A TOPOLOGICAL LAGRANGIAN FOR MONOPOLES ON FOUR-MANIFOLDS

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ABSTRACT

We present a topological quantum field theory which corresponds to the moduli problem associated to Witten’s monopole equations for four-manifolds. The construction of the theory is carried out in purely geometrical terms using the Mathai-Quillen formalism, and the corresponding observables are described. These provide a rich set of new topological quantities.
In a recent work Witten [1] has shown that Donaldson theory [2,3] with gauge group $SU(2)$ is equivalent to a new moduli problem which involves an abelian Yang-Mills connection and a spinor coupled in a pair of “monopole equations”. This result is a consequence of previous work on $N = 2$ and $N = 4$ Yang-Mills theory [6,4,5]. The equivalence discovered by Witten is very powerful and allows to write explicit expressions for Donaldson polynomials. An immediate task which arises from his work is the search for a topological quantum field theory related to the new moduli problem presented in [1]. The observables of such topological quantum field theory could provide new topological invariants which could contain important topological information.

The aim of this paper is to construct the topological quantum field theory corresponding to the new moduli problem proposed in [1]. This will be done using the Mathai-Quillen formalism [7]. The resulting theory turns out to be an abelian Donaldson-Witten theory, which as it is widely known can be obtained from the twisting of $N = 2$ Yang-Mills theory, coupled to a twisted version of the $N = 2$ hypermultiplet [8,9,10]. The resulting type of topological model has been studied previously in [11, 12]. Related topological quantum field theories have been analyzed in [13], and their connection to the moduli problem presented in [1] has been recently considered in [14].

The Mathai-Quillen formalism allows one to construct the action of a topological quantum field theory starting from moduli problems formulated in purely geometrical terms. Moduli problems are often stated in the following form: given a moduli space $\mathcal{M}$ and a vector bundle over $\mathcal{M}$, $\mathcal{V}$, one defines the basic equations of the problem as sections of this vector bundle. Typically one is interested in computing the Euler characteristic of this bundle, or, equivalently, its Thom class. In the case at hand, because of the gauge symmetry of the theory, one also has the action of a group $\mathcal{G}$ on both, the manifold $\mathcal{M}$ and the vector bundle. Rather than compute the Euler characteristic of the bundle itself one wants to get rid of the gauge degrees of freedom and compute the Euler characteristic of the quotient bundle obtained “dividing by $\mathcal{G}$”: $\mathcal{V}/\mathcal{G} \rightarrow \mathcal{M}/\mathcal{G}$. In the same way, the
section $s : \mathcal{M} \rightarrow \mathcal{V}$ is taken to be gauge-equivariant and hence one can define the associated section $\hat{s} : \mathcal{M}/G \rightarrow \mathcal{V}/G$.

As in the Donaldson-Witten case, for monopoles on four-manifolds the vector bundle is trivial and can be written as $\mathcal{V} = \mathcal{M} \times \mathcal{F}$, where $\mathcal{F}$ is the fibre on which a $\mathcal{G}$-invariant metric is defined. If one considers the moduli space $\mathcal{M}$ as a principal bundle with group $\mathcal{G}$ the quotient bundle is the associated vector bundle $\mathcal{E} = \mathcal{M} \times_{\mathcal{G}} \mathcal{F}$. This is the situation analyzed in [15], where the results are particularized for Donaldson-Witten theory. A more general situation, involving non-trivial vector bundles, is considered in [16, 17].

To define the Mathai-Quillen form of the associated bundle $\mathcal{E}$ one needs a connection on it. If the space $\mathcal{M}$ has a $G$-invariant metric defined on it there is a natural way to construct it as follows [15]: consider on the principal bundle $\mathcal{M} \rightarrow \mathcal{M}/G$ the connection defined by declaring the horizontal subspaces to be the orthogonal ones to the vertical subspaces. The latest are just the gauge orbits given by the action of the group $\mathcal{G}$. This connection on the principal bundle $\mathcal{M}$ induces a connection on the associated bundle $\mathcal{E}$ in the standard way, and this is just the connection that one needs in the construction of the topological lagrangian. Notice, however, that in more general situations (mainly when the vector bundle of the moduli problem is not a trivial one, as it happens in topological string theory) one must add another connection to the previous one [16, 17].

