Abstract

Segal spaces are simplicial spaces with a higher categorical compositional structure, and ever since their associated model structure has been introduced by Charles Rezk, complete Segal spaces have become one of the standard models for $(\infty, 1)$-category theory. This paper is a study of Segal spaces with invertible arrows, first considered as fibrant objects in a model structure by Julie Bergner under the name of Bousfield-Segal spaces. We show that Bergner’s model structure is a left Bousfield localization of Rezk’s model structure for Segal spaces, and note that complete Bousfield-Segal spaces in fact have been studied under various different names in the literature. It follows that complete Bousfield-Segal spaces indeed yield a model for both $\infty$-groupoids and Homotopy Type Theory.

1 Introduction

In [1, 6], Julie Bergner introduced a model structure for (complete) Bousfield-Segal spaces, a version of complete Bousfield-Segal spaces itself originated in Bousfield’s work [3] under the name “very special bisimplicial sets of type 0”. The notion was defined in the last section of her paper, proposing a model structure whose fibrant objects are to be thought of as $\infty$-groupoidal “Segal-like” spaces. Her first approach to invertible Segal spaces – via simplicial presheaves on a simplex-category $I$ – was later shown to model the homotopy theory of $(\infty, 1)$-categories with an involution rather than the homotopy theory of $\infty$-groupoids ([2]), while, to this date, the study of the model structure for Bousfield-Segal spaces has not been developed any further. The primary topic of this paper is to study the model structure for (complete) Bousfield-Segal spaces in the style of [13] and relate it to Rezk’s model structure for (complete) Segal spaces.

In the work of Bergner, Bousfield-Segal spaces are Reedy fibrant bisimplicial sets $X$ equipped with a higher categorical fraction operation
\[
\lmod{X_1(x, z) \times X_1(x, y)} \to X_1(y, z)
\]
induced by associated Bousfield maps, in a very similar way as Segal spaces are Reedy fibrant bisimplicial sets $Z$ equipped with a composition operation
\[
o: Z_1(y, z) \times Z_1(x, y) \to Z_1(x, z)
\]
induced by associated Segal maps. While acyclicity of the Segal maps corresponds to a right lifting property against the inner horn inclusions, acyclicity of the Bousfield maps corresponds to a right lifting property against the left horn inclusions. While it may not be obvious how Bousfield-Segal spaces give rise to an $\infty$-groupoidal composition operation, it is a classical result of ordinary group theory that fraction operations (subject to suitable axioms) and group structures yield equivalent data on any given set. Accordingly, we will construct a model structure for Segal spaces with invertible arrows and show that it coincides with the model structure for Bousfield-Segal spaces as introduced by Bergner.

In order to explain how we obtain a model structure for Segal spaces with invertible arrows from Rezk’s model structure for Segal spaces in a very natural way, let us recall that the category $\textbf{Gpd}$ of (small) groupoids arises as a localization of the category $\textbf{Cat}$ of (small) categories. If by $I$ we denote the free groupoid generated by the walking arrow [1] (that is the “walking isomorphism”), then $\textbf{Gpd}$ is the localization of $\textbf{Cat}$ at the inclusion $e_1: [1] \to I$. Likewise, the category of simplicial groupoids is a localization of the category of simplicial categories. The model structure for Kan complexes can be obtained similarly as the left Bousfield localisation of the model structure for quasi-categories (either at $Ne_1: \Delta^1 \to N I$ or at the left horn inclusions), such that Kan complexes...
are understood as quasi-categories with invertible edges. Modelling higher category theory in the category $sS$ of bisimplicial sets, Charles Rezk introduced model structures $(sS,S)$ and $(sS,CS)$ for Segal spaces and complete Segal spaces, respectively, in [15]. The homotopy theory associated to the latter is a model for $(\infty, 1)$-category theory equivalent to the one associated to the model category for quasi-categories. We hence will see that the model structure for Bousfield-Segal spaces -- as introduced by Bergner in [1] -- is a left Bousfield localization of the model structure for Segal spaces at a canonical map induced by the inclusion $e : [1] \to I$. In particular, we will see that the model structure $(sS, CB)$ for complete Bousfield-Segal spaces is a model for $\infty$-groupoids equivalent to the one associated to Kan complexes, as stated in [1, Theorem 6.12] (without proof). We will further see that $(sS, CB)$ also supports a model of Homotopy Type Theory with univalent universes in the sense of [19]. In fact, these last two results also follow directly from the work of Rezk, Schwede and Shipley [17] (or respectively Dugger [8] and [20], or respectively Cisinski [6]), since it is fairly easy to prove that a Bousfield-Segal space is complete if and only if it is locally constant.

Therefore, Section 2 recalls the Reedy model structure $(sS, R_n)$ on bisimplicial sets and some of its associated Joyal-Tierney calculus. Section 3 introduces Bousfield-Segal spaces in the sense of [1]. Here, we explain how every Bousfield-Segal space $X$ comes equipped with a fraction operation (unique up to homotopy) which induces an associated homotopy groupoid $Ho_B(X)$. In Section 4 we will show that such a fraction operation on a Bousfield-Segal space $X$ induces an invertible composition operation on $X$, proving that every Bousfield-Segal space is in fact a Segal space and the associated model structure $(sS, B)$ for Bousfield-Segal spaces as introduced by Bergner is a left Bousfield localization of $(sS, S)$. We will also see that the homotopy category $Ho(X)$ of a Bousfield-Segal space $X$ associated to it as a Segal space (following [15, 5.5]) is a groupoid and coincides with the construction $Ho_B(X)$. Hence, many of Rezk’s results in [15] and Joyal and Tierney’s results in [13] carry over to the model structure for (complete) Bousfield-Segal spaces. In Section 5 we use this to describe Bousfield-Segal spaces as the Segal spaces with invertible edges in a precise way. In Section 6 we study complete Bousfield-Segal spaces, give various characterizations of such and show that the diagonal functor is part of a Quillen equivalence to the Quillen model structure $(S, Kan)$ using, for instance, the work of Joyal and Tierney [13]. We will see that complete Bousfield-Segal spaces are exactly the Reedy fibrant locally constant bisimplicial sets, and so it follows that their associated model structure $CB$ is contained in different classes of well understood model structures treated in the literature of [17], [7] and [6] respectively. Using the results in the cited literature, it follows that $(sS, CB)$ is a type theoretic model category with as many univalent fibrant universes as there are inaccessible cardinals. We will also give a direct proof of right properness, and deduce from Rezk’s work in [15] that $(sS, CB)$ is a cartesian closed model category.

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2 Preliminaries on bisimplicial sets

A bisimplicial set $X \in sS$ can be understood as a functor $X : \Delta^{op} \times \Delta^{op} \to \text{Set}$, and whenever done so, will be denoted by $X_{\bullet \bullet}$ to highlight its two components. Currying to the left and to the right yields a simplicial object in $S$, whose evaluation at an object $[n] \in \Delta^{op}$ is the $n$-th row $X_{\bullet n}$ and the $n$-th column $X_n := X_{n \bullet}$ respectively.

The box product and its adjoints

To recall some constructions which are very convenient in describing the generating sets for the model structures on bisimplicial sets we are interested in, we briefly summarise some constructions from [13, Section 2].

By left Kan extension of the Yoneda embedding $y : \Delta \times \Delta \to sS$ along the product of Yoneda embeddings $y \times y : \Delta \times \Delta \to S \times S$ one obtains a bicontinuous functor $- \Box - : S \times S \to sS$, often
called the \textit{box product}. The box product is divisible on both sides, i.e. gives rise to adjoint pairs
\[ A \boxtimes \_ : S \leftrightarrow sS : A \setminus \_ \]
and
\[ \_ \boxtimes B : S \leftrightarrow sS : \_ / B \]
for all simplicial sets \( A \) and \( B \). In particular, for any bisimplicial set \( X \) the simplicial set \( \Delta^n \setminus X \cong X_n \) is the \( n \)-th column and \( X / \Delta^n \cong X_{\cdot n} \) is the \( n \)-th row of \( X \). Vice versa, for a given \( X \in sS \), the induced functors
\[ \_ \setminus X : S^{op} \leftrightarrow S : \_ / X \]
are mutually right adjoint, i.e. both pairs \((\_ \setminus X, X / \_ )\) and \((X / \_ , \_ \setminus X)\) are adjoint pairs.

Considering the Leibniz construction (see e.g. [18, Definition 4.4]) for the box product and its dual, we get a functor
\[ \_ \boxtimes' \_ : S^{[1]} \times S^{[1]} \to (sS)^{[1]} \]
on the arrow-categories, taking a pair of arrows \( u : A \to B \), \( v : A' \to B' \) in \( S \) to the natural map
\[
\begin{align*}
\begin{array}{c}
A \boxtimes A' \xrightarrow{u \boxtimes u} A \boxtimes B'\\
\downarrow \Box u \\
B \boxtimes A' \\
\downarrow \Box u \\
B \boxtimes B'
\end{array}
& \cong \begin{array}{c}
A \boxtimes A' \xrightarrow{v \boxtimes v} A \boxtimes B'\\
\downarrow \Box v \\
B \boxtimes A' \\
\downarrow \Box v \\
B \boxtimes B'
\end{array}
\end{align*}
\]
in \( sS \). The functor \( \_ \boxtimes' \_ \) is divisible on both sides, too, the respective right adjoints for a given map \( f \in sS \) are denoted by
\[ \langle f \setminus \_ \rangle, \langle \_ / f \rangle : (sS)^{[1]} \to S^{[1]} \]

\textbf{Proposition 2.1} ([13, Proposition 2.1]). For any two maps \( u, v \in S \) and \( f \in sS \), we have
\[ (u \boxtimes v) \upharpoonright f \iff u \upharpoonright \langle f / v \rangle \iff v \upharpoonright \langle u \setminus f \rangle. \]

In analogy to [13, Lemma 2.11] we have the following lemma.

\textbf{Lemma 2.2.} For every triple \( A, B, C \in S \) the diagonal \( d^* \) yields
\[ d^*(A \boxtimes B) = A \times B \quad \text{and} \quad A \setminus d_* C \cong C^A \cong d_* C / A. \]

More precisely, these equations also hold for morphisms, such that we obtain isomorphisms of bifunctors.

\textbf{Proof.} The first equation is clear. The other two are easily derived from the adjunctions associated to the three left adjoints \( d^* \), \( A \times \_ \) and \( A \boxtimes \_ \).

\textbf{The vertical and horizontal Reedy model structures}

It is well known that the Reedy and injective model structures on \( sS \) coincide since the simplex category \( \Delta \) is an elegant Reedy category (in fact it is the archetype of such a Reedy category). We loosely follow the language and structure of [13] and call this model structure the \textit{vertical Reedy model} structure, denoted by \( R_v \). Its cofibrations are the (pointwise) monomorphisms, its weak equivalences the pointwise weak equivalences and its fibrations the maps with the right lifting property to acyclic cofibrations.

For \( n \geq 0 \) we denote by \( \delta_n : \partial \Delta^n \hookrightarrow \Delta^n \) the \( n \)-th boundary inclusion of the \( n \)-simplex \( \Delta^n \in S \) and, for \( 0 \leq i \leq n \), by \( h^n_i : \Lambda^n_i \hookrightarrow \Delta^n \) the corresponding \( i \)-th horn inclusion. Recall that the set \( \{\delta_n | n \geq 0\} \) of boundary inclusions generates the class of cofibrations and the set \( \{h^n_i | 0 \leq i \leq n\} \) of horn inclusions generates the class of acyclic cofibrations in the Quillen model structure \((S, \text{Kan})\).

In terms of the general calculus of Reedy structures as presented for example in [11, Section 5.2], the object \( \partial \Delta^n \setminus X \) is the \( n \)-th matching object of \( X \). Hence, by [11, Theorem 5.2.5], a map \( f : X \to Y \) in \((sS, R_v)\) is an (acyclic) \( v \)-fibration if and only if the associated maps
\[ \langle \delta_m \setminus f \rangle : X_n \to Y_n \times_{(\partial \Delta^n \setminus Y)} (\partial \Delta^n \setminus X) \]

is divisible on both sides, too, the respective right adjoints for a given
are (acyclic) Kan fibrations in \( S \). Then it is easy to see that the class of cofibrations \( C_v \) of \((sS, R_v)\) is generated by the set
\[
\mathcal{I}_v := \{ \delta_n \Box \delta_n : (\Delta^n \Box \partial \Delta^n) \cup_{\partial \Delta^n \Box \partial \Delta^n} (\partial \Delta^n \Box \Delta^n) \rightarrow (\Delta^n \Box \Delta^n) \mid 0 \leq m, n \},
\]
and the class \( \mathcal{W}_v \cap C_v \) of acyclic cofibrations is generated by the set
\[
\mathcal{J}_v := \{ \delta_i \Box h_i^m : (\Delta^n \Box \Lambda_i^m) \cup_{\partial \Delta^n \Box \Lambda_i^m} (\partial \Delta^n \Box \Delta^n) \rightarrow (\Delta^n \Box \Delta^n) \mid 0 \leq i \leq m, n \}.
\]

**Proposition 2.3 ([13, Proposition 2.5]).** A map \( f \in sS \) is a fibration in \((sS, R_v)\), say a v-fibration, if and only if it satisfies one of the following equivalent conditions:

1. \( \langle \delta_m \setminus f \rangle \) is a Kan fibration for all \( m \geq 0 \).
2. \( \langle u \setminus f \rangle \) is a Kan fibration for all monomorphisms \( u \in S \).
3. \( \langle f/h_i^m \rangle \) is a trivial Kan fibration for all \( 0 \leq i \leq n \).
4. \( \langle f/v \rangle \) is a trivial Kan fibration for all anodyne maps \( v \in S \).

\[
\square
\]

The projection \( p_2 : \Delta \times \Delta \rightarrow \Delta \) onto the second component and the corresponding inclusion \( \iota_2 = ([0], \text{id}) : \Delta \rightarrow \Delta \times \Delta \) constitute an adjoint pair \( p_2 \dashv \iota_2 \), and hence give rise to an adjoint pair
\[
p_2^* : S \leftrightarrow sS : \iota_2^*,
\]
with \( (p_2^* A)_n = A \) for all \( n \geq 0 \), and \( \iota_2^* X = X_0 \) the 0th column of \( X \). We obtain a simplicial enrichment of \( sS \) via \( \text{Hom}_2(X, Y) := \iota_2^* (Y^X) \) for bisimplicial sets \( X \) and \( Y \).

