New no-scalar-hair theorem for black-holes

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Abstract

A new no-hair theorem is formulated which rules out a very large class of non-minimally coupled finite scalar dressing of an asymptotically flat, static, and spherically symmetric black-hole. The proof is very simple and based in a covariant method for generating solutions for non-minimally coupled scalar fields starting from the minimally coupled case. Such method generalizes the Bekenstein method for conformal coupling and other recent ones. We also discuss the role of the finiteness assumption for the scalar field.

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I. INTRODUCTION

Black-hole solutions are very rigid in gravitational physics. We know that the Schwarzschild solution is the only asymptotically flat and spherically symmetric solution of the vacuum Einstein equations. The no-hair conjecture[1] states that the exterior region of a black-hole admits only fields for which there is a geometrical Gauss-like law, as electromagnetic fields for example. Early no-hair theorems excluding for the exterior region of a black hole minimally coupled Klein-Gordon[2], massive vectors[3], and spinor[4] fields have stressed the conjecture.

The problem of the existence of scalar hairs for black-holes has received some attention recently. Although we know that scalar fields are not elementary fields in nature, they commonly arise in effective actions. In fact, some scalar actions have been considered recently in astrophysical contexts, see for instance[5]. However, with the conformally coupled case as the only exception[6–8], only minimally coupled scalar fields have been examined. In[9] it is presented a new theorem which rules out a multicomponent scalar hair with non-quadratic Lagrangian, but with minimal coupling to gravity. As it is stressed in [9], scalar fields effective actions are obtained by integrating the functional integral of the elementary fields in nature over some of the fields, and more complicated actions involving non-minimally coupling should arise.

The purpose of the present work is to point toward the filling of this gap by presenting a theorem that excludes finite scalar hairs of any asymptotically flat, static, and spherically symmetric black-hole solution of the system described by the action

\[ S[g, \phi] = \int d^4x \sqrt{-g} \left\{ f(\phi)R - h(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right\}, \]

with \( f(\phi) \) and \( h(\phi) > 0 \). We adopt all the conventions of[10]. Many physically relevant theories belong to the class described by[11]. Maybe the most popular non-minimal coupling for the scalars fields corresponds to the choice \( f(\phi) = 1 - \xi\phi^2 \) and \( h(\phi) = 1 \). The case \( \xi = \frac{1}{6} \) corresponds to the conformal coupling case, and the Bekenstein method[11] allows us to
construct its exact solutions from the solutions of the minimally coupled case ($\xi = 0$). A method for generating solutions for arbitrary $\xi$ is presented in \cite{12}. The extension of Bekenstein method for $n$-dimensions ($n > 3$) was obtained recently in \cite{5}, and used to study conformal scalar hairs \cite{3,4} and gravitational waves \cite{3}. Dilaton-like gravity is given by $f(\phi) = \frac{1}{4}h(\phi) = e^{-2\phi}$. The general model of Bergman, Wagoner and Nordtved discussed in \cite{10} corresponds to the choice $f(\phi) = \phi$ and $h(\phi) = \frac{\omega(\phi)}{\phi}$, from which Brans-Dicke theory is obtained from the limit $\omega$ constant.

The paper is organized as follows. In the section \cite{II} we present a covariant method for generating solutions for the system described by (1). This method will be the central point for the formulation of the theorem, which is presented in the same section. In the section \cite{III} we analyse as particular cases the Brans-Dicke theory and one of its generalizations in order to shed light in the role of the finiteness assumption for the scalar field and its relation to naked singularities. The last section is devoted to some concluding remarks, in particular a comparison between our results and recent ones.

