Ordering of two small parameters in the shallow water wave problem

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Abstract

The classical problem of irrotational long waves on the surface of a shallow layer of an ideal fluid moving under the influence of gravity as well as surface tension is considered. A systematic procedure for deriving an equation for surface elevation for a prescribed relation between the orders of the two expansion parameters, the amplitude parameter $\alpha$ and the long wavelength (or shallowness) parameter $\beta$, is developed. Unlike the heuristic approaches found in the literature, when modifications are made in the equation for surface elevation itself, the procedure starts from the consistently truncated asymptotic expansions for unidirectional waves, a counterpart of the Boussinesq system of equations for the surface elevation and the bottom velocity, from which the leading-order and higher order equations for the surface elevation can be obtained by iterations. The relations between the orders of the two small parameters are taken in the form $\beta = O(\alpha^n)$ and $\alpha = O(\beta^m)$ with $n$ and $m$ specified to some important particular cases. The analysis shows, in particular, that some evolution equations, proposed before as model equations in other physical contexts (such as the Gardner equation, the modified Korteweg-de Vries (KdV) equation and the so-called fifth-order KdV equation), can emerge as the leading-order equations in the asymptotic expansion for the unidirectional water waves, on equal footing with the KdV equation. The results related to the higher orders of approximation provide a set of consistent higher order model equations for unidirectional water waves which replace the KdV equation with higher order corrections in the case of non-standard ordering when the parameters $\alpha$ and $\beta$ are not of the same order of magnitude. The shortcomings of certain models used in the literature become apparent as a result of the subsequent analysis. It is also shown that various model equations obtained by assuming a prescribed relation $\beta = O(\alpha^n)$ between the orders of the two small parameters can be equivalently treated as obtained by applying transformations of variables which scale out the parameter $\beta$, in favor of $\alpha$. It allows us to consider the nonlinearity-dispersion balance, epitomized by the
The behavior of surface gravity waves on shallow water has been a subject of intensive research. In particular, the famous Korteweg–de Vries (KdV) equation, which is the prototypical example of an exactly solvable soliton equation, was first introduced as a unidirectional nonlinear wave equation obtained via asymptotic expansion around simple wave motion of the Euler equations for shallow water.

The system of equations describing the long, small-amplitude wave motion in shallow water with a free surface [1–4] involves two independent small parameters: $\alpha$, which measures the ratio of wave amplitude to undisturbed fluid depth, and $\beta$, which measures the square of the ratio of fluid depth to wave length, and no relationship between orders of magnitude of $\alpha$ and $\beta$ follows from the statement of the problem. The KdV equation,

$$\eta_t + \eta_x + \frac{1}{6} \alpha \eta_{xx} + \frac{1}{6} \beta \eta_{3x} = 0,$$

(1)

emerges at first order (in both parameters $\alpha$ and $\beta$) in the asymptotic expansion as an equation for the surface elevation $\eta$ associated with the right-moving wave. The derivation assumes (sometimes tacitly) that $\beta = O(\alpha)$. It is evident that in the case, when $\alpha$ and $\beta$ differ in their orders of magnitude, the leading-order equation maintaining the balance between linear dispersion and nonlinear steepening, which is the primary physical mechanism for the propagation of solitary shallow water waves, should change its form. The same holds true for the equations which (like the higher order KdV equations) address higher order effects.

A heuristic approach to deriving model equations for unidirectional water waves is frequently used when some additional terms are included in the equation for the surface elevation based on relations between the orders of parameters. However, this may lead to inconsistencies. For example, the assumption $\alpha \geq \beta > \alpha^2$ is made in [5], while the terms involving $\alpha$, $\beta$, $\alpha^2$, $\alpha \beta$ and $\beta^2$ are kept and, accordingly, the terms involving $\alpha^3$, $\alpha^2 \beta$, $\alpha \beta^2$ and $\beta^3$ are neglected. It is readily seen that the relation $\beta = \alpha^r$ with $3/2 \leq r \leq 2$ which satisfies the above inequality is in conflict with the truncation made: the neglected term $\sim \alpha^3$ is as important as the retained term $\sim \beta^2$. In [6], based on an inequality $O(\beta) < O(\alpha)$, the truncation is made such that the terms involving $\alpha$, $\beta$, $\alpha^2$, $\alpha \beta$ and $\beta^2$ are kept and the terms involving $\alpha^4$, $\alpha^3 \beta$ and $\beta^3$ are neglected. However, this choice of truncation is questionable since there exists no relationship between the orders of $\beta$ and $\alpha$ of the form $\beta = O(\alpha^r)$ (or $\alpha = O(\beta^r)$) for which such a truncation is consistent. Indeed, assuming $\beta = O(\alpha^r)$ with $r > 1$ (which is compatible with $O(\beta) < O(\alpha)$), one can see that the two requirements $\alpha^4 < \alpha^2 \beta$ and $\beta^2 < \alpha^2 \beta$ lead to conflicting results: $r < 2$ and $r > 2$, respectively. Thus, such a heuristic approach does not provide a reliable way to determine even a form of the equation for surface elevation and, what is more, it does not allow determining coefficients of the equation. It is well known that the solution properties may strongly depend on the relations between the coefficients—the higher order KdV equations can be mentioned in this respect (see e.g. [7–9] and references therein).

In general, to arrive at a consistent model equation for water waves, the ordering of terms should be made in the original asymptotic expansion for unidirectional water waves based on a prescribed relationship between orders of magnitude of $\alpha$ and $\beta$. Then a consistent
truncation of the expansion can be made and the related leading-order and higher order

evolution equations can be defined. In this paper, such a procedure for deriving an equation for

surface elevation for a prescribed relation between the orders of the two expansion parameters

$\alpha$ and $\beta$ is developed. It makes possible a systematic study of different particular cases

and corresponding leading-order and higher order equations. The following special cases are

considered: $\beta = O(\alpha^2)$, $\alpha = O(\beta^3)$, $\beta = O(\alpha^3)$ and $\alpha = O(\beta^3)$. The analysis is aimed

deriving an equation for the surface elevation having a form of an evolution equation;

therefore, equations which, like the Benjamin–Bona–Mahoney (BBM) equation [10], contain

time derivatives in the higher order terms are excluded from consideration. The results

of the analysis show, in particular, that some evolution equations proposed before as model

equations in other physical contexts can play the role of a model equation at the leading order

of the asymptotic expansion for the unidirectional water waves on equal footing with the KdV

equation. Some of these equations, both integrable and non-integrable, are known to have a

rich structure of solitary wave solutions which differ in their properties from the KdV solitons.

Thus, the leading-order soliton dynamics in the unidirectional water wave problem can differ

from the one described by the KdV equation. The equations obtained in the higher orders of

approximation, in general, also differ from the KdV equation with higher order corrections.

