On the semigroup generated by the renormalized Nelson Hamiltonian

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Abstract

We consider the renormalized Nelson model at a fixed total momentum $P$: $H_{\text{ren}}(P)$; The Hamiltonian $H_{\text{ren}}(P)$ is defined through an infinite energy renormalization. We prove that $e^{-\beta H_{\text{ren}}(P)}$ is positivity improving for all $P \in \mathbb{R}^3$ and $\beta > 0$ in the Fock representation.

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1 Introduction

In a celebrated paper [29], Nelson studies the Hamiltonian, which describes the interaction of $N$ particles with a massive Bose field. He constructs a model without the ultraviolet cutoff through an infinite energy renormalization. We expect that his observation provides a hint to understand renormalization procedures in more complicated models; His model is nowadays called the Nelson model, and has been actively studied.

For example, Fröhlich studies the Nelson model at a fixed total momentum [9, 10]; asymptotic completeness is addressed in [2, 7]; existence of a ground state is proved in [3, 16, 34]; functional integral representations are constructed in [15, 18, 21], and so on [1, 12, 14, 17, 30, 38].

The cutoff Nelson Hamiltonian reads

$$H_\Lambda = -\frac{1}{2}\Delta - g \int_{\mathbb{R}^3} dk \frac{\chi_\Lambda(k)}{\sqrt{\omega(k)}} (e^{ik \cdot x} a(k) + e^{-ik \cdot x} a(k)^*) + H_f$$

acting in

$$L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ is the bosonic Fock space over $L^2(\mathbb{R}^3)$. Recall that

$$\mathcal{F} = \sum_{n \geq 0} \mathcal{L}^{\text{sym}}_{2n}(\mathbb{R}^3),$$
where $L^2_{\text{sym}}(\mathbb{R}^{3n}) = \{ \varphi \in L^2(\mathbb{R}^{3n}) \mid \varphi(k_1, \ldots, k_n) = \varphi(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) \text{ a.e. } \forall \sigma \in \mathfrak{S}_n \}$ and $L^2_{\text{sym}}(\mathbb{R}^3) = \mathbb{C}$ (where $\mathfrak{S}_n$ is the permutation group on a set $\{1, 2, \ldots, n\}$). The single particle Schrödinger operator $-\frac{1}{2}\Delta$ is the Hamiltonian of the free particle, where $\Delta$ is the 3-dimensional Laplacian. The annihilation- and creation operators of the field, $a(k)$ and $a(k)^*$, satisfy the standard commutation relations:

$$[a(k), a(k')^*] = \delta(k - k'), \quad [a(k), a(k')] = 0, \quad k, k' \in \mathbb{R}^3. \quad (1.4)$$

The field energy $H_f$ is given by

$$H_f = \int_{\mathbb{R}^3} dk \omega(k) a(k)^* a(k). \quad (1.5)$$

The dispersion relation $\omega(k)$ is given by

$$\omega(k) = \sqrt{k^2 + m^2}, \quad m > 0. \quad (1.6)$$

The ultraviolet cutoff function $\chi_\Lambda (\Lambda > 0)$ is defined by

$$\chi_\Lambda(k) = \begin{cases} 
1, & |k| \leq \Lambda \\
0, & |k| > \Lambda. 
\end{cases} \quad (1.7)$$

The prefactor $g$ is a coupling strength between the particle and the field. Without loss of generality, we may assume that

$$g > 0. \quad (1.8)$$

The interaction is infinitesimally small relative to the free Hamiltonian. Hence, by the Kato-Rellich theorem, $H_\Lambda$ is self-adjoint on the domain $\text{dom}(-\Delta) \cap \text{dom}(H_f)$ and bounded from below.

The generator of translations is the total momentum operator

$$P_{\text{tot}} = -i\nabla + P_t \quad (1.9)$$

with $P_t = \int_{\mathbb{R}^3} dk ka(k)^* a(k)$. The total momentum is conserved, namely, $e^{iaP_{\text{tot}}} H_\Lambda = H_\Lambda e^{iaP_{\text{tot}}}$ for all $a \in \mathbb{R}^3$. Therefore, $H_\Lambda$ admits the direct integral decomposition

$$UH_\Lambda U^* = \int_{\mathbb{R}^3} \mathbb{S} H_\Lambda(P) dP, \quad (1.10)$$

$$H_\Lambda(P) = \frac{1}{2} (P - P_t)^2 - g \int_{\mathbb{R}^3} dk \frac{\chi_\Lambda(k)}{\sqrt{\omega(k)}} (a(k) + a(k)^*) + H_f, \quad (1.11)$$

where $U$ is some unitary operator on $L^2(\mathbb{R}^3) \otimes \mathfrak{F}$. $H_\Lambda(P)$ acts in $\mathfrak{F}$. By the Kato-Rellich theorem again, $H_\Lambda(P)$ is self-adjoint on the domain $\text{dom}(P_t^2) \cap \text{dom}(H_f)$ and bounded from below for all $P \in \mathbb{R}^3$. $H_\Lambda(P)$ is called the cutoff Nelson Hamiltonian at a fixed total momentum $P$.

Let

$$E_\Lambda = -g^2 \int_{\mathbb{R}^3} dk \frac{\chi_\Lambda(k)}{\omega(k)(\omega(k) + k^2/2)}. \quad (1.12)$$
Notice that $E_\Lambda \to -\infty$ as $\Lambda \to \infty$. We define

$$H_{\text{ren},\Lambda} = H_\Lambda - E_\Lambda, \quad H_{\text{ren},\Lambda}(P) = H_\Lambda(P) - E_\Lambda.$$  \hspace{1cm} (1.13)

Nelson’s result is stated as follows.

**Theorem 1.1 (Removal of UV cutoff [29])**  
(i) There exists a self-adjoint operator $H_{\text{ren}}$ bounded from below such that $H_{\text{ren},\Lambda}$ converges to $H_{\text{ren}}$ in strong resolvent sense as $\Lambda \to \infty$.

(ii) For all $P \in \mathbb{R}^3$, there exists a self-adjoint operator $H_{\text{ren}}(P)$ bounded from below such that $H_{\text{ren},\Lambda}(P)$ converges to $H_{\text{ren}}(P)$ in strong resolvent sense as $\Lambda \to \infty$.

In this study, we are interested in the renormalized Nelson Hamiltonian at a fixed total momentum: $H_{\text{ren}}(P)$.

Following Fröhlich [9, 10], we introduce a convex cone $\mathfrak{F}^+$ by

$$\mathfrak{F}^+ = \sum_{n \geq 0} L^2_{\text{sym}}(\mathbb{R}^{3n})^+,$$  \hspace{1cm} (1.14)

where $L^2_{\text{sym}}(\mathbb{R}^{3n})^+ = \{ \varphi \in L^2_{\text{sym}}(\mathbb{R}^{3n}) | \varphi(k_1, \ldots, k_n) \geq 0 \text{ a.e.} \}$ with $L^2_{\text{sym}}(\mathbb{R}^0)^+ = \mathbb{R}^+ = \{ r \in \mathbb{R} | r \geq 0 \}$. To state our results, the following terminologies are needed.

**Definition 1.2**  
• A vector $\varphi \in \mathfrak{F}$ is called **positive** if $\varphi \in \mathfrak{F}^+$;

• A vector $\varphi = \sum_{n \geq 0} \varphi_n \in \mathfrak{F}$ is called **strictly positive** if $\varphi_n(k_1, \ldots, k_n) > 0$ a.e. for all $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$;

• We say that a bounded linear operator $A$ is **positivity preserving** if $A \mathfrak{F}^+ \subseteq \mathfrak{F}^+$;

• A bounded linear operator $A$ is called **positivity improving** if $A\varphi$ is strictly positive whenever $\varphi$ is positive and $\varphi \neq 0$. ◦

Our main theorem is the following.

**Theorem 1.3** $e^{-\beta H_{\text{ren}}(P)}$ is positivity improving for all $P \in \mathbb{R}^3$ and $\beta > 0$.

The following corollary immediately follows from Theorems 1.3 and 2.10.

**Corollary 1.4** Suppose that $E(P) = \inf \text{spec}(H_{\text{ren}}(P))$ is an eigenvalue. Then $E(P)$ is a simple eigenvalue with a strictly positive eigenvector.

**Remark 1.5**  
(i) By applying methods in [9, 19], we can prove that $E(P)$ is actually an eigenvalue, provided that $|P| < 1$.

(ii) Theorem 1.3 remains true when we consider the Hamiltonian $H_{\text{ren}}(P)$ with $\omega$ and $\chi_\Lambda$ replaced by $\omega_0(k) = |k|$ and $\chi_\Lambda^\sigma = \chi_\Lambda - \chi_\sigma$, where the infrared cutoff $\sigma$ is chosen so that $0 < \sigma < \Lambda$. (Note that when $\sigma = 0$, we have to take extra care for the infrared problem, see, e.g., [3, 18, 34]. We will examine such a case in [27].) ◦
In order to explain our achievement, let us introduce the modified Nelson Hamiltonian by

\[ H_\varrho(P) = \frac{1}{2}(P - P_1)^2 - g \int_{\mathbb{R}^3} dk \frac{\varrho(k)}{\sqrt{\omega(k)}} (a(k) + a(k)^*) + H_f, \tag{1.15} \]

where \( \varrho(k) \) is real-valued. Under the assumptions

\[ \omega^{-1/2} \varrho, \quad \omega^{-1} \varrho \in L^2(\mathbb{R}^3), \tag{1.16} \]

\( H_\varrho(P) \) is self-adjoint on \( \text{dom}(P_1^2) \cap \text{dom}(H_f) \), and bounded from below for all \( P \in \mathbb{R}^3 \). In a famous paper [9], Fröhlich has shown that, if \( \varrho(k) > 0 \) a.e. \( k \), then \( e^{-\beta H_\varrho(P)} \) is positivity improving for all \( P \in \mathbb{R}^3 \) and \( \beta > 0 \) in the Fock representation. His idea has been applied to the polaron problem successfully [11, 28, 37]. In particular, it has been proven in [23, 24, 25] that the semigroup generated by the Fröhlich Hamiltonian without ultraviolet cutoff is positivity improving for all \( P \in \mathbb{R}^3 \). Note that, in [35, 36], Sloan has proved that the semigroup generated by the two-dimensional polaron model without ultraviolet cutoff is positivity improving for \( P = 0 \); His beautiful method is different from Fröhlich’s approach, and is applicable in the Schrödinger representation. The primary reason for these successes is that no energy renormalization is needed, when we remove the ultraviolet cutoff from the polaron models.

In contrast to the polaron problem, the Hamiltonian \( H_{\text{ren}}(P) \) is defined through an infinite energy renormalization. By this obstacle, Fröhlich’s original method only tells us that \( e^{-\beta H_{\text{ren}}(P)} \) is positivity preserving for all \( P \in \mathbb{R}^3 \) and \( \beta > 0 \). It has been a long standing problem to prove that \( e^{-\beta H_{\text{ren}}(P)} \) is positivity improving for all \( P \in \mathbb{R}^3 \). To overcome this difficulty, we apply operator theoretic correlation inequalities studied in [22, 23, 24, 25, 26]. In our previous works on the polaron models [23, 24, 25], we have clarified that this approach is very useful for studies on the semigroup generated by the operator. In the present paper, we further develop this method so that we can get over a difficulty arising from the infinite energy renormalization.

