A NOTE ON THE CRITICAL POINTS OF THE TORSION FUNCTION

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Abstract. Let $\Omega \subset \mathbb{C}$ be a bounded domain. In this note, we use complex variable methods to study the number of critical points of the function $v = v_\Omega$ that solves the elliptic problem $\Delta v = -2$ in $\Omega$, with boundary values $v = 0$ on $\partial \Omega$. We provide an upper bound on the number of critical points of $v$ when $\Omega$ belongs to a special class of domains in the plane, namely, domains for which the boundary $\partial \Omega$ is contained in $\{ z : |z|^2 = f(z) + \overline{f(z)} \}$, where $f(z)$ is a rational function. We furnish examples of domains where this bound is attained. We also prove a bound on the number of critical points in the case when $\Omega$ is a quadrature domain, and conclude the note by stating some open problems and conjectures.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a finitely-connected bounded domain, and consider the solution $v$ to the following boundary value problem

$$
\begin{cases}
\Delta v(z) = -2, & z \in \Omega \\
v(z) = 0, & z \in \partial \Omega.
\end{cases}
$$

It is a simple fact that such a solution $v$ is unique. In this paper, we study the number of critical points of $v$ in $\Omega$, i.e., the number of solutions to $\nabla v(z) = 0$ for $z \in \Omega$. Observe that $v$ is a smooth superharmonic function in $\Omega$ that vanishes on the boundary. By the minimum principle, $v > 0$ in $\Omega$ and therefore attains its maximum on $\overline{\Omega}$ at a point $z_0 \in \Omega$. This guarantees that there is at least one solution to $\nabla v(z) = 0$. By superharmonicity we also have that $v$ does not have any minima in $\Omega$ (only maxima and saddles). Note further that by the Hopf lemma, $\nabla v$ cannot vanish on the boundary $\partial \Omega$. Since $\partial \Omega$ is a level set of $v$ this implies that $\nabla v$ is parallel to the inward pointing normal. This allows a simple application of index theory in order to relate the number of maxima to the number of saddles (given the connectivity of $\partial \Omega$), see Theorem 2.1 below. In particular, this produces a lower bound on the number of critical points of $v$ in terms of the connectivity of $\Omega$.

The problem of counting critical points of $v$, as well as the more general program of counting the critical points of solutions to elliptic PDE, is classical, see the survey [24].

Further motivation for studying the critical points of the solution to (1) comes from recent studies on localization of eigenfunctions [12]. Namely, solutions to (1) have been proposed as a “universal landscape” governing the localization phenomena observed for eigenfunctions of the Laplacian. The landscape (graph of $v$) is used to construct a valley network (a cell complex with vertices given by saddles and edges constructed from steepest descent curves along with arcs from $\partial \Omega$), and each cell serves as a candidate region for localized high-amplitude activity exhibited by one of the eigenfunctions. The complexity of the valley network is closely tied to the number of critical points (one may even use the maxima themselves as rudimentary predictors for locating high-amplitude regions). Thus, it would be useful to have results estimating the number of critical points of $v$ in terms of basic geometric features of the domain $\Omega$.

When $\Omega$ is simply-connected the function $v$ is the so-called torsion function of $\Omega$, used in elasticity theory to model the stress in a cylindrical bar with uniform cross section $\Omega$ undergoing a twisting force. Note that the torsion function from elasticity theory is defined differently in the multiply-connected case (having distinct Dirichlet conditions on separate boundary components). Since we
are primarily motivated by [2], we take vanishing Dirichlet condition on all boundary components, but we will refer to \( v \) as the “torsion function” throughout the paper even when \( \Omega \) is multiply-connected.

When \( \Omega \) is convex, \( v \) has a unique critical point. It appears that very little is known beyond this. It would be desirable to bound the number of critical points of \( v \) in terms of geometric features of \( \partial \Omega \). For instance, for \( \Omega \) simply-connected and \( \partial \Omega \) smooth, is there an upper bound on the number of critical points of \( v \) in terms of the number of oscillations of \( \partial \Omega \) (defined as oscillations of the distance between \( \partial \Omega \) and a fixed reference point)?

Alternatively, one may restrict attention to an interesting class of domains admitting an algebraic description (accompanied by a notion of degree or order) and attempt to bound the number of critical points of \( v \) in terms of the order of \( \partial \Omega \). Some natural classes that come to mind are quadrature domains, lemniscate domains, or the examples discussed in [13] for which \( \nabla u \) turns out to be the complex conjugate of a rational function in \( z \), where \( u \) is the harmonic part of \( v \) defined below. In this note, we focus primarily on the latter class of domains, which we refer to as TR-domains.

