Problems in Lie Group Theory

Luis J. Boya
Departamento de Física Teórica
Universidad de Zaragoza, E-50009 Zaragoza, Spain
e-mail: luisjo@posta.unizar.es

Abstract

The theory of Lie groups and representations was developed by Lie, Killing, Cartan, Weyl and others to a degree of quasi-perfection, in the years 1870-1930. The main topological features of compact simple Lie groups were elucidated in the 40s by H. Hopf, Pontriagin and others. The exceptional groups were studied by Chevalley, Borel, Freudenthal etc. in 1949-1957. Torsion in the exceptional groups was considered by Toda, Adams etc. in the 80s. However, one can still ask some questions for which the answer is either incomplete or absent, at least to this speaker. We would like to raise and discuss some of them in this communication.
1 Systematic Construction

The list of compact simple Lie groups contains the classical groups, related to \( O(n) \), \( U(n) \) and \( Sp(n) \), and the five exceptional structures \( G_2 \), \( F_4 \), \( E_6 \), \( E_7 \), and \( E_8 \). We shall consider the Lie group and its Lie algebra simultaneously, referring at times to \( G \) and \( L(G) \) respectively [1, 2].

Given a Lie group in a series \( G(n) \) (e.g. the orthogonal, unitary . . . ), how is the group \( G(n+1) \) constructed? For the orthogonal series \( (B_n \text{ and } D_n \text{ in Cartan’s notation}) \) the answer is simple: given \( O(n) \) acting on itself, that is, the adjoint \( (adj) \) representation, and the vector representation, \( n \), it turns out that there is an onto map

\[
\mathbf{n} \otimes \mathbf{n} \rightarrow \text{adj } L(O(n))
\]

which satisfies Jacobi identity. Hence in the direct sum \( \text{adj} + \text{vect} \), there is hidden the structure of \( L(O(n+1)) \)

\[
\text{Adj } O(n) + \text{Vect } O(n) \rightarrow \text{Adj } O(n+1)
\]

Dimensions match, but one has to check the Jacobi identity. In the simplest case \( O(2) \rightarrow O(3) \), if \( \{e\} \) describes \( O(2) \) and \( \{x, y\} \) the vector representation, defining \( [x, y] = e \), the only thing to check is \( [e, [x, y]] + \text{cycl.} = 0 \), which is trivial. By induction, if \( L_{ij} \) describes \( O(n) \) and \( n \) is the vectorial, defining \( L_{k,n+1} = x_k \), where \( x \) is the vectorial \( n \), one checks all Jacobi’s are fulfilled, in a case-by-case procedure.

For the unitary series \( SU(n) \) one adds the trivial \( U(1) \) plus the real part of the vector, as \( n \times n^* = \text{adj} + \text{Id} \), and the balance is

\[
\text{Adj } SU(n) + \text{Id} + n + n^* = \text{Adj } SU(n+1)
\]

and checking the Jacobi’s is tedious but it works.

For the symplectic series \( Sp(n) = C_n \), instead of \( U(1) \) one adds \( Sp(1) \) to the vector, of complex dimension \( 2n \)

\[
\text{Adj } Sp(n) + \text{Adj } Sp(1) + 2(n + n^*) = \text{Adj } Sp(n + 1)
\]
and again checking Jacobi’s is messy. (I would like to see more clearly why one should add once \( Sp(1) \)).

For the exceptional groups, the \( F_4 \) & \( E \) series one repeats the orthogonal (\( F_4 \)), unitary (\( E_6 \)), symplectic (\( E_7 \)) and orthogonal (\( E_8 \)) method above starting from an orthonormal group and the real part of its Spin representation, to wit (Adams [3]):

\[
\text{Adj } SO(9) + \text{Spin}(9) \rightarrow \text{Adj } F_4 \quad (36 + 16 = 52) \quad (5)
\]

\[
\text{Adj } SO(10) + \text{Spin}(10) + \text{Id} \rightarrow \text{Adj } E_6 \quad (45 + 1 + 2 \cdot 16 = 78) \quad (6)
\]

\[
\text{Adj } SO(12) + \text{Spin}(12) + Sp(1) \rightarrow \text{Adj } E_7 \quad (66 + 3 + 2 \cdot 32 = 133) \quad (7)
\]

\[
\text{Adj } SO(16) + \text{Spin}(16) \rightarrow \text{Adj } E_8 \quad (120 + 148 = 248) \quad (8)
\]