**Proposition 2.4 ([13, Propositions 2.4 and 2.6]).** The simplicial enrichment \( \text{Hom}_2(X, Y) \) on \( sS \) turns \((sS, R_v)\) into a simplicial model category.

\[
\square
\]

It is immediate that properness of \((S, \text{Kan})\) implies properness of \((sS, R_v)\). The permutation \( \sigma := \langle p_2, p_1 \rangle : \Delta \times \Delta \rightarrow \Delta \times \Delta \) induces an isomorphism \( \sigma^* : sS \rightarrow sS \) which transports the vertical Reedy model structure into the horizontal Reedy model structure \( R_h \) with
\[
C_h = \{ \text{monomorphisms in } sS \}
\]
and
\[
\mathcal{W}_h = \{ f : X \rightarrow Y \mid f_{\bullet \cdot} : X_{\bullet \cdot} \rightarrow Y_{\bullet \cdot} \text{ is a weak homotopy equivalence for all } n \geq 0 \}.
\]
Its cofibrations and acyclic cofibrations are generated by the sets
\[
\mathcal{I}_h = \mathcal{I}_v
\]
and
\[
\mathcal{J}_h = \{ h_i^m \Box \delta_n : (\Delta^n \Box \partial \Delta^n) \cup_{\partial \Delta^n \Box \Lambda_i^m} (\Lambda_i^n \Box \Delta^n) \rightarrow (\Delta^n \Box \Delta^n) \mid 0 \leq i \leq m, n \}
\]
respectively. A map is a weak equivalence in \((sS, R_h)\) if and only if it is a rowwise weak homotopy equivalence in \( S \).

In analogy to the pair \( p_2 \dashv \iota_2 \), we have an adjunction between the projection to the first component and the corresponding inclusion
\[
p_1^* : S \leftrightarrow sS : \iota_1^*
\]
with \( (p_1^* A)_{\bullet n} = A \) for all \( n \geq 0 \), and \( \iota_1^* X = X_{\bullet 0} \) the 0th row of \( X \).

Joyal and Tierney show in [13] that \((sS, R_v)\) naturally comes equipped with two orthogonal projections, a Quillen adjunction \( p_1^* : (sS, R_v) \rightarrow (S, \text{Kan}) \) on the one hand, and a mere adjunction \( p_2^* : sS \rightarrow S \) on the other. In order to construct a homotopy theory of \((\infty, 1)\)-categories in \( sS \), they localize \((sS, R_v)\) at a suitable set of maps such that the horizontal projection \( p_2^* : sS \rightarrow S \) becomes a Quillen adjunction (and in fact a Quillen equivalence) to the Joyal model structure \((S, \text{Qcat})\). In this spirit, they are interested in the rowwise “categorical” homotopy theory in \( sS \). In order to construct a homotopy theory of \( \infty \)-groupoids, they localize \((sS, R_v)\) at a larger class of maps such that the horizontal projection \( p_2^* : sS \rightarrow S \) becomes a Quillen adjunction (and in fact a Quillen equivalence) to the model structure for \( \text{Kan complexes} \) \((S, \text{Kan})\). Therefore, we are interested in the rowwise “homotopical” homotopy theory, while discussing the categorical statements in [13] only so much as they help us to establish their groupoidal counterparts.
3 Bousfield-Segal spaces

Let \( i_n : I_n \hookrightarrow \Delta^n \) be the \( n \)-th spine-inclusion, i.e.

\[
I_n = \bigcup_{i=n}^{\cdots} j_i[\Delta^1]
\]

for \( j_i : [1] \to [n], 0 \mapsto i, 1 \mapsto i + 1 \). Localizing \( (sS, R_v) \) at the set of horizontally constant diagrams

\[
S := \{ p_i^*(i_n) : p_i^*(I_n) \hookrightarrow p_i^*(\Delta^n) \mid 2 \leq n \}
\]
yields the left-proper combinatorial simplicial model structure \( (sS, S) := \mathcal{L}_S(sS, R_v) \) whose fibrant objects are the Segal spaces as defined in [15, Section 4.1] and [13, Definition 3.1]. By construction, these are \( v \)-fibrant bisimplicial sets \( X \) such that the maps

\[
(p_i^*(i_n))^* : \text{Hom}_2(p_i^*(\Delta^n), X) \to \text{Hom}_2(p_i^*(I_n), X)
\]

are weak homotopy equivalences for all \( n \geq 2 \). In other words, these are \( v \)-fibrant bisimplicial sets \( X \) such that the maps \( i_n \setminus X : \Delta^n \setminus X \to I_n \setminus X \) are weak homotopy equivalences for all \( n \geq 2 \). Recall that we have \( \Delta^n \setminus X \cong X_n \) and note that \( I_n \setminus X \cong X_1 \times X_0 \cdots \times X_0 X_1 \) is the pullback taken along the boundaries \( d_0 \setminus X \) and \( d_1 \setminus X \) successively. In the following, we denote this pullback by \( X_1 \times \overset{s}{X_0} \cdots \times \overset{s}{X_0} X_1 \) or \((X_1/X_0)_n^*, \) where \( n \) is the number of components. Then we define the Segal maps

\[
\xi_n : X_n \to X_1 \times \overset{s}{X_0} \cdots \times \overset{s}{X_0} X_1
\]

via \( \xi_n := i_n \setminus X \) for \( n \geq 2 \), such that Segal spaces are the \( v \)-fibrant bisimplicial sets whose associated Segal maps are acyclic fibrations. One can think of Segal spaces \( X \) as horizontal simplicial collections of Kan complexes

\[
\begin{array}{cccccc}
X_0 & s & X_1 & s & X_2 & \cdots \\
\downarrow s_0 & \downarrow s_0 & \downarrow s_0 & \downarrow s_0 & \downarrow s_0 & \downarrow s_0 \\
X_0 & s & d & X_2 & s & d & \cdots \\
\downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 \\
X_0 & s & d & X_2 & s & d & \cdots \\
\downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 & \downarrow d_0 \\
\end{array}
\]

where \( X_0 \) is the space of objects and \( X_1 \) is the space of morphisms. It comes equipped with a horizontal weak composition via the Segal maps just like quasi-categories are simplicial collections of sets which come equipped with a weak composition via the lifts of inner horn inclusions. In analogy to Kan complexes, which are quasi-categories with lifts for left horn inclusions, we consider Segal spaces with the corresponding lifting property. Namely, for the map \( \gamma_i : [1] \to [n] \) with \( 0 \mapsto 0, 1 \mapsto i \) let

\[
C_{0,n} := \bigcup_{0<i} \gamma_i[\Delta^1]
\]

be the 1-skeletal cone whose pinnacle is the initial vertex \( 0 \in \Delta^n \). We will refer to its edges as the initial edges of \( \Delta^n \) and let \( \iota_{0,n} : C_{0,n} \hookrightarrow \Delta^n \) denote the canonical inclusion. Localizing \( (sS, R_v) \) at the set of horizontally constant diagrams

\[
B := \{ p_i^*(\iota_{0,n}) : p_i^*(C_{0,n}) \to p_i^*(\Delta^n) \mid n \geq 2 \}
\]
yields a model structure \( (sS, B) := \mathcal{L}_B(sS, R_v) \). This model structure was considered in [1, 6], Bergner calls its fibrant objects Bousfield-Segal spaces. Note that a \( v \)-fibrant bisimplicial set \( X \) is \( B \)-local if and only if the fibrations \( i_n \setminus X : \Delta^n \setminus X \to C_{0,n} \setminus X \) are weak homotopy equivalences. Here, \( \Delta^n \setminus X \cong X_1 \times X_0 \cdots \times X_0 X_1 \) is the \( n \)-fold fibre product of \( X_1 \) over \( X_0 \) along \( d_1 \) everywhere. We distinguish this pullback notationally by \( X_1 \times_{X_0} \cdots \times_{X_0} X_1 \) or \((X_1/X_0)_n^* \). We define the Bousfield maps

\[
\beta_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

of \( X \) via \( \beta_n := \iota_{0,n} \setminus X \).

**Definition 3.2.** Let \( X \) be a \( v \)-fibrant bisimplicial set \( X \). We say that \( X \) is a Bousfield-Segal space (B-space for short) if the Bousfield maps

\[
\beta_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1
\]

are weak homotopy equivalences for all \( n \geq 2 \).
Given a B-space $X$, the Bousfield maps $\beta_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ are acyclic fibrations between Kan complexes, in particular the map $\beta_2$ exhibits a section $\mu_2 : X_1 \times_{X_0} X_1 \to X_2$ and thus the composite map

$$-/ - : X_1 \times_{X_0} X_1 \xrightarrow{\mu_2} X_2 \xrightarrow{d_0} X_1.$$ 

From now on we refer to this map as the fraction operation associated to $X$.

**Notation 3.4.** For vertices $x \in X_{00}$ we write $1_x := s_0 x$ and for $v, w \in X_{00}$ we write $v \sim w$ if $[v] = [w] \in \pi_0 X_n$. Given a Reedy fibrant bisimplicial set $W$ and points $x, y \in W$, the hom-space $W(x, y)$ denotes the pullback of $(d_1, d_0) : W_1 \to W_0 \times W_0$ along $(x, y) \in W_{00} \times W_{00}$.

**Lemma 3.5.** For any B-space $X$ and $x, y, z \in X_{00}$, the fraction operation restricts to a map

$$-/ - : X_1(x, y) \times X_1(x, z) \to X_1(z, y).$$

On the horizontal Kan complexes $X_{\bullet m}$ it maps edges as follows,

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{g} & & \downarrow{\mu_2(f, g)} \\
  z & \xrightarrow{f/g} & y \\
  \downarrow{f} & & \downarrow{f/g} \\
  \end{array}
\]

Then
1. $f/f \sim 1_y$ for all vertices $f : x \to y$ in $X_1$,
2. $f/1_x \sim f$ for all vertices $f : x \to y$ in $X_1$,
3. $f/g \sim (f/h)/(g/h)$ for all vertices $(f, g, h) \in X_1 \times_{X_0} X_1 \times_{X_0} X_1$.

**Proof.** Straightforward calculation.

The maps $\mu_2$ and $d_0$ are natural transformations of simplicial sets, hence $-/-$ descends to homotopy classes. Therefore, for the family of sets

$$\text{Ho}_B(X) := \langle \pi_0 X_1(x, y) | x, y \in X_{00} \rangle$$

indexed over the set of vertices $X_{00}$ we obtain the following corollary.

**Corollary 3.6.** The family of sets $\text{Ho}_B(X)$ comes equipped with an operation

$$-/ - : \text{Ho}_B(X)(x, y) \times \text{Ho}_B(X)(x, z) \to \text{Ho}_B(z, y)$$

satisfying
1. $[f]/[f] = [1_y]$ for all $f \in X_1(x, y)$,
2. $[f]/[1_x] = f$ for all $f \in X_1(x, y)$,
3. $[f]/[g] = ([f]/[h])/([g]/[h])$ for all $(f, g, h) \in X_1(x, y) \times X_1(x, z) \times X_1(x, w)$.

**Proof.** This fraction operation on B-spaces is referred to in [1, Section 6] and in its essence also already used in [3].
4 Bousfield-Segal spaces are $B$-local Segal spaces

Despite the suggestive name it is not clear a priori that Bousfield-Segal spaces as defined in the previous section are in fact Segal spaces. In this section we dispose of this potential ambiguity in notation and show that Bousfield-Segal spaces and $B$-local Segal spaces are the exact same thing.

Therefore, we start with the following combinatorial lemma which is essential to later calculations. Let

$$k_n : C_{0,n} \rightarrow \Lambda^0_n$$

be the canonical inclusion of simplicial sets, such that $t_{0,n} = h^n_0 \circ k_n$.

The proof of the next lemma is a variation of [13, Lemma 3.5] which is a similar statement for essential edges.

**Lemma 4.1.** Let $A \subseteq S$ be a saturated class of morphisms. Suppose further that $A$ has the right cancellation property for monomorphisms, i.e. $vu \in A$ and $u \in A$ imply $v \in A$ for all monomorphisms $u, v \in S$. Then $(h^n_n)_{n \geq 2} \subseteq A$ if and only if $(t_{0,n})_{n \geq 2} \subseteq A$.

**Proof.** The inclusion $t_{0,n} : C_{0,n} \hookrightarrow \Delta^n$ factors through the inclusions

$$C_{0,n} \xrightarrow{k_{2,n}} \Lambda^0_{2,n} \xrightarrow{h^n_{2,n}} \Delta^n,$$

so it suffices to show that $k_n \in A$ for all $n \geq 2$ for both directions.