II. THE THEOREM

The proof of our theorem centers in a covariant method for generating solutions for the Euler-Lagrange equations of (1) starting from the well known solutions for the minimally coupled case,

$$\bar{S}[\bar{g}, \bar{\phi}] = \int d^4x \sqrt{-\bar{g}} \left\{ \bar{R} - \bar{g}^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} \right\}.$$  \hspace{1cm} (2)

The method uses a conformal transformation, and generalizes the Bekenstein one \cite{11} and the proposed in the Ref. \cite{12}. Such a kind of method has a long history, and the Ref. \cite{14}, for instance, presents a good set of references on the subject. A method of this type was also used in \cite{13} to show that the action given by $\int d^4x \sqrt{-g} \left\{ F(\phi, R) - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}$ is equivalent to an Einstein-Hilbert action plus minimally coupled self-interacting scalar fields, equivalent in the sense that there is a conformal transformation and $\phi$-redefinition connecting them.
The Euler-Lagrange equations obtained from (1) are
\[ f(\phi)R_{\mu\nu} - h(\phi)\partial_\mu\phi\partial_\nu\phi - D_\mu D_\nu f(\phi) - \frac{1}{2}g_{\mu\nu}\Box f(\phi) = 0, \]  
\[ 2h(\phi)\Box \phi + h'(\phi)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + f'(\phi)R = 0, \]  
where the prime denotes derivation with respect to \( \phi \). Equations (3) are clearly much more complicated than the Euler-Lagrange equations derived from (2), namely
\[ \bar{R}_{\mu\nu} - \partial_\mu\bar{\phi}\partial_\nu\bar{\phi} = 0, \]
\[ \Box \bar{\phi} = 0. \]  
In order to realize how the solutions of (3) and (4) are related, consider the conformal transformation \( g_{\mu\nu} = \Omega^2\bar{g}_{\mu\nu} \). Under a conformal transformation, the scalar of curvature transforms as \( R(\Omega^2\bar{g}_{\mu\nu}) = \Omega^{-2}\bar{R} - 6\Omega^{-3}\Box \Omega \), and with the choice
\[ f(\phi) = \Omega^{-2}, \]  
on one gets from (1)
\[ S[\Omega^2\bar{g}, \phi] = \int d^4x \sqrt{-\bar{g}} \left\{ R - \left( \frac{3}{2} \left( \frac{d}{d\phi} \ln f(\phi) \right)^2 + \frac{h(\phi)}{f(\phi)} \right) \bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right\}. \]  
Now, defining \( \tilde{\phi}(\phi) \) as
\[ \tilde{\phi}(\phi) = \int_a^\phi d\xi \sqrt{\frac{3}{2} \left( \frac{d}{d\xi} \ln f(\xi) \right)^2 + \frac{h(\xi)}{f(\xi)}}, \]  
with arbitrary \( a \), we get the desired result, \( S[\Omega^2\bar{g}, \phi(\tilde{\phi})] = \tilde{S}[\bar{g}, \tilde{\phi}] \). Due to the assumption of \( f \) and \( h \) positive, the right-handed side of (7) is a monotonically increasing function of \( \phi \), which guarantees the existence of the inverse \( \phi(\bar{\phi}) \). The constant \( a \) is determined by the boundary conditions of \( \phi \) and \( \bar{\phi} \). Also, we have that \( \lim_{\bar{\phi} \to \infty} \phi(\bar{\phi}) = \infty \).

The transformation given by eq. (3) and (7), therefore, maps a solution \((g_{\mu\nu}, \phi)\) of (3) to a solution \((\bar{g}_{\mu\nu}, \bar{\phi})\) of (4). The transformation is independent of any assumption of symmetries, and in this sense is covariant. We can easily infer that the transformation is one-to-one in general, in the sense that any solution of (3) is mapped in an unique solution of (4). Also,
the transformation preserves symmetries, what means that if $\bar{g}_{\mu\nu}$ admits a Killing vector $\xi$ such that $\mathcal{L}_\xi \bar{\phi} = 0$, then $\xi$ is also a Killing vector of $g_{\mu\nu}$ and $\mathcal{L}_\xi \phi = 0$. From this, one concludes if we know all solutions $(\bar{g}_{\mu\nu}, \bar{\phi})$ with a given symmetry we automatically know all $(g_{\mu\nu}, \phi)$ with the same symmetry. This is the base of the proof.

