ADAPTIVE DISTRIBUTED METHODS UNDER COMMUNICATION CONSTRAINTS

By Botond Szabó∗†, and Harry van Zanten∗

Leiden University and University of Amsterdam

We study distributed estimation methods under communication constraints in a distributed version of the nonparametric signal-in-white-noise model. We derive minimax lower bounds and exhibit methods that attain those bounds. Moreover, we show that adaptive estimation is possible in this setting.

1. Introduction. In this paper we take up the study of the fundamental possibilities and limitations of distributed methods for high-dimensional, or nonparametric problems. The design and study of such methods has attracted substantial attention recently. This is for a large part motivated by the ever increasing size of datasets, leading to the necessity to analyze data while distributed over multiple machines and/or cores. Other reasons to consider distributed methods include privacy considerations or the simple fact that in some situations data are physically collected at multiple locations.

By now a variety of methods are available for estimating nonparametric or high-dimensional models to data in a distributed manner. A (certainly incomplete) list of recent references includes the papers [12, 18, 11, 4, 13, 14, 8]. The number of more theoretical papers on the fundamental performance of such methods is still rather limited however. In the paper [15] we recently introduced a distributed version of the canonical signal-in-white-noise model to serve as a benchmark model to study aspects like convergence rates and optimal tuning of distributed methods. We used it to compare the performance of a number of distributed nonparametric methods recently introduced in the literature. The study illustrated the perhaps intuitively obvious fact that in order to achieve an optimal bias-variance trade-off, or, equivalently, to find the correct balance between over- and under-fitting, distributed methods need to be tuned differently than methods that handle all data at once. Moreover, our comparison showed that some of the proposed methods are more successful at this than others.

A major challenge and fundamental question for nonparametric distributed methods is whether or not it is possible to achieve a form of adaptive inference. In other words, whether we can design methods that do automatic, data-driven tuning in order to achieve the optimal bias-variance trade-off. We illustrated by example in [15] that naively using methods that are known to achieve optimal adaptation in non-distributed settings, can easily lead to sub-optimal performance in the distributed case. In the recent paper [19], which considers the same distributed signal-in-white-noise model and was written independently and at the same time as the present paper, it is in fact conjectured that adaptation in this model is not possible.

In order to study convergence rates and adaptation for distributed methods in a meaningful way the class of methods should be restricted somehow. Indeed, if there is

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no limitation on communication or computation, then we could simply communicate all data from the various local machines to a central machine, aggregate it, and use some existing adaptive, rate-optimal procedure. In this paper we consider a setting in which the communication between the local and the global machines is restricted, much in the same way as the communication restrictions imposed in [18] in a parametric framework and recently in the simultaneously written paper [19] in the context of the distributed signal-in-white-noise model we introduced in [15].

In this distributed model with communication constraints we can derive minimax lower bounds for the best possible rate that any distributed procedure can achieve under smoothness conditions on the true signal. Technically this essentially relies on an extension of the information theoretic approach of [18] to the infinite-dimensional setting (this is different from the approach taken in [19], which relies on results from [17]). It turns out there are different regimes, depending on how much communication is allowed. On the one extreme end, and in accordance with intuition perhaps, if there is enough communication allowed, it is possible to achieve the same convergence rates in the distributed setting as in the non-distributed case. The other extreme case is that there is so little communication allowed that combining different machines does not help. Then the optimal rate under the communication restriction can already be obtained by just using a single local machine and discarding the others. The interesting case is the intermediate regime. For that case we show there exists an optimal strategy that involves grouping the machines in a certain way and letting them work on different parts of the signal.

These first results on rate-optimal distributed estimators are not adaptive, in the sense that the optimal procedures depend on the regularity of the unknown signal. The same holds true for the procedure obtained in parallel in [19]. In this paper we go a step further and show that contrary perhaps to intuition, and contrary to the conjecture in [19], adaptation is in fact possible. Indeed, we exhibit in this paper an adaptive distributed method which involves a very specific grouping of the local machines, in combination with a Lepski-type method that is carried out in the central machine. We prove that the resulting distributed estimator adapts to a range of smoothness levels of the unknown signal and that, up to logarithmic factors, it attains the minimax lower bound.

Although our analysis is theoretical, we believe it contains interesting messages that are ultimately very relevant for the development of applied distributed methods in high-dimensional settings. First of all, we show that depending on the communication budget, it might be advantageous to group local machines and let different groups work on different aspects of the high-dimensional object of interest. Secondly, we show that it is possible to have adaptation in communication restricted distributed settings, i.e. to have data-driven tuning that automatically achieves the correct bias-variance trade-off. We note however that although our proof of this fact is constructive, the method we exhibit appears to be still rather unpractical. We view our adaptation result primarily as a first proof of concept, that hopefully invites the development of more practical adaptation techniques for distributed settings.

1.1. Notations. For two positive sequences $a_n, b_n$ we use the notation $a_n \lesssim b_n$ if there exists an universal positive constant $C$ such that $a_n \leq C b_n$. Along the lines $a_n \asymp b_n$ denotes that $a_n \lesssim b_n$ and $b_n \lesssim a_n$ hold simultaneously. Furthermore we write $a_n \ll b_n$ if $a_n/b_n = o(1)$. In the proofs we use the notation $C$ and $c$ for universal constants which value can differ from line to line. Let us denote by $\lceil a \rceil$ and $\lfloor a \rfloor$ the upper and lower integer value of the real number $a$, respectively. The sum $\sum_{i=a}^{b} x_j$
for $a, b$ positive real number denotes the sum $\sum_{i \in \mathbb{N} : a \leq i \leq b} x_j$. For a set $A$ let $\#(A)$ or $|A|$ denote the size of the set. For $f \in L_2[0, 1]$ we denote the standard $L_2$-norm as $\|f\|_2^2 = \int_0^1 f(x)^2 dx$, while for bounded functions $\|f\|_\infty$ denotes the $L_\infty$-norm.

The function sign : $\mathbb{R} \mapsto \{0, 1\}$ evaluates to 1 on $(-\infty, 0)$ and 0 on $[0, \infty)$. Furthermore, we use the notation $\text{mean}\{a_1, \ldots, a_n\} = (a_1 + \ldots + a_n)/n$.

2. Main results. As in [15] we work with the distributed version of the signal-in-white-noise model which, we regard as the natural ‘benchmark model’ in this case, playing the same role as the usual signal-in-white-noise model in the non-distributed case (e.g. [16, 7, 10]). We assume that we have $m$ ‘local’ machines and in the $i$th machine we observe the random function $(X_t^{(i)} : t \in [0, 1])$ satisfying the stochastic differential equation

\begin{align}
    dX_t^{(i)} = f_0(t) + \sqrt{\frac{m}{n}} dW_t^{(i)}, \quad t \in [0, 1],
\end{align}

where $W^{(1)}, \ldots, W^{(m)}$ are independent Wiener processes and $f_0 \in L_2[0, 1]$ (which is the same for all machines) is the unknown functional parameter of interest. For our theoretical results we will assume that the unknown true function $f_0$ belongs to some regularity class. We work in our analysis with Besov smoothness classes, more specifically we assume that for some degree of smoothness $s > 0$ we have $f_0 \in B^s_{2,\infty}(L)$ or $f_0 \in B^s_{\infty,\infty}(L)$. The first class is of Sobolev type, while the second one is of Hölder type. For precise definitions, see Appendix B. Each local machine carries out (parallel to the others) a local statistical procedure and transmits the results to a central machine, which produces an estimator for the signal $f_0$ by somehow aggregating the messages received from the local machines.

We study these distributed procedures under communication constraints between the local machines and the central machine. We allow each local machine to send at most $B^{(i)}$ bits on average to the central machine. More formally, a distributed estimator $\hat{f}$ is a measurable function of $m$ binary strings, or messages, passed down from the local machines to the central machine. We denote by $Y^{(i)}$ the finite binary string transmitted from machine $i$ to the central machine, which is a measurable function of the local data $X^{(i)}$. For a class of potential signals $\mathcal{F} \subset L_2[0, 1]$, we restrict the communication between the machines by assuming that for numbers $B^{(1)}, \ldots, B^{(m)}$, it holds that $\mathbb{E}_{f_0}[l(Y^{(i)})] \leq B^{(i)}$ for every $f_0 \in \mathcal{F}$ and $i = 1, \ldots, m$, where $l(Y)$ denotes the length of the string $Y$. We denote the resulting class of communication restricted distributed estimators $\hat{f}$ by $\mathcal{F}_{\text{dist}}(B^{(1)}, \ldots, B^{(m)}; \mathcal{F})$. The number of machines $m$ and the communication constraints $B^{(i)}$ are allowed to depend on the overall signal-to-noise ratio $n$, in fact that is the interesting situation. To alleviate the notational burden somewhat we do not make this explicit in the notation however.

2.1. Distributed minimax lower bounds for the $L_2$-risk. The first theorem we present gives a minimax lower bound for distributed procedures for the $L_2$-risk, uniformly over Sobolev-type Besov balls, see Appendix B for rigorous definitions.

\textbf{Theorem 2.1.} Consider $s, L > 0$ and communication constraints $B^{(1)}, \ldots, B^{(m)} > 0$. Let the sequence $\delta_n = o(1)$ be defined as the solution to the equation

\begin{align}
    \delta_n = \min \left\{ \frac{m}{n \log_2 n}, \frac{m}{n \sum_{i=1}^m [\log_2(n)\delta_n^{1+s}] B^{(i)} \wedge 1} \right\}.
\end{align}
Then in distributed signal-in-white-noise model (2.1) we have that

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B^{(1)}, \ldots, B^{(m)}; B_{2, \infty}^s(L))} \sup_{f_0 \in B_{2, \infty}^s(L)} \mathbb{E}_{f_0} \| \hat{f} - f_0 \|_2^2 \gtrsim \delta_n^{2s}. $$

\textbf{Proof.} See Section 3.1 \hfill \square

We briefly comment on the derived result. First of all note that the quantity $\delta_n$ in (2.2) is well defined, since the left-hand side of the equation is increasing, while the right-hand side is decreasing in $\delta_n$. The proof of the theorem is based on an application of a version of Fano’s inequality, frequently used to derive minimax lower bounds. More specifically, as a first step we find as usual a large enough finite subset of the whole space. This is done by finding the ‘effective resolution level’ $j_n$ in the wavelet representation of the function of interest and perturbing the corresponding wavelet coefficients, while setting the rest of the coefficients to zero. This effective resolution level for non-distributed models (e.g. [7]). However, in our distributed setting the effective resolution changes to $(1 + 2s)^{-1} \log 2 n$, which can be substantially different from the non-distributed case, as it strongly depends on the number of transmitted bits. The dependence on the expected number of transmitted bits enters the formula by using a variation of Shannon’s source coding theorem. Many of the information-theoretic manipulations in the proof are an extended and adapted version of the approach introduced in [18] in context of distributed methods with communication constraints over parametric models.

To understand the result it is illustrative to consider the special case that the communication constraints are the same for all machines, i.e. $B^{(1)} = \cdots = B^{(m)} = B$ for some $B > 0$. We can then distinguish three regimes: (i) the case $B \geq n^{1/(1 + 2s)} / \log_2 n$; (ii) the case $(n \log_2(n)/m)^{2 + 2s} (1/(1 + 2s) \leq B < n^{1/(1 + 2s)} / \log_2 n$; and (iii) the case $B < (n \log_2(n)/m)^{2 + 2s} (1/(1 + 2s)$.

In regime (i) we have a large communication budget and by elementary computations we get that the minimum in (2.2) is taken in the second fraction and hence that $\delta_n = 1/n$. This means that in this case the derived lower bound corresponds to the usual non-distributed minimax rate $n^{-2s/(1 + 2s)}$. In the other extreme case, regime (iii), the minimum is taken at the first term in (2.2) and $\delta_n = m/(n \log_2 n)$, so the lower bound is of the order $(n \log_2(n)/m)^{-2s/(1 + 2s)}$. This rate is, up to the $\log_2 n$ factor, equal to the minimax rate corresponding to the noise level $n/m$. Consequently, in this case it does not make sense to consider distributed methods, since by just using a single machine the best rate can already obtained (up to a logarithmic factor). In the intermediate case (ii) it is straightforward to see that $\delta_n = (nB \log_2 n)^{(1+2s)/(2+2s)}$, It follows that if $B = o(n^{1/(1 + 2s)} / \log_2 n$, i.e. if we are only allowed to communicate ‘strictly’ less than in case (i), then the lower bound is strictly worse than the minimax rate corresponding to the non-distributed setting.

The findings above are summarized in the following corollary.

\textbf{Corollary 2.2.} Consider $s, L > 0$ and a communication constraint $B > 0$.

(i) If $B \geq n^{1/(1 + 2s)} / \log_2 n$, then

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B, \ldots, B; B_{2, \infty}^s(L))} \sup_{f_0 \in B_{2, \infty}^s(L)} \mathbb{E}_{f_0} \| \hat{f} - f_0 \|_2^2 \gtrsim n^{-2s}. $$


(ii) If \((n \log_2(n)/m^{2+2s})^{1/(1+2s)} \leq B < n^{1/(1+2s)}/\log_2 n\), then
\[
\inf_{f \in \mathcal{F}_{dist}(B,\ldots,B;B_{2,\infty}(L))} \sup_{f_0 \in B_{2,\infty}(L)} \mathbb{E}_{f_0} \|f - f_0\|_2^2 \geq \left(\frac{n^{1/(1+2s)}}{B \log_2 n}\right)^{2s/(1+2s)} n^{-2s/(1+2s)}.
\]
(iii) If \((n \log_2(n)/m^{2+2s})^{1/(1+2s)} > B\)
\[
\inf_{f \in \mathcal{F}_{dist}(B,\ldots,B;B_{2,\infty}(L))} \sup_{f_0 \in B_{2,\infty}(L)} \mathbb{E}_{f_0} \|f - f_0\|_2^2 \geq \left(\frac{n \log_2 n}{m}\right)^{-2s/(1+2s)}.
\]

2.2. Non-adaptive rate-optimal distributed procedures for \(L_2\)-risk. Next we show that the derived lower bounds are sharp by exhibiting distributed procedures that attain the bounds (up to logarithmic factors). We note that it is sufficient to consider only the case \(B \geq (n \log_2(n)/m^{2+2s})^{1/(1+2s)}\), since otherwise distributed techniques do not perform better than standard techniques carried out on one of the local servers.

The estimators we provide start by a standard expansion of the observed sample paths on every machine in an appropriate wavelet basis (cf. e.g. [7]). In \(L_2[0,1]\) we consider Daubechies wavelets \(\varphi_{j,k}, j \in \mathbb{N}, k \in K_j\), of regularity \(s_{\text{max}} > s\). Here \(K_j\) is the set of wavelet basis function at resolution level \(j\) and note that \(|K_j| = 2^j\). (For details, see Appendix B.) Then, slightly abusing notation, for every machine \(i\) we define the observed noisy wavelet coefficients by
\[
X_{jk}^{(i)} = \int_0^1 \varphi_{j,k}(t) dX_t^{(i)}.
\]
Note that it follows from (2.1) that for all \(j \in \mathbb{N}\) and \(k \in K_j\) we have
\[
X_{jk}^{(i)} = f_{0,jk} + \sqrt{\frac{m}{n}} Z_{jk}^{(i)},
\]
where \(f_{0,jk} = \int_0^1 f_0(t) \varphi_{j,k}(t) dt\) are the wavelet coefficients of the true signal \(f_0\) and the \(Z_{jk}^{(i)}\) are independent standard normal variables.

