An Analytical Approximation to the Seismic Response of Light Linear Nonstructural Components in Resonance With Linear Structures

Julio Cesar Miranda (✉ julio.miranda@jacobs.com)
Jacobs Engineering Group Inc

Research Article

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Posted Date: August 17th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-799351/v1

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An analytical approximation to the seismic response of light linear nonstructural components in resonance with linear structures

Julio C. Miranda

JACOBS, 1737 N. First Street, Suite 300, San Jose CA 95112, julio.miranda@jacobs.com.

ABSTRACT

This paper considers the analytical calculation of the seismic response of light nonstructural components resonant with the buildings to which they are affixed. The mechanical systems thus conformed are presumed to be linearly elastic and classically damped, such that a traditional modal analysis can be carried out. Intended to estimate the upper limit of the response, the procedure developed is indicated for resonance with the fundamental mode, a condition which usually controls the response of the components. The correlation of the two closely spaced modes resulting from the dynamics of the component-building system, is accounted for through a spectral analysis. Comparison of the results predicted by the procedure described in this paper with limited numerical applications, corroborates that it is successful in predicting the response of the components when these are tuned to low order modes of the carrying structure. However, as also seen, the procedure is inappropriate to calculate the response of the components when their tuning involves higher modes of the supporting structure. Given the successful numerical forecasting of the response of the components, and given the compact form of the proposed equations, the feasibility of developing their codified form merits further investigation.

KEYWORDS: structural dynamics; seismic design; building technology; building design; seismic code provisions; non-structural components

1. INTRODUCTION

The determination of the response to seismic excitation of nonstructural components is important for the design of the components themselves, as well as for the design of their anchorage to the supporting structure. Such calculation is more relevant in the case of components resonating with any of the modes of the building to which they are affixed, since it is well known that the response becomes then more incisive. In some industrial sectors, such as for microelectronics fabrications, the equipment cost amply exceeds the cost of the sheltering building, making it very likely that its malfunction or destruction as a consequence of seismic action will result in very significant economic losses. In other sectors, such as for medical care, the destruction or functional failure of some critical components, could paralyze the operation of the facilities at the most pressing moment. Thus, in view of its importance, the problem has been under study for a long time. Penzien et al [9], were among the firsts to research the subject. Sackman et al [11] studied interactive nonstructural components-buildings systems, and proposed for the first time analytical methods to evaluate their response to transient ground motion. Igusa et al [5] used random methods to analytically derive the response of components to white noise. These studies notwithstanding, the migration of the research findings to codes of practice has been slow, surely reflecting the complexity of the subject. Alternatively, the seismic response of
the mechanical systems conformed by the nonstructural components-buildings can be obtained through response history calculations. However, such modality requires considerable resources and time, such that its applicability in current everyday structural engineering practice is not acceptable. In the United States of America, the practical calculation of the seismic forces acting on nonstructural components is enforced through the application of Standard ASCE 7-16 [6]. Chapter 13 of this document proposes an empirical formula that predicts accelerations linearly proportional to the height of the floors above the base of the structure. This formula is applicable to nonstructural components that are essentially rigid, where this condition is defined as to having periods smaller than 0.06 seconds. Thus, in using this expression, elastic nonstructural components may attain a maximum horizontal acceleration at the roof, of up to three times the effective Peak Ground Acceleration (PGA). For nonstructural components that possess more flexibility, such as those mounted on antivibration devices, the formula dictates a maximum horizontal acceleration at the roof, of up to 7.5 times the effective PGA. That is, the formula allows for a dynamic amplification factor of the floor response equal to 2.5. The determination of the dynamic amplification factor corresponding to intermediate periods, i.e. between rigid and flexible components, requires engineering judgment. The procedure just described is eminently practical, very useful, and simple to apply. However, it elicits important criticism; the proposed formula ignores basic structural dynamics parameters such as component and building periods, and component and building damping. Also, the records upon which it is based, are indicative of floor accelerations, and do not reflect the dynamic amplification experienced by the nonstructural components themselves. The amplification factor of 2.5 mentioned above, is a suggested factor, see Soong et al [15].

Given the noted limitations of the ASCE 7 formula, currently a number of studies are being performed to improve the procedure to calculate the horizontal acceleration acting on nonstructural components, see for example NIST [13], Fathali et al [16], NIST [17], and Lizundia [18]. Some other regulations, such as the Eurocode 8, [20], take into consideration the periods of the components and fundamental building modes for the purpose of tuning calibration, but just like ASCE 7-16, this regulation assumes a geometrical shape for determining the component accelerations along the height of the structure.

In view of this state of knowledge, and since for engineering applications the maximum of the response is the primary interest, in this paper an analytical approximation to the upper boundary of the response of mechanical systems conformed by nonstructural component and supporting buildings, is proposed. Under the assumption that the component is resonant with any of the modes of the supporting structures, well established structural dynamics spectral methods are used to determine the response under these conditions. It is assumed that the mass of the components is very small in relation to the effective mass of the structural resonant modes. Given the state of current practice and knowledge, the assumption of equality of damping for the components and the structures is also made. It is then, numerically demonstrated that the formula proposed is adequate to capture the responses involving the lower modes, in particular that of the fundamental mode. For higher modes, the proposed formulas predict unsatisfactory results. However, since the modal energy associated with the fundamental mode usually controls, assuming resonance with such mode will provide the maximum response of interest, and that is the direction this paper follows.
The analytical platform adopted for this endeavor, relies on a model of relative modal kinetic and strain energies, developed previously by Miranda [8] for the study of Tuned Mass Dampers to be used for the seismic response reduction in buildings. The derived analytical model provides a spectral analysis of the mechanical system conformed by the component and the structure, with resulting equations that are analytical, yet compact and simple. It shall be noted that while in this paper the calculation of the maximum response in acceleration is emphasized, the maximum response in relative displacement, or in relative velocity, can also be obtained by simple substitution of the respective spectra in the response equations.

