ON THE DECOMPOSITION NUMBERS OF THE REE GROUPS $^2F_4(q^2)$ 
IN NON-DEFINING CHARACTERISTIC

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Abstract. We compute the $\ell$-modular decomposition matrices of the simple Ree groups $^2F_4(q^2)$, where $q^2 = 2^{2n+1}$ and $n$ is a positive integer, for all primes $\ell > 3$ up to some entries in the unipotent characters. Using these matrices we determine the smallest degree of a non-trivial irreducible $\ell$-modular representation of $^2F_4(q^2)$ for all primes $\ell > 3$. We also obtain results on the 3-modular decomposition matrices of $^2F_4(q^2)$.

1. Introduction

This paper deals with the modular representation theory of the simple Ree groups $^2F_4(q^2)$ where $q^2 = 2^{2n+1}$ and $n$ is a positive integer. Very valuable information on the modular representations of these groups in non-defining characteristic was obtained by G. Hiß. In [24] and [25], he determined the Brauer trees of all blocks of $^2F_4(q^2)$ with a cyclic defect group. Furthermore, he computed the $\ell$-modular decomposition numbers of the non-simple group $^2F_4(2)$ and of its derived subgroup, the simple Tits group $^2F_4(2)'$, for all odd primes $\ell$, see [22].

In this article, we consider the $\ell$-modular representation theory of the simple Ree groups $^2F_4(q^2)$, $q^2 = 2^{2n+1}$, $n > 0$, for all odd primes $\ell$ such that the Sylow $\ell$-subgroups of $^2F_4(q^2)$ are not cyclic. For all such primes $\ell > 3$, we compute the $\ell$-modular decomposition matrices of $^2F_4(q^2)$ up to several entries in the unipotent characters and one entry in the non-unipotent characters. For fixed $q$, this determines all irreducible $\ell$-modular Brauer characters of $^2F_4(q^2)$ up to at most four. As a corollary, we show that for all $\ell > 3$ the $\ell$-modular decomposition matrix of $^2F_4(q^2)$ has a lower unitriangular shape and that the reductions modulo $\ell$ of all ordinary cuspidal unipotent characters of $^2F_4(q^2)$ are irreducible Brauer characters. This was conjectured by G. Hiß and M. Geck in a more general context, see [15, Conjecture 3.4].

Most of the methods we use to determine the decomposition numbers are elementary in the sense that they only require calculations with characters. The main ingredients are the character table of $^2F_4(q^2)$ which was computed by G. Malle [33] and is contained in the CHEVIE library [16], and the character tables of the proper parabolic subgroups of $^2F_4(q^2)$ which were determined in [19], [21] and which are also available as CHEVIE files. We produce projective characters of $^2F_4(q^2)$ by inducing projective characters of the proper parabolic subgroups and by tensoring defect 0 characters with ordinary characters. Then, by computing scalar products of these projectives with ordinary irreducible characters, we obtain an approximation to the decomposition matrix which already implies the lower unitriangular shape of these matrices. To determine the decomposition numbers below the diagonal we use relations which are obtained by expressing Brauer characters as linear combinations of the elements of certain basic sets of Brauer characters.

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Additionally, we use some less elementary tools like Hecke algebra methods, arguments from modular Harish-Chandra theory and a theorem of M. Geck, G. Hiß and G. Malle on the cuspidality of the modular Steinberg character. For the non-unipotent blocks we use C. Bonnafé’s and R. Rouquier’s modular version of the Jordan decomposition of characters. All calculations concerning scalar products and induction and restriction of characters are carried out using CHEVIE.

The situation is significantly different for $\ell = 3$, since $3$ is a bad prime for a root system of type $F_4$ and the Sylow 3-subgroups of $2F_4(q^2)$ are non-abelian. However, our methods are still good enough to compute the 3-modular decomposition matrices of $2F_4(q^2)$ except for several entries in six rows corresponding to ordinary unipotent irreducible characters. In particular, we are able to show that the 3-modular decomposition matrices of $2F_4(q^2)$ have a lower unitriangular shape, too.

The $\ell$-modular decomposition numbers for $\ell \geq 3$ which we are not able to determine are multiplicities of certain cuspidal Brauer characters in the reduction modulo $\ell$ of ordinary unipotent characters of large degree, including the Steinberg character. For these unknown multiplicities we only get upper bounds depending on $q$. The problems in the determination of these multiplicities seem to be a usual phenomenon in the calculation of the decomposition matrices of finite groups of Lie type in non-defining characteristic; see for example [13] and [23]. Maybe module theoretic arguments as in [40] might eventually lead to more information on these numbers.

As an application, we obtain new information on the degrees of modular irreducible representations of $2F_4(q^2)$. Let $G = 2F_4(q^2)$, $q^2 = 2^{2n+1}$, $n > 0$ and $k$ be an algebraically closed field of characteristic $\ell \geq 0$. We denote the smallest degree of a nontrivial irreducible $kG$-representation by $d_\ell(G)$. Using the Brauer trees in [25], F. Lübeck [31] was able to determine $d_\ell(G)$ for all primes $\ell$ such that the Sylow $\ell$-subgroups of $G$ are cyclic. In fact, for such $\ell$ he proved $d_\ell(G) = d_0(G)$ where the latter is known due to G. Malle’s ordinary character table of $G$. Using our $\ell$-modular decomposition matrices, we can extend F. Lübeck’s result to all primes $\ell > 3$: We show that $d_\ell(G) = d_0(G)$ holds for all primes $\ell > 3$. The idea of the proof is similar to [18]: The decomposition matrices determine almost all irreducible Brauer characters of $G$ and in particular their degrees. In most cases (depending on $q$ and $\ell$) only three or four of these degrees are irreducible Brauer characters remain unknown. From the upper bounds for the decomposition numbers, we can derive that these unknown degrees are larger than $d_0(G)$. This last step turns out to be surprisingly difficult because there are some ordinary unipotent irreducible characters of $G$ whose degrees do not differ “very much” from the degree of the Steinberg character; for more details see Section 6.

This paper is organized as follows: In Section 2, we introduce notation for characters, Lusztig series and blocks. In Sections 3 and 4, we state our main results on the decomposition matrices of $2F_4(q^2)$, which are then proved in Section 5. In Section 6, we describe the consequences for the degrees of modular representations of $2F_4(q^2)$ in odd characteristics. Scalar products, relations and decomposition matrices are given in several Appendices.

2. Notation and Setup

In this section, we introduce the general setup and notation which will be used throughout this paper.

2.1. Group theoretical setup. We use the notation and setup from [21, Section 2] and [19, Sections 3 and 4]. In particular, $n > 0$ is an integer and $q = \sqrt{2^{2n+1}}$. Let $\Phi$ be the
root system of type $F_4$ described in [21, Section 2], so $\Phi$ has simple roots $r_1, r_2, r_3, r_4$ and Dynkin diagram:

\[
\begin{array}{cccc}
  r_1 & r_2 & r_3 & r_4 \\
\end{array}
\]

Fix a simply connected linear algebraic group $G$ defined over an algebraically closed field of characteristic $2$ with root system $\Phi$ and a Frobenius morphism $F$ such that $G := G^F$ is the Ree group $2F_4(q^2)$ defined over the field $\mathbb{F}_{q^2}$. Let $B$ be an $F$-stable Borel subgroup and $P_a$ and $P_b$ be $F$-stable maximal parabolic subgroups of $G$ as in [19] and [21]. The finite group $G = 2F_4(q^2)$ has the order

\[|G| = q^{24} \phi_1^2 \phi_2^2 \phi_4^2 \phi_8^2 \phi_{12}^2 \phi_{24}^2,\]

where $\phi_1 = q - 1, \phi_2 = q + 1, \phi_4 = q^2 + 1, \phi_8 = q^2 + \sqrt{2}q + 1, \phi_8' = q^2 - \sqrt{2}q + 1, \phi_{12} = q^4 - q^2 + 1, \phi_{24} = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1, \phi_{24}' = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$. Furthermore, we set $\phi = \phi_6 \phi_8''$ and $\phi_{24} = \phi_{24}' \phi_{24}''$. The Borel subgroup $B := B^F$ and the maximal parabolic subgroups $P_a := P_a^F$ and $P_b := P_b^F$ of $G$ containing $B$ have the orders

\[|B| = q^{24} \phi_1^2 \phi_2^2, \quad |P_a| = q^{24} \phi_1^2 \phi_2^2 \phi_4, \quad |P_b| = q^{24} \phi_1^2 \phi_2^2 \phi_8'' \phi_8'.\]

As in [21] and [19], we write $T$, $L_a$ and $L_b$ for the Levi subgroups and $U, U_a$ and $U_b$ for the unipotent radicals of $B, P_a$ and $P_b$, respectively. So $T$ is a maximally split torus of $G$, and the Levi subgroups $L_a, L_b$ have the structure $L_a \cong \mathbb{Z}_{q^2-1} \times SL_2(q^2) \cong GL_2(q^2)$ and $L_b \cong \mathbb{Z}_{q^2-1} \times Sz(q^2)$, respectively. Representatives for the conjugacy classes of $G, B, P_a$ and $P_b$ are given in Appendix A of [19] and [21], together with the corresponding class fusions.

2.2. Ordinary and modular representations. In this whole article, $\ell$ is an odd prime number and $(K, R, k)$ a splitting $\ell$-modular system for all subgroups of $G = 2F_4(q^2)$. The word “character” will always mean an ordinary character associated with a representation over $K$. For subgroups $H$ of $G$, we write $\text{Irr}(H)$ for the set of ordinary irreducible characters of $H$. All Brauer characters, blocks, decomposition numbers will be taken with respect to the fixed prime number $\ell$. The restriction of any class function $\vartheta$ of $G$ to the set of $\ell$-regular elements will be denoted by $\bar{\vartheta}$. In particular, if $\chi$ is an ordinary character, then $\bar{\chi}$ is a Brauer character. Sometimes, it will be convenient to extend such functions by zero to the $\ell$-singular elements.

2.3. Characters of the Ree groups. The ordinary irreducible characters of $G = 2F_4(q^2)$ were computed by G. Malle and are contained in the CHEVIE library [16]. A construction of the unipotent irreducible characters of $G$ is described in [33]. There are 21 unipotent irreducible characters of $G$ and we denote them by $\chi_1, \chi_2, \ldots, \chi_{21}$. This notation coincides with the numbering of the irreducible characters of $G$ in CHEVIE. In Table A.1 in the Appendix, we collect some information on the unipotent characters including a dictionary between our notation and the notation in [8], [25] and [33].

The character $\chi_1$ is the trivial character, $\chi_{21}$ is the Steinberg character. The characters in the principal series are $\chi_1, \chi_4, \chi_5, \chi_6, \chi_7, \chi_{18}$ and $\chi_{21}$. The characters $\chi_2$ and $\chi_9$ are constituents of the Harish-Chandra induction of one of the cuspidal unipotent characters of the Levi subgroup $\mathbb{Z}_{q^2-1} \times Sz(q^2)$ of $P_b$. The characters $\chi_3$ and $\chi_{20}$ are constituents of the Harish-Chandra induction of the other cuspidal unipotent character of this Levi subgroup. The remaining unipotent characters of $G$ are cuspidal. The characters $\chi_2$ and $\chi_3$ are complex conjugate to each other, and the same holds for the pairs $(\chi_{11}, \chi_{12}), (\chi_{13}, \chi_{14})$. 


$(\chi_{15}, \chi_{16}), (\chi_{19}, \chi_{20})$. Since not all of these unipotent irreducible characters of $G$ are uniquely determined by their degree we also provide some character values in the last two columns of Table A.1. The $\varepsilon_4$ is a complex fourth root of unity, see [21, Table 5].

The set of unipotent irreducible characters of $G$ can be partitioned into certain subsets, called families, see [8, Section 12.3]. The families of unipotent characters of $G = 2F_4(q^2)$ were determined by G. Lusztig in [32, Section 14.2]. There are 7 families: four of them have 1 element, two have 2 elements and one has 13 elements. The distribution of the unipotent irreducible characters of $G$ into families is indicated by the horizontal lines in Table A.1.

Our notation for the non-unipotent irreducible characters of $G$, is motivated by the Jordan decomposition of characters. Let $(G^*, F^*)$ be dual to $(G, F)$ and let $G^* := G^{*F^*}$. For every semisimple element $s \in G^*$, there is a set $\mathcal{E}(G, s) \subseteq \text{ Irr}(G)$, called the Lusztig series associated with $s$, and

$$\text{ Irr}(G) = \bigcup_s \mathcal{E}(G, s),$$

where $s$ runs over a set of representatives for the semisimple conjugacy classes of $G^*$, is a partition of $\text{ Irr}(G)$. For every semisimple element $s \in G^*$, there is a bijection between $\mathcal{E}(G, s)$ and the set of unipotent irreducible characters of the centralizer $C_{G^*}(s)$. There are 18 types $g_1, g_2, \ldots, g_{18}$ of Lusztig series of $G$, and the Dynkin type of the corresponding centralizers is described in [20, Table B.9]. We write $\chi = \chi_{i, \lambda}$ for the irreducible character in the Lusztig series of type $g_i$ of $G$ corresponding to the unipotent character $\lambda \in \text{ Irr}(C_{G^*}(s))$. The degree of $\chi_{i, \lambda}$ is $\chi_{i, \lambda}(1) = \lambda(1) \cdot |G^*: C_{G^*}(s)|^{1/2}$, where $\mathcal{E}(G, s)$ is of type $g_i$. Details on the non-unipotent irreducible characters of $G$ are given in Table A.2 in the Appendix.

2.4. Characters of the parabolic subgroups. The notation we use for the irreducible characters of the proper parabolic subgroups $B, P_a$ and $P_b$, is the same as in [21, Table A.5, A.6] and [19, Tables A.4, A.5, A.8, A.9]. That is, we denote the irreducible characters of $B$ by $b\chi_1, \ldots, b\chi_{58}$, the irreducible characters of $P_a$ by $p_a\chi_1, \ldots, p_a\chi_{40}$ and the irreducible characters of $P_b$ by $p_b\chi_1, \ldots, p_b\chi_{56}$ (maybe depending on some parameters $k$ or $l$).

2.5. Induction and restriction. Let $H$ be a finite group and $H_1$ a subgroup of $H$. If $\chi$ is a character of $H_1$, we write $\chi^H$ for the induced character; and if $\chi$ is a character of $H$, we write $\chi_{H_1}$ for the restriction of $\chi$ to $H_1$. Using the class fusions in [21, Table A.4], [19, Tables A.1, A.3, A.7] and CHEVIE, we can easily compute the induction and restriction of characters between the subgroups $G, B, P_a$ and $P_b$.

2.6. Blocks and basic sets. The distribution of ordinary irreducible characters of $G$ into blocks is compatible with the Lusztig series in the following sense: For every semisimple element $s \in G^*$ of order prime to $\ell$, the set

$$\mathcal{E}_\ell(G, s) := \bigcup_{t \in C_{G^*}(s)_{\ell}} \mathcal{E}(G, st) \subseteq \text{ Irr}(G),$$

where $C_{G^*}(s)_{\ell}$ is the set of elements of $\ell$-power order of the centralizer $C_{G^*}(s)$, is a union of blocks. For the Ree groups, this is shown in [24, p. 113], see the remarks after the proof of Lemma D.3.4. Note that a similar compatibility between blocks and geometric conjugacy classes holds in a much more general context, see for example [5, Theorem 2.2] or [7, Theorem 9.12]. The blocks in $\mathcal{E}_\ell(G, 1)$ are called the unipotent blocks of $G$. These are the blocks of $G$ containing at least one unipotent character.
Let $B$ be a union of blocks of $G$. A basic set in $B$ is a set of linearly independent Brauer characters in $B$ such that every Brauer character in $B$ is a linear combination with integer coefficients of the elements in this set. A basic set is called ordinary, if it consists of the restrictions of some ordinary characters to the $\ell$-regular elements of $G$. In this case, we identify these ordinary characters with the elements of the basic set. So, in order to describe the decomposition matrix of $B$, it is enough to describe the decomposition numbers of the characters in an ordinary basic set and the relations expressing the restrictions of the remaining ordinary characters in $B$ to the $\ell$-regular elements as linear combinations of the characters in the basic set. For more details on blocks and basic sets see [15, Section 3].

3. Decomposition Matrices for $\ell > 3$

As in Section 2 let $G = 2F_4(q^2)$, $q^2 = 2^{2n+1}$, $n > 0$, and let $\ell$ be an odd prime number. In [24], [25], G. Hiß determined the Brauer trees of all blocks of $G$ with cyclic defect group. In particular, if the Sylow $\ell$-subgroups of $G$ are cyclic, then the $\ell$-modular decomposition numbers of $G$ can be read off from these Brauer trees. In this section, we describe the $\ell$-modular decomposition numbers of $G$ for all remaining primes $\ell > 3$. Some of our methods and the presentation are inspired by [15] and [28].

We fix a prime number $\ell > 3$ and an $\ell$-modular splitting system $(K, R, k)$ for all subgroups of $G$ as in Subsection 2.2. All references to Brauer characters, blocks, decomposition numbers etc., in this section will refer to this fixed prime $\ell$. Of course, we get non-trivial decomposition matrices only if $\ell$ divides the group order. From now on, we assume that $\ell$ divides $|G| = q^{24} \overline{\phi_1}^2 \phi_4 \phi_8^2 \phi_8'' \phi_8' \phi_{24} \phi_{24}''$, where $\overline{\phi_1} := \phi_1 \phi_2 = q^2 - 1$. The condition $\ell > 3$ implies that $\ell$ divides exactly one of the factors $\phi_1, \phi_4, \phi_8, \phi_8', \phi_8'', \phi_{12}, \phi_{24}', \phi_{24}''$. So, we are in the generic situation studied in [3], [4]. Note that $\overline{\phi_1}, \phi_4, \phi_8, \phi_8', \phi_8'', \phi_{12}, \phi_{24}', \phi_{24}''$ correspond to $(\ell 2)$-cyclotomic polynomials in the sense of [3, 3F]. If $\ell$ divides $\phi_{12}, \phi_{24}'$ or $\phi_{24}''$, then the Sylow $\ell$-subgroups of $G$ are cyclic and the decomposition numbers can be read off from the Brauer trees in [24], [25]. So, we only have to consider the cases

$$(1) \quad \ell \mid q^2 - 1, \quad \ell \mid q^2 + 1, \quad \ell \mid q^2 + \sqrt{2}q + 1, \quad \ell \mid q^2 - \sqrt{2}q + 1.$$

For every semisimple $\ell'$-element $s \in G^*$, the set $E_\ell(G, s)$ is a union of blocks of $G$, see Subsection 2.6. The assumption $\ell > 3$ implies that $\ell$ is a good prime for $G$ in the sense of [10, p. 125]. Therefore, we can deduce from a general result of M. Geck and G. Hiß [14] that $E(G, s)$ is an ordinary basic set for $E_\ell(G, s)$.