As in the Donaldson-Witten theory, we will use the Cartan model for the equivariant cohomology which gives the BRST symmetry of the theory. Hence we will deal with the Cartan model of the Mathai-Quillen form. This is an equivariant differential form of the fibre $\mathcal{F}$ which can be written as:

$$U = e^{-|x|^2} \int D\chi \exp\left( \frac{1}{4} \langle \chi, \Omega \chi \rangle + i \langle dx, \chi \rangle \right). \tag{1}$$

In this expression, $x$ denotes a (commuting) vector coordinate for the fibre $\mathcal{F}$, $\chi$ a Grassman coordinate and the bracket a $\mathcal{G}$-invariant metric on $\mathcal{F}$. $\Omega$ is the universal curvature which acts on the fibre according to the action of the group $\mathcal{G}$. Now,
in order to obtain a differential form on the base space \( \mathcal{M}/G \) we must use the Chern-Weil homomorphism which has the effect of substituting \( \Omega \) by the actual curvature on \( \mathcal{M} \) and thus gives a basic differential form on \( \mathcal{M} \times \mathcal{F} \). However, in the Cartan model, due to the relation between the Cartan model and the Weil model for equivariant cohomology, one needs to make an horizontal projection in order to obtain a closed form on \( \mathcal{E} \). In other words, the differential form on \( \mathcal{M} \times \mathcal{F} \) must be evaluated on the horizontal subspace of \( \mathcal{M} \). Once we do that, we have a form on \( \mathcal{E} \) which descends to a form on \( \mathcal{M}/G \) by simply taking the pullback by the section \( \hat{s} \). This has the effect of substituting the coordinate \( x \) by the section \( \hat{s} \).

Let us describe in detail how to construct the connection on \( \mathcal{M} \) and how to enforce the horizontal projection. The gauge orbits are given by the vertical tangent space on the principal bundle with group \( G \), which is given by a map from the Lie algebra of the group \( G \), which we denote by \( \text{Lie}(G) \), to the tangent space to \( \mathcal{M} \),

\[
C : \text{Lie}(G) \longrightarrow T\mathcal{M}.
\]  

We will assume that both \( \text{Lie}(G) \) and \( \mathcal{M} \) are provided with metrics (in the case of \( \text{Lie}(G) \) this is simply an appropriate generalization of the Cartan-Killing form) so we can consider the adjoint operator \( C^\dagger \) and the operator \( R = C^\dagger C \). The connection one-form is given by [15],

\[
\Theta = R^{-1}C^\dagger.
\]  

As the Cartan representative acts on horizontal vectors, we can write the curvature as

\[
\Omega = d\Theta = R^{-1}dC^\dagger.
\]  

Now, to enforce the horizontal projection we should have to integrate over the vertical degrees of freedom which amounts to an integration over the Lie group. Alternatively, we can introduce a “projection form” [17] which, besides of projecting on the horizontal direction, automatically involves the Weil homomorphism.
which substitutes the universal curvature by the actual curvature on the bundle (4). The projection form also allows to write the correlation functions on the quotient moduli space \( \mathcal{M}/\mathcal{G} \) as integrals over \( \mathcal{M} \). Taking into account all these facts, and after some suitable manipulations, we obtain the following expression for the Thom class of the bundle \( \mathcal{E} \):

\[
\int D\eta D\chi D\phi D\lambda \exp \left( -|s|^2 + \frac{1}{4} \langle \chi, \phi \chi \rangle_g + i \langle ds, \chi \rangle + i \langle dC^\dagger, \lambda \rangle_g - i \langle \phi, R\lambda \rangle_g + i \langle C^\dagger \theta, \eta \rangle_g \right).
\]

(5)

Here, \( \phi, \lambda \) are commuting Lie algebra variables and \( \eta \) is a Grassmann one. The variables \((P, \theta)\) (the first one is commuting and present in \( s \), the second one is Grassmann) are the usual superspace coordinates for the integration of differential forms on \( \mathcal{M} \). The bracket with the subscript \( g \) is the Cartan-Killing form of \( \text{Lie}(\mathcal{G}) \). This expression is to be understood as a differential form on \( \mathcal{M} \) which when integrated out with the measure \( DP D\theta \) gives the Euler characteristic of \( \mathcal{E} \).

