EQUALITY CONDITIONS OF DATA PROCESSING INEQUALITY FOR $\alpha$-$z$ RÉNYI RELATIVE ENTROPIES

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Abstract. The $\alpha$-$z$ Rényi relative entropies

$$D_{\alpha,z}(\rho||\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{\frac{1}{1-\alpha}} \rho^{\frac{1}{z}} \sigma^{\frac{1}{1-\alpha}} \right)^z,$$

are a two-parameter family of Rényi relative entropies that are quantum generalizations of the classical $\alpha$-Rényi relative entropies. In [Zha20] Zhang decided the full range of $(\alpha, z)$ for which the Data Processing Inequality (DPI)

$$D_{\alpha,z}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D_{\alpha,z}(\rho||\sigma),$$

is valid. Here $\rho$ and $\sigma$ are two arbitrary faithful quantum states, and $\mathcal{E}$ is any quantum channel. In this paper we give algebraic conditions of $(\rho, \sigma, \mathcal{E})$ for the equality in DPI. For the full range of parameters $(\alpha, z)$, we give necessary conditions and sufficient conditions. For most parameters we give equivalent conditions. This generalizes and strengthens the results in [LRD17].

1. Introduction

The Data Processing Inequality (DPI) plays a fundamental role in the quantum information theory. It states that for the quantum relative entropy (usually known as Umegaki relative entropy [Ume62]) defined by

$$D(\rho||\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)],$$

we have

$$D(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D(\rho||\sigma).$$

(1.1)

Here and in what follows $\rho$ and $\sigma$ are always two arbitrary faithful quantum states, and $\mathcal{E}$ is always a quantum channel. This inequality suggests that after the operation of a quantum channel, it becomes much harder to distinguish two quantum states.

DPI has been studied for various generalizations of Umegaki relative entropy $D$. Usually this is equivalent to the joint convexity/concavity of certain trace functionals, which has become an active topic since Lieb’s pioneering work [Lie73] resolving the conjecture of Wigner, Yanase and Dyson [WY63]. In this paper, the quantum relative entropies that we are concerned with are the so-called $\alpha$-$z$ Rényi relative entropies $D_{\alpha,z}$, first introduced by Audenaert and Datta [AD15]:

$$D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{\frac{1}{1-\alpha}} \rho^{\frac{1}{z}} \sigma^{\frac{1}{1-\alpha}} \right)^z, \quad \alpha \in (-\infty, 1) \cup (1, \infty), \quad z > 0.$$

In general the definition of $D_{\alpha,z}(\rho||\sigma)$ can be extended to quantum states $\rho$ and $\sigma$ such that $\text{supp}(\rho) \subset \text{supp}(\sigma)$, where $\text{supp}(x)$ denotes the support of $x$; see [AD15]. In this paper for simplicity we always assume that they are faithful. The family of $\alpha$-$z$ Rényi relative entropies

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$D_{\alpha,z}$ is a quantum generalization of the classical $\alpha$-Rényi relative entropies [Rén61]. It unifies two other important quantum analogues of $\alpha$-Rényi relative entropies

$$D_{\alpha}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha}),$$

and the so-called sandwiched Rényi relative entropies [MLDS*13, WWY14]

$$\tilde{D}_{\alpha}(\rho||\sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\sigma^{\frac{1-\alpha}{2}} \rho \sigma^{\frac{1-\alpha}{2}})^\alpha,$$

by taking $z = 1$ and $z = \alpha$, respectively. Note that both of $D_{\alpha}$ and $\tilde{D}_{\alpha}$ admit the Umegaki relative entropy $D$ as a limit case when $\alpha \to 1$.

In [Zha20] Zhang identified all the pairs $(\alpha, z)$ for which DPI for $\alpha$-$z$ Rényi relative entropy $D_{\alpha,z}$ (the following (1.2)) is valid.

**Theorem 1.1.** [Zha20, Theorem 1.2] The $\alpha$-$z$ Rényi relative entropy $D_{\alpha,z}$ satisfies the Data Processing Inequality

$$D_{\alpha,z}(E(\rho)||E(\sigma)) \leq D_{\alpha,z}(\rho||\sigma),$$

where $\rho, \sigma$ are any faithful quantum states over $\mathcal{H}$, $E : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is any quantum channel, and $\mathcal{H}$ is any finite dimensional Hilbert space, if and only if one of the following holds

1. $0 < \alpha < 1$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
2. $1 < \alpha \leq 2$ and $\frac{2}{\alpha} \leq z \leq \alpha$;
3. $2 \leq \alpha < \infty$ and $\alpha - 1 \leq z < \alpha$.

Remark that a quantum channel $E$ in [Zha20] is meant to be a completely positive trace preserving (CPTP) map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$. In this paper $E$ is a quantum channel if it is CPTP from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ for some finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The main results in [Zha20], including the above theorem, are still valid for quantum channels in this more general sense.

In this paper, we are interested in the equality condition of DPI for $D_{\alpha,z}$ (1.2). Petz [Pet86, Pet88] proved that the equality in DPI for $D$ (1.1) is captured for the triple $(\rho, \sigma, E)$ if and only if there exists a quantum channel $R$, usually known as the recovery map, such that it reverses the action of $E$ over $\{\rho, \sigma\}$:

$$R \circ E(\rho) = \rho \text{ and } R \circ E(\sigma) = \sigma.$$

The “if” part is trivial, by applying DPI (1.1) again to $(E(\rho), E(\sigma), R)$. The “only if” direction is much more difficult and usually yields deeper result than DPI; see for example the work of Carlen and Vershynina [CV18, CV20a] on stability of DPI and other related results [FR15, JRS+18, SBT17, Sha14, Sut18]. It is a natural question to ask the existence of such recovery maps for other quantum relative entropies. The existence of recovery map is known for $D_{\alpha}$ with $\alpha \in (0, 2]$ [Pet86, Pet88, HMPB11], which is the full range of $\alpha$ for which DPI for $D_{\alpha}$ is valid, and for $\tilde{D}_{\alpha}$ with $\alpha \in (\frac{3}{4}, 1) \cup (1, \infty)$ [Jen17, Jen18, HM17], which is the full range of $\alpha$ for which DPI for $\tilde{D}_{\alpha}$ is valid. A related notion is the sufficiency (of channels). We refer to [Pet86, Pet88, HM17, LRD17, Jen17, Jen18] for more discussions on the sufficiency.

In [LRD17] Leditzy, Rouzé and Datta proved that in DPI for $\tilde{D}_{\alpha}$

$$\tilde{D}_{\alpha}(E(\rho)||E(\sigma)) \leq \tilde{D}_{\alpha}(\rho||\sigma),$$

with $\alpha \in (\frac{3}{4}, 1) \cup (1, \infty)$ (which is the full range of $\alpha$ for (1.3) to hold), the equality is captured for $(\rho, \sigma, E)$ if and only if

$$\sigma^\gamma(\sigma^\gamma \rho \sigma^\gamma)^{\alpha-1} \sigma^\gamma = E^1 \left( E(\sigma)^\gamma |E(\sigma)^\gamma E(\rho)E(\sigma)^\gamma|^{\alpha-1} E(\sigma)^\gamma \right),$$

where $\gamma$ is any appropriate parameter.
where $\gamma = \frac{1-\alpha}{z}$ and $E^\dagger$ is the adjoint of $E$ with respect to the Hilbert-Schmidt inner product. Note that “if” part is obvious, and the difficulty lies in the “only if” part. It is not clear whether one can deduce the existence of a recover map for $D_\alpha$ from this algebraic condition (1.4) except for $\alpha = 2$. We shall explain the case $\alpha = 2$ in Remark 1.3.