For all primes $\ell > 3$ satisfying one of the conditions (1), we are going to describe the decomposition numbers of all ordinary irreducible characters in the unipotent blocks $E_\ell(G, 1)$. For the non-unipotent blocks, we are only going to describe the decomposition numbers of the irreducible characters in the ordinary basic sets $E_\ell(G, s)$, $s \neq 1$. In particular, this determines all non-unipotent irreducible Brauer characters of $G$ (up to a single exception in the case $\ell \mid q^2 + 1$, see Subsections 3.2 and 3.5). All of the statements in the following subsections will be proved in Section 5. Note that in the decomposition matrices and tables of scalar products in the Appendices B, C and D, zeros are replaced by dots.

3.1. The case $\ell \mid q^2 - 1$. We begin with some comments on the decomposition matrices in Appendices C and D: The decomposition numbers of the unipotent blocks $E_\ell(G, 1)$ are given in Table C.1. The first column of this table contains notation for the ordinary
irreducible characters in \( E_\ell(G, 1) \). The 21 irreducible Brauer characters in the unipotent blocks are denoted by \( \phi_1, \ldots, \phi_{21} \).

Some information on the distribution of \( \phi_1, \ldots, \phi_{21} \) into modular Harish-Chandra series is presented in the second row of Table C.1; for a definition of modular Harish-Chandra series see [15, Section 2]. There are 13 such series, corresponding to the Levi subgroups \( T \), \( L_b \) and \( G \). The Levi subgroup \( T \) has a unique cuspidal unipotent Brauer character, and the Levi subgroup \( L_b \) has exactly two cuspidal unipotent Brauer characters. The corresponding modular Harish-Chandra series of \( G \) contain, respectively, 7, 2, 2 irreducible Brauer characters. Additionally, \( G \) has 10 cuspidal unipotent Brauer characters.

The columns labeled by \( ps \) correspond to the irreducible Brauer characters in the principal series. The Levi subgroup \( L_b \) has two cuspidal unipotent Brauer characters \( \varphi_a \) and \( \varphi_b \); these are the restrictions of the ordinary irreducible characters of degree \( \sqrt{\sqrt{2} (q^2 - 1)} \) to the set of \( \ell \)-regular elements. The columns labeled by \( 2B_2[a] \) and \( 2B_2[b] \) correspond to the irreducible Brauer characters in the modular Harish-Chandra series of \( G \) associated with \( \varphi_a, \varphi_b \), respectively. The remaining columns (labeled by “c”) correspond to the cuspidal unipotent Brauer characters.

The decomposition numbers of the non-unipotent irreducible characters are given in Tables D.1, D.2, D.4-D.8 in the Appendix. In these tables, only the decomposition numbers of the ordinary basic sets \( E_\ell(G, s) \), \( s \neq 1 \), are listed. The left most column of the tables contains notation for the irreducible Brauer characters in \( E_\ell(G, s) \). The second row of the tables contains notation for the irreducible Brauer characters in the corresponding blocks. Each of the blocks described by Table D.8 has only one irreducible Brauer character.

**Theorem 3.1.** Let \( \ell \) be a prime dividing \( q^2 - 1 \). The \( \ell \)-modular decomposition numbers of \( G = 2F_4(2^{2n+1}) \), \( n > 0 \), are given by Tables C.1 and D.1, D.2, D.4-D.8 in the Appendix.

**3.2. The case \( 3 \neq \ell | q^2 + 1 \).** The decomposition numbers of the unipotent blocks \( E_\ell(G, 1) \) are given in Table C.2. The 21 irreducible Brauer characters \( \phi_1, \ldots, \phi_{21} \) in the unipotent blocks are distributed into 16 Harish-Chandra series. The Levi subgroups \( T \) and \( L_a \) each have a unique cuspidal unipotent Brauer character, and the Levi subgroup \( L_b \) has exactly two cuspidal unipotent Brauer characters. The corresponding modular Harish-Chandra series of \( G \) contain, respectively, 4, 1, 2, 2 irreducible Brauer characters. Additionally, \( G \) has 12 cuspidal unipotent Brauer characters.

Again, the principal series is abbreviated by \( ps \). The Levi subgroup \( L_a \) has a unique cuspidal unipotent Brauer character (the modular Steinberg character, see [17]) and we denote the corresponding modular Harish-Chandra series of \( G \) by \( A_1 \). The Levi subgroup \( L_b \) has two cuspidal unipotent Brauer characters: the restrictions of the ordinary irreducible characters of degree \( \sqrt{\sqrt{2} (q^2 - 1)} \) to the \( \ell \)-regular elements. The corresponding modular Harish-Chandra series are denoted by \( 2B_2[a] \) and \( 2B_2[b] \), respectively. The remaining columns correspond to cuspidal unipotent Brauer characters.

The decomposition numbers of the non-unipotent irreducible characters are given in Tables D.1-D.3 and D.5-D.8. The decomposition numbers of all ordinary characters in \( E_\ell(G, s) \), where \( s \in G^* \) is semisimple of type \( q_3 \), are given in Table D.3. For all other non-unipotent blocks, only the decomposition numbers of the ordinary basic sets are given.

**Theorem 3.2.** Let \( \ell > 3 \) be a prime dividing \( q^2 + 1 \). The \( \ell \)-modular decomposition numbers of \( G = 2F_4(2^{2n+1}) \), \( n > 0 \), are given by Tables C.2 and D.1-D.3, D.5-D.8 in the Appendix. There are the following bounds:

(i) \( 2 \leq a \leq \sqrt{\frac{q^2 - 2}{3}} \).
Let theorem 3.3. only the decomposition numbers of the ordinary basic sets $E$ reducible characters are given in Tables D.1, D.2, D.4-D.8 in the Appendix. In these tables, cuspidal unipotent Brauer characters. The decomposition numbers of the non-unipotent irreducible characters of degree $\ell$ $G$ see [17, Theorem 4.2]. We denote the corresponding modular Harish-Chandra series of cuspidal unipotent Brauer characters.

There are the following bounds:

(i) $1 \leq b \leq \frac{q^2 + \sqrt{2q}}{4}$. If $\ell \neq 11$ or $n \equiv 27 \mod 55$, then $b \geq 2$.

(ii) $0 \leq c \leq \frac{q^2 - \sqrt{2q}}{4}$. If $\ell \neq 11$ or $n \equiv 27 \mod 55$, then $c \geq 2$.

(iii) $2 \leq d \leq \frac{q^2}{3}$.

(iv) $2 \leq e \leq \frac{d^2}{2}$. If $\ell \neq 11$ or $n \equiv 27 \mod 55$, then $e \geq 3$.

(vi) $2 \leq a' \leq \frac{q^2 - \sqrt{2q}}{4}$.

3.3. The case $\ell | q^2 + \sqrt{2q} + 1$. The decomposition numbers of the unipotent blocks $E_t(G, 1)$ are given in Table C.3. The 21 irreducible Brauer characters $\phi_1, \ldots, \phi_{21}$ in the unipotent blocks are distributed into 18 Harish-Chandra series. The Levi subgroup $T$ has a unique cuspidal unipotent Brauer character, and the Levi subgroup $L_b$ has exactly three cuspidal unipotent Brauer characters. The corresponding modular Harish-Chandra series of $G$ contain, respectively, 4, 1, 1, 1 irreducible Brauer characters. Additionally, $G$ has 14 cuspidal unipotent Brauer characters.

The columns labeled by $s$ correspond to the irreducible Brauer characters in the principal series. The Levi subgroup $L_b$ has two cuspidal unipotent Brauer characters $\varphi_a, \varphi_b$, the restrictions of the ordinary irreducible characters of degree $\frac{q}{\sqrt{q}}(q^2 - 1)$ to the set of $\ell$-regular elements. Additionally, the modular Steinberg character $\varphi_{St}$ of $L_b$ is cuspidal, see [17, Theorem 4.2]. We denote the corresponding modular Harish-Chandra series of $G$ by $^2B_3[a], ^2B_2[b]$ and $^2B_2[St]$, respectively. The remaining columns correspond to the cuspidal unipotent Brauer characters. The decomposition numbers of the non-unipotent irreducible characters are given in Tables D.1, D.2, D.4-D.8 in the Appendix. In these tables, only the decomposition numbers of the ordinary basic sets $E(G, s), s \neq 1$, are listed.

**Theorem 3.3.** Let $\ell$ be a prime dividing $q^2 + \sqrt{2q} + 1$. The $\ell$-modular decomposition numbers of $G = ^2F_4(2^{2n+1}), n > 0$, are given by Tables C.3 and D.1, D.2, D.4-D.8 in the Appendix. There are the following bounds:

(i) $0 \leq a \leq \frac{q^2 + 3\sqrt{2q} + 4}{12}$.
(ii) $0 \leq b \leq \frac{q^2 + \sqrt{2q}}{4}$.
(iii) $0 \leq c \leq \frac{q^2 - 2}{3}$.
(iv) $0 \leq d \leq \frac{\sqrt{q}(q^2 - 2)}{24}$.
(v) $0 \leq e \leq \frac{\sqrt{q}(q^2 - 2)}{8}$.
(vi) $0 \leq g \leq \frac{\sqrt{q}(q^2 - 2)}{8}$.
(vii) $1 \leq h \leq \frac{\sqrt{q}}{4}$.
(viii) $0 \leq i \leq \frac{\sqrt{q}(q^2 + 1)}{6}$.
(ix) $1 \leq j \leq \frac{\sqrt{q}}{4}$.
(x) $0 \leq r \leq \frac{q^2 - 4}{12}$.
(xi) $1 \leq s \leq \frac{q^2 + \sqrt{2q}}{4}$. If $\ell \neq 5$ or $n \equiv 7 \mod 20$, then $s \geq 2$.
(xii) $0 \leq t \leq \frac{q^2}{4}$.
(xiii) $1 \leq u \leq \frac{q^2 + 3\sqrt{2q} + 4}{12}$.
(xiv) $0 \leq v \leq \frac{q^2 + 2}{3}$. 
For small $q$, some of the lower and upper bounds in Theorem 3.3 coincide. Thus, we obtain the following obvious consequence:

**Corollary 3.4.** Suppose $n = 1$, that is $G = ^2F_4(8)$, and $\ell | q^2 + \sqrt{2}q + 1$. Then $\ell = 13$ and for the decomposition numbers in Theorem 3.3 we have $h = j = 1$ and $x = 2$.

### 3.4. The case $\ell | q^2 - \sqrt{2}q + 1$.

This case is similar to the case $\ell | q^2 + \sqrt{2}q + 1$. The decomposition numbers of the unipotent blocks $E(G, 1)$ are given in Table C.4. The irreducible Brauer characters $\phi_1, \ldots, \phi_{21}$ in the unipotent blocks are distributed into 18 Harish-Chandra series. The Levi subgroup $T$ has a unique cuspidal unipotent Brauer character, and the Levi subgroup $L_0$ has three cuspidal unipotent Brauer characters. The corresponding modular Harish-Chandra series of $G$ contain, respectively, 4, 1, 1, 1 irreducible Brauer characters. Additionally, $G$ has 14 cuspidal unipotent Brauer characters.

As usual, the columns labeled by $ps$ correspond to the irreducible Brauer characters in the principal series. As in the case $\ell | q^2 + \sqrt{2}q + 1$, we denote the modular Harish-Chandra series of $G$ coming from $L_0$ by $^2B_2[a]$, $^2B_2[b]$ and $^2B_2[st]$. They belong to the cuspidal Brauer characters $\varphi_a$, $\varphi_b$ of degree $\sqrt{2}/2(q^2 - 1)$ and the modular Steinberg character $\varphi_{st}$ of $L_0$, respectively. The columns labeled by “c” correspond to the cuspidal unipotent Brauer characters. The decomposition numbers of the non-unipotent irreducible characters are given in Tables D.1, D.2, D.4-D.8 in the Appendix.

**Theorem 3.5.** Let $\ell$ be a prime dividing $q^2 - \sqrt{2}q + 1$. The $\ell$-modular decomposition numbers of $G = ^2F_4(2^{\nu+1})$, $\nu > 0$, are given by Tables C.4 and D.1, D.2, D.4-D.8 in the Appendix. There are the following bounds:

1. $1 \leq w \leq q^2 + \sqrt{2}q + 4/4$.
2. $1 \leq c \leq q^2 - \sqrt{2}q$. 
3. $0 \leq a \leq q^2 - 3\sqrt{2}q + 12/12$.
4. $0 \leq b \leq q^2 - \sqrt{2}q$. 
5. $0 \leq c \leq q^2 - 2/4$.
6. $0 \leq d \leq \sqrt{2}q(q^2 - 2)/24$. 
7. $0 \leq e \leq \sqrt{2}q - 1/4$. 
8. $0 \leq f \leq \sqrt{2}q(q^2 + 2)/8$. 
9. $0 \leq h \leq \sqrt{2}q(q^2 + 2)/20$. 
10. $0 \leq i \leq \sqrt{2}q(q^2 + 1)/6$. 
11. $0 \leq j \leq \sqrt{2}q - 1/4$. 
12. $0 \leq k \leq \sqrt{2}q - 4/12$. 
13. $1 \leq s \leq q^2 - \sqrt{2}q$. If $\ell \neq 5$ or $n \equiv 2$ or $n \equiv 17$ mod 20, then $s \geq 2$. 
14. $0 \leq t \leq q^2 - 3\sqrt{2}q + 12/12$. 
15. $0 \leq u \leq q^4$. 
16. $0 \leq v \leq q^4 - 2/4$. 
17. $0 \leq w \leq q^2 - \sqrt{2}q + 4/4$. 
18. $0 \leq x \leq q^2 - \sqrt{2}q - 2$. 

Corollary 3.6. Suppose \( n = 1 \), that is \( G = 2F_4(8) \), and \( \ell | q^2 - \sqrt{2}q + 1 \). Then \( \ell = 5 \) and for the decomposition numbers in Theorem 3.5 we have \( a = e = j = t = x = 0, s = 1 \).

3.5. Remarks on the decomposition matrices.

(a) If \( \ell \) divides \( q^2 - 1 \), then \( \ell \) is a linear prime for \( G \) in the sense of [24, Section 6] and most of the statements in Theorem 3.1 are immediate consequences of results of G. Hiß; see [24, Theorem 6.3.7]. One can see from Table C.1 that each unipotent block of \( G \) coincides with exactly one modular Harish-Chandra series.

(b) Tables D.1, D.2 and D.5–D.8 are valid for all odd primes \( \ell \), in particular for \( \ell = 3 \) and those primes \( \ell \) where the Sylow \( \ell \)-subgroups of \( G \) are cyclic. For \( \ell = 3 \) this will be discussed in the next section, for the remaining odd primes this follows from Theorems 3.1, 3.2, 3.3, 3.5 and [24]. Example: the otherwise in Table D.7 means for all odd primes \( \ell \) not dividing \( q^4 + 1 \).

(c) Together with the ordinary character table of \( G \) in CHEVIE, the decomposition matrices in Appendices C and D determine all irreducible \( \ell \)-modular Brauer characters of \( G \) for all primes \( \ell \neq 2, 3 \) except for three irreducible Brauer characters (two unipotent and one non-unipotent) in case \( \ell | q^2 + 1 \) and four irreducible Brauer characters in case \( \ell | q^4 + 1 \).

(d) For all non-unipotent blocks, the tables in Appendix D describe only the decomposition numbers of the corresponding ordinary basic sets with one exception: If \( s_5 \in G^+ \) is semisimple of type \( q_5 \) and \( 3 \neq \ell | q^2 + 1 \), then Table D.3 contains information on the decomposition numbers of all ordinary characters in \( E_{\ell}(G, s_5) \). The reason for this is that we were not able to compute the decomposition number \( a' \) in \( \chi_{5, S_1} \) and we hope that this additional information might be useful in the determination of \( a' \) using arguments similar to those in [34].

(e) The decomposition numbers of \( 2F_4(2) \) and the simple Tits group \( 2F_4(2)' \) were calculated for all odd primes \( \ell \) by G. Hiß [22]. The case of defining characteristic was handled by F. Veldkamp [39]. These decomposition matrices are contained in the GAP library [12].

3.6. On a conjecture of M. Geck and G. Hiß. In the following corollary we verify a conjecture of M. Geck and G. Hiß in the special case of the Ree groups \( G = 2F_4(q^2) \), see [15, Conjecture 3.4].

Corollary 3.7. Let \( \ell \) be a good prime for a root system of type \( F_4 \) and let \( \mathcal{M}(F_1) = \{\chi_1\} \), \( \mathcal{M}(F_2) = \{\chi_2, \chi_3\} \), \( \mathcal{M}(F_3) = \{\chi_4\} \), \( \mathcal{M}(F_4) = \{\chi_5, \ldots, \chi_{17}\} \), \( \mathcal{M}(F_5) = \{\chi_{18}\} \), \( \mathcal{M}(F_6) = \{\chi_{19}, \chi_{20}\} \), \( \mathcal{M}(F_7) = \{\chi_{21}\} \) be the families of unipotent irreducible characters of \( G = 2F_4(2^{2n+1}) \), \( n > 0 \), see Subsection 2.3. Furthermore, let \( D \) be the part of the decomposition matrix of \( G \) corresponding to the rows labeled by the ordinary unipotent irreducible characters.

(a) The irreducible Brauer characters in \( E_{\ell}(G, 1) \) can be labeled such that \( D \) has the following shape:

\[
D = \begin{pmatrix}
D_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & D_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & D_3 & 0 & 0 & 0 & 0 \\
* & * & * & D_4 & 0 & 0 & 0 \\
* & * & * & * & D_5 & 0 & 0 \\
* & * & * & * & * & D_6 & 0 \\
* & * & * & * & * & * & D_7
\end{pmatrix}
\]
where \( D_i \) is the identity matrix of size \( |\mathcal{M}(F_i)| \times |\mathcal{M}(F_i)| \) for \( 1 \leq i \leq 7 \).

(b) If an ordinary unipotent character \( \chi \in \text{Irr}(G) \) is cuspidal, then \( \bar{\chi} \) is an irreducible Brauer character.

