Functional Inequalities for Stable-Like Dirichlet Forms

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Abstract

Let $V \in C^2(\mathbb{R}^d)$ such that $\mu_V(dx) := e^{-V(x)} dx$ is a probability measure, and let $\alpha \in (0, 2)$. Explicit criteria are presented for the $\alpha$-stable-like Dirichlet form

$$E_{\alpha,V}(f, f) := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+\alpha}} dy e^{-V(x)} dx$$

to satisfy Poincaré-type (i.e., Poincaré, weak Poincaré and super Poincaré) inequalities. As applications, sharp functional inequalities are derived for the Dirichlet form with $V$ having some typical growths. Finally, the main result of [15] on the Poincaré inequality is strengthened.

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1 Introduction

Functional inequalities are powerful and efficient tools to analyze Markov semigroups and their generators, see e.g. [28] for a general theory of functional inequalities and applications. In particular, the Nash/Sobolev inequalities are corresponding to uniform heat

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kernel upper bounds of the semigroup, the log-Sobolev inequality is equivalent to Nelson’s
hypercontractivity ([16]) of the semigroup, the super log-Sobolev inequality (also called
the log-Sobolev inequality with parameter) is equivalent to the supercontractivity and
in some cases implies the ultracontractivity of the semigroup, the Poincaré inequality is
equivalent to the exponential convergence of the semigroup, and the weak Poincaré in-
equality characterizes various convergence rates of the semigroup slower than exponential,
see e.g. [8, 12, 9, 17, 25] for details. As a general version of functional inequalities stronger
than the Poincaré one, the super Poincaré inequality is equivalent to the uniform integra-
bility of the semigroup, and also the absence of the essential spectrum of the generator if
the semigroup has an asymptotic density, see [23, 24, 10, 26] for details.

To establish functional inequalities, many explicit criteria have been proved for diffusion
processes and Markov chains, but rare is known for Lévy type jump processes. Of
course, using subordination techniques, functional inequalities for a class of jump pro-
cesses can be deduced from known ones of diffusion processes, see [2, 27, 22, 11] and [21,
Chapter 12.3] (in an abstract setting) for details. However, in general it is difficult (and
impossible in many cases) to identify a Lévy type jump process as subordination of a diffu-
sion process. So, it is necessary to provide general criteria to verify functional inequalities
for Lévy type jump processes. We remark that using harmonic analysis technique, a suf-
ficient condition for the Poincaré inequality to hold, see (1.8) below, has been presented
in [15]. As pointed out after Corollary 1.5 below, this condition excludes many typical
examples which possess the even stronger super Poincaré inequality. The purpose of this
paper is to find out sharp and easy to check sufficient conditions for general functional
inequalities of stable-like jump processes.

To make the paper easy to follow, let us start with a simple example, i.e. the Ornstein-
Uhlenbeck process driven by the \( \alpha \)-stable process. Let \( \Delta \) be the Laplacian on \( \mathbb{R}^d \).
Consider the Ornstein-Uhlenbeck operator

\[
A_\alpha f(x) := -\left(-\Delta\right)^{\alpha/2} f(x) - \langle x, \nabla f(x) \rangle, \quad f \in C_0^\infty(\mathbb{R}^d)
\]

for \( \alpha \in (0, 2) \). Then the associated Markov semigroup has a unique invariant (but not re-
versible, see [11]) probability measure \( \mu_\alpha \), which is identified by the Fourier transforma-
tion

\[
\hat{\mu}_\alpha(\xi) := \int_{\mathbb{R}^d} e^{i(x,\xi)} \mu_\alpha(dx) = e^{-\frac{1}{4}|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.
\]

For any \( f \in C_0^\infty(\mathbb{R}^d) \), the set of all smooth functions on \( \mathbb{R}^d \) with compact support, we
have (see [14, Proposition 4.1] or [18, (1.9)])

\[
E_\alpha(f, f) := -\int_{\mathbb{R}^d} f A_\alpha f \, d\mu_\alpha = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+\alpha}} \, d\mu_\alpha(dx).
\]

Let \( \mathcal{D}(\mathcal{E}_\alpha) = \{ f \in L^2(\mu_\alpha) : \mathcal{E}_\alpha(f, f) < \infty \} \). According to [18, Example 3.2(2)], the
semigroup \( P_t^\alpha \) generated by \( A_\alpha \) is not hyperbounded, i.e. \( \| P_t^\alpha \|_{L^p(\mu_\alpha) \to L^q(\mu_\alpha)} = \infty \) for any
t \( \geq 0 \) and \( q > p \geq 1 \). Therefore, the log-Sobolev inequality of \( \mathcal{E}_\alpha \) does not hold. In fact,
since
\[
\frac{1}{c(1 + |x|^2)^{(d+\alpha)/2}} \, dx \leq \mu_\alpha(dx) \leq \frac{c}{(1 + |x|^2)^{(d+\alpha)/2}} \, dx
\]
holds for some constant \(c > 1\), see e.g. [3, Theorem 2.1] or [6, (1.5)], Corollary 1.2 below provides a stronger statement, i.e. the super Poincaré inequality is not available neither. Recall that the log-Sobolev inequality
\[
\mu_\alpha(f^2 \log f^2) \leq C \mathcal{E}_\alpha(f, f), \quad f \in \mathcal{D}(\mathcal{E}_\alpha), \mu_\alpha(f^2) = 1
\]
holds for some constant \(C > 0\) if and only if the super Poincaré inequality
\[
\mu_\alpha(f^2) \leq r \mathcal{E}_\alpha(f, f) + \exp \left( c(1 + r^{-1}) \right) \mu_\alpha(|f|)^2, \quad r > 0, \quad f \in \mathcal{D}(\mathcal{E}_\alpha)
\]
holds for some constant \(c > 0\). On the other hand, Corollary 1.2(1) implies that the Poincaré inequality
\[
\mu_\alpha(f^2) \leq C \mathcal{E}_\alpha(f, f), \quad f \in \mathcal{D}(\mathcal{E}_\alpha), \mu_\alpha(f) = 0
\]
holds for some constant \(C > 0\), which has been open for a long time. Therefore, for this typical example, the best possibility among functional inequalities mentioned above is the Poincaré inequality.

Now, as a generalization of (1.1), we consider
\[
\mathcal{E}_{\alpha,V}(f, g) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dy \mu_V(dx),
\]
\[
\mathcal{D}(\mathcal{E}_{\alpha,V}) := \left\{ f \in L^2(\mu_V) : \mathcal{E}_{\alpha,V}(f, f) < \infty \right\},
\]
where \(V\) is a measurable function on \(\mathbb{R}^d\) such that
\[
\mu_V(dx) := \int_{\mathbb{R}^d} e^{-V(x)} dx, \quad e^{-V(x)} = \mu_V(dx)
\]
is a probability measure. Then \((\mathcal{E}_{\alpha,V}, \mathcal{D}(\mathcal{E}_{\alpha,V}))\) is a symmetric Dirichlet form on \(L^2(\mu_V)\). Let \(P_t^{\alpha,V}\) be the associated Markov semigroup. Let
\[
h(r) = \inf_{|x| \leq r} e^{V(x)}, \quad H(r) = \sup_{|x| \leq r} e^{V(x)},
\]
\[
\Phi(r) = \inf_{|x| \geq r} \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}}, \quad \Phi^{-1}(r) = \inf \left\{ s \geq 0 : \Phi(s) \geq r \right\}, \quad r > 0,
\]
where we set \(\inf \emptyset = \infty\) by convention. Moreover, let
\[
\Psi_1(r) = \left( \sup_{|x| \leq r} \frac{(1 + |x|)^{d+\alpha}}{e^{V(x)}} \right) \sup_{x \in \mathbb{R}^d} \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}}
\]
\[
\Psi_2(r) = \frac{1}{\mu_V(B(0, r))^2} \sup_{x \in B(0, r)} \int_{B(0, r)} |y - x|^{d+\alpha} e^{-2V(y)} dy, \quad r > 0.
\]
The main result of the paper is the following
Theorem 1.1. Let $\int_{\mathbb{R}^d} e^{-V(x)} dx < \infty$ such that $\mu_V$ is a well defined probability measure.

1. If $e^{-V} \in C^2_b(\mathbb{R}^d)$ such that
   \begin{equation}
   \limsup_{r \to \infty} \left\{ r^{d+\alpha-1} \sup_{|x| \geq r-1} |\nabla e^{-V}(x)| + r^{d+\alpha-2} \sup_{|x| \geq r-1} e^{-V(x)} \right\} = 0
   \end{equation}
   and $\Phi(0) > 0$, then the Poincaré inequality
   \begin{equation}
   \mu_V(f^2) \leq C_{\alpha,V}(f,f), \quad f \in D(\mathcal{E}_{\alpha,V}), \mu_V(f) = 0
   \end{equation}
   holds for some constant $C > 0$.

