Several characterizations of left Köthe rings

Shadi Asgari\(^2\) · Mahmood Behboodi\(^1,2\) · Somayeh Khedrizadeh\(^1\)

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Abstract
We study classical Köthe’s problem, concerning the structure of non-commutative rings with the property that: “every left module is a direct sum of cyclic modules”. In 1934, Köthe showed that left modules over Artinian principal ideal rings are direct sums of cyclic modules. A ring \(R\) is called a left Köthe ring if every left \(R\)-module is a direct sum of cyclic \(R\)-modules. In 1951, Cohen and Kaplansky proved that all commutative Köthe rings are Artinian principal ideal rings. During the years 1961–1965, Kawada solved Köthe’s problem for basic finite-dimensional algebras: Kawada’s theorem characterizes completely those finite-dimensional algebras for which any indecomposable module has a square-free socle and a square-free top, and describes the possible indecomposable modules. But, so far, Köthe’s problem is open in the non-commutative setting. In this paper, we classified left Köthe rings into three classes one contained in the other: left Köthe rings, strongly left Köthe rings and very strongly left Köthe rings, and then, we solve Köthe’s problem by giving several characterizations of these rings in terms of describing the indecomposable modules. Finally, we give a new generalization of Köthe–Cohen–Kaplansky theorem.

Keywords  Left Köthe rings · Köthe’s problem · Finite representation type · Square-free modules · Kawada rings

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Mahmood Behboodi
mbehbood@iut.ac.ir
Shadi Asgari
sh_asgari@ipm.ir
Somayeh Khedrizadeh
s.khedrizadeh@math.iut.ac.ir

1 Department of Mathematical Sciences, Isfahan University of Technology, 84156-83111, Isfahan, Iran
2 School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395-5746, Tehran, Iran

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1 Introduction

An old problem in non-commutative ring theory is to determine rings $R$ whose left modules are direct sums of cyclic modules (see [18, Question 15.8] and [29, Appendix B, Problem 2.48]). In 1934, Köthe [23] showed that over an Artinian principal ideal ring, each module is a direct sum of cyclic modules. A ring for which any left (resp., right) module is a direct sum of cyclic modules, is now called a left (resp., right) Köthe ring, and characterizing such rings is called Köthe’s problem. In 1941, Nakayama [27, 28] introduced the notion of generalized uniserial rings as a generalization of Artinian principal ideal rings, and proved that generalized uniserial rings are Köthe rings (a ring $R$ is called a generalized uniserial ring, if $R$ has a unit element and every left ideal $Re$ as well as every right ideal $eR$ generated by a primitive idempotent element $e$ possesses only one composition series). However, as shown by Nakayama, the rings of this type are not general enough for solving Köthe’s problem (see [28, Page 289]). In 1951, Cohen and Kaplansky [9] proved that all commutative Köthe rings are Artinian principal ideal rings. Thus, by combining the above results it is obtained that:

**Theorem 1.1** (Köthe,[23]) An Artinian principal ideal ring is a Köthe ring.

**Theorem 1.2** (Köthe–Cohen–Kaplansky, [9, 23]) A commutative ring $R$ is a Köthe ring if and only if $R$ is an Artinian principal ideal ring.

However, a left Artinian principal left ideal ring $R$ need not be a left Köthe ring, even if $R$ is a local ring (see [11, page 212, Remark (2)]). During the years 1961–1965, Kawada [19–22] solved Köthe’s problem for basic finite-dimensional algebras. Kawada’s theorem characterizes completely those finite-dimensional algebras for which any indecomposable module has a square-free socle and a square-free top (called Kawada algebras), and describes the possible indecomposable modules. In fact, Kawada’s theorem, contains a set of 19 conditions which characterize Kawada algebras, as well as the list of the possible indecomposable modules (see [19], for more details of 19 conditions). This seems to be the most elaborate result of the classical representation theory. Faith [11] characterized all commutative rings whose proper factor rings are Köthe rings. Behboodi et al. [7] showed that if $R$ is an Abelian left Köthe ring, then $R$ is an Artinian principal right ideal ring (a ring in which all idempotents are central is called Abelian). So, it is obtained that

**Theorem 1.3** ([7]) An Abelian ring $R$ is a Köthe ring if and only if $R$ is an Artinian principal ideal ring.

This generalizes Köthe–Cohen–Kaplansky theorem. In this paper, we solve Köthe’s problem, for a ring without any conditions, by giving several characterizations in terms of describing the indecomposable modules. Also, we give a generalization of Theorem 1.3.

Throughout this paper, all rings have an identity element and all modules are unital. Let $M$ be an $R$-module. The **socle** of $M$, denoted by $soc(M)$, is the sum of all simple submodules of $M$. If there is no simple submodules in $M$ we put $soc(M) = 0$. Dual to the socle the **radical** of $M$, denoted by $Rad(M)$, is the intersection of all maximal submodules of $M$. If $M$ has no maximal submodules we set $Rad(M) = M$. If $M$ has a proper submodule which contains all other proper submodules, then $M$ is called a **local module** (i.e., $Rad(M)$ is a maximal submodule of $M$). A ring is a **local ring** if and only if $R$ (or $R_R$) is a local module. For any module $M$, the **top** of $M$, denoted by $top(M)$, is the factor module $M/Rad(M)$. Also, for a ring $R$ we denote by $J(R)$ the Jacobson radical of $R$. The module $M$ is called **square-free** if it does not contain a direct sum of two nonzero isomorphic submodules. In Sect. 2, we
provide several characterizations for a left Köthe ring. Among these, it is proved that $R$ is a left Köthe ring if and only if $R$ is of finite representation type and every (finitely generated) indecomposable left $R$-module has a cyclic top (see Theorem 2.6). In the sequel, we introduce and study the following concepts. Using them, we obtain more characterizations of Köthe rings.

**Definition 1.4** We say that a ring $R$ is a:
- **strongly left (resp., right) Köthe ring** if every nonzero left (resp., right) $R$-module is a direct sum of modules with nonzero square-free cyclic top.
- **very strongly left (resp., right) Köthe ring** if every nonzero left (resp., right) $R$-module is a direct sum of modules with simple top.
- **strongly Köthe ring** if $R$ is both a strongly left and a strongly right Köthe ring.
- **very strongly Köthe ring** if $R$ is both a very strongly left and right Köthe ring.

It is shown that these types of rings are left Köthe (see Theorem 2.6), so we have the following inclusion relationships:

Very strongly left Köthe rings $\subseteq$ Strongly left Köthe rings $\subseteq$ Left Köthe rings.

Several equivalent conditions for the concept of a strongly left Köthe ring are given in Sect. 3 (see Theorem 3.4). We show that the notions of a left Köthe ring and a strongly left Köthe ring are the same for left quasi-duo rings (see Theorem 3.5). In Sect. 4, we characterize very strongly left Köthe rings and prove that very strongly Köthe rings are exactly Artinian serial rings (see Theorems 4.1 and 4.7). It is shown that for Abelian rings, the notions of a Köthe ring, a strongly Köthe ring, a very strongly Köthe ring and an Artinian principal ideal ring coincide (see Proposition 4.6). Moreover, the equivalency of the notions of a very strongly Köthe ring and an Artinian serial ring is a generalization of Theorem 1.3, and so it is a new generalization of Köthe–Cohen–Kaplansky theorem.