For readers’ convenience, we give a brief outline of the proof of Theorem 1.3 here. For every \( \kappa > 0 \), let \( B_\kappa \) be the ball of radius \( \kappa \) in \( \mathbb{R}^3 \) centered at the origin. Let \( \tilde{\mathcal{F}}^{\leq \kappa} \) be the Fock space over \( L^2(B_\kappa) \) and let \( \tilde{\mathcal{F}}^{> \kappa} \) be the Fock space over \( L^2(B_\kappa^c) \), where \( B_\kappa^c \) is the complement of \( B_\kappa \). The Fock space \( \tilde{\mathcal{F}} \) can be factorized as

\[ \tilde{\mathcal{F}} = \tilde{\mathcal{F}}^{\leq \kappa} \otimes \tilde{\mathcal{F}}^{> \kappa}. \tag{1.17} \]

Corresponding to (1.17), \( H_{\text{ren}}(P) \) can be decomposed as

\[ H_{\text{ren}}(P) = H_{\text{ren}}^{\leq \kappa}(P) \otimes 1 + C_\kappa \otimes K_\kappa, \tag{1.18} \]

where \( \otimes \) indicates the form sum. The local part \( H_{\text{ren}}^{\leq \kappa}(P) \) acts in \( \tilde{\mathcal{F}}^{\leq \kappa} \), while \( K_\kappa \) lives in \( \tilde{\mathcal{F}}^{> \kappa} \). \( C_\kappa \) is the cross-term. In Section 4, we will prove the following: To show that \( e^{-\beta H_{\text{ren}}(P)} \) improves the positivity in \( \tilde{\mathcal{F}} \), it suffices to show that \( e^{-\beta H_{\text{ren}}^{\leq \kappa}(P)} \) improves the positivity in \( \tilde{\mathcal{F}}^{\leq \kappa} \) and \( e^{-\beta K_\kappa} \) preserves the positivity in \( \tilde{\mathcal{F}}^{> \kappa} \) for all \( \kappa > 0 \). On the other hand, we can apply Fröhlich’s idea to see that \( e^{-\beta H_{\text{ren}}^{\leq \kappa}(P)} \) improves the positivity in \( \tilde{\mathcal{F}}^{\leq \kappa} \). In this way, we obtain Theorem 1.3. The most difficult part in the above is the reduction of the positivity improvingness of \( e^{-\beta H_{\text{ren}}(P)} \) to the properties of \( e^{-\beta H_{\text{ren}}^{\leq \kappa}(P)} \).
This procedure can be achieved by extending Faris’ idea in [8] as we will see in Section 4.

Path measure methods have been actively studied, and made remarkable progress [15, 21, 18]. As far as we are aware, this methods can only cover a case where \( P = 0 \); To be precise, it can be proved by a functional integral formula that \( e^{-\beta H_{\text{ren}}(0)} \) is positivity improving in the Schrödinger representation. Note that this methods work for \( P = 0 \) only. In contrast to this, our methods work for all \( P \in \mathbb{R}^3 \), and are effective in the Fock representation. On the other hand, path measure methods can treat the Hamiltonian \( H_{\text{ren}} + V \) with an external potential \( V : \mathbb{R}^3 \to \mathbb{R} \). By using ideas in [26], our approach can also cover this case only if \( V \) is assumed to be ferromagnetic\(^3\). We will discuss this problem in [27]. In conclusion, our operator theoretic and path measure methods complement each other and both have specific advantages.

Recently, Griesemer and Wünsch reported an interesting finding of the domain property of the renormalized Nelson Hamiltonian in [12]. Namely, they showed that the domain of the Nelson model satisfies \( \text{dom}(H_{\text{ren}}) \cap \text{dom}(H_0) = \{0\} \), where \( H_0 = -\Delta + H_f \). Fortunately, this anomalous property unaffects our arguments in the present paper. To be more precise, the point of our proof is the reduction of the problem to the local properties as we mentioned above; this step is essentially based on the algebraic relation (1.18), and detailed information on the domain is unnecessary for our proof.

The organization of the present paper is as follows: In Section 2, we briefly review some basic properties of operator theoretic correlation inequalities. Section 3 is devoted to study useful properties of the second quantized operators. In Section 4, we prove Theorem 1.3 by applying operator theoretic correlation inequalities. In Appendix A, we give a list of fundamental facts that are used in the main sections.

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2 Operator theoretic correlation inequalities

2.1 Positivity preserving operators

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathfrak{P} \) be a convex cone in \( \mathcal{H} \). We say that \( \mathfrak{P} \) is self-dual if

\[
\mathfrak{P} = \{ x \in \mathcal{H} \mid \langle x|y \rangle \geq 0 \ \forall \ y \in \mathfrak{P} \}.
\]

Henceforth, we always assume that \( \mathfrak{P} \neq \{0\} \). The following properties of \( \mathfrak{P} \) are well-known [4, 5]:

**Proposition 2.1** We have the following:

\(^3\)Roughly speaking, we say that \( V \) is ferromagnetic if \( \hat{V}(k) < 0 \), where \( \hat{V} \) is the Fourier transformation of \( V \).
Lemma 2.3

(i) \( \mathcal{P} \cap (-\mathcal{P}) = \{0\} \).

(ii) There exists a unique involution \( j \) in \( \mathcal{H} \) such that \( jx = x \) for all \( x \in \mathcal{P} \).

(iii) Each element \( x \in \mathcal{H} \) with \( jx = x \) has a unique decomposition \( x = x_+ - x_- \), where \( x_+, x_- \in \mathcal{P} \) and \( (x_+|x_-) = 0 \).

(iv) \( \mathcal{H} \) is linearly spanned by \( \mathcal{P} \).

Definition 2.2

- A vector \( x \) is said to be positive w.r.t. \( \mathcal{P} \) if \( x \in \mathcal{P} \). We write this as \( x \geq 0 \text{ w.r.t. } \mathcal{P} \).
- A vector \( x \in \mathcal{P} \) is called strictly positive w.r.t. \( \mathcal{P} \) whenever \( (x|y) > 0 \) for all \( y \in \mathcal{P} \setminus \{0\} \). We write this as \( x > 0 \text{ w.r.t. } \mathcal{P} \).
- Let \( \mathcal{H}_R = \{x \in \mathcal{H} | jx = x\} \), where \( j \) is given in Proposition 2.1. Let \( x, y \in \mathcal{H}_R \). If \( x - y \in \mathcal{P} \), then we write this as \( x \geq y \text{ w.r.t. } \mathcal{P} \). ◊

Example 1 For each \( d \in \mathbb{N} \), we set

\[
L^2(\mathbb{R}^d)_+ = \{ f \in L^2(\mathbb{R}^d) | f(u) \geq 0 \text{ a.e. } u \}. \tag{2.2}
\]

\( L^2(\mathbb{R}^d)_+ \) is a self-dual cone in \( L^2(\mathbb{R}^d) \). \( f \geq 0 \text{ w.r.t. } L^2(\mathbb{R}^d)_+ \) if and only if \( f(u) \geq 0 \) a.e. \( u \). On the other hand, \( f > 0 \text{ w.r.t. } L^2(\mathbb{R}^d)_+ \) if and only if \( f(u) > 0 \text{ a.e. } u \). ◊

Let \( \mathcal{W} \) be a dense subspace of \( \mathcal{H} \) such that \( \mathcal{W} \cap \mathcal{P} \neq \{0\} \). Set

\[
\mathcal{L}(\mathcal{W}) = \{ A : \text{linear operator s.t. } \mathcal{W} \subseteq \text{dom}(A) \cap \text{dom}(A^*) \text{, } A\mathcal{W} \subset \mathcal{W}, A^*\mathcal{W} \subset \mathcal{W} \}. \tag{2.3}
\]

The following lemma is easy to check:

Lemma 2.3 We have the following:

(i) \( \mathcal{L}(\mathcal{W}) \) is a linear space.

(ii) If \( A, B \in \mathcal{L}(\mathcal{W}) \), then \( AB \in \mathcal{L}(\mathcal{W}) \).

(iii) If \( A \in \mathcal{L}(\mathcal{W}) \), then \( A^* \in \mathcal{L}(\mathcal{W}) \).

(iv) If \( A \in \mathcal{L}(\mathcal{W}) \), then \( \text{dom}(A) \cap \mathcal{P} \supseteq \mathcal{W} \cap \mathcal{P} \neq \{0\} \).

(v) If \( A \in \mathcal{L}(\mathcal{W}) \), then \( \text{dom}(A) \cap \mathcal{H}_R \supseteq \mathcal{W} \cap \mathcal{H}_R \neq \{0\} \).

Definition 2.4

- Let \( A \in \mathcal{L}(\mathcal{W}) \). If \( A(\text{dom}(A) \cap \mathcal{P}) \subseteq \mathcal{P} \), then we write this as \( A \geq 0 \text{ w.r.t. } \mathcal{P} \). Remark that, by Lemma 2.3 (iv), this definition is meaningful.

In this case, we say that \( A \) preserves the positivity w.r.t. \( \mathcal{P} \).

- Let \( A, B \in \mathcal{L}(\mathcal{W}) \). Suppose that \( A(\text{dom}(A) \cap \mathcal{H}_R) \subseteq \mathcal{H}_R \) and \( B(\text{dom}(B) \cap \mathcal{H}_R) \subseteq \mathcal{H}_R \). If \( (A - B)(\text{dom}(A) \cap \text{dom}(B) \cap \mathcal{P}) \subseteq \mathcal{P} \), then we write this as \( A \geq B \text{ w.r.t. } \mathcal{P} \). ◊

\(^2\text{In concrete applications in Sections 3 and 4 we will see that } \mathcal{W} \text{ satisfies a much stronger condition: } \mathcal{W} \cap \mathcal{P} = \mathcal{P}.\)
Remark 2.5 Suppose that $A$ and $B$ are bounded. Then $A \succeq B$ w.r.t. $\mathcal{P}$ if and only if $\langle x|Ay \rangle \geq \langle x|By \rangle$ for all $x, y \in \mathcal{P}$. ♦

Example 2 Let $F$ be a multiplication operator on $L^2(\mathbb{R}^d)$ by the function $F(u)$. Assume that $\|F\|_\infty < \infty$. If $F(u) \geq 0$ a.e., then $F \succeq 0$ w.r.t. $L^2(\mathbb{R}^d)_+$. ♦

Lemma 2.6 Let $A, A_1, A_2, B, B_1, B_2 \in \mathcal{L}(\mathcal{W})$. We have the following:

(i) If $0 \preceq A$ and $0 \preceq B$ w.r.t. $\mathcal{P}$, then $0 \preceq AB$ w.r.t. $\mathcal{P}$.

(ii) If $0 \preceq A_1 \preceq B_1$ and $0 \preceq A_2 \preceq B_2$ w.r.t. $\mathcal{P}$, then $0 \preceq aA_1 + bA_2 \preceq aB_1 + bB_2$ w.r.t. $\mathcal{P}$ for all $a, b \in \mathbb{R}^+$.

(iii) Suppose that $\mathcal{P} \cap \text{dom}(A)$ is dense in $\mathcal{P}$. If $0 \preceq A$ w.r.t. $\mathcal{P}$, then $0 \preceq A^* w.r.t. \mathcal{P}.$

Proof. (i) and (ii) are easy to see.

(iii) Let $x \in \text{dom}(A) \cap \mathcal{P}$ and let $y \in \text{dom}(A^*) \cap \mathcal{P}$. Then we have

$$\langle x|A^*y \rangle = \langle Ax|y \rangle \geq 0.$$  \hfill (2.4)

Because $\text{dom}(A) \cap \mathcal{P}$ is dense in $\mathcal{P}$, \eqref{2.4} holds true for all $x \in \mathcal{P}$. Thus, $A^*y \geq 0$, which implies that $A^* \succeq 0$ w.r.t. $\mathcal{P}$. □

Let $\mathcal{B}(\mathfrak{H})$ be the set of all bounded linear operators on $\mathfrak{H}$.

