Abstract. Fix a non-negative integer $g$ and a positive integer $I$ dividing $2g - 2$. For any Henselian, discretely valued field $K$ whose residue field is perfect and admits a degree $I$ cyclic extension, we construct a curve $C/K$ of genus $g$ and index $I$. We can in fact give a complete description of the finite extensions $L/K$ such that $C$ has an $L$-rational point. Applications are discussed to the corresponding problem over number fields. S. Sharif, in his 2006 Berkeley thesis, has independently obtained similar (but not identical) results. Our proof, however, is different: via deformation theory, we reduce to the problem of finding suitable actions of cyclic groups on finite graphs.

Some terminological conventions: by a variety (resp. a curve) over a field $K$ we will mean a finite-type $K$-scheme which is smooth, projective and geometrically integral (resp. of dimension one). By a variety (resp. a curve) over a field $k$ we will mean a finite-type $k$-scheme which is geometrically integral (resp. of dimension one) but possibly incomplete or singular. If $V$ is a variety defined over $K$ and $L/K$ is a field extension, we say that $L$ splits $V$ if $V(L) \neq \emptyset$.

1. Introduction

Given a variety $V$ defined over a field $K$, one would like to determine whether $V$ has a $K$-rational point, and if it does not, to say something about $S(V)$, the set of finite field extensions $L/K$ for which $V$ acquires an $L$-rational point. This is a very difficult problem: e.g., it is believed by many (but unproved) that there is no algorithm for the task of deciding whether a variety $V/\mathbb{Q}$ has a $\mathbb{Q}$-rational point.

In order to quantify the second part of the question, it is natural to introduce the **index** $I(V)$ of a variety $V(K)$: it is the greatest common divisor of all degrees of closed points on $V$ (so $V(K) \neq \emptyset \implies I(V) = 1$, but not conversely). For a curve $C/K$, the index is equal to the least positive degree of a line bundle on $K$. If $C$ has genus $g$, then the canonical bundle $\Omega_{C/K}^1$ has degree equal to $2g - 2$ and $I(C) \mid 2g - 2$.

In general, to know $I(V)$ is much less than to know $S(V)$: we need not know any particular splitting field $L/K$, nor even the least possible degree of a splitting field (a quantity called the **m-invariant** $m(V)$ in [Cl2]). For instance, it follows from the Weil bound for curves that every variety over a finite field has $I(V) = 1$; nevertheless computing $S(V)$ or even $m(V)$ is still a nontrivial task.

As usual, when the “direct problem” of computation of an invariant is sufficiently difficult, it is natural to consider as well the “inverse problem” of which invariants actually arise. In particular, we may well ask:

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1^Note that this holds – vacuously – even when $g = 1$. 

Question 1. Fix a field $K$. For which pairs $(g, I) \in \mathbb{N} \times \mathbb{Z}^+$ does there exist a curve $C_{/K}$ with $I(C) = I$?

As above, the existence of the canonical divisor implies that a necessary condition is $I \mid 2g - 2$. This condition is, of course, not sufficient: e.g., we have $I(C) = 1$ for all curves when $K$ is finite or PAC\footnote{A field $K$ is \textbf{Pseudo Algebraically Closed} if every variety $V_{/K}$ has a rational point. This includes separably closed fields, but there are many others: see [FJ].} whereas if $K$ is $\mathbb{R}$ (or is real-closed, or pseudo-real closed) we have $I(C) \mid 2$.

In contemporary arithmetic geometry, the fields of most interest are those which are infinite and finitely generated (“IFG”).

Conjecture 2. Let $K$ be an IFG field. Then for any $g$ and $I$ with $I \mid 2g - 2$, there exists a curve $C_{/K}$ of genus $g$ with $I(C) = I$.

Remark 1.1: Conjecture 2 is true when $g = 0$; this is a relatively easy exercise involving quaternion algebras which we leave to the reader.

Remark 1.2: The main result of [Cl1] is that Conjecture 2 holds for $g = 1$ when $K$ is a number field. It ought to be possible to extend this result to positive characteristic global fields (using the period-index obstruction map in flat cohomology) and then to all IFG fields (using isotrivial elliptic curves). Indeed, in genus one it is natural to make a much stronger conjecture: see [CS, Conjecture 1].

