Level-rank duality of untwisted and twisted D-branes

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Abstract

Level-rank duality of untwisted and twisted D-branes of WZW models is explored. We derive the relation between D0-brane charges of level-rank dual untwisted D-branes of $\widehat{\text{su}}(N)_K$ and $\widehat{\text{sp}}(n)_k$, and of level-rank dual twisted D-branes of $\widehat{\text{su}}(2n + 1)_{2k+1}$. The analysis of level-rank duality of twisted D-branes of $\widehat{\text{su}}(2n + 1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\widehat{\text{sp}}(n)_k$. We also demonstrate level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes in each of these cases.

\textsuperscript{1}Research supported in part by the NSF under grant PHY-0456944
\textsuperscript{2}Research supported in part by the DOE under grant DE–FG02–92ER40706
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1 Introduction

D-branes on group manifolds have been the subject of much work, both from the algebraic and geometric points of view \[1\]–\[26\]. (For a review, see ref. \[27\]. See also ref. \[28\].) Algebraically, these D-branes correspond to the allowed boundary conditions for a Wess-Zumino-Witten (WZW) model on a surface with boundary \[29\].

Much can be learned about D-branes by studying their charges, which are classified by K-theory or, in the presence of a cohomologically nontrivial $H$-field background, twisted K-theory \[30\]. The charge group for D-branes on a simply-connected group manifold $G$ with level $K$ is given by the twisted K-group \[10, 12, 31, 32, 33, 18\]

$$K^*(G) = \bigoplus_{i=1}^m \mathbb{Z}_x, \quad m = 2^{\text{rank } G - 1} \quad (1.1)$$

where $\mathbb{Z}_x \equiv \mathbb{Z}/x\mathbb{Z}$ with $x$ an integer depending on $G$ and $K$. For $\hat{\mathfrak{su}}(N)_K$, for example, $x$ is given by \[10\]

$$x_{N,K} \equiv \frac{N + K}{\gcd\{N + K, \text{lcm}\{1, \ldots, N - 1\}\}}. \quad (1.2)$$

One of the $\mathbb{Z}_x$ factors in the charge group corresponds to the charge of untwisted (symmetry-preserving) D-branes. For $\mathfrak{su}(N)$ with $N > 2$, another of the $\mathbb{Z}_x$ factors corresponds to D-branes twisted by the charge-conjugation symmetry. For the D-branes corresponding to the remaining factors, see refs. \[10, 12, 18\].

WZW models with classical Lie groups possess an interesting property called level-rank duality: a relationship between various quantities in the $\hat{\mathfrak{su}}(N)_K$, $\hat{\mathfrak{so}}(N)_K$, or $\hat{\mathfrak{sp}}(n)_k$ model, and corresponding quantities in the level-rank dual $\hat{\mathfrak{su}}(K)_N$, $\hat{\mathfrak{so}}(K)_N$, or $\hat{\mathfrak{sp}}(k)_n$ model \[34\]–\[37\]. Implications of level-rank duality for boundary Kazama-Suzuki models were explored in ref. \[24\].

In ref. \[38\], we began the study of level-rank duality in boundary WZW theories, and in particular the level-rank duality of untwisted D-branes of $\hat{\mathfrak{su}}(N)_K$. In this paper, we extend this work to untwisted D-branes of the $\hat{\mathfrak{sp}}(n)_k$ WZW model, and to twisted D-branes of $\hat{\mathfrak{su}}(2n + 1)_{2k + 1}$, which are closely related to the untwisted D-branes of $\hat{\mathfrak{sp}}(n)_k$. We focus on two aspects of this duality: the relation between the D0-brane charges of level-rank dual D-branes, and the level-rank duality of the spectrum of an open string stretched between untwisted or twisted D-branes (i.e., the coefficients of the open-string partition function). For untwisted D-branes, these coefficients are given by the fusion coefficients of the bulk WZW theory \[29\], so duality of the untwisted open-string partition function follows from the well-known level-rank duality of the fusion rules \[31, 33, 36\]. For twisted D-branes, the open-string partition function coefficients may be calculated in terms of the modular-transformation matrices of twisted affine Lie algebras \[11, 14, 16\]. In this paper, we show that the spectrum of an open string stretched between twisted D-branes of $\hat{\mathfrak{su}}(2n + 1)_{2k + 1}$ is level-rank dual.

In section 2, we review some salient features of untwisted D-branes of WZW models. Section 3 describes the level-rank duality of the charges of untwisted D-branes of $\hat{\mathfrak{su}}(N)_K$ for all values of $N$ and $K$ (our results in ref. \[38\] were restricted to $N + K$ odd), and of the untwisted open-string partition function. Section 4 describes the level-rank duality of the charges of untwisted D-branes of $\hat{\mathfrak{sp}}(n)_k$, and of the untwisted open-string partition function.
Twisted D-branes of WZW models are reviewed in section 5, and section 6 is devoted to demonstrating the level-rank duality of the charges of twisted D-branes of $\hat{\mathfrak{su}}(2n+1)_{2k+1}$, and of the twisted open-string partition function. Concluding remarks constitute section 7.

2 Untwisted D-branes of WZW models

In this section, we review some salient features of Wess-Zumino-Witten models and their untwisted D-branes.

The WZW model, which describes strings propagating on a group manifold, is a rational conformal field theory whose chiral algebra (for both left- and right-movers) is the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}_K$ at level $K$. The Dynkin diagram of $\hat{\mathfrak{g}}_K$ has one more node than that of the associated finite-dimensional Lie algebra $\mathfrak{g}$. Let $(m_0, m_1, \ldots, m_n)$ be the dual Coxeter labels of $\hat{\mathfrak{g}}_K$ (where $n = \text{rank} \, \mathfrak{g}$) and $h^\vee = \sum_{i=0}^{n} m_i$ the dual Coxeter number of $\mathfrak{g}$. The Virasoro central charge of the WZW model is then $c = K \dim \mathfrak{g} / (K + h^\vee)$.

The building blocks of the WZW conformal field theory are integrable highest-weight representations $V_\lambda$ of $\hat{\mathfrak{g}}_K$, that is, representations whose highest weight $\lambda \in P^+_K$ has non-negative Dynkin indices $(a_0, a_1, \ldots, a_n)$ satisfying

$$\sum_{i=0}^{n} m_i a_i = K.$$  \hspace{1cm} (2.1)

With a slight abuse of notation, we also use $\lambda$ to denote the highest weight of the irreducible representation of $\mathfrak{g}$ with Dynkin indices $(a_1, \ldots, a_n)$, which spans the lowest-conformal-weight subspace of $V_\lambda$.

For $\hat{\mathfrak{su}}(n+1)_K = (A_n^{(1)})_K$ and $\hat{\mathfrak{sp}}(n)_K = (C_n^{(1)})_K$, the untwisted affine Lie algebras with which we will be principally concerned, we have $m_i = 1$ for $i = 0, \ldots, n$, and $h^\vee = n + 1$. It is often useful to describe irreducible representations of $\mathfrak{g}$ in terms of Young tableaux. For example, an irreducible representation of $\mathfrak{su}(n+1)$ or $\mathfrak{sp}(n)$ whose highest weight $\lambda$ has Dynkin indices $a_i$ corresponds to a Young tableau with $n$ or fewer rows, with row lengths

$$\ell_i = \sum_{j=1}^{n} a_j, \quad i = 1, \ldots, n.$$  \hspace{1cm} (2.2)

Let $r(\lambda) = \sum_{i=1}^{n} \ell_i$ denote the number of boxes of the tableau. Representations $\lambda$ corresponding to integrable highest-weight representations $V_\lambda$ of $\hat{\mathfrak{su}}(n+1)_K$ or $\hat{\mathfrak{sp}}(n)_K$ have Young tableaux with $K$ or fewer columns.

