Recursive Construction for a Class of Radial Functions I — Ordinary Space

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A class of spherical functions is studied which can be viewed as the matrix generalization of Bessel functions. We derive a recursive structure for these functions. We show that they are only special cases of more general radial functions which also have a, properly generalized, recursive structure. Some explicit results are worked out.

I. INTRODUCTION

In 1957, Harish–Chandra derived a famous formula for a certain class of group integrals. Let $G$ be a compact semi–simple Lie group and let $a$ and $b$ be elements of its Cartan subalgebra $\mathcal{H}$, then

$$\int_{U \in G} d\mu(U) \exp(\text{tr} U^{-1}aUb) = \frac{1}{|W|} \sum_{w \in W} \exp(\text{tr} w(a)b) \Pi(w(a))\Pi(w(b)).$$

Here, $d\mu(U)$ is the invariant measure, $\Pi(a)$ is the product of all positive roots of $\mathcal{H}$ and $W$ is the Weyl reflection group of $G$ with $|W|$ elements $w$.

This result depends crucially on the condition that $a$ and $b$ are in the Cartan subalgebra $\mathcal{H}$. In other words, $U^{-1}aUb$ has to be in $G$. If one replaces $a$ and $b$ in the integral on the left hand side with more general matrices $x$ and $k$ which are not in $\mathcal{H}$, formula (1.1) is not valid anymore. The spherical functions introduced by Gelfand form an important class of such integrals which are, in general, not covered by Harish–Chandra’s result (1.1). In another work, Harish–Chandra studies in great detail the harmonic analysis involving these spherical functions. In a more physics oriented contribution, Olshanetsky and Perelomov discussed them in the framework of quantum integrable systems.

Here, we wish to address spherical functions of the following kind: we take $x$ and $k$ as diagonal matrices containing the eigenvalues of a Hamiltonian in a matrix representation. The Hamiltonian is diagonalized by the integration matrix $U$. In particular, we assume that the Hamiltonian $U^{-1}xU$ or, equivalently, $UkU^{-1}$ is real–symmetric, Hermitean or Hermitean self–dual. Thus, $G$ is the orthogonal, the unitary or the unitary–symplectic group. We will refer to these spherical functions as matrix Bessel functions. We notice that the unitary case is special: since it so happens that the eigenvalues $x$ and $k$ do lie in the Cartan subalgebra $\mathcal{H}$, the result (1.1) applies and coincides with the Itzykson–Zuber formula. In the orthogonal and the unitary–symplectic cases, however, formula (1.1) is not valid.

We choose the term matrix Bessel function for the spherical functions to be discussed here, because they can be viewed as a natural extension of the ordinary vector Bessel functions. However, due to the rich features of the spherical functions, other extensions relating to ordinary Bessel functions are equally natural. Related functions have been discussed and terms similar to matrix Bessel functions have already been used by Hertz and by Okounkov and Olshanski. Kontsevich introduced the matrix Airy functions.

Duistermaat and Heckman developed a stationary phase approach involving localization for a class of spherical functions, see also the treatise by Szabo.

Remarkably, our matrix Bessel functions are only special cases of more general objects which we call radial functions. Moreover, there is an important connection to the Calogero–Sutherland models which we will discuss separately, see below.

The matrix Bessel functions are of considerable interest for applications in physics. They appear in Random Matrix Theory which models spectral fluctuations of complex systems, such as quantum chaotic ones. In particular, they are the kernels of Dyson’s Brownian motion describing crossover transitions between different symmetry or invariance classes. Unfortunately, only the case of broken time–reversal invariance can be treated explicitly with the help of the Itzykson–Zuber formula. In the physically important cases of conserved time–reversal invariance, the kernels are not known analytically, as argued above. Muirhead discusses spherical functions in the framework of multivariate statistical theory. In his book, an expansion in terms of Jack polynomials for the orthogonal case can be found. Such an expansion for arbitrary Dyson index was recently worked out by Okounkov and Olshanski.

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The goal of the present paper is to explore the structure of the radial functions which contain the matrix Bessel functions as special cases. In particular, we show how explicit results can be obtained. The paper is organized as follows. In Sec. II, we briefly review some properties of the vector Bessel functions. In doing so we wish to help the reader in developing an intuition for the matrix Bessel functions which we introduce in Sec. III. In Sec. IV, we state and derive a fundamental recursive structure for matrix Bessel functions. We show in Sec. V that this recursion is an iterative solution of general radial functions which contain group integrals defining the matrix Bessel functions as special case. Secs. IV and V are our main results. In Sec. VI we illustrate how the recursion can lead to closed and explicit formulae. Because of its special importance, we discuss the connection to Calogero–Sutherland models separately in Sec. VII. In Sec. VIII, we summarize and conclude. Various aspects and calculations are collected in the appendix.

II. VECTOR BESSEL FUNCTIONS REVISITED

Before turning to the matrix case, we compile, for the convenience of the reader, some well known results for the vector case.

In a real, $d$ dimensional space with $d = 2, 3, 4, \ldots$, we consider a position vector $\vec{r} = (x_1, \ldots, x_d)$ and a wave vector $\vec{k} = (k_1, \ldots, k_d)$. The plane wave $\exp \left( i \vec{k} \cdot \vec{r} \right)$ satisfies the wave equation

$$\Delta \exp \left( i \vec{k} \cdot \vec{r} \right) = -\vec{k}^2 \exp \left( i \vec{k} \cdot \vec{r} \right)$$

(2.1)

where we define the Laplacean as in the physics literature,

$$\Delta = \frac{\partial^2}{\partial r^2} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$  

(2.2)

The zeroth order Bessel function in this space is the angular average of the plane wave,

$$\chi^{(d)}(kr) = \int d\Omega \exp \left( i \vec{k} \cdot \vec{r} \right),$$

(2.3)

over the solid angle $\Omega$, defining the orientation of either $\vec{r}$ or $\vec{k}$. In our context, it is advantageous to take $\Omega$ as the solid angle of $\vec{k}$. Obviously, only the relative angle between $\vec{r}$ and $\vec{k}$ matters and $\chi^{(d)}(kr)$ can only depend on the product of the lengths $r = |\vec{r}|$ and $k = |\vec{k}|$ of the two vectors. We normalize the measure $d\Omega$ with the volume $2\pi^{d/2}/\Gamma(d/2)$ of the unit sphere, i.e. we have

$$\int d\Omega = 1.$$  

(2.4)

Thus, by construction, we also have

$$\chi^{(d)}(0) = 1.$$  

(2.5)

It is convenient to view $\vec{r}$ as the azimuthal direction of the coordinate system in which we measure $\Omega$. Thus, in these spherical coordinates, one finds $\vec{k} \cdot \vec{r} = kr \cos \vartheta$ where $\vartheta$ is the azimuthal angle. The measure $d\Omega$ contains $\sin^{d-2} \vartheta$ and one has

$$\chi^{(d)}(kr) = \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \int_0^\pi \exp \left( i kr \cos \vartheta \right) \sin^{d-2} \vartheta d\vartheta$$

$$= 2^{(d-2)/2} \Gamma(d/2) \frac{J_{(d-2)/2}(kr)}{(kr)^{(d-2)/2}}$$

(2.6)

where $J_\nu(z)$ is the standard Bessel function of order $\nu$. The functions (2.6) are often referred to as zonal functions.

There is a remarkable difference for the functions $\chi^{(d)}(kr)$ if one compares even and odd dimensions. For example, one has in $d = 2$ dimensions $\chi^{(2)}(kr) = J_0(kr)$ and in $d = 3$ dimensions $\chi^{(3)}(kr) = (\pi/2)^{1/2}J_{1/2}(kr)/(kr)^{1/2} = j_0(kr)$ with the spherical Bessel function $j_0(z)$ of zeroth order. In $d = 2$ dimensions, $J_0(z)$ is a complicated infinite series
in the argument $z$, in $d = 3$ dimensions, however, $j_0(z)$ is the simple ratio $j_0(z) = \sin z/z$. One easily sees how this generalizes. Upon introducing $\xi = \cos \vartheta$ as integration variable in Eq. (2.8), one finds the representation

$$\chi^{(d)}(kr) = \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \int_{\xi=1}^{\xi=+1} \exp(ikr\xi) \left(1 - \xi^2\right)^{(d-3)/2} \frac{d\xi}{\sin kr}. \tag{2.7}$$

In dimensions $d \geq 3$, this can be cast into the form

$$\chi^{(d)}(kr) = \frac{2\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \sum_{\mu=0}^{\infty} \left(\frac{(d-3)/2}{\mu}\right) \frac{\partial^{2\mu}}{\partial(kr)^{2\mu}} \sin kr. \tag{2.8}$$

For even $d$, the exponent $(d-3)/2$ is a fraction $-1/2, +1/2, +3/2, \ldots$, and the function $(1 - \xi^2)^{(d-3)/2}$ in the integrand in Eq. (2.7) is an infinite power series. This yields, for $d = 4, 6, 8, \ldots$, the complicated power series (2.8) involving an infinite number of inverse powers of $kr$. However, if $d$ is odd, the exponent $(d-3)/2$ is an integer $0, 1, 2, \ldots$, and the function $(1 - \xi^2)^{(d-3)/2}$ is a finite polynomial of order $(d-3)/2$ in $\xi^2$. Thus, $\chi^{(d)}(kr)$ acquires a comparatively simple structure, because it only contains a finite number of inverse powers of $kr$. Formally, this means that for odd $d$ all binomial coefficients for $\mu > (d-3)/2$ are zero.

The differential equation for the functions $\chi^{(d)}(kr)$ is easily obtained by averaging Eq. (2.1) over the solid angle $\Omega$ of $\vec{k}$, i.e. by integrating both sides,

$$\Delta \int d\Omega \exp \left(ik \cdot \vec{r}\right) = -k^2 \int d\Omega \exp \left(ik \cdot \vec{r}\right). \tag{2.9}$$

We notice that the Laplacean $\Delta$ commutes with the integral, because the former is in the space of the position vector, the latter in the space of the wave vector. Moreover, the integral trivially commutes with $\vec{k}^2 = k^2$. Hence, one arrives at

$$\Delta_r \chi^{(d)}(kr) = -k^2 \chi^{(d)}(kr). \tag{2.10}$$

Since $\chi^{(d)}(kr)$ depends exclusively on radial variables, we replaced the full Laplacean $\Delta$ with its radial part

$$\Delta_r = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}. \tag{2.11}$$

In general, there are two fundamental solutions $\chi^{(d)}_+(kr)$ and $\chi^{(d)}_-(kr)$ of the differential equation (2.10) which behave as $\exp(\pm ikr)/(kr)^{(d-1)/2}$ for large arguments $kr$. Thus, to obtain the full solutions, one can make the Hankel ansatz

$$\chi^{(d)}(kr) = \frac{\exp(\pm ikr)}{(kr)^{(d-1)/2}} w^{(d)}_{\pm}(kr). \tag{2.12}$$

Here, $w^{(d)}_{\pm}(kr)$ is a function with the property $w^{(d)}_{\pm}(kr) \to 1$ for $kr \to \infty$. The differential equation follows easily from Eq. (2.11) and is given by

$$\left(\frac{\partial^2}{\partial r^2} \pm ik \frac{\partial}{\partial r} - \frac{d-1}{2} \left(\frac{d-1}{2} - 1\right) \frac{1}{r^2}\right) w^{(d)}_{\pm}(kr) = 0. \tag{2.13}$$

For $d \geq 3$, one uses the ansatz as an asymptotic power series

$$w^{(d)}_{\pm}(kr) = \sum_{\mu=0}^{\infty} \frac{a_\mu}{(\pm kr)^\mu} \tag{2.14}$$

which yields a recursion for the coefficients

$$a_{\mu+1} = \frac{1}{i2(\mu+1)} \left(\mu(\mu+1) - \frac{d-1}{2} \left(\frac{d-1}{2} - 1\right)\right) a_\mu, \tag{2.15}$$

with the starting value $a_0 = 1$. A special situation occurs when the integer running index $\mu$ reaches the critical value $\mu_c = (d-3)/2$. If $d$ is odd, $\mu_c$ is integer and the recursion terminates at $\mu = \mu_c$, i.e. one has $a_\mu = 0$, $\mu > \mu_c$. Thus, the asymptotic series becomes a finite polynomial in inverse powers of $kr$. However, if $d$ is even, $\mu_c$ is half-odd integer and the series cannot terminate, it is always infinite. This explains the different structure of the Bessel functions in even and odd dimensional spaces from the viewpoint of the differential equation.

In App. A we discuss an alternative integral representation which has an interesting analogue in the matrix space.
We compile the basics features of the matrix spaces we want to work with in Sec. IIIA, before we define the matrix Bessel functions as group integrals in Sec. IIIB.

Two general aspects are shifted into the appendix. First, we present an interesting alternative integral representation in App. E. Second, the matrix Bessel functions play a crucial role in harmonic analysis or, equivalently, in Fourier–Bessel analysis in matrix spaces. For the general theory, we refer the reader to Harish–Chandra’s treatise in Ref. [20] and to Helgason’s book [21]. However, to achieve our goal of being explicit, we collect, for the convenience of the reader, some results for the Fourier–Bessel analysis of invariant functions in matrix spaces in App. [22].

A. Basics and Notation

We introduce $N\times N$ matrices $H$ whose elements $H_{nm}$, $n, m = 1,\ldots,N$ are real, complex or quaternion variables. In other words, each element $H_{nm}$ has $\beta$ real components $H_{nm}^{(\alpha)}$, $\alpha = 0,\ldots, (\beta - 1)$ with $\beta = 1, 2, 4$, respectively,

$$H_{nm} = \sum_{\alpha=0}^{\beta-1} H_{nm}^{(\alpha)} \tau^{(\alpha)} .$$

(3.1)

Here, we use the basis $\tau^{(\alpha)}$, $\alpha = 0,\ldots, (\beta - 1)$. We have $\tau^{(0)} = 1$ for the real case with $\beta = 1$. For the complex case with $\beta = 2$, we have $\tau^{(0)} = 1$ and $\tau^{(1)} = i$. Finally, we have

$$\tau^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau^{(1)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ,$$

$$\tau^{(2)} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad \tau^{(3)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

(3.2)

in the quaternion case for $\beta = 4$ where the $\tau^{(\alpha)}$, $\alpha = 1, 2, 3$ are the Pauli matrices. We notice that the total $H$ is a $2N \times 2N$ matrix for $\beta = 4$. However, here and in the following, the dimensions that we use always refer to the number of matrix elements such as $H_{nm}$. These are scalar for $\beta = 1$ and quaternion for $\beta = 4$. The label $\beta$ is often referred to as Dyson index.

