Computing the Jones index of quadratic permutation endomorphisms of $\mathcal{O}_2$

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Abstract

We compute the index of the inclusions of type $\text{III}_{1/2}$ factors arising from endomorphisms of the Cuntz algebra $\mathcal{O}_2$ associated to the rank-two permutation matrices.

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1 Introduction

Subfactor theory started with the breakthrough work by V. Jones [12]. In particular, given a subfactor \( N \subset M \), i.e. an inclusion of von Neumann algebras with trivial center, it is possible to associate a numerical invariant \([M:N]\) called the index of \( N \) in \( M \). Roughly this invariant resembles the index of a subgroup, and it reduces to that in very special situations, however in general it takes values in the Jones set \( J := \{4\cos^2(\pi/n) \mid n = 3, 4, 5, \ldots\} \cup [4, \infty] \). The theory was developed first for subfactors of type \( \text{II}_1 \), i.e. those admitting a faithful tracial state, but it has been generalized since to general inclusions of factors by Kosaki and Longo [20, 21]. In this more general framework the index \( \text{Ind}_E(N \subset M) \) depends on the choice of a faithful conditional expectation \( E : M \to N \). In turn, one can always find a unique conditional expectation that minimizes the index. Subfactor theory has found important applications in Quantum Field Theory and Conformal Field Theory (see e.g. [22, 16] for a small sample) and in the study of the tensor categories arising from (compact) groups and quantum groups. The square root of the minimal index is usually called statistical dimension, as it gives back the DHR dimension of sectors in QFT/CFT and the quantum dimension in the context of the representation theory of quantum groups. When \( M \) is an infinite factor the dimension provides a semi-ring homomorphism \( \text{Sect}_0(M) \to \sqrt{J} \) where \( \text{Sect}_0(M) := \text{End}_0(M)/\text{Inn}(M) \) and \( \text{End}_0(M) \) denotes the family of unital normal \( * \)-endomorphisms of \( M \) with finite index, i.e. those for which \( \text{Ind}(\rho(M) \subset M) < \infty \). It has been also speculated that the Jones index could replace the Fredholm index in a truly quantum form of the celebrated Atiyah-Singer theory [24]. Anyhow, computing indices in specific situations is often challenging and sometimes worthwhile. For a non-technical but broad overview of the subject including lot of important connections with other areas we refer the reader to [13].

For a number of years already, endomorphisms of the Cuntz algebras \( \mathcal{O}_n \) have been subject of intensive investigations. In addition to the intrinsic value of such study, the reason for this wide interest in endomorphisms of \( \mathcal{O}_n \) might be twofold. Firstly, they provide a nice framework in which explicit computations and analysis of non-trivial examples is feasible. Secondly, they link in many an interesting way with several aspects of theory of operator

\[^1\text{That is, additive for the natural direct sum of sectors and multiplicative for their composition.}\]
algebras such as subfactors and index theory, entropy, dynamical systems, wavelets and quantum field theory.

A very special class of such endomorphisms are the so-called permutation endomorphisms, i.e. those arising from permutation unitaries in the core UHF-subalgebra. In particular, Kawamura [18, 19] analyzed systematically some properties like properness, irreducibility and branching laws of rank-two (or “quadratic”) permutation endomorphisms of $\mathcal{O}_2$. More recently Skalski and Zacharias computed Voiculescu’s topological entropy [29] of the same endomorphisms [26], thus extending Choda’s computation of the entropy of the canonical shift of $\mathcal{O}_n$ [2].

The main purpose of this note is to add another bit of information, namely the computation of the index in the sense of Jones-Kosaki-Longo [12, 20, 21, 22] of the normal extension of these endomorphisms. More precisely, we will mostly focus on the $III_{1/2}$ factors obtained as the weak closure of $\mathcal{O}_2$ in the GNS representation of the canonical KMS state with respect to the (rescaled) gauge action. After Jones posed the problem in [14], general methods for computing the index of an inclusion of factors associated to a localized endomorphism of a Cuntz algebra $\mathcal{O}_n$ were first foreseen by Longo [23] using the theory of sectors. Later such methods were developed in full strength in [4] for the type $III$ situation and in the unpublished PhD dissertation of Akemann for the type $I_{1}$ case [1] (cf. [15, Example 5.1.6]), see also [3, 9].