Next, we repeatedly use a particular method to transmit a finite-bit approximation of a real number from a local machine to the central machine. To explain how it works, take an arbitrary number \(x \in \mathbb{R}\) not equal to zero and consider its binary scientific representation
\[
x = (-1)^{\text{sign}(x)} \ast 0.1\ldots \ast 2^{1+\lfloor \log_2 |x| \rfloor}.
\]
Note that the sign of \(x\) can be encoded by one bit, the sign of the exponent again by one bit and the absolute value of the exponent by \(1 + \lfloor \log_2(1 + \lfloor \log_2 |x| | \rfloor) \rfloor\) bits, since it is a nonnegative integer. Then we construct our approximation \(y\) of \(x\) by taking the first \(\lfloor 0.5 \log_2 n \rfloor\) bits after the dot in the scientific representation of \(x\) and multiplying it by \((-1)^{\text{sign}(x)} \ast 2^{1+\lfloor \log_2 |x| \rfloor}\). Note that \(x = 0\) can be transmitted by a single bit. Abusing our notation slightly, we use the notation \(y\) both for the real number approximating \(x\) and for the binary string encoding it. The length of the string is denoted by \(l(y)\). We summarize the approximation scheme in Algorithm 1 (with \(D = 1/2\)).

Observe that the length of the constructed approximation \(y\) of the number \(x\) (viewed as a binary string) is bounded from above by \(3 + \log_2(1 + \lfloor \log_2 |x| | \rfloor) + \lfloor \log_2 \frac{1}{2} x \rfloor\).
coefficients we simply average the transmitted approximations to obtain the estimated wavelet and 

\[ \hat{c} > 0 \]

for some large enough constant \( c \). Alternatively, we also have that

\[ \hat{c} = \text{first} \left( \left\lfloor \frac{D \log_2 n}{|\alpha|} \right\rfloor \right) \text{digits of} \ x. \]

The final estimator \( \hat{c} \) for \( x \) is defined as Algorithm 1 below.

**Algorithm 1** Transmitting a finite-bit approximation of a number

1: procedure TransApprox\((x)\)
2: \( \text{For} \ x = 0 \ \text{transmit} \ 0. \)
3: \( \text{Else write} \ x = (-1)^{\text{sign}(x)} \cdot 0.1... \cdot 2^{1+|\log_2 |x|}. \)
4: \( \text{Transmit:} \ \text{sign}(x), \text{sign}(1 + |\log_2 |x|), \lfloor 1 + |\log_2 |x| \rfloor. \)
5: \( \text{Transmit:} \ \bar{y} = \text{first} \left( \left\lfloor \frac{D \log_2 n}{|\alpha|} \right\rfloor \right) \text{digits of} \ x. \)
6: \( \text{Construct:} \ y = (-1)^{\text{sign}(x)} \cdot 0.\bar{y} \cdot 2^{(1-\text{sign}(1+|\log_2 |x|)) \cdot \lfloor 1+|\log_2 |x| \rfloor}. \)

0.5 log2 \( n \) bits. The following lemma asserts that if \( X \) is a normally distributed random variable with bounded mean and variance \( m/n \), then the expected length \( \mathbb{E}[l(Y)] \) of the constructed binary string approximating \( X \) is less than log2 \( n \) (for sufficiently large \( n \)) and the approximation is sufficiently close to \( X \).

**Lemma 2.3.** Assume that \( X \sim N(\mu, m/n) \), with \( |\mu| \leq L \). Then the approximation \( Y \) of \( X \) given in Algorithm 1 satisfies that

\[
0 \leq (X - Y)/X \leq n^{-D} \quad \text{and} \quad \mathbb{E}[l(Y)] \leq (D + o(1)) \log_2(n).
\]

Furthermore we also have that

\[
\mathbb{P}(l(Y) \leq D \log_2 n + 3 + \log_2 \log_2 n) \geq 1 - cn^{-3/2},
\]

for some large enough constant \( c > 0 \).

**Proof.** See Section 3.4.

After these preparations we can exhibit procedures attaining (nearly) the theoretical limits obtained in Corollary 2.2.

We first consider the case (i) that \( B \geq n^{1/(1+2s)}/\log_2 n \). In this case each local machine \( i = 1, \ldots, m \) transmits the approximations \( Y_{jk}^{(i)} \) (given in Algorithm 1) of the first \( n^{1/(1+2s)} \wedge (B/\log_2 n) \) observed coefficients \( X_{jk}^{(i)} \), i.e. for all indices \( j \in \mathbb{N} \) and \( k \in K_j \) such that \( 2^j + k \leq n^{1/(1+2s)} \wedge (B/\log_2 n) \). Then in the central machine we simply average the transmitted approximations to obtain the estimated wavelet coefficients

\[
\hat{f}_{jk} = \begin{cases} \frac{1}{m} \sum_{i=1}^{m} Y_{jk}^{(i)}, & \text{if} \ 2^j + k \leq n^{1/(1+2s)} \wedge (B/\log_2 n), \\ 0, & \text{else}. \end{cases}
\]

The final estimator \( \hat{f} \) for \( f_0 \) is the function in \( L_2[0, 1] \) with these wavelet coefficients, i.e. \( \hat{f} = \sum \hat{f}_{jk} \phi_{jk} \). The method is summarized as Algorithm 2 below.

**Algorithm 2** Nonadaptive \( L_2 \)-method, case (i)

1: In the local machines:
2: \( \text{for} \ i = 1 \ \text{to} \ m \) do:
3: \( \text{for} \ 2^j + k = 1 \ \text{to} \ n^{1/(1+2s)} \wedge (B/\log_2 n) \) do
4: \( Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)}) \)
5: In the central machine:
6: \( \text{for} \ 2^j + k = 1 \ \text{to} \ n^{1/(1+2s)} \wedge (B/\log_2 n) \) do
7: \( \hat{f}_{jk} := \text{mean}(Y_{jk}^{(i)} : 1 \leq i \leq m) \).
8: Construct: \( \hat{f} = \sum \hat{f}_{jk} \phi_{jk} \).

The following theorem asserts that the constructed estimator indeed attains the lower bound in case (i) (up to a logarithmic factor).
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**Theorem 2.4.** Let \( s, L > 0 \) and suppose that \( B \geq n^{1/(1+2s)} / \log_2 n \). Then the distributed estimator \( \hat{f} \) described in Algorithm 2 belongs to \( \mathcal{F}_{dist}(B, \ldots, B; B_{2,\infty}^2(L)) \) and satisfies

\[
\sup_{f_0 \in B_{2,\infty}^2(L)} \mathbb{E} f_0 \| \hat{f} - f_0 \|^2 \lesssim n^{-\frac{1}{1+2s}} \vee (B/\log_2 n)^{-2s}.
\]

**Proof.** See Section 3.2

Next we consider the case (ii) of Corollary 2.2, i.e. the case that the communication restriction satisfies \( (n \log_2(n)/m^{2+2s})^{1/(1+2s)} \leq B < n^{1/(1+2s)} / \log_2 n \). Using Algorithm 2 in this case would result in a highly sub-optimal procedure, as we prove at the end of Section 3.3. It turns out that under this more severe communication restriction we can do much better if we form different groups of machines that work on different parts of the signal.

We introduce the notation \( \eta = \lfloor (n^{1/2} \log_2 n/B) (1+2s)/(2+2s) \rfloor \) and note that in this case \( 1 \ll \eta \ll m \). Then we group the local machines into \( \eta \) groups and let the different groups work on different parts of the signal as follows: the machines with numbers \( 1 \leq i \leq m/\eta \) each transmit the approximations \( Y_{jk}^{(i)} \) of the observations \( X_{jk}^{(i)} \) for \( 1 \leq 2^i + k \leq B/\log_2 n \); the next machines, with numbers \( m/\eta < i \leq 2m/\eta \), each transmit the approximations \( Y_{jk}^{(i)} \) for \( B/\log_2 n < 2^i + k \leq 2B/\log_2 n \), and so on. The last machines with numbers \( (\eta - 1)m/\eta < i \leq m \) transmit the \( Y_{jk}^{(i)} \) for \( (\eta - 1)B/\log_2 n < 2^i + k \leq \eta B/\log_2 n \). Then in the central machine we average the corresponding transmitted noisy coefficients in the obvious way. Formally, using the notation \( \mu_{jk} = \lfloor (2^i + k) \log_2 n/B \rfloor - 1 \), the aggregated estimator \( \hat{f} \) is the function with wavelet coefficients given by

\[
\hat{f}_{jk} = \begin{cases} 
  \text{mean}(Y_{jk}^{(i)} : \frac{\mu_{jk}m}{\eta} < i \leq \frac{(\mu_{jk}+1)m}{\eta}), & \text{if } 2^i + k \leq \frac{\eta B}{\log_2 n}, \\
  0, & \text{else}.
\end{cases}
\]

The procedure is summarized as Algorithm 3.

**Algorithm 3** Nonadaptive \( L_2 \)-method, case (ii)

1: In the local machines:
2: for \( \ell = 1 \) to \( \eta \) do
3: for \( i = \lfloor (\ell - 1)m/\eta \rfloor + 1 \) to \( \lfloor \ell m/\eta \rfloor \) do
4: for \( 2^i + k = \lfloor (\ell - 1)B/\log_2 n \rfloor + 1 \) to \( \lfloor \ell B/\log_2 n \rfloor \) do
5: \( Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)}) \)
6: In the central machine:
7: for \( 2^i + k = \lfloor \eta B/\log_2 n \rfloor \) do
8: \( f_{jk} := \text{mean}(Y_{jk}^{(i)} : \frac{\mu_{jk}m}{\eta} < i \leq \frac{(\mu_{jk}+1)m}{\eta}) \)
9: Construct: \( \hat{f} = \sum f_{jk} \varphi_{jk} \).

The following theorem asserts that this estimator attains the lower bound in case (ii) (up to a logarithmic factor).

**Theorem 2.5.** Let \( s, L > 0 \) and suppose that \( (n \log_2(n)/m^{2+2s})^{1/(1+2s)} \leq B < n^{1/(1+2s)} / \log_2 n \). Then the distributed estimator \( \hat{f} \) described in Algorithm 3 belongs
to $F_{\text{dist}}(B, \ldots, B; B_{2, \infty}^s(L))$ and satisfies

$$\sup_{f_0 \in B_{s, \infty}^\infty(L)} \mathbb{E}_{f_0} \| \hat{f}_n - f_0 \|^2 \lesssim M_n \left( \frac{n^{1/(1 + 2s)}}{B \log_2 n} \right)^{2s} n^{-2s/(1 + 2s)},$$

with $M_n = \left( \log_2 n \right)^{4s}$. 

**Proof.** See Section 3.3.

2.3. Distributed minimax results for $L_\infty$-risk. When we replace the $L_2$-norm by the $L_\infty$-norm and correspondingly change the type of Besov balls we consider, we can derive a lower bound similar to Theorem 2.1 (see Appendix B for the rigorous definition of Besov balls).

**Theorem 2.6.** Consider $s, L > 0$ and communication constraints $B^{(1)}, \ldots, B^{(m)} > 0$. Let the sequence $\delta_n = o(1)$ be defined as the solution to the equation (2.2). Then in distributed signal-in-white-noise model (2.1) we have that

$$\inf_{\hat{f} \in F_{\text{dist}}(B^{(1)}, \ldots, B^{(m)}; B_{s, \infty}^\infty(L))} \sup_{f_0 \in B_{s, \infty}^\infty(L)} E_{f_0} \| \hat{f} - f_0 \|_{\infty} \gtrsim \left( \frac{n}{\log n} \right)^{s/(1 + 2s)} \vee n^{-s/(1 + 2s)},$$

**Proof.** See Section 3.5.

The proof of the theorem is very similarly to the proof of Theorem 2.6. The first term on the right hand side follows from the usual non-distributed minimax lower bound. For the second term we use the standard version of Fano’s inequality. We again consider a large enough finite subset of $B_{s, \infty}^\infty(L)$. The effective resolution level in the non-distributed case is $(1 + 2s)^{-1} \log_2 (n/\log n)$ in case of the $L_\infty$-norm. Similar to the $L_2$ case the effective resolution level changes to $(1 + 2s)^{-1} \log \delta_n^{-1}$, which can be again substantially different from the non-distributed case. The rest of the proof follows the lines of that of Theorem 2.6.

We can draw similar conclusions for the $L_\infty$-norm as for the $L_2$-norm. If we do not transmit a sufficient amount of bits (at least $n^{1/(1 + 2s)}$ up to a log $n$ factor) from the local machines to the central one then the lower bound from the theorem exceeds the minimax risk corresponding the non-distributed case. Furthermore by transmitting the sufficient amount of bits $Cn^{1/(1 + 2s)} \log n$ corresponding to the class $B_{s, \infty}^\infty(L)$, the lower bound will coincide with the non-distributed minimax estimation rate. One can also show that slightly modified versions of the algorithms given in the $L_2$-case can reach the lower bound (up to a logarithmic factor) in the $L_\infty$-case as well, hence the derived results are (nearly) sharp.

2.4. Adaptive distributed estimation. The (almost) rate-optimal procedures considered so far have in common that they are non-adaptive, in the sense that they all use the knowledge of the regularity level $s$ of the unknown functional parameter of interest. In this section we exhibit a distributed algorithm attaining the lower bounds (up to a logarithmic factor) across a whole range of regularities $s$ simultaneously. In the non-distributed setting it is well known that this is possible, and many adaptation methods exist, including for instance the block Stein method, Lepski’s method and Bayesian adaptation methods (e.g. [16, 7]). In the distributed case the matter is more complicated. Using the usual adaptive tuning methods in the local
machines will typically not work (see [15]) and in fact it was recently conjectured that adaptation, if at all possible, would require more communication than is allowed in our model (see [19]).

We will show however that in our setting, if all machines have the same communication restriction given by $B > 0$, it is possible to adapt to regularities $s$ ranging in the interval $[s_{\min}, s_{\max})$, where

$$s_{\min} = \arg\inf_{s > 0} \left\{ \left( n \log_2(n)/m^{2+2s} \right)^{1/2s} \leq B \right\}$$

and $s_{\max}$ is the regularity of the considered Daubechies’ wavelet and can be chosen arbitrarily large. Note that $s_{\min}$ is well defined. If $s \in [s_{\min}, s_{\max})$, then we are in one of the non-trivial cases (i) or (ii) of Corollary 2.2. We will construct a distributed method which, up to logarithmic factors, attains the corresponding lower bounds, without using knowledge about the regularity level $s$.

In the non-adaptive case we saw that different strategies were required to attain the optimal rate, case (ii) requiring a particular grouping of the local machines. The cut-off between cases (i) and (ii) depends however on the value of $s$, so in the present adaptive setting we do not know beforehand in which of the two cases we are. In order to tackle this problem we introduce a somewhat more involved grouping of the machines, which basically gives us the possibility to carry out both strategies simultaneously. This is combined with a modified version of Lepski’s method, carried out in the central machine, ultimately leading to (nearly) optimal distributed concentration rates for every regularity class $s \in [s_{\min}, s_{\max})$, simultaneously.