Let us consider now the moduli problem of monopoles on four-manifolds proposed in [1]. Let \( X \) be a spin four-manifold, endowed with a Riemannian metric \( g_{ij} \). Denote by \( S^+ \) and \( S^- \) the positive and negative chirality spin bundles on \( X \), respectively. Consider in addition a complex line bundle \( L \) with an associated \( U(1) \) connection. Let \( \mathcal{A} \) denote the moduli space of these abelian connections, and \( \Gamma(X, S^+ \otimes L) \) the sections of the product bundle \( S^+ \otimes L \), i.e., positive chirality spinors taking values in \( L \). The moduli space of our problem is thus \( \mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes L) \). The vector bundle over \( \mathcal{M} \) is a trivial one with fibre \( \mathcal{F} = \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L) \), where the first factor denotes the self-dual differential forms of degree 2 on \( X \). As in the Donaldson-Witten case, the group \( \mathcal{G} \) is the group of gauge transformations of the principal \( U(1) \)-bundle associated to the connection \( A \), whose action on the moduli space is given locally by:

\[
g^*(A_i) = A_i + ig^{-1}\partial_i g, \quad g^*(M_\alpha) = gM_\alpha,
\]

where \( M \in \Gamma(X, S^+ \otimes L) \) and \( g \) takes values in \( U(1) \). The group of gauge transformations also acts on the fibre \( \mathcal{F} \), but we must use \( g^{-1} \), as the construction
of an associated vector bundle imposes. Also notice that there is no action on the factor $\Omega^{2,\ast}(X)$, for the group is abelian. The Lie algebra of the group $G$ is just $\text{Lie}(G) = \Omega^0(X)$, as the Lie algebra of $U(1)$ is $\mathbb{R}$. Now we need metrics on both the moduli space and the vector bundle. The tangent space to the moduli space at the point $(A,M)$ is just $T(A,M) = T_A A \oplus T_M \Gamma(X,S^+ \otimes L) = \Omega^1(X) \oplus \Gamma(X,S^+ \otimes L)$, for $\Gamma(X,S^+ \otimes L)$ is a vector space. The metric on $M$ is given by:

$$
\langle (\psi, \mu), (\theta, \nu) \rangle = \int_X \psi \wedge \ast \theta + \frac{1}{2} \int_X e(\bar{\mu}^\alpha \nu_\alpha + \mu_\alpha \bar{\nu}^\alpha),
$$

where $e = \sqrt{g}$. For spinors we use the following notation. If $\mu_\alpha = (a, b)$, $\bar{\mu}^\alpha$ is chosen as $\bar{\mu}^\alpha = (a^*, b^*)$. Notice that throughout this work the signature of the metric $g_{ij}$ is Euclidean. Spinor indices are lowered and raised using the invariant tensor $C_\alpha\beta$ as in [18] ($C_\alpha\beta$ are the entries of the Pauli matrix $\sigma^2$, for example, $\bar{\mu}_\alpha = \bar{\mu}^\beta C_\beta\alpha = (i b^*, -i a^*)$). An expression analogous to (7) gives the inner product on the vector bundle $V$. Notice that we are considering both $S^+ \otimes L$ and $S^- \otimes L$ as real vector spaces of dimension four.

Let us now introduce the section associated to the “monopole equations” in [1]. Recall that the product of two spinors can be decomposed in terms of a 0-form and a self-dual 2-form. This allows one to write:

$$
\textit{s}(A,M) = \left( \frac{1}{\sqrt{2}}(F^+_{\alpha\beta} + \frac{i}{2} M_{\alpha\beta}), D_{\alpha\dot{\alpha}} M_{\alpha} \right).
$$

