Abstract. This article studies the behavior of the Maronna robust scatter estimator $\hat{C}_N \in \mathbb{C}^{N \times N}$ of a sequence of observations $y_1, \cdots, y_n$ which is composed of a $K$ dimensional signal drawn in a heavy tailed noise, i.e $y_i = A_N s_i + x_i$ where $A_N \in \mathbb{C}^{N \times K}$ and $x_i$ is drawn from elliptical distribution. In particular, we prove that as the population dimension $N$, the number of observations $n$ and the rank of $A_N$ grow to infinity at the same pace and under some mild assumptions, the robust scatter matrix can be characterized by a random matrix $\hat{S}_N$ that follows a standard random model. Our analysis can be very useful for many applications of the fields of statistical inference and signal processing.

1. Introduction

Estimation of covariance matrices is at the heart of the theory of multivariate statistical analysis [12]. Its importance can be seen from its broad range of applications including financial data analysis, statistical signal processing, and wireless communication. A natural way to estimate covariance matrices is represented by the sample covariance matrix. Given $n$ observations $y_1, \cdots, y_n$, of size $N$, independent, and identically distributed (i.i.d) then the sample covariance matrix is given by $\frac{1}{n} \sum_{i=1}^{n} y_i y_i^*$. The popularity of the sample covariance matrix essentially comes from its low-complexity and the existence of a good understanding of its behaviour in two asymptotic regimes: $n$ goes to infinity while $N$ is fixed when $N$ and $n$ go to infinity with the same pace. Recent advances in the theory of large random matrices have made it clear that in the second asymptotic regime, the sample covariance matrix is no longer consistent. Conventional estimation methods that are based on the use of the sample covariance matrix are thus inefficient when the number of observations and their dimension become commensurate and large. Such scenario naturally arises in current array processing applications where the trend is to employ large antenna arrays. Based on a deep understanding of the behaviour of the sample covariance matrix, a new wave of detection methods [4, 5, 13] and subspace estimation techniques [11, 14, 18] has recently emerged. Although consistent, these methods are bound by the fact that they still fundamentally rely on the sample covariance matrix, their consistency being obtained by resorting to a deep analysis of its asymptotic behaviour. Nevertheless, the use of the sample covariance matrix can lead to poor performances, especially when observations are drawn from heavy tailed distributions or contain outliers. In such situations, the use of robust covariance estimators has been acknowledged as an efficient solution to combat the presence of outliers. Although references to robust techniques are traced back to the eighties with the works of Huber [9] and Maronna [10], the study of their performance has been often restricted to the conventional regime where the number of observations is too large as compared to their dimensions. It
was only recently that new tools have been developed in [6, 7, 8] which allow to analyse
the behaviour of robust Maronna’s scatter estimators. The main contributors are Couillet
et al. who established that the robust scatter estimator can be well-approximated in the
asymptotic regime by a random matrix that follows a standard random model. One of the
key advantages of this result, is that it allows to bring back the asymptotic analysis of robust-
catters to that of an other random object for which an important load of results already
exist.

Despite their high value, these works have been derived only for the case of pure noise
observations. While the case of a low rank signal observations can be dealt with by resorting
to easy adaptations of the approach of [8], handling high-rank signal observations is much
more challenging. Building on the tools developed in [8], we propose in this work to analyse
this difficult scenario. We show that in this case the adaption of the method in [8] is not
immediate and necessitates the development of additional appropriate tools. Some of the
required results that were of independent interest were submitted in an other work which
can be found in [1].

Notations: In the remainder of this work, we shall denote $\lambda_1(X) \leq \cdots \leq \lambda_N(X)$ the real
eigenvalues of $n \times n$ Hermitian matrix $X$. The notation $\|\cdot\|$ will refer to the spectral norm
of matrices and Euclidean norm for vectors, while $\ast$ sill stand for the complex conjugate
operator. The derivative of a differentiable function $f$ will be denoted by $f'$.

2. Assumptions and Main results

We start by introducing the data model under study. We consider $n$ sample vectors
$y_1, \cdots, y_n \in \mathbb{C}^N$ satisfying:

$$y_i = A_N s_i + x_i, \ i = 1, \cdots, n,$$

where $A_N$ is a $N \times K$ deterministic matrix and $x_1, \cdots, x_n$ are random vectors defined as:

$$x_i = \sqrt{\tau_i} w_i$$

with the scalars $\tau_1, \cdots, \tau_n \in \mathbb{R}_+$. Let $\tilde{N} = K + N$. We denote by $c_N = \frac{N}{n}$ and considers the
following assumptions:

**Assumption A-1.** For each $N$, $c_N < 1$, $c_N \geq 1$, and

$$c_- < \lim\inf_n c_N < \lim\sup_n c_N < c_+,$$

with $0 < c_- < c_+ < 1$.

This paper studies the asymptotic behaviour of the Maronna’s M-robust scatter estimator
in the regime of Assumption [4]. We recall that the Maronna’s M-robust estimator which we
denote by $\hat{C}_N$ is given by the unique solution in $Z$ of the following equation:

$$Z = \frac{1}{n} \sum_{i=1}^{n} u \left( \frac{1}{N} y_i^* Z^{-1} y_i \right) y_i y_i^*.$$  \hspace{1cm} (2.1)

where function $u(\cdot)$ satisfies the following properties:

**Assumption A-2.** i) Function $u(\cdot) : [0, \infty) \to [0, \infty)$ is non-negative continuous and
non-increasing,

ii) The function $\phi(\cdot) : x \mapsto xu(x)$ is increasing, bounded and continuously differentiable
with $\lim_{x \to \infty} \phi(x) \equiv \phi_{\infty} > 1$ and $\phi' > 0$. 


iii) $\phi_\infty < c_+^{-1}$.

and the scalars $\tau_i$ are such that:

**Assumption A-3.**  
  i) The random empirical measure $\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\tau_i}$ converges weakly to $\nu$ which satisfies $\int x \nu(x) = 1$,
  
i) There exists $\epsilon < 1 - \phi_\infty^{-1} < 1 - c_+$ and $m > 0$ such that for all large $n$ a.s., $\nu([0,m]) < \epsilon$.

The conditions in Assumption 2 are the same as those considered in [8]. It is worth observing that Assumption 2-ii) is different from the one considered by Maronna in [10], in that $\phi$ is not allowed to be constant on any open interval. However, Assumption 2-iii) is much more adapted to the high-dimensional regime than Assumption (D) p.53 of [10], which requires that $\phi_\infty > N$.

Assumption 3 is different from the original assumption in [8] as we assume here the weak convergence of the empirical measure $\nu_n$. However, one can easily see by the Portmanteau lemma that Assumption 3 will bring about the same useful requirements, namely the a.s. tightness of $\{\nu_n\}_{n=1}^\infty$, i.e., for each $\eta > 0$, there exists $M > 0$ such that for probability one, $\nu_n([M,\infty)) < \eta$, along with the absence of a heavy mass concentrating close to zero ($\nu_n([0,m)) < \epsilon$ for $n$ large enough a.s.).

The statistical hypothesis on $y_1, \cdots, y_n$ is detailed below:

**Assumption A-4.**  
  i) $w_1, \cdots, w_n \in \mathbb{C}^N$ are independent invariant complex zero-mean vectors with for each $i$, $\|w_i\|^2 = N$ and are independent of $\tau_1, \cdots, \tau_n$,
  
i) $s_i \sim \mathcal{CN}(0, I_K), i = 1, \cdots, n$ are independent standard Gaussian distributed vectors,
  
iii) Define $B_N = A_N A_N^*$, then $\lim \sup \|B_N\| < \infty$ and $\lim \inf \frac{1}{N} \text{Tr} B_N > 0$.

In addition to the above assumptions, the following hypothesis might be required:

**Assumption A-5.** For each $a > b > 0$, a.s.,

$$\limsup_{t \to \infty} \limsup_{n \to \infty} \nu_n(t, \infty) - \phi(at) - \phi(bt) = 0.$$ 

**Theorem 2.1** (Uniqueness). Let Assumptions 2-4 hold true. Then, for all large $n$, [2.1] admits a unique solution $C_N$. Moreover, $C_N$ is the limit of the sequence $Z(t)$ given by:

$$Z(t+1) = \frac{1}{n} \sum_{i=1}^n u_i \left( \frac{1}{N} y_i^* (Z_t)^{-1} y_i \right) y_i y_i^*,$$

where $Z^{(0)} \succeq 0$.

**Theorem 2.2.** Let Assumptions 2-5 hold. Let $\hat{C}_N$ be given by Theorem 2.1 when uniquely defined. Then,

$$\|\hat{C}_N - \hat{S}_N\| \xrightarrow{a.s.} 0$$

where

$$\hat{S}_N = \frac{1}{n} \sum_{i=1}^n \nu(\delta_i) y_i y_i^*.$$
and $\delta_1, \ldots, \delta_n$ are the unique positive solutions in $x_1, \ldots, x_n$ to the following system of equations

$$x_i = \frac{1}{N} \text{Tr} (B_N + \tau_i I_N) \left( \frac{1}{n} \sum_{j=1}^{n} \frac{v(x_j)(B_N + \tau_j I_N)}{1 + c N \psi(x_j)} \right),$$

with the functions $v : x \rightarrow (u \circ g^{-1})(x)$, $\psi(\cdot) : x \rightarrow x v(x)$ and $g(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$, $x \mapsto x/(1 - c N \phi(x))$.

**Corollary 2.3.** Let Assumptions [4][3] hold true. Let $\hat{C}_N$ be the solution of (2.1) when uniquely defined. Assume further that the empirical distribution $F_{\hat{N}}$ converges in distribution to $F^B$, a cumulative distribution function and $c_N \rightarrow c$. Set $\chi_\infty$ and $\gamma_\infty$ the unique solutions to the following system of equations:

$$\chi_\infty = \int_0^{\gamma_\infty} \int \frac{y}{1 + c N \psi(\chi_\infty + \tau_i \gamma_\infty)} \nu(dt) F^B(dy) \tag{2.3}$$

$$\gamma_\infty = \int_0^{\gamma_\infty} \frac{1}{1 + c N \psi(\chi_\infty + \tau_i \gamma_\infty)} \nu(dt) F^B(dy). \tag{2.4}$$

Then,

$$\left\| \hat{C}_N - S_N \right\| \xrightarrow{a.s.} 0$$

where $S_N = \frac{1}{n} \sum_{i=1}^{n} v(\chi_\infty + \tau_i \gamma_\infty) y_i y_i^*$.

**Proof.** Let $\delta_1, \ldots, \delta_n$ be the solution of the system of equations (2.2). Let $T_N$ be given by:

$$T_N = \left( \frac{1}{N} \sum_{j=1}^{n} \frac{v(\delta_j)(B_N + \tau_j I_N)}{1 + c N \psi(\delta_j)} \right)^{-1} \tag{2.2}$$

Let $\hat{\chi}_N = \frac{1}{N} \text{Tr} B_N T_N$ and $\hat{\gamma}_N = \frac{1}{N} \text{Tr} T_N$. Then,

$$\delta_j = \hat{\chi}_N + \tau_j \hat{\gamma}_N, \quad j = 1, \ldots, n.$$

Noticing that $\hat{\chi}_N$ and $\hat{\gamma}_N$ satisfy:

$$\hat{\chi}_N = \frac{1}{N} \text{Tr} B_N \left( \frac{1}{n} \sum_{j=1}^{n} \frac{v(\hat{\chi}_N + \tau_j \hat{\gamma}_N)(B_N + \tau_j I_N)}{1 + c N \psi(\hat{\chi}_N + \tau_j \hat{\gamma}_N)} \right)^{-1} \tag{2.3}$$

$$\hat{\gamma}_N = \frac{1}{N} \text{Tr} \left( \frac{1}{n} \sum_{j=1}^{n} \frac{v(\hat{\chi}_N + \tau_j \hat{\gamma}_N)(B_N + \tau_j I_N)}{1 + c N \psi(\hat{\chi}_N + \tau_j \hat{\gamma}_N)} \right)^{-1}, \tag{2.4}$$

it is not difficult to see that solving the system of the $n$ equations in (2.2) can be reduced to determining the solutions of a two equations system, whose solutions are $\hat{\chi}_N$ and $\hat{\gamma}_N$. The control of $\delta_j$ in Lemma 4.6 allow us to ensure that $\hat{\chi}_N$ and $\hat{\gamma}_N$ are uniformly bounded for enough large $n$ a.s. Hence, there exists a subsequence over which $\hat{\gamma}_N$ and $\hat{\chi}_N$ converge to $\gamma_\infty$ and $\chi_\infty$. Taking the limits of both sides of (2.3) and (2.4), we obtain

$$\chi_\infty = \int_0^{\gamma_\infty} \int \frac{y}{1 + c N \psi(\chi_\infty + \tau_i \gamma_\infty)} \nu(dt) F^B(dy) \tag{2.5}$$

$$\gamma_\infty = \int_0^{\gamma_\infty} \frac{1}{1 + c N \psi(\chi_\infty + \tau_i \gamma_\infty)} \nu(dt) F^B(dy). \tag{2.6}$$
Such limits are unique since the solutions of the systems of equations (2.5) and (2.6) are unique in case they exist. The existence and unicity of the solutions of (2.5) and (2.6) essentially relies on showing that the following function

\[ h : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \]

\[ (x_1, x_2) \mapsto (h_1(x_1, x_2), h_2(x_1, x_2)) \triangleq \left( \int_0^{\infty} \frac{v(x_1 + tx_2)}{1 + c\psi(x_1 + tx_2)} F_B(dy), \int_0^{\infty} \frac{v(x_1 + tx_2)(y + t)}{1 + c\psi(x_1 + tx_2)} F_B(dy) \right) \]

is a standard interference function \cite{20}, i.e. it satisfies the three conditions of positivity, monotonicity and scalability that have been used in the proof of Theorem 2.1.

\[ \Box \]

3. Numerical analysis

In order to assess the accuracy of our results, we represent in Fig. 1 the empirical estimate of the mean squared error (MSE) between the robust scatter estimate and \( \hat{S}_N \) with respect to \( N \)

\[ \text{MSE} = E \left\| \hat{S}_N - \hat{C}_N \right\|^2 \]

when \( n = 3N \) and \( B_N = A_N A_N^* \) with \( A_N \) is \( N \times \frac{N}{2} \) having independent standard Gaussian entries with zero mean and variance \( \frac{1}{K} \). We set \( u(t) = \frac{1 + \alpha}{1 + \alpha} \), and \( \alpha = 0.5 \). We note that the MSE decreases with \( N \), thereby supporting the convergence of \( \hat{C}_N \) to \( \hat{S}_N \).

\[ \text{Figure 1. MSE with respect to } N. \]

4. Proofs

4.1. Heuristic Analysis. The study of the asymptotic behaviour of robust scatter matrices requires careful attention. The difficulty essentially lies in the rank-1 matrices present in the sum of (2.1) being dependent through \( \hat{C}_N \). At first sight, this observation might make us
think that the asymptotic analysis of \( \hat{C}_N \) is out of the framework of the standard random matrix theory. However, a careful investigation of the expression of \( \hat{C}_N \) can lead us to replace \( \hat{C}_N \) by a random object, whose analysis using the theory of random matrices is quite standard.

Hereafter, we develop some heuristics that will lead to determine the asymptotic random equivalent of \( \hat{C}_N \). We believe that beyond their interest to the considered scenario, these heuristics can facilitate the understanding of the asymptotic behaviour of robust estimation techniques in the regime where the number of observations is of the same order of the size of the population covariance matrix.

Building on the ideas of [8], we will first start by deriving a new rewriting of \( \hat{C}_N \) that will also be extensively used in section 4.2 devoted to the exposition of the rigorous proofs. Let \( \hat{C}_{(i)} \) be the matrix \( \hat{C}_N \) where we remove \( \frac{1}{n}u((\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i)y_iy_i^* \), i.e.,
\[
\hat{C}_{(i)} = \hat{C}_N - \frac{1}{n}u\left(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i\right)y_iy_i^*.
\]
Applying the identity:
\[
(A - tzz^*)^{-1}z = \frac{A^{-1}z}{1 - tz^*A^{-1}z}
\]
for any invertible \( A \), vector \( z \) and scalar \( t \) such that \( A - tzz^* \) is invertible, we obtain:
\[
\frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i = \frac{\frac{1}{N}y_i^*\hat{C}_Ny_i}{1 - \frac{1}{n}y_i^*\hat{C}_N^{-1}y_iu(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i)}
= \frac{\frac{1}{N}y_i^*\hat{C}_Ny_i}{1 - c_N\phi(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i)}
= g_N\left(\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i\right),
\]
where \( g_N : [0, \infty) \to [0, \infty), x \mapsto \frac{x}{1 - c_N\phi(x)} \). As \( \phi \) is increasing and \( \phi_\infty < c_N^{-1} \), function \( g_N \) is positive increasing and maps \([0, \infty) \) to \([0, \infty) \). It is therefore invertible with inverse denoted by \( g_N^{-1} \). We have thus:
\[
\frac{1}{N}y_i^*\hat{C}_N^{-1}y_i = g_N^{-1}\left(\frac{1}{N}y_i^*\hat{C}_{(i)}^{-1}y_i\right).
\]
We can therefore express \( \hat{C}_N \) as:
\[
\hat{C}_N = \frac{1}{n} \sum_{j=1}^{n} \left(u \circ g_N^{-1}\right)\left(\frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j\right)y_jy_j^*
= \frac{1}{n} \sum_{j=1}^{n} v\left(\frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j\right)y_jy_j^*
\]
with \( v = u \circ g_N^{-1} \) positive and non-increasing.

This new rewriting of \( \hat{C}_N \) is of fundamental importance. It has two major advantages. First, it reveals that \( \hat{C}_N \) is uniquely determined by \( q_j = \frac{1}{N}y_j^*\hat{C}_{(j)}^{-1}y_j \), \( j = 1, \ldots, n \). This can be seen by noticing that a solution \( \hat{C}_N \) to \([2.1]\) exists and is unique if and only if the
the terms interesting insights about the asymptotic behaviour of \( \hat{C}_N \). In effect, it is not difficult to understand that \( y_i \) is weakly dependent on \( \hat{C}_N \), since \( \hat{C}_N \) depends on \( y_i \) only through the terms \( \frac{1}{N} y_i^* C_N^{-1} y_j, j \neq i \). Standard results from random matrix theory will thus lead to \( q_i = \frac{1}{N} y_i^* C_N^{-1} y_i \sim \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_N^{-1} \), which tends to imply that \( q_i \) scales with \( \tau_i \).

