Directed Polymer Melts
and
Quantum Critical Phenomena

Randall D. Kamien and David R. Nelson

Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138

The statistical mechanics of directed line-like objects, such as directed polymers in an external field, strands of dipoles in both ferro- and electrorheological fluids, and flux lines in high-$T_c$ superconductors bears a close resemblance to the quantum mechanics of bosons in $2 + 1$ dimensions. We show that single component and binary mixture critical phenomena in these systems are in the universality class of three dimensional uniaxial dipolar ferromagnets and ferroelectrics. Our results also apply to films of two superfluid species undergoing phase separation well below their $\lambda$-points near $T = 0$. In the case of directed polymers and electrorheological fluids we analyze the effects of free ends occurring in the sample as well as a novel directionally-dependent compressibility.

**Keywords:** quantum critical phenomena; directed polymers; critical mixing; phase transitions
1. Introduction and Summary

There has recently been renewed interest in the analogy between the classical statistical mechanics of lines and the quantum mechanics of bosons in $2 + 1$ dimensions. Configurations of directed lines can be mapped onto the world lines of a collection of bosons in one fewer dimension. Examples include flux lines in high temperature superconductors, polymer nematics in a strong external field, and ferro- and electrorheological fluids.

These systems can be modeled by a free energy of the form

$$F = \int dz d^2r \left[ \frac{h}{2} \dot{t}^2 + \frac{c}{2} (\nabla_\perp \delta \rho)^2 + \frac{\alpha}{2} (\partial_z \delta \rho)^2 + \frac{b}{2} \delta \rho^2 + \frac{w}{3!} \delta \rho^3 + \frac{c}{4!} \delta \rho^4 \right]$$

The partition function

$$Z = \int [dt] [d\delta \rho] e^{-F/k_B T}$$

is subject to the constraint

$$\partial_z \delta \rho + \nabla_\perp \cdot \mathbf{t} = 0,$$

where $\delta \rho(\mathbf{r}, z) = \rho(\mathbf{r}, z) - \rho_0$ is a cross-sectional areal density deviation from its average value $\rho_0$ in a plane perpendicular to $z$, and $\mathbf{t}(\mathbf{r}, z)$ is a tangent field in the same plane. Given a microscopic description of directed polymers, via their trajectories defined by single-valued functions $\{r_j(z)\}$ along the $z$-axis, $\rho(\mathbf{r}, z)$ and $\mathbf{t}(\mathbf{r}, z)$ are defined by (see Figure 1)

$$\rho(\mathbf{r}, z) = \sum_i \delta^2[\mathbf{r} - \mathbf{r}_i(z)]$$

and

$$\mathbf{t}(\mathbf{r}, z) = \sum_i \frac{d\mathbf{r}_i(z)}{dz} \delta^2[\mathbf{r} - \mathbf{r}_i(z)].$$

We assume that, as external parameters such as field strength, osmotic pressure, temperature, solvent quality, etc. are varied, the polynomial part of (1.1) in $\delta \rho$ exhibits a line of first order phase transitions from a high to low density phase, terminating in a second order phase transition. Along the critical isochore, $w = 0$, and $b$ then changes sign at the mean field critical point. Here and throughout, we distinguish the $z$ direction as the direction along which the lines are aligned, and denote by $\nabla_\perp$ the derivative perpendicular to the $z$-axis. Additionally, we use boldface to denote vectors in the perpendicular plane. In the case of quantum mechanical bosons, we think of $\hat{z}$ as a time-like axis along which particle world lines are extended. Planck’s constant $\hbar$ plays the role of “temperature” in
The constraint (1.3) in the context of polymers corresponds to the absence of polymer free ends \[3,4\], that is, conservation of polymer number as we would move up the \(z\)-axis. In the context of quantum mechanical bosons, this constraint merely sets \(t\) to be the canonical momentum density of the particles \[4\]. Models similar to (1.1) were first posed as Landau-Ginzburg type theories \[2,6-8\] but were later shown to be a direct consequence of the boson mapping \[4\].

In fact, this model is equivalent to a three dimensional uniaxial dipolar ferroelectric (or ferromagnet) near its critical point, first studied in 1969 by Larkin and Khmel’nitski˘ı \[9\]. To see this, we use the constraint (1.3) to solve for the longitudinal part of \(t\). Upon integrating out the transverse part of \(t\) we obtain an effective free energy

\[
F_{\text{eff}} = \frac{1}{2} \int \frac{dq_{\perp} \, dq_{z}}{2\pi} \left( \frac{h q_{z}^{2}}{q_{\perp}^{2}} + eq_{\perp}^{2} + \alpha q_{z}^{2} + b \right) |\delta \rho(q_{\perp}, q_{z})|^{2} + \frac{c}{4!} \int dz d^{2}r (\delta \rho)^{4},
\]

where the quadratic part has been written in Fourier space. Here and henceforth, we restrict our attention to the critical isochore, and set \(w = 0\). Equation (1.6) is the familiar form of the Landau theory for a uniaxial Ising ferromagnet or ferroelectric with a mean field critical point at \(b = 0\) \[9\].

A model very similar to (1.1) applies to polymer nematics in which the external field is turned off, so that the alignment represents spontaneously broken symmetry instead of being externally imposed \[3,4,6,8\]. In this case, \(t(r, z)\) represents a nematic director field and the term \(\frac{1}{2} h t^{2}\) is replaced by the usual nematic gradient free energy \[10\]. Simple power counting arguments for this model suggest that nonclassical critical behavior results below \((3 + 1)\)-dimensions. A complete treatment, however, would require a discussion of the effects of hairpin turns in the polymer configurations \[4\]. We restrict our attention here to polymer nematics in a strong external field, and to length scales less than the spacing between hairpins, to avoid these complications.

1.1. Electrorheological Fluids and the Effect of Free Ends

The effective attraction for polymers in a nematic solvent is similar to the fluctuation induced attraction of strings of colloidal spheres discussed by Halsey and Toor \[11\]. This attractive part of the interchain potential in electrorheological fluids can be offset by a hard-core repulsion at short distances, and may be adjusted by varying the dielectric
constant of the solvent. Since an electric field is required to create directed strings of dielectric spheres, hairpin configurations will be greatly suppressed in electrorheological fluids, as there is no longer an up-down symmetry. We then expect that (1.1) is especially apt for these systems, once free ends are taken into account. Halsey and Toor [12], by considering the bulk energetics of the system, have argued that at sufficiently large fields, the electrorheological strings will phase separate, leading to two-phase coexistence. The critical point discussed here would then describe a transition to a uniform density.

To study this point further, consider the free energy at fixed potential due to the electric field

\[ U = -\frac{1}{8\pi} \int dV \epsilon(n(\vec{x}))\vec{E}^2(\vec{x}) \]  

(1.7)

where \( n \) is the full three-dimensional colloid density, and \( \epsilon(n) \) is the effective dielectric constant. There is an overall minus sign because this is the energy at fixed \( \vec{E} \). Since \( \epsilon''(n) > 0 \) [13], it is energetically favorable to phase separate into dense and dilute regions. However, at finite temperature, configurational entropy plays a role. Certainly, at fixed electric field, the temperature can be increased until entropy dominates and there is only one uniform phase of directed lines. However, entropy also favors shorter chains at such temperatures, which could obscure the entanglement effects which are the main focus of this paper (see below).

