Lorentz-violating modification of Dirac theory based on spin-nondegnerate operators

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The Standard-Model Extension (SME) parameterizes all possible Lorentz-violating contributions to the Standard Model and General Relativity. It can be considered as an effective framework to describe possible quantum-gravity effects for energies much below the Planck energy. In the current paper, the spin-nondegnerate operators of the SME fermion sector are the focus. The propagators, energies, and solutions to the modified Dirac equation are obtained for several families of coefficients including nonminimal ones. The particle energies and spinors are computed at first order in Lorentz violation and, with the optical theorem, they are shown to be consistent with the propagators. The optical theorem is then also used to derive the matrices formed from a spinor and its Dirac conjugate at all orders in Lorentz violation. The results are the first explicit ones derived for the spin-nondegnerate operators. They will prove helpful for future phenomenological calculations in the SME that rely on the footing of quantum field theory.

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I. INTRODUCTION

Quantum-gravity effects may induce minuscule violations of Lorentz invariance. This is motivated by a number of articles that have been published during the past 25 years. Lorentz symmetry violation was shown to occur in string-theory models \[1\text{–}5\], loop quantum gravity \[6, 7\], noncommutative theories \[8, 9\], models describing a small-scale structure of spacetime \[10\text{–}12\], quantum field theories on spacetimes with nontrivial topology \[13, 14\], and last but not least, Hořava-Lifshitz gravity \[15\].

Although there are specific models in various approaches to quantum gravity, it is highly challenging to extract generic physical statements or principles from such models. After all, theoretical observations are very specific to a particular model under consideration and it is not clear at all whether the same or similar results will be obtained within a different model. Understanding the physics of model after model always requires starting a calculation from scratch and studying more general models may turn out to be impractical. Besides, it is not clear at all where to look for Lorentz violation specifically, insofar there is no preference of any particle sector or any prototype of quantum gravity.

For these reasons, it is much more reasonable to have a general, effective framework available including all possible Lorentz-violating terms that are consistent with coordinate invariance and the gauge structure of the Standard Model of elementary particles. Such a framework is provided by the Standard-Model Extension (SME), which is a collection of all Lorentz-violating contributions in both the Standard Model and General Relativity \[16\text{–}18\]. Each Lorentz-violating term is decomposed into field operators and component coefficients that can be interpreted as background fields permeating the vacuum. The SME allows for obtaining experimental predictions that are independent of any specific underlying model. The big advantage is that large regions of coefficient space can be tested with a single experiment only, i.e., in principle many distinct models are covered by doing so. Since every particle sector is contained, a large variety of experiments can be considered ranging from precise measurements of hyperfine splitting in hydrogen to observations of ultra-high energy cosmic rays. In the physics community, there seems to be a greater interest in searches for \(CPT\) violation than for Lorentz violation. In this context, one has to remark that \(CPT\) violation implies Lorentz violation \[19\], which is why all \(CPT\)-violating operators are contained in the SME automatically.

Lorentz-violating operators are classified according to their mass dimension. The finite number of operators with mass dimensions of 3 and 4 are contained in the minimal SME \[17\]. The remaining infinite number of higher-dimensional operators are part of the nonminimal SME \[20\text{–}22\]. Note that the nonlinear nonminimal gravity sector was discussed in \[23\], where the recent paper \[24\] contains a classification of the nonminimal coefficients in the linearized gravity sector. The nonminimal SME is a natural generalization of the minimal framework. It must be kept in mind that nonminimal operators become more dominant for increasing energies. Also, due to the negative mass dimension of the commutativity tensor in noncommutative field theories, such models can only be mapped to coefficients of the nonminimal SME, which justifies its consideration.

With the construction of the (minimal) SME, searches for violations of Lorentz invariance in nature have had their revival. This has led to a steadily increasing number of cutting-edge experiments testing Lorentz invariance, which enlarges the set of constraints on Lorentz violation yearly. At the same time, the sensitivity for detecting Lorentz invariance has been augmenting at
a fast pace, leading to an improvement of constraints by several orders of magnitude within few years, such as in the neutrino sector [25].

The theoretical properties of the SME at tree-level were investigated in a large series of papers [26–42], while radiative corrections were studied in [43–61]. The authors of [26] examined properties of the minimal fermion sector in general, whereas [28] is dedicated to the minimal $a$ and $b$ coefficients. Furthermore, in [38] the (nonminimal) fermion operators that are degenerate with respect to particle spin are on the focus. This concerns the $a$, $c$, $f$, and $m$ coefficients. So far, the modified propagators and particle spinors have not been stated explicitly in the literature for the spin-nondegenerate cases, which involves the $b$, $d$, $H$, and $g$ coefficients. The goal of the current paper is to fill this gap. Since all fields will be defined in Minkowski spacetime, it suffices to consider explicit Lorentz symmetry breaking. Hence, the Lorentz-violating background fields are introduced by hand and they lack any dynamics. Note that in curved spacetimes, this procedure is not sufficient, but one either has to break Lorentz symmetry spontaneously [62–65] or one must work in an extended geometrical framework. Finsler geometry seems to be very promising in this context and it has been on the focus for a couple of years [66–80].

The paper is organized as follows. In Sec. II, we state the propagators for the operators that break spin degeneracy. The results are valid for all possible choices of $b$, $d$, $H$, and $g$ coefficients, no matter whether they are part of the minimal or nonminimal SME. In Sec. III, both the dispersion relations and the solutions of the modified Dirac equation are given at first order in Lorentz violation for specific choices of minimal and nonminimal coefficients. The method of obtaining these solutions, which was developed in [22], will be reviewed. Section IV is dedicated to obtaining the matrices constructed from a spinor and its Dirac conjugate. These objects are indispensable in phenomenological calculations within quantum field theory. In this context, the optical theorem is employed as a tool to obtain the matrices and also to check consistency between the propagators and the first-order solutions of the Dirac equation. It is well-known that additional time derivatives in Lorentz-violating theories lead to several issues both in the minimal and the nonminimal SME. How these can be resolved for specific cases will be outlined in Sec. V. Last but not least, the results are summarized and discussed in Sec. VI. For demonstration purposes, exact spinors for a limited number of coefficients can be found in App. A. For completeness, in App. B we give the spinors and the propagator for the spin-degenerate operators. Besides, we will state a couple of specific spinor matrices in App. D. Natural units are used with $\hbar = c = 1$, unless otherwise stated. For typesetting purposes, momentum components will often have lower indices, but these should always be understood as components of the contravariant momentum, though.

II. SME FERMION SECTOR AND PROPAGATORS

The construction of the minimal SME fermion sector was initiated in [16, 17]. Ultimately, the Lagrange density of both the minimal and the nonminimal sector is expressed as follows [22]:

$$L = \frac{1}{2} \bar{\psi} \left( \gamma^\mu i \partial_\mu - m_\psi 1_4 + \hat{Q} \right) \psi + \text{H.c.} \quad (2.1)$$

Here, $\psi$ is a Dirac spinor and $\bar{\psi} \equiv \psi^1 \gamma^0$ is its Dirac conjugate. The fermion mass is denoted as $m_\psi$ to distinguish it from one of the Lorentz-violating operators. Furthermore, $\gamma^\mu$ are the standard

\footnote{Note that even for these coefficients there are particular choices that have a single dispersion relation for particles and antiparticles, respectively. Currently such a framework is under consideration and the outcomes will be reported in a forthcoming paper.}
Dirac matrices obeying the Clifford algebra \( \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4 \), with the Minkowski metric \( \eta_{\mu\nu} \) and the identity matrix \( \mathbb{1}_4 \) in spinor space. Lorentz-violating contributions are contained in \( \hat{Q} \), which is a \( 4 \times 4 \) matrix in spinor space as well. All fields are defined in Minkowski spacetime with metric signature \((+, -, -, -)\). In the nonminimal SME, \( \hat{Q} \) is an expansion in terms of derivatives \( \partial_\mu \), in position space, or momenta, \( p_\mu = i\partial_\mu \), in momentum space. In spinor space, \( \hat{Q} \) is decomposed into the 16 Dirac bilinears:

\[
\hat{Q} = \hat{\mathcal{S}} + i\hat{\mathcal{P}}\gamma_5 + \hat{\mathcal{V}}^\mu\gamma_\mu + \hat{\mathcal{A}}^\mu\gamma_5\gamma_\mu + \frac{1}{2}\hat{\mathcal{T}}^{\mu\nu}\sigma_{\mu\nu},
\]

with the chiral Dirac matrix \( \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \) and the commutator \( \sigma_{\mu\nu} \) of two Dirac matrices: \( \sigma_{\mu\nu} \equiv i/2[\gamma_\mu, \gamma_\nu] \). The scalar, pseudoscalar, and vector operators \( \hat{\mathcal{S}}, \hat{\mathcal{P}}, \) and \( \hat{\mathcal{V}} \) contain the \( a, c, e, f, m, \) and \( m_5 \) coefficients, cf. Eq. (7) in [22]. The \( m_5 \) coefficients can be absorbed into the physical fields by a chiral transformation but they are usually stated for completeness. The remaining ones were subject to studies in [38]. In the latter reference, certain properties of quantum field theories, based on these coefficients, were examined. Such frameworks do not break spin degeneracy, which means that the characteristics of a particle do not depend on the direction of the spin projection along the quantization axis. Hence, there is only a single dispersion relation and a single spinor for both particles and antiparticles. We will encounter this behavior in the course of the article.

Before solving the modified Dirac equation, we are interested in the propagators of the frameworks that rest on the operators of Eq. (2.3). In the context of quantum field theory, a propagator

\[
\hat{\mathcal{A}}^\mu = \hat{d}^\mu - \hat{b}^\mu, \quad \hat{\mathcal{T}}^{\mu\nu} = \hat{g}^{\mu\nu} - \hat{H}^{\mu\nu},
\]

\[
\hat{b}^\mu = \sum_{d \text{ odd}} b^{(d)\mu\alpha_1...\alpha_{d-3}}p_{\alpha_1}...p_{\alpha_{d-3}},
\]

\[
\hat{d}^\mu = \sum_{d \text{ even}} d^{(d)\mu\alpha_1...\alpha_{d-3}}p_{\alpha_1}...p_{\alpha_{d-3}},
\]

\[
\hat{H}^{\mu\nu} = \sum_{d \text{ odd}} H^{(d)\mu\alpha_1...\alpha_{d-3}}p_{\alpha_1}...p_{\alpha_{d-3}},
\]

\[
\hat{g}^{\mu\nu} = \sum_{d \text{ even}} g^{(d)\mu\alpha_1...\alpha_{d-3}}p_{\alpha_1}...p_{\alpha_{d-3}},
\]
describes a virtual particle that is generated at one spacetime point and annihilated at another one. Propagators play a role in interaction processes where virtual particles occur. Thereby, due to the physical propagator poles, the contribution to the probability amplitude becomes large whenever the momentum of any virtual particle is nearly on-shell. The propagator \( iS \) is the Green’s function of the free-field equations in momentum space, i.e., it is the inverse of the Dirac operator, \( S^{-1} \), that appears in the field equations (multiplied by an additional factor of \( i \) according to the conventions of [81]). In momentum space the latter is given by Eq. (4) of [22]:

\[
S^{-1} = \gamma^\mu p_\mu - m_\psi \mathbb{1}_4 + \mathcal{Q}.
\]  

(2.4)

The Dirac operator is a \( 4 \times 4 \) matrix in spinor space. Thus, to obtain the propagator this matrix has to be inverted, for which the 16 Dirac bilinears \( \{\Gamma_A\} \equiv \{\mathbb{1}_4, \gamma^\mu, i\gamma^5\gamma^\mu, \sigma^{\mu\nu}\} \) are indispensable. Any complex \( 4 \times 4 \) matrix can be expanded in terms of \( \{\Gamma_A\} \). Hence, for the inverse of the Dirac operator we propose the Ansatz

\[
iS = \frac{i}{\Delta} \left( \xi_{\mu} \gamma^\mu + \sum_{\Lambda} \mathbb{1}_4 + \mathcal{Y} \gamma^5 + \zeta_{\mu} \gamma^5 \gamma^\mu + \psi_{\mu\nu} \sigma^{\mu\nu} \right),
\]

where \( \Delta \) is the overall denominator of the propagator. There are now two possibilities of proceeding. First, Eq. (2.5) can be inserted into \( S^{-1} S = SS^{-1} = \mathbb{1}_4 \). This delivers a system of 16 linear equations in the 16 parameters \( \{\xi, \psi\} / \Delta \) that are themselves functions of the four-momentum components, the particle mass, and the Lorentz-violating coefficients. It is no obstacle to solving this system with computer algebra. The second possibility is to keep in mind that the Dirac bilinears obey the orthogonality relation \( \text{Tr}(\Gamma_A \Gamma_B) = 4 \delta_A^B \), where the appropriate Lorentz indices are lowered to obtain the dual basis \( \{\Gamma_A\} \). Multiplying the inverse of the Dirac operator with each of the Dirac bilinears, each parameter can be obtained relying on the orthogonality condition.

A. Propagator for \( \hat{A}^\mu \)

Now the propagator is obtained for the \( b \) and \( d \) coefficients, that are comprised in the observer Lorentz pseudovector \( \hat{A}^\mu \), according to Eq. (2.3a). The Dirac operator is given by Eq. (2.4), where \( \mathcal{Q} = \hat{A}^\mu \gamma_5 \gamma_\mu \). Either of the two procedures outlined above leads to the result

\[
\hat{\Xi}_A = m_\psi \left( p^2 - m_\psi^2 - \hat{A}^2 \right),
\]

(2.6a)

\[
\hat{\mathcal{Y}}_A = 0,
\]

(2.6b)

\[
\hat{\zeta}_A^\mu = (p^2 - m_\psi^2 + \hat{A}^2) p^\mu - 2(p \cdot \hat{A}) \hat{A}^\mu,
\]

(2.6c)

\[
\hat{\zeta}_A^\mu = 2(p \cdot \hat{A}) p^\mu - (p^2 + m_\psi^2 + \hat{A}^2) \hat{A}^\mu,
\]

(2.6d)

\[
\hat{\psi}_A^{\mu\nu} = m_\psi \epsilon^{\mu\nu\rho\sigma} p_\rho \hat{A}_\sigma,
\]

(2.6e)

\[
\Delta_A = (p + \hat{A})^2 (p - \hat{A})^2 - 2m_\psi (p^2 - \hat{A}^2) + m_\psi^4,
\]

(2.6f)

with the four-dimensional totally antisymmetric Levi-Civita symbol, \( \epsilon^{\mu\nu\rho\sigma} \), where \( \epsilon^{0123} = 1 \). Several remarks are in order. First, the physical dispersion relations are poles of the propagator, i.e.,
they are zeros of the global denominator $\Delta$. The result for $\Delta$ corresponds to the determinant of the Dirac operator that was obtained in Eq. (4) of [70]. For component coefficients that are not associated with additional time derivatives, this is a fourth-order polynomial in $p_0$. Under that condition, there are two particle dispersion relations and (after reinterpretation) two antiparticle dispersion relations. Hence, the usual spin degeneracy of the particle energy is broken for this framework. Note that with additional time derivatives, the number of poles even increases. Second, for vanishing Lorentz violation, $\hat{A}^\mu = 0$, the standard propagator $iS = i(p + m_\psi)/(p^2 - m_\psi^2)$ is recovered. Third, the propagator, as it stands, is valid for both the minimal and nonminimal framework because additional powers of the four-momentum, which are contained in $\hat{A}^\mu$, do not modify the overall propagator structure.

