The Era of Gravitational Astronomy and Gravitational Field of Non-Rotating Single Point Particle in General Relativity

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Abstract—Utilizing various gauges of the radial coordinate, we give a General Relativistic (GR) description of static spherically symmetric spacetimes with a massive point source and vacuum outside this singularity. We show that in GR there exists a two-parameter family of such solutions to the Einstein equations which are physically distinguishable and describe the gravitational field of a single massive point particle with positive proper mass $M_0$ and positive Keplerian mass $M < M_0$. In particular, we show that the widespread Hilbert form of the Schwarzschild solution, which depends only on the Keplerian mass $M$ and describes Black Holes (BH), does not solve the Einstein equations with a massive point particle stress-energy tensor. Novel normal coordinates for the gravitational field and a new physical class of gauges are proposed, thus achieving a correct description of a point mass source in GR. We also introduce a gravitational mass defect of a point particle and determine the dependence of the solutions on this mass defect. The result can be described as a change of the Newton potential $\varphi_N = -G_N M_0 / r$ to a modified one $\varphi_G = -G_N M / r + G_N M / c^2 \ln (M_0 / M)$ and the corresponding modification of the four-interval. We show that the proper 3D flat space, where these two potentials can be compared, is the tangent space above the position of the massive point source. In addition, we present invariant characteristics of the physically and geometrically different classes of spherically symmetric static spacetimes created by a point mass. Our findings are important for description of Extremely Compact Objects (ECOs) studied in relation with possible echoes in Gravitational Waves (GW) recently discovered by the LIGO/VIRGO collaboration.

Keywords: General Relativity, gauge conditions, the Schwarzschild problem, massive point source, proper mass, the Keplerian mass, mass defect, gauge classes of solutions, modified gravitational potential, invariants, extremely compact objects, black holes

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1. INTRODUCTION

LIGO detection of Gravitational Waves (GW) is the most important finding in gravity after Sir Isaac Newton’s discovery of the gravitational field. Indeed, Sir Isaac Newton discovered the gravitational field attached to bodies. LIGO discovered the gravitational field detached from bodies and freely spreading in space. The last being a qualitatively different novel gravitational phenomenon.

Without any doubts, with the LIGO/VIRGO observations of more than 11 GW events (see the references at the WEB address [1]), a new era in fundamental physics started [2]. Finally, the gravitational astronomy opened a novel window to the Universe and started to give us hitherto unreachable knowledge of the Nature.

These achievements deserve extraordinary careful analysis of all issues, which appear for the first time and are accompanied by many uncertainties and unknowns. Prior ruthless examination of all facts, hypotheses, assumptions, and interpretations, we cannot be sure of what we really see in this newly opened window to the Universe.

We certainly may refer to the already existing models and theoretical achievements like General Relativity (GR) well tested in other physical domains or to a variety of strictly speaking still hypothetical Black Hole (BH) models.

However, the most important in the new situation is the newly appearing opportunity to examine experimentally the theoretical assumptions, as well as to look for new developments and unexpected physical phenomena.
The used methods for processing the LIGO data were good enough to discover GW without any doubts but failed to recover the most important details needed to establish definitely the right theory of the observed GW and the physical nature of their sources.

For example, these methods were too crude to recover Quasi Normal Modes (QNMs) as fingerprints of BH, see, for example, [2, 5, 6, 8–11, 13, 49] and a huge amount of references therein.

The used methods also turned out to be not enough to establish the existence or absence of echoes as fingerprints of Extremely Compact Objects (ECOs)\(^1\) which are not BH [14–20].

Between others, GW150914 is the event with the largest signal-to-noise-ratio in the QNMs ringing domain. None of the methods, recently proposed for finding echoes, have yet found a significant echo signal in this event [17] or discarded firmly their existence.

As a result, we still cannot refute firmly many of the alternative theoretical explanations for the sources of GW events observed by the LIGO/VIRGO collaboration.

For recent reviews on the above problems, as well as on a series of other ones, see the talk by Alan Weinstein at the First LIGO Open Data Workshop 2018 [1] and a large review paper on the general situation in gravity [2].

The most recent observation of the shadow of the ECO at the center of the elliptic galaxy M87 proves the extremal compact character of the central object [21] but it does not give any indication of the presence or absence of an event horizon of this object.

In the present paper, we consider in a mathematically correct way the most compact object—a non-rotating massive point particle in GR.

We dub this subject “the Schwarzschild problem”, since the first attempt to solve it was made by Schwarzschild in the famous paper [3] titled “On the Gravitational Field of a Mass Point according to Einstein’s Theory”, as early as in 1916.

It is physically clear that this model is a good approximation for description of any spherically symmetric non-rotating matter object of finite dimension, as seen from a large-enough distance.

This means that if one measures physical quantities like masses, mass defects, etc., of bodies of finite dimension, being at large-enough distance from the body, the measured values are the same as for the corresponding point particles.

The correct mathematical description of “to be simplest” the Schwarzschild problem in GR is not a trivial issue. It turns to be essential for solution of some of the above mentioned problems related with ECOs.

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\(^1\) In the last few years the abbreviation “ECO” is used for Exotic Compact Objects without an event horizon, which are not vacuum BH solutions of gravitational equations. We use here this abbreviation in a broader sense, including BH in the class of ECOs = Extremely Compact objects.

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Our approach to the problem is based on our previous papers [22, 23], which have already a more than sixteen years history. Those papers were reported at various seminars and discussed with many people privately.

Before the discovery of the GW, these papers did not attract adequate attention of the general audience, which is happy with the most-well-studied Hilbert form of the Schwarzschild solution (see details in the text below.).

However, in the last few months one can see increasing interest in the paper [22] in the scientific nets like Academia, Research Gate, etc, see also [24]. This stimulated the actualization of this paper and its adaptation to the present-day hot topics of fundamental physics.

The most significant finding of this approach is a novel two-parameter-family of exact solutions of the Einstein equations for a massive point particle, which are regular in the standard domain of the radial variable \(r\geq0\).

As all other matter objects, in GR these supercompact ones have no event horizon and are defined by positive proper mass \(M_0 > 0^2\) and positive Keplerian mass \(M < M_0\) of the matter-point-particle with mass-defect \(M_0 − M > 0\), or gravitational binding energy \(E_{\text{binding}} = (M_0 − M)c^2 > 0^3\).

The mass-defect of the matter bodies is described more conveniently by the mass ratio \(\rho = M/M_0 \in (0, 1)\). It was introduced in the Schwarzschild problem for the first time in [22].

The ratio \(\rho\) is of fundamental significance for all ECOs that are not BHs, and must be included in their physical description. In particular, it restricts from below the physically admissible values of the luminosity distance \(\rho \in [\rho_{\text{min}}, \infty)\) of the ECO.

The minimal luminosity distance \(\rho_{\text{min}}\) is a coordinate independent quantity that defines the luminosity of the point object at a small distance from its center. In GR, the center of the such ECO is placed in position with luminosity radius value \(\rho_{\text{min}} > \rho_{\text{G}}\). Hence, the point sources with the same intensity and larger \(\rho_{\text{min}}\) are darker. In GR, when one approaches the position of a massive point source, its luminosity goes to a finite limit \(\sim 1/\rho_{\text{min}}^2\).

Let \(\rho_{\text{G}} = 2G_NM\) be the gravitational radius of the massive point particle with the fixed Keplerian mass \(M\).

\(^2\) The proper mass \(M_0 > 0\) is defined, as usual, as the mass of the physical system with gravitational interaction turned off. See, for example, the Landau–Lifshitz book in [25]. In the neutron star physics the term “barion” mass is used often for the proper mass. This is not precisely the case since the proper mass can be associated not only with baryons.

\(^3\) Further one we use units in which the velocity of light is \(c = 1\).
Sometimes this radius is called also the Schwarzschild radius. Here and further on $G_N$ is the Newton gravitational constant.

The gravitational radius appears for the first time in the model of invisible superheavy stars, invented by John Michell (1783) and Pierre-Simon Laplace (1793) [4] in the Newton gravity. Here we present their results in modern notation:

Let light photon starts move with velocity $c = 1$ in the radial direction from the surface of a spherical star with the radius $R_*$, and let $R_{\text{max}}$ be the radial distance to the point, where the photon stops and turns back to the star, due to gravitational pull.

Then, in the Newton gravity, the relation

$$1/R_* = 1/\rho_G + 1/R_{\text{max}}$$

takes place. It is a direct consequence of conservation of the photon energy in the Newton gravitational field. The mean density of the star mass is

$$\mu_* = \frac{1}{k} \frac{3(n + 1)^3}{\rho_G},$$

where $k = \frac{8\pi G_N}{c^2}$ is the Einstein constant and $n = \rho_G / R_{\text{max}}$.

For $n = 1$, the radius of the superheavy star is $R_* = \rho_G / 2$ and the light can spread only inside the gravitational radius $\rho_G$.

Nevertheless, the Mitchell–Laplace superheavy star, being made of some matter, is not a BH, which is a vacuum solution of the Einstein Eqs. (1).

For example, if the central ECO in our Galaxy is a Mitchell–Laplace superheavy star, its mean density will be $\sim 9 \times 10^7$ g/cm$^3$, i.e., about 0.01 of the density of a white dwarf. In the same case, the central ECO in M87 will have a lower mean density $\sim 4 \times 10^{-3}$ g/cm$^3$, i.e., about 4 times the density of the water, or about the main density of a typical planet as the Earth.

We do not think that the Mitchell–Laplace superheavy star model gives a correct description of the observed real superheavy objects. Of course, the correct model has to be based on GR or on its proper relativistic generalization.

Here we only wish to stress three important facts: (1) As a limiting case of GR, the Newton model contains some part of the physical truth, as well as the corresponding quantities, which appeared in GR. (2) The Mitchell–Laplace superheavy star model is not a predecessor of BH, which are a vacuum phenomenon in GR. The Mitchell–Laplace model is just a model of matter body, which is invisible in the electromagnetic spectrum, and does not describe a hole in the space-time. (3) The Mitchell–Laplace superheavy star model does not require some more exotic matter, than the one, we find in GR models of the observed objects.

In the present paper, we shall show that in GR description of a massive point particle with fixed Keplerian mass $M$ and minimal value of the luminosity radius $\rho_{\text{min}}$, its proper mass $M_0$ can be represented in the form

$$M_0 = M/\sqrt{1-\rho_G/\rho_{\text{min}}} \Leftrightarrow \rho_{\text{min}} = \rho_G/(1-\rho^2).$$

These basic formulas relate the minimal luminosity distance $\rho_{\text{min}} > \rho_G$ with the mass ratio $\rho \in (0, 1)$ of the point mass. According to those relations, the massive point sources of the same intensity but with a bigger $\rho$ (i.e. of a smaller mass defect $M_0 - M > 0$) are darker.

Above the corresponding minimal value $\rho_{\text{min}}$ of the luminosity distance, the Birkhoff theorem is strictly valid for all solutions of the Schwarzschild problem.

For BHs, being vacuum solutions of Eq. (1) of the finite Keplerian mass $M > 0$, proper mass $M_0 = 0$ has never been introduced. The Schwarzschild BH solutions, as vacuum solutions, have no defect of mass and in this case the ratio of masses $\rho = \infty$ is a senseless quantity.

As we shall see in detail, in GR the BH solutions form a specific gauge sector of spherically symmetric solutions. This gauge sector is completely different from the one of massive point sources of gravity in GR, both physically and mathematically.

In the last few years, a special kind of ECOs is intensively discussed in the literature, see the review paper [20] and a large number of references therein. These ECOs were introduced as alternatives to BH and, especially, to study possible echoes in GW during the QNM-ringing phase of binary merger. For this purpose, the ECOs need to have a luminosity radius well-below the luminosity radius of the photon sphere, i.e., $\rho_{\text{min}} < \rho_{\text{ps}} = 3/2 \rho_G$.

For appearance of observable echoes, the condition $\rho_{\text{min}} = \rho_G(1+ \epsilon)$ with $0 < \epsilon \ll 1$ is preferable [14–20]. Then, $\rho \sim \sqrt{\epsilon}$ and we obtain $M_0 \sim M/\sqrt{\epsilon} \gg M$. Hence, the binding energy $M_0-M \sim M(1/\sqrt{\epsilon}-1) \gg M$. Such ECOs are strongly bounded by gravity and are very stable.

When the Keplerian mass $M > 0$ is fixed, for $\epsilon \to 0$ we obtain a limiting case with $\rho_{\text{min}} = \rho_G$, $\rho = 0$, infinite $M_0$ and infinite binding energy. This is not a BH, since at $\rho = \rho_G$ we have a reflecting mirror described by the corresponding jump ($\sim M_0 \to \infty$) of the metric for $\rho < \rho_{\text{min}}$.

The value $\rho = 1$ indicates the absence of mass defect and is not admissible in GR. In this case, we have a non-physical absolutely dark noncompact object with $\rho_{\text{min}} = \infty$ and zero binding energy. This object is gravitationally not stable.

We present here an updated and corrected version of the paper [22], pointing out some of its relations with modern hot topics of fundamental physics.
The basic character of the Schwarzschild problem for gravitational physics prompts us to keep the careful and detailed character of our investigation.

The long history of the Schwarzschild problem requires paying attention to different mathematical and physical achievements of a large number of authors, ignoring possible misunderstandings and mistakes in many of the cited papers that become obvious when they are assessed from a modern point of view.

2. GAUGES IN GENERAL RELATIVITY

The Einstein equations:

\[ G^\mu_\nu = k T^\mu_\nu, \quad \mu, \nu = 0,1,2,3; \]  

(1)
determine the solution of a given physical problem up to four arbitrary functions, i.e., up to a choice of coordinates. This reflects the well-known fact that GR is a specific kind of gauge theory.

According to the standard textbooks [25], the fixing of the gauge in GR in a holonomic frame is represented by a proper choice of the quantities

\[ \Gamma^\mu_\nu = \nabla^\mu (\sqrt{|g|} g^{\lambda\mu} \partial_\lambda \rho), \]  

(2)

which emerge when one expresses the 4D d’Alembert operator in the form \( g^{\mu\nu} \nabla_\mu \nabla_\nu = g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma^\rho_\mu_\nu) \).

We shall call the change of the gauge fixing expressions (2), without any preliminary conditions on the analytical behavior of the used functions, gauge transformations in a broad sense. This way we essentially expand the class of admissible gauge transformations in GR.

Of course, this will alter some of the well-known mathematical results in the commonly used mathematical scheme of GR.

We think that a careful analysis of such a wider framework for the gauge transformations in GR can help us to clarify some long-standing physical problems.

After all, in the physical applications, the mathematical constructions are to reflect in an adequate way the properties of the real physical objects.

It is well known that in the gauge theories we may have different solutions with the same symmetry in the base space.

In general, these solutions belong to different gauge sectors owning different geometrical, topological and physical properties.

For a given sector of solutions, there exists a class of regular gauges, which alter the form of the solution without changing its essential characteristics, i.e. living in the solution in the same sector.

In contrast, by performing a singular gauge transformation one can change geometrical, topological and physical properties of the solution, making a transition to another gauge sector.

In a geometrical sense the solutions of Eqs. (1) define different 4D pseudo-Riemannian manifolds \( M^{(1,3)}(g_{\mu\nu}(x)) \).

A subtle point in the GR formalism is that transitions from a given physical solution to an essentially different one is traditionally dubbed a “change of coordinates”.

From the point of view of gauge theory this is confusing, because the choice of space-time coordinates in GR means simultaneously two different procedures: (1) choosing a gauge sector in which the solution lives; and at the same time (2) fixing the (regular) gauge in this sector.

