DENSENESS OF MINIMAL HYPERSURFACES FOR GENERIC METRICS

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Abstract. For almost all Riemannian metrics (in the \( C^\infty \) Baire sense) on a closed manifold \( M^{n+1} \), \( 3 \leq (n+1) \leq 7 \), we prove that the union of all closed, smooth, embedded minimal hypersurfaces is dense. This implies there are infinitely many minimal hypersurfaces thus proving a conjecture of Yau (1982) for generic metrics.

1. Introduction

Minimal surfaces are among the most extensively studied objects in Differential Geometry. There is a wealth of examples for many particular ambient spaces, but their general existence theory in Riemannian manifolds is still rather mysterious. A motivating conjecture has been:

Conjecture (Yau [17], 1982): Every closed Riemannian three-manifold contains infinitely many smooth, closed, immersed minimal surfaces.

In this paper we settle the generic case, and in fact prove that a much stronger property holds true: there are infinitely many closed embedded minimal hypersurfaces intersecting any given ball in \( M \).

Main Theorem: Let \( M^{n+1} \) be a closed manifold of dimension \( (n+1) \), with \( 3 \leq (n+1) \leq 7 \). Then for a \( C^\infty \)-generic Riemannian metric \( g \) on \( M \), the union of all closed, smooth, embedded minimal hypersurfaces is dense.

Besides some specific metrics (e.g. [8]), the existence of infinitely many closed, smooth, embedded minimal hypersurfaces was only known for manifolds of positive Ricci curvature \( M^{n+1} \), \( 3 \leq (n+1) \leq 7 \), as proven by the last two authors in [12]. Before that the best result was due to Pitts (1981, [13]), who built on earlier work of Almgren ([1]) to prove there is at least one closed embedded minimal hypersurface. In [12] it was shown the existence of at least \( (n+1) \) such hypersurfaces.

The main ingredient in the proof of our Main Theorem is the Weyl law for the volume spectrum conjectured by Gromov ([5]) and recently proven by the last two authors jointly with Liokumovich in [9]. We also need the Morse index estimates proven by the last two authors in [10], for minimal
hypersurfaces constructed by min-max methods. Finally, we use an idea of the first author ([7]) who proved an analogous denseness result for closed geodesics (not necessarily embedded) in surfaces. The argument of [7] is based on a different kind of asymptotic law, involving spectral invariants in Embedded Contact Homology ([3]).

The volume spectrum of a compact Riemannian manifold $(M^{n+1}, g)$ is a nondecreasing sequence of numbers $\{\omega_k(M,g) : k \in \mathbb{N}\}$ defined variationally by performing a min-max procedure for the area functional over multiparameter sweepouts. The first estimates for these numbers were proven by Gromov in the late 1980s [4] (see also Guth [6]).

The main result of [9] used in this paper is:

**Weyl Law for the Volume Spectrum** (Liokumovich, Marques, Neves, 2016): There exists a universal constant $a(n) > 0$ such that for any compact Riemannian manifold $(M^{n+1}, g)$ we have:

$$\lim_{k \to \infty} \frac{\omega_k(M,g) k^{-\frac{1}{n+1}}}{\omega_k(M,g)^{\frac{n}{n+1}}} = a(n)\text{vol}(M,g)^{\frac{n}{n+1}}.$$ 

The dimensional restriction in the Main Theorem is due to the fact that in higher dimensions min-max (even area-minimizing) minimal hypersurfaces can have singular sets. We use Almgren-Pitts theory ([1], [13]), which together with Schoen-Simon regularity ([14]) produces smooth minimal hypersurfaces when $3 \leq (n+1) \leq 7$. We expect that the methods of this paper can be generalized to handle the higher-dimensional singular case.

2. Preliminaries

We denote by $\mathcal{Z}_n(M; \mathbb{Z}_2)$ the space of modulo two $n$-dimensional flat chains $T$ in $M$ with $T = \partial U$ for some $(n+1)$-dimensional modulo two flat chain $U$ in $M$, endowed with the flat topology. This space is weakly homotopically equivalent to $\mathbb{RP}^\infty$ (see Section 4 of [11]). We denote by $\bar{X}$ the generator of $H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$. The mass ($n$-dimensional volume) of $T$ is denoted by $M(T)$.