If \( v \) is the torsion function, it is easy to see then that \( u(z) = v(z) + \frac{|z|^2}{2} \) solves

\[
\begin{align*}
\Delta u(z) &= 0, \quad z \in \Omega \\
u(z) &= \frac{|z|^2}{2}, \quad z \in \partial \Omega.
\end{align*}
\]

Therefore, critical points of \( v \) are the same as solutions to

\[\nabla u(z) = z, \quad z \in \Omega.\]

This is an anti-holomorphic fixed point problem, placing it within a class of problems that has seen many breakthroughs over the past fifteen years utilizing anti-holomorphic dynamics (the primary technique in these studies is an adaptation of the Fatou theorem from holomorphic dynamics).

In [19], the authors used anti-holomorphic dynamics to address a question of Sheil-Small [29] on the number of zeros of harmonic polynomials. An extension of this result to the setting of harmonic rational functions in [18] settled a conjecture [28] in astronomy (concerning gravitational lensing by point-mass configurations). Transcendental dynamics were used in [17] to study another problem in gravitational lensing involving elliptical galaxies (a sharp result was obtained in [4]). Utilizing quasiconformal surgery, the use of anti-holomorphic dynamics as an indirect technique was further adapted to study the topology of quadrature domains in [21]. Anti-holomorphic dynamics involving elliptic functions were used in [5] to give much shorter and greatly simplified proofs of the results in [22], [9], [23] on the number of critical points of Green’s function on a torus. Transcendental dynamics were used again in [6] to completely classify the number of zeros of certain complex equations involving polynomials and logarithms. Prior to these applications, the study of anti-holomorphic dynamics had been initiated in [10] (see also [27], [26]).

Among the studies mentioned above, the study [5] on the critical points of Green’s function on a torus is especially relevant. Taking a sublevel set of (the negation of) Green’s function (in order to remove a neighborhood of the logarithmic pole) and adding a constant so that the function vanishes on the boundary of the sublevel set, we obtain obtain the torsion function of a domain on a torus, and the results of [5] give a complete description of the possible number of critical points.

1.1. Outline of the paper. We review some preliminary results in Section 2. In Section 3 we prove an upper bound on the number of critical points of \( v \) for so-called TR-domains, and in Section 4 we prove an upper bound for certain quadrature domains. We consider some additional examples in Section 5 and we conclude with some open problems in Section 6.

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2. Preliminaries

For the next two results we refer to [2] Thm. 3.3 and Cor. 3.4.

**Theorem 2.1.** Let \( \Omega \) be a bounded domain in the plane whose boundary \( \partial \Omega \) consists of \( k \) connected components each of class \( C^{1,\alpha} \). If the critical points of the torsion function \( v \) are isolated, then \( S - M = k - 2 \), where \( S \) and \( M \) denote the number of saddles and maxima (respectively) of \( v \).

**Theorem 2.2.** Let \( \Omega \) be a bounded simply-connected domain in the plane with boundary \( \partial \Omega \) of class \( C^{1,\alpha} \). Then the number \( N \) of critical points of the torsion function \( v \) is finite.

If the assumption that \( \Omega \) is simply-connected is removed then the critical set may be infinite. For instance, in the case of an annulus the critical set is a circle. More generally, prescribing an arbitrary non-singular analytic Jordan curve \( \gamma \), one may construct as follows an annular neighborhood \( \Omega \) of \( \gamma \) whose torsion function \( v \) has \( \gamma \) as its critical set. Let \( S(z) \) denote the Schwarz function of \( \gamma \). Then \( S(z) \) is an anti-conformal reflection function that fixes \( \gamma \).

The above pathological examples might seem plentiful since we were free to prescribe the analytic curve \( \gamma \) of critical points. However, we point out that these examples are unstable under perturbation. Namely, following a standard argument from Morse theory [25, Lemma 2.21] we can obtain a Morse function (Morse functions have isolated nondegenerate critical points) by adding an arbitrarily small linear perturbation to \( v \). Since the perturbation is linear, the perturbed \( v \) still solves the same PDE, and with an appropriate corresponding perturbation of \( \Omega \) the vanishing boundary condition is also satisfied. Using further tools from Morse theory [25, Lemma 2.26], one can also show that the “good” domains whose torsion function \( v \) is a Morse function (and hence has isolated nondegenerate critical points) are stable under perturbation. We do not pursue these genericity results in detail as they are not needed for the primary class of domains we consider (where we can more directly rule out pathological examples, see the proof of Theorem 3.2).

3. The number of critical points of \( v \): an upper bound for TR-domains

Let us consider the special class of domains that were introduced in [13], for which the solution to the Dirichlet problem with data \(|z|^2\) is the real part of a function whose derivative is rational in \( z \).

**Definition 3.1.** We say that a domain \( \Omega \) is a **TR-domain** (“torsionally rational domain”) if the boundary \( \partial \Omega \) of \( \Omega \) is a subset of

\[
\{ z : |z|^2 = f(z) + \overline{f}(z) \},
\]

where the derivative \( f'(z) \) of \( f(z) \) is a rational function. Given a domain \( \Omega \) of this type, we define the **order** of \( \Omega \) to be the degree of the derivative \( f'(z) \) of the associated function \( f \).