### 2 Characterizations of left Köthe rings

In this section, we solve left Köthe’s problem by giving several characterizations in terms of describing the indecomposable modules. Recall that a submodule $N$ of an $R$-module $M$ is called an $RD$-pure submodule if $N \cap rM = rN$ for all $r \in R$. Every direct summand of $M$ is a $RD$-pure submodule (we note that in [8], Chase used the term “pure submodule” instead of “RD-pure submodule”).

The following lemmas play an important role in the sequel.

**Lemma 2.1** (Chase [8, Theorem 3.1]) Let $R$ be a ring, and $A$ an infinite set of cardinality $\zeta$ where $\zeta \geq \text{card}(R)$. Set $M = \prod_{\alpha \in A} R(\alpha)$, where $R(\alpha) \cong R$ is a left $R$-module. Suppose that $M$ is an $RD$-pure submodule of a left $R$-module of the form $N = \bigoplus \beta N_\beta$, where each $N_\beta$ is generated by a subset of cardinality less than or equal to $\zeta$. Then $R$ must satisfy the descending chain condition (DCC) on principal right ideals.

**Lemma 2.2** (B. Zimmermann-Huisgen-W. Zimmermann [39, Page 2]) For a ring $R$ the following statements are equivalent:

1. Each left $R$-module is a direct sum of finitely generated modules.
2. There exists a cardinal number $\aleph$ such that each left $R$-module is a direct sum of $\aleph$—generated modules.
(3) Each left $R$-module is a direct sum of indecomposable modules ($R$ is left pure semisimple).

Let $M$ be a left $R$-module. By [37, Proposition 21.6 (4)], if $\text{Rad}(M)$ is a superfluous submodule of $M$, then $M$ is finitely generated if and only if $\text{top}(M)$ is finitely generated. Similar to this, we have the following lemma.

**Lemma 2.3** Let $M$ be a left $R$-module such that $\text{Rad}(M)$ is a superfluous submodule of $M$. Then $M$ is cyclic if and only if $\text{top}(M)$ is cyclic.

**Proof** Let $\text{top}(M) = M/\text{Rad}(M)$ be cyclic with the generating element $x + \text{Rad}(M)$, where $x \in M$. Then for each $m \in M$, $m + \text{Rad}(M) = rx + \text{Rad}(M)$ for some $r \in R$. Hence, $m - rx \in \text{Rad}(M)$ and so $m = (m - rx) + rx \in \text{Rad}(M) + Rx$. Thus, $\text{Rad}(M) + Rx = M$ and so $M = Rx$, since $\text{Rad}(M)$ is superfluous in $M$. \hfill $\square$

Recall that a ring $R$ is semiperfect if $R/J(R)$ is left semisimple and idempotents in $R/J(R)$ can be lifted to $R$. So, for example, local rings are semiperfect. From [1, (15.16), (15.19) and (27.1)] it follows that a left (or right) Artinian ring is semiperfect. It is worthy of note that in a semiperfect ring, the radical is the unique largest ideal containing no nonzero idempotents (see [1, (15.12)])

A semiperfect ring $R$ can be written as a direct sum of indecomposable cyclic left $R$-modules, written as $R = (Re_1)^{(t_1)} \oplus \cdots \oplus (Re_n)^{(t_n)}$, where $(Re_i)^{(t_i)}$ denotes the direct sum of $t_i$ copies of $Re_i$ and $Re_i \not\cong Re_j$ whenever $i \neq j$. If $A = Re_1 \oplus \cdots \oplus Re_n$, then the category of left $R$-modules and left $A$-modules are Morita equivalent. The semiperfect ring $R$ is said to be basic if $t_1 = \cdots = t_n = 1$, i.e., there are no isomorphic modules in any decomposition of $R$ into direct sum of indecomposable left $R$-modules. In fact, a semiperfect ring $R$ is basic if the quotient ring $R/J(R)$ is a direct sum of division rings.

Also, a ring $R$ is left (right) perfect in case each of its left (right) modules has a projective cover. It follows from [1, Proposition 27.6] that left perfect rings and right perfect rings are both semiperfect. However, left perfect rings need not be right perfect (see [1, Exercise(28.2)]). The pioneering work on perfect rings was carried out by H. Bass in 1960 and most of the main characterizations of these rings are contained in his celebrated paper [6].

Recall that an idempotent $e \in R$ is primitive in case it is nonzero and cannot be written as a sum $e = e' + e''$ of nonzero orthogonal idempotents. A left (right) ideal of $R$ is primitive in case it is of the form $Re$ ($eR$) for some primitive idempotent $e \in R$. The endomorphism ring of $Re$ is isomorphic to $eRe$.

We recall that a set $\{e_1, \ldots, e_n\}$ of idempotents of a semiperfect ring $R$ is called basic in case they are pairwise orthogonal, $Re_i \not\cong Re_j$ for each $i \neq j$ and for each finitely generated indecomposable projective left $R$-module $P$, there exists $i$ such that $P \cong Re_i$. Clearly, the cardinal of any two basic sets of idempotents of a semiperfect ring $R$ are equal. An idempotent $e$ of a semiperfect ring $R$ is called a basic idempotent of $R$ in case $e$ is the sum $e = e_1 + \cdots + e_m$ of a basic set $e_1, \ldots, e_m$ of primitive idempotents of $R$.

Let $M$ and $N$ be $R$-modules. A monomorphism $N \rightarrow M$ is called an embedding of $N$ to $M$, and we denote it by $N \hookrightarrow M$. The following proposition, play a key role in this paper.

**Proposition 2.4** Let $R$ be a semiperfect ring and $M$ be a finitely generated left $R$-module with the projective cover $P(M)$. Then $R \cong \oplus_{i=1}^n (Re_i)^{(t_i)}$, where $n, t_i \in \mathbb{N} \text{ and } \{e_1, \ldots, e_n\} \text{ is a basic set of idempotents of } R, R/J(R) \cong \oplus_{i=1}^n (Re_i/J(Re_i))^{(t_i)}$, top$(M) \cong \oplus_{i=1}^n (Re_i/J(Re_i))^{(s_i)}$, for some $s_1, \ldots, s_n \in \mathbb{N} \cup \{0\}$, and $P(M) \cong \oplus_{i=1}^n (Re_i)^{(s_i)}$. Consequently $\text{top}(M)$ is an $R/J$-module and,
(i) the following are equivalent for $M$:

1. $M$ is a cyclic $R$-module.
2. $\text{top}(M)$ is a cyclic module.
3. $M \cong Re/Ie$ for some idempotent $e \in R$ and for some left ideal $I \subseteq J(R)$.
4. $\text{Re}$ is equal to the projective cover of $M$ for some idempotent $e \in R$.
5. By the notations above, we have $s_i \leq t_i$ for all $i = 1, \ldots, n$.
6. $\text{top}(M) \hookrightarrow R/J(R)$. In particular, $\text{top}(M)$ is square-free if and only if $s_i \in \{0, 1\}$ for all $i = 1, \ldots, n$.

(ii) The following are equivalent for $M$:

1. $M$ is a local module.
2. $\text{top}(M)$ is a simple module.
3. $P(M)$ is an indecomposable module.
4. $P(M) = Re$ for some local idempotent $e \in R$.
5. $P(M)$ is a projective cover of a simple module.

Proof The first statement is by [1, Corollary 15.18, Theorems 27.6, 27.13 and Proposition 27.10].

(i) (1) $\iff$ (2). It is by Lemma 2.3, since $M$ is finitely generated, and so by [37, Proposition 21.6 (4)], $\text{Rad}(M)$ is superfluous in $M$.

