We propose an optical implementation of the Gaussian continuous-variable quantum cloning machines. We construct a symmetric $N \rightarrow M$ cloner which optimally clones coherent states and we also provide an explicit design of an asymmetric $1 \rightarrow 2$ cloning machine. All proposed cloning devices can be built from just a single non-degenerate optical parametric amplifier and several beam splitters.

Quantum information theory was originally developed in the context of the discrete quantum variables, with the central notion of a single qubit as a unit of quantum information. Recently, however, various concepts of the quantum-information processing have been extended to the domain of continuous quantum variables. The established continuous-variable protocols comprise the quantum state teleportation [7], quantum error correction [3], quantum cryptography [3], quantum computation [4], entanglement purification [5], quantum dense coding [6], quantum teleportation [1], quantum error correction [2], and, finally, quantum cloning [7].

In this paper we propose a simple optical implementation of the Gaussian continuous-variable cloning machine introduced by Cerf et al. [7]. It is well known that the exact cloning of an unknown quantum state is impossible [8]. The cloning machine thus produces approximate copies of the state subject to cloning. We first design an asymmetric $1 \rightarrow 2$ cloning machine which produces two copies of coherent state with different fidelities. Then we show how to build a symmetric $N \rightarrow M$ cloner which prepares $M$ identical approximate clones of $N$ available copies and optimally clones coherent states. It turns out that the cloning machines can be built from a single non-degenerate optical parametric amplifier (NOPA, two-mode squeezer) and several beam splitters.

We note that the experimental realization of the Gaussian cloning machine has been recently discussed in [9] where three NOPAs were used to build the machine. It is, however, impossible to exactly construct the machine solely from two-mode squeezers. Therefore, the device proposed in [9] is only approximate, and the desired cloning transformation is achieved only in the limit of infinite squeezing. Such problems are avoided if one employs beam splitters together with a two-mode squeezer.

Let us begin with $1 \rightarrow 2$ cloners. Suppose that the mode $c$ initially contains an unknown quantum state that we want to clone. At the output of the cloning machine, two duplicates of the input state $|\psi\rangle_c$ emerge at the modes $a$ and $c$. The implementation of cloning also requires an ancilla mode $b$. At the input, modes $a$ and $b$ are in vacuum states and the total three mode state can be written as $|0\rangle_a|0\rangle_b|\psi_{in}\rangle_c$. The unitary cloning transformation performed by the cloning machine yields the output state

$$|\psi_{out}\rangle = e^{-i(U+V)} e^{-i\chi Y}|0\rangle_a|0\rangle_b|\psi_{in}\rangle_c.$$

Here

$$U = i(ca^\dagger - c^\dagger a), \quad V = i(cb - c^\dagger b^\dagger), \quad Y = i(ab-a^\dagger b^\dagger),$$

$$a, b, c (a^\dagger, b^\dagger, c^\dagger)$$ denote annihilation (creation) operators of three different modes of propagating optical field, and $\chi$ is a parameter controlling the asymmetry of the two clones. If $\chi = (\ln 2)/2$ then the cloning is symmetric. It is thus convenient to define the new parameter $\gamma = \chi - (\ln 2)/2$.

Notice that in Ref. [7] the cloning transformation is defined as $\exp[-i(U + V)]$ and the modes $a$ and $b$ enter the cloning machine in an entangled two-mode squeezed state $\exp(-i\chi Y)|0\rangle_a|0\rangle_b$. Here we assume that $a$ and $b$ are initially in the vacuum state and we define the cloning transformation as

$$C = \exp[-i(U + V)] \exp(-i\chi Y).$$

The Hermitian operators $V$ and $Y$ are generators of two-mode squeeze transformations which can be performed by NOPAs. $U$ is the generator of a mixing transformation, which can be implemented with the use of a beam splitter. It is easy to verify that $U$, $V$ and $Y$ form a closed commutator algebra and they thus generate a three parameter sub-group of the symplectic group $Sp(6, R)$.