We assume that the matrix $H$ is real symmetric, Hermitean or Hermitean self–dual in the three cases $\beta = 1, 2, 4$. We always write $H^\dagger = H$ to indicate this symmetry. There are $N$ independent real variables $H_{nn} = H_{nn}^{(0)}$, $n = 1,\ldots,N$ on the diagonal and $\beta N(N - 1)/2$ independent real variables $H_{nm}^{(\alpha)}$, $\alpha = 0,\ldots, (\beta - 1)$, $1 \leq n < m \leq N$ outside the diagonal. We write the volume element of $H$ in the form

$$d[H] = \prod_{n=1}^{N} dH_{nn}^{(0)} \prod_{n<m} \prod_{\alpha=0}^{\beta-1} dH_{nm}^{(\alpha)}$$

(3.3)

The matrix $H$ is diagonalized by the matrix $U$, with columns $U_n$, $n = 1,\ldots,N$. Depending on the value of $\beta$, the matrix $U$ is either orthogonal, unitary or unitary–symplectic. Following Gilmore’s notation [23], we write $U \in U(N; \beta)$ with $U(N; 1) = SO(N)$, $U(N; 2) = U(N)$ and $U(N; 4) = USp(2N)$. The volume of these groups is given by

$$\text{vol}U(N; \beta) = \prod_{n=1}^{N} \frac{2\pi^{\beta n/2}}{\Gamma(\beta n/2)} = \frac{2^N \pi^{\beta N(N+1)/4}}{\prod_{n=1}^{N} \Gamma(\beta n/2)} .$$

(3.4)

We use it to normalize the invariant measure $d\mu(U)$ of $U \in U(N; \beta)$ to unity,

$$\int d\mu(U) = 1 .$$

(3.5)

The $N$ real eigenvalues $x_n$, $n = 1,\ldots,N$ of $H$ are ordered in the diagonal matrix $x$. We have $x = \text{diag} (x_1,\ldots,x_N)$ for $\beta = 1$ and $\beta = 2$. For $\beta = 4$, the eigenvalues are doubly degenerate and we have $x = \text{diag} (x_1, x_1,\ldots,x_N, x_N)$. Physically, this doubling of the eigenvalues is due to Kramer’s degeneracies. Thus, the diagonalization reads

$$H = U^\dagger x U , \quad \text{with} \quad H_{nm} = U^\dagger_{n} x U_{m} .$$

(3.6)
The diagonalizing matrix \( U \) has the property \( U^{-1} = U^\dagger \). The volume element in eigenvalue–angle coordinates is given by

\[
d\![H] = C_N^{(\beta)} |\Delta N(x)|^{\beta} d[x] d\mu(U)
\]

(3.7)

where \( d[x] \) denotes the product of all differentials \( dx_n \). We have introduced the Vandermonde determinant

\[
\Delta_N(x) = \prod_{n<m} (x_n - x_m) .
\]

(3.8)

The normalization constant

\[
C_N^{(\beta)} = \pi^{\beta N(N-1)/4} N! \prod_{n=1}^{N} \Gamma(\beta n/2)
\]

(3.9)

obtains from the constants given in Mehta’s book27 and from Eq. (3.4).

To avoid inconveniences and to ensure a compact notation, we define the trace \( \text{Tr} \) and the determinant \( \text{Det} \) with \( \text{Tr} = \text{tr} \) and \( \text{Det} = \text{det} \) for \( \beta = 1, 2 \) and with

\[
\text{Tr} K = \frac{1}{2} \text{tr} K \quad \text{and} \quad \text{Det} K = \sqrt{\text{det} K}
\]

(3.10)

in the case \( \beta = 4 \) for a matrix \( K \) with quaternion entries. If \( k \) denotes the diagonal matrix of the eigenvalues of a real–symmetric, Hermitian or Hermitian self–dual matrix, it is also useful to define the associate matrix \( \hat{k} \). In all three cases \( \beta \), it is the \( N \times N \) matrix \( \hat{k} = \text{diag} (k_1, k_2, \ldots, k_N) \), i.e. we have \( \hat{k} = k \) for \( \beta = 1, 2 \) and no degeneracies for \( \beta = 4 \).

**B. Integral Definition and Differential Equation**

As in the case of vector Bessel functions, we start in the matrix case with the plane wave. For two matrices \( H \) and \( K \) with the same symmetries \( H^\dagger = H \) and \( K^\dagger = K \), we introduce the matrix plane wave as \( \exp(i \text{Tr} HK) \) where the trace is the proper scalar product in the matrix space. The matrix plane wave has the property

\[
\frac{1}{(2\pi)^N \pi^{\beta N(N-1)/2}} \int d\![H] \exp(i \text{Tr} HK) = \delta(K)
\]

(3.11)

where \( \delta(K) \) is the product of the \( \delta \) distributions of all independent variables. We define the matrix gradient \( \partial/\partial H \) and the Laplacean operator

\[
\Delta = \text{Tr} \left( \frac{\partial^2}{\partial H^2} \right) = \sum_{n=1}^{N} \frac{\partial^2}{\partial H_{nn}^2} + \frac{1}{2} \sum_{n<m}^{\beta-1} \sum_{\alpha=0}^{\beta n/2} \frac{\partial^2}{\partial H_{nm}^2}
\]

(3.12)

which acts on the matrix plane wave as

\[
\Delta \exp(i \text{Tr} HK) = -\text{Tr} K^2 \exp(i \text{Tr} HK) .
\]

(3.13)

We notice that, for \( \beta = 4 \), inconvenient factors of two would occur if we used \( \text{tr} \) instead of \( \text{Tr} \).

Analogously to vector Bessel functions, we define the matrix Bessel functions as the angular average

\[
\Phi_N^{(\beta)}(x, k) = \int d\mu(U) \exp(i \text{Tr} HK) .
\]

(3.14)

The diagonal matrix \( k \) contains the eigenvalues of \( K \) which is diagonalized by a matrix \( V \) such that \( K = V^\dagger k V \). Due to the invariance of the measure \( d\mu(U) \), the matrix \( V \) is absorbed and the functions \( \Phi_N^{(\beta)}(x, k) \) depend on the eigenvalues \( x \) and \( k \) only,

\[
\Phi_N^{(\beta)}(x, k) = \int d\mu(U) \exp(i \text{Tr} U^\dagger x U k) .
\]

(3.15)
Thus, in the scalar product $\text{Tr} HK$, solely the relative angles between $H$ and $K$ matter. The matrix Bessel functions are symmetric in the arguments,

$$
\Phi_N^{(\beta)}(x, k) = \Phi_N^{(\beta)}(k, x)
$$

(3.16)

and normalized to unity,

$$
\Phi_N^{(\beta)}(x, 0) = 1 \quad \text{and} \quad \Phi_N^{(\beta)}(0, k) = 1.
$$

(3.17)

due to Eq. (3.5). These are spherical functions in the sense of Ref.

As in the vector case, the differential equation is obtained by averaging Eq. (3.13) over the relative angles,

$$
\Delta \int d\mu(V) \exp(i\text{Tr} HK) = -\text{Tr} K^2 \int d\mu(V) \exp(i\text{Tr} HK) .
$$

(3.18)

Again, the Laplacean $\Delta$ commutes with the integral, because the former is in the space of the matrix $H$, the latter over the diagonalizing matrix $V$ of $K$. The integral also commutes with $\text{Tr} K^2 = \text{Tr} k^2$. Due to the symmetry between $H$ and $K$, the integral is obviously identical to the definition (3.13) and we find

$$
\Delta x \Phi_N^{(\beta)}(x, k) = -\text{Tr} k^2 \Phi_N^{(\beta)}(x, k).
$$

(3.19)

Since the matrix Bessel function $\Phi_N^{(\beta)}(x, k)$ depends only on the relative variables, i.e. on the eigenvalues, we replaced the full Laplacean with its radial part $\Delta x$. Because of the transformation rule (3.7), it reads

$$
\Delta x = \sum_{n=1}^{N} \frac{1}{\Delta N(x)^{2\beta}} \frac{\partial}{\partial x_n} \Delta N(x)^{\beta} \frac{\partial}{\partial x_n}
$$

$$
= \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \sum_{n<m}^{N} \beta \left( \frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right) (3.20)
$$

We notice that these steps are fully parallel to the corresponding discussion in Sec. I. Importantly, due to the symmetry (3.16), the functions $\Phi_N^{(\beta)}(x, k)$ must also solve the differential equation in the $k_n$, $n = 1, \ldots, N$ which results from Eq. (3.19) by exchanging $x$ and $k$. Obviously, this a very restrictive requirement.

Comparing the radial operator (3.14) in the vector case and the radial operator (3.20), we see that it is the $\beta$ that corresponds to the spatial dimension $d$ or, more precisely, to $d - 1$. The role played by the matrix dimension $N$ is a different one. To illustrate this, we study the two simplest cases. First, we can formally set $N = 1$ and find from the definition (3.14) that $\Phi_1^{(\beta)}(x, k) = \exp(i x_1 k_1)$ where $H_{11} = x_1$ and $K_{11} = k_1$. In this case, the matrix $U$ has dropped out trivially. This reflects simply that the scalar product $\vec{k} \cdot \vec{r}$ is linear in the relative solid angle $\Omega$ between the vectors whereas the scalar product $\text{Tr} HK$ is quadratic in the relative diagonalizing matrix $U$. The corresponding radial Laplacean $\Delta x$ for $N = 1$ is identical to the Cartesian $\Delta$. Therefore, the case $N = 1$ is too trivial to give any further insight. Second, we set $N = 2$ and find straightforwardly from the differential equation (3.19)

$$
\Phi_2^{(\beta)}(x, k) = \exp\left(\frac{(x_1 + x_2)(k_1 + k_2)}{2}\right) \chi^{(\beta+1)} \left(\frac{(x_1 - x_2)(k_1 - k_2)}{2}\right) (3.21)
$$

where $\chi^{(d)}$ is the vector Bessel function in $d$ dimensions as defined in Eq. (2.34). This functions appears in the solution, because the differences $x_1 - x_2$ and $k_1 - k_2$ directly correspond to the lengths $|\vec{r}|$ and $|\vec{k}|$. In higher matrix dimensions $N$, this simple correspondence is lost. However, we will see in great detail that the features of the functions $\Phi_N^{(\beta)}(x, k)$, in particular whether or not explicit solutions can be constructed, are stronger influenced by $\beta$ than by $N$.

Another point in this context deserves to be underlined. In the vector case, the differential equation (2.10) and the solution (2.11) were constructed for integer dimensions $d$. However, both equations are also well defined for any real and positive $d$. Similarly, we observe in the matrix case that the differential equation (3.19) was derived for the cases $\beta = 1, 2, 4$. However, neither itself nor its solution (3.21) for $N = 2$ are confined to these cases $\beta = 1, 2, 4$, they are valid for any real and positive $\beta$. Thus, the cases $\beta = 1, 2, 4$ which correspond to a matrix model, i.e. to the defining integral (3.14) of the matrix Bessel functions, are only special cases of a much more general problem, namely finding the solutions of the differential equation (3.19) for every integer $N$ and for arbitrary real values of $\beta$. We will return to this in Sec. V.
The normalization constant obtains from results in Gilmore’s book. This remarkable feature is the main result of this section. We emphasize that the radial spaces do not lie in the manifolds covered by the groups $U(N; \beta)$. However, we will show that the group integral (4.1) can be exactly mapped onto a recursive structure which acts exclusively in the radial space. This remarkable feature is the main result of this section.

Under rather general circumstances, the matrix Bessel functions $\Phi_N^{(\beta)}(x, k)$ can be calculated iteratively by the explicit recursion formula

$$\Phi_N^{(\beta)}(x, k) = \int d\mu(x', k) \exp(i(x - x't)) \Phi_{N-1}^{(\beta)}(x', k)$$

where $\Phi_{N-1}^{(\beta)}(x', k)$ is the group integral (4.1) over $U(N - 1; \beta)$. We have introduced the diagonal matrix $k = \text{diag}(k_1, \ldots, k_{N-1})$ for $\beta = 1, 2$ and $k = \text{diag}(k_1, k_{N-1})$ for $\beta = 4$. Importantly, the $N - 1$ integration variables $x'_n$, $n = 1, \ldots, N - 1$, ordered in the diagonal matrix $x' = \text{diag}(x'_1, \ldots, x'_{N-1})$ for $\beta = 1, 2$ and $x' = \text{diag}(x'_1, x'_2, \ldots, x'_{N-1})$ for $\beta = 4$ are arguments of $\Phi_N^{(\beta)}(x', k)$. Moreover, we notice that their further appearance in the exponential is a simple one due to the trace.

The coordinates $x'$ are constructed in the spirit of, but they are different from, the Gelfand–Tzetlin coordinates of Refs. 11 and 12. To clearly distinguish these two sets of coordinates from each other, we refer to the latter as angular Gelfand–Tzetlin coordinates and to the variables $x'$ as radial Gelfand–Tzetlin coordinates. The difference is at first sight minor, but of crucial importance. In the angular case, $x$ is in the Cartan subalgebra belonging to $U(N; \beta)$. In the radial case, however, $x$ is in the radial space of the eigenvalues of the real–symmetric, Hermitean or Hermitean self–dual matrix $H$, which are the arguments of the functions (4.1). While the angular Gelfand–Tzetlin coordinates never leave the group space, the radial ones establish an exact and unique relation between the group and the radial space. The radial Gelfand–Tzetlin coordinates re–parametrize the sphere that is described by the $N^{th}$ column $U_N$ of the matrix $U \in U(N; \beta)$. The recursion formula (4.2) can only be constructed in the radial coordinates $x'$, but not in the angular ones. The radial and the angular Gelfand–Tzetlin coordinates are, in general, different. They happen to coincide for $\beta = 2$, i.e. for the unitary group $U(N)$. This illustrates, in the framework of our recursion formula, the special rôle played by the unitary group.

The invariant measure $d\mu(x', x)$ is, apart from phase angles, the invariant measure $d\mu(U_N)$ on the sphere in question, expressed in the radial coordinates $x'$. It only contains algebraic functions and reads explicitly

$$d\mu(x', x) = \frac{2^{N-1} \Gamma(N/2)}{\pi^{N/2}(\beta-2)(\beta-4)/6} \frac{\Delta_{N-1}(x') \Delta_{N-1}(x)}{\Delta_N(x)} \left( - \prod_{n,m} (x_n - x'_m) \right)^{(\beta-2)/2} d[x'].$$

The normalization constant obtains from results in Gilmore’s book. It ensures normalization to unity according to Eq. (3.3). The domain of integration is compact and given by

$$x_n \leq x'_n \leq x_{n+1} \quad , \quad n = 1, \ldots, (N - 1),$$

reflecting a “betweenness condition” for the radial Gelfand–Tzetlin coordinates. This is why no absolute value signs appear in the measure (4.3).
The general recursion formula (4.2) states an iterative way for constructing the matrix Bessel function $\Phi_N^{(2)}(x, k)$ for arbitrary $N$ from the matrix Bessel function $\Phi_2^{(2)}(x, k)$ for $N = 2$ which can usually be obtained trivially. We remark that the recursion formula allows one to express the matrix Bessel functions in the form

$$
\Phi_N^{(2)}(x, k) = \int \prod_{n=1}^{N-1} d\mu(x^{(n)}, x^{(n-1)})
\exp \left( i(\text{Tr} x^{(n-1)} - \text{Tr} x^{(n)})k_{N-n+1} \right)
\exp \left( i|_{1}^{(N-1)-1}k_{1} \right)
$$

(4.5)

where we have introduced the radial Gelfand–Tzetlin coordinates $x^{(n)}_m$, $m = 1, \ldots, N - n$ on $N - 1$ levels $n = 1, \ldots, (N - 1)$. We define $x^{(0)} = x$ and $x^{(1)} = x'$.

