HAMILTON TRIANGLE OF A TRIANGLE IN THE ISOTROPIC PLANE

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ABSTRACT

In this paper we introduce the concept of the Hamilton triangle of a given triangle in an isotropic plane and investigate a number of important properties of this concept. We prove that the Hamilton triangle is homological with the observed triangle and with its contact and complementary triangles. We also consider some interesting statements about the relationships between the Hamilton triangle and some other significant elements of the triangle, like e.g. the Euler and the Feuerbach line, the Steiner ellipse and the tangential triangle.

KEYWORDS

standard triangle, Hamilton triangle, isotropic plane

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1. INTRODUCTION

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, absolute line $\omega$, and one point on that line, absolute point $\Omega$. The lines through the point $\Omega$ are isotropic lines, and the points on the line $\omega$ are isotropic points. Points with the same abscissa, i.e., which lie on the same isotropic line, are called parallel.

For two non-parallel points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the isotropic distance is defined by $d(P_1, P_2) := x_2 - x_1$. The isotropic distance is directed. For two parallel points $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$, the isotropic span is defined by $s(P_1, P_2) := y_2 - y_1$.

A triangle is called allowable if none of its sides is isotropic ([9]). If we choose the coordinate system in such a that the circumscribed circle of an allowable triangle $ABC$ has the equation $y = x^2$, and therefore its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where

$$a + b + c = 0,$$  \hspace{1cm} (1.1)
then we say that the triangle $ABC$ is in the standard position, or that $ABC$ is a standard triangle, for short. Its sides $BC$, $CA$, and $AB$ have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. Denoting
\[ p := abc, \quad q := bc + ca + ab, \]
we get e.g.
\[ q = bc - a^2. \tag{1.3} \]
To prove geometric facts for all allowable triangles it is sufficient to provide a proof for a standard triangle $[6]$.

The midpoints $A_m, B_m,$ and $C_m$ of the sides $BC, CA,$ and $AB$ determine the so-called complementary triangle $A_mB_mC_m$ of the triangle $ABC$ (Figure 1). In $[6]$, it is shown that for the standard triangle $ABC$ we get
\[
A_m = \left( -\frac{a}{2}, -\frac{1}{2}(q + bc) \right), \quad B_m = \left( -\frac{b}{2}, -\frac{1}{2}(q + ca) \right), \quad C_m = \left( -\frac{c}{2}, -\frac{1}{2}(q + ab) \right) \tag{1.4}
\]
and e.g. the line $B_mC_m$ has the equation
\[ y = -ax - q + \frac{bc}{2}. \tag{1.5} \]

The standard triangle $ABC$ has the centroid $G = (0, -\frac{2}{3}q)$ and its Euler line $E$ is given by the equation $x = 0$.

Isotropichy lines $h_a, h_b, \text{ and } h_c$ associated with sides $BC, CA,$ and $AB$ are isotropic lines passing through vertices $A, B,$ and $C$. The points $A_h = BC \cap h_a, B_h = CA \cap h_b,$ and $C_h = AB \cap h_c$ are vertices of the triangle called the orthic triangle of the triangle $ABC$. In case of the standard triangle $ABC$, the line $B_hC_h$ is given by the equation
\[ y = 2ax - q + 2bc. \tag{1.6} \]
The lines $AA_h, BB_h, CC_h$ pass through the absolute point $\Omega$, the points $BC \cap B_hC_h, CA \cap C_hA_h, AB \cap A_hB_h,$ lie on a line, call it $H$, and we say that the triangles $ABC$ and $A_hB_hC_h$ are homological. The center of this homology is the absolute point, and the line $H$ is the axis of homology (Figure 1). The line $H$ is the orthic line of the triangle $ABC$ and in case of the standard triangle $ABC$ its equation is $y = -\frac{2}{3}q$.