**Proof.** The condition that \( \ell \) is good for \( F_4 \) is equivalent with \( \ell > 3 \). So the claim follows from the decomposition matrices in Appendix C and the Brauer trees in [24], [25]. \( \square \)

4. **On the Decomposition Matrices for \( \ell = 3 \)**

In this section, we provide some information on the 3-modular decomposition numbers of \( G = F_4(q^2), \) \( q^2 = 2^{2n+1}, \) \( n > 0 \). We fix a 3-modular splitting system \((K, R, k)\) for all subgroups of \( G \) as in Subsection 2.2. All references to Brauer characters, blocks, decomposition numbers in this section will refer to characteristic 3. The prime number 3 is a bad prime for \( G \) in the sense of [10, p. 125] and it divides \( \phi_4 \) and \( \phi_{12} \) and does not divide \( \bar{\phi}_1 \phi_6 \phi_8 \phi_{24} \phi_{24}'. \) The Sylow 3-subgroups of \( G \) are non-abelian.

For every semisimple element \( s \in G^* \) of order prime to 3, the set \( E_3(G, s) \) is a union of blocks. However, since 3 is a bad prime for \( G \), the results on ordinary basic sets in [14] do not apply in this situation. In fact, the set \( E(G, 1) \) of ordinary unipotent irreducible characters of \( G \) is not a basic set for \( E_3(G, 1) \). Using the explicit knowledge of the character table of \( G \), is is not difficult to see that

\[
\{ \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{18}, \chi_{19}, \chi_{20}, \chi_{21} \}
\]

is a basic set for \( E_3(G, 1) \). There are also basic sets consisting entirely of unipotent characters, but we had to use this basic set with the non-unipotent character \( \chi_{5,1} \) in order to show that the decomposition matrix has a unitriangular shape. Again, using the explicit knowledge of the character table of \( G \), it is easy to determine the relations expressing the restrictions of the remaining ordinary characters in \( E_3(G, 1) \) to the 3-regular elements as linear combinations of the characters in the basic set, see Table B.13. The following statements will be proved in Section 5.

We begin with some comments on the decomposition numbers of the unipotent blocks \( E_3(G, 1) \) which are given in Table C.5. There are 19 irreducible Brauer characters in the unipotent blocks; notation for these Brauer characters is given in the first row of Table C.5. These irreducible Brauer characters are distributed into 14 Harish-Chandra series corresponding to the Levi subgroups \( T, L_a, L_b \) and \( G \). The Levi subgroups \( T \) and \( L_a \) each have a unique cuspidal unipotent Brauer character, and the Levi subgroup \( L_b \) has exactly two cuspidal unipotent Brauer characters. The corresponding modular Harish-Chandra series of \( G \) contain, respectively, 4, 1, 2, 2 irreducible Brauer characters. Additionally, \( G \) has 10 cuspidal unipotent Brauer characters.

The principal series is abbreviated by \( ps \). The Levi subgroup \( L_a \) has a unique cuspidal unipotent Brauer character, the modular Steinberg character; see [17]. We denote the corresponding modular Harish-Chandra series of \( G \) by \( A_1 \). The Levi subgroup \( L_b \) has two cuspidal unipotent Brauer characters \( \varphi_a, \varphi_b \), the restrictions of the ordinary unipotent irreducible cuspidal characters to the 3-regular elements. We denote the corresponding modular Harish-Chandra series by \( B_2[a], B_2[b] \), respectively. The remaining columns correspond to the cuspidal unipotent Brauer characters.

The situation for the non-unipotent characters of \( G \) is less complicated. Using the character table of \( G \), one can see that for all 3'-elements \( s \neq 1 \), the set \( E(G, s) \) is an ordinary basic set for \( E_3(G, s) \). The decomposition numbers of the non-unipotent irreducible characters are given in Tables D.1, D.2, D.5-D.8 in the Appendix. In these tables, only the decomposition numbers of the ordinary basic sets \( E(G, s), s \neq 1 \), are listed.
Theorem 4.1. The 3-modular decomposition numbers of $G = 2F_4(2^{2n+1})$, $n > 0$, are given by Tables C.5 and D.1, D.2 and D.5-D.8 in the Appendix. There are the following bounds:

(i) $2 \leq a \leq q^2$.

(ii) $0 \leq b \leq \frac{q^2 + \sqrt{2q}}{4}$. If $n \equiv 1$ or 4 mod 6, then $b \geq 1$. If $n \equiv 4$ or 13 mod 18, then $b \geq 2$.

(iii) $0 \leq c \leq \frac{q^2 - \sqrt{2q}}{4}$. If $n \equiv 1$ or 4 mod 6, then $c \geq 1$. If $n \equiv 4$ or 13 mod 18, then $c \geq 2$.

(iv) $2 \leq d \leq q^4$.

(v) $1 \leq e \leq \frac{q^2 + 2}{4}$. If $n \equiv 1$ or 4 mod 6, then $e \geq 2$. If $n \equiv 4$ or 13 mod 18, then $e \geq 3$.

(vi) $0 \leq x_7 \leq \frac{q^2}{2}$.

(vii) $0 \leq x_8 \leq \frac{q^2 + 3\sqrt{2q} + 4}{12}$.

(viii) $0 \leq x_{10} \leq \frac{q^2 - 2}{6}$.

(ix) $1 \leq x_{15} \leq \frac{q^2 + 1}{4}$.

(x) $0 \leq x_{18} \leq q^2(q^2 - 1)$.

(xi) $1 \leq x_{21} \leq q^6$.

Theorem 4.1 will be proved in Section 5.

Corollary 4.2. Suppose $n = 1$, that is $G = 2F_4(8)$, and $\ell = 3$. Then for the decomposition number $c$ in Theorem 4.1 we have $c = 1$.

Proof. This follows from the bounds in Theorem 4.1. \qed

Corollary 4.3. Let $\ell$ be an odd prime and $G = 2F_4(2^{2n+1})$ where $n \geq 0$. After a suitable arrangement of rows and columns, the $\ell$-modular decomposition matrix of $G$ has a lower unitriangular shape.

Proof. For $n = 0$, this follows from the decomposition matrices of $2F_4(2)$ in GAP. For $n > 0$, it is a consequence of the decomposition matrices in Theorems 3.1-3.3, 3.5, 4.1 and the Brauer trees in [25]. \qed

4.1. Remarks on the 3-modular decomposition matrices. It seems to be necessary to use methods different from the ones in this paper to improve the bounds for the decomposition numbers in Theorem 4.1 substantially. We induced projective characters of $B$ and several maximal subgroups of $G$, computed tensor products of projective characters and irreducible characters of $G$ and considered restrictions of modules to 3-blocks of the parabolic subgroup $P_a$. However, in this way we were not able to derive better upper bounds for the decomposition numbers in Theorem 4.1.

5. Proofs

In this section, we prove the theorems of Sections 3 and 4. We begin by describing some information and general methods which will be used in the proofs of these theorems.
5.1. Projective characters. One of the main ingredients in the proof of the theorems in Sections 3 and 4 is the construction of projective characters. When we speak of a projective character of $G$ we mean an ordinary character which is a sum of characters of the projective indecomposable $kG$-modules (PIMs). Via Brauer reciprocity, we can interpret the decomposition numbers of $G$ as the scalar numbers of the projective characters corresponding to the PIMs with the ordinary irreducible characters of $G$. We are going use the following methods to produce projective characters:

- Every character of defect 0 of $G$ is projective.
- If $\chi, \psi$ are characters of $G$ and $\psi$ is projective, then the product $\chi \otimes \psi$ is projective.
- If $\psi$ is a projective character of a subgroup $H$ of $G$, then the induced character $\psi^G$ is projective. This is particularly useful, if $H$ is an $\ell'$-subgroup, since then every character of $H$ is projective.
- If $L$ is one of the Levi subgroups $T, L_\alpha, L_\beta$ and $\psi$ a projective character of $L$, then the Harish-Chandra induced character $R^G_L(\psi)$ is projective; see [24, Lemma 4.4.3].

In particular, if $\gamma$ is the Gelfand-Graev character of $L$, then $R^G_L(\gamma)$ is projective.

The character tables of $B, P_\alpha, P_\beta, G$ are available as CHEVIE files and we have computer programs (written by C. K"ohler and the author), based on the class fusions in [19] and [21], for induction and restriction of characters between these parabolic subgroups. Using these programs and CHEVIE, we can easily compute induced and Harish-Chandra induced characters as well as tensor products and scalar products of characters of the various parabolic subgroups of $G$.

5.2. Hecke algebras. Another ingredient in the proof of the theorems in Sections 3 and 4 are the decomposition numbers of the Hecke algebra $\mathcal{H}$ corresponding to the permutation module on the cosets of the Borel subgroup $B$ in $G$. By [11, Corollary 4.10], these decomposition numbers form a submatrix of the decomposition matrix of $\mathcal{E}_L(G, 1)$.

The computation of the decomposition numbers of $\mathcal{H}$ is similar to the proof of [13, Proposition 5.1]: Let $1_B$ be the trivial $RB$-module, and $V := 1_B^G$ the corresponding $RG$-permutation module with ordinary character $\vartheta$. Computing scalar products with CHEVIE, we get

$$\vartheta = \chi_1 + \chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + \chi_{18} + \chi_{21}.$$

The Weyl group $W_1$ of $G$ (as a group with a BN-pair) has Coxeter generators $w_a, w_b$ and is isomorphic to a dihedral group of order 16; see [21, Section 2]. The Hecke algebra $\mathcal{H} := \text{End}_{RG}(V)$ of $V$ has a natural $R$-basis $\{T_w | w \in W_1\}$ satisfying the relations

$$T_\gamma T_w = \begin{cases} T_{w\gamma w}, & \text{if } l(w, \gamma) > l(w), \\ p_\gamma T_{w\gamma w} + (p_\gamma - 1) T_w, & \text{if } l(w, \gamma) < l(w) \end{cases}$$

for all $\gamma \in \{a, b\}$ and $w \in W_1$. Here, $l$ denotes the Coxeter length and we have used the abbreviation $T_\gamma = T_{w, \gamma}$. The parameters $p_a = q^2$ and $p_b = q^4$ are given in [24, p. 66]. As described in [9, § 67C], one can construct irreducible representations $\text{sgn}, \sigma_1, \omega_2, S_1, S_{-1}$ and $S_0$ of $\mathcal{H}$ affording irreducible representations of $\mathcal{H}_K := K \otimes_R \mathcal{H}$ (by extending scalars), which will be denoted in the same way. The 1-dimensional of these representations are

$$\text{ind} : \begin{cases} T_a \mapsto q^2, \\ T_b \mapsto q^4 \end{cases}, \quad \text{sgn} : \begin{cases} T_a \mapsto -1, \\ T_b \mapsto -1, \end{cases}, \quad \sigma_1 : \begin{cases} T_a \mapsto -1, \\ T_b \mapsto q^4, \end{cases}, \quad \sigma_2 : \begin{cases} T_a \mapsto q^2, \\ T_b \mapsto -1, \end{cases}$$
and for $\varepsilon = 0, \pm 1$ there are the following 2-dimensional representations:

$$S_\varepsilon : \quad T_a \mapsto \begin{pmatrix} q^2 + \varepsilon \sqrt{2}q + 1 & 0 \\ q^2 & -1 \end{pmatrix}, \quad T_b \mapsto \begin{pmatrix} -1 & q^2 \\ 0 & q^4 \end{pmatrix}.$$  

There is a natural bijection (“Fitting correspondence”) between the isomorphism classes of irreducible representations of $H_K$ and the irreducible constituents of $\vartheta$. The representations ind, sgn, $\sigma_1$, $\sigma_2$, $S_1$, $S_{-1}$ and $S_0$ correspond to the unipotent characters $\chi_1$, $\chi_{21}$, $\chi_4$, $\chi_{18}$, $\chi_5$, $\chi_6$ and $\chi_7$, respectively. This can be seen for example by computing generic degrees using [8, Theorem 10.11.5].

**Proposition 5.1.** For $\ell \mid \tilde{\phi}_4\phi_8\phi_6'$, the $\ell$-modular decomposition numbers of the Hecke algebra $H$ are given by Table 1.

**Proof.** The decomposition numbers of the 1-dimensional representations are clear. For the 2-dimensional ones, they follow easily from considering simultaneous eigenspaces. □

|       | $\ell \mid q^2 - 1$ | $\ell \mid q^2 + 1$ | $\ell \mid q^2 + \sqrt{2}q + 1$ | $\ell \mid q^2 - \sqrt{2}q + 1$ |
|-------|---------------------|---------------------|-------------------------------|-------------------------------|
| ind   | 1 . . . . . . . .   | 1 . . . . . . . .   | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| $\sigma_1$ | . 1 . . . . . .     | 1 . . . . . . . .   | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| $S_1$  | . . 1 . . . . . .   | 1 . . . . . . . .   | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| $S_{-1}$ | . . . 1 . . . . .   | . 1 . . . . . . . . | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| $S_0$  | . . . . . 1 . . .   | 1 . . . . . . . .   | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| $\sigma_2$ | . . . . . 1 . .     | . 1 . . . . . . . . | 1 . . . . . . . . .          | 1 . . . . . . . . .          |
| sgn   | . . . . . . . . 1   | . . . . . . . . 1   | . . . . . . . . . . . . .   | . . . . . . . . . . . . .   |

5.3. **Relations.** Let $\ell > 3$ be a prime number. By Subsection 2.6, the set of unipotent irreducible characters of $G$ is an ordinary basic set for the unipotent blocks $E_\ell(G, 1)$. In particular, for every non-unipotent irreducible character $\chi \in E_\ell(G, 1)$, there are $a_{\chi,j} \in \mathbb{Z}$ such that

$$\bar{\chi} = \sum_{j=1}^{21} a_{\chi,j} \cdot \chi_j. \quad (2)$$

Here, we have identified $\chi_j$ and $\bar{\chi}_j$. We call (2) a relation with respect to the basic set of unipotent characters. Such relations can be used to derive lower bounds for the decomposition numbers of the unipotent characters or to prove the indecomposability of certain projective modules. For the relevant primes $\ell$, the relations with respect to the basic set of unipotent characters are given in Tables B.9-B.12 in the Appendix.

The rows of these tables are labeled by the non-unipotent irreducible characters in $E_\ell(G, 1)$ and the numbering of the columns corresponds to the unipotent irreducible characters of $G$. The entry in the row corresponding to $\chi \in E_\ell(G, 1)$ and the $j$-th column is equal to $a_{\chi,j}$. In these tables zeros are replaced by dots. The data in Tables B.9-B.12 can easily be computed from the character table of $G$ in CHEVIE.
5.4. **Jordan decomposition of Brauer characters.** Our main tool to determine the decomposition numbers of the non-unipotent blocks of $G$ is C. Bonnafé’s and R. Rouquier’s modular version of the Jordan decomposition of characters [2, Theorem 11.8]. Let $\ell$ be an odd prime. For $1 \leq i \leq 18$, let $s_i \in G^*$ be a semisimple $\ell$'-element of type $g_5$, see [20, Table B.10]. We consider the centralizer of $s_i$ in the algebraic group $G^*$. Since the center $Z(G)$ is connected, the centralizer $C_{G^*}(s_i)$ is also connected. Now suppose $i \neq 5$. Then, $C_{G^*}(s_i)$ can be realized as the centralizer of a semisimple element of an order not divisible by 2 and 3. So by [7, Proposition 13.16], the centralizer $C_{G^*}(s_i)$ is a Levi subgroup of $G^*$ and we can apply [2, Theorem 11.8]. Thus, for all $i \neq 5$, Lusztig induction induces a 1-1-correspondence between the sets $E_\ell(G, s_i)$ and $E_\ell(C_{G^*}(s_i), 1)$ of ordinary irreducible characters, and with respect to this correspondence the decomposition matrices of $E_\ell(G, s_i)$ and $E_\ell(C_{G^*}(s_i), 1)$ coincide (after a suitable ordering of the columns).

The Dynkin type of $C_{G^*}(s_i)$ is given in [20, Table B.9], see also [24, Lemma D.3.2]. For $i \neq 1, 5$, it is $2B_2$, $A_1$ or $A_0$ and so the decomposition numbers of $E_\ell(C_{G^*}(s_i), 1)$ are known, see [6] and [27]. Thus, Bonnafé’s and Rouquier’s Jordan decomposition gives us the decomposition numbers for all non-unipotent blocks of $G$ except for $E_\ell(G, s_5)$. Note that the semisimple element $s_5$ has order 3 and is isolated in the sense of [1, 1.B]. This follows from [1, Corollary 1.4] and the data in [20, Table B.9]. So, $C_{G^*}(s_5)$ is not contained in a Levi subgroup of a proper parabolic subgroup of $G^*$ and Bonnafé’s and Rouquier’s theorem does not apply.

5.5. **Proof of Theorem 3.1.** Suppose $\ell \mid q^2 - 1$. By [24, Proposition 6.3.4], $\ell$ is a linear prime for $G$, so that we can apply [24, Theorem 6.3.7]. We begin with the decomposition numbers of the unipotent blocks of $G$ in Table C.1. The numbers in the right most column are obtained by counting the elements of $\ell$-power order in $G^*$, which can easily be done using the representatives in [20, Table B.10]. The decomposition numbers in Table C.1 follow from [24, Theorem 6.3.7] and the relations in Table B.9.

The modular Harish-Chandra series can be determined as follows: By Subsection 5.2, the Brauer characters $\phi_1, \phi_4, \phi_5, \phi_6, \phi_7, \phi_{18}, \phi_{21}$ are the irreducible Brauer characters in the principal series. By [26, Lemma 4.3], the Brauer characters $\phi_i, 8 \leq i \leq 17$, are cuspidal. The Levi subgroup $L_4$ has 4 unipotent irreducible Brauer characters: the restrictions of the trivial character and of the Steinberg character and the restrictions of the two ordinary cuspidal unipotent characters of degree $\frac{q}{2}(q^2 - 1)$ to the $\ell$-regular elements (this follows from the decomposition numbers in [6]). The first two of these Brauer characters are in the principal series, the latter two are cuspidal Brauer characters. Again by [26, Lemma 4.3], we know that $\phi_2, \phi_3, \phi_{19}, \phi_{20}$ are not cuspidal. Now, the claim about the Harish-Chandra series of these four characters follows from the remarks in Subsection 2.3.