We note that even though these domains have a special form, they are dense in the space of Jordan domains with smooth boundary, see Proposition 3.3 below. Also, note that for a TR-domain, the torsion function is given by \( v(z) = \text{Re} f(z) - \frac{|z|^2}{2} \). The following is directly related to
a result of Khavinson and Neumann [18] on the number of solutions of an equation of the form
\[ r(z) + \bar{z} = 0, \]
where \( r \) is a rational function of degree \( n \geq 2 \).

**Theorem 3.2.** Suppose \( \Omega = \{z \in \mathbb{C} : f(z) + \bar{f(z)} - |z|^2 > 0\} \) is a smooth TR-domain of order \( n \geq 2 \), and connectivity \( k \). If \( \Omega \) is not an annulus, then the number \( N \) of critical points of \( v \) satisfies
\[ N \leq 4n + k - 6. \]

**Proof.** The assumption that \( \Omega \) is not an annulus guarantees that the critical set is finite. Indeed, suppose the critical set is infinite. Then \( \nabla v = f'(z) - z \) must vanish on an analytic arc \( \gamma \) (see the proof of Corollary 3.4 in [2]). We notice that the Schwarz function of \( \gamma \) is \( f'(z) \). Since \( f'(z) \) is rational, this implies that \( \gamma \) is a circle [11, Ch. 10] and, after a complex linear change of variables, we have \( f'(z) = 1/z \). Hence \( u(z) = \log |z| + C \), for \( C \) a constant, so that the set \( |z|^2/2 = u(z) \) defines an annulus.

The locations where maxima of \( v \) occur correspond to non-repelling fixed points of the function \( \overline{F(z)} \), where \( F(z) = f'(z) \). Indeed, if \( z_0 \) is a critical point then we have that \( \nabla v = 0 \), or equivalently \( f'(z_0) = z_0 \). i.e., \( z_0 \) is a fixed point of \( F(z) \). Moreover, if \( v \) attains a maximum at \( z_0 \) then the determinant of its Hessian matrix is non-negative at \( z_0 \). The Hessian matrix of \( v \) is the Jacobian matrix of the harmonic mapping \( \overline{F(z)} - z \), and the Jacobian determinant of this harmonic mapping is \( |F'(z)|^2 - 1 \). Thus, if a maximum occurs at \( z_0 \), then \( |F'(z_0)|^2 \leq 0 \), or equivalently, \( |F'(z_0)| \leq 1 \). i.e., \( z_0 \) is a non-repelling fixed point of \( F(z) \).

This along with [18] Prop. 1 implies that the number \( M \) of non-repelling fixed points of \( \overline{F(z)} \) is at most \( 2n - 2 \). Applying Theorem 2.1 we then have
\[ S = M + k - 2 \]
\[ \leq (2n - 2) + k - 2 \]
\[ = 2n + k - 4, \]
where \( S \) denotes the number of saddles in \( \Omega \). Thus, the total number \( N \) of critical points satisfies
\[ N = M + S \leq 2n - 2 + 2n + k - 4 = 4n + k - 6, \]
as desired. \( \square \)

**Remark 3.3.** Requiring that the boundary is smooth in Theorem 3.1 is not too restrictive of a condition. The boundary \( \partial \Omega \) of a TR-domain is given by \( \{z : \text{Re}(f) = |z|^2\} \), i.e., a level curve of a real analytic function \( \psi \). If \( \partial \Omega \) is non-singular, i.e., \( \nabla \psi \) does not vanish on \( \partial \Omega \), then \( \partial \Omega \) is smooth.

**Extremal examples:** We exhibit a TR-domain of order \( n \geq 2 \) and connectivity \( k = n + 1 \) for which the above bound is attained, so that the number of critical points is exactly \( 5n - 5 \). Our construction is an adaptation of the extremal examples from gravitational lensing that were constructed by S. Rhie [28]. For simplicity we assume \( n \geq 3 \) (the case \( n = 3 \) can be dealt with separately). Let
\[ w(z) = \epsilon \log |z| + \sum_{j=1}^{n-1} \log |z - a_j| - \frac{|z|^2}{2}, \]
where \( a_j = ae^{2\pi i j/(n-1)} \) for \( j = 1, \ldots, n - 1 \), with \( a > 0 \) small (and depending on \( n \), see [7]), and with \( a \gg \epsilon > 0 \). For \( T \gg 1 \), consider the set \( \Omega_T = \{z : w(z) + T > 0\} \). All the conditions in Definition 3.1 hold, so that \( \Omega_T \) is a bounded TR-domain of order \( n \) and connectivity \( n + 1 \). A simple verification yields that the torsion function of \( \Omega_T \) is \( v(z) = w(z) + T \), and \( f'(z) = \frac{\xi}{z} + \sum_{j=1}^{n-1} \frac{1}{z - a_j} \) is the associated rational function. As we observed before, the equation \( \nabla v(z) = 0 \) is the same as \( f'(z) = \frac{\xi}{z} \). By the known result [28], this latter equation has \( 5n - 5 \) solutions in the plane. A priori these critical points may not all be contained in \( \Omega_T \), but we can ensure that this is the case by
making $T$ larger. Therefore for this particular choice, the number of critical points of $v$ is exactly $5n - 5$ which concludes the proof.