Lemma 2.7 \cite{20} Let $A, B, C, D \in \mathcal{B}(\mathfrak{H})$ and let $a, b \in \mathbb{R}$.

(i) If $A \succeq B \succeq 0$ and $C \succeq D \succeq 0$ w.r.t. $\mathcal{P}$, then $AC \succeq BD \succeq 0$ w.r.t. $\mathcal{P}$.

(ii) If $A \succeq 0$ w.r.t. $\mathcal{P}$, then $A^* \succeq 0$ w.r.t. $\mathcal{P}$.

Proof. (i) By Lemma 2.6 (i), we have

$$AC - BD = \begin{cases} A_{\preceq 0} \left( C_{\preceq 0} \right) + \left( A_{\preceq 0} - B_{\preceq 0} \right) \left( D_{\succeq 0} \right)_{\preceq 0} \succeq 0 & \text{w.r.t. } \mathcal{P}. \end{cases}$$

(ii) follows from Lemma 2.6 (iii). □

Proposition 2.8 Let $\mathfrak{A} = \{ A \in \mathcal{B}(\mathfrak{H}) \mid A \succeq 0 \text{ w.r.t. } \mathcal{P} \}$. Then $\mathfrak{A}$ is a weakly closed convex cone.

Proof. Let $\{ A_n \}_{n=1}^\infty$ be a sequence in $\mathfrak{A}$. Assume that $A_n$ weakly converges to $A$. Take $x, y \in \mathcal{P}$ arbitrarily. Because $\langle x|A_n y \rangle \geq 0$ for all $n \in \mathbb{N}$, we have $\langle x|Ay \rangle \geq 0$, which implies that $A \succeq 0$ w.r.t. $\mathcal{P}$. Thus, $\mathfrak{A}$ is weakly closed. □
2.2 Positivity improving operators

Definition 2.9 Let \( A \in \mathcal{B}(\mathcal{H}) \). We write \( A \succ 0 \) w.r.t. \( \mathcal{P} \), if \( Ax > 0 \) w.r.t. \( \mathcal{P} \) for all \( x \in \mathcal{P}\setminus\{0\} \). In this case, we say that \( A \) improves the positivity w.r.t. \( \mathcal{P} \). ♦

The following theorem plays an important role.

Theorem 2.10 (Perron–Frobenius–Faris) Let \( A \) be a self-adjoint positive operator on \( \mathcal{H} \). Suppose that \( 0 \prec e^{-\beta A} \) w.r.t. \( \mathcal{P} \) for all \( \beta \geq 0 \), and that \( \inf \text{spec}(A) \) is an eigenvalue. Let \( P_A \) be the orthogonal projection onto the closed subspace spanned by eigenvectors associated with \( \inf \text{spec}(A) \). Then, the following are equivalent:

(i) \( \dim \text{ran} P_A = 1 \) and \( P_A \succ 0 \) w.r.t. \( \mathcal{P} \).

(ii) \( 0 \prec e^{-\beta A} \) w.r.t. \( \mathcal{P} \) for all \( \beta > 0 \).

(iii) For each \( x, y \in \mathcal{P}\setminus\{0\} \), there exists a \( \beta > 0 \) such that \( \langle x | e^{-\beta A} y \rangle > 0 \).

Proof. See, e.g., [8, 23, 33]. ✷

Remark 2.11 (i) is equivalent to the following: The eigenvalue \( \inf \text{spec}(A) \) is simple with a strictly positive eigenvector. ♦

3 Second quantized operators

We briefly summarize necessary results concerning the second quantized operators. As to basic definitions, we refer to [6] as an accessible text.

3.1 Basic definitions

Let \( \mathcal{H} \) be a complex Hilbert space. The bosonic Fock space over \( \mathcal{H} \) is defined by

\[
\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{F}^{(n)}(\mathcal{H}), \quad \mathcal{F}^{(n)}(\mathcal{H}) = \mathcal{H}^\otimes n, \tag{3.1}
\]

where \( \mathcal{H}^\otimes n \) is the \( n \)-fold symmetric tensor product of \( \mathcal{H} \) with convention \( \mathcal{H}^\otimes 0 = \mathbb{C} \). \( \mathcal{F}^{(n)}(\mathcal{H}) \) is called the \( n \)-boson subspace. A finite particle subspace \( \mathcal{F}_\text{fin}(\mathcal{H}) \) is defined by

\[
\mathcal{F}_\text{fin}(\mathcal{H}) = \left\{ \phi = \sum_{n \geq 0} \varphi_n \in \mathcal{F}(\mathcal{H}) \bigg| \exists N \in \mathbb{N}_0 \text{ such that } \varphi_n = 0 \text{ for all } n \geq N \right\}. \tag{3.2}
\]

We denote by \( a(f) (f \in \mathcal{H}) \) the annihilation operator on \( \mathcal{F}(\mathcal{H}) \), its adjoint \( a(f)^* \), called the creation operator, is defined by

\[
a(f)^* \varphi = \sum_{n \geq 1} \sqrt{n} S_n (f \otimes \varphi_{n-1}) \tag{3.3}
\]

for \( \varphi = \sum_{n \geq 0} \varphi_n \in \text{dom}(a(f)^*) \), where \( S_n \) is the symmetrizer on \( \mathcal{F}^{(n)}(\mathcal{H}) \). The annihilation- and creation operators satisfy the canonical commutation relations (CCRs)

\[
[a(f), a(g)] = \langle f | g \rangle, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*] \tag{3.4}
\]
on \( \mathcal{F}_\text{fin}(\mathcal{H}) \).
Let $C$ be a contraction operator on $\mathcal{H}$, that is, $\|C\| \leq 1$. Then we define a contraction operator $\Gamma(C)$ on $\mathcal{F}(\mathcal{H})$ by

$$\Gamma(C) = \sum_{n \geq 0} C^\otimes n$$

with $C^\otimes 0 = 1$, the identity operator.

For a self-adjoint operator $A$ on $\mathcal{H}$, let us introduce

$$d\Gamma(A) = 0 \oplus \sum_{n \geq 1} \sum_{n \geq k \geq 1} 1^\otimes (k-1) \otimes A \otimes 1^\otimes (n-k)$$

acting in $\mathcal{F}(\mathcal{H})$. Then $d\Gamma(A)$ is essentially self-adjoint. We denote its closure by the same symbol.

If $A$ is positive, then one has

$$\Gamma(e^{-tA}) = e^{-td\Gamma(A)}, \quad t \geq 0.$$  

The following proposition is well-known.

**Proposition 3.1** Let $A$ be a positive self-adjoint operator. For each $f \in \text{dom}(A^{-1/2})$, we have the following operator inequalities:

$$a(f)^* a(f) \leq \|A^{-1/2}f\|^2 (d\Gamma(A) + 1),$$  

$$a(f) a(f)^* \leq \|A^{-1/2}f\|^2 (d\Gamma(A) + 1),$$  

$$d\Gamma(A) + a(f) + a(f)^* \geq -\|A^{-1/2}f\|^2.$$  

### 3.2 Fock space over $L^2(\mathbb{R}^3)$

In this study, the bosonic Fock space over $L^2(\mathbb{R}^3)$ is important. We simply write it as

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^3)).$$  

The $n$-boson subspace $\mathcal{F}^{(n)} = L^2(\mathbb{R}^3)^{\otimes n}$ is naturally identified with $L^2_{sym}(\mathbb{R}^{3n})$. Hence

$$\mathcal{F} = \mathbb{C} \oplus \sum_{n \geq 1} L^2_{sym}(\mathbb{R}^{3n}).$$  

The annihilation- and creation operators are symbolically expressed as

$$a(f) = \int_{\mathbb{R}^3} dk f(k) a(k), \quad a(f)^* = \int_{\mathbb{R}^3} dk f(k) a(k)^*.$$  

If $F$ is a multiplication operator by the function $F(k)$, then $d\Gamma(F)$ is formally written as

$$d\Gamma(F) = \int_{\mathbb{R}^3} dk F(k) a(k)^* a(k).$$  

Note that $d\Gamma(F) \upharpoonright L^2_{sym}(\mathbb{R}^{3n})$ is a multiplication operator by the function $F(k_1) + \cdots + F(k_n)$.  

9
3.3 The Fröhlich cone

Let $\mathfrak{F}_+$ be a convex cone defined by \[14\]. We begin with the following lemma:

**Lemma 3.2** $\mathfrak{F}_+$ is a self-dual cone in $\mathfrak{F}$.

**Proof.** It suffices to show that $L^2_{\text{sym}}(\mathbb{R}^{3n})_+$ is a self-dual cone for all $n \in \mathbb{N}_0$. To this end, we set $\mathfrak{P} = L^2_{\text{sym}}(\mathbb{R}^{3n})_+$. It is easy to check that $\mathfrak{P} \subseteq \mathfrak{P}^\dagger$. To prove the converse, we note the following fact: Let $\psi \in L^2_{\text{sym}}(\mathbb{R}^{3n})$. $\psi \geq 0$ w.r.t. $L^2_{\text{sym}}(\mathbb{R}^{3n})_+$ if and only if $\langle f_1 \otimes \cdots \otimes f_n | \psi \rangle \geq 0$ (3.15) for all $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)_+$, where $(f_1 \otimes \cdots \otimes f_n)(k_1, \ldots, k_n) = f_1(k_1) \cdots f_n(k_n)$. But it is easy to prove (3.15) for each $\psi \in \mathfrak{P}^\dagger$. $\blacksquare$

**Definition 3.3** ([9, 10]) The self-dual cone $\mathfrak{F}_+$ is called the Fröhlich cone. $\bowtie$

**Lemma 3.4** We have the following:

(i) $a(f)$ and $a(f)^* \in L(\mathfrak{F}_{\text{fin}})$ for all $f \in L^2(\mathbb{R}^3)$.

(ii) If $F$ is a multiplication operator such that $\|F\|_\infty \leq 1$, then $\Gamma(F) \in L(\mathfrak{F}_{\text{fin}})$.

By using the above lemma, we can discuss operator inequalities given in Section 2.

The following lemma will be useful.

**Lemma 3.5** Let $\mathfrak{F}_{\text{fin},+} = \mathfrak{F}_{\text{fin}} \cap \mathfrak{F}_+$. Then $\mathfrak{F}_{\text{fin},+} = \mathfrak{F}_+$, where the bar indicates the closure in the strong topology.

We summarize properties of operators on $\mathfrak{F}$ below. All propositions were proven in [24]. For reader’s convenience, we will provide proofs.

**Proposition 3.6** Let $C$ be a contraction operator on $L^2(\mathbb{R}^3)$. If $C \geq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$, then we have $\Gamma(C) \geq 0$ w.r.t. $\mathfrak{F}_+$.

**Proof.** Let $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)_+$. Because $C \geq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$, we have $C f_j \geq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$, which implies that $C f_1 \otimes \cdots \otimes C f_n \geq 0$ w.r.t. $L^2_{\text{sym}}(\mathbb{R}^{3n})_+$. Thus,

$$\langle f_1 \otimes \cdots \otimes f_n | C^\otimes n | \psi \rangle = \langle C f_1 \otimes \cdots \otimes C f_n | \psi \rangle \geq 0$$

(3.16)

for all $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)_+$, which implies that $C^\otimes n \geq 0$ w.r.t. $L^2_{\text{sym}}(\mathbb{R}^{3n})_+$. $\square$

**Proposition 3.7** Let $B$ be a positive self-adjoint operator. If $e^{-tB} \geq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$ for all $t \geq 0$, then $e^{-t\Gamma(B)} \geq 0$ w.r.t. $\mathfrak{F}_+$ for all $t \geq 0$.

