Scattering amplitudes in non-Fermi liquid systems

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I. INTRODUCTION

The purpose of this paper is to address the calculation of scattering amplitudes in systems exhibiting non-Fermi liquid physics. In such systems, scattering processes involving electrons (more precisely, the quasiparticles adiabatically connected with the bare electrons in the Landau Fermi liquid theory, with charge $e$ and spin 1/2) can give rise to the appearance of other kinds of quasiparticles. A characteristic example of this phenomenon (to be discussed in details below) occurs when electrons are injected from a Fermi liquid into a Luttinger liquid, and can give rise to various combinations of Laughlin quasiparticles and quasiholes [1]. Another example occurs in the spin 1/2 two channel Kondo problem: at the low energy fixed point, the amplitude for an electron to be scattered into an electron (or any finite combination of electrons and holes) is actually zero, hence giving rise to 'unitarity puzzles', which are solved by recognizing that the scattering takes place entirely into the spinors sector [2].

The determination of these amplitudes is conceptually a very important question, as one may argue they are the essence of non Fermi liquids physics. Technically however, it is a very difficult one, which has been tackled mostly in perturbation theory around the high energy fixed point [3]. The purpose of this paper is to present some non perturbative attempts at calculating these amplitudes in a case where the problem is integrable - and especially simple.

Our initial attempts were based on the technique of form-factors [4], and this paper builds on earlier works in this area [5, 6, 7]. As we will see, the answer to the physical questions of interest involves the determination of correlation functions in a massless field theory with a boundary interaction. In the paper [6] it was shown that such correlation functions could be determined with remarkable accuracy in the case of operators with no anomalous dimension, like the current and the stress energy tensor. In the paper [7] preliminary attempts were made to determine similar correlators in the case of vertex operators with non trivial, anomalous dimensions. From a technical point of view, the present work is a follow up of that paper, and will use similar regularization techniques. It will turn out however that for the specific questions asked - for instance, what is the probability that an electron from a Fermi liquid tunnels into a (chiral) Luttinger liquid, and can give rise to various combinations of Laughlin quasiparticles and quasiholes [1]. The determination of these amplitudes is of direct relevance to the case of a contact between a Fermi liquid and a Luttinger liquid [9].

The two main applications are presented: the problem of an impurity in a one dimensional Luttinger liquid, and the problem of a point contact between a Fermi liquid and a Luttinger liquid. These two problems are related with the boundary sine-Gordon model, with which we assume the reader to be familiar. Mostly for technical reasons, we restrict in this paper to the case $g = \frac{1}{2}$. Although this is a “free fermion theory”, the amplitudes of interest involve operators which are not local in terms of the fermions, and whose correlators are already quite complex. The physical features are not expected to depend strongly on $g$, so we expect our results to shed light on the general situation; moreover, $g = \frac{1}{2}$ is of direct relevance to the case of a contact between a Fermi liquid and a $\nu = \frac{1}{3}$ edge [8].

In the case of the impurity in a Luttinger liquid, or, equivalently, tunneling between two identical chiral Luttinger liquids (section 2), we discuss both the reflection and transmission amplitude for a single Laughlin quasiparticle. In the case of tunneling between a chiral Luttinger liquid and a Fermi liquid (section 3), we discuss the $e \rightarrow e$, $lqp \rightarrow lqp$ and the $e \rightarrow e_{edge}$ amplitudes. Here $e_{edge}$ denotes the edge electron, which can also be considered as the bound state of three Laughlin quasiparticles.

Exact expressions are obtained in all cases. For the ease of the reader, we gather these expressions here. In the
FIG. 1: Tunneling between two identical chiral Luttinger liquids. What we call reflection corresponds to a lqp remaining within the same liquid, and what we call transmission amplitude corresponds to a lqp jumping from one liquid to the other.

first situation we have

\[ G_T(p,p_B) \propto p^{-1/2} \left( 1 + \frac{ip}{p_B} \right)^{-1/2} \]  
Transmission of lqp

\[ G_R(p,p_B) = 0, \quad p_B \neq 0 \]  
Reflection of lqp  

while in the second

\[ G(p/p_B) = F\left(\frac{1}{2}, \frac{1}{2}; 1; -i \frac{p}{p_B}\right) \]  
\[ 1e \rightarrow 1e \]

\[ G(p,p_B) \propto p^{-2/3} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}; -i \frac{p}{p_B}\right) \]  
\[ 1lqp \rightarrow 1lqp \]

\[ G(p,p_B) \propto p^{1/4} \left[ \frac{\Gamma(1/2) \Gamma(3/4)}{\Gamma(5/4)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{4}; -i \frac{p}{p_B}\right) + \frac{\Gamma(1/2) \Gamma(-1/4)}{\Gamma(1/4)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}; -i \frac{p}{p_B}\right) \right] \]  
\[ 1e \rightarrow 1e_{edge} \]

Some technical issues are relegated to the appendices.

To conclude this introduction, we would like to point out that the mere questions we want to answer have to be defined very carefully, a task we do not completely undertake in this paper. Indeed, in the massless theories we are discussing, there are ambiguities in the set of scattering states one uses as a “physical basis”. For the one dimensional Fermi liquid, the choice of electrons and holes is canonical, but for general Luttinger liquids, this is not so. For instance, for the \( g = \frac{1}{3} \) case, a choice based on charges \( \frac{1}{3} \) (Laughlin quasihole) and \( -1 \) (edge electron) quasians has been studied in great details \[10\]. Other choices, eg using charges \( \frac{1}{3} \) and \( -\frac{1}{3} \) (Laughlin quasiparticle) quasians, should be possible as well: none appears more fundamental than the other, but the choice has to be carefully specified whenever one discusses, for instance, unitarity. Whatever the choice, the vertex operators (in the bosonized formalism) one uses as “creation/annihilation” operators of these quasians are not one particle operators in the usual sense. They do not simply add or subtract a single quasi-particle from a many particle state, but have a more complicated action due to their non trivial commutators. This can make the mere definition of a scattering amplitude quite non trivial \[10\].

II. IMPURITY WITHIN A LUTTINGER LIQUID

A. Generalities

We consider a problem with two identical chiral Luttinger liquids and tunneling between them induced by a gate voltage - the set up is represented on the figure \[11\].
The Lagrangian reads

$$L = \frac{1}{4\pi} \partial_x \phi_1(\partial_t - \partial_x)\phi_1 + \frac{1}{4\pi} \partial_x \phi_2(\partial_t - \partial_x)\phi_2 + v\delta(x) \cos[\sqrt{2} (\phi_1 - \phi_2)],$$

where the fields $\phi_1, \phi_2$ are associated with each of the two Luttinger liquids. By forming linear combinations

$$\phi' = \frac{\phi_1 + \phi_2}{\sqrt{2}}, \quad \phi = \frac{\phi_1 - \phi_2}{\sqrt{2}}$$

the field $\phi'$ decouples, while the dynamics of the $\phi$ field is determined by the Lagrangian

$$L = \frac{1}{4\pi} \partial_x \phi(\partial_t - \partial_x)\phi + v\delta(x) \cos(\sqrt{2}\nu\phi).$$

The coupling constant $v$ describes the strength of the tunneling process. The normalization is such that one has the propagator ($T$ the time ordering operator)

$$\langle T \phi(x, t)\phi(0, 0) \rangle = -\frac{1}{4\pi} \ln(x - t + i\epsilon \text{sign}(t)).$$

It is then convenient to fold this problem onto the half line $x \in [-\infty, 0]$, with action (here $\Phi = \phi(x, t) + \phi(-x, t)$ is a non chiral boson)

$$A = \frac{1}{8\pi} \int_{-\infty}^{0} dx \left[ (\partial_x \Phi)^2 - (\partial_t \Phi)^2 \right] + v \cos[\sqrt{2}\nu\Phi(0)].$$

The boundary interaction has physical dimension $d = \nu$. In the following we consider mostly the case $\nu = \frac{1}{2}$.

The two amplitudes we wish to consider are what we call the reflection and transmission amplitudes for Laughlin quasiparticles. The creation operator of a Laughlin quasiparticle in the first Luttinger liquid is proportional to $e^{-i\sqrt{2}\nu\phi_1}$, while the transmission amplitude (corresponding to a Laughlin quasiparticle jumping from one chiral Luttinger liquid to the other) reads

$$G_R = \langle T e^{i\sqrt{2}\nu(\phi_2(x_2 > 0, t_2) - \phi_1(x_1 < 0, t_1))} \rangle$$

while the transmission amplitude (corresponding to a Laughlin quasiparticle jumping from one chiral Luttinger liquid to the other) reads

$$G_T = \langle T e^{i\sqrt{2}\nu(\phi_2(x_2 > 0, t_2) + \phi_1(x_1 < 0, t_1))} \rangle$$

The main technical difference between these two amplitudes is the fact that exponentials of the field $\phi$ have opposite signs in the reflection case and identical signs in the transmission case. Restricting to $\nu = \frac{1}{2}$, the two objects we have to determine are thus

$$g_{R,T}(1, 2) = \langle T \exp(\pm i\phi(x_2 > 0, t_2)/2) \exp(-i\phi(x_1 < 0, t_1)/2) \rangle.$$

To proceed, we first use the mapping to the folded theory on the half line. A field at $x < 0$ is an ‘in’, right moving field, while a field at $x > 0$ now becomes an ‘out’, left moving field. The new correlators are then given by

$$g_{R,T}(1, 2) = \langle T \exp(\pm i\phi_L(-x_2, t_2)/2) \exp(-i\phi_R(x_1, t_1)/2) \rangle.$$

The general approach to compute such correlators is the massless form-factors technique. Let us therefore recall some basic facts about integrable boundary field theories. The Hamiltonian is diagonalized using a quasiparticle basis, with left and right massless particles $e = \pm p = e^\beta$, carrying a label $\pm$ corresponding to soliton or antisoliton. The bulk scattering of these quasiparticles is $S = -1$. Their boundary scattering is

$$R^+ = R^- = \frac{e^{\beta - \beta_B}}{e^{\beta - \beta_B} + i}, \quad R^+_+ = R^-_- = \frac{i}{e^{\beta - \beta_B} + i}$$

where $e^{\beta_B} \propto v^2$. 
The form-factors of the vertex operators can in principle be obtained by taking the massless limit of the formulas in [11]. Subtleties arise in taking this limit however. Let us first recall the massive result. Restricting to $|\alpha| < 1$, the basic formula in [11] reads, setting $U_\alpha = e^{i\alpha \Phi}$,

$$\frac{\langle 0 | U_\alpha(x, t) | \beta_{2n}, \ldots, \beta_1 \rangle^{+ \ldots + - \ldots -}}{\langle 0 | U_\alpha | 0 \rangle} = (-)^n(n-1)/2 \left( \sin \frac{\pi \alpha}{i} \right)^n \times \exp \left( -i \sum_{k=1}^{2n} \left( t \cosh \theta_k - x \sinh \theta_k \right) \right) \times \prod_{1 \leq k < j \leq n} \sin \frac{\theta_k - \theta_j}{2} \sinh \frac{\theta_{n-k} - \theta_{n-j}}{2}$$

The vacuum expectation value itself is obtained using techniques of [12] and is of the form $\langle 0 | U_\alpha | 0 \rangle = (am)^2 C(\alpha)$ where $a$ is a UV cut-off and $C(\alpha)$ a numerical constant whose expression is exactly known. Notice that the form factors are neutral, that is they are non zero only acting on states whose total solitonic charge vanishes exactly.