2. If $e^{-V} \in C^2_b(\mathbb{R}^d)$ such that \((1.3)\) holds and $\Phi(r) \uparrow \infty$ as $r \uparrow \infty$, then there exist constants $c_1, c_2 > 0$ such that the super Poincaré inequality
   \begin{equation}
   \mu_V(f^2) \leq r e_{\alpha,V}(f,f) + \beta(r) \mu_V(|f|^2), \quad r > 0, f \in D(\mathcal{E}_{\alpha,V})
   \end{equation}
   holds for
   \[ \beta(r) = c_1 \left( 1 + r^{-d/\alpha} \left\{ h \circ \Phi^{-1}(c_2r^{-1}) \right\}^{-1-d/\alpha} \left\{ H \circ \Phi^{-1}(c_2r^{-1}) \right\}^{2+d/\alpha} \right), \quad r > 0. \]

3. There exists a universal constant $c > 0$ such that the weak Poincaré inequality
   \begin{equation}
   \mu_V(f^2) \leq \tilde{\beta}(r) e_{\alpha,V}(f,f) + r \|f\|_\infty^2, \quad r > 0, f \in D(\mathcal{E}_{\alpha,V}), \mu_V(f) = 0
   \end{equation}
   holds for
   \[ \tilde{\beta}(r) := \inf \left\{ (c \Psi_1(R)) \wedge \Psi_2(R) : \mu_V(B(0,R)^c) \leq \frac{r}{1+r} \right\} < \infty, \quad r > 0. \]

Although we assume in Theorem 1.1(1)-(2) that $e^{-V}$ is at least $C^2$-smooth, the assertions work also for singular case by using perturbation results of functional inequalities, see [5]. To illustrate this result, below we consider some typical families of $V$ with different type growths: for faster growth of $V$ one derives stronger functional inequality. When we apply Theorem 1.1(3) to derive weak Poincaré inequalities for theses families of $V$, the function $\Psi_1$ in the definition of $\tilde{\beta}$ is better than $\Psi_2$. On the other hand, however, $\Psi_2$ is always finite but in some cases $\Psi_1$ is infinite. So, in general these two functions are not comparable.

According to (1.2), in the following result $\mu_V$ is a natural extension to $\mu_\alpha$, i.e. when $\varepsilon = \alpha$ a Poincaré type inequality for $\mathcal{E}_{\alpha,V}$ and $\mu_V$ is equivalent to that for $\mathcal{E}_{\alpha}$ and $\mu_\alpha$. In particular, as mentioned above, this result implies that $\mathcal{E}_{\alpha}$ satisfies the Poincaré inequality but not the super Poincaré inequality.

Corollary 1.2. Let $V(x) = \frac{1}{2}(d + \varepsilon) \log(1 + |x|^2)$, $\varepsilon > 0$.

1. The Poincaré inequality \((1.4)\) holds for some constant $C > 0$ if and only if $\varepsilon \geq \alpha$.  

(2) The super Poincaré inequality holds for some function \( \beta : (0, \infty) \to (0, \infty) \) if and only if \( \varepsilon > \alpha \), and in this case there exists a constant \( c > 0 \) such that the inequality holds with
\[
\beta(r) = c \left( 1 + r^{-\frac{d}{\alpha}} \frac{(d+\varepsilon)(2\alpha+d)}{\alpha(\varepsilon-\alpha)} \right), \quad r > 0,
\]
and equivalently,
\[
\|P_t^\alpha V\|_{L^1(\mu_V) \to L^\infty(\mu_V)} \leq \lambda \left( 1 + t^{-\frac{d}{\alpha}} \frac{(d+\varepsilon)(2\alpha+d)}{\alpha(\varepsilon-\alpha)} \right), \quad r > 0
\]
holds for some constant \( \lambda > 0 \).

(3) If \( \varepsilon \in (0, \alpha) \), then there exists a constant \( c > 0 \) such that the weak Poincaré inequality holds for
\[
\tilde{\beta}(r) = c \left( 1 + r^{-(\alpha-\varepsilon)/\varepsilon} \right), \quad r > 0.
\]
Consequently, there exists a constant \( \lambda > 0 \) such that
\[
\|P_t^\alpha V - \mu_V\|_{L^\infty(\mu_V) \to L^2(\mu_V)} \leq \frac{\lambda}{t^{\varepsilon/(\alpha-\varepsilon)}}, \quad t > 0.
\]
This \( \tilde{\beta} \) is sharp in the sense that does not hold if \( \lim_{r \to 0} r^{(\alpha-\varepsilon)/\varepsilon} \tilde{\beta}(r) = 0 \).

Since \( \varepsilon = \alpha \) in Corollary 1.2 is the critical situation for the Poincaré inequality, we consider below lower order perturbations of the corresponding \( V \).

**Corollary 1.3.** Let \( V(x) = \frac{1}{2}(d + \alpha) \log(1 + |x|^2) + \varepsilon \log \log(e + |x|^2), \quad \varepsilon \in \mathbb{R} \).

(1) The super Poincaré inequality holds for some \( \beta \) if and only if \( \varepsilon > 0 \), and in this case it holds with
\[
\beta(r) = \exp \left[ c \left( 1 + r^{-1/\varepsilon} \right) \right]
\]
for some constant \( c > 0 \), so that when \( \varepsilon > 1 \),
\[
\|P_t^\alpha V\|_{L^1(\mu_V) \to L^\infty(\mu_V)} \leq \exp \left[ \lambda \left( 1 + t^{-1/(\varepsilon-1)} \right) \right], \quad t > 0
\]
holds for some constant \( \lambda > 0 \).

(2) The super Poincaré inequality in (1) is sharp in the sense that does not hold if
\[
\lim_{r \to 0} r^{1/\varepsilon} \log \beta(r) = 0.
\]

(3) The log-Sobolev inequality
\[
\mu_V(f^2 \log f^2) \leq C \mathcal{E}_\alpha V(f, f), \quad f \in \mathcal{D}(\mathcal{E}_\alpha V), \mu_V(f^2) = 1
\]
holds for some constant \( C > 0 \) if and only if \( \varepsilon \geq 1 \).
(4) The Poincaré inequality (1.4) holds for some constant \( C > 0 \) if and only if \( \varepsilon \geq 0 \), and there exists a universal constant \( c > 0 \) such that for \( \varepsilon < 0 \) the weak Poincaré inequality (1.6) holds with

\[
\tilde{\beta}(r) = c \left( 1 + \log^{-\varepsilon} (1 + r^{-1}) \right), \quad r > 0.
\]

Consequently, for \( \varepsilon < 0 \) there exist constants \( \lambda_1, \lambda_2 > 0 \) such that

\[
\|P_t^{\alpha,V} - \mu_V\|_{L^\infty(\mu_V) \to L^2(\mu_V)} \leq \exp \left[ \lambda_1 - \lambda_2 t^{1/(1-\varepsilon)} \right], \quad t > 0.
\]

This \( \tilde{\beta} \) is sharp in the sense that for \( \varepsilon < 0 \) the weak Poincaré inequality (1.6) does not hold if \( \lim_{r \to 0} \tilde{\beta}(r) \log^{\varepsilon}(1 + r^{-1}) = 0 \).

Below we consider a family of \( V \) with slower growth such that \( \mu_V \) is a probability measure, for which merely the weak Poincaré inequality is available.

**Corollary 1.4.** Let \( V(x) = \frac{d}{2} \log(1 + |x|^2) + \varepsilon \log \log(e + |x|^2), \; \varepsilon > 1 \). Then there exist some constants \( c_1, c_2 > 0 \) such that the weak Poincaré inequality (1.6) holds with

\[
\tilde{\beta}(r) = c_1 \exp \left[ c_2 r^{-1/(\varepsilon-1)} \right].
\]

Consequently, there exists some constant \( \lambda > 0 \) such that

\[
\|P_t^{\alpha,V} - \mu_V\|_{L^\infty(\mu_V) \to L^2(\mu_V)} \leq \lambda \left[ \log(1 + t) \right]^{1-\varepsilon}, \quad t > 0.
\]

This \( \tilde{\beta} \) is sharp in the sense that the weak Poincaré inequality (1.6) does not hold if \( \lim_{r \to 0} r^{1/(\varepsilon-1)} \log \tilde{\beta}(r) = 0 \).

Finally, we consider two families of \( V \) with stronger growths than all those presented above, so that the rather stronger super Poincaré inequality is available.