(ii) (1) $\implies$ (3). Assume that $M$ is a cyclic left $R$-module. Since $R$ is a semiperfect ring, $M$ has a projective cover by [1, Corollary 27.6]. Also by [1, Lemma 27.3], $M \cong Re/Ie$ for some idempotent $e \in R$ and some left ideal $I \subseteq J(R)$, and the natural map $\text{Re} \to \text{Re}/Ie \to 0$ is a projective cover of $M$.

(iii) (3) $\implies$ (4). By [1, Lemma 27.3], for some idempotent $e$ of $R$ and some left ideal $I \subseteq J(R)$, the natural map $\text{Re} \to \text{Re}/Ie \to 0$ is a projective cover of $M$.

(iv) (4) $\implies$ (1). Since $e$ is an idempotent in $R$, so $\text{Re}$ is a direct summand of $R_R$. Then $\text{Re} = P(M)$, and hence $M$ is cyclic.

(v) (1) $\implies$ (6). If $M$ is cyclic, then by the first statement, $\text{top}(M)$ is a cyclic $R/J(R)$-module, and it follows that $\text{top}(M)$ is isomorphic to a direct summand of $R/J(R)$.

(vi) (6) $\implies$ (1). It is clear.

(vii) (6) $\implies$ (5). It is clear by the first statement.

(viii) (5) $\implies$ (2). If $s_i \leq t_i$ for all $i = 1, \ldots, n$, then $\text{top}(M)$ is isomorphic to a direct summand of $R/J(R)$ and hence $\text{top}(M)$ is a cyclic module.

(ii) (1) $\implies$ (2) is clear.

(ii) (2) $\implies$ (1). Since $\text{top}(M)$ is simple, $\text{Rad}(M)$ is the unique maximal submodule of $M$. Finally, since $M$ is nonzero and finitely generated, every proper submodule of $M$ is contained in a maximal submodule, and therefore $\text{Rad}(M)$ is the greatest proper submodule of $M$.

The rest of the proof (ii) is by [17, Theorem 1.9.4, Proposition 1.9.6]. $\square$

Corollary 2.5 Let $R$ be a semiperfect ring and $M$ be a finitely generated left $R$-module with the projective cover $P(M)$. If $\text{top}(M)$ is square-free, then $P(M)$ (and so $M$) is cyclic.

Proof If $\text{top}(M)$ is square-free, then by Proposition 2.4, we have:

$$\text{top}(M) \cong (Re_{i_1}/J(R)e_{i_1}) \oplus \cdots \oplus (Re_{i_k}/J(R)e_{i_k}),$$

where $e_{i_1}, \ldots, e_{i_k}$ are distinct elements of $\{e_1, \ldots, e_n\}$. Also, by Proposition 2.4, there exists a projective cover $f : P = Re_{i_1} \oplus \cdots \oplus Re_{i_k} \to M$. Clearly $P$ is a direct summand of $R$,

and so $P$ and $M \cong P/\text{Ker}(f)$ are cyclic modules. $\square$
Recall that a ring $R$ is of finite representation type (or finite type) if $R$ is Artinian and there is a finite number of the isomorphism classes of finitely generated indecomposable left (and right) $R$-modules. The ring $R$ is said to be of left bounded (representation) type, if it is left Artinian and there is a finite upper bound for the lengths of the finitely generated indecomposable modules in $R$-Mod. By [37, Proposition 54.3], $R$ is of left bounded representation type if and only if $R$ is of finite representation type. Moreover, A subset $X$ of a ring $R$ is called left $T$-nilpotent if, for every sequence $x_1, x_2, \ldots$ of elements in $X$, there is an $n \in \mathbb{N}$ with $x_1 x_2 \cdots x_n = 0$. Similarly, a right $T$-nilpotent subset of $R$ is defined. The importance of the concept of $T$-nilpotency is due to the fact that the radical $J(R)$ of a ring $R$ is left $T$-nilpotent precisely when the “Nakayama’s Lemma” [1, Corollary 15.13] holds for all left modules, finitely generated or not. Recall that a ring $R$ is called left Max-ring if every left $R$-module contains a maximal submodule.

We are now in a position to provide the following properties and characterizations of left Köthe rings.

**Theorem 2.6** The following statements are equivalent for any ring $R$:

1. $R$ is a left Köthe ring.
2. Every nonzero left $R$-module is a direct sum of modules with nonzero cyclic top.
3. $R$ is left perfect and every left $R$-module is a direct sum of modules with cyclic top.
4. $R$ is left Artinian and every left $R$-module is a direct sum of modules with cyclic top.
5. $R$ is left pure semisimple and every left $R$-module is a direct sum of modules with cyclic top.
6. $R$ is of finite representation type and every (finitely generated) indecomposable left $R$-module has a cyclic top.
7. $R$ is of finite representation type and top($U$) $\rightarrow$ $R/J(R)$ for each indecomposable left $R$-module $U$.
8. $R \cong \bigoplus_{i=1}^{n} (Re_i)^{(t_i)}$, where $n$, $t_i \in \mathbb{N}$, $i = 1, \ldots, n$, $\{Re_1, \ldots, Re_n\}$ is a complete set of representatives of the isomorphism classes of indecomposable projective left $R$-module, and each indecomposable left $R$-module is isomorphic to $Re/Ie$ for some idempotent $e \in R$ and for some left ideal $I \subseteq J(R)$.
9. $R$ is of finite representation type and $R \cong \bigoplus_{i=1}^{n} (Re_i)^{(t_i)}$, where $n$, $t_1, \ldots, t_n \in \mathbb{N}$, $\{Re_1, \ldots, Re_n\}$ is a complete set of representatives of the isomorphism classes of indecomposable projective left $R$-modules and top($U$) $\cong \bigoplus_{i=1}^{n} (Re_i/J(R)e_i)^{(u_i)}$, for some $u_i \in \mathbb{N} \cup \{0\}$, where $u_i \leq t_i$ for all $i = 1, \ldots, n$ for each indecomposable left $R$-module $U$.

**Proof** (1) $\Rightarrow$ (4). By the hypothesis, every $R$-module is a direct sum of cyclic modules. Clearly, the top of every cyclic module is cyclic. Also, $R$ is left Artinian by [8, Theorem 4.4].

(4) $\Rightarrow$ (2). Since $R$ is left Artinian, so by [1, Corollary 28.8], $R$ is a left perfect ring. Then by [1, Theorem 28.4] and [1, lemma 28.3], every $R$-module with cyclic top has a nonzero top.

