We show that non-perturbative entities such as solitons and instantons saturate bounds on entropy when the theory saturates unitarity. Simultaneously, the entropy becomes equal to the area of the soliton/instanton. This is strikingly similar to black hole entropy despite absence of gravity. We explain why this similarity is not an accident. We present a formulation that allows to apply the entropy bound to instantons. The new formulation also eliminates apparent violations of the Bekenstein entropy bound by some otherwise-consistent unitary systems. We observe that in QCD, an isolated instanton of fixed size and position violates the entropy bound for strong 't Hooft coupling. At critical 't Hooft coupling the instanton entropy is equal to its area.

1. Unitarity and Entropy

The Bekenstein entropy bound\(^1\) (see also, \([2]\])

\[
S_{\text{max}} = MR,
\]

puts an upper limit on the amount of information stored in a system of energy \(M\) and size \(R\). Throughout the paper, in most of the expressions for bounds we shall drop the obvious numerical coefficients in order to focus on scaling. We shall restore them whenever required. The Bekenstein bound (1) is independent of gravity. Despite this, it attracted enormous attention in the context of gravity. The reason is that in gravity the bound is saturated by black holes and this saturation comes in form of the area,\(^2\)

\[
S_{\text{max}} = MR = (RM_p)^d - 1.
\]

Here \(M\) is the mass of a black hole in \(d+1\) dimensional space-time, \(R\) is its radius and \(M_p\) is the Planck mass.

In a recent paper,\(^4\) we have posed the following questions:

- What is the physical meaning of saturation of the Bekenstein bound in non-gravitational quantum field theories?
- Is the area-form of saturation unique to gravity?

We have attempted answering these questions in the following sequence. First, we showed that both properties (saturation of the bound as well as its area form) take place already in renormalizable quantum field theories. For example, we have observed such a behaviour for a 't Hooft-Polyakov monopole\(^5\) in gauge theories and for a baryon\(^6\) in QCD with large number of colors.\(^7\) We have observed that these objects saturate the Bekenstein entropy bound exclusively when the theory saturates the bound on unitarity. Simultaneously, the entropy acquires the form of the area. Putting it shortly:

\[
\text{Bekenstein = unitarity = area.}
\]

The explanation given in \([4]\) to the above phenomenon is that the key role both in saturation of the entropy bound as well as in its area-form is played by an inter-particle coupling constant \(\alpha\) that controls perturbative unitarity of the theory. The idea is that the objects that share the property (3) are maximally packed in the sense of \([8]\).

- In any consistent \(d+1\)-dimensional effective field theory the entropy \(S\) of a self-sustained non-perturbative solitonic state of radius \(R\) satisfies the following bound

\[
S \leq \frac{1}{\alpha}\,.
\]

where \(\alpha\) is a relevant dimensionless quantum coupling constant evaluated at energy scale \(1/R\).
- This bound is saturated when the theory saturates unitarity.
- There always exists a well-defined physical scale \(f\) that plays the role of a symmetry-breaking order parameter and also sets the coupling strengths in the low energy theory.
• The coupling $\alpha$ evaluated at the scale $1/R$ is equal to the inverse area measured in units of $f$:

$$\frac{1}{\alpha} = (Rf)^{d-1}. \quad (5)$$

This ensures the area form of the entropy at the point of saturation:

$$S_{\text{max}} = \frac{1}{\alpha} = (Rf)^{d-1}. \quad (6)$$

In each example, the scale $f$ can be determined unambiguously. For example, in gravity it is given by the Planck mass $f = M_p$, which sets the strength of the graviton-graviton interaction. On the other hand, for topological solitons such as 't Hooft-Polyakov monopole the same scale is set by the vacuum expectation value of the Higgs field, $f = v$. At the same time, in case of a baryon/skyrmion $f$ is set by the pion decay constant $f_{\pi}$. We thus encounter the following universal pattern at the saturation point,

$$\text{max. entropy} = \frac{1}{\text{coupling}} = \text{area}. \quad (7)$$

For all isolated solitons existing in Lorentzian space-times the bound (4) fully agrees with the standard Bekenstein bound (1). However, the form (4) allows for a consistent generalization of the concept of the entropy bound to Euclidean field configurations such as instantons that describe virtual processes rather than states. To such entities the standard formulation of Bekenstein bound (1) cannot be applied directly since their energy is not defined.

We proceed in the following steps. First, we generalize the analysis of [4] to solitons in 5D theory. We consider examples of scale-independent solitons in theories with global and gauge symmetries. The forms of these solitons are identical to instantons in corresponding 4D theories. We show that 5D solitons saturate both bounds, (1) and (4), simultaneously and together with unitarity. We check explicitly that at the saturation point the entropy is equal to the area, in full accordance to (6).

Next, using the formulation of the bound (4) and the connection between the 5D soliton and 4D instanton, we generalize the concept of the Bekenstein entropy bound to an instanton. Note, we are not talking about the entropy for a gas of instantons but rather the entropy of a single isolated instanton of fixed size and position.