Suppose $(t_{0,n})_{n \geq 2} \subseteq A$. Then clearly $k_2 = \text{id}_{C_{0,2}}$ is contained in $A$. For $n \geq 2$, we construct $k_{n+1}$ from the inclusions $t_{0,n}$ for $m \leq n$ by a recursive pasting procedure. Therefore, let $n \geq 2$ and assume that the inclusion

$$C_{0,n} \hookrightarrow C_{0,n} \cup \bigcup_{0 < j \leq i} d^j[\Delta^{n-1}]$$

is contained in $A$ for every $0 < i \leq n$. Note that for $n = 2$ this is trivial and for $i = n$ this inclusion is $k_n$. We now show that the inclusion

$$C_{0,n+1} \hookrightarrow C_{0,n+1} \cup \bigcup_{0 < j \leq i} d^j[\Delta^n]$$

is contained in $A$ for every $0 < i \leq n + 1$. There is a pushout square

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{d^i} & d^i[\Delta^n] \cap C_{0,n+1} \\
\downarrow t_{0,n} & & \downarrow t_{0,n} \\
\Delta^n & \xrightarrow{d^i} & d^i[\Delta^n] \cap C_{0,n+1} \cup d^i[\Delta^n]
\end{array}$$

(7)

where the boundaries $d^i$ in the left square are isomorphisms, because the coboundary $d^i : [n] \rightarrow [n+1]$ is a monomorphism. This implies that the inclusion

$$t_{(0,1,n+1)} : C_{0,n+1} \hookrightarrow C_{0,n+1} \cup d^i[\Delta^n]$$

is contained in $A$. Similarly, note that for $0 < i \leq n$ the boundaries

$$\begin{array}{ccc}
C_{0,n} \cup \bigcup_{0 < j \leq i} d^j[\Delta^{n-1}] & \xrightarrow{d^{i+1}} & d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0 < j \leq i} d^j[\Delta^n]) \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{d^{i+1}} & d^{i+1}[\Delta^n]
\end{array}$$

are isomorphisms. Indeed, the upper boundary $d^{i+1}$ is an isomorphism, because

$$d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0 < j \leq i} d^j[\Delta^n]) = (d^{i+1}[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{0 < j \leq i} (d^{i+1}[\Delta^n] \cap d^j[\Delta^n])$$

$$= (d^{i+1}[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{0 < j \leq i} d^{i+1}d^j[\Delta^{n-1}]$$

$$= (d^{i+1}C_{0,n} \cup \bigcup_{0 < j \leq i} d^j[\Delta^{n-1}])$$.
By assumption, the inclusion \( C_{0,n} \hookrightarrow C_{0,n} \cup \bigcup_{0<j\leq i} d^i[\Delta^n] \) is contained in \( A \). But then, by the right cancellation property of \( A \), the inclusion \( C_{0,n} \cup \bigcup_{0<j\leq i} d^i[\Delta^n] \hookrightarrow \Delta^n \) is contained in \( A \), too. Therefore, since the square

\[
d^{i+1}[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{0<j\leq i} d^i[\Delta^n]) \xrightarrow{\gamma} C_{0,n+1} \cup \bigcup_{0<j\leq i} d^i[\Delta^n] \\
d^{i+1}[\Delta^n] \xrightarrow{\gamma} C_{0,n+1} \cup \bigcup_{0<j\leq i+1} d^j[\Delta^n]
\]

is a pushout, the inclusion

\[
\iota_{(0,i+1,n+1)}: C_{0,n+1} \cup \bigcup_{0<j\leq i} d^j[\Delta^n] \hookrightarrow C_{0,n+1} \cup \bigcup_{0<j\leq i+1} d^j[\Delta^n]
\]

is contained in \( A \) for every \( 0 < i \leq n + 1 \). But then the composition

\[
\iota_{(0,i+1,n+1)} \circ \iota_{(0,2,n+1)} \circ \iota_{(0,1,n+1)}: C_{0,n+1} \hookrightarrow \bigcup_{0<j\leq i+1} d^j[\Delta^n]
\]

is contained in \( A \) which finishes the induction. In particular, \( k_{n+1} \) as the composition of all \( \iota_{(0,i,n+1)} \) for \( 0 < i \leq n + 1 \) is contained in \( A \).

For the other direction, assume that \((h_0^n)_{n\geq 2} \subseteq A\). For \( n = 2 \), we have \( C_{0,2} = \Delta_2^0 \) and \( h_0^0 = \iota_{0,2} \), hence \( F_0,2 \) is contained in \( A \). Suppose \( n \geq 2 \) and \( t_{0,m} \in A \) for all \( 2 \leq m \leq n \). As we have seen above, by Diagrams (7) and (8), this implies \( k_{n+1} \in A \). This in turn implies \( t_{0,n+1} \in A \), because \( t_{0,n+1} = h_0^{n+1} \circ k_{n+1} \).

**Corollary 4.2.** Let \( X \in s\mathbf{S} \) be \( v \)-fibrant. Then the following two statements are equivalent.

1. \( t_{0,n} \setminus X \) is an acyclic fibration for all \( n \geq 2 \).
2. \( h_0^n \setminus X \) is an acyclic fibration for all \( n \geq 2 \).

Both conditions imply that \( k_n \setminus X \) is an acyclic fibration for all \( n \geq 2 \).

**Proof.** Let \( X \) be \( v \)-fibrant. The class

\[
A := \{ f \in \mathbf{S} \mid f \text{ is a monomorphism and } f \setminus X \text{ is an acyclic fibration}\}
\]

has the right cancellation property for monomorphisms and is saturated by Proposition 2.1 and the fact that the class of monomorphisms in \( \mathbf{S} \) is saturated. Therefore, (1) and (2) are equivalent by Lemma 4.1. Further, in the proof of Lemma 4.1 we have seen that \( A \) contains \((k_n)_{n\geq 2}\) whenever it contains \((t_{0,n})_{n\geq 2}\) or \((h_0^n)_{n\geq 2}\), so the last part follows immediately.

Now, let \( X \) be a Bousfield-Segal space and recall the notation from (4) and (5) for its associated Segal and Bousfield maps respectively. Then its Bousfield maps \( \beta_n: X_n \rightarrow (X_i/X_0)_B^n \) are acyclic fibrations and in order to show that \( X \) is a Segal space, we have to infer that its Segal maps \( \xi_n: X_n \rightarrow (X_i/X_0)_S^n \) are acyclic, too. We have seen in the previous section that \( X \) comes equipped with a fraction operation \( / \): \( (X_1/X_0)_B^n \rightarrow X_1 \) and hence, for \( n \geq 2 \), with induced maps \( \kappa_n := (\pi_1, \pi_2/\pi_1, \ldots, \pi_n/\pi_{n-1}) \) as follows.

\[
\kappa_n: (X_1/X_0)_B^n \rightarrow (X_1/X_0)_S^n \\
(f_1, \ldots, f_n) \mapsto (f_1, f_2/f_1, \ldots, f_n/f_{n-1})
\]

We want to use these \( \kappa_n \) as a comparison between the Bousfield maps and the Segal maps of \( X \), therefore note that there are maps

\[
\gamma_n: (X_1/X_0)_S^n \rightarrow (X_1/X_0)_B^n \\
(f_1, \ldots, f_n) \mapsto (f_1, f_2/(1f_1f_1/f_1), \ldots, f_i/(1f_if_{i-1}/f_{i-1}), \ldots f_n/f_{n-1})_{i \geq 0}
\]

in the converse direction constructed by recursion on \( n \geq 2 \). In the following sequence of lemmas we show that the maps \( \gamma_n \) are homotopy inverses to \( \kappa_n \). The proofs are long but consist mainly of elementary computations, therefore some parts will only be outlined. All left out details can be found in [21, Section 4.4].

**Lemma 4.3.** Let \( X \) be a Bousfield-Segal space. Then there are homotopies
1. $H^\gamma_2$: $\text{id} \sim \gamma_2 \circ \kappa_2$,
2. $H^\kappa_2$: $\text{id} \sim \kappa_2 \circ \gamma_2$

which are constant on vertices (i.e. the homotopies are constant after applying the boundaries $X_1 \times X_0 \to X_0 \times X_0 \times X_0$).

To distinguish the various projections present, given a simplicial set $W$, we distinctly denote the first projection $W \times \Delta^1 \to W$ by $\text{pr}_1$ and thus the constant homotopy $W \times \Delta^1 \to Z$ from a map $g: W \to Z$ to itself simply by $g\text{pr}_1$.

**Proof.** For part (1) we have to prove that there is a homotopy $H^\gamma_2 = (H^1_2, H^2_2)$ between the identity and

$$\gamma_2 \circ \kappa_2: X_1 \times X_0 \times X_1 \to X_1 \times X_0 \times X_1$$

$$(f_1, f_2) \mapsto (f_1, (f_2/f_1)/(1_{d_1 f_1}/f_1)).$$

That means we have to construct homotopies

* $H^1_2: (X_1 \times X_0 \times X_1) \times \Delta^1 \to X_1$ between $\pi_1$ and $\pi_1 \gamma_2 \kappa_2 = \pi_1$,
* $H^2_2: (X_1 \times X_0 \times X_1) \times \Delta^1 \to X_1$ between $\pi_2$ and $\pi_2 \gamma_2 \kappa_2 = (\pi_2/\pi_1)/(1_{d_1 \pi_1}/\pi_1)$

whose “whiskering” with $d_1$ coincide on the base $X_0$. Since $X_1 \times X_0 \times X_1$ is a homotopy pullback, in quasi-categorical terms this is exactly the necessary construction in order to show that the map $\gamma_2 \kappa_2$ is a vertex in the contractible space

$$\text{Hom}_{S/(d_0,d_1)}(X_1 \times X_0 \times X_1, X_1 \times X_0 \times X_1)$$

for $S$ the quasi-category of spaces and the diagram $(d_0, d_1): \Lambda^2_1 \to S$ given by the boundaries $d_0, d_1: X_1 \to X_0$.

Clearly, the constant homotopy $H^1_2 = \pi_1 \text{pr}_1$ does half the deal. Once can construct $H^2_2$ via the section and right-homotopy inverse $\mu_2$ of $\beta_2$ and a section $\mu_0$ to the map

$$h_0^3 \setminus X: X_3 \sim X_2 \times X_1 \times X_2 \times X_1 X_2$$

which is an acyclic fibration by Corollary 4.2.

Namely, the two sections induce a map $I: X_1 \times X_0 \times X_1 \to X_2$ as the composite of

$$X_1 \times X_0 \times X_1 \to X_2 \times X_1 \times X_2 \times X_1 \times X_2 \xrightarrow{\mu_0} X_3 \xrightarrow{d_0} X_2$$

$$(f_1, f_2) \mapsto (\mu_2(1_{f_1}, f_1, f_2(f_1), s_0 f_2)) \mapsto I(f_1, f_2).$$

On the horizontal simplicial sets $X_m$, the composite $I$ assigns pairs of edges $(f_1, f_2)$ to 2-simplices in $X_m$ in the following way.

By construction, we have $I(f_1, f_2) \in \beta^{-1}_2 (f_2/f_1, 1/f_1)$ for every tuple $(f_1, f_2) \in X_{1m} \times X_{1m}$. Since $\mu_2$ is also a homotopy right-inverse to $\beta_2$, there is a homotopy

$$H: X_2 \times \Delta^1 \to X_2$$

from the identity to $\mu_2 \beta_2$ over $X_1 \times X_0 \times X_1$. This induces a homotopy $H^2_2: (X_1 \times X_0 \times X_1) \times \Delta^1 \to X_1$ as the composite of the top maps in the following diagram.

$$\begin{array}{ccc}
X_1 \times X_0 \times X_1 & \rightarrow & X_2 \\
\downarrow id & & \downarrow \mu_2 \beta_2 \\
X_1 \times X_0 \times X_1 & \rightarrow & X_2 \\
\downarrow \beta^1_2 & & \downarrow \beta^2_2 \\
(X_1 \times X_0 \times X_1) \times \Delta^1 & \rightarrow & X_1 \\
\end{array}$$

Straightforward elementwise calculation shows that $H^2_2$ is a homotopy between $H^2_2|_{(0)} = d_0 I = \pi_2$ and $H^2_2|_{(1)} = d_0 \mu_2 \beta_2 I = \pi_2 \gamma_2 \kappa_2$ such that $d_1 H^2_2 = d_1 \pi_2 \text{pr}_1$. But $d_1 \pi_1$ and $d_1 \pi_2$ coincide on $X_1 \times X_0 \times X_1$. Similarly we get $d_0 H^2_1 = d_0 \pi_1 \text{pr}_1$ and $d_0 H^2_2 = d_0 \pi_2 \text{pr}_1$.

For part (2), again, we have to construct homotopies
* $L_2^1: (X_1 \times_{X_0} X_1) \times \Delta^1 \to X_1$ between $\pi_1$ and $\pi_1 \kappa_2 \gamma_2 = \pi_1$,
* $L_2^2: (X_1 \times_{X_0} X_1) \times \Delta^1 \to X_1$ between $\pi_2$ and $\pi_2 \kappa_2 \gamma_2 = [\pi_2/(1_{d_1} \pi_1)]/\pi_1$

such that the boundary conditions are satisfied. Just as in the first case, the constant homotopy $L_2 = \pi_1 pr_1$ will do. Towards a formula for the homotopy $L_2^2$, consider the map $J: X_1 \times_{X_0} X_1 \to X_2$ defined as the composite

$$X_1 \times_{X_0} X_1 \to X_2 \times X_1 \xrightarrow{\mu_3} X_2 \xrightarrow{\delta_0} X_2$$

$$(f_1, f_2) \mapsto (\mu_2(1,1/f_1), \mu_2(f_2,1/f_1), s_0 f_2)) \mapsto J(f_1, f_2).$$

On the horizontal simplicial sets $X_m$, it assigns pairs of edges $(f_1, f_2)$ to 2-simplices in $X_m$ in the following way.

Further, as a witness of the relation $(f^{-1})^{-1} \sim f$, consider the map $K: X_1 \to X_2$ defined as the composite

$$X_1 \to X_2 \times_{X_1} X_2 \times_{X_1} X_2 \xrightarrow{\mu_3} X_3 \xrightarrow{\delta_0} X_2$$

$$f_1 \mapsto (s_1 f_1, \mu_2(1, f_1), s_0 f_1)) \quad \mapsto \quad K(f_1),$$

These two maps yield homotopies $H_J := H \circ (J, id)$ and $H_K := \tilde{H} \circ (K, id)$, where $H: id \sim \mu_2 \beta_2$ is the homotopy introduced above and $\tilde{H}$ denotes the flipped homotopy from $\mu_2 \beta_2$ to the identity over $X_1 \times_{X_0} X_1$.
Via $H_K$ we obtain a new homotopy $H'_K: (X_1 \times^S_{X_0} X_1) \times DU \to X_1 \times^S_{X_0} X_1$ as follows.

The outer rectangle commutes, because $\bar{H}$ is a homotopy over $\beta_2$. By construction we have $H'_K|_{\{i\}} = \langle \pi_2/(1/\pi_1), d_0 H_K|_{\{i\}} \rangle$, and thus the composition

$$(X_1 \times^S_{X_0} X_0) \times DU \xrightarrow{H'_K} X_1 \times_{X_0}^B X_1 \xrightarrow{\mu_2} X_2$$

is a homotopy beginning at $\mu_2 H'_K|_{\{0\}} = H_J|_{\{1\}}$. Therefore, the pushforward of the concatenation of $H_J$ with $\mu_2 H'_K$ along $d_0$ — that is $d_0(H_J * \mu_2 H'_K)$ — is a homotopy between

$$d_0 H_J|_{\{0\}} = d_0 J = \pi_2$$

and $d_0 \mu_2 H'_K|_{\{1\}} = d_0 \mu_2 (\pi_2/(1/\pi_1), \pi_1) = (\pi_2/(1/\pi_1)) \circ \pi_1 = \pi_2 \mu_2 \gamma_2$.

