0. Introduction.

The purpose of this paper is to investigate the relationship between between stable holomorphic vector bundles on a compact complex surface and the same such objects on a modification (blowup) of the surface. In large part, the paper is a continuation of the work in [B2] where it was shown that a holomorphic bundle on a compact complex surface admits an irreducible Hermitian-Einstein connection if and only if the bundle is stable, the notions of stability and Hermitian-Einstein both being with respect the same $\bar{\partial}\partial$-closed positive $(1, 1)$-form: this is a generalization of Donaldson’s result [D2] where it is assumed that the form is $d$-closed and defines an integral cohomology class (i.e., the algebraic case).

Much of the underlying motivation for this work comes from its potential applications to topology (though no such applications are considered here). Donaldson has proved fundamental results on the topology of smooth 4-manifolds by defining topological invariants of moduli spaces of solutions of the anti-self-dual Yang-Mills equations on a Riemannian 4-manifold, and showing that these are differential invariants of the manifold itself. In the case of an algebraic surface with Hodge metric, the result of [D2] mentioned above identifies the Yang-Mills moduli spaces with moduli spaces of stable vector bundles, and these spaces and/or invariants can be computed using techniques standard in complex analysis. In this way Donaldson has been able to prove some of his most remarkable results [D4], [D5], and others have built upon his work ([FM1], [FM2], [K1], [OV], to cite just a few. See also [FM3] for a comprehensive account of developments, and [FS] for a calculation of the Donaldson invariants of a blownup 4-manifold in terms of those of the manifold). This interaction between real analysis, complex analysis and topology provides a rich area for investigation, and parts of this paper are directly concerned particularly with the interplay between the real and the complex analysis.

The proof of the main result of [B2] is a modification of that given by Donaldson [D1] to prove the same theorem in the case of Riemann surfaces. The differences in the proofs arise from the appearance of certain singularities in the two-dimensional case, and a successful way around these singularities is to blow up the surface and pull back. In so doing, various relationships between bundles and sheaves on the surface and its blowups are uncovered, and these relationships turn out to be directly related to other aspects of gauge theory and/or complex analysis which are themselves of independent interest.

A study of degenerating sequences of stable bundles on the projective plane leads naturally to conjecture whether blowups can be used to compactify moduli spaces of stable bundles in general. In [B4], it is shown that sequences of stable bundles, identified with sequences of Hermitian-Einstein connections have convergent subsequences after pulling back to blowups, at least when weak limits are stable. This leads to a the definition of a natural topology on moduli spaces stable bundles over a surface and its blowups, and the proof of the compactness of the generic such space is presented here.

The paper is organized as follows: §1 introduces notation, definitions and central background material, and gives some useful lemmas concerning “invariants” of stable holomorphic bundles. In §2 a local description and characterization of bundles on the blowup of the ball in $\mathbb{C}^2$ at the origin is given. A holomorphic version of Taubes’ “cut-and-paste” construction for gauge fields [T] is given, enabling a global description of bundles on the blowup of an arbitrary complex surface in terms of bundles on the original surface. Also included in this section is a short discussion of the relationship to—and between—associated constructions of Serre and Schwarzenberger.

Questions of stability are considered in the third section from a purely complex-analytic viewpoint, and a detailed description of the conditions required for bundles on a blowup $\tilde{X} \to X$ to be stable is
The analysis in the third section encounters pathological sheaves which are semi-stable but not stable. Using the cut-and-paste method, a mechanism for “stabilising” such sheaves is given in §4. A similar method also provides a simple way to desingularise singular points in moduli spaces.

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1. Preliminaries.

The purpose of this section is to re-cap on, and to expand upon the basic notation, definitions and results of [B2]. Further details can be found in that reference.

Let $X$ be a compact complex surface and let $\omega$ be a $\partial\overline{\partial}$-closed positive (1,1)-form on $X$: it is a theorem of Gauduchon [Gau] that every positive (1,1)-form has a unique positive conformal rescaling such the rescaled form is $\partial\overline{\partial}$-closed and gives the same volume $V := Vol(X, \omega) := \frac{i}{2} \int_X \omega^2$. With such a form $\omega$, the degree $\deg(L) = \deg(L, \omega)$ of a holomorphic line bundle $L$ on $X$ is unambiguously defined by the formula

$$\deg(L) := \frac{i}{2\pi} \int_X f_L \wedge \omega,$$

where $f_L$ is the curvature of any hermitian connection on $L$. The degree depends only on $c_1(L)$ if and only if $b_1(X)$ is even, and when this is the case $\omega$ is cohomologous modulo the image of $\partial + \overline{\partial}$ to a closed form which itself is unique up to the image of $\partial\overline{\partial}$: ([B2], Proposition 2).

If $E$ is a holomorphic $r$-bundle on $X$, set $\deg(E) := \deg(det E)$ and $\mu(E) := \deg(E)/r$; the latter is called the normalized degree or slope of $E$. A hermitian connection on $E$ is Hermitian-Einstein if the curvature $F$ satisfies $F = i\lambda 1$ where $\tilde{F} := * (\omega \wedge F) =: \Lambda F$, $\lambda = \frac{r-2\pi}{V} \cdot \mu(E)$ and $1$ is the identity endomorphism of $E$. The bundle $E$ is (semi-) stable if $\mu(S) < (\leq) \mu(E)$ for every coherent subsheaf $S \subset E$ with $0 < \text{rank}(S) < r$. As mentioned in the introduction, the main result of [B2] is that a bundle admits an irreducible Hermitian-Einstein connection if and only if it is stable, this generalizing the same result proved by Donaldson [D2] in the case that $(X, \omega)$ is algebraic. A bundle admitting a Hermitian-Einstein connection is a direct sum of stable bundles all of the same normalized degree; i.e., is quasi-stable.

If $E$ has a Hermitian-Einstein connection with curvature $F$, the equation $\omega \wedge (F - \frac{1}{r} tr F 1) = 0$ and the skew-Hermitian property of $F$ give $tr(F - \frac{1}{r} tr F 1)^2 = |F - \frac{1}{r} tr F 1|^2 dV$. Since the former 4-form is a representative for the characteristic class $8\pi^2 (c_2 - \frac{r-1}{2r} c_1^2)(E)$, this motvates defining the charge of $E$, $C(E)$, for an arbitrary $r$-bundle $E$ by the formula

$$C(E) := (c_2 - \frac{r-1}{2r} c_1^2)(E) = \frac{1}{8\pi^2} \int_X tr(F - \frac{1}{r} tr F 1)^2.$$


This number is non-negative for any bundle admitting a Hermitian-Einstein connection, and when this is the case, is identically zero only if the induced Hermitian-Einstein connection on the adjoint bundle is flat; (cf. [L]). Note that the charge is invariant under tensoring by line bundles: $C(E \otimes L) = C(E)$ for any such $L$. In general, $C(E \otimes A) = aC(E) + rC(A)$, where $a, r$ are the ranks of $A, E$ respectively.

Recall that a coherent analytic sheaf $\mathcal{S}$ is torsion-free if and only if the canonical morphism $\mathcal{S} \to \mathcal{S}^{**}$ is injective, and $\mathcal{S}$ is by definition reflexive if this map is an isomorphism; recall also that the singularity sets of such sheaves are of codimension at least 2 and 3 respectively; ([OSS], II.1.1). For exact sequences $0 \to A \to B \to C \to 0$ of locally free sheaves on $X$ it is easy to check that the charges are related by

$$C(\mathcal{B}) = C(\mathcal{A}) + C(\mathcal{C}) - \frac{b}{2ac} \left[ \frac{a}{b} c_1(\mathcal{B}) - c_1(\mathcal{A}) \right]^2,$$

where $a, b$ and $c$ are the ranks of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ respectively. The definition of charge extends to torsion-free sheaves $\mathcal{S}$ of rank $r$ by means of the formula

$$C(\mathcal{S}) := C(\mathcal{S}^{**}) + h^0(\mathcal{S}^{**}/\mathcal{S}),$$

which is consistent with a definition of $c_2(\mathcal{S})$ extending that of the Chern character on bundles in such a way that the Hirzebruch-Riemann-Roch formula

$$h^0(\mathcal{S}) - h^1(\mathcal{S}) + h^2(\mathcal{S}) = \chi(\mathcal{S}) = -C(\mathcal{S}) + \frac{1}{2r} c_1^2(\mathcal{S}) + \frac{1}{2} c_1(\mathcal{S}) \cdot c_1(X) + r\chi(\mathcal{O}_X)$$

remains valid. If $\mathcal{C}$ is only torsion-free, it follows from this definition that (1.2) remains valid and this in turn implies that the formula (1.2) holds for arbitrary torsion-free sheaves $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. Note that a torsion-free sheaf is (semi-)stable iff its double-dual is.

If $b_1(X)$ is odd, the intersection form on $H^2(X, \mathbb{R})$ restricted to $H^{1,1}(X)$ is negative definite ([BPV], Theorem IV.2.13) and the last term on the right in (1.2) therefore contributes positively to the sum. If $b_1(X)$ is even, the intersection form on $H^{1,1}(X)$ has one positive eigenvalue and the rest are all negative. In either case, $\omega$ defines a positive definite hermitian form on $H^{1,1}(X)$ by setting $\|f\|^2 := V^{-1} |(f, \omega)|^2 - (f, f)$, where $(f, g) := \int_X f \wedge g$; (recall $V = (\omega, \omega)/2$ throughout). Equation (1.2) can therefore be written

$$C(\mathcal{B}) = C(\mathcal{A}) + C(\mathcal{C}) + \frac{b}{2ac} \left[ \frac{a}{b} c_1(\mathcal{B}) - c_1(\mathcal{A}) \right]^2 - \frac{b}{2ac} \nu_B(\mathcal{A})^2 / V,$$

where $\nu_B(\mathcal{A}) := a [\mu(\mathcal{B}) - \mu(\mathcal{A})]$. By induction on rank, it follows the charge is non-negative for any torsion-free semi-stable sheaf. Note that if $b_1(X)$ is odd it follows by induction from (1.2) (and the existence of Hermitian-Einstein connections on stable bundles) that the charge is non-negative for any torsion-free coherent analytic sheaf, semi-stable or otherwise.

The function $\nu_{\bullet}(\bullet)$ plays an important role in the proof of the main result of [B2]. It has a number of simple but useful properties, three of which are summarised for convenience in the following lemma.

**Lemma 1.5.**

(a) If

$$\begin{array}{cccccc}
0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{C} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \to & \mathcal{A}' & \to & \mathcal{B}' & \to & \mathcal{C}' & \to & 0
\end{array}$$

is a commutative diagram with exact rows such that the vertical arrows are inclusions, then

$$\nu_B(\mathcal{B}') = \nu_{\mathcal{A}}(\mathcal{A}') + \nu_{\mathcal{C}}(\mathcal{C}') + \left( \frac{a'}{a} - \frac{c'}{c} \right) \nu_B(\mathcal{A});$$

(b) If $\mathcal{A}$ and $\mathcal{B}$ are locally free and if $\mathcal{B}$ is stable and $\mathcal{A} \subset \mathcal{B}$ minimizes $\nu_B$ over all proper non-zero subsheaves of $\mathcal{B}$ then $\mathcal{A}$ is stable and in addition, the quotient $\mathcal{C} = \mathcal{B}/\mathcal{A}$ is both torsion-free and stable;
(c) If $E$ is a holomorphic bundle equipped with a Hermitian metric and $A \subset E$ has torsion-free quotient $C$ then off the singular set of $C$ the second fundamental form $\beta \in A^{0,1} \otimes \text{Hom}(C, A)$ of the induced Hermitian connection lies in $L^{2p}(X)$ for any $p < 2$, and

$$\nu_E(A) = -\frac{1}{2\pi} \int_X tr_A(i F(E) + \lambda_E 1 \omega) \wedge \omega + \|\beta\|_{L^2(\omega)}^2.$$  

The proof of (c) is given in [B2] (remark (f) p.634). Part (b) is the same as Lemma 2 of the same reference; the proof follows immediately from (1.6) which itself is a straightforward calculation. The existence of such $A \subset B$ minimizing $\nu_B$ when the latter is stable is proved in Lemma 4 of [B2], which provides one of the key steps in the proof of the main result there by enabling the argument to proceed by induction on rank; (it is also proved in that lemma that there always exists $A \subset B$ maximizing $\mu$ over admissible subsheaves regardless of the stability or otherwise of $B$). For a stable bundle $E$ it follows that $\nu_E(*)$ is bounded above and away from 0 on the set of subsheaves of $E$, from which it follows immediately that stability is an open condition on the metric.

When $b_1(X)$ is even, this bound on the slopes of subsheaves can be made uniform in $E$ as the next result shows:

**Lemma 1.8.** Suppose that $b_1(X)$ is even. For any $r_0, C_0 > 0$ there exists $\delta_0 = \delta_0(r_0, C_0) > 0$ with the following property: if $E$ is a semi-stable torsion-free sheaf of rank $r \leq r_0$ and charge $C(E) \leq C_0$ admitting a subsheaf $A \subset E$ with $\nu_E(A) < \delta_0$, then $\nu_E(A) = 0$.

**Proof:** Since $C(E^*) \leq C(E)$ and $\nu_{E^*}(A^*) = \nu_E(A)$ it suffices to prove the result with “torsion-free” in the hypotheses replaced by “locally free”. For subsheaves $A \subset E$ of the same rank as $E$ the result follows from Corollary 2 of [B2], so it can also be supposed that all such subsheaves have rank strictly less than that of the ambient bundle.

If there is no stable bundle of rank $\leq r_0$ and charge $\leq C_0$ which admits a proper non-zero reflexive subsheaf, then $\nu_E$ is identically 0 for all such bundles. Otherwise, there is a sequence $\{E_i\}$ of bundles admitting such subsheaves $A_i \subset E_i$ with $\{\nu_{E_i}(A_i)\}$ strictly decreasing. By Lemma 4 of [B2], it can be assumed that each $A_i$ minimizes $\nu_{E_i}$ over the proper non-zero subsheaves of $E_i$, and by Lemma 1.5(b), both $A_i$ and the quotient $C_i := E_i/A_i$ are torsion-free and stable. Since $C(A_i)$ and $C(C_i)$ are therefore non-negative and $\nu_{E_i}(A_i) = \nu_{E_i}(C_i)$ is decreasing, (1.4) and the bound on the rank and charge of $E_i$ give a uniform bound on $\|a_i c_1(E_i) - r_i c_1(C_i)\|$, so there is a subsequence with $r_i$ and $a_i c_1(E_i) - r_i c_1(C_i)$ constant.

Since $b_1(X)$ is even, the degree is topological so $\nu_{E_i}(A_i)$ is constant on the subsequence, implying that the original sequence is finite. It follows that there exists $\delta > 0$ such that $\nu_E(A) \geq \delta$ for any proper non-zero subsheaf $A$ of a stable torsion-free sheaf $E$ of rank $\leq r_0$ and charge $\leq C_0$. Set $\delta_0 := \delta$.

If now $E$ is semi-stable (of rank $\leq r_0$ and charge $\leq C_0$) and $D \subset E$ satisfies $\nu_E(D) < \delta_0$, then $E$ cannot be stable so there exists non-trivial $A \subset E$ with torsion-free quotient $C$ such that $\mu(A) = \mu(E)$. Since $E$ is semi-stable, so too are both $A$ and $C$ and moreover they also satisfy the hypotheses of the lemma by (1.4). If $C'$ is the image of the composition $D \to E \to C$ and $A'$ is the kernel, the result follows from (1.6) using induction on $\text{rank}(E)$.

**Remark:** The result is false if $b_1(X)$ is not even: moduli spaces of stable 2-bundles with trivial determinant on a Hopf surface are explicitly computed in [BH], and the description there shows that there are stable 2-bundles of charge 1 possessing subsheaves of degree arbitrarily close to 0.

2. Vector bundles on a blowup.

The purpose of this section is to investigate the nature of holomorphic vector bundles on a neighbourhood of a blown-up point in a complex surface. Two approaches are taken: the first starts from the splitting type of such bundles on the exceptional divisor, whereas the second is more global in nature, identifying such bundles with a class of bundles on the complex projective plane.

