Elliptic Algebra and Integrable Models for Solitons on Noncommutative Torus \( \mathcal{T} \)

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Abstract

We study the algebra \( A_n \), the basis of the Hilbert space \( H_n \) in terms of \( \theta \) functions of the positions of \( n \) solitons. Then we embed the Heisenberg group as the quantum operator factors in the representation of the transfer matrices of various integrable models. Finally we generalize our result to the generic \( \theta \) case.

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1 Solitons on noncommutative plane

In the noncommutative plane \( R^2 \), the coordinates \( x^1 \) and \( x^2 \) satisfy the following relation:

\[
[x^i, x^j] = i\theta_{ij},
\]

(1)

here \( \theta \) is a constant. The algebra \( \mathcal{A} \) associated with this space is generated by the functions of \( x^1 \) and \( x^2 \). The functional form of the algebra \( \mathcal{A} \) is defined by the Moyal \(*\) product

\[
f * g(x) = e^{i\epsilon_{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)g(y)|_{x=y}.
\]

(2)

The derivative \( \partial_i \) is the infinitesimal translation automorphism of the algebra \( \mathcal{A} \):

\[
x^i \rightarrow x^i + \epsilon^i,
\]

(3)

where \( \epsilon^i \) is a c-number. For algebra \( \mathcal{A} \) this automorphism is internal:

\[
\partial_i f(x) = i\theta_{ij}[x^j, f(x)] = i\theta_{ij}[x^j, f(x)],
\]

(4)

here \( \theta_{ij} = \theta \epsilon_{ij} \)

The operator form of \( \mathcal{A} \) is generated by Weyl Moyal transformation.

\[
a^\dagger = \frac{1}{\sqrt{2\theta}}(x^1 + ix^2), \quad a = \frac{1}{\sqrt{2\theta}}(x^1 - ix^2),
\]

(5)

which obey

\[
[a, a^\dagger] = 1.
\]

(6)
Since $a$ and $a^\dagger$ satisfy the commutation relations of the creation and annihilation operators, we can identify the function $f(x^1, x^2)$ as the functions of $a$ and $a^\dagger$ acting on the standard Fock space $\mathcal{H}$ of the creation and annihilation operators:

$$\mathcal{H} = \{|0\>, |1\>, \cdots, |n\>, \cdots\}. \quad (7)$$

where $|0\rangle$ and $|n\rangle$ satisfy:

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a^\dagger a|n\rangle = n|n\rangle. \quad (8)$$

The Weyl Moyal transformation maps the ordinary commutative functions onto operators in the Fock space $\mathcal{H}$:

$$f(x) = f(z = x^1 - ix^2, \bar{z} = x^1 + ix^2) \mapsto \hat{f}(a, a^\dagger) = \int \frac{d^2xd^2y}{(2\pi)^2} f(x) e^{ip(\sqrt{2}\theta a - z) + p(\sqrt{2}\theta a^\dagger - \bar{z})}, \quad (9)$$

where:

$$p = \frac{y^1 + iy^2}{2}, \quad \bar{p} = \frac{y^1 - iy^2}{2}. \quad (10)$$

It is easy to see that if $f \mapsto \hat{f}$, $g \mapsto \hat{g}$,

then

$$f \ast g \mapsto \hat{f}\hat{g} \quad (12)$$

and

$$\int d^2xf(x) \rightarrow \pi\theta Tr \hat{f}(a, a^\dagger). \quad (13)$$

The translations of $R^2$ are generated by $\hat{\partial}_i$ which are isomorphism to $\mathcal{A}$ while applying on the Fock space $\mathcal{H}$:

$$\hat{\partial}_i \longleftrightarrow i\theta_{ij}\hat{x}_j. \quad (14)$$

In paper [4], Harvey, Kraus and Larsen introduced a quasi-unitary operator to generate various soliton solutions in noncommutative geometry. In noncommutative plane $R^2$, this operator is defined as

$$T \equiv \frac{a^\dagger}{\sqrt{a^\dagger a}}. \quad (15)$$

Acting this operator $T$ on the basis of the Hilbert space $\mathcal{H}$, we have