The Euler circle $K_e$ of the triangle $ABC$ touches its inscribed circle $K_i$ at the so-called Feuerbach point of this triangle ([1]). Their common tangent at this point is the so-called Feuerbach line $F$ of the triangle $ABC$ (Figure 1). In case of the standard triangle $ABC$, the Feuerbach point is $\Phi = (0, -q)$, the Feuerbach line $F$ is given by $y = -q$, and its inscribed circle $K_i$ is given by the equation
\[ y = \frac{1}{4}x^2 - q. \tag{1.7} \]

Tangent lines of the circle circumscribed to the triangle $ABC$ at vertices $A, B, C$ form the so-called tangential triangle $A_1B_1C_1$ of the triangle $ABC$. These two triangles are homological. The center of this homology is the symmedian center $K$ and its axis of homology is the Lemoine line $L$ of the triangle $ABC$ (Figure 1). The isotropic line through the symmedian center is the Brocard diameter $B$ of the triangle $ABC$. These facts are considered in $[5]$, where it is shown that for the standard triangle $ABC$ the Brocard diameter and the Lemoine line are given by $x = \frac{3p}{2q}$ and
\[ y = \frac{3p}{q}x + \frac{q}{3}, \tag{1.8} \]
respectively.
If $G$ is the centroid of the triangle $ABC$, then the homothecy $(G, -2)$ maps each line to its anticomplementary line. As $G = (0, -\frac{2}{3}q)$, the line $x = -\frac{2}{3}p$ is the anticomplementary line of the Brocard diameter.

Points $A_i, B_j, C_k$, where the incircle touches $BC, CA$ and $AB$ respectively, determine the contact triangle $A_iB_jC_k$ of the triangle $ABC$. According to [2], for the standard triangle $ABC$, we have e.g.

$$A_i = (-2a, bc - 2q)$$

and the line $B_jC_k$ has the equation

$$y = \frac{a}{2}x - q - bc.$$  

(1.10)

The triangles $ABC$ and $A_iB_jC_k$ are homological. The center of this homology is the Gergonne point-$\Gamma$ of the triangle $ABC$, while the axis of homology is the harmonic line of the point $\Gamma$. The Gergonne point $\Gamma$ is

$$\Gamma = \left( -\frac{3p}{q}, -\frac{4}{3}q \right).$$

(1.11)

its harmonic line is given by the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}q.$$  

(1.12)

and it is also the Steiner axis of the triangle $ABC$ ([12]).

For each triangle $ABC$ there is an ellipse which touches the lines $BC, CA, AB$ at points $A_m, B_m, C_m$ respectively. This is the inscribed Steiner ellipse $S$ of the triangle $ABC$ (Figure 1). According to [12], for the standard triangle $ABC$ this ellipse has the equation

$$4q^2x^2 - 36pxy - 12qy^2 - 24pqx - 16q^2y + 9p^2 - 4q^3 = 0.$$  

(1.13)
Owing to [7], two points are inverse to each other with respect to a circle if and only if they parallel and the circle contains their midpoint.

2. HAMILTON TRIANGLE AND SOME OTHER SIGNIFICANT ELEMENTS OF A TRIANGLE

By analogy to the Euclidean case, the triangle \( UVW \), whose vertices are the intersections of the corresponding sides of the complementary and the contact triangle of the triangle \( ABC \), \( U = B_mC_m \cap B_iC_i \), \( V = C_mA_m \cap C_iA_i \), \( W = A_mB_m \cap A_iB_i \), will be called the Hamilton triangle of the triangle \( ABC \), and the lines \( A_iU, B_iV, C_iW \) which join the corresponding vertices of the contact and the Hamilton triangle will be called the Hamilton lines of the triangle \( ABC \) (Figure 2).

**THEOREM 2.1.** Vertices of the Hamilton triangle \( UVW \) of the standard triangle \( ABC \) are

\[
U = \left( \frac{bc}{a}, -q - \frac{bc}{2} \right), \quad V = \left( \frac{ca}{b}, -q - \frac{ca}{2} \right), \quad W = \left( \frac{ab}{c}, -q - \frac{ab}{2} \right),
\]

(2.1)

the sides \( VW, WV, UW \) are given by

\[
y = \frac{bc}{2a}x + \frac{bc}{2} - q, \quad y = \frac{ca}{2b}x + \frac{ca}{2} - q, \quad y = \frac{ab}{2c}x + \frac{ab}{2} - q,
\]

(2.2)

and the Hamilton lines of the triangle \( ABC \) have the following equations

\[
y = -\frac{a}{2}x - q, \quad y = -\frac{b}{2}x - q, \quad y = -\frac{c}{2}x - q.
\]

(2.3)

**Proof.** From equations (1.5) of \( B_mC_m \) and (1.10) of \( B_iC_i \) we obtain the coordinates of the point \( U \)

\[
x = \frac{bc}{a}, \quad y = -q - \frac{bc}{2}.
\]

The obtained point \( U \) lies e.g. on the second line (2.2). The first line in (2.3) passes through points \( A_i \) from (1.9), and \( U \) from (2.1). Analogous facts hold for the remaining equalities (2.1) – (2.3). \( \square \)

**THEOREM 2.2.** Hamilton lines of a triangle pass through its Feuerbach point (Figure 2).

**Proof.** The lines in (2.3) obviously pass through the point \( \Phi = (0, -q) \). \( \square \)
Theorem 2.2 states that the contact and the Hamilton triangle of the triangle \(ABC\) are homological with the center of homology \(\Phi\). We would like to answer the following question: What is the axis of this homology?