It is more difficult to verify the existence of examples (conjectured below) that attain the bound for each $n$ and with every connectivity $k = 1, 2, \ldots, n + 1$, but based on computer simulations a promising strategy is as follows: first perturb the coefficients of the logarithmic terms in (2) in order to break the symmetry (and ensure the saddles have distinct heights), and then obtain domains of each connectivity by gradually decreasing the level $T$ so that the connectivity of $\Omega$ experiences $n$ bifurcations at $n$ different heights corresponding to saddle points. Such bifurcations could, a priori, disconnect $\Omega$, and the difficulty is to show that $\Omega$ in fact remains connected as $T$ crosses the heights of the $n$ highest saddles.

**Conjecture 3.4.** For every $n \geq 2$, and for each $k = 1, 2, \ldots, n + 1$ there exists a TR-domain $\Omega$ of order $n$ and connectivity $k$ whose torsion function has exactly $4n + k - 6$ critical points in $\Omega$.

Note that $k$ cannot exceed $n + 1$ since each bounded component of the complement of a TR-domain contains a singularity of $f'$ (see [13, Thm. 3]).

We show some examples of extremal domains and their torsion functions in Figures [1] and [2]. The domain on the left in Figure [1] is given by

$$\Omega_L = \left\{ z \in \mathbb{C} : |z|^2/2 - \frac{3}{4} \sum_{k=0}^{2} \log |z - e^{2k\pi i/3}| < 1/2 \right\},$$

and the torsion function is $v_L(z) = 1/2 - |z|^2/2 + \frac{3}{4} \sum_{k=0}^{2} \log |z - e^{2k\pi i/3}|.$

The domain on the right in Figure [1] is given by

$$\Omega_R = \left\{ z \in \mathbb{C} : |z|^2/2 - \frac{3}{4} \sum_{k=0}^{2} \log |z - e^{2k\pi i/3}| < 2/5 \right\},$$

and the torsion function is $v_R(z) = 2/5 - |z|^2/2 + \frac{3}{4} \sum_{k=0}^{2} \log |z - e^{2k\pi i/3}|.$

The domain shown in Figure [2] is given by

$$\Omega = \left\{ z \in \mathbb{C} : |z|^2/2 - \frac{1}{3} \sum_{k=0}^{4} \log |z - e^{2k\pi i/5}| - \frac{9}{40} \log |z| = 1/2 \right\},$$

and the torsion function is

$$v(z) = 1/2 - |z|^2/2 + \frac{1}{3} \sum_{k=0}^{4} \log |z - e^{2k\pi i/5}| + \frac{9}{40} \log |z|.$$

**Proposition 3.5.** Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain whose boundary $\partial \Omega$ is analytic. Then, for any $\varepsilon > 0$ there is a TR-domain $D$ such that

$$d_H(D, \Omega) < \varepsilon,$$

where $d_H(A, B)$ denotes the Hausdorff distance between the sets $A$ and $B$.

**Proof.** Let $w$ be the torsion function of $\Omega$. Then $u(z) = w(z) + \frac{|z|^2}{2}$ solves

$$(3) \quad \begin{cases} \Delta u(z) = 0, & z \in \Omega \\ u(z) = \frac{|z|^2}{2}, & z \in \partial \Omega. \end{cases}$$

Since $\partial \Omega$ is analytic, $u$ has a harmonic extension (using the Schwarz function) to a bounded simply connected domain $\tilde{\Omega}$ which contains $\Omega$ as a proper subdomain. Let $g = u + iv$ be an analytic completion of $u$ in $\tilde{\Omega}$. Let $2\mu = \text{dist}(\tilde{\Omega}, \partial \tilde{\Omega})$ and $\Omega_1 = \{ z \in \tilde{\Omega} : \text{dist}(z, \Omega) < \mu \}$. Let $\delta > 0$ be a
constant to be chosen later. By Runge’s theorem, we can find a rational function \( R(z) \) whose poles lie outside \( \Omega_1 \) with \( \|g - R\|_{\Omega_1} < \delta \). By Cauchy’s estimate, \( \|g' - R'\|_{\Omega} < \frac{\delta}{\mu} \). In particular,

\[
\|u - \text{Re} R\|_{\bar{\Omega}} < \delta, \quad \|\nabla u - \nabla \text{Re} R\|_{\bar{\Omega}} < \frac{\delta}{\mu}.
\]

