Levi-Civita Ricci-flat metrics on compact complex manifolds

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Abstract. In this paper, we study the geometry of compact complex manifolds with Levi-Civita Ricci-flat metrics and prove that compact complex surfaces admitting Levi-Civita Ricci-flat metrics are Kähler Calabi-Yau surfaces or Hopf surfaces.

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1. Introduction

Einstein manifolds and Einstein metrics are fundamental topics in math physics and differential geometry. It is well-known that, the background Riemannian metric of a Kähler-Einstein metric is Einstein. However, when the ambient complex manifolds are not Kähler, the relationships between Hermitian metrics and their background Riemannian metrics are complicated and somewhat mysterious. Let \((X, h)\) be a Hermitian manifold and \(g\) be the background Riemannian metric. On the Hermitian holomorphic tangent bundle \((T^{1,0}X, h)\), there are two typical metric compatible connections:

(A) the Chern connection \(\nabla\), i.e. the unique connection \(\nabla\) compatible with the Hermitian metric and also the complex structure \(\overline{\partial}\);
(B) the Levi-Civita connection \(\nabla^{LC}\), i.e. the restriction of the complexified Levi-Civita connection on \(T_CX\) to the holomorphic tangent bundle \(T^{1,0}X\).

The Chern connection is the key object in complex geometry and the Levi-Civita connection \(\nabla^{LC}\) is a representative of the Riemannian geometry. It is well-known that when \((X, h)\) is not Kähler, \(\nabla\) and \(\nabla^{LC}\) are not the same. The complex geometry of the Chern connection is extensively investigated in the literatures by using various methods (e.g. \([3, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 25, 26]\)).
In [13], we introduced the first Aeppli-Chern classes for holomorphic line bundles. Let $L \rightarrow X$ be a holomorphic line bundle over $X$. The first Aeppli-Chern class is defined as
\[ c_{1}^{AC}(L) = \left[-\sqrt{-1}\partial\bar{\partial}\log h\right]_{A} \in H^{1,1}_{A}(X) \]
where $h$ is an arbitrary smooth Hermitian metric on $L$ and the Aeppli cohomology is
\[ H_{A}^{p,q}(X) := \frac{\text{Ker} \partial \bar{\partial} \cap \Omega^{p,q}(X)}{\text{Im} \partial \cap \Omega^{p,q}(X) + \text{Im} \bar{\partial} \cap \Omega^{p,q}(X)} \]
For a complex manifold $X$, $c_{1}^{AC}(X)$ is defined to be $c_{1}^{AC}(K^{-1}_X)$ where $K^{-1}_X$ is the anti-canonical bundle of $X$. Note that, for a Hermitian line bundle $(L, h)$, the classes $c_{1}(L)$ and $c_{1}^{AC}(L)$ have the same $(1,1)$-form representative $\Theta^h = -\sqrt{-1}\partial\bar{\partial}\log h$ in different cohomological classes.

It is well-known that on a Hermitian manifold $(X, \omega)$, the first Chern Ricci curvature
\[ \text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \det(\omega) \]
represents the first Chern class $c_{1}(X)$. As an analog, we proved in [13, Theorem 1.1] that the first Levi-Civita Ricci curvature $\mathcal{Ric}(\omega)$ represents the first Aeppli-Chern class $c_{1}^{AC}(X)$. Hence, it is very natural to study (non-Kähler) Calabi-Yau manifolds by using the first Aeppli-Chern class $c_{1}^{AC}(X)$ and the first Levi-Civita Ricci curvature $\mathcal{Ric}(\omega)$.

The classification of various Ricci-flat manifolds are important topics in differential geometry. The following result is fundamental and well-known, and we refer to the nice paper [19] of V. Tosatti for discussions on Bott-Chern classes and Chern Ricci-flat metrics.

**Theorem 1.1.** Let $X$ be a compact complex surface. Suppose $X$ admits a Chern Ricci-flat Hermitian metric $\omega$, i.e. $\text{Ric}(\omega) = 0$. Then $X$ is minimal and it is exactly one of the following

1. Enriques surfaces;
2. bi-elliptic surfaces;
3. $K3$ surfaces;
4. 2-tori;
5. Kodaira surfaces.

As shown in [13], the Levi-Civita Ricci-flat condition $\mathcal{Ric}(\omega) = 0$ is equivalent to
\[ \text{Ric}(\omega) = \frac{1}{2}(\partial\partial^{*} \omega + \bar{\partial}\bar{\partial}^{*} \omega), \]
where $\text{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n)$ is the Chern Ricci curvature. The equation (1.1) is not of Monge-Ampère type since there are also non-elliptic terms on the right hand side. As it is well-known, it is particularly challenging to solve such equations. By
using conformal methods, functional analysis and explicit constructions, we obtain the following result analogous to Theorem 1.1, which also generalizes the previous results in [14].

**Theorem 1.2.** Let $X$ be a compact complex surface. Suppose $X$ admits a Levi-Civita Ricci-flat Hermitian metric $\omega$, i.e. $\mathfrak{Ric}(\omega) = 0$. Then $X$ is minimal. Moreover, it lies in one of the following

1. Enriques surfaces;
2. bi-elliptic surfaces;
3. $K3$ surfaces;
4. 2-tori;
5. Hopf surfaces.

**Remark 1.3.** Note that, Enriques surfaces, bi-elliptic surfaces, $K3$ surfaces and 2-tori are Kähler Calabi-Yau surfaces. It is obvious that the Kähler Calabi-Yau metrics are Levi-Civita Ricci-flat. However, by using Yau’s theorem, there exist many non-Kähler Levi-Civita Ricci-flat metrics on each Kähler Calabi-Yau manifold.

**Remark 1.4.** It is easy to see that, for a Kodaira surface $X$, it has $c^B_1(X) = c_1(X) = c^A_1(X) = 0$. By Theorem 1.1, it has a Chern Ricci-flat metric. However, we can see from Theorem 1.2 that it can not support a Levi-Civita Ricci-flat metric.

**Remark 1.5.** It is well-known, it is very difficult to write down explicitly Ricci-flat metrics. In this paper, we obtain Levi-Civita Ricci-flat metrics on Hopf surfaces by explicit constructions. It also worths to point out that we only construct Levi-Civita Ricci-flat metrics on Hopf surfaces of class 1 (see Theorem 1.6). We conjecture that all Hopf surfaces can support Levi-Civita Ricci-flat metrics. On the other hand, every Hopf surface $X$ is a non-Kähler Calabi-Yau manifold, i.e. $c_1(X) = 0 \in H^2(X, \mathbb{R})$. However, $X$ can not support a Chern Ricci-flat Hermitian metric, i.e. a Hermitian metric $\omega$ with $\text{Ric}(\omega) = 0$. In this view point, the existence of Levi-Civita Ricci-flat Hermitian metrics on Hopf surfaces is quite exceptional.