The massless limit is obtained by letting rapidities go to infinity setting $\theta = \pm \theta_0 \pm \beta$ where $\theta_0 \to \infty$, and letting at the same time the mass go to zero with $mc^2 \theta_0 \to 2$ (the choice of this constant is non universal, and can be absorbed in a global shift of the rapidities). For example, let us take the massless limit of the form-factor by choosing $\alpha$- particles among $n$ particles with charge $+1$ in the R movers, and $q$ to be $L$ movers (with $p + q = n$) and similarly, $u$ particles among the $n$ particles with charge $-1$ to be $R$ movers, $v$ particles to be $L$ movers (with $u + v = n$). As $\theta_0 \to \infty$, the leading behaviour of the ratio (13) behaves as

$$\exp[\alpha \theta_0 (p - q - u + v)] \left\langle \prod_{k,j} e^{\theta_0 p_k e^{\theta_0 q_j}} \right\rangle$$

This can be rewritten as $e^{\theta_0 [|\alpha|^2 - (\alpha + q - v)^2]}$. Taking the prefactor $\langle 0 | U_\alpha | 0 \rangle$ into account, it thus follows that the form-factors all vanish in the massless limit. The calculation of correlators is thus bound to be a complex task: what happens is that, while the form-factors vanish in that limit, the integrals over rapidities all diverge at low energy due to the presence of infinitely many soft modes, an “IR catastrophe”, making the contributions to correlators of undetermined form $0 \times \infty$. The proper way to proceed would be to keep track of this divergence carefully, by making calculations at finite mass, and then considering the limit when the length scales get much smaller than the inverse of this mass. It is not clear however how many form-factors would have to be taken into account in order to get accurate results - probably an infinity - and this procedure clearly is not efficient.

The procedure we will use below bypasses this difficulty, and consists in working directly within the massless theory, essentially by considering ratios of correlators, which gets rid of the overall normalization problem. It turns out that the ratios (14) have different behaviours depending on $\alpha$ and the balance of charges. In particular, the limit of the ratio is always finite when the charges are individually balanced in the left and right sector. In that case, the ratio factorizes, and one can individually define L and R versions as (setting $V^R_{\alpha, \text{resp.} L} = e^{i\alpha \Phi_{R, L}}$)

$$\frac{\langle 0 | V^R_{\alpha, \text{resp.} L}(x, t) | \beta_{2n}, \ldots, \beta_1 \rangle^{+ \ldots + - \ldots -}}{\langle 0 | V_{\alpha} | 0 \rangle} = (-)^n(n-1)/2 \left( \sin \frac{\pi \alpha}{i} \right)^n \times \exp \left( -i \sum_{k=1}^{2n} \sum_{j=1}^{n} e^{\beta_j} \right) \times \prod_{1 \leq k < j \leq n} \sin \frac{\beta_k - \beta_j}{2} \sinh \frac{\beta_{n-k} - \beta_{n-j}}{2}$$

For $|\alpha| < \frac{1}{2}$, this is in fact the only case where the ratio has a non zero limit. Our procedure thus seems well defined when $|\alpha| < \frac{1}{2}$. When $\frac{1}{2} < |\alpha| < 1$, some of the ratios (14) diverge in the massless limit, and handling out the form-factors series is thus probably even more difficult. We have not tackled this case, but see the appendix for analytical techniques.

The case $|\alpha| = \frac{1}{2}$ is special. In addition to the case where L and R sectors are separately neutral, the ratio (14) also has a finite limit when the balance of charges is $+1$ in the R sector and $-1$ in the L sector. For instance

$$\frac{\langle 0 | V^{L}_{1/2} V^{R}_{1/2}(\beta_{2n}, \beta_1)^{+ \ldots + - \ldots -}}{\langle 0 | V_{1/2} | 0 \rangle} = \frac{2}{i} \exp \left( -i(t-\alpha) e^{\beta_2} - i(t+\alpha) e^{\beta_1} \right)$$

while the same expression for $V^{-1/2}$ vanishes in the massless limit.
Finally, the case $|\alpha| = 1$ is also special, even in the massive case. This is because $\sin \alpha \pi$ vanishes, while the constant $C(\alpha)$ diverges. The best way to understand what happens is to make connection with fields with $\alpha = 1$ and fermion operators in a free fermion theory. In the massless limit, one finds that $V_{R,L}^{\pm 1} \propto \psi_{R,L}^{\dagger} (\text{resp. } \psi_{R,L}^{\dagger})$. This is discussed in more details below.

The normalization of multiparticle states in (14) is $<\beta | \beta'> = 2\pi \delta(\beta - \beta')$. We denote below the creation operators $Z_{L,R}^{\dagger}(\beta)$ such that $\langle \beta | Z_{L,R}^{\dagger}(\beta) | 0 \rangle = \langle \beta > ^{a}_{R,L}$. Left and right creation/annihilation operators anticommute.

The boundary interaction can be conveniently handled by moving to an Euclidian description, such that the impurity interaction becomes a boundary interaction, the boundary lying along the imaginary time axis. One then makes a Wick rotation, and quantizes the theory along the boundary axis instead. The coordinate along the boundary, $y$, is related with the time by $y = it$. The whole effect of the boundary interaction is then taken into account by the boundary state $\hat{13}$, whose expression in the integrable basis is extremely simple:

$$|B> = \exp \left[ \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} K^{ab}(\beta_B - \beta) Z_{L}^{ab}(\beta) Z_{R}^{b}(\beta) \right]$$

Here, the $K$ matrix is simply related with the $R$ matrix and given by

$$K^{++} = K^{--} = \frac{i e^{\beta - \beta_B}}{e^{\beta - \beta_B} + 1}, \quad K^{+-} = K^{-+} = \frac{-i}{e^{\beta - \beta_B} + 1} \quad (17)$$

**B. The transmission amplitude**

A calculation involving up to two particles gives then the following expression, where we neglected some trivial normalization factors (note that the $L$ particles are only annihilated by the $L$ field and the $R$ ones by the $R$ field)

$$g_T(1,2) = \left( \exp(-i \phi_L(-x_2, y_2)/2) \exp(-i \phi_R(x_1, y_1)/2) \right)$$

$$\propto \left\{ 1 + \int \frac{d\beta}{2\pi} \exp\left[ e^\beta(x_1 - x_2 + i(y_1 - y_2)) \right] \frac{2}{1 + e^\beta} \right.$$\n
$$\left. + \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} \exp\left[ e^{\beta_1}(x_1 - x_2 + i(y_1 - y_2)) \right] \right.$$\n
$$\left. \times \left[ \frac{1}{1 + e^{\beta_1}} \frac{1}{1 + e^{\beta_2}} - \frac{e^{\beta_1}}{1 + e^{\beta_1}} - \frac{e^{\beta_2}}{1 + e^{\beta_2}} \right] \right\} \quad (18)$$

where $\hat{\beta} = \beta - \beta_B$, and the dots denote processes involving three and more particles in the expansion of the boundary state. As mentioned briefly earlier, this integral is divergent at low energies $\beta_1 \to -\infty$ due to the proliferation of low energy particles, and the expression as it stands cannot be used. Integrals converge at high energy since $x_2 > 0$, $x_1 < 0$.

To proceed, we use a technique $\hat{4}$ experimented in this context in $\hat{7}$. Denote $p_B = e^{\beta_B}$ and observe that the low energy limit is also the limit $v \to \infty(p_B \to \infty)$, where the field $\phi$ sees Dirichlet boundary conditions, and thus we do know that the correlator should go as

$$g_T(1,2)_{p_B=\infty} \propto \frac{1}{(x_2 - x_1 + i(y_2 - y_1))^{1/4}}$$

corresponding to a totally diagonal scattering. The trick to obtain finite results will thus be to consider not $g_T$ itself but the ratio $g_T(1,2)/g_T(1,2)_{p_B=\infty}$. Writing to first non trivial order $g_T(1,2) = 1 + I_1 + \ldots$, we thus consider the new approximation to the correlator

$$g_T(1,2) \approx \frac{1}{(x_2 - x_1 + i(y_2 - y_1))^{1/4}} \left[ 1 + I_1(p_B) + \ldots \right] \quad (19)$$

and expand the ratio to first order (ie, two $\beta$ integrals)

$$g_T(1,2) = \frac{1}{(x_2 - x_1 + i(y_2 - y_1))^{1/4}} \left[ 1 + I_1(p_B) - I_1(\infty) + \ldots \right] \quad (20)$$

This procedure happens to give finite integrals, as is easily seen at this order, since the subtraction of $I_1(\infty)$ cancels the divergence of the first integral in (18). It follows that, to leading order, the full correlator of interest reads, after the Wick rotation

$$G_T(1,2) = \frac{1}{(x_2 - x_1 + i(y_2 - y_1))^{1/2}} \left\{ 1 - \frac{1}{\pi} \int_{0}^{\infty} \frac{du}{p_B^2 + u^2} e^{-u(x_2 - x_1 + i(y_2 - y_1))} + \ldots \right\} \quad (21)$$
After a few simple manipulations using the Laplace representation of the square root prefactor, we end up with

$$G_T(p, p_B) = \frac{1}{\sqrt{\pi p}} \left[ 1 - \frac{1}{\pi} \int_0^{p/p_B} \frac{du}{1 + u} \left( 1 - \frac{p_B}{p} u \right)^{-1/2} + \ldots \right]$$

(22)

In the low energy (that is, low energy of incident particles) limit $p/p_B \to 0$, corresponding to total transmission, $G_T \propto p^{-1/2}(1 + \text{cst} \frac{p_B}{p})$, as expected. In the high energy limit $p/p_B \to \infty$, $G_T \propto p^{1/2}(1 + \text{cst} \ln(p_B/p))$. This latter result clearly does not make much sense, as one expects $G_T$ to vanish in this limit.

