TRIANGULAR PICTURE FUZZY LINGUISTIC INDUCED ORDERED WEIGHTED AGGREGATION OPERATORS AND ITS APPLICATION ON DECISION MAKING PROBLEMS

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Abstract. The primary goal of this paper is to solve the investment problem based on linguistic picture decision making method under the linguistic triangular picture linguistic fuzzy environment. First to define the triangular picture linguistic fuzzy numbers. Further, we define operations on triangular picture linguistic fuzzy numbers and their aggregation operator namely, triangular picture fuzzy linguistic induce OWA (TPFLIOWA) and triangular picture fuzzy linguistic induce OWG (TPFLIOGW) operators. Multi-criteria group decision making method is developed based on TPFLIOWA and TPFLIOGW operators and solve the uncertainty in the investment problem. We study the applicability of the proposed decision making method under triangular picture linguistic fuzzy environment and construct a descriptive example of investment problem. We conclude from the comparison and sensitive analysis that the proposed decision making method is more effective and reliable than other existing models.

1. Introduction. Zadeh’s [56] developed the idea of fuzzy set (FS) by assign membership grade to elements of a set in the interval of [0, 1]. In fuzzy set each element “ni” of the domain set consist on one grade (positive membership degree) “µ(ni)”, which belong to [0, 1], and hesitancy degree of the fuzzy set is equal to “1 − µ(ni)”. Fuzzy set theory applied to many fields i.e. fuzzy decision making problems [4, 6]. But fuzzy set theory failed to explain the complete uncertainty in the real life problem due to non-membership grade. Then, in 1986, Atanassov [3] developed the concept of intuitionistic fuzzy set (IFS), which have two membership degree, one is positive and second is negative membership degree.

But in intuitionistic fuzzy set Atanassov discussed only two category of responses i.e. “yes” and “no”, but we have three types of responses in case of selection, for example “yes”, “no” and “refusal”, and the complicated answer is “refusal”. So, to overcome this business, Cuo [8, 9] introduced a novel concept of picture fuzzy

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183
set (PFS), which dignified in three different functions presenting the positive, neutral and negative membership degrees. Cuong [10], discussed some characteristic of PFSs and also their distance measures are approved. Cuong and Hai [11] defined first time fuzzy logic operators and introduced fundamental operations for fuzzy derivation forms in the PF logic. Cuong et al. [12] examined the characteristic of picture fuzzy t-norm and t-conorm. Phong et al. [23] explored certain configuration of picture fuzzy relations. Wei et al. [48, 49, 50] defined many procedure to compute the closeness between PFSs. Presently, many researchers have developed more models in the PFSs condition: Correlation coefficient of PFS are proposed by Sing [31] and apply it to clustering analysis. Son et al. [32, 35] provided time arrangement calculation and temperature estimation on the basis of PFSs domain. Son [33, 34] defined picture fuzzy (PF) separation measures, generalized PF distance measures and PF association measures, and combined it to tackle grouping examination under PFSs condition. A novel fuzzy derivation structure on PFS are defined by Son et al. [40] to improved the performance of the classical fuzzy inference system. Thong et al. [36, 37] utilized the PF clustering approach for complex data and particle clump optimization. Wei [47] exhibit PF aggregation operators and tested it to MADM problem for ranking EPR framework. Using the PF weighted cross-entropy concept, Wei [46] studied basic leadership technique and used this technique to rank the alternative. Based on PFSs, Yang et al. [54] defined adjustable soft discernibly matrix and tested it in decision making. Garg [16] design aggregation operations on PFSs and tested it for MCDM problems. Peng et al. [22] proposed an picture fuzzy set algorithm and tested it in decision making. For other researches on PFS, the readers are see [25, 38, 39]. To handle MAGDM problems, Ashraf et al. [2] give two techniques, firstly the aggregation operators to aggregate the PF information and secondly, the TOPSIS method to aggregate the picture fuzzy information. Bo and Zhang [5] studied some operations of picture fuzzy relations such as type-2 inclusion relation, type-2 union, type-2 intersection and type-2 complement operations and also defined the anti-reflexive kernel, symmetric kernel, reflexive closure and symmetric closure of a PF relation. Ashraf et al. [1] extended the structure of cubic sets to the picture fuzzy sets. He also defined the notion of positive-internal, neutral-internal, negative-internal and positive-external, neutral-external and negative-external cubic picture fuzzy sets.

Shu et al. [29] defined a triangular intuitionistic fuzzy number (TIFN) in a similar way, as the fuzzy number defined by Dubois and Prade [14]. Recently, Li [19] defined a new definition of the triangular intuitionistic fuzzy number which has an logically reasonable interpretation and some applications are given in Roostae et al. [28] and Robinson and Amirtharaj [26]. Liang et al. [20] proposed a new method of MCDM problem using TIFN based on geometric average operators of TIFN. Morgan et al., [15] defined some technique to find the distance between IF sets on the basis of $\alpha$-cuts. Nowadays, the TIFNs as a kind of typical IFNs, have attracted more and more considerations through a large amount of literature. A number of achievements on TIFNs have been proposed. Roughly, these achievements may be divided into two types: ranking methods of TIFNs and their applications to decision making problems and aggregation operators of TIFNs and their applications to decision making problems are briefly reviewed in the following, respectively. The first type is the ranking of TIFNs. Li et al. [18] presented a ranking method based on the notion of a ratio of the value index to the ambiguity index and tested it to MADM problems in which the ratings of alternatives on attributes are expressed
with TIFNs. Zhang et al. [58] defined the distance measure of triangular IFNs and gave the ranking to the alternative based on TOPSIS method. Wan et al. [41, 42, 43] proposed the notion of the weighted possibility mean, variance, and covariance of TIFNs, and they proposed some ranking methods of TIFNs and applied them to deal with MADM problems. Based on the possibility method, Wan and Dong [44] introduced a technique for MAGDM based on TIFNs with incomplete preference information. The second type is the aggregation operators of TIFN. Zhang and Liu [57] defined the concepts of TIFN in which the positive membership and negative membership degree are represented by triangular fuzzy number, and also proposed the weighted geometric and the weighted averaging operators and used these operators for the decision-making problems. Robinson and EC [27] investigated triangular intuitionistic fuzzy ordered weighted averaging (TIFOWA) and triangular intuitionistic fuzzy hybrid averaging (TIFHA) operators. Chen and Li [7] defined a distance measure between two TIFNs to aid in finding attribute weights, and they presented the weighted arithmetic averaging operator on TIFNs, and then contact a dynamic MADM model with TIFNs. Wang et al. [45] defined an arithmetic operations and logic operators for TIFNs and applied them to fault analysis of a printed circuit board assembly system. Yu and Xu [55] defined the notion of intuitionistic multiplicative triangular fuzzy (IMTF) set and intuitionistic multiplicative triangular fuzzy number, and then discussed their operational laws and some desirable properties. Based on the operational laws, they developed a series of aggregation operators for IMTF information. Combining the fuzzy measure and Choquet integral, Wan and Dong [30] defined the TIF Choquet integral aggregation operator and investigated some desirable properties for this operator.

In this article, we defined the triangular picture fuzzy linguistic induced ordered weighted averaging and geometric aggregation operators. The objective of this study is to solve the input arguments taken the form of triangular picture fuzzy linguistic information. These are the new aggregation operators that uses the main characteristics of the picture fuzzy ordered weighted averaging (PFOWA) and picture fuzzy ordered weighted geometric (PFOWG) operators. Thus, the properties of triangular picture fuzzy linguistic induced ordered weighted averaging (TPFLOWA) and triangular picture fuzzy linguistic induced ordered weighted geometric (TPFLOWG) operators can be discussed. Finally, we developed a new algorithm based on the proposed operators and solved an example.

The reminder of the article is described as follows. In Sec. 2, we briefly discussed the basic knowledge about the extension of fuzzy linguistic set. In Sec. 3, we present TPFLOWA and TPFLOWG operators and its properties. Based on the developed operators an algorithm for multi-criteria decision making problem are developed in Sec. 4. In Sec. 5, makes some discussions on application of proposed method and their comparison with existing approach, and conclusion are drawn in Sec. 6.