The main result of this paper is a generalization and strength of Leditzky-Rončé-Datta’s result. On the one hand, we prove that if the equality of DPI for $D_{\alpha,z}$ is captured for some $(\rho, \sigma, E)$, then necessarily an algebraic condition (1.8), as a generalization of (1.4), is valid (see Theorem 1.2 (i)). Remark that when $\alpha \neq z$, this necessary condition might not be sufficient.

On the other hand, we give two other algebraic conditions (1.9) and (1.10), which are sufficient for equality of DPI for $D_{\alpha,z}$ (see Theorem 1.2 (ii)). Moreover, for certain parameters (roughly speaking “non-endpoint” case), these sufficient conditions are also necessary (see Theorem 1.2 (iii) and (iv)). This is even new when $\alpha = z$.

Our main result is the following

**Theorem 1.2.** Let $(\alpha, z)$ be as in Theorem 1.1 and set $p := \frac{\alpha}{z}$ and $q := \frac{1-\alpha}{z}$. For two faithful quantum states $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ and a quantum channel $E : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ put

$$x := \sigma^{\frac{1}{2}}(\sigma^{p} \rho^{p} \sigma^{q})^{-\frac{1}{p}} \sigma^{\frac{1}{p}} = \rho^{-\frac{1}{p}}(\rho^{p} \sigma^{q} \rho^{q})^{\frac{1}{p}} \rho^{-\frac{1}{p}},$$

and

$$y := E(\sigma)^{\frac{1}{2}} \left( E(\sigma) \frac{1}{2} E(\rho)^{p} E(\sigma)^{\frac{1}{2}} \right)^{-\frac{1}{p}} E(\sigma)^{\frac{1}{p}}$$

$$= E(\rho)^{-\frac{1}{p}} \left( E(\rho) \frac{1}{2} E(\sigma)^{q} E(\rho)^{\frac{1}{2}} \right)^{-\frac{1}{p}} E(\rho)^{-\frac{1}{p}}.$$

Consider the following statements

1. the inequality in DPI (1.2) becomes an equality:

$$D_{\alpha,z}(E(\rho)||E(\sigma)) = D_{\alpha,z}(\rho||\sigma);$$

2. there holds the identity

$$x = E^\dagger(y);$$

where $E^\dagger$ is the adjoint of $E$ with respect to the Hilbert-Schmidt inner product;

3. there holds the identity

$$E \left( x^{\frac{1}{2}} \rho^{p} x^{\frac{1}{2}} \right)^{\frac{1}{2}} = \left( y^{\frac{1}{2}} E(\rho)^{p} y^{\frac{1}{2}} \right)^{\frac{1}{2}};$$

4. there holds the identity

$$E \left( x^{-\frac{1}{2}} \sigma^{q} x^{-\frac{1}{2}} \right)^{\frac{1}{2}} = \left( y^{-\frac{1}{2}} E(\sigma)^{q} y^{-\frac{1}{2}} \right)^{\frac{1}{2}}.$$

Then we have

(i) both (3) and (4) imply (1);

(ii) (1) $\Rightarrow$ (2), and when $\alpha = z$ (or equivalently $p = 1$): (2) $\Rightarrow$ (1);

(iii) if $\alpha \neq z$ (or equivalently $p \neq 1$), then (1) $\Rightarrow$ (3);

(iv) if $1 - \alpha \neq \pm z$ (or equivalently $q \neq \pm 1$), then (1) $\Rightarrow$ (4).

**Remark 1.3.** Consider the map $R_{\alpha,z} : \mathcal{B}(\mathcal{H})^{++} \rightarrow \mathcal{B}(\mathcal{H})^{++}$ determined by

$$\sigma^{\frac{1}{2}}(\sigma^{p} R_{\alpha,z}(\omega)^{p} \sigma^{q})^{-\frac{1}{p}} \sigma^{\frac{1}{p}} = E^\dagger \left[ E(\sigma)^{\frac{1}{2}} \left( E(\sigma) \frac{1}{2} \omega^{p} E(\sigma)^{\frac{1}{2}} \right)^{-\frac{1}{p}} E(\sigma)^{\frac{1}{p}} \right].$$
Clearly $R_{\alpha, z}(\mathcal{E}(\sigma)) = \sigma$, since $\mathcal{E}^\dagger(1_K) = 1_K$. When (1.8) holds, we also have $R_{\alpha, z}(\mathcal{E}(\rho)) = \rho$. In particular, if $\alpha = z = 2$, or equivalently $(p, q) = (1, -\frac{1}{2})$, then

\begin{equation}
R_{2, 2}(\omega) = \sigma^{\frac{1}{2}}\mathcal{E}^{\dagger}\left(\mathcal{E}(\sigma)^{-\frac{1}{2}}\omega\mathcal{E}(\sigma)^{-\frac{1}{2}}\right)\sigma^{\frac{1}{2}},
\end{equation}

is a quantum channel and thus a recovery map.

**Remark 1.4.** In particular, when $\alpha \neq z$ and $1 - \alpha \neq \pm z$, or equivalently $p \neq 1$ and $q \neq \pm 1$, we have (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Here the equivalence of (1.9) in (3) and (1.10) in (4) is obtained via (1). It will be interesting to find a direct proof for (3) $\Leftrightarrow$ (4).

This paper is organized as follows. In Section 2 we give some lemmas for the proof of main result Theorem 1.2. Some of them are of independent interest. In Section 3 we give the proof of Theorem 1.2.

**Notations.** In this paper $\mathbb{R}$ (resp. $\mathbb{N}$ and $\mathbb{C}$) denotes the set of all real numbers (resp. natural numbers and complex numbers).

We use $\mathcal{H}, \mathcal{H}'$ and $\mathcal{K}$ to denote finite-dimensional (complex) Hilbert spaces. For a finite-dimensional Hilbert space $\mathcal{H}$ we use $1_\mathcal{H}$ to denote the identity operator over $\mathcal{H}$. We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators over $\mathcal{H}$, that is, all complex matrices of size $\dim \mathcal{H} \times \dim \mathcal{H}$. We denote by $\mathcal{B}(\mathcal{H})^+$ (resp. $\mathcal{B}(\mathcal{H})^{++}$) to denote the subfamily of positive (resp. positive invertible) elements of $\mathcal{B}(\mathcal{H})$, that is, all positive semi-definite (resp. positive definite) matrices of size $\dim \mathcal{H} \times \dim \mathcal{H}$. By $\mathcal{B}(\mathcal{H})^\times$ we mean the subcollection of invertible elements in $\mathcal{B}(\mathcal{H})$. We use the usual trace $\text{Tr}$ on a matrix algebra. By a faithful quantum state we mean an invertible positive operator over $\mathcal{H}$ (or a positive definite matrix of size $\dim \mathcal{H} \times \dim \mathcal{H}$) with unit trace. For an operator $T$ on a matrix algebra, we denote by $T^\dagger$ its adjoint with respect to the Hilbert-Schmidt inner product. For any $K \in \mathcal{B}(\mathcal{H})$, $|K| = (K^* K)^{\frac{1}{2}}$ denotes its modulus.

By a quantum channel we mean a completely positive trace preserving map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ for some finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Recall that a map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is completely positive if $\mathcal{E} \otimes 1_{\mathbb{C}^n} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathbb{C}^n)$ is positive for all $n \geq 1$.

**Note added.** After completion of this paper, the author has been informed that in a recent preprint [CV20b], Anna Vershynina and Sarah Chehade have obtained necessary and sufficient conditions on a partial range of $(\alpha, z)$. Their conditions appear to be different from ours. It will be interesting to compare these conditions.