It is well known that a rigorous study of the systems under consideration, requires a complex approach for the determination of the eigenvalues and eigenvectors, in view of non-classical damping. However, it is discussed below that the calculation errors due to this issue are small, and therefore tolerable for practical engineering purposes.

2. THEORETICAL DEVELOPMENT

![Two-degree-of-freedom mechanical system](image)

**Figure 1.** Two-degree-of-freedom mechanical system

This theoretical development applies to mechanical systems that can be idealized as the two-degree-of-freedom system depicted on Figure 1, where the upper part represents a nonstructural component with a mass $M_U$, a damping constant $C_U$, and a stiffness $K_U$. The lower part is characterized by a mass $M_L$, a damping constant $C_L$, and a stiffness $K_L$. The last set of parameters represents effective properties corresponding to the mode of vibration of the structure to which the nonstructural component is attached to. In this development it is assumed that the nonstructural component is in resonance with any of the modes of the structure. For the sake of brevity, and as long as it is not confusing, the nonstructural component will be referred to as the “component”, and the structural mode being considered will be referred to as the “building”. The system is presumed to possess low classical damping. Hence, given a ground acceleration $a(t)$, the equation of motion for the system thus described is:

$$M \ddot{x}(t) + C \dot{x}(t) + Kx(t) = -Mr(t)$$  \hspace{1cm} (1)
where $x(t)$ is a vector of relative displacements, with amplitudes $x_U$ and $x_L$ for the upper and lower mass respectively. The vector $r$ is a vector of ones, and $M$, $C$, and $K$, are the mass, damping, and stiffness matrices of the system, respectively, that are written:

$$M = \begin{bmatrix} M_U & 0 \\ 0 & M_L \end{bmatrix}$$  \hspace{1cm} (2)$$

$$C = \begin{bmatrix} C_U & -C_U \\ -C_U & C_U + C_L \end{bmatrix}$$  \hspace{1cm} (3)$$

$$K = \begin{bmatrix} K_U & -K_U \\ -K_U & K_U + K_L \end{bmatrix}$$  \hspace{1cm} (4)$$

It is pertinent now, to define the following parameters:

$$\omega^2_U = \frac{K_U}{M_U}$$  \hspace{1cm} (5)$$

$$\omega^2_L = \frac{K_L}{M_L}$$  \hspace{1cm} (6)$$

$$\Omega = \frac{\omega_U}{\omega_L}$$  \hspace{1cm} (7)$$

$$\mu = \frac{M_U}{M_L}$$  \hspace{1cm} (8)$$

Equation (5) provides the circular frequency of the component, $\omega_U$, when this is considered independently as a system with one-degree-of-freedom. Likewise, Equation (6) provides the circular frequency of the building, $\omega_L$, as this is considered to be a one-degree-of-freedom system. Equation (7), usually denominated tuning, represents the ratio between the circular frequencies of the component and the building. For the purposes of this paper, $\Omega$ is equal to one. Equation (8) represents the ratio $\mu$ between the masses of the component and the building. The mass ratios to be considered in this work are small, of the order of 0.001.

For systems with the type of damping being considered, the modal solution of Equation (1) passes through the resolution of its eigenproblem, given by:

$$[K - \omega^2_j M]X_j = 0$$  \hspace{1cm} (9)$$

where $X_j$ and the square of $\omega_j$ represent the $j$th real eigenvector, and its corresponding $j$th real eigenvalue, respectively. The parameter $\omega_j$ is the circular frequency for the $j$th mode. Thus, a modal solution of Equation (1) yields the response of its $i$th degree of freedom $R_i(t)$, in terms of either relative displacement, relative velocity, or absolute acceleration, given by:
\[ R_j(t) = \sum_{j=1}^{N} \gamma_j X_{ij} r_j(t) \]  

where \( r_j(t) \) is the response to \( a(t) \) of a single-degree-of-freedom system with damping and frequency corresponding to the \( j \)th mode, and \( \gamma_j \) is the modal participation factor for the \( j \)th mode, given by:

\[ \gamma_j = \frac{\sum_{i=1}^{N} M_i X_{ij}}{\sum_{i=1}^{N} M_i X_{ij}^2} \]  

The term \( X_{ij} \) in the above equations indicates the \( i \)th amplitude of the \( j \)th mode, and \( M_i \) indicates the mass assigned to the \( i \)th degree-of-freedom. The summations in Equations (10) and (11) extend to the \( N \) degrees of freedom. Proceeding with a spectral calculation of the response, it is noted that in view of the small mass ratio, the two modal frequencies for the system depicted on Figure 1 will be very close to each other. For this reason, double summation equations will be used, Gupta [4]. Such family of methods were initially proposed by Rosenblueth et al [10]. A widely used modality is the Complete Quadratic Combination (CQC), developed by Der Kiureghian [3], and Wilson et al [14]. Thus, an approximation to the response \( R \) of the system being considered is given by:

\[ R \approx \left( \sum_{i=1}^{2} \sum_{j=1}^{2} \rho_{ij} R_i R_j \right)^{1/2} \]  

Or:

\[ R \approx \sqrt{R_1^2 + R_2^2 + 2\rho_{12} R_1 R_2} \]  