2. Preliminaries.

Definition 2.1. [8] Let $\tilde{N} \neq 0$, be a fixed set. Then, a picture fuzzy set $\rho$ in $\tilde{n}$ is presented as

$$\rho = \{(\tilde{n}, \mu_\rho(\tilde{n}), \eta_\rho(\tilde{n}), \nu_\rho(\tilde{n})) | \tilde{n} \in \tilde{N}\}, \quad (1)$$

where the functions $\mu_\rho(\tilde{n}) : \tilde{N} \rightarrow [0, 1], \eta_\rho(\tilde{n}) : \tilde{N} \rightarrow [0, 1], \nu_\rho(\tilde{n}) : \tilde{N} \rightarrow [0, 1]$ satisfy the condition: $0 \leq \mu_\rho(\tilde{n}), \eta_\rho(\tilde{n}), \nu_\rho(\tilde{n}) \leq 1$. Furthermore, $\mu_\rho(\tilde{n}), \eta_\rho(\tilde{n})$ and $\nu_\rho(\tilde{n})$ represent the positive, neutral and negative
degree of the element $\tilde{n} \in \tilde{N}$ to the set $\rho$, respectively. In PFS, the degree of refusal is denoted by $\pi_{\rho}(\tilde{n}) = 1 - \mu_{\rho}(\tilde{n}) - \eta_{\rho}(\tilde{n}) - \nu_{\rho}(\tilde{n})$.

**Definition 2.2.** Let $\tilde{N} \neq 0$, be a universal set. Then, a picture fuzzy number $\tilde{\rho} \in \tilde{N}$ is denoted as $\tilde{\rho} = \{ (\rho^\mu, \rho, \rho^n; \mu_\rho), (\rho^n, \rho, \rho^\mu; \eta_\rho), (\rho^\nu, \rho, \rho^\nu; \nu_\rho) \}$, whose positive, neutral and negative membership functions are defined as follows:

$$
\mu_{\rho}(\tilde{n}) = \begin{cases} 
\frac{f^1_{\rho}(\tilde{n})}{\rho - \rho^0}, & \rho^0 \leq \tilde{n} < \rho \\
\mu_{\rho}, & \tilde{n} = \rho \\
f^0_{\rho}(\tilde{n}), & \rho < \tilde{n} \leq \rho^0 \\
0, & \text{otherwise}
\end{cases},
$$

(2)

$$
\eta_{\rho}(\tilde{n}) = \begin{cases} 
g^1_{\rho}(\tilde{n}), & \rho^0 \leq \tilde{n} < \rho \\
\eta_{\rho}, & \tilde{n} = \rho \\
g^0_{\rho}(\tilde{n}), & \rho < \tilde{n} \leq \rho^0 \\
0, & \text{otherwise}
\end{cases},
$$

(3)

$$
\nu_{\rho}(\tilde{n}) = \begin{cases} 
h^1_{\rho}(\tilde{n}), & \rho^\nu \leq \tilde{n} < \rho \\
\nu_{\rho}, & \tilde{n} = \rho \\
h^0_{\rho}(\tilde{n}), & \rho < \tilde{n} \leq \rho^\nu \\
1, & \text{otherwise}
\end{cases},
$$

(4)

where $f^1_{\rho}, g^1_{\rho}$ and $h^1_{\rho}$ are continuous monotone increasing functions and $f^0_{\rho}, g^0_{\rho}$ and $h^0_{\rho}$ are continuous monotone decreasing functions. $f^1_{\rho}, g^1_{\rho}, f^0_{\rho}, g^0_{\rho}$ and $h^0_{\rho}$ are the left and right basis functions of the positive, neutral and negative membership functions, respectively. The values $\mu_\rho, \eta_\rho$ and $\nu_\rho$, respectively, represent the maximum degrees of the positive, neutral and negative membership where $0 \leq \mu_\rho \leq 1, 0 \leq \eta_\rho \leq 1, 0 \leq \nu_\rho \leq 1$ and $0 \leq \mu_\rho + \eta_\rho + \nu_\rho \leq 1$.

In this work $\tilde{N} = E$, and the positive, neutral and negative membership functions of the triangular picture fuzzy numbers are defined as, respectively,

$$
\mu_{\rho}(\tilde{n}) = \begin{cases} 
\frac{(\tilde{n} - \rho^n)\mu_\rho}{\rho^n - \rho}, & \rho^n \leq \tilde{n} < \rho \\
\mu_{\rho}, & \tilde{n} = \rho \\
\frac{(\rho^n - \tilde{n})\mu_\rho}{\rho^n - \rho}, & \rho \leq \tilde{n} < \rho^n \\
0, & \text{otherwise}
\end{cases},
$$

$$
\eta_{\rho}(\tilde{n}) = \begin{cases} 
\frac{\rho - \tilde{n} + \eta_\rho(\tilde{n} - \rho^n)}{\rho - \rho^n}, & \rho^n \leq \tilde{n} < \rho \\
\eta_{\rho}, & \tilde{n} = \rho \\
\frac{\tilde{n} - \rho + \eta_\rho(\rho^n - \tilde{n})}{\rho^n - \rho}, & \rho \leq \tilde{n} < \rho^n \\
0, & \text{otherwise}
\end{cases},
$$
The values $\mu_\rho$, $\eta_\rho$, and $\nu_\rho$, respectively, denote the maximum membership degree of positive, neutral, and negative, where $0 \leq \mu_\rho \leq 1$, $0 \leq \eta_\rho \leq 1$, and $0 \leq \nu_\rho \leq 1$, and $0 \leq \mu_\rho + \eta_\rho + \nu_\rho \leq 1$.

For convenience, we represent $\tilde{\rho} = ([\rho_1, \rho_2, \rho_3]; \mu_\rho, \eta_\rho, \nu_\rho)$ be a triangular picture fuzzy number.

2.1. Picture fuzzy linguistic variables. Assume that $\tilde{S} = \{s_i | i = 0, 1, 2, 3, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, where $\alpha$th linguistic variable and the cardinality of $\tilde{S}$ are denoted by $s_\alpha$ and $g$, respectively with $g > 0$. e.g. the nine terms set as follows [52].

$\tilde{S} = \{s_0 = non,s_1 = extremely\ low, s_2 = very\ low, s_3 = low, s_4 = medium, s_5 = High, s_6 = very\ high, s_7 = extremely\ high, s_8 = perfect\}$. In which $s_m < s_n$, if $m < n$, and the following characteristics must be satisfy linguistic term set [17, 21].

(1) Negation operator is defined as, $Neg(s_m) = s_n, n = g - m$

(2) Maximum($s_m, s_n$) = $s_m$, if $s_m \geq s_n$

(3) Minimum($s_m, s_n$) = $s_m$, if $s_m \leq s_n$

Definition 2.3. [24] Let $\tilde{N} \neq 0$, be a fixed set. Then, the picture fuzzy linguistic set $\rho$ is defined as:

$$\rho = \left\{ \tilde{n}, (s_\theta(\tilde{n}), \mu_\rho(\tilde{n}), \eta_\rho(\tilde{n}), \nu_\rho(\tilde{n})) \mid \tilde{n} \in \tilde{N} \right\},$$

which is characterized by a linguistic term $s_\theta(\tilde{n}) \in \tilde{g}$, a positive degree $\mu_\rho(\tilde{n}) \in [0, 1]$, a neutral degree $\eta_\rho(\tilde{n}) \in [0, 1]$ and a negative degree $\nu_\rho(\tilde{n}) \in [0, 1]$ of the element $\tilde{n}$ to $\rho$ with the following condition:

$$\mu_\rho(\tilde{n}), \eta_\rho(\tilde{n}), \nu_\rho(\tilde{n}) \leq 1, \forall \tilde{n} \in \tilde{N}.$$  

$$\pi_\rho(\tilde{n}) = 1 - \mu_\rho(\tilde{n}) - \eta_\rho(\tilde{n}) - \nu_\rho(\tilde{n})$$

is known is the refusal degree of $\tilde{n}$ to $s_\theta(\tilde{n})$ for all $\tilde{n} \in \tilde{N}$.

Definition 2.4. Let $\tilde{\tilde{\rho}} = \left\{ s_\alpha, s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$ be a triangular picture fuzzy linguistic set, where $s_\alpha, s_\beta, s_\gamma \in \tilde{S}$, $s_\alpha \leq s_\beta \leq s_\gamma$, the values $\mu_{\tilde{\tilde{\rho}}}$, $\eta_{\tilde{\tilde{\rho}}}$, and $\nu_{\tilde{\tilde{\rho}}}$ represent the maximum degrees of positive, neutral and negative membership, respectively, and satisfy $0 \leq \mu_{\tilde{\tilde{\rho}}} \leq 1, 0 \leq \eta_{\tilde{\tilde{\rho}}} \leq 1, 0 \leq \nu_{\tilde{\tilde{\rho}}} \leq 1$, and $0 \leq \mu_{\tilde{\tilde{\rho}}} + \eta_{\tilde{\tilde{\rho}}} + \nu_{\tilde{\tilde{\rho}}} \leq 1$.