B. Propagator for $\hat{T}^{\mu\nu}$

The second framework considered is based on a nonvanishing observer two-tensor $\hat{T}^{\mu\nu}$ in Eq. (2.3a), which contains both the $H$ and the $g$ coefficients. The Dirac operator includes the modification $\hat{Q} = \hat{T}^{\mu\nu}\sigma_{\mu\nu}/2$. The propagator can be obtained in analogy to the previous framework and its structure is a bit more involved:

$$\hat{\Xi}_{\hat{T}} = m_\psi(p^2 - m_\psi^2 + 2X),$$

$$\hat{\Upsilon}_{\hat{T}} = -2im_\psi Y,$$

$$\hat{\xi}^\mu_{\hat{T}} = (p^2 - m_\psi^2 - 2X)p^\mu - 2V^\mu,$$

$$\hat{\zeta}^\mu_{\hat{T}} = 2m_\psi\tilde{\hat{T}}^{\mu\nu}p^\nu,$$

$$\hat{\psi}^{\mu\nu}_{\hat{T}} = \left[X - \frac{1}{2}(p^2 + m_\psi^2)\right]\hat{\tau}^{\mu\nu} + Y\hat{\tau}^{\mu\nu} + \hat{T}^{\mu\nu}p_\rho p^\rho - p^\mu\hat{\tau}^{\nu\rho}p_\rho,$$

$$\Delta_{\hat{T}} = (p^2 - m_\psi^2 - 2X)^2 + 4(Y^2 - V \cdot p - 2m_\psi^2 X),$$

$$V^\mu \equiv \tilde{\hat{T}}^{\mu\nu}\hat{\sigma}_\nu p^\sigma,$$

$$X \equiv \frac{1}{4}\hat{\tau}^{\mu\nu}\hat{T}_{\mu\nu}, \quad Y \equiv \frac{1}{4}\tilde{\hat{T}}^{\mu\nu}\hat{T}_{\mu\nu}, \quad \tilde{\hat{T}}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\hat{\tau}_{\rho\sigma}.$$

Again several remarks are in order. First, the zeros of the overall denominator $\Delta$ correspond to the physical dispersion relations. Note that $\Delta$ is equal to the determinant of the Dirac operator, which is stated in Eq. (15) of [70]. Such as for $\hat{A}^\mu$, the denominator $\Delta$ has at least four zeros. Second, the propagator has a pseudoscalar part that is characterized by a nonvanishing quantity, $Y$, linked to the dual tensor of $\hat{T}^{\mu\nu}$.

It is interesting to observe that computing the propagators is much easier than obtaining the spinors, as we shall see. The reason is that propagators are off-shell objects, where $p^0$ is a free quantity. However, the spinor solutions of the Dirac equation are valid on-shell. Exact dispersion relations can become highly complicated even for simple choices of component coefficients, for example when they produce third-order polynomials in $p^0$. It may not even possible to give
solutions of polynomials with order higher than four in a closed form. Hence, it is not surprising that the complexity of the spinors follows the complexity of the dispersion laws.

III. DISPERSION RELATIONS AND SPINORS

The Lorentz-violating background field does not only modify the propagators but it changes the physical energy-momentum correspondences as well. We have already observed this from the modified overall denominator in the propagator, whose poles correspond to these dispersion relations on the one hand. On the other hand, it is possible to obtain the dispersion relations directly from the determinant of the Dirac operator that changes due to Lorentz violation. The modified Dirac operator is stated in Eq. (2.4) with the Lorentz-violating modification $\hat{Q}$. In the current section, the Dirac equation will be solved for multiple frameworks that are considered as exemplary. There are several procedures to do so.

The first method is to simply consider the Dirac equation in momentum space as a homogeneous linear system of equations whose solutions are the Dirac spinors. For the spinor solutions to be nontrivial, the determinant of the Dirac operator has to vanish. This condition sets the hitherto arbitrary zeroth four-momentum component to allowed energy levels. Since the Dirac operator is a 4×4 matrix in spinor space, the polynomial in $p^0$ is at least of degree 4. There are both positive and negative-energy values. The positive ones $E_>$ can be interpreted as the energies for free particles, i.e., $p^0 = E_>(p)$ in this case. The negative ones $E_<$ indicate the appearance of antiparticles, which are reinterpreted according to the procedure of Feynman and Stückelberg to give physically meaningful results. The reinterpretation works such that antiparticles are considered as particles propagating backward in time, which leads to four-momentum components with opposite signs. Hence, for antiparticles we have that $p^0 = -E_<(-p) > 0$ [22, 26, 38]. With $p^0$ set to the allowed energy values, the system of equations has nontrivial solutions for the spinors. For antiparticles, the Feynman-Stückelberg reinterpretation has to be applied not only to the negative-energy solutions but to the whole Dirac operator. Solving the resulting system produces the spinors for antiparticles.

The advantage of this general procedure is to deliver energies and spinors that are exact in Lorentz violation. However, the complexity of both the energies and the spinors rises drastically the more controlling coefficients are taken into account. The reason is that, even at second order in Lorentz violation, more and more possible products of distinct coefficients can be formed with an increasing number of coefficients. For the purpose of demonstration two isotropic cases will be treated in App. A according to the method outlined.

It makes sense to obtain such exact solutions for theoretical reasons, e.g., to investigate the physical consistency of the framework under consideration. However, for phenomenological studies it often suffices to restrain particle energies, spinors, etc. to first order in Lorentz violation. The particle and antiparticle energies can then be obtained from the first-order Hamiltonians given by Eqs. (60) and (64) of [22], respectively. The method of computing the spinors is presented in Sec. III.A of the latter paper. In this context, a unitary matrix $U$ is constructed to block-diagonalize the Dirac operator, which decouples positive and negative-energy states. In the standard case, the matrix $U$ is given by the product of two matrices, $V$ and $W$, with

$$V = \frac{I_4 + \gamma_0 \gamma_5}{\sqrt{2}}, \quad W(p) = \frac{E_0 + m_\psi + p \cdot \gamma}{\sqrt{2E_0(E_0 + m_\psi)}},$$

(3.1)
where $E_0 = \sqrt{p^2 + m^2}$ is the standard dispersion relation for fermions [22]. Note that for certain Lorentz-violating frameworks such as those based on the $a$, $c$, $e$ and $m$ coefficients, the matrix $U$ that block-diagonalizes the Dirac operator still has this form, where just the four-momentum or the particle mass is replaced by a suitable combination involving the controlling coefficients (for details cf. App. [13]). This works at all orders in Lorentz violation. However, constructing $U$ for the $b$, $d$, $H$, and $g$ coefficients is much more involved, which is why the result is only known at first order in Lorentz violation:

$$U = \left( \mathbb{I}_4 + \frac{1}{4E_0} [\gamma_5, VW \gamma_0 \hat{Q} W^\dagger V^\dagger] \right) VW,$$

(3.2)

where $[\bullet, \bullet]$ denotes the ordinary commutator of two matrices [22]. After block-diagonalization, the Dirac equation is brought into the form $(E - H)U \psi = 0$, with the energy $E$, the Hamiltonian $H$, and the spinor $\psi$. This is an eigenvalue problem for the energies and the eigenvectors $U \psi$. Having these eigenvectors at hand allows for computing the spinors by multiplying the eigenvectors with $U^\dagger$. The approach developed is similar to the Foldy-Wouthuysen procedure [22] in spirit. Note that all computations will be carried out with the Dirac matrices in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix},$$

(3.3a)

using the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(3.3b)

and the identity matrix $\mathbb{I}_2$ in two dimensions.

A. General properties of particle and antiparticle energies

In the standard case and for some Lorentz-violating frameworks such as those based on the $a$ and $c$ coefficients, the particle energy is degenerate with respect to the particle spin, i.e., the energy is the same no matter in what direction along the quantization axis spin points. However, the frameworks based on the $b$, $d$, $H$ and $g$ coefficients are nondegenerate with respect to the particle spin. Therefore, a particle has two possible energy states that are denoted as $E^{(\pm)}(\mu \nu ...)$, where the ordering of the energies is chosen along the lines of App. B in [16]. We use a notation for the particle energies similar to that introduced in [26].

Since there exist two distinct positive energies, there are two distinct negative ones that will be called $E^{(\pm)}_<(\mu \nu ...)$. Based on the transformation properties of controlling coefficients under charge conjugation $C$, the positive energies are related to the negative ones as follows [26]:

$$E^{(\pm)}_<(\mu \nu ...) = -E^{(\mp)}_>(-\mu \nu ...),$$

(3.4)

Hence, only the $d$ and $H$ coefficients appear with opposite signs on both sides of the equation where the signs of the $b$ and $g$ coefficients remain unaffected. This is due to the fact that the $d$ and $H$ coefficients are odd under $C$, whereas the $b$ and $g$ coefficients are even (see Table P31 in [25]).

The ordering of the energies is reversed as well according to App. B of [16] and Eq. (16) of [26]. These negative energies are physically meaningless and they have to be reinterpreted according to
Feynman and Stückelberg. By doing so, the signs of the spatial momentum components and of the energy, which corresponds to the zeroth four-momentum component, have to be reversed:

\[ -E_>(\pm)(-p, -d^{\mu\nu\cdots}, -H^{\mu\nu\cdots}) \rightarrow E_>(\mp)(p, -d^{\mu\nu\cdots}, -H^{\mu\nu\cdots}) \equiv E_v(\mp)(p) , \]

where we employ the notation of [26] for the antiparticle energies as well. Hence, ultimately the antiparticles have the latter positive energy values with both the signs of the \(d\) and \(H\) coefficients reversed. As the \(d\) and \(H\) coefficients violate \(C\), the particle energies are related to the antiparticle energies as follows:

\[ E_u(\pm)(p, d^{\mu\nu\cdots}, H^{\mu\nu\cdots}) = E_u(\mp)(p, -d^{\mu\nu\cdots}, -H^{\mu\nu\cdots}) = E_v(\mp)(p) . \]  

(3.6)

**B. Obtaining the spinors**

The great advantage of using the method of obtaining the spinors, which was outlined in Sec. III will be explained as follows. First of all, it suffices to compute such a spinor for one particular nonzero component coefficient, such as for the minimal coefficient \(b_3^{(3)}\), defining a preferred direction pointing along the third spatial axis of the coordinate system. Having obtained this result, allows for generalizing it to the pseudovector \(\hat{A}\) and even to the appropriate nonminimal frameworks.

For example, with the particle spinors for the coefficient \(b_3^{(3)}\) at hand, we can replace \(b_3^{(3)}\) by \(- (d_3^{(4)} - b_3^{(3)}) \equiv -A_3^{(3)}\), cf. Eq. (7) in [22], which is the generalization within the minimal SME. In a further step at first order in Lorentz violation, the latter \(A_3^{(3)}\) is promoted to an operator, which corresponds to generalizing it to the nonminimal SME:

\[ A^{(3)3} \mapsto \hat{A}^3 \equiv \sum_{d \text{ odd}} A^{(d)3\alpha_1 \cdots \alpha_{d-3}p_{\alpha_1} \cdots p_{\alpha_{d-3}}} . \]  

(3.7)

Similarly, this works for the tensor coefficients \(H\) and \(g\). As a first step, it is reasonable to obtain the spinors for a particular nonzero \(H\) coefficient, such as for \(H_{01}^{(3)}\). The results are then generalized within the minimal SME according to Eq. (7) of [22], i.e., we replace \(H_{01}^{(3)}\) by \(- (g_{01}^{(4)} - H_{01}^{(3)}) \equiv -\mathcal{T}_{01}^{(3)}\).

The generalization to the nonminimal SME amounts to promoting \(\mathcal{T}_{01}^{(3)}\) to an operator:

\[ \mathcal{T}^{(3)01} \mapsto \hat{\mathcal{T}}_{01} \equiv \sum_{d \text{ odd}} \mathcal{T}^{(d)01\alpha_1 \cdots \alpha_{d-3}p_{\alpha_1} \cdots p_{\alpha_{d-3}}} . \]  

(3.8)

So it is wise to start with the simplest component coefficients in this context, which are the minimal \(b\) and \(H\) coefficients and to generalize the spinors obtained based on the previous description. Note that for the spinors at first order in Lorentz violation it is sufficient to replace all additional \(p^0\) components in the Lorentz-violating terms of the particle and antiparticle spinors by the standard energy \(E_0\). It has to be kept in mind, though, that the simple generalization outlined above only works at first order in Lorentz violation. The exact spinors are expected to involve complicated combinations of distinct coefficients of both the minimal and nonminimal SME such as \(b_3^{(3)}\) and \(d_3^{(6)1111}\).

**C. Normalization**

The positive-energy spinors (for particles) are denoted as \(u^{(\alpha)}\) and the negative-energy spinors (reinterpreted for antiparticles according to Feynman and Stückelberg) are denoted as \(v^{(\alpha)}\). Additional subscripts \(E_u^{(\pm)}\) will indicate whether a spinor is associated to an energy \(E_u^{(+)}\) or \(E_u^{(-)}\). This
distinction is necessary because of broken spin degeneracy. For the latter reason, both the two-particle spinors and the antiparticle spinors are automatically orthogonal to each other. Furthermore, the spinors should be normalized. We choose a normalization such that the spinors satisfy
\[ u^{(\alpha)^\dagger} u^{(\alpha)} = 2E^{(\alpha)}_u, \tag{3.9a} \]
\[ v^{(\alpha)^\dagger} v^{(\alpha)} = 2E^{(\alpha)}_v. \tag{3.9b} \]

Note that α is not summed over on the left-hand sides. This normalization has a great advantage when dealing with the optical theorem. There is then no additional global adjustment necessary in the optical theorem to make it work out. We will observe this in Sec. [IV] Except of a factor $2/m_\psi$ on the right-hand sides of these conditions, our normalization corresponds to the normalization of spinors chosen for the minimal SME in [26].