(2) $\Rightarrow$ (3). Set $\tilde{R} = R/J(R)$, and $M = \prod_{a \in A} \tilde{R}^{(a)}$ where $A$ is an infinite set of cardinality $\zeta \geq \text{card}(\tilde{R})$, and $\tilde{R}^{(a)} \cong \tilde{R}$ is a left $\tilde{R}$-module. Clearly, every left $\tilde{R}$-module is also a direct sum of modules with cyclic top. Thus, $M = \prod_{a \in A} \tilde{R}^{(a)} = \bigoplus_{\lambda \in \Lambda} N_{\lambda}$, where $\Lambda$ is an index set and each $N_{\lambda}$ has a cyclic top. Since $J(\tilde{R}) = 0$, one can easily see that $\text{Rad}(M) = 0$ (see for instance, [1, Page 174, Exercises 3]), and hence by [37, Proposition 21.6 (5)], we conclude that $\text{Rad}(N_{\lambda}) = 0$ for each $\lambda \in \Lambda$. It follows that $N_{\lambda}$ is cyclic for each $\lambda \in \Lambda$. Thus, $M$ is a direct sum of cyclic left $R$-modules. Since the cardinality of each cyclic left $R$-module is at most $\text{card}(\tilde{R}) \leq \zeta$, by Lemma 2.1, $\tilde{R}$ must satisfy the descending chain condition on
principal right ideals. By Bass [6, Theorem P], the ring $\bar{R}$ is a semisimple ring, i.e., $R/J(R)$ is a semisimple ring. Now, by our hypothesis, $Rad(N_\lambda) \neq N_\lambda$ for each $\lambda \in \Lambda$. Since each $N_\lambda$ has a cyclic top, $Rad(M) = \bigoplus_{\lambda} Rad(N_\lambda) \neq \bigoplus_{\lambda} N_\lambda = M$ by [37, Proposition 21.6 part (5)]. It follows that every nonzero left $R$-module contains a maximal submodule, i.e., $R$ is a left max-ring. So $J(R)$ is left $T$-nilpotent by [1, Remark 28.5], and hence by [1, Theorem 28.4], $R$ is a left perfect ring.

(3) $\Rightarrow$ (1). Since $R$ is a left perfect ring, by [1, Theorem 28.4], every nonzero left $R$-module contains a maximal submodule. It follows that for each left $R$-module $M$, every proper submodule of $M$ is contained in a maximal submodule. Now let $M$ be a module with cyclic top. Then by [37, Proposition 21.16 (3)], $Rad(M)$ is a superfluous submodule of $M$, and so by Lemma 2.3, $M$ is cyclic. Thus, every left $R$-module is a direct sum of cyclic modules, i.e., $R$ is a left Köthe ring.

(4) $\Rightarrow$ (3). Since $R$ is left Artinian, $R$ is a left prefect ring by [1, Corollary 28.8].

(1) $\Rightarrow$ (5). Since $R$ is a left Köthe ring, $R$ is a left pure semisimple ring by Lemma 2.2. Now it is clear that each cyclic left $R$-module has a cyclic top.

(5) $\Rightarrow$ (6). Since $R$ is a left pure semisimple, by [8, Theorem 4.4], $R$ is a left Artinian ring and every finitely generated indecomposable left $R$-module is cyclic. Thus, $R$ is of finite length and $length(R)$ is a finite upper bound for the lengths of the finitely generated indecomposable left $R$-modules. i.e., $rR$ is bounded. Now, by [37, Proposition 54.3]), $R$ is of finite representation type.

(6) $\Leftrightarrow$ (7). It is by Proposition 2.4 (i).

(7) $\Rightarrow$ (8). It is by Proposition 2.4 (i) and [1, Proposition 27.10].

(8) $\Rightarrow$ (9). It is by Proposition 2.4 (i).

(9) $\Rightarrow$ (1). Since $R$ is of finite representation type, by [37, Proposition 54.3] and Lemma 2.2, every left $R$-module is a direct sum of (finitely generated) indecomposable modules. Moreover, our assumption and Proposition 2.4 (i) imply that indecomposable components are cyclic.

The left Köthe property is not a Morita invariant property. In fact, there exists a ring $R$ and a positive integer $n \geq 2$ such that the matrix ring $M_n(R)$ is a left Köthe ring but $R$ is not a left Köthe ring (see [12, Proposition 4.7, Remark 4.8]). In the last result of this section we give a characterization of a ring $R$ for which any Morita equivalent to it is a left Köthe ring:

**Proposition 2.7** The following statements are equivalent for a ring $R$:

1. Any ring Morita equivalent to $R$ is a left Köthe ring.
2. $R$ is a semiperfect ring and the basic ring $A$ of $R$ is a left Köthe ring.

**Proof** (1) $\Rightarrow$ (2). By the hypothesis, $R$ is a left Köthe ring, and so $R$ is left Artinian. Thus by [1, Proposition 27.14], the basic ring $A_0$ of $R$ is Morita equivalent to $R$. Hence by assumption, $A_0$ is a left Köthe ring.

(2) $\Rightarrow$ (1). Let $T$ be a ring Morita equivalent to $R$. Then by [1, Proposition 27.14], $T$ is Morita equivalent to $A$, and hence by [12, Corollary 4.2], $T$ is a left Köthe ring. □

### 3 Characterizations of strongly left Köthe rings

In this section, we study strongly left Köthe rings. Also, we give more characterizations of a left Köthe ring $R$, whenever $R$ is left quasi-duo. In the sequel, we denote by $R$-Mod (resp., $R$-mod) the category of left $R$-modules (resp., finitely generated left $R$-modules), and by
Mod-$R$ (resp., mod-$R$) the category of right $R$-modules (resp., finitely generated right $R$-modules). For an $R$-module $M$, $E(M)$ denotes the injective hull of $M$. Now, we explain some concepts regarding the functors rings of the finitely generated modules of $R$-Mod, which will be used extensively in this section. Let $\{V_\alpha\}_A$ be a family of finitely generated $R$-modules and $V = \bigoplus_A V_\alpha$. For any $N \in R$-Mod we define:

$$\hat{\text{Hom}}(V, N) = \{ f \in \text{Hom}(V, N) \mid f(V_\alpha) = 0 \text{ for almost all } \alpha \in A \}.$$  

For $N = V$, we write $\hat{\text{Hom}}(V, V) = \hat{\text{End}}(V)$. Note that these constructions do not depend on the decomposition of $V$ (see [37, Chap. 10, Sect. 51] for more details).

**Remark 3.1** Let $R$ be a ring of finite representation type, and $V := V_1 \oplus \cdots \oplus V_n$, where $\{V_1, \ldots, V_n\}$ is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left $R$-modules. Clearly, the functor ring $T = \hat{\text{End}}(V)$ is equal to $\text{End}_R(V)$, and in this case, the ring $T := \text{End}_R(V)$ is called the left Auslander ring of $R$. By [4, Proposition 3.6], $T$ is an Artinian ring and so $\text{soc}(T_T)$ is an essential submodule of $T_T$ and $T$ contains only finitely many non-isomorphic types of simple modules. Also by [37, Proposition 46.7], the functor $\text{Hom}_R(V, \_)$ establishes an equivalence between the category of left $R$-modules and the full subcategory of finitely generated projective left $T$-modules, which preserves and reflects finitely generated left $R$-modules and finitely generated indecomposable left $R$-modules correspond to finitely generated indecomposable projective $T$-modules (see [37, Proposition 51.7(5)]). In fact, each finitely generated indecomposable projective $T$-module is local and hence it is the projective cover of a simple $T$-module. Therefore $\text{Hom}_R(V, \_)$ yields a bijection between a minimal representing set of finitely generated, indecomposable left $R$-modules and the set of projective covers of non-isomorphic simple left $T$-modules.