**Proof.** By [6] and Proposition 3.6, we obtain the desired assertion. $\square$

**Proposition 3.8** If $f \geq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$, then $a(f)^* \geq 0$ and $a(f) \geq 0$ w.r.t. $\mathfrak{F}_+$.
Remark that \( \psi \) we have \( \phi \).

Because \( \phi \), Under the identifications \( F \), we have \( \phi \) \( \phi \).

\( \phi \) \( \phi \).

Choose \( \phi \).

Because \( \phi \), we have \( \phi \) \( \phi \).

Proposition 3.8, we have \( \phi \).

Proof. \( \phi \).

Because \( \phi \), we have \( \phi \) \( \phi \).

Remark that \( \phi \).

Because \( \phi \), Under the identifications \( F \), we have \( \phi \) \( \phi \).

Proposition 3.9 (Ergodicity) For each \( f \in L^2(\mathbb{R}^3) \), let \( \phi(f) \) be a linear operator defined by

\[
\phi(f) = a(f) + a(f)^*.
\]

Note that \( \phi(f) \) is essentially self-adjoint. We denote its closure by the same symbol. If \( f > 0 \) w.r.t. \( \mathcal{F}_+ \), that is, \( f(k) > 0 \) a.e. \( k \), then \( \phi(f) \) is ergodic in the sense that, for any \( \varphi, \psi \in \mathcal{F}_+ \setminus \{0\} \), there exists an \( n \in \mathbb{N}_0 \) such that \( \langle \varphi | \phi(f)^n \psi \rangle > 0 \).

Proof. Choose \( \varphi, \psi \in \mathcal{F}_+ \setminus \{0\} \), arbitrarily. We can express \( \varphi \) and \( \psi \) as

\[
\varphi = \sum_{n \geq 0} \varphi_n, \quad \psi = \sum_{n \geq 0} \psi_n.
\]

Because \( \varphi \) and \( \psi \) are non-zero, there exist \( p, q \in \mathbb{N}_0 \) such that \( \varphi_p \neq 0 \) and \( \psi_q \neq 0 \). Under the identifications

\[
\varphi_p = \sum_{n \geq 0} \delta_{np} \varphi_n, \quad \psi_q = \sum_{n \geq 0} \delta_{nq} \psi_n,
\]

we have \( \varphi \geq \varphi_p \) and \( \psi \geq \psi_q \) w.r.t. \( \mathcal{F}_+ \), where \( \delta_{mn} \) is the Kronecker delta. By Proposition \( \ref{prop:ergodic} \), we have

\[
\langle \varphi | \phi(f)^{p+q} \psi \rangle \geq \langle \varphi_p | \phi(f)^{p+q} \psi_q \rangle.
\]

Because \( \phi(f)^p \geq a(f)^p \) and \( \phi(f)^q \geq a(f)^q \) w.r.t. \( \mathcal{F}_+ \), we have

the RHS of \( \ref{eq:ergodicity} \) \( \geq \langle a(f)^p \varphi_p | a(f)^q \psi_q \rangle \).

Remark that

\[
a(f)^p \varphi_p = \sqrt{p!} (f^{\otimes p} | \varphi_p \rangle \Omega, \quad a(f)^q \psi_q = \sqrt{q!} (f^{\otimes q} | \psi_q \rangle \Omega,
\]

where \( \Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \) is the Fock vacuum. Since \( \langle f^{\otimes p} | \varphi_p \rangle > 0 \) and \( \langle f^{\otimes q} | \psi_q \rangle > 0 \), we get, by \( \ref{eq:ergodicity} \) and \( \ref{eq:ergodicity2} \),

\[
\langle \varphi | \phi(f)^{p+q} \psi \rangle \geq \sqrt{p!} \sqrt{q!} \langle f^{\otimes p} | \varphi_p \rangle \langle f^{\otimes q} | \psi_q \rangle > 0.
\]

Thus we are done.
3.4 Local properties

Let $B_\kappa$ be a ball of radius $\kappa$ in $\mathbb{R}^3$ centered at the origin and let $\chi_\kappa$ be a function on $\mathbb{R}^3$ defined by $\chi_\kappa(k) = 1$ if $k \in B_\kappa$ and $\chi_\kappa(k) = 0$ otherwise. Then as a multiplication operator, $\chi_\kappa$ is an orthogonal projection on $L^2(\mathbb{R}^3)$ and $Q_\kappa = \Gamma(\chi_\kappa)$ is an orthogonal projection on $\mathfrak{F}$ as well. We remark the following properties:

- If $\kappa_1 \geq \kappa_2$, then $Q_{\kappa_1} \geq Q_{\kappa_2}$.
- $Q_\kappa$ strongly converges to 1 as $\kappa \to \infty$.

Let us define the local Fock space by

$$\mathfrak{F}^{\leq \kappa} = Q_\kappa \mathfrak{F}.$$  \hspace{1cm} (3.25)

Since $\chi_\kappa L^2(\mathbb{R}^3) = L^2(B_\kappa)$, $\mathfrak{F}^{\leq \kappa}$ can be identified with $\mathfrak{F}(L^2(B_\kappa))$. In what follows, $\mathfrak{F}_\mathrm{fin}$ denotes $\mathfrak{F}(L^2(B_\kappa))$.

**Proposition 3.10** For each $\kappa \geq 0$, we set $Q_\perp = 1 - Q_\kappa$. Then we have the following:

(i) $Q_\kappa \geq 0$ w.r.t. $\mathfrak{F}_+$. 

(ii) $Q_\perp \geq 0$ w.r.t. $\mathfrak{F}_+$. 

**Proof.** (i) immediately follows from Proposition 3.6.

(ii) Under the identification (3.12), we see

$$(Q_\kappa \varphi_n)(k_1, \ldots, k_n) = \left[ \prod_{j=1}^n \chi_\kappa(k_j) \right] \varphi_n(k_1, \ldots, k_n)$$  \hspace{1cm} (3.27)

for each $\varphi_n \in L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$. Hence

$$(Q_\perp \varphi_n)(k_1, \ldots, k_n) = \left\{ 1 - \prod_{j=1}^n \chi_\kappa(k_j) \right\} \varphi_n(k_1, \ldots, k_n).$$  \hspace{1cm} (3.28)

If $\varphi_n(k_1, \ldots, k_n) \geq 0$ a.e., then the right hand side of (3.28) is positive for a.e. $k_1, \ldots, k_n$ because $1 - \prod_{j=1}^n \chi_\kappa(k_j) \geq 0$. This means that $Q_\perp \geq 0$ w.r.t. $\mathfrak{F}_+$.

We remark the following:

$$a(f)Q_\kappa = a(\chi_\kappa f) = \int_{|k| \leq \kappa} dk \overline{f(k)} a(k),$$  \hspace{1cm} (3.29)

$$Q_\kappa a(f)^* = a(\chi_\kappa f)^* = \int_{|k| \leq \kappa} dk f(k) a(k)^*,$$  \hspace{1cm} (3.30)

$$d\Gamma(F)Q_\kappa = d\Gamma(\chi_\kappa F) = \int_{|k| \leq \kappa} dk F(k) a(k)^* a(k).$$  \hspace{1cm} (3.31)

By these facts, we obtain the following proposition.
Proposition 3.11 We have the following:

(i) \([Q_\kappa, a(f)] = Q_\kappa a((1 - \chi_\kappa) f)\) on \(\mathfrak{F}_\text{fin}\).

(ii) \([Q_\kappa, d\Gamma(F)] = 0\) on \(\text{dom}(d\Gamma(F))\).

Next let us introduce a natural self-dual cone in \(\mathfrak{F}^{\leq \kappa}\). To this end, define

\[
\mathfrak{F}_{n,+}^{\leq \kappa} = \{ \varphi \in L^2_{\text{sym}}(B_\kappa^{\times n}) \mid \varphi(k_1, \ldots, k_n) \geq 0 \text{ a.e.} \}\]  

(3.32)

with \(\mathfrak{F}_{0,+}^{\leq \kappa} = \mathbb{R}_+\). Each \(\mathfrak{F}_{n,+}^{\leq \kappa}\) is a self-dual cone in \(L^2(B_\kappa)^{\otimes n} = L^2_{\text{sym}}(B_\kappa^{\times n})\).

Definition 3.12 The \textit{local Fröhlich cone} is defined by

\[
\mathfrak{F}_{+}^{\leq \kappa} = \sum_{n \geq 0} \mathfrak{F}_{n,+}^{\leq \kappa}. 
\]

(3.33)

\(\mathfrak{F}_{+}^{\leq \kappa}\) is a self-dual cone in \(\mathfrak{F}^{\leq \kappa}\). As before, we define \(\mathfrak{F}_{\text{fin},+}^{\leq \kappa} = \mathfrak{F}_{\text{fin}}^{\leq \kappa} \cap \mathfrak{F}_{+}^{\leq \kappa}\). Note that \(\mathfrak{F}_{\text{fin},+}^{\leq \kappa} = \mathfrak{F}_{+}^{\leq \kappa}\). ♦

Proposition 3.13 Propositions 3.6, 3.7, 3.8 and 3.9 are still true even if one replaces \(L^2(\mathbb{R}^3)^{+}, \mathfrak{F}_{+}^{\leq \kappa}\) and \(\mathfrak{F}_{\text{fin},+}^{\leq \kappa}\) by \(L^2(B_\kappa)^{+}, \mathfrak{F}_{+}^{\leq \kappa}\) and \(\mathfrak{F}_{\text{fin},+}^{\leq \kappa}\), respectively.

3.5 Decomposition properties

Let \(h_1\) and \(h_2\) be complex Hilbert spaces. Remark the following factorization property:

\[
\mathfrak{F}(h_1 \oplus h_2) = \mathfrak{F}(h_1) \otimes \mathfrak{F}(h_2). 
\]

(3.34)

Corresponding to this, we have the following:

• For each \(f \in h_1, g \in h_2\),

\[
a(f \oplus g) = a(f) \otimes 1 + 1 \otimes a(g). 
\]

(3.35)

• Let \(A\) and \(B\) be self-adjoint operators. We have

\[
d\Gamma(A \oplus B) = \{ d\Gamma(A) \otimes 1 + 1 \otimes d\Gamma(B) \}^{-}, 
\]

(3.36)

where \(\{ \cdots \}^{-}\) indicates the closure of \(\{ \cdots \}\).

• Let \(C\) and \(D\) be contraction operators. We have

\[
\Gamma(C \oplus D) = \Gamma(C) \otimes \Gamma(D). 
\]

(3.37)

For each \(\kappa > 0\), we have the following identification:

\[
L^2(\mathbb{R}^3) = L^2(B_\kappa) \oplus L^2(B_\kappa^c), 
\]

(3.38)

where \(B_\kappa^c\) indicates the complement of \(B_\kappa\). Using (3.34) and (3.38), we have

\[
\mathfrak{F} = \mathfrak{F}^{\leq \kappa} \otimes \mathfrak{F}^{> \kappa}, 
\]

(3.39)
where $\mathcal{F}^{\geq} = \mathcal{F}(L^2(B^c))$. Thus, we have

$$\mathcal{F} = \sum_{n \geq 0} \mathcal{F}^{\geq} \otimes L^2_{\text{sym}}((B^c)^{\times n})$$

where $L^2_{\text{sym}}((B^c)^{\times n}) : = \mathbb{C}$. The following lemma will be useful.