To ensure that $L^\dagger_1$ and $d_0(H_J * \mu_2 H'_K)$ yield a homotopy $L = (L^\dagger_1, d_0(H_J * \mu_2 H'_K))$ into the pullback $X_1 \times^S_{X_0} X_1$, we have to choose the concatenation $H_J * \mu_2 H'_K$ constant over $d_1$ and $d_0$. Namely, the fact that $d_0 L^\dagger_1 = d_0 \sigma_1$ requires us to check that $d_1 L^\dagger_1 = d_0 f_1$ holds for all triples $(f_1, f_2, \sigma) \in (X_1 \times^S_{X_0} X_1) \times DU$. This is satisfied indeed by the homotopies $d_0 H_J$ and $d_0 \mu_2 H'_K$, i.e. one computes $d_1 d_0 H_J(f_1, f_2, \sigma) = d_0 f_1$ and $d_1 d_0 \mu_2 H'_K(f_1, f_2, \sigma) = d_0 f_1$. Also $d_0 d_0 H_J = d_0 \pi_2 \sigma_1$ and $d_0 d_0 \mu_2 H'_K = d_0 \pi_2 H'_K = d_0 \pi_2 \sigma_1$ hold by the same line of equations. Defining

$$Q := \left( ((X_1 \times^S_{X_0} X_1) \times DU) \cup ((X_1 \times^S_{X_0} X_1) \times \{1\}) \times DU \right),$$

these computations render the diagram commutative. Thus, we have a diagram

$$(X_1 \times^S_{X_0} X_1) \times DU \xrightarrow{H_J \pi_2 * \mu_2 H'_K \pi_2} X_2 \xrightarrow{d_1 \pi_1 \sigma_1} ((X_1 \times^S_{X_0} X_1) \times DU) \cup ((X_1 \times^S_{X_0} X_1) \times \{1\}) \times DU \xrightarrow{\pi_2 \sigma_1} X_0 \times X_0$$

in the slice $S/(X_0 \times X_0)$. Observe that $d_0 d_0 \times d_0 d_0: X_2 \to X_0 \times X_0$ is a Kan fibration by Lemma 2.3, since $d^0 d^1 \sqcup d^0 d^1: \Delta^0 \sqcup \Delta^0 \to \Delta^2$ is a cofibration and $X$ is $v$-fibrant. Therefore, we obtain a lift

$$(X_1 \times^S_{X_0} X_1) \times DU \xrightarrow{H_J \pi_2 * \mu_2 H'_K \pi_2} X_2 \xrightarrow{d_1 \pi_1 \sigma_1} ((X_1 \times^S_{X_0} X_1) \times DU) \cup ((X_1 \times^S_{X_0} X_1) \times \{1\}) \times DU \xrightarrow{\pi_2 \sigma_1} X_0 \times X_0$$
Then by Lemma 4.3 there are homotopies \( \pi \kappa \) between \( S \). Then one constructs homotopies \( H \pi \) between well. This can be done similar to the construction of \((iii)\) \((ii)\) \( H \pi \). By assumption, \( \gamma \kappa \) holds, we can define the homotopy \( \gamma \kappa \). For the diagonal \( \Delta : \Delta \times \Delta \to \Delta \times \Delta \), set
\[ L_2^1 := d_0 H_{JK}(<id, \Delta>): (X_1 \times X_0) X_1 \to \Delta \times \Delta \to X_2. \]
Then \( d_1 L_2^1 = d_0 d_0 H_{JK}(<id, \Delta>) = d_0 \pi_1 pr_1^2 <id, \Delta> = \pi_1 pr_1 \) holds by construction. Note that
\[ d_0 L_2^2 = d_0 \pi_2 pr_1 \quad (9) \]
holds, too, and so \( H_2^{\kappa} := (L_2^1, L_2^2) \) is a homotopy as required. □

**Lemma 4.5.** Let \( X \) be a Bousfield-Segal space. Then the maps \( \kappa_n \) and \( \gamma_n \) are mutually homotopy inverse for all \( n \geq 2 \), i.e. for all \( n \geq 2 \) there are homotopies
1. \( H_n^{\kappa} : <id, \kappa_n \circ \gamma_n, \gamma_n \circ \kappa_n> \)
2. \( H_n^{\gamma} : <id, \kappa_n \circ \gamma_n> \)

In the following, given a product \( A_1 \times \cdots \times A_n \) and a sequence of numbers \( \{i_1, \ldots, i_k\} \) between 1 and \( n \), the map \( \pi(i_1, \ldots, i_k) : A_1 \times \cdots \times A_n \to A_{i_1} \times \cdots \times A_{i_k} \) denotes the projection into the components specified by the sequence. Given a number \( m \leq n \), \( m \) denotes the sequence of all numbers \( 1 \leq k \leq n \) with \( k \neq m \) and \( \pi_m \) denotes the corresponding projection.

**Proof.** By Lemma 4.3 there are homotopies \( H_2^{\gamma} : <id, \gamma_2 \gamma_2> \) and \( H_2^{\kappa} : <id, \gamma_2 \kappa_2> \) which are constant on vertices. Suppose further for all \( 2 \leq m \leq n \) there are homotopies \( H_m^{\kappa} : <id, \kappa_m \gamma_m> \) and \( H_m^{\gamma} : <id, \gamma_m \kappa_m> \) such that
\[ \begin{align*}
(i) \quad & \pi(i_1, \ldots, i_m) H_n^{\kappa} = H_m^{\kappa} (\pi(i_1, \ldots, i_m), \id_{\Delta^1}) \quad \text{and} \quad \pi(i_1, \ldots, i_m) H_n^{\gamma} = H_m^{\gamma} (\pi(i_1, \ldots, i_m), \id_{\Delta^1}) \quad \text{for all} \ m \leq n, \\
(ii) \quad & d_i \pi_m H_n^{\kappa} = d_i \pi_m H_n^{\gamma} = d_i \pi_m \pr_1 \quad \text{for all} \ i \in \{0, 1\} \quad \text{and} \ m \leq n, \quad \text{i.e. the homotopy is constant on vertices}, \\
(iii) \quad & d_0 \pi_n H_n^{\kappa} = d_0 \pi_n H_n^{\gamma} = d_0 \pi_n \pr_1, \quad \text{i.e. the homotopy is constant on the last vertex.}
\end{align*} \]
Then one constructs homotopies \( H_n^{\kappa} \) and \( H_n^{\gamma} \) satisfying conditions (i), (ii) and (iii) by recursion. Towards a formula for \( H_n^{\kappa} : <id, \gamma_{n+1} \kappa_{n+1}> \) for part (1), construct homotopies
\[ \begin{align*}
* H_n^{\kappa} : (X_1/X_0)^n \times \Delta^1 \to (X_1/X_0)^n \quad \text{between} \quad \pi_{(n+1)} \gamma_{n+1} \kappa_{n+1}, \\
* H_n^{\gamma} : (X_1/X_0)^n \times \Delta^1 \to X_1 \quad \text{between} \quad \pi_{n+1} \quad \text{and} \quad \pi_{n+1} \gamma_{n+1} \kappa_{n+1}
\end{align*} \]
such that the homotopies
\[ d_1 \pi_{n+1} H_{n+1}^{\kappa}, d_1 H_{n+1}^{\gamma} : (X_1/X_0)^n \times \Delta^1 \to X_0 \]
coincide. Recalling the definitions of \( \gamma_{n+1} \) and \( \kappa_{n+1} \), we note that on the first \( m \)-many components we have \( \pi_{(n+1)} \gamma_{n+1} \kappa_{n+1} = \gamma_n \kappa_m \pi_{(n+1)} \). So \( H_{n+1}^{\kappa} \) defined as
\[ H_n^{\kappa} \pi_{(n+1)} : \pi_{(n+1)} \sim \pi_{(n+1)} \gamma_{n+1} \kappa_{n+1} \]
gives us the first homotopy. Towards a formula for \( H_{n+1}^{\kappa} \), note that on the \( (n+1) \)-st component we have
\[ \pi_{n+1} \gamma_{n+1} \kappa_{n+1} = (\pi_{n+1} / \pi_n) / (1 d_i \pi_i / \pi_n \gamma_n \kappa_m \pi_{(n+1)}). \]
By assumption, \( \gamma_n \kappa_n \) is homotopic to the identity, so we only have to construct a homotopy between \( \pi_{n+1} \gamma_{n+1} \kappa_{n+1} \) and \( (\pi_{n+1} / \pi_n) / (1 / \pi_n) \), and make sure that the homotopies concatenate well. This can be done similar to the construction of \( H_2^{\kappa} \) in Lemma 4.3, using validity of conditions (i) and (ii) for \( n \).

For part (2), towards a formula for the homotopy \( H_n^{\kappa} \), once again we have to construct homotopies
\[ \begin{align*}
* L_n^{1} : (X_1/X_0)^n \times \Delta^1 \to (X_1/X_0)^n \quad \text{between} \quad \pi_{(n+1)} \quad \text{and} \quad \pi_{(n+1)} \kappa_{n+1} \gamma_{n+1}, \\
* L_n^{2} : (X_1/X_0)^n \times \Delta^1 \to X_1 \quad \text{between} \quad \pi_{n+1} \quad \text{and} \quad \pi_{n+1} \kappa_{n+1} \gamma_{n+1},
\end{align*} \]
such that
\[ \begin{align*}
d_0 \pi_n L_n^{1} : (X_1/X_0)^n \times \Delta^1 \to X_0 \\
d_1 L_n^{2} : (X_1/X_0)^n \times \Delta^1 \to X_0
\end{align*} \]
coincide and conditions (i) and (iii) are satisfied. As in the prior case, because \( \pi_{(n+1)} \kappa_{n+1} \gamma_{n+1} = \kappa_n \gamma_n \pi_{(n+1)} \) holds, we can define the homotopy \( L_n \) simply to be
\[ H_n^{\kappa} \pi_{(n+1)} : <id, \pi_{(n+1)} \sim \pi_{(n+1)} \kappa_{n+1} \gamma_{n+1}> \]

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Towards a formula for the homotopy $L_{n+1}^2$ on the $(n+1)$-st component, note that

$$\pi_{n+1} \kappa_{n+1} \gamma_{n+1} = \left(\pi_{n+1}/(1/\pi_{n+1})\right)/\pi_n \gamma_{n+1} = \left(\pi_{n+1}/(1/\pi_n \gamma_{n+1})\right)/\pi_n \gamma_{n+1} \pi_{(n+1)} = \pi n \gamma_{(n+1)}.$$ 

Therefore, simply set $L_{n+1}^2$ to be

$$\pi_2 H_{n+1}^\gamma (\{\pi_n \gamma_{n+1}, \pi_{n+1}\}, \id_{\Delta^1}) : (X_1/X_0)^{n+1}_S \times \Delta^1 \to X_1,$$

such that $L_{n+1}^2|_0 = \pi_2 (\pi_n \gamma_{n+1}, \pi_{n+1}) = \pi_n$ and $L_{n+1}^2|_{1} = \pi_{n+1} \kappa_{n+1} \gamma_{n+1}$. By condition (iii) and Lemma 4.3, one can show that the boundary conditions $d_0 \pi_n L_{n+1}^1 = d_0 \pi_n H_{n+1}^\gamma (\{\pi_{n+1}\}, \id) = d_0 \pi_n \gamma_{n+1}$ and $d_1 L_{n+1}^1 = \pi_n \gamma_{n+1}$ hold. So we are left to verify conditions (i) and (iii) for $H_{n+1}^\gamma := (L_{n+1}^1, L_{n+1}^2)$. But condition (i) is immediate by the definition of $L_{n+1}^1$ and the inductive hypothesis, and condition (iii) holds because $d_0 \pi_n H_{n+1}^\gamma = d_0 \pi_n \gamma_{n+1}$. So the induction succeeds. This finishes the proof.

So we have seen that, if $X$ is a B-space, the maps

$$\kappa_n : (X_1/X_0)_B^n \to (X_1/X_0)_S^n$$

are homotopy equivalences. The following lemma shows that this comparison of pullbacks in fact yields a comparison between the Bousfield maps and the Segal maps of $X$.

**Lemma 4.6.** Let $X$ be a Bousfield-Segal space. Then for every $n \geq 2$ there is a section and homotopy right-inverse $\mu_n$ of $\beta_n$ such that the square

$$\begin{array}{ccc}
X_n & \xrightarrow{\mu_n \beta_n} & X_n \\
\downarrow{\beta_n} & & \downarrow{\zeta_n} \\
(X_1/X_0)_B^n & \xrightarrow{\kappa_n} & (X_1/X_0)_S^n
\end{array}$$

commutes.