D. Energies and spinors for $\hat{A}^\mu$

We start computing the spinors for the pseudovector operator $\hat{A}^\mu$. All results are understood to be valid for a positive expansion parameter, i.e., for a positive combination of Lorentz-violating coefficients and four-momentum components. If this combination is negative, the labels of the dispersion relations and the spinor solutions just have to be switched. We will elaborate on this at the very end of Sec. [IV].

1. Full “isotropic” operator

It is always a good advice to start with an isotropic framework. First, this is due to practical reasons, since computations are expected to be much simpler when there is no direction dependence. Second, from a phenomenological point of view, controlling coefficients that are associated with preferred spacelike directions are often constrained more strictly in comparison to their isotropic counterparts. The leading-order terms in the dimensional expansion of the $b$, $d$, and $g$ coefficients are chosen to be isotropic, which will be indicated by the standard notation of a ring diacritic (cf. Sec. IV.B in [22]). For the minimal $b$ coefficients, only the zeroth component of the coefficient vector leads to an isotropic dispersion relation, i.e., $\hat{b} \equiv b^{(3)0}$. The minimal $d$ coefficient matrix has to be chosen as a diagonal (traceless) matrix with the spatial components equal to each other, i.e.,
\[ d^{(4)}_{\mu\nu} = \hat{d} \times \text{diag}(1, 1/3, 1/3, 1/3)^{\mu\nu}, \]
where $\hat{d} \equiv d^{(4)00}$. Last but not least, the minimal $g$ coefficients produce isotropic expressions if all spatial coefficients with a totally antisymmetric permutation of indices are equal modulo the sign of the permutation: $g^{(4)ijk} = \hat{g} \times \varepsilon^{ijk}$, with $\hat{g} \equiv g^{(4)123}$. There is no isotropic choice for the minimal $H$ coefficients. Respecting these leading-order choices, we define the total “isotropic” operators as follows:

\[ \hat{\hat{A}} \equiv \hat{\hat{d}}^0 - \hat{\hat{b}}, \tag{3.10a} \]
\[ \hat{\hat{b}} \equiv \sum_{d \text{ odd}} b^{(d)00a_1...a_{d-3}} p_{a_1} \cdots p_{a_{d-3}}, \tag{3.10b} \]
\[ \hat{\hat{d}}^\mu \equiv \hat{\hat{d}} \times \text{diag} \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)^{\mu a_1} p_{a_1}, \quad \hat{\hat{d}} \equiv \sum_{d \text{ even}} d^{(d)00a_2...a_{d-3}} p_{a_2} \cdots p_{a_{d-3}}. \tag{3.10c} \]
\[\hat{g}^{ij} \equiv \hat{g} \times \varepsilon^{ijk} p_k, \quad \hat{g} \equiv \sum_{d \text{ even}} g^{(d)123\ldots\alpha_{d-3}} p_{\alpha_2} \cdots p_{d-3}. \]  
(3.10d)

Including the higher-dimensional contributions, these choices are certainly no longer isotropic. However, they will still be denoted with a ring diacritic to remind the reader of the isotropic nature of the leading-order (minimal) terms. The modified dispersion relations, at first order in the controlling coefficients, read

\[E_u^{(\pm)}(0) = E_0 \left[ 1 \pm \frac{|p|}{E_0^2} \left( \frac{4}{3} E_0 \hat{d} - \hat{b} + m_\psi \hat{g} \right) \right]. \]  
(3.11)

Relying on the second procedure reviewed at the beginning of the current section, the particle spinors at first order in Lorentz violation are calculated and cast into the following shape:

\[u^{(1.2)}(X) = \hat{N}_u^{(1.2)} \hat{U} \left( \hat{A} - \frac{\hat{g}^2}{m_\psi} \right), \]  
(3.12a)

\[\hat{U}(X) = \begin{pmatrix} \phi^+ \\ \hat{\chi}^+ \end{pmatrix} = \begin{pmatrix} m_\psi \pm |p| \\ \pm i |p| \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \hat{\chi}^+ \end{pmatrix} X, \]  
(3.12b)

\[\hat{\chi}^\pm = [E_0 + m_\psi \pm |p|] \begin{pmatrix} p_3 \mp |p| \\ p_1 + ip_2 \end{pmatrix}, \]  
(3.12c)

\[\hat{\chi}^\pm = [E_0 + m_\psi \pm |p|] \begin{pmatrix} p_3 \mp |p| \\ p_1 - ip_2 \end{pmatrix}, \]  
(3.12d)

with the normalization factors

\[\hat{N}_u^{(1.2)} = \frac{1}{4 \sqrt{|p| (|p| \pm p_3) (E_0 + m_\psi)}} \left[ 2 \pm \frac{|p|}{E_0^2} \left( \frac{4}{3} E_0 \hat{d} - \hat{b} + m_\psi \hat{g} \right) \right]. \]  
(3.12e)

The isotropic spinors only depend on the magnitude of the momentum, as expected. The spinors contain a small number of functions dependent on the energy, momentum, and mass with different combinations of signs. Thereby, the Lorentz-violating contribution does not introduce any terms with a different structure. For vanishing Lorentz violation the spinors do not reproduce the standard results given in most textbooks. However, two issues must be taken into account in this context. Firstly, in most textbooks, the spinors are given for the Dirac representation of the Dirac matrices, whereas here the chiral representation is used, cf. Eq. (3.3). Secondly, the standard method to solving the Dirac equation with a decomposition into two-component spinors does not work for the \(b, d, H,\) and \(g\) coefficients. Therefore, the structure of these spinors for vanishing Lorentz violation is more involved than the structure of the usual spinors obtained for the chiral representation of Dirac matrices. Nevertheless, the more complicated standard spinors, deduced from Eq. (3.12a), satisfy the standard Dirac equation. Additionally, the Lorentz-violating term is suppressed by the ratio of the fermion mass and the standard particle energy \(E_0\). The combination of \(m_\psi/E_0^2\) and the Lorentz-violating operators is dimensionless to make the second summand match the mass dimension of the first.

Note that the widely used field-theory model by Myers and Pospelov [83] can be mapped to effective dimension-5 \(a\) and dimension-6 \(g\) coefficients, cf. Eqs. (27), (156), and (157) in [22].
Therefore, it is possible to treat this particular model based on the combined results of [22, 38] and the current article. Choosing a special observer frame, with a purely timelike preferred direction \( u^\mu = (1, 0, 0, 0)^\mu \), allows for identifying the Myers-Pospelov Lagrangian with a subset of the general isotropic SME fermion sector for \( d = 5, 6 \), which is given by Eqs. (97), (98) of [22]. The model of [83] is a special case of the latter equations, where the isotropic effective coefficients \( \hat{a}^{(5)}_0 \) and \( \hat{g}^{(6)}_1 \) are nonvanishing only. Hence, there is a single \( d = 5 \) and a single \( d = 6 \) isotropic degree of freedom out of the eight possible ones for dimension 5 and 6 (see Table III in [22]). The degree of freedom \( \hat{g}^{(6)}_1 \) is a combination of the nonvanishing component coefficients \( b^{(5)000}_5 \equiv b_5 \) and \( g^{(6)ijk00}_6 \equiv \varepsilon^{ijk}g_6 \). Then, the above energies and spinors are valid with

\[
\hat{b} = b_5 E_0^2, \quad \hat{d} = 0, \quad \hat{g} = g_6 E_0^2.
\]

(3.13)

2. Anisotropic operator \( \hat{A}^i \) with \( i = \{1, 2\} \)

After establishing the isotropic result, we intend to consider anisotropic Lorentz violation in the realm of the pseudovector \( \hat{A}^\mu \). The preferred direction is chosen to either point along the first or the second spatial axis of the coordinate system, as both cases can be treated in one go. The particle spinors are obtained in the same manner as before, based on the perturbative method reviewed at the very beginning of the current section. First of all, the particle dispersion relations at first order in Lorentz violation are given by

\[
E_u^{(\pm)}|\hat{A}^i = E_0 \left( 1 \pm \frac{S_i}{E_0} \hat{A}^i \right), \quad S_i = \sqrt{p_i^2 + m_\psi^2}. \tag{3.14}
\]

Quantities such as \( S_i \) are characteristic for anisotropic frameworks and they will appear at various places in the spinors. For all cases that follow, we will only state the first particle spinor \( u^{(1)} \) that is connected to the particle energy \( E_u^{(+)} \). The remaining spinors can be computed from \( u^{(1)} \) by simple transformations, cf. Sec. III D 4. In this context the spinors for the components \( \hat{A}^1 \) and \( \hat{A}^2 \) are closely related to each other, being expressed in terms of a master function \( \hat{U} \):

\[
\begin{align*}
\begin{pmatrix} \hat{A}_u^{(1)} & \hat{A}_u^{(2)} \end{pmatrix} & = \hat{N}^{(1)}_{u} \hat{U} \left( \begin{pmatrix} p, \hat{A}^1, S_1 \end{pmatrix} \right), & \left( 3.15a \right) \\
\begin{pmatrix} \hat{A}_u^{(1)} & \hat{A}_u^{(2)} \end{pmatrix} & = \hat{N}^{(1)}_{u} \begin{pmatrix} -i\hat{U}_1 & -i\hat{U}_2 & -i\hat{U}_3 \\ \hat{U}_1 & \hat{U}_2 & \hat{U}_3 \end{pmatrix} \begin{pmatrix} p_1 = p_2, p_2 = -p_1, p_3, \hat{A}^2, S_2 \end{pmatrix}, & \left( 3.15b \right) \\
\hat{U}(p, X, S_i) & = \begin{pmatrix} \hat{\phi}_+ \\ \hat{\phi}_- \end{pmatrix} + \frac{1}{2E_0^2} \begin{pmatrix} \hat{\phi}_+ \\ \hat{\phi}_- \end{pmatrix} X, & \left( 3.15c \right) \\
\hat{\phi}_\pm & = \begin{pmatrix} \hat{A}_\pm \\ \hat{B}_\pm \end{pmatrix}, \quad \delta \hat{\phi}_\pm = \pm ip_2 \begin{pmatrix} \hat{A}_\pm \\ -\hat{B}_\pm \end{pmatrix} \pm p_3 \begin{pmatrix} -\hat{B}_\pm \\ \hat{A}_\pm \end{pmatrix}, & \left( 3.15d \right) \\
\hat{A}_\pm & = (\pm p_3 - E_0)(S_i \pm p_1) + m_\psi(\pm ip_2 - S_i), & \left( 3.15e \right) \\
\hat{B}_\pm & = m_\psi(E_0 + m_\psi \pm p_3) + (p_1 + ip_2)(p_1 \pm S_i), & \left( 3.15f \right)
\end{align*}
\]
with the normalization factor
\[
\tilde{N}_u^{(1)}(p, X, S_i) = \frac{1}{4\sqrt{(E_0 + m_\psi)S_i^2 - p_1p_3S_i}} \left( 2 + \frac{S_i}{E_0} X \right).
\] (3.15g)

The structure of the spinors is evidently a bit more involved in comparison to the “isotropic” case of Sec. III D 1. The spinors can still be expressed via two different functions including two sign choices for each. The transformations \(p_1 = p_2, p_2 = -p_1\) have to be applied to the normalization factor as well, when computing the spinor for \(\tilde{T}^{02}\). However, note that the momentum components contained in the Lorentz-violating operator itself stay unaffected.

3. Anisotropic operator \(\tilde{A}^3\)

For a framework with a nonzero \(\tilde{A}^3\), the energies correspond to Eq. (3.14), where just the Lorentz-violating operator and the quantity \(S_3\) have to be adapted accordingly:
\[
E_u^{(\pm)}|\tilde{A}^3 = E_0 \left( 1 \pm \frac{S_3}{E_0} \tilde{A}^3 \right), \quad S_3 = \sqrt{p_3^2 + m_\psi^2}.
\] (3.16)

The spinors for this framework can be expressed in a very convenient form:
\[
u^{(1)}|\tilde{A}^3 = \tilde{N}_u^{(1)} \bar{U}(\tilde{A}^3),
\] (3.17a)
\[
\bar{U}(X) = \begin{pmatrix} \bar{\phi}_+ X \\ \bar{\phi}_- \end{pmatrix} + \frac{p_1 + ip_2}{2E_0^2} \begin{pmatrix} \bar{\delta} \bar{\phi}_+ \\ \bar{\delta} \bar{\phi}_- \end{pmatrix} X,
\] (3.17b)
\[
\bar{\phi}_+ = \begin{pmatrix} \tilde{A}_+ \\ \tilde{B}_+ \end{pmatrix}, \quad \delta \bar{\phi}_\pm = \pm \varepsilon \cdot \bar{\phi}_\pm, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\] (3.17c)
\[
\tilde{A}_\pm = (E_0 - S_3)(m_\psi \mp p_3 - S_3),
\] (3.17d)
\[
\tilde{B}_\pm = \pm (p_1 + ip_2)(m_\psi \pm p_3 - S_3),
\] (3.17e)

with the normalization factor
\[
\bar{N}_u^{(1)} = \frac{1}{4\sqrt{S_3(E_0 - S_3)(S_3 - m_\psi)}} \left( 2 + \frac{S_3}{E_0} \tilde{A}^3 \right).
\] (3.17f)

Here, the asterisk means complex conjugation and \(\varepsilon\) is the matrix representation of the two-dimensional Levi-Civita symbol. Hence, the spinors can again be expressed completely in terms of two functions, where additional sign choices must be taken into account. The Lorentz-violating contribution involves a global ratio of certain momentum components over the standard particle energy squared.