While the procedure (2) solves an inessential local gauge fixing problem: change of the labels of the spacetime points, the above procedure (1) fixes the essential global characteristics of the solution, i.e. their physical properties and physical meaning.

One can understand better this peculiarity of GR in the framework of its modern differential geometrical description as a gauge theory, using the principal frame bundle \( F(M^{(1,3)}(g_{\mu\nu}(x))) \). (See, for example, the second edition of the monograph [26, 27] and references therein.)

The change of coordinates on the base space \( M^{(1,3)}(g_{\mu\nu}(x)) \) induces automatically a nontrivial change of the frames on the fibre of frames.

Singular coordinate transformations can produce a change of the gauge sector of the solution, because they may change the topology of the frame bundle, adding new singular points and/or singular sub-manifolds or removing some of the existing ones.

For example, singular transformations can alter the fundamental group of the base space \( M^{(1,3)}(g_{\mu\nu}(x)) \), the holonomy group of the affine connection on \( F(M^{(1,3)}(g_{\mu\nu}(x))) \), etc.

Thus, one sees that coordinate changes in our broad sense may be much more than a pure alteration of the labels of space-time points.

If one works in the framework of the theory of smooth real manifolds, ignoring the analytical properties of the solutions in the complex domain, it seems that one is allowed to change the gauge without altering the physical problem in the real domain, i.e. without change of the boundary conditions, as well as without introduction of new singular points in the real domain, or change of the character of the existing ones in this domain.

Such special type of regular gauge transformations in GR describes the diffeomorphisms of the real manifold \( M^{(1,3)}(g_{\mu\nu}(x)) \).
This manifold is already fixed by the initial choice of the gauge. Hence, the real manifold \( M^{(1,3)}(g_{\mu\nu}(x)) \) is actually described by a class of equivalent gauges, which correspond to all diffeomorphisms of this manifold and are related with regular gauge transformations in the real domain.

From the physical point of view one must make difference between changes of variables in the 4D space-time manifold \( M^{(1,3)}(g_{\mu\nu}(x)) \) and changes of variables in the 3D space \( M^{(3)}(-g_{mn}(r)) \), \( m, n = 1, 2, 3 \).

While in the last 3D case we indeed have a change of coordinates, which at first glance looks like a formal mathematical procedure, more general changes of coordinates in the first 4D case include changes of the reference frame (i.e. the reference physical body), which is certainly a physical procedure.

Further on we will consider only changes of coordinates in the Riemannian 3D manifold \( M^{(3)}(-g_{mn}(r)) \) in a fixed reference frame.

The transitions between some specific real 3D manifolds \( M^{(3)}(-g_{mn}(r)) \), which are not diffeomorphic, can be produced by the use of proper singular gauge transformations.

If we start from an everywhere smooth in the real domain manifold, after a singular transformation we will have a new 3D manifold with some different singularities, which possibly lie only in the complex domain.

The new singularities are to describe physical phenomena in the problem at hand as the corresponding properties of its solutions.

The inclusion of such singular transformations in our consideration yields the necessity to introduce gauge transformations in a broad sense. They are excluded from present-day standard considerations by the commonly used assumption that in GR one has to allow only diffeomorphic mappings of real manifolds.

Similar singular transformations are well known in gauge theories of other fundamental physical interactions: electromagnetic interactions, electroweak interactions, chromodynamics.

For example, in the gauge theories singular gauge transformations describe transitions between solutions in topologically different gauge sectors. Singular gauge transformations are used in the theory of Dirac monopole, vortex solutions, t’Hooft–Polikov monopoles, Yang–Mills instantons, etc. See, for example, [27, 28] and references therein.

The simplest example is the singular gauge transformation of the 3D vector potential: \( A(r) \to A(r) + V\varphi(r) \) in electrodynamics, defined in the Cartesian coordinates \( x, y, z \) by the singular gauge function \( \varphi = \alpha \arctan(y/x) \) (\( \alpha = \text{const} \)).

Suppose that before the transformation we had a 3D flat Euclidean space \( M^{(3)}(-g_{mn}(r) = \delta_{mn}) = C^{(3)}(\delta_{mn}) \). Then this singular gauge transformation removes the whole axis \( OZ \) out of the 3D space \( C^{(3)}(\delta_{\mu\nu}) \) changing the topology of this part of the base space. As a result the quantity \( \oint C d\mathbf{r} A(r) \), which is gauge invariant under regular gauge transformations, now changes its value to \( \oint C d\mathbf{r} A(r) + 2\pi N \alpha \), where \( N \) is the winding number of the cycle \( C \) around the axis \( OZ \).

Under such singular gauge transformation the solution of some initial physical problem will be transformed into a solution of a completely different physical problem.

At present, the role of singular gauge transformations in the above physical theories is well understood.

In contrast, we still have no systematic study of the classes of physically, geometrically and topologically different solutions in GR created by singular gauge transformations even in the simple case of static spherically symmetric space-times with only one point singularity, although the first solution of this type was discovered first by Schwarzschild more than one century ago, i.e. as early as 1916 [3].

Moreover, in GR at present there is still no clear understanding of both the above gauge problem and its physical consequences.

Here we present some initial steps toward the clarification of the role of different GR gauges in a broad sense for spherically symmetric static space-times with point singularity at the center of symmetry and vacuum outside this singularity.

### 3. GR Static Spherically Symmetric Space-Times with a Single Point Source of Gravity at Rest

#### 3.1. The Three Dimensional Riemannian Geometry of the Point Particle Problem

The single point particle with proper rest mass \( M_0 \) can be treated as a 3D entity. Its proper frame of reference is most suitable for description of the static space-time with this single particle in it.

We prefer to reduce the problem of single point source of gravity in GR to a 1D mathematical problem, considering the dependence of the corresponding functions on the only essential variable—the radial variable \( r \). This can be achieved in the following way.

The spherical symmetry of the 3D space reflects adequately the non-rotating-point character of the source of gravity.

The spherically symmetric 3D Riemannian space \( M^{(3)}(-g_{mn}(r)) \subseteq M^{(1,3)}(g_{\mu\nu}(x)) \), \( m, n = 1, 2, 3 \), can be described using the standard spherical coordinates.
The role of GR is that the Einstein Eqs. (1) transform the Euclidean space \( \mathcal{E}^{(3)}(-\eta_{mn}(x^1,x^2,x^3)) \) to some curved Riemannian one \( \mathcal{M}^{(3)}(-g_{mn}(x^1,x^2,x^3)) \) with a squired 3D infinitesimal distance
\[
dl^2 = -g_{rr}(r)dr^2 + \rho(r)^2d\Omega^2, \tag{3}\]
with beforehand unknown functions \( g_{rr}(r) < 0 \) and \( \rho(r) > 0 \), which have to be obtained from the Einstein Eq. (1).

The physical and geometrical meaning of the radial coordinate \( r \) in space \( \mathcal{M}^{(3)}(-g_{mn}(r)) \) is not defined by the spherical symmetry of the problem and is unknown a priori [29].

The only clear thing is that its value \( r = 0 \) corresponds by construction to the center of the symmetry, where one must place the physical source of the gravitational field\(^5\)

This fact permits us to identify the Euclidean space \( \mathcal{E}^{(3)} \) with the tangent space above the place of the point source of gravity, which clarifies its geometric meaning from a point of view of modern differential geometry.

Now the physical meaning of the space \( \mathcal{E}^{(3)}(-\eta_{mn}) \) becomes clear. This Euclidean space is precisely the tangent 3D space to \( \mathcal{M}^{(3)}(-g_{mn}) \) at the point \( r = 0 \) where we are performing all our measurements with our laboratory equipment staying in the rest reference frame of the point particle.

The space \( \mathcal{E}^{(3)}(-\eta_{mn}) \) is a subspace of the tangent space to the 4D basic manifold \( \mathcal{M}^{(4)}(g_{\mu\nu}(x)) \) above the proper point with \( r \)-arbitrary fixed in the rest frame of the point particle.

In this 4D tangent space the special relativity is locally valid, together with all established physical laws.

GR permits us to relate the measurements in this reference system with measurements in other ones above other points in the base space \( \mathcal{M}^{(4)}(g_{\mu\nu}) \).

The radial 3D geometrical distance from the point source at the center of the symmetry is
\[
l(r) = \frac{r}{\sqrt{-g_{rr}(r)}}. \tag{4}\]

The quantity \( \rho \) in Eq. (3) has a clear geometrical and physical meaning: It is well known that \( \rho \) defines the area
\[
A = 4\pi\rho^2 \tag{5}\]
of a centered at \( r = 0 \) sphere with a luminosity radius \( \rho \) and the length of a big circle on it
\[
l_{p} = 2\pi\rho. \tag{6}\]

Relations (5) and (6) only resemble the Euclidean geometry ones since the variable \( \rho \) does not measure geometrical distances in the curved space \( \mathcal{M}^{(3)}(-g_{mn}) \).

One can refer to the quantity \( \rho \) as an “area radius”, or as an optical “luminosity distance”, because the luminosity of distant physical objects is reciprocal to \( A \).

This consideration shows that the area radius \( \rho \) is an invariant quantity and its value does not depend on the choice of coordinates in the 3D Riemannian space \( \mathcal{M}^{(3)}(-g_{mn}) \).

Most probably this property stimulated Hilbert to choose the luminosity distance \( \rho \) as a radial coordinate in the Schwarzschild problem [30].

We shall also mention that in 3D geometry with metric (3) the 3D volume of a ball with \( r \)-radius \( r \) and the center at the origin is
\[
V^{(3)}(r) = 4\pi\int_{0}^{r} \sqrt{-g_{rr}(r)}\rho(r)rdr. \tag{7}\]

### 3.2. The Spacetime Pseudo-Riemannian Geometry and Solutions of the Einstein Equations Outside the Point Source

Now going to the geometry of the 4D pseudo Riemannian manifold \( \mathcal{M}^{(4)}(g_{\mu\nu}(x)) \) of signature \(+, -, -, -\) we have to stress that there exists unambiguous choice of a global time \( t \) due to the requirement to use a static metric when we are in the rest frame of the point particle placed at the geometrical point \( r = 0 \).

This yields a familiar form of the space-time interval [33]:
\[
ds^2 = g_{rr}(r)dt^2 + g_{rr}(r)dr^2 - \rho(r)^2d\Omega^2 \tag{8}\]
with unknown functions \( g_{rr}(r) > 0, g_{rr}(r) < 0, \rho(r) \), which have to be obtained from equations of the physical theory of gravity.

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\(^4\) The natural idea to use such auxiliary Euclidean space in the problem under consideration can be found already in [3], but Schwarzschild applied different coordinates in it.

\(^5\) In the present article, we assume that in the Universe other sources of the gravitational field outside the center of symmetry do not exist.
Note that in the frame of free falling clocks, if $\rho_{\text{fixed}}$ is some arbitrary fixed value of the luminosity distance, at which the clocks start to fall with zero velocity, the expression

$$(\rho_2 - \rho_1)/(1 - 2M/\rho_{\text{fixed}})^{1/2}$$

measures the 3D geometrical distance between the geometrical points 2 and 1 on a radial geodesic line [31].

Nevertheless, even in this frame the absolute value of the variable $\rho$ remains not fixed by the 3D distance measurements.

In the static spherically symmetric case the choice of spherical coordinates and static metric dictates the form of three of the gauge fixing coefficients (2): $\Gamma_{t} = 0$, $\Gamma_{\theta} = -\cot \theta$, $\Gamma_{\phi} = 0$, but the form of the quantity $\Gamma_{r}$ is still not fixed. Here and further on, the prime denotes differentiation with respect to the variable $r$.

We refer to the freedom of choice of the function $\rho(r)$ as “rho-gauge freedom”, and to the choice of the $\rho(r)$ function as “rho-gauge fixing” in a broad sense.

In the present article, we will not use more general gauge transformations in a broader sense than the rho-gauge ones.

In our 1D approach to the problem at hand all possible mathematical complications, due to the use of such a wide class of transformations, can be easily controlled.

The overall action for the aggregate of a point particle and its gravitational field is $\mathcal{A}_{\text{tot}} = \mathcal{A}_{\text{GR}} + \mathcal{A}_{M_0}$. Neglecting the surface terms, one can represent the Hilbert–Einstein action

$$\mathcal{A}_{\text{GR}} = -\frac{1}{16\pi G_{n}} \int d^{4}x \sqrt{|g|} R,$$

and the mechanical action

$$\mathcal{A}_{M_0} = -M_0 \int ds$$

of the point source with proper mass $M_0$ as integrals with respect to the time $t$ and the radial variable $r$ of the following Lagrangian densities:

$$\mathcal{L}_{\text{GR}} = \frac{1}{2G_{n}} \left( 2\rho \left( \sqrt{g_{rr}} \right)^{2} + \left( \rho^{2} \right) \sqrt{g_{tt}} \right) - \frac{\rho^{2} \sqrt{g_{tt}}}{\sqrt{g_{rr}}} + \sqrt{g_{tt}} \sqrt{g_{rr}} \delta(r),$$

$$\mathcal{L}_{M_0} = -M_0 \sqrt{g_{rr}} \delta(r).$$

Here $G_{n}$ is the Newton gravitational constant, $\delta(r)$ is the 1D Dirac function [32]. (In the units $c = 1$.)

As a result of the rho-gauge freedom, the field variable $\sqrt{g_{rr}}$ is not a true dynamical variable but rather plays the role of a Lagrange multiplier, which is needed in the description of the constrained dynamics.

This auxiliary variable enters into the Lagrangian $\mathcal{L}_{\text{GR}}$ in a nonlinear manner, in contrast to the case of the standard Lagrange multipliers.

The corresponding Euler–Lagrange equations read:

$$\left( \frac{2 \rho \left( \sqrt{g_{rr}} \right)^{2}}{\sqrt{g_{rr}}} - \frac{\rho^{2} \delta}{2G_{n}} \right) \frac{d \rho}{\sqrt{g_{rr}}} = \left( \frac{\rho^{2} \sqrt{g_{tt}}}{\sqrt{g_{rr}}} \delta \right) \sqrt{g_{rr}} = 0,$$

$$2 \rho \left( \sqrt{g_{rr}} \right)^{2} + \left( \rho^{2} \right) \sqrt{g_{tt}} - \sqrt{g_{tt}} \sqrt{g_{rr}} \delta = 0,$$

where the symbol “$\equiv$" denotes a weak equality in the sense of the theory of constrained dynamical systems.

If one ignores the point source of the gravitational field, thus considering only the domain $r > 0$ where $\delta(r) \equiv 0$, one obtains the standard vacuum solution of this system [33]:

$$g_{tt}(r) = 1 - \rho_{G}/\rho(r), \quad g_{rr}(r)g_{tt}(r) = -(\rho(r))^{2},$$

where $\rho_{G} = 2G_{n}M$ is the Schwarzschild luminosity radius, $M$ is the gravitational (Keplerian) mass of the source, and $\rho(r)$ is an arbitrary ‘$c^{1}$’ function.

### 4. SOME EXAMPLES OF DIFFERENT RADIAL GAUGES

In the general case of metric (8) with arbitrary function $\rho(r)$ one finds the relation $\Gamma = -\varphi'$. On the solutions of Einstein equations

$$\mathcal{V}(r) = \ln \left( \frac{\tilde{\rho}^{2}}{\rho (\rho - \rho_{G})} \right),$$

#### 4.1. Examples of Radial Gauges

**for Static Spherically Symmetric Solutions of the Einstein Equations**

In the literature one can find different choices of the function $\rho(r)$ for the Schwarzschild problem.