**Proof.** Recall that acyclic fibrations $p : X \to Y$ between cofibrant objects $X, Y$ always exhibit a section $s$ together with a homotopy $H : ps \sim id$ over $p$ as for example shown in [10, Proposition 7.6.11.(2)]. First, in order to find such $\mu_n$ such that (10) commutes, we construct a distinguished factorization $\beta_n = \beta_n^{0,1} \circ \beta_n^{1,2}$ such that we control the essential edges under the resulting homotopy inverses $\mu_n^{0,2}$ and $\mu_n^{1,1}$. Note that in order to render the square in (10) commutative, we do not need to care about the output of $\mu_n \beta_n$ at any edges but the initial and essential ones. Hence it suffices to control the specific 2-simplices which, given adjacent initial edges $f_i$ and $f_{i+1}$, generate the essential edges $f_{i+1}/f_i$. Therefore, we consider the factorization

$$C_{0,n} \xrightarrow{t_{0,n}} \bigcup_{0 < i < n} \Delta^2 \hookrightarrow \Delta^n$$

of $t_{0,n} : C_{0,n} \hookrightarrow \Delta^n$, where $\Delta^2 \subseteq \Delta^n$ is given by the 2-simplex $\sigma_i \in \Delta^2$ with $d_0 \sigma_i = \Delta^{(0,i)}$ the edge from 0 to $i$ and $d_2 \sigma_i = \Delta^{(i,0)}$ the edge from 0 to $i$. For $0 < j \leq n$, let

$$t_{0,j} : C_{0,2} \to C_{0,n} \cup \bigcup_{0 < i < j} \Delta^2 \subset \Delta^n$$

be the inclusion given by $\Delta^{(0,i)} \hookrightarrow \Delta^{(0,j+i-1)}$. For any such $j \leq n$, we have

$$C_{0,2} \xrightarrow{t_{0,j}} C_{0,n} \cup \bigcup_{0 < i < j} \Delta^2 \xrightarrow{id} C_{0,n} \cup \bigcup_{0 < i < j} \Delta^2 \xrightarrow{\Gamma} C_{0,n} \cup \bigcup_{0 < i < j} \Delta^2$$

and $C_{0,n} \cup \bigcup_{0 < i < n} \Delta^2 \subset C_{0,2} \cup \bigcup_{0 < i < n} \Delta^2$. Since $a_n$ is a finite composition of cobase changes of $t_{0,2}$, it induces an acyclic fibration

$$a_n \setminus X : (X_2/X_1)_B^{n-1} \xrightarrow{\sim} (X_1/X_0)_B^n,$$

13
where $(X_2/X_1)^{n−1}_{B} := X_2 \times X_1 \times \cdots \times X_1, X_2 \equiv \bigcup_{0<i<n} \Delta^2 \setminus X$ is the pullback consecutively taken along adjacent initial edges. We denote this fibration by $\beta_n^{2,1}$. Further, the spine inclusion $\iota_n: I_n \hookrightarrow \Delta^n$ also factors via

$$I_n \xrightarrow{\iota_n} \bigcup_{0<i<n} \Delta^2_i \hookrightarrow \Delta^n,$$

so in order to show that the Segal maps $\zeta_n = \iota_n \setminus X$ are weak equivalences, by 2-for-3 it suffices to show that the inclusion $I_n \setminus X$ yields an acyclic fibration $I_n \setminus X$. Note that $I_n \setminus X$ is the map

$$\langle d_0 \pi_1, d_0 \pi_2, d_0 \pi_1, \ldots, d_0 \pi_{n−1} \rangle: (X_2/X_1)^{n−1}_{B} \to (X_1/X_0)^{n}_{B}.$$  \((11)\)

One can show acyclicity of the fibration $l_n \setminus X$ by constructing a weak homotopy equivalence $\mu_n^{2,1}: (X_1/X_0)^{n}_{B} \simto (X_2/X_1)^{n−1}_{B}$ for every $n \geq 2$ such that the triangle

$$\xymatrix{ (X_1/X_0)^{n}_{B} \ar[r]^{\mu_n^{2,1}} \ar[d]_{l_n \setminus X} & (X_2/X_1)^{n−1}_{B} \ar[d]_{\zeta_n^{2}} \ar[r]_{\kappa_n} \ar@/_2pc/[rr]_{\beta_n} & (X_1/X_0)^{n}_{B} \ar[d]_{l_n \setminus X} \ar[r]_{\zeta_n^{2}} \ar@/_2pc/[rr]_{\beta_n} & (X_1/X_0)^{n}_{B} \ar[d]_{l_n \setminus X} \ar[r]_{\zeta_n^{2}} \ar@/_2pc/[rr]_{\beta_n} & \cdots \ar[r] \ar@/_2pc/[rr]_{\beta_n} & \cdots }$$  \((12)\)

commutes. Since the $\kappa_n$ are homotopy equivalences by Lemma 4.5, the statement follows again by 2-for-3. The maps $\mu_n^{2,1}$ are constructed by recursion, starting with $\mu_2^{2,1} := \mu_2$ and defining $\mu_n^{2,1}$ by successive application of $\mu_2$ to pairs of adjacent components.

**Theorem 4.7.** Every Bousfield-Segal space is a Segal space. In particular, the model structures $(sS, B)$ and $L_B(sS, S)$ coincide.

**Proof.** Let $X$ be a Bousfield-Segal space. By Lemma 4.6, there is a section $\mu_n$ of $\beta_n$ such that the square

$$\xymatrix{ X_n & X_n \ar[l]_{\beta_n} \ar[d]_{\zeta_n} \ar[r]^{\mu_n} \ar[r] \ar@/_2pc/[rr]_{\beta_n} & X_n \ar[d]_{\zeta_n} \ar[r]_{\zeta_n} \ar@/_2pc/[rr]_{\beta_n} & \cdots \ar[r] \ar@/_2pc/[rr]_{\beta_n} & \cdots }$$

commutes. But $\beta_n$ and $\mu_n$ are weak homotopy equivalences, and so is $\kappa_n$ by Lemma 4.5. Hence, the Segal maps $\zeta_n$ are weak homotopy equivalences by 2-for-3 and $X$ is a Segal space. This means that every fibrant object in $(sS, B)$ is also fibrant in $\mathcal{L}_B(sS, S)$. But fibrant objects in $\mathcal{L}_B(sS, S)$ are $v$-fibrant and $B$-local by construction, so the left Bousfield localizations $(sS, B)$ and $\mathcal{L}_B(sS, S)$ have the same class of fibrant objects and hence coincide.

Theorem 4.7 implies that all constructions from [15] apply to the class of B-spaces. In particular every B-space $X$ comes equipped with a homotopy category $\text{Ho}(X)$ as constructed in [15, 5.5]. Recall the groupoid $\text{Ho}(X)$ associated to $X$ in Proposition 3.7.

**Corollary 4.8.** For any B-space $X$, the categories $\text{Ho}(X)$ and $\text{Ho}(B)(X)$ coincide. In particular, $\text{Ho}(X)$ is a groupoid.

**Proof.** Let $X$ be a B-space. Clearly the families $\text{Ho}(B)(X)$ and $\text{Ho}(X)$ of sets coincide and have the same identity, so we have to show that the corresponding compositions $o_B$ and $o_S$ coincide, too. By Theorem 4.7, let $\eta_2$ be a section to $\xi_2: X_2 \simto X_1 \times X_1$, such that $o_S := d_1 \eta_2$ is a composition for the Segal space $X$. For any two morphisms $f \in X(x, y)$ and $g \in X(y, z)$, the inner 3-horn map $\eta_2(f, g) \cup \mu_2(1_x, f) \cup \mu_2(g, 1_z/f): \Delta^2 \to X_{\eta_2}$ of the form

![Diagram](https://via.placeholder.com/150)

has a lift $L(f, g): \Delta^3 \to X_{\eta_2}$. Both the simplex

![Diagram](https://via.placeholder.com/150)
and \( s_0(g \circ_S f) \) lie in the fibre \( \beta_2^{-1}(g \circ_S f, 1_x)_0 \). But \( \beta_2 \) is a trivial fibration, hence \( d_1 L(f, g) \) and \( s_0(g \circ_S f) \) lie in the same connected component of \( X_2 \). Therefore, by naturality of \( d_0 \), we have

\[
[g \circ_B f] = [d_0 d_1 L(f, g)] = [d_0 s_0(g \circ_S f)] = [g \circ_S f]
\]

in \( \pi_0X_1(x, z) = \text{Ho}(X)(x, z) = \text{Ho}_B(X)(x, z) \).

\[\square\]

5 Further characterizations

In this section we prove a few basic properties of \( B \)-spaces and characterize \( B \)-spaces as those Segal spaces with invertible edges.

**Proposition 5.1.** A Segal space \( X \) is a Bousfield-Segal space if and only if the Bousfield map

\[
\beta_2 = \iota_{0,2} \setminus X : X_2 \to X_1 \times_{X_0} X_1
\]

is an acyclic fibration. In particular, the model structures \( \mathcal{L}_{\iota_{0,2}}(\mathbf{sS}, S) \) and \( (\mathbf{sS}, B) \) coincide.

**Proof.** As both model structures are left Bousfield localizations of the same Reedy structure, we only have to compare their fibrant objects. Clearly, every \( B \)-space is fibrant in \( \mathcal{L}_{\iota_{0,2}} \) and conversely, we have to show that fibrant objects in \( \mathcal{L}_{\iota_{0,2}}(\mathbf{sS}, S) \) are \( p_1^1 \iota_{0,n} \)-local for all \( n \geq 2 \). Consider the class

\[
A := \{ f \in \mathbf{S} \mid f \text{ is a monomorphism and } p_1^1 f \text{ is an acyclic cofibration in } \mathcal{L}_{\iota_{0,2}}(\mathbf{sS}, S) \}.
\]

\( A \) is saturated and has the right cancellation property for monomorphisms, by construction \( \mathbf{S} \cup \{ \iota_{0,2} \} \) is a subset of \( A \). In a similar fashion to the proof of Lemma 4.1, we show \( \iota_{0,n} \in A \) by induction on \( n \). Suppose \( \iota_{0,m} \in A \) for all \( 2 \leq m \leq n \). In the proof of Lemma 4.1 we have seen that under these assumptions we have

\[
(C_{0,n+1} \hookrightarrow (C_{0,n+1} \cup \bigcup_{0 < j \leq n+1} d^j[\Delta^n])) \in A.
\]

The same proof, just replacing the boundary \( d^1 : [n] \to n+1 \) with the boundary \( d^2 \) in Diagram (7) and continuing the line of reasoning accordingly, shows that the inclusion

\[
C_{0,n+1} \hookrightarrow (C_{0,n+1} \cup \bigcup_{1 < j \leq n+1} d^j[\Delta^n]) \quad (13)
\]

is contained in \( A \), too. We observe that the 0-face of the codomain of the map (13) is

\[
d^0[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{1 < j \leq n+1} d^j[\Delta^n]) = (d^0[\Delta^n] \cap C_{0,n+1}) \cup \bigcup_{1 < j \leq n+1} (d^0[\Delta^n] \cap d^j[\Delta^n])
\]

\[
\quad = \bigcup_{1 < j \leq n+1} d^0 d^j[\Delta^{n-1}]
\]

\[
\quad = \bigcup_{1 < j \leq n+1} d^0 d^{j-1}[\Delta^{n-1}]
\]

\[
\quad = \bigcup_{0 < j \leq n} d^0 d^j[\Delta^{n-1}]
\]

\[
\quad \cong d_0 \left( \bigcup_{0 < j \leq n} d^j[\Delta^{n-1}] \right)
\]

\[
\quad = \Lambda^n_0.
\]

Thus we have isomorphisms

\[
\begin{array}{ccc}
\Lambda^n_0 & \xrightarrow{\cong} & d^0[\Delta^n] \cap (C_{0,n+1} \cup \bigcup_{1 < j \leq n+1} d^j[\Delta^n]) \\
\delta_0^n & \xrightarrow{\text{id}} & \bigcup_{0 < j \leq n} d^j[\Delta^{n-1}] \\
\Delta^n \xrightarrow{\text{id}} & \xrightarrow{\text{id}} & d^0[\Delta^n]
\end{array}
\]
and induced inclusions

\[
\begin{array}{ccc}
C_{0,n} \xrightarrow{k_n} \Delta_0^n \xrightarrow{i_{0,n}} \Delta^n & \xrightarrow{\tau} & C_{0,n+1} \cup \bigcup_{1 \leq j \leq n+1} d^j[\Delta^n] \\
\end{array}
\]

(14)

The maps \(i_{0,m}\) are contained in \(A\) for \(m \leq n\) by assumption and so is the map \(k_n\) by the proof of Lemma 4.1. Hence, by the right cancellation property of \(A\), the inclusion \(h_0^n\) is contained in \(A\), too, and so is the pushout along the bottom map in Diagram (14). Lastly, the inner horn inclusions \(h_i^{n+1}: \Delta_i^{n+1} \hookrightarrow \Delta^{n+1}\) for \(0 < i < n + 1\) are contained in \(A\) by [13, Lemma 3.5] and hence the composition

\[
i_{0,n+1}: C_{0,n+1} \to C_{0,n+1} \cup \bigcup_{1 \leq j \leq n+1} d^j[\Delta^n] \xrightarrow{\lambda_i^{n+1}} \Delta^{n+1}
\]

is contained in \(A\), too, since every component of it is contained in \(A\).

Similar to the choice of fraction operations \(\_ / \_\) for B-spaces, giving a Segal space \(X\) and a section \(\eta_2: X_1 \times_X^S X_1 \to X_2\) to the Segal map \(\xi_2\) determines a composition operation \(\_ \circ \_\) via

\[
d_1\eta_2: X_1(x,y) \times_X^S X_1(y,z) \to X_1(x, z)
\]

as we have seen already in Corollary 4.8. This yields a commuting triangle

\[
\begin{array}{ccc}
X_1 \times_X^S X_1 & \xrightarrow{\lambda_2} & X_1 \times_B X_1 \\
\eta_2 \downarrow & & \beta_2 \downarrow \\
X_2 & \xrightarrow{\lambda_2} & X_1 \times_B X_1
\end{array}
\]

for the map \(\lambda_2(f,g) = (g \circ f, f)\). Let \(X_{\text{hovequiv}} \subseteq X_1\) denote the full sub-Kan complex of homotopy equivalences in \(X\) whose edges are those which become isomorphisms in \(\text{Ho}X\).

**Corollary 5.2.** A Segal space \(X\) is a B-space if and only if either of the following equivalent conditions is satisfied.

1. The map \(\lambda_2\) is a weak homotopy equivalence.
2. Its associated homotopy category \(\text{Ho}X\) is a groupoid.

**Proof.** Part (1) follows immediately from Proposition 5.1 and the 2-for-3 property. For part (2), let \(X\) be a Segal space and assume \(\text{Ho}X\) is a groupoid. By Proposition 5.1 it suffices to show that the Bousfield map

\[
\beta_2: X_2 \to X_1 \times_X X_1
\]

is a weak equivalence. But the fact that \(\text{Ho}X\) is a groupoid implies that \(X_{\text{hovequiv}} = X_1\) and so the statement follows immediately from [15, Lemma 11.6].

**Remark 5.3.** Note that the homotopy category \(\text{Ho}X\) of a given Segal space \(X\) is a groupoid if and only if the quasi-category \(X_{\ast 0}\) is a Kan complex. This in turn holds if and only if all rows \(X_{\ast 0}\) are Kan complexes. So we see that Bousfield-Segal spaces are exactly the Segal spaces horizontally fibrant in the projective model structure over \((S, \text{Kan})\).

