**Strong Solutions to a Beta-Wishart Particle System**

Benjamin Jourdain\(^1\) · Ezéchiel Kahn\(^1\)

Received: 6 April 2020 / Revised: 15 February 2021 / Accepted: 25 May 2021 / Published online: 9 July 2021

© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

**Abstract**

The purpose of this paper is to study the existence and uniqueness of solutions to a stochastic differential equation (SDE) coming from the eigenvalues of Wishart processes. The coordinates are non-negative, evolve as Cox–Ingersoll–Ross (CIR) processes and repulse each other according to a Coulombian like interaction force. We show the existence of strong and pathwise unique solutions to the system until the first multiple collision and give a necessary and sufficient condition on the parameters of the SDEs for this multiple collision not to occur in finite time.

**Keywords** Stochastic differential equations · Diffusions with gradient drift · Singular interaction · Random matrices

**Mathematics Subject Classification** 60H10 · 60J60 · 60B20 · 60G17 · 60J70

**1 Introduction**

Let \( \alpha \geq 0, \gamma \in \mathbb{R}, \beta > 0, n \geq 1, \) and \( \mathbf{B} = (B^1_t, \ldots, B^n_t) \) be a \( n \)-dimensional Brownian motion. Our SDEs system of interest is the following:

\[
\frac{d\lambda_i}{\sqrt{\lambda_i}} = 2\sqrt{\lambda_i} dB^i_t + \left( \alpha - 2\gamma \lambda_i + \beta \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt \tag{1}
\]

\[
= 2\sqrt{\lambda_i} dB^i_t + \left( \alpha - (n-1)\beta - 2\gamma \lambda_i + 2\beta \lambda_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) dt
\]

\[^1\text{CERMICS, Ecole des Ponts, INRIA, Marne-la-Vallée, France}\]

Ezéchiel Kahn
ezechiel.kahn@enpc.fr

Benjamin Jourdain
benjamin.jourdain@enpc.fr
for all \( i \in \{1, \ldots, n\} \),

\[
0 \leq \lambda^1_i < \cdots < \lambda^n_i, \text{ a.s.}\]

(2)

almost everywhere.

The strict inequalities (3) allow the interaction terms in the system (1) to make sense. We will look for continuous solutions to the SDEs (1). Thus, by continuity, we have for all \( t \geq 0 \), \( 0 \leq \lambda^1_i \leq \cdots \leq \lambda^n_i \) a.s. While, for \( n = 1 \), the system reduces to the Cox–Ingersoll–Ross SDE

\[
d\lambda^1_t = 2\sqrt{\lambda^1_t} dB^1_t + \alpha dt - 2\gamma\lambda^1_t dt,
\]

the coordinates are repulsed by a Coulombian interaction when \( n \geq 2 \).

Our goal is to study the existence and uniqueness of strong solutions to this system of SDEs in a general setting for the parameters \( \alpha, \beta \) and \( \gamma \), especially in the case \( \beta < 1 \) and \( \alpha - (n - 1)\beta \in (0, 1) \) which is not covered to our knowledge by the literature. The reader will find a summary of the conditions on the parameters for which the system (1) admits a unique strong solution in Table 1.

The difficulty in proving the existence of solutions to (1) comes from the fact that there are singularities both when a particle touches zero, where the derivative of the square root diffusion coefficient explodes, and when two particles touch each other. If we define \( D = \{0 < \lambda^1 < \lambda^2 < \cdots < \lambda^n\} \), a collision occurs when the process \( \Lambda = (\lambda^1_t, \ldots, \lambda^n_t) \), hits the boundary \( \partial D \) made of the union of \( \{\lambda^i = \lambda^{i+1}\} \) for \( i \in \{1, \ldots, n - 1\} \) and \( \{\lambda^1 = 0\} \). A multiple collision occurs when two of these sets are reached at the same time and we will speak about "collision between particles" when two particles touch each other. Because of the singularity at the origin, it is not enough to show that there is no collision between the particles to prove the existence of a solution, as it is, for instance, the case for the Dyson Brownian motions which satisfy up to a change of time

\[
dx^i_t = \sqrt{2} dB^i_t + \beta \sum_{j \neq i} \frac{dt}{x^i_t - x^j_t}, \text{ for all } i \in \{1, \ldots, n\}.
\]

When \( \beta \geq 1 \), there is no collision between these particles in finite time and (4) admits a unique strong solution (see [23]).

Our results about the SDEs (1) are the following. In Proposition 2.4, we state that \( k(\alpha - (n - k)\beta) < 2 \) is a necessary and sufficient condition for multiple collisions between \( k \) particles to occur at position zero in finite time. Our main result Theorem 2.2 gives the existence and uniqueness of solutions to the SDEs (1) when \( \beta \in (0, 1) \) and \( \alpha - (n - 2)\beta \geq 1 \) (so that there is no multiple collision at the origin). In Proposition 2.6, we state that when \( \beta < 1 \), then every pair of neighbor particles collides in finite time, and we explicit in Proposition 2.7 the unique stationary probability measure of the SDEs (1).

The paper is organized as follows. The remaining of the introduction is devoted to the bibliographical background of this work. In Sect. 2, we state our main results. We prove in Sect. 3 some useful properties of the solutions to the system (1), before checking Proposition 2.4 and Theorem 2.2 in Sect. 4. We prove the rest of the results in Sect. 5. Section 6 is Appendix stating some well-known results that we use in our proofs.

\[\text{Springer}\]
The system (1) originally comes from the following matrix valued SDE: let $m \geq n$ and $(M_t)$ be a stochastic process taking its values in the space of $m \times n$ matrices with real entries verifying the following stochastic differential equation

$$dM_t = dW_t - \gamma M_t dt, \quad M_0 = m_0,$$

where $W$ is a $m \times n$ matrix filled with independent Brownian motions, $m_0$ is a $m \times n$ deterministic matrix. The entries of the matrix $M$ are independent Ornstein–Uhlenbeck processes just as the one considered in [3]. Such a process is called a Wishart process, and it was shown in [2] and [3] that the eigenvalues of $M^\dagger M$ (where $\dagger$ is the transpose operator) satisfy the system of SDEs

$$d\lambda^i_t = 2\sqrt{\lambda^i_t} dB^i_t + m dt - 2\gamma \lambda^i_t dt + \sum_{j \neq i} \frac{\lambda^j_t + \lambda^i_t}{\lambda^i_t - \lambda^j_t} dt$$

for all $i \in \{1, \ldots, n\}$,

where $B^1, \ldots, B^n$ are independent Brownian motions. The reader will find in [17] an analysis of the complex analog of Bru’s model. The system (1) generalizes the matrix inherited system by the introduction of an intensity of the interaction factor $\beta > 0$ before the sum and by the replacement of the integer parameter $m \in \mathbb{N}^* = \{1, 2, \ldots\}$ by $\alpha > 0$. This replacement is analogous to the generalization of the dynamics of the square of the norm of an $m$-dimensional real Brownian motion to the square $\alpha$-dimensional Bessel dynamics (see, for instance, [26, chapter XI]).

Like in the correspondence between square Bessel processes and Bessel processes, let us consider the square root change of variables $x^i_t = \sqrt{\lambda^i_t}$ and set $X = (x^1_t, \ldots, x^n_t)_t$. We apply Itô’s formula, formally after the stopping time $\inf \{s \geq 0 : \lambda^1_s = 0\}$ since the square root is not twice continuously differentiable at 0, and obtain

$$dx^i_t = dB^i_t + \left(\frac{\alpha - 1}{2} \frac{1}{x^i_t} - \gamma x^i_t + \frac{\beta}{2x^i_t} \sum_{j \neq i} \frac{(x^j_t)^2 + (x^j_t)^2}{(x^j_t)^2 - (x^i_t)^2}\right) dt$$

for all $i \in \{1, \ldots, n\}$

$$= dB^i_t + \left(\frac{\alpha - (n - 1)\beta - 1}{2} \frac{1}{x^i_t} - \gamma x^i_t + \beta x^i_t \sum_{j \neq i} \frac{1}{(x^j_t)^2 - (x^i_t)^2}\right) dt$$

$$0 \leq x^1_t < \cdots < x^n_t, \text{ a.s., } dt - a.e. \quad (6)$$

When $X$ is a solution to (2), then $((x^1_t)^2, \ldots, (x^n_t)^2)_t$ is a solution to (1), but it is not necessarily true the other way round.

The system (5) can be rewritten as a gradient diffusion

$$dx^i_t = dB^i_t - \partial_i V(x^1_t, \ldots, x^n_t) dt$$

for all $i \in \{1, \ldots, n\}, \quad (7)$
with potential

\[ V(x_1, \ldots, x_n) = -\sum_{i=1}^{n} \left\{ \frac{\alpha - (n - 1)\beta - 1}{2} \ln |x^i| - \frac{1}{2}\gamma(x^i)^2 + \frac{\beta}{4} \sum_{j \neq i} \left( \ln |x^i - x^j| + \ln |x^i + x^j| \right) \right\}. \]  

(8)

Systems of interacting particles following equations of the type

\[ dx^i_t = b_i(x^i_t)dt + \sigma_i(x^i_t)dB^i_t - \partial_i V(x^1_t, \ldots, x^n_t)dt, \]

(9)

where \( V : \mathbb{R}^n \mapsto (-\infty, +\infty] \) is a lower semi continuous convex function such that \( D = \{ x \in \mathbb{R}^n : V(x) < +\infty \} \) is a non-empty convex set and \( V \) is continuously differentiable in \( D \) have been studied by many authors. For instance, in the peculiar case \( \sigma_i(x^i) = \sigma > 0, b_i(x^i) = 0 \) and \( V(x^1, \ldots, x^n) = -\frac{\beta}{2} \sum_{i=1}^{n} \sum_{j \neq i} \ln |x^i - x^j| + \frac{\theta}{2} \sum_{i=1}^{n} (x^i)^2 \) with \( \beta \geq \frac{\sigma^2}{2} \) and \( \theta > 0 \), the existence and uniqueness of a strong solution to (9) were derived in [23].

**Link with the multivalued stochastic differential equations theory** The systems of type (9) with \( b_i \) and \( \sigma_i \) Lipschitz and \( (x_1^0, \ldots, x_n^0) \in \bar{D} \) were deeply studied by Cépa and Lépingle, for instance, in [6] and [7] where they apply Cépa’s multivalued stochastic differential equations theory developed in [8]. This theory treats the existence and uniqueness of solutions to multivalued SDEs associated with a convex function defined on a domain of \( \mathbb{R}^n \). In [22], Lépingle applied this theory to a constrained Brownian motion between reflecting and repellent walls of Weyl chambers. The boundary behavior of the convex function dictates the behavior of the process on these same boundaries (hitting or not the boundary in finite time, reflection on the boundary, etc.).

Our SDEs of interest rewritten in the form (7) thanks to the square root change of variables can be seen this way, and we will exploit this connection in the paper.

**Link with radial Dunkl processes** (see [10] for a more complete description of the theory)

Let us define a reduced root system \( R \) by a finite set in \( \mathbb{R}^n \setminus \{0\} \) spanning \( \mathbb{R}^n \) such that

- for all \( \alpha \in R \), \( R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \),
- for all \( \alpha \in R \), \( \sigma_\alpha(R) = R \),

where \( \sigma_\alpha \) is the reflection with respect to the hyperplane orthogonal to \( \alpha \). A simple system \( \Delta \) is a basis of \( \mathbb{R}^n \) which induces a total ordering in \( R \) the following way : a root \( \alpha \in R \) is positive if it is a positive linear combination of elements of \( \Delta \). A simple system \( \Delta \) being fixed, we can thus define \( R_+ \) as the set of positive roots of \( R \). When \( \sigma_i = 1 \) and \( V \) takes the form

\[ V : x \mapsto -\sum_{\alpha \in R_+} k(\alpha) \ln(\langle \alpha, x \rangle), \quad x \in D, \]
where $D$ is the positive Weyl chamber defined by

$$D = \{ x \in \mathbb{R}^n, \langle \alpha, x \rangle > 0 \ \forall \alpha \in R_+ \},$$

Demni proved in [10, Theorem 1] the existence and uniqueness of a solution to (9) on the domain $D$ when $k(\alpha) > 0$ for all $\alpha \in R_+$. To do so, he applied Cépa’s multivalued stochastic differential equations theory. This system corresponds to (1) for some choice of $R_+$ and $k$. Indeed, when the root system is of so-called $B_n$-type, it is defined by

$$R = \{ \pm e_i, \pm e_j, 1 \leq i < j \leq n \},$$
$$\Delta = \{ e_{i+1} - e_i, 1 \leq i \leq n - 1, e_1 \},$$
$$R_+ = \{ e_i, 1 \leq i \leq n, e_j \pm e_i, 1 \leq i < j \leq n \},$$
$$D = \{ x \in \mathbb{R}^n, 0 < x^1 < \cdots < x^n \},$$

which with the right choice of $k$ gives Eq. (5) with $\gamma = 0$ (see (8)). The condition $k(\alpha) > 0$ for all $\alpha \in R_+$ implies $\alpha - (n - 1)\beta > 1$. We seek here to obtain the existence of a solution to (1) while relaxing the last inequality.

**Link with other works** The reader will find in Graczyk and Malecki [14] and [15] a treatment of equations of the form

$$d\lambda_t^i = \sigma_i(\lambda_t^i) dB_t^i + \left( b_i(\lambda_t^i) + \sum_{j \neq i} \frac{H_{i,j}(\lambda_t^i, \lambda_t^j)}{\lambda_t^i - \lambda_t^j} \right) dt, \text{ for all } i \in \{1, \ldots, n\}$$

$$\lambda_t^1 \leq \cdots \leq \lambda_t^n, \quad t \geq 0,$$

where the functions $\sigma_i, b_i$ and $H_{i,j}$ are assumed continuous, with $H_{i,j}$ non-negative and symmetric in the sense that $H_{i,j}(x, y) = H_{j,i}(y, x)$ for all $x, y \in \mathbb{R}$. The system (1) is recovered in the particular case when $H_{i,j}(\lambda^i, \lambda^j) = \beta(\lambda^i + \lambda^j), \sigma_i(\lambda^i) = 2\sqrt{\lambda^i}$ and $b_i(\lambda^i) = \alpha - 2\gamma \lambda^i$. According to [15, Sect. 6.4], (1) admits a strong solution on the time interval $[0, +\infty)$ when $\beta \geq 1$. In this regime, the authors proved that there is no collision between the particles. They also demonstrated the pathwise uniqueness of the solutions for every $\beta > 0$, as recalled in Lemma 3.1.

The same authors studied in [16] the system of SDEs defined for all $i \in \{1, \ldots, n\}$ by

$$d\lambda_t^i = 2\sqrt{\lambda_t^i} dB_t^i + \tilde{\beta} \left( \tilde{\alpha} + \sum_{j \neq i} \frac{|\lambda_t^j| + |\lambda_t^j|}{\lambda_t^i - \lambda_t^j} \mathbb{1}_{[\lambda_t^i \neq \lambda_t^j]} \right) dt$$

$$\lambda_t^1 \leq \cdots \leq \lambda_t^n, \quad \forall t \geq 0.$$

They prove existence of a unique strong solution for all $\tilde{\alpha} \in \mathbb{R}$ and $\tilde{\beta} \geq 1$. The main difference here with our problem (1) is the relaxation of the hard edge at zero by the replacement of $\sqrt{\lambda_t^i}$ by $\sqrt{|\lambda_t^i|}$ in the diffusion coefficient. This enables the authors to establish the existence and uniqueness of a strong solution for all $\tilde{\alpha} \in \mathbb{R}$, whereas in our framework, as we will see in Remark 2.1, we necessarily need $\alpha \geq (n - 1)\beta$ for

Springer
a solution to exist on the whole interval $\mathbb{R}_+$. The introduction of the absolute value in the diffusion coefficient was presented earlier in [13] in the case of squared Bessel processes. Moreover, our work differs from [16] by the fact that our main objective is to tackle the $0 < \beta < 1$ regime.

In all these references, $\beta$ is identified as a fundamental parameter, its position relative to 1 governing the possibility of collisions between the particles.

2 Main Results

The form (2) of the SDEs hints that $\alpha - (n - 1)\beta$ is a fundamental parameter impacting the existence of solutions. Consequently, we will study the range of values of this coefficient for which the system has a solution. For instance, if we assume $\alpha - (n - 1)\beta < 0$, we have:

$$\begin{align*}
d\lambda_i^1 &= 2\sqrt{\lambda_i^1} dB_i^1 + \left( \alpha - (n - 1)\beta - 2\gamma \lambda_i^1 + 2\beta \lambda_i^1 \sum_{j \neq i} \frac{1}{\lambda_i^1 - \lambda_j^1} \right) dt \\
&\leq 2\sqrt{\lambda_i^1} dB_i^1 - 2\gamma \lambda_i^1 dt.
\end{align*}$$

Then, according to the pathwise comparison theorem of Ikeda and Watanabe (that we recall in Theorem 6.2):

$$\lambda_i^1 \leq r_t \text{ a.s. for all } t \geq 0,$$

where

$$r_t = \lambda_0^1 + 2 \int_0^t \sqrt{r_s} dB_s^1 - 2 \int_0^t \gamma r_s ds \text{ for all } t \geq 0,$$

which is a CIR process (see, for instance, [21, Theorem 6.2.2]). By standard results on CIR processes recalled in Lemma 6.1, we can conclude that the stopping time $T = \inf\{t \geq 0 : r_t = 0\}$ verifies $\mathbb{P}(T < \infty) = 1$ for $\gamma \geq 0$, and $0 < \mathbb{P}(T < \infty) < 1$ for $\gamma < 0$. On $\{T < \infty\}$, after $T$, $r$ stays at zero indefinitely. As the drift in (11) is strictly negative when $\lambda_i^1 = 0$ and therefore when $r_t = 0$, it will stay strictly negative on a time interval of positive measure. Consequently, the system of SDEs has no global solution.