Assume that \( q_i, i = 1, \cdots, n \) can be approximated by \( \delta_i \) where \( \delta_i \) does not depend on the random vector \( w_i \). Then, because of rank-1 perturbation arguments leading to replace \( \hat{C}_N^{-1} \) with \( \hat{C}_N^{-1} \), we have:

\[
q_i = \frac{\delta_i}{\frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_N^{-1}(i)} \sim \frac{\delta_i}{\frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \hat{C}_N^{-1}}.
\]

On the other hand, from the asymptotic equivalence between \( q_i \) and \( \delta_i \), we expect \( \hat{C}_N \) to be asymptotically equivalent to \( \frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^* \). As we will see later, without inducing a major error, one can assume that \( y_i \), \( i = 1, \cdots, n \) are Gaussian. The asymptotic behaviour of \( \frac{1}{n} \sum_{i=1}^n v(\delta_i) y_i y_i^* \) can be thus studied using results from [19]. If Theorem 1 in [19] is applicable, then \( \delta_i \) should satisfy:

\[
1 \sim \frac{1}{N} \text{Tr} \left( \frac{B_N + \tau_i I_N}{\delta_i} \hat{C}_N^{-1} \right) \frac{1}{n} \sum_{j=1}^n v(\delta_j) (B_N + \tau_j I_N) (B_N + \tau_i I_N)^{-1}, \tag{4.1}
\]

where \( e_1, \cdots, e_n \) are the fixed point solutions to the following system of equations:

\[
e_i = \frac{v(\delta_i)}{n} \text{Tr}(B_N + \tau_i I_N) \left( \frac{1}{n} \sum_{j=1}^n v(\delta_j) (B_N + \tau_j I_N) \right)^{-1}.
\]

Multiplying both sides of (4.1), we thus get:

\[
e_i \sim c_N \delta_i v(\delta_i) = c_N v(\delta_i).
\]

Plugging the above equations into (4.1), we obtain that \( \delta_1, \cdots, \delta_n \) are solutions to the following system of equations:

\[
\delta_i = \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \left( \frac{1}{n} \sum_{j=1}^n v(\delta_j) (B_N + \tau_j I_N) \right)^{-1}, \quad i = 1, \cdots, n.
\]

4.2. Rigorous Proofs. The main differences of our work with respect to the one in [8] lies in the considered data model. While [8] assumes purely noise observations drawn from elliptical distributions, we consider in the present work, sequence of time observations that are given by the sum of a heavy-tailed noise and a Gaussian distributed vector modeling the "signal" part of the observations. In practice, the estimation of the covariance matrix
of the available observations can help infer precious information on the signal of interest. From a theoretical standpoint, if the useful data live in a low-dimensional space, the same approach considered in [8] can be pursued with only minor changes. Although less popular, high rank data models, occurring when $K$ scales with $N$, are more attractive for several applications of array processing concerning distributed source localization [17]. They are also more difficult to handle, since the use of the approach of [8] poses many technical difficulties, when $B_N$ is allowed to be of high rank. This can be easily seen by noticing that our heuristic computations involve solving a system of $n$ equations while those of [8] requires only solving the fixed point of a single equation. One can easily convince oneself that in the context of interest, it is much more difficult to get insights into the behaviour of the $n$ solutions of the underlying system. Before delving into the core of the proof, we need first to introduce in the sequel some preliminary results that will help adapt the techniques of [8] to our particular context.

4.2.1. Preliminary Results.

Function $u$ and Related Functions. The robust-scatter estimator is parametrized by function $u$, which significantly impacts its performance. This intuition is further confirmed by theoretical analysis, showing that a number of sequence of functions in relation with $u$ naturally arise. This section aims at presenting the list of these functions along with some of their most important properties. We first summarize in the following table some of the results that has been established in [8].

| Function $u(x)$ | Properties            |
|-----------------|-----------------------|
|                 | Non-increasing        |
|                 | Positive              |
|                 | Continuous            |
| $\phi(x) \triangleq xu(x)$ | Increasing            |
|                 | Positive              |
|                 | Continuous            |
|                 | Bounded with $\phi_{\infty} < c^{-1}$ |

| Sequence of Functions $g_N(x) \triangleq \frac{x}{1-c_N \phi(x)}$ | Properties            |
|---------------------------------------------------------------------|-----------------------|
|                                                                    | Increasing            |
|                                                                    | Positive              |
|                                                                    | Continuous            |
|                                                                    | Unbounded             |

| Sequence of Functions $v_N(x) \triangleq u \circ g_N^{-1}(x)$ | Properties            |
|-----------------------------------------------------------------|-----------------------|
|                                                                  | Non-Increasing        |
|                                                                  | Positive              |
|                                                                  | Continuous            |

| Sequence of Functions $\psi_N(x) \triangleq xv(x)$ | Properties            |
|----------------------------------------------------|-----------------------|
|                                                    | Increasing            |
|                                                    | Positive              |
|                                                    | Continuous            |
|                                                    | Bounded with $\psi_{\infty} = \frac{\phi_{\infty}}{1-c_N \phi_{\infty}}$ |

In addition to the aforementioned properties, we need to prove the following results, which will be used in our proofs.
Lemma 4.1. Let $u(\cdot)$ and $\phi(\cdot)$ be two functions satisfying assumption \[2\] Then, we have, for all $x, y \geq 0$,
\[
\frac{\phi(x) - \phi(y)}{x - y} \leq u(x) \leq u(0).
\]
In other words, $\phi(\cdot)$ is Lipschitz with Lipschitz constant $u(0)$.

Proof. We have:
\[
\frac{\phi(x) - \phi(y)}{x - y} = \frac{xu(x) - yu(y)}{x - y} = \frac{xu(x) - yu(x) + yu(x) - yu(y)}{x - y}.
\]
Since $u(\cdot)$ is non-increasing:
\[
yu(x) - u(y) \leq 0.
\]
Therefore,
\[
\frac{\phi(x) - \phi(y)}{x - y} \leq \frac{xu(x) - yu(x)}{x - y} = u(x) \leq u(0).
\]

Lemma 4.2. Let $u$ and $\phi$ be two functions satisfying assumption \[2\] Then, we have, for all $x \geq 0$:
\[
\phi(x) \leq u(0)x.
\]
Moreover, for all $x \in [0, m]$,
\[
u(m)x \leq \phi(x) \leq u(0)x.
\]

Proof. The first statement follows from the previous lemma by setting $y = 0$. To prove the second, notice that when $x \leq m$, $u(x) \geq u(m)$, thereby showing that $\phi(x) \geq xu(m)$ whenever $x \in [0, m]$.

Remark 4.1. As it has been proven in \[8\], functions $x \mapsto \psi(x)$ and $x \mapsto v(x)$ share respectively the same properties as $x \mapsto \phi(x)$ and $x \mapsto u(x)$. As a consequence, we can prove that $x \mapsto \psi(x)$ is Lipschitz with constant lipschitz $v(0) = u(0)$. The constant Lipschitz being independent on $n$, we conclude that $(\psi_N)$ form an equicontinuous family of functions and as such converge uniformly on $[0, \infty)$. Moreover,
\[
\psi(x) \leq v(0)x
\]
and $\psi(x) \geq v(m)x$ whenever $x \in [0, m]$.

Lemma 4.3. Let $g_N(\cdot) : x \mapsto \frac{x}{1 - c_N \phi(x)}$. Denote by $g_N^{-1}(\cdot)$ the inverse function corresponding to $g_N$. Then, for all $y \geq z \geq 0$, we have:
\[
g_N^{-1}(y) - g_N^{-1}(z) \leq (y - z)(1 - c_N \phi(g_N^{-1}(y)))
\]
In particular, $g_N^{-1}$ is lipschitz on $[0, \infty)$ with constant lipschitz $1$. Besides, functions $\left(x \mapsto \frac{\psi_N(x)}{1 + c_N \psi_N(x)}\right)_{N=1}^\infty$ are Lipschitz and converge uniformly on $[0, \infty)$. 

Proof. Let \( z \geq 0 \). From the relation \( g_N^{-1}(y) = y - c_N\phi(g_N^{-1}(y)) \), we have:
\[
g_N^{-1}(y) - g_N^{-1}(z) = y - z + c_N\phi(g_N^{-1}(z)) - c_Ny\phi(g_N^{-1}(y))
\]
\[
= y - z + c_N z \left( \phi(g_N^{-1}(z)) - \phi(g_N^{-1}(y)) \right) + c_N(z - y)\phi \left( g_N^{-1}(y) \right).
\]
Since \( g_N^{-1} \) is increasing, \( \phi(g_N^{-1}(z)) - \phi(g_N^{-1}(y)) \leq 0 \). Hence,
\[
g_N^{-1}(y) - g_N^{-1}(z) \leq (y - z)(1 - c_N\phi(g_N^{-1}(y))) \leq (y - z).
\]
Finally, after simple calculations, we can prove that:
\[
\frac{\psi_N(x)}{1 + c_N\psi_N(x)} = \phi \circ g_N^{-1}(x).
\]
Therefore, \( x \mapsto \frac{\psi_N(x)}{1 + c_N\psi_N(x)} \) is Lipschitz with constant lipschitz equal to \( u(0) \). This constant being independent on \( n \), the sequence of functions \( \frac{\psi_N(x)}{1 + c_N\psi_N(x)} \) converge uniformly on \( [0, \infty) \).

**Useful results.** As previously stated, the difficulty of studying the robust-scatter estimator lies in the control of the asymptotic behaviour of \( q_i \) and \( \delta_i \). The proof of Theorem 2.1 and Theorem 2.2 will require us to show that \( q_i \) and \( \delta_i \) scale with \( \tau_i \) and to control quadratic forms involving matrix \( \frac{1}{n} \sum_{i=1}^{n} f(\tau_i)y_iy_i^* \) where \( f \) is a certain functional. To this end, we develop in this section two key results that will underlie the proof of the main theorems.

**Proposition 4.4.** Let \( (B_N) \) be a sequence of \( N \times N \) hermitian positive matrices satisfying Assumption 4-iii. In addition, let \( \tau_i, i = 1, \ldots, n \) be positive random variables satisfying assumption 4-ii). Consider \( (f_N) \) a sequence of piece-wise continuous positive bounded functions defined on \( [0, \infty) \) that has at least one subsequence converging uniformly. We assume that functions \( t \mapsto f_N(t) \) satisfy the following additional properties:

- Function \( t \mapsto f_N(t) \) grows at most linearly, i.e there exists \( \alpha, \beta > 0 \) such that:
  \[
  \sup_N f_N(t) \leq \alpha \quad \forall t \geq 0,
  \]
  \[
  \sup_N f_N(t) \leq \beta t \quad \forall t \geq 0.
  \]
- \( \int f_N(t)\nu(dt) = 1 \)
- \( \liminf_N \inf_{t \in [0, \infty)} f_N(t) > 0 \).

Then the following equation in \( x \):
\[
\int \frac{F^{B_N}(dy)}{\int f_N(t)\nu(dt)} = 1 \quad (4.2)
\]
admits a unique positive solution which we denote by \( \eta_N \). Then, there exists a sequence \( (r^{-}_N) \) with \( \liminf r^-_N > 0 \) such that:
\[
r^{-}_N \frac{1}{N} \text{Tr } B_N \leq \eta_N \leq \frac{1}{N} \text{Tr } B_N.
\]
Moreover, we have:
\[
c_N - |\epsilon_{n,j}| \leq \frac{1}{N} \text{Tr } \frac{B_N + \tau_j I}{\tau_j + \eta_N} \left( \frac{1}{N} \sum_{i=1}^{n} f_N(\tau_i) \frac{B_N + \tau_i I}{\tau_i + \eta_N} \right)^{-1} \leq c_N + |\epsilon_{n,j}|, \quad (4.3)
\]
where \( \max_{1 \leq j \leq n} |\epsilon_{n,j}| \) converges almost surely to zero.
Proof: We start by showing that (4.2) admits a unique solution \( \eta_N \). It is clear that function \( h_N : x \mapsto \int \frac{F_B N(dy)}{\int m + \beta y f_N(t) \nu(dt)} \) is increasing and continuous on \((0, +\infty)\) with the limit at \( x \to 0^+ \) less than 1, while the limit when \( x \to +\infty \) is +\( \infty \). Therefore, there exists a unique \( \eta_N \) that satisfies (4.2). It is easy to check that \( \eta_N \) is less than the maximum eigenvalue of \( B_N \).

Therefore, we can restrict the domain of \( h_N \) to the set \([0, \|B_N\|]\). Since \( y \mapsto \int \frac{1}{\int m + \beta y f_N(t) \nu(dt)} \) is convex, applying the Jensen inequality, we obtain:

\[
\int \frac{F_B N(dy)}{\int m + \beta y f_N(t) \nu(dt)} \geq \frac{1}{\int \frac{\frac{\beta}{m} B_N + t}{t + x} f_N(t) \nu(dt)}.
\]

Setting \( x = \eta_N \), the above inequality becomes:

\[
1 \geq \frac{1}{\int \frac{\frac{\beta}{m} B_N + t}{t + \eta_N} f_N(t) \nu(dt)}.
\] (4.4)

Therefore, \( \eta_N \leq \frac{1}{N} \text{Tr} B_N \) because otherwise, (4.4) would not hold.

The proof of the lower-bound inequality is more delicate. Let \( m \) be as in Assumption [3] and denote by \( h_{m,N} \) the following map:

\[
h_{m,N} : [0, \|B_N\|] \to \mathbb{R}^+, \quad x \mapsto \frac{1}{\int y \int_0^\infty \frac{f_N(t)}{t + \mu} \nu(dt) + \int_0^\infty \frac{\frac{\beta}{m} f_N(t) T_N}{t + \mu} \nu(dt)} F_B N(dy).
\]

Functions \( h_N \) and \( h_{m,N} \) are both increasing, while \( h_{m,N}(x) \geq h_N(x) \quad \forall x \in [0, \|B_N\|] \).

Furthermore, we can easily check that:

\[
\lim_{x \to 0^+} h_{m,N}(x) < 1 \quad \text{and} \quad \lim_{x \to +\infty} h_{m,N}(x) = +\infty.
\]

Therefore, there exists \( \eta_{N,m} \) solution in \( x \) to the equation \( h_{m,N}(x) = 1 \). Moreover, we have \( h_N(\eta_{N,m}) \leq 1 \), and thus \( \eta_N \geq \eta_{N,m} \). On the other hand, \( h_{m,N} \) is differentiable with derivative \( h_{m,N}'(x) \) given by:

\[
h_{m,N}'(x) = \frac{y \int m^+ \int_{m+} f_N(t) \nu(dt) + \int_0^{+\infty} \frac{\beta}{m} f_N(t) T_N \nu(dt)}{\nu \left( \int_{m+} f_N(t) \nu(dt) + \int_0^{+\infty} \frac{\beta}{m} f_N(t) T_N \nu(dt) \right)^2} F_B N(dy).
\]

Let \( a_{m,N} = \int \frac{f_N(t)}{t} \nu(dt) \). Hence, if \( 0 \leq x \leq \|B_N\| \),

\[
h_{m,N}'(x) \leq \frac{y \int m^+ \int_{m+} f_N(t) \nu(dt) + \int_0^{+\infty} \frac{\beta}{m} f_N(t) T_N \nu(dt)}{\nu \left( \int_{m+} f_N(t) \nu(dt) + \int_0^{+\infty} \frac{\beta}{m} f_N(t) T_N \nu(dt) \right)^2} F_B N(dy) \]

\[
\leq \frac{\beta \frac{\|B_N\|^2}{m} + 1}{m^2 a_{m,N}^2}.
\]

The mean value theorem implies that:

\[
1 - h_{m,N}(0) \leq \frac{\beta \left( \frac{\|B_N\|^2}{m} + 1 \right)^3}{m^2 a_{m,N}^2},
\] (4.5)
where

\[ h_{m,N}(0) = \int y \frac{1}{\int_{m}^{\infty} \frac{f_N(y)}{t} + 1} F^{B_N}(dy) = \frac{1}{N} \text{Tr} (a_{m,N} B_N + I_N)^{-1}. \]

As a consequence,

\[ 1 - h_{m,N}(0) = \frac{1}{N} \text{Tr} I_N = \frac{1}{N} \text{Tr} (a_{m,N} B_N + I_N)^{-1} \]

\[ = \frac{a_{m,N}}{N} \text{Tr} B_N (a_{m,N} B_N + I_N)^{-1} \geq \frac{a_{m,N}}{a_{m,N} \| B_N \| + 1}. \quad (4.6) \]

Combining (4.5) and (4.6), we therefore get:

\[ \eta_N \geq \frac{m^2 a_{m,N}}{(a_{m,N} \| B_N \| + 1) (1 + \| B_N \|)^3} \beta. \]

Note that it is easy to prove that \( r_N \triangleq \frac{m^2 a_{m,N}^3}{(a_{m,N} \| B_N \| + 1) (1 + \| B_N \|)^3} \beta \leq 1 \), since \( ma_{m,N} \leq 1 \) and \( \frac{a_{m,N}}{\beta} < 1 \). Moreover, \( \text{lim inf} \ r_N > 0 \) as \( r_N \geq \frac{m^2 a_{m,N}^3}{\beta (\| B_N \| + 1)} \) and \( \text{lim inf} \ a_{m,N} \geq \text{lim inf} \inf_{x \in [m, \infty)} f_N(x) \int_{m}^{\infty} \frac{1}{t} \nu(dt) > 0 \).

We will now proceed proving the inequalities in (4.3). Let \( \lambda_1^N \leq \cdots \leq \lambda_N^N \) be the eigenvalues of \( B_N \). We have:

\[ \frac{1}{N} \text{Tr} B_N + \tau_j I \left( \frac{1}{N} \sum_{i=1}^{n} f_N(\tau_i) \frac{B_N + \tau_i I}{\tau_i + \eta_N} \right)^{-1} = \frac{1}{N} \sum_{k=1}^{n} \frac{\lambda_k^N + \tau_j}{\tau_j + \eta_N} \sum_{i=1}^{n} f_N(\tau_i) \frac{\lambda_k^N + \tau_j}{\tau_i + \eta_N} \]

\[ = \frac{e_N}{\tau_j + \eta_N} \int \frac{y F^{B_N}(dy)}{\tau_j + \eta_N} + \frac{c_N \tau_j}{\tau_j + \eta_N} \int \frac{F^{B_N}(dy)}{\tau_j + \eta_N} + \frac{e_N}{\tau_j + \eta_N} \int \frac{dF^{B_N}(t)}{\tau_j + \eta_N} \]

\[ \epsilon_{n,1,2}. \quad (4.7) \]

where

\[ \epsilon_{n,1} = \int y \left( \frac{1}{n} \sum_{i=1}^{n} f_N(t) \frac{\tau_i + \eta_N}{\tau_i + \eta_N} \right) F^{B_N}(dy) \quad (4.8) \]

\[ \epsilon_{n,2} = \int \left( \frac{1}{n} \sum_{i=1}^{n} f_N(t) \frac{\tau_i + \eta_N}{\tau_i + \eta_N} \right) F^{B_N}(dy). \quad (4.9) \]

As \( \eta_N \) is the unique solution of (4.2), \( \int \frac{y F^{B_N}(dy)}{\tau_j + \eta_N} f_N(t) \nu(dt) \) can be further simplified as:

\[ \int \frac{y F^{B_N}(dy)}{\tau_j + \eta_N} f_N(t) \nu(dt) = \frac{y \int f_N(t) \nu(dt)}{\int f_N(t) \nu(dt)} \int f_N(t) \nu(dt) + \frac{1}{\int f_N(t) \nu(dt)} \int f_N(t) \nu(dt) \int \frac{y F^{B_N}(dy)}{\tau_j + \eta_N} f_N(t) \nu(dt) \]

\[ = \eta_N, \quad (4.10) \]
where (a) follows due to the fact that $\int_{\mathbb{R}^d} \frac{1}{1 + t + \eta N} f_N(t) \nu(dt) F^B_N(dy) = 1$. Substituting (4.10) into (4.7), we get:

$$\frac{1}{N} \text{Tr} \left[ B_N + \tau_j I \right] \left( \frac{1}{N} \sum_{i=1}^{n} f(\tau_i) B_N + \tau_i I \right) \left( \frac{1}{\tau_i + \eta N} \right)^{-1} \leq c_N + \frac{c_N \tau_j}{\tau_j + \eta N} \epsilon_{n,1} + \frac{cN \tau_j}{\tau_j + \eta N} \epsilon_{n,2}.$$

Therefore, the result immediately follows once we prove that $\max_{1 \leq j \leq n} \frac{1}{\tau_j + \eta N} |\epsilon_{n,1}|$ and $|\epsilon_{n,2}|$ converge almost surely to zero. We will only control $\frac{1}{\tau_j + \eta N} \epsilon_{n,1}$. The control of $\epsilon_{n,2}$ can be obtained using the same arguments. We have:

$$\frac{|\epsilon_{n,1}|}{\tau_j + \eta N} \leq \sup_y \left| \int \frac{y + t}{t + \eta N} f_N(t) \nu(dt) - \frac{1}{n} \sum_{i=1}^{n} f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta N} \right| \times \frac{1}{\eta N} \int \frac{y F^B_N(dy)}{\left( \frac{1}{n} \sum_{i=1}^{n} f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta N} \right) \int \frac{y + t}{t + \eta N} f_N(t) \nu(dt)}.$$

Since $f_N(t) \leq \frac{\beta t}{t + \eta N} \leq \beta$ and $\frac{f_N(t)}{t + \eta N} \leq \frac{t}{t + \eta N} \leq \alpha$, sequences $\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt)$ and $\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt)$ are bounded. One can extract a subsequence $(n)$ such that: $\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt)$ and $\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt)$ converge. Over this subsequence, $t \mapsto \frac{f_N(t)}{t + \eta N}$ converge uniformly to $t \mapsto \frac{f(t)}{t + \eta N}$ and as such:

$$\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt) \xrightarrow{a.s.} 0.$$

Moreover, since $\liminf \eta_N \geq \liminf r_N \frac{1}{N} \text{Tr} B_N > 0$, $\eta \neq 0$. The sequence of functions $t \mapsto \frac{f_N(t)}{t + \eta N}$ converge also uniformly to $t \mapsto \frac{f(t)}{t + \eta N}$, thereby yielding:

$$\int \frac{f_N(t)}{t + \eta N} \nu_N(dt) - \int \frac{f(t)}{t + \eta N} \nu(dt) \xrightarrow{a.s.} 0.$$

It remains thus to check that $\frac{1}{\eta N} \int_{\mathbb{R}^d} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{f_N(\tau_i)}{\tau_i + \eta N} \int \frac{y + t}{t + \eta N} f_N(t) \nu(dt) \right) \frac{y F^B_N(dy)}{\int \frac{y + t}{t + \eta N} f_N(t) \nu(dt)}$ is almost surely bounded.