**THEOREM 2.3.** The axis of homology of the contact and the Hamilton triangle of the allowable triangle \(ABC\) is the line anticomplementar to the Brocard diameter of the triangle \(ABC\) (Figure 2).

**Proof.** From the first equalities (1.10) and (2.2) we obtain the equation for the abscissa of the point \(B_1C_1 \cap VW\)

\[
\left( \frac{a}{2} - \frac{bc}{2a} \right) x = \frac{3}{2} bc, \quad \text{i.e.,} \quad (a^2 - bc)x = 3p,
\]

which, because of (1.3), gives \(x = -\frac{3p}{4}\). Therefore, the axis of homology is the line with equation \(x = -\frac{3p}{4}\), anticomplementary to the Brocard diameter of the triangle \(ABC\). □

**THEOREM 2.4.** The standard triangle \(ABC\) and its Hamilton triangle are homological. The center of homology is the point at infinity of the Steiner axis of the triangle \(ABC\), and the axis of homology is the line anticomplementar to the Brocard diameter of the triangle \(ABC\), i.e., the points \(A, B,\) and \(C\) lie on the lines \(VW, WU,\) and \(UV\) respectively (Figure 2).

**Proof.** Due to (1.3) and (1.2), the line \(AU\) through \(A = (a, a^2)\) and \(U\) from (2.1) has the slope \(-\frac{3p}{4q}\) as does the line (1.12). By (1.1), \(B = (b, b^2)\) and \(C = (c, c^2)\) obviously lie on the line \(y = -ax - bc\). According to (1.3), the point \((-a, -q)\) also lies on this line as well as on the line \(VW\) with the first equation in (2.2), and on the Feuerbach line of the triangle \(ABC\). The point \(A = (a, a^2)\) also satisfies the first equation in (2.2) because of (1.3). □

For the Euclidean case see [3] and [11].

The following theorem is a generalization of the Euclidean case, see [11].

**THEOREM 2.5.** The complementary and the Hamilton triangle of the standard triangle \(ABC\) are homological. The midpoint of the centroid and the Gergonne point of the triangle \(ABC\) is the center of homology and its Euler line is the axis of homology (Figure 2).

**Proof.** Let us consider the line

\[
2(q - 3bc)y = 2aqx - 2q^2 + 3bcq + 3b^2c^2.
\]

The points \(A_m\) and \(U\) from (1.4) and (2.1) lie on this line since, due to (1.3),

\[
2aq \left( -\frac{a}{2} \right) - 2q^2 + 3bcq + 3b^2c^2 = q(q - bc) - 2q^2 + 3bcq + 3b^2c^2
\]

\[
= -(q - 3bc)(q + bc),
\]

\[
2aq \cdot \frac{bc}{a} - 2q^2 + 3bcq + 3b^2c^2 = -2q^2 + 5bcq + 3b^2c^2
\]

\[
= -(q - 3bc)(2q + bc).
\]

However, by (1.3), we get

\[
2aq \left( -\frac{3p}{2q} \right) - 2q^2 + 3bcq + 3b^2c^2 = -3bc(bc - q) - 2q^2 + 3bcq + 3b^2c^2
\]

\[
= -2q(q - 3bc)
\]

and the point

\[
S = \left( -\frac{3p}{2q}, -q \right)
\]

(2.4)
lies on this line. The point \( S \) is the midpoint of points \( G = (0, -\frac{2}{3}q) \) and \( \Gamma \) from (1.11). With \( x = 0 \) from equation (1.5) and the first equation in (2.2) of lines \( B_mC_m \) and \( VW \) we get \( y = \frac{bc}{2} - q \), and therefore

\[
B_mC_m \cap VW = \left( 0, \frac{bc}{2} - q \right).
\]

This point obviously lies on the Euler line of the triangle \( ABC \). \( \square \)

Similarly, as in the Euclidean case (\([10]\)), we have:

**Theorem 2.6.** Let \( A_mB_mC_m \) be the complementary triangle and \( VW \) the Hamilton triangle of the standard triangle \( ABC \). If \( B'_m \) and \( C'_m \) are points symmetrical to \( B_m \) and \( C_m \) with respect to points \( W \) and \( V \), then the centroid \( G_a \) of points \( A_m, B'_m, C'_m \) lies on the line \( VW \).