A higher order calculation is perfectly possible, and would extend the degree of validity of the form-factors result towards high energies, while giving an essentially exact result at low energies. For any finite order however, the form-factors result seem to always become unreliable at sufficiently high energy, and behave like the first order in that limit. Rather than pursuing this direction (which might be the only one available for other values of $g$), it is better to observe that the quantity $G_T(1, 2)$ can be evaluated in closed form. This follows from a chain or arguments relating it to a very elegant calculation of the one point function of the spin operator in the Ising model with a boundary magnetic field carried out by Chatterjee and Zamolodchikov [15]. The same kind of argument was used in the study of the Friedel oscillations in [14]. There, it was shown that that

$$g_T = \langle \exp \left[ \pm \frac{i}{2} (\phi_L + \phi_R)(x, y) \right] \rangle = \langle \sigma(x) \rangle_h \langle \sigma(x) \rangle_\infty$$

(23)

where $\langle \sigma \rangle_h$ denotes the one point function of the spin operator in the Ising model with boundary field $h \propto v$ (the right hand side does not depend on $y$ when the two vertex operators are inserted at the same (imaginary) time), and $x$ is the distance from the boundary. This is similar but different from the usual result in the bulk [16, 17].

$$\langle \sigma(z, \bar{z}) \sigma(0, 0) \rangle^2 = \langle 0 | \sin(\Phi(z, \bar{z})/2) \sin(\Phi(0))/2 \rangle | 0 \rangle$$

$$\langle \mu(z, \bar{z}) \mu(0, 0) \rangle^2 = \langle 0 | \cos(\Phi(z, \bar{z})/2) \cos(\Phi(0))/2 \rangle | 0 \rangle .$$

(24)

This result immediately extends to the case where the right and the left field are inserted at different points, since the correlator depends on $x_2 - x_1 + i(y_2 - y_1)$ only:

$$g_T = \langle \exp(-i\phi_L(-x_2, y_2)/2) \exp(-i\phi_R(x_1, y_1)/2) \rangle$$

$$= \langle \sigma((x_2 - x_1 + i(y_2 - y_1))/2) \rangle_h \langle \sigma((x_2 - x_1 + i(y_2 - y_1))/2) \rangle_\infty$$

(25)

Using form factors, one can expand the one point function of the Ising spin operator as

$$\langle \sigma(x) \rangle_h = \sum_{n=0}^\infty \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ \frac{d\beta_i}{2\pi} \tanh \frac{\beta_i}{2} e^{2\pi s_i} \right\} \prod_{i<j} \left( \tanh \frac{\beta_i - \beta_j}{2} \right)^2$$

(26)

and one can also check the factorization (23) of $g$ directly using the form-factors expansion as described in [14]. The case of infinite magnetic field corresponds to $h \to \infty$ and since $p_B = e^{\beta_B} = 4\pi h^2$ it also corresponds to $p_B \to \infty$ in this formula. In that case the result is known to be $1/x^{1/3}$ from boundary conformal field theory. Recall on the other hand the result that

$$\langle \sigma(x) \rangle_h \propto x^{3/8} \sqrt{\frac{p_B}{4\pi}} e^{p_B x} K_0(p_B x)$$

(27)

one thus find that

$$G_T(1, 2) \propto \sqrt{\frac{p_B}{4\pi}} e^{p_B x/2} K_0(p_B x/2), \quad x = x_2 - x_1 + i(y_2 - y_1)$$

(28)

where the factor $1/2$ came up in the argument of the Bessel function since in the initial Ising case, our variable $x$ would be twice the value in [27]. We can now use the integral representation

$$K_0(z) = \frac{e^{-z}}{\sqrt{2}} \int_0^\infty \frac{e^{-zt}}{\sqrt{t}} \left( 1 + \frac{t}{2} \right)^{-1/2} dt$$

to write

$$G_T(1, 2) \propto \int_0^\infty e^{-xp^{-1/2}} \left( 1 + \frac{p}{p_B} \right)^{-1/2} dp$$

(29)
giving rise to the stunningly simple momentum dependent transmission amplitude

\[ G_T(p) \propto p^{-1/2} \left(1 + \frac{p}{p_B}\right)^{-1/2}. \]  \hspace{1cm} (30)

Finally, we perform a shift (in the rapidity variable) of the integration contour to get back to the theory in real time, quantized with \( x \) as the space axis, and the interaction at the origin, which amounts to \( p \to ip \):

\[ G_T(p) \propto p^{-1/2} \left(1 + \frac{ip}{p_B}\right)^{-1/2}. \]  \hspace{1cm} (31)

Note that this result could also be obtained by quantizing the theory in the crossed channel. In that case, the boundary interaction appears because asymptotic states are of the form \( |\beta_R + R(\beta)|\beta_L\), and leads to expressions with the shifted contour, and \( K \) replaced by \( R \). This correspondence is discussed in details in [17].

At low energy, \( G_T \propto p^{-1/2}(1 + \text{cst} \frac{L}{p_B}) \) and at high energies \( G_T \propto p^{-1/2} \sqrt{\frac{2B}{p}} \): this is the expected behaviour. Indeed, in the limit of small \( p_B \) the perturbative expansion should be linear in \( v \propto \sqrt{p_B} \), and one thus expects the expansion to go as \( \sqrt{p_B/p} \). On the other hand, at large \( p_B \), one approaches the IR fixed point (total transmission), along the operators \( \cos \tilde{\Phi} \) of dimension \( d = 2 \), and \( \langle \tilde{\Phi} \rangle \) (here, \( \tilde{\Phi} = \phi_R - \phi_L \) is the dual of the field \( \phi \), and the theory is defined on the half line). The coupling of these operators must therefore have dimension \(-1\), and thus be proportional to \( v^{-2} \) since \( v \) has dimension \( 1/2 \). The perturbative series starts linearly in the coupling, that is linear in \( p/p_B \), as we just found.

It is interesting to recall that the Bessel function \( K_0 \) only admits an asymptotic expansion in powers of the argument

\[ e^z K_0(z) \approx \sqrt{\frac{\pi}{2z}} \left[ \sum_{k=0}^{\infty} \frac{1}{(2z)^k} \frac{\Gamma(k+1/2)}{k!(k+1/2)} \right]. \]

In our problem, this corresponds to the expansion near the IR fixed point (low energy of incident particles, large \( p_B \)), which is thus only asymptotic for the correlator (in fact, the expansion of \( K_0 \) is good only up to exponential terms \( e^{-z} \). Here this corresponds to terms \( e^{-z/(1/v^2)} \), which are non perturbative in term of the IR coupling constant \( 1/v^2 \). All these difficulties disappear however in the momentum dependent amplitude, whose expansion is now convergent, thanks to the Laplace transform.

Let us get back finally to the expression for \( \langle \sigma(x) \rangle_h \). The leading order in \( h \) is proportional to \( h x^{3/8} \ln(xh^2) \). In perturbation theory meanwhile, the leading order involves an integral over the boundary (in the Euclidian formalism) \( \int dy (x^2 + y^2)^{-7/8} \) which must be regularized by the introduction of an IR cut-off. The situation is similar to the massive bulk Ising model or the bulk sine-Gordon model at \( \beta^2 = 4\pi \) (see [16] and references therein). An additional renormalization is required, leading to the replacement of the cut-off by the inverse of the mass - here, by the inverse coupling constant \( 1/p_B \). Alternatively, one could instead consider the correlator at finite temperature. \( 1/T \) then acts as an IR cut-off, and the scaling limit of \( \langle \sigma \rangle \) is entirely well defined. Taking the limit \( T \to 0 \) then gives the result mentioned above [11].

C. The reflection amplitude

The reflection amplitude is an intriguing quantity. Let us think of it first by considering the vicinity of the IR fixed point, which is is approached along the operator \( \cos \tilde{\Phi} \). In an IR perturbation theory, the average of the quantity \( \exp[i(\phi_R - \phi_L)/2] \) will vanish to every order, because a charge neutral exponential can never be formed. Therefore, if the reflection amplitude happened to be non zero, if would be entirely due to non perturbative effects! In fact, we think this amplitude is exactly zero. This can be argued more solidly by considering the UV point of view. The form of the IR perturbation shows that, right at the UV fixed point, the field at the boundary obeys \( \phi_R - \phi_L(0) = (2n+1)\pi \), \( n \) an integer. It follows that the quantity \( \exp[i(\phi_R - \phi_L)/2] \) changes sign when one goes from one well of the boundary potential to the next. Since a proper conformal invariant boundary condition involves a sum over all wells, the average \( \langle \exp[i(\phi_R - \phi_L)/2] \rangle \) has to vanish right at the fixed point. This observation can be made more formal by appealing to the formalism of conformal boundary conditions (notations follow those of [16]; \( \alpha_n \) are the usual bosonic modes; \( |w, k \rangle \) the zero modes). The Dirichlet state is given by consideration of the IR fixed point (radius \( r = \frac{1}{\sqrt{p}} \))

\[ |B_D(\Phi_0)\rangle = \frac{1}{\sqrt{2r\sqrt{\pi}}} \sum_{k=-\infty}^{\infty} e^{-ik\Phi_0/r} \exp \left[ -\sum_{n=1}^{\infty} \frac{\alpha-n\alpha-n}{n} \right] |0, k\rangle \]  \hspace{1cm} (32)
The Neumann state compatible with this is

$$\left| B_N(\Phi_0) \right| = \frac{1}{2} \sqrt{2r \sqrt{\pi}} \sum_{w=-\infty}^{\infty} e^{-2i\pi r w \Phi_0} \exp \left[ \sum_{n=1}^{\infty} \frac{\alpha-n}{n-\alpha} \frac{\alpha-n}{n} \right] [w, 0]$$

(33)

While the operator \(\exp [i(\phi_L - \phi_R)]\) corresponds to \(w = 1\), and hence can have a non trivial one point function with \(N\) boundary conditions, the operator \(\exp [i(\phi_L - \phi_R)/2]\) would correspond formally to \(w = 1/2\). The corresponding state is not in the decomposition of the \(N\) boundary state, and therefore the one point function has to vanish.