Let us now present evidence for Conjecture 2 for curves of higher genus ($g \geq 2$). The following result uses the author’s work on the genus one case together with some simple covering considerations (essentially those suggested to the author by Bjorn Poonen in 2003) to attain a solution for “small indices.”

Theorem 3. Let $K$ be a number field, $g \in \mathbb{N}$ and $k \in \mathbb{Z}^+$ with $k \mid g - 1$. Then there exists a curve $Y_{/K}$ of genus $g$ and index $I = \frac{g - 1}{k}$.

Proof: Since $K$ has characteristic different from 2, it is especially easy to see that there exist (hyperelliptic) curves over $K$ of all genera with $K$-rational (Weierstrass) points; we may therefore assume that $I > 1$. By Remark 1.2 there exists a curve $X_{/K}$ of genus one and index $I$. By [Cl3, Prop. 11], there exist linearly equivalent $K$-rational divisors $D_1$ and $D_2$ on $C$, both effective of degree $kI$, such that the support of $D_1 - D_2$ has cardinality $2kI$. Let $f \in K(X)$ be the rational function (uniquely determined up to a constant multiple) with divisor $D_1 - D_2$, and let $\varphi : Y \to X$ be the branched covering corresponding to the extension of function fields $K(X)(\sqrt{f})/K(X)$. By the Riemann-Hurwitz formula, $Y$ has genus $g = kI + 1$.

As for any covering of curves, we have $I(Y) \geq I(X) = I$; conversely, any point $P$ in the support of $D_1$ has degree $I$ and is a ramification point for $\varphi$, so its unique preimage $\tilde{P} \in Y(K)$ has degree $I$. Thus we have $I(Y) = m(Y) = I$.

In this note we are mainly concerned with presenting a technique for constructing curves $C$ (and perhaps higher-dimensional varieties) over \textit{local} fields for which the set $S(C)$ can be explicitly computed. Here is our main result:
Main Theorem. Let $K$ be a Henselian discretely valued field with perfect residue field. Assume that there is a cyclic, degree $I$, unramified extension $K_1/K$ ("hypothesis $K_1$"). Then for any non-negative integer $g$ such that $I \mid 2g - 2$, there exists a curve $C/K$ with the following properties:

a) If $I$ is odd or $g = 1$, then a finite extension $L/K$ splits $C$ iff $L \supset K_1$.

b) Otherwise, a finite extension $L/K$ splits $C$ iff either of the following holds:

- $L \supset K_1$;
- $L \supset K_{1/2}$ – the unique subextension of $K_1/K$ of degree $I/2$ – and $2 \nmid \epsilon(L/K)$.

Remark 1.3: So if $K$ satisfies hypothesis $K_1$ for all $I$, then for all $I \mid 2g - 2$ there is a curve $C/K$ of genus $g$ and index $I$. In particular this holds when the residue field is finitely generated, when we may construct $K_1$ by adjoining suitable roots of unity.

Remark 1.4: Some hypothesis on the existence of unramified extensions is necessary, since if $k$ is algebraically closed, then $I(C) \mid g - 1$ [BL, Remark 1.8]. On the other hand, one could ask for a classification of all possible indices under the milder assumption that $K$ admits a quadratic unramified extension, e.g. in the case $K = \mathbb{R}((T))$. We shall not pursue this here.

Corollary 4. Let $K$ be an infinite, finitely generated field. For any $g \in \mathbb{N}$, there exists a finite extension $L/K$ and a genus $g$ curve $C/L$ with $I(C) = 2g - 2$.

Proof of Corollary 4. Such a field $K$ admits a discrete valuation $v$ with finitely generated residue field $k$, so by Remark 1.3 we may apply our Main Theorem to the Henselization $K_v$ of $K$. We thus get a curve $C$ of genus $g$ and index $I$ defined over $K_v$, which is an algebraic extension of $K$; it is evidently defined over some finite extension $L$ of $K$.