**Corollary 1.5.** (1) Let \( V(x) = \log^{1+\varepsilon}(1 + |x|^2), \; \varepsilon > 0 \). Then there exists a constant \( c > 0 \) such that (1.5) holds for

\[
\beta(r) = c \left( 1 + cr^{-2(\alpha+d)/\alpha} \exp \left[ c \log^{1/(1+\varepsilon)}(1 + r^{-1}) \right] \right), \quad r > 0.
\]

Consequently, there exists a constant \( \lambda > 0 \) such that

\[
\|P_t^{\alpha,V}\|_{L^1(\mu_V) \to L^\infty(\mu_V)} \leq \lambda + \lambda t^{-2(\alpha+d)/\alpha} \exp \left[ \lambda \log^{1/(1+\varepsilon)}(1 + t^{-1}) \right], \quad t > 0.
\]

(2) Let \( V(x) = (1 + |x|^2)^\varepsilon, \; \varepsilon > 0 \). Then there exists a constant \( c > 0 \) such that the super Poincaré inequality (1.5) holds for

\[
\beta(r) = c \left( 1 + r^{-2(\alpha+d)/\alpha} \log^{(2\alpha+d)(d+\alpha)/(2\varepsilon \alpha)}(1 + r^{-1}) \right), \quad r > 0,
\]

and consequently,

\[
\|P_t^{\alpha,V}\|_{L^1(\mu_V) \to L^\infty(\mu_V)} \leq \lambda \left( 1 + t^{-2(\alpha+d)/\alpha} \log^{(2\alpha+d)(d+\alpha)/(2\varepsilon \alpha)}(1 + t^{-1}) \right), \quad t > 0
\]

holds for some constant \( \lambda > 0 \).
We remark that the following sufficient condition for \( E_{\alpha,V} \) to satisfy the Poincaré inequality has been presented in [15]: \( V \in C^2(\mathbb{R}^d) \) such that

\[
(1.8) \quad \lim_{|x| \to \infty} \{ \delta |\nabla V|^2 - \Delta V \} = \infty \quad \text{for some constant } \delta \in (0, 1/2).
\]

Obviously, this condition does not hold for \( V \) in Corollaries [1.2-1.5(1)]. In the situation of Corollary [1.5(2)], (1.8) holds if and only if \( \varepsilon > \frac{1}{2} \). In this case, using the argument of [15], we are able to confirm the super Poincaré inequality for (see Theorem 5.1 below)

\[
\beta(r) = \exp \left[ c \left( 1 + r^{-2\varepsilon/(\alpha(2\varepsilon-1))} \right) \right], \quad r > 0
\]

for some constant \( c > 0 \), which is however much worse than the one given in Corollary [1.4(2)]. We also mention that sufficient conditions for a (non-symmetric) \( L^2 \)-generator of Lévy driven Ornstein-Uhlenbeck processes to satisfy Poincaré inequality have been investigated in [13, Section 5], where the proof is based on exact asymptotics for a distribution density of certain Lévy functionals; however, extensions to the present setting are not yet available.

The proof of Theorem 1.1 is based on Lyapunov type conditions considered in [4]. To verify these conditions, we first characterize in Section 2 the infinitesimal generator of \( E_{\alpha,V} \), then present complete proofs of the above results in Section 3 and Section 4. Finally, in Section 5 we present a result on the super Poincaré inequality using a weaker version of condition (1.8) by allowing \( \delta \) to approach 1, such that the main result in [15] on the Poincaré inequality is strengthened.

## 2 The infinitesimal generator of \( E_{\alpha,V} \)

We first introduce some facts concerning the Dirichlet form and generator of the \( \alpha \)-stable process. Let

\[
\mathcal{C}_\alpha = \{ f \in C^2(\mathbb{R}^d) : \| \nabla f \|_\infty < \infty, |f| \leq C(1 + |\cdot|^r) \text{ holds for some } C > 0, r \in (0, \alpha) \}.
\]

For any \( f \in \mathcal{C}_\alpha \), there exist constants \( C > 0 \) and \( r \in (0, \alpha) \) such that

\[
|f(x + z) - f(x) - \langle \nabla f, z \rangle 1_{\{|z| \leq 1\}}|\frac{1}{|z|^{d+\alpha}} \leq \sup_{B(x,1)} \| \nabla^2 f \| \frac{C(1 + |x|^r + |z|^r)}{|z|^{d+\alpha}} 1_{\{|z| > 1\}}.
\]

Then for \( f \in \mathcal{C}_\alpha \),

\[
(2.1) \quad -(-\Delta)^{\alpha/2} f := C_\alpha \int_{\mathbb{R}^d} \left( f(\cdot + z) - f - \langle \nabla f, z \rangle 1_{\{|z| \leq 1\}} \right) \frac{dz}{|z|^{d+\alpha}}
\]
is a well-defined locally bounded measurable function, where $C_\alpha > 0$ is a constant such that (see [20, Example 32.7]),

\[(2.2)\quad \frac{2}{C_\alpha} \int_{\mathbb{R}^d} \left( f (-\Delta)^{\alpha/2} g \right)(x) \, dx = \mathcal{E}_\alpha^{(0)}(f, g), \quad f, g \in C_0^2(\mathbb{R}^d), \]

where

\[\mathcal{E}_\alpha^{(0)}(f, g) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dy \, dx.\]

Next, for $f \in C_0^2(\mathbb{R}^d)$ and $g \in \mathcal{C}_\alpha$, there exist constants $C, R > 0$ and $r \in (0, \alpha)$ such that

\[|f(x) - f(y)| \cdot |g(x) - g(y)| \leq C|x - y|^2 1_{\{|x-y| \leq R\}} + C(|x|^r + |y|^r + 1)1_{\text{supp}f \times \text{supp}g}(x, y)1_{\{|x-y| > R\}},\]

so that $\mathcal{E}_\alpha^{(0)}(f, g) \in \mathbb{R}$ is well-defined.

Moreover, since for any function $g \in \mathcal{C}_\alpha$ there exist $\{g_n\}_{n \geq 1} \subset C_0^2(\mathbb{R}^d)$ such that $\|\nabla g_n\|_{\infty} \leq C, |g_n| \leq C(1 + |\cdot|^r)$ holds for some constants $C > 0$ and $r \in (0, \alpha)$, and that $g_n \to g, \nabla g_n \to \nabla g$ and $\nabla^2 g_n \to \nabla^2 g$ uniformly on compact sets, (2.2) implies that

\[(2.3)\quad \frac{2}{C_\alpha} \int_{\mathbb{R}^d} \left( f (-\Delta)^{\alpha/2} g \right)(x) \, dx = \mathcal{E}_\alpha^{(0)}(f, g), \quad f \in C_0^2(\mathbb{R}^d), g \in \mathcal{C}_\alpha.\]

Finally, if $e^{-V} \in C_0^1(\mathbb{R}^d)$ and $g \in \mathcal{C}_\alpha$, then

\[|g(x) - g(y)| \cdot |e^{-V(x)} - e^{-V(y)}| \leq C|x - y|^2 1_{\{|x-y| \leq 1\}} + \frac{C(1 + |x|^r + |y|^r)}{|x - y|^{d+\alpha}} 1_{\{|x-y| > 1\}}\]

holds for some constant $C > 0$ and $r \in (0, \alpha)$. Therefore, in conclusion, if $e^{-V} \in C_0^2(\mathbb{R}^d)$ and $g, ge^{-V} \in \mathcal{C}_\alpha$, then

\[(2.4)\quad L_{\alpha,V} g := \frac{2}{C_\alpha} \left( ge^V (-\Delta)^{\alpha/2} e^{-V} - e^V (-\Delta)^{\alpha/2} (e^{-V} g) \right) \]

\[ - e^V \int_{\mathbb{R}^d} \frac{(g - g(y))(e^{-V} - e^{-V(y)})}{| \cdot - y |^{d+\alpha}} \, dy\]

gives rise to a locally bounded measurable function.