We assign the instanton a micro-state entropy given by the entropy of its counterpart soliton in a theory with one dimension higher. Assigning an entropy to an instanton may sound unusual since the instanton does not describe a “real” object but rather a tunneling process. Nevertheless, using the connection between an instanton and a soliton in one dimension higher gives a well-defined physical meaning to this assignment.

Next, using the form (4) we impose the Bekenstein bound on an instanton in a self-consistent way. We then discover that the instanton entropy saturates the bound and assumes the form of the area when the theory saturates the bound on unitarity. This happens when the ’t Hooft coupling assumes a critical value order-one. This gives us confidence in meaningfulness of generalizing the concept of entropy to virtual states and in imposing the entropy bound on such states.

Finally, in appendix, we provide an explicit example illustrating how the formulation (4) eliminates an apparent violation of the Bekenstein bound (1) by a fully consistent unitary theory.

### 2. Scalar No-Scale Soliton in 5D

As first example we consider a scale-invariant soliton in 5D theory without gauge redundancy. The simplest Lagrangian which leads to such a soliton has the following form,

$$L_\sigma = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{4} g^2 \sigma^4 \quad (8)$$

Here $\sigma$ is a real scalar field. The parameter $g^2 > 0$ is a five-dimensional coupling constant that has the dimensionality $[g^2] = [\text{mass}]^{-1}$. The canonical dimensionality of the scalar field is $[\sigma] = [\text{mass}]^2$. The non-perturbative solutions of 4D version of the above theory are very well-known.\[10\] Due to an attractive self-coupling the energy of the theory (8) is unbounded from below. Despite this, it represents a useful prototype model for testing our ideas. We shall later move to a gauge theory that does not suffer from such a problem.

The dimensionless four-point quantum coupling at energy scale $E$ is given by

$$\alpha \equiv g^2 E. \quad (9)$$

Due to a dimensionful gage coupling the theory violates perturbative unitarity above the following cutoff scale,

$$\Lambda \sim \frac{1}{g^2}. \quad (10)$$

That is, above the scale $\Lambda$ the dimensionless coupling $\alpha$ becomes strong. The above theory has a time-independent spherically symmetric solution of the following form,

$$\sigma = \frac{\sqrt{8}}{g} \frac{R}{x^2 + R^2}, \quad (11)$$

where $x$ is the space four-vector. The form of the solution is identical to an instanton that appears in 4D version of the theory\[10\] where it describes a tunneling process. The mass of the soliton is given by

$$M_{\text{sol}} = \frac{8 \pi^2}{3 g^2}, \quad (12)$$

and is independent of the localization radius $R$. Correspondingly the moduli space of the soliton consists of translation and dilatation zero modes that can be regarded as Goldstone bosons of corresponding broken symmetries. These modes are not sufficient for endowing the soliton with a large entropy that can be increased parametrically. In order to achieve such an entropy, we need to increase the number of localized zero modes. We shall accomplish this by assuming that $\sigma$-field transforms as a large irreducible representation of some internal “flavor” symmetry group.
For example, $\sigma_\alpha$, $\alpha = 1, 2, \ldots, N$ can form an $N$-dimensional representation of a global $SO(N)$-symmetry. Correspondingly, the Lagrangian now takes the form,

$$L_\sigma = \frac{1}{2} \partial_\mu \sigma_\alpha \partial^\mu \sigma_\alpha + \frac{1}{4} g^2 (\sigma_\alpha \sigma_\alpha)^2.$$  

(13)

Note, with the enlargement of symmetry the unitarity bound becomes,

$$N \lesssim \frac{1}{a} \equiv \frac{1}{(E g^2)}.$$  

(14)

Correspondingly, the cutoff $\Lambda$ is no longer defined by (10) but instead by the following relation,

$$N \sim \frac{\Lambda^{-1}}{g^2}.$$  

(15)

The soliton solution (11) goes through almost unchanged. However, it becomes highly degenerate:

$$\sigma_\alpha = \frac{\sqrt{8}}{g} \frac{R}{x^2 + R^2} \frac{a_\alpha}{\sqrt{N_\alpha}},$$  

(16)

where $a_\alpha$ are arbitrary subject to the following constraint,

$$\sum_{\alpha = 1}^{N} \alpha_\alpha^2 = N_\alpha.$$  

(17)

The mass of the soliton is given by the same expression (12). In classical theory the value of the parameter $N_\alpha$ is irrelevant. However, in quantum theory it acquires an important meaning, as we shall explain.

The above degeneracy can be understood in two equivalent languages. The first is the language of a moduli space. Since the soliton became embedded into the $SO(N)$-space, its moduli space got increased. Indeed, an arbitrary orthogonal $SO(N)$-transformation,

$$\sigma_\alpha \rightarrow O_{ab} \sigma_\beta,$$  

(18)

that acts on the soliton non-trivially, gives again a valid solution with exactly the same mass $M_{\text{sol}}$ and size $R$. Thus, the soliton’s internal moduli space becomes $SO(N)/SO(N-1)$ which has a topology of an $S_{N-1}$-sphere. Thus, we have $N-1$ moduli parameterizing the location of the soliton on this $N-1$-dimensional sphere.

