On the symmetries of integrable systems with boundaries

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Abstract

We employ appropriate realizations of the affine Hecke algebra and we recover previously known non-diagonal solutions of the reflection equation for the $U_q(\hat{gl}_n)$ case. With the help of linear intertwining relations involving the aforementioned solutions of the reflection equation, the symmetry of the open spin chain with a particular choice of the left boundary is exhibited. The symmetry of the corresponding local Hamiltonian is also explored.

1 Introduction

In order to construct an open spin chain one needs two basic building blocks, namely the $R$ matrix acting on $V\otimes^2$, satisfying the Yang-Baxter equation

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2), \quad (1.1)$$

and the $K$ matrix acting on $V$, and obeying the reflection equation

$$R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \quad (1.2)$$

We shall consider henceforth a particular solution of the Yang–Baxter equation, that is the $R$ matrix associated to $U_q(\hat{gl}_n)$

$$R(\lambda) = a(\lambda) \sum_{i=1}^n \hat{e}_{ii} \otimes \hat{e}_{ii} + b(\lambda) \sum_{i\neq j=1}^n \hat{e}_{ii} \otimes \hat{e}_{jj} + c \sum_{i\neq j=1}^n e^{-\text{sgn}(i-j)\lambda} \hat{e}_{ij} \otimes \hat{e}_{ji} \quad (1.3)$$

where

$$a(\lambda) = \sinh(\lambda + i\mu), \quad b(\lambda) = \sinh \lambda, \quad c = \sinh i\mu. \quad (1.4)$$

2 Affine Hecke algebra and solutions of the reflection equation

Given the structural similarity between the defining relations of the affine Hecke algebra, and the Yang–Baxter and reflection equations we shall employ representations of the affine

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Hecke algebra in order to derive solutions of the reflection equation \[4, 5, 6\]. We shall henceforth restrict our attention to quotients of the affine Hecke algebra \[4, 5, 6\].

**Definition 1.** A quotient of the affine Hecke algebra, called the $B$-type Hecke algebra $B_N(q, Q)$, is defined by generators $U_l$, $l \in \{1, \ldots, N-1\}$, $U_0$ satisfying:

\[
U_l U_l = \delta U_l, \quad U_l U_{l+1} U_l - U_l U_{l+1} U_l - U_{l+1} = 0, \quad |j-l| > 1,
\]

\[
U_0 U_0 = \delta_0 U_0, \quad [U_j, U_l] = 0, \quad |j-l| > 1.
\]

$\delta = -q - q^{-1}$ ($q = e^{i\mu}$), $\delta_0 = -(Q+Q^{-1})$ and $\kappa = qQ^{-1} + q^{-1}Q$. We are free to renormalize $U_0$ and consequently $\delta_0$ and $\kappa$. In fact, if we relax the first of the relations (2.2) we recover the affine Hecke algebra. Moreover, the algebra defined by the generators $U_l$ ($l \neq 0$) is the Hecke algebra $H_N(q)$.

It was shown in \[4, 5\] that tensor representations of quotients of the affine Hecke algebra provide solutions to the reflection equation. For our purposes here we shall make use of the following \[6\]:

**Proposition 1.** Tensor representations of $H_N(q)$ that extend to $B_N(q, Q)$, $\rho : B_N(q, Q) \rightarrow \text{End}(V \otimes V)$ provide solutions to the reflection equation, i.e.

\[
K(\lambda) = x(\lambda)I + y(\lambda)\rho(U_0),
\]

with

\[
x(\lambda) = -\delta_0 \cosh(2\lambda + i\mu) - \kappa \cosh 2\lambda - \cosh 2i\mu \zeta, \quad y(\lambda) = 2 \sinh 2\lambda \sinh i\mu.
\]

**Proof:** The proof is straightforward. The values of $x(\lambda)$ and $y(\lambda)$ can be found by direct computation, by substituting the ansatz \[2.1\] in \[1.2\] and also using equations \[2.1\]–\[2.3\].