A compact complex surface $X$ is called a Hopf surface if its universal covering is analytically isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Its fundamental group $\pi_1(X)$ is a finite extension of an infinite cyclic group generated by a biholomorphic contraction which takes the form $(z, w) \to (az, bw + \lambda z^m)$ where $a, b, \lambda \in \mathbb{C}$, $|a| \geq |b| > 1$, $m \in \mathbb{N}^*$ and $\lambda(a - b^m) = 0$. There are two different cases:

1. the Hopf surface $H_{a,b}$ of class 1 if $\lambda = 0$;
2. the Hopf surface $H_{a,b,\lambda,m}$ of class 0 if $\lambda \neq 0$ and $a = b^m$.

Let $H_{a,b} = \mathbb{C}^2 \setminus \{0\}/\sim$ where $(z, w) \sim (az, bw)$ and $|a| \geq |b| > 1$. We set $k_1 = \log |a|$ and $k_2 = \log |b|$. Define a real smooth function $\Phi(z, w) = e^{k_1 \frac{b}{a}z + \frac{b}{a}w}$ where $\theta(z, w)$ is a real smooth function defined by $|z|^2 e^{-\frac{k_1}{2k_2} \theta} + |w|^2 e^{-\frac{k_2}{k_1} \theta} = 1$. We construct explicitly Levi-Civita Ricci-flat metrics on $H_{a,b}$ by perturbations and conformal changes.
Theorem 1.6. On the Hopf surface $H_{a,b}$ of class 1, the Hermitian metric

\begin{equation}
\omega = \Delta^3 \left( \frac{1}{\Phi} \sqrt{-1} \partial \bar{\partial} \Phi \right) - \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \left( \log \Phi \right)
\end{equation}

is Levi-Civita Ricci-flat, i.e. $\text{Ric}(\omega) = 0$, where

\[ \Delta = \alpha |z|^2 \Phi^{-\alpha} + (2 - \alpha) |w|^2 \Phi^{\alpha-2} \quad \text{and} \quad \alpha = \frac{2k_1}{k_1 + k_2}. \]

Remark 1.7. If $a = b$, $H_{a,a}$ is exactly the usual diagonal Hopf surface. In this case, the Levi-Civita Ricci-flat metric constructed in Theorem 1.6 is the same as that constructed in [13, Theorem 6.2] or [14, Theorem 7.3].

It is well-known that, on a compact Kähler manifold $X$, the Kähler Ricci-flat metrics are all Einstein flat metrics.

Question 1.8. On a compact complex manifold $X$, does there exist some Levi-Civita Ricci-flat (non-Kähler) Hermitian metric such that the background Riemannian metric is Einstein?

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2. Preliminaries

2.1. Chern connection on complex manifolds. Let $(X, \omega_g)$ be a compact Hermitian manifold. There exists a unique connection $\nabla$ on the holomorphic tangent bundle $T^{1,0}X$ which is compatible with the Hermitian metric and also the complex structure of $X$. This connection $\nabla$ is called the Chern connection. The Chern connection $\nabla$ on $(T^{1,0}X, \omega_g)$ has curvature components

\begin{equation}
R_{\overline{jk} \ell} = -\frac{\partial^2 g_{\overline{k \ell}}}{\partial z^j \partial \overline{z}^k} + g^pq \frac{\partial g_{\overline{k \ell}}}{\partial z^j} \frac{\partial g_{p \overline{q}}}{\partial \overline{z}^k}.
\end{equation}

The (first) Chern-Ricci form $\text{Ric}(\omega_g)$ of $(X, \omega_g)$ has components

\[ R_{ij} = g^{k\overline{l}} R_{ij \overline{k} \ell} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \overline{z}^j}, \]

which also represents the first Chern class $c_1(X)$ of the complex manifold $X$. The Chern scalar curvature $s_C$ of $(X, \omega_g)$ is given by

\begin{equation}
s_C = \text{tr}_{\omega_g} \text{Ric}(\omega_g) = g^{\overline{k} \ell} R_{ij \overline{k} \ell}.
\end{equation}

The total Chern scalar curvature of $\omega_g$ is

\begin{equation}
\int_X s_C \omega^n_g = n \int \text{Ric}(\omega_g) \wedge \omega_g^{n-1},
\end{equation}

where $n$ is the complex dimension of $X$. 
2. Bott-Chern classes and Aeppli classes. The Bott-Chern cohomology and the Aeppli cohomology on a compact complex manifold $X$ are given by

$$H^{p,q}_{BC}(X) := \frac{\text{Ker} \cap \Omega^{p,q}(X)}{\text{Im} \partial \cap \Omega^{p,q}(X)} \quad \text{and} \quad H^p_A(X) := \frac{\text{Ker} \bar{\partial} \cap \Omega^{p,q}(X)}{\text{Im} \partial \cap \Omega^{p,q}(X)}.$$ 

Let $\text{Pic}(X)$ be the set of holomorphic line bundles over $X$. As similar as the first Chern class map $c_1 : \text{Pic}(X) \to H^{1,1}_A(X)$, there is a first Aeppli-Chern class map

$$c^A_1 : \text{Pic}(X) \to H^{1,1}_A(X).$$

Given any holomorphic line bundle $L \to X$ and any Hermitian metric $h$ on $L$, its curvature form $\Theta_h$ is locally given by $-\sqrt{-1} \partial \bar{\partial} \log h$. We define $c^A_1(L)$ to be the class of $\Theta_h$ in $H^{1,1}_A(X)$. For a complex manifold $X$, $c^A_1(X)$ is defined to be $c^A_1(K_X^{-1})$ where $K_X^{-1}$ is the anti-canonical line bundle. The first Bott-Chern class $c^B_1(X)$ can be defined similarly.

2.3. The Levi-Civita connection on the holomorphic tangent bundle. Let’s recall some elementary settings (e.g., [13, Section 2]). Let $(M, g, \nabla)$ be a 2n-dimensional Riemannian manifold with the Levi-Civita connection $\nabla$. The tangent bundle of $M$ is also denoted by $T\mathbb{R}_M$. Let $T\mathbb{C}M = T\mathbb{R}M \otimes \mathbb{C}$ be the complexification. We can extend the metric $g$ and the Levi-Civita connection $\nabla$ to $T\mathbb{C}M$ in the $\mathbb{C}$-linear way. Let $(M, g, J)$ be an almost Hermitian manifold, i.e., $J : T\mathbb{R}M \to T\mathbb{R}M$ with $J^2 = -1$, and for any $X, Y \in T\mathbb{R}M$, $g(JX, JY) = g(X, Y)$. The Nijenhuis tensor $N_J : \Gamma(M, T\mathbb{R}M) \times \Gamma(M, T\mathbb{R}M) \to \Gamma(M, T\mathbb{R}M)$ is defined as

