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INTERPOLATION THEOREMS FOR SELF-ADJOINT OPERATORS

SHIJUN ZHENG

ABSTRACT. We prove a complex and a real interpolation theorems on Besov spaces and Triebel-Lizorkin spaces associated with a self-adjoint operator $L$, without assuming the gradient estimate for its spectral kernel. The result applies to the cases where $L$ is a uniformly elliptic operator or a Schrödinger operator with electromagnetic potential.

1. Introduction and main result

Interpolation of function spaces has played an important role in classical Fourier analysis and PDEs [1, 6, 12, 15, 18]. Let $L$ be a selfadjoint operator in $L^2(\mathbb{R}^n)$. Then, for a Borel measurable function $\phi: \mathbb{R} \to \mathbb{C}$, we define $\phi(L)$ using functional calculus. In [15, 11, 2, 17] several authors introduced and studied the Besov spaces and Triebel-Lizorkin spaces associated with Schrödinger operators. In this note we present an interpolation result on these spaces for $L$.

Let $\{\varphi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ be a dyadic system satisfying (i) supp $\varphi_0 \subset \{x : |x| \leq 1\}$, supp $\varphi_j \subset \{x : 2^{j-2} \leq |x| \leq 2^j\}$, $j \geq 1$, (ii) $|\varphi_j^{(k)}(x)| \leq c_k 2^{-kj}$ for all $j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (iii) $\sum_{j=0}^\infty |\varphi_j(x)| \approx 1$, $\forall x$. Let $\alpha \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The inhomogeneous Besov space associated with $L$, denoted by $B^\alpha_{p,q}(\mathcal{L})$, is defined to be the completion of $\mathcal{S}(\mathbb{R}^n)$, the Schwartz class, with respect to the norm

$$\|f\|_{B^\alpha_{p,q}(\mathcal{L})} = \left(\sum_{j=0}^\infty 2^{j\alpha q} \|\varphi_j(L)f\|_{L^p}^q\right)^{1/q}.$$
Similarly, the inhomogeneous Triebel-Lizorkin space associated with $\mathcal{L}$, denoted by $F_p^{s,q}(\mathcal{L})$, is defined by the norm
\[
\|f\|_{F_p^{s,q}(\mathcal{L})} = \left\| \left( \sum_{j=0}^{\infty} 2^{jq|\varphi_j(\mathcal{L}) f|^q} \right)^{1/q} \right\|_{L^p}.
\]

The following assumption on the kernel of $\phi_j(\mathcal{L})$ is fundamental in the study of function space theory. Let $\phi(\mathcal{L})(x,y)$ denote the integral kernel of $\phi(\mathcal{L})$.

**Assumption 1.1.** Let $\phi_j \in C^\infty_0(\mathbb{R})$ satisfy conditions (i), (ii) above. Assume that there exist some $\varepsilon > 0$ and a constant $c_n > 0$ such that for all $j$

\[
|\phi_j(\mathcal{L})(x,y)| \leq c_n \frac{2^{nj/2}}{(1 + 2^{j/2}|x-y|)^{n+\varepsilon}}.
\]

This is the same condition assumed in [28, 18] except that we drop the gradient estimate condition on the kernel. This is the case when $\mathcal{L}$ is a Schrödinger operator $-\Delta + V$, $V \geq 0$ belonging to $L^1_{\text{loc}}(\mathbb{R}^n)$ [14, 19] or $\mathcal{L}$ is a uniformly elliptic operator in $L^2(\mathbb{R}^n)$ [8, Theorem 3.4.10].

In what follows, $[A,B]_{\theta}$ denotes the usual complex interpolation between two Banach spaces; $(A,B)_{\theta,r}$ the real interpolation, see Section 2. The notion $T: X \to Y$ means that the linear operator $T$ is bounded from $X$ to $Y$.

