A note about the linearized stability of a class of nonlinear fractional differential systems

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Abstract

In this paper, we discuss about the linearized stability of the trivial solution for a class of nonlinear Caputo fractional differential systems of order \( \alpha \in (1, 2) \). We show that some recent existing results in this direction are wrong. To complete the literature we also give a criterion to test the asymptotic stability for the trivial solution of this system.

1 Introduction

One of the most fundamental problems in the qualitative theory of fractional differential equations is stability theory. Two main methods are usually used to investigate the stability of solution of nonlinear fractional systems as the linearization method and the Lyapunov function method. There have been many publications on the Lyapunov function and we refer the reader to [8] or [7] for a survey. However, up to now, there are few researches concerning the linearization method.

Recently, some papers studying about the linearized stability of nonlinear fractional differential systems of order \( \alpha \in (1, 2) \) are published, see e.g., Chen et al [1], Chen et al [2], Zhang et al [11], and B.K. Lenka and S. Banerjee [6]. Unfortunately, these papers [1, 2, 11, 6] contain serious flaws in the proof of the linearized stability theorem. Namely, there are two common flaws in these papers:

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• **Incorrect application of the Gronwall lemma:** The main theorems in [1] and [11] are [1, Theorem 2] and [11, Theorem 1], respectively. In order to obtain the proof, the authors need to apply the Gronwall lemma (see [1, line 4, column 2, p. 604], [11, line -6, column 2, p. 103]), but the multiplier function in the inequality they want to apply the Gronwall lemma does depend on the variable $t$ besides the variable $\tau$ of the integration. This circumstance makes their application of the Gronwall lemma invalid.

• **Wrong estimation of Mittag-Leffler functions:** The key in the proof of the main results in [2] and [6] (see [2, Theorem 1] and [6, Theorem 2]) is the estimation for the Mittag-Leffler functions $E_{\alpha,1}(\cdot)$, $E_{\alpha,2}(\cdot)$ and $E_{\alpha,\alpha}(\cdot)$, see [2, Estimates (16), (17), p. 636] and [6, Estimates (25), (26), p. 4]. Hence, Lemma 2(ii) [2, Lemma 2(ii)] plays an essential important role. However, this result is wrong. Indeed, consider the case $\alpha \in (1,2)$ and $A$ is a negative real number $-\lambda$, where $\lambda > 0$. According to Theorem [10, Theorem 1.4, pp. 33–34], we have the following expansion as $t \to \infty$:

$$E_{\alpha,\beta}(-\lambda t^\alpha) = \frac{1}{\Gamma(\beta - \alpha)\lambda t^\alpha} + \mathcal{O}(|t|^{2\alpha}).$$

This implies that for $\beta \in \{1,2\}$, there exists a positive number $N$ small enough such that

$$|E_{\alpha,\beta}(-\lambda t^\alpha)| > \frac{N}{t^\alpha}, \quad (1)$$

for $t$ is large enough. Furthermore, if $\beta = \alpha$, from the asymptotic expansion of the Mittag-Leffler function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ above, we obtain

$$E_{\alpha,\alpha}(-\lambda t^\alpha) = \mathcal{O}(|t|^{2\alpha})$$

as $t \to \infty$, which implies that

$$|E_{\alpha,\alpha}(-\lambda t^\alpha)| > \frac{N}{t^{2\alpha}}, \quad (2)$$

for $t$ is large enough. Combining (1) and (2) shows that for $t$ sufficiently large

$$|E_{\alpha,\beta}(-\lambda t^\alpha)| > \exp(-\lambda t),$$

where $\beta \in \{1,\alpha, 2\}$. Thus, Lemma 2 in [2] is not true which leads that the proof of theorems [2, Theorem 1] and [6, Theorem 2] is invalid.
In this paper, we consider system of nonlinear Caputo fractional differential equations of order $\alpha \in (1, 2)$:

$$^C D^\alpha_0+ x(t) = Ax(t) + f(x(t)), \quad (3)$$

where $t \geq 0$, $A \in \mathbb{R}^{d \times d}$, $f$ is local Lipschitz continuous in a neighborhood of the origin and satisfies

$$f(0) = 0, \quad \lim_{r \to 0} \ell_f(r) = 0, \quad \text{(4)}$$

in which

$$\ell_f(r) := \sup_{\|x\|,\|y\| \leq r, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$ 

By using the Linearization method, we will give a criterion to test the asymptotic stability for the trivial solution of system (3).