B. Derivation

We introduce a matrix $V = \text{diag}(\vec{V}, V_0)$ with $\vec{V} \in U(N-1; \beta)$ and $V_0 \in U(1; \beta)$ such that $V \in U(N-1; \beta) \otimes U(1; \beta) \subset U(N; \beta)$ and multiply the right hand side of the definition (4.1) with

$$
1 = \int d\mu(V) = \int d\mu(V_0) \int d\mu(\vec{V})
$$

(4.6)

The invariance of the Haar measure $d\mu(U)$ allows us to replace $U$ with $UV^\dagger$ and to write

$$
\Phi_N^{(2)}(x, k) = \int d\mu(V_0) \int d\mu(\vec{V}) \int d\mu(U) \exp(i\text{Tr} U^\dagger xU V^\dagger kV)
$$

(4.7)

We collect the first $N - 1$ columns $U_n$ of $U$ in the $N \times (N - 1)$ rectangular matrix $B$ such that $B = [U_1 U_2 \cdots U_{N-1}]$ and $U = [BU_N]$. We notice that

$$
B^\dagger B = 1_{N-1} \\
BB^\dagger = \sum_{n=1}^{N-1} U_n U_n^\dagger = 1_N - U_N U_N^\dagger
$$

(4.8)

As already stated in Sec. III A, the elements of a vector or a matrix are scalar for $\beta = 1, 2$ and quaternion for $\beta = 4$. In this sense, we also write $1_N$ as the unit matrix for $\beta = 4$ because its elements are $\tau^{(0)}$. By defining the $(N - 1) \times (N - 1)$ square matrices $\vec{H} = B^\dagger xB$ and $\vec{K} = \vec{V}^\dagger k\vec{V}$ we may rewrite the trace in Eq. (4.7) as

$$
\text{Tr} U^\dagger xU V^\dagger kV = \text{Tr} \vec{H} \vec{K} + H_{NN}k_N
$$

(4.9)

with $H_{NN} = U_N^\dagger xU_N$ according to Eq. (3.4). We notice that $V_0$ has dropped out. Since the first term of the right hand side of Eq. (4.9) depends only on the first $N - 1$ columns $U_n$ collected in $B$ and the second term depends only on $U_N$, we use the decomposition

$$
d\mu(U) = d\mu(B) d\mu(U_N)
$$

(4.10)

of the measure to cast Eq. (4.7) into the form

$$
\Phi_N^{(2)}(x, k) = \int d\mu(U_N) \exp(iH_{NN}k_N) \int d\mu(\vec{V}) \int d\mu(B) \exp(i\text{Tr} \vec{H} \vec{K})
$$

(4.11)

where we have already done the trivial integration over $V_0$.

The difficulty to overcome lies in the decomposition (4.10). While $d\mu(U_N)$ is simply the invariant measure on the sphere described by $U_N$, the measure $d\mu(B)$ is rather complicated. Pictorially speaking, the degrees of freedom in $d\mu(B)$ have always to know that they are locally orthogonal to $U_N$. Thus, $d\mu(B)$ depends on $U_N$. Luckily, there is one distinct set of coordinates that is perfectly suited to this situation. It is the system of the radial Gelfand–Tzetlin coordinates. We construct it by transferring the methods of Ref. 13 for the angular case to the radial case.
The $N \times N$ matrix $(1 - U_N U_N^\dagger)$ is a projector onto the $(N-1) \times (N-1)$ space obtained from the original $N \times N$ space by slicing off the vector $U_N$. We project the radial coordinates $x$ onto this space and study its spectrum. The defining equation reads

\[(1 - U_N U_N^\dagger) x (1 - U_N U_N^\dagger) E'_n = x'_n E'_n \quad n = 1, \ldots, N-1. \tag{4.12}\]

Equation (4.12) determines the $N-1$ radial Gelfand–Tzetlin coordinates $x'_n$ and the corresponding vectors $E'_n$ as eigenvalues and eigenvectors of the matrix $(1 - U_N U_N^\dagger) x (1 - U_N U_N^\dagger)$ which has the rank $N-1$. Since we have by construction $U_N^\dagger E'_n = 0$, we may as well write

\[(1 - U_N U_N^\dagger) x E'_n = x'_n E'_n, \quad n = 1, \ldots, N-1. \tag{4.13}\]

The eigenvalues $x'_n$, $n = 1, \ldots, N$ are obtained from the characteristic equation

\[0 = \text{Det} \left( (1 - U_N U_N^\dagger) x - x'_n \right) = \text{Det} \left( x - x'_n \right) \text{det} \left( 1 - (x - x'_n)^{-1} U_N U_N^\dagger x \right)\]

\[= \text{Det} \left( x - x'_n \right) \left( 1 - U_N^\dagger \frac{x}{x - x'_n} U_N \right)\]

\[= -x'_n \text{Det} \left( x - x'_n \right) \text{Tr} U_N^\dagger \frac{1}{x - x_n} U_N. \tag{4.14}\]

Together with the normalization $\text{Tr} U_N^\dagger U_N = 1$, this yields the $N$ equations

\[1 = \text{Tr} U_N^\dagger U_N = \sum_{n=1}^{N} \sum_{\alpha=0}^{\beta-1} U_{nN}^{(\alpha)2}\]

\[0 = \text{Tr} U_N^\dagger \frac{1}{x - x'_n} U_N = \sum_{m=1}^{N} \sum_{\alpha=0}^{\beta-1} U_{mN}^{(\alpha)2} \frac{1}{x_m - x'_n}, \quad n = 1, \ldots, N-1. \tag{4.15}\]

In these formulae, the trace $\text{Tr}$ is only needed in the symplectic case. We notice that the equations for the variables $x'$ depend on the variables $x$ as parameters. We emphasize once more that $x$ in these equations is in the radial space and, in general, not in the Cartan subalgebra of $U(N; \beta)$.

At this point, it is not clear yet why the introduction of the radial Gelfand–Tzetlin coordinates is at all helpful. The great advantage will reveal itself when we express the matrix $\widetilde{H}$ and the matrix element $H_{NN}$ in the trace (4.11) in these coordinates. To this end, we first multiply Eq. (4.12) from the right with $E'_n$ and sum over $n$,

\[(1 - U_N U_N^\dagger) x (1 - U_N U_N^\dagger) = \sum_{n=1}^{N-1} x'_n E'_n E'_n \tag{4.16}\]

where we used the completeness relation

\[\sum_{n=1}^{N-1} E'_n E'_n + U_N U_N^\dagger = 1_N. \tag{4.17}\]

Taking the trace of the spectral expansion (4.16) we find immediately

\[\text{Tr} x - \text{Tr} x' = \text{Tr} U_N^\dagger x U_N = H_{NN} \tag{4.18}\]

This is a remarkably simple result. An analogous expression exists for the $NN$ matrix element of the unitary group in the theory of angular Gelfand–Tzetlin coordinates for the unitary group. Here we have shown that Eq. (4.18) is a general feature in every radial space.

We now turn to the $(N-1) \times (N-1)$ matrix $\widetilde{H}$. Its $N-1$ eigenvalues $y_n$, $n = 1, \ldots, N-1$ are determined by the characteristic equation.
0 = \det \left( \tilde{H} - y_n \right) = \det \left( B^\dagger x B - y_n \right) = -\frac{1}{y_n} \det \left( B B^\dagger x - y_n \right) = -\frac{1}{y_n} \det \left( (1_N - U_N U_N^\dagger) x - y_n \right) \quad (4.19)

where we used Eq. (4.8) and re-expressed a \((N - 1) \times (N - 1)\) determinant as a \(N \times N\) determinant. The comparison of Eq. (4.19) with Eq. (4.14) shows that, most advantageously, we have \(y_n \equiv x'_n, \ n = 1, \ldots, N - 1\). Thus we may write

\[
\tilde{H} = B^\dagger x B = \tilde{U}^\dagger x' \tilde{U} \quad (4.20)
\]

by introducing the \((N - 1) \times (N - 1)\) square matrix \(\tilde{U}\) which diagonalizes \(\tilde{H}\). Obviously, \(\tilde{U}\) must be a complicated function of the \(N \times (N - 1)\) rectangular matrix \(B\), i.e. of the columns \(U_n, \ n = 1, \ldots, N - 1\). However, all we need to know is that \(\tilde{U}\) must be in the group \(U(N - 1; \beta)\) because, by construction, \(\tilde{H}\) has the symmetry \(\tilde{H}^\dagger = \tilde{H}\).

Collecting everything, we cast Eq. (4.11) into the form

\[
\Phi_N^{(\beta)} (x, k) = \int d\mu(x', x) \exp \left( i(Tr x - Tr x') k_N \right) \int d\mu(V) \int d\mu(B) \exp (iTr \tilde{U}^\dagger x' \tilde{U} V^\dagger k V) . \quad (4.21)
\]

We may now use the invariance of the Haar measure \(d\mu(V)\) to absorb \(\tilde{U}\) such that

\[
\Phi_N^{(\beta)} (x, k) = \int d\mu(x', x) \exp \left( i(Tr x - Tr x') k_N \right) \int d\mu(V) \exp (iTr x' V^\dagger k V) \int d\mu(B) . \quad (4.22)
\]

Thus, the integration over \(B\) is trivial and yields unity due to our normalization. The remaining integration over \(V\) gives precisely the matrix Bessel function \(\Phi_N^{(\beta)} (x', \tilde{k})\). This completes the derivation of the recursion formula in Sec. IV A. The reader experienced with group integration has realized that the introduction of the matrix \(V = \text{diag}(\tilde{V}, \tilde{V}_0)\) was not strictly necessary. Alternatively, one could have shown that the measure \(d\mu(B)\) can be identified with \(d\mu(U)\) and have done the corresponding integral. However, we believe that the introduction of \(V\) makes this part of the derivation more transparent.

C. Invariant Measure

The invariant measure \(d\mu(U_N)\) has to be expressed in terms of the radial coordinates \(x'\). To this end, we first have to solve Eq. (4.13) for the moduli squared of the vector \(U_N\) as a function of the new coordinates \(x'\). Since Eq. (4.13) for the total moduli square for all \(\beta\) coincides with the equation for the angular Gel'fand–Tzetlin coordinates of the unitary group, we can use the results as derived in [42]. We have in the three cases

\[
[U_N]_n^2 = \sum_{\alpha=0}^{\beta-1} (U^{(\alpha)}_n N)^2 = \prod_{m=1}^{N-1} (x_n - x'_m) \prod_{m \neq n} (x_n - x_m) . \quad (4.23)
\]

The betweenness condition (4.4) follows from the positive definiteness of this expression. We parametrize the remaining degrees of freedom of \(U_n N\) in the cases \(\beta = 2, 4\). We set \(U_{nN}^{(0)} = \cos \gamma_n\) and \(U_{nN}^{(1)} = \sin \gamma_n\) in the case \(\beta = 2\) and

\[
U_{nN} = \begin{bmatrix}
\cos \psi_n \exp(i\gamma_n^{(1)}) & \sin \psi_n \exp(i\gamma_n^{(2)}) \\
-\sin \psi_n \exp(-i\gamma_n^{(2)}) & \cos \psi_n \exp(-i\gamma_n^{(1)})
\end{bmatrix} \quad (4.24)
\]

for \(\beta = 4\) in the basis (3.2). The invariant length element reads
\[ \text{Tr} \, dU_N^\dagger dU_N = \sum_{n=1}^N \sum_{\alpha=0}^{\beta-1} (dU_n^{(\alpha)})^2 \]
\[ = \sum_{n=1}^N \left( \frac{1}{4|U_{nN}|^2} (d|U_{nN}|^2)^2 + \sum_{i=1}^{\beta/2} |U_{nN}|^2 (d\gamma_n^{(i)})^2 + \delta_{\beta4}|U_{nN}|^2 (d\cos \phi_n)^2 \right). \]  

(4.25)

To express the differential \(d|U_{nN}|^2\) in terms of the \(dx_n'\), we again take advantage of the results in Refs. [9,12]
\[ \sum_{n=1}^N \frac{1}{4|U_{nN}|^2} (d|U_{nN}|^2)^2 = \sum_{n=1}^{N-1} \prod_{m=1}^{N-1} (x_m' - x_m')(dx_n')^2. \]  

(4.26)

From these equations, we can read off the metric \(g\) in the basis of the coordinates \(x_n'\), \(\gamma_n^{(i)}\) and \(\psi_n\). Conveniently, it is diagonal. The determinant of \(g\) is given by
\[ \det g = \frac{\Delta^{N-1}_{N-2}(x')}{\Delta^N_{N-2}(x)} \prod_{n,m} (x_n' - x_m')^{(\beta-2)}, \]  

(4.27)

which yields the invariant measure \(dp(U_N)\) in terms of the \(x_n'\) and of the additional coordinates \(\gamma_n^{(j)}\) and \(\psi_n\). These angles can be integrated out trivially. This yields Eq. (4.3).

V. RADIAL FUNCTIONS FOR ARBITRARY \(\beta\)

Remarkably, the recursion introduced in the previous section, is the iterative solution of the radial equation for arbitrary values of \(\beta\). Thus, the matrix Bessel functions are special cases of more general functions which we want to refer to as radial functions. We give the precise formulation of the problem in Sec. V A and show in Sec. V B that the recursion is the general iterative solution. In Sec. V C, we discuss a Hankel ansatz for the radial functions.

A. Definition by the Differential Equation

In Sec. III B, we defined the matrix Bessel function through the group integral (3.14) or, equivalently, the group integral (3.15). This definition confines the dimension \(\beta\) to the values \(\beta = 1, 2, 4\), corresponding to the groups \(U(N; \beta)\). However, discussing the simplest case \(N = 2\), we already saw in Sec. III B that \(\Phi_2^{(\beta)}\) is well defined for arbitrary values of \(\beta\). This was a simple consequence of the explicit form (3.21) which expresses \(\Phi_2^{(\beta)}\) in terms of the Bessel function \(\chi^{(\beta+1)}\). The latter is known to be well defined for arbitrary \(\beta\). Hence, we conclude that the cases \(\beta = 1, 2, 4\) which relate to matrices and groups are embedded into a space of far more general functions.

It seems natural that this phenomenon also extends to \(N > 2\). The problem has to be posed as follows: We seek the solutions \(\Phi_N^{(\beta)}\) of the differential equation
\[ \Delta_x \Phi_N^{(\beta)}(x, k) = -\sum_{n=1}^N k_n^2 \Phi_N^{(\beta)}(x, k) \]  

(5.1)

where the operator is given by
\[ \Delta_x = \sum_{n=1}^N \frac{\partial^2}{\partial x_n^2} + \sum_{n<m} \frac{\beta}{x_n - x_m} \left( \frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right). \]  

(5.2)

Here, \(\beta\) is arbitrary. For technical reasons, however, we restrict ourselves for the time being to real and positive values of \(\beta\). We make no reference whatsoever to matrices, eigenvalues and groups. To emphasize this, we view \(x\) and \(k\) as sets of \(N\) variables \(x_n, n = 1, \ldots, N\) and \(k_n, n = 1, \ldots, N\) for every positive \(\beta\). We do not use traces.

We require that the solutions are symmetric in the argument.
\[ \Phi_N^{(\beta)}(x, k) = \Phi_N^{(\beta)}(k, x) \]  

(5.3)

and normalized

\[ \Phi_N^{(\beta)}(0, k) = 1 \quad \text{and} \quad \Phi_N^{(\beta)}(x, 0) = 1 \]  

(5.4)

at the origin \( x = 0 \) and \( k = 0 \).