**Proposition 2.1.** Assume that $e^{-V} \in C_0^2(\mathbb{R}^d)$. For any $f \in C_0^2(\mathbb{R}^d)$ and $g \in \mathcal{C}_\alpha$ such that $e^{-V} g \in \mathcal{C}_\alpha$,

\[\mathcal{E}_{\alpha,V}(f, g) = - \int_{\mathbb{R}^d} f L_{\alpha,V} g \, d\mu_V.\]

**Proof.** Since $fe^V, fge^V \in C_0^2(\mathbb{R}^d)$ and $e^{-V} g, e^{-V} \in \mathcal{C}_\alpha$, it follows from (2.3) that

\[- \int_{\mathbb{R}^d} f L_{\alpha,V} g \, d\mu_V = \frac{2}{C_\alpha} \int fe^V (-\Delta)^{\alpha/2} (e^{-V} g) \, d\mu_V - \frac{2}{C_\alpha} \int fge^V (-\Delta)^{\alpha/2} e^{-V} \, d\mu_V\]

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According to Proposition 2.1, the operator \((L_{\alpha,V}, C^2_0(\mathbb{R}^d))\) is symmetric on \(L^2(\mu_V)\); on the other hand, \((\mathcal{E}_{\alpha,V}, C^\infty_0(\mathbb{R}^d))\) is closable and it is easy to see that its closure coincides with \((\mathcal{E}_{\alpha,V}, \mathcal{D}(\mathcal{E}_{\alpha,V}))\). Moreover, combining (2.1) with (2.4), we obtain the following result with explicit expression of \(L_{\alpha,V}\).

**Proposition 2.2.** Assume that \(e^{-V} \in C^2_0(\mathbb{R}^d)\). For any \(f \in \mathcal{C}_\alpha\) such that \(fe^{-V} \in \mathcal{C}_\alpha\),

\[
L_{\alpha,V} f(x) = \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle 1_{\{|z| \leq 1\}} \right) \frac{1 + e^{V(x) - V(x + z)}}{|z|^{d+\alpha}} \, dz \\
+ \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle (e^{V(x) - V(x + z)} - 1) \frac{dz}{|z|^{d+\alpha}}.
\]

**Proof.** By (2.1) we have

\[
L_{\alpha,V,1} f(x) := \frac{2}{C_\alpha} \left( f(x)e^{V(x)}(-\Delta)^{\alpha/2}e^{-V}(x) - e^{V(x)}(-\Delta)^{\alpha/2}(e^{-V}f)(x) \right) \\
= 2 \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle 1_{\{|z| \leq 1\}} \right) \frac{e^{V(x) - V(x + z)}}{|z|^{d+\alpha}} \, dz \\
+ 2 \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle (e^{V(x) - V(x + z)} - 1) \frac{dz}{|z|^{d+\alpha}}.
\]

On the other hand,

\[
L_{\alpha,V,2} f(x) := e^{V(x)}\int_{\mathbb{R}^d} \frac{(f(x) - f(y))(e^{-V(x)} - e^{-V(y)})}{|x - y|^{d+\alpha}} \, dy \\
= \int_{\mathbb{R}^d} \left( f(y) - f(x) \right) \frac{e^{-V(y) + V(x)} - 1}{|y - x|^{d+\alpha}} \, dy
\]
\[
\lim_{\varepsilon \to 0} \left[ \int_{|z| \geq \varepsilon} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle 1_{\{|z| \leq 1\}} \right) \frac{e^{-V(x+z)+V(x)} - 1}{|z|^{d+\alpha}} \, dz \right. \\
+ \int_{|z| \geq \varepsilon} \langle \nabla f(x), z \rangle 1_{\{|z| \leq 1\}} \frac{e^{-V(x+z)+V(x)} - 1}{|z|^{d+\alpha}} \, dz \right] \\
= \int_{\mathbb{R}^d} \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle 1_{\{|z| \leq 1\}} \right) \frac{e^{-V(x+z)+V(x)} - 1}{|z|^{d+\alpha}} \, dz \\
+ \int_{\{|z| \leq 1\}} \langle \nabla f(x), z \rangle \left( e^{-V(x+z)+V(x)} - 1 \right) \frac{1}{|z|^{d+\alpha}} \, dz.
\]

Combining both equalities above with (2.4), we prove the desired assertion. \qed

Finally, the following result confirms the Lyapunov condition used in [4] for the study of super Poincaré inequalities.

**Proposition 2.3.** Assume \( e^{-V} \in C^2_b(\mathbb{R}^d) \) and that (1.3) holds. Let \( \alpha_0 \in (0, 1 \wedge \alpha) \) and let \( \phi \in C^\infty(\mathbb{R}^d) \) such that \( \phi(x) = 1 + |x|^{\alpha_0} \) for \( |x| \geq 1 \). Then \( e^{-V}, \phi, e^{-V} \in \mathcal{C}_\alpha \). If moreover \( \Phi(0) > 0 \), then there exist constants \( r_0, C_1, C_2 > 0 \) such that

\[
L_{\alpha,V} \phi(x) \leq -C_1 \Phi(|x|) \phi(x) + C_2 1_{\{|x| \leq r_0\}}, \quad x \in \mathbb{R}^d.
\]

**Proof.** By (1.3) and the choice of \( \phi \), it is easy to see that \( \phi, e^{-V} \phi \in \mathcal{C}_\alpha \). Since \( L_{\alpha,V} \phi \) is locally bounded, we only need to verify the conclusion for \( |x| \) large enough.

Using the facts that \( 2 \langle x, z \rangle = |x + z|^2 - |x|^2 - |z|^2 \) for all \( x, z \in \mathbb{R}^d \), and \( b^{\alpha_0} - a^{\alpha_0} \leq \alpha_0 a^{\alpha_0 - 1}(b - a) \) for any \( a, b \geq 0 \), we get that for \( |x| \) large enough,

\[
\int_{\{|z| \leq 1\}} \left( \phi(x + z) - \phi(x) - \langle \nabla \phi(x), z \rangle \right) \frac{dz}{|z|^{d+\alpha}} \\
\leq \alpha_0 |x|^{\alpha_0 - 1} \int_{\{|z| \leq 1\}} \left( |x + z| - |x| - \frac{\langle x, z \rangle}{|x|} \right) \frac{dz}{|z|^{d+\alpha}} \\
= \frac{1}{2} \alpha_0 |x|^{\alpha_0 - 2} \int_{\{|z| \leq 1\}} \left( 2|x + z| \cdot |x| - 2|x|^2 - |x + z|^2 + |x|^2 + |z|^2 \right) \frac{dz}{|z|^{d+\alpha}} \\
= \frac{1}{2} \alpha_0 |x|^{\alpha_0 - 2} \int_{\{|z| \leq 1\}} \left( |z|^2 - (|x| - |x + z|)^2 \right) \frac{dz}{|z|^{d+\alpha}} \\
\leq \frac{1}{2} \alpha_0 |x|^{\alpha_0 - 2} \int_{\{|z| \leq 1\}} \frac{dz}{|z|^{d+\alpha-2}} \\
\leq 1.
\]

Let \( c_1 = \sup_{|z| \leq 1} \phi(z) \). Then \( \phi(x) \leq c_1 + 1 + |x|^{\alpha_0} \) holds for all \( x \in \mathbb{R}^d \). Combining this with \( \phi(x) = 1 + |x|^{\alpha_0} \) for \( |x| \geq 1 \), and the triangle inequality \( (a + b)^{\alpha_0} \leq a^{\alpha_0} + b^{\alpha_0} \) for \( a, b \geq 0 \), we obtain that for \( |x| \) large enough

\[
\int_{\{|z| > 1\}} \left( \phi(x + z) - \phi(x) \right) \frac{1}{|z|^{d+\alpha}} \, dz
\]
\[
\begin{align*}
\leq & \int_{\{|z|>1\}} \left( c_1 + |x + z|^{\alpha_0} - |x|^{\alpha_0} \right) \frac{1}{|z|^{d+\alpha}} \, dz \\
\leq & \int_{\{|z|>1\}} \left( c_1 + |z|^{\alpha_0} \right) \frac{dz}{|z|^{d+\alpha}} \\
= & \, c_2 < \infty.
\end{align*}
\]

Therefore, for \(|x|\) large enough,

(2.5) \[
\int_{\mathbb{R}^d} \left( \phi(x + z) - \phi(x) - \nabla \phi(x) \cdot z 1_{\{|z| \leq 1\}} \right) \frac{dz}{|z|^{d+\alpha}} \leq 1 + c_2.
\]