Recent numerical simulations by Tao [14] suggest that at fixed temperature there are actually two critical electric fields, \( E_C^1 \) and \( E_C^2 \) (\( E_C^2 < E_C^1 \)). Below \( E_C^2 \) the fluid consists primarily of a gas of monomers. Above \( E_C^1 \) the colloidal particles clump together and phase separate into “columns” and dilute chains as Halsey and Toor’s argument would suggest. The columns may in fact be crystalline [14,15]. Between \( E_C^2 \) and \( E_C^1 \) there are dense strings of colloidal spheres with random positions in any constant-\( z \) cross section. The simulation suggests a first order phase transition from the random line phase to the clumped phase. At lower densities, experimental measurements do not suggest crystalline order in the columns [16]. Thus, at low enough densities we can have coexistence of liquid-like columns and a gas of chains, possibly terminating in a critical point.

One shortcoming of our model in its present form is that the chains it describes must start and stop at the boundaries of the sample. Both directed polymers and colloidal chains should be allowed to have free ends within the sample. As discussed in Section 4,
free ends may be incorporated by relaxing the constraint (1.3) and using the free energy

$$F = \int dzd^2x \left[ \frac{b}{2} t^2 + \frac{c}{2} (\nabla \delta \rho)^2 + \frac{\alpha}{2} (\partial_z \delta \rho)^2 + \frac{b}{2} \delta \rho^2 + \frac{c}{4!} \delta \rho^4 + \frac{G}{2} (\partial_z \delta \rho + \nabla \cdot \mathbf{t})^2 \right]$$

(1.8)

By taking \( G \to \infty \), we recover our original model. Additionally, it can be shown that \( G \sim \ell^{-1} \) as \( \ell \to \infty \), where \( \ell \) is the chain length. The crossover from elongated objects to point objects when the chains are finite will be discussed in detail in Section 4.

1.2. Phase Separation in the Flux Liquid Phase

Flux lines in superconductors may also be described by an analogy with bosons. However, in the absence of magnetic monopoles, flux lines cannot have free ends inside the sample. Their behavior, then, is much closer to the behavior quantum mechanical bosons, as the constraint (1.3) is always satisfied.

Calculations to date for fluctuating flux lines have assumed a purely repulsive pair potential. Muzikar and Pethicke, however, have argued that an effective attractive interaction arises under some circumstances near \( H_{c1} \) in the extreme type II limit. This interaction would convert the continuous onset of flux penetration at \( H_{c1} \) to a first order transition. An even more intriguing situation would arise if the interaction between the flux lines had two distinct minima (see the inset to Figure 3). In this case, there should be a first order transition at \( H_{c1} \), and a line separating two coexisting flux liquid phases, as in Figure 2. Our theory applies to the critical point marking the terminus of this first order phase boundary. The critical isochore is shown as a dotted line in the figure.

1.3. Quantum Critical Phenomena

Our analysis of course, can also be applied directly to boson systems at \( T = 0 \), which are equivalent to infinitely long directed polymers. In the case of a single particle species, we would be describing the terminus of a line of first order transitions separating superfluids coexisting at two different densities. How this situation might come about is illustrated in Figure 3. The inset shows an unusual pair potential with two distinct minima describing particles which, like helium, are light enough to avoid crystallization at \( T = 0 \) at low pressures. The two minima lead to two distinct liquid phases with increasing pressure. At
low pressures, the stable zero temperature liquid has near neighbors occupying the deeper outermost minimum. As pressure is increased, there is a first order phase transition to a denser liquid with the innermost minimum occupied, followed eventually by crystallization at even higher pressure. The two liquids must be superfluids at $T = 0$. In Figure 3, the parameters have been adjusted so that the critical point for the dilute and dense superfluids $S_1$ and $S_2$ occurs below the dashed $\lambda$-line describing the transition to a normal liquid with increasing temperature. There is a second critical point associated with the coexistence of this normal liquid with a dilute gas.

To obtain the critical point of interest to us here, we must increase $\hbar$ (or decrease the mass) at $T = 0$ until the two superfluid liquid phases become indistinguishable as at the point $c$ in the figure. Although this gedanken experiment is clearly rather esoteric, some control over quantum fluctuations may be achievable for two-dimensional helium films, where out of plane fluctuations alter the effective two-body potential. By varying the type of substrate, it may be possible to create a purely repulsive effective potential and a novel superfluid gas phase at $T = 0$ \[18\]. We should stress, however, that we do not expect the unusual phase diagram of Figure 3 for either purely repulsive potentials or conventional helium potentials with a single minimum.

The density operator of the bosons described above is given by (1.4). Upon differentiating with respect to $z$ we have

$$
\partial_z \rho(r, z) = -\sum_i \frac{dR_i(z)}{dz} \cdot \nabla_\perp r \delta^2[r - r_i(z)] = -\nabla_\perp \cdot \sum_i \frac{dR_i(z)}{dz} \delta^2[r - r_i(z)] \quad (1.9)
$$

The final sum on the right hand side is just (1.4), so we have derived (1.3). The quantity $t(r, z) = \sum_i \frac{dR_i(z)}{dz} \delta^2[r - r_i(z)]$ is just the momentum operator of the bosons \[1, 4\]. On a more formal level, note that the “time dependence” (i.e. the $z$ dependence) of the quantum operator (1.4) is given by the (Wick rotated) Heisenberg equation of motion,

$$
\partial_z \rho(r, z) = [\rho(r, z), \mathcal{H}], \quad (1.10)
$$

where $\mathcal{H}$ is the underlying quantum Hamiltonian. Upon identifying (1.1) with the integral over imaginary time of the coarse-grained hydrodynamic Lagrangian associated with this Hamiltonian (as is appropriate for finite temperature quantum statistical mechanics), we recover (1.3) from (1.10) upon using the commutation relation between $\rho(r, z)$ and $t(r, z)$,

$$
[rho(r, z), t(r', z)] = \frac{k_B T}{\hbar} \nabla_\perp \delta^2(r - r'). \quad (1.11)
$$
In the hydrodynamic limit considered here, this commutator becomes the classical “Poisson bracket” relation between density and momentum, which arises because the momentum operator is the generator of translations $[19]$. The constraint (1.3) thus directly reflects the non-commutivity of the underlying position and momentum operators.

Note that our results represent a new universality class for Ising-like quantum critical phenomena. A related type of quantum critical phenomena, extensively studied in the past $[20]$, arises for the Ising model in a transverse magnetic field. The quantum Hamiltonian, defined on a hypercubical lattice in $d$-dimensions is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma^z_i \sigma^z_j - h \sum_j \sigma^x_j$$

where the $\{\vec{\sigma}_i\}$ are Pauli spin matrices. The ordered state at $T = 0$ is destroyed for sufficiently large $h$. This disordering transition is known to be in the universality class of the $(d + 1)$-dimensional classical Ising model $[20]$. Equation (1.12) describes a quantum lattice gas, with the world lines of the “particles” (up spins, say) in a $T = 0$ path integral formulation representing the directed polymers discussed in this paper. Because the $\sigma^z_j$ is a spin flip operator, these “polymers” stop and start at random in $(d + 1)$-dimensions. The appropriate coarse-grained Landau description would be similar to (1.1) without, however, the constraint (1.3). The spin flip operator in (1.12) does not respect this conservation law, so it is not surprising that we find a different universality class for boson critical phenomena. Introducing free ends in our model has the same effect as the spin flip operator in (1.12). As discussed above, we then find a crossover from the $(d + 1)$-dimensional uniaxial dipolar Ising critical phenomena to the $(d + 1)$-dimensional Ising behavior expected for (1.12).