We thus proved the following result:

**Remark 2.1** A necessary condition for the existence of a global solution to (1) is $\alpha - (n - 1)\beta \geq 0$.

We will thus assume this condition in the remaining of the paper. This condition is of course not necessary in the framework of (10) (see [16]).

It is proved in [14, Corollary 8] with condition $0 \leq \lambda_0^1 < \cdots < \lambda_0^n$ that for $\beta \geq 1$, the SDEs (1) has a unique global strong solution and that there actually is no collision between particles.
Demni proved in [10, Sect. 5.1] by applying Cépa’s multivalued equations theory [8] that under the conditions
\[ \alpha - (n - 1)\beta > 1, \quad \beta > 0, \quad \gamma = 0 \quad \text{and} \quad 0 < \lambda_0^1 < \cdots < \lambda_0^n, \]
the system of SDEs (5) admits a unique strong solution, and that for \( 0 < \beta < 1 \) and \( \alpha - (n - 1)\beta > 1 \), there is collision between any neighbor particles \( \lambda^i \) and \( \lambda^{i+1} \) for \( i \in \{1, \ldots, n-1\} \) in finite time almost surely. The condition \( \alpha - (n - 1)\beta > 1 \) ensures the convexity of the potential \( V \) defined in (8) which is needed in Cépa’s multivalued equations theory. In Theorem 2.2 and Proposition 2.6, we tackle, respectively, the existence problem and the collision problem on wider ranges for the parameters \( \alpha - (n - 1)\beta \) and \( \gamma \).

Our following result applies in the case \( \alpha - (n - 1)\beta \leq 1 \):

**Theorem 2.2** Let us assume \( \beta < 1, \alpha - (n - 1)\beta > 0 \). Let the initial condition \( \Lambda_0 = (\lambda_0^1, \ldots, \lambda_0^n) \) be independent from the Brownian motion \( B \) such that \( 0 \leq \lambda_0^1 \leq \cdots \leq \lambda_0^n \) a.s. and \( \lambda_0^0 > 0 \) a.s.

Then, the system of SDEs (1) has a unique strong solution defined on the time interval
\[ [0, \lim_{\epsilon \to 0} \zeta_\epsilon), \]
where, for \( \epsilon > 0 \),
\[ \zeta_\epsilon = \inf \{ t \geq 0 : \lambda_t^1 \leq \epsilon \quad \text{and} \quad \lambda_t^2 - \lambda_t^1 \leq \epsilon \}. \]

Moreover,

(i) for \( \gamma \in \mathbb{R} \), if \( \alpha - (n - 1)\beta \geq 1 - \beta \) then \( \lim_{\epsilon \to 0} \zeta_\epsilon = \infty \) a.s.

(ii) for \( \gamma \geq 0 \), if \( \alpha - (n - 1)\beta < 1 - \beta \) then \( \lim_{\epsilon \to 0} \zeta_\epsilon < \infty \) a.s.

(iii) There is no double collision between particles, i.e.,
\[ \mathbb{P}\left[ \exists t \in (0, \lim_{\epsilon \to 0} \zeta_\epsilon) : \lambda_t^i = \lambda_t^{i+1} \quad \text{and} \quad \lambda_t^j = \lambda_t^{j+1} \quad \text{for some} \quad 1 \leq i < j \leq n - 1 \right] = 0. \]

Theorem 2.2 is proved in Sect. 4.

**Remark 2.3** This result states the existence of a unique strong solution \( (\lambda_t^1, \ldots, \lambda_t^n) \) to (1) defined on the time-interval \( \mathbb{R}_+ \) for \( \alpha - (n - 1)\beta \geq 1 - \beta \) and on \( [0, \lim_{\epsilon \to 0} \zeta_\epsilon) \) if \( 0 < \alpha - (n - 1)\beta < 1 - \beta \). When \( \lim_{\epsilon \to 0} \zeta_\epsilon < +\infty \), then, according to the first step in the proof of assertion (ii) of Proposition 2.4, the solution can be continuously extended to the closed time interval \( [0, \lim_{\epsilon \to 0} \zeta_\epsilon) \). From the definition of \( \zeta_\epsilon \), then \( \lambda_{\lim_{\epsilon \to 0} \zeta_\epsilon}^1 = \lambda_{\lim_{\epsilon \to 0} \zeta_\epsilon}^2 = 0 \). The next step would be to find how to start back from \( (\lambda_{\lim_{\epsilon \to 0} \zeta_\epsilon}^1, \ldots, \lambda_{\lim_{\epsilon \to 0} \zeta_\epsilon}^n) \) to define a solution on the whole interval \( \mathbb{R}_+ \).

The disjunction \( (i) - (ii) \) comes from the application with \( k = 2 \) of the following proposition, which gives a condition for \( k \) particles to collide at the position zero.
Proposition 2.4 (Multiple collision at zero) Let $k \in \{1, \ldots, n\}$. Let the initial condition $\Lambda_0 = (\lambda_0^1, \ldots, \lambda_0^n)$ be independent from the Brownian motion $B$ such that $0 \leq \lambda_0^1 \leq \cdots \leq \lambda_0^n$ a.s. Then,

(i) if $\gamma \geq 0$, (1) has a global solution $\Lambda = (\lambda_t^1, \ldots, \lambda_t^n)$, and $k(\alpha - (n - k)\beta) < 2$, then

$$\mathbb{P}(\exists t \geq 0 : \lambda_t^1 + \lambda_t^2 + \cdots + \lambda_t^k = 0) = 1,$$

(ii) if $\gamma \in \mathbb{R}$, $\lambda_0^k > 0$ a.s., (1) has a local solution $\Lambda = (\lambda_t^1, \ldots, \lambda_t^n)$ defined up to a stopping time $T$ and $k(\alpha - (n - k)\beta) \geq 2$, then there is no collision of $k$ particles at zero almost surely. More precisely,

$$\mathbb{P}(T = +\infty, \exists t \geq 0 : \lambda_t^1 + \lambda_t^2 + \cdots + \lambda_t^k = 0) + \mathbb{P}(T < +\infty, \inf_{t \in [0, T)} \lambda_t^1 + \lambda_t^2 + \cdots + \lambda_t^k = 0) = 0.$$

Remark 2.5 Under the assumptions made in the first assertion of Proposition 2.4 but with $\gamma < 0$, we can in fact prove that

$$\mathbb{P}(\exists t \geq 0 : \lambda_t^1 + \lambda_t^2 + \cdots + \lambda_t^k = 0) \in (0, 1),$$

using Lemma 6.1 point 3. As a consequence, under the assumptions made in (ii) of Theorem 2.2 but with $\gamma < 0$,

$$\mathbb{P}(\lim_{\epsilon \to 0} \zeta_\epsilon < \infty) \in (0, 1).$$

This proposition is proved in the beginning of Sect. 4.

Demni proved in [10] that for $\beta < 1$ and $\alpha - (n - 1)\beta > 1$, a collision between the particles $\lambda_i$ and $\lambda_i + 1$ occurs in finite time almost surely for all $i \in \{1, \ldots, n - 1\}$. We strengthen here this result by showing that for $\beta < 1$ and $\alpha - (n - 1)\beta > 0$, every particle touches its neighbor particles in finite time almost surely.

Proposition 2.6 Let $\beta \in (0, 1)$, $\alpha > 0$ and the initial condition $\Lambda_0 = (\lambda_0^1, \ldots, \lambda_0^n)$ be independent from the Brownian motion $B$ such that $0 \leq \lambda_0^1 \leq \cdots \leq \lambda_0^n$. Let us assume that either $\gamma > 0$ and there is a global solution to (1) or $\gamma \geq 0$, $\alpha - (n - 1)\beta \geq 1 - \beta$ and $\lambda_0^2 > 0$ a.s. (so that, by Theorem 2.2, there also exists a global solution to (1)). Then, for all $i \in \{2, \ldots, n\}$, the stopping time

$$T^{(i)} = \inf\{t > 0 : \lambda_t^i = \lambda_t^{i-1}\}$$

is such that

$$\mathbb{P}\left\{T^{(i)} < \infty\right\} = 1.$$
To check this result, we prove that
\[
\int_0^{+\infty} \lambda_1^s \, ds = +\infty \text{ a.s. (12)}
\]
using the next result.

**Proposition 2.7** Let us assume \( \gamma > 0 \) and \( \alpha - (n - 1)\beta > 0 \). The unique stationary probability measure of the system of SDEs (1) is \( \rho_{\text{inv}} \) with density with respect to the Lebesgue measure
\[
d\rho_{\text{inv}}(\lambda^1, \ldots, \lambda^n) = \frac{1}{Z} \times \prod_{i=1}^{n} \left( (\lambda^i)^{\frac{\alpha - (n - 1)\beta}{2}} - 1 \right) e^{-\gamma \lambda^i \prod_{j \neq i} |\lambda^j - \lambda^i|^{\beta/2}} 1_{\{0 \leq \lambda^1 \leq \cdots \leq \lambda^n\}} d\lambda^1 \ldots d\lambda^n,
\]
where \( Z \) is a normalizing constant. More precisely, if a solution to (1) is such that the distribution of \( \Lambda_t \) does not depend on \( t \), then this distribution is \( \rho_{\text{inv}} \). Conversely, (1) admits a unique solution \( \Lambda \) starting from \( \Lambda_0 \) distributed according to \( \rho_{\text{inv}} \) and independent from the Brownian motion \( B \). Moreover, for all \( t \in \mathbb{R}^+ \), \( \Lambda_t \) is distributed according to \( \rho_{\text{inv}} \).

**Remark 2.8** To prove (12), one could consider applying the ergodic theorem, but we were not able to prove that the process defined by the system of SDEs (1) is a Markov process. The difficulty came from the choice of the state space: when \( \lambda^2 = 0 \), we do not know how to prove the existence of a solution to (1). If the state space is \( \{0 \leq \lambda^1 \leq \cdots \leq \lambda^n, \lambda^2 > 0\} \) we do not know how to prove that for all \( t > 0 \), \( \mathbb{P}(\lambda^2_t > 0) = 1 \) when \( \alpha - (n - 1)\beta < 1 - \beta \).

**Remark 2.9** The density of \( \rho_{\text{inv}} \) generalizes one of the beta-Laguerre ensembles which writes
\[
\frac{1}{Z} \prod_{i=1}^{n} \left( (\lambda^i)^{\frac{m\beta - (n-1)\beta}{2}} - 1 \right) e^{-\frac{\beta}{2} \lambda^i \prod_{j \neq i} |\lambda^j - \lambda^i|^{\beta/2}} 1_{\{0 \leq \lambda^1 \leq \cdots \leq \lambda^n\}},
\]
by replacement of \( m\beta \) by \( \alpha \) in the exponent of \( \lambda^i \) and of \( \frac{\beta}{2} \) by \( \gamma \) in the exponential factor. The reader will find in [12] a documentation on beta-Laguerre ensembles. This ensemble is related to a random matrix model. The parameters \( m \) and \( n \) are then, respectively, the number of lines and columns of the underneath random matrix model, and \( \beta = 1, 2 \) or 4 depending on the dimension of the underlying algebra (\( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \)).

Table 1 shows, for \( \gamma \geq 0 \), the conditions on the coefficients of the SDEs (1) for the existence of strong solutions.
Table 1  Conditions on the coefficients of SDEs (1) for the existence of strong solutions when $\gamma \geq 0$

| $\alpha - (n-1)\beta$ | $\beta \geq 1$ | $\beta < 1$ |
|------------------------|----------------|----------------|
| $< 0$                  | Defined on $[0, \lim_{\epsilon \to 0} \inf \{t \geq 0 : \lambda^1_t \leq \epsilon\}$), no collision, Proposition 5.3 | Defined on $[0, \lim_{\epsilon \to 0} \inf \{t \geq 0 : \lambda^1_t \leq \epsilon\}$) which is finite a.s., Proposition 5.3 |
| $\in (0, 1-\beta)$    | empty interval | Defined on $[0, \lim_{\epsilon \to 0} \inf \{t \geq 0 : \lambda^1_t \leq \epsilon \}$ and $\lambda^2_t - \lambda^1_t < \epsilon\}$, which is finite a.s., Theorem 2.2 |
| $\geq 1 - \beta$      | Defined on $\mathbb{R}_+$, no collision between particles, proved in [14] with separate initial condition | Defined on $\mathbb{R}_+$, collision in finite time but no multiple collision at zero, Theorem 2.2 |
| $> 1$                  | Defined on $\mathbb{R}_+$, no collision between particles, proved in [14] with separate initial condition or in [10] with $\gamma = 0$ | Defined on $\mathbb{R}_+$, collision between particles in finite time, [10], Proposition 2.6 |
| $\geq 2$               | Defined on $\mathbb{R}_+$, no collision, proved in [14] with separate initial condition or in [10] with $\gamma = 0$ | Defined on $\mathbb{R}_+$, collision between particles in finite time, $\inf \{t > 0 : \lambda^1_t = 0\} = +\infty$ a.s., Proposition 2.4 |

3 Properties of the Solutions

One can first remark, and it will be useful in several proofs, that the sum of the $n$ coordinates of a solution $(\lambda^1_t, \ldots, \lambda^n_t)_t$ to SDEs (1) follows a CIR dynamics. Indeed,

$$
\begin{align*}
\frac{d}{dt}\left(\sum_{i=1}^{n} \lambda^i_t\right) &= 2\sum_{i=1}^{n} \lambda^i_t dB^i_t - 2\gamma \sum_{i=1}^{n} \lambda^i_t dt + n\alpha dt \\
&= 2\left(\sum_{i=1}^{n} \lambda^i_t dW_t - 2\gamma \sum_{i=1}^{n} \lambda^i_t dt + n\alpha dt\right) \\
&= 2\sqrt{\sum_{i=1}^{n} \lambda^i_t} dW_t - 2\gamma \sum_{i=1}^{n} \lambda^i_t dt + n\alpha dt, \\
\end{align*}
$$

(13)

where $W$ defined by $W_0 = 0$ and $dW_t = \sum_{i=1}^{n} \left(\frac{1}{\sqrt{\sum_{j=1}^{n} \lambda^j_t}} \right) dB^i_t$ is a Brownian motion according to Lévy’s characterization.

The pathwise uniqueness part of the next Lemma is proved in [15, Theorem 5.3], but we reproduce the proof for the sake of completeness.

Lemma 3.1 Let $\gamma \in \mathbb{R}$. The solutions to (1) are pathwise unique.

Moreover, if $Z = (z^1_t, \ldots, z^n_t)$ and $\tilde{Z} = (\tilde{z}^1_t, \ldots, \tilde{z}^n_t)_t$ are two global solutions to (1) with the same driving Brownian motion and verifying

$$
\mathbb{E} \sum_{i=1}^{n} \left| z^i_0 + \tilde{z}^i_0 \right| < +\infty,
$$

$\square$ Springer
then for all $t \geq 0$,

$$\sum_{i=1}^{n} \mathbb{E}|z^i_t - \tilde{z}^i_t| \leq \left( \sum_{i=1}^{n} \mathbb{E}|z^i_0 - \tilde{z}^i_0| \right) \exp(-2\gamma t). \quad (14)$$

**Proof** Let $Z$ and $\tilde{Z}$ be two solutions to (1). Because of the square root diffusion coefficient, the local time of $z^i_t - \tilde{z}^i_t$ at 0 is zero ([26, Lemma 3.3 p.389]). Applying the Tanaka formula and then the integration by parts formula to compute $de^{2\gamma s}|z^i_s - \tilde{z}^i_s|$, we get

$$d e^{2\gamma s} \sum_{i=1}^{n} |z^i_s - \tilde{z}^i_s| = 2 \sum_{i=1}^{n} e^{2\gamma s} \sqrt{z^i_s - \tilde{z}^i_s} dB^i_s + \beta e^{2\gamma s} \sum_{i=1}^{n} \text{sgn}(z^i_s - \tilde{z}^i_s) \sum_{j \neq i} \left( \frac{z^i_s + z^j_s}{z^i_s - z^j_s} \right) ds$$

$$\leq 2 \sum_{i=1}^{n} e^{2\gamma s} \sqrt{z^i_s - \tilde{z}^i_s} dB^i_s,$$

where $\text{sgn}(x) = 1$ if $x > 0$ and $\text{sgn}(x) = -1$ if $x \leq 0$. The inequality (15) comes from the fact that we have for all $i < j$:

$$\left[ \frac{z^i_s + z^j_s}{z^i_s - z^j_s} - \frac{\tilde{z}^i_s + \tilde{z}^j_s}{\tilde{z}^i_s - \tilde{z}^j_s} \right] (\text{sgn}(z^i_s - \tilde{z}^i_s) - \text{sgn}(z^j_s - \tilde{z}^j_s))$$

$$= 2 \frac{z^i_s (\tilde{z}^i_s - z^j_s) + z^j_s (\tilde{z}^i_s - z^i_s)}{(z^i_s - \tilde{z}^i_s)(\tilde{z}^i_s - z^j_s)} (\text{sgn}(z^i_s - \tilde{z}^i_s) - \text{sgn}(z^j_s - \tilde{z}^j_s))$$

$$= -2 \frac{z^i_s |\tilde{z}^i_s - z^j_s| + z^j_s |\tilde{z}^i_s - z^i_s|}{(z^i_s - \tilde{z}^i_s)(\tilde{z}^i_s - z^j_s)} |\text{sgn}(z^i_s - \tilde{z}^i_s) - \text{sgn}(z^j_s - \tilde{z}^j_s)|$$

$$\leq 0,$$

as the denominator is non-negative. Let the two solutions $Z$ and $\tilde{Z}$ be, respectively, defined on $[0, T]$ and $[0, \tilde{T}]$, where $T$ and $\tilde{T}$ are stopping times for a filtration $(\mathcal{F}_t)_{t \geq 0}$ with respect to which $B$ is a Brownian motion and $Z_0$ and $\tilde{Z}_0$ are $\mathcal{F}_0$-measurable.