Let $X$ be a finite dimensional simplicial complex. A continuous map $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is called a $k$-sweepout if

$$\Phi^*(\bar{X}^k) \neq 0 \in H^k(X; \mathbb{Z}_2).$$

We say $X$ is $k$-admissible if there exists a $k$-sweepout $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ that has no concentration of mass, meaning

$$\lim_{r \to 0} \sup \{M(\Phi(x) \cap B_r(p)) : x \in X, p \in M\} = 0.$$ 

The set of all $k$-sweepouts $\Phi$ that have no concentration of mass is denoted by $\mathcal{P}_k$. Note that two maps in $\mathcal{P}_k$ can have different domains.

In [12], the last two authors defined
Definition: The k-width of \((M, g)\) is the number
\[
\omega_k(M, g) = \inf_{\Phi \in \mathcal{P}_k} \sup \{ M(\Phi(x)) : x \in \text{dmn}(\Phi) \},
\]
where \(\text{dmn}(\Phi)\) is the domain of \(\Phi\).

Lemma 2.1. The k-width \(\omega_k(M, g)\) depends continuously on the metric \(g\) (in the \(C^0\) topology).

Proof. Suppose \(g_i\) is a sequence of smooth Riemannian metrics that converges to \(g\) in the \(C^0\) topology. Given \(\varepsilon > 0\), let \(\Phi : X \to Z_n(M; \mathbb{Z}_2)\) be a \(k\)-sweepout of \(M\) that has no concentration of mass (this condition does not depend on the metric) and such that
\[
\sup \{ M_g(\Phi(x)) : x \in X \} \leq \omega_k(M, g) + \varepsilon,
\]
where \(M_g(T)\) is the mass of \(T\) with respect to \(g\).

Since
\[
\omega_k(M, g_i) \leq \sup \{ M_{g_i}(\Phi(x)) : x \in X \}
\leq (\sup_{v \neq 0} \frac{g_i(v, v)}{g(v, v)})^\frac{2}{n} \sup \{ M_g(\Phi(x)) : x \in X \}
\leq (\sup_{v \neq 0} \frac{g_i(v, v)}{g(v, v)})^\frac{2}{n} (\omega_k(M, g) + \varepsilon),
\]
and \(\varepsilon > 0\) is arbitrary, we get \(\limsup_{i \to \infty} \omega_k(M, g_i) \leq \omega_k(M, g)\). Similarly, one can prove \(\liminf_{i \to \infty} \omega_k(M, g_i) \geq \omega_k(M, g)\).

The proof of the next Proposition is essentially contained in Section 1.5 of [10], but we prove it here for the sake of completeness. It follows from the index estimates of the last two authors ([10]) and a compactness theorem of Sharp ([15]).

Proposition 2.2. Suppose \(3 \leq (n + 1) \leq 7\). Then for each \(k \in \mathbb{N}\), there exist a finite disjoint collection \(\{\Sigma_1, \ldots, \Sigma_N\}\) of closed, smooth, embedded minimal hypersurfaces in \(M\), and integers \(\{m_1, \ldots, m_N\} \subseteq \mathbb{N}\), such that
\[
\omega_k(M, g) = \sum_{j=1}^{N} m_j \text{vol}_g(\Sigma_j),
\]
and
\[
\sum_{j=1}^{N} \text{index}(\Sigma_j) \leq k.
\]

Proof. Choose a sequence \(\{\Phi_i\}_{i \in \mathbb{N}} \subset \mathcal{P}_k\) such that
\[
\lim sup_{i \to \infty} \{ M(\Phi_i(x)) : x \in X_i = \text{dmn}(\Phi_i) \} = \omega_k(M, g).
\]
Denote by \(X_i^{(k)}\) the \(k\)-dimensional skeleton of \(X_i\). Then \(H^k(X_i, X_i^{(k)}; \mathbb{Z}_2) = 0\) and hence the long exact cohomology sequence gives that the natural
pullback map from $H^k(X_i; \mathbb{Z}_2)$ into $H^k(X_i^{(k)}; \mathbb{Z}_2)$ is injective. This implies $(\Phi_i|_{X_i^{(k)}}) \in P_k$. The definition of $\omega_k$ then implies

$$\lim_{i \to \infty} \sup \{M(\Phi_i(x)) : x \in X_i^{(k)}\} = \omega_k(M, g).$$

The interpolation machinery developed by the last two authors ([12], item (ii) of Corollary 3.12) implies that we can suppose $\Phi_i : X_i^{(k)} \to \mathbb{Z}_n(M, \mathbb{Z}_2)$ is continuous in the $F$-metric (see Section 2.1 of [12]) for every $i$.