Where the sub-indexes correspond to the order of the modes of the system conformed by the components and the building. The parameter \( \rho_{ij} \) is the modal correlation factor. Chopra [2], has compared the correlation factor as indicated by the CQC method, with the corresponding factor proposed by Rosenblueth et al [10], and showed that both methods yield essentially the same results. This is more apparent when the frequencies are very close, as is the case for the systems being considered. However, it is noted that the correlation factor equations proposed by Rosenblueth et al [10] are simpler to manipulate, and are therefore chosen for the development of this paper. Thus, the modal correlation factor is written as:

\[ \rho_{12} = [1 + \left( \frac{\omega_1 \sqrt{1 - \xi^2_1} - \omega_2 \sqrt{1 - \xi^2_2}}{\frac{\omega_1}{\xi_1} + \frac{\omega_2}{\xi_2}} \right) \]^2 \]  

Where \( \omega_j \) and \( \xi_j \) represent the circular frequencies and coefficients of damping, respectively, for the modes of the system. The equivalent coefficients of damping \( \frac{3}{2} \xi_j \) take into account the
response reduction due to the finite duration $s$ of the white noise signal to which the earthquake is assimilated, and is written as:

$$\bar{\xi}_j = \xi_j + \frac{2}{\omega_j s}$$  \hspace{1cm} (15)

Assuming that the duration of the earthquake motion is sufficiently long, and that the modal circular frequencies are such that the product $\omega_j s$ is much larger than two, the second term to the right of equation (15) may be ignored, with the following identity being thus obtained:

$$\bar{\xi}_j \approx \xi_j$$  \hspace{1cm} (16)

Then, with consideration of Equation (16), and normalizing with respect to the circular frequency of the building, Equation (14) becomes:

$$\rho_{12} = \frac{(\frac{\omega_1}{\omega_L} + \frac{\omega_2}{\omega_L})^2}{(\frac{\omega_1}{\omega_L} + \frac{\omega_2}{\omega_L})^2 + (\frac{\omega_1}{\omega_L} \sqrt{1-\xi_1^2} + \frac{\omega_2}{\omega_L} \sqrt{1-\xi_2^2})^2}$$  \hspace{1cm} (17)

Based on the work by Miranda [8], a summary of which is presented in Appendix A, relative modal energy considerations are made to study the systems represented in Figure 1. Thus, the parameters $\alpha_j$ are defined as the modal kinetic energy ratios between the upper and lower masses while the system is freely vibrating in its $X_j$ mode. Similarly, the parameters $\beta_j$ are defined as the modal strain energy ratios between the upper and lower springs, while the system is freely vibrating in its $X_j$ mode. It is shown in appendix, that for a given tuning these four parameters are interrelated such that the knowledge of one of them is sufficient to determine the dynamic state of the system. Miranda [8], identified that for the condition of resonance, the following parameters apply:

$$\Omega = 1$$  \hspace{1cm} (18)

Then, from Equations (A.5), (A.6), (A.8), and (A.9), the corresponding relative modal energy parameters are:

$$\alpha_1 = \frac{1}{\alpha_2} = \beta_2 = \frac{1}{\beta_1} = \left(\frac{\sqrt{4 + \mu} + \sqrt{\mu}}{2}\right)^2$$  \hspace{1cm} (19)

For the small mass ratios under consideration, these parameters can be approximated as:

$$\alpha_1 = \beta_2 = 1 + \sqrt{\mu}$$  \hspace{1cm} (20)

And:
\[ \alpha_2 = \beta_i = 1 - \sqrt{\mu} \]  

(21)

For small mass ratios, Equations (20) and (21) indicate that important modal energy transfer, either kinetic or strain, is happening between the upper and lower parts, as the alpha and beta ratios are close to one. Ignoring coupling of the modal equations due to damping, the generalized damping, \( C_j \), for the \( j \)th mode may be approximated as:

\[ C_j^* = X_j^T C X_j = 2\xi_j \omega_j X_j^T M X_j \]  

(22)

And for the component and the building considered separately, it can be written that:

\[ C_U = 2\xi_U \omega_U M_U \]  

(23)

And:

\[ C_L = 2\xi_L \omega_L M_L \]  

(24)

Where \( \xi_U \) and \( \xi_L \) are the coefficients of damping for the component and the building, respectively. Replacing Equations (23) and (24) into (22) results in equation (A.17), which may be shown to be a first order approximation to the modal damping coefficients, Adhikari et al [1] based on Rayleigh [7]. This equation applies as long as the damping provided separately by the component and the building are not much different than as established by Equation (A.18). Hence, at resonance, in view of equation (18) and the small mass ratio, equation (A.17) is further approximated as:

\[ \xi_j = \frac{1}{2} \frac{\omega_j}{\omega_L} (\xi_U + \xi_L) \]  

(25)

Replacing the last equation, plus Equations (A.13) and (A.14) into (17), and in view of the small mass ratios being considered, the modal correlation factor is written as:

\[ \rho_{12} = \frac{4(\xi_U + \xi_L)^2}{8 + 4\mu + 2[(\xi_U + \xi_L)^2 - \sqrt{16 - 4(2 + \mu)(\xi_U + \xi_L)^2 + (\xi_U + \xi_L)^4}]} \]  

(26)

Which after further manipulations, is reduced to:

\[ \rho_{12} = \frac{(\xi_U + \xi_L)^2}{\mu + (\xi_U + \xi_L)^2} \]  

(27)
Now using Equations (20), (21), and equation (A.12), it may be shown that for small values of the mass ratio, the modal frequencies of the component-building system may be approximated by:

\[
\frac{\omega_1}{\omega_L} = 1 - \frac{\sqrt{\mu}}{2}
\]  

(28)

And:

\[
\frac{\omega_2}{\omega_L} = 1 + \frac{\sqrt{\mu}}{2}
\]  

(29)