It is worth noting that the above differences from the standard model are not due to taking

the surface tension into account. New equations and dynamics arise even in the classical

formulation, when capillary effects are neglected, if the ordering is non-standard ($\beta$ and $\alpha$ are

not of the same order of magnitude). Including surface tension in general does not alter the

structure of the leading-order and higher order equations, only some specific cases, like the

case $\tau = 1/3$ of the standard analysis, should be considered separately.

The paper is organized as follows. In section 2 following the introduction, we present the

statement of the problem, the basic equations and the outline of the procedure. The main ideas

of the analysis are described in more detail in section 3, where the procedure is presented for

the best studied case of $\beta = O(\alpha)$. The cases when the relation $\beta = O(\alpha)$ does not hold

are studied in the subsequent section 4. In the said section, the analysis is restricted to the

pure gravity waves in order to better explain the main points and also to demonstrate that the

differences from the standard model are not due to taking the surface tension into account. The

results for the gravity-capillary waves are listed in the appendix. In section 5, the concluding

remarks are given and an alternative interpretation of the results, based on a transformation of

variables that scales out the parameter $\beta$, in favor of $\alpha$, is discussed.

2. Outline of the procedure

Consider the standard system of equations describing the two-dimensional irrotational wave

motion of an inviscid incompressible fluid in a channel with the flat horizontal rigid bottom and

the free surface under the influence of gravity as well as surface tension. After an appropriate

choice of non-dimensional variables, the equations of motion and boundary conditions can

be reduced to the system written in terms of the velocity potential $\phi(x, y, t)$ and the surface

elevation $\eta(x, t)$, see e.g. [1]

$$
\beta \phi_{xx} + \phi_{yy} = 0, \quad 0 \leq y \leq 1 + \alpha \eta \quad (2)
$$

$$
\phi_y = 0, \quad y = 0 \quad (3)
$$

$$
\eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_y = 0, \quad y = 1 + \alpha \eta \quad (4)
$$

$$
\phi_t + \frac{1}{2} \alpha \phi_x^2 + \frac{1}{2} \beta \phi_y^2 + \eta - \tau \beta \frac{\eta_{xx}}{(1 + \alpha^2 \beta \eta_t^2)^{3/2}} = 0, \quad y = 1 + \alpha \eta, \quad (5)
$$
where \( t \) is time and \( x \) and \( y \) are respectively the horizontal and vertical coordinates, with \( y = 0 \) being the bottom. The non-dimensional variables are defined as follows (after non-dimensionalizing, the tildes have been omitted):

\[
\tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{H}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{t}{L/\sqrt{gH}}, \quad \tilde{\phi} = \frac{\phi}{L(a/H)\sqrt{gH}},
\]

(6)

where \( g \) is the acceleration due to gravity, \( H \) is the upstream mean depth and \( a \) and \( L \) are typical values of the amplitude and of the wavelength of the waves. Equations (2)–(5) contain three non-dimensional parameters: the amplitude parameter \( \alpha = \frac{a}{H} \), the wavelength parameter \( \beta = \frac{\beta}{L} \) and the Bond number \( \tau = \frac{T}{\sqrt{gH}} \), where \( T \) is the surface tension coefficient and \( \rho \) is the density of water.

Equations (2) and (3) are satisfied by making a standard substitution

\[
\phi = \sum_{m=0}^{\infty} \frac{(-\beta)^m}{(2m)!} \frac{\partial^{2m} f(x, t)}{\partial x^{2m}} y^{2m},
\]

(7)

where \( f(x, t) = \phi|_{y=0} \). Substituting (7) into the surface conditions (4) and (5) and differentiating (5) with respect to \( x \) yields a system of two equations for the surface elevation \( \eta(x, t) \) and the horizontal velocity at the bottom \( w(x, t) = f_1 \) in the form of infinite series with respect to \( \beta \). We are interested in considering weakly nonlinear small amplitude waves in a shallow water, so we will treat \( \alpha \) and \( \beta \) as small parameters.

In the zero order in both \( \alpha \) and \( \beta \), the system of equations for \( w \) and \( \eta \) reads

\[
\eta_t + w_x = 0, \quad w_t + \eta_x = 0,
\]

so both \( w \) and \( \eta \) satisfy the linear wave equation \( \zeta_{tt} - \zeta_{xx} = 0 \) which describes waves traveling in two directions. A wave moving to the right corresponds in this order of approximation to \( w = \eta \) and \( \eta_t + \eta_x = 0 \). To derive the equations describing right-moving waves in higher orders in \( \alpha \) and \( \beta \), we can, along the lines of [1], reduce the system of equations for \( w \) and \( \eta \) to an asymptotically equivalent set of equations consisting of a relationship between the horizontal velocity \( w \) and the surface elevation \( \eta \) and an evolution equation for the elevation. To do this, we set

\[
w = \sum_{i,j=0}^{\infty} R_{ij} \alpha^i \beta^j,
\]

(8)

where \( R_{ij} \) depend on \( \eta \) and its \( x \)-derivatives, and possibly some nonlocal variables, with \( R_{00} = \eta_t \). We require that \( \eta \) satisfy an evolution equation of the form

\[
\eta_t = \sum_{i,j=0}^{\infty} S_{ij} \alpha^i \beta^j,
\]

(9)

where \( S_{00} = -\eta_{tt} \), and in general \( S_{ij} \) depend on \( \eta \) and its \( x \)-derivatives. The functions \( R_{ij} \) and \( S_{ij} \) are determined from the requirement of consistency of (8) and (9) with the above system of PDEs for \( w \) and \( \eta \). To implement this, an iterative procedure starting from the zero order of approximation and continuing to the higher orders is applied; see the subsequent sections for details. In each order, the \( t \)-derivatives of \( \eta \) are replaced by their expressions from the lower order equations.

However, it is obvious that truncating the asymptotic expansions and keeping only the terms up to a certain order requires knowledge of the relationship between the orders of magnitude of the two small parameters \( \alpha \) and \( \beta \), because otherwise it is impossible to determine which terms should be retained and which can be neglected. A commonly used assumption is that \( \alpha \) and \( \beta \) have the same order of magnitude \( (\beta = O(\alpha)) \). Then, choosing, for example, \( \alpha \) to be a primary parameter and retaining the terms up to \( O(\alpha^3) \), we arrive at the so-called \( n \)-th-order Boussinesq system [1, 11]. If the first-order Boussinesq system is considered and the
corresponding order expansions are taken for (8) and (9), then equation (9) for the elevation \( \eta \) takes the form of the KdV equation (1), cf [1], see also the following section. The same procedure continued to next orders results in the KdV equation with higher order corrections. If the relationship \( \beta = O(\alpha) \) does not hold, then one needs an alternative assumption relating the orders of \( \alpha \) and \( \beta \) to make a truncation of the expansions consistent.

