A REMARK ON THE RADIAL MINIMIZER OF THE GINZBURG-LANDAU FUNCTIONAL

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Abstract. Denote by $E_{\varepsilon}$ the Ginzburg-Landau functional in the plane and let $\hat{u}_{\varepsilon}$ be the radial solution to the Euler equation associated to the problem \( \min \{ E_{\varepsilon}(u, B_1) : u|_{\partial B_1} = (\cos \vartheta, \sin \vartheta) \} \). Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain with the same area as $B_1$. Denoted by

\[
K = \left\{ v = (v_1, v_2) \in H^1(\Omega; \mathbb{R}^2) : \int_{\Omega} v_1 \, dx = \int_{\Omega} v_2 \, dx = 0, \int_{\Omega} |v|^2 \, dx \geq \int_{B_1} |\hat{u}_{\varepsilon}|^2 \, dx \right\},
\]

we prove

\[
\min_{v \in K} E_{\varepsilon}(v, \Omega) \leq E_{\varepsilon}(\hat{u}_{\varepsilon}, B_1).
\]

1. Introduction

The Ginzburg-Landau energy has as order parameter a vectorial field $u \in H^1(\Omega; \mathbb{R}^2)$ and is defined as

\[
E_{\varepsilon}(u, \Omega) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{4\varepsilon^2} \int_{\Omega} (|u|^2 - 1)^2 \, dx,
\]

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $\varepsilon > 0$. This kind of functionals has been originally introduced as a phenomenological phase-field type free-energy of a superconductor, near the superconducting transition, in absence of an external magnetic field. Moreover these functionals have been used in superfluids such as Helium II. In this context $u$ represents the wave function of the superfluid part of liquid and the parameter $\varepsilon$, which has the dimension of a length, depends on the material and its temperature. The Ginzburg-Landau functionals have deserved a great attention by the mathematical community too. Starting from the paper [4] by Bethuel, Brezis and Hélein, many mathematicians have been interested in studying minimization problems for the Ginzburg-Landau energy with several constraints, also because, besides the physical motivation, these problems appear as the simplest nontrivial examples of vector field minimization problems.

In [4] the authors consider Dirichlet boundary conditions $g \in C^1(\partial \Omega; S^1)$ and study the asymptotic behavior, as $\varepsilon \to 0$, of minimizers $u_{\varepsilon}$, which clearly satisfies the following Euler equation

\[
\begin{cases}
-\Delta u_{\varepsilon} = \frac{1}{\varepsilon^2} u_{\varepsilon} (1 - |u_{\varepsilon}|^2) & \text{in } \Omega \\
u_{\varepsilon} = g & \text{on } \partial \Omega.
\end{cases}
\]

It turns out that the value $d = \deg(g, \partial \Omega)$ (i.e., the Brouwer degree or winding number of $g$ considered as a map from $\partial \Omega$ into $S^1$) plays a crucial role in the asymptotic analysis of $u_{\varepsilon}$.

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1
In the case $\Omega = B_1$ (the unit ball in $\mathbb{R}^2$ centered at the origin), $g(x) = x$, it is natural to look for radial solutions to (1.1). Indeed, in [9, 5, 12] the authors prove, among other things, that (1.1) has a unique radial solution, that is a solution of the form
\begin{equation}
\tilde{u}_\varepsilon(x) = \tilde{f}_\varepsilon(|x|) (\cos \vartheta, \sin \vartheta)
\end{equation}
with $\tilde{f}_\varepsilon \geq 0$. Moreover $\tilde{f}_\varepsilon'$ > 0; thus, summarizing, $\tilde{f}_\varepsilon$ is the solution to the following problem
\begin{equation}
\begin{cases}
- \tilde{f}_\varepsilon'' - \frac{\tilde{f}_\varepsilon'}{r} + \frac{\tilde{f}_\varepsilon}{r^2} = \frac{1}{\varepsilon^2} \tilde{f}_\varepsilon \left( 1 - \tilde{f}_\varepsilon^2 \right) & \text{in } (0, 1) \\
\tilde{f}_\varepsilon(0) = 0, \; \tilde{f}_\varepsilon(1) = 1, \; \tilde{f}_\varepsilon \geq 0, \; \tilde{f}_\varepsilon' > 0.
\end{cases}
\end{equation}

It is conjectured that the radial solution (1.2) is the unique minimizer of $E_\varepsilon$ on $B_1$. In [14] (see also [13]) the author gives a partial answer to such a conjecture, proving that $\tilde{u}_\varepsilon$ is stable, in the sense that the quadratic form associated to $E_\varepsilon(\tilde{u}_\varepsilon, B_1)$ is positive definite.

Other types of boundary conditions, for instance prescribed degree boundary conditions, have been considered in [3, 7].

In this paper we let $\Omega$ vary among smooth domains with fixed area and prove that the map $\tilde{u}_\varepsilon$ in (1.2) provides an upper bound for the energy $E_\varepsilon$ on the class $\mathcal{K}$ we are going to introduce.