For our purposes it is convenient to work in the Heisenberg picture, where an arbitrary operator $X$ transforms according to $X \rightarrow C^\dagger X C$. After some algebra we obtain the cloning transformation in the Heisenberg picture,

$$a_{out} = c_{in} + e^{\gamma/2}(a_{in} - b_{in}^\dagger),$$

$$b_{out} = -\sqrt{2}\sinh \gamma a_{in}^\dagger + \sqrt{2}\cosh \gamma b_{in} - c_{in}^\dagger,$$

$$c_{out} = c_{in} - e^{-\gamma/2}(a_{in} + b_{in}^\dagger).$$

It was shown in [10] that the symmetric cloning machine corresponding to $\gamma = 0$ in Eq. (4) is optimal for cloning of coherent states.
In this sense, the asymmetric cloner is optimal.

A generic symplectic cloning transformation has the form

\[ \frac{d,e,f,g}{a} \] have the same fidelity \( F \) as defined in Eq. (6).

The cloning transformation (6) does not prefer any direction in the phase space so that clones of position and momentum states suffer from the same errors. This leads to the constraints \([d,e] = [f,g] = 0\). If this holds, then the Wigner functions of the output modes \( a_{\text{out}} \) and \( c_{\text{out}} \) are convolutions of the Wigner function of \(|\psi_{\text{in}}\rangle\), with a phase independent Gaussian. One can show for generic cloning transformation (6) that the product of noise photons is limited from below by

\[ \langle n_{\text{ch}}\rangle_a \langle n_{\text{ch}}\rangle_c \geq \frac{1}{4} \] (7)

and the transformation (6) achieves the lower bound 1/4. In this sense, the asymmetric cloner is optimal.

The Q-functions of the cloned coherent states read

\[ Q_{a,c}(\alpha) = \frac{1}{(\langle n_{\text{ch}}\rangle_{a,c} + 1)\pi} \exp \left(-\frac{|\alpha - \xi|^2}{\langle n_{\text{ch}}\rangle_{a,c} + 1} \right) \] (8)

and the fidelities of the two clones of the coherent state \(|\xi\rangle\) can be expressed as

\[ F_a = \langle \xi | a_{\text{out}} | \xi \rangle = \pi Q_a(\xi) = \frac{2}{e^{2\gamma} + 2}, \]
\[ F_c = \langle \xi | c_{\text{out}} | \xi \rangle = \pi Q_c(\xi) = \frac{2}{e^{-2\gamma} + 2}. \] (9)

The fidelities are invariant for all coherent states. A symmetric cloning machine has \( \gamma = 0 \) and here both copies have the same fidelity \( F_a = F_c = 2/3 \) [3].

The explicit construction of the cloning machine can be based on simple group-theoretical considerations. Since the cloning transformation \( I \) belongs to the three-parametric sub-group of the group \( Sp(6,\mathbb{R}) \), we need three optical elements (beam splitters and two-mode squeezers) to build a device performing this transformation. We construct the cloning machine from two beam splitters and one squeezing element because beam splitters are much simpler and cheaper optical elements than two-mode squeezers.

The cloning transformation \( C \) can be factorized as

\[ C = e^{-iuv}e^{-i\gamma w}e^{-iwU}, \] (10)

where \( u, v, w \) are unknown parameters that should be determined. The factorization of a complicated symplectic transformation into a sequence of simpler evolutions is a powerful technique which allows us to get some physical insight into the nature of \( C \). Recently, this approach has been applied to the analysis of nonlinear optical couplers [11]. Here we adopt the same strategy to construct the cloning machine. A physical device corresponding to the factorization (10) is depicted in Fig. 1. The input modes \( a_{\text{in}} \) and \( c_{\text{in}} \) are mixed at the first beam splitter (BS1) and \( w \) is the mixing angle of this beam splitter. Subsequently, the modes \( a_1 \) and \( b_1 \), are squeezed in NOPA and \( v \) is the squeezing parameter. Finally, the modes \( a_1 \) and \( c_2 \) are re-combined at BS2 whose mixing angle is denoted by \( w \).

Remarkably, the cloning machine depicted in Fig. 1 is essentially a Mach-Zehnder interferometer with a NOPA placed in one of its arms. The proposed scheme is also closely related to the generic decomposition of linear optical circuits into sequence of \( N \)-port linear interferometers, \( N \) single-mode squeezers, and another \( N \)-mode interferometer put forward by Braunstein [13]. Our device has the added convenience of using a single two-mode squeezing element, which leads to the simplest possible structure.