Notice that \( 8+1, 8+2, 8+4 \) and \( 8+8 \) appear. In this sense the octonions show up as a “second coming” of the reals, completed with the spin, not the vector irrep. One checks that the antisymmetric product of the spin irrep. contains the adjoint; for example for \( F_4 \), \( dim(\text{Spin}(9) \equiv \Delta) = 16 \), and \( \Delta \wedge \Delta = 36 \) (a 2-form = \( \text{Adj } O(9) \)) + 84 (a 3-form). This expresses that the \( F_4, E_6, E_7 \) corresponds to the octo, octo-complex, octo-quater and octo-octo bi-rings, as the Freudenthal Magic Square confirms [4]. For an explicit calculation of the Jacobi identity in the last case see [5].

For \( G_2 \), which has the independent definition as the automorphism group of octonions, we have the defining sequence \( SU(3) \rightarrow G_2 \rightarrow S^6 \subset R^7 \), as \( G_2 \) acts transitive on the 6-sphere of unit imaginary octonions, and then

\[
\text{Adj } SU(3) + n + n^* \rightarrow G_2 \quad (8 + 2 \cdot 3 = 14) \quad (9)
\]

whereas the “conventional” series \( SU(3) \rightarrow SU(4), \text{dim} = 15 \), would include a \( U(1) \) factor, as above in the unitary series; this “unimodular” character of \( G_2 \) is connected, through exceptional holonomy manifolds, with the compactification in M-theory \( 11 \rightarrow 4 \) [6].
So the problem of systematic construction of Lie groups is essentially solved. One would like however to understand better the Jacobi identities, and also why “other” combinations $adj + rep$ do not lead to new groups . . .

The Weyl group, the automorphisms, the center etc. of these groups are fairly well understood; Borel \cite{7} raises the question, for the $Spin(4n)$ groups, $n > 2$, whether the two spin irreps $\Delta_L$ and $\Delta_R$, which are obviously isomorphic, are also isomorphic to the vector irrep: the three have the structure $Spin(4n)/Z_2$, (the center of $Spin(4n)$ is $Z_2^2$), but only for $n = 2$ is the isomorphism clear (triality principle), and for $n = 1$ they are clearly not isomorphism, $Spin(3) \times SO(3) \neq SO(4)$; it is remarkable that $Spin(4n)$ has no faithful irreps, as any group with more than one involutive central element.

Another, more fundamental question, is the geometry associated to the exceptional groups, the $E$-series at least. Are we happy with $G_2$ as the automorphism group of the octonions, $F_4$ as the isometry of the octonionic projective plane, $E_6$ (in a noncompact form) as the collineations of the same, and $E_7$ resp. $E_8$ as examples of symplectic resp. metasymplectic geometries \cite{4,8}? Many people think this leaves much to be desired . . . one would like to understand the exceptional groups at the level we understand the classical groups, as automorphism groups of some natural geometric objects.

A recent paper by Atiyah \cite{9} sheds some light in the question. The first row of the Magic Square \cite{8} is just (compact form)

\begin{align}
B_1 : & \quad RP^2 = O(3)/O(1) \times O(2) \\
A_2 : & \quad CP^2 = U(3)/U(2) \times U(1) \\
C_3 : & \quad HP^2 = Sp(3)/Sp(2) \times Sp(1) \\
F_4 : & \quad OP^2 = F_4/Spin(9)
\end{align}

Now there is a sense in complexifying the four planes. The new projective planes are still homogeneous spaces, giving the second row of the Magic Square, and a real understanding of $E_6$. We have the series Group $\rightarrow$ Space $\rightarrow$ Isotropy given by
\[ SU(3) \rightarrow CP^2 \rightarrow U(2) \]  
\[ (SU(3) \cdot SU(3)) \rightarrow (CP^2)^2 \rightarrow (U(2))^2 \]  
\[ SU(6) \rightarrow Gr^c_{6,2} \rightarrow S[U(2) \cdot U(4)] \]  
\[ E_6 \rightarrow X \rightarrow Spin(10) \cdot U(1) \]

We lack a clear picture of \( X \). We use only the compact form. The complexified quaternionic plane coincides with the grassmanian of planes in \( C^6 \). Notice the naturality of the isotropy groups. We expect eagerly this analysis to be extended to the last two rows of the Magic Square.