Recall that an injective left $R$-module $Q$ is a cogenerator in the category of left $R$-modules if and only if it cogenerates every simple left $R$-module, or equivalently, $Q$ contains every simple left $R$-module as a submodule (up to isomorphism) (see [37, Proposition 16.5]. Also, a cogenerator $Q$ is called minimal cogenerator if $Q \cong E(\oplus_{i \in I} S_i)$, where $I$ is an index set and $\{S_i \mid i \in I\}$ is a complete set of representatives of the isomorphism classes of simple left $R$-modules. Let $R$ and $S$ be two rings. An additive contravariant functor $F : R$-mod$\rightarrow$mod-$S$ is called duality if it is an equivalence of categories (see [37, Chap. 9]). Moreover, a ring $R$ is called a left Morita ring if there is an injective cogenerator $RU$ in $R$-Mod such that for $S = \text{End}(RU)$, the module $US$ is an injective cogenerator in Mod-$S$ and $R \cong \text{End}(US)$. The following result summarizes all the facts presented above on rings of finite representation type.

**Proposition 3.2** Let $R$ be a ring of finite representation type, $U := U_1 \oplus \cdots \oplus U_n$, where $\{U_1, \ldots, U_n\}$ is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left $R$-modules. $Q \cong E(S_1) \oplus \cdots \oplus E(S_m)$, where $\{S_i \mid 1 \leq i \leq m\}$ is a complete set of representatives of the isomorphism classes of simple left $R$-modules, $T := \text{End}_R(U)$ and $S = \text{End}_R(Q)$. Then

(a) The functor $\text{Hom}_R(U, \_)$ establishes an equivalence between the category of left $R$-modules and the full subcategory of finitely generated projective left $T$-modules.

(b) The functor $\text{Hom}_R(\_, Q) : R$-Mod$\rightarrow$Mod-$S$ is a duality with the inverse duality $\text{Hom}_S(\_, Q) : \text{Mod}$-$S$ $\rightarrow$ R-Mod.

(c) $S$ is a right Artinian ring and $\{\text{Hom}_R(U_1, Q), \ldots, \text{Hom}_R(U_n, Q)\}$ is a complete set of representatives of the isomorphism classes of finitely generated indecomposable right $S$-modules.
(d) The left Auslander ring $T$ of $R$ is isomorphic to the right Auslander ring $T'$ of $S$ (thus $R$ is left Morita to $S = \text{End}_R(Q)$).

(e) If $P$ is an indecomposable projective left $T$-module, then $P \cong \text{Hom}_R(U, M)$ for some indecomposable left $R$-module $M$.

(f) If $M$ is a finitely generated (indecomposable) left $R$-module with square-free top, then the right $S$-module $\text{Hom}_R(M, Q)$ has a square-free socle.

(g) If $M$ is an indecomposable left $R$-module with square-free top, then $M$ is cyclic with square-free top and so the right $S$-module $\text{Hom}_R(M, Q)$ has a square-free socle.

(h) If $M$ is an indecomposable left $R$-module with simple top, then the right $S$-module $\text{Hom}_R(M, Q)$ has a simple socle.

(i) For each finitely generated left $R$-module $X$, as $S$-modules, we have:

$$\text{soc}(\text{Hom}_R(X, Q)) \cong \text{Hom}_R(\text{top}(X), Q).$$

**Proof** (a). Since $R$ is of finite representation type, $U = \bigoplus A U_\alpha$ is a generator in $R\text{-Mod}$ by Remark 3.1 and [37, Chap. 10, Sect. 52]. The rest of proof is obtained by [32, Page 507, Part(10)].

(b) and (c). By [37, Proposition 47.15].

(e). It is by [37, Chap. 10, Sect. 52].

(d). See [37, 52.9, Exercises, Exercise 1].

The facts (f), (g), (h) and (i) are by (b), (c) and [37, Proposition 47.3] (we see that the functor $\text{Hom}_R(-, Q) : R\text{-Mod} \rightarrow \text{Mod-S}$ is a duality with the inverse duality $\text{Hom}_S(-, Q) : \text{Mod-S} \rightarrow R\text{-Mod}$). Also by [1, Proposition 24.5] for each finitely generated left $R$-module $X$, the lattice of all $R$-submodules of $M$ and the lattice of all $S$-submodules of $\text{Hom}_R(M, Q)$ are anti-isomorphic. Hence, for each finitely generated left $R$-module $X$, $\text{soc}(\text{Hom}_R(X, Q)) \cong \text{Hom}_R(\text{top}(X), Q)$, and $\text{top}(\text{Hom}(X, Q)) = \frac{\text{Hom}(X, Q)}{\text{Rad}(\text{Hom}(X, Q))} \cong \text{Hom}(\text{soc}(X), Q)$ as $S$-modules (see [1, 24. Exercises, Exercises 6]).

Recall that a semiperfect ring $R$ is said to be a left (resp., right) $QF$-2 ring if every indecomposable projective left (resp., right) $R$-module has a simple essential socle (see [14]). This definition and the fact that simple modules are square-free motivated us to define the following concept.

**Definition 3.3** We say that a semiperfect ring $R$ is a generalized left (resp., right) $QF$-2 ring if every indecomposable projective left (resp., right) $R$-module has a square-free essential socle. The ring $R$ is called a generalized $QF$-2 ring if $R$ is both a generalized left and a generalized right $QF$-2 ring.

We are now in a position to provide the following properties and characterizations of strongly left Köthe rings.

**Theorem 3.4** The following statements are equivalent for a ring $R$:

1. $R$ is a strongly left Köthe ring.
2. Every nonzero left $R$-module is a direct sum of modules with nonzero socle and nonzero square-free cyclic top.
3. $R$ is left perfect, and every left $R$-module is a direct sum of modules with square-free top.
4. $R$ is left Artinian and every left $R$-module is a direct sum of modules with square-free top.
5. $R$ is left pure semisimple, and every left $R$-module is a direct sum of modules with square-free top.
(6) $R$ is of finite representation type, and every (finitely generated) indecomposable left $R$-module has a square-free top.

(7) $R$ is of finite representation type, and for the left Morita dual ring of $R$, every (finitely generated) indecomposable right module has a square-free socle.

(8) $R$ is of finite representation type, and the left Auslander ring of $R$ is a right generalized $QF-2$ ring.

(9) $R$ is of finite representation type, and for each indecomposable left $R$-module $U$, $\text{top}(U) \leftrightarrow (\text{Re}_1/J(R)e_1) \oplus \cdots \oplus (\text{Re}_n/J(R)e_n)$, where $e_1, \ldots, e_n$ is a basic set of primitive idempotents for $R$.

(10) $R \cong \bigoplus_{i=1}^{n} (\text{Re}_i)^{(t_i)}$, where $n$, $t_i \in \mathbb{N}$, $\{\text{Re}_1, \ldots, \text{Re}_n\}$ is a complete set of representatives of the isomorphism classes of indecomposable projective left $R$-modules, each indecomposable left $R$-module has a square-free top, and it is isomorphic to $\text{Re}_1/e$ for some idempotent $e \in R$ and a left ideal $I \subseteq J(R)$.

**Proof**

(1) $\Rightarrow$ (2). Let $0 \ne M = \bigoplus_{\lambda} M_{\lambda}$, where $\Lambda$ is an index set and each $M_{\lambda}$ is a left $R$-module with nonzero square-free cyclic top. Hence, $\text{Rad}(M_{\lambda}) \ne M_{\lambda}$ for each $\lambda \in \Lambda$. Since each $M_{\lambda}$ has a cyclic top, $\text{Rad}(M) = \bigoplus_{\lambda} \text{Rad}(M_{\lambda}) \ne \bigoplus_{\lambda} M_{\lambda} = M$ by [37, Proposition 21.6 part(5)]. So every nonzero left $R$-module contains a maximal submodule. It follows that for each left $R$-module $M$, every proper submodule of $M$ is contained in a maximal submodule. Now let $M$ be a module with square-free cyclic top. Then by [37, Proposition 21.16 (3)], $\text{Rad}(M)$ is a superfluous submodule of $M$, and so by Lemma 2.3, $M$ is cyclic. Thus, every left $R$-module is a direct sum of cyclic modules, i.e., $R$ is a left Köthe ring.