In a UV perturbation theory, the same sum over wells (which is compatible with the perturbation) will give rise to the same result. (Notice how the situation for the quantity \(\exp [i(\phi_R + \phi_L)/2]\) is entirely different. In the UV, because the perturbation is \(\cos \Phi/2 = \cos(\phi_R + \phi_L)/2\), neutral combinations can be formed, and one finds non trivial perturbative contributions. In the IR, the field obeys now \((\phi_R + \phi_L)/2 = (2n + 1)\pi\), and thus the exponential of interest is invariant when one goes from one well to the next.)

Finally, the vanishing can also be understood in terms of form-factors. In the massive theory, the operator \(\exp [i(\phi_L - \phi_R)/2]\) is not neutral, and increases the \(L - R\) charge by one unit. The non zero form-factors therefore will have to involve an odd number of particles. Since the boundary state is a superposition of an even number of particles only, the one point function of this operator must vanish identically. A close correspondence with the disorder operator in the Ising model is discussed in the appendix.

Of course, right at the UV fixed point, one can “by hand” restrict to one well of the potential, and find a non zero amplitude, of the expected form \(G_{\epsilon}(p) \propto p^{-1/2}\). It is tempting to force this amplitude to be non zero away from the fixed point as well.

To do so, one can again use the form-factors approach. As explained in the appendix, the non zero value right at the UV fixed point can be obtained by adding to the boundary state a particle of vanishing energy (and momentum). It is reasonable to try to define a non zero amplitude near the UV fixed point by generalizing the addition of this particle at finite value of the coupling. This leads to the result

$$g_R = \langle \exp(i\phi_L(-x_2, y_2)/2) \exp(-i\phi_R(x_1, y_1)/2) \rangle = \frac{1}{(x_2 - x_1 + i(y_2 - y_1))^1/8} \langle \sigma(x_2 - x_1 + i(y_2 - y_1)/2) \rangle_\delta$$

(34)

where the second quantity has the form factor expansion

$$\langle \sigma(x) \rangle_\delta = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left[ \frac{d\beta_i}{2\pi} \tanh \frac{\beta_B - \beta_i}{2} e^{-2x e^{\beta_i}} \right] \prod_{i<j} \left( \tanh \frac{\beta_i - \beta_j}{2} \right)^2$$

(35)

and the minus signs have been generated by the introduction of the particle at zero energy.

Of course, this expansion is once again divergent. Moreover, in contrast with the case of the transmission amplitude, we do not even know the value in the IR, so we cannot divide by it to form a finite quantity. We can however observe that the logarithmic derivative of the correlator is regularizable by the same procedure previously used for \(\langle \sigma \rangle_\delta\). The first two terms read for instance

$$\frac{\partial}{\partial \beta_B} \ln \langle \sigma(x, p_B) \rangle_\delta = - \int_0^{\infty} du \frac{p_B}{\pi (p_B + u)^2} e^{-2x u}$$

$$- \int_0^{\infty} du \frac{u}{\pi^2} (u + v) p_B \left( \frac{p_B^2 - uv}{(p_B + u)^2} - 1 \right) e^{-2x (u + v)} + \ldots$$

(36)

(where recall \(p_B = e^{\beta_B}\)). The integrals converge now at the origin, including when \(p_B = 0\). Call the series in the right hand side \(\Sigma(xp_B)\). We can thus try to write

$$\langle \sigma(x, p_B) \rangle_\delta = \frac{1}{x^{1/8}} \exp \left[ \int_0^{p_B} \frac{dp_B'}{p_B'} \Sigma(x p_B') \right]$$

Although right at \(p_B = 0\) this expression reproduces the right result, there is a problem as soon as \(p_B \neq 0\). This is because, at every order we have explored, \(\Sigma(z)\) goes to a finite quantity as \(z \to 0\), and thus the integral diverges at the origin! This can be studied in more details at the first order already, where the exponential reads, for small \(p_B\),

$$\exp \left[ - \int_0^{p_B} \frac{dp_B'}{p_B'} \frac{1}{\pi} \right]$$

To make the integral finite we have to introduce a new (IR) cut-off \(\epsilon\), with dimension of temperature = length\(^{-1}\), so we get

$$\exp \left( -\frac{1}{\pi} \ln(p_B/\epsilon) \right) = (\epsilon/p_B)^{1/\pi}$$
If we now let the cut-off go to zero, we see that the amplitude again vanishes. A similar result was noticed independently by N. Sandler (private communication). This result is confirmed by higher order calculations. We think it is a consequence of the ill defined nature of the reflection amplitude within field theory, and can be interpreted by saying that this reflection amplitude is identically zero for any \( p_B \neq 0 \), in the scaling limit. The only way to define this amplitude is to keep an additional (IR) cut-off, and then, to first order in the form-factors expansion, one finds

\[
\langle \sigma(x) \rangle_h = \frac{1}{x^{1/8}} \left( \frac{\epsilon}{p_B} \right)^{1/\pi} \exp \left[ \frac{1}{\pi} \int_0^\infty \frac{dz}{z(1+z)} \left( e^{-2zxp_B} - 1 \right) \right]
\]  

(37)

It is in fact possible to obtain the exact value of the one point function by using more results about the Ising model, as detailed in the appendix:

\[
\langle \sigma(x) \rangle_h = \frac{1}{x^{1/8}} (\epsilon x)^{1/2} \sqrt{2p_Bx} (2\Psi' - \Psi)
\]  

(38)

where again \( \Psi = \Psi(1/2, 1; z) = \frac{e^{\pi z/2}}{\sqrt{\pi}} K_0(z/2) \), \( z = 2p_Bx \). From this the result follows that \( \langle \sigma \rangle_h \propto e^{1/2} x^{3/8} \) at large values of \( p_B x \).

To each order we have explored, the value of the exponent at the origin was \( \frac{1}{2} \), that is, every contribution but the first one seemed to vanish exactly. Meanwhile, the exact value expected from the formula \( (38) \) is \( \frac{3}{2} \). Most likely what happens is that the form-factors expansion does not converge uniformly at the origin - an identical feature takes place in the case of the ordinary one point function \( \langle \sigma \rangle_h \).

Going back now to the physical amplitude we find

\[
G_R \propto \frac{1}{x^{1/2}} (\epsilon x)^{1/2} \sqrt{2p_Bx} (2\Psi' - \Psi)
\]  

(39)

In particular it follows that \( G_R(p) \propto p^{-1/2} (\epsilon/p)^{1/2} \) in the IR limit.

D. Other amplitudes

What one may call an “Andreev process” meanwhile would correspond to a Laughlin quasiparticle incident on the point contact giving rise to an electron in the same chiral Luttinger liquid, and a Laughlin quasihole in the other chiral Luttinger liquid. The amplitude for this process is

\[
G_A = \left\langle e^{-i\phi_2(x_2>0,t_2)/\sqrt{2}} e^{i\phi_1(x_2>0,t_2)} e^{-i\phi_1(x_1<0,t_1)/\sqrt{2}} \right\rangle
\]  

(40)

The interacting part of this correlator, after using even odd combinations and folding, becomes

\[
g_A = \left\langle e^{i\phi_L/2} e^{-i\phi_R/2} e^{i\phi_L} \right\rangle
\]  

(41)

and the argument has the form \( \frac{\phi_L + \phi_R}{2} + \phi_L \). This time, the exponential is invariant when going from one well to the next, so the average will be non trivial. The amplitude vanishes both in the UV and IR limits, but has non vanishing perturbative contributions in the vicinity of either fixed point. All indications are that this amplitude is well defined in the scaling limit.

In fact, the momentum dependent amplitude \( G_A(p,q) \) (with say \( q \) the amplitude of the outgoing electron) has the same dimension as the would be amplitude for the reflection process \((39)\). It is likely that the regularized reflection amplitude we have defined previously \((39)\) coincides with the limit of \( G_A(p,q) \) as \( q \) goes to zero (and then \( q \) has to be identified with \( \epsilon \)).

Note finally that by folding one can map the geometry of two \( R \) moving Luttinger liquids to the geometry of a single non chiral Luttinger liquid. This can actually be done in several ways: the most natural is illustrated on the figure \( \ref{fig:2} \). In this case, what we called before transmission amplitude becomes a backscattering amplitude (small at small \( v \)) and what we called reflection amplitude becomes transmission amplitude (small at large \( v \)).

III. POINT CONTACT BETWEEN A FERMI AND A LUTTINGER LIQUID

A. Set-up

A fundamental component in the experimental study of Luttinger liquids is the presence of a point contact with a three dimensional reservoir (in fact, a semi infinite three dimensional reservoir, whose boundary is a plane \( \{ \} \). The
analysis of the situation is somewhat similar to the Kondo problem. Electrons in the 3d reservoir can be organized not into plane waves but into spherical waves, and one finds that only the \( s = 0 \) wave interacts with the Luttinger liquid (the Kondo impurity in the other case). Because there is a single mode, it can be considered as well as arising from a one dimensional conductor, and one can reduce the problem to the one of a point contact between a 1D Fermi liquid and a 1D Luttinger liquid. The situation then looks as in the figure 3.

By unfolding, one can map this problem onto the one of two chiral (say right moving) liquids with an interaction localized at the origin, corresponding to the bottom part of \( \Phi \). The translation between the two is obvious (and different from the folding in figure 2): for instance, a process where an electron comes in and bounces within the Fermi liquid corresponds in the unfolded picture to an electron going through on the top line.