Definition 2.5. Let $\tilde{\tilde{\rho}} = \left\{ s_\alpha, s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$, $\tilde{\tilde{\rho}}_1 = \left\{ s_\alpha, s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$, $\tilde{\tilde{\rho}}_2 = \left\{ s_\alpha, s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$ be the triangular picture fuzzy linguistic sets, and $\lambda_1, \lambda_2 \in [0, 1]$, then there operation laws are defined as:

(1) $\tilde{\tilde{\rho}}_1 \oplus \tilde{\tilde{\rho}}_2 = \left\{ [s_\alpha \times s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$

(2) $\tilde{\tilde{\rho}}_1 \odot \tilde{\tilde{\rho}}_2 = \left\{ [s_\alpha \times s_\beta, s_\gamma; \mu_{\tilde{\tilde{\rho}}}, \eta_{\tilde{\tilde{\rho}}}, \nu_{\tilde{\tilde{\rho}}} \right\}$.
(3) \( \lambda \circ \tilde{\varphi} = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; 1 - (1 - \mu_{\tilde{\varphi}}) \lambda, \eta_{\tilde{\varphi}} \lambda, \nu_{\tilde{\varphi}} \lambda \right\}; \)

(4) \( \tilde{\varphi}^\lambda = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; \mu_{\tilde{\varphi}} \lambda, 1 - (1 - \eta_{\tilde{\varphi}}) \lambda, 1 - (1 - \nu_{\tilde{\varphi}}) \lambda \right\}; \)

(5) \( \lambda \circ (\tilde{\varphi}_1 \oplus \tilde{\varphi}_2) = \lambda \circ \tilde{\varphi}_1 \oplus \lambda \circ \tilde{\varphi}_2 \)

(6) \( (\tilde{\varphi}_1 \ominus \tilde{\varphi}_2)^\lambda = \tilde{\varphi}_1^\lambda \ominus \tilde{\varphi}_2^\lambda. \)

Inspired by [32, 13], we defined the following concepts;

**Definition 2.6.** Let \( \tilde{\varphi} = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; \mu_{\tilde{\varphi}}, \eta_{\tilde{\varphi}}, \nu_{\tilde{\varphi}} \right\} \) be the triangular picture fuzzy linguistic set, then we defined the value index as:

\[
\mathcal{V}(\tilde{\varphi}) = \frac{\mathcal{V}_\mu(\tilde{\varphi}) + \mathcal{V}_\eta(\tilde{\varphi}) + \mathcal{V}_\nu(\tilde{\varphi})}{3},
\]

and the ambiguity index as:

\[
\mathcal{R}(\tilde{\varphi}) = \frac{\mathcal{R}_\mu(\tilde{\varphi}) + \mathcal{R}_\eta(\tilde{\varphi}) + \mathcal{R}_\nu(\tilde{\varphi})}{3},
\]

such that \( \mathcal{V}_\mu(\tilde{\varphi}) = \frac{(\lambda \alpha + \lambda \beta + \lambda \gamma) \mu_{\tilde{\varphi}}}{6}, \mathcal{V}_\eta(\tilde{\varphi}) = \frac{(\lambda \alpha + \lambda \beta + \lambda \gamma)(1 - \eta_{\tilde{\varphi}})}{6} \) and \( \mathcal{V}_\nu(\tilde{\varphi}) = \frac{(\lambda \alpha + \lambda \beta + \lambda \gamma)(1 - \nu_{\tilde{\varphi}})}{6} \) are the indexes of the positive, neutral and negative membership function of \( \tilde{\varphi} \), where as \( \mathcal{R}_\mu(\tilde{\varphi}) = \frac{(\lambda \alpha - \lambda \beta) \mu_{\tilde{\varphi}}}{3}, \mathcal{R}_\eta(\tilde{\varphi}) = \frac{(\lambda \alpha - \lambda \beta)(1 - \eta_{\tilde{\varphi}})}{3} \) and \( \mathcal{R}_\nu(\tilde{\varphi}) = \frac{(\lambda \alpha - \lambda \beta)(1 - \nu_{\tilde{\varphi}})}{3} \) are the ambiguity indexes of the positive, neutral and negative membership function of \( \tilde{\varphi} \), respectively.

**Definition 2.7.** Let \( \tilde{\varphi}_1 = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; \mu_{\tilde{\varphi}_1}, \eta_{\tilde{\varphi}_1}, \nu_{\tilde{\varphi}_1} \right\}, \tilde{\varphi}_2 = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; \mu_{\tilde{\varphi}_2}, \eta_{\tilde{\varphi}_2}, \nu_{\tilde{\varphi}_2} \right\} \) be the three triangular picture fuzzy linguistic variables. Then, the g-value of triangular picture fuzzy linguistic variables \( \tilde{\varphi} \) is defined as:

\[
G(\tilde{\varphi}) = \mathcal{V}(\tilde{\varphi}) - \mathcal{R}(\tilde{\varphi}),
\]

then, we have

(1) \( G(\tilde{\varphi}_1) < G(\tilde{\varphi}_2) \), then \( \tilde{\varphi}_1 \) is smaller than \( \tilde{\varphi}_2 \), denoted by \( \tilde{\varphi}_1 \prec_{PF} \tilde{\varphi}_2. \)

(2) \( G(\tilde{\varphi}_1) = G(\tilde{\varphi}_2) \), then

(a) If \( \mathcal{V}(\tilde{\varphi}_1) < \mathcal{V}(\tilde{\varphi}_2) \), \( \tilde{\varphi}_1 \) is smaller than \( \tilde{\varphi}_2 \), denoted by \( \tilde{\varphi}_1 \prec_{PF} \tilde{\varphi}_2 \)

(b) If \( \mathcal{V}(\tilde{\varphi}_1) = \mathcal{V}(\tilde{\varphi}_2) \), \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) represent the same information, i.e., \( \tilde{\varphi}_1 \equiv_{PF} \tilde{\varphi}_2. \)

3. TPFLIOWA and TPFLIOWG operators.

3.1. Triangular picture fuzzy linguistic induced ordered weighted averaging operator.

**Definition 3.1.** Let \( \tilde{S} \) be the collection of triangular picture fuzzy linguistic numbers, and \( \tilde{\varphi} = \left\{ \left[ \lambda \alpha, \lambda \beta, \lambda \gamma \right]; \mu_{\tilde{\varphi}}, \eta_{\tilde{\varphi}}, \nu_{\tilde{\varphi}} \right\} \in \tilde{\varphi}(i = 1, 2, ..., n) \), then a triangular picture fuzzy linguistic induced ordered weighted averaging (TPFLIOWA) operator of dimension \( n \) is a mapping \( \Phi_{TPFLIOWA} : R^n \times \tilde{S} \rightarrow \tilde{S} \), where the associated weights
are $\mathbb{N} = (N_1, N_2, \ldots, N_n)^T$, with $N_j \in [0, 1]$ and $\sum_{j=1}^{n} N_j = 1$, and it is defined to aggregate the set of second arguments of a list of $n$ pairs $\{(\tilde{a}_1, s_{\rho_1}), (\tilde{a}_2, s_{\rho_2}), (\tilde{a}_3, s_{\rho_3}), \ldots, (\tilde{a}_n, s_{\rho_n})\}$ according to the following expression:

$$\Phi_{TPFLIOWA}\{(\tilde{a}_1, \tilde{\psi}_1), (\tilde{a}_2, \tilde{\psi}_2), (\tilde{a}_3, \tilde{\psi}_3), \ldots, (\tilde{a}_n, \tilde{\psi}_n)\} = PF \sum_{i=1}^{n} N_i \circ \tilde{\psi}_{\sigma(i)}$$  \hspace{1cm} (9)

where $\sigma : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is a permutation, and $\tilde{\psi}_{\sigma(i)}$ is the $i$th biggest $\tilde{a}_i$, $\tilde{a}_i$ is the order inducing variable and $\tilde{\psi}_i$ denoted the argument variable in the form of individual triangular picture fuzzy linguistic variable.