Let $Y$ be a discrete set of points in a complex surface $X$ and let $\tilde{X} \xrightarrow{\pi} X$ be the blowup of $X$ along $Y$. The exceptional divisor $\tilde{Y} = \pi^{-1}(Y) \subset \tilde{X}$ is defined by a section of a certain holomorphic line bundle, and
since this line bundle restricts to $\mathcal{O}(-1)$ on each component of $\tilde{Y}$, the notation $\mathcal{O}(-1)$ will be used to denote (the sheaf of sections of) this line bundle. If $\mathcal{I}_Y \subset \mathcal{O}_X$ denotes the ideal sheaf of $Y$ and $\mathcal{N}_Y = (\mathcal{I}_Y/\mathcal{I}_Y^2)^*$ is the normal bundle of $Y$ in $X$, then it is straightforward to show that the direct images of the sheaves $\mathcal{O}(n)$ under $\pi$ are canonically given by

$$
\pi_* \mathcal{O}_X(n) = \begin{cases} 
\mathcal{I}_Y^n & \text{if } n \geq 0 \\
\mathcal{O} & \text{if } n \leq 0 
\end{cases}
$$

(2.1)

(where $\pi^n = R^n\pi_*$ denotes the $n$-th direct image under $\pi$) with all other direct images vanishing.

Let $U \subset X$ be a small ball in $X$ such that $U \cap Y$ is the singleton $x_0$, let $\tilde{U} \to U$ be the blowup of $U$ at $x_0$, and let $L_0 = \pi^{-1}(x_0)$ be the exceptional line. If $E$ is a holomorphic $r$-bundle on $U$, then the restriction of $E$ to $L_0$ splits as a sum of line bundles, and the nature of this splitting determines much about the bundle itself, as will be demonstrated in the results which follow.

**Lemma 2.3.** Suppose $E|_{L_0} = \oplus_{i=1}^r \mathcal{O}(a_i)$.

(a) If $a_i \leq 0$ for all $i$ then $\pi_* E$ is locally free;

(b) If $\pi_* E$ is locally free then $\sum a_i \leq 0$;

(c) If $a_i = 0$ for all $i$ then $E = \pi^* \pi_* E$ and is trivial on $\tilde{U}$;

(d) If $a_i > -2$ for all $i$ then $\pi_* E = 0$.

**Proof:** The space $\tilde{U}$ can be viewed as a closed subspace of $U \times \mathbb{P}_1$, defined by a section of the line bundle $\mathcal{O}(1)$. The bundle $\tilde{E}$ on $\tilde{U}$ can be extended (non-uniquely) to a bundle $E'$ on $U \times \mathbb{P}_1$ simply by extending a transition function on the intersection of a pair of Stein sets covering $\tilde{U}$. Thus there is an exact sequence

$$
0 \to E'(-1) \to E' \to \tilde{E} \to 0
$$

(2.4)

on $U \times \mathbb{P}_1$, (where the notation $E(n)$ denotes $E \otimes \mathcal{O}(n)$ throughout).

If $a_i < 0$ for all $i$ then $H^0(\{x\} \times \mathbb{P}_1, E') = 0$ for $x = x_0$ and hence for all $x$ in a neighbourhood of $x_0$ by semi-continuity of cohomology. From the base-change theorem ([BS], Theorem 3.4) it follows that $\pi_* E' = 0 = \pi_* E'(1)$ and the sheaves $\pi_* \tilde{E}'$ and $\pi_* \tilde{E}'(-1)$ are locally free on $U$. From the direct image of (2.4)

$$
0 \to \pi_* E \to \pi_* \tilde{E}'(-1) \to \pi_* \tilde{E}' \to \pi_* \tilde{E} \to 0
$$

it therefore follows from Lemma II.1.1.10 of [OSS] that $\pi_* \tilde{E}$ is reflexive, and hence locally free. Moreover, taking direct images of the exact sequence $\tilde{E} \otimes (0 \to \mathcal{O}_U(1) \to \mathcal{O}_U \to \mathcal{O}_L \to 0)$ on $\tilde{U}$ and using the fact that $\pi_* (\tilde{E}|_{L_0})$ vanishes, it follows that $\pi_* \tilde{E}(1) = \pi_* E$ is also reflexive, which proves (a).

To prove (b), suppose det $\tilde{E} = \mathcal{O}(a)$ for some $a > 0$. If $\pi_* \tilde{E}$ is locally free, then by (2.1) so too is $\pi_* \tilde{E}''$ for $\tilde{E}'' := \tilde{E} \oplus \mathcal{O}(-a)$, and therefore $\pi_* \pi_* \tilde{E}''$ is a bundle on the blowup. The canonical sheaf homomorphism $\pi_* \pi_* \tilde{E}'' \to \tilde{E}$ is an isomorphism off the exceptional divisor, but since both bundles have trivial determinant the homomorphism must in fact be an isomorphism everywhere. This is not possible since the pull-back is trivial on the exceptional divisor, whereas $\tilde{E}$ is not.

If $\tilde{E}$ is trivial on $L_0$, then $\pi_* \tilde{E}$ is locally free by (a) and the argument of the last paragraph shows that $\tilde{E} \simeq \pi_* \pi_* \tilde{E}$, proving (c).

Finally, if $a_i > -2$ for all $i$ then $H^1(\{x\} \times \mathbb{P}_1, E')$ vanishes at $x = x_0$, and therefore for all $x$ in a neighbourhood of $x_0$, again by semi-continuity of cohomology. Taking direct images of (2.4) this time shows that $\pi_* \tilde{E}$ vanishes near $x_0$.  

With $\tilde{E}$ as in Lemma 2.3, suppose that $a_1 \leq \ldots \leq a_r$. The obstruction to extending a section in $\Gamma(L_0, \tilde{E}(-a_1))$ to $U$ lies in $H^1(U, E(1-a_1))$, a group which vanishes by part (d) of the lemma. More generally, any finite number of such sections which are independent over $L_0$ will also be so near $L_0$. Dualising $\tilde{E}$, if $\lambda_1, \ldots, \lambda_k \in \Gamma(L_0, \tilde{E}^*(a_r))$ are independent, then they can be extended to sections of $\tilde{E}^*(a_r)$ over a
neighbourhood of $L_0$ in $\tilde{U}$ which are linearly independent at each point in this neighbourhood. Rewriting $\tilde{E}|_{L_0}$ in the form $\tilde{E}|_{L_0} = \oplus_{i=1}^m V_i(b_i)$ where $V_i$ is a vector space with $b_1 < b_2 < \ldots < b_m$, the last statement implies that the projection $\tilde{E}|_{L_0} \to V(b_m)$ extends to an epimorphism $\tilde{E} \to V(b_m)$ in a neighbourhood of $L_0$. Using induction on rank, this gives the following local description of bundles on the blowup of a two-dimensional ball at a point:

**Proposition 2.5.** Suppose $\tilde{E}|_{L_0} = \oplus_{i=1}^m V_i(b_i)$ where $V_i$ is a $d_i$-dimensional vector space and $b_1 < b_2 < \ldots < b_m$. Then in a neighbourhood of $L_0$ in $\tilde{U}$ there is a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_m = \tilde{E}$$

of $\tilde{E}$ by vector bundles $F_k$ such that $F_k/F_{k-1} \simeq V_k(b_k)$ and such that $F_k|_{L_0} = \oplus_{i=1}^k V_i(b_i)$. \hfill \square

**Corollary 2.7.** If $\tilde{E}|_{L_0} = \oplus_{i=1}^r O(a_i)$ with $|a_i - a_j| \leq 2$ for all $i$, then $\tilde{E} \simeq \oplus_{i=1}^r O(a_i)$ in a neighbourhood of $L_0$.

**Proof:** A bundle $F$ on $\tilde{U}$ which is given as an extension $0 \to V(a) \to F \to W(b) \to 0$ for some vector spaces $V,W$ is determined by an element of $H^1(\tilde{U}, \bigvee V(a-b))$. If the extension splits on $\pi^{-1}(x_0)$ then this class lies in the image of $H^1(\tilde{U}, \bigvee V(a-b+1)) \to H^1(\tilde{U}, \bigvee V(a-b))$, and if $a-b \geq -2$ then the former group vanishes, by (2.1). \hfill \square

The preceding discussion provides some insight into well-known constructions of Serre [Ser] and of Schwarzenberger [Sch], both described in [OSS]. A bundle $\tilde{E}$ on $\tilde{X}$ is the pull-back of a bundle $E$ from $X$ if and only if $\tilde{E}$ restricts trivially to every component of the exceptional divisor $\tilde{Y}$; if this is the case, then necessarily $E = \pi_* \tilde{E}$. Let $L_1, L_2$ be line bundles on $X$ and let $L_i$ also denote $\pi^* L_i$ on $\tilde{X}$. Extensions of the form $\tilde{X} : 0 \to L_1(1) \to \tilde{E} \to L_2(1) \to 0$ are classified by $H^1(\tilde{X}, L_1 L_2^{-}(2))$ which can be computed from the Leray spectral sequence for $\pi$ using (2.1). This gives an exact sequence

$$0 \to H^1(X, L_1 L_2^{*}) \to H^1(\tilde{X}, L_1 L_2^{*}(2)) \to H^0(Y, \bigvee \det N_Y) \to H^2(X, L_1 L_2^{*})$$

and the bundle $\tilde{E}$ corresponding to an element of $H^1(\tilde{X}, L_1 L_2^{-}(2))$ is trivial on $\pi^{-1}(x_0)$ if and only if the element of $L_1 L_2^{*} \otimes \det N_{Y,x_0}$ obtained from (2.8) is non-zero. Using (2.1) again, the bundle $E$ is given by an exact sequence $0 \to L_1 \to E \to L_2 \otimes \bigvee Y \to 0$, so $E \otimes L_1^{*}$ has a section vanishing precisely at $Y$ and $\mathcal{N}_Y$ has been extended to the bundle $E^* \otimes L_2$. If $X$ is compact, the Chern classes of $E$ are given by $c_1(E) = c_1(L_1) - c_1(L_2)$ and $c_2(E) = c_1(L_1) c_1(L_2) + PD([Y])$ where $PD([Y])$ denotes the Poincaré dual of $[Y]$.

More generally, this construction applies when $Y$ is an arbitrary codimension 2 locally complete intersection in a complex manifold $X$ provided that $\det \mathcal{N}_Y$ can be extended to a line bundle on $X$. Indeed, with the exception of Corollary 2.7, all of the results of this section so far presented remain valid if $X$ has arbitrary dimension and $Y$ is a codimension two locally complete intersection; modifications to the proofs above are straightforward.

If $X$ is compact the charge on $\tilde{E}$ can be estimated in terms of its splitting on $L_0 = \pi^{-1}(x_0)$ and the charge on its direct image:

**Proposition 2.9.** Let $\tilde{X} \xrightarrow{\pi} X$ be the blowup of the compact surface $X$ at $x_0$, with $L_0 = \pi^{-1}(x_0)$. If $\tilde{E}$ is an $r$-bundle on $\tilde{X}$ such that $\tilde{E}|_{L_0} = \oplus_{i=1}^r O(a_i)$, then for $a := \sum_i a_i$ and $E := (\pi_* \tilde{E})^{**}$ it follows

$$C(E) + \frac{1}{2} \sum_{i=1}^r |a_i - a/r| + \frac{\bar{a}}{2r} (2r - \bar{a} - n) \leq C(\tilde{E}) \leq C(E) + \frac{1}{2} \sum_{i=1}^r (a_i - a/r)^2$$

where $\bar{a} \equiv a \pmod{r}$, $0 \leq a < r$ and $n := \#\{a_i \mid a_i \leq a/r\}$. Moreover, equality holds in the first case iff $\tilde{E} \simeq (\pi_* E)(k)$ for some $k \in \mathbb{Z}$ and in the second case iff $\tilde{E} \simeq \oplus O(a_i)$ in a neighbourhood of $L_0$.

**Proof:** Since $\chi(\tilde{E}) = \chi(\pi_* \tilde{E}) - \chi(\pi_* E)$ and $c_1(\tilde{X}) = \pi^* c_1(X) + c_1(\mathcal{O}(1))$ the Riemann-Roch formula gives

$$C(\tilde{E}) = C(\pi_* \tilde{E}) + \chi(\pi_* E) - \frac{a(a+r)}{2r} = C(E) + \dim(E/\pi_* E) + \chi(\pi_* E) - \frac{a(a+r)}{2r}.$$
If \( a_i \leq 0 \) for all \( i \) and \( \tilde{E} \) splits in a neighbourhood of \( L_0 \) as a direct sum of line bundles and then \( \pi_1 \tilde{E} \) is locally free and \( \chi(\pi_1 \tilde{E}) = - \sum \chi(O(a_i)) + \sum \chi(\pi_1 O(a_i)) = (1/2)\sum a_i(a_i + 1) \), giving \( C(\tilde{E}) = C(E) + (1/2) \sum (a_i - a_i/r)^2 \) in this case. If \( \tilde{E} \) does not split in a neighbourhood of \( L_0 \) then \( \chi(\pi_1 \tilde{E}) \) is strictly less than the corresponding number in the split case—this follows easily by induction on rank using Proposition 2.5.

Since (2.10) is invariant under tensoring \( \tilde{E} \) by \( O(-k) \), this proves the upper bound.

To obtain the lower bound, first twist \( \tilde{E} \) by a suitable line bundle so that \( 0 \leq a < r \). Using Lemma 2.3 as in the proof of Proposition 2.5, in a neighbourhood of \( L_0 \) the bundle \( \tilde{E} \) can be written as an extension

\[
0 \to A \to \tilde{E} \to B \to 0
\]

which splits on \( L_0 \) and where \( A|_{L_0} = \oplus \{O(a_i)\} \) and \( B|_{L_0} = \oplus O(a_i) \). Since the extension splits on \( L_0 \), it lies in the image of \( H^1(U, B^* \otimes A(1)) \), so there is a bundle \( \tilde{E}_0 \) on \( U \) which is given by a compatible extension

\[
0 \to A \to \tilde{E}_0 \to E_0 \to 0.
\]

Compatibility of the extension implies that there is an exact sequence

\[
0 \to \tilde{E}_0 \to \tilde{E} \to A_{L_0} \to 0
\]

and \( \pi_1 \tilde{E}_0(1) = \pi_1 \tilde{E}(1) \). Using Lemma 2.3, \( \tilde{E}_0 \) is another triple of such objects such that \( \tilde{E}_0 \) is isomorphic to \( \tilde{E} \). Hence \( \chi(\pi_1 \tilde{E}_0(1)) \geq \chi(\pi_1 \tilde{E}(1)) = - \sum a_i\leq 0 \), since from (2.11) applied to \( \tilde{E}(1) \) it follows

\[
C(\tilde{E}) = C(\tilde{E}_0(1)) \geq C(E) + \sum a_i \leq 0 |a_i| + a(r - a)/2r.
\]

Averaging the two gives

\[
C(\tilde{E}) \geq \frac{1}{2} \sum_{i=1}^{r} |a_i| + \frac{a(r - a)}{2r} \quad \text{if} \quad 0 \leq a < r.
\]

The expression on the left of (2.12) follows immediately from this in the general case by simple calculation. Clearly equality holds in (2.12) iff \( a_i = 0 \) for all \( i \).

The lower bound in (2.10) is sharp: given any splitting type \( \oplus O(a_i) \) for a bundle \( \tilde{E} \) on \( L_0 \), an argument by induction on rank shows that there is a bundle \( \tilde{E}_0 \) which splits minimally on \( L_0 \) which is given as an extension \( 0 \to A(1) \to \tilde{E}_0 \to B \to 0 \) where \( A \) and \( B \) are bundles of the form described above. The inequalities (2.10) appear in [FM2] (Remark 5.4) in the case of rank 2 bundle with \( c_1 = 0 \).

As before, let \( X \) be a complex surface and let \( X \to \tilde{X} \) be the blowup of \( X \) at the point \( x_0 \in X \).

The local description provided by Proposition 2.5 can be combined with a holomorphic version of Taubes’ “cut-and-paste” construction [T] to provide a global description of bundles on \( \tilde{X} \).

Let \( \tilde{E} \) be a holomorphic \( r \)-bundle on \( \tilde{X} \), and let \( U \subset X \) be a neighbourhood of \( x_0 \) isomorphic to a ball. Then \( \tilde{E} \) can be viewed as comprised of two pieces, namely the bundle \( (\pi_1 \tilde{E})^{\otimes} \) on \( X \) and the bundle \( \tilde{E}|_{\tilde{U}} \) on \( \tilde{U} = \pi^{-1}(U) \); the two pieces are glued together by means of the isomorphism \( \pi_1 \tilde{E} \simeq (\pi_1 \tilde{E})^{\otimes} \) over \( U \setminus \{x_0\} \approx \tilde{U} \cdot L \).