Next, since \(|x + z|^{\alpha_0} - |x|^{\alpha_0} \leq |z|^{\alpha_0}\), and for large enough \(|x|\),

\[
1_{\{|x+z| \leq |x|\}} \left( |x + z|^{\alpha_0} - |x|^{\alpha_0} \right) \leq 1_{\{|x+z| \leq 1\}} \left( |x + z|^{\alpha_0} - |x|^{\alpha_0} \right),
\]

there exists a constant \(c_3 > 0\) such that for \(|x|\) large enough,

\[
\begin{align*}
\int_{\{|z|>1\}} & \left( \phi(x + z) - \phi(x) \right) \frac{e^{V(x) - V(x+z)}}{|z|^{d+\alpha}} \, dz \\
\leq & \, e^{V(x)} \int_{\{|z|>1\}} \left( c_1 + |x + z|^{\alpha_0} - |x|^{\alpha_0} \right) \frac{e^{-V(x+z)}}{|z|^{d+\alpha}} \, dz \\
\leq & \int_{\{|z|>1, |x+z| \leq |x|\}} \left( c_1 + |x + z|^{\alpha_0} - |x|^{\alpha_0} \right) \frac{e^{V(x) - V(x+z)}}{|z|^{d+\alpha}} \, dz \\
& + \int_{\{|z|>1, |x+z| > |x|\}} \left( c_1 + |z|^{\alpha_0} \right) \frac{e^{V(x) - V(x+z)}}{|z|^{d+\alpha}} \, dz \\
\leq & \, e^{V(x)} \left( \inf_{\{|z| \leq 1\}} e^{-V(z)} \right) \int_{\{|z|>1, |x+z| \leq |x|\}} \left( c_1 + 1 - |x|^{\alpha_0} \right) \frac{dz}{|z|^{d+\alpha}} \\
& + e^{V(x)} \left( \sup_{\{|z| \geq 1\}} e^{-V(z)} \right) \left[ c_1 \int_{\{|z|>1, |x+z| > |x|\}} \frac{dz}{|z|^{d+\alpha}} + \int_{\{|z|>1, |x+z| > |x|\}} \frac{dz}{|z|^{d+\alpha-\alpha_0}} \right] \\
\leq & \, -\frac{e^{V(x)} |x|^{\alpha_0}}{2} \left( \inf_{\{|z| \leq 1\}} e^{-V(z)} \right) \int_{\{|z|>1, |x+z| \leq |x|\}} \frac{dz}{|z|^{d+\alpha}} \\
& + \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \Phi(0) \left[ c_1 \int_{\{|z|>1\}} \frac{dz}{|z|^{d+\alpha}} + \int_{\{|z|>1\}} \frac{dz}{|z|^{d+\alpha-\alpha_0}} \right]
\end{align*}
\]
we conclude that

\[ L \]

Combining this with (2.5) and (2.6), and using the expression of \( L_{\alpha,v} \) in Proposition 2.2, we conclude that

\[
L_{\alpha,v} \phi(x) \leq -\frac{c_3}{8} \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \phi(x) \leq -\frac{c_3}{8} \Phi(|x|) \phi(x)
\]
holds for large enough \(|x|\).
3 Proof of Theorem 1.1

In the spirit of [4, Theorem 2.10], to derive functional inequalities using the Lyapunov condition confirmed in Proposition 2.3, we need only to verify the corresponding local inequality. So, we first present two lemmas concerning the local super Poincaré inequality and the local Poincaré inequality.

**Lemma 3.1.** There exists a constant \( c > 0 \) such that for any \( s, r > 0 \) and any \( f \in C_0^\infty(\mathbb{R}^d) \),

\[
\int_{B(0,r)} f(x)^2 e^{-V(x)} \, dx \leq \frac{1}{h(r)} \left( s \int_{B(0,r)} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \, dx \right)^2 + c_H(r)^{2+d/\alpha} (1 + s^{-d/\alpha}) \left( \int_{B(0,r)} |f(x)| \, dx \right)^2.
\]

**Proof.** Note that the Sobolev inequality of dimension \( 2d/\alpha \) holds for fractional Laplacians uniformly on balls, e.g. see [7, Section 2]. Then, according to [28, Corollary 3.3.4] (see also [24, Theorem 4.5]), there exists a constant \( c_1 > 0 \) such that

\[
\int_{B(0,r)} f^2(x) \, dx \leq \frac{1}{h(r)} \int_{B(0,r)} f^2(x) \, dx \leq \frac{1}{h(r)} \left( s \int_{B(0,r)} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \, dx \right)^2 + c_1 (1 + s^{-d/\alpha}) \left( \int_{B(0,r)} |f(x)| \, dx \right)^2.
\]

This implies the desired assertion by replacing \( s \) with \( sh(r)H(r)^{-1} \). \( \square \)

**Lemma 3.2.** For any \( r > 0 \) and \( f \in C_0^\infty(\mathbb{R}^d) \),

\[
(3.1) \quad \mu_V(f^2 1_{B(0,r)}) \leq \Psi_2(r) e_{c,V}(f, f) + \frac{\mu_V(f 1_{B(0,r)})^2}{\mu_V(B(0,r))}.
\]

Consequently, the weak Poincaré inequality (1.6) holds for

\[
(3.2) \quad \tilde{\beta}(r) := \inf \left\{ \frac{\Psi_2(R)}{\mu_V(B(0,R)^c)} : \mu_V(B(0,R)^c) \leq \frac{r}{1+r} \right\} < \infty, \quad r > 0.
\]
Proof. By the Cauchy-Schwarz inequality,
\[
\int_{B(0,r)} \left( f(x) - \frac{1}{\mu_V(B(0,r))} \int_{B(0,r)} f(x) \mu_V(dx) \right)^2 \mu_V(dx)
\]
\[
= \int_{B(0,r)} \left( \frac{1}{\mu_V(B(0,r))} \int_{B(0,r)} (f(x) - f(y)) \mu_V(dy) \right)^2 \mu_V(dx)
\]
\[
\leq \frac{1}{\mu_V(B(0,r))^2} \int_{B(0,r)} \left( \int_{B(0,r)} (f(x) - f(y))^2 \frac{e^V(y)}{|y-x|^{d+\alpha}} \mu_V(dy) \right) \frac{|y-x|^{d+\alpha}}{e^V(y)} \mu_V(dx)
\]
\[
\leq \Psi_2(r) \int_{B(0,r) \times B(0,r)} \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} dy \mu_V(dx).
\]

So, the inequality (3.1) holds, which implies the desired weak Poincaré inequality according to [17, Theorem 3.1] or [28, Theorem 4.3.1].

Since \( \mu_V(\mathbb{R}^d) < \infty \), most likely we have \( \int_{\mathbb{R}^d} e^{-2V(y)} dy < \infty \), so that \( \Psi_2(R) \leq c_1 R^{d+\alpha} \) holds for some constant \( c_1 > 0 \) and all \( R \geq 1 \). In this case there exists a constant \( c > 0 \) such that \( \tilde{\beta} \) in (3.2) satisfies
\[
\tilde{\beta}(r) \leq c + c \inf \left\{ R^{d+\alpha} : \mu_V(B(0,R)^c) \leq \frac{r}{1+r} \right\} < \infty, \quad r > 0.
\]

In many cases this \( \tilde{\beta} \) is however not sharp, for instance, in the proofs of Corollaries 1.2 and 1.3 we will use \( \Psi_1 \) rather than \( \Psi_2 \) in Theorem 1.1.5 to derive sharp estimates on \( \tilde{\beta} \).

Proof of Theorem 1.1. First, according to Proposition 2.3 we have
\[
1_{B(0,r)^c} \leq \frac{1}{C_1 \Phi(r)} \frac{-L_{\alpha,V} \phi}{\phi} + \frac{C_2}{C_1 \Phi(r)} 1_{B(0,r_0)}, \quad r \geq r_0.
\]

Then, for any \( f \in C_0^\infty(\mathbb{R}^d) \),
\[
\mu_V(f^2 1_{B(0,r)^c}) \leq \frac{1}{C_1 \Phi(r)} \mu_V \left( f^2 - \frac{L_{\alpha,V} \phi}{\phi} \right) + \frac{C_2}{C_1 \Phi(r)} \mu_V(f^2 1_{B(0,r_0)}).
\]

By Proposition 2.2 and the fact that
\[
\left( \frac{f^2(x)}{\phi(x)} - \frac{f^2(y)}{\phi(y)} \right) (\phi(x) - \phi(y)) = f^2(x) + f^2(y) - \left( \frac{\phi(y)}{\phi(x)} f^2(x) + \frac{\phi(x)}{\phi(y)} f^2(y) \right)
\]
\[
\leq f^2(x) + f^2(y) - 2 |f(x)f(y)|
\]
\[
\leq (f(x) - f(y))^2,
\]
we obtain
\[
\mu_V \left( f^2 - \frac{L_{\alpha,V} \phi}{\phi} \right) \leq \mathcal{E}_{\alpha,V}(f, f).
\]
Therefore, (3.3) implies

\[ \mu_V(f^2 1_{B(0,r)}) \leq \frac{1}{C_1 \Phi(r)} \mathcal{E}_{\alpha,V}(f,f) + \frac{C_2}{C_1 \Phi(r)} \mu_V(f^2 1_{B(0,r_0)}), \quad r \geq r_0. \]  

We are now to prove (1) and (2) in Theorem 1.1 respectively.