Remark 1.5: Only a few days after the results of this paper were first obtained I received a copy of the 2006 Berkeley thesis of S.I. Sharif [Sh], which contains closely related results.

Theorem 5. (Sharif, [Sh])

a) Let $K$ be a locally compact discretely valued field of characteristic different from 2. Then for any $(g, I) \in \mathbb{N} \times \mathbb{Z}^+$ with $I \mid 2g - 2$, there exists a curve $C/K$ of genus $g$ and index $I$.

b) For any $g \in \mathbb{N}$ and $I \in \mathbb{Z}^+$ such that $4 \nmid I \mid 2g - 2$, there exists a number field $K = K(g, I)$ and a curve $C/K$ of genus $g$ and index $I$.

By Remark 1.3, part a) of Theorem 5 is a special case of our Main Theorem, whereas part b) is similar in spirit to Theorem 3 and Corollary 4 but not directly comparable to either one. On the other hand, Sharif’s results go further than those presented here in that he also considers the possible values of the period $P$ (the least positive degree of a $K$-rational divisor class), and – comparing with the restrictions on period and index obtained by Lichtenbaum – his constructions give the complete list of possible values of $(g, P, I)$ for curves over locally compact fields of characteristic different from 2.

The strategy of Sharif’s proof – namely, construction of degree two covers of curves of genus one and two via $p$-adic theta functions – is quite different from the proof of our Main Theorem.
Let us now say something about how we shall prove the Main Theorem. There are
three steps: (i) we observe that if a curve $C/K$ admits a regular model $C/R$ with
semistable special fiber, then the set $\mathcal{S}(C)$ of finite extensions $L/K$ with $C(L) \neq \emptyset$
depends only on the special fiber. This reduces us to the problem of determining
differential points on a semistable curve over the special fiber, and the next idea is
that the “more singular” the special fiber, the easier it is to analyze its set of ra-
tional points. In particular, if we restrict to the case of totally degenerate special
fibers – in which each geometric component has genus zero – then the information
we want can be gleaned from the action of the Galois group $\mathfrak{g}_k = \text{Gal}(\bar{K}/k)$ on the
dual graph whose vertices are the components and edges are the intersection points.
(ii) A fundamental result in the deformation theory of one-dimensional complete
intersections (due essentially to Grothendieck) tells us that every semistable curve
over $k$ lifts to a regular scheme over the valuation ring $R$ with smooth generic fiber;
complementing this with the elementary observation (due to A. Pál) that every
cubic graph is the dual graph of a semistable curve over $k$, we are reduced to (iii) a
combinatorial problem involving the construction of a family of finite graphs with
suitable Euler characteristic and automorphisms by a finite cyclic group. It turns
out that one solution to the problem is obtained by using a well-known family
of graphs, the Möbius ladders. It seems enlightening to present the construction
in terms of Cayley graphs, and we do so here, although we have written up the
endgame in such a way as to offer the reader unfamiliar with the formalism of Cay-
ley graphs a choice: he may either learn this material here or bypass it in favor of
a simple, concrete description.

It is natural to try to generalize the results to a broader class of algebraic vari-
eties, e.g. varieties $V/K$ admitting a regular SNC model (in particular all algebraic
curves), or varieties whose special fiber is completely degenerate. I also suspect that
there should be a more elegant approach via non-Archimedean uniformization.

I have decided not to try to work out these generalizations in the present
note, because (i) it seems they would require significantly more technical appa-
ratus, whereas our present methods – with the exception of the aforementioned
deformation-theoretic result that we treat as a “black box” – are rather elemen-
tary; and (ii) because our work overlaps substantially with S. Sharif’s thesis, it
seems best to record once and for all our work that was done independently of [Sh],
so that our subsequent work can draw freely on both sources.

2. LOCAL POINTS ON SEMISTABLE CURVES

The results of this section may be well-known to some, but – especially in the ab-
sence of any satisfactory reference – we prefer to work them out in detail.