3. Procedure for the case of \( \beta = O(\alpha) \)

In order to explain how the forms of expansions (8) and (9) are determined up to a certain order through an iterative procedure, we will first present the procedure for the best studied case of \( \beta = O(\alpha) \). (We can set \( \beta = \alpha \) without loss of generality.) We will also consider the problem without surface tension to make the analysis as transparent as possible. Then equation (5) is replaced by the following equation:

\[
\phi_t + \frac{1}{2} \alpha \phi_{xx} + \frac{1}{2} \alpha \phi_x^2 + \eta = 0, \quad y = 1 + \alpha \eta. \tag{10}
\]

If in the system of equations for \( w \) and \( \eta \) the terms in the second power of \( \alpha \) are retained and the higher order terms are dropped, then we arrive at the second-order Boussinesq system:

\[
\eta_t + \omega_x + \alpha \left( \omega \eta_x \right)_x + \alpha^2 \left( -\frac{1}{2} (\eta \omega_{xx})_x + \frac{1}{240} \omega_{x3} x \right) = 0 \tag{11}
\]

\[
w_t + \omega_x + \alpha \left( \omega \omega_x - \frac{1}{2} \omega_{xx} \right) + \alpha^2 \left( -\left( \eta \omega_{xx} \right)_x + \frac{1}{5} \omega_x \omega_{xx} - \frac{1}{3} \frac{1}{5} \omega_{xxx} + \frac{1}{24} \omega_{x4} \right) = 0. \tag{12}
\]

In the lowest (zero) order, the system (11), (12) reads \( \eta_t + \omega_x = 0, \quad w_t + \omega_x = 0 \), and the equivalent system (8), (9) describing a right-moving wave is reduced to \( w = \eta, \quad \eta_t + \omega_x = 0 \). In the next-order iteration, we look for a solution for \( w \) corrected to first order as

\[
w = \eta + \alpha Q^{(1)}, \tag{13}
\]

where \( Q^{(1)} \) is a function of \( \eta \) and its \( x \)-derivatives, and substitute (13) into equations (11) and (12) with the terms of order higher than \( O(\alpha) \) dropped. Upon the substitution, the equations in question become

\[
\eta_t + \omega_x + \alpha \left( \omega \eta_x \right)_x + \alpha^2 \left( -\frac{1}{2} (\eta \omega_{xx})_x + \frac{1}{240} \omega_{x3} x \right) = 0, \tag{14}
\]

\[
\eta_t + \omega_t + \alpha \left( \omega \omega_x - \frac{1}{2} \omega_{xx} \right) + \alpha^2 \left( -\left( \eta \omega_{xx} \right)_x + \frac{1}{5} \omega_x \omega_{xx} - \frac{1}{3} \frac{1}{5} \omega_{xxx} + \frac{1}{24} \omega_{x4} \right) = 0. \tag{15}
\]

The function \( Q^{(1)} \) is sought such that the two equations (14) and (15) agree (up to the first order in \( \alpha \)) upon expressing all the \( t \)-derivatives of \( \eta \) in terms of its \( x \)-derivatives using the zero-order equation \( \eta_t + \omega_x = 0 \). This yields \( Q^{(1)} = -\frac{1}{4} \eta^2 + \frac{1}{2} \eta \omega_{2x} \). Then equations (13) and (15) become

\[
w = \eta + \alpha \left( -\frac{1}{4} \eta^2 + \frac{1}{2} \eta \omega_{2x} \right), \quad \eta_t + \omega_x + \alpha \left( \frac{3}{4} \eta \omega_{2x} + \frac{1}{2} \eta_{3x} \right) = 0. \tag{16}
\]

The equation for \( \eta \) is reduced to the KdV equation in a standard form

\[
\eta_t + 6 \eta \eta_x + \eta_{3x} = 0 \tag{17}
\]

by the change of variables \( (x, t) \rightarrow (\hat{x}, t) \), where

\[
\hat{x} = \sqrt{\frac{2}{3}} (x - t), \quad \hat{t} = \frac{1}{4} \sqrt{\frac{2}{3}} \alpha t. \tag{18}
\]

At the next step, the above expression for \( w \) is corrected to second order in the form

\[
w = \eta + \alpha \left( -\frac{1}{4} \eta^2 + \frac{1}{2} \eta \omega_{2x} \right) + \alpha^2 Q^{(2)}. \tag{19}
\]

It is substituted into (11) and (12), and then all the \( t \)-derivatives of \( \eta \) are replaced by their expressions through the \( x \) derivatives using the lower order equation, namely the second
leads to an equation for \( \beta \) hold are studied. Two important special cases, \( \beta = 4 \).

Examples of ordering 

wave moving to the right is specialized. Systems are reduced to the same high-order KdV equation for the surface elevation (21). 

is not the case: it can be readily checked that all those different but asymptotically equivalent derived from the Boussinesq equations under the assumption of unidirectionality. However, this the Boussinesq equations, should result in a freedom in the equation for the surface elevation (21) of this paper, corrections including the terms up to the ninth-order spatial derivatives are calculated for the case of \( \alpha \neq O(\beta^2) \).

It should be emphasized once again that our procedure is aimed at deriving equations for \( \eta \) which have the form of an evolution equation, and, accordingly, all the \( t \)-derivatives in the terms of the order higher than zero are replaced by their expressions through the \( x \)-derivatives. Therefore, applying this procedure cannot yield equations which, like the BBM equation [10], contain the time derivatives in the higher order terms.

It is also worth noting that there exists a possibility of introducing certain freedom into the Boussinesq system. For example, a class of Boussinesq systems which are formally equivalent to the system displayed in (11)–(12) can be derived using other variables instead of the horizontal velocity at the bottom \( w \) and employing the lower order equations in higher order terms [1, 15, 11]. It might seem that this freedom, revealing itself as free parameters present in the Boussinesq equations, should result in a freedom in the equation for the surface elevation derived from the Boussinesq equations under the assumption of unidirectionality. However, this is not the case: it can be readily checked that all those different but asymptotically equivalent systems are reduced to the same high-order KdV equation for the surface elevation (21) if the wave moving to the right is specialized.

4. Examples of ordering

In this section, examples of a non-standard ordering when the relation \( \beta = O(\alpha^2) \) does not hold are studied. Two important special cases, \( \beta = O(\alpha^2) \) and \( \alpha = O(\beta^2) \), are considered in more detail. The analysis is restricted to the pure gravity waves in order to better explain the main points and demonstrate that the differences from the standard model are not due to taking the surface tension into account. The results for the problem including surface tension are presented in the appendix.