**Example 5.4.** Let \(X\) be a bisimplicial set, let \(\partial: X_n \to \partial \Delta^n \setminus X\) denote its \(n\)-th matching object and \(\text{Sub}_{\Delta^n} X_1\) denote those subobjects \(Y \subseteq X_1\) which factor the degeneracy \(s_0: X_0 \to X_1\). The evaluation

\[
(\_ 1)_1: \text{Sub}(X) \to \text{Sub}_{\Delta^n} X_1
\]

of subobjects of \(X\) has a fully faithful right adjoint \(G_1\) whose value at a subobject \(Y \subseteq X_1\) can be thought of as the largest subobject \(K\) of \(X\) such that \(K_1 \subset Y\). For \(Y \in \text{Sub}_{\Delta^n} X_1\), its values are recursively given by

\[
G_1(Y)_0 := X_0, \quad G_1(Y)_1 := Y \subseteq X_1
\]
and

\[ \partial \Delta^n \setminus G_1(Y) \hookrightarrow \partial \Delta^n \setminus X \]

for \( n \geq 2 \). The boundaries are directly inherited from \( X \) while \( s_0 : G_1(Y)_0 \to G_1(Y)_1 \) is given by requiring that \( s_0 : X_0 \hookrightarrow X_1 \) factors through \( Y \). Assuming that the degeneracies \( s_k : G_1(Y)_{n-1} \hookrightarrow G_1(Y)_n \) for \( 0 \leq k < n \) are defined, for \( 0 \leq i < n+1 \) let \( \partial s_i := (s_{i-1}d_0, \ldots, s_{i-1}d_{i-1}, 1, 1, s_id_{i+2}, \ldots, s_id_n) \), so we obtain

\[
\begin{array}{c}
\xymatrix{ \partial \Delta^n \setminus G_1(Y) \ar[r]^\partial \ar[d] & \partial \Delta^n \setminus X \\
G_1(Y)_n \ar[r]_{s_i} & X_n } \\
G_1(Y)_{n+1} \ar[u]_{\partial s_i} \ar[r]_\partial & X_{n+1} \ar[u]_\partial \end{array}
\]

(15)

according to the corresponding simplicial identities. Then \( G_1(Y) \) satisfies all simplicial identities as they do hold for \( X \) and the natural map \( G_1(Y)_n \to X_n \) is monic. Hence, \( G_1(Y) \) is a simplicial object in \( \mathbf{S} \).

For any Segal space \( X \in \mathbf{sS} \), the subobject \( X_{\text{hoequiv}} \subseteq X_1 \) contains the image of the degeneracy \( s_0 : X_0 \hookrightarrow X_1 \), so we can define

\[
\text{Core}(X) := G_1(X_{\text{hoequiv}}) \subseteq X,
\]

the core of \( X \). Then it is easy to show that for any Segal space \( X \), the bisimplicial set \( \text{Core}(X) \) is a Bousfield-Segal space.

**Remark 5.5.** For a quasi-category \( \mathbf{C} \), let \( \mathbf{C}^\simeq \subseteq \mathbf{C} \) denote the set of equivalences in \( \mathbf{C} \). Given a Segal space \( X \), recall that \( X_{\text{hoequiv}} \) is the sub-Kan complex of \( X_1 \) generated by the set

\[
\{(f : x \to y) \in X_1 \mid \exists g, h \in X_1 : (gf \sim 1_x) \in X_1(x, x) \text{ and } (fh \sim 1_y) \in X_1(y, y)\}.
\]

This equals \( X_{\text{hoequiv}}^{\simeq} \) by Reedy fibrancy of \( X \). In fact for each \( i \in \mathbb{N} \) the sets \( (X_{\text{hoequiv}})^i \) and \( X_i^{\simeq} \) coincide and thus, denoting the nerve of the free groupoid over the category \([n]\) by \( F[n] \), we see that

\[
\text{Core}(X)_{nm} = s\mathbf{S}(NF[n] \boxdot \Delta^m, X).
\]

**Corollary 5.6.** If \( X \) is a \( B \)-space, then \( X/A \) is a Kan complex for every \( A \in \mathbf{S} \). In particular, every row of \( X \) is a Kan complex.

**Proof.** Let \( A \in \mathbf{S} \) and \( X \) be a \( B \)-space. \( X \) is a Segal space by Theorem 4.7 and hence the simplicial set \( X/A \) is a quasi-category by [13, Corollary 3.6]. We know that \( h^n \setminus X \) is an acyclic fibration for all \( n \geq 2 \) by Corollary 4.2 and thus has the right lifting property against the cofibration \( 0 \hookrightarrow A \).

By Proposition 2.1 it follows that \( X/A \) has the right lifting property against all left horn inclusions and thus is an \( \{h^n \mid n \geq 2\} \)-local quasi-category, i.e. left fibrant. In other words, \( X/A \) is a Kan complex.

**Notation 5.7.** Let \( J := N(I) \) be the nerve of the interval object

\[
I = 0 \xrightarrow{\subseteq} 1
\]

in the category of groupoids, \( e : 1 \to J \) the inclusion \( * \hookrightarrow 0 \), \( !_J : J \to 1 \) its terminal map and \( e_1 : \Delta^1 \to J \) the canonical inclusion.

\( J \) is the “freely walking isomorphism” and maps out of it determine the core of a quasi-category. Rezk showed in [15, Theorem 6.2] that every Segal space \( W \) induces a weak equivalence \( e_1 \setminus W : J \setminus W \to W_{\text{hoequiv}} \), where \( W_{\text{hoequiv}} \subseteq W_1 \) is the subsimplicial set of homotopy equivalences in \( W \).

**Proposition 5.8.** The bisimplicial map \( p_1^*: 1 \to p_1^*: J \) is an acyclic cofibration in \((\mathbf{sS}, B)\). If \( X \) is a \( B \)-space, then \( X_{\text{hoequiv}} = X_1 \) and the canonical inclusion \( e_1 : \Delta^1 \hookrightarrow J \) induces an acyclic fibration \( e_1 \setminus X : J \setminus X \to X_1 \).
Proof. We have seen in Corollary 4.7 that the homotopy category of a B-space $X$ is a groupoid, thus $X_{\text{hoequiv}} = X_1$. Therefore the statement follows directly from [15, Theorem 6.2, Section 11]. Indeed, in his proof of [15, Theorem 6.2], Rezk actually gives an explicit description of the inclusion $e_1$ as a transfinite composition of pushouts of the class $\{b_n^0 \mid n \geq 2\}$.

6 Complete Bousfield-Segal spaces

Localizing the model structure $(s\mathcal{S}, S)$ for Segal spaces at the set

$$C := \{p_1^*: p_1^*J \to p_1^*1\}$$

defines the model structure $(s\mathcal{S}, CS) := \mathcal{L}_C(s\mathcal{S}, S)$ which originally was presented in [15] and is further studied in [13, Section 4]. Its fibrant objects are the C-local Segal spaces – the complete Segal spaces – i.e. the Segal spaces $X$ such that the map

$$c \setminus X: J \setminus X \to X_0$$

is an acyclic fibration. Analogously, localizing $(s\mathcal{S}, B)$ at the set $C$ yields the simplicial, left-proper and combinatorial model category

$$(s\mathcal{S}, CB) := \mathcal{L}_C(s\mathcal{S}, B) = \mathcal{L}_B(s\mathcal{S}, CS).$$

Definition 6.1. We say that $X \in s\mathcal{S}$ is a complete B-space if $X$ is a $B$-local complete Segal space. That is if and only if $X$ is fibrant in $(s\mathcal{S}, CB)$.

Recall that a Segal space $X$ is complete if and only if the simplicial set $X_{\text{hoequiv}}$ is a path object for $X_0$. If $X$ is a B-space, we have seen that $X_{\text{hoequiv}} = X_1$ in Proposition 5.8, so in that case $X$ is complete if and only if the object $X_1$ is a path space for $X_0$.

Example 6.2. Rezk introduces the classifying diagram $N_R(\mathcal{C})$ of a category $\mathcal{C}$ in [15, Section 3.5]. For $I[n]$ the free groupoid generated by the category $[n]$, its formula is given by

$$N_R(\mathcal{C})_{mn} = \text{Hom}_{\mathcal{Cat}}([m] \times I[n], \mathcal{C}) = \text{Hom}^*_\mathcal{S}(\Delta^m \Box N(I[n]), d_n N(\mathcal{C}))$$

where $d_n$ denotes the right adjoint to the diagonal $d^*: \mathcal{S} \to \mathcal{S}$. Rezk shows in [15, Proposition 6.1] that the classifying diagram $N_R(\mathcal{C})$ of a category $\mathcal{C}$ is a complete Segal space. It follows that the classifying diagram $N_R(\mathcal{G})$ of a groupoid $\mathcal{G}$ is a complete B-space.

Indeed, it is only left to show that $N_R(\mathcal{G})$ is B-local. But for a groupoid $\mathcal{G}$, we have $\text{Hom}_{\mathcal{Cat}}([m] \times I[n], \mathcal{G}) \cong \text{Hom}_{\mathcal{Cat}}([m] \times [n], \mathcal{G})$, hence $N_R(\mathcal{G}) \cong d_n N(\mathcal{G})$. Therefore, $t_{0,n} \setminus N_R(\mathcal{G}) = N(\mathcal{G})^{n,n}$ by Lemma 2.2. But $N(\mathcal{G})$ is a Kan complex and $t_{0,n}$ is anodyne, hence $N(\mathcal{G})^{n,n}$ is an acyclic fibration and $N_R(\mathcal{G})$ is a B-space by Definition 3.2.

Remark 6.3. We have seen in Theorem 4.7 that $(s\mathcal{S}, B)$ and $\mathcal{L}_B(s\mathcal{S}, CS)$ coincide, so the equality $(s\mathcal{S}, CB) = \mathcal{L}_B(s\mathcal{S}, CS)$ obviously holds, too. Thus, informally understanding the localization $\mathcal{L}$ at a set of maps as a partial function on the collection of model categories $\mathcal{M}$ together with a set of maps in $\mathcal{M}$, the genealogy of the considered model structures so far looks as follows.

There is a much more direct and concise proof of the bottom equality as some diagram chasing – which will be omitted here – shows that the inner horn inclusions can be obtained from the set $l$ of left horn inclusions together with the map $c: 1 \to J$ by closure under finite pushouts, compositions and left cancellation of monomorphisms, using the following lemma.

Lemma 6.4. Let $X \in s\mathcal{S}$ be $v$-fibrant. Then the following are equivalent.

1. $X$ is a complete $B$-space,
2. \( \iota_{0,n} \backslash X \) is an acyclic fibration for all \( n \geq 2 \) and \( c \backslash X \) is a trivial fibration,
3. \( h^n_0 \backslash X \) is an acyclic fibration for all \( n \geq 1 \),
4. \( \iota_{0,n} \backslash X \) is acyclic fibration for all \( n \geq 2 \) and \( X \) is a complete Segal space,
5. \( h^n_0 \backslash X \) is an acyclic fibration for all \( 0 \leq k < n \),
6. \( u \backslash X \) is an acyclic fibration for all anodyne maps \( u \in S \).
7. \( X/\delta_n \) is a Kan fibration for all \( n \geq 0 \),
8. \( X/v \) is a Kan fibration for all monomorphisms \( v \in S \).

**Proof.** (1) \( \iff \) (2) holds by definition. Towards proving (2) \( \iff \) (3), observe that, by Lemma 4.2, both conditions (2) and (3) imply that \( X \) is a B-space and hence that \( e_1 \backslash X \) is a weak equivalence by Lemma 5.8. So, keeping in mind that

\[
\begin{array}{ccc}
1 & \xrightarrow{c} & J \\
\downarrow h^0_0 & \downarrow & \downarrow e_1 \\
\Delta^1 & \xrightarrow{} & \Delta^1
\end{array}
\] (16)

commutes, the map \( h^0_0 \backslash X \) is a weak homotopy equivalence if and only if its section \( c \backslash X \) is such. This gives (2) \( \iff \) (3).

The equivalence of conditions (2), (4) and (5) follows from Theorem 4.7 and Lemma 4.2 similarly.

The equivalence of conditions (5) to (8) follows from Proposition 2.1. Note here that whenever \( X \) is a complete B-space and \( u: A \rightarrow B \) is a monomorphism in \( S \), the map \( X/u \) is a left fibration between the Kan complexes \( X/A \) and \( X/B \) by Corollary 5.6 and part (5). But left fibrations between Kan complexes are Kan fibrations, see [14, Lemma 2.1.3.3]. This gives (5) \( \iff \) (7). The equivalence of conditions (6), (7) and (8) is a direct application of Proposition 2.1. \( \Box \)

**Remark 6.5.** Lemma 6.4 shows that localizing at the left, right or all outer horn inclusions yield the same model structure. Indeed, the right horn inclusions are anodyne, so the maps \( h^n_0 \backslash X \) are acyclic fibrations for every complete B-space \( X \). Hence, the maps \( p^n_k h^n_0 \) are B-equivalences already.

The map \( s_0: \Delta^1 \rightarrow \Delta^0 \) is anodyne, hence \( s_0 \backslash X: X_0 \rightarrow X_1 \) is a weak homotopy equivalence for every complete B-space \( X \). Vice versa, we have seen that the map \( e_1 \backslash X: J \backslash X \rightarrow X_1 \) is an acyclic fibration for every B-space \( X \), and clearly the maps \( \Delta^1 \xrightarrow{s_0} J \xrightarrow{e_1} \Delta^0 \) compose to \( s_0 \). Thus we can factor the degeneracy \( s_0 \backslash X \) via

\[
\begin{array}{ccc}
X_0 & \xrightarrow{s_0} & X_1 \\
\downarrow \iota_{J \backslash X} & \downarrow & \downarrow c_1 \backslash X \\
J \backslash X & \rightarrow & \Delta^1
\end{array}
\]

and see that a B-space \( X \) is complete if and only if \( s_0 \backslash X: X_0 \rightarrow X_1 \) is a weak equivalence.

**Corollary 6.6.** Let \( X \in sS \). Then the following are equivalent.

1. \( X \) is a complete B-space,
2. \( X \) is a complete Segal space and \( e_1 \backslash X: J \backslash X \rightarrow X_1 \) is a weak equivalence,
3. \( X \) is a complete Segal space and \( \lambda_2: X_1 \times X_0 \rightarrow X_1 \times X_0, X_1 \mapsto (f,g) \mapsto (g \circ f, f) \) is a weak equivalence for any choice of composition “\( \circ \)” as in Corollary 5.2,
4. \( X \) is a complete Segal space and its associated homotopy category \( \text{Ho}X \) is a groupoid.