### 1.4. Directed Polymer Blends and Quantum Binary Mixtures

By extending the original model to allow for binary mixtures, we can study the consolute point of two different directed polymer species. In particular, we can determine the critical exponents near the demixing point of directed polymer blends and compare them to those for point particles. Analogous phenomena should occur in binary mixtures of two different superfluids. Among the possible candidates would be films of $He^6$ and $He^4$ or possibly, a superfluid monolayer of spin aligned hydrogen mixed with superfluid $He^4$. As the mass ratio, $m_1/m_2$, of the two species is increased towards unity with $\hbar > 0$, quantum fluctuations should eventually cause the two superfluids to mix. The same effect may be achievable by varying the properties of the substrate or formally, by increasing $\hbar$. 

6
A phase diagram as a function of temperature, composition and $\bar{h}$ is shown in Figure 4. Note the incomplete phase separation appearing for nonzero $\bar{h}$. The theory applies to the critical point at $T = 0$. In Section 5 we will analyze such binary mixtures in greater detail. Although we shall concentrate on $2 + 1$ quantum systems, our results are easily adapted to $3 + 1$ dimensional quantum problems which are well described by mean field theory (see Section 4).

1.5. Outline

In Section 2 we formulate the Landau theory that will be the focus of this paper. We follow the derivation developed in [2] for flux lines, which leads to a coherent state functional integral. From there we describe the hydrodynamic theory with and without free ends as in [4]. In appendix A we present a similar derivation for two interacting polymer species.

In Section 3 we describe the correlation functions of this theory which could be measured by X-ray or neutron diffraction experiments. We also discuss the meaning of the direction dependent compressibility which characterizes these directed line systems.

In Section 4 we study the critical behavior of our model via an expansion in $2 - \epsilon + 1$ dimensions. We reproduce the results of Larkin and Khmel’nitskii for $\epsilon = 0$ when there are no free ends [9], and find that free ends are a relevant perturbation leading to the critical behavior of a $(d + 1)$-dimensional Ising model with short range interactions.

Finally, in Section 5 we discuss the critical mixing of two polymer or quantum boson species. Using the free energy derived in appendix A, we show that, just as in the point-particle case, the critical mixing of line-like objects is in the same universality class as single component critical phenomena.

2. Coherent State Functional Integrals and Landau Theory

The derivation of (1.1) from a more microscopic free energy for directed lines follows the presentations in [2] and [4]. We only outline the main points here, first reviewing the theory for chains which span the system. Internal free ends are then introduced by adding a source to the boson coherent state field theory [3,4].
We start from the following path integral partition function for the $N$ chains,

$$Z_N = \frac{1}{N!} \int \prod_{i=1}^{N} [dr_i] \exp \left\{ -\frac{1}{k_B T} \int_0^L dz \left[ \sum_{i=1}^{N} \frac{g}{2} \left( \frac{dr_i}{dz} \right)^2 + \sum_{i<j} V[r_i(z) - r_j(z)] \right] \right\}$$

(2.1)

where we have restricted our attention to interactions between lines through an equal “time” potential $V[r_i(z) - r_j(z)]$.

The boson field theory is derived in three steps. One first writes the transfer matrix associated with (2.1) in terms of a many-particle real-time Schrödinger equation for the polymer positions, and then second quantizes this equation using bosons. Finally, we pass to a coherent state functional integral representation of the second quantized formalism.

The grand canonical partition sum which results is

$$Z_{gr} \equiv \sum_{N=0}^{\infty} e^{L \mu N/k_B T} Z_N = \int [d\psi][d\psi^*] e^{-S[\psi^*, \psi]} ,$$

(2.2)

where $\psi(r, z)$ is a complex boson field. The boson action $S$ reads

$$S = \int_0^L dz \int d^2r \left[ \psi^*(r, z) \left( \partial_z - D \nabla_\perp^2 - \bar{\mu} \right) \psi(r, z) + \frac{1}{2} \int d^2r' |\psi(r, z)|^2 |\psi(r', z)|^2 \right]$$

(2.3)

where $D = k_B T/2g$, $\bar{\mu} = \mu/k_B T$ and $\bar{v} = V/k_B T$. Standard manipulations show that the density of flux lines is

$$\rho(r, z) = |\psi(r, z)|^2 .$$

(2.4)

2.1. Entangled Limit with Free Ends

We now assume that all phases are dense enough to be entangled and expand around the mean density $\rho_0$. For polymers of length $\ell$, entanglement means $\sqrt{D \ell} >> \rho_0^{-1/2}$, while for quantum systems, we require very low temperatures so that the thermal de Broglie wavelength is large compared to the particle spacing. We parameterize the field $\psi$ by $\psi = \sqrt{\rho} e^{i\theta}$, a change of variables with unit functional determinant, and find from (2.3) that,

$$S = \int dz d^2r \left[ \frac{D(\nabla_\perp \rho)^2}{4\rho} + D\rho(\nabla_\perp \theta)^2 + i\rho \partial_z \theta - \bar{\mu} \rho + \frac{\bar{v}}{2} \rho^2 \right]$$

(2.5)
Upon introducing an auxiliary field $t(r,z)$ conjugate to $\nabla_\perp \theta(r,z)$ via a Hubbard-Stratonovich transformation, and integrating out $\theta(r,z)$, we arrive at the partition sum (1.2). See Appendix A for the analogous transformation applied to binary mixtures of quantum bosons.

We can add a (Poisson distribution) of free ends to the theory by adding to the action (2.3) a term of the form

$$\delta S = -\eta \int dzd^2r \left[ \psi(r,z) + \psi^*(r,z) \right]$$

$$= -\int dzd^2r 2\eta\sqrt{\rho}\cos\theta$$ \hspace{1cm} (2.6)

The mean polymer length $\ell$ is related to $\eta$ by $2\eta\ell/k_B T = \sqrt{\rho_0}$ [4]. We may expand this term in powers of $\theta$ if the fluctuations in $\theta$ are small, which they will be if the polymers are sufficiently dense and entangled. To leading order in $\theta$, equation (2.6) becomes

$$\delta S \approx \text{const.} + \int dzd^2r \eta\sqrt{\rho}\theta^2$$ \hspace{1cm} (2.7)

The Hubbard-Stratonovich transformation now leads to

$$S = \int dzd^2r \left\{ \frac{D_\perp^2}{\rho} + \frac{D(\nabla_\perp \rho)^2}{4\rho} - \bar{\mu}\rho + \frac{v}{2}\rho^2 + \frac{1}{4\eta\sqrt{\rho}} [\partial_z \rho + \nabla_\perp t]^2 \right\}$$ \hspace{1cm} (2.8)

Upon expanding around an average density $\rho_0$, we arrive at the theory for finite length polymers (1.8), with $G = 1/4\eta\sqrt{\rho_0}$ and $h = 2D/\rho_0$.