Starting from the case of \( \beta = O(\alpha^2) \), we first write down the system of equations for \( w \) and \( \eta \) obtained by keeping all the terms of the order not higher than \( \beta^2, \beta \alpha^2 \) and \( \alpha^4 \):

\[
\eta_t + \eta_x + \alpha \eta \eta_x - \frac{1}{2} \beta w_{3x} - \frac{1}{2} \alpha \beta (\eta w_{2x})_x - \frac{1}{4} \alpha^2 \beta (\eta^2 w_{2x})_x + \frac{1}{120} \beta^2 w_{5x} = 0 \tag{22}
\]

\[
w_t + \eta_t + \alpha w \eta_x - \frac{1}{2} \beta w_{2x} + \alpha \beta (- \eta w_{2})_x + \frac{1}{4} \alpha^2 (\eta^2 w_{2})_x - \frac{1}{2} \beta w_{3x} + \frac{1}{3} \beta^2 w_{4x} = 0 \tag{23}
\]

Next, we apply the iterative procedure described in the previous section to determine the form of the unidirectional wave equations (8) and (9) for the case of \( \beta = O(\alpha^2) \). The resulting
equations, with the terms up to $O(\alpha^2)$ retained, read (we have used the square brackets to gather the terms having the same order of magnitude)

$$w = \eta - \frac{\eta^2}{4} + \left[ \alpha^2 \frac{\eta^3}{8} + \beta \eta_{x} \right] + \left[ -\alpha^3 \frac{5\eta^4}{64} + \alpha \beta \left( \frac{3\eta^2}{16} + \eta_{2x} \right) \right]$$

$$+ \left[ \alpha^4 \eta^5 \left( \frac{\eta^2}{8} + \frac{3\eta_2}{32} + \frac{3\eta}{16}\right) + \beta^2 \eta_{4x} \right], \quad z = \int \eta^3 \, dx.$$  \hspace{1cm} (24)

$$\eta_t + n_x + \frac{3}{2} \alpha \eta n_x + \left[ \frac{3}{8} \alpha^2 \eta^2 n_x + \frac{1}{6} \beta \eta_{3x} \right] + \left[ \frac{3}{16} \alpha^3 \eta^3 n_x + \alpha \beta \left( \frac{23}{24} \eta_2 n_{2x} + \frac{5}{12} \eta_{3x} \right) \right]$$

$$+ \left[ \frac{15}{128} \alpha^4 \eta^4 n_x + \alpha^2 \beta \left( \frac{5}{16} \eta^2 n_{3x} + \frac{23}{16} \eta_2 \eta_{2x} + \frac{19}{32} \eta^4 \right) + \frac{19}{360} \beta^2 \eta_{5x} \right] = 0.$$  \hspace{1cm} (25)

To make the things even more clear, rewrite the last equation taking $\beta = B \alpha^2$ ($B = O(1)$) and ordering the terms according to powers of $\alpha$. We obtain

$$\eta_t + n_x + \frac{3}{2} \alpha \eta n_x + \alpha^2 \left( -\frac{3}{8} \eta^2 n_x + \frac{1}{6} B \eta_{3x} \right) + \alpha^3 \left[ \frac{1}{16} \eta^3 n_x + B \left( \frac{23}{24} \eta_2 n_{2x} + \frac{5}{12} \eta_{3x} \right) \right]$$

$$+ \alpha^4 \left[ -\frac{155}{128} \eta^2 n_x + B \left( \frac{5}{16} \eta^2 n_{3x} + \frac{23}{16} \eta_2 \eta_{2x} + \frac{19}{32} \eta^4 \right) + \frac{19}{360} B^2 \eta_{5x} \right] = 0.$$  \hspace{1cm} (26)

It is immediate that an equation involving both nonlinearity and dispersion is obtained at the leading order, which is now second in $\alpha$ and first in $\beta$. Therefore, this leading-order equation contains an extra term $-\frac{1}{2} \alpha^2 \eta^2 \eta_x$, and reads as follows:

$$\eta_t + n_x + \frac{3}{2} \alpha \eta n_x + \frac{1}{8} \alpha^2 \eta^2 n_x + \frac{1}{6} \beta \eta_{3x} = 0.$$  \hspace{1cm} (27)

Thus, if $\beta = O(\alpha^2)$, then the leading-order equation is not the KdV equation but the Gardner equation which is a linear combination of the KdV and of the modified KdV equation. The Gardner equation has appeared in the literature in other physical contexts; in particular, it was derived in asymptotic theory for internal waves in a two-layer liquid with a density jump at the interface [16, 17]. Our derivation shows that the Gardner equation emerges in the classical water wave problem as the leading-order equation in the case of $\beta = O(\alpha^2)$. The Gardner equation is integrable and possesses solitary wave solutions, but the Gardner solitons may differ in their properties from their KdV counterparts, see e.g. [18].

The Gardner equation (27) can be transformed into the modified KdV equation

$$\tilde{\eta}_t = \tilde{\eta}_{3x} + 6 \tilde{\eta}_x \tilde{\eta}_x,$$  \hspace{1cm} (28)

where $\tilde{\eta}$ is a shifted variable and $(\xi, \tilde{t})$ are the rescaled variables in a moving frame. Equation (28) is well known to be integrable, see e.g. [2, 3]. In addition to standard soliton solutions, it has solutions in the form of ‘breather solitons’ and also solutions describing breather-soliton interactions. In view of the fact that the transformation from (27) to (28) includes a shift of the dependent variable, soliton solutions of equation (28) for $\tilde{\eta}$ can be relevant for the original problem in terms of $\eta$ if the flows with hydraulic jumps are considered.

The form of the higher order corrections to the leading-order Gardner equation is also evident from equation (25) (or (26)). Note that equation (25) has the differential structure of a combination of the Gardner equation (27) and its first commuting flow; this feature is similar to what is observed for the KdV with a higher order correction in the case of $\beta = O(\alpha)$.

In [6], the so-called second- and third-order approximations of water wave equations are studied for the case $O(\beta) < O(\alpha)$ of [19] specified to $\beta \sim \alpha^2$. The comparison of these equations with (25) and (26) shows that the ordering (and hence the truncation) used in [6] are invalid. In particular, in the second-order approximation equation the terms involving
\(\alpha \beta\) are present, but the same order term involving \(\alpha^3\) is missing. Likewise, in the third-order approximation equation the terms involving \(\alpha^2 \beta\) are retained, but the same order terms involving \(\alpha^4\) and \(\beta^2\) are omitted.