As a first step in our adaptive procedure we divide the machines into groups. Let us first take the first $\lfloor m/2 \rfloor$ machines and denote the set of their index numbers by $I$. Then the remaining $\lceil m/2 \rceil$ machines are split into $\hat{n} = \lfloor (1 + 2s_{\min} - 1) \log_2 n \rfloor$ equally sized groups (for simplicity each group has $\lfloor m/2 \rfloor/\hat{n}$ machines and the leftovers are discarded). The corresponding sets of indices are denoted by $I_0, I_1, \ldots, I_{\hat{n}-1}$. Note that $|I_t| = m/\log n$, for $t \in \{0, \ldots, \hat{n} - 1\}$. Then the machines in the group $I$ transmit the approximations $Y_{jk}^{(i)}$ (with $D = 5/2$ in Algorithm 1) of the observed coefficients $X_{jk}^{(i)}$, for $1 \leq j \leq j_{B,n} - 1, k \in K_j$, with

$$j_{B,n} := \lfloor \log_2(B/(3 \log_2 n)) \rfloor,$$

to the central machine. The machines in group $I_t, t \in \{0, \ldots, \hat{n} - 1\}$, will be responsible for transmitting the coefficients at resolution level $j = j_{B,n} + t$. First for every $t \in \{0, \ldots, \hat{n} - 1\}$, the machines in group $I_t$ are split again into $2^t$ equal size groups (for simplicity each group has $\lceil m/2 \rceil/\hat{n}$ machines and the leftovers are discarded again), denoted by $I_{t,1}, I_{t,2}, \ldots, I_{t,2^t}$. A machine $i$ in one of the groups $I_{t,\ell}$ for $\ell \in \{1, \ldots, 2^t\}$ transmits the approximations $Y_{jk}^{(i)}$ (again with $D = 5/2$ in Algorithm 1) of the noisy coefficients $X_{jk}^{(i)}$, for $j = j_{B,n} + t$ and $(\ell - 1)2^{j_{B,n}} \leq k < \ell 2^{j_{B,n}}$ to the central machine.

In the central machine we first average the transmitted approximations of the corresponding coefficients. We define

$$(2.4) \quad Y_{jk} = \begin{cases} |I|^{-1} \sum_{i \in I} Y_{jk}^{(i)} & \text{if } j < j_{B,n}, k \in K_j, \\ |I_{t,\ell}|^{-1} \sum_{i \in I_{t,\ell}} Y_{jk}^{(i)} & \text{if } j_{B,n} \leq j < j_{B,n} + \hat{n}, k \in K_j. \end{cases}$$

Using these coefficients we can construct for every $j$ the preliminary estimator

$$\hat{f}(j) = \sum_{t \leq j-1} \sum_{k \in K_t} Y_{jk} \varphi_{jk}.$$
This gives us a sequence of estimators from which we select the appropriate one using a modified version of Lepski’s method. We consider \( J = \{ 1, ..., \frac{\log_2 n}{1 + 2s_{\text{min}}} \} \) and define \( \hat{j} \) as

\[
(2.5) \quad \hat{j} = \min \{ j \in J : \| \hat{f}(j) - \hat{f}(l) \|_2 \leq \tau^{2^{l}}/n_l, \forall l > j, l \in J \},
\]

for some sufficiently large parameter \( \tau > 1 \) (\( \tau = 2^{0.8_{\text{max}}} \) is sufficiently large, but smaller choices are possible) and \( n_j = |I_j - j_{B,n,1}| n/m \approx \frac{n B_{\text{Lepski}}}{2^j (\log_2 n)^2} \), for \( j \geq j_{B,n} \) and \( n_j = |I| n/m \approx n \) for \( j < j_{B,n} \). Then we construct our final estimator \( \hat{f} \) simply by taking \( \hat{f} = \hat{f}(\hat{j}) \).

We summarize the above procedure (without discarding servers for achieving equal size subgroups) in Algorithm 4, below.

---

**Algorithm 4 Adaptive L2-method**

1. **In the local machines:**
   2. for \( i = 1 \) to \( \lfloor m/2 \rfloor \) do
   3. for \( j = 1 \) to \( j_{B,n} - 1 \) do
   4. \( j_k = 0 \) to \( 2^{l} - 1 \) do
   5. \( Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)}) \)
   6. for \( t = 0 \) to \( \bar{n} - 1 \) do
   7. for \( \ell = 1 \) to \( 2^l \) do
      8. for \( i = \lfloor m/2 \rfloor + t \left( \left\lfloor \frac{m/2}{2} \right\rfloor + (\ell - 1) \left( 2^{-t} \left( \left\lfloor \frac{m/2}{2} \right\rfloor + 1 \right) \right) \right) \)
      \( \leq \lfloor \frac{m/2}{2} \rfloor + t \left( \left\lfloor \frac{m/2}{2} \right\rfloor + (\ell - 1) \left( 2^{-t} \left( \left\lfloor \frac{m/2}{2} \right\rfloor + 1 \right) \right) \right) \) do
   9. for \( j = j_{B,n} \) to \( j_{B,n} + \bar{n} - 1 \) do
   10. for \( k = 0 \) to \( 2^{l} - 1 \) do
   11. \( Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)}) \)

12. **In the central machine:**
13. **(1) Averaging the local observations:**
14. for \( j = 1 \) to \( j_{B,n} - 1 \) do
15. \( j_k = 0 \) to \( 2^{l} - 1 \) do
16. \( y_{jk} := \text{mean}(Y_{jk}^{(i)} : i \leq m/2) \)
17. for \( t = 0 \) to \( \bar{n} - 1 \) do
18. \( j := j_{B,n} + t \)
19. for \( \ell = 1 \) to \( 2^l \) do
20. for \( k = (\ell - 1)2^{j_{B,n}} \) to \( \ell 2^{j_{B,n}} - 1 \) do
21. \( y_{jk} := \text{mean}(Y_{jk}^{(i)} : \left\lfloor \frac{m/2}{2} \right\rfloor + t \left( \left\lfloor \frac{m/2}{2} \right\rfloor + (\ell - 1) \left( 2^{-t} \left( \left\lfloor \frac{m/2}{2} \right\rfloor + 1 \right) \right) \right) \leq \left\lfloor \frac{m/2}{2} \right\rfloor + t \left( \left\lfloor \frac{m/2}{2} \right\rfloor + (\ell - 1) \left( 2^{-t} \left( \left\lfloor \frac{m/2}{2} \right\rfloor + 1 \right) \right) \right) \) do
22. \( y_{jk} := \text{mean}(Y_{jk}^{(i)} : \left\lfloor \frac{m/2}{2} \right\rfloor + t \left( \left\lfloor \frac{m/2}{2} \right\rfloor + (\ell - 1) \left( 2^{-t} \left( \left\lfloor \frac{m/2}{2} \right\rfloor + 1 \right) \right) \right) \) do

23. **(2) Lepski’s method:**
24. for \( j = 1 \) to \( \lfloor \log_2 n/(1 + 2s_{\text{min}}) \rfloor \) do
25. \( \hat{f}(j) := \sum_{i \leq j} \sum_{k \in K_j} y_{jk} \varphi(j_k) \)
26. Let \( \hat{j} := \lfloor \log_2 n/(1 + 2s_{\text{min}}) \rfloor \), \( \text{stop} := \text{FALSE} \)
27. while \( \text{stop} \neq \text{FALSE} \) and \( \hat{j} \geq 1 \) do
28. Let \( l = \hat{j} + 1 \)
29. while \( \text{stop} \neq \text{FALSE} \) and \( l \leq \lfloor \log_2 n/(1 + 2s_{\text{min}}) \rfloor \) do
30. if \( \| \hat{f}(j) - \hat{f}(l) \|_2 \leq \tau^{2^l}/n_l \) then
31. \( l := l + 1 \)
32. else \( \text{stop} := \text{TRUE} \)
33. if \( \text{stop} \neq \text{FALSE} \) then
34. \( \hat{j} := \hat{j} - 1 \)
35. Construct: \( \hat{f} = \hat{f}(\hat{j}) \).
Theorem 2.7. For every $L, s > 0$ the distributed method $\hat{f}$ described above belongs to $\mathcal{F}_{\text{dist}}(B_1, \ldots, B; B_{2, \infty}(L))$. Furthermore for all $L > 0$ there exists a positive constant $C > 0$ such that for all $s \in [s_{\min}, s_{\max})$

$$
\sup_{f_0 \in B_{2, \infty}(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \leq C \left\{ \begin{array}{ll}
    n^{-2s/(1+2s)}, & \text{if } B \geq C_L n^{1/(1+2s)} \log_2 n, \\
    B/n, & \text{if } n^\frac{1}{1+s} (\log_2 n)^\frac{2s}{1+s} \leq B < C_L n^{1/(1+2s)} \log_2 n, \\
    \left(\frac{nB}{\log_2 n}\right)^{-\frac{2s}{1+s}}, & \text{if } B < n^{1/(1+2s)} (\log_2 n)^\frac{2s}{1+s},
\end{array} \right.
$$

with $C_L = 12 L^{2/(1+2s)}$.

Proof. See Section 3.6. \qed

3. Proofs.

3.1. Proof of Theorem 2.1. Note that without loss of generality we can multiply $\delta_n$ with an arbitrary constant. In the proof we define $\delta_n$ as the solution to

$$
\delta_n = 2^{-15} L^{-2} \min \left\{ \frac{m}{n \log_2 n}, \frac{m}{n \sum_{i=1}^{m} \delta_n \log_2(n) B(i) \wedge 1} \right\}.
$$

We prove the desired lower bound for the minimax risk using a modified version of Fano’s inequality, given in Theorem A.6. As a first step we construct a finite subset $\mathcal{F}_0 \subset B_{2, \infty}(L)$. We use the wavelet notation outlined in Appendix B. Define $j_n = \lfloor (\log_2 \delta_n^{-1})/(1 + 2s) \rfloor$. For $\beta \in \{-1, 1\}^{|K_{j_n}|}$, let $f_\beta \in L_2[0, 1]$ be the function with wavelet coefficients

$$
f_{\beta, jk} = \begin{cases} L \beta_k \delta_n^{1/2}, & \text{if } j = j_n, k \in K_{j_n}, \\ 0, & \text{else}. \end{cases}
$$

Now define $\mathcal{F}_0 = \{ f_\beta : \beta \in \{-1, 1\}^{|K_{j_n}|} \}$. Note that $\mathcal{F}_0 \subset B_{2, \infty}(L)$, since

$$
\|f_\beta\|_{B_{2, \infty}}^2 = \sup_j 2^{2sj} \sum_{k \in K_j} f_{\beta, jk}^2 = L^2 2^{(2s+1)j_n} \delta_n \leq L^2.
$$

For this set of functions $\mathcal{F}_0$, the maximum and minimum number of elements in balls of radius $t > 0$, given by

$$
N_t^{\max} = \max_{f_\beta \in \mathcal{F}_0} \left\{ \# \{ f_\beta' \in \mathcal{F}_0 : \|f_\beta - f_\beta'\|_2 \leq t \} \right\},
$$

$$
N_t^{\min} = \min_{f_\beta \in \mathcal{F}_0} \left\{ \# \{ f_\beta' \in \mathcal{F}_0 : \|f_\beta - f_\beta'\|_2 \leq t \} \right\},
$$

satisfy $N_t^{\max} = N_t^{\min}$ and $N_t^{\max} = \sum_{i=0}^{\lfloor |K_{j_n}| \rfloor} \binom{|K_{j_n}|}{i} < |\mathcal{F}_0|/2$ for $t = \frac{\ell^2}{4m_n \tau^2} < |K_{j_n}|/2$ (and therefore $N_t^{\max} < |\mathcal{F}_0| - N_t^{\min}$).

Now recall that $X = (X^{(1)}, \ldots, X^{(m)})$ is the data available at the local machines and $Y = (Y^{(1)}, \ldots, Y^{(m)})$ are the binary messages transmitted to the central machine. We can consider the Markov chain $F \rightarrow X \rightarrow Y$, where $F$ is a uniform random
element in $\mathcal{F}_0$. It then follows from Theorem A.6 (with $t^2 = 2L^2\delta_n|K_{jn}|/3$ and $d(f, g) = \|f - g\|_2$) that

\begin{equation}
\inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B^{(1)}, \ldots, B^{(m)}; B^{(1)}_{2^\infty}(L))} \sup_{f_0 \in B^{(1)}_{2^\infty}(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim L^2\delta_n|K_{jn}| \left(1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N^\text{max}_t)}\right),
\end{equation}

where $I(F; Y)$ is the mutual information between the random variables $F$ and $Y$.

To lower bound the right-hand side, first note that $N^\text{max}_t = \sum_{i=1}^{\bar{t}} \left(\lfloor K_{jn} \rfloor \right) < 2^{\left(\lfloor K_{jn} \rfloor \right)} \leq 2(e|K_{jn}|/\bar{t})^{\bar{t}}$ and therefore, for $\bar{t} = |K_{jn}|/6$ (i.e. $t^2 = 2L^2\delta_n|K_{jn}|/3$),

\[ \log(|\mathcal{F}_0|/N^\text{max}_t) \geq |K_{jn}| \log(2(6e)^{-1/6}2^{-1/|K_{jn}|}) \geq |K_{jn}|/6. \]

Hence, using $|K_{jn}| = \delta_n^{-1/2}$ we see that to derive the statement of the theorem from (3.3) it is sufficient to show that

\begin{equation}
I(F; Y) \leq \delta_n^{-1/(1+2s)} / 8 + O(1).
\end{equation}

To do so, first note that for $\delta_n \leq m/(2^{11}L^2n\log_2 n)$ the conditions of Lemma A.13 are satisfied hence by applying the lemma (with $\delta^2 = L^2\delta_n$ and $d = \delta_n^{-1/2}$) we get

\begin{align*}
I(F; Y) &\leq 2L^2n\delta_n m^{-1} \min_{i=1}^m \left\{ 2^{10} \log(m\delta_n^{-1/2}) H(Y^{(i)}), \delta_n^{-1/2} \right\} + 4 \log 2 \\
&\leq 2L^2n\delta_n m^{-1} \delta_n^{-1/2} \sum_{i=1}^m \left( 2^{11} \log(n)\delta_n^{-1/2} B^{(i)} \wedge 1 \right) + O(1),
\end{align*}

where the last inequality follows from Lemma A.15. Since from the definition of $\delta_n$ it follows that

\[ \delta_n \leq \frac{2^{-4}L^{-2}mn^{-1}}{\sum_{i=1}^m \left[ 2^{11} \log(n)\delta_n^{-1/2} B^{(i)} \wedge 1 \right]}, \]

the right-hand side of (3.5) is further bounded by $2^{-3}\delta_n^{-1/2} + O(1)$, finishing the proof of assertion (3.4) and concluding the proof of the theorem.