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2. Some lemmas

**Lemma 2.1.** Let $\alpha_i, \beta_i, i = 1, 2$ be real numbers such that $\alpha_1 \beta_2 \neq \alpha_2 \beta_1$. Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Then for $K \in \mathcal{B}(\mathcal{H})^\times$, the pair $(A, B) \in \mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++}$ that solves the equations

\begin{equation}
A^{\alpha_1} = KB^{\beta_1}K^* \quad \text{and} \quad A^{\alpha_2} = KB^{\beta_2}K^*,
\end{equation}

is unique and takes the form

\begin{equation}
A = |K^*|^{\frac{2(\beta_1 - \beta_2)}{\alpha_2 \beta_1 - \alpha_1 \beta_2}} \quad \text{and} \quad B = |K|^{\frac{2(\alpha_1 - \alpha_2)}{\alpha_2 \beta_1 - \alpha_1 \beta_2}}.
\end{equation}
Lemma 2.2. Let $X, Y$ be two convex sets and $f$ be any real function on $X \times Y$. For $n \geq 2$, take any $(x_j)_{1 \leq j \leq n} \subset X$ and any $\lambda_j > 0$, $1 \leq j \leq n$ such that $\sum_{j=1}^{n} \lambda_j = 1$. Set $x_0 := \sum_{i=1}^{n} \lambda_j x_j$. 

Proof. It is easy to see that the pair $(A, B)$ in \eqref{2.2} really solves \eqref{2.1}. This is a consequence of the following identities:

\begin{equation}
|K|^2 = (KK^*)^\alpha = K(K^*K)^{\alpha-1}K^* = K|K|^{2(\alpha-1)}K^*, \quad \alpha \in \mathbb{R}.
\end{equation}

In fact, applying \eqref{2.3} to $\alpha = \frac{\alpha_1(\beta_1-\beta_2)}{\alpha_2(\beta_1-\beta_2)}$, we get

\[A^{\alpha_1} = K|K|^\frac{2\alpha_1(\beta_1-\beta_2)}{\alpha_2(\beta_1-\beta_2)}K^* = K|K|^{\frac{2(\alpha_1-\alpha_2)\beta_1}{2(\alpha_1-\alpha_2)}K^* = KB^{\beta_1}K^*,\]

and applying \eqref{2.3} to $\alpha = \frac{\alpha_2(\beta_1-\beta_2)}{\alpha_2(\beta_1-\beta_2)}$, we obtain

\[A^{\alpha_2} = K|K|^\frac{2\alpha_2(\beta_1-\beta_2)}{\alpha_2(\beta_1-\beta_2)}K^* = K|K|^{\frac{2(\alpha_2-\alpha_2)\beta_2}{2(\alpha_2-\alpha_2)}K^* = KB^{\beta_2}K^*.\]

For the proof of \eqref{2.3}, observe first that it is obvious for all $\alpha \in \mathbb{R}$. Then one can prove that it is valid for all $\alpha \in \mathbb{R}$ using functional calculus and Weierstrass approximation theorem.

It remains to show that \eqref{2.2} is the only solution of \eqref{2.1}. Note first that this is trivial when $\alpha_1 = \alpha_2$ or $\beta_1 = \beta_2$. In fact, if $\alpha_1 = \alpha_2$, then one has

\[KB^{\beta_1}K^* = A^{\alpha_1} = A^{\alpha_2} = KB^{\beta_2}K^*.\]

Since $\beta_1 \neq \beta_2$, we have $B = 1_{E_1}$. Thus $A = (KK^*)^\frac{\alpha_1}{\alpha_2} = |K|^\frac{\alpha_1}{\alpha_2}$ and this finishes the proof for $\alpha_1 = \alpha_2$. The case $\beta_1 = \beta_2$ can be proved similarly.

Now assume that $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. By \eqref{2.1},

\[KB^{\beta_1-\beta_2}K^{-1} = A^{\alpha_1-\alpha_2} = (K^*)^{-1}B^{\beta_1-\beta_2}K^*.\]

It follows that

\[K^*KB^{\beta_1-\beta_2} = B^{\beta_1-\beta_2}K^* K,\]

and

\[KK^*A^{\alpha_1-\alpha_2} = KB^{\beta_1-\beta_2}K^* = A^{\alpha_1-\alpha_2}KK^*.\]

Since $\alpha_1 \neq \alpha_2$, we obtain that $A^\gamma$ commutes with $|K|^\gamma$ for any $\gamma \in \mathbb{R}$. Similarly, $B^\gamma$ commutes with $|K|$ for any $\gamma \in \mathbb{R}$ because $\beta_1 \neq \beta_2$. Thus one has

\[A^{2\alpha_1} = KB^{\beta_1}K^*KB^{\beta_1}K^* = K(B^{2\beta_1}|K|^2K^*),\]

and

\[A^{2\alpha_2} = KB^{\beta_2}K^*KB^{\beta_2}K^* = K(B^{2\beta_2}|K|^2K^*).\]

Then by induction one can show that for all integers $n \geq 1$

\[A^{n\alpha_1} = K(B^{n\beta_1}|K|^{2n-2}K^*)\quad \text{and} \quad A^{n\alpha_2} = K(B^{n\beta_2}|K|^{2n-2}K^*).
\]

By functional calculus and Weierstrass approximation theorem, for all $\gamma_1, \gamma_2 \in \mathbb{R}$:

\[A^{\gamma_1\alpha_1} = K(B^{\gamma_1\beta_1}|K|^{2\gamma_1-2}K^*)\quad \text{and} \quad A^{\gamma_2\alpha_2} = K(B^{\gamma_2\beta_2}|K|^{2\gamma_2-2}K^*).
\]

Choosing $\gamma_1 = \alpha_2$ and $\gamma_2 = \alpha_1$, we have

\[K(B^{\alpha_1\beta_2}|K|^{2\alpha_1-2}K^*) = A^{\alpha_1\alpha_2} = K(B^{\alpha_1\beta_2}|K|^{2\alpha_1-2})K^*.
\]

This, together with the assumption $\alpha_1 \beta_2 \neq \alpha_2 \beta_1$, yields that

\[B = |K|^{\frac{2(\alpha_1-\alpha_2)}{2(\alpha_1-\alpha_2)}}.
\]

Similarly we obtain

\[A = |K|^{-1}K^{\frac{2(\beta_1-\beta_2)}{2(\beta_1-\beta_2)}} = |K|^\frac{2(\beta_1-\beta_2)}{2(\beta_1-\beta_2)}.
\]

Hence the only solution of \eqref{2.1} is \eqref{2.2} and the proof is finished. \qed
(1) Suppose that for any $x \in X$, $\max_{y \in Y} f(x, y)$ exists and is attained by a unique element $y_x \in Y$. If for any $y \in Y$, the function $x \mapsto f(x, y)$ is convex, then the function $g(x) := \max_{y \in Y} f(x, y)$ is convex:

$$g(x_0) \leq \sum_{j=1}^{n} \lambda_j g(x_j).$$

Moreover, if the equality is captured, we have

$$f(x_0, y_{x_0}) = \sum_{j=1}^{n} \lambda_j f(x_j, y_{x_0}),$$

and

$$y_{x_0} = y_{x_j} \text{ for } 1 \leq j \leq n.$$

(2) Suppose that for any $x \in X$, $\min_{y \in Y} f(x, y)$ exists and is attained by a unique element $y_x \in Y$. If $(x, y) \mapsto f(x, y)$ is jointly convex, then the function $g(x) := \min_{y \in Y} f(x, y)$ is convex:

$$g(x_0) \leq \sum_{j=1}^{n} \lambda_j g(x_j).$$

Moreover, if the equality is captured, then we have

$$y_{x_0} = \sum_{j=1}^{n} \lambda_j y_{x_j}.$$
assumption of minimizers. If the equality in (2.7) is captured, then necessarily we have
\[ f(x_0, y_{x_0}) = f \left( x_0, \sum_{j=1}^{n} \lambda_j y_{x_j} \right) \]

By uniqueness of minimizers, one has \( y_{x_0} = \sum_{j=1}^{n} \lambda_j y_{x_j} \). \( \square \)

The following lemma is a variant of [Zha20, Theorem 3.3]. It takes the advantage that minimizers/maximizers in the variational formulas are unique. The uniqueness will be crucial in the proof of Theorem 1.2, as indicated in the previous lemma.