For that, first note that since $\eta_N = \int \frac{y}{t + \eta N} f_N(t) \nu(dt)$, we have:

$$\frac{1}{\eta N} \int \frac{1}{n} \sum_{i=1}^{n} f_N(\tau_i) \frac{y + \tau_i}{\tau_i + \eta N} \int \frac{y + t}{t + \eta N} f_N(t) \nu(dt) dF^B_N \leq \frac{1}{n} \sum_{i=1}^{n} \frac{f_N(\tau_i)}{\tau_i + m + \eta N} f_N(t) m + \eta_N.$$

As $\eta_N \leq \lambda N$ and $\liminf_{t \in [m, +\infty)} f_N(t) > 0$, $\frac{1}{n} \sum_{i=1}^{n} \frac{f_N(\tau_i)}{\tau_i + m + \eta N}$ is almost surely bounded away from zero, thereby implying the desired result. The control of $\epsilon_{n,2}$ could be done using the same arguments. □
The second ingredient that will be of extensive use in the proof of the theorem is provided by the following key-lemma.

**Lemma 4.5.** Let Assumptions [4.14] hold true. Let \((f_N)\) be a sequence functions satisfying the conditions of proposition [4.4]. Denote by \(\eta_N\) the unique solution in \(x\) to the following equation:

\[
\int_0^\infty \frac{F_{BN}}{F_{BN}(t)\nu(dt)} = 1.
\]

Consider \(e_1, \cdots, e_n\) the unique solutions to the following system of equations:

\[
e_k = \frac{f_N(\tau_k)}{n} \text{Tr} \frac{B_N + \tau_k I_N}{\tau_k + \eta_N} \left( \frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1}.
\]

Then, the following statements hold true:

i) \(\max_{1 \leq j \leq n} \left| \frac{f_N(\tau_j)}{N(\tau_j + \eta_N)} y_j^* \left( \frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j - e_N^{-1} e_j \right| \to 0, \ a.s.

ii) If \(T_N = \left( \frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)}{\tau_i + \eta_N} \right)^{-1}\). Then, we have:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{N(\tau_j + \eta_N)} y_j^* \left( \frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right)^{-1} y_j - \frac{1}{N} \text{Tr} \frac{(B_N + \tau_j I_N)}{\tau_j + \eta_N} T_N \right| \to 0, \ a.s.
\]

iii) If \(\|f_N\|_\infty < \frac{1}{e_N}\), then, there exists \(\epsilon_n \downarrow 0\) such that for \(n\) large enough, \(a.s.

\[
\max_{1 \leq k \leq n} e_k \leq \frac{\epsilon_n \|f_N\|_\infty}{1 - \|f_N\|_\infty e_N} + \epsilon_n.
\]

**Proof.** The proof of the first two items is based on Lemma [4.15] and Lemma [4.16] in Appendix 4.2.2. For these Lemmas to be applicable, we need to check that \(\lim \inf_{N} \min_{1 \leq j \leq n} \lambda_1 \left( \frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right) > 0\). To this end, first note that:

\[
\min_{1 \leq j \leq n} \lambda_1 \left( \frac{1}{n} \sum_{i \neq j} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right) \geq \min_{1 \leq j \leq n} \lambda_1 \left( \frac{1}{n} \sum_{i \neq j, \tau_i \geq m} \frac{f_N(\tau_i)}{\tau_i + \eta_N} y_i y_i^* \right).
\]

The right-hand side of the above equality is almost surely bounded above zero since \(f_N(\tau_i)^{\frac{B_N + \tau_j I_N}{\tau_j + \eta_N}} \geq \inf_{t \in [m, \infty)} f_N(t)^{\frac{m}{m + \eta_N}}\) and \(\eta_N\) is almost surely bounded by proposition 4.4. We conclude thus by resorting to Lemma 4.14 in Appendix 4.2.2.

In order to prove the last statement, let \(j_0\) be the index of the maximum element in \(\{e_1, \cdots, e_n\}\). We therefore have:

\[
e_{j,0} \leq \|f_N\|_\infty (1 + e_{j_0}) \frac{1}{n} \text{Tr} \frac{(B_N + \tau_{j_0} I_N)}{\tau_{j_0} + \eta_N} \left( \frac{1}{n} \sum_{i=1}^n \frac{f_N(\tau_i)}{\tau_i + \eta_N} \right)^{-1}
\]

\[
(\text{a}) \leq \|f_N\|_\infty (1 + e_{j_0}) (e_N + \epsilon_n),
\]
where (a) follows from proposition [4.4]. Besides, scalars \( e_1, \ldots, e_n \) being the limits of almost surely bounded random quantities are bounded. Therefore,

\[
e_{j_0} \leq \frac{\|f_N\|_{\infty}e_N}{1 - \|f_N\|_{\infty}e_N} + \epsilon_n'
\]

where \( \epsilon_n' \downarrow 0 \).

4.2.2. Proof of the Main Theorems. With the above preliminary results at hand, we are now in position to provide the proofs of Theorem [2.1] and Theorem [2.2].

Proof of Theorem 2.1: Asymptotic Existence of the Robust Scatter Estimator. Theorem [2.1] establishes the existence of the robust scatter estimate for large \( n \) and \( N \). In particular, it implies that for each realization, there exists \( n_0 \) and \( N_0 \) large such that for all \( n \) and \( N \) greater than \( n_0 \) and \( N_0 \), equation (2.1) admits a unique solution. Although we believe that a stronger result showing the existence of the robust scatter estimate for well-behaved set of samples can be established using probably the same kind of techniques as in [5], we have chosen in this paper to show Theorem 2.1 under the setting of the asymptotic regime. The reason is that the techniques used in that proof will be key to understanding some aspects of the asymptotic behaviour of the robust scatter estimate, thereby paving the way towards the proof of Theorem 2.2.

The proof of Theorem 2.1 follows the same lines as in [8]. Define \( h = (h_1, \ldots, h_n) \) with:

\[
h_j : \mathbb{R}_+^n \to \mathbb{R}_+
\]

\[
(x_1, \ldots, x_n) \mapsto \frac{1}{N} y_j^* \left( \frac{1}{n} \sum_{i=1}^{n} v(x_i)y_i y_i^* \right)^{-1} y_j.
\]

As it has already been mentioned, in order to prove that \( C_N \) is uniquely defined for \( n \) large enough a.s., it suffices to show that the system of equations in \( x_1, \ldots, x_n \)

\[
x_j = h_j(x_1, \ldots, x_n), j = 1, \ldots, n
\]

admits a unique solution \( q_1, \ldots, q_n \) a.s. for \( n \) large enough. To this end, we will show that \( h \) is a standard interference function, i.e, it satisfies the following three conditions:

a) Positivity: For each \( q_1, \ldots, q_n \geq 0 \), and each \( i, h_i(q_1, \ldots, q_n) > 0 \),

b) Monotonicity: For each \( q_1 \geq q_1', \ldots, q_n \geq q_n' \) and each \( i, h_i(q_1, \ldots, q_n) \geq h_i(q_1', \ldots, q_n') \),

c) Scalability: For all \( \alpha > 1 \), and \( q_1, \ldots, q_n \geq 0, \alpha h_i(q_1, \ldots, q_n) > h_i(\alpha q_1, \ldots, \alpha q_n) \).

Item a) can be easily shown by noticing that matrix \( \frac{1}{n} \sum_{i=1, i \neq j}^{n} v(q_i)y_i y_i^* \) is invertible almost surely and is positive definite, while the monotonicity follows immediately from the fact that \( h_j \) is non-decreasing of each \( q_i \). As for the scalability, we can assume without loss of generality that there exists \( q_i > 0 \) as the results holds trivially when \( q_1 = \cdots = q_n = 0 \). With this assumption at hand, we rewrite \( h_j(q_1, \ldots, q_n) \) as:

\[
h_j(q_1, \ldots, q_n) = \frac{1}{N} y_j^* \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi_N(q_i)}{q_i} y_i y_i^* \right)^{-1} y_j.
\]
As $\psi_N$ is increasing, $\psi_N(\alpha q) > \psi_N(q)$ for $\alpha > 1$ and $q > 0$. Hence,

$$h_j(\alpha q_1, \ldots, \alpha q_n) = \frac{\alpha}{N} y_j^* \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi_N(\alpha q_i)}{q_i} y_i y_j^* \right)^{-1} y_j.$$

If there exists at least $q_i > 0$, we therefore get:

$$h_j(\alpha q_1, \ldots, \alpha q_n) < \frac{\alpha}{N} y_j^* \left( \frac{1}{n} \sum_{i=1, i \neq j}^{n} \frac{\psi_N(q_i)}{q_i} y_i y_j^* \right)^{-1} y_j = \alpha h_j(q_1, \ldots, q_n).$$

We have thus established that $h$ is a standard interference function. Referring to the results of [20], it remains to show that there exists vector $(q_1, \ldots, q_n)$ such that for all $i = 1, \ldots, n$, $q_i > h_i(q_1, \ldots, q_n)$ a.s. for $n$ large enough, a statement which is known as the feasibility condition.

In order to establish the feasibility condition, let $q^*_N$ be chosen so that:

$$\int_0^{+\infty} \psi_N(q^*_N t) \nu(dt) = \frac{1 + \kappa}{1 - c_N \phi_\infty},$$

for some sufficiently small $0 \leq \kappa \leq 1$ satisfying:

$$1 + \kappa \leq \phi_\infty (1 - \nu \{0\}).$$

This is possible since,

$$\lim_{q \to +\infty} \int_0^{+\infty} \psi_N(q t) \nu(dt) = \psi_\infty (1 - \nu \{0\}) = \frac{\phi_\infty (1 - \nu \{0\})}{1 - c_N \phi_\infty}.$$ 

We will prove that $q^*_N$ is a bounded sequence. To this end, we will proceed by contradiction. Assume that there exists a sequence $(n)$ such that $\lim_{n \to +\infty} q(n) = +\infty$. Since the sequence of functions $\psi_N$ converge uniformly, one can extract a subsequence $(p)$ from $(n)$ such that: $c(p) \to c^*$ and $\psi(p) \to \phi_\infty$. Therefore, the sequence of functions $(f_p : q \mapsto \int_0^{+\infty} \left( 1 - c(p) \phi_\infty \right) \psi(p)(qt) \nu(dt))$ converge uniformly to $f^* : q \mapsto \int_0^{+\infty} (1 - c^* \phi_\infty) \psi^*(qt) \nu(dt)$. Hence,

$$\lim_{n \to +\infty} f_p(q) \to \lim_{x \to +\infty} f^*(x) = \psi_\infty (1 - c^* \phi_\infty) (1 - \nu \{0\}) = \phi_\infty (1 - \nu \{0\}) > 1 + \kappa,$$

which is in contradiction with the fact that:

$$\int_0^{+\infty} (1 - c_N \phi_\infty) \psi_N(q^*_N t) \nu(dt) = 1 + \kappa.$$ 

Now, consider $\eta_N$ the unique solution of:

$$1 = \int \frac{\int_0^{+\infty} \psi_N(q^*_N x) \nu(dx) F^{B_N}(dy)}{\int_0^{+\infty} \psi_N(q^*_N t) \nu(dt)} \nu(dt).$$

Such $\eta_N$ exists and is unique by Proposition 4.4. Set $q_i = q^*_N (\tau_i + \eta_N)$. We will prove that this choice of $q_j, j = 1, \ldots, n$, guarantees:

$$\frac{h_j(q_1, \ldots, q_n)}{q_j} \leq 1,$$
Let $j_0$ be the index of the maximum element in \{e_1, \ldots, e_n\}. Then, there exists $\epsilon_n \downarrow 0$ such that for all $j = 1, \ldots, n$

$$h_j(q_1, \ldots, q_n) = \frac{1}{q_j} \frac{1}{N(\tau_j + \eta_N)} \frac{1}{y_j} \left( \frac{1}{n} \right) \left( \frac{1}{\sum_{i \neq j} \psi_N(q_j^+ \tau_i)} y_i y_i^* \right) \frac{1}{y_j} \leq \frac{1}{N(\tau_j + \eta_N)} \int_0^{\infty} \psi_N(q_j^+ x) \nu(dx) y_j \left( \frac{1}{n} \right) \left( \frac{1}{\sum_{i \neq j} \psi_N(q_j^+ \tau_i)} y_i y_i^* \right) \frac{1}{y_j},$$

where $\overline{\psi}_N^+(x) = \frac{\psi_N(q_j^+ x)}{\psi_N(q_j^+ t) \nu(dt)}$. From item \textit{iii)} of Lemma 4.5, we have:

$$\max_{1 \leq k \leq n} \left| \frac{1}{N(\tau_j + \eta_N)} \frac{1}{y_j} \left( \frac{1}{n} \right) \left( \frac{1}{\sum_{i \neq j} \overline{\psi}_N^+(q_j^+ \tau_i)} \right) y_i y_i^* \right| \to 0,$$

where $T_N = \left( \frac{1}{n} \sum_{i=1}^n \overline{\psi}_N^+(\tau_i) (B_N + \tau_i I_N) \right)^{-1}$ with $e_1, \ldots, e_n$ are the unique solutions to the following system of equations:

$$e_k = \frac{\overline{\psi}_N^+(\tau_k)}{n} \left( \frac{1}{\sum_{i=1}^n \overline{\psi}_N^+(\tau_i) (B_N + \tau_i I_N)} \right)^{-1} \left( B_N + \tau_k I_N \right) \left( \frac{1}{\sum_{i=1}^n \overline{\psi}_N^+(\tau_i) (B_N + \tau_i I_N)} \right)^{-1} \left( \frac{1}{n} \right) \left( \frac{1}{\sum_{i \neq j} \overline{\psi}_N^+(q_j^+ \tau_i)} \right) \frac{1}{y_i y_i^*}.$$

As $\|\overline{\psi}_N^+\|_{\infty} = \frac{\psi_{\infty}}{\int_0^{\infty} \psi_N(q_j^+ t) \nu(dt)} = \frac{\phi_{\infty}}{1 + \kappa} \leq \frac{1}{n (1 + \kappa)}$, we obtain from item \textit{iii)} of Lemma 4.5:

$$\max_{1 \leq k \leq n} e_k \leq \frac{c_N}{1 - c_N} \|\overline{\psi}_N^+\|_{\infty} + \epsilon_n = \frac{c_N \phi_{\infty}}{1 + \kappa - c_N \phi_{\infty}} + o(1).$$

where $o(1)$ refers to some sequences converging almost surely to zero as $n$ grow to infinity. Plugging (4.14) into (4.13), and using the fact that:

$$\left( \frac{1}{N} \right) \left( \frac{1}{\sum_{i=1}^n \overline{\psi}_N^+(\tau_i) (B_N + \tau_i I_N)} \right)^{-1} \leq 1 + o(1),$$

we finally get:

$$h_j(q_1, \ldots, q_n) \leq \frac{1 - c_N \phi_{\infty}}{1 + \kappa - c_N \phi_{\infty}} + o(1),$$

thereby establishing that:

$$h_j(q_1, \ldots, q_n) < q_j$$

a.s. for $n$ large enough.
Proof of Theorem 2.2: Asymptotic Convergence of the Robust-Scatter Estimator. The proof of Theorem 2.2 heavily relies on the new rewriting of the robust-scatter estimate as:

\[ \hat{C}_N = \frac{1}{n} \sum_{i=1}^{n} v(q_i) y_i y_i^*, \]  

(4.15)

where \( q_1, \ldots, q_n \) are the unique solutions of the following system of equations:

\[ q_j = y_j^* \left( \frac{1}{n} \sum_{i=1, i \neq j}^{n} v(q_i) y_i y_i^* \right)^{-1} y_j, \]

their existence and uniqueness in the asymptotic regime being established in the proof of Theorem 2.1. From the rewriting of \( \hat{C}_N \) in (4.15), it appears that an in-depth study of the asymptotic behaviour of \( q_1, \ldots, q_n \) can be a good starting point. As mentioned in our heuristic analysis, one intuitively expects the \( q_1, \ldots, q_n \) to approach in the asymptotic regime \( \delta_1, \ldots, \delta_n \), the solutions of the following system of equations:

\[ \delta_i = \frac{1}{N} \text{Tr}(B_N + \tau_i I_N) \left( \frac{1}{n} \sum_{j=1}^{n} \frac{v(\delta_j)(B_N + \tau_j I_N)}{1 + c_N \psi_N(\delta_j)} \right)^{-1}. \]

This intuition underlies the proof of Theorem 2.2. In particular, we will prove that:

\[ f_i = \frac{v(q_i)}{v(\delta_i)}, \quad i = 1, \ldots, n \]

satisfy:

\[ \max_{1 \leq i \leq n} |f_i - 1| \xrightarrow{\alpha_n} 0. \]  

(4.16)

This in particular will allow us to state that \( \hat{C}_N \) can be approximated by \( \hat{S}_N = \frac{1}{n} \sum_{i=1}^{n} v(\delta_i) y_i y_i^* \).

The importance of this finding lies in the fact that unlike \( \hat{C}_N \), \( \hat{S}_N \) follows a classical random matrix model, thereby opening up possibilities of exploiting an important load of available results. Prior to proceeding into the proof of the convergence stated in (4.16), we first need to introduce the following key lemmas that allow to identify the intervals within which lie almost surely quantities \( q_1, \ldots, q_n \) and \( \delta_1, \ldots, \delta_n \). We start by handling terms \( \delta_1, \ldots, \delta_n \).