In this expression $D_{\alpha\dot{\alpha}}$ is the Dirac operator. In our notation, $D_{\alpha\dot{\alpha}} M_{\beta} = \sigma^j_{\alpha\dot{\alpha}}(\theta_j + i A_j) M_{\beta}$ where the matrices $\sigma^j$ are $\sigma^j = (1, i \sigma^1, i \sigma^2, i \sigma^3)$, being $\sigma^1$, $\sigma^2$ and $\sigma^3$ the Pauli matrices. In (8) $F^+_{\alpha\beta}$ is the self-dual part of the gauge field-strength:

$$
F^+_{\alpha\beta} = (p^+(F))_{\alpha\beta} = C^{\dot{\alpha}\dot{\beta}}(\sigma^i)_{\alpha\dot{\alpha}}(\sigma^j)_{\beta\dot{\beta}} \frac{1}{2}(F_{ij} - \frac{1}{2e} \epsilon_{ijkl} F_{kl}),
$$

being $C^{\dot{\alpha}\dot{\beta}}$ the matrix $-\sigma^2$, and $\epsilon_{ijkl}$ the totally antisymmetric tensor density. In (9) $p^+(Z)$ denotes the projection of a two-form $Z$ into its self-dual part. The factor $1/\sqrt{2}$ in (8) has been introduced for convenience as will be explained below.
Once the underlying geometry of the model has been presented we will construct the topological lagrangian from the general expression (5). First of all we must describe explicitly the tangent vertical space by means of the operator $C$.

This is simply obtained from (6), and it reads:

$$C(\epsilon) = (-d\epsilon, i\epsilon M) \in \Omega^1(X) \oplus \Gamma(X, S^+ \otimes L), \quad \epsilon \in \Omega^0(X). \quad (10)$$

To obtain the adjoint of this operator we must use the Cartan-Killing form on $\text{Lie}(G) = \Omega^0(X)$, which is just the usual product of differential forms on $X$. One finds:

$$C^\dagger(\psi, \mu) = -d^*\psi + \frac{i}{2}(\bar{\mu^a M_\alpha} - \overline{M^a} \mu_\alpha), \quad (11)$$

and hence the operator $R$ is given by

$$R = d^*d + \overline{M^a} M_\alpha. \quad (12)$$

Another operator we need to write the lagrangian is $ds : T_{(A,M)}M \rightarrow F$. To obtain it we must linearize the monopole equations. The result is:

$$ds(\psi, \mu) = \left(\frac{1}{\sqrt{2}}((p^+(d\psi))_{\alpha \beta} + \frac{i}{2}((\overline{M_{(\alpha}} M_{\beta)}) + D_{\alpha \hat{a}} \mu^a + i\psi_{\alpha \hat{a}} M^a\right), \quad (13)$$

where $p^+$ is the projection defined in (9). The maps $C$ and $ds$ are important because they give the instanton deformation complex:

$$0 \rightarrow \Omega^0(X) \xrightarrow{C} \Omega^1(X) \oplus \Gamma(X, S^+ \otimes L) \xrightarrow{ds} \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L) \rightarrow 0. \quad (14)$$

The index of this complex is precisely minus the dimension of the tangent space to the zero locus of the section $\hat{s}$, which is the moduli space of solutions to the monopole equations modulo gauge transformations. These two operators are also the operators appearing in the fermion kinetic terms of the lagrangian, as one can see in (5), and the referred index computes the difference of zero modes of the
fermion fields. To obtain the index of this complex, notice that we can drop out the terms of order zero of the involved operators, for their leading symbol is not changed. In this way we obtain an equivalent complex which factorizes into the complex for the Dirac operator (the leading term for $ds$) and the complex

$$0 \rightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{p^+ d} \Omega^{2,+}(X) \rightarrow 0$$

where $p^+$ is the projection into the sel-dual part defined in (9). The index we are looking for is simply the index of the second complex minus the index of the Dirac operator multiplied by two. As $X$ is four-dimensional, we obtain for the last one $c_1(L)^2 - p_1(X)/12$. For the second complex, the index is $b_0 - b_1 + b_2^+$, which can be written as $(\chi + \sigma)/2$, where $\chi$ is the Euler characteristic of $X$ and $\sigma = b_2^+ - b_2^-$ its signature. Now, using the Hirzebruch signature formula, we have $\sigma = p_1(X)/3$. Taking all this into account, we obtain [1]:

$$\text{index } T = \frac{2\chi + 3\sigma}{4} - c_1^2(L).$$

In order to write the topological quantum field theory associated to the moduli problem we must indicate the field content and the topological symmetry. These are determined by the geometrical structure we have been developing. For the moduli space we have commuting fields $P = (A, M) \in \mathcal{M} = A \times \Gamma(X, S^+ \otimes L)$, with ghost number 0 and their superpartners, representing a basis of differential forms on $\mathcal{M}$, $\theta = (\psi, \mu)$, with ghost number 1. Now, we must introduce fields for the fibre (corresponding to the $\chi$ variable in (5)), which we denote by $(\chi_{ij}, v_\alpha) \in \Omega^{2,+}(X) \oplus \Gamma(X, S^- \otimes L)$, with ghost number 1. It is also useful in the construction of the action from gauge fermions to introduce auxiliary commuting fields with the same geometrical content, $(H_{ij}, h_\alpha)$. The gauge symmetry makes necessary to introduce three fields in $\text{Lie}(G)$, as we have remarked in writing (5). The field $\phi$, with ghost number 2, is a commuting one. It roughly corresponds to the universal curvature and enters in the equivariant cohomology of $\mathcal{M}$. The fields $\lambda, \eta$, with
ghost number $-2$ and $-1$, respectively, come from the projection form, as explained in [17]. The BRST cohomology of the model is:

\begin{align}
[Q, A_i] &= \psi_i, \\
\{Q, \psi_i\} &= \partial_i \phi, \\
[Q, \phi] &= 0 \\
\{Q, \chi_{ij}\} &= H_{ij}, \\
[Q, H_{ij}] &= 0, \\
[Q, \lambda] &= \eta, \\
[Q, M_\alpha] &= \mu_\alpha, \\
\{Q, \mu_\alpha\} &= -i\phi M_\alpha, \\
\{Q, \mu_\alpha\} &= -i\phi M_\alpha, \\
\{Q, v_\dot{\alpha}\} &= h_\dot{\alpha}, \\
[Q, h_\dot{\alpha}] &= -i\phi v_\dot{\alpha}, \\
\{Q, \eta\} &= 0.
\end{align}

(17)

For the fields on the base space the BRST operator is the Cartan differential for equivariant cohomology $Q = d - \iota_\phi$, where $\iota_\phi$ denotes the interior product on differential forms. The fields $\psi_i$ represent a basis of differential forms and they can be interpreted formally as $dA_i$ (we can also see them as a basis of tangent vectors). Notice also that $Q^2 A_i = -\mathcal{L}_\phi A_i$, where $\mathcal{L}_\phi$ denotes the Lie derivative generated by $\phi$. The same considerations apply to the fields $M_\alpha, \mu_\alpha$. For the fibre variables we close the algebra up to a gauge transformation generated by $-\phi$ (recall that the group acts on the fibre with $g^{-1}$). We are now in the position of writing out the action of the theory. Let us consider first the last five terms in the exponential of the Thom class (5),
\[-i \langle \phi, R \lambda \rangle_g = -i \int_X \lambda \wedge \ast d^* d\phi - i \int_X e\phi \lambda \overline{M}^\alpha M_\alpha, \]

\[i \langle (\chi, v), ds \rangle = \frac{i}{\sqrt{2}} \int_X \chi \wedge \ast p^+ d\psi - \frac{1}{2\sqrt{2}} \int_X e\chi^{\alpha\beta} (\overline{M}_{(\alpha\beta)} + \bar{\mu}_{(\alpha\beta)}) + \frac{i}{2} \int_X e(\overline{M}^\alpha \psi_{\alpha\dot{a}} \psi^\dot{a} - \bar{\psi}^\dot{a} \psi_{\alpha\dot{a}} M^\alpha), \]

\[i \langle C^\dagger (\psi, \mu_\alpha), \eta \rangle_g = -i \int_X \eta \wedge \ast d\psi - \frac{1}{2} \int_X e\eta (\bar{\mu}^\alpha M_\alpha - \overline{M}^\alpha \mu_\alpha), \]

\[\frac{1}{4} \langle (\chi, v), \phi (\chi, v) \rangle_g = -\frac{i}{4} \int_X e\bar{\phi} \psi_{\alpha\dot{a}}, \]

\[i \langle dC^\dagger, \lambda \rangle_g = \int_X e\lambda \bar{\mu}^\alpha \mu_\alpha. \quad (18)\]

