Pfaff systems, currents and hulls

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June 21, 2016

Abstract

Let $S$ be a Pfaff system of dimension 1, on a compact complex manifold $M$. We prove that there is a positive $\ddbar$-closed current $T$ of bidimension $(1, 1)$ and of mass 1 directed by the Pfaff system $S$. There is no integrability assumption. We also show that local singular solutions always exist. Under a transversality assumption of $S$ on the boundary of an open set $U$, we prove the existence in $U$ of positive $\ddbar$-closed currents directed by $S$ in $U$.

Using $i\partial\bar{\partial}$-negative currents, we discuss Jensen measures, local maximum principle and hulls with respect to a cone $\mathcal{P}$ of smooth functions in the Euclidean complex space, subharmonic in some directions. The case where $\mathcal{P}$ is the cone of plurisubharmonic functions is classical. We use the results to describe the harmonicity properties of the solutions of equations of homogeneous, Monge-Ampère type. We also discuss extension problems of positive directed currents.

Classification AMS 2010: Primary: 37A30, 57R30; Secondary: 58J35, 58J65, 60J65.

Keywords: Pfaff systems, hulls, directed current

1 Introduction

Positive currents play an important role in complex dynamics, especially in the theory of singular foliations by Riemann surfaces. The analog of invariant measures for discrete dynamical systems turns out to be the positive $\ddbar$-closed currents directed by the foliation. The existence of such currents is proved in [1]. See also [28], [18] and [10].

Surprisingly enough, it turns out that such currents do exist without any integrability condition.

Theorem 1.1. Let $(M, \omega)$ be a compact complex Hermitian manifold of dimension $k$. Let $(\alpha_1, \ldots, \alpha_{k-1})$ be continuous $(1, 0)$-forms on $M$. Then, there exists a positive current $T$ on $M$, of mass 1 and of bidimension $(1, 1)$, such that $T \wedge \alpha_j = 0$ for $j = 1, \ldots, k - 1$, and $i\partial\bar{\partial}T = 0$.
The proof is a variation of the proof in [1]. This should be the first step in order to study the global dynamics of some Pfaff systems.

In the second part of the article, we revisit some results about polynomial convexity and hulls from [13]. We develop the study of convexity in the context of $\Gamma$-directed currents. The main new tool is a refinement of a maximum principle proved in [1] for plurisubharmonic functions with respect to a positive current $T$, such that $i\partial\bar{\partial}T \leq 0$.

For simplicity we study the convexity theory mostly in $\mathbb{C}^k$. For $z \in \mathbb{C}^k$, let $\Gamma_z$ be a closed cone of $(1,0)$-vectors and assume $\Gamma := \cup_{z \in \mathbb{C}^k} \Gamma_z$ is closed.

Let $\mathcal{P}_T$ denote the cone of smooth functions $u$, such that for all $\xi \in \Gamma_z$,

$$\langle i\partial\bar{\partial}u(z), i\xi \wedge \bar{\xi} \rangle \geq 0.$$ 

When for all $z$, $\Gamma_z$ is the cone of all $(1,0)$-vectors, we get $\mathcal{P}_0$ the cone of smooth plurisubharmonic functions (psh for short).

It is natural to introduce the polar of $\mathcal{P}_T$, i.e. the positive currents $T$, of bidimension $(1,1)$, such that for every cutoff function $\chi$, and for every $u \in \mathcal{P}_T$, $\langle \chi T, i\partial\bar{\partial}u \rangle \geq 0$.

Such currents are the $\Gamma$-directed currents. Notions of positive $\Gamma$-directed currents are used in [28] and [1].

There is a natural notion of hull of a compact $K$ with respect to $\mathcal{P}_T$. We define

$$\hat{K}_\Gamma := \left\{ z \in \mathbb{C}^k : u(z) \leq \sup_{K} u, \text{ for every } u \in \mathcal{P}_T \right\}.$$ 

Let $\mu_z$ be a Jensen measure representing $z \in \hat{K}_\Gamma$, with respect to $\mathcal{P}_T$. Then, there is a $\Gamma$-directed current $T_z \geq 0$, with compact support, such that $i\partial\bar{\partial}T_z = \mu_z - \delta_z$. Here $\delta_z$ denotes the Dirac mass at $z$. This permits to get local singular solutions for Pfaff systems and more generally for distributions of cones in the tangent bundle. The computation $\mathcal{P}_T$ hulls appears as a problem in control theory.

The representation of Jensen measures is proved for $\mathcal{P}_0$ in [13]. It has been extended by Harvey-Lawson [21] to hulls in calibrated geometries.

The previous result permits to give a description of $\hat{K}_\Gamma \setminus K$ as a union of supports of currents $T_z$. They replace somehow, the analytic varieties with boundary in $K$, which are known not to always exist.

We also prove an extension of Rossi’s local maximum principle. In the context of polynomial convexity the statement is as follows.

Let $\bar{K}$ denote the polynomial convex hull of $K$. Let $V$ be a neighborhood of a point $z_0 \in \hat{K} \setminus K$. Then for every psh function $u$,

$$u(z_0) \leq \sup \left\{ u(\zeta) : \zeta \in (\bar{K} \cap \partial V) \cup (V \cap K) \right\}.$$ 

Surprisingly enough, we get a similar statement for $\mathcal{P}_T$, and a new proof of Rossi’s maximum principle, without using any deep several complex variables.
result, as it is classical, see [25] and [22]. We use the maximum principle for appropriate currents. This turns out to be a consequence of Green’s formula.

To study the above notion of hull, we show that if \( \Gamma_z \neq 0 \) for every \( z \) in an open set \( U \), then \( U \) is contained in the \( \Gamma \)-hull of its boundary. As a consequence, under a transversality assumption of the cones \( \Gamma_z \) to \( \partial U \), we prove the existence of positive \( \partial \bar{\partial} \)-closed currents directed by \( \Gamma \) in \( U \).

We then develop the Perron method for Monge-Ampère equations in the context of \( \Gamma \)-hulls. Let \( B \) denote the unit ball in \( C^k \). In the classical context, Bedford and Taylor [2] proved that any smooth function \( v \) on \( \partial B \) can be extended as a psh function \( u \) in \( B \) and the extension is in \( C^{1,1} \). Moreover, \( (i\partial \bar{\partial}u)^k = 0 \) in \( B \). It turns out, that even when \( v \) is a real polynomial, the corresponding function \( u \) is not necessarily of class \( C^2 \), see [20]. The fact that \( u \) is only in \( C^{1,1} \), is crucial in applications.

It was observed by the author, long ago, that for such functions, there is not always a non-trivial holomorphic disc \( D_z \) through \( z \in B \), such that \( u \) restricted to \( D_z \) is harmonic. The example is based on the existence of non-trivial polynomial hulls without analytic structure.

A recent paper by Ross-Nyström [24] describes a large class of geometric examples, with a similar phenomenon. This raise the following question. Let \( u \) be a continuous psh function in the closed unit ball \( \bar{B} \) of \( C^k \). Assume \( (i\partial \bar{\partial}u)^k = 0 \) in \( B \). What are the analytic objects on which, the function \( u \) is harmonic? We show here, that there is always a Jensen measure \( \mu_z \) and a positive current \( T_z \), of bidimension \((1,1)\), such that \( i\partial \bar{\partial} T_z = \mu_z - \delta_z \) in \( B \) and \( i\partial \bar{\partial} u \wedge T_z = 0 \). So the function \( u \) is \( T_z \)-harmonic.

In Section 5, we show that a function which is locally a supremum of functions of \( P_\Gamma \), is globally a supremum of such functions. This type of results is discussed in [20] in the context of function algebras.

In the last section, we introduce \( P_\Gamma \)-pluripolar sets and we study extension Theorems for positive \( P_\Gamma \)-directed currents.

Acknowledgements. It is a pleasure to thank T.C. Dinh and M. Paun for their interest and comments.

2 Pfaff equations on complex manifolds

Denote by \( D \) the unit disc in \( \mathbb{C} \). A holomorphic map \( \varphi_a : D \to M, \varphi_a(0) = a \), is called a holomorphic disc. It is non degenerate at \( a \) if the derivative at 0 is non zero.

**Theorem 2.1.** Let \( X \) be a compact set in a complex Hermitian manifold \( (M, \omega) \) of dimension \( k \). Let \( E \) be a locally pluripolar set in \( M \) (not necessarily closed). Assume \( X \setminus E = X \). Let \( (\alpha_1, \ldots, \alpha_m) \) be continuous \((1,0)\)-forms on \( M \). Assume that for each \( a \in X \setminus E \), there is a holomorphic disc, with image in \( X \), \( \varphi_a : D \to M \), such that \( \varphi_a \) is...
$D \to X$, $\varphi_a(0) = a$, non degenerate at $a$, such that $(\varphi_a^* \alpha_j)(a) = 0$, for $j = 1, \ldots, m$. Then there exists a positive current $T$, supported on $X$, of mass $1$ and of bidimension $(1,1)$, such that $T \wedge \alpha_j = 0$ for $j = 1, \ldots, m$, and $i\partial \bar{\partial} T = 0$.