An alternative language for describing the soliton degeneracy is of Goldstone modes. The non-zero expectation value of the $\sigma$-field breaks spontaneously the global $SO(N)$-symmetry down to $SO(N-1)$. Consequently, there emerge $N-1$ Goldstone modes localized within the soliton.

The situation that we got is fully analogous to the example of a ’t Hooft-Polyakov monopole constructed in [4]. There too, a global $SO(N)$-symmetry was spontaneously broken within the monopole core. Therefore, the counting of the micro-state degeneracy goes in the same way. Of course, classically, the degeneracy of the moduli space is infinite, since every point on it counts as a different Goldstone vacuum of the soliton. This is also a manifestation of the fact that in $\hbar = 0$ limit the entropy is infinite, as it should be. Of course, in quantum theory the entropy of a localized soliton becomes finite.

The micro-state degeneracy can be deduced from the effective Hamiltonian describing the vacuum structure of the soliton. This can be written in the following simple form,[4]

$$\hat{H} = X \left( \sum_{\alpha = 1}^{N} \hat{a}_\alpha^\dagger \hat{a}_\alpha - N_\alpha \right).$$  

(19)

where $X$ is a Lagrange multiplier which enforces the constraint (17) and the parameter $N_\alpha$ will be determined below. All the non-zero frequency modes have been excluded as they do not contribute into the ground-state structure but only into the excited states. Therefore, the operators $\hat{a}_\alpha^\dagger, \hat{a}_\alpha$ represent the creation and annihilation operators of the zero frequency moduli. They satisfy the usual commutation relations $[\hat{a}_\alpha^\dagger, \hat{a}_\beta] = \delta_{\alpha \beta}$. Obviously, the quantities $a_\alpha^2$ entering in (17) and (16) must be understood as the expectation values of the number operators $\hat{n}_\alpha \equiv \hat{a}_\alpha^\dagger \hat{a}_\alpha$. The micro-state degeneracy is then given by all possible distributions of these numbers subject to the constraint (17). The total number of such states is given by the binomial coefficient

$$n_{x_\alpha} = \left( \frac{N_\alpha + N - 1}{N_\alpha} \right).$$  

(20)

In order to evaluate this expression, we need to determine the value of the parameter $N_\alpha$ which of course must depend on $R$. For fixing it, notice that we can describe the soliton as the bound-state of $N_\alpha$ quanta of the field $\sigma$ and treat it in Hartree approximation essentially following the method applied by Witten to baryons/skyrmions in [6]. 1 Indeed, the bound-state has the size $R$ and is stabilized due to an attractive interaction measured by the coupling $g^2$. This means that the positive kinetic energy of each quantum, which scales as $\sim 1/R$, is exactly balanced by the negative potential energy of attraction from the rest, which scales as $\sim g^2 N_\alpha$. We thus conclude,

$$N_\alpha \sim \frac{R}{g^2}.$$  

(21)

Now, comparing the above expression to (14), it is clear that whenever $N$ saturates the unitarity bound at energy $1/R$, we have

unitarity limit $\rightarrow N_\alpha \sim N.$  

(22)

Or equivalently, this equality takes place whenever the soliton size becomes equal to the unitarity cutoff scale $R = 1/\Lambda$. In this case, from (20) using Stirling’s approximation we get that the micro-state entropy of the soliton in the unitarity limit becomes equal to,

$$S_{\text{sol}} = \ln(n_{x_\alpha}) \sim N.$$  

(23)

1 Here we use the bound-state view for a simple estimate, otherwise not relying on it. More detailed steps in understanding corpuscular structure of solitons/instantons were taken in [11].
We are now ready to test our claims. The first task is to see explicitly that the above entropy saturates the Bekenstein bound (1). Next, we wish to see that this saturation is in full agreement with (4). Thirdly, we must show that at the saturation point the entropy assumes the form of the area according to (6).

Taking into account the expression for the soliton mass (12), the Bekenstein bound (1) reads,

$$S_{\text{max}} \sim \frac{R}{g}. \quad (24)$$

Notice, this is exactly the bound (4) with $a = g^2 E$ evaluated at the scale $E = 1/R$. Thus, the two bounds fully agree. They become saturated when (24) becomes equal to (23). This happens when

$$N \sim \frac{R}{g}. \quad (25)$$

As it was already said, by taking into account (14), the expression (25) means that $N$ saturates the unitarity bound and at the same time $R = \Lambda^{-1}$. Indeed, for $R = \Lambda^{-1}$ the equation (25) takes the form (14). Thus, we see that the soliton saturates the Bekenstein bound (1) (and simultaneously (4)) precisely when its size $R$ becomes equal to the cutoff scale $\Lambda^{-1}$ fixed by the unitarity bound (14).

Let us now investigate whether at the saturation point the entropy takes the form of the area (6). For this, we first need to identify the scale $f$. This scale is determined by the order parameter that breaks the $SO(N)$-symmetry spontaneously. The latter is given by the maximal value that the field $\sigma$ reaches in the core of the soliton and is equal to

$$\sigma_{\text{max}} = \sigma(0) = \frac{\sqrt{g}}{gR}. \quad (26)$$

Since, $\sigma$ has dimensionality of $[\text{mass}]^2$, it is clear that the relevant scale in the problem is

$$f = \frac{1}{(gR)^{3/2}}. \quad (27)$$

This is the correct scale that must be used for measuring the surface area of the soliton. Expressing then (25) through (27) it is clear that the entropy at the saturation point can be written as the surface area in units of $f$:

$$S_{\text{sol}} \sim (Rf)^{3}. \quad (28)$$

Note, the surface area of a soliton in 5D is a three-dimensional surface. So, (28) is in full accordance with (6).