**Lemma 2.3.** Let \( r_i > 0 \), \( i = 0, 1, 2 \). Suppose that \( \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} \). Then for any \( X, Y \in \mathcal{B}(\mathcal{H})^\times \) we have
\[
\text{Tr}|XY|^{2r_0} = \min_{H \in \mathcal{B}(\mathcal{H})^+} \left\{ \frac{r_0}{r_1} \text{Tr}(XH X^*)^{r_1} + \frac{r_0}{r_2} \text{Tr}(Y^* H^{-1} Y)^{r_2} \right\},
\]
and
\[
\text{Tr}|XY|^{2r_1} = \max_{H \in \mathcal{B}(\mathcal{H})^+} \left\{ \frac{r_1}{r_0} \text{Tr}(XH X^*)^{r_0} - \frac{r_1}{r_2} \text{Tr}(Y^* H^{1-t} Y)^{r_2} \right\}.
\]
Moreover, the minimizer in (2.9) is unique and takes the form
\[
\mathcal{H} = X^{-1}|Y^* X^*|^{\frac{2r_0}{2r_1}} (X^{-1})^* = Y|XY|^{-\frac{1}{2r_1}} Y^*,
\]
and similarly the maximizer in (2.10) is unique and takes the form
\[
\overline{\mathcal{H}} = X^{-1}|Y^* X^*|^{\frac{2r_1}{2r_0}} (X^{-1})^* = Y|XY|^{\frac{1}{2r_0}} Y^*.
\]
In particular, for \( A \in \mathcal{B}(\mathcal{H})^{++}, K \in \mathcal{B}(\mathcal{H})^\times \) and \( 0 < s < 1 < t < \infty \), we have
\[
\text{Tr}(K^* A^t K)^\frac{1}{t} = \max_{Z \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{s} \text{Tr}(K^* A^s K Z^{1-s}) - \frac{1-s}{s} \text{Tr}Z \right\},
\]
with the unique maximizer being \( Z = (K^* A^s K)^\frac{1}{s} \), and
\[
\text{Tr}(K^* A^s K)^\frac{1}{s} = \min_{Z \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{t} \text{Tr}(K^* A^t K Z^{1-t}) + \frac{t-1}{t} \text{Tr}Z \right\},
\]
with the unique minimizer being \( Z = (K^* A^t K)^\frac{1}{t} \).

**Remark 2.4.** Note that when \( s = 1 \) or \( t = 1 \), we still have (2.13) or (2.14), respectively. The variational formulas are trivial and certainly maximizers/minimizers are not unique.

**Proof of Lemma 2.3.** We first check that (2.13) and (2.14) follow from (2.10) and (2.9), respectively. In fact, taking
\[
(r_0, r_1, r_2, X, Y) = \left( 1, \frac{1}{s}, \frac{1}{1-s}, A^s K, 1_\mathcal{H} \right),
\]
in (2.10), we get
\[
\text{Tr}(K^* A^s K)^\frac{1}{s} = \max_{Z \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{s} \text{Tr}(K^* A^s K Z^{1-s}) - \frac{1-s}{s} \text{Tr}Z \right\},
\]
and the unique maximizer is \( Z = (K^* A^s K)^\frac{1}{s} \). Similarly, taking
\[
(r_0, r_1, r_2, X, Y) = \left( \frac{1}{l}, \frac{1}{t-1}, A^t K, 1_\mathcal{H} \right),
\]
in (2.9), we obtain
\[
\begin{align*}
\text{Tr}(K^*A^tK)^\dagger & = \min_{H \in B(\mathcal{H})^++} \left\{ \frac{1}{t} \text{Tr}(K^*A^tKH) + \frac{t-1}{t} \text{Tr}H^{2/7} \right\} \\
& = \min_{Z \in B(\mathcal{H})^++} \left\{ \frac{1}{t} \text{Tr}(K^*A^tKZ^{-1}) + \frac{t-1}{t} \text{Tr}Z \right\},
\end{align*}
\]
and the unique minimizer is \( Z = (K^*A^tK)^\dagger \). Now it remains to show (2.9) and (2.10). By [Zha20, Theorem 3.3], we have
\[
\text{Tr}(XY)^{2\gamma_0} = \min_{Z \in B(\mathcal{H})^+} \left\{ \frac{\gamma_0}{\gamma_1} \text{Tr}|XZ|^{2\gamma_1} + \frac{\gamma_0}{\gamma_2} \text{Tr}|Z^{-1}Y|^{2\gamma_2} \right\},
\]
and
\[
\text{Tr}(XY)^{2\gamma_1} = \max_{Z \in B(\mathcal{H})^+} \left\{ \frac{\gamma_1}{\gamma_0} \text{Tr}|XZ|^{2\gamma_0} - \frac{\gamma_1}{\gamma_2} \text{Tr}|Y^{-1}Z|^{2\gamma_2} \right\}.
\]
Replacing \( Z \in B(\mathcal{H})^+ \) with \( H = ZZ^* \in B(\mathcal{H})^{-1} \) in (2.15) and (2.16), one has (2.9) and (2.10), respectively. We refer to [Zha20, Theorem 3.3] for the proof of variational formulas (2.15) and (2.16), which is essentially based on Hölder's inequality. We remark that the minimizers in (2.15) (resp. maximizers in (2.16)) are not unique. For example, if \( Z \) is a minimizer in (2.15), then so is \( ZU \) for unitary \( U \).

It is an easy computation that \( \underline{H} \) (resp. \( \overline{H} \)) is really a minimizer in (2.9) (resp. a maximizer in (2.10)). Actually one has
\[
\begin{align*}
\text{Tr}(XH^tX^*)^{\gamma_1} & = \text{Tr}|Y^*X^*|^{2\gamma_0} = \text{Tr}|XY|^{2\gamma_0} = \text{Tr}(Y^*H^{-1}Y)^{\gamma_2}, \\
\text{Tr}(X\overline{H}^tX^*)^{\gamma_0} & = \text{Tr}|Y^*X^*|^{2\gamma_1} = \text{Tr}|XY|^{2\gamma_1} = \text{Tr}(Y^{-1}\overline{H}(Y^{-1})^*)^{\gamma_2}.
\end{align*}
\]
Then it remains to prove that \( \underline{H} \) (resp. \( \overline{H} \)) is the only minimizer in (2.9) (resp. maximizer in (2.10)).