The general asymptotically flat, static, and spherically symmetric solution $(\bar{g}_{\mu\nu}, \bar{\phi})$ of (4) is known (See [16] for some properties of the solution and references). It is given by two-parameter $(\lambda, r_0)$ family of solutions

$$\bar{\phi} = \sqrt{2(1 - \lambda^2)} \ln \mathcal{R},$$

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\mathcal{R}^{2\lambda} dt^2 + \left(1 - \frac{r_0^2}{r^2}\right)^2 \mathcal{R}^{-2\lambda} \left(dr^2 + r^2 d\Omega^2\right), \quad (8)$$

where $\mathcal{R} = \frac{r - r_0}{r + r_0}$. The parameter $\lambda$ can take values in $[-1, 1]$ in principle, but we neglect the negative range because the solution will have a negative ADM mass [16]. For $\lambda = 1$, the solution is the usual exterior vacuum Schwarzschild solution with the horizon at $r'_0 = 4r_0$, as one can check by using the coordinate transformation $r' = r \left(1 + \frac{\phi}{r}\right)^{2\lambda}$. For $0 \leq \lambda < 1$, (8) does not represent a black-hole due to that the surface $r = r_0$ is not a horizon, i.e. a regular null surface, but it is instead a naked singularity, as we can check, for instance, by calculating the scalar of curvature

$$\bar{R} = \frac{8r_0^2 r^4}{(r + r_0)^{2(2+\lambda)}} \times \frac{1 - \lambda^2}{(r - r_0)^{2(2-\lambda)}}. \quad (9)$$

In total accordance with the original scalar no-hair theorem [2], we see the only black-hole solution of (8) is that one for which $\lambda = 1$ and consequently $\phi = 0$, i.e. the usual Schwarzschild solution.

Any asymptotically flat, static, and spherically symmetric solution of (3) can be obtained from (8) by means of the transformations (5) and (7). This provides us with a two-parameters family of $(g_{\mu\nu}, \phi)$ solutions. The discussed properties of the transformation (7) and the expression for $\bar{\phi}(r)$ in (8) lead to the conclusion that the only solution with $\phi$ finite in the surface $r = r_0$ is that one for which $\phi$ is constant for $r > r_0$. In this case, (8) is only a rigid scale transformation, and the solution $(g_{\mu\nu}, \phi = a)$ is the usual Schwarzschild solution. This
is the desired result, which we formulate for clearness as follows.

**Theorem.** The only asymptotically flat, static, and spherically symmetric exterior solution of the system governed by the action

$$S = \int d^4x \sqrt{-g} \left\{ f(\phi) R - h(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}, \quad f(\phi), h(\phi) > 0.$$  

with $\phi$ everywhere finite is the Schwarzschild solution.

It is important to note that the used conformal transformation forbids that $f(\phi) \to \infty$ for any $r \neq r_0$.

Our approach can be extended in a straightforward way to other dimensions. The transformations (5) and (7) can be defined for any dimension $n > 2$. They shall be replaced by

$$f = \Omega^{2-n},$$

$$\bar{\phi}(\phi) = \int_\phi^\infty d\xi \sqrt{\frac{n-1}{n-2}} \left( \frac{d}{d\xi} \ln f(\xi) \right)^2 + \frac{h(\xi)}{f(\xi)}.$$  

(10)

The general asymptotically flat, static, and spherically symmetric solution for any space-time dimension $n > 3$ of (4) is known [16]. Its expression in isotropic coordinates is given by

$$\bar{\phi} = \sqrt{\frac{n-2}{n-3}} (1 - \lambda^2) \ln \mathcal{R}_n,$$

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -\mathcal{R}_n^{2\lambda} dt^2 + \left( 1 - \frac{r_0^{2n-6}}{r^{2n-6}} \right)^{\frac{2}{n-3}} \mathcal{R}_n^{\frac{2\lambda}{n-3}} \left( dr^2 + r^2 d\Omega^2 \right),$$  

(11)

where $\mathcal{R}_n = \frac{r^{n-3} - r_0^{n-3}}{r^{n-3} + r_0^{n-3}}$ and $d\Omega$ denotes the metric of the unitary $(n-2)$ sphere. The behavior of the solution (11) is similar to the four dimensional case. The only true black-hole solution is the usual one ($\lambda = 1$), due to the fact that the hyper-surface $r = r_0$ is not a regular one if $\lambda \neq 1$, as one can see from the expression for the scalar of curvature

$$\bar{R} = \frac{4(n-2)(n-3)r_0^{2(n-3)}r^{2(n-4)}}{(r^{n-3} + r_0^{n-3})^{2(n-2\lambda)/(n-3)}} \times \frac{1 - \lambda^2}{(r^{n-3} - r_0^{n-3})^{2(n-2\lambda)/(n-3)}}.$$  

(12)

By applying (10) and the same arguments used to the four dimensional case we can extend our theorem for any space-time dimension $n > 3$. 