We will only consider WZW theories with a diagonal closed-string spectrum:

$$\mathcal{H}_{\text{closed}} = \bigoplus_{\lambda \in P^+_K} V_\lambda \otimes V_{\lambda^*}$$  \hspace{1cm} (2.3)

where $V$ represents right-moving states, and $\lambda^*$ denotes the representation conjugate to $\lambda$. The partition function for this theory is

$$Z_{\text{closed}}(\tau) = \sum_{\lambda \in P^+_K} |\chi_\lambda(\tau)|^2$$  \hspace{1cm} (2.4)
where

$$\chi_{\lambda}(\tau) = \text{Tr}_{V_{\lambda}} q^{L_0-\frac{c}{24}}$$

(2.5)

is the affine character of the integrable highest-weight representation $V_{\lambda}$. The affine characters transform linearly under the modular transformation $\tau \to -\frac{1}{\tau}$,

$$\chi_{\lambda}(-1/\tau) = \sum_{\mu \in P^+_K} S_{\mu\lambda} \chi_{\mu}(\tau),$$

(2.6)

and the unitarity of $S$ ensures the modular invariance of the partition function (2.4).

Next we turn to consider D-branes in the WZW model [1]-[26]. These D-branes may be studied algebraically in terms of the possible boundary conditions that can consistently be imposed on a WZW model with boundary. We consider boundary conditions that leave unbroken the $\hat{g}_K$ symmetry, as well as the conformal symmetry, of the theory, and we label the allowed boundary conditions (and therefore the D-branes) by $\alpha, \beta, \cdots$. The partition function on a cylinder, with boundary conditions $\alpha$ and $\beta$ on the two boundary components, is then given as a linear combination of affine characters of $\hat{g}_K$

$$Z_{\alpha\beta}^{\text{open}}(\tau) = \sum_{\lambda \in P^+_K} n_{\beta\lambda}^\alpha \chi_{\lambda}(\tau).$$

(2.7)

This describes the spectrum of an open string stretched between D-branes labelled by $\alpha$ and $\beta$.

In this section, we consider a special class of boundary conditions, called untwisted (or symmetry-preserving), that result from imposing the restriction

$$\left[ J^a(z) - \mathcal{J}^a(\bar{z}) \right] \bigg|_{z=\bar{z}} = 0$$

(2.8)

on the currents of the affine Lie algebra on the boundary $z = \bar{z}$ of the open string worldsheet, which has been conformally transformed to the upper half plane. Open-closed string duality allows one to correlate the boundary conditions (2.8) of the boundary WZW model with coherent states $|B\rangle \in \mathcal{H}^{\text{closed}}$ of the bulk WZW model satisfying

$$\left[ J^a_m + \mathcal{J}^a_{-m} \right] |B\rangle = 0,$$

(2.9)

where $J^a_m$ are the modes of the affine Lie algebra generators. Solutions of eq. (2.9) that belong to a single sector $V_{\mu} \otimes \mathcal{V}_{\mu^*}$ of the bulk WZW theory are known as Ishibashi states $|\mu\rangle \in \mathcal{H}^{\text{closed}}$, and are normalized such that

$$\langle \mu | q^H | \nu \rangle = \delta_{\mu\nu} \chi_{\mu}(\tau),$$

(2.10)

where $H = \frac{1}{2} \left( L_0 + \mathcal{L}_0 - \frac{1}{12} c \right)$ is the closed-string Hamiltonian. For the diagonal theory (2.3), Ishibashi states exist for all integrable highest-weight representations $\mu \in P^+_K$ of $\hat{g}_K$.

A coherent state $|B\rangle$ that corresponds to an allowed boundary condition must also satisfy additional (Cardy) conditions (2.9), among which are that the coefficients $n_{\beta\lambda}^\alpha$ in eq. (2.7) must be non-negative integers. Solutions to these conditions are labelled by integrable highest-weight representations $\lambda \in P^+_K$ of the untwisted affine Lie algebra $\hat{g}_K$, and
are known as (untwisted) Cardy states $|\lambda\rangle_C$. The Cardy states may be expressed as linear combinations of Ishibashi states

$$|\lambda\rangle_C = \sum_{\mu \in P_+^K} \frac{S_{\lambda\mu}}{\sqrt{S_{0\mu}}} |\mu\rangle_I$$

(2.11)

where $S_{\lambda\mu}$ is the modular transformation matrix given by eq. (2.6), and 0 denotes the identity representation. Untwisted D-branes of $\hat{g}_K$ correspond to $|\lambda\rangle_C$ and are therefore also labelled by $\lambda \in P_+^K$.

The partition function of open strings stretched between untwisted D-branes $\lambda$ and $\mu$

$$Z_{\lambda\mu}^{\text{open}}(\tau) = \sum_{\nu \in P_+^K} n_{\mu\nu} \chi_{\nu}(-1/\tau)$$

(2.12)

may alternatively be calculated as the closed-string propagator between untwisted Cardy states $Z_{\lambda\mu}^{\text{open}}(\tau) = C\langle \langle |\lambda\rangle^{\tilde{g}H} |\mu\rangle \rangle_C$, $\tilde{g} = e^{{2\pi i}(1/\tau)}$.

Combining eqs. (2.13), (2.11), (2.10), (2.6), and the Verlinde formula [40], we find

$$Z_{\lambda\mu}^{\text{open}}(\tau) = \sum_{\rho \in P_+^K} S_{\lambda\rho}^* S_{\mu\rho} S_{0\rho} \chi_{\rho}(-1/\tau) = \sum_{\nu \in P_+^K} \sum_{\rho \in P_+^K} S_{\mu\rho} S_{\nu\rho} S_{\lambda\rho}^* S_{0\rho} \chi_{\nu}(\tau) = \sum_{\nu \in P_+^K} N_{\mu\nu} \chi_{\nu}(\tau).$$

(2.14)

Hence, the coefficients $n_{\mu\nu} \chi_{\nu}$ in the open-string partition function (2.12) are simply given by the fusion coefficients $N_{\mu\nu}$ of the bulk WZW model.

Finally, an untwisted D-brane labelled by $\lambda \in P_+^K$ can be considered a bound state of D0-branes [41, 5, 8, 9, 10, 12]. It possesses a conserved D0-brane charge $Q_{\lambda}$ given by $(\text{dim } \lambda)_g$, but the charge is only defined modulo some integer [9, 10, 12, 21]. For D-branes of $\hat{\mathfrak{su}}(N)_K$, for example, this integer is given by eq. (1.2), thus

$$Q_{\lambda} = (\text{dim } \lambda)_{\hat{\mathfrak{su}}(N)} \mod x_{N,K} \quad \text{for } \hat{\mathfrak{su}}(N)_K$$

(2.15)

is the charge of the untwisted D-brane labelled by $\lambda$.

\section{Level-rank duality of untwisted D-branes of $\hat{\mathfrak{su}}(N)_K$}

In ref. [38], the relation between the charges of untwisted D-branes of the $\hat{\mathfrak{su}}(N)_K$ model and those of the level-rank-dual $\hat{\mathfrak{su}}(K)_N$ model was ascertained for odd values of $N + K$. In this section, we extend these results to all values of $N$ and $K$.

Since charges of $\hat{\mathfrak{su}}(N)_K$ D-branes are only defined modulo $x_{N,K}$, and those of $\hat{\mathfrak{su}}(K)_N$ D-branes modulo $x_{K,N}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\text{gcd}\{x_{N,K}, x_{K,N}\}$. Without loss of generality we will henceforth assume that $N \geq K$, in which case $\text{gcd}\{x_{N,K}, x_{K,N}\} = x_{N,K}$.