4. Second particle spinor and antiparticle spinors

For the pseudovector operator \(\tilde{A}^\mu\), both particle spinors are related to each other by changing the sign of certain quantities. For the isotropic case, the latter is the magnitude \(|p|\) of the momentum
and for the anisotropic cases, these are the quantities $S_i$:

\begin{align}
E_u^{(-)} \circ (|p|) &= E_u^{(+)} \circ (-|p|), \quad u^{(2)} |E_u^{(-)} p| = u^{(1)} |E_u^{(+)} p|, \\
E_u^{(-)} |\hat{A}^i (S_i) &= E_u^{(+)} |\hat{A}^i (-S_i), \quad u^{(2)} |E_u^{(-)} \hat{A}^i (S_i) &= u^{(1)} |E_u^{(+)} \hat{A}^i (-S_i). \tag{3.18b}
\end{align}

Hence, the objects $|p|$ and $S_i$ are essential as they control both the particle energies and the spinor types. The antiparticle spinors can be computed from the particle spinors by applying the charge conjugation matrix $C$ to them, cf. Eq. (21) of [25]. Independently of any representation, the charge-conjugated spinor reads $\psi_c = C \psi^T$, where $C = B (\gamma^0)^T$ with $-(\gamma^\mu)^* = B^{-1} \gamma^\mu B$. In the chiral representation, we find that $B = i \gamma^2$ and thus $\psi_c = i \gamma^2 (\gamma^0)^T \psi^a = i \gamma^2 \psi^a$. Therefore, the charge-conjugated spinors in the special frameworks considered result in

\begin{equation}
v^{(1,2)} |E_u^{(\pm)} = \begin{pmatrix}
\begin{array}{c}
u_4^{(2,1)} \
-\nu_3^{(2,1)} \
-\nu_2^{(2,1)} \
\nu_1^{(2,1)}
\end{array}
\end{pmatrix}^* \begin{pmatrix}
-d^{\mu\nu\ldots}
\end{pmatrix}, \quad N_v^{(1,2)} = N_u^{(2,1)}(-d^{\mu\nu\ldots}), \tag{3.19}
\end{equation}

where $N_v^{(1,2)}$ are the spinor normalization constants for the antiparticle spinors. So the spinor components just have to be rearranged including additional signs at appropriate positions (followed by a complex conjugation). Recall that the signs of the $d$ coefficients have to be reversed as well, since the latter are odd under charge conjugation, cf. Table P31 in [25].

### E. Energies and spinors for $\hat{T}^{\mu\nu}$

The second type of frameworks that shall be considered are based on the two-tensor operator $\hat{T}^{\mu\nu}$ that comprises both the $H$ and $g$ coefficients. The structure of the spinors is expected to be more complicated than the previous results for $\hat{A}^\mu$. We consider only one of the six nonzero components of $\hat{T}^{\mu\nu}$ at a time.

#### 1. Operator $\hat{T}^{0i}$ with $i = \{1, 2\}$

The spinor results for $\hat{T}^{0i}$, with $i = \{1, 2\}$, are observed to be related to each other. Once the first particle spinor for one of the two operators is known, the spinor for the other is obtained by rearranging its components and by relabelling the momentum components. A similar behavior was observed for the anisotropic $\hat{A}^{1,2}$ in Sec. III D 2. First, the modified particle dispersion relations read

\begin{equation}
E_u^{(\pm)} |\hat{T}^{0i} = E_0 \left(1 \pm \frac{S_i}{E_0^2} \hat{T}^{0i}\right), \quad S_i = \sqrt{p^2 - p_i^2}. \tag{3.20}
\end{equation}

In contrast to the quantities $S_i$ that were defined in the context of the pseudoscalar operator $\hat{A}^\mu$, such as in Eq. (3.14), the current $S_i$ do not depend on the fermion mass, but only on certain momentum components. The spinors associated to the particle energy $E_u^{(+)}$ for both operators are based on a single master function $\hat{U}$ and they are given by:

\begin{equation}
\begin{aligned}
u^{(1)} |E_u^{(+)} = \hat{N}^{(1)} \hat{U} \left(p, \hat{T}^{01}, S_1\right),
\end{aligned} \tag{3.21a}
\end{equation}
\[ u^{(1)}_{\tilde{E}^{(+)}} = \hat{N}^{(1)}_{u} \begin{pmatrix} -i\hat{U}_1 \\ \hat{U}_2 \\ -i\hat{U}_3 \\ \hat{U}_4 \end{pmatrix} \begin{pmatrix} p_1 = p_2, p_2 = -p_1, p_3, \tilde{T}^{02}, S_2 \end{pmatrix}, \]  
\text{(3.21b)}

\[ \hat{U}(p, X, S_i) = \left( \frac{\delta \hat{\phi}_+}{\delta \hat{\phi}_-} \right) X, \]  
\text{(3.21c)}

\[ \hat{\phi}_\pm = \begin{pmatrix} \hat{A}_\mp \\ \hat{B}_\pm \end{pmatrix}, \quad \delta \hat{\phi}_\pm = i p_1 \begin{pmatrix} \hat{A}_\mp \\ \hat{B}_\pm \end{pmatrix} \pm i m_\psi \begin{pmatrix} \hat{A}_\mp \\ \hat{B}_\pm \end{pmatrix}, \]  
\text{(3.21d)}

\[ \hat{A}_\pm = i(E_0 + m_\psi)(p_2 - S_i) \pm p_3(p_1 - iS_i), \]  
\text{(3.21e)}

\[ \hat{B}_\pm = p_3(E_0 + m_\psi \pm p_3) \pm (p_2 - ip_1)(p_2 - S_i), \]  
\text{(3.21f)}

with the normalization factor
\[ \hat{N}^{(1)}_{u} = \frac{1}{4} \sqrt{\frac{S_i + p_2}{p_3^2 S_i(E_0 + m_\psi)}} \left( 2 + \frac{S_i}{E_0^2} \tilde{T}^{03} \right). \]  
\text{(3.21g)}

Again, the transformation \( p_1 = p_2, p_2 = -p_1 \) must be applied to the normalization factor as well, but the momentum components within the Lorentz-violating operator should not be touched, cf. Sec. III D 2. Furthermore, we have observed that these results are very similar to the spinors for \( \hat{A}^{1,2} \) in structure. There are the following correspondences between the parameters that appear in both spinors:
\[ \hat{A}_{1,2} = i\hat{A}_{1,2}(p_1 \leftrightarrow \pm p_2, p_3 \leftrightarrow \pm m_\psi), \]  
\text{(3.22a)}

\[ \hat{B}_{1,2} = \pm \hat{A}_{1,2}(p_1 \leftrightarrow \pm p_2, p_2 \leftrightarrow \mp p_1, p_3 \leftrightarrow \mp m_\psi, m_\psi \leftrightarrow \pm p_3). \]  
\text{(3.22b)}

2. Operator \( \tilde{T}^{03} \)

The particle energies for the operator \( \tilde{T}^{03} \) are given by Eq. (3.20) with \( i = 3 \). The first particle spinor is simpler than the previous two:
\[ u^{(1)}_{\tilde{T}^{03}} = \tilde{N}^{(1)}_{u} \hat{U} \begin{pmatrix} \tilde{T}^{03}, S_3 \end{pmatrix}, \]  
\text{(3.23a)}

\[ \tilde{U}(p, X, S_3) = \left( \frac{\delta \tilde{\phi}_+}{\delta \tilde{\phi}_-} \right) X, \]  
\text{(3.23b)}

\[ \tilde{\phi}_\pm = \begin{pmatrix} (ip_1 + p_2)\tilde{A}_\pm \\ S_3\tilde{A}_\mp \end{pmatrix}, \quad \delta \tilde{\phi}_\pm = \begin{pmatrix} (ip_1 + p_2)\tilde{B}_\pm \\ S_3\tilde{B}_\mp \end{pmatrix}, \]  
\text{(3.23c)}

\[ \tilde{A}_\pm = E_0 \pm iS_3 \mp p_3 + m_\psi, \]  
\text{(3.23d)}

\[ \tilde{B}_\pm = ip_3(\tilde{A}_\mp - m_\psi) + im_\psi(p_3 \mp \tilde{A}_\mp), \]  
\text{(3.23e)}
with the normalization constant
\[ \bar{N}_u^{(1)} = \frac{1}{4|S_3| \sqrt{E_0 + m_\psi}} \left(2 + \frac{S_3}{E_0^2} \hat{T}^{03}\right). \] (3.23f)

Two functions including sign choices are sufficient to parameterize the solution of the Dirac equation. Note the absolute-value bars around \( S_3 > 0 \) in the normalization. They are stated explicitly to indicate that the normalization does not change sign for the antiparticle spinors when \( S_3 \) is replaced by \(-S_3\), cf. Sec. III E 5 below. This is also the only case where a quantity \( S_i \) appears outside of a square root function in the standard part of the normalization factor.

3. Operator \( \hat{T}^{i3} \) with \( i = \{1, 2\} \)

Last but not least, we want to state the spinor solutions for the \( \hat{T}^{\mu\nu} \) operator with two spatial indices nonvanishing. The subsequent results are simpler than the previous ones for \( \hat{T}^{0i} \) with one timelike index. First of all, the energy levels at first order in Lorentz violation can be expressed as before, with anisotropic quantities \( S_{ij} \) that here depend on the particle mass:
\[ E_u(\pm) \hat{T}^{ij} = E_0 \left(1 \pm \frac{S_{ij}}{E_0} \hat{T}^{ij}\right), \quad S_{ij} = \sqrt{p_i^2 + p_j^2 + m_\psi^2}. \] (3.24)

The spinors for \( \hat{T}^{13} \) and \( \hat{T}^{23} \) are again related to each other, which makes it possible to express them via a single master function \( \bar{U} \). A similar behavior was encountered for \( \hat{A}^{1,2} \) in Sec. III D 2 and for \( \hat{T}^{01,02} \) in cf. Sec. III E 1.

\[ u(1)|\hat{T}^{23}_{E_u(1)} = \bar{N}_u^{(1)} \bar{U}(\mathbf{p}, \hat{T}^{23}, S_{23}), \] (3.25a)

\[ u(1)|\hat{T}^{13}_{E_u(1)} = \bar{N}_u^{(1)} \begin{pmatrix} i\bar{U}_1 \\ i\bar{U}_2 \\ i\bar{U}_3 \\ \bar{U}_4 \end{pmatrix} \begin{pmatrix} p_1 = -p_2, p_2 = p_1, p_3, \hat{T}^{13}, S_{13} \end{pmatrix}, \] (3.25b)

\[ \bar{U}(\mathbf{p}, X, S_{ij}) = \begin{pmatrix} \bar{\phi}_+ \\ \bar{\phi}_- \end{pmatrix} + \frac{p_1}{2E_0^2} \left(\begin{pmatrix} \bar{\phi}_+ \\ \bar{\phi}_- \end{pmatrix} X, \right. \] (3.25c)

\[ \bar{\phi}_\pm = \pm \begin{pmatrix} \bar{A}_\pm \\ \bar{B}_\pm \end{pmatrix}, \] (3.25d)

\[ \bar{A} = \frac{p_1 p_3 + (E_0 + m_\psi) S_{ij}}{E_0(E_0 + m_\psi) - p_1(p_1 + ip_2)}, \] (3.25e)

\[ \bar{A}_\pm = p_1 - ip_2 \pm (E_0 + m_\psi \mp p_3) \bar{A}, \] (3.25f)

\[ \bar{B}_\pm = (p_1 + ip_2) \bar{A} \pm (E_0 + m_\psi \pm p_3), \] (3.25g)

with the normalization factor
\[ \bar{N}_u^{(1)} = \frac{\sqrt{E_0 + m_\psi - p_1 p_3/S_{ij}}}{4(E_0 + m_\psi)} \left(2 + \frac{S_{ij}}{E_0^2} \hat{T}^{ij}\right). \] (3.25h)
In contrast to the spinors for $\hat{T}^{0i}$, only two two-dimensional vectors are sufficient to construct the spinors at first order in Lorentz violation. The same will be true for $\hat{T}^{12}$ below.

### 4. Operator $\hat{T}^{12}$

For the last operator to be considered, the energies are given by Eq. (3.24) with $\{i, j\} = \{1, 2\}$. The first particle spinor reads

$$u^{(1)}\big|_{\hat{T}^{12}} = N^{(1)}_u \tilde{U}(p, \hat{T}^{12}, S_{12}),$$  \hspace{1cm} (3.26a)

$$\tilde{U}(p, X, S_i) = \left(\tilde{\phi}^+ \phi^- + \frac{p_3}{2E_0} \left(\tilde{\phi}^- \phi^+ \right) X \right),$$  \hspace{1cm} (3.26b)

$$\tilde{\phi}_\pm = \pm \left(\begin{array}{c} -\ddot{A}_\pm \\ \dddot{B}_\pm \end{array} \right),$$  \hspace{1cm} (3.26c)

$$\ddot{A} = \frac{(E_0 + m_\psi)(E_0 - S_{12}) - p_3^2}{(p_1 + ip_2)p_3},$$  \hspace{1cm} (3.26d)

$$\dddot{A}_\pm = p_1 - ip_2 \pm (E_0 + m_\psi \mp p_3)\ddot{A},$$  \hspace{1cm} (3.26e)

$$\dddot{B}_\pm = (p_1 + ip_2)\dddot{A}_\pm \pm (E_0 + m_\psi \pm p_3),$$  \hspace{1cm} (3.26f)

$$\hat{N}^{(1)}_u = \sqrt{\frac{E_0 + m_\psi + S_{12} + E_0m_\psi/S_{12}}{4(E_0 + m_\psi)} \left(2 + \frac{S_{12}}{E_0^2}\hat{T}^{12}\right)}.$$  \hspace{1cm} (3.26g)

Evidently, the form of the spinor is analogous to the results for $\hat{T}^{01}$ and $\hat{T}^{02}$. Furthermore, $\dddot{A}_\pm$, $\dddot{B}_\pm$ have the same form as the previous $\dddot{A}_\pm$, $\dddot{A}_\pm$. However, $\dddot{A} \neq \dddot{A}$ and the structure of the normalization factors differs from each other, too.