It is seen that Equations (28) and (29) symmetrically bracket the frequency of the building from below and above. Given the proximity of these two frequencies, and given Equation (25), small errors are committed if the following approximations are accepted:

\[
(SA)_{j}^{2}(\omega_1, \xi_j) \cong (SA)_{2}^{2}(\omega_2, \xi_2) \cong (SA)_1(\omega_1, \xi_1)(SA)_2(\omega_2, \xi_2) \cong (SA)^2(\omega_L, \frac{\xi_u + \xi_L}{2})
\]  

(30)

Where the parameters \((SA)_j\) and \((SA)\) are the absolute acceleration spectral ordinates corresponding to the jth mode, and to the building, respectively. It may be observed that the latter corresponds to the average modal frequency, and to the average of the modal damping coefficients, of the component and building system respectively. Now, from Equations (A.15) and (A.16), as well as from Equations (20) and (21), the participation factors may be approximated as:

\[
\gamma_1 = \frac{2 + \sqrt{\mu}}{4}
\]  

(31)

\[
\gamma'_1 = \frac{2 - \sqrt{\mu}}{4}
\]  

(32)

At this point, it will be noted that in using the method shown in appendix, the resolution of the eigenproblem proposed by equation (9) is bypassed, since the mode shapes and frequencies are obtained through alternate means. Thus, given the modes of the system component-building from Equation (A.3), and using the double summation procedure described above, the maximum absolute acceleration for the component may be approximated as:

\[
R_u \approx \frac{(SA)(\omega_2, \frac{\xi_u + \xi_L}{2})}{2} \frac{2}{\sqrt{2[\mu + (\xi_u + \xi_L)^2]}}
\]  

(33)
The derivation of Equation (33) is the main objective of this paper. And as already stated in this development, this equation applies to systems for which the effects of non-proportional damping may be ignored; and to long seismic motions for which the product $\omega_0^2$ is much larger than two; and to components that resonate with one of the modes of the carrying structure. In the measure that the real conditions differentiate from the considerations discussed, Equation (33) will reduce its validity. In a similar way, the maximum acceleration of the building may be approximated as:

$$R_L \approx \frac{\sqrt{\mu + 2(\xi_U + \xi_L)^2}}{\sqrt{2[\mu + (\xi_U + \xi_L)^2]}} (SA)(\omega_L, \frac{\xi_U + \xi_L}{2})$$ \hspace{1cm} (34)

Equation (34) indicates that a response reduction may be obtained for low damped buildings that are resonant with highly damped components, through the effect of increased average damping on the spectral ordinates. This approach has already been proposed by Villaverde [12]. As seen above, this approach is valid only for components with light masses.

3. EFFECTIVE MASS RATIO AND EFFECTIVE SPECTRAL ACCELERATION

![Figure 2. Nonstructural component-building system](image)

Assuming only horizontal displacements at floor levels, Figure 2 represents a component affixed to the $i$th degree-of-freedom of a $N$ degree-of-freedom structure. This is contrasted with Figure 1 where the component is shown affixed to a resonant mode of the carrying structure. For the purpose of this study, and for both representations to be equivalent, the lower mass, the mass ratio, and the spectral absolute acceleration values considered up to now, require to be replaced by their corresponding effective values. Sackman et al [11], have demonstrated that the effective lower mass, $M_L^{effective}$, which is the mass corresponding to the building, is given by:
\[ (M_L)_{\text{effective}} = \frac{\phi_j^T M_s \phi_j}{(\phi_j)^2} = \frac{(M_S)^*_j}{(\phi_j)^2} \]  

(35)

Where \( \phi_j \) is the resonant mode of the structure, and \( \phi_j \) corresponds to the location of the \( i \)th degree-of-freedom to which the non-structural component is affixed to. \( M_s \) is the diagonal mass matrix for the structure. It may be observed that \( \phi_j^T M_s \phi_j \) is the generalized mass corresponding to the \( j \)th resonant mode, also written as \( (M_S)^*_j \). Therefore, the effective mass ratio, \( \mu_{\text{effective}} \), is written as:

\[ \mu_{\text{effective}} = \frac{M_U}{(M_L)_{\text{effective}}} = \frac{M_U (\phi_j)^2}{(M_S)^*_j} \]  

(36)

Likewise, Sackman et al [11] have demonstrated that the effective spectral absolute acceleration, \( A_{\text{effective}} \), is given by:

\[ (SA)_{\text{effective}} = \phi_j^T M_s r / \phi_j^T M_s \phi_j \]  

(37)

Where it is noted that, \( \phi_j^T M_s r / \phi_j^T M_s \phi_j \) is the participation factor for the resonant mode \( \phi_j \), and \( r \) is a vector of ones. It is understood then, that in the previous equations the mass ratio to consider shall be the one defined in Equation (36), and that the spectral absolute acceleration to consider shall be the one defined in Equation (37). Since the objective is to calculate the maximum response, the spectral absolute acceleration shall always be considered positive.

4. EFFECT OF NON-PROPORTIONALITY OF DAMPING

It is necessary to judge upon the validity of equation (33) in view of ignoring the coupling of the modal equations due to non-classical damping. For this objective, Igusa et al [5], have proposed a “non-classical damping parameter” \( \delta \), that using the nomenclature of this paper is written as:

\[ \delta = (\xi_L - \xi_U) \frac{1}{\Omega} \]  

(38)

These authors calculated the mean square response of resonant components to a white noise, exactly initially, that is taking into account the effects of non-proportional damping. And subsequently, in an approximated manner ignoring the resulting coupling. They found that given an error \( e \), exact procedures should be used if:

\[ \delta^2 > e[\mu + (\xi_U + \xi_L)^2] \]  

(39)

This means that for resonant systems exact procedures are to be used if:

\[ (\xi_U - \xi_L)^2 > e[\mu + (\xi_U + \xi_L)^2] \]  

(40)
As a way of illustration, for values of $\mu=0.001$, $\xi_U=0.03$, $\xi_L=0.05$, and accepting an error in the calculation of the response of 10%, the left side of Equation (40) yields 0.0004, whereas the right side becomes equal to 0.00074. Therefore, for this case Equation (40) does not hold, and thus the mean square response of the component can be calculated ignoring modal coupling due to non-proportional damping, within the error allowance. If the particular condition given by equation (A.18) applies, Equation (38) becomes zero, and Equations (39) and (40) are never fulfilled since the system would be classically damped.