Hence by [8, Theorem 4.4], $R$ is a left Artinian ring, and so every nonzero left $R$-module has a nonzero socle.

(2) $\Rightarrow$ (3). By Theorem 2.6, $R$ is a left Köthe ring. Thus by [8, Theorem 4.4], $R$ is a left Artinian ring and so by [1, Corollary 28.8], $R$ is a left perfect ring.

(3) $\Rightarrow$ (5). Since $R$ is a left perfect ring, by Corollary 2.5, $R$ is a left Köthe ring. So by Theorem 2.6, $R$ is left pure semisimple.

(5) $\Rightarrow$ (4). It is clear, since by [8, Theorem 4.4], every left pure semisimple ring is a left Artinian ring.

(4) $\Rightarrow$ (1). Since $R$ is a left Artinian ring, $R$ is a left perfect ring by [1, Corollary 28.8]. Also, by Corollary 2.5, $R$ is a left Köthe ring. Since every left $R$-module is a direct sum of modules with square-free top, we conclude that $R$ is a strongly left Köthe ring.

(5) $\Rightarrow$ (6). By [8, Theorem 4.4], $R$ is a left Artinian ring. Hence $R$ is a left prefect ring. Also, by Corollary 2.5, $R$ is a left Köthe ring. Thus, $R$ is of finite length and since every (finitely generated) indecomposable left $R$-module is cyclic, $R$ is of left bounded representation type, and hence by [37, Proposition 54.3], $R$ is of finite representation type.

In the next steps of proof, for the finite representation type ring $R$, by Proposition 3.2, we have the following assumptions.

(i) The left Auslander ring of $R$ is $T = \text{End}_R(U)$, where $U = U_1 \oplus \cdots \oplus U_n$ and $\{U_1, \ldots, U_n\}$ is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left $R$-modules.

(ii) The left Morita dual ring of $R$ is $S = \text{End}(Q)$, where $Q \cong E(S_1) \oplus \cdots \oplus E(S_m)$, $\{S_i | 1 \leq i \leq m\}$ is a complete set of representatives of the isomorphism classes of simple left $R$-modules.

(iii) $\{\text{Hom}_{R}(U_1, Q), \ldots, \text{Hom}_{R}(U_n, Q)\}$ is a complete set of representatives of the isomorphism classes of finitely generated indecomposable right $S$-modules.
(6) ⇒ (7). Since \( \{\text{Hom}_R(U_1, Q), \ldots, \text{Hom}_R(U_n, Q)\} \) is a complete set of representatives of the isomorphism classes of finitely generated indecomposable right \( S \)-modules, and since each \( U_i \) has a square-free top, by Proposition 3.2 (g), each indecomposable right \( S \)-module \( \text{Hom}_R(U_i, Q) \) has a square-free socle.

(7) ⇒ (8). Since \( R \) is of finite representation type, by Proposition 3.2 (d), the left Auslander ring \( T \) of \( R \) is isomorphic to the right Auslander ring \( T' \) of \( S \). Thus, every indecomposable projective right \( T \)-module has a square-free socle.

(8) ⇒ (6). By Proposition 3.2 (d), \( T \cong T' \), where \( T' \) is the right Auslander ring of \( S \). By Proposition 3.2 (e) and (c), each of \( \text{Hom}(U_1, Q), \ldots, \text{Hom}(U_n, Q) \) has a square-free socle. By Proposition 3.2 (i), we have \( \text{soc}(\text{Hom}(U_i, Q)) \cong \text{Hom}_R(\text{top}(U_i), Q) \) and since each \( \text{soc}(\text{Hom}(U_i, Q)) \) is square-free, we conclude that \( \text{top}(U_i) \) is also square-free for each \( i \). Thus \( R \) is of finite representation type and every (finitely generated) indecomposable left \( R \)-module has a square-free top.

(6) ⇒ (9). It is by Proposition 2.4 (i).

(9) ⇒ (1). It is by Lemma 2.3.

(6) ⇒ (10). It is by Propositions 2.4 and [1, Proposition 27.10] and also by [1, Lemma 27.3]. Also, for \( 1 \leq i \leq n \), every indecomposable module \( Re_i/le_i \) has a square-free top.

(10) ⇒ (1). Since every indecomposable left \( R \)-module is isomorphic to the cyclic module \( Re_i/le_i \) for some \( 1 \leq i \leq n \) and a left ideal \( I \subseteq J(R) \), it is enough to show that \( R \) is of finite representation type. Since \( R \) is an Artinian ring, each \( Re_i \) has finite length, and then we set \( n = \max\{\text{length}(Re_i) \mid 1 \leq i \leq n\} \). By the assumption every indecomposable module \( M \) is a factor of \( Re_i \), thus \( \text{length}(M) \leq n \) and hence \( R \) is of bounded representation type. So by [37, Proposition 54.3], \( R \) is of finite representation type. Then every left \( R \)-module is a direct sum of indecomposable cyclic modules and each of indecomposable cyclic module \( Re/le \) has a square-free top.

A ring \( R \) is said to be left duo (resp., left quasi-duo) if every left ideal (resp. maximal left ideal) of \( R \) is an ideal. Obviously, left duo rings are left quasi-duo. Other examples of left quasi-duo rings include, for instance, the commutative rings, the local rings, the rings in which every non-unit element has a (positive) power that is a central element, the endomorphism rings of uniserial modules, the power series rings and the rings of upper triangular matrices over any of the above-mentioned rings (see [38]). It is easy to see that if a ring \( R \) is left duo (resp. left quasi-duo), so is any factor ring of \( R \). By a result of [38], a ring \( R \) is left quasi-duo if and only if \( R/J(R) \) is left quasi-duo, and if \( R \) is left quasi-duo, then \( R/J(R) \) is a subdirect product of division rings. The converse is not true in general (see [24, Example 5.2]), but the converse is true when \( R \) has only a finite number of simple left \( R \)-modules up to isomorphisms (see [24, Page 252]). Consequently, a semilocal ring \( R \) is left (right) quasi-duo if and only if \( R/J(R) \) is a direct product of division rings. Note that any basic ring is a left and a right quasi-duo ring.

The following theorem shows that for any left quasi-duo ring \( R \) the concepts of “left Köthe” and “strongly left Köthe” coincide.

**Theorem 3.5** The following statements are equivalent for a left quasi-duo ring \( R \):

1. \( R \) is a left Köthe ring.
2. \( R \) is a strongly left Köthe ring.
3. \( R \) is of finite representation type and every indecomposable left \( R \)-module has a cyclic top.
4. \( R \) is of finite representation type and every indecomposable left \( R \)-module has a square-free top.
(5) $R$ is of finite representation type and for the left Morita dual ring of $R$, every (finitely generated) indecomposable right module has a square-free socle.

(6) $R$ is of finite representation type and the left Auslander ring of $R$ is a generalized right $QF$-2 ring.