**Proof.** The equivalence of (1) and (3) is Corollary 5.2. It is clear that (1) implies (2), while the converse also follows from Proposition 5.2. Namely, it suffices to show that \( h^0_0 \backslash X \) is a weak equivalence. But the functor \( \Delta^0 \backslash X \) sends every map in the diagram

\[
\begin{array}{ccc}
\Delta^0 \backslash X & \xrightarrow{d^0} & \Delta^1 \backslash X \\
\downarrow d^0 \downarrow \downarrow & \downarrow \downarrow & \downarrow \downarrow \\
\Delta^1 & \xrightarrow{h^n_0} & \Delta^2
\end{array}
\]
to an acyclic fibration, since $h^i_0 \setminus X$ is the Segal map $\zeta_2$ and $d^i = h^i_{1-i}$ is part of Diagram (16). Hence, the composition $\iota_1 \setminus X : \Delta^2 \setminus X \to \Delta^0 \setminus X$ is an acyclic fibration, too, and by 2-for-3, every retraction $\iota_i \setminus X$ of the map $!_{\Delta^2} \setminus X$ for $i \in \{0, 1, 2\}$ is an acyclic fibration. Thus, considering the diagram

$$
\begin{array}{ccc}
\Delta^0 & \to & \Delta^1 \\
\downarrow^{d^i} & & \downarrow^{\zeta} \\
\Delta^1 & \to & X^0_0 \\
\end{array}
$$

we see that $h^i_0 \setminus X$ is a weak equivalence indeed, again by 2-for-3. Clearly, (3) implies (4). Vice versa, Rezk noted in [15, Corollary 6.6] that (4) holds if and only if $X$ is a complete Segal space and $s_0 \setminus X : X_0 \to X_1$ is a weak equivalence.

**Remark 6.7.** Corollary 6.6 yields both a bisimplicial analogy to Joyal’s criterion for a quasi-category to be a Kan complex – for instance as presented in [14, Proposition 1.2.4.3 and 1.2.5.1] – and an $\infty$-categorical analogy to the fact that the category $\text{Gpd}$ is the reflective localization of $\text{Cat}$ at the map $e_1 : [1] \hookrightarrow I$.

**Remark 6.8.** Along the lines of the characterization of v-fibrations in Proposition 2.3, one can obtain a characterization of h-fibrations by simply swapping the components in the brackets $⟨ \_ , \_ ⟩$ and $⟨ \_ / \_ ⟩$ respectively. Indeed, Lemma 6.4 shows that a bisimplicial set $X$ is a complete B-space if and only if it is simultaneously v-fibrant and h-fibrant. This observation is all it will take to show that $(sS, \text{CB})$ is right proper later in this section.

**Remark 6.9.** A map $f : X \to Y$ between complete B-spaces $X$ and $Y$ is a weak equivalence in $(sS, \text{CB})$ if and only if it is a levelwise weak homotopy equivalence. Rezk’s result in [15, Proposition 7.6] shows that this in turn holds if and only if $f$ is a Dwyer-Kan equivalence, i.e. an equivalence on the associated homotopy categories and fully faithful on mapping spaces.

**Proposition 6.10.** Let $f : X \to Y$ be a v-fibration between complete B-spaces. Then the map $(f/v) : X/T \to Y/T \times_{Y/S} X/S$

is a Kan fibration for every monomorphism $v : S \to T$ in $S$.

**Proof.** By [13, Lemma 4.3], $(f/v)$ is a quasi-fibration. But quasi-fibrations between Kan complexes are Kan fibrations.

**Corollary 6.11.** The model category $(sS, \text{CB})$ is also a left Bousfield localization of $(sS, R_h)$.

**Proof.** The cofibrations in both cases are precisely the monomorphisms. In order to show that $W_h \subseteq W_{\text{CB}}$ holds, it suffices to show that the identity id: $(sS, R_h) \to (sS, \text{CB})$ is a left Quillen functor, because all objects in $(sS, R_h)$ are cofibrant. Equivalently, we may show that the identity id: $(sS, \text{CB}) \to (sS, R_h)$ preserves fibrations between fibrant objects. These are exactly the v-fibrations between complete B-spaces, and such are h-fibrations by Proposition 6.10. Indeed, $(sS, \text{CB})$ is the Bousfield localization of $(sS, R_h)$ at $\{p^2_{0,n} | n \geq 2\} \cup \{p_2^c\}$, although with respect to the enrichment $\text{Hom}_1(X,Y) := \iota_1^*(Y^X)$ by “orthogonal” argumentation.

In analogy to [13, Proposition 4.6], Corollary 6.11 implies the following.

**Corollary 6.12.** The box product $\Box'$: $(S, \text{Kan}) \times (S, \text{Kan}) \to (sS, \text{CB})$ is a left Quillen bifunctor.

**Proof.** Let $u, v \in S$ be cofibrations. By general argumentation about Reedy model structures, specifically [13, Proposition 7.36], $u\Box' v$ is a cofibration. If furthermore $v$ is anodyne, $u\Box' v$ is acyclic in $(sS, R_v)$, and so it is acyclic in $(sS, \text{CB})$. Now, suppose $u$ is anodyne. We shall show that $u\Box' v$ has the right lifting property with respect to fibrations between complete B-spaces. But given a v-fibration $f : X \to Y$ between complete B-spaces, the map $(f/v)$ is a Kan fibration by Proposition 6.10. Therefore $u \triangledown (f/v)$ holds and hence $u\Box' v \triangledown f$.

In [13, Theorem 4.11] it is shown that the pair $(p_1^*, !_1) : (S, \text{Qcat}) \to (sS, \text{CS})$

from (3) is a Quillen equivalence. That means a complete Segal space $X$ is determined by the quasi-category $X_{\bullet 0}$ and the homotopy theory of complete Segal spaces is equivalent to the homotopy theory of quasi-categories.
Theorem 6.13. The pair 
\[(p_1^*, \iota_1^*): (S, \text{Kan}) \to (\mathbf{sS}, \text{CB})\]
is a Quillen equivalence.

Proof. By [13, Theorem 4.11] and [10, 3.3.20.(i)] the pair
\[(p_1^*, \iota_1^*): \mathcal{L}_l(S, \text{Qcat}) \to \mathcal{L}_{L\mathbf{p}l}(s\mathbf{S}, \text{CS})\]
is a Quillen equivalence, where \(l\) denotes the set of left \(n\)-horn inclusions for \(n \geq 2\). But \(\mathcal{L}_l(S, \text{Qcat}) = (S, \text{Kan})\) and so we are left to show that the model structures \(\mathcal{L}_{L\mathbf{p}l}(s\mathbf{S}, \text{CS})\) and \((s\mathbf{S}, \text{CB})\) coincide. Every object in \((S, \text{Kan})\) is cofibrant, so
\[\mathcal{L}_{L\mathbf{p}l}(s\mathbf{S}, \text{CS}) = \{p_1^*h^n_0 \mid n \geq 2\} \subset (s\mathbf{S}, \text{CB}).\]
By Lemma 4.2, a \(v\)-fibrant object \(X \in s\mathbf{S}\) is \(p_1^*\)-local if and only if it is \(B\)-local, and hence the model categories \(\mathcal{L}_{L\mathbf{p}l}(s\mathbf{S}, \text{CS})\) and \(\mathcal{L}_B(s\mathbf{S}, \text{CS}) = (s\mathbf{S}, \text{CB})\) coincide. \(\Box\)

Theorem 6.14. The diagonal \(d^*: s\mathbf{S} \to \mathbf{S}\) is part of a Quillen equivalence
\[(d^*, d_*): (s\mathbf{S}, \text{CB}) \to (S, \text{Kan}).\]

Proof. The pair \((d^*, d_*): (s\mathbf{S}, \text{R}_s) \to (S, \text{Kan})\) is a Quillen pair, since \(d^*\) preserves monomorphisms and pointwise weak equivalences by the Realization Lemma ([9, IV, Proposition 1.7]). Hence, the right adjoint \(d_*\) takes Kan fibrations to \(v\)-fibrations. Thus, in order to show that the right adjoint \(d_*: (s\mathbf{S}, \text{CB}) \to (S, \text{Kan})\) maps Kan fibrations between Kan complexes to \(B\)-fibrations ([13, Proposition 7.15]), it suffices to show that \(d_*\) maps Kan complexes to complete \(B\)-spaces. Given a Kan complex \(X\), the maps \(\iota_{0,n} \setminus d_1A = A^{0\to n}\) and \(c \setminus d_2A = A^c\) are acyclic fibrations, because \(\iota_{0,n}\) and \(c\) are acyclic and \((S, \text{Kan})\) is cartesian. Hence, \(d_*\) is a complete \(B\)-space.

So all three pairs \((d^*, d_*), (p_1^*, \iota_1^*)\) and \((\text{id}, \text{id})\) are Quillen pairs, and note that \(d^*p_1^* = \text{id}: \mathbf{S} \to \mathbf{S}\) and \(\iota_1^*d_* = \text{id}: \mathbf{S} \to \mathbf{S}\). Therefore, the statement follows from Corollary 6.13 by 2-for-3. \(\Box\)

Remark 6.15. The fact that the diagonal induces an equivalence on homotopy categories as shown in Proposition 6.14 is exactly the content of [3, Theorem 3.1] for “very special bisimplicial sets” of type \(n = 0\).

In fact, Theorem 6.13 and Theorem 6.14 have already been shown in the literature multiple times, under different names for the model structure \(\text{CB}\) using different techniques. That is, it turns out that the model structure \((s\mathbf{S}, \text{CB})\) coincides with the canonical, realization or \(\text{hocolim}\) model structure on \(s\mathbf{S}\) as introduced in [17] and [8] respectively. Therefore, recall the following definition.

Definition 6.16. A bisimplicial set \(X\) is said to be homotopically (or locally) constant if the map \(X(f): X_n \to X_n\) is a weak equivalence for every function \((f: n \to m) \in \Delta\).

Lemma 6.17. A \(v\)-fibrant bisimplicial set \(X\) is a complete Bousfield-Segal space if and only if \(X\) is homotopically constant.

Proof. Clearly, \(X\) is homotopically constant if and only if all boundary and degeneracy maps of \(X\) are weak homotopy equivalences. This in turn holds if and only if all boundary maps of \(X\) are weak homotopy equivalences (since the degeneracies are sections of the boundaries). If \(X\) is homotopically constant, it is easy to see that all the \(X_n\) and all pullbacks \((X_{1/X_0})^n_B\) are contractible by right properness of \((S, \text{Kan})\), so the Bousfield maps are weak homotopy equivalences. Complete-ness follows trivially. Vice versa, if \(X\) is a complete \(B\)-space, we have seen that the degeneracy \(s_0: X_0 \to X_1\) is a weak homotopy equivalence, and hence so are the boundaries \(d_i: X_1 \to X_0\). This implies contractibility of the pullbacks \((X_{1/X_0})^n_B\) and, since the Bousfield maps are weak homotopy equivalences, therefore contractibility of the Kan complexes \(X_n\). Thus, all boundaries of \(X\) are weak homotopy equivalences. \(\Box\)

In [17], given a model category \(\mathcal{M}\), the model structure on \(\mathcal{M}^{\Delta^{op}}\) whose fibrant objects are exactly the homotopically constant Reedy fibrant simplicial objects is called the canonical model structure on \(\mathcal{M}^{\Delta^{op}}\). So Lemma 6.17 shows that \((s\mathbf{S}, \text{CB})\) is the canonical model structure on \(s\mathbf{S}\). By [17, Theorem 3.6] this implies that the projection \(\iota_1^*: (s\mathbf{S}, \text{CB}) \to (S, \text{Kan})\) onto the first column is part of a Quillen equivalence. Also, recall the isomorphism \(\sigma^*: s\mathbf{S} \cong s\mathbf{S}\) induced by the permutation \(\sigma: \Delta \times \Delta \to \Delta \times \Delta\) swapping the components \(([n], [m]) \mapsto ([m], [n])\). Using the notation from
Section 2, note that $\sigma^*[W_i] = W_h$, $\sigma^*[C_j] = C_h = C$ and even $\sigma^*[I_i] = I_h$ and $\sigma^*[J_i] = J_h$ as $\sigma^*$ preserves colimits. Furthermore, for all objects $A, B \in sS$ the isomorphism satisfies

$$\text{Hom}_2(\sigma^*A, \sigma^*B) := \iota_2^*(\sigma^*B^\sigma^A) = \iota_2^*\sigma^*(B^A) = \iota_1^*(B^A) =: \text{Hom}_1(A, B)$$

and $\text{Hom}_1$ turns $(sS, R_h)$ into a simplicial model category. Let

$$CB^\perp := \{p_2^*c \cup \{p_2^{*n}a_n \mid n \geq 2\},$$

so we can build the Bousfield localization $L_{CB^\perp}(sS, R_h)$.

Note that the model structures $(sS, CB)$ and $L_{CB^\perp}(sS, R_h)$ are isomorphic, so that all arguments presented so far are symmetric with respect to the vertical and horizontal direction. Hence, the fact that the first row projection $\iota_2^*: sS \to S$ is part of a Quillen equivalence as stated in Theorem 6.13 also follows from Lemma 6.17 and the general observations in [17, Theorem 3.6] (or [8] respectively).

**Remark.** It is easy to show that the model structures $(sS, CB)$ and $L_{CB^\perp}(sS, R_h)$ in fact coincide. This means that a bisimplicial set $X$ is a (vertical) complete $B$-space if and only if $\sigma^*X$ is a (vertical) complete $B$-space. Indeed, the model categories presented so far are symmetric with respect to the vertical and horizontal direction. Hence, the fact that the first row projection $\iota_2^*: sS \to S$ is part of a Quillen equivalence as stated in Theorem 6.13 also follows from Lemma 6.17 and the general observations in [17, Theorem 3.6] (or [8] respectively).

Note that this does not follow from Corollary 6.11, because $(sS, R_h)$ is not $S$-enriched via $\text{Hom}_2(X, Y) = \iota_2^*(Y^X)$, but via $\text{Hom}_1(X, Y) := \iota_1^*(Y^X)$.