### 2 Topology of Lie groups: Sphere structure

The gross topology of Lie groups is well-known. The non-compact case reduces to compact times an euclidean space (Malcev-Iwasawa theorem). The compact case is reduced to a finite factor, a Torus, and a semisimple compact Lie group. H. Hopf determined in 1.941 that the real homology of simple compact Lie groups is that of a product of odd spheres; for example

\[ H_*(G_2; R) = H_*(S^3 \times S^{11}; R) \]  

The exponents of a Lie group are the numbers \( i \) such \( S^{2i+1} \) is an allowed sphere; e.g. for \( U(n) \), they are \( 0, 1, \ldots, n - 1 \).

Can one see this sphere structure directly? The author has shown that in many cases the defining representation provides a basis for induction, starting from \( A_1 = B_1 = C_1 \). For example, \( SU(2) = Sp(1) = Spin(3) \) is exactly \( S^3 \), and as \( SU(3) \) acts in the defining (vector) irrep in \( C^3 = R^6 \supset S^5 \), the bundle

\[ SU(2) \rightarrow SU(3) \rightarrow S^5 \]

is a principal bundle. As \( \pi_4(S^3) = Z_2 \) classifies \( SU(2) \) bundles over \( S^5 \), and no simple Lie group is a product, \( SU(3) \) is the unique non-trivial bundle over
$S^5$ with fiber $SU(2)$, in the sense that the square is trivial; hence we dare write

$$SU(3) = S^3 \times S^5$$

(20)
as a finite twisted product. For the unitary and symplectic series the method works perfectly (see [10]); indeed, this is connected with the fact that neither the U-series nor the Sp-series have torsion [7]. The exponents are succesive in $U(n)$ and jump by two in $Sp(n)$.

But for the orthogonal series one has to consider some other manifolds besides spheres, with the same real homology; is this an imperfection? No! It accounts for the torsion. The place of spheres is played by some Stiefel manifold, and this introduces (precisely) $2$-torsion: in fact, $Spin(n)$, $n \geq 7$ and $SO(n)$, $n \geq 3$, have $2$-torsion. The low cases $Spin(3, 4, 5, 6)$ coincide with $Sp(1)$, $Sp(1) \times Sp(1)$, $Sp(2)$ and $SU(4)$, and have no torsion.

For the exceptional groups, let us start with the smallest, $G_2$; the structure diagram is [11]

\[
\begin{array}{ccccccc}
SU(2) & = & SU(2) & \downarrow & \downarrow & \downarrow & \downarrow \\
 & & & & & & \\
SU(3) & \rightarrow & G_2 & \rightarrow & S^6 & \rightarrow & \\
 & & & & & & \\
 & & & & & & \\
S^5 & \rightarrow & M_{11} & \rightarrow & S^6 & \\
\end{array}
\]

(21)

where $M_{11}$ is again a Stiefel manifold, with real homology like $S^{11}$, but with $2$-torsion. Hence

$$H_*(G_2; R) = H_*(S^3 \times S^{11}; R),$$

(22)
the correct result. For $F_4$ we do not get the sphere structure from any irrep, and in fact $F_4$ has $2$- and $3$-torsion. Does the $3$-torsion of $F_4$ come from the Euler Triplet, i.e. Euler number of $(F_4/Spin(9) = Moufang Plane^1) = 3$? [12].

\[1\]The projective plane over octonions was discovered by Ruth Moufang in 1.932. It makes little sense to call it the Cayley plane; A. Cayley was not even the first discoverer of the octonions!
There is no torsion in the U- and Sp-series, 2-torsion only in $Spin(n)$ and $G_2$, as referred to above. Now 2- and 3-Torsion appears in $F_4$ (as mentioned), $E_6$ and $E_7$. We have here no comment to offer; in particular, there is no clue that we see for torsion on the twisted sphere product and the natural actions of these groups, nor for $E_8$. But...

$E_8$ has 2-, 3- and 5-torsion [2]! Where on earth the 5-torsion comes from? I should pinpoint two hints: (i): The Coxeter number $(\dim - \text{rank})/\text{rank}$ of $E_8$ is $30 = 2 \cdot 3 \cdot 5$, in fact a mnemonic for the exponents of $E_8$ is: they are the coprimes up to 30, namely $(1, 7, 11, 13, 17, 19, 23, 29)$; (ii) The first perfect numbers are 6, 28 and 496, associated to the primes 2, 3 and 5 (these Mersenne numbers are $2^{p-1} \cdot (2^p - 1)$, $p$ and $2^p - 1$ primes). And the reader will recall that $496 = \dim O(32) = \dim E(8) \times E(8)$. Why the square? It happens also in $O(4)$, $\dim = 6$ (prime 2), as $O(4) \sim O(3) \times O(3)$; even $O(8)$ (prime 3) is like $S^7 \times S^7 \times G_2$.