**Proof.** The difference with Theorem 1.4 in [1] is that we do not assume here that the holomorphic disc $\varphi_a$ satisfies $(\varphi_a^* \alpha_j)(z) = 0$, for every $z$ in $D$. So we do not assume that the system $S$ has local solutions, which are holomorphic discs.

Recall that $E$ is locally pluripolar, if for every $a \in M$, there is a neighborhood $U$ of $a$ and a psh function $v$ in $U$ such that $E \cap U \subset \{z \in U : v(z) = -\infty\}$. Let

$$C := \left\{ T : T \geq 0 \text{ bidimension } (1,1), \text{ supported on } X, \int T \wedge \omega = 1, \right.$$

$$T \wedge \alpha_j = 0, \ 1 \leq j \leq m \}. \right.$$  

Since the $\alpha_j$ are continuous, $C$ is a convex compact set in the space of bidimension $(1,1)$-currents. Let

$$Y := \{ i\partial \bar{\partial} u, \ u \text{ test smooth function on } M \}^\perp. \right.$$  

The space $Y$ is the space of the $i\partial \bar{\partial}$-closed currents. We have to show that $C \cap Y$ is nonempty. Suppose the contrary. The compact convex set $C$ is in the dual of a reflexive space. The Hahn-Banach theorem implies that we can strongly separate $C$ and $Y$. Hence, there is $\delta > 0$ and a test function $u$ such that $(i\partial \bar{\partial} u, T) \geq \delta$ for every $T \in C$. There is a point $z_0 \in X$ where $u$ attains its maximum. Choose $r > 0$, such that $E \cap B(z_0, r) \subset \{v = -\infty\}$, $v$ being psh in $B(z_0, r)$. Then the function $u_1(z) := u(z) - (\delta/4)|z - z_0|^2$ has a maximum in $B(z_0, r)$, at $z_0$. We get also, that for every Dirac current in $C$, $(i\partial \bar{\partial} u_1, T) \geq \delta/4$.

For $\epsilon > 0$ small enough, $u_2 := u_1 + \epsilon v$ will still have a maximum at a point $a \in X \setminus E$, near $z_0$. Consider a non degenerate holomorphic disc with center at $a$, contained in $X$, such that $\varphi_a^*(a) = 0$, for $1 \leq j \leq m$. So the disc is tangent at $a$, to the distribution $(\alpha_j)_{1 \leq j \leq m}$. Then $u_2 \circ \varphi_a$ has a maximum at $0$. Since $\varphi_a$ is non degenerate at $a$, $i\partial \bar{\partial} (u_1 \circ \varphi_a) > 0$ at $0$, and hence in a neighborhood of $0$. Indeed, we just have to apply the above estimate to a positive Dirac current directed by $\varphi_a(0)$. Adding $v$, preserves the strict subharmonicity. This contradicts the local maximum principle for the subharmonic function $u_2 \circ \varphi_a$ in a disc. Hence $C \cap Y = \emptyset$. The theorem is proved.  

**Remark 2.2.** The condition on the continuity of $(\alpha_j)_{1 \leq j \leq k-1}$ can be relaxed. It is enough to assume that $z \mapsto i\alpha_j \wedge \bar{\alpha}_j$ is lower-semi continuous (lsc for short) in the convex salient cone of positive $(1,1)$ forms. Then if $T_n \in C$ converge to $T$,

$$\liminf_n \int T_n \wedge i\alpha_j \wedge \bar{\alpha}_j \chi = 0$$
for any positive cutoff function \( \chi \). It follows that the convex set \( C \) in the proof of Theorem 2.1 is closed.

**Examples 2.3.** 1. Let \( u \) be a function in \( \mathbb{P}^2 \) of class \( \mathcal{C}^2 \). Consider the positive current \( S := i\partial u \wedge \overline{\partial u} \). It is directed by the Pfaff form \( \alpha := \partial u \). It is shown in [18] that if \( u \) is real non-constant, then \( i\partial S \) is not identically zero. According to Theorem [11], there is a current \( T \) positive and \( \partial \bar{\partial} \)-closed, directed by \( i\partial u \wedge \overline{\partial u} \). This means that on a set where \( \partial u \) is non-zero, there is a positive \( \sigma \) finite measure \(|T|\), such that on that set
\[
T = i|T|\partial u \wedge \overline{\partial u}
\]
is \( \partial \bar{\partial} \)-closed. In particular if \( i\partial u \wedge \overline{\partial u} \) vanishes only on a finite set of points we get the above representation globally.

2. Let \( X \) be a compact set in \( M \). Assume that \( f \) is a holomorphic self-map in \( M \), such that \( f(X) = X \). Assume moreover that \( X \) is hyperbolic in the following dynamical sense. For every \( a \in X \), not contained in a given pluripolar set \( E \), there is a local holomorphic stable manifold \( W^s_a \) manifold of positive dimension, contained in \( X \). The stable manifold \( W^s_a \), is defined as
\[
W^s_a := \{ z \in M : \text{dist}(f^n(z), f^n(a)) < C\rho^N, \ N \geq 1 \},
\]
where \( \rho < 1 \). We assume that \( W^s_a \) contains a holomorphic disc \( D^s_a \) of positive radius and that these discs \( D^s_a \) vary continuously. This is a situation studied in smooth dynamical systems. The proof of Theorem 2.1 shows that there is \( T \geq 0 \) directed by these discs and \( \partial \bar{\partial} \)-closed.

An example of this situation is the one of a hyperbolic Hénon map \( f \) in \( \mathbb{C}^2 \). The set \( X \) is the closure in \( \mathbb{P}^2 \) of set of points with bounded orbits. The set \( E \), is just the unique indeterminacy point of the extension of \( f \) to \( \mathbb{P}^2 \). The situation is studied in detail in [16].

3. Consider \((M, V, E)\) where \( M \) is a compact complex manifold, \( E \) a closed locally pluripolar set and \( V \) is a holomorphic sub-bundle of the restriction of the tangent bundle \( TM \) to the complement of \( E \) in \( M \). If there is no non-zero positive \( \partial \bar{\partial} \)-closed current supported on \( E \), then, we get positive bidimension \((1,1)\), \( \partial \bar{\partial} \)-closed currents directed by \( V \). They should be useful in the study of the Kobayashi metric.

### 3 Convexity and currents in \( \mathbb{C}^k \)

#### 3.1 \( \Gamma \)-directed currents in \( \mathbb{C}^k \), \( \Gamma \)-hulls

Let \( \mathcal{P}_0 \) denote the convex cone of smooth psh functions in \( \mathbb{C}^k \). A function \( u \in \mathcal{P}_0 \) iff \( \langle i\partial \overline{\partial} u(z), i\xi_z \wedge \overline{\xi_z} \rangle \geq 0 \) for every \((1,0)\)-vector \( \xi_z \).

A function is psh if \( u = \lim \downarrow u_\varepsilon \), where \( u_\varepsilon \) are smooth psh functions. We want to study hulls with respect to convex cones containing \( \mathcal{P}_0 \). We will obtain in
this setting, new proofs of classical results like Rossi’s local maximum principle. All the notions can be developed for an open set $U \subset \mathbb{C}^k$ or more generally a manifold $N$ admitting a smooth strictly psh function $\rho$.

We now introduce the cone $\mathcal{P}_\Gamma$. For every $z \in \mathbb{C}^k$, let $\Gamma_z$ be a closed cone of $(1,0)$-vectors. We assume that $\Gamma_z$ always contains non-zero vectors and that $\Gamma := \bigcup_{z \in \mathbb{C}^k} \Gamma_z$ is closed. Define

$$\mathcal{P}_\Gamma := \{ u \in C^2 : \langle i\partial \bar{\partial} u(z), i\xi_z \wedge \bar{\xi}_z \rangle \geq 0 \text{ for every } \xi_z \in \Gamma_z \}.$$ 

The cone $\mathcal{P}_\Gamma$ is convex and contains the cone of smooth psh functions. We define,

$$\overline{\mathcal{P}}_\Gamma := \{ u : u = \lim \downarrow u_\epsilon, \ u_\epsilon \in \mathcal{P}_\Gamma \}.$$ 

We now consider the positive currents of bidimension $(1,1)$, $\Gamma$-directed.

$$\mathcal{P}_\Gamma^0 := \{ T \geq 0, \ \text{of bidimension} \ (1,1), \ T \wedge i\partial \bar{\partial} u \geq 0 \ \text{for every} \ u \in \mathcal{P}_\Gamma \}.$$ 

The cone $\mathcal{P}_\Gamma^0$ is closed for the weak-topology of currents. It is the closed convex hull of the Dirac currents it contains. If $\mathcal{P}_\Gamma = \mathcal{P}_0$, then $\mathcal{P}_\Gamma^0$ are all positive currents. In general it is a smaller class of currents.