### 5. Second particle spinor and antiparticle spinors

For the tensor operator $\hat{T}^{\mu\nu}$ the first particle spinors only were given previously. The second particle spinor can be obtained from the first in an easy way in replacing all quantities $S_i$ or $S_{ij}$ by their counterparts with opposite sign:

$$E^{(-)}_u \big|_{\hat{T}^{0i}}(S_i) = E^{(+)}_u \big|_{\hat{T}^{0i}}(-S_i), \quad E^{(-)}_u \big|_{\hat{T}^{ij}}(S_{ij}) = E^{(+)}_u \big|_{\hat{T}^{ij}}(-S_{ij}),$$  \hspace{1cm} (3.27a)

$$u^{(2)}\big|_{\hat{T}^{0i}}^{(-)}(S_i) = u^{(1)}\big|_{\hat{T}^{0i}}^{(+)}(-S_i), \quad u^{(2)}\big|_{\hat{T}^{ij}}^{(-)}(S_{ij}) = u^{(1)}\big|_{\hat{T}^{ij}}^{(+)}(-S_{ij}).$$  \hspace{1cm} (3.27b)

In analogy to the pseudovector operator, the types of energy and spinor are controlled by the quantities $S_i$ and $S_{ij}$. Also, the antiparticle spinors for $\hat{T}^{\mu\nu}$ are determined completely by the components of the particle spinors. With the charge-conjugated spinor $\psi_c = i\gamma^2\psi^*$ in the chiral
TABLE I: Collection of modified particle energies and spinors.

representation, we obtain them directly from Eq. (21) of [26]:

\[
v^{(1,2)}|_{E^{(\pm)}_u} = \begin{pmatrix} u^{(2,1)}_1 \\
-u^{(2,1)}_2 \\
-u^{(2,1)}_3 \\
u^{(2,1)}_4 \end{pmatrix}^* (-H^{\mu\nu...}), \quad N_{\psi}^{(1,2)} = N_{\psi}^{(2,1)} (-H^{\mu\nu...}). \tag{3.28}
\]

The signs of the \( H \) coefficients also have to be reversed, since these are odd under charge conjugation, cf. Table P31 in [25].

F. Additional observations

It is interesting to note that some of the spinors can be cast into a very simple shape by factoring the Lorentz-invariant contribution out of each component. This can be carried out for the “isotropic case,” for each of the spacelike components of the pseudovector operator \( \hat{A}^\mu \), and for the two-tensor operator components \( \hat{T}^{03}, \hat{T}^{12} \). For these cases, it is possible to express each spinor component of the first particle spinor in the following form:

\[
u_{i}^{(1)}|_{E^{(\pm)}_u} = \begin{pmatrix} \hat{\phi}_{-} \\
\hat{\chi}_{+} \end{pmatrix}_i (1 - \theta \gamma_5), \quad \theta = \frac{1}{2E_0^2} \left( \hat{A} - \frac{g^2}{m_\psi} P^\mu \right), \tag{3.29a}\]

\[
u_{i}^{(1)}|_{E^{(\pm)}_u} = \begin{pmatrix} \hat{\phi}_{+} \\
\hat{\phi}_{-} \end{pmatrix}_i (1 - \iota \gamma_5), \quad \iota = \frac{\hat{A}^1}{2E_0} \left( \frac{S_1}{E_0} p_3 + i p_2 \right), \tag{3.29b}\]

\[
u_{i}^{(1)}|_{E^{(\pm)}_u} = \begin{pmatrix} \hat{\phi}_{+} \\
\hat{\phi}_{-} \end{pmatrix}_i (1 - \kappa \gamma_5), \quad \kappa = \frac{\hat{A}^3}{2E_0}, \tag{3.29c}\]

| Coeff. \( X \) | \( E^{(\pm)}_u / E_0 \) | Spinors | Definitions |
|---|---|---|---|
| \((4/3)E_0\hat{d} - \hat{b} + m_\psi \hat{g}\) | 1 ± (\(|\mathbf{p}|/E_0^2\)) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{A}^1 \) | 1 ± \((S_i/E_0^2)\) | \( S_i \equiv \sqrt{p_i^2 - p_i^2} \) |
| \( \hat{A}^2 \) | Eq. (3.15a) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{A}^3 \) | Eq. (3.17a) | \( S_i \equiv \sqrt{p_i^2 - p_i^2} \) |
| \( \hat{T}^{01} \) | 1 ± \((S_i/E_0^2)\) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{T}^{02} \) | Eq. (3.21a) | \( S_i \equiv \sqrt{p_i^2 - p_i^2} \) |
| \( \hat{T}^{03} \) | Eq. (3.23a) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{T}^{13} \) | 1 ± \((S_{ij}/E_0^2)\) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{T}^{23} \) | Eq. (3.25a) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
| \( \hat{T}^{12} \) | Eq. (3.26a) | \( S_i \equiv \sqrt{p_i^2 + m_\psi^2} \) |
\[ u^{(1)}_i \hat{T}^{03}_{E^{\pm}} = \left( \frac{\tilde{\phi}^+}{\tilde{\phi}^-} \right)_i (1 - \tau \gamma_i), \quad \tau = \frac{\hat{T}^{03}}{2E_0} \left( \frac{S_3}{E_0} p^3 + i m_\psi \right), \]  
(3.29d)

\[ u^{(1)}_i \hat{T}^{12}_{E^{\pm}} = \left( \frac{\tilde{\phi}^+}{\tilde{\phi}^-} \right)_i (1 - \omega \gamma_i), \quad \omega = \frac{\hat{T}^{12}}{2E_0} S \]  
(3.29e)

with the vectors

\[ \mathcal{V} = \begin{pmatrix} |p| + E_0 \\ |p| + E_0 \\ |p| - E_0 \\ |p| - E_0 \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} 1/(p^3 - E_0) \\ 1/(p^3 + E_0) \\ 1/(p^3 + E_0) \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} S + E_0 \\ S - E_0 \\ S + E_0 \\ S - E_0 \end{pmatrix} \]  
(3.29f)

Note the indices \( i \) on both sides of the relations, which refers to a particular component \( i \) of the spinors and defined vectors, respectively. What is characteristic for the stated spinors is the exceptionally simple form of the Lorentz-violating terms. For the second and the fourth of these spinors, only the Lorentz-violating operator \( \hat{X}^{0...} \), are further interesting quantities. They are measures for how much the Lorentz-violating part of the spinors is suppressed compared to the Lorentz-invariant part. The quantities \( \Omega_u \) are simple for all operators considered and they read

\[ \Omega_u = \frac{\sqrt{u^\dagger u |\hat{X}^{\mu...}=0}}{u^\dagger u}, \quad \Omega_v = \sqrt{v^\dagger v |\hat{X}^{\mu...}=0} \]  
(3.30)

where in the latter equal indices are not summed over. The results for both particle spinors correspond to each other. Furthermore, \( \Omega_v = \Omega_u (-d^{\mu\nu}, -H^{\mu\nu}) \) for antiparticles. The findings tell us that Lorentz-violating effects may be additionally suppressed for \(|p| \ll m_\psi\), i.e., for decreasing momentum. First of all, all Lorentz-violating operators have mass dimension 1. As \( \Omega_u \) is dimensionless, the operator has to be divided by a dimensionful scale, where for vanishing momentum the only one is the fermion mass. This holds for the “isotropic case” and the two-tensor operator with one timelike index. For the remaining frameworks, there is an additional suppression factor \( p_i/m_\psi \), with the largest momentum component \( p_i \).

Finally, another issue related to the spinors shall be mentioned at this point. There is a matrix transformation that connects the chiral representation of \( \gamma \) matrices to the Dirac representation \( \gamma^5 \). It can be written in the form \( M_D = V M_{ch} V^{-1} \), where \( M_{ch} \) and \( M_D \) are \( \gamma \) matrices in the chiral and the Dirac representation, respectively. The matrix \( V \) here corresponds to the matrix \( V \) in Eq. (3.1), contained in the transformation that diagonalizes the Dirac operator (with \( \gamma^0 \) and \( \gamma^5 \).
themselves in the chiral representation). The spinors of both representations are then related by \( \psi_D = V \psi_{\text{ch}} \). Hence, all the spinors obtained in this paper can be transformed from the chiral to the Dirac representation according to this rule. We carried this out and considered the limits of the spinors for zero Lorentz violation, where the particle and antiparticle states become spin-degenerate. It was verified successfully that there exist proper linear combinations of the particle spinors \( u^{(\alpha)} \) and of the antiparticle spinors \( v^{(\alpha)} \), such that the standard solutions of the Dirac equation are obtained:

\[
\sum_{\alpha=1,2} \varsigma^{(\alpha)} s u^{(\alpha)} = \left( \sigma \cdot p / (E_0 + m_\psi) \cdot \phi^{(s)} \right), \quad \phi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

\[
\sum_{\alpha=1,2} \tau^{(\alpha)} s v^{(\alpha)} = \left( \sigma \cdot p / (E_0 + m_\psi) \cdot \chi^{(s)} \right), \quad \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

with \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \), where \( \sigma^i \) are the Pauli matrices. The parameters \( \varsigma^{(\alpha)} s \) and \( \tau^{(\alpha)} s \) are chosen appropriately.

### IV. SPINOR MATRICES AND OPTICAL THEOREM

For calculations in high-energy physics that are based on quantum field theory, the spinors are often not needed directly. Instead, when computing matrix element squares spinors are always combined in the form of \( uu \), where the latter is a \( 4 \times 4 \) matrix in spinor space. In the Lorentz-invariant case, it often suffices to calculate unpolarized cross sections, which means that one has to average over initial particle spins. As such a scattering or decay process is a quantum process, the final state is not predictable, which is why the final particle spins must be summed over and the phase space has to be integrated out. Hence, in the standard case the following well-known expressions (with proper normalization of the spinors) are usually extremely helpful:

\[
\sum_{\alpha=1,2} u^{(\alpha)} u^{\dagger (\alpha)} = p + m_\psi \mathbb{I}_4,
\]

\[
\sum_{\alpha=1,2} v^{(\alpha)} v^{\dagger (\alpha)} = p - m_\psi \mathbb{I}_4.
\]

For the Lorentz-violating case with broken spin degeneracy, there are essential differences. First, due to the modified kinematics in Lorentz-violating frameworks, a small part of the phase space of otherwise forbidden particle processes may open and render them possible. Examples are Cherenkov-type processes in vacuo and decays of photons into electron-positron pairs. Since in a theory with broken spin degeneracy both the particles and antiparticles can have two distinct energies, such a process may be allowed only for one of these energies. Under this circumstance, sums over both particle spins in the matrix element square do not have to be carried out as only a single spin state contributes. So what we are interested in are expressions such as \( u^{(\alpha)} u^{\dagger (\alpha)} \) or \( v^{(\alpha)} v^{\dagger (\alpha)} \), with a particular \( \alpha \). From now on these matrices will be referred to as “spinor matrices.” However, for calculational purposes, the matrices themselves are not that useful. Instead, such a matrix should be expressed in terms of the 16 Dirac bilinears. The matrix structure is supposedly more complicated than what we encountered in Eq. (4.1), where the Dirac matrices and the
There are two possibilities of computing the spinor matrices. The first takes into account the spinors that were determined in the previous section. The matrices are calculated directly based on these spinors and they are expressed in terms of the Dirac bilinears. The resulting expressions will be valid at first order in Lorentz violation. The second possibility is to use the optical theorem. The latter gives a relationship between the imaginary part of a forward scattering amplitude to the total cross sections of all processes that are energetically allowed by cutting the propagators in the diagram representing the forward scattering amplitude. The validity of the optical theorem was demonstrated at tree-level for various sets of Lorentz-violating frameworks, such as both minimal and nonminimal modifications of the photon and the fermion sector [31, 32, 36–39]. Therefore, it is not expected to lose its validity for the range of coefficients considered within the current paper. Nevertheless, a cross check will be carried out at first order in Lorentz violation using the spinor matrices obtained from the spinors directly.

To make use of the optical theorem, a particular scattering process is needed. Thus, we have to introduce an interaction and we decide to couple the modified fermions to standard photons. In this context it is reasonable to mention the very recent paper [85], where nonminimal Lorentz violation in the fermion sector is constrained by Penning trap experiments. To do so, fermions are minimally coupled to electromagnetic fields. This amounts to replacing the particle derivative at all occurrences by the covariant derivative \( D_\mu = \partial_\mu - eA_\mu \), where \( A_\mu \) is the vector potential and \( e \) the elementary charge. However, in contrast to the partial derivative, the covariant derivative does not commute, which introduces additional interactions for the nonminimal terms. In principle, when computing the cross section or decay rate of a particular process, all these additional interactions must be taken into account. Since we are only interested in investigating the validity and implications of the optical theorem for fermions and since this calculation will be purely formal, the exact form of the interaction is not important and we can even resort to the standard one. The fermion propagator and the external particles, which are described by spinors, lie in the center of our considerations.

The particular process to be analyzed is one of the two possible contributions of Compton scattering with an incoming photon and electron scattering and producing another photon-electron pair, cf. Fig. 1. The second contribution with both vertices interchanged is not needed in this formal analysis. The left-hand side of the equation, presented in the latter figure, contains the forward
scattering amplitude of this process, where the initial and the final state correspond to each other. Based on the optical theorem, the imaginary part of the left-hand side is linked to the total cross section of a photon absorption process shown on the right-hand side. The left-hand side of the equation contains a propagator where the right-hand side involves external fermions and, therefore, spinors that are combined into spinor matrices.