### 3. Correlations and Compressibility

From the free energy (1.8) we can, by considering only quadratic terms, easily find the hydrodynamic form of the structure function which could be measured by either neutron or x-ray diffraction. Here and henceforth we change units so that $e = 1$. The Fourier-transformed density-density correlation function is then

$$S(q_\perp, q_z) = \left\langle \frac{\delta \rho(q_\perp, q_z) \delta \rho(-q_\perp, -q_z)}{\rho_0^2} \right\rangle$$

$$= \frac{k_B T (h + Gq_\perp^2)}{(Gaq_\perp^2 + Gh + \alpha h) q_z^2 + (Gb + h) q_z^2 + Gq_\perp^4 + hb}.$$ \hspace{1cm} (3.1)

Note that $G$ drops out of (3.1) upon setting $h = 0$. This would have to be the case in the boson analogy, since turning $h = 0$ neglects quantum mechanics by decoupling the momentum from the position. The constraint (1.3) is unimportant in this classical limit.
In the limit $G \to \infty$, we recover the correlations typical of infinite polymers with no heads or tails from (3.1),
\[
S(q_\perp, q_z) = \frac{k_B T q_\perp^2}{(h + \alpha q_\perp^2) q_z^2 + b q_\perp^2 + q_\perp^4}.
\] (3.2)

For small momenta, we can examine the curves of constant $S(q_\perp, q_z)$. It is easy to see that these curves will go into the origin on straight lines, i.e. $q_z \sim q_\perp$. This is a generic feature expected to hold for all systems with the morphology of directed polymers, where the direction is explicitly chosen by an external field. Recalling that the limit as $|q| \to 0$ of the structure function is related to the compressibility of the sample [22], we can determine the compressibility of a sample of directed lines, which will depend on the direction of the compression [23]. Upon considering a compressional wave at an angle $\phi$ from the $z$ axis, we take the momentum to be $q_z = q \cos \phi$ and $q_\perp = q \sin \phi$, and find for small $q$,
\[
S(q_\perp, q_z) \sim_{q \to 0} \frac{k_B T}{b + h (\cot \phi)^2} = k_B T \chi_T(\phi),
\] (3.3)

where $\chi_T$ is the compressibility, and is a function of the direction $\phi$ in which the compression is made. Note again that the $h = 0$ limit is especially simple: The compressibility no longer depends on the compression angle if tangent fluctuations are ignored. We then recover the result for an isotropic fluid of point particles, namely $\chi_T = b^{-1}$. The compressibility enters the linear response relation:
\[
\langle \delta \rho(q_\perp, q_z) \rangle = \chi(q_\perp, q_z) \delta \sigma(q_\perp, q_z),
\] (3.4)

where $\delta \sigma$ is a sinusoidally varying force directed along $\vec{q} = (q_\perp, q_z)$. The unusual $\phi$-dependence of this response function may be understood as follows: Equations (3.4) and (3.3) predict that $\langle \delta \rho \rangle = 0$ for compressions when $\phi \to 0$. This result arises because squeezing the system along the $z$-axis will not change the number of polymers piercing any constant $z$-slice. In general, the linear response (3.4) depends on the directions of the compression relative to $\hat{z}$, as parameterized by $\phi$.

Returning to finite length polymers, and hence to finite $G$, we repeat the above analysis and find a $\phi$-independent result:
\[
S(q_\perp, q_z) \sim_{q \to 0} \frac{k_B T}{b}.
\] (3.5)

Because at sufficiently large distances (larger than the average polymer length $\ell$) finite length polymers will ultimately behave like point particles, the long-wavelength properties of the sample will not exhibit any character of the directed constituents. At intermediate scales, the factors of $O(G q_\perp^2)$ may not be neglected in (3.1), and we return to the direction dependent result (3.3).
4. Renormalization Group and Finite Length Polymers

In the last section we saw that at long-wavelengths the line-like nature of the directed melt disappears for finite chain lengths. Here we construct a renormalization group which allows us to study this crossover in detail. When $G \to \infty$, we recover the results of Larkin and Khmel’nitškiĭ [9]. For $G$ finite, however, the system crosses over to the behavior of a $(d + 1)$-dimensional Ising model with short range interactions [20]. We shall treat the $z$-direction separately from the directions in the plane perpendicular to $z$. We let $d$ be the dimension of the bosons so that the corresponding directed polymers exist in $(d + 1)$-dimensions (see Figure 1).

Consider first the graph which renormalizes the four-point interaction $\delta \rho^4$. To interpret this graph we need to invert a $(d + 1) \times (d + 1)$ matrix to get the propagator, as there are $d$ degrees of freedom in $t$ and an additional degree of freedom from the density fluctuations, $\delta \rho$. However, as mentioned earlier, the transverse modes of $t$ decouple. To make this explicit, we decompose $t$ in momentum space,

$$t(q_\perp, q_z) = t_L + t_T,$$

with

$$t_{L,j} = \frac{q_j q_k}{q_\perp^2} t_{j,k} \quad (4.2)$$

and

$$t_{T,j} = \left( \delta_{jk} - \frac{q_j q_k}{q_\perp^2} \right) t_k \quad (4.3)$$

In (4.2) and (4.3) the indices $j$ and $k$ only run over the $d$-dimensional space perpendicular to $z$ and we use the summation convention. Additionally we take $t_{L,j} = -iq_j \pi(q_\perp, q_z)$. With this parameterization, the fourier transform of the divergence of $t$ is simply $iq_\perp \cdot t = q_\perp^2 \pi$, and the fourier transform of the $t^2$ term in (1.8) is $\int dq_z dq_\perp [q_\perp^2 \pi^2 + t_{T}^2 - \frac{1}{q_\perp^2} (q_\perp \cdot t_T)^2]$. Thus we can integrate out $t_T$ from (1.8), and we are left with a theory only involving $\delta \rho$ and $\pi$.

Inverting the $2 \times 2$ quadratic form which remains gives the following useful propagators:

$$\langle |\delta \rho(q_\perp, q_z)|^2 \rangle = \frac{h + Gq_\perp^2}{(G\alpha q_\perp^2 + G\pi + \alpha h) q_\perp^2 + (Gb + h) q_\perp^2 + Gq_\perp^4 + \alpha h} \quad (4.4a)$$

$$\langle |\pi(q_\perp, q_z)|^2 \rangle = \frac{q_\perp^2 + b + \alpha q_\perp^4 + Gq_\perp^2}{q_\perp^2 (G\alpha q_\perp^2 + G\pi + \alpha h) q_\perp^2 + (Gb + h) q_\perp^2 + Gq_\perp^4 + \alpha h} \quad (4.4b)$$

$$\langle \delta \rho^*(q_\perp, q_z) \pi(q_\perp, q_z) \rangle = \frac{iGq_z}{(G\alpha q_\perp^2 + G\pi + \alpha h) q_\perp^2 + (Gb + h) q_\perp^2 + Gq_\perp^4 + \alpha h} \quad (4.4c)$$
After inserting these propagators into the graph shown in Figure 5, we integrate out the frequency-like variable $q_z$ and find the renormalized four point interaction associated with equation (1.8) is $c_R = c + \delta c + O(c^3)$, where

$$\delta c = -\frac{3c^2}{8} \frac{2}{\Gamma\left(\frac{d}{2}\right)(4\pi)^{d/2}} \int \frac{q_{\perp}^{d-1}dq_{\perp}}{q_{\perp}^2} \frac{\sqrt{1 + \frac{h}{Gq_{\perp}^2}}}{\sqrt{h + \alpha q_{\perp}^2 + \alpha \frac{b}{G}}} \left(1 + \frac{b}{q_{\perp}^2}\right)^3$$