Conversely, given \( r \)-bundles \( E_0 \) on \( X \) and \( E_1 \) on \( \tilde{U} \), the two can be glued together by means of an isomorphism \( \rho: \pi_1 E_1 \to E_0 \) over \( U \setminus \{x_0\} \), extending uniquely to \( U \) as an isomorphism \( (\pi_1 E_1)^{\otimes} \to E_0|_{U} \) to define a bundle \( E_0 \#_\rho E_1 \) on \( \tilde{X} \). If \( (E_0', E_1', \rho') \) is another triple of such objects such that \( E_0 \#_\rho E_1 \simeq E_0' \#_{\rho'} E_1' \), then by Hartogs’ theorem the isomorphism \( E_0|_{X \setminus \{x_0\}} \simeq E_0'|_{X \setminus \{x_0\}} \) extends to an isomorphism \( \phi_0 \) over \( X \), and if \( \phi_1 \) induces the induced isomorphism \( E_1 \to E_1' \), it follows that \( \rho' = \phi_0 \rho \phi_1^{-1} \).

If \( \rho_0: (\pi_1 E_1)^{\otimes} \to E_0|_{U} \) is fixed and \( \rho \) is any other such isomorphism, then \( \rho \rho_0^{-1} \) is an automorphism of \( E_0 \) over \( U \). Hence the following description is obtained:

**Proposition 2.13.** Let \( E_0 \) be a bundle on \( X \) and \( E_1 \) be a bundle on a neighbourhood \( \pi^{-1}(U) \) of \( \pi^{-1}(x_0) \) in the blowup \( \tilde{X} \) of \( X \) at \( x_0 \) and \( \rho_0: (\pi_1 E_1)^{\otimes} \to E_0|_{U} \) be given. Then isomorphism classes of vector bundles \( \tilde{E} \) on \( \tilde{X} \) such that \( \pi_1 \tilde{E} \simeq E_0 \) and \( \tilde{E} \simeq E_1 \) in a neighbourhood of \( \pi^{-1}(x_0) \) are parameterized by the stalk of the skyscraper sheaf \( Aut(E_0)/\rho_0 \pi_1 Aut(E_1) \rho_0^{-1} \) at \( x_0 \), modulo the left action of \( \Gamma(X, Aut(E_0)) \).

**Remarks:**

1. If \( E_1 = \bigoplus O(a_i) \), the space \( Aut((\pi_1 E_1)^{\otimes})/\pi_1 Aut(E_1) \) is easily identified with the total space of some homogeneous vector bundle over a flag manifold. Two simple examples which will be of some relevance subsequently are the cases \( E_1 = O(1) \oplus \mathcal{O}^{-r} \) and \( E_1 = O(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{-r-2} \), for which the corresponding skyscrapers are respectively \( F_1(C^r) \) and the total space of the bundle \( 2O(1, 1) \) over \( F_{1,r-1}(C^r) \). (the zero section corresponds to those bundles which extend to \( \widetilde{P}_2 \) as a direct sum of line bundles).

2. This description provides a simple way to construct bundles on \( \tilde{X} \) from bundles on \( X \), but in contrast with the construction of Serre/Schwarzenberger the bundles produced this way are all non-trivial on the exceptional divisor. However, if \( \sum a_i = 0 \) the generic deformation of the bundle \( \tilde{E} \) (or \( E_1 \)) will be trivial.
on $L$, and the earlier construction can be seen as a deformation of a bundle restricting to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $L$; (the existence of such deformations is discussed further in §4).

An effective description of the spaces of holomorphic bundles on $\tilde{X}$ requires such a description for the spaces of bundles in a neighbourhood of $L_0$, but that given by Proposition 2.5 has some redundancy: the filtration (2.6) is not uniquely determined. However, by gluing a bundle on a neighbourhood of $L_0$ to the trivial bundle on $\mathbb{P}_2$ the classification problem becomes that of determining the bundles on $\mathbb{P}_2$ which are trivial in a neighbourhood of the line $L_\infty$ at infinity. A trivialisation of such a bundle in a neighbourhood of $L_\infty$ is determined by its restriction to $L_\infty$, so isomorphism classes of pairs $(E_1, \rho)$ where $\rho$ is a trivialisation of $(\pi_*E_1)^{**}$ in a neighbourhood of $x_0$ correspond to isomorphism classes of pairs $(\tilde{E}, \varphi)$ where $\tilde{E}$ is a bundle on $\mathbb{P}_2$ such that $(\pi_*\tilde{E})^{**}$ is trivial and $\varphi$ is a trivialisation of $\tilde{E}$ on $L_\infty$.

The space $\mathbb{P}_2$ is isomorphic to the Hirzebruch surface $H_1$ and bundles on this space have been studied in [B1]. Using the lemma of §1 of that reference, a monad description of all holomorphic bundles on $H_1$ trivial on $L_\infty$ is easily given, and the precise condition on such monads for the corresponding bundles to be trivial in a neighbourhood of $L_\infty$ is easily calculated; ([B5]).

3. Structure of moduli spaces I.

The “cut-and-paste” construction of the previous section makes no reference to questions of stability, an issue which is considered in this section. When the discussion is not limited to a single bundle but rather to whole moduli spaces, the results generally apply only in the case that $b_1(X)$ is even; the reason for this can be traced to the failure of Lemma 1.8 when $b_1(X)$ is odd.

The definition of stability requires a hermitian metric, and throughout this section such a metric (positive (1,1)-form) $\omega$ is a fixed on $X$. The metrics to be used on blowups of $X$ are the same as those used in [B2], the construction of which will be briefly recalled here for convenience.

Let $\tilde{X} \to X$ be the blowup of $X$ at $x_0 \in X$. Let $L := \pi^{-1}(x_0)$ be the exceptional divisor so $\pi^*\omega$ is everywhere non-negative and is degenerate only in directions tangent to $L$. Let $\sigma$ be $i/2\pi$ times the curvature form of any hermitian connection on the line bundle $\mathcal{O}(-L) =: \mathcal{O}(1)$ restricting positively to $L$, and let $\omega_c := \pi^*\omega + \sigma$ for $\epsilon > 0$; (recall $\mathcal{O}(L)|_{L_\infty} \cong \mathcal{O}_L(-1)$). It follows that if $\epsilon$ is sufficiently small then $\omega_c$ defines a positive form in a neighbourhood of $L$; if $\sigma$ is compactly supported in $\tilde{X}$ then $\omega_c$ is everywhere positive for sufficiently small $\epsilon$. If $\omega$ is $\partial\bar{\partial}$-closed and $\sigma$ is compactly supported, it follows from the fact that $L$ has self-intersection $-1$ that $Vol(\tilde{X}, \omega_c) = Vol(X, \omega) - \epsilon^2/2 = V - \epsilon^2/2$, and if $\omega$ is $d$-closed, then so too is $\omega_c$.

A useful model to keep in mind is the following: if $x_0$ corresponds to the origin in local holomorphic coordinates $\{z^\alpha\}$, the orientation-reversing map $z^\alpha \mapsto z^\alpha/|z|^2$ lifts to the blowup to define an isomorphism of a neighbourhood of $L$ with a neighbourhood of a line in $\mathbb{C}P_2$, realising $\tilde{X}$ as the connected sum $\tilde{X} \simeq_{diffeo} X \# \mathbb{C}P_2$. Under this diffeomorphism, the pullback of $\omega_1 = (i/2)\partial\bar{\partial}(|z|^2 + \log |z|^2)$ is conformal to the Fubini-Study metric. The form $\omega_d$ can be taken to be $(i/2)\partial\bar{\partial}\log(\psi(|z|^2))$ where $\psi(t)$ is a smooth function which is the identity near 0 and a positive constant for $t \geq t_0$. Pulling back under the “dilations” $z \mapsto \epsilon^{-1/2}z$ and rescaling by $\epsilon$ gives the metric $\omega_\epsilon$, “stretching out” the neck of the connected sum as in [D3].

Now let $\tilde{X} \to X$ be a modification of $X$ consisting of $n$ successive blowups (at simple points), and let $\sigma_i$ be a closed smooth (1,1)-form on $\tilde{X}$ corresponding as in the last paragraph to the $i$-th blowup. Let $\mathbb{R}_+^n := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid \alpha_i > 0, \ i = 1, \ldots, n\}$ and for $\alpha \in \mathbb{R}_+^n$ let $\rho_\alpha := \sum \alpha_i \sigma_i$, so $\rho_\alpha, \rho_0 = -\sum \alpha_i^2 = -|\alpha|^2$ and $\omega_\alpha := \pi^*\omega + \rho_\alpha$ is positive for $|\alpha|$ sufficiently small; (this definition differs slightly from that in [B2] where $\rho$ has the opposite sign). A vector $\alpha \in \mathbb{R}_+^n$ is called suitable if $\omega_\alpha$ is a positive form on $\tilde{X}$.

Let $\tilde{X} \to X$ be a blowup of the compact surface $X$, equipped with a metric of the form $\omega_\alpha$ as above. Let $\tilde{E}$ be a holomorphic bundle on $\tilde{X}$, and let $E \subset \tilde{E}$ be a bundle of rank $\alpha$ included in $\tilde{E}$ as a subsheaf. By definition of $\nu_\alpha^{\bullet}(\bullet)$,
\begin{equation}
\nu_\alpha^{\bullet}(\tilde{A}, \omega_\alpha) = \nu_{\pi_*\tilde{E}}(\pi_*\tilde{A}) + \rho_\alpha \left[ \frac{\alpha}{r} c_1(\tilde{E}) - c_1(\tilde{A}) \right].
\end{equation}

If $\tilde{A}$ has torsion-free quotient $\tilde{C}$ of rank $c > 0$ then (1.2) and (1.4) give
\begin{align}
C(\tilde{E}) = C(\tilde{A}) + C(\tilde{C}) \\
+ \frac{r}{2ac} \left[ \left\| \frac{\alpha}{r} c_1(\pi_*\tilde{E}) + c_1(\pi_*\tilde{A}) \right\|^2 + \left\| \alpha \right\|^2 \right] Q^{-1} \nu_{\pi_*\tilde{E}}(\pi_*\tilde{A})^2,
\end{align}

\( Q = V - \epsilon^2/2 \).
where \( x \|_q^2 := -(x - \pi^*x) \cdot (x - \pi^*x) \) for \( x \in H^2(\bar{X}, \mathbb{Q}) \).

Suppose now that \( \bar{E} \) is semi-stable but not stable with respect to \( \omega_{\alpha} \), and that \( \bar{A} \) destabilises \( \bar{E} \). Then from (3.1) it follows that \( \nu_{\pi, \bar{E}}(\pi_*\bar{A}) = -\rho \cdot \frac{1}{\pi} c_1(\bar{E}) - c_1(\bar{A}) \) so (3.2) implies

\[
C(\bar{E}) \geq C(\bar{A}) + C(\bar{C}) + \frac{r}{2\alpha c} \left[ \frac{a c_1(\pi_*\bar{E}) - c_1(\pi_*\bar{A})}{\alpha} + (1 - V^{-1} |\alpha|^2) \right] c_1(\bar{E}) - c_1(\bar{A}) \left| q \right|^2 .
\]

(3.3)

Since \( \bar{A} \) destabilises \( \bar{E} \) for the metric \( \omega_{\alpha} \), it follows from the semi-stability of \( \bar{E} \) that both \( \bar{A} \) and \( \bar{C} \) are also semi-stable (with respect to this metric), implying \( C(\bar{A}), C(\bar{C}) \) are non-negative. Hence (3.3) yields a uniform bound on \( \| (a/r) c_1(\bar{E}) - c_1(\bar{A}) \| \) which involves only \( C(\bar{E}) \) and \( r \) (if \( |\alpha| \) is suitably bounded from above).

With these preparations in hand, the following result summarises most of the important relationships between stability on \( X \) and \( \bar{X} \). Parts of it appear in [FM2] (Theorem 5.5) and [Br] (Theorem 4) in the case of bundles of rank 2.

**Proposition 3.4.** Let \( \bar{E} \) be an \( r \)-bundle on \( \bar{X} \).

(a) If \( \bar{E} \) is \( \omega_{\alpha} \)-stable for all sufficiently small \( \epsilon > 0 \) then \( \pi_*\bar{E} \) is \( \omega \)-semi-stable;

(b) If \( \bar{E} = \pi^*E \) for some bundle \( E \) on \( X \) and \( \bar{E} \) is \( \omega_{\alpha} \)-stable, then \( E \) is \( \omega \)-stable;

(c) If \( \bar{E} \) is \( \omega_{\alpha} \)-semi-stable and \( \pi_*\bar{E} \) is \( \omega \)-semi-stable, then \( \bar{E} \) is \( \omega_{\alpha} \)-semi-stable for all \( \epsilon \) in \( (0, 1] \);

(d) If \( \pi_*\bar{E} \) is stable, then \( \bar{E} \) is \( \omega_{\alpha} \)-stable for all suitable \( \alpha \in \mathbb{R}^+ \) sufficiently small. In fact, if \( \nu_{\pi, \bar{E}}(\bar{A}) \geq \delta > 0 \) for all \( A \subset \pi_*\bar{E} \) with non-zero torsion-free quotient, then \( \bar{E} \) is \( \omega_{\alpha} \)-stable for all suitable \( \alpha \in \mathbb{R}^+ \) such that \( |\alpha| < \delta \sqrt{\frac{2V}{3\delta^2 + rVC(\bar{E})}} \).

**Proof:**

(a) By Lemma 2.3, after twisting \( \bar{E} \) with a suitable line bundle it can be supposed that \( \pi_*\bar{E} \) is locally free. If \( A \subset \pi_*\bar{E} \) has torsion-free quotient, then \( \pi^*A \subset \pi^*\pi_*\bar{E} \subset \bar{E} \). If \( \bar{A} \) is the maximal normal extension of \( \pi^*A \) in \( \bar{E} \) then \( \mu(\pi^*A, \omega_{\alpha}) \leq \mu(\bar{A}, \omega_{\alpha}) \). Replacing \( \alpha \) by \( \alpha \) in (3.1) and letting \( \epsilon \to 0 \) gives \( \nu_{\pi, \bar{E}}(\bar{A}) \geq 0 \).

(b) If \( A \subset \pi_*\bar{E} \) has torsion-free quotient and \( \bar{A} \subset \bar{E} \) is the maximal normal extension of \( \pi^*A \) in \( \bar{E} = \pi^*E \), then \( \mu(\bar{A}, \omega_{\alpha}) \leq \mu(\bar{A}, \omega_{\alpha}) < \mu(\bar{E}, \omega_{\alpha}) \). A similar argument to (a) gives \( \nu_{\pi, \bar{E}}(\bar{A}) \geq 0 \).

(c) Suppose \( A \subset \pi_*\bar{E} \) has torsion-free quotient. If \( \nu_{\pi, \bar{E}}(\bar{A}, \omega_{\alpha}) < 0 \) for some \( \epsilon \) in \( (0, 1] \) then since \( \nu_{\pi, \bar{E}}(\pi_*\bar{A}) \geq 0 \) by hypothesis it would follow that \( \epsilon |\alpha| \cdot \frac{1}{\pi} c_1(\bar{E}) - c_1(\bar{A}) \geq 0 \); this would imply \( \nu_{\pi, \bar{E}}(\bar{A}, \omega_{\alpha}) < 0 \) also, contradicting the hypotheses. If \( \bar{A} \) destabilises \( \bar{E} \) for \( \omega_{\alpha} \) then both \( \nu_{\pi, \bar{E}}(\pi_*\bar{A}) \) and \( \rho_\alpha \cdot \frac{1}{\pi} c_1(\bar{E}) - c_1(\bar{A}) \) must be zero. Otherwise, one must be strictly positive and therefore \( \bar{E} \) must be strictly stable with respect to \( \omega_{\alpha} \) for all \( \epsilon \) in \( (0, 1] \).

(d) If \( A \subset \pi_*\bar{E} \) has torsion-free quotient, the first Chern class of \( \bar{A} \) restricted to any irreducible component of the exceptional divisor is bounded above by a constant depending on the maximum of the first Chern classes of the line bundles in the decomposition of \( \bar{E} \) on that component. The proof of Lemma 5 of [B2] now applies to show that if \( \bar{E} \) has at least one non-trivial subsheaf, then there exists such a subsheaf \( \bar{A} \) with torsion-free quotient \( C \) which maximizes \( \mu(\bar{A}, \omega_{\alpha}) \) for any sufficiently small \( \epsilon > 0 \). It then follows from (3.1) that \( \bar{E} \) is stable with respect to \( \omega_{\alpha} \) for all \( \epsilon \) sufficiently small.