$$T|n\rangle = |n + 1\rangle, \quad \langle n|T^\dagger = \langle n + 1|. \quad (16)$$

and

$$T|n\rangle\langle n|T^\dagger = |n + 1\rangle\langle n + 1|. \quad (17)$$

This means that

$$TP_nT^\dagger = P_{n+1}, \quad (18)$$

where $P_n = |n\rangle\langle n|$ denote the projection operator onto the $n$-th states and $P_n^2 = P$. Thus we have

$$TT^\dagger|n\rangle = |n\rangle, (n \leq 1) \quad \text{and} \quad TT^\dagger|0\rangle = 0, \quad (19)$$

and

$$TT^\dagger = 1 - |0\rangle\langle 0| = 1 - P_0. \quad (20)$$

$T$ is the quasiunitary soliton generating operator.
In the noncommutative torus $\mathcal{T}$, the algebra $A$ is generated by the Wilson Loop $\hat{U}_i$, ($i = 1, 2$). The arbitrary element $a \in A$ is
\[
a = \sum_{j_1 j_2} c_{j_1 j_2} U_1^{j_1} U_2^{j_2} \tag{21}\]

For the periodicities $l$ and $2\pi l \tau$ of the torus, the generators of the algebra $A$ are
\[
U_1 = e^{i \theta x^1}, \quad U_2 = e^{i (\tau_2 x^1 - \tau_1 x^2)}. \tag{22}\]

Since $[x^1, x^2] = i \theta$ locally, so
\[
\hat{U}_1 \hat{U}_2 = \hat{U}_2 \hat{U}_1 e^{i 2 \pi \tau \theta}. \tag{23}\]

Now let us consider the integral torus case $\frac{\tau_2}{2 \pi} \theta^2 = A \in \mathbb{N}$ (or $\mathbb{Z}_+$) i.e. the normalized area $A$ of the torus is an integer. Then the Wilson loop $U_1$ and $U_2$ are commutative
\[
U_1 U_2 = U_2 U_1. \tag{24}\]

We orbifold $\mathcal{T}$ into $\mathcal{T}_{n \times n} = \mathcal{T}_n$ by introducing
\[
W_i = (U_i)^{\frac{1}{n}}, \tag{25}\]

then on $\mathcal{H}_n$, the Hilbert space on $\mathcal{T}_n$, we will have noncommutative algebra $A_n$ generated by
\[
W_1 W_2 = W_2 W_1 e^{2 \pi i \theta \frac{n}{n}} \equiv W_2 W_1 \omega \tag{26}\]

which satisfy
\[
W_1^n = W_2^n = 1 \tag{27}\]

where $\omega = e^{2 \pi i \frac{n}{n}}$.

The Basis vectors of the Hilbert space $\mathcal{H}_n$ are
\[
V_a = \sum_{b=1}^{n} F_{-a,b}, (a = 1, 2, \ldots, n), \quad F_{\alpha} = F_{\alpha_1, \alpha_2} = e^{i \pi n \alpha_1} \prod_{j=1}^{n} \sigma_{\alpha_1, \alpha_2} (z_j - \frac{1}{n} \sum_{k=1}^{n} z_k), \tag{28}\]

here $\alpha \equiv (\alpha_1, \alpha_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$, and
\[
\sigma_\alpha(z) = \theta \left[ \frac{z + \alpha_1}{z + \frac{1}{n}}, \frac{z + \alpha_2}{z + \frac{1}{n}} \right](z, \tau). \tag{29}\]

The $\theta$ function can be transformed to a operator form by the Weyl Moyal transformation:
\[
\theta(z) = \sum_{m} e^{i \pi m^2 \tau + 2 \pi i m z} \to \theta(\hat{z}) = \sum_{m} e^{i \pi m^2 \tau} : U_1^{m} U_2^{m} : \tag{30}\]