2 Main results

Let $\mathcal{O}_2 = C^*(s_1, s_2)$ be the Cuntz algebra [7], that is the $C^*$-algebra generated by two isometries $s_1, s_2$ such that $s_1s_1^* + s_2s_2^* = 1$. Recall [8] that for any unitary $u \in \mathcal{O}_2$ there exists a unital $*$-endomorphism $\lambda_u$ of $\mathcal{O}_2$ such that $\lambda_u(s_i) = us_i$, $i = 1, 2$. Also, we denote by $\omega$ and $\pi_\omega$ the faithful canonical KMS state $\tau \circ E$ on $\mathcal{O}_2$ and its GNS representation, respectively, where $E$ is the conditional expectation of $\mathcal{O}_2$ onto its core UHF subalgebra $\mathcal{F}_2 \simeq \otimes_{i=1}^{\infty} M_2$ obtained by averaging over the gauge action of $\mathbb{T}$ and $\tau$ is the unique trace on $\mathcal{F}_2$.

In the case of rank-two endomorphisms $\lambda_u$ of $\mathcal{O}_2$ (that is, when $u \in M_2 \otimes M_2 \subset \mathcal{F}_2$) the minimal index $\text{Ind}(\lambda_u)$ [10] of the inclusions of A.F.D. $III_{1/2}$ factors $\pi_\omega(\lambda_u(\mathcal{O}_2))'' \subset \pi_\omega(\mathcal{O}_2)''$ satisfies [23, 4]

$$\text{Ind}(\lambda_u) \leq \left[ \pi_\omega(\mathcal{F}_2)'': \pi_\omega(\lambda_u(\mathcal{F}_2))'' \right] \leq 2^2 = 4$$
and thus all inclusions are extremal and \( \text{Ind}(\lambda_u) \) coincides with the Jones index of the inclusion of A.F.D. \( \text{II}_1 \) factors \( \pi_\omega(\lambda_u(\mathcal{F}_2))'' \subset \pi_\omega(\mathcal{F}_2)'' \). Moreover, according to the general analysis in [4, 11], in the above situation \( \text{Ind}(\lambda_u) \) is necessarily an integer and more precisely takes values in the set \( \{1, 2, 4\} \).

The main result of this section is the computation of the precise values of the indices \( \text{Ind}(\rho_\sigma) \) for all the endomorphisms \( \rho_\sigma := \lambda_{u_\sigma} \) of \( \mathcal{O}_2 \) induced by permutation matrices \( u_\sigma \in M_2 \otimes M_2 \subset \mathcal{F}_2 \). These values are plotted in the last column of Table 1 (below), modelled on those in [18, 19, 26], where for multiindices \( \alpha \) and \( \beta \) we denote \( s_\alpha s_\beta^* \) by \( s_{\alpha, \beta} \). Notice that by the results in [18, 19] among these 24 endomorphisms there are only 16 distinct inner equivalence classes. Earlier computations included the well-known canonical endomorphism of \( \mathcal{O}_2 \), here denoted \( \rho_{23} \), and also \( \rho_{12} \) [17, Theorem 1.3 (ii)].

**Automorphisms of \( \mathcal{O}_2 \).** There are four automorphisms of \( \mathcal{O}_2 \) [18], namely \( \rho_{id}, \rho_{(12)(34)}, \rho_{(13)(24)} \) and \( \rho_{(14)(23)} \), which clearly give index 1.