**Theorem 3.2.** Let $(\tilde{a}_i, \tilde{\psi}_i) (i = 1, 2, \ldots, n)$ are the set of $TPFLIOWA$ pairs, $\tilde{\psi}_i$ in $(\tilde{a}_i, \tilde{\psi}_i)$ is referred to as the triangular picture fuzzy linguistic number denoted by $\tilde{\psi} = \langle [s_{\alpha}, s_{\beta}, s_{\gamma}], \mu_{\tilde{\psi}}, \eta_{\tilde{\psi}}, \nu_{\tilde{\psi}} \rangle$, then the aggregated value utilizing the $TPFLIOWA$ operator is also a triangular picture fuzzy linguistic number, and

$$\Phi_{TPFLIOWA}\{(\tilde{a}_1, \tilde{\psi}_1), (\tilde{a}_2, \tilde{\psi}_2), (\tilde{a}_3, \tilde{\psi}_3), \ldots, (\tilde{a}_n, \tilde{\psi}_n)\}$$  \hspace{1cm} (10)

where $\tilde{\psi}_{\sigma(i)} = \langle [s_{\alpha_{\sigma(i)}}, s_{\beta_{\sigma(i)}}, s_{\gamma_{\sigma(i)}}], \mu_{\tilde{\psi}_{\sigma(i)}}, \eta_{\tilde{\psi}_{\sigma(i)}}, \nu_{\tilde{\psi}_{\sigma(i)}} \rangle$, and $\mathbb{N} = (N_1, N_2, \ldots, N_n)^T$ is an associating weights with $N_i \in [0, 1]$ and $\sum_{i=1}^{n} N_i = 1$.

**Proof.** The first result follows from the Definitions (4) and (5). We prove Equation 10 only by the mathematical induction method:

For $n = 2$, since $\tilde{\psi}_1 = \langle [s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1}], \mu_{\tilde{\psi}_1}, \eta_{\tilde{\psi}_1}, \nu_{\tilde{\psi}_1} \rangle$, $\tilde{\psi}_2 = \langle [s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2}], \mu_{\tilde{\psi}_2}, \eta_{\tilde{\psi}_2}, \nu_{\tilde{\psi}_2} \rangle$, then,

$$N_1 \circ \tilde{\psi}_1 \oplus N_2 \circ \tilde{\psi}_2 = \langle \left[ \begin{array}{c} \sum_{i=1}^{2} N_i \circ s_{\alpha_i} \\ \sum_{i=1}^{2} N_i \circ s_{\beta_i} \\ \sum_{i=1}^{2} N_i \circ s_{\gamma_i} \end{array} \right] : 1 - \prod_{i=1}^{2} \left( 1 - \mu_{\tilde{\psi}_i} \right), \prod_{i=1}^{2} \eta_{\tilde{\psi}_i}, \prod_{i=1}^{2} \nu_{\tilde{\psi}_i} \rangle,$$

assume that Equation 10, is true for $n = k, k \in \mathbb{N}$, that is

$$= \left[ \begin{array}{c} \sum_{i=1}^{k} N_i \circ s_{\alpha_i} \\ \sum_{i=1}^{k} N_i \circ s_{\beta_i} \\ \sum_{i=1}^{k} N_i \circ s_{\gamma_i} \end{array} \right] : 1 - \prod_{i=1}^{k} \left( 1 - \mu_{\tilde{\psi}_i} \right), \prod_{i=1}^{k} \eta_{\tilde{\psi}_i}, \prod_{i=1}^{k} \nu_{\tilde{\psi}_i} \rangle.$$

Then, when $n = k + 1$, using the operational laws in Definition (5), we have

$$N_1 \circ \tilde{\psi}_1 \oplus N_2 \circ \tilde{\psi}_2 \oplus \ldots \oplus N_k \circ \tilde{\psi}_k \oplus N_{k+1} \circ \tilde{\psi}_{k+1}$$
Proof. Let
\[ \Phi_{TPFLIOWA}(\tilde{a}_i, \tilde{\gamma}_i, \ldots, \tilde{a}_n, \tilde{\gamma}_n) = PF \Phi_{TPFLIOWA}(\tilde{a}_1, \tilde{\gamma}_1, \ldots, \tilde{a}_n, \tilde{\gamma}_n). \]

Proposition 2. (Idempotency). If \( \tilde{\varphi}_i = PF \tilde{\varphi} \) for all \( \tilde{\varphi}_i, \tilde{\varphi} \in \tilde{S}, \) \( i = 1, 2, \ldots, n, \) where
\[ \tilde{\varphi} = \left\{ [s_1, s_2, s_3]; \mu_{\tilde{\varphi}}, \eta_{\tilde{\varphi}}, \nu_{\tilde{\varphi}} \right\} \]
then
\[ \Phi_{TPFLIOWA}(\tilde{a}_1, \tilde{\gamma}_1, \tilde{a}_2, \tilde{\gamma}_2, \ldots, \tilde{a}_n, \tilde{\gamma}_n) = PF \tilde{\varphi}. \] (11)
Boundedness. Let

\(\Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\}, \tilde{a}_2, \tilde{\varphi}_2\}, ... (\tilde{a}_n, \tilde{\varphi}_n)\} =_{PF} \tilde{\varphi}.

So,

\[ \Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\}, (\tilde{a}_2, \tilde{\varphi}_2),...,(\tilde{a}_n, \tilde{\varphi}_n)\} =_{PF} \tilde{\varphi}. \]

\(\square\)

Proposition 3. (Monotonicity). Let \(\{(\tilde{a}_1, \tilde{\varphi}_1), (\tilde{a}_2, \tilde{\varphi}_2), (\tilde{a}_3, \tilde{\varphi}_3),...,(\tilde{a}_n, \tilde{\varphi}_n)\}\) and \(\{(\tilde{a}_1, \tilde{\varphi}_1), (\tilde{a}_2, \tilde{\varphi}_2), (\tilde{a}_3, \tilde{\varphi}_3),...,(\tilde{a}_n, \tilde{\varphi}_n)\}\) are two triangular picture fuzzy linguistic variable vector, if \(\tilde{\varphi}_i \preceq_{PF} \tilde{\varphi}_i\) for all \(i = 1, 2, ..., n\), then

\[ \Phi_{TPFLIOA}\{(\tilde{a}_1, \tilde{\varphi}_1), (\tilde{a}_2, \tilde{\varphi}_2),...,(\tilde{a}_n, \tilde{\varphi}_n)\} \preceq_{PF} \Phi_{TPFLIOA}\{(\tilde{a}_1, \tilde{\varphi}_1), (\tilde{a}_2, \tilde{\varphi}_2),...,(\tilde{a}_n, \tilde{\varphi}_n)\}. \]  

\((12)\)

\(\square\)

Proposition 4. (Boundedness). Let \(\tilde{\varphi}_m =_{PF} \min_i(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3,...,\tilde{\varphi}_n)\), \(\tilde{\varphi}_M = \max_i(\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3,...,\tilde{\varphi}_n)\), \(\tilde{\varphi}_m, \tilde{\varphi}_M\) then

\[ \tilde{\varphi}_m \preceq_{PF} \Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\},...,(\tilde{a}_n, \tilde{\varphi}_n)\} \preceq_{PF} \tilde{\varphi}_M. \]

\((13)\)

Proof. Since \(\tilde{\varphi}_m \preceq_{PF} \tilde{\varphi}_A, \preceq_{PF} \tilde{\varphi}_M\) and \(\sum_{i=1}^{n} N_i = 1\), according to proposition 1-4, and we have

\[ \Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\},...,(\tilde{a}_n, \tilde{\varphi}_n)\} =_{PF} N_1 \circ \tilde{\varphi}_{\sigma(1)} \oplus N_2 \circ \tilde{\varphi}_{\sigma(2)} \oplus ... \oplus N_n \circ \tilde{\varphi}_{\sigma(n)} \]

\[ \preceq_{PF} N_1 \circ \tilde{\varphi}_m \oplus N_2 \circ \tilde{\varphi}_m \oplus ... \oplus N_n \circ \tilde{\varphi}_m \]

\[ =_{PF} (N_1 + N_2 + ... + N_n) \tilde{\varphi}_m \]

\[ =_{PF} \tilde{\varphi}_m. \]

\[ \Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\},...,(\tilde{a}_n, \tilde{\varphi}_n)\} =_{PF} N_1 \circ \tilde{\varphi}_{\sigma(1)} \oplus N_2 \circ \tilde{\varphi}_{\sigma(2)} \oplus ... \oplus N_n \circ \tilde{\varphi}_{\sigma(n)} \]

\[ \preceq_{PF} N_1 \circ \tilde{\varphi}_M \oplus N_2 \circ \tilde{\varphi}_M \oplus ... \oplus N_n \circ \tilde{\varphi}_M \]

\[ =_{PF} (N_1 + N_2 + ... + N_n) \tilde{\varphi}_M \]

\[ =_{PF} \tilde{\varphi}_M. \]

