Analytic Wavefunctions for Collective Modes in Fractional Quantum Hall Fluids

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We show model wavefunctions for neutral collective modes in fractional quantum Hall (FQH) states have simple analytic forms obtained from judiciously reducing the powers of selected pairs in the ground state Jastrow factor. This scheme of “pair excitations” works for the magneto-roton modes of single-component Abelian and non-Abelian FQH states, as well as neutral fermion mode for the Moore-Read (MR) state. The analytic wavefunctions enable computations in the thermodynamic limit previous inaccessible to numerics, and the long wavelength limit of the neutral energy gap of the magneto-roton modes can be interpreted as fusion of charges in two-dimensional plasma picture, extending the plasma analogy to neutral excitations. A lattice diagrammatic method of representing these many-body wavefunctions and FQH elementary excitations is also presented.

The fractional quantum Hall effect (FQHE) is one of the prime examples where strong interaction between electrons dictates the dynamics. A very fruitful approach for such a strongly correlated system is to look for model wavefunctions and model Hamiltonians, where physical interpretations are more transparent, than are adiabatically connected to experimentally accessible systems. For the ground states and charged excitations of FQHE at several filling factors, wavefunctions can be written down analytically. We can thus identify the properties of FQHE in the thermodynamic limit, and reinterpret the wavefunctions in analogy to two-dimensional plasmas, or as conformal blocks of conformal field theory (CFT). In general for these approaches, incompressibility of FQHE is always assumed, and the dynamics is not explicitly discussed.

Incompressibility of FQHE is defined by the neutral collective excitations. Such excitations are important for understanding which topological phases of FQHE can be stabilized, and also very relevant to recent development of fractional Chern insulator, where the issue of incompressibility and its mapping to FQHE is a areas of active research. The first formal treatment of neutral excitations came from single mode approximation (SMA) for the magneto-roton mode, where good model wavefunctions of density wave excitations can be constructed numerically up to the momentum of roton minimum. However in practice the long wavelength limit is not accessible due to limitation of the system size. It is now understood that the collective modes can be thought as excitons of composite fermions, and in the long wavelength limit it is a “spin 2” quadrupole excitation; as momentum increases it relaxes into a dipole excitation beyond the roton minimum. Interestingly, even though the underlying phenomenological pictures of the collective modes can be different, it is found numerically that exactly the same set of model wavefunctions are found with different approaches, with very rich algebraic structures. This suggests a natural way of representing these model wavefunctions for the neutral excitations as well, with no need of explicit variational parameters.

In this Letter, we present analytic wavefunctions that are identical to those numerically generated in, and calculate the energy gap of the quadrupole excitations in the thermodynamic limit. We start by presenting the wavefunctions of the collective modes for fermionic Laughlin state at filling factor υ = 1/m in the lowest Landau level (LLL), where m is odd. On the sphere the ground state is the Laughlin wavefunction in total angular momentum L = 0 sector, or the fermionic Jack polynomial in the I=0/1,0,00... state. By stripping away the single particle normalization factor the holomorphic part of the wavefunction is the same on all genus manifold. Thus we label the state with its total angular momentum on the sphere. Taking z = x + iy, z* = x − iy, where l_B = h/eB is the magnetic length, the unique ground state is given by ψ_l = Π_l<k(z_k − z_l)m e−1/2 ∑_k,z_k e^{i q_\ell \cdot (\mathbf{R}_k − \mathbf{R}_l)}V_l

where L_n(x) is the nth Laguerre polynomial and R_k is the guiding center coordinate of the kth particle. Physically, V_l is the short range interaction that projects into the two-body Hilbert space with their relative angular momentum smaller than m. The family of collective modes at different angular momentum sector (we omit the exponential part of the wavefunction, which is irrelevant in LLL) is as follows:

\[ V_{ij} = \int \frac{d^2 q}{2\pi} \sum_{n=0}^{m-1} L_n(q\hat{l}_B) e^{-\frac{1}{2}q^2\hat{l}_B^2} e^{i q \cdot (\tilde{R}_i - \tilde{R}_j)} \] (1)
Here $\mathcal{A}$ indicates antisymmetrization over all particle indices, and $\prod'_{i<j}$ means products of only pairs $\{ij\}$ that do not appear in the prefactors to the left of it. Thus the $L=2$ state, which is the quadrupole excitation in the thermodynamic limit[23], is obtained from the ground state by reducing the power of one pair of particles (which we choose arbitrarily as particle 1 and 2) by two, followed by antisymmetrizing over all particles. This scheme naturally forbids an $L=1$ state by pair excitation, since if we reduce the power of one pair of particles by one, antisymmetrization kills the state.