**Proof** \((1) \Rightarrow (2).\) Since $R$ is a left Köthe ring, By [8, Theorem 4.4], $R$ is of finite representation type and each indecomposable left $R$-module is cyclic. Thus, we can assume that \(\{R/I_1, \ldots, R/I_n\}\) is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left $R$-modules, where $I_1, \ldots, I_n$ are left ideals of $R$. But $R$ has only a finite number of simple left $R$-modules (up to isomorphisms) and $R$ is left quasi-duo, so by [24, Page 252], $R/J(R)$ is a finite direct product of division rings and for each maximal (left) ideals $P_1 \neq P_2$ of $R$, $R/P_1 \not\cong R/P_2$. Now by [1, Corollary 15.18], we have $\text{Rad}(R/I_i) = J(R/I_i)$ for each $1 \leq i \leq n$. Thus,

$$\text{top}(R/I_i) = \frac{R/I_i}{J(R/I_i)} \cong \frac{R}{J(R) + I_i} \cong \bigoplus_{I_i \leq P_j \in \text{Max}(R)} \frac{R}{P_j} \cong \bigcap_{I_i \leq P_j \in \text{Max}(R)} P_j.$$

So, every indecomposable left $R$-module has a square-free top. Hence by Theorem 3.4, the proof is complete.

\((2) \Rightarrow (1).\) Since $R$ is of finite representation type, we can assume that \(\{U_1, \ldots, U_n\}\) is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left $R$-modules. Since by our hypothesis each $U_i$ (1 $\leq i \leq n$) has a square-free top, each $U_i$ is cyclic by Corollary 2.5, and hence every left $R$-module is a direct sum of cyclic modules, i.e., $R$ is a left Köthe ring.

\((2) \iff (4) \iff (5) \iff (6).\) By Theorem 3.4.

\((1) \iff (3).\) By Theorem 2.6. \(\square\)

### 4 Characterizations of very strongly left Köthe rings

The rings $R$ satisfying the following \((\ast)\) condition were first studied by Tachikawa [33] in 1959, by using duality theory.

\((\ast)\) $R$ is a right Artinian ring and every finitely generated indecomposable right $R$-module is local.

But, Singh and Al-Bleahed [30] have studied rings $R$ satisfying \((\ast)\) without using duality. Since each local module is cyclic indecomposable and it has a simple top, the condition \((\ast)\) is a stronger condition than the strongly right Köthe condition. In fact, we will see that the rings with \((\ast)\) condition are exactly very strongly right Köthe rings. In the first result we give several characterizations of very strongly left Köthe rings.

**Theorem 4.1** The following statements are equivalent for any ring $R$:

1. $R$ is a very strongly left Köthe ring.
2. Every nonzero left $R$-module is a direct sum of modules with nonzero socle and simple top.
3. $R$ is left perfect, and every left $R$-module is a direct sum of modules with simple top.
4. $R$ is left Artinian, and every left $R$-module is a direct sum of modules with simple top.
5. $R$ is left pure semisimple, and every left $R$-module is a direct sum of modules with simple top.
(6) \( R \) is of finite representation type, and every (finitely generated) indecomposable left \( R \)-module has a simple top.

(7) Every left \( R \)-module is a direct sum of lifting modules.

(8) Every left \( R \)-module is a direct sum of local modules.

(9) \( R \) is of finite representation type, and for the left Morita dual ring of \( R \) every (finitely generated) indecomposable right module has a simple socle.

(10) \( R \) is of finite representation type, and the left Auslander ring of \( R \) is a right \( QF-2 \) ring.

(11) \( R \) is of finite representation type, and for each indecomposable left \( R \)-modules \( U \), \( \text{top}(U) \cong R e_i / J(R)e_i \) where \( e_i \in \{ e_1, \ldots, e_n \} \) and \( \{ e_1, \ldots, e_n \} \) is a basic set of primitive idempotents for \( R \).

(12) \( R \) is an Artinian right serial ring, \( R \cong \oplus_{i=1}^{n} (R e_i)^{(t_i)} \), where \( \{ R e_1, \ldots, R e_n \} \) is a complete set of representatives of the isomorphism classes of indecomposable projective left \( R \)-modules with \( n \), \( t_i \in \mathbb{N} \), and each \( R e_i \) has a simple top and each indecomposable left \( R \)-module is isomorphic to \( R e_i / I e_i \) for some \( 1 \leq i \leq n \) and a left ideal \( I \subseteq J(R) \).

**Proof** The equivalence of part (1) to (6) is obtained by Theorem 3.4, since every module with simple top is a module with square-free cyclic top.

(6) \( \iff \) (7). It is by [15, Theorem 2.1].

(1) \( \iff \) (8). It is by Proposition 2.4 (ii).

In the next steps of the proof, for the finite representation type ring \( R \), by Proposition 3.2, we have the following assumptions.

(i) The left Auslander ring of \( R \) is \( T = \text{End}_R(U) \), where \( U = U_1 \oplus \cdots \oplus U_n \) and \( \{ U_1, \ldots, U_n \} \) is a complete set of representatives of the isomorphism classes of finitely generated indecomposable left \( R \)-modules.

(ii) The left Morita dual ring of \( R \) is \( S = \text{End}(Q) \), where \( Q \cong E(S_1) \oplus \cdots \oplus E(S_m) \), where \( \{ S_i \mid 1 \leq i \leq m \} \) is a complete set of representatives of the isomorphism classes of simple left \( R \)-modules.

(iii) \( \{ \text{Hom}_R(U_1, Q), \ldots, \text{Hom}_R(U_n, Q) \} \) is a complete set of representatives of the isomorphism classes of finitely generated indecomposable right \( S \)-modules.

(8) \( \Rightarrow \) (9). By Proposition 3.2 (c) and (e), it is enough to show that each of indecomposable right \( S \)-modules \( \text{Hom}_R(U_1, Q), \ldots, \text{Hom}_R(U_n, Q) \), has a simple socle. By our hypothesis, each \( U_i \) has a simple top. By Proposition 3.2 (i), we have \( \text{soc}(\text{Hom}(U_i, Q)) \cong \text{soc}(\bigoplus_{k=1}^{m} \text{Hom}(S_i, Q)) = \text{Hom}(\text{top}(U_i), Q) \) is simple. It follows that every indecomposable right \( S \)-module \( \text{Hom}(U_i, Q) \) has a simple socle.

(9) \( \Rightarrow \) (10). By Proposition 3.2 (d), \( T \cong T' \), where \( T' \) is the right Auslander ring of \( S \). Since each of indecomposable right \( S \)-module has a simple socle, so every finitely generated indecomposable projective right \( T \)-module has a simple socle, i.e., \( T \) is a right \( QF-2 \) ring.

(10) \( \Rightarrow \) (6). By Proposition 3.2 (d), \( T \cong T' \), where \( T' \) is the right Auslander ring of \( S \). By Proposition 3.2 (e) and (c), each of \( \text{Hom}(U_1, Q), \ldots, \text{Hom}(U_n, Q) \) has a simple socle. By Proposition 3.2 (i), we have \( \text{soc}(\text{Hom}(U_i, Q)) \cong \text{Hom}_R(\text{top}(U_i), Q) \) and since each \( \text{soc}(\text{Hom}(U_i, Q)) \) is simple, we conclude that \( \text{top}(U_i) \) is also simple for each \( i \). Thus, \( R \) is of finite representation type and every (finitely generated) indecomposable left \( R \)-module has a simple top.

(6) \( \Rightarrow \) (11). By Proposition 2.4 (ii)
(11) ⇒ (6). It is clear.