In order to derive analytical formulas for the parameters \( u, v, w \) as functions of \( \gamma \) we consider the propagation of the output modes backward through the cloning machine. When we transfer the output modes \( a_{\text{out}} \) and \( c_{\text{out}} \) back through BS2, we get

\[ a_1 = \cos(w)a_{\text{out}} - \sin(w)c_{\text{out}}, \]
\[ c_2 = \cos(w)c_{\text{out}} + \sin(w)a_{\text{out}} \] (11)

On inserting (11) into (12), we have

\[ a_1 = \frac{1}{\sqrt{2}} \left(e^\gamma \cos w + e^{-\gamma} \sin w\right) a_{\text{in}} \]
\[ + \frac{1}{\sqrt{2}} \left(e^{-\gamma} \sin w - e^\gamma \cos w\right) b_{\text{in}}^1 \]
\[ + (\cos w - \sin w) c_{\text{in}}, \] (12)

and a similar expression holds for \( c_2 \). Before mixing at the second beam splitter, mode \( a_1 \) is not coupled to \( b_{\text{in}} \). Therefore the coefficient in front of \( b_{\text{in}}^1 \) in Eq. (12) must be zero and we obtain

![FIG. 1: Continuous-variable quantum cloning machine. The device consists of two beam splitters BS1 and BS2 and a non-degenerate parametric amplifier (NOPA). M denote auxiliary mirrors.](image-url)
After squeezing, the mode $c_1$ is mixed with $a_{1,\text{in}}$ at the beam splitter $\text{BS}_1$. The mode $a_{1,\text{out}}$ contains the first approximate copy. The output $c_2$ is split at the $\text{BS}_2$ and so on, until we reach the last beam splitter $\text{BS}_{M-1}$. The transformation performed by the $j$th beam splitter can be written as [13]

$$a_{j,\text{out}} = \sqrt{\frac{1}{M-j+1}} c_j + \sqrt{\frac{M-j+1}{M-j}} a_{j,\text{in}},$$

$$c_{j+1} = \sqrt{\frac{M-j}{M-j+1}} c_j - \sqrt{\frac{1}{M-j+1}}a_{j,\text{in}}.$$  \hspace{1cm} (19)

It is easy to verify that

$$a_{j,\text{out}} = c_{\text{in}} - \sqrt{\frac{M-1}{M}}b_{\text{in}} + \sum_{k=1}^{j} \beta_{jk}a_{k,\text{in}},$$

$$c_{M} = c_{\text{in}} - \sqrt{\frac{M-1}{M}}b_{\text{in}} + \sum_{k=1}^{M-1} \beta_{Mk}a_{k,\text{in}},$$ \hspace{1cm} (20)

where the coefficients $\beta_{jk}$ can be determined from [19].

If $c_{\text{in}}$ is prepared in the coherent state $|\xi\rangle$ then the $Q$-function of all $M$ clones can be expressed as

$$Q(\alpha) = \frac{1}{\pi} \frac{M}{2M-1} \exp\left( -\frac{M}{2M-1}|\alpha-\xi|^2 \right),$$ \hspace{1cm} (21)

and the cloning fidelity reads $F = \frac{M}{2M-1}$, which is exactly the upper bound of the fidelity for a $1 \rightarrow M$ cloner [10]. Thus we have proven that the machine shown in Fig. 2 is the optimal one.

We are now in the position to address the most complicated case, the optimal $N \rightarrow M$ cloner. The key observation is to notice that the noise, which spoils the quality of each copy, stems from the amplification in the NOPA. If $N$ copies of the coherent state subject to cloning are available, then it would be optimal to first collect all the signal into a single mode, then to amplify this mode in a NOPA and, finally, to distribute the amplified signal among $M$ output modes in the same manner as in Fig. 2. A scheme of such an $N \rightarrow M$ cloner is given in Fig. 3. The $N$ available copies of the quantum state subject to cloning are stored in modes $c_{1,\text{in}}, \ldots, c_{N,\text{in}}$. The first interferometer IF$_1$ contains a chain of $N-1$ beam splitters. Let us denote by $d_{j+1}$ a mode propagating inside the IF$_j$ between $j$-th and $j+1$-th beam splitters. The unitary transformation performed by IF$_1$ can be written down as