Near the critical point, $b \approx 0$ and so when $G = \infty$, $\delta c$ diverges in the infrared for $d \leq 2$. This infrared divergence suggests an $\epsilon$-expansion around $d = 2$, where $\epsilon = 2 - d$. Our approach follows [24], which is a variation of the dynamic renormalization group of [25]. First we integrate out a momentum shell in $q_{\perp}$ from $\Lambda e^{-\ell}$ to $\Lambda$, but integrate freely over $q_z$. We then rescale our variables so that the ultraviolet cutoff in the perpendicular direction is held fixed. After rescaling $r$ and $t$ accordingly, we are left with the same theory but with different coupling constants. When this procedure is iterated, $c$ is driven toward a fixed point which describes the universal long wavelength critical behavior for large $G$.

4.1. Momentum Shell Integration

We must first integrate out the transverse momentum in the range $\Lambda e^{-\ell} < q_{\perp} < \Lambda$. This can be done straightforwardly by expanding the functional integral in $c$. Care must be taken to account for all possible contractions of the operators in the expansion. The symmetry factors can be found in the usual way for Wick expansions. It is important to note that diagrams renormalize the remaining low momentum modes, and are not simply expectation values. Graphs which renormalize the quadratic and quartic contributions to Landau theory are shown in Figure 6. Upon carrying out this procedure, we arrive at the following relations for the intermediate values of the coupling constants in (1.8)

$$h' = h$$

$$\alpha' = \alpha$$

$$b' = b + \frac{c}{4\sqrt{G}} \int_{\Lambda e^{-\ell}}^{\Lambda} \frac{d^dq_{\perp}}{(2\pi)^d} \left[\frac{\sqrt{h + Gq_{\perp}^2}}{\sqrt{b + q_{\perp}^2}} \frac{1}{\sqrt{h + \alpha q_{\perp}^2 + \alpha \frac{b}{G}}}\right]$$

$$c' = c \left(1 - \frac{3c}{8\sqrt{G}} \int_{\Lambda e^{-\ell}}^{\Lambda} \frac{d^dq_{\perp}}{(2\pi)^d} \left[\frac{\sqrt{h + Gq_{\perp}^2}}{\sqrt{(b + q_{\perp}^2)^3}} \frac{1}{\sqrt{h + \alpha q_{\perp}^2 + \alpha \frac{b}{G}}}\right]\right)$$

$$G' = G$$
where we have gone to one-loop order. In (4.6) we have already integrated over \( q_z \).

We rescale perpendicular lengths by \( R'_\perp = R_\perp e^{-\ell} \) and the \( z \) direction by \( Z' = Ze^{-\int_0^\ell \gamma(\ell')d\ell'} \). In momentum space, these rescalings read \( q'_\perp = q_\perp e^{\ell} \) and \( q'_z = q_z e^{\int_0^\ell \gamma(\ell')d\ell'} \), where the function \( \gamma(\ell) \) is to be determined.

When doing the first momentum shell integration, the coupling constants were independent of length scale. However, they then acquire a momentum dependence because we have absorbed the large momentum effects into them. The correct renormalized theory is a coupled set of integral equations where the coupling constants are taken to be scale dependent. An alternative, but equivalent approach is to integrate over a small momentum range where the coupling constants are approximately fixed and then repeat the entire calculation iteratively. This leads to the usual differential renormalization group equations.

4.2. Recursion Relations and Critical Behavior

We now choose an infinitesimal momentum shell \( e^{-\delta} \), and take the limit \( \delta \to 0 \). This leads to differential renormalization group equations which can be integrated to produce the couplings appropriate for a cutoff \( \Lambda e^{-\ell} \) in the perpendicular direction. We use units such that \( \Lambda = 1 \) in the following. We also define the renormalized “reduced temperature”

\[
r = b + \frac{c}{2\sqrt{h + \alpha}} A_d, \tag{4.7}
\]

which vanishes at the critical point in the fluctuation-corrected theory. The constant \( A_d = 2/\left[\Gamma\left(\frac{d}{2}\right)(4\pi)^{d/2}\right] \) is a geometrical factor. The differential recursion relations valid near \( d = 2 \) are

\[
\frac{dh(\ell)}{d\ell} = h \left( 4 - 2\gamma \right) \tag{4.8a}
\]

\[
\frac{d\alpha(\ell)}{d\ell} = \alpha \left( 2 - 2\gamma \right) \tag{4.8b}
\]

\[
\frac{dr(\ell)}{d\ell} = r \left( 2 - \frac{c}{16\pi\sqrt{h + \alpha + \alpha G}} \right) + \frac{ch}{16\pi G\sqrt{h + \alpha + \alpha G}} \tag{4.8c}
\]

\[
\frac{dc(\ell)}{d\ell} = c \left( 4 - d - \gamma - \frac{3c}{16\pi\sqrt{h + \alpha + \alpha G}} \right) \tag{4.8d}
\]

\[
\frac{dG(\ell)}{d\ell} = G \left( 2 - 2\gamma \right) \tag{4.8e}
\]
We set $\gamma = 2$ in order to hold $h$ fixed, and note that $\alpha$ is an irrelevant variable. We have expanded about $G^{-1} = 0$. Examining these recursion relations, we see that perturbation theory amounts to an expansion in two parameters, namely $\bar{c} = c/\sqrt{h + \alpha h}$ and $(\bar{G})^{-1} = hG^{-1}$. These couplings have the recursion relations

\[
\frac{d\bar{c}}{d\ell} = \bar{c} \left( \epsilon - \frac{3\bar{c}}{16\pi} \right) \tag{4.9a}
\]

\[
\frac{d}{d\ell} \left( \frac{h}{G} \right) = 2 \left( \frac{h}{G} \right) \tag{4.9b}
\]

where $\epsilon = 2 - d$.