**Theorem 1.2** (complex interpolation). Suppose that $\mathcal{L}$ is a selfadjoint operator satisfying Assumption [17]. Let $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $s_0, s_1 \in \mathbb{R}$ and

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.
\]

(a) If $1 < p_i < \infty$, $1 < q_i < \infty$, $i = 0, 1$, then
\[
[F_{p_0}^{s_0,q_0}(\mathcal{L}), F_{p_1}^{s_1,q_1}(\mathcal{L})]_\theta = F_p^{s,q}(\mathcal{L}).
\]

(b) If $1 \leq p_i \leq \infty$, $1 \leq q_i \leq \infty$, $i = 0, 1$, then
\[
[B_{p_0}^{s_0,q_0}(\mathcal{L}), B_{p_1}^{s_1,q_1}(\mathcal{L})]_\theta = B_p^{s,q}(\mathcal{L}).
\]

c) If $T: F_{p_0}^{s_0,q_0}(\mathcal{L}) \to F_{p_0}^{s_0,q_0}(\mathcal{L})$ and $T: F_{p_1}^{s_1,q_1}(\mathcal{L}) \to F_{p_1}^{s_1,q_1}(\mathcal{L})$, then $T: F_p^{s,q}(\mathcal{L}) \to F_p^{s,q}(\mathcal{L})$, where $s, \tilde{s}, \tilde{p}, \tilde{q}$ and $s_i, \tilde{s}_i, \tilde{p}_i, \tilde{q}_i$, satisfy the same relations as those for $s, p, q$ and $s_i, p_i, q_i$, $1 < p_i, q_i < \infty$. Similar statement holds for $B_p^{s,q}(\mathcal{L})$.

Complex interpolation method originally was due to Calderón [4] and Lions and Peetre [16]; see also [13, 24]. The classical interpolation theory for Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$ has been given
systematic treatments in [20], [3], and [25, 26]. There are interesting discussions on interpolation theory in [20] and [26, 25, 22] for generalized Besov spaces associated with differential operators, which requires certain Riesz summability for \( L \) that seems a nontrivial condition to verify. Nevertheless, we would like to mention that the Riesz summability, the spectral multiplier theorem and the decay estimate in (1) are actually intimately related [10, 18].

The real interpolation result for \( B_{\alpha,q}^p(\mathbb{R}^n) \), \( F_{\alpha,q}^p(\mathbb{R}^n) \) can be found in [20] and [26, 25]. Following the proof as in the classical case, but applying the estimate in (1) instead of spectral multiplier result, we obtain

**Theorem 1.3** (real interpolation). Suppose that \( L \) satisfies Assumption [1, 2]. Let \( 0 < \theta < 1, 1 \leq r \leq \infty, s = (1 - \theta)s_0 + \theta s_1, s_0 \neq s_1. \)

(a) If \( 1 \leq p < \infty, 1 \leq q_1, q_2 \leq \infty, \) then
\[
(F_{p}^{s_0,q_0}(L), F_{p}^{s_1,q_1}(L))_{\theta,r} = B_{p}^{s,r}(L).
\]

(b) If \( 1 \leq p, q_1, q_2 \leq \infty, \) then
\[
(B_{p}^{s_0,q_0}(L), B_{p}^{s_1,q_1}(L))_{\theta,r} = B_{p}^{s,r}(L).
\]

The homogeneous spaces \( \dot{B}_{p}^{\alpha,q}(L) \) and \( \dot{F}_{p}^{\alpha,q}(L) \) can be defined using \( \{\varphi_j\}_{j=0}^{\infty} \) in (i) to (iii), instead of \( \{\varphi_j\}_{j=0}^{\infty} \). Then the analogous results of Theorem 1.2 and Theorem 1.3 hold.

2. INTERPOLATION FOR \( L \)

Theorem 1.2 and Theorem 1.3 are part of the abstract interpolation theory for \( L \). In this section we present the outline of their proofs. It was mentioned in [22] that the interpolation associated with \( L \) is a “subtle and difficult” subject, which normally relies on the very property of \( L \).

2.1. Complex interpolation. The proof of Theorem 1.2 is similar to that given in [25] in the Fourier case. The insight is that the three line theorem (involved in Riesz-Thorin or Calderón’s constructive proof for \( L^p \) spaces) reflects the fact that the value of an analytic function in the interior of a domain is determined by its boundary values.

**Definition 2.2.** Let \( (A_0, A_1) \) be an interpolation couple, i.e., \( A_0, A_1 \) are (complex) Banach spaces, linearly and continuously embedded in a Hausdorff space \( \mathcal{H} \). The space \( A_0 \cap A_1 \) is endowed with the norm \( \|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}, \overline{a}, 0, 1\} \). The space \( A := A_0 + A_1 \) is endowed with the norm
\[
\|a\|_A = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1\}.
\]
Let $S = \{ z \in \mathbb{C} : 0 \leq \Re z \leq 1 \}$ and $\bar{S}$ its closure. Denote $F$ the class of all $A$-valued functions $f(z)$ on $\bar{S}$ such that $z \mapsto f(z) \in A$ is analytic in $S$ and continuous on $\bar{S}$, satisfying

(i) \[ \sup_{z \in \bar{S}} \| f(z) \|_A \text{ is finite}. \]

(ii) The mapping $t \mapsto f(j + it) \in A_j$ are continuous from $\mathbb{R}$ to $A_j$, $j = 0, 1$.