(1) According to [24, Theorem 4.5 and Theorem 3.2], the local super Poincaré inequality in Lemma 3.1 implies that the associated Markov semigroup on \( B(0, r) \) has a uniformly bounded density, and hence the spectrum of the associated generator is discrete. Moreover, it is easy to see that the Dirichlet form on \( B(0, r) \) is irreducible so that 0 is a simple eigenvalue of the generator, we conclude that the spectral gap exists. Equivalently, for any \( r > 0 \) there exists a constant \( C(r) > 0 \) such that the local Poincaré inequality

\[ \mu_V(f^2 1_{B(0,r)}) \leq C(r) \int_{\mathbb{R}^d} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \]  

holds for all \( f \in C_0^\infty(\mathbb{R}^d) \) with \( \mu_V(f 1_{B(0,r)}) = 0 \). This, together with (3.5) implies the defective Poincaré inequality

\[ \mu_V(f^2) \leq c_1 \mathcal{E}_{\alpha,V}(f,f) + c_2 \mu_V(|f|^2) \]

for some constants \( c_1, c_2 > 0 \); and due to [17, Theorem 3.1], (3.6) also implies the weak Poincaré inequality of \( \mathcal{E}_{\alpha,V} \). According to [17, Proposition 1.3], these two inequalities then imply the desired Poincaré inequality.

(2) Now, assume that \( \Phi(r) \uparrow \infty \) as \( r \uparrow \infty \). By Lemma 3.1 there exists a constant \( c > 0 \) such that

\[ \mu_V(f^2 1_{B(0,r)}) \leq s \mathcal{E}_{\alpha,V}(f,f) + \beta(r,s) \mu_V(|f|^2), \quad s, r > 0, f \in C_0^\infty(\mathbb{R}^d) \]

holds for

\[ \beta(r,s) := \frac{cH(r)^{2+d/\alpha}}{h(r)^{1+d/\alpha}} (1 + s^{-d/\alpha}). \]

Combining this with (3.5) and (3.6) with \( r = r_0 \), we may find a constant \( c_0 > 0 \) such that, for any \( r \geq r_0 \),

\[ \mu_V(f^2) = \mu_V(f^2 1_{B(0,r)}) + \mu_V(f^2 1_{B(0,r^c)}) \leq \left( s + \frac{c_0}{\Phi(r)} \right) \mathcal{E}_{\alpha,V}(f,f) + (c_0 + \beta(r,s)) \mu_V(|f|^2), \quad s > 0, f \in C_0^\infty(\mathbb{R}^d). \]

Letting \( s_0 = c_0/\Phi(r_0) \) and taking \( r = \Phi^{-1}(c_0/s) \), which is larger than \( r_0 \) if \( s \in (0, s_0) \), we obtain

\[ \mu_V(f^2) \leq 2s \mathcal{E}_{\alpha,V}(f,f) + \{c_0 + \beta(\Phi^{-1}(c_0/s), s)\} \mu_V(|f|^2), \quad s \in (0, s_0), f \in C_0^\infty(\mathbb{R}^d). \]

Replacing \( s \) by \( s/2 \), we

\[ \mu_V(f^2) \leq s \mathcal{E}_{\alpha,V}(f,f) + \{c_0 + \beta(\Phi^{-1}(2c_0/s), s/2)\} \mu_V(|f|^2), \quad s \in (0, 2s_0), f \in C_0^\infty(\mathbb{R}^d). \]
Noting that
\[ \beta(\Phi^{-1}(2c_0/s), s/2) = \frac{c \{ H \circ \Phi^{-1}(2c_0/s) \}^{2+d/\alpha}}{\{ h \circ \Phi^{-1}(2c_0/s) \}^{1+d/\alpha}(1 + 2^d/\alpha s^{-d/\alpha})}, \]
this implies the super Poincaré inequality with the desired \( \beta \) for some constants \( c_1, c_2 > 0 \) and all \( s \in (0, 2s_0) \). Then the inequality holds also for \( s \geq 2s_0 \) with a possibly large constant \( c_1 \) by taking \( \beta(s) = \beta(2s_0) \) for \( s \geq 2s_0 \).

(3) Let \( V_0(x) = \frac{d+\alpha}{2} \log(1 + |x|^2) \). Then Theorem 1.1(1) implies that the Poincaré inequality
\[ (3.7) \]
\[ \mu_{V_0}(f^2) \leq C \mu_{V_0}(\Gamma(f, f)), \quad f \in C_0^\infty(\mathbb{R}^d), \mu_{V_0}(f) = 0 \]
holds for some constant \( C > 0 \), where
\[ \Gamma(f, f)(x) := \int_{\mathbb{R}^d} \frac{|f(y) - f(x)|^2}{|y-x|^{d+\alpha}} dy, \quad x \in \mathbb{R}^d. \]

For any \( R > 0 \) and any \( f \in C_0^\infty(\mathbb{R}^d) \), it follows from (3.7) that
\[ \int_{B(0,R)} \left( f(x) - \frac{1}{\mu_V(B(0,R))} \int_{B(0,R)} f(x) \mu_V(dx) \right)^2 \mu_V(dx) \]
\[ = \inf_{a \in \mathbb{R}} \int_{B(0,R)} \left( f(x) - a \right)^2 e^{-V(x)} dx \]
\[ \leq \int_{B(0,R)} \left( f(x) - \mu_{V_0}(f) \right)^2 e^{-V(x)} dx \]
\[ \leq \left( \sup_{|x| \leq R} \left( 1 + |x|^{2(d+\alpha)/2} e^{V(x)} \right) \right) \int_{B(0,R)} \left( f(x) - \mu_{V_0}(f) \right)^2 e^{-V_0(x)} dx \]
\[ \leq C \left( \sup_{|x| \leq R} \left( 1 + |x|^{2(d+\alpha)/2} e^{V(x)} \right) \right) \int_{\mathbb{R}^d} \Gamma(f, f)(x) e^{-V_0(x)} dx \]
\[ \leq c \Psi_1(R) \int_{\mathbb{R}^d} \Gamma(f, f)(x) e^{-V(x)} dx. \]

That is,
\[ \mu_V(f^2 1_{B(0,R)}) \leq c \Psi_1(R) \mathcal{E}_{\alpha,V}(f, f) + \frac{\mu_V(f 1_{B(0,R)})^2}{\mu_V(B(0,R))}. \]

Combining this with Lemma 3.2 we obtain
\[ \mu_V(f^2 1_{B(0,R)}) \leq \{ (c \Psi_1(R)) \wedge \Psi_2(R) \} \mathcal{E}_{\alpha,V}(f, f) + \frac{\mu_V(f 1_{B(0,R)})^2}{\mu_V(B(0,R))}. \]

The required weak Poincaré inequality then follows from [17, Theorem 3.1] or [28, Theorem 4.3.1].

Remark 3.1. The formula (3.4) for diffusion operators is easily derived by using a chain rule, e.g. see [21, (2.2)]; and the proof of it for symmetric jump processes is based on the large derivation, see [4, Lemma 2.12]. Our proof here is more straightforward.
4 Proofs of Corollaries

In all these Corollaries, the sufficiency for the Poincaré/super Poincaré/weak Poincaré
inequalities will be confirmed by Theorem 1.1. To verify the necessary, we will make use
of the reference functions $g_n \in C^\infty(\mathbb{R}^d), n \geq 1$, such that $|\nabla g_n| \leq 2/n$ and

$$
\begin{aligned}
g_n(x) &= 0, \quad \text{if } |x| \leq n, \\
&\in [0, 1], \quad \text{if } |x| \in [n, 2n], \\
&= 1, \quad \text{if } |x| \geq 2n.
\end{aligned}
$$

Then there exists a constant $c > 0$ independent of $n$ such that

$$
\Gamma(g_n, g_n)(x) := \int_{\mathbb{R}^d} \frac{|g_n(y) - g_n(x)|^2}{|x - y|^{d+\alpha}} \, dy \
\leq \frac{4}{n^2} \int_{|x-y| \leq n} \frac{1}{|y-x|^{d+\alpha-2}} \, dy + \int_{|x-y| \geq n} \frac{1}{|x-y|^{d+\alpha}} \, dy \
\leq \frac{c}{n^\alpha}, \quad n \geq 1.
$$

(4.1)

Proof of Corollary 1.2. Obviously, for any $\varepsilon > 0$, the function

$$
V(x) := \frac{d + \varepsilon}{2} \log(1 + |x|^2), \quad x \in \mathbb{R}^d
$$

satisfies condition (1.3).