The two last terms are obtained as follows. For the term involving \(\langle (\chi, v), \phi (\chi, v) \rangle_g\) one must take into account the action of \(\text{Lie}(G)\) on \(v_{\dot{a}}\), which lives in the fibre: \(\phi (v_{\dot{a}}) = -i\phi v_{\dot{a}}\). On the 2-forms, the action of \(\text{Lie}(G)\) is trivial, because the Lie algebra of \(U(1)\) is abelian. To compute \(dC^\dagger\), which is a 2-form on the moduli space, one can evaluate it on a basis of tangent vectors using the expression \(dC^\dagger (\mu_1, \mu_2) = \mu_1 (C^\dagger (\mu_2)) - \mu_2 (C^\dagger (\mu_1)) - C^\dagger ([\mu_1, \mu_2])\), and take into account that, as \(\mu_1, \mu_2\) are constant vector fields, their Lie bracket is 0. Finally, we must compute the section term in (5). It takes the form,

\[|s(A, M)|^2 = \frac{1}{2} \int_X e(F^{+\alpha\beta} + \frac{i}{2} \overline{M}^\alpha \overline{M}^\beta)(F^{\alpha\beta}_+ + \frac{i}{2} \overline{M}^\alpha M_\beta) + \int_X e D_{\alpha\dot{a}} \overline{M}^\alpha D_{\dot{a}} M^\dot{a} M^\beta \]

\[= \int_X e [g^{ij} D_i \overline{M}^\alpha D_j M_\alpha + \frac{1}{4} R M^\alpha M_\alpha + \frac{1}{2} F^{+\alpha\beta} F_{\alpha\beta}^+ - \frac{1}{8} \overline{M}^\alpha \overline{M}^\beta \overline{M}^\alpha \overline{M}^\beta], \quad (19)\]

where \(R\) is the scalar curvature (not to be confused with the operator \(R\) in (3)) on \(X\). To obtain the second expression in this equation one can either write explicitly the form of the Dirac operator, or, alternatively, one can integrate by parts, use the relation \(D_{\alpha\dot{a}} D_{\dot{a}} M^\dot{a} M^\beta = (g^{ij} D_i D_j - \frac{1}{4} R) M_\alpha + iF^{+\alpha\beta} \overline{M}^\beta\), and then integrate back.
by parts. Notice that if one denotes the components of \( M_\alpha \) by \( M_\alpha = (a, b) \), the last factor in (19) is in fact \( \frac{1}{2}(|a|^2 + |b|^2)^2 \), and therefore it is positive definite. The factor \( i\overline{M}^\alpha F^+_{\alpha\beta} M^\beta \) has cancelled in the sum, and then each term in the second expression for \( |s(A, M)|^2 \) in (19) is positive definite except the one involving the scalar curvature. This was the reason of choosing the factor \( 1/\sqrt{2} \) in (8). The advantage of this form of the bosonic sector in the action is that one can apply vanishing theorems which improve the analysis of the space of solutions of the monopole equations [1, 5].

The action resulting after adding all the terms in (18) to (19) is manifestly topological, for it is the field theoretical representation of the Thom class of the bundle \( \mathcal{E} \). Let us show how it can be obtained in a more standard way from a BRST symmetry (\( i.e. \) a nilpotent \( Q \) operator up to gauge transformations) and an appropriate choice of gauge fermion \( \Psi \) such that the action resulting from (18) and (19) is \( -\{Q, \Psi\} \) after introducing auxiliary fields. This approach to topological quantum field theories can be regarded from the traditional BRST point of view initiated in [19] and reviewed in [20], or from a modern perspective as described in [17]. We will follow in this paper the latter. In a topological field theory with gauge symmetry there exists a localization gauge fermion which comes directly from the Cartan model representative of the Thom class (5) with additional auxiliary fields \((H, h)\),

\[
\Psi_{\text{loc}} = -i\langle (\chi, v), s(A, M) \rangle - \frac{1}{4} \langle (\chi, v), (H, h) \rangle,
\]

(20)

and a projection gauge fermion which implements the horizontal projection,

\[
\Psi_{\text{proj}} = i\langle \lambda, C^\dagger(\psi, \mu) \rangle_g.
\]

(21)