We consider the incoming electron to be in the spin-state associated to the energy $E_u^{(+)}$; it is then described by the spinor $u^{(1)}$. The standard Feynman rules for the external photons and the vertices can be employed to write up the matrix element $\mathcal{M} \equiv \mathcal{M}(e^- \gamma \rightarrow e^- \gamma)$ of the forward scattering amplitude (cf. also [81]):

$$\mathcal{M} = -\int \frac{d^4 p}{(2\pi)^4} \delta^{(4)}(k_1 + p_1 - p)e^{2i\pi^{(\alpha)}(p_1)\gamma^\mu S(p)\gamma^\mu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}(k_1)\varepsilon^\dagger(\lambda)(k_1)}, \quad (4.2)$$

where Lorentz violation just sits in the modified fermion propagator $S$. Furthermore, $\varepsilon^{(\lambda)}$ is the standard polarization vector of a photon in the polarization state $\lambda$. The $\delta$ function directly behind the integration measure enforces energy-momentum conservation. The fermion propagator is either taken from Eq. (2.6) or Eq. (2.7), dependent on whether Lorentz violation based on $A^\mu$ or $T^{\mu\nu}$ is considered. However, for this formal calculation, it will be chosen generically. The Feynman-type propagator has to be used in the amplitude, which means that an infinitesimal imaginary part must be added to the denominator:

$$\frac{1}{\Delta + i\epsilon} = \mathscr{Z}\left(\frac{1}{p^0 - E_u^{(+)} + i\epsilon}\right)\left(\frac{1}{p^0 - E_u^{(-)} + i\epsilon}\right)\left(\frac{1}{p^0 - E_\lambda^{(+)} - i\epsilon}\right)\left(\frac{1}{p^0 - E_\lambda^{(-)} - i\epsilon}\right), \quad (4.3)$$

with a global prefactor $\mathscr{Z}$ that does not depend on $p^0$. The imaginary number $\epsilon$ prevents singularities emerging for a vanishing $\Delta$, which is exactly what produces the imaginary part of the forward scattering amplitude. The denominator has four roots in $p^0$, which are composed of two positive ones $E_u^{(\pm)}$ and two negative ones $E_\lambda^{(\pm)}$. The imaginary parts are added to the poles such that the two positive poles are shifted to the lower complex half plane and the two negative poles are shifted to the upper one. In Sec. V it will be investigated under which circumstances the form of Eq. (4.3) fails to be valid and how such a case can be treated. Energy-momentum conservation of the process shall allow $\Delta$ to vanish for $p^0 = E_u^{(+)}$ only. To evaluate the expression, the following identity is used

$$\frac{1}{p^0 - E_u^{(+)} + i\epsilon} = \mathcal{P}\frac{1}{p^0 - E_u^{(+)}} - i\pi\delta\left(p^0 - E_u^{(+)}\right), \quad (4.4)$$

where $\mathcal{P}$ is the principle value. The first term is real, i.e., it does not contribute to the imaginary part of $\mathcal{M}$. The second term does so only and it forces $p^0$ to be equal to the energy $E_u^{(+)}$ when evaluating the integral over $p^0$. That corresponds to cutting the propagator in the Feynman diagram on the left-hand side of Fig. 1 into two lines, which represent on-shell fermions described by the spinor $u^{(1)}$. The imaginary part of the forward scattering amplitude can then be cast into the form

$$2\text{Im}(\mathcal{M}) = \int \frac{d^3 p}{(2\pi)^3 2E_u^{(+)}(k_1 + p_1 - p)e^{2i\pi^{(\alpha)}(p_1)\gamma^\mu S(p)\gamma^\mu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}(k_1)\varepsilon^\dagger(\lambda)(k_1)}|_{p^0 = E_u^{(+)}}$$

$$\times \left(\tilde{\xi}^{\mu\gamma_\mu} + \tilde{\Xi} + \tilde{\zeta}^{\mu\gamma_5\gamma_\mu} + \tilde{\psi}^{\mu\nu}\sigma_{\mu\nu}\right)|_{p^0 = E_u^{(+)}}$$
Then, the forward scattering amplitude

\[ \times \gamma^\mu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}_\mu(k_1)\bar{\varepsilon}^{(\lambda)}(k_1), \]  

(4.5a)

with the generic function

\[ \mathcal{G}(a, b, c) \equiv \frac{2p^0}{\mathcal{F}(p^0 - a)(p^0 - b)(p^0 - c)}. \]  

(4.5b)

The latter function, evaluated at \( p^0 = E_u^{(+)} \), is the remainder of the denominator \( \Delta \). The additional factor of \( 2p^0 \) is introduced to cancel the factor of twice the particle energy that has been generated in the denominator of the integral measure. Thus, what remains from the propagator is the scalar function \( \mathcal{G} \) and the matrix structure, which is expressed in terms of the Dirac bilinears and which is evaluated at \( p^0 = E_u^{(+)} \). In principle, it is not difficult to take into account an additional prefactor in the denominator of Eq. (4.3), which is independent of \( p^0 \). Such a prefactor would now appear in the denominator of Eq. (4.5b).

Due to the validity of the optical theorem, Eq. (4.5a) must be equal to the total cross section \( \sigma \equiv \sigma(e^-\gamma \rightarrow e^-) \) of the process on the right-hand side of Fig. [4.1]. Thereby, the initial electron is in the spin-state associated with the energy \( E_u^{(+)} \), for which reason it is described by the spinor \( u^{(1)} \). With \( \tilde{\mathcal{M}} \equiv \mathcal{M}(e^-\gamma \rightarrow e^-) \), we obtain:

\[ \sigma = \int \frac{d^3p}{(2\pi)^3E_u^{(+)}(k_1 + p_2 - p)|\tilde{\mathcal{M}}|^2} \]
\[ = \int \frac{d^3p}{(2\pi)^3E_u^{(+)}(k_1 + p_2 - p)} \]
\[ \times \left( i\varepsilon^{(1)}(p)\gamma^\nu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}(k_1) \right) \]
\[ \times i\varepsilon^{(1)}(p)\gamma^\mu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}(k_1), \]
\[ = \int \frac{d^3p}{(2\pi)^3E_u^{(+)}(k_1 + p_1 - p)} \]
\[ \times e^{2\varepsilon^{(1)}(p_1)\gamma^\nu u^{(\alpha)}(p_1)\bar{\varepsilon}^{(\lambda)}(k_1)} \gamma^\mu u^{(\alpha)}(p_1)\varepsilon^{(\lambda)}(k_1)\bar{\varepsilon}^{(\lambda)}(k_1). \]  

(4.6)

A comparison of the latter final result with Eq. (4.5a) allows for deriving an expression for the spinor matrix formed from \( u^{(1)} \). The whole computation can be performed for an electron in the spin state connected to \( E_u^{(-)} \) in an analogous way leading to the spinor matrix obtained from \( u^{(2)} \). The results will be given below. Similarly, it is possible to derive spinor matrices for the antiparticle spinor \( v^{(1)} \). To do so, the same process is considered with only the electron replaced by a positron. Then, the forward scattering amplitude \( \mathcal{M} \equiv \mathcal{M}(e^+\gamma \rightarrow e^+\gamma) \) reads

\[ \mathcal{M} = \int \frac{d^4p}{(2\pi)^4} \delta^{(4)}(k_1 + p_1 - p)e^{2\varepsilon^{(1)}(p_1)\gamma^\nu S(-p)\gamma^\mu v^{(\alpha)}(p_1)}\varepsilon^{(\lambda)}(k_1)\bar{\varepsilon}^{(\lambda)}(k_1). \]  

(4.7)

The propagator has to be evaluated at the negative of the four-momentum to take into account that the momentum of the internal line flows in a direction opposite to the arrow on this line. Besides, a global prefactor of \( -1 \) has to be considered to account for the interchange of fermionic operators when applying Wick’s theorem [38]. First of all, we evaluate the denominator of the propagator for the four-momentum components with their signs reversed:

\[ \Delta_{\tilde{A},\tilde{A}}(-p) = \left( -p^0 - E_u^{(+)} \right) \left( -p^0 - E_u^{(-)} \right) \left( -p^0 - E_<(^{+)} \right) \left( -p^0 - E_<(^{+}) \right) |_{p \rightarrow -p}. \]
\[
2\text{Im}(\mathcal{M}) = \int \frac{d^3 p}{(2\pi)^3 2E_v^{(+)}} \delta^{(4)}(k_1 + p_1 - p)e^2 \bar{\pi}^{(\alpha)}(p_1) \gamma^\nu \\
\quad \times \left[ -\mathcal{C} \left( E_v^{(-)}, E_v^{(+)}, \{0\}^{\mu\nu}, H^{\mu\nu} \right) \right] \bigg|_{p^\mu = E_v^{(+)}} \\
\quad \times \left( \tilde{\xi}^\mu \gamma_\mu + \tilde{\Xi} + \tilde{\xi}^\alpha \gamma_\mu + \tilde{\psi}_\mu^\alpha \sigma_{\mu\nu} \right) \bigg|_{p^\mu = -E_v^{(+)}} \\
\quad \times \gamma^\mu v^{(\alpha)}(p_1) \varphi^{(\lambda)}(k_1) \bar{\varphi}^{(\lambda)}(k_1), \tag{4.9}
\]

with the scalar function \( \mathcal{C} \) defined in Eq. (4.5b). The remainder of the antifermion propagator \( S(-p) \) contains the latter scalar function and the matrix structure of the propagator, which is evaluated at the antiparticle energy \( E_v^{(+)} \) and with all spatial momentum components replaced by their counterparts with opposite sign. The optical theorem says that Eq. (4.9) is related to the cross section \( \tilde{\sigma} \equiv \sigma(e^+\gamma \rightarrow e^+) \), with the amplitude \( \tilde{\mathcal{M}} = \mathcal{M}(e^+\gamma \rightarrow e^+) \), given as

\[
\tilde{\sigma} = \int \frac{d^3 p}{(2\pi)^3 2E_v^{(+)}} \delta^{(4)}(k_1 + p_2 - p)|\tilde{\mathcal{M}}|^2 \\
= \int \frac{d^3 p}{(2\pi)^3 2E_v^{(+)}} \delta^{(4)}(k_1 + p_1 - p) \\
\quad \times ie\bar{\pi}^{(\alpha)}(p_1)\gamma^\mu v^{(1)}(p)\varphi^{(\lambda)}(k_1) \left( ie\bar{\pi}^{(\alpha)}(p_1)\gamma^\nu v^{(1)}(p)\varphi^{(\lambda)}(k_1) \right) \dagger \\
= \int \frac{d^3 p}{(2\pi)^3 2E_v^{(+)}} \delta^{(4)}(k_1 + p_1 - p) \\
\quad \times e^2 \bar{\pi}^{(\alpha)}(p_1)\gamma^\mu \left[ v^{(1)}(p)\pi^{(1)}(p) \right] \gamma^\nu v^{(\alpha)}(p_1)\varphi^{(\lambda)}(k_1) \bar{\varphi}^{(\lambda)}(k_1), \tag{4.10}
\]

The calculation can be carried out one-to-one for the antiparticle energy \( E_v^{(-)} \). With these results at hand, we are able to simply read the spinor matrices \( v^{(\alpha)} \pi^{(\alpha)} \) off by comparing Eqs. (4.9) and (4.10) directly to each other.

It is possible to treat the frameworks of the pseudovector coefficients \( \hat{A}_\mu \) and the two-tensor coefficients \( \hat{\mathcal{T}}^{\mu\nu} \) in one go. The spinor matrices for particles and antiparticles finally read as follows:

\[
u^{[1,2]}(\pi^{[1,2]})|_{E_v^{(\pm)}} = \mathcal{C}(E_v^{(\mp)}, E_v^{(+)}, E_v^{(-)})|_{p^\mu = E_v^{(\pm)}} \\
\quad \times \left( \tilde{\xi}^\mu \gamma_\mu + \tilde{\Xi} + \tilde{\xi}^\alpha \gamma_\mu + \tilde{\psi}_\mu^\alpha \sigma_{\mu\nu} \right) \bigg|_{p^\mu = E_v^{(\pm)}}, \tag{4.11a}
\]
\begin{equation}
\begin{aligned}
    v^{(1,2)}(\pi^{(1,2)})|_{E_v^{(\pm)}} &= -\mathcal{G} \left( E_v^0, E_v^\pm(-d^{\mu\nu...}, -\bar{H}^{\mu\nu...}), E_v^0(-d^{\mu\nu...}, -\bar{H}^{\mu\nu...}) \right)
    \times \left( \tilde{\gamma}^\mu \gamma + \tilde{\gamma} \gamma^\mu \right)
    \left| \begin{array}{c}
        p^\mu = E_v^{(\pm)} \\
    \end{array} \right.
    \left( \frac{1}{2} E_v^{(\alpha)} u^{(\alpha)}(\pi^{(\alpha)})|_{E_v^{(\alpha)}} + \frac{1}{2} E_v^{(\alpha)}(-p) \bar{v}^{(\alpha)}(-p)|_{E_v^{(\alpha)}} \right) \gamma^0 = \mathds{1}_4.
    \end{aligned}
\end{equation}

with the scalar function \( \mathcal{G} \) of Eq. (4.5b). The matrix coefficients have to be taken accordingly either from the propagator for \( \bar{A}^\mu \) or \( \bar{T}^{\mu\nu} \). Several remarks are in order. First, since these expressions result from the exact propagators given in Sec. II A, they are valid at all orders in Lorentz violation. Second, they hold for on-shell particles, which is why all \( p^0 \) must be replaced by the appropriate energies \( E_u^{(\pm)} \) for particles and \( E_v^{(\pm)} \) for antiparticles that are associated to the spinors. Third, the results, as they stand, are only valid when there are no controlling coefficients that produce additional time derivatives. How to treat such cases will be described in the latter Sec. V. Fourth, the validity of Eq. (23) in [26], at first order in Lorentz violation, employing the spinor results directly. Furthermore, we were able to demonstrate the validity of Eq. (23) in [26], at first order in Lorentz violation, employing the spinor results directly. Finally, note that for the “isotropic” case of Sec. III D 1, the \( b \) and \( d \) coefficients have to be treated separately from the \( g \) coefficients, as the first are comprised by \( \bar{A}^\mu \) and the latter are contained in \( \bar{T}^{\mu\nu} \). Besides the general expressions stated above, several special results at first order in Lorentz violation will be given in App. I to make the behavior of the spinor matrices more transparent.

We will not give explicit results that are exact in Lorentz violation as they do not provide any further insight. Once a specific result is needed for practical purposes, it should be possible to derive it from the general formulas of Eq. (4.11), with the help of computer algebra.