An important property of the cone $\mathcal{P}_\Gamma$ that we will use frequently is the following. Let $u_1, \ldots, u_m$ be functions in $\mathcal{P}_\Gamma$ and let $H$ be a smooth function in $\mathbb{R}^m$, convex and increasing in each variable. Then $H(u_1, \ldots, u_m) \in \mathcal{P}_\Gamma$. If $H$ is non-smooth, then $H(u_1, \ldots, u_m) \in \overline{\mathcal{P}}_\Gamma$.

Let $T$ be a positive $\Gamma$-directed current of bidimension $(1,1)$. A function $v$ is $T$-subharmonic, if there is a decreasing sequence $(v_n)$ of smooth functions such that $i\partial \bar{\partial} v_n \wedge T \geq 0$ and $v_n \to v$ in $L^2(\sigma_T)$, where $\sigma_T$ is the trace measure of $T$. The notion can be localized.

A bounded function $u$ is $T$-harmonic, in an open set $U$, if there is a monotone sequence of smooth functions $u_n$, $|u_n| \leq C$ in $U$, such that $\lim \downarrow u_n = u$ or $\lim \uparrow u_n = u$, and $0 \leq i\partial \bar{\partial} u_n \wedge T \to 0$. We will write $i\partial \bar{\partial} u \wedge T = 0$.

We will introduce the notion of hull with respect to $\mathcal{P}_\Gamma$. Let $K$ be a compact set in $\mathbb{C}^k$. We define the hull $\hat{K}_\Gamma$ of $K$ with respect to $\mathcal{P}_\Gamma$ as follows:

$$\hat{K}_\Gamma := \{ z \in \mathbb{C}^k : \ u(z) \leq \sup_{K} u \ \text{for every} \ u \in \mathcal{P}_\Gamma \}.$$ 

When $\mathcal{P}_\Gamma$ is the cone $\mathcal{P}_0$ of smooth psh functions, then $\hat{K}_\Gamma = \hat{K}$, the polynomially convex hull of $K$. We always have that $\hat{K}_\Gamma \subset \hat{K}$. The hull $\hat{K}_\Gamma$ is unchanged if we replace $\mathcal{P}_\Gamma$ by $\overline{\mathcal{P}}_\Gamma$.

We recall few facts from Choquet’s theory, with some refinement by Edwards [6], [14]. Gamelin [19] gives an exposition of their results.

On a compact space $M$, consider a convex cone $\mathcal{E}$ of upper-semicontinuous (usc for short) functions with values in $[-\infty, \infty)$. We assume that $\mathcal{E}$, contains the constants and functions separating points.
A representing measure for \( x \in M \), relatively to \( E \) is a probability measure \( \nu_x \) such that \( \varphi(x) \leq \nu_x(\varphi) \) for every \( \varphi \in E \). It is clear that the set \( J_x(E) \) of representing measures is a convex compact set. The elements of \( J_x(E) \) are called Jensen measures. Define
\[
\partial \varepsilon M := \{ x \in M : J_x(E) = \delta_x \}.
\]
A result by Choquet says that every function in \( E \) attains its maximum at a point of \( \partial \varepsilon M \). Since \( E \) is convex, there is a smallest closed set on which every function attains its maximum. It is called the Shilov boundary \( S(M) \) and \( S(M) = \partial \varepsilon M \).

It is easy to check that the set \( J_x(E) \) is unchanged, if we replace \( E \), by the smallest cone containing \( E \) and stable by finite sup. This cone is also convex, see [14].

**Theorem 3.1.** (Choquet-Edwards) Let \( M \) be a compact set and \( E \) a convex cone of continuous functions from \( M \) to \([-\infty, \infty) \) separating points and containing the constants. Let \( X \) be a compact set in \( M \) containing the Shilov boundary \( S(M) \). For each lsc function \( u : X \to (-\infty, \infty] \), define
\[
\hat{u}(z) := \sup \{ v(z) : v \in E, \ v \leq u \text{ on } X \}.
\]
Then
\[
\hat{u}(z) = \inf \left\{ \int ud\mu_z : \mu_z \text{ Jensen measure for } z \text{ w.r.t. } E, \text{ with support in } X \right\}.
\]

### 3.2 Maximum principle

We now turn to the description of the hull \( \hat{K}_\Gamma \) in terms of \( \Gamma \)-directed currents. The following maximum principle which is a refinement of a result in [1] will be useful.

**Theorem 3.2.** Let \( U \) be a relatively compact open set in \( C^k \). Let \( T \in \mathcal{P}_0^\Gamma \).

i) Assume \( i\partial \bar{\partial} T \leq 0 \), in \( U \). Then for \( z \in \text{supp}(T) \cap U \) we have for any \( v \in \overline{\mathcal{P}}_\Gamma \),
\[
v(z) \leq \sup_{\partial U \cap \text{supp}(T)} v.
\]

ii) Assume \( i\partial \bar{\partial} T = \nu^+ - \nu^- \), with \( \nu^\pm \) positive measures. Suppose \( u \) is a bounded \( T \)-harmonic function. Let \( v \in \overline{\mathcal{P}}_\Gamma \). Assume \( v \leq u \) on \( (\partial U \cap \text{supp}(T)) \cup (U \cap \text{supp}(\nu^+)) \). Then \( v \leq u \) on \( U \cap \text{supp}(T) \).

**Proof.** i) We can assume that \( v \) is smooth. Suppose \( v < 0 \) on \( \partial U \cap \text{supp}(T) \) and that \( v(p) > 0 \), where \( p \) is a point in \( U \cap \text{supp}(T) \). The same inequalities hold if we replace \( v \) by \( v_1(z) := v(z) + \epsilon |z - p|^2 \), with \( \epsilon > 0 \) small enough. We can assume that \( v_1 < 0 \), in a neighborhood \( U_1 \) of \( \partial U \cap \text{supp}(T) \). So the intersection of the
support of \( v_1^+ \) and of \( U \cap \text{supp}(T) \) is compact. Hence we get by integration by parts that:

\[
0 < \int i\partial\bar{\partial}v_1^+ \wedge T = \int v_1^+ i\partial\bar{\partial}T \leq 0.
\]

A contradiction.

We make the argument more explicit. Let \( \rho \) be a cutoff function equal to one in a neighborhood of the support of \( v_1^+ \) in \( \text{supp}(T) \) and which varies on \( \text{supp}(T) \) only on \( U_1 \). If we multiply \( v_1^+ \) with \( \rho \), we get a genuine function with compact support. By integration by part we obtain the above relation. Observe that the function \( \rho \) does not appear. Hence the maximum principle holds for \( \mathcal{P}_\Gamma \) with respect to \( T \).

Since the functions of \( \mathcal{P}_\Gamma \), are decreasing limits of functions in \( \mathcal{P}_\Gamma \) we get also the maximum principle for functions in \( \mathcal{P}_\Gamma \).

ii) As above assume \( v - u < 0 \) on \( (\partial U \cap \text{supp}(T)) \cup (U \cap \text{supp}(\nu^+)) \). Suppose that for \( p \in U \cap \text{supp}(T) \), \( (v - u)(p) > 0 \). Observe that \( \text{supp}(\nu^+) \) is contained in \( \text{supp}(T) \).

Let \( |u_n| \leq c \), be a sequence of smooth functions, such that \( 0 \leq i\partial\bar{\partial}u_n \wedge T \to 0 \) and \( u = \lim \uparrow u_n \). The function \((v - u)\) is usc, hence we can assume that \( v \) is smooth. So for \( n \) large enough, we still have \( v - u_n < 0 \) near \( (\partial U \cap \text{supp}(T)) \cup (U \cap \text{supp}(\nu^+)) \). Replacing \( v \), by \( v + \epsilon|z - p|^2 \), with \( \epsilon > 0 \) small enough, we can also assume that: \( \lim_n i\partial\bar{\partial}(v - u_n) \wedge T \geq \epsilon i\partial\bar{\partial}|z|^2 \wedge T \). Hence

\[
0 < \lim_n \int i\partial\bar{\partial}(v - u_n)^+ \wedge T = \lim_n \int (v - u_n)^+(\nu^+ - \nu^-) \leq \lim_n \int (v - u_n)^+ \nu^+ = 0.
\]

A contradiction. The case where \( u = \lim \downarrow u_n \) is even simpler. So the maximum principle holds.

For the polynomially convex case of the following Corollary see \[1\] and \[11\] Proposition 2.5.