In this section, we pay attention only to some of the mathematical achievements of the cited articles and ignore their incorrect interpretations, due to the lack in understanding of the gauge dependence of the solutions of the Einstein Eq. (1).

1. The original Schwarzschild gauge [3]:

$$\rho(r) = \rho_{G} \left( \frac{r}{\rho_{G}} \right)^{3}.$$

It produces

$$\mathcal{V}(r) = \ln \left( \frac{\tilde{\rho}^{2}}{\left( r^{3} + \rho \right)^{2} \left( r^{3} + \rho^{2} \right)^{3/2}} \right).$$
2. The Hilbert gauge [30]:
\[ \rho(r) = r. \]

It gives
\[ \tilde{\phi}(r) = \ln \left( \frac{\overline{p}^2}{r (r - \rho_G)} \right). \]

This simple choice of the function \( \rho(r) \) is often related incorrectly with the original Schwarzschild article [3] (see for example [34]).

In this case the coordinate \( r \) coincides with the luminosity distance \( p \) and the physical domain \( r \in [0, \infty) \) contains an event horizon at \( \rho_H = \rho_G \).

This unusual circumstance forces one to develop a nontrivial theory of BH for the Hilbert gauge—see for example [25, 35] and references therein.

3. The Droste gauge [36]:

The function \( \rho \) is given implicitly by the relations
\[ \rho/\rho_G = \cosh^2 \psi \geq 1, \]
\[ r/\rho_G = \psi + \sinh \psi \cosh \psi. \]

Now \( \tilde{\phi}(r) = \ln \left( \frac{\overline{p}^2/\rho^2}{\sinh(\psi) \cosh(\psi)^3} \right) \).

For this solution the variable \( r \) has a clear geometrical meaning: it measures the 3D-radial distance \( l = r \) (see Eq. (4)) to the center of the spherical symmetry.

4. The Weyl gauge [37]:
\[ \rho(r) = \frac{\rho_G}{4} \left( \sqrt{r/\rho_G} + \sqrt{\rho_G/r} \right)^2 \geq \rho_G, \]
\[ \tilde{\phi}(r) = \ln \left( \frac{\overline{p}^2}{(r - \rho_G/4)(r + \rho_G/4)} \right). \]

In this gauge the 3D-space \( \mathcal{M}^{(3)}(\{-\mathbb{R}_{mn}(r)\}) \) becomes conformally flat. The coordinate \( r \) is the radial variable in the corresponding Euclidean 3D space.

5. The Einstein–Rosen gauge [38]:

In the original article [38] the variable \( u^2 = \rho - \rho_G \) was used. To have a proper dimension, we replace it by \( r = u^2 \geq 0 \). Hence,
\[ \rho(r) = r + \rho_G \geq \rho_G, \]
and
\[ \tilde{\phi}(r) = \ln \left( \frac{\overline{p}^2}{(r - \rho_G/2)(r + \rho_G/2)} \right). \]

6. The isotropic \( t-x \) gauge, defined according to the formula
\[ x = \rho + \rho_G \ln \left( \frac{\rho}{\rho_G} - 1 \right), \rho \geq \rho_G. \]

The variable \( x \in (-\infty, \infty) \) is called the tortoise coordinate. The inverse relation is described by the Lambert \( W \) function in the form
\[ \rho(x) = \rho_G(W + 1), W = \text{Lambert} \ W \left( \exp \left( \frac{x}{\rho_G} - 1 \right) \right). \]

Then, only the combination \( (dr^2 - dx^2) \) appears in the 4D-interval \( ds^2 \) and
\[ \phi(x) = 2 \ln \left( \frac{\overline{p}/\rho_G}{W + 1} \right). \]

7. Several different rho-gauges in GR were introduced by Pugachev and Gun’ko and, independently, by Menzel, see [33].

We will not consider in detail them and restrict ourselves only to mentioning the idea of using the same additional condition in the flat Minkowski space \( c^{(1,3)}(\frac{1}{\rho}(x)) \) and in the curved one \( \mathcal{M}^{(1,3)}(\{\rho_{\mu
u}(x)\}) \).

This approach defines the choice of the coordinates in these two different spacetimes in a coherent way. Using a zero tensor of affine deformation
\[ D_{\mu\nu} = \Gamma_{\mu\nu}(\rho_{\alpha\beta}(x), \rho_{\gamma\delta}(x)) \]
\[ - \Gamma_{\mu\nu}(\eta_{\alpha\beta}(x), \eta_{\gamma\delta}(x)) = 0 \]
in the spherical coordinates \( x \), Pugachev and Gun’ko have derived once more the original Schwarzschild solution, see the above point 1.

Note that the relation \( D_{\mu\nu} = 0 \) is not a gauge condition of the type of Eq. (2).

In two successive papers (see in [33]) Kohler introduced a new gauge condition
\[ \nabla_{\alpha}^\text{flat} \nabla_{\beta}^\text{flat} \left( g^{\alpha\gamma} \eta^{\gamma\delta} \nabla_{\gamma}^\text{flat} \left( \frac{g(x)}{\eta(x)} \right) \right) = 0. \]

and studied some of its applications.

Here \( \nabla_{\alpha}^\text{flat} \) is the covariant derivative with respect to the flat spacetime metric \( \eta_{\alpha\beta}(x) \) and \( \eta(x) = \det(\eta_{\alpha\beta}(x)), g(x) = \det(\eta_{\alpha\beta}(x)) \).

8. The coherent gauge:

Let us apply the idea of coherent gauge directly to the quantities \( \Gamma_{\mu\nu} \) using the gauge condition
\[ \Gamma_{\rho \nu} = \frac{2}{r} \]
in the curved spacetime \( \mathcal{M}^{(1,3)}(\{\rho_{\mu\nu}(x)\}) \). It is identical to the rho-gauge fixing of spherical coordinates in the Minkowski spacetime \( c^{(1,3)}(\frac{1}{\rho}(x)) \).

Thus, the spherical coordinates \( t, r, \theta, \phi \) in the curved spacetime \( \mathcal{M}^{(1,3)}(\{\rho_{\mu\nu}(x)\}) \) are fixed in a complete coherent way with the flat spacetime spherical ones.

Then we obtain a novel form of the 4D interval for a point source
\[ ds^2 = e^{2\rho_{\rho\phi}(r)} \left( dr^2 - \frac{dr^2}{N(r)^4} - r^2 d\Omega^2 \right), \]
where
\[ N(r) = (2\rho_{\rho\phi})^{-1}(e^{2\rho_{\rho\phi}} - 1). \]
$N(r) \sim 1 + \mathcal{O}\left(\frac{\rho}{r}\right)$—for $r \to \infty$ and $N(r) \sim \frac{r}{\rho_G}$—for $r \to +0$. In this specific gauge

$$\rho(r) = \frac{r}{N(r)} = \frac{\rho_G}{1 - e^{2\phi_0(r)}},$$

(18)

and $\rho(r) \sim r$—for $r \to \infty$, $\rho(r) \to \rho_G$—for $r \to +0$.

Let us consider in more detail the properties of this not well-known exact solution to Eqs. (1).

The most remarkable property of this flat spacetime gauge is the role of the classical Newton gravitational potential

$$\phi_N(r) = -\frac{G_NM}{r},$$

(19)

in the above exact GR solution. As usual, the component

$$g_{tt} = e^{2\phi_0(r)} - 1 + 2\phi_0(r) + O((\phi_0(r)) \text{ for } r \to \infty.$$

Hence, the new form of the metric is asymptotically flat for $r \to \infty$.

For the exact solution at hand the center $r = 0$ is an essential singular point of the function $g_{tt}$. This strong singularity is the left boundary of the physical intervals $r \in (0, \infty)$ and $\rho \in (0, \infty)$.

For $r \to +0$ the limit of the squared 4D-interval is

$$ds^2 \to \rho_G^2 d\Omega^2.$$  

(20)

As a result, the area of the sphere centered at $r = 0$ has a finite limit

$$\lim_{r \to +0} A(r) = 4\pi \rho_G^2.$$  

(21)

When one approaches the center $r = 0$ from the right, the 4D volume and the 3D volume approach zero because

$$\lim_{r \to +0} \sqrt{|g|} = \lim_{r \to +0} \left(e^{2\phi_0(r)}N(r)^{-1}\rho_G^2 \sin \theta \right) = 0,$$

and

$$\lim_{r \to +0} \sqrt{|g|} = \lim_{r \to +0} \left(e^{2\phi_0(r)}N(r)^{-1}\rho_G^2 \sin \theta \right) = 0.$$

The only functionally independent algebraic invariant, the Kretschman one

$$dI_0^2 = \frac{1}{48} R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} = \frac{1}{4\rho_G^4} \left(1 - e^{2\phi_0(r)}\right)^6 \rho_G^2 = \frac{\rho_G^2}{4\rho_G^4},$$

(22)

is finite everywhere, including the center $r \to +0$.

The same is true for the simplest differential invariant

$$dI_1^2 = \frac{1}{720} N_{\lambda\mu\nu\kappa} \nabla_{\lambda} R_{\mu\nu\lambda\kappa} = \frac{e^{2\phi_0(r)}(1 - e^{2\phi_0(r)})^8}{4\rho_G^6},$$

(23)

with a zero limit when approaching the center $r \to +0$, as well as for higher order differential invariants of the type

$$dI_n^2 = \nabla_{\lambda_1} \cdots \nabla_{\lambda_n} R_{\mu\nu\lambda_1} \cdots R_{\mu\nu\lambda_n},$$

(24)

which have a much more complicated explicit form.

Nevertheless, for odd numbers $n$ the invariants $dI_n^2$ have finite values in this limit. We directly checked this fact for $n = 0, 1, 2, \ldots, 9$.

At this point one has to stress two facts: (i) For static spherically symmetric solutions of the Einstein Eqs. (1) in the domain $r > 0$ the event horizon $\rho_H = \rho_G$ exists under the Hilbert choice of the function $\rho(r) \equiv r$ but not for all other gauges discussed above.

This demonstrates that the existence of the mathematical Schwarzschild BH strongly depends on the choice of the rho-gauge in a broad sense.

(ii) The choice of the function $\rho(r)$ can change drastically the character of the singularity at the place of the point source of the gravitational field in GR.

### 4.2. Some Further Notes and Problems

New interesting gauge conditions for the radial variable $r$ and corresponding gauge transformations were introduced and investigated in [39]. In more recent articles [40] it was shown that using a nonstandard-gauge for the application of GR in stellar physics, one can describe ECOs with arbitrary large mass, density and size. In particular, one is able to shift the value of the Openheimer–Volkoff limiting mass $0.7M_\odot$ of neutron stars to an essentially larger one: $3.8M_\odot$.

This may shed a new light on these astrophysical problems and once more shows that the choice of rho-gauge (i.e. the choice of the coordinate $r$) can have real physical consequences [41].

Thus, we see that by analogy with the classical electrodynamics and non-abelian gauge theories of general type, in GR we must use a more refined terminology and the corresponding mathematical constructions. We already call the choice of the function $\rho(r)$ “a rho-gauge fixing in a broad sense”.

Now we see that different functions $\rho(r)$ may describe different physical solutions of the Einstein Eqs. (1) with the same spherical symmetry in the presence of only one mater point at the center of symmetry.

As we have seen, the mathematical properties of the singular point may depend on the choice of the rho-gauge in a broad sense. Now the problem is to clarify the physical meaning of the singular points with different mathematical characteristics.

Some of the static spherically symmetric solutions with a singularity at the center can be related using regular gauge transformations.
For example, choosing the Hilbert gauge we obtain a black hole solution of the vacuum Einstein Eqs. (1). Then we can perform a regular gauge transformation to harmonic coordinates, defined by the relations $\Gamma_i = 0$, $\Gamma^i_x = 0$, $\Gamma^x_y = 0$, $\Gamma^y_z = 0$, preserving the existence of the event horizon, i.e. staying in the same gauge sector.

In the next sections, we will find geometrical criteria for answering the question: When do different representations of static spherically symmetric solutions with point singularity describe diffeomorphic space-times?

The definition of the regular gauge transformations allows transformations that do not change the number and the character of singular points of the solution and the behavior of the solution at the boundary of the physical domain but can place these points and the boundary in new positions.

5. GENERAL FORM OF THE GRAVITATIONAL FIELD EQUATIONS IN THE HILBERT GAUGE AND THEIR SOLUTIONS

According to Lichnerowicz [26, 43, 44], the physical spacetimes $\mathcal{M}^{(1,3)}(g_{\mu\nu}(x))$ of general type must be smooth manifolds at least of class $C^2$ and the metric coefficients $g_{\mu\nu}(x)$ have to be at least of class $C^3$, i.e. at least three times continuously differentiable functions of the coordinates $x$. When one considers $g_{\mu\nu}(r)$ and $\rho(r)$ not as distributions but as usual functions of this class, one is allowed to use the rules of the analysis of functions of one variable $r$. Especially, one can multiply these functions and their derivatives of the corresponding order, raise them at various powers, define functions like log, exp and other mathematical functions of $g_{\mu\nu}(r)$, $\rho(r)$ and the corresponding derivatives. In general, these operations are forbidden for distributions [32].

In addition, making use of the standard rules for operating with the Dirac $\delta$-function [32] and accepting (for simplicity) the following assumptions: (1) the function $\rho(r)$ is a monotonic function of the real variable $r$; (2) $\rho(0) = \rho_{\text{min}}$ and the equation $\rho(r) = \rho_{\text{min}}$ has only one real solution: $r = 0$; one obtains the following alternative form of Eq. (12):

$$
\sqrt{-g_{pp}} - \frac{1}{\sqrt{-g_{pp}}} \frac{d}{d \ln \rho} \left( \rho \left( \frac{1}{-g_{pp}} - 1 \right) \right) \\
= 2\sigma_0 G_N M_0 \delta(\rho - \rho_{\text{min}}),
$$

$$
d^2 \ln g_{tt} + \frac{1}{2} \left( \frac{d \ln g_{\rho\rho}}{d \ln \rho} \right)^2 + \left( 1 + \frac{d \ln g_{tt}}{d \ln \rho} \right) \frac{d \ln g_{\rho\rho}}{d \ln \rho} = 0,
$$

$$
\frac{d \ln g_{tt}}{d \rho} + g_{pp} + 1 \equiv 0.
$$

Note that the only remnant of the function $\rho(r)$ in the system, Eq. (25), is the numbers $\rho_{\text{min}}$ and $\sigma_0 = \text{sign}(\rho'(0))$, which enter only into the first equation related to the source of gravity. Here $g_{tt}$ and $g_{pp} = g_{\rho\rho}/(\rho')^2$ are considered as functions of the independent variable $\rho \in [\rho_{\text{min}}, \infty)$.

The form of Eq. (25) shows why one is tempted to consider the Hilbert gauge as a preferable one: In this form the arbitrary function $\rho(r)$ “disappears” and it seems that one can not pay attention to the problem of the choice of gauge.

This is obviously a wrong conclusion from both mathematical and physical reasons.

One usually ignores the general case of an arbitrary value $\rho_{\text{min}} \neq 0$, accepting, without any justification, the value $\rho_{\text{min}} = 0$, which seems to be natural in the Hilbert gauge. Indeed, if we erroneously consider the luminosity distance as a measure of the real geometrical distance to the point source of gravity in the 3D space, we have to accept the value $\rho_{\text{min}} = 0$ for the position of the point source. Otherwise, the $\delta$-function term in Eq. (25) will describe a shell with the radius $\rho_{\text{min}} \neq 0$, instead of a point source.