We have in particular the following Lemma:

Lemma 4.6. Let:

\[ h_j : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \]

\[ (x_1, \ldots, x_n) \mapsto \begin{cases} \frac{1}{N} \text{Tr}(B_N + \tau_j I) \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(x_i)(B_N + \tau_i I)}{x_i(1 + c_N \psi(x_i))} \right)^{-1} & \text{if } \exists x_i \neq 0 \\ \frac{1}{N} \text{Tr}(B_N + \tau_j I) \left( \frac{1}{n} \sum_{i=1}^{n} v(0)(B_N + \tau_i I) \right)^{-1} & \text{otherwise.} \end{cases} \]

Then, for all large \( n \), there exists a unique vector \( (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n_+ \) such that:

\[ h_j(\delta_1, \ldots, \delta_n) = \delta_j, \quad \forall \ 1 \leq j \leq n. \]  

(4.17)

Besides, vector \( (\delta_1, \ldots, \delta_n) \) is given by:

\[ (\delta_1, \ldots, \delta_n) = \lim_{t \to +\infty} (\delta_1^t, \ldots, \delta_n^t), \]
Moreover, \( \eta \) that, almost surely for \( n \) large enough, we have:

\[
\delta_N(\tau_j + \eta_N) \leq \delta_j \leq \delta_N^+(\tau_j + \eta_N^+), \quad 1 \leq j \leq n.
\]

Moreover, \( \eta_N^+ \) and \( \eta_N^- \) satisfy:

\[
\eta_N^+ = O\left(\frac{1}{N} \text{Tr} B_N\right) \quad \text{and} \quad \eta_N^- = O\left(\frac{1}{N} \text{Tr} B_N\right).
\]

**Proof.** Similar to the proof of Theorem 2.1, we can show along the same lines that \( h = (h_1, \ldots, h_n) \) is a standard interference function. It remains to prove the existence of \((\delta_1, \ldots, \delta_n)\) such that \( h_j(\delta_1, \ldots, \delta_n) < \delta_j \). To this end, take \( \delta_j = \xi(\tau_j + \eta_N) \), where \( \eta_N \) is the unique solution to the following equation:

\[
\int \frac{E^{B_N}(dy)}{\int_{\mathbb{R}^3} \nu(dt)} = 1.
\]

Such \( \eta_N \) exists according to proposition 4.4. Then:

\[
\lim_{\xi \to +\infty} \frac{h_j(\delta_1, \ldots, \delta_n)}{\delta_j} = \frac{1 + c_N \psi_{\infty}}{\psi_{\infty}} \frac{1}{N} \text{Tr} \left(\frac{B + \tau_j I}{\tau_j + \eta_N} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{B + \tau_i I}{\tau_i + \eta_N} \right)^{-1}.
\]

(4.18)

Again, the limit in the above equation (4.18) can be controlled using proposition 4.4, thereby yielding:

\[
\lim_{\xi \to +\infty} \frac{h_j(\delta_1, \ldots, \delta_n)}{\delta_j} \leq \frac{1 + c_N \psi_{\infty}}{\psi_{\infty}} + \epsilon_n = \frac{1}{\phi_{\infty}} + \epsilon_n
\]

where \( \epsilon_n \downarrow 0 \) almost surely. As \( \phi_{\infty} > 1 \), one can conclude that there exists \( \delta_1, \ldots, \delta_n \), such that for enough large \( N \),

\[
h_j(\delta_1, \ldots, \delta_n) < \delta_j.
\]

We are now in position to prove the uniform boundedness of \( \delta \). For that, consider \( \theta > 0 \) such that \( \theta < \frac{\phi_{\infty}}{1 + c_{-} \phi_{\infty}} \). Let \( M_\theta \) be chosen such that \( \nu(M_\theta, +\infty) < \theta \) and \( M_\theta \) is greater than the limit support of \( \|B_N\| \). Set \( \delta_N^- \) and \( \delta_N^+ \) so that the following conditions are fulfilled:

\[
\int_{0}^{\infty} \frac{\psi_{N}(\delta_N^+(t))}{1 + c_N \psi(\delta_N^+(t))} \nu(dt) > 1, 
\]

(4.19)

\[
\int_{0}^{M_\theta} \psi_{N}(\delta_N^-(t + M_\theta)) \nu(dt) < \frac{1}{2}.
\]

(4.20)

Such choices are possible since

- \( \lim_{\delta^+ \to +\infty} \int_{0}^{\infty} \frac{\psi_{N}(\delta^+(t))}{1 + c_N \psi(\delta^+(t))} \nu(dt) = (1 - \nu(\{0\})) \psi_{\infty} = (1 - \nu(\{0\})) \phi_{\infty} > 1, \)

- \( \lim_{\delta^- \to 0^+} \int_{0}^{M_\theta} \psi(\delta^-(t + M_\theta)) \nu(dt) = 0, \)
Moreover, we can check that one can choose \( \delta_{N}^- \) and \( \delta_{N}^+ \) such that \( \liminf_{N} \delta_{N}^- > 0 \) and \( \limsup_{N} \delta_{N}^+ < +\infty \). As a matter of fact, building on the same reasoning used to show that \( \limsup_{N} \alpha_{N} < +\infty \) in the proof of Theorem 2.1, we take \( \delta_{N}^+ \) and \( \delta_{N}^- \) the positive reals that verify:

\[
\int_{0}^{+\infty} \frac{\psi(\delta_{N}^+ t)}{1 + c_N \psi(\delta_{N}^+ t)} \nu(dt) = 1 + \kappa \\
\int_{0}^{M_0} \psi \left( \delta_{N}^-(t + M_0) \right) \nu(dt) = \kappa,
\]

where \( 0 \leq \kappa < \frac{1}{2} \) satisfies \( 1 + \kappa < (1 - \nu \{ 0 \}) \phi_{\infty} \). Assume that \( \liminf_{N} \delta_{N}^- = 0 \). There exists a sequence \( (n) \) such that \( \lim_{n \to +\infty} \delta_{N}^- = 0 \). Since the sequence of functions \( \psi_N \) converge uniformly, one can extract a subsequence \( (p) \) from \( (n) \) such that \( c(p) \to c^* \) and \( \psi_p \) converge uniformly to \( \psi^* \). Therefore, the sequence of functions \( f_{p} : \alpha \mapsto \int_{0}^{M_0} \psi (\alpha (t + M_0)) \nu(dt) \) converge uniformly to \( f^* : \alpha \mapsto \int_{0}^{M_0} \psi (\alpha (t + M_0)) \nu(dt) \). Hence:

\[
\lim_{n \to +\infty} f_{p}(\delta_{N}^-) \to \lim_{x \to 0} f^*(x) = 0.
\]

which is in contradiction with the fact that \( f_{p}(\delta_{N}^-) = \kappa \). The same method can be used to prove that \( \limsup_{N} \delta_{N}^- < \infty \). Consider now the function \( f^+ : t \mapsto \frac{\psi(\delta_{N}^+ t)}{1 + c_N \psi(\delta_{N}^+ t)} \) in the domain \( t \in [0, \infty) \). Define \( \eta_{N}^+ \) the unique solution to the following equation:

\[
1 = \int \frac{F_{B_N} \left( dy \right) \int_{0}^{+\infty} f^+(x) \nu(dx)}{\int_{0}^{+\infty} \frac{y \nu(t)}{\psi(t) + \eta_{N}^+}} f^+(t) \nu(dt).
\]

Similarly, define on \( \mathbb{R}^+ \) the function \( f^- : t \mapsto \psi_{\infty} 1_{\{ t \geq M_0 \}} + \psi(2\delta^-(t + M_0)) 1_{\{ t < M_0 \}} \). Let \( \eta_{N}^- \) be the unique solution to the following equation:

\[
1 = \int \frac{F_{B_N} \left( dy \right) \int_{0}^{+\infty} f^-(x) \nu(dx)}{\int_{0}^{+\infty} \frac{y \nu(t)}{\psi(t) + \eta_{N}^-}} f^-(t) \nu(dt).
\]

Note that from proposition 4.4 \( \eta_{N}^+ \) and \( \eta_{N}^- \) are well-defined and satisfy:

\[
\eta_{N}^+ = \mathcal{O} \left( \frac{1}{N} \operatorname{Tr} B_N \right), \quad \eta_{N}^- = \mathcal{O} \left( \frac{1}{N} \operatorname{Tr} B_N \right).
\]

Set for all \( i \), \( \delta_{i}^0 = \delta_{N}^-(\tau_i + \eta_{N}^+) \). Define recursively the sequences:

\[
\delta_{i}^{j+1} = \frac{1}{N} \operatorname{Tr} \left( B_N + \tau_j I_N \right) \left( \frac{1}{n} \sum_{i=1}^{n} \psi(\delta_{i}^j)(B_N + \tau_j I_N) \right)^{-1}.
\]

From the previous analysis, \( \delta_i = \lim_{t \to +\infty} \delta_t^i \). To prove the uniform boundedness of \( \delta_i \), one can proceed by induction on \( t \). For \( t = 0 \), the result is true. Let \( t \in \mathbb{N}^* \) and assume that \( \delta_j^k \leq \delta_{N}^-(\tau_j + \eta_N^+) \) holds true for any \( k \leq t \) and \( j = 1, \ldots, n \). We propose to prove it for \( k = t + 1 \). We have:

\[
\frac{\delta_{j}^{t+1}}{\delta_{N}^-(\tau_j + \eta^+)} = \frac{1}{N} \operatorname{Tr} \left( B + \tau_j I \right) \left( \frac{1}{n} \sum_{i=1}^{n} \psi(\delta_{i}^j)(B + \tau_j I) \right)^{-1}.
\]
From the induction assumption along with the fact that \(x \mapsto \frac{\psi(x)}{x(1+cN\psi(x))}\) is non-increasing, we obtain:

\[
\frac{\psi(\delta_{t+1}^j)}{\delta_{t+1}^j (1 + cN\psi(\delta_{t+1}^j))} \geq \frac{\psi(\delta_N^j(\tau_i + \eta_N^j))}{\delta_N^j(\tau_i + \eta_N^j)(1 + cN\psi(\delta_N^j(\tau_i + \eta_N^j))}.
\]

Hence,

\[
\frac{\delta_{t+1}^j}{\delta_N^j(\tau_j + \eta_N^j)} \leq \frac{1}{N} \text{Tr} \left( B_N + \tau_j I_N \right) \left( \frac{1}{n} \sum_{i=1}^{n} f^+(\tau_i) \left( \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^+} \right)^{-1} \right)
\]

From Remark 4.1 along with Lemma 4.3, function \(t \mapsto f^+(t)\) satisfies the assumptions of proposition 4.4. We have therefore,

\[
\frac{f^{+\infty}(x)\nu(dx)}{N} \text{Tr} \left( B_N + \tau_j I_N \right) \left( \frac{1}{n} \sum_{i=1}^{n} f^+(\tau_i) \left( \frac{B_N + \tau_i I_N}{\tau_i + \eta_N^+} \right)^{-1} \right) \leq 1 + \epsilon_{n,j}, \; \forall 1 \leq j \leq n.
\]

where \(\max_j |\epsilon_{n,j}|\) converges to zero almost surely. Equation 4.19 guarantees that \(f^{+\infty}(x)\nu(dx) > 1\), thereby showing, that almost surely for \(n\) large enough:

\[
\frac{\delta_{t+1}^j}{\delta_N^j(\tau_j + \eta_N^j)} \leq 1.
\]

We will now prove the lower-bound inequality. Similarly, consider for all \(i, \delta_{t}^i = 2\delta_N^i(\tau_i + \eta_N^i)\). The sequence:

\[
\delta_{t+1}^j = \frac{1}{N} \text{Tr} \left( B + \tau_j I \right) \left( \frac{1}{n} \sum_{i=1}^{n} \psi(\delta_{t}^i) \left( \frac{B + \tau_i I}{\delta_{t}^i(1 + cN\psi(\delta_{t}^i))} \right)^{-1} \right)
\]

converges to \(\delta_{t}^j\) as \(t \to +\infty\). In the same way as for the upper-bound inequality, we will show the result by induction on \(t\). For \(t = 0\), the result is true. Let \(t \in \mathbb{N}^+\) and assume that \(\delta_{k}^j \geq \delta_N^j(\tau_j + \eta_N^j)\) holds true for any \(k \leq t\) and \(j = 1, \cdots n\). We propose to prove the result
for \( k = t + 1 \). Similar to above, using the fact that \( x \mapsto \frac{\psi(x)}{x} \) is non-increasing, we have:

\[
\frac{\delta_{j}^{t+1}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} = \frac{1}{N} \text{Tr} \cdot \frac{B_{N} + \tau_{j}I_{N}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \frac{1}{n} \sum_{i=1}^{n} \psi(\delta_{i}) (B_{N} + \tau_{i}I_{N}) \right)^{-1} \]

\[
\geq \frac{1}{N} \text{Tr} \cdot \frac{B_{N} + \tau_{j}I_{N}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \sum_{\tau_{i} \geq M_{\theta}} \psi(\delta_{N}^{-} (\tau_{i} + \eta_{N})) (B_{N} + \tau_{i}I_{N}) \frac{1}{(\tau_{i} + \eta_{N})} \right)^{-1} \]

\[
= \frac{1}{N} \text{Tr} \cdot \frac{B_{N} + \tau_{j}I_{N}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \sum_{\tau_{i} \geq M_{\theta}} \psi(\delta_{N}^{-} (\tau_{i} + \eta_{N})) (B_{N} + \tau_{i}I_{N}) \frac{1}{(\tau_{i} + \eta_{N})} \right)^{-1} \]

\[
= \frac{1}{N} \text{Tr} \cdot \frac{B_{N} + \tau_{j}I_{N}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \sum_{\tau_{i} \leq M_{\theta}} f^{-} (\tau_{i}) \frac{B_{N} + \tau_{i}I_{N}}{\tau_{i} + \eta_{N}} \right),
\]

where \((a)\) follows from the fact that \( \eta_{N}^{-} \leq \|B_{N}\| \leq M_{\theta} \). Again, from remark \[4.1\] and Lemma \[4.3\], function \( t \mapsto f^{-}(t) \int f^{-}(x) \nu(dx) \) satisfies the assumptions of proposition \[4.4\]. We have therefore,

\[
\frac{1}{N} \text{Tr} \cdot \frac{B_{N} + \tau_{j}I_{N}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \frac{1}{n} \sum_{i=1}^{n} f^{-} (\tau_{i}) \frac{B_{N} + \tau_{i}I_{N}}{\tau_{i} + \eta_{N}} \right)^{-1} \geq 1 - |\epsilon_{n,j}|,
\]

where \( \max_{j} |\epsilon_{n,j}| \) converges to zero almost surely. On the other hand,

\[
\int_{0}^{+\infty} f^{-}(x) \nu(dx) \leq \frac{\phi_{\infty}}{1 - c_{+} \phi_{\infty}} + \int_{0}^{M_{\theta}} \psi(\delta_{N}^{-} (t + M_{\theta})) < 1,
\]

and hence, almost surely, for enough large \( n \),

\[
\frac{\delta_{j}^{t+1}}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \geq \frac{1}{N} \text{Tr} \cdot \frac{B + \tau_{j}I}{\delta_{N}^{-} (\tau_{j} + \eta_{N})} \left( \sum_{i=1}^{n} f^{-} (\tau_{i}) \frac{B_{N} + \tau_{i}I_{N}}{\tau_{i} + \eta_{N}} \right) > 1.
\]

\[
\square
\]

The following refinement of Lemma \[4.6\] will be required in the proof of the asymptotic convergence of the robust-scatter estimator.

**Lemma 4.7.** Let \((\kappa, M_{\kappa})\) be couples indexed by \( \kappa \) with \( 0 < \kappa < 1 \) and \( M_{\kappa} > 0 \) such that \( \nu(M_{\kappa}, + \infty) < \kappa \). Then, for sufficiently small \( \kappa \) the following system of equations:

\[
\delta_{j} = \frac{1}{N} \text{Tr} \cdot \left( B_{N} + \tau_{j}I_{N} \right) \left( \frac{1}{n} \sum_{\tau_{i} \leq M_{\kappa}} \psi(\delta_{i}) (B_{N} + \tau_{i}I_{N}) \frac{1}{\delta_{i} (1 + c_{N} \psi(\delta_{i}))} \right)^{-1}, \quad \forall 1 \leq j \leq n
\]

(4.22)
has a unique vector solution \((\delta^+_N, \cdots, \delta^+_N)\) for all large \(n\) a.s, and there exists \(\delta^-_{N,0}, \delta^+_N, 0\) with \(\limsup \delta^+_N < \infty\) and \(\liminf \delta^-_{N,0} > 0\) and \(\eta^-_{N,0}, \eta^+_{N,0}\) such that for all \(\kappa < \kappa_0\) small:

\[
\delta^-_{N,\kappa}(\tau_i + \eta_{N,\kappa}^-) \leq \delta^\kappa \leq \delta^+_{N}(\tau_i + \eta^+_{N,\kappa}), \quad i = 1, \cdots, n
\]

for all large \(n\) a.s. Moreover, \(\eta^+_{N,\kappa}\) and \(\eta^-_{N,\kappa}\) satisfies:

\[
\eta^+_{N,\kappa} = O\left(\frac{1}{N} \text{Tr} B_N\right) \quad \text{and} \quad \eta^-_{N,\kappa} = O\left(\frac{1}{N} \text{Tr} B_N\right)
\]

**Proof.** The same proof as that of Lemma 4.7 holds by taking \(\kappa_0\) smaller than \(\theta\) and choosing \(\delta^+_{N}\) so that it satisfies:

\[
\int_0^{M_{\kappa_0}} \frac{\psi(\delta^+_{N} t)}{1 + c_N \psi(\delta^+_{N} t)} \nu(dt) > 1
\]

while \(\delta^-_{N}\) is set in the same way as before. \(\Box\)

We will now provide similar results for the random quantities \(q_1, \cdots, q_n\). In particular, we have the following Lemma:

**Lemma 4.8.** Let \(q_t \triangleq y^t \tilde{C}_t^{-1} y_t, i = 1, \cdots, n\). There exists \(q^+_N, q^-_N, \alpha^+_N, \alpha^-_N > 0\) with \(\limsup_N q^+_N < +\infty\) and \(\liminf_N q^-_N > 0\) such that, for all large \(n\) a.s.,

\[
q^-_N (\tau_i + \alpha^-_N) \leq q_t \leq q^+_N (\tau_i + \alpha^+_N), \quad i = 1, \cdots, n.
\]

**(4.23)**

**Proof.** The proof is based on the same tools as those used to show Lemma 4.6. The single difference is that the random quantities \(q_t\) involve quadratic forms which will be treated by resorting to Lemma 4.5. First recall that \(q_1, \cdots, q_n\) are given by:

\[
(q_1, \cdots, q_n) = \lim_{t \to +\infty} (q_1^t, \cdots, q_n^t)
\]

with \((q_1^0, \cdots, q_n^0)\) chosen arbitrarily in \(\mathbb{R}^n_+\) and:

\[
q_j^{t+1} = \frac{1}{N} y_j \left( \frac{1}{n} \sum_{i=1, i \neq j}^n \frac{\psi_N(q_i^t)}{q_i^t} y_i y_i^* \right)^{-1} y_j.
\]

Similar to the proof of Theorem 2.1, consider \((q^+_N)\) so that \(\int_0^{+\infty} \psi_N(q^+_N t) \nu(dt) > \frac{1}{1 - c_N \psi_{\infty}}\)

and \(\limsup q^+_N < +\infty\). Let \(\alpha^+_N\) be the unique solution of:

\[
1 = \int \int_0^{+\infty} \frac{\psi(q^+_N + x) \nu(dx)}{\psi(q^+_N t)^{\frac{m+1}{2} + \alpha_N} \nu(dt)} F_{BN}(dy).
\]

Set \(q^+_N = q^+_N (\tau_i + \alpha_N)\). We will prove by induction on \(t\) that \(q^t_i \leq q^+_N (\tau_i + \alpha_N)\). For \(t = 0\),

the result holds true. Assume now that for all \(k \leq t\):

\[
q^k_i \leq q^+_N (\tau_i + \alpha_N),
\]
and let us show that $q_{i}^{t+1} \leq q_{N}^{+}(\tau_{i} + \alpha_{N})$. We have:

$$
\frac{q_{j}^{t+1}}{q_{N}^{+}((\tau_{j} + \alpha_{N})} = \frac{1}{N q_{N}^{+}(\tau_{j} + \alpha_{N})} y_{j} \left( \frac{1}{n} \sum_{i \neq j} \psi(q_{i}^{t}) y_{i} y_{i}^{*} \right)^{-1} y_{j} \leq \frac{1}{N(\tau_{j} + \alpha_{N})} y_{j} \left( \frac{1}{n} \sum_{i \neq j} \psi(q_{N}^{+}((\tau_{j} + \alpha_{N})} y_{i} y_{i}^{*} \right)^{-1} y_{j}.
$$