It is proved in \[17\] Corollary 2.6, that the complement of the support of a positive current of bidimension \((p,p)\) such that \( i\partial\bar{\partial}T \leq 0 \) is locally \( p\)-pseudoconvex. If \( p = 1 \), this is just the usual pseudoconvexity. Other notions of pseudoconvexity are used in \[11\] to prove similar results.

**Corollary 3.3.** Let \( K \) be a compact set in \( \mathbb{C}^k \). Let \( T \geq 0 \) be a \( \Gamma \)-directed current of bidimension \((1,1)\) with compact support. Assume \( i\partial\bar{\partial}T \leq 0 \) on \( \mathbb{C}^k \setminus K \). Then \( \text{supp}(T) \subset \mathring{K}_\Gamma \).

**Proof.** Let \( B \) be a large ball containing \( \text{supp}(T) \). Let \( K_r \) be the \(-\)neighborhood of \( K \). We can choose for \( U \) the set \( B \setminus K_r \), with \( r \) arbitrarily small. If \( x \in \text{supp}(T) \setminus K \) and \( u \in \mathcal{P}_\Gamma \), then it follows from the above maximum principle that \( u(x) \leq \sup_K u \). So \( x \in \mathring{K}_\Gamma \).

We now extend a result from \[13\] to hulls with respect to \( \mathcal{P}_\Gamma \).
Theorem 3.4. Let $K$ be a compact set in $\mathbb{C}^k$, contained in a ball $B$. Assume $x \in \hat{K}_\Gamma$. Let $\mu_x$ be a Jensen measure on $K$ representing $x$ for $\mathcal{P}_\Gamma$. There is a positive $\Gamma$-directed current $T_x$, with support in $B$, such that

$$i\partial\bar{\partial}T_x = \mu_x - \delta_x.$$ 

In particular, for every $\varphi \in \mathcal{P}_\Gamma$, we have the Jensen’s formula:

$$\int \varphi d\mu_x = \varphi(x) + \langle T_x, i\partial\bar{\partial}\varphi \rangle.$$ 

Proof. Consider the cone $\mathcal{P}_\Gamma(B)$ of smooth functions on $B$, such that, for all $z$ in $B$, and $\xi_z \in \Gamma_z$

$$\langle i\partial\bar{\partial}u(z), i\xi_z \wedge \bar{\xi}_z \rangle \geq 0.$$ 

It is easy to check, that $\mu_x$ is a Jensen measure for $\mathcal{P}_\Gamma(B)$. We can assume that $u$ extends smoothly to $\mathbb{C}^k$ and add a smooth psh function $v$ vanishing on a slightly smaller ball and such that $u + v$ is in $\mathcal{P}_\Gamma$. Define

$$C := \{i\partial\bar{\partial}T : T \geq 0, \Gamma\text{-directed, supp}(T) \subset B\}.$$ 

Clearly, $C$ is a convex set of distributions. Since $\langle i\partial\bar{\partial}T, |z|^2 \rangle = \langle T, i\partial\bar{\partial}|z|^2 \rangle$, it follows that any bounded set of $C$ is compact. Suppose $\mu_x - \delta_x \not\in C$. Then by Hahn-Banach, there is a smooth function $u$ such that $\int ud\mu_x - u(x) < 0$. Moreover for every $T \in \mathcal{P}_\Gamma^0$, supported on $B$, $\langle i\partial\bar{\partial}T, u \rangle \geq 0$. So every $T \in \mathcal{P}_\Gamma^0$, with supp$(T) \subset B$ satisfies $\langle T, i\partial\bar{\partial}u \rangle \geq 0$.

The last inequality says that $u \in \mathcal{P}_\Gamma(B)$. A contradiction, with the fact that $\mu_x$ is a Jensen measure representing $x$ for $\mathcal{P}_\Gamma(B)$.

The Jensen’s formula is then clear. 

Corollary 3.5. We have: $\hat{K}_\Gamma = \bigcup \text{supp}(T) \cup K$, the first union being taken over all $T \geq 0$, with compact support, $\Gamma$-directed, such that $i\partial\bar{\partial}T \leq 0$, in $\mathbb{C}^k \setminus K$.

Proof. We have seen that if $i\partial\bar{\partial}T \leq 0$, on $\mathbb{C}^k \setminus K$, then supp$(T) \subset \hat{K}_\Gamma$. The previous theorem shows that for every $x \in \hat{K}_\Gamma \setminus K$, there is such a $T$ containing $x$ in its support. 

Corollary 3.6. (Local maximum principle for $\hat{K}_\Gamma$.) Let $U \subseteq \mathbb{C}^k$ be an open set. Then for every $v \in \overline{\mathcal{P}_\Gamma}$ and $x \in U \cap \hat{K}_\Gamma$ we have:

$$v(x) \leq \sup_{(\partial U \cap K) \cup (U \cap K)} v.$$ 

Proof. We can assume that $x \in U \setminus K$. Let $\mu_x$ be a Jensen measure for $x$, supported on $K$. Let $T_x$ be a positive $\Gamma$-directed current such that $i\partial\bar{\partial}T_x = \mu_x - \delta_x$. We then apply the maximum principle for $T_x$ and $\mathcal{P}_\Gamma$. 

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Observe that when $\mathcal{P}_\Gamma = \mathcal{P}_0$, the cone of psh functions, we get Rossi’s local maximum principle (see [23] or [22, p. 74]). Indeed, $\hat{K}$ is the hull of $K$ with respect to $\mathcal{P}_0$.

**Remark 3.7.** For Rossi’s local maximum principle, we can also use the family $\frac{n}{m} \log |\rho|$, with $\rho$ polynomial and $n, m$ integers, without introducing general psh functions and using implicitly that the two notions of hulls are the same, see [22].

We show next that under appropriate assumptions a ball is in the $\mathcal{P}_\Gamma$-hull of its boundary.

**Theorem 3.8.** (Semi-local solutions for directed systems) Let $N$ be a complex manifold, of dimension $k$. Let $\Gamma := \bigcup_{z \in N} \Gamma_z$ be a closed family of cones as in Subsection 3.1 such that $\Gamma_z \neq 0$, for every $z$ in $N$. Assume moreover that $N$ admits a smooth exhaustion function $\rho$ such that for all non-zero $\xi_z \in \Gamma_z$,

$$\langle i\partial \bar{\partial} \rho(z), i\xi_z \wedge \bar{\xi}_z \rangle > 0.$$

Let $U$ be a relatively compact open set in $N$ and let $K = \partial U$. Then $U \subset \hat{K}_\Gamma$. In particular for every $x \in U$, there is a probability measure $\mu_x$ on $\partial U$ and a positive, bidimension $(1,1)$ current $T_x$, with compact support, such that

$$i\partial \bar{\partial} T_x = \mu_x - \delta_x.$$

**Proof.** We consider the cone $\mathcal{P}_\Gamma$ of smooth $\Gamma$-psh functions. We want to prove that, if $K := \partial U$, then $\overline{U} \subset \hat{K}_\Gamma$. Suppose the contrary. Then, there is $x \in U$ and $u \in \mathcal{P}_\Gamma$, such that $u < 0$ on $\partial U$ and $u(x) > 0$. For $0 < \delta \ll 1$, the function $u_1 := u + \delta \rho$ has the same properties. Suppose $u_1$ has a maximum at $a \in U$. We then consider a holomorphic disc $D_a$ tangent at $\xi_a \in \Gamma_a \setminus \{0\}$.

The restriction of $u_1$ to $D_a$ has a positive Laplacean in a neighborhood of $a$, and a maximum at $a$. This is a contradiction. Hence, $x \in \hat{K}_\Gamma$. It suffices then to consider a Jensen measure $\mu_x$ supported on $\partial U$ and to adapt the proof of Theorem 3.4. Using that $\rho$ is an exhaustion function, we can construct first a $\mathcal{P}_\Gamma$-convex set containing $U$. We can compose $\rho$ with convex increasing functions to get an approximation result as in Theorem 3.4.

**Remark 3.9.** As a special case we get a similar statement for Pfaff-systems. Let $(\alpha_j)_{1 \leq j \leq m}$ be $m$ continuous $(1,0)$-forms on $N$. Assume that at any point $z \in N$, the subspace

$$\Gamma_z := \{\xi_z : \alpha_j(z)\xi_z = 0, \quad 1 \leq j \leq m\}$$

is of dimension at least 1. Let $U$ be a relatively compact open set in $N$. Then for every $x \in U$, there is a probability measure $\mu_x$ on $\partial U$, and a positive, bidimension $(1,1)$ current $T_x$, such that

$$T_x \wedge \alpha_j = 0, \quad 1 \leq j \leq m \quad \text{and} \quad i\partial \bar{\partial} T_x = \mu_x - \delta_x.$$
We now address the question of the local existence of $\partial \bar{\partial}$-closed currents. The approach is similar to the one used in [13] to study the polynomial hull of totally real compact sets.