Let $C/K$ be a curve. Suppose that $C$ has semistable reduction: that is, there
exists a regular arithmetic surface $C$ over the valuation ring $R$ of $K$ with generic
fiber isomorphic to $C$ and with special fiber a semistable curve $C/k$. Recall that a
semistable curve $C/k$ is a one-dimensional projective $k$-scheme which is reduced,
geometrically connected, and whose only singularities are ordinary double points.

Write $C/k = \sum_{i=1}^{N} C_i$, so that the $C_i$ are the $k$-irreducible components (which
we will henceforth call the components).
There is a natural $g_k$-action on the set of components. We will say that a component $C_i$ is defined over a finite field extension $l/k$ if $g_l$ fixes $C_i$. There is evidently a unique minimal such field extension (necessarily Galois over $k$), which we denote by $l_i$. Let $d_i = [l_i : k_i]$ and $d = \gcd d_i$.

For any reduced finite-type scheme $S/k$, define its nonsingular index $I^{ns}(S)$ to be the index of the nonsingular locus $S^{ns}$. Put $I_i := I^{ns}(C_i)$.

**Theorem 6.** Let $C/R$ be a regular arithmetic surface with generic fiber $C/K$ and semistable special fiber $C/k$. Then

$$I(C/K) = I^{ns}(C/k) = \gcd\left\{d_i \cdot I_i \right\}.$$

The proof will come later in this section. More information can be obtained under the following hypothesis:

(A) For every finite extension $l/k$, every component $C_i$ which is defined over $l$ has an $l$-rational point which is not a nodal point of $C$.

**Proposition 7.** Maintain the hypotheses of Theorem 6 and assume also (A).

a) If $d$ is odd, then a finite extension $L/K$ splits $C$ iff its maximal unramified extension $L'/K$ splits $C$ iff its residue extension contains $l_i$ for some $i$.

b) If $d$ is even, then a finite extension $L/K$ splits $C$ iff either

(i) the residue extension $l$ contains $l_i$ for some $i$; or

(ii) the residue extension $l/k$ is such that $g_l$ stabilizes a pair of intersecting components, and $e(L/K)$ is even.

Remark: Let us say that we are in **Case 1** if $d$ is odd or condition (bii) does not occur, and that we are in **Case 2** if $d$ is even and condition (bii) occurs.

It will be convenient to introduce the (so-called) dual graph $\mathcal{G} = (V, E)$, an undirected, connected, finite graph whose vertices are the components of $C$, and where vertices $C_i$ and $C_j$ are linked by $C_i \cdot C_j$ edges. The natural action of the Galois group $g_k$ on components and on singular geometric points gives rise to an action of $g_k$ on $\mathcal{G}$ by graph-theoretical automorphisms.

**Proof** of Proposition 7. By a well-known version of Hensel’s Lemma (e.g. [JL]), $C(K) \neq \emptyset$ iff $C$ has a smooth $k$-rational point. In the present case this occurs iff some vertex $C_i$ of $\mathcal{G}$ is fixed by the action of $g_k$. Since regularity of a model is unaffected by unramified base change, an unramified extension $L/K$ splits $C$ iff its residue extension $l$ contains $l_i$ for some $i$.

Let us now consider the effect of making a totally ramified base extension $L/K$. We can form the arithmetic surface $C' := C \otimes_R S$ (where $S$ is the valuation ring of $R$), and the special fiber is still $C/R = C/k$, but $C'$ will no longer be regular (unless $C/k$ was smooth), because the complete strict local ring at a singularity will now be of the form $S'[[x, y]]/(xy - \pi^e)$, where $\pi$ is a uniformizer for $L$ and $e = e(L/K)$ is the relative ramification index. The remedy is well-known – we must blowup $e - 1$ times to replace the intersection point with a chain of $e - 1$ rational curves. However, keep in mind that we are really blowing up a closed point whose residue field may be larger than $k$, or in other words, we are simultaneously blowing up each point in the $g_k$-orbit of the given singular point, and there is, in an evident way, an
induced $g_k$-action on these chains. In order for there to be a smooth $l$-rational point after the blowing-up process which was not there before that process, necessary and sufficient conditions are: first, for some chain to be $g_k$-stable – in other words we need for a pair $\{C_i, C_j\}$ of intersecting components to be preserved by $g_k$; and second, for the chain to have odd length, i.e., for $e$ to be even. This completes the proof of Proposition 7.