### 3.2. Proof of Theorem 2.4.

In the procedure as a first step we take the first $n^{1/(1+2s)} \wedge (B/\log_2 n)$ noisy coefficients, i.e. $X^{(i)}_{jk}, 2^j + k \leq n^{1/(1+2s)} \wedge (B/\log_2 n)$, and transmit their approximation $Y^{(i)}_{jk}$, cf. Algorithm 1 with $D = 1/2$. Note that for any $f \in B^{(1)}_{2^\infty}(L)$ we have that $\int f_j^2 \leq 2^{2s} \sum_k f_j^2 \leq L^2$, hence in view of Lemma 2.3 (with $|\mu| = |f_{0,jk}| \leq L$) the approximation satisfies

\[ 0 \leq (X^{(i)}_{jk} - Y^{(i)}_{jk})/X^{(i)}_{jk} \leq 1/\sqrt{n} \quad \text{and} \quad \mathbb{E}_f [l(Y^{(i)}_{jk})] \leq (1/2 + o(1))\log_2 n. \]

Therefore we need at most $(1/2 + o(1))B$ bits in expected value to transmit $\{Y^{(i)}_{jk} : 2^j + k \leq n^{1/(1+2s)} \wedge (B/\log_2 n)\}$, hence $f_n \in \mathcal{F}_{\text{dist}}(B, \ldots, B; B^{(1)}_{2^\infty}(L)).$

Next for convenience we introduce the notation $\varepsilon^{(i)}_{jk}$ for the error term $Y^{(i)}_{jk} = (1 - \varepsilon^{(i)}_{jk})X^{(i)}_{jk}$, which satisfies $\varepsilon^{(i)}_{jk} \in [0, n^{-1/2}]$. The estimator $\hat{f}$ is given by its wavelet
coefficients \( \hat{f}_{jk}, j \in \mathbb{N}, k \in K_j \). For \( 2^j + k > n^{1/(1+2s)} \land (B/\log n) \) we have \( \hat{f}_{jk} = 0 \), while for \( 2^j + k \leq n^{1/(1+2s)} \land (B/\log n) \),

\[
\hat{f}_{jk} = \frac{1}{m} \sum_{i=1}^{m} Y_{jk}^{(i)} = \frac{1}{m} \sum_{i=1}^{m} X_{jk}^{(i)} (1 - \epsilon_{jk}^{(i)}) = f_{0,jk} (1 - \epsilon_{jk}) + \frac{1}{\sqrt{n}} Z_{jk} - W_{jk},
\]

where \( \epsilon_{jk} = m^{-1} \sum_{i=1}^{m} \epsilon_{jk}^{(i)} \in [0, n^{-1/2}] \), \( Z_{jk} \sim N(0, 1) \) and \( W_{jk} = (nm)^{-1/2} \sum_{i=1}^{m} Z_{jk}^{(i)} \epsilon_{jk}^{(i)} \), satisfying

\[
E_{f_0} |W_{jk}| \leq n^{-1/2} E|Z_{jk}^{(i)}| = \sqrt{2/\pi n}^{-1/2}, \quad E_{f_0} W_{jk}^2 \leq n^{-1} E(Z_{jk}^{(i)})^2 = n^{-1}.
\]

For convenience we also introduce the notation \( j_n = \lfloor \log_2(n \frac{1}{m/\eta} \land (B/\log n)) \rfloor \). Then the risk is bounded from above by

\[
E_{f_0} ||\hat{f} - f_0||_2^2 \leq \sum_{j \geq j_n} \sum_{k \in K_j} f_{0,jk}^2 + \sum_{j=1}^{j_n} \sum_{k \in K_j} \left( E_{f_0} \left( \frac{1}{\sqrt{n}} Z_{jk} - W_{jk} \right)^2 + f_{0,jk} E_{f_0} \epsilon_{jk}^2 + E_{f_0} |f_{0,jk} \epsilon_{jk}| (|W_{jk}| + n^{-1/2} |Z_{jk}|) \right)
\]

\[
\lesssim \sum_{j \geq j_n} 2^{-2js} \sup_{j \geq j_n} \sum_{k \in K_j} f_{0,jk}^2 + \sum_{j=1}^{j_n} \sum_{k \in K_j} n^{-1} \lesssim 2^{-2js} n^{s} + 2^{jn}/n \lesssim n^{-2s/(1+2s)} \land (B/\log n)^{-2s},
\]

where we have used that for \( f_0 \in B_{2,\infty}^s(L) \) we have \( |f_{0,jk}| \leq L \) for any \( j \geq 1, k \in K_j \).

### 3.3. Proof of Theorem 2.5

First recall that for every \( f_0 \in B_{2,\infty}^s(L) \) we have \( f_{0,jk}^2 \leq L^2, j \geq 1, k \in K_j \). Therefore, in view of Lemma 2.3 (with \( D = 1/2 \)) we have \( E_{f_0}[|Y_{jk}^{(i)}|] \leq (1/2 + o(1)) \log n \). Since each machine transmits at most \( |B/\log n| + 1 \) coefficients we get that the total amount of transmitted bits per machine is bounded from above by \( B \) (for large enough \( n \)), hence \( \hat{f} \in \mathcal{F}_{\text{dist}}(B, \ldots, B; B_{2,\infty}^s(L)) \).

Let \( A_{jk} = \{ [\mu_{jk} m/\eta] + 1, \ldots, [\mu_{jk} + 1 m/\eta] \} \) be the collection of machines transmitting the \((j, k)\)th coefficient and note that \#(\( A_{jk} \)) \( \asymp m/\eta \). Then our aggregated estimator \( \hat{f} \) satisfies for \( 2^j + k \leq n/\log n \) (i.e. the total number of different coefficients transmitted) that

\[
\hat{f}_{jk} = \frac{1}{\#(A_{jk})} \sum_{i \in A_{jk}} Y_{jk}^{(i)} = f_{0,jk} (1 - \tilde{\epsilon}_{jk}) + \sqrt{m/\#(A_{jk})} Z_{jk} - \tilde{W}_{jk},
\]

where \( \tilde{\epsilon}_{jk} = \frac{1}{\#(A_{jk})} \sum_{i \in A_{jk}} \epsilon_{jk}^{(i)} \in [0, n^{-1/2}] \), \( \tilde{W}_{jk} = \frac{\sqrt{m/\#(A_{jk})}}{\sqrt{n} \#(A_{jk})} \sum_{i \in A_{jk}} Z_{jk}^{(i)} \epsilon_{jk}^{(i)} \) satisfying

\[
E_{f_0} |\tilde{W}_{jk}| \leq n^{-1/2} E|Z_{jk}^{(i)}| = \sqrt{2/\pi n}^{-1/2}, \quad E_{f_0} \tilde{W}_{jk}^2 \leq n^{-1} E(Z_{jk}^{(i)})^2 = n^{-1},
\]

and \( Z_{jk} \sim N(0, 1) \).

Let \( j_n = \lfloor \log_2(n B/\log n) \rfloor \). Then similarly to (3.6) the risk of the aggregated
estimator is bounded as
\[
\mathbb{E}_{f_0} \| \hat{f} - f_0 \|^2 \leq \sum_{j=j_n}^{\infty} \sum_{k \in K_j} f_{0,jk}^2 + \sum_{j=1}^{j_n} \sum_{k \in K_j} \mathbb{E}_{f_0} \left\{ \mathbb{E}_{f_0} \left( \sqrt{\frac{m}{n \#(A_{jk})}} Z_{jk} - W_{jk} \right)^2 \right\} + \mathbb{E}_{f_0} \left\{ \mathbb{E}_{f_0} \left( \sqrt{\frac{m}{n \#(A_{jk})}} Z_{jk} \right)^2 \right\}
\]
\[
\leq \sum_{j=j_n}^{\infty} \sup_{j \geq j_n} \sum_{k \in K_j} f_{0,jk}^2 + \sum_{j=1}^{j_n} \sum_{k \in K_j} \eta/n
\]
\[
\leq \left( \frac{B \eta}{\log_2 n} \right)^{-2s} + \frac{B \eta^2}{n \log_2 n} \approx (nB/\log_2 n)^{-\frac{2s}{2+2s}}
\]
(3.7)
concluding the proof of the theorem.

Finally we show that Algorithm 1 is in general suboptimal in this case. Consider the function \( f_0 \in B_2^2(n,1) \) with wavelet coefficients \( f_{0,jk} = 2^{-j(j+1/2)} \), \( j \in \mathbb{N} \), \( k \in K_j \), and take \( j_n = \lfloor \log_2 (B/\log_2 n) \rfloor \), then
\[
\mathbb{E}_{f_0} \| \hat{f} - f_0 \|^2 \geq \sum_{j \geq j_n} \sum_{k \in K_j} f_{0,jk}^2 \geq \sum_{k \in K_{j_n}} 2^{-j_n(2s+1)}
\]
\[
\geq \left( \frac{B}{\log_2 n} \right)^{-2s} = M_n \left( \frac{n^{1/(1+2s)}}{B \log_2 n} \right)^{\frac{2s}{2+2s}} n^{-\frac{2s}{2+2s}}
\]
where the multiplication factor \( M_n = \left( \frac{n^{(\log_2 n)^{3+2s}}}{B^{1+2s}} \right)^{\frac{2s}{2+2s}} \) term tends to infinity and can be of polynomial order, yielding a highly sub-optimal rate.

3.4. Proof of Lemma 2.3. One can easily see by construction that
\[
0 \leq (X - Y)/X \leq n^{-D}
\]
(3.8)
Next note that the expected number of transmitted bits is bounded from above by
\[
\mathbb{E}(3 + \log_2 (1 + |1 + \log_2 X|)) + D \log_2 n = 3 + D \log_2 n + \mathbb{E} \log_2 |1 + \log_2 X|.
\]
The expected value \( \mathbb{E} \log_2 (1 + \log_2 X) \) is further bounded from above by
\[
2 \left( \int_0^{n^{-1}} + \int_1^{n^{-1}} \right) \frac{\log_2 (1 + \log_2 (1/x))}{\sqrt{2\pi m/n}} e^{-\frac{y-x \rho^2}{2m/n}} dy + 2 \int_{\log (n)}^{\infty} \frac{\log_2 (1 + \log_2 (x))}{\sqrt{2\pi m/n}} e^{-\frac{y-x \rho^2}{2m/n}} dx
\]
\[
\lesssim \sqrt{n/m} \int_0^{n^{-1}} \log (1/x) dx + \log (\log n) \int_{n^{-1}}^{1} \frac{1}{\sqrt{2\pi m/n}} e^{-\frac{y-x \rho^2}{2m/n}} dy
\]
\[
+ \int_{\log (n)}^{\infty} \frac{x}{\sqrt{2\pi m/n}} e^{-\frac{y-x \rho^2}{2m/n}} dx
\]
\[
\lesssim (\log n)(nm)^{-1/2} + \log \log n + \sqrt{\frac{2m}{n\pi}} e^{-\frac{\rho^2}{2m/n}} + \mu \left( 1 - 2 \Phi \left( -\frac{\mu}{\sqrt{m/n}} \right) \right)
\]
\[
\lesssim \log \log n.
\]
We can conclude that \( \mathbb{E} (Y) \leq (D + o(1)) \log n \).
Finally, note that by similar computation
\[
\mathbb{P}(\log_2(1 + 1 + \log_2 |X|) > \log_2(1 + \log_2 n^2)) \leq \mathbb{P}(|X| < n^{-2}) + \mathbb{P}(|X| > n^2)
\]
\[
\leq 2 \left( \int_0^{n^{-2}} + \int_{n^2}^{\infty} \right) \sqrt{\frac{n}{2\pi m}} e^{-\frac{(x-n)^2}{2m/n}} dx
\]
\[
\lesssim m^{-1/2}n^{-3/2} + \sqrt{n} e^{-n^3/(4m)} = o(n^{-3/2}),
\]
finishing the proof of the lemma.

3.5. Proof of Theorem 2.6. First of all we note that in the non-distributed case where all the information is available in the global machine the minimax $L_{\infty}$-risk is $(n/\log n)^{-\frac{1}{1+2s}}$. Since the class of distributed estimators is clearly a subset of the class of all estimators this will be also a lower bound for the distributed case. The rest of the proof goes similarly to the proof of Theorem 3.1.

First we construct a finite subset $\mathcal{F}_0 \subset B_{\infty,\infty}^s(L)$ and then give a lower bound for the minimax risk over it. Let us denote by $\tilde{K}_j$ the largest set of Daubechies wavelets with disjoint supports. Note that $|K_j| \geq c_0 2^j$ (for large enough $j$ and sufficiently small $c_0 > 0$). Let us again multiply $\delta_n$ with a sufficiently small constant and work with this $\delta_n$ in the rest of the proof

\begin{equation}
\delta_n := c_0 2^{-13}L^{-2} \min \left\{ \frac{m}{n \log_2 n}, \frac{m}{n \sum_{j=1}^m \left\{ \left[ \delta_n \log_2 n \right]B(n) \land 1 \right\}} \right\}.
\end{equation}

Let $j_n = \lfloor (\log_2 \delta_n^{-1})/(1 + 2s) \rfloor$ and for $\beta \in \{-1, 1\}^{\tilde{K}_{j_n}}$ let $f_\beta \in L_2[0, 1]$ be the function with wavelet coefficients

\[ f_{\beta,j,k} = \left\{ \begin{array}{ll} L\delta_n^{1/2} \beta_k, & \text{if } j = j_n, k \in \tilde{K}_{j_n}, \\ 0, & \text{else,} \end{array} \right. \]

Now let $\mathcal{F}_0 = \{ f_{\beta} : \beta_k \in \{-1, 1\}, k \in \tilde{K}_{j_n} \}$.

Note that each function $f_\beta \in \mathcal{F}_0$ belongs to the set $B_{\infty,\infty}^s(L)$, since

\[ \|f_\beta\|_{B_{\infty,\infty}^s} = \sup_j 2^{(s+1/2)j} \sup_{k \in \tilde{K}_j} f_{\beta,j,k}^2 = 2^{(s+1/2)j_n} \sup_{k \in \tilde{K}_{j_n}} \|L\delta_n^{1/2} = L2^{(s+1/2)j_n} \delta_n^{1/2} \leq L. \]

Furthermore, if $f_\beta \neq f_{\beta'}$, then there exists a $k \in \tilde{K}_{j_n}$ such that $\beta_k \neq \beta'_k$. Then due to the disjoint support of the corresponding Daubechies' wavelets $\varphi_{j_n,k}$, $k \in \tilde{K}_{j_n}$ the $L_{\infty}$-distance between the two functions is bounded from below by

\[ \|f_\beta - f_{\beta'}\|_{\infty} \geq |f_{j_n,k} - f'_{j_n,k'}| \cdot \|\varphi_{j_n,k'}\|_{\infty} \gtrsim 2^{j_n/2 + 1} \delta_n^{1/2} \gtrsim \delta_n^{-\frac{1}{1+2s}}. \]

Now let $F$ be a uniform random variable on the set $\mathcal{F}_0$. Then in view of Fano's inequality (see Theorem A.5 with $\delta = \delta_n^{s/(1+2s)}$ and $p = 1$) we get that

\[ \inf_{\hat{f} \in \mathcal{F}_{\text{distr}}(B(1), \ldots, B(m); B_{\infty,\infty}^s(L))} \sup_{f_0 \in B_{\infty,\infty}^s(L)} \mathbb{E}_{f_0} \left( \|\hat{f} - f_0\|_{\infty} \right) \gtrsim \delta_n^{-\frac{1}{1+2s}} \left( 1 - \frac{I(F; Y) + \log 2}{\log_2 |\mathcal{F}_0|} \right). \]

Hence, since $\log_2 |\mathcal{F}_0| \gtrsim |\tilde{K}_{j_n}| \gtrsim c_0 2^{j_n} = c_0 \delta_n^{-1/(1+2s)}$, it remains to show that

\[ I(F; Y) \leq (c_0/2) \delta_n^{-1/(1+2s)}. \]
In view of Lemma A.13 (applied with \( \delta = \delta_n^{1/2}, d = |\bar{K}_n| = c_0 \delta_n^{-1/2} \), \( X = X^{(i)}, Y = Y^{(i)}, i = 1, \ldots, m \), and noting that \( \delta_n \leq m/(2^{11}L_2 n \log_2 n) \) hence the conditions are fulfilled)

\[
I(F; Y) \leq 2L^2 n \delta_n m^{-1} \delta_n^{-1/2} \sum_{i=1}^{m} \left( 2^{10} \log(n) \delta_n^{-1/2} H(Y^{(i)} \cap c_0) + 4 \log 2, \right) \leq 2L^2 n \delta_n m^{-1} \delta_n^{-1/2} \sum_{i=1}^{m} \left( \log(n) \delta_n^{-1/2} B(i) + 1 \right) + O(1),
\]

where the last line follows from Theorem A.15. We conclude the proof by using the definition of \( \delta_n \), see (3.9).

3.6. Proof of Theorem 2.7. First recall that for every \( s,L > 0 \) and \( f_0 \in B_{2,\infty}^s(L) \) we have \( f_{0,jk}^2 \leq L^2, j \geq 1, k \in K_j \). Therefore, in view of Lemma 2.3 (with \( D = 5/2 \)) we have \( E_{f_0}[l(Y^{(i)j})] \leq (5/2 + o(1)) \log_2 n \). Since the machines in group \( I \) and the machines in \( I_{t,\ell}, \ell \in \{0, \ldots, n-1\}, t \in \{1, \ldots, 2^t\} \) transmit at most \( 2^{[\log_2(B/(3\log_2 n)]} \) coefficients we have that in expected value at most

\[
2^{[\log_2(B/(3\log_2 n)]}(5/2 + o(1)) \log_2 n \leq B
\]

bits are transmitted per machine (for \( n \) large enough). Therefore the estimator indeed belongs to \( F_{\text{dist}}(B, \ldots, B; B_{2,\infty}^s(L)) \).