These are not real problems, but features for which we should expect a better explanation.

3 Other topological features

3.1 Capicua

The sphere structure of compact simple Lie groups has a curious “capicua”\footnote{This catalan word (cap i cua = head and tail) is much more expressive that the greek palindrome. Capicua RNA sequences are very important in the RNA world which presumably ruled on earth before organized life.} form: the exponents are symmetric from each end; for example, for $E_6$ and $E_7$:

\begin{equation}
\text{exponents of } E_6 : 1, 4, 5, 7, 8, 11. \text{ Differences : } 3, 1, 2, 1, 3 \tag{23}
\end{equation}

\begin{equation}
\text{exponents of } E_7 : 1, 5, 7, 9, 11, 13, 17. \text{ Differences : } 4, 2, 2, 2, 2, 4 \tag{24}
\end{equation}

This question was raised by Chevalley [13], and still (I) do not understand it.
3.2 Supersymmetry

The real homology algebra of a simple Lie group is a Grassmann algebra, as it is generated by odd (i.e., anticommutative) elements. However, from them we can get, in the enveloping algebra, multilinear symmetric forms, one for each generator; the construction is standard \[13\]; in physics they are called the Casimir invariants, in mathematics the invariants of the Weyl group.

For example, for $SU(3)$, we have the quadratic and the cubic invariant \[14\]

$$I_2(x, y) = Tr \text{ad} x \text{ad} y, \quad I_3(x, y, z) = \{x, y \lor z\} \quad (25)$$

Is this a fact of life, or an indication of a hidden odd-even symmetry (supersymmetry)? It was remarked already in \[10\].

There is probably a more profound relation with supersymmetry, which we are just starting to notice; it was discovered by P. Ramond \[15\], and the mathematical basis is being clarified by B. Kostant \[16\]. The first example is with the pair $F_4 - B_4$: it turns out that $F_4/B_4$ is the Moufang plane $OP^2$ and

$$\dim \text{Weyl}(F_4)/\text{Weyl}(B_4) = 1152/384 = \text{Euler } (OP^2) = 3 \quad (26)$$
as $b_0 = b_8 = b_{16} = 1$, others $b$s = 0 in $OP^2$. Kostant now says: any irrep of $F_4$ generates three of $B_4$, in many cases supersymmetric, that is, the dimensions of spinors (faithful irrep of $Spin(9)$) and tensors (faithful of $SO(9) = Spin(9)/\mathbb{Z}_2$) match; for example (negative signs for spinors)

The identical irrep of $F_4$ generates the triplet : $+44 - 128 + 84$ of $Spin(9)$

$$\quad (27)$$
corresponding, in physics, to the 11-dimensional maximal Supergravity multiplet with graviton, gravitino and the 3-form $C$, which in $M$-theory is radiated by the M2 brane. Ramond found many triplets with fermi/bose ($= \text{spinorial vs. vectorial irreps}$) matching, and also many matchings in the dimensions of the Casimir invariants (but not perfect!).

We think this is a very important development; it points out to explain, for the first time, the existence of supersymmetry in physics! The theory \[16\]
is that for any pair $H \subset G$ of Lie groups, $G$ semisimple and $H$ reductive subgroup (i.e. it can contain $U(1)$ subgroups), the $Pin(2N)$ irrep, $2N = \dim G - \dim H$, splits in $\chi$ basic irreps of $H$, where

$$\chi = \text{Euler}(G/H) = \dim[\text{Weyl}(G)/\text{Weyl}(H)]$$

The analogous analysis for $CP^2 = SU(3)/U(2)$ and $HP^2 = Sp(3)/[Sp(1) \times Sp(2)]$, also with Euler number = 3, corresponds to the supersymmetric hypermultiplet and Yang-Mills multiplet, respectively [15]. That the 3 simply connected projective planes display the standard Susy, scalar in 6D, vector in 10D and tensor in 11D is thrilling!