Then $F$ is a Banach space with the norm \[ \| f \|_F = \max_j \{ \sup_t \| f(j + it) \|_{A_j} \}. \]

For $0 < \theta < 1$ we define the interpolation space $[A_0, A_1]_\theta$ as 

$$[A_0, A_1]_\theta := \{ a \in A : \exists f \in F \text{ with } f(\theta) = a \}. $$

Then $[A_0, A_1]_\theta$ is a Banach space equipped with the norm \[ \| a \|_\theta := \inf \{ \| f \|_F : f \in F \text{ and } f(\theta) = a \}. \]

2.3. Outline of the proof of Theorem 1.2. Let $\{ \phi_j \}, \{ \psi_j \}$ satisfy the conditions in (i)-(iii) and \( \sum_j \psi_j(x) \phi_j(x) = 1 \). Define the operators $S : f \mapsto \{ \phi_j(\mathcal{L}) f \}$, and $R : g \mapsto \sum_j \psi_j(\mathcal{L}) g$. The proof for part (a) follows from the commutative diagram

$$
\begin{array}{c}
F^s_q(\mathcal{L}) \\
\downarrow Id \\
F^s_q(\mathcal{L})
\end{array}
\xrightarrow{S}
\begin{array}{c}
L^p(\ell^q) \\
\downarrow Id \\
L^p(\ell^q)
\end{array}
\xleftarrow{R}
\begin{array}{c}
F^s_q(\mathcal{L})
\end{array}
$$

and Lemma 2.4 and Lemma 2.5 which are interpolation results for Banach space valued $L^p$ and $\ell^q$ spaces [25].

Lemma 2.4. Let $0 < \theta < 1$, $1 \leq p_0, p_1 < \infty$ and \( p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1} \). Let $A_0, A_1$ be Banach spaces. Then

\[ [L^{p_0}(A_0), L^{p_1}(A_1)]_\theta = L^p([A_0, A_1]_\theta). \]

If $p_1 = \infty$, then (2) holds with $L^{p_1}(A_1)$ replaced by $L^\infty_0(A_1)$, the completion of simple $A_1$-valued functions with the esssup norm.

As in [25], denote $\ell^q(A_j)$ the space of functions consisting of $a = \{ a_j \}$, $a_j \in A_j$ ($A_j$ being Banach spaces) equipped with the norm \[ \| a \|_{\ell^q(A_j)} = \left( \sum_j \| a_j \|_{A_j}^q \right)^{1/q} \].
Lemma 2.5. Let $0 < \theta < 1$, $1 \leq q_0, q_1 < \infty$ and $q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$. Let $A_j$ be Banach spaces, $j \in \mathbb{N}$. Then

\begin{equation}
\{\ell^{q_0}(A_j), \ell^{q_1}(B_j)\}_\theta = \ell^q([A_j, B_j]_\theta).
\end{equation}

If $q_1 = \infty$, then

\begin{equation}
\{\ell^{q_0}(A_j), \ell^{\infty}(B_j)\}_\theta = \ell^q([A_j, B_j]_\theta) = \{\ell^{q_0}(A_j), \ell^{\infty}(B_j)\}_\theta,
\end{equation}

where $\ell^{\infty}(B_j) := \{c_j \in \ell^{\infty}(B_j) : \|c_j\|_{B_j} \to 0 \text{ as } j \to \infty\}$.

If $1 \leq q_0, q_1 < \infty$, (3) also follows from Lemma 2.4 as a special case where the underlying measure space can be taken as $(X, \mu) = \mathbb{Z}$. If $q_1 = \infty$, then the remark in [25, Subsection 1.18.1] shows that the second statement in Lemma 2.5 is also true.

In the diagram above in order to show $S, R$ are continuous mappings, we need the following well-known lemma.

Lemma 2.6. Let $h(x)$ be a monotely nonincreasing, radial function in $L^1(\mathbb{R}^n)$. Let $h_j(x) = 2^{mn/2}h(2^{j/2}x)$ be its scaling. Then for all $f$ in $L^1_{\text{loc}}(\mathbb{R}^n)$

\[|\int h_j(x - y)f(y)dy| \leq c_n\|h\|_1 Mf(x),\]

where $Mf$ denotes the usual Hardy-Littlewood maximal function.