The representation of the Hecke algebra that provides the $R$ matrix \[11\] is given by \[9\], $\rho : H_N(q) \rightarrow \text{End}((C^n) \otimes V)$ such that

\[
\rho(U_l) = I \otimes \ldots \otimes U \otimes \ldots \otimes I
\]

acting non-trivially on $V_l \otimes V_{l+1}$, with

\[
U = \sum_{i, j=1}^{n} (\hat{e}_{ij} \otimes \hat{e}_{ji} - q^{-\text{sgn}(i-j)} \hat{e}_{ii} \otimes \hat{e}_{jj}).
\]

The above representation may be extended to the $B$-type Hecke algebra \[6\] $\rho : B_N(q, Q) \rightarrow \text{End}((C^n) \otimes V)$ with \[2.6\], \[2.7\] and

\[
\rho(U_0) = \frac{1}{2i \sinh i\mu} U_0 \otimes I \ldots \otimes I
\]

acting non-trivially on $V_1$ and

\[
U_0 = -Q^{-1} \hat{e}_{11} - Q \hat{e}_{nn} + \hat{e}_{1n} + \hat{e}_{n1}.
\]
It is convenient to set \( Q = i e^{i \mu m} \), then by substituting (2.9) in the ansatz (2.4) we obtain the \( n \times n \) \( K \) matrix:

\[
\begin{align*}
K_{11}(\lambda) &= e^{2\lambda} \cosh i \mu m - \cosh 2i \mu \zeta, \\
K_{1n}(\lambda) &= K_{n1}(\lambda) = -i \sinh 2\lambda, \\
K_{jj}(\lambda) &= \cosh(2\lambda + i \mu \zeta) - \cosh 2i \mu \zeta, \quad j \in \{2, \ldots, n-1\}. 
\end{align*}
\] (2.10)

The later matrix coincides with the one found in ’95 by Abad and Rios, subject to parameter identifications \[6\].

3 The reflection algebra and the open spin chain

Having at our disposal c-number solutions of (1.2) we may build the more general form of solution of (1.2) as argued in \[7\]. To achieve this it is necessary to define the following objects:

\[
\mathcal{L}(\lambda) = e^{\lambda} \mathcal{L}^+ - e^{-\lambda} \mathcal{L}^-,
\] (3.1)

with the matrices \( \mathcal{L}^+ \) (\( \mathcal{L}^- \)) being upper (lower) triangular, and \( \mathcal{L} \) satisfies:

\[
R_{ab}(\lambda_1 - \lambda_2) \mathcal{L}_a(\lambda_1) \mathcal{L}_b(\lambda_2) = \mathcal{L}_b(\lambda_2) \mathcal{L}_a(\lambda_1) R_{ab}(\lambda_1 - \lambda_2).
\] (3.2)

Define also

\[
\hat{\mathcal{L}}(\lambda) = \mathcal{L}^{-1}(-\lambda).
\] (3.3)

The more general solution of (1.2) is then given by \[7\]:

\[
\mathcal{K}(\lambda) = \mathcal{L}(\lambda - \Theta) \ (K(\lambda) \otimes I) \ \hat{\mathcal{K}}(\lambda + \Theta),
\] (3.4)

where \( K \) is the c-number solution of the reflection equation, \( \Theta \) is a constant and hereafter will be considered zero. The entries of \( \mathcal{K} \) are elements of the so called reflection algebra \( \mathcal{R} \), with exchange relations dictated by the algebraic constraints (1.2), (see also \[7\]).

One may easily show that all the elements of the reflection algebra ‘commute’ with the solutions of the reflection equation (see also \[8\]). Let \( \pi_\lambda \) be the evaluation representation \( \pi_\lambda : \mathcal{A} \rightarrow \End(\mathcal{C}^m) \). Then by acting with the evaluation representation on the second space of (3.4) it follows

\[
\pi_\lambda(K_{ij}(\lambda')) K(\lambda) = K(\lambda) \pi_{-\lambda}(K_{ij}(\lambda')), \quad i, j \in \{1, \ldots, n\}.
\] (3.5)

The reflection algebra is also endowed with a coproduct inherited essentially from \( \mathcal{A} \) \[8\] \[6\], i.e. \( \Delta : \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{A} \), such that

\[
\Delta(K_{ij}(\lambda)) = \sum_{k,l=1}^{n} K_{kl}(\lambda) \otimes L_{ik}(\lambda) \ \hat{\mathcal{L}}_{ij}(\lambda) \quad i, j \in \{1, \ldots, n\}.
\] (3.6)

