SCALING LIMITS OF PERMUTATIONS AVOIDING LONG DECREASING SEQUENCES

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Abstract. We determine the scaling limit for permutations conditioned to have longest decreasing subsequence of length at most \( d \). These permutations are also said to avoid the pattern \((d + 1)d\cdots21\) and they can be written as a union of \( d \) increasing subsequences. We show that these increasing subsequences can be chosen so that, after proper scaling, and centering, they converge in distribution. As the size of the permutations tends to infinity, the distribution of functions generated by the permutations converges to the eigenvalue process of a traceless \( d \times d \) Hermitian Brownian bridge.

1. Introduction

In this paper, we consider random permutations without long decreasing subsequences as a model of non-intersecting paths. It is a classical result that if the longest decreasing subsequence of a permutation \( \pi \) has length \( d \) then \( \pi \) can be written as the union of \( d \) increasing subsequences. The origins of this result are hard to trace, but it goes back at least to [20] where it is already noted as something that is not hard to see. Our main result is that if \( \sigma \) is uniformly random permutation of \([n] = \{1, 2, \ldots, n\}\) conditioned on its longest decreasing subsequence having length at most \( d \), then these decreasing subsequences can be chosen so that, after linearly interpolating, scaling, and centering, they converge in distribution, as \( n \) tends to infinity, to the eigenvalue process of a traceless \( d \times d \) Hermitian Brownian bridge. Our results fall in the intersection of two lines of research that have received significant interest in the recent literature – properties of random pattern-avoiding permutations and limit theorems for non-intersecting paths.

Let \( S_n \) denote the set of permutations of length \( n \). For \( k \leq n \), \( \rho \in S_k \) and \( \tau \in S_n \) we say \( \tau \) contains the pattern \( \rho \) if there exists \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that for \( 1 \leq r < s \leq k \), \( \tau(i_r) < \tau(i_s) \) if and only if \( \rho(r) < \rho(s) \). The permutation \( \tau \) avoids \( \rho \) if it does not contain the pattern \( \rho \). We denote the subset of \( S_n \) that avoid all permutations in a set \( A \subset S_k \) by \( Av_n(A) \).

Taking \( \rho_d = (d + 1)d\cdots21 \), the decreasing pattern of length \( d + 1 \), we have that \( Av_n(\rho_d) \) is the set of permutations of \([n]\) whose longest decreasing subsequence has length at most \( d \).

Permutations whose longest decreasing subsequence as length at most \( d \) (or whose longest increasing sequence has length at most \( d \)) have a long history both of being studied directly and of appearing in the study of other mathematical objects. For example, permutations avoiding the pattern 123 seem to have first been considered by MacMahon [34], who showed that they are counted by the Catalan numbers \( C_n = \frac{1}{n+1} \binom{2n}{n} \). Later, Knuth showed that permutations avoiding any fixed pattern of length three are also counted by the Catalan numbers, as are the number of rectangular standard Young tableaux with 2 rows and 2\( n \) boxes. Regev [46] used the RSK correspondence to give an asymptotic formula for the cardinality of \( Av_n(\rho_d) \) for \( d \geq 2 \). Also using RSK, Novak [43] extended Knuth’s result in an asymptotic sense to show that for any \( d \), the cardinality of \( Av_{dn}(\rho_d) \) is asymptotically equal to the rectangular standard Young tableaux with \( d \) rows and \( dn \) boxes. Permutations whose longest decreasing subsequence as length at most \( d \) have also appeared in

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random matrix theory and integrable probability. For example, [45] shows that if $n > d$, then the $2n$'th moment of the trace of random $d$-dimensional unitary matrix equals the number of permutations of $[n]$ whose longest increasing subsequence is at most $d$. Further connections to integrals over classical groups are established in, for example, [3]. In [17] it was shown that the number of configurations in certain random-turns vicious walker models with $d$ walks and $n$ steps was equal to the number of permutations of $[n]$ whose longest increasing subsequence is at most $d$. These results generally rely on the RSK algorithm giving a bijection from these permutations to pairs of Young tableaux with the same shape with at most $d$ rows.

Recently there has been significant interest in understanding, for a fixed set of patterns $A$, the behavior of a uniformly random element of the set $Av_n(A) \subset S_n$ as $n \to \infty$, see e.g. [1, 4, 5, 8, 9, 7, 13, 22, 23, 24, 26, 27, 33, 35, 36, 37, 38, 39, 40, 41] for a sample of the available results. Most of this literature focuses on permutations avoiding short patterns and significant attention has been given to understanding uniformly random elements of $Av_n(123)$ and $Av_n(321)$, see e.g. [13, 22, 23, 24, 26, 40]. Early work in this area was motivated by studying the longest increasing subsequence problem for pattern avoiding permutations [11, 13, 48]. The longest increasing subsequence of a uniformly random element of $Av_n(\sigma)$ when $\sigma$ is a permutation of length 3 was studied in [13] using exact enumeration methods. The main result of [48] shows that with appropriate centering and scaling the limit of the shape of the Young tableaux obtained by applying RSK to a uniformly random element of $Av_n(\rho_d)$ is given by the eigenvalues of a $d \times d$ GUE matrix conditioned to have trace 0. Similar results for the shape of the tableaux obtained by applying RSK to a random word were obtained in [49, 28]. A common feature of all of these results is the combinatorial nature of the analysis, relying on generating functions, bijections with Dyck paths, or colored trees, or similar well understood combinatorial structures. In contrast, our methods here rely on an approximate bijection (in a sense made clear in our analysis) that allows us to closely couple the graph of a uniformly random permutation whose longest decreasing subsequence has length at most $d$ with graph of a bridge of a random walk in $\mathbb{R}^d$ conditioned to remain in a certain cone. Using this coupling, we are able to leverage recent results on scaling limits of random walks in cones [12] to establish our results.

Non-intersecting paths and their connections to random matrices, which go back to Dyson [15], have also featured prominently in the physics, combinatorics, and probability literature, see e.g. [2, 16, 18, 21, 29, 44]. Non-intersecting Brownian bridges specifically have arisen in several contexts [10, 11, 30, 42]. Random matrices conditioned to have trace equal to 0 have also previously appeared in the literature [6, 28, 31, 45]. In this case, because we are working with Gaussian processes, conditioning to have trace equal to 0 can be easily thought of as projection, which allows for the transfer of many results. In these applications, the models of non-intersecting paths that have been studied have an integrable structure and the ability to analyze the exact formulas that come out of this plays a central role. In contrast, the model of non-intersecting paths that comes from random permutations without long decreasing subsequences is not known to be integrable. The non-intersecting paths derived from a permutation $\sigma$ are essentially an emergent phenomenon. It is easy to see them in simulations for large $n$, but the increasing subsequences that $\sigma$ divides into have disjoint domains, so it is not obvious what it means for them not to intersect.

2. Main Results

2.1. From permutations to functions. Every $\sigma \in Av_n(\rho_d)$ defines a $d$-tuple of non-intersecting functions on $C([0,1])$ as follows. First we note that each element in $\sigma \in Av_n(\rho_d)$ gives a natural partition of $[n]$ into $d$ sets, $\{A^i(\sigma)\}_{i \in [d]}$ as follows. Define

$$A^i(\sigma) = \{i : \forall i' < i \text{ such that } \sigma(i') > \sigma(i)\}$$
and for $1 < i \leq d$

$$A^i(\sigma) = \{ i : \exists i' < i \text{ and } i' \notin \bigcup_{j < i} A^j(\sigma) \text{ such that } \sigma(i') > \sigma(i) \}.$$  

Thus the sequence $A^1(\sigma)$ consists of all $i$ which are the left right maxima of $\sigma$. The sequence $A^2(\sigma)$ consists of all $i$ which are the left right maxima of $\sigma$ after removing the elements $(i, \sigma(i))$ with $i \in A^1(\sigma)$, etc.

Next define $d$ sequences

$$\alpha^l(\sigma) = \{(i, \sigma(i))\}_{i \in A^l(\sigma)}$$

for $l \in [d]$. These sequences give a unique way to construct pairs of words $\omega_\sigma \in [d]^n \times [d]^n$ where $\omega_\sigma(i) = (l_1, l_2)$ if $i \in A^l_1$ and $\sigma^{-1}(i) \in A^l_2$. The pair of words $\omega_\sigma$ can be seen by projecting the labels of the points $\alpha^l(\sigma)$ either horizontally or vertically (see Figure 1).

For any sequence $\alpha = \{(a(i), b(i))\}_{i=1}^m$ and $n$ with $1 \leq a(1) < a(2) < \cdots < a(m) \leq n$ we can form a continuous function $f(\alpha)$ on $[0, 1]$ by linearly interpolating between the points $(0, 0), (1, 0)$ and

$$\left\{ \left( \frac{a(i) - b(i)}{n + 1}, \frac{a(i)}{\sqrt{2dn}} \right) \right\}_{i=1}^m.$$

For $\sigma \in A_{n}(\rho_d)$ we take the $d$ sequences $\{\alpha^l(\omega)\}_{l \in [d]}$, and form

$$P_\sigma = \left( f(\alpha^1(\sigma)), \cdots, f(\alpha^d(\sigma)) \right).$$

If follows from our definition of $A^i$ that $f(\alpha^1(\sigma)) \geq f(\alpha^2(\sigma)) \geq \cdots \geq f(\alpha^d(\sigma))$, so that $P_\sigma$ is a family of non-intersecting paths (See Figure 2). Our main result is an invariance principle for these paths.

### 2.2. Traceless Dyson Brownian bridge.

In order to state our main result formally we need to introduce the limiting object, which is the process ranked eigenvalues of a $d$ by $d$ Hermitian Brownian bridge conditioned to have trace equal to 0 for all time.

Let $\{Z_{dd}(t)\}_{t=1}^d$ be standard Brownian bridges conditioned so that $\sum_{i=1}^d Z_{ii} = 0$. Let $\{Z_{ij}\}_{i<j \leq d}^d$ be independent standard complex Brownian bridges (i.e. $\sqrt{2}\Re(Z_{ij})$ and $\sqrt{2}\Im(Z_{ij})$ are independent standard Brownian bridges). Finally let $Z_{ji} = \bar{Z}_{ij}$. We use these random variables to define
Figure 2. On the left is a labeled plot of a permutation $\sigma \in \mathcal{A}v_{20}(\rho_3)$. On the right is the corresponding collection of paths

$$P_\sigma = (f(\alpha^1(\sigma)), f(\alpha^2(\sigma)), f(\alpha^3(\sigma)))$$

with $f(\alpha^1(\sigma))$ in green, $f(\alpha^2(\sigma))$ in red and $f(\alpha^3(\sigma))$ in blue.

The process $\Lambda(Z) = (\Lambda(Z_1(t)), 0 \leq t \leq 1)$ is what appears as the limiting object in our main result. We sometimes refer to it as the traceless Dyson Brownian bridge.

Theorem 2.1. If $\sigma$ is a uniformly random element of $\mathcal{A}v_n(\rho_d)$ then, as $n \to \infty$, the following convergence holds in distribution with respect to the supremum norm topology on $C([0,1], \mathbb{R}^d)$:

$$P_\sigma \overset{\text{dist}}{\to} \Lambda(Z).$$

This is proved in Section 7.2. Note that the map from $\sigma$ to $P_\sigma$ is not canonical. (For instance we could have defined the functions by the orthogonal distance to the diagonal instead of the vertical distance.) It is easy to modify our result to prove an appropriate limit theorem for the modified function. Figure 3 gives an example of a large permutation $\sigma \in \mathcal{A}v_n(654321)$ and the corresponding $P_\sigma$.

2.3. Connection to previous results. Theorem 2.1 can be viewed as an extension of Theorem 1.2 of [23] as follows. That theorem considered the case of $d = 2$. Let $(e_t)_{t \in [0,1]}$ be standard Brownian excursion. It proves that

- $\sqrt{2} f(\alpha^1(\sigma)) \overset{\text{dist}}{\to} (e_t)$ and
- $f(\alpha^1(\sigma)) + f(\alpha^2(\sigma)) \overset{\text{dist}}{\to} 0$. 

The following Hermitian matrix valued process:

$$Z(t) = (Z_{ij}(t))_{1 \leq i,j \leq d}.$$ 

For a Hermitian matrix $M$, we let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_d(M)$ be the eigenvalues of $M$ ranked in non-increasing order. Furthermore, we define

$$\Lambda(M) = (\lambda_1(M), \lambda_2(M), \cdots, \lambda_d(M)).$$

Notice that

$$\sum_{i=1}^{d} \lambda_i(Z_t) = \sum_{i=1}^{d} Z_{ii}(t) = 0.$$

for all $t \in [0,1]$.

The process $\Lambda(Z) = (\Lambda(Z(t)), 0 \leq t \leq 1)$ is what appears as the limiting object in our main result. We sometimes refer to it as the traceless Dyson Brownian bridge.
Figure 3. A permutation $\sigma \in \Av_{100000}$ and the corresponding collection of scaled paths $P_\sigma$.

Figure 4. On the left is a permutation $\sigma$ in $\Av_{100000}(321)$. On the right are the functions $f(\alpha^1(\sigma))$ and $f(\alpha^2(\sigma))$. They are approximated by a Brownian excursion $e_t$ and $-e_t$, respectively.

The convergence is in the supremum norm topology on $C([0,1],\mathbb{R})$.

To derive this result from Theorem 2.1 we let $(B_j(t))_{t \in [0,1]}$ be independent standard Brownian bridges. Then

$$Z \overset{d}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} B_1 & B_2 + iB_3 \\ B_2 - iB_3 & -B_1 \end{pmatrix}.$$ 

A direct computation shows that

$$\lambda_1(Z) \overset{d}{=} \frac{1}{\sqrt{2}} \sqrt{B_1^2 + B_2^2 + B_3^2} \quad \text{and} \quad \lambda_2(Z) \overset{d}{=} -\frac{1}{\sqrt{2}} \sqrt{B_1^2 + B_2^2 + B_3^2}.$$ 

Using the identity in law between Brownian excursion and the 3-dimensional Bessel bridge \cite[Chapter XII]{BesselBridge} shows that $(\lambda_1(Z), \lambda_2(Z)) \overset{d}{=} (2^{-1/2}e_t, -2^{-1/2}e_t)_{t \in [0,1]}$, where $(e_t)_{t \in [0,1]}$ is a standard Brownian excursion. This is the conclusion of \cite[Theorem 1.2]{BrownianExcursion} (with a slightly different normalization). See Figure 4 for an example.

3. Outline

In Section 2 we showed one way to take a permutation $\sigma \in \Av_n(\rho_d)$ to get an $\mathbb{R}^d$ valued function $P_\sigma$ on $[0,1]$. In Section 4 we will show a different way to map $\Av_n(\rho_d)$ to $\mathbb{R}^d$ valued functions on $[0,1]$. In particular, we first take $\sigma \in \Av_n(\rho_d)$ and map it to a path $s_{\omega_\sigma}$ on $\mathbb{Z}^d$ and then space
time scaling pushes this forward to an \( \mathbb{R}^d \) valued function \( \hat{s}_{\omega} \) on \([0, 1]\). If \( \sigma \in \mathcal{Av}_n(\rho_d) \) is uniformly random, we will analyze \( s_{\omega_{\sigma}} \) using techniques developed to study random walks in cones. We are interested in a cone in \( \mathbb{Z}^d \) that, by a slight abuse of terminology, we call the Weyl Chamber

\[
\text{Weyl} = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_1 \geq \cdots \geq x_d\}.
\]

Ideally we would have liked our paper to have consisted of the following steps.

- Find a random walk \( S \) such that if \( \sigma \in \mathcal{Av}_n(\rho_d) \) is uniformly random then \( s_{\omega_{\sigma}} \) is distributed like \( \{S_i\}_{i=1}^n \) conditioned to stay in the Weyl Chamber and start and end at the origin.
- Prove that for every \( \sigma \in \mathcal{Av}_n(\rho_d) \) the functions \( P_\sigma \) and \( \hat{s}_{\omega_{\sigma}} \) are close in the supremum norm.
- Prove that the scaling limit of a bridge of the random walk \( S \) conditioned to remain in the Weyl Chamber is distributed like the eigenvalues of a traceless Dyson Brownian bridge.
- Combine these three statements to prove our main theorem.

Unfortunately the claims in the first two bullet points above cannot be accomplished. Fortunately some appropriate modification of each of these steps is achievable. And these modifications are sufficiently strong to allow us to prove Theorem \ref{main theorem}.

In Section 4 we show how to take \( \sigma \in \mathcal{Av}_n(\rho_d) \) and map it to a path \( s_{\omega_{\sigma}} \) on \( \mathbb{Z}^d \). We also introduce a random walk \( S(t) \) such that \( s_{\omega_{\sigma}} \) is in the range of \( S(t) \) and discuss some typical properties of the paths of \( s_{\omega_{\sigma}} \).

In Section 5 we do much of the work connecting pattern avoiding permutations and the paths of random walks. First in Lemma 5.1 we define a subset of \( \sigma \in \mathcal{Av}_n(\rho_d) \) such that \( s_{\omega_{\sigma}} \) spends most of the time in the Weyl Chamber. Next in Lemma 5.3 we define a subset of paths in the Weyl Chamber that start and end at the origin where for each \( s_{\omega_{\sigma}} \) in this subset, there is a \( \sigma \in \mathcal{Av}_n(\rho_d) \) such that \( s_{\omega_{\sigma}}(m) = s_{\omega}(m) \) for most \( m \). Finally in Lemma 5.4 we define a set of \( \sigma \in \mathcal{Av}_n(\rho_d) \) such that the functions \( P_\sigma \) and \( \hat{s}_{\omega_{\sigma}} \) are close in the supremum norm.

In Section 7 we show that the size of the above subsets is \( 1 - \epsilon \) times the size of the respective spaces. Thus we have the adaptations of the first three bullet points. We also show how to combine these results with the scaling limit of our random walk in the Weyl Chamber to prove Theorem \ref{main theorem}.

Sections 6, 8, and 9 are auxiliary sections. In Section 6 we define some technical lemmas about random walks close to the Weyl Chamber that will be used in Section 7. These lemmas are proven in Section 9. In Section 8 we adapt the previous literature to show that the scaling limit of our random walk in the Weyl Chamber is distributed like the eigenvalues of a traceless Hermitian Brownian bridge. The proofs in Section 8 are independent of the results in the rest of the paper.

4. Notation

4.1. Paths on \( \mathbb{Z}^d \). Let \( e_l \) be the \( d \)-dimensional vector with a one in the \( l \)th coordinate and zero everywhere else. For the norm on \( \mathbb{Z}^d \), we use the \( L^1 \) norm.

**Definition 4.1.** Let \( I \) be a connected subset of \( \mathbb{N} \). A path is a function \( s : I \to \mathbb{Z}^d \) where

\[
s(t + 1) - s(t) \in \{e_i - e_j\}_{1 \leq i, j \leq d}
\]

for all \( t \) such that \( t, t + 1 \in I \) and

\[
s(t) \in \left\{ (z_1, \cdots, z_d) \in \mathbb{Z}^d : \sum_{i=1}^d z_i = 0 \right\}
\]

for all \( t \in I \).

As \( s(t + 1) - s(t) = \{e_i - e_j\}_{1 \leq i, j \leq d} \) as long as \( s(t) \in \left\{ (z_1, \cdots, z_d) \in \mathbb{Z}^d : \sum_{i=1}^d z_i = 0 \right\} \) for one \( t \in I \) then it satisfies the condition of being in the codimension 1 subspace for all \( t \).
Figure 5. On the left is permutation $\sigma \in \text{Av}_{20}(\rho_3)$ with corresponding projections. On the right are the coordinate projections of $s_{\omega}(i) = \sum_{j=1}^{i} e_{a(j)} - e_{b(j)}$.

We consider a path $s$ to be a lattice path on a lattice whose vertices are the points of $\mathbb{Z}^d$ and whose edges are given by the relation $x \sim y$ if $x - y \in \{e_i - e_j\}_{1 \leq i,j \leq d}$. In particular, we define the boundaries of sets relative to this lattice.

**Definition 4.2.** For $A \subseteq \mathbb{Z}^d$, the boundary of $A$ is

$$\partial A = \{x \in A : x + e_i - e_j \in A^c \text{ for some } 1 \leq i,j \leq d\}.$$

**4.2. Defining our probability space.** Let $\Omega_N = [d]^N \times [d]^N$ and let $\mathbb{P} = \mu^N \times \mu^N$ be the product measure on $\Omega_N$, where $\mu$ is the uniform distribution on $[d]$. This will be our probability space for much of the paper. (We will also consider uniform distribution on a number of subsets of permutations.) For $\omega \in \Omega_N$ we write $\omega = (a,b)$ where $a = (a(t))_{t=1}^{\infty}$ is the projection onto the first sequence and $b = (b(t))_{t=1}^{\infty}$ is the projection onto the second sequence. When it will not cause confusion, we will use $\omega$ both for an arbitrary element in $\Omega_N$ and the canonical random variable given by the identity map on $(\Omega_N, \mathbb{P})$. We also let $\omega(t) = (a(t), b(t))$. Observe that if $\omega$ is distributed like $\mathbb{P}$ then the random variables $\{\omega(t)\}_{t \in \mathbb{N}}$ are i.i.d. with

$$\mathbb{P}(\omega(t) = (i,j)) = 1/d^2$$

for all $i,j \in [d]$.