To see that \( \underline{H} \) given in (2.11) is the unique minimizer in (2.9), we set
\[
(2.17) \quad f_H(X, Y) := \frac{\gamma_0}{\gamma_1} \text{Tr}(XH^tX^*)^{\gamma_1} + \frac{\gamma_0}{\gamma_2} \text{Tr}(Y^*H^{-1}Y)^{\gamma_2}.
\]
Fix \( X, Y \in B(\mathcal{H})^+ \) and put \( \varphi(H) := f_H(X, Y) \). Then for any minimizer \( H \) of \( \varphi \), the differential \( D\varphi \) of \( \varphi \) must vanish at \( H \). In fact, for any self-adjoint \( Z \in B(\mathcal{H}) \), we have \( H + tZ \in B(\mathcal{H})^+ \) for \( t \in \mathbb{R} \) with \(|t| \) small enough. For such \( t > 0 \):
\[
\frac{1}{t} (\varphi(H + tZ) - \varphi(H)) \geq 0,
\]
and for such \( t < 0 \):
\[
\frac{1}{t} (\varphi(H + tZ) - \varphi(H)) \leq 0.
\]
Then for any self-adjoint \( Z \in B(\mathcal{H}) \) we have
\[
\lim_{t \to 0} \frac{1}{t} (\varphi(H + tZ) - \varphi(H)) = 0,
\]
where \( D\varphi(H) \in B(\mathcal{H}) \) is a self-adjoint element given by
\[
D\varphi(H) = \gamma_0 \left[ X^*(XH^tX^*)^{\gamma_1-1}X - (Y^{-1})^*(Y^{-1}H(Y^{-1})^*)^{\gamma_2-1}Y^{-1} \right].
\]
Recall that \( \gamma_0 \neq 0 \). Choose \( Z = D\varphi(H) \) and we obtain \( D\varphi(H) = 0 \). That is,
\[
X^*(XH^tX^*)^{\gamma_1-1}X = (Y^{-1})^*(Y^{-1}H(Y^{-1})^*)^{\gamma_2-1}Y^{-1} = (Y^{-1})^*(Y^*H^{-1}Y)^{\gamma_2+1}Y^{-1}.
\]
Set $A := Y^*H^{-1}Y$, $B := XX^*$ and $K := Y^*X^*$. Then we have

$$A = KB^{-1}K^*$$

Applying Lemma 2.1 to $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, r_2 + 1, -1, r_1 - 1)$, it follows that (note that $\alpha_1 \beta_2 - \alpha_2 \beta_1 = r_1 + r_2 \neq 0$)

$$A = |K^*|^{\frac{2p_1}{r_1 + r_2}} = |XY|^\frac{2p_1}{r_2}$$

Hence

$$H = X^{-1}B(X^*)^{-1} = X^{-1}|Y^*X|^\frac{2p_1}{r_2} (X^*)^{-1} = Y A^{-1} Y^* = Y|XY|^{-\frac{2p_1}{r_2}} Y^*,$$

which proves the uniqueness of minimizers.

The proof for the maximizer is similar. Using the above argument, for fixed $X, Y \in \mathcal{B}(\mathcal{H})^{++}$, any maximizer $H$ in (2.10) solves

$$r_1 [X^*(XX^*)^{r_0 - 1}X - (Y^{-1})^*(Y^{-1}H(Y^{-1}))^{r_2 - 1}Y^{-1}] = 0,$$

where the left hand side is the differential of

$$H \mapsto \frac{r_1}{r_0} \text{Tr}(XX^*)^{r_0 - 1} - \frac{r_1}{r_2} \text{Tr}(Y^{-1}H(Y^{-1}))^{r_2}.$$

Since $r_1 \neq 0$, $H$ satisfies

$$X^*(XX^*)^{r_0 - 1}X = (Y^{-1})^*(Y^{-1}H(Y^{-1}))^{r_2 - 1}Y^{-1}.$$

Set $A := XX^*$, $B := Y^{-1}H(Y^{-1})$ and $K := XY$. Then we have

$$A = KBK^*$$

Applying Lemma 2.1 to $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1 - r_0, 1, -r_2)$, it follows that (note that $\alpha_1 \beta_2 - \alpha_2 \beta_1 = r_0 - r_2 \neq 0$)

$$A = |K^*|^{\frac{2p_1}{r_2}} = |Y^*X|^\frac{2p_1}{r_2}$$

Hence

$$H = X^{-1}A(X^*)^{-1} = X^{-1}|Y^*X|^\frac{2p_1}{r_2} (X^*)^{-1} = YBY^* = Y|XY|^{\frac{2p_1}{r_2}} Y^*,$$

which proves the uniqueness of maximizers.

For convenience of later use, we collect a classical convexity/concavity result in next lemma. The concavity is due to Lieb [Lie73], and the convexity is due to Ando [And79]. We refer to [NEE13] for a unifying and simple proof.

**Lemma 2.5.** [Lie73, And79] For any $K \in \mathcal{B}(\mathcal{H})^*$, the function

$$\mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++} \ni (A, B) \mapsto \text{Tr}(K^*APK^{1 - p}),$$

is

1. jointly concave if $0 < p \leq 1$;
2. jointly convex if $-1 \leq p < 0$.
3. Proof of main result

In this section we prove our main result Theorem 1.2. The proof is inspired by the arguments in [LRD17]. For convenience let us denote by \( \Psi_{p,q} \) the trace functionals inside \( \alpha \)-z Rényi relative entropies \( D_{\alpha,z} \):

\[
\Psi_{p,q}(A,B) := \text{Tr}(A^{\frac{p}{2}} B^{\frac{q}{2}}) = \text{Tr}(B^{\frac{q}{2}} A^{\frac{p}{2}}) = \text{Tr}(A^{\frac{p}{2}} B^{\frac{q}{2}}).
\]

Recall that \( (p, q) = \left( \frac{\alpha}{z}, \frac{1 - \alpha}{z} \right) \).

**Proof of Theorem 1.2.** (i) To show \( (3) \Rightarrow (1) \), note that by definitions of \( x \) (1.5) and \( y \) (1.6) we have

\[
\left( \rho^{\frac{p}{2}} x \rho^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \rho^{\frac{p}{2}} \sigma^{q} \rho^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.1}
\]

and

\[
\left( \mathcal{E}(\rho)^{\frac{p}{2}} y \mathcal{E}(\rho)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \mathcal{E}(\rho)^{\frac{p}{2}} \mathcal{E}(\sigma)^{q} \mathcal{E}(\rho)^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.2}
\]

These two identities, together with (1.9) in (2), yield that

\[
\Psi_{p,q}(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \text{Tr} \left( \mathcal{E}(\rho)^{\frac{p}{2}} \mathcal{E}(\sigma)^{q} \mathcal{E}(\rho)^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.2}
\]

\[
= \text{Tr} \left( \mathcal{E}(\rho)^{\frac{p}{2}} y \mathcal{E}(\rho)^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{1.9}
\]

\[
= \text{Tr} \left( x^{\frac{p}{2}} \rho^{p} x^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.1}
\]

\[
= \Psi_{p,q}(\rho, \sigma),
\]

where in the fourth equality we also used the fact that \( \mathcal{E} \) is trace-preserving.

The implication \( (4) \Rightarrow (1) \) is similar. Note that by (1.5) and (1.6) one has

\[
\left( \sigma^{\frac{p}{2}} x^{-1} \sigma^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \sigma^{\frac{p}{2}} \rho^{p} \sigma^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.3}
\]

and

\[
\left( \mathcal{E}(\sigma)^{\frac{p}{2}} y^{-1} \mathcal{E}(\sigma)^{\frac{p}{2}} \right)^{\frac{1}{p}} = \left( \mathcal{E}(\sigma)^{\frac{p}{2}} \mathcal{E}(\rho)^{p} \mathcal{E}(\sigma)^{\frac{p}{2}} \right)^{\frac{1}{p}} \tag{3.4}
\]
These, together with (1.10) in (3), imply that
\[ \Psi_{p,q}(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \text{Tr} \left( \frac{\mathcal{E}(\rho)^{\frac{1}{2}} y^{-\frac{1}{2}} \mathcal{E}(\sigma)^{\frac{1}{2}}}{y} \right)^{\frac{1}{2}} \]
\[ = \text{Tr} \left( y^{-\frac{1}{2}} \mathcal{E}(\rho)^{\frac{1}{2}} y^{-\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ = \text{Tr} \left( x^{-\frac{1}{2}} \mathcal{E}(\rho) x^{-\frac{1}{2}} \right)^{\frac{1}{2}} \]
\[ = \text{Tr} \left( \frac{\sigma_{x}^{\frac{1}{2}} \mathcal{E}(\rho) \sigma_{x}^{\frac{1}{2}}}{x} \right)^{\frac{1}{2}} \]
Again, in the fourth equality we also used the fact that \( \mathcal{E} \) preserves the trace.

