Spatially Structure Spatial Problem of the Stressed-Deformed State of a Structural Inhomogeneous Rod

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Abstract. The basic design relations for thermal-force spatial bending with tension, transverse shear and torsion were obtained for a spatially hospismatic rod of rectangular cross section, composed of quasi-homogeneous parts (phases), which were made of various structural materials. Approximations of the transverse shears functions and a membrane analogy were used for shear deformations during torsion, using the Tymoshenko hypotheses. As a result, obtained relations allow one to perform approximate formulations and solutions of various boundary-value direct and inverse problems, including: identifying the stress-strain state of a composite rod under thermal power, evaluating its strength and stiffness, identifying rational geometric and structural parameters of the inhomogeneous structure of the rod, and optimization problems. Expressions were obtained for the stiffness characteristics of the zeroth, first, and second orders in bending with tension, shear and tensional stiffness of the section, which allowed us to formulate the boundary value problems of spatial deformation of composite rods.

1. Introduction

Inhomogeneous constructions have a number of undeniable advantages over homogeneous ones, constructed with the help of rational design principles [1–4]. The main ones are: a) ability to adapt more effectively to given physical fields [5]; b) ability to regulate the stress-strain state; c) obtaining savings in mass and cost of construction materials.

Due to the complexity of the study object - a spatially deformable inhomogeneous rod under force and thermal effects - various special cases of design schemes are usually considered in the literature: flat problems [1, 6–8], simple types of action on individual design elements [9, 10], or rods of simple geometric shape with particular cases of heterogeneity under the complications strengthening [9]. A specific feature of heterogeneous structures is their sensitivity to thermal influences [5, 11, 12].

Refined calculation methods are constructed without using hypotheses of the rods’ theory in a number of works. So in the research work [13], the expansion of the original problem solution in a series with respect to the strains derivatives of the accompanying homogeneous rod problem was applied to construct the engineering theory of the inhomogeneous rods resistance. In a zeroth approximation, the proposed method gives a classical solution. In [14, 15], the method of asymptotic splitting was used to study spatial problems. These approaches [13–15] are very valuable for evaluating various approximate methods, as well as numerical [7, 16–18] and experimental [8, 19] studies. In the practical use of heterogeneous structures, it is now required to create engineering methods of calculation that have acceptable laboriousness and versatility, which makes it possible to evaluate the stress-strain state under combined spatial effects. In this research work, we consider an
approximate statement of the problem and a method for constructing the basic design relationships for the thermo-force spatial deformation problem of an inhomogeneous rod.

In Figure 1, in the \(xyz\) coordinate system, a straight rod is shown that experiences spatial bending in combination with tension (compression), torsion, and stationary thermal stress.

![Figure 1. The design scheme of the rod.](image)

The core is composed of \(s\) quasihomogeneous parts (phases), made of various structural materials: metals, concrete, engineering plastics, wood, artificial dispersed composites, etc. providing ideal interphase contact without discontinuities in deformations at the phase boundaries The material of the \(k\)-th phase is characterized by elastic modul \(E_k\), \(G_k\), coefficient of thermal expansion \(\alpha_k\), and a stationary temperature field \(t_k(x, y, z)\), identified on the basis of solving the heat conduction problem. The rod has a prismatic shape, a rectangular cross section and a structure symmetrical with respect to the \(xy, xz\) planes.

We take the description of rod’s deformed state of the Tymoshenko hypothesis with additional reason for the deployment of sections under torsion, considering the rod to be thin. We will consider the construction of the basic relations separately for cases of spatial bending, transverse shear, and torsion.

2. Materials and Methods

2.1. Spatial thermo-force bending with tension

We take the displacements’ functions of \(u\), \(v\), \(w\) in the direction of the \(x\), \(y\), \(z\) axes, deformations \(\varepsilon\), and shears \(\gamma\) according to the expressions

\[
\begin{align*}
u(x, y, z) &= u_0 - \theta_z y + \theta_y z, \\
v(x, y, z) &= v_0, \\
w(x, y, z) &= w_0,
\end{align*}
\]

\[
\begin{align*}
\kappa_y(x) &= \frac{d\theta_y}{dx}, \\
\kappa_z(x) &= \frac{d\theta_z}{dx}, \\
\gamma_{xy}(x, y, z) &= \frac{d\theta_y}{dx} - \frac{\varepsilon}{x} = \gamma_{xy}(x), \\
\gamma_{xz}(x, y, z) &= \frac{d\theta_z}{dx} + \frac{\varepsilon}{z} = \gamma_{xz}(x), \\
\gamma_{yz} &= 0,
\end{align*}
\]

where \(u_0(x), \ v_0(x), \ w_0(x)\) – moving points of the geometric axis; \(\theta_y(x), \ \theta_z(x)\) – rotation angles of the sections relative to the axes \(y, z\).