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

The almost complex structure $J$ is called integrable if $N_J \equiv 0$ and then we call $(M, g, J)$ a Hermitian manifold. We can also extend $J$ to $T\mathbb{C}M$ in the $\mathbb{C}$-linear way. Hence for any $X, Y \in T\mathbb{C}M$, we still have $g(JX, JY) = g(X, Y)$. By Newlander-Nirenberg’s theorem, there exists a real coordinate system $\{x^i, x^j\}$ such that $z^i = x^i + \sqrt{-1}x^j$ are local holomorphic coordinates on $M$. Moreover, we have $T\mathbb{C}M = T^{1,0}M \oplus T^{0,1}M$ where

$$T^{1,0}M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\} \quad \text{and} \quad T^{0,1}M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\}.$$ 

Since $T^{1,0}M$ is a subbundle of $T\mathbb{C}M$, there is an induced connection $\nabla^\text{LC}$ on the holomorphic tangent bundle $T^{1,0}M$ given by

$$\nabla^\text{LC} = \pi \circ \nabla : \Gamma(M, T^{1,0}M) \xrightarrow{\nabla} \Gamma(M, T\mathbb{C}M \otimes T\mathbb{C}M) \xrightarrow{\pi} \Gamma(M, T\mathbb{C}M \otimes T^{1,0}M).$$

Let $h = (h_J)$ be the corresponding Hermitian metric on $T^{1,0}M$ induced by $(M, g, J)$. It is obvious that $\nabla^\text{LC}$ is a metric compatible connection on the Hermitian holomorphic vector bundle $(T^{1,0}M, h)$, and we call $\nabla^\text{LC}$ the Levi-Civita connection on the
complex manifold $M$. It is obvious that, $\nabla^{\text{LC}}$ is determined by the following relations

$$
\nabla^{\text{LC}} \frac{\partial}{\partial z^k} := \Gamma^p_{ik} \frac{\partial}{\partial z^p} \quad \text{and} \quad \nabla^{\text{LC}} \frac{\partial}{\partial \bar{z}^k} := \Gamma^p_{jk} \frac{\partial}{\partial \bar{z}^p},
$$

where

$$
\Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{i\ell}}{\partial z^j} + \frac{\partial h_{j\ell}}{\partial z^i} - \frac{\partial h_{ij}}{\partial z^\ell} \right), \quad \text{and} \quad \Gamma^k_{ij} = \frac{1}{2} h^{k\ell} \left( \frac{\partial h_{j\ell}}{\partial \bar{z}^i} - \frac{\partial h_{i\ell}}{\partial \bar{z}^j} \right).
$$

The curvature tensor $\mathfrak{R} \in \Gamma(M, \Lambda^2 \mathcal{O}_M \otimes T^{*1,0}M \otimes T^{1,0}M)$ of $\nabla^{\text{LC}}$ is given by

$$
\mathfrak{R}(X, Y)s = \nabla^{\text{LC}}_X \nabla^{\text{LC}}_Y s - \nabla^{\text{LC}}_Y \nabla^{\text{LC}}_X s - \nabla^{\text{LC}}_{[X,Y]} s
$$

for any $X, Y \in T\mathcal{O}_M$ and $s \in T^{1,0}M$. A straightforward computation shows that the curvature tensor $\mathfrak{R}$ has $(1,1)$ components

$$
\mathfrak{R}^{\ell}_{ij} = - \left( \frac{\partial \nabla^\ell_{ik}}{\partial z^j} - \frac{\partial \nabla^\ell_{jk}}{\partial z^i} + \nabla^\ell_{ik} \nabla^\ell_{js} - \nabla^\ell_{jk} \nabla^\ell_{si} \right).
$$

The (first) Levi-Civita Ricci curvature $\mathfrak{Ric}(\omega_h)$ of $(T^{1,0}M, \omega_h, \nabla^{\text{LC}})$ is

$$
\mathfrak{Ric}(\omega_h) = \sqrt{-1} \mathfrak{R}^{(1)}_{ij} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \mathfrak{R}^{(1)}_{ij} = \mathfrak{R}^k_{ij}.\]

The Levi-Civita scalar curvature $s_{\text{LC}}$ of $\nabla^{\text{LC}}$ on $T^{1,0}M$ is

$$
s_{\text{LC}} = h^i \mathfrak{R}^{\ell}_{i\ell}.\]

2.4. Special manifolds. Let $X$ be a compact complex manifold with complex dimension $n \geq 2$. A Hermitian metric $\omega_g$ is called a Gauduchon metric if $\partial \bar{\partial} \omega_g^{-1} = 0$. It is proved by Gauduchon ([9]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to scaling). A Hermitian metric $\omega_g$ is called a balanced metric if $d\omega_g^{-1} = 0$ or equivalently $d^* \omega_g = 0$. On a compact complex surface, a balanced metric is also Kähler, i.e. $d\omega_g = 0$. It is well-known many Hermitian manifolds can not support balanced metrics, e.g. Hopf surface $S^3 \times S^1$. It is also obvious that balanced metrics are Gauduchon.

3. Geometry of the Levi-Civita connections

3.1. Some computational formulas. In this subsection, we recall some elementary and well-known computational lemmas on Hermitian manifolds.

Lemma 3.1. Let $(X, \omega)$ be a compact Hermitian manifold and $\omega = \sqrt{-1} h_{ij} dz^i \wedge d\bar{z}^j$. (3.1)

$$
\partial^* \omega = -\sqrt{-1} \Lambda (\partial) \omega = -2\sqrt{-1} \Gamma^k_{jk} d\bar{z}^j \quad \text{and} \quad \bar{\partial}^* \omega = \sqrt{-1} \Lambda (\partial) \omega = 2\sqrt{-1} \Gamma^k_{ik} dz^i.
$$
Proof. By the well-known Bochner formula (e.g. [12]),
\[
[D, L] = \sqrt{-1} (\partial + \tau)
\]
where \( \tau = [\Lambda, \partial \omega] \), we see
\[
\partial^* \omega = 2 \sqrt{-1} \Gamma_{ik}^l \partial z^i.
\]
\( \square \)

Let \( T \) be the torsion tensor of the Hermitian metric \( \omega \), i.e.

\[
T^{ij}_k = h^k_{\ell j} (\partial h_{i \ell} \partial z^i - \partial h_{i \ell} \partial z^j).
\]

Corollary 3.2. [14, Corollary 4.2] Let \((X, \omega)\) be a compact Hermitian manifold. Let \( s \) be the Riemannian scalar curvature of the background Riemannian metric \( \omega \). Then

\[
s = 2 s_C + \left( \langle \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega, \omega \rangle - 2 |\partial^* \omega|^2 \right) - \frac{1}{2} |T|^2,
\]

\[
s_{LC} = s_C - \frac{1}{2} \langle \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega, \omega \rangle = s_C - \langle \partial \partial^* \omega, \omega \rangle.
\]

3.2. The first Aeppli-Chern class and Levi-Civita connections. The following result is obtained in [13, Theorem 1.2] (see also [14, Theorem 4.1]).