5. COMPARISON WITH NUMERICAL RESULTS

For the purpose of comparison between the results derived from the theory developed above, and the numerical results, it will be assumed that the damping coefficient of the components is equal to the damping coefficient of the buildings. This assumption is made in view of limitations of current commercial software used for dynamic analysis, and is not a condition imposed by the methodology. This is also consistent with the state of knowledge about the damping for nonstructural components, and with common practice. Therefore, accepting this equality, the response of the component and the building, is now written as:

$$R_U \approx \frac{(SA)_{\text{effective}}(\omega_L, \xi_L)}{\sqrt{2(\mu_{\text{effective}} + 4\xi_L^2)}}$$

And:

$$R_L \approx \frac{\sqrt{\mu_{\text{effective}} + 8\xi_L^2}}{\sqrt{2(\mu_{\text{effective}} + 4\xi_L^2)}}(SA)_{\text{effective}}(\omega_L, \xi_L)$$

It is apparent that the validity of the equality of the coefficients of damping is debatable. In general, the damping of the buildings would be expected to be larger than that of the components. However, under this condition of equality it must be noted that Equations (41) and (42), besides all of the approximations made, are conceptually exact since they fulfill equation (A.18), indicating the adequacy of using classical modes.

Figure 3 depicts a mechanical system having four horizontal degrees of freedom representing the idealization of a building. Once a component is affixed to any level, the system will have five-degree-of-freedom as shown, for example, on Figure 4 and Figure 6. To assess the adequacy of Equation (41) for predicting the response of the component while resonant with any of the modes of the structure under spectral excitation, analyses were carried out using the software STAAD [19]. The system component-building will be considered to have a damping ratio of 5% for all modes, while subjected to an earthquake idealized by the acceleration spectrum shown in Figure 5. Using the nomenclature of ASCE 7-16 [6], this spectrum is defined by the parameters $S_{DS}=1.0g$ and $S_{D1}=0.4g$, where $g$ is the acceleration of gravity, and is representative of seismic motion in rigid soil. For these calculations, STAAD will use the CQC
method discussed above. Likewise, the coefficient of damping for the components and the building, is taken as 5%.

![Figure 3. Structural system with four-degree-of-freedom](image)

Referencing Figure 3, it will be assumed that each of the masses $m$ of the structure is equal to 444.8 kN/g (100 kips/g). It will also be assumed that the lateral stiffness of each of the four stories is equal to 59,384.31 kN/m (339.11 kips/inch). With these mass and stiffness values, it is noted that the period of the fundamental mode is equal to 0.5 seconds. The calculations also assume that the mass of the affixed components is equal to 0.01$m$. The components will be provided with a stiffness such that they are sequentially in resonance with each of the modes of the supporting structure, the dynamic characteristics of which are shown on Table 1. It is noted that the mode components are listed from the top down. It may also be noted that the modes have been normalized such that they are one at the roof of the building.

![Figure 4. Structural system with five degree-of-freedom with the component affixed to the top level](image)
For the component located at the top level, corresponding to the system shown in Figure 4, Table 2 shows the effective mass ratios, and effective spectral excitation corresponding to each of the structural modes on Table 1. It may be seen that for this case, the effective mass ratios decrease with the order of the corresponding modes.

**TABLE 1:** Modal characteristics of the four-degree-of-freedom structure shown in Figure 3

| LEVEL | MODE 1 | MODE 2 | MODE 3 | MODE 4 |
|-------|--------|--------|--------|--------|
| 4     | 1.000  | 1.000  | 1.000  | 1.000  |
| 3     | 0.879  | 0.000  | -1.347 | -2.532 |
| 2     | 0.653  | -1.000 | -0.532 | 2.879  |
| 1     | 0.347  | -1.000 | 1.532  | -1.879 |
| PERIOD | 0.500  | 0.174  | 0.113  | 0.092  |
| CIRCULAR FREQUENCY ($\omega_j$) | 12.566 | 36.183 | 55.437 | 68.000 |
| PARTICIPATION FACTOR ($\gamma_j$) | 1.241  | -0.333 | 0.120  | -0.028 |
| SPECTRAL ACCELERATION (SA)$_j$ | 0.800g | 1.000g | 1.000g | 1.000g |

**Figure 5.** 5% Damping Response spectrum
**TABLE 2:** Effective mass ratios, and effective spectral acceleration for the five-degree-of-freedom structure shown in Figure 4

|                  | MODE 1 | MODE 2 | MOD 3 | MODE 4 |
|------------------|--------|--------|-------|--------|
| EFFECTIVE MASS RATIO $\mu_{\text{effective}}$ | 0.00431 | 0.00333 | 0.00184 | 0.00052 |
| EFFECTIVE SPECTRAL ACCELERATION $A_{\text{effective}}$ | 0.99291g | 0.33333g | 0.11985g | 0.02766g |