**Lemma 3.14** Let $\psi = \sum_{n \geq 0} \psi_n(k_1, \ldots, k_n) \in \mathcal{F}$. For each $\kappa > 0$, we have

$$Q_\kappa \psi = \psi_\kappa \otimes \Omega^{>\kappa}, \quad (3.41)$$

where $\Omega^{>\kappa}$ is the Fock vacuum in $\mathcal{F}^{>\kappa}$ and

$$\psi_\kappa = \sum_{n \geq 0} \left[ \prod_{\ell=1}^{n} \chi_\kappa(k_\ell) \right] \psi_n(k_1, \ldots, k_n). \quad (3.42)$$

A natural self-dual cone in $\mathcal{F}^{>\kappa}$ is given by

$$\mathcal{F}^{>\kappa} = \sum_{n \geq 0} L^2_{\text{sym}}((B^c)^{\times n})_+, \quad (3.43)$$

where $L^2_{\text{sym}}((B^c)^{\times n})_+ : = \mathbb{R}_+$. As before, we set $\mathcal{F}^{>\kappa}_\text{fin} = \mathcal{F}_\text{fin}(L^2(B^c))$ and $\mathcal{F}^{>\kappa}_{\text{fin},+} = \mathcal{F}^{>\kappa}_\text{fin} \cap \mathcal{F}^{>\kappa}$. 

**Proposition 3.15** Propositions 3.6, 3.7, 3.8 and 3.9 are still true even if one replaces $L^2(\mathbb{R}^3)$, $\mathcal{F}_+$ and $\mathcal{F}^{>\kappa}_{\text{fin},+}$ by $L^2(B^c)$, $\mathcal{F}^{>\kappa}_+$ and $\mathcal{F}^{>\kappa}_{\text{fin},+}$, respectively.

The self-dual cone $\mathcal{F}_+$ can be expressed as

$$\mathcal{F}_+ = \mathcal{F}^{<\kappa}_+ \oplus \sum_{n \geq 1} \mathcal{F}^{<\kappa}_+ \otimes L^2_{\text{sym}}((B^c)^{\times n})_+, \quad (3.44)$$

where

$$\mathcal{F}^{<\kappa}_+ \otimes L^2_{\text{sym}}((B^c)^{\times n})_+ = \{ \psi \in \mathcal{F}^{<\kappa} \otimes L^2_{\text{sym}}((B^c)^{\times n}) \left| \psi(k_1, \ldots, k_n) \geq 0 \right. \text{ w.r.t. } \mathcal{F}^{<\kappa}_+ \text{ a.e.} \}. \quad (3.45)$$

**Theorem 3.16** We have the following:

(i) $Q_\kappa \mathcal{F}_+ = \mathcal{F}^{<\kappa}_+$. 

(ii) $\mathcal{F}_+ = \bigcup_{\kappa > 0} \mathcal{F}^{<\kappa}_+$. 

Proof. (i) This immediately follows from (3.44).

(ii) With the identification $\mathcal{F}^{<\kappa}_+ = \mathcal{F}^{<\kappa}_+ \oplus \{0\}$, we know that $\mathcal{F}_+ \supset \mathcal{F}^{<\kappa}_+$ by (3.44). Hence, $\mathcal{F}_+ \supset \bigcup_{\kappa > 0} \mathcal{F}^{<\kappa}_+$. 

Let $\psi \in \mathcal{F}_+$. For each $\kappa > 0$, we know that $Q_\kappa \psi \in \mathcal{F}^{<\kappa}_+$ by (3.41). Because $Q_\kappa$ strongly converges to 1 as $\kappa \to \infty$, we conclude that $\psi \in \bigcup_{\kappa > 0} \mathcal{F}^{<\kappa}_+$. □
Lemma 3.17 Let \( \psi \in \mathcal{F} \). The following (i) and (ii) are equivalent:

(i) \( \psi \geq 0 \) w.r.t. \( \mathcal{F}_+ \).

(ii) \( \langle \xi \otimes \eta | \psi \rangle \geq 0 \) for all \( \xi \in \mathcal{F}_{+}^{\leq \kappa} \) and \( \eta \in \mathcal{F}_{+}^{> \kappa} \).

Proof. (ii) \( \Rightarrow \) (i): Without loss of generality, we may assume that \( \psi \in \mathcal{F}_{\text{fin}} \). Thus, it suffices to consider the case where \( \psi = \psi_1 \otimes \psi_2 \) with \( \psi_1 \in \mathcal{F}_{\text{fin}}^{\leq \kappa} \) and \( \psi_2 \in \mathcal{F}_{\text{fin}}^{> \kappa} \). Because \( \langle \xi \otimes \eta | \psi \rangle = \langle \xi | \psi_1 \rangle \langle \eta | \psi_2 \rangle \geq 0 \), we can choose \( \psi_1 \) and \( \psi_2 \) such that \( \psi_1 \geq 0 \) w.r.t. \( \mathcal{F}_{+}^{\leq \kappa} \) and \( \psi_2 \in \mathcal{F}_{+}^{> \kappa} \). Thus, we conclude that \( \psi \geq 0 \) w.r.t. \( \mathcal{F}_+ \).

(i) \( \Rightarrow \) (ii): By arguments similar to those in the above, it suffices to consider the case \( \psi = \psi_1 \otimes \psi_2 \) with \( \psi_1 \in \mathcal{F}_{+}^{\leq \kappa} \) and \( \psi_2 \in \mathcal{F}_{+}^{> \kappa} \). In this case, we easily check that \( \langle \xi \otimes \eta | \psi \rangle \geq 0 \) for all \( \xi \in \mathcal{F}_{+}^{\leq \kappa} \) and \( \eta \in \mathcal{F}_{+}^{> \kappa} \). \( \square \)

Proposition 3.18 Let \( A \in \mathcal{B}(\mathcal{F}_{+}^{\leq \kappa}) \) and \( B \in \mathcal{B}(\mathcal{F}_{+}^{> \kappa}) \). If \( A \geq 0 \) w.r.t. \( \mathcal{F}_{+}^{\leq \kappa} \) and \( B \geq 0 \) w.r.t. \( \mathcal{F}_{+}^{> \kappa} \), then \( A \otimes B \geq 0 \) w.r.t. \( \mathcal{F}_+ \).

Proof. Let \( \xi \in \mathcal{F}_{+}^{\leq \kappa} \) and let \( \eta \in \mathcal{F}_{+}^{> \kappa} \). By the assumption, we have \( A^* \xi \geq 0 \) and \( B^* \eta \geq 0 \). Thus, by Lemma 3.17,

\[
\langle \xi \otimes \eta | A \otimes B \psi \rangle = \langle (A^* \xi) \otimes (B^* \eta) | \psi \rangle \geq 0.
\]

(3.46)

By Lemma 3.17 again, we have \( A \otimes B \psi \geq 0 \) w.r.t. \( \mathcal{F}_+ \). \( \square \)

4 Proof of Theorem 1.3

4.1 Decomposition of \( H_{\text{ren,}\Lambda}(P) \)

In what follows, we always assume that \( \kappa < \Lambda \). Let \( F \) be a real-valued measurable function on \( \mathbb{R}^3 \). Suppose that \( F(k) \) is finite for almost everywhere. Then \( d\Gamma(F) \) is essentially self-adjoint. For each \( \kappa > 0 \), we set \( F_{\leq \kappa} = \chi_\kappa F \) and \( F_{> \kappa} = (1 - \chi_\kappa) F \). By (4.38) and (4.39), we have

\[
d\Gamma(F) = \{ d\Gamma(F_{\leq \kappa}) \otimes 1 + 1 \otimes d\Gamma(F_{> \kappa}) \}^-.
\]

(4.1)

Keeping this fact in mind, we set

\[
P_{\leq \kappa}^{i,j} = d\Gamma(k_j \chi_\kappa), \quad P_{> \kappa}^{i,j} = d\Gamma(k_j (1 - \chi_\kappa)), \quad j = 1, 2, 3.
\]

(4.2)

Remark the following formulas:

\[
d\Gamma(\omega) = d\Gamma(\omega_{\leq \kappa}) \otimes 1 + 1 \otimes d\Gamma(\omega_{> \kappa}),
\]

\[
P_{i,j} = \{ P_{\leq \kappa}^{i,j} \otimes 1 + 1 \otimes P_{> \kappa}^{i,j} \}^-,
\]

\[
a(f) = a(\chi_\kappa f) \otimes 1 + 1 \otimes a((1 - \chi_\kappa) f).
\]

(4.3)

(4.4)

(4.5)

Let

\[
E_\kappa^\Lambda = - g^2 \int_{\mathbb{R}^3} dk \frac{\chi_\kappa^\Lambda(k)}{\omega(k) \{ \omega(k) + k^2/2 \}}, \quad \chi_\kappa^\Lambda = \chi_\Lambda - \chi_\kappa.
\]

(4.6)
Note that \( E^\Lambda_\kappa = E_\Lambda - E_\kappa \), where \( E_\Lambda \) is defined by (1.12), while \( E_\kappa \) is defined by (1.12) with \( \Lambda \) replaced by \( \kappa \). Using (4.3), (4.4) and (4.5), we have

\[
H_{\text{ren},\Lambda}(P) = H^{\leq \kappa}_{\text{ren}}(P) \otimes 1 + 1 \otimes K_{\kappa,\Lambda} - (P - P^{\leq \kappa}_f) \cdot P^{> \kappa}_f, 
\]

where

\[
H^{\leq \kappa}_{\text{ren}}(P) = \frac{1}{2} (P - P^{\leq \kappa}_f)^2 - g \int_{\mathbb{R}^3} dk \frac{\chi_k(k)}{\sqrt{\omega(k)}} (a(k) + a(k)^*) + d\Gamma(\omega^{\leq \kappa}) - E_\kappa, 
\]

\[
K_{\kappa,\Lambda} = \frac{1}{2} (P^{> \kappa}_f)^2 - g \int_{\mathbb{R}^3} dk \frac{\chi^\Lambda_k(k)}{\sqrt{\omega(k)}} (a(k) + a(k)^*) + d\Gamma(\omega^{> \kappa}) - E^\Lambda_\kappa,
\]

and

\[
(P - P^{\leq \kappa}_f) \cdot P^{> \kappa}_f = \sum_{j=1}^3 (P_j - P^{\leq \kappa}_{f,j}) \otimes P^{> \kappa}_{f,j}.
\]

### 4.2 \( e^{-\beta H_{\text{ren}}(P)} \) is positivity preserving w.r.t. \( \mathfrak{F}^+ \)

In this subsection, we will show the following proposition.

**Proposition 4.1** For all \( P \in \mathbb{R}^3 \) and \( \beta \geq 0 \), we have \( e^{-\beta H_{\text{ren}}(P)} \triangleright 0 \) w.r.t. \( \mathfrak{F}^+ \).

#### 4.2.1 Proof of Proposition 4.1

**Lemma 4.2** We have the following:

(i) \( e^{-\beta d\Gamma(\omega)} \triangleright 0 \) w.r.t. \( \mathfrak{F}^+ \) for all \( \beta \geq 0 \).

(ii) \( e^{-\beta (P - P_f)^2/2} \triangleright 0 \) w.r.t. \( \mathfrak{F}^+ \) for all \( P \in \mathbb{R}^3 \) and \( \beta \geq 0 \).

**Proof.** (i) Note that \( e^{-\beta \omega} \triangleright 0 \) w.r.t. \( L^2(\mathbb{R}^3)_+ \) for all \( \beta \geq 0 \). By Proposition 3.7, we obtain (i).