In a particular experiment, different tunneling terms will be induced by the realization of the point contact. At low energy - or low temperature - the decoupled fixed point, where the two electronic liquids are decoupled, is stable. To leading order, the tunneling that kicks in as energy is increased is a process where one electron from the Fermi liquid is exchanged for an “edge” electron in the Luttinger liquid. The amplitude for this term is traditionally called \( \Gamma \). In the limit of small \( \Gamma \), the Lagrangian reads

\[
L = \frac{1}{4\pi} \partial_x \phi_a (\partial_t - \partial_x)\phi_a + \frac{1}{4\pi} \partial_x \phi_b (\partial_t - \partial_x)\phi_b + \Gamma \delta(x) \cos \left[ \frac{1}{\sqrt{\nu}} (\phi_a - \phi_b) \right],
\]  

where the field \( \phi_a \) is associated with the Luttinger liquid, and \( \phi_b \) with the Fermi liquid. This Lagrangian can be brought in a simpler form by the definition of new rotated fields \( \varphi_{a,b} \):

\[
L = \frac{1}{4\pi} \partial_x \varphi_a (\partial_t - \partial_x)\varphi_a + \frac{1}{4\pi} \partial_x \varphi_b (\partial_t - \partial_x)\varphi_b + \Gamma \delta(x) \cos \left[ \frac{1}{\sqrt{g'}} (\varphi_a - \varphi_b) \right]
\]

where

\[
\frac{1}{g'} = \frac{\nu + 1}{2\nu}
\]
We will restrict here to $\nu = 1/3$, which gives rise to $g' = 1/2$. The fields can now be gathered into an even combination which is free, and an odd combination that sees a boundary sine-Gordon type interaction.

The perturbation induced by the tunneling term has dimension 2, and is irrelevant. This means that at low energies, there is essentially no tunneling between the initial Fermi and Luttinger liquids: one is at the fixed point where transmission is zero. This crucial feature of the problem, induces considerable complications. Indeed, suppose one were to consider an experimental realization starting from the trivial situation of decoupled wires at low energy, and increasing the energy to explore the non trivial scattering processes. In term of renormalization group, this would translate into going against the RG flow, since the low energy fixed point is stable, and the coupling between the wires is an irrelevant perturbation. As is well known, such a situation does not have a well defined universal limit, as the RG trajectory is influenced by the many other irrelevant operators present in the problem. In our case, this means for instance that one also has to take into account the density density coupling, which also has dimension 2, as well as other fields maybe, and the results will be affected by the balance of these different terms. A priori, only extreme fine tuning would allow one to reach the perfect transmitting fixed point at high energy.

Fortunately, there seems to be a way to realize the high energy fixed point $\Gamma = \infty$. Indeed, in a very interesting paper, C. Kane pointed out that if tunneling between the two liquids takes place through an impurity, it is possible, by tuning two parameters only, to achieve perfect resonance that is, to have perfect tunneling between the Fermi and Luttinger liquids. Away from the resonance, deviations are controlled by the single relevant operator at the resonant fixed point, and the problem becomes formally identical to the by now well known problem of edge states tunneling at $\nu = g'$. In this new situation, at high energy, there is maximum tunneling between the Fermi and the Luttinger liquid which are strongly coupled, while at low energy, there is no remaining tunneling, the two are decoupled, and the scattering is entirely diagonal. This is illustrated on the figure 4.

The problem in the vicinity of the perfectly transmitting fixed point can be formulated in terms of dual fields, with Lagrangian

$$L = \frac{1}{4\pi} \partial_x \tilde{\phi}_+ (\partial_t - \partial_x) \tilde{\phi}_+ + \frac{1}{4\pi} \partial_x \tilde{\phi}_- (\partial_t - \partial_x) \tilde{\phi}_- + \tilde{\Gamma}(x) \cos \sqrt{2g'} \tilde{\phi}_-$$

and $\tilde{\Gamma} \propto \Gamma^{-1/2}$. In this case, the perturbation is relevant, with dimension 1/2. As energy is lowered, physical processes where electrons and Laughlin quasiparticles are incident on the point contact and scatter into various combinations of outgoing electrons and Laughlin quasiparticles will have energy dependent amplitudes.

In the following we denote the field $\tilde{\phi}_-$ simply by $\phi$. We will also denote $\tilde{\phi}_+$ by $\phi'$. We fold the model so that instead of having right movers on the full line, we have left and right movers on the half line only, with the Lagrangian

$$L = \frac{1}{8\pi} \int_{-\infty}^{d} dx \left[ (\partial_x \Phi)^2 - (\partial_t \Phi)^2 \right] + \tilde{\Gamma} \cos \frac{\Phi(0)}{2}$$

and $\tilde{\Gamma} \propto \Gamma^{-1/2}$. In this case, the perturbation is relevant, with dimension 1/2. As energy is lowered, physical processes where electrons and Laughlin quasiparticles are incident on the point contact and scatter into various combinations of outgoing electrons and Laughlin quasiparticles will have energy dependent amplitudes.

In states are now right moving, and out states are left moving. As before, the perturbation has dimension 1/2, and is relevant.

The most general quantities we are interested in read, in the original $\phi_{a,b}$ variables,

$$G(4,3,2,1) = \langle T e^{i \sqrt{\nu} \phi_{a}^{out}(4)} e^{i q \phi_{b}^{out}(3)} e^{-i \sqrt{\nu} \phi_{a}^{in}(2)} e^{-i m \phi_{b}^{in}(1)} \rangle$$

FIG. 4: Tunneling between the Fermi liquid and the $\nu = 1/3$ Luttinger liquid as a function of the energy
These correlators correspond either to what may happen is that an infinity of processes have non vanishing amplitudes at non zero \( \tilde{\Gamma} \), with the sum of the individual term vanishes, but the sum remains equal to 1. We do not know how to explore this situation in more detail.

For simplicity, we will mostly restrict to correlators of the form

\[
G(m, n) = \langle T \exp \left( -\alpha \phi_L(x_3, t_3) \right) \exp \left( i \phi_R(x_1, t_1) \right) \rangle
\]

where \( x_3 > 0, x_1 < 0 \), and

\[
g(m, n) = \langle T \exp \left( i \phi_L(-x_3, t_3) \right) \exp \left( i \phi_R(x_1, t_1) \right) \rangle.
\]

These correlators correspond either to \( p = n = 0, q = m \) or \( q = m = 0, p = n \) and describe the scattering of \( q \) electrons into \( q \) electrons, or \( p \) Laughlin quasiparticles into \( p \) Laughlin quasiparticles.

In the limit \( \tilde{\Gamma} = 0 \) (high energy fixed point), the field \( \phi \) sees Neumann boundary conditions, and \( g \) vanishes by charge neutrality. This corresponds to the Fermi liquid and the Luttinger liquid being strongly coupled, so that all processes of finite amplitude involve non diagonal scattering, like electrons scattering into Laughlin quasiparticles. In the limit \( \tilde{\Gamma} \to \infty \), on the contrary, the field sees Dirichlet boundary conditions, the scattering of electrons or Laughlin quasiparticles is diagonal, and the correlator takes a finite value,

\[
g(m, n) \to \frac{1}{(x_3 - x_1 - t_3 + t_1 + i \epsilon \text{ sign}(t_3 - t_1))^{\alpha^2}}. \quad \tilde{\Gamma} \to \infty
\]

In general, this correlator will exhibit a cross-over behaviour, which we will determine using the form-factors technique. Knowledge of the correlators will then allow us to determine the amplitude of interest.
B. The case $2e \rightarrow 2e$

The case $\alpha = 1$ is special, and much simpler. Indeed, as discussed in the first part, the operator $V_{\pm}$ behaves like a fermion field, $V_{\pm} \sim \psi^{(1)}$ in a free fermion theory, the $-$ soliton like a fundamental fermion, the $+$ soliton like a hole,

$$
< 0 | \psi_L | \beta >_{L-} = \mu e^{\frac{\beta}{2} e^{-ie^\beta(t+x)}}; \quad L+ < \beta | \psi_L | 0 > = \mu e^{\frac{\beta}{2} e^{ie^\beta(t+x)}}
$$

$$
< 0 | \psi_R | \beta >_{R-} = \mu e^{\frac{\beta}{2} e^{-ie^\beta(t-x)}}; \quad R+ < \beta | \psi_R | 0 > = \mu e^{\frac{\beta}{2} e^{ie^\beta(t-x)}}
$$

(53)

where $\mu$ is a normalization factor ($\mu = \sqrt{2\pi}$ if one wants the fermions two point function to be normalized to unity). Similarly, for the dagger operators,

$$
< 0 | \psi_L^\dagger | \beta >_{L+} = \mu e^{\frac{\beta}{2} e^{-ie^\beta(t+x)}}; \quad L- < \beta | \psi_L^\dagger | 0 > = \mu e^{\frac{\beta}{2} e^{ie^\beta(t+x)}}
$$

$$
< 0 | \psi_R^\dagger | \beta >_{R+} = \mu e^{\frac{\beta}{2} e^{-ie^\beta(t-x)}}; \quad R- < \beta | \psi_R^\dagger | 0 > = \mu e^{\frac{\beta}{2} e^{ie^\beta(t-x)}}
$$

(54)

It follows immediately that, for $\alpha = 1$, using the same technique as in the first part,

$$
g(3, 1) = < 0 | \psi_L(3) \psi_R^\dagger(1) | B > \equiv < \psi_L(3) \psi_R^\dagger(1) > \propto \int_{-\infty}^{\infty} \frac{e^{\beta} \beta}{1 + e^{\beta - \beta_B}} \frac{d\beta}{2\pi} e^{\beta (x_1 - x_3 + iy_1 - y_3)}
$$

(55)

Writing now the integral representation

$$
G(3, 1) = \int_0^{\infty} dp \ G(p, p_B) e^{p(x_1 - x_3 + iy_1 - y_3)}
$$

(56)

we arrive, after convolution, at the following formula

$$
G(p, p_B) \propto -\frac{p_B}{2} \left[ (p + p_B)^2 \ln(1 + p/p_B) - \frac{3p^2 + 2pp_B}{2} \right]
$$

(57)

such that

$$
G(p, p_B) \approx -\frac{p^3}{6}, \quad \frac{p}{p_B} \rightarrow 0
$$

$$
G(p, p_B) \approx -\frac{1}{2}p_B \ln p_B, \quad \frac{p}{p_B} \rightarrow \infty
$$

(58)

where we recall that the high energy limit is $p_B \rightarrow 0, \tilde{\Gamma} \rightarrow 0$.