For $\omega = (a,b) \in \Omega_N$. we define the path $s_{\omega}$ by

$$s_{\omega}(i) = \sum_{j=1}^{i} e_{a(j)} - e_{b(j)}.$$

Note that if $\omega$ has distribution $\mathbb{P}$, then $s_\omega$ is a lazy random walk such that $s_\omega(t+1) - s_\omega(t) = 0$ with probability $1/d$ and $s_\omega(t+1) - s_\omega(t) = e_j - e_k$ with probability $1/d^2$ for each $i \neq j$ with $1 \leq i,j \leq d$.

For a given starting time $t \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ we use the notation $\mathbb{P}_{(t,x)}$ for the conditional probability on the set $\{\omega \in \Omega_N : s_\omega(t) = x\}$ and $\mathbb{E}_{(t,x)}$ for the expectation with respect to $\mathbb{P}_{(t,x)}$. This can also be used when $t$ is a stopping time. For the remainder of the paper, we will implicitly restrict to $(t,x)$ such that $\mathbb{P}(s_\omega(t) = x) > 0$. 
For a finite set \( A \) and a function \( f \) defined on \( A \) we use the notation \( (f(a) : a \in A) \) for \[
\frac{1}{|A|} \sum_{a \in A} \delta_{f(a)}
\]
where \( \delta_{f(a)} \) is a point mass at \( f(a) \) and \( |A| \) is the cardinality of \( A \).

4.3. \( \text{CW}((0,0),(n,0)) \). For \( \omega \in \Omega_N \) with \( \omega(i) = (a(i),b(i)) \) and \( l \in [d] \) define
\[
\text{count}_a^I(m) = \sum_{j=1}^m 1_{a(j)=l}, \quad \text{and} \quad \text{count}_b^I(m) = \sum_{j=1}^m 1_{b(j)=l}.
\]
Define the path \( s_\omega \) by setting the \( l \)th coordinate of \( s_\omega(m) \) to be
\[
\text{diff}^l(m) = \text{count}_a^I(m) - \text{count}_b^I(m).
\]
Let
\[
\Omega_n = \{ (a,b) \in [d]^n \times [d]^n : \text{diff}^l(n) = 0 \ \forall \ l \in [d] \}.
\]
For any \( n \) we can, with a slight abuse of notation, think of \( \Omega_n \) as the subset of \( \Omega_N \) given by all infinite sequences that extend a finite sequence in \( \Omega_n \). Similarly we can have a set of \( \Omega_N \) that is defined by only finitely many coordinates and think of this as a finite object.

Definition 4.3. We define \( \text{CW}((i,v),(j,w)) \) to be the set of \( \omega \in \Omega_N \) such that \( s_\omega(i) = v, s_\omega(j) = w \) and \( s_\omega(\ell) \in \text{Weyl} \) for \( i \leq \ell \leq j \).

4.4. The function \( P_\omega \). For the sequence, \( a \), and an interval \( I \subset [n] \) with \( I = [i',j'] \) we let \( \text{count}_a^I(I) = \sum_{i \in I} 1_{a(i)=l} \) and define \( \text{count}_b^I(I) \) similarly for the sequence \( b \). For \( i > 0 \) define the function \( \text{pos}_a^I(i) := \inf \{ t : \text{count}_a^I(t) = i \} \) if \( i \leq \text{count}_a^I(n) \) and \( \text{pos}_a^I(i) := n \) otherwise. Similarly define \( \text{pos}_b^I(i) \).

Note that any \( \omega \in \Omega_n \) defines \( d \) increasing sequences \( \alpha^1, \ldots, \alpha^d \), in the following way. Let
\[
\alpha^l(m) = (\text{pos}_a^l(m), \text{pos}_b^l(m)).
\]
The condition that all the \( \text{diff}^l(n) = 0 \) ensures that the sum of the lengths of the sequences is \( n \). Using the function \( f(\alpha) \) defined at the beginning of section 2 we convert these \( d \) sequences into piecewise linear functions. In this way any element \( \omega \in \Omega_n \) generates a \( \mathbb{R}^d \) valued function \( P_\omega \).

4.5. Petrov conditions. We now describe a family of events that are moderate deviation conditions on \( \Omega_N \). We refer to these as the Petrov conditions and they play a critical role in the paper.

Definition 4.4 (Petrov Conditions). Fix \( m \in \mathbb{N} \). For \( \omega \in \Omega_N \), we say \( \omega \) has property Petrov\((m)\) if the following properties are satisfied for each \( l \in [d] \):

- For all intervals \( [i,j] \subseteq [0,m] \) with \( j - i > m^{-1} \)
  1. \( |\text{count}_a^I(j) - \text{count}_a^I(i) - \frac{1}{d} (j - i)| < (2d)^{-2} (j - i)^6 \),
  2. \( |\text{count}_b^I(j) - \text{count}_b^I(i) - \frac{1}{d} (j - i)| < (2d)^{-2} (j - i)^6 \).

- For all intervals \( [i,j] \subseteq [0,m] \) with \( j - i < m^4 \)
  1. \( |\text{count}_a^I(j) - \text{count}_a^I(i) - \frac{1}{d} (j - i)| < (2d)^{-2} m^{25} \),
  2. \( |\text{count}_b^I(j) - \text{count}_b^I(i) - \frac{1}{d} (j - i)| < (2d)^{-2} m^{25} \).

Note that any interval of length less than \( \kappa \) alone cannot cause Petrov\((m)\) to not be satisfied. The following lemma extends the Petrov conditions to the functions \( \text{pos}_a^I \) and \( \text{pos}_b^I \).

Lemma 4.5. Let \( [i,j] \subseteq [0,m] \) with \( \max\{\text{pos}_a^I(j), \text{pos}_b^I(j)\} \leq m \). If \( |j - i| > m^{-1} \), then
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There exists Standard moderate deviation estimates imply that there exists a proof.

\[ \{ k \} \begin{array}{l}
\end{array} \]

Then, by (4), we have \( |\text{count}^l_a(t) - \text{count}^l_a(s) - d^{-1}(t-s)| < (2d)^{-2}(t-s)^6. \)

By the Triangle Inequality

\[ d^{-1}(t-s) - (\text{count}^l_a(t) - \text{count}^l_a(s)) < (2d)^{-2}(t-s) \]

or

\[ (t-s)(1-(2d)^{-1}) < d(\text{count}^l_a(t) - \text{count}^l_a(s)) \]

giving, in terms of \( i \) and \( j \),

\[ \text{pos}^l_a(j) - \text{pos}^l_a(i) <= 2d(j-i). \]

Rewriting inequality (3) in terms of \( i \) and \( j \) gives

\[ |(j-i) - d^{-1}(\text{pos}^l_a(j) - \text{pos}^l_a(i))| < (2d)^{-2}(\text{pos}^l(j) - \text{pos}^l(i))^6. \]

Then, by (4), we have

\[ |(j-i) - d^{-1}(\text{pos}^l_a(j) - \text{pos}^l_a(i))| < (2d)^{-2}|2d(j-i)|^6 < (2d)^{-1}|j-i|^6. \]

The exact same argument works for \( \text{pos}^b_b \), and a similar argument works when \( j-i < m^4 \), finishing the proof.

Throughout this paper we assume the results of Lemma 4.5 when we cite the Petrov conditions.

Fix \( n \in \mathbb{N} \). For a pair of sequences \( \omega = (a, b) \), let \( \omega^* = (a^*, b^*) \) denote the pair given by the reverse of the first \( n \) elements of the two sequences. That is \( a^*(i) = a(n+1-i) \) and \( b^*(i) = b(n+1-i) \).

Similarly for a path \( s = \{ s(i) \}_{i=0}^n \) on \( \mathbb{Z}^d \) of length \( n \) let \( s^* \) denote the reverse of the path \( \{ s(n-i) \}_{i=0}^n \).

Although it does not matter we can set \( \omega^*(m) = \omega(n) \) for all \( m > n \).

**Definition 4.6.** We say \( \omega \) has property Petrov\(^*\)(m) if \( \omega^* \) has property Petrov(m).

**Lemma 4.7.** There exists \( \gamma > 0 \) and \( C \) such that for all \( t < m \) and \( x \in \mathbb{Z}^d \)

\[ \mathbb{P}_{(t,x)}(\text{Petrov}(m)^C) \leq Ce^{-m^\gamma}. \]

\[ \mathbb{P}_{(t,x)}(\text{Petrov}(m)^C \mid \Omega_n) \leq Ce^{-m^\gamma}. \]

**Proof.** Standard moderate deviation estimates imply that there exists \( \gamma' \) such that the set of \( (a, b) \in [d]^m \times [d]^m \) in Petrov\(^C\)(m) is bounded by \( Ce^{-m^\gamma} \). See Lemma 2.4 of [22] for more details.

We get a similar bound when we condition on \( (a, b) \in \Omega_n \) as follows. As \( \mathbb{P}_{(0,0)}(\Omega_n) \geq n^{-k} \) for some \( k \) if \( m > n^1 \) then we only need to lower \( \gamma \). If \( m \leq n^1 \) then the local central limit theorem implies that conditioning on any sequence \( \{ (a(i), b(i)) \}_{i=1}^n \) the probability of being in \( \Omega_n \) differs by at most a factor of 2. Thus

\[ \mathbb{P}_{(t,x)}(\text{Petrov}(m)^C \mid \Omega_n) \leq C'e^{-m^\gamma}. \]

□
Lemma 4.8. Let \( \omega \in \Omega_N \) satisfy Petrov(m). The following hold:

\[
\begin{align*}
(6) \quad & |\text{count}_a^l(m) - \text{count}_b^l(m)| < d^{-2}m^6, \\
(7) \quad & |s_\omega(m)| < d^{-1}m^6, \\
\end{align*}
\]

If \([j, j'] \subset [0, m]\) and \(|j - j'| > m^3\) then

\[
(8) \quad |s_\omega(j') - s_\omega(j)| < d^{-1}|j' - j|^6. 
\]

Proof. If \( \omega \) satisfies Petrov(m) then \( |\text{count}_a^l(m) - d^{-1}m| < (2d)^{-2}m^6 \). Similarly \( |\text{count}_b^l(m) - d^{-1}m| < (2d)^{-2}m^6 \) and thus by the triangle inequality

\[
|\text{count}_a^l(m) - \text{count}_b^l(m)| \leq |\text{count}_a^l(m) - d^{-1}m| + |\text{count}_b^l(m) - d^{-1}m| < d^{-2}m^6, 
\]

proving (6). This provides a uniform bound for each of the \( d \) coordinates of \( s_\omega(m) \) and thus also proves (7).

For Equation (8) we have

\[
|s_\omega(j') - s_\omega(j)| \leq d \max_{i \in [d]} |\text{count}_a^i(j') - \text{count}_b^i(j') - (\text{count}_a^i(j) - \text{count}_b^i(j))| \\
\leq d \max_{i \in [d]} |\text{count}_a^i(j') - \text{count}_a^i(j) - (\text{count}_b^i(j') - \text{count}_b^i(j))| \\
\leq d \left( (d^{-1}(j' - j)) + (2d)^{-2}|j' - j|^6 - d^{-1}(j' - j) + (2d)^{-2}|j' - j|^6 \right) \\
\leq d^{-1}|j' - j|^6. 
\]

Lemma 4.9. Let \( \omega \in \Omega_N \) satisfy Petrov(m). For \( l \in [d] \), let \( i \leq m \) be such that \( \text{count}_a^l(m) = \text{count}_b^l(i) \). Then \( m - i < m^6 \) A similar statement holds if \( \text{count}_a^l(i) = \text{count}_b^l(m) \).

Proof. By Lemma 4.8 \( |\text{count}_a^l(m) - \text{count}_b^l(m)| < d^{-2}m^6 \). Under our assumptions we may replace \( \text{count}_a^l(m) \) with \( \text{count}_b^l(i) \), giving

\[
|\text{count}_b^l(i) - \text{count}_b^l(m)| < d^{-2}m^6. 
\]

By Petrov(m) we have

\[
|\text{count}_b^l(m) - \text{count}_b^l(i) - d^{-1}(m - i)| < (2d)^{-2}m^6. 
\]

Thus combined we have

\[
|d^{-1}(m - i)| < (2d)^{-2}m^6 + d^{-2}m^6 < d^{-1}m^6. 
\]

Multiplying by \( d \) finishes the proof.

Lemma 4.10. Let \( \omega \in \Omega_N \) satisfy Petrov(m). Then for all \( l \)

\[
\text{pos}_a^l(\text{count}_a^l(m)) \in (m - m^{19}, m] \\
\]

and

\[
\text{pos}_b^l(\text{count}_b^l(m)) \in (m - m^{19}, m]. 
\]

Proof. Consider the intervals \( I_1 = (m - m^3 - m^{19}, m - m^{19}] \) and \( I_2 = (m - m^3 - m^{19}, m] \). Both intervals have size at least \( m^3 \) and therefore by the Petrov conditions

\[
|\text{count}_a^l(I_1) - d^{-1}m^3| < (2d)^{-2}m^{18} 
\]

while

\[
|\text{count}_a^l(I_2) - d^{-1}(m^3 + m^{19})| < (2d)^{-2}|m^3 + m^{19}|^6 < d^{-2}m^{18}. 
\]

These together imply that the interval \( I_3 = (m - m^{19}, m] \) satisfies

\[
(9) \quad \text{count}_a^l(I_3) = \text{count}_a^l(I_2) - \text{count}_a^l(I_1) \geq d^{-1}m^{19} - 2(d)^{-2}m^{18} > 0. 
\]
The value \( \text{pos}_l^k(\text{count}_a^l(m)) \) is the position of the last occurrence of \( l \) at or before position \( m \).

Inequality (9) shows that there is at least one \( l \) somewhere in \( I_3 \).

Inequality (8) shows that there is at least one \( l \) in \( I_3 \) and thus this last occurrence must occur somewhere in \( I_3 \). The same argument holds for \( \text{pos}_l^k(\text{count}_a^l(m)) \).

5. From permutations to paths close to the Weyl Chamber

5.1. Definitions. Remember that we have defined

\[
\text{Weyl} = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_1 \geq \cdots \geq x_d\}.
\]

Heuristically we think of the path \( s_{\omega_\sigma} \) associated with a permutation in \( \sigma \in \mathcal{A}_n(\rho_d) \) as being a path in Weyl. But the reality is that it may not necessarily stay within Weyl. (See Figure 6.) And not every \( \omega \) such that \( s_\omega(t) \in \text{Weyl} \) for all \( t \) is the image \( s_{\omega_\sigma} \) for some \( \sigma \in \mathcal{A}_n(\rho_d) \).

Because of this we need to consider other family of paths.

We define \( \text{Weyl}_k \) to be Weyl shifted so that its apex is at \((dk, (d - 1)k, \ldots, k, 0)\). That is

\[
\text{Weyl}_k = \{(dk, (d - 1)k, \ldots, k, 0) + \text{Weyl} : (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_i \geq x_{i+1} + k, \forall i = \{1, \ldots, d - 1\}\}.
\]

Recall the definition of \( CW((i, v), (j, v')) \) from Definition 4.3. Based on this we define

- \( CW^-(((i, v), (j, v')) \) to be all \( \omega \in \Omega_N \) such that
  (1) \( s_\omega(i) = v, s_\omega(j) = v' \),
  (2) \( s_\omega(m) \in \text{Weyl}_{m, 4} \) for all \( m \in [i, j] \) and
  (3) for every \( m \in [i, j] \), every \( l \in [d] \) and every interval \([i', j'] \subset [i + 1, m] \) the conditions from the definition of Petrov(\( m \)) (Definition 4.4) are satisfied.

Note that the definition can be checked by knowing \( s_\omega(i) \) and \( \omega(k) \) for \( k \in [i + 1, j] \). This definition is very useful because in Lemma 5.3 we show that if \( s_\omega \in CW^-((t, x), (n - t^*, x^*)) \) (for certain choices of \( t, t^*, x \) and \( x^* \)) then it can be extended so that it equals \( s_{\omega_\sigma} \) for some \( \sigma \in \mathcal{A}_n(\rho_d) \). We also define

- \( CW^+(((i, v), (j, v')) \) to be all \( \omega \in \Omega_N \) such that \( s_\omega(i) = v, s_\omega(j) = v' \), and \( s_\omega(m) \in \text{Weyl}_{m, 4} \) for all \( m \in [i, j] \).

- \( CW^{++}((i, v), (j, v')) \) to be all \( \omega \in \Omega_N \) such that \( s_\omega(i) = v, s_\omega(j) = v' \), and for all \( m \in [i, j] \) either \( s_\omega(m) \in \text{Weyl}_{m, 4} \) or there exists \( l \in [d] \) and interval \([i', j'] \subset [i, m] \) such that the conditions from the definition of Petrov(\( m \)) (Definition 4.4) are not satisfied.

In Lemma 5.1 we will show that if \( \sigma \in \mathcal{A}_n(\rho_d) \) then the associated path \( s_{\omega_\sigma} \in CW^-((0, 0), (n, 0)) \).

In all the proceeding notation we can replace \( v \) and \( v' \) by * which represents taking a union. For example

\[
CW^-((j, *), (n, *)) = \bigcup_{v, w} CW^-((j, v), (n, v'))
\]

and

\[
CW^+(j, *), (n, v')) = \bigcup_{v} CW^+(j, v), (n, v'))
\]

Fix \( n \). We also want to consider symmetric version of these sets. By the definition of \( CW \) it is already symmetric. By this we mean that if \( \omega \in CW((0, 0), (n, 0)) \) then so is \( \omega^* \). The corresponding statements are not true for \( \omega \in CW^-((0, 0), (n, 0)) \) or \( \omega \in CW^+(0, 0), (n, 0) \).

We define

\[
SCW^+++((j, v), (k, w))
\]

to be all paths \( \omega \) such that \( s_\omega(j) = v, s_\omega(k) = w \), and for all \( i \in [j, n/2] \) either Petrov(\( i \)) fails or

\[
d(s_\omega(i), \text{Weyl}) \leq Ci^4
\]
Figure 6. A permutation $\sigma \in \mathcal{A}_{20}(\rho_3)$ whose associated path $s_{\omega_\sigma}$ is not in Weyl.

and for all $i \in [n/2, k]$ either Petrov*$(i)$ fails or

$$d(s_\omega(i), \text{Weyl}) \leq C(n - i)^4.$$  

The connection between $\mathcal{A}_n(\rho_d)$ and $\text{SCW}^+((0, 0), (n, 0))$ is summarized in the following lemma.

**Lemma 5.1.** For any $\sigma \in \mathcal{A}_n(\rho_d)$, $\omega_\sigma \in \text{SCW}^+((0, 0), (n, 0))$.

**Proof.** We will show that if $d(s_\omega(i), \text{Weyl}) > i^4$ and Petrov$(i)$ occurs we have a contradiction. For $\omega_\sigma$ to be distance greater than $i^4$ from Weyl, there must be some some $1 \leq l < d$ such that,

$$\text{count}_a^l(i) - \text{count}_b^l(i) - \text{count}_a^{l+1}(i) + \text{count}_b^{l+1}(i) < -i^4.$$ 

Let

$$j = \text{pos}_a^{l+1}(\text{count}_a^l(i)),$$

the closest position of an $l + 1$ in $a$ that occurs at or before $i$. Let

$$k = \text{pos}_a^l(\text{count}_a^l(j)),$$

the closest position of an $l$ that occurs strictly before position $j$ (which is the position of an $l + 1$) in $a$. With these definitions we have

$$\sigma(k) = \text{pos}_a^l(\text{count}_a^l(k)) > \text{pos}_b^{l+1}(\text{count}_a^{l+1}(j)) = \sigma(j).$$

It is possible that $k$ is the position of the closest $l$ that occurs at or before $i$ in $A$. However it is also possible that some $l$ occurs between $j$ and $i$. In either case, $\text{pos}_a^l(\text{count}_a^l(i))$ is the position of the closest $l$ to $i$ and $\text{pos}_b^l(\text{count}_a^l(i)) \geq \sigma(k) > \sigma(j)$.

We have three cases to consider. Either

1. $\sigma(j) < \sigma(k) \leq \text{pos}_a^l(\text{count}_a^l(i)) \leq i$,
2. $\sigma(j) < i < \text{pos}_a^l(\text{count}_a^l(i))$, or
3. $i \leq \sigma(j) < \sigma(k) \leq \text{pos}_b^l(\text{count}_a^l(i))$.

The arguments for the first and last case are exactly the same with a slight change in the definition of $j$ and $k$. The argument for the middle case requires only a slightly modified approach. We will proceed by considering the first case (see Figure 7), leaving the other two cases to the reader.