Setting \( \tau(z) = \text{Re} R(z) - \frac{1}{2} |z|^2 \), we obtain from the above displayed equation that

\[
\|w - \tau\|_{\bar{\Omega}} < \delta, \quad \|\nabla w - \nabla \tau\|_{\bar{\Omega}} < \frac{\delta}{\mu}.
\]
Now let $\varepsilon > 0$ be given. We want to show that there exists a TR-domain which approximates $\Omega$ within Hausdorff distance $\varepsilon$. Recall that $\{ w = 0 \} = \partial \Omega$ and $\{ w > 0 \} = \Omega$. From (4), we get

$$\Omega_\delta = \{-\delta < w < \delta\}$$

contains a component of $\{ \tau = 0 \}$. Hopf lemma tells us that $|\nabla w| \neq 0$ near $\partial \Omega$. Choose $\delta = \delta(\varepsilon) > 0$ small enough so that

$$|\nabla w| \geq C > 0 \text{ on } \Omega_\delta, \quad d_H(\Omega_\delta, \partial \Omega) < \varepsilon.$$

Such a choice is possible because of the non vanishing nature of the gradient which implies continuity of the level sets for small $\delta$. Combining this with the second estimate from (4), we obtain that $|\nabla \tau| \neq 0$ on $\Omega_\delta$. In particular $\{ \tau = 0 \}$ is a Jordan curve that bounds a simply connected TR-domain, say $D$. By further reducing $\delta$ if necessary, we obtain from (5) that $d_H(D, \Omega) < \varepsilon$.

\[\square\]

4. Quadrature domains

Recall that a bounded domain $\Omega \subset \mathbb{C}$ is called a quadrature domain if it admits a formula expressing the area integral of any integrable analytic function $g$ as a finite sum of weighted point evaluations of $g$ and its derivatives:

$$\int_D g(z) dA(z) = \sum_{m=1}^L \sum_{k=0}^{n_m} a_{m,k} g^{(k)}(z_m),$$

where the points $z_m \in \Omega$ and constants $a_{m,k}$ are each independent of $g$. The order of the quadrature domain $\Omega$ is defined as the sum $\sum_{m=1}^L n_m$. Quadrature domains have been studied extensively in function theory, potential theory, and fluid dynamics, see [16] for a survey.

We conjecture that the size of the critical set of $v$ is bounded by a function that is linear in the order $n$ of $\Omega$ and the connectivity $k$ of $\Omega$.

**Conjecture 4.1.** Let $\Omega$ be a quadrature domain of order $n$ and connectivity $k$. The number $N$ of critical points of $v$ satisfies $N \leq 2n - 2 + k$.

The topology of quadrature domains was studied in [21] where it was shown that $k \leq 2n - 4$ for $n \geq 3$. Thus, if the above conjecture is true, then we have $N \leq 4n - 6$, for $n \geq 3$.

Let us describe a heuristic for constructing examples that attain the conjectured bound (intuition for the conjecture is based on our belief that this construction is extremal). We recall an interpretation of quadrature domains in terms of logarithmic potential theory. If $\Omega$ is a quadrature domain (and suppose the quadrature formula holds for harmonic test functions as well as analytic ones) then the logarithmic potential generated by area measure over $\Omega$ coincides (outside of $\Omega$) with that which is generated by a finite system of point charges (possibly including dipoles and higher order multi-poles) situated at the quadrature nodes $z_1, z_2, ..., z_L$. With this formulation, the order of the quadrature domain is the number of poles (counting multi-poles with multiplicity). Consider a collection of $n$ disjoint disks with each tangent to at least one other disk. The logarithmic potential generated by the area measure on these disks is equivalent to that of a system of point charges. Slightly increasing each point charge by $\varepsilon > 0$, the corresponding quadrature domain (that generates the same logarithmic potential as the system of increased charges) contains the closure of the original set of disks, and for $\varepsilon > 0$ small we expect the torsion function $v$ to have $n$ maxima. The number $S$ of saddles is then given by $S = n + k - 2$ so that the total number of critical points attains the conjectured bound of $2n + k - 2$. 
The following result provides an upper bound for $N$ (yet with quadratic growth in $n$) in the special case when $\Omega$ is simply-connected and there is a single node appearing in the quadrature formula, i.e., we have $L = 1$ in (6). In this case it is known \cite{1} that the conformal mapping is a polynomial of degree $n$, and by making a translation we may assume that the point evaluation in (6) is at $\phi(0) = 0$.

**Theorem 4.2.** Let $\Omega$ be a smooth, simply-connected quadrature domain of order $n$ with $L = 1$, i.e., the conformal mapping $\phi : \mathbb{D} \to \Omega$ is a polynomial of degree $n$. Then the number $N$ of critical points of $v$ in $\Omega$ satisfies $N \leq (2^n - 1)^2$.