**Reducible endomorphisms of \( \mathcal{O}_2 \).** There are ten reducible endomorphisms of \( \mathcal{O}_2 \) [18]. As \( \pi_\omega(\lambda_u(\mathcal{O}_2)') \cap \mathcal{O}_2 \subset \pi_\omega(\lambda_u(\mathcal{O}_2)') \cap \pi_\omega(\mathcal{O}_2)' \), the corresponding inclusions of factors are reducible too and the index is necessarily \( \geq (1 + 1)^2 = 4 \). Thus for all these endomorphisms the index is 4.

In the sequel we follow closely the discussion in [4, Section 6]. Given a rank-two permutation matrix \( u_\sigma \) we define a selfadjoint subspace \( \Xi \) of \( \mathcal{F}_2 := \text{span}\{s_is_j^*, 1 \leq i, j \leq 2\} \cong M_2 \) as \( \bigcap_{s \in \mathbb{N}} (K^s)^* \mathcal{F}_2 K^s \), where \( K = \text{span}\{u_\sigma s_1, u_\sigma s_2\} \subset \mathcal{O}_2 \). We will need the next result, that gathers together few facts from [4, Section 6].

**Theorem 2.1.** Let \( u \) be a unitary in \( \mathcal{F}_2 := M_2 \otimes M_2 \) and assume that the corresponding subspace \( \Xi \) satisfies \( \Xi^2 \subset \Xi \). Suppose in addition that

(a) \( \omega(a\lambda_u(b)) = \omega(a)\omega(b) \) for all \( a \in \Xi, b \in \mathcal{F}_2 \)

(b) \( E_1(u^*au) = \omega(a)1 \) for every \( a \in \Xi \), where \( E_1 : \pi_\omega(\mathcal{O}_2)'' \to \mathcal{F}_2 \) is the \( \omega \)-invariant conditional expectation.

2 A few words on the notation: in the situation under consideration these matrices \( u_\sigma \) are naturally parametrized by permutations of the set \( \{1, 2\}^2 \); after identifying \( \{11, 12, 21, 22\} \) with \( \{1, 2, 3, 4\} \), they are conveniently indexed by elements in the usual symmetric group on four elements. Thus for instance \( \rho_{12} \) is the endomorphism of \( \mathcal{O}_2 \) induced by the permutation matrix associated to the cycle \( \sigma_{12} \equiv (12) \in \mathfrak{S}_4 \).

3 If \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, 2\}^k \) then \( s_\alpha := s_{\alpha_1} \ldots s_{\alpha_k} \).
Then
\[ \text{Ind}(\lambda_u) = \dim(\Xi). \]

**Proof.** It follows directly from [4, Proposition 6.1] and [4, Theorem 6.5 (c)] (or [4, Corollary 6.6]). □

**The transformation induced by** \( \sigma_{12} \). One can check that the subspace \( \Xi \subset M_2 \) is the linear span of 1 and \( s_1s_2^* + s_2s_1^* \). Moreover, the conditions (a) and (b) in Theorem 2.1 are easily verified and thus \( \text{Ind}(\rho_{12}) = 2 \).

**The transformation induced by** \( \sigma_{13} \). One can check that the subspace \( \Xi \subset M_2 \) is the linear span of \( s_1s_1^* \) and \( s_2s_2^* \). Moreover, the conditions (a) and (b) in Theorem 2.1 are easily verified and thus \( \text{Ind}(\rho_{13}) = 2 \).

**The transformation induced by** \( \sigma_{142} \). One can check that in this case \( \Xi = F_1 \). Moreover, the conditions (a) and (b) in Theorem 2.1 are easily verified and thus \( \text{Ind}(\rho_{142}) = 4 \). In alternative, one can apply [4, Corollary 7.6] with \( U = u_{142} \) and \( V = -u_{134} \). We omit the lengthy but straightforward calculation. Notice that the restriction of \( \rho_{142} \) to \( F_2 \) can easily be shown to be reducible.