If \( \alpha \) is suitable and satisfies the inequality of (d) and \( \bar{E} \) is not \( \omega_{\alpha} \)-stable, then since stability is an open condition on the metric there exists \( \epsilon \) in \( (0, 1] \) such that \( \bar{E} \) is \( \omega_{\alpha} \)-semi-stable but not stable. Then \( \delta \leq \nu_{\pi, \bar{E}}(\bar{A}) = -\rho \cdot \frac{1}{\pi} (a/r) c_1(\bar{E}) - c_1(\bar{A}) \leq |\alpha| \| c_1(\bar{E}) - c_1(\bar{A}) \| q \). Replacing \( \alpha \) by \( \alpha \) in (3.3) gives the bound \( \| c_1(\bar{E}) - c_1(\bar{A}) \| q \leq rVC(\bar{E})/2(2V - 2\alpha^2) \), which gives the contradiction \( \delta < \delta \) after a little algebra.

A simple corollary of the proposition is that if \( \bar{E} = E_0 \#_\rho E_1 \) is a bundle on \( \bar{X} \) constructed by the gluing construction of the last section and if \( E_0 \) on \( X \) is \( \omega \)-stable, then \( \bar{E} \) is \( \omega_{\alpha} \)-stable for all suitable \( \alpha \in \mathbb{R}^+ \) sufficiently small. The gluing construction, combined with the existence of Hermitian-Einstein connections on stable bundles, can thus be viewed as a holomorphic interpretation of Donaldson’s “connected sums of connections” theorem [D3] where one of the summands is (a connected sum of copies of) \( \mathbb{F}_2 \).

By Lemma 1.8, Proposition 2.9 and part (d) of Proposition 3.4, when \( b_1(X) \) is even it follows that for any bundle \( \bar{E} \) on \( \bar{X} \) of rank \( r \) and charge \( \leq C_0 \) such that \( \pi_*\bar{E} \) is stable, \( \bar{E} \) is \( \omega_{\alpha} \)-stable for all suitable \( \alpha \in \mathbb{R}^+ \).
satisfying a uniform bound independent of $E$. In particular, the pull-backs from $X$ of stable bundles of bounded ranks and charge are all stable with respect to the same metrics on the blowup. The restriction that $b_1(X)$ be even gives the following strengthening of Proposition 3.4:

**Proposition 3.5.** Suppose that $b_1(X)$ is even. For any $r_0, C_0 > 0$ there exists $\epsilon_0 = \epsilon_0(r_0, C_0, \omega)$ with the property that any bundle on a blowup $\bar{X}$ of $X$ of rank $\leq r_0$ and charge $\leq C_0$ which is stable with respect to $\omega_\alpha$ for some suitable $\alpha \in \mathbb{R}_+^n$ satisfying $|\alpha_0| < \epsilon_0$ is stable with respect to $\omega_\alpha$ for all $\alpha \in (0, \epsilon_0/|\alpha|)$. Moreover, any bundle which is semi-stable with respect to $\omega_\alpha$ and has semi-stable direct image is semi-stable with respect to $\omega_\alpha$ for all $\alpha \in (0, \epsilon_0/|\alpha|)$.

**Proof:** Set $\epsilon_1 := \delta_0 \sqrt{2V/(2\delta_0^n + rVC(\bar{E}))}$ where $\delta_0$ is as in Lemma 1.8. Suppose that there exists suitable $\alpha_1 \in \mathbb{R}_+^n$ with $|\alpha_1| < \epsilon_1$ and a bundle $\bar{E}_1$ of rank $r_1 \leq r_0$ on $\bar{X}$ with $C(\bar{E}_1) \leq C_0$ which is $\omega_\alpha_1$-stable but which is not $\omega_{\delta_1\alpha_1}$-stable for some $\delta_1 \in (0, \epsilon_1/|\alpha_1|)$, where $\delta_1\alpha_1$ is suitable. Since stability is an open condition on the metric, by altering $\delta_1$ if necessary it can be supposed that $\bar{E}_1$ is semi-stable but not stable with respect to $\omega_{\delta_1\alpha_1}$, and hence that there exists $A_1 \subset \bar{E}_1$ destabilising as in the discussion preceding Proposition 3.4. Set $\epsilon_2 := (1/2)\min\{|\alpha_1|, \delta_1|\alpha_1|\}$ and repeat this procedure, generating a sequence $\{(\epsilon_i, \alpha_i, \bar{E}_i, A_i, \delta_i)\}$ by iteration. By construction,

$$\nu_{\pi_*\bar{E}_i}(\pi_*\bar{A}_i) + \rho_{\alpha_i} \cdot [(a_i/r_i)c_1(\bar{E}_i) - c_1(\bar{A}_i)] > 0$$  

and

$$\nu_{\pi_*\bar{E}_i}(\pi_*\bar{A}_i) + \delta_i \rho_{\alpha_i} \cdot [(a_i/r_i)c_1(\bar{E}_i) - c_1(\bar{A}_i)] = 0,$$

with $\delta_1|\alpha_1| < \epsilon_i < \epsilon_{i-1}/2$. Since $1 \leq a_i \leq r_i \leq r_0$ and $C(\bar{E}_i) \leq C_0$ the uniform bound on the norms $\| (a_i/r_i)c_1(\bar{E}_i) - c_1(\bar{A}_i) \|$ provided by (3.3) implies that there is a subsequence for which $(a_i/r_i)c_1(\bar{E}_i) - c_1(\bar{A}_i)$ is constant. Since $b_1(X)$ is even, $\nu_{\pi_*\bar{E}}$ is topological and therefore constant on this subsequence. If the subsequence is infinite, then $\epsilon_i \to 0$, so (3.6)(b) implies $\nu_{\pi_*\bar{E}}(\pi_*\bar{A}_i)$ is eventually 0 on this subsequence and therefore so too is $\rho_{\alpha_i} \cdot [(a_i/r_i)c_1(\bar{E}_i) - c_1(\bar{A}_i)]$; this however contradicts (3.6)(a). Thus the subsequence must be finite, implying the original sequence terminated, which in turn implies the first statement of the proposition.

To prove the second statement, suppose that $\bar{E}$ is $\omega_\alpha$-semi-stable for some suitable $\alpha \in \mathbb{R}_+^n$ with $|\alpha| < \epsilon_0$, and that $\pi_*\bar{E}$ is semi-stable. If $\bar{E}$ is $\omega_\alpha$-stable then the first part of the proposition applies, so it can be assumed that $\bar{E}$ is not stable. If $\bar{A} \subset \bar{E}$ has torsion-free quotient and destabilises $\bar{E}$ with respect to $\omega_\alpha$, then $\nu_{\pi_*\bar{E}}(\pi_*\bar{A}) = -\rho_{\alpha} \cdot [(a/r)c_1(\bar{E}) - c_1(\bar{A})] / \omega_\alpha$ is not zero. Since $\pi_*\bar{E}$ is semi-stable, the condition $\| (a/r)c_1(\bar{E}) - c_1(\bar{A}) \|_Q < \delta_0$ by construction of $\epsilon_0$ and by (3.3), so by Lemma 1.8 it follows $\nu_{\pi_*\bar{E}}(\pi_*\bar{A}) = 0 = \rho_{\alpha} \cdot [(a/r)c_1(\bar{E}) - c_1(\bar{A})]$ for all $\alpha$. $\square$

In general, it is not the case that the moduli spaces of $\omega_\alpha$-stable holomorphic structures on a given topological bundle over $X$ are independent of $\alpha \in \mathbb{R}_+^n$ once $|\alpha|$ is sufficiently small. If $\bar{E}$ is a bundle on $\bar{X}$ of rank $\leq r_0$ and charge $\leq C_0$ which is stable with respect to $\omega_\alpha$ but not stable with respect to $\omega_\beta$ for $|\alpha|, |\beta| \leq \epsilon_0$, then it follows easily as in the proof of Proposition 3.5 that there exists $c \in H^2(X, \mathbb{Z})^\perp \subset H^2(\bar{X}, \mathbb{Z}) \cap H^{1,1}(\bar{X})$ with $\|c\|_Q \leq \sqrt{\rho_0VC_0/(2\delta_0^n - 2\delta_0^n)}$ such that $\rho_\alpha \cdot c > 0$ and $\rho_\beta \cdot c \leq 0$, namely $c = (ac_1(\bar{E}) - rc_1(\bar{A})) - (a\rho_\alpha c_1(\pi_*\bar{E}) + r\rho_\alpha c_1(\pi_*\bar{A}))$ for some destabilising $\bar{A} \subset \bar{E}$. The moduli spaces will be independent of suitable $\alpha \in \mathbb{R}_+^n$ satisfying $|\alpha| < \epsilon_0$ provided that $\alpha$ remains within one of the finitely many chambers of $\mathbb{R}_+^n$, cut out by the equations $\rho_\alpha \cdot c = 0$ for $c \in H^2(X, \mathbb{Z})^\perp$ with $\|c\|_Q \leq \sqrt{\rho_0VC_0/(2\delta_0^n - 2\delta_0^n)}$. Such a “chamber structure” for moduli spaces is quite well-known—see, e.g., [D4], [K2].

Moduli spaces also depend non-trivially on $|\alpha|$ in general: it is not hard to construct examples of bundles on a blowup $\bar{X}$ which are stable with respect to $\omega_\alpha$ for some suitable $\alpha$, but which fail to be stable with respect to $\omega_\alpha$ for some $\epsilon \in (0, 1)$.

4. Stabilisation and desingularisation.

The appearance of sheaves and bundles which are semi-stable but not stable represents a divergence between the real analytical and the complex analytical descriptions of moduli: whereas isomorphism classes of stable bundles and irreducible Hermitian-Einstein connections are in one-to-one correspondence, this fails
to be true as soon as stable is replaced by semi-stable: for example, if $A$ and $C$ are stable bundles and $\mu(A) = \mu(C)$, then any extension of the form $0 \to A \to E \to B \to 0$ defines a bundle $E$ with $\mu(E) = \mu(C)$ and which is always semi-stable but not stable. The bundle $E$ admits a Hermitian-Einstein connection if and only if the extension splits. In this section, the gluing construction of §2 is used to provide a mechanism for “stabilising” a semi-stable bundle (or torsion-free sheaf). The methods, which are strictly sheaf-theoretical, can also be used to by-pass some of the technical difficulties encountered in [D5] to show that moduli spaces of stable bundles of sufficiently large charge on a blowup of a surface have open sub-sets of which are smooth; this is indicated in the second half of the section. Finally, the same methods are used to show that bundles on a blowup which are topologically trivial on the exceptional divisor can be approximated (off a finite set) by bundles which are holomorphically trivial on the divisor.

In general, a semi-stable sheaf $S$ on the compact surface $X$ determines a semi-stable bundle $\Sigma(S)$ admitting a Hermitian-Einstein connection as follows: $\Sigma(S) := \Sigma(S^*)$ and if $A \subset S$ has $\mu(A) = \mu(S)$, then $\Sigma(S) := \Sigma(A) \oplus \Sigma(S/A)$. It is straightforward to verify by induction on rank that this prescription is well-defined and uniquely determines the bundle $\Sigma(S)$. This bundle has the same rank and determinant as $S$ and never has greater charge; it is a direct sum of stable bundles all of the same slope (i.e., is quasi-stable), and there are non-zero holomorphic maps $S \to \Sigma(S)$, $\Sigma(S) \to S^*$.

It is also convenient to introduce the notation $B(E)$ for a semi-stable bundle $E$ to denote the set of points $x \in X$ for which there is a semi-stable bundle $A$ with $\mu(A) = \mu(E)$ and a sheaf inclusion $A \to E$ such that $A_x \to E_x$ not of maximal rank; (note that the quotient $E/A$ must be torsion-free else semi-stability of $E$ will be violated by the maximal normal extension of $A$ in $E$). Again, it is easily verified by induction on the rank of $E$ that $B(E)$ is finite.

Let $E$ be a semi-stable $r$-bundle on $X$, and let $A$ be a semi-stable $a$-bundle with $\mu(A) = \mu(E)$ for which there is a map $A \to E$ inducing a sheaf inclusion. Pick a point $x_0 \in X \setminus B(E)$ and let $\bar{X} \to X$ be the blowup of $X$ at $x_0$. If $E_1$ is any $r$-bundle on a neighbourhood of $L := \pi^{-1}(x_0)$ and $\rho: E \to (\pi_*E_1)^\ast$ is an isomorphism over this neighbourhood, then if $E$ is in fact stable it follows from Proposition 3.4 that the bundle $\bar{E} := E\#\pi_1$ is stable with respect to $\omega$, for all sufficiently small $\epsilon > 0$; the more delicate and interesting case is when $E$ is not stable which is henceforth assumed.

Now take $E_1 = O(1) \oplus O(-1)$. Up to isomorphism, the bundle $\bar{E}$ is determined by a non-zero element, $\varphi$ say, of the vector space $E_{x_0}$, with any non-zero multiple giving an isomorphic bundle: the correspondence is given explicitly by taking direct images of the sequence $0 \to \bar{E} \to \bar{E}(1) \to \bar{E}(1)L \to 0$ and using the fact that $\pi_*\bar{E}(1)$ is locally free, hence equal to $E$.

Let $\tilde{A} \subset E$ have torsion-free quotient $\tilde{C}$ and satisfy $\mu(\pi_*\tilde{A}) = \mu(E)$, and consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \pi_*\tilde{A} & \to & \pi_*\tilde{E} & \to & \pi_*\tilde{C} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\pi_*\tilde{A})^\ast & \to & (\pi_*\tilde{E})^\ast & \to & (\pi_*\tilde{C})^\ast & \to & 0 \\
\end{array}
\]

Since $\pi_*\tilde{E} = 0$, the cokernel of $\pi_*\tilde{E} \to \pi_*\tilde{C}$ is $\pi_*\tilde{A}$. Since $x_0 \not\in B(E)$, $(\pi_*\tilde{A})^\ast \subset (\pi_*\tilde{E})^\ast = E$ is a sub-bundle at $x_0$ and the lower row must be exact near there, implying that $(\pi_*\tilde{A})^\ast/\pi_*\tilde{A} \to (\pi_*\tilde{E})^\ast/\pi_*\tilde{E}$ is injective and that $(\pi_*\tilde{E})^\ast/\pi_*\tilde{E} \to (\pi_*\tilde{C})^\ast/\pi_*\tilde{C}$ is surjective.

If the kernel of $\varphi: E_{x_0} \to \tilde{C}$ does not contain the image of $(\pi_*\tilde{A})^\ast$, then the restriction of $\varphi$ to $(\pi_*\tilde{A})^\ast$ is non-zero and since $(\pi_*\tilde{E})^\ast/\pi_*\tilde{E} = \mathbb{C}$, this implies that $(\pi_*\tilde{A})^\ast/\pi_*\tilde{A} = \mathbb{C}$ also. It follows in this case that $\pi_*\tilde{A} = 0$ and $\pi_*\tilde{C} = (\pi_*\tilde{C})^\ast$ implying (since $\pi_*\tilde{C} = 0$) that $\tilde{C}$ is locally free near $L$, and by Lemma 2.3, that if $\tilde{C}|_L = \sum O(c_i)$ then $\sum c_i \leq 0$. Since $E_1|_L \to \tilde{C}|_L$ is onto, all $c_i$ must be non-negative, so $c_i = 0$ for all $i$, and $\tilde{A}|_L = O(1) \oplus \sum O$.

On the other hand, if the kernel of $\varphi$ does contain the image of $(\pi_*\tilde{A})^\ast$ then $(\pi_*\tilde{A})^\ast$ and there is an exact sequence $0 \to \pi_*\tilde{A} \to C \to (\pi_*\tilde{C})^\ast/\pi_*\tilde{C} \to 0$. Since $\pi_*\tilde{A}$ is locally free, Lemma 2.3 implies that near $L$, $\det \tilde{A} = O(a)$ for some $a \leq 0$, and since $\det \tilde{E} = O(1)$, the same lemma implies that $\pi_*\tilde{C}$ cannot be locally free near $x_0$. In the exact sequence $0 \to \pi_*\tilde{E}(\tilde{C}^\ast/\tilde{C}) \to (\pi_*\tilde{C})^\ast/\pi_*\tilde{C} \to (\pi_*\tilde{C}(\tilde{C}^\ast))\ast/\pi_*\tilde{C}(\tilde{C}^\ast) \to 0$ the last term is therefore non-zero, implying the same of the middle term. Thus from the previous exact sequence it now follows that $C = \pi_*\tilde{C}(\tilde{C}^\ast)^\ast/\pi_*\tilde{C} \to (\pi_*\tilde{C}(\tilde{C}^\ast))\ast/\pi_*\tilde{C}(\tilde{C}^\ast) \to 0$.