(ii) Note that

\[
e^{-\beta (P - P_f)^2/2} = \sum_{n \geq 0} \binom{n}{\@} e^{-\beta (P - k_1 - \cdots - k_n)^2/2}.
\]

Since each multiplication operator \( e^{-\beta (P - k_1 - \cdots - k_n)^2/2} \) preserves the positivity w.r.t. \( L^2_{\text{sym}}(\mathbb{R}^{3n})_+ \), we conclude (ii). \( \square \)

**Lemma 4.3** \( e^{-\beta H_{\text{ren},\Lambda}(P)} \triangleright 0 \) w.r.t. \( \mathfrak{F}^+ \), for all \( P \in \mathbb{R}^3 \), \( \beta \geq 0 \) and \( \Lambda > 0 \).

**Proof.** By Proposition 3.8 and Lemma 4.2, we can apply Proposition A.1 with \( A = \frac{1}{2} (P - P_f)^2 + d\Gamma(\omega) \) and \( B = -\{a(f) + a(f)^*\}, \ f = g \frac{\chi_\Lambda}{\sqrt{\omega}} \).

**Proof of Proposition 4.1**

Because \( e^{-\beta H_{\text{ren},\Lambda}(P)} \) strongly converges to \( e^{-\beta H_{\text{ren}}(P)} \), the assertion follows from Proposition 2.8 and Lemma 4.3. \( \square \)
4.3 $e^{-\beta H_{\text{fin}}^\leq(P)}$ is positivity improving w.r.t. $\mathcal{F}_+^\leq\kappa$

Our goal here is to prove the following.

**Proposition 4.4** For all $P \in \mathbb{R}^3$, $\beta > 0$ and $\kappa > 0$, we have $e^{-\beta H_{\text{fin}}^\leq(P)} \triangleright 0$ w.r.t. $\mathcal{F}_+^\leq\kappa$.

### 4.3.1 Proof of Proposition 4.4

Using arguments similar to those in the proof of Lemma 4.2, we have the following.

**Lemma 4.5** We have the following:

(i) $e^{-\beta d(\omega^\leq\kappa)} \triangleright 0$ w.r.t. $\mathcal{F}_+^\leq\kappa$ for all $\beta > 0$.

(ii) $e^{-\beta (P-P^\leq\kappa)^2/2} \triangleright 0$ w.r.t. $\mathcal{F}_+^\leq\kappa$ for all $P \in \mathbb{R}^3$ and $\beta \geq 0$.

**Lemma 4.6** For all $P \in \mathbb{R}^3$, $\beta > 0$ and $\kappa > 0$, we have $e^{-\beta H_{\text{fin}}^\leq(P)} \triangleright 0$ w.r.t. $\mathcal{F}_+^\leq\kappa$.

**Proof.** By Proposition 3.13 and Lemma 4.5, we can apply Proposition A.1 with $A = \frac{1}{2}(P-P^\leq\kappa)^2 + d(\omega^\leq\kappa)$ and $B = \{-a(F) + a(F)^*\}$, $F = g \frac{\chi}{\omega}$. □

**Proof of Proposition 4.4**

Let $F = g \frac{\chi}{\omega}$. Because $F(k) > 0$ on $\Omega_\kappa$, $\phi(F) = a(F) + a(F)^*$ is ergodic w.r.t. $\mathcal{F}_+^\leq\kappa$ by Proposition 3.11. Let $\varphi, \psi \in \mathcal{F}_+^\leq\kappa \setminus \{0\}$. We can express $\varphi$ and $\psi$ as $\varphi = \sum_{n \geq 0}^{\oplus} \varphi_n$ and $\psi = \sum_{n \geq 0}^{\oplus} \psi_n$. Since $\varphi$ and $\psi$ are non-zero, there exist $n_1, n_2 \in \mathbb{N}_0$ such that $\varphi_{n_1} \neq 0$ and $\psi_{n_2} \neq 0$. By the identifications similar to (3.20) and the ergodicity of $\phi(F)$, there exists an $l \in \mathbb{N}_0$ such that

$$\langle \varphi_{n_1}\psi_{n_2} \rangle > 0.$$  \hspace{1cm} (4.12)

Since $\varphi \geq \varphi_{n_1}$ and $\psi \geq \psi_{n_2}$ w.r.t. $\mathcal{F}_+^\leq\kappa$, we have

$$\langle \varphi | e^{-\beta H_{\text{fin}}^\leq(P)} | \psi \rangle \geq \langle \varphi_{n_1} | e^{-\beta H_{\text{fin}}^\leq(P)} | \psi_{n_2} \rangle$$  \hspace{1cm} (4.13)

for all $\beta \geq 0$, by Lemma 4.6. Let $H_0 = \frac{1}{2}(P-P^{\leq\kappa})^2 + d(\omega^{\leq\kappa})$. By the Duhamel formula, we obtain

$$e^{-\beta H_{\text{fin}}^\leq(P)} = \sum_{j=0}^{\ell} D_j + R_\ell \quad \text{on } \mathcal{F}_+^\leq\kappa,$$  \hspace{1cm} (4.14)

where

$$D_j = \int_0^{t} \ldots \int_0^{t-s_1} \int_0^{t-s_1} \ldots \int_0^{t-\sum_{i=1}^{j-1} s_i} ds_1 \times$$

$$\times e^{-s_1 H_0} \phi(F) e^{-s_2 H_0} \ldots e^{-s_j H_0} \phi(F) e^{-(t-\sum_{i=1}^{j} s_i) H_0},$$  \hspace{1cm} (4.15)

$$R_\ell = \int_0^{t} \ldots \int_0^{t-s_1} \int_0^{t-s_1} \ldots \int_0^{t-\sum_{i=1}^{\ell} s_i} ds_{\ell+1} \times$$

$$\times e^{-s_1 H_0} \phi(F) e^{-s_2 H_0} \ldots e^{-s_\ell H_0} \phi(F) e^{-(t-\sum_{i=1}^{\ell+1} s_i) H_{\text{fin}}^\leq(P)}.$$  \hspace{1cm} (4.16)
Because \( e^{-sH_0} \geq 0 \) and \( \phi(F) \geq 0 \) w.r.t. \( \mathbb{S}_2^\kappa \), we know that \( \langle \varphi_{n_1}|D_f^\psi_{n_2}\rangle \geq 0 \). Similarly, by Lemma 4.6 we have \( \langle \varphi_{n_1}|R_f^\psi_{n_2}\rangle \geq 0 \). Hence,

\[
\langle \varphi_{n_1}|e^{-\beta H_0^n}(P)|\psi_{n_2}\rangle \geq \langle \varphi_{n_1}|D_f^\psi_{n_2}\rangle. \tag{4.17}
\]

Let \( G(s_1, \ldots, s_t) = \langle \varphi_{n_1}|e^{-s_1H_0}\phi(F)e^{-s_2H_0}\ldots e^{-s_tH_0}\phi(F)e^{-(t-\sum_{i=1}^t s_i)H_0}^n|\psi_{n_2}\rangle \). By (4.12), we see that \( G(0, \ldots, 0) > 0 \). Because \( G(s_1, \ldots, s_t) \) is positive and continuous, we have

\[
\langle \varphi_{n_1}|D_f^\psi_{n_2}\rangle = \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-\sum_{i=1}^{t-1} s_i} ds_t G(s_1, \ldots, s_t) > 0. \tag{4.18}
\]

Combining (4.13), (4.17) and (4.15), we arrive at \( \langle \varphi|e^{-\beta H_0^n}(P)\psi\rangle > 0 \) for all \( \beta > 0 \). \( \square \)

### 4.4 Basic properties of \( K_{\kappa,\Lambda} \)

In this subsection, we will show the following.

**Proposition 4.7** For all \( \kappa > 0 \), there exists a self-adjoint operator \( K_\kappa \) bounded from below such that

(i) \( e^{-\beta K_{\kappa,\Lambda}} \) strongly converges to \( e^{-\beta K_\kappa} \) for all \( \beta \geq 0 \), as \( \Lambda \to \infty \);

(ii) \( e^{-\beta K_\kappa} \geq 0 \) w.r.t. \( \mathbb{S}_2^\kappa \) for all \( \beta \geq 0 \).

#### 4.4.1 Proof of Proposition 4.7 (i)

We will apply Nelson’s idea [29]. Choose \( k \) such that \( \kappa < K < \Lambda \). Let

\[
\beta(k) = g \frac{1 - \chi_k(k)}{\omega(k)^{1/2} \left\{ \omega(k) + k^2/2 \right\}}. \tag{4.19}
\]

We define an anti-self-adjoint operator \( T \) by

\[
T = \{ a(G) - a(G)^* \}^-, \quad G = \beta \chi_\kappa. \tag{4.20}
\]

The unitary operator \( e^T \) is called the *Gross transformation*, which was introduced in [13]. We can check the following (For notational simplicity, we give somewhat formal expressions here.):

- \( e^TP_t^{\kappa} e^{-T} = P_t^{\kappa} + A + A^* \), where \( A = (A_1, A_2, A_3) \) with \( A_j = a(k_j G) \).
- \( e^T a(k) e^{-T} = a(k) + G(k) \).

Let \( \tilde{K}_{\kappa,\Lambda} = e^T K_{\kappa,\Lambda} e^{-T} \). Using the above facts, we obtain the following:

\[
\tilde{K}_{\kappa,\Lambda} = \frac{1}{2} (P_t^{\kappa})^2 + P_t^{\kappa} \cdot A + A^* \cdot P_t^{\kappa} + \frac{1}{2} A^2 + \frac{1}{2} A^* + H_1 + d\Gamma(\omega^{\kappa}) - E^K_\kappa,
\]

where

\[
H_1 = -g \int_{\mathbb{R}^3} dk \frac{\chi^K_\kappa(k)}{\sqrt{\omega(k)}} (a(k) + a(k)^*). \tag{4.22}
\]
We set
\[ \mathcal{J} = \frac{1}{2}(P^{>\kappa})^2 + d\Gamma(\omega^{>\kappa}). \] (4.23)

Let us define a quadratic form \( B_\Lambda \) on \( \text{dom}(\mathcal{J}^{1/2}) \times \text{dom}(\mathcal{J}^{1/2}) \) by
\[ B_\Lambda(\varphi, \psi) = \sum_{j=1}^{3} \left\{ \langle P^{>\kappa}_{i,j}\varphi|A_j\psi \rangle + \langle A_j\varphi|P^{>\kappa}_{i,j}\psi \rangle + \frac{1}{2}\langle A_j^*\varphi|A_j\psi \rangle + \frac{1}{2}\langle A_j\varphi|A_j^*\psi \rangle \right\} + \langle \varphi|H_I\psi \rangle. \] (4.24)

We easily check that
\[ \langle \varphi|K_{\kappa,\Lambda}\psi \rangle = \langle \mathcal{J}^{1/2}\varphi|\mathcal{J}^{1/2}\psi \rangle + B_\Lambda(\varphi, \psi), \quad \varphi, \psi \in \text{dom}(\mathcal{J}^{1/2}). \] (4.25)

Let \( G_\infty = \beta(1 - \chi_\kappa) \) and let \( A_\infty = a(kG_\infty) \). We define a quadratic form \( B_\infty \) on \( \text{dom}(\mathcal{J}^{1/2}) \times \text{dom}(\mathcal{J}^{1/2}) \) by replacing \( A \) with \( A_\infty \) in (4.24).

**Lemma 4.8** Let \( C(K) \) be a positive number defined by
\[ C(K)^2 = \int_{\mathbb{R}^3} dk \frac{1 - \chi_K(k)}{\{\omega(k) + k^2/2\}^2}. \] (4.26)

For all \( \varepsilon > 0 \), there exists a constant \( D_{K,\varepsilon} > 0 \) such that
\[ |B_\infty(\varphi, \varphi)| \leq \{6C(K) + 6C(K)^2 + \varepsilon\}||\mathcal{J} + 1||^{1/2}\varphi||^2 + D_{K,\varepsilon}\|\varphi\|^2 \] (4.27)
for all \( \varphi \in \text{dom}(\mathcal{J}^{1/2}) \).