In summary, we observe that an instanton-like soliton in 5D exhibits exactly the same tendency as was observed earlier for ’t Hooft-Polyakov monopoles and baryons in [4]: the saturation of the Bekenstein bound (1) is equivalent to the saturation of the bound (4) and both are saturated together with the unitarity bound (15). The corresponding entropy exhibits the area-law (28) in agreement with the general relation (6).

3. No-Scale Gauge Monopole

We shall now discuss a gauge soliton in 5D. We first consider a simplest model that contains such a soliton. This is a theory with a gauged $SO(3)$ symmetry with a triplet of gauge fields $A_\mu^a$ where $a = 1, 2, 3$ is an $SO(3)$-index. Since we are in five space-time dimensions, the indexes $\mu, \nu$ take values 0,1,2,3,4. The Lagrangian has the following form:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (29)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g e^{abc} A_\mu^b A_\nu^c$. The parameter $g$ is a five-dimensional gauge coupling constant that has a dimensionality $[g] = [\text{mass}]^{-1}$. The canonical dimensionality of five-dimensional gauge field is $[A_\mu^a] = [\text{mass}]^{-1}$. The dimensionless four-point coupling at energy scale $E$ is given by the same expression (9) as in the previous example. The same is true about the cutoff scale $\Lambda$ above which the coupling $\alpha$ becomes strong and violates unitarity. It is given by (10).

The above theory admits a time-independent localized monopole-like soliton solution with the topological charge given by $Q = \frac{1}{8\pi} \int d^4x \varepsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu} F_{\rho\sigma\tau}$. This soliton is identical to an instanton solution in four-dimensional Euclidean space.[9] Therefore, when lifted to a five-dimensional space-time with the Lorentzian signature, it describes a localized monopole-like soliton. These 5D solitons and their relation to 4D instantons are well known and were studied previously in various contexts, in particular, in the context of large extra dimensions.[11]

For definiteness, we shall focus on the case $Q = 1$. The solution has the well-known form:

$$A_\mu^a = \frac{2}{g} \eta_\mu^a \frac{x^\mu}{x^2 + R^2} \quad (30)$$

$$F_{\mu\nu} = -\frac{4}{g} \eta_\mu^a \eta_\nu^b \frac{R^2}{(x^2 + R^2)} \delta^{ab}. \quad (29)$$

where $\eta_\mu^a$ are ’t Hooft’s parameters that shall not be displayed explicitly. As in the case of the soliton in previous example, the mass of the gauge monopole is independent of $R$ and is given by the similar expression,

$$M_{\text{mon}} = \frac{8\pi^2}{g^2}. \quad (31)$$

The zero mode spectrum of the above monopole consists of a standard set of translation, dilatation and orientation moduli, eight in total. These are not sufficient for delivering a microstate entropy that we could increase in a controllable way. For achieving the latter goal, we must increase the number of bosonic and/or fermionic gapless modes localized within the monopole as it was done in [4].

4. Bosonic Zero Modes

In this case we need to enlarge the monopole moduli space by embedding the $SU(2)$ gauge symmetry as a subgroup into a larger $SU(N)$ symmetry group. This embedding does not change
the expression for the monopole mass (31). However, the unitarity constraint becomes (14) with the unitarity cutoff scale \( \Lambda \) given by (15). In order to fix the numerical coefficients, we shall define a five-dimensional analog of the ’t Hooft coupling in the following way.

\[
\lambda_t \equiv \frac{N \alpha^2 E}{8 \pi^2}.
\]

(32)

Obviously, for strong \( \lambda_t \), the theory violates perturbative unitarity.

The above embedding results into the appearance of additional bosonic zero modes. These modes parameterize the orientation of the monopole’s SU(2)-subgroup within SU(N). They correspond to global SU(N)-transformations that act non-trivially on the monopole. Therefore, the orientation moduli space is SU(N)/SU(N - 2) \( \times \) U(1). However, the U(1)-factor is only partially residing within the stability group of the monopole. At large \( N \) this moduli-space has dimensionality \( \sim 4N \). This space is identical to a standard moduli space of an instanton of SU(N) gauge theory in 4D. The novelty here is that we are interpreting this degeneracy in terms of the Goldstone phenomenon and giving it a meaning of the micro-state entropy. Next, we are correlating the resulting entropy bound with the unitarity and the area of the system.