The proof will require the following alternate form of Bezout's theorem.

**Theorem 4.3.** Let $A$ and $B$ be polynomials in the real variables $x$ and $y$ with real coefficients with $\deg A = n_1$ and $\deg B = n_2$. If $A$ and $B$ have finitely many common zeros in a domain $\Omega$ then they have at most $n_1 \cdot n_2$ common zeros in $\Omega$.

**Proof of Theorem 4.2.** The result follows from the exact same argument as given in the proof of \cite[Thm. 2]{30}, but instead of considering all real zeros of a hypothetical common factor $C(x, y)$, one considers only on the real zeros of $C(x, y)$ within $\Omega$. \hfill $\square$

**Proof of Theorem 4.3.** Let $u$ denote the solution to (3), and recall that the critical points of $v$ are solutions to $\nabla u = z$. We use the conformal mapping $\phi$ to pull back $u$ to a harmonic function $\hat{u}(w) = u(\phi(w))$ that solves

$$\begin{cases}
\Delta \hat{u}(w) = 0, & w \in \mathbb{D} \\
\hat{u}(w) = \frac{|\phi(w)|^2}{2}, & w \in \partial \mathbb{D}.
\end{cases}$$

As shown in \cite[proof of Thm. 1.1]{14}, the solution to this Dirichlet problem satisfies $\hat{u}(w) = P(w) + \overline{P(w)}$, where $P$ is a polynomial of degree $n - 1$.

The solutions of $\nabla v = 0$, or $\nabla u(z) = z$ correspond, in $w$ coordinates, to solutions of

$$(\nabla u)(\phi(w)) = \phi(w).$$

Since $P'(w) = \nabla \hat{u}(w) = (\nabla u)(\phi(w)) \cdot \overline{\phi'(w)}$, we rewrite the above as

$$\frac{P'(w)}{\phi'(w)} = \phi(w),$$

or, upon clearing the denominator,

$$P'(w) - \phi(w)\overline{\phi'(w)} = 0. \tag{7}$$

The critical points of $v$ in $\Omega$ correspond to the zeros of (7) in the unit disk $\mathbb{D}$. Since $\Omega$ is simply-connected, Theorem 2.2 implies that the number of such zeros is finite. Taking real and imaginary parts in (7), we arrive at a system of two real polynomial equations in two variables with each polynomial being of degree $2n - 1$. Applying Theorem 4.3 we see that there are at most $(2n - 1)^2$ many solutions of (7) in $\mathbb{D}$ and hence $v$ has at most this many critical points in $\Omega$. \hfill $\square$

### 5. Additional Examples

In the following example, we determine the number of critical points of $v$ for a one-parameter family of simply-connected quadrature domains of order $n = 2$. 

5.1. Neumann’s oval. Let $\Omega \subset \mathbb{C}$ be the domain whose boundary is given by
\[
\{(x, y) : (x^2 + y^2)^2 = a^2(x^2 + y^2) + 4x^2\}
\]
for some parameter $a > 0$. Let $R = \frac{a + \sqrt{a^2 + 4}}{2}$. Note that $R > 1$. It is known that $\phi(z) = \frac{1 - R^4 + \sqrt{(R^4 - 1)^2 + 4R^4z^2}}{2Ra}$ maps $\Omega$ conformally onto the unit disk $\mathbb{D}$ and that the torsion function of $\Omega$ is given by [15]
\[
v(z) = \frac{R^4 - 1}{2R^2} + \frac{\text{Re}(z\phi(z))}{R} - \frac{|z|^2}{2}.
\]
A routine computation gives
\[
\nabla v(z) = \frac{2R^2z}{\sqrt{(R^4 - 1)^2 + 4R^4z^2}} - z.
\]
(8)
We claim that there exists $R_0 > 1$ such that if $1 < R \leq R_0$, $\nabla v$ has exactly 3 zeros (counting multiplicity), all of which are real, and if $R > R_0$ there is only one zero, at the origin.

For this, we let $z = t \in \mathbb{R}$ in equation (8). Then we obtain $t \left(2R^2 - \sqrt{(R^4 - 1)^2 + 4R^4t^2} \right) = 0$ whose solutions are $t = 0$, and the roots of
\[
t^2 = \frac{4R^4 - (R^4 - 1)^2}{4R^4}.
\]
It now follows that if $4R^4 - (R^4 - 1)^2 \geq 0$, i.e., if $1 < R \leq R_0 = \sqrt{1 + \sqrt{2}}$, equation (8) has three real zeros. Namely, one positive root, one negative root, and a root at 0 if $1 < R < R_0$, and a root of multiplicity 3 at the origin $t = 0$, if $R = R_0$. If $R > R_0$ it is clear that the only real root is at 0. It remains to be shown that equation (8) has no nonreal zeros.