Finally, we treat the decomposition numbers for the non-unipotent blocks in Tables D.1, D.2 and D.4-D.8. Let $s \neq 1$ be a semisimple $\ell$'-element of $G^*$. If $s$ is not of type $g_5$, then the decomposition numbers of $E_\ell(G, s)$ are clear by Bonnafé’s and Rouquier’s Jordan decomposition, see Subsection 5.4. This proves the decomposition numbers in Tables D.1, D.2 and D.5-D.8. The decomposition numbers in Table D.4 follow from the fact that $\ell$ is a linear prime and [24, Theorem 6.3.7]. This completes the proof of Theorem 3.1.

5.6. **Proof of Theorem 3.2.** Suppose $\ell > 3$ and $\ell \mid q^2 + 1$. We start with the decomposition numbers of the unipotent blocks of $G$ in Table C.2. As in the proof of Theorem 3.1, the numbers in the right most column are obtained by counting the elements of $\ell$-power order in $G^*$, which can easily be done using the representatives in [20, Table B.10]. Note that under the assumptions of the theorem, we have $\frac{1}{48}(\ell^f - 1)(\ell^f - 11) > 0$ if and only if
\( \ell \neq 11 \) or \( n \equiv 27 \mod 55 \) where \( q^2 = 2^{2n+1} \). Consequently, the relation in the last row of Table B.10 exists if and only if \( \ell \neq 11 \) or \( n \equiv 27 \mod 55 \).

Using the relations in Table B.10, the rows of the decomposition matrix of \( E_\ell(G,1) \) corresponding to the non-unipotent irreducible characters can be written as linear combinations of the rows corresponding to the unipotent irreducible characters. So, it is sufficient to determine the decomposition numbers of the unipotent irreducible characters \( \chi_1, \ldots, \chi_{21} \).

We construct projective characters \( \Psi_1, \ldots, \Psi_{21} \) of \( G \) according to Table B.2 in the Appendix and compute the scalar products \( (\chi_i, \Psi_j)_G \) for \( 1 \leq i, j \leq 21 \) using CHEVIE. These scalar products are given in Table B.1 in the Appendix. We already see from Table B.1 that the decomposition matrix of \( E_\ell(G,1) \) has a lower unitriangular shape giving us a natural bijection between the set of ordinary unipotent irreducible characters and the set of irreducible Brauer characters in \( E_\ell(G,1) \).

For \( 1 \leq i \leq 21 \), let \( \phi_i \) be the irreducible Brauer character corresponding to \( \chi_i \) and \( \Phi_i \) the character of the corresponding PIM.

From Table B.1, we already get the assertions about all \( \Phi_i, i \neq 1,4,7,8,9,17,18 \) in the decomposition matrix Table C.2. Let \( \alpha \) be the multiplicity of \( \chi_{18} \) in \( \Phi_{17} \) and \( b, c, d, e \) the multiplicity of \( \chi_{21} \) in \( \Phi_8, \Phi_9, \Phi_{17}, \Phi_{18} \), respectively. The entries in Table B.1 lead to the upper bounds for \( a, b, c, d, e \) in Theorem 3.2, and the lower bounds follow from the decomposition numbers of the non-unipotent characters in the last three rows of Table C.2, since decomposition numbers are non-negative. So we have shown all assertions about \( \Phi_8, \Phi_9, \Phi_{17}, \Phi_{18} \) in Theorem 3.2. Furthermore, [11, Corollary 4.10] and Proposition 5.1 imply the assertions about \( \Phi_1 \) and \( \Phi_7 \).

So, we are only left with the decomposition numbers in the fourth column of Table C.2. All of them are clear from Table B.1 except for the decomposition numbers \( (\chi_7, \Phi_4)_G, (\chi_{21}, \Phi_4)_G \in \{0, 1\} \). Since \( \Phi_7 \) is not a summand of \( \Psi_4 \), we get \( (\chi_7, \Phi_4)_G = 1 \).

To determine the decomposition number \( (\chi_{21}, \Phi_4)_G \), we collect some information on the modular Harish-Chandra series of \( G \). By Proposition 5.1, we already know that \( \phi_1, \phi_5, \phi_6, \phi_7 \) are the Brauer characters in the principal series. The Levi subgroup \( L_b \) has two cuspidal unipotent irreducible Brauer characters: the restrictions of the two ordinary cuspidal unipotent characters of degree \( \frac{q^2}{\sqrt{2}}(q^2-1) \) to the set of \( \ell \)-regular elements (this follows from the decomposition numbers in [6] and [26, Lemma 4.3]). We write \( \varphi_a \) and \( \varphi_b \) for these two Brauer characters and \( \Phi_a, \Phi_b \) for the characters of the corresponding PIMs of \( L_b \). Let \( 2B_2[a] \), \( 2B_2[b] \) be the modular Harish-Chandra series of \( G \) corresponding to \( \varphi_a \) and \( \varphi_b \), respectively. Let \( u(\Phi_a) \) be the character of the unipotent quotient of \( \Phi_a \) and \( u(R_{L_a}^G(\Phi_a)) \) the character of the unipotent quotient of the Harish-Chandra induced character \( R_{L_a}^G(\Phi_a) \), see [26, Section 6]. By [26, Lemma 6.1], Harish-Chandra induction commutes with taking unipotent quotients. Using the class fusions in [19] and CHEVIE, we compute

\[
u(\Phi_a) = R_{L_a}^G(\Phi_a) = R_{L_a}^G(u(\Phi_a)) = \rho_a, \chi_2(0)^G = \chi_2 + \chi_{19}.
\]

Thus, from [26, Section 5] we see that \( \phi_2 \) and \( \phi_{19} \) are the only irreducible Brauer characters in the series \( 2B_2[a] \). Analogously, by computing \( u(R_{L_a}^G(\Phi_b)) \) we see that \( \phi_3 \) and \( \phi_{20} \) are the only irreducible Brauer characters in the series \( 2B_2[b] \).

Next, we consider the modular Harish-Chandra series \( A_1 \). Let \( \Phi_{St} \) be the character of the PIM corresponding to the modular Steinberg character of \( L_a \). Using CHEVIE, we can compute the unipotent quotient

\[
u(\Phi_{St}) = R_{L_a}^G(u(\Phi_{St})) = \rho_a, \chi_2(0)^G = \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_{21}.
\]

By the construction in [21, Section 4], the character \( \rho \chi_8^G \) is the Gelfand-Graev character of \( G \). Hence, Table B.2 implies that \( \phi_{21} \) is the modular Steinberg character of \( G \) and it follows from [17, Theorem 4.2] that \( \phi_{21} \) is cuspidal. From the decomposition (3) we get
that $\phi_4$ is the only irreducible Brauer character of $G$ in the Harish-Chandra series $A_1$ and $(\chi_{21}, \Phi_4)_G = 1$. Consequently, the remaining Brauer characters $\phi_i$ for $8 \leq i \leq 18$ have to be cuspidal. This proves all assertions about the unipotent blocks in Theorem 3.2.

Finally, we deal with the decomposition numbers of the non-unipotent irreducible characters of $G$ in Tables D.1-D.3 and D.5-D.8. Let $s \neq 1$ be a semisimple $\ell$-element of $G^*$. If $s$ is not of type $g_5$, then the decomposition numbers of $E(G, s)$ are clear by Bonnafé’s and Rouquier’s Jordan decomposition, see Subsection 5.4. This proves the decomposition numbers in Tables D.1, D.2 and D.5-D.8. The decomposition numbers in Table D.3 can be determined as follows: Suppose $s_5 \in G^*$ is a semisimple $\ell$-element of type $g_5$. We get an approximation to the decomposition matrix of $E(G, s_5)$ from the following scalar products of basic set characters with projective characters:

| $R^G_{L_a}(\gamma_{L_a})$ | $p_0 \chi_{22}^G$ | $\psi_{\chi_{15}}^G$ |
|---------------------------|-------------------|-------------------|
| $\chi_{5,1}$              | 1                 | 1                 |
| $\chi_{5,q^2(q^2-1)}$    | $q^0$             | 1                 |
| $\chi_{5,St}$             | $q^2 - \sqrt{2q} + 1$ | 1                 |

Here, $\gamma_{L_a}$ is the Gelfand-Graev character of $L_a$, the character $\psi_{\chi_{15}}^G$ is the Gelfand-Graev character of $G$ and $p_0 \chi_{22}$ is projective since $\ell \nmid |P_0|$. This already gives the upper bound for $a'$ in Theorem 3.2. Let $\Phi_{5,1}, \Phi_{5,2}, \Phi_{5,3}$ be the characters of the PIMs corresponding to the irreducible Brauer characters $\phi_{5,1}, \phi_{5,2}, \phi_{5,3}$. We have the following relations on the $\ell$-regular elements:

$$
\tilde{\chi}_{6,1} = \tilde{\chi}_{5,1} + \tilde{\chi}_{5,q^2(q^2-1)},
$$

$$
\tilde{\chi}_{6,St} = -\tilde{\chi}_{5,q^2(q^2-1)} + \tilde{\chi}_{5,St},
$$

$$
\tilde{\chi}_{15,1} = -\tilde{\chi}_{5,1} - 2 \cdot \tilde{\chi}_{5,q^2(q^2-1)} + \tilde{\chi}_{5,St}.
$$

Via these relations, we can express the decomposition numbers of $\chi_{6,1}, \chi_{6,St}, \chi_{15,1}$ in terms of the decomposition numbers of $\chi_{5,1}, \chi_{5,q^2(q^2-1)}$ and $\chi_{5,St}$. Since decomposition numbers are non-negative, we get $(\chi_{5,St}, \Phi_{5,1})_G = 1$ and $a' \geq 2$. This completes the proof of Theorem 3.2. □

5.7. Proof of Theorem 3.3. Suppose $\ell | q^2 + \sqrt{2q} + 1$. The decomposition numbers of the unipotent blocks of $G$ in Table C.3 can be determined as follows: As in the proof of Theorem 3.1, the numbers in the right most column are obtained by counting the elements of $\ell$-power order in $G^*$, which can easily be done using the representatives in [20, Table B.10]. Note that under the assumptions of the theorem, we have $\frac{1}{m} ((\ell^f-1)(\ell^f-5) > 0$ if and only if $\ell \neq 5$ or $n \equiv 7$ or $n \equiv 12$ mod 20 where $q^2 = 2^{2n+1}$. Consequently, the relation in the last row of Table B.11 exists if and only if $\ell \neq 5$ or $n \equiv 7$ or $n \equiv 12$ mod 20.

Using the relations in Table B.11, the rows of the decomposition matrix of $E_1(G, 1)$ corresponding to the non-unipotent irreducible characters can be written as linear combinations of the rows corresponding to the unipotent irreducible characters. So, it is sufficient to determine the decomposition numbers of the unipotent irreducible characters $\chi_1, \ldots, \chi_{21}$. We construct projective characters $\Psi_1, \ldots, \Psi_{21}$ of $G$ according to Table B.4 and compute the scalar products $(\chi_i, \Psi_j)_G$ for $1 \leq i, j \leq 21$ using CHEVIE. In Table B.4, $\Phi_a$ and $\Phi_b$ are the characters of the projective covers of the unipotent irreducible Brauer characters $\varphi_a$ and $\varphi_b$ of $L_b$, respectively, and $\Phi_{St}$ is the character of the projective cover of the modular Steinberg character $\varphi_{St}$ of $L_b$; see the comments in Subsection 3.3. For the calculations in CHEVIE, we only deal with the unipotent quotients. The scalar products $(\chi_i, \Psi_j)_G$ are given in Table B.3 in the Appendix. The symbol $*$ in this table means some non-negative
integer depending on \(q\). We already see from Table B.3 that the decomposition matrix of \(E_1(G, 1)\) has a lower unitriangular shape giving us a natural bijection between the set of ordinary unipotent irreducible characters and the set of irreducible Brauer characters in \(E_1(G, 1)\). For \(1 \leq i \leq 21\), let \(\phi_i\) be the irreducible Brauer character corresponding to \(\chi_i\) and \(\Phi_i\) the character of the corresponding PIM.

From Table B.3, we already get the assertions about \(\Phi_i\) for \(i = 6, 7, 8, 15, 16, 21\) in the decomposition matrix Table C.3. Furthermore, [11, Corollary 4.10] and Proposition 5.1 imply the assertions about \(\Phi_1\) and \(\Phi_4\) and we see that \(\phi_1, \phi_4, \phi_6\) and \(\phi_7\) are the irreducible Brauer character in the principal series.

We introduce abbreviations for some of the decomposition numbers: Let \(a\) and \(c\) be the multiplicity of \(\chi_{18}\) in \(\Phi_9\) and \(\Phi_{17}\), respectively. Furthermore, let \(r, s, v, w\) be the multiplicity of \(\chi_{21}\) in \(\Phi_9, \Phi_{10}, \Phi_{17}, \Phi_{18}\), respectively. We construct three additional projective characters of \(G\):

\[
\Psi'_{13} := \chi_3 \otimes \chi_8, \quad \Psi'_{14} := \chi_2 \otimes \chi_8, \quad \Psi'_{18} := \chi_2 \otimes \chi_6.
\]

These characters are projective since \(\chi_6\) and \(\chi_8\) have defect 0. Using CHEVIE, we compute the scalar products \((\Psi'_i, \chi_j)_G\) for \(i = 13, 14, 18\) and \(1 \leq j \leq 21\), see Table 2. For all pairs \((\Psi'_i, \chi_j)\) not listed in this table, we have \((\Psi'_i, \chi_j)_G = 0\). The scalar products for \(\Psi'_{13}, \Psi'_{14}\) imply:

\[
(\chi_{18}, \Phi_{13})_G = (\chi_{20}, \Phi_{13})_G = (\chi_{18}, \Phi_{14})_G = (\chi_{19}, \Phi_{14})_G = 0.
\]

**TABLE 2.** The scalar products \((\Psi'_i, \chi_j)_G\).

| \(\chi_7\) | \(\chi_{13}\) | \(\chi_{14}\) | \(\chi_{18}\) | \(\chi_{19}\) | \(\chi_{20}\) | \(\chi_{21}\) |
|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \(\Psi'_{13}\) | 0 | \(q\) \(\sqrt{2}\) | 0 | \(q^2 + \sqrt{2}q\)/4 | 0 | \(\sqrt{2}\) \(q^2 + \sqrt{2}q\)/4 |
| \(\Psi'_{14}\) | 0 | 0 | \(q\) \(\sqrt{2}\) | 0 | \(q^2 + \sqrt{2}q\)/4 | \(\sqrt{2}\) \(q^2 + \sqrt{2}q\)/4 |
| \(\Psi'_{18}\) | \(q\) \(\sqrt{2}\) | \(q\) \(\sqrt{2}\) | 0 | \(q^2 + \sqrt{2}q\)/2 | \(q^2 + \sqrt{2}q\)/2 | \(\sqrt{2}\) \(q^2 + \sqrt{2}q\)/8 |

The pairs \((\chi_{11}, \chi_{12}), (\chi_{13}, \chi_{14}), (\chi_{19}, \chi_{20})\) are pairs of complex conjugate characters, and so are the pairs \((\Phi_{11}, \Phi_{12}), (\Phi_{13}, \Phi_{14}), (\Phi_{19}, \Phi_{20})\). On the other hand, \(\chi_9, \chi_{17}\) and \(\chi_{18}\) are real-valued, and so are \(\Phi_9, \Phi_{17}\) and \(\Phi_{18}\). Hence, we have the following identities of decomposition numbers:

\[
(\chi_{18}, \Phi_{11})_G = (\chi_{18}, \Phi_{12})_G =: b, \quad (\chi_{19}, \Phi_{9})_G = (\chi_{20}, \Phi_{9})_G =: d,
\]
\[
(\chi_{19}, \Phi_{11})_G = (\chi_{20}, \Phi_{12})_G =: e, \quad (\chi_{20}, \Phi_{11})_G = (\chi_{19}, \Phi_{12})_G =: g,
\]
\[
(\chi_{19}, \Phi_{13})_G = (\chi_{20}, \Phi_{14})_G =: h, \quad (\chi_{19}, \Phi_{17})_G = (\chi_{20}, \Phi_{17})_G =: i,
\]
\[
(\chi_{19}, \Phi_{18})_G = (\chi_{20}, \Phi_{18})_G =: j, \quad (\chi_{21}, \Phi_{11})_G = (\chi_{21}, \Phi_{12})_G =: t,
\]
\[
(\chi_{21}, \Phi_{13})_G = (\chi_{21}, \Phi_{14})_G =: u, \quad (\chi_{21}, \Phi_{19})_G = (\chi_{21}, \Phi_{20})_G =: x.
\]

From the scalar products in Table B.3, we readily get the upper bounds in Theorem 3.3 except for the decomposition numbers \(h, j, u, w\). The scalar products with the projectives in Table B.4 lead to upper bounds for these remaining four decomposition numbers. However, these bounds are not good enough for our purposes. Instead, we use the projectives in Table 2. From this table, we get

\[
\Psi'_{13} = \frac{q}{\sqrt{2}} \Phi_{13} + A \cdot \Phi_{19} + B \cdot \Phi_{21} + \Phi,
\]
where $A, B$ are non-negative integers and $\Phi$ is a projective character belonging to non-unipotent blocks. Thus, we get: $$\frac{q^2+\sqrt{2q}}{4} = \frac{q}{\sqrt{2}} \cdot h + A \geq \frac{q}{\sqrt{2}} \cdot h$$ and so $h \leq \frac{\sqrt{2q}}{4} + \frac{1}{2}$.

Since $h$ is an integer, it follows $h \leq \frac{\sqrt{2q}}{4}$. For $\Psi'_{18}$, we get

$$\Psi'_{18} = \frac{q}{\sqrt{2}} \Phi_7 + \frac{q}{\sqrt{2}} \Phi_{13} + \frac{q}{\sqrt{2}} \Phi_{18} + A \cdot \Phi_{19} + B \cdot \Phi_{20} + C \cdot \Phi_{21} + \Phi,$$

where $A, B, C$ are non-negative integers and $\Phi$ is a projective character belonging to non-unipotent blocks. Then, we get: $$\frac{q^2+\sqrt{2q}}{4} \geq \frac{q}{\sqrt{2}} \cdot j$$ which implies $j \leq \frac{\sqrt{2q}}{2}$. In a similar way, one can obtain the upper bounds for $u$ and $w$ from $\Psi'_{13}$ and $\Psi'_{18}$. The lower bounds for the decomposition numbers $a, b, \ldots, x$ follow from the decomposition numbers of the non-unipotent characters in the last five rows of Table C.3, since decomposition numbers are non-negative. This proves all assertions about $\Phi_i$ for $i = 9, 10, 11, 12, 13, 14, 17, 18, 19, 20$ in the decomposition matrix Table C.3.