Indeed, the moduli space can be understood as the degeneracy of the soliton vacuum due to spontaneous breaking of SU(N) global symmetry by the monopole. This is fully analogous to the breaking of global SO(N) symmetry by a soliton considered in the previous example. Consequently, up to order-one factors, the counting of the entropy is very similar. Thus, the micro-state degeneracy is again described by the effective Hamiltonian of the form (33) with the number of zero modes now being of order 4\( N \)

\[
\hat{H} = X \left( \sum_{a=1}^{4N} \hat{a}^\dagger_a \hat{a}_a - N_{\text{mon}} \right),
\]

(33)

and the parameter \( N_{\text{mon}} \) given by (21). However, we would like to be more precise. We shall make a guess and choose \( N_{\text{mon}} \) to be a dimensionless quantity constructed out of the size of the soliton \( R \) and the topological invariant

\[
N_{\text{mon}} = R \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = R \frac{32 \pi^2}{g^2}.
\]

(34)

The number of resulting micro-states at large-\( N \) is given by the expression analogous to (20),

\[
n_{\text{st}} \simeq \left( \frac{N_{\text{mon}} + 4N}{N_{\text{mon}}} \right).
\]

(35)

Interestingly, this qualitatively matches what would be the pre-factor for the contribution of 4D instanton of size \( R \) into the vacuum-vacuum transition probability. In particular, for weak ’t Hooft coupling the equation (35) can be written as

\[
n_{\text{st}} \sim \frac{1}{(N!)^4} \left( \frac{8 \pi^2 R}{g^2 e^{\lambda_t}} \right)^{4N}.
\]

(36)

The visual similarity with the pre-factor in a standard instanton transition probability\( ^{12,19} \) is clear. The underlying physical reason for this connection shall become more transparent below, after we extend the notion of entropy to instantons.

The entropy of the monopole/instanton saturates the Bekenstein bound when the ’t Hooft coupling becomes order one. In order to make this more precise, let us evaluate the monopole entropy for \( \lambda_t = 1 \). First, taking into account (32) and (46) we see that in this limit \( N_{\text{mon}} = 4N = \frac{R^{12}\epsilon^4}{g^2} \). Next, plugging this in (35) and using Stirling’s approximation, we find that the monopole entropy for \( \lambda_t = 1 \) takes the following form

\[
S_{\text{mon}} = \ln(n_{\text{st}}) \approx \frac{64 \pi^2 R}{g^2} \ln(2)
\]

\[
\approx 1.3 \left( \frac{4 \pi}{3} M_{\text{mon}} R \right)
\]

(37)

where \( M_{\text{mon}} \) is given by (31). The last expression in the brackets is nothing but a 5D Bekenstein bound on an entropy of an object of mass \( M_{\text{mon}} \) and the size \( R \),

\[
S_{\text{max}} = \frac{4 \pi}{3} M_{\text{max}} R.
\]

(38)

At the same time, of course, this expression is equal to the Bekenstein-Hawking entropy of a would-be static black hole with the same parameters but in 5D theory with gravity.

We thus observe that the monopole/instanton entropy saturates the Bekenstein bound when ’t Hooft coupling is close to one. Note, since the sensitivity to the value of \( \lambda_t \) is exponential, the pre-factor 1.3 simply indicates that the saturation takes place when \( \lambda_t \) is very close to one. Equivalently, we can write,

\[
S_{\text{mon}}(\lambda_t \approx 1) = S_{\text{max}}.
\]

(39)

Simultaneously, the entropy acquires a form of the area measured in units of the scale (28) with the scale \( f \) given by (27). Notice, the scale \( f \) has a very well defined physical meaning, as it represents an order parameter of spontaneous breaking of the global SU(N)-symmetry by the monopole. Equivalently, it sets the interaction strength of the orientation moduli. In this sense, there is no ambiguity in defining the physical meaning of the scale \( f \). Thus, the area form of the entropy at the saturation point is unambiguous.

It is interesting to map the entropy of a monopole on an entropy of a would-be black hole of the same mass and size. For this, we write the monopole entropy in form of the black hole entropy by explicitly separating 1/4:

\[
S_{\text{mon}} = \frac{A}{4G}.
\]

(40)

where \( A \equiv 2 \pi^2 R^3 \) is the monopole surface area in 5D, and the parameter \( G \) has the following form,

\[
G \equiv \frac{3}{64 \pi} g^2 R^2 \equiv f^{-3}.
\]

(41)

As said above, \( G \) has a very transparent physical meaning as it sets the coupling strength of orientation moduli that represent the
Goldstone bosons of broken global symmetry. In this sense it is fully analogous to the Newton’s constant that sets the interaction strength among gravitons. Correspondingly, the scale $f$ plays the role of the Planck mass.

5. Fermion Zero Modes

An alternative possibility for increasing the micro-state entropy is to populate the 5D monopole by fermion zero modes, as it was done in [4]. This is accomplished via coupling the gauge field to a large number of fermion species. Let us assume that fermions form an $N$-dimensional representation of a global flavor symmetry group. Due to existence of $\sim N$ fermion species the unitarity bound is given by (14) as in the previous case. Correspondingly, the cutoff scale $\Lambda$ is determined by (15).

As it is well-known, in the instanton/monopole background fermions give rise to localized zero modes. A detailed construction of fermion zero modes in the instanton backgrounds can be found in several excellent reviews.[12] For us, the important fact is that the number of zero modes scales as $\sim N$ and can be made arbitrarily large by increasing the number of flavors.

