Gravitoelectromagnetism in a complex Clifford algebra

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Abstract

A linear vector model of gravitation is introduced in the context of quantum physics as a generalization of electromagnetism. The gravitoelectromagnetic gauge symmetry corresponds to a hyperbolic unitary extension of the usual complex phase symmetry of electromagnetism. The reversed sign for the gravitational coupling is obtained by means of the pseudoscalar of the underlying complex Clifford algebra.

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1 Introduction

The formal correspondence between Newton’s law of gravitation and the electrostatic Coulomb law has motivated many attempts to formulate gravitation in an analogy to electromagnetism. Maxwell himself [1] (and later Heaviside [2, 3]) turned his attention to the possibility of formulating the theory of gravitation in a form corresponding to the electromagnetic equations. However, the problem of the negative energy of the gravitational field, due to the mutual attraction of material bodies, appeared too serious to him to further follow this approach. Holzmüller [4] and Tisserand [5, 6] postulated that the gravitational force of the Sun had an additional magnetic component. This postulated gravitomagnetic component could be adjusted to reproduce the excess perihelion precession of Mercury. However, the solution of the relativistic Kepler problem, without any additional interaction terms, explains only one sixth of the discrepancy of 43 arc-seconds per century (see, e.g., Rindler [7]).

Some decades later Einstein predicted in his general relativity that gravitation is mediated by a second degree tensor field associated with the metric of spacetime. Einstein’s general relativity provided an explanation of the excess motion of Mercury’s perihelion in terms of a relativistic gravitoelectric correction to the Newtonian gravitational potential of the Sun [8]. Though Einstein’s theory of gravitation is substantially different to Maxwell’s vector gravitation, it could be shown that there are formal similarities. Bel [9] and Penrose [10] have shown that the linearized Einstein equations, perturbed about flat space-time, can be written in a form that looks similar to Maxwell’s equations. These
gravitoelectromagnetic field equations have been derived in different forms by several authors. For more information and further references about this topic it is referred to the following articles [11, 12, 13, 14, 15, 16, 17].

Despite the success of Einstein’s general relativity there was again interest in a Maxwell-like vector gravitation theory in the seventies and eighties of the last century [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. The simplest way to cross over from electromagnetism to gravity consists here in substituting for the electrical charge and positive dielectric constant in the Maxwell equations either the imaginary gravitational charges (Majerník [18, 20]) or the negative permittivity (Brillouin [28]). Then the Coulomb and Newton laws for two charges get the same form, with the only difference, the opposite sign of the forces. The field energy density is, however, necessarily negative (see also Richterek and Majerník [29]). Majerník [30] reproduces in this context the perihelion-shift when taking into account the self-gravity of field energy of a gravitational field. He [31] has also shown that by means of a coupling between the gravitational and electromagnetic fields all well-known tests of Einstein’s theory of gravitation connected with the propagation of light in gravitational field can be correctly calculated. The consistency of the vector theory of gravitation has been examined also by Singh [32]. He modified the Hamiltonian for the two-body interaction by a term for the self-interaction between particle velocity and its vector potential to explain the precession of the perihelion of a planet, the deflection of light in the gravitational field of a star, and the gravitational red shift, as predicted by the results of the general theory of relativity.

Motivated by these attempts to reproduce post-Newtonian gravitational effects by a vector theory of gravitation, a Maxwell-like model for gravitoelectromagnetism is proposed in this work. The model differs not only from Einstein’s general relativity, but also from the known Maxwell-like approaches. It is introduced in the terminology of quantum physics as a straightforward extension of the \( U(1, \mathbb{C}) \) gauge symmetry of electromagnetism into the hyperbolic unitary group \( U(1, \mathbb{H}) \), including electromagnetism and gravitation.

A larger list of references regarding applications of the hyperbolic numbers \( \mathbb{H} \), defined as an extension of the real or the complex numbers, has been given in [33, 34]. However, it should be mentioned again that they have been applied also to general relativity, where the hyperbolic numbers are also known as paracomplex or split-complex numbers. The connection between differential geometry and the hypercomplex number systems has been shown originally by Bianchi [47], and recently outlined again by Catoni et al. [48]. Paracomplex projective models and harmonic maps were investigated by Erdem [55, 56, 57]. A survey on paracomplex geometry, para-Hermitian, and para-Kaehler manifolds has been given by Cruceanu et al. [38, 39]. Solutions of Minkowskian sigma models generated by hyperbolic numbers were considered by Lambert et al. [40, 41]. Zhong investigated hyperbolic complex linear symmetry groups and their local gauge transformation actions [42]. He generated new solutions of the stationary axisymmetric Einstein equations with hyperbolic numbers [43]. Furthermore, the hyperbolic complexification of Hopf algebras [44]. Moffat [45] has interpreted the hyperbolic number as fermion number. This interpretation has led to fundamental explanation of stability of fermionic matter. Kunstatter et al. [46] investigated in this context a generalized theory of gravitation, based on a nonsymmetric metric in a four-dimensional real manifold.