(8) ⇒ (12). By [30, Theorem 2.4], $R$ is an Artinian right serial ring. Also, by [1, Proposition 27.10], $R \cong \bigoplus_{i=1}^{n} (Re_i)^{(t_i)}$ where $n$, $t_i \in \mathbb{N}$, $i = 1, \ldots, n$ and $\{e_1, \ldots, e_n\}$ is a basic set of idempotents of $R$. By Proposition 2.4 (ii), each $Re_i$ is local for $i = 1, \ldots, n$. Now let $M$ be a local left $R$-module. Since any local module is cyclic, by [1, Lemma 27.3], $M \cong Re/Je$ for some idempotent $e \in R$ and some left ideal $I \subseteq J(R)$, and the natural map $Re \to Re/Je \to 0$ is a projective cover of $M$. Since $M$ is indecomposable cyclic left $R$-module with simple top, by Proposition 2.4 (ii), $Re$ is an indecomposable direct summand of $R$ and so by [1, Corollary 7.4], $e$ is a primitive idempotent. It follows that by Proposition 1, Proposition 27.10, $Re \cong Re_i$ for some $i (1 \leq i \leq n)$ and $M \cong Re_i/Je_i$, also $\{Re_1, \ldots, Re_n\}$ is a (finite) complete set of representatives of the isomorphism classes of indecomposable projective left $R$-modules. By Proposition 2.4 (ii), $top(M) = Re_i/J(R)e_i$ is simple for $1 \leq i \leq n$ and proof is complete.

In the rest of paper, as an application, we combine some results of Singh and Al-Bleahed [30] with our main theorems and obtain some interesting results on local Köthe rings and Abelian Köthe rings. Also we give a new generalization of Köthe–Cohen–Kaplansky theorem. The following lemmas are helpful.

**Lemma 4.2** ([16, Proposition 3]) Let $R$ be a left Artinian ring. Then $R$ is a finite product of local rings if and only if $R$ is Abelian.

**Lemma 4.3** (See [30, Theorem 2.12]) Let $R$ be a left Artinian local ring such that every finitely generated indecomposable left $R$-module is local. Then either $J(R)^2 = 0$ or $R$ is an uniserial ring.

**Theorem 4.4** The following statements are equivalent for a local ring $R$:

1. $R$ is a (very strongly) left Köthe ring.
2. $R$ is of finite representation type and the left Auslander ring $T$ of $R$ is a right QF-2 ring.
3. Either $R$ is an Artinian principal ideal ring or $R$ is local with

$$J(R) = soc(RR) = S_1 \oplus \cdots \oplus S_n$$

and $\{R/I_k | I_k = S_1 \oplus \cdots \oplus S_k, 1 \leq k \leq n\}$ are mutually non-isomorphic finitely generated indecomposable left $R$-modules.

**Proof** It follows by Theorem 4.1 and [30, Theorem 3.8].

**Corollary 4.5** The following statements are equivalent for a local ring $R$:

1. $R$ is a Köthe ring.
2. $R$ is a strongly Köthe ring.
(3) \( R \) is a very strongly Köthe ring.
(4) Any Morita equivalent to \( R \) is a Köthe ring.
(5) \( R \) is of finite representation type and the left (right) Auslander ring \( T \) of \( R \) is a QF-2 ring.
(6) \( R \) is an Artinian principal ideal ring.

**Proof** It follows by Proposition 2.7, Theorem 4.4 and [30, Theorem 3.8]. \( \square \)

**Proposition 4.6** The following statements are equivalent for an Abelian ring \( R \):

1. Any Morita equivalent to \( R \) is a Köthe ring.
2. \( R \) is an Artinian principal ideal ring.
3. \( R \cong R_1 \times \cdots \times R_k \), where \( k \), \( n_1, \ldots, n_k \in \mathbb{N} \) and \( R_i \) is a local Artinian principal ideal rings for each \( 1 \leq i \leq k \).
4. \( R \) is a Köthe ring.
5. \( R \) is a strongly Köthe ring.
6. \( R \) is a very strongly Köthe ring.
7. Every left \( R \)-module is semidistributive.
8. Every right \( R \)-module is semidistributive.
9. Every left \( R \)-module is serial.
10. Every right \( R \)-module is serial.
11. \( R \) is an Artinian serial ring.
12. \( R \) is isomorphic to a finite product of Artinian uniserial rings.

**Proof** (1) \( \Leftrightarrow \) (4). This is obtained by Proposition 2.7 and the fact that Artinian Abelian rings are basic.

(2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (12). See [7, Corollary 3.3].

(3) \( \Leftrightarrow \) (4). It is obtained by Lemma 4.2, [35, Theorem 3.6] and the fact that in any Artinian local ring the Jacobson radical is the unique prime ideal.

(4) \( \Leftrightarrow \) (5) \( \Leftrightarrow \) (6). These follow from Lemma 4.2 and Corollary 4.5.

(4) \( \Leftrightarrow \) (7) \( \Leftrightarrow \) (8). These are obtained by [36, Theorem 11.6].

(9) \( \Leftrightarrow \) (10) \( \Leftrightarrow \) (11). These follow from [37, Proposition 55.16].

(12) \( \Rightarrow \) (11) is clear and (11) \( \Rightarrow \) (2) is obtained by [37, Proposition 55.1(2)]. \( \square \)

An Artinian ring \( R \) is said to have left colocal type if every finitely generated indecomposable left \( R \)-module has a simple socle. Such rings and algebras have been investigated by several authors including Makino [25], Sumioka [31, 32], Tachikawa [33, 34], and a special case by Fuller [13]. Artinian serial rings clearly have finite representation type, as well as right and left colocal type.

Also, an Artinian ring \( R \) is said to have a self-(Morita) duality if there is a Morita duality \( D \) between \( R \)-mod, the category of finitely generated left \( R \)-modules, and \( \text{mod-} R \), the category of finitely generated right \( R \)-modules. Since \( R \) is Artinian, by what Morita [26] and Azumaya [5] have shown: \( R \) has a self-duality \( D \) if and only if there is an injective cogenerator \( R E \) of \( R \)-mod and a ring isomorphism \( v : R \rightarrow \text{End}_{R}(R E) \) (which induces a right \( R \)-structure on \( E \) via \( x.r = xv(r) \) for \( x \in E \) and \( r \in R \)) such that the dualities \( D \) and \( \text{Hom}_R(\text{---}, R E_R) \) are naturally equivalent. In the last result we give more equivalent conditions for a very strongly Köthe ring. In particular, we show that \( R \) is a very strongly Köthe ring if and only if \( R \) is an Artinian serial ring. This equivalence is a generalization of Theorem 1.3, and so it is a new generalization of Köthe–Cohen–Kaplansky theorem.

**Theorem 4.7** The following conditions are equivalent for a ring \( R \):
(1) $R$ is a very strongly Köthe ring.

(2) $R$ is an Artinian serial ring.

(3) $R$ is of finite representation type and has colocal type (i.e., every finitely generated indecomposable left $R$-module has a simple socle).

(4) Every left and right $R$-module is a direct sum of uniform modules.

(5) Every left and right $R$-module is a direct sum of extending modules.

(6) $R$ is of finite representation type and the left (right) Auslander ring of $R$ is a $QF$-2 ring.

(8) Every left $R$-module is a direct sum of (finitely generated) indecomposable modules with simple socle and simple top.

**Proof** The equivalence (2) ⇔ (5) follows from [10, Corollary 2], and the others are obtained by Theorem 4.1 and the fact that every Artinian serial ring has self-duality.

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