For any given $R$, the existence of $N$ fermionic zero modes creates $n_{st} = 2^N$ degenerate micro-states in the monopole spectrum. Correspondingly, the monopole acquires a micro-state entropy give by,

$$S_{mon} = \ln(n_{st}) \sim N.$$  (42)

From here, it is obvious that all the effects observed in the previous examples repeat themselves. This can be easily seen by taking into account (14) and (15) and comparing (42) with (24). Thus again the monopole saturates the entropy bounds (1) and (4) together with unitarity. The area form of the entropy (28) at the saturation point also remains intact.

6. Instanton in 4D

We shall now move to 4D and consider an instanton instead of a soliton. It is given by exactly the same solution (30) but “downgraded” to 4D Euclidean space. Therefore, we have to take into account the change of dimensionalities of the parameters. We shall denote them by the same symbols as in 5D. However, we must remember that $g$ now is dimensionless and $A^\mu$ has a dimensionality of mass. Correspondingly, we define the four-dimensional analog of the ’t Hooft coupling as

$$\lambda \equiv N \frac{g^2}{8\pi^2}.$$  (43)

Note, in 4D we wish to keep the theory asymptotically-free. So we must assume that the fermion content is chosen appropriately. In the present case, for simplicity, we shall ignore the fermion flavors.

We shall now undertake the following two steps. First, we shall assign entropy to an instanton. Secondly, we shall try to understand what is the upper bound on this entropy and what is the physical meaning of its saturation.

We assign the entropy to the instanton by a direct generalization of the entropy of its 5D counterpart soliton. Namely, the entropy of the instanton will be counted as the log of the number of micro-states (46) due to the existence of zero mode bosons and/or fermions, exactly as it was counted for the 5D monopole. So we shall assign to an instanton the following number of “micro-states”

$$n_{inst} \equiv \left( \frac{N_{inst} + 4N}{N_{inst}} \right)^4.$$  (44)

where taking into account the dimensionality of the coupling constant we take,

$$N_{inst} = \int d^4x e^{i\sigma^\mu\nu\alpha\beta} F^\mu\nu F^\alpha\beta = \frac{32\pi^2}{g^2}.$$  (45)

From the perspective of Lorentzian space, (44) counts the number of micro-processes that cost the same Euclidean action $I_{inst} = \frac{4\pi}{g^2}$. For weak ’t Hooft coupling the “degeneracy” (44) can be approximated as,

$$n_{st} \sim \frac{1}{(N!)^2} \left( \frac{8\pi^2}{g^2} e^k \right)^4.$$  (46)

The coupling $g^2$ is evaluated at the scale $R$. This matches the expected measure for the contribution of a given size instanton into the vacuum-vacuum transition probability.[12,19] This is remarkable since the correlation between the degeneracy of actual states and their contribution into the virtual processes is rather non-trivial. The interpretation of the transition probabilities in terms of instanton entropy can be highly instructive. Namely, the question that we would like to ask is:

What is the physical significance of the violation of the entropy bound by an instanton?

In order to answer this question, we first need to solve the following dilemma. Since we are in Euclidean space, there exists no notion of energy. So the Bekenstein entropy bound (1) is not well-defined and cannot be applied to an instanton directly. However, the bound (4) is well-defined and can be applied. This bound is saturated when the ’t Hooft coupling (43) becomes order-one. Indeed, taking $\lambda_1 = 1$ we get

$$S_{inst} = \ln(n_{st}) \simeq \frac{64\pi^2}{g^2} \ln(2),$$  (47)

where $g^2$ has to be evaluated at the energy scale $1/R$.

Equivalently, we can say that an instanton of a given size $R$ saturates the entropy bound (4) when the counterpart monopole in 5D saturates the ordinary Bekenstein bound as described by (37). The fact that the entropy saturation takes place when ’t Hooft coupling becomes order one, cannot be a simple coincidence and must be revealing some deep underlying physics.

Similarly to the 5D monopole, at the saturation point the entropy takes the form of the area.

$$S_{mon} = \frac{A}{4G}.$$  (48)
where \( A \equiv 4\pi R^2 \) is the area and the parameter \( G \) has the form,

\[
G \equiv \frac{\alpha^2 R^2}{64\pi \ln(2)} \equiv f^{-2}.
\]

(49)

As in 5D case, the parameter \( G \) has a very well defined physical meaning as it sets the coupling strength of the Goldstone modes. In this respect it is fully analogous to Newton’s constant in gravity.

Notice, we get the area law as for a localized object in 4D Minkowski space-time. The physical meaning of this is easy to understand. An instanton describes a tunnelling process. We can connect this process to 5D monopole discussed in the previous section. For this, we can imagine that the 4D Minkowski gauge theory resides on a 4D slice of the 5D gauge theory. Now, the 5D theory houses monopoles. These monopoles can tunnel through the 4D surface. This “passing-by” virtual monopole is “seen” by a 4D observer as instanton.2

In such a picture, an each passing-through event can be attributed a particular location in 4D space. Of course, by 4D Poincare invariance we effectively integrate over all possible locations and sizes whenever we compute instanton contribution into the physical observables. However, this is not important for the present discussion since we are interested in the entropy of an instanton with fixed size and location. Then, the area of a sphere surrounding each event is a two-dimensional surface.