So, we still have to complete the decomposition numbers in columns 2, 3 and 5 of the decomposition matrix. The relation corresponding to $\chi_{10,1}$ and the fact that decomposition numbers are non-negative implies $(\chi_5, \Phi_2)_G = (\chi_5, \Phi_3)_G = 1$. Then, $j \geq 1$ implies that $\Phi_{18}$ cannot be subtracted from $\Psi_2$, $\Psi_3$ and $\Psi_5$, and we get $(\chi_{18}, \Phi_2)_G = (\chi_{18}, \Phi_3)_G = (\chi_{18}, \Phi_5)_G = 1$. From the relations corresponding to $\chi_{10, 2} \Phi(a^2-1)_a$ and $\chi_{10, 2} \Phi(a^2-1)_b$ and $\chi_{10, St}$, we get $(\chi_{19}, \Phi_2)_G = (\chi_{20}, \Phi_3)_G = (\chi_{21}, \Phi_2)_G = (\chi_{21}, \Phi_3)_G = 1$.

By [17, Theorem 4.2], the modular Steinberg character of $G$ is cuspidal. So, the construction of $\Psi_5$ in Table B.4 implies that the projective cover of $\phi_{21}$ is not a summand of the projective module corresponding to $\Psi_5$. Hence, $(\chi_{21}, \Phi_5)_G = 1$. We have already seen that $\phi_1$, $\phi_4$, $\phi_5$ are the irreducible Brauer characters in the principal series. By the construction of the projectives in Table B.4, it is clear that $\phi_2$, $\phi_3$, $\phi_5$ are the only irreducible Brauer characters in the Harish-Chandra series $^2B_2[a], ^2B_2[b], ^2B_2[St]$, respectively. This finishes the proof of Theorem 3.3 for the unipotent blocks.

Next, we deal with the non-unipotent blocks in Tables D.1, D.2 and D.4-D.8. Let $s \neq 1$ be a semisimple $\ell$-element of $G^*$. If $s$ is not of type $g_5$, then the decomposition numbers of $E(G, s)$ are clear by Bonnafé’s and Rouquier’s Jordan decomposition, see Subsection 5.4. This proves the decomposition numbers in Tables D.1, D.2 and D.5-D.8. The decomposition numbers in Table D.4 follow from the fact that $\chi_{5,1}, \chi_{5,q^2(q-1)^2}, \chi_{5,St}$ have defect 0. This completes the proof of Theorem 3.3.  

5.8. Proof of Theorem 3.5. The proof is similar to the proof of Theorem 3.3. So we only give a brief sketch. Suppose $\ell \nmid q^2 - \sqrt{2q} + 1$. The numbers in the right most column of the decomposition matrix Table C.4 are obtained by counting the elements of $\ell$-power order in $G^*$. Under the assumptions of the theorem, we have $$\frac{1}{\ell} (\ell^k - 1) (\ell^k - 5) > 0$$ if and only if $\ell \neq 5$ or $n \equiv 2$ or $n \equiv 17 \mod 20$ where $q^2 = 2^{2n+1}$. Thus, the relation in the last row of Table B.12 exists if and only if $\ell \neq 5$ or $n \equiv 2$ or $n \equiv 17 \mod 20$.

We construct projective characters $\Psi_1, \ldots, \Psi_{21}$ of $G$ according to Table B.6 and compute the scalar products $(\chi_i, \Psi_j)_G$ for $1 \leq i, j \leq 21$ using CHEVIE. Here, $\Phi_a$ and $\Phi_b$ are the characters of the projective covers of the unipotent irreducible Brauer characters $\varphi_a$ and $\varphi_b$ of $L_b$, respectively, and $\Phi_{St}$ is the character of the projective cover of the modular Steinberg character $\varphi_{St}$ of $L_b$; see the comments in Subsection 3.4. In this way, we obtain the approximation Table B.5 to the decomposition matrix of the unipotent characters. The unitriangular shape gives a natural bijection between the set of ordinary unipotent irreducible characters and the set of irreducible Brauer characters in $E_\ell(G, 1)$. For $1 \leq i \leq 21$, 

□
let $\phi_i$ be the irreducible Brauer character corresponding to $\chi_i$ and $\Phi_i$ the character of the corresponding PIM.

From Table B.5, we get the assertions about $\Phi_1$ for $i = 5, 7, 9, 15, 16, 21$ in the decomposition matrix Table C.4. Furthermore, [11, Corollary 4.10] and Proposition 5.1 imply the corresponding PIM.

Products of these projectives with the ordinary unipotent characters of $(\Psi_\Phi)$ are projective since $\chi_5, \chi_9$ and $\Phi_3 \chi_{25}$ have defect 0. Using CHEVIE, we compute the scalar products of these projectives with the ordinary unipotent characters of $G$, see Table 3. All scalar products $(\Psi'_i, \chi_j)$ and $(\Psi''_i, \chi_j)$ not listed in this table are zero.

**TABLE 3. The scalar products $(\Psi'_i, \chi_j)_G$ and $(\Psi''_i, \chi_j)_G$.**

| $\chi_7$ | $\chi_{11}$ | $\chi_{12}$ | $\chi_{18}$ | $\chi_{19}$ | $\chi_{20}$ | $\chi_{21}$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\Psi'_i$ | $\frac{q}{\sqrt{2}}$ | 0 | 0 | $q^2 \sqrt{2}q^2$ | 0 | $\frac{\sqrt{2}q(\sqrt{2}q - 3\sqrt{2}q + 4)}{4}$ |
| $\Psi''_i$ | $\frac{q}{\sqrt{2}}$ | 0 | 0 | $\frac{q}{\sqrt{2}}$ | 0 | $\frac{q^2 - \sqrt{2}q}{4}$ |
| $\Psi''_i$ | 0 | 0 | $\frac{q}{\sqrt{2}}$ | 0 | $\frac{q^2 - \sqrt{2}q}{4}$ | $\frac{\sqrt{2}q(q^2 - 3\sqrt{2}q + 4)}{4}$ |
| $\Psi''_i$ | $\frac{q}{\sqrt{2}}$ | 0 | $\frac{q}{\sqrt{2}}$ | $\frac{q^2 + 2}{2}$ | $\frac{q^2 - \sqrt{2}q}{4}$ | $\frac{\sqrt{2}q(q^2 - 3\sqrt{2}q + 4)}{4}$ |

The scalar products for $\Psi'_i$ and $\Psi''_i$ imply:

$$(\chi_{18}, \Phi_{11})_G = (\chi_{20}, \Phi_{11})_G = (\chi_{18}, \Phi_{12})_G = (\chi_{19}, \Phi_{12})_G = 0.$$ 

From the scalar products in Table B.5 and Table 3 we get the upper bounds for the decomposition numbers $a, b, \ldots, v, w$ in the same way as in the proof of Theorem 3.3. The only difference is the upper bound for $x$ which can be derived as follows: From Table 3, we get

$$\Psi''_i = \Phi_{11} + A \cdot \Phi_{19} + B \cdot \Phi_{21} + \Phi,$$

where $A, B$ are non-negative integers and $\Phi$ is a projective character belonging to non-unipotent blocks. So we get: $e + A = \frac{q}{\sqrt{2}}$ and $t + A \cdot x + B = \frac{q^2 - \sqrt{2}q}{4}$. The first equation and the upper bound $e \leq \frac{\sqrt{2}q - 4}{4}$ imply $A \geq \frac{q}{2\sqrt{2}} + 1$ and the second equation gives:

$$A \cdot x \leq t + A \cdot x + B = \frac{q^2 - \sqrt{2}q}{4}.$$ 

Hence $x \leq \frac{q^2 - \sqrt{2}q}{4} \leq \frac{q}{2\sqrt{2}} - 3 + \frac{12}{\sqrt{2}q + 4}$. Since $x$ is an integer, it follows $x \leq \frac{q}{2\sqrt{2}} - 2$. The proof of the remaining statements is analogous to the proof of Theorem 3.3. □

5.9. Proof of Theorem 4.1. Suppose $\ell = 3$. As in the proof of Theorem 3.1, the numbers in the right most column of Table C.5 are obtained by counting the elements of 3-power order in $G^*$. Note that under the assumptions of the theorem, we have $\frac{1}{2}(3^\ell - 3) > 0$ if and only if $n \equiv 1$ or $n \equiv 4 \mod 6$ where $q^2 = 2^{2n+1}$. Consequently, the relations corresponding to $\chi_{6,1}$ and $\chi_{8,9}$ in Table B.13 exist if and only if $n \equiv 1$ or $n \equiv 4 \mod 6$. Furthermore, $\frac{1}{16}(3^\ell - 3)(3^\ell - 9) > 0$ if and only if $n \equiv 4$ or $n \equiv 13 \mod 18$. So the relation in the last row of Table B.13 exists if and only if $n \equiv 4$ or $n \equiv 13 \mod 18$.

We choose the ordinary basic set

$$\{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{18}, \chi_{19}, \chi_{20}, \chi_{21}\}$$
of $E_3(G, 1)$ as in Section 4. Using the relations in Table B.13, the rows of the decomposition matrix of $E_3(G, 1)$ corresponding to the irreducible characters not belonging to the above basic set can be written as linear combinations of the rows corresponding to the irreducible characters in the above basic set. So, we only have to determine the decomposition numbers of the irreducible characters in the above basic set.

We construct projective characters $\Psi_i$ and $\Psi_{5,1}$ of $G$ according to Table B.8 in the Appendix and compute the scalar products of these projective characters with the ordinary characters in the above basic set using CHEVIE, see Table B.7 in the Appendix. We already get from Table B.7 that the decomposition matrix of $\Phi_{\ell}(G, 1)$ follows from Bonnafé’s and Rouquier’s Jordan decomposition, see Table B.7 in the Appendix. We Thus, get from Table B.8 that the decomposition matrix of $\Phi_{\ell}(G, 1)$ has a lower untriangular shape giving us a natural bijection between the above ordinary basic set and the set of irreducible Brauer characters in $E_4(G, 1)$. Let $\phi_i$ be the irreducible Brauer character corresponding to $\chi_i$ and $\psi_{5,1}$ be the irreducible Brauer character corresponding to $\chi_{5,1}$. We write $\Phi_i$ and $\Phi_{5,1}$ for the corresponding PIMs, respectively.

From Table B.7, we get the assertions about all PIMs except for $\Phi_i$, $i = 1, 4, 7, 8, 10, 15, 18$ and $\Phi_{5,1}$ in the decomposition matrix Table C.5. Let $a$ be the multiplicity of $\chi_{18}$ in $\Phi_{15}$ and $b, c, d, e$ the multiplicity of $\chi_{21}$ in $\Phi_8$, $\Phi_{10}$, $\Phi_{15}$, $\Phi_{18}$, respectively. Furthermore, let $x_i$ be the multiplicity of $\chi_i$ in $\Phi_{5,1}$ for $i = 7, 8, 10, 15, 18, 21$. The assertions about $\Phi_{5,1}$ follow from $\Psi_{5,1}$ by subtracting defect 0 characters. The entries in Table B.7 lead to the upper bounds for $a, b, c, d, e, x_i$ in Theorem 4.1, and the lower bounds follow from the decomposition numbers of the characters not in the basic set in the lower part of Table C.5. So we have shown all assertions about $\Phi_{5,1}, \Phi_{10}, \Phi_{15}, \Phi_{18}$ in Theorem 4.1. Furthermore, [11, Corollary 4.10] and Proposition 5.1 imply the assertions about $\Phi_i$ and $\Phi_7$. So, we are only left with the decomposition numbers

$$(\chi_{5,1}, \Phi_4)_G, (\chi_7, \Phi_4)_G, (\chi_{21}, \Phi_4)_G, (\chi_{10}, \Phi_8)_G \in \{0, 1\}.$$ 

The relation for $\chi_9$ in Table B.13 and the fact that decomposition numbers are non-negative imply $(\chi_{5,1}, \Phi_4)_G = (\chi_{10}, \Phi_8)_G = 1$. Since $\Phi_7$ is not a summand of $\Psi_4$, we can deduce $(\chi_7, \Phi_4)_G = 1$. The remaining assertions about the unipotent blocks, in particular, the modular Harish-Chandra series and $(\chi_{21}, \Phi_4)_G = 1$ can be shown analogously to the proof of Theorem 3.2.

The decomposition numbers of the non-unipotent irreducible characters of $G$ in Tables D.1, D.2, D.5-D.8 follow from Bonnafé’s and Rouquier’s Jordan decomposition, see Subsection 5.4. Note that for $\ell = 3$, one does not have to deal with blocks corresponding to semisimple elements of type $g_5$. This completes the proof of Theorem 4.1. □

6. Degrees of Irreducible Representations

For a finite group $S$ and an algebraically closed field $k$ of characteristic $\ell \geq 0$, we write $d_\ell(S)$ for the smallest degree of a nonlinear irreducible $kS$-representation. In [36, Problem 1.1], P. Tiep proposed the following problem:

*Given a finite quasisimple group $S$ and $\ell$, determine $d_\ell(S)$ and all nontrivial irreducible $kS$-representations of degree $d_\ell(S)$."

For related problems and applications of smallest degrees see [36] and [38].

6.1. Smallest degrees of the Ree groups. In this section, we consider $d_\ell(G)$ for the simple Ree groups $G = 2E_6(q^2)$, $q^2 = 2^{2n+1}$, $n > 0$. Of course, due to G. Malle’s ordinary character table, the smallest degree $d_\ell(G)$ is known for $\ell = 0$ and all primes $\ell$ not dividing the order of $G$. For the defining characteristic, one has $d_2(G) = 26$, see [30]. Using the Brauer trees in [25], F. Lübeck [31] showed that $d_\ell(G)$ coincides with $d_0(G)$ for all primes
\( \ell \) such that the Sylow \( \ell \)-subgroups of \( G \) are cyclic. In Theorem 6.1, we are going to extend this result to all primes \( \ell > 3 \).

**Theorem 6.1.** Let \( G = 2^F_4(q^2) \), \( q^2 = 2^{2n+1} \), \( n > 0 \). For every prime \( \ell > 3 \) one has

\[
d_\ell(G) = d_0(G) = \frac{q}{\sqrt{2}}(q^2 - 1)(q^2 + 1)^2(q^4 - q^2 + 1)
\]

and up to isomorphism, there are two \( kG \)-modules of this degree. Their Brauer characters are \( \bar{\chi}_2 \) and \( \bar{\chi}_3 \).

Before we start with the proof of the theorem, we describe the general strategy: The decomposition matrices in Appendices C and D show that \( \bar{\chi}_2 \) and \( \bar{\chi}_3 \) are irreducible Brauer characters of degree \( d_0(G) \). Furthermore, these decomposition matrices determine almost all irreducible Brauer characters of \( G \) and in particular their degrees, see Remark 3.5 (c). We show that the remaining unknown degrees are larger than \( d_0(G) \). To achieve this, we use techniques similar to those for Steinberg’s triality groups in [18]: The lower and upper bounds for the unknown decomposition numbers in Theorems 3.1, 3.2, 3.3, 3.5 lead to lower bounds for the unknown degrees and eventually show, that these degrees are larger than \( d_0(G) \).

However, there are significant differences to [18] which are due to the fact that the degrees of the ordinary unipotent characters \( \chi_{18}, \chi_{19}, \chi_{20}, \chi_{21} \) are “asymptotically close together”. Let us have a closer look at the case \( \ell | q^2 + \sqrt{2}q + 1 \) and the unknown degree of the modular Steinberg character \( \phi_{21} \). From the decomposition matrix Table C.3, we get:

\[
\deg(\phi_{21}) = \chi_{21}(1) - x \cdot (\phi_{19}(1) + \phi_{20}(1)) - w \cdot \phi_{18}(1) - v \cdot \phi_{17}(1) - \cdots - \phi_{2}(1).
\]

By the decomposition matrix Table C.3 we know the degrees \( \phi_1(1), \ldots, \phi_{17}(1) \). Assume that we have already proved sufficiently good lower and upper bounds for the unknown degrees \( \phi_{18}(1), \phi_{19}(1), \phi_{20}(1) \). Plugging in the upper bounds for the decomposition numbers \( r, s, t, u, v, w, x \) given in Theorem 3.3 and the upper bounds for \( \phi_{18}(1), \phi_{19}(1), \phi_{20}(1) \) into the right hand side of (4) leads to a lower bound for \( \deg(\phi_{21}) \). Unfortunately, this bound turns out to be negative and does not give any information at all. To overcome this difficulty, we use certain dependencies between the various decomposition numbers which are derived from projective characters.

**Proof.** Because of the Brauer trees in [24], [25] and F. Lübeck’s result [31], we only have to consider the cases

\[
\ell | q^2 - 1, \quad \ell | q^2 + 1, \quad \ell | q^2 + \sqrt{2}q + 1, \quad \ell | q^2 - \sqrt{2}q + 1.
\]

We only demonstrate the case \( \ell | q^2 + \sqrt{2}q + 1 \). The case \( \ell | q^2 - \sqrt{2}q + 1 \) is similar, the other cases are much easier. Suppose \( \ell \) is a prime dividing \( q^2 + \sqrt{2}q + 1 \). In this case, Theorem 3.3 determines all decomposition numbers of \( G \), except for several decomposition numbers in the ordinary characters \( \chi_{18}, \chi_{19}, \chi_{20}, \chi_{21} \). Thanks to the unitriangular shape of the decomposition matrix, this gives us the degrees of all irreducible Brauer characters of \( G \) except \( \phi_{18}(1), \phi_{19}(1), \phi_{20}(1), \phi_{21}(1) \). We see \( \phi_{2}(1) = \phi_{3}(1) = d_0(G) \) and that all other known degrees are strictly bigger than \( d_0(G) \). So, for the four unknown degrees we have to show \( \phi_{18}(1), \phi_{19}(1), \phi_{20}(1), \phi_{21}(1) > d_0(G) \). From the decomposition matrix Table C.3, we get

\[
\phi_{18}(1) = \chi_{18}(1) - \sum_{i=1}^{3} \phi_i(1) - \phi_3(1) - a \cdot \phi_9(1) - b \cdot (\phi_{11}(1) + \phi_{12}(1)) - c \cdot \phi_{17}(1).
\]
Plugging in the known degrees and the upper bounds for \( a, b, c \) in Theorem 3.3 into the right hand side of (5), we see \( \deg(\phi_{18}) > d_0(G) \). Furthermore, plugging in \( a = b = c = 0 \) into (5) we get an upper bound for \( \deg(\phi_{18}) \). The proof of \( \phi_{19}(1) > d_0(G) \) is similar:

From the decomposition matrix Table C.3, we get

\[
\phi_{19}(1) = \chi_{19}(1) - \phi_2(1) - d \cdot \phi_9(1) - e \cdot \phi_{11}(1) - g \cdot \phi_{12}(1) - h \cdot \phi_{13}(1) - i \cdot \phi_{17}(1) - j \cdot \phi_{18}(1).
\]

Plugging in the known degrees and the upper bounds for \( 22 \) FRANK HIMSTEDT

\( \phi_1, \ldots, \phi_{22} \) are given explicitly.