In the sequel, we want to refer to the functions \( \Phi_N^{(\beta)}(x, k) \) for arbitrary \( \beta \) as radial functions while we reserve the term matrix Bessel functions to the cases \( \beta = 1, 2, 4 \) where the direct connection to matrices and Lie groups exists.

### B. Recursive Solution

We claim that the solutions are, for arbitrary \( \beta \), given as an iteration in \( N \) by the recursion formula

\[ \Phi_N^{(\beta)}(x, k) = \int d\mu(x', x) \exp \left( i \left( \sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x' \right) k_N \right) \Phi_{N-1}^{(\beta)}(x', \tilde{k}) \]  

(5.5)

where \( \Phi_{N-1}^{(\beta)}(x', \tilde{k}) \) is the solution of the differential equation (5.2) for \( N - 1 \). Here, \( \tilde{k} \) denotes the set of variables \( k_n, n = 1, \ldots, (N - 1) \) and \( x' \) the set of integration variables \( x_n, n = 1, \ldots, (N - 1) \). The integration measure

\[ d\mu(x', x) = G_N^{(\beta)} \frac{\Delta_{N-1}(x')}{\Delta_{N-1}^{(\beta)}(x)} \left( -\prod_{n=m}^{N} (x_n - x'_m) \right)^{(\beta - 2)/2} d[x'] \]  

(5.6)

is the continuation of Eq. (4.3) to arbitrary positive \( \beta \). The normalization constant

\[ G_N^{(\beta)} = 2^{N-1} \frac{\Gamma(N\beta/2)}{\Gamma(N/2)} \]  

(5.7)

is also the continuation of the constant in Eq. (4.3). We calculate it in App. B. As in the cases \( \beta = 1, 2, 4 \), the inequalities

\[ x_n \leq x'_n \leq x_{n+1}, \quad n = 1, \ldots, (N - 1) \]  

(5.8)

define the domain of integration.

We stress that we derived the recursion formula (5.3) in Sec. IV for the cases \( \beta = 1, 2, 4 \). To prove that it is the iterative solution for arbitrary positive \( \beta \), we show that it solves the differential equation (5.1). The keystone for the proof is the identity

\[ \Delta_k \Phi_N^{(\beta)}(x, k) = -k_N^2 \Phi_N^{(\beta)}(x, k) \]

\[ + \int d\mu(x', x) \exp \left( i \left( \sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x' \right) k_N \right) \Delta_{x'} \Phi_{N-1}^{(\beta)}(x', \tilde{k}) \]  

(5.9)

which is derived in App. B. Equation (5.3) establishes a not immediately obvious, but nevertheless natural connection between, on one hand, the action of the Laplacean \( \Delta_k \) in the \( N \) variables \( x_n \) on the radial function in \( N \) dimensions, i.e. on the recursion integral (5.3), and, on the other hand, the recursion integral over the Laplacean \( \Delta_{x'} \) in the \( N - 1 \) variables \( x' \) acting on the radial function in \( N - 1 \) dimensions. There is a compensation term which is just \(-k_N^2 \Phi_N^{(\beta)}(x, k)\). Thus, we can prove the eigenvalue equation (5.1) by induction: assuming that it is correct for \( N - 1 \), identity (5.9) implies Eq. (5.1) for \( N \). The induction starts with \( N = 2 \) where the eigenvalue equation (5.1) is clearly valid for arbitrary \( \beta \) as shown in Sec. II B by deriving the explicit solution (3.21).

The symmetry relation (5.3) is non-trivial. In the matrix cases \( \beta = 1, 2, 4 \), it is obvious from the integral definitions (3.14) and (8.13). For arbitrary \( \beta \), we cannot use this argument, we only have the recursion (5.3). In App. B, we prove the symmetry relation (5.3) by an explicit change of variables.

The normalization \( \Phi_N^{(\beta)}(x, 0) = 1 \) in Eq. (5.4) follows directly from the normalization of the measure (5.6). The symmetry relation (5.3) then also yields \( \Phi_N^{(\beta)}(0, k) = 1 \).
Regarding the domain of $\beta$, a comment is in order. We have seen in Sec. that for $N = 2$ the matrix Bessel function is well defined for arbitrary complex $\beta$. This should also be true for our recursion formula \(5.3\). However, for $\beta \leq 0$ non-integrable singularities arise at the boundaries in the integral in Eq. \(5.3\). At the same time the normalization constant becomes zero for $\beta = 0, -2, -4, \ldots$ compensating the singularities of the integral. This makes the recursion formula for $\beta \leq 0$ not ill-defined but it gets more difficult to treat. Therefore, we have restricted ourselves to positive values of $\beta$.

In the work of Okounkov and Olshanski\(30\) an expansion of the radial functions for arbitrary $\beta$ in Jack polynomials is derived. The series run over sets of partitions $\{\mu\}$. These authors also derive a recursion formula for the Jack polynomials depending on one set of continuous variables $x$, say, and belonging to such partitions $\{\mu\}$. It is related to, but different from ours which involves two sets of continuous variables $x$ and $k$. The crucial difference rests in the exponential function which is present in our formula \(5.3\), but not in the formula of Ref.\(30\). Importantly, it is this exponential term which makes sure that the symmetry condition \(5.3\) is fulfilled on all levels of the recursion. Since the Jack polynomials themselves do not obey such a symmetry condition, there is no exponential term in the recursion formula of Ref.\(30\). However, it must be possible to derive the recursion formula for the radial functions from the one for the Jack polynomials. An interesting, although probably not very elegant approach would be the following: If one inserted the recursion formula for the Jack polynomials into the expansion \(5.3\) of the radial functions in terms of these Jack polynomials, one ought to see that the series over the partitions can, at least partly, be resummed to yield the exponential function present in the recursion formula \(5.3\). This is remarkable and could be very helpful for the application of Jack polynomials, because, in general, resummations over partitions are known to be difficult and involved. For the connection to Calogero–Sutherland models, we refer the reader to Sec. \(VII\).

C. Hankel Ansatz

In the spirit of Eq. \(2.12\) for the vector case, we make a Hankel ansatz for our radial functions for arbitrary positive $\beta$. We also do this in view of the applications in Sec. \(VI\). Since the sum over the $k_n^2$ on the right hand side of the eigenvalue equation \(5.1\) is invariant under all permutations of the $k_n$ or, equivalently, their indices $n$, we can label a set of solutions $\Phi^{(\beta)}_{N,\omega}(x, k)$ by an element $\omega$ of the permutation group $S_N$ of $N$ objects. For these solutions, we make the ansatz

$$
\Phi^{(\beta)}_{N,\omega}(x, k) = \frac{\exp \left( i \sum_{n=1}^{N} x_n k_{\omega(n)} \right)}{|\Delta_N(x) \Delta_N(k)|^{\beta/2}} W^{(\beta)}_{N,\omega}(x, k), \quad (5.10)
$$

where $\omega(k)$ is the diagonal matrix constructed from $k$ by permuting the $k_n$, or the indices $n$. The full solution $\Phi^{(\beta)}_{N}(x, k)$, satisfying the constraints \(5.3\) and \(5.4\), is then, apart from possible normalization constants, given as the linear combination

$$
\Phi^{(\beta)}_{N}(x, k) = \frac{1}{N!} \sum_{\omega \in S_N} (-1)^{\pi(\omega)} \Phi^{(\beta)}_{N,\omega}(x, k), \quad (5.11)
$$

of the functions \(5.11\). Here, $\pi(\omega)$ is the parity of the permutation.

We find for the function $W^{(\beta)}_{N,\omega}(x, k)$ the differential equation

$$
L_{x,\omega(k)} W^{(\beta)}_{N,\omega}(x, k) = 0, \quad (5.12)
$$

where the operator is given by

$$
L_{x,\omega(k)} = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + i2 \sum_{n=1}^{N} k_{\omega(n)} \frac{\partial}{\partial x_n} - \beta \left( \frac{\beta - 1}{2} \right) \sum_{n<m} \frac{1}{(x_n - x_m)^2}. \quad (5.13)
$$

This differential equation generalizes Eq. \(2.13\) to the matrix case for $\beta = 1, 2, 4$ and, furthermore, the latter to general radial function for arbitrary $\beta$.

Again, due to the symmetry \(5.3\), the differential equation \(5.12\) must also hold if $x$ and $\omega(k)$ are interchanged. It is the last term of the operator $L_{x,\omega(k)}$ that makes the differential equation \(5.12\) so difficult. This shows that the case $\beta = 2$ corresponding to unitary matrices $U \in U(N)$ is special: the last term vanishes and we simply have $W^{(2)}_{N,\omega}(x, k) = 1$. This is the Itzykson–Zuber case \(13\). For arbitrary $\beta$, it is obvious from the differential operator that...
$W_{N,\omega}(x, k) \to 1$ if $|x_n - x_m| \to \infty$ for all pairs $n < m$. Once more, this must also be true if $|k_n - k_m| \to \infty$. Thus, we expect that $W_{N,\omega}(x, k)$ is some kind of asymptotic series, generalizing Eq. (2.14) in the vector case.

Hence, we re–derive a known result by concluding that the leading contribution in an asymptotic expansion of the functions (5.10) is given by

$$
\Phi_{N,\omega}^{(\beta)}(x, k) \sim \frac{\exp \left( i \sum_{n=1}^{N} x_n k_{\omega(n)} \right)}{|\Delta_N(x)\Delta_N(k)|^{\beta/2}}. \tag{5.14}
$$

According to Eq. (5.14), this means that

$$
\Phi_{N}^{(\beta)}(x, k) \sim \frac{\det[\exp(ix_n k_{\omega(n)})]_{n,m=1,\ldots,N}}{|\Delta_N(x)\Delta_N(k)|^{\beta/2}} \tag{5.15}
$$

is the asymptotic behavior of the radial functions $\Phi_{N}^{(\beta)}(x, k)$ if the differences $|x_n - x_m|$ and $|k_n - k_m|$ are large for all pairs $n < m$.

The functions $W_{N,\omega}^{(\beta)}(x, k)$ are translation invariant, i.e. they depend only on the differences $(x_n - x_m)$. We show this in App. 3. Due to the symmetry, this argument carries also over to $k$ and $W_{N,\omega}^{(\beta)}(x, k)$ depends only on the differences $(k_n - k_m)$ as well. Moreover, the symmetry implies that it depends only on the products $(k_{\omega(n)} - k_{\omega(m)})(x_n - x_m)$.

Collecting all these pieces of information, we make the ansatz

$$
W_{N,\omega}^{(\beta)}(x, k) = \sum_{(\mu)} a_{\mu_1\mu_2\cdots\mu_{(N-1)}N} \prod_{n<m} ( (k_{\omega(n)} - k_{\omega(m)})(x_n - x_m) )^{\mu_{nm}} \tag{5.16}
$$

with coefficients $a_{\mu_1\mu_2\cdots\mu_{(N-1)}N}$ that depend on $N(N-1)/2$ integer indices $\mu_{nm}$, as many as there are differences. The summation is over the set of these indices. The presence of the $k_n$ makes it very difficult to solve Eq. (5.12) with the ansatz (5.14). In the vector case, one easily sees that the differential equation (2.13) in $r$ can be transformed into an equation in the dimensionless variables $kr$ such that $k$ does not appear anymore. This leads to the simple recursion (2.13) for the coefficients. Here, in the matrix case, the $k_n$ cannot easily be absorbed and the recursion formulae for the coefficients will depend on the $k_n$ in a non–trivial way. However, in some simple cases, it is possible to solve them. These difficulties were an important motivation for us to develop the methods which we introduced in Sec. V.

VI. APPLICATIONS

Can we obtain explicit formula for the radial functions by using the recursion formula (5.5) ? — At least in some cases, this ought to be possible. Here, we present our first attempts.

For the sake of completeness, we comment once more on the special case $\beta = 2$, i.e. the unitary case. Obviously, the measure (5.6) simplifies enormously. This is so because the radial Gelfand–Tzetlin coordinates coincide with the angular ones. Thus, the case $\beta = 2$ is identical to the re–derivation of the Itzykson–Zuber integral by Shatashvili³.

We now consider the orthogonal case $\beta = 1$. The recursion formula reads

$$
\Phi_{N}^{(1)}(x, k) = G_N^{(1)} \int \frac{\Delta_{N-1}(x')}{\sqrt{-\prod_{n,m}(x_n - x'_m)}} \exp \left( i \sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x' \right) k_N \Phi_{N-1}^{(1)}(x', \bar{k}) d|x'|. \tag{6.1}
$$

The square roots appearing in the measure make a further evaluation very difficult. As obvious from the trivial case $N = 2$, given in Eq. (5.21), the function $\Phi_{N}^{(1)}(x, k)$ will be an infinite series for all values of $N$. However, we expect that, due to the different construction, this series is different from the expansion in zonal functions which was obtained by Muirhead³.

Obviously, there is a pattern emerging. The integration measure (5.6) is purely rational for all even and positive values of $\beta$. This is reminiscent of the situation for vector Bessel functions in odd dimensions $d$, which consist of a finite number of terms, as discussed in Sec. IV. Hence, we conjecture that the radial functions $\Phi_{N}^{(\beta)}(x, k)$ can also be written as a finite sum, exclusively containing exponential and rational functions.
For all other values of $\beta$, the measure \((6.6)\) is algebraic, but not rational, and the radial functions $\Phi^{(4)}_{N}(x,k)$ must be *infinite* series. Nevertheless, these infinite series contain exponential and rational functions. Thus, they are different from expansions in terms of zonal polynomials.