It is instructive to assess the dynamic characteristics of the component-building system for component resonance with each of the building modes, with the component located at the top level as shown in Figure 4. Such characteristics are presented in Table 3 and Table 4. It can be observed on these two tables, that each mode is split into two modes with very close frequencies, that are denominated mode $a$ and mode $b$. It may be verified that the frequencies for these split modes approximate from below and above the frequency of the original structural mode, in agreement with Equations (28) and (29). For example, for the split first mode it is obtained:

$$\omega_{1,a} = (1 - \sqrt{\mu_{\text{effective}}/2})\omega_1 = 12.154 \text{ rad/sec}$$

(43)

And:

$$\omega_{1,b} = (1 + \sqrt{\mu_{\text{effective}}/2})\omega_1 = 12.979 \text{ rad/sec}$$

(44)

It is also observed that the participation factor for each mode has been divided, approximately, in the proportions indicated by Equations (31) and (32). For example, for the split first mode, these proportions are:

$$\gamma_{1,a} = \frac{2 + \sqrt{\mu_{\text{effective}}}}{4} = 0.516$$

(45)

And:

$$\gamma_{1,b} = \frac{2 - \sqrt{\mu_{\text{effective}}}}{4} = 0.484$$

(46)

It can also be observed in Tables 3 and 4, that as a result of the significant energy transfer between the component and the building, as indicated in Equations (20) and (21), the modal amplitude at the location of the component is quite important with respect to the other modal amplitudes. The results of the spectral analysis is shown on Table 5, indicating the seismic response experienced by the component as calculated with STAAD, and as predicted by Equation (41). It is shown on the second line of this table, that Equation (41) accurately forecasts the
seismic response of the components, with the exception of the fourth mode. It is also seen in this
table that Equation (41) significantly underestimates the results provided by STAAD for the
response of the fourth mode. Given that the modal energy of the building decreases in the
measure that the order of modes increases, and given the significant energy interaction between
the component and the building, it is to be expected in consequence that the response of the
components reduces with higher modes. It is noted that Equation (41), successfully predicts the
upper bound of the response, which is controlled by the fundamental mode.

TABLE 3: Modal characteristics for the five degree-of-freedom mechanical system
shown in Figure 4 and resonant with the first and second modes

| LEVEL | MODE 1.a | MODE 1.b | MODE 2.a | MODE 2.b |
|-------|----------|----------|----------|----------|
| COMPONENT | 15.773 | -14.717 | 18.522 | -16.265 |
| 4 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | 0.869 | 0.890 | -0.120 | 0.112 |
| 2 | 0.641 | 0.666 | -1.125 | -0.895 |
| 1 | 0.339 | 0.356 | -1.068 | -0.953 |
| PERIOD | 0.517 | 0.484 | 0.179 | 0.169 |
| CIRCULAR FREQUENCY \( (\omega_j) \) | 12.157 | 12.981 | 35.144 | 37.265 |
| PARTICIPATION FACTOR \( (\gamma_j) \) | 0.631 | 0.611 | -0.165 | -0.167 |
| SPECTRAL ACCELERATION \( (SA)_j \) | 0.775g | 0.827g | 1.000g | 1.000g |

TABLE 4: Modal characteristics for the five degree-of-freedom mechanical system
shown in Figure 4 and resonant with the third and fourth modes

| LEVEL | MODE 3.a | MODE 3.b | MODE 4.a | MODE 4.b |
|-------|----------|----------|----------|----------|
| COMPONENT | 26.976 | -20.425 | 63.251 | -31.586 |
| 4 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | -1.868 | -0.958 | -4.669 | -1.491 |
| 2 | -0.517 | -0.559 | 5.879 | 1.446 |
| 1 | 2.001 | 1.215 | -3.991 | -0.881 |
| PERIOD | 0.116 | 0.111 | 0.093 | 0.091 |
| CIRCULAR FREQUENCY \( (\omega_j) \) | 54.376 | 56.754 | 67.430 | 69.038 |
| PARTICIPATION FACTOR \( (\gamma_j) \) | 0.055 | 0.063 | -0.010 | -0.015 |
| SPECTRAL ACCELERATION \( (SA)_j \) | 1.000g | 1.000g | 1.000g | 1.000g |
**TABLE 5:** Comparison of the response of the components for effective values $\mu$ as given on Table 2, $\xi_U = 0.05$ and $\xi_L = 0.05$, for resonance with each mode of the carrying structure

|       | MODE 1 | MODE 2 | MODE 3 | MODE 4 |
|-------|--------|--------|--------|--------|
| FROM STAAD | 5.88g  | 2.06g  | 0.80g  | 0.24g  |
| FROM EQ (41) | 5.87g  | 2.04g  | 0.78g  | 0.19g  |
| STAAD/EQ(41) | 1.00   | 1.01   | 1.03   | 1.26   |

Consider now Figure 6, where it is shown that the component is affixed to the second level of the supporting structure. For this condition, Table 6 shows the corresponding effective mass ratios, and effective spectral accelerations. It will be noted that the effective mass ratios for modes two and four increase with respect to the previous mode, instead of decreasing with the mode order. Overall, the effective masses corresponding to this component location, tend to be smaller than the corresponding values for the component located at the top of the building. Likewise, the effective spectral acceleration for this component location tend to be smaller than those corresponding to the component located at the top of the building.