Since $\tilde{A} \to \tilde{E}$ is now a bundle map, it follows that if $\tilde{A}$ splits on $L$ as $\sum O(a_i)$, then all $a_i$ must be $\leq 1$. Since $\pi_*\tilde{A} = 0$, all $a_i$ must also be $\geq -1$, so by Corollary 2.7 it follows that $\tilde{A}$ splits as $\sum O(a_i)$ in a neighbourhood of $L$. Since $\pi_*\tilde{A}$ is locally free, it must therefore be the case that all $a_i$ are in fact $\leq 0$. Now
replace the top row of the diagram above with the exact sequence $0 \to \pi_xA(-1) \to \pi_xE(-1) \to \pi_xC(-1)$, the cokernel of the last arrow being $\pi_x^1A(-1)$. Since the middle term is now locally free, it follows that the kernel of $\pi_xC(-1) \to \pi_x^1A(-1)$ is isomorphic to $(\pi_xC(-1))^\vee$, so $\pi_xC(-1)$ is locally free and $\pi_x^1A(-1) = 0$. Consequently $A$ is trivial and $\bar{C} = O(1) \oplus \sum O$ near $L$ in this case.

Thus the quotient $\bar{C}$ is always locally free near $L$ and $\bar{A}|_L = \sum O$ or $O(1) \oplus \sum O$ according to whether the kernel of $\varphi$ does or does not contain the image of $(\pi_x^1A)^\vee$ respectively. If it does not, then $\mu(A, \omega) = \mu(\pi_xA) = \alpha/a = \mu(E) = \alpha/a < \mu(E) = \alpha/r = \mu(\bar{E}, \omega_\alpha)$, so $\bar{E}$ would be stable for all $\alpha > 0$ sufficiently small if every such $\bar{A}$ could be guaranteed to fall into the second category.

In general, it is not be possible guarantee that $(\pi_xA)^\vee$ should not be contained in the kernel of $\varphi$. However, the following somewhat technical lemma shows that if the construction is repeated at a number of points, a uniform upper bound on the number of such “bad” points can be given:

**Lemma 4.1.** Suppose $x_1, \ldots, x_n \in X \setminus B(E)$. Then there are linear maps $\varphi_i : E_{x_i} \to \mathbb{C}$, $i = 1, \ldots, n$ with the following property: for any stable bundle $A$ on $X$ of rank $a$ with $\mu(A) = \mu(E)$ and any non-zero map $A \to E$, the number of maps $\varphi_i$ for which $A_{x_i} \subset \ker \varphi_i$ is at most $m - 1$, where the multiplicity of $A$ in $\Sigma(E)$ is $ma - c$, $0 \leq c < a$.

**Proof:** By induction on the rank $r$ of $E$. Without loss of generality, there is a semi-stable bundle $K$ with $\mu(K) = \mu(E)$ and a map $K \to E$ such that the quotient $B$ is a torsion-free stable sheaf of rank $0 < b < r$.

By the inductive hypotheses, maps $\varphi^K_i$ with the requisite properties exist for $K$.

Suppose first that the extension $0 \to K \to E \to B \to 0$ is non-trivial. If $A$ is a stable bundle of rank $a$ with $\mu(A) = \mu(E)$ and there is a non-zero map $A \to E$, then by stability of $A$ and $B$ the composition $A \to B$ is either 0 or an isomorphism, but the latter is ruled out by the assumption that the extension does not split. Thus any such $A$ maps into $K$ and since $\Sigma(E) = \Sigma(K) \oplus B^{**}$, any extensions of the maps $\varphi^K_i$ to $E_{x_i}$ will satisfy the requirements of the lemma.

Suppose now that the extension does split, so $B$ is in fact locally free. Fix maps $\varphi^K_i$ as above and choose maps $\varphi^B_i : B_{x_i} \to \mathbb{C}$; set $\varphi_i := \varphi^K_i + \varphi^B_i$. If $\text{Hom}(B, K) = 0$ then for any $A$ as in the last paragraph, the map $A \to B$ is either 0 or an isomorphism and $A \to K$ is then an inclusion or 0 respectively, so the conclusion of the lemma is again satisfied. If, on the other hand, there does exist a non-zero homomorphism from $B$ into $K$, let $mb - c$ be the multiplicity of $B$ in $\Sigma(K)$ for some integers $m, c$ with $0 \leq c < b$. By the inductive hypothesis, for any non-zero map $B \to K$ the image of $B$ is contained in kernels of at most $m - 1$ of the maps $\varphi^K_i$. Suppose then that for each choice of maps $\varphi^B_i : B_{x_i} \to \mathbb{C}$ there is a map $B \to K$ such that more than $m - 1$ of the maps $\varphi_i$ restricted to the image of $B$ are identically 0; let there be $m + k$ of them generically. Such a map $B \to K$ is uniquely determined since the non-zero difference between any two would give a map $B \to K$ whose images in $K_{x_i}$ would be contained in more than $m - 1$ of the subspaces $\ker \varphi^K_i$. Thus one obtains a map $\Psi : \bigoplus_{i=1}^n B^*_{x_i} \to \Gamma(X, \text{Hom}(B, K))$ (which is clearly linear) such that, for some $i_1, \ldots, i_{m+k}$, the compositions $\varphi^K_i \circ \Psi(\varphi^B_{i_1}, \ldots, \varphi^B_{i_{m+k}}) + \varphi^B_i$, $i = i_1, \ldots, i_{m+k}$ are all identically 0 as maps $B \to \mathbb{C}$. Equivalently, $\Psi$ can be viewed as a linear map $\bigoplus_{i=1}^n B^*_{x_i} \to K$ such that, after restriction to the points $\{x_i\}$, the composition $\text{End}(B_{x_i}) = B_{x_i} \oplus B^*_{x_i} \to K \to \mathbb{C}$ is just $-\text{trace}$, $i = i_1, \ldots, i_{m+k}$. If, for some $i$ the map $B \otimes B^*_{x_i} \to K$ annihilates every section restricting to a trace-free endomorphism at $x_i$, then it is easy to see that $B$ must have rank 1. Hence regardless of the rank of $B$ there is an inclusion $B \otimes \bigoplus_{i=1}^{m+k} B^*_{x_{i_j}} \to K$, giving an embedding of $(m+k)b$ copies of $B$ in $K$. Since the multiplicity of $B$ in $K$ is $mb - c$, it follows that $c = 0 = k$ and the multiplicity of $B$ in $E = K \oplus B$ is $mb + 1 = (m+1)b - (b - 1)$. Moreover, for generic $\varphi^K_i$, the homomorphisms $\varphi_i$ have the required property. \hfill $\Box$

Suppose that $E$ is as above and that $\Sigma(E) = \bigoplus V_i \otimes A_i$ where $V_i$ is a $d_i$-dimensional vector space and $A_i$ is a stable $a_i$-bundle with $\mu(A_i) = \mu(E)$ and with $A_i \not\cong A_j$ for $i \neq j$. Pick any $n$ points $x_1, \ldots, x_n \in X \setminus B(E)$ and glue in the bundle $E_1$ at each of these points in the generic fashion described by Lemma 4.1 to obtain a bundle $E$ on the blowup $\bar{X} \to X$ of $X$ at all $x_1, \ldots, x_n$. If $\sigma_i$ represents $-L_i$ let $\rho := \sum \sigma_i$ and for $\epsilon > 0$ take $\omega_i := \pi^*\omega + \epsilon\rho$.

Let $\bar{A} \subset E$ have rank $a$, have torsion-free quotient, and maximize $\mu(*, \omega)$ over the admissible subsheaves of $E$ by $\bar{A}$ for all $\epsilon > 0$ sufficiently small. Since $\mu(\bar{A}, \omega) = \mu(\pi_x\bar{A}) + \epsilon\rho - c_1(\bar{A})/\alpha$, letting $\epsilon$ go to 0 shows that $\mu(\bar{A}, \omega) = \mu(E)$. By the discussion preceding Lemma 4.1, $\sigma_i c_1(A)$ is either 0 or $-1$ according respectively to whether or not $(\pi_xA)^\vee$ is contained in the kernel of the linear form defining the gluing of $E$ to $E_1$ at $x_i$. By Lemma 4.1 itself, the former occurs at most $k := \max \{d_i/a_i \mid \leq r \}$ times. Since $\mu(\bar{E}, \omega) = \mu(E) - cn/r$, it follows that $\mu(\bar{A}, \omega_i) < \mu(\bar{E}, \omega_i)$ if $n > r\kappa$, in which case $\bar{E}$ is stable with respect to $\omega$ for all $\epsilon > 0$ sufficiently small.
To summarise:

**Proposition 4.2.** Let $E$ be a semi-stable $r$-bundle with $\Sigma(E) = \bigoplus_i V_i \otimes A_i$ where $V_i$ is a $d_i$-dimensional vector space and $A_i$ is a stable $a_i$-bundle with $\mu(A_i) = \mu(E)$ and with $A_i \neq A_j$ for $i \neq j$. If $n > r[\max\{d_i/a_i\}]$ then for any choice of $n$ points in $X$ there is a bundle $\tilde{E}$ on the blowup $\tilde{X} \to X$ of $X$ at these points such that $\tilde{E}$ restricts to $O(1) \oplus \bigoplus i = 1^{n-1} O$ on each component of the exceptional divisor, $(\pi^*E)^* = E$, and $\tilde{E}$ is stable with respect to $\omega_e := \pi^*\omega + \epsilon \sum_{i=1}^n \sigma_i$ for all $\epsilon > 0$ sufficiently small.

**Remarks:**

1. Since $\pi_*E(-1)$ is locally free, there is a map $\pi^*E \to \tilde{E}(-1)$ obtained by pulling-back and composing with the canonical map $\pi_\dual$ there is a map $E$ Chern class (and determinant) as

2. If $r = 1$ the entire procedure is considerably simplified. In particular, if $E_0$ is given by an extension $0 \to L_1 \to E_0 \to L_2 \to J \to 0$ for some line bundles $L_1$, $L_2$ with $deg(L_1) = deg(L_2)$ and some sheaf of ideals $J$ with $\mathcal{O}/J$ supported at a finite set, then if $L_1 \neq L_2$ or the extension does not split, it suffices to blow up at one point and take to be a non-trivial extension $0 \to \pi^*L_1(1) \to \tilde{E} \to \pi^*L_2 \otimes J \to 0$. If $E_0 = L_1 \otimes L_2$ then it suffices to blow up at 3 points and take a non-zero extension $0 \to \pi^*L_1(1,-1,1) \to \tilde{E} \to \pi^*L_1(0,0,0) \to 0$.

3. Instead of gluing-in the bundle $E_1 = \mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ to construct $\tilde{E}$, another natural choice is to glue in the bundle $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}^{r-2}$ (assuming of course that $r \geq 2$). The new bundle now has the same first Chern class (and determinant) as $E_0$, and charge one unit greater (as opposed to $(r-1)/2r$ units greater).

An analysis similar to that which was given above to prove Proposition 4.2 should be possible in this case, but the calculations are more involved and have not as yet been carried out.

As mentioned in the introduction to this section, the gluing construction also yields a simple way to by-pass the technical difficulties of [D5] which can occur because of potential singularity of moduli spaces of stable bundles.

For a stable $r$-bundle $E$ on the compact surface $(X,\omega)$ with Hermitian-Einstein connection $\nabla$, the cokernel of $\nabla^*: \Lambda^1(End_0 E) \to \Lambda^2(End_0 E)$ vanishes iff $H^2(X, End_0 E)$ does, where $End_0$ denotes trace-free endomorphisms. Thus $\nabla$ is a smooth point in the moduli space of irreducible Hermitian-Einstein connections iff $E$ is a smooth point in the moduli space of stable bundles. By Serre duality, $H^2(X, End_0 E) \cong H^0(X, End_0 E \otimes K_X^*)$ where $K_X$ is the canonical bundle of $X$. If $s \in H^0(X, End_0 E \otimes K_X)$ is a non-zero section, pick a point $x_0$ at which $s$ is not zero, and blowup $X$ at $x_0$. Now take $E_0 = E_1 = O(-1) \oplus O(-1) \oplus O^{r-2}$ and construct a bundle $\tilde{E} = E_0 \# E_1$ on $\tilde{X}$ as in §2. Since $K_X \cong (\pi^*K_X)(-1)$ ([BPV, Theorem I.9.1]), Lemma 2.3(a) implies that $\pi_\alpha(End_0 E \otimes K_X(-1))$ is locally free, and therefore it is isomorphic to $End_0 E \otimes K_X$ since it agrees with this sheaf away from $x_0$. The direct image of the sequence

$$End_0 \tilde{E} \otimes K_X 0 \to End_0 \tilde{E} \otimes K_X \to End_0 \tilde{E} \otimes K_X(-1) \to End_0 \tilde{E} \otimes K_X \otimes O_{\tilde{X}} \to 0$$

thus gives $0 \to \pi_*End_0 \tilde{E} \otimes K_X \to End_0 E \otimes K_X \to C_{x_0}$, and it is straightforward to check that the under the composition $H^0(X, End_0 E \otimes K_X) \cong H^0(\tilde{X}, End_0 \tilde{E} \otimes K_{\tilde{X}}(-1)) \to H^0(L_0, E \otimes K_{\tilde{X}}(-1)) \cong \mathbb{C}$ the section $s$ is not mapped to zero for generic $\rho$. For such $\rho$, it follows that the dimension of $H^2(\tilde{X}, End_0 \tilde{E})$ is one less than that of $H^2(X, End_0 E)$ so after performing this operation at enough points the bundle $\tilde{E}$ on $\tilde{X}$ will satisfy $H^2(\tilde{X}, End_0 \tilde{E}) = 0$. By semi-continuity of cohomology the same will be true for any bundle on $\tilde{X}$ sufficiently near $\tilde{E}$, and by Proposition 3.4 the bundle $\tilde{E}$ will be stable with respect to $\omega_e$ for all suitable $\alpha$ sufficiently close to 0.

More generally, if $L$ is any line bundle on $X$ the same technique shows that by blowing up at least $h^2((End_0 E) \otimes L)$ generic points in $X$, the generic bundle $\tilde{E} = E \# E_1$ on the blowup satisfies $H^2((End_0 \tilde{E}) \otimes \pi^*L) = 0$.

To summarise:

**Proposition 4.3.** If $E$ is an $r$-bundle on $X$ and $L$ is a holomorphic line bundle, then for any set $T$ of $n \geq h^2(X, (End_0 E) \otimes L)$ points in general position there is a blowup $\tilde{X}$ of $X$ centered at $T$ together with a bundle $\tilde{E}$ on $\tilde{X}$ restricting to $O(1) \oplus O(-1) \oplus O^{r-2}$ on each component of the exceptional divisor satisfying $(\pi_*E)^* = E$ and $H^2(\tilde{X}, (End_0 \tilde{E}) \otimes \pi^*L) = 0$. \qed
If $X'$ is a blowup of $X$ with exceptional divisor $D'$, and $E'$ is a bundle on $X'$ which is topologically trivial on $D'$, the cokernel of $H^1(X', \text{End}_0 E') \rightarrow H^1(D', \text{End}_0 E')$ is the kernel of the epimorphism $H^2(X', (\text{End}_0 E')(-D')) \rightarrow H^2(X', \text{End}_0 E')$. If the former group vanishes, every small deformation of $E'$ on $D'$ is induced by a small deformation of $E'$ on $X'$. Since every bundle on $\mathbb{P}_1$ with 0 first Chern class has arbitrarily small deformations which are holomorphically trivial, it follows in this case that $E'$ has arbitrarily small deformations which are holomorphic pull-backs from $X$.

If $\text{deg}(K_X, \omega) < 0$ and $E'$ is $\omega_\alpha$-stable for sufficiently small suitable $\alpha$, $\text{End}_0 E' \otimes K_X(D')$ has negative degree with respect to $\omega_\alpha$, and therefore $H^2(X', (\text{End}_0 E')(-D')) = 0$ by duality. If $\text{deg}(K_X, \omega) \geq 0$, applying Proposition 4.3 (blowing up points of $X' \setminus D'$) yields a blowup $\tilde{X}'$ of $X'$ with exceptional divisor $D$ together with a bundle $\tilde{E}'$ on $\tilde{X}'$ such that $H^2(\tilde{X}', (\text{End}_0 E')(-D')) = 0$. From the previous paragraph it follows that there are arbitrarily small deformations of $\tilde{E}'$ which restrict to holomorphically trivial bundles on $D'$. The behaviour of the deformations on $D$ cannot be controlled, but for sufficiently small perturbations of $\tilde{E}'$, the bundles must be either trivial or split as $O(1) \oplus O(-1) \oplus O^{-2}$ on each component by semi-continuity of cohomology.