Let $\psi_{q_{N}}^{+}(x) = \frac{\psi(q_{N}^{+}(x))}{\int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx)}$. From item ii) in Lemma 4.5, we have:

$$
\max_{1 \leq j \leq n} \left| \frac{1}{N(\tau_{j} + \alpha_{N})} \int_{0}^{\infty} \psi(q_{N}^{+}(x)\nu(dx) \right| y_{j}^{*} \left( \frac{1}{n} \sum_{i \neq j} \psi_{q_{N}}^{+}(\tau_{i}(B_{N} + \tau_{i}I_{N}) \right)^{-1} y_{j} = \frac{1}{N(\tau_{j} + \alpha_{N})} \int_{0}^{\infty} \psi(q_{N}^{+}(x)\nu(dx)
$$

where $T_{N} = \left( \frac{1}{n} \sum_{i=1}^{n} \psi_{q_{N}}^{+}(\tau_{i}(B_{N} + \tau_{i}I_{N}) \right)^{-1}$ with $e_{1}, \ldots, e_{n}$ the unique solutions to the following system of equations:

$$
eq \left( \psi_{q_{N}}^{+}(\tau_{k}) \right)_{n} \left( \frac{1}{n} \sum_{i=1}^{n} \psi_{q_{N}}^{+}(\tau_{i}(B_{N} + \tau_{i}I_{N}) \right)^{-1} = \left( \frac{1}{n} \sum_{i=1}^{n} \psi_{q_{N}}^{+}(\tau_{i}(B_{N} + \tau_{i}I_{N}) \right)^{-1}
$$

The limit of the convergence in (4.24) can be bounded as:

$$
\frac{1}{N} \int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) \leq (1 + \max_{1 \leq k \leq n} e_{k}) \frac{1}{N} \int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) \leq (1 + \max_{1 \leq k \leq n} e_{k}) + \epsilon_{n,j}
$$

where $\max_{1 \leq j \leq n} |\epsilon_{n,j}|$ converges to zero almost surely (inequality (a) being a by-product of (4.3) in proposition 4.4). Finally, from item iii) of Lemma 4.5 we get:

$$
\int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) \leq \frac{1}{N} \int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) \leq \frac{1}{N} \int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) - c_{N}\psi_{\infty} + \epsilon_{n,j}
$$

with $\epsilon_{n,j}$ converging to zero almost surely. Since $q_{N}^{+}$ satisfies:

$$
\int_{0}^{\infty} \psi(q_{N}^{+}(x))\nu(dx) > \frac{1}{1 - c_{N}\psi_{\infty}} = 1 + c_{N}\psi_{\infty},
$$

we obtain that:

$$
\frac{q_{j}^{t+1}}{q_{N}^{+}(\tau_{j} + \alpha_{N})} < 1
$$

for $n$ large enough a.s. In order to prove the lower bound in (4.23), the same reasoning as the one used in the previous lemma applies. In particular, it suffices to set $\theta > 0$ and $M_{\theta}$ such that $\theta < \frac{\phi_{\infty}}{2(1 - c_{\infty})}$, $\nu(M_{\theta}, +\infty) < \theta$ and $M_{\theta} \geq \|B_{N}\|$. Taking $q_{N}^{+}$ such that:

$$
\int_{0}^{M_{\theta}} \psi_{N}(q_{N}^{+}(t + M_{\theta}))\nu(dt) < \frac{1}{2}
$$
and setting $\alpha^{-}_N$ to the unique solution of the following equation:

$$1 = \int_0^{+\infty} f^{-}(x)\nu(dx) \int_0^{+\infty} \frac{y+t}{t+\alpha^{-}_N} f^{-}(t)\nu(dt) F^{B_N}(dy)$$

with $f^{-}: t \mapsto \psi\nu1_{t\geq M_\theta} + \psi(2q_N(t + M_\theta))1_{t\leq M_\theta}$, we can establish by induction on $t$ and using the same steps as in the control of the lower bound of $\delta_i$ that:

$$\frac{q_i}{q_N(\tau_i + \alpha^{-}_N)} > 1, \quad i = 1, \ldots, n.$$

\[\square\]

The determination of an interval in which lies all quantities $\delta_1, \ldots, \delta_n$ is of utmost important in that it allows us to control the quadratic forms:

$$\frac{\psi(\delta_i)}{N\delta_j} y_j^* \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j,$$

and

$$\frac{\psi(\delta_i)}{N\delta_j} y_j^* \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j,$$

where $\delta_1, \ldots, \delta_n$ and $\delta_1^*, \ldots, \delta_n^*$ are solutions of equations (4.22) and (4.17). In particular, we have the following two lemmas which easily follows from item-i of Lemma 4.5.

**Lemma 4.9.** Let Assumptions 1-4 hold true. Then,

$$\max_{1 \leq j \leq n} \left| \frac{\psi(\delta_j)}{N\delta_j} y_j^* \left( \sum_{i=1}^{n} \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j) \right| \xrightarrow{a.s.} 0.$$

**Lemma 4.10.** Let $(\kappa, M_\kappa)$ be couples indexed by $\kappa$ with $0 < \kappa < 1$, and $M_\kappa > 0$ such that $\nu(M_\kappa, \infty) < \kappa$. Then, for all $\kappa < \kappa_0$, we have:

$$\max_{1 \leq i, j \leq n} \left| \frac{\psi(\delta_i)}{N\delta_j} y_j^* \left( \sum_{i=1, i \neq j}^{n} \frac{\psi(\delta_i)}{\delta_i} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j) \right| \xrightarrow{a.s.} 0,$$

where $\delta_i^*, i = 1, \ldots, n$ are defined the solutions of (4.22).

With these results at hand, we are now in position to prove $f_i = \frac{\psi(q_i)}{\nu(\delta_i)}$ satisfies:

$$\max_{1 \leq i \leq n} |f_i - 1| \xrightarrow{a.s.} 0.$$

As in [3], we will distinguish two cases: the case where all $\tau_i$s are bounded and that of unbounded $\tau_i$. The proof is merely based on the same techniques with only some modifications and will be detailed for sake of completeness.

**All $\tau_i$s are Bounded.** Assume that there exists a constant $M$ such that $\tau_i \leq M$ for all $i = 1, \ldots, n$. Define $f_i = \frac{\psi(q_i)}{\nu(\delta_i)} > 0$. Without loss of generality, we assume that $f_1 \leq \cdots \leq f_n$. 
We have:

\begin{align}
f_j &= \frac{v(\frac{1}{N} y_j^* \tilde{C}_j^{-1} y_j)}{v(\delta_j)} \\
&= \frac{v \left( \frac{1}{N} y_j^* \left( \frac{1}{n} \sum_{i \neq j} v(q_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j)} \\
&= \frac{v \left( \frac{1}{N} y_j^* \left( \frac{1}{n} \sum_{i \neq j} f_i v(\delta_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j)} \\
&\leq \frac{v \left( \frac{1}{N} y_j^* \left( \frac{1}{n} \sum_{i \neq j} f_n v(\delta_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j)} \\
&= \frac{v \left( \frac{\delta_j}{f_n v(\delta_j)} \psi(\delta_j) \left( \frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j)}.
\end{align}

In a similar way, we also have:

\begin{align}
f_1 &\geq \frac{v \left( \frac{\delta_j}{f_n v(\delta_j)} \psi(\delta_j) \left( \frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j)}.
\end{align}

From Lemma 4.9, let \( 0 < \epsilon_n < 1 \) with \( \epsilon_n \downarrow 0 \) such that for all large \( n \), a.s. and for all \( 1 \leq j \leq n \):

\begin{align}
\psi(\delta_j) - \epsilon_n < \frac{\psi(\delta_j)}{N \delta_j} y_j^* \left( \frac{1}{n} \sum_{i \neq j} v(\delta_i) y_i y_i^* \right)^{-1} y_j \leq \psi(\delta_j) + \epsilon_n.
\end{align}

In particular, since \( v \) is non-increasing, taking \( j = n \) in (4.26) and applying the left-hand inequality in (4.27), we obtain:

\begin{align}
f_n < \frac{v \left( \frac{\delta_n}{f_n v(\delta_n)} \max(\psi(\delta_n) - \epsilon_n, 0) \right)}{v(\delta_n)}.
\end{align}

Assume now that for some \( \ell > 0 \), \( f_n > 1 + \ell \) infinitely often. Therefore, there exists a sequence \( (n) \) over which \( f(n) > 1 + \ell \) for \( n \) large enough. We distinguish two cases. First, assume that \( \lim \inf \delta_n = 0 \). There exists a sequence \( (p) \) obtained from a subsequence of \( (n) \) over which \( \lim_{n \to +\infty} \delta_(p) = 0 \).

From (4.28), we have:

\begin{align}
\lim_{n \to +\infty} f_(p) \leq \lim_{n \to +\infty} \frac{v \left( \frac{1}{f(\delta(p))} \max(\psi(\delta(p)) - \epsilon(\delta(p)), 0) \right)}{v(\delta(p))} = 1.
\end{align}

which is in contradiction with \( f(\delta(p)) > 1 + \ell \). Therefore, for (4.28) to hold, we must have \( \lim \inf \delta_n > \delta_{\min} \). Since all \( \tau \)-s are bounded, \( (\delta_n)_n \) is also a bounded sequence. One can thus extract a subsequence \( (q) \) extracted from \( (p) \) over which \( \delta_(q) \to x > 0 \) and \( c_N \to c \). Let
\[ \psi(x) = \lim_{c \to c} \psi(x) \] and write (4.28) in the following equivalent form:

\[
(1 - \frac{\epsilon(q)}{\psi(\delta(q))}) \frac{\psi(\delta(q))}{\psi(\frac{\delta(q)}{f(q)} (1 - \frac{\epsilon(q)}{\psi(\delta(q))}))} < 1. \tag{4.29}
\]

We therefore have:

\[
\lim_{n \to +\infty} (1 - \frac{\epsilon(q)}{\psi(\delta(q))}) \psi(\frac{\delta(q)}{f(q)} (1 - \frac{\epsilon(q)}{\psi(\delta(q))})) \geq \lim_{n \to +\infty} (1 - \frac{\epsilon(q)}{\psi(\delta(q))}) \frac{\psi(\delta(q))}{\psi(\frac{\delta(q)}{f(q)} (1 - \frac{\epsilon(q)}{\psi(\delta(q))}))} = \frac{\psi_c(x)}{\psi_c((1 + \ell)^{-1}x)} > 1,
\]

which is in contradiction with (4.28). Symmetrically, we obtain that for \( \epsilon_n \downarrow 0 \) and for large \( n \) a.s.,

\[ f_1 > \frac{v(f(q_1))}{v(\delta(q_1))}(\psi(q_1) + \epsilon_n) \]

which is equivalent to:

\[ f_1 \frac{v(q_1)}{v(q_1)}(\psi(q_1) + \epsilon_n). \]

We conclude using the same reasoning as above that for each \( \ell > 0 \) small \( f_1 \geq 1 - \ell \) for all large \( n \) a.s. so that finally, we have:

\[ \max_{1 \leq i \leq n} |f_i| = 1 \stackrel{\text{a.s.}}{\longrightarrow} 0. \]

The uniform boundedness of \( \tau_i \) implies that of \( q_i \) and \( \delta_i \), thereby ensuring that:

\[ \max_{1 \leq i \leq n} |v(q_i) - v(q_i)| \stackrel{\text{a.s.}}{\longrightarrow} 0. \]

Hence, for any \( \ell > 0 \), arbitrarily small, we have for all large \( n \),

\[
(1 - \ell)\frac{1}{n} \sum_{i=1}^{n} \frac{\psi(q_i)}{\delta_i} y_i y_i^* \leq \frac{1}{n} \sum_{i=1}^{n} v(q_i) y_i y_i^* \leq (1 + \ell)\frac{1}{n} \sum_{i=1}^{n} \frac{\psi(q_i)}{\delta_i} y_i y_i^*.
\]

Since the spectral norm of \( \frac{1}{n} \sum_{i=1}^{n} \frac{\psi(q_i)}{\delta_i} y_i y_i^* \) is almost surely bounded and \( \ell \) is arbitrary, we conclude that:

\[ \| \hat{C}_N - \hat{S}_N \| \stackrel{\text{a.s.}}{\longrightarrow} 0. \]

**Unbounded** \( \tau_i \). We now relax the boundedness assumption on the support of the distribution of \( \tau_i \). We will follow the same technique used in [8]. Similarly to [8], let \((\kappa, M_\kappa)\) be couples indexed by \( \kappa \) such that for all large \( n \), we have \( \nu_n(M_\kappa, +\infty) < \kappa \leq \kappa_0 \), for \( \kappa_0 \) small enough, and \( M_\kappa \geq \limsup \|B_N\| \). Denote by \( C_\kappa = \{ i, \tau_i \leq M_\kappa \} \) with cardinality \( |C_\kappa| \). Then,

\[ \frac{|C_\kappa|}{n} = 1 - \nu_n(M_\kappa, +\infty) \geq 1 - \kappa. \]

In the sequel, we will differentiate the indexes in \( C_\kappa \) from those in \( C^c_\kappa \). Define \( f_1^\kappa, \ldots, f_n^\kappa \) as:

\[ f_i^\kappa = \frac{v(q_i)}{v(q_i)}. \]
where $\delta_1^\kappa, \ldots, \delta_n^\kappa$ are the solutions of the system of equations (4.22) given in Lemma 4.7. Let $j \in \mathcal{C}_\kappa$ and denote by $f_j^\kappa = \min_{i \in \mathcal{C}_\kappa} f_i^\kappa$ and $f_j^\kappa = \max_{i \in \mathcal{C}_\kappa} f_i^\kappa$. We have:

$$f_j^\kappa = \frac{v(q_j)}{v(\delta_j)} \left( \frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} f_i^\kappa v(\delta_i^\kappa) y_i y_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} v(q_i) y_i y_i^* \right)^{-1} y_j$$

$$= \frac{v \left( \frac{1}{n} y_j^* \left( \frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} f_i^\kappa v(\delta_i^\kappa) y_i y_i^* + \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} v(q_i) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j^\kappa)} \leq \frac{v \left( \frac{1}{n} y_j^* \left( \frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j^\kappa)} ,$$

where we used in the first inequality the fact that $q_i \geq q_N(\tau_i + \alpha^-_N)$. Since $f_j^\kappa = \frac{v(q_j)}{v(\delta_j)} = \frac{\psi(q_j) \delta_j^\kappa}{\psi(\delta_j) \varphi} = \frac{\psi(q_j) \delta_j^\kappa}{\psi(\delta_j) \varphi j}$, we obtain:

$$\frac{v(q_N(\tau + \alpha^-_N))}{\psi(\delta_{N-\kappa})} \leq f_j^\kappa \leq \frac{\psi(q_N(\tau + \alpha^-_N)))}{\psi(\delta_{N-\kappa})} \delta_j^\kappa \kappa \psi(q_N(\tau + \alpha^-_N))) q_N(\tau + \alpha^-_N) .$$

The above inequalities imply that $f_j^\kappa$ is almost surely bounded irrespective of $\kappa$ small enough. To see that, note that if $\lim \inf \psi(\tau) = 0$, the left inequality ensures that $\lim \inf f_j^\kappa > 0$ while if $\lim \sup \psi(\tau) = \infty$, the second inequality ensures that $\lim \sup f_j^\kappa < \infty$. As a consequence, we can assume that $f_j^\kappa > f_\kappa$ for all large $n$ and for all $\kappa$ small enough. From this observation, for all large $n$, a.s. we have:

$$f_j^\kappa \leq \frac{v \left( \frac{1}{n} y_j^* \left( \frac{1}{n} \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* \right)^{-1} y_j \right)}{v(\delta_j^\kappa)} \leq \frac{v \left( \psi(\delta_j^\kappa) \sum_{i \neq j, i \in \mathcal{C}_\kappa} v(\delta_i^\kappa) y_i y_i^* \left( y_j + w_{j,n} \right) \right)}{v(\delta_j^\kappa)} ,$$

where we defined similarly to [8] $w_{j,n}$ as:

$$w_{j,n} = \frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* (D_{\kappa,j} + C_\kappa)^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* y_j^* D_{\kappa,j}^{-1} y_j$$

with

$$D_{\kappa,j} = \sum_{i \in \mathcal{C}_\kappa, i \neq j} v(\delta_i^\kappa) y_i y_i^*, \quad C_\kappa = \sum_{i \in \mathcal{C}_\kappa} \frac{1}{\tau_i + \alpha^-_N} y_i y_i^*$$

Using the resolvent identity $D^{-1} - F^{-1} = D^{-1}(F - D)F^{-1}$ (for any invertible matrices $D$ and $F$) along with Cauchy-Schwartz inequality, we obtain:

$$|w_{n,j}| \leq \left( \frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* (D_{\kappa,j} + C_\kappa)^{-1} C_\kappa (D_{\kappa,j} + C_\kappa)^{-1} y_j \sqrt{\frac{\psi(\delta_j^\kappa)}{N \delta_j^\kappa} y_j^* y_j^* D_{\kappa,j}^{-1} C_\kappa D_{\kappa,j}^{-1} y_j} \right).$$
Note that for $\kappa$ small enough, matrix $D_{\kappa,j}$ is invertible. Besides, from assumption $3\rightarrow ii)$, for $\kappa$ small enough and for enough large $n$, $\nu_n([m,M_n]) \geq c_\ast$. Using Lemma 14 in Appendix 4.2.2 we conclude that there exists $C_1$ such that $\min_1 \lambda_1(D_{\kappa,j}) \geq C_1$. Since matrix $C_\kappa$ has a bounded spectral norm, Theorem 15 in Appendix 4.2.2 along with the rank-1 perturbation Lemma 16, Lemma 2.6] yields:

$$\max_{j \in \mathcal{C}_\kappa} \left| \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} C_{\kappa} D_{\kappa,j}^{-1} y_j - \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} \text{Tr}(B_N + \tau_j I_N) D_{\kappa}^{-1} C_\kappa D_{\kappa}^{-1} \right| \xrightarrow{a.s.} 0,$$

where $D_\kappa = D_{\kappa,j} + \frac{1}{n} v(\delta_n^\kappa) y_j^* y_j^*$. From $|\text{Tr} XY| \leq \|X\| \text{Tr} Y$ for positive definite $Y$, we have:

$$\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} \text{Tr}(B_N + \tau_j I_N) D_{\kappa}^{-1} C_\kappa D_{\kappa}^{-1} \leq \frac{\|B_N + \tau_j I_N\|}{C_1^2 \delta_N^\kappa \nu_\infty(\tau_j + \eta_\kappa^{-\kappa})} \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} y_i^* y_i \text{Tr} B_N + \tau_j I_N \xrightarrow{a.s.} 0.$$

Since $\frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \frac{1}{N} \sum_{i \in \mathcal{C}_\kappa} \psi_i \nu_n(M_{\kappa},+\infty)$ for all $\kappa \leq \kappa_0$ and for all large $n$ a.s., we have:

$$\max_{j \in \mathcal{C}_\kappa} \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* D_{\kappa,j}^{-1} C_{\kappa} D_{\kappa,j}^{-1} y_j \leq K_1 \nu_n(M_{\kappa},+\infty),$$

where $K_1$ is a constant that does not depend on $\kappa \leq \kappa_0$. In the same way, we can control the term $\frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* (D_{\kappa,j} + C_{\kappa})^{-1} C_\kappa (D_{\kappa,j} + C_{\kappa})^{-1} y_j$. Finally, we conclude that:

$$\max_{j \in \mathcal{C}_\kappa} |w_{j,n}| \leq K \nu_n(M_{\kappa},+\infty)$$

(4.31)

for some constant $K$ independent of $\kappa \leq \kappa_0$. Quantities $w_{j,n}$ being controlled for $j \in \mathcal{C}_\kappa$, we can now proceed in a similar way as in the case of the bounded $\tau_i$ case. Lemma 33 implies that for any fixed $\kappa > 0$ there exists a sequence $\delta_n^\kappa \downarrow 0$ such that a.s. for $n$ large enough,

$$\max_{j \in \mathcal{C}_\kappa} \left| \frac{\psi(\delta_j^\kappa)}{N\delta_j^\kappa} y_j^* \left( \frac{1}{n} \sum_{i \in \mathcal{C}_\kappa} \psi(\delta_i^\kappa) \frac{\delta_i^\kappa}{\delta_j^\kappa} y_i y_i^* \right)^{-1} y_j - \psi(\delta_j^\kappa) \right| \leq \epsilon_n^\kappa.$$  