\section*{Level-rank duality of untwisted D-brane charges}

Given a Young tableau $\lambda$ corresponding to an integrable highest-weight representation of $\hat{\mathfrak{su}}(N)_K$ (with $N - 1$ or fewer rows, and $K$ or fewer columns), its transpose $\bar{\lambda}$ corresponds to
an integrable highest-weight representation of $\tilde{\text{su}}(K)_N$. (The map between representations of $\tilde{\text{su}}(N)_K$ and $\tilde{\text{su}}(K)_N$ is not one-to-one, but the map between cominimal equivalence classes of representations is. These equivalence classes are generated by the simple-current symmetry $\sigma$ of $\tilde{\text{su}}(N)_K$, which takes $\lambda$ into $\lambda' = \sigma(\lambda)$, where the Dynkin indices of $\lambda'$ are $a'_i = a_{i-1}$ for $i = 1, \ldots, N - 1$, and $a'_0 = a_{N-1}$.)

For odd $N + K$, the relation between $Q_\lambda$, the charge of the untwisted $\tilde{\text{su}}(N)_K$ D-brane labelled by $\lambda$, and $\tilde{Q}_\lambda$, the charge of the level-rank-dual $\tilde{\text{su}}(K)_N$ D-brane labelled by $\tilde{\lambda}$, was shown to be [38]

$$\tilde{Q}_\lambda = (-1)^{r(\lambda)}Q_\lambda \mod x_{N,K}, \quad \text{for } N + K \text{ odd}.$$  

(3.1)

where $r(\lambda)$ is the number of boxes in the tableau $\lambda$. In this section, we show that for the case of even $N + K$, the charges obey

$$\tilde{Q}_\lambda = Q_\lambda \mod x_{N,K}, \quad \text{for } N + K \text{ even (except for } N = K = 2^m).$$  

(3.2)

In the remaining case, we conjecture the relation

$$\tilde{Q}_\lambda = \begin{cases} (-1)^{r(\lambda)/N}Q_\lambda \mod x_{N,N}, & \text{when } N \mid r(\lambda) \\ Q_\lambda \mod x_{N,N}, & \text{when } N \not\mid r(\lambda) \end{cases} \quad \text{for } N = K = 2^m$$  

(3.3)

for which we have numerical evidence, but (as of yet) no complete proof.

Proof of eq. (3.2): We proceed as in ref. [38] by writing the dimension of an arbitrary irreducible representation $\lambda$ of $\text{su}(N)$ (with row lengths $\ell_i$ and column lengths $k_i$) as the determinant of an $\ell_1 \times \ell_1$ matrix (eq. (A.6) of ref. [44])

$$(\dim \lambda)_{\text{su}(N)} = \left| \left( \dim \Lambda_{k_i+j-i} \right)_{\text{su}(N)} \right|, \quad i, j = 1, \ldots, \ell_1$$  

(3.4)

where $\Lambda_s$ is the completely antisymmetric representation of $\text{su}(N)$, whose Young tableau is $s$. The maximum value of $s$ appearing in eq. (3.4) is $k_1 + \ell_1 - 1$, which is bounded by $N + K - 2$ for integrable highest-weight representations of $\tilde{\text{su}}(N)_K$. The representations $\Lambda_0$ and $\Lambda_N$ both correspond to the identity representation, with dimension 1. For $1 \leq s \leq N - 1$, $\Lambda_s$ are the fundamental representations of $\text{su}(N)$, with $\left( \dim \Lambda_s \right)_{\text{su}(N)} = \binom{N}{s}$. We define dim $\Lambda_s = 0$ for $s < 0$ and for $s > N$.

In ref. [38], we showed that

$$(\dim \Lambda_s)_{\text{su}(N)} = \begin{cases} (-1)^s(\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x_{N,K}, & \text{for } s \leq N + K - 2, \text{ except } s = N \\ (-1)^{K-1}(\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x_{N,K}, & \text{for } s = N \end{cases}$$  

(3.5)

where $\tilde{\Lambda}_s$ is the completely symmetric representation of $\text{su}(K)$, whose Young tableau is $\overline{s}$. (We define dim $\tilde{\Lambda}_s = 0$ for $s < 0$.) When $N + K$ is odd, eq. (3.5) becomes simply

$$(\dim \Lambda_s)_{\text{su}(N)} = (-1)^s(\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x_{N,K} \text{ for all } s \leq N + K - 2.$$  

This was used in ref. [38] to yield eq. (3.1).

Now we turn to the case of even $N + K$, first considering $N > K$. In eq. (1.2), the factor $\text{lcm}\{1, \ldots, N - 1\}$ then contains $(N + K)/2$, so $x_{N,K}$ is at most 2. It is easy to see that
\( x_{N,K} = 2 \) if \( N + K = 2^m \), and \( x_{N,K} = 1 \) otherwise. For \( x_{N,K} \leq 2 \), the minus signs in eq. (3.5) are irrelevant (since \( n = -n \mod 2 \)), so we may simply write

\[
(\dim \Lambda_s)_{\text{su}(N)} = (\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x_{N,K}, \quad \text{for } s \leq N+K-2, \text{ with } N+K \text{ even and } N > K.
\]  

(3.6)

We will use this below.

Next we consider \( N = K \). We begin by observing that if \( N \) is a power of a prime \( p \), then \( x_{N,N} = 4 \) if \( p = 2 \), and \( x_{N,N} = p \) if \( p > 2 \). If \( N \) contains more than one prime factor, then \( x_{N,N} = 1 \). In the latter case, eq. (3.2) is trivially satisfied, so we need only consider \( N = K = p^m \), where \( p \) is prime. Let us obtain the relation between \( (\dim \Lambda_s)_{\text{su}(p^m)} \) and \( (\dim \tilde{\Lambda}_s)_{\text{su}(p^m)} \) by considering three separate cases:

- **0 ≤ s ≤ N − 1:**
  
  By examining the factors of \( p \) (prime) in the numerator and denominator of \( (\dim \Lambda_s)_{\text{su}(p^m)} = \left(\frac{p^m}{s}\right) \), one can establish that if \( p^{l-1} \) divides \( s \) but \( p^l \) does not (for any \( l \leq m \)), then \( p^{m-l+1} \) divides \( \left(\frac{p^m}{s}\right) \). Thus \( (\dim \Lambda_s)_{\text{su}(p^m)} = 0 \mod p \) for \( 1 \leq s \leq N - 1 \). Combining this with eq. (3.5), we have

\[
(\dim \Lambda_s)_{\text{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\text{su}(p^m)} \mod x_{N,N}, \quad \text{for } 1 \leq s \leq N - 1.
\]  

(3.7)

This is trivially extended to \( s = 0 \).

- **s < 0, or N + 1 ≤ s ≤ 2N − 2:**
  
  In this case,

\[
(\dim \Lambda_s)_{\text{su}(p^m)} = (\dim \tilde{\Lambda}_s)_{\text{su}(p^m)} \mod x_{N,N}, \quad \text{for } s < 0, \text{ or } N + 1 \leq s \leq 2N - 2,
\]  

(3.8)

is valid because the l. h. s. vanishes, and so, by eq. (3.5), the r. h. s. either vanishes or is a multiple of \( x_{N,N} \).

- **s = N:**
  
  The remaining case yields

\[
(\dim \Lambda_N)_{\text{su}(p^m)} = (-1)^N(\dim \tilde{\Lambda}_N)_{\text{su}(p^m)} \mod x_{N,N}
\]  

(3.9)

which is in accord with the other cases when \( p \) is a prime other than 2.

We combine these results with eq. (3.6) to write

\[
(\dim \Lambda_s)_{\text{su}(N)} = (\dim \tilde{\Lambda}_s)_{\text{su}(K)} \mod x_{N,K}, \quad \text{for } s \leq N + K - 2,
\]

for \( N + K \) even (except \( N = K = 2^m \)).