The structure of this paper is as follows: in Section 2, we recall some background on fractional calculus and fractional differential equations. Section 3 is devoted to the main result about linearized asymptotic stability for fractional differential equations. We conclude this introductory section by introducing some notations which are used throughout the paper.

First, let $\mathbb{R}^d$ be endowed with the max norm, i.e., $\|x\| = \max(|x_1|, \ldots, |x_d|)$ for all $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$. For $r > 0$, denote $B_{\mathbb{R}^d}(0, r) := \{x \in \mathbb{R}^d : \|x\| \leq r\}$. The set of nonnegative real number is denoted by $\mathbb{R}_{\geq 0}$. The space of continuous functions $\xi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^d$ such that

$$\|\xi\|_{\infty} := \sup_{t \in \mathbb{R}_{\geq 0}} \|\xi(t)\| < \infty$$

is defined by $(C_{\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^d), \cdot, \|\cdot\|)$. Furthermore, the ball having radius $r$ centering at 0 is written as $B_{C_{\infty}}(0, r) := \{\xi \in C_{\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^d) : \|\xi\|_{\infty} \leq r\}$. Finally, we use the notation $\Lambda^\alpha_{\delta}$ to denote the sector

$$\Lambda^\alpha_{\delta} := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| > \frac{\alpha \pi}{2}\}$$

for a given positive parameter $\alpha$. 

3
2 Preliminaries

In this section, we recall a short introduction about fractional differential equations.

Let $\alpha \in (1, 2)$ and $-\infty \leq a < b \leq \infty$. The Caputo fractional derivative $\mathcal{C}D_{a+}^\alpha x$ of order $\alpha$ of a function $x \in C^2([a, b]; \mathbb{R})$ is defined by

$$\mathcal{C}D_{a+}^\alpha x(t) := \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-\tau)^{1-\alpha} D^2 x(\tau) \, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function, $D = \frac{d^2}{dt^2}$ is the usual second derivative.

While the Caputo fractional derivative of a $d$-dimensional vector function $x(t) = (x_1(t), \cdots, x_d(t))^T$ is defined component-wise as

$$(\mathcal{C}D_{a+}^\alpha x)(t) := (\mathcal{C}D_{a+}^\alpha x_1(t), \cdots, \mathcal{C}D_{a+}^\alpha x_d(t))^T.$$

Since the function $f(\cdot)$ is local Lipschitz continuous, [4, Theorem 6.5] implies unique existence of local solutions of the initial value problems (3),

$$x(0) = x_0, \quad \frac{dx}{dt}|_{t=0} = \bar{x}$$

for any $x, \bar{x} \in \mathbb{R}^d$. Let $\varphi : I \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $t \mapsto \varphi(t; x, \bar{x})$, denote the solution of (3), $x(0) = x$, $\frac{dx}{dt}|_{t=0} = \bar{x}$ on its maximal interval of existence $I = [0, t_{\text{max}}(x, \bar{x})]$ with $0 < t_{\text{max}}(x, \bar{x}) \leq \infty$. We now give the notions of stability and asymptotic stability of the trivial solution of (3).

**Definition 1.** The trivial solution of (3) is called:

- **stable** if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x, \bar{x} \in B_{\mathbb{R}^d}(0, \delta)$ we have $t_{\text{max}}(x, \bar{x}) = \infty$ and

$$\|\varphi(t; x, \bar{x})\| \leq \varepsilon \quad \text{for } t \geq 0.$$

- **unstable** if it is not stable.

- **attractive** if there exists $\hat{\delta} > 0$ such that $\lim_{t \to \infty} \varphi(t; x, \bar{x}) = 0$ whenever $x, \bar{x} \in B_{\mathbb{R}^d}(0, \hat{\delta})$.

The trivial solution is called **asymptotically stable** if it is both stable and attractive.