The hyperbolic numbers are used in this work to represent the \( \mathbb{R}^{3,0} \) paravec-
tor algebra, which has been introduced by Sobczyk [49] for the representation of relativistic vectors. Baylis has shown that the theory of electrodynamics can be fully expressed in terms of this algebra. In his textbook [50] a wide range of explicit physical applications of the $R_{3,0}$ algebra can be found. Inserting the hyperbolic unit into this formalism implies, that the algebra can be complexified further to provide the complex Clifford algebra $\mathcal{C}_{3,0}$. It has been proposed in [51] to use this algebra to represent physical operators, like the mass operator, in their most general form.

2 Hyperbolic algebra

The commutative ring of hyperbolic numbers $z \in H$ can be defined as an extension of the complex numbers

$$z = x + iy + jv + ijw, \quad x, y, v, w \in \mathbb{R},$$

where the hyperbolic unit $j$ has the property $j^2 = 1$. In the terminology of Clifford algebras the hyperbolic numbers defined in this way are represented by $\mathcal{C}_{1,0}$, i.e., they correspond to the universal one-dimensional complex Clifford algebra (see Porteous [52]).

Beside the grade involution, two anti-involutions play a major role in the description of Clifford algebras and their structure, conjugation and reversion. Conjugation changes the sign of the complex and the hyperbolic unit

$$\bar{z} = x - iy - jv + ijw.$$  \hspace{1cm} (2)

Reversion, denoted as $z^\dagger$, changes only the sign of the complex unit. Anti-involutions reverse the ordering in the multiplication, e.g., $(ab)^\dagger = b^\dagger a^\dagger$. This becomes important when non-commuting elements of an algebra are considered. In physics, reversion is denoted as hermitian conjugation. Note, that in [51] it has been suggested to relate hermiticity in the physical sense to the conjugation anti-involution. With respect to conjugation the square of the hyperbolic number can be calculated as

$$z\bar{z} = x^2 + y^2 - v^2 - w^2 + 2ij(xw - yv).$$  \hspace{1cm} (3)

The hyperbolic numbers are used to form the hyperbolic paravector algebra. A Minkowski vector $x^\mu = (x^0, x^i) \in R^{3,1}$ is represented in terms of the hyperbolic algebra as

$$x = x^\mu e_\mu.$$  \hspace{1cm} (4)

The basis elements $e_\mu = (e_0, e_i)$ include the unity and the Pauli algebra multiplied by the hyperbolic unit $j$

$$e_\mu = (1, j\sigma_i).$$  \hspace{1cm} (5)

The only non-trivial expressions that can be generated by multiplication of the basis elements are $j\sigma_i$, $i\sigma_i$, and $ij$. Together with the unity they form the eight-dimensional algebra $R_{3,0}$. The algebra can be complexified with either the hyperbolic or the complex unit, which provides the additional elements $i$, $j$, $\sigma_i$, and $ij\sigma_i$. The full structure is equivalent to the universal complex Clifford
algebra $\mathcal{C}_{3,0}$. The complexified algebra includes sixteen real dimensions. The pseudoscalar of the hyperbolic algebra corresponds to

$$ij = e_0\bar{e}_1e_2\bar{e}_3 .$$

(6)

The scalar product of two Minkowski vectors is defined as

$$x \cdot y = \frac{1}{2}(x\bar{y} + y\bar{x}) .$$

(7)

The basis elements of the $\mathcal{C}_{3,0}$ paravector algebra can be considered as the basis vectors of the relativistic vector space. These basis elements form a non-cartesian orthogonal basis with respect to the scalar product

$$e_\mu \cdot e_\nu = g_{\mu\nu} ,$$

(8)

where $g_{\mu\nu}$ is the metric tensor of the Minkowski space.