We explain the procedure for the identity irrep of $SU(n+1)$; for instance let us consider in $CP^n = SU(n+1)/U(n)$ the diagram

$$
\begin{array}{cccc}
Z_2 & = & Z_2 \\
\downarrow & & \downarrow \\
U(n) & \rightarrow & Spin(2n) & \rightarrow & X \\
\parallel & & \downarrow & & \downarrow \\
U(n) & \rightarrow & SO(2n) & \rightarrow & X/2
\end{array}
$$

(29)

where the nature of $X$ needs not concern us (it is $CP^3$ for $n = 3$). The irrep of $Pin(2n)$ of dim $2^n$ splits, regarding $U(n)$, in all the antisymmetric forms ($p$-forms); for example for $n = 4$

$$\Delta_L + \Delta_R = [0] + [1] + [1^2] + [1^3] + [1^4]$$

(30)

$$2^4 = 16 = 1^+ + 4^- + 6^+ + 4^- + 1^+$$

(31)

The $U(1)$ factor in $U(n) = [SU(n) \times U(1)]/Z_n$ gives the sign (grading) which amounts to a generalization of supersymmetry; notice the number of summands, 5, is Euler($CP^4$). Kostant character formula [16] ammounts to substracting, instead of adding, the two spin irreps, and tensoring by any irrep of $SU(n+1)$: the alternating multiplet in the r.h.s, which makes sense in the Grothendieck representation ring $R(G)$, exhibits a generalized supersymmetry. For extensions to the symplectic and octonionic cases see [17].
4 Representations

For finite groups there is a duality, a kind of Fourier equivalence, between conjugacy classes and irreducible representations in the following sense (Frobenius): the number is the same, and the irreps are obtained from the central idempotents in which the classes, as centrals in the group algebra, are decomposed spectrally. In particular the group algebra splits in sum of matrix algebras, one per class, containing the irreps as many times as the dimension: Burnside’s formula reflects this \[13\]:

\[
\text{ord } G = \sum d_i^2
\]

where \(i\) runs through the irreps \(\leftrightarrow\) classes.

The question we want to pose now is this: how does this correspondence from conjugacy classes vs. irreps generalize for compact Lie groups? For compact abelian Lie groups \(A\) it is very clear: it is Pontriagin duality, trading the circle by the integers as many times as \(rkA\).

It should be possible to explain the gross features of the discrete lattice of irreps of a compact simple Lie group by the geometry of the compact manifold (or rather, orbifold) of the conjugacy classes. How?

I can only recall the simplest case, \(SU(2)\); the set of classes is a segment (labelled by the rotation angle), and the irreps are the nonnegative integers. For a group of rank \(r\), the set of classes is a compact “manifold” of dimension \(r\), and consequently the lattice of irreps is labeled by \(r\) integers. How does one go beyond this? One would like to do Fourier analysis in the center (class functions) of the \(L^1(G)\) convolution algebra of functions, and distillate the lattice of irreps...

The irreps of the classical and exceptional structures have a set of order \(r\) again called primitive irreps, in the sense that the ring \(R(G)\) is generated by them; they correspond one to one to the nodes of the Dynkin diagram. Our next question is: How are these irreps selected?

For the A-, B-, C- and D-series, it is fairly clear: they are the natural (or defining) irrep, plus some \(p\)-forms, traceless in the C-series, plus the
spin(s) irreps. in the O-series. Indeed the different root in the symplectic series $C_n$ is the irrep $[1^n] - [1^{n-2}]$ with (complex) singularized dimension $2(2n + 1)(2n) \ldots (n+3)/n!$.

Now for the primitive irreps of the exceptional groups [20]: for $G_2$ are the natural irrep (7) and the adjoint (14). For $F_4$, the natural (3x3 octonionic hermitean traceless, 26) and the adjoint (52). The other two are: $273 = 26 \cdot 25/2 - 52$ and $1274 = 52 \cdot 51/2 - 52 = (26 \cdot 25 \cdot 24/3! - 52)/2$, natural constructs in both cases.

For $E_6$ the natural is 27 (given by the $3 \times 3$ hermitean complex-octonionic Jordan algebra), and the adjoint is 78. As the center is $Z_3$, there are complex irreps [19] (this is why $E_6$ is a candidate for Grand Unified Theories, as it can accommodate chiral multiplets). Besides $27^*$, $351 = 27 \cdot 26/2$, $351^*$ and the real $2925 = 27 \cdot 26/2 \cdot 3$, exterior products. Also $2925 = 78 \cdot 77/2 - 78$.