\textbf{Theorem 1.1.} Let $\varepsilon > 0$ and $\Omega \subset \mathbb{R}^2$ be a smooth, bounded domain such that $|\Omega| = |B_1|$. Denoted by
\begin{equation}
\mathcal{K} = \left\{ v = (v_1, v_2) \in H^1(\Omega; \mathbb{R}^2) : \int_\Omega v_1 \, dx = \int_\Omega v_2 \, dx = 0, \; \int_\Omega |v|^2 \, dx \geq \int_{B_1} |\tilde{u}_\varepsilon|^2 \, dx \right\},
\end{equation}
it holds
\begin{equation}
\min_{v \in \mathcal{K}} E_\varepsilon(v, \Omega) \leq E_\varepsilon(\tilde{u}_\varepsilon, B_1).
\end{equation}

\textbf{2. Proof of Theorem 1.1}

Define the following continuous extension of $\tilde{f}_\varepsilon$
\begin{equation}
f_\varepsilon(r) = \begin{cases}
\tilde{f}_\varepsilon(r) & \text{if } 0 \leq r \leq 1 \\
1 & \text{if } r > 1
\end{cases}
\end{equation}
and the correspondent vector field extending $\tilde{u}_\varepsilon$ to the whole $\mathbb{R}^2$
\begin{equation}
\phi_\varepsilon(x) = (\phi_{\varepsilon,1}(x), \phi_{\varepsilon,2}(x)) = f_\varepsilon(|x|) (\cos \vartheta, \sin \vartheta).
\end{equation}

It is possible (see [10], see also [11]) to choose the origin in such a way that
\begin{equation}
\int_\Omega \phi_{\varepsilon,1} \, dx = \int_\Omega \phi_{\varepsilon,2} \, dx = 0.
\end{equation}

Note that $\phi_\varepsilon \in \mathcal{K}$. Indeed, besides (2.1), it holds
\begin{equation}
\int_\Omega |\phi_\varepsilon|^2 \, dx = \int_{\Omega \cap B_1} |\tilde{u}_\varepsilon|^2 \, dx + |\Omega \setminus B_1| \geq \int_{B_1} |\tilde{u}_\varepsilon|^2 \, dx,
\end{equation}
since $|\tilde{u}_\varepsilon| \leq 1$ in $B_1$. A direct computation yields

$$E_\varepsilon(\phi_\varepsilon, \Omega) = \frac{1}{2} \int_{\Omega} \left( f'_\varepsilon(|x|)^2 + \frac{f_\varepsilon(|x|)^2}{|x|^2} \right) dx + \frac{1}{4\varepsilon^2} \int_{\Omega} \left( f_\varepsilon(|x|)^2 - 1 \right)^2 dx$$

$$= \int_{\Omega} B_\varepsilon(|x|) dx,$$

where

$$B_\varepsilon(r) = \frac{1}{2} \left( f'_\varepsilon(r)^2 + \frac{f_\varepsilon(r)^2}{r^2} \right) + \frac{1}{4\varepsilon^2} \left( f_\varepsilon(r)^2 - 1 \right)^2.$$

Using (1.3) it is straightforward to verify that

$$B'_\varepsilon(r) = \frac{2}{\varepsilon^2} f_\varepsilon(r) f'_\varepsilon(r) \left( 1 - f_\varepsilon(r)^2 \right) - \frac{1}{r} \left( f'_\varepsilon(r) - \frac{f_\varepsilon(r)}{r} \right)^2, \quad 0 < r < 1,$$

while, when $r > 1$, it holds $B_\varepsilon(r) = \frac{1}{2r}$. Thus $B_\varepsilon(r)$ is a decreasing function in $(0, +\infty)$. By Hardy-Littlewood inequality (see for instance [10]) we finally get

$$E_\varepsilon(\phi_\varepsilon, \Omega) = \int_{\Omega} B_\varepsilon(|x|) dx \leq \int_{B_1} B_\varepsilon(|x|) dx = E_\varepsilon(\tilde{u}_\varepsilon, B_1)$$

and hence (1.3).

**Remark 2.1.** The appearance of the function $\tilde{u}_\varepsilon$ (i.e., the candidate to be the unique minimizer of $E_\varepsilon$ in $B_1$ under the Dirichlet boundary condition $g(x) = x$ in (1.4) as an upper bound of the energy in the class $K$ could seem odd. On the other hand such a phenomenon looks more natural once one notices an analogy between the problem under consideration and the maximization problem of the first nontrivial eigenvalue $\mu_1(\Omega)$ of the Neumann Laplacian among sets with prescribed area. As well-known, if $\Omega$ is a smooth, bounded domain of $\mathbb{R}^2$, $\mu_1(\Omega)$ can be variationally characterized as

$$\mu_1(\Omega) = \left\{ \int_{\Omega} |\nabla z|^2 : z \in H^1(\Omega; \mathbb{R}), \int_{\Omega} |z|^2 dx = 1, \int_{\Omega} z dx = 0 \right\}.$$

If $|\Omega| = |B_1|$ the celebrated Szegő-Weinberger inequality in the plane (see [16], see also [15] 2 11 18 11) states

$$\mu_1(\Omega) \leq \mu_1(B_1).$$

Moreover, $\mu_1(B_1)$ is achieved by the functions $J_1(j'_{1,1} |x|) \cos \vartheta$ or $J_1(j'_{1,1} |x|) \sin \vartheta$, where $J_1$ is the Bessel function of the first kind and $j'_{1,1}$ is the first zero of its derivative. The role played by $J_1$ in (2.2) is now played by the function $f_\varepsilon^\prime$.

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