So

\[ \tilde{\varphi}_m \preceq_{PF} \Phi_{TPFLIOA}\{\tilde{a}_1, \tilde{\varphi}_1\},...,(\tilde{a}_n, \tilde{\varphi}_n)\} \preceq_{PF} \tilde{\varphi}_M. \]

\(\square\)
Remark 1. If \( N = (N_1, N_2, \ldots, N_n)^T = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^T \), then we get the TPFLIA operator.

\[
\Phi_{TPFLIA} = \left\{ (\bar{a}_1, \bar{v}_1), (\bar{a}_2, \bar{v}_2), \ldots, (\bar{a}_n, \bar{v}_n) \right\} = \sum_{i=1}^{n} \frac{1}{n} \circ \bar{v}_i = \bar{v} = \sum_{i=1}^{n} \frac{1}{n}. \tag{14}
\]

Remark 2. If \( \mu_i = 1, \nu_i = 0, \forall i(1, 2, \ldots, n) \), then we obtain the TFLIA operator.

\[
\Phi_{TPFLIA} = \left\{ (\bar{a}_1, \bar{v}_1), (\bar{a}_2, \bar{v}_2), \ldots, (\bar{a}_n, \bar{v}_n) \right\} = Pf \sum_{i=1}^{n} N_i \circ \bar{v}_{\sigma(i)}. \tag{15}
\]

where \( \bar{v}_{\sigma(i)} = (\tilde{s}_{\sigma_{i}(1)}, \tilde{s}_{\sigma_{i}(2)}, \ldots, \tilde{s}_{\sigma_{i}(n)}) \) is the \( ith \) largest of the fuzzy linguistic variable \( \tilde{s}_1, \ldots, \tilde{s}_n \), and if \( \bar{a}_i \geq \bar{a}_{i+1} \) for all \( i \), then, the ordered position of \( \bar{a}_i \) is the same as the ordered position of \( \bar{v}_i \), the TFLIA operator reduced to the TFLIA operator.

Remark 3. If \( \bar{a}_i > \bar{a}_{i+1} \forall i, \) and the ordered position of \( \bar{a}_i \) is the same as the ordered position of \( \bar{v}_i \), then, the TPFLIA operator is gained:

\[
\Phi_{TPFLIA} = \left\{ (\bar{a}_1, \bar{v}_1), (\bar{a}_2, \bar{v}_2), \ldots, (\bar{a}_n, \bar{v}_n) \right\} = Pf \sum_{i=1}^{n} N_i \circ \bar{v}_{\sigma(i)}. \tag{16}
\]

3.2 Triangular picture fuzzy linguistic induced ordered weighted geometric operator.

Definition 3.3. Let \( \tilde{S} \) are the collection of triangular picture fuzzy linguistic variables and \( \tilde{v} = \left\{ (\tilde{s}_{\bar{a}_1}, \tilde{s}_{\bar{a}_2}, \ldots, \tilde{s}_{\bar{a}_n}); \mu_\tilde{v}, \eta_\tilde{v}, \nu_\tilde{v} \right\} \in \tilde{v}(1, 2, \ldots, n) \). Then, a triangular picture fuzzy linguistic induced ordered weighted geometric (TPFLIGW) operator of dimension \( n \) is a mapping \( \Phi_{TPFLIGW} : R^n \times \tilde{S} \rightarrow \tilde{S} \), in which the associated weighting vector are \( N = (N_1, N_2, \ldots, N_n)^T \), so that \( N_j \in [0, 1] \) and \( \sum_{j=1}^{n} N_j = 1 \), and it is defined to aggregate the set of second arguments of a list of \( n \) pairs \( \{ (\bar{a}_1, \tilde{s}_{\bar{a}_1}), (\bar{a}_2, \tilde{s}_{\bar{a}_2}), \ldots, (\bar{a}_n, \tilde{s}_{\bar{a}_n}) \} \), according to the following expression:

\[
\Phi_{TPFLIGW} = \left\{ (\bar{a}_1, \tilde{v}_1), (\bar{a}_2, \tilde{v}_2), \ldots, (\bar{a}_n, \tilde{v}_n) \right\} = Pf \prod_{i=1}^{n} \tilde{v}_{\sigma(i)}. \tag{17}
\]

where \( \sigma : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n) \) is a permutation, \( \forall i, \tilde{v}_{\sigma(i-1)} > \tilde{v}_{\sigma(i)} \) and \( \tilde{v}_{\sigma(i)} \) is \( \tilde{v}_i \) value of the TPFLIGW pair \( (\tilde{a}_{\sigma(i)}, \tilde{v}_{\sigma(i)}) \) having the ith largest \( \tilde{a}_i \), \( \tilde{a}_i \) is the order inducing variable and \( \tilde{v}_i \) is the argument variable expressed in the individual form of triangular picture fuzzy linguistic variable.

Theorem 3.4. Let \( (\tilde{a}_i, \tilde{v}_i)(i = 1, 2, \ldots, n) \) be the set of TPFLIGW pairs, \( \tilde{v}_i \) in \( (\tilde{a}_i, \tilde{v}_i) \) is referred to as the triangular picture fuzzy linguistic variable denoted by \( \tilde{v} = \left\{ (\tilde{s}_{\bar{a}_1}, \tilde{s}_{\bar{a}_2}, \ldots, \tilde{s}_{\bar{a}_n}); \mu_\tilde{v}, \eta_\tilde{v}, \nu_\tilde{v} \right\} \), then their aggregated value by using the TPFLIGW operator is also an triangular picture fuzzy linguistic variable and

\[
\Phi_{TPFLIGW} = \left\{ (\bar{a}_1, \tilde{v}_1), (\bar{a}_2, \tilde{v}_2), \ldots, (\bar{a}_n, \tilde{v}_n) \right\} = Pf \prod_{i=1}^{n} \tilde{v}_{\sigma(i)}. \tag{18}
\]
where $\tilde{\psi}_{\sigma(i)} = \langle s_{\sigma(i)}, s_{\beta(i)}, s_{\gamma(i)}; \mu_{\tilde{\psi}_{\sigma(i)}}, \eta_{\tilde{\psi}_{\sigma(i)}}, \nu_{\tilde{\psi}_{\sigma(i)}} \rangle$, and the associated weighting vector are $\mathbb{N} = (N_1, N_2, ..., N_n)^T$, with $N_i \in [0,1]$ and $\sum_{i=1}^{n} N_i = 1$.

**Proposition 5.** (Commutativity). Let $\{(\tilde{a}_1, \tilde{\psi}_1), (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\}$ be any permutation of the triangular picture fuzzy linguistic variable collection, then

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\} =_{PF} \Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, ..., (\tilde{a}_n, \tilde{\psi}_n)\},$$

(19)

**Proposition 6.** (Idempotency). If $\tilde{\psi}_i =_{PF} \tilde{\psi} \forall \tilde{\psi}, \tilde{\psi} \in \tilde{\psi}, i = 1, 2, ..., n$, and let

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\} =_{PF} \tilde{\psi}.$$  

(20)

**Proposition 7.** (Monotonicity). Let $\{(\tilde{a}_1, \tilde{\psi}_1), (\tilde{a}_2, \tilde{\psi}_2), (\tilde{a}_3, \tilde{\psi}_3), ..., (\tilde{a}_n, \tilde{\psi}_n)\}$ and $\{(\tilde{a}_1, \tilde{\psi}_1), (\tilde{a}_2, \tilde{\psi}_2), (\tilde{a}_3, \tilde{\psi}_3), ..., (\tilde{a}_n, \tilde{\psi}_n)\}$ are two triangular picture fuzzy linguistic variable collection, if $\tilde{\psi}_i \preceq_{PF} \tilde{\psi}_i$ for all $i = 1, 2, ..., n$, then

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, ..., (\tilde{a}_n, \tilde{\psi}_n)\} \preceq_{PF} \Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, ..., (\tilde{a}_n, \tilde{\psi}_n)\}.$$  

(21)

**Proposition 8.** (Boundedness). Let $\tilde{\psi}_m =_{PF} \min\{\tilde{\psi}_1, \tilde{\psi}_2, ..., \tilde{\psi}_n\}$, \( \tilde{\psi}_M = \max\{\tilde{\psi}_1, \tilde{\psi}_2, ..., \tilde{\psi}_n\}, \) then

$$\tilde{\psi}_m \preceq_{PF} \Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, ..., (\tilde{a}_n, \tilde{\psi}_n)\} \preceq_{PF} \tilde{\psi}_M.$$  

(22)

**Remark 4.** If $\mathbb{N} = (N_1, N_2, ..., N_n)^T = (\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N})^T$, then we obtain the TPFLIOG operator.