Theorem 3.3. Let \((X, \omega)\) be a compact Hermitian manifold. Then the first Levi-Civita Ricci form \( \mathfrak{Ric}(\omega) \) represents the first Aeppli-Chern class \( c_1^{AC}(X) \) in \( H^{4,4}(X) \). Moreover, we have the Ricci curvature relation

\[
\mathfrak{Ric}(\omega) = \mathfrak{Ric}(\omega) - \frac{1}{2} \langle \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega, \omega \rangle.
\]

Lemma 3.4. Let \((X, \omega)\) be a compact Hermitian manifold with complex dimension \( n \). Suppose \( f \in C^\infty(X, \mathbb{R}) \) and \( \omega_f = e^f \omega \). Then we have

\[
\overline{\partial} f \omega_f = \overline{\partial} \omega + \sqrt{-1} (n - 1) \partial f \quad \text{and} \quad \overline{\partial} \bar{\partial}^* f \omega_f = \overline{\partial} \bar{\partial}^* \omega - \sqrt{-1} (n - 1) \partial \overline{\partial} f.
\]

where \( \overline{\partial} \), \( \overline{\partial} \) are the adjoint operators with respect to the metric \( \omega \) and \( \omega_f \) respectively. In particular, we have

\[
\mathfrak{Ric}(e^f \omega) = \mathfrak{Ric}(\omega) - \sqrt{-1} \partial \overline{\partial} f.
\]

Definition 3.5. The Kodaira dimension \( \kappa(L) \) of a line bundle \( L \) is defined to be

\[
\kappa(L) := \lim sup_{m \to +\infty} \frac{\log \dim C H^0(X, L^\otimes m)}{\log m}
\]

and the Kodaira dimension \( \kappa(X) \) of \( X \) is defined as \( \kappa(X) := \kappa(K_X) \) where the logarithm of zero is defined to be \(-\infty\).

Theorem 3.6. Let \((X, \omega)\) be a compact Hermitian manifold of complex dimension. Let \( \omega_f = e^f \omega \) be the Gauduchon metric in the conformal class of \( \omega \). Then we have

\[
\int_X s_f \cdot \omega_f^n = n \int_X \mathfrak{Ric}(\omega_f) \wedge \omega_f^{n-1} = \int_X e^{(n-1)f} \cdot s_{LC} \cdot \omega^n + n \| \overline{\partial} \omega_f \|^2_{\omega_f}.
\]
Proof. By Lemma 3.4 and Theorem 3.3, we have
\[
\text{Ric}(\omega_f) - \frac{\partial \partial^* \omega_f + \overline{\partial \partial^*} \omega_f}{2} = \text{Ric}(\omega) - \frac{\partial \partial^* \omega + \overline{\partial \partial^*} \omega}{2} - \sqrt{-1} \partial \bar{\partial} f
\]
\[
= \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} f.
\]
Moreover, we have
\[
\int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = \int_X \left( \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} f + \frac{\partial \partial^* \omega_f + \overline{\partial \partial^*} \omega_f}{2} \right) \wedge \omega_f^{n-1}
\]
\[
= \int_X \text{Ric}(\omega) \wedge \omega_f^{n-1} + \frac{1}{2} \left( \|\partial \bar{\partial} \omega_f\|^2_{\omega_f} + \|\overline{\partial \partial^*} \omega_f\|^2_{\omega_f} \right)
\]
\[
= \frac{1}{n} \int_X e^{(n-1)f} \cdot s_{\text{LC}} \cdot \omega^n + \|\partial \bar{\partial} \omega_f\|^2_{\omega_f}.
\]
\[
(3.8)
\]

Theorem 3.7. Let \(X\) be a compact complex manifold. Suppose \(\omega\) is a Hermitian metric with \(s_{\text{LC}} \geq 0\). Then either

1. \(\kappa(X) = -\infty\); or
2. \(\kappa(X) = 0\) and \((X, \omega)\) is conformally balanced with \(K_X\) a holomorphic torsion, i.e. \(K_X^\otimes m = \mathcal{O}_X\) for some \(m \in \mathbb{Z}^+\).

Proof. Let \(\omega_f = e^f \omega\) be the Gauduchon metric in the conformal class of \(\omega\). Then by formula (3.7), the total Chern scalar curvature of \(\omega_f\) is
\[
\int_X s_f \cdot \omega_f^n = n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} \geq n \|\partial \bar{\partial} \omega_f\|^2_{\omega_f}
\]
since the Levi-Civita scalar curvature \(s_{\text{LC}} \geq 0\). Suppose \(\overline{\partial \partial^*} \omega_f \neq 0\), then
\[
\int_X s_f \cdot \omega_f^n > 0.
\]
By [25, Corollary 3.3], we have \(\kappa(X) = -\infty\). On the other hand, if \(\overline{\partial \partial^*} \omega_f = 0\), i.e. \((X, \omega)\) is conformally balanced. Then the total Chern scalar curvature of the Gauduchon metric
\[
\int_X s_f \cdot \omega_f^n \geq 0.
\]
Then by [25, Theorem 1.4], we have \(\kappa(X) = -\infty\) or \(\kappa(X) = 0\), and when \(\kappa(X) = 0\), \(K_X\) is a holomorphic torsion.
\[
\Box
\]
4. Compact complex manifolds with Levi-Civita Ricci-flat metrics

Let’s recall that, a Levi-Civita Ricci-flat metric is a Hermitian metric satisfying
\[ \text{Ric}(\omega) = 0, \]
or equivalently, by formula (3.5)
\[ \text{Ric}(\omega) = \frac{\partial\bar{\partial}\ast \omega + \bar{\partial}\partial\ast \omega}{2}. \]

It is easy to see that

**Corollary 4.1.** Let \( X \) be a compact complex manifold. Then
\[ c_1^{BC}(X) = 0 \implies c_1(X) = 0 \implies c_1^{AC}(X) = 0. \]

The first obstruction for the existence of Levi-Civita Ricci-flat Hermitian metric is the top first Chern number:

**Corollary 4.2.** Suppose \( c_1^{AC}(X) = 0 \), then the top intersection number \( c_1^n(X) = 0. \)
In particular, if \( X \) has a Levi-Civita Ricci-flat Hermitian metric \( \omega \), then \( c_1^n(X) = 0. \)

**Proof.** By definition, if \( c_1^{AC}(X) = 0 \), then
\[ \text{Ric}(\omega) = \bar{\partial}A + \partial B \]
where \( A \) is a \((1,0)\)-form and \( B \) is a \((0,1)\)-form. Hence
\[ c_1^n(X) = \int_X (\text{Ric}(\omega))^n = \int_X (\text{Ric}(\omega))^{n-1} \wedge (\bar{\partial}A + \partial B) = 0 \]
since \( \text{Ric}(\omega) \) is both \( \partial \) and \( \bar{\partial} \)-closed. \( \square \)

**Theorem 4.3.** Let \( X \) be a compact complex manifold with \( \kappa(X) = -\infty \). If \( X \) has a Levi-Civita Ricci-flat Hermitian metric \( \omega \), then \( X \) must be a non-Kähler manifold.