The integral force factors in the cross section of the \(s\)-phase rod are given by the formulas

\[
\left[ N, M_z, M_y \right](x) = \sum_{k=1}^{s} \int \int \int [\sigma_{x}^{(k)}, -y\sigma_{y}^{(k)}, z\sigma_{z}^{(k)}]dA,
\]

where \(A_k\) – area \(k\)-th of the phase in normal section.

Substituting in (2) the expression of the Duhamel-Neumann law

\[
\sigma_{i}^{(k)} = E_k \left[ \varepsilon_i - \alpha_k t_k \right],
\]

(3)
taking into account (1) for deformation \( \varepsilon \), we obtain physical relations connecting the integral force factors with generalized deformations during thermoelastic bending with tension

\[
\begin{align*}
D_0 \varepsilon_0 - D_z \kappa_z + D_y \kappa_y &= N - N_t, \\
-2D_0 \varepsilon_0 + D_z \kappa_z - D_y \kappa_y &= M_z - M_{zt}, \\
D_y \varepsilon_0 - D_y \kappa_z + D_y \kappa_y &= M_y - M_{yy}.
\end{align*}
\]

(4)

Integral Temperature Power Factors

\[
N_t = -\sum_k E_k \alpha_k \int t_k dA, \quad M_{zt} = \sum_k E_k \alpha_k \int t_k y dA, \quad M_{yy} = -\sum_k E_k \alpha_k \int t_k z dA
\]

(5)

represent forces in an inhomogeneous section arising in the presence of a temperature field and the absence of deformations \( \varepsilon_0 = \kappa_z = \kappa_y = 0 \).

The coefficients of the generalized deformations in (4) form a stiffness matrix with components

\[
D_0 = \sum_k E_k \int dA, \quad \begin{bmatrix} D_{zz} & D_{zy} & D_{yy} \end{bmatrix} = \sum_k E_k \int \begin{bmatrix} y^2, yz, z^2 \end{bmatrix} dA
\]

(6)

zero \( D_0 \), first \( (D_z, D_y) \) and second \( (D_{zz}, D_{zy}, D_{yy}) \) geometric orders. In the research work [3], stiffnesses \( D_z, D_y \) are called stiffness of mutual influence: axial deformation on bending moments and curvatures \( \kappa_z, \kappa_y \), on longitudinal force.

Dividing in the system of (4) the first equation by \( D_0 \), the second by \( D_{zz} \), and the third by \( D_{yy} \), we obtain

\[
\begin{align*}
\varepsilon_0 - y_0 \kappa_z + z_0 \kappa_y &= \frac{N - N_t}{D_0}, \\
\frac{y_0}{i_z^2} \varepsilon_0 + \kappa_z - \frac{z_0}{i_z^2} \kappa_y &= \frac{M_z - M_{zt}}{D_{zz}}, \\
\frac{z_0}{i_y^2} \varepsilon_0 - \frac{y_0}{i_y^2} \kappa_z + \kappa_y &= \frac{M_y - M_{yy}}{D_{yy}}.
\end{align*}
\]

(7)

\[
y_0 = \frac{D_z}{D_0}, \quad z_0 = \frac{D_y}{D_0}, \quad i_z = \frac{D_{zz}}{D_0}, \quad i_y = \frac{D_{yy}}{D_0}, \quad i_{zz} = \frac{D_{zz}}{D_0}
\]

(8)

The stiffness parameters (8) of the inhomogeneous section are introduced here: \( y_0, z_0 \) are the coordinates of the section stiffness center; \( i_z, i_y \) axial, and centrifugal \( i_{zz} \) stiffness radii of the cross section.