Suppose that $z \notin \mathbb{R}$ is a critical point of $v$, i.e,
\[
2R^2z = \bar{z}\sqrt{(R^4 - 1)^2 + 4R^4z^2}.
\]
(9)
We first show that $z$ cannot be purely imaginary. Indeed, substituting $z = it$, $t \neq 0$, into equation (9) yields $2R^2 = -\sqrt{(R^4 - 1)^2 - 4R^4t^2}$ which is clearly impossible, since $t$ is supposed to be a real number. Now assume that $z = te^{i\theta}$ is a critical point of $v$, where $\theta \in (0, 2\pi)$, with $\theta \notin \{\frac{\pi}{2}, \pi\}$. Substituting in equation (9) and squaring, we obtain
\[
4R^4 = e^{-i4\theta}(R^4 - 1)^2 + 4R^4t^2 e^{i2\theta}.
\]
Equating the imaginary parts in equation (10), we obtain $(R^4 - 1)^2 \sin(4\theta) = -4R^4t^2 \sin(2\theta)$. Writing $\sin(4\theta)$ as $2\sin(2\theta)\cos(2\theta)$ and remembering that $\theta \notin \{0, \frac{\pi}{2}, \pi\}$, we obtain
\[
\cos(2\theta) = -\frac{2t^2R^4}{(R^4 - 1)^2}.
\]
(11)
Similarly, equating the real parts of equation (10), and solving for $\cos(2\theta)$ yields,
\[
\cos(2\theta) = \frac{1}{4(R^4 - 1)^2} \left( -4R^4t^2 \pm \sqrt{(4R^4t^2)^2 + 8(R^4 - 1)^2 \left[(R^4 - 1)^2 + 4R^4\right]} \right).
\]
(12)
Clearly equations (11) and (12) are incompatible. This shows that all critical points of $v$ are real and finishes the proof of our claim.
Remark 5.1. The Neumann’s oval is an example of a Hippopede. From the classical theory of plane curves it is known that Hippopedes can be parametrized in polar coordinates [20, p. 145], showing that the Neumann’s oval is star-shaped with respect to the origin. Our example above therefore furnishes a star-shaped domain in which there are two maxima (near the 2 nodes) for the torsion function. In particular, this answers a question in the survey [24, p. 19], where the author asks if there is a unique maximum for the torsion function of a star-shaped domain. As we mentioned before, in the case of convex domains the existence of a unique critical point is well known.

Lemma 5.2. For \( \varepsilon > 0 \) denote \( N_\varepsilon = (-1, 1) \times (-\varepsilon, \varepsilon) \), and let \( M \gg 1 \). Let \( w : N_\varepsilon \rightarrow \mathbb{R} \) satisfy

\[
\begin{cases}
\Delta w(z) = -2, & z \in N_\varepsilon \\
w(z) = 0, & z \in \text{top and bottom sides}, \\
w(z) = M, & z \in \text{left and right sides}.
\end{cases}
\]

Then

\[
\sup_{y \in [-\varepsilon, \varepsilon]} w(0, y) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\]

We will refer to \( N_\varepsilon \) as an \( \varepsilon \)-neck.

Proof. Let \( w_1(z) = w(z) + y^2 - \varepsilon^2 \). Then \( \Delta w_1(z) = 0 \) in \( N_\varepsilon \), with boundary values

\[
\begin{cases}
w_1(z) = 0, & z \in \text{top and bottom sides}, \\
w_1(z) \leq M, & z \in \text{left and right sides}.
\end{cases}
\]

If \( h \) is defined on \( N_{2\varepsilon} \) by

\[
h(x, y) = C_\varepsilon \cosh\left(\frac{\pi x}{4\varepsilon}\right) \cos\left(\frac{\pi y}{4\varepsilon}\right)
\]

where \( C_\varepsilon = \frac{2M}{\cosh(\pi \varepsilon)} \), then \( h \) is positive harmonic in \( N_{2\varepsilon} \). Furthermore, it is easy to see that on \( \partial N_\varepsilon \), \( w_1 \leq h \). By the maximum principle, this forces \( w_1 \leq h \) in \( N_\varepsilon \). This in turn implies,

\[
w(z) \leq h(z) + \varepsilon^2 - y^2 \leq h(z) + \varepsilon^2, \quad z \in N_\varepsilon.
\]

In particular, taking \( z = (0, y) \in N_\varepsilon \), we obtain

\[
w(0, y) \leq C_\varepsilon + \varepsilon^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0
\]

As an application, consider \( \Omega \), a dumbbell shaped domain consisting of two almost discs, attached by an \( \varepsilon \)-neck for a very small \( \varepsilon > 0 \). Let \( v \) be the torsion function on \( \Omega \). Then taking \( M \) to be a very large number and applying the Lemma, we obtain \( v \) is very small in the middle of this neck. Hence \( v \) has at least one maximum in each disc, for a total of at least 3 critical points (2 maxima and a saddle) in \( \Omega \).