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\} = \left( \prod_{i=1}^{n} \tilde{\psi}_i \right)^{\frac{1}{N}}.$$  

(23)

**Remark 5.** If $\mu_{\tilde{\psi}_{\sigma(i)}} = 1$, $\eta_{\tilde{\psi}_{\sigma(i)}} = 0$, $\nu_{\tilde{\psi}_{\sigma(i)}} = 0$, $\forall i = 1, 2, ..., n$, then, we obtain the TPLIOG operator.

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\} =_{PF} \prod_{i=1}^{n} \tilde{\psi}_{\sigma(i)}.$$  

(24)

where $\tilde{\psi}_{\sigma(i)} = \langle s_{\sigma(i)}, s_{\beta(i)}, s_{\gamma(i)} \rangle$ is the element largest of the fuzzy linguistic variable $\tilde{\psi}_i$. And if $\tilde{a}_i \preceq \tilde{a}_{i+1}$ for all $i$, then, the ordered position of $\tilde{a}_i$ is the same as the ordered position of $\tilde{\psi}_i$, the TPFLIOWG operator reduced to the TPLIOG operator.

**Remark 6.** If $\tilde{a}_i > \tilde{a}_{i+1}$, and the ordered position of $\tilde{a}_i$ is the same as $\tilde{\psi}_i$, then the TPFLIOWG operator obtains.

$$\Phi_{TPFLIOWG}\{\tilde{a}_1, \tilde{\psi}_1\}, (\tilde{a}_2, \tilde{\psi}_2), ..., (\tilde{a}_n, \tilde{\psi}_n)\} =_{PF} \prod_{i=1}^{n} \tilde{\psi}_{\sigma(i)}.$$  

(25)
4. Algorithm for multi-criteria group decision-making under the triangular picture fuzzy linguistic environment. Based on the TPFLIOWA or the TPFLIOWG operators, we consider the criteria group decision making problems in which the experts and criteria weights take the form of real numbers, and the criteria values occur in the form of triangular picture fuzzy linguistic environment.

Let \( \mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_m) \) be a discrete set of alternatives, and \( C = (c_1, c_2, ..., c_n) \) be the set of criteria, whose weight vector is \( \varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T \), where \( \varpi_j \in [0, 1] \) and \( \sum_{j=1}^{n} \varpi_j = 1 \). The set of decision makers are \( \mathcal{R} = (R_1, R_2, ..., R_P)^T \), with \( R_j \in [0, 1] \) and \( \sum_{j=1}^{P} R_j = 1 \). Suppose that the decision makers compare these alternatives with respect to a single criterion by using the triangular picture fuzzy linguistic values \( \tilde{\wp}_{ij}^{(k)} \) of the alternative \( \mathcal{R}_i \in \mathcal{R} \), with respect to the criteria \( c_i \in C \), and construct the triangular picture fuzzy linguistic decision matrices \( \tilde{e}^{(k)} = (\tilde{e}_{ij}^{(k)})_{m \times n} \), where \( \tilde{e}_{ij}^{(k)} = \left( \left[ \bar{S}_{aij}^{(k)}, \underline{S}_{aij}^{(k)}, \bar{S}_{aij}^{(k)} \right]; \mu_{\tilde{e}_{ij}^{(k)}}, \eta_{\tilde{e}_{ij}^{(k)}}, \nu_{\tilde{e}_{ij}^{(k)}} \right) \), \( \mu_{\tilde{e}_{ij}^{(k)}}, \eta_{\tilde{e}_{ij}^{(k)}}, \nu_{\tilde{e}_{ij}^{(k)}} \in [0, 1] \), \( \mu_{\tilde{e}_{ij}^{(k)}} + \eta_{\tilde{e}_{ij}^{(k)}} + \nu_{\tilde{e}_{ij}^{(k)}} \leq 1 \).

**Step 1.** Find the total preference triangular picture fuzzy linguistic values \( \tilde{\wp}_{i}^{(k)} \) of the alternative \( \mathcal{R}_i \) by using the TPFLIOWA operator under inducing variables \( \bar{a}_i \in \bar{a}, (j = 1, 2, ..., n) \).

\[
\tilde{\wp}_{i}^{(k)} = \left( \left[ \bar{S}_{ai}^{(k)}, \underline{S}_{ai}^{(k)}, \bar{S}_{ai}^{(k)} \right]; \mu_{\tilde{e}_{i}^{(k)}}, \eta_{\tilde{e}_{i}^{(k)}}, \nu_{\tilde{e}_{i}^{(k)}} \right)
= PF_{TPFLIOWA} \left( (\tilde{\wp}_{i1}^{(k)}, ..., \tilde{\wp}_{in}^{(k)}) \right)
\]  

(26)

where weighting vector is \( \varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T \), where \( \varpi_j \in [0, 1] \) and \( \sum_{j=1}^{n} \varpi_j = 1 \).

**Step 2.** Using the TPFLIOWA operator:

\[
\tilde{\wp}_{i}^{(k)} = \left( \left[ \bar{S}_{ai}^{(k)}, \underline{S}_{ai}^{(k)}, \bar{S}_{ai}^{(k)} \right]; \mu_{\tilde{e}_{i}^{(k)}}, \eta_{\tilde{e}_{i}^{(k)}}, \nu_{\tilde{e}_{i}^{(k)}} \right) = PF_{TPFLIOWA} \left( \tilde{\wp}_{1}^{(1)}, ..., \tilde{\wp}_{n}^{(P)} \right)
\]

(27)

\[
\tilde{\wp}_{i}^{(k)} = PF \sum_{k=1}^{P} R_k \tilde{\wp}_{i}^{(k)}
\]

(28)

(29)

to obtain the collective total preference triangular picture fuzzy linguistic variables \( \tilde{\wp}_{i} \) of the alternative \( \mathcal{R}_i \), where \( \bar{R} = (R_1, R_2, ..., R_P)^T \) be the weights of decision makers, with \( R_k \in [0, 1] \), \( \sum_{j=1}^{P} R_j = 1 \), and \( \tilde{\wp}_{i}^{(k)} \) be the \( k \)th biggest of the triangular picture fuzzy linguistic variable \( \tilde{\wp}_{i}^{(k)} \).

**Step 3.** Find the \( \mathcal{S}(\tilde{\wp}_{i}) \) and \( \mathcal{R}(\tilde{\wp}_{i}) \), and G-value \( G(\tilde{\wp}_{i}) \) of the total picture fuzzy triangular linguistic variables \( \tilde{\wp}_{i} \).