Normal stresses (3) using (1), (7) for an inhomogeneous cross section of a general form are representable in the form

\[
\sigma_{ik}^{(k)} = \frac{E_k}{D_0} \left[ \frac{N - N_t}{D_0} \left( 1 - \frac{i_z^4}{i_z^2} \right) + \left( \frac{i_z^2}{i_z} y_0 - y_0 \right) \frac{i_z^2}{i_y} + \left( \frac{i_z^2}{i_z} z_0 - z_0 \right) \frac{z_0}{i_z^2} \right]
\]
\[
\frac{M_z - M_\alpha}{D_{zz}} \left[ \frac{\partial^2}{\partial y^2} z_0 - y_0 + \left( 1 - \frac{y_0}{i_y} \right) y + \left( \frac{y_0 z_0 - i_{yz}}{i_y} \right) \frac{z}{i_z} \right] + \\
\frac{M_y - M_\alpha}{D_{yy}} \left[ \frac{\partial^2}{\partial z^2} y_0 - z_0 + \left( y_0 z_0 - i_{yz} \right) \frac{y}{i_z} + \left( 1 - \frac{y_0}{i_y} \right) \frac{z}{i_z} \right] - E_k \alpha_k \frac{t_k}{i_z}
\]

(9)

The matrix of the stiffness system (4) takes a diagonal form in the case of a symmetric section adopted in this work, or making the transition to the main central stiffness axes, which ensure the conditions, met for the stiffness of mutual influence \( D_y = D_z = D_{yz} = 0 \), and therefore, according to (8), \( y_0 = z_0 = i_{yz} = 0 \), \( D = 1 \), the system (7) splits into three independent equations, which subsequently allows one to solve boundary value problems for two plane bends and tension separately. In this case, the normal stresses (9) acting in the k-th phase under the force and thermal effects can be found by the formula

\[
\sigma^{(k)}_x = E_k \left\{ \frac{N - N_1}{D_0} \frac{M_z - M_\alpha}{D_{zz}} y + \frac{M_y - M_\alpha}{D_{yy}} z - \alpha_k \frac{t_k}{i_z} \right\},
\]

it is consistent with the formulas obtained in [6].

Noteworthy is the fact that the value of stiffness matrix determinant is invariant under the parallel transfer of the \( y, z \) axes of the coordinate system. In order to use systems (4) and (7), respectively, we have

\[
\begin{vmatrix}
D_0 & -D_z & D_y \\
-D_z & D_{zz} & -D_{yz} \\
D_y & -D_{yz} & D_{yy}
\end{vmatrix} = D_0 D_{yy} D_{zz}, \quad \det \begin{vmatrix}
1 & -y_0 & z_0 \\
\frac{y_0}{i_z^2} & 1 & \frac{z_0}{i_z^2} \\
\frac{z_0}{i_y^2} & -\frac{i_{yz}}{i_z^2} & 1
\end{vmatrix} = 1.
\]

2.2 Spatial transverse shear

For the Timoshenko rod we assume approximation of the shifts in the form

\[
\begin{align*}
\gamma_{xy}(x, y, z) &= a_y(x) f_y(y), \\
\gamma_{xz}(x, y, z) &= a_z(x) f_z(z)
\end{align*}
\]

(10)

with amplitudes \( a_y, a_z \), and given distribution functions, satisfying the boundary conditions on the surfaces

\[
\begin{align*}
f_y(y) &= 0, \quad y = \pm h/2, \\
f_z(z) &= 0, \quad z = \pm b/2.
\end{align*}
\]

A similar approximation was applied for plates in [2]. Using Hooke's Law and Equilibrium Conditions
we obtain approximate formulas for stresses in the $k$-th phase
\[
\tau_{xy}^{(k)}(z, y, z) = \frac{Q_y G_k f_y(y)}{A_k} \quad \text{and} \quad \tau_{xz}^{(k)}(z, y, z) = \frac{Q_z G_k f_z(z)}{A_k}.
\]