This construction can be generalized to a region comprised of \( n \) almost disks held together by \( n - 1 \) \( \varepsilon \)-necks. This will yield that the torsion function on such a domain has at least \( 2n - 1 \) critical points.

6. Concluding Remarks

Critical points of \( v \) and oscillations of \( \partial \Omega \). Let us restrict to the case \( \Omega \) is simply-connected, and also assume that \( \partial \Omega \) is smooth and real-analytic. Then the conformal mapping \( \phi : \mathbb{D} \rightarrow \Omega \) from the unit disk extends to be analytic in \( \mathbb{D} \). Let \( n \) denote the number of local maxima of \( |\phi| \) over \( \partial \mathbb{D} \). Geometrically, \( n \) counts oscillations of the distance from the boundary \( \partial \Omega \) to the origin. We thus find the following problem attractive, since an affirmative answer would give an estimate based on a purely geometric feature of \( \Omega \).
**Problem 6.1.** Let \( \Omega \) be as above. Prove an upper bound for the number \( N \) of critical points of \( v \) in terms of \( n \).

We conjecture that \( N \) increases at most linearly with \( n \), and one specific guess for a bound of this form (without yet sufficient evidence to state as a conjecture) is

\[
N \leq 2n - 1.
\]

Let us explain the intuition behind the guess (13). Let \( u \) be the harmonic function that solves (3), and let \( F(z) \) denote the complex conjugate of \( \nabla u \). The assumptions on \( \partial \Omega \) imply that \( F(z) \) extends analytically to a neighborhood of \( \overline{\Omega} \). The number of critical points of \( F \) in \( \Omega \) is at most \( n - 1 \). Indeed, using the conformal mapping \( \phi : \mathbb{D} \to \Omega \) we pull back \( u \) to a harmonic function \( \hat{u} \) that solves

\[
\begin{aligned}
\Delta \hat{u}(w) &= 0, \quad w \in \mathbb{D} \\
\hat{u}(w) &= \frac{|\phi(w)|^2}{2}, \quad w \in \partial \mathbb{D}.
\end{aligned}
\]

It is known [3] that the number of critical points of \( \hat{u} \) in \( \mathbb{D} \) is at most \( n - 1 \), where as above \( n \) denotes the number of local maxima of the boundary data. If it could be shown that the number \( M \) of maxima of \( F \) is bounded by the number \( n - 1 \) of critical points of \( F \) as in the anti-holomorphic dynamics schema, then the bound (13) would indeed follow from Theorem 2.1.

Such an application of anti-holomorphic dynamics could be realized by addressing the following problem in quasiconformal surgery.

**Problem 6.2.** Find conditions on \( \Omega \) that guarantee \( F(z) \) extends to \( \mathbb{C} \) as a quasirational mapping of degree \( n \).

By “quasirational” we mean that the extended mapping \( F \) is quasiconformally conjugate to a rational mapping [8]. Such an extension allows iteration of \( F \) and application of anti-holomorphic dynamics.

The following problem concerns a weaker extension of \( F \) that would yet result in a rough upper bound (explained below).

**Problem 6.3.** Find conditions that guarantee \( F(z) \) extends to \( \mathbb{C} \) as a smooth (but not necessarily complex-analytic) proper orientation-preserving mapping of degree \( n \) and asymptotic to \( z^n \) as \( z \to \infty \).

Suppose such an extension holds. Then we can estimate the number of critical points of \( v \) as follows. The critical points of \( v \) are solutions of the fixed point problem

\[
F(z) = z
\]

which are also solutions of the fixed point problem

\[
F(F(z)) = z,
\]

or equivalently, the zero-finding problem

\[
F(F(z)) - z = 0.
\]

Since \( F(F(z)) - z \) is orientation-preserving, we can count the number of zeros in a large disk using the argument principle. The winding number is \( n^2 \), since \( F(F(z)) \) is asymptotic to \( z^{n^2} \) on the boundary of a large disk. Hence, \( v \) has at most \( n^2 \) critical points.

**Multiply connected domains with convex boundary components.** The following conjecture is inspired partly by experimenting with \( TR \)-domains.

**Conjecture 6.4.** Suppose \( \Omega \) is a \( k \)-connected domain with each boundary component being a smooth convex curve. Then for \( k \geq 3 \) the number \( N \) of critical points of \( v \) satisfies

\[
N \leq 5(k - 2).
\]
Magnanini’s conjecture. We conclude by recalling the following conjecture stated by R. Magnanini (see [24] for discussion).

**Conjecture 6.5.** The number $N$ of critical points of $v$ in $\Omega$ satisfies

$$N \leq 2m - 1,$$

where $m$ denotes the number of maxima in $\Omega$ of the function $d_{\partial\Omega}(z)$ defined as the distance from $z$ to $\partial\Omega$.

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