Actually, according to our previous consideration, the point source has to be described using the function $\rho(r)$. The physical source of gravity is placed at the point $r = 0$ by definition. There is no reason to change this initial position of the source, or the interpretation of the variables in the problem at hand.

To what value of the luminosity distance $\rho_{\text{min}} = \rho(0)$ corresponds the real position of the point source is not known a priori. This depends strongly on the choice of the rho-gauge.

One cannot exclude such nonstandard behavior of the physically reasonable rho-gauge function $\rho(r)$, which leads to some value $\rho_{\text{min}} \neq 0$. Physically, this means that instead of infinity, the luminosity of the point source will go to a finite value, when the distance to the source goes to zero, as it happens to the Pugachev—Gun'ko—Menzel solution, see Section 4.2.

This very interesting new possibility appears in curved space-times due to their unusual geometrical properties. It may have a great impact for physics and deserves further careful investigation.

If one accepts the value $\rho_{\text{min}} = 0$, one has to note that the Hilbert-gauge geometrical singularity at $\rho = 0$ will be space-like, not time-like.

Indeed, because of the well-known change of the sighs of the components $g_{tt}(r)$ and $g_{rr}(r)$ in the interior of BH, i.e., for $\rho \in [0 \ldots \rho_{\text{min}}]$ the signature of the metric becomes $\{-, +, -, -\}$ and the physical meaning of the variables $t$ and $\rho$ in this domain is interchanged: Now the “interior time” is $\rho$ and interior “radial variable” is $t$.

This is a quite unusual nonphysical property for a physical source of gravity.
In this case, there is no reason at all to speak about the “center of spherical symmetry”, as people often do.

Actually, it is well known that in this domain the solution of Eqs. (1) is not static.

Moreover, in the case of the Hilbert gauge the singular point \( \rho = 0 \) physically defines the future-time-infinity in the BH interior and does not define at all the center of spherical symmetry.

The choice of values of the parameters \( \rho_{\min} \) and \( \sigma \) was discussed in [45]. There, a new argument in favor of the choice \( \rho_{\min} = \rho_\text{G} \) and \( \sigma = 1 \) was given for different physical problems with spherical symmetry using an analogy with the Newton theory of gravity.

In contrast, in the present article we shall analyze the physical meaning of different values of the parameter \( \rho_{\min} \). They turn out to be related with the gravitational mass defect.

The solution of the subsystem formed by the last two equations of the system (25) is given by the well-known functions

\[
g_{\rho\rho}(\rho) = 1 - \rho_\text{G}/\rho, \quad g_{\theta\theta}(\rho) = -1/g_{tt}(\rho).
\]  

(26)

Note that in this subsystem one of the equations is a field equation but the other one is a constraint.

However, these functions do not solve the first of Eqs. (25) for any value of \( \rho_{\min} \), if \( M_0 \neq 0 \).

Indeed, for these functions the left hand side of the first equation equals identically zero and does not have a \( \delta \)-function-type of singularity, in contrast to the right-hand side. Hence, the first field equation remains unsolved, if \( M_0 \neq 0 \).

Thus, we see that:

1. Outside the singular point, i.e. for \( \rho > \rho_{\min} \), the Birkhoff theorem is strictly valid and we have the standard Hilbert form (26) of the solution, when expressed through the luminosity distance \( \rho \).

2. The assumption that \( g_{\rho\rho}(\rho), g_{\theta\theta}(\rho) \) and \( \rho(\rho) \) are usual \( C^2 \) smooth functions, instead of distributions, yields a contradiction, if \( M_0 \neq 0 \).

This way one is not able to describe correctly the gravitational field of a massive point source of gravity in GR.

For this purpose the first derivative with respect to the variable \( \rho \) at least of one of the metric coefficients \( g_{\mu\nu} \) must have a finite jump, needed to reproduce the Dirac \( \delta \)-function in the energy-momentum tensor of the massive point particle.

3. The widespread form of the Schwarzschild solution in the Hilbert gauge (26) does not describe a gravitational field of a massive point source and corresponds to the case of vacuum solution with \( M_0 = 0 \).

The Hilbert solution does not belong to any physical gauge sector of the gravitational field created by a massive point source in GR.

Obviously, this vacuum solution has a pure geometrical-topological nature and may be used in the attempts to reach a pure geometrical description of “matter without matter” [46].

In the standard approach to this solution no proper mass distribution has ever been introduced, since it is identically zero.

The above consideration confirms the conclusion, which was reached in [42] using the isotropic gauge for the Hilbert solution.

In contrast to [42], in the next Sections we will show how one can include in GR neutral particles with nonzero proper mass \( M_0 \). The corresponding new solutions are related to the Hilbert one via singular gauge transformations and belong to a different gauge sector of GR as a nonlinear gauge theory of gravity.

6. ON THE SCHWARZSCHILD SINGULARITY

Usually, the Schwarzschild singularity \( \rho = \rho_\text{G} \) is erroneously related only with the singularity of the metric (8) with components (26), or with algebraic invariants of the Riemannian tensor.

The presence of such a singularity in metric is coordinate dependent and can be removed from the metric by proper coordinate transformations, see details in [25].

The absence of this singularity in the algebraic invariants of the Riemannian tensor does not prove that it cannot be present in other geometrical or physical quantities, as we will see in this section.

6.1. A Simple Pure Geometrical Meaning of the Schwarzschild Singularity

From Eq. (3) one obtains the coordinate-independent relation

\[
\frac{dl}{d\rho} = 1/\sqrt{1-\rho_\text{G}/\rho}.
\]  

(27)

It shows that for the Schwarzschild solution the function \( l(\rho) \) has infinite derivative at the Schwarzschild luminosity radius \( \rho_\text{G} \).

Obviously, this is a geometrical singularity. It is invariant and can not be removed by any coordinate transformation in the spacetime.

This consideration recovers part of the true geometrical meaning of the notion of the Schwarzschild singularity.

Relation (27) can also be rewritten in the following geometrical form\(^6\)

\[
\frac{dA}{dl} = 8\pi\sqrt{\rho(\rho-\rho_\text{G})}.
\]  

(28)

\(^6\)Taking into account Eq. (5), we see that actually relation (28) represents an invariant ordinary differential equation of first order for the function \( A(l) \) with only one parameter \( \rho_\text{G} \) in it.
It shows that in the case of the Schwarzschild solution the point \( \rho = \rho_G \) is a stationary point of the area function \( A(l) \) and stresses a special character of the corresponding value of the area \( A_G = 16\pi G_N^2 M^2 \). This value is well known from its role in the theory of BH.

The above property of the area \( A \) justifies the geometrical meaning of the Schwarzschild singularity.

6.2. The Schwarzschild Singularity in the Regge–Wheeler and Zirilli Equations for QNMs of GW

As stressed in the Introduction, QNMs of GW are a very hot topic in the era of gravitational astronomy.

The first description of these QNMs was given by the Regge–Wheeler and Zerilli radial differential equations [47–49]

\[
g_\rho (\rho) \frac{d}{d\rho} \left( g_\rho (\rho) \frac{d\hat{H}}{d\rho} \right) + (\omega^2 - V(\rho))\hat{H}(\rho) = 0, \tag{29}\]

where \( \rho \in (0, \infty) \) and for \( L = l(l + 1), l = 2, 3, \ldots \) the functions

\[
V_{RW}(\rho) = g_\rho(\rho) \left( \frac{L}{\rho^2} - \frac{2}{\rho^3} \right), \tag{30a}
\]

\[
V_Z(\rho) = g_\rho(\rho) \times \frac{L(L - 2)^3 \rho^3 + 3((L - 2)^2 + 3/2)^2 + 9/4}{\rho^3((L - 2)^2 + 3)^2} \tag{30b}
\]

define the Regge–Wheeler (RW) and Zerilli (Z) potentials \( \hat{H}(\rho) \) in Eq. (29), respectively.

It is transparent that the Schwarzschild singularity \( \rho = \rho_G \) is a regular singularity (a simple pole) of these second order differential equations of Frobenius type.

This singularity is an additional one to the other two singularities: the regular singular point \( \rho = 0 \) and the irregular singular point \( \rho = \infty \).

Due to the presence of these three unremovable by radial-coordinate-change singular points, which are the only ones in the whole complex domain of the variable \( \rho \), the exact solutions \( \hat{H}(\rho) \) of the above basic equations can be represented in terms of the confluent Heun functions, as shown for the first time in [50–52].

If one wrongly supposed, that the Schwarzschild singularity \( \rho = \rho_G \) can be removed from Eqs. (29), (30), this would lead to a completely different type of the exact QNM solutions and spectra of GW in GR.

Solving the two-singularity boundary problem for Eqs. (29), (30) in the physical domain \( \rho \in (\rho_G, \infty) \), in [50–52], we obtained with very good precision (much higher than the previously published ones) the frequencies of the BH QNMs: \( \omega_{nl}^2, n = 0, \ldots, l = 2, \ldots \) of two types of GW (axial and polar) in GR.

These frequencies are physical quantities and their values are defined by the existing three singularities in Eqs. (29), (30).

One can extract the authentic values of these physical quantities from the observational data by the LIGO/VIRGO collaboration, see [1, 2, 13] and references therein.

Of course, the physical properties of the spectrum of QNMs do not depend on the choice of coordinates in the Riemann space \( M^3\{ 1 - g_{mn}(x) \} \).

At the same time, the physical spectrum of QNMs depends essentially on the presence of the real Schwarzschild Singularity \( \rho = \rho_G \) in Eqs. (29), (30), together with the two others: \( \rho = 0 \) and \( \rho = \infty \), as well as on the absence of any additional singularities.

We have to remind the reader that the Regge–Wheeler and Zerilli equations are linear approximations to the Einstein Eqs. (1) for weak GW on the Schwarzschild background.

It is extremely interesting that in domain of QNM ringing the nonlinearity of the Einstein equations is not seen. For example, in GW from binary BH merger: “Rather than nonlinearities, times around the peak are dominated by ringdown overtones—the quasinormal modes with the fastest decay rates, but also the highest amplitudes near the waveform peak”, see the very recent paper [13], and the references therein.

Since the linear QNM solutions of the problem approximate very well the numerical solutions in full GR in the domain of QNM ringing of BH, and give a reasonable description of observational data for GW in this domain [13], we can expect that the above conclusions about existing of the unavoidable physical Schwarzschild singularity in the QNM problem take place also for the corresponding exact solution of the Einstein Eqs. (1).

6.3. Invariants of the Riemann Tensor and the Schwarzschild Singularity

It is well known that the algebraic invariants of the Riemann tensor do not see the Schwarzschild singularity \( \rho = \rho_G \). For metric (8) with components (26) there exists only one functionally independent of them—the Kretschman invariant [25, 56]

\[
df_0^2 = \frac{1}{48} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{1}{4} \frac{G^2}{\rho^2}. \tag{31}\]

Since the pure algebraic invariants of the tensor \( R_{\alpha\beta\gamma\delta} \) do not fix completely the geometry, their consideration is not sufficient to recover all gauge-invariant spacetime properties. For this purpose one must con-
sider a large enough number of high-order differential invariants of the Riemann tensor [53].

As early as in [54], it was recognized that the simplest differential invariant of metric (8) with components (26)

$$dI_r^2 = \frac{1}{720} \nabla^2 \partial_{i} \partial_{j} \psi \nabla^2 \partial_{i} \partial_{j} \psi + \frac{1}{4} \frac{\rho_G}{\rho} \left( 1 - \frac{\rho_G}{\rho} \right)$$

(32)

saw as a zero point the Schwarzschild singularity.

This is a complete coordinate-independent result.

As a byproduct, each of Eqs. (31) or (32) proves once more the invariant geometrical nature of the luminosity radius \(\rho\), now in the sense of 4D spacetime geometry.

As seen, the differential invariant (32) changes its sign when one crosses the horizon \(\rho = \rho_G\).

Physically this means that one can experimentally fix the crossing of the Schwarzschild singularity and disproves the wide-spread statement that "nothing special", i.e. measurable, happens when one crosses the horizon of a BH in classical GR.

For the same result for the Kerr metric see [55] and references therein.

A preliminary, still unpublished study shows that the result can be generalized to a much wider class of solutions of the Einstein Eqs. (1) with horizons.

The higher order differential invariants

$$dI^2 = \nabla^2 \partial_{i} \partial_{j} \cdots \nabla^2 \partial_{i} \partial_{j} \psi$$

(33)

with odd number \(n = 1, 3, \ldots\) also see the horizon \(r = r_G\) as a zero point. The differential invariants (33) with even \(n = 0, 2, \ldots\) have a finite values at the horizon \(r = \rho_G\).

The full investigation of this problem is not trivial and at present is not carried out even for the Schwarzschild problem.

Obviously, the well-known and wide-accepted opinion that the Schwarzschild singularity is a pure coordinate one is not correct.

This incorrect statement does not solve all geometrical and physical problems related with the Schwarzschild singularity. The whole story is still with an open end.

7. NORMAL COORDINATES
FOR GRAVITATIONAL FIELD
OF A MASSIVE POINT PARTICLE
IN GENERAL RELATIVITY

An obstacle for the description of the gravitational field of a point source at the initial stage of development of GR was the absence of an adequate mathematical formalism.

Even after the development of the correct theory of mathematical distributions [32] there still exists an opinion that this theory is inapplicable to GR because of the nonlinear character of the Einstein Eqs. (1).

For example, the author of [44] emphasizes that "the Einstein equations, being non-linear, are defined essentially, only within framework of functions. The functionals, introduced in ... physics and mathematics (Dirac's \(\delta\)-function, "weak" solutions of partial differential equations, distributions of Schwartz) are suitable only for linear problems, since their product is not, in general, defined".

In a more recent article [57] the authors have considered singular lines and surfaces using mathematical distributions.

Those authors have stressed that "there is apparently no viable treatment of point particles as concentrated sources in GR". See also [26] and references therein.

Here we propose a novel approach to this problem, based on a specific choice of the field variables.

Let us represent the metric (8) of the problem at hand in a specific form:

$$ds^2 = e^{2\lambda} dt^2 - e^{-2\lambda} dr^2 - e^{2\lambda} d\theta^2 - e^{2\lambda} d\phi^2$$

(34)

where \(\lambda_1(r), \lambda_2(r)\) and \(\overline{\omega}(r)\) are unknown functions of the variable \(r\). The constant \(\overline{\rho}\) is the unit for luminosity distance \(\rho = \overline{\rho}e^{-\lambda_1 + \lambda_2}\).

By ignoring the surface terms in the corresponding integrals, one obtains the gravitational and point particle actions in the form:

$$S_{GR} = \frac{1}{2\alpha} \int dt \int dr \left( e^\lambda (-\overline{\rho} \lambda_1) + e^\lambda (-\overline{\rho} \lambda_2) + e^\lambda e^{2\lambda} \right)$$

(35)

Thus, we see that the field variables \(\lambda_1(r), \lambda_2(r)\) and \(\overline{\omega}(r)\) play the role of normal fields' coordinates in our problem. The field equations read:

$$\Delta \lambda_1 = e^\lambda (\overline{\rho} \lambda_1 - \rho \rho_0 \lambda_1)$$

$$\Delta \lambda_2 = e^\lambda (\overline{\rho} \lambda_2 - \rho \rho_0 \lambda_2)$$

(36)

where \(\Delta = e^{-\lambda} \frac{d}{dr} \left( e^\lambda \frac{d}{dr} \right)\) is related to the radial part of the 3D-Laplacian, \(\Delta = -\sqrt{3} g_{ij} \partial_i \partial_j \times \left( \sqrt{3} \right) \) or \(\Delta = \overline{\lambda} + \frac{1}{\rho^2} \Delta \overline{\omega} = -\overline{\lambda} \Delta \overline{\omega}.

