The mean-field approximation model of company’s income growth

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Abstract

We introduce a mean-field type approximation for description of company’s income statistics. Utilizing huge company data we show that a discrete version of Langevin equation with additive and multiplicative noises can appropriately describe the time evolution of a company’s income fluctuation in statistical sense. The Zipf’s law of income distribution is shown to be hold in a steady-state widely, and country-dependence of income distribution can also be nicely implemented in our numerical simulation.

Key words: Econophysics, Company’s Income, Zip’s Law
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1 Introduction

Studies on wealth distribution can be traced back more than 100 years. It is widely known that Pareto published the first report on the individual income distribution in 1897 [1]. He showed that the probability density distribution of income follows a power law distribution in the high-income range. For smaller income the cumulative probability distribution in log-log plot clearly deviates from the power law. This range of individual income is reported to be approximated by a log normal distribution [2] or an exponential distribution [3,4,5].

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In recent years huge data bases of income data are accessible to anyone due to the progress of computer technology, and a new type of research focusing on macroscopic statistical laws deriving from microscopic economics data is prosperous [6,7].

Similar analysis of income statistics has been done also for companies. Company’s data is finely described more than the individual because the accounting of the company is directly related to the tax revenues and the information is essential for investors. It is well-known that the size distribution of companies is also characterized by a power law distribution [8]. Distribution of sales follows a log-normal distribution [9] and the growth of sales shows sporadic violent fluctuations [10,11]. In the case of Japanese companies income the cumulative probability distribution clearly follows a power law with exponent −1 so-called the Zip’s law [12]. The Zipf law of company statistics is also known about the distribution of the number of employees [13].

The wealth distribution among individual investors is theoretically studied using a generalized Lotka-Volterra model [14,15] and the relation between the exponent of wealth distribution and the number of agents in the market is discussed in detail [16]. It is pointed out that a multiplicative process plays an essential role in the dynamics of wealth growth. There are other types of models that realize the distribution of individual income by introducing exchanges of wealth between agents [17,18]. Similar approach was already introduced in the study of company size distribution by modeling the competition among companies [19].

It is reported that Zipf’s law of Japanese income distribution has been maintained over 30 years with very high precision [6,20]. This is a special feature compared with the case of personal income distribution [21] in which the exponent apparently changes year by year. It is likely that the income statistics of companies can be regarded as a steady stochastic process maintaining the power law distribution. In the study of statistical physics it is known that one of the simplest stochastic processes producing a steady power law distribution is the discrete version of Langevin equation with additive and multiplicative noises [22, 23]. A preliminary study on modeling company’s income statistics based on such type of Langevin equation has produced promising results explaining the basic power law distribution from the statistics of growth rates [6].

In this paper we analyze databases of companies’ income of three countries following this formulation. We introduce a kind of mean-field type evolution equation of a company’s annual income that involves a multiplicative growth rate and an additive noise term, having the identical form with the discrete Langevin equation. Precise parameter fittings can be done based on this modeling and our numerical simulations can reproduce the present income distri-
2 Income Distribution

At first we investigate statistical laws of company’s income by analyzing the data (about 15,000 companies all over the world except USA [24], about 15,000 companies in USA [25], and Japanese companies whose annual incomes exceed 40 million yen, that is, about 80,000 companies [26]). Here, income is defined by the total incoming cash flows minus outgoing ones before taxation, therefore, it can take a negative value although major companies’ incomes are nearly always positive.

Incomes of each company show large fluctuations year by year as shown in Fig.1. It is not rare that this year’s income is increased tenfold or reduced to one-tenth from last year’s income. Namely, it is an amazing fact that the total distribution of income distribution is keeping the identical power law for more than 30 years as each component company’s income changes significantly and even there are many companies disappeared during this period. Power law distribution is also observed in case of USA and other countries. In Fig.2 we plot the distributions of income in Japan, UK, and USA, and in each case the distribution is confirmed to be close to a power law with exponent -1. For USA we also plotted the distribution of negative incomes by their absolute values for comparison. It is a non-trivial fact that the negative income distribution also follows the similar distribution having a power law tail [3].