(4.32)

Combining (4.30), (4.31) and (4.32) we then have for all large $n$ a.s. and for all $j \in \mathcal{C}_\kappa$,

$$f_j^\kappa \leq \frac{v\left(\frac{\delta_j^\kappa}{\psi(\delta_j^\kappa)} \max \left( \psi(\delta_j^\kappa) - \epsilon_n^\kappa - K \nu_n(M_{\kappa},+\infty),0 \right) \right)}{v(\delta_j^\kappa)}$$

which for $j = \pi$ becomes:

$$f_\pi^\kappa \leq \frac{v\left(\frac{\delta_\pi^\kappa}{\psi(\delta_\pi^\kappa)} \max \left( \psi(\delta_\pi^\kappa) - \epsilon_n^\kappa - K \nu_n(M_{\kappa},+\infty),0 \right) \right)}{v(\delta_\pi^\kappa)}.$$  

(4.33)

Assume that $\lim sup f_\pi > 1 + \ell$ for some $\ell > 0$. Let us restrict the sequence $f_\pi$ to those indexes for which $f_\pi > 1 + \ell$. Similar to the case of bounded $\tau_i$, we can see that (4.33) implies that $\liminf \delta^\kappa_\pi > \delta^\kappa_{\text{min}}$, a bound which can be chosen independent of $\kappa \leq \kappa_0$. In effect, from (4.33) we have:

$$f_\pi^\kappa \leq \frac{v(0)}{v(\delta_\pi^\kappa)}$$
which is equivalent to:

\[
v(\delta^c_\pi) \leq \frac{v(0)}{\ell + 1},
\]

or also:

\[
\delta^c_\pi \geq v^{-1}\left(\frac{v(0)}{\ell + 1}\right).
\]

Using the definition of \(\psi\), (4.33) reads for \(\kappa\) sufficiently small:

\[
\psi\left(\frac{\delta^c_\pi}{f_\pi} \left(1 - \frac{c^s_\kappa}{\psi(\delta^c_\pi )} - \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )}\right)\right) \geq 1
\]

or also for \(n\) large enough:

\[
\psi(\delta^c_\pi ) \left(1 - \frac{c^s_\kappa}{\psi(\delta^c_\pi )} - \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )}\right) \leq 1.
\]

Hence,

\[
\frac{\psi(\delta^c_\pi ) - \psi\left((f_\pi^{-1})^{-1}\right) \left(1 - \frac{c^s_\kappa}{\psi(\delta^c_\pi )} - \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )}\right)}{\psi\left((f_\pi^{-1})^{-1}\right)} \leq \frac{c^s_\kappa}{\psi(\delta^c_\pi )} + \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )}.
\]

or equivalently:

\[
\frac{\psi(\delta^c_\pi ) - \psi\left((f_\pi^{-1})^{-1}\delta^c_\pi\right)}{c^s_\kappa + K\nu(M_\kappa, \infty)} \leq \frac{\psi\left((f_\pi^{-1})^{-1}\delta^c_\pi\right)}{\psi(\delta^c_\pi )} \left(1 - \frac{c^s_\kappa}{\psi(\delta^c_\pi )} - \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )}\right).
\]

Since \(f_\pi^s > 1 + \ell\), \(\psi\left((f_\pi^{-1})^{-1}\delta^c_\pi\right) < \psi(\delta^c_\pi\).

Therefore, for \(\kappa\) chosen sufficiently small so that:

\[
1 - \frac{c^s_\kappa}{\psi(\delta^c_\pi )} - \frac{K\nu(M_\kappa, \infty)}{\psi(\delta^c_\pi )} \leq \frac{1}{2},
\]

we have:

\[
\frac{\psi(\delta^c_\pi ) - \psi\left((f_\pi^{-1})^{-1}\delta^c_\pi\right)}{c^s_\kappa + K\nu(M_\kappa, \infty)} \leq 2. \tag{4.34}
\]

Since \(\delta^c_\pi\) belongs to the interval \(\left[\delta_{\text{min}}, \limsup \delta^c_N(M_\kappa) + \eta^+_N\right]\) for \(n\) large enough, taking the limit of (4.34) over some subsequences over which \(\delta^c_\pi \to x_\kappa \in \left[\delta_{\text{min}}, \limsup \delta^c_N(M_\kappa) + \eta^+_N\right]\), \(c_\pi \to c\) and \(\nu(M_\kappa, \infty)\) converges, we obtain:

\[
\psi_c(x_\kappa) - \psi_c\left(\frac{x_\kappa}{1 + \ell}\right) \leq 2, \tag{4.35}
\]

where \(\psi_c = \lim_{c_\pi \to c} \psi_N\). We now operate on \(\kappa\). If \(\limsup_{x_\kappa \to 0} x_\kappa < \infty\), the left-hand side of (4.35) goes to \(+\infty\) as \(\kappa \to 0\) so that starting from \(\kappa\) sufficiently small and taking the limit over \(n\) on the considered subsequence raises a contradiction. If instead \(\limsup_{x_\kappa \to 0} x_\kappa = +\infty\), then since \(x_\kappa \leq 2 \limsup \delta^c_M\), we have:

\[
\frac{\psi_c(x_\kappa) - \psi_c\left(\frac{x_\kappa}{1 + \ell}\right)}{\lim_{n} \nu_n(M_\kappa, \infty)} \geq \frac{\psi_c(x_\kappa) - \psi_c\left(\frac{x_\kappa}{1 + \ell}\right)}{\lim_{n} \nu_n(\frac{x_\kappa}{\limsup \delta^c_\pi}, \infty)}.
\]
Let \( y_n = g_{c_n}^{-1}(x_n) \) with \( g_c(x) = \frac{x}{1-c \phi g_c} \). Recall that \( \psi_c = \frac{\phi c^{-1} g_c}{1-c \phi g_c} \). Then,

\[
\psi_c(x_n) - \psi_c\left(\frac{x_n}{n+1}\right) = \frac{\phi \circ g_c^{-1}(x_n) - \phi \circ g_c^{-1}\left(\frac{x_n}{n+1}\right)}{\limsup_{n \to \infty} \delta_n}.
\]

Using the same reasoning as with \( w \), \( \nu_n \) sufficiently small and then consider the limit over \( n \) on the subsequence under consideration raises a contradiction. Consequently, we must have \( \limsup_{n \to \infty} \nu_n \frac{x_n}{\limsup_{n \to \infty} \delta_n} \).

Since \( y_n \to \infty \) as \( x_n \to \infty \), from Assumption 3 the right-hand side must go to \( \infty \) as \( x_n \to \infty \).

Let \( y \) be 

\[
\lim_{\nu_n} \psi_c(y_n - \psi_c(y_n)) \to \infty \quad \text{sufficiently small and then consider the limit over} \quad n \quad \text{on the subsequence under consideration raises a contradiction.}
\]

Consequently, we must have \( \limsup_{n \to \infty} f^{\kappa}_{\ell} \leq 1 + \ell \) a.s. A similar reasoning allows to show that \( \liminf_{n \to \infty} f^{\kappa}_{\ell} \geq 1 - \ell \) a.s. for any given \( \ell > 0 \). We conclude thus:

\[
\max_{j \in C_{\kappa}} \left| f^{\kappa}_{j} - 1 \right| \to 0.
\]

We will now deal with \( f^{\kappa}_{j} \) for \( j \in C_{\kappa} \). Recall that \( f_{j} \) is given by:

\[
f_{j} = \frac{v\left(\frac{1}{n} \sum_{i \in C_{\kappa}} f^{\kappa}_{i} v(\delta^{\kappa}_{i})y_{i}y_{i}^{*} + \frac{1}{n} \sum_{i \in C_{\kappa}, i \neq j} v(q_{i})y_{i}^{*}\right)^{-1} y_{j}}{v(\delta^{\kappa}_{j})}
\]

\[
= v\left(\frac{\delta^{\kappa}_{j}}{v(\delta^{\kappa}_{j})} \frac{1}{n} \sum_{i \in C_{\kappa}} f^{\kappa}_{i} v(\delta^{\kappa}_{i})y_{i}y_{i}^{*} + \frac{1}{n} \sum_{i \in C_{\kappa}, i \neq j} v(q_{i})y_{i}^{*}\right)^{-1} y_{j} + \tilde{w}_{j,n} \right\}
\]

where

\[
\tilde{w}_{j,n} = \frac{v(\delta^{\kappa}_{j})}{N \delta^{\kappa}_{j}} y_{j} \left(\tilde{D}_{\kappa} + \tilde{C}_{\kappa,j}\right)^{-1} y_{j} - \frac{v(\delta^{\kappa}_{j})}{N \delta^{\kappa}_{j}} y_{j} \tilde{D}_{\kappa}^{-1} y_{j}
\]

with:

\[
\tilde{D}_{\kappa} = \frac{1}{n} \sum_{i \in C_{\kappa}} v(\delta^{\kappa}_{i})y_{i}y_{i}^{*}
\]

\[
\tilde{C}_{\kappa,j} = \frac{1}{n} \sum_{i \in C_{\kappa}, i \neq j} v(\delta^{\kappa}_{i})y_{i}y_{i}^{*}.
\]

Using the same reasoning as with \( w_{j,n} \), we can show that for \( \kappa \) sufficiently small and \( n \) large enough,

\[
\max_{j \in C_{\kappa}} \left| \tilde{w}_{j,n} \right| \leq K \nu_{n}(M_{\kappa}, \infty) \leq K \kappa
\]
with $K$ independent of $\kappa \leq \kappa_0$. On the other hand, since $\max_{i \in C_n} |f_i^\kappa - 1| \xrightarrow{a.s.} 0$, we have:

$$\frac{\psi(\delta_j^\kappa) 1}{\delta_j^\kappa} \frac{1}{N} y_j^i \left( \frac{1}{n} \sum_{i \in C_n} f_i^\kappa v(\delta_{ij}^\kappa) y_i y_i^* \right)^{-1} y_j - \frac{\psi(\delta_j^\kappa) 1}{\delta_j^\kappa} \frac{1}{N} y_j^i \left( \frac{1}{n} \sum_{i \in C_n} v(\delta_{ij}^\kappa) y_i y_i^* \right)^{-1} y_j \xrightarrow{a.s.} 0.$$  

As a consequence, for $\kappa$ sufficiently small and $n$ large enough:

$$\max_{j \in C_n} \left| \frac{\psi(\delta_j^\kappa) q_j}{\delta_j^\kappa} - \psi(\delta_j^\kappa) \right| \leq \kappa',$$

where $\lim_{\kappa \to 0} \kappa' = 0$. Now, write $f_j^\kappa$ as:

$$f_j = \frac{\psi \left( \frac{\delta_j^\kappa}{\psi(\delta_j^\kappa)} - \frac{1}{\psi(\delta_j^\kappa)} q_j \right)}{\psi(\delta_j^\kappa)}.$$

Then, one can easily note that:

$$\lim_{\kappa \to 0} \lim_{n \to \infty} \max_{j \in C_n} \left\{ |f_j^\kappa - 1| \right\} \to 0.$$

Combining the results for $j \in C_n$ and $j \in C_n^c$, we conclude that for each $\ell > 0$, there exists $\kappa > 0$ small enough such that a.s.,

$$(1 - \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^* \leq \frac{1}{n} \sum_{i=1}^n v(q_i) y_i y_i^* \leq (1 + \ell) \frac{1}{n} \sum_{i=1}^n \frac{\psi(\delta_i^\kappa)}{\delta_i^\kappa} y_i y_i^*.$$

It remains thus to show that for each $\varepsilon > 0$, there exists $\kappa_0$ such that for any $\kappa \leq \kappa_0$ and all large $n$,

$$\max_j \left| 1 - \frac{\delta_j}{\delta_j^\kappa} \right| \leq \varepsilon.$$

Recall that $(\delta_1^\kappa, \cdots, \delta_n^\kappa)$ are given by:

$$(\delta_1^\kappa, \cdots, \delta_n^\kappa) = \lim_{t \to \infty} (\delta_1^\kappa(t), \cdots, \delta_n^\kappa(t))$$

with $\delta_i^\kappa(0), \cdots, \delta_n^\kappa(0)$ are arbitrary and:

$$\delta_j^\kappa(t + 1) = \frac{1}{N} \text{Tr} (B_N + \tau_j I_N) \left( \frac{1}{n} \sum_{\tau_i \leq M_n} \frac{\phi \circ g_N^{-1}(\delta_i^\kappa(t))}{\delta_i^\kappa(t)} (B_N + \tau_i I_N) \right)^{-1},$$

where we used the relation $\frac{\psi_N^\kappa}{1 + N \psi_N^\kappa} \phi \circ g_N^{-1}$. Set for $t = 0$, $\delta_i^\kappa = \delta_i, i = 1, \cdots, n$. We will prove by induction on $t$ that $\delta_j \leq \delta_j^\kappa(t)$ for all $j = 1, \cdots, n$, thereby showing that $\delta \leq \delta_j^\kappa$. Obviously, the desired result holds for $t = 0$. Assume now that for all $t \leq k$, $\delta_j^\kappa(t) \geq \delta_j$, and let us show that $\delta_j^\kappa(k + 1) \geq \delta_j$. Since $x \mapsto \frac{\phi \circ g_N^{-1}(x)}{x}$ is non-increasing and $\delta_i^\kappa(k) \geq \delta_i$, we have:

$$\frac{\phi \circ g_N^{-1}(\delta_i^\kappa(k))}{\delta_i^\kappa(k)} \leq \frac{\phi \circ g_N^{-1}(\delta_i)}{\delta_i}.$$
Hence,

\[ \delta^\kappa_j (k + 1) = \frac{1}{N} \text{Tr}(B_N + \tau_j I_N) \left( \frac{1}{n} \sum_{\tau_i \leq M} \frac{1}{\delta_j^\kappa (k)} \frac{\phi \circ g_{N^{-1}}^{-1} (\delta_j^\kappa)}{\delta_j^\kappa} (B_N + \tau_i I_N) \right)^{-1} \]

\[ \geq \frac{1}{N} \text{Tr}(B_N + \tau_j I_N) \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\delta_j^\kappa} (B_N + \tau_i I_N) \right)^{-1} \]

\[ = \delta_j. \]

We are now in position to control the convergence of \( \max_{1 \leq j \leq n} \left| \frac{1}{\delta_j} - \frac{1}{\delta_j} \right| \) as \( \kappa \to 0 \). In particular, we recall that we need to prove that for each \( \varepsilon > 0 \), there exists \( \kappa_0 \) such that:

\[ \max_j \left| \frac{1}{\delta_j} - \frac{1}{\delta_j} \right| \leq \varepsilon. \]

To this end, define the maps \( T_N, T_N^M \) as:

\[ T_N : (x_1, \cdots, x_n) \mapsto \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\delta_j^\kappa} (B_N + \tau_i I_N) \right)^{-1} \]

and

\[ T_N^M : (x_1, \cdots, x_n) \mapsto \left( \frac{1}{n} \sum_{\tau_i \leq M} \frac{1}{\delta_j^\kappa} (B_N + \tau_i I_N) \right)^{-1}. \]

From Lemma 4.7 and 4.6 it is easy to see that the spectral norms of \( T_N(\delta_1, \cdots, \delta_n) \) and \( T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) \) are uniformly bounded. Note that:

\[ \| T_N(\delta_1, \cdots, \delta_n) - T_N^M(\delta_1, \cdots, \delta_n) \| \leq \left\| T_N^M(\delta_1, \cdots, \delta_n) \right\| \left( \frac{\phi}{\delta_j^\kappa} \limsup \frac{\| B_N \|}{\delta_j^\kappa \liminf \delta_j^\kappa} \right) \nu_n(M, \infty). \]

and

\[ \| T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) - T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) \| \leq \left\| T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) \right\| \left( \frac{\phi}{\delta_j^\kappa} \liminf \frac{\| B_N \|}{\delta_j^\kappa \liminf \delta_j^\kappa} \right) \nu_n(M, M, \kappa) \]

for any \( \kappa \geq M \). Setting \( M \) large enough so that \( \liminf \nu_n(m, M) > 0 \), we get:

\[ \max \left( \left\| T_N^M(\delta_1, \cdots, \delta_n) \right\|, \left\| T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) \right\| \right) \leq \frac{\max \left( \limsup (m + \eta_\kappa^+), \limsup (m + \eta_\kappa^+)^{+} \right)}{m \phi \circ g_{N^{-1}}^{-1} (m - \limsup \nu_n(M, \infty))}. \]

Therefore, one can fix \( M \) sufficiently large in such a way that:

\[ \limsup_N \left\| T_N(\delta_1, \cdots, \delta_n) - T_N^M(\delta_1, \cdots, \delta_n) \right\| \leq \frac{\varepsilon}{3} \] (4.36)

and

\[ \limsup_N \left\| T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) - T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) \right\| \leq \frac{\varepsilon}{3}. \] (4.37)

With this value of \( M \) at hand, we will now prove that:

\[ \lim_{\kappa \to 0} \limsup_N \left\| T_N^M(\delta_1^\kappa, \cdots, \delta_n^\kappa) - T_N^M(\delta_1, \cdots, \delta_n) \right\| = 0. \]
To this end, we will work out the differences $\frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa}$. We have:

$$
\frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa} =
\frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \leq M_n} \left( \phi \circ g_N^{-1}(\delta_i) - \phi \circ g_N^{-1}(\delta_i^\kappa) \right) \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
+ \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \geq M_n} \phi \circ g_N^{-1}(\delta_i) \right] \left[ \frac{1}{\delta_i^\kappa} - \frac{1}{\delta_i} \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
+ \frac{1}{N} \text{Tr} \frac{B_N + \tau_j I_N}{\delta_j^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \geq M_n} \phi \circ g_N^{-1}(\delta_i) \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
\leq \alpha_{1,j} + \alpha_{2,j} + \alpha_{3,j}.
$$

Note that $\alpha_2$ can be bounded as:

$$
\alpha_{2,j} \leq \max_i \left| \frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa} \right|.
$$

Let $j_0$ be the index of the maximum element of $\frac{\delta_j^\kappa - \delta_j}{\delta_j^\kappa}, i = 1, \ldots, n$. Therefore:

$$
-\alpha_{1,j_0} \leq \alpha_{3,j_0}
$$

or equivalently,

$$
\frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \leq M_n} \left( \phi \circ g_N^{-1}(\delta_i) - \phi \circ g_N^{-1}(\delta_i^\kappa) \right) \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
\leq \frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \geq M_n} \phi \circ g_N^{-1}(\delta_i) \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
$$

Hence,

$$
\frac{1}{N} \text{Tr} \frac{B_N + \tau_{j_0} I_N}{\delta_{j_0}^\kappa} T_N^{M^*}(\delta_1^\kappa, \ldots, \delta_n^\kappa) \left[ \frac{1}{n} \sum_{\tau_i \leq M} \left( \phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i) \right) \right] (B_N + \tau_i I_N) T_N(\delta_1^\kappa, \ldots, \delta_n)
\leq \| T_N(\delta_1, \ldots, \delta_n) \| \phi_{\infty} \nu_n(M_n, \infty) \limsup \| B_N \| \liminf \eta_N \delta_N
$$
The right-hand side in the above inequality converges to zero as \( \kappa \to 0 \). This is possible if and only if:

\[
\frac{1}{n} \sum_{\tau_i \leq M} \left( \frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} \right) (B_N + \tau_i I_N) \xrightarrow{\kappa \to 0} 0. \tag{4.38}
\]