Next we show that the estimator \( \hat{f} \) achieves the minimax rate. First let us introduce the notation \( \hat{j} = \lfloor (\log_2(n)/(1 + 2s_{\min}) \rfloor \) and \( \hat{\epsilon}^{(i)j} = (X^{(i)j} - Y^{(i)j})/X^{(i)j}_{jk} \in [0, n^{-5/2}] \). Then note that for \( j \leq \hat{j} \) and \( k \in K_j \) the aggregated quantities \( Y_{j,k} \) defined in (2.4) are equal to

\[
Y_{j,k} = \frac{1}{|J_{jk}|} \sum_{i \in J_{jk}} X^{(i)j} (1 - \hat{\epsilon}^{(i)j}) = f_{0,jk} + n_j^{-1/2} Z_{jk} - W_{jk} - \hat{\epsilon}_{jk} f_{0,jk},
\]

where

\[
J_{jk} = \begin{cases} I, & \text{if } j < j_{B,n}, k \in K_j, \\ J_{j-B,n,\ell}, & \text{if } j = j_{B,n}, (\ell - 1)2^{j_{B,n}} \leq k < \ell 2^{j_{B,n}}, \end{cases}
\]

where \( \hat{\epsilon}_{jk} = n_j^{-1} \sum_{i \in J_{jk}} \hat{\epsilon}^{(i)j} \in [0, n^{-5/2}] \), \( W_{jk} = (m/n)^{1/2} n_j^{-1} \sum_{i \in J_{jk}} Z_{jk}^{(i)j} \), satisfying \( E_{f_0}|W_{jk}| = O(n^{-5/2}) \) and \( E_{f_0} W_{jk}^2 = O(n^{-5}) \), \( Z_{jk} \overset{iid}{\sim} N(0,1) \), and recall that \( n_j = n|J_{jk}|/m \) for every \( j \leq \hat{j} \), \( k \in K_j \). Recall also that \( n_j \approx n B/(2^t (\log_2 n)^2) \) for \( j \geq j_{B,n} \) and \( n_j \approx n \) for \( j < j_{B,n} \).

Note that the squared bias satisfies

\[
\|E_{f_0} \hat{f}(j) - f_0\|^2 \lesssim \|K(f_0, j) - f_0\|^2 + n^{-5}(\|f_0\|^2 + 1) \lesssim 2^{-2 j} \|f_0\|_{B_{2,\infty}^s}^2 + n^{-5},
\]

where \( K(f_0, j) = \sum_{l=1}^{j} \sum_{k \in K_l} f_{0,jk} \hat{f}_{jk} \). Let us introduce the notation \( B(j, f_0) = 2^{-2 j} \|f_0\|_{B_{2,\infty}^s}^2 \) and define the optimal choice of the parameter \( j \) (the optimal resolution level) as

\[
j^* = \min \{ j \in J : B(j, f_0) \leq 2^j/n_j \}.
\]
Note that since the right hand side is monotone increasing and the left hand side is monotone decreasing in $j$, we have that

$$B(j, f_0) \leq 2^{j'}/n_j, \text{ for } j \geq j^* \text{ and } B(j, f_0) > 2^{j'}/n_j, \text{ for } j < j^*.$$  

Therefore

$$2^{j'-1}/n_{j'-1} < B(j^* - 1, f_0) = 2^{2s} B(j^*, f_0) \leq 2^{2s} 2^{j'}/n_{j^*}.$$  

Let us distinguish three cases according to the value of $j^*$. If $j^* < j_{B,n}$ then $n_{j^*-1} = n_{j^*} \propto n$ and therefore $2^{j'} \propto n^{1/(1+2s)}$ (using the definition $B(j^*, f_0) = 2^{-2j's}\|f_0\|_{B_{2,\infty}^s}$). Note that the inequality $j^* < j_{B,n}$ is implied by $B(j_{B-1}, f_0) \leq 2^{2n-1}/n_{B,n-1}$, which in turns holds if $2^{2n-1} \geq (n\|f_0\|_{B_{2,\infty}^s})^{1/(1+2s)}$. Therefore we can conclude that $B \geq 12L^2/(1+2s)n^{1/(1+2s)} \log_2 n$ implies the inequality $j^* < j_{B,n}$. If $j^* = j_{B,n}$, then $n_{j^*} \propto n/\log_2 n, n_{j^*-1} \propto n$ and therefore $(n/\log_2 n)^{1/(1+2s)} \leq 2^{j'} \propto B/\log_2 n \lesssim n^{1/(1+2s)}$. Finally, if $j^* > j_{B,n}$, then $n_{j^*-1} \propto n_{j^*} \propto nB/(2^{j'} \log_2^2 n)$ and therefore $2^{j'} \propto (nB/\log_2^2 n)^{1/(2+2s)}$. We summarize these findings in the following display

\[
2^{j'} = \begin{cases} 
  n^{1/(1+2s)}, & \text{if } B \geq C_L n^{1/(1+2s)} \log_2 n, \\
  B/\log_2 n, & \text{if } n^{1/(1+2s)} \log_2 n < B \leq C_L n^{1/(1+2s)} \log_2 n, \\
  (nB/\log_2^2 n)^{1/(2+2s)}, & \text{if } B < n^{1/(1+2s)} \log_2^2 n, 
\end{cases}
\]

and

\[
n_{j^*} \geq \begin{cases} 
  n, & \text{if } B \geq C_L n^{1/(1+2s)} \log_2 n, \\
  n/\log_2 n, & \text{if } n^{1/(1+2s)} \log_2 n \leq B \leq C_L n^{1/(1+2s)} \log_2 n, \\
  (nB/\log_2^2 n)^{1/(2+2s)}, & \text{if } B < n^{1/(1+2s)} \log_2^2 n, 
\end{cases}
\]

where $C_L = 12L^2/(1+2s)$.

Finally, we also note that for $\ell \leq j$ we have $n_{\ell} \geq n_{j}$ and hence

\[
\mathbb{E}_{f_0} \| \tilde{f}(j) - E_{f_0} \tilde{f}(j) \|_2^2 \leq \sum_{\ell \leq j-1} \sum_{k \in K_j} \left( \frac{2}{n_{\ell}} \mathbb{E}_{f_0} Z_{\ell k}^2 + 2 \mathbb{E}_{f_0} W_{\ell k}^2 + \tilde{f}_{0,\ell k}^2 \mathbb{E}_{f_0} e_{\ell k}^2 + 2 |f_{0,\ell k}| \mathbb{E} (n_{\ell}^{-1/2} |Z_{\ell k}| + |W_{\ell k}|) |\varepsilon_{\ell k}| \right) \\
\lesssim \sum_{\ell \leq j-1} \sum_{k \in K_j} n_{\ell}^{-1} \leq 2^j/n_j.
\]

Let us split the risk into two parts

\[
\mathbb{E}_{f_0} \| f_0 - \tilde{f} \|_2^2 = \mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 \mathbb{I}_{j > j^*} + \mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j \leq j^*},
\]

and deal with each term on the right-hand side seperately. First note that

\[
\mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j \leq j^*} \lesssim 2 \mathbb{E}_{f_0} \| \tilde{f}(j^*) - \tilde{f}(j) \|_2^2 1_{j \leq j^*} + 2 \mathbb{E}_{f_0} \| \tilde{f}(j^*) - f_0 \|_2^2 \\
\lesssim \tau 2^{j^*}/n_{j^*} + \| \mathbb{E}_{f_0} \tilde{f}(j^*) - f_0 \|_2^2 + \mathbb{E}_{f_0} \| \tilde{f}(j^*) - \mathbb{E}_{f_0} \tilde{f}(j^*) \|_2^2 \\
\lesssim 2^{j^*}/n_{j^*} + 2^{-2j's},
\]

for $j^* < j_{B,n}$.

\[
\mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j > j^*} \lesssim \mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j > j^*}. 
\]

where $C_L = 12L^2/(1+2s)$.

Finally, we also note that for $\ell \leq j$ we have $n_{\ell} \geq n_{j}$ and hence

\[
\mathbb{E}_{f_0} \| \tilde{f}(j) - E_{f_0} \tilde{f}(j) \|_2^2 \leq \sum_{\ell \leq j-1} \sum_{k \in K_j} \left( \frac{2}{n_{\ell}} \mathbb{E}_{f_0} Z_{\ell k}^2 + 2 \mathbb{E}_{f_0} W_{\ell k}^2 + \tilde{f}_{0,\ell k}^2 \mathbb{E}_{f_0} e_{\ell k}^2 + 2 |f_{0,\ell k}| \mathbb{E} (n_{\ell}^{-1/2} |Z_{\ell k}| + |W_{\ell k}|) |\varepsilon_{\ell k}| \right) \\
\lesssim \sum_{\ell \leq j-1} \sum_{k \in K_j} n_{\ell}^{-1} \leq 2^j/n_j.
\]

Let us split the risk into two parts

\[
\mathbb{E}_{f_0} \| f_0 - \tilde{f} \|_2^2 = \mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j > j^*} + \mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j \leq j^*},
\]

and deal with each term on the right-hand side seperately. First note that

\[
\mathbb{E}_{f_0} \| f_0 - \tilde{f}(j) \|_2^2 1_{j \leq j^*} \lesssim 2 \mathbb{E}_{f_0} \| \tilde{f}(j^*) - \tilde{f}(j) \|_2^2 1_{j \leq j^*} + 2 \mathbb{E}_{f_0} \| \tilde{f}(j^*) - f_0 \|_2^2 \\
\lesssim \tau 2^{j^*}/n_{j^*} + \| \mathbb{E}_{f_0} \tilde{f}(j^*) - f_0 \|_2^2 + \mathbb{E}_{f_0} \| \tilde{f}(j^*) - \mathbb{E}_{f_0} \tilde{f}(j^*) \|_2^2 \\
\lesssim 2^{j^*}/n_{j^*} + 2^{-2j's},
\]

for $j^* < j_{B,n}$.
We deal with the five terms on the right hand side separately. Note that the functions get for $l$ for some universal sufficiently small constant $c > 0$. Then by the definition of $\hat{\mathbf{f}}_j$ and note that for every $l \geq j$ we have

$$\mathbb{P}_0(\hat{j} = j) \leq \sum_{l=j}^{j_{\max}} \mathbb{P}_0(\|f_j - \mathbf{f}(l)\|_2^2 > \tau 2^l/n_l).$$

Note that the left hand side term in the probability in view of Parseval’s inequality can be given in the form

$$\|f_j - \mathbf{f}(l)\|_2^2 = \sum_{l=j}^{l_{\max}} \sum_{k \in K_l} \left( f_{0,\ell k} (1 - \varepsilon_{\ell k}) + n_\ell^{-1/2} Z_{\ell k} - W_{\ell k} \right)^2 \leq \sum_{l=j}^{l_{\max}} \sum_{k \in K_l} \left( 2 f_{0,\ell k}^2 + 2 n_\ell^{-1} Z_{\ell k}^2 + 2 n_\ell^{-1/2} Z_{\ell k} f_{0,\ell k} \right)^2 + 3 W_{\ell k}^2 + 2 n_\ell^{-1/2} |Z_{\ell k} \varepsilon_{\ell k}|.$$
Next recall that $E_{f_0}W_{\ell k}^2 = O(n^{-5})$ and then by applying Markov’s and Chebyshev’s inequalities we get that

$$
P_{f_0}\left( \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} \frac{2^l}{\sqrt{n_l}} |Z_{\ell k}| > \frac{2^l}{2n_l} \right) \leq 4 \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} n^{-5} n_l^{-1} E_{f_0}Z_{\ell k}^2 2^{2l/n_l^2} = O(n^{-4}),$$

and

$$
P_{f_0}\left( \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} 3W_{\ell k}^2 > \frac{2^l}{2n} \right) \leq 6 \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} E_{f_0}W_{\ell k}^2 2^{2l/n} = O(n^{-4}).$$

Furthermore, note that

$$
P_{f_0}\left( \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} Z_{\ell k}^2 > \frac{\tau - 3}{2} \frac{2^l}{n_l} \right) \leq \sum_{\ell=j}^{l-1} P_{f_0}\left( \sum_{k \in K_\ell} (Z_{\ell k}^2 - 1) > \frac{\tau - 5}{4} \frac{2^l}{n_l} \right)$$

and

$$
P_{f_0}\left( \sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} Z_{\ell k} f_{0,\ell k} > \frac{\tau - 3}{2} \frac{2^l}{n_l} \right) \leq \sum_{\ell=j}^{l-1} P_{f_0}\left( |Z(j^-)| > \frac{\tau - 3}{2} \frac{2^l}{\sqrt{n_l}} \right),$$

where $Z(j^-)$ is a centered Gaussian random variable with variance

$$
\sum_{\ell=j}^{l-1} \sum_{k \in K_\ell} f_{0,\ell k}^2 \leq B(j^*, f_0).
$$

Then by Theorem 3.1.9 of [7] the first probability in the preceding display is bounded from above by a multiple of

$$
\sum_{\ell=j}^{l-1} \exp\{-D\frac{(\tau - 5)^2 2^{2l}}{(\tau - 4)^2} \} \leq \exp\{-\tilde{c}\tau 2^l \},
$$

for some universal, small enough constants $\tilde{c}, D > 0$ (the constants $D = 2^{-6}$ and $\tilde{c} = 2^{-8}$ are sufficiently small if $\tau \geq 10$) and the second probability by

$$
\sum_{\ell=j}^{l-1} \exp\{-D'\frac{(\tau - 3)^2 2^{2l}}{n_l B(j^*, f_0)} \} \leq \exp\{-c'\tau 2^l \},
$$

again for some universal, small enough constants $c', D' > 0$ (the constants $D' = 2^{-4}$ and $c' = 2^{-2}$ are sufficiently small if $\tau \geq 9$). Those we obtain that

$$
P_{f_0}(\hat{j} = j) \lesssim \sum_{l=j}^{j_{\max}} \sum_{\ell=j}^{l} \exp\{-c\tau 2^l \} + O(n^{-4}), \quad \text{for all } j > j^*,$$

for some universal constant $c > 0$ ($c = 2^{-8}$ is sufficiently small), finishing the proof of the statement. \qed

**APPENDIX A: INFORMATION THEORETIC RESULTS**

**A.1. Basic definitions and results.** A classical reference for basic concepts and results from information theory is the second chapter of [2]. Statements are only proved for discrete variables in Chapter 2 of [2], but several are valid more generally and are used more generally in this paper. All logarithms are base $e$ here.
In this section, where we recall notations and basic results, \((X, Y, Z)\) is a random triplet in a space \(\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}\) that is nice enough, so that regular versions of all conditional distributions exist (for instance a Polish space). We denote the joint distribution by \(P_{X,Y,Z}\), the (regular version of the) conditional distribution of \(X\) given \(Y = y\) by \(P_{X|Y=y}\), etcetera. If \(X\) and \(Y\) are discrete we denote by \(p_{(X,Y)}\) the joint probability mass function (pmf) of \((X, Y)\), by \(p_X\) and \(p_Y\) the marginal pmf of \(X\) and \(Y\), and by \(p_{X|Y=y}(x) = p_{(X,Y)}(x, y)/p_Y(y)\) the conditional pmf of \(X\) given \(Y = y\). For probability measures \(P\) and \(Q\) on the same space we define the Kullback-Leibler divergence as usual as \(K(P, Q) = \int (\log dP/dQ) \, dP\) if \(P \ll Q\), and as +\(\infty\) if not.