To prove (ii) - (iv), we shall simply investigate the equality condition in the proof of Theorem 1.1 from [Zha20]. For this recall that for each quantum channel \( \mathcal{E} : B(\mathcal{H}) \to B(\mathcal{K}) \), using Stinespring’s Theorem [Sti55], there exist a finite-dimensional Hilbert space \( \mathcal{H}' \), a pure state \( \delta \) over \( \mathcal{H}' \otimes \mathcal{K} \), and a unitary operator \( U \) over \( \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{K} \) such that for any quantum state \( \omega \) over \( \mathcal{H} \)
\[ \mathcal{E}(\omega) = \text{Tr}_{12} U(\omega \otimes \delta) U^* \]
where \( \text{Tr}_{12} \) denotes the partial trace over the first two factors \( \mathcal{H} \otimes \mathcal{H}' \) of \( \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{K} \). For a detailed proof, see [Wol12, Theorem 2.5]. Put \( d = \dim \mathcal{H} \otimes \mathcal{H}' \). Recall that ([Wol12, Example 2.1]) the discrete Heisenberg-Weyl group over \( \mathcal{H} \otimes \mathcal{H}' \) consists of unitaries \( U_{k,l} \), \( 1 \leq k, l \leq d \) over \( \mathcal{H} \otimes \mathcal{H}' \) defined by
\[ U_{k,l} := \sum_{r=1}^{d} \eta^{r|k+r|} \langle r | \] with \( \eta := e^{\frac{2\pi i}{d}} \),
where addition inside the ket is modulo \( d \). One can easily check that for any \( \rho \in B(\mathcal{H} \otimes \mathcal{H}') \) with \( \text{Tr}\rho = 1 \):
\[ \frac{1}{d^2} \sum_{k,l=1}^{d} U_{k,l} \rho U_{k,l}^* = \frac{1}{d} \mathcal{H} \otimes \mathcal{H}' \]
For convenience, let us denote: \( \{ u_j \}_{j=1}^{d^2} := \{ U_{k,l} \}_{k,l=1}^{d} \). Then combining (3.5) and (3.6), we get
\[ \frac{1}{d^2} \sum_{j=1}^{d^2} (u_j \otimes 1_\mathcal{K}) U(\omega \otimes \delta) U^*(u_j^* \otimes 1_\mathcal{K}) \]
In particular, we have
\[ \frac{1}{d^2} \sum_{j=1}^{d^2} V_j \text{ and } \frac{1}{d^2} \sum_{j=1}^{d^2} W_j \]
where
\[ V_j = (u_j \otimes 1_\mathcal{K}) U(\rho \otimes \delta) U^*(u_j^* \otimes 1_\mathcal{K}) \]
and
\begin{equation}
W_j = (u_j \otimes 1_K)U(\sigma \otimes \delta)U^*(u_j^* \otimes 1_K).
\end{equation}

Note that
\begin{equation}
\Psi_{p,q}\left(\frac{1}{d}H \otimes H', \frac{1}{d}E(\rho) \otimes \sigma \otimes \delta \otimes \delta\right) = \Psi_{p,q}(E(\rho), E(\sigma)),
\end{equation}
and for $1 \leq j \leq d^2$,
\begin{equation}
\Psi_{p,q}(V_j, W_j) = \Psi_{p,q}(\rho, \sigma).
\end{equation}

In view of (3.8), (3.11) and (3.12), the identity (1.7) in (1) is equivalent to
\begin{equation}
\Psi_{p,q}\left(\frac{1}{d^2} \sum_{j=1}^{d^2} V_j, \frac{1}{d^2} \sum_{j=1}^{d^2} W_j\right) = \frac{1}{d^2} \sum_{j=1}^{d^2} \Psi_{p,q}(V_j, W_j).
\end{equation}

Recall that $x$ and $y$ are given in (1.5) and (1.6), respectively. For $1 \leq j \leq d^2$, put
\begin{equation}
(H_0, L_0) := \left(\frac{1}{d}H \otimes H', y \otimes \rho y \right)^{\frac{1}{2}}, \left(\frac{1}{d}E(\rho) \otimes \sigma \otimes \delta \otimes \delta\right)^{\frac{1}{2}},
\end{equation}
and
\begin{equation}
(K_j, L_j) := \left(H_0 \otimes V_j^* H_0^*\right)^{\frac{1}{2}}, \left(H_0 \otimes W_j^* H_0^*\right)^{\frac{1}{2}}.
\end{equation}

We claim that from (1) we have
(a) for any $1 \leq j \leq d^2$,
\begin{equation}
H_0 = H_j;
\end{equation}
(b) if $\alpha \neq z$, or equivalently $p \neq 1$, then
\begin{equation}
K_0 = \frac{1}{d^2} \sum_{j=1}^{d^2} K_j;
\end{equation}
(c) if $1 - \alpha \neq \pm z$, or equivalently $q \neq \pm 1$, then
\begin{equation}
L_0 = \frac{1}{d^2} \sum_{j=1}^{d^2} L_j.
\end{equation}

Then the desired results (ii) - (iv) will follow from (a) - (c). We first use (a) to prove (ii), then use (a) and (b) to prove (iii), and finally use (a) and (c) to prove (iv). The claimed (a) - (c) will be shown later.

In view of (3.14) and (3.15), (3.18) is nothing but
\begin{equation}
1_{H \otimes H'} \otimes y = (u_j \otimes 1_K)U(x \otimes 1_{H' \otimes K})U^*(u_j^* \otimes 1_K),
\end{equation}
for $1 \leq j \leq d$. Since each $u_j$ is unitary, we have
\begin{equation}
1_{H \otimes H'} \otimes y = U(x \otimes 1_{H' \otimes K})U^*.
\end{equation}
It follows that
\begin{equation}
U^*(1_{H \otimes H'} \otimes y)U(1_{H} \otimes \delta) = x \otimes \delta.
\end{equation}
Taking the partial trace over the last two factors $\mathcal{H}^\prime \otimes \mathcal{K}$ of $\mathcal{H} \otimes \mathcal{H}^\prime \otimes \mathcal{K}$, we obtain
\begin{equation}
\text{Tr}_{23} [U^*(1_{\mathcal{H}^\prime} \otimes y)U(1_{\mathcal{H}} \otimes \delta)] = x.
\end{equation}
Note that by (3.5), the adjoint $E^\dagger$ of $E$ is given by
\begin{equation}
E^\dagger(\cdot) = \text{Tr}_{23} [U^*(1_{\mathcal{H}^\prime} \otimes \cdot)U(1_{\mathcal{H}} \otimes \delta)].
\end{equation}
So we have proved
\begin{align*}
E^\dagger(y) &= x,
\end{align*}
which finishes the proof of (1) $\Rightarrow$ (2). When $\alpha = z$, the implication of (2) $\Rightarrow$ (1) follows immediately from (1.8) and the definition of $E^\dagger$. Hence (ii) is proved.