Function \( x \mapsto \phi \circ g_N^{-1}(x) \) is continuously differentiable. Therefore, by the mean value theorem,

\[
\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i) = \left( \phi \circ g_N^{-1} \right)'(\xi_i^\kappa) \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa},
\]

where \( \left( \phi \circ g_N^{-1} \right)' \) denotes the derivative of \( \phi \circ g_N^{-1} \) and \( \xi_i^\kappa \in [\delta_i, \delta_i^\kappa] \). Now, since for \( n \) large enough, \( \min_{i, \tau_i \leq M} \delta_i \geq a \triangleq \liminf \delta_N \eta_N \) and \( \max_{i, \tau_i \leq M} \delta_i^\kappa \leq b \triangleq \limsup \delta_N^+ (\eta_N^+ + M) \), we obtain:

\[
\frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} \geq \inf_{x \in [a, b]} \left( \phi \circ g_N^{-1} \right)'(x) \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa}.
\]

Since \( \inf_{x \in [a, b]} (\phi \circ g_N^{-1})' > 0 \) by Assumption 2(ii), we get:

\[
\frac{1}{n} \sum_{\tau_i \leq M} \frac{\delta_i^\kappa - \delta_i}{\delta_i^\kappa} (B_N + \tau_i I_N) \xrightarrow{\kappa \to 0} 0. \tag{4.39}
\]

Using the convergences \( 4.38 \) and \( 4.39 \), we can prove that:

\[
\lim_{\kappa \to 0} \sup_N \left\| T_N^M(\delta_1^\kappa, \ldots, \delta_n^\kappa) - T_N^M(\delta_1, \ldots, \delta_n) \right\| \to 0.
\]

This can be easily seen by noting that:

\[
T_N^M(\delta_1, \ldots, \delta_n) - T_N^M(\delta_1^\kappa, \ldots, \delta_n^\kappa) = T_N^M(\delta_1, \ldots, \delta_n) \frac{1}{n} \sum_{\tau_i \leq M} \left( \frac{\phi \circ g_N^{-1}(\delta_i^\kappa) - \phi \circ g_N^{-1}(\delta_i)}{\delta_i^\kappa} + \phi \circ g_N^{-1}(\delta_i) \left( \frac{1}{\delta_i^\kappa} - \frac{1}{\delta_i} \right) \right) (B_N + \tau_i I_N)
\]

\[
\times T_N^M(\delta_1^\kappa, \ldots, \delta_n^\kappa).
\]

One can thus choose \( \kappa_0 \) in such a way that for all \( \kappa \leq \kappa_0 \)

\[
\limsup_N \left\| T_N^M(\delta_1^\kappa, \ldots, \delta_n^\kappa) - T_N^M(\delta_1, \ldots, \delta_n) \right\| \leq \frac{\varepsilon}{3}.
\]

From \( 4.36 \) and \( 4.37 \), we therefore get for all \( \kappa \leq \kappa_0 \)

\[
\limsup_N \left\| T_N(\delta_1, \ldots, \delta_n) - T_N^M(\delta_1^\kappa, \ldots, \delta_n^\kappa) \right\| \leq \varepsilon.
\]

In an equivalent way, we therefore have, for each \( \varepsilon > 0 \), there exists \( \kappa_0 \) such that for any \( \kappa \leq \kappa_0 \) and all large \( n \),

\[
\max_{1 \leq i \leq n} \left| \frac{1}{\delta_i^\kappa} - \frac{1}{\delta_i} \right| \leq \varepsilon.
\]

Using this result, we will show that for each \( \ell > 0 \), there exist \( \kappa > 0 \) small enough, such that a.s.,

\[
(1 - \ell) \frac{1}{n} \sum_{i=1}^{n} \psi_N(\delta_i) y_i y_i^\ast \leq \frac{1}{n} \sum_{i=1}^{n} \psi_N(\delta_i^\kappa) y_i y_i^\ast \leq (1 + \ell) \frac{1}{n} \sum_{i=1}^{n} \psi_N(\delta_i) y_i y_i^\ast.
\]

To this end, it suffices to show that for each \( \varepsilon > 0 \) and \( \kappa \) small enough:

\[
\max_{1 \leq i \leq n} \left| \psi_N(\delta_i) - \psi_N(\delta_i^\kappa) \right| \leq \varepsilon.
\]
If this was not true, then one can find a sequence \((n)\) over which:

\[
\max_{1 \leq i \leq (n)} \left| \psi_{N(n)}(\delta_i) - \psi_{N(n)}(\delta^*_i) \right| \geq \epsilon. \tag{4.40}
\]

for any small \(\kappa\). Since the sequence function \(\psi_N\) converge uniformly, one can extract a subsequence \((p)\) from \((n)\) such that \(c(p) \to c^*\) and \(\psi(p)\) converge uniformly to \(\psi^*\). On the other hand, we know that for any arbitrarily small \(r\) there exists \(\kappa_0\) such that for any \(\kappa \leq \kappa_0\) and for all large \(n\),

\[
\max_{1 \leq j \leq n} \left| \frac{1 - \delta_j}{\delta^*_j} \right| \leq r.
\]

or also, for all \(j = 1, \ldots, n\),

\[
\delta_j \geq (1 - r)\delta^*_j.
\]

Let \(x_0\) be such that \(\psi^*(x_0(1 - r)) \geq \psi(x_0(1 - \epsilon/3))\). Since \(\psi^*\) is increasing and bounded at infinity by \(\psi(x)\), we have, for any \(x, y \geq x_0(1 - r)\),

\[
|\psi^*(x) - \psi^*(y)| \leq \frac{\epsilon}{3}.
\]

Consider the indices \(i\) such that \(\delta^*_i \geq x_0\), and thus \(\delta_i \geq x_0(1 - r)\). Take \(n\) large enough such that:

\[
||\psi_N - \psi^*|| \leq \frac{\epsilon}{3}.
\]

Then, for those indices, one can prove that:

\[
\max_{1 \leq j \leq (n)} \left| \psi_{N(n)}(\delta_j) - \psi_{N(n)}(\delta^*_j) \right| \leq \max_{\delta^*_j \geq x_0} \left| \psi_{N(n)}(\delta_j) - \psi^*(\delta_j) \right|
\]

\[
+ \left| \psi^*(\delta_j) - \psi^*(\delta^*_j) \right| + \left| \psi^*(\delta^*_j) - \psi_{N(n)}(\delta^*_j) \right| \leq \epsilon.
\]

Consider now the indices \(i\) such that \(\delta^*_i \leq x_0\). For those indices, we have:

\[
\psi^*(\delta^*_i) - \psi^*(\delta_i)u(0) (\leq \delta^*_i - \delta_i)
\]

\[
= u(0) \frac{\delta^*_i - \delta_i}{x_0} x_0
\]

\[
\leq u(0) x_0 \frac{\delta^*_i - \delta_i}{\delta^*_i}.
\]

Taking \(r \leq \frac{\epsilon}{x_0u(0)}\), we will get:

\[
|\psi^*(\delta^*_i) - \psi^*(\delta_i)| \leq \epsilon
\]

which is in contradiction with (4.40). We therefore have for each \(\ell > 0\),

\[
(1 - \ell)^2 \frac{1}{n} \sum_{i=1}^{n} \frac{\psi_{N}(\delta_i)}{\delta_i} y_i y^*_i \preceq \hat{C}_N \preceq (1 + \ell)^2 \frac{1}{n} \sum_{i=1}^{n} \frac{\psi_{N}(\delta_i)}{\delta_i} y_i y^*_i
\]

which therefore implies that \(\|\hat{C}_N - \hat{S}_N\| \xrightarrow{a.s.} 0\). This completes the proof.
APPENDIX: TECHNICAL LEMMAS

This appendix gathers some technical Lemmas that will help control quadratic forms of the type:

\[ z_i^* R_j \left( \frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i^* \right)^{-1} R_j z_j, \]

where \( z_1, \ldots, z_n \) are independent random vectors with size \( N \times 1 \) and \( R_1, \ldots, R_n \) are \( n \) matrices of size \( N \times N \) independent of \( z_1, \ldots, z_n \) and whose eigenvalues are bounded above and below by constants independent of \( n \) and \( N \). The control of this quadratic form can be performed using the most well-known trace Lemma of Silverstein et al. [2, Lemma 2.7], provided that we can guarantee that the infimum of the set \( S \) of smallest eigenvalues of matrices \( \frac{1}{n} \sum_{i=1, i \neq j}^n R_i z_i z_i^* R_i^* \) being above zero almost surely was implicitly raised in [19] where this fact was assumed because no immediate answer can be provided in general. It was only recently that we have provided a rigorous proof thereof under the Gaussian setting [11]. In this appendix, we extend this result to the random vector model of the present work, i.e. \( z_i = [s_i, w_i] \) with \( s_i \sim \mathcal{CN}(0, I_K) \) and \( w_i \) zero-mean unitarily invariant satisfying \( ||w_i||^2 = N \). The control of the infimum of the set \( S \) will be shown along the same lines of the proof of Lemma 1 in [7].

In the sequel, we will start by bounding the maximum eigenvalue of \( \frac{1}{n} \sum_{i=1}^n R_i z_i z_i^* R_i^* \) when \( z_1, \ldots, z_n \) are Gaussian random vectors. To this end, we will start by introducing the following concentration inequality, the proof of which is provided for sake of completeness.

**Lemma 1.1.** Let \( \gamma_1, \ldots, \gamma_n \) be \( n \) independent random variables having an exponential distribution with rate parameter 1, and \( (\alpha_i)_{i=1}^n \) be positive scalars. Then, there exists \( C \) such that for any \( t > 0 \):

\[
\mathbb{P} \left[ \sum_{i=1}^n \alpha_i \gamma_i > t \right] \leq C \exp \left( -\min \left( \frac{t}{4 \sum_{i=1}^n \alpha_i^2}, \frac{t}{4 \max_{1 \leq i \leq n} \alpha_i} \right) \right).
\]

**Proof.** Let \( s \) be a positive scalar such that \( s < \frac{1}{2 \alpha_i} \) for all \( i = 1, \ldots, n \). Then:

\[
\mathbb{P} \left[ \sum_{i=1}^n \alpha_i \gamma_i > t \right] = \mathbb{P} \left[ \exp \left( s \sum_{i=1}^n \alpha_i \gamma_i \right) > \exp(ts) \right] \leq \frac{\exp(-ts)}{\prod_{j=1}^n (1 - s\alpha_j)}.
\]

Now, using the inequality \( -\log(1 - x) \leq x + x^2 \) for \( x \leq \frac{1}{2} \), we get:

\[
\frac{1}{1 - s\alpha_j} \leq \exp(s\alpha_j) \exp(s^2 \alpha_j^2) \leq \exp(\frac{1}{2}) \exp(s^2 \alpha_j^2),
\]

where \( \alpha_j \) are the eigenvalues of \( R_j \), and \( R_j \) is the maximum eigenvalue of \( R_j \).
thereby yielding:

\[
P \left[ \sum_{i=1}^{n} \alpha_i \gamma_i > t \right] \leq \exp \left( \frac{1}{2} \exp \left( -ts + \frac{n}{\sum_{i=1}^{n} \alpha_i^2} \right) \right).
\]

Two cases have to be considered. If \( \frac{t \max_{1 \leq i \leq n} \alpha_i}{\sum_{i=1}^{n} \alpha_i^2} \leq \frac{1}{2} \). Then, setting \( s = \frac{t}{\sum_{i=1}^{n} \alpha_i^2} \) yields:

\[
P \left[ \sum_{i=1}^{n} \alpha_i \gamma_i > t \right] \leq C \exp \left( -\frac{t^2}{4 \sum_{i=1}^{n} \alpha_i^2} \right)
\]  

(.41)

where \( C = \exp \left( \frac{1}{2} \right) \). Otherwise, if \( \frac{t \max_{1 \leq i \leq n} \alpha_i}{\sum_{i=1}^{n} \alpha_i^2} \geq \frac{1}{2} \). Then, \( \sum_{i=1}^{n} \alpha_i^2 \leq t \max_{1 \leq i \leq n} \alpha_i \). Set \( s = \frac{1}{2 \max_{1 \leq i \leq n} \alpha_i} \), then we have:

\[
P \left[ \sum_{i=1}^{n} \alpha_i \gamma_i > t \right] \leq C \exp \left( -\frac{t}{4 \max_{1 \leq i \leq n} \alpha_i} \right)
\]  

(.42)

Gathering (.41) and (.42) yields the desired result. \( \square \)

With this Lemma at hand, we will now control the maximum eigenvalue of \( \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i \).

We have in particular, the following result:

**Lemma .12.** Let \( z_1, \cdots, z_n \) be \( n \) independent \( \mathbb{N} \times 1 \) standard Gaussian vectors. Consider \( (R_i)_{i=1}^{n} \) a family of \( \mathbb{N} \times \mathbb{N} \) matrices with uniformly bounded spectral norm, i.e.,

\[
\lim \sup_{n} \max_{1 \leq i \leq n} \| R_i \| < +\infty.
\]

Let \( \Sigma \) be given by:

\[
\Sigma = \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i^*
\]

Then, there exists a constant \( K_{\text{max}} \) such that a.s. for \( n \) large enough,

\[
\| \Sigma \| < K_{\text{max}}.
\]

**Proof.** The proof relies on the observation that:

\[
\| \Sigma \| = \max_{\| a \| = 1} a^* \Sigma a.
\]

Based on the result of Lemma [11], a concentration inequality involving the term \( a^* \Sigma a \) can be established. Define \( u_i = \frac{R_i a}{\| R_i a \|} \) and expand \( a^* \Sigma a \) as:

\[
a^* \Sigma a = \frac{1}{n} \sum_{i=1}^{n} a^* R_i z_i z_i^* R_i a
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} (a^* R_i^* a) |u_i^* z_i|^2.
\]

Since \( u_i \) is unitary, the random quantity \( u_i^* z_i \) is a Gaussian random variable with zero mean and variance 1. Hence, \( |u_i^* z_i|^2, i = 1, \cdots, n \) is a sequence of \( n \) independent exponential
distributed random variables with rate parameter 1. Applying Lemma .11 we get:

\[ P[a^* \Sigma a > t] = P \left[ \sum_{i=1}^{n} (a^* R_i R_i^* a) |u_i^* z_i|^2 > nt \right] \leq C \exp \left( - \min \left( \frac{n^2 t^2}{4 \sum_{i=1}^{n} a^* R_i R_i^* a}, \frac{nt}{4 \max_{1 \leq i \leq n} a^* R_i R_i^* a} \right) \right), \]

where \( C \) is some constant independent of \( n \) and \( N \). For \( t \geq 1 \), we therefore have:

\[ P[a^* \Sigma a > t] \leq C \exp \left( - \frac{nt}{4 \max_{1 \leq i \leq n} \| R_i \|^2} \right). \tag{.43} \]

With the above inequality at hand, we are now in position to control the behaviour of the spectral norm of \( \Sigma \). For that, we will resort to the well-known \( \epsilon \)--net argument. Let \( S \) be an \( \frac{1}{2} \)--net of the unit sphere of \( \mathbb{C}^N \). Using Lemma 2.3.2 of [16], we have:

\[ P \left[ \| \Sigma \| \geq t \right] \leq P \left[ \bigcup_{a \in S} a^* \Sigma a > \frac{t^2}{4} \right], \]

Using (43), we obtain that each of the probabilities of \( P \left[ a^* \Sigma a > \frac{t^2}{4} \right] \) is bounded by \( C \exp(- \frac{ct^2 n}{4}) \) for \( t \) and \( n \) large enough with \( c \) some constant independent of \( n \). On the other hand, the cardinality of \( S \) is of order \( \left( O(1) \right)^n \). By taking \( t \) large enough, this term can be absorbed into the exponential gain of \( C \exp(- \frac{ct^2 n}{4}) \). For some \( t \) large enough the event \( \left\{ \| \Sigma \| < \frac{t}{4} \right\} \) holds with overwhelming probability. Setting \( K_{\text{max}} \geq t \). We have thus a.s. for \( n \) large enough,

\[ \| \Sigma \| < K_{\text{max}}. \]

\[ \square \]

All the above results are derived under the assumption of a Gaussian setting. Before going further into the proofs of the main lemmas of this appendix, we will show that the considered random model of the paper is equivalent to a Gaussian model. In particular, we have the following result:

**Lemma .13.** Let \( z_1, \ldots, z_n \in \mathbb{C}^N \) be \( n \) independent and identically distributed vectors such that \( z_i = [s_i^T, w_i^T]^T \) where \( s_i \) and \( w_i \) are independent and distributed as:

- \( s_i \sim \mathcal{CN}(0, I_K) \)
- \( w_i \) is unitarily invariant zero-mean vector such that \( \| w_i \| = N \).

Write \( w_i \) as \( w_i = \frac{\sqrt{N} \tilde{w}_i}{\| \tilde{w}_i \|} \) with \( \tilde{w}_i \sim \mathcal{CN}(0, I_N) \), and denote by \( \Sigma \) the \( N \times N \) matrix given by:

\[ \Sigma = \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i^* \]
where \( R_i = [R_{i,1}, R_{i,2}] \), \( R_{i,1} \in \mathbb{C}^{N \times K} \) and \( R_{i,2} \in \mathbb{C}^{N \times N} \) are some deterministic matrices with uniformly bounded norm. Let \( \tilde{z}_i = [s_i^T, \tilde{w}_i^T]^T \), and \( \tilde{\Sigma} \) be given by:

\[
\tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} R_i \tilde{z}_i \tilde{z}_i^* R_i^*
\]

Then, in the asymptotic regime,

\[
\|\Sigma - \tilde{\Sigma}\| \xrightarrow{a.s.} 0.
\]

**Proof.** Notice that \( \Sigma \) can be written as:

\[
\Sigma = \frac{1}{n} \sum_{i=1}^{n} R_i D_i \tilde{z}_i \tilde{z}_i^* D_i R_i^*
\]

where

\[
D_i = \begin{bmatrix} I_K & 0 \\ 0 & \frac{\sqrt{N}}{\|w_i\|} I_N \end{bmatrix}.
\]

Then,

\[
\Sigma - \tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} R_i (D_i - I) \tilde{z}_i \tilde{z}_i^* (D_i - I) R_i^* + \frac{1}{n} \sum_{i=1}^{n} R_i \tilde{z}_i \tilde{z}_i^* (D_i - I) R_i^*
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} R_i (D_i - I) \tilde{z}_i \tilde{z}_i^* R_i^*
\]

\[
= \Theta_1 + \Theta_2 + \Theta_3.
\]

In the sequel, we will prove that the spectral norms of \( \Theta_i, i = 1, 2, 3 \) converge to zero almost surely. We will treat only the term \( \Theta_2 \) since the treatment of \( \Theta_1 \) and \( \Theta_3 \) relies on the same arguments. Expanding \( \Theta_2 \) using \( R_i = [R_{i,1}, R_{i,2}] \), we get:

\[
\Theta_2 = \frac{1}{n} \sum_{i=1}^{n} R_{i,1} s_i \tilde{w}_i^* \left( \frac{\sqrt{N}}{\|w_i\|} - 1 \right) R_{i,2}^* + \frac{1}{n} \sum_{i=1}^{n} R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2}^* \left( \frac{\sqrt{N}}{\|w_i\|} - 1 \right)
\]

\[\triangleq \Theta_{1,2} + \Theta_{2,2}.
\]

Let us control \( \Theta_{1,2} \).