The variation of the total action with respect to the auxiliary variable \(\overline{\omega}\) gives the constraint:

$$e^\lambda (-\overline{\rho} \lambda_1) + e^\lambda (-\overline{\rho} \lambda_2) = 0.$$

(37)

One can have some doubts about the correctness of the above derivation of the field Eqs. (36), because here we use the Weyl trick applying the spherical symmetry directly to the action functional, not to the Ein-
stein Eqs. (1). The correctness of the result of this procedure is proved in Appendix A.

Therefore, if one prefers, one can consider the Lagrangian densities (11), or the actions (35) as auxiliary tools for formulating our one-dimensional problem as a variational problem, defined by the reduced spherically symmetric the Einstein Eqs. (1).

The variational approach makes transparent the role of various differential equations, which govern the problem. It leads to a simple application of the theory of constrained dynamical systems in the problem at hand.

8. REGULAR GAUGES
AND GENERAL REGULAR SOLUTIONS OF THE PROBLEM

The advantage of the above normal fields’ coordinates is that when expressed through them, the field Eqs. (36) are linear with respect to the derivatives of the unknown functions Eqs. (36) are linear with respect to the derivatives of

The variational approach makes transparent the role of various differential equations, which govern the problem. It leads to a simple application of the theory of constrained dynamical systems in the problem at hand.

In terms of the metric components the basic regular gauge condition reads

\[ \rho^4 g_{rr} + \bar{\rho}^4 g_{\tau\tau} = 0, \] (38)

and gives

\[ \Gamma_\tau = 0. \]

Under this gauge the field Eqs. (36) acquire the simplest quasi-linear form:

\[ \phi_1''(r) = \left( \frac{G_N M_0}{\rho^2} e^{\Phi(r)} \right) \delta(r), \quad \phi_2''(r) = \frac{1}{\bar{\rho}^2} e^{2\Phi(r)}. \] (39)

The constraint (37) acquires a simple form:

\[-(\bar{\rho} \phi_1')^2 + (\rho \phi_2')^2 - e^{2\Phi} u = 0. \] (40)

As can be easily seen, the basic regular gauge \( \bar{\rho}(r) \neq 0 \) has the unique property to split the system of field Eqs. (36) and the constraint (37) into three independent relations.

The new field Eqs. (41) have a general solution

\[ \phi_1(r) = \frac{G_N M_0}{\rho^2} e^{\Phi(0)} (\Theta(r) - \Theta(0)) + \phi_1^0(r) + \phi_1(0), \]

\[ \phi_2(r) = -\ln \left( \frac{1}{\sqrt{2e_2}} \sinh \left( \frac{\sqrt{2e_2} (r_0 - r)}{\rho} \right) \right). \]

The first expression in Eq. (41) represents a distribution \( \phi_1(r) \). In it \( \Theta(r) \) is the Heaviside step function.

Here and further on we use the additional assumption \( \Theta = 0 \). It gives a specific regularization of the products, degrees and functions of the distribution \( \Theta(r) \) and makes them definite.

For example: \( (\Theta(r))^2 = \Theta(r) \), \( (\Theta(r))^3 = \Theta(r) \), \( \Theta \), ...

\( \Theta(r) \delta(r) = \delta(r) \), \( f(r \Theta(r)) = f(r) \Theta(r) - f(0) \Theta(r) - \Theta(0) \)—for any function \( f(r) \) with convergent Taylor series expansion around the point \( r = 0 \), and so on.

This is the only simple regularization of distributions we need in the present article.

The second expression \( \phi_2(r) \) in Eq. (41) is a usual function of the variable \( r \). The symbol \( r_0 \) is used as abbreviation for the constant expression

\[ r_0 = \text{sgn}(\Phi'(0))r_0 \] (42)

are the values of the corresponding first integrals (42) of the differential Eqs. (39) for a given solution (41).
Then for the regular solutions (41) the condition (40) reads:
\[ \varepsilon_1 + \varepsilon_2 + \frac{G_{N}M_0}{\beta^2} e^{\theta(r)} (\Theta(r) - \Theta(0)) \leq 0. \] (43)

An unexpected property of this relation is that it cannot be satisfied for any value of the variable \( r \in (-\infty, \infty) \), because \( \varepsilon_{1,2} \) are constants.

The constraint (42) can be satisfied either on the interval \( r \in [0, \infty) \), or on the interval \( r \in (-\infty, 0) \).

If from physical reasons we choose it to be valid at only one point \( r^* \in [0, \infty) \), this relation will be satisfied on the whole interval \( r \in [0, \infty) \) and this interval will be the physically admissible real domain of the radial variable.

Thus one can see that our approach gives a unique possibility to derive the admissible real domain of the variable \( r \) from the dynamical constraint (40), i.e., this dynamical constraint yields a geometrical constraint on the values of the radial variable.

As a result, in the physical domain the values of the first integrals (42) are related by the standard equation
\[ \varepsilon_{\text{tot}} = \varepsilon_1 + \varepsilon_2 \leq 0, \] (44)
which reflects the fact that our variation problem is invariant under re-parametrization of the independent variable \( r \).

At the end, as a direct consequence of relation (44) one obtains the inequality \( \varepsilon_2 = -\varepsilon_1 > 0 \), because in the real physical domain \( r \in [0, \infty) \) we have \( \varepsilon_1 = -1/2 \beta^2 \varphi'_0(r) = \text{const} < 0 \).

For the function \( \rho_{\text{reg},0}(r) \geq 0 \), which corresponds to the basic regular gauge, one easily obtains
\[ \rho_{\text{reg},0}(r) = \rho_G \left( 1 - \exp \left( 4 \frac{r - r_0}{\rho_G} \right) \right)^{-1}. \] (45)

Now one has to impose several additional conditions on the solutions (41):
(i) The requirement to have asymptotically flat space-time.

The limit \( r \to r_0 \) corresponds to the limit \( \rho \to \infty \).

For solutions (41) we have the property \( g_n(r)/\rho^2(r) = 1 \). The only nontrivial asymptotic condition is \( g_n(r) = 1 \).

It gives \( \varphi'_0(r_0) + \varphi_0(r) = 0 \).

(ii) The requirement to have the correct Keplerian mass \( M \), as seen by a distant observer.

Excluding the variable \( r > 0 \) from \( g_n(r) = e^{2\theta(r)} \) and \( \rho(r) = \beta e^{\Phi(r)} \), for solutions (41) one obtains \( g_n = 1 + \text{const}/\rho \), where
\[ \text{const} = 2 \text{sgn}(\rho) \text{sgn}(\varphi'_0(0)) \beta^2 \varphi_0(0) = -2G_N M \] (iii) The consistency of the previous conditions with the relation \( g_n g_{rr} + \rho^2 = 0 \) gives
\[ \text{sgn}(r_0) = \text{sgn}(\rho) = \text{sgn}(\varphi'_0(0)) = \text{sgn}(\varphi_0(0)) = 1. \]

(iv) The most suitable choice of the unit \( \beta \) is
\[ \beta = G_N M = \frac{\rho_G}{2}. \]

As a result, all initial constants become functions of the two parameters in the problem—\( r_0 \) and \( M \):
\[ \varphi_0(0) = -\frac{r_0}{G_N M}, \]
\[ \varphi_2(0) = -\ln \left( \sinh \left( \frac{r_0}{G_N M} \right) \right), \]
\[ \varphi'_0(0) = \frac{1}{G_N M}, \]
\[ \varphi'_2(0) = \coth \left( \frac{r_0}{G_N M} \right). \]

(v) The gravitational defect of the mass of a point particle.

Representing the proper mass \( M_0 \) of the point particle in the form
\[ M_0 = \int_0^{r_0} M_0 \delta(r) dr = 4\pi \int_{0}^{r_0} \sqrt{-g_{rr}(r)} \rho^2(r) \rho'(r) \mu(r) dr, \]
one obtains for the mass distribution of the point particle the expression
\[ \mu(r) = M_0 \delta(r) \left( 4\pi \sqrt{-g_{rr}(r)} \rho^2(r) \rho'(r) \right) = M_0 \delta_s(r), \]
where
\[ \delta_s(r) = \delta(r) \left( 4\pi \sqrt{-g_{rr}(r)} \rho^2(r) \rho'(r) \right) \]
\[ = \delta_s(\rho) = \text{inv}[\rho \nRightarrow \rho] \]
is the 1D invariant under one-to-one changes \( \rho \nRightarrow r \) Dirac delta function.

The Keplerian gravitational mass \( M \) can be calculated using the components of the massive point particle stress-energy tensor in the Tolman formula \( M = \int d^3 r \sqrt{-g} (T_0^0 - T_i^1 - T_2^2 - T_3^3) \) [25]:
\[ M = \int_0^{r_0} M_0 \sqrt{g_{rr}(r)} \delta(r) dr = M_0 \sqrt{g_{00}(0)}. \] (46)

As a result, we reach the important relations:
\[ g_n(0) = e^{2\theta(0)} = \exp \left( -2 \frac{r_0}{G_N M} \right) = \left( \frac{M}{M_0} \right)^2 \leq 1, \]
\[ r_0 = G_N M \ln \left( \frac{M_0}{M} \right) \geq 0. \] (47)
Note that due to our convention $\Theta(0) = 1$ the component $g_{rr}(r)$ is a continuous function in the interval $r \in [0, \infty)$ and $g_{rr}(0) = g_{rr}(+0)$ is a well defined quantity.

The ratio
\[
\vartheta = \frac{M}{M_0} = \sqrt{g_{rr}(0)} \in [0,1]
\]    \hspace{1cm} (48)


describes the gravitational mass defect of the point particle as a second physical parameter in the problem, see the Introduction.

The Keplerian mass $M$ and the ratio $\vartheta$ are the two parameters that define completely the solutions (41).

For the initial constants of the problem one obtains:
\[
\varphi_1(0) = \ln \vartheta, \quad \varphi_2(0) = -\ln \frac{1 - \vartheta^2}{2\vartheta},
\]
\[
\varphi_1'(0) = \frac{1}{G_NM}, \quad \varphi_2'(0) = \frac{1}{G_NM} \frac{1 + \vartheta^2}{1 - \vartheta^2}.
\]    \hspace{1cm} (49)

Thus we arrive at the following form of the solutions (41):
\[
\begin{align*}
\varphi_1(r) &= \frac{r\Theta(r)}{G_N M} - \ln(1/\vartheta), \\
\varphi_2(r) &= -\ln \left( \frac{1}{2} \left[ \frac{1}{2} g e^{-r/G_N M} - \frac{1}{2} g e^{r/G_N M} \right] \right),
\end{align*}
\]    \hspace{1cm} (50)

and the rho-gauge fixing function
\[
\rho_{\text{ref}}(r) = \rho_G \left( 1 - \vartheta^2 \exp \left( 4r/\rho_G \right) \right)^{-1}.
\]    \hspace{1cm} (51)

An unexpected feature of this two parametric variety of solutions for the gravitational field of a massive point particle is that each solution must be considered only in the finite interval
\[
r \in [0, G_N M \ln(1/\vartheta)]
\]
in the Euclidean 3D space $\mathcal{E}^{(3)}$ (see Section 3.1) since we must have $\rho_{\text{ref}}(r) > 0$. Then, we have a monotonic increase of the luminosity distance in the interval
\[
\rho_{\text{ref}}(r) \in [\rho_{\text{min}}, \infty),
\]

for $r$ in the above interval.

It is easy to check that away from the source, i.e., for $r > 0$, these solutions coincide with the solution (26) in the Hilbert gauge. Hence, outside the source the solutions (50) acquire the well-known standard form, when represented using the variable $\rho$.

This means that the solutions (50) strictly respect the generalized Birkhoff theorem. For a massive point source of gravity its proper generalization requires only justification of the physical domain of the variable $\rho$. It is remarkable that for the solutions (50) the minimal value of the luminosity distance is
\[
\rho_{\text{min}} = \rho_G / (1 - \vartheta^2) \geq \rho_G.
\]    \hspace{1cm} (52)

This changes the Gauss theorem and leads to important physical consequences for the physical properties of the massive ECOs, see the Introduction.

One such consequence is that in this case one must apply the Birkhoff theorem only in the interval $\rho \in [\rho_{\text{min}}, \infty)$, if spherical symmetry takes place.

9. REGULAR GAUGE MAPPING

OF THE INTERVAL $r \in [0, \rho_{\text{min}}]$ ONTO THE INTERVAL $r \in [0, \infty]$ OF A NEW AUXILIARY RADIAL VARIABLE $\tilde{r}$

As we have stressed in the previous section, the solutions (50) for the gravitational field of a point particle must be considered only in the physical domain of the auxiliary variable $r \in [0, \rho_{\text{min}}]$.

It is not convenient to work with such an unusual radial variable $r$.

Besides, the interpretation of the flat spacetime $\mathcal{E}^{(3)}$ as a tangent space above the point $r = 0$ requires to include all its points in our consideration.

Remembering that the used until now variable $r$ is a radial coordinate in the initial auxiliary Euclidean space $\mathcal{E}^{(3)}$, one can easily overcome this problem using an additional regular rho-gauge transformation
\[
r \mapsto \rho_{\text{ref}}^{-1} \tilde{r} \mapsto \rho_{\text{ref}}^{-1} \tilde{r} + 1,
\]    \hspace{1cm} (53)

with an arbitrary scale $\tilde{r}$ of the new auxiliary radial variable $r$.

Thus, we replace the initial auxiliary space $\mathcal{E}^{(3)}$ with another copy of this linear space, making use of the fractional linear transformation (53).

Note that this fractional linear change of the variable $r$ does not alter the number and the character of the singular points of the solutions in the whole compactified complex plane $\mathcal{E}^{(3)}_{\rho_G}$.

The transformation (53), being indeed a change of the labels in a specific tangent space, is physically inessential but important for clear understanding of the obtained mathematical results.

To avoid unnecessary notation for the radial coordinate, in the present article we use the same notation $r$ for these two different radial variables.

The transformation (53) simply places the point $r = r_{\infty}$ at the infinity $r = \infty$, at the same time preserving the initial place of the origin $r = 0$.

Now the new variable $\tilde{r}$ varies in the standard interval $r \in [0, \infty)$. Assuming $r > 0$ and taking into account
that \( \tilde{r}(0) = 1 \), in this interval the regular solutions (50) acquire the form

\[
\varphi_i(r) = -\ln \left( \frac{1}{\varrho} \left( 1 - \frac{r/\tilde{r}}{r^{2}/\tilde{r} + 1} \right) \right),
\]

\[
\varphi_2(r) = -\ln \left( \frac{1}{\varrho} \left( \frac{1}{\varrho^{1/2}} - \varrho^{1/2} \right) \right),
\]

and the rho-gauge fixing function reads

\[
\rho_{\text{reg}}(r) = \rho_G \left( 1 - \frac{1}{\varrho^{1/2}} \right)^{-1}.
\]

The last expression shows that the mathematically admissible interval of the values of the ratio \( \varrho \) is the open interval \((0, 1)\). This is so, because for \( \varrho = 0 \) and for \( \varrho = 1 \) we would have impermissible trivial gauge-fixing functions \( \rho_{\text{reg}}(r) \equiv 1 \) and \( \rho_{\text{reg}}(r) \equiv 0 \), respectively.

### 10. Transformation of the Hilbert Solution of the Schwarzschild Problem to the Solution with Massive Point Source in GR

After all, we are ready to describe the singular character of the coordinate transition from the Hilbert form of the Schwarzschild solution (26) to the regular one (54).