Again, it is instructive to assess the dynamic characteristics of the component-building system for component resonance with each of the building modes. Table 7 and Table 8 indicate such characteristics for component resonance with each of the modes, while the component is assigned to the second level as shown of Figure 6. Again, it can be observed on these two tables, that each mode is split into two modes with very close frequencies, that are denominated mode $a$ and mode $b$. The modal amplitude for the component is annotated at the second line of the tables. It may be verified that the frequencies for these split modes approximate from below and above the frequency of the original structural mode, in agreement with Equations (28) and (29). For example, for the first split mode it is obtained:

**TABLE 6:** Effective mass ratios, and effective spectral acceleration for the five degree-of-freedom structure shown in Figure 6

|       | MODE 1 | MODE 2 | MODE 3 | MODE 4 |
|-------|--------|--------|--------|--------|
| EFFECTIVE MASS RATIO $\mu_{\text{Effective}}$ | 0.00184 | 0.00333 | 0.00052 | 0.00431 |
| EFFECTIVE SPECTRAL ACCELERATION $A_{\text{Effective}}$ | 0.64807g | 0.33333g | 0.06377g | 0.07964g |
Figure 6. Structural system with five-degree-of-freedom, component affixed to the second level

\[ \omega_{1,a} = (1 - \frac{\sqrt{\mu_{\text{effective}}}}{2})\omega_1 = 12.297 \text{ rad / sec} \]  

And:

\[ \omega_{1,b} = (1 + \frac{\sqrt{\mu_{\text{effective}}}}{2})\omega_1 = 12.836 \text{ rad / sec} \]

It is also observed that the participation factor for each mode of the structure has been divided, approximately, in the proportions indicated by Equations (31) and (32). For example, for the split first mode these proportions are:

\[ \gamma_{1,a} = \frac{2 + \sqrt{\mu_{\text{effective}}}}{4} = 0.511 \]

And:

\[ \gamma_{1,b} = \frac{2 - \sqrt{\mu_{\text{effective}}}}{4} = 0.489 \]

It is observed in Tables 7 and 8, that again as a result of the significant energy interaction between the component and the building, as indicated in Equations (20) and (21), the modal amplitude at the location of the component is quite important with respect to the other modal amplitudes. The results of the spectral analysis is shown in Table 9, indicating the seismic response experienced by the component, as calculated with STAAD, and as predicted by Equation (41). It is show on the second line of this table, that Equation (41) accurately forecasts the seismic response of the components for modes one, two, and four. However, the result for mode three is not acceptable. It is also seen in this table that Equation (41), for this location of the component, tends to underestimates the results provided by STAAD for resonances with the
Given that the modal energy of the building decreases in the measure that the order of modes increases, and given the significant energy interaction between the component and the building, it is to be expected in consequence that the response of the components reduces with higher modes. It is noted that Equation (41), successfully predicts the upper bound of the response, which is controlled by the fundamental mode. It is also noted that the upper bound of the response at this level, is inferior to the upper bound of the response in the case where the component is affixed to the top floor.

**TABLE 7:** Modal characteristics for the five degree-of-freedom mechanical system shown in Figure 6 and resonant with the first and second modes

| LEVEL | MODE 1.a | MODE 1.b | MODE 2.a | MODE 2.b |
|-------|----------|----------|----------|----------|
| COMPONENT | 15.939 | -14.554 | -18.389 | 16.150 |
| 4 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | 0.885 | 0.874 | 0.058 | -0.057 |
| 2 | 0.667 | 0.638 | -0.939 | -1.054 |
| 1 | 0.354 | 0.341 | -1.051 | -0.937 |
| PERIOD | 0.511 | 0.490 | 0.179 | 0.169 |
| CIRCULAR FREQUENCY ($\omega_j$) | 12.296 | 12.833 | 35.119 | 37.203 |
| PARTICIPATION FACTOR ($\gamma_j$) | 0.626 | 0.615 | -0.175 | -0.158 |
| SPECTRAL ACCELERATION ($SA_j$) | 0.783g | 0.818g | 1.000g | 1.000g |

**TABLE 8:** Modal characteristics for the five degree-of-freedom mechanical system shown in Figure 6 and resonant with the third and fourth modes

| LEVEL | MODE 3.a | MODE 3.b | MODE 4.a | MODE 4.b |
|-------|----------|----------|----------|----------|
| COMPONENT | -27.941 | 19.433 | 34.904 | -54.795 |
| 4 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | -1.293 | -1.399 | -2.320 | -2.785 |
| 2 | -0.621 | -0.442 | 2.064 | 3.969 |
| 1 | 2.116 | 1.109 | -1.563 | -2.224 |
| PERIOD | 0.115 | 0.112 | 0.095 | 0.089 |
| CIRCULAR FREQUENCY ($\omega_j$) | 54.794 | 56.040 | 65.931 | 70.392 |
| PARTICIPATION FACTOR ($\gamma_j$) | 0.060 | 0.057 | -0.019 | -0.010 |
| SPECTRAL ACCELERATION ($SA_j$) | 1.000g | 1.000g | 1.000g | 1.000g |
TABLE 9: Comparison of the response of the components for effective values $\mu$ as given on Table 6, $\xi_U = 0.05$ and $\xi_L = 0.05$, for resonance with each mode of the carrying structure

| MODE 1 | MODE 2 | MODE 3 | MODE 4 |
|--------|--------|--------|--------|
| FROM STAAD | 4.25g  | 2.13g  | 0.72g  | 0.47g  |
| FROM EQ (41) | 4.21g  | 2.04g  | 0.44g  | 0.47g  |
| STAAD/EQ(41) | 1.01   | 1.04   | 1.64   | 1.00   |

At this point, through limited numerical calculations, the analytical formula given by Equation (41) has been demonstrated to be effective in predicting the response of components that are resonant with the lower structural modes. But in particular, and since the fundamental mode governs the response, the formula furnishes an upper bound of the response, if resonant with such mode.