Before beginning the proof of Theorem 6 we will record two quick lemmas.

Lemma 8. If $V_{jk}$ is a finite-type reduced scheme and $l/k$ is a finite field extension, then $I(V_{jk}) \mid [l:k] \cdot I(V_{jl})$.

Proof: Let $D_l$ be an $l$-rational zero-cycle on $V$ of degree $I(V_{jl})$; its trace from $l$ down to $k$ is a $k$-rational zero-cycle of degree $[l:k] \cdot I(V_{jl})$.

Lemma 9. The nonsingular index $I^a(V)$ of a variety $V_{jk}$ is a birational invariant. In particular the nonsingular index $I^a(C)$ of a curve $C_{jk}$ is unchanged by the removal of finitely many closed points.

Proof: If $k$ is finite, then much more is true: it follows from the Weil bounds for curves over finite fields that for all finite-type geometrically integral schemes $V_{/F_q}$ $I^a(V) = 1$. The argument is well-known to field arithmeticians: see [FJ]. When $k$ is infinite, see [CFJ] p.8.

Remark 2.1: With $I$ instead of $I^a$, the conclusion of Lemma 9 does not follow: take $X^2 + Y^2 + Z^2 = 0$ over $\mathbb{R}$.

We come now to the proof of Theorem 6.

Step 1: We will show that $I^a(C_{jk}) = \gcd_i d_i I_i$. Indeed for any $i, 1 \leq i \leq N$ $I^a(C_{jk}) = I(C_{jk}^i) \mid [l_i:k] \cdot I(C_{jk}^i) \mid d_i I_i$

( these divisibilities use Lemmas 8 and 9), so $I^a(C_{jk}) \mid \gcd_i d_i I_i$. For the converse, $I^a(C_{jk})$ is the gcd of all degrees of field extensions $l/k$ such that $C$ has a smooth $l$-rational point $P$. Thus $l$ must be a field of definition for the component on which $P$ lies – i.e., $l_i \subset l$, and then clearly $I_i \mid [l : I_i]$.

Step 2: It is clear from Hensel’s Lemma that $I(C_{jk}) \mid I^a(C_{jk})$; more precisely, the fields $l$ for which $C$ acquires a smooth $l$-rational point correspond to the unramified splitting fields. It remains to account for the possibility that $L$ splits $K$ when its maximal unramified subextension $K'$ does not. As in the proof of Proposition 7, this can only happen when there is a $k'$-rational singular point $q$ on $C$ and $e(L/K)$ is even. Suppose first that $q$ is the intersection of distinct components $\{C_i, C_j\}$. Then the pair of components is stabilized by $g_{k'}$, and $q$ is a smooth point on each component, so $I^a(C_{jk'}) \mid 2$, so that there is an unramified splitting field $K''/K$ with $[K'':K] \mid [L : K]$. The other possibility is that $q$ is a nodal singularity on a single component $C_i$ which is defined over $k$. But then the preimage of $q$ in the normalization $\tilde{C}_i$ of $C_i$ is a rational divisor of degree 2, which by Lemma 9 can be “moved” to a rational divisor of degree 2 with support disjoint from the singular locus, hence projecting down to a rational divisor of degree 2 on $C_i$, i.e., $I_i \mid 2$ and again we have found an unramified splitting field $K''$ such that $[K'':K] \mid [L : K]$.
This completes the proof of Theorem 6.

Remark 2.2: The proof of Theorem 6 shows that curves $C_{/K}$ with semistable reduction are “unramified” in a strong sense: not only do they possess unramified splitting fields, but also the index can be calculated using only unramified extensions. However, in the absence of Hypothesis (A), the least degree of a field extension $L/K$ such that a curve $C_{/K}$ with semistable reduction acquires rational points will not in general be attainable by an unramified extension. See [Cl2] for a naturally occurring example (involving the Shimura curves $X^0_d(N)$) where in order to show that a semistable curve has points over a quartic base extension, a ramified base change was used in a perhaps essential way.