Evidently the decay estimate in [1] and Lemma 2.6 imply the continuity of $S$ and $R$, in light of the $L^p(\ell^q)$-valued maximal inequality.

The proof for $B_p^{s,q}(\mathcal{L})$ in part (b) proceeds in a similar way.

2.7. Real interpolation. Peetre’s $K$-functional [21] is defined as

\[K(t, a) := K(t, a; A_0, A_1) = \inf\|a_0\|_{A_0} + t\|a_1\|_{A_1},\]

where the infimum is taken over all representations of $a = a_0 + a_1$, $a_i \in A_i$. Let $0 < q \leq \infty, 0 < \theta < 1$. For a given interpolation couple $(A_0, A_1)$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is given by

\[(A_0, A_1)_{\theta,q} = \{a \in A_0 + A_1 : \|a\|_{(A_0, A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta q}K(t, a)^{\frac{q}{\theta}} dt\right)^{1/q} < \infty\}\]

with usual modifications if $q = \infty$.

Proof of Theorem 1.3 is similar to [26, Subsection 2.4.2] and [3, Theorem 6.4.5]. Define $\ell^{s,q}(A) = \{a = \{a_j\} : \|a\|_{\ell^{s,q}(A)} = \|\{2^{js}\|a_j\|_{A}\|_{\ell^q} < \infty\}$. For Besov spaces it follows from

\[(\ell^{s_0,q_0}(A_0), \ell^{s_1,q_1}(A_1))_{\theta,q} = \ell^{s,q}((A_0, A_1)_{\theta,q}),\]

$s = (1 - \theta)s_0 + \theta s_1, q^{-1} = (1 - \theta)q_0^{-1} + \theta q_1^{-1}$ and the commutative diagram for $B_p^{s,q}(\mathcal{L})$. Consult [26, 25] or [20, Chapter 5, Theorem 6]; both of their proofs rely on retraction method. Also see [3] for a different
proof in the special case involving Sobolev spaces. In the general case
[3] suggests using a more concrete characterization of the $K$-functional
for the Lorentz space $L^{pq}$.

For the $F$-space the proof follows from the commutative diagram for
$F^{s,q}(\mathcal{L})$ and

$$(L^{p_0}(A_0, w_0), L^{p_1}(A_1, w_1))_{\theta,p} = L^p((A_0, A_1)_{\theta,p}, w),$$

where $p^{-1} = (1 - \theta)p_0^{-1} + \theta p_1^{-1}$, $w = w_0^{1-\theta} w_1^\theta$, $w_0, w_1$ being two weight
functions [20, Chapter 5].

2.8. Schrödinger operators with magnetic potential. From [14],
[28] or [18] we know that if the heat kernel of $L$ satisfies the upper
Gaussian bound

$$(5) \quad |e^{-t\mathcal{L}(x, y)}| \leq c_n t^{-n/2} e^{-c|x-y|^2/t}$$

then the kernel decay in Assumption 1.1 holds. Let

$$H = -\sum_{j=1}^n (\partial_{x_j} + ia_j)^2 + V,$$

where $a_j(x) \in L^2_{loc}(\mathbb{R}^n)$ is real-valued, $V = V_+ - V_-$ with $V_+ \in L^1_{loc}(\mathbb{R}^n)$,
$V_- \in K_n$, the Kato class [23]. Proposition 5.1 in [7] showed that (5)
is valid for $-\Delta + V$ if $V_+ \in K_n$ and $\|V_+\|_{K_n} < \gamma_n := \pi^{n/2}/\Gamma(n/2 - 1)$,
n $\geq 3$, whose proof evidently works for $V_+ \in L^1_{loc}$. By the diamagnetic inequality [23, Theorem B.13.2], we see that (5) also holds for $H$
provided $\|V_-\|_{K_n} < \gamma_n$, $n \geq 3$.

As another example, a uniformly elliptic operator is given by

$$\mathcal{L} = -\sum_{j,k=1}^n \partial_{x_j}(a_{jk}\partial_{x_k}),$$

where $a_{jk}(x) = a_{kj}(x) \in L^\infty(\mathbb{R}^n)$ are real-valued and satisfy the ellipticity condition $(a_{jk}) \approx I_n$. Then [19, Theorem 1] tells that (5) is true
provided that the infimum of its spectrum $\inf \sigma(\mathcal{L}) = 0$.

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