**Step 4.** According to the Definition 2.7, give ranking the alternatives \( \mathcal{R}_i \).

Now, we apply the TPFLIOWG operator with the following steps:

**Step 1.** Find the total preference triangular picture fuzzy linguistic values \( \tilde{\wp}_{i}^{(k)} \) of the alternative \( \mathcal{R}_i \) by using the TPFLIOWG operator under inducing variables
\( \tilde{a}_t \in \tilde{a}, (j = 1, 2, ..., n). \)

\[
\tilde{\psi}_t^{(k)} = \left\{ \left[ \frac{s_{\alpha_j}^{(k)}}{s_{\beta_j}^{(k)}}, \frac{s_{\gamma_j}^{(k)}}{s_{\delta_j}^{(k)}}, \mu_{\tilde{\psi}_t}^{(k)}, \eta_{\tilde{\psi}_t}^{(k)}, \nu_{\tilde{\psi}_t}^{(k)} \right], \tilde{\psi}_t^{(k)} \right\} \tag{30}
\]

\[
= PF_{TPFLIOWG} \left\{ (\tilde{a}_1, \tilde{\psi}_t^{(1)}), (\tilde{a}_2, \tilde{\psi}_t^{(2)}), ..., (\tilde{a}_n, \tilde{\psi}_t^{(n)}) \right\} \tag{31}
\]

where weighting vector are \( \vec{\Sigma} = (\Sigma_1, \Sigma_2, ..., \Sigma_n)^T \), where \( \Sigma_j \in [0, 1] \) and \( \sum_{j=1}^{n} \Sigma_j = 1. \)

**Step 2.** Using the TPFLIOWG operator:

\[
\tilde{\psi}_t = \left\{ \left[ \frac{s_{\alpha_j}^{(k)}}{s_{\beta_j}^{(k)}}, \frac{s_{\gamma_j}^{(k)}}{s_{\delta_j}^{(k)}}, \mu_{\tilde{\psi}_t}^{(k)}, \eta_{\tilde{\psi}_t}^{(k)}, \nu_{\tilde{\psi}_t}^{(k)} \right], \tilde{\psi}_t \right\} \tag{32}
\]

\[
= PF_{TPFLIOWG} \left\{ \tilde{\psi}_t^{(1)}, \tilde{\psi}_t^{(2)}, ..., \tilde{\psi}_t^{(p)} \right\} \tag{33}
\]

\[
= PF \prod_{i=1}^{p} \left( \tilde{\psi}_t^{(k)} \right)^{\Sigma_k} \tag{34}
\]

to obtain the collective total preference triangular picture fuzzy linguistic variables \( \tilde{\psi}_t \) of the alternative \( \mathcal{R}_t \), where \( \vec{\Sigma} = (\Sigma_1, \Sigma_2, ..., \Sigma_p)^T \) be the weights of the decision makers, with \( \Sigma_k \in [0, 1], \sum_{j=1}^{m} \Sigma_j = 1 \), and \( \tilde{\psi}_t^{(k)} \) be the \( k \)th biggest of the triangular picture fuzzy linguistic variable \( \tilde{\psi}_t^{(k)} \).

**Step 3.** Find the \( \mathfrak{S}(\tilde{\psi}_t) \) and \( \mathfrak{R}(\tilde{\psi}_t) \), and G-value \( G(\tilde{\psi}_t) \) of the total triangular picture fuzzy linguistic variables \( \tilde{\psi}_t \).

**Step 4.** According to the Definition 2.7, give ranking to the alternatives \( \mathcal{R}_t \).

5. **Numerical example.** Assume that there is an investment company which operate in Europe and North America to analyze its general strategy for the coming year and wants to provide the perfect strategy. The investment company examine the following criteria \( c_i (i = 1, ..., 5) \) for the possible alternatives are (1) \( c_1 \) is the risk analysis (2) \( c_2 \) is the growth analysis (3) \( c_3 \) is the social-political impact analysis (4) \( c_4 \) is the environmental impact analysis (5) \( c_5 \) is the other.

The five feasible alternatives \( \mathcal{R}_i (i = 1, ..., 5) \) are developed by using the triangular picture fuzzy linguistic variables by the three decision makers under the criteria \( c_i (i = 1, ..., 5) \), where the weighting vector are \( \vec{\Sigma} = (0.35, 0.40, 0.25)^T, \vec{\varphi} = (0.2, 0.1, 0.25, 0.3, 0.15)^T \), respectively. Thus, design the triangular picture fuzzy linguistic decision matrices \( \tilde{E} \) as follows:

\[
\tilde{E} = \left( \tilde{\psi}_{ij} \right)_{5 \times 5}, (k = 1, 2, 3) \tag{35}
\]

| \( \tilde{E} \) | 1 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| \( 0.5, 0.1, 0.4 \) | \( 0.6, 0.1, 0.3 \) | \( 0.7, 0.1, 0.2 \) | \( 0.8, 0.1, 0.1 \) | \( 0.9, 0.1, 0.0 \) | \( 1.0, 0.1, 0.0 \) |
| \( 0.4, 0.2, 0.3 \) | \( 0.5, 0.2, 0.3 \) | \( 0.6, 0.2, 0.3 \) | \( 0.7, 0.2, 0.3 \) | \( 0.8, 0.2, 0.3 \) | \( 0.9, 0.2, 0.3 \) |
| \( 0.4, 0.3, 0.0 \) | \( 0.5, 0.3, 0.0 \) | \( 0.6, 0.3, 0.0 \) | \( 0.7, 0.3, 0.0 \) | \( 0.8, 0.3, 0.0 \) | \( 0.9, 0.3, 0.0 \) |
| \( 0.3, 0.4, 0.1 \) | \( 0.4, 0.4, 0.1 \) | \( 0.5, 0.4, 0.1 \) | \( 0.6, 0.4, 0.1 \) | \( 0.7, 0.4, 0.1 \) | \( 0.8, 0.4, 0.1 \) |
| \( 0.2, 0.3, 0.2 \) | \( 0.3, 0.3, 0.2 \) | \( 0.4, 0.3, 0.2 \) | \( 0.5, 0.3, 0.2 \) | \( 0.6, 0.3, 0.2 \) | \( 0.7, 0.3, 0.2 \) |
| \( 0.1, 0.2, 0.3 \) | \( 0.2, 0.2, 0.3 \) | \( 0.3, 0.2, 0.3 \) | \( 0.4, 0.2, 0.3 \) | \( 0.5, 0.2, 0.3 \) | \( 0.6, 0.2, 0.3 \) |
Step 1. Find the $\widetilde{\varphi}^{(k)}$ (individual total preference triangular picture fuzzy linguistic information values) of the $\mathcal{R}_i$(alternative) by using the decision information given in the triangular picture fuzzy linguistic decision matrix $\widetilde{E}^{(k)}$ and the TPFLIOWA operator.

$$
\begin{align*}
\widetilde{e}_1^1 &= ([8.40, 8.45, 8.50], 0.454, 0.098, 0.354), \\
\widetilde{e}_1^2 &= ([8.45, 8.50, 8.60], 0.554, 0.132, 0.288), \\
\widetilde{e}_1 &= ([8.15, 8.30, 8.50], 0.331, 0.194, 0.326), \\
\widetilde{e}_2 &= ([8.30, 8.40, 8.50], 0.505, 0.218, 0.213), \\
\widetilde{e}_3^1 &= ([8.25, 8.30, 8.50], 0.462, 0.131, 0.282), \\
\widetilde{e}_3 &= ([8.20, 8.30, 8.50], 0.446, 0.261, 0.121), \\
\widetilde{e}_4 &= ([8.20, 8.30, 8.50], 0.396, 0.313, 0.303), \\
\widetilde{e}_5 &= ([8.30, 8.40, 8.50], 0.379, 0.178, 0.194), \\
\end{align*}
$$

Step 2. Find the cumulative total preference picture fuzzy linguistic information values $\bar{e}_i$ ($i = 1, \ldots, m$) of the alternative $\mathcal{R}_i$ by using the TPFLOWA operator.

$$
\begin{align*}
\bar{e}_1 &= PFPFLW\left(\widetilde{e}_1^1, \widetilde{e}_1^2, \widetilde{e}_1\right) = ([8.00, 8.11, 8.21], 0.422, 0.136, 0.289), \\
\bar{e}_2 &= PFPFLW\left(\widetilde{e}_2, \widetilde{e}_2, \widetilde{e}_2\right) = ([8.34, 8.43, 8.57], 0.496, 0.171, 0.248), \\
\bar{e}_3 &= PFPFLW\left(\widetilde{e}_3^1, \widetilde{e}_3, \widetilde{e}_3^2\right) = ([8.29, 8.36, 8.52], 0.308, 0.234, 0.268), \\
\bar{e}_4 &= PFPFLW\left(\widetilde{e}_4^1, \widetilde{e}_4, \widetilde{e}_4\right) = ([8.28, 8.36, 8.42], 0.484, 0.160, 0.195), \\
\bar{e}_5 &= PFPFLW\left(\widetilde{e}_5^1, \widetilde{e}_5, \widetilde{e}_5^2\right) = ([8.85, 8.38, 8.47], 0.427, 0.231, 0.185).
\end{align*}
$$

Step 3. Find the $\mathbb{Z}(\bar{e}_i)$(value index) and the $\mathbb{R}(\bar{e}_i)$(ambiguity index), $G(\bar{e}_i)$(g-value) of the total triangular picture fuzzy linguistic variables $\bar{e}_i$ ($i = 1, \ldots, m$) as follows:

$$
\mathbb{Z}(\bar{e}_1) = 1.699, \mathbb{R}(\bar{e}_1) = 0.490, G(\bar{e}_1) = 1.209.
$$
and the Definition 2.6, we have

\[ G(\tilde{e}_2) = 0.399, G(\tilde{e}_3) = 1.151, G(\tilde{e}_4) = 0.380, G(\tilde{e}_5) = 0.902, \]

According to the Definition 2.6, we get

\[ \tilde{e}_1 \succ PF \tilde{e}_2 \succ PF \tilde{e}_4 \succ PF \tilde{e}_5 \succ PF \tilde{e}_3 \]

### Step 4.

Give the ranking to the alternatives \( \mathcal{R}_i = (\kappa = 1, ..., 5) \), using the the g-value value \( G(\tilde{e}_i) \) of total triangular picture fuzzy linguistic variables \( \tilde{e}_i (\kappa = 1, ..., 5) \) and the Definition 2.6, we have \( \tilde{e}_1 \succ PF \tilde{e}_2 \succ PF \tilde{e}_4 \succ PF \tilde{e}_5 \succ PF \tilde{e}_3 \) and thus the most desirable alternative is \( \mathcal{R}_1 \).