The open spin chain may be constructed following the generalized QISM \[7\]. We first need to define

\[
T_0(\lambda) = \mathcal{L}_{0N}(\lambda) \ldots \mathcal{L}_{01}(\lambda), \quad \hat{T}_0(\lambda) = \hat{\mathcal{L}}_{01}(\lambda) \ldots \hat{\mathcal{L}}_{0N}(\lambda)
\] (3.7)
usually the space denoted by 0 is called auxiliary, whereas the spaces denoted by 1, \ldots, N are called quantum. The general tensor type solution of the \( T \) takes the form

\[
T_0(\lambda) = T_0(\lambda) K_0^{(r)}(\lambda) T_0(\lambda).
\] (3.8)

Notice that relations similar to (3.5) may be derived for \( T \), i.e.

\[
(\pi_\lambda \otimes \text{id} \otimes \sigma_0) \Delta'(N+1) (K_{ij}(\lambda')) T(\lambda) = T(\lambda) (\pi_{-\lambda} \otimes \text{id} \otimes \sigma_0) \Delta'(N+1) (K_{ij}(\lambda')),
\] (3.9)

where

\[
\Delta' = \Pi \circ \Delta, \quad \Pi : a \otimes b \to b \otimes a.
\] (3.10)

We can now introduce the transfer matrix of the open spin chain \( T \), which may be written as

\[
t(\lambda) = \text{Tr}_0 \left\{ M_0 K_0^{(\lambda)} T_0(\lambda) \right\},
\] (3.11)

where \( M_{ij} = e^{i \mu (n_1 - 2j+1)} \delta_{ij} \), \( K(\lambda) \) is a solution of the reflection equation (1.2), and here we consider \( K^{(\lambda)} = I \).

It can be proved using the fact that \( T \) is a solution of the reflection equation (1.2) that

\[
\left[ t(\lambda), \ t(\lambda') \right] = 0,
\] (3.12)

which ensures that the open spin chain (3.11) is integrable.

**The Hamiltonian:** It is useful to write down the Hamiltonian of the open spin chain. For this purpose we should restrict our attention in the case where the evaluation representation \( \pi_0 \) acts on the quantum spaces of the spin chain well, then \( L(\lambda) \to R(\lambda) \), \( \hat{L}(\lambda) \to R(\lambda) = R'(\lambda) \) (\( t \) denotes total transposition). The Hamiltonian is given by

\[
\mathcal{H} = -\frac{(\sinh i \mu)^{-2N+1}}{4x(0)} \left( \text{tr}_0 M_0 \right)^{-1} \frac{d}{d\lambda} t(\lambda)|_{\lambda=0}
\] (3.13)

and it may be written exclusively in terms of the generators of \( B_N \) as

\[
\mathcal{H} = -\frac{1}{2} \sum_{i=1}^{N-1} \rho(\mu i) - \frac{\sinh i \mu y'(0)}{4x(0)} \rho(\mu 0) + c \rho(1).
\] (3.14)

4 The boundary symmetry

From the asymptotic behaviour of \( T \) we obtain the ‘boundary non-local’ charges (entries of \( T(\lambda \to \infty) \))

\[
T_{11}^{+}(N) = 2 \cosh i \mu \sum_{j=1}^{N-1} T_{1j}^{+}(N) \hat{T}_{j1}^{+}(N) - iT_{11}^{+}(N) \hat{T}_{11}^{+}(N) - iT_{n1}^{+}(N) \hat{T}_{n1}^{+}(N) + e^{i \mu m} \sum_{j=2}^{N-1} T_{1j}^{+}(N) \hat{T}_{j1}^{+}(N)
\]

\[
T_{11}^{+}(N) = e^{i \mu m} \sum_{j=1}^{N-1} T_{1j}^{+}(N) \hat{T}_{j1}^{+}(N) - iT_{11}^{+}(N) \hat{T}_{11}^{+}(N) - iT_{n1}^{+}(N) \hat{T}_{n1}^{+}(N),
\]
\[ T^{-1}_{ij} = e^{i \mu m} \sum_{j=1}^{n-1} T^{-1}_{ij} \hat{T}^{-1}_{ij} - iT^{-1}_{in} \hat{T}^{-1}_{in}, \quad i \in \{2, \ldots, n\} \]
\[ T^{-1}_{kl} = e^{i \mu m} \sum_{j=\max(k,l)}^{n-1} T^{-1}_{kj} \hat{T}^{-1}_{jl}, \quad k, l \in \{2, \ldots, n-1\}, \quad (4.1) \]

where \( T^{-1}_{ij}, \hat{T}^{-1}_{ij} \) are the entries of \( T(\lambda \to \infty), \hat{T}(\lambda \to \infty) \) \([17]\), i.e. tensor product realizations of \( U_q(\mathfrak{gl}_n) \) (non–affine). The charge associated to the affine generators is omitted here for brevity, but it is presented in \([6]\). The non–local charges \( (4.1) \) form the ‘boundary quantum algebra’ satisfying exchange relations entailed from \( (1.2) \) as \( \lambda_i \to \infty \) \([6]\). Boundary non-local charges were also identified in \([9]\) in the isotropic case \( q = 1 \) for two distinct types of boundary conditions, corresponding to boundary or twisted Yangians.