**Theorem 3.10.** Let $N$ be a complex manifold, with a smooth positive exhaustion function $\rho$, as in Theorem 3.8. Let $\Gamma := \bigcup_z \Gamma_z$, be a closed set on the tangent bundle of $N$. We assume that each fiber is a cone and $\Gamma_z \neq 0$, for every $z$ in $N$. Let $U$ be a relatively compact open set in $N$, with connected and smooth boundary defined by $r < 0$. The function $r$ is smooth in a neighborhood of $\overline{U}$ and $dr$ does not vanish on $\partial U$. Assume also that $\overline{U}$ is $\mathcal{P}_\Gamma$-convex.

Assume that there is $z$ in $\partial U$ such that every vector in $\Gamma_z$ is transverse to $\partial U$. Then there is a positive $\Gamma$-directed current $T$ of mass one in $U$ such that $\langle i\partial \bar{\partial} T, \rangle = 0$ in $U$.

**Proof.** Let $V$ denote the non-empty open set of points $a$ in $\partial U$ such that every vector in $\Gamma_a$ is transverse to $\partial U$. For $a$ in $V$ define $C(a)$ as the convex compact set of positive $(1,1)$ $\Gamma$-directed currents $S$, of mass one, supported on $\overline{U}$ and such that $i\partial \bar{\partial} S = \lim (\nu^+_n - \nu^-_n)$. Here, $\nu^+_n$ are positive measures supported on $\partial U$ and $\nu^-_n$ are positive measures whose support converges to the point $a$.

For $S$ in $C(a)$, $i\partial \bar{\partial} S$ is supported on $\partial U$. We need to show that the currents in $C(a)$ are not supported on $\partial U$. Let $a = \lim a_n$ with $a_n$ in $U$. It follows from Theorem 3.8, that there is a sequence $T'_n$ of positive $(1,1)$ $\Gamma$-directed currents and a sequence $\mu_n$ of Jensen measures supported on $\partial U$ such that

$$i\partial \bar{\partial} T'_n = \mu_n - \delta_{a_n}.$$

Define $T_n = c_n T'_n$, where $c_n$ is a constant such that $T_n$ is of mass one. Assume $T = \lim T_n$. We show that $T$ has non-zero mass on $U$. Suppose on the contrary that $T$ is supported on $\partial U$. We prove first that it has zero mass on $V$. Let $\theta$ be a nonnegative test function supported in a neighborhood of a point $x$ in $V$ and identically one near $x$.

From the transversality hypothesis on $V$ it follows that:

$$\langle \theta T, i\partial \bar{\partial} r^2 \rangle = \langle \theta T, 2i\partial r \wedge \bar{\partial} r \rangle,$$

is strictly positive if $T$ has positive mass near $x$ and $T$ is directed by vectors transverse to $V$. The vanishing of $r^2$ to second order on $\partial U$ implies that:

$$\langle \theta T, i\partial \bar{\partial} r^2 \rangle = \langle \theta i\partial \bar{\partial} T, r^2 \rangle = c_n \langle \mu_n - \delta_{a_n}, \theta r^2 \rangle.$$

Since $\mu_n$ is supported on the boundary, the limit is non-positive.

So the positive current $T$ is supported on the compact $Z = \partial U \setminus V$. Since $a$ is not in $Z$, we then have that $i\partial \bar{\partial} T \geq 0$. This is only possible if $T = 0$. Hence $T$ gives mass to $U$. Indeed, if $\rho$ is strictly psh negative in a neighborhood of $Z$, then

$$\langle T, i\partial \bar{\partial} \rho \rangle = \langle i\partial \bar{\partial} T, \rho \rangle.$$
which is negative. So $T = 0$.

Similarly, if $a, a'$ are two points in $V$, then $C(a)$, is disjoint from $C(a')$. If $T$ belongs to the intersection, it is positive and $i\partial \overline{\partial}T \geq 0$. Hence it is equal to zero. 

Remark 3.11. Let $K$ be a compact set on $\partial U$. Assume that $\hat{K}_G$ is non-trivial and contained in $\overline{U}$. Assume also that $\Gamma$ is transverse to $\partial U$. Then there is a positive $\Gamma$-directed current $T$ of mass 1, supported on $\hat{K}_G$, such that $i\partial \overline{\partial}T = 0$ in $U$. Indeed, since currents with negative $i\partial \overline{\partial}$ cannot be compactly supported in $U$, there is a sequence $a_n$ in $U \cap \hat{K}_G$, such that $a = \lim a_n$ is in $K$. We can then apply the scaling in the proof of the Theorem 3.11, in order to construct positive currents.

Corollary 3.12. Let $N$ and $U$ be as in Theorem 3.10. Let $F$ be a foliation by Riemann surfaces in a neighborhood of $\hat{U}$. Assume the leaves are transverse to $\partial U$ on a non empty open set $V$. Then there exists a positive current $T$ directed by $F$ of mass 1 on $U$, satisfying $i\partial \overline{\partial}T = 0$ in $U$.

If the singularities of the foliation are isolated points and each singularity is non degenerate, then the current $T$ is $L^2$ regular. More precisely, there is a $(0, 1)$-form $\tau$ such that $\partial T = \tau \wedge T$ and $\tau$ is $L^2$ integrable with respect to $T$ on $U$.

Proof. For the first part, it suffices to apply Remark 3.11. For the second part, it suffices to use the decomposition on the flow boxes and to to observe as in [18], thanks to the Ahlfors Lemma, that the metric $i\tau \wedge \overline{\tau}$ is bounded by the Poincaré metric. Moreover as shown in [10], Proposition 4.2, the current $T$ has finite mass with respect to the Poincaré metric. The result follows.

Remark 3.13. In [1], $L^2$ estimates for the $\overline{\partial}$ equation on $L^2$-regular currents are proved. One can observe that to get the results there, it is enough to assume that the weights $\phi$ are $T$-subharmonic.

Example 3.14. Consider in $\mathbb{C}^4$ the forms.

$$
\alpha := dz_2 - z_3 dz_1, \quad \beta := dz_3 - z_4 dz_1.
$$

The system $\{\alpha, \beta\}$ does not satisfy the Frobenius condition. Moreover through every point there are infinitely many holomorphic discs. Let $f$ be any holomorphic function. Consider the disc defined by $z_2 := f(z_1)$, $z_3 := f'(z_1)$ and $z_4 := f''(z_1)$. These discs are tangent to $\{\alpha, \beta\}$. The example is just the complex analog of the Engel normal form in Engel’s condition, [3, p. 50].

More generally it is enough to assume that the ideal generated by 2 forms $\{\alpha, \beta\}$ satisfy Engel’s condition at each point, [3, p. 50]. Let $\Gamma_z$ be the space of vectors $\xi_z$ such that, $\alpha(z)\xi_z = 0, \beta(z)\xi_z = 0$. Our description of hulls applies. The hulls are non-trivial. We then get global dynamical systems of interest.
Example 3.15. Let $\rho$ be a smooth function in $\mathbb{C}^k$. Let $\Gamma_z := \ker \partial \rho(z)$. Let $h$ be a positive smooth function on $\mathbb{R}$. Define $T = h(\rho)i\partial \rho \wedge \overline{\partial} \rho$. If the function $h$ is non-decreasing and $\rho$ is psh, then $i\partial \overline{\partial} T \leq 0$. So hulls are non-trivial.

For an arbitrary $\rho$, a function $u$ is in $\mathcal{P}_T$, iff $i\partial \overline{\partial} u \wedge i\partial \rho \wedge \overline{\partial} \rho$ is positive as a $(2, 2)$-form.

If the function $\rho$ is strictly psh, then there are no non-trivial $\Gamma$-holomorphic discs, i.e. non-constant holomorphic discs tangent to the distribution $\Gamma_z$.

Example 3.16. The most obvious $\Gamma$-directed currents are $\Gamma$-directed holomorphic discs, when they exist. More precisely, if $\phi : D \to \mathbb{C}^k$ is holomorphic, with $\phi(\zeta) = z$, then $\phi'(\zeta) \in \Gamma_z$ for each $\zeta \in D$. The current $[\phi_* (D)]$ is $\Gamma$-directed, when it is of finite mass. Let $\lambda$ denote the Lebesgue measure on the unit circle.