If $f = 0$, system (3) reduces to a linear system

$$\mathcal{C}D_{0+}^\alpha x(t) = Ax(t). \quad (5)$$
For the initial condition \( x(0) = x, \frac{dx(t)}{dt} \big|_{t=0} = \bar{x} \), system (5) has the unique solution as \( E_{\alpha,1}(t^\alpha A)x + tE_{\alpha,2}(t^\alpha A)\bar{x} \), where the Mittag-Leffler function \( E_{\alpha,\beta}(A) \) is defined as
\[
E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.
\]

The following theorem gives a representation for bounded solutions of (3).

**Theorem 2** (Variation of constants formula for fractional differential systems). Consider system (3). Assume that \( f(\cdot) \) is global Lipschitz continuous in \( \mathbb{R}^d \). Then, the solution \( \varphi(t; x, \bar{x}) \) of (3) with the initial condition \( \varphi(0; x, \bar{x}) = x, \frac{d\varphi}{dt} \big|_{t=0} = \bar{x} \) satisfies
\[
\varphi(t; x, \bar{x}) = E_{\alpha,1}(t^\alpha A)x + tE_{\alpha,2}(t^\alpha A)\bar{x} + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha}((t - \tau)^\alpha A)f(\varphi(\tau; x, \bar{x})) \, d\tau
\]
for any \( t \geq 0 \) and \( x, \bar{x} \in \mathbb{R}^d \).

**Proof.** See [9].

To complete this section, we list here some interesting properties of the Mittag-Leffler functions which will be used to prove the main our result.

**Lemma 3.** For \( \alpha \in (1, 2) \) as shown above, let \( \lambda \in \Lambda^*_\alpha \). Then, the following statements hold:

(i) the functions \( E_{\alpha,1}(\lambda t^\alpha) \) and \( tE_{\alpha,\alpha}(\lambda t^\alpha) \) decay at the infinity, i.e.,
\[
\lim_{t \to \infty} |E_{\alpha,1}(\lambda t^\alpha)| = 0, \quad \lim_{t \to \infty} |tE_{\alpha,2}(\lambda t^\alpha)| = 0;
\]

(ii) there exists a positive constant \( M(\alpha, \lambda) \) and a positive number \( t_0 \) such that
\[
|t^{\alpha - 1} E_{\alpha,\alpha}(\lambda t^\alpha)| < \frac{M(\alpha, \lambda)}{t^{\alpha + 1}} \quad \text{for any } t > t_0;
\]

(iii) there exists a positive constant \( C(\alpha, \lambda) \) such that
\[
\sup_{t \geq 0} \int_0^t |(t - \tau)^{\alpha - 1} E_{\alpha,\alpha}(\lambda(t - \tau)^\alpha)| \, d\tau < C(\alpha, \lambda).
\]
Proof. (i) Using the asymptotic expansion of Mittag-Leffler functions, see [10, Theorem 1.4, pp. 33-34]. The proof of (ii) and (iii) is similar to the proof of [3, Proposition 2.4]. Hence, we omit it.

3 Linearized asymptotic stability for fractional differential equations

Consider system (3):

\[ C^\alpha_0 x(t) = Ax(t) + f(x(t)). \]

We now state the main result of this paper.

Theorem 4. Assume that the spectrum \( \sigma(A) \) of the matrix \( A \) is contained in the sector \( \Lambda_{\alpha}^s \), i.e., \( \sigma(A) \subset \Lambda_{\alpha}^s \) and the function \( f \) is global Lipschitz continuous in \( \mathbb{R}^d \) and satisfies condition (4). Then the trivial solution of system (3) is asymptotically stable.