Since
\[
W_i : U_1^{m} U_2^{m} := \omega^{\pm m} : U_1^{m} U_2^{m} : \tag{31}\]

we have
\[
W_1 V_a(z_1, \ldots, z_n) = (\prod_{i=1}^{n-1} T^{(i)}_n) T^{(n)}_{-\tau} V_a(z_1, \ldots, z_n) \tag{32}\]
\[
W_2 V_a(z_1, \ldots, z_n) = V_a(z_1 + \frac{1}{n}, \ldots, z_n + \frac{1}{n} - 1). \tag{33}\]
where
\[ T_a^{(i)} f(z) = e^{\pi i a^2 \tau + 2 \pi i a z_i} f(z_1, \ldots, z_n). \] (34)

Substituting the expressions of \( V_a \) we get
\[ W_1 V_a(z_1, \ldots, z_n) = V_{a-1}(z_1, \ldots, z_n), \quad W_2 V_a(z_1, \ldots, z_n) = e^{-2 \pi i a^2} V_a(z_1, \ldots, z_n). \] (35)

Then the algebra
\[ \mathcal{A}_n = \{ W^\alpha \equiv W^{\alpha_1 \alpha_2} = W_1^{\alpha_1} W_2^{\alpha_2} \} \] (36)
is realized as the \( 2^n \times 2^n \) Heisenberg matrices \( I^\alpha \),
\[ (I_\alpha)_{ab} = \delta_{\alpha + \alpha_1, a_2} \omega_{ba2} \] (37)

Corresponding to the \( \partial_t \) on \( R^2 \), we have a \( su_n(T_n) \) acting on \( \mathcal{H}_n \) \[ su_n(T_n) : \{ E_\alpha | \alpha \neq (0, 0) \}. \] (38)

Here
\[ E_\alpha = (-1)^{\alpha_1} \sigma_\alpha(0) \sum_j \prod_{k \neq j} \sigma_\alpha(z_{jk}) \left[ \frac{I}{n} \sum_{i \neq j} \frac{\sigma_\alpha'(z_{ji})}{\sigma_\alpha(z_{ji})} - \partial_j \right], \alpha = (\alpha_1, \alpha_2) \neq (0, 0) \equiv (n, n), \] (39)and
\[ E_0 = - \sum_j \partial_j, \] (40)

where \( z_{jk} = z_j - z_k \), \( \partial_j = \frac{\partial}{\partial z_j} \). The commutation relation between \( E_\alpha \) and \( E_\gamma \) is
\[ [E_\alpha, E_\gamma] = (\omega^{-\alpha_2 \gamma_1} - \omega^{-\alpha_1 \gamma_2}) E_{\alpha + \gamma}, \] (41)
or in more common basis, let \( E_{ij} \equiv \sum_{\alpha \neq 0} (I^\alpha)_{ij} E_\alpha \), we have
\[ [E_{jk}, E_{lm}] = E_{jm} \delta_{kl} - E_{lk} \delta_{jm}. \] (42)

This commutation rule can also be obtained from the quasiclassical limit of the representation of the Sklyanin algebra [19].

Since the Wilson loops \( W_1 \) and \( W_2 \) acting on the noncommutative covering torus \( T \) is to shift \( z_i \) to \( (z_i + \frac{1}{n} - \delta_{in} \tau) \) and \( (z_i + \frac{1}{n} - \delta_{in} \tau) \) respectively, we can get the automorphism of \( E_\beta \in su_2(T) \) by noncommutative gauge transformation \( w^\alpha \in A \)
\[ W_1 E_\alpha(z_i) W_1^{-1} = \omega^{-\alpha_2} E_\alpha(z_i), \] (43)
\[ W_2 E_\alpha(z_i) W_2^{-1} = \omega^{\alpha_1} E_\alpha(z_i). \] (44)

Let \( E_\alpha \in g \) to act on \( V_a \), we find that
\[ E_\alpha V_a = \sum_b (I_\alpha)_{ba} V_a. \] (45)

Next, we know that
\[ W_\alpha V_a = \sum_b (I_\alpha)_{ba} V_b, \] (46)
so on \( \mathcal{H}_n \), we establish the isomorphism:
\[ su_n(T) \leftrightarrow A; \quad E_\alpha \leftrightarrow W_\alpha. \] (47)

The operator form of the projection operators becomes
\[ \frac{1}{n} \sum_\beta W^{0 \beta}(I_\beta)_{ii} = P_i = |V_i\rangle \langle V_i| \] (48)
and the ABS operators is simply
\[ E_{10} \cong W_1 = \sum_a |V_{a+1}\rangle \langle V_a| \] (49)
3 The integrable models for the solitons on noncommutative torus \( \mathcal{T} \)

In this section, we will embed the \( su_n(\mathcal{T}) \) derivative operators as the "quantum" operator factors in the representation of the transfer matrix (Lax operator) of the various integrable models i.e.