**Proof.** Let \( \omega \) be a Hermitian metric with \( 2\text{Ric}(\omega) = 0 \). By formula (3.5), we have
\[ \text{Ric}(\omega) = \frac{\partial\bar{\partial}\ast \omega + \bar{\partial}\partial\ast \omega}{2}. \]

Note that \( \text{Ric}(\omega) \) is \( \partial \)-closed and \( \bar{\partial} \)-closed, and so we have
\[ \bar{\partial}\partial\ast \omega = 0 \]
Suppose \( X \) is a Kähler manifold, then by \( \partial\bar{\partial} \)-Lemma on \( X \), the \( \bar{\partial} \)-closed and \( \partial \)-exact \((1,1)\)-form \( \partial\bar{\partial}\ast \omega \) is \( \bar{\partial} \)-exact, i.e. there exists a smooth function \( \varphi \) such that
\[ \partial\bar{\partial}\ast \omega = \bar{\partial}\partial\varphi. \]

Therefore,
\[ \text{Ric}(\omega) = \sqrt{-1}\bar{\partial}\partial F \]
where \( F = -\frac{\varphi}{2\sqrt{-1}} \in C^\infty(X, \mathbb{R}) \). It is obvious that the Hermitian metric \( e^F \omega \) is Chern Ricci-flat, i.e. \( c_1^{BC}(X) = c_1(X) = 0 \) is unitary flat. Hence \( X \) is a Kähler Calabi-Yau manifold. In particular, \( \kappa(X) = 0 \). This is a contradiction. \( \square \)
Remark 4.4. On a compact Kähler Calabi-Yau manifold $X$ with $\dim_{\mathbb{C}} X = n \geq 2$, the Levi-Civita Ricci-flat metrics are not necessarily Kähler. Indeed, let $\omega_{\text{CY}}$ be a Calabi-Yau Kähler metric on $X$. Then for any non constant smooth function $f \in C^\infty(X, \mathbb{R})$, we can construct a non-Kähler Levi-Civita Ricci-flat metric. By Yau’s theorem, there exists a Kähler metric $\omega_0$ such that

$$\omega_0^n = e^{-f} \omega_{\text{CY}}^n.$$  

Let $\omega = e^f \omega_0$. Then $\omega$ is a non-Kähler metric with Levi-Civita Ricci-flat curvature. Indeed,

$$\mathcal{Ric}(\omega) = \mathcal{Ric}(\omega) - \frac{\partial \partial^* \omega + \overline{\partial \partial^* \omega}}{2}$$

$$= \mathcal{Ric}(\omega_0) - n \sqrt{-1} \partial \overline{\partial} f - \frac{\partial \overline{\partial}^0 \omega_0 + \overline{\partial} \partial^0 \omega_0}{2} + (n-1) \sqrt{-1} \partial \overline{\partial} f$$

$$= \mathcal{Ric}(\omega_{\text{CY}}) + \sqrt{-1} \partial \overline{\partial} f - n \sqrt{-1} \partial \overline{\partial} f - \frac{\partial \overline{\partial}^0 \omega_0 + \overline{\partial} \partial^0 \omega_0}{2} + (n-1) \sqrt{-1} \partial \overline{\partial} f$$

$$= 0,$$

where we use Lemma 3.4 in the second identity.

Theorem 4.3 has the following variant:

Corollary 4.5. Let $X$ be a compact Kähler manifold. If $\kappa(X) = -\infty$, then $X$ has no Levi-Civita Ricci-flat Hermitian metric.

As an application, we obtain

Theorem 4.6. Let $X$ be a compact complex surface with $\kappa(X) \geq 0$. Suppose $X$ admits a Hermitian metric with $s_{\text{LC}} \geq 0$. Then $X$ is a minimal Kähler surface of Calabi-Yau type, i.e. $X$ is exactly one of the following

1. an Enriques surface;
2. a bi-elliptic surface;
3. a $K3$ surface;
4. a torus.

Proof. By Theorem 3.7, we know $\kappa(X) = 0$ and the canonical line bundle $K_X$ is a holomorphic torsion, i.e. $K_X^m = \mathcal{O}_X$ for some $m \in \mathbb{Z}^+$. Since $\dim X = 2$, by Theorem 3.7 again, $X$ is a balanced surface and so it is Kähler. It is easy to see that, $X$ is minimal. According to the Kodaira-Enriques’ classification, $X$ is either an Enriques surface, a bi-elliptic surface, a $K3$ surface or a torus. All these surfaces are Kähler surfaces of Calabi-Yau type, and all Kähler Calabi-Yau metrics are Levi-Civita Ricci-flat. □
5. The proof of Theorem 1.2

In this section, we prove Theorem 1.2, i.e.

**Theorem 5.1.** Let $X$ be a compact complex surface. Suppose $X$ admits a Levi-Civita Ricci-flat Hermitian metric $\omega$. Then $X$ is minimal. Moreover, it lies in one of the following

1. Enriques surfaces;
2. bi-elliptic surfaces;
3. $K_3$ surfaces;
4. 2-tori;
5. Hopf surfaces.

**Proof.** Let $(X, \omega)$ be a compact complex surface with Levi-Civita Ricci-flat metric $\omega$. Then we have $c_1^{AC}(X) = 0$ and $s_{LC} = 0$. By Theorem 3.7, $\kappa(X) = -\infty$ or $\kappa(X) = 0$. We shall show $X$ is a minimal surface.

Suppose $\kappa(X) = 0$, by Theorem 4.6, we know $X$ is a minimal Kähler Calabi-Yau surface, i.e. $X$ is exactly one of the following

1. a Enriques surface;
2. a bi-elliptic surface;
3. a $K_3$ surface;
4. a torus.

Suppose $\kappa(X) = -\infty$. Let $X_{\text{min}}$ be the minimal model of $X$, then $X_{\text{min}}$ lies in one of the following classes:

1. minimal rational surfaces;
2. ruled surfaces of genus $g \geq 1$;
3. surface of class VII$_0$.

If $X_{\text{min}}$ is in (1) or (2), we know $X$ is projective. Since $\kappa(X) = -\infty$, by Corollary 4.5, $X$ has no Levi-Civita Ricci-flat metric. Hence $X_{\text{min}}$ is not in (1) or (2). Suppose $X_{\text{min}}$ lies in (3), i.e. of class VII$_0$. A class VII$_0$ surface is a minimal compact complex surface with $b_1 = 1$ and $\kappa(X) = -\infty$. It is well-known that the first Betti number $b_1$ of compact complex surfaces are invariant under blowing-ups, i.e. $b_1(X) = 1$. By [2, Theorem 2.7 on p.139], we know

$$b_1(X) = h^{1,0}(X) + h^{0,1}(X), \quad \text{and} \quad h^{1,0}(X) \leq h^{0,1}(X)$$

hence $h^{0,1}(X) = 1$. Since $\kappa(X) = \kappa(X_{\text{min}}) = -\infty$, by Serre duality, we have

$$h^{0,2}(X) = h^{2,0}(X) = h^0(X, K_X) = 0.$$ 

Therefore, by the Euler-Poincaré characteristic formula, we get

$$\chi(\mathcal{O}_X) = 1 - h^{0,1}(X) + h^{0,2}(X) = 0.$$
On the other hand, by the Noether-Riemann-Roch formula,

\[ \chi(O_X) = \frac{1}{12} (c_1^2(X) + c_2(X)) = 0, \]

we obtain

\[ c_2(X) = -c_1^2(X). \]