At this point, we will make a couple of final statements on certain conditions for the validity of the previous results. For the pseudovector operator \( \bar{A}^\mu \), the spinors, the spinor matrices of Eq. (4.11), and the completeness relation of Eq. (4.13) are valid for controlling coefficients chosen such that the following conditions are fulfilled:

\begin{align}
    0 &\leq \bar{A}^\mu, \\
    0 &\leq \bar{A}^\mu(d^{\mu\nu...} \mapsto -d^{\mu\nu...}), \\
    0 &\leq \bar{A}^\mu(d^{\mu\nu...} \mapsto -d^{\mu\nu...}, p \mapsto -p).
\end{align}
The first of those ensures that the particle energies and spinors are numbered such as indicated throughout the paper. The second grants the same for the antiparticle energies and spinors. The third applies to the completeness relation and makes certain that the numbering of the antiparticle energies and spinors stays consistent. There exist controlling coefficients and momentum components such that all three conditions are satisfied. For the two-tensor operator, there are similar three conditions:

\[
0 \leq \hat{T}^{\mu\nu}, \quad (4.15a)
\]

\[
0 \leq \hat{T}^{\mu\nu}(H^{\mu\nu\cdots} \mapsto -H^{\mu\nu\cdots}), \quad (4.15b)
\]

\[
0 \leq \hat{T}^{\mu\nu}(H^{\mu\nu\cdots} \mapsto -H^{\mu\nu\cdots}, p \mapsto -p). \quad (4.15c)
\]

The reasons for choosing these conditions are analogous to the reasons outlined for \(\hat{A}^{\mu}\). However, \(\hat{T}^{\mu\nu}\) differs from \(\hat{A}^{\mu}\) crucially. Although each of the conditions above can be fulfilled for \(\hat{A}^{\mu}\), this is not possible for \(\hat{T}^{\mu\nu}\). Either the first two are true, but not the third or vice versa. The reason for this is that the \(g\) coefficients are contracted with one additional power of the momentum compared to the \(H\) coefficients. So, once the first two conditions are valid, changing the signs of the momentum components obscures the third condition. This does not occur for the pseudovector \(\hat{A}^{\mu}\), since it is the \(d\) coefficients that are contracted with one additional four-momentum relative to the \(b\) coefficients. Hence, once the first two conditions are valid for \(\hat{A}^{\mu}\), the third is not compromised by changing the signs of the momentum components. Now let us assume that the first two conditions for \(\hat{T}^{\mu\nu}\) hold. The consequence is then that the antiparticle energies have to be switched in the completeness relation of Eq. (4.13).

V. ADDITIONAL TIME DERIVATIVES

The results established in the previous section with the help of the optical theorem are only valid in case there are not any controlling coefficients that introduce additional time derivatives into the Lagrangian. Such time derivatives lead to an unconventional time evolution of the physical states [86]. Furthermore, for the nonminimal SME they increase the degree of the polynomial in \(p^0\) that follows from the determinant of the Dirac operator. First, this may lead to additional spurious dispersion relations that do not have Lorentz-invariant equivalents. Second, the structure of the denominator \(\Delta\) in propagators is modified drastically, which renders the previous proof based on the optical theorem invalid [37, 38].

A. Minimal fermion sector

One possibility of remedying this behavior at least for the minimal SME was outlined in [86]. The authors of the latter paper suggest introducing a new spinor \(\chi\), which is linked to the former spinor \(\psi\) via a transformation with an invertible matrix \(A\): \(\psi = A\chi\). The matrix \(A\) shall be constructed such that \(A^\dagger \gamma^0 \Gamma^0 A = 1_4\). Since \(\Gamma^0\) is linked to additional time derivatives in the minimal fermion sector all such time derivatives can be removed in this way.

The procedure will be exemplified within the minimal fermion sector with a nonzero coefficient \(d^{(4)}_{00}\). The matrix \(A\) is found by solving the matrix equation \(A^\dagger \gamma^0 (\gamma^0 + d^{(4)}_{00} \gamma^5 \gamma^0) A = 1_4\). Making
the Ansatz of a diagonal $A$ with real coefficients allows for a convenient solution of the system
and it gives:

$$A = \text{diag} \left( \frac{1}{\sqrt{1 + (d^{(4)00})^2}}, \frac{1}{\sqrt{1 + (d^{(4)00})^2}}, \frac{1}{\sqrt{1 - (d^{(4)00})^2}}, \frac{1}{\sqrt{1 - (d^{(4)00})^2}} \right).$$

(5.1)

Introducing the new spinor $\chi$ into the Dirac operator leads to

$$\mathcal{L} = \frac{1}{2} \chi S'^{-1} \chi + \text{H.c.}, \quad S'^{-1} = \gamma^0 A^\dagger \gamma^0 (\gamma^\mu \partial_{\mu} - m_\psi 1_4 + \hat{Q}) A.$$

(5.2)

Note that the dispersion relations are not modified by this transformation but the propagator
is. The new propagator $S'$ can be shown to be of the following shape where the purely timelike four-vector
$\lambda^\mu = (1, 0, 0, 0)^\mu$ is introduced for convenience:

$$S' = \frac{1}{\Delta} \left( \hat{\xi}'_{\mu} \gamma^\mu + \hat{\Xi}' \gamma^5 + \hat{\zeta}'_{\mu} \gamma^5 \gamma^\mu + \hat{\psi}'_{\mu\nu} \sigma^\mu_{\nu} \right),$$

(5.3a)

$$\hat{\Xi}' = (A_{11} A_{33})^{-1} \hat{\Xi},$$

(5.3b)

$$\hat{\Gamma}' = (A_{11} A_{33})^{-1} \hat{\Gamma},$$

(5.3c)

$$\hat{\xi}'^\mu = \hat{\xi}^\mu + \frac{1}{27} (\alpha \lambda^\mu + \beta p^\mu),$$

(5.3d)

$$\hat{\zeta}'^\mu = \hat{\zeta}^\mu + \frac{1}{9} (\gamma \lambda^\mu + \delta p^\mu),$$

(5.3e)

$$\hat{\psi}'_{\mu\nu} = (A_{11} A_{33})^{-1} \hat{\psi}_{\mu\nu},$$

(5.3f)

with the helpful quantities

$$\alpha = 4p^0 (d^{(4)00})^2 \left\{ 8(d^{(4)00})^2 (p \cdot \lambda)^2 + \left[ 9 + (d^{(4)00})^2 \right] p^2 + 9m_\psi^2 \right\},$$

(5.3g)

$$\beta = -(d^{(4)00})^2 \left\{ 8 \left[ 9 + (d^{(4)00})^2 \right] (p \cdot \lambda)^2 - \left[ 9 - (d^{(4)00})^2 \right] p^2 + 9m_\psi^2 \right\},$$

(5.3h)

$$\gamma = 8p^0 (d^{(4)00})^3 \left[ 4(p \cdot \lambda)^2 - p^2 \right],$$

(5.3i)

$$\delta = -d^{(4)00} \left\{ 16(d^{(4)00})^2 (p \cdot \lambda)^2 + \left[ 9 - (d^{(4)00})^2 \right] p^2 - 9m_\psi^2 \right\}.$$  

(5.3j)

Hence, the scalar, pseudoscalar, and the tensor parameters are just multiplied with a global prefactor,
which corresponds to the inverse product of two matrix components. The modification of the vector
and pseudovector parameters is more involved, though. As expected, the denominator $\Delta$
remains unchanged since it corresponds to the determinant of the Dirac operator modulo a global
prefactor.

As the Dirac operator has changed the spinors will change as well. The new spinors are given
by $\chi = A^{-1} \psi$ where the spinors $\psi$ for particles at first order in Lorentz violation are stated in
Eq. (3.12a) and for antiparticles they are obtained from Eq. (3.19). Hence, each of the already
known spinors just has to be multiplied with the inverse of $A$, which is also a diagonal matrix.
Note that the normalization must be adapted as well. Based on these spinors, the spinor matrices at first order in Lorentz violation are derived:

\[
\begin{align*}
\sigma^{(1,2)}_1 \pi^{(1,2)} &= u_\pm (\xi^\mu \gamma_\mu + \Xi l_4 \pm \psi^\mu \gamma_\mu \pm \psi^\mu \gamma_\mu \sigma^\mu) \big|_{p^0 = E_0}, \\
u^{(1,2)}_1 \pi^{(1,2)} &= v_\pm (\xi^\mu \gamma_\mu - \Xi l_4 \pm \psi^\mu \gamma_\mu \mp \psi^\mu \gamma_\mu \sigma^\mu) \big|_{p^0 = E_0},
\end{align*}
\]

\(\xi^\mu = \frac{p^\mu}{2},\)

\(\zeta^0 = \frac{|p|}{2},\quad \zeta = \frac{E_0 p}{|p| 2},\)

\(\psi^{0i} = 0,\quad \psi^{ij} = \frac{m_\psi}{4|p|} \varepsilon^{ijk} p^k,\)

\(\Xi = \frac{m_\psi}{2},\)

\(v_\pm = 1 \pm \frac{4|p|}{3 E_0} \delta^{(4)00}.\)

Using the results for the modified propagator of Eq. (5.3) and the previously stated spinor matrices allows for demonstrating the validity of the optical theorem at first order in Lorentz violation. A reasonable assumption is that the optical theorem is valid at all orders in Lorentz violation, cf. [31, 32, 36–39]. Then the spinor matrices are given by Eq. (4.11) based on the modified propagator quantities stated in Eq. (5.3).

\section*{B. Nonminimal fermion sector}

In the nonminimal SME, there is an infinite number of component coefficients that are contracted with additional time derivatives. This leads to problems similar to the ones discussed in the previous section. One of the simplest examples within the scope of the paper is the operator \(b^{(5)000}(i\partial^0)^2\) that is contracted with two time derivatives. Such nonminimal frameworks can have spurious dispersion relations that do not have equivalents in the standard theory:

\[
\begin{align*}
(p^0)^{(\pm)} &= \frac{1}{b^{(5)000}} \pm |p| - \frac{1}{2} (2p^2 + m_\psi^2) b^{(5)000} + \ldots, \\
(p^0)^{(\pm)} &= -\frac{1}{b^{(5)000}} \pm |p| + \frac{1}{2} (2p^2 + m_\psi^2) b^{(5)000} + \ldots.
\end{align*}
\]

Dispersion relations like these could be interpreted as Planck-scale effects. However, since the realm of applicability of the SME is expected to be far below the Planck energy (see also the issues connected to highly boosted observer frames in [26]) such dispersion relations are usually discarded in any analysis. Note that the validity of the optical theorem is obscured by the occurrence of spurious dispersion relations [37, 38].

In contrast to the minimal fermion sector, contributions with additional time derivatives in the nonminimal sector do not modify the matrix structure in spinor space, which is why the additional time derivatives cannot be simply removed with a matrix transformation along the
lines of the previous section. One method of dealing with such terms at least at first order in Lorentz violation was introduced in \[37, 38\]. It amounts to replacing all zeroth four-momentum components in the Lorentz-violating terms of the Dirac operator by the standard dispersion relation \( p^0 = E_0 = \sqrt{p^2 + m^2} \). This does not modify the physical dispersion relations at first order but it removes the spurious dispersion relations given by Eq. (5.5). As the current framework is isotropic the spinor solutions of Eq. (3.12a) must be taken with all \( p^0 \) in the Lorentz-violating terms replaced by \( E_0 \), which was already indicated in Sec. III B anyhow. Additionally, the same should be carried out in the propagator where it has to be kept in mind that this modified propagator can only be used in the proof of the optical theorem. In general, a propagator describes virtual (off-shell) particles, which is why \( p^0 \) does not satisfy the dispersion relation in this context. Based on that simple modification, the validity of the optical theorem is again established at first order in Lorentz violation. Hence, the spinor matrices are obtained from Eq. (4.11) with \( p^0 = E_0 \) in the Lorentz-violating contributions. The corresponding explicit results will be stated in App. D.

VI. CONCLUSIONS

In the current paper, we have examined the fermion sector of the Standard-Model Extension for the spin-nondegenerate Lorentz-violating operators. This concerned the \( b, d, H, \) and \( g \) coefficients where the first two are contained in the pseudovector \( \hat{A}^\mu \) and the latter two are contained in the two-tensor \( \hat{T}^{\mu\nu} \). We obtained the modified propagators for both \( \hat{A}^\mu \) and \( \hat{T}^{\mu\nu} \). These results are exact in Lorentz violation and they are valid for coefficients of both the minimal and the nonminimal SME.

Also, the dispersion relations and solutions of the modified Dirac equation, i.e., the spinors for particles and antiparticles were computed at first order in Lorentz violation for particular families of both minimal and nonminimal coefficients in \( \hat{A}^\mu \) and \( \hat{T}^{\mu\nu} \). With the optical theorem, the propagators and the spinors were checked to be consistent with each other at first order in Lorentz violation. Under the assumption that the optical theorem is valid at tree-level at all orders in Lorentz violation, which is reasonable based on earlier investigations, the spinor matrices \( u \bar{u} \) and \( v \bar{v} \) were extracted from the propagator at all orders in Lorentz violation.

The expressions obtained will prove useful for both future theoretical and phenomenological studies. First, the spinor solutions of the Dirac equation and especially their nonrelativistic limits may be employed for phenomenology in fermion systems where the electron spin is essential and cannot be neglected. Second, the spinor matrices and propagators are needed for computations in high-energy physics that are carried out in the context of quantum field theory. Such investigations are currently in progress and the findings will be reported elsewhere.

VII. ACKNOWLEDGMENTS

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Appendix A: Exact spinors for special frameworks

In this part of the appendix, we demonstrate how to obtain exact solutions of the modified Dirac equation. Doing this with computer algebra is unproblematic as long as there is a single nonzero controlling coefficient only. However, the complexity of both the modified dispersion relations and the spinor solutions rises drastically with an increasing number of coefficients. This is the main reason why most of the computations were restricted to first order in Lorentz violation, which allows to obtaining the energies and spinors for large sets of coefficients — even including nonminimal ones. For brevity, we will study isotropic cases here only.