**Proof.** Using (3.8) and (3.9), we have \( ||A_\infty^{\#}\varphi|| \leq ||\omega^{-1/2}k_jG||||\mathcal{J} + 1||^{1/2}\varphi|| \), where \( a^{\#} = a \) or \( a^* \). Because \( ||\omega^{-1/2}k_jG|| \leq C(K) \), we obtain
\[ ||A_\infty^{\#}\varphi|| \leq C(K)||\mathcal{J} + 1||^{1/2}\varphi||, \quad \varphi \in \text{dom}(\mathcal{J}^{1/2}). \] (4.28)

On the other hand, we have
\[ ||P^{>\kappa}_{i,j}\varphi|| \leq ||\mathcal{J} + 1||^{1/2}\varphi||, \quad \varphi \in \text{dom}(\mathcal{J}^{1/2}). \] (4.29)

By using (4.28) and (4.29), we can estimate the terms involving \( A \) and \( P^{>\kappa}_{i,j} \).

In order to estimate \( \langle \varphi|H_I\psi \rangle \), we observe, by (3.8) and (3.9) again,
\[ ||\varphi|| < D||\varphi||||\mathcal{J} + 1||^{1/2}\varphi||, \] (4.30)
where \( D = 2g\left(\int dk \frac{\chi_K(k)}{\omega^2}\right)^{1/2} \). Using \( ab \leq \varepsilon a^2 + b^2/4\varepsilon \), we obtain
\[ ||\varphi|| \leq \varepsilon||\mathcal{J} + 1||^{1/2}\varphi||^2 + \frac{D}{4\varepsilon}\|\varphi\|^2. \] (4.31)

Thus we are done. \( \square \)
Choose $K$ sufficiently large as $6C(K) + 6C(K)^2 < 1$. By the KLMN theorem [32, Theorem X. 17] and Lemma 4.8 there exists a unique self-adjoint operator $\tilde{K}_\kappa$ such that

$$\langle \varphi | \tilde{K}_\kappa | \psi \rangle = \langle J^{1/2} \varphi | J^{1/2} \psi \rangle + B_\infty (\varphi, \psi).$$

(4.32)

Note that $\tilde{K}_\kappa$ is bounded from below.

**Lemma 4.9** We have

$$|B_\infty (\varphi, \varphi) - B_\Lambda (\varphi, \varphi)| \leq \left\{ 6C(\Lambda) + 12C(K)C(\Lambda) \right\} \|(J + 1)^{1/2} \varphi\|^2$$

(4.33)

for all $\varphi \in \text{dom}(J^{1/2})$, where $C(K)$ and $C(\Lambda)$ are defined by (4.26).

**Proof.** By (3.8) and (3.9), we have

$$\|(A^\#_{\infty,i} - A^\#_i) \varphi\| \leq \|\omega^{-1/2} k_j \beta (1 - \chi_{\kappa} - \chi_{\Lambda})\| \|(J + 1)^{1/2} \varphi\|$$

$$\leq C(\Lambda) \|(J + 1)^{1/2} \varphi\|, \ \varphi \in \text{dom}(J^{1/2}).$$

(4.34)

Using (4.28), (4.29) and (4.34), we can prove (4.33). $\square$

**Proof of Theorem 4.7 (i)**

Note that $C(\Lambda) \to 0$ as $\Lambda \to \infty$. By Lemma 4.9 and [31, Theorem VIII. 25], $\tilde{K}_{\kappa, \Lambda}$ converges to $\tilde{K}_\kappa$ in norm resolvent sense as $\Lambda \to \infty$. Let $T_\infty = \{ a(G_\infty) - a(G_\infty)^* \}^-$.

Because $e^T$ strongly converges to $e^{T_\infty}$, we obtain the desired result. $\square$

**4.4.2 Proof of Proposition 4.7 (ii)**

Using arguments similar to those in the proof of Lemmas 4.2 and 4.3, we can show the following lemma.

**Lemma 4.10** $e^{-\beta K_{\kappa, \Lambda}} \triangleright 0$ w.r.t. $\mathcal{F}_+^{\kappa}$ for all $\beta \geq 0$, $\kappa > 0$ and $\Lambda > 0$.

**Proof of Proposition 4.7 (ii)**

By Proposition 4.7 (i), $e^{-\beta K_{\kappa, \Lambda}}$ strongly converges to $e^{-\beta K_{\kappa}}$ as $\Lambda \to \infty$. Using Proposition 4.8 and Lemma 4.10 we conclude Proposition 4.7 (ii). $\square$

**4.5 A key theorem**

Let

$$L_\kappa = H^{\leq \kappa}_{\text{ren}} (P) \otimes 1 + 1 \otimes K_\kappa.$$  

(4.35)

Our purpose in this subsection is to prove the following theorem.

**Theorem 4.11** The following (i) and (ii) are mutually equivalent:

(i) $e^{-\beta H_{\text{ren}} (P)} \triangleright 0$ w.r.t. $\mathcal{F}_+$ for all $\beta > 0$.

(ii) For each $\varphi, \psi \in \mathcal{F}_+ \setminus \{ 0 \}$, there exist $\beta \geq 0$ and $\kappa > 0$ such that $\langle \varphi | e^{-\beta L_\kappa} \psi \rangle > 0$.  

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4.5.1 Proof of Theorem 4.11

Let $A$ and $B$ be self-adjoint operators, and let $E_A$ and $E_B$ be their spectral measures. Assume that $E_A$ and $E_B$ commute with each other: $E_A(I)E_B(J) = E_B(J)E_A(I)$ for all $I, J \in \mathcal{B}^1$, the Borel sets of $\mathbb{R}$. We can decompose $A$ as $A = A_+ - A_-$, where $A_+$ and $A_-$ are positive and negative parts of $A$, respectively. Similarly, we have $B = B_+ - B_-.$

For each $n \in \mathbb{N}$, we set

$$ (AB)[n] = A_+B_+ + A_-B_- - \left( A_+E_A[0,n]B_-E_B[-n,0] + A_-E_A[-n,0]B_+E_B[0,n] \right). $$

(4.36)

Note that $E_A[-n,0] = E_{A_+}[0,n]$ and $E_B[-n,0] = E_{B_-}[0,n]$. Thus, we have

$$ (AB)[n] \geq -2n^2, $$

(4.37)

$$ (AB)[n] \geq (AB)[n+1]. $$

(4.38)

Similarly, we define

$$ (AB)[n] = A_+E_A[0,n]B_+E_B[0,n] + A_-E_A[-n,0]B_-E_B[-n,0] - (A_+B_- + A_-B_+). $$

(4.39)

We have

$$ (AB)[n] \leq 2n^2, $$

(4.40)

$$ (AB)[n] \leq (AB)[n+1]. $$

(4.41)

For each $\kappa > 0$, we define a sequence of self-adjoint operators $\{C^{\pm\kappa}_{\kappa,n}\}_{n=1}^{\infty}$ by

$$ C^{+\kappa}_{\kappa,n} = -3 \sum_{j=1}^{3} \left( (P_j - P^{\leq\kappa}_l)P^{<\kappa}_{l,j} \right)[n]. $$

(4.42)

Similarly, we define

$$ C^{-\kappa}_{\kappa,n} = -3 \sum_{j=1}^{3} \left( (P_j - P^{\leq\kappa}_l)P^{<\kappa}_{l,j} \right)[n]. $$

(4.43)

Let $C_\kappa = -(P - P^{\leq\kappa}_l)P^{>\kappa}_l$. Note that $C^{+\kappa}_{\kappa,n}\varphi$ converges to $C_\kappa \varphi$ as $n \to \infty$ for each $\varphi \in \text{dom}(C_\kappa)$. By (4.37), (4.38), (4.40) and (4.41), we have

$$ C^{+\kappa}_{\kappa,n} \leq 6n^2, $$

(4.44)

$$ C^{+\kappa}_{\kappa,n} \leq C^{+\kappa}_{\kappa,n+1}, $$

(4.45)

$$ C^{-\kappa}_{\kappa,n} \geq -6n^2, $$

(4.46)

$$ C^{-\kappa}_{\kappa,n} \geq C^{-\kappa}_{\kappa,n+1}. $$

(4.47)

Lemma 4.12 For all $n \in \mathbb{N}$ and $s \geq 0$, we have the following:

(i) $e^{-sC^{-\kappa}_{\kappa,n}}$ is bounded and $e^{-sC^{-\kappa}_{\kappa,n}} \geq 0$ w.r.t. $\mathfrak{F}_+$. 

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(ii) $e^{sC_{\kappa,n}^+}$ is bounded and $e^{sC_{\kappa,n}^-} \geq 0$ w.r.t. $\mathcal{F}_+$. 

Proof. (i) By (4.46), $e^{-sC_{\kappa,n}^-}$ is bounded for all $s \geq 0$. We can express $e^{-sC_{\kappa,n}^-}$ as

$$e^{-sC_{\kappa,n}^-} = \sum_{\ell \geq 0} F_{\ell},$$

(4.48)

where $F_{\ell}$ is some multiplication operator on $L^2_{\text{sym}}(\mathbb{R}^3)$. We easily see that the function $F_{\ell}$ is positive. Thus, $F_{\ell} \geq 0$ w.r.t. $L^2_{\text{sym}}(\mathbb{R}^3)$ for all $\ell \in \mathbb{N}$, which implies (i). Similarly, we can prove (ii). \[ \Box \]

Lemma 4.13 Let $\varphi, \psi \in \mathcal{F}_+$. 

(i) If $\langle \varphi | \psi \rangle = 0$, then $\langle \varphi | e^{-sC_{\kappa,n}^-} \psi \rangle = 0$ for all $n \in \mathbb{N}$, $s \geq 0$ and $\kappa > 0$.

(ii) If $\langle \varphi | \psi \rangle = 0$, then $\langle \varphi | e^{sC_{\kappa,n}^+} \psi \rangle = 0$ for all $n \in \mathbb{N}$, $s \geq 0$ and $\kappa > 0$.

Proof. (i) We can express $\varphi$ and $\psi$ as

$$\varphi = \sum_{\ell \geq 0} \varphi_{\ell}, \quad \psi = \sum_{\ell \geq 0} \psi_{\ell}.$$ 

(4.49)

Note that $\varphi_{\ell}$ and $\psi_{\ell}$ are positive functions in $L^2_{\text{sym}}(\mathbb{R}^3)$. The condition $\langle \varphi | \psi \rangle = 0$ is equivalent to the condition $\langle \varphi_{\ell} | \psi_{\ell} \rangle = 0$ for all $\ell \in \mathbb{N}_0$. Recall the expression (4.48). Because $F_{\ell}$ is positive and bounded, we conclude that $\langle \varphi_{\ell} | F_{\ell} \psi_{\ell} \rangle = 0$, which implies that $\langle \varphi | e^{-sC_{\kappa,n}^-} \psi \rangle = \sum_{\ell = 0}^{\infty} \langle \varphi_{\ell} | F_{\ell} \psi_{\ell} \rangle = 0$. Similarly, we can prove (ii). \[ \Box \]

Lemma 4.14 $e^{-\beta L_{\kappa}} \geq 0$ w.r.t. $\mathcal{F}_+$ for all $P \in \mathbb{R}^3$ and $\beta \geq 0$.