Physically, the case $\alpha = 1$ corresponds to $q = m = 2$, i.e., a process where a wave packet of charge $2e$ comes from the left in the Fermi liquid, and goes access to a charge $2e$ wave packet still in the Fermi liquid. In the limit $\tilde{\Gamma} = \infty$, this process occurs with probability 1, and it occurs with probability 0 in the limit $\tilde{\Gamma} = 0$.

The meaning of $G$ as a probability in between these two limits requires some more discussion. The problem boils down to what one wishes to call a wave packet of charge $2e$. In the bosonized theory, it is convenient to think of this as the object created by $e^{i\tilde{\phi}_L}$, which in the original Fermi liquid corresponds to the quantity $\chi \partial \chi$ (the fermion $\chi$ is the fermion field in the Fermi liquid, and has no direct relation with the fermionic field $\psi$ which appears in the reformulation of the boundary sine-Gordon theory at the value $\beta^2 = 4\pi$). When written in terms of modes, $\chi \partial \chi$ reads as an superposition of states with two electrons at different momenta, weighed by some energy dependent coefficient. The corresponding creation operators have commutation relations of the form $[a^\dagger(k), a(k')] \sim k^2 \delta(k - k')$, and it is convenient to use these states as normalized states for charge $2e$ wave packets. With this convention, the probability that such a wave packet goes through unaltered is

$$
P_{2 \rightarrow 2} = \left| \frac{G(ip, p_B)}{G(ip, \infty)} \right|^2 = 9 \left[ ix(1 + ix)^2 \ln \left( 1 + \frac{1}{ix} \right) - \frac{3ix}{2} - (ix)^2 \right]^2
$$

(59)

with $x = \frac{p_B}{p}$. This probability is represented in figure [4].
FIG. 6: Probabilities for the processes $2e \rightarrow 2e$ and $e \rightarrow e$.

**C. The case $e \rightarrow e$**

The case $\alpha = \frac{1}{2}$, which corresponds to a process where an electron comes out as an electron, is of course much more complicated to study, and bears close resemblance to the transmission amplitude of the first part. A closed form expression is again possible, and one finds that, in imaginary time,

$$G(3, 1) \propto x^{-1/2} \sqrt{\frac{p_B}{4\pi}} e^{p_B x^2/2} K_0(p_B x/2), \quad x = x_3 - x_1 + i(y_3 - y_1)$$

(60)

This provides a closed form expression for the Laplace transform

$$G(p/p_B) = \frac{1}{\pi} \int_0^1 \frac{dt}{\sqrt{t(1-t)}} \sqrt{1 + \frac{p}{p_B} t}$$

(61)

which it is convenient to rewrite, to study the small $p_B$ limit, as

$$G(p/p_B) = \frac{1}{\pi \sqrt{p_B/p}} \int_0^{p/p_B} \frac{dt}{\sqrt{t(1+t)}} \sqrt{1 - p_B t/p}$$

(62)

From this we see that the leading corrections goes in fact as $\sqrt{p_B/p} \ln(p_B/p)$ - in agreement with a perturbative calculation near the UV fixed point. The large $p_B$ limit can be studied as well, with

$$G(p/p_B) = \sum_{k=0}^{\infty} \left( \frac{p}{p_B} \right)^k$$

(63)

such that $G = 1$ when $p/p_B = 0$. The leading order is linear in $p/p_B$, again in agreement with perturbative calculations near the IR fixed point. It is convenient to reexpress $G$ as a hypergeometric function

$$G(p/p_B) = F \left( \frac{1}{2}, \frac{1}{2} + \frac{1}{2} + \frac{1}{2}; -\frac{p}{p_B} \right).$$

(64)

Standard formulas then allow for a complete expansion in the large $p_B$ limit. For further use, we quote the general result

$$\frac{\Gamma(1/2)}{\Gamma(c)} F \left( \frac{1}{2}, \frac{1}{2}, c; -\frac{p}{p_B} \right) = \frac{(p_B/p)^{1/2}}{\Gamma(c - 1/2)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2 + n) \Gamma(3/2 - c + n)}{\Gamma(1/2) \Gamma(3/2 - c)} \left( \frac{p}{p_B} \right)^{-n} \left[ \ln(p/p_B) + h_n(c) \right]$$

(65)
where
\[ h_n(c) = 2\psi(1 + n) - \psi(n + 1/2) - \psi(c - n - 1/2) \] (66)

In the present case, \( G \), after analytic continuation, can directly be interpreted as the scattering amplitude for an incident electron into an outgoing electron, and \( |G|^2 \) is therefore the probability of this process
\[ P_{1\rightarrow 1} = |G(ip/p_B)|^2 = \left| F\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{p}{ip}\right)\right|^2 \] (67)
The result is also shown in figure 6.

Note that we can also extract from these results expressions for the density of states inside the Fermi liquid. This density has a constant part and an oscillatory part, and thus reads, as a function of the distance \( x \) from the point contact:
\[ \rho(x) \propto \text{cst} + e^{ipx} \int dp \, G(ip/p_B)e^{ipx} + \text{h.c.} \] (68)
This will be discussed in more details elsewhere.

### D. The case \( lqp \rightarrow lqp \)

When a Landau quasi-particle (lqp) scatters into a Landau quasi-particle, the amplitude is related to a slightly different kind of correlator,
\[ G(4, 2) = g(4, 2) \langle T \, \exp(i\alpha/\sqrt{3})\phi_L^*(x_4, t_4) \exp(-i\alpha/\sqrt{3})\phi_R(x_2, t_2) \rangle \]
\[ = g(4, 2) \frac{1}{(x_4 - x_2 - t_4 + t_2 + i\epsilon \text{ sign}(t_4 - t_2))^{\alpha^2/3}} \] (69)
where \( x_4 > 0, x_2 < 0 \) and
\[ g(4, 2) = \langle T \, \exp(-i\alpha\phi_L(-x_4, t_4)) \exp(-i\alpha\phi_R(x_2, t_2)) \rangle . \] (70)
The case \( lqp \rightarrow lqp \) corresponds again to \( \alpha = 1/2 \), that is an expression which is identical, in the \( \phi \) sector, to the amplitude for the electron electron process. One can then deduce the lqp amplitude from the electron amplitude
\[ G(4, 2) \propto \int_0^\infty p^{-2/3} e^{-px} dp \int_0^1 (1 - u)^{-7/6} u^{-1/2} \left( 1 + \frac{p}{p_B} u \right)^{-1/2} du \] (71)
where \( x = x_4 - x_2 + i(y_4 - y_2) \). To obtain the latter result, we used a Laplace representation of \( x^{1/6} \), which is divergent, and can be made finite by the analytical continuation (continuation of a \( \Gamma \) function in this case). Similarly, \( \Gamma(1/2) \) is divergent, and can be made finite by expanding in powers of \( p/p_B \) at low energy, and using \( B(\Gamma) \) functions regularization. One finds then
\[ G(p, p_B) \propto p^{-2/3} \sum_{n=0}^\infty \frac{(p/p_B)^n}{p_B} \Gamma(-1/6)\Gamma(n + 1/2) \Gamma(n + 1/3) \] (72)
By applying Stirling’s formula at large \( n \), one can check that the series converges for \( p/p_B < 1 \). At large energy meanwhile, one finds the leading behaviour
\[ G(p, p_B) \propto p^{-2/3} \left( \frac{p_B}{p} \right)^{1/2} \ln(p_B/p) \] (73)
again in agreement with perturbative calculations. The complete large energy expansion can be obtained by recognizing that
\[ G(p, p_B) \propto p^{-2/3} F\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{p}{p_B}\right) \] (74)
and then using (65).

A graph of the associated probability (normalized by the low energy limit) i.e. \( |F(\frac{1}{2}, \frac{1}{2}, 1; -\frac{p}{p_B})|^2 \) is provided in figure 7.
This is still another case, corresponding to $m = 1, p = 3$. The amplitude is therefore

$$G(4, 1) = g(4, 1) \frac{1}{(x_4 - x_1 - t_4 + t_1 + i\epsilon \text{sign}(t_4 - t_1))^{3/4}},$$  \hspace{1cm} (75)

where $x_4 > 0, x_1 < 0$ and

$$g(4, 1) = \langle T \exp \left(-\frac{3i}{2} \phi_L(-x_4, t_4)\right) \exp \left(\frac{i}{2} \phi_R(x_1, t_1)\right) \rangle.$$

$$\hspace{1cm} (76)$$

This amplitude is expected to vanish both at the UV and IR fixed points, and take non trivial values in between. Note that the amplitude is invariant under the shifts $\phi_R \rightarrow \phi_R + \pi$ and $\phi_L \rightarrow \phi_L - \pi$ corresponding to the UV boundary state, and $\phi_R \rightarrow \phi_R + 2\pi$, $\phi_L \rightarrow \phi_L + 2\pi$ corresponding to the IR boundary state. One can similarly check that the leading relevant (resp. irrelevant) operator give it a non trivial value away from the UV (resp. IR) fixed point. More sophisticated arguments presented in the appendix give rise to the following expression

$$G(4, 1) \propto x^{-2} \int_0^\infty e^{-y(1 - 2y)}y^{-1/2} \left(1 + \frac{y}{p_Bx}\right)^{-1/2} dy$$

$$\hspace{1cm} (77)$$

from which the momentum dependent amplitude follows

$$G(p, p_B) \propto p^{1/4} \int_0^1 (1 - u)^{-1/4}u^{-1/2}\left(1 + \frac{p}{p_Bu}\right)^{-1/2} du$$

$$\hspace{1cm} (78)$$

Note that this correlator vanishes in the low and high energy limits, as expected from previous considerations about the fixed points. An expansion at low energy follows from $\Gamma$ function regularization again

$$G(p, p_B) \propto p^{1/4} \sum_0^\infty (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \left[2 \frac{\Gamma(-1/4)\Gamma(1/2 + n)}{\Gamma(1/4 + n)} - \frac{\Gamma(-1/4)\Gamma(3/2 + n)}{\Gamma(5/4 + n)}\right] \left(\frac{p}{p_B}\right)^n$$

$$\hspace{1cm} (79)$$

The leading behaviour at high energy is

$$G(p, p_B) \propto p^{1/4} \left(\frac{p_B}{p}\right)^{1/2} \ln(p_B/p)$$

$$\hspace{1cm} (80)$$
Once again, one can recognize a sum of two gamma functions

\[ G(p, p_B) \propto p^{1/4} \left[ \frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(5/4)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{4}, -\frac{p}{p_B}\right) + \frac{\Gamma(1/2)\Gamma(-1/4)}{\Gamma(1/4)} F\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, -\frac{p}{p_B}\right) \right] \tag{81} \]

and then use formula (65) to obtain the complete high energy expansion.