The \textit{mutual information} between \(X\) and \(Y\) and the \textit{conditional mutual information} between \(X\) and \(Y\), given \(Z\), are defined as

\[
I(X; Y) = K(P_{(X,Y)}, P_X \times P_Y), \quad I(X; Y | Z = z) = K(P_{(X,Y)|Z=z}, P_X|Z=z \times P_Y|Z=z), \quad I(X; Y | Z) = \int I(X; Y | Z = z) \, dP_Z(z).
\]

The (conditional) mutual information is nonnegative and symmetric in \(X\) and \(Y\).

If \(X\) and \(Y\) are discrete we define the \textit{entropy} of \(X\) and the \textit{conditional entropy} of \(X\) given \(Y\) by

\[
H(X) = -\sum_x p_X(x) \log p_X(x) \quad \text{and} \quad H(X | Y) = -\sum_{x,y} p_{(X,Y)}(x, y) \log p_{X|Y=y}(x).
\]

Entropy and conditional entropy are nonnegative. For discrete \(X\) and \(Y\), it holds that \(I(X; Y) = H(X) - H(X | Y)\). Hence, since mutual information is nonnegative, \(H(X | Y) \leq H(X)\) (conditioning reduces entropy). If \(X\) is a discrete variable on a finite set \(\mathcal{X}\) then \(H(X) \leq \log |\mathcal{X}|\), with equality if \(X\) is uniformly distributed on \(\mathcal{X}\). We denote by \(H(p) = -p \log p - (1 - p) \log(1 - p)\) the entropy of a Bernoulli variable with parameter \(p \in (0,1)\). The function \(p \rightarrow H(p)\) is a concave function that is symmetric around \(p = 1/2\). Its maximum value, attained at \(p = 1/2\), equals \(\log 2\).

We now recall a number of basic identities for mutual information. First of all, we have the following rule for a general, not necessarily discrete random triplet.

**Proposition A.1 (Chain rule for mutual information).** We have

\[
I(X; (Y, Z)) = I(X; Y | Z) + I(X; Z).
\]

We call the triplet \((X, Y, Z)\) a \textit{Markov chain}, and write \(X \rightarrow Y \rightarrow Z\), if the joint distribution disintegrates as

\[
dP_{(X,Y,Z)}(x, y, z) = dP_X(x)dP_Y|X=x(y)dP_Z|Y=y(z).
\]

In this situation we have the following result, which relates the information in the different links of the chain. Again, discreteness of the variables is not necessary for this result.
Proposition A.2 (Data-processing inequality). If $X \rightarrow Y \rightarrow Z$ is a Markov chain, then
\[ I(X; Y) = I(X; Y | Z) + I(X; Z). \]

In particular
\[ I(X; Z) \leq I(X; Y). \]

In case of independence, mutual information is sub-additive in the following sense.

Proposition A.3 (Role of independence). If $Y$ and $Z$ are conditionally independent given $X$, then
\[ I(X; (Y, Z)) = I(X; Y) + I(X; Z) - I(Y; Z) \leq I(X; Y) + I(X; Z). \]

Finally we recall Fano’s inequality [6] which we use in the following form in this paper.

Proposition A.4 (Fano’s lemma). Let $X \rightarrow Y \rightarrow \hat{X}$ be a Markov chain, where $X$ and $\hat{X}$ are random elements in a finite set $\mathcal{X}$ and $X$ has a uniform distribution on $\mathcal{X}$. Then
\[ P(X \neq \hat{X}) \geq 1 - \frac{\log 2 + I(X; Y)}{\log |\mathcal{X}|}. \]

A.2. Lower bounds for estimators using processed data. In this section we consider a situation in which we have a random element $X$ in $\mathcal{X}$ with a distribution $P_f$ depending on a parameter $f$ in a semimetric space $(\mathcal{F}, d)$. Moreover, we assume that we have a Markov chain $X \rightarrow Y$ defined through a Markov transition kernel $Q(dy | x)$ from $\mathcal{X}$ to some space $\mathcal{Y}$. Note that this includes the case that $Y$ is simply a measurable function $Y = \psi(X)$ of the full data $X$. We view $Y$ as a transformed, or processed version of the full data $X$. We are interested in lower bounds for estimators $\hat{f}$ for $f$ that are only based on the processes data. The collection of all such estimators, i.e. measurable functions of $Y$, is denoted by $E(Y)$.

The usual approach of relating lower bounds for estimation to lower bounds for testing multiple hypotheses, in combination with Fano’s lemma, gives the following useful result in our setting.

Theorem A.5. If $\mathcal{F}$ contains a finite set $\mathcal{F}_0$ of functions that are $2\delta$-separated for the semimetric $d$, then
\[ \inf_{f \in \mathcal{F}_0} \sup_{\hat{f} \in \mathcal{E}(Y)} \mathbb{E}_f d^p(\hat{f}, f) \geq 2^p \left( 1 - \frac{\log 2 + I(F; Y)}{\log |\mathcal{F}_0|} \right). \]

for all $p > 0$. Here $I(F; Y)$ is the mutual information between $F$ and $Y$ in the Markov chain $F \rightarrow X \rightarrow Y$, where $F$ has a uniform distribution on $\mathcal{F}_0$ and $X | f \sim P_f$.

We also use a slight modification of this basic result, where the condition that the functions in $\mathcal{F}_0$ are separated is replaced by a condition on the minimum and maximum number of elements in $\mathcal{F}_0$ that are contained in small balls. Given a finite set $\mathcal{F}_0$, we use the notations
\[ N_f^{\max} = \max_{\tilde{f} \in \mathcal{F}_0} \left\{ \# \{ \tilde{f} \in \mathcal{F}_0 : d(f, \tilde{f}) \leq t \} \right\}, \]
\[ N_f^{\min} = \min_{\tilde{f} \in \mathcal{F}_0} \left\{ \# \{ \tilde{f} \in \mathcal{F}_0 : d(f, \tilde{f}) \leq t \} \right\}. \]
The following theorem is a slight extension of Corollary 1 of [5]. In the latter corollary it is implicitly assumed that $Y$ is a discrete random variable (the conditional entropy $H(F \mid Y)$ is considered), while we can allow continuous random variables as well. For self-containedness we provide the proof below.

**Theorem A.6.** If $F$ contains a finite set $\mathcal{F}_0$ and $|\mathcal{F}_0| - N_t^{\min} > N_t^{\max}$, then

$$
\inf_{f \in \mathcal{E}(Y)} \sup_{f \in \mathcal{F}} \mathbb{E}_f d^p(\hat{f}, f) \geq t^p \left(1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N_t^{\max})}\right)
$$

for all $p,t > 0$. Here $I(F; Y)$ is the mutual information between $F$ and $Y$ in the Markov chain $F \rightarrow X \rightarrow Y$, where $F$ has a uniform distribution on $\mathcal{F}_0$ and $X \mid f \sim \mathbb{P}_f$.

**Proof.** We have

$$
\inf_{f \in \mathcal{E}(Y)} \sup_{f \in \mathcal{F}} \mathbb{E}_f d^p(\hat{f}, f) \geq t^p \inf_{f \in \mathcal{E}(Y)} \sup_{f \in \mathcal{F}_0} \mathbb{P}_f(d(\hat{f}, f) > t)
$$

therefore it is sufficient to show that

$$
\inf_{f \in \mathcal{E}(Y)} \mathbb{P}(Y,F)(d(\hat{f}, F) > t) \geq 1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N_t^{\max})}.
$$

By definition,

$$
I(F; Y) = \int \log \frac{dP_{(F,Y)}}{d(P_F \times P_Y)} dP_{(F,Y)}.
$$

By disintegration, $dP_{(F,Y)}(f,y) = dP_Y(y) dP_{F \mid Y=y}(f)$, so that by Fubini,

$$
(A.1) \quad I(F; Y) = \int \left( \int \log \frac{dP_{F \mid Y=y}}{dP_F} dP_{F \mid Y=y} \right) dP_Y(dy).
$$

Now $P_F$ is the uniform distribution on $\mathcal{F}_0$. Hence, it has density $p(f) = 1/|\mathcal{F}_0|$ w.r.t. the counting measure $df$ on $\mathcal{F}_0$. Define $p(f \mid y) = P_{F \mid Y=y}(\{f\})$, so that $P_{F \mid Y=y}$ has density $p(f \mid y)$ w.r.t. $df$. The KL-divergence $K(P_{F \mid Y=y}, P_F)$ in the inner integral can then be written as

$$
\int p(f \mid y) \log \frac{p(f \mid y)}{p(f)} df.
$$

Similarly $\hat{f}$ has some density $\hat{p}(\hat{f})$ w.r.t. the counting measure $d\hat{f}$ on $\mathcal{F}_0$ and we define $\hat{p}(\hat{f} \mid y) = P_{\hat{F} \mid Y=y}(\{\hat{f}\})$.

Next note that by the data-processing inequality, see Proposition A.2, we have $I(F; Y) - I(F; \hat{f}) \geq 0$. Then

$$
0 \leq I(F; Y) - I(F; \hat{f}) = \int \int p(f \mid y) \log p(f \mid y) df \ dP_Y(dy) - \int p(f) \log p(f) df
$$

$$
- \left( \int \int p(f, \hat{f}) \log p(f, \hat{f}) df \ d\hat{f} - \int p(f) \log p(f) df \right)
$$

$$
= \int \int p(f \mid y) \log p(f \mid y) df \ dP_Y(dy) + H(F \mid \hat{f}).
$$
Next note that since $F$ is uniform on $\mathcal{F}_0$ (see Section A.1),
\[
\int \int p(f, y) \log p(f|y) df dy = I(F; Y) - H(F) = I(F; Y) - \log(|\mathcal{F}_0|).
\]

We can summarize the above results as
\[(A.2) \quad H(F|\hat{f}) \geq \log(|\mathcal{F}_0|) - I(F; Y).
\]

Since by assumption the conditions of Proposition 1 of [5] hold, we have
\[
H(F|\hat{f}) \leq H(p_t) + p_t \log \frac{|\mathcal{F}_0| - N^\text{min}_t}{N^\text{max}_t} + \log N^\text{max}_t,
\]
where $p_t = P_{(F,Y)}(d(\hat{f}, F) > t)$. Then noting that $\log p_t \leq \log 2$ and by combining the preceding display with (A.2) we get that
\[
\log(|\mathcal{F}_0|) - I(F; Y) \leq 2 + p_t \log \frac{|\mathcal{F}_0| - N^\text{min}_t}{N^\text{max}_t} + \log N^\text{max}_t.
\]

Reformulation of the inequality yields
\[
P_{(Y,F)}(d(\hat{f}, F) > t) \geq \frac{\log(|\mathcal{F}_0|/N^\text{max}_t)}{\log ((|\mathcal{F}_0| - N^\text{min}_t)/N^\text{max}_t)} - \frac{I(F; Y) + \log 2}{\log ((|\mathcal{F}_0| - N^\text{min}_t)/N^\text{max}_t)} \geq 1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N^\text{max}_t)},
\]
which completes the proof.

In the next subsections we give bounds for the mutual information under various assumptions on the random variables $X$ and $f$.

**A.3. Bounding $I(F; Y)$: bounded likelihood ratios.** We consider again a Markov chain $F \rightarrow X \rightarrow Y$. We assume that $F$ has a uniform distribution on a finite set $\mathcal{F}_0$ and that $X | (F = f) \sim \mathbb{P}_f$.

In this section we assume that there exists a constant $C \geq 1$ and a set $\mathcal{X}'$ that has full mass under $\mathbb{P}_f$ for every $f \in \mathcal{F}_0$, such that
\[(A.3) \quad \sup_{x \in \mathcal{X}'} \max_{f_1, f_2 \in \mathcal{F}_0} \frac{d\mathbb{P}_{f_1}}{d\mathbb{P}_{f_2}}(x) \leq C.
\]

This condition bounds the information in the first link $F \rightarrow X$ of the chain. As a result, it becomes possible to derive an upper bound on $I(F; Y)$ in terms of the constant $C$ and the information $I(X; Y)$ in the other link of the chain.

The following theorem is a slight extension of Lemma 3 of [18] (without the independence assumption, see later) where we allow the random variables $X$ and $Y$ to be continuous as well, unlike in Lemma 3 of [18], where it was implicitly assumed that they are discrete (by using the entropy $H(X)$ and $H(X|Y)$ in the proof). However, in our manuscript $X$ is continuous and therefore the above mentioned lemma does not apply directly.

**Theorem A.7.** Assume that (A.3) holds for $C \geq 1$. Then
\[
I(F; Y) \leq 2C^2(C - 1)^2 I(X; Y).
\]
Proof. In view of \((A.1)\)
\[
I(F; Y) = \int \int p(f \mid y) \log \frac{p(f \mid y)}{p(f)} df dP_Y(dy).
\]
\((A.4)\)

Since KL-divergence is nonnegative, we have
\[
- \int p(f) \log \frac{p(f \mid y)}{p(f)} df \geq 0.
\]
It follows that the inner integral in \((A.4)\) is bounded by
\[
\int (p(f \mid y) - p(f)) \log \frac{p(f \mid y)}{p(f)} df.
\]
Since \(|\log(a/b)| \leq |a - b|/(a \wedge b)|

we have the further bound
\[
K(P_{F \mid Y = y}, P_F) \leq \int \frac{|(p(f \mid y) - p(f))^2}{p(f \mid y) \wedge p(f)} df.
\]
\((A.5)\)

We will see ahead that the denominator in the integrand is always strictly positive.

We also define \(p(f \mid x) = P_{F \mid X = x} \{f\}\). Then by conditioning on \(X\) we see that
\[
p(f \mid y) = \int p(f \mid x) dP_X \mid Y = y(x), \quad p(f) = \int p(f \mid x) dP_X(x).
\]

By subtracting these relations and using also that
\[
0 = \int p(f)(dP_X \mid Y = y(x) - dP_X(x)),
\]
we obtain
\[
|p(f \mid y) - p(f)| = \left| \int (p(f \mid x) - p(f))(dP_X \mid Y = y(x) - dP_X(x)) \right|
\]
\((A.5)\)

\[
\leq 2 \sup_{x \in \mathcal{X}'} |p(f \mid x) - p(f)| \|P_X \mid Y = y - P_X\|_{TV}.
\]

Now by Bayes’ formula and the assumption \((A.3)\) on the likelihood,
\[
p(f \mid x)/p(f) = \frac{1}{\int dP_{f'/f}(x)p(f') df'} \in [1/C, C]
\]
\((A.6)\)

for all \(x \in \mathcal{X}'\). But then also
\[
(1 - C)p(f) \leq (1/C - 1)p(f) \leq p(f \mid x) - p(f) \leq (C - 1)p(f),
\]
that is, \(|p(f \mid x) - p(f)| \leq (C - 1)p(f)|. Also note that since
\[
p(f \mid y) = \int p(f \mid x) dP_X \mid Y = y(dx),
\]
\((A.6)\) implies that \(p(f \mid y)/p(f) \in [1/C, C]\) as well. In particular, \(p(f) \leq C p(f \mid y)\).
Together, we get
\[
|p(f \mid x) - p(f)| \leq C(C - 1)(p(f) \wedge p(f \mid y))
\]
for all $x \in \mathcal{X}'$. Combining with what we had above, we get

$$|p(f \mid y) - p(f)| \leq 2C(C - 1)(p(f) \wedge p(f \mid y))\|P_{X \mid Y = y} - P_X\|_{TV},$$

and hence

$$\frac{|(p(f \mid y) - p(f)|^2}{p(f \mid y) \wedge p(f)} \leq 4C^2(C - 1)^2 p(f)\|P_{X \mid Y = y} - P_X\|^2_{TV}.$$ 

Integrating w.r.t. $f$ this gives the bound

$$K(P_F, P_{F \mid Y = y}) \leq 4C^2(C - 1)^2\|P_{X \mid Y = y} - P_X\|^2_{TV}.$$ 

Use Pinsker’s inequality (e.g. [16], p. 88) and integrate w.r.t. $P_Y$ to arrive at the statement of the theorem. \qed

**A.4. Bounding $I(F; Y)$: general case.** We consider the same setting as in the preceding section, that is, we have Markov chain $F \rightarrow X \rightarrow Y$ and we assume that $F$ has a uniform distribution on a finite set $\mathcal{F}_0$ and that $X \mid f \sim P_f$. In this section we drop the condition that we have a uniform bound on the likelihood ratio. The bound for the mutual information $I(F; Y)$ then takes the following form.