Consider now the case \(\alpha = O(\beta^2)\). Then the basic system of equations for \(w\) and \(\eta\) obtained by keeping the terms up to \(O(\beta^4)\) (or \(O(\alpha^3))\) has the form

\[
\eta_t + w_x - \frac{1}{3} \beta w_{3x} + \alpha(\eta w)_x + \frac{1}{120} \beta^2 w_{5x} - \frac{1}{3} \alpha \beta (\eta w_{2x})_x - \frac{1}{5} \alpha \beta^2 (\eta w_{2x})_x + \frac{1}{3} \alpha \beta^2 (\eta w_{4x})_x + \frac{1}{3} \beta^2 \beta^4 w_{9x} = 0
\]

\[w_t + \eta_x - \frac{1}{3} \beta w_{2x} + \alpha w w_x + \frac{1}{24} \beta^2 w_{4x} + \alpha \beta (- (\eta w)_x) + \frac{1}{3} \eta w, w_{2x} - \frac{1}{3} w w_{3x} - \frac{1}{3} \alpha \beta \frac{1}{3} (\eta w_{3x})_x + \frac{1}{12} w_{2x} w_{3x} - \frac{1}{3} w_{4x} \]

\[
- \frac{1}{3} \eta \eta_{4x} + \alpha \beta \left( \frac{1}{2} (\eta w_{3x})_x + \frac{1}{12} w_{2x} w_{3x} - \frac{1}{3} w_{4x} \right) + \frac{1}{40} \beta^4 \beta^4 w_{8x} = 0.
\]

An equivalent system of the unidirectional wave equations (8) and (9) truncated to keep terms up to \(O(\beta^4)\) is (the meaning of the square brackets is the same as in (25))

\[
w = \eta + \frac{1}{3} \beta \eta_{2x} + \left[ - \frac{1}{2} \alpha \eta^2 + \frac{1}{16} \beta \eta^2 \eta_{4x} \right] + \left[ \frac{1}{12} \alpha \beta (3 \eta^2 + 8 \eta \eta_{2x}) + \frac{61}{1800} \beta \eta^2 \eta_{6x} \right] + \left[ \frac{1}{6} \alpha \eta^2 \eta_{4x} + \frac{1}{24} \beta^2 (\frac{121}{30} \eta^2 \eta_{4x} + \frac{1099}{144} \eta \eta_{3x} + \frac{203}{70} \eta \eta_{4x}) + \frac{1261}{1134000} \beta^4 \eta^2 \eta_{8x} \right]
\]

\[
\times \eta_t + \eta_x + \frac{1}{6} \beta \eta_{3x} + \left[ \frac{1}{2} \alpha \eta \eta_{4x} + \frac{19}{500} \beta^2 \eta^2 \eta_{6x} \right] + \left[ \frac{1}{2} \eta \eta_{4x} + \frac{1}{24} \beta^2 (\frac{23}{30} \eta \eta_{3x} + \frac{54}{500} \eta \eta_{5x} + \frac{55}{60} \beta \eta^2 \eta_{5x} + \frac{181}{13140} \beta^4 \eta_{9x} \right]
\]

\[
+ \left[ \frac{1}{2} \alpha \eta \eta_{4x} + \frac{1}{24} \beta^2 \left( \frac{121}{30} \eta_{2x} \eta_{3x} + \frac{1079}{144} \eta \eta_{4x} + \frac{19}{500} \beta \eta^2 \eta_{5x} \right) + \frac{11813}{1314400} \beta^4 \eta^2 \eta_{9x} \right] = 0.
\]

It is immediate that for \(\alpha = O(\beta^2)\) the equation including at leading order both nonlinearity and dispersion is

\[
\eta_t + \eta_x + \frac{1}{6} \beta \eta_{3x} + \frac{3}{8} \alpha \eta \eta_{4x} = 0.
\]

By the change of variables

\[
\tilde{\xi} = \sqrt{\frac{3\alpha}{2\beta}} (x - t), \quad \tilde{t} = \frac{1}{4} \sqrt{\frac{3\alpha}{2\beta}},
\]

equation (31) can be reduced to the following equation:

\[
\eta_t + 6\eta \eta_{t} + \eta_{3x} + M \eta_{5x} = 0,
\]

\[
M = \frac{19}{30} \alpha.
\]

This equation, which is frequently referred to as the \textit{fifth-order KdV equation}, has been derived in [20] (with the parameter \(M\) defined in a different way) as a model equation for the gravity-capillary shallow water waves of small amplitude when the Bond number is close to but just less than 1/3. It has been extensively studied since then, see e.g., [21], and, although it is not integrable via the inverse scattering transform, it is known to have a rich structure of solitary wave solutions—in particular, the existence of nonlocal solitary waves with propagating oscillatory tails and of asymmetric solitary waves has been established. Our analysis shows that the fifth-order KdV equation (33) arises as the leading-order equation in the classical water wave problem \textit{without surface tension} when \(\alpha = O(\beta^2)\).

We will also present without derivation the leading-order equation for the case \(\beta = O(\alpha^3)\) obtained by retaining the terms which are at most cubic in \(\alpha\). It reads

\[
\eta_t + \eta_x + \frac{3}{8} \alpha \eta \eta_{4x} = \frac{3}{8} \alpha^2 \eta^2 \eta_{4x} + \frac{3}{16} \alpha^3 \eta^3 \eta_{4x} + \frac{6}{500} \eta^2 \eta_{3x} \eta_{7x} = 0.
\]

This equation can be transformed into

\[
\eta_t = \eta^3 \eta_{4x} \eta_{9x}.
\]

which belongs to the type \(K(m, n)\) introduced by Rosenau and Hyman [22] with \(m = 4\) and \(n = 1\). Equation (35) is non-integrable but admits soliton-like traveling wave solutions in some range of wave velocities.
5. Discussion

We have presented a procedure for systematic derivation of the leading-order and higher order evolution equations for the surface elevation of unidirectional shallow water waves. This procedure is based on a consistent ordering of terms in the original asymptotic expansions for a prescribed relationship between orders of magnitude of two small parameters $\alpha$ and $\beta$. Our results provide a set of consistent model equations for unidirectional water waves which replace the KdV equation and the higher order KdV equations in the cases when the parameters $\alpha$ and $\beta$ are not of the same order of magnitude. Some of the equations emerging in our analysis as the leading-order equations in the asymptotic expansion for the unidirectional water waves have been proposed before as model equations in other physical contexts (e.g., the Gardner equation, the modified KdV equation and the so-called fifth-order KdV equation). In the higher orders of approximation, a variety of evolution equations which can serve as higher order models for unidirectional water waves on equal footing with the higher order KdV equations are found. Our analysis also reveals that certain model equations used in the literature are questionable since they have been obtained as a result of an improper ordering which is invalid for any relationship among orders of $\alpha$ and $\beta$.

The present analysis is based on assuming a prescribed relationship between orders of magnitude of two small parameters $\alpha$ and $\beta$. However, the results can be interpreted in another alternative way along the lines of the analysis presented in [23]. The main concern of the analysis of [23] is to demonstrate that the condition $\beta = O(\alpha)$ is not necessary for having a balance between nonlinearity and dispersion characteristic of the KdV equation and that the KdV balance is possible for any $\beta$ provided that $\alpha \to 0$. To this end the variables are transformed in such a way that the parameter $\beta$ is scaled out, in favor of $\alpha$, which leads to a prescription, in asymptotic terms, of the region of time and space where the balance occurs and so the KdV equation is valid. This conceptual shift from a relationship between orders of magnitude of the two small parameters to distances and times needed for achieving the balance between nonlinearity and dispersion provides a new view which is more relevant to applications in nature.