To furnish our conjecture about the form of the radial functions for even and positive values of $\beta$ with an illustrative example, we turn to the unitary–symplectic case $\beta = 4$. To simplify the notation we avoid the imaginary unit by writing

$$
\Phi^{(4)}_{N}(-ix,k) = \int_{U \in USp(2N)} \exp(\text{Tr } u^{-1} xuk) \, d\mu(U) ,
$$

where $x = \text{diag} (x_1,x_1,\ldots, x_N,x_N)$ and $k = \text{diag} (k_1,k_1,\ldots,k_N,k_N)$ are diagonal matrices with Kramers degeneracies. The starting point of the recursion is the smallest non–trivial case $N = 2$, i.e. the group $USp(4)$. We obtain after an elementary calculation

$$
\Phi^{(4)}_{2}(-ix,k) = G^{(4)}_{2} \sum_{\omega \in S_{2}} \left( \frac{1}{\Delta_{2}^{3}(x) \Delta_{2}^{3}(\omega(k))} - \frac{2}{\Delta_{2}^{3}(x) \Delta_{2}^{3}(\omega(k))} \right) \exp(\text{Tr } x\omega(k)) .
$$

The sum runs over the elements of the permutation group $S_{N}$ for $N = 2$. Inserting Eq. \((6.3)\) into the recursion formula, we find for $USp(6)$, the next step in the recursion,

$$
\Phi^{(4)}_{3}(-ix,k) = G^{(4)}_{3} G^{(4)}_{2} \sum_{\omega \in S_{2}} \int_{x_1}^{x_2} dx'_{1} \int_{x_2}^{x_3} dx'_{2} \int_{x_3}^{x_{3}} \frac{\prod_{i=1}^{3} \prod_{j=1}^{2} (x_{i} - x_{j}')}{\Delta_{3}^{3}(x) \Delta_{2}^{3}(k)} \exp \left( (\text{Tr } x - \text{Tr } x') k_{3} + \text{Tr } x\omega(k) \right) \left( \frac{1}{\Delta_{2}(x')} - \frac{2}{\Delta_{2}(x') \Delta_{2}(\omega(k))} \right) .
$$

(6.4)

Although the integrand is finite everywhere, in particular at $x'_{1} = x'_{2} = x_{2}$, the denominators $\Delta_{2}(x')$ and $\Delta_{2}^{2}(x')$ raise a technical difficulty. The key to remove them is to use the identity

$$
\frac{2}{\Delta_{2}^{3}(x')} = - \left( \frac{\partial}{\partial x'_{1}} - \frac{\partial}{\partial x'_{2}} \right) \frac{1}{\Delta_{2}(x')}
$$

(6.5)

and to observe that the product $\prod_{i=1}^{3} \prod_{j=1}^{2} (x_{i} - x_{j}')$ annihilates all boundary terms. Hence, we can integrate by parts and arrive at

$$
\Phi^{(4)}_{3}(-ix,k) = G^{(4)}_{3} G^{(4)}_{2} \sum_{\omega \in S_{2}} \frac{1}{\Delta_{3}^{3}(x) \Delta_{3}^{3}(\omega(k))} \int_{x_{1}}^{x_{2}} dx'_{1} \int_{x_{2}}^{x_{3}} dx'_{2} \int_{x_{3}}^{x_{3}} \frac{\prod_{i=1}^{3} (x_{j} - x'_{i}) (x_{j} - x_{2})}{\Delta_{2}(x')} \exp \left( (\text{Tr } x - \text{Tr } x') k_{3} + \text{Tr } x\omega(k) \right) ,
$$

(6.6)

where no denominator is left. Due to the permutation symmetry of the original integral, we can restrict ourselves to the unity element $e$ of the permutation group in the further evaluation of Eq. \((6.6)\). Thus we need only to consider the limits $x'_{i} \rightarrow x_{i}$, $i = 1,2$ while integrating by parts. After collecting orders in $k$ we find

$$
\Phi^{(4)}_{3,e}(-ix,k) = G^{(4)}_{3} G^{(4)}_{2} \frac{1}{\Delta_{3}^{3}(x) \Delta_{3}^{3}(k)} \left( - \Delta_{3}(x) \Delta_{3}(k) + 2 \sum_{i<j} \left( \frac{\Delta_{3}(x) \Delta_{3}(k)}{(x_{i} - x_{j})(k_{i} - k_{j})} \right) - 4 \sum_{i<j} (x_{i} - x_{j})(k_{i} - k_{j}) + 12 \right) \exp(\text{Tr } xk) .
$$

(6.7)
By introducing the composite variables
\[ z_{\omega(ij)} = (x_i - x_j)(k_{\omega(i)} - k_{\omega(j)}) \quad i, j = 1, \ldots, 3, \quad \omega \in S_3, \] (6.8)
we can express \( \Phi_3^{(4)}(-ix, k) \) in a compact form as
\[
\Phi_3^{(4)}(-ix, k) = G_3^{(4)} G_2^{(4)} \sum_{\omega \in S_3} \frac{1}{\Delta_3^3(x) \Delta_3^3(\omega(k))} \left( 4 + \prod_{i<j} (2 - z_{\omega(ij)}) \right) \exp(\text{Tr} x\omega(k)) .
\] (6.9)

So far, we have not been able to extend this procedure to all values of \( N \).

However, we succeeded in calculating \( \Phi_4^{(4)}(-ix, k) \), i.e. the case of the group \( USp(8) \), by an hybrid method which combines informations obtained from the recursion with an Hankel ansatz as described in the previous section. We extend the right hand side of Eq. (6.9) for \( N = 3 \) to \( N = 4 \) and use this expression as an ansatz for the function \( W_{N,\omega}^{(4)}(x, k) \). As it turns out, a correction term is needed and, furthermore a correction to the correction. Fortunately, there is a structure to this. We give the details in App. [3]. We emphasize that the knowledge of \( \Phi_3^{(4)}(-ix, k) \) is essential for this hybrid procedure, in particular the fact, that \( \Phi_3^{(4)}(-ix, k) \) contains only linear terms in every composite variable \( z_{\omega(ij)} \). Up to a normalization, \( \Phi_4^{(4)}(-ix, k) \) is given by
\[
\Phi_4^{(4)}(-ix, k) = \sum_{\omega \in S_4} \frac{1}{\Delta_4^3(x) \Delta_4^3(\omega(k))} \left( \prod_{i<j} (2 - z_{\omega(ij)}) + \frac{1}{2} \sum_{1 < m < n} \prod_{i<j \neq m} (2 - z_{\omega(ij)}) + \frac{1}{4} \sum_{1 < m < n} \prod_{i<j; k} (2 - z_{\omega(ij)}) \right) \exp(\text{Tr} x\omega(k)) .
\] (6.10)

Comparing this result with Eq. (6.9) we notice that, once more, the composite variables \( z_{\omega(ij)} \) enter only linearly in the polynomial part of \( \Phi_4^{(4)}(-ix, k) \). Similarly, the spherical Bessel function \( j_1(z) \), which is the counterpart of \( \Phi_N^{(4)}(x, k) \) in the vector case given in Eqs. (2.6) and (3.21), has a polynomial part linear in \( z \). We expect that such analogies are also present for higher values of \( \beta \) and the dimension \( d \).

Formula (6.11) indicates a general structure for \( \Phi_N^{(4)}(x, k) \). The leading term is always the generating function of the elementary symmetric functions in \( z \). To this term combinations of other symmetric functions are added, where certain combinations of indices are cut out.

**VII. CONNECTION TO CALOGERO–SUTHERLAND MODELS**

The radial functions are related to, but different from, the eigenfunctions which are usually employed in models of the Calogero–Sutherland type. Since this issue is so important for applications and so often raised in discussions, we briefly collect the main points.

The radial Laplace operator defined in Eq. (5.3) is closely related to the Calogero–Sutherland Hamiltonians. In general one can always cast a Fokker–Planck operator in a Hamilton operator by adjunction with the square root of the stationary probability distribution defined through \( \Delta_P(x) = 0 \). Choosing \( P_{\text{eq}}(x) = |\Delta_N(x)|^{-\beta/2} \), the operator (5.3) can be associated with the Hamiltonian
\[
H_D = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \frac{\beta(\beta - 2)}{4} \sum_{n < m} \frac{1}{(x_n - x_m)^2} .
\] (7.1)

It describes a scattering system with a continuous spectrum, the large time behavior is determined by the states near the ground state. Apart from a sign, this operator \( H_D \) coincides with the operator \( L_{x,0} \) in Eq. (5.13) for \( k = 0 \). We also notice that the interaction vanishes for \( \beta = 2 \).
To have a well defined thermodynamic limit one often confines the motion of the particles to a circle. This yields the Calogero–Sutherland Hamiltonian

\[ H_{CS} = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \frac{\beta(\beta - 2)}{4} \sum_{n<m}^{N} \frac{(\pi/N)^2}{\sin^2(\pi(x_n - x_m)/N)} \]  

(7.2)

which can also be derived directly from Dyson’s circular ensembles. Another way of confining the particles is by a harmonic potential. This leads to the Calogero Hamiltonian

\[ H_C = -\sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} + \frac{\beta(\beta - 2)}{4} \sum_{n<m}^{N} \frac{1}{(x_m - x_m)^2} + \frac{1}{16} \sum_{n=1}^{N} x_n^2 \]  

(7.3)

In the thermodynamic limit the particle density \( R_1(x) \) of the ground state is described by Wigner’s semi-circle law. The mean particle level spacing \( D = 1/R_1(0) \) scales as \( D \propto 1/\sqrt{N} \). Therefore in the thermodynamic limit the harmonic confining term in Eq. (7.3) vanishes on the scale of the mean level spacing. On this unfolded scale the correlation functions become independent of the confinement mechanism. The three Hamiltonians \( H_C, H_D \) and \( H_{CS} \) are known to be integrable systems for arbitrary \( \beta \). However, the three values \( \beta = 1, 2, 4 \) are distinguished, since they establish a connection to the random matrix ensembles. Indeed, for these values of \( \beta \) they belong to a much wider class of integrable systems, which can be constructed by means of the root space of a simple Lie algebra or – still more generally – of a Kac–Moody algebra. This class comprises Hamiltonians which can be derived by an adjunction procedure from a Laplace–Beltrami operator of a group acting in a symmetric space. This space has positive curvature for \( H_{CS} \) and zero curvature for \( H_D, H_C \). In Refs. it was pointed out, that the Dorokhov–Mello–Pereyra–Kumar equation for scattering matrices with broken time reversal symmetry corresponds to a Laplace–Beltrami operator in a symmetric space of negative curvature.

Eigenfunctions \( \psi^{(\beta)}_{N,E}(x) \) of the Hamiltonians \( H_C, H_D, H_{CS} \) with eigenenergy \( E \) for arbitrary \( \beta \) are known. Essentially, these solutions are products of the ground state wave function and symmetric polynomials in the coordinates \( x \) of the \( N \) particles. In case of the Calogero–Sutherland Hamiltonian \( H_{CS} \), these polynomials are the Jack polynomials [11]. In this approach, the energy eigenvalues \( E \) are labeled by a partition of length \( N \). The crucial difference to the matrix Bessel functions is that the Jack polynomials are symmetric polynomials in one set of variables \( x \) only whereas the matrix Bessel functions \( \Phi^{(\beta)}_N(x,k) \) are symmetric in two sets of variables \( x \) and \( k \). Importantly, they are, in addition, symmetric under interchange of the two sets of variables, \( \Phi^{(\beta)}_N(x,k) = \Phi^{(\beta)}_N(k,x) \). This is reflected in the fact that the operator \( L_{x\cdot x}(k) \) emerging in the Hankel ansatz depends on \( k \) while \( H_D \) does not. Due to their symmetry, the matrix Bessel functions \( \Phi^{(\beta)}_N(x,k) \) are, at least for \( H_D \), the more natural eigenfunctions. This becomes obvious in the fact that, to obtain orthogonality conditions, one has to sum the \( \psi^{(\beta)}_{N,E}(x) \) over an infinite number of partitions. On the other hand, orthogonality relations are an inherent feature of the \( \Phi^{(\beta)}_N(x,k) \) due to their meaning in the Fourier–Bessel analysis, as discussed in App. 

In other words, the functions \( \psi^{(\beta)}_{N,E}(x) \) can be viewed as a basis in an expansion of the \( \Phi^{(\beta)}_N(x,k) \). We can consider the variables \( k \) as a set of real numbers corresponding to the energies \( E \) labeling the eigenstates of \( H_D \). The matrix Bessel functions (9.13) are solutions of the Schrödinger equation with Hamiltonian \( H_D \) for the coupling parameters \( \beta = 1, 2, 4 \). The recursion formula (9.15) represents an analytic continuation of these integral solutions to arbitrary positive \( \beta \). All these functions \( \Phi^{(\beta)}_N(x,k) \) have, for arbitrary \( \beta \) additional features, such as the symmetry in \( x \) and \( k \), which have no analogue in the functions \( \psi^{(\beta)}_{N,E}(x) \). The merit of our recursion formula lies in the fact that, a priori, no infinite summation is required to obtain functions of the type \( \Phi^{(\beta)}_N(x,k) \). Nevertheless Forrester and Nagao showed that such resummed expressions can successfully be used in certain cases. They treated the case of Poissonian initial conditions for the Calogero–Sutherland Hamiltonian \( H_{CS} \) and derived exact expressions for the correlation functions for arbitrary \( \beta \) for one or two particles. This is also related to the works of Muirhead and Pandey.

**VIII. SUMMARY AND CONCLUSION**

We presented a recursive construction for certain spherical functions. We referred to them as matrix Bessel functions because, first, they are a natural extension of vector Bessel functions in the sense that the integration over a group corresponds to the integration over a solid angle and, second, they satisfy a partial differential equation generalizing
the Bessel ordinary differential equation. For matrices, the index $\beta$ labeling the groups appears analogously to the dimension $d$ in the case of vectors. The introduction of radial Gelfand–Tzetlin coordinates, which are related to but different from the ordinary angular ones, was crucial for the recursion. The Cayley transformation ought to provide a connection between the angular and the radial Gelfand–Tzetlin coordinates. As evident from its construction, the recursion maps an integral over a group fully onto an iteration which exclusively takes place in the radial space.

Remarkably, the recursion turned out to be far more general than was to be expected, at first sight, from the proof which involved Lie groups. We showed that our recursion is also the iterative solution of the corresponding partial differential equation for arbitrary values of $\beta$. We introduced the term radial functions for this generalization of matrix Bessel functions. We expect that one has to employ the theory of quantum groups to give a group theoretical derivation of the recursion formula for arbitrary values of $\beta$.

Using the recursion formula, we discussed the structure of radial Bessel functions. We conjectured that, for even $\beta$, they can be written as finite sums involving only exponential and rational functions. We illustrated that by working out, for $\beta = 4$, the cases of $N = 3$ and $N = 4$ distinct eigenvalues. Further evaluation of explicit formulae for arbitrary $N$ and, maybe, for all even $\beta$ does not seem impossible. Work is in progress. The extension of the stationary phase approach by Duistermaat and Heckman to higher orders could, for even $\beta$, be an alternative to derive such explicit results, because the expansion terminates. In this context, we mention that the radial function for higher values of $\beta$ are, to some extent, but not fully, the higher order radial functions for lower values of $\beta$. This also generalizes the situation for ordinary Bessel functions. However, there are many more higher order radial functions, they are not at all exhausted by this mapping between values of $\beta$.

In the present contribution, we only focussed on ordinary spaces, i.e. spaces which are built upon commuting numbers. In a second study, we also address superspaces which involve commuting and anticommuting variables.

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APPENDIX A: ALTERNATIVE INTEGRAL REPRESENTATION FOR VECTOR BESSEL FUNCTIONS

The Bessel functions $\chi^{(d)}(kr)$ are defined as an integral over angles in Eq. (2.1), but they can also be written as integrals over the entire or half real axis. To make possible an instructive comparison with the matrix case, we quote and re–derive the representation

$$\chi^{(d)}(kr) = \frac{\Gamma(d/2)}{i^d \pi^{(d-2)/2}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \frac{kr}{t + \frac{1}{t}} \right) \frac{dt}{t^{d/2}}. \tag{A.1}$$

The singularities have to be treated properly.