5. Structure of moduli spaces II.

Let $X$ be a compact complex surface, equipped with a $\bar{\partial}\partial$-closed positive $(1,1)$-form $\omega$. Consider a sequence $\{E_i\}$ of stable holomorphic bundles of fixed topological type and degree, identified with a sequence $\{A_i\}$ of Hermitian-Einstein connections on a fixed unitary bundle. For this sequence of connections, the $L^2$ norms of the curvatures are uniformly bounded, so by Uhlenbeck’s theorem [Sed], [U1] there is a finite set $S \subset X$ and gauge transformations such that the gauge-transformed sequence (also denoted $\{A_i\}$) converges weakly in $L^2_{1,\text{loc}}(X, \mathcal{S})$ to a connection $\mathcal{A}$ defining a finite-action Hermitian-Einstein connection. Ellipticity of the Hermitian-Einstein equations combined with Donaldson’s argument in the proof of Corollary 23 of [D2] shows that a subsequence is converging weakly in $L^p_{1,\text{loc}}(X, \mathcal{S})$ for any $p$ and by bootstrapping and diagonalisation, that a subsequence converges strongly in $\mathcal{C}^k$ on compact subsets of $X \setminus S$ for any $k$. Since a Hermitian-Einstein connection can be twisted locally by a Hermitian-Einstein connection on a trivial bundle so that the new connection has $\lambda = 0$, it follows from the Removability of Singularities theorem [U2] that the limit extends across $S$ to define a new Hermitian-Einstein connection on a bundle over $X$, and therefore a new semi-stable bundle $E$ with $\Sigma(E) = E$. The new holomorphic bundle $E$ has the same rank and first Chern class as the bundles in the sequence, its determinant is the limit of the determinants, but its charge is at least one less for each point in $S$ where the curvature has “bubbled”.

Following [D5], this type of “convergence” for sequences of connections is referred to as weak convergence (on $X \setminus S$), and sequences of stable holomorphic bundles of the same degree and topological type converge weakly (on $X \setminus S$) (with respect to $\omega$) if the corresponding irreducible Hermitian-Einstein connections converge weakly.

In dealing with limits of sequences of stable bundles, arguments are greatly simplified whenever it is known that a weak limit is itself stable, rather than just semi-stable. The stabilisation construction given in the previous section is designed to meet this type of need, and when combined with Lemma 2.2 of [B4] (semi-continuity of cohomology on $H^0$ for weak limits), the upshot is Lemma 5.1 below. Recall that the notation $\Lambda F$ denotes $*(\omega \wedge F)$ and if $A$ is a connection on a unitary bundle and $g$ is a complex automorphism (an intertwining operator) on that bundle, $g \cdot A$ is the connection with $(0,1)$ part $\partial_g \cdot A = g \circ \partial_A \circ g^{-1}$ and $(1,0)$ part $\partial_{\bar{g}} \cdot A = (g^*)^{-1} \circ \partial_A \circ g^*$.

**Lemma 5.1.** Let $\{E_i\}$ be a sequence of stable bundles on $X$ corresponding to a sequence $\{A_i\}$ of Hermitian-Einstein connections on a fixed $U(r)$-bundle and let $S \subset X$ be a finite set such that the sequence converges weakly on $X \setminus S$ to a Hermitian-Einstein connection $\mathcal{A}$ defining a quasi-stable bundle $E$. If $(\tilde{X}, \tilde{E})$ is a stabilisation of $E$ with the none of the blown-up points lying in $S$, then there is a subsequence with stabilisations $\tilde{E}_{i\epsilon}$ on $\tilde{X}$ converging weakly to $\tilde{E}$ on $\tilde{X} \setminus \pi^{-1}(S)$. Here stability on $\tilde{X}$ is with respect to $\omega_\epsilon = \omega_{\epsilon \alpha_0}$ for $\alpha_0 := (1, \ldots, 1)$ and $\epsilon > 0$ sufficiently small.

**Proof:** Let $T \subset X \setminus B(E)$ be the finite set used to stabilize $E$. Since there is no bubbling of curvature near the points of $T$, a sequence of integrable connections $\tilde{A}_i$ corresponding to bundles $\tilde{E}_i$ with $\pi_* \tilde{E}_i = E_i$ can be found such that $\tilde{A}_i$ agrees with $\pi^* A_i$ outside a fixed neighbourhood $U$ of the exceptional divisor and which converge smoothly inside this neighbourhood to a connection inducing $\tilde{E}$ there, so the $\tilde{A}_i$ converge weakly in $L^1_{1,\text{loc}}(\tilde{X} \setminus \pi^{-1}(S))$ to a connection inducing $\tilde{E}$. 


If \( b_1(X) \) is even then by Proposition 3.5, all of the bundles \( \bar{E}_i \) and \( \bar{E} \) will be stable with respect to the same metric \( \omega \) for sufficiently small \( \epsilon > 0 \). However, by construction of \( \bar{E} \) and \( \bar{E}_i \), a reflexive subsheaf \( A_i \) minimising \( \nu_{\bar{E}_i}((*,\omega)) \) for all sufficiently small \( \epsilon \) and sufficiently large \( i \) will satisfy the same type of splitting behaviour as that described in the discussion preceding Proposition 4.2, and therefore there exists \( \epsilon_0 > 0 \) such that \( \bar{E}_i \) is \( \omega_i \)-stable for all \( i \) and all \( \epsilon < \epsilon_0 \), regardless of the parity of \( b_1(X) \).

By weak compactness on \( \bar{X} \) there is a finite set \( \bar{S} \subset \bar{X} \) such that, after gauge transformations a subsequence of the corresponding \( \omega_i \)-Hermitian-Einstein connections \( \bar{A}_i \) inducing \( \bar{E}_i \) converges weakly in \( L^p_{1,\text{loc}}(\bar{X} \setminus \bar{S}) \) to a Hermitian-Einstein \( \bar{A} \) connection on \( \bar{X} \) which, after removal of singularities, defines a semi-stable bundle \( \bar{E}' \) there. By Lemma 2.2 of [B4], after rescaling if necessary the automorphisms \( g_i \) intertwining \( \bar{A}_i \) with \( \bar{A}_i \) give rise to a non-zero holomorphic map \( \bar{E} \to \bar{E}' \), but since the former is stable and the two bundles have the same degree, it must be an isomorphism. This implies that the sequences \( \{g_i\} \) are both uniformly bounded in \( C^0(\bar{X}) \), and by Lemma 2.1 of that same reference it follows that \( \bar{S} = \pi^{-1}(S) \) and that the two sequences bubble the same amount of charge at each point of \( S \).

From the complex analytic viewpoint, Propositions 3.4 and 3.5 indicate that on a blowup \( \bar{X} \) of \( X \) there exist “stable” moduli spaces \( \mathcal{M}(\bar{X}, \mathcal{E}_{\text{top}}, \omega_\alpha) \) of stable bundles as \( \epsilon \) tends to 0, at least when \( b_1(X) \) is even. Given a bundle \( \bar{E} \) in one of these moduli spaces, it is natural to enquire about the behaviour of the corresponding sequences of Hermitian-Einstein connections and to determine to what extent the behaviour of this sequence reflects the complex analytic picture of a bundle on \( X \) glued to a bundle on a neighbourhood of the exceptional divisor according the prescription of \( \S 2 \). These questions are at least partially answered by Corollary 3.3 below where it is shown that the corresponding connections converge on \( \bar{X} \) and that the behaviour of the sequence near the exceptional divisor is described in Proposition 5.4.

**Proposition 5.2.** Let \( \pi: \bar{X} \to X \) be a modification of \( X \) with exceptional divisor \( D \subset \bar{X} \), and let \( \{\omega_i\} \) be a sequence of \( \partial\bar{\partial}\)-closed positive \((1,1)\)-forms on \( \bar{X} \) converging smoothly to \( \pi^*\omega \). Let \( \bar{E} \) be a holomorphic \( r \)-bundle on \( \bar{X} \) and let \( \{A_i\} \) be a sequence of smooth hermitian connections on \( \bar{E} \) such that

1. \( |A_i\bar{F}(A_i) - \sqrt{-1}\bar{\theta}_i 1|_{L^1(\bar{X},\omega_i)} \to 0 \), for \( \lambda_i = -2\pi \mu(\bar{E},\omega_i)/\text{Vol}(\bar{X},\omega_i) \), and
2. there is a finite set \( S \subset \bar{X} \), \( p > 4 \) and \( k \geq 1 \) such that \( \pi_*A_i \) converges weakly in \( L^p_{k,\text{loc}}(\bar{X} \setminus (S \cup \pi(D)), \omega) \) to a connection \( A_\infty \) with \( \|F(A_\infty)\|_{L^2(\bar{X})} < \infty \).

Then

(a) \( \pi_*\bar{E} \) is semi-stable;
(b) the quasi-stable bundle \( E_\infty \) on \( X \) defined by \( A_\infty \) (after removal of singularities) is isomorphic to \( \Sigma(\pi_*\bar{E}) \);
(c) after suitable gauge transformations, a subsequence of \( \{\pi_*A_i\} \) converges weakly in \( L^p_k \) and strongly in \( C^{k-1} \) on compact subsets of \( X \setminus (\pi(D) \cup B((\pi_*\bar{E}), \epsilon)) \).

**Proof:** If \( L \) is an irreducible component of \( D \), the curvature form \( f_i \) of the Hermitian-Einstein connection on \( \mathcal{O}(L) \) corresponding to the metric \( \omega_i \) satisfies \( \|f_i\|_{L^2(\bar{X},\omega_i)} = 4\pi^2 [1 - \deg(\mathcal{O}(L), \omega_i)^2/\text{Vol}(\bar{X},\omega_i)] \), which converges to \( 4\pi^2 \). Ellipticity of the Hermitian-Einstein equations and the convergence of the sequence \( \{\omega_i\} \) to \( \pi^*\omega \) implies that (a subsequence of) \( \{f_i\} \) converges weakly in \( L^2(X) \) and smoothly on compact subsets of \( X \setminus \pi(L) \) to a finite action solution of the Hermitian-Einstein equations which, by removable singularities, extends to \( \bar{X} \) to define a holomorphic line bundle of degree 0 there. By Lemma 2.2 of [B4] and Hartogs’ theorem, this line bundle has a non-zero holomorphic section, but since the degree of the bundle is 0, this section is covariantly constant and therefore defines a global trivialisation—all of the curvature has concentrated along \( L \). It follows from this that if \( \bar{E} \) is twisted by a suitable line bundle trivial off \( D \) and the sequence \( \{A_i\} \) is correspondingly twisted by the sequence of Hermitian-Einstein connections on these line bundles, then the hypotheses of the proposition remain true for the new sequence and it can be supposed from now on that \( \pi_*\bar{E} \) is locally free.

By Lemma 2.2 of [B4] and Hartogs’ theorem again there is a non-zero holomorphic map \( E_\infty \to \pi_*\bar{E} \), so if either bundle is stable this map is an isomorphism. In particular, this occurs if \( \bar{E} \) is a line bundle.

(a) If \( A \subset \pi_*\bar{E} \) has rank \( a \) and torsion-free quotient then \( \nu_{\pi_*\bar{E}}(A) = \nu_{\bar{E}}(\pi^*A, \omega_i) - \frac{1}{2}(|\omega_i - \pi^*\omega| \cdot \det(\bar{E} \otimes \pi^*A^*)) \). Since \( \nu_{\bar{E}}(\pi^*A, \omega_i) \geq \nu_{\bar{E}}(\tilde{A}, \omega_i) \) where \( \tilde{A} \) is the maximal normal extension of \( \pi^*A \) in \( \bar{E} \) and the latter term has non-negative limit by hypothesis (ii) and (1.7), the convergence of \( \omega_i \) to \( \pi^*\omega \) implies \( \nu_{\pi_*\bar{E}}(A) \geq 0 \).
(b) If either $E_\infty$ or $\pi_*\tilde{E}$ is stable, this is proved already by the remarks above. If $\pi_*\tilde{E}$ is not stable, there exists a subsheaf $A \subset \pi_*\tilde{E}$ with torsion free quotient $C$ such that $\nu_{\pi_*\tilde{E}}(A) = 0$. Pulling back to $\tilde{X}$ and taking the maximal normal extension gives a subsheaf $\tilde{A} \subset \tilde{E}$ with torsion-free quotient $\tilde{C}$ such that $\pi_*\tilde{A} = A$ and $C \subset \pi_*\tilde{C}$. Desingularise the sequence $0 \to A \to \tilde{E} \to \tilde{C} \to 0$ as in §3 of [B2] to obtain a modification $\tilde{X}' \overset{\nu}{\to} \tilde{X} \overset{\pi}{\to} X$ and a sequence of bundles $0 \to \tilde{A}' \to \pi'^*\tilde{E} \to \tilde{C}' \to 0$ on $\tilde{X}'$ with $\pi'_*\tilde{A}' = \tilde{A}$ and $\tilde{C} \subset \pi'_*\tilde{C}'$.

Choose metrics $\omega'_i = \pi'^*\omega_i + \rho_{\alpha_i}$ on $\tilde{X}'$ with $\rho_{\alpha_i}$ converging smoothly to $0$, and make this convergence sufficiently fast so that $\omega'_i \wedge F(\pi'^*A_i) - \sqrt{-1}X'_i \omega'^2_i$ converges to $0$ in $L^1(\tilde{X}', \omega'_i)$; this is possible since the connections $A_i$ are smooth (cf. the final remark in §2 of [B2], p.631). From (1.7) and (i) it follows that if $\beta_i$ is the second fundamental form of $\tilde{A}'$ in $\pi'^*\tilde{E}$ for the hermitian connection $\pi'^*A_i$, then $\beta_i$ is converging to $0$ in $L^2(\tilde{X}', \omega'_i)$. This implies that the hypotheses of the proposition are satisfied by the induced connections on $\tilde{A}'$ and $\tilde{C}'$. If $A_\infty, C_\infty$ are the holomorphic bundles on $X$ defined by the limiting (Hermitian-Einstein) connections on $\tilde{A}', \tilde{C}'$, then since $\beta_i$ → $0$ it follows from the construction that $E_\infty = A_\infty \oplus C_\infty$. By induction on rank, $A_\infty = \Sigma((\pi'^*A_i), C_\infty = \Sigma((\pi'^*C_i))$, giving $E_\infty = A_\infty \oplus C_\infty = \Sigma((\pi'^*A_i)) \oplus \Sigma((\pi'^*C_i)) = \Sigma(A) \oplus \Sigma(C) = \Sigma(\pi_*E)$, proving (b) in general.

(c) By passing to a subsequence it can be supposed that the sequence $\{g_i\}$ of intertwining operators from which the holomorphic map $\pi_*E \to E_\infty$ is constructed is converging strongly in $C^k$ on any compact subset of $X \setminus (S \cup (\pi(D))$. If $\pi_*E$ is stable then this map is an isomorphism and the sequence $\{g_i^{-1}\}$ must also be bounded in $C^k$ on compact subsets of $X \setminus (S \cup (\pi(D))$. Then Lemma 2.1 of [B3] implies that curvature can only concentrate on $\pi(D)$ itself, which proves (c) in this case. The general case now follows by induction on rank using the proof of (b) when $\pi_*E$ is not stable, since any point of $\tilde{X}$ in the center of the modification $\tilde{X}' \to \tilde{X}$ is mapped into $B(\pi_*E)$ by $\pi$.

**Corollary 5.3.** If $\tilde{E}$ on $\tilde{X}$ is stable with respect to $\omega_n$, then after suitable gauge transformations, a subsequence of the corresponding sequence $\{\nabla_i\}$ of Hermitian-Einstein connections converges smoothly on compact subsets of $X \setminus (\pi(D) \cup B((\pi_*E)^*))$ to a Hermitian-Einstein connection inducing $\Sigma(\pi_*E)$.

**Proof:** On any compact subset of $X \setminus (\pi(D)$ the $L^2$ norm of $F(\nabla_i)$ with respect to $\omega$ is uniformly bounded. The weak compactness arguments of [Sed], [U1] still apply in this setting to obtain a subsequence of $\{\nabla_i\}$ (after gauge transformations) converging weakly in $L^2$ on compact subsets of $X \setminus (S \cup (\pi(D))$ for some finite set $S \subset X$. Ellipticity of the Hermitian-Einstein equations together with boot-strapping and diagonalisation then give a subsequence converging smoothly on compact subsets of this complement, and the conclusion then follows from the proposition.