We first consider the subspace of theories with $G = \infty$. In this case there is a stable, nontrivial fixed point for positive $\epsilon$. Though we have already noted that this theory is identical to the uniaxial ferroelectric, and hence should have all the same critical behavior, a direct calculation of these exponents provides a useful check of our renormalization group method. In particular we can calculate the logarithmic corrections to the compressibility discussed in Section 3 in the critical dimension of $2 + 1 = 3$. Upon setting $\epsilon = 0$ in (4.9a) we solve for $\bar{c}(\ell)$ and find that

\[
\bar{c}(\ell) = \frac{\bar{c}(0)}{1 + \frac{3\bar{c}(0)}{16\pi} \ell} \tag{4.10}
\]

which leads via (1.8) to

\[
r(\ell) = r(0)e^{2\ell} \left( 1 + \frac{3\bar{c}(0)}{16\pi} \ell \right)^{-1/3} \tag{4.11}
\]

Since at this order in $\epsilon$ there is no nontrivial rescaling of the fields, we have

\[
\chi(q_\perp, q_z; r_0) = \int dz dx e^{iq_\perp \cdot x - i q_z z} \langle \rho(x, z) \rho(0, 0) \rangle_{\ell=0} = \int e^{\int_0^\ell \gamma(\ell') d\ell'} dz' e^{d\ell} dx' \times \left( e^{-\frac{1}{2} \int_0^\ell \gamma(\ell') d\ell' (2-d)/2} e^{iq_\perp \cdot x' - i q_z z'} \langle \rho'(x', z') \rho'(0, 0) \rangle_{\ell} \right) = e^{2\ell} \chi_\ell \left( e^{\int_0^\ell \gamma(\ell') d\ell'} q_\perp, q_z; r(\ell) \right) \tag{4.12}
\]

We now choose $\ell = \ell^*$ large enough so that $r(\ell^*)$ has grown to $O(1)$. Then we have from (4.11)

\[
1 = r(\ell^*) = r_0 (3\bar{c}_0/16\pi)^{-1/3} e^{2\ell^*} (\ell^*)^{-1/3} \tag{4.13}
\]
so that
\[ e^{2\ell^*} \sim r_0^{-1} |\ln r_0|^{1/3} \] (4.14)
and thus
\[ \chi(q_\perp, q_\parallel; r_0) = \frac{|\ln r_0|^{1/3}}{r_0} \chi(\xi_\perp q_\perp, \xi_\parallel q_\parallel; 1) \] (4.15)
where \( \xi^2 \sim \xi \sim (\ln r_0)^{1/3}/r_0 \), in agreement with the results of [4]. Note the anisotropically diverging correlation lengths produced by the directed line-like nature of the degrees of freedom. The two-loop calculation of [26] leads to the result \( \xi \sim \xi_\perp \frac{2^2}{\pi^2} \) to \( O(\epsilon^2) \).

Recalling the discussion in the last section of the compressibility, we have finally
\[ \chi(q \sin \phi, q \cos \phi; r_0) \sim \frac{|\ln r_0|^{1/3}}{r_0} \frac{1}{1 + h(\frac{|\ln r_0|^{1/3}}{r_0})(\cot \phi)^2} \] (4.16)
which should be compared to the mean field result (3.3). Note that the effective direction of the compressional wave changes as we go to longer length scales. Near the critical point we always find \( \chi(\phi) \sim \frac{1}{h(\cot \phi)^2} \), which appears to be a universal result.

Similarly, in the ordered phase, we can calculate the specific heat exponent by flowing along a renormalization group trajectory and matching on to an effective free energy at long-wavelengths (A more complicated calculation shows that this behavior occurs above \( T_C \) as well). In the ordered phase, the bulk free energy is approximately \(-3\Omega(r^2/2c)\), where \( \Omega \) is the total volume of the system. We then have
\[ F_\ell = -3\Omega \frac{r(\ell)^2}{2c(\ell)} \]
\[ = -\frac{3}{2} e^{-d\ell} e^{-\int_0^\ell \gamma(\ell')d\ell'} \frac{r(\ell)^2}{c(\ell)} \] (4.17)
Again choosing \( \ell = \ell^* \) as in (4.13) we find that
\[ F_\ell = -\frac{3}{2} e^{-d\ell^*} e^{-\int_0^{\ell^*} \gamma(\ell')d\ell'} \frac{1^2}{c(\ell^*)} \sim e^{-(d+2)\ell^*} \ell^* \sim r_0^2 |\ln r_0|^{1/3} \] (4.18)
Differentiating twice with respect to \( r_0 \), we recover the correct specific heat-like behavior at the critical point.

Going back to the full theory we now allow \( G^{-1} \) to take on finite values. In this case we are driven away from our fixed point and \( G^{-1} \) flows towards \( \infty \). This reflects a crossover to an anisotropic but point-like phase where the constraint (1.3), and hence the line-like nature of the polymers is unimportant. Because t decouples from the density fluctuations the quantum mechanical bosons behave classically at long wavelengths. A similar crossover occurs in polymer nematics when hairpins are introduced into the theory and the previously directed polymers become isotropic [4].
5. Mixing Exponents

We now consider directed polymer blends, or, equivalently, binary mixtures of two superfluids. As with binary mixing in classical systems of point particles, the mixing exponents are identical to the liquid-gas exponents. We demonstrate this equivalence by showing that the Landau-Ginzburg theory controlling the mixing fraction is the same as that for a single component critical point, (1.1). For simplicity, we restrict our attention to the limit $G \to \infty$. In appendix A we derive the quadratic part of the free energy of two polymer species using the boson representation. To these terms we add nonlinear couplings to account for the configurational entropy of each polymer species, multiparticle interactions, etc., and consider the model free energy

$$F = \int dz d^d x \left[ \frac{h_1}{2} t_1^2 + \frac{h_2}{2} t_2^2 + h_{12} t_1 \cdot t_2 + \frac{1}{2} (\nabla_\perp \delta \rho_1)^2 + \frac{1}{2} (\nabla_\perp \delta \rho_2)^2 \right]$$

subject to the constraints

$$\partial_z \delta \rho_j + \nabla_\perp \cdot t_j = 0 \quad j = 1, 2. \quad (5.1)$$

We have neglected gradient terms of the form $(\partial_z \delta \rho_i)^2$, because these turn out to be irrelevant at the demixing point. Terms proportional to $\delta \rho_1 \delta \rho_2^2$ and $\delta \rho_2 \delta \rho_1^2$ could have been added, but do not affect our results in any essential way. Additionally, we assume that the $(\nabla_\perp \delta \rho_i)^2$ terms have been diagonalized so that the term $\nabla_\perp \delta \rho_1 \cdot \nabla_\perp \delta \rho_2$ is not present. It is useful to pass to sum and difference variables, i.e. $\delta \rho_\pm = (\delta \rho_1 \pm \delta \rho_2)/\sqrt{2}$. We only keep terms in the free energy up to second order in the noncritical mode $\delta \rho_\pm$, but keep all terms up to quartic order in $\delta \rho_\pm$. We also define fields $t_\pm = (t_1 \pm t_2)/\sqrt{2}$ which are the tangents associated with the sum and difference variables. The free energy now takes the form

$$F = \int dz d^d x \left[ \frac{h_+}{2} t_+^2 + \frac{h_-}{2} t_-^2 + h_{+\pm} t_+ t_- + \frac{1}{2} (\nabla_\perp \delta \rho_+)^2 + \frac{1}{2} (\nabla_\perp \delta \rho_-)^2 \right]$$

subject to the constraints now reading

$$\partial_z \delta \rho_\pm + \nabla_\perp \cdot t_\pm = 0. \quad (5.2)$$

(5.3)
We can again integrate out the transverse parts of \( t_\pm \), and use the constraint (5.4) to solve for the longitudinal parts. The resulting quadratic part of the free energy reads (in momentum space)

\[
F_{\text{quad}} = \int \frac{d\varphi_z}{2\pi} \frac{d^d q_\perp}{(2\pi)^d} \left[ \frac{1}{2} \left( h_+ \frac{q_\perp^2}{q_\perp^2} + q_\perp^2 + b_+ \right) |\delta \rho_+|^2 + \frac{1}{2} \left( h_+ \frac{q_\perp^2}{q_\perp^2} + q_\perp^2 + b_- \right) |\delta \rho_-|^2 + h_+ \frac{q_\perp^2}{q_\perp^2} \delta \rho_+ \delta \rho_- + b_+ \delta \rho_+ \delta \rho_- \right]
\]