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III. AN EXPLICIT EXAMPLE

A closer look in an explicit example will help us to understand the role of the assumption of finiteness of the scalar field. We see from (8) and (9) that for the minimal coupling, the finiteness of \( \bar{\phi} \) is related to the regularity of the horizon. The scalar field diverges in the surface \( r = r_0 \) for \( \lambda \neq 1 \), in this case the scalar of curvature has a non-removable singularity, what confirms that such surface is not a regular one, but it corresponds to a naked singularity. We will see that this is the case for some non-minimal couplings also. To this end, let us consider the Brans-Dicke theory, for which \( f(\phi) = \phi \) and \( h(\phi) = \frac{\omega}{\phi} \). Using the transformation (5) and (4) we can construct its general asymptotically flat, static, and spherical symmetric solution starting from the minimally coupled solution \((\bar{g}_{\mu\nu}, \bar{\phi})\),

\[
g_{\mu\nu} = \phi^{-1} \bar{g}_{\mu\nu}, \quad \bar{\phi} = \sqrt{\frac{3}{2} + \omega \int_{a}^{\phi} \frac{d\xi}{|\xi|}}. \tag{13}
\]

The expression for \( \bar{\phi} \) is divergent for \( a = 0 \). Also, if we choose \( a > 0 \), then \( \phi \) must be positive too to avoid the singularity. Let us take the solution \((\bar{g}_{\mu\nu}, -\bar{\phi})\) of (4), and consider \( \phi \in [a, \infty), a > 0 \). In this case we have

\[
\left(\frac{\phi}{a}\right)^{\frac{3}{2} + \omega} = \left(\frac{r + r_0}{r - r_0}\right)^{\sqrt{2(1 - \lambda^2)}}. \tag{14}
\]

The expression (14) hides a subtleness in the limit of large \( \omega \), maybe the most important limit in Brans-Dicke theory; recent solar system experiments has been established \( \omega > 600 \) [10]. In the limit \( \omega \to \infty \), the left-handed side of (14) can be 1 or \( \infty \), according to if \( \phi = a \) or \( \phi > a \). Due to the fact that the right-handed side is bounded for any \( \lambda \) and for \( r > r_0 \), the consistency of the equation imply that \( \lambda \) must be 1 and \( \phi = a \) in the limit \( \omega \to \infty \). This would guarantee that one gets the General Relativity in the limit \( \omega \to \infty \). Taking this into account we have from (13)

\[
\phi = a R^{-k}, \quad \text{ads}^2 = -R^{2\lambda+k} dt^2 + \left(1 - \frac{r_0^2}{r^2}\right)^2 R^{-2\lambda+k} \left(dr^2 + r^2 d\Omega^2\right). \tag{15}
\]
where \( k = \sqrt{\frac{4(1-\lambda^2)}{3+2\omega}} \). The two-parameter \((\lambda, r_0)\) family of solutions \((13)\) corresponds to the general asymptotically flat, static, and spherically symmetrical solution of the Brans-Dicke theory.

Our theorem states that the only black-hole solution of \((13)\) with finite \(\phi\) is the Schwarzschild one, but, at first sight, we can think that the null-surface \(r = r_0\) might be a horizon for some \(\lambda\) or \(\omega\). We can check that such surface is not a regular null-surface, but instead it is a naked singularity for any solution with non-constant \(\phi\). To this end, consider the scalar of curvature obtained from \((3)\)

\[
R = \frac{\omega}{\phi^2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi = \frac{4r_0^2 r^4}{(r + r_0)^{4+2\lambda-k}} \times \frac{\omega k^2}{(r - r_0)^{4-2\lambda+k}}.
\]

(16)