3 The Mean-Field Approximation Model of Income Growth

As mentioned above we assume the following form of stochastic time evolution of income $I$ for each company,

$$I(t + 1) = \alpha(t) \cdot B(t : I) \cdot I(t) + f(t),$$

where, $B(t : I)$ is a multiplicative noise representing the income growth, $f(t)$ is the additive noise, and the coefficient $\alpha(t)$ specifies the sign of income, namely, it takes either 1 or -1 with empirically determined probability. It is known that the steady power law realizes if the multiplicative factor $B(t : I)$ and the additive noise $f(t)$ randomly fluctuate independently with respect
to $I(t)[22]$. By introducing the coefficient $\alpha(t)$ we can take into account the cases with negative incomes when a company slumps suddenly. We find the occurrence of such cases even for very large companies with probability about 3%.

4 The Growth coefficient $B(t : I)$ in Model

We estimate the growth coefficient $B(t : I)$ by investigating the distribution of $(R(t) \equiv I(t + 1)/I(t)$ in the range of large $I(t)$ because $f(t)/I(t)$ is expected to be very small in this range. In Fig.3 we show the growth rate distributions of income from 1989 up to 1995 in USA. It is found from our data analysis that the coefficient $B(t : I)$ is dependent on income $I(t)$. In order to take into account this size dependence we normalize the growth rate of income by $\sigma(I)/\sigma_0$, where $\sigma(I)$ is the standard deviation of logarithmic growth rates $\log(I(t + 1)/I(t))$ and $\sigma_0$ is the standard deviation observed in the large income range. The curves of distribution of normalized growth rates $R^\prime(t)$ for different income categories collapse nicely to a curve as shown in Fig.4.

5 Coefficients $\alpha(t)$ and $f(t)$ in Model

For estimation of the additive noise term $f(t)$ we can utilize the sign-change probability of income in the following way. As shown in Fig.5 the observed sign-change probability is nearly constant for large $|I(t)|$, while its value increases for smaller $|I(t)|$. This empirical fact can be approximated by the simple assumption that the statistics of $f(t)$ is independent of $I(t)$ and its standard deviation is given by the value of $I$ at which the sign-change probability begins to bend, that is, about 400 in the case of Fig.5. Actually this model gives the sign-change probability as shown by the dotted curves in Fig.5.

Now our basic model equation is given as

$$I(t + 1) = \alpha(t) \cdot \frac{R^\prime(t)^{\alpha(t)}}{\sigma_0} \cdot I(t) + f(t).$$  \hspace{1cm} (2)

where the function of $\alpha(t)$ for USA companies, for example, is given by

$$\alpha(t) = 1 \text{ with probability } 0.97(I(t) > 0), 0.75(I(t) < 0)$$

$$= -1 \text{ with probability } 0.03(I(t) > 0), 0.25(I(t) < 0).$$  \hspace{1cm} (3)
6 Monte Carlo Simulations and Theoretical Analysis

We now perform Monte Carlo simulation of the growth of income of each company using the values of coefficients estimated from the real data for each country. The initial value of income of any company used in our simulation is 100 and the number of companies is 60000. The time development of distributions of both positive and negative incomes in the case of USA is shown in Fig.6. The simulation result with $t = 50$ (meaning 50 years) matches the actual present distribution nicely. Results for income distributions for UK and Japan are plotted in Fig.7 together with the results of $t=50$ for USA. In the case of Japan a steady-state distribution realizes at about 25 years and in the case of UK the best fit curve is obtained for $t = 100$ by comparing with Fig.2. In Japanese case it is confirmed that growth coefficient distribution does not depend on the income range, namely, the coefficient $\sigma(I)$ of Eq.(2) is treated as a constant and the mean value of the growth rate is very close to 1 [6]. It is proved theoretically that in such case the income distribution at the steady state is characterized by a power law with exponent -1 by applying the formulas of random multiplicative Langevin equation relating the growth rate distribution and the steady-state distribution [22].

$$< B(t : I)^\beta > = 1 \quad , \quad P(I) \propto I^{-\beta} \quad , \quad \text{with} \beta = 1 \quad (4)$$

This theoretical result is consistent with the fact that the observed power exponent is about -1 for over 30 years in Japan.

7 Discussion

We can expect future income distribution by assuming that future income growth rate distribution to be the same as that of past income growth because this model is time evolution equation. For example, the company’s income distribution in USA is expected to be still growing, namely, assuming that the present company growth rate distribution being kept identical then our simulation result predicts that the income distribution will keep growing for more than 100 years.

For practical purposes our model may contribute to a policy of tax revenues or evaluation of investment strategy. It is possible to estimate the statistical outcome of investment for a group of companies, actually our preliminary simulation results suggests that investment for companies with intermediate income is most profitable with intermediate risks.