At this point we are in the position to integrate out \( \delta \rho_+ \), resulting in

\[
F'_{\text{quad}}[\delta \rho_-] = \frac{1}{2} \int \frac{d\varphi_z}{2\pi} \frac{d^d q_\perp}{(2\pi)^d} \left[ h_+ \frac{q_\perp^2}{q_\perp^2} + q_\perp^2 + b_- - \left( h_+ \frac{q_\perp^2}{q_\perp^2} + b_- \right) \right] |\delta \rho_-|^2.
\]

Upon expanding the denominator of (5.6) in \( q_\perp \) and \( q_z \), keeping only relevant and marginally relevant terms, and reintroducing \( t_- \) we finally have

\[
F' = \int dz d^d x \left[ \frac{1}{2} \left( h_- - \frac{2b_- h_- + b_+^2 h_+}{b_+^2} \right) t_-^2 + \frac{1}{2} \left( 1 + \frac{b_-^2}{b_+^2} \right) (\nabla_\perp \delta \rho_-)^2 \right.
\]

\[
\left. + \frac{1}{2} \left( b_- - \frac{b_-^2}{b_+} \right) \delta \rho_-^2 + \frac{w}{3!} \delta \rho_-^3 + \frac{c}{4!} \delta \rho_-^4 \right]
\]

subject to

\[
\partial_\varphi \delta \rho_- + \nabla_\perp \cdot t_-= 0.
\]

As in the single component model, the critical mixing line is gotten by setting \( w_- = 0 \). This could be done by altering the chemical potentials of the two species, for instance. Our model is thus equivalent to the original model (1.1), except for the unimportant coupling \( \alpha (\partial_\varphi \delta \rho_-)^2 \). We conclude that the critical mixing of line-like objects is also in the universality class of the three-dimensional uniaxial ferroelectric.

6. Acknowledgments

It is a pleasure to acknowledge helpful conversation with L. Balents, P. Clark, M. Cole, B.I. Halperin, T. Hwa, P. Le Doussal, G. McKinley, R.B. Meyer, O. Narayan and T. Witten during the course of this investigation. One of us (RDK), would like to acknowledge the support of a National Science Foundation Graduate Fellowship. This work was supported by the National Science Foundation, through Grant DMR91-15491 and through the Harvard Materials Research Laboratory.
Appendix A. Derivation of Hydrodynamics for Binary Mixtures

Our starting point is a pair of boson field theories, each describing a distinct superfluid or polymer species \[2\]. We then couple them through their densities, as well as their “currents” or tangent fields. More precisely the partition function is

\[
Z = \int [d\psi^*_1][d\psi^*_2][d\psi_1][d\psi_2] e^{-S[\psi^*_1, \psi_1, \psi^*_2, \psi_2]}
\]  

(A.1)

where the action is

\[
\frac{F}{k_B T} \equiv S = S_1[\psi^*_1, \psi_1] + S_2[\psi^*_2, \psi_2] + S_{12}[\psi^*_1, \psi_1, \psi^*_2, \psi_2].
\]  

(A.2)

The individual terms read

\[
S_j = \int dz d^dx \left[ \psi_j^* \left( \partial_j - D_j \nabla^2 - \bar{\mu}_j \right) \psi_j + \frac{B_j}{4} |\psi_j|^4 \right]
\]  

(A.3)

while the interaction is

\[
S_{12} = \int dz d^dx \left[ B_{12} |\psi_1|^2 |\psi_2|^2 + \eta_{12} (\psi_1^* \nabla_\perp \psi_1 - \psi_1 \nabla_\perp \psi_1^*) \cdot (\psi_2^* \nabla_\perp \psi_2 - \psi_2 \nabla_\perp \psi_2^*) \right]
\]  

(A.4)

The change of variables

\[
\psi_j = \sqrt{\rho_j} e^{i\theta_j}
\]  

(A.5)

leads to

\[
S = \int dz d^dx \left\{ i \rho_1 \partial_z \theta_1 + \frac{D_1 (\nabla_\perp \rho_1)^2}{4 \rho_1} + D_1 \rho_1 (\nabla_\perp \theta_1)^2 - \bar{\mu}_1 \rho_1 + \frac{B_1}{4} \rho_1^2 \\
+ i \rho_2 \partial_z \theta_2 + \frac{D_2 (\nabla_\perp \rho_2)^2}{4 \rho_2} + D_2 \rho_2 (\nabla_\perp \theta_2)^2 - \bar{\mu}_2 \rho_2 + \frac{B_2}{4} \rho_2^2 \\
+ B_{12} \rho_1 \rho_2 + \eta_{12} \rho_1 \rho_2 \nabla_\perp \theta_1 \cdot \nabla_\perp \theta_2 \right\}
\]  

(A.6)

We now introduce two fields which we shall see are proportional to the tangent fields of the individual species. We start with the identity

\[
\exp \left\{ \int dz d^dx \left[ D_1 \rho_1 (\nabla_\perp \theta_1)^2 + D_2 \rho_2 (\nabla_\perp \theta_2)^2 + \eta_{12} \rho_1 \rho_2 \nabla_\perp \theta_1 \cdot \nabla_\perp \theta_2 \right] \right\}
\]  

\[
= \int [dp_1][dp_2] \exp \left\{ \int dz d^dx \left[ \frac{D_2 \rho_2 p_1^2 + D_1 \rho_1 p_2^2 - 2\eta_{12} \rho_1 \rho_2 p_1 \cdot p_2}{D_1 D_2 \rho_1 \rho_2 - \eta_{12}^2 \rho_1^2 \rho_2^2} \right] + i p_1 \cdot \nabla_\perp \theta_1 + i p_2 \cdot \nabla_\perp \theta_2 \right\}
\]  

(A.7)
Upon using this identity to replace the terms quadratic in $\theta$ appearing in (A.6) and integrating over $\theta_j$, we find that

$$Z = \int [d\rho_1][d\rho_2][dp_1][dp_2] \delta[\partial_z \rho_1 + \nabla_\perp \cdot p_1] \delta[\partial_z \rho_2 + \nabla_\perp \cdot p_2] e^{-S'[\rho_1, \rho_2, p_1, p_2]} \tag{A.8}$$

where

$$S' = \int dzd^4x \left\{ \frac{D_2 \rho_2 p_1^2 + D_1 \rho_1 p_2^2 - 2\eta_{12} \rho_1 \rho_2 p_1 \cdot p_2}{D_1 D_2 \rho_1 \rho_2 - \eta_{12}^4 \rho_1^2 \rho_2^2} \right\} + \frac{D_1 (\nabla_\perp \rho_1)^2}{4\rho_1} + \frac{D_2 (\nabla_\perp \rho_2)^2}{4\rho_2} \right\} \right. \\
- \frac{\mu_1 \rho_1 - \mu_2 \rho_2 + \frac{B_1}{4} \rho_1^2 + \frac{B_2}{4} \rho_2^2 + B_{12} \rho_1 \rho_2}{4\rho_1 4\rho_2} \right\} \tag{A.9}$$