However, the analysis of [23] is restrictive in the sense that the transformation of variables introduced in [23] may result only in the problem which leads to the KdV equation to leading order as $\alpha \to 0$. In what follows, we show that it is not because of some intrinsic properties of the water wave equations but simply due to a specific character of the transformation used in [23]. We extend the analysis of [23] by introducing a generalized transformation dependent on a parameter $n$ (the transformation of [23] becomes a particular case). This generalized transformation, like the transformation introduced in [23], results in the system of equations which contains only one small parameter $\alpha$. Specifying the transformation parameter $n$ to different values allows us to obtain a variety of different problems and a variety of the corresponding leading-order equations (like the Gardner equation, the fifth-order KdV equation and so on) including the KdV equation. As a matter of fact, each problem obtained from the original one by applying the transformation for a particular value of $n$ can be equivalently obtained by assuming the relationship $\beta = O(\alpha^n)$ between orders of magnitude of the small parameters. The former approach allows us to consider the nonlinearity–dispersion balance, epitomized by the soliton equations, as existing for any $\beta$, provided that $\alpha \to 0$, but imposes conditions on the regions of space and time in which the soliton dynamics (the KdV dynamics, the Gardner dynamics, the fifth-order KdV dynamics and so on) are expected to occur.

In [23], the transformations eliminating $\beta$ are applied to the original system of equations in terms of velocities $(u, w)$, pressure $p$ and elevation $\eta$, and then the system of equations in terms of $\phi$ and $\eta$ is obtained from the transformed equations. Therefore, in our analysis, we will
also deal with the original equations (although the same could be done for equations (2)–(5) in terms of $\phi$ and $\eta$). The system of equations of a two-dimensional irrotational wave, with effects of surface tension negligible, after non-dimensionalizing takes the form

$$
\begin{align*}
\alpha u_x + \alpha (uw_x + wu_y) &= -p_x, \\
\beta (w_t + \alpha (uw_x + wu_y)) &= -p_y,
\end{align*}
$$

(36)

$$
\begin{align*}
u_x + w_y &= 0, \\
u_y - \beta w_x &= 0,
\end{align*}
$$

(37)

$$
w = 0 \quad \text{on} \quad y = 0,
$$

(38)

$$
p = \eta, \quad w = \eta + \alpha u_n_x \quad \text{on} \quad y = 1 + \alpha \eta.
$$

(39)

(In the notation of [23], $y \rightarrow z$, $\beta \rightarrow \delta^2$, $\alpha \rightarrow \epsilon$.) The scales for $(x, y, t)$ are as in (6) and the scales for $u, w$ and $p$ are respectively

$$
\left(\frac{a}{H}\right)\sqrt{gH}, \quad \left(\frac{a}{L}\right)\sqrt{gH} \quad \text{and} \quad \rho g a.
$$

The following transformations are applied to equations (36)–(39) in [23]:

$$
\begin{align*}
x &\rightarrow \sqrt{\frac{\beta}{\alpha^n}} x, \\
y &\rightarrow y, \\
t &\rightarrow \sqrt{\frac{\beta}{\alpha^n}} t, \\
p &\rightarrow p, \\
\eta &\rightarrow \eta, \\
w &\rightarrow \alpha^{n/2} \sqrt{\frac{\beta}{\alpha^n}} w.
\end{align*}
$$

(40)

As a result, the system (36)–(39) reduces to the system of equations

$$
\begin{align*}
u_t + \alpha (uw_x + wu_y) &= -p_x, \\
\alpha^n (w_t + \alpha (uw_x + wu_y)) &= -p_y,
\end{align*}
$$

(41)

$$
u_x + w_y &= 0, \\
u_y - \alpha^n w_x &= 0,
$$

(42)

$$
w = 0 \quad \text{on} \quad y = 0,
$$

(43)

$$
p = \eta, \quad w = \eta + \alpha u_n x \quad \text{on} \quad y = 1 + \alpha \eta,
$$

(44)

which are the same as (36)–(39), but with $\beta$ replaced by $\alpha^n$, for arbitrary $\beta$. From the analysis made in section 3 of this paper (and from an equivalent analysis of [23]), it is evident that equations (41)–(44) constitute the representation that leads to the KdV equation (17) to leading order as $\alpha \rightarrow 0$.

As was explained above, the transformations (40) can be generalized. The generalized transformations are

$$
\begin{align*}
x &\rightarrow \sqrt{\frac{\beta}{\alpha^n}} x, \\
y &\rightarrow y, \\
t &\rightarrow \sqrt{\frac{\beta}{\alpha^n}} t, \\
p &\rightarrow p, \\
\eta &\rightarrow \eta, \\
w &\rightarrow \alpha^{n/2} \sqrt{\frac{\beta}{\alpha^n}} w,
\end{align*}
$$

(45)

where $n$ is arbitrary. Applying the transformations (45) to equations (36)–(39) results in the system

$$
\begin{align*}
u_t + \alpha (uw_x + wu_y) &= -p_x, \\
\alpha^n (w_t + \alpha (uw_x + wu_y)) &= -p_y,
\end{align*}
$$

(46)

$$
u_x + w_y &= 0, \\
u_y - \alpha^n w_x &= 0,
$$

(47)

$$
w = 0 \quad \text{on} \quad y = 0,
$$

(48)

$$
p = \eta, \quad w = \eta + \alpha u_n_x \quad \text{on} \quad y = 1 + \alpha \eta,
$$

(49)

which is (36)–(39), with $\beta$ replaced by $\alpha^n$.

It is clear that the same problem (46)–(49) results also from the assumption $\beta = O(\alpha^n)$ which allows us to replace $\beta$ by $\alpha^n$ without loss of generality. As a matter of fact, all the results of this paper are related to the problem (46)–(49), independent of the approach through which it is obtained. The analysis made in sections 3 and 4 indicates that equations (46)–(49) specified to different values of $n$ give rise to different equations to leading order as $\alpha \rightarrow 0$. The
KdV equation arises as a particular case for \( n = 1 \). Other particular cases might be the Gardner equation \( (n = 2) \), the fifth-order KdV equation \( (n = 1/2) \) and the \( K(4,1) \)-type equation in the sense of \( [22] \) \( (n = 3) \). In the case, when the system (46)--(49) is treated as obtained via the transformations (45), the results are valid under some conditions on the regions of space and time where thus the corresponding soliton dynamics are expected to occur. It should be emphasized, however, that although the system (46)--(49) can be equivalently obtained either by applying the transformations (45) or by assuming \( \beta = O(\alpha^\omega) \), these two approaches represent alternative views which cannot be combined.