Our main aim now is to derive the conserved quantities commuting with the transfer matrix \([6, 10]\). Recall that we focus here on the case where the left boundary is trivial i.e. \( K(l) = I \).

**Proposition 2.** The boundary charges \( (4.1) \) in the fundamental representation commute with the generators of \( \mathcal{B}_N \) given in the representation \([2, 4, 22]\) i.e.
\[ \left[ \rho(\mathcal{U}_l), \pi_{0}^{\otimes N}(T_{ij}^{-1}(N)) \right] = 0, \quad l \in \{0, \ldots, N-1\}. \quad (4.2) \]

**Proof:** Recall that all the boundary charges are expressed in terms of the \( U_q(\mathfrak{gl}_n) \) generators it immediately follows that \([22]\) is valid for all \( l \in 1, \ldots N-1 \). It may be also proved by inspection, and given the tensor product form of the non-local charges, that \([22]\) is valid for \( l = 0 \) as well.

**Corollary:** The open spin chain Hamiltonian \([3.14]\) commutes with the fundamental representation of the boundary charges \( (4.1) \)
\[ \left[ H, \pi_{0}^{\otimes N}(T_{ij}^{-1}(N)) \right] = 0. \quad (4.3) \]

This is evident from the form of \( H \) \([3.14]\), which is expressed solely in terms of the representation \( \rho(\mathcal{U}_l) \) of \( \mathcal{B}_N \).

Also a general statement on the symmetry of the open transfer matrix can be made:

**Proposition 3.** The open transfer matrix \([6, 8]\) commutes with all the non-local charges \( T_{ij}^{-1}(N) \) \((4.1)\), i.e.
\[ \left[ t(\lambda), T_{ij}^{-1}(N) \right] = 0. \quad (4.4) \]

**Proof:** The proof relies primarily on the existence of the generalized intertwining relations \([6, 8, 9, 10]\). We shall use as a paradigm the \( U_q(\mathfrak{sl}_2) \) case \([10]\), although the proof may be generalized in a straightforward manner for the \( U_q(\mathfrak{gl}_n) \) case \([6]\). In the \( U_q(\mathfrak{sl}_2) \) case there exist only one non-trivial boundary charge (non–affine) \([10]\) i.e.
\[ T_{11}^{-1}(N) = q^{-\frac{1}{2}} K(N) E(N) + q^{\frac{1}{2}} K(N) F(N) + x_1(K(N))^2 - x_1I, \quad (4.5) \]
where \( K(N), E(N), F(N) \) are the the \( N \) coproducts of the \( U_q(\mathfrak{sl}_2) \) generators \([3]\). Let
\[ T(\lambda) = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix} \quad (4.6) \]
then using (3.9) as $\lambda' \to \infty$, and (4.6) the following important algebraic relations are entailed

$$\left[T^{+}_{11}(N), A_1\right] = e^{-i\mu(B - C)}, \quad \left[T^{+}_{11}(N), A_2\right] = -e^{i\mu(B - C)}$$

(4.7)

$$\left[T^{+}_{11}(N), C\right]_{q^{-1}} = A_2 - A_1 + x_1(q - q^{-1})C$$

(4.8)

$$\left[T^{+}_{11}(N), B\right]_q = A_1 - A_2 + x_1(q^{-1} - q)B,$$

(4.9)

where we define $[X, Y]_q = qXY - q^{-1}YX$, and $x_1$ is a constant depending on the boundary parameters $m$, $\zeta$. Recall that $K^{(l)}(\lambda) = I$ and $M = diag(q, q^{-1})$, then the transfer matrix can be written as:

$$t(\lambda) = e^{i\mu A_1} + e^{-i\mu A_2}.$$  

(4.10)

Finally, by virtue of (4.10) and recalling the exchange relations (4.7)–(4.9) it follows that:

$$\left[t(\lambda), T^{+}_{11}(N)\right] = 0$$

(4.11)

and this concludes our proof.

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