For $E_7$, 56 defining and 133 adjoint; then $1539 = 56 \cdot 57/2 - 1$, $27664 = 56 \cdot 55 \cdot 54/2 \cdot 3 - 56$; $365750 = 56 \cdot 55 \cdot 54 \cdot 53/4! - 56 \cdot 55/2$ all correspond to antisymmetric traceless products, as $E_7$ is symplectic. $8645 = 133 \cdot 132/2 - 133$ is also natural. There is still the 912, for which T. Smith (Atlanta, GA) (personal communication) proposes $912 = 16 \times 56 + 16 \times 1$.

Finally, for $E_8$, 248 is both the natural and the adjoint (unique case among Lie groups). All except 3875 and 147250 are obtained, again, from traceless p-forms on 248 [21].

**Dedictory**

The man who made group theory and representations accesible and useful for physics was Eugene P. Wigner, whose 100th birthday we gather here to celebrate. I offer this modest contribution to his memory.

**Acknowledgements**

I have talked many of the topics presented here with M. Santander (Valladolid), who was very helpful. Also discussions with A. Segui (Zaragoza) and J. Mateos (Salamanca) are gratefully acknowledged.
References

[1] The books on the subject are legion. I quote two:
   J.F. Cornwell, *Group Theory in Physics*. Vols. I and II. Academic Press, San Diego, CA 1984.
   J.A. Azcárraga and J.M. Izquierdo, *Lie groups, Lie Algebras, Cohomology and some applications in Physics*. Cambridge U.P., Cambridge 1995.

[2] A modern text on topological questions is:
   M. Mimura and H. Toda, *Topology of Lie Groups I and II*. College Press, University of Beijing, China, 1998.

[3] J. F. Adams, *Lectures on Exceptional Lie groups*. University of Chicago P., Chicago, 1996.

[4] B. Rosenfeld, *Geometry of Lie Groups*. Ch. VII. Kluwer Academic, Dordrecht 1997.

[5] M. Green, J.H. Schwarz and E. Witten, *Superstring Theory I*, p.344. Cambridge U.P., Cambridge 1987.

[6] M. Duff, *M-theory on manifolds of G2 holonomy: the first twenty years*. Talk at the Supergravity 25. hep-th 0201062.

[7] A. Borel, *Topology of Lie groups and characteristic classes*. Bull. Am. Math. Soc. 61, 397-432 (1955), p. 414.

[8] H. Freudenthal, *Lie Groups in the Foundations of Geometry*. Adv. Math. 1, 145-190 (1964).

[9] M. Atiyah and J. Berndt, *Projective Planes, Severi varieties and Spheres*. math-DG 0206135.

[10] L.J. Boya, *The Geometry of Lie Groups*. Rep. Math. Phys. (Poland) 30, 149-162 (1991).

[11] L.J. Boya, *Octonions and M-Theory*. Talk at the 24 ICGTMP, Paris. July 2002. hep-th 0301037.

[12] L. Brink, P. Ramond and X. Xiong, *Supersymmetry and Euler multiplets*. hep-th 0207253
[13] C. Chevalley, *The Betti numbers of exceptional Lie Groups*. Int. Cong. Math. Harvard, MA 1950, Vol. II, 21-24.

[14] L. Michel and L.A. Radicati, *The geometry of the octet*. Ann. Ins. Henri Poincare 18, 185-214 (1973).

[15] T. Pengpan and P. Ramond, *M(ysterious) patterns in SO(9)*. Phys. Rep. 315, 137-152 (1999).

P. Ramond, *Exceptional Groups in Physics*. hep-th 0301050.

[16] B. Gross, B. Kostant, P. Ramond and S. Sternberg, *The Weyl character formula, the half-spin representations, and equal rank subgroups*. math-RT 9808133.

B. Kostant, *A Cubic Dirac Operator and the Emergence of Euler Number Multiplets of Representations for Equal Rank Subgroups*. Duke Math. Journ. 100, 447-501 (1999).

[17] L.J. Boya, in preparation. For background material see J. C. Baez, *The octonians*, Bull. Amer. Math. Soc. 39, 145 (2002).

[18] H. Weyl, *The Theory of Groups and Quantum Mechanics*. Methuen, London 1931 (also by Dover).

[19] L.J. Boya, *Representations of Lie Groups*. Rep. Math. Phys. 12, 351-354 (1993).

[20] N. Bourbaki, *Groupes et algèbres de Lie*. Ch. 7. Hermann, Paris 1975.

[21] Tony Smith pointed out to me that in J. F. Adams, *The fundamental representations of E8*, Contemp. Math. 37, 1-10, Amer. Math. Soc. Providence, R I (1985), there is a nice discussion of the primitive irreps of $E_8$. 