The position vector in the $d$ dimensional space is $\vec{r} = r \vec{e}_r$, where the vector $\vec{e}_r$ parametrizes the unit sphere. To integrate over its orientation, i.e. over the solid angle $\Omega$, one can re–express the measure as

$$d\Omega = \frac{\Gamma(d/2)}{\pi^{d/2}} \delta \left( e_r^2 - 1 \right) d^de_r. \tag{A.2}$$

Here, the vector $\vec{e}_r$ is re–interpreted: its components live on the entire real axis and the domain of integration is the full $d$ dimensional space with the Cartesian measure $d^de_r$. The $\delta$ distribution confines the vector to the unit sphere. Writing this distribution as a Fourier transform, we obtain from Eq. (2.1)

$$\chi^{(d)}(kr) = \frac{\Gamma(d/2)}{\pi^{d/2}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int d^de_r \exp \left( it (e_r^2 - 1) \right) \exp \left( i\vec{k} \cdot \vec{e}_r \right)$$

$$= \frac{\Gamma(d/2)}{2\pi} \int_{-\infty}^{+\infty} \exp(-it) \frac{\exp(-it)}{(it)^{d/2}} \exp \left( -\frac{(kr)^2}{4t} \right) dt \tag{A.3}$$

where the integral over $\vec{e}_r$ gave a Gaussian in $d$ dimensions. The contour for the integration over $t$ has to be chosen appropriately. Upon a trivial change of variables, this result yields Eq. (A.1)
APPENDIX B: ALTERNATIVE INTEGRAL REPRESENTATION FOR MATRIX BESSEL FUNCTIONS

The matrix Bessel functions \( \Phi^{(\beta)}_N(x,k) \) for \( \beta = 1, 2, 4 \) can be be written in an alternative way. Although we can hardly believe that this representation is completely new, we could not find it in the literature. Similarly to Eq. (A.1) in the vector case, we can write

\[
\Phi^{(\beta)}_N(x,k) = A^{(\beta)}_N \int d[\hat{T}] \exp (i \text{Tr} \hat{T}) \text{Det}^{-\beta/2} \left( x \otimes \hat{k} - T \otimes 1_N \right)
\]  

(B.1)

where \( 1_N \) is the \( N \times N \) unit matrix. The normalization constant is given by

\[
A^{(\beta)}_N = \frac{i^{\beta N^2/2} \pi^{\beta N(N-1)/4}}{\beta^{N+\beta N(N-1)/2}} \prod_{n=1}^N \Gamma(\beta n/2) .
\]  

(B.2)

The matrix \( T \) in Eq. (B.1) is real symmetric, Hermitian or Hermitian self-dual, respectively, for \( \beta = 1, 2, 4 \). The measure \( d[\hat{T}] \) is Cartesian and given by Eq. (13). All independent variables in \( T \) are integrated over the entire real axis. To ensure convergence, the diagonal elements of \( T \) have to be given a proper imaginary increment. We notice that \( x \) and \( T \) are \( N \times N \) matrices for \( \beta = 1, 2 \) and \( 2N \times 2N \) for \( \beta = 4 \) with doubly degenerated eigenvalues. The matrix \( \hat{k} \) is, in all three cases \( \beta \), just the \( N \times N \) matrix \( \hat{k} = \text{diag} \left( k_1, k_2, \ldots, k_N \right) \), as defined following Eq. (3.14).

The integral representation (B.1) leads to an interesting integral equation for the matrix Bessel functions,

\[
\Phi^{(\beta)}_N(x,k) = B^{(\beta)}_N \text{Det}^{-1-\beta/2} x \int d[t] \Delta_N(t)^{\beta} \frac{\Phi^{(\beta)}_N(x,t)}{\prod_{n,m} (k_m - t_n)^{\beta/2}} ,
\]  

(B.3)

where the normalization constant reads

\[
B^{(\beta)}_N = \frac{i^{\beta N^2/2} \Gamma^{N} \left( \beta/2 \right)}{(2\pi)^N N!} .
\]  

(B.4)

The \( t_n \) in Eq. (B.3) have a proper imaginary increment and their domain of integration is the real axis. Due to the symmetry relation (B.10), the variables \( x \) and \( k \) can be interchanged in Eqs. (B.1) and (B.3).

It is not difficult to see from the integral equation (B.3) that the product in the denominator of its right hand side can, in the case \( \beta = 2 \) be written as

\[
\frac{B^{(2)}_N}{\prod_{n,m} (k_m - t_n)} = \frac{\text{det} \left[ \delta(x_n - t_m) \right]_{n,m=1,\ldots,N}}{\Delta_N(k) \Delta_N(t)^{1/2}} .
\]  

(B.5)

For \( \beta \neq 2 \), the term \( \text{Det}^{-1-\beta/2} x \) contributes. Nevertheless, the product still shares features with a \( \delta \) distribution.

To derive this alternative integral representation, we proceed analogously to Eq. (A.2) by re-writing the invariant measure of \( U \in U(N;\beta) \) using \( \delta \) distributions. The invariance simply means that all columns \( U_n, n = 1, \ldots, N \) are orthonormal, \( \text{Tr} U_n^\dagger U_m = \delta_{nm} \). The trace \( \text{Tr} \) is only needed for \( \beta = 4 \), because the entries of \( U \) are quaternions in this case. Thus, we may write

\[
d\mu(U) = M^{(\beta)}_N d[U] \prod_{n=1}^N \delta \left( \text{Tr} U_n^\dagger U_n - 1 \right) \prod_{n<m} \delta \left( \text{Tr} U_n^\dagger U_m \right)
\]  

(B.6)

where \( d[U] \) is the Cartesian measure of all entries of \( U \) and the integration is for all variables over the entire real axis. The constant \( M^{(\beta)}_N \) will be determined later. Ullah[40] used such forms for the measure to work out certain probability density functions. The bilinear forms in the \( \delta \) distributions have \( \beta \) components for \( n \neq m \),

\[
U_n^\dagger U_m = \sum_{\alpha=0}^{\beta-1} \left[ U_n^\dagger U_m \right]^{(\alpha)} \tau^{(\alpha)} .
\]  

(B.7)

We notice that \( \left[ U_n^\dagger U_m \right]^{(\alpha)} = 0 \) for \( \alpha > 0 \) in the case \( n = m \), because the length of every vector is real. Thus, because of Eq. (B.6), the \( \delta \) distributions in the measure (B.6) have to be products of \( \delta \) distributions for every non-zero component \( \left[ U_n^\dagger U_m \right]^{(\alpha)} \). We now introduce Fourier representations.
\[
\delta \left( [U_n^\dagger U_m]^{(\alpha)} \right) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dT_{nn}^{(\alpha)} \exp \left( -i2 \left[ U_n^\dagger U_m \right]^{(\alpha)} T_{nn}^{(\alpha)} \right)
\]
\[
\delta \left( [U_n^\dagger U_n]^{(0)} - 1 \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dT_{nn}^{(0)} \exp \left( -i \left( [U_n^\dagger U_n]^{(0)} - 1 \right) T_{nn}^{(0)} \right)
\]
(B.8)

for \( n \neq m \) and \( n = m \), respectively. The Fourier variables form the elements
\[
T_{nm} = \sum_{\alpha=0}^{\beta-1} T_{nm}^{(\alpha)} T^{(\alpha)}.
\]
(B.9)
of a matrix \( T \) which is real–symmetric, Hermitean or Hermitean self–dual according to \( \beta = 1, 2, 4 \). We notice that the diagonal elements \( T_{nm} = T_{nn} \) are always real.
\[
\delta \left( \text{Tr} U_n^\dagger U_m \right) = \frac{1}{\pi^{\beta}} \int d^\beta T_{nm} \exp \left( -i \text{Tr} U_n^\dagger (T_{nm} \otimes 1_N) U_m \right)
\]
\[
\delta \left( \text{Tr} U_n^\dagger U_n - 1 \right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dT_{nn} \exp \left( i \text{Tr} T_{nn} - i \text{Tr} U_n^\dagger (T_{nn} \otimes 1_N) U_n \right)
\]
(B.10)
for \( n \neq m \) and \( n = m \), as above. Just as the trace \( \text{Tr} \), the direct product is only needed in the case \( \beta = 4 \).

We order the columns \( U_n \), \( n = 1, \ldots, N \) of the matrix \( U \) in a vector \( \vec{U} = (U_1, U_2, \ldots, U_N)^T \) with \( N^2 \) elements. For \( \beta = 1, 2 \), the elements are scalars, for \( \beta = 4 \), they are quaternions. Collecting everything, we can re–write the measure \( \mu(U) \) in the form
\[
d\mu(U) = \frac{M_N^{(\beta)} d[U]}{(2\pi)^N N^\beta (N-1)/2} \int d[T] \exp \left( i \text{Tr} T - i \text{Tr} \vec{U}^\dagger (T \otimes 1_N) \vec{U} \right).
\]
(B.11)

To use this in the integral \( \text{B.15} \) for the matrix Bessel functions \( \Phi_N^{(\beta)}(x, k) \), we also take advantage of the relation
\[
\text{Tr} U^{-1} x U k = \text{Tr} \vec{U}^\dagger (x \otimes k) \vec{U}
\]
(B.12)
which allows us to write
\[
\Phi_N^{(\beta)}(x, k) = \frac{M_N^{(\beta)}}{(2\pi)^N \pi^\beta N (N-1)/2} \int d[U] \int d[T] \exp \left( i \text{Tr} T \right)
\]
\[
\exp \left( i \text{Tr} \vec{U}^\dagger (x \otimes k - T \otimes 1_N) \vec{U} \right)
\]
\[
= \frac{M_N^{(\beta)} i^\beta N^2 N^\beta /2}{(2\pi)^N} \int d[T] \exp \left( i \text{Tr} T \right)
\]
\[
\Det^{-\beta/2} \left( x \otimes k - T \otimes 1_N \right) .
\]
(B.13)

Thus, the integration over \( U \) could be done as a Gaussian one and gave the result \( \text{B.1} \). Obviously, the Gaussian integrals over \( \vec{U} \) only converge, if the diagonal elements of \( T \) have a proper imaginary increment.

Formula \( \text{B.13} \) yields immediately the integral equation \( \text{B.3} \). Upon making the change of variables
\[
T = x^{1/2} T' x^{1/2}, \quad \text{implying} \quad d[T] = \Det^{1+\beta(N-1)/2} x d[T'],
\]
(B.14)
we bring \( x \) into the exponential function and remove it from the determinant. We diagonalize \( T' = V'^{-1} t' V' \) and find
\[
\Det^{-\beta/2} \left( x \otimes k - T \otimes 1_N \right) = \Det^{-\beta/2} x \prod_{n,m} (k_m - t_n')^{-\beta/2}.
\]
(B.15)
The integral over \( V' \) is then just the integral definition \( \text{B.15} \) of the matrix Bessel function \( \Phi_N^{(\beta)}(x, t') \) and we arrive at Eq. \( \text{B.3} \).
The normalization constants remain to be derived. Conveniently, they nicely relate to a special form of Selberg’s integral which is given in Eq. (17.5.2) of Mehta’s book.

\[ J_N = \int d[t] |\Delta_N(t)|^{2\gamma} \prod_{n=1}^{N} (a_1 + it_n)^{-b_1} (a_2 - it_n)^{-b_2} \]

\[ = \frac{(2\pi)^N}{(a_1 + a_2)^{(b_1+b_2)N-\gamma N(N-1)-N}} \prod_{n=0}^{N-1} \frac{\Gamma(1 + (n+1)\gamma)\Gamma(b_1 + b_2 - (N+n-1)\gamma - 1)}{\Gamma(1 + \gamma)\Gamma(b_1 - n\gamma)\Gamma(b_2 - n\gamma)}. \]  

(B.16)

We now put \( x = 0 \) or \( k = 0 \) and have \( \Phi_N^{(\beta)}(0,k) = 1 \) or \( \Phi_N^{(\beta)}(x,0) = 1 \) on the left hand side of Eq. (B.13). We diagonalize \( T = V^{-1}tV \) and use the invariance of the integral. Employing the measure \((B.4)\) and the constant \( C_N^{(\beta)} \) given in Eq. (B.3), we find the condition

\[ 1 = \frac{M_N^{(\beta)} C_N^{(\beta)} \pi^{\beta N/2}}{(2\pi)^N} \int d[t] |\Delta_N(t)|^{\beta} \prod_{n=1}^{N} \exp(i\eta_n) \prod_{n=1}^{N} \eta_n^{\beta N/2}. \]  

(B.17)

We map this onto Selberg’s integral \((B.16)\) by setting \( \gamma = \beta/2, b_1 = \beta N/2 \) and \( a_2 = b_2 \), by using

\[ \lim_{a_2 \to \infty} \frac{a_2^{a_2}}{(a_2 - i\eta_n)^{a_2}} = \exp(i\eta_n) \]  

(B.18)

and by considering \( a_2^{N\alpha} J_N \) in the limits \( a_1 \to 0 \) and \( a_2 \to \infty \). With the help of some standard asymptotic formulae for the \( \Gamma \) function, we obtain \( M_N^{(\beta)} \) and, eventually, the constants \( A_N^{(\beta)} \) and \( B_N^{(\beta)} \) in Eqs. \((B.2)\) and \((B.4)\).

**APPENDIX C: FOURIER–BESSEL ANALYSIS**

The Fourier–Bessel Analysis involving matrix Bessel functions was discussed by Harish–Chandra in a general and formal way. To show the connection to our results, we summarize here some essential features of the Fourier–Bessel analysis on an explicit level.

We write the Fourier transform of a function \( f(H) \) as

\[ F(K) = D_N^{(\beta)} \int d[H] \exp(i\text{Tr}HK) f(H) \]  

(C.1)

where the matrices \( H \) and \( K \) have the same symmetries. If we choose a symmetric normalization,

\[ D_N^{(\beta)} = \frac{1}{(2\pi)^{N/2}\pi^{\beta N(N-1)/4}}. \]  

(C.2)

we can write the inverse transform as

\[ f(H) = D_N^{(\beta)} \int d[K] \exp(-i\text{Tr}KH) F(K). \]  

(C.3)

We notice that, according to Eq. \((B.11)\), the Fourier transform of the constant \( D_N^{(\beta)} \) is the \( \delta \) distribution \( \delta(K) \) and vice versa.

If \( f \) is an invariant function such that \( f(H) = f(x) \), its Fourier transform turns out to be invariant as well, \( F(K) = F(k) \). Introducing eigenvalue–angle coordinates, we easily find

\[ F(k) = D_N^{(\beta)} C_N^{(\beta)} \int d[x] |\Delta_N(x)|^\beta \Phi_N^{(\beta)}(x,k) f(x) \]  

(C.4)

for the Fourier transform and
Formulae \((\text{C.6})\) and \((\text{C.7})\) can be viewed as special cases of these results.

The Fourier decomposition

\[ y = \int d[k] |\Delta_N(k)|^\beta \Phi_N^{(\beta)*}(k, x) F(k) \]  

for its inverse. We now insert the transform \((\text{C.4})\) into the inverse \((\text{C.5})\) and conclude that

\[
\left( D_N^{(\beta)} C_N^{(\beta)} \right)^2 \int d[k] |\Delta_N(k)|^\beta \Phi_N^{(\beta)}(x, k) \Phi_N^{(\beta)*}(k, y) = \frac{\det[\delta(x_n - y_m)]_{n,m=1,\ldots,N}}{|\Delta_N(x)\Delta_N(y)|^{\beta/2}} .
\]  

This is the analogue of Hankel’s expansion of the \(\delta\) distribution. From Eq. \((\text{C.6})\), the formula

\[
\int d\mu(U) \delta (U^\dagger x U - y) = \frac{1}{C_N^{(\beta)}} \frac{\det[\delta(x_n - y_m)]_{n,m=1,\ldots,N}}{|\Delta_N(x)\Delta_N(y)|^{\beta/2}} .
\]  

obtains. To see this, we introduce a matrix \(G\) having the same symmetries as \(H\) and write

\[ \delta(H - G) = (D_N^{(\beta)})^2 \int d[K] \exp(-i\text{Tr} \, K \, (H - G)) \]  

Averaging over the diagonalizing matrix \(U\) of \(H\) yields

\[
\int d\mu(U) \, \delta(H - G) = (D_N^{(\beta)})^2 \int d[K] \Phi_N^{(\beta)*}(x, k) \exp(i\text{Tr} \, K \, G) ,
\]  

by using the invariance of the measure. We now introduce eigenvalue–angle coordinates for \(K\) and do the integral over \(V\), the diagonalizing matrix of \(K\),

\[
\int d\mu(U) \, \delta(H - G) = (D_N^{(\beta)})^2 C_N^{(\beta)} \int d[k] |\Delta_N(k)|^\beta \Phi_N^{(\beta)*}(x, k) \Phi_N^{(\beta)}(k, y) ,
\]  

where we have, once more, employed the invariance of the measure. Since the right hand side of this equation does only depend on the eigenvalues \(y\) of \(G\), we may replace \(G\) on the left hand side with \(y\). Together with Eq. \((\text{C.4})\), this gives formula \((\text{C.7})\).