By Lemma 4.10 we have

$$i - k < 2i^{19}.$$
Lemma 4.8 implies \(\text{count}^l_a(i)\) and \(\text{count}^l_b(i)\) differ by at most \(d^{-2}i^6\). We are assuming that \(\text{pos}^l_b(\text{count}^l_b(i)) \leq i\). Thus by Lemma 4.5 we may use Petrov(i) to claim

\[
|\text{pos}^l_b(\text{count}^l_a(i)) - \text{pos}^l_b(\text{count}^l_b(i)) - d(\text{count}^l_a(i) - \text{count}^l_b(i))| < |\text{count}^l_a(i) - \text{count}^l_b(i)|^6 \leq \frac{1}{2^i}^{36},
\]

and therefore

\[
(13) \quad \text{pos}^l_b(\text{count}^l_a(i)) - \text{pos}^l_b(\text{count}^l_b(i)) = d(\text{count}^l_a(i) - \text{count}^l_b(i)) + \epsilon^l_i
\]

where \(|\epsilon^l_i| \leq \frac{1}{2^i}i^{36}\). Similarly,

\[
(14) \quad \text{pos}^{l+1}_b(\text{count}^{l+1}_a(i)) - \text{pos}^{l+1}_b(\text{count}^{l+1}_b(i)) = d(\text{count}^{l+1}_a(i) - \text{count}^{l+1}_b(i)) + \epsilon^{l+1}_i,
\]

with \(|\epsilon^{l+1}_i| \leq \frac{1}{2}i^{36}\).

Combining (13) and (14) with (11) we have

\[
\left[\text{pos}^l_b(\text{count}^l_a(i)) - \text{pos}^l_b(\text{count}^l_b(i))\right] - \left[\text{pos}^{l+1}_b(\text{count}^{l+1}_a(i)) - \text{pos}^{l+1}_b(\text{count}^{l+1}_b(i))\right] = d \left(\text{count}^l_a(i) - \text{count}^l_b(i) - \text{count}^{l+1}_a(i) + \text{count}^{l+1}_b(i)\right) + \epsilon^l_i + \epsilon^{l+1}_i
\]
\[
< -di^4 + i^{36}
\]
\[
< -i^4.
\]

(15)

On the other hand by Lemma 4.10

\[
(16) \quad 0 < i - \text{pos}^l_b(\text{count}^l_b(i)) < i^{19} \quad \text{and} \quad 0 < i - \text{pos}^{l+1}_b(\text{count}^{l+1}_b(i)) < i^{19}.
\]
This map Mat is clearly 1-1. For notational convenience we often refer to \( \Omega \)
set of matrices which are the image of \( \Omega \) argument will work using Petrov
\( \sigma \) for
(17)

\[
\begin{align*}
\text{Proof.} & \text{ This follows from the definition of minimal and the construction of sets } A^i \text{ in Section 2}.
\end{align*}
\]

By construction \( \text{pos}_b^t(\text{count}_a^i(i)) \geq \text{pos}_b^t(\text{count}_a^t(k)) > \text{pos}_b^{t+1}(\text{count}_a^{t+1}(i)) \), so along with (16) we have

\[
\begin{align*}
\left[ \text{pos}_b^t(\text{count}_a^i(i)) - \text{pos}_b^t(\text{count}_a^t(i)) \right] & - \left[ \text{pos}_b^{t+1}(\text{count}_a^{t+1}(i)) - \text{pos}_b^{t+1}(\text{count}_b^{t+1}(i)) \right] \\
& \geq \left[ \text{pos}_b^t(\text{count}_a^t(k)) - \text{pos}_b^{t+1}(\text{count}_a^{t+1}(i)) \right] \\
& + \left[ \text{pos}_b^{t+1}(\text{count}_b^{t+1}(i)) - \text{pos}_b^t(\text{count}_b^t(i)) \right] \\
& > 0 + \left[ -21^{19} \right].
\end{align*}
\]

Both inequalities (15) and (17) cannot simultaneously be true. Therefore we can conclude that for \( \sigma \in Av_n(\rho_d) \) both \( d(s_\omega(i), \text{Weyl}) > i^4 \) and Petrov(i) cannot both be true. A symmetric argument will work using Petrov^*(i) for \( i > \lfloor n/2 \rfloor \).

For any \( \omega \in \Omega_n \) we have the matrix given by

\[
\text{Mat}(\omega)(i,j) = \begin{cases} 
1 & \text{if } (i,j) = (\text{pos}_a^l(m), \text{pos}_b^l(m)) \text{ for some } m \text{ and } l \\
0 & \text{else}.
\end{cases}
\]

This map Mat is clearly 1-1. For notational convenience we often refer to \( \Omega_n \) when we consider the set of matrices which are the image of \( \Omega_n \) under the map Mat.

We say the \( ij \)th entry of Mat(\( \omega \)) is proper if Mat(\( \omega \)) \( ij \) = \( l \) and

- for every \( 0 < l' < l \), there exists \( i' < i \) and \( j' > j \) such that Mat(\( \omega \)) \( i'j' \) = \( l' \), and
- for every \( l' \geq l > 0 \) and \( i' < i \) and \( j' > j \), Mat(\( \omega \)) \( i'j' \) \( \neq \) \( l' \).

We say Mat(\( \omega \)) is proper if every nonzero \( ij \)th entry is proper. See Figure 8 for an example of a proper labeling.

For \( \sigma \in Av_n(\rho_d) \), we let \( \omega_\sigma \in \Omega_n \) be the pair of sequences given by projection of the non-zero entries of the matrix Mat(\( \sigma \)) onto the x and y axis. Conversely for \( \omega \in \Omega_n \) let \( \sigma_\omega \) denote the permutation in \( S_n \) where \( i, \sigma_\omega(i) \) is constructed by finding unique values \( t \) and \( l \) such that \( \text{pos}_a^t(t) = i \) and \( \text{pos}_b^t(t) = \sigma_\omega(i) \). We say \( \omega \in \Omega_n \) is minimal if and only if there exists a \( \sigma \in Av_n(\rho_d) \) such that \( \omega = \omega_\sigma \).

**Lemma 5.2.** \( \omega \in \Omega_n \) is minimal if and only if Mat(\( \omega \)) is proper.

**Proof.** This follows from the definition of minimal and the construction of sets \( A^i \) in Section 2.

For \( \omega \in \Omega_n \) and times \( t, t^* \leq \lfloor n/2 \rfloor \) we define the decomposition of \( \omega \) by

\[
\omega = \omega^1 \oplus \omega^2 \oplus \omega^3.
\]
where $\omega^1$ denotes the beginning of $\omega$ until time $t$, $\omega^2$ the portion of $\omega$ from $t$ to $n - t^*$, and $\omega^3$ the portion of $\omega$ from $n - t^*$ to $n$.

**Lemma 5.3.** Fix $n$, $L > 0$ and let $x, y \in \text{Weyl}_L$. Let $t$ and $t^*$ be bounded by both $4.1^L$ and $n/2$. Let $\omega \in \text{SCW}^-(t, x), (n - t^*, y))$. Let $\sigma \in \mathcal{A}_n(\rho_d)$ be a permutation such that $\omega_\sigma$ satisfies Petrov($t$) and Petrov$^*(t^*)$, $s_{\omega^1}(t) = x$ and $s_{\omega^3}(n - t^*) = y$. Finally, let $\omega' \in \Omega_n$ be given by

$$\omega' = \omega^1 \oplus \omega^2 \oplus \omega^3.$$ 

That is, the pair of sequences whose initial component until position $t$ and final component from position $n - t^*$ until $n$ are both obtained from $\omega_\sigma$ and whose middle component is obtained from $\omega$. Then there exists $\sigma \in \mathcal{A}_n(\rho_d)$ such that $\omega' = \omega_\sigma$.

**Proof.** We will first show that Mat($\omega'$) is proper. By Lemma 5.2 this is sufficient to prove the lemma. We work by contradiction and suppose $M = \text{Mat}(\omega')$ is not proper. Then there exists $1 < l \leq d$ and $(i, j)$ such that $M_{ij} = l$ and all points labeled $l - 1$ that occur to the left of $i$ occur below $j$.

If $(i, j) \in [0, t]^2$ then $M_{ij} = l$ if and only if Mat($\omega_\sigma$)$_{ij} = l$. Thus properness of the entries of Mat($\omega_\sigma$) ensures properness of $M_{ij}$ in this range.

Now consider points $(i, j)$ not in $[0, t]^2$. First we will assume that $t \leq j \leq n/2$ and $i \leq j$. If $M_{ij} = l$ then

$$\text{count}^a_l(i) = \text{count}^l_j(j).$$

If $\sigma_{\omega'} \in \text{SCW}^-((t, x), (n - t^*, y))$ then

$$j^4 < d(\sigma_{\omega'}, \partial \text{Weyl}) \leq \min_{l'} \{\text{diff}_l(j) - \text{diff}^{l+1}_l(j)\}.$$  

Suppose that no point above and to the left of $(i, j)$ is labeled $l - 1$. This implies that

$$\text{count}^a_l(i) = \text{count}^b_l(j)$$

since $(i, j)$ is labeled $l$. Similarly $\text{count}^a_l(i) = \text{count}^b_l(j)$, as the point $(i, j)$ is labeled $l$. By Lemma 4.9, if Petrov($j$) occurs, then $|j - i| < j^6$ and for all $1 \leq l' \leq d$,

$$d^{-1}(j - i) - (2d)^{-2}j^{36} \leq \text{count}^a_l(j) - \text{count}^a_l(i) \leq d^{-1}(j - i) + (2d)^{-2}j^{36}.$$ 

As $d(s_{\omega'}, \text{Weyl}) > j^4$

$$j^4 < \text{diff}_l^{-1}(j) - \text{diff}_l(j) = \text{count}^a_l(j) - \text{count}^b_l(j) - (\text{count}^b_l(j) - \text{count}^a_l(i)) \leq d^{-1}(j - i) + (2d)^{-2}j^{36} - (d^{-1}(j - i) - (2d)^{-2}j^{36}) \leq j^{36}.$$ 

This is a contradiction to (19) as it implies that if Petrov($j$) occurs and $d(s_{\omega'}, \text{Weyl}) > j^4$, then $\text{count}^a_l(i) \neq \text{count}^b_l(j)$.

A similar argument works if we assume $t \leq j \leq i \leq n/2$. We may also apply the same argument for $(i, j)$ such that $t^* \leq n - j \leq n/2$ or $t^* \leq n - i \leq n/2$. We also must consider the cases when $i \leq n/2 \leq j$ or $j \leq n/2 \leq i$. Those two cases are very similar to the argument above and we leave the details to the reader.

The only points not covered are those in $[0, t] \times [n - t^*, n]$ or $[n - t^*, n] \times [0, t]$. The conditions Petrov($t$) and Petrov$^*(t^*)$ guarantee that all points in these regions are labeled $0$ and thus cannot make $M(\omega')$ non-proper. Then we may conclude that Mat($\omega'$) is proper and therefore $\omega'$ is minimal. Thus Lemma 5.2 implies there exists $\sigma \in \mathcal{A}_n(\rho_d)$ such that $\omega' = \omega_\sigma$. 


Figure 9. A comparison of $P_{\sigma}(t)$ (left) and $s_{\omega_{\sigma}}(t)$ (right) for the permutation $\sigma \in \mathcal{Av}_{20}(\rho_3)$ from the permutation from Figure 1.

Figure 10. A comparison of $P_{\sigma}(t)$ (left) and $\hat{s}_{\omega_{\sigma}}(t)$ (right) for a permutation $\sigma \in \mathcal{Av}_{100000}(\rho_3)$. Lemma 5.4 gives conditions on $\omega \in \Omega_n$ which insures that if $n$ is large then the two sets of functions are close.

For any function $s : [n] \to \mathbb{Z}^d$ let $\hat{s}$ be the scaled and linearly interpolated function on $[0, 1]$ given by $\hat{s}(t) = \frac{s(\lfloor nt \rfloor)}{\sqrt{2n/d}}$. The following lemma gives conditions such that $\hat{s}_{\omega_{\sigma}}(t)$ and $P_{\sigma}(t)$ are close in the sup norm. (See Figure 10 for a comparison). Let $D$ be the $L^1$ norm on $\mathbb{R}^d$.

**Lemma 5.4.** Fix $T, T'$ and let $n$ be sufficiently large. Let $\sigma \in \mathcal{Av}_n(\rho_d)$ with $\omega_{\sigma} \in SCW^-(T, \ast, (n-T', \ast))$. Then

$$\sup_{t \in [0, 1]} D(P_{\sigma}(t), \hat{s}_{\omega_{\sigma}}(t)) \leq n^{-1}.$$  

**Proof.** For $t \leq T/n$ the maximum over $l \in [d]$ of the component $f(\alpha^l(\sigma))(t)$ is at most $4T/\sqrt{n}$ and therefore $|P_{\sigma}(t)| < 4dT/\sqrt{n}$. Thus for $T$ small ($< n^{-39}$) and $n$ large we have $|P_{\sigma}(t)| < 4dn^{-11} < \frac{1}{2}n^{-1}$. By Lemma 4.8, Petrov($T$) guarantees that $|s_{\sigma}(nt)| < |(nt)|^6$, thus for $t \leq T/n < n^{-61}$, $|s_{\sigma}(nt)| < n^{25}$. Scaling by $\sqrt{2n/d}$ to obtain $\hat{s}_{\omega_{\sigma}}(t)$ we see that $\hat{s}_{\omega_{\sigma}}(t) < \frac{1}{2}n^{-1}$. Combining these bounds shows that for small $t$,

$$D(P_{\sigma}(t), \hat{s}_{\omega_{\sigma}}(t)) \leq n^{-1}.$$
For \( t \in [T/n, 1/2] \), the \( l \)th component of \( P_\sigma(t) \) is obtained by linear interpolation between the points
\[
\left( \frac{1}{(2dn)^{1/2}} \left( \text{pos}_a^l(\text{count}_a^l(i)) - \text{pos}_a^l(\text{count}_b^l(i)) \right) \right).
\]
The \( l \)th component of \( \hat{s}_{\omega}(t) \) is given by the values \( \frac{1}{(2n/d)^{1/2}}(\text{count}_a^l(i) - \text{count}_b^l(i)) \).

By the Petrov conditions
\[
\frac{1}{(2dn)^{1/2}}(\text{pos}_a^l(\text{count}_a^l(i)) - \text{pos}_a^l(\text{count}_b^l(i)))
= \frac{1}{(2dn)^{1/2}}(\text{pos}_a^l(\text{count}_a^l(i)) - \text{pos}_a^l(\text{count}_b^l(i)) + \text{pos}_b^l(\text{count}_b^l(i)) - \text{pos}_a^l(\text{count}_a^l(i)))
= \frac{1}{(2dn)^{1/2}}(d(\text{count}_a^l(i) - \text{count}_b^l(i)) + \epsilon^l(i))
\]

where by Lemmas 4.8 and 4.10
\[
|\epsilon^l(i)| \leq \left( |\text{count}_a^l(i) - \text{count}_b^l(i)|^6 + d|2i^3| \right) < n^4.
\]
Thus for large enough \( n \) and for any \( t \in [0, 1/2] \)
\[D(P_\sigma(t), \hat{s}_{\omega}(t)) \leq n^{-1}.\]
For \( t \in [1/2, 1] \) a similar argument shows that for both \( n/2 < nt < n - T' \) and \( n - T' \leq nt \leq n \),
\[D(P_\sigma(t), \hat{s}_{\omega}(t)) \leq n^{-1} \] for large enough \( n \), and therefore the bound holds for all \( t \in [0, 1] \).

6. Random walks close to the Weyl Chamber

In this section we state some results about this walk conditioned to remain close to a cone that we will need. The proofs of these results are intricate and are delayed until Sections 8 and 9.

The following proposition allows us to import lemmas from [12] even though our random walk \( s_\omega(t) \) does not satisfy all of the hypothesis in [12].

**Proposition 6.1.** Suppose that \( \omega \) is distributed like \( \mathbb{P} \). There is a linear transformation \( H : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1} \), invertible on the span of \( \{e_i - e_j \}_{1 \leq i, j \leq d} \), such that the random walk \( H(s_\omega) \) and the interior of the cone generated by \( H(Weyl) \) satisfy the hypotheses of [12].

See Proposition 8.3 for a detailed statement with a specific choice for \( H \). For our purposes, the existence of \( H \) is typically more important than any particular choice for it.

We define the sequence
\[T_l = \min\{\inf\{t : s_\omega(t) \in \text{Weyl}_2\}, [4.1^l]\}.\]

For \( z \in \mathbb{Z}^d \) we define
\[U(z) = \prod_{i<j}(z_i - z_j).\]

The function \( U \) is harmonic for \( s_\omega \), see [32].

**Lemma 6.2.** For any \( l > l_0 > L, x \in \partial\text{Weyl}_{2l_0}, T < 4.1^{l_0}, \text{ and } |x| \leq 2T,\)
\[(21) \sum_{x'} U(x')\mathbb{P}(T,x) (\omega \in \text{CW}^+((T,x),(T_l,x'))) \leq KU(x)\]
where \( K = 1 + \prod_{j=0}^{\infty}(1 + .01 \cdot (.95)^{l_0+j})\)

**Proof.** This is proved in Lemma 9.8 in Section 9. 

For any \( l \geq l' > L, T < 4.1^L, |x| \leq T \text{ and } x \in \partial\text{Weyl}_{2L} \) let \( \hat{E}_1(l', T, x) \) be the event that
Lemma 6.3. For any \( l > l_0 > L, T \geq \lfloor 4.1l^0 \rfloor \) and \( x \) with \( |x| \leq 2T \) and \( x \notin \text{Weyl}_{2l_0} \)

\[
\sum_{x'} U(x') \mathbb{P}_{(T,x)}(\omega \in CW^+((T,x),(T_l,x'))) \leq K(2T)^{d(d-1)/2}
\]

where \( K = \prod_{j=0}^{\infty} (1 + .02 \cdot (0.95)^{l_0+j}) \).

**Proof.** This is proved in Lemma 9.9 in Section 9.

Lemma 6.4. There exists a function \( H(L) = o(1) \) such that for any \( L, l > L, T < 4.1L, |x| \leq 2T \) and \( x \in \partial \text{Weyl}_{2l} \)

\[
\sum_{y} U(y) \mathbb{P}_{(T,x)}(\omega \in CW^+((T,x),(T_l,y)) \setminus CW^-(T,x),(T_l,y))) \leq H(L)U(x).
\]

**Proof.** This is proved in Lemma 9.11 in Section 9.

Choose \( L_f \) to be the smallest integer such that

\[
4^{L_f} > n^{1-.08/d(d-1)}.
\]

Also let \( \delta = .02/d(d-1) < .01 \). Then \( 2^{L_f} > n^{5-2\delta} > n^{-4\delta} \).

For any \( j, k < n/2 \) and \( x, y \in \mathbb{Z}^d \) define \( SCW^-((j,x),(n-k,y)) \) to be all paths \( \omega \) such that

\[ s_\omega(i) \in CW^-((j,x),(\lfloor n/2 \rfloor,*)) \]

\[ s_\omega(n-i) \in CW^-((k,y),(\lfloor n/2 \rfloor,*)) \]

\( SCW^+((j,x),(n-k,y)) \) and \( SCW^{++}((j,x),(n-k,y)) \) are defined in an analogous way.

Lemma 6.5. For any \( \epsilon > 0 \) there exists \( l \) such that if \( x, y \in \text{Weyl}_{2l} \) and \( T,T^* \leq (4.1)^l \) then for any \( n \) sufficiently large

\[
\mathbb{P}_{(T,x)} \left( SCW^-((T,x),(n-T^*,y)) \mid SCW^{++}((T,x),(n-T^*,y)) \right) > 1 - \epsilon
\]

which implies

\[
\mathbb{P}_{(T,x)} \left( SCW^-((T,x),(n-T^*,y)) \mid CW((T,x),(n-T^*,y)) \right) > 1 - \epsilon.
\]

**Proof.** This is proved in Lemma 9.14 (and Corollary 9.15) in Section 9.

Lemma 6.6. There exist \( C'' \) such that for all \( n, R \in [n/2,n], T \leq n/4 \) and for all \( x, y \in \text{Weyl}_{n,5-2\delta} \) such that \( |x|, |y| \leq n^{5-\delta} \),

\[
\mathbb{P}_{(T,x)}(\omega \in CW((T,x),(R,y))) \geq C'' U(x)U(y)n^{-d(d-1)/2} \cdot n^{-(d-1)/2}.
\]

There also exists \( C''' \) such that for all \( n, R \in [n/2,n] \) and for all \( x, y \in \text{Weyl}_{n,5-2\delta} \) such that \( |x|, |y| \leq n^{5-\delta} \),

\[
\mathbb{P}_{(T,x)}(\omega \in CW((T,x),(R,y))) \leq C''' U(x)U(y)n^{-d(d-1)/2} \cdot n^{-(d-1)/2}.
\]

**Proof.** This is proved in Lemma 9.13 in Section 9.
Corollary 6.7. Fix $\epsilon > 0$ and $K$. Let $M$ be a measure on quadruples $(s, x)$ and $(t, y)$ such that with probability one have $0 \leq s, t \leq K$, $x, y \in$ Weyl and $|x|, |y| \leq K$. For $n > 2K$ define $M_n$ to be the measure generated by picking $(s, x)$ and $(t, y)$ according to $M$ and then sampling $\omega$ from $CW((s, x), (n - t, y))$. There is a $K'$ such that for any $M$

\[
d_{K'}(M_n, (s_\omega : \omega \in CW((0, 0), (n, 0)))) < \epsilon.
\]

Proof. This is proved in Corollary [8.11] in Section 8.