**Corollary 5.3** describes the behaviour of sequences of Hermitian-Einstein connections away from the exceptional divisor $D$. To describe the behaviour near $D$ consider the case that $\tilde{X}$ is the blowup of $X$ at a single point $x_0$ and fix a metric on $\tilde{X}$ of the form $\omega = \pi^*\omega + \sigma$ where $\sigma$ restricts to the Fubini-Study metric on $L_0 := \pi^{-1}(x_0)$. Then as $\epsilon \to 0$ the corresponding sequence of Hermitian-Einstein connections on the line bundle $O(1)$ over $\tilde{X}$ converges smoothly on compact subsets of $\tilde{X} \setminus L_0$ to the trivial flat connection.

Near $L_0$ on the other hand, restrict attention to a small neighbourhood of $x_0$ isomorphic to $B_\sigma$ and choose holomorphic coordinates $(z^0, z^1)$ in that neighbourhood so that $\omega(x_0)$ is standard in these coordinates. If $\omega$ is the standard flat metric $\delta := (i/2)\partial\overline{\partial}|z|^2$ near $x_0$ then under the dilation $z \mapsto \sqrt{\epsilon}z$ the metric $\delta_1$ pull-back to $\delta_0$; in general, if $\omega$ is arbitrary the pull-back of $\omega_\epsilon$ differs from $\delta_0$ by a term of order $\epsilon^{3/2}$ $(\epsilon^2$ if $d\omega(x_0) = 0$) and the coordinates are appropriately chosen) as $\epsilon \to 0$.

From the local description of bundles on a blowup given in Proposition 2.5 it is easy to see that the pullbacks of a natural transition function for $\tilde{E}$ under the maps $\lambda_i$ given by $B(x_i, r_0/\sqrt{\epsilon}) \ni z \mapsto \sqrt{\epsilon}z \in B(x_0, r_0)$ converge smoothly to a transition function for a direct sum of line bundles on $\mathbb{C}^2$.

The following result can be proved using Lemma 5.1, Corollary 5.3 and the stabilisation procedure of §4:

**Proposition 5.4.** Let $\tilde{X} \overset{\nu}{\to} X$ be the blowup of $X$ at $x_0$ with $L := \pi^{-1}(x_0)$, and let $\tilde{E}$ be a bundle with $\tilde{E}|_L = \oplus_i O(a_i)$. If $\tilde{E}$ admits an $\omega_\epsilon$-Hermitian-Einstein connection $\nabla_\epsilon$ for all $\epsilon > 0$ sufficiently small, then as $\epsilon \to 0$ the pull-back of $\nabla_\epsilon$ to $B(x_0, r_0/\sqrt{\epsilon})$ converges smoothly on compact subsets of $\mathbb{C}^2$ after suitable gauge transformations to the direct sum of the standard anti-self-dual connections on $O(a_i)$, where anti-self-duality is with respect to the standard metric $\delta_1$ on $\mathbb{C}^2$ in suitable holomorphic coordinates.
6. **Compactification of moduli spaces.**

Let $X$ be a compact complex surface equipped with a positive $\bar{\partial}\bar{\partial}$-closed $(1,1)$-form $\omega$, and let $\{E_i\}$ be a sequence of stable bundles on $X$ converging weakly to a quasi-stable bundle $E$. Blow-up $X$ along $S$ to $\tilde{X} \to X$ and pull back the sequence $\{E_i\}$. Assume temporarily that every bundle in the pulled-back sequence is stable with respect to the same metric $\omega_\alpha$, for some suitable $\alpha$ sufficiently small—this will be true if either $b_1(X)$ is even (by Proposition 3.5) or if $E$ is stable. (If $\tilde{A}_i \subset \pi^*E_i$ maximises $\mu(\bullet, \omega_{\alpha})$ for all $\alpha > 0$ sufficiently small, then $\pi_*\tilde{A}_i$ is semi-stable. If $\nu_{\pi^*E_i}(\tilde{A}_i, \omega_{\epsilon, \alpha}) = 0$ for $\epsilon_i \to 0$, then as in $\S$ 3, the ranks, first Chern classes, charges, and degrees of $\pi_*\tilde{A}_i$ are all uniformly bounded, and the same holds for any stable $A'_i \subset \pi_*\tilde{A}_i$ of the same degree. Passing to a subsequence, taking a weak limit and applying semi-continuity of cohomology then yields a semi-stable bundle $A$ with $\text{rank}(A) < \text{rank}(E)$, $\mu(A) = \mu(E)$ and a non-zero holomorphic map $A \to E$, contradicting stability of $E$.)

By weak compactness on $\tilde{X}$ the new sequence converges weakly off some finite set $\tilde{S} \subset \tilde{X}$ to a semi-stable bundle $\tilde{E}$ and by semi-continuity of cohomology there is a non-zero holomorphic map $(\pi_*E)^{**} \to E$. If either is stable, then this map is an isomorphism and it follows from Lemma 2.1 of [B4] that $\tilde{S} \subset \pi^{-1}S$; moreover $C(\tilde{E}) \geq C(E)$ by Proposition 2.9.

This argument suggests that there is another compactification of moduli spaces tied more closely to the complex analysis, distinct from the more usual ones ([Gie], [M]): if $\tilde{X} \to X$ is a blowup of $X$ at $x_0$, sequences of stable bundles $\tilde{E}_i$ on $\tilde{X}$ are easily constructed such that each is trivial on the exceptional line and such that the sequence converges smoothly on $X$ to a bundle $E$ which is stable but which is now non-trivial on the exceptional line. Thus the theorems of Uhlenbeck [U1], [U2] on compactness and removability of singularities in the space of Yang-Mills connections are reinterpreted as the well-known phenomenon of jumping of holomorphic structures. It is natural to conjecture that this is the only type of catastrophe which can occur, and that, given a degenerating sequence of Hermitian-Einstein connections on a bundle over $X$, after a finite number blowups and pull-backs, a strongly convergent subsequence can be found. Unfortunately, there is no guarantee that after blowing up and pulling back, the new sequence of Hermitian-Einstein connections will not bubble precisely the same amount of charge as the original and that iterating the process will eventually lead to a convergent subsequence. Indeed, by a process of diagonalisation, sequences of connections can be constructed for which this blowing-up and pulling-back procedure will not terminate after finitely many steps. Despite this, provided that some flexibility is introduced into the blowing-up process, a form of compactness does hold, as is indicated by the following result proved in [B4]:

**Theorem 6.1.** Let $X$ be a compact complex surface equipped with a positive $\bar{\partial}\bar{\partial}$-closed $(1,1)$-form $\omega$. Let $\{A_i\}$ be a sequence of Hermitian-Einstein connections on a fixed unitary bundle $E_{\text{top}}$ of rank $r$ over $X$ such that the corresponding holomorphic bundles $E_i$ are stable and are of uniformly bounded degree. Suppose that $E_i$ converges weakly to $E$ off $S \subset X$. Then for some subsequence $\{E_{i_j}\} \subset \{E_i\}$:

1. There is a sequence of blowups $\tilde{X}_{i_j}$ of $X$ converging to a blowup $\tilde{X} \to X$ of $X$ with exceptional divisor $\pi^{-1}(S)$ and with $c_2(\tilde{X}) = c_2(X) + 2C(E_{\text{top}}) - 2C(E) - 1$;
2. There are complex automorphisms $\gamma_{i_j}$ of $\pi^*E_{\text{top}}$ such that $|\gamma_{i_j}| + |g_{i_j}^{-1}|$ is uniformly bounded on compact subsets of $X \setminus \pi^{-1}(S)$ and $\{|g_{i_j}\}$ is converging uniformly in $C^3$ on such subsets;
3. $\{g_{i_j} \cdot (\pi^*_{i_j}A_{i_j})\}$ converges uniformly in $C^2$ to a smooth connection on $\pi^*E_{\text{top}}$ over $\tilde{X}$ which defines a holomorphic bundle $\tilde{E}$ such that $(\pi_*\tilde{E})^{**} = E$;
4. If $E$ is stable then for any suitable $\alpha$ with $|\alpha|$ sufficiently small, the connections $g_{i_j} \cdot (\pi^*_{i_j}A_{i_j})$ can be taken to be $(\pi^*_{i_j}\omega + \rho_\alpha)$-Hermitian-Einstein, and $\tilde{E}$ is $\omega_{\alpha}$-stable.

If $E$ is not stable, but $b_1(X)$ is even and $\text{rank}(E_{\text{top}}) = 2$ there is sequence of blowups $\{X_{i_j}\}$ consisting of at most $2C(E_{\text{top}}) - 1$ individual blowups converging to a blowup $\tilde{X}$ such that, for some suitable $\alpha$, $\pi^*_{i_j}E_{i_j}$ is $\omega_{\epsilon, \alpha}$-stable for all $\epsilon \in (0, 1]$ and the subsequence $\{\pi^*_{i_j}E_{i_j}\}$ converges strongly to a bundle $\tilde{E}$ on $\tilde{X}$, stable with respect to $\omega_{\epsilon, \alpha}$-stable for all $\epsilon \in (0, 1]$. The convergence of a sequence of blowups should be interpreted as the convergence of a sequence of integrable complex structures on the same underlying smooth manifold $X \# n\mathbb{P}_2$ endowed with a fixed Riemannian
metric. In the current setting, these complex structures are isomorphic on the complement of a fixed open set with strictly pseudoconvex boundary.

A priori, there is no reason why the last statement of the theorem cannot be extended to bundles of arbitrary rank, but as yet a proof is lacking. The complications arise, as always, from the presence of bundles which are semi-stable but not stable.

Theorem 6.1 suggests a variety of different compactifications for moduli of stable bundles—for example, adding torsion-free semi-stable sheaves which are direct images of stable bundles on blowups is an obvious candidate. One other which has some interesting properties is described below.

Let $E_{\text{top}}$ be a unitary $r$-bundle over $(X, \omega)$ and let $\mathcal{M}(X, E_{\text{top}})$ denote the space of isomorphism classes of quasi-stable holomorphic structures on $E_{\text{top}}$. Consider the set of pairs $(\tilde{X}, \tilde{E})$ where

1. $\tilde{X}$ is a blowup of $X$;
2. $\tilde{E}$ is a holomorphic bundle on $\tilde{X}$ topologically isomorphic to $\pi^* E_{\text{top}}$ such that $\pi_* E$ is semi-stable;
3. $\tilde{E}$ is $\omega_\alpha$-quasi-stable for all suitable $\alpha$ in an open set of such;
4. If $b_1(X)$ is odd, $\deg(\pi_* \tilde{E}, \omega) = 0$.

Note that the third condition implies that if $\tilde{E}$ is not stable, then it is a direct sum of stable bundles each of which is topologically trivial on the exceptional divisor. Note also that the requirement that $\pi_* \tilde{E}$ be semi-stable implies that $\tilde{E}$ is $\omega_\alpha$-(quasi-)stable for all $\epsilon \in (0, 1]$ if $\tilde{E}$ is $\omega_\alpha$-(quasi-)stable.

If $b_1(X)$ is odd, it follows from the proof of Proposition 2 in [B2] that for any $t \in \mathbb{R}$ there is a line bundle $L_t$ on $X$ with $c_1(L_t) = 0$ and $\deg(L_t, \omega) = t$, so the fourth condition simply prevents this type of non-compactness.

On the set of pairs $(\tilde{X}, \tilde{E})$ satisfying the above conditions, define an equivalence relation $\sim$ by setting $(\tilde{X}_1, \tilde{E}_1) \sim (\tilde{X}_2, \tilde{E}_2)$ if there is a joint blowup $\tilde{X}_{12}$ such that $\pi_2^* \tilde{E}_1 \cong \pi_2^* \tilde{E}_2$ on $\tilde{X}_{12}$, and let $\overline{\mathcal{M}}(X, E_{\text{top}})$ denote the set of equivalence classes. A topology is defined on $\overline{\mathcal{M}} = \mathcal{M}(X, E_{\text{top}})$ by defining $\{(\tilde{X}_i, \tilde{E}_i)\} \subset \overline{\mathcal{M}}$ to converge to $[(\tilde{X}, \tilde{E})]$ iff $[(\tilde{X}_i, \tilde{E}_i)]$ can be represented by a sequence of blowups $\tilde{X}_i$ converging to $\tilde{X}$ with $\alpha$-quasi-stable bundles $\tilde{E}_i$ on $\tilde{X}_i$ converging strongly to $\tilde{E}$ on $\tilde{X}$.

If $(\tilde{X}, \tilde{E}) \in \overline{\mathcal{M}}$ with $\tilde{E}$ quasi-stable with respect to $\omega_{\alpha'}$ for all $\alpha'$ in a neighbourhood of $\alpha$ and $A_0$ is the $\omega_{\alpha'}$-Hermitian-Einstein connection on $E_{\text{top}}$ inducing $\tilde{E}$, it is easily verified that the image in $\overline{\mathcal{M}}$ of a set of the form $U(A_0, c_1, c_2) := \{A_0 + a\}$ where $A_0 + a$ is $\omega_{\alpha'}$-Hermitian-Einstein, $||a||_{C^1(\omega)} < c_1$ with $E(A_0 + a)$ $\omega_{\alpha'}$-quasi-stable for $|\alpha' - \alpha| < c_2$ is open in $\overline{\mathcal{M}}$ and that every open set containing $(\tilde{X}, \tilde{E})$ contains such a open neighbourhood. Hence the collection of such sets forms a base for the topology on $\overline{\mathcal{M}}$, from which it is clear that this topology is second countable.

Theorem 6.1 implies that at least under certain circumstances, every sequence in $\mathcal{M}$ has a subsequence converging in $\overline{\mathcal{M}}$. To attempt to use this construct a compactification of $\mathcal{M}$ clearly requires a bound on the number of blowups required to represent classes in $\overline{\mathcal{M}}$, a bound which is provided by Proposition 6.3 below:

**Lemma 6.2.** For any $r_0, C_0$ there exists $n = n(r_0, C_0, \omega)$ such that any semi-stable bundle $E$ of rank $\leq r_0$ and charge $\leq C_0$ satisfies $h^2(X, \text{End}_0 E) \leq n$.

**Proof:** If not, there exists a sequence $\{E_i\}$ of semi-stable bundles of bounded rank and charge such that $h^2(X, \text{End}_0 E_i) \rightarrow \infty$. It can be assumed without loss of generality that each $E_i$ is quasi-stable since it follows easily by induction on rank that $h^2(\text{End}_0 E) \leq h^2(\text{End}_0 \Sigma(E))$ for any semi-stable $E$. The bundles $\text{End}_0 E_i$ have uniformly bounded ranks ($\leq r_0^2 - 1$) and charges ($\leq 2r_0 C_0$) with trivial determinants, and each is also quasi-stable. By passing to a subsequence if necessary, it can be assumed that $\text{End}_0 E_i$ is converging weakly to a quasi-stable bundle $B$ off some finite set $S \subset X$. Passing to another subsequence and using Theorem 6.1, there is a sequence of blowups $\{\tilde{X}_i\}$ converging to a blowup $\tilde{X}$ such that a sequence of smooth integrable connections representing $\pi_0^* \text{End}_0 E_i$ converges strongly in $C^2$ to define a bundle $\tilde{B}$ on $\tilde{X}$ with $(\pi_0, \tilde{B})^{**} = B$. Then $h^2(\tilde{X}_i, \pi_0^* \text{End}_0 E_i) = h^0(\tilde{X}_i, (\pi_0^* \text{End}_0 E_i) \otimes K_{\tilde{X}_i}) = h^0(X, (\text{End}_0 E_i) \otimes K_X) = h^2(X, \text{End}_0 E_i) \rightarrow \infty$, contradicting the fact that $h^2(\tilde{X}_i, \text{End}_0 E_i) \leq h^2(\tilde{X}, \tilde{B})$ for $i$ sufficiently large, by standard semi-continuity of cohomology.

**Proposition 6.3.** There is an integer $N = N(X, \omega, E_{\text{top}})$ such that every class in $\overline{\mathcal{M}}$ can be represented by a bundle on a blowup consisting of at most $N$ individual blowups. If $\deg(K_X, \omega) < 0$ then $N \leq 2C(E_{\text{top}}) - 1$.

**Proof:** Suppose $(\tilde{X}, \tilde{E}) \in \overline{\mathcal{M}}$ and $\tilde{E}$ is $\omega_\alpha$-quasi-stable for some suitable $\alpha$, and let $\tilde{S} \subset \tilde{X}$ be the exceptional divisor. Pick a set of $r^2 + 1$ points $T_0 \subset X \setminus \pi(\text{supp}(\rho_0))$ and stabilise $(\pi_* \tilde{E})^{**} =: E_0$ to a bundle $E_0'$ on
a blowup $X'$ of $X$ centered at $T_0$. Then for $\beta = \epsilon_0(1,1,\ldots,1)$, it follows easily that the corresponding “stabilisation” $E'$ of $E$ on $\tilde{X}$ is $E_{\alpha_0,\beta_0}$-stable for all $\epsilon, \delta > 0$ sufficiently small.