Note also that \( c_2(X) \) is the Euler characteristic \( e(X) \) of \( X \), i.e.

\[ c_2(X) = e(X) = 2 - 2b_1(X) + b_2(X) = b_2(X) \]

and so \( c_1^2(X) = -b_2(X) \leq 0 \). Suppose \( X \) has a Levi-Civita Ricci-flat Hermitian metric, then we have \( c_1^{AC}(X) = 0 \). By Corollary 4.2, we have \( c_1^2(X) = 0 \). Therefore \( b_2(X) = 0 \). It is well-known that, blowing-ups increase the second Betti number at least by 1, hence we have \( X = X_{\text{min}} \). We complete the proof of the statement that: if a compact complex surface admits a Levi-Civita Ricci-flat metric, then it is a minimal surface.

There are three classes of surfaces of VII\(_0\):

- class VII\(_0\) surfaces with \( b_2 > 0 \);
- Inoue surfaces: a class VII\(_0\) surface has \( b_2 = 0 \) and contains no curves;
- Hopf surfaces: its universal covering is \( \mathbb{C}^2 - \{0\} \), or equivalently a class VII\(_0\) surface has \( b_2 = 0 \) and contains a curve.

1. A class VII\(_0\) surface \( X \) with \( b_2 > 0 \) has no Levi-Civita Ricci-flat metrics. Indeed, by a similar computation as before, we know \( c_1^2(X) = -b_2 < 0 \) which contradicts to Corollary 4.2.

2. On an Inoue surface \( X \), there is no Levi-Civita Ricci-flat Hermitian metrics. This is essentially proved in [14, Theorem 7.2]. For the reader’s convenience, we include a sketched proof here. It is well-known ([10]) that an Inoue surface is a quotient of \( \mathbb{H} \times \mathbb{C} \) by a properly discontinuous group of affine transformations where \( \mathbb{H} \) is the upper half-plane. There are three types of Inoue surfaces:

- **(A)** Inoue surfaces \( S_M \). Let \( M \) be a matrix in \( \text{SL}_3(\mathbb{Z}) \) admitting one real eigenvalue \( \alpha > 1 \) and two complex conjugate eigenvalues \( \beta \neq \overline{\beta} \). Let \( (a_1, a_2, a_3) \) be a real eigenvector of \( M \) corresponding to \( \alpha \) and let \( (b_1, b_2, b_3) \) be an eigenvector of \( M \) corresponding to \( \beta \). Then \( X = S_M \) is the quotient of \( \mathbb{H} \times \mathbb{C} \) by the group of affine automorphisms generated by

\[ g_0(w, z) = (\alpha w, \beta z), \quad g_i(w, z) = (w + a_i, z + b_i), \quad i = 1, 2, 3. \]

- **(B)** Inoue surfaces \( X = S_{N, p, q, r, t}^+ \) are defined as the quotient of \( \mathbb{H} \times \mathbb{C} \) by the group of affine automorphisms generated by

\[ g_0(w, z) = (\alpha w, z + t), \quad g_i(w, z) = (w + a_i, z + b_i w + c_i), \quad i = 1, 2 \]

\[ g_3(w, z) = \left( w, z + \frac{b_1 a_2 - b_2 a_1}{r} \right), \]
where \( (a_1, a_2) \) and \( (b_1, b_2) \) are the eigenvectors of some matrix \( N \in \text{SL}_2(\mathbb{Z}) \) admitting real eigenvalues \( \alpha > 1, \alpha^{-1} \). Moreover \( t \in \mathbb{C} \) and \( p, q, r \) are integers, and \( (c_1, c_2) \) depends on \( (a_i, b_i), p, q, r \).

(C) Inoue surfaces \( X = S^+_N, p, q, r, t \) have unramified double cover which are Inoue surfaces of type \( S^+_N, p, q, r, t \).

Suppose—to the contrary—that there exists a Levi-Civita Ricci-flat Hermitian metric \( \omega \) on the Inoue surface \( X \). Let \( \omega_f = e^{f} \omega \) be the Gauduchon metric in the conformal class of \( \omega \), then by formula (3.7), the total Chern scalar curvature of \( \omega_f \) is

\[
\int_X s_f \cdot \omega_f^2 = 2 \int_X \text{Ric}(\omega_f) \land \omega_f = 2 \| \partial_f \omega_f \|_{\omega_f}^2 \geq 0.
\]

We shall show that on each Inoue surface, there exists a smooth Gauduchon metric with non-positive but not identically zero first Chern-Ricci curvature. Indeed, it is easy to see that the metric \( h^{-1} = [\text{Im}(w)]^{-1} (dw \land dz) \odot (d\overline{w} \land d\overline{z}) \) (resp. \( h^{-1} = [\text{Im}(w)]^{-2} (dw \land dz) \odot (d\overline{w} \land d\overline{z}) \)) is a globally defined Hermitian metric on the anticanonical bundle of \( S_M \) (resp. \( S^+_N, p, q, r, t \)) (e.g. [4, Section 6]). Hence, the Chern Ricci curvature of \( S_M \) is

\[
-\sqrt{-1} \partial \overline{\partial} \log h^{-1} = \sqrt{-1} \partial \overline{\partial} \log [\text{Im}(w)] = -\frac{1}{4} \frac{dw \land d\overline{w}}{[\text{Im}(w)]^2},
\]

which also represents \( c_1^{\text{BC}}(X) \). By Theorem [17, Theorem 1.3], there exists a Gauduchon metric \( \omega_G \) with

\[
\text{Ric}(\omega_G) = -\frac{1}{4} \frac{dw \land d\overline{w}}{[\text{Im}(w)]^2} \leq 0.
\]

Hence, for any Gauduchon metric \( \omega \), one has

\[
\int_X \text{Ric}(\omega) \land \omega = \int_X \text{Ric}(\omega_G) \land \omega < 0
\]

which is a contradiction to (5.1). We can deduce similar contradictions for \( S^+_N, p, q, r, t \).