Proof. By Propositions 3.18, 4.4 and 4.7, we obtain the assertion in the lemma. \[ \Box \]

Lemma 4.15 We have the following:

(i) $L_{\kappa} + C_{\kappa,n}^-$ converges to $H_{\text{ren}}(P)$ in strong resolvent sense as $n \to \infty$, where $\dot{\pm}$ in dicates the form sum.

(ii) $H_{\text{ren}}(P) - C_{\kappa,n}^+$ converges to $L_{\kappa}$ in strong resolvent sense as $n \to \infty$.

Proof. (i) Let us define a sequence of closed, positive quadratic form $\{t_n\}_{n=1}^{\infty}$ by

$$t_n(\varphi, \psi) = \langle \varphi | [L_{\kappa} + C_{\kappa,n}^- + \text{Const.}] \psi \rangle,$$

(4.50)

where $\text{Const.}$ is chosen such that $t_n$ is uniformly positive. By (4.37), we have $t_1 \geq t_2 \geq \cdots \geq t_n \geq \cdots$ and $\lim_{n \to \infty} t_n(\varphi, \varphi) = t_\infty(\varphi, \varphi)$, where $t_\infty$ is a quadratic form associated with $H_{\text{ren}}(P)$. Thus, by [31] Theorem S. 16], we obtain (i).

Similarly, we can prove (ii) by applying [31] Theorem S. 16]. \[ \Box \]

Proof of Theorem 4.11
Taking assumption (i), we have \( e^{-\beta L_n} \psi = 0 \) for all \( \beta \geq 0 \) and \( \kappa > 0 \). We will show that \( K(\psi) = \{ 0 \} \). Let \( \varphi \in K(\psi) \):

\[
\langle \varphi | e^{-\beta L_n} \psi \rangle = 0 \quad \text{for all } \beta \geq 0 \quad \text{and } \kappa > 0.
\]

By Lemma 4.14 (i) and Lemma 4.14, we have \( \langle e^{-sC_{\kappa,n}} \varphi | e^{-\beta L_n} \psi \rangle = 0 \) for all \( n \in \mathbb{N} \), \( s \geq 0 \), \( \beta \geq 0 \) and \( \kappa > 0 \), which implies that \( e^{-sC_{\kappa,n}} K(\psi) \subseteq K(\psi) \). On the other hand, it is easy to check that \( e^{-tL_n} K(\psi) \subseteq K(\psi) \) for all \( t \geq 0 \). Hence, \( (e^{-\beta L_n/\ell} e^{-\beta C_{\kappa,n}/\ell})^t K(\psi) \subseteq K(\psi) \) for all \( \ell \in \mathbb{N} \). Taking \( \ell \to \infty \), we obtain that \( e^{-\beta(L_n+C_{\kappa,n})} K(\psi) \subseteq K(\psi) \) for all \( n \in \mathbb{N} \) and \( \beta \geq 0 \) by [31] Theorem S. 21].

Taking \( n \to \infty \), we arrive at \( e^{-\beta H_{\text{ren}}(P)} K(\psi) \subseteq K(\psi) \) for all \( \beta \geq 0 \) by Lemma 4.15 (i). Therefore, for each \( \varphi \in K(\psi) \), it holds that \( \langle \varphi | e^{-\beta H_{\text{ren}}(P)} \psi \rangle = 0 \) for all \( \beta \geq 0 \). By the assumption (i), \( \varphi \) must be 0.

(ii) \( \implies \) (i): We will provide a sketch. For each \( \psi \in \mathfrak{F}_+ \setminus \{ 0 \} \), we set \( J(\psi) = \{ \varphi \in \mathfrak{F}_+ | \langle \varphi | e^{-\beta L_n} \psi \rangle = 0 \quad \text{for all } \beta \geq 0 \} \). Using arguments similar to those in the previous part, we can show that \( e^{-\beta L_n} J(\psi) \subseteq J(\psi) \) for all \( \beta \geq 0 \) and \( \kappa > 0 \). Thus, for each \( \varphi \in J(\psi) \), we obtain \( \langle \varphi | e^{-\beta L_n} \psi \rangle = 0 \) for all \( \beta \geq 0 \) and \( \kappa > 0 \). By the assumption (ii), \( \varphi \) must be 0, which implies \( J(\psi) = \{ 0 \} \). Thus, for each \( \varphi, \psi \in \mathfrak{F}_+ \setminus \{ 0 \} \), there exists a \( \beta \geq 0 \) such that \( \langle \varphi | e^{-\beta H_{\text{ren}}(P)} \psi \rangle > 0 \). Applying Theorem 2.10 we conclude (i).

\section*{4.6 Completion of proof of Theorem 1.3}

\textbf{Proposition 4.16} For all \( P \in \mathbb{R}^3 \) and \( \kappa > 0 \), we have

\[
e^{-\beta L_n} \geq \langle \Omega^{\kappa} | e^{-\beta K_{\kappa} \Omega^{\kappa}} e^{-\beta H_{\text{ren}}(P)} \otimes 1Q_\kappa^n \rangle
\]

w.r.t. \( \mathfrak{F}_+ \), where \( \Omega^{\kappa} \) is the Fock vacuum in \( \mathfrak{F}^{\kappa} \).

\textbf{Proof.} By Proposition 3.10 it holds that \( Q_\kappa \geq 0 \) and \( Q_\kappa^\perp \geq 0 \) w.r.t. \( \mathfrak{F}_+ \). Thus, by Lemma 4.14

\[
e^{-\beta L_n} \geq Q_\kappa e^{-\beta L_n} Q_\kappa \quad \text{w.r.t. } \mathfrak{F}_+ \text{ for all } \beta \geq 0.
\]

By Lemma 4.14 we have

\[
\langle \varphi | Q_\kappa e^{-\beta L_n} Q_\kappa \psi \rangle = \langle \varphi \otimes \Omega^{\kappa} | e^{-\beta L_n} \psi \otimes \Omega^{\kappa} \rangle
\]

\[
= \langle \Omega^{\kappa} | e^{-\beta K_{\kappa} \Omega^{\kappa}} \langle \varphi | e^{-\beta H_{\text{ren}}(P)} \psi \rangle \rangle
\]

\[
= \langle \Omega^{\kappa} | e^{-\beta K_{\kappa} \Omega^{\kappa}} \langle \varphi | e^{-\beta H_{\text{ren}}(P)} \otimes 1Q_\kappa \psi \rangle \rangle,
\]

which implies that \( Q_\kappa e^{-\beta L_n} Q_\kappa = \langle \Omega^{\kappa} | e^{-\beta K_{\kappa} \Omega^{\kappa}} e^{-\beta H_{\text{ren}}(P)} \otimes 1Q_\kappa \rangle \). Here, we used the fact that \( Q_\kappa e^{-\beta H_{\text{ren}}(P)} \otimes 1Q_\kappa = e^{-\beta H_{\text{ren}}(P)} \otimes 1Q_\kappa \), which follows from Proposition 3.11.

\textbf{Lemma 4.17} \( \langle \Omega^{\kappa} | e^{-\beta K_{\kappa} \Omega^{\kappa}} \rangle > 0 \) for all \( \beta \geq 0 \) and \( \kappa > 0 \).

\textbf{Proof.} Because \( \ker(e^{-\beta K_{\kappa}}) = \{ 0 \} \) by Proposition 1.17, the assertion is easy to check.

\textbf{Proof of Theorem 1.3}
Proof. This proposition is already proved in [23], see also [24, 25, 26]. For readers’ convenience, we provide a proof.

Remark that we have the following norm convergent expansion:

\[
\langle \varphi | e^{-\beta H_{\text{cov}}(P)} \otimes 1_Q \psi \rangle = \langle \varphi_\kappa | e^{-\beta H_{\text{cov}}(P)} \psi_\kappa \rangle,
\]

(4.55)

where \( \varphi_\kappa \) and \( \psi_\kappa \) are defined by (3.42). Of course, \( \varphi_\kappa \neq 0 \) and \( \psi_\kappa \neq 0 \). By Proposition 4.4, the right hand side of (4.55) is strictly positive, provided that \( \beta > 0 \). Because \( \langle \Omega^\kappa | e^{-\beta K_\kappa \Omega^\kappa} \rangle > 0 \) by Lemma 4.17, we know that the right hand side of (4.55) is strictly positive. By Theorem 4.11, we finally conclude that \( e^{-\beta H_{\text{cov}}(P)} \geq 0 \) w.r.t. \( \mathfrak{F}_+ \) for all \( P \in \mathbb{R}^3 \) and \( \beta > 0 \). □

A A useful proposition

In this appendix, we will review a useful result concerning the operator inequalities introduced in Section 2.

Proposition A.1 Let \( A \) be a positive self-adjoint operator and let \( B \) be a symmetric operator. Assume the following:

(i) \( B \) is \( A \)-bounded with relative bound \( a < 1 \), i.e., \( \text{dom}(A) \subseteq \text{dom}(B) \) and \( \|Bx\| \leq a\|Ax\| + b\|x\| \) for all \( x \in \text{dom}(A) \).

(ii) \( 0 \leq e^{-tA} \) w.r.t. \( \mathfrak{F} \) for all \( t \geq 0 \).

(iii) \( 0 \leq -B \) w.r.t. \( \mathfrak{F} \).

Then \( 0 \leq e^{-t(A+B)} \) w.r.t. \( \mathfrak{F} \) for all \( t \geq 0 \).

Proof. This proposition is already proved in [23], see also [24, 25, 26]. For readers’ convenience, we provide a proof.

For each \( \varepsilon > 0 \), we set \( B_\varepsilon = e^{-\varepsilon A} Be^{-\varepsilon A} \). By (i) and (iii), \( B_\varepsilon \) is bounded and \(-B_\varepsilon \geq 0 \) w.r.t. \( \mathfrak{F} \). Let us consider a self-adjoint operator \( C_\varepsilon = A + B_\varepsilon \). By the Duhamel formula, we have the following norm convergent expansion:

\[
e^{-tC_\varepsilon} = \sum_{n=0}^{\infty} D_n,
\]

(4.1)

\[
D_n = \int_{S_n(t)} e^{-s_1 A}(-B_\varepsilon) e^{-s_2 A}(-B_\varepsilon) \cdots e^{-s_n A}(-B_\varepsilon) e^{-(\beta - \sum_{j=1}^{n} s_j) A},
\]

(4.2)

where \( \int_{S_n(t)} = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \) and \( D_0 = e^{-tA} \). Since \(-B_\varepsilon \geq 0 \) and \( e^{-tA} \geq 0 \) w.r.t. \( \mathfrak{F} \) for all \( t \geq 0 \), it holds that, by Lemma 2.7,

\[
e^{-s_1 A}(-B_\varepsilon) e^{-s_2 A} \cdots e^{-s_n A}(-B_\varepsilon) e^{-(t - \sum_{j=1}^{n} s_j) A} \geq 0,
\]

(4.3)

provided that \( s_1 \geq 0, \ldots, s_n \geq 0 \) and \( t - s_1 - \cdots - s_n \geq 0 \). Thus, by Proposition 2.8, we obtain \( D_n \geq 0 \) w.r.t. \( \mathfrak{F} \) for all \( n \geq 0 \). Accordingly, by (4.1), we have \( e^{-tC_\varepsilon} \geq D_{n=0} = e^{-tA} \geq 0 \) w.r.t. \( \mathfrak{F} \) for all \( t \geq 0 \) and \( \varepsilon \geq 0 \). Because \( e^{-tC_\varepsilon} \) strongly converges to \( e^{-t(A+B)} \) as \( \varepsilon \to +0 \), we conclude that \( e^{-t(A+B)} \geq 0 \) w.r.t. \( \mathfrak{F} \) for all \( t \geq 0 \) by Proposition 2.8 □
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