6. CONCLUSIONS

An approximated analytical model for the rational study of the seismic response of linear elastic light nonstructural components resonant with a structural mode of the carrying building, has been proposed. Given the assumed smallness of the mass of the components with respect to the mass of the resonant mode, it may be assumed that an almost perfect energetic balance between the component and the resonant mode exists. Given this behavior, a spectral analysis leads to equations that predict the response of the components as a function of the dynamics of the component themselves, and of the dynamics of the resonant modes, obviating in this manner the necessity of performing response history calculations. The comparison between some numerical calculations, assuming equality of coefficients of damping for the component and the resonant building mode, demonstrates that the equations derived in this paper can within practical precision, capture the response of the components when these are resonant with the energy rich lower modes. However, the precision decreases for resonance with the higher modes of the building, in which case the response of the components is substantially underestimated. Additional research is needed to assess this reduction in precision. Such observation notwithstanding, the proposed analytical model successfully defines an upper limit of the response when resonance is induced with the fundamental mode. Finally, since the equations derived in this paper are compact, they could be extended to a format compatible with current building regulations, for their potential use in construction codes, providing in that manner expressions that can be rational yet sufficiently practical.

APPENDIX A

Consider the non-damped structural system shown on Figure 1. Assuming that it freely vibrates in its $X_j$ mode, the parameters $\eta_j$ are defined as the ratio between the kinetic energy induced at the top mass and the kinetic energy induced at the lower mass, such that:
\[ \alpha_j = \frac{M_U X_{Uj}^2}{M_L X_{Lj}^2} \]  

(A.1)

In the equation above, the subscripts \( U \) and \( L \) denote the upper and lower subsystems, respectively. In a similar manner, define \( \beta_j \) as the ratios between the strain energy stored by the upper spring and the strain energy stored by the lower spring, such that:

\[ \beta_j = \frac{K_U (X_{Uj} - X_{Lj})^2}{K_L X_{Lj}^2} \]  

(A.2)

For these two equations, if the lower modal components are normalized to one, the modes may be written as:

\[ X_j^T = \begin{bmatrix} \pm \frac{\alpha_j}{\mu} & 1 \end{bmatrix} \]  

(A.3)

or as:

\[ X_j^T = \begin{bmatrix} 1 \pm \frac{1}{\Omega} \sqrt{\frac{\beta_j}{\mu}} & 1 \end{bmatrix} \]  

(A.4)

Enforcing the orthogonality with respect to the mass and stiffness matrices, it may be shown that the following equations are obtained:

\[ \alpha_1 \alpha_2 = 1 \]  

(A.5)

\[ \Omega^2 = \frac{1}{1 + \sqrt{\mu \alpha_1} - \sqrt{\mu \alpha_2} - \mu} \]  

(A.6)

\[ \Omega = \frac{(\sqrt{\mu \beta_2} - \sqrt{\mu \beta_1}) + \sqrt{(\sqrt{\mu \beta_1} + \sqrt{\mu \beta_2})^2 + 4}}{2(1 + \mu)} \]  

(A.7)

And:

\[ \beta_1 \beta_2 = 1 \]  

(A.8)

From the upper components of the modes given by Equations A.3 and A.4, the energy ratios are related in accordance with the following expressions:
\[ \beta_j = \Omega^2 (\sqrt{\alpha_j} \mu \sqrt{\mu})^2 \] (A.9)

For conservative systems, the maximum modal kinetic energy \( E_K \) is equal to the maximum modal strain energy \( E_S \), therefore in terms of modal energy for the upper and lower components it may be written that for the \( j \)th mode, the following equation holds:

\[ (E_{KUj} + E_{KLj})_{\text{max.}} = (E_{SUj} + E_{SLj})_{\text{max.}} \] (A.10)

Or:

\[ \omega_j^2 M_L (1 + \alpha_j) = K_L (1 + \beta_j) \] (A.11)

Thus, the modal frequencies are given by the following expression:

\[ \left( \frac{\omega_j}{\omega_L} \right)^2 = \frac{1 + \beta_j}{1 + \alpha_j} \] (A.12)

It may be shown that the modal frequencies comply with the following expressions:

\[ \left( \frac{\omega_L}{\omega_L} \right) \left( \frac{\omega_2}{\omega_L} \right) = \Omega \] (A.13)

And:

\[ \left( \frac{\omega_L}{\omega_L} \right)^2 + \left( \frac{\omega_2}{\omega_L} \right)^2 = 1 + (1 + \mu) \Omega^2 \] (A.14)

Also, the participation factors may be shown to be:

\[ \gamma_1 = \frac{\sqrt{\alpha_2} + \sqrt{\mu}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} = \frac{\sqrt{\beta_2}}{\sqrt{\beta_1} + \sqrt{\beta_2}} \] (A.15)

And:

\[ \gamma_2 = \frac{\sqrt{\alpha_1} - \sqrt{\mu}}{\sqrt{\alpha_1} + \sqrt{\alpha_2}} = \frac{\sqrt{\beta_1}}{\sqrt{\beta_1} + \sqrt{\beta_2}} \] (A.16)

From the equality of the global modal energy dissipation to the summation of the modal energy dissipation by each of the two subsystems composing the system in Figure 1, it may be shown that the modal damping can be approximated by the expression:
It may be demonstrated that the modal coupling due to damping, for the system under consideration, vanishes if the following relationship is complied with:

\[
\Omega = \frac{\xi_u}{\xi_L} \quad (A.18)
\]

If such was the case, the modal vectors derived above will decouple the modal equations for the systems being studied.

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\[
\xi_j = \frac{\beta_j}{\Omega (1 + \beta_j)} \frac{\omega_j}{\omega_L} \xi_u + \frac{1}{(1 + \beta_j)} \frac{\omega_j}{\omega_L} \xi_L \quad (A.17)
\]
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