(3.10)

Inserting this in eq. (3.4), we find

\[
(\dim \lambda)_{\text{su}(N)} = \left| (\dim \tilde{\Lambda}_{ki+j-i})_{\text{su}(K)} \right| \mod x_{N,K}, \quad \text{for } N + K \text{ even (except } N = K = 2^m \).
\]  

(3.11)
By an alternative formula for the dimension of a representation (eq. (A.5) of ref. [44]), the r.h.s. is the dimension of a representation of \( su(K) \) with row lengths \( k_i \) and column lengths \( \ell_i \), that is, the transpose representation \( \tilde{\lambda} \), hence

\[
(dim \lambda)_{su(N)} = (dim \tilde{\lambda})_{su(K)} \mod x_{N,K}, \quad \text{for } N + K \text{ even (except } N = K = 2^m). \tag{3.12}
\]

from which eq. (3.2) follows.\(^4\)

### Level-rank duality of the untwisted open string spectrum

In ref. [35, 36], it was shown that the fusion coefficients \( N_{\mu\nu}^{\lambda} \) of the bulk \( \hat{su}(N)_K \) WZW model are related to those of the \( \hat{su}(K)_N \) WZW model, denoted by \( \tilde{N} \), by

\[
N_{\mu\nu}^{\lambda} = \tilde{N}_{\sigma}^{\mu} \sigma^{\Delta(\tilde{\lambda})} \tilde{\lambda}_{\sigma}^{\Delta(\tilde{\nu})} \tag{3.13}
\]

where \( \Delta = \left[ r(\mu) + r(\nu) - r(\lambda) \right]/N \).

Since by eq. (2.14) the fusion coefficients \( N_{\mu\nu}^{\lambda} \) are equal to the coefficients \( n_{\mu\nu}^{\lambda} \) of the open-string partition function (2.12), it follows that if the spectrum of an \( \hat{su}(N)_K \) open string stretched between untwisted D-branes \( \lambda \) and \( \mu \) contains \( n_{\mu\nu}^{\lambda} \) copies of the highest-weight representation \( V_\nu \) of \( \hat{su}(N)_K \), then the spectrum of an \( \hat{su}(K)_N \) open string stretched between untwisted D-branes \( \tilde{\lambda} \) and \( \tilde{\mu} \) contains an equal number of copies of the highest-weight representation \( V_{\sigma-\Delta(\tilde{\nu})} \) of \( \hat{su}(K)_N \).

### 4 Level-rank duality of untwisted D-branes of \( \hat{Sp}(n)_k \)

In this section, we examine the relation between untwisted D-branes of the \( \hat{Sp}(n)_k \) model and those of the level-rank-dual \( \hat{Sp}(k)_n \) model.

Untwisted D-branes of \( \hat{Sp}(n)_k \) are labelled by integrable highest-weight representations \( V_\lambda \) of \( \hat{Sp}(n)_k = (C_n^{(1)})_k \). The D0-brane charge of D-branes of \( \hat{Sp}(n)_k \) are defined modulo the integer \( 21 \) [17]

\[
x = \frac{n + k + 1}{\gcd\{n + k + 1, \lcm\{1, 2, 3, \ldots, n, 1, 3, 5, \ldots, 2n - 1\}\}}
\]

\[
= \frac{n + k + 1}{\gcd\{n + k + 1, \frac{1}{2}\lcm\{1, 2, \ldots, 2n\}\}}
\]

\[
= \frac{2(n + k + 1)}{\gcd\{2(n + k + 1), \lcm\{1, 2, \ldots, 2n\}\}}
\]

\[
= x_{2n+1,2k+1} \tag{4.1}
\]

\(^3\)If \( \lambda \) has \( \ell_1 = K \), then the transpose \( \tilde{\lambda} \) contains leading columns of \( K \) boxes. In that case, one can apply the formula \( 14 \) \( Q_{\sigma(\lambda)} = (-1)^{N-1} Q_\lambda \mod x_{N,K} \) several times to relate \( \lambda \) to a tableau with no rows of length \( K \) before using eq. \( 3.3 \). The minus sign is irrelevant when \( x_{N,K} \leq 2 \), and vanishes when \( N \) is an odd prime.

\(^4\)Since minus signs are irrelevant when \( x_{N,K} \leq 2 \), eq. \( 3.1 \) actually holds for all \( N \neq K \), not just odd \( N + K \). Equation \( 3.1 \) is not valid, however, when \( N = K \). This is most easily seen by considering representations of \( \hat{su}(N)_N \) whose tableaux are invariant under transposition, and whose dimensions are not multiples of \( x \), such as the adjoint of \( \hat{su}(3)_3 \).
where $x_{2n+1,2k+1}$ is given by eq. (1.2). That is,

$$Q_\lambda = (\dim \lambda)_{\text{sp}(n)} \mod x_{2n+1,2k+1} \quad \text{for} \quad \tilde{\text{sp}}(n)_k$$

is the charge of the untwisted $\tilde{\text{sp}}(n)_k$ D-brane labelled by $\lambda$, where $(\dim \lambda)_{\text{sp}(n)}$ is the dimension of the $\text{sp}(n)$ representation $\lambda$. As we showed in the previous section, for $n \neq k$, we have $x_{2n+1,2k+1} = 2$ if $n + k + 1 = 2^m$, and $x_{2n+1,2k+1} = 1$ otherwise. For $n = k$, we have $x_{2n+1,2n+1} = p$ if $2n + 1 = p^m$, and $x_{2n+1,2n+1} = 1$ if $2n + 1$ contains more than one prime factor.

Since charges of $\tilde{\text{sp}}(n)_k$ D-branes are only defined modulo $x_{2n+1,2k+1}$, and those of $\tilde{\text{sp}}(k)_n$ D-branes modulo $x_{2k+1,2n+1}$, comparison of charges of level-rank-dual D-branes is only possible modulo $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\}$. Without loss of generality we henceforth assume that $n \geq k$, in which case $\gcd\{x_{2n+1,2k+1}, x_{2k+1,2n+1}\} = x_{2n+1,2k+1}$.

Level-rank duality of untwisted D-brane charges

Given a Young tableau $\lambda$ corresponding to an integrable highest-weight representation of $\tilde{\text{sp}}(n)_k$ (with $n$ or fewer rows and $k$ or fewer columns), its transpose $\tilde{\lambda}$ corresponds to an integrable highest-weight representation of $\tilde{\text{sp}}(k)_n$. The mapping between representations is one-to-one, in contrast to the case of $\tilde{\text{su}}(N)_k$.

We will show that the relation between $Q_\lambda$, the charge of the $\tilde{\text{sp}}(n)_k$ D-brane labelled by $\lambda$, and $\tilde{Q}_{\tilde{\lambda}}$, the charge of the level-rank-dual $\tilde{\text{sp}}(k)_n$ D-brane labelled by $\tilde{\lambda}$, is given by

$$\tilde{Q}_{\tilde{\lambda}} = Q_\lambda \mod x_{2n+1,2k+1}.$$  

(4.3)

The relation (4.3) is nontrivial only when $x_{2n+1,2k+1} > 1$, that is, when $n \neq k$ with $n + k + 1 = 2^m$, or when $n = k$ with $2n + 1 = p^m$.