**Proof.** Since \( B'_m = 2W - B_m \) from (1.4) and (2.1) we get the coordinates of the point \( B'_m \)

\[
\frac{2ab}{c} + \frac{b}{2} = \frac{bc + 4ab}{2c}, \quad -2q - ab + \frac{1}{2}q + \frac{1}{2}ca = \frac{1}{2}(ca - 2ab - 3q),
\]

so we have

\[
B'_m = \left( \frac{bc + 4ab}{2c}, \frac{1}{2}(ca - 2ab - 3q) \right), \quad C'_m = \left( \frac{bc + 4ac}{2b}, \frac{1}{2}(a - 2ca - 3q) \right). \tag{2.5}
\]

Coordinates of the centroid \( G_a \) of points \( A_m, B'_m, \) and \( C'_m \) from (1.4) and (2.5), are given by (1.1) and (1.3)

\[
3x = -\frac{a}{2} + \frac{1}{2bc}(b^2c + 4ab^2 + bc^2 + 4ac^2)
\]

\[
= -\frac{a}{2} + \frac{1}{2bc}(bc(b + c) + 4a(b + c)^2 - 8abc)
\]

\[
= -\frac{a}{2} + \frac{1}{2bc}(4a^3 - 9abc) = \frac{2a}{bc}(bc - q) - 5a = -3a - \frac{2aq}{bc},
\]

\[
3y = \frac{1}{2}(-q - bc + ca - 2ab - 3q + ab - 2ca - 3q)
\]

\[
= \frac{1}{2}(-bc - ca - ab - 7q) = -4q,
\]

and therefore

\[
G_a = \left( -a - \frac{2aq}{3bc}, -q \right).
\]

The point \( G_a \) lies on the first line (2.2) because

\[
\frac{bc}{2a} \left( -a - \frac{2aq}{3bc} \right) + \frac{bc}{2} - q = -\frac{1}{3}q - q = -\frac{4}{3}q. \]

\( \square \)

Similarly, as in the Euclidean case (\([3]\)), we have:

**Theorem 2.7.** The Hamilton triangle of the standard triangle is an autopolar triangle with respect to its inscribed circle and its inscribed Steiner ellipse (Figure 3).

**Proof.** The polar line of the point \( (x_o, y_o) \) with respect to the circle (1.7) is the line \( y + y_o = \frac{1}{2}x_o - 2q \), which in the case of the point \( U \) from (2.1) becomes

\[
y - q = \frac{bc}{2a}x - 2q,
\]

and it is the first equation in (2.2), i.e., the polar of the point \( U \) is the line \( VW \). The polar of the point \( (x_o, y_o) \) with respect to the ellipse (1.13) is the line

\[
4q^2x_o - 18p(y_o + x_o)y - 12qy_o - 12pq(x + x_o) - 8q^2(y + y_o) + 9p^2 = 0.
\]
In particular, for the point $U$ from (2.1), because of (1.2) and (1.3), the coefficients of $x$, $y$ and 1 in this equation are

$$4q^2x_0 - 18py_0 - 12pq = 4 \frac{bc}{a} q^2 + 18p \left( q + \frac{bc}{2} \right) - 12pq$$

$$= \frac{bc}{a} (4q^2 + 6a^2q + 9ap)$$

$$= \frac{bc}{a} [4q^2 + 6q(bc - q) + 9bc(bc - q)]$$

$$= - \frac{bc}{a} (2q^2 + 3bcq - 9b^2c^2),$$

$$-18px_0 - 12qy_0 - 8q^2 = -18b^2c^2 + 12q \left( q + \frac{bc}{2} \right) - 8q^2$$

$$= 2(2q^2 + 3bcq - 9b^2c^2),$$

$$-12pqx_0 - 8q^2y_0 - 9p^2 - 4q^3 = -12b^2c^2q + 8q^2 \left( q + \frac{bc}{2} \right) + 9p^2 - 4q^3$$