(3). A compact complex surface \( X \) is called a Hopf surface if its universal covering is analytically isomorphic to \( \mathbb{C}^2 \setminus \{0\} \). It has been proved by Kodaira that its fundamental group \( \pi_1(X) \) is a finite extension of an infinite cyclic group generated by a biholomorphic contraction which takes the form

\[
(z, w) \rightarrow (az, bw + \lambda z^m)
\]

where \( a, b, \lambda \in \mathbb{C}, |a| \geq |b| > 1, m \in \mathbb{N}^* \) and \( \lambda(a - b^m) = 0 \). Hence, there are two different cases:

(I) the Hopf surface \( H_{a, b} \) of class 1 if \( \lambda = 0 \);
(II) the Hopf surface \( H_{a, b, \lambda m} \) of class 0 if \( \lambda \neq 0 \) and \( a = b^m \).
In the following, we consider the Hopf surface of class 1. Let $H_{a,b} = \mathbb{C}^2 \setminus \{0\} \sim (z, w)$ where $(z, w) \sim (az, bw)$ and $|a| \geq |b| > 1$. We set $k_1 = \log |a|$ and $k_2 = \log |b|$. Define a real smooth function

$$
\Phi(z, w) = e^{\frac{k_1 + k_2}{2\pi} \theta}
$$

where $\theta(z, w)$ is a real smooth function defined by

$$
|z|^2 e^{-\frac{k_1 \theta}{\pi}} + |w|^2 e^{-\frac{k_2 \theta}{\pi}} = 1.
$$

This is well-defined since for fixed $(z, w)$ the function $t \rightarrow |z|^2 |a|^t + |w|^2 |b|^t$ is strictly increasing with image $\mathbb{R}_+$. Let $\alpha = \frac{2k_1}{k_1 + k_2}$ and so $1 \leq \alpha < 2$. Then the key equation (5.4) is equivalent to

$$
|z|^2 \Phi^{-\alpha} + |w|^2 \Phi^{\alpha - 2} = 1.
$$

It is easy to see that

$$
\theta(az, bw) = \theta(z, w) + 2\pi, \quad \text{and} \quad \Phi(az, bw) = |a||b|\Phi(z, w).
$$

We define a quantity

$$
\Delta = \alpha|z|^2 \Phi^{-\alpha} + (2 - \alpha)|w|^2 \Phi^{\alpha - 2}.
$$

In the next theorem, we construct precisely Levi-Civita Ricci-flat metrics on Hopf surfaces of class 1.

**Theorem 5.2.** On the Hopf surface $H_{a,b}$ of class 1, the Hermitian metric

$$
\omega = \Delta^3 \left( \frac{1}{\Phi} \sqrt{-1} \partial \bar{\partial} \Phi - \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \log \Phi \right)
$$

is Levi-Civita Ricci-flat, i.e. $\mathfrak{R}ic(\omega) = 0$.

**Remark 5.3.** The proof of Theorem 5.2 is carried out in the next section. We should point out the construction follows from the ideas in [13, Theorem 6.2] and [14, Theorem 7.3]. More precisely, when $a = b$, we have $\alpha = 1$, $\Delta = 1$ and $\Phi = |z|^2 + |w|^2$. In this case, the Levi-Civita Ricci-flat metric constructed in Theorem 5.2 is exactly the same as the metrics constructed in [13, Theorem 6.2] and [14, Theorem 7.3].

6. The construction of Levi-Civita Ricci-flat metrics on Hopf surfaces of type 1

In this section, we prove Theorem 5.2.

**Lemma 6.1.** $|z|^2 \Phi^{-\alpha}$ and $|w|^2 \Phi^{\alpha - 2}$ are well-defined on $H_{a,b}$.

**Proof.** Indeed,

$$
|az|^2 \Phi^{-\alpha}(az, bw) = |a|^2 |a|^{-\alpha} |b|^{-\alpha} \cdot |z|^2 \Phi^{-\alpha}(z, w)
$$
and

$$|a|^2|a|^{-a}|b|^{-a} = e^{k_1(2-\alpha)} e^{-k_2\alpha} = 1.$$ 

Similarly, we can show \(|w|^2\Phi^{2-\alpha}\) is well-defined on \(H_{a,b}\). \(\square\)

Lemma 6.2. \(\sqrt{-1}\partial\bar{\partial}\log \Phi\) has a semi-positive matrix representation

\begin{equation}
\left( \begin{array}{c}
\frac{1}{\Delta^3 \Phi^2} \left[ (\alpha - 2)^2 |w|^2 \alpha (\alpha - 2) \bar{\omega}z \\
\alpha (\alpha - 2) \bar{\omega}w \end{array} \right] \\
\frac{1}{\alpha^2 |z|^2} \left[ \bar{\omega}z \Phi^{2a-2} \right]
\end{array} \right),
\end{equation}

and \(\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi\) has a matrix representation

\begin{equation}
\left( \begin{array}{c}
\frac{1}{\Delta^2 \Phi^{2\alpha - 2}} \left[ |z|^2 \bar{\omega}w \Phi^{2a-2} \right] \\
\frac{1}{|w|^2 \Phi^{4a-4}} \left[ \frac{\alpha^2 |z|^2 \Phi^2}{\Delta^3 \Phi^2} \right]
\end{array} \right).
\end{equation}

Proof. See [24, Appendix]. \(\square\)

As motivated by [13, Theorem 6.2] and [14, Theorem 7.3], we consider the \((1,1)\)-form

\begin{equation}
\omega_\lambda = \frac{\sqrt{-1}\partial\bar{\partial}\Phi}{\Phi} + \lambda \sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi
\end{equation}

It also takes the form

$$\omega_\lambda = (1 + \lambda) \sqrt{-1}\partial\bar{\partial}\log \Phi + \frac{\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi}{\Phi^2}$$

and it has the matrix representation

\begin{equation}
\left( \begin{array}{c}
\frac{1}{\Delta^3 \Phi^2} \frac{(1 + \lambda) \alpha (\alpha - 2) |w|^2}{\Delta^2 \Phi^{2\alpha - 2}} \left[ |z|^2 \bar{\omega}w \Phi^{2a-2} \right] \\
\frac{1}{\Delta^3 \Phi^2} \frac{\alpha^2 |z|^2 \Phi^2}{\Delta^3 \Phi^2} \left[ \frac{1}{\Delta^2 \Phi^{2\alpha - 2}} \right]
\end{array} \right).
\end{equation}

Since \(\det(\sqrt{-1}\partial\bar{\partial}\log \Phi) = \det(\sqrt{-1}\partial\Phi \wedge \bar{\partial}\Phi) = 0\), the determinant

$$\det(\omega_\lambda) = \frac{(1 + \lambda) \alpha (\alpha - 2) |w|^2}{\Delta^2 \Phi^{2\alpha - 2}} \left[ |z|^2 \bar{\omega}w \Phi^{2a-2} \right] + \frac{\alpha^2 |z|^2 \Phi^2}{\Delta^3 \Phi^2} \left[ \frac{1}{\Delta^2 \Phi^{2\alpha - 2}} \right]$$

$$= \frac{(1 + \lambda) \alpha |z|^2}{\Delta^5} \left( \frac{(\alpha - 2) |w|^2 \Phi^{0 - 2\alpha}}{\Phi^4} - \frac{\alpha^2 |z|^2}{\Phi^4} \right)$$

$$+ \frac{(1 + \lambda) \alpha |z|^2}{\Delta^5} \left( \frac{(\alpha - 2) |w|^2 \Phi^{0 - 2\alpha}}{\Phi^4} - \frac{\alpha^2 |z|^2}{\Phi^4} \right)$$

$$= \frac{1 + \lambda}{\Delta^5} \cdot \frac{1}{\Phi^{2 + 2\alpha}} (\alpha |z|^2 + (2 - \alpha) |w|^2 \Phi^{2a - 2})^2.$$ 