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Appendix. Results for the case of nonzero surface tension

Since the multiplier \((1 - 3\tau)\) appears in the coefficients of the highest derivatives in the leading-order equations, the case of \( \tau = 1/3 \) should be considered separately. We will assume that \( \tau \neq 1/3 \) and moreover that \(|\tau - 1/3| \) is not small. Indeed, if \(|\tau - 1/3| \ll 1 \), one has to introduce yet another small parameter \( \epsilon = \tau - 1/3 \) and consider the asymptotic expansion with respect to \( \epsilon \) as well, see e.g. \( [20] \).

A.1. \( \beta = O(\alpha^2) \)

We will consider the case of \( \beta = O(\alpha^2) \) keeping the terms that are at most quartic in \( \alpha \). Following the procedure described above, we obtain

\[
\begin{align*}
    w &= \eta - \frac{\alpha \eta^2}{4} + \frac{\alpha^2 \eta^3}{8} + \frac{\beta}{6} (2 - 3\tau) \eta_{2x} - \frac{5}{64} \alpha^3 \eta^4 + \alpha \beta \left( \frac{3}{16} \eta^2 + 4(2 + \tau) \eta_{2x} \right) \\
    &\quad + \frac{7\alpha^2 \eta^5}{128} + \frac{\alpha^2 \beta}{32} (2 - 3\tau) \eta^2 \eta_{2x} + \frac{3}{16} (1 - 7\tau) \eta_{2x}^2 + 6(1 - \tau) z \\
    &\quad - \frac{\beta^2}{120} (-12 + 20\tau + 15\tau^2) \eta_{4x},
\end{align*}
\]

(A.1)

\[
\begin{align*}
    \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x &= \frac{3}{8} \alpha^2 \eta^2 \eta_x + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{3}{16} \alpha^3 \eta^3 \eta_x \\
    &\quad + \frac{\alpha \beta}{24} (23 + 15\tau) \eta_x \eta_{2x} + \frac{2}{15} (5 - 3\tau) \eta \eta_{3x} - \frac{15}{128} \alpha^4 \eta^4 \eta_x \\
    &\quad + \frac{\alpha^2 \beta}{32} (2(5 + \tau) \eta^2 \eta_{3x} + 2(23 - 5\tau) \eta \eta_x \eta_{2x} + (19 - 13\tau) \eta_x^3) \\
    &\quad - \frac{\beta^2}{360} (-19 + 30\tau + 45\tau^2) \eta_{5x} = 0.
\end{align*}
\]

(A.2)

where \( z = \int \eta_t^4 \, dx \).

If we keep in (A.2) the terms of order not greater than \( O(\alpha^2) \) to retain the dispersion and nonlinearity at the leading order, then it reads

\[
\begin{align*}
    \eta_t + \eta_x + \frac{3}{2} \alpha \eta \eta_x &= \frac{3}{8} \alpha^2 \eta^2 \eta_x + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} = 0.
\end{align*}
\]

(A.3)
This is nothing but the Gardner equation which for $\tau = 0$ coincides with equation (27) discussed in section 4. This equation can be reduced to the modified KdV equation

$$\eta_t = K\eta_{3x} + 6\eta^2\eta_x,$$

where $K = \text{sign}(\beta(1 - 3\tau))$.

A.2. $\beta = O(\alpha^3)$

Keeping the terms up to the order of $\alpha^5$ we have

$$w = \eta - \frac{\alpha^2\eta^2}{4} + \frac{\alpha^2\eta^2}{8} - \frac{5\alpha^2\eta^4}{64} + \frac{\beta(2 - 3\tau)}{6} \eta_{2x} + \frac{7\alpha^2\eta^5}{128} + \frac{\alpha\beta}{4} \left(2(\tau + \tau)\eta_{2x} + (3 + 7\tau)\frac{\eta^2}{2}\right)$$

$$- \frac{21\alpha^2\eta^6}{512} + \frac{\alpha^2\beta}{16} \left(3(\tau - 1)\frac{\eta^2}{2} - 3(\tau - 1)\tau + (2 - 3\tau)^2\eta^2_{2x}\right),$$

(A.4)

$$\eta_t + \alpha\eta + \frac{3}{2} \alpha^2\eta_{xt} - \frac{3}{8} \alpha^2\eta^2\eta_x + \frac{3}{16} \alpha^3\eta^3\eta_x + \frac{\beta(1 - 3\tau)}{6} \eta_{3x}$$

$$- \frac{15}{128} \alpha^3\eta^4\eta_x + \frac{\alpha\beta}{24} ((23 + 15\tau)\eta_{2x} - 2(3\tau - 5)\eta_{3x}) + \frac{21}{256} \alpha^3\eta^5\eta_x$$

$$+ \frac{\alpha^2\beta}{32} ((-13\tau + 19)\eta^2_x + 2(23 - 5\tau)\eta_{2x}\eta_{2x} + 2(\tau + 5)\eta^2_{2x}) = 0,$$

(A.5)

where again $z = \int \eta_t^2 dx$.

If we consider the leading-order equation, i.e. restrict ourselves to the terms which are at most cubic in $\alpha$, then (A.4) becomes

$$\eta_t + \alpha\eta + \frac{3}{2} \alpha^2\eta_{xt} - \frac{3}{8} \alpha^2\eta^2\eta_x + \frac{3}{16} \alpha^3\eta^3\eta_x + \frac{\beta(1 - 3\tau)}{6} \eta_{3x} = 0.$$

(A.6)

This equation can be further transformed into

$$\eta_t = M\eta^3\eta_x + \eta_{3x},$$

(A.7)

where $M = \text{sign}(\beta(1 - 3\tau))$. Equation (A.7) belongs to the type $K(m, n)$ introduced by Rosenau and Hyman [22] with $m = 4$ and $n = 1$. It is non-integrable, but it is readily seen to admit soliton-like traveling wave solutions for $M = 1$ in a certain range of wave velocities.