Proof of eq. (4.3): We may write the dimension of an arbitrary irreducible representation $\lambda$ of $\text{sp}(n)$ as the determinant of an $\ell_1 \times \ell_1$ matrix (Prop. (A.44) of ref. [44]; see also ref. [37])

$$(\dim \lambda)_{\text{sp}(n)} = \left| \begin{array}{cccc}
\chi_k & (\chi_{k+1} + \chi_{k-1}) & \cdots & (\chi_{k+\ell_1-1} + \chi_{k-\ell_1+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_{k-i+1} & (\chi_{k-i+2} + \chi_{k-i}) & \cdots & (\chi_{k+\ell_1-i} + \chi_{k-\ell_1+i+2}) \\
\vdots & \vdots & \ddots & \vdots \\
\end{array} \right| , \quad i, j = 1, \ldots, \ell_1$$

(4.4)

where $\chi_s = (\dim \Lambda_s)_{\text{sp}(n)}$, with $\Lambda_s$ the completely antisymmetric representation of $\text{sp}(n)$, whose Young tableau is $\square \bigsqcup \square \cdots \bigsqcup \square$. The maximum value of $s$ appearing in eq. (4.4) is $k_1 + \ell_1 - 1$, which is bounded by $n + k - 1$ for integrable highest-weight representations of $\tilde{\text{sp}}(n)_k$. The representation $\Lambda_0$ corresponds to the identity representation with dimension 1. For $1 \leq s \leq n$, $\Lambda_s$ are the fundamental representations of $\text{sp}(n)$. (We define $(\dim \Lambda_s)_{\text{sp}(n)} = 0$ for $s < 0$ and for $s > n$.) Also, let $\tilde{\Lambda}_s$ be the completely symmetric representation of $\text{sp}(k)$, whose Young tableau is $\square \cdots \bigsqcup \square$. (We define $(\dim \tilde{\Lambda}_s)_{\text{sp}(k)} = 0$ for $s < 0$.)

Next, we may use the branching rules $(\Lambda_s)_{\text{su}(2n+1)} = \oplus_{t=0}^s (\Lambda_t)_{\text{sp}(n)}$ (for $s \leq n$) and $(\tilde{\Lambda}_s)_{\text{su}(2n+1)} = \oplus_{t=0}^s (\tilde{\Lambda}_t)_{\text{sp}(n)}$ of $\text{su}(2n + 1) \supset \text{sp}(n)$ to relate the dimensions of the fundamental representations of $\text{sp}(n)$ to those of the fundamental representations of $\text{su}(2n + 1)$:

$$(\dim \Lambda_s)_{\text{sp}(n)} = (\dim \Lambda_s)_{\text{su}(2n+1)} - (\dim \Lambda_{s-1})_{\text{su}(2n+1)},$$

9
\[
\text{(dim }\tilde{\Lambda}_s\text{)}_{\text{sp}(k)} = (\text{dim }\tilde{\Lambda}_s)_{\text{su}(2k+1)} - (\text{dim }\tilde{\Lambda}_{s-1})_{\text{su}(2k+1)}.
\]

Using this together with eq. (3.10), we have

\[
(\text{dim }\Lambda_s)_{\text{sp}(n)} = (\text{dim }\tilde{\Lambda}_s)_{\text{sp}(k)} \mod x_{2n+1,2k+1}, \quad \text{for } s \leq 2n + 2k.
\]

We use this in eq. (4.4) to obtain

\[
(\text{dim }\lambda)_{\text{sp}(n)} = \left| \begin{array}{cccc}
\tilde{\chi}_{k_1} & (\tilde{\chi}_{k_1+1} + \tilde{\chi}_{k_1-1}) & \cdots & (\tilde{\chi}_{k_1+\ell_1-1} + \tilde{\chi}_{k_1-\ell_1+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{\chi}_{k_{i-1}+1} & (\tilde{\chi}_{k_{i-1}+2} + \tilde{\chi}_{k_{i-1}-2}) & \cdots & (\tilde{\chi}_{k_{i-1}+\ell_{i-1}-1} + \tilde{\chi}_{k_{i-1}-\ell_{i-1}+2}) \\
\vdots & \vdots & \ddots & \vdots
\end{array} \right| \mod x_{2n+1,2k+1}
\]

where \(\tilde{\chi}_s = (\text{dim }\tilde{\Lambda}_s)_{\text{sp}(k)}\). By an alternative formula for the dimension of a representation (Prop. (A.50) of ref. [44]), the r.h.s. is the dimension of a representation of \(sp(k)\) with row lengths \(k_i\) and column lengths \(\ell_i\), that is, the transpose representation \(\hat{\lambda}\), hence

\[
(\text{dim }\lambda)_{\text{sp}(n)} = (\text{dim }\tilde{\lambda})_{\text{sp}(k)} \mod x_{2n+1,2k+1},
\]

from which eq. (4.3) follows. \(QED\).

**Level-rank duality of the untwisted open string spectrum**

In ref. [36], it was shown that the fusion coefficients \(N_{\mu\nu}^\lambda\) of the bulk \(\hat{sp}(n)_k\) WZW model are related to those of the \(\hat{sp}(k)_n\) WZW model by

\[
N_{\mu\nu}^\lambda = \tilde{N}_{\mu\nu}^\hat{\lambda}.
\]

Since the fusion coefficients \(N_{\mu\nu}^\lambda\) are equal to the coefficients \(n_{\mu\nu}^\lambda\) of the open-string partition function, it follows that if the spectrum of an \(\hat{sp}(n)_k\) open string stretched between untwisted D-branes \(\lambda\) and \(\mu\) contains \(n_{\mu\nu}^\lambda\) copies of the highest-weight representation \(V_\nu\) of \(\hat{sp}(n)_k\), then the spectrum of an \(\hat{sp}(k)_n\) open string stretched between untwisted D-branes \(\hat{\lambda}\) and \(\hat{\mu}\) contains an equal number of copies of the highest-weight representation \(\hat{V}_\nu\) of \(\hat{sp}(k)_n\).

**5 Twisted D-branes of WZW models**

In this section we review some aspects of twisted D-branes of the WZW model, drawing on refs. [2] [3] [4] [10]. As in section 2, these D-branes correspond to possible boundary conditions that can imposed on a boundary WZW model.

A boundary condition more general than eq. (2.8) that still preserves the \(\hat{g}_K\) symmetry of the boundary WZW model is

\[
\left[ J^a(z) - \omega \overline{\mathcal{J}}^a(\overline{z}) \right]\bigg|_{z = \overline{z}} = 0,
\]

where \(\omega\) is an automorphism of the Lie algebra \(g\). The boundary conditions (5.1) correspond to coherent states \(|B\rangle^\omega\in \mathcal{H}_{\text{closed}}\) of the bulk WZW model that satisfy

\[
\left[ J^a_m + \omega \overline{\mathcal{J}}^a_{-m} \right] |B\rangle^\omega = 0, \quad m \in \mathbb{Z}.
\]
The $\omega$-twisted Ishibashi states $|\mu\rangle^\omega_I$ are solutions of eq. (5.2) that belong to a single sector $V_\mu \otimes V^{\omega(\mu)^*}$ of the bulk WZW theory, and whose normalization is given by

$$\omega_I \langle \langle \mu|qH|\nu \rangle \rangle^\omega_I = \delta_{\mu\nu} \chi_\mu(\tau), \quad q = e^{2\pi i \tau}.$$  \hspace{1cm} (5.3)

Since we are considering the diagonal closed-string theory (2.3), these states only exist when $\mu = \omega(\mu)$, so the $\omega$-twisted Ishibashi states are labelled by $\mu \in \mathcal{E}^\omega$, where $\mathcal{E}^\omega \subset P^+_K$ are the integrable highest-weight representations of $\hat{g}_K$ that satisfy $\omega(\mu) = \mu$. Equivalently, $\mu$ corresponds to a highest-weight representation, which we denote by $\pi(\mu)$, of $\hat{g}$, the orbit Lie algebra [42] associated with $\hat{g}_K$.