Now we prove (iii) from (3.18) in (a) and (3.19) in (b). Note first that
\begin{align*}
K_j &= \left( H_0^\dagger V_p^j H_0^\dagger \right)^\dagger \\
&= \left( H_j^\dagger V_p^j H_j^\dagger \right)^\dagger \\
&= (u_j \otimes 1_{\mathcal{K}})U \left( \left( x^\dagger \rho^p x^\dagger \right)^\dagger \otimes \delta \right) U^*(u_j^* \otimes 1_{\mathcal{K}}),
\end{align*}
where the last equality follows from the definitions of $V_j$ (3.9) and $H_j$ (3.15). Plugging this and (3.16) into (3.19), and using (3.7), one has
\begin{align*}
\frac{1_{\mathcal{H} \otimes \mathcal{H}^\prime}}{d} \otimes \left( y^\dagger \mathcal{E}(\rho)^p y^\dagger \right)^\dagger &= \frac{1}{d^2} \sum_{j=1}^{d^2} (u_j \otimes 1_{\mathcal{K}})U \left( \left( x^\dagger \rho^p x^\dagger \right)^\dagger \otimes \delta \right) U^*(u_j^* \otimes 1_{\mathcal{K}}) \\
&= \frac{1_{\mathcal{H} \otimes \mathcal{H}^\prime}}{d} \otimes \mathcal{E} \left( \left( x^\dagger \rho^p x^\dagger \right)^\dagger \right),
\end{align*}
From this we infer that
\begin{align*}
\left( y^\dagger \mathcal{E}(\rho)^p y^\dagger \right)^\dagger = \mathcal{E} \left( \left( x^\dagger \rho^p x^\dagger \right)^\dagger \right),
\end{align*}
which is nothing but (1.9) in (3). So (1) $\Rightarrow$ (3) and this proves (iii). Using (3.18) in (a) and (3.20) in (c), one can prove (iv) analogously.

Now it remains to prove our claim: (a) - (c). For this we set
\begin{equation}
f_H(A, B) : = \frac{p}{p+q} \text{Tr}(A^\dagger H A^\dagger)^\dagger + \frac{q}{p+q} \text{Tr}(B^\dagger H^{-1} B^\dagger)^\dagger \\
= \frac{p}{p+q} \text{Tr}(H^\dagger A^p H^\dagger)^\dagger + \frac{q}{p+q} \text{Tr}(H^{-\dagger} B^q H^{-\dagger})^\dagger.
\end{equation}
Note that for $(\alpha, z)$ in Theorem 1.1, for which DPI is valid, we have either
\begin{equation}
0 < p, q \leq 1,
\end{equation}
or
\begin{equation}
1 \leq p \leq 2, -1 \leq q < 0 \text{ and } (p, q) \neq (1, -1).
\end{equation}

**Case 1:** $(p, q)$ satisfies (3.24). For $A, B \in B(\mathcal{H})^{++}$, apply (2.9) in Lemma 2.3 to
\begin{align*}
(r_0, r_1, r_2, X, Y) &= \left( \frac{1}{p+q}, \frac{1}{p}, \frac{1}{q}, A^\dagger, B^\dagger \right),
\end{align*}
and we get
\begin{equation}
\Psi_{p,q}(A, B) = \min_{H \in B(\mathcal{H})^{++}} f_H(A, B),
\end{equation}
with the unique minimizer being
\[ (A, B) = \left( \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\rho), \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\sigma) \right) \]
the associated unique minimizer is \( H_0 \) given in (3.14), and for \( (A, B) = (V_j, W_j) \) the associated unique minimizer is \( H_j \) given in (3.15).

If \( 0 < p, q < 1 \), then we have by (2.13) in Lemma 2.3 that
\[
f_{H}(A, B) = \frac{p}{p+q} \max_{K, L \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{p} \text{Tr} \left( H^{\frac{1}{2}} A^p H^{\frac{1}{2}} K^{1-p} \right) - \frac{1}{p} \text{Tr} K \right\} + \frac{q}{p+q} \max_{L \in \mathcal{B(\mathcal{H})}^{++}} \left\{ \frac{1}{q} \text{Tr} \left( H^{\frac{1}{2}} B^q H^{\frac{1}{2}} L^{1-q} \right) - \frac{1}{q} \text{Tr} L \right\}
\]
(3.28)

\[
= \frac{1}{p+q} \max_{K, L \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{p} \text{Tr} \left( H^{\frac{1}{2}} A^p H^{\frac{1}{2}} K^{1-p} \right) - \frac{1}{p} \text{Tr} K \right\} + \frac{1}{p+q} \text{Tr} \left( H^{\frac{1}{2}} B^q H^{\frac{1}{2}} L^{1-q} \right) - \frac{1}{p+q} \text{Tr} L \right\}
\]
(3.29)

\[
= \frac{1}{p+q} g_H(A, B, K, L),
\]
with the unique maximizer being
\[
(\mathcal{K}, \mathcal{L}) = \left( \left( H^{\frac{1}{2}} A^p H^{\frac{1}{2}} \right)^{\frac{1}{2}}, \left( H^{\frac{1}{2}} B^q H^{\frac{1}{2}} \right)^{\frac{1}{2}} \right).
\]
In particular, for
\[
(H, A, B) = \left( H_0, \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\rho), \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\sigma) \right)
\]
the associated unique maximizer is \((K_0, L_0)\) given in (3.16), and for \( (H, A, B) = (H_0, V_j, W_j) \) the associated unique maximizer is \((K_j, L_j)\) given in (3.17).

Since \( 0 < p, q < 1 \), by Lieb’s concavity theorem (Lemma 2.5 (1)), \( g_H \) is jointly concave for any \( H \in \mathcal{B}(\mathcal{H})^{++} \). Then from Lemma 2.3, (3.28) and (3.26), both \( f_H \) and \( \Psi_{p,q} \) are jointly concave. By (3.13), which is equivalent to (1.7) in (1), and Lemma 2.2 (1) we have
\[
H_0 = H_j, \quad 1 \leq j \leq d^2,
\]
which proves (a), and
\[
f_{H_0} \left( \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\rho), \frac{1_{H \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\sigma) \right) = \frac{1}{d^2} \sum_{j=1}^{d^2} f_{H_0}(V_j, W_j).
\]
By (3.30) and Lemma 2.2 (2) we have
\[
(K_0, L_0) = \left( \frac{1}{d^2} \sum_{j=1}^{d^2} K_j, \frac{1}{d^2} \sum_{j=1}^{d^2} L_j \right),
\]
(3.31)
which proves (b) and (c).

If \( 0 < p < 1 \) and \( q = 1 \), then \( g_H = g_H(A, B, K, *) \) is independent of \( L \). The above argument still applies to \( H \) and \( K \), thus in this case one can still prove (a) and (b). Similarly, if \( p = 1 \) and \( 0 < q < 1 \), then \( g_H = g_H(A, B, *, L) \) is independent of \( K \). In this case one can still prove (a) and (c), since the above argument works well for \( H \) and \( L \).
**Case 2:** \((p, q)\) satisfies (3.25). The proof is similar to that of **Case 1**. For \(A, B \in \mathcal{B}(\mathcal{H})^{++}\), apply (2.10) in Lemma 2.3 to

\[
(r_0, r_1, r_2, X, Y) = \left( \frac{1}{p}, \frac{1}{p+q}, \frac{1}{-q}, A^\frac{p}{2}, B^\frac{q}{2} \right),
\]

and we get

\[
(3.32) \quad \Psi_{p,q}(A, B) = \max_{H \in \mathcal{B}(\mathcal{H})^{++}} f_H(A, B),
\]

with the unique maximizer being

\[
(3.33) \quad \mathcal{T} = A^{-\frac{p}{2}} \left( A^\frac{p}{2} B^q A^\frac{p}{2} \right)^{-\frac{q}{2}} A^{-\frac{p}{2}} A^\frac{p}{2} \left( B^q A^p B^q \right)^{-\frac{q}{2}} B^\frac{q}{2}.
\]

In particular, for

\[
(A, B) = \left( \frac{1_{\mathcal{H} \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\rho), \frac{1_{\mathcal{H} \otimes \mathcal{H}}}{d} \otimes \mathcal{E}(\sigma) \right)
\]

the associated unique maximizer is \(H_0\) given in (3.14), and for \((A, B) = (V_j, W_j)\) the associated unique maximizer is \(H_j\) given in (3.15).