Proof. Suppose that \( A \) has the eigenvalues as \( \hat{\lambda}_1, \ldots, \hat{\lambda}_m \). Let \( T \) be the nonsingular matrix which transforms \( A \) into the Jordan normal form, i.e.,

\[ T^{-1}AT = \text{diag}(A_1, \ldots, A_n), \]

where for \( i = 1, \ldots, n \), the block \( A_i \) is of the following form

\[ A_i = \lambda_i \text{id}_{d_i \times d_i} + \eta_i N_{d_i \times d_i}, \]

with \( \eta_i \in \{0, 1\} \), \( \lambda_i \in \sigma(A) = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_m\} \), and the nilpotent matrix \( N_{d_i \times d_i} \) is given by

\[
N_{d_i \times d_i} := \begin{pmatrix}
  0 & 1 & 0 & \cdots & 0 \\
  0 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & 0 & 1 \\
  0 & 0 & \cdots & 0 & 0
\end{pmatrix}_{d_i \times d_i}
\]

Let \( \gamma \) be an arbitrary but fixed positive number. Using the transformation \( P_i := \text{diag}(1, \gamma, \ldots, \gamma^{d_i-1}) \), we obtain that

\[
P_i^{-1} A_i P_i = \lambda_i \text{id}_{d_i \times d_i} + \gamma_i N_{d_i \times d_i},
\]
\( \gamma_i \in \{0, \gamma \} \). Hence, under the transformation \( y := (TP)^{-1} x \) system (3) becomes

\[
C D_0^\alpha y(t) = \text{diag}(J_1, \ldots, J_n)y(t) + h(y(t)),
\]

where \( J_i := \lambda_i \text{id}_{d_i \times d_i} \) for \( i = 1, \ldots, n \) and the function \( h \) is given by

\[
h(y) := \text{diag}(\gamma_1 N_{d_1 \times d_1}, \ldots, \gamma_n N_{d_n \times d_n})y + (TP)^{-1} f(TPy).
\]

We have the following two important remarks:

1. The map \( x \mapsto \text{diag}(\gamma_1 N_{d_1 \times d_1}, \ldots, \gamma_n N_{d_n \times d_n})x \) is a Lipschitz continuous function with Lipschitz constant \( \gamma \). Moreover, by (4) and (8) we have

\[
h(0) = 0, \quad \lim_{r \to 0} \ell_h(r) = \begin{cases} 
\gamma & \text{if there exists } \gamma_i = \gamma, \\
0 & \text{otherwise.}
\end{cases}
\]

2. The type of stability of the trivial solution of equations (3) and (7) are the same, i.e., they are both stable, attractive or unstable. Hence, to show that the trivial solution of (3) is asymptotically stable, it is sufficient to do it for (7).

We now consider system (7). For any \( x, \bar{x} \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n} \), we define the Lyapunov-Perron operator \( T_{x, \bar{x}} : C_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^d) \to C_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^d) \) associated with (7) as follows:

\[
(T_{x, \bar{x}} \xi)(t) = ((T_x \xi)^1(t), \ldots, (T_x \xi)^n(t))^T,
\]

where for \( i = 1, \ldots, n, \)

\[
(T_x \xi)^i(t) = E_\alpha(t^\alpha J_i)x^i + tE_{\alpha,2}(t^\alpha J_i)\bar{x}^i + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha} ((t - \tau)^\alpha J_i) h^i(\xi(\tau)) d\tau
\]

\[
= E_\alpha(\lambda_i t^\alpha)x^i + tE_{\alpha,2}(\lambda_i t^\alpha)\bar{x}^i + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha,\alpha}(\lambda_i (t - \tau)^\alpha) h^i(\xi(\tau)) d\tau.
\]

It is easy to see that if \( \varphi(\cdot) \) is a fixed point of \( T_{x, \bar{x}} \) then it is also a solution of (7) satisfying \( \varphi(0) = x, \frac{d\varphi(t)}{dt}|_{t=0} = \bar{x} \). For each \( i = 1, \ldots, m \), from Lemma 3(iii), we can find a constant \( C(\alpha, \lambda_i) \) such that