One has that \(4 - 2\lambda + k > 0\) for \(\lambda \in [0,1]\) and for \(\omega \in [0, \infty)\), and thus \((13)\) has a non-removable singularity for any \(\lambda \neq 1\) and \(\omega \neq 0\). We see that the only true black-hole solution is that one for which \(\lambda = 1\), i.e. again the Schwarzschild solution with \(\phi = a\), as it was predicted by the theorem. The case \(\omega = 0\) can be ruled out by analyzing the singularities of quadratic invariants, as for example \(R_{\mu\nu}R^{\mu\nu}\), that can be written through \((3)\) by means of \(\phi\). We notice that the first ho-hair theorem for Brans-Dicke theory is due to Hawking \([17]\), and that Bekenstein also proved recently the absence of scalar hairs in Brans-Dicke theory by using his novel no-hair theorem for minimally coupled scalar fields with non-quadratic Lagrangian \([1]\).

We can extend this result for theories such that \(\omega(\phi)\) is a \(C^1\) function and \(\lim_{\phi \to \infty} \omega(\phi) = \omega_c\). For such a case, we can evaluate an asymptotic expression for the scalar of curvature valid for the vicinity of the horizon, and it will lead us to the conclusion that the only black-hole solution also for this case is the Schwarzschild one. From \((7)\) one can see that for \(\lim_{\phi \to \infty} \omega(\phi) = \omega_c\) and \(r \to r_0\) we have

\[
\phi(r) \approx a R^{-\sqrt{\frac{4(1-\lambda^2)}{3+2\omega_c}}}
\]

(17)

From \((17)\) we have that the expression for \(R\) valid for \(r \to r_0\) is the same one of \((16)\), from which we conclude that there is no scalar hair in the model of Bergman, Wagoner and Nordtved with \(\lim_{\phi \to \infty} \omega(\phi) = \omega_c\). The result is valid for any space-time dimension \(n > 3\).
We can easily apply analogous arguments to prove the absence of scalar hair in dilaton gravity for any space-time dimension \( n > 3 \).

**IV. FINAL REMARKS**

In spite of the theorem’s broad assumptions, there are situations that it does not cover. In situations where the divergence of the scalar field is not related to a naked singularity it is possible, in principle, to exist a scalar hair. This is the case of the Bekenstein conformal scalar hair \([11]\), that obviously escapes from the theorem’s assumptions due to the divergence of the scalar field in the horizon. Such divergence is not related to any space-time singularity, and for an observer that does not interact directly with the scalar field the divergence is physically harmless.

A recent result due to Zannias \([8]\) also stresses the relevant role of the divergence of the scalar field in the existence of hairs. In our approach, the finiteness of \( \phi \) guarantees that the only null-surface of \( g_{\mu\nu} \) corresponds to \( r = r_0 \). If \( \phi \) diverges for some point of the space-time, say \( r_1 \), the conformal factor \( \Omega(r_1) \) in \((5)\) vanishes and consequently \( g_{00}(r_1) = 0 \), what would induce another null-surface for \( r = r_1 \). This is precisely what happens with the Bekenstein conformal hair. However, in principle one can find out case by case asymptotic expressions for the geometrical quantities, as we did in the Sect. \([11]\), and to control the regions very close to the horizons.

We finish noting that two recent works are devoted to problems similar to the ones discussed here. In \([18]\) Heusler studies with great detail the case of self-gravitating nonlinear sigma models, for which the action would be given in our notation by

\[
S[g, \phi^i] = \int d^4x \sqrt{-g} \left\{ R - h_{jk}(\phi^i)g^{\mu\nu}\partial_\mu\phi^j\partial_\nu\phi^k + W(\phi^i) \right\},
\]

\((18)\)

where \( i \in (1, \ldots, N) \). He proved that the only asymptotically flat, static, and spherically symmetric black-hole solution of \((18)\) is the Schwarzschild one. Sudarsky \([19]\) considered the case where \( h_{jk}(\phi^i) = \delta_{jk} \), getting the same result in a simpler way. These results are in
agreement with our theorem since the case $N = 1$ and $W(\phi) = 0$ corresponds to our $f = 1$ case. However, we believe that our proof is much more simpler.

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