$$= 4q^3 + 4bcq^2 - 12b^2c^2q + 9b^2c^2(bc - q)$$

$$= 4q^3 + 4bcq^2 - 21b^2c^2q + 9b^3c^3$$

$$= (2q - bc)(2q^2 + 3bcq - 9b^2c^2),$$

respectively, so this equation becomes

$$- \frac{bc}{a} x + 2y + 2q - bc = 0,$$

and this is the first equation in (2.2). □

**Figure 3.**

**Theorem 2.8.** Lengths of the sides of the allowable triangle $ABC$ and its Hamilton triangle $UVW$ satisfy the equality $d(V, W) \cdot d(W, U) \cdot d(U, V) = d(B, C) \cdot d(C, A) \cdot d(A, B)$, and the ratio between their areas is $-2 : 1$. 
which implies the circumscribed circle of the Hamilton triangle of the standard triangle.

**THEOREM 2.10.**

\[
d(V, W) = \frac{ab}{c} - \frac{ca}{b} = \frac{a}{bc}(b^2 - c^2) = -\frac{a^2}{bc}(b - c) = \frac{a^2}{bc} \cdot d(B, C),
\]

and analogously

\[
d(W, U) = \frac{b^2}{ca} \cdot d(C, A), \quad d(U, V) = \frac{c^2}{ab} \cdot d(A, B),
\]

which implies \(d(V, W) \cdot d(W, U) \cdot d(U, V) = d(B, C) \cdot d(C, A) \cdot d(A, B).\) For the areas \(\triangle\) and \(\triangle'\) of triangles \(ABC\) and \(UVW\) we obtain

\[
2\triangle = \begin{vmatrix}
  a & a^2 & 1 \\
  b & b^2 & 1 \\
  c & c^2 & 1
\end{vmatrix} = (b - c)(c - a)(a - b),
\]

\[
2\triangle' = \begin{vmatrix}
  bc & -q & -bc \\
  ab & -q & -ab \\
  cb & -q & -cb
\end{vmatrix} = \begin{vmatrix}
  bc & -bc \\
  ab & -ab \\
  cb & -cb
\end{vmatrix} = \frac{1}{2} c a \begin{vmatrix}
  b & 1 \\
  ab & 1
\end{vmatrix} = \frac{1}{2} (b c (b - c) + c a (c - a + a b (a - b))
\]

\[
= \frac{1}{2} (b - c)(c - a)(a - b) = -\frac{1}{2} \cdot 2\triangle = -\triangle.
\]

**THEOREM 2.9.** The circumscribed circle of the Hamilton triangle of the standard triangle \(ABC\) (Figure 3) is given by the equation

\[
y = -\frac{1}{2} x^2 + \frac{q^2}{2p} x - \frac{q}{2}.
\]

**Proof.** If we consider for example the point \(U\) from (2.1), then because of (1.3) we get

\[
\frac{1}{2} \cdot \frac{b^2 c^2}{a^2} + \frac{q^2}{2p} \cdot \frac{bc}{a} - \frac{q}{2} = -\frac{1}{2a^2} (b^2 c^2 - q^2 + a^2 q)
\]

\[
= -\frac{1}{2a^2} ((bc + q) a^2 + a^2 q)
\]

\[
= \frac{1}{2} (bc + 2q) = -q - \frac{bc}{2}.
\]

We will mention two more results concerning the Hamilton triangle of the triangle \(ABC\).

**THEOREM 2.10.** Let \(U_1 = C_m A_m \cap A_i B_i\), \(U_2 = A_m B_m \cap C_i A_i\), \(V_1 = A_m B_m \cap B_i C_i\), \(V_2 = B_m C_m \cap A_i B_i\), \(W_1 = B_m C_m \cap C_i A_i\), and \(W_2 = C_m A_m \cap B_i C_i\) be the intersections of the non-corresponding sides of the complementary triangle \(A_mB_mC_m\) and the contact triangle \(A_i B_i C_i\) of the allowable triangle \(ABC\). Points \(U_1\) and \(U_2\), \(V_1\) and \(V_2\), \(W_1\) and \(W_2\), are inverse to each other, with respect to the inscribed circle of the triangle, and are parallel to points \(A, B,\) and \(C\) respectively (Figure 3).

For the Euclidean case, see [4].