If \(1 < p \leq 2\) and \(0 < q < 1\), we have by (2.13) and (2.14) in Lemma 2.3 that

\[
(3.34) \quad f_H(A, B) = \frac{p}{p+q} \min_{K, L \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{p} \mathrm{Tr} \left( H^\frac{p}{2} A^p H^\frac{p}{2} K^{1-p} \right) + \frac{p-1}{p} \mathrm{Tr}K \right\}
\]

\[
= \frac{-q}{p+q} \max_{K, L \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{-q} \mathrm{Tr} \left( H^\frac{q}{2} B^{-q} H^\frac{q}{2} L^{1+q} \right) - \frac{1+q}{-q} \mathrm{Tr}L \right\}
\]

\[
= \min_{K, L \in \mathcal{B}(\mathcal{H})^{++}} \left\{ \frac{1}{p+q} \mathrm{Tr} \left( H^\frac{p}{2} A^p H^\frac{p}{2} K^{1-p} \right) + \frac{p-1}{p+q} \mathrm{Tr}K \right\}
\]

\[
\quad = \min_{K, L \in \mathcal{B}(\mathcal{H})^{++}} h_H(A, B, K, L),
\]

with the unique minimizer being

\[
(3.35) \quad (K, L) = \left( \left( H^\frac{p}{2} A^p H^\frac{p}{2} \right)^{\frac{1}{p}}, \left( H^\frac{q}{2} B^{-q} H^\frac{q}{2} \right)^{\frac{1}{-q}} \right).
\]

In particular, for

\[
(H, A, B) = \left( H_0, \frac{1_{\mathcal{H} \otimes \mathcal{H}'}}{d} \otimes \mathcal{E}(\rho), \frac{1_{\mathcal{H} \otimes \mathcal{H}'}}{d} \otimes \mathcal{E}(\sigma) \right)
\]

the associated unique minimizer is \((K_0, L_0)\) given in (3.16), and for \((H, A, B) = (H_0, V_j, W_j)\) the associated unique minimizer is \((K_j, L_j)\) given in (3.17).

Since \(1 < p \leq 2\) and \(0 < -q < 1\), by Lieb’s concavity theorem (Lemma 2.5 (1)) and Ando’s convexity theorem (Lemma 2.5 (2)), \(h_H\) is jointly convex for any \(H \in \mathcal{B}(\mathcal{H})^{++}\). Then from Lemma 2.3, (3.28) and (3.26), both \(f_H\) and \(\Psi_{p,q}\) are jointly convex. Hence we can deduce (a) - (c) from (3.13), which is equivalent to (1.7) in (1), and Lemma 2.2 as we did in **Case 1**.

Again as in **Case 1**, we can use the same argument to prove (a) and (c) when \(p = 1\) and \(0 < -q < 1\), and prove (a) and (b) when \(1 < p \leq 2\) and \(-q = 1\).  \(\square\)
References

[AD15] K. M. R. Audenaert and N. Datta. $\alpha$-Rényi relative entropies. *J. Math. Phys.*, 56(2):022202, 16, 2015.

[And79] T. Ando. Concavity of certain maps on positive definite matrices and applications to Hadamard products. *Linear Algebra Appl.*, 26:203–241, 1979.

[CV18] E. A. Carlen and A. Vershynina. Recovery and the data processing inequality for quasi-entropies. *IEEE Trans. Inform. Theory*, 64(10):6929–6938, 2018.

[CV20a] E. A. Carlen and A. Vershynina. Recovery map stability for the data processing inequality. *Journal of Physics A: Mathematical and Theoretical*, 53(3):035204, 2020.

[CV20b] S. Chekade and A. Vershynina. Saturating the data processing inequality for $\alpha$–z rényi relative entropy. *arXiv:2006.07726*, 2020.

[FR15] O. Fawzi and R. Renner. Quantum conditional mutual information and approximate Markov chains. *Comm. Math. Phys.*, 340(2):575–611, 2015.

[HM17] F. Hiai and M. Mosonyi. Different quantum $f$-divergences and the reversibility of quantum operations. *Rev. Math. Phys.*, 29(7):1750023, 80, 2017.

[HMPB11] F. Hiai, M. Mosonyi, D. Petz, and C. Bény. Quantum $f$-divergences and error correction. *Rev. Math. Phys.*, 23(7):691–747, 2011.

[Jen17] A. Jenčová. Rényi relative entropies and noncommutative $L_p$-spaces II. *arXiv:1707.00047*, 2017.

[Jen18] A. Jenčová. Rényi relative entropies and noncommutative $L_p$-spaces. *Ann. Henri Poincaré*, 19(8):2513–2542, 2018.

[JRS+18] M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter. Universal recovery maps and approximate sufficiency of quantum relative entropy. *Ann. Henri Poincaré*, 19(10):2955–2978, 2018.

[Lie73] E. H. Lieb. Convex trace functions and the Wigner-Yanase-Dyson conjecture. *Adv. Math.*, 11:267–288, 1973.

[LRD17] F. Leditzky, C. Rouzé, and N. Datta. Data processing for the sandwiched Rényi divergence: a condition for equality. *Lett. Math. Phys.*, 107(1):61–80, 2017.

[MLDS+13] M. Müller-Lennert, F. Dupuis, O. Sæther, S. Fehr, and M. Tomamichel. On quantum Rényi entropies: a new generalization and some properties. *J. Math. Phys.*, 54(12):122203, 20, 2013.

[MO15] M. Mosonyi and T. Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. *Comm. Math. Phys.*, 334(3):1617–1648, 2015.

[NEE13] I. Nikoufar, A. Ebadian, and G. M. Eshaghi. The simplest proof of Lieb concavity theorem. *Adv. Math.*, 248:531–533, 2013.

[Pet86] D. Petz. Sufficient subalgebras and the relative entropy of states of a von Neumann algebra. *Comm. Math. Phys.*, 105(1):123–131, 1986.

[Pet88] D. Petz. Sufficiency of channels over von Neumann algebras. *The Quarterly Journal of Mathematics*, 39(1):97–108, 1988.

[Rényi61] A. Rényi. On measures of entropy and information. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, Vol. I, pages 547–561. Univ. California Press, Berkeley, Calif., 1961.

[SBT17] D. Sutter, M. Berta, and M. Tomamichel. Multivariate trace inequalities. *Comm. Math. Phys.*, 352(1):37–58, 2017.

[Sha14] N. Sharma. Equality conditions for quantum quasi-entropies under monotonicity and joint-convexity. In *2014 Twentieth National Conference on Communications (NCC)*, pages 1–6. IEEE, 2014.

[Sti55] W. F. Stinespring. Positive functions on $C^*$-algebras. *Proc. Amer. Math. Soc.*, 6:211–216, 1955.

[Sut18] D. Sutter. *Approximate quantum Markov chains*, volume 28 of *SpringerBriefs in Mathematical Physics*. Springer, Cham, 2018.

[Ume62] H. Umegaki. Conditional expectation in an operator algebra. IV. Entropy and information. *Kodai Math. Sem. Rep.*, 14:59–85, 1962.

[Wol12] M. M. Wolf. Quantum channels & operations: Guided tour. 2012.

[WWY14] M. M. Wilde, A. Winter, and D. Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Comm. Math. Phys.*, 331(2):593–622, 2014.

[WY63] E. P. Wigner and M. M. Yanase. Information contents of distributions. *Proc. Nat. Acad. Sci. U.S.A.*, 49:910–918, 1963.

[Zha20] H. Zhang. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture. *Adv. Math.*, 365:107053, 2020.