We denote by $\Pi_i$ the homotopy class of $\Phi_i$ as defined in [10]. This is the class of all maps $\Phi'_i : X_i^{(k)} \to \mathbb{Z}_n(M, \mathbb{Z}_2)$, continuous in the $F$-metric, that are homotopic to $\Phi_i$ in the flat topology. In particular, $(\Phi'_i)^*(\lambda^k) = \Phi_i^*(\lambda^k)$. Continuity in the $F$-metric implies no concentration of mass, hence every such $\Phi'_i$ is also a $k$-sweepout.

Therefore the min-max number (defined in Section 1 of [10])

$$L(\Pi_i) = \inf_{\Phi'_i \in \Pi_i} \sup_{x \in X_i^{(k)}} \{M(\Phi'_i(x))\}$$

satisfies

$$\omega_k(M, g) \leq L(\Pi_i) \leq \sup \{M(\Phi_i(x)) : x \in X_i^{(k)}\}$$

and in particular

$$\lim_{i \to \infty} L(\Pi_i) = \omega_k(M, g).$$

Theorem 1.2 of [10] now implies the existence of a finite disjoint collection $\{\Sigma_{i,1}, \ldots, \Sigma_{i,N_i}\}$ of closed, smooth, embedded minimal hypersurfaces in $M$, and integers $\{m_{i,1}, \ldots, m_{i,N_i}\} \subset \mathbb{N}$, such that

$$L(\Pi_i) = \sum_{j=1}^{N_i} m_{i,j} \text{vol}_g(\Sigma_{i,j}),$$

and

$$\sum_{j=1}^{N_i} \text{index}(\Sigma_{i,j}) \leq k.$$

The monotonicity formula for minimal hypersurfaces in Riemannian manifolds implies that there exists $\delta > 0$, depending only on $M$, such that the volume of any closed minimal hypersurface is greater than or equal to $\delta$. Hence the number of components $N_i$ and the multiplicities $m_{i,j}$ are uniformly bounded. The Compactness Theorem of Sharp (Theorem 2.3 of [15]) implies that there exists a finite disjoint collection $\{\Sigma_1, \ldots, \Sigma_N\}$ of closed, smooth, embedded minimal hypersurfaces in $M$, satisfying

$$\sum_{j=1}^{N} \text{index}(\Sigma_j) \leq k,$$
and integers \(\{m_1, \ldots, m_N\} \subset \mathbb{N}\) such that, after passing to a subsequence,
\[
\sum_{j=1}^{N_0} m_{i,j} \cdot \Sigma_{i,j} \rightarrow \sum_{j=1}^{N} m_j \cdot \Sigma_j
\]
as varifolds. Hence \(\omega_k(M, g) = \sum_{j=1}^{N} m_j \text{vol}_g(\Sigma_j)\), and the proof of the proposition is finished.

\[\square\]

**Proposition 2.3.** Let \(\Sigma\) be a closed, smooth, embedded minimal hypersurface in \((M^{n+1}, g)\). Then there exists a sequence of metrics \(g_i\) on \(M\), \(i \in \mathbb{N}\), converging to \(g\) in the smooth topology such that \(\Sigma\) is a nondegenerate minimal hypersurface in \((M^{n+1}, g_i)\) for every \(i\).