$(sS, CB)$ is a model of univalent type theory

The model structure $CB$ is a cofibrantly generated model structure on the presheaf category $sS$ whose cofibrations are exactly the monomorphisms. This means that the simplicial model category $(sS, CB)$ defines a *Cisinski model category*. Therefore, by [20, Theorem 5.1], in order for $(sS, CB)$ to support a homotopy type theoretical interpretation, we only have to show that $(sS, CB)$ is right proper and that it supports an infinite sequence of univalent universes. In this section we discuss two ways to show this.

The first of these ways can be covered rather swiftly. In [6, Section 1], Cisinski introduces the *locally constant model structure* $(|A^{op}, S|, lc)$ on simplicial presheaves over any elegant Reedy category $A$. It is a left Bousfield localization of the injective model structure whose fibrant objects are exactly the homotopically constant Reedy fibrant objects $X \in |A^{op}, S|$, i.e. those Reedy fibrant objects such that the $f$-action $X(f): X(b) \to X(a)$ is a homotopy weak equivalence for all maps $f: a \to b$ in $A$. Hence, Lemma 6.17 shows that $(sS, lc) = (sS, CB)$. In [5], he shows that $(|A^{op}, S|, lc)$ is always right proper and in [6, Proposition 1.1] he shows that the model category contains a fibrant univalent universe classifying $\kappa$-small maps for every inaccessible cardinal $\kappa$ large enough.

The following presents the second way. We obtain a sequence of univalent fibrant universes for $(sS, CB)$ from [20, Theorem 3.2] (fibrancy of the universes follows by [21, Corollary 2.5.9]) if we can show that there is a set of generating acyclic cofibrations for $(sS, CB)$ with representable codomain. Having obtained the model structure $(sS, CB)$ by left Bousfield localization, it a priori is very hard to present a well behaved set of generating acyclic cofibrations. But recall that the authors of [17] show that the fibrations in the canonical model structure $(sS, CB)$ are exactly the equi-fibred Reedy fibrations. For such, a set of generating acyclic cofibrations is given in [17, Proposition 8.5] by $J_{CB} = J_h \cup \mathcal{J}''$ for

$$J_h = \{s_n^{\delta_i} \cap \delta_m: (\Delta^m \square \delta^\Delta^m) \cup_{\Delta^n \square \delta^\Delta^m} (\Delta^n \square \Delta^m) \to (\Delta^n \square \Delta^m) \mid 0 \leq i \leq m, n\},$$

$$\mathcal{J}'' := \{\delta_n \cap \delta_i^m: (\Delta^m \square \delta^\Delta^m) \cup_{\delta^\Delta^n \square \Delta^m} (\Delta^m \square \delta^\Delta^m) \to (\Delta^n \square \Delta^m) \mid n \geq 0, m \geq i \geq 0\}.$$

The box products $\Delta^m \square \Delta^m$ are exactly the representables in $sS$, thus a set of generating acyclic cofibrations with representable codomain exists indeed.

Also, even though right properness of $(sS, CB)$ follows from the general considerations on fundamental localizers in [5] as mentioned above, there is a direct hands on proof for right properness in this special case. The rest of this subsection presents this proof.
Therefore, we simply use the fact that fibrant objects in \((s\mathcal{S}, \text{CB})\) are exactly the objects fibrant both in the horizontal and the vertical Reedy structures as noted in Remark 6.8, and that both these Reedy structures are right proper. Recall that a model category \(\mathcal{M}\) is right proper if and only if the pullback of any acyclic cofibration with fibrant codomain along fibrations is a weak equivalence. This is shown in [4, Lemma 9.4] for example.

Recall the sets \(\mathcal{J}_v\) and \(\mathcal{J}_h\) from Section 2 which generate the acyclic cofibrations in \((s\mathcal{S}, R_v)\) and in \((s\mathcal{S}, R_h)\) respectively.

**Lemma 6.18.** The class of acyclic cofibrations with fibrant codomain in \((s\mathcal{S}, \text{CB})\) is exactly the class of maps in the saturation of \(\mathcal{J}_v \cup \mathcal{J}_h\) with fibrant codomain, i.e.

\[(W_{\text{CB}} \cap \mathcal{C})/\text{CB-spaces} = \mathcal{N}(\mathcal{J}_v \cup \mathcal{J}_h) / \text{CB-spaces}.\]

**Proof.** As \((s\mathcal{S}, \text{CB})\) is a left Bousfield localization of both \((s\mathcal{S}, R_v)\) and \((s\mathcal{S}, R_h)\), we have

\[\mathcal{J}_v \cup \mathcal{J}_h \subseteq W_{\text{CB}} \cap \mathcal{C},\]

so one direction is clear. Vice versa, let \(j: A \rightarrow B\) be a weak CB-equivalence with \(B\) a complete B-space. Note that \((\mathcal{J}_v \cup \mathcal{J}_h)^\mathcal{N}\) is the intersection of the set \(\mathcal{F}_v\) of v-fibrations and the set \(\mathcal{F}_h\) of h-fibrations, and hence the pair \(\mathcal{N}(\mathcal{J}_v \cup \mathcal{J}_h)/\mathcal{F}_v \cap \mathcal{F}_h\) is a weak factorization system on \(s\mathcal{S}\) by general category theory. Pick a factorization \(A \xrightarrow{k} C \xrightarrow{j} B\) of \(j\) with \(k \in \mathcal{N}(\mathcal{J}_v \cup \mathcal{J}_h)^\mathcal{N}\) and \(q \in \mathcal{F}_v \cap \mathcal{F}_h\),

\[
\begin{array}{ccc}
A & \xrightarrow{k} & C \\
\downarrow j & & \downarrow q \\
B & \xrightarrow{} & B.
\end{array}
\]

Since \(B\) is a complete B-space, \(C\) is now both v-fibrant and h-fibrant, hence a complete B-space, too. But a map between complete B-spaces is a CB-fibration if and only if it is a v-fibration. This in turn holds if and only if it is an h-fibration as can be seen by [13, Proposition 7.21]. Hence, we obtain a lift for the square (*) which exhibits \(j\) as retract of \(k\). Therefore, \(j \in \mathcal{N}(\mathcal{J}_v \cup \mathcal{J}_h)^\mathcal{N}\). \(\square\)

**Corollary 6.19.** Every acyclic cofibration in \((s\mathcal{S}, \text{CB})\) into a complete B-space is the transfinite composition of acyclic v- and h-cofibrations.

\(\square\)

**Lemma 6.20.** The class of morphisms which are mapped into a weak CB-equivalence via pullback along some fixed map \(p\) is saturated.

**Proof.** In the language of [20, Section 3], this holds in virtue of the “exactness” properties of Grothendieck toposes, i.e. pullbacks in \(s\mathcal{S}\) commute with pushouts, transfinite compositions and retracts in such a way that the proof becomes a straightforward induction. \(\square\)

Now, we easily can derive right properness as anticipated.

**Theorem 6.21.** The model category \((s\mathcal{S}, \text{CB})\) is right proper.

**Proof.** By Lemma 6.18 and Lemma 6.20 it remains to check that a pullback square of the form

\[
\begin{array}{ccc}
P & \xrightarrow{j} & D \\
p \downarrow & & \downarrow j \\
X \xrightarrow{p} \Delta^m \sqcup \Delta^n \\
\end{array}
\]

with a CB-fibration \(p\) and \(j \in \mathcal{J}_v \cup \mathcal{J}_h\) exhibits the arrow \(p^* j\) to be a weak CB-equivalence. But \(\mathcal{F}_{\text{CB}}\) is a subset of \(\mathcal{F}_v \cap \mathcal{F}_h\), so \(p\) is both a v-fibration and an h-fibration. Both Reedy structures \((s\mathcal{S}, R_v)\) and \((s\mathcal{S}, R_h)\) are right proper due to the right properness of \((\mathcal{S}, \text{Kan})\). Therefore, \(p^* j \in W_v \cup W_h\). But both \(W_v\) and \(W_h\) are contained in \(W_{\text{CB}}\), since the model structure \(CB\) is a left Bousfield localization of both. This finishes the proof. \(\square\)

\((s\mathcal{S}, \text{CB})\) is a model topos ([16]) in virtue of the Quillen equivalence to \((\mathcal{S}, \text{Kan})\) from Theorem 6.14, but note that the localization \((s\mathcal{S}, R_v) \rightarrow (s\mathcal{S}, \text{CB})\) is not left exact.
Proposition 6.22. The localization \((sS, R_v) \to (sS, CB)\) is not left exact.

Proof. Since every map between non-empty (discrete simplicial) sets is a Kan fibration, every map \(S \to T\) of simplicial sets induces a Reedy fibration \(p_1^* S \to p_1^* T\) of bisimplicial sets. Let

\[
\begin{array}{ccc}
P & \longrightarrow & C \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]

be a cartesian square in \(S\) such that \(C \to B\) is a weak homotopy equivalence and its pullback \(P \to A\) is not. Then

\[
\begin{array}{ccc}
p_1^* P & \longrightarrow & p_1^* C \\
\downarrow & & \downarrow \\
p_1^* A & \longrightarrow & p_1^* B
\end{array}
\]

is cartesian in \(sS\), \(p_1^* C \to p_1^* B\) is a weak equivalence in \((sS, CB)\) and \(p_1^* A \to p_1^* B\) is a Reedy fibration (although \(A \to B\) is not a Kan fibration). Then \(p_1^* P \to p_1^* A\) cannot be a weak equivalence in \((sS, CB)\), because \(p_1^\ast\) is the left adjoint of a Quillen equivalence and hence reflects weak equivalences between cofibrant objects. In particular, the square cannot be homotopy cartesian in \((sS, CB)\). But it certainly is homotopy cartesian in \((sS, R_v)\), because \(p_1^* A \to p_1^* B\) is a Reedy fibration and the Reedy model structure is right proper.

The localization nevertheless is semi-left exact in the sense of [21, Chapter 7], since we just have shown that \((sS, CB)\) is right proper in Section 6 ([21, Lemma 7.3.9.(1)]). But note that semi-left exactness also follows from commutativity of the square of left Quillen functors

\[
\begin{array}{ccc}
(SC, Qcat) & \overset{p_1^*}{\longrightarrow} & (sS, CS) \\
\downarrow & & \downarrow \\
(S, Kan) & \overset{id}{\longrightarrow} & (sS, CB)
\end{array}
\]

and the fact that, first, both horizontal maps are Quillen equivalences by [13, Theorem 4.11] and Theorem 6.13, and second, that the localization \(id \colon (S, Qcat) \to (S, Kan)\) is semi-left exact by [21, Lemma 7.3.9.(1)] since \((S, Kan)\) is right proper. Thus, conversely, this gives yet another proof of right properness of \((sS, CB)\).

Cartesian closedness

In this short last section we prove cartesian closure of the simplicial model category \((sS, CB)\). The result follows easily from Rezk’s combinatorial arguments for cartesian closure of the model category \((sS, CS)\) for complete Segal spaces.

Lemma 6.23. If \(X\) and \(Y\) are complete B-spaces, then so is the exponential \(Y^X\).

Proof. Knowing that \(X\) and \(Y\) are in particular complete Segal spaces, the exponential \(Y^X\) is a complete Segal space by [15, Theorem 7.2]. We are left to show that \(Y^X\) is B-local. Equivalently, we may show that the maps \(\langle p_1^* \iota_{0,n}, id_X \rangle\) : \(p_1^* C_{0,n} \times X \to p_1^* \Delta^n \times X\) are weak CB-equivalences for every complete B-space \(X\). Now, the maps \(p_1^* \iota_{0,n}\) are weak CB-equivalences by Definition 6.1 and we have shown in Theorem 6.21 that \((sS, CB)\) is right proper. Therefore \(\langle p_1^* \iota_{0,n}, id_X \rangle\) is a weak CB-equivalence due to the fibranity of \(X\).

Lemma 6.24. If \(X\) is a complete B-space, then so is the exponential \(X^{p_1^* \Delta^1}\).

Proof. Recall from [15, Theorem 6.2] that the map \(p_1^* e_1 : p_1^* \Delta^1 \to p_1^* J\) is an acyclic cofibration in \((sS, CS)\) (and hence so is \((sS, CB)\)). The Reedy structure \((sS, R_v)\) is cartesian closed and \(p_1^* e_1\) is a cofibration, so \(X^{p_1^* e_1}\) is clearly a v-fibration. Rezk shows in [15, Theorem 7.1] that the model structure \((sS, CS)\) is cartesian closed, hence the objects \(X^{p_1^* J}\) and \(X^{p_1^* \Delta^1}\) are complete Segal spaces. The constant bisimplicial set \(p_1^* J\) is strictly B-local itself (i.e. its Bousfield maps are isomorphisms), because \(J\) is a Kan complex. Let

\[
r_J : p_1^* J \overset{\tau_J}{\longrightarrow} R_v p_1^* J \overset{\tau_{CS}}{\longrightarrow} R_{CS} p_1^* J
\]
be the composition of fibrant replacements in \((sS, R_v)\) and \((sS, CS)\) respectively. Then \(R_v p_1^* J\) is a B-space and \(r_{CS}\) is a Dwyer-Kan equivalence in the sense of \([15, 7.4]\) by \([15, \text{Theorem 7.7}]\). Hence, the homotopy category of the complete Segal space \(R_{CS} p_1^* J\) is a groupoid. Therefore, \(R_{CS} p_1^* J\) is a complete B-space by Corollary 6.6. It follows that the exponential \(X^{R_{CS} p_1^* J}\) is a complete B-space as we have just shown in Lemma 6.23. The maps \(r_J\) and \(p_1^* e_1\) are acyclic cofibrations in \((sS, CS)\), hence the exponential

\[
X^{r_J p_1^* e_1} : X^{R_{CS} p_1^* J} \to X^{p_1^* \Delta^1}
\]

is an acyclic fibration from a complete B-space to a complete Segal space. Hence, \(X^{p_1^* \Delta^1}\) is B-local by \([10, \text{Proposition 3.3.15}(1)]\) and thus a complete B-space.

**Proposition 6.25.** The model structure \((sS, CB)\) is cartesian closed.

**Proof.** This follows immediately from \([15, \text{Proposition 9.2}]\) and Lemma 6.24.

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