**The remaining transformations.** One has \( \rho_{34} = \rho_{(12)(34)} \circ \rho_{12}, \rho_{1324} = \text{Ad}(u_{(13)(24)}) \circ \rho_{12}, \rho_{1423} = \text{Ad}(u_{(13)(24)}) \circ \rho_{34}, \rho_{24} = \rho_{13} \circ \rho_{(13)(24)}, \rho_{1234} = \text{Ad}(u_{(13)(24)}) \circ \rho_{24} \) and \( \rho_{1432} = \text{Ad}(u_{(13)(24)}) \circ \rho_{13} \) and thus, by multiplicativity, \( \text{Ind}(\rho_{34}) = \text{Ind}(\rho_{1324}) = \text{Ind}(\rho_{1423}) = \text{Ind}(\rho_{24}) = \text{Ind}(\rho_{1234}) = \text{Ind}(\rho_{1432}) = 2 \). Also, one has \( \rho_{134} = \rho_{142} \circ \rho_{(13)(24)} \), thus \( \text{Ind}(\rho_{134}) = \text{Ind}(\rho_{142}) = 4 \).
of any general direct relation between their indices (the seventh column of computations and the framework in \cite{26}. Certainly the (minimal) index is rank-two permutation endomorphisms of $O_2$.

In Table 1 above we collected the currently known key characteristics of the rank-two permutation endomorphisms of $O_2$.\footnote{In the fourth column, inn, irr, red and out stand for “inner automorphism”, “irreducible endomorphism”, “reducible endomorphism” and “outer automorphism”, respectively, as endomorphisms of $O_2$.}

Certainly the (minimal) index is

\begin{table}[h]
\centering
\begin{tabular}{lllllll}
\hline
$\rho_\sigma$ & $\rho_\sigma(s_1)$ & $\rho_\sigma(s_2)$ & property & $ht(\rho_\sigma)$ & $ht(\rho_\sigma(s_1))$ & $\text{Ind}(\rho_\sigma)$ \\
\hline
$\rho_{id} = \text{id}$ & $s_1$ & $s_2$ & \text{inn} & 0 & 0 & 1 \\
$\rho_{12}$ & $s_{12,1} + s_{11,2}$ & $s_2$ & \text{irr} & $\log 2$ & 0 & 2 \\
$\rho_{13}$ & $s_{21,1} + s_{12,2}$ & $s_{11,1} + s_{22,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 2 \\
$\rho_{14}$ & $s_{22,1} + s_{12,2}$ & $s_{21,1} + s_{11,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{23}$ & $s_{11,1} + s_{21,2}$ & $s_{21,1} + s_{12,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{24}$ & $s_{11,1} + s_{22,2}$ & $s_{21,1} + s_{12,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 2 \\
$\rho_{34}$ & $s_1$ & $s_{22,1} + s_{21,2}$ & \text{irr} & $\log 2$ & 0 & 2 \\
$\rho_{123}$ & $s_{12,1} + s_{21,2}$ & $s_{11,1} + s_{22,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{132}$ & $s_{21,1} + s_{11,2}$ & $s_{12,1} + s_{22,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{142}$ & $s_{22,1} + s_{11,2}$ & $s_{21,1} + s_{12,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{134}(\simeq \rho_{142})$ & $s_{21,1} + s_{12,2}$ & $s_{22,1} + s_{11,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{143}$ & $s_{22,1} + s_{12,2}$ & $s_{11,1} + s_{21,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{234}$ & $s_{11,1} + s_{21,2}$ & $s_{22,1} + s_{12,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{243}(\simeq \rho_{123})$ & $s_{11,1} + s_{22,2}$ & $s_{12,1} + s_{21,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{1234}(\simeq \rho_{243})$ & $s_{12,1} + s_{21,2}$ & $s_{12,1} + s_{21,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 2 \\
$\rho_{1324}(\simeq \rho_{12})$ & $s_{21,1} + s_{11,2}$ & $s_{12,1} + s_{21,2}$ & \text{irr} & $\log 2$ & 0 & 2 \\
$\rho_{1342}(\simeq \rho_{34})$ & $s_{21,1} + s_{11,2}$ & $s_{22,1} + s_{12,2}$ & \text{red} & $\log 2$ & $\log 2$ & 4 \\
$\rho_{1423}(\simeq \rho_{34})$ & $s_{22,1} + s_{12,2}$ & $s_1$ & \text{irr} & $\log 2$ & 0 & 2 \\
$\rho_{1432}(\simeq \rho_{13})$ & $s_{22,1} + s_{11,2}$ & $s_{12,1} + s_{21,2}$ & \text{irr} & $\log 2$ & $\log 2$ & 2 \\
$\rho_{(12)(34)}(\simeq \rho_{(13)(24)})$ & $s_{12,1} + s_{11,2}$ & $s_{22,1} + s_{21,2}$ & \text{out} & 0 & 0 & 1 \\
$\rho_{(13)(24)}$ & $s_2$ & $s_1$ & \text{out} & 0 & 0 & 1 \\
$\rho_{(14)(23)}(\simeq \text{id})$ & $s_{22,1} + s_{21,2}$ & $s_{12,1} + s_{11,2}$ & \text{inn} & 0 & 0 & 1 \\
\hline
\end{tabular}
\end{table}