Let $M > 0$ and

$$\tau_M = \inf \{ t \in [0, T \wedge \tilde{T}] : \tilde{z}^n_t + z^n_t \geq M \},$$

with the convention $\inf \emptyset = T \wedge \tilde{T}$. As $Z$ and $\tilde{Z}$ are continuous and assumed well defined on, respectively, $[0, T]$ and $[0, \tilde{T}]$, $\tau_M \uparrow T \wedge \tilde{T}$ when $M \uparrow \infty$.

As for all $i \in \{1, \ldots, n\}$:

$$\mathbb{E} \int_0^{t \wedge \tau_M} e^{4\gamma s} |\sqrt{z^i_s} - \sqrt{\tilde{z}^i_s}|^2 ds < \infty.$$
the stochastic integrals \( \int_{0}^{t \wedge \tau_{M}} e^{2\gamma s} \sqrt{z_{i}^{s} - \sqrt{\tilde{z}_{i}^{s}}} dB^{i}_{s} \) have zero expectation so that integrating (15) on \([0, t \wedge \tau_{M}]\) and taking expectations, we obtain that

\[
\mathbb{E} e^{2\gamma (t \wedge \tau_{M})} \sum_{i=1}^{n} |z_{i}^{t \wedge \tau_{M}} - \tilde{z}_{i}^{t \wedge \tau_{M}}| \leq \mathbb{E} \sum_{i=1}^{n} |z_{0}^{i} - \tilde{z}_{0}^{i}|.
\]

When \( Z_{0} = \tilde{Z}_{0} \), we deduce that for all \( t \geq 0 \) and \( M > 0 \)

\[
\sum_{i=1}^{n} \mathbb{E} |z_{i}^{t \wedge \tau_{M}} - \tilde{z}_{i}^{t \wedge \tau_{M}}| = 0,
\]

and conclude using Fatou’s Lemma to take the limit \( M \to \infty \), that for all \( t \geq 0 \)

\[
\sum_{i=1}^{n} \mathbb{E} |z_{i}^{t \wedge T \wedge \tilde{T}} - \tilde{z}_{i}^{t \wedge T \wedge \tilde{T}}| = 0,
\]

which concludes the proof of pathwise uniqueness.

Let us now prove (14) by assuming that \( Z \) and \( \tilde{Z} \) are defined globally, with integrable initial conditions, i.e.

\[
\mathbb{E} \sum_{i=1}^{n} \left[ z_{0}^{i} + \tilde{z}_{0}^{i} \right] < +\infty.
\]

As Eq. (13) shows that the sum of the coordinates of \( Z \) and \( \tilde{Z} \) is both CIR processes, we deduce from Lemma 6.1 point 4 that

\[
\mathbb{E} \left[ \int_{0}^{t} e^{4\gamma s} \left( \sum_{i=1}^{n} z_{i}^{s} + \sum_{i=1}^{n} \tilde{z}_{i}^{s} \right) ds \right] < \infty,
\]

so that \( \mathbb{E} \sum_{i=1}^{n} \int_{0}^{t} e^{2\gamma s} |\sqrt{z_{i}^{s}} - \sqrt{\tilde{z}_{i}^{s}}| dB^{i}_{s} = 0 \). Integrating (15) between 0 and \( t \) and taking expectations, we conclude that

\[
\sum_{i=1}^{n} \mathbb{E} |z_{i}^{t} - \tilde{z}_{i}^{t}| \leq \left( \sum_{i=1}^{n} \mathbb{E} |z_{0}^{i} - \tilde{z}_{0}^{i}| \right) \exp(-2\gamma t).
\]

\( \square \)

**Proof of Proposition 2.7** We start by proving the uniqueness of the invariant distribution. Let us first show that an invariant distribution \( \rho_{\text{inv}} \) has a finite first order moment. To do so, one can remark (see (13)) that the image of \( \rho_{\text{inv}} \) by the sum of the \( n \) coordinates is invariant for the CIR process

\[
dr_{t} = 2\sqrt{r_{t}} dW_{t} + (n\alpha - 2\gamma r_{t}) dt.
\]
It is known (see, for instance, [5]) that the invariant distribution of such a process is a gamma law of positive parameters, whose density is

\[ r \mapsto \frac{\gamma_{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} r^{\frac{\alpha}{2}-1} e^{-\gamma r}, \]

which has a finite first order moment. We can thus first apply the second part of Lemma 3.1 for two solutions to (1) starting, respectively, according to two invariant distributions to deduce that these two invariant distributions are equal.

The candidate density, obtained by the square of each coordinate change of variables from the density proportional to \( e^{-2V(x_1, \ldots, x^n)} \) with potential \( V \) defined in (8), candidate to be stationary for the gradient diffusion (5), writes

\[
    f_{\text{inv}}(\lambda^1, \ldots, \lambda^n) = \frac{e^{-2V(\sqrt{\lambda^1}, \ldots, \sqrt{\lambda^n})}}{Z\sqrt{\lambda^1 \cdots \lambda^n}}
    = \frac{1}{Z} \prod_{i=1}^{n} \left( (\lambda_i^{\alpha/2}) e^{-\gamma \lambda_i^1} \prod_{j \neq i} |\lambda_j - \lambda_i|^{\beta/2} \right)^{1 \{0 \leq \lambda_1^1 \leq \cdots \leq \lambda^n\}}
\]

with \( Z \) a normalizing constant. The second factor is indeed integrable since for \( 0 \leq \lambda^1 \leq \cdots \leq \lambda^n \),

\[
    \prod_{i=1}^{n} \left( (\lambda_i^{\alpha/2}) e^{-\gamma \lambda_i^1} \prod_{j \neq i} |\lambda_j - \lambda_i|^{\beta/2} \right) \leq (\lambda^n)^{(n-1)^2 \beta + \alpha - 2} e^{-\gamma \lambda^n} \prod_{i=1}^{n-1} \left( (\lambda_i^{\alpha/2}) e^{-\gamma \lambda_i^1} \right),
\]

and \( \alpha - (n-1)\beta > 0 \implies \frac{\alpha - 2 - (n-1)\beta}{2} > -1 \). Let us check that the probability measure \( \rho_{\text{inv}} \) with density \( f_{\text{inv}} \) with respect to the Lebesgue measure solves the Fokker–Planck equation in the sense of distributions

\[ \mathcal{A}^* \rho_{\text{inv}} = 0, \]

where \( \mathcal{A} \) is the infinitesimal generator associated with the dynamics (1):

\[
    \mathcal{A} = \sum_{i=1}^{n} \left( \alpha - 2\gamma \lambda_i^1 + \beta \sum_{j \neq i} \frac{\lambda_j^1 + \lambda_j^i}{\lambda_j^1 - \lambda_j^i} \right) \frac{\partial}{\partial \lambda_i^1} + 2 \sum_{i=1}^{n} \lambda_i^1 \frac{\partial^2}{\partial (\lambda_i^1)^2}. \]

For a test function \( \phi : \mathbb{R}^n \mapsto \mathbb{R} \), compactly supported and twice continuously differentiable, since \( f_{\text{inv}} \) vanishes for \( \lambda^1 = 0, \lambda^i = \lambda^{i+1} \) when \( i \in \{1, \ldots, n-1\} \),
and for $\lambda^n \to +\infty$, we obtain by integration by parts that for $i \in \{1, \ldots, n\}$

$$\int_{0 \leq \lambda^1 \leq \cdots \leq \lambda^n} \lambda^i \frac{\partial^2 \phi}{\partial (\lambda^i)^2} (\lambda^1, \ldots, \lambda^n) f_{inv}(\lambda^1, \ldots, \lambda^n) d\lambda^1 \ldots d\lambda^n$$

$$= -\int_{0 \leq \lambda^1 \leq \cdots \leq \lambda^n} \frac{\partial \phi}{\partial \lambda^i} (\lambda^1, \ldots, \lambda^n) \left( f_{inv}(\lambda^1, \ldots, \lambda^n) + \lambda^i \frac{\partial f_{inv}}{\partial \lambda^i} (\lambda^1, \ldots, \lambda^n) \right) d\lambda^1 \ldots d\lambda^n.$$

Since

$$\left( \alpha - 2\gamma \lambda^i + \beta \sum_{j \neq i} \frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j} - 2 \right) f_{inv}(\lambda^1, \ldots, \lambda^n) - 2\lambda^i \frac{\partial f_{inv}}{\partial \lambda^i} (\lambda^1, \ldots, \lambda^n)$$

$$= f_{inv}(\lambda^1, \ldots, \lambda^n) \left[ \alpha - 2\gamma \lambda^i + \beta \sum_{j \neq i} \frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j} - 2 \right. \right.$$  

$$+ 2\lambda^i \left( \frac{1}{2\lambda^i} + \frac{2}{2\sqrt{\lambda^i}} \frac{\partial V}{\partial \lambda^i}(\sqrt{\lambda^1}, \ldots, \sqrt{\lambda^n}) \right) \right.$$  

$$= f_{inv}(\lambda^1, \ldots, \lambda^n) \left[ \alpha - 2\gamma \lambda^i + \beta \sum_{j \neq i} \frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j} - 2 + 1 \right.$$  

$$+ 2\sqrt{\lambda^i} \left( -\frac{\alpha - 1}{2} \frac{1}{\sqrt{\lambda^i}} + \gamma \sqrt{\lambda^i} - \frac{\beta}{2\sqrt{\lambda^i}} \sum_{j \neq i} \frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j} \right) \right],$$

where the last factor vanishes, we conclude that $\int_{0 \leq \lambda^1 \leq \cdots \leq \lambda^n} A\phi(\lambda^1, \ldots, \lambda^n) f_{inv}(\lambda^1, \ldots, \lambda^n) d\lambda^1 \ldots d\lambda^n = 0$.

To deduce the existence of a weak solution to (1) whose marginals follow the law $\rho_{inv}$, we may apply [28, Theorem 2.5], as soon as

$$\int_{0 \leq \lambda^1 \leq \cdots \leq \lambda^n} \sum_{i=1}^n \left\{ 2\lambda^i + \left| \alpha - 2\gamma \lambda^i + \beta \sum_{j \neq i} \frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j} \right| \right\} d\rho_{inv}(\lambda^1, \ldots, \lambda^n) < +\infty.$$

This property can be proved by remarking that in the definition (17) of $f_{inv}$, the exponential factors ensure integrability at infinity while the factor $|\lambda^j - \lambda^i|^\beta/2$ makes the singularity of the denominator of the interaction term $\frac{\lambda^i + \lambda^j}{\lambda^i - \lambda^j}$ integrable. By pathwise uniqueness proved in Lemma 3.1, this weak solution to (1) is a strong solution. \qed
4 Proof of Theorem 2.2

We start this section by the proof of Proposition 2.4 since this result is crucial in the proof of Theorem 2.2.

Proof of Proposition 2.4 (i) To prove this assertion, we study for all \( k \in \{1, \ldots, n\} \) the process \( \lambda^{1} + \cdots + \lambda^{k} \) and show that, as its interaction terms with the particles \( \lambda^{k+1}, \ldots, \lambda^{n} \) are non-positive, it is smaller than a CIR process hitting zero in finite time.

Let us define \( W^{k} \) by \( W^{k}_{0} = 0 \) and

\[
dW^{k}_{t} = \sum_{i=1}^{k} \left( \mathbb{1}_{\left\{ \sum_{j=1}^{k} \lambda^{j}_{t} \neq 0 \right\}} \frac{\sqrt{\lambda^{i}_{t}}}{\sqrt{\sum_{j=1}^{k} \lambda^{j}_{t}}} \right) dB^{i}_{t} + \mathbb{1}_{\left\{ \sum_{j=1}^{k} \lambda^{j}_{t} = 0 \right\}} \frac{1}{\sqrt{k}} dB^{k}_{t}.
\]

According to Lévy’s characterization, \( W^{k} \) is a Brownian motion. For \( t \geq 0 \),

\[
d(\lambda^{1}_{t} + \cdots + \lambda^{k}_{t}) = 2\sqrt{\lambda^{1}_{t} + \cdots + \lambda^{k}_{t}} dW^{k}_{t} - 2\gamma (\lambda^{1}_{t} + \cdots + \lambda^{k}_{t}) dt + k(\alpha - (n - k)\beta) dt
\]

\[
+ 2\beta \sum_{i=1}^{k} \sum_{j=k+1}^{n} \frac{1}{\lambda^{i}_{t} - \lambda^{j}_{t}} dt
\]

\[
\leq 2\sqrt{\lambda^{1}_{t} + \cdots + \lambda^{k}_{t}} dW^{k}_{t} - 2\gamma (\lambda^{1}_{t} + \cdots + \lambda^{k}_{t}) dt + k(\alpha - (n - k)\beta) dt.
\] (18)

By the pathwise comparison theorem of Ikeda and Watanabe (that we recall in Theorem 6.2),

\[
\lambda^{1}_{t} + \cdots + \lambda^{k}_{t} \leq r_{t} \quad \text{for all } t \geq 0 \text{ a.s.,}
\]

where

\[
r_{t} = \lambda^{1}_{0} + \cdots + \lambda^{k}_{0} + 2 \int_{0}^{t} \sqrt{r_{s}} dW^{k}_{s} - 2\gamma \int_{0}^{t} r_{s} ds + k(\alpha - (n - k)\beta) t
\] (19)

is a CIR process. Applying Lemma 6.1 with \( a = k(\alpha - (n - k)\beta) \), \( b = 2\gamma \) and \( \sigma = 2 \), which satisfy \( a < \frac{\sigma^{2}}{2} \) and \( b \geq 0 \), we can conclude.

(ii) Before proving the assertion by backward induction on \( k \), let us first check that, whatever \( \alpha \geq 0, \beta > 0 \) and \( \gamma \in \mathbb{R} \), a local solution to \( (1) \) defined up to a stopping time \( T \) is actually continuous and solves \( (1) \) on the closed time interval \([0, T]\) on \( \{T < \infty\} \). For all \( k \in \{1, \ldots, n\} \), we define the Brownian motion \( W^{k} \) the
following way:

\[ dW^k_t = \mathbb{1}_{\{t \in [0, T)\}} \sum_{i=1}^{k} \left( \mathbb{1}_{\sum_{j=1}^{k} \lambda^j_i \neq 0} \frac{\sqrt{\lambda^j_i}}{\sqrt{\sum_{j=1}^{k} \lambda^j_i}} + \mathbb{1}_{\sum_{j=1}^{k} \lambda^j_i = 0} \frac{1}{\sqrt{k}} \right) dB^i_t \]

\[ + \mathbb{1}_{\{t \geq T\}} \frac{1}{\sqrt{k}} \sum_{i=1}^{k} dB^i_t. \]

For \( k = n \) the inequality (18) is an equality for \( t < T \), and according to Lemma 6.1, the CIR process \( r \) satisfying (19) is defined globally and for \( t \in [0, T) \) we have \( \lambda^1_r + \cdots + \lambda^n_r = r_t \). Moreover, for \( k \in \{1, \ldots, n\} \) and \( t \in [0, T) \),

\[ \sum_{i=1}^{k} \lambda^i_t = \sum_{i=1}^{k} \lambda^0_i + 2 \sum_{i=1}^{k} \int_{0}^{t} \sqrt{\lambda^i_s} dB^i_s \]

\[ -2\gamma \int_{0}^{t} \sum_{i=1}^{k} \lambda^i_s ds + k(\alpha - (n - k)\beta)t \]

\[ -2\beta \int_{0}^{t} \sum_{i=1}^{k} \lambda^i_s \sum_{j=k+1}^{n} \frac{1}{\lambda^j_s - \lambda^i_s} ds. \]

On \( \{T < \infty\} \), \( \int_{0}^{T} \sum_{i=1}^{n} \lambda^i_s ds = \int_{0}^{T} r_s ds < +\infty \) so that the second and third terms in the right-hand side are continuous functions of \( t \) on the time-interval \([0, T] \). Since the fourth term in the right-hand side is also obviously continuous, the integrand in the last term in the right-hand side is non-negative and the left-hand side is also non-negative, we deduce that \( \int_{0}^{T} \sum_{i=1}^{k} \lambda^i_s \sum_{j=k+1}^{n} \frac{1}{\lambda^j_s - \lambda^i_s} ds < +\infty \) so that the last term in the right-hand side is also continuous on the time interval \([0, T] \). Therefore, still on \( \{T < \infty\} \), \( (\lambda^1_t, \lambda^2_t + \lambda^2_t, \ldots, \lambda^n_t + \cdots + \lambda^n_t) \) admits a limit as \( t \to T^- \) and so does \( (\lambda^1_T, \lambda^2_T, \ldots, \lambda^n_T) \) as this limit, we conclude that (1) is satisfied on the closed time interval \([0, T] \) on \( \{T < \infty\} \).