$$d_{j+1} = \sqrt{\frac{j}{j+1}} d_j + \sqrt{\frac{1}{j+1}}c_{j+1,\text{in}},$$

$$c_{j+1,\text{out}} = \frac{1}{\sqrt{j+1}} d_j - \sqrt{\frac{j}{j+1}} c_{j+1,\text{in}},$$  \hspace{1cm} (22)

where $j = 1, \ldots, N-1$, $d_1 \equiv c_{1,\text{in}}$ and $d_N \equiv c_{1,\text{out}}$. If the $N$ modes $c_{j,\text{in}}$ are all in the same coherent state $|\xi\rangle$ then
and $\langle n_{ch}\rangle$ chaotic photons. The fidelity can be obtained as
\begin{equation}
F = \frac{1}{\langle n_{ch}\rangle + 1} = \frac{MN}{MN + M - N},
\end{equation}
which is exactly the upper bound on fidelity of $N \rightarrow M$ cloner derived in [10], hence the machine shown in Fig. 3 is an optimal $N \rightarrow M$ cloner of coherent states.

In summary, we have proposed an optical implementation of the continuous-variable cloning machine. The asymmetric $1 \rightarrow 2$ cloner is a Mach-Zhender interferometer with a NOPA in one of its arms. The symmetric $N \rightarrow M$ cloner consists of an $N$-port linear interferometer followed by a NOPA and an $M$-port interferometer.

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FIG. 3. Symmetric $N \rightarrow M$ quantum cloning machine. The $N$-port interferometer IF$_1$ contains a sequence of $N - 1$ beam splitters. Similarly, the $M$-port interferometer IF$_2$ is a chain of $M - 1$ beam splitters. Mode $c_{1,\text{out}}$ is amplified in non-degenerate parametric amplifier (NOPA).

The $N - 1$ output modes $c_{2,\text{out}}, \ldots, c_{N,\text{out}}$ are in vacuum state and all the signal is collected in mode $c_{1,\text{out}}$ which is in the coherent state with amplitude $\sqrt{N}\xi$.

In the NOPA, the mode $c_{1,\text{out}}$ is amplified by a factor of $\sqrt{M/N}$, and we have
\begin{align}
    c_1 &= \sqrt{\frac{M}{N}} c_{1,\text{out}} - \sqrt{\frac{M-N}{N}} b_{\text{in}}^\dagger, \\
    b_{\text{out}} &= \sqrt{\frac{M}{N}} b_{\text{in}} - \sqrt{\frac{M-N}{N}} c_{1,\text{out}}^\dagger.
\end{align}

Finally, the amplified signal is fed into a $M$-port interferometer IF$_2$ where it is split into $M$ output modes, thereby preparing $M$ approximate clones of the coherent state $|\xi\rangle$. The interferometer IF$_2$ is exactly the sequence of $M - 1$ beam splitters shown in Fig. 3. If we use $e_j$ to denote a mode propagating between the $j$th and the $j + 1$th beam splitter inside IF$_2$ then the unitary transformation performed by IF$_2$ is given by the formulas [19] where we have to replace $c_j$ with $e_j$. To avoid possible confusion, we rewrite it explicitly,
\begin{align}
    a_{j,\text{out}} &= \sqrt{\frac{1}{M-j+1}} e_j + \sqrt{\frac{M-j}{M-j+1}} a_{j,\text{in}}, \\
    e_{j+1} &= \sqrt{\frac{M-j}{M-j+1}} e_j - \sqrt{\frac{1}{M-j+1}} a_{j,\text{in}}.
\end{align}

The $M$ approximate copies emerge in modes $a_{j,\text{out}}$, $j = 1, \ldots, M - 1$, and $e_M$. The amplification (23) spoils the signal with $(M-N)/N$ chaotic photons which are equally distributed among the $M$ copies. Thus each copy contains $\langle n_{ch}\rangle = (M-N)/(MN)$ chaotic photons. Assuming an input coherent state $|\xi\rangle$, all copies are prepared in thermalized coherent state with the complex amplitude $\xi$. 

\[\text{[Diagram]}\]