The following theorem is a slight extension of Lemma 4 of [18] (without the independence assumption, see later) where we allow again that the random variables $X$ and $Y$ are continuous as well, unlike in Lemma 4 of [18], where it was implicitly assumed that they are discrete.

**Theorem A.8.** For all $C \geq 1$ we have

$$I(F; Y) \leq \log 2 + \frac{\log |\mathcal{F}_0|}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} P_f \left( \max_{f_1, f_2 \in \mathcal{F}_0} \frac{dP_{f_1}}{dP_{f_2}}(X) > C \right) + 2C^2(C - 1)^2 I(X; Y).$$

**Proof.** Define the set

$$B = \left\{ x : \mathcal{X} : \max_{f_1, f_2 \in \mathcal{F}_0} \frac{dP_{f_1}}{dP_{f_2}}(x) \leq C \right\}$$

and the indicator variable $E = 1_{X \in B}$. With this notation the statement of the theorem reads

$$I(F; Y) \leq \log 2 + P_{(X,F)}(E = 0) \log |\mathcal{F}_0| + 2C^2(C - 1)^2 I(X; Y),$$

where $P_{(X,F)}$ is the probability measure defined by the Markov chain, i.e. the measure under which $F$ is uniform and $X \mid (F = f) \sim P_f$.

By the chain rule and the fact that the mutual information is nonnegative, $I(F; (Y, E)) = I(F; Y) + I(F; E \mid Y) \geq I(F; Y)$. On the other hand, applying the chain rule with $Y$ and $E$ reversed shows that $I(F; (Y, E)) = I(F; Y \mid E) + I(F; E)$. Hence, we have the inequality

$$I(F; Y) \leq I(F; Y \mid E) + I(F; E).$$

The second term on the right involves only discrete variables and can be bounded by $H(E)$. This is the entropy of a Bernoulli variable, which is bounded by $\log 2$. The first term equals

$$pI(F; Y \mid E = 1) + (1 - p)I(F; Y \mid E = 0),$$
where \( p = P_X(E = 1) \). Below we prove that

\[
A.7 \quad I(F; Y \mid E = 1) \leq 2C^2(C - 1)^2 I(X, Y \mid E = 1).
\]

By the chain rule, \( I(X; Y \mid E) = I(Y; (X, E)) - I(Y; (X, E)) \). Since \( E \) is a function of \( X \), the last quantity equals \( I(Y; X) \). Next, observe that it follows from the definitions that

\[
I(F; Y \mid E = 0) = H(F \mid E = 0) - \int \int p(f|y) \log \frac{1}{p(f|y)} df \, dP_Y \mid E = 0(y) \leq H(F \mid E = 0).
\]

Since \( F \mid E = 0 \) lives in the finite set \( \mathcal{F}_0 \), this is further bounded by \( \log |\mathcal{F}_0| \).

It remains to establish \( A.7 \). This essentially follows from conditioning on \( E = 1 \) everywhere in the proof of Theorem A.7. Indeed, conditioning in the first part of the proof shows that

\[
I(F; Y \mid E = 1) \leq \int \left( \int \frac{|p(f|y, E = 1) - p(f|E = 1)|^2}{p(f|y, E = 1) \wedge p(f|E = 1)} \, df \right) \, dP_Y \mid E = 1(dy)
\]

and

\[
|p(f|y, E = 1) - p(f|E = 1)| \leq 2 \sup_{x \in B} \left| p(f|x, E = 1) - p(f|E = 1) \right| \times \|P_X|Y=y, E=1 - P_X|E=1\|_{TV}.
\]

Now observe that since the likelihood ratio is uniformly bounded by \( C \) for \( x \in B \), Bayes formula implies that

\[
p(f|x, E = 1) \quad p(f|E = 1) \in [1/C, C]
\]

for all \( x \in B \). We can then follow the rest of the proof of Theorem A.7 and arrive at \( A.7 \).

\[
\square
\]

A.5. Bounding \( I(F; Y) \): extra independence assumption. Next we consider one additional assumption on the structure of the problem. We assume that the data \( X \) is a \( d \)-dimensional vector of the form \( X = (X_1, \ldots, X_d) \), and that \( F \) is a \( d \)-dimensional vector as well such that for all coordinates \( j \in \{1, \ldots, d\} \), it holds that \( F_j \) and \( X_j \mid (F_j = f_j) \) are independent of \( F_{-j} \). More precisely, we assume that for the marginal conditional density of \( X_j \) it holds that

\[
p(x_j|f) = p(x_j \mid f_j)
\]

for every \( j \). Note that this is an assumption on the statistical model for the data \( X \) and is not related to the distribution of \( F \).

The following theorem is an extended version of Lemma 3 of [18] (now also with the independence assumption) as it holds also for continuous random variables \( X \) and \( Y \), unlike the result derived in [18].

**Theorem A.9.** Suppose that \( F \) and \( X \) are \( d \)-dimensional and that \( X_j \mid F \) only depends on \( F_j \) (i.e. \( p(x_j|f) = p(x_j|f_j) \)). Moreover, suppose that for the marginal densities \( p(x_j \mid f_j) \) it holds that

\[
\sup_{x_j} \max_{f \neq f'} \frac{p(x_j|f_j)}{p(x_j|f'_j)} \leq C
\]
for a constant $C \geq 1$. Then

$$I(F; Y) \leq 2C^2(C - 1)^2 \sum_{i=1}^{d} I(X_i; Y \mid F_{1:i-1}).$$

**Proof.** By the chain rule,

$$I(F; Y) = \sum_{j=1}^{d} I(F_j; Y \mid F_{1:j-1}).$$

So consider term $i$ in the sum.

By definition of conditional mutual information and Fubini’s theorem,

$$I(F_j; Y \mid F_{1:j-1}) = \int p(f_{1:j-1}) \left( \int p(y \mid f_{1:j-1}) \left( \int p(f_j \mid y, f_{1:j-1}) \log \frac{p(f_j \mid y, f_{1:j-1})}{p(f_j \mid f_{1:j-1})} dy \right) df_j \right) df_{1:j-1}.$$ 

We are first going to analyze the inner integral. So fix $f_{1:j-1}$ for now. To simplify the notation somewhat we are going to write densities that are conditional on $f_{1:j-1}$ by $\tilde{p}$ instead of $p$. So then the inner integral becomes

$$\int \tilde{p}(f_j \mid y) \log \frac{\tilde{p}(f_j \mid y)}{\tilde{p}(f_j)} df_j.$$

Since KL-divergence is nonnegative, it follows that the inner integral is bounded by

$$\int (\tilde{p}(f_j \mid y) - \tilde{p}(f_j)) \log \frac{\tilde{p}(f_j \mid y)}{\tilde{p}(f_j)} df_j.$$

In view of the inequality $|\log(a/b)| \leq |a - b|/(a \wedge b)$ we obtain the further bound

$$\int \frac{|(\tilde{p}(f_j \mid y) - \tilde{p}(f_j)|^2}{\tilde{p}(f_j \mid y) \wedge \tilde{p}(f_j)} df_j.$$

(we will see ahead that the denominator in the integrand is always strictly positive). By conditioning on $x_j$ (and still on $f_{1:j-1}$) we see that

(A.8) \hspace{1em} \tilde{p}(f_j \mid y) = \int \tilde{p}(f_j \mid x_j) \tilde{p}(x_j \mid y) dx_j, \quad \tilde{p}(f_j) = \int \tilde{p}(f_j \mid x_j) \tilde{p}(x_j) dx_j.

By subtracting these relations and using also that $0 = \int \tilde{p}(f_j)(\tilde{p}(x_j \mid y) - \tilde{p}(x_j)) dx_j$, we obtain

$$|\tilde{p}(f_j \mid y) - \tilde{p}(f_j)| = \left| \int (\tilde{p}(f_j \mid x_j) - \tilde{p}(f_j))\tilde{p}(x_j \mid y) - \tilde{p}(x_j)) dx_j \right|$$

(A.9) \hspace{1em} \leq 2 \sup_{x_j \in X_j} |\tilde{p}(f_j \mid x_j) - \tilde{p}(f_j)| \|\tilde{P}_{X_j \mid Y = y} - \tilde{P}_{X_j}\|_{TV}.

Now by Bayes’ formula,

$$\frac{\tilde{p}(f_j \mid x_j)}{\tilde{p}(f_j)} = \frac{\tilde{p}(x_j \mid f_j)}{\int \tilde{p}(x_j \mid f'_j) df'_j}. $$
But by the conditional independence assumption, \( \tilde{p}(x_j \mid f_j) = p(x_j \mid f_{1:j}) = p(x_j \mid f_j) \). Hence, by the assumed bound on the marginal likelihood-ratio, \( \tilde{p}(f_j \mid x_j) / \tilde{p}(f_j) \in [1/C, C] \) for all \( x_j \in X' \). But then also

\[
(1 - C)\tilde{p}(f_j) \leq (1/C - 1)\tilde{p}(f_j) \leq \tilde{p}(f_j \mid x_j) - \tilde{p}(f_j) \leq (C - 1)\tilde{p}(f_j),
\]

that is, \( |\tilde{p}(f_j \mid x_j) - \tilde{p}(f_j)| \leq (C - 1)\tilde{p}(f_j) \). Also note that the first identity in (A.8) implies that \( \tilde{p}(f_j \mid y) / \tilde{p}(f_j) \in [1/C, C] \) as well. In particular, \( \tilde{p}(f_j) \leq C\tilde{p}(f_j \mid y) \).

Together, we get

\[
|\tilde{p}(f_j \mid x_j) - \tilde{p}(f_j)| \leq C(C - 1)(\tilde{p}(f_j) \land \tilde{p}(f_j \mid y))
\]

for all \( x_j \in X' \). Combining with what we had above, we get

\[
|\tilde{p}(f_j \mid y) - \tilde{p}(f_j)| \leq 2C(C - 1)(\tilde{p}(f_j) \land \tilde{p}(f_j \mid y))\|	ilde{P}_{X_j \mid Y = y} - \tilde{P}_{X_j}\|_\text{TV},
\]

and hence

\[
\frac{|(\tilde{p}(f_j \mid y) - \tilde{p}(f_j)|^2}{\tilde{p}(f_j \mid y) \land \tilde{p}(f_j)} \leq 4C^2(C - 1)^2\tilde{p}(f_j) \|	ilde{P}_{X_j \mid Y = y} - \tilde{P}_{X_j}\|_\text{TV}^2.
\]

Integrating w.r.t. \( f_j \) this gives the bound

\[
\int \tilde{p}(f_j \mid y) \log \frac{\tilde{p}(f_j \mid y)}{\tilde{p}(f_j)} \, df_j \leq 4C^2(C - 1)^2\|	ilde{P}_{X_j \mid Y = y} - \tilde{P}_{X_j}\|_\text{TV}^2.
\]

By Pinsker’s inequality,

\[
\|	ilde{P}_{X_j \mid Y = y} - \tilde{P}_{X_j}\|_\text{TV}^2 \leq \frac{1}{2} K(\tilde{P}_{X_j \mid Y = y}, \tilde{P}_{X_j}).
\]

Hence, by multiplying by \( \tilde{p}(y) \) and integrating we find that

\[
\int \tilde{p}(y) \left( \int \tilde{p}(f_j \mid y) \log \frac{\tilde{p}(f_j \mid y)}{\tilde{p}(f_j)} \, df_j \right) dy \leq 2C^2(C - 1)^2I(X_j; Y \mid F_{1:j-1} = f_{1:j-1}).
\]

Multiplying by \( p(f_{1:j-1}) \) and integrating gives

\[
I(F_j; Y \mid F_{1:j-1}) \leq 2C^2(C - 1)^2I(X_j; Y \mid F_{1:j-1}).
\]

\( \square \)

We also have the version of the preceding result for the case that we do not have the likelihood ratio bound everywhere. This result is an extended version of Lemma 4 of [18].

**Theorem A.10.** Suppose that \( F \) and \( X \) are \( d \)-dimensional and that \( F_j \) and \( X_j \mid F_j \) are independent of \( F_{-j} \). For \( C \geq 1 \), define

\[
B_j = \{ x_j : \max_{f \neq f'} \frac{p(x_j \mid f_j)}{p(x_j \mid f'_j)} \leq C \}
\]

for a constant \( C \geq 1 \). Then

\[
I(F; Y) \leq \sum_{j=1}^d \left( \log 2 \sqrt{P_{X_j}(X_j \notin B_j)} + \log |F_0| P_{X_j}(X_j \notin B_j) \right) + 2C^2(C - 1)^2I(X; Y)
\]
we have that
\[ I(F; Y) = \sum_{j=1}^{d} I(F_j; Y | F_{1:j-1}). \]

Now for fixed \( j \) we argue as in Theorem A.8, but conditional on \( F_{1:j-1} \), to get
\[ I(F_j; Y | F_{1:j-1}) \leq H(1_{X_j \in B_j} | F_{1:j-1}) + \log |\mathcal{F}_0|P_{X_j}(X_j \not\in B_j) + 2C^2(C - 1)^2 I(X_j, Y | F_{1:j-1}). \]

Since conditioning decreases entropy,
\[ H(1_{X_j \in B_j} | F_{1:j-1}) \leq H(1_{X_j \in B_j}) \leq (\log 2) \sqrt{P_{X_j}(X_j \not\in B_j)}. \]

Combining the preceding computations we get that
\[ I(F, Y) \leq \sum_{j=1}^{d} \left( (\log 2) \sqrt{P_{X_j}(X_j \not\in B_j)} + \log |\mathcal{F}_0|P_{X_j}(X_j \not\in B_j)
+ 2C^2(C - 1)^2 I(X_j, Y | F_{1:j-1}) \right). \]

Then the statement of the theorem follows from Lemma A.11 (below) and by applying the chain rule of information, i.e.
\[ \sum_{j=1}^{d} I(X_j; Y | F_{1:j-1}) \leq \sum_{j=1}^{d} I(X_j; Y | X_{1:j-1}) = I(X; Y). \]

\[ \square \]

**Lemma A.11.** Under the assumption that \( X_j | F_j \) and \( F_j \) are independent of \( F_{-j} \) we have that
\[ I(X_j; Y | F_{1:j-1}) \leq I(X_j; Y | X_{1:j-1}). \]

**Proof.**
\[ I(X_j; Y | F_{1:j-1}) = \int \int p(x_j, y | f_{1:j-1}) \log \frac{p(x_j, y | f_{1:j-1})}{p(x_j | f_{1:j-1})p(y | f_{1:j-1})} dx_j dy p(f_{1:j-1}) df_{1:j-1} \]
\[ = \int \int \int p(x_j, y, f_{1:j-1}) \log p(x_j | y, f_{1:j-1}) dx_j dy df_{1:j-1} - \int \int \int p(x_j, f_{1:j-1}) \log p(x_j | f_{1:j-1}) dx_j df_{1:j-1}. \]
\[ (A.10) \]