As a new field of physics our approach paves the way to bridge macro-economics
and micro-economics by the concepts and methods developed in statistical physics based on actual data. It may be impossible even in the future to describe fully the activity of a company in terminology of physics, however, the overall statistical properties can be nicely approximated by a rather simple stochastic equation familiar in physics as demonstrated in this paper. We believe this new field will be promising for development of both physics and economics.

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References

[1] V. Pareto, Le Cours d’Economie Politique, Macmillan, London, (1897).
[2] E. W. Montroll and M. F. Shlesinger, J. Stat. Phys. 32 (1983) 209-230.
[3] H. Aoyama, Y. Nagahara, M-P. Okazaki, W. Souma, H. Takayasu, and M. Takayasu, Fractals, 8 (2000), 293-300.
[4] A. Dragulescu and V. M. Yakovenko, The European Physical Journal B 20 (2001) 585.
[5] A. Dragulescu and V. M. Yakovenko, Physica A 299 (2001) 213.
[6] T. Mizuno, M. Katori, H. Takayasu and M. Takayasu, in Empirical Science of Financial Fluctuations – The Advent of Econophysics, (Springer Verlag, Tokyo, 2002) 321-330.
[7] Y. Fujiwara, et al. cond-mat/0208398.
[8] Y. Ijiri and H. Simon, Skew distributions and the sizes of business firms (in Amsterdam, North Holland, 1977).
[9] J. Voit, Advances in Complex Systems, 4, No. 1 (2001) 149-162.
[10] M. H. R. Stanley, et al, Nature 379 (1996) 804-806.
[11] Y. Lee, L. A. N. Amaral, D. Canning, M. Meyer, H. E. Stanley, Phys Rev Lett 81 (1998) 3275-3278.
[12] K. Okuyama, M. Takayasu and H. Takayasu, Physica A 269 (1999) 125-131.
[13] R. L. Axtell, Zipf Distribution of U.S. Firm Sizes, Science 293 (2001) 1818-1820.
[14] S. Solomon, in Computational Finance 97 (Kluwer Academic Publishers 1998).
[15] S. Solomon and P. Richmond, Physica A 299 (2001) 188-197.

[16] Z. F. Huang and S. Solomon, Physica A 294 (2001) 503-513.

[17] S. Ispolatov, et al, Eur. Phys. J. B 2 (1998) 267-276.

[18] N. Scafetta, et al, cond-mat/0209373

[19] H. Takayasu, and K. Okuyama, Fractals 6 (1998), 67-79.

[20] H. Takayasu, and M. Takayasu, *Econophysics – Toward Scientific Reconstruction of Economy* (in Japanese, Nikkei, Tokyo, 2001).

[21] W. Souma, in *Empirical Science of Financial Fluctuations – The Advent of Econophysics*, (Springer Verlag, Tokyo, 2002) 343-352.

[22] H. Takayasu, A-H. Sato, M. Takayasu, Phys Rev Lett 79 (1997) 966-969.

[23] D. Sornette, Linear stochastic dynamics with nonlinear fractal properties, Physica A 250, 295-314 (1998)

[24] Moody’s international company data 1989-1995. Big companies all over the world more than 11,000 companies.

[25] Moody’s company data 1989-1995. Big companies in U.S.A more than 10,000 companies.

[26] The data, “Japanese companies the best about 80,000” which published in 1997-2000, and “Company’s income ranking in Japan” which published in 1970-1996, by a publisher Diamond Inc. in Tokyo, Japan.
Fig. 1. An example of income fluctuation of a Japanese company for 30 years in log-scale. White and dark squares show incomes and corresponding rankings, respectively.

Fig. 2. Log-log plot of cumulative distributions of positive company’s incomes in Japan, UK, USA and negative income in USA. The power exponent of the dashed line is -1. In the case of UK the distribution for low income is distorted due to lack of data.

Fig. 3. Log-log plot of the probability density of income growth rates of USA companies. The unit of income $I(t)$ in the figure is 1000$. 
Fig. 4. Normalized income growth rate distribution of USA companies. The dashed lines show approximation by power laws used in the numerical simulation.

Fig. 5. The sign change probability of income as a function of income size. Dark triangles show sign change probability from positive income to negative income, and white triangles show the probability of opposite cases.

Fig. 6. Simulation results of income growth for USA. The line of USA for $t = 50$ agrees with the actual distribution in Fig.2.
Fig. 7. The simulation results for Japan, UK and USA.