**Proof.** If \(\tilde{g} = \exp(2\phi)g\), then the second fundamental form of \(\Sigma\) with respect to \(\tilde{g}\) is given by (Besse [2], Section 1.163)
\[
A_{\Sigma, \tilde{g}} = A_{\Sigma, g} - g \cdot (\nabla \phi)^\perp,
\]
where \((\nabla \phi)^\perp(x)\) is the component of \(\nabla \phi\) normal to \(T_x \Sigma\). The Ricci curvatures are related by (see Besse [2], Theorem 1.159):
\[
Ric_{\tilde{g}} = Ric_g - (n-1)(Hess_g \phi - d\phi \otimes d\phi) - (\Delta_g \phi + (n-1)|\nabla \phi|^2) \cdot g.
\]

Suppose both \(\phi\) and \(\nabla \phi\) vanish on \(\Sigma\). Then \(\tilde{g}|\Sigma = g|\Sigma\) and \(A_{\Sigma, \tilde{g}} = A_{\Sigma, g}\).

In particular, \(\Sigma\) is also minimal with respect to \(\tilde{g}\) and \(|A_{\Sigma, \tilde{g}}|^2 = |A_{\Sigma, g}|^2\). A unit normal \(N\) to \(\Sigma\) with respect to \(g\) is also a unit normal to \(\Sigma\) with respect to \(\tilde{g}\) and
\[
Ric_{\tilde{g}}(N, N) = Ric_g(N, N) - (n-1)Hess_g \phi(N, N) - \Delta_g \phi.
\]
Since \(\nabla \phi = 0\) on \(\Sigma\), we have \(\Delta_g \phi = Hess_g \phi(N, N)\) on \(\Sigma\) and therefore
\[
Ric_{\tilde{g}}(N, N) = Ric_g(N, N) - nHess_g \phi(N, N).
\]

Let \(\eta : M \rightarrow \mathbb{R}\) be a smooth function such that is equal to 1 in \(V_\delta(\Sigma)\) and equal to zero in \(M \setminus V_{3\delta}(\Sigma)\), where \(V_r(\Sigma) = \{x \in M : d_g(x, \Sigma) \leq r\}\). We choose \(\delta > 0\) sufficiently small so that the function \(x \mapsto d_g(x, \Sigma)^2\) is smooth in \(V_{3\delta}(\Sigma)\). We define \(h(x) = \eta(x)d_g(x, \Sigma)^2\) for \(x \in V_{3\delta}(\Sigma)\) and \(h(x) = 0\) for \(x \in M \setminus V_{3\delta}(\Sigma)\), so \(h : M \rightarrow \mathbb{R}\) is a smooth function that coincides with \(x \mapsto d_g(x, \Sigma)^2\) in some small neighborhood of \(\Sigma\).

Let \(g_i = \exp(2\phi_i)g\), where \(\phi_i = \frac{1}{i}h\). Since \(h(x) = d_g(x, \Sigma)^2\) in a neighborhood of \(\Sigma\), we have that, on \(\Sigma\), \(\phi_i = 0\), \(\nabla \phi_i = 0\) and \(Hess_g \phi_i(N, N) = \frac{2}{i}\), and \(\Sigma\) is minimal with respect to \(g_i\).

Therefore
\[
Ric_{g_i}(N, N) + |A_{\Sigma, g_i}|^2_{g_i} = Ric_g(N, N) + |A_{\Sigma, g}|^2_g - \frac{2n}{i}.
\]

The Jacobi operator acting on normal vector fields is given by the expression
\[
L_{\Sigma, g}(X) = \Delta_{\Sigma, g}X + (Ric_g(N, N) + |A_{\Sigma, g}|^2_g)X.
\]
Since \( g_{i|\Sigma} = g|\Sigma \), we have \( \Delta_{\Sigma, g_i} X = \Delta_{\Sigma, g} X \) and hence

\[
L_{\Sigma, g_i}(X) = L_{\Sigma, g}(X) - \frac{2n}{i} X.
\]

We conclude that

\[
\text{spec}(L_{\Sigma, g_i}) = \text{spec}(L_{\Sigma, g}) + \frac{2n}{i}.
\]

Hence \( \Sigma \) is nondegenerate with respect to \( g_i \) for every sufficiently large \( i \).

3. Proof of the Main Theorem

We denote by \( \mathcal{M} \) the space of all smooth Riemannian metrics on \( M \), endowed with the \( C^\infty \) topology.