Theorem 6.8. Suppose that $x \in \mathcal{C}$. Then for all bounded, continuous functions $f : D([0, 1], \mathbb{R}^d) \to \mathbb{R}$ we have

\[
\mathbb{E} \left( f \left( \frac{x + s_\omega(n)}{\sqrt{2n/d}} \right) \bigg| \tau_x > n, s_\omega(n) = 0 \right) \to \mathbb{E}[f(\Lambda(Z))].
\]

Proof. This is proved in Theorem [8.10] in Section 8.

7. The scaling limit.

In this section we connect the set of $Av_n(\rho_d)$ with paths on $\mathbb{Z}^d$. We fix $n$ and $d$ and (as we will be adding subscripts and superscripts) we write $Av_n(\rho_d)$ as PAP. Then we calculate the scaling limit. We will frequently use the set we defined in [2]

\[
\Omega_n = \left\{ (a, b) \in [d]^n \times [d]^n : \# \{ i : a_i = l \} = \# \{ i' : b_{i'} = l \} \ \forall \ l \in [d] \right\}.
\]

Consider $\omega = \{(a, b)\} \in \Omega_n$. In Section 4 we showed how a pair of sequences $\omega$ maps to a path $s_\omega$ in $\mathbb{Z}^d$.

We can compute the embedding of PAP into $\Omega_n$ and the map from $\Omega_n$ to paths on $\mathbb{Z}^d$ to get a map from a PAP to paths on $\mathbb{Z}^d$. For $\sigma \in$ PAP we let $s_{\omega_\sigma}$ denote this path. If we linearly interpolate and scale $s_{\omega_\sigma}$ properly we then get an ordered collection of $d$ functions on $[0, 1]$ that start and end at 0. We call this scaled function $\hat{s}_{\omega_\sigma}$. We also have another map $P_\sigma$ from PAP to a collection of $d$ functions on $[0, 1]$ that start and end at 0. We will show that the scaled version of $s_{\omega_\sigma}$ is usually very close to $P_\sigma$. In fact we will show that $P_\sigma$ and $\hat{s}_{\omega_\sigma}$ are sufficiently close so that when $\sigma$ is chosen uniformly from $Av_n(\rho_d)$ they have the same scaling limit.

Fix some large integer $L$. Define

\[
R_L = \inf \{ t : \text{ Petrov}(t) \cap \omega(t) \in \text{ Weyl}_{2L} \}
\]

and

\[
R_L^* = \inf \{ t : \text{ Petrov}^*(t) \cap \omega^*(t) \in \text{ Weyl}_{2L} \}.
\]

Note that $R_L$ is a stopping time and $R_L^*$ is a stopping time for the reverse walk.

For $n >> 4.1^L$ we divide the set of PAP into three disjoint subsets. Define

\[
PAP_1 = \left\{ \sigma : \max(R_L(\omega_\sigma), R_L^*(\omega_\sigma)) < [4.1^L], \omega_\sigma \in SCW^-(\{R_L, \ast\}(n - R_L, \ast)) \right\},
\]

\[
PAP_2 = \left\{ \sigma : \max(R_L(\omega_\sigma), R_L^*(\omega_\sigma)) < [4.1^L], \omega_\sigma \notin SCW^-(\{R_L, \ast\}(n - R_L^*, \ast)) \right\}
\]

and

\[
PAP_3 = \left\{ \sigma : \max(R_L(\omega_\sigma), R_L^*(\omega_\sigma)) = [4.1^L] \right\}.
\]

Our strategy is (roughly) as follows. We will show that if $L$ is large then the scaling limit of $s_{\omega_\sigma}$ for $\sigma \in PAP_1$ is close to the traceless Dyson Brownian Bridge (TDBB). From Lemma 5.4 we showed that the scaled version, $\hat{s}_{\omega_\sigma}$, of $s_{\omega_\sigma}$ is close to $P_\sigma$ for $\sigma \in PAP_1$. Thus we can determine the scaling limit of $P_\sigma$ for $\sigma \in PAP_1$. Then we will show that for any $\epsilon > 0$ there exists an $L$ such that
If $|\text{PAP}| > (1 - \epsilon)|\text{PAP}|$. Thus the scaling of $P_{\sigma}$ for $\sigma \in \text{PAP}$ is the same as the scaling of $P_{\sigma}$ for $\sigma \in \text{PAP}$. We use the connection with random walks in a cone to show that they both are given by TDBB.

7.1. $|\text{PAP}_3| << |\text{PAP}|$.

**Lemma 7.1.** There exists $c > 0$ such that for all $n$ $|\text{PAP}| \geq cd^n n^{-(d^2 - 1)/2}$.

**Proof.** This follows from [46, Theorem 2.10] where an exact enumeration is given. \hfill \blacksquare

Let $f(m) = \sum_{k > 4.1} k^{2d^2} e^{-ck\beta}$. Given $\epsilon$ and $d$ choose $L$ such that

(25) $(4.1L)^2 f(L) < \epsilon$

and

(26) $f(L)^2 < \epsilon$.

Let $\beta = .15$. Let $L^\dagger$ be the largest $l$ such that $4.1^l < n/10$.

Consider the following events:

- $E^0 = \{R_L \leq 4.1^L \text{ and } \omega \in \text{CW}^{++}((0, 0), (R_L, *))\}$.
- $E_k^1 = \{R_L = k \text{ and } \omega \in \text{CW}^{++}((0, 0), (R_L, *))\}$.
- $F = \{R_L \geq n^{1-\beta} \text{ and } \omega \in \text{CW}^{++}((0, 0), (n^{1-\beta}, *))\}$.

We can also similarly define $E^0, E_k^1$ and $F$ based on the path $S^*$.

**Lemma 7.2.** If $\sigma \in \text{PAP}$ and $R_L, R_L^* < n/2$ then

$\omega_{\sigma} \in \text{CW}^{++}((0, 0), (R_L, *))$,

and

$\omega_{\sigma} \in \text{SCW}^{++}((R_L, *), (n - R_L^*, *))$.

**Proof.** By Lemma 5.1 we have that

$\omega_{\sigma} \in \text{SCW}^{++}((0, 0), (n, 0))$.

Thus for all $t \leq n/2$

$\omega_{\sigma} \in \text{CW}^{++}((0, 0), (t, *))$.

As $R_L < n/2$ this proves the first claim. The third claim is proven in an identical manner. The proof of the second claim is very similar to the proof of Lemma 5.1. We leave the proof to the reader. \hfill \blacksquare

**Lemma 7.3.** If $\sigma \in \text{PAP}$ has $\max(R_L, R_L^*) \geq 4.1^L$ for $\omega_{\sigma}$ then $\omega_{\sigma}$ must be in one of the following sets:

1. $F$
2. $F^*$
3. $\exists k \in ([4.1^L], n^{1-\beta}) \text{ such that } E_k^1 \cap E_k^{0, *} \cap \text{SCW}^{++}((k, 0), (n, 0))$
4. $\exists k' \in ([4.1^L], n^{1-\beta}) \text{ such that } E_0^1 \cap E_k^{1, *} \cap \text{SCW}^{++}((k', 0), (n - k', *))$ or
5. $\exists k, k' \in ([4.1^L], n^{1-\beta}) \text{ such that } E_k^1 \cap E_k^{1, *} \cap \text{SCW}^{++}((k, 0), (n - k', 0))$. 

Proof. If \( \max(R_L, R^*_L) \leq n^{1-\beta} \) then by Lemma 7.2 we have that
\[
\omega_\sigma \in CW^+((0,0), (R_L, *))
\]
\[
\omega_\sigma \in SCW^+((R_L, *), (n - R^*_L, *))
\]
and
\[
\omega^*_\sigma \in CW^+((0,0), (R^*_L, *)).
\]
Using this plus the requirement that \( \max(R_L, R^*_L) \) is bounded below by \([4.1^L]\) we get that at least one of \( E^1_k \cap E^{0, *}, E^0 \cap E^{1, *}_k \), or \( E^1_k \cap E^{1, *}_k \) must occur for some \([4.1^L] \leq k, k' \leq n^{1-\beta} \). If \( \max(R_L, R^*_L) \geq n^{1-\beta} \) then at least one of \( F \) or \( F^* \) occurs.

Lemma 7.4. There exist \( C_1, c_2 \) and \( \delta_1 > 0 \) such that
\[
\mathbb{P}_{(0,0)}(F), \mathbb{P}_{(0,0)}(F^*) \leq C_1 e^{-c_2 n^{\delta_1}}.
\]

Proof. Let \( m \) be the last time before \( R_L \) such that Petrov\((m)\) fails. If \( m^5 > n \) then by Lemma 4.7 this occurs with probability at most \( c_3 e^{-m/5} < c_3 e^{-n^{\gamma'}} \). If \( m^5 \leq n \) then from \( n^2 \) until \( n^{85} = n^{1-\beta} \), \( s_{\omega_\sigma} \) must be within \( n^4 \) of the boundary of Weyl. By standard arguments about random walk, the probability the fluctuation over the time scale \( n^{85} = n^8 n^{.05} \) is bounded by \( e^{-c_4 n^{.05}} \) for some positive constant \( c_4 \). Then choose \( \delta_1, C_1 \) and \( c_2 \) so that \( \min(c_3 e^{-\gamma'}, e^{-c_4 n^{.05}}) < C_1 e^{-c_2 n^{\delta_1}} \). By symmetry the second inequality also holds.

Lemma 7.5. There exists \( C < \infty \) such that for all \( x, y \in \text{Weyl}_{2L^1} \), \( T_{L^1} \) and \( T_{L^1}^* \)
\[
\mathbb{P}_{(T_{L^1}, x)}(SCW^+((T_{L^1}, x), (n - T_{L^1}^*, y))) \leq C U(x) U(y) n^{-(d-1)/2}.
\]

Proof. If a path is in \( SCW^+((T_{L^1}, x), (n - T_{L^1}^*, y)) \) then either it is in \( \text{Weyl}_{-n^4} \) from \( T_{L^1} \) to \( T_{L^1}^* \) or Petrov\((m)\) fails for some \( m > 2L^1 > n^2 \). We will show that the probability of the former is bounded by \( C' U(x) U(y) n^{-(d-1)/2} \). By Proposition 6.1 and Lemma 28 in [12] this probability is bounded by \( C(x,y)n^{-(d-1)/2} \). The function \( C(x,y) \) is bounded by \( C U(x + (n^4, \ldots, n^4) + x_0) U(y + (n^4, \ldots, n^4) + x_0) \) for some fixed \( x_0 \). (See the note at the bottom of page 10 in [12] or Theorem 4 of [50].) As every coordinate of \( x \) and \( y \) are at least \( 2L^1 > n^4 \) then
\[
U(x + (n^4, \ldots, n^4) + x_0) U(y + (n^4, \ldots, n^4) + x_0) \leq 2^{d(d-1)} U(x) U(y).
\]
Thus the probability of this event is at most \( C U(x) U(y) n^{-(d-1)/2} \).

By Lemma 4.7 the probability that Petrov\((m)\) fails for some \( m \) in this region is at most \( e^{-c n^{\gamma'}} \) for some \( \gamma' > 0 \). The lemma follows by putting these two estimates together with the union bound.

Lemma 7.6. There exists \( r, C \) and \( \alpha > 0 \) such that for all \( k \in [4.1^L], T_{L^1} \)
\[
\mathbb{E}_{(0,0)} \left( 1_{E_k \cap \{ k \leq T_{L^1} \}} U(s_{\omega_k}(T_{L^1})) \right) \leq C k^r e^{-c k^{\alpha}}.
\]

Proof. Fix \( k > 4.1^L \). First we show that \( \mathbb{P}_{(0,0)}(E_k) \leq e^{-c k^\alpha} \). This works in exactly the same way as Lemma 7.4. Let \( m \) be the largest integer less than or equal to \( k \) such that Petrov\((m)\) does not occur. If \( m \geq k^{1/5} \) we use the bound on the probability of Petrov\((m)\) failing. If \( m \leq k^{1/5} \) then the path stays close to the boundary of Weyl without entering Weyl_{2L} between \( k^{1/5} \) and \( k - 1 \). This also has low probability, which can be bounded by a similar argument to that used in Lemma 7.4.

If \( E_k \) occurs and \( k < T_{L^1} \) then we let \( x = s_{\omega_k}(k) \). Then we have \( |x| \leq 2k \) and \( x \notin \text{INT}(\text{Weyl}_{2L}) \). Thus Lemma 6.3 applies. The portion of the path after \( k \) is independent of the portion before \( k \). So
The equality is because the portion of the path after $k$ is independent of the portion before $k$. For the next line we have $|x| \leq 2k$ and $x \not\in INT(Weyl_{2L})$. Thus Corollary 6.3 justifies the first inequality.

**Lemma 7.7.** There exists $c > 0$ such that for all $L$ and $n$ (and thus all $L^\dagger$)

$$\mathbb{E}_{(0,0)} \left( 1_{E^0} U(s_\omega(T_{L^1})) \right) \leq c(2 \cdot 4.1^L)^{d(d-1)/2}$$

and

$$\mathbb{E}_{(0,0)} \left( 1_{E^0} U(s_\omega(T_{L^1})) \right) \leq c(2 \cdot 4.1^L)^{d(d-1)/2}.$$

**Proof.** Every point $z$ such that $s_\omega(T_{R_n}) = z$ has $U(z) \leq (2 \cdot 4.1^L)^{d(d-1)/2}$. Then we apply Lemmas 6.2 and 6.4. The second statement follows in the same way.

**Lemma 7.8.** There exists $C''$, $c$, $r'$ and $\alpha$ such that

$$\sum_{k \geq 4.1^L} \mathbb{P}_{(0,0)} \left( E_k^1 \cap E_0^{0,*} \cap SCW^{++}((T_{L^1},*),(n - T_{L^1}^r, *)) \right) \leq n^{-(d-1)/2} \sum_{k \geq 4.1^L} C^r k^r e^{-ck^\alpha}.$$

**Proof.** Fix $k$. We sample $\omega$ as follows. First we sample $\omega$ from 0 to $T_{L^1}$ then we sample $\omega^*$ from 0 to $T_{L^1}^r$ then we sample $\omega$ from $T_{L^1}$ to $n - T_{L^1}^r$. We only get a valid path in the desired set if $\omega \in E_k^1$, $\omega \in E_0^{0,*}$ and

$$s_\omega(n - T_{L^1}^r) = s_\omega^*(T_{L^1}^r).$$

The first two of these events are independent and the third is independent conditioned on $s_\omega(T_{L^1}) = x$. Thus using Lemma 6.6 and Lemmas 7.6 and 7.7 we get

$$\mathbb{P}_{(0,0)} \left( E_k^1 \cap E_0^{0,*} \cap SCW^{++}((T_{L^1},*),(n - T_{L^1}^r, *)) \right) = \sum_{x,y} \mathbb{P}_{(0,0)} \left( E_k^1 \cap \{ k \leq T_{L^1} \} \cap s_\omega(T_{L^1}) = x \right) \mathbb{P}_{(0,0)} \left( E_{0,0}^* \cap s_\omega^*(T_{L^1}^r, *) = y \right) \cdot \mathbb{P}_{(0,0)} \left( \omega \in SCW^{++}((T_{L^1},x),(n - T_{L^1}^r,y) \mid s_\omega(T_{L^1}) = x \right) \leq \sum_{x,y} \mathbb{P}_{(0,0)} \left( E_k^1 \cap \{ k \leq T_{L^1} \} \cap s_\omega(T_{L^1}) = x \right) \cdot \mathbb{P}_{(0,0)} \left( E_{0,0}^* \cap s_\omega^*(T_{L^1}^r, *) = y \right) \cdot CU(x)U(y) n^{-(d-1)/2} \leq n^{-(d-1)/2} \left( \sum_x 1_{E_k^1 \cap \{ k \leq T_{L^1} \} \cap s_\omega(T_{L^1}) = x} U(x) \right) \left( \sum_y 1_{E_{0,0}^* \cap s_\omega^*(T_{L^1}^r,*) = y} U(y) \right) \leq (n^{-(d-1)/2})(C^r k^r e^{-ck^\alpha})(2k)^{d(d-1)/2} \leq C'' k^r e^{-ck^\alpha} n^{-(d-1)/2}.$
Summing up over $k$ gives us
\[
\sum_{k>4.1^L} \mathbb{P}_{(0,0)}(E_k \cap E_{0,*} \cap SCW^{++}((T_{L^1},*),(n-T_{L^1}^*,*))) \leq n^{-(d-1)/2} \sum_{k\geq4.1^L} C^{m} k^{r} e^{-ck^\alpha}.
\]

**Lemma 7.9.**

\[
\sum_{k,k'>4.1^L} \mathbb{P}_{(0,0)}(E_k \cap E_k' \cap SCW^{++}((T_L^1,*),(n-T_L^*,*))) \leq n^{-(d-1)/2} \sum_{k,k'\geq4.1} C^{2} k^{r} e^{-ck^\alpha} (k')^{r'} e^{-ck'^\alpha}.
\]

**Proof.** The proof of this goes in exactly the same way as the proof of Lemma 7.8. ■

**Lemma 7.10.** $|PAP_3| = o(|PAP|)$.

**Proof.** By Lemma 7.1

\[|PAP| \geq c d^{2n} n^{-(d-1)/2}.\]

By Lemma 7.3 we have that

\[PAP_3 \subseteq \bigcup_{i=1}^{5} PAP_{3,i},\]

where the sets $PAP_{3,i}$ are the sets defined in the statement of Lemma 7.3. By Lemma 7.4 we get that for any $\epsilon > 0$ we can find a large $L$ such that for all $n$ sufficiently large

\[|PAP_{3,1}|, |PAP_{3,2}| \leq \epsilon d^{2n} n^{-(d-1)/2}.\]

If $\sigma \in PAP$ but not in the union of sets in Lemma 7.8 then Petrov($m^C$) or Petrov($m^C$) must occur for some $m > n^4$. This has probability at most $C e^{-n^\alpha}$ for some positive $C$ and $\alpha$. Thus by Lemma 7.8 we get that for any $\epsilon > 0$ we can find a large $L$ such that for all $n$ sufficiently large

\[|PAP_{3,3}|, |PAP_{3,4}| \leq \epsilon d^{2n} n^{-(d-1)/2}.\]

Similarly by Lemma 7.9 we get that for any $\epsilon > 0$ we can find a large $L$ such that for all $n$ sufficiently large

\[|PAP_{3,5}| \leq \epsilon d^{2n} n^{-(d-1)/2}.\]

Thus the lemma follows. ■

By Lemma 5.1 the path that is the image of a permutation in $PAP$ is in $SCW^{++}$. We will show that the cardinality of

\[SCW^{++}((T_L,*),(n-T_L^*,*)) \cap \{\max(T_L, T_L^*) < \lfloor 4.1^L \rfloor\}\]

is $1 + o(1)$ times the cardinality of $SCW^{-}((T_L,*),(n-T_L^*,*)) \cap \{\max(T_L, T_L^*) < \lfloor 4.1^L \rfloor\}$.

The image of the set $PAP_1$ is not exactly the set of paths that we can use our previous results to calculate the scaling limit. But the size of the symmetric difference between the set we will describe and $PAP_1$ will be small in comparison with $|PAP_1|$. Thus the two sets will have the same scaling limits. We first describe which paths we want to exclude from $PAP_1$. We want paths that satisfy the Petrov conditions at both $T_L$ and $T_L^*$. Let

\[PAP_1' = \left\{ \sigma : \sigma \in PAP_1, \omega \text{ satisfies Petrov}(T_L), \omega^* \text{ satisfies Petrov}(T_L^*) \right\}\]

**Lemma 7.11.** $|PAP \setminus PAP_1'| = o(1)|PAP|$.
Proof. By Lemma 7.10 and our partition
\[ \text{PAP} = \text{PAP}_1 \cup \text{PAP}_2 \cup \text{PAP}_3 \]
it is enough to bound \(|(\text{PAP}_1 \cup \text{PAP}_2) \setminus \text{PAP}_1^*|\). If
\[ \omega_{\sigma} \in (\text{PAP}_1 \cup \text{PAP}_2) \setminus \text{PAP}_1^* \]
then
\[ s_{\omega_{\sigma}} \in SCW^+((T_L, *), (n - T_L, *)) \]
with \(T_L, T_L^* < |(4.1)^L|\) and either
1. Petrov\((T_L)^C\),
2. Petrov\((n - T_L^*)^C\) or
3. \(\omega_{\sigma} \notin SCW^-((T_L, *), (n - T_L^*, *))\).