We now expand $\rho_i = \rho_{0,i} + \delta \rho_i$ around the minimum of the bulk free energy and rescale $\delta \rho_j$ so that the kinetic term in the action is $\frac{1}{2k_B T} (\nabla_\perp \delta \rho_j)^2$ to obtain

$$S' = \frac{1}{k_B T} \int dzd^4x \left\{ \frac{h_1}{2} t_1^2 + \frac{h_2}{2} t_2^2 + h_{12} t_1 \cdot t_2 + \frac{1}{2} (\nabla_\perp \delta \rho_1)^2 + \frac{1}{2} (\nabla_\perp \delta \rho_2)^2 \right\} + \frac{b_1}{2} \delta \rho_1^2 + \frac{b_2}{2} \delta \rho_2^2 + b_{12} \delta \rho_1 \delta \rho_2 \tag{A.10}$$

where $t_j = \sqrt{D_j/2\rho_{j,0}} p_j$ and the $h_j$ and $b_j$ can be read off from (A.9) in the expansion of $\rho_j$ around $\rho_{j,0}$. The delta-functionals in (A.8) impose the constraints

$$\partial_z \delta \rho_j + \nabla_\perp \cdot t_j = 0. \tag{A.11}$$
References

[1] R.P. Feynman, Phys. Rev. 91: 1291 (1953); see also R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965) and R.P. Feynman, Statistical Mechanics, (Benjamin/Cummings, Reading, MA, 1972).

[2] D.R. Nelson, Phys. Rev. Lett. 60: 1973 (1988); D.R. Nelson and H.S. Seung, Phys. Rev. B 39: 9153 (1989); D.R. Nelson and P. Le Doussal, Phys. Rev. B 42: 10113 (1990).

[3] P. Le Doussal and D.R. Nelson, Europhys. Lett. 15: 161 (1991).

[4] R.D. Kamien, P. Le Doussal and D.R. Nelson, Phys. Rev. A 45: 8727 (1992).

[5] For a review, see D.R. Nelson, Physica A 177: 220 (1991).

[6] P.G. de Gennes, J. Phys. (Paris) Lett. 36L: 55 (1975).

[7] M.C. Marchetti and D.R. Nelson, Phys. Rev. B 42: 9938 (1990); Physica C 174: 40 (1991).

[8] J.V. Selinger and R.F. Bruinsma, Phys. Rev. A 43: 2910 (1991).

[9] A.I. Larkin and D.E. Khmel’nitškiǐ, JETP 29: 1123 (1969); see also A. Aharony, Phys. Rev. B 8: 3363 (1973); B 9: 3946 (E) (1974).

[10] P.G. de Gennes, Physics of Liquid Crystals (Oxford University, London, 1974).

[11] T.C. Halsey and W. Toor, J. Stat. Phys. 61: 1257 (1990).

[12] T.C. Halsey and W. Toor, Phys. Rev. Lett. 65: 2820 (1990).

[13] For a system of spheres arranged in a cubical lattice or amorphously, the Clausius-Mosotti relation holds, so \( \epsilon(n) = (1 + 2v\gamma n)/(1 - v\gamma n) \), where \( \gamma \) is the molecular polarizability and \( v \) is the molecular volume. In this case \( \epsilon''(n) \) is explicitly positive. A virial expansion for the bulk dielectric constant in terms of the density can be done in general. The first virial term leads to the Clausius-Mosotti relation, and the second term is generally quite small. Thus \( \epsilon''(n) \) is expected to be positive even if the spheres are not arranged either cubically or randomly. See G.C. Maitland, M. Rigby, E.B. Smith and W. Wakeham, Intermolecular Forces, (Clarendon Press, Oxford, 1981).

[14] R. Tao, Southern Illinois University Preprint, (1992).

[15] T. Chen, R.N. Zitter and R. Tao, Southern Illinois University Preprint, (1992).

[16] A.P. Gast and C.F. Zukoski, Advances in Colloid and Interface Sci. 30: 153 (1989).

[17] P. Muzikar and C. Pethicke, Phys. Rev. B 24: 2533 (1981).

[18] E. Cheng, M.W. Cole and P.B. Shaw, J. Low Temp. Phys. 79: 49 (1991); see, however, recent work by C. Carrero and M. Cole for evidence against this hypothesis.

[19] D. Forster, Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions, (Benjamin/Cummings, Reading, MA, 1975); see also the treatment of bosons in pp. 6-8 of A.A. Abrikosov, L.P. Gorkov and I.E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (Dover, New York, 1963).

[20] A.P. Young, J. Phys. C 8: L309 (1975) and references therein.
[21] See also Terry Hwa, unpublished

[22] See for instance, R.K. Pathria, *Statistical Mechanics*, (Pergamon, Exeter, Great Britain, 1972).

[23] We thank Tom Witten for conversations on this point.

[24] D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. A **16**: 732 (1977).

[25] B.I. Halperin, P.C. Hohenberg, and S.K. Ma, Phys. Rev. Lett. **29**: 1548 (1972).

[26] E. Brézin and J. Zinn-Justin, Phys. Rev. B **13**: 251 (1976).
Figure Captions

Fig. 1. Hydrodynamic volume averaging over directed lines in \((d+1)\) dimensions which leads to the coarse-grained density and tangent fields used in this paper.

Fig. 2. Speculative phase diagram for Type II superconductors with an attractive interaction between flux lines which allows for coexisting dense and dilute entangled flux liquids. At sufficiently high temperatures, this line terminates in a critical point. The critical isochore, which marks the extension of this line into the single phase region, is shown as a dotted line.

Fig. 3. Pressure-temperature phase diagram for a quantum fluid with the unusual pair potential shown in the inset. The outer minimum dominates in the superfluid phase \(S_1\), while the inner minimum is primarily occupied in the higher pressure superfluid \(S_2\). Phases \(S_1\) and \(S_2\) are separated from a normal liquid \(N\) by the dashed \(\lambda\)-line. Points \(a\) and \(b\) are conventional critical points. By enhancing quantum fluctuations at \(T = 0\) (say, by increasing \(\hbar\)), one arrives at the critical point \(c\) discussed in the paper.

Fig. 4. Phase diagram for a binary mixture of two superfluid species as a function of the concentration of species B, temperature and Planck’s constant. This paper addresses the nature of the critical point which occurs at \(T = 0\).

Fig. 5. Hartree graph which diverges for \(d < 2\). We integrate here over all internal momenta.

Fig. 6. Graphs which contribute to the renormalization group calculation. In these graphs, the internal lines are integrated over all values of \(q_z\) but \(q_\perp\) is only integrated in a momentum shell between \(\Lambda e^{-\ell}\) and \(\Lambda\). Figure (a) is a graph which renormalizes the propagator, and figure (b) is a graph which renormalizes the four-point coupling \(c\).