3. Proof of the Main Theorem

As the reader has probably guessed, the curves $C_{/K}$ referred to in the Main Theorem will be such that they admit regular models whose special fibers are semistable and satisfy hypothesis (A), so information about their splitting fields will reduce to an analysis of the dual graph. In fact, we we will place ourselves in a situation in which we need only construct the dual graph and not the arithmetic surface itself. This is done via the following two results:

**Theorem 10.** For any semistable curve $C_{/k}$, there exists a regular arithmetic surface whose special fiber is isomorphic to $C$ and whose generic fiber is a curve over $K$.

*Proof:* This is (I gather) a standard result in deformation theory. A relatively accessible reference is [Vi, 4.4].

Thus it suffices to construct singular curves over the residue field $k$. We will in fact construct totally degenerate semistable curves, namely with each component of geometric genus 0. For this:

**Lemma 11.** Let $G$ be any connected graph in which each vertex has degree at most 3. Let $G$ be a finite group acting on $G$ by automorphisms. Given a field $k$, a Galois extension $l/k$ and an isomorphism $\mathfrak{g}_{l/k}$ with $G$, there is a totally degenerate semistable curve $C_{/k}$ whose dual graph is isomorphic to $G$, under an isomorphism which identifies the Galois action on $G$ with the action of $G$.

*Proof:* This is shown in [Pa] under the hypothesis that $k$ is infinite but without the hypothesis that $G$ have degree 3. The infinitude of $k$ is used precisely to ensure that the intersection points of the graph can be identified with $k$-rational points of $\mathbb{P}^1(k)$. Since $\#\mathbb{P}^1(k) \geq 3$ for all $k$, the argument goes through verbatim with the hypothesis of degree at most 3.

Recall that the arithmetic genus of a totally degenerate semistable curve $C_k$ (which is the genus of any smooth lift $C_{/K}$ in the usual sense) is just $1 - \chi(G)$, where $\chi$ is the Euler characteristic of the dual graph in the usual topological sense, computable as the number of vertices minus the number of edges.

The curves constructed by Lemma 11 satisfy hypothesis (A) unless the residue field $k$ is $\mathbb{F}_2$. More precisely, what we need is that for each finite extension $l/k$,
every component which is defined over \( l \) has at most \( \#l \) singular points. Since our graphs have degree at most 3, the only problematic case is when \( k = \mathbb{F}_2 \) and \( I = 1 \) (because if \( I > 1 \), we only want points over an extension with larger residue field). But this is a trivial case: it is enough, for instance, to find a nonsingular curve \( C/\mathbb{F}_2 \), of genus \( g \), and with \( C(\mathbb{F}_2) \neq \emptyset \). Or, staying with the same graph-theoretical strategy, we need only to find, for all \( g \geq 0 \), a connected graph with Euler characteristic \( 1 - g \), in which each vertex has degree at most 3, and at least one vertex has degree at most 2. Of course such graphs exist: for \( g = 0 \) take the graph with one vertex and no edges (the dual graph of \( \mathbb{P}^1 \)), and for \( g \geq 1 \) we can build such a graph out of \( g \) “coathangers” (the graph with vertex set \( \{0, 1, 2, 3\} \) and \( 0 \sim 1 \), \( 0 \sim 2 \), \( 0 \sim 3 \), \( 2 \sim 3 \)). Henceforth we will assume that \( I > 1 \).

Let \( G \) be a group and \( S \subset G \) such that \( S = S^{-1} \), \( 1 \notin S \), and \( \langle S \rangle = G \). We define the Cayley graph \( \text{Cay}(G, S) \), a simple (no loops, no multiple edges) undirected graph whose vertex set is \( G \) itself, and with

\[
g \sim g' \iff \exists s \in S \mid gs = g'.
\]

Note the following (almost tautological) properties of \( \text{Cay}(G, S) \): (i) it is connected; each vertex has degree \( \#S \); (iii) it admits a left \( G \)-action which is free on vertices, and free on edges unless \( S \) contains an element of order 2.