There are at most \(4.1^L\) choices for each of these \(T_L\) and \(T_L^*\) and \(4.1^{Ld}\) choices for \(s_{\omega_{\sigma}}(T_L)\) and \(s_{\omega_{\sigma}}(T_L^*)\). By Lemma 4.7 for each of these \(L, d\) and \(k, k'\) the probability that Petrov\((k)^C\) or Petrov\((k')^C\) occurs is \(2e^{-2^L}\) for some \(c > 0\). Each of the possible \(s_{\omega_{\sigma}}(T_L)\) and \(s_{\omega_{\sigma}}(T_L^*)\) have \(U(s_{\omega_{\sigma}}(T_L))\) and \(U(s_{\omega_{\sigma}}(T_L^*))\) and most \(4.1^{Ld}d\). For each of these sets we apply Lemma 7.5. Thus the number of permutations satisfying one of the first two conditions is at most
\[ (4.1^{Ld})^2(4.1^L)^2(2e^{-2^L})d^2n^{-2} \leq \epsilon d^2n^{-(d^2-1)/2}. \]
This implies that
\[ |\text{PAP}_1 \setminus \text{PAP}_1^*| \leq |\text{PAP}_1|. \]

Now we count the permutations satisfying the first two conditions but not the third. Call this set \(Y\). Partition \(Y\) by the time and position at times \(T_L\) and \(n - T_L^*\), that is
\[ Y = \bigcup_{t, t^*, v, w} \{ \sigma: T_L = t, T_L^* = t^*, s_{\omega_{\sigma}}(T_L) = v, s_{\omega_{\sigma}}(n - T_L^*) = w \} \cap Y. \]

Call a set on the right hand side to be \(H(t, t^*, v, w)\). Further partition the set on the right hand side \(H(t, t^*, v, w)\) into \(H_1(t, t^*, v, w)\) and \(H_2(t, t^*, v, w)\) be the collection of pairs of sequences that are obtained from permutations in \(\text{PAP}_1\) (and thus \(\text{PAP}_1^*\)) or \(\text{PAP}_2\) respectively. By Lemma 6.5 \((1 - \epsilon)\)\(|H(t, t^*, v, w)|\) of pairs of sequences in \(H(t, t^*, v, w)\) will be in \(SCW^-((T_L, v), (n - T_L^*, w))\) and therefore in \(H_1(t, t^*, v, w)\), hence \(|H_2(t, t^*, v, w)| \leq \epsilon |H(t, t^*, v, w)|\). By Lemma 5.1 the pair of sequences associated to a permutation in \(\text{PAP}_1\) is always in \(SCW^{++}\), thus
\[ |H_1(t, t^*, v, w)| \geq (1 - \epsilon)|H(t, t^*, v, w)|. \]
Combined this shows that \(|H_2(t, t^*, v, w)| \leq \frac{\epsilon}{1 - \epsilon}|H_1(t, t^*, v, w)|\). This bound is uniform over all \(v, w, T_L,\) and \(T_L^*\) such that Petrov\((T_L)\) and Petrov\((T_L^*)\) hold and \(v, w \in \text{Weyl}_{2L}\), thus
\[ |\text{PAP}_2| \leq \frac{\epsilon}{1 - \epsilon}|\text{PAP}_1^*|. \]

Combining (28) and (29) with Lemma 7.10 proves the lemma. \(\blacksquare\)

**Lemma 7.12.** For every \(\sigma \in \text{PAP}_1\) there exists \(t, t^* \leq |4.1^L|, x, x^* \in \text{Weyl}_{2L}\) and \(\omega \in [d]^{\{t+1, \ldots, n-t^*\}}\) such that
1. \(T_L(s_{\omega_{\sigma}}) = t,\)
2. \(T_L^*(s_{\omega_{\sigma}}) = t^*;\)
3. \(s_{\omega_{\sigma}}(t) = x;\)
4. \(s_{\omega_{\sigma}}(t^*) = x^*;\)
5. \(\tilde{\omega}^1(j) = \omega_{\sigma}(j)\) for all \(j \in \{1, \ldots, t\}\) which satisfies Petrov\((t)\),
6. \(\tilde{\omega}^2(j) = \omega_{\sigma}(j)\) for all \(j \in \{t + 1, \ldots, n - t^*\}\), and
7. \(\tilde{\omega}^3(j) = \omega_{\sigma}(n + 1 - j)\) for all \(j \in \{1, \ldots, t^*\}\) which satisfies Petrov\((t^*)\).
For \( \sigma \neq \sigma' \) this collection of these seven objects is different.

**Proof.** The existence of these objects follow from the definition of \( \text{PAP}_1' \). If \( \sigma \neq \sigma' \) then \( \omega_{\sigma} \neq \omega_{\sigma'} \). Thus one of the last three must be different as well. \[ \square \]

**Lemma 7.13.** For any \( \epsilon > 0 \) there exist \( L \) and \( K \) satisfying the following. Fix any \( t, t^*, x, x^* \), \( \tilde{\omega}^1 \) and \( \tilde{\omega}^3 \). Let \( A = A(t, t^*, x, x^*, \tilde{\omega}^1, \tilde{\omega}^3) \) be the set of \( \sigma \in \text{PAP}_1' \) associated with these six objects. The fraction of \( \sigma \in A \) with

\[
\omega_{\sigma} \in \text{SCW}^-((t, x), (n - t^*, x^*))
\]

is at least \( 1 - \epsilon \).

**Proof.** By Lemma 6.5

\[ (30) \quad \left| \text{SCW}^-((t, x), (n - t^*, x^*)) \right| > (1 - \epsilon) \left| \text{SCW}^+((t, x), (n - t^*, x^*)) \right| \]

For every \( \sigma \in A \) there exists \( \tilde{\omega}^2 \in [d]^{t+1, t^*} \times [d]^{t+1, n-t^*} \). By Lemma 5.1 we have

\[ (31) \quad \tilde{\omega}^1 \oplus \tilde{\omega}^2 \oplus \tilde{\omega}^3 \in \text{SCW}^+((0, 0), (n, 0)). \]

By (31) we have that \( \tilde{\omega}^2 \in \text{SCW}^+((t, x), (n - t^*, x^*)) \). Thus every \( \sigma \in A \) corresponds with a unique element in the set on the right hand side of (30).

By Lemma 5.3 we have that if

\[
\tilde{\omega}^2 \in \text{SCW}^-((t, x), (n - t^*, x^*))
\]

then \( \tilde{\omega}^1 \oplus \tilde{\omega}^2 \oplus \tilde{\omega}^3 \) corresponds to a permutation in \( \text{PAP} \). It is easy to check that this permutation is also in \( \text{PAP}_1' \). Thus every element in the set on the left hand side of (30) corresponds with a unique \( \sigma \in A \).

Combining the conclusions of these two paragraphs with (30) completes the proof. \[ \square \]

Recall the definition of the pseudo-metric \( d_K \). Fix \( K \) and \( n > 2K \) and two paths \( s \) and \( s' \) in \( \mathbb{Z}^d \). We say \( d_K(s, s') = 0 \) if \( s(t) = s'(t) \) for all \( t \in [K, n - K] \). Otherwise we say \( d_K(s, s') = 1 \). We can extend \( d_K \) to distributions on paths. For two measures on paths \( \mu \) and \( \mu' \), we set \( d_K(\mu, \mu') \) to be the infimum over all joinings \( \nu \) of \( \mu \) and \( \mu' \) of \( \mathbb{E}_\nu(d_K(s, s')) \).

**Lemma 7.14.** For any \( \epsilon > 0 \) there exist \( L \) and \( K \) such that the following is true. Fix any \( t, t^*, x, x^*, \tilde{\omega}^1 \) and \( \tilde{\omega}^3 \). Let \( A = A(t, t^*, x, x^*, \tilde{\omega}^1, \tilde{\omega}^3) \) be the set of \( \omega \in \text{PAP}_1' \) associated with these six objects. Then there exists \( K \) such that

\[
d_K\left( (s_{\omega_\sigma} : \sigma \in A), (s_\omega : \omega \in CW((0, 0), (n, 0))) \right) < \epsilon.
\]

**Proof.** Let

\[
A_1 = \{ \sigma \in A : \tilde{\omega}^2 \in \text{SCW}^-((t, x), (n - t^*, x^*)) \}.
\]

By Lemma 7.13 we get \( |A_1| > (1 - \epsilon)|A| \). As \( A_1 \subseteq A \) the previous statement implies that for any \( K \)

\[ (32) \quad d_K((s_{\omega_\sigma} : \sigma \in A_1), (s_\omega : \sigma \in A)) < \epsilon. \]

By Lemma 6.5 we get

\[ (33) \quad d_K\left( (s_{\omega_\sigma} : \sigma \in A_1), (s_\omega : \omega \in CW((t, x), (n - t^*, x^*))) \right) < \epsilon. \]

By Corollary 6.7 there exists \( K \) such that

\[ (34) \quad d_K\left( (s_\omega : \omega \in CW((0, 0), (n, 0))), (s_\omega : \omega \in CW((t, x), (n - t^*, x^*))) \right) < \epsilon. \]

Combining (32), (33) and (34) proves the lemma. \[ \square \]
7.2. Proof of our main theorem. We are now prepared to prove our main result, which we restate for convenience.

**Theorem.** If $\sigma$ is a uniformly random element of $A_n(\rho_d)$ then, as $n \to \infty$, the following convergence holds in distribution with respect to the supremum norm topology on $C([0,1], \mathbb{R}^d)$:

$$P_\sigma \xrightarrow{\text{dist}} \Lambda(Z).$$

**Proof of Theorem 2.1.** By Theorem 6.8 random walk in a cone has a scaling limit of traceless Dyson Brownian bridge. By Lemma 7.11 it remains to show that the distribution $(P_\sigma : \sigma \in \text{PAP}_1')$ has the same scaling limit as a random walk in a cone.

By Lemma 7.12 we can write $\text{PAP}_1'$ as a disjoint union of sets parameterized by $t, x, t^*, x^*, \tilde{\omega}^1$ and $\tilde{\omega}^3$. Thus we can write $(s_{\omega_{\sigma}} : \sigma \in \text{PAP}_1')$ as a linear combination of pieces of the form

$$(s_{\omega_{\sigma}} : \sigma \in \text{PAP}_1', t, x, t^*, x^*, \tilde{\omega}^1, \tilde{\omega}^3).$$

By Lemma 7.14 each of those pieces can be coupled with $CW((0,0),((n,0)))$ to show

$$d_K\left((s_{\omega_{\sigma}} : \sigma \in \text{PAP}_1', t, x, t^*, x^*, \tilde{\omega}^1, \tilde{\omega}^3), (s_{\omega} : \omega \in CW((0,0),((n,0))))\right) < \epsilon.$$

Scaling these paths we get a joining $\nu$ of $(\hat{s}_{\omega} : \omega \in CW((0,0), (n,0)))$ and $(\hat{s}_{\omega_{\sigma}} : \sigma \in \text{PAP}_1')$ such that with probability at least $1 - \epsilon$ the paths are within $Cn^{-5}$ in the supremum norm.

By Lemmas 7.13 at least $1 - \epsilon$ fraction of the $\sigma \in \text{PAP}_1'$ satisfy the hypothesis of Lemma 5.4. By Lemma 5.4 for those $\sigma$ we have that

$$\sup_{t \in [0,1]} D(P_\sigma(t), \hat{s}_{\omega_{\sigma}}(t)) \leq n^{-1}.$$

Thus the coupling $\nu$ gives a coupling of $(\hat{s}_{\omega} : \omega \in CW((0,0), (n,0)))$ and $(P_\sigma : \sigma \in \text{PAP}_1')$ such that with probability at least $1 - 2\epsilon$ the paired paths are within $Cn^{-5} + n^{-1}$ in the supremum norm. By the opening paragraph this completes the proof. \hfill \blacksquare

8. Walks in the Weyl Chamber

Recall that if $\omega$ has distribution $\mathbb{P}$, then $s_\omega$ is a lazy random walk such that $s_\omega(t) - s_\omega(t - 1) = 0$ with probability $1/d$ and $s_\omega(t) - s_\omega(t - 1) = e_i - e_k$ with probability $1/d^2$ for each $i \neq j$ with $1 \leq i, j \leq d$. To simplify notation, in this section we let $S_t = S(t) = s_\omega(t)$, $X_t = s_\omega(t) - s_\omega(t - 1)$ and define $\mathbb{P}_x$ to be the law of $S + x$ for $x \in \mathbb{R}^d$.

For any $m$ recall the definition of Weyl$_m$ in [10].

8.1. Results from the literature. In this section we recall some useful theorems from the literature.

**Theorem 8.1 ([32]).** The function

$$U(x) = \prod_{i=1}^{d} \prod_{j=i+1}^{d} (x_i - x_j)$$

is harmonic for the random walk $S$.

One of our main tools will be the results from [12] about random walks in cones. The random walk we are interested in does not satisfy the hypotheses of [12] but can be transformed into one that does by an appropriate linear transformation. In particular, our random walk takes place on a $(d - 1)$-dimensional subspace of $\mathbb{R}^d$ and the its covariance matrix is not the identity. We now explain how to fix this for our random walk.
Using both classical and easy to show. The second property is an easy computation and the third and fourth properties follow from orthogonality.  

Let \( \mathbf{u} \in \mathbb{R}^d \) be the unit vector defined by \( u_i = (2d - 2\sqrt{d})^{-1/2} \) for \( 1 \leq i \leq d - 1 \) and

\[
u_d = \frac{1 - \sqrt{d}}{\sqrt{2d - 2\sqrt{d}}}
\]

Using \( u \), we define the linear transformation \( H_u : \mathbb{R}^d \to \mathbb{R}^d \) by

\[H_u(x) = x - 2 \langle x, u \rangle u,
\]

where \( \langle \cdot, \cdot \rangle \) is the usual inner product on \( \mathbb{R}^d \).

**Proposition 8.2.** \( H_u \) has the following properties:

1. \( H_u \) is an orthogonal involution.
2. If \( \mathbf{1} \) is the all ones vector then \( H_u(\mathbf{1}) = \sqrt{d}e_d \), where \( e_d \) is the \( d \)'th standard basis vector in \( \mathbb{R}^d \).
3. If \( x \in \mathbb{R}^d \), then \( \langle x, \mathbf{1} \rangle = 0 \) if and only if \( H_u(x)_d = 0 \).
4. \( H_u \) is a bijection between \( \mathcal{C} \) and the cone

\[
\mathcal{C} = \left\{ x \in \mathbb{R}^d : x_1 > x_2 > \cdots > x_{d-1} > 1/\sqrt{d-1} \sum_{i=1}^{d-1} x_i, x_d = 0 \right\}.
\]

**Proof.** \( H_u \) is a Householder transformation, and the fact that it is an orthogonal transformation is both classical and easy to show. The second property is an easy computation and the third and fourth properties follow from orthogonality.

Let \( \pi_{d-1} : \mathbb{R}^d \to \mathbb{R}^{d-1} \) be the natural projection onto the first \( d - 1 \) coordinates, so that \( \pi_{d-1}(x_1, x_2, \ldots, x_{d-1}, x_d) = (x_1, x_2, \ldots, x_{d-1}) \). Define

\[
X_i = (d/2)^{1/2} \pi_{d-1} H_u(X_i) \quad \text{and} \quad \tilde{S}_n = (d/2)^{1/2} \pi_{d-1} H_u(S_n).
\]

Let us introduce two lattices, \( \mathcal{L} = \mathbb{Z}^d \cap \mathcal{C} \) and \( \tilde{\mathcal{L}} = (d/2)^{1/2} \mathcal{C} \). Note that \((S_n)_{n \geq 0}\) is a random walk on the lattice \( \mathcal{L} \) and \( \pi_{d-1} \circ H_u \) is an isometry from \( \mathcal{H} \) to \( \mathcal{L} \). Using this, and letting \( v_\mathcal{L} \) (\( v_{\tilde{\mathcal{L}}} \)) be the volume of a fundamental cell of \( \mathcal{L} \) (\( \tilde{\mathcal{L}} \)) with respect to the appropriate dimensional Hausdorff measure, we see that the volume \( v_\mathcal{L} = (d/2)^{-\left( (d-1)/2 \right)^2} v_{\tilde{\mathcal{L}}} \).

**Proposition 8.3.** The random walk \( \tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n \) and the cone \( \tilde{\mathcal{C}} \) satisfy the hypotheses of [12]. In particular letting \( \tilde{X}_i = (X_{i,1}, \ldots, X_{i,d-1}) \), we see that \( \mathbb{E}(X_{i,j}) = 0 \), \( \mathbb{E}(X_{i,j}^2) = 1 \), and \( \mathbb{E}(X_{i,j}X_{i,k}) = 0 \) for \( j \neq k \). Moreover, the random variable \( \tilde{X}_i \) is supported on the \( \tilde{\mathcal{L}} \) and, additionally, \( \tilde{X}_i \) satisfies the lattice assumptions of [12] with respect to \( \tilde{\mathcal{L}} \).

**Proof.** The assertions about \( \tilde{\mathcal{C}} \) and \( \tilde{\mathcal{L}} \) are immediate. The claims about the means and (co)variances of the coordinates of \( \tilde{X}_i \) are a straightforward computation.

Based this result, it is now straightforward translate the results of [12] to our current context.

Let \( B \) be a standard Brownian motion in \( \mathbb{R}^d \) and let \( v_1, v_2, \ldots, v_d \) be an orthonormal basis for \( \mathbb{R}^d \) such that \( v_1 = d^{-1/2}(1, 1, \ldots, 1) \). Then we can express \( B \) as

\[
B = \sum_{i=1}^{d} B_i v_i
\]
where $B_1, \ldots, B_d$ are independent, standard, one dimensional Brownian motions. Since
\[
\left\langle \sum_{i=2}^{d} B_{i} v_{i}, (1,1,\ldots,1) \right\rangle = \sqrt{d} \left\langle \sum_{i=2}^{d} B_{i} v_{i}, v_{1} \right\rangle = 0,
\]
leaving $B^0 = B_2 u_2 + \cdots + B_d u_d$, we see that $\mathbb{P}(B^0_t \in \mathcal{H}$ for all $t) = 1$. Furthermore, it is easy to see that if we let $\tau^B_x = \inf \{ t : x + B_t \notin C \}$ and $\tau^0_x = \inf \{ t : x + B^0_t \notin \tilde{C} \}$, then $\tau^0_x = \tau^B_x$ since adding or subtracting $B_1 v_1$ preserves the relative order of the coordinates. Furthermore, since $H_u$ is an isometry, we see that $\bar{\tau} = \pi_{d-1} H_u(B) = \pi_{d-1} H_u(B^0)$ is a standard Brownian motion on $\mathbb{R}^{d-1}$. Thus, if we let $\tau^B_y = \inf \{ t : y + B \notin \pi_{d-1} H_u(C) \}$ and $x = H_u((y,0))$ (where $(y,0) = (y_1, \ldots, y_{d-1},0)$) then we have that $\tau^B_y = \tau^B_x$. It follows from [19, Equation (20)], that
\[
\mathbb{P}(\tau^B_y > t) = \mathbb{P}(\tau^B_x > t) \sim \mathbb{P} U(x)^{(d-1)/4} = \mathbb{P} U(H_u((y,0)))^{(d-1)/4},
\]
where
\[
\mathbb{P} = \frac{1}{\prod_{i<j}(j-i)} \frac{2^{3d/2}}{(2\pi)^{d/2}} \frac{\prod_{k=1}^{d} [\Gamma((k/2) + 1)]^2}{\prod_{i<j}(j-i)}.
\]

Letting $\tau_x = \inf \{ n : x + S_n \notin C \}$ and $\bar{\tau}_x = \inf \{ n : x + S_n \notin \pi_{d-1} H_u(C) \}$, we translate [12, Theorem 1] to our present context, to get the following result.

**Theorem 8.4** ([12], Theorem 1). There exists a strictly positive function $\tilde{V}$ such that for all $x$ in the interior of $\pi_{d-1} H_u(C)$ we have
\[
\mathbb{P}(\bar{\tau}_x > n) \sim \mathbb{P} V(x)n^{-(d-1)/4}
\]
Consequently, for $x \in \tilde{C}$, we have
\[
\mathbb{P}(\tau_x > n) \sim \mathbb{P} (\tilde{V})^{(d/2)1/2}(\pi_{d-1} H_u(x))n^{-(d-1)/4}.
\]

Motivated by this theorem, we define $V(x) = \tilde{V} (\tilde{(d/2)1/2}(\pi_{d-1} H_u(x)))$. Translating [12, Theorem 5] into the present context thus gives the following result.

**Theorem 8.5.** For $x \in C$,
\[
\sup_{y \in C} \left| \frac{\tilde{v}}{\tilde{v}} \tilde{v}^{(d-1)(d+1)/4} \mathbb{P}(x + S_n = y, \tau_x > n) - \mathbb{P}(x + S_n = y) \tilde{h}_O U \left( \frac{\sqrt{d}}{\sqrt{2n}} y \right) e^{-d|y|^2/4n} \right| \to 0,
\]
where $h_0$ is chosen so that $p(y) = h_0 U(H_u((y,0)))e^{-|y|^2/2}$ is a probability density function with respect to Lebesgue measure on $\pi_{d-1} H_u(C)$.