To simplify the notation, let us introduce dimensionless variables \( z = \varphi/\varrho_G \) and \( \zeta = r/\tilde{r} \). Then Eq. (55) shows that the essential part of the change of the coordinates is described, in both directions, by the functions

\[
z(\zeta) = \left( 1 - \frac{2}{\varrho_G \zeta^2} \right)^{-1}, \quad \text{and}
\]

\[
\zeta(z) = \ln \left( \frac{1}{\varrho_G^2} \right) - 1, \quad \varrho \in (0, 1).
\]

Obviously, the function \( z(\zeta) \) is regular at the place of the point source \( \zeta = 0 \); it has a simple pole at \( \zeta = \infty \) and an essential singular point at \( \zeta = -1 \). At the same time the inverse function \( \zeta(z) \) has logarithmic branch points at the Hilbert-gauge center of symmetry \( z = 0 \) and at the event horizon \( z = 1 \).

Thus, we see how one produces the Hilbert-gauge singularities at \( \varrho = 0 \) and at \( \varrho = \varrho_H = \varrho_G \), starting from a regular solution. The derivative

\[
d\zeta/dz = \frac{\ln \left( \frac{1}{\varrho_G^2} \right)}{z(z-1)(\ln z - \ln(z-1))^2}
\]

approaches infinity at these two points, hence the singular character of the change in the whole complex domain of the variables.

Now we have a complete description of the change of the coordinates and singularities of this change in the complex domain of the radial variable.

The restriction of the change of the radial variables on the corresponding physical interval outside the source: \( \zeta \in (0, \infty) \leftrightarrow z \in (1/(1 - \varrho^2), \infty) \), is a regular one.

It becomes obvious that the Hilbert solution (26) and the solution (54) belong to mathematically and physically different gauge sectors of GR as a nonlinear gauge theory.

### 11. Final Form of the GR Solution of the Einstein Equations With Single Massive Point Source

Expressions (54) and (55) still depend on the choice of the units for the new auxiliary variable \( r \).

We must fix the scale \( r \) in the form

\[
\tilde{r} = \varrho_G \ln \left( \frac{1}{\varrho_G^2} \right) = G_N M \ln \left( \frac{M_0}{M} \right)
\]

to ensure the validity of the standard asymptotic expansion:

\[
g_{tt} \sim 1 - \varrho_G^2 / r + O \left( \left( \varrho_G/r \right)^2 \right),
\]

when \( r \to \infty \), i.e., the Eddington condition [29].

This way the GR results for a single massive point particle, as described in the new auxiliary Euclidian 3D space \( E(3) \) can be correctly compared with the results of the same problem in the Newton theory of gravity in the flat 3D space \( E(3) \).

Now in the basic regular gauge \( \varphi(r) = 0 \) the solutions for a massive point particle read

\[
\varphi_1(r) = -\frac{G_N M}{r + G_N M \ln \left( \frac{M_0}{M} \right)} = \varphi_G(r),
\]

\[
\varphi_2(r) = \ln \left( \frac{2}{e^{-\varrho_G^2} - e^{\varrho_G^2}} \right).
\]

Using the modified (Newton-like) gravitational potential \( \varphi_G(r) \) (58), we can write down the final form of the 4D interval defined by the new regular solutions:

\[
ds^2 = e^{2\varphi_G(r)} \left( dt^2 - \frac{dr^2}{N_G(r)^4} - \rho(r)^4 d\Omega^2 \right),
\]

where the coefficient

\[
N_G(r) = \left( 2\varphi_G \right)^{-1} (e^{2\varrho_G} - 1) > 0,
\]

and the optical luminosity distance

\[
\rho(r) = \varrho_G \left( 1 - e^{2\varrho_G} \right) = \frac{r + G_N M \ln \left( \frac{M_0}{M} \right)}{N_G(r)}.
\]
These basic formulas describe in a more usual way our regular solutions of the Einstein equations for \( r \in [0, \infty) \) and show that:

1. In GR we have two types of regular solutions of the massive point particle problem:

   (A) Normal solutions for which \( \varrho \in (0, 1) \) and binding energy is positive \( E_{\text{binding}} = M_0 - M > 0 \).

   For them the modified newtonian gravitational potential (58) is a regular bounded function in the whole interval \( 0 \leq r \leq \infty \) without singularity at the origin \( r = 0 \). Its singular point \( r_{\text{sing}} = \ln \varrho < 0 \) is in the nonphysical domain \( r < 0 \).

   In the language of Newton’s gravity, the positivity of binding energy reflects the fact that the effective gravitational force is an attractive one and has a negative potential energy.

   (B) Abnormal solutions for which \( \varrho \in (1, \infty) \) and binding energy is negative \( E_{\text{binding}} = M_0 - M \leq 0 \).

   For them the modified newtonian gravitational potential (58) is a singular function in the physical domain \( r \in [0, \infty) \) and its singular point is \( r_{\text{sing}} = \ln \varrho \geq 0 \). For the value \( \varrho = 1 \) we have \( r_{\text{sing}} = 0 \), as in the case of Newton’s gravity.

   Despite the fact that the effective gravitational force for abnormal solutions is also an attractive one, the positivity of potential energy in the interval \( r \in [0, r_{\text{sing}}) \) (in Newtonian language), as well as the positivity of binding energy, together make transparent the nonphysical character of the abnormal solutions.

   Therefore, we do not consider them in more detail in the present paper.

2. For normal regular solutions \( \varphi_\text{Gr}(0) = \ln \varrho < 0 \) is finite and

   \[
   ds^2|_{r=0} = \varrho^2 dr^2 - \frac{\varrho^2 (\ln \varrho)^4}{(1 - \varrho^2)^3} dr^2 - \frac{\varrho^2}{(1 - \varrho^2)^3} d\Omega^2.
   \]

3. In the limit \( \varrho \to 1 \) for any value of \( r \in [0, \infty) \) we obtain \( \varphi_\text{Gr}(r, M, M_0) \to 0 \) and at all 3D space points

   \[
   g_\varphi(r) \to 1, \quad g_\sigma(r) \to -1, \quad \rho(r) \to \infty.
   \]

   Due to the last result, the 4D geometry does not have a meaningful limit when \( \varrho \to 1 \), i.e., massive point solutions without mass defect are not admissible in GR.

4. In the limit \( \varrho \to 0 \) our normal solutions tend to the Kohler–Pugachev–Gun’ko–Menzel ones, see Section 4.1.9. In this case, as in the Newton gravity \( \varphi_\text{Gr}(0) = \ln \varrho \to -\infty \).

5. For both normal and abnormal solutions in the limit \( r \to r_{\text{sing}} \) we obtain

   \[
   \lim_{r \to r_{\text{sing}}} ds^2 = \rho^2_0 d\Omega^2.
   \]

This result shows that the singular 2D surface \( r = r_{\text{sing}} \) in \( M^3 \{ -\text{g}_\text{min} \} \) is not an event horizon.

It is not hard to see that the new class of regular solutions does not contain the Hilbert one, see Section 4.1.2. The Hilbert solution cannot be considered also as a limit of any kind of the new regular solutions.

Thus we see that the Schwarzschild BH solutions and our regular solutions belong to different gauge sectors of GR.

12. TOTAL ENERGY OF THE SYSTEM OF A POINT SOURCE AND ITS GRAVITATIONAL FIELD

In the problem at hand we have an extreme example of an “island universe”. In it a privileged reference system and a well defined global time exist.

It is well known that under these conditions the energy of the gravitational field can be defined unambiguously [25].

Moreover, we can calculate the total energy of the aggregate of a mechanical particle and its gravitational field.

Indeed, the canonical procedure produces the total Hamilton density

\[
H_{\text{tot}} = \sum_{\mu=1}^{12} \pi_\mu \dot{\varphi}_\mu - L_{\text{tot}} = \frac{1}{2G_N} \left( -\frac{\rho^2 \varphi_\varphi^2}{(1 - \varrho^2)^2} + M_0 \varphi_\varphi \delta(r) \right). 
\]

Using the constraint (44) and the first of the relations (49), one immediately obtains for the total energy of the GR universe with one point particle in it:

\[
E_{\text{tot}} = \int_0^\infty H_{\text{tot}} dr = M = \varrho M_0 \leq M_0. 
\]

This result completely agrees with the strong equivalence principle of GR. The gravitational energy, i.e., the total energy of the gravitational field, created by a point particle, is the negative quantity:

\[
E_{gr} = E_{\text{tot}} - E_0 = M - M_0 = -M_0(1 - \varrho) < 0.
\]

The above consideration gives a clear physical explanation of the gravitational mass defect for a point particle and justifies the physical meaning of our normal regular solutions.

13. INVARIANT CHARACTERISTICS OF THE SOLUTIONS WITH A POINT SOURCE

13.1. Local Singularities of Point Sources

Using the invariant ID Dirac function, one can write down the first of Eqs. (36) in the form of the exact relativistic Poisson equation:

\[
\Delta_\varphi \varphi_\varphi(r) = 4\pi G_N M_0 \delta(r). 
\]

This equation is a specific realization of Fock’s idea (see Fock in [25]) using our normal field coordinates.
It can also be rewritten in transparent 3D form:

$$\Delta \rho_{\text{in}} = 4\pi G_{\text{N}} M_{\text{sch}}^{(0)}(r).$$  \hspace{1cm} (66)

The use of the invariant Dirac $\delta_{\text{in}}$-function has the advantage that under diffeomorphisms of the space $\mathcal{M}^{(0)}(\{g_{\mu\nu}(r)\})$ the singularities of the right-hand side of the relativistic Poisson equation (65) remain unchanged. Hence, we have the same singularity at the place of the source for the whole class of physically equivalent rho-gauges.

Then one can distinguish the physically different solutions of the Einstein Eqs. (1) with spherical symmetry by investigating the asymptotic in the limit $r \to +0$ of the coefficient

$$\gamma(r) = \frac{1}{4\pi \rho(r)} = \sqrt{1 - \rho_{\text{H}}/\rho(r)} \frac{d \rho}{dr}$$

in front of the usual 1D Dirac $\delta(r)$-function in the representation $\delta_{\text{in}}(r) = \gamma(r) \delta(r)$ of the invariant one.

For regular solutions (50), (54) the limit $r \to +0$ of this coefficient is a constant

$$\gamma(0) = \frac{1}{4\pi \rho_{\text{G}}} \frac{(1 - \varrho)^{2}}{\varrho(2 \ln \varrho)^{2}} = \frac{1 - \varrho}{4\pi \rho_{\text{G}}} F(\varrho),$$  \hspace{1cm} (67)

which describes the intensity of the invariant $\delta$-function and leads to the formula $\delta_{\text{in}}(r) = \gamma(0) \delta(r)$. The graph of the universal function $F(\varrho)$ is shown in Fig. 1.

Now we see that:

1. The condition $\varrho \in (0, 1)$ ensures a correct sign of the intensity $\gamma(0)$, i.e., the property $\gamma(0) \in (0, \infty)$, and thus, the negativity of the total energy $E_{\text{tot}}$ of the gravitational field of a point particle, according the previous Section 12.

2. For $\varrho = 0$ we have $\gamma(0) = \infty$, and for $\varrho = 1$: $\gamma(0) = 0$. Hence, in these two cases we have non-physical values of the intensity $\gamma(0)$.

Once more, our conclusion is that the physical interval of values of the ratio $\varrho$ for regular solutions is the open interval $(0, 1)$. This is consistent with the mathematical analysis of this problem, given in the previous sections.

Consider the same situation for other known solutions of the Schwarzschild problem, listed in Section 4.1.

(i) For the original the Schwarzschild solution one obtains

$$\gamma_{\text{Schwarzschild}}(0) = \infty.$$  

The corresponding Eq. (68) gives two nonphysical solutions: $\varrho_{\text{in}} = 0$ and $\varrho_{\text{in}} = \infty$; see Fig. 1. Hence, this solution for a massive point particle is not a physical one but may be considered as a limiting case of the physical solution.

(ii) As a result of the alteration of the physical meaning of the variable $\rho = r$ inside the sphere of the radius $\rho_{\text{H}}$, in the Hilbert gauge the coefficient $\gamma(r)$ tends to imaginary infinity for $r \to 0$:

$$\gamma_{\text{Hilbert}}(r) = \frac{i \rho_{\text{G}}}{4\pi r^{3/2}} \to i\infty.$$  

This is in sharp contrast to the real asymptotic of all other solutions at hands and shows once more that the Schwarzschild BH belong to a different gauge class of the Schwarzschild problem.

(iii) The Droste solution is a regular one:

$$\gamma_{\text{Droste}}(0) = \frac{1}{4\pi \rho_{\text{G}}}.$$  

From Eq. (67) one obtains the physical value of the gravitational mass defect

$$\varrho_{\text{Droste}} = 0.0134612785,$$

which belongs to the normal physical interval $\varrho \in (0, 1)$ and shows that the Droste solution is a regular one and can be transformed to (50), or (54) by a proper regular rho-gauge transformation.

There exists another root of the same equation $\varrho_{\text{Droste}} = 0.0134612785$, which belongs to the abnormal branch and does not correspond to a physical solution.

(iv) For the Einstein—Rosen solution, as a solution for a massive point particle, one obtains the nonphysical value

$$\gamma_{\text{Einstein—Rosen}}(0) = 0,$$
and nonphysical value of the mass defect ratio \( \varrho_{\text{Einstein-Rosen}} = 0 \).

(v) The case of the Weyl solution is more complicated. This solution resembles a nonphysical regular solution with \( \gamma_{\text{Weyl}}(0) = 0 \) and nonphysical value of the mass defect ratio \( \varrho_{\text{Weyl}} = 0 \).

Actually, the exact invariant \( \delta \)-function for this solution can be written in the form (see the Appendix B):

\[
\delta_{\text{w},w}(r) = \frac{16}{\pi \varrho_G} \left( \frac{r}{\varrho_G} \right)^4 \delta(r) + \left( \frac{\varrho_G}{r} \right)^4 \left( \delta \left( \frac{\varrho_G}{r} \right) \right). \tag{68}
\]

This distribution acts as a zero-functional on the standard class of smooth test functions with compact support, which are finite at the points \( r = 0, \infty \).

Formula (68) shows that:

(a) The Weyl solution describes a problem with spherical symmetry in the presence of two point sources: one with \( \varrho_0 = 1 \) at the point \( r = 0 \) and the other one with \( \varrho_0 = 1 \) at the point \( r = \infty \).

Thus, this solution is the first exact analytical two-particle-like solution of the Einstein Eqs. (1).

Unfortunately, as a two-point solution, the Weyl solution is physically trivial: it does not describe the dynamics of two point particles at a finite distance, but rather gives only a static state of two particles at infinite distance.

(b) To make these statements correct, one has to compactify the conformally flat 3D space \( \mathcal{M}^{(3)}(g_{\mu\nu}(r)) \) joining it to the infinite point \( r = \infty \). Then, both the Weyl solution and its source will be invariant with respect to the inversion \( r \rightarrow \varrho_G/r \).

(vi) The singularity of the coefficient \( \gamma(r) \) in front of the usual 1D Dirac \( \delta \)-function in the coherent gauge solution is stronger than any other in other listed solutions:

\[ \gamma_{\text{KPGM}}(0) = \infty. \]

The corresponding Eq. (68) gives two nonphysical solutions: \( \varrho_0 = 0 \) and \( \varrho_2 = \infty \), see Fig. 1. Those solutions can be considered as a limiting case of the regular physical solutions (50), or (54).

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13.2. Invariants of the Space-Times of Single Massive Point Particle

It is well known that the scalar invariants of the Riemann tensor allow a manifestly coordinate independent description of the geometry of space-time manifold.