Taking into account (10), (11), (12), the stiffness of inhomogeneous section under shear, which are the part of the physical relations
\[
D_{Qy} \gamma_{y0} = Q_y, \quad D_{Qz} \gamma_{z0} = Q_z,
\]
can be written as
\[
D_{Qy} = \frac{1}{k_y} \sum_{k=1}^{s} G_k \int f_y dA, \quad D_{Qz} = \frac{1}{k_z} \sum_{k=1}^{s} G_k \int f_z dA,
\]
with
\[
k_y = \frac{\gamma_{y0}}{a_y}, \quad k_z = \frac{\gamma_{z0}}{a_z}.
\]

The parameters $k_y, k_z$ are the averaging coefficients of the shear strain functions (10), introducing generalized shears in $\gamma_{y0}, \gamma_{z0}$ kinematic relations (1). In particular, the value $k_y = 1$ corresponds to the adoption as $\gamma_{y0}$ maximum shift. In the case of integral averaging
\[
k_y = \frac{1}{h} \int_{-h/2}^{h/2} f_y dy, \quad \gamma_{y0} = k_y a_y.\]
Taking, for example, a parabolic function $f_y = (2y/h)^2$, it gives the value $k_y = 2/3$.

For rods with a simple geometric phase shape, for example, layered, tangential stresses $\tau_{xy}$ can be found more strictly from the equilibrium conditions of the movable section parts
\[
\tau_{xy}^{(k)}(z, y, z) = \frac{Q_y D^{sec}_{xz}}{D_{xz}} \sum_{j=1}^{s} G_z(y) b_j(y), \quad \tau_{xz}^{(k)}(z, y, z) = \frac{Q_z D^{sec}_{xy}}{D_{xy}} \sum_{j=1}^{s} G_y(z) b_j(z),
\]
with
\[
D^{sec}_{z}(y) = \int_{y-h/2}^{y+h/2} E(y, z) y dy dz, \quad D^{sec}_{y}(z) = \int_{z-h/2}^{z+h/2} E(y, z) z dy dz.
\]

Here, the second factors are introduced that contain the shear module and reflect the dependence on the second coordinate in the section in comparison with the formulas given in [6] for a plane problem. The summation in $\tau_{xz}^{(k)}$ formula is performed over the phases intersected by the horizontal level $y$ with numbers $j \in j_y$, calculating the stress. Similarly, in formula for $\tau_{xy}^{(k)}$, it is calculated for phases intersected by level $z$ with numbers $j \in j_z$.

2.3. The rod torsion
Displacements during torsion will be obtained as a result of rigid rotation of the section and its deployment [20]
Here $\theta_x$ – x-axis rotation angle, $\kappa_x$ – rod axis torsion, $\psi$ – deployment function of Saint-Venant. Taken into account (15), the components of shear strain and shear stresses take the form of

$$\gamma_{xy}(x, y, z) = \kappa_x \left( \frac{\partial \psi}{\partial y} - z \right), \quad \gamma_{xz}(x, y, z) = \kappa_x \left( \frac{\partial \psi}{\partial z} + y \right),$$

$$\gamma_{yz} = 0,$$

$$\tau^{(k)}_{xy} = G_k \kappa_x \left( \frac{\partial \psi}{\partial y} - z \right), \quad \tau^{(k)}_{xz} = G_k \kappa_x \left( \frac{\partial \psi}{\partial z} + y \right).$$

Further, torsional stresses will be determined using the Prandtl stress function $\phi$ [20], based on the expression of stresses in a homogeneous rod

$$\tau_{xy} = \frac{\partial \phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y},$$

we represent them for an inhomogeneous rod in the form of

$$\tau^{(k)}_{xy} = G_k \kappa_x \frac{\partial \phi}{\partial z}, \quad \tau^{(k)}_{xz} = -G_k \kappa_x \frac{\partial \phi}{\partial y}. \quad (16)$$

Here, $\phi$ is the approximation of the torsional strain function, uniform for all phases of the cross section. Let us take, the function of deflection of a flexible membrane $w$ fixed to the section contour ($w \rightarrow \phi$) according to the membrane analogy. We approximate the membrane deflections by the expression

$$\phi = bh[\text{ch}(\alpha_{0z}) - \text{ch}(\alpha_{0z} \xi_z) \text{ch}(\alpha_{0y}) - \text{ch}(\alpha_{0y} \xi_y)], \quad (17)$$

$$\xi_y = 2y/h, \quad \xi_z = 2z/h,$$

satisfying the required condition $\phi = 0$ at the boundary $\xi_y = \xi_z = \pm 1$. The set scale parameters $\alpha_{0z}$, $\alpha_{0y}$ determine the intervals of hyperbolic functions in (17), used for approximation in a rectangular region of a section. In contrast to the trigonometric approximation, the form (17) gives the physically correct sign of the second stresses’ derivatives $\partial^2 \tau_{xz} / \partial z^2$, $\partial^2 \tau_{xy} / \partial y^2$.