**Proposition 3.1.** Suppose \( 3 \leq (n + 1) \leq 7 \), and let \( U \subset M \) be a nonempty open set. Then the set \( \mathcal{M}_U \) of all smooth Riemannian metrics on \( M \) such that there exists a nondegenerate, closed, smooth, embedded, minimal hypersurface \( \Sigma \) that intersects \( U \) is open and dense in the \( C^\infty \) topology.

**Proof.** Let \( g \in \mathcal{M}_U \) and \( \Sigma \) be like in the statement of the proposition. Because \( \Sigma \) is nondegenerate, a standard application of the Inverse Function Theorem implies that for every Riemannian metric \( g' \) sufficiently close to \( g \), there exists a unique nondegenerate closed, smooth, embedded minimal hypersurface \( \Sigma' \) close to \( \Sigma \). This implies \( \mathcal{M}_U \) is open.

It remains to show the set \( \mathcal{M}_U \) is dense. Let \( g \) be an arbitrary smooth Riemannian metric on \( M \) and \( \mathcal{V} \) be an arbitrary neighborhood of \( g \) in the \( C^\infty \) topology. By the Bumpy Metrics Theorem of White (Theorem 2.1, [16]), there exists \( g' \in \mathcal{V} \) such that every closed, smooth immersed minimal hypersurface with respect to \( g' \) is nondegenerate. If one of these minimal hypersurfaces is embedded and intersects \( U \) then \( g' \in \mathcal{M}_U \), and we are done.

Hence we can suppose that every closed, smooth, embedded minimal hypersurface with respect to \( g' \) is contained in the complement of \( U \). Since \( g' \) is bumpy, it follows from Sharp (Theorem 2.3 and Remark 2.4, [15]) that the set of connected, closed, smooth, embedded minimal hypersurfaces in \( (M, g') \) with both area and index bounded from above by \( q \) is finite for every \( q > 0 \). Therefore the set

\[
\mathcal{C} = \left\{ \sum_{j=1}^{N} m_j \text{vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^{N} \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^{N} \text{ disjoint collection} \right\}
\]

of closed, smooth, embedded minimal hypersurfaces in \( (M, g') \) is countable.

Choose \( h : M \to \mathbb{R} \) a smooth nonnegative function such that \( \text{supp}(h) \subset U \) and \( h(x) > 0 \) for some \( x \in U \). Define \( g'(t) = (1 + th)g' \) for \( t \geq 0 \), and let \( t_0 > 0 \) be sufficiently small so that \( g'(t) \in \mathcal{V} \) for every \( t \in [0, t_0] \). Notice that \( g'(t) = g' \) outside some compact set \( K \subset U \) for every \( t > 0 \).
We have $\text{vol}(M, g'(t_0)) > \text{vol}(M, g')$. It follows from the Weyl Law for the Volume Spectrum (see Introduction) that there exists $k \in \mathbb{N}$ such that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$. Assume by contradiction that for every $t \in [0, t_0]$, every closed, smooth, embedded minimal hypersurface in $(M, g'(t))$ is contained in $M \setminus U$. Since $g'(t) = g'$ outside $K \subset U$ we conclude from Proposition 2.2 that $\omega_k(M, g'(t)) \in C$ for all $t \in [0, t_0]$. But $C$ is countable and we know from Proposition 2.1 that the function $t \mapsto \omega_k(M, g'(t))$ is continuous. Hence $t \mapsto \omega_k(M, g'(t))$ is constant in the interval $[0, t_0]$. This contradicts the fact that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$.

Therefore we can find $t \in [0, t_0]$ such that there exists a closed, smooth, embedded minimal hypersurface $\Sigma$ with respect to $g'(t)$ that intersects $U$. Since $g'(t) \in V$, Proposition 2.3 implies there exists a Riemannian metric $g'' \in V$ such that $\Sigma$ is minimal and nondegenerate with respect to $g''$. Therefore $g'' \in V \cap M_U$ and we have finished the proof of the Proposition.

**Proof of the Main Theorem.** Let $\{U_i\}$ be a countable basis of $M$. Since, by Proposition 3.1, each $M_{U_i}$ is open and dense in $M$ the set $\bigcap_i M_{U_i}$ is $C^\infty$ Baire-generic in $M$. This finishes the proof. □

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