The problem of a single point particle can be considered as an extreme case of a perfect fluid, which consists of only one particle.

According to the article [56], the maximal number of independent real invariants for such a fluid is 9. These are the scalar curvature \( R \) and the standard invariants \( r_1, w_1, w_2, m_3, m_5 \). (See [56] for their definitions.)

The invariants \( R, r_1 \) and \( m_3 \) are real numbers and the invariants \( w_1, w_2 \) and \( m_5 \) are, in general, complex numbers. The form of these invariants for spherically symmetric space-times in normal field variables is presented in Appendix C. Using these formulas we obtain for the regular solutions (50) in the basic gauge the following simple invariants:

\[ I_1 = \frac{1}{8\varrho_G} \left( 1 - \frac{2}{\varrho_G^2} \right)^4 \delta \left( \frac{r}{\varrho_G} \right) \]

\[ = \frac{\pi}{2\varrho} \left( 1 - \frac{2}{\varrho_G^2} \right)^3 \delta \left( \frac{r}{\varrho_G} \right) = \frac{1}{2} R(r). \tag{69a} \]

\[ I_2 = 0, \tag{69b} \]

\[ I_3 = \frac{\Theta(r/\varrho_G) - 1}{8\varrho_G^2 \varrho^2} \]

\[ \times (1 - \frac{2}{\varrho_G^2} e^{4r/\varrho_G})^4 = \frac{\varrho_G^2}{8\varrho} \Theta(r/\varrho_G) - 1 \]

\[ I_4 = \frac{4\varrho_G^2}{\varrho^2} \frac{\Theta(r/\varrho_G) - 1}{4\varrho_G^2} \]

\[ \times (1 - \frac{2}{\varrho_G^2} e^{4r/\varrho_G})^3 = \frac{\varrho_G^2}{4\varrho} \Theta(r/\varrho_G). \tag{69d} \]

During the calculation of the expressions \( I_{3,4} \) we have used once more our assumption \( \Theta(0) = 1 \).

As can be easily seen:

(1) The invariants \( I_{1,...,4} \) of the Riemann tensor are well-defined distributions and:

(i) This is in accordance with the general expectations described in the articles [58], where one can find a correct mathematical treatment of distributional valued curvature tensors in GR.

As we saw, the manifold \( \mathcal{M}^{(3)}(g_{\mu\nu}(x)) \) for our regular solutions has a geometrical singularity at the point, where the physical massive point particle is placed.

(ii) The metric for the regular solutions is globally continuous, but its first derivative with respect to the radial variable \( r \) has a finite jump at the point source.

This jump is needed for a correct mathematical description of the delta-function distribution of matter of the massive point source in the right-hand side of the Einstein Eqs. (1).

(iii) Three of the invariants (69) are independent for the values \( r \in (-\infty, \infty) \) on the real axes and this is the maximal number of the independent invariants in the problem of a single point source of gravity.

On the real physical interval \( r \in [0, \infty) \) one has \( I_3 = 0 \) and we remain with only two independent invariants.
For \( r \in (0, \infty) \) the only independent invariant is \( I_4 \), as is well known from the case of the Hilbert solution.

(2) The scalar invariants \( I_{1, \ldots, 4} \) have the same form as in Eq. (69) for the regular solutions in the representation (54) obtained via the fractional linear mapping (53).

(3) All other geometrical invariants \( r_1(r), w_{1,3}(r), m_{3,5}(r) \) include degrees of the Dirac \( \delta \)-function and are not well defined distributions.

Therefore, the choice of the simple invariants \( I_{1, \ldots, 4} \) is essential and allows us to use solutions of the Einstein equations, which are distributions.

To the best of author’s knowledge, the requirement to make the correct use of the theory of distributions possible is a novel criteron for the choice of invariants of the curvature tensor and seems not been used until now.

It is curious to know whether it is possible to apply this new criteron in more general cases, which differ from the case of static spherically symmetric space-times.

13.3. Existence of Event Horizon for Static Spherically Symmetric Space-Times

According to the well-known theorems by Hawking, Penrose, Israel and many other investigators, the only solution with a regular horizon not only in GR but in the theories with scalar field(s) and in more general theories of gravity is the Hilbert one [35].

As we have seen, this vacuum solution has a pure geometrical nature and does not describe a gravitational field of a point particle with proper mass \( M_p \neq 0 \).

The strong mathematical results like the well-known no hear theorems, etc., for the case of metrics with an event horizon are based on the assumption that such horizon indeed exists.

These mathematical results do not contradict the ones obtained in the present article, because the other solutions, which we have considered together with the Hilbert one, do not have an event horizon at all.

Indeed, for the point \( \rho_H \) at which \( g_r(\rho_H) = \pm \infty \) one obtains the equation

\[
\rho(\rho_H) = \rho_G.
\]

The last equation does not have any solution \( \rho_H \in \mathbb{C}^{(1)} \) for the regular solutions (50), (54). The absence of a horizon in the physical domain \( r \in [0, \infty) \) is obvious in the representation (60) of the regular solutions.

For the other classical solutions one obtains from the above equation

- Schwarzschild solution: \( \rho_H = 0 \);
- Hilbert solution: \( \rho_H = \rho_G \);
- Droste solution: \( \rho_H = 0 \);
- Weyl solution: \( \rho_H = \rho_G \);
- Einstein–Rosen solution \( \rho_H = 0 \);
- Isotropic \((t,x)\) solution: \( \rho_H = -\infty \);
- coherent gauge solution: \( \rho_H = 0 \).

As one can see, only for the Hilbert and Weil solutions the metric component \( g_{rr}(r) \) has a simple pole at the finite point \( \rho_H = \rho_G \), i.e., the solution has an event horizon at the finite point in the physical domain.

In the cases when \( \rho_H = 0 \), we have a physical massive point particle at this point, if the mass ratio \( \varrho \neq 0, 1 \).

14. UNUSUAL 3D GEOMETRY OF MASSIVE POINT SOURCE IN GR

It is curious that for a metric given by a regular solution the 3D-volume of a ball with a small radius \( r_b < \rho_G \) centered at the source is

\[
\text{Vol}(r_b) = \frac{4}{3} \pi \rho_G^3 \left( \frac{12 \varrho}{(1 - \varrho^2)^4} \rho_G + \mathcal{O} \left( \frac{r_b}{\rho_G} \right) \right).
\]

It goes to zero linearly with respect to \( r_b \to 0 \), in contrast to the Euclidean case, where \( \text{Vol}(r_b) \sim r_b^3 \).

This happens, because for the regular solutions (50), (54) in the limit \( r \to 0 \) we obtain

\[
\sqrt{\frac{3}{4}} | \mathbf{g} | = \rho(r)^2 \sin \theta \to \rho_{\text{min}}^2 \sin \theta \neq 0.
\]

Nevertheless,

\[
\lim_{r_b \to 0} \text{Vol}(r_b) = 0,
\]

and this legitimates the use of the term “a point source of gravity” in the problem at hand: the source can be surrounded by a sphere with an arbitrary small volume \( \text{Vol}(r_b) \) in it and with an arbitrary small radius \( r_b \).

In contrast, when \( r_b \to 0 \) the area of the ball surface has a finite limit

\[
\lim_{r_b \to 0} A(r_b) = \frac{4 \pi \rho_G^2}{(1 - \varrho^2)^2} > 4 \pi \rho_G^2,
\]

and the length of a big circle on this surface tends to a finite number

\[
\lim_{r_b \to 0} l(r_b) = \frac{2 \pi \rho_G}{1 - \varrho^2} > 2 \pi \rho_G.
\]

Such unusual geometry, created by the massive point sources in GR, may have interesting physical consequences.

For example, spacetimes defined by the regular solutions (50) and (54) have a unique property: when one approaches the point source, its luminosity remains finite, see also the formula (35).

This leads to an extremely important modification of the Gauss theorem in the corresponding 3D spaces.

After all, this modification may solve the well-known problem of the classical divergences in the field theory, because, as we saw, GR offers a natural cut-off parameter \( \tilde{r} = \rho_G/\ln(1/\varrho^2) \) for the fields created by massive point particles.
If this will be confirmed by more detailed calculations, the price one must pay for the possibility to overcome the old classical divergences problem will be to accept the fact that the massive point objects in GR can have a nonzero surface, having at the same time a zero volume and a zero size: \( r = 0 \) and \( l = 0 \).

On the other hand, as we have seen, the inclusion of point particles in GR is impossible without such unusual geometry.

The Hilbert solution no longer offers an attractive alternative for description of a source of gravity as a physical massive point placed at the coordinate point \( \rho = 0 \) and with usual properties of a geometrical point.

Hence, in GR we have no possibility to introduce the notion of a physical point particle with nonzero proper mass without a proper change of the standard definition for a point particle as a mathematical object, which has simultaneously zero size, zero surface and zero volume.

In GR we are forced to remove at least one of these three usual requirements as a consequence of the concentration of the proper mass \( M_0 > 0 \) at only one 3D space geometrical point.

From a physical point of view the novel properties of a massive physical point particle, at which we arrived in this article, is obviously preferable.

Moreover, it seems natural to have an essential difference between the geometry of the “empty” spacetime points and that of the matter points in GR since in the last case, keeping the finite proper mass \( M_0 > 0 \) into one geometrical point, we will obviously have an infinite mass density at this geometrical point.

According to the basic idea of GR, such an infinite mass density must produce an essentially strong deformation of the spacetime and, as we have shown here in detail, it creates a very unusual spacetime geometry.

The squared 4D infinitesimal distance at this point becomes

\[
d s^2 \bigg| _r = 0 = \rho^2 d t^2 - \frac{\rho^2 (\ln \rho)^4}{(1 - \rho^4)} d r^2 - \frac{\rho^2}{(1 - \rho^4)} d \Omega^2.
\]

This formula shows explicitly also the existence of a conic singularity with an angle excess around the massive point particle source of the gravitational field.

Alike geometry for such essentially different objects like an empty mathematical point and a massive physical point is possible only in the classical physics, where the spacetime geometry does not depend on the matter. Only under the last assumption there is no geometrical difference between pure mathematical space points and matter points.

The intriguing novel situation in GR, described in detail in the present article, deserves further careful analysis.

The appearance of the above nonstandard geometry in the point mass problem in GR was discussed without much details for the first time by Marcel Brillouin [60].

Our detailed consideration of the regular solutions (50), (54) confirms his preliminary guess on the character of the singularity of the gravitational field of massive point particles in GR.

15. CONCLUDING REMARKS

The most important result of the present article is the explicit demonstration of the fact that there exist infinitely many different static solutions of the Einstein Eqs. (1) with spherical symmetry, a mass point singularity placed at the center of symmetry, and vacuum outside this singularity, and with the same Keplerian mass \( M \).

These solutions fall into different gauge classes, which describe physically and geometrically different space-times.

Several of them were discovered at the early stages of the development of GR. Up to now they are often considered as equivalent representations of some “unique” solution, which depends on only one parameter—the Keplerian mass \( M \).

As shown in the present paper, this is not the case.

As a consequence of the Birkhoff theorem, when expressed in the same variables, the solutions with the same mass \( M \) indeed coincide in their common domain of coexistence—outside the singularities.

In contrast, they may have different behavior at the corresponding singular points.

A correct description of a massive point source of gravity is impossible, if using most of these classical solutions. The only exceptions are the Droste solution and the coherent gauge solution introduced in the present paper.

Using novel normal coordinates for the gravitational field of a single point particle with the proper mass \( M_0 > 0 \), we were able to describe correctly the massive point source of gravity in GR. The singular gauge transformations yield the possibility to overcome the restriction to have a zero proper mass \( M_0 = 0 \) for neutral point particles in GR [42]. It turns out that this problem has a two-parametric family of regular solutions.

One of the parameters—the proper mass \( M_0 \) of the point source, can be obtained in the form of surface integral, integrating both sides of Eq. (66) on the whole 3D space \( M^{(3)} \{ - g_{mn} \} \):

\[
M_0 = \frac{1}{4 \pi G_N} \int_{M^{(3)}} d^3 r \sqrt{|g|} \partial_j (\ln \sqrt{g_{ij}}) \int_{\partial M^{(3)}} d^2 r \sqrt{|g|} g_{ij} \partial_j (\ln \sqrt{g_{ij}}).
\]
The second parameter is the Keplerian mass $M$ described, as shown in Section 12, by the total energy:

$$M = \int_{\mathcal{M}} d^3r \mathcal{H}_{\text{tot}} = \int_{\mathcal{M}} d^3r \sqrt{g} M_\nu \delta_{(3)}^g (r).$$

These two parameters define the gauge class of the given regular solution.

Obviously, the parameters $M$ and $M_0$ are invariant under regular static gauge transformations, i.e., under diffeomorphisms of the 3D space $\mathcal{M}^3 \approx \mathcal{M}^3(\geq m_s)$. It is an analytical manifold with a strong singularity at the place of the massive point particle.

For every regular solution of the Einstein Eqs. (1) both parameters are finite and positive and satisfy the gravitational mass ratio for this solution

$$\frac{\rho}{\rho_0} \approx 0.0134612785.$$

For correct understanding of these statements one has to take into account that the above Dirac idea is expressed just in terms of the luminosity radius $\rho$. For correct understanding of these statements one has to take into account the above Dirac idea is expressed just in terms of the luminosity radius $\rho$. We are forced to cut the Hilbert form of the vacuum Schwarzschild solution at the value $\rho = \rho_\text{Sch} / (1 - \rho^2)$ because of the presence of the matter point mass. Its presence ultimately requires a definite jump of the first derivatives of the metric. The jump is needed to make the Einstein tensor coherent with the energy-momentum stress tensor of the point particle. This cutting can be considered as a further development and physical clarification of Dirac’s idea [59]:

"Each particle must have a finite size no smaller than the Schwarzschild radius. I tried for some time to work with a particle with radius equal to the Schwarzschild radius, but I found great difficulties, because the field at the Schwarzschild radius is so strongly singular, and it seems that a more profitable line of investigation is to take a particle bigger than the Schwarzschild radius. I tried for some time to work with a particle with radius equal to the Schwarzschild radius, but I found great difficulties, because the field at the Schwarzschild radius is so strongly singular, and it seems that a more profitable line of investigation is to take a particle bigger than the Schwarzschild radius and to try to construct a theory for such particle interacting with the gravitational field."

For the class of regular solutions written in the form (50), (54), or (60), the non-physical interval of the optical luminosity distance $\rho \in [0, \rho_G/(1 - \rho^2)]$, which includes the luminosity radius $\rho_H = \rho_G$, must be considered as an “optical illusion”.

All pure mathematical objects, which belong to this interval, are in the imaginary domain “behind” the real physical source of the gravitational field and have to be considered as a specific kind of optical “mirage”.

This is in agreement with Dirac’s conclusion about the non-physical character of the domain $\rho \in [0, \rho_G]$ [59] for point particles and extends this conclusion to the whole non-physical interval of the optical luminosity distance $\rho \in [0, \rho_G/(1 - \rho^2)] \supset [0, \rho_G]$ for a given regular solution.

The metric defined by a regular solution is globally continuous in the whole domain $r \rightarrow -\infty, \infty$, but its first derivatives have a jump at the position of the matter point source. The second derivative of the metric is a distribution described by the Dirac function.
An alternative development, which is much closer to the Dirac approach to the above problem, can be found in the recent articles [61], as well as in the latest developments [14–20], where the general framework of ECOs, which are not BH, is developed in relation with possible echoes in gravitational waves from binary merger. Our approach sheds essentially new light on these new problems.