Let us now prove the assertion by backward induction on \( k \) noticing that \( \min_{1 \leq t \leq n} \ell(\alpha - (n - \ell)\beta) = k(\alpha - (n - k)\beta) \). The idea is to show that the process \( \lambda^1 + \cdots + \lambda^k \) is not smaller than a CIR process which never hits zero. For \( k = n \), if \( n\alpha \geq 2 \), then Lemma 6.1 applied with \( a = n\alpha \), \( b = 2\gamma \) and \( \sigma = 2 \) which satisfy \( a \geq \frac{\sigma^2}{2} \), implies that the CIR process \( r \) defined by (19) and which coincides with \( \lambda^1 + \cdots + \lambda^n \) on \([0, T] \) does not hit 0.

Let us now assume that for some \( k \in \{1, \ldots, n-1\} \), \( k(\alpha - (n - k)\beta) \geq 2 \) and that the desired property holds for the sum of the \( k + 1 \) first coordinates. Since \( (k + 1)(\alpha - (n - k - 1)\beta) > k(\alpha - (n - k)\beta) \geq 2 \), the probability for the \( k + 1 \) smallest particles to collide at zero is null. Consequently, the probability for the \( k \) smallest particles to collide at zero is the probability for exactly the \( k \) smallest
particles to collide at zero and not the \((k + 1)\)th which stays away from zero. We thus have:

\[
\begin{align*}
\mathbb{P} \left( T = \infty, \exists t \geq 0 : \lambda_1^1 + \lambda_1^2 + \cdots + \lambda_1^k = 0 \right) \\
\quad = \lim_{\epsilon \downarrow 0} \mathbb{P} \left( T = \infty, \exists t \geq 0 : \lambda_1^1 + \lambda_1^2 + \cdots + \lambda_1^k = 0 \text{ and } \lambda_{k+1}^k - \lambda_k^k \geq \epsilon \right) \\
\quad = \lim_{\epsilon \downarrow 0} \mathbb{P} \left( T = \infty, \exists t \geq 0 : \lambda_1^1 + \lambda_1^2 + \cdots + \lambda_1^k + 1_{\{\lambda_{k+1}^k - \lambda_k^k < \epsilon\}} = 0 \right).
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{P} \left( T < \infty, \inf_{t \in [0, T)} (\lambda_1^1 + \cdots + \lambda_k^k) = 0 \right) \\
\quad = \lim_{\epsilon \downarrow 0} \mathbb{P} \left( T < \infty, \inf_{t \in [0, T)} (\lambda_1^1 + \cdots + \lambda_k^k + 1_{\{\lambda_{k+1}^k - \lambda_k^k < \epsilon\}}) = 0 \right).
\end{align*}
\]

For \(\epsilon > 0\), starting with

\[
\tau_0^\epsilon = \inf \{ t \in [0, T) : \lambda_{k+1}^k - \lambda_k^k \geq \epsilon \},
\]

let us define inductively for \(j \in \mathbb{N}\)

\[
\begin{align*}
\sigma_j^\epsilon & = \inf \{ t \in (\tau_j^\epsilon, T) : \lambda_{k+1}^k - \lambda_k^k \leq \frac{\epsilon}{2} \}, \\
\tau_{j+1}^\epsilon & = \inf \{ t \in (\sigma_j^\epsilon, T) : \lambda_{k+1}^k - \lambda_k^k \geq \epsilon \},
\end{align*}
\]

with the convention \(\inf \emptyset = T\). As the function \(t \mapsto \lambda_{k+1}^k - \lambda_k^k\) is continuous on \([0, T)\) and even on \([0, T]\) when \(T < +\infty\),

\[
\sigma_j^\epsilon, \tau_{j+1}^\epsilon \underset{j \to +\infty}{\longrightarrow} T \text{ and } \max \{ j \in \mathbb{N} : \tau_j^\epsilon < T \} < +\infty \text{ a.s. on } \{ T < \infty \}.
\]

As \(\lambda_{k+1}^k - \lambda_k^k < \epsilon\) on \([0, \tau_0^\epsilon)\) and \(\sigma_j^\epsilon, \tau_{j+1}^\epsilon\) for \(j \in \mathbb{N}\), we deduce that

\[
\begin{align*}
\{ T < \infty, \inf_{t \in [0, T)} (\lambda_1^1 + \cdots + \lambda_k^k + 1_{\{\lambda_{k+1}^k - \lambda_k^k < \epsilon\}}) = 0 \} \\
\quad = \{ T < \infty, \exists j \in \mathbb{N}^*, \inf_{t \in [\tau_j^\epsilon, \sigma_j^\epsilon)} (\lambda_1^1 + \cdots + \lambda_k^k) = 0 \}
\end{align*}
\]

and

\[
\begin{align*}
\{ T = \infty, \exists t \geq 0 : \lambda_1^1 + \lambda_1^2 + \cdots + \lambda_k^k + 1_{\{\lambda_{k+1}^k - \lambda_k^k < \epsilon\}} = 0 \} \\
\quad = \{ T = \infty, \exists j \in \mathbb{N}^*, \exists t \in [\tau_j^\epsilon, \sigma_j^\epsilon) : \lambda_1^1 + \lambda_1^2 + \cdots + \lambda_k^k = 0 \}.
\end{align*}
\]
Therefore, it is enough to check that
\[
\mathbb{P}\left( \left\{ T < +\infty, \exists j \in \mathbb{N}^*, \inf_{t \in [\tau_j^\epsilon, \sigma_j^\epsilon]} (\lambda_1^j + \cdots + \lambda_k^j) = 0 \right\} \right) + \mathbb{P}\left( \left\{ T = +\infty, \exists j \in \mathbb{N}^*, \exists t \in [\tau_j^\epsilon, \epsilon_j^\epsilon) : \lambda_1^j + \lambda_2^j + \cdots + \lambda_k^j = 0 \right\} \right) = 0.
\]

We have for all \( t \in [\tau_j^\epsilon, \epsilon_j^\epsilon) \), \( \lambda_k^{j+1} - \lambda_k^j > \frac{\epsilon}{2} \) and
\[
d(\lambda_1^j + \cdots + \lambda_k^j) = 2\sqrt{\lambda_1^j + \cdots + \lambda_k^j} dW_k^t - \gamma (\lambda_1^j + \cdots + \lambda_k^j) dt
\]
\[+ k(\alpha - (n - k)\beta) dt + 2\beta \sum_{i=1}^{k} \lambda_i^j \sum_{j=k+1}^{n} \frac{1}{\lambda_i^j - \lambda_j^j} dt
\]
\[\geq 2\sqrt{\lambda_1^j + \cdots + \lambda_k^j} dW_k^t - \gamma (\lambda_1^j + \cdots + \lambda_k^j) dt
\]
\[+ k(\alpha - (n - k)\beta) dt - \frac{4}{\epsilon} \beta (n - k) \sum_{i=1}^{k} \lambda_i^j dt
\]
\[\geq 2\sqrt{\lambda_1^j + \cdots + \lambda_k^j} dW_k^t - \left( \gamma + \frac{4}{\epsilon} \beta (n - k) \right) (\lambda_1^j + \cdots + \lambda_k^j) dt
\]
\[+ k(\alpha - (n - k)\beta) dt.
\]

We can then define on \( \{ \tau_j^\epsilon < \infty \} \) the process \( r_j^j \) by \( r_0^j = \left( \lambda_1^j_{\tau_j^\epsilon} + \cdots + \lambda_k^j_{\tau_j^\epsilon} \right) \)
\[\mathbb{1}_{\{ \tau_j^\epsilon < T \}} + \mathbb{1}_{\{ \tau_j^\epsilon = T \}} \] and for all \( t \geq 0 \) :
\[
dr_j^j = 2\sqrt{r_j^j} dW_k^{t+\tau_j^\epsilon} + \left[ - \left( \gamma + \frac{4}{\epsilon} \beta (n - k) \right) r_j^j + k(\alpha - (n - k)\beta) \right] dt
\]
\[= 2\sqrt{r_j^j} d(W_k^{t+\tau_j^\epsilon} - W_k^{\tau_j^\epsilon} + W_k^{\tau_j^\epsilon})
\]
\[+ \left[ - \left( \gamma + \frac{4}{\epsilon} \beta (n - k) \right) r_j^j + k(\alpha - (n - k)\beta) \right] dt
\]
\[= 2\sqrt{r_j^j} dW_k^t + \left[ - \left( \gamma + \frac{4}{\epsilon} \beta (n - k) \right) r_j^j + k(\alpha - (n - k)\beta) \right] dt,
\]
where conditionally on \( \{ \tau_j^\epsilon < \infty \} \), by the strong Markov property, \((\hat{W}_t^k = W_k^{t+\tau_j^\epsilon} - W_k^{\tau_j^\epsilon})_{t \geq 0}\) is a Brownian motion independent from \( \mathcal{F}_{\tau_j^\epsilon} \). Conditionally on \( \{ \tau_j^\epsilon < \infty \} \), the process \( r_j^j \) is a CIR process defined globally according to Lemma 6.1 with \( a = k(\alpha - (n - k)\beta) \) and \( \sigma = 2 \) which satisfy \( a \geq \frac{\sigma^2}{2} \), and it stays positive on \( \mathbb{R}_+ \).
We can conclude since by (20) and Theorem 6.2, for all \(t \in [\tau_j^\epsilon, \sigma_j^\epsilon)\),

\[
\lambda_t^1 + \cdots + \lambda_t^k \geq r_j^i \tau_{t \! - \! t_i^j}.
\]

\(\Box\)

**Proof of Theorem 2.2** As explained in introduction, the main difficulty in proving this result comes from the fact that we have to deal with both singularities when a particle hits zero and when two particles collide at the same time. For \(\epsilon > 0\), our method precisely consists in separating these difficulties by defining two new SDEs \((\hat{A}_\epsilon)\) and \((B_\epsilon)\) which each remove one type of singularity and coincide with (1) on domains that cover \(\{t \geq 0 : 0 \leq \lambda_t^1 \leq \cdots \leq \lambda_t^n \text{ and } \lambda_t^1 \vee (\lambda_t^2 - \lambda_t^1) \geq \epsilon\}\). This allows us to build a solution to (1) by piecing together solutions to \((\hat{A}_\epsilon)\) and \((B_\epsilon)\).

Let us consider in this proof the Brownian motion \(B = (B_1^t, \ldots, B_n^t)\), \(\mathcal{F}_t = \sigma((\lambda_0^1, \ldots, \lambda_0^n), (B_s^t)_{s \leq t})\) for all \(t \geq 0\), and the SDEs defined by (1). Let us define for all \(\epsilon > 0\) the following SDEs:

\[
d\hat{\lambda}_t^{i,\epsilon} = 2\sqrt{\hat{\lambda}_t^{i,\epsilon}} dB_t^i + \left[ \alpha - (n - 1)\beta + 1 - \left( 0 \lor \frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}} \left( \sqrt{\hat{\lambda}_t^{i,\epsilon}} - \frac{\sqrt{\epsilon}}{2\sqrt{\epsilon}} \right) \right) \land 1 \right] dt
\]

\[
-2\gamma \hat{\lambda}_t^{i,\epsilon} + 2\beta \sum_{j \neq i} \frac{1}{\hat{\lambda}_t^{i,\epsilon} - \hat{\lambda}_t^{j,\epsilon}} dt
\]

\(0 \leq \hat{\lambda}_t^{1,\epsilon} < \cdots < \hat{\lambda}_t^{n,\epsilon}\), a.s., \(dt \to a.e.\)

\[
\left\{ \begin{array}{l}
 d\tilde{\lambda}_t^{1,\epsilon} = 2\sqrt{\tilde{\lambda}_t^{1,\epsilon}} dB_t^1 + (\alpha - (n - 1)\beta) dt - 2\gamma \tilde{\lambda}_t^{1,\epsilon} dt \\
 - 2\beta \sum_{j \neq 1} \frac{\tilde{\lambda}_t^{1,\epsilon} \land \epsilon}{(\tilde{\lambda}_t^{j,\epsilon} - \tilde{\lambda}_t^{1,\epsilon} \land \epsilon) \lor \epsilon} dt \\
 \forall i \in \{2, \ldots, n\}, d\tilde{\lambda}_t^{i,\epsilon} = 2\sqrt{\tilde{\lambda}_t^{i,\epsilon}} dB_t^i + (\alpha - (n - 1)\beta + 1) dt - 2\gamma \tilde{\lambda}_t^{i,\epsilon} dt + 2\beta \sum_{j \geq 2, j \neq i} \frac{\tilde{\lambda}_t^{i,\epsilon}}{\tilde{\lambda}_t^{j,\epsilon} - \tilde{\lambda}_t^{1,\epsilon} \land \epsilon \lor \epsilon} dt - \left[ 0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{\tilde{\lambda}_t^{i,\epsilon}} - \frac{\sqrt{\epsilon}}{2\sqrt{\epsilon}} \right) \right] dt + 2\beta \frac{\tilde{\lambda}_t^{i,\epsilon}}{(\lambda_t^{1,\epsilon} - \tilde{\lambda}_t^{1,\epsilon} \land \epsilon) \lor \epsilon} dt \\
\end{array} \right. 
\]
\[ 0 \leq \tilde{\lambda}_{t}^{1,e} \text{ and } 0 \leq \tilde{\lambda}_{t}^{2,e} < \cdots < \tilde{\lambda}_{t}^{n,e}, \text{ a.s., } dt - a.e. \]

These systems are built such as:

\[ (\hat{A}_\epsilon) \text{ coincides with (1) on } \{ t, \hat{\lambda}_{t}^{1,e} \geq \epsilon/2 \}, \]
\[ (B_\epsilon) \text{ coincides with (1) on } \{ t, \hat{\lambda}_{t}^{1,e} \leq \epsilon \text{ and } \tilde{\lambda}_{t}^{2,e} - \tilde{\lambda}_{t}^{1} \geq \epsilon \}. \]

Lemmas 4.1 and 4.2 give existence of global pathwise unique strong solutions to \((\hat{A}_\epsilon)\) and \((B_\epsilon)\) for any random initial condition with ordered non-negative coordinates and independent from the driving Brownian motion.

For \( \xi \in \mathbb{R}^n_+ \) deterministic with ordered coordinates, let \( \hat{\Lambda}^{\epsilon,T,:,:}_\epsilon \) denote the process solution to \((\hat{A}_\epsilon)\) on \([T, +\infty)\) starting from \( \xi \) at time \( T \) and equal to 0 on \((-\infty, T)\). Likewise, \( \tilde{\Lambda}^{\epsilon,T,:,:}_\epsilon \) denotes the process solution to \((B_\epsilon)\) on \([T, +\infty)\) starting from \( \xi \) at time \( T \) and equal to 0 on \((-\infty, T)\). We define by induction:

\[
\tau_0^{\epsilon} = 0; \\
\text{let us note } \Lambda^{(0)} = (\lambda_t^{(0),1}, \ldots, \lambda_t^{(0),n})_t = \hat{\Lambda}^{\epsilon,0,\Lambda_0}; \\
\tau_1^{\epsilon} = \mathbb{1}_{\{\lambda_0^{(0),1} \geq \epsilon\}} \inf \left\{ t \geq 0 : \lambda_t^{(0),1} \leq \epsilon/2 \right\}; \\
\text{let us note } \Lambda^{(1)} = (\lambda_t^{(1),1}, \ldots, \lambda_t^{(1),n})_t = \hat{\Lambda}^{\epsilon,\tau_1^{\epsilon},\Lambda^{(0)}_1}; \\
\tau_2^{\epsilon} = \inf \left\{ t \geq \tau_1^{\epsilon} : \lambda_t^{(1),1} \geq \epsilon \right\}; \\
\vdots \\
\text{let us note } \Lambda^{(2j)} = (\lambda_t^{(2j),1}, \ldots, \lambda_t^{(2j),n})_t = \hat{\Lambda}^{\epsilon,\tau_{2j}^{\epsilon},\Lambda^{(2j-1)}_t}; \\
\tau_{2j+1}^{\epsilon} = \inf \left\{ t \geq \tau_{2j}^{\epsilon} : \lambda_t^{(2j),1} \leq \epsilon/2 \right\}; \\
\text{let us note } \Lambda^{(2j+1)} = (\lambda_t^{(2j+1),1}, \ldots, \lambda_t^{(2j+1),n})_t = \hat{\Lambda}^{\epsilon,\tau_{2j+1}^{\epsilon},\Lambda^{(2j)}_t}; \\
\tau_{2j+2}^{\epsilon} = \inf \left\{ t \geq \tau_{2j+1}^{\epsilon} : \lambda_t^{(2j+1),1} \geq \epsilon \right\}; \\
\vdots
\]

and as for all \( i \in \mathbb{N} \), the \( \tau_i^{\epsilon} \) defined before are stopping times for the filtration \( (\mathcal{F}_t)_{t \geq 0} \), the random vectors \( \mathbb{1}_{\{\tau_i^{\epsilon} < +\infty\}} \Lambda^{(i)}_t \) are \( \mathcal{F}_t^{\epsilon} \)-measurable, the construction makes sense.