Substituting the stresses (16), written with function (17), into the equilibrium condition for the torque

$$M_f(x) = \sum \int \int \tau^{(k)}_{xz} y - \tau^{(k)}_{xy} z dA,$$

torsional shear stress formula

$$\tau^{(k)}_{xz} = \frac{2G_k M_f}{D_t} h \alpha_{0z} \text{sh}(\alpha_{0z} \xi_z)[\text{ch}(\alpha_{0z}) - \text{ch}(\alpha_{0z} \xi_z)],$$

$$\tau^{(k)}_{xy} = -\frac{2G_k M_f}{D_t} h \alpha_{0z} \text{sh}(\alpha_{0z} \xi_z)[\text{ch}(\alpha_{0z}) - \text{ch}(\alpha_{0z} \xi_z)]$$

and torsional stiffness
\[ D_x = 2 \sum_{k=1}^{4} G_k \left[ b a_0, \text{sh}(a_0, \xi_z) \left[ \text{ch}(a_0, \xi_z) - \text{ch}(a_0, \xi_z) \right] - h a_0, \text{sh}(a_0, \xi_z) \left[ \text{ch}(a_0, \xi_z) - \text{ch}(a_0, \xi_z) \right] \right] \text{d}A, \]  

included in the physical generalized equality under torsion

\[ D_x \kappa_x = M_x. \]  

2.4. The formulation of the boundary value problem

As it is known, the physical rod structure does not affect the form of the equilibrium conditions and its formed kinematic relations. Let us imagine a system of differential equations in displacements for the considered types of deformation with stiffness characteristics (6), (14), (18) and physical relations (4) taking into account symmetry, (13), (19)

\[
\begin{align*}
\frac{d}{dx} \left( D_0 \frac{du}{dx} \right) &= q_x - \frac{dN_x}{dx}, \\
\frac{d^2}{dx^2} \left[ D_{s1} \left( \frac{d^2 v}{dx^2} - \frac{d^2 y_0}{dx^2} \right) \right] &= q_y - \frac{d^2 M_{s1}}{dx^2}, \\
\frac{d^2}{dx^2} \left[ D_{s2} \left( \frac{d^2 w}{dx^2} - \frac{d^2 y_0}{dx^2} \right) \right] &= q_z - \frac{d^2 M_{s2}}{dx^2}, \\
\frac{d}{dx} \left( D_x \frac{d\theta_x}{dx} \right) &= m_x.
\end{align*}
\]

The system is supplemented by twelve kinematic and static conditions corresponding to the fixing method the ends of the rod. The latter are written, using physical dependencies (4), (13), (19).

3. Results

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\frac{d}{dx} \left( D_0 \frac{du}{dx} \right) &= q_x - \frac{dN_x}{dx}, \\
\frac{d^2}{dx^2} \left[ D_{s1} \left( \frac{d^2 v}{dx^2} - \frac{d^2 y_0}{dx^2} \right) \right] &= q_y - \frac{d^2 M_{s1}}{dx^2}, \\
\frac{d^2}{dx^2} \left[ D_{s2} \left( \frac{d^2 w}{dx^2} - \frac{d^2 y_0}{dx^2} \right) \right] &= q_z - \frac{d^2 M_{s2}}{dx^2}, \\
\frac{d}{dx} \left( D_x \frac{d\theta_x}{dx} \right) &= m_x.
\end{align*}
\]

The system is supplemented by twelve kinematic and static conditions corresponding to the fixing method the ends of the rod. The latter are written, using physical dependencies (4), (13), (19).

The obtained relations allow one to perform approximate formulations and solutions of various boundary-value direct and inverse problems, including: identifying the stress-strain state of a composite rod under thermal power, evaluating its strength and stiffness, identifying rational
geometric and structural parameters of the inhomogeneous structure of the rod, and optimization problems.

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