**Theorem 8.6.** For $x \in C$,
\[
\sup_{y \in C} \left| \frac{\tilde{v}}{\tilde{v}} \mathbb{P}(x + S_n = y|\tau_x > n) - \tilde{h}_O U \left( \frac{\sqrt{d}}{\sqrt{2n}} y \right) e^{-d|y|^2/4n} \right| \to 0.
\]

We remark that this local central limit theorem should be interpreted has taking place on $\mathcal{H}$ relative to the $(d-1)$-dimensional Hausdorff measure. In particular,
\[
\nu(x) = (d/2)^{(d-1)/2} h_0 U \left( \frac{\sqrt{d}}{\sqrt{2n}} y \right) e^{-d|y|^2/4n},
\]
is a probability density function with respect to the $(d-1)$-dimensional Hausdorff measure and we obtain the following corollary.
Corollary 8.7. Let $A \subseteq \mathcal{H}$ be an open set. Then
\[ \lim_{n \to \infty} P\left(\frac{x + S_n}{\sqrt{n}} \in A \middle| \tau_x > n\right) = \int_A \nu(u)du, \]
where $dx$ is the $(d-1)$-dimensional Hausdorff measure on $\mathcal{H}$.

Theorem 8.8 ([12], Theorem 6). For $x, y \in \mathcal{C}$,
\[ P(x + S_n = y, \tau_x > n) \sim \frac{\sqrt{\frac{1}{2}N^2}}{n^{(d-1)(d+1)/4}} \int_{C} \left(\frac{d}{2}\right)^{d-1} h_0 U \left(\frac{\sqrt{d}}{\sqrt{2}}\right)^2 e^{-d|u|^2/4} du \]
If $t \in (0, 1)$ and $D \subseteq \mathcal{C}$ then, letting $[t]$ be the integer part of $t$,
\[ P\left(\frac{x + S_{[n]}}{\sqrt{n}} \in D \middle| x + S_n = y, \tau_x > n\right) \to \frac{\int_{D \cap C} U \left(\frac{\sqrt{d}}{\sqrt{2}}\right)^2 e^{-d|u|^2/4(1-t)} du}{\int_{C} U \left(\frac{\sqrt{d}}{\sqrt{2}}\right)^2 e^{-d|u|^2/4(1-t)} du}. \]

For the next result, we let $S_{x,y,k,n} = (S_{x,y,k,n})_{j=0}^{n-2k}$ be a random variable whose distribution is given by
\[ P(S_{x,y,k,n} \in A) = P\left((S_{k+j})_{j=0}^{n-2k} \in A \middle| S_0 = x, S_n = y, \tau_x > n\right). \]

Theorem 8.9. If $x \in \mathcal{C}$ and $y, x', y' \in (x + \mathbb{Z}^m) \cap \mathcal{C}$ are then
\[ \lim_{k \to \infty} \sup_{n \to \infty} d_{TV}(S_{x,y,k,n}, S_{x',y',k,n}) = 0. \]

Proof. If $s = (s_j)_{j=0}^{n-2k}$ is a path in $(x + \mathbb{Z}^m) \cap \mathcal{C}$, then the Markov property of $(S_n)_{n \geq 0}$ and a time reversal argument imply that
\[ P(S_{x,y,k,n} = s) = \frac{P_s(0)((S_{j})_{j=0}^{n-2k} = s)P_x(S_k = s_0, \tau_x > k)P_y(S_k = s_{n-2k}, \tau_y > k)}{P_x(S_n = y, \tau_x > n)}. \]

Consequently,
\[ \frac{P(S_{x,y,k,n} = s)}{P(S_{x',y',k,n} = s)} = \frac{P_x(S_k = s_0, \tau_x > k) P_y(S_k = s_{n-2k}, \tau_y > k)}{P_{x'}(S_n = y, \tau_x > n)} \frac{P_x(S_k = s_0, \tau_x > k) P_{y'}(S_k = s_{n-2k}, \tau_{y'} > k)}{P_{x'}(S_n = y, \tau_x > n)} \]

Given $\epsilon > 0$, using Corollary 8.7 we find a bounded, open set $A$ such that $d(A, \partial \mathcal{C}) > 0$,
\[ \int_A \nu(x)dx > 1 - \epsilon \]
and
\[ \lim_{n \to \infty} \max_{z \in \{x, x', y, y'\}} \left| P\left(\frac{z + S_n}{\sqrt{n}} \in A \middle| \tau_z > n\right) - \int_A \nu(u)du \right| = 0. \]

It follows from Theorem 8.5 that
\[ \max_{z \in \{x, x', y, y'\}} \sup_{y \in (n^{-1/4}(x+\mathbb{Z})) \cap A} \left| \frac{n^{(d-1)(d+1)/4} P(z + S_n = \sqrt{n}y, \tau_z > n)}{R_{x} V(z) h_0 U \left(\frac{\sqrt{d}}{\sqrt{2}}\right)^2 e^{-d|y|^2/4}} - 1 \right| \to 0. \]

Consequently, we see that
\[ \lim_{k \to \infty} \max_{z \in \{x, x', y, y'\}} \sup_{w \in (n^{-1/4}(x+\mathbb{Z})) \cap A} \max_{\{u \in (x+\mathbb{Z}) \cap (\sqrt{n}A)\}} \frac{V(w) P_z(S_k = u, \tau_u > k)}{V(z) P_w(S_k = u, \tau_u > k)} - 1 = 0. \]

Similarly,
\[ \frac{P_x(S_n = y, \tau_x > n)}{P_{x'}(S_n = y', \tau_x' > n)} \sim \frac{V(x) V(y)}{V(x') V(y')}. \]
Therefore,
\[
\lim_{k \to \infty} \lim_{n \to \infty} \max_{s_{n,k}, s_{n,k}^0, s_{n,k}' \in (x+L) \cap (\sqrt{k}A)} \left| \frac{P(S, y, k, n = s_{n,k}) - P(S', y', k, n = s_{n,k}')} {P(S, y, k, n = s_{n,k})} - 1 \right| \\
= \lim_{k \to \infty} \lim_{n \to \infty} \max_{s_{n,k}, s_{n,k}^0, s_{n,k}' \in (x+L) \cap (\sqrt{k}A)} \left| \frac{P_x(S_k = 0, \tau_x > k) - P_{y'}(S_k = 0, \tau_{y'} > k)} {P_x(S_k = 0, \tau_x > k)} - 1 \right| \\
= \lim_{k \to \infty} \max_{s, w \in (x+L) \cap (\sqrt{k}A)} \left| \frac{P_x(S_k = z, \tau_x > k) - P_{y'}(S_k = w, \tau_{y'} > k)} {P_x(S_k = z, \tau_x > k)} - 1 \right| \\
= \left| \frac{V(x)} {V'(y')} - 1 \right| = 0.
\]

Hence, for \(k\) and \(n\) sufficiently large,
\[
\sum_{s_{n,k}} \left| P(S, y, k, n = s_{n,k}) - P(S', y', k, n = s_{n,k}) \right| \\
\leq 4\varepsilon + \sum_{s_{n,k}, s_{n,k}^0, s_{n,k}' \in (x+L) \cap (\sqrt{k}A)} \left| P(S, y, k, n = s_{n,k}) - P(S', y', k, n = s_{n,k}') \right| \\
\leq 4\varepsilon + \max_{s_{n,k}, s_{n,k}^0, s_{n,k}' \in (x+L) \cap (\sqrt{k}A)} \left| P(S, y, k, n = s_{n,k}) - P(S', y', k, n = s_{n,k}') \right| - 1 \leq 5\varepsilon,
\]
which completes the proof.

**Theorem 8.10.** Suppose that \(x \in \mathcal{C}\). Then for all bounded, continuous functions \(f : D([0, 1], \mathbb{R}^d) \to \mathbb{R}\) we have
\[
E \left( f \left( \frac{x + S_{[nt]}} {\sqrt{2nt/d}} \right) \bigg| \tau_x > n, S(t) = 0 \right) \to E[f(\Lambda(Z))].
\]

**Proof.** Suppose that \(y \in \hat{\mathcal{C}}\). For \(0 < t = 1\) define
\[
\tilde{S}(n)(t) = \frac{y + S_{[nt]}} {\sqrt{n}}.
\]

By [14] Theorem 4] there is a process \(\tilde{B}^0 = (\tilde{B}^0(t), 0 < t \leq 1)\), called Brownian excursion in \(\hat{\mathcal{C}}\), such that for bounded and continuous \(f : D([0, 1], \mathbb{R}^{d-1}) \to \mathbb{R}\),
\[
E \left( f \left( \tilde{S}(n)(t) \right) \bigg| \tilde{\tau}_y > n, \tilde{S}(n) = 0 \right) \to E[f(\tilde{B}^0)].
\]

Consequently, if for \(x \in \mathcal{C}\) we define
\[
S(n)(t) = \frac{x + S_{[nt]}} {\sqrt{2nt/d}},
\]
then for bounded and continuous \(f : D([0, 1], \mathbb{R}^d) \to \mathbb{R}\) we have
\[
E \left( f \left( S(n)(t) \right) \bigg| \tau_x > n, S(t) = 0 \right) \to E[f(H_u(\tilde{B}^0, 0))].
\]

It remains to identify the law of \(H_u(\tilde{B}^0, 0)\) as the law of the eigenvalues of a traceless Hermitian Brownian bridge. The law of \(\tilde{B}^0\) is specified in terms of Brownian motion conditioned to remain in \(\hat{C}\) for all time, denoted \(\tilde{B}^0\), which in turn is defined as an \(h\)-transform of Brownian motion killed on exiting \(\hat{C}\). Since \(H_u\) is an isometry, we have that we have that \(\tilde{B}^0 = \pi_{d-1}H_u(B^0_>)\).
Suppose that $0 < t < 1$ and $f : D([0,1], \mathbb{R}^d) \to \mathbb{R}$ is a bounded continuous function such that $f(g)$ depends only on the restriction of $g$ to $[0,t]$. From \cite{[14]} Equations (18), (26) [accounting for a time change suppressed in (35)] and the Brownian scaling invariance of $B^0_t$, we have that there is a constant $C_t$ such that

$$
\mathbb{E}[f(H_\alpha(\bar{B}_t^0, 0))] = C_t \mathbb{E}[f(B^0_{\infty}(:t))e^{-|B^0_\infty(t)|^2/(2(1-t))}],
$$

so the result follows from Equation \[(57).\]

Fix $K'$ and $n > 2K'$. We put a pseudo-metric on paths $s$ and $s'$ in $\mathbb{Z}^d$. We say $d_{K'}(s, s') = 0$ if $s(t) = s'(t)$ for all $t \in [K', n - K']$. Otherwise we say $d_{K'} = 1$.

**Lemma 8.11.** Fix $\epsilon > 0$ and $K$. Let $M$ be a measure on quadruples $(s, x)$ and $(t, y)$ such that with probability one have $0 \leq s, t \leq K$, $x, y \in \text{Weyl}$ and $|x|, |y| \leq K$. For $n > 2K$ define $\hat{M}_n$ to be the measure generated by picking $(s, x)$ and $(t, y)$ according to $M$ and then sampling $\omega$ from $CW((s, x), (n - t, y))$. There is a $K'$ such that for any $M$

$$
d_{K'}(\hat{M}_n, (s_\omega : \omega \in CW((0, 0), (n, 0)))) < \epsilon.
$$

**Proof.** The measure $\hat{M}_n$ generates a measure $M^*$ on quadruples $(K, x)$ and $(K, y)$. The measure on $CW((0, 0), (n, 0))$ also generates $N^*$, a measure on quadruples of the same form. Both of these measures are linear combinations of a finite number of point masses on quadruples $(K, x)$ and $(K, y)$. By Theorem \[8.9\] we can find a $K''$ that works for any two choices of $(K, x)$ and $(K, y)$ and $(K, x')$ and $(K, y')$. The lemma follows from taking any coupling of $M^*$ and $N^*$, applying Theorem \[8.9\] and taking linear combinations.

9. **Proofs of results about random walks close to the Weyl chamber**

In this section we lay out our main theorems about the paths that are the image of pattern avoiding permutations. We show, in some very strong sense, a relationship between the distribution of a uniformly chosen pattern avoiding permutation and the distribution of a random walk in Weyl.

For any $k \in N$ we define the pseudo metric $d_k$ on paths of length $n$ by

$$
d_k(s, s') = \#\{i \in (k, n - k) : s(i) \neq s'(i)\}.
$$

Remember that for $x \in \mathbb{Z}^d$ we defined

$$
U(x) = \prod_{i < j} (x_i - x_j),
$$

which is harmonic for the random walk $s_\omega(t)$.

We start with the following argument that proves (after a minor alteration) that for any $x \in \text{Weyl}, T \in \mathbb{Z}$ and $l$

$$
\mathbb{P}(T, x)(\omega \in CW((T, x), (T_l, *)) \leq U(x)(2^l)^{-d(d - 1)/2}.
$$

Let $T^*_l$ be the minimum time $t$ greater than or equal to $T$ such that $s_\omega(t) \notin \text{Weyl}$ or $s_\omega(t) \in \text{Weyl}_{2^l}$. This is a stopping time. By the optional stopping time theorem and the fact that $U(x) = 0$ for all $x \in \partial\text{Weyl}$ we have that

$$
U(x) = U(T^*_l) = \sum_{x'} U(x') \mathbb{P}(T, x)(s_\omega(T^*_l) = x') \geq \mathbb{P}(T, x)(T^*_l \in \text{Weyl}_{2^l}) \min_{x' \in \text{Weyl}_{2^l}} U(x').
$$
As 

$$\text{for all } x' \in \text{Weyl}_2^l \text{ we have}$$

$$P(T, x) (T^*_l \in \text{Weyl}_2^l) \leq U(x)(2^l)^{d(d-1)/2}$$

Proposition 6.1 and results in [12] imply that the lower bound is within a constant factor of this upper bound.

This is not rigorous because $s_\omega$ changes in two coordinates every time that it changes. Thus we can have that

$$s_\omega(T^*_l) \notin \partial \text{Weyl} \cup \text{Weyl}_2^l$$

and $U(s_\omega(T^*_l)) < 0$. To account for this we need to bound

$$\sum_{x' : U(x') < 0} U(x') P(T, x) (s_\omega(T^*_l) = x')$$

from below. We give a bound that is (in absolute value) much smaller than $U(x)$.

All of the bounds in this section are some variant of this argument. We define a stopping time $T^*$ such that with very high probability either

1. $s_\omega(T^*) \in \text{Weyl}_2^l$, or
2. $|s_\omega(T^*)|$ is small.

Then we bound $P(s_\omega(T^*) \in \text{Weyl}_2^l)$ with the optional stopping time theorem.

Now we make the preceding argument rigorous and strengthen it. We define the sequence

$$T_l = \min \{ \inf \{ t : s_\omega(t) \in \text{Weyl}_2^l \}, [4.1^l] \}.$$

Let $B(t)$ be Brownian motion on the submanifold of $\mathbb{R}^d$ such that the sum of all the coordinates is zero. We choose $\gamma$ such that

$$P(B(1) \notin \text{Weyl}_2^l) < \gamma < 1.$$

**Lemma 9.1.** For any $l$ sufficiently large and any $t \leq 4.1^l$ and $x$ with $d(x, \text{Weyl}) \leq t^4$

$$P_{(t, x)} (\omega \in CW((t, x), (t + 4^l, *)) \cap T_l > t + 4^l) < \gamma$$

and

$$P_{(t, x)} (\omega \in CW^+((t, x), (t + 4^l, *)) \cap T_l > t + 4^l) < \gamma.$$  

**Proof.** For sufficiently large $l$, if $d(s_\omega(t), \text{Weyl}) \leq t^4$ and $s_\omega(t + 4^l) \in \text{Weyl}_{2^{l+1}}$ then $s_\omega(t + 4^l) \in \text{Weyl}_{2^l}$ and $T_l \leq t + 4^l$.

$$P_{(t, x)} (T_l > t + 4^l) \leq P(s_\omega(t + 4^l) \in \text{Weyl}_{2^{l+1}}).$$

By the convergence of random walk to Brownian motion

$$P_{(t, x)} (s_\omega(t + 4^l) \in \text{Weyl}_{2^{l+1}}) \leq P(B(1) \notin \text{Weyl}_2^l) + \epsilon < \gamma$$

for $\epsilon$ sufficiently small and for all large $l$. \[\blacksquare\]

**Lemma 9.2.** For $l$ sufficiently large, for all $j$, all $T < 4.1^l$, and all $x \in \text{Weyl}$,

$$P_{(T, x)} (\omega \in CW((T, x), (T, *)) \cap T_l - T \geq j4^l | s_\omega(T) = x) \leq \gamma^j,$$

and

$$P_{(T, x)} (\omega \in CW^+((T, x), (T, *)) \cap T_l - T \geq j4^l | s_\omega(T) = x) \leq \gamma^j.$$  

**Proof.** Note that $T_l \leq 4.1^l$ so we only need to check this for $j \leq (4.1/4)^l$. Thus this lemma follows from repeated applications of Lemma 9.1. \[\blacksquare\]
The following lemma is a consequence of standard moderate deviations bounds, see e.g. [22, Lemmas 5.1-2].

**Lemma 9.3.** There exist constants $\Theta, \theta > 0$ such that for all $j, l$,
\[ P((0,0) \left( \max_{0 \leq i \leq j2^l} |s_\omega(i)| \geq j2^l \right) \leq \Theta e^{-\theta j}. \]

Consequently,

**Lemma 9.4.** There exists $\beta \in (0, 1)$ such that for all $l$ sufficiently large, $T \leq 4.1^l$, $x \in \text{Weyl} \setminus \text{INT}(\text{Weyl}_0)$,
\[ P(T,x) \left( |s_\omega(T_l) - x| \geq j2^l \cap \omega \in \text{CW}((T,x),(T_l,*)) \right) \leq C\beta^j. \]

and
\[ P(T,x) \left( |s_\omega(T_l) - x| \geq j2^l \cap \omega \in \text{CW}^+(T,x),(T_l,*) \right) \leq C\beta^j. \]

**Proof.** If $|s_\omega(T_l) - x| > j2^l$ then either
1. $T_l - T > j4^l$ or
2. $T_l - T \leq j4^l$ and $|s_\omega(T_l) - x| > j2^l$.

From the first claim in Lemma [9.2] we have that for $l$ sufficiently large and for all $j$ and all $x \in \text{Weyl}$
\[ P(T,x) \left( \omega \in \text{CW}((T,x),(T_l,*)), T_l - T \geq j4^l \right) \leq \gamma^j. \]

Thus the first event happen with probability bounded by $\gamma^j$. By Lemma [9.3] the probability of the second event is bounded, uniformly in $x$, $j$, and $l$ by $Ce^{-cj}$ for some appropriate constants. Taking $\beta = \max(e^{-c}, \gamma)$ completes the proof of the first claim.

The calculation for $\text{CW}^+$ is done in the same manner. We just use the second part of Lemma [9.2] instead of the first.

Let $x \in \mathbb{Z}^2$ and $T < 4.1^l_0$. We define a sequence of stopping times, $R_l$. Let $R_l = T_l$ if
\[ \omega \in \text{CW}^+(T,x),(T_l,*). \]

Otherwise we set $R_l$ to be the smallest $r$ such that
\[ \omega \notin \text{CW}^+(T,x),(r,*). \]

Choose $L$ such that
\[ \sum_{\lambda \in \mathbb{N}} 9^L (2\lambda + 1)^{-1+d(d-1)/2}C\beta^{\lambda-1} \leq .01 \cdot (.95)^L, \]
and
\[ \sum_{m=L}^{\infty} 13^m(-1+d(d-1)/2)\gamma(1.025)^m \leq 1. \]

**Lemma 9.5.** For any $l_0 > L$, $x \in \partial \text{Weyl}_{2^{l_0}}$ and $T < 4.1^{l_0}$
\[ \sum_{y} |U(y)|P(T,x)(s_\omega(R_{l_0}+1) = y, R_{l_0+1} \neq T_{l_0+1}) \leq .01 \cdot (.95)^{l_0}U(x). \]

**Proof.** If $|y - x| \leq \lambda 2^{l_0}$ then for each $i < i'$
\[ |y_i - y_{i'}| \leq x_i - x_{i'} + \lambda 2^{l_0} \leq (2\lambda + 1)(x_i - x_{i'}). \]

If in addition $d(y, \text{Weyl}) \leq (4.1^{l_0+1})^4$ then for one $i, i'$ we have
\[ |y_i - y_{i'}| \leq d(y, \text{Weyl}) \leq (4.1^{l_0+1})^4. \]
Also for any \( \lambda \in \mathbb{N} \) by the portion of Lemma 9.4 for \( CW^+ \)
\[
\sum_{y: |y-x| \leq (\lambda-1)2^0, \lambda2^0} \mathbb{P}_{(T,x)}(s_\omega(R_{l_0+1}) = y, R_{l_0+1} \neq T_{l_0+1}) \leq C\beta^{-1}. 
\]

The last two inequalities are by (40) and (42).

Now we prove two slight variants of Lemma 9.5.