We finally define for all \( \epsilon > 0 \) and \( t \geq 0 \):

\[
\mathcal{Z}_t^{\epsilon} = \mathbb{1}_{\{0 \leq t \leq \tau_1^{\epsilon}\}} \Lambda^{(0)}_t + \sum_{i=1}^{+\infty} \mathbb{1}_{\{\tau_i^{\epsilon} < t \leq \tau_{i+1}^{\epsilon}\}} \Lambda^{(i)}_t.
\]
For all \( i \in \mathbb{N} \) and for \( t \in [\tau_{2i+1}^\epsilon, \tau_{2i+2}^\epsilon) \), the equation for the smallest coordinate in \((B_\epsilon)\) and the non-negativity of \( \lambda_t^{(2i+1),1} \) give

\[
d\lambda_t^{(2i+1),1} \leq 2\sqrt{\lambda_t^{(2i+1),1}} dB_t^1 + (\alpha - (n - 1)\beta)dt. \tag{21}
\]

Then, according to the pathwise comparison theorem of Ikeda and Watanabe (that we recall in Theorem 6.2), for all \( i \in \mathbb{N}^* \) and for all \( t \in [\tau_{2i+1}^\epsilon, \tau_{2i+2}^\epsilon) \)

\[
\lambda_t^{(2i+1),1} \leq r_{2i+2} - r_{2i+1},
\]

where for all \( t \geq 0 \)

\[
r_{2i+2} = \frac{\epsilon}{2} + 2 \int_{\tau_{2i+1}^\epsilon}^{\tau_{2i+2}^\epsilon} \sqrt{r_s} dB_s^1 + (\alpha - (n - 1)\beta)t,
\]

which is a CIR process.

For each \( i \in \mathbb{N}^* \), the delay \( \tau_{2i+2} - \tau_{2i+1}^\epsilon \) is larger than the time needed by the CIR process \( r_{2i+2} \) to go from \( \frac{\epsilon}{2} \) to \( \epsilon \). Moreover, the times for the processes \( r_{2i+2} \) to go from \( \frac{\epsilon}{2} \) to \( \epsilon \) are iid positive random variables. Consequently, there is no accumulation of the stopping times \( \tau_{2i}^\epsilon \) which go to infinity as \( j \to \infty \).

The stochastic process \( Z^\epsilon \) is thus defined globally.

We recall that \((\hat{A}_\epsilon)\) and \((B_\epsilon)\), respectively, coincide with (1) when \( \hat{\lambda}_t^{1,\epsilon} \geq \frac{\epsilon}{2} \), and when \( \hat{\lambda}_t^{1,\epsilon} \leq \epsilon \) and \( \hat{\lambda}_t^{2,\epsilon} - \hat{\lambda}_t^{1,\epsilon} \geq \epsilon \). On the other hand, on \([\tau_{2i}^\epsilon, \tau_{2i+1}^\epsilon] \), \( Z^\epsilon \) evolves according to \((\hat{A}_\epsilon)\) and \( Z^{1,\epsilon} \geq \frac{\epsilon}{2} \) while on \([\tau_{2i+1}^\epsilon, \tau_{2i+2}^\epsilon] \), \( Z^\epsilon \) evolves according to \((B_\epsilon)\) and \( Z^{1,\epsilon} \leq \epsilon \). By induction on \( i \), we deduce that \( Z^\epsilon \) is a solution to (1) until

\[
\inf \left\{ t \in \bigcup_{i \in \mathbb{N}} [\tau_{2i+1}^\epsilon, \tau_{2i+2}^\epsilon] \mid Z_t^{2,\epsilon} - Z_t^{1,\epsilon} \leq \epsilon \right\} 
\geq \inf \{ t \geq 0 : Z_t^{1,\epsilon} \leq \epsilon \text{ and } Z_t^{2,\epsilon} - Z_t^{1,\epsilon} \leq \epsilon \} =: \zeta_\epsilon.
\]

From Lemmas 4.1 and 4.2, we have:

\[
\mathbb{P}\{\exists i \in \mathbb{N}, \exists t \in (\tau_{2i}^\epsilon, \tau_{2i+1}^\epsilon] : Z_t^{j,\epsilon} = Z_t^{j+1,\epsilon} \text{ and } Z_t^{k,\epsilon} = Z_t^{k+1,\epsilon} \text{ for some } 0 \leq j < k \leq n - 1 \} = 0,
\]

where by convention \( Z^{0,\epsilon} \equiv 0 \), and

\[
\mathbb{P}\{\exists i \in \mathbb{N}, \exists t \in (\tau_{2i+1}^\epsilon, \tau_{2i+2}^\epsilon] : Z_t^{j,\epsilon} = Z_t^{j+1,\epsilon} \text{ and } Z_t^{k,\epsilon} = Z_t^{k+1,\epsilon} \text{ for some } 2 \leq j < k \leq n - 1 \} = 0.
\]
On the time intervals \([\tau_{2i+1}^\varepsilon \land \zeta \varepsilon, \tau_{2i+2}^\varepsilon \land \zeta \varepsilon]\), we have \(Z_t^{i,\varepsilon} = Z_{t}^{i+1,\varepsilon}\) and \(Z_t^{j,\varepsilon} = Z_{t}^{j+1,\varepsilon}\) for some \(1 \leq i < j \leq n-1\). Consequently,

\[
P\{\exists t \in (0, \zeta \varepsilon) : Z_t^{i,\varepsilon} = Z_{t}^{i+1,\varepsilon} \text{ and } Z_t^{j,\varepsilon} = Z_{t}^{j+1,\varepsilon} \text{ for some } 1 \leq i < j \leq n-1\} = 0. \tag{22}
\]

As the solutions to Eq. (1) are pathwise unique (see Lemma 4.2), for \(n \in \mathbb{N}^\ast\), the processes \(Z_{t}^{1/\varepsilon}\) and \(Z_{t}^{n/\varepsilon}\) coincide on \(\left[0, \zeta_{1/\varepsilon} \land \zeta_{n/\varepsilon}\right]\). Thus, \(\zeta_{1/\varepsilon} \land \zeta_{n/\varepsilon} = \zeta_{1/\varepsilon}\) and the sequence \((\zeta_{1/\varepsilon})_{n \in \mathbb{N}^\ast}\) is non-decreasing. Moreover, for all \(n \in \mathbb{N}^\ast\), \(Z_{t}^{n/\varepsilon}\) verifies (22).

Consequently, we can define for all \(t \in [0, \lim_{\varepsilon \to 0} \zeta \varepsilon)\)

\[
\Lambda_t = Z_t^{1/\varepsilon} \mathbb{1}_{\{0 \leq t \leq \zeta_{1/\varepsilon}\}} + \sum_{n \geq 1} Z_t^{1/\varepsilon} \mathbb{1}_{\left\{\zeta_{1/\varepsilon} < t \leq \zeta_{1/\varepsilon} + \lambda_{n/\varepsilon}^{-1}\right\}}. \tag{23}
\]

which is a solution to the SDEs (1) on \([0, \lim_{\varepsilon \to 0} \zeta \varepsilon)\) verifying \((iii)\) of Theorem 2.2.

Finally, as the solutions to (1) are pathwise unique (Lemma 3.1), we can apply the Yamada–Watanabe theorem (see, for instance, [26, Theorem 1.7 p.368]) to deduce the existence of strong solutions to the equation.

Since on \(\{\zeta \varepsilon < +\infty\}, \lambda_{1/\varepsilon}^1 + \lambda_{1/\varepsilon}^2 = 2\lambda_{1/\varepsilon}^1 + \lambda_{1/\varepsilon}^2 - \lambda_{1/\varepsilon}^1 \leq 3\varepsilon\), on \(\lim_{\varepsilon \to 0} \zeta \varepsilon < +\infty\) we have \(\inf_{t \in [0, \lim_{\varepsilon \to 0} \zeta \varepsilon)} \lambda_{t}^1 + \lambda_{t}^2 = 0\). We then use Proposition 2.4 \((ii)\) with \(k = 2\) to conclude that \(\lim_{\varepsilon \to 0} \zeta \varepsilon = +\infty\) when \(\alpha - (n - 1)\beta \geq 1 - \beta\) which is \((i)\) from Theorem 2.2.

For \(\alpha - (n - 1)\beta < 1 - \beta\) and \(\mu \geq 0\), let us prove assertion \((ii)\) of Theorem 2.2.

Following the steps of the proof of Proposition 2.4 with \(k = 2\) until (18), we have for all \(0 \leq t < \lim_{\varepsilon \to 0} \zeta \varepsilon:\)

\[
d(\lambda_{t}^1 + \lambda_{t}^2) \leq 2\sqrt{\lambda_{t}^1 + \lambda_{t}^2} dW_{t}^2 - 2\gamma(\lambda_{t}^1 + \lambda_{t}^2) dt + 2(\alpha - (n - 2)\beta) dt,
\]

where by Lévy’s characterization, \(W^2\) defined by \(W_0^2 = 0\) and \(dW_t^2 = \mathbb{1}_{\left\{0 \leq t < \lim_{\varepsilon \to 0} \zeta \varepsilon\right\}} \frac{\sqrt{\lambda_{t}^1 dB_{t}^1 + \sqrt{\lambda_{t}^2 dB_{t}^2}}}{\sqrt{\lambda_{t}^1 + \lambda_{t}^2}} + \mathbb{1}_{\left\{t \geq \lim_{\varepsilon \to 0} \zeta \varepsilon\right\}} \sqrt{\frac{\lambda_{t}^1 + \lambda_{t}^2}{2}}\) is a Brownian motion.

By the pathwise comparison theorem of Ikeda and Watanabe (that we recall in Theorem 6.2),

\[
\lambda_{t}^1 + \lambda_{t}^2 \leq r_t \text{ for all } 0 \leq t < \lim_{\varepsilon \to 0} \zeta \varepsilon \text{ a.s.}
\]

where for all \(t \geq 0\)

\[
r_t = \lambda_{0}^1 + \lambda_{0}^2 + 2 \int_{0}^{t} \sqrt{r_s} dW_s^2 - 2\gamma \int_{0}^{t} r_s ds + 2(\alpha - (n - 2)\beta) t
\]
is a CIR process. Applying Lemma 6.1 with $a = 2(\alpha - (n - 2)\beta)$, $b = 2\gamma$ and $\sigma = 2$ which satisfy $a < \frac{\sigma^2}{2}$ and $b \geq 0$, we conclude that the hitting time of 0 by $r$ is finite almost surely. Consequently, $\mathbb{P}\left(\lim_{\epsilon \to 0} \zeta_\epsilon = +\infty\right) = 0$ which concludes the proof of Theorem 2.2 (ii). \hfill \Box

**Lemma 4.1** Let us assume $\alpha - (n - 1)\beta > 0$. The system

$$d\hat{\Lambda}^{i,\epsilon}_t = 2\sqrt{\hat{\Lambda}^{i,\epsilon}_t} d\hat{B}_t^i + \left[\alpha - (n - 1)\beta + 1 - 0 \lor \frac{2\sqrt{2}}{\sqrt{\epsilon}} \left(\sqrt{\hat{\Lambda}^{i,\epsilon}_t} - \frac{\sqrt{\epsilon}}{2\sqrt{2}}\right) \right]_t$$

$$(\hat{\Lambda}_\epsilon)$$

$$_{0 \leq \hat{\Lambda}^{1,\epsilon}_t < \ldots < \hat{\Lambda}^{n,\epsilon}_t, \ a.s., \ dt - a.e.}$$

has a global pathwise unique strong solution $(\hat{\lambda}^{1,\epsilon}_i, \ldots, \hat{\lambda}^{1,\epsilon}_n)$ starting from any random initial condition $\hat{\Lambda}_0 = (\lambda^{1,0}_1, \ldots, \lambda^{n,0}_n)$ independent from $B$ such that $0 \leq \lambda^{1,0}_1 \leq \cdots \leq \lambda^{n,0}_n$ a.s.

Moreover,

$$\mathbb{P}\{\exists t > 0 : \hat{\Lambda}^{i,\epsilon}_t = \hat{\Lambda}^{i+1,\epsilon}_t \text{ and } \hat{\Lambda}^{j,\epsilon}_t = \hat{\Lambda}^{j+1,\epsilon}_t \text{ for some } 0 \leq i < j \leq n - 1\} = 0,$$

where by convention $\hat{\Lambda}^{0,\epsilon}_t \equiv 0$.

**Proof** Let us consider $F_t = \sigma \left(\left(\sqrt{\lambda^{1,0}_1}, \ldots, \sqrt{\lambda^{n,0}_n}\right), \left(B_s\right)_{s \leq t}\right)$ and the system of SDEs defined by

$$dx^{i,\epsilon}_t = dB^{i}_t + \frac{\alpha - (n - 1)\beta - 0 \lor \frac{2\sqrt{2}}{\sqrt{\epsilon}} \left(x^{i,\epsilon}_t - \frac{\sqrt{\epsilon}}{2\sqrt{2}}\right) \lor 1}{2x^{i,\epsilon}_t} dt - \gamma x^{i,\epsilon}_t dt$$

$$(A_\epsilon)$$

for $i \in \{1, \ldots, n\}$

$$_{0 \leq x^{1,\epsilon}_t < \ldots < x^{n,\epsilon}_t, \ a.s., \ dt - a.e.}$$

with random initial condition $(\sqrt{\lambda^{1,0}_1}, \ldots, \sqrt{\lambda^{n,0}_n})$ such that $0 \leq \sqrt{\lambda^{1,0}_1} \leq \cdots \leq \sqrt{\lambda^{n,0}_n}$.

We are going to apply Cepa’s multivalued equations theory ([8]) to conclude that there exists a unique strong solution to $(A_\epsilon)$. \hfill \endproof

\textcopyright Springer
To do so, we define

\[ D = \{0 < x^1 < x^2 < \cdots < x^n\}, \]

\[ \Phi_{\gamma} : (x^1, \ldots, x^n) \in \mathbb{R}^n \rightarrow \begin{cases} -\sum_{i=1}^{n} \left[ \frac{\alpha - (n-1)\beta}{2} \ln |x^i| \right] \\
\quad -\frac{\gamma}{2}(x^i)^2 + \frac{\beta}{4} \sum_{j \neq i} (\ln |x^i - x^j|) \\
\quad + \ln |x^i + x^j| \end{cases} \text{ if } x \in D \\
\quad +\infty \text{ if } x \notin D, \]

\[ g : (x^1, \ldots, x^n) \in \mathbb{R}^n \rightarrow \begin{pmatrix} -\frac{1}{2} \sqrt{\lambda_1} \left( x^1_{t, \epsilon} - \sqrt{\epsilon} \right) \wedge 1 \\
-\frac{1}{2} \sqrt{\lambda_n} \left( x^n_{t, \epsilon} - \sqrt{\epsilon} \right) \wedge 1 \end{pmatrix}, \]

for all \( t \geq 0 \), \( \tilde{X}_t \in \bar{D} \) a.s. \( \tilde{X}_0 = (\sqrt{\lambda_1^0}, \ldots, \sqrt{\lambda_n^0}) \).

where \( \tilde{X} \) is a continuous adapted to \((\mathcal{F}_t)_{t \geq 0}\) process, \( L \) is a continuous non-decreasing adapted to \((\mathcal{F}_t)_{t \geq 0}\) process with \( L_0 = 0 \) verifying

\[ L_t = \int_0^t 1_{\{\tilde{X}_s \in \partial D\}} dL_s, \]

and \( \nu(x) \in \pi(x) \) (\( \pi(x) \) is the set of unitary outward normals to \( \partial D \) at \( x \in \partial D \)). The solution to equation \((A_\epsilon)\) follows the conditions : for all \( t > 0 \)

\[ \mathbb{E} \left[ \int_0^t 1_{\{\tilde{X}_s \in \partial D\}} ds \right] = 0, \]

\[ \mathbb{E} \left[ \int_0^t |\nabla \Phi_{\gamma}(\tilde{X}_s)| ds \right] < \infty. \]
Let us now prove that the boundary process $L$ is equal to zero.