The elliptic Gaudin model on noncommutative space \( \mathbb{T} \) is defined by the transfer matrix (quantum Lax operator):

\[
L_{ij}^G(u) = \sum_{\alpha \neq (0,0)} w_\alpha(u) E_{\alpha}(I_{\alpha})_{ij}
\]

where \( w_\alpha(u) = \frac{\partial(0)_{\alpha}(u)}{\sigma_{\alpha}(0)} \) and \( E_{\alpha} \) and \( I_{\alpha} \) are the generators of \( su(n) \) (or \( A_{n-1} \) Weyl) and \( G_{\mathcal{H}}(n) \) respectively. This transfer matrix can also be obtained as the nonrelativistic limit of the Ruijsenaars-Macdonald operators. The common eigenfunctions and eigenvalues of Gaudin model is solved in terms of the Bethe ansatz \[21\]. Now we substitute the difference representation of \( su(n) \) \( E_{\alpha} \) \[33\] into \[50\], we get a factorized \( L \) of the Gaudin model

\[
L_G(u)_j^i = E_0 + \sum_{\alpha \neq (0,0)} E_{\alpha} I_{\alpha} j^i = \sum_k \phi(u, z)_j^k \phi^{-1}(u, z)_j^k \partial_u - l \sum_k \partial_u \phi(u, z)_j^k \phi^{-1}(u, z)_j^k,
\]

where the factors are the vertex face intertwiner

\[
\phi(u, z)_j^i = \theta \left[ \frac{\frac{1}{2} - \frac{1}{n}}{z_j - z_k + \frac{n-1}{2}}, n \right].
\]

For the Gaudin model on noncommutative torus, the \( z_i \) is the orgin (position) of the \( i \)-th soliton, \( \partial_i \) as its infinitesimal translation is equivalently to \( \{z_i^*\} \).

Next, the elliptic Calogero Moser model is defined by the Himiltonian:

\[
H = \sum_{i=1}^n \partial_i^2 + \sum_{i \neq j} g \varphi(z; j)
\]

where \( \varphi(z) = \partial^2 \sigma(z) \). The corresponding Lax operator is

\[
L_{CM}(u)_j^i = (p_i - \frac{l}{n} \delta^i_j) \ln(\Delta(z)) \partial_j^i - \frac{l}{n} \sigma'(0) (1 - \delta^i_j) \frac{\sigma(u + z_{ji})}{\sigma(u) \sigma(z_{ji})}
\]

This Lax operator can be gauge transformed into the factorized \( L \) \[51\] of the Gaudin model by the following matrix:

\[
G(u; z)_j^i \equiv \phi(u; z)_j^i \prod_{l \neq j} \theta_{l; \frac{1}{2}}(z_{ji})
\]

The C.M. model gives the dynamics of a long distance interaction between \( n \)-bodies located at \( z_i \) \( (i = 1, \cdots, n) \). On noncommutative torus, it gives the dynamics of \( n \) solitons and \( z_i \) becomes the position of the center of the \( i \)-th soliton. According to \[9\], the interaction between \( n \)-solitons is the Laplacian of a Kähler potential \( K \), which is the logarithm of a Vandermonde determinant. Actually we have

\[
\sum_{i \neq j} \varphi(z; j) = \sum_i \partial_i^2 \log \prod_{j \neq k} \sigma(z_j - z_k) \equiv \sum_i \partial_i^2 K(u, z)
\]

and

\[
e^K(u, z) = \prod_{j \neq k} \sigma(z_j - z_k) \sigma(nu + \frac{n-1}{n}) = \text{det}(\phi^*_j) \equiv \sigma(nu + \frac{n-1}{2}) \prod_{i \neq j} \sigma(z_i - z_j).
\]
The variable $u$ of the marked torus is the spectral parameter or evaluation parameter of Lax matrix $K^i_j$.