(iv) If \( G \) is finite,

\[
\chi(\text{Cay}(G, S)) = \#G \left( 1 - \frac{\#S}{2} \right).
\]

(v) If \( \rho : H \hookrightarrow G \) is an embedding, then \( H \) acts on \( \text{Cay}(G, S) \), freely on vertices and freely on edges unless \( \rho(H) \cap S \) contains an element of order 2.

Now let \( G_I = \langle \sigma \mid \sigma^I = 1 \rangle \), and identify \( G_I \) with \( g_{K/I} = g_{K/I/k} \).

When \( g = 0 \) and \( I = 2 \), we can take \( G = \text{Cay}(G_2, \{\sigma\}) \), the unique connected graph with two vertices and one edge: \( \chi = 1 \). Since the generator has order 2, the unoriented edge gets stabilized, so by Proposition 4 we get the “Case 2” splitting behavior indicated in the theorem.

When \( g = 1 \) and \( I > 2 \), we take \( G = \text{Cay}(G, \{\sigma, \sigma^{-1}\}) \), the \( I \)-cycle: \( \chi = 0 \). Here \( G \) acts freely on the edges, so by Proposition 4 we get the “Case 1” splitting behavior indicated in the theorem.

When \( g = 1 \) and \( I = 2 \) we can take \( G \) to be the 2-cycle, and let \( G_I \) act by “180 degree rotation”, i.e., by swapping both vertices and both edges: \( \chi = 0 \), Case 1.

This graph is nonsimple and a fortiori not a Cayley graph according to our setup.\(^3\)

If we insist on seeing a Cayley graph construction, fix \( N > 1 \) and let \( \rho : G_2 \hookrightarrow G_{2N} \) be the embedding \( \sigma \mapsto \sigma^N \).

When \( g > 1 \) and \( I = 2g - 2 \), take \( G = \text{Cay}(G_I, \{\sigma, \sigma^{-1}, \sigma^{g-1}\}) \). Or, in plainer terms, start with the \( 2g - 2 \) cycle and connect each pair of antipodal points by an edge: these “spokes” do not ruin the obvious \( G_{2g-2} \) action by rotations.

\(^3\)On the other hand, the Cayley graph construction admits several variants. The 2-cycle is a Cayley graph for \( G_2 \) according to the conventions of e.g. [dlH].
This second description is graph-theoretically correct (which is, of course, all that matters for us) but geometrically wrong: the graph does not really live in the Euclidean plane because the spokes would have to meet at the center of the circle, adding an unwanted (and $G_I$-fixed) vertex. Indeed, when $g = 4$ the graph is precisely the complete bipartite graph $K_{3,3}$, and for larger $g$ the graph contains $K_{3,3}$ as a topological subgraph, so these graphs are not embeddable in the Euclidean plane.\footnote{A real algebraic geometer might be tempted to point out that the graph naturally lives in the blowup of $\mathbb{RP}^2$ at a single point.} So here is a “better” geometric description: take a rectangle of length $g$ and height 1, subdivide the top and bottom sides into $g - 1$ equal parts, and draw in the $g + 1$ equidistant vertical lines linking each vertex on the top to its corresponding bottom vertex. The resulting graph has $2g$ vertices and $g + 2(g - 1)$ edges. Now identify the right and left sides of the rectangle with a half-twist, getting a graph embedded isometrically into the Möbius band with $2g - 2$ vertices and $g + 2(g - 1) - 1 = 3g - 3$ edges with a natural action of the cyclic group $G_I$ by unit length horizontal rotations. In any case we have $\chi = 1 - g$, and $G_I$ acts freely on vertices but not on edges, Case 2.

When $g > 1$ and $1 < I \mid 2g - 2$, take the above graph and the embedding $\rho : G_I \hookrightarrow G_{2g - 2}, \sigma \mapsto \sigma^{2g-2-I}$. If $I$ is odd, we are in Case 1; if $I$ is even, Case 2.