One can also introduce the current $\tau^\phi := \phi_* (\log \frac{1}{|z|})$. It is shown in [13, Example 4.9] that if $\phi$ is bounded and $\phi(0) = x$, then the current $\tau^\phi$ has finite mass and:

$$i\partial \overline{\partial} \tau^\phi = (\phi)_* (\lambda) - \delta_x.$$  \hspace{1cm} (1)

Moreover the mass of $\tau^\phi$ is equivalent to

$$\|\tau^\phi\| = \int (1 - |\zeta|) |\phi'(\zeta)|^2 d\lambda(\zeta).$$

This is precisely equivalent to the Nevanlinna characteristic of $\phi$, $T(\phi, 1)$, considered as a map from $D$ to $\mathbb{P}^k$. Jensen’s formula gives:

$$T(\phi, 1) = \int_0^1 \frac{dt}{t} \int_D \phi^* (\omega) = \int_D \log \frac{1}{|z|} \phi^* (\omega) = \frac{1}{2\pi} \int_0^{2\pi} \log (1 + |\phi|^2) (e^{i\theta}) d\theta - \log (1 + |\phi(0)|^2).$$

A family of maps is uniformly bounded, in the Nevanlinna class, if there is a constant $C$ such that for every $\phi$ in the family, $\limsup_{r < 1} \int_0^{2\pi} \log (1 + |\phi|^2) (r e^{i\theta}) d\theta \leq C$. Then the family $\tau^\phi$ is a relatively compact family of $\Gamma$-directed currents. This permits in particular to prove equation (1).

The currents $\tau^\phi := \phi_* (\log \frac{1}{|z|})$, have been used in Foliation Theory and holomorphic dynamical systems. They permit to prove ergodic theorems and to study rigidity properties, see [18], [8], [10].

Remark 3.17. In [29], Wold gives an independent proof of a weaker estimate of the mass of the currents $\tau^\phi := \phi_* (\log \frac{1}{|z|})$ in the above example. He then deduces Theorem 3.4 in the context of polynomial hulls as a consequence of the Bu-Schachermeyer, Poletsky Theorem ([11, 23]). Their theorem says that if $x$ is in the polynomial hull of a compact $K$, then for every Jensen measure $\nu_x$ on $K$, there is a sequence of $\Gamma$-holomorphic discs $\phi_n : D \to \mathbb{C}^k$ such that $\phi_n(0) = x$ and $(\phi_n)_*(\lambda) \to \nu_x$. 

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Following this approach, Drinovec-Forstneric in [12], proved Theorem 3.4 when the cone $\Gamma$ is the null cone in $\mathbb{C}^3$, i.e. a vector $a = (a_1, a_2, a_3)$ is in $\Gamma_z$ iff $\sum_{j=1}^3 a_j^2 = 0$. Their proof requires to construct first $\Gamma$-discs, satisfying a B-Schachermeyer, Poletsky type Theorem, for the null-cone.

4 Homogeneous $\Gamma$-Monge-Ampère equation and Dirichlet problem

Let $\Omega \Subset \mathbb{C}^k$ be a smooth strictly pseudoconvex domain. Let $u$ be a continuous function on $\partial \Omega$. Using Perron’s method, Bremermann considered the following envelope. He defined:

$$\hat{u} := \sup \{ v : v \text{ psh in } \Omega, \ v \leq u \text{ on } \partial \Omega \}.$$ 

It is easy to show that $\hat{u}$ is continuous in $\overline{\Omega}$, psh in $\Omega$. Bedford and Taylor [2] showed that $(i\partial \bar{\partial} \hat{u})^k = 0$ in $\Omega$. To explain the property, it was tempting to think that through every point $x \in \Omega$, there is a non-trivial holomorphic disc $D_x$, such that $\hat{u}|_{D_x}$ is harmonic. That would explain that in some sense there is a “zero eigenvalue” for $i\partial \bar{\partial} u$. The author has given an example showing that it is not always the case. More precisely, let $B$ denote the unit ball in $\mathbb{C}^k$. There is $u \in C^{1,1}(B)$, such that $(i\partial \bar{\partial} \hat{u})^k = 0$ and the restriction of $u$ to any holomorphic disc through zero, is not harmonic. We show below, that indeed the result becomes true if we replace discs through $x$, by positive currents $T$, such that $i\partial \bar{\partial} T \leq 0$, strictly negative at $x$.

We will consider a compact set $K$, such that every point of $K$ is in the Jensen boundary for $\mathcal{P}_{\Gamma}$. More precisely, if $\mu_x$ is a Jensen measure for $x$, supported on $K$, then $\mu_x = \delta_x$. Let $u$ be a lsc bounded function on $K$. We define on $\hat{K}_{\Gamma}$ the function

$$\hat{u}_\Gamma(x) := \sup \{ v(x) : v \in \mathcal{P}_{\Gamma}, \ v \leq u \text{ on } K \}.$$ 

We will call $\hat{u}_\Gamma$, the solution of the $\Gamma$-Monge-Ampère problem. Our assumption implies that $\hat{u}_\Gamma$ is lsc. The Choquet-Edwards theorem implies that

$$\hat{u}_\Gamma(x) = \inf_{\mu_x \in J_x} \int u d\mu_x.$$ 

So in particular, $\hat{u}_\Gamma = u$ on $K$.

**Theorem 4.1.** Let $K, \hat{K}_{\Gamma}, u, \hat{u}_\Gamma$ as above. Then for every $x \in \hat{K}_{\Gamma} \setminus K$, there is a Jensen measure $\mu_x$, and a positive current $T_x \geq 0, \Gamma$-directed, of bidimension $(1, 1)$, such that $i\partial \bar{\partial} T_x = \mu_x - \delta_x$ in $\mathbb{C}^k \setminus K$. Moreover, $i\partial \bar{\partial} \hat{u}_\Gamma \wedge T_x = 0$ on $\hat{K}_{\Gamma} \setminus K$.

Conversely, let $\tilde{u}$ be a lsc function on $\hat{K}_{\Gamma}$ such that $\tilde{u} = \lim \uparrow u_n$, where each $u_n$ is a finite sup of functions in $\mathcal{P}_{\Gamma}$. Assume that for every $x \in \hat{K}_{\Gamma} \setminus K$, there is a positive current $S_x$ with $x \in \text{supp}(S_x)$, such that $i\partial \bar{\partial} S_x \leq 0$ on $\mathbb{C}^n \setminus K$, and such that $i\partial \bar{\partial} \tilde{u} \wedge S_x = 0$ on $\hat{K} \setminus K$. Then $\tilde{u} = \hat{u}_\Gamma$, with $u := \tilde{u}|_K$. 

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Proof. The Choquet-Edwards theorem implies that for every \( x \in \hat{K}_{\Gamma} \setminus K \), there is a Jensen measure \( \mu_x \) on \( K \), such that \( \hat{u}_\Gamma(x) = \int u \, d\mu_x \). Theorem 3.4 implies that there is a \( \Gamma \)-directed current \( T_x \geq 0 \), such that \( i\partial \bar{\partial} T_x = \mu_x - \delta_x \). We show that \( \langle i\partial \bar{\partial} T_x, \hat{u}_\Gamma \rangle = 0 \). The function \( \hat{u}_\Gamma \) is an increasing limit of a sequence \( v_n \in \mathcal{P}_\Gamma \).

By Jensen’s formula, \( 0 \leq T_x \wedge i\partial v_n \to 0 \). It follows that, \( i\partial \bar{\partial} \hat{u}_\Gamma \wedge T_x = 0 \).

We have used implicitly that we can replace the supremum by a composition with an appropriate smooth convex function.

For the converse, let \( u := \tilde{u} |_K \). Let \( v \in \mathcal{P}_\Gamma \), such that \( v \leq \tilde{u} \) on \( K \). Since \( \tilde{u} \) satisfies the hypothesis of the maximum principle, with respect to \( S_x \), we get that \( v \leq \tilde{u} \) on \( S_x \). So \( \hat{u}_\Gamma \leq \tilde{u} \).

It is clear that \( \tilde{u} \leq \hat{u}_\Gamma \).

Remarks 4.2. 1. In the classical Monge-Ampère case, i.e when \( \mathcal{P}_\Gamma = \mathcal{P}_0 \), it is enough to assume that \( \bigcup \text{supp}(S_x) \) is of full measure in the domain \( \Omega \). We then get \( \hat{u} = \tilde{u} \) on a set of full measure and hence \( \hat{u} = \tilde{u} \) by pluri-subharmonicity.

2. Assume that for every Jensen measure \( \nu_x \) on \( K \), there is a sequence of \( \Gamma \)-holomorphic discs \( \phi_n : \overline{D} \to \mathbb{C}^k \) such that \( \langle i\partial \bar{\partial} T_x, \nu_x \rangle = 0 \). This is the case when \( \mathcal{P}_\Gamma = \mathcal{P}_0 \). We then have the following interpretation of the harmonicity of \( \hat{u} \) on \( T_x \), in the context of the classical Monge-Ampère equation.

Let \( \mu_x \) be a Jensen measure on \( \partial \Omega \), such that \( \int \hat{u} d\mu_x - \hat{u}(x) = 0 \). Let \( \phi_n \) the sequence of holomorphic maps associated to \( \mu_x \). In the present situation, it is easy to show that the function \( \hat{u} \), is continuous.