For the convolution in matrix space of two functions \(f_1(H)\) and \(f_2(H)\), we straightforwardly find the generalization of the standard convolution theorem,

\[
f(H) = \int d[G] f_1(G) f_2(H - G) = \int d[K] \exp(-i\text{Tr} \, K \, G) F_1(K) F_2(K) \]  

where \(G\) has the same symmetries as \(H\). The functions \(F_1(K)\) and \(F_2(K)\) are the Fourier transforms of \(f_1(H)\) and \(f_2(H)\), respectively. If the functions are invariant, the second of Eqs. \((\text{C.11})\) acquires the form

\[
f(x) = C_N^{(\beta)} \int d[k] |\Delta_N(k)|^\beta \Phi_N^{(\beta)*}(x, k) F_1(k) F_2(k) .
\]  

On the other hand, we find from the first of Eqs. \((\text{C.11})\)

\[
f(x) = C_N^{(\beta)} \int d[y] |\Delta_N(y)|^\beta f_1(y) \hat{f}_2(x, y)
\]

where \(y\) are the eigenvalues of \(G\). This formula is a convolution in the curved space of the eigenvalues. The second function is given by

\[
\hat{f}_2(x, y) = \int d\mu(U) \, f_2(x - U^\dagger y U) .
\]

We insert the Fourier integral for \(F_1(k)\) according to Eq. \((\text{C.4})\) into Eq. \((\text{C.12})\), compare with Eq. \((\text{C.13})\) and obtain the Fourier decomposition

\[
\hat{f}_2(x, y) = D_N^{(\beta)} C_N^{(\beta)} \int d[k] |\Delta_N(k)|^\beta \Phi_N^{(\beta)*}(x, k) F_2(k) \Phi_N^{(\beta)}(k, y) .
\]  

Formulae \((\text{C.4})\) and \((\text{C.7})\) can be viewed as special cases of these results.
We make the notation more compact by defining
\[ d\tilde{\mu}(x', x) = d\mu(x', x) \exp \left( i \left( \sum_{n=1}^{N} x_n - \sum_{n=1}^{N-1} x'_n \right) k_N \right), \] (D.1)
where the measure is given in Eq. (5.6). To prove the identity (5.9), we write the integral using Θ functions. The left hand side of Eq. (5.9) reads
\[ \Delta_x \int \tilde{\mu}(x', x) \Phi_N^{(\beta)}(x', \tilde{k}) \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) \, dx', \] (D.2)
where now the integration domain is the real axis for all variables. Thus, we can directly calculate the action of the operator \( \Delta_x \) onto the integral. We find
\[
\begin{align*}
\Delta_x \int \tilde{\mu}(x', x) \Phi_N^{(\beta)}(x', \tilde{k}) & \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) \, dx' \\
& = \int \Phi_N^{(\beta)}(x', \tilde{k}) \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) \\
& \quad \left( \Delta_x^{(-)} + \beta \sum_{n \neq m} \frac{1}{(x'_n - x'_m)^2} - k_N^2 \right) \tilde{\mu}(x', x) \, dx' \\
& + \int \Phi_N^{(\beta)}(x', \tilde{k}) \tilde{\mu}(x', x) \Delta_x \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) \, dx' \\
& + 2 \int \Phi_N^{(\beta)}(x', \tilde{k}) \sum_{n=1}^{N} \frac{\partial}{\partial x_n} \tilde{\mu}(x', x) \frac{\partial}{\partial x_n} \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) \, dx',
\end{align*}
\] (D.3)
where we define the operator
\[ \Delta_x^{(-)} = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_n^2} - \sum_{n<m} \beta \left( \frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right). \] (D.4)
By a series of integrations by parts, the operator \( \Delta_x^{(-)} \) acting on \( \tilde{\mu}(x', x) \) is transformed to \( \Delta_x^{(-)} \) acting only on \( \Phi(x', \tilde{k}) \).

At taking the derivative of the Θ functions, we notice that only adjacent levels contribute, because otherwise terms like \( \Theta(x_i - x_j) \) with \( i < j \) arise which annihilate the integral due to the chosen ordering. Therefore, we can write
\[
\begin{align*}
\frac{\partial}{\partial x_n} \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) = \\
\prod_{\Theta_{\neq nn', \neq (n-1)n'}} \left( \delta(x_n - x'_n) \Theta(x_{n-1}' - x_n) \\
- \delta(x_{n-1}' - x_n) \Theta(x_n - x_{n-1}') \right),
\end{align*}
\] (D.5)
where \( \prod_{\Theta_{\neq nn', \neq (n-1)n'}} \) is short–hand for the product on the left hand side of Eq. (D.5) without the two factors \( \Theta(x_{n-1}' - x_n) \Theta(x_n - x_{n-1}') \). Importantly, this product is symmetric in \( x_{n-1}' \) and \( x_n' \). The second derivatives yield
\[
\begin{align*}
\frac{\partial}{\partial x_n} \prod_{i>j} \Theta(x_i - x'_j) \prod_{j \geq l} \Theta(x_j' - x_l) = \prod_{\Theta_{\neq nn', \neq (n-1)n'}} \left( \delta'(x_n - x'_n) \Theta(x_{n-1}' - x_n) \\
+ \delta'(x_{n-1}' - x_n) \Theta(x_n - x_{n-1}') \\
+ \delta(x_{n-1}' - x_n) \delta(x_n - x_{n-1}') \right).
\end{align*}
\] (D.6)
The last term vanishes upon integration, since it is symmetric in \( x_{n-1}' \) and \( x_n' \), whereas the rest of the integrand is antisymmetric due to the Vandermonde determinant \( \Delta_{N-1}(x') \) in the measure (5.6). Differentiation with respect to \( x_n' \) gives
\[
\frac{\partial}{\partial x_n} \prod_{i>j} \Theta(x_i - x_j) \prod_{j \geq l} \Theta(x'_j - x_l) = \\
\prod_{n'(n+1), \Theta \neq n'n} \left( \delta(x'_n - x_{n+1}) \Theta(x_n - x'_n) - \delta(x_n - x'_n) \Theta(x'_n - x_{n+1}) \right). 
\]  

Integration by parts of the first term of the right hand side of Eq. (D.3) yields

\[
\Delta_x \int \tilde{\mu}(x', x) \Phi^{(\beta)}_{N-1}(x', \tilde{k}) \prod_{n \neq n'} \Theta(x_n - x_n') \prod_{n \neq m} \Theta(x_m - x_n') d[x'] = \\
\int \tilde{\mu}(x', x) \Delta_x \Phi^{(\beta)}_{N-1}(x', \tilde{k}) d[x'] - k_N^2 \int \tilde{\mu}(x', x) \Phi^{(\beta)}_{N-1}(x', \tilde{k}) d[x'] \\
+ 2 \int \Phi^{(\beta)}_{N-1}(x', \tilde{k}) \sum_{n=1}^{N-1} \left( \prod_{n \neq n'} \Theta(x_n - x_n') \right) \left( \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x'_n} + \frac{\beta}{2} \sum_{m \neq n} \frac{1}{x_n - x_m} - \frac{\beta}{2} \sum_{m \neq n} \frac{1}{x'_n - x_m} \right) \tilde{\mu}(x', x) d[x'] . 
\]

Inserting in Eq. (D.8) the function \( \tilde{\mu}(x', x) \) as given in Eq. (D.1) and Eq. (5.6) we find after a straightforward calculation

\[
\Delta_x \int \tilde{\mu}(x', x) \Phi^{(\beta)}_{N-1}(x', \tilde{k}) d[x'] = \\
\int \tilde{\mu}(x', x) \Delta_x \Phi^{(\beta)}_{N-1}(x', \tilde{k}) d[x'] - k_N^2 \int \tilde{\mu}(x', x) \Phi^{(\beta)}_{N-1}(x', \tilde{k}) d[x'] \\
+ 2 \int \Phi^{(\beta)}_{N-1}(x', \tilde{k}) \sum_{n=1}^{N-1} \left( \prod_{n \neq n'} \Theta(x_n - x_n') \right) g(x_n; x'_n; x, x') \\
\left( \delta(x_n - x'_n) \Theta(x'_n - x_{n+1}) + \delta(x'_n - x_{n+1}) \Theta(x_n - x'_n) \right) \tilde{\mu}(x', x) d[x'],
\]

with

\[
g(x_n; x'_n; x, x') = (\beta/2 - 1) \left( \sum_{m=1}^{N-1} \frac{1}{x_n - x_m} - \sum_{m \neq n} \frac{1}{x_n - x_m} \right) \\
g(x'_n; x_n; x, x') = (\beta/2 - 1) \left( \sum_{m \neq n} \frac{1}{x'_n - x_m} - \sum_{m=1}^{N} \frac{1}{x'_n - x_m} \right) .
\]

We now can perform the integration of the \( \delta \) distributions in Eq. (D.9). We notice that the difference \( g(x'_n; x_n; x, x') - g(x'_n; x_n; x, x') \) vanishes linearly, whenever \( x'_n \) approaches one of the boundaries of its integration domain. Thus the second integral in Eq. (D.9) yields zero as long as the measure diverges less than \( (x_n - x'_n)^{-1} \) when \( x_n \) approaches \( x . \) This is always the case for \( \beta > 0 \). Collecting everything, we arrive at the identity (E.3).

**APPENDIX E: SYMMETRY OF THE RADIAL FUNCTIONS FOR ARBITRARY \( \beta \)**

Applying the recursion formula (5.5) to all \( N - 1 \) levels, we can extend Eq. (1.5) to arbitrary \( \beta \) and write

\[
\Phi^{(\beta)}_{N}(x, k) = \int \prod_{n=1}^{N-1} d\mu(x^{(n)}, x^{(n-1)})
\]

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where \( x^{(0)} = x \). Analogously, we also find

\[
\Phi_N^{(\beta)}(k, x) = \int \prod_{n=1}^{N-1} d\mu(k^{(n)}, k^{(n-1)}) \exp\left( i \left( \sum_{m=1}^{N-n-1} k^{(n-1)} m - \sum_{m=1}^{N-n} k^{(n)} m \right) x^{(n)} \right) \exp\left( i x^{(N-1)} k^{(1)} \right)
\]

(E.1)

with \( k^{(0)} = k \) for the solution of the differential equation which results from Eq. (5.1) by interchanging \( x \) and \( k \). We have to show that these two radial functions (E.1) and (E.2) are identical. To this end, we change in Eq. (E.1) on the \( n \)th level the variables \( x^{(n)} = \sum_{m=1}^{N-n-1} x^{(n-1)} m - \sum_{m=1}^{N-n} k^{(n)} m \) \( \Phi_N^{(\beta)}(k, x) \) at all. Equation (E.4) is just the normalization of a column of a matrix \( k \) because it is independent of \( \beta \). The first equality sign goes back to the radial Gelfand–Tzetlin coordinates. We may use this piece of information, because it is independent of \( \beta \). Hence, to satisfy this when changing the variables, we must have

\[
\prod_{i \neq m} (x^{(n-1)} m - x^{(n-1)} i) = \prod_{i \neq m} (k^{(n-1)} m - k^{(n-1)} i)
\]

(E.3)

for \( n = 1, \ldots, (N-1) \). These are, on the \( n \)th level, \( N-n+1 \) equations for making a change of \( N-n \) variables. However, one has

\[
\sum_{m=1}^{N-n+1} r^{(n)} m = \frac{N-n+1}{2}
\]

(E.4)

on all levels which eliminates one of the \( N-n+1 \) equations.

Of course, the substitution (E.3) is motivated by the radial Gelfand–Tzetlin coordinates which we introduced to construct the recursion formula for \( \beta = 1, 2, 4 \). In this case, the \( r^{(n)} m \) are the moduli squared of a column of a matrix \( U \in U(N-n; \beta) \). Here, we do not use this connection to matrices and groups. We simply view Eq. (E.3) as a standard change of variables in an integral. We underline that Eq. (E.3) does not involve \( \beta \) at all. Equation (E.4) is just the normalization of a column of \( U \) for \( \beta = 1, 2, 4 \). Since it is independent of \( \beta \), it also holds for arbitrary \( \beta \). One can also verify Eq. (E.3) by a direct calculation.

The original domains of integration are \( x^{(n-1)} m \leq x^{(n-1)} i \). In these boundaries, the \( r^{(n)} m \) are positive definite. Hence, to satisfy this when changing the variables, we must have \( k^{(n)} m \leq k^{(n)} i \leq k^{(n)} m+1 \) for the new domains of integration.

To work out the measure in the new variables \( k^{(n)} m \), we interpret Eq. (E.3) as a change to the integration variables \( r^{(n)} m \), too. This yields immediately

\[
\frac{\Delta_{N-n}(x^{(n)})}{\Delta_{N-n+1}(x^{(n-1)})} d[x^{(n)}] = \frac{\Delta_{N-n}(k^{(n)})}{\Delta_{N-n+1}(k^{(n-1)})} d[k^{(n)}].
\]

(E.5)

The first equality sign goes back to the radial Gelfand–Tzetlin coordinates. We may use this piece of information, because it is independent of \( \beta \). The second equality sign is simply due to Eq. (E.3). Using this result, we find for the full and \( \beta \) dependent measure

\[
\frac{\Delta_{N-n}(x^{(n)})}{\Delta_{N-n+1}(x^{(n-1)})} d[x^{(n)}] = \frac{\Delta_{N-n}(k^{(n)})}{\Delta_{N-n+1}(k^{(n-1)})} d[k^{(n)}]
\]

(E.5)
They are the moduli squared of the coordinates on the unit sphere in the complex natural to use the following type of hyper spherical coordinates algebraic. This completes the proof of the symmetry relation (5.3) for arbitrary as Eq. (E.7) does not involve where the integral over \( \vartheta \)

For \( \beta = 1, 2, 4 \), this result is a direct consequence of the invariance of the group measure \( d\mu(U) \). Here, we have derived it for arbitrary \( \beta \). This, in turn, implies that the invariance of the group measure \( d\mu(U) \) is embedded into and reflects much more general features.