Using the \( Q \)-transformations (17) we obtain the localization and the projection
This action turns out to be:

\[
\{Q, \Psi_{\text{loc}}\} = \{Q, \int X e \left[ - \chi^{\alpha \beta} \left( \frac{i}{\sqrt{2}} (F_{\alpha \beta}^+ + \frac{i}{2} \overline{M} (\alpha M\beta) + \frac{1}{4} H_{\alpha \beta}) \right) \right. \\
\left. - \frac{i}{2} (\bar{v}^\dot{\alpha} D_{\dot{\alpha} \dot{\alpha}} M^\alpha + \overline{M}^\alpha D_{\dot{\alpha} \dot{\alpha}} v^\dot{\alpha}) - \frac{1}{8} (\bar{v}^\dot{\alpha} h_{\dot{\alpha}} - \bar{h}_{\dot{\alpha}} v^\dot{\alpha}) \right] \} \\
= \int X e \left[ - \frac{i}{\sqrt{2}} H^{\alpha \beta} (F_{\alpha \beta}^+ + \frac{i}{2} \overline{M} (\alpha M\beta) + \frac{1}{8} \chi^{\alpha \beta} ((p^+ (d\psi))_{\alpha \beta} + \frac{i}{2} (\bar{\mu}_{(\alpha M\beta)} + \overline{M} (\alpha M\beta) ) \\
- \frac{1}{4} H^{\alpha \beta} H_{\alpha \beta} - \frac{i}{2} (\bar{v}^\dot{\alpha} D_{\dot{\alpha} \dot{\alpha}} M^\alpha + \overline{M}^\alpha D_{\dot{\alpha} \dot{\alpha}} h^\dot{\alpha}) + \frac{i}{2} (\bar{v}^\dot{\alpha} D_{\dot{\alpha} \dot{\alpha}} h^\dot{\alpha} - \bar{h}^\dot{\alpha} v^\dot{\alpha}) \\
+ \frac{1}{2} (\overline{M}^\alpha \psi_{\dot{\alpha} \dot{\alpha}} v^\dot{\alpha} - \bar{v}^\dot{\alpha} \psi_{\dot{\alpha} \dot{\alpha}} M^\alpha) - \frac{1}{4} (\bar{h}^\dot{\alpha} h_{\dot{\alpha}} + i \phi \bar{v}^\dot{\alpha} v_{\dot{\alpha}}) \right], \\
\text{(22)}
\]

\[
\{Q, \Psi_{\text{proj}}\} = \{Q, - \int X e [i \lambda \wedge \ast d^* \psi + \frac{1}{2} e \lambda (\bar{\mu}^\alpha M\alpha - \overline{M}^\alpha \mu^\alpha)] \} \\
= - \int X e [i (\eta \wedge \ast d^* \psi + \lambda \wedge \ast d^* d\phi) + \frac{1}{2} e \eta (\bar{\mu}^\alpha M\alpha - \overline{M}^\alpha \mu^\alpha) \\
- e \lambda (\bar{\mu}^\alpha \mu^\alpha - i \phi \overline{M}^\alpha M\alpha)] . \\
\text{(23)}
\]

The sum of (22) and (23) is just the same as the sum of the terms in (18) plus \(-|s(A, M)|^2\) as given in (19) once the auxiliary fields \(H_{\alpha \beta}\) and \(h_{\dot{\alpha}}\) have been integrated out. This is indeed the exponent appearing in the Thom class (5) which must be identified as minus the action, \(-S\), of the topological quantum field theory. This action turns out to be:

\[
S = \int X e \left[ g^{ij} \overline{M}^\alpha D_i M\alpha + \frac{1}{4} R \overline{M}^\alpha M\alpha + \frac{1}{2} F_{\alpha \beta}^+ F_{\alpha \beta}^+ - \frac{1}{8} \overline{M} (\alpha M\beta) \overline{M} (\alpha M\beta) \right] \\
+ i \int X e \left( \lambda \wedge \ast d^* d\phi - \frac{1}{\sqrt{2}} \chi \wedge \ast p^+ d\psi + \eta \wedge \ast d\psi \right) \\
+ \int X e \left( i \phi \lambda \overline{M}^\alpha M\alpha + \frac{1}{2 \sqrt{2}} \chi^{\alpha \beta} (\overline{M}_{(\alpha M\beta)} + \bar{\mu}_{(\alpha M\beta)}) - \frac{i}{2} (\bar{v}^\dot{\alpha} D_{\dot{\alpha} \dot{\alpha}} h^\dot{\alpha} - \bar{h}^\dot{\alpha} v^\dot{\alpha}) \\
- \frac{1}{2} (\overline{M}^\alpha \psi_{\dot{\alpha} \dot{\alpha}} v^\dot{\alpha} - \bar{v}^\dot{\alpha} \psi_{\dot{\alpha} \dot{\alpha}} M^\alpha) + \frac{1}{2} \eta (\bar{\mu}^\alpha M\alpha - \overline{M}^\alpha \mu^\alpha) + \frac{i}{4} \phi \bar{v}^\dot{\alpha} v_{\dot{\alpha}} - \lambda \bar{\mu}^\alpha \mu^\alpha \right) . \\
\text{(24)}
\]