Now, we used TPFLIOWG operator to find the most desirable alternative.

### Step 1.

Using the decision information stated in the triangular picture fuzzy linguistic decision matrix \( \tilde{E} \) and the TPFLIOWG operator, to find the individual total preference triangular picture fuzzy linguistic information values \( \tilde{e}^{(k)} \) of the alternative \( \mathcal{R}_i \):

\[
\begin{align*}
\tilde{e}_1 &= ( \tilde{e}^{(1)}_1, \tilde{e}^{(2)}_1, \tilde{e}^{(3)}_1 ) = ( [0.395, 0.119, 0.501], [0.496, 0.156, 0.300], [0.496, 0.156, 0.300] ), \\
\tilde{e}_2 &= ( \tilde{e}^{(1)}_2, \tilde{e}^{(2)}_2, \tilde{e}^{(3)}_2 ) = ( [0.283, 0.272, 0.359], [0.521, 0.257, 0.182], [0.521, 0.257, 0.182] ), \\
\tilde{e}_3 &= ( \tilde{e}^{(1)}_3, \tilde{e}^{(2)}_3, \tilde{e}^{(3)}_3 ) = ( [0.224, 0.190, 0.298], [0.432, 0.209, 0.298], [0.432, 0.209, 0.298] ), \\
\tilde{e}_4 &= ( \tilde{e}^{(1)}_4, \tilde{e}^{(2)}_4, \tilde{e}^{(3)}_4 ) = ( [0.400, 0.230, 0.420], [0.429, 0.221, 0.179], [0.429, 0.221, 0.179] ), \\
\tilde{e}_5 &= ( \tilde{e}^{(1)}_5, \tilde{e}^{(2)}_5, \tilde{e}^{(3)}_5 ) = ( [0.211, 0.247, 0.367], [0.269, 0.240, 0.374], [0.269, 0.240, 0.374] ).
\end{align*}
\]

### Step 2.

Find the cumulative total preference triangular picture fuzzy linguistic information values \( \tilde{e}_i (\kappa = 1, ..., m) \) of the alternative \( \mathcal{R}_i \) by using the TPFLOWG operator.

\[
\begin{align*}
\tilde{e}_1 &= PF \Phi_{TPFLOWG}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = ( [0.346, 0.185, 0.401], [0.427, 0.218, 0.319], [0.427, 0.218, 0.319] ), \\
\tilde{e}_2 &= PF \Phi_{TPFLOWG}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = ( [0.247, 0.409, 0.317], [0.416, 0.239, 0.316], [0.416, 0.239, 0.316] ), \\
\tilde{e}_3 &= PF \Phi_{TPFLOWG}(\tilde{e}_3, \tilde{e}_4, \tilde{e}_5) = ( [0.333, 0.224, 0.257], [0.333, 0.224, 0.257] ).
\end{align*}
\]

### Step 3.

Find the value index, ambiguity index \( \mathfrak{M}(\tilde{e}_i), \mathfrak{R}(\tilde{e}_i) \) respectively, and g-value \( G(\tilde{e}_i) \) of the total triangular picture fuzzy linguistic variables \( \tilde{e}_i (\kappa = 1, ..., m) \) as follows:

\[
\begin{align*}
\mathfrak{M}(\tilde{e}_1) &= 1.395, \mathfrak{R}(\tilde{e}_1) = 0.362, G(\tilde{e}_1) = 1.033, \\
\mathfrak{M}(\tilde{e}_2) &= 1.225, \mathfrak{R}(\tilde{e}_2) = 0.422, G(\tilde{e}_2) = 0.803.
\end{align*}
\]
According to the Definition 2.6, we get;\

\[\Re(\mathcal{e}_3) = 0.439, \Im(\mathcal{e}_3) = 0.961, G(\mathcal{e}_3) = 0.522,\]
\[\Re(\mathcal{e}_4) = 0.473, \Im(\mathcal{e}_4) = 1.041, G(\mathcal{e}_4) = 0.568,\]
\[\Re(\mathcal{e}_5) = 0.422, \Im(\mathcal{e}_5) = 1.154, G(\mathcal{e}_5) = 0.732.\]

### Step 4.

Give ranking to the alternatives \(\Re_i(i = 1, ..., 5)\), using the g-value

\[G(\mathcal{e}_i)\] of total triangular picture fuzzy linguistic variables \(\mathcal{e}_i(i = 1, ..., 5)\) and the

Definition 2.6, we have \(\mathcal{e}_1 \succ_{PF} \mathcal{e}_2 \succ_{PF} \mathcal{e}_5 \succ_{PF} \mathcal{e}_4 \succ_{PF} \mathcal{e}_3\) and thus the most desirable alternative is \(\Re_1\).

#### 5.1. Comparing with the other method.

Comparing our developed approach with the approach developed by S. Xian and W. Xue [53].

In the comparison analysis, we solved the example introduced in this paper, using the approach introduced by Xian and Xue [53]. Because the approach of [53] deals with those problems in which the criteria information in the form of TILNs, so, we only omit the neutral term in TPFLNs and convert them to TILNs.

| Method       | Ranking                                      |
|--------------|----------------------------------------------|
| TPFLIOWA     | \[\mathcal{e}_1 \succ_{PF} \mathcal{e}_2 \succ_{PF} \mathcal{e}_4 \succ_{PF} \mathcal{e}_5 \succ_{PF} \mathcal{e}_3\] |
| TPFLIOWG     | \[\mathcal{e}_1 \succ_{PF} \mathcal{e}_2 \succ_{PF} \mathcal{e}_5 \succ_{PF} \mathcal{e}_4 \succ_{PF} \mathcal{e}_3\] |
| IFLIOWA [53] | \[\mathcal{e}_1 \succ_{IF} \mathcal{e}_2 \succ_{IF} \mathcal{e}_4 \succ_{IF} \mathcal{e}_5 \succ_{IF} \mathcal{e}_3\] |
| IFLIOWG [53] | \[\mathcal{e}_1 \succ_{IF} \mathcal{e}_2 \succ_{IF} \mathcal{e}_5 \succ_{IF} \mathcal{e}_4 \succ_{IF} \mathcal{e}_3\] |

**Comparison Analysis:** From Table 4, we notice that the first alternative in each ordering is the best alternative, and the TPFLIOWA operator attain the same optimal alternative as the TPFLIOWG, IFLIOWA and IFLIOWG operators. But other results present different ranking by the four operators. However, the ranking results gained by the TPFLIOWA and TPFLIOWG operator display more flexibility with the linguistic value of the attribute by utilizing the triangular picture fuzzy linguistic information.

#### 6. Conclusion.

Generally, we used the induced aggregation operators to aggregate the information occur in the form of numerical values, but existing aggregation operators fail to arrange triangular picture fuzzy linguistic information. So, for this type of data, we developed some new operators like as TPFLIOWA and TPFLIOWG operators and review some advisable properties of the operators, like as commutativity, Idempotency and monotonicity, also tested these operators to decision making investment problem with triangular picture fuzzy linguistic information. Lastly, we solved a investment problem to present the application of the developed method. Our future work as to study the application of the developed operators in different fields, e.g., medical diagnostic, control systems and intelligent decision system.

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