1. Isotropic $b$ coefficients

The first framework shall be characterized by a nonzero $b^{(3)0}$. The particle dispersion relations are obtained directly from the determinant of the Dirac operator:

$$E^{(\pm)}_u = \sqrt{p^2 + m_\psi^2 \mp 2b^{(3)0}|p| + (b^{(3)0})^2}.$$  \hspace{1cm} (A.1)

There are two distinct ones, as expected, and they differ from each other at first order in Lorentz violation. Plugging these energies into the Dirac equation and solving it leads to the particle spinor solutions:

$$u^{(1,2)}|_{E^{(\pm)}_u}(p) = N^{(1,2)}_u \left( \begin{array}{c} (p_3 \pm |p|)(E^{(\pm)}_u - p_3 + b^{(3)0}) - (p_1^2 + p_2^2) \\ (p_1 + ip_2)(E^{(\pm)}_u \mp |p| + b^{(3)0}) \\ m_\psi(p_3 \pm |p|) \\ m_\psi(p_1 + ip_2) \end{array} \right).$$ \hspace{1cm} (A.2)

The antiparticle spinors are connected to the particle spinors according to Eq. (3.19). The normalizations for the particle spinors are chosen such that $(u^{(\alpha)})^\dagger u^{(\beta)} = 2E^{(\alpha)}_u \delta_{\alpha\beta}$ and in analogy for the antiparticle spinors. This also requires that spinors for different energies are orthogonal to each other, which is granted by the hermiticity of the Dirac operator. Furthermore, such a normalization allows for a convenient check of the optical theorem, which we saw in Sec. [IV]. For this particular case the normalizations read as follows:

$$N^{(1,2)}_u = \frac{1}{\sqrt{2|p|(p_3 \pm |p|)(E^{(\pm)}_u \mp |p| + b^{(3)0})}}, \quad N^{(2,1)}_u = N^{(1,2)}_u.$$ \hspace{1cm} (A.3)

2. Isotropic $g$ coefficients

In contrast to the $H$ coefficients, there is an isotropic case for the $g$ coefficients. It is characterized by a totally antisymmetric choice of nonzero coefficients: $g^{(4)i}j^k \equiv \varepsilon^{ij}g_1$, where $i$, $j$, and $k$ are spatial indices. There are again two dispersion relations that differ from each other at first order in Lorentz violation:

$$E^{(\pm)}_u = \sqrt{(1 + g_1^2)p^2 \pm 2g_1m_\psi|p| + m_\psi^2}.$$ \hspace{1cm} (A.4)
Note that the $g$ coefficients are dimensionless. The particle spinor solutions can be obtained such as before:

$$u^{(1,2)}_{\mu u}(p) = N_u^{(1,2)} \begin{pmatrix} \left(-p_1^2 + p_2^2 \pm (p_3 - E_u^{(\pm)})(|p| \pm p_3)\right) \\ (p_1 + ip_2)(E_u^{(\pm)} \mp |p|) \\ (p_3 \mp |p|)(m_\psi \pm g_1 |p|) \\ (p_1 + ip_2)(m_\psi \pm g_1 |p|) \end{pmatrix},$$

and the antiparticle spinor solutions are connected to these based on Eq. (3.28). The normalization factors are simply expressed in terms of the particle energies:

$$N_u^{(1,2)} = \frac{1}{\sqrt{2|p|(E_u^{(\pm)} \mp |p|)}}, \quad N_v^{(1,2)} = N_u^{(2,1)}.$$  (A.6)

The exact spinors in both subsections were checked to correspond to the results in Sec. III D 1 at first order in Lorentz violation.

**Appendix B: Spinors for spin-degenerate operators**

For completeness, we state the spinors, normalization factors, and the propagator for the spin-degenerate cases that are encoded in the scalar operator $\hat{S}$ and the vector operator $\hat{V}^\mu$. The Dirac operator can be diagonalized with the matrix $U = V \cdot W$ where $V$ and $W$ are given by Eq. (3.1). In these matrices all $E_0$ have to be replaced by the exact dispersion relation $E_u$ and all occurrences of $m_\psi$ by $m_\psi - \hat{S}$ and $p^\mu$ by $(p + \hat{V})^\mu$ [22]. In the chiral representation the particle spinors at all orders in Lorentz violation can be cast into the following form:

$$\frac{1}{N_u} u^{(1)}|_{E_u} = (m_\psi - \hat{S}) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -(p + \hat{V})^1 + i(p + \hat{V})^2 \\ E_u + \hat{V}^0 + (p + \hat{V})^3 \\ (p + \hat{V})^1 - i(p + \hat{V})^2 \\ E_u + \hat{V}^0 - (p + \hat{V})^3 \end{pmatrix},$$

$$\frac{1}{N_u} u^{(2)}|_{E_u} = (m_\psi - \hat{S}) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} E_u + \hat{V}^0 - (p + \hat{V})^3 \\ -(p + \hat{V})^1 - i(p + \hat{V})^2 \\ E_u + \hat{V}^0 + (p + \hat{V})^3 \\ (p + \hat{V})^1 + i(p + \hat{V})^2 \end{pmatrix},$$

where $E_u$ are the exact particle energies. The antiparticle spinors are obtained from the particle spinors by charge conjugation (cf. Sec. III D 4):

$$v^{(1,2)}|_{E_v} = \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix} \begin{pmatrix} u_1^{(2,1)} \\ -u_3^{(2,1)} \\ -u_2^{(2,1)} \\ u_1^{(2,1)} \end{pmatrix}^* (-a^{\mu\nu\ldots}, -e^{\mu\nu\ldots}), \quad E_v = E_u(-a^{\mu\nu\ldots}, -e^{\mu\nu\ldots}).$$

(B.2)

with the signs of the $C$-odd $a$ and $e$ coefficients reversed (see Tab. P31 in [25]). The normalization factors for particles and antiparticles read

$$N_u = \sqrt{\frac{E_u}{(E_u + \hat{V}^0 + m_\psi - \hat{S})^2 + (p + \hat{V})^2}}; \quad N_v = N_u(-a^{\mu\nu\ldots}, -e^{\mu\nu\ldots}).$$

(B.3)
where $\hat{\mathbf{\nabla}}$ is the spatial part of the vector operator $\hat{\nabla}^{\mu}$. Last but not least, the propagator can be expressed as follows:

$$iS = \frac{i(\slashed{p} + \mathbf{\hat{\nabla}} + (m_\psi - \mathbf{\hat{S}})\mathbb{1}_4)}{(p + \mathbf{\hat{\nabla}})^2 - (m_\psi - \mathbf{\hat{S}})^2}. \quad \text{(B.4)}$$

**Appendix C: Spinors for pseudoscalar operator $\hat{f}$**

The spinors for the pseudoscalar operator $\hat{Q} = i\hat{f}\gamma^5$ with $\hat{f} = \hat{f}^\nu p_\nu$ shall be stated as well.

We do not consider the chiral mass term $i\hat{m}\gamma^5$, which in principle also adds to the pseudoscalar operator but can be rotated away by a chiral transformation in many cases \[22\]. The operator $\hat{f}$ has the peculiarity that its contributions to the dispersion relation are of quadratic order at least, i.e., $E_u = E_0 + \hat{f}^2/(2E_0)$. However, the spinors contain first-order terms in Lorentz violation:

$$\frac{1}{N_u} u^{(1)}|_{E_u} = \left(\begin{array}{c}
\sigma^3 \phi_- \\
\phi_+
\end{array}\right) - \frac{i\hat{f}}{2E_0} \left(\begin{array}{c}
\phi_+ \\
-\sigma^3 \phi_-
\end{array}\right), \quad \text{(C.1a)}$$

$$\frac{1}{N_u} u^{(2)}|_{E_u} = \left(\begin{array}{c}
-i\sigma^2 \phi_+ \\
\sigma^1 \phi_-
\end{array}\right) - \frac{i\hat{f}}{2E_0} \left(\begin{array}{c}
\sigma^1 \phi_- \\
-i\sigma^2 \phi_+
\end{array}\right), \quad \text{(C.1b)}$$

$$\phi_\pm = \left(\begin{array}{c}
A_\pm \\
B
\end{array}\right), \quad \text{(C.1c)}$$

$$A_\pm = E_0 + m_\psi \pm p_3, \quad \text{(C.1d)}$$

$$B = p_1 + ip_2, \quad \text{(C.1e)}$$

with the Pauli matrices $\sigma^i$. The antiparticle spinors can be obtained as usual where the signs of the $C$-odd coefficients $f^\mu$ must be reversed (see Tab. P31 in \[25\]):

$$v^{(1,2)}|_{E_v} = \left(\begin{array}{c}
u_4^{(2,1)} \\
-u_3^{(2,1)} \\
-u_2^{(2,1)} \\
-u_1^{(2,1)}
\end{array}\right)^* \left(\begin{array}{c}
-f^{\mu\nu} \\
E_v = E_u.
\end{array}\right), \quad \text{(C.2)}$$

Last but not least, the normalization factors of the spinors and antispinors read

$$N_u = \frac{1}{\sqrt{2(E_0 + m_\psi)[1 + \hat{f}^2/(4E_0^2)]}}, \quad N_v = N_u(-f^{\mu\nu}), \quad \text{(C.3)}$$

and the propagator is given by

$$iS = \frac{i(\slashed{p} + m_\psi \mathbb{1}_4 + i\hat{f}\gamma^5)}{p^2 - m_\psi^2 - \hat{f}^2}. \quad \text{(C.4)}$$

In the denominator the square of $\hat{f}$ appears again demonstrating that $\hat{f}$ contributes to the dispersion relation quadratically. The latter result corresponds to Eq. (6.49) in \[38\].
Appendix D: Special results for spinor matrices

Finally, in this current section some explicit results for spinor matrices \( u^{(\alpha)} \) for particles and \( v^{(\alpha)} \) for antiparticles shall be presented. They are based on the general expressions stated in Eq. (4.11) and will be given for the purpose of illustration. Two cases for the pseudovector operator \( \hat{A}^\mu \) and one case for the two-tensor operator \( \hat{T}^{\mu\nu} \) will be under consideration.

1. Pseudovector operator

The first case is characterized by isotropic Lorentz violation and it was examined in Sec. III D 1. The spinor matrices for particles and antiparticles are explicitly given by

\[
u_{(1,2)}|_{E_y^{(\pm)}} = \xi_{\pm}^\mu \gamma_\mu + \Xi_{\pm} \gamma_5 \gamma_\mu + \psi_{\pm}^{\mu\nu} \sigma_{\mu\nu}, \quad \text{and} \quad \nu_{(1,2)}|_{E_y^{(\pm)}} = \xi_{\pm}^\mu \gamma_\mu - \Xi_{\pm} \gamma_5 \gamma_\mu - \psi_{\pm}^{\mu\nu} \sigma_{\mu\nu},
\]

for particles and antiparticles, respectively.

\[
\xi_{\pm}^\mu = \frac{p_\mu^\pm}{2}, \quad \xi_{\pm}^\mu = \frac{p_\mu}{2}\left(1 \mp \hat{b}^{(3)0}\right),
\]

\[
\zeta_{\pm}^\mu = \frac{1}{2}\left(\pm |p| - \hat{b}^{(3)0}\right), \quad \zeta_{\pm}^\mu = \pm \frac{p_\mu}{2|p|} - \hat{b}^{(3)0}
\]

\[
\psi_{0i}^\pm = 0, \quad \psi_{ij}^\pm = \pm \frac{m_\psi}{4|p|}\epsilon_{ijk}p^k.
\]

Due to isotropy, no momentum component is preferred in these expressions. Furthermore, all parameter functions decompose into a vector or tensor part and a scalar part that depends on the magnitude of the momentum only. Note that the signs of \( \Xi \) for particles and antiparticles are opposite, as expected.

The second framework is characterized by a nonvanishing \( \hat{A}^3 \), which makes it anisotropic, cf. Sec. III D 3. With the preferred purely spacelike direction \( \lambda^\mu = (0, 0, 1, 0)^\mu \), the spinor matrices can be expressed conveniently:

\[
u_{(1,2)}|_{E_y^{(\pm)}} = \frac{p_\mu^\pm}{2} + \frac{\hat{A}^3}{2S_3}(p \cdot \lambda)\lambda^\mu,
\]

\[
\zeta_{\pm}^\mu = \frac{1}{2}\left[\pm \left(\frac{(p \cdot \lambda)^2}{S_3} - S_3\right) - \hat{A}^3\right] \lambda^\mu \pm \frac{p_\mu}{2S_3} - \hat{b}^{(3)0}
\]

\[
\psi_{0i}^\pm = \pm \frac{m_\psi}{4S_3}\epsilon^{ijk}p^j, \quad \psi_{ij}^\pm = \pm \frac{m_\psi}{4S_3}\epsilon^{ijk}\lambda^k,
\]
Since this case is anisotropic with the preferred direction pointing along the third spatial axis of the coordinate system the third spatial momentum component is preferred. This manifests in expressions depending on $S_3 = \sqrt{p_3^2 + m_\psi^2}$ and the scalar product $p \cdot \lambda$. Furthermore, due to the preferred spatial axis, the quantity $\Xi$ becomes nondegenerate for the particle and antiparticle spinors, respectively.

2. Two-tensor operator

As an example for the two-tensor operator $\tilde{T}^{\mu\nu}$, a nonzero $\tilde{T}^{01}$ is considered, cf. Sec. [III E 1]. For convenience, we introduce the three-dimensional vectors $p^{(3)} \equiv (0,p_2,p_3)$ and $\tilde{p}^{(3)} = \varepsilon^{1ij}p^{(3)j} = (0,p_3,-p_2)$. The parameters of the spinor matrices can then be written in a handy form:

$$u^{(1,2),u^{(1,2)}}_{E_\mu}(\pm) = \xi_{\pm}^{\mu} \gamma_\mu + \Xi f_{4} + \zeta_{\pm}^{\mu} \gamma_\mu + \psi_{\pm}^{\mu\nu} \sigma_{\mu\nu} \big|_{p = E_\mu}^{(\pm)} ,$$  

$$v^{(1,2),v^{(1,2)}}_{E_\mu}(\pm) = \xi_{\pm}^{\mu} \gamma_\mu - \Xi f_{4} - \zeta_{\pm}^{\mu} \gamma_\mu + \psi_{\pm}^{\mu\nu} \sigma_{\mu\nu} \big|_{p = E_\mu}^{(\pm)} ,$$

$$\xi_{\pm}^{\mu} = \frac{p^{\mu}}{2} \pm \frac{\tilde{T}^{01}}{2S_1} (0,p^{(3)})^{\mu} ,$$  

$$\zeta_{\pm}^{\mu} = \frac{\pm m_\psi}{2S_1} (0,\tilde{p}^{(3)})^{\mu} ,$$

$$\psi_{01}^{\pm} = \frac{1}{4} (\pm S_1 + \hat{T}^{01}) , \quad \psi_{0i}^{\pm} = \mp p^{(3)i} \frac{p_{1}}{4S_1} , \quad \psi_{ij}^{\pm} = \pm \varepsilon^{ijk}p^{(3)k} \frac{p_{0}}{4S_1} ,$$  

$$\Xi = \frac{m_\psi}{2} ,$$

with $S_1$ from Eq. (3.20). Note that it is possible to express the vectors in $\xi_{\pm}^{\mu}$ and $\zeta_{\pm}^{\mu}$ directly in terms of the two-tensor operator under consideration, i.e.,

$$\left(0,\tilde{p}^{(3)}\right)^{\mu} = - \frac{\tilde{T}^{\mu\nu}p_\nu}{\hat{T}^{01}} , \quad \left(0,p^{(3)}\right)^{\mu} = \frac{\varepsilon^{01\mu\nu}\hat{T}^{\nu\rho}p_\rho}{\hat{T}^{01}} .$$

This concludes the examples for the spinor matrices that shall be stated explicitly.
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