The $L=3$ state is generated by pairing particle 1 with another particle and reducing their pair power by one. It is now clear how the modes in other momentum sectors are generated. Naturally for a total of $N$ particles, the family of collective modes ends at $L=N$, agreeing with the scheme in [23]. Indeed all wavefunctions here satisfy the highest weight condition, and the states relax to the ground state far away from the excited pairs; these are exactly the conditions we used to numerically generate the unique model wavefunction in each momentum sector.

An intuitive way to visualize the family of collective modes is to map the particles onto a lattice, where each lattice site represents a particle. Since for FQHE we have a quantum fluid instead of a solid, every two lattice sites interact with each other. The number of bonds between each pair of lattice sites equal to the power of the pair of particles in the wavefunction. As an example we consider the simplest Laughlin state at $\nu=1/3$, so for the ground state every two lattice sites are connected by three bonds, as shown in Fig. 1.

The collective modes are obtained by breaking the bonds between lattice sites, as shown in Fig. 2. We can view the entire family of collective modes as elementary excitations centered around a single red lattice site. Note the lattice pattern uniquely defines the many-body wavefunction, and different types of “elementary excitations” can be identified with different patterns of bond-breaking around a single lattice site.

The same scheme applies to MR state. It is instructive to first see how the MR ground state is obtained. The Laughlin wavefunction at half filling is given by $J_{1010101}^2(z_1) = \prod_{i<j}(z_i - z_j)^2$. For fermions this is not a valid state; instead the ground state was constructed by a pairing mechanism, which is also a Jack polynomial $J_{\lambda}^3_{1100110011}$... The pairing reduces the power of each pair of particles by one. Explicitly we have for $2n$ particles

$$\prod_{i<j}(z_i - z_j)^2 \rightarrow \mathcal{A}((z_1 - z_2)(z_3 - z_4)\cdots(z_{2n-1} - z_{2n})) \prod_{i<j}(z_i - z_j)^2 = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}(z_i - z_j)^2$$

where the last line of Eq. (3) is the familiar Pfaffian for the MR ground state. This suggests lattice representation of MR state and its magneto-roton mode with the same scheme, as shown in Fig. 3.

We can also use the same scheme to generate the neutral fermion mode for the MR states with odd number of particles. In this case, starting from the Bosonic Laughlin wavefunction at half filling, every two particles form a pair except for one particle. Naturally the “ground state” of the neutral fermion mode is given by
For even number of electrons we have $N_e = 2n$ and Eq. (6) is the MR ground state. The magneto-roton modes are given by

$$\psi_{mr}^{L=k+2} = \prod_{i<j} (z_i - z_j)^2 \mathcal{A}[P_{13} \cdots P_{1+2k}]$$

(7)

For odd number of electrons we have $N_e = 2n + 1$ and Eq. (6) is the MR quasihole state of Eq. (4). The neutral fermion modes are given by

$$\psi_{mr}^{L=2+k} = \prod_{i<j} (z_i - z_j)^2 \mathcal{A}[P_{N_e,1} \cdots P_{N_e,1+2k}]$$

(8)

The analytic wavefunction is useful in calculating the magneto-roton mode energy gap in the long wavelength limit. For the Laughlin state, the energy gap is given by

$$\epsilon_q \rightarrow 0 = \lim_{N_e \rightarrow \infty} \frac{\langle \psi_{L=2}\mid V \mid \psi_{L=2} \rangle}{\langle \psi_{L=2} \mid \psi_{L=2} \rangle}$$

(9)

We already know from [23] that in $L = 2$ and $L = 3$ sector, SMA is exact for the magneto-roton mode model wavefunction. Defining the guiding center ladder operators as $b_i^\dagger = z_i, b_i = \partial_{z_i}$, we have

$$\psi_{L=2}^t = \frac{1}{2m(m-1)} \sum_i (b_i)^2 \psi_t$$

(10)