Solutions of eq. (5.2) that also satisfy the Cardy conditions are denoted $\omega$-twisted Cardy states $|\alpha\rangle^\omega_C$, where the labels $\alpha$ take values in some set $\mathcal{B}^\omega$. The $\omega$-twisted Cardy states may be expressed as linear combinations of $\omega$-twisted Ishibashi states

$$|\alpha\rangle^\omega_C = \sum_{\mu \in \mathcal{E}^\omega} \frac{\psi_{\alpha\pi(\mu)^*}(\rho)}{\sqrt{S_{0\rho}}} |\mu\rangle^\omega$$ \hspace{1cm} (5.4)

where $\psi_{\alpha\pi(\mu)^*}(\rho)$ are some as-yet-undetermined coefficients. The $\omega$-twisted D-branes of $\hat{g}_K$ correspond to $|\alpha\rangle^\omega_C$ and are therefore also labelled by $\alpha \in \mathcal{B}^\omega$. These states (apparently) correspond [4] to integrable highest-weight representations of the $\omega$-twisted affine Lie algebra $\hat{g}_K^\omega$ (but see ref. [19]).

The partition function of open strings stretched between $\omega$-twisted D-branes $\alpha$ and $\beta$

$$Z_{\alpha\beta}^{\text{open}}(\tau) = \sum_{\lambda \in P^+_K} n_{\beta\lambda}^\alpha \chi_\lambda(\tau)$$ \hspace{1cm} (5.5)

may alternatively be calculated as the closed-string propagator between $\omega$-twisted Cardy states

$$Z_{\alpha\beta}^{\text{open}}(\tau) = \omega_C \langle \langle \alpha|\tilde{q}H|\beta \rangle \rangle^\omega_C, \quad \tilde{q} = e^{2\pi i (-1/\tau)}.$$ \hspace{1cm} (5.6)

Combining eqs. (5.6), (5.4), (5.3), and (2.6), we find

$$Z_{\alpha\beta}^{\text{open}}(\tau) = \sum_{\rho \in \mathcal{E}^\omega} \frac{\psi_{\alpha\pi(\rho)^*} S_{\lambda\rho}}{S_{0\rho}} \chi_\rho(-1/\tau) = \sum_{\lambda \in P^+_K} \sum_{\rho \in \mathcal{E}^\omega} \frac{\psi_{\alpha\pi(\rho)^*} S_{\lambda\rho} \psi_{\beta\pi(\rho)}}{S_{0\rho}} \chi_\lambda(\tau).$$ \hspace{1cm} (5.7)

Hence, the coefficients of the open-string partition function (5.5) are given by

$$n_{\beta\lambda}^\alpha = \sum_{\rho \in \mathcal{E}^\omega} \frac{\psi_{\alpha\pi(\rho)^*} S_{\lambda\rho} \psi_{\beta\pi(\rho)}}{S_{0\rho}}.$$ \hspace{1cm} (5.8)

Finally, the coefficients $\psi_{\alpha\pi(\rho)}$ relating the $\omega$-twisted Cardy states and $\omega$-twisted Ishibashi states may be identified [4] with the modular transformation matrices of characters of twisted affine Lie algebras [46], as may be seen, for example, by examining the partition function of an open string stretched between an $\omega$-twisted and an untwisted D-brane [14, 16].

11
6 Level-rank duality of twisted D-branes of $\hat{\text{su}}(2n + 1)_{2k+1}$

The finite Lie algebra $\text{su}(N)$ possesses an order-two automorphism $\omega_c$ arising from the invariance of its Dynkin diagram under reflection. This automorphism maps the Dynkin indices of an irreducible representation $a_i \rightarrow a_{N-i}$, and corresponds to charge conjugation of the representation. This automorphism lifts to an automorphism of the affine Lie algebra $\hat{\text{su}}(N)_K$, leaving the zero$^\text{th}$ node of the extended Dynkin diagram invariant, and gives rise to a class of $\omega_c$-twisted D-branes of the $\hat{\text{su}}(N)_K$ WZW model (for $N > 2$). Since the details of the $\omega_c$-twisted D-branes differ significantly between even and odd $N$, and we will restrict our attention to the $\omega_c$-twisted D-branes of the $\hat{\text{su}}(2n + 1)_{2k+1}$ WZW model.

First, recall that the $\omega_c$-twisted Ishibashi states $|\mu\rangle_{\omega_c}^\omega$ are labelled by self-conjugate integrable highest-weight representations $\mu \in \mathcal{E}^\omega$ of $(A^{(1)}_{2n})_{2k+1}$. Equation (2.1) implies that the Dynkin indices $(a_0, a_1, a_2, \cdots, a_{n-1}, a_n, a_{n-1}, \cdots, a_1)$ of $\mu$ satisfy

$$a_0 + 2(a_1 + \cdots + a_n) = 2k + 1. \quad (6.1)$$

In ref. [42], it was shown that the self-conjugate highest-weight representations of $(A^{(1)}_{2n})_{2k+1}$ are in one-to-one correspondence with integrable highest weight representations of the associated orbit Lie algebra $\tilde{g} = (A^{(2)}_{2n})_{2k+1}$, whose Dynkin diagram is

![Dynkin diagram](image)

with the integers indicating the dual Coxeter label $m_i$ of each node. The representation $\mu \in \mathcal{E}^\omega$ corresponds to the $(A^{(2)}_{2n})_{2k+1}$ representation $\pi(\mu)$ with Dynkin indices $(a_0, a_1, \cdots, a_n)$. Consistency with eq. (6.1) requires that the dual Coxeter labels are $(m_0, m_1, \cdots, m_n) = (1, 2, 2, \cdots, 2)$, and hence we must choose as the zero$^\text{th}$ node the right-most node of the Dynkin diagram above. The finite part of the orbit Lie algebra $\tilde{g}$, obtained by omitting the zero$^\text{th}$ node, is thus $C_n$. (Note that $C_n$ is the orbit Lie algebra of the finite Lie algebra $A_{2n}$ [42].)

Observe that, by eq. (6.1), $a_0$ must be odd, and that the representation $\pi(\mu)$ of the orbit algebra $\tilde{g}$ is in one-to-one correspondence [42] [15] [16] with the integrable highest-weight representation $\pi(\mu)'$ of the untwisted affine Lie algebra $(C^{(1)}_n)_k$ with Dynkin indices $(a'_0, a'_1, \cdots, a'_n)$, where $a'_0 = \frac{1}{2}(a_0 - 1)$ and $a'_i = a_i$ for $i = 1, \cdots, n$.

Next, the $\omega_c$-twisted Cardy states $|\alpha\rangle_C^{\omega_c}$ (and therefore the $\omega_c$-twisted D-branes) of the $(A^{(1)}_{2n})_{2k+1}$ WZW model are (apparently) labelled [4] by the integrable highest-weight representations $\alpha \in \mathcal{B}^{\omega_c}$ of the twisted Lie algebra $\hat{\text{su}}^{\omega_c}(2n)_{2k+1} = (A^{(2)}_{2n})_{2k+1}$ (but see ref. [19]). We adopt the same convention as above for the labelling of the nodes of the Dynkin diagram (consistent with refs. [16] [16] but differing from refs. [18] [14]). Thus, the Dynkin indices $(a_0, a_1, \cdots, a_n)$ of the highest weights $\alpha$ must also satisfy eq. (6.1), and the $\omega_c$-twisted D-branes are therefore characterized [16] [19] by the irreducible representations of $C_n = \text{sp}(n)$ with Dynkin indices $(a_1, \cdots, a_n)$ (also denoted, with a slight abuse of notation, by $\alpha$). The charge of the $\omega_c$-twisted D-brane of $\hat{\text{su}}(2n + 1)_{2k+1}$ labelled by $\alpha$ is given by [17]

$$Q^{\omega_c}_{\alpha} = (\dim \alpha)_{\text{sp}(n)} \mod x_{2n + 1, 2k + 1} \quad \text{for} \quad \hat{\text{su}}(2n + 1)_{2k+1}. \quad (6.2)$$
The periodicity of the charge is the same as that of all D-branes of $\hat{s}\hat{u}(2n+1)_{2k+1}$.