This Ruijsenaars operators are related to the quantum Dunkle operators and the $q$-deformed Kniznik Zamolodchikov Bernard equations. The eigenfunctions could be also expressed in terms of double Bloch wave as the algebraic geometric methods [22]. We will show this in the more familiar formalism of the elliptic quantum group.

4 The $Z_n \times Z_n$ Heisenberg group in case of the general $\theta$

For the generic $\theta$ case, as in paper [23] we find that $\theta \tau = \eta$, here $\eta$ is the crossing parameter and the $Z_n \times Z_n$ Heisenberg group of shift of solitons is realized by the Sklyanin algebra $S_{\tau,\eta}$. The noncommutative algebra $A$ is realized as Elliptic quantum group $E_{\tau,\eta}$. The evaluation module of $E_{\tau,\eta}$ is expressed by the Boltzmann weight of the IRF model.

$$R(u, \lambda) = \sum_{i=1}^{n} E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} \alpha(u, \lambda_{ij}) E_{i,j} \otimes E_{j,i} + \sum_{i \neq j} \beta(u, \lambda_{ij}) E_{i,j} \otimes E_{j,i}$$

(58)

where

$$\alpha(u, \lambda) = \frac{\theta(u) \theta(\lambda + \eta)}{\theta(u - \eta) \theta(\lambda)}$$

$$\beta(u, \lambda) = \frac{\theta(u + \lambda) \theta(\eta)}{\theta(u - \eta) \theta(\lambda)}$$

(59)

It satisfies the dynamical YBE:

$$R(u_1, u_2, \lambda - \eta h^{(3)})^{12} R(u_1, \lambda)^{13} R(u_2, \lambda - \eta h^{(1)})^{23} = R(u_2, \lambda)^{23} R(u_1, \lambda - \eta h^{(2)})^{13} R(u_1 - u_2, \lambda)^{12}$$

(60)

where $R(u, \lambda - \eta h^{(3)})^{12}$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(u, \lambda - \eta \mu) \otimes I_d$ if $v_3$ has weight $\mu$.

The elliptic quantum group $E_{\tau,\eta}(sl_n)$ is an algebra generated by a meromorphic function of a variable $h$ and a matrix $L(z, \lambda)$ with noncommutative entries:

$$R(u_1 - u_2, \lambda - \eta h^{(3)})^{12} L(u_1, \lambda)^{13} L(u_2, \lambda - \eta h^{(1)})^{23} = L(u_2, \lambda)^{23} L(u_1, \lambda - \eta h^{(2)})^{13} R(u_1 - u_2, \lambda)^{12}.$$ 

(61)

Here $L(z, \lambda)$ gives an evaluation representation of the quantum group

$$L(u, \lambda)^{ij}_k = \frac{\sigma_0(u + \frac{\xi}{n} - \eta \delta - \eta a_{kj} - \frac{a-1}{2}) \prod_{i \neq j} \sigma_0(-\frac{\xi}{n} + \eta a_{ji})}{\sigma_0(u - \eta \delta - \frac{n-1}{2})}$$

(62)

The Transfer matrix of IRF is expressed by the Ruijsenaars operators which gives the dynamics of solitons

$$T(u) f(\lambda) = \sum_{i=1}^{N} L_{ii}(u, \lambda) f(\lambda - \eta h)$$

(63)

and the Ruijsenaars Macdonald operator $M$ is

$$M = \sum_{i} \prod_{j: j \neq i} \frac{\theta(\lambda_i - \lambda_j + b \eta)}{\theta(\lambda_i - \lambda_j)} T_i$$

(64)

So we have

$$T_i f(\lambda) = f(\lambda_i - \eta \delta)$$

(65)

Then the Hilbert space of non-commutative torus becomes the common eigenvectors of the transfer matrix.

The wave functions have the form

$$\psi = \prod_i e^{\xi_i z_i} \prod \theta(z_i + t_i - \eta)$$

(66)

which will be twisted by $\eta$ when $z_i$ changed by Wilson loop $U_1, U_2$. 

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