In the present case, we do not know how to normalize the associated probability, which is expected to vanish in both low and high energy limits - this would presumably require a more complete understanding of unitarity issues. Qualitatively, we expect that it should behave as \(|G(ip, p_B)/p^{1/4}|^2\), which is represented in figure 8.

\section*{IV. CONCLUSIONS}

The results we have obtained in closed form are but a very small step towards a general understanding of the scattering amplitudes. Extensions of this work to other problems would involve in general considerable technical difficulties, if only because the form-factors approach is so hard to implement in the massless case. On top of this, even the form-factors themselves might not be known, or only known in a very involved form - clearly, a new idea is needed there if progress is to be made.

A more rewarding direction of development would probably be to restrict to the present \(g = 1/2\) case, and consider instead the question of unitarity, and how the probability spreads among multiparticle processes as energy is varied. It also remains to be seen whether a direct determination of scattering amplitudes is experimentally feasible.

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\section*{APPENDIX A: DISORDER OPERATOR AND FLOW CONSIDERATIONS.}

The one point function of the disorder operator in the Ising model with a boundary magnetic field vanishes all along the flow. This follows immediately from the fact that all non vanishing form-factors for \(\mu\) involve odd numbers of particles, while the boundary state is a superposition of an even number of particles only.
Here, a word of caution is necessary. Right at \( h = 0 \), the proper boundary state in the massive theory is in fact a superposition of states with odd and states with even numbers of particles:

\[
|B >_{\text{free}} = (1 + Z^I(\theta)) \exp \left[ \frac{1}{2} \int_{-\infty}^{\infty} K_{\text{free}}(\theta) Z^I(-\theta) Z^I(\theta) \right]
\]

One has \( K_{\text{free}} = -i \coth \frac{\theta}{2} \), and the presence of \( Z^I(0) \) is related with the pole of \( K \) at \( \theta = 0 \). So, right at the free boundary conditions fixed point, the one point function of the disorder operator is non zero. In the massless limit, the zero momentum particle becomes a zero momentum zero energy particle, and becomes almost invisible except for the fact that it changes odd into even number of particles, and introduces some sign factors. One then finds (this is discussed in more details below)

\[
\langle \mu(x) \rangle_{h=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left\{ \frac{d\beta_i}{2\pi} e^{-2x\beta_i} \right\} \prod_{i<j} \left( \tanh \frac{\beta_i - \beta_j}{2} \right)^2
\]

where the term \( \prod_{i} \tanh \frac{2n-\beta}{2} \) which is there equal to \((-1)^n\), is cancelled by the \((-1)^n\) term arising from the presence of an additional particle at vanishing energy and momentum. This formal expression coincides with \( \langle \sigma(x) \rangle_{h=\infty} \), and thus \( \langle \mu(x) \rangle_{h=0} \propto x^{-1/8} \).

For any non zero \( h \) however, there is no one particle state in \( |B >_{\text{free}} \), and the one point function of \( \mu \) vanishes exactly.

In this appendix we first would like to discuss this result further. To do so, we first can try to follow the route of \[8\], and derive a differential equation for \( \langle \mu > \). Recall that in the usual case, Chatterjee and Zamolodchikov find

\[
\left[ \frac{d^2}{dY^2} + \left( \frac{1}{Y} - 4 \right) \frac{d}{dY} + \left( -\frac{1}{2Y} + \frac{9}{16Y^2} \right) \right] \langle \sigma \rangle = 0, \quad Y = 2p_B x
\]

\( (x \text{ taken positive by convention here}) \) and \( p_B = 4\pi \hbar^2 \). Setting \( \langle \sigma \rangle = Y^{3/8} \Psi \), one finds the equation

\[
Y\Psi'' + (1 - Y)\Psi' - \frac{\Psi}{2} = 0
\]

whose physical solution is proportional to the degenerate hypergeometric function \( \Psi = \Psi(1/2, 1, Y) \).

In the case of the disorder operator, the same logic as in \[8\] can be followed, with proper modifications due to the different monodromies involves. We find instead the equation

\[
\left[ \frac{d^2}{dY^2} + \left( \frac{1}{Y} - 4 \right) \frac{d}{dY} + \left( -\frac{1}{2Y} + \frac{1}{16Y^2} \right) \right] \langle \mu \rangle = 0, \quad Y = 2p_B x
\]

And, setting \( \langle \mu \rangle = Y^{-1/8} \tilde{\Psi} \), one has this time

\[
Y\tilde{\Psi}'' - Y\tilde{\Psi}' = 0
\]

with only solution (that does not grow exponentially at large distance) \( \tilde{\Psi} \) a constant. The only physical solution is thus \( \langle \mu > = 0 \), since we know it has to vanish in the limit of large fields, where the boundary spins are fixed (the case \( p_B = 0 \) can be exceptional, since then the variable in the differential equation is not well defined).

This calculation is formally the same as the calculation of the one point function of the spin \( \langle \sigma \rangle \) for an Ising model where the boundary perturbation, instead of being the boundary field, would induce the following equations of motion: (if \( t \) is the coordinate along the boundary as in \[13\])

\[
\left( \frac{d}{dt} + ip_B \right) \psi(t) = - \left( \frac{d}{dt} - ip_B \right) \tilde{\psi}(t)
\]

Note the crucial minus sign on the right hand side - in the usual problem of Ising model with a boundary magnetic field, the rhs comes with a plus sign, and the physics is the flow from free to fixed boundary conditions. It is easy to check that switching the sign in this equation leads to switching the sign of the reflection matrix, which reads in this case

\[
\tilde{R} = -i \tanh \left( \frac{\beta - \beta_B}{2} - \frac{i\pi}{4} \right)
\]

(in the following we set now \( p_B = 4\pi \hbar^2 \)). In the UV, one finds \( R = -i \) and in the IR \( R = i \), so the role of the free and fixed boundary conditions are exchanged: this is the flow ‘dual’ to the ordinary one, which can be understood for instance by using a low and high temperature graphical expansion of the Ising model partition functions.
Here, one has to be a bit careful. From entropy considerations, one does not expect a flow from fixed to free boundary conditions to be possible. Rather, what probably happens is a flow from the superposition of two boundary states (fixed + and fixed −) towards free boundary conditions. The ratio of degeneracies is then \( g_{UV}/g_{IR} = 2 \times \sqrt{2} = \sqrt{2} \), the same as for the usual flow from fixed to free. In such a flow, the \( Z_2 \) symmetry is never broken, and therefore \(<\sigma>_h = 0 \) always. The free energy in this flow is in fact the same (as a function of \( T, p_B \)) as the free energy in the ordinary flow - a feature compatible with the fact that the free energy in the integrable approach only depends on the derivative of the log of the \( R \) matrix. In this point of view the UV fixed point appears extremely singular, as it is a superposition of two independent states - for any \( p_B \neq 0 \), these two states are coupled, and “cutting the system in \( (Z_2) \) half” does not make physical sense. Only at the “double fixed” fixed point can one do so, and define \(<\sigma>_{h=0} \propto e^{-1/8} \).

Now we can ask about the boundary state associated with (A8). In the usual case, the proper way to discuss this is to go to the massive theory. This can be done here in various ways. An interesting one is to make a detour through the massive Kondo model [24], that is the theory of massive fermions in the bulk, with a Kondo-like interaction at a boundary magnetic field, while the other has the following \( R \) matrix

\[
\tilde{R} = -i \coth \left( \frac{i\pi}{4} - \frac{\theta}{2} \right) \frac{1 + \frac{i\lambda}{2\pi} + i \sinh \theta}{1 + \frac{i\lambda}{2\pi} - i \sinh \theta}
\]

(A9)

Here one sees that \( \tilde{R} = R_{\text{free}} \) at large value of the coupling \( h \), while \( \tilde{R} = R_{\text{fixed}} \) at small value of the coupling. For any value of the coupling, the \( K \) matrix has a pole at the origin, indicating the presence of a zero momentum particle in the boundary state. Note that in the massless limit the two reflection matrices become \( R = i \tanh \left( \frac{\beta - \beta_m}{2} - \frac{i\pi}{4} \right) \), and \( \tilde{R} = -i \tanh (\frac{\beta - \beta_m}{2} - \frac{i\pi}{4}) \). The model thus decomposes into two Ising models, one with a flow from free to fixed, and one with a flow from “double fixed” to free (note that the ratio of \( g \) factors is thus \( g_{UV}/g_{IR} = \sqrt{2} \times \sqrt{2} = 2 \) as required for the spin 1/2 Kondo model.) The boundary state for the latter flow always contain a zero momentum, zero energy particle.

To proceed, let us consider the form factor

\[
\langle 0|\mu|0, \theta_2, \ldots, \theta_1 \rangle = \prod_{i=1}^{2n} \tanh(\theta_i/2) \prod_{i<j=1}^{2n} \tanh(\theta_{ij}/2)
\]

(A10)

In the massless limit, \( n \) particles will become \( R \) movers, \( n \) will become \( L \) movers. Whatever particles we choose for this, the particle at rapidity zero will introduce a factor \((-1)^n\), and multiplying this factor, we will have the massless limit of the product \( \prod_{i<j=1}^{2n} \tanh(\theta_{ij}/2) \), which will be a factor of similar terms for \( R \) and \( L \) components independently, times a sign factor which depends on which particles become \( R \) and which particles become \( L \). The minus sign hence generated by the particle at vanishing rapidity exactly cancels the minus sign coming from the \( \tilde{R} \) matrix, so we can write at the end

\[
\langle \mu \rangle_h = \langle \sigma \rangle_h
\]

(A11)

hence establishing that the one point function of the disorder operator in the dual flow is the same as the one point function of the order operator in the original flow.