Observe also that the $\omega_c$-twisted D-branes $\alpha \in B^{\omega_c}$ are in one-to-one correspondence with integrable highest-weight representations $\alpha'$ of the untwisted affine Lie algebra $(C_n^{(1)})_k$ with Dynkin indices $(a'_0, a'_1, \ldots, a'_n)$, where $a'_0 = \frac{1}{2}(a_0 - 1)$ and $a'_i = a_i$ for $i = 1, \ldots, n$. That is, both the $\omega_c$-twisted Ishibashi states and the $\omega_c$-twisted Cardy states of $\hat{s}\hat{u}(2n+1)_{2k+1}$ are classified by integrable representations of $\hat{sp}(n)_k$.

Recall from eq. (5.8) that the coefficients of the partition function of open strings stretched between $\omega_c$-twisted D-branes $\alpha$ and $\beta$ are given by

$$n_{\beta\lambda}^\alpha = \sum_{\rho \in E^{\omega_c}} \psi^\ast_{\alpha \pi(\rho)} S_{\lambda \rho} \psi_{\beta \pi(\rho)} S_{0 \rho}$$

where $\alpha$, $\beta \in B^{\omega_c}$, $\lambda \in P^+_k$, and $\pi(\rho)$ is the representation of the orbit Lie algebra $(A_{2n}^{(2)})_{2k+1}$ that corresponds to the self-conjugate representation $\rho$ of $\hat{s}\hat{u}(2n+1)_{2k+1}$. The coefficients $\psi_{\alpha \pi(\rho)}$ are given [4, 14, 16] by the modular transformation matrix of the characters of $(A_{2n}^{(2)})_{2k+1}$. These in turn may be identified [42, 15, 16] with $S'_{\alpha' \pi(\rho)_r}$, the modular transformation matrix of $(C_n^{(1)})_k = \hat{sp}(n)_k$, so

$$n_{\beta\lambda}^\alpha = \sum_{\rho \in E^{\omega_c}} S'_{\alpha' \pi(\rho)} S_{\lambda \rho} S'_{\beta' \pi(\rho)_r} S_{0 \rho}^\ast.$$  

(6.4)

We will use this below to demonstrate level-rank duality of $n_{\beta\lambda}^\alpha$.

**Level-rank duality of twisted D-brane charges**

It is now straightforward to show the equality of charges of level-rank-dual $\omega_c$-twisted D-branes of $\hat{s}\hat{u}(2n+1)_{2k+1}$. As seen above, the $\omega_c$-twisted $\hat{s}\hat{u}(2n+1)_{2k+1}$ D-brane labelled by $\alpha$ is in one-to-one correspondence with an integrable highest-weight representation $\alpha'$ of $\hat{sp}(n)_k$, and has the same charge (6.2) as the untwisted $\hat{sp}(n)_k$ D-brane labelled by $\alpha'$ [12], including periodicity. The integrable highest-weight representation $\alpha'$ of $\hat{sp}(n)_k$ is level-rank-dual to the integrable highest-weight representation $\tilde{\alpha}'$ of $\hat{sp}(k)_n$ obtained by transposing the Young tableau corresponding to $\alpha'$, and the charges of the corresponding untwisted D-branes obey

$$(\dim \alpha')_{\hat{sp}(n)} = (\dim \tilde{\alpha}')_{\hat{sp}(k)} \mod x_{2n+1,2k+1},$$

(6.5)

as shown in sec. 4. Therefore the $\omega_c$-twisted D-branes of $\hat{s}\hat{u}(2n+1)_{2k+1}$ are in one-to-one correspondence with the $\omega_c$-twisted D-branes of $\hat{s}\hat{u}(2k+1)_{2n+1}$, and the charges of level-rank-dual $\omega_c$-twisted D-branes obey

$$Q_{\alpha}^{\omega_c} = \tilde{Q}_{\tilde{\alpha}}^{\omega_c} \mod x_{2n+1,2k+1}$$

(6.6)

where the map between $\omega_c$-twisted D-branes is given by transposition of the associated $\hat{sp}(n)_k$ tableaux.

**Level-rank duality of the twisted open string spectrum**

The coefficients of the partition function of open strings stretched between $\omega_c$-twisted D-branes $\alpha$ and $\beta$ are real numbers so we may write (6.4) as

$$n_{\beta\lambda}^\alpha = \sum_{\rho \in E^{\omega_c}} S'_{\alpha' \pi(\rho)} S_{\lambda \rho} S'_{\beta' \pi(\rho)_r} S_{0 \rho}^\ast.$$  

(6.7)
Under level-rank duality, the \( \hat{su}(N)_K \) modular transformation matrices transform as \[35, 36\]

\[
S_{\lambda \mu} = \sqrt{\frac{K}{N}} e^{-2\pi i r(\lambda) r(\mu) / NK} S_{\bar{\lambda} \bar{\mu}}^*
\]

(6.8)

and the (real) \( \hat{sp}(n)_k \) modular transformation matrices transform as \[36\]

\[
S'_{\alpha' \beta'} = \bar{S}'_{\alpha' \beta'} = \bar{S}_{\alpha' \beta'}^*
\]

(6.9)

where \( \bar{S} \) and \( \bar{S}' \) denote the \( \hat{su}(K)_N \) and \( \hat{sp}(k)_n \) modular transformation matrices respectively, \( \bar{\mu} \) is the transpose of the Young tableau corresponding to the \( \hat{su}(N)_K \) representation \( \mu \), and \( \bar{\alpha'} \) is the transpose of the Young tableau corresponding to the \( \hat{sp}(n)_k \) representation \( \alpha' \). These imply

\[
n_{\beta \lambda}^{\alpha} = \sum_{\rho \in \mathcal{E}^w} \frac{\hat{S}'_{\alpha' \beta'} \hat{S}_{\bar{\lambda} \bar{\mu}} S_{\bar{\lambda} \bar{\rho}} S_{\lambda \rho}}{\bar{S}_{0 \bar{\rho}}} e^{2\pi i r(\lambda) r(\mu)/(2n+1)(2k+1)}
\]

= \[
\sum_{\rho \in \mathcal{E}^w} \frac{\bar{\psi}_{\bar{\alpha} r(\rho)} \bar{\psi}_{\bar{\lambda} r(\rho)} \bar{\psi}_{\bar{\rho}}}{\bar{S}_{0 \bar{\rho}}} e^{2\pi i r(\lambda) r(\mu)/(2n+1)(2k+1)} .
\]

(6.10)

Let \( \hat{\rho} \) be the self-conjugate \( \hat{su}(2k + 1)_{2n+1} \) representation that maps to the \( \hat{sp}(k)_n \) representation \( \pi(\hat{\rho})' \), which is the transpose of the \( \hat{sp}(n)_k \) representation \( \pi(\rho)' \). In other words, the representation \( \pi(\hat{\rho}) \) of the orbit algebra is identified with \( \pi(\hat{\rho}) \). Now \( \hat{\rho} \) is not equal to \( \bar{\rho} \) (the transpose of \( \rho \)), which is generally not a self-conjugate representation, but they are in the same cominimal equivalence class,

\[
\bar{\rho} = \sigma^{r(\rho)/(2n+1)}(\hat{\rho}) ,
\]

(6.11)

which we prove at the end of this section. Equation (6.11) implies that \[35, 36\]

\[
\bar{S}_{\lambda \bar{\rho}} = e^{-2\pi i r(\lambda) r(\rho)/(2n+1)(2k+1)} \bar{S}_{\lambda \rho}^*
\]

(6.12)

so that eq. (6.10) becomes

\[
n_{\beta \lambda}^{\alpha} = \sum_{\hat{\rho}} \frac{\bar{\psi}_{\bar{\alpha} \hat{\rho}} \bar{\psi}_{\bar{\lambda} \hat{\rho}} \bar{\psi}_{\bar{\rho}}}{\bar{S}_{0 \bar{\rho}}} = \bar{n}_{\beta \lambda}^{\alpha} ,
\]

(6.13)

proving the level-rank duality of the coefficients of the open-string partition function of \( \omega_{c-} \)-twisted D-branes of \( \hat{su}(2n + 1)_{2k+1} \). That is, if the spectrum of an \( \hat{su}(2n + 1)_{2k+1} \) open string stretched between \( \omega_{c-} \)-untwisted D-branes \( \alpha \) and \( \beta \) contains \( n_{\beta \lambda}^{\alpha} \) copies of the highest-weight representation \( V_{\lambda} \) of \( \hat{su}(2n + 1)_{2k+1} \), then the spectrum of an \( \hat{su}(2k + 1)_{2n+1} \) open string stretched between \( \omega_{c-} \)-twisted D-branes \( \bar{\alpha} \) and \( \bar{\beta} \) contains an equal number of copies of the highest-weight representation \( V_{\bar{\lambda}} \) of \( \hat{su}(2k + 1)_{2n+1} \).