(1) If $\varepsilon \geq \alpha$, we have $\Phi(0) > 0$, so that the Poincaré inequality follows from Theorem
1.1(1). To disprove the Poincaré inequality for $\varepsilon \in (0, \alpha)$, let us take the reference function
$g_n$ introduced above. Obviously,

$$
\mu_V(g_n)^2 \geq \frac{c_1}{n^\varepsilon}, \quad \mu_V(g_n)^2 \leq \frac{c_2}{n^{2\varepsilon}}, \quad n \geq 1
$$

hold for some constants $c, c_1, c_2 > 0$. Combining this with (4.1) we see that

$$
\lim_{n \to \infty} \frac{\delta_{\alpha, V}(g_n, g_n)}{\mu_V(g_n)^2 - \mu_V(g_n)^2} \leq \lim_{n \to \infty} \frac{cn^{-\alpha}}{c_1n^{-\varepsilon} - c_2n^{-2\varepsilon}} = 0
$$

provided $\varepsilon \in (0, \alpha)$. Thus, for any constant $C > 0$, the Poincaré inequality (1.4) does not
hold.

(2) We first prove that if $\varepsilon \leq \alpha$, then for any $\beta : (0, \infty) \to (0, \infty)$ the super Poincaré
inequality (1.5) does not hold. Indeed, if this inequality holds, then

$$
\frac{c_1}{n^\varepsilon} \leq \frac{cr}{n^\alpha} + \frac{c_2 \beta(r)}{n^{2\varepsilon}}, \quad r > 0, n \geq 1
$$

holds for some constants $c, c_1, c_2 > 0$. Since $\varepsilon \in (0, \alpha]$, we obtain

$$
c_1 \leq \lim_{n \to \infty} \frac{cr}{n^{\alpha-\varepsilon}} + \lim_{n \to \infty} \frac{c_2 \beta(r)}{n^\varepsilon} \leq cr, \quad r > 0.
$$

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Letting $r \to 0$ we conclude that $c_1 \leq 0$, which is however impossible.

Next, let $\varepsilon > \alpha$, we aim to confirm the super Poincaré inequality with the desired function $\beta(r)$. It is easy to see that

$$
h(r) = 1, \quad H(r) = (1 + r^2)^{(d+\varepsilon)/2}, \quad r > 0
$$

and $\Phi(r) \geq c_3 r^{\varepsilon - \alpha}$ for $r$ large so that

$$
\Phi^{-1}(c_2 r^{-1}) \leq c_4 r^{-1/(\varepsilon - \alpha)} \quad \text{for } r > 0 \text{ small.}
$$

Hence, the function $\beta$ given in Theorem 1.1 (2) satisfies

$$
\beta(r) \leq c \left( 1 + r^\frac{d}{\alpha} \frac{(d+\varepsilon)(2\alpha + d)}{\alpha(\varepsilon - \alpha)} \right), \quad r > 0
$$

for some constant $c > 0$. The equivalence of the concrete super Poincaré inequality and the corresponding bound of $\|P_t^{\alpha,V}\|_{L^2(\mu_V)} \to L^2(\mu_V)$ then follows from [24, Theorem 4.5(2)] (see also [28, Theorem 3.3.15(2)]).

(3) It is easy to see that $\Psi_1(R) = O(R^{\alpha - \varepsilon})$ for large $R$. Then the desired weak Poincaré inequality follows from Theorem 1.1(3). According to [14, Corollary 2.4(2)] (see also [28, Theorem 4.1.5(2)]), we have the claimed bound of $\|P_t^{\alpha,V} - \mu_V\|_{L^\infty(\mu_V)} \to L^2(\mu_V)$. On the other hand, for $g_n$ presented in the beginning of this section, we have $\|g_n\|_{L^\infty} \leq 1$, $\mu_V(g_n^2) - \mu_V(g_n)^2 \geq c_1 n^{-\varepsilon}$ for some constant $c_1 > 0$, and due to (4.1), $\mathbb{E}_{\alpha,V}(g_n, g_n) \leq cn^{-\alpha}$. Then (1.6) implies that

$$
\frac{c}{n^\alpha} \beta(r) \geq \frac{c_1}{n^{\varepsilon}} - r, \quad r > 0.
$$

Taking $r_n = \frac{c_1}{2n^{\varepsilon}}$ which goes to zero as $n \to \infty$, we obtain

$$
\liminf_{n \to \infty} r_n^{(\alpha-\varepsilon)/\varepsilon} \beta(r_n) > 0.
$$

Thus, (1.6) does not hold if $\lim_{r \to 0} r^{(\alpha-\varepsilon)/\varepsilon} \beta(r) = 0$. \hfill \square

**Proof of Corollary 1.3.** Since when $\varepsilon > 0$ we have $\Phi(0) > 0$, the Poincaré inequality holds due to Theorem 1.1(1). According to e.g. [24, Corollary 1.3(1)], the super Poincaré inequality with $\beta(r) = \exp \left( c (1 + r^{-1}) \right)$ for some constant $c > 0$ is equivalent to the log-Sobolev inequality [17] for some constant $C > 0$, we conclude that (1) and (2) imply (3). So, it suffices to prove (1), (2) and (4).

(1) As in the proof of Corollary 1.2(2), when $\varepsilon \leq 0$ the super Poincaré inequality does not hold. Let $\varepsilon > 0$. We have

$$
e^{-V(x)} = \frac{1}{(1 + |x|^2)^{(d+\alpha)/2} \log^\varepsilon (e + |x|^2)}, \quad x \in \mathbb{R}^d.
$$

Then it is easy to see that

$$
h(r) = 1, \quad H(r) = (1 + r^2)^{(d+\alpha)/2} \log^\varepsilon (e + r^2)
$$

with
and 
\[ \Phi(r) \geq c_0 \log^\varepsilon(1 + r^2), \quad r > 0 \]
holds for some constant \( c_0 > 0 \). So, there exists a constant \( c_3 > 0 \) such that 
\[ \Phi^{-1}(c_2 r^{-1}) \leq \exp[c_3 r^{-1/\varepsilon}], \quad r > 0, \]
and hence, the function \( \beta \) given in Theorem 1.1(2) satisfies 
\[ \beta(r) \leq \exp[c(1 + r^{-1/\varepsilon})], \quad r > 0 \]
for some constant \( c > 0 \). When \( \varepsilon > 1 \), the equivalence of the concrete super Poincaré inequality and the corresponding bound of 
\[ \|P_{t}^{\alpha,V} \|_{L^1(\mu_V) \to L^2(\mu_V)} \]
then follows from [24, Theorem 4.5(1)] (see also [28, Theorem 3.3.15(1)]).

(2) It is easy to see that 
\[ \mu_V(g_n^2) \geq \frac{c_1}{n^\alpha \log^\varepsilon(e + n)}, \quad \mu_V(|g_n|)^2 \leq \frac{c_2}{n^{2\alpha} \log^{2\varepsilon}(e + n)}, \quad n \geq 1 \]
hold for some constants \( c_1, c_2 > 0 \). Combining this with (1.1) and (1.5), we obtain 
\[ \frac{c_1}{\log^\varepsilon(e + n)} \leq cr + \frac{c_2 \beta(r)}{n^{\alpha} \log^{2\varepsilon}(e + n)}, \quad r > 0. \]
Taking \( r_n = \frac{c_1}{2c} \log^{-\varepsilon}(e + n) \), we derive 
\[ \beta(r_n) \geq \frac{c_1}{2} n^\alpha \log^\varepsilon(e + n), \quad n \geq 1. \]

Therefore, 
\[ \liminf_{n \to \infty} r_n^{1/\varepsilon} \log \beta(r_n) \geq \alpha > 0. \]
Thus, the proof of (2) is done.

(4) Let \( \varepsilon < 0 \). Then there exist constants \( C, c > 0 \) such that 
\[ \Psi_1(R) \leq C \log^{-\varepsilon}(e + R), \quad \mu_V(B(0,R)^c) \leq cR^{-\alpha} \log^{\varepsilon}(e + R), \quad R > 0. \]
So, the desired weak Poincaré inequality follows from Theorem 1.1(3), and the corresponding convergence rate of 
\[ \|P_t^{\alpha,V} - \mu_V\|_{L^\infty(\mu_V) \to L^2(\mu_V)} \]
follows from [17, Corollary 2.4(1)]. Finally, the sharpness of \( \tilde{\beta} \) can be easily verified using reference functions \( g_n, n \geq 1 \).