A last comment about the dual flow. Although we do not know the action corresponding to the one point function near the fixed boundary conditions fixed point (and thus the parameter \( h \) is only formal), we can find it near the IR (free) boundary conditions fixed point by using the arguments of [7]. Since the \( \tilde{R} \) matrix is analytic, it follows that this action involves only terms of the form \( \psi \partial^{2n-1} \psi \). Since on the other hand the correlation functions of the spin operator with these (conserved) quantities all vanish with free boundary conditions, it follows that, in a perturbative expansion near the free boundary conditions (IR) fixed point, \( \langle \sigma \rangle_h \) has to vanish identically. Of course, it may well be that the expansion of this quantity has a vanishing radius of convergence, wiht non perturbative contributions - the foregoing arguments have established that this is not the case however.

As discussed in the main text, a regularized version of \( \langle \mu \rangle_h \) can be obtained by adding to the boundary state a fermion at vanishing energy and momentum. The corresponding expression in terms of form-factors is given in [55].

An exact expression for this can be obtained by considering, following Chatterjee Zamolodchikov [8], the correlator, 

\[
< \chi(z) \mu(w, \bar{w}) > \text{ where } \chi \equiv (\partial_z + i\lambda) \psi \text{ for } Im \ z > 0 \text{ which we assume in the following. To start, we consider in fact a simpler correlator, } < \psi(z) \mu(w, \bar{w}) > . \text{ We calculate this correlator using the form factor formalism, after insertion of the boundary } |B >. \text{ The } n^{th} \text{ order term then reads}
\]

\[
\frac{1}{n!} \int_{-\infty}^{\infty} d\beta_i (-i) \tanh \left( \frac{\beta_i}{2} \right) (\beta_i B - \beta_i) (-1)^{n(n-1)/2} < 0|\psi(z) \mu(w, \bar{w})|\beta_1, \ldots, \beta_n ; \beta_1, \ldots, \beta_n >_{R,...,R,L,...,L}
\]
where the $(-1)^{(n-1)/2}$ originates from reordering the $n^{th}$ term in the expansion of the boundary state. Note that
this term is exactly cancelled by the sign coming from the massless limit of the $\mu$ form factor.

The disorder operator has non vanishing form factors only on odd numbers of particles. The fermion is simply able
to destroy one particle. Suppose $\mu$ destroy the particles $\beta_2, \ldots, \beta_n$ in the R channel and $\beta_1, \ldots, \beta_n$ in the R channel. Then $\psi$ the remaining particle $\beta_1$ in the R channel, so the matrix elements $<0|...>$ will give the contribution

$$e^{\beta_1/2}i^n \prod_{i<j=2}^{n} \tanh \frac{\beta_i - \beta_j}{2} \prod_{k<l=1}^{n} \tanh \frac{\beta_k - \beta_l}{2} \exp \left[-2x \left(e^{\beta_2} + \ldots + e^{\beta_n}\right)\right] \exp[-(z-w)e^{\beta_1}]$$

and $x = \frac{w+\bar{w}}{2}$.

Let us now look at the large $z$ behaviour ($Im\, z > 0$) of the corresponding integral. In that limit, the integral is
dominated by the region where $\beta_1 \to -\infty$, and we are left with the integral over $\beta_2, \ldots, \beta_n$ of the expression

$$i^n \prod_{i<j=2}^{n} \tanh \frac{\beta_i - \beta_j}{2} (-1)^n \prod_{k<l=1}^{n} \tanh \frac{\beta_k - \beta_l}{2} \exp \left[-2x \left(e^{\beta_2} + \ldots + e^{\beta_n}\right)\right]$$

while the integral over $\beta_1$ produces a factor $1/\sqrt{z}$. The crucial factor $(-1)^n$ comes from the limit of the form factors
contributions $\tanh \frac{\beta_i - \beta_j}{2}$. The factor coming from the boundary state meanwhile reduces to $\prod_{i=2}^{n} \tanh \frac{\beta_i - \beta_2}{2}$.

Now we could also have $\psi$ annihilate any of the other $\beta_2, \ldots, \beta_n$ R particles instead. If this so happens, commuting
$\psi$ through the particles on the left of the ones it annihilates generates minus signs. These however will be cancelled
exactly by the minus signs coming from the limit of the product of $\tanh$ factors. It thus simply follows that we get a
multiplicity of $n$ for the term we have just analyzed, and therefore at large $z$

$$< \chi(z)\mu(w, \bar{w}) > \propto \lambda \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\beta_i}{2\pi} \left(\tanh \frac{\beta_B - \beta_i}{2}\right)^2$$

$$(-1)^n \prod_{i<j=1}^{n} \left(\tanh \frac{\beta_i - \beta_j}{2}\right)^2 \exp \left[-2x \left(e^{\beta_1} + \ldots + e^{\beta_n}\right)\right]$$

and the proportionality factor is independent of $\lambda$. Of course as always the generic terms in the series are divergent,
and we do not learn much (in particular, the $Z$ power law dependence is not reliable) until we regularize by taking
ratios.

Using \[8\], we can on the other hand calculate the large $z$ behaviour by computing $B$ in the right hand side of their
equation (24). Putting things together we obtain the final result that

$$\left(\sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\beta_i}{2\pi} (\tanh \frac{\beta_B - \beta_i}{2})^2 \prod_{i<j=1}^{n} \left(\tanh \frac{\beta_i - \beta_j}{2}\right)^2 \exp \left[-2x \left(e^{\beta_1} + \ldots + e^{\beta_n}\right)\right]\right)$$

/ (same with $\beta_B = \infty$) $= -\sqrt{z} \left[2d\Psi(1/2, 1; z) - \Psi(1/2, 1; z)\right], \quad z = 2p_B x$ \hspace{1cm} (A12)

Numerical checks of this result (obtained by carrying out the form factors expansion to fourth order) are presented
in the figure. The agreement is not spectacular, but maybe this is expected, as successive orders in the form-factors
expansion all add up with the same (positive) sign, while in the usual case they form an alternating series. Note the
divergence at vanishing $p_B$.

**APPENDIX B: ANALYTICAL RELATIONS.**

This appendix details a simple extension of the method of Chatterjee-Zamolodchikov \[8\] to determine correlators
directly in the sine-Gordon model. Let us start from

$$A = \frac{1}{8\pi} \int_{-\infty}^{0} dx \left[(\partial_x \Phi)^2 - (\partial_t \Phi)^2\right] + \nu \cos \frac{1}{2} \Phi(x = 0, t). \hspace{1cm} (B1)$$

Fermionizing (we revert to the more standard notation $\psi, \bar{\psi})\psi_{\pm}(z) = e^{\pm i\phi(z)}$ and $\bar{\psi}_{\pm}(z) = e^{\mp i\phi(z)}$, we get an action consisting of two decoupled Ising models having an interaction at the boundary (see \[27\] for details). Taking
the variation with respect to the boundary action, we find boundary conditions for the fermionic fields

$$\psi_{\pm}(z) + \psi_{\mp}(z) = \bar{\psi}_{\pm}(z) + \bar{\psi}_{\mp}(z) \hspace{1cm} (B2)$$

$$i\partial_t (\psi_+ - \psi_-) + v(\psi_+ - \psi_-) = -i\partial_t (\bar{\psi}_+ - \bar{\psi}_-) + v(\bar{\psi}_+ - \bar{\psi}_-) \hspace{1cm} (B3)$$
The first equation for instance follows from analyticity together with the OPE around $\langle g \rangle$ giving an exact result for $\chi$ leading to the simple $\chi_+ = \psi_+ + \psi_-$ (B4)

$\chi_-(z) = \frac{i}{2} \partial_z (\psi_+ - \psi_-) + v(\psi_+ - \psi_-)$ (B5)

with $z = \frac{1}{4}(t + ix)$. The boundary conditions insure that the fields are analytic in the full complex plane. In terms of the bosonic field they read

$\chi_+(z) = 2 \cos \phi(z)$ (B6)

$\chi_-(z) = -\partial_z \sin \phi(z) + iv \sin \phi(z)$. (B7)

Now, given the OPE of the operators $\chi_{\pm}$ with the vertex operators $V_{1/2,-1/2}$ we expect that

$\langle \chi_+(z) e^{i(\phi(w) - i\phi(\bar{w}))/2} (z - w)^{1/2} (z - \bar{w})^{1/2} \rangle = B_+(w, \bar{w})$ (B8)

$\langle \chi_-(z) e^{i(\phi(w) - i\phi(\bar{w}))/2} (z - w)^{1/2} (z - \bar{w})^{1/2} \rangle = \frac{A_-}{(z - w)} + B_-(w, \bar{w})$ (B9)

The first equation for instance follows from analyticity together with the OPE around $z = w$

$\chi_+(z) V_{1/2,-1/2}(w, \bar{w}) \approx \frac{1}{(z - w)^{1/2}} [V_{-1/2,-1/2} + (z - w)(V_{3/2,-1/2} - i\partial\phi V_{-1/2,-1/2} + \cdots)]$ (B10)

a similar OPE around $z = \bar{w}$, and the condition $\chi_+ \approx \frac{1}{2}$ at infinity.

Expanding the first equation in $B_+$ in powers of $z - w$ leads to the determination of $B_+ = (w - \bar{w})^{1/2} \langle V_{-1/2,-1/2}(w, \bar{w}) \rangle$. The comparison with the expansion in $(z - \bar{w})$ leads to a tautology relating the one point function of $V_{1/2,1/2}$ and that of $V_{-1/2,-1/2}$.

Expanding to higher orders gives non trivial relations. For example, to next order in $(z - w)$ we have

$\langle - \frac{B(w, \bar{w})}{2(w - \bar{w})^{3/2}} \rangle = \langle V_{3/2,-1/2} \rangle - i \langle \partial\phi V_{-1/2,-1/2} \rangle$ (B12)

leading to the simple

$\langle V_{3/2,-1/2} \rangle = -\frac{1}{2\pi} \langle V_{-1/2,-1/2} \rangle - 2 \frac{\partial}{\partial x} \langle V_{-1/2,-1/2} \rangle$ (B13)

giving an exact result for $\langle V_{3/2,-1/2} \rangle$ in terms of $\langle V_{-1/2,-1/2} \rangle$. Since the latter quantity is known from $\chi$, the former follows.

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