**Proof of eq. (6.11):** Let \( \rho \), a self-conjugate representation of \( \hat{su}(2n + 1)_{2k+1} \), have Dynkin indices

\[
\rho = (2k + 1 - 2\ell_1, a_1, \ldots, a_n, a_n, \ldots, a_1)
\]

(6.14)
where \( \ell_1 = \sum_{i=1}^n a_i \). The Young tableau for \( \rho \) has \( r(\rho) = (2n+1)\ell_1 \) boxes. The representation \( \pi(\rho)' \) of \( \widehat{\text{sp}}(n)_k \) that corresponds to \( \rho \) has Dynkin indices \( (k-\ell_1, a_1, \ldots, a_n) \). Let the transpose representation \( \pi(\rho)' \) of \( \widehat{\text{sp}}(k)_n \) have Dynkin indices \( (n-\ell_1, a_1, \ldots, a_n) \), with \( \ell_1 = \sum_{i=1}^k a_i \).

The representation \( \hat{\rho} \) of \( \widehat{\text{su}}(2k+1)_{2n+1} \) that corresponds to \( \pi(\rho)' \) has Dynkin indices \( (2n+1-2\ell_1, a_1, \ldots, a_k, a_k, \ldots, a_1) \). Finally, the representation \( \sigma^\ell(\hat{\rho}) \) has Dynkin indices

\[
\sigma^\ell(\hat{\rho}) = (a_{\ell_1}, a_{\ell_1-1}, \ldots, a_1, 2n+1-2\ell_1, a_1, \ldots, a_{\ell_1}, 0, \ldots, 0) \tag{6.15}
\]

where the last \( 2(k-\ell_1) \) entries vanish since \( a_i = 0 \) for \( i > \ell_1 \).

Since \( \pi(\rho)' \) and \( \pi(\rho)' \) are transpose representations, with row lengths \( \ell_i = \sum_{j=i}^n a_j \) and \( \tilde{\ell}_i = \sum_{j=i}^k a_j \) respectively, their index sets, defined by \( 35, 36 \)

\[
I = \{ \ell_i - i + n + 1 \mid 1 \leq i \leq n \}, \quad \mathcal{T} = \{ n + i - \tilde{\ell}_i \mid 1 \leq i \leq \ell_1 \} \tag{6.16}
\]

satisfy

\[
I \cup \mathcal{T} = \{ 1, 2, \ldots, n + \ell_1 \}, \quad I \cap \mathcal{T} = 0 \tag{6.17}
\]

where we have used \( \tilde{\ell}_i = 0 \) for \( i > \ell_1 \).

To prove that the Young tableau of \( \sigma^\ell(\hat{\rho}) \) is the transpose of \( \rho \), we must show that the index sets \( 35, 36 \)

\[
J = \{ \lambda_i - i + 2n + 2 \mid 1 \leq i \leq 2n+1 \}, \quad \mathcal{J} = \{ 2n+1 + i - \lambda_i \mid 1 \leq i \leq 2k+1 \} \tag{6.18}
\]

(where \( \lambda_i \) and \( \tilde{\lambda}_i \) are the row lengths of \( \rho \) and \( \sigma^\ell(\hat{\rho}) \) respectively, and \( \lambda_{2n+1} = \tilde{\lambda}_{2k+1} = 0 \))

satisfy

\[
J \cup \mathcal{J} = \{ 1, 2, \ldots, 2n + 2k + 2 \}, \quad J \cap \mathcal{J} = 0. \tag{6.19}
\]

Using eqs. (6.14) and (6.15), one gets

\[
J = J_1 \cup J_2 \cup J_3, \quad \mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3, \tag{6.20}
\]

\[
J_1 = \{ \ell_i - i + \ell_1 \mid 1 \leq i \leq n \}, \quad \mathcal{J}_1 = \{ \tilde{\ell}_i - i + \ell_1 + 1 \mid 1 \leq i \leq \ell_1 \},
\]

\[
J_2 = \{ n + \ell_1 + 1 \}, \quad \mathcal{J}_2 = \{ 2n + \ell_1 + 1 + i - \tilde{\ell}_i \mid 1 \leq i \leq \ell_1 \},
\]

\[
J_3 = \{ 2n + 2 + \ell_1 + \ell_i - i \mid 1 \leq i \leq n \}, \quad \mathcal{J}_3 = \{ 2n + 2\ell_1 + 1 + i \mid 1 \leq i \leq 2k - 2\ell_1 + 1 \},
\]

where \( \ell_i \) and \( \tilde{\ell}_i \) are the row lengths of the \( \widehat{\text{sp}}(n)_k \) and \( \widehat{\text{sp}}(k)_n \) representations \( \pi(\rho)' \) and \( \pi(\rho)' \).

Using eq. (6.17), one observes that

\[
J_1 \cup \mathcal{J}_1 = \{ 1, 2, \ldots, n + \ell_1 \}, \quad J_1 \cap \mathcal{J}_1 = 0,
\]

\[
J_2 = \{ n + \ell_1 + 1 \},
\]

\[
J_3 \cup \mathcal{J}_2 = \{ n + \ell_1 + 2, \ldots, 2n + 2\ell_1 + 1 \}, \quad J_3 \cap \mathcal{J}_2 = 0,
\]

\[
\mathcal{J}_3 = \{ 2n + 2\ell_1 + 2, \ldots, 2n + 2k + 2 \}, \tag{6.21}
\]

which establishes eq. (6.19). QED.
7 Conclusions

In this paper, we have continued our analysis, begun in ref. [38], of level-rank duality in boundary WZW models. We examined the relation between the D0-brane charges of level-rank dual untwisted D-branes of $\hat{\mathfrak{su}}(N)_K$ and $\hat{\mathfrak{sp}}(n)_k$, and of level-rank dual $\omega_c$-twisted D-branes of $\hat{\mathfrak{su}}(2n + 1)_{2k+1}$. We also demonstrated the level-rank duality of the spectrum of an open string stretched between untwisted or $\omega_c$-twisted D-branes in each of these theories. The analysis of level-rank duality of $\omega_c$-twisted D-branes of $\hat{\mathfrak{su}}(2n + 1)_{2k+1}$ is facilitated by their close relation to untwisted D-branes of $\hat{\mathfrak{sp}}(n)_k$.

It is expected that level-rank duality will also be present in the boundary WZW models and D-branes of other level-rank dual groups. Also, the level-rank duality of bulk $\hat{\mathfrak{su}}(N)_K$ WZW models presumably has consequences for the twisted D-branes of boundary $\hat{\mathfrak{su}}(N)_K$ models even when $N$ and $K$ are not odd. The level-rank map between the twisted D-branes in these cases is expected, however, to be more complicated than for $\hat{\mathfrak{su}}(2n + 1)_{2k+1}$. We leave this to future work.

Further, it would be interesting to derive the level-rank dualities described in this paper directly from K-theory.

Acknowledgments

SGN would like to thank M. Gaberdiel for illuminating comments.
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