Next, we prove several inequalities between the unknown decomposition numbers, \( a, b, c, \ldots, w, x \) and get:

\[
\deg(\phi_{21}) = q^{24} - \sqrt{2} q^{23} + (2 x j - w) q^{22} + (\frac{xh}{2} - \frac{t}{4} + \frac{wa}{12} - \frac{t}{2} + \frac{wb}{2} - \frac{2 x j c}{3} + \frac{xe}{2} + \frac{x d}{6} - x j b) q^{20} + P(q, a, b, c, \ldots, w, x),
\]

where \( P(q, a, b, c, \ldots, w, x) \) is a polynomial in \( q \) and the unknown decomposition numbers \( a, b, c, \ldots, w, x \) with coefficients in \( \mathbb{Q}[\sqrt{2}] \) such that the degree of \( P(q, a, b, c, \ldots, w, x) \), when considered as a polynomial in \( q \), is at most 19. Note that \( P(q, a, b, c, \ldots, w, x) \) can be given explicitly.

Next, we prove several inequalities between the unknown decomposition numbers, which allow us to bound the bold face expressions in (6) from below. We are going to use some of the projective characters \( \Psi_3 \) and \( \Psi'_3 \) constructed in the proof of Theorem 3.3. Scalor products of these projective characters with some ordinary characters are given in Tables 2 and B.3. We write \( \Phi_i \) for the character of the PIM corresponding to \( \phi_i \).

From Table 2, we get

\[
\Psi_3 = \Psi'_3 \cdot \Phi_{13} + A \cdot \Phi_{19} + B \cdot \Phi_{21} + \Phi,
\]

where \( A, B \) are non-negative integers and \( \Phi \) is a projective character belonging to non-unipotent blocks. Thus, we get:

\[
\frac{q}{\sqrt{2}} \cdot h + A = q^2 + \sqrt{2} q + \frac{q}{2} \cdot u + A \cdot x + B = \sqrt{2} q (q^2 + 3 \sqrt{2} q + 4).
\]

The first equation implies \( A = \frac{q^2 + \sqrt{2} q}{4} - \frac{q}{\sqrt{2}} \cdot h \) and from the second equation, we then get

\[
\frac{q}{\sqrt{2}} \cdot u + A \cdot x \leq \frac{q}{\sqrt{2}} \cdot u + A \cdot x + B = \frac{\sqrt{2} q (q^2 + 3 \sqrt{2} q + 4)}{24}.
\]

So

\[
-\frac{u}{2} \geq \frac{\sqrt{2} A x}{2 q} - \frac{q^2 + 3 \sqrt{2} q + 4}{24} = \frac{\sqrt{2} q x}{8} + \frac{x h}{4} - \frac{q^2}{24} - \frac{\sqrt{2} q}{8} - \frac{1}{6}.
\]

Analogously, using the projective characters \( \Psi'_{18}, \Psi_{11}, \Psi_9, \Psi_{17} \), we obtain:

\[
-w \geq u + \frac{3 \sqrt{2} q x}{4} + \frac{x}{2} - x h - 2 x j + \sqrt{2} x - \frac{q^2}{4} - \frac{\sqrt{2} q}{4} - 1,
\]
\[- \frac{t}{2} \geq \frac{wq^2}{8} + \frac{w\sqrt{2}q}{8} - \frac{wb}{2} + \frac{x\sqrt{2}q^3}{8} - \frac{xe}{2} - \frac{xj\sqrt{2}q}{4} + \frac{xj}{2} - \frac{q^4}{8},\]

\[- \frac{r}{12} \geq \frac{wq^2}{144} + \frac{w\sqrt{2}q}{48} + \frac{w}{36} - \frac{wa}{144} + \frac{x\sqrt{2}q^3}{72} - \frac{x\sqrt{2}q}{6} - \frac{xj\sqrt{2}q}{24} - \frac{xj}{18} + \frac{xja}{6} - \frac{q^4}{144} + \frac{1}{36},\]

\[- \frac{v}{3} \geq \frac{wq^2}{9} - \frac{2w}{9} - \frac{wc}{3} + \frac{x\sqrt{2}q^3}{9} + \frac{x\sqrt{2}q}{9} - \frac{2xi}{3} - \frac{2xj\sqrt{2}q}{9} + \frac{4xj}{9} + \frac{2xjc}{3} - \frac{q^4}{9} - \frac{2}{9}.\]

By replacing the bold face \(-w, -\frac{t}{2}, -\frac{r}{12}, -\frac{u}{2}, -\frac{v}{3}\) in (6) by the right hand sides of (7) and (8), we obtain a lower bound for \(\deg(\varphi_{21})\). Again, this lower bound can be written as a polynomial \(Q(q, a, b, c, \ldots, w, x)\) in \(q\) and \(a, b, \ldots, x\) with coefficients in \(Q(\sqrt{2})\). We consider two cases:

Case 1: \(n > 1\). Expand the polynomial \(Q(q, a, b, c, \ldots, w, x)\), and in all terms with a positive coefficient, replace \(a, b, \ldots, x\) by the lower bounds in Theorem 3.3; in all terms with a negative coefficient, replace \(a, b, \ldots, x\) by the upper bounds in Theorem 3.3. In this way, we obtain another lower bound for \(\deg(\varphi_{21})\) which is now a polynomial in \(q\) only, and from this bound we get \(\deg(\varphi_{21}) > d_0(G)\).

Case 2: \(n = 1\), that is \(G = 2F_4(8)\). First, substitute \(q = \sqrt{8}\) in \(Q(q, a, b, c, \ldots, w, x)\). Then, expand this polynomial in \(a, b, \ldots, x\), and in all terms with a positive coefficient, replace \(a, b, \ldots, x\) by the lower bounds in Theorem 3.3, in all terms with a negative coefficient, replace \(a, b, \ldots, x\) by the upper bounds in Theorem 3.3. Note that by Theorem 3.3 and Corollary 3.4, one can use \(h = j = 1\), \(x = 2\) and \(s \geq 2\) in this case. In this way, we obtain:

\[\deg(\varphi_{21}) \geq \frac{11769507827}{3} > 64638 = d_0(G).\]

So, all unknown degrees of irreducible Brauer characters are strictly bigger than \(d_0(G)\) and the claim follows.

6.2. Remarks on Theorem 6.1.

(a) For fixed \(q\), the bounds in Theorem 4.1 imply that the degrees of all nontrivial 3-modular irreducible Brauer characters \(\neq \bar{\chi}_2, \bar{\chi}_3\) of \(G\) are larger than \(d_0(G)\), except for possibly \(\deg(\varphi_{18})\) or \(\deg(\varphi_{21})\).

(b) Theorem 6.1 improves the bounds of V. Landazuri, G. Seitz, A. Zalesskii, P. Tiep in [29], [35], [37] for \(\ell > 3\).
### Appendix A: Notation for Characters

Table A.1. Notation, degrees and families of the unipotent irreducible characters of $G = 2 F_4(q^2)$. The notation we use is the CHEVIE notation in the left most column. For a definition of $\phi_j$, $\phi'_j$, $\phi''_j$ see Subsection 2.1; see also [19, Table 4].

| CHEVIE | Notation | Degree | Conjugacy Class | Value |
|--------|-----------|--------|-----------------|-------|
| $\chi_1$ | $1$ | $\xi_1$ | $\chi_1$ | $1$ |
| $\chi_2$ | $2 B_2[a]$, $1$ | $\xi_5$ | $\chi_5$ | $\frac{q}{\sqrt{2}} \phi_1 \phi_2 \phi_2^2 \phi_1 12$ | $c_{1,11}$ | $- \frac{q^3}{\sqrt{2}} + \varepsilon q^2$ |
| $\chi_3$ | $2 B_2[b]$, $1$ | $\xi_6$ | $\chi_6$ | $\frac{q}{\sqrt{2}} \phi_1 \phi_2 \phi_2^2 \phi_1 12$ | $c_{1,11}$ | $- \frac{q^3}{\sqrt{2}} - \varepsilon q^2$ |
| $\chi_4$ | $\varepsilon'$ | $\xi_2$ | $\chi_2$ | $\phi^2 \phi_12 \phi_{24}$ |
| $\chi_5$ | $\rho_2'$ | $\xi_9$ | $\chi_9$ | $\frac{q^4}{\sqrt{2}} \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_6$ | $\rho_2'$ | $\xi_10$ | $\chi_{10}$ | $\frac{q^4}{2} \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_7$ | $\rho_2$ | $\xi_{11}$ | $\chi_{11}$ | $\phi^2 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_8$ | cusp | $\xi_{12}$ | $\chi_{12}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_9$ | cusp | $\xi_{13}$ | $\chi_{13}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_{10}$ | cusp | $\xi_{14}$ | $\chi_{14}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_{11}$ | cusp | $\xi_{15}$ | $\chi_{15}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{1,11}$ | $- \frac{q^3}{ \frac{4}{3} + \varepsilon q^2}$ |
| $\chi_{12}$ | cusp | $\xi_{16}$ | $\chi_{16}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{1,11}$ | $- \frac{q^3}{ \frac{4}{3} - \varepsilon q^2}$ |
| $\chi_{13}$ | cusp | $\xi_{17}$ | $\chi_{17}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{1,11}$ | $- \frac{q^3}{ \frac{4}{3} + \varepsilon q^2}$ |
| $\chi_{14}$ | cusp | $\xi_{18}$ | $\chi_{18}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{1,11}$ | $- \frac{q^3}{ \frac{4}{3} + \varepsilon q^2}$ |
| $\chi_{15}$ | cusp | $\xi_{19}$ | $\chi_{19}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{5,3}$ | $\frac{q^2}{ \frac{6}{5} - \frac{1}{3} + \varepsilon q^2}$ |
| $\chi_{16}$ | cusp | $\xi_{20}$ | $\chi_{20}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ | $c_{5,3}$ | $\frac{q^2}{ \frac{6}{5} - \frac{1}{3} - \varepsilon q^2}$ |
| $\chi_{17}$ | cusp | $\xi_{21}$ | $\chi_{21}$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12 \phi_{24}^2$ |
| $\chi_{18}$ | $\phi''$ | $\xi_3$ | $\chi_3$ | $\phi^{10} \phi_1 \phi_2 \phi_{24}$ |
| $\chi_{19}$ | $2 B_2[a]$, $\varepsilon$ | $\xi_7$ | $\chi_7$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12$ | $c_{1,3}$ | $\varepsilon q^2 \phi^2 \phi_{24}$ |
| $\chi_{20}$ | $2 B_2[b]$, $\varepsilon$ | $\xi_8$ | $\chi_8$ | $\phi^2 \phi_1 \phi_2 \phi_2^2 \phi_1 12$ | $c_{1,3}$ | $-\varepsilon q^2 \phi^2 \phi_{24}$ |
| $\chi_{21}$ | $\varepsilon$ | $\xi_4$ | $\chi_4$ | $\phi^{24}$ |
Table A.2. Notation, degrees and numbers of the non-unipotent irreducible characters of $G = ^2 E_4(q^2)$. Dependencies on parameters $k, l$ are omitted. For a definition of $\phi_j$, $\phi'_j$, $\phi''_j$, see Subsection 2.1.

| Character | Notation in CHEVIE | Degree | Number of Characters |
|----------|--------------------|--------|---------------------|
| $\chi_{2,1}$ | $\chi_{22}$ | $\phi_1^0 \phi_2^0 \phi_1 \phi_2 \phi_{12} \phi_{24}$ | $\frac{1}{2} (q^2 - 2)$ |
| $\chi_{2, \sqrt{2}(q^2 - 1)_a}$ | $\chi_{23}$ | $\frac{q}{\sqrt{2}} \phi_1 \phi_2 \phi_3^2 \phi_4 \phi_{12} \phi_{24}$ | |
| $\chi_{2, \sqrt{2}(q^2 - 1)_b}$ | $\chi_{24}$ | $\frac{q}{\sqrt{2}} \phi_1 \phi_2 \phi_3 \phi_4 \phi_{12} \phi_{24}$ | |
| $\chi_{2, St}$ | $\chi_{25}$ | $q^2 \phi_1^2 \phi_2 \phi_{12} \phi_{24}$ | |
| $\chi_{3,1}$ | $\chi_{26}$ | $\phi_4 \phi_5 \phi_1 \phi_2 \phi_{12} \phi_{24}$ | $\frac{1}{2} (q^2 - 2)$ |
| $\chi_{3, St}$ | $\chi_{27}$ | $\phi_4^2 \phi_5 \phi_1 \phi_2 \phi_{12} \phi_{24}$ | |
| $\chi_{4,1}$ | $\chi_{28}$ | $\phi_1^3 \phi_2^0 \phi_1 \phi_2 \phi_{12} \phi_{24}$ | $\frac{1}{16} (q^4 - 10q^2 + 16)$ |
| $\chi_{5,1}$ | $\chi_{29}$ | $\phi_1 \phi_2 \phi_3^2 \phi_4 \phi_{12} \phi_{24}$ | |
| $\chi_{5, St}$ | $\chi_{30}$ | $q^2 \phi_1 \phi_2 \phi_3^2 \phi_4 \phi_{12} \phi_{24}$ | |
| $\chi_{6,1}$ | $\chi_{32}$ | $\phi_1 \phi_2 \phi_3^2 \phi_4 \phi_{12} \phi_{24}$ | $\frac{1}{2} (q^2 - 2)$ |
| $\chi_{6, St}$ | $\chi_{33}$ | $q^2 \phi_1 \phi_2 \phi_3^2 \phi_4 \phi_{12} \phi_{24}$ | |
| $\chi_{7,1}$ | $\chi_{34}$ | $\phi_1 \phi_2 \phi_4 \phi_5 \phi_{12} \phi_{24}$ | $\frac{1}{4} (q^4 - 2q^2)$ |
| $\chi_{8,1}$ | $\chi_{35}$ | $\phi_1 \phi_2 \phi_3^2 \phi_4 \phi_5 \phi_{12} \phi_{24}$ | $\frac{1}{4} (q^4 - \sqrt{2}q^2 - 2q^2 + 2\sqrt{2}q)$ |
| $\chi_{8, St}$ | $\chi_{36}$ | $q^2 \phi_1 \phi_2 \phi_3^2 \phi_4 \phi_5 \phi_{12} \phi_{24}$ | |
| $\chi_{9,1}$ | $\chi_{37}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{8} (q^4 + \sqrt{2}q^3 - 2q^2 - 2\sqrt{2}q)$ |
| $\chi_{10,1}$ | $\chi_{38}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{4} (q^4 + 2\sqrt{2}q^3 - 2q^2 + 4\sqrt{2}q)$ |
| $\chi_{10, St}$ | $\chi_{39}$ | $q^2 \phi_1 \phi_2 \phi_3^2 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | |
| $\chi_{11,1}$ | $\chi_{40}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_7 \phi_{12} \phi_{24}$ | $\frac{1}{8} (q^4 - 2q^2)$ |
| $\chi_{12,1}$ | $\chi_{41}$ | $\phi_1 \phi_2 \phi_3^2 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{8} (q^4 - 2q^2 + 4\sqrt{2}q^3 - 2q^2 + 4\sqrt{2}q)$ |
| $\chi_{13,1}$ | $\chi_{42}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{50} (q^4 - 2\sqrt{2}q^3 - 2q^2 + 4\sqrt{2}q)$ |
| $\chi_{14,1}$ | $\chi_{43}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{50} (q^4 - 4\sqrt{2}q^3 - 2q^2 - 4\sqrt{2}q)$ |
| $\chi_{15,1}$ | $\chi_{44}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{12} (q^4 - 10q^2 + 16)$ |
| $\chi_{16,1}$ | $\chi_{45}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{4} (q^4 - 2q^2)$ |
| $\chi_{17,1}$ | $\chi_{46}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{12} (q^4 + q^2 - \sqrt{2}q^2 - \sqrt{2}q)$ |
| $\chi_{18,1}$ | $\chi_{47}$ | $\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6 \phi_{12} \phi_{24}$ | $\frac{1}{12} (q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q)$ |
### Appendix B: Scalar Products, Projective Characters and Relations

**Table B.1.** Scalar products of the unipotent irreducible characters of $G$ with some projective characters for $3 \neq \ell | q^2 + 1$. See Table B.2 for a construction of these projectives.

| $\chi_1$ | $\Psi_1$ | $\Psi_2$ | $\Psi_3$ | $\Psi_4$ | $\Psi_5$ | $\Psi_6$ | $\Psi_7$ | $\Psi_8$ | $\Psi_9$ | $\Psi_{10}$ | $\Psi_{11}$ | $\Psi_{12}$ | $\Psi_{13}$ | $\Psi_{14}$ | $\Psi_{15}$ | $\Psi_{16}$ | $\Psi_{17}$ | $\Psi_{18}$ | $\Psi_{19}$ | $\Psi_{20}$ | $\Psi_{21}$ |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\chi_2$ | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_3$ |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_4$ | 1      |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_5$ |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_6$ |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_7$ | 1      |        | 1      |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_8$ |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_9$ |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_{10}$ |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_{11}$ |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_{12}$ |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |        |
| $\chi_{13}$ |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |        |
| $\chi_{14}$ |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |        |
| $\chi_{15}$ |        |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |        |
| $\chi_{16}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |        |
| $\chi_{17}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |        |        |
| $\chi_{18}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $\frac{q^2 - 2}{3}$ | 1 |        |        |        |
| $\chi_{19}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |        |
| $\chi_{20}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | 1      |        |        |
| $\chi_{21}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $\frac{q^4 + 2}{3}$ | $q^2 + 2$ |        | 1        |
Table B.2. Projective characters of $G$ for $3 \neq \ell \mid q^2 + 1$.

| Projective | Construction | Comments                  |
|------------|--------------|---------------------------|
| $\Psi_1$   | $\rho_1 \chi_1(0)^G - \chi_5 - \chi_6$ | $\ell \mid |P_b|$         |
| $\Psi_2$   | $\chi_2$     | defect 0                  |
| $\Psi_3$   | $\chi_3$     | defect 0                  |
| $\Psi_4$   | $\rho_8 \chi_8(0)^G - \chi_5 - \chi_6$ | $\ell \mid |P_b|$         |
| $\Psi_5$   | $\chi_5$     | defect 0                  |
| $\Psi_6$   | $\chi_6$     | defect 0                  |
| $\Psi_7$   | $\rho_4 \chi_4(0)^G - \chi_5 - \chi_6$ | $\ell \mid |P_b|$         |
| $\Psi_8$   | $\rho_8 \chi_8 - \chi_{10}$      | $\ell \mid |P_b|$         |
| $\Psi_9$   | $\rho_8 \chi_{22} - \chi_{10}$    | $\ell \mid |P_b|$         |
| $\Psi_{10}$| $\chi_{10}$  | defect 0                  |
| $\Psi_{11}$| $\chi_{11}$  | defect 0                  |
| $\Psi_{12}$| $\chi_{12}$  | defect 0                  |
| $\Psi_{13}$| $\chi_{13}$  | defect 0                  |
| $\Psi_{14}$| $\chi_{14}$  | defect 0                  |
| $\Psi_{15}$| $\chi_{15}$  | defect 0                  |
| $\Psi_{16}$| $\chi_{16}$  | defect 0                  |
| $\Psi_{17}$| $\rho_8 \chi_{20} - \frac{\sqrt{2q}}{6}(q^2 + 1)\chi_{19} - \frac{\sqrt{2}}{6}(q^2 + 1)\chi_{20}$ | $\rho_8 \chi_{20}$ has defect 0 |
| $\Psi_{18}$| $\rho_8 \chi_{15} - \chi_5 - \chi_6$ | $\ell \mid |P_b|$         |
| $\Psi_{19}$| $\chi_{19}$  | defect 0                  |
| $\Psi_{20}$| $\chi_{20}$  | defect 0                  |
| $\Psi_{21}$| $B \chi_8^G$ | $\ell \mid |B|$           |
Table B.3. Scalar products of the unipotent irreducible characters of $G$ with some projective characters for $\ell \mid q^2 + \sqrt{2}q + 1$.