This action is invariant under the modified BRST transformations which are ob-
tained from (17) after integrating out the auxiliary fields. It contains the standard
gauge fields of a twisted $N = 2$ vector multiplet, or Donaldson-Witten fields, cou-
pled to the “matter” fields of the twisted $N = 2$ hypermultiplet.

The observables of the theory are built out of products of BRST invariant
operators which are cohomologically non-trivial. These observables are based on
forms which can be grouped into families labeled by a positive integer $n$. These
forms can be obtained solving the standard descent equations [3] or using the $G_i$
operators in [12]. One can also use the method explained in [21]. They turn out
to be:

$$
\Theta^n_0 = \binom{n}{0} \phi^n, \quad \Theta^n_1 = \binom{n}{1} \phi^{n-1} \psi,
$$
$$
\Theta^n_2 = \binom{n}{2} \phi^{n-2} \psi \wedge \psi + \binom{n}{1} \phi^{n-1} F,
$$
$$
\Theta^n_3 = \binom{n}{3} \phi^{n-3} \psi \wedge \psi \wedge \psi + 2 \binom{n}{2} \phi^{n-2} \psi \wedge F,
$$
$$
\Theta^n_4 = \binom{n}{4} \phi^{n-4} \psi \wedge \psi \wedge \psi \wedge \psi + 3 \binom{n}{3} \phi^{n-3} \psi \wedge \psi \wedge F + \binom{n}{2} \phi^{n-2} F \wedge F.
$$

The ghost number of the $i$-form $\Theta^n_i$ is $2n - i$ for $i = 0, 1, 2, 3, 4$. We have not found
non-trivial observables involving the “matter” fields. The forms (25) verify the
descent equations,

$$
[Q, \Theta^n_i] = d \Theta^n_{i-1},
$$

and therefore one can associeate a $Q$-invariant operator to each of them in the
following way. Let $x$ denote a point in $X$, and $\gamma_j$ a $j$-cycle for $j = 1, 2, 3$. The
$Q$-invariant operators have the form:

$$
O_0(n, x) = \Theta^n_0(x),
$$
$$
O_1(n, \gamma_1) = \int_{\gamma_1} \Theta^n_1, \quad O_3(n, \gamma_3) = \int_{\gamma_3} \Theta^n_3,
$$
$$
O_2(n, \gamma_2) = \int_{\gamma_2} \Theta^n_2, \quad O_4(n) = \int_{X} \Theta^n_4.
$$
Observables are built out of products of these operators. In order to have non-trivial contributions the ghost number of these products must match the ghost-number anomaly in the theory, which coincides with the index calculated in (16). This is a necessary condition to get a non-trivial vacuum expectation value but certainly is not sufficient.

In [1] Witten showed that in certain situations in which the ghost-number anomaly vanishes the sum of the partition function of the theory over classes of $U(1)$ bundles such that the index in (16) vanishes is related to Donaldson invariants. The construction of the topological quantum field theory provides a rich set of operators whose vacuum expectation values might lead to interesting topological invariants in more general situations. Certainly, it opens the possibility of discovering new topological invariants unrelated to Donaldson invariants.

The theory constructed in this work can be generalized in several directions. One would correspond to the abelian $U(1)^n$ case with “matter” fields carrying different charges. This generalization is rather straightforward after our construction for the simple $U(1)$ case. More interesting but certainly not so simple is its non-abelian counterpart. The construction of the non-abelian generalization can be carried out also in the framework of the Mathai-Quillen formalism using technics which are similar to the ones used in this paper. Work in this direction will be reported elsewhere.

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