**Lemma 9.6.** For any \( l_0 > L, T < [4.1^{l_0}] \) and \( x \notin \text{Weyl}_{2^{l_0}} \)
\[
\sum_y |U(y)| \mathbb{P}_{(T,x)}(s_\omega(R_{l_0+1}) = y, R_{l_0+1} \neq T_{l_0+1}) \leq 2^{l_0} \max(|x|, 2^{l_0})^{-1+d(d-1)/2}. 
\]

**Proof.** The proof of this lemma is very similar to the proof of Lemma 9.5. Instead of bounding \( U(y) \) with the estimate \( |y_u - y_i| \leq |x_u - x_i| + \lambda2^{l_0+1} \) we use the estimate \( |y_u - y_i| \leq |x| + \lambda2^{l_0+1} \) to find that
\[
\sum_y |U(y)| \mathbb{P}_{(T,x)}(s_\omega(R_{l_0+1}) = y, R_{l_0+1} \neq T_{l_0+1}) 
\]
\[
\leq \sum_{\lambda \in \mathbb{N}} (4.1^{l_0+1})^4(|x| + \lambda2^{l_0+1})^{-1+d(d-1)/2}C\beta^{-1} 
\]
\[
\leq \max(|x|, 2^{l_0})^{-1+d(d-1)/2} (4.1^{l_0+1})^4 \sum_{\lambda \in \mathbb{N}} (1 + \lambda2^{l_0+1})^{-1+d(d-1)/2}C\beta^{-1} 
\]
\[
\leq 2^{l_0} \max(|x|, 2^{l_0})^{-1+d(d-1)/2}. 
\]

Let \( x \in \mathbb{Z}^d \) and \( T < 4.1^{l_0} \). We define a stopping time, \( \hat{R}_{l_0} \), by setting \( \hat{R}_{l_0} \) to be the smallest \( r > T \) such that
\[
d(s_\omega(r), \partial \text{Weyl}) \leq r^{-4} 
\]
if that is less than \( T_{l_0} \) and otherwise we set \( \hat{R}_{l_0} = T_{l_0} \).
Lemma 9.7. For any $l_0 > L$, $x \in \partial Weyl_{2l_0}$ and $T < 4.1^{l_0}$

$$\sum_{y} |U(y)|\mathbb{P}_{(T,x)}(s_\omega(\hat{R}_{l_0+1}) = y, \hat{R}_{l_0+1} \neq T_{l_0+1}) \leq 0.01 \cdot (0.95)^{l_0}U(x).$$

Proof. The proof is identical to Lemma 9.5. ■

Now we repeatedly apply these three lemmas.

Lemma 9.8. For any $l > l_0 > L$, $x \in \partial Weyl_{2l_0}$, $T < 4.1^{l_0}$, and $|x| \leq 2T$,

$$(43) \quad \sum_{x'} U(x')\mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_1, x'))) \leq KU(x)$$

where $K = 1 + \prod_{j=0}^{\infty}(1 + 0.01 \cdot (0.95)^{l_0+j})$

Proof. We prove the lemma by induction. Set

$$H_{l_0,l}(T, x) = \sum_{x'} U(x')\mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_1, x'))) \quad (44)$$

We will inductively show that for all $k > l_0$

$$H_{l_0,k}(T, x) \leq U(x) \prod_{j=0}^{k-l_0-1} (1 + 0.02 \cdot (0.95)^{l_0+j})$$

which implies the lemma.

Observe that, since $R_k$ is a bounded stopping time and $U(s_\omega(t))$ is a martingale, it follows from the optional stopping theorem that for each $k > l_0$,

$$(45) \quad U(x) = H_{l_0,k}(T, x) + \sum_{y} U(y)\mathbb{P}_{(T,x)}(s_\omega(R_k) = y, R_k \neq T_k).$$

Taking $k = l_0 + 1$ and applying Lemma 9.5, the sum in Equation (45) is at most $0.01(0.95)^{l_0}U(x)$ and

$$H_{l_0,l_0+1}(T, x) \leq U(x)(1 + 0.01(0.95)^{l_0}) \leq U(x)(1 + 0.02(0.95)^{l_0}).$$

This establishes Equation (44) in the case $k = l_0 + 1$.

To extend this to all $k > l_0$, we take the difference of Equation (45) for consecutive values of $k$ to to find that

$$H_{l_0,k+1}(T, x) - H_{l_0,k}(T, x) = \sum_{y} U(y)\mathbb{P}_{(T,x)}(s_\omega(R_k) = y, R_k \neq T_k) - \mathbb{P}_{(T,x)}(s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1})$$

Decomposing $(s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1})$ based on whether the path exits $CW^+$ before or after $T_k$ we see that, almost surely under $\mathbb{P}(\cdot | s_\omega(T) = x)$,

$$(s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1}) = (s_\omega(R_k) = y, R_k \neq T_k) \cup (s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1}, R_k = T_k),$$

and the union is disjoint. Consequently,

$$H_{l_0,k+1}(T, x) - H_{l_0,k}(T, x) = -\sum_{y} U(y)\mathbb{P}_{(T,x)}(s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1}, R_k = T_k)$$

$$= -\sum_{y} \sum_{y'} U(y)\mathbb{P}_{(T,x)}(s_\omega(R_{k+1}) = y, s_\omega(R_k) = y', R_{k+1} \neq T_{k+1}, R_k = T_k).$$
The proof is virtually identical to Lemma 9.8. Proof. Let

\[ K = \left| \left( T_k \right) \right| \]

so by Lemma 9.6 the sum is at most

\[ 2^L \left( |x| + 2(41) \right)^{k(-1+2(d-1)/2)\gamma(1.025)^{k-1}} \]

For each \( y' \) for which the contribution is not zero and \( y' \in \partial Weyl_{10}^0 \) we get

\[ \sum_y |U(y)| \mathbb{P}_{(T,x)}(s_\omega(R_k) = y', s_\omega(R_{k+1}) = y, R_k = T_k, R_{k+1} \neq T_{k+1}) \leq \sum_y |U(y)| \mathbb{P}_{(R_k, y')} (s_\omega(R_{k+1}) = y, R_{k+1} \neq T_{k+1}) \mathbb{P}_{(T,x)}(s_\omega(R_k) = y') \leq (.01)(.95)^k U(y') \mathbb{P}_{(T,x)}(s_\omega(R_k) = y') \]

When we sum over all \( y' \) we get at most

\[ (.01)(.95)^k H_{l_0, k}(T, x) \]

Combining these two estimates we get

\[ H_{l_0, k+1}(T, x) - H_{l_0, k}(T, x) \leq (.01)(.95)^k H_{l_0, k}(T, x) + 2^L \left( |x| + 2(41) \right)^{k(-1+2(d-1)/2)\gamma(1.025)^{k-1}} \]

Solving this first order linear recurrence with variable coefficients, and using that \( U(x) \geq 2^L \) and \( |x| \leq 4.1^{l_0} \), gives the bound

\[ H_{l_0, n}(T, x) \leq \left( \prod_{k=l_0}^{n-1} \left( 1 + (.01)(.95)^k \right) \right) \left( U(x) + \sum_{m=l_0}^{n-1} 2^L \left( |x| + 2(41) \right)^{m(-1+2(d-1)/2)\gamma(1.025)^{m-1}} \right) \leq \left( 1 + \prod_{k=0}^{\infty} (1 + (.01)(.95)^{l_0+k}) \right) U(x), \]

as desired.

For any \( l' > L, T < 4.1^L, |x| \leq T \) and \( x \in \partial Weyl_{2L} \) let \( \hat{E}_1(l', T, x) \) be the event that

1. \( \omega \in CW^+(T, x), (T_l, \ast) \)
2. there exists \( t \in (T_{l-1}, T_l) \) such that \( d(s_\omega(t), \partial Weyl) \leq t^4 \) and
3. \( d(s_\omega(t), \partial Weyl) > t^4 \) for all \( t \in (T_l, T_l) \).

Lemma 9.9. For any \( l > l_0 > L, T \geq [4.1^{l_0}] \) and \( x \) with \( |x| \leq 2T \) and \( x \notin \text{Weyl}_{2l_0} \)

\[ \sum_{x'} U(x') \mathbb{P}_{(T,x)}(\omega \in CW^+(T, x), (T_l, x')) \leq K(2T)^{d(1-\gamma)/2} \]

where \( K = \prod_{j=0}^{\infty} (1 + .02 \cdot (.95)^{l_0+j}) \).

Proof. The proof is virtually identical to Lemma 9.8.

Lemma 9.10. For any \( T < 4.1^L, |x| \leq 2T, x \in \partial Weyl_{2L} \) and \( l \geq l' > L \), we have that

\[ \sum_y U(y) \mathbb{P}_{(T,x)}(\omega \in \hat{E}_1(l', T, x), s_\omega(T_l) = y) \leq .04K^2 \cdot (.95)^{l'-1} U(x). \]
Proof. First we apply Lemma 9.8 to get

\[ \sum_{x'} U(x') \mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_{l-1}, x'))) \leq KU(x). \]  \hfill (48)

We define two stopping times. First let \( \hat{T}_1 \) be the minimum of \( t > T_{l-1} \) such that

\[ d(s_\omega(t), \partial \text{Weyl}) \leq t^4 \quad \text{or} \quad s_\omega(t) \in \text{Weyl}_{2'} \quad \text{or} \quad 4.1'. \]

Second let \( \hat{T}_2 \) be the minimum of \( t > T_{l-1} \) such that

\[ d(s_\omega(t), \text{Weyl}) \geq t^4 \quad \text{or} \quad s_\omega(t) \in \text{Weyl}_{2'} \quad \text{or} \quad 4.1'. \]

If \( \hat{E}_1(l', T, x) \) occurs then the first stopping time is achieved by the first condition. Let \( F_1 \) be the event that the first stopping time is achieved by the first condition.

\[
\sum_z U(z) \mathbb{P}_{(T,x)}(\omega \in F_1, s_\omega(\hat{T}_1) = z)
\]

\[
= \sum_{x'} \sum_z \mathbb{P}(\omega \in F_1, s_\omega(\hat{T}_1) = z \mid \omega \in CW^+((T, x), (T_{l-1}, x')))
\]

\[
= \sum_{x' \not\in \partial \text{Weyl}_{2'}-1} \mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_{l-1}, x')) \mathbb{P}(\omega = z \mid \omega \in CW^+((T, x), (T_{l-1}, x')))
\]

\[
+ \sum_{x' \in \partial \text{Weyl}_{2'}-1} \mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_{l-1}, x')) \mathbb{P}(\omega = z \mid \omega \in CW^+((T, x), (T_{l-1}, x')))
\]

\[
\leq \mathbb{P}(T_{l-1} = [4.1'^{-1}] \cdot \sup U(s_\omega(\hat{T}_1))\right)
\]

\[
+ \sum_{x' \in \partial \text{Weyl}_{2'}-1} \mathbb{P}_{(T,x)}(\omega \in CW^+((T, x), (T_{l-1}, x')))(.01(95)^{l'-1}U(x'))
\]

\[
\leq \gamma^{.025l'^{-1}} (2 \cdot 4.1'^d (d-1)/2 + .01K \cdot (.95)^{l'-1}U(x)
\]

\[
\leq .02K \cdot (.95)^{l'-1}U(x).
\]

The first equality is the decomposition of the event based on the value \( s_\omega(T_{l-1}) \). The second half of the first inequality comes from Lemma 9.7. The second inequality comes from (48).

Let \( F_2 \) be the event that the second stopping time is achieved by the first condition. A similar calculation (using Lemma 9.5 instead of Lemma 9.8) shows that

\[
\sum_z U(z) \mathbb{P}_{(T,x)}(\omega \in F_2, s_\omega(\hat{T}_1) = z) \leq .02K \cdot (.95)^{l'-1}U(x).
\]
Now let \( l' = l \). If \( \omega \in \hat{E}_1(l', T, x) \) then \( \omega \in F_1 \setminus F_2 \). As \( U \) is harmonic then \( U(s_\omega(\hat{t})) \) is a martingale. As \( \hat{T}_2 \) is a stopping time
\[
|\sum_{x'} U(x') \mathbb{P}_{(T, x)}(\omega \in CW^+((T, x), (\hat{T}_2, x')))|
\]
\[
= |\sum_{x''} U(x'') \mathbb{P}_{(T, x)}(\omega \in CW^+((T, x), (\hat{T}_1, x'')))|
\]
\[
\leq |\sum_{z} U(z) \mathbb{P}_{(T, x)}(\omega \in F_1, s_\omega(\hat{T}_1) = z)|
+ |\sum_{z} U(z) \mathbb{P}_{(T, x)}(\omega \in F_2, s_\omega(\hat{T}_1) = z)|
\]
\[
\leq .04K \cdot (.95)^{l'-1} U(x).
\]
Thus the lemma is proven for \( l = l' \).

For \( l > l' \) we need to another decomposition of the path based on where it is at \( T_l \) and apply Lemma 9.8.

**Lemma 9.11.** There exists a function \( H(L) = o(1) \) such that for any \( L, l > L, T < 4.1^L, |X| \leq 2T \) and \( x \in \partial \text{Weyl}_L \),
\[
(49) \sum_y U(y) \mathbb{P}_{(T, x)}(\omega \in CW^+((T, x), (T_l, y)) \setminus CW^-((T, x), (T_l, y))) \leq H(L) U(x).
\]

**Proof.** Define the sum on the left hand side of \((49)\) to be \( \tilde{M} \). Let \( J \) be the largest \( j \) such that \( T_j = 4.1^j \) and let \( M \) be the largest \( t \) such that Petrov(\( t \))^C occurs. If there is no such \( j \) (resp. \( t \)) then we say \( J = \infty \) (\( M = \infty \)). If
\[
\omega \in CW^+((T, x), (T_l, y)) \setminus CW^-((T, x), (T_l, y))
\]
then either
\begin{enumerate}[label=(\arabic*)]
  
  \item \( J = M = \infty \),
  
  \item \( T_{J+1} > M \) or
  
  \item \( M \geq T_{J+1} \).
\end{enumerate}

We define the events \( E_1, E_2 \) and \( E_3 \) to be events that
\[
\omega \in CW^+((T, x), (T_l, y)) \setminus CW^-((T, x), (T_l, y))
\]
and (respectively) the first, second or third of those possibilities occurs. For \( i = 1, 2, 3 \) we write
\[
(50) \tilde{M}_i = \sum_y U(y) \mathbb{P}_{(T, x)}(1_{E_i}, \omega \in CW^+((T, x), (T_l, y)) \setminus CW^-((T, x), (T_l, y))).
\]

Note that
\[
E_1 \subset \bigcup_{l' \in (L, L)} \hat{E}_1(l', T, x) \subset CW^+((T, x), (T_l, *)).
\]
As \( x \in \partial \text{Weyl}_{2L} \) by the above and Lemma 9.10 we can bound
\[
\tilde{M}_1 \leq \sum_{l' \in (L, L)} \sum_y U(y) \mathbb{P}_{(T, x)}(\hat{E}_1(l', T, x), s_\omega(T_l) = y)
\]
\[
\leq \sum_{l' \in (L, L)} .04K^2 \cdot (.95)^{l'-1} U(x)
\]
\[
\leq K^2(.95)^L U(x).
\]
If \( E_2 \cap \{ J = j \} \) occurs then
\begin{enumerate}[label=(\arabic*)]
  
  \item \( \omega \in CW^+((T_{J+1}, *), (T_l, *)) \)
\end{enumerate}
(2) \( s_\omega(T_{j+1}) \in \partial \text{Weyl} \) and 
(3) \( T_{j+1} < \lfloor 4.1^{j+1} \rfloor \).

Thus

\[
U(s_\omega(T_{j+1})) \leq (2 \cdot 4.1)^{(j+1)d(d-1)/2}
\]

and by Lemma 9.2 for \( j > L \) we get that

\[
\sum_{y'} U(y') \mathbb{P}_{(T,x)}(E_2, J = j, s_\omega(T_{j+1}) = y') \leq \mathbb{P}_{(T,x)}(E_2, J = j) \leq \mathbb{P}_{(T,x)}(T_j = \lfloor 4.1^j \rfloor) \leq \gamma^{1.025^{j-1}}.
\]

We then have,

\[
\sum_{y''} U(y'') \mathbb{P}_{(T,x)}(E_2, J = j, s_\omega(T_1) = y'') \leq K \sum_y U(y) \mathbb{P}_{(T,x)}(E_2, J = j, s_\omega(T_{j+1}) = y) \leq K(2 \cdot 4.1)^{(j+1)d(d-1)/2} \sum_{y'} \mathbb{P}_{(T,x)}(E_2, J = j, s_\omega(T_{j+1}) = y') \leq K(2 \cdot 4.1)^{(j+1)d(d-1)/2} \gamma^{1.025^{j-1}} \leq (.95)^j
\]

The first inequality comes from Lemma 9.8. The second comes from (51) and the third from (52). The fourth holds because \( j \geq L \) and \( L \) is large. Thus

\[
\tilde{M}_2 = \sum_{j \in \{L, L\}} \sum_{y''} U(y'') \mathbb{P}_{(T,x)}(E_2, J = j, s_\omega(T_1) = y'') \leq \sum_{l' \in \{L, L\}} (.95)^j \leq 20(.95)^L
\]

The bound for \( \tilde{M}_3 \) is similar to the bound for \( \tilde{M}_2 \). If \( E_3 \cap \{M = m\} \) let \( J^* \) be such that \( T_{J^*+1} < M \leq T_{J^*+2} \) occurs. Then

\[
(1) \ \omega \in CW^+((T_{J^*+2},*),(T_1,*)), \\
(2) \ T_{J^*} < \lfloor 4.1^{J^*+2} \rfloor \text{ and} \\
(3) \ s_\omega(T_{J^*+2}) \in \partial \text{Weyl}_{2,J^*+2}.
\]

Also \( 2^{J^*+1} \leq m \leq 4.1^{J^*+2} \). So for any fixed \( m \) there are at most \( \log(m) \) possibilities for \( J^* \). From this we get that \( T_{J^*+2} \leq m^3 \). This implies that \( U(T_{J^*+2}) \leq (2m^3)^{d(d-1)/2} \) and there are at most \( (2m^3)^{d(d-1)/2} \) possibilities for \( T_{J^*+2} \).
For any $m$

\[
\sum_j \sum_y U(y') \mathbb{P}_{(T,x)}(E_3, J^* = j, M = m, s_\omega(T_i) = y')
= \sum_j \sum_z \sum_y U(y') \mathbb{P}_{(T,x)}(E_3, J^* = j, M = m, s_\omega(T_i) = y', s_\omega(T_{j+2}) = z)
= \sum_j \sum_z \sum_y U(y') \mathbb{P}_{(T,x)}(E_3, J^* = j, s_\omega(T_i) = y' \mid s_\omega(T_{j+2}) = z, M = m)
\leq \sum_j \sum_z \sum_y U(y') \mathbb{P}_{(T,x)}(CW^+((T_{j+2}, z), (T_i, y')) \mid s_\omega(T_{j+2}) = z) \mathbb{P}(\text{Petrov}(m)^C)
\leq \sum_j \sum_z KU(z)e^{-m^\delta}
\leq \log(m)(2m^3)d(d-1)/2K(2m^3)d(d-1)/2e^{-m^\delta}
\leq e^{-m^\delta/2}.
\]

The first inequality comes from (1) and the Markov property of $S$. The second comes from Lemma 9.8 and Lemma 4.7. The third inequality comes from the estimates in the above paragraph.

So we get

\[
\tilde{M}_3 \leq \sum_{m \in [L, l)} \sum_j \sum_y U(y') \mathbb{P}_{(T,x)}(E_3, J^* = j, M = m, s_\omega(T_i) = y')
\leq \sum_{m \in [L, l)} e^{-m^\delta/2}
\leq 20(.95)^L.
\]

Combining these three bounds we get

\[
\tilde{M} \leq \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3
\leq K^2(.95)^L U(x) + 20(.95)^L + 20(.95)^L
\leq 50K^2(.95)^L U(x)
\]

Setting $H(L) = 50K^2(.95)^L$ completes the proof. \hfill \blacksquare

9.1. **Symmetric versions of these sets of paths.** Choose $L_f$ to be the smallest integer such that

\[
4^{L_f} > n^{1-.08/d(d-1)}.
\]

Also let $\delta = .02/d(d-1) < .01$. Then $2^{L_f} > n^{5-2\delta} > n^{49}$. Thus

For any $j, k < n/2$ and $x, y \in \mathbb{Z}^d$ define $SCW^-((j, x), (n - k, y))$ to be any path such that

1. $s_\omega(i) \in CW^-((j, x), ([n/2], *))$
2. $s_\omega(n - i) \in CW^-((k, y), ([n/2], *))$

$SCW^+((j, x), (n - k, y))$ and $SCW^{++}((j, x), (n - k, y))$ are defined in an analogous way.