7. Black Holes

Although in the present paper we focus on non-gravitational theories, we must comment on an obvious connection with the black hole entropy (2). As it was already noticed in [8] the entropy of a black hole of size \( R \) can be written in the form (6) where \( \alpha \) must be understood as the gravitational coupling at the scale \( 1/R \):

\[
\alpha g^2 = \frac{1}{(RM_H)^{d-1}}.
\]

(50)

This fact suggests that the remarkable similarity between solitons/instantons/baryons/skyrmions on one hand and black holes on the other exhibited at the saturation point lies in the fact that all these seemingly-different entities are maximally packed composite objects. [6] Such objects consist of maximal occupation number of quanta compatible with the strength of the coupling \( \sim 1/\alpha \). It is then natural that the entropy capacity of such objects fully saturates the unitarity bound of the system for a given \( \alpha \).

It is also clear why the Planck mass plays the role analogous to symmetry breaking parameter. Notice, the canonically normalized graviton field reaches value \( \sim M_p \) near the black hole horizon as seen by an external observer in Schwarzschild coordinates. This is analogous to the field \( \sigma \) reaching the values set by the scale \( f \). The same is true about the baryon that can be viewed as the skyrmion soliton.[14, 15] There too the maximal value of the order parameter is set by the pion decay constant \( f \).

As already pointed out in [4], yet another similarity in black hole case is the existence of the species length-scale \( \Lambda^{-1} = N^{\frac{d}{2}}/M_p \). [17] This is the size of a smallest black hole which saturates the bounds (1) and (4) on information storage capacity. At the same time, at the same scale, the gravitational interaction saturates unitarity.

8. Discussions

In the present paper we gained an additional support for the results obtained in [4] and made some further steps. First, we continue to observe that manifestations of the Bekenstein entropy bound that are usually attributed to gravity are universal. They are shared by non-perturbative objects in generic consistent theories regardless of renormalizability and/or the presence of gravity. We observe that the saturation of the entropy bound and its area form are linked with the saturation of unitarity.

For soliton- and instanton-like non-perturbative objects the entropy bound can be formulated in the form (4) in which the mass of the object does not enter explicitly. Instead, the bound is set by the inverse of the coupling constant. This form makes the connection between the saturation of the entropy bound and unitarity more transparent. Through the relation (6) it also gives a quantum field theoretic explanation to the area-form of the entropy at the saturation point.

In appendix we highlight the importance of the coupling constant in formulation of the entropy bound (4). Namely, we provide an explicit example of manifestly unitary theory in which the Bekenstein bound in its original formulation (1) is seemingly violated, however the formulation (4) restores the consistency.

Another novelty is that we have generalized the concept of the entropy bound to an instanton of a fixed size and location. For this, we first needed to assign the entropy to an instanton. We did this in two equivalent ways. First, we assign to an instanton an entropy that would be carried by its counterpart soliton in a space with one dimension higher. An alternative way is to use the formulation (4) which for solitons and baryons fully agrees with Bekenstein bound (1). However, since the bound (4) does not include any dependence on the mass, we can directly apply it to an instanton. We then observe that the instanton saturates the bound (4) when its counterpart soliton in a theory one dimension higher saturates the Bekenstein bound (1). Both saturations are synchronized with the saturations of unitarity in respective theories and take the forms of the respective areas. The saturation of the entropy bound takes place when the ’t Hooft coupling reaches the critical value order one.

It would be interesting to understand if this criticality is somehow related with possible phase transitions in large-N QCD, for example, of the type suggested by Gross and Witten[20] and by Wadia,[21] or with other non-analitics at large-N.[22]

One lesson we are learning is that the solitons and instantons are no less holographic[16] than black holes. Our observations indicate that saturation of the entropy bound and its area-form go well beyond gravity and are defined by fundamental aspects of quantum field theory such as unitarity and asymptotic freedom.

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2 In the present discussion the embedding of 4D theory into 5D is just a mental exercise for understanding the connection, with no real physical meaning implied for the fifth dimension. The embedding however can be given a direct physical meaning in specific constructions such as [13] or [18].
Appendix: Coupling and Entropy Bound

Here we would like to discuss the importance of the coupling for formulation of the entropy bound. This should provide an additional justification for writing the bound in the form (4). In order to do this, it is the simplest to give an explicit example of a consistent field theory that naively violates the entropy bound formulated in the original Bekenstein form (1) but the violation is avoided after we take into account the formulation (4).

We note that apparent violations of the Bekenstein bound in gravitational context and attempts of resolving such violations where discussed previously, e.g., in [23,24]. The formulation (4) can be useful in this respect also, as it avoids violation of the bound in any unitary theory.

Consider any unitary theory that admits a large number of non-interacting localized solitons. For example, let us focus on a set of $N$-copies of $SO(3)$ gauge theories in 4D, where $j = 1, 2, \ldots, N$ is the label of the group. Assume that each copy is Higgsed by an accompanying triplet scalar field $\Phi_j$, where $a_j = 1, 2, 3$ is the gauge index of the respective $SO(3)$. In this way, each $SO(3)$ is Higgsed down to an $U(1)$-subgroup. As a result, we obtain a `t Hooft-Polyakov magnetic monopole residing in each theory. For simplicity, let us assume that the parameters of the theories such as gauge couplings $\epsilon_j = \epsilon$ and masses are equal and the theories do not talk to each other. That is, the Higgs vacuum expectation values are all equal $\langle \Phi_j \rangle = \nu$ and so are the gauge boson masses $m_j = \nu = m$.