\[
\sup_{t \geq 0} \int_0^t |(t - s)^{\alpha - 1} E_{\alpha,\alpha}(\lambda_i (t - s)^\alpha)| ds < C(\alpha, \lambda_i).
\]
Put $C(\alpha, \lambda) := \max_{1 \leq i \leq m} C(\alpha, \hat{\lambda}_i)$ and choose $\gamma$ such that $\gamma C(\alpha, \lambda) < \frac{1}{2}$. Due to Remark 1 above, for every $\varepsilon > 0$ is small enough (we can choose $\varepsilon$ such that $C(\alpha, \lambda) \ell_h(\varepsilon) < 1$), we have
\[
q := C(\alpha, \lambda) \ell_h(\varepsilon) < 1.
\]
Take
\[
\delta = \frac{\varepsilon(1 - q)}{\max_{1 \leq i \leq m} \sup_{t \geq 0} |E_{\alpha,1}(\hat{\lambda}_i t^\alpha)| + \max_{1 \leq i \leq m} \sup_{t \geq 0} t |E_{\alpha,2}(\hat{\lambda}_i t^\alpha)|}.
\]
For any $x, \bar{x} \in B_{\mathbb{R}^d}(0, \delta)$, we see that $T_{x, \bar{x}}(B_{C_{\infty}}(0, \varepsilon)) \subset B_{C_{\infty}}(0, \varepsilon)$. Moreover, $T_{x, \bar{x}}$ is contractive in $B_{C_{\infty}}(0, \varepsilon)$. Indeed, for any $\xi, \hat{\xi} \in B_{C_{\infty}}(0, \varepsilon)$, we have
\[
(T_{x, \bar{x}}\xi)(t) - (T_{x, \bar{x}}\hat{\xi})(t) = \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}((t - \tau)^\alpha J)[h(\tau) - h(\hat{\tau})] \, d\tau,
\]
where this difference has the $i$-component as
\[
\int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda_i (t - \tau)^\alpha) [h^i(\tau) - h^i(\hat{\tau})] \, d\tau.
\]
Hence, using Remark 1 and Lemma 3(iii), we obtain the estimate
\[
\|T_{x, \bar{x}}\xi - T_{x, \bar{x}}\hat{\xi}\|_{\infty} \leq C(\alpha, \lambda) \ell_h(\max(\|\xi\|_{\infty}, \|\hat{\xi}\|_{\infty})) \|\xi - \hat{\xi}\|_{\infty},
\]
\[
\leq q \|\xi - \hat{\xi}\|_{\infty},
\]
which implies that $T_{x, \bar{x}}$ is contractive. On the other hand, for every $t \geq 0$, we have
\[
\|(T_{x, \bar{x}})\xi(t)\| \leq \|E_\alpha(t^\alpha J)x\| + \|tE_{\alpha,2}(t^\alpha J)\bar{x}\|
\]
\[
+ \int_0^t \|(t - \tau)^{\alpha-1} E_{\alpha,\alpha}((t - \tau)^\alpha J) h^i(\tau)\| \, d\tau
\]
\[
\leq \delta \left( \max_{1 \leq i \leq n} \sup_{t \geq 0} \|E_{\alpha,1}(\lambda_i t^\alpha)x^i\| + \max_{1 \leq i \leq n} \sup_{t \geq 0} \|tE_{\alpha,2}(\lambda_i t^\alpha)x^i\| \right)
\]
\[
+ C(\alpha, \lambda) \ell_h(\varepsilon) \|\xi\|_{\infty}
\]
\[
\leq \varepsilon,
\]
for any $\xi \in B_{C_{\infty}}(0, \varepsilon)$. Thus $T_{x, \bar{x}}(B_{C_{\infty}}(0, \varepsilon)) \subset B_{C_{\infty}}(0, \varepsilon)$. By Banach’s fixed point theorem, there exists a unique fixed point $\varphi(\cdot) \in B_{C_{\infty}}(0, \varepsilon)$ of $T_{x, \bar{x}}$. This point is also a solution of $f(t)$ with the initial condition $\varphi(0) = x$. 8
\[
\frac{d\varphi(t)}{dt}_{|t=0} = \tilde{x}. \text{Since the initial value problem for (7) has unique solution in } B_{R_{\delta}}(0, \varepsilon), \text{this shows that the trivial solution 0 is stable. To complete the proof of the theorem, we have to show that the trivial solution 0 is attractive. Suppose that } \varphi(t) = (\varphi^1(t), \ldots, \varphi^n(t)) \text{is the solution of (7) which satisfies } \varphi(0) = x, \text{ and } \frac{d\varphi(t)}{dt}_{|t=0} = \bar{x} \text{ for the arbitrary vectors } x, \bar{x} \in B_{R_{\delta}}(0, \delta). \text{From the estimate (9), we see that } \|\varphi\|_\infty \leq \varepsilon. \text{Denote } a := \limsup_{t \to \infty} \|\varphi(t)\|, \text{then } a \in [0, \varepsilon]. \text{Let } \tilde{\varepsilon} \text{be an arbitrary positive number. There exists } T(\tilde{\varepsilon}) > 0 \text{such that}
