Chaos around a Hénon-Heiles-inspired exact perturbation of a black hole

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Abstract

A solution of the Einstein’s equations that represents the superposition of a Schwarzschild black hole with both quadrupolar and octopolar terms describing a halo is exhibited. We show that this solution, in the Newtonian limit, is an analog to the well known Hénon-Heiles potential. The integrability of orbits of test particles moving around a black hole representing the galactic center is studied and bounded zones of chaotic behavior are found.

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There are two main lines of research of chaotic behavior in General Relativity: one deals with chaos associated to the existence of generic singularities in homogeneous cosmological models, as in Bianchi IX model [1]. In this case, Einstein’s equations themselves – although suitably reduced to a simple toy model – form the dynamical system being considered. Invariance of the relativistic theory against arbitrary coordinate changes, particularly time changes, challenges the proper concept of chaos in this approach [2]. The other line assumes a given geometry and looks for chaotic behavior of geodesic motion in this background. In this case, the geometry can be taken as either an approximate or an exact solution of Einstein’s equations. Examples of chaotic geodesic motion in exact geometries are considered in [3–5]. On the other hand, a linearized treatment of the geometry is made in [3] and [7], to model the interaction of a black hole with a weak gravitational wave and a weak gravitational dipole moment, respectively.

Firstly in this letter we consider – with some astronomical motivation – an exact solution of Einstein’s equations that represents a Schwarzschild black hole together with quadrupolar and octopolar contributions of arbitrary strength. Since the multipolar contribution can be switched off, this solution is considered to represent a perturbation of a black hole. Secondly, we address the question whether this exact perturbation breaks the integrability of test motions around the black hole. Some previous results [6,7] deal with linearized perturbations of the Schwarzschild background which, although interesting, make the perturbation “strong enough” to push it out of the full theory. This is an important limitation since we know that chaos is rather the rule than the exception in the general context of dynamical systems and that only a small change in the dynamics could drastically alter the motion portrait.

These facts motivate us to look for a solution of vacuum Einstein’s equations whose additional controlable parameters are linked to multipolar structures, originated in the model from exterior matter distributions. To this end, our search has been inspired by the well known Hénon-Heiles potential (HHP) – classical in astronomy and also in dynamical systems– one of the most studied paradigms of chaotic behavior [8–10]. It was originally intended to model a Newtonian axially symmetric galactic potential. In fact, we have found
a solution that is a general relativistic analog of HHP and moreover seems to be a less idealized model for an axially symmetric galaxy with a massive core plus an exterior halo of quadrupoles and octopoles.

Finally, we shall see that the quadrupolar terms only are not sufficient to cause chaotic geodesic motion. The chaotic behavior occurs only due to the presence of the octopolar terms, which break the reflection symmetry of the potential about the middle plane of the galaxy, in the same manner that the cubic terms in the HHP give rise to the nonintegrability of the motion.

We start with the Weyl metric describing static axially symmetric space-times

$$ds^2 = e^{2\nu}c^2dt^2 - e^{-2\nu}[e^{2\gamma}(dz^2 + dr^2) + r^2d\phi^2].$$  \hspace{1cm} (1)

$\nu$ and $\gamma$ are functions of $r$ and $z$ only. The Einstein’s equations reduce in this case to the usual Laplace equation $\nu_{rr} + \nu_r/r + \nu_{zz} = 0$ and the quadrature $d\gamma = r[(\nu_r)^2 - (\nu_z)^2]dr + 2r\nu_r\nu_zdz$. Despite the simplicity of the Einstein’s equations in the coordinates $r$ and $z$, they are rather deceiving; better coordinates are the prolate spheroidal $u$ and $v$ defined as

$$u = \frac{1}{2L}[\sqrt{r^2 + (z + L)^2} + \sqrt{r^2 + (z - L)^2}], \quad u \geq 1,$$

$$v = \frac{1}{2L}[\sqrt{r^2 + (z + L)^2} - \sqrt{r^2 + (z - L)^2}], \quad -1 \leq v \leq 1,$$  \hspace{1cm} (2)

where $L$ is an arbitrary constant. In these coordinates Laplace’s equation can be solved in terms of standard Legendre polynomials. In particular, we are interested in the solution

$$\nu(u, v) = a_0Q_0(u) + b_2P_2(u)P_2(v) + b_3P_3(u)P_3(v).$$  \hspace{1cm} (3)

The first term is the monopole one, which is related to the Schwarzschild solution with $a_0 = -1$ and $L = GM/c^2 = R_0$. The remaining terms represent the multipolar structure of a halo. We impose the additional condition of elementary flatness over $r = 0$ for $|z| > L$, i.e., $\gamma(u, v) = 0$ in this region of the $z$ axis, thus eliminating conical singularities there.