In the thermodynamic limit, the normalization constant of the above two modes are thus related to the long wavelength expansion of the ground state guiding center structure factor:

$$S_q = \frac{1}{N_e} \langle \langle \delta \tilde{p}_q \delta \tilde{p}_{-q} \rangle_0 - \langle \delta \tilde{p}_q \rangle_0 \langle \delta \tilde{p}_{-q} \rangle_0 \rangle$$

$$= - \frac{\bar{s}}{4m} (g^{ab} q_a q_b)^2 + O(q^0)$$

(11)

where $\bar{s} = \frac{1-m}{2}$ is the guiding center spin [22, 24], and $g^{ab}$ is the guiding center metric [25]. We thus have

$$\langle \psi_{L=2}\mid V \mid \psi_{L=2} \rangle = - \frac{\bar{s} N_e}{2m^2(m-1)^2}$$

(12)

The numerator of Eq. (11) can be calculated with plasma analogy. Note in Eq. (11), before antisymmetrization the term only has one pair of particles with relative angular momentum smaller than $m$. We thus have

$$\langle \psi_{L=2}\mid V \mid \psi_{L=2} \rangle = \frac{N_e (N_e - 1)}{2N^2} \langle \tilde{\psi} | P_{12} V_{12} P_{12} | \tilde{\psi} \rangle$$

(13)
where $N$ is the normalization constant of the Laughlin state. We note that $V_{12}$ projects out states with relative angular momentum $(z_1-z_2)^m$, which can be integrated over. The numerator is thus equivalent to evaluating the norm of the following wavefunction:

$$\tilde{\psi} = \prod_{i=2}^{N-1} \left( \frac{1}{\sqrt{2}} z_1 - z_i \right)^{2m} \prod_{1 < i < j < N-1} (z_i - z_j)^m$$

which can be evaluated as the free energy of two-dimensional one-component plasma (OCP) on a disk with radius $R^2 = \frac{mN_e}{2\pi}$ and elementary charge $e = 2\sqrt{\pi m k_B T}$, where particle 1 interacts with the rest of the particles with charge $2e$. We thus obtain

$$\epsilon_{q \to 0} = -\frac{2m(m-1)^2}{\pi \bar{s}_{mr}^2} e^{-\frac{F_3-Q_1}{k_B T}}$$

Both $\bar{F}_2$ and $\bar{F}$ are free energies of OCP in the thermodynamic limit ($N_e \to \infty$), where $\bar{F}$ is for $N_e$ particles, each with charge $e$ with logarithmic two-body interactions together with a neutralizing background of radius $R$; for $\bar{F}_2$, we have the same neutralizing background with $N_e-2$ particles of charge $e$, and exactly one particle with charge $2e$. Thus $\bar{F}_2 - \bar{F}$ is the free energy cost of fusing two particles of charge $e$ to create a particle of charge $2e$, which is an $O(1)$ effect.

Similar calculations can be carried out for the magneto-roton mode in the MR state. Analogous to Eq. (10) we have $\psi_{mr}^{L=2} = \frac{1}{2\pi} \sum_i b^*_i \psi_{mr}$, and in the long wavelength limit we have

$$\epsilon_{q \to 0} = \frac{24}{\pi \bar{s}_{mr}^2} e^{-\frac{F_3-Q_1}{k_B T}}$$

where $\bar{s}_{mr} = -2$ is the guiding center spin for the MR state, and $\bar{F}_{II}$ is the standard two-component plasma free energy for the MR ground state. The charge for interaction between the two components is given by $Q_1 = \pm \sqrt{3k_B T}$, while the charge for interaction between one component and the neutralizing background is given by $Q_2 = 2\sqrt{k_B T}$. $\bar{F}_3 - \bar{F}_{II}$ is the free energy cost of fusing three particles for each component to create one particle with charge $3Q_2$ but with the same $\pm Q_1$.

The evaluation of the long wavelength gap of the neutral fermion mode is less transparent. The difficulty lies with evaluating the normalization constant of $\psi_{mr}^{L=2}$, which is not known if in the thermodynamic limit the gap should be inversely proportional to the guiding center spin. On the other hand $\langle \psi_{mr}^{L=2}, V_{\text{bdy}} \psi_{mr}^{L=2} \rangle$ can be mapped to 2-component plasma as well, and we obtain

$$\epsilon_{q \to 0} \sim e^{-\frac{F_3-Q_1}{k_B T}}$$

Here $\bar{F}_{II}$ is the free energy of the 2-component plasma similar to that of $\bar{F}_{II}$ with only one difference: there is exactly one more particle carrying charge $Q_2$ that interacts with the neutralizing background, and its $Q_1$ charge is zero. This is how an unpaired fermion in the MR state is interpreted in the plasma analogy. Furthermore, $\bar{F}_3 - \bar{F}_{II}$ is the energy cost of fusing the unpaired fermion with one pair of two other fermions, creating a particle with charge $Q_2 = 6\sqrt{k_B T}$ but again with zero $Q_1$.