For all $m \in \{1, \ldots, n\}$, for all $t \geq 0$, we have with $C = \frac{\alpha - (n-1)\beta}{2}$:

$$
\mathbb{E} \int_0^t \left| \frac{C}{x_{s,m,e}^m} + \beta x_{s,m,e}^m \sum_{j \neq m} \frac{1}{(x_{s,m,e}^m)^2 - (x_{s,j,e}^m)^2} \right| ds < \infty. \tag{25}
$$

Let us prove by backward induction on $m$ that

for all $1 < m \leq n$ and for all $t \geq 0$, $\mathbb{E} \int_0^t \left| \frac{1}{x_{s,m,e}^m} \right| + \sum_{l < m} \left| \frac{x_{s,m,e}^m}{(x_{s,m,e}^m)^2 - (x_{s,l,e}^m)^2} \right| ds < \infty.

(H_m)

• $m = n$

As all the terms in the absolute value of (25) have the same sign for $m = n$, we deduce the individual integrability.

• Let $1 < m \leq n$ and let us assume $(H_j)$ for all $j \in \{m + 1, \ldots, n\}$.

We have:

$$
\frac{C}{x_{s,m,e}^m} + \beta x_{s,m,e}^m \sum_{j \neq m} \frac{1}{(x_{s,m,e}^m)^2 - (x_{s,j,e}^m)^2} = \frac{C}{x_{s,m,e}^m} + \beta x_{s,m,e}^m \sum_{j < m} \frac{1}{(x_{s,m,e}^m)^2 - (x_{s,j,e}^m)^2} - \beta \sum_{j > m} \frac{x_{s,m,e}^m}{(x_{s,j,e}^m)^2 - (x_{s,m,e}^m)^2}.
$$

Let us remark that for all $s \geq 0$

$$
\frac{C}{x_{s,m,e}^m} + \beta x_{s,m,e}^m \sum_{j < m} \frac{1}{(x_{s,m,e}^m)^2 - (x_{s,j,e}^m)^2} \leq \frac{C}{x_{s,m,e}^m} + \beta x_{s,m,e}^m \sum_{j \neq m} \frac{1}{(x_{s,m,e}^m)^2 - (x_{s,j,e}^m)^2} + \beta \sum_{j > m} \frac{x_{s,j,e}^m}{(x_{s,j,e}^m)^2 - (x_{s,m,e}^m)^2}, \tag{26}
$$

by the triangle inequality and since $x_{s,j,e}^m \geq x_{s,m,e}^m$ for $j > m$.

By (25) and the induction hypothesis for $j \in \{m + 1, \ldots, n\}$, each term in the right-hand side of (26) is integrable, which ends the induction argument.

Consequently, for all $1 \leq l < m \leq n$ and for all $t \geq 0$ we have

$$
\mathbb{E} \int_0^t \frac{1}{x_{s,m,e}^m - x_{s,l,e}^l} ds = \mathbb{E} \int_0^t \frac{x_{s,m,e}^m + x_{s,l,e}^l}{(x_{s,m,e}^m)^2 - (x_{s,l,e}^l)^2} ds \leq 2 \mathbb{E} \int_0^t \frac{x_{s,m,e}^m}{(x_{s,m,e}^m)^2 - (x_{s,l,e}^l)^2} ds < \infty. \tag{27}
$$
As in the second part of the proof of [7, Theorem 2.2] (equation (2.40)), using the occupation times formula and (27), we have for \(1 \leq l < m \leq n, t \geq 0\)

\[
\int_{0}^{+\infty} \frac{L_{t}^{a}(x_{m,e} - x_{l,e})}{a} da = \int_{0}^{t} \frac{d(x_{m,e} - x_{l,e})_{s}}{x_{s}^{m,e} - x_{s}^{l,e}} ds < +\infty,
\]

and

\[
\int_{0}^{+\infty} \frac{L_{t}^{a}(x_{1,e})}{a} da = \int_{0}^{t} \frac{d(x_{1,e})_{s}}{x_{s}^{1,e}} = \int_{0}^{t} \frac{1}{x_{s}^{1,e}} ds < +\infty,
\]

where \(L_{t}^{a}(\mathcal{X})\) is the local time at time \(t\) and on level \(a\) for a real continuous semi-martingale \(\mathcal{X}\). Since the function \(a \mapsto \frac{1}{a}\) is not integrable at 0 and \((L_{t}^{a}(\mathcal{X}))\) is cadlag at \(a\) by [26, Theorem 1.7 p.225], one deduces that \(L_{t}^{0}(x_{m,e} - x_{l,e}) = L_{t}^{0}(x_{1,e}) = 0\).

From there, the reasoning made in the proof of [7, Theorem 2] allows to conclude that the boundary process \(L\) is equal to zero.

Then, setting \(\hat{A}^{i,e} = (x^{i,e})^{2}\) for all \(i \in \{1, \ldots, n\}\) we obtain a global solution to \((\hat{A}_{e})\). Following the approach used in the proof of Lemma 4.2 to demonstrate pathwise uniqueness for the slightly more complicated system \((B_{e})\), we obtain that the solutions to \((\hat{A}_{e})\) are pathwise unique. The Yamada–Watanabe Theorem (see, for instance, [26, Theorem 1.7 p.368]) allows to conclude that \((\hat{A}_{e})\) has a pathwise unique global strong solution.

Let us now prove (24).

Let us consider the system defined by \((A_{e})\) with initial condition \(0 \leq \sqrt{\lambda_{1,0}^{1}} \leq \cdots \leq \sqrt{\lambda_{1,0}^{n}}\).

Let us define for all \(\epsilon > 0, M > 0\)

\[
\tau_{M} = \inf\{t \geq 0 : \exists i \in \{1, \ldots, n\}, x_{i,\epsilon}^{t} \geq M\},
\]

and for \(t \in [0; \tau_{M}) : \Theta(t) = (\theta_{1}(t), \ldots, \theta_{n}(t))\) with

\[
\forall i \in \{1, \ldots, n\}, \theta_{i}(t) = -\frac{0 \lor \frac{2\sqrt{\epsilon}}{\sqrt{\epsilon}} \left(x_{i,\epsilon}^{t} - \frac{\epsilon}{2\sqrt{2}}\right) \land 1}{2x_{i,\epsilon}^{t}} + \gamma x_{i,\epsilon}^{t},
\]

and for all \(t \geq 0\)

\[
Z(t) = \exp\left\{-\int_{0}^{t \wedge \tau_{M}} \Theta(u) \cdot dB_{u} - \frac{1}{2} \int_{0}^{t \wedge \tau_{M}} ||\Theta(u)||^{2} du\right\}.
\]
We have for all \( i \in \{1, \ldots, n\} \),

\[
\theta_i^2(t) = \left( -\frac{0 \vee \frac{2\sqrt{2}}{\sqrt{\epsilon}} \left( x_{i,e}^t - \frac{\sqrt{\epsilon}}{2\sqrt{2}} \right) \wedge 1}{2x_{i,e}^t} + \gamma x_{i,e}^t \right)^2
\]

\[
\leq 2 \left( -\frac{0 \vee \frac{2\sqrt{3}}{\sqrt{\epsilon}} \left( x_{i,e}^t - \frac{\sqrt{\epsilon}}{2\sqrt{2}} \right) \wedge 1}{2x_{i,e}^t} \right)^2 + 2(\gamma x_{i,e}^t)^2
\]

\[
\leq \frac{1}{\epsilon} + 2\gamma^2 M^2.
\]

We thus have

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^{t \wedge \tau_M} \|\Theta(u)\|^2 du \right\} \right] < \infty \text{ for all } t \geq 0.
\]

Then, according to Novikov’s criterion (see, for instance, [18, Proposition 5.12 p.198]), \( Z \) is a \( \mathbb{P} \)-martingale, and \( \mathbb{E}[Z(t)] = 1 \). Consequently, recalling that \( \mathcal{F}_t = \sigma\left(\sqrt{\lambda_0}, \ldots, \sqrt{\lambda_0}, (B_s)_{s \leq t}\right) \) and defining \( Q \) such that

\[
\frac{dQ}{d\mathbb{P} | \mathcal{F}_t} = Z(t),
\]

and

for all \( i \in \{1, \ldots, n\} \), \( \tilde{\mathcal{B}}^{i,M}_t = B^{i,M}_t + \int_0^{t \wedge \tau_M} \theta_i(s) ds \)

\[
= B^{i,M}_t - \int_0^{t \wedge \tau_M} 0 \vee \frac{2\sqrt{3}}{\sqrt{\epsilon}} \left( x_{i,e}^s - \frac{\sqrt{\epsilon}}{2\sqrt{2}} \right) \wedge 1\]

\[
+ \gamma x_{i,e}^s ds, \; t \geq 0,
\]

\( \tilde{\mathcal{B}}^M = (\tilde{\mathcal{B}}^{1,M}, \ldots, \tilde{\mathcal{B}}^{n,M})_t \) is a \( Q \)-Brownian motion according to the Girsanov theorem (see, for instance, [18, Proposition 5.4 p.194]).

Thus, \( (A_\epsilon) \) can be rewritten in terms of \( \tilde{\mathcal{B}}^M \) as

\[
dx_{i,e}^t = d\tilde{\mathcal{B}}^{i,M}_t + \frac{\alpha - (n - 1)\beta}{2x_{i,e}^t} dt - \gamma x_{i,e}^t \mathbb{1}_{\{t \geq \tau_M\}} dt
\]

\[
+ \beta x_{i,e}^t \sum_{j \neq i} \frac{dt}{(x_{i,e}^t)^2 - (x_{j,e}^t)^2} \text{ for } i \in \{1, \ldots, n\}.
\]
By the same arguments as in the beginning of the proof, the normally reflected SDE
\[
\begin{align*}
&d\tilde{X}_t = d\tilde{B}^M_t - \nabla \Phi_0(\tilde{X}_t)dt - \nu(\tilde{X}_t)d\tilde{L}_t \quad \text{for all } t \geq 0 \\
&\forall t \geq 0, \; \tilde{X}_t \in \tilde{D} \; \text{a.s.,} \\
&\tilde{X}_0 = (\sqrt{\lambda^1_0}, \ldots, \sqrt{\lambda^n_0})
\end{align*}
\]

admits a global solution and the term \(\nu(\tilde{X}_t)d\tilde{L}_t\) is zero. We can apply [22, Theorem 3.1] to the SDE (29) to conclude that its solutions cannot have multiple collisions. This last SDE can be rewritten
\[
\begin{align*}
d\tilde{x}_i^{i,e} &= d\tilde{B}_i^M + \frac{\alpha - (n - 1)\beta}{2\tilde{x}_i^{i,e}}dt + \beta\tilde{x}_i^{i,e} \sum_{j \neq i} \frac{dt}{(\tilde{x}_i^{i,e})^2 - (\tilde{x}_j^{i,e})^2} \quad \text{for } i \in \{1, \ldots, n\} \\
0 \leq \tilde{x}_1^{1,e} < \cdots < \tilde{x}_n^{n,e}, \; \text{a.s., } dt - a.e.
\end{align*}
\]

By pathwise uniqueness deduced from the application of Lemma 3.1 after the square of each coordinate change of variable, the solutions to this last system of SDEs coincide with the solutions to \((\tilde{A}_\epsilon, M)\) on \([0, \tau_M]\), which implies that there is no collision of \((x_1^{i,e}, \ldots, x_n^{n,e})\) on \([0, \tau_M]\) under the probability \(\mathbb{Q}\). There is thus no multiple collision of \((x_1^{i,e}, \ldots, x_n^{n,e})\) under the probability \(\mathbb{P}\) on \([0, \tau_M]\):
\[
\mathbb{P}\{\exists t \in [0, \tau_M) : \tilde{x}_i^{i,e} = \tilde{x}_i^{i+1,e} \text{ and } \tilde{x}_j^{j,e} = \tilde{x}_j^{j+1,e} \text{ for some } 0 \leq i < j \leq n - 1\} = 0.
\]

Since it is true for all \(M > 0\), and since as the system of SDEs \((A_\epsilon)\) admits a continuous global solution, \(\tau_M \xrightarrow{M \to +\infty} +\infty \mathbb{P} - a.s.,\) we have the result for \((A_\epsilon)\), and thus for \((\hat{A}_\epsilon)\), which concludes the proof. \(\square\)

**Lemma 4.2** Let us assume \(\alpha - (n - 1)\beta > 0\). The system of SDEs \((B_\epsilon)\) with random initial condition \((\tilde{\lambda}_1^{1,e}, \ldots, \tilde{\lambda}_n^{n,e})\) such that \(0 \leq \tilde{\lambda}_1^{1,e} \leq \cdots \leq \tilde{\lambda}_n^{n,e} \) a.s. and independent from the Brownian motion \(B = (B_1^1, \ldots, B_n^1)\), has a global pathwise unique strong solution \((\tilde{\lambda}_1^{1,e}, \ldots, \tilde{\lambda}_n^{n,e})\).

Moreover,
\[
\mathbb{P}\{\exists t > 0 : \tilde{\lambda}_i^{i,e} = \tilde{\lambda}_i^{i+1,e} \text{ and } \tilde{\lambda}_j^{j,e} = \tilde{\lambda}_j^{j+1,e} \text{ for } 2 \leq i < j \leq n - 1\} = 0. \quad (30)
\]
Proof Let us consider a Brownian motion \( \tilde{B} = (\tilde{B}^1_t, \ldots, \tilde{B}^n_t) \) and the system

\[
\begin{align*}
    d\tilde{\lambda}_i^{1,e} &= 2\sqrt{\tilde{\lambda}_i^{1,e}} d\tilde{B}^1_t + (\alpha - (n-1)\beta)dt - 2\gamma \tilde{\lambda}_i^{1,e} dt \\
    \forall i \in \{2, \ldots, n\}, d\tilde{\lambda}_i^{i,e} &= 2\sqrt{\tilde{\lambda}_i^{i,e}} d\tilde{B}^i_t + (\alpha - (n-1)\beta + 1)dt \\
    &\quad - \left[ 0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{\tilde{\lambda}_i^{i,e} - \frac{\epsilon}{2}} \right) \land 1 \right] dt \\
    &\quad - 2\gamma \tilde{\lambda}_i^{i,e} dt + 2\beta \sum_{j \geq 2, j \neq i} \frac{\tilde{\lambda}_i^{j,e} - \tilde{\lambda}_i^{i,e}}{\tilde{\lambda}_i^{j,e} - \tilde{\lambda}_i^{i,e}} dt. \quad (31)
\end{align*}
\]

Let us remark that \( \tilde{\lambda}_1 \) is a CIR process, and that the coordinates \( i \in \{2, \ldots, n\} \) satisfy an autonomous system of SDEs for \( n - 1 \) particles similar to equation \((A_{\epsilon})\). The only differences are the coefficient \((n-1)\beta\) in \((A_{\epsilon})\) which remains \((n-1)\beta\) here and is thus unaffected by the change in the number of particles, and the terms \( \left[ 0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{\tilde{\lambda}_i^{i,e} - \frac{\epsilon}{2}} \right) \land 1 \right] \) in \((A_{\epsilon})\) which becomes \( \left[ 0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{\tilde{\lambda}_i^{i,e} - \frac{\epsilon}{2\sqrt{2}}} \right) \land 1 \right] \) here. We can still consider the root process \((\sqrt{\tilde{\lambda}_2^{2,e}}, \ldots, \sqrt{\tilde{\lambda}_n^{n,e}})\), and apply the same method as in the proof of Lemma 4.1 to prove the existence of a strong solution to this subsystem. Equation (31) also has a strong solution (Lemma 6.1), and consequently, the whole \( n \)-particles system considered here admits a global strong solution.