We now collect all these intermediate results and plug them into Eq. (E.1). Apart from the expressions in the exponential functions, we have full agreement with the right hand side of Eq. (E.2). Hence, it remains to be shown that the change of variables (E.3) leads to the identity exponential functions, we have full agreement with the right hand side of Eq. (E.2). Hence, it remains to be shown that the change of variables (E.3) leads to the identity

\[
\sum_{n=1}^{N-1} \left( \sum_{m=1}^{N-n+1} x_m^{(n-1)} - \sum_{m=1}^{N-n} x_m^{(n)} \right) k_{N-n+1} + x_1^{(N-1)} k_1 = \sum_{n=1}^{N-1} \left( \sum_{m=1}^{N-n+1} k_m^{(n-1)} - \sum_{m=1}^{N-n} k_m^{(n)} \right) x_{N-n+1} + k_1^{(N-1)} x_1. \tag{E.7}
\]

Since the symmetry relation (5.3) holds for \( \beta = 1, 2, 4 \), we know that Eq. (E.7) must be true in these cases. However, as Eq. (E.7) does not involve \( \beta \) at all, it must also be valid for arbitrary \( \beta \). Inserting this into the right hand side of Eq. (E.1), we recover Eq. (E.2) as desired. We notice that this line of arguing cannot be spoiled by any other contribution to the argument of the exponential functions, because all other terms in the integrand are purely algebraic. This completes the proof of the symmetry relation (5.3) for arbitrary \( \beta \).

**APPENDIX F: CALCULATION OF THE NORMALIZATION CONSTANT \( G_N^{(\beta)} \)**

In the previous App. E, we introduced the coordinates \( r'_n = r_n^{(1)}, n = 1, \ldots, N \) on the first level of the recursion. They are the moduli squared of the coordinates on the unit sphere in the complex \( N \) dimensional space. Thus, it is natural to use the following type of hyper spherical coordinates

\[
\sqrt{r'_n} = \cos \vartheta_n \prod_{\nu=1}^{n-1} \sin \vartheta_\nu, \quad n = 1, \ldots, (N - 1),
\]

\[
\sqrt{r'_N} = \sin \vartheta_{N-1} \prod_{\nu=1}^{N-2} \sin \vartheta_\nu \tag{F.1}
\]

where the positive semidefiniteness of the \( r'_n \) restricts the domain of integration to \( 0 \leq \vartheta_n < \pi/2, n = 1, \ldots, (N - 1) \). Thus, we integrate over a \((2^N)\)th segment of the unit sphere. The measure

\[
d\mu(r') = \prod_{n=1}^{N-1} \sin^{2(N-n)-1} \vartheta_n \cos \vartheta_n d\vartheta_n \tag{F.2}
\]
is, apart from the phase angles, the measure on the unit sphere. Collecting everything, we have

\[
1 = \int d\mu(x', x) = G_N^{(\beta)} \int \left( \prod_{n=1}^{N} \sqrt{r'_n} \right)^{\beta-2} d\mu(r')
\]

\[
= G_N^{(\beta)} N \prod_{n=1}^{N-1} \int_0^{\pi/2} \sin^{(N-n)\beta-1} \vartheta_n \cos^{\beta-1} \vartheta_n d\vartheta_n
\]

\[
= G_N^{(\beta)} N \prod_{n=1}^{N-1} \frac{\Gamma((N-n)\beta/2)\Gamma(\beta/2)}{2\Gamma((N-n+1)\beta/2)} = G_N^{(\beta)} \frac{\Gamma^N(\beta/2)}{2^{N-1}\Gamma(N\beta/2)} \tag{F.3}
\]

where the integral over \( \vartheta_n \) is just Euler’s integral of the first kind.

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APPENDIX G: TRANSLATION INVARIANCE OF $W^{(β)}_{N,ω}(X, K)$

We shift every $x_n$ in the the recursion formula (5.3) for arbitrary $β$ by a constant $\bar{x}$ and obtain

$$\Phi_N^{(β)}(x + \bar{x}, k) = \int d\mu(x', x + \bar{x}) \exp \left( i \left( \sum_{n=1}^{N} x + N\bar{x} - \sum_{n=1}^{N-1} x' \right) k_N \right) \Phi_{N-1}^{(β)}(x', \bar{k})$$

with $x_n + \bar{x} \leq x_n' \leq x_{n+1} + \bar{x}$ as the domains of integration. The change of variables $x_n' \rightarrow x_n' + \bar{x}$ removes $\bar{x}$ from the measure given in Eq. (5.6) and the domains of integration, we find

$$\Phi_N^{(β)}(x + \bar{x}, k) = \exp (i\bar{x}k_N) \int d\mu(x', x) \exp \left( i \left( \sum_{n=1}^{N} x - \sum_{n=1}^{N-1} x' \right) k_N \right) \Phi_{N-1}^{(β)}(x' + \bar{x}, \bar{k}) .$$

(G.2)

We want to employ an induction. We assume that the radial functions for arbitrary $β$ have the property

$$\Phi_N^{(β)}(x + \bar{x}, k) = \exp \left( i\bar{x} \sum_{n=1}^{N} k_n \right) \Phi_{N-1}^{(β)}(x, k) .$$

(G.3)

If this is correct for $N - 1$, formula (G.2) implies that it is also true for $N$. The induction starts with $N = 2$ where the correctness of Eq. (G.3) is immediately obvious from the explicit solution (3.21) for arbitrary $β$. Thus, Eq. (G.3) is valid for all $N$.

Since the $k_n$ are arbitrary and since the sum over all $k_n$ is invariant under the permutations $ω(k)$, the property (G.3) must also be true for every function $Φ_N^{(β)}(x, ω(k))$ with $ω ∈ S_N$. We compare this with the expression

$$\Phi_{N,ω}^{(β)}(x + \bar{x}, k) = \exp \left( i\bar{x} \sum_{n=1}^{N} k_n \right) \frac{\exp \left( i \sum_{n=1}^{N} x_n k_n^{(ω)} \right)}{|Δ_N(x)Δ_N(κ)|^{β/2}} W^{(β)}_{N,ω}(x + \bar{x}, k)$$

(G.4)

which results from the Hankel ansatz (5.10). Hence, we conclude that we necessarily have

$$W^{(β)}_{N,ω}(x + \bar{x}, k) = W^{(β)}_{N,ω}(x, k) .$$

(G.5)

This is the translation invariance.

APPENDIX H: CALCULATION OF $φ^{(4)}(X, K)$

We perform the calculation for $φ^{(4)}_{4}(-ix, k)$ to avoid inconvenient factors of $i$. The operator $L_{x,ω(k)}$ defined in Eq. (6.13) splits into two parts. The first part

$$\bar{Δ}_{x,ω(k)} = \sum_{n=1}^{N} \frac{∂^2}{∂x_n^2} - 4 \sum_{n<m} \frac{1}{(x_n - x_m)^2} .$$

(H.1)

doess not change the order in $k$, while the second one,

$$Δ_{x,ω(k)} = 2 \sum_{n=1}^{N} k_{ω(n)} \frac{∂}{∂x_n} ,$$

(H.2)

raises the order in $k$ by one. Since we can restrict ourselves to one element of the permutation group, we discuss only the identity permutation in the sequel. The symmetry of $x$ and $k$ together with the result for $φ^{(4)}_{4}(x, k)$ suggests one to try an expansion in the composite variable $z_{ij}$ as defined in Eq. (6.8). To this end we define the elementary symmetric functions
\[ e_\nu(z) = \sum_{i_1,j_1 < i_2,j_2 < \ldots < i_\nu,j_\nu} \prod_{l=1}^\nu z_{i_l,j_l}. \]  

(H.3)

Here, we assume the following ordering of the composite index \( \{i_l,j_l\} \), \( i_l < j_l \). We say \( \{i_l,j_l\} < \{i_m,j_m\} \) if \( i_l < i_m \) or \( i_l = i_m \) and \( j_l < j_m \). All indices run to \( N \). The highest order elementary symmetric function is of order \( N(N-1)/2 \) and is given by \( \Delta_N(x) \Delta_N(k) \). The asymptotic formula (6.14) yields the leading term for large arguments. It is the starting point for a recursion in powers of \( z^{-1} \).

\[ W_N^{(4)}(z) = \sum_{\nu=0}^{N(N-1)/2} p_\nu(z^{-1}), \]  

(H.4)

where \( p_\nu(z) \) is a symmetric function of order \( \nu \) in \( x_i \) and \( k_i \). We investigate the action of the two operators defined in Eq. (H.1), (H.2) and find

\[ \Lambda_{x,k} e_\nu(z^{-1}) = -2 \sum_{n\prec m} N \frac{1}{(x_n-x_m)^2} e_{\nu-1}(z_{\neq nm}^{-1}) \]  

(H.5)

\[ \tilde{\Lambda}_{x,k} e_\nu(z^{-1}) = -4 \sum_{n\prec m} N \frac{1}{(x_n-x_m)^2} e_{\nu}(z_{\neq nm}^{-1}) \]

\[ -2 \sum_{\substack{n\prec m \, k\neq m \, k\neq m \, k\neq m}} N \frac{1}{(x_n-x_m)^2} z_{nk}^{-1} z_{mk}^{-1} e_{\nu-2}(z_{\neq nm}^{-1}) \]  

(H.6)

The function \( e_\nu(z_{\neq nm}) \) is the elementary symmetric function \( e_\nu(z) \) with all terms containing \( z_{nm} \) omitted. For \( \nu = 0,1,2 \) we simply have \( p_\nu(z^{-1}) = (-2)^\nu e_\nu(z^{-1}) \). For \( \nu \geq 3 \) the last term in Eq. (H.5) causes corrections to the elementary symmetric functions. This arises due to the mixed derivatives which have to be taken into account in the action of \( \tilde{\Lambda}_{x,k} \) onto \( e_\nu(z^{-1}) \) for \( \nu \geq 3 \). Because of this term the Hankel Ansatz seems becomes increasingly cumbersome as higher values of \( N \) are considered, since more and more correction terms have to be constructed. So far, the construction was only possible for \( N = 4 \). To construct the correction terms explicitly for the case \( N = 4 \), we define a new set of symmetric functions as follows

\[ f_\nu(z^{-1}) = \sum_{k<l} N z_{kl}^{-1} z_{km}^{-1} z_{lm}^{-1} e_{\nu-3}(z_{\neq kl}^{-1} z_{\neq km}^{-1} z_{\neq lm}^{-1}) \]  

(H.7)

Again we have to investigate the action of \( \Lambda_{x,k} \) and \( \tilde{\Lambda}_{x,k} \) on \( f_\nu(z^{-1}) \). We find

\[ \tilde{\Lambda}_{x,k} f_3(z^{-1}) = -4 \sum_{n\prec m} N \frac{1}{(x_n-x_m)^2} f_3(z_{\neq nm}^{-1}) \]  

(H.8)

and

\[ \Lambda_{x,k} f_3(z^{-1}) = -2 \sum_{\substack{n\prec m \, k\neq m \, k\neq m \, k\neq m}} N \frac{1}{(x_n-x_m)^2} z_{nk}^{-1} z_{mk}^{-1} \]  

(H.9)

thus \( f_3(z^{-1}) \) is the desired correction term. We have

\[ p_3(z^{-1}) = -2^3 \left( e_3(z^{-1}) + \frac{1}{2} f_3(z^{-1}) \right) \]  

(H.10)

Fortunately, due to Eq. (H.8) in the next step the correction term itself has not to be corrected and we find

\[ p_4(z^{-1}) = 2^4 \left( e_4(z^{-1}) + \frac{1}{2} f_4(z^{-1}) \right) \]  

(H.11)
Up to now these results are valid for arbitrary $N$. The action of $\Delta_{x,k}$ onto the symmetric function $f_4(z^{-1})$ is not as simple as Eq. \( H.8 \). After a series of manipulations we arrive at

\[
\Delta_{x,k} f_4(z^{-1}) = -4 \sum_{n < m}^N \frac{1}{(x_n - x_m)^2} f_4(z_{nm}^{-1}) \\
-2 \sum_{n < m}^N \frac{1}{(x_n - x_m)^2} z_{nk}^{-1} z_{mk}^{-1} f_2(z_{nm}^{-1}) .
\]  

\( H.12 \)

\( H.13 \)

The contribution \( H.8 \) has to be added to this expression stemming from the action of $\Delta_{x,k}$ onto $e_4(z^{-1})$. On the other hand we calculate

\[
\Lambda_{x,k} f_5(z^{-1}) = -2 \sum_{n < m}^N \frac{1}{(x_n - x_m)^2} f_4(z_{nm}^{-1}) \\
-2 \sum_{n < m}^N \frac{1}{(x_n - x_m)^2} z_{nk}^{-1} z_{mk}^{-1} e_2(z_{nm}^{-1}) .
\]  

\( H.14 \)

\( H.15 \)

Thus, we have to find yet another correction term to compensate the second term in Eq. \( H.13 \). We define

\[
f_5'(z^{-1}) = \sum_{i_1 < i_2 < i_3 < i_4} \prod_{r < j} z_{i_r i_j}^{-1} \sum_{r < j} z_{i_r i_j} = \sum_{l < m} (z_{jl}^{-1} z_{jm}^{-1} z_{kl}^{-1} z_{km}^{-1})
\]  

\( H.16 \)

and see that $\Lambda_{x,k} f_5'(z^{-1})$ yields exactly the desired second term of Eq. \( H.13 \). Pushing forward this procedure becomes more complicated step by step. There seems to be no obvious way of constructing the additional terms. Apparently for higher orders the correction terms also involve an increasing amount of indices. Nevertheless for $N = 4$ we are already at the end of the recursion. Then the general expression

\[
p_5(z^{-1}) = -2^5 \left( e_5(z^{-1}) + \frac{1}{2} f_5(z^{-1}) + \frac{1}{4} f_5'(z^{-1}) \right)
\]  

\( H.17 \)

reduces to

\[
p_5(z^{-1}) = -72 e_5(z^{-1}) .
\]  

\( H.18 \)

The last step can readily be done, since the action of $\Delta_{x,k}$ onto $e_5(z^{-1})$ is already known by Eq. \( H.6 \). Thus we arrive at

\[
p_6(z^{-1}) = 288 e_6(z^{-1}) .
\]  

\( H.19 \)

Importantly, we have

\[
\Delta_{x,k} e_6(z^{-1}) = \Delta_{x,k} \frac{1}{\Delta_4(z) \Delta_4(k)} = 0 .
\]  

\( H.20 \)

That means, the sequence finishes after the sixth step. Collecting everything and observing that, for $N = 4$, $f_5(z) = 2e_5(z)$ and $f_6(z) = 4e_6(z)$, we obtain

\[
W_4^{(4)}(x, k) = \sum_{\nu=1}^6 (-2)^\nu e_\nu(z^{-1}) + \sum_{\nu=3}^6 (-2)^\nu f_\nu(z^{-1}) - 8 e_5(z^{-1}) + 96 e_6(z^{-1}) .
\]  

\( H.21 \)

This can be rewritten more in a more compact way as

\[
W_4^{(4)}(x, k) = \frac{1}{\Delta_4(x) \Delta_4(k)} \left( \prod_{i < j} (2 - z_{ij}) + \right.
\]
\[ \frac{1}{2} \sum_{l<m<n} \prod_{i<j \neq k \neq l \neq m \neq n} (2 - z_{ij}) + \frac{1}{4} \sum_{l<m} \prod_{i<j \neq k \neq l \neq m \neq n} (2 - z_{ij}) \]  

(H.22)

which yields Eq. (6.10).