3 Further comments

The fourth column, inn, irr, red and out stand for “inner automorphism”, “irreducible endomorphism”, “reducible endomorphism” and “outer automorphism”, respectively, as endomorphisms of $O_2$. Certainly the (minimal) index is
related to the Pimsner-Popa entropy [25]. From this point of view, it is quite intriguing that the inequality in [26, Theorem 2.1] is completely analogous to the one already known for the index [23] (see also [4], bottom of p.373), thus suggesting the existence of some relation between (square root of) index and topological entropy, cf. [27, Section 10]. Notice that the index pertains to von Neumann algebras while the topological entropy refers to $C^*$-algebras, so it would probably make sense also to investigate Watatani’s notion of index [30] in a purely $C^*$-framework to spell out more details.

It is also worth pointing out that in the situation under consideration both the quantities $\text{ht}(\rho_{\sigma})$ and $\text{ht}(\rho_{\sigma}|_{\mathcal{D}_2})$ studied in [26] attain only two values, 0 and $\log 2$. Which value occurs depends on whether the given endomorphism or its restriction to the diagonal is actually an automorphism or not, a fact that certainly deserves more thorough investigations. Indeed, as shown in [26], $\text{ht}(\rho_{\sigma}) = 0$ if and only if $\rho_{\sigma}$ is an automorphism of $\mathcal{O}_2$. Furthermore, the four proper endomorphisms $\rho_{\sigma}$ with $\text{ht}(\rho_{\sigma}|_{\mathcal{D}_2}) = 0$ are precisely those four proper endomorphisms which restrict to automorphisms of the diagonal $\mathcal{D}_2$. This is easily verified by the method developed in [5, 28] (see Theorem 3.4, Corollary 4.6 and Subsection 5.1 in [5]). Also note that the four automorphisms of $\mathcal{O}_2$ appearing in Table 1 all are of finite order (order two) and thus their topological entropy is 0 [29, Proposition 4.2]. It is shown in [5, Subsection 5.2] that all rank-three automorphisms of $\mathcal{O}_2$ (and there are exactly 48 of them) have also finite orders. The first infinite order automorphisms appear in rank-four [5, Subsection 5.3] and it would be interesting to determine their topological entropies. In the case of $\mathcal{O}_n$ with $n \geq 3$ infinite order automorphisms appear already at rank two and the question of determining their entropies remains open too.

Regarding index computations, for a fixed value of $n$, say $n = 2$ in our case, the next step of complexity would be to look at the $2^3!$ rank-three permutation endomorphisms. Among those, the ones providing genuine automorphisms of $\mathcal{O}_2$ or $\mathcal{D}_2$ have been completely classified in [5]. Still, one can not exclude the possibility that some proper endomorphisms of $\mathcal{O}_2$ extend to automorphisms of the associated factor. Some examples of this kind were announced for the case $n = 3$ and $k = 2$ in [1].

\footnote{The irreducibility of such endomorphisms is automatic [5, Proposition 1.1].}
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