The group $SU(2,H)$ corresponds to the spin group of the hyperbolic algebra and its elements can be used to express rotations and boosts of the paravectors. The rotation of a paravector can be represented as

$$x \rightarrow x' = RxR^\dagger ,$$

(9)

For the boosts one finds the transformation rule

$$x \rightarrow x' = BxB^\dagger .$$

(10)

Rotations and boosts are given as

$$R = \exp(-i\sigma_i\theta^i/2) , \quad B = \exp(j\sigma_i\xi^i/2) .$$

(11)

The infinitesimal generators of a Lorentz transformation can be identified as

$$J_i = \sigma_i/2 , \quad K_i = ij\sigma_i/2 .$$

(12)

The generators satisfy the Lie algebra of the Lorentz group.

Boosts are invariant under reversion $B^\dagger = B$, whereas the conjugated boost corresponds to the inverse $\bar{B} = B^{-1}$. For rotations reversion and conjugation correspond both to the inverse $R^\dagger = \bar{R} = R^{-1}$. This relationship indicates that in non-relativistic physics the hermiticity of operators can be defined either with respect to reversion or conjugation. The effect of conjugation, reversion, and graduation on the used hypercomplex units is displayed in Table II.

Note, that graduation is an involution, which does not reverse the ordering in a product, i.e., $\hat{a}\hat{b} = \hat{a}\hat{b}$. Conjugation, reversion, and graduation are related by $\bar{a} = \hat{a}$.

This was a brief summary of the most important facts. A more detailed representation of the hyperbolic algebra can be found in [33].

3 Maxwell-like model of gravitation

The electromagnetic vector potential is attractive for unequal and repulsive for equal charges. For gravitation one expects the potential to be attractive for equal charges and repulsive for unequal charges. The proposal made here is
Table 1: Effect of conjugation, reversion, and graduation on the used hypercomplex units.

| a  | ă | a⁺ | ă⁺ |
|----|---|----|----|
| e₀ | + | +  | +  |
| e₁ | − | +  | −  |
| σ₁ | + | +  | +  |
| i  | − | −  | +  |
| j  | − | +  | −  |

to extend the $U(1, C)$ gauge symmetry of electromagnetism to the hyperbolic numbers. The $U(1, H)$ phase transformation is thus written as

$$C = \exp \left(-i(A + ij\Lambda_g)\right).$$

(13)

This global gauge transformation is unitary with respect to conjugation

$$CC = 1.$$  

(14)

The phase transformation can be extended to local gauge transformations $C(x)$. The mass operator \[33\] is then modified to be invariant under these transformations.

$$M^2 = (p - V(x))(\bar{p} - \bar{V}(x)),$$  

(15)

where $p = i\partial^\mu e_\mu$ corresponds to the momentum operator. The vector potential $V(x) = V^\mu(x)e_\mu$ is a combination of electromagnetic and gravitoelectromagnetic contributions

$$V(x) = eA(x) + ijgA_g(x),$$  

(16)

where $g$ denotes the gravitoelectric charge. Eq. (15) implies a coupling between the electromagnetic and the gravitoelectromagnetic fields. To simplify the considerations this coupling is neglected in the following.

The mass operator is acting on the spinor field with the squared mass of the state as its eigenvalue

$$M^2\psi(x) = m^2\psi(x).$$  

(17)

The hyperbolic spinor is represented here as a two-component column spinor $\psi^i \in \mathbb{H}^2$. This implies that the Pauli algebra is given in terms of the Pauli matrices. The bar symbol indicates that the correlation, which maps the elements of the spinor to its dual space, is represented with transposition and conjugation as given in Eq. (2). Note, that the spinor can be represented also in an algebraic form \[34\].

The mass equation for the electromagnetic vector potential is defined as

$$M^2A(x) = -J(x),$$  

(18)

with the mass operator $M^2 = p\bar{p}$ and the current $J(x) = J^\mu(x)e_\mu$. Explicitly, this equation can be written as \[33\]

$$M^2A = -\nabla \cdot E - \partial^\mu C$$
$$-j(\nabla \times B - \partial^\mu E - \nabla C)$$
$$-i(\nabla \times E + \partial^\mu B)$$
$$+ij\nabla \cdot B = -\rho - jJ,$$  

(19)
where \( C = \partial_\mu A^\mu \) disappears in the Lorentz gauge. Note, that the Pauli algebra is implicitly part of the three-dimensional vectors, e.g., \( E = E^i \sigma_i \).