\[
\|\Theta_{1,2}\| = \sup_{||a||=1,||b||=1} \left| \frac{1}{n} \sum_{i=1}^{n} a^* R_{i,1} s_i \tilde{w}_i^* R_{i,2}^* b \left( \frac{\sqrt{N}}{\|w_i\|} - 1 \right) \right|
\]

\[
\leq \max_i \left| \frac{\sqrt{N}}{\|w_i\|} - 1 \right| \sup_{||a||=1,||b||=1} \left| \frac{1}{n} \sum_{i=1}^{n} a^* R_{i,1} s_i \tilde{w}_i^* R_{i,2}^* b \right|
\]

\[
\leq \max_i \left| \frac{\sqrt{N}}{\|w_i\|} - 1 \right| \sup_{||a||=1} \left( \frac{1}{n} \sum_{i=1}^{n} a^* R_{i,1} s_i s_i^* R_{i,1}^* a \right) \sup_{||b||=1} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{w}_i^* R_{i,2} b b^* R_{i,2}^* \tilde{w}_i \right)
\]

\[
\leq \max_i \left| \frac{\sqrt{N}}{\|w_i\|} - 1 \right| \left( \frac{1}{n} \sum_{i=1}^{n} R_{i,1} s_i s_i^* R_{i,1}^* \right) \left( \frac{1}{n} \sum_{i=1}^{n} R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2}^* \right)
\]
Using the facts that \( \max_i \left| \frac{\sqrt{N}}{\|w_i\|} - 1 \right| \) converges almost surely to zero and \( \| \frac{1}{n} \sum_{i=1}^{n} R_{i,1} s_i s_i^* R_{i,1} \| \) and \( \| \frac{1}{n} \sum_{i=1}^{n} R_{i,2} \tilde{w}_i \tilde{w}_i^* R_{i,2} \| \) are almost surely bounded as a result of Lemma 12, we have:

\[
\| \Theta_{1,2} \| \overset{a.s.}{\to} 0.
\]

Similarly, we can also prove that:

\[
\| \Theta_{2,2} \| \overset{a.s.}{\to} 0,
\]

thereby implying that:

\[
\| \Theta_2 \| \overset{a.s.}{\to} 0.
\]

Using the same notations of Lemma 13, consider \( \Sigma_j \) and \( \tilde{\Sigma}_j \) the \( N \times N \) matrices given by:

\[
\Sigma_j = \frac{1}{n} \sum_{i=1, i \neq j}^{n} R_i z_i z_i^* R_i
\]

\[
\tilde{\Sigma}_j = \frac{1}{n} \sum_{i=1, i \neq j}^{n} R_i \tilde{w}_i \tilde{w}_i^* R_i.
\]

Arguing along the same lines as in the proof of Lemma 13, we can show that:

\[
\max_{1 \leq j \leq n} \left\| \Sigma_j - \tilde{\Sigma}_j \right\| \overset{a.s.}{\to} 0.
\]

This observation is essential to facilitate the proof of the first main result of this Appendix which is about showing that:

\[
\min_{1 \leq j \leq n} \left\{ \lambda_1 (\Sigma_j) : j = 1, \ldots, n \right\} > \epsilon
\]

In fact, from the convergence inequality in (44), we can see that the proof can be reduced to showing this result when \( \Sigma_j \) is replaced with \( \tilde{\Sigma}_j \). The proof of the following Lemma will rely on this observation:

Lemma 14. Let \( z_1, \ldots, z_n \in \mathbb{C}^N \) be \( n \) independent and identically distributed vectors such that \( z_i = [s_i^T, w_i^T]^T \) where \( s_i \) and \( w_i \) are independent and distributed as:

- \( s_i \sim \mathcal{CN}(0, I_K) \)
- \( w_i \) is unitarily invariant zero-mean vector such that \( \|w_i\| = N \).

Define matrices \( \Sigma \) and \( \Sigma_j \) as:

\[
\Sigma = \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i
\]

\[
\Sigma_j = \frac{1}{n} \sum_{i=1, i \neq j}^{n} R_i z_i z_i^* R_i,
\]

where \( (R_i)_{i=1}^{n} \) are \( N \times N \) matrices satisfying:

\[
\liminf_{N} \min_{1 \leq i \leq n} \lambda_1 (R_i R_i^*) > 0
\]

and

\[
\limsup_{N} \max_{1 \leq i \leq n} \lambda_N (R_i R_i^*) < +\infty,
\]
Consider the asymptotic regime of Assumption 1. Therefore, there exists \( \epsilon > 0 \) such that for all large \( n \) a.s.,
\[
\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon.
\]

Proof. It is clear from the discussion before the statement of the above lemma that we can assume \( z_1, \ldots, z_n \) to be standard Gaussian vectors. Under the Gaussian setting, the fact that the smallest eigenvalue of \( \Sigma \) is almost surely bounded above zero can be deduced from corollary 5 of our work in [1]. It remains thus to treat that of \( \Sigma_j \). To this end, we will resort to the same kind of the arguments as those used in the proof of [7, Lemma 1]. Notice first that we can assume without loss of generality that \( \lambda_1(\Sigma_j) \neq \lambda_1(\Sigma) \), for all \( j = 1, \cdots, n \). By definition, the eigenvalues of \( \Sigma_j \) are solutions in \( \lambda \) of the following equation:
\[
\det(\Sigma_j - \lambda I_N) = 0.
\]
Developing the above equation, we obtain:
\[
\det \left( \Sigma_j - \lambda I_N \right) = \det \left( \Sigma - \frac{1}{n} R_j z_j z_j^* R_j^* - \lambda I_N \right) \\
= \det Q(\lambda) \det \left( I_N - Q(\lambda) \frac{1}{n} R_j z_j z_j^* R_j^* Q(\lambda) \frac{1}{2} \right) \\
= \det Q(\lambda) \left( 1 - \frac{1}{n} z_j^* R_j^* Q(\lambda) R_j z_j \right),
\]
where \( Q(\lambda) = (\Sigma - \lambda I_N)^{-1} \). If \( \lambda \) is an eigenvalue of \( \Sigma_j \) different from that of \( \Sigma \), then necessarily:
\[
\frac{1}{n} z_j^* R_j^* Q(\lambda) R_j z_j = 1.
\]
Building on the ideas of [7], we propose to study the behaviour of function:
\[
f_{n,j}(x) = \frac{1}{n} z_j^* R_j^* Q(x) R_j z_j,
\]
in a neighborhood of zero. The result of the lemma follows if we prove that there exists \( \xi > 0 \) such that \( f_{n,j}(x) < 1 \) for all \( x \in [0, \xi] \) and \( j = 1, \cdots, n \) a.s. for \( n \) large enough. From our recent result in [1], we know that there exists \( \eta > 0 \) such that a.s. for \( n \) large enough, \( \lambda_1(\Sigma) > \eta \). Functions \( x \mapsto f_{n,j}(x) \) being increasing in the interval \([0, \eta]\), it suffices thus to show that there exists \( \xi \) in \([0, \eta]\) such that \( f_{n,j}(\xi) < 1 \) a.s. for \( n \) large enough.

Let us start by analyzing the behaviour of \( f_{n,j}(x) \) for \( x < 0 \). Define \( Q_j(x) \) as \( Q_j(x) = (\Sigma_j - x I_N)^{-1} \). Using the matrix inversion relation: \( a^*(A + a a^*)^{-1} = a^* A^{-1} a / (1 + a^* A^{-1} a) \) for \( a \in \mathbb{C}^{N \times 1} \) and \( A \) any \( N \times N \) invertible matrix, we obtain:
\[
f_{n,j}(x) = \frac{1}{n} z_j^* R_j^* Q_j(z) R_j z_j \\
= 1 - \frac{1}{1 + \frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j}.
\]
Now, for \( x < 0 \), using the trace lemma of Silverstein et al. [2, Lemma 2.7] in conjunction with the rank-one perturbation Lemma [15, Lemma 2.6], we can prove that:
\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} z_j^* R_j^* Q_j(x) R_j z_j - \frac{1}{n} \text{Tr} Q(x) R_j R_j^* \right| \xrightarrow{a.s.} 0.
\]
and thus:
\[
\max_{1 \leq j \leq n} \left| f_{n,j}(x) - 1 + \frac{1}{1 + \frac{1}{n} \text{Tr} Q(x) R_j R_j^*} \right| \xrightarrow{a.s.} 0.
\]
Therefore, for \( \epsilon < \min_{1 \leq j \leq n} \frac{1}{1 + \lambda N(R_j R_j^*)} \) and \( x < 0 \), we have for \( n \) large enough a.s.,

\[
\forall j = 1, \ldots, n \quad f_{n,j}(x) \leq 1 - \frac{1}{1 + \frac{1}{n} \operatorname{Tr} R_j R_j^*} + \epsilon. \tag{.45}
\]

On the other hand, since the smallest eigenvalue of \( \Sigma \) is greater than \( \eta \), we have for \( n \) large enough, a.s.

\[
\frac{1}{n} \operatorname{Tr} R_j R_j^* Q \leq \frac{\lambda N(R_j R_j^*)}{\eta} \tag{.46}
\]

Plugging (.46) into (.45), we obtain that for each \( x < 0 \) we have for \( n \) large enough, a.s.

\[
\forall j = 1, \ldots, n \quad f_{n,j}(x) \leq 1 - \epsilon. \tag{.47}
\]

Now, we will consider the analysis of functions \( f_{n,j}(x) \) on the open interval \( U = (-\frac{\eta}{2}, \frac{\eta}{2}) \).

Note that on this interval, functions \( x \mapsto f_{n,i}(x), i = 1, \ldots, n \) are well-defined and continuously differentiable. Moreover, for each \( x \in U \), we have:

\[
f'_{n,j}(x) = \frac{1}{n} z_j^* R_j^* Q^2 (x) R_j z_j \leq \frac{1}{n} z_j^* R_j^* R_j z_j \leq \frac{1}{n} \lambda_1(\Sigma) \frac{1}{\eta^2} z_j.
\]

Moreover, we have:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} z_j^* R_j^* R_j z_j - \frac{1}{n} \operatorname{Tr} R_j R_j^* \right| \overset{a.s.}{\rightarrow} 0.
\]

The above convergence along with the fact that \( \lambda_1(\Sigma) > \eta \) for \( n \) large enough a.s. yields:

\[
0 < f'_{n,j}(x) < \frac{2}{n} \limsup_{n \to \infty} \max_{1 \leq j \leq n} \left\{ \frac{1}{n} \operatorname{Tr} R_j R_j^* \right\} \triangleq K.
\]

By bounding the derivatives of functions \( f_{n,j} \) over \( U \), we have by the mean value theorem:

\[
\forall x \in [0, \frac{\eta}{2}], \quad f_{n,j}(x) < f_{n,j}(-x) + 2x K'.
\]

Set \( \xi = \min \left( \frac{\eta}{2}, \frac{x}{K} \right) \). Then, we know from (.47) that:

\[
f_{n,j}(-\xi) \leq 1 - \frac{1}{1 + \frac{1}{n} \operatorname{Tr} R_j R_j^*}.
\]

Combining this inequality with the fact that \( f_{n,j}(\xi) < f_{n,j}(-\xi) + 2x K' \), yields:

\[
f_{n,j}(\xi) \leq 1 - \frac{\epsilon}{4K},
\]

thereby finishing the proof. \[ \square \]

We are now in position to state the following key results of this appendix:

**Lemma .15.** Let \( z_1, \ldots, z_n \in \mathbb{C}^N \) be \( n \) independent and identically distributed vectors such that \( z_i = [s_i^T, w_i^T]^T \) where \( s_i \) and \( w_i \) are independent and distributed as:

- \( s_i \sim \mathcal{CN}(0, I_K) \)
- \( w_i \) is unitarily invariant zero-mean vector such that \( \|w_i\| = N/\kappa \).

Let \( (A_{N,j})_{j=1}^n \) be random matrices independent of \( z_1, \ldots, z_n \) and \( \kappa \) be a positive constant. Then,

\[
\max_{1 \leq j \leq n} \frac{1}{n} \|A_j\| \leq \kappa \left| \frac{1}{N} z_j^* A_{N,j} z_j - \frac{1}{N} \operatorname{Tr} A_{N,j} \right| \overset{a.s.}{\rightarrow} 0.
\]
Proof. Write \(w_i = \frac{\sqrt{N} \hat{w}_i}{\|\hat{w}_i\|}\) with \(\hat{w}_i \in \mathbb{C}^{N-K}\). Let \(D_j\) be the diagonal matrix given by:

\[
D_j = \begin{bmatrix}
I_K & 0 \\
0 & \frac{\sqrt{N}}{\|\hat{w}_i\|}
\end{bmatrix}
\]

Let \(\hat{z}_i = [s_i^T, \hat{w}_i^T]^T\). Then:

\[
1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j = 1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* D_j A_{N,j} D_j \hat{z}_j
\]

\[
= 1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* (D_j - I_{N}) A_{N,j} (D_j - I_{N}) \hat{z}_j + 1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* A_{N,j} (D_j - N) \hat{z}_j
\]

\[
+ 1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* (D_j - I_{N}) A_{N,j} \hat{z}_j + 1_{\|A_j\| \leq \kappa} \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j.
\] (48)

Since \(\max_j \left| 1 - \frac{\sqrt{N}}{\|\hat{w}_i\|} \right| \xrightarrow{a.s} 0\),

\[
\max_{1 \leq j \leq n} \|D_j - I_N\| \xrightarrow{a.s} 0.
\]

Therefore, it is easy to see that the maximum over \(j\) of the first three terms in (48) converge to zero almost surely. The problem unfolds thus to the control of \(1_{\|A_{N,j}\| \leq \kappa} \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j\). Let \(E_{\hat{z}_j}\) denote the expectation with respect to \(\hat{z}_j\). From the trace Lemma of Silverstein et al [2] Lemma 2.7) applied for \(p > 2\), we obtain:

\[
E_{\hat{z}_j} \left[ 1_{\|A_{N,j}\| \leq \kappa} \left( \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j - \frac{1}{N} \text{Tr} A_{N,j} \right)^p \right] \leq \frac{1_{\|A_j\| \leq \kappa} K_p}{N^{\frac{p}{2}}} \left[ \left( \frac{\zeta^2}{\sqrt{N}} \text{Tr} A_{N,j} \right)^\frac{p}{2} + \frac{\zeta p}{N^{\frac{p}{2}}} \text{Tr} A_{N,j}^p \right]
\]

where \(\zeta\) is any upper-bound on \(E \left[ \|\hat{z}_{i,j}\|^p \right]\) and \(K_p\) a constant dependent only in \(p\). Since \(1_{\|A_{N,j}\| \leq \kappa} \|A_{N,j}\| \leq \kappa\), we have:

\[
E_{\hat{z}_j} \left[ 1_{\|A_{N,j}\| \leq \kappa} \left( \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j - \frac{1}{N} \text{Tr} A_{N,j} \right) \right] \leq \frac{K_p p}{N^{\frac{p}{2}}} \left( \frac{\zeta^2}{N^{\frac{1}{2}}} \right)
\]

This bound being independent of \(A_{N,j}\), we can take the expectation with respect to \(A_{N,j}\) to obtain:

\[
E \left[ 1_{\|A_{N,j}\| \leq \kappa} \left( \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j - \frac{1}{N} \text{Tr} A_{N,j} \right)^p \right] \leq \frac{K_p p}{N^{\frac{p}{2}}} \left( \frac{\zeta^2}{N^{\frac{1}{2}}} \right) \cdot
\]

Therefore,

\[
\max_{1 \leq j \leq n} 1_{\|A_{N,j}\| \leq \kappa} \left| \frac{1}{N} \hat{z}_j^* A_{N,j} \hat{z}_j - \frac{1}{N} \text{Tr} A_{N,j} \right| \xrightarrow{a.s} 0.
\]

\(\square\)

**Lemma .16.** Let \(z_1, \ldots, z_n \in \mathbb{C}^N\) be \(n\) independent and identically distributed vectors such that \(z_i = [s_i^T, w_i^T]^T\) where \(s_i \in \mathbb{C}^K\) and \(w_i \in \mathbb{C}^{N-K}\) are independent and distributed as:

- \(s_i \sim \mathcal{CN}(0, I_K)\)
- \(w_i\) is unitarily invariant zero-mean vector such that \(\|w_i\| = N\) where \(N \leq N\).

Denote by \((R_i)\) a family of \(N \times N\) deterministic matrix that satisfy

\[
\limsup_n \max_{1 \leq i \leq n} \lambda_N(R_i R_i^*) < +\infty
\]
Consider the asymptotic regime of Assumption 1. Let \( \Sigma_j \) be given as:

\[
\Sigma_j = \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i^*
\]

Assume that there exists \( \epsilon > 0 \) such that for all large \( n \) a.s.,

\[
\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon
\]

Then, for any \( \Theta_N \in \mathbb{C}^{N \times N} \) with bounded spectral norm,

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \left( \frac{1}{n} \sum_{j=1}^{n} R_j R_j^* \right)^{-1} \right| \overset{a.s.}{\longrightarrow} 0 \quad (49)
\]

where \( e_1, \cdots, e_n \) are the unique solutions to the following system of equations:

\[
e_k = \frac{1}{n} \text{Tr} R_k R_k^* \left( \frac{1}{n} \sum_{j=1}^{n} R_j R_j^* \right)^{-1}.
\]

Proof. To prove Lemma 16, we show first that:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} R_j R_j^* \Sigma_j^{-1} \right| \overset{a.s.}{\longrightarrow} 0
\]

From the resolvent identity, we have:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} \right| = \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} (\Sigma - \Sigma_j) \Sigma_j^{-1} \right|
\]

\[
\leq \max_{1 \leq j \leq n} c_N \left( \frac{\|\Theta\|^2 R_j^2}{\lambda_1(\Sigma) \min_{1 \leq j \leq n} \lambda_1(\Sigma_j^{-1})^2} \right) \left( \frac{z_j^* z_j}{n^2} \right) \left( \frac{\max_{1 \leq j \leq n} s_j^* s_j}{n^2} \right) \]

Since there exists \( \epsilon > 0 \) such that for all large \( n \) a.s.

\[
\lambda_1(\Sigma) \geq \min_{1 \leq j \leq n} \lambda_1(\Sigma_j) > \epsilon
\]

and \( \max_{1 \leq j \leq n} \frac{1}{n} s_j^* s_j \) is almost surely bounded, we have:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} \right| \overset{a.s.}{\longrightarrow} 0 \quad (50)
\]

Similarly to the previous Lemma, write \( w_i = \frac{\sqrt{N} \tilde{w}_i}{\|\tilde{w}_i\|} \), and let \( \tilde{z}_i = [s_i^T, \tilde{w}_i^T]^T \). Denote by \( \tilde{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} R_i z_i z_i^* R_i^* \). Then from Lemma 13:

\[
\|\Sigma - \tilde{\Sigma}\| \overset{a.s.}{\longrightarrow} 0.
\]
Therefore,

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma^{-1} - \frac{1}{n} \text{Tr} \Theta \tilde{\Sigma}^{-1} \right| = \max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma^{-1} \left( \tilde{\Sigma} - \Sigma \right) \tilde{\Sigma}^{-1} \right|.
\]

\[
\leq c_N \left\| \tilde{\Sigma} - \Sigma \right\| \max_{1 \leq j \leq n} \left\| \Theta \right\| \left\| \Sigma^{-1} \right\| \left\| \tilde{\Sigma}^{-1} \right\| \overset{a.s.}{\longrightarrow} 0. \tag{.51}
\]

Hence, plugging (50) into (49), we get:

\[
\max_{1 \leq j \leq n} \left| \frac{1}{n} \text{Tr} \Theta \Sigma_j^{-1} - \frac{1}{n} \text{Tr} \Theta \tilde{\Sigma}_j^{-1} \right| \overset{a.s.}{\longrightarrow} 0.
\]

The asymptotic convergence of \(\frac{1}{n} \text{Tr} \Theta (\tilde{\Sigma} - z I_N)^{-1}\) has been studied in [19] for \(z \in \mathbb{C}_+\). Since the smallest eigenvalue of \(\tilde{\Sigma}\) is almost surely away from zero, we can extend the convergence results for \(z = 0\) by using the same arguments as those presented in [8, footnote in page 20]. □

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