Let us define for all \( \epsilon > 0 \) and for \( t \geq 0 \): \( \Theta(t) = (\theta_1(t), \ldots, \theta_n(t)) \) with

\[
\theta_1(t) = -\frac{\beta}{\sqrt{\tilde{\lambda}_1^{1,e}}} \sum_{j \neq 1} \frac{\tilde{\lambda}_i^{1,e} \land \epsilon}{(\tilde{\lambda}_i^{j,e} - \tilde{\lambda}_i^{1,e} \land \epsilon) \lor \epsilon},
\]

\[
\forall i \in \{2, \ldots, n\}, \theta_i(t) = \frac{\beta}{\sqrt{\tilde{\lambda}_i^{i,e}}} \frac{\tilde{\lambda}_i^{i,e}}{(\tilde{\lambda}_i^{i,e} - \tilde{\lambda}_i^{1,e} \land \epsilon) \lor \epsilon},
\]

and for all \( t \geq 0 \)

\[
Z(t) = \exp \left\{ \int_0^t \Theta(u) \cdot d\tilde{B}_u - \frac{1}{2} \int_0^t ||\Theta(u)||^2 du \right\}.
\]
We have
\[
\theta_i^2(t) = \frac{\beta^2}{\tilde{\lambda}_{i, \epsilon}} \left( \sum_{j > 1} \frac{\tilde{\zeta}_{j, \epsilon}^1 \wedge \epsilon}{\tilde{\lambda}_{i, \epsilon}} \right)^2 \\
\leq \frac{\beta^2 (\tilde{\lambda}_{i, \epsilon}^1 \wedge \epsilon)^2}{\tilde{\lambda}_{i, \epsilon}^1} \left( \sum_{j > 1} \frac{1}{\epsilon} \right)^2 \\
\leq \frac{(n - 1)^2 \beta^2}{\epsilon}. \quad (32)
\]

For all \(1 < i \leq n\),
\[
\theta_i^2(t) = \frac{\beta^2}{\tilde{\lambda}_{i, \epsilon}} \frac{(\tilde{\lambda}_{i, \epsilon})^2}{((\tilde{\lambda}_{i, \epsilon}^i - \tilde{\lambda}_{1, \epsilon}^1 \wedge \epsilon) + \tilde{\lambda}_{i, \epsilon}^1 \wedge \epsilon)} \\
\leq \frac{\beta^2 (\tilde{\lambda}_{i, \epsilon}^1 - \tilde{\lambda}_{1, \epsilon}^1 \wedge \epsilon + \tilde{\lambda}_{i, \epsilon}^1 \wedge \epsilon)}{((\tilde{\lambda}_{i, \epsilon}^i - \tilde{\lambda}_{1, \epsilon}^1 \wedge \epsilon) \vee \epsilon)^2} \\
\leq \frac{2 \beta^2}{\epsilon}. \quad (33)
\]

We thus have
\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^t ||\Theta(u)||^2 du \right\} \right] < \infty \text{ for all } t \geq 0.
\]

Then, according to Novikov’s criterion (see, for instance, [18, Proposition 5.12 p.198]), \(\mathbb{Z}\) is a \(\mathbb{P}\)-martingale, and \(\mathbb{E}[\mathbb{Z}(t)] = 1\) for all \(t \geq 0\). Consequently, defining for all \(t \geq 0\), \(\tilde{\mathcal{F}}_t = \sigma \left( (\tilde{\mathcal{B}}_s)_{s \leq t}, (\tilde{\zeta}_{0, \epsilon}^1, \ldots, \tilde{\zeta}_{n, \epsilon}^n) \right)\) and \(\mathbb{Q}\) such that
\[
\frac{d\mathbb{Q}}{d\mathbb{P}|\tilde{\mathcal{F}}_t} = \mathbb{Z}(t),
\]
and
\[
\tilde{B}_1^1 = \tilde{B}_1^1 - \int_0^t \theta_1(s) ds \\
= \tilde{B}_1^1 + \int_0^t \frac{\beta}{\tilde{\lambda}_s^1} \sum_{j \neq 1} \frac{\tilde{\zeta}_{j, \epsilon}^1 \wedge \epsilon}{\tilde{\lambda}_s^1 - \tilde{\lambda}_s^1 \wedge \epsilon} \wedge \epsilon) ds,
\]
for all \(i \in \{2, \ldots, n\}\), \(\tilde{B}_i^i = \tilde{B}_i^i - \int_0^t \theta_i(s) ds \)
\[
= \tilde{B}_i^i - \int_0^t \frac{\beta}{\tilde{\lambda}_s^i} \frac{\tilde{\zeta}_{i, \epsilon}^i \wedge \epsilon}{\tilde{\lambda}_s^i - \tilde{\lambda}_s^i \wedge \epsilon} \wedge \epsilon ds, \quad 0 \leq t,
\]
\(\mathbb{Q}\) Springer
\( \tilde{B} = (\tilde{B}_1^t, \ldots, \tilde{B}_n^t) \), is a \( \mathbb{Q} \)-Brownian motion according to the Girsanov theorem (see, for instance, [18, Proposition 5.4 p.194]).

Consequently, \( (B_\epsilon) \) has a global weak solution.

We now have to prove the pathwise uniqueness of the solutions to \( (B_\epsilon) \). The differences with Lemma 3.1 are the term

\[
0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{z_{i, s}^\epsilon} - \frac{\sqrt{\epsilon}}{2} \right) \land 1,
\]

and the interaction terms between the first particle and the others.

Let \( Z = (z_1^t, \ldots, z_n^t) \) and \( \tilde{Z} = (\tilde{z}_1^t, \ldots, \tilde{z}_n^t) \), be two global solutions to \( (B_\epsilon) \) with \( Z_0 = \tilde{Z}_0 \) independent from the same driving Brownian motion \( B = (B_1^t, \ldots, B_n^t) \).

Let \( M > 0 \) and

\[
\tau_M = \inf \{ t \geq 0 : z_1^t + z_1^t \land z_1^t \geq M \}.
\]

As \( Z \) and \( \tilde{Z} \) are continuous and assumed well defined on \( \mathbb{R}_+ \), \( \tau_M \uparrow \infty \) when \( M \uparrow \infty \).

The local time of \( z_i^t - \tilde{z}_i^t \) at 0 is zero ([26, Lemma 3.3 p.389]). Applying the Tanaka formula to the process \( z_i^t - \tilde{z}_i^t \) stopped at \( \tau_M \) and summing over \( i \),

\[
\sum_{i=1}^{n} |z_i^{t \wedge \tau_M} - \tilde{z}_i^{t \wedge \tau_M}| = 2 \sum_{i=1}^{n} \int_{0}^{t \wedge \tau_M} |\sqrt{z_i^s} - \sqrt{\tilde{z}_i^s}| dB_i^s - 2\gamma \int_{0}^{t \wedge \tau_M} \sum_{i=1}^{n} |z_i^s - \tilde{z}_i^s| ds
+ 2\beta \int_{0}^{t \wedge \tau_M} \sum_{i=2}^{n} \text{sgn}(z_i^s - \tilde{z}_i^s) \sum_{j \geq 2, j \neq i} \left( \frac{z_i^s}{z_i^s - z_j^s} - \frac{\tilde{z}_i^s}{z_i^s - \tilde{z}_j^s} \right) ds
+ 2\beta \int_{0}^{t \wedge \tau_M} \sum_{i=2}^{n} \left\{ \text{sgn}(z_i^s - \tilde{z}_i^s) \left( \frac{z_i^s}{z_i^s - \tilde{z}_i^s} \lor \epsilon \right) - \frac{\tilde{z}_i^s}{(z_i^s - \tilde{z}_i^s) \lor \epsilon} \right\} ds
- \int_{0}^{t \wedge \tau_M} \sum_{i=2}^{n} \text{sgn}(z_i^s - \tilde{z}_i^s) \left\{ 0 \lor \frac{2}{\sqrt{\epsilon}} \left( \sqrt{z_i^s} - \frac{\sqrt{\epsilon}}{2} \right) \land 1 \right\} ds.
\]

As the processes are stopped at \( \tau_M \), the expectation of the stochastic integrals is zero. As in the proof of Lemma 3.1, the terms (34) are not positive. To deal with the
expectation of (35), one remarks that the function 

\[(x, y) \mapsto \frac{x}{(x - y \wedge \epsilon) \vee \epsilon}\]

is Lipschitz on \([0, M]^2\).

As for the term (36), the function \(f : z \mapsto \left[0 \vee \frac{2}{\sqrt{\epsilon}} \left(\sqrt{z} - \frac{\epsilon}{2}\right) \wedge 1\right]\) defined on \(\mathbb{R}_+\) is constant on \([0, \frac{\epsilon}{4}]\) and on \([\epsilon, +\infty)\) and is differentiable on \((\frac{\epsilon}{4}, \epsilon)\) with

\[f'(z) = \frac{1}{\sqrt{\epsilon} z} \leq \frac{2}{\epsilon}\]

for all \(z \in (\frac{\epsilon}{4}, \epsilon)\).

Consequently, \(f\) is Lipschitz with constant \(\frac{2}{\epsilon}\). Then, for all \(M > 0\) there exists a constant \(K_M \geq 0\) depending on \(M\) such that for all \(t \geq 0\)

\[
\sum_{i=1}^{n} \mathbb{E}|z^i_{t \wedge \tau_M} - \tilde{z}^i_{t \wedge \tau_M}| \leq K_M \mathbb{E}\left[\int_0^{t \wedge \tau_M} \sum_{i=1}^{n} |z^i_s - \tilde{z}^i_s| ds\right]
\]

\[
\leq K_M \int_0^{t} \sum_{i=1}^{n} \mathbb{E}|z^i_{s \wedge \tau_M} - \tilde{z}^i_{s \wedge \tau_M}| ds.
\]

The Grönwall Lemma allows to conclude that for all \(M > 0\) and \(t \geq 0\)

\[
\sum_{i=1}^{n} \mathbb{E}|z^i_{t \wedge \tau_M} - \tilde{z}^i_{t \wedge \tau_M}| = 0.
\]

Using Fatou’s Lemma to take the limit \(M\) going to infinity, we deduce that for all \(t \geq 0\)

\[
\sum_{i=1}^{n} \mathbb{E}|z^i_t - \tilde{z}^i_t| = 0,
\]

which concludes the proof on the existence and pathwise uniqueness.

Let us now prove (30). To do so, we can use the same method used to prove (24) on the square root of the coordinates \((\tilde{\lambda}^2, \epsilon, \ldots, \tilde{\lambda}^n, \epsilon)_t\), which solve, as explained in the beginning of the proof, a system of SDEs similar to \((A_\epsilon)\) for \(n - 1\) particles. \(\square\)

### 5 Proof of the Other Results

Let us now prove the result on the collision time between particles. To do so, we study the difference between two neighbor coordinates and bound it from above by a time changed Bessel process hitting zero in finite time.
Proof of Proposition 2.6 For $i \in \{2, \ldots, n\}$,

$$
\begin{align*}
    d(\lambda_i^j - \lambda_i^{j-1}) &= 2\sqrt{\lambda_i^j} dB_i^j - 2\sqrt{\lambda_i^{j-1}} dB_i^{j-1} + 2\gamma (\lambda_i^{j-1} - \lambda_i^j) dt \\
    &+ 2\beta \frac{\lambda_i^j + \lambda_i^{j-1}}{\lambda_i^j - \lambda_i^{j-1}} dt + 2\beta \sum_{j \neq [i,i-1]} \frac{\lambda_i^j}{(\lambda_i^j - \lambda_i^j)(\lambda_i^{j-1} - \lambda_i^{j-1})} dt \\
    &\leq 2\sqrt{\lambda_i^j} dB_i^j - 2\sqrt{\lambda_i^{j-1}} dB_i^{j-1} + 2\beta \frac{\lambda_i^j + \lambda_i^{j-1}}{\lambda_i^j - \lambda_i^{j-1}} dt, \\
    (37)
\end{align*}
$$

because

$$
2\gamma (\lambda_i^{j-1} - \lambda_i^j) \leq 0,
$$

and the contribution of the greater and smaller coordinates is non-positive (in the sum, the numerator of the ratio factor is always non-positive and the denominator always non-negative). Let us fix $i \in \{2, \ldots, n\}$ and let us define the continuous local martingale

$$
M_i^j = 2 \int_0^t \sqrt{\lambda_s^j} dB_s^j - \sqrt{\lambda_s^{j-1}} dB_s^{j-1} \quad \text{for } t \geq 0,
$$

and

$$
\langle M^j \rangle_t = 4 \int_0^t (\lambda_s^j + \lambda_s^{j-1}) ds \quad \text{for } t \geq 0,
$$

with generalized inverse

$$
C_i^j = \inf\{s \geq 0 : \langle M^j \rangle_s > t\}.
$$

The process $(\langle M^j \rangle_t)$ is continuous, and according to Lemma 5.2, \lim_{t \to \infty} \langle M^j \rangle_t = +\infty$ a.s., which implies that for all $t \geq 0$, $C_i^j < \infty$. We can apply the Dambins–Dubins–Schwarz Theorem (see, for instance, [26, Theorem 1.6 p.181]) to conclude that the process defined by

$$
(B_i^{(j)})_t = \left(2 \int_0^{C_i^j} \sqrt{\lambda_s^j} dB_s^j - \sqrt{\lambda_s^{j-1}} dB_s^{j-1}\right)_t
$$

is a $(\mathcal{F}_{C_i^j})_t$-Brownian motion.

We define for all $t \in \mathbb{R}_+ : D_i^j = \lambda^j_{C_i^j} - \lambda^{j-1}_{C_i^j}$. By (37) and the definition of $B^{(j)}$, we have:

$$
\begin{align*}
    dD_i^j &= d(\lambda^j_{C_i^j} - \lambda^{j-1}_{C_i^j}) \\
    &\leq dB_t^{(j)} + \frac{\beta}{2(\lambda^j_{C_i^j} - \lambda^{j-1}_{C_i^j})} dt = dB_t^{(i)} + \frac{\beta}{2D_i^j} dt.
\end{align*}
$$
Let us define the Bessel process

$$r^i_t = \lambda^i_0 - \lambda^i_0^{-1} + B^{(i)}_t + \frac{\beta}{2} \int_0^t \frac{1}{r^i_s} ds.$$ 

Then,

$$d(r^i_t - D^i_t) \geq \frac{\beta}{2} \frac{D^i_t - r^i_t}{r^i_tD^i_t} dt,$$

and as long as neither $r^i$ nor $D^i$ touches 0

$$d\left( e^{\frac{\beta}{2} \int_0^t \frac{ds}{r^i_s} (r^i_t - D^i_t) \right) \geq 0,$$

and $r^i_t \geq D^i_t$ thanks to the equality $r^i_0 = D^i_0$. As the trajectories are continuous, as soon as $r^i$ reaches 0, which is the case in finite time almost surely when $\beta \leq 1$ (see, for instance, [26, Chapter XI, (ii) p.442]), so does $D^i$, which concludes the proof. \(\square\)

**Remark 5.1** It turns out that whatever $\gamma \in \mathbb{R}$ and $\alpha, \beta > 0$, we can conclude that $\int_0^{+\infty} (\lambda^n_s + \lambda^{n-1}_s) ds = +\infty$ a.s. as soon as (1) admits a global solution. Since the coordinates are ordered, for all $t \geq 0$

$$\int_0^t (\lambda^n_s + \lambda^{n-1}_s) ds \geq \frac{2}{n} \int_0^t \left( \sum_{i=1}^n \lambda^i_s \right) ds.$$ 

Let us define $W$ by $W_0 = 0$ and $dW_t = \sum_{i=1}^n \left( \mathbb{1}_{\{\sum_{j=1}^n \lambda^j_t \neq 0\}} \frac{\sqrt{\lambda^i_t}}{\sqrt{\sum_{j=1}^n \lambda^j_t}} \right) dB^i_t$. According to Lévy’s characterization, $W$ is a Brownian motion. Then, using the equality (13) in introduction, $(\sum_{i=1}^n \lambda^i_s)_{s \geq 0}$ is a CIR process. Proposition 6.2.4 in [21] gives an expression of the Laplace transform of integrated CIR processes: for any $\mu > 0$,

$$\mathbb{E}\left[ e^{-\mu \int_0^t \sum_{i=1}^n \lambda^i_s ds} \right] = \exp(-2\alpha \phi(\mu)) \exp\left( -\left( \sum_{i=1}^n \lambda_0^i \right) \psi(\mu) \right),$$

where

$$\phi(\mu) = -\frac{1}{2} \ln \left( \frac{2\sqrt{\gamma^2 + 2\mu} e^{(\gamma - \sqrt{\gamma^2 + 2\mu}) t}}{(\sqrt{\gamma^2 + 2\mu} - \gamma) e^{-2t\sqrt{\gamma^2 + 2\mu}} + \sqrt{\gamma^2 + 2\mu + \gamma}} \right) \xrightarrow{t \to +\infty} +\infty,$$

and $\psi(\mu) = \frac{\mu \left( e^{2t\sqrt{\gamma^2 + 2\mu}} - 1 \right)}{\sqrt{\gamma^2 + 2\mu - \gamma} + e^{2t\sqrt{\gamma^2 + 2\mu}}(\sqrt{\gamma^2 + 2\mu + \gamma})} \xrightarrow{t \to +\infty} \frac{\mu}{\sqrt{\gamma^2 + 2\mu + \gamma}}.$
Thus,
\[ E\left[e^{-\mu \int_0^t \sum_{i=1}^n \lambda_i^1 ds}\right] = \lim_{t \to +\infty} E\left[e^{-\mu \int_0^t \sum_{i=1}^n \lambda_i^1 ds}\right] = 0, \]
so that
\[ \int_0^{+\infty} \sum_{i=1}^n \lambda_i^1 ds = +\infty \text{ a.s.} \]
and we can conclude.

**Lemma 5.2** Let us assume that \( \alpha, \beta > 0 \) with \( \alpha - (n - 1)\beta \geq 0 \) and that \( \Lambda = (\lambda_1^1, \ldots, \lambda_n^1) \) is a global solution to (1). If either \( \gamma > 0 \) or \( \gamma \leq 0, \alpha - (n - 1)\beta \geq 1 - \beta \) and \( \lambda_2^0 > 0 \) a.s., then
\[ \int_0^{+\infty} \lambda_2^0 ds = +\infty \text{ a.s.} \]

**Proof** The pathwise uniqueness result in Lemma 3.1 ensures that the solution is strong. Therefore, it is enough to deal with the case of a deterministic initial condition \( \Lambda_0 = (\lambda_1^0, \ldots, \lambda_n^0) \). We thus suppose without loss of generality that \( E\sum_{i=1}^n \lambda_i^0 < \infty \) still with \( \lambda_2^0 > 0 \) a.s. in the second setting.