It is now proposed that the equations for the gravitoelectromagnetic fields have exactly the same form as for the electromagnetic fields. The reversion of sign for the gravitoelectromagnetic coupling is realized in the hyperbolic algebra with the following mechanism. A Lagrangian of the form

\[
\mathcal{L}(x) = \bar{\psi} M^2 \psi - m^2 \bar{\psi} \psi
\]

(20)
is assumed for the spinor field. Eqs. (15) and (16) then imply that the gravitoelectromagnetic current is proportional to the gravitoelectric charge multiplied by the pseudoscalar of the hyperbolic algebra

\[
J \propto e, \quad J_g \propto ijg.
\]

(21)

From Eq. (18) it follows that also the gravitoelectromagnetic vector potential is proportional to the pseudoscalar

\[
A_g \propto ijg.
\]

(22)

Reinserting this relationship into Eqs. (15) and (16) leads to a reversion of sign for the gravitoelectromagnetic coupling by the square of the pseudoscalar \((ij)^2 = -1\). This mechanism provides an attractive potential for equally charged particles.

Note, that the model is thus very close to the one of Majerník [18]. Instead of current and fields proportional to the complex unit \(i\), the corresponding quantities are related here to the pseudoscalar of the hyperbolic algebra \(ij\).

4 Post-Newtonian effects

The presented model benefits like all Maxwell-like approaches from the close analogy to electrodynamics and classical field theory. For the description of the gravitational field of big macroscopic objects like planets and stars common classical methods can be applied. In this context the concept of Majerník is used and the coupling of the electromagnetic charge is replaced in the formulas by the imaginary mass, here by \(ijm\).

Based on this concept the Lagrangian of a planet moving in the potential of the Sun is introduced in the framework of relativistic classical mechanics as

\[
\mathcal{L} = -mc^2 \sqrt{1 - \beta^2} - ijm\phi + \frac{ijm}{c} A \cdot v,
\]

(23)

where the velocity of light \(c\) is displayed explicitly, \(\beta = v/c\), and \(m\) denotes the mass of the planet. Following the approach of Singh [32], the spatial vector potential is introduced as

\[
A = \frac{v\phi}{c}.
\]

(24)

The potential of the Sun is defined in complete analogy to electromagnetism using the replacement rule for the electric charge

\[
\phi = \frac{ijM}{r},
\]

(25)
where $M$ denotes the mass of the Sun. The hyperbolic complex contributions now drop out in Eq. (23) by $(ij)^2 = -1$ and the Lagrangian becomes identical to the Lagrangian used by Singh.

One can then follow his calculations showing that the total energy equation of the Sun planet system is identical to the solution obtained from Einstein’s general relativity, which is used to derive the precession of the perihelion of a planet. Similarly, one can follow the arguments of Singh to explain the deflection of light in the gravitational field of a star, and the gravitational redshift.

5 Discussion

Despite all known concerns with respect to a vector theory of gravitation it is proposed to reconsider a Maxwell-like form of the interaction. The gravitoelectromagnetic interaction is introduced in the context of quantum physics in analogy to electromagnetism with the help of a hyperbolic unitary gauge symmetry. A gravitoelectric charge is introduced, which stands in contrast to the concept of a mass charge in the theories of Newton, Maxwell-Heaviside, and Einstein. This gravitoelectric charge is related to a non-compact symmetry group and therefore might appear in nature in a complete different form than the electric charge.

If the model can be further justified this would lead to a significant simplification of the physical theories. The combined $U(1, H)$ gauge symmetry of electromagnetism and gravitation is naturally included as a substructure in the internal $SU(4, H)$ symmetry, which is induced by the generalized mass operator proposed in [51]. This symmetry is isomorphic to the $SU(4, C) × SU(4, C)$ gauge group of the Pati-Salam model [55], with the consequence that the Pati-Salam model could be considered not only as a model for a unified theory of the standard model, but also as a unified theory for all known interactions.

It is assumed that the proposed model is the simplest way to introduce gravitation into the concept of the hyperbolic $C_{3,0}$ representation of physics. Due to the success of Einstein’s general relativity, the presented Maxwell-like model should appear as a substructure within general relativity in the macroscopic context.

It is possible to represent general relativity in an algebraic form as an extension of the $R_{1,3}$ Dirac algebra to curved spacetime. For a detailed discussion it is referred to Lasenby et al. and Hestenes [56, 57, 58]. Similar considerations could be made also for the $R_{3,0}$ and $C_{3,0}$ algebras, respectively. It has been shown by Bianchi that the structure constants of hypercomplex numbers can be written in the same way as the coefficients of connection [47]. Following the concept of Singh and with the help of an appropriate algebraic representation of gravitation, it might be possible to derive further direct relationships between Einstein’s general relativity and the Maxwell-like approach presented in this work.

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