The integration of $\gamma$ is guaranteed by the fact that $\nu$ is a solution of Laplace’s equation. We find
\[ 2\nu = (1 + \epsilon) \log \left(\frac{u - 1}{u + 1}\right) + \nu_Q(u, v) + \nu_O(u, v), \]
\[ 2\gamma = (1 + \epsilon)^2 \log \left(\frac{u^2 - 1}{u^2 - v^2}\right) + \gamma_Q(u, v) + \gamma_O(u, v) + \gamma_{OO}(u, v), \]  
(4)

where \( \epsilon \) is an arbitrary constant and

\[ \nu_Q = (Q/3)(3u^2 - 1)(3v^2 - 1), \]
\[ \nu_O = (O/5)uv(5u^2 - 3)(5v^2 - 3), \]
\[ \gamma_Q = -4Q(1 + \epsilon)u(1 - v^2) + (Q^2/2)[9u^4v^4 - 10u^4v^2 - 10u^2v^4 \]
\[ + 12u^2v^2 + u^4 + v^4 - 2u^2 - 2v^2 + 1], \]
\[ \gamma_O = -2O(1 + \epsilon)[3u^2v - 3u^2v^3 + v^3 - \frac{9}{5}v + \frac{4}{5}] + 2O^2[\frac{75}{8}u^6v^6 - \frac{117}{8}u^6v^4 \]
\[ - \frac{117}{8}u^4v^6 + \frac{45}{8}u^6v^2 + \frac{45}{8}u^2v^6 + \frac{189}{8}u^4v^4 - \frac{387}{40}u^4v^2 - \frac{387}{40}u^2v^4 + \frac{891}{200}u^2v^2 \]
\[ - \frac{3}{8}u^6 - \frac{3}{8}v^6 + \frac{27}{40}u^4 + \frac{27}{40}v^4 - \frac{81}{200}u^2 - \frac{81}{200}v^2 + \frac{21}{200}], \]
\[ \gamma_{OO} = 2QO[9u^5v^5 - 12u^5v^3 + 3u^5v - 12u^3v^5 + \frac{84}{5}u^3v^3 - \frac{24}{5}u^3v + 3uv^5 - \frac{24}{5}uv^3 + \frac{9}{5}uv]. \]

\( Q \) and \( O \) are respectively the quadrupole and the octopole strengths. The terms proportional to \( Q \) and \( O \) in \( \gamma_Q \) and \( \gamma_O \), respectively, represent nonlinear interactions between the black hole and the multipoles. For \( \epsilon = 0 \) and \( L = R_0 \) we have the Schwarzschild solution perturbed by quadrupoles and octopoles and, if \( \epsilon = -1 \), we are left only with the multipolar structure. Note that only the odd terms in \( v \) break the reflection symmetry about the plane \( z = 0 \) and all them are linked only to the octopole strength. The particular case \( \epsilon = O = 0 \) and \( L = R_0 \) is considered in [11].

To see that the multipolar structure in the solution (4) and (5) is a general relativistic analog of HHP, we write the solution in the usual Schwarzschild coordinates \((t, \rho, \theta, \phi)\)

\[ z = (\rho - L) \cos \theta, \quad r = \sqrt{\rho(\rho - 2L)} \sin \theta. \]  
(6)

For \( \epsilon = 0 \) and \( L = R_0 \) in (4) plus (5), we get

\[ ds^2 = (1 - \frac{2R_0}{\rho})e^{(\nu_Q + \nu_O)}c^2dt^2 - e^{(\gamma_Q + \gamma_O + \gamma_{OO} - \nu_Q - \nu_O)}[(1 - \frac{2R_0}{\rho})^{-1}d\rho^2 + \rho^2d\theta^2] \]
\[ + e^{-(\nu_Q + \nu_O)}\rho^2 \sin^2 \theta d\phi^2. \]  
(7)

In the Newtonian approximation only \( g_\mu \) in (7) is important. We expand it to the first order in the strengths \( Q \) and \( O \) and identify a Newtonian potential \( \Phi_N \) through
\[ g_{tt} = 1 + \frac{2}{c^2} \Phi_N \] .