Let us first deal with the case \( \gamma > 0 \). To do so, let \( \tilde{\Lambda}_0 \) be distributed according to \( \rho_{inv} \) and independent from the Brownian motion \( B \) and let \( \tilde{\Lambda} = (\tilde{\lambda}_1^1, \ldots, \tilde{\lambda}_n^1) \) be the solution to (1) starting from \( \tilde{\Lambda}_0 \) given by Proposition 2.7. For all \( t \geq 0 \), \( \tilde{\Lambda}_t \) is distributed according to \( \rho_{inv} \). Let us show that
\[ \mathbb{P}\left( \int_0^{+\infty} \tilde{\lambda}_2^0 ds = +\infty \right) = 1. \]

Since \( (\tilde{\Lambda}_{t+1})_{t \geq 0} \) is a solution to (1) starting from \( \tilde{\Lambda}_1 \) distributed according to \( \rho_{inv} \) for the Brownian motion \( (B_{t+1} - B_1)_{t \geq 0} \) and pathwise uniqueness implies weak uniqueness, \( (\tilde{\Lambda}_{t+1})_{t \geq 0} \) has the same distribution as \( (\tilde{\Lambda}_t)_{t \geq 0} \). Thus, \( \int_0^{+\infty} \tilde{\lambda}_2^0 ds \) has the same distribution as \( j_{t}^{+\infty} \tilde{\lambda}_2^0 ds \). Consequently, since
\[ e^{-\int_0^{+\infty} \tilde{\lambda}_2^0 ds} \leq e^{-\int_0^{+\infty} \tilde{\lambda}_2^0 ds} \text{ a.s.}, \]
\[ e^{-\int_0^{+\infty} \tilde{\lambda}_2^0 ds} = e^{-\int_0^{+\infty} \tilde{\lambda}_2^0 ds} \text{ a.s.}, \]
which also writes
\[ \left( 1 - e^{\int_0^{+\infty} \tilde{\lambda}_2^0 ds} \right) = 0 \text{ a.s.} \]

As
\[ \rho_{inv}(\{x \in \mathbb{R}^n, x_1^1 > 0\}) = 1, \]

\( \square \) Springer
we have
\[ \tilde{\lambda}_0 > 0 \text{ a.s.} \]
and
\[ e^{\int_0^{+\infty} \tilde{\lambda}_s ds} > 1 \text{ a.s.,} \]
and one can deduce that
\[ e^{-\int_0^{+\infty} \tilde{\lambda}_s ds} = 0 \text{ a.s.} \]
so that
\[ \int_0^{+\infty} \tilde{\lambda}_s ds = +\infty \text{ a.s.} \]

By (14), for all \( s \geq 0 \)
\[ \mathbb{E}|\lambda_s - \tilde{\lambda}_s| \leq \left( \sum_{i=1}^{n} \mathbb{E}|\lambda_s^i - \tilde{\lambda}_s^i| \right) \exp(-2\gamma s), \]
and then,
\[ \mathbb{E} \left| \int_0^{+\infty} \lambda_s ds - \int_0^{+\infty} \tilde{\lambda}_s ds \right| \leq \int_0^{+\infty} \mathbb{E} \left| \lambda_s - \tilde{\lambda}_s \right| ds \]
\[ \leq \int_0^{+\infty} \sum_{i=1}^{n} \mathbb{E}|\lambda_s^i - \tilde{\lambda}_s^i| \exp(-2\gamma s) ds. \]

As \( \gamma > 0 \), the right-hand side is finite. Since
\[ \int_0^{+\infty} \tilde{\lambda}_s ds = +\infty \text{ a.s.,} \]
we can deduce that
\[ \int_0^{+\infty} \lambda_s ds = +\infty \text{ a.s.} \]

Let us now suppose that \( \gamma \leq 0 \) and \( \alpha - (n - 1)\beta \geq 1 - \beta \). Let \( \hat{\gamma} > 0 \) and \( \hat{\Lambda} \)
 denote the global solution to (1) with \( \gamma \) replaced by \( \hat{\gamma} \) starting from \( \hat{\Lambda}_0 = \Lambda_0 \) given by Theorem 2.2. By the previous reasoning,
\[ \int_0^{+\infty} \hat{\lambda}_s ds = +\infty \text{ a.s.} \]
For $M > 0$, let

$$
\tau_M = \inf\{t \geq 0 : \lambda^n_i + \hat{\lambda}^n_i \geq M\},
$$

with the convention $\inf\emptyset = +\infty$. As $\Lambda$ and $\hat{\Lambda}$ are continuous global solutions, $\tau_M \uparrow +\infty$ when $M \uparrow \infty$. Reasoning like in the proof of Lemma 3.1, we obtain

$$
E \left[ \sum_{i=1}^{n} (\hat{\lambda}^i_{t \wedge \tau_M} - \lambda^n_i)^+ \right] = E \left[ \int_0^{t \wedge \tau_M} \sum_{i=1}^{n} \mathbb{1}_{[\hat{\lambda}^i_s > \lambda^n_i]} \left(-2\hat{\lambda}^i_s + 2\gamma \hat{\lambda}^i_s + \beta \sum_{j \neq i} (\hat{\lambda}^i_s + \hat{\lambda}^j_s) - \hat{\lambda}^i_s - \hat{\lambda}^j_s \right) ds \right]
$$

$$
\leq \beta E \left[ \int_0^{t \wedge \tau_M} \sum_{i=1}^{n} \mathbb{1}_{[\hat{\lambda}^i_s > \lambda^n_i]} \sum_{j \neq i} (\hat{\lambda}^i_s + \hat{\lambda}^j_s - \hat{\lambda}^i_s - \hat{\lambda}^j_s) ds \right]
$$

$$
\leq 0.
$$

(38)

The last inequality comes from the fact that, as in (16), we have for all $i < j$:

$$
\left[ \frac{\hat{\lambda}^i_s + \hat{\lambda}^j_s - \lambda^n_i - \lambda^n_j}{\hat{\lambda}^i_s - \hat{\lambda}^j_s} \right] (\mathbb{1}_{[\hat{\lambda}^i_s > \lambda^n_i]} - \mathbb{1}_{[\hat{\lambda}^j_s > \lambda^n_j]})
$$

$$
= -2 \frac{\lambda^n_j |\hat{\lambda}^i_s - \lambda^n_i| + \lambda^n_i |\lambda^n_j - \hat{\lambda}^j_s|}{(\hat{\lambda}^i_s - \hat{\lambda}^j_s)(\lambda^n_i - \lambda^n_j)} [\mathbb{1}_{[\hat{\lambda}^i_s > \lambda^n_i]} - \mathbb{1}_{[\hat{\lambda}^j_s > \lambda^n_j]}] \leq 0,
$$

as the denominator is non-negative.

Using Fatou’s Lemma to take the limit $M$ going to infinity, we deduce that for all $t \geq 0$

$$
E \left[ \sum_{i=1}^{n} (\hat{\lambda}^i_t - \lambda^n_i)^+ \right] = 0,
$$

and thus, for all $i \in \{1, \ldots, n\}$,

$$
\hat{\lambda}^i_t < \lambda^n_i \text{ a.s.}
$$

Since $\int_0^{+\infty} \hat{\lambda}^i_t ds = +\infty \text{ a.s.}$, we conclude that

$$
\int_0^{+\infty} \lambda^n_i ds = +\infty \text{ a.s.}
$$

 Proposition 5.3  Let us assume $\alpha - (n - 1)\beta < 0$, $\beta > 0$ and $0 \leq \lambda^n_0 \leq \cdots \leq \lambda^n_n$. 

$\square$ Springer
The system (1) has a unique strong solution defined on the time interval \([0, \lim_{\epsilon \to 0} S_\epsilon)\) where for all \(\epsilon > 0\),

\[ S_\epsilon = \inf\{t \geq 0 : x^{1,\epsilon}_t \leq \epsilon\}. \]

Proof  Let \(\epsilon > 0\) and let us consider the system

\[
\begin{align*}
dx^{i,\epsilon}_t &= dB^i_t + \frac{\alpha - (n - 1)\beta - 1}{2} \frac{1}{x^{i,\epsilon}_t} dt - \gamma x^{i,\epsilon}_t dt \\
&\quad + \beta x^{i,\epsilon}_t \sum_{j \neq i} \frac{dt}{(x^{i,\epsilon}_t)^2 - (x^{j,\epsilon}_t)^2} \quad \text{for } i \in \{1, \ldots, n\}
\end{align*}
\]

with initial condition \(0 \leq x^{1,\epsilon}_0 < \cdots < x^{n,\epsilon}_0\) a.s. \(dt - a.e.\)

As the function \(x \mapsto \frac{1}{x^{i,\epsilon}_0} \leq \cdots \leq \frac{1}{x^{n,\epsilon}_0}\) is globally Lipschitz, we can apply Cepa’s multivalued equations theory (\([8]\)) to conclude that there exists a unique strong solution to \((C_\epsilon)\) defined globally. Moreover, one can remark that the solution to \((C_\epsilon)\) is a solution to (5) on \([0, \inf\{t \geq 0 : x^{1,\epsilon}_t \leq \sqrt{\epsilon}\})\) for all \(\epsilon > 0\). By taking the square of each coordinate, we obtain a solution to (1) up to \(S_\epsilon\) for all \(\epsilon > 0\). By pathwise uniqueness for (1) (see Lemma 3.1), these solutions are consistent and we can conclude. \(\square\)

Acknowledgements  We thank Djalil Chafai for numerous fruitful discussions. We also thank the referees for their remarks which helped us to improve the first version of the manuscript.

6 Appendix

The next lemma deals with the existence and uniqueness to the CIR SDE and with the probability for the solution to hit zero. It is proved, for instance, in [21, Theorem 6.2.2 and Proposition 6.2.3]. The point 4 comes directly from [5].

Lemma 6.1  Let \(a \geq 0\), \(b, \sigma \in \mathbb{R}\). Suppose that \(W\) is a standard Brownian motion defined on \(\mathbb{R}_+\). For any real number \(x \geq 0\), there is a unique continuous, adapted process \(X\), taking values in \(\mathbb{R}_+\), satisfying \(X_0 = x\) and

\[
dX_t = (a - bX_t)dt + \sigma \sqrt{X_t}dW_t \text{ on } [0, \infty).
\]

Moreover, if we denote by \(X^x\) the solution to this SDE starting at \(x\) and by \(\tau_0^x = \inf\{t \geq 0 : X^x_t = 0\}\),

1. If \(a \geq \sigma^2/2\), we have \(\mathbb{P}(\tau_0^x = \infty) = 1\), for all \(x > 0\).
2. If \(0 \leq a < \sigma^2/2\) and \(b \geq 0\), we have \(\mathbb{P}(\tau_0^x < \infty) = 1\), for all \(x > 0\).
3. If \(0 \leq a < \sigma^2/2\) and \(b < 0\), we have \(0 < \mathbb{P}(\tau_0^x < \infty) < 1\), for all \(x > 0\).
4. For all $s > t$,

$$
\mathbb{E}[r_s | r_t] = r_t e^{-b(s-t)} + \frac{a}{b} (1 - e^{-b(s-t)}).
$$

The following result is the Ikeda–Watanabe Theorem, which allows to compare two Itô processes if their starting points and their drift coefficients are comparable, and if their diffusion coefficients are regular enough. It is proved, for instance, in [25, Theorem V.43.1 p.269].

**Theorem 6.2 (Ikeda–Watanabe)** Suppose that, for $i = 1, 2$,

$$
X_t^i = X_0^i + \int_0^t \sigma(X_s^i) dB_s + \int_0^t \beta_s^i ds,
$$

and that there exist $b : \mathbb{R} \mapsto \mathbb{R}$, such that

$$
\beta_s^1 \geq b(X_s^1), \quad b(X_s^2) \geq \beta_s^2.
$$

Suppose also that

1. $\sigma$ is measurable and there exists an increasing function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$
\int_0^+ \rho(u)^{-1} du = \infty,
$$

and for all $x, y \in \mathbb{R}$,

$$
(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|);
$$

2. $X_0^1 \geq X_0^2$ a.s.;

3. $b$ is Lipschitz.

Then, $X_t^1 \geq X_t^2$ for all $t$ a.s.

**References**

1. Anderson, G.W., Guionnet, A., Zeitouni, O.: An Introduction to Random Matrices, Cambridge Studies in Advanced Mathematics, vol. 118. Cambridge University Press, Cambridge (2010)

2. Bru, M.-F.: Diffusions of perturbed principal component analysis. J. Multivar. Anal. 29(1), 127–136 (1989)

3. Bru, Marie-France: Wishart processes. J. Theoret. Probab. 4(4), 725–751 (1991)

4. Chybiryakov O: Processus de Dunkl et relation de Lamperti, June (2006), Thèse Paris VI

5. Cox, J.C., Ingersoll, J.E., Ross, S.: A theory of the term structure of interest rates. Econometrica 53(2), 385–407 (1985)

6. Cèpa, E., Lépingle, D.: Diffusing particles with electrostatic repulsion. Probab. Theory Relat. Fields 107(4), 429–449 (1997)

7. Cèpa, E., Lépingle, D.: Brownian particles with electrostatic repulsion on the circle: Dyson’s model for unitary random matrices revisited. ESAIM Probab. Stat. 5, 203–224 (2001)
8. Cépa E: Équations différentielles stochastiques multivoques, Séminaire de Probabilités, XXIX, Lecture Notes in Math., vol. 1613, pp. 86–107. Springer, Berlin, (1995)
9. Demni, N.: The laguerre process and generalized Hartman-Watson law. Bernoulli 13(2), 556–580 (2007)
10. Demni, N.: Radial dunkl processes: existence, uniqueness and hitting time. C. R. Math. Acad. Sci. Paris 347(19–20), 1125–1128 (2009)
11. Dyson, F.J.: A Brownian-motion model for the eigenvalues of a random matrix. J. Math. Phys. 3, 1191–1198 (1962)
12. Forrester, P.J.: Log-gases and Random Matrices. London Mathematical Society Monographs Series, vol. 34. Princeton University Press, Princeton (2010)
13. Göring-Jaeschke, A., Yor, M.: A survey and some generalizations of bessel processes. Bernoulli 9(2), 313–349 (2003)
14. Graczyk, P., Malecki, J.: Multidimensional Yamada-Watanabe theorem and its applications to particle systems. J. Math. Phys. 54(2), 021503 (2013)
15. Graczyk, P., Malecki, J.: Strong solutions of non-colliding particle systems. Electron. J. Probab. 19(119), 21 (2014)
16. Graczyk, P., Malecki, J.: On squared bessel particle systems. Bernoulli 25(2), 828–847 (2019)
17. König, W., O’Connell, N.: Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. Electron. Comm. Probab. 6, 107–114 (2001)
18. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Graduate Texts in Mathematics, vol. 113, 2nd edn. Springer, New York (1991)
19. Katori, M., Tanemura, H.: Noncolliding processes, matrix-valued processes and determinantal processes [translation of mr2561146]. Sugaku Expos. 24(2), 263–289 (2011)
20. Katori, M., Tanemura, H.: Noncolliding squared Bessel processes. J. Stat. Phys. 142(3), 592–615 (2011)
21. Lamberton, D., Lapeyre, B.: Introduction to stochastic calculus applied to finance. Chapman & Hall/CRC Financial Mathematics Series, 2nd edn. Chapman & Hall/CRC, Boca Raton (2008)
22. Lépingle, D.: Boundary behavior of a constrained Brownian motion between reflecting-repellent walls. Probab. Math. Statist. 30(2), 273–287 (2010)
23. Rogers, L., Chris, G., Zhan, S.: Interacting Brownian particles and the Wigner law. Probab. Theory Relat. Fields 95(4), 555–570 (1993)
24. Rösler, M., Voit, M.: Markov processes related with Dunkl operators. Adv. Appl. Math. 21(4), 575–643 (1998)
25. Rogers, L.C.G., David, W.: Diffusions, Markov Processes, and Martingales. Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). Itô calculus, Reprint of the second (1994) edition
26. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Grundlachen der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 3rd edn. Springer, Berlin (1999)
27. Schapira, B.: The Heckman-Opdam Markov processes. Probab. Theory Relat. Fields 138(3–4), 495–519 (2007)
28. Trevisan, D.: Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. Electron. J. Probab. 21(22), 41 (2016)
29. Voit, M. Woerner JHC.: Functional central limit theorems for multivariate bessel processes in the freezing regime, (2019), arXiv:1901.08390
30. Yor, M., Chybiryakov, O., Gallardo, L.: Dunkl processes and their radial parts relative to a root system. In: Graczyk, P., Rösler, M., Yor, M. (eds.) Harmonic and Stochastic Analysis of Dunkl Processes. Hermann, Paris (2008). Travaux en cours

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.