Since $\pi^{\ast}(\text{End}_0 E_0^{\ast}(-1)) = \text{End}_0 E_0$, it follows that $h^2(X', \text{End}_0 E_0^{\ast}) = h^2(X, \text{End}_0 E_0)$. Consequently, $h^2(X', \text{End}_0 E' \otimes \mathcal{O}(\tilde{S})) = h^0(X', \text{End}_0 E' \otimes \pi^{\ast} K_X \otimes \mathcal{O}(2\tilde{S})) \leq h^0(X', \text{End}_0 E_0 \otimes K_X) = h^2(X, \text{End}_0 E_0) \leq n$, where $n$ is a uniform bound as specified by Lemma 6.2. If $\deg(K_X, \omega) < 0$ then $H^2(X, \text{End}_0 E_0) = 0$ for any semi-stable $E$, so take $n$ in this case.

Pick a finite set $T_1 \subset \tilde{X}' \setminus \text{supp}(\rho_0 + \rho_3)$ consisting of at most $n$ points, blow up $X$ at these points to $X_1$ and construct a bundle $E_1$ on $X_1$ such that $(\pi_1, E_1)^{\ast} = E_0$ with $E_1$ restricting to $\mathcal{O}(1-\delta) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{\tau-2}$ on every component of the exceptional divisor and with $h^2(X_1, \text{End}_0 E_1) = 0$. There are corresponding bundles $E_1'$ on the blowup of $X'$ along $T_1$ with $(\pi_1', E_1')^{\ast} = E_1$ and $E_1'$ on the blowup $\tilde{X}'_1$ of $X'$ along $T_1$ satisfying $(\pi_1', E_1')^{\ast} = E_1'$. Moreover, if $\epsilon_0$ is fixed and sufficiently small with $\alpha_0 = \epsilon_\alpha\alpha_0$ and $\beta_0 := \epsilon_\beta\beta_0$, for any $\gamma$ with $|\gamma|$ sufficiently small it follows $E_1$ is $E_{\alpha_0,\beta_0,\gamma}$-stable and $E_1'$ is $E_{\beta_0,\gamma}$-stable.

By construction, $H^2(\tilde{X}'_1, \text{End}_0 E_1' \otimes \mathcal{O}(\tilde{S})) = 0$, so the bundle $E_1'$ has arbitrarily small deformations which are holomorphically trivial on $\tilde{S}$. Hence there is a sequence $\{E_{1,i}'\}$ of bundles on $X_1'$ such that $\pi^{\ast}E_{1,i}'$ converges smoothly to $E_1'$. Since this bundle is $E_{\alpha_0,\beta_0,\gamma}$-stable for all $\delta \in (0,1]$, by passing to a subsequence it can be supposed that the same is true of $\pi^{\ast}E_{1,i}'$, and therefore $E_{1,i}'$ must be $E_{\alpha_0,\beta_0,\gamma}$-stable for $\delta \in (0,1]$. By passing to another subsequence if necessary, it can be assumed that $\{E_{1,i}'\}$ is converging weakly to $\omega_{\alpha_0,\beta_0,\gamma}$-quasi-stable bundle $E_1'$ off some finite set $T_2 \subset \tilde{X}_1'$, and by semi-continuity of cohomology and $\omega_{\beta_0}$-stability of $E_0'$ it follows that $(\pi_1, E_1')^{\ast} = E_0'$.

By construction, $C(E_{1,i}) \leq C(E_{\text{top}}) + (r^2 + 1)/2r + n$, and $C(E_1')$ must be at least $(r^2 + 1)/2r$ since $c_1(E_1')$ restricts to 0 on each component of $\pi^{\ast}(1-\delta)(T_0)$. By Theorem 6.1 therefore, after passing to another subsequence if necessary there is a sequence of blowups $\tilde{X}_{2,i}$ of $X_1'$ converging to a blowup $X_2'$ centered at $T_2$ such that, if $\tilde{X}_{2,i} \to X_2'$ is the blowing-down map, $\pi^{\ast}_{2,i} E_{1,i}'$ converges to a bundle $E_2'$ on $X_2'$ satisfying $(\pi_2, E_2')^{\ast} = E_1'$, with $c_2(X_2') \leq c_2(X_1') + 2(C(E_{\text{top}}) + n) - 1$. The convergence on this sequence of blowups is with respect to a metric of the form $\omega_{\beta_0,\gamma}$ for $|\xi|$ small; fix one such $\xi$.

A subsequence of the sequence of joint blowups $\tilde{X}_{2,i}$ of $X_{2,i}$ and $X$ has a subsequence converging to a joint blowup $\tilde{X}_2$ of $X_2$ and $\tilde{X}$, Abusing notation slightly, the pull-back $\pi^{\ast}E_2'$ of $E_2'$ to $\tilde{X}_2$ is $E_{\alpha_0,\beta_0,\gamma}$-stable for $\epsilon > 0$ sufficiently small, so by semi-continuity of cohomology the pull-backs of the bundles $E_{1,i}'$ must converge to $\pi^{\ast}E_2'$ with respect to $E_{\alpha_0,\beta_0,\gamma}$-stable for $\delta > 0$ sufficiently small, so with respect to this metric the pull-backs of $E_{1,i}'$ converge to $\pi^{\ast}E_2'$. By semi-continuity of cohomology there is a non-zero holomorphic map $\pi^{\ast}E_2' \to \pi^{\ast}E_1'$, giving a non-zero map $\pi_2, E_2' = \pi_2,\pi_1^{\ast} E_1' \to \pi_2,\pi_1^{\ast} E_2' = \pi_1^{\ast} E_1'$. On taking double-duals, this gives a non-zero holomorphic map $E_1' \to E_2'$; since $E_1'$ is $E_{\beta_0,\gamma}$-stable, this map is an isomorphism, and therefore, since $\det(\pi^{\ast}E_2') = \det(\pi^{\ast}E_1')$, it follows that the map $\pi^{\ast}E_2' \to \pi^{\ast}E_1'$ must be an isomorphism.

Let $X_3$ be the blowup of $X$ obtained by blowing down the components of the exceptional divisor in $X_2'$ over $T_0$ and $T_1$, so $c_2(X_3) = c_2(X_2') - n - (r^2 + 1) \leq c_2(X) + 2C(E_{\text{top}}) + n - 1$, and let $E_3 := (\pi_1^{\ast} E_2')^{\ast}$. Taking double-duals of the direct image under $\pi^{\ast}$ of the isomorphism $\pi^{\ast} E_2' \simeq \pi^{\ast} E_1'$ gives $\pi^{\ast} E_3 = \pi^{\ast} E_2$, so $(X_3, E_3) \sim (\tilde{X}, E) \in \overline{\mathcal{M}}$. Note that $\pi^{\ast} E_2$ is $\omega_{\epsilon_\beta}$-quasi-stable for sufficiently small $\delta > 0$, which implies that $\pi^{\ast} E_3$ has the same property; this in turn implies that $E_3$ is $\omega_{\epsilon_\beta}$-quasi-stable for small enough $\delta$. Since $\xi$ is assumed to be generic, if $E_3$ is not actually stable, each stable summand must be a topological pull-back from $X$, so $E_3$ is $\omega_{\epsilon_\beta}$-quasi-stable for all $\xi$ near $\xi$. \qed

**Theorem 6.4.**

1. $\mathcal{M} \subset \overline{\mathcal{M}}$ is an open set, dense if $\deg(K_X, \omega) < 0$;
2. If $b_1(X)$ is even and rank($E_{\text{top}}$) = 2 then $\mathcal{M}$ is compact;
3. If every semi-stable bundle $E$ on $X$ satisfying $\text{rank}(E) = \text{rank}(E_{\text{top}})$, $c_1(E) = c_1(E_{\text{top}})$ and $C(E) \leq C(E_{\text{top}})$ is stable then $\overline{\mathcal{M}}$ is a compact Hausdorff space, smooth if $\deg(K_X, \omega) < 0$.

Note that if $b_1(X)$ is even and $c_1(E_{\text{top}}) \in H^2(X, \mathbb{Z})$ is not a torsion class (where $r = \text{rank}(E_{\text{top}})$), the hypothesis in the third statement holds for generic $\omega$. Even if $c_1(E_{\text{top}}) = 0$, by fixing a base point $x_0 \in X$ and blowing up at this point, the $\mathbb{P}_{r-1}$-bundle over $\mathcal{M}_{\text{stable}} \subset \mathcal{M}$ with fibre $\mathbb{P}(E_{\text{top}})$ at $E \in \mathcal{M}_{\text{stable}}$ is isomorphic to the space of $\omega_{\epsilon_\beta}$-stable bundles on $X$ restricting to $\mathcal{O}(1) \oplus \mathcal{O}^{r-1}$ on $\pi^{-1}(x_0)$ with direct image topologically isomorphic to $E_{\text{top}}$.

If $b_1(X)$ is odd, the hypothesis in the third statement is quite a strong restriction since it will only be satisfied if $c_2(E_{\text{top}}) < 0$, for if $E$ is a holomorphic $r$-bundle and $b_1(X)$ is odd, for some line bundle $L$ with
c_1(L) = 0 and deg(L, ω) = deg(E, ω)/r the bundle \((L^* \otimes \det E) \oplus \mathcal{O}_{r}^{-1}\) is quasi-stable, not stable, has rank and first Chern class equal to that of \(E\) and has charge \(C(E) - c_2(E)\).

**Proof of Theorem 6.4**: 1. If \(\{(\tilde{X}_i, \tilde{E}_i)\} \subset \overline{\mathcal{M}} \setminus \mathcal{M}\) is converging to \((\tilde{X}, \tilde{E}) \in \mathcal{M}\), there is an irreducible component \(L\) of the exceptional divisor in \(\tilde{X}_i\) to which \(\tilde{E}_{i_j}\) restricts non-trivially for every \(i_j\) in some subsequence. Then \(H^0(L, \tilde{E}_{i_j}(-1)) \neq 0\) so by semi-continuity of cohomology it follows \(H^0(L, (\tilde{E}(-1)) \neq 0\), implying \(\tilde{E} \in \overline{\mathcal{M}} \setminus \mathcal{M}\) and hence that \(\mathcal{M}\) is an open set.

If \(\text{deg}(K_X, \omega) < 0\) then as seen at the end of §4, each \((\tilde{X}, \tilde{E}) \in \overline{\mathcal{M}}\) has arbitrarily small deformations restricting trivially to the exceptional divisor in \(\tilde{X}\).

2.3. Suppose \(\{(\tilde{X}_i, \tilde{E}_i)\} \subset \overline{\mathcal{M}}\). By Proposition 6.2, it can be assured that for each \(i\), \(c_2(\tilde{X}_i) \leq c_2(X) + N\) and by passing to a subsequence, it can then be assumed that \(c_2(X_i) = c_2(X) + n\) is constant. It is easy to see (by induction on \(n\) for example), that there is a subsequence such that \(\tilde{X}_i\) converges to a blowup \(\tilde{X}\) of \(X\).

There are finitely many classes in \(c \in H^2(X, \mathbb{Z}) \subset H^2(\tilde{X}, \mathbb{Z})\) satisfying \(-c \cdot c \leq r^3 C(E_{\text{top}})\) and therefore only finitely different classes of metrics on \(\tilde{X}_i\) of the form \(\omega_{i, \alpha} := \pi_i^* \omega + \rho_i\) with respect to which \(\tilde{E}_i\) can be quasi-stable for all \(\alpha\) in an open set. By passing to a subsequence, one such class of metrics can be fixed, and then for some new subsequence a corresponding sequence of metrics can be found converging to a non-degenerate metric \(\omega_{\alpha}\) on \(\tilde{X}\).

The \(\omega_{i, \alpha}\)-Hermitian-Einstein connection on \(\tilde{E}_i\) has curvature bounded in \(L^2(\omega_{i, \alpha})\), and hence in \(L^2(\omega_{\alpha})\) since the metrics are converging and hence compare uniformly. Regarding the sequence of connections as a sequence of unitary connections on the fixed bundle \(\pi^* E_{\text{top}}\) over the smooth Riemannian manifold \((X \# n \mathbb{F}_2, \omega_{\alpha})\) with uniformly \(L^2\)-bounded curvature, Uhlenbeck’s weak compactness result implies that after passing to another subsequence there is a finite set of points where the curvature is bubbling, with the sequence converging (after gauge transformations) weakly in \(L^2_{1, \text{loc}}\) on the complement. Convergence of the sequence of metrics \(\{\omega_{i, \alpha}\}\) and ellipticity of the Hermitian-Einstein equations imply that for some subsequence, this convergence can be bootstrapped to uniform convergence in \(C^2\) on compact subsets of the complement of the bad set, converging weakly to an \(\omega_{\alpha}\)-quasi-stable limit \(\tilde{E}\) on \(\tilde{X}\).

By the same arguments as in the proof of Theorem 1.3 of [B4], after passing to another subsequence if necessary, there is a further sequence of blowups \(\{\tilde{X}'_i\}\) such that \(\tilde{X}'_i\) is a blowup of \(\tilde{X}\) consisting of at most \(2(C(E_{\text{top}}) - C(\tilde{E})) - 1\) blowups, \(\pi_i^* \tilde{E}_i\) is \(\omega_{i, \alpha, \beta}\)-quasi-stable and some sequence of smooth integrable connections inducing \(\pi_i^* \tilde{E}_i\) converges strongly on \(\tilde{X}'\) to a bundle \(\tilde{E}'\) satisfying \((\pi_i^* \tilde{E}_i)^* = \tilde{E}\).

Under the hypotheses of the third statement of the theorem, \(\pi_* \tilde{E}\) must be stable and therefore so too is \(\tilde{E}\) with respect to \(\omega_{\alpha}\) for any suitable \(\alpha\) with \(|\alpha|\) sufficiently small by Proposition 3.4. Hence so too is \(\tilde{E}'\) if \(|\beta|\) is small enough, proving that \(\overline{\mathcal{M}}\) is sequentially compact (and hence compact, by second countability of the topology) in this case. Furthermore, if a sequence \(\{(\tilde{X}_i, \tilde{E}_i)\} \subset \overline{\mathcal{M}}\) converges to \((\tilde{X}_1, \tilde{E}_1)\) and \((\tilde{X}_{1,i}, \tilde{E}_{1,i}) \sim (\tilde{X}_{2,i}, \tilde{E}_{2,i})\) with \(\{(\tilde{X}_{2,i}, \tilde{E}_{2,i})\}\) converging to \((\tilde{X}_2, \tilde{E}_2)\), the sequence of joint blowups \(\tilde{X}_{12,i}\) converges to a joint blowup \(\tilde{X}_{12}\) of \(X\) and \(\pi_1^* \tilde{E}_2, \pi_2^* \tilde{E}_1\) must both be stable with respect to any metric on \(\tilde{X}_{12}\) of the form \(\omega_{\alpha, \beta}\) for \(|\alpha| + |\beta|\) sufficiently small. Since semi-continuity of cohomology gives a non-zero holomorphic map between these two pull-backs, this map must be an isomorphism. Thus \((\tilde{X}_1, \tilde{E}_1) \sim (\tilde{X}_2, \tilde{E}_2)\) and therefore the topology on \(\overline{\mathcal{M}}\) is Hausdorff. If \(\text{deg}(K_X, \omega) < 0\) then \(\text{deg}(\text{End}_0 \tilde{E} \otimes K_{\tilde{X}}, \omega_{\alpha}) < 0\) for any bundle \(\tilde{E}\) on a blowup \(\tilde{X}\) stable with respect to \(\omega_{\alpha}\) for \(|\alpha|\) small enough, so \(H^0(\tilde{X}, \text{End}_0 \tilde{E}) = 0 = H^2(\tilde{X}, \text{End}_0 \tilde{E})\) for any such bundle, implying that the space of deformations of \(\tilde{E}\) is smooth near \(\tilde{E}\).

If \(b_1(X)\) is even and \(\text{rank}(E_{\text{top}}) = 2\), the same arguments as given in §7 of [B4] to prove Theorem 1.4 of that reference (i.e., the last statement of Theorem 6.1 above) can be repeated and yield the same conclusion as that above, i.e., that \(\overline{\mathcal{M}}\) is compact in this case.

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