Then, using Jensen’s formula we get:

\[
\limsup_n \int_0^1 dt \int_{D_t} i\partial \bar{\partial} (\hat{u} \circ \phi_n) = \lim_n \int_0^{2\pi} (\hat{u} \circ \phi_n)(e^{i\theta})d\theta - \hat{u}(x) = \int \hat{u} d\mu_x - \hat{u}(x) = 0.
\]

Observe that \( i\partial \bar{\partial} (\hat{u} \circ \phi_n) \geq 0 \). Hence we see that \( \hat{u} \) is asymptotically harmonic on the holomorphic discs \( \phi_n : \overline{D} \to \mathbb{C}^k \).

5 Localization

We now address a localization problem. Let \( v \) be a continuous function on \( \hat{K}_{\Gamma} \).

Assume that \( v \) is locally a supremum of functions in \( \mathcal{P}_\Gamma \). Is it globally the supremum of functions in \( \mathcal{P}_\Gamma \)? In particular, is the \( \Gamma \)-Monge-Ampère problem, a local problem?

More generally, consider a convex cone \( \mathcal{P} \) of \( C^2 \)-functions in \( \mathbb{C}^k \), containing a smooth strictly psh function, the constants and separating points. We assume also that if \( u \in \mathcal{P} \) and \( h \) is a convex increasing function, then \( h \circ u \in \mathcal{P} \). This is a
crucial assumption on \( \mathcal{P} \). For a compact \( K \), we can define the hull \( \hat{K}_P \) of \( K \) with respect to \( \mathcal{P} \). We will assume that \( \hat{K}_P \), is always compact. A basic example is as follows.

Let \( R \) be a strongly positive \((p,p)\) form or current. Define the cone of functions,

\[
\mathcal{P}_R := \{ u \in C^2 : i\partial \bar{\partial} u \wedge R \geq 0 \}.
\]

The cone \( \mathcal{P}_R \) satisfies the following property. Let \( u_1, \ldots, u_m \) be functions in \( \mathcal{P}_R \) and let \( H \) be a function in \( \mathbb{R}^m \), convex and increasing in each variable, then \( H(u_1, \ldots, u_m) \) is in \( \mathcal{P}_R \). We can consider

\[
\mathcal{P}^0 := \{ T : T \geq 0, \langle \chi T, \partial \bar{\partial} u \rangle \geq 0 \text{ for every } u \in \mathcal{P} \},
\]

where \( \chi \) is an arbitrary cutoff function. The elements of \( \mathcal{P}^0 \) are bi-dimension \((1,1)\) currents. We describe a very special case of the above situation.

Suppose, the current \( R \) is strongly positive and closed, with no non-constant holomorphic discs in its support \([13]\). Then there will be no non-trivial holomorphic disc \( \varphi : D \to \mathbb{C}^k \), such that \( u \circ \varphi \) is subharmonic for every \( u \) in \( \mathcal{P}_R \). But hulls are non-trivial. The case where \( R \) is strongly positive and \( i\partial \bar{\partial} \)-negative, in an open set \( U \), is also of interest.

The previous theory is valid for \( \mathcal{P} \). The maximum principle for \( \mathcal{P} \), with respect to \( T \in \mathcal{P}^0 \), is valid if \( i\partial \bar{\partial} T \leq 0 \). In particular, if we consider the cone \( \mathcal{P}_R \) and we assume that \( i\partial \bar{\partial} R \leq 0 \), the maximum principle holds for functions on support \( R \) which are decreasing limits of functions in \( \mathcal{P}_R \).

If \( x \in \hat{K}_P \), we have Jensen measures \( \mu_x \) for \( x \), with support in \( K \). We can also solve the equation \( i\partial \bar{\partial} T = \mu_x - \delta_x \), with \( T \geq 0 \), in \( \mathcal{P}^0 \). Hence get the local maximum principle for \( \hat{K}_P \).

We now introduce the Monge-Ampère problem, with respect to \( \mathcal{P} \). Let \( u \) be a bounded lsc function on \( K \). Define \( \hat{u} \) on \( \hat{K}_P \) as

\[
\hat{u} := \sup \{ v : v \leq u \text{ on } K, \ v \in \mathcal{P} \}.
\]

**Proposition 5.1.** Let \( K \) be a compact set in \( \mathbb{C}^k \) and let \( u \) be a bounded lsc function on \( K \). Let \( U \) be an open set in \( \hat{K}_P \). Define \( v := \hat{u}|_{\partial U \cup (U \cap K)} \). Let \( \hat{v} \) denote the solution of the Monge-Ampère problem on \( U \), with respect to \( \mathcal{P}|_U \). Then \( \hat{v} = \hat{u} \) on \( U \). In particular, if for \( \varphi \in \mathcal{P} \), \( \varphi \leq \hat{u} \) on \( \partial U \cup (U \cap K) \), then \( \varphi \leq \hat{u} \) on \( U \).

**Proof.** Let \( \varphi \in \mathcal{P} \). Suppose \( \varphi \leq \hat{u} \) on \( \partial U \cup (U \cap K) \). We have to show that \( \varphi \leq \hat{u} \) on \( U \). Let \( x \in U \setminus K \) and \( \mu_x \) a Jensen measure for \( x \) with support on \( K \). Let \( T_x \in \mathcal{P}^0 \), \( T_x \geq 0 \), \( i\partial \bar{\partial} T_x = \mu_x - \delta_x \), be as in Theorem 4.1. Since \( \hat{u} \) is \( T_x \)-harmonic and \( \varphi \leq \hat{u} \) on \( \partial U \cup (U \cap K) \), the maximum principle implies that \( \varphi(x) \leq \hat{u}(x) \).

The following result was proved in the context of function algebras in \([20]\).
Theorem 5.2. Let $u$ be a continuous function on $\hat{K}_P$. Assume that every point of $K$ is a Jensen boundary point for $\mathcal{P}$. Suppose that for every $x \in \hat{K}_P$, there is a neighborhood $U$ such that on $U$, $u$ is a supremum of functions in $\mathcal{P}|_U$. Then $u$ is a supremum on $\hat{K}_P$ of functions in $\mathcal{P}$.

Proof. Define for $z \in \hat{K}_P$,

$$\hat{u}(z) := \sup \left\{ v(z) : v \in \mathcal{P}, v \leq u \text{ on } \hat{K}_P \right\}.$$ 

We want to show that $\hat{u} = u$. The Choquet-Edwards Theorem implies that $\hat{u} = u$ on $K$.

Let $\alpha := \sup \{ u(z) - \hat{u}(z) : z \in \hat{K}_P \}$ and $E := \{ z : z \in \hat{K}_P : u(z) - \hat{u}(z) = \alpha \}$. Suppose $\alpha > 0$. We need to find a contradiction. The set $E$ is compact. Let $z_0 \in E$ be a point in the Jensen boundary of $\mathcal{P}|_E$. By hypothesis, there is a neighborhood $U$ of $z_0$ such that for $z \in U$, $u(z) = \sup \{ v(z) : v \in \mathcal{P}|_U \}$. Since $z_0$ is in the Jensen boundary of $\mathcal{P}|_E$, there is $v_0 \in \mathcal{P}$, with $v_0 < 0$ on $E \setminus U$, and $v_0(z_0) > 0$.

We can choose $c > 0$ large enough such that $-c\alpha + \sup v_0 < 0$ on $\hat{K}_P$ and $c(u - \hat{u} - \alpha) + v_0 < 0$ on $\partial U$. So $c(u - \alpha) + v_0 < cu$ on $\hat{K}_P$ and $c(u - \alpha) + v_0 < c\hat{u}$ on $\partial U$.

The function, $c(u - \alpha) + v_0$, is a supremum on $U$ of functions on $\mathcal{P}|_U$ and it is dominated by $c\hat{u}$ on $\partial U$. We can apply Proposition 5.1 for $K = \hat{K}_P$. We get that $c(u - \alpha) + v_0 < c\hat{u}$ on $U$. This contradicts that $v(z_0) > 0$ and $c(u - \alpha - \hat{u})(z_0) = 0$. \qed

6 Extension of directed currents

The problem of extension of positive closed currents through analytic varieties or through complete pluripolar set has been studied quite extensively, see [27], [15], [26], [9].

Here we replace , the classical complete pluripolar sets, by complete pluripolar sets with respect to a cone $\mathcal{P}$. A positive current $T$ of bi-degree $(p, p)$ is $\mathcal{P}$-directed iff the current $T \wedge i\partial\bar{\partial}u$, is positive for every $u$ in $\mathcal{P}$. We obtain extension results for positive $\Gamma$-directed currents, $\partial\bar{\partial}$-closed and of bounded mass in the complement of a closed complete $\mathcal{P}_\Gamma$-pluripolar set.

In this section we first discuss the notion of pluripolar sets with respect to a cone $\mathcal{P}$. We then we give few extension results, see Theorem 6.4.