In conclusion, analytic wavefunctions for both the magneto-roton modes and the neutral fermion modes are presented. The energy gap of the quadrupole excitation in the thermodynamic limit can be related to the free energy cost of the fusion of charges in the plasma energy, and is inversely proportional to the guiding center spin which characterizes its topological order. This is the first time that the plasma analogy is extended to neutral excitations of FQHE, and the analogy not only applies to the wavefunctions, but also to the dynamics as well. The lattice diagrams we presented uniquely defines the many-body wavefunctions; one would also conjecture the diagrams are useful in determining the many-body wavefunctions of multi-roton excitations. Since the collective mode in the long wavelength limit is buried in the multi-roton continuum, it is important to calculate the decay rate of the collective mode. Numerical calculation has been performed to show that even in the continuum the decay rate of the collective mode is very small. This opens up the possibility of experimental detection of these modes. A more detailed analysis of the decay rate of collective modes will be presented elsewhere.

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[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
[2] S. M. Girvin, A. H. MacDonald and P. M. Platzman, Phys. Rev. Lett. 54, 581 (1985); Phys. Rev. B 33, 2481 (1986).
[3] S.A. Parameswaran, R. Roy and S.L. Sondhi, Phys. Rev. B. 85, 241308 (2012).
[4] T. Scaffidi and G. Moller, Phys. Rev. Lett. 109 246805 (2012).
[5] Rahul Roy, arXiv: 1208.2055
[6] Zhao Liu and E.J. Bergholtz, Phys. Rev. B 87, 035306 (2013).
[7] Ying-Hai Wu, J.K. Jain and Kai Sun, Phys. Rev. B. 86, 165129 (2012).
[8] Yang-Le Wu, N. Regnault and B. A. Bernevig, Phys. Rev. Lett. 110, 106802 (2013).
[9] Ching Hua Lee, R. Thomale and Xiao-Liang Qi, arXiv:
[10] A. Pinczuk, B.S. Dennis, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 70, 3983 (1993).
[11] C.J. Mellor et al, Phys. Rev. Lett. 74, 2339 (1995).
[12] U. Zeitler et al, Phys. Rev. Lett. 82, 5333 (1999).
[13] Moonsoo Kang, A. Pinczuk, B.S. Dennis, L.N. Pfeiffer and K.W. West, Phys. Rev. Lett. 86, 2637 (2001).
[14] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[15] G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
[16] N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).
[17] G. Moller, A. Wojs and N.R. Cooper, Phys. Rev. Lett. 107, 036803 (2011)
[18] B. A. Bernevig and F. D. M Haldane, Phys. Rev. Lett. 102, 066802 (2009)
[19] B. A. Bernevig and F. D. M Haldane, Phys. Rev. Lett. 100, 246802 (2008)
[20] R.K. Kamilla, X.G. Wu and J.K. Jain, Phys. Rev. B. 54, 4873 (1996)
[21] G.J. Sreejith, A. Wojs and J.K. Jain, Phys. Rev. Lett. 107 136802 (2011).
[22] I.D. Rodriguez, A. Sterdyniak, M. Hermanns, J.K. Slingerland, N. Regnault, Phys. Rev. B. 85 035128 (2012)
[23] Bo Yang, Zi-Xiang Hu, Z. Papic and F. D. M. Haldane, Phys. Rev. Lett. 108, 256807 (2012).
[24] F. D. M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
[25] Bo Yang, Z. Papic, E.H. Rezayi, R.N. Bhatt and F.D.M. Haldane, Phys. Rev. B. 85, 165318(2012).
[26] Parsa Bonderson, Victor Gurarie and Chetan Nayak, Phys. Rev. B 83, 075303 (2011)
[27] Bo Yang, Z. Papic and F.D.M. Haldane, work in progress.