A.3. $\alpha = O(\beta^2)$

In this case we have

$$w = \eta + \frac{\beta}{6} (2 - 3\tau)\eta_{2x} + \beta^2 \left(\frac{1}{10} - \frac{\tau}{6} - \frac{\tau^2}{8}\right) \eta_{4x} - \frac{\alpha\eta^2}{4}$$

$$+ \beta^3 (488 - 756\tau - 630\tau^2 - 945\tau^3) \frac{\eta_{6x}}{15120}$$

$$+ \alpha\beta \left((3 + 7\tau) \frac{\eta^2}{16} + (2 + \tau) \frac{\eta_{2x}}{4}\right)$$

$$+ \beta^4 \left(\frac{1261}{113400} - \frac{61\tau}{3780} - \frac{\tau^2}{80} - \frac{\tau^3}{48} - \frac{5\tau^4}{128}\right) \eta_{8x}$$

$$+ \alpha\beta^2 \left(\frac{326 + 435\tau + 315\tau^2}{720} \frac{\eta^2_{2x}}{20} + (28 - 20\tau + 5\tau^2) \frac{\eta_{4x}}{80}\right)$$

$$+ (1091 + 480\tau + 945\tau^2) \frac{\eta_{6x}}{1440} + \frac{\alpha^2\eta^3}{8},$$

(A.8)
\[ \eta_t + \eta_x + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{3}{2} \alpha \eta \eta_x + \frac{\beta^2}{360} (19 - 30\tau - 45\tau^2) \eta_{5x} \]
\[ + \frac{\alpha \beta}{24} (23 + 15\tau) \eta_x \eta_{2x} + 2(5 - 3\tau) \eta \eta_{3x} \]
\[ - \frac{\beta^3}{15120} (-275 + 399\tau + 315\tau^2 + 945\tau^3) \eta_x \]
\[ - \frac{3}{8} \alpha^2 \eta^2 \eta_x + \alpha \beta^2 \left( \frac{317 + 270\tau + 441\tau^2}{1440} \right) \eta_{2x} \eta_{3x} \]
\[ + \left( 1079 - 150\tau + 855\tau^2 \right) \frac{\eta \eta_{4x}}{1440} \]
\[ - \left( -57 + 50\tau + 15\tau^2 \right) \frac{\eta \eta_{5x}}{240} \]
\[ + \beta^4 \left( \frac{11813}{1814400} - \frac{55\tau}{6048} - \frac{19\tau^2}{2880} - \frac{\tau^3}{96} - \frac{5\tau^4}{128} \right) \eta_{9x} = 0. \quad (A.9) \]

Note that in the second order in \( \beta \), which is the leading order in this case, equation (A.9) for \( \eta \) can be transformed into the KdV equation only if \( 19 - 30\tau - 45\tau^2 = 0 (\tau \approx 0.4) \) when the term with \( \eta_{5x} \) vanishes. If \( 19 - 30\tau - 45\tau^2 \neq 0 \), then (A.9) in second order in \( \beta \) reads
\[ \eta_t + \eta_x + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{3}{2} \alpha \eta \eta_x + \frac{\beta^2}{360} (19 - 30\tau - 45\tau^2) \eta_{5x} = 0. \]

We can get rid of the term \( \eta_x \) by passing from \( x \) to \( x' = x - t \), so upon omitting the prime at \( x \) the equation under study becomes
\[ \eta_t + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{3}{2} \alpha \eta \eta_x + \frac{\beta^2}{360} (19 - 30\tau - 45\tau^2) \eta_{5x} = 0. \]

Next, let \( x = Ax, \ t = Bt, \ \eta = C\tilde{\eta} \), where \( B = \frac{360 \alpha^2}{(\beta(-19 + 30\tau + 45\tau^2)/10800\alpha^2)} \), \( C = -\beta^2(-19 + 30\tau + 45\tau^2)/(10800\alpha^2) \). Then upon omitting tildes at \( x, t \) and \( \eta \) we obtain the so-called fifth-order KdV equation (see section 4) in the form
\[ \eta_t = \eta \eta_x + K \eta_{3x} + \eta_{5x}, \quad (A.10) \]
where \( K = (3\tau - 1)60\alpha^2/\beta(-19 + 30\tau + 45\tau^2) \). Assuming \( \tau \neq 1/3 \), we can set \( A = ((\beta(-19 + 30\tau + 45\tau^2)/(60(3\tau - 1)))^{1/2} \), and then \( K = \text{sign}((\beta(-19 + 30\tau + 45\tau^2))/(3\tau - 1)). \)

\subsection{A.4. \( \alpha = O(\beta^3) \)}

If \( \alpha = O(\beta^3) \), then we obtain
\[ w = \eta + \frac{\beta}{6} (2 - 3\tau) \eta_{2x} + \beta^2 \left( \frac{1}{10} - \frac{\tau}{6} \frac{\tau^2}{8} \right) \eta_{4x} \]
\[ - \frac{\alpha \eta^2}{4} - \frac{\beta^3}{15120} (-488 + 756\tau + 630\tau^2 + 945\tau^3) \eta_{6x} \]
\[ + \beta^4 \left( \frac{1261}{113400} - \frac{61\tau}{3780} - \frac{\tau^2}{80} - \frac{\tau^3}{48} - \frac{5\tau^4}{128} \right) \eta_{8x} \]
\[ + \frac{\alpha \beta}{16} (3 + 7\tau) \eta_x^2 + 4(2 + \tau) \eta \eta_{2x} \]
\[ - \frac{\beta^5}{39916800} (-159264 + 221936\tau + 161040\tau^2 + 249480\tau^3) \]
\[
\eta_t + \eta_x + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{\beta^2}{360} (19 - 30\tau - 45\tau^2) \eta_{5x} \\
+ \frac{3\alpha}{2} \eta_x \eta_{xx} - \frac{\beta^3}{15 \times 120} (-275 + 399\tau + 315\tau^2 + 945\tau^3) \eta_{7x}
\]
\[
+ \frac{\alpha \beta^2}{1440} (\frac{11813}{1814400} - \frac{55}{6048}\tau - \frac{19}{2880}\tau^2 - \frac{\tau^3}{96} - \frac{5\tau^4}{128}) \eta_{9x}
\]
\[
+ \frac{\alpha \beta^5}{24} (2(3\tau - 5)\eta \eta_{3x} + (23 + 15\tau)\eta_x \eta_{2x})
\]
\[
+ \frac{\beta^5}{3 \times 9196800} (-95265 - 129943\tau - 90750\tau^2 - 131670\tau^3 - 259875\tau^4 - 1091475\tau^5) \eta_{11x}
\]
\[
+ \frac{\alpha \beta^2}{288} (317 + 270\tau + 441\tau^2) \eta_{2x} \eta_{3x} + (1079 - 150\tau + 855\tau^2) \frac{\eta_x \eta_{4x}}{1440}
\]
\[
- (57 + 50\tau + 15\tau^2) \frac{\eta_{5x}}{240}
\]

\begin{equation}
(A.12)
\end{equation}

In this case, the leading order is 3, and upon introducing a new variable \( x' = x - t \) the leading-order equation for \( \eta \) reads

\[
\eta_t + \frac{\beta}{6} (1 - 3\tau) \eta_{3x} + \frac{\beta^2}{360} (19 - 30\tau - 45\tau^2) \eta_{5x} \\
+ \frac{3\alpha}{2} \eta_x \eta_{xx} - \frac{\beta^3}{15 \times 120} (-275 + 399\tau + 315\tau^2 + 945\tau^3) \eta_{7x} = 0.
\]

\begin{equation}
(A.13)
\end{equation}

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