**Proof.** The lines \(C_m A_m\) and \(A_i B_i\), analogous to lines \(B_m C_m\) and \(B_i C_i\), with equations (1.5) and (1.10), are given by

\[
y = -bx - q + \frac{ca}{2}, \quad y = \frac{c}{2} x - q - ab,
\]

from where \(x = a, y = \frac{ca}{2} - ab - q\), so we get

\[
U_1 = \left( a, \frac{ca}{2} - ab - q \right), \quad U_2 = \left( a, \frac{ab}{2} - ca - q \right).
\]
The points $U_1$ and $U_2$ are parallel to the point $A$, and the ordinate of their midpoint is
\[
\frac{1}{2} \left( -\frac{ca}{2} - \frac{ab}{2} - 2q \right) = \frac{1}{2} \left( -\frac{q}{2} + \frac{bc}{2} - 2q \right) = \frac{1}{4} (bc - 5q) = \frac{1}{4} (a^2 - 4q),
\]
and this midpoint $\left( a, \frac{1}{4}a^2 - q \right)$ lies on the circle (1.7), i.e., the points $U_1$ and $U_2$ are inverse to each other.

**THEOREM 2.1.** If $L, M, N$ are the intersections of the corresponding sides of the orthic triangle $A_6B_6C_6$ and the contact triangle $A_1B_1C_1$ of the allowable triangle $ABC$, then the triangles $ABC$ and $LMN$ are homological. The center of this homology lies on the orthic line of the triangle $ABC$ and its axis is parallel to the Lemoine line of this triangle.

In the Euclidean case, the center of homology is the Feuerbach point of the triangle $ABC$ (see [8]).

**Proof.** First of the three analogous points
\[
L = \left( -\frac{2bc}{a}, -q - 2bc \right),
\]
\[
M = \left( \frac{2ca}{b}, -q - 2ca \right),
\]
\[
N = \left( -\frac{2ab}{c}, -q - 2ab \right)
\]
lies on lines (1.6) and (1.10) and therefore $L = B_6C_6 \cap B_1C_1$. The line
\[(3bc - q)y = 3px - a^2q\]
passes through points $A = (a, a^2)$ and $L$, since, from (1.2) and (1.3) we obtain
\[(3bc - q)a^2 - 3pa + a^2q = 0,
(3bc - q)(-q - 2bc) - 3p \left( -\frac{2bc}{a} \right) + a^2q = q^2 - bcq - 6b^2c^2 + 6\frac{p}{a}bc + a^2q
= q(q - bc + a^2) = 0.
\]
Also, this line obviously passes through the point
\[
T = \left( -\frac{2q}{9p}, -\frac{q}{3} \right)
\]
because
\[
(3bc - q) \left( -\frac{q}{3} \right) - 3p \left( -\frac{2q^2}{9p} \right) + a^2q = q(q - bc + a^2) = 0.
\]
Likewise, the lines $BM$ and $CN$ pass through the point $T$. This point lies on the orthic line of the triangle $ABC$ given by $y = -\frac{q}{2}$. The line
\[y = -\frac{bc}{a} - q + 2bc\]
passes through points $M$ and $N$ from (2.6) because e.g. by (1.1), for the point $M$ we get
\[-\frac{bc}{a} \left( -\frac{2ca}{b} \right) = -q + 2bc = -q + 2bc + 2c^2 = -q - 2ca.
\]
The point
\[
\left( \frac{3p}{q}, a, -q - \frac{3ap}{q} \right)
\]
lies on this line as well as on the line BC because, according to (1.2) and (1.3), we get
\[
\frac{bc}{a} \left( \frac{3p}{q} - a \right) - q + 2bc = -\frac{3b^2c^2}{q} - q + 3bc
\]
\[
= -\frac{3bc}{q}(q + a^2) - q + 3bc = -q - \frac{3ap}{q},
\]
and therefore
\[
-a \left( \frac{3p}{q} - a \right) - bc = -q - \frac{3ap}{q}.
\]
Hence, the point (2.8) is the intersection BC ∩ MN. It lies on the line
\[
y = \frac{3p}{q}x - q - \frac{9p^2}{q^2}
\]
(2.9)
because
\[
\frac{3p}{q} \left( \frac{3p}{q} - a \right) - q - \frac{9p^2}{q^2} = -q - \frac{3ap}{q},
\]
and analogous points CA ∩ NL and AB ∩ LM lie on this line as well. Line (2.9) is parallel to the line (1.8), i.e., to the Lemoine line of the triangle ABC. □

**COROLLARY 2.12.** In case of the standard triangle ABC, the center and the axis of homology of the triangles ABC and LMN from Theorem 2.11 are given by (2.7) and (2.9), and the vertices of the triangle LMN are given by (2.6).

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