Proof of Corollary 1.4. There exist constants \( C, c > 0 \) such that 
\[ \Psi_1(R) \leq \frac{C R^\varepsilon}{\log^\varepsilon(e + R)}, \quad \mu_V(B(0,R)^c) \leq c \left( \log(e + R)^{-\varepsilon} \right), \quad R > 0. \]
So, the desired weak Poincaré inequality follows from Theorem 1.1(3) and the corresponding convergence rate of 
\[ \|P_t^{\alpha,V} - \mu_V\|_{L^\infty(\mu_V) \to L^2(\mu_V)} \]
follows from [17 Corollary 2.4(3)]. Similar to the part (4) in the proof of Corollary 1.3 the sharpness of \( \tilde{\beta} \) can be easily verified using reference functions \( g_n, n \geq 1 \).
Proof of Corollary 1.5. For the super Poincaré inequality with desired $\beta$, we need to prove for small $r > 0$, since we may always take $\beta$ to be deceasing in the super Poincaré inequality.

1. Since
\[ e^{V(x)} = \exp \left[ \log^{1+\varepsilon}(1 + |x|^2) \right], \]

it is easy to see that $h(r) = 1$, $H(r) = \exp \left[ \log^{1+\varepsilon}(1 + r^2) \right]$ and
\[ \Phi(r) = \frac{H(r)}{(1 + r)^{d+\alpha}} \geq \exp \left[ \frac{1}{2} \log^{1+\varepsilon}(1 + r) \right], \]
holds for some constant $r_0 > 0$. So,
\[ H \circ \Phi^{-1}(c_2r^{-1}) = \left\{ \Phi \circ \Phi^{-1}(c_2r^{-1}) \right\} \cdot \left\{ 1 + \Phi^{-1}(c_2r^{-1}) \right\}^{d+\alpha} \leq c r^{-1} \exp \left[ c \log^{1/(1+\varepsilon)} r^{-1} \right] \]
holds for some constant $c > 0$ and small $r > 0$. Then (1.5) with the desired $\beta$ for small $r > 0$ follows from Theorem 1.1(2), and the corresponding bound of $\|P_{t}^{\alpha,V}\|_{L^1(\mu_V) \rightarrow L^\infty(\mu_V)}$ then follows from e.g. [24, Theorem 4.4].

2. Since $e^{V(x)} = \exp[(1 + |x|^2)^\varepsilon]$, it is easy to see that $h(r) = 1$, $H(r) = \exp \left[ (1 + r^2)^\varepsilon \right]$ and
\[ \Phi(r) = \frac{\exp[(1 + r^2)^\varepsilon]}{1 + r^{d+\alpha}}, \]
holds for some constant $r_0 > 0$. Then there exists a constant $c > 0$ such that for small enough $r > 0$,
\[ H \circ \Phi^{-1}(c_2r^{-1}) = \left\{ \Phi \circ \Phi^{-1}(c_2r^{-1}) \right\} \cdot \left\{ 1 + \Phi^{-1}(c_2r^{-1}) \right\}^{d+\alpha} = c_2 r^{-1} \left( \Phi^{-1}(c_2r^{-1})^{d+\alpha} \right) \leq cr^{-1} \log^{(d+\alpha)/(2\varepsilon)}(1 + r^{-1}). \]
Therefore, the super Poincaré inequality with the desired $\beta(r)$ for small enough $r > 0$ follows from Theorem 1.1(2), and the corresponding bound of $\|P_{t}^{\alpha,V}\|_{L^1(\mu_V) \rightarrow L^\infty(\mu_V)}$ then follows from [24, Theorem 4.4].

5 Super Poincaré inequalities implied by (1.8)

This section aims to establish the super Poincaré inequality using condition (1.8), so that the assertion in [15] for the Poincaré inequality is strengthened. As already indicated in Section 1 that the resulting super Poincaré inequality is normally worse than that presented in Theorem 1.1.

For fixed $V \in C^2(\mathbb{R}^d)$ such that $\mu_V$ is a probability measure, let $h, H$ be as in Theorem 1.1 and let
\[ W_{\delta}(r) = \inf_{|x| \geq r} \left( \delta |\nabla V(x)|^2 - \Delta V(x) \right), \]
\[ r > 0. \]
Theorem 5.1. Let $V \geq 0$. If there exists a constant $\delta \in (0, 1)$ such that

$$\lim_{|x| \to \infty} \{\delta |\nabla V|^2 - \Delta V\} = \infty,$$

Then there exist constants $c_1, c_2 > 0$ such that the super Poincaré inequality \cite{15} holds for

$$\beta(r) = c_1 \left(1 + r^{-d/\alpha} H^{2+d/2} \left(W^{-1}(c_2 r^{-2/\alpha})\right) h^{-1+d/2} \left(W^{-1}(c_2 r^{-2/\alpha})\right)\right), \ r > 0.$$ 

In particular, if $V(x) = (1 + |x|^2)^\varepsilon$ for $\varepsilon > \frac{1}{2}$, then there exists a constant $c > 0$ such that the super Poincaré inequality holds for

$$\beta(r) = \exp \left(c(1 + r^{-2\varepsilon/(\alpha(2\varepsilon-1))})\right), \ r > 0.$$ 

Proof. We only prove the first assertion, since the second one is a simple consequence. Let

$$L_V f = \Delta f - \langle \nabla V, \nabla f \rangle, \ f \in C^2(\mathbb{R}^d).$$

Then

$$\mathcal{E}_V(f, g) := \int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle \, d\mu_V = - \int_{\mathbb{R}^d} f L_V g \, d\mu_V, \ f, g \in C_0^\infty(\mathbb{R}^d).$$

Hence, the Friedrichs extension $(L_V, \mathcal{D}(L_V))$ of $(L_V, C_0^\infty(\mathbb{R}^d))$ in $L^2(\mu_V)$ is a negatively definite self-adjoint operator. Let $(-L_V)^{\alpha/2}$ be the associated fractional operator. Let $\varphi = e^{(1-\delta)V}$. We have

$$\frac{L_V \varphi}{\varphi} = -(1 - \delta)\left(\delta |\nabla V|^2 - \Delta V\right).$$

Then, by the assumption on $V$ and \cite{13} Theorem 2.10], there exist constants $c_3, c_4 > 0$ such that the super Poincaré inequality

$$\mu_V(f^2) \leq r \mathcal{E}_V(f, f) + \beta_V(r)\mu(|f|)^2, \ r > 0, f \in C_0^\infty(\mathbb{R}^d), \mu(f) = 0$$

holds for

$$\beta_V(r) := c_3 \left(1 + r^{-d/2} H^{2+d/2} \left(W^{-1}(c_4 r^{-1})\right) h^{-1+d/2} \left(W^{-1}(c_4 r^{-1})\right)\right), \ r > 0.$$ 

According to \cite{27} Corollary 2.1 or the proof of \cite{22} Proposition 9], this implies

$$\mu_V(f^2) \leq r \int_{\mathbb{R}^d} f (-L_V)^{\alpha/2} f \, d\mu_V + \frac{8}{\alpha} \beta_V \left((r/4)^{2/\alpha}\right) \mu_V(|f|)^2, \ r > 0$$

for all $f \in C_0^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$. A close inspection of the arguments in \cite{13} Section 3] (see \cite{19} Lemma 3.2 and Lemma 3.3 for details) yields that there is a constant $C > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^d)$ with $\mu_V(f) = 0$,

$$\int_{\mathbb{R}^d} f (-L_V)^{\alpha/2} f \, d\mu_V \leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, d\mu_V(dx) = C \mathcal{E}_{\alpha, V}(f, f).$$

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Combining this with (5.1), we obtain

\[ \mu_V(f^2) \leq r \varepsilon_{\alpha,V}(f,f) + \frac{8}{\alpha} \beta_V \left( \frac{r}{4C} \right)^{2/\alpha} \mu_V(|f|)^2, \quad r > 0, f \in C^\infty_0(\mathbb{R}^d), \mu_V(f) = 0. \]

Then the desired assertion follows immediately. \(\square\)

Similarly, combining the proof above with [17, Theorem 3.1] and [27, Corollary 2.2] (or the proof of [22, Proposition 9]), we have the following result for weak Poincaré inequalities for stable-like Dirichlet forms, which is normally less sharp than that given in Theorem 1.1.

**Theorem 5.2.** For any \( V \in C^2(\mathbb{R}^d) \) such that \( \mu_V \) is a probability measure, there exist constants \( c_1, c_2 > 0 \) such that the weak Poincaré inequality (1.6) holds for

\[ \tilde{\beta}(r) = c_1 U(c_2r^{\alpha/2})^2 \exp \left( 2\delta_{U(c_2r^{\alpha/2})}(V) \right), \quad r > 0, \]

where

\[ U(r) = \inf \left\{ s > 0 : \int_{|x| > s} e^{-V(x)} \, dx \leq r/(1 + r) \right\} \quad \text{and} \quad \delta_r(V) = \sup_{|x| \leq r} V(x). \]

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