See Table B.4 for a construction of these projectives.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $x_{11}$ | $x_{12}$ | $x_{13}$ | $x_{14}$ | $x_{15}$ | $x_{16}$ | $x_{17}$ | $x_{18}$ | $x_{19}$ | $x_{20}$ | $x_{21}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 1     |       |       |       |       |       |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     |       |       |       |       |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     |       |       |       |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     |       |       |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     |       |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     |       |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     |       |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
Table B.4. Projective characters of $G$ for $\ell \mid q^2 + \sqrt{2}q + 1$.

| Projective | Construction | Comments              |
|------------|--------------|-----------------------|
| $\Psi_1$  | $p_a \chi_1(0)^G - \chi_6 - \chi_7$ | $\ell \nmid |P_a|$ |
| $\Psi_2$  | $R_{L_a}(\Phi_a) - \chi_6 - \chi_7$ | Harish-Chandra induced projective |
| $\Psi_3$  | $R_{L_b}(\Phi_b) - \chi_6 - \chi_7$ | Harish-Chandra induced projective |
| $\Psi_4$  | $p_a \chi_2(0)^G - \chi_6 - \chi_7$ | $\ell \nmid |P_a|$ |
| $\Psi_5$  | $R_{L_b}(\Phi_{St}) - \chi_6 - \chi_7$ | Harish-Chandra induced projective |
| $\Psi_6$  | $\chi_6$ | defect 0 |
| $\Psi_7$  | $\chi_7$ | defect 0 |
| $\Psi_8$  | $\chi_8$ | defect 0 |
| $\Psi_9$  | $p_a \chi_{24}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{10}$ | $p_a \chi_{18}^G - \chi_8$ | $p_a \chi_{18}$ has defect 0 |
| $\Psi_{11}$ | $p_a \chi_{16}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{12}$ | $p_a \chi_{17}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{13}$ | $p_a \chi_{14}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{14}$ | $p_a \chi_{15}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{15}$ | $\chi_{15}$ | defect 0 |
| $\Psi_{16}$ | $\chi_{16}$ | defect 0 |
| $\Psi_{17}$ | $p_a \chi_{20}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{18}$ | $p_a \chi_{11}^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{19}$ | $p_a \chi_{9}^G$ | $p_a \chi_{9}$ has defect 0 |
| $\Psi_{20}$ | $p_a \chi_{10}^G$ | $p_a \chi_{10}$ has defect 0 |
| $\Psi_{21}$ | $B \chi_8^G$ | $\ell \nmid |B|$ |
| $\chi_1$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | $\psi_5$ | $\psi_6$ | $\psi_7$ | $\psi_8$ | $\psi_9$ | $\psi_{10}$ | $\psi_{11}$ | $\psi_{12}$ | $\psi_{13}$ | $\psi_{14}$ | $\psi_{15}$ | $\psi_{16}$ | $\psi_{17}$ | $\psi_{18}$ | $\psi_{19}$ | $\psi_{20}$ | $\psi_{21}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_2$ | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_3$ | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_4$ | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_5$ | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_6$ | 1 | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_7$ | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_8$ | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_9$ | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_{10}$ | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_{11}$ | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_{12}$ | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_{13}$ | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . |
| $\chi_{14}$ | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . |
| $\chi_{15}$ | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . |
| $\chi_{16}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . |
| $\chi_{17}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . |
| $\chi_{18}$ | 1 | . | . | . | 1 | $q^2 - 3\sqrt{2q + 4}$ | . | * | $q^2 - \sqrt{2q + 4}$ | $q^2 - \sqrt{2q}$ | $q^2 - \sqrt{2q}$ | $q^2 - \sqrt{2q}$ | $q^2 - \sqrt{2q}$ | $q^2 - \sqrt{2q}$ | $q^2 - \sqrt{2q}$ | 1 | . | . | . | . | . | . | . |
| $\chi_{19}$ | . | 1 | . | . | $\sqrt{2q(q^2 - 2)}$ | . | * | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 - 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | * | 1 | . | . | . | . | . | . | . |
| $\chi_{20}$ | . | . | 1 | . | $\sqrt{2q(q^2 - 2)}$ | . | * | $\sqrt{2q(q^2 - 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | $\sqrt{2q(q^2 + 2)}$ | * | . | 1 | . | . | . | . | . | . | . |
| $\chi_{21}$ | . | . | . | 1 | . | 1 | . | * | $\frac{q^2 + 2}{3}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | . | . | . |
| $\chi_{22}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

Table B.5. Scalar products of the unipotent irreducible characters of $G$ with some projective characters for $\ell | q^2 - \sqrt{2q + 1}$. See Table B.6 for a construction of these projectives.
TABLE B.6. Projective characters of $G$ for $\ell \mid q^2 - \sqrt{2q} + 1$.

| Projective | Construction | Comments |
|------------|--------------|----------|
| $\Psi_1$   | $\rho_a \chi_1(0)^G - \chi_5 - \chi_7$ | $\ell \nmid |P_a|$ |
| $\Psi_2$   | $R_{L_a}^G(\Phi_a)$ | Harish-Chandra induced projective |
| $\Psi_3$   | $R_{L_b}^G(\Phi_b)$ | Harish-Chandra induced projective |
| $\Psi_4$   | $\rho_a \chi_2(0)^G - \chi_5 - \chi_7$ | $\ell \nmid |P_a|$ |
| $\Psi_5$   | $\chi_5$ | defect 0 |
| $\Psi_6$   | $R_{L_b}^G(\Phi_{St}) - \chi_5 - \chi_7$ | Harish-Chandra induced projective |
| $\Psi_7$   | $\chi_7$ | defect 0 |
| $\Psi_8$   | $\rho_a \chi_23^G$ | $\ell \nmid |P_a|$ |
| $\Psi_9$   | $\chi_9$ | defect 0 |
| $\Psi_{10}$ | $\rho_a \chi_22^G - \chi_9$ | $\rho_a \chi_22$ has defect 0 |
| $\Psi_{11}$ | $\rho_a \chi_16^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{12}$ | $\rho_a \chi_17^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{13}$ | $\rho_a \chi_14^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{14}$ | $\rho_a \chi_15^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{15}$ | $\chi_15$ | defect 0 |
| $\Psi_{16}$ | $\chi_16$ | defect 0 |
| $\Psi_{17}$ | $\rho_a \chi_20^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{18}$ | $\rho_a \chi_11^G$ | $\ell \nmid |P_a|$ |
| $\Psi_{19}$ | $\rho_a \chi_9^G$ | $\rho_a \chi_9$ has defect 0 |
| $\Psi_{20}$ | $\rho_a \chi_10^G$ | $\rho_a \chi_10$ has defect 0 |
| $\Psi_{21}$ | $B \chi_8^G$ | $\ell \nmid |B|$ |
Table B.7. Scalar products of the basic set characters of $G$ with some projective characters for $\ell = 3$. See Table B.8 for a construction of these projectives.

| $\chi$  | $\Psi_1$ | $\Psi_2$ | $\Psi_3$ | $\Psi_4$ | $\Psi_{5,1}$ | $\Psi_5$ | $\Psi_6$ | $\Psi_7$ | $\Psi_8$ | $\Psi_{10}$ | $\Psi_{12}$ | $\Psi_{13}$ | $\Psi_{14}$ | $\Psi_{15}$ | $\Psi_{18}$ | $\Psi_{19}$ | $\Psi_{20}$ | $\Psi_{21}$ |
|---------|----------|----------|----------|----------|-------------|---------|---------|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $\chi_1$ | 1        | .        | .        | .        | .           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_2$ | .        | 1        | .        | .        | .           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_3$ | .        | .        | 1        | .        | .           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_4$ | 1        | .        | .        | 1        | 1           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{5,3}$ | .        | .        | 1        | .        | .           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_5$  | .        | .        | .        | *        | 1           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_6$  | .        | .        | .        | *        | .           | 1        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_7$  | 1        | .        | .        | 1        | $\frac{q^2}{2}$ | 1        | .        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_8$  | .        | .        | .        | $\frac{q^2 + 3\sqrt{2}q + 1}{q^2}$ | .        | .        | 1        | .        | .        | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_9$  | .        | .        | .        | $\frac{q^2 + 2}{q}$ | .        | .        | 1        | 1        | .        | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{10}$ | .        | .        | .        | *        | .           | .        | .        | 1        | .        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{11}$ | .        | .        | .        | *        | .           | .        | .        | .        | 1        | .            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{12}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | 1            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{13}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | 1            | .            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{14}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | .            | 1            | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{15}$ | .        | .        | .        | $\frac{q^2 + 1}{q^3}$ | .        | .        | .        | .        | .        | .            | $\frac{q^2 + 1}{q^3}$ | .            | .            | .            | .            | .            | .            | .            |
| $\chi_{16}$ | .        | .        | .        | $\frac{q^2(q^2 - 1)}{3}$ | .        | .        | 1        | .        | .        | .            | .            | $\frac{q^2 + q^2}{3}$ | 1            | .            | .            | .            | .            | .            | .            |
| $\chi_{17}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | .            | .            | .            | .            | $\frac{q^2 + q^2}{3}$ | 1            | .            | .            | .            |
| $\chi_{18}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | .            | .            | .            | $\frac{q^2 + q^2}{3}$ | 1            | .            | .            | .            | .            | .            |
| $\chi_{19}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | .            | .            | .            | .            | $\frac{q^2 + q^2}{3}$ | 1            | .            | .            | .            |
| $\chi_{20}$ | .        | .        | .        | *        | .           | .        | .        | .        | .        | .            | .            | .            | .            | .            | .            | 1            | .            | .            | .            |
| $\chi_{21}$ | .        | .        | 1        | $q^6$    | .        | 1        | $\frac{q^2 + \sqrt{2}q}{3}$ | $\frac{q^2 - \sqrt{2}q}{3}$ | .        | .            | $\frac{q^2 + q^2}{3}$ | $\frac{q^2 + 2}{3}$ | .        | .            | 1            | .            | .            | .            | .            | .            | .            |
Table B.8. Projective characters of $G$ for $\ell = 3$.

| Projective | Construction | Comments |
|------------|--------------|----------|
| $\Psi_1$  | $\rho_6 \chi_1(0)^G - \chi_5 - \chi_6$ | $3 \mid |P_b|$ |
| $\Psi_2$  | $\chi_2$      | defect 0 |
| $\Psi_3$  | $\chi_3$      | defect 0 |
| $\Psi_4$  | $\rho_6 \chi_8(0)^G - \chi_5 - \chi_6$ | $3 \mid |P_b|$ |
| $\Psi_{5,1}$ | $\rho_6 \chi_{39}^G$ | $3 \mid |P_b|$ |
| $\Psi_5$  | $\chi_5$      | defect 0 |
| $\Psi_6$  | $\chi_6$      | defect 0 |
| $\Psi_7$  | $\rho_6 \chi_4(0)^G - \chi_5 - \chi_6$ | $3 \mid |P_b|$ |
| $\Psi_8$  | $\rho_6 \chi_{18}^G$ | $3 \mid |P_b|$ |
| $\Psi_{10}$ | $\rho_6 \chi_{22}^G$ | $3 \mid |P_b|$ |
| $\Psi_{11}$ | $\chi_{11}$ | defect 0 |
| $\Psi_{12}$ | $\chi_{12}$ | defect 0 |
| $\Psi_{13}$ | $\chi_{13}$ | defect 0 |
| $\Psi_{14}$ | $\chi_{14}$ | defect 0 |
| $\Psi_{15}$ | $\sum_{k=1}^{(q^2+1)/3} \rho_6 \chi_{38}(k)^G - \sqrt{2} q (q^2 + 1)^2 \cdot (\chi_{19} + \chi_{20})$ | $3 \mid |P_b|$ |
| $\Psi_{18}$ | $\rho_6 \chi_{15}^G - \chi_5 - \chi_6$ | $3 \mid |P_b|$ |
| $\Psi_{19}$ | $\chi_{19}$ | defect 0 |
| $\Psi_{20}$ | $\chi_{20}$ | defect 0 |
| $\Psi_{21}$ | $B \chi_8^G$ | $3 \mid |B|$ |
### Table B.9

Relations (with respect to the basic set of unipotent characters) for the irreducible characters in the unipotent blocks in case $\ell \mid q^2 - 1$. For details on the labeling and entries of this table, see Subsection 5.3.

| Characters | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| $\{\chi_{2,1}\}$ | 1 | . | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $\{\chi_{2,\sqrt{q^2-1}}\}$ | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . |
| $\{\chi_{2,\sqrt{q^2-1}}\}$ | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | 1 |
| $\{\chi_{3,1}\}$ | 1 | . | . | 1 | 1 | 1 | . | . | . | . | . | . | . | 1 | . | . | . | . | . | 1 |
| $\{\chi_{3,St}\}$ | . | . | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\{\chi_{4,1}\}$ | 1 | . | . | 1 | 2 | 2 | 2 | . | . | . | . | . | . | . | . | . | . | . | 1 | . | 1 |

### Table B.10

Relations (with respect to the basic set of unipotent characters) for the irreducible characters in the unipotent blocks in case $3 \neq \ell \mid q^2 + 1$. For details on the labeling and entries of this table, see Subsection 5.3.

| Characters | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| $\{\chi_{6,1}\}$ | -1 | . | 2 | . | . | -1 | 1 | 1 | . | . | . | . | . | . | . | . | -2 | 1 | . | . | . |
| $\{\chi_{6,St}\}$ | . | . | -1 | . | . | 1 | -1 | -1 | . | . | . | . | . | . | . | 2 | -2 | . | . | 1 |
| $\{\chi_{15,1}\}$ | 1 | . | . | -3 | . | . | 2 | -2 | -2 | . | . | . | . | 4 | -3 | . | . | 1 |
Table B.11. Relations (with respect to the basic set of unipotent characters) for the irreducible characters in the unipotent blocks in case $\ell \mid q^2 + \sqrt{2}q + 1$. For details on the labeling and entries of this table, see Subsection 5.3.

| Characters | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\{\chi_{10,1}\}$ | $-1$ | $-1$ | $-1$ | $-1$ | 1 | . . | 1 | 1 | 1 | 1 | . . | . . | . . | . . | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{10,\sqrt{2}(q^2-1)_a}\}$ | . | $-1$ | . | $-1$ | 1 | . . | 1 | . | 1 | $-1$ | . . | $-1$ | $-1$ | 1 | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{10,\sqrt{2}(q^2-1)_b}\}$ | . | . | $-1$ | $-1$ | 1 | . . | 1 | . | 1 | . | $-1$ | . . | $-1$ | $-1$ | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{10,St}\}$ | . . . | . | $-1$ | . . | $-1$ | $-1$ | $-1$ | 1 | . . . | 1 | $-1$ | $-1$ | 1 |
| $\{\chi_{14,1}\}$ | 1 | 2 | 2 | 3 | $-4$ | . . | $-4$ | $-2$ | $-3$ | $-3$ | 1 | 1 | . . | 2 | 3 | $-2$ | $-2$ | 1 |

Table B.12. Relations (with respect to the basic set of unipotent characters) for the irreducible characters in the unipotent blocks in case $\ell \mid q^2 - \sqrt{2}q + 1$. For details on the labeling and entries of this table, see Subsection 5.3.

| Characters | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\{\chi_{8,1}\}$ | $-1$ | 1 | 1 | $-1$ | . | 1 | . | 1 | . | 1 | . | 1 | 1 | . | . . | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{8,\sqrt{2}(q^2-1)_a}\}$ | . | $-1$ | . | 1 | $-1$ | $-1$ | . | 1 | . | $-1$ | . . | 1 | 1 | 1 | . . | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{8,\sqrt{2}(q^2-1)_b}\}$ | . | . | $-1$ | 1 | $-1$ | $-1$ | . . | 1 | $-1$ | . . | 1 | 1 | $-1$ | . . | . . | . . | . . | . . | . . | . . | . . |
| $\{\chi_{8,St}\}$ | . . . | . . | $-1$ | . | $-1$ | . | $-1$ | . . | $-1$ | $-1$ | . . . | 1 | 1 | 1 | 1 |
| $\{\chi_{13,1}\}$ | 1 | $-2$ | $-2$ | 3 | . | $-4$ | . | $-4$ | $-2$ | 1 | 1 | $-3$ | $-3$ | . . | 2 | 3 | 2 | 2 | 1 |
TABLE B.13. Relations (with respect to the ordinary basic set for the unipotent blocks) in case $\ell = 3$. For details on the labeling and entries of this table, see Section 4.