By (5.6), we have

$$\alpha |z|^2 + (2 - \alpha) |w|^2 \Phi^{2a - 2} = \Delta \Phi^\alpha$$

and so

\begin{equation}
\det(\omega_\lambda) = \frac{1 + \lambda}{\Delta^3 \Phi^2}.\end{equation}
It is easy to see from (6.3) that, when \( \lambda > -1 \),

\[
\omega_\lambda = \frac{-1}{\Phi} \partial \overline{\partial} \Phi + \lambda \sqrt{-1} \partial \overline{\partial} \log \Phi
\]

is a Hermitian metric. Let \( \partial^* \) and \( \overline{\partial}^* \) be the adjoint operators taken with respect to the metric \( \omega_\lambda \), and \( \Lambda \) be the dual operator of \( \omega_\lambda \wedge \cdot \).

**Lemma 6.3.** We have

\[
\frac{\partial \partial^* \omega_\lambda + \partial \partial^* \omega_\lambda}{2} = \partial \partial^* \omega_\lambda = \frac{-1}{1 + \lambda} \partial \overline{\partial} \log \Phi.
\]

**Proof.** The metric \( \omega_\lambda \) has local matrix representation

\[
(h_{ij}) = \begin{pmatrix}
\frac{1 + \lambda}{\Phi} \Phi_{11} & \frac{\lambda \Phi_{12}}{\Phi^2} & \frac{1 + \lambda}{\Phi^2} \\
\frac{1 + \lambda}{\Phi} \Phi_{21} & \frac{\lambda \Phi_{12}}{\Phi^2} & \frac{1 + \lambda}{\Phi^2} \\
\frac{1 + \lambda}{\Phi} \Phi_{31} & \frac{\lambda \Phi_{12}}{\Phi^2} & \frac{1 + \lambda}{\Phi^2}
\end{pmatrix}
\]

and its inverse matrix representation is

\[
(h^{ij}) = \frac{\Phi^2 \Delta^3}{1 + \lambda} \begin{pmatrix}
\frac{1 + \lambda}{\Phi} \Phi_{11} + \frac{\lambda \Phi_{22}}{\Phi^2} & -\frac{1 + \lambda}{\Phi^2} \Phi_{11} + \frac{\lambda \Phi_{12}}{\Phi^2} \\
-\frac{1 + \lambda}{\Phi^2} \Phi_{22} + \frac{\lambda \Phi_{12}}{\Phi^2} & \frac{1 + \lambda}{\Phi^2} \Phi_{22}
\end{pmatrix}.
\]

By Lemma 3.1, we have

\[
\partial^* \omega_\lambda = -\sqrt{-1} \Lambda \overline{\partial} \omega_\lambda = -2 \sqrt{-1} (\Gamma_\lambda)^k_{jk} d\bar{z}^i.
\]

A straightforward computation shows that

\[
\frac{\partial h_{ij}}{\partial \overline{z}^k} = \frac{\partial}{\partial \overline{z}^k} \left( \frac{1 + \lambda}{\Phi} \Phi_{ij} - \frac{\lambda}{\Phi^2} \Phi_{ji} \Phi_{ij} \right)
= -\frac{1 + \lambda}{\Phi^2} \Phi_{ij} \Phi_{ji} + \frac{1 + \lambda}{\Phi} \Phi_{ji} + \frac{2 \lambda}{\Phi^3} \Phi_{ij} \Phi_{ji} - \frac{\lambda}{\Phi^2} \Phi_{ji} \Phi_{ij} - \frac{\lambda}{\Phi^2} \Phi_{ij} \Phi_{ji}.
\]

Hence, we have

\[
(\Gamma_\lambda)^k_{ij} = \frac{1}{2} h^{ijij} \left( \frac{\partial h_{ij}}{\partial \overline{z}^k} - \frac{\partial h_{ij}}{\partial \overline{z}^k} \right)
= \frac{1}{2} h^{ijij} \left( -\frac{1 + \lambda}{\Phi^2} \Phi_{ij} \Phi_{ji} + \frac{1 + \lambda}{\Phi^2} \Phi_{ji} \Phi_{ij} \right)
\]

and

\[
\partial^* \omega_\lambda = -2 \sqrt{-1} (\Gamma_\lambda)^k_{ij} d\overline{z}^i = -\sqrt{-1} h^{ijij} \frac{1}{\Phi^2} \left( -\Phi_{ij} \Phi_{ji} + \Phi_{ji} \Phi_{ij} \right) d\overline{z}^i
\]

\[
= -\sqrt{-1} h^{ijij} \frac{1}{\Phi^2} \left( -\Phi_{ij} \Phi_{ji} + \Phi_{ji} \Phi_{ij} \right)
= -\sqrt{-1} h^{ijij} \frac{1}{\Phi^2} \left( \Phi_{ij} \Phi_{ji} + \Phi_{ji} \Phi_{ij} \right)
= \frac{\Phi_{ij} \sqrt{-1}}{\Phi} \frac{1}{1 + \lambda} d\overline{z}^i = \frac{\sqrt{-1}}{1 + \lambda} \partial \overline{\partial} \log \Phi.
\]

Therefore, we get (6.6). \( \square \)
By formulas (3.5), (6.5) and (6.6), we obtain
\[
\mathfrak{Ric}(\omega_{\lambda}) = -\sqrt{-1} \partial \overline{\partial} \det(\omega_{\lambda}) - \frac{\partial \overline{\partial}^* \omega_{\lambda} + \overline{\partial} \partial^* \omega_{\lambda}}{2} \\
= \left(2 - \frac{1}{1 + \lambda}\right) \sqrt{-1} \partial \overline{\partial} \log \Phi + 3 \sqrt{-1} \partial \overline{\partial} \log \Delta.
\]
In particular, we can take \( \lambda = -\frac{1}{2} \), and obtain
\[
(6.7) \quad \mathfrak{Ric}(\omega_{-\frac{1}{2}}) = 3 \sqrt{-1} \partial \overline{\partial} \log \Delta.
\]

Theorem 6.4. Let
\[
\omega = \Delta^3 \omega_{-\frac{1}{2}} = \Delta^3 \left(\frac{\sqrt{-1}}{\Phi} \partial \overline{\partial} \Phi - \frac{1}{2} \sqrt{-1} \partial \overline{\partial} \log \Phi\right).
\]
Then we have
\[
(6.8) \quad \mathfrak{Ric}(\omega) = 0.
\]

Proof. By Lemma 3.4 and formula (6.7), we have
\[
(6.9) \quad \mathfrak{Ric}(\omega) = \mathfrak{Ric}(\omega_{-\frac{1}{2}}) - \sqrt{-1} \partial \overline{\partial} \log \Delta^3 = 0.
\]

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