The geometry around physical massive point particles is essentially different from the geometry around the “empty” geometrical points.

This unusual geometry, discussed for the first time by Brillouin, may offer a new way for overcoming the classical fields divergence problems. The new possibility differs from the one used in [42] for charged point particles. It is based on the existence of the natural cut-off parameter \( \tilde{r} = G_\text{N}M(c^2 \ln(M_0/M))^{-1} \) and on a new interpretation of the relation between the singular mass distribution of a point particle and the geometry of the space-time around this particle. This relation is described here for the first time in full detail (see also [22]).

According to the remarkable comment by Poincaré [62], real problems can never be considered as solved or unsolved but rather they are always more or less solved.

The strong physical singularity at the “event horizon” of the Hilbert form of solution of the Schwarzschild problem in the presence of a massive point source, stressed by Dirac, as well as further physical consequences, which one can derive from the new regular solutions (54) of this more than one century old problem, will be discussed in separate articles, see, for example [63].

The corresponding development for a charged massive point source can be found in [64].

Here we will add some more remarks:

Our consideration shows that the observed by the astronomers extremely compact dark objects ECOs, called by them currently black holes (BHs) without any direct evidences for the existence of real event horizons [65], cannot be described theoretically by solutions (26) in the Hilbert gauge, if we assume that these objects are made of matter of some nonzero proper mass \( M_0 > 0 \).

At present, the only real fact is the existence of massive invisible ECOs with the Keplerian mass \( M \approx 5M_\odot \), which is too large with respect to the conventional understanding of the star physics.

Concerning these unusual ECOs, most probably one actually has to solve a much more general problem.

Since a significant amount of invisible dark matter, whichmanifests itself only due to its gravitational field, is observed in the Nature at very different scales: in the clusters of galaxies, in the halos of the galaxies, at the center of our galaxy, and as compact dark components in some binary star systems, one is tempted to look for some universal explanation of all these phenomena. Obviously, such universal explanation cannot be based on the Schwarzschild solution in the Hilbert gauge and one has to look for some other theoretical approach. A similar idea was pointed out independently in the recent articles [66].

At the end, we wish to mention some additional open problems, both mathematical and physical, connected with the gravitational field of point particles.

Our consideration was essentially restricted to the real domain of variables. The only exception was the description of the singular change (56) in the radial variable.

It is well known that the natural domain for studying the solutions of holomorphic differential equations is the complex one. Complete knowledge of solutions of these equations is impossible without a description of all singularities of the solutions in the complex domain.

Therefore, looking for a complete analysis of the solutions of some differential equation of \( n \)th order \( f(w^n)(z), w^{n-1}(z), w(z), z = 0 \) for a function \( w(z), z \in \mathbb{C}(\mathbb{R}) \), where \( f(\ldots) \) is a holomorphic function of the corresponding complex variable, one has to consider these solutions as holomorphic functions of the complex variable \( z \). Now the study of the singular points of the solutions in the whole complex domain becomes the most important issue.

For the Einstein Eqs. (1) a full four-dimensional complex analysis is impossible at present, because the corresponding mathematical methods are not developed enough.

For the problem of single massive point source in GR, in its 1D formulation used in this article, one can apply the classical complex analysis. Moreover, using the well-known complex representation of distributions (see Bremermann in [32]), it is not difficult to generalize the classical results for the case when the differential equation has on the right-hand side a distribution that depends on the variable \( z \). Such a term may lead to a discontinuity of the solution \( w(z) \) or its derivatives. The corresponding complex analysis of the solution remains an important open mathematical problem, together with the problem of finding and classifying all static solutions of the Einstein equations with one matter point singularity and spherical symmetry.

The analysis of QNMs of waves of different spin on the Schwarzschild or Kerr BH background in complex domain of the radial variable \( \rho \) was the basic idea for successful high-precision computations in [50–52, 55].
To find criteria for an experimental and/or observational probe of different solutions of the Einstein equations with spherical symmetry becomes the most important physical problem. Since there exist physical and geometrical differences between the solutions of this type, one can find a real way of distinguishing them experimentally.

Especially, a problem of the present day is to answer the question of which of the spherically symmetric solutions gives the right description of the observed ECOs if one hopes that the ECOs can be considered in a good approximation as spherically symmetric objects with some Keplerian mass $M > 0$ and some mass ratio $\varrho \in (0, 1)$.

In this case, according to the Birkhoff theorem, we must use the new solutions (61) for describing their gravitational field in the outer vacuum domain, outside the ECOs.

For the same reason, the solutions (60) must be used also for describing the vacuum gravitational field around spherically symmetric bodies of finite size $R$, made of usual matter.

Due to the specific character of the correction of the Newton law of gravitation defined by the term $G_N M c^{-2} / \ln \left( \frac{M_0}{M} \right)$ in the potential $\phi_G(r; M, M_0)$ (58), the difference between our solutions (61) and the Hilbert one is in higher order relativistic terms, when
\[
\ln \left( \frac{M_0}{M} \right) \sim 1.
\]

This difference will not influence the rho-gauge invariant local effects like gravitational red-shift, perihelion shift, deviation of light rays, time-dilation of electromagnetic pulses, and so on, but it may be essential for the quantities, which depend directly on the precise form of the potential $\phi_G(r; M, M_0)$ (58) and especially on the minimal admissible value $\rho_{\text{min}} = \frac{\rho_G}{1 - \varrho^2} \geq \rho_G$.

For all bodies in the solar system the term $G_N M c^{-2} / \ln \left( \frac{M_0}{M} \right)$ is very small in comparison with their physical radius $R$: $(\rho_G/2R)_{\text{Earth}} \sim 10^{-10}$, $(\rho_G/2R)_{\odot} \sim 10^{-6}$, etc.

Nevertheless, the small differences between the Newton potential and the modified one may be observed in the near future in the high precision measurements under the programs like APOLLO, LATOR, etc. (see, for example, [67]) if we will be able to find the proper quantities, which depend on the precise form of the potential $\phi_G(r; M, M_0)$ (58).

A better possibility for observation of deviations from Newton’s gravitational potential and in four-interval may be offered by neutron stars, since for them $(\rho_G/2R)_{\text{m}} \sim 0.17–0.35$.

The results of the present article may have an important impact on the problem of gravitational collapse in GR and the experimental study of GW, which is already accessible for us [1, 2].

Up to now most of the investigations known to the author presuppose to have BH as a final state of the collapsing object, i.e. the Hilbert solution, or its rotating Kerr generalization with a more general kind of the event horizon, in the spirit of the Roger Penrose cosmic censorship hypotheses and the Kip Thorn hoop conjecture.

These two conjectures have not yet been proven, after more than 50 years of hard efforts in this direction. They both disregard the unusual spacetime geometry created by a massive point with a mass defect, see Section 14.

Both conjectures presuppose a jump between the solutions of the Einstein equations, which belong to different gauge classes. Such a topological jump cannot be obtained in a continuous way, for example, during the exhaustion of a star’s nuclear fuel, which is a continuous process from a macroscopic point of view.

In the last decade, more attention was paid also to the final states with necked singularities. The existence of a two-parametric class of regular static solutions without an event horizon opens a new perspective for these investigations.

From a conceptual point of view, one can stress that the existence of different gauge classes of solutions in GR, which is certainly a nonlinear gauge theory, is still not enough recognised and carefully used in different kinds of its applications.

Most probably, the proper understanding of the origin of the mass defect of a point particle, introduced in the present article, will be reached by a correct description of the gravitational collapse.

Without any doubts, the inclusion of the new static spherically symmetric regular solutions of the Einstein equations in the corresponding investigations will open new perspectives for further developments in GR, as well as in its modern generalizations.

A proper generalization of our approach to rotating massive-point-particle sources in GR is urgently needed.

Application of the new regular solutions for studying dynamics of several point particles in GR may also be of critical importance for present-day experimental applications of GR, especially in theoretical studies of GW, radiated by such systems of particles [68].
DIRECT DERIVATION OF FIELD EQUATIONS FROM THE EINSTEIN EQUATIONS

Using the normal field variables, introduced in Section 5, one can write down the nonzero mixed components $G^i_r$ of the Einstein tensor in the basic regular gauge ($\varpi(r) = 0$) as follows:

$$G_i^r = \frac{e^{\phi_0 - 4\varpi}}{\rho^2} \left( 2\rho^2 \phi'' - 2\rho^2 \varphi'' - (\varpi \phi')^2 + (\varpi \varphi')^2 + e^{2\phi_0} \right),$$

$$G_r^r = \frac{e^{2\phi_0 - 4\varpi}}{\rho^2} \left( (\varpi \phi')^2 + (\varpi \varphi')^2 + e^{2\phi_0} \right),$$

$$G_0^0 = G_\vartheta^\vartheta = \frac{e^{2\phi_0 - 4\varpi}}{\rho^2} \left( (\varpi \varphi')^2 - (\varpi \phi')^2 - (\varpi \varpi')^2 \right).$$

(A1)

Taking into account that the only nonzero component of the energy-momentum tensor is $T_\varrho^\varrho$, one immediately obtains the field Eqs. (39) and the constraint (44). A slightly more general consideration with an arbitrary rho-gauge fixing function yields Eqs. (36) and constraint (38).

A similar derivation produces Eqs. (12), which present another version of the field equations in spherically symmetrical space-times. In this case:

$$G_i^r = \frac{1}{g_{rr}} \left( -\left( \frac{\varpi}{\rho} \right)' + 3 \left( \frac{\varpi}{\rho} \right)^2 + 2 \frac{\varpi}{\rho} \left( \frac{\sqrt{g_{rr}}}{\rho} \right) + \frac{1}{\rho^2} \right),$$

$$G_r^r = \frac{1}{g_{rr}} \left( -\left( \frac{\varpi}{\rho} \right)' + 2 \frac{\varpi}{\rho} \left( \frac{\sqrt{g_{rr}}}{\rho} \right) + \frac{1}{\rho^2} \right),$$

$$G_0^0 = G_\vartheta^\vartheta = \frac{1}{g_{rr}} \left( -\left( \frac{\varpi}{\rho} \right)' - \left( \frac{\varpi}{\rho} \right)^2 - \frac{\varpi}{\rho} \left( \frac{\sqrt{g_{rr}}}{\rho} \right) \right) + \frac{\varpi}{\rho} \left( \frac{\sqrt{g_{rr}}}{\rho} \right)^2 - \frac{\varpi}{\rho} \left( \frac{\sqrt{g_{rr}}}{\rho} \right)^2 \right).$$

(A2)

Due to the limits $r_0^0(\rho) \to \infty$ and $r_\vartheta^\vartheta(\rho) \to 0$ for $\rho \to \infty$, one can write down the invariant $\delta$-function in the case of the Weyl gauge in the form:

$$\delta_{\varrho,\vartheta}(\rho) = \frac{\sqrt{1 - \rho_G/\rho}}{4\pi \rho^4} \delta \left( \frac{1}{\rho} \right).$$

(B1)

Now taking into account that the equation $1/\rho(r) = 0$ has two solutions: $r = 0$ and $1/r = 0$ and using the standard formula for expansion of the distribution $\delta(1/\rho(r))$, we obtain easily the representation (68).

INDEPENDENT NONZERO INVARIANTS OF RIEMANN CURVATURE IN NORMAL FIELD COORDINATES

Using the representation (34) of the metric in the normal field variables and after some algebraic manipulations, one obtains the following expressions for the possibly independent nonzero invariants of the Riemann curvature tensor in the problem at hand:

$$R = -2e^{2(\phi_0 - \varpi)} (\phi'' - 2E_2 - E_3),$$

$$\eta_1 = \frac{1}{4} e^{4(\phi_0 - 2\varpi)} \left( 2 \left( \phi'' - 2E_2 - E_3 \right)^2 + \left( \phi'' + E_3 \right)^2 \right),$$

$$w_1 = \frac{C_1^2}{6}, \quad w_2 = -\frac{C_2^3}{36},$$

$$m_3 = \frac{C_2^2}{12} e^{4(\phi_0 - 2\varpi)} \left( \left( \phi'' - 2E_2 - E_3 \right)^2 + 2 (\phi'' + E_3)^2 \right),$$

$$m_3 = \frac{C_2^2}{12} e^{4(\phi_0 - 2\varpi)} \left( \left( \phi'' - 2E_2 - E_3 \right)^2 + 2 (\phi'' + E_3)^2 \right),$$

where $E_2 := \phi'' - e^{2\phi_0}/\rho^2$ and $E_3 := \phi'' - e^{2\phi_0}/\rho^2$.

As we can see, the invariants $w_{1,2}$ and $m_3$ in our problem are real. Hence, in it we may have at most 6 independent invariants.
In addition \((w_1)^3 = 6 (w_2)^2\) and the metric (34) falls into the class II of the Petrov classification [56]. As a result, the number of independent invariants is at most 5.

Instead of the invariants \(w_1\) and/or \(w_2\), we use as an independent invariant
\[
C := \mathcal{E}^{2\theta - 4\phi} \left(2\phi'' + 6\phi' \left(\phi'_1 - \phi'_2\right) - E_2 - 2E_3\right). \tag{C2}
\]

It has the following advantages:

(i) It is linear with respect of the derivative \(\phi''\). This property makes the expression (C2) meaningful in the cases when \(\phi''\) is a distribution;

(ii) It is a homogeneous function of first degree with respect to the Weyl conformal tensor \(C_{\mu\nu\lambda\rho}\);

(iii) It is proportional to an eigenvalue of a proper tensor, as pointed, for example, by Landau and Lifshitz [25].

(iv) The difference between the scalar curvature \(R\) and the invariant (C2) yields a new invariant
\[
D := R - C = 3\mathcal{E}^{2\theta - 4\phi} \left(2\phi' \left(\phi'_1 - \phi'_2\right) + E_2\right), \tag{C3}
\]

which does not include the derivative \(\phi''\).

Due to this property, the invariant \(D\) in the case of regular solutions (50), (54) will not contain the Dirac \(\delta\)-function. One can use the new invariant \(D\) as an independent one, instead of the invariant \(C\).

As seen from Eq. (C1), the following functions of the invariants \(r_1, m_3\) and \(m_5\):
\[
e^{2(\theta - 2\phi)} \left(\phi'' + E_3\right) = \sqrt{4(m_3/w_1 - r_1)/3}, \tag{C4}
\]
\[
e^{2(\theta - 2\phi)} \left(2\phi'' - E_2 - E_3\right) = \sqrt{2(4r - m_3/w_1)/3},
\]
are linear with respect to the derivative \(\phi''\). Using proper linear combinations of these invariants and the scalar curvature \(R\), one can easily check that the quantities \(E_2 e^{2(\theta - 2\phi)}\) and \(E_3 e^{2(\theta - 2\phi)}\) are independent invariants, too.

Thus, our final result is that, in general, for the metric (34) in the basic regular gauge \(\mathcal{G} \equiv 0\) the Riemann curvature tensor has the following four independent invariants:

\[
I_1 := e^{2(\theta - 2\phi)} \phi'' ,
\]
\[
I_2 := e^{2(\theta - 2\phi)} \left(\phi'' - e^{2\phi}/p^2\right) ,
\]
\[
I_3 := e^{2(\theta - 2\phi)} \left(-\phi''^2 + \phi'_1^2 + 2\phi'_2 e^{2\phi}/p^2\right)/2 ,
\]
\[
I_4 := e^{2(\theta - 2\phi)} \phi'_1 \phi'_2/2 .
\]

These invariants are linear with respect to the second derivatives of the functions \(\phi_{1,2}\)—a property, which is of critical importance when we have to work with distributions \(\phi''_{1,2}\).
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