Then, we get the following situation. The masses of the monopole species $M_j = m_j/e^2$ as well as their sizes $R_j = 1/m_j$ are equal to each other. $M_j = M_{\text{mon}}$, $R_j = R$. Since the theories are decoupled, the monopoles from different sectors experience no interaction and they can be placed at arbitrary distances from each other. In particular, we can place them right on top of one another and create a stack of $N$ monopoles. The size of the stack is $R$ but the mass scales as $M = N M_{\text{mon}}$. Thus, the maximal entropy permitted by the Bekenstein bound (1) scales linearly with $N$: $S_{\text{max}} = N (MR)$. On the other hand the naive entropy scales as $S_{\text{mon}} \sim N^2$ due to the number of moduli that parameterize the relative positions. Their number is growing as $\sim N^2$. So, it appears that by taking $N$ sufficiently large the entropy of the monopole stack can violate the Bekenstein bound (1).

However, the issue is more subtle. Although the entropy formally grows unbounded with $N$, the information content stored in such micro-states is “sterile” and therefore meaningless. Indeed, in order to read-out the quantum information stored in a state of displacement moduli, a device must exist that can measure such displacements. However, in quantum field theory there exist no external devices. All the “devices” are manufactured out of the quantum fields. Thus, in order to measure the quantum information stored in the monopole displacements, an agent is required in form of a quantum field that interacts with all the monopole species. If such an agent is absent, the information is unreadable. The Bekenstein bound formulated in the form (1) cannot capture this difference explicitly, since it is independent of the strength of the coupling.

On the other hand, the bound formulated as (4) does capture the difference. Indeed, in the absence of any mediator quantum field, the inter-monopole coupling strength $\alpha_j$ vanishes. So written in the form (4)

\[
S_{\text{max}} < \frac{1}{\alpha_j} \quad \text{(A.1)}
\]

the bound is maintained for arbitrary large $N$ and consistency is restored. The advantage of the bound (4) is that it accounts for the fact that in order to read-out the information and/or to maintain the bound-state the coupling must be non-zero.

Let us now create a measuring device. The role of it can be played by a field that couples to all the Higgs triplets. For example, a gauge-singlet scalar $\chi$ and/or a Majorana gauge-singlet fermion $\psi$ with the following couplings,

\[
\sum_{j} g^2_{ij} |\chi|^2 (\Phi_j^a \Phi_j^b) + \tilde{g}_{ij} \Phi_j^a \tilde{\psi}_j^b \psi. \quad \text{(A.2)}
\]

would do the job. Notice, in order to be connected via a gauge-singlet fermion $\psi$, an each $j$-sector must also contain a gauge-triplet fermion $\psi^a$. There is no need for this in the scalar case.

For simplicity, we can assume all the $g$ and $\tilde{g}$ to be of the same order $g \sim \tilde{g} \sim g$. Then, unitarity puts the following bound on this coupling-strength

\[
\sum_{j} g^2_j \sim g^2 N \leq 1. \quad \text{(A.3)}
\]

Now, the coupling to the connectors induces the coupling among the different monopole species already at one-loop level due to the exchanges by $\chi$ and $\psi$. This can be seen from the following part of one-loop effective Coleman-Weinberg potential,

\[
V = \frac{1}{64 \pi^2} \left( M^4_j \ln \frac{M^4_j}{\mu^2} - M^4_v \ln \frac{M^4_v}{\mu^2} \right), \quad \text{(A.4)}
\]

where $M^2_j \equiv \sum a g^2_j \Phi_j^a \Phi_j^b$, $M^2_v \equiv \sum a \tilde{g}^2_j \Phi_j^a \tilde{\psi}_j^b$ and $\mu$ is a renormalization scale.

This potential induces the coupling among the $i$ and $j$ monopole species of the strength $\alpha_{ij} = g^2_{ij} - g^2_i - g^2_j \sim g^2$. By a suitable choice of parameters this interaction can be made attractive, repulsive or neutral. Thus, the effective inter-monopole coupling by the unitarity bound (3) scales as $\alpha_{ij} \leq 1/N^2$. So the entropy bound formulated as (4) is not violated by the stack of monopoles as long as the system is unitary.

Finally, we notice that at the saturation point $e^2 = g^2 = 1/N$ the area law (6) is satisfied. Indeed, the scale $f$ is given by $f^2 = Nv^2$ and we have,

\[
S_{\text{max}} = \frac{1}{\alpha} = N^2 = (Rf)^2. \quad \text{(A.5)}
\]

Thus, although increasing $N$ increases the entropy of the stack of monopoles without increasing its size $R$, nevertheless the area measured in units of the order parameter $f$ increases with $N$ in such a way that it always matches the entropy at the saturation point.
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Conflict of Interest

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