4. Final remarks

Remark 4.1: The genus 0 case can also be handled using the correspondence between genus 0 curves and quaternion algebras together with the structure theorem for Brauer groups over Henselian fields with perfect residue field \cite[Ch. XIII]{Ser}. It is somewhat amusing to remark that in the case of a finite residue field, our analysis gives a geometric proof of the well-known fact that a quaternion algebra over a locally compact field is split by every quadratic extension. It seems interesting that the behavior in genus zero, which is often seen as an anomalous case -- e.g., by virtue of the nonuniqueness of a minimal regular model -- is completely in line with the behavior for $g \geq 2$. It is the case of genus one which is truly exceptional.

Remark 4.2: Consider now the case of a curve $C/K$ admitting an $R$-model such that the reduced subscheme of the special fiber is semistable: such an arithmetic surface $C_K$ is called an SNC model of its generic fiber $C_K$. (It is known that every curve admits a regular SNC model.) In this case, writing $e_i$ for the multiplicity of $C_i$, I find it quite likely that Theorem 2 generalizes in the form

$$I(C/K) = \gcd_i (d_i \cdot e_i \cdot I_i).$$

At least in the case of finite residue field (so $I_i = 1$ for all $i$), this formula follows by combining work of \cite{CTS} and \cite{BL} (as is observed in \cite[p. 22]{PS}). As far as I can see, the proofs -- which are more technically elaborate than in the semistable case -- should go through for any perfect residue field. However, unlike the case of semistable reduction, the special fiber does not itself determine the set $S(C)$ of all possible splitting fields.

Remark 4.3: The proverbial alert reader will have noticed that we defined the index $I(V)$ only for a nonsingular, projective, geometrically irreducible variety $V/K$. The...
definition that we gave makes sense for any finite-type $K$-scheme $S$ but – in view of Remark 4.2 and the fact that this quantity depends only on the reduced subscheme $S^\text{red}$ – seems to be the wrong definition for singular schemes. From the arguments of §2 – and in particular the proof of Theorem 10 – it follows easily that if $S/K$ is the generic fiber of a regular model $S/R$ whose special fiber $S_k = \bigcup S_i$ is semistable (i.e., is a reduced finite union of varieties whose singular points are analytically isomorphic to transversely intersecting hyperplanes), then $I(S/K) = \gcd_i (d_i I_i)$ where $S_i$ is defined over $l_i$, $d_i = [l_i : k_i]$, and $I_i$ is the index of $(S_i)/l_i$. In other words, $\gcd_i (d_i I_i)$ is a suitable definition of the index of a reduced finite-type scheme over a perfect field.

As mentioned above in the case of curves, it seems quite plausible that if $S/k$ is a SNC $k$-scheme such that $S_k = \bigcup S_i$ and $e_i$ is the multiplicity of the component $S_i$, then defining the index $I'(S/k)$ to be $\gcd_i (d_i e_i I_i)$, then $I(S/K) = I'(S/k)$, i.e., the index is preserved by specialization when the model is regular, the generic fiber is smooth and the special fiber is SNC. One wonders about a more general “index specialization theorem.”

**Conjecture 12.** Assume $K$ is a Henselian discrete valuation field with perfect residue field $k$. Let $V/K$ be a smooth, projective, geometrically integral finite-type $K$-scheme and assume that $V$ admits some regular $R$-model, with special fiber $V_k$. Then $I(V/K) = I'(V_k)$.

Remark 4.4: Suppose that $V/K$ is an $n$-dimensional variety admitting a regular model whose special fiber is completely degenerate, in the sense that it is reduced and every geometric component is isomorphic to $\mathbb{P}^n$, and such that the $k$-fold intersections are transverse and isomorphic to a disjoint union of $\mathbb{P}^{n-(k-1)}$'s. Such varieties arise naturally: they include all varieties uniformized by Drinfeld’s $n$-dimensional upper halfspace. For such a variety one can compute the set $S(V)$ in terms of the combinatorics of the dual (simplicial) complex, a generalization of the dual graph of a curve. What we lack in this context is an analogue of Theorem 10 in general, higher-dimensional varieties (even smooth ones) do not lift smoothly to characteristic 0. How to choose the combinatorial geometry of the special fiber so as to permit a smooth lifting as well as some description of which classes of higher-dimensional varieties can have completely degenerate reduction are interesting questions to which we hope to return in a later work.

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