Remember that (8) are valid only for \( \rho \geq 2R_0 \). If we suppose that there exists some region between the horizon event \( \rho = 2R_0 \) of the galactic core and the external multipolar structure simulating the halo, in which the Schwarszchild coordinates are approximately the Euclidean spherical ones, then there we have \( z \approx \rho \cos \theta \) and \( \rho^2 \approx z^2 + R^2 \) with \( R \) being the radial cylindrical coordinate. Expanding (8) and using this approximation, we arrive at

\[ \Phi_N = -\frac{GM}{\rho} + \frac{Qc^2}{2R_0^2}[2z^2 - R^2 + f(\rho, z)] + \frac{Qc^2}{2R_0^2}[2z^3 - 3zR^2 + g(\rho, z)] , \] (9)

where

\[
\begin{align*}
f(\rho, z) &= -\frac{R_0^2}{3}(3\frac{z^2}{\rho^2} - 1)(12\frac{\rho}{R_0} - 14 + 4\frac{R_0}{\rho}), \\
g(\rho, z) &= -\frac{R_0^2}{5}z(5\frac{z^2}{\rho^2} - 3)(25\frac{\rho}{R_0} - 42 + 26\frac{R_0}{\rho} - 4\frac{R_0^2}{\rho^2}) .
\end{align*}
\] (10)

For comparison we quote HHP from [8]

\[ \Phi_{HH} = \frac{1}{2}[z^2 + R^2 + 2zR^2 - \frac{2}{3}z^3] . \] (11)

The quadrupolar and octopolar terms explicitly shown in (9) are qualitatively the same as those appearing in HHP, with one difference that will be of importance in the latter discussion: in HHP the quadrupolar terms constitute an (isotropic) bidimensional oscillator, while in our case we have a saddle point. Once HHP is considered to be by far more interesting by its dynamical richness allied to simplicity than by actually modeling any concrete astronomical situation, the similarity above justifies the parallelism. Moreover, the quadratic and cubic terms in (9) are indubitably in the extended Hénon-Heiles family of potentials [9].

In Fig. 1 we show Poincaré sections across the plane \( z = 0 \) in Weyl coordinates for some time-like geodesic orbits for the metric (9) and (8). In Fig. 1(a) we show a section for the unperturbed Schwarszchild background \( (\epsilon = 0, Q = O = 0) \), where the complete integrability is seen. In Fig. 1(b) we switch on the octopolar terms over Fig. 1(a) \( (\epsilon = 0, \)
\( Q = 0 \) and \( O \neq 0 \). This breaks the reflection symmetry about the middle plane and hence also the integrability of the motion. We also have checked that this chaotic manifestation is not explained by the signal structure of the eigenvalues of the curvature tensor as proposed in [4]. We see that the octopole terms are a powerful source of chaos since the strength we have used is only \( O = 1.5 \times 10^{-7} \) in nondimensional units. In Fig. 1(c) we switch on the quadrupolar terms over Fig. 1(a) (\( \epsilon = 0, Q \neq 0 \) and \( O = 0 \)). We observe that integrability is preserved, yet in a more involved toroidal structure in phase space. Finally, Fig. 1(d) exhibits chaotic behavior due to the full perturbation of the Schwarzschild background (\( \epsilon = 0, Q \neq 0 \) and \( O \neq 0 \)). We note that although the quadrupolar terms do not cause chaos, their presence only with strength \( Q = 2.2 \times 10^{-6} \) does the system more sensible to the octopolar terms.

By switching the black hole off (\( \epsilon = -1 \)) – with or without the octopoles – we were not able to find regions of bounded motion that include points outside the middle plane. This lack of nonplanar bounded regions when solely the multipolar halo is taken into account is probably related to the repulsive quadratic \( z \)-term present in the expansion [3].

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FIGURES

FIG. 1. Poincaré sections (in Weyl coordinates) in the plane $z = 0$ for timelike geodesic orbits in different geometries. All test particles (with mass $\mu$) have energy $E = 0.975 \mu c^2$ and angular momentum $L = 3.8 R_0 \mu c$. In all figures $\epsilon = 0$ and the nondimensional variables are defined by $X \equiv r/R_0$ and $dT \equiv ds/R_0$: (a) unperturbed Schwarzschild metric ($Q = O = 0$). (b) Schwarzschild metric perturbed by octopolar terms ($Q = 0, O = 1.5 \times 10^{-7}$). (c) Schwarzschild metric perturbed by quadrupolar terms ($Q = 2.2 \times 10^{-6}, O = 0$). (d) Schwarzschild metric plus the full perturbation ($Q = 2.2 \times 10^{-6}, O = 1.5 \times 10^{-7}$).