\]
\[
\|\varphi(t)\| \leq (a + \tilde{\varepsilon}) \quad \text{for any } t \geq T(\tilde{\varepsilon}).
\]
For each } i = 1, \ldots, n, \text{we will estimate } \limsup_{t \to \infty} \|\varphi^i(t)\|. \text{According to Lemma 3(ii), we obtain}
\[
\limsup_{t \to \infty} \left\| \int_0^{T(\tilde{\varepsilon})} (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(\lambda_i(t - \tau)^\alpha) h_i(\varphi(\tau)) d\tau \right\|
\leq \max_{t \in [0, T(\tilde{\varepsilon})]} \|h_i(\varphi(t))\| \limsup_{t \to \infty} \int_0^{T(\tilde{\varepsilon})} M(\alpha, \lambda_i) \frac{1}{(t - \tau)^{\alpha + 1}} d\tau
\leq 0.
\]
\[
\text{Therefore, from the fact that } \varphi^i(t) = (T_{x, \bar{x}} \varphi)^i(t) \text{and } \lim_{t \to \infty} E_{\alpha}(\lambda_i t^\alpha) = \lim_{t \to \infty} t E_{\alpha, 2}(\lambda_i t^\alpha) = 0, \text{we have}
\]
\[
\limsup_{t \to \infty} \|\varphi^i(t)\| = \limsup_{t \to \infty} \left\| \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(\lambda_i(t - \tau)^\alpha) h_i(\varphi(\tau)) d\tau \right\|
\leq \ell_h(\varepsilon) C(\alpha, \lambda_i)(a + \tilde{\varepsilon}),
\]
where we use the estimate
\[
\left\| \int_{T(\tilde{\varepsilon})}^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(\lambda_i(t - \tau)^\alpha) d\tau \right\| = \left\| \int_0^{t-T(\tilde{\varepsilon})} u^{\alpha - 1} E_{\alpha, \alpha}(\lambda_i u^\alpha) du \right\|
\leq C(\alpha, \lambda_i),
\]
see Lemma 3(iii), to obtain the inequality above. This implies
\[
a \leq \max \left\{ \limsup_{t \to \infty} \|\varphi^1(t)\|, \ldots, \limsup_{t \to \infty} \|\varphi^n(t)\| \right\}
\leq \ell_h(\varepsilon) C(\alpha, \lambda)(a + \tilde{\varepsilon}).
\]
Let } \tilde{\varepsilon} \to 0, \text{we have}
\[
a \leq \ell_h(\varepsilon) C(\alpha, \lambda)a.
\]
Due to the fact that } \ell_h(\varepsilon) C(\alpha, \lambda) < 1, \text{we get that } a = 0 \text{and the proof is complete.} \]
From Theorem 4 we can easily obtain a more general stability result for the case of locally Lipschitz $f$. Namely, we have

**Theorem 5** (Asymptotic Stability of Fractional Differential Equations). Consider the nonlinear fractional differential equation (3)

$${}^C D_0^\alpha x(t) = Ax(t) + f(x(t)),$$

where the spectrum $\sigma(A)$ of $A$ satisfies $\sigma(A) \subset \Lambda^\alpha$ and $f$ is a local Lipschitz continuous in a neighborhood of the origin and satisfies (4). Then, the trivial solution of (3) is asymptotic stable.

**Proof.** A closer look at the proof of Theorem 4 we see that we only need to use the behavior of $f$ in a neighborhood of the origin to assure our arguments leading to the asymptotic stability assertion. Therefore, we may substitute $f$ by a globally Lipschitz function $\hat{f}$ which coincides with $f$ in a neighborhood of the origin (clearly such a function $\hat{f}$ exists) and apply Theorem 4 to a new system with $\hat{f}$ to get asymptotic stability of the new system which will then lead to the asymptotic stability of the initial system with $f$ as well. \qed

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