| Characters        | 1 | 2 | 3 | 4 | 5, 1 | 5 | 6 | 7 | 8 | 10 | 11 | 12 | 13 | 14 | 15 | 18 | 19 | 20 | 21 |
|-------------------|---|---|---|---|------|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $\chi_9$          | 1 | . | . | -1| 1    | . | . | -1| 1 | .   | .  | .  | .  | .  | .  | .  | .  | .  |    |
| $\chi_{16}$       | . | . | . | . | .    | . | . | . | . | .   | .  | .  | .  | .  | .  | .  | 1  | .  | .  |    |
| $\chi_{17}$       | -1| . | . | 1 | -1   | . | . | . | . | .   | .  | .  | .  | 1  | .  | .  | .  | .  |    |
| $\chi_{5,q^2(q^2-1)}$ | 2 | . | . | -1| 2    | . | -1| 1 | . | .   | .  | .  | -2 | 1  | .  | .  |    |    |
| $\chi_{5,St}$     | -1| . | . | 1 | -1   | . | . | . | . | .   | .  | .  | -1 | .  | 1  |    |    |
| $\{\chi_{6,1}\}$ | 2 | . | . | -1| 3    | . | -1| 1 | . | .   | .  | -2 | 1  | .  | .  |    |    |
| $\{\chi_{6,St}\}$| -3| . | . | 2 | -3   | . | 1 | -1| . | .   | .  | 2  | -2 | .  | 1  |    |    |
| $\{\chi_{15,1}\}$| -5| . | . | 3 | -6   | . | 2 | -2| . | .   | .  | 4  | -3 | .  | 1  |    |    |    |
APPENDIX C: DECOMPOSITION NUMBERS OF UNIPOTENT CHARACTERS

Table C.1. The decomposition numbers of the unipotent blocks of $G$ for $\ell \mid q^2 - 1$. In the right most column, $\ell^f$ is the largest power of $\ell$ dividing $q^2 - 1$.

| Series | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{16}$ | $\phi_{17}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\chi_1$ | $1$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_2$ | $-$ | $1$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_3$ | $-$ | $1$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_4$ | $-$ | $-$ | $-$ | $1$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_5$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_6$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_7$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_8$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_9$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{10}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{11}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{12}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{13}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{14}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{15}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{16}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{17}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{18}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{19}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{20}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| $\chi_{21}$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $1$ |
| \{\chi_{2,1}\} | \phi_1 | \phi_2 | \phi_3 | \phi_4 | \phi_5 | \phi_6 | \phi_7 | \phi_8 | \phi_9 | \phi_{10} | \phi_{11} | \phi_{12} | \phi_{13} | \phi_{14} | \phi_{15} | \phi_{16} | \phi_{17} | \phi_{18} | \phi_{19} | \phi_{20} | \phi_{21} | \text{Number} |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \{\chi_{2,1}\} | 1 | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{2,\sqrt{q^2\ell_f}}(q^2-1)_a\} | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{2,\sqrt{q^2\ell_f}}(q^2-1)_b\} | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{2,St}\} | . | . | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | 1 | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{3,1}\} | 1 | . | . | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | 1 | . | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{3,St}\} | . | . | . | 1 | 1 | 1 | 1 | . | . | . | . | . | . | . | . | . | . | 1 | \begin{align*} \frac{1}{2}(\ell_f - 1) \end{align*} |
| \{\chi_{4,1}\} | 1 | . | . | 1 | 2 | 2 | 2 | . | . | . | . | . | . | . | . | . | . | 1 | \begin{align*} \frac{1}{16}(\ell_f - 1)(\ell_f - 7) \end{align*} |
ON THE DECOMPOSITION NUMBERS OF $^2E_6(q^2)$

| Series | $p^g \cdot B_2[A_1^{2B_2[B_2]}]$, $A_1^{2B_2[B_2]}$ | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{16}$ | $\phi_{17}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number |
|--------|----------------------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| X1     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X2     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X3     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X4     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X5     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X6     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X7     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X8     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X9     |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X10    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X11    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X12    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X13    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X14    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X15    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X16    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X17    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X18    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X19    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| X20    |                                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| $\{X_{13} \} \setminus \{X_{15} \}$ |                  | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| $\{X_{15} \}$ |                            | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
TABLE C.3. The decomposition numbers of the unipotent blocks of $G$ for $\ell | q^2 + \sqrt{2}q + 1$. In the right most column, $\ell^f$ is the largest power of $\ell$ dividing $q^2 + \sqrt{2}q + 1$.

| Series | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{16}$ | $\phi_{17}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number |
|--------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\chi_1$ | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_2$ | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_3$ | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_4$ | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_5$ | 1        | 1        | 1        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_6$ | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_7$ | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_8$ | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_9$ | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{10}$ | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{11}$ | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{12}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{13}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{14}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          |          | 1        |
| $\chi_{15}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          |          | 1        |
| $\chi_{16}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          | 1        |
| $\chi_{17}$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |          |          |          |          |          |          |          | 1        |
| $\chi_{18}$ | 1        | 1        | 1        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |
| $\chi_{19}$ | .        | 1        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |
| $\chi_{20}$ | .        | .        | 1        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |
| $\chi_{21}$ | .        | 1        | 1        | 1        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        | v        | w        | x        | x        | .        | .        | .        | 1        | 1        |
\begin{table}[h]
\centering
\renewcommand{\arraystretch}{1.2}
\begin{tabular}{cccccccccccccc}
\hline
& \(\phi_1\) & \(\phi_2\) & \(\phi_3\) & \(\phi_4\) & \(\phi_5\) & \(\phi_6\) & \(\phi_7\) & \(\phi_8\) & \(\phi_9\) & \(\phi_{10}\) & \(\phi_{11}\) & \(\phi_{12}\) & \(\phi_{13}\) & \(\phi_{14}\) & \(\phi_{15}\) & \(\phi_{16}\) & \(\phi_{17}\) & \(\phi_{18}\) & \(\phi_{19}\) & \(\phi_{20}\) & \(\phi_{21}\) & Number \\
\hline
\{\chi_{10,1}\} & \ldots & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{1}{4}(\ell^f - 1) \\
\{\chi_{10,1}(q^2-1)_a\} & \ldots & \ldots & \ldots & * & * & * & * & \ldots & * & j - 1 & 1 & \ldots & \frac{1}{4}(\ell^f - 1) \\
\{\chi_{10,1}(q^2-1)_b\} & \ldots & \ldots & \ldots & * & * & * & h - 1 & \ldots & * & j - 1 & 1 & \ldots & \frac{1}{4}(\ell^f - 1) \\
\{\chi_{10,St}\} & \ldots & 1 & \ldots & * & s - 1 & * & * & u - h & \ldots & * & 1 - 2j + w & x - 1 & x - 1 & 1 & \frac{1}{4}(\ell^f - 1) \\
\{\chi_{14,1}\} & \ldots & \ldots & \ldots & s - 2 & * & * & * & \ldots & * & x - 2 & x - 2 & 1 & \frac{(\ell^f - 1)(\ell^f - 5)}{96} \\
\hline
\end{tabular}
\end{table}
Table C.4. The decomposition numbers of the unipotent blocks of $G$ for $\ell | q^2 - \sqrt{2}q + 1$. In the rightmost column, $\ell^f$ is the largest power of $\ell$ dividing $q^2 - \sqrt{2}q + 1$.

| Series | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{16}$ | $\phi_{17}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|          |
| $\chi_1$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_2$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_3$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_4$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_5$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_6$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_7$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_8$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_9$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{10}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{11}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{12}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{13}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{14}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{15}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{16}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{17}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{18}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{19}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{20}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
| $\chi_{21}$ | 1       |         |         |         |         |         |         |         |         |             |             |             |             |             |             |             |             |             |             |             |             | 1          |
### Table C.4. (continued)

|                  | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_9$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{16}$ | $\phi_{17}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number            |
|------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|----------------|
| $\{\chi_{8,1}\}$ | .        | 1        | 1        | .        | 1        | 1        | .        | 1        | 1        | .            | .            | .            | .            | .            | .            | .            | .            | .            | .            | .            | .            | $\frac{1}{4}(\ell^f - 1)$ |
| $\{\chi_{8,\sqrt{2}}(q^2-1)_a\}$ | .        | .        | .        | .        | .        | *        | .        | *        | *        | .            | .            | .            | .            | *            | *            | .            | *            | 1            | .            | .            | .            | $\frac{1}{4}(\ell^f - 1)$ |
| $\{\chi_{8,\sqrt{2}}(q^2-1)_b\}$ | .        | .        | .        | .        | .        | .        | *        | .        | *        | .            | *            | *            | .            | .            | .            | *            | *            | 1            | .            | .            | .            | $\frac{1}{4}(\ell^f - 1)$ |
| $\{\chi_{8,St}\}$ | .        | 1        | 1        | .        | 1        | .        | *        | s−1      | *        | *        | *            | .            | *            | *            | 1            | .            | .            | .            | *            | *            | 1            | $\frac{1}{4}(\ell^f - 1)$ |
| $\{\chi_{13,1}\}$ | .        | .        | .        | .        | .        | .        | *        | s−2      | *        | *        | *            | .            | *            | *            | 1            | .            | .            | .            | *            | *            | 1            | $\frac{(\ell^f - 1)(\ell^f - 5)}{56}$ |
Table C.5. The decomposition numbers of the unipotent blocks of $G$ for $\ell = 3$. In the right most column, $3^f$ is the largest power of 3 dividing $q^2 + 1$.

| Series | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\phi_4$ | $\phi_{5,1}$ | $\phi_5$ | $\phi_6$ | $\phi_7$ | $\phi_8$ | $\phi_{10}$ | $\phi_{11}$ | $\phi_{12}$ | $\phi_{13}$ | $\phi_{14}$ | $\phi_{15}$ | $\phi_{18}$ | $\phi_{19}$ | $\phi_{20}$ | $\phi_{21}$ | Number |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\chi_1$ | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_2$ | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_3$ | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_4$ | 1 | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_{5,1}$ | . | . | . | 1 | 1 | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_5$ | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_6$ | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_7$ | 1 | . | . | 1 | $x_7$ | . | . | 1 | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_8$ | . | . | . | $x_8$ | . | . | . | 1 | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_9$ | . | . | . | $x_{10}$ | . | . | . | 1 | 1 | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_{11}$ | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | . | 1 |
| $\chi_{12}$ | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | . | . | 1 |
| $\chi_{13}$ | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | . | 1 |
| $\chi_{14}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | . | 1 |
| $\chi_{15}$ | . | . | . | $x_{15}$ | . | . | . | . | . | . | . | . | . | . | 1 | . | . | . | 1 |
| $\chi_{16}$ | . | . | . | $x_{16}$ | . | . | . | 1 | . | . | . | . | . | . | . | . | 1 | . | 1 |
| $\chi_{17}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | 1 |
| $\chi_{18}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | . | . | 1 |
| $\chi_{19}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 | 1 |
| $\chi_{20}$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | 1 |
| $\chi_{21}$ | . | . | . | 1 | $x_{21}$ | . | . | 1 | b | c | . | . | . | . | d | e | . | 1 | 1 |
| \chi | \phi_1 | \phi_2 | \phi_3 | \phi_4 | \phi_5 | \phi_6 | \phi_7 | \phi_8 | \phi_9 | \phi_{10} | \phi_{11} | \phi_{12} | \phi_{13} | \phi_{14} | \phi_{15} | \phi_{16} | \phi_{17} | \phi_{18} | \phi_{19} | \phi_{20} | \phi_{21} | Number |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \chi_9 | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{16} | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_7 | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{5,q^2} & \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{5,St} & \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{6,1} & \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{6,8} & \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
| \chi_{10,1} & \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | \cdot | 1 |
APPENDIX D: DECOMPOSITION NUMBERS OF NON-UNIPOTENT CHARACTERS

**TABLE D.1.** The decomposition numbers of the irreducible characters in $E(G,s_2) = \{\chi_{2,1}, \chi_2, \sqrt{2}, \chi_2, q\sqrt{2}(q^2-1)_a, \chi_2, q\sqrt{2}(q^2-1)_b, \chi_2, St\}$ where $s_2 \in G^*$ is a semisimple $\ell'$-element of type $g_2$.

| Character | $\ell \mid q^2 + \sqrt{2}q + 1$ | $\ell \mid q^2 - \sqrt{2}q + 1$ | otherwise |
|-----------|-------------------------------|-------------------------------|------------|
| $\chi_{2,1}$ | 1 | 1 | 1 |
| $\chi_2, \sqrt{2}(q^2-1)_a$ | 1 | . | 1 |
| $\chi_2, \sqrt{2}(q^2-1)_b$ | . | 1 | . |
| $\chi_2, St$ | 1 | 1 | 1 |

**TABLE D.2.** The decomposition numbers of the irreducible characters in $E(G,s_3) = \{\chi_{3,1}, \chi_{3, St}\}$ where $s_3 \in G^*$ is a semisimple $\ell'$-element of type $g_3$.

| Character | $\ell \mid q^2 - 1$ | $\ell \mid q^2 + 1$ | otherwise |
|-----------|----------------------------|----------------------------|------------|
| $\chi_{3,1}$ | 1 | 1 | 1 |
| $\chi_{3, St}$ | 1 | 1 | 1 |

**TABLE D.3.** The decomposition numbers of the irreducible characters in $E(G,s_5) = \{\chi_{5,1}, \chi_5, q^2(q^2-1), \chi_5, St, \chi_{6,1}, \chi_6, St, \chi_{15,1}\}$ where $s_5 \in G^*$ is a semisimple $\ell'$-element of type $g_5$ and $3 \neq \ell \mid q^2 + 1$. In the right most column, $\ell^f$ is the largest power of $\ell$ dividing $q^2 + 1$.

| Character | $3 \neq \ell \mid q^2 + 1$ | otherwise |
|-----------|----------------------------|------------|
| $\chi_{5,1}$ | 1 | 1 |
| $\chi_{5, q^2(q^2-1)}$ | . | 1 |
| $\chi_{5, St}$ | 1 | $a'$ |
| $\{\chi_{6,1}\}$ | 1 | 1 |
| $\{\chi_{6, St}\}$ | 1 | $a' - 1$ |
| $\{\chi_{15,1}\}$ | . | $a' - 2$ | \(\ell^f - 1\) | \(\ell^f - 1\) | \(\frac{1}{6}(\ell^f - 1)(\ell^f - 2)\)
ON THE DECOMPOSITION NUMBERS OF $^2E_4(q^2)$

Table D.4. The decomposition numbers of the irreducible characters in $E(G, s_5) = \{\chi_{5, 1}, \chi_{5, q^2(q^2-1)}, \chi_{5, St}\}$ where $s_5 \in G^*$ is a semisimple $\ell'$-element of type $g_5$ and $\ell \nmid q^2 + 1$.

|         | $\ell \mid q^2 - 1$ | $\ell \mid q^4 - q^2 + 1$ | otherwise |
|---------|---------------------|-----------------------------|-----------|
| $\phi_{5, 1}$ | 1                   | 1                           | 1         |
| $\phi_{5, 2}$ | .                   | 1                           | .         |
| $\phi_{5, 3}$ | .                   | .                           | .         |

Table D.5. The decomposition numbers of the irreducible characters in $E(G, s_6) = \{\chi_{6, 1}, \chi_{6, St}\}$ where $s_6 \in G^*$ is a semisimple $\ell'$-element of type $g_6$.

|         | $\ell \mid q^2 - 1$ | $\ell \mid q^2 + 1$ | otherwise |
|---------|---------------------|---------------------|-----------|
| $\phi_{6, 1}$ | 1                   | 1                   | 1         |
| $\phi_{6, 2}$ | .                   | .                   | .         |

Table D.6. The decomposition numbers of the irreducible characters in $E(G, s_8) = \{\chi_{8, 1}, \chi_{8, \sqrt{q^2-1}a}, \chi_{8, \sqrt{q^2-1}b}, \chi_{8, St}\}$ where $s_8 \in G^*$ is a semisimple $\ell'$-element of type $g_8$.

|         | $\ell \mid q^2 + \sqrt{2q} + 1$ | $\ell \mid q^2 - \sqrt{2q} + 1$ | otherwise |
|---------|---------------------------------|---------------------------------|-----------|
| $\phi_{8, 1}$ | 1                               | 1                               | 1         |
| $\phi_{8, 2}$ | .                               | .                               | .         |
| $\phi_{8, 3}$ | .                               | .                               | .         |
| $\phi_{8, 4}$ | .                               | .                               | .         |
TABLE D.7. The decomposition numbers of the irreducible characters in $E(G,s_{10}) = \{\chi_{10,1}, \chi_{10,\sqrt{2}(q^2-1)_a}, \chi_{10,\sqrt{2}(q^2-1)_b}, \chi_{10,St}\}$ where $s_{10} \in G^*$ is a semisimple $\ell'$-element of type $g_{10}$.

| $\chi_{10,1}$ | $\ell | q^2 + \sqrt{2}q + 1$ $\phi_{10,1} \phi_{10,2} \phi_{10,3} \phi_{10,4}$ | $\ell | q^2 - \sqrt{2}q + 1$ $\phi_{10,1} \phi_{10,2} \phi_{10,3} \phi_{10,4}$ | otherwise $\phi_{10,1} \phi_{10,2} \phi_{10,3} \phi_{10,4}$ |
|----------------|------------------------------------------------|------------------------------------------------|-----------------------------------------------|
| $\chi_{10,\sqrt{2}(q^2-1)_a}$ | . . 1 . | 1 . . . | 1 . . . |
| $\chi_{10,\sqrt{2}(q^2-1)_b}$ | . . 1 . | . 1 . . | . 1 . . |
| $\chi_{10,St}$ | 1 1 1 | 1 . 1 . | . . 1 . |

TABLE D.8. The decomposition numbers of the irreducible characters in $E(G,s_i) = \{\chi_{i,1}\}$ where $s_i \in G^*$ is a semisimple $\ell'$-element of type $g_i$ and $i \in \{4, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18\}$. These are the basic sets corresponding to the regular semisimple $\ell'$-elements of $G^*$.

| $\phi_{i,1}$ | $\ell \text{ odd}$ |
|-------------|----------------|
| $\chi_{i,1}$ | 1 |

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