**Lemma 9.12.** There exist $C$ such that for all $n$, all $T, T^* > n^{5-2\delta}$, $T + T^* < n/2$ and for all $x, y \in \text{Weyl}_{n, 5-2\delta}$ such that $|x|, |y| \leq n^{5-\delta}$

\[
\mathbb{P}_{(T,x)}(\omega \in SCW^{++}((T, x), (n - T^*, y)) \setminus SCW^-(T, x), (n - T^*, y)))
\leq CU(x)U(y)n^{-(d-1)/2} \cdot n^{-(d-1)/2} \cdot n^{-\delta}.
\]
Proof. The conditions on \(x\) and \(y\) imply that \(U(x), U(y) \geq n^{(5-2\delta)d(d-1)/2}\) and

\[ U(x)U(y)n^{-d(d-1)/2} \geq n^{-2(d-1)d\delta} \geq Cn^{-0.04}. \]

So it sufficient to show that \([5.4]\) is less than \(Cn^{-0.05} \cdot n^{-(d-1)/2}\).

There are two ways a path could be in

\[ SCW^+((T, x), (n - T^*, y)) \setminus SCW^-((T, x), (n - T^*, y)). \]

First it could fail a Petrov condition. The values of \(T\) and \(T^*\) are such that probability of this is at most \(e^{-n^c}\) for some \(c > 0\). The other possibility is that the path gets close to \(\partial\text{Weyl}\) but never goes too far away from Weyl. To bound the probability of this we break this event up into parts. First we note that either there exists \(t \leq n/2\) such that \(d(s_\omega(t), \partial\text{Weyl}) \leq t^4\) or there exists \(t \geq n/2\) such that \(d(s_\omega(t), \partial\text{Weyl}) \leq (n - t)^{-4}\). We consider the first possibility. The latter case is identical.

Let \(T_1\) be the first time after \(T\) where \(d(s_\omega(T_1), \partial\text{Weyl}) \leq t^4\). By assumption \(T_1 \leq n/2\). Also let \(T_2\) be the first time after \(T_1\) such that \(d(s_\omega(T_2), \partial\text{Weyl}) \geq n^{45}\). We will split the case up into two cases. The first is that \(T_2 < n - T^* - .01n\). Start a path at \(s_\omega(T_1)\). By a martingale argument (restricting the path to the dimension where it is initially closest to \(\partial\text{Weyl}\)) the probability that it hits \(\text{Weyl}_{4.5}\) before the distance to Weyl is at least \(n^4\) is at most \(2n^{-0.05}\). Now for any value of \(s_\omega(T_2)\) the probability that the path hits \(y\) at time \(n - T^*\) is comparable to \(\mathbb{P}(s_\omega(n - T^* - T_2) = x - y)\).

Thus the probability that this happens is at most

\[ C2n^{-0.05}n^{-(d-1)/2}. \]

Otherwise the path spent an interval of time of at least \(4n\) without venturing more than \(n^{45}\) from \(\partial\text{Weyl}\). The probability of this is at most \(e^{-n^c}\) for some \(c > 0\). We can see this by breaking up the time interval into pieces of length \(n^9\). In each interval the path has a bounded from below positive probability of straying more than \(n^{45}\) from \(\partial\text{Weyl}\). The bound on the probability of each event is independent of whether the preceding events occurred.

Combining these estimates proves the lemma. 

\[ \blacksquare \]

Lemma 9.13. There exist \(C''\) such that for all \(n, R \in [n/2, n], T \leq n/4\) and for all \(x, y \in \text{Weyl}_{n, 5-2\delta}\) such that \(|x|, |y| \leq n^{5-\delta}\),

\[ \mathbb{P}(T, x)(\omega \in \text{CW}((T, x), (R, y))) \geq C''U(x)U(y)n^{-(d-1)/2} \cdot n^{-(d-1)/2}. \]

There also exists \(C'''\) such that for all \(n, R \in [n/2, n]\) and for all \(x, y \in \text{Weyl}_{n, 5-2\delta}\) such that \(|x|, |y| \leq n^{5-\delta}\)

\[ \mathbb{P}(T, x)(\omega \in \text{CW}((T, x), (R, y))) \leq C'''U(x)U(y)n^{-(d-1)/2} \cdot n^{-(d-1)/2}. \]

Proof. Define \(\tau_x\) to be the minimum time \(t\) such that \(x + s_\omega(t) \notin \text{Weyl}\).

Find \(b > 0\) and \(V \subset \mathbb{R}^d\) satisfying the following. Let \(V_n = \mathbb{Z}^d \cap \sqrt{n}V\). For all \(r, s \in V_n\) we have

\[ \mathbb{P}(r + s_\omega(.1n) = s) > bn^{-(d-1)/2}. \]

Get \(a, C\) from Proposition 6.1 and \([12]\) Lemma 29. Find \(u\) such that \(Ce^{-au^2} < b/2\). Let \(V'\) be a translate of \(V\) such that

\(d(V', \partial\text{Weyl}) > u\).

Let \(V'_n = \mathbb{Z}^d \cap \sqrt{n}V'\). Then by Proposition 6.1 and Lemma 29 of \([12]\) for all \(n\) sufficiently large and all \(r, s \in V'_n\) we have

\[ \mathbb{P}(r + s_\omega(.1n) = s, \tau_r < .1n) < Ce^{-au^2} n^{-(d-1)/2} < (b/2)n^{-(d-1)/2}. \]

Then for all \(n\) sufficiently large and all \(r, s \in V'_n\) we have

\[ \mathbb{P}(r + s_\omega(.1n) = s, \tau_r \geq .1n) > (b/2)n^{-(d-1)/2}. \]
Let $x, y \in \text{Weyl}_{n,5-\delta}$. By Proposition 6.1 and Lemma 20 of [12] we have that there exists $C' > 0$ such that for all $R \in [n/2, n]$

$$\mathbb{P}(\tau_x \geq .45n - (n - R)/2, x + s_\omega(.45n - (n - R)/2) \in V'_n) \geq C'U(x)n^{-d(d-1)/4}$$

and

$$\mathbb{P}(\tau_y \geq .45n - (n - R)/2, y + s_\omega(.45n - (n - R)/2) \in V'_n) \geq C'U(y)n^{-d(d-1)/4}.$$ 

Putting this together we have that for all $x, y \in \text{Weyl}_{n,5-\delta}$

$$\mathbb{P}(\tau_x > R, x + s_\omega(R) = y) \geq \mathbb{P}(\tau_x \geq .45n - (n - R)/2, x + s_\omega(.45n - (n - R)/2) \in V'_n) 
\cdot \mathbb{P}(\tau_y \geq .45n - (n - R)/2, y + s_\omega(.45n - (n - R)/2) \in V'_n) 
\cdot \min_{r,s \in V'_n} \mathbb{P}(\tau_r \geq .1n, r + s_\omega(.1n) = s) 
\geq C'U(x)n^{-d(d-1)/4} \cdot C'U(y)n^{-d(d-1)/4} \cdot (b/2)n^{-(d-1)/2} 
\geq C''U(x)U(y)n^{-(d-1)/2} \cdot n^{-d/2}.$$ 

As this holds for all $n$ sufficiently large we can find a constant for which it holds for all $n$. The upper bound follows from Proposition 6.1 and Lemma 28 of [12].

**Lemma 9.14.** For any $\epsilon > 0$ there exists $l$ such that if $v, v' \in \text{Weyl}_{2}$ and $T, T^* \leq (4.1)^l$ then for any $n$ sufficiently large

$$\mathbb{P}_{(T,v)}\left(\text{SCW}^-((T,v),(n-T^*,v')) \mid \text{SCW}^++((T,v),(n-T^*,v'))\right) > 1 - \epsilon.$$ 

This implies

$$\mathbb{P}_{(T,v)}\left(\text{SCW}^-((T,v),(n-T^*,v')) \mid \text{CW}((T,v),(n-T^*,v'))\right) > 1 - \epsilon.$$ 

**Proof.** Recall $L_f$ defined in (53). If

$$\omega \in \text{SCW}^+((T,v),(n-T^*,v')) \setminus \text{SCW}^-((T,v),(n-T^*,v'))$$

then either

1. $T_{L_f} > .01n$
2. $T'_{L_f} > .01n$
3. $|s_\omega(T_{L_f})| > n^{5-\delta}$ or
4. $|s_\omega(T'_{L_f})| > n^{5-\delta}$

or neither of those happen but at least one of the following occurs.

5. $\omega \in \text{SCW}^+((T,v),(n-T^*,v')) \setminus \text{CW}^-((T,v),(T_{L_f},*))$
6. $\omega \in \text{SCW}^+((T,v),(n-T^*,v')) \setminus \text{CW}^-((T^*,v'),(T_{L_f},*))$
7. $\omega \in \text{SCW}^+((T,v),(n-T^*,v')) \setminus \text{SCW}^-((T_{L_f},*),(n-T^*,v'))$ and $\omega \in \text{CW}^-((T,v),(T_{L_f},*))$

and

$$\omega^* \in \text{CW}^-((T^*,v'),(T'_{L_f},*)).$$ 

Thus it is sufficient to show that each of these seven sets have probability that is small in comparison with the probability of $\text{CW}^-((T,v),(n-T^*,v'))$. By the lower bound in Lemma 9.13 this is of order $U(v)U(v')n^{-(d-1)/2}n^{-(d-1)/2}$.

First we show that the events in (1) and (2) and (3) and (4) have probability at most $Ce^{-n^n}$ for some $C$ and $\eta > 0$. The probability of the event in (1) is at most $Ce^{-n^n}$ by Lemma 9.2. The probability of the event in (3) is at most $Ce^{-n^n}$ by Lemma 9.4. The argument for the events in (2) and (4) are the same by symmetry.
Next we bound the probability of the event in \([5]\). We break \(\partial \text{Weyl}_{L_f}^i\) into disjoint sets
\[
D_i = \{ x \in \partial \text{Weyl}_{L_f}^i : U(x)2^{-L_fd(d-1)/2} \in [i, i+1) \}
\]
for \(i \in \mathbb{N}\).

For each \(i\) and \(j\) and \(x \in D_i\) and \(y \in D_j\) by the upper bound in Lemma \([9,13]\)
\begin{equation}
\Pr(T_{L_f}x)(\omega \in \text{CW}((T_{L_f}, x), (n - T_{L_f}^*, y))) \leq C(i + 1)2^{L_fd(d-1)/2}j2^{L_fd(d-1)/2}n^{-(d-1)/2}n^{-d(d-1)/2}.
\end{equation}
By Lemma \([9,12]\) and the second half of Lemma \([9,13]\) we get
\begin{equation}
\Pr(T_{L_f}x)(\omega \in \text{SCW}^+(T_{L_f}, x), (n - T_{L_f}^*, y))) \leq C'_{ij}2^{L_fd(d-1)/2}j2^{L_fd(d-1)/2}n^{-(d-1)/2}n^{-d(d-1)/2}.
\end{equation}
This bound is uniform over all \(x \in D_i\) and \(y \in D_j\) and value of \(T_{L_f}\) and \(T_{L_f}^*\).

By Lemma \([9,11]\) we get
\[
\sum_i \Pr(T_{L_f}x)(\omega \in \text{CW}^+(T, v), (n - T_{L_f}^*, *))) \leq C'(i)^22^{L_fd(d-1)/2} \leq tU(v)
\]
and by Lemma \([9,11]\) and Lemma \([9,8]\)
\[
\sum_i \Pr(\omega^* \in \text{CW}^+(T^*, w), (T_{L_f}^*, *)) \in \text{SCW}^+(T_{L_f}^*, v))\Pr(T_{L_f}x)(\omega \in \text{CW}^+(T, v), (n - T_{L_f}^*, v')) \leq C'(i)^22^{L_fd(d-1)/2} \leq C'(i)^2U(v).
\]

We can sample the paths with \(s_\omega(T) = v\) and \(s_\omega(n - T^*) = v'\) as follows. First we sample \(\omega^*\) with \(s_\omega(T^*) = v'\) and find \(T_{L_f}^*\). Then we can (independently) sample \(\omega\) from time \(T\) to \(n - T_{L_f}^*\). If \(s_\omega(T^*) = s_\omega(n - T_{L_f}^*)\) then we concatenate the paths.

\begin{align*}
\Pr(T_{L_f}x)(\omega \in \text{SCL}(T, v), (n - T^*, v')) & \leq \sum_{i,j} \Pr(T_{L_f}x)(T_{L_f} = t, T_{L_f}^* = t^*, \omega \in \text{CW}^+(T, v), (n - T^*, v)) \\
& \leq \sum_{i,j} \Pr(T_{L_f}x)(\omega \in \text{CW}^+(T, v), (n - T^*, v')) \leq \sum_{i,j} \Pr(T_{L_f}x)(\omega \in \text{SCL}(T, v), (n - T^*, v')) \leq \sum_{i,j} \Pr(T_{L_f}x)(\omega \in \text{SCL}(T, v), (n - T^*, v'))
\end{align*}
The last line comes from (56). Let

$$
\Pi = CW^+((T, v), (T_{L_f}, *)) \setminus CW^-((T, v), (T_{L_f}, *))
$$

and

$$
\Pi^* = CW^+((T^*, v'), (T_{L_f}^*, *)) \setminus CW^-((T^*, v'), (T_{L_f}^*, *)).
$$

Using the independence of \( \omega \) and \( \omega^* \) we find that

$$
P_{(T,v)}(\omega \in SCW^+((T, v), (n - T^*, v')) \setminus SCW^-(((T, v), (n - T^*, v')))
\leq C'n^{-d(d-1)/2}n^{-(d-1)/2} \sum_i j2^{L_f(d-1)/2} P(\omega \in \Pi, s_{\omega}(T_{L_f}) \in D_i)
\leq C''eU(v)U(v')n^{-d(d-1)/2}n^{-(d-1)/2}
$$

We use independence to calculate the probability of

$$
\omega \in CW^+((T, v), (T_{L_f}, *)) \cap s_{\omega}(T_{L_f}) \in D_i
$$

and

$$
\omega^* \in CW^+((T^*, v'), (T_{L_f}^*, *)) \cap s^*_{\omega^*}(T_{L_f}^*) \in D_j
$$

and the maximum of \( x \in D_i \) and \( y \in D_j \) of the probability of

$$
SCW^+((T_{L_f}, x), (n - T_{L_f}^*, y)) \setminus SCW^-(((T_1, *), (n - T_{L_f}^*, *)))
$$

and summing up over \( i \) and \( j \) as before.

By Lemma 9.11 we have

$$
\sum_i (i + 1)2^{L_f(d-1)/2} P(T, v)(s_{\omega}(T_{L_f}) \in D_i, \omega \in CW^+((T, v), (T_{L_f}, *)) \leq 2U(v)2^{-L_f(d-1)/2}
$$

and the maximum of \( x \in D_i \) and \( y \in D_j \) of the probability of

$$
SCW^+((T_{L_f}, x), (n - T_{L_f}^*, y)) \setminus SCW^-(((T_1, *), (n - T_{L_f}^*, *)))
$$

is at most

$$
C(i + 1)2^{L_f(d-1)/2} \sum_j (j + 1)2^{L_f(d-1)/2} P(s^*_{\omega^*}(T_{L_f}) \in D_j, \omega \in CW^+((T^*, v'), (T_{L_f}^*, *)) \leq 2U(v')2^{-L_f(d-1)/2}
$$

which is small in comparison with the probability of \( CW((T, v), (n - T^*, v')) \) and thus in comparison with \( SCW^+((T, v), (n - T^*, v')) \).

The final inequality follows because

$$
CW((T, v), (n - T^*, v')) \subset SCW^+((T, v), (n - T^*, v'))
$$

\[\square\]

**Corollary 9.15.** For any \( \epsilon > 0 \) there exists \( K \) and \( l \) such that if \( v, v' \in \text{Weyl}_2 \) and \( T, T^* \leq (4.1)^l \) then there exists \( K \) such that for any \( n \) sufficiently large

$$
d_K(SCW^-((T, v), (n - T^*, v'))), CW((T, v), (n - T^*, v')) < \epsilon.
$$
Proof. This follows from Lemma 8.11 and Lemma 9.14 as follows. By Lemma 8.11 for any \( \epsilon > 0 \) we can find \( M \) such that

\[
d_M(CW((T, v), (n - T^*, v')), CW((0, 0), (n, 0))) < \epsilon.
\]

By Lemma 9.14 for any \( \epsilon > 0 \) we can find \( M' \) such that

\[
d_{M'}(SCW^-((T, v), (n - T^*, v')), CW((0, 0), (n, 0))) < \epsilon.
\]

Putting \( K = \max\{M, M'\} \) and using the triangle inequality proves the lemma. \( \blacksquare \)

10. Appendix: Traceless Hermitian Brownian Motion

In this section we recount some elementary facts about traceless Hermitian Brownian motion an its connection to non-intersecting paths. These results are well-known without the traceless condition and the transfer of the traceless case is straightforward.

Let \( \{H_{ii}\}_{i=1}^d \) be standard Brownian motions and let \( \{H_{ij}\}_{1 \leq i < j \leq d} \) be independent standard complex Brownian motions. For \( j < i \), let \( H_{ij} = \bar{H}_{ji} \). Hermitian Brownian motion is the matrix valued process \( H = (H_{ij})_{i,j=1}^d \). Traceless Hermitian Brownian motion can then be constructed by projecting \( H \) onto the space of matrices with trace equal to 0. In particular, if we define

\[
H^0 = H - \frac{Tr(H)}{d} I,
\]

where \( I \) is the identity matrix, the \( H^0 \) is a traceless Hermitian Brownian motion (independent of \( Tr(H) \)) whose distribution is the same as the distribution of \( H \) conditioned to have trace equal to 0 for all time. Consequently, on the level of eigenvalues the projection onto traceless matrices results in each eigenvalue being shifted by the same amount.

The equivalent relation holds for \( d \)-dimensional Brownian motion conditioned on its coordinates summing to 0. Indeed, if we let \( v_1, v_2, \ldots, v_d \) be an orthonormal basis for \( \mathbb{R}^d \) such that \( v_1 = d^{-1/2}(1, 1, \ldots, 1) \). Then we can standard Brownian motion on \( \mathbb{R}^d \) as

\[
B = \sum_{i=1}^d B_i v_i
\]

where \( B_1, \ldots, B_d \) are independent, standard, one dimensional Brownian motions. Conditioning the coordinates of \( B \) to sum to 0 is equivalent to projecting onto the subspace orthogonal to \( v_1 \), so that

\[
B^0 = B - B_1 v_1 = \sum_{i=2}^d B_i v_i
\]

is distributed like \( B \) conditioned on its coordinates summing to 0.

Dyson [15] found that, if we let \( B_> \) be distributed like \( B \) conditioned to remain in the cone

\[
C_> = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 > x_2 > \cdots > x_d \right\},
\]

for all time, which can be defined using an \( h \)-transform, then \( B_> =_d \Lambda(H) \). Let \( B^0_> \) be distributed like \( B^0 \) conditioned to remain in \( C_> \) for all time. Since subtracting \( B_1(t) v_1 \) does not change whether or not \( B(t) \in C_> \) and subtracting \( \frac{Tr(H)}{d} I \) induces the same shift (in distribution) on the the eigenvalues of \( H \) that subtracting \( B_1 v_1 \) induces on the coordinates of \( B \), it is easy to that we also have \( B^0_> =_d \Lambda(H^0) \), see e.g. [6 Proposition 2].

To obtain our limiting object \( Z \), it remains to condition \( H^0 \) to be 0 at time 1. The easiest way to do this is by conditioning each entry to be 0 at time 1, which results in the definition of \( Z \) we gave in terms of Brownian bridge. Since all norms on finite dimensional spaces are equivalent, this
is the same as conditioning the spectral norm of $H^0$ to be 0 at time 1. In particular, for $G$ bounded and continuous we have

$$E G(Z) = \lim_{\epsilon \downarrow 0} E \left[ G(H^0) \mid |H^0(1)| < \epsilon \right].$$

Since $M \mapsto \Lambda(M)$ is continuous, using the identity $B^0_\omega =_d \Lambda(H^0)$, we see that for $F$ bounded and continuous we have that

$$E F(\Lambda(Z)) = \lim_{\epsilon \downarrow 0} E \left[ F(\Lambda(H^0)) \mid |H^0(1)| < \epsilon \right] = \lim_{\epsilon \downarrow 0} E \left[ F(B^0_\omega) \mid |B^0_\omega(1)| < \epsilon \right].$$

This shows that $\Lambda(Z)$ has the same distribution as a bridge of $B^0_\omega$ from 0 to 0. For later use, we remark that a straightforward computation using the Markov property and the transition densities for $B^0_\omega$ (see [6]) shows that if $F : D([0,1], \mathbb{R})$ is bounded and continuous and $F(g)$ depends only on the restriction of $g$ to $[0, t]$ for some $0 < t < 1$, then there is a constant $C_t$ such that

$$E[F(\Lambda(Z))] = \lim_{\epsilon \downarrow 0} E \left[ F(B^0_\omega) \mid |B^0_\omega(1)| < \epsilon \right] = C_t E \left[ F(B^0_\omega (\cdot \wedge t)) e^{-|B^0_\omega(t)|^2/(2(1-t))} \right].$$

(57)

This shows that for $0 < t < 1$, the law of $(\Lambda(Z_s), 0 \leq s \leq t)$ is absolutely continuous with respect to the law of $(B^0_\omega(s), 0 \leq s \leq t)$.

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