Next we note that since \( X_j \) is independent of \( F_{1:j-1} \) we have \( p(x_j) = p(x_j | f_{1:j-1}) \), furthermore since \( X_j \) and \( X_{1:j-1} \) are independent we get \( p(x_j) = p(x_j | x_{1:j-1}) \). Besides, we show below that
\[ \int \int \int p(x_j, y, f_{1:j-1}) \log p(x_j | y, f_{1:j-1}) dx_j dy df_{1:j-1} \]
\[ \leq \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log p(x_j | y, f_{1:j-1}, x_{1:j-1}) dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ (A.11) \]
Combining the preceding assertions we get that the right hand side of (A.10) is further bounded from above by

\[ \int \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log p(x_j | y, f_{1:j-1}, x_{1:j-1}) dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ - \int \int \int p(x_j, x_{1:j-1}) \log p(x_j | x_{1:j-1}) dx_j dx_{1:j-1} \]
\[ = \int \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log p(x_j | y, f_{1:j-1}, x_{1:j-1}) dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ - \int \int \int p(x_j, y, x_{1:j-1}) \log p(x_j | x_{1:j-1}) dx_j dy dx_{1:j-1} \]
\[ = I(X_j; Y | X_{1:j-1}), \]

where in the first equation we used the Markov property of the chain \( F \mapsto X \mapsto Y \) combined with the independence of \( X_{j:d} \) and \( F_{1:j-1} \), i.e.

\[ p(x_j | y, f_{1:j-1}, x_{1:j-1}) = \frac{p(x_j, y | f_{1:j-1}, x_{1:j-1})}{p(y | f_{1:j-1}, x_{1:j-1})} = \frac{p(x_j, y | x_{1:j-1})}{p(y | x_{1:j-1})} = p(x_j | y, x_{1:j-1}). \]

It remained to verify the inequality (A.11).

\[ \int \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log p(x_j | y, f_{1:j-1}, x_{1:j-1}) dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ - \int \int \int p(x_j, y, f_{1:j-1}) \log p(x_j | y, f_{1:j-1}) dx_j dy df_{1:j-1} \]
\[ = \int \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log \frac{p(x_j | y, f_{1:j-1}, x_{1:j-1})}{p(x_j | y, f_{1:j-1})} dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ = \int \int \int \int p(x_j, y, f_{1:j-1}, x_{1:j-1}) \log \frac{p(x_j, f_{1:j-1} | y, x_{1:j-1})}{p(x_j | y, f_{1:j-1}) p(x_{1:j-1} | y, f_{1:j-1})} dx_j dy df_{1:j-1} dx_{1:j-1} \]
\[ = I(X_j; X_{1:j-1} | Y, F_{1:j-1}) \geq 0. \]

\[ \square \]

A.6. Bounding \( I(F; Y) \): decomposable Markov chain. Suppose now in addition that the data can be decomposed as \( X = (X^{(1)}, \ldots, X^{(m)}) \) and that under \( P_f \), the \( X^{(i)} \) are independent and \( X^{(i)} \sim P^{(i)}_f \). This is intended to describe a setting in which the data is distributed over \( m \) different local machines. The machines have independent data and each machine has its local model \( (P^{(i)}_f) : f \in \mathcal{F} \). Next we have for every \( i \) a Markov chain \( X^{(i)} \rightarrow Y^{(i)} \) defined by some Markov transition kernel \( Q^{(i)} \). In other words, every machine processes or transforms its local data in some way. Next the processed data is aggregated into a vector \( Y = (Y^{(1)}, \ldots, Y^{(m)}) \). As before we consider the collection \( \mathcal{E}(Y) \) of all estimators that are measurable functions of this aggregated, processed data \( Y \). In this distributed setting we have by Proposition A.3 that

\[ (A.12) \quad I(F; Y) \leq \sum_i I(F; Y^{(i)}). \]

Then the statement of Theorem A.12 can be reformulated as
THEOREM A.12. Let us assume that the data $X$ is decomposable as above and suppose that $F$ and $X$ are $d$-dimensional and that $X_j | F_j$ and $F_j$ are independent of $F_{-j}$. For $C \geq 1$, define

$$B_j = \{ x_j : \max_{f \neq f'} p(x_j | f_j) / p(x_j | f'_j) \leq C \}$$

for a constant $C \geq 1$. Then

$$I(F; Y) \leq \sum_{i=1}^{m} \sum_{j=1}^{d} \left( (\log 2) \sqrt{P_{X_j}^i(X_j^i \notin B_j) + \log |F_0| P_{X_j}^i(X_j^i \notin B_j)} \right)$$

$$+ 2C^2(C - 1)^2 \sum_{i=1}^{m} I(X_i; Y_i).$$

Here $I(X_i^i; Y_i^i)$ is the mutual information between $X_i^i$ and $Y_i^i$ in the Markov chain $F \rightarrow X_i^i \rightarrow Y_i^i$, where $F$ has a uniform distribution on $F_0$ and $X_i^i | (F = f) \sim P_f^i = P_{X_i}^i | F = f$. 

PROOF. The statement of the theorem follows by combining assertion (A.12) with Theorem A.12. 

A.7. Bounding $I(F; Y)$: Gaussian example. In this example we further assume that the random variables $X_j^i$ are conditionally Gaussian, i.e. $X_j^i | (F_j = f_j) \sim N(f_j, m/n)$, for $f_j = \delta \beta_j$ with $\beta_j \in \{-1, 1\}$ and that the observation $X$ satisfies the independence and decomposable assumptions introduced in Sections A.5 and A.6, respectively. Then we can give a more concrete upper bound for the mutual information $I(F; Y)$.

Lemma A.13. Let $F = \delta \beta$, with $\delta^2 \leq 2^{-10} m/(n \log(md))$ and $\beta$ a uniformly distributed random variable over $\{-1, 1\}^d$. Furthermore, suppose that $X = (X_1^1, ..., X^m)$, the $X_i^i$'s are independent, $d$-dimensional random variables and that $X_j^i | (F = f) \sim P_f^i = N(f_j, m/n)$. Then

$$I(F; Y) \leq \sum_{i=1}^{m} 2\delta^2 \min_{m/n} \left\{ 2^{10 \log(md) H(Y_i), d} \right\} + 4 \log 2,$$

where $I(F; Y)$ is the mutual information between $F$ and $Y$ in the Markov chain $F \rightarrow X \rightarrow Y$. 

PROOF. Let us introduce the notation $a^2 = 2^4 \log(md)m/n$ and note that

$$\sup_{|x| \leq a} \varphi_{\delta, m/n}(x) \leq \sup_{|x| \leq a} e^{n[(x-\delta)^2-(x+\delta)^2]/2} \leq \sup_{|x| \leq a} e^{2n|\delta|} \leq e^{2n|\delta|},$$

where $\varphi_{\mu, \sigma^2}$ denotes the density function of a normal distribution with mean $\mu$ and variance $\sigma^2$. Furthermore, let us introduce the notation $B_j = \{|x_j| \leq a\}, j = 1, ..., d$. Then by Theorem A.12 (with $F_0 = \{f = \delta \beta : \beta \in \{-1, 1\}^d\}$) we have that

$$I(F; Y) \leq dm(\log 2) \sqrt{P_{X_j}^i(X_j^i \notin B_j) + d^2 m P_{X_j}^i(X_j^i \notin B_j)}$$

(A.13)

$$+ 2C^2(C - 1)^2 \sum_{i=1}^{m} I(X_i; Y_i).$$
with \( C = e^{2\beta\sqrt{\log(md)n/m}} \). Next note that for \( Z \sim N(0,1) \)

\[
P_{X_j^{(i)}}(X_j^{(i)} \notin B_j) \leq P(|Z| \geq a - \delta) \leq 2e^{-\frac{(a-\delta)^2}{2m}} \leq 2e^{-\frac{\sqrt{\log(md)}}{4m}} \leq 2(md)^{-4},
\]

and the inequality \( I(X^{(i)}; Y^{(i)}) \leq H(Y^{(i)}) \) holds. Then by plugging in the above inequalities into (A.13) and using the inequalities \( e^x \leq 1 + 2x \) for \( x \leq 0.4 \) and \( C^2 \leq 2 \) we get that

\[
I(F;Y) \leq \sqrt{2}(\log 2)(md)^{-1} + 2(\log 2)m^{-3}d^{-2} + 2^{11}d^2\frac{\log(md)n}{m} \sum_{i=1}^m H(Y^{(i)}).
\]

Furthermore, from the data-processing inequality and the convexity of the KL divergence

\[
I(F;Y) \leq \sum_{i=1}^m I(F;Y^{(i)}) \leq \sum_{i=1}^m I(F;X^{(i)})
\]

\[
\leq \sum_{i=1}^m \frac{1}{|F_0|^2} \sum_{f,f' \in F_0} K(\|P^{(i)}_f - P^{(i)}_{f'}\|_2)
\]

\[
= \sum_{i=1}^m \frac{\delta^2}{2m/n |F_0|^2} \sum_{f,f' \in F_0} \|\beta - \beta'\|_2^2
\]

\[
\leq 2nd\delta^2.
\]

\[\square\]

**Remark A.14.** We note that the Gaussianity of the problem was only applied in Lemma A.13, above. The rest of the information theoretic computations are distribution free and apply for a wider range of models. This leaves room for possible extensions of the minimax lower bounds for other problems as well.

**A.8. Entropy of a finite binary string.** In the proof of Theorem 2.1 we need to bound the entropy of transmitted finite binary string \( Y^{(i)} \). Since we do not want to restrict ourselves only to prefix codes, we can not use a standard version of Shannon’s source coding theorem for this purpose. Instead we use the following result.

**Lemma A.15.** Let \( Y \) be a random finite binary string. Its expected length satisfies the inequality

\[
H(Y) \leq 2E|Y| + 1.
\]

**Proof.** Let \( N = l(Y) \) and consider a random binary string \( U \) with distribution \( U|N = n \sim \text{Unif}([0,1]^n) \). Then for \( S \) the collection of finite binary strings,

\[
K(Y,U) = \sum_{s \in S} \mathbb{P}(Y = s) \log \frac{\mathbb{P}(Y = s)}{\mathbb{P}(U = s)}
\]

\[
= \sum_{s \in S} \mathbb{P}(Y = s) \log \frac{1}{\mathbb{P}(U = s)} - H(Y)
\]

\[
= \sum_{n} \sum_{s \in \{0,1\}^n} \mathbb{P}(Y = s) \log \frac{1}{\mathbb{P}(U = s)} - H(Y).
\]
Now for every $n$ and $s \in \{0,1\}^n$, we have $P(U = s) = P(U = s \mid N = n)P(N = n) = 2^{-n}P(N = n)$. It follows that

$$\sum_{s \in \{0,1\}^n} P(Y = s) \log \frac{1}{P(U = s)} = P(N = n) \log \frac{2^n}{P(N = n)}.$$  

Hence,

$$K(Y, U) \leq (\log 2) \mathbb{E} N + H(N) - H(Y)$$  

The non-negativity of the KL-divergence thus implies that $H(Y) \leq \mathbb{E} N + H(N)$. To complete the proof we show that $H(N) \leq \mathbb{E}N + 1$. To do so consider the index set $I = \{i : P(N = i) \geq e^{-1}\}$ and note that the function $x \mapsto x \log(1/x)$ is monotone increasing for $x \leq e^{-1}$. Then

$$H(N) = \sum_{i \in I} P(N = i) \log \frac{1}{P(N = i)} + \sum_{i \in I^c} P(N = i) \log \frac{1}{P(N = i)}$$

$$\leq \sum_{i \in I} P(N = i)i + \sum_{i \in I^c} e^{-1}i \leq \mathbb{E} N + 1.$$  

This completes the proof.  

**APPENDIX B: DEFINITIONS AND NOTATIONS FOR WAVELETS**

In this section we give a brief introduction to wavelets. A more detailed and elaborate description of wavelets can be found for instance in [9, 7].

In our work we consider the Cohen, Daubechies and Vial construction of compactly supported, orthonormal, $N$-regular wavelet basis of $L_2[0,1]$, see for instance [1]. First for any $N \in \mathbb{N}$ one can follow Daubechies’ construction of the father $\varphi(.)$ and mother $\psi(.)$ wavelets with $N$ vanishing moments and bounded support on $[0,2N - 1]$ and $[-N + 1,N]$, respectively, see for instance [3]. Then we obtain the basis functions

$$\{ \varphi_{j0,m}, \psi_{jk} : m \in \{0,...,2^{j0-1}\}, \ j > j_0, \ k \in \{0,...,2^{j-1}\} \},$$

for some sufficiently large resolution level $j_0 > 0$. We also denote by $K_j = \{0,1,...,2^j - 1\}$ the collection indices at resolution level $j > 0$. Note that $|K_j| = 2^j$. The basis functions (on $x \in [0,1]$) are given as $\psi_{jk}(x) = 2^{j/2}\psi(2^jx - k)$, for $k \in [N - 1, 2^j - N]$, and $\varphi_{j0,k}(x) = 2^{j0}\varphi(2^{j0}x - m)$, for $m \in [0,2^{j0} - 2N]$, while for other values of $k$ and $m$, the basis functions are specially constructed, to form a basis with the required smoothness property. For convenience we introduce the notation $\psi_{j0,k} := \varphi_{j0,k}$ for $k = 0,...,2^{j0-1}$. This does not mean, however, that $\varphi_{j0,k}(x) = 2^{-1}\psi_{j0+1,k}(2^{-1}x)$. Then the function $f \in L_2[0,1]$ can be represented in the form

$$f = \sum_{j = j_0}^{\infty} \sum_{k \in K_j} f_{jk} \psi_{jk}.$$  

Note that in view of the orthonormality of the wavelet basis the $L_2$-norm of the function $f$ is equal to

$$\|f\|_2^2 = \sum_{j = j_0}^{\infty} \sum_{k \in K_j} f_{jk}^2.$$  


For notational convenience we will take $j_0$ to be 1 in our paper, this can be done without loss of generality.

Next we introduce the Besov spaces we are considering in our analysis. Let us define the Besov (Sobolev-type) norm for $s \in (0, N)$ as

$$\|f\|_{s,2} = \max \{ 2^{j_0 s} \| \langle f, \varphi_{j_0} \rangle \|_2, \sup_{j > j_0} 2^{j s} \| \langle f, \psi_j \rangle \|_2 \},$$

where $\langle f, g \rangle$ is the standard $L_2$-inner product of the functions $f, g \in L_2[0,1]$. Then the Besov space $B_{2,\infty}^s$ and Besov ball $B_{2,\infty}^s(L)$ of radius $L > 0$ are defined as

$$B_{2,\infty}^s = \{ f \in L_2[0,1] : \|f\|_{s,2} < \infty \}, \quad \text{and} \quad B_{2,\infty}^s(L) = \{ f \in L_2[0,1] : \|f\|_{s,2} < L \},$$

respectively. We note that the present Besov space is larger than the standard Sobolev space where instead of the supremum one would take the sum over the resolution levels $j$. Then we introduce the Besov (Hölder-type) norm for $s \in (0, N)$ as

$$\|f\|_{s,\infty} = \sup_{j \geq j_0} \{ 2^{(s+1)/2} |f_{jk}| \}.$$

Then similarly to before we define the Besov space $B_{\infty,\infty}^s$ and Besov ball $B_{\infty,\infty}^s(L)$ of radius $L > 0$ as

$$B_{\infty,\infty}^s = \{ f \in L_2[0,1] : \|f\|_{s,\infty} < \infty \}, \quad \text{and} \quad B_{\infty,\infty}^s(L) = \{ f \in L_2[0,1] : \|f\|_{s,\infty} < L \},$$

respectively. For $s \neq N$ these spaces are equivalent to the classical Hölder spaces, while for integer $s$ they are equivalent to the so called Zygmund spaces, see [1].

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**Mathematical Institute**
Leiden University
Niels Bohrweg 1
2333 CA Leiden
The Netherlands
E-mail: b.t.szabo@math.leidenuniv.nl

**Korteweg-de Vries Institute for Mathematics**
University of Amsterdam
Science Park 107
1098 XG Amsterdam
The Netherlands
E-mail: hvzanten@uva.nl