On an open connected set $U$ in $\mathbb{C}^k$, we consider a cone $\mathcal{P}$ of smooth functions, as in section 5. The two main cases, we have in mind are the cones $\mathcal{P}_\Gamma$ considered in section 3, and the cones $\mathcal{P}_R$, associated to a positive current $R$, as in section 5. We define

$$\overline{\mathcal{P}} := \{ u : u = \lim \downarrow u_\epsilon, \ u_\epsilon \in \mathcal{P} \}.$$ 

We will consider only functions which are not identically $-\infty$. 

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A set $F$ in $U$ is $\mathcal{P}$-pluripolar, if it is contained in \{z $\in U$ : $v(z) = -\infty$\}, where $v$ is a function in $\overline{\mathcal{P}}$. The set $F$ is complete $\mathcal{P}$-pluripolar, if $F = \{z \in U : v(z) = -\infty\}$. We can also define the notions of locally $\mathcal{P}$-pluripolar, or locally complete $\mathcal{P}$-pluripolar.

**Example 6.1.** Let $\rho$ be a smooth non-negative function in $U$. Let $\Gamma_z := \ker \partial \rho(z)$. Assume that the function $\rho$ is in $\mathcal{P}$, i.e. $i\partial \overline{\partial} \rho \wedge i\partial \rho \wedge \overline{\partial} \rho$ is positive. Then the set $\rho = 0$, is $\mathcal{P}$-pluripolar. It is associated to the function $\log(\rho)$, which is in $\overline{\mathcal{P}}$. Similarly for any $c > 0$ the set $(\rho \leq c)$, is $\mathcal{P}$-pluripolar. It is associated to the function $(\rho - c)^+$. So the $\mathcal{P}$-pluripolar sets, can be quite different from the classical ones.

As in [26], we have the following result.

**Proposition 6.2.** Let $F$ be a complete $\mathcal{P}$-pluripolar set in $U$. For any $V \subseteq U$, there is a sequence $(u_n)$ of functions in $\mathcal{P}|_V$, $0 \leq u_n \leq 1$, vanishing in a neighborhood of $V \cap F$, and such that, $(u_n)$ converges pointwise to 1 in $U \setminus F$. When $F$ is closed we can choose $(u_n)$ increasing.

**Proof.** Choose a sequence $(v_n)$ in $\mathcal{P}$, decreasing to a function $v$ in $\overline{\mathcal{P}}$, such that $F = \{z \in U : v(z) = -\infty\}$. We can assume that $v_n \leq 0$ on $V$. Let $h$ be a convex increasing function on $\mathbb{R}$ vanishing near 0 and such that $h(1) = 1$.

Define $u_j = h(exp(v_j/n_j))$, with $n_j$ increasing to infinity. The sequence $(u_j)$ satisfies the stated properties. If $F$ is closed, we can first construct a function $u$ in $\overline{\mathcal{P}}$, smooth out of $F$, continuous and such that $F = \{z \in U : u(z) = -\infty\}$. Then $u_n = h(exp(u/n))$, satisfies the required properties.

We start with the following proposition which is a version of the Chern-Levine-Nirenberg inequality [5], [7]. In what follows, $\omega$ denotes a Kähler $(1,1)$-form on $U$. If $T$ is a current of order zero, the mass of $T$ on a Borel set $K \subset U$ is denoted by $\|T\|_K$. If $T$ is a positive or a negative $(p,p)$-current, $\|T\|_K$ is equivalent to $|\int_K T \wedge \omega^{k-p}|$. We identify these two quantities. Observe however that the mass estimates (resp. the extension results) for $(p,p)$-currents, can be easily reduced to similar question for bi-dimension $(1,1)$-currents. It is enough to wedge, with an appropriate power of $\omega$. This is also valid for $\mathcal{P}$-directed currents.

**Proposition 6.3.** Let $U$ be an open subset $\mathbb{C}^k$. Let $K$ and $L$ be compact sets in $U$ with $L \Subset K$. Assume that $T$ is a positive current on $U$ of bi-degree $(p,p)$, $\mathcal{P}$-directed. Assume also, that $i\partial \overline{\partial} T$ has order zero. Then there exists a constant $c_{K,L} > 0$ such that for every smooth function $u$ in $\mathcal{P}$, we have the following estimates

$$\int_L i\partial u \wedge \overline{\partial} u \wedge T \wedge \omega^{k-p-1} \leq c_{K,L}\|u\|_{\mathcal{L}_\infty(K)}^2 (\|T\|_K + \|i\partial \overline{\partial} T\|_K),$$

and

$$\|i\partial \overline{\partial} u \wedge T\|_L \leq c_{K,L}\|u\|_{\mathcal{L}_\infty(K)} (\|T\|_K + \|i\partial \overline{\partial} T\|_K),$$

where $c_{K,L} > 0$ is a constant independent of $u$ and $T$. 

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The Proposition is proved in [9] for psh functions, see also [26]. The proof can be easily adapted to functions in $\mathcal{P}$ and to $\mathcal{P}$-directed currents.

**Theorem 6.4.** Let $F$ be a closed subset in $U$. Let $T$ be a positive $(p,p)$-current $\mathcal{P}$-directed on $U \setminus F$. Assume that $F$ is locally complete $\mathcal{P}$-pluripolar and that $T$ has locally finite mass near $F$. Assume also that there exists a positive $(p+1,p+1)$-current $S$ with locally finite mass near $F$ such that $i\partial \bar{\partial} T \leq S$ on $U \setminus F$. Then $i\partial \bar{\partial} T$ has locally finite mass near $F$. If $\tilde{T}$ and $\partial \tilde{T}$ denote the extensions by zero of $T$ and $\partial \tilde{T}$ on $U$, then the current $i\partial \bar{\partial} T - i\partial \bar{\partial} \tilde{T}$ is positive. If moreover the current $dT$, is of order zero and of bounded mass near $F$, then $\tilde{dT} = d\tilde{T}$.

**Proof.** The Theorem is proved in [9] when $\mathcal{P}$, is the cone of psh functions. The needed modifications are straightforward. We just sketch the strategy. We consider the sequence $(u_n)$ of functions constructed in Proposition 6.2. We have $u_n T \to \tilde{T}$. The following formula is an elementary calculus identity:

$$u_n(i\partial \bar{\partial} T) - i\partial \bar{\partial} u_n \wedge T = i\partial \bar{\partial}(u_n T) - i\partial(\partial u_n \wedge T) + i\partial(\bar{\partial} u_n \wedge T).$$

The previous identity implies that

$$u_n(i\partial \bar{\partial} T - S) - i\partial \bar{\partial} u_n \wedge T = -u_n S + i\partial \bar{\partial}(u_n T) - i\partial(\partial u_n \wedge T) + i\partial(\bar{\partial} u_n \wedge T).$$

One shows, using Proposition 6.3, that $\partial u_n \wedge T \to 0$. Then, the right hand side converges to $-\tilde{S} + i\partial \bar{\partial} \tilde{T}$ where $\tilde{S}$ is the trivial extension by zero of $S$ on $U$. Since both terms on the left hand side are negative currents, their limit values are negative. We then deduce that $-\tilde{S} + i\partial \bar{\partial} \tilde{T}$ is negative, and hence $i\partial \bar{\partial} T$ has bounded mass near $F$. It follows that $u_n(i\partial \bar{\partial} T - i\partial \bar{\partial} \tilde{T}) = i\partial \bar{\partial} u_n \wedge T + o(1)$.

So, the left hand side of the previous equation, converges to $i\partial \bar{\partial} T - \partial \tilde{T}$. Since $\lim i\partial \bar{\partial} u_n \wedge T$ is positive, it follows that $i\partial \bar{\partial} T - i\partial \bar{\partial} \tilde{T}$ is positive. The others assertions are proved similarly.

**Corollary 6.5.** Let $F$ be a closed complete $\mathcal{P}$-pluripolar subset of $U$. Let $(u_n)$, be a sequence of functions in $\mathcal{P}$, as constructed in Proposition 6.2. Let $T$ be a bidegree $(p,p)$ positive $\partial \bar{\partial}$-closed, $\mathcal{P}$-directed current in $U$. Assume $T$ has no mass on $F$. Then, for every compact $K$ in $U$ we have that

$$\int_K i\partial u_n \wedge \bar{\partial} u_n \wedge T \wedge \omega^{k-p-1}$$

and

$$\int_K i\partial \bar{\partial} u_n \wedge T \wedge \omega^{k-p-1},$$

converge to zero, when $n$ goes to infinity.
Proof. The current $T$ has no mass on $F$, hence it is equal to its trivial extension through $F$. It follows that, $\lim i\partial\bar{\partial}u_n \wedge T = \tilde{i}\partial\bar{\partial}T - \tilde{i}\partial\bar{\partial}\tilde{T} = 0$. So, we get the second relation in the corollary. If we apply this relation to the sequence $(u^2_n)$, we get the first assertion. 

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