On the physics of cold MHD winds from oblique rotators.

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Received date today; accepted date yesterday

Abstract. I show that the self-consistent solution of the problem of MHD plasma flow in magnetosphere of an oblique rotator with an initially split-monopole magnetic field is reduced to the solution of the similar problem for the axisymmetric rotator. All properties of the MHD cold plasma flows from the axisymmetric rotators with the initially split-monopole magnetic field are valid for the oblique rotators as well. Rotational losses of the oblique rotator do not depend on the inclination angle and there is no temporal evolution of this angle. Self-consistent analytical and numerical solutions for the axisymmetric plasma flows obtained earlier show that the rotators can be divided on fast rotators (σ_0/U_0^2 > 1) and slow rotators (σ_0/U_0^2 < 1), where σ_0 is the ratio of the Poynting flux to the matter energy flux in the flow at the equator on the surface of the star, U_0 = γ_0v_0/c, v_0 and γ_0 being the initial velocity and the Lorentz-factor of the plasma. The self-consistent approximate analytical solution for the plasma flow from the oblique rotator is obtained under the condition σ_0/U_0^2 ≪ 1. Implications of these results for radio pulsars are discussed. In particular, I argue that all radio pulsars are apparently the slow rotators ejecting the Poynting dominated relativistic wind.

Key words: MHD – pulsars; jets and outflows

1. Introduction

An analysis of the relativistic plasma flow is necessary for understanding the processes taking place in radio pulsar’s magnetospheres, compact galactic objects and in AGN’s (Arons 1996; Mirabel 1996; Pelletier et al. 1996). In the present paper, as in previous one (Bogovalov 1997), we concentrate our attention totally on the problem of the relativistic plasma flow in the conditions typical for radio pulsars.

In spite of systematic research in the field of the physics of radio pulsars in the last years, the structure of the magnetosphere and the mechanisms for the acceleration of plasma in these objects to large extent remain vague (Lyubarsky 1995). One of the most important unsolved problem of the physics of radio pulsars is the problem of energetics of the ejected wind of relativistic plasma. For example, the kinetic energy of the plasma accelerated in the inner magnetosphere of the Crab pulsar is not sufficient to explain the energetics of the relativistic wind exiting the synchrotron nebula surrounding the pulsar (Arons 1996).

The basis of the theory of the plasma production in radio pulsar magnetospheres was initiated by Sturrock (1974). This theory was developed in more detail for different conditions on the stellar surface by Ruderman & Sutherland (1975) and by Arons (1981). Primary electrons are accelerated in the so called “electrostatic gaps” and produce dense relativistic plasma of secondary particles with Lorentz-factor γ_0 ∼ 10^2 – 10^3. This plasma screens the accelerating electric field and limits the potential drop by the value γ_{gap}mc^2 with γ_{gap} ∼ 10^7. The total flux of the kinetic energy of all particles appears much less than the total rotational losses of the fast rotating radio pulsars due to this limitation. Almost all energy is carried out by the electromagnetic field. The wind with similar characteristics is formed in the outer gap model (Cheng, Ho & Ruderman 1986). The wind of relativistic plasma from radio pulsars can be characterized by the ratio of the Poynting flux to the flux of the kinetic energy of the plasma σ. The plasma is Poynting flux dominated when σ > 1 and is kinetic energy dominated if σ < 1. Electrostatic gaps give σ > 1000 for the Crab pulsar. At the same time interpretation of observations of the Crab Nebula compels us to conclude that this ratio at large distances from the pulsar is very small. This conclusion is based on the assumption that the subsonic flow of the plasma terminated by the shock wave at the interaction of the wind with the interstellar medium can be considered as the flow of ideal plasma with the only dissipative process of synchrotron cooling. Under this assumption the observed expansion of the outer edge of the Crab Nebula, the synchrotron and TeV gamma-ray emission of the nebula produced via Inverse Compton Scattering of the relativistic particles on 2.7-K MBR emission can be explained under unique choice σ = 3 · 10^{-3} (Atoyan & Aharonian 1996).
Recently Begelman (1998) has revisited the key assumption of the theory by Kennel & Coroniti (1984). He argued that even a wind with $\sigma \sim 1$ in the pre-shock region can give the observable properties of the Crab Nebula if one takes into account dissipative processes in the nebula. It follows from the theory of relativistic MHD shocks that a wind with $\sigma \sim 1$ in the pre-shock region creates the flow in the post-shock region with $\sigma > 1$. But this flow must be strongly unstable. The instability provides an effective transformation of the magnetic field into the kinetic energy of the plasma accompanied by acceleration of particles in the nebula. It is important that in both (Kennel & Coroniti or Begelman) scenario it follows that there exists some unspecified mechanism for the transformation of the Poynting flux into the flux of the kinetic energy as the plasma travels from the star to infinity. This mechanism is apparently the basic mechanism for the plasma acceleration since it ensures the transformation of at least 50% (in the Begelman scenario) or 99.7% (in the Kennel & Coroniti model) of the rotational energy of the neutron star into the kinetic energy of the relativistic wind.

One of the possible mechanisms of the acceleration is the magnetic acceleration of the plasma by the rotating magnetosphere. Unfortunately this mechanism appeared non effective for the axisymmetric rotators with typical pulsar’s parameters. No effective acceleration was found neither in nearest zone (Bogovalov 1997), nor in far zone of the rotator (Bogovalov & Tsinganos 1999). At the same time up to now there is no solution describing the MHD plasma flow from the oblique rotator. Usually it is believed that the inclination of the magnetic moment of the star to the axis of rotation could be important for the plasma acceleration. In this paper we study this possibility in the model of the oblique rotator with the initially split-monopole magnetic field on the surface of the star.

2. Basic assumptions

The axis of rotation of real radio pulsars is not directed along the magnetic moment. The solution of the problem of the plasma flow in magnetosphere of this object is extremely complicated. To simplify the problem several models of radio pulsars were proposed. The sequence of these models is presented in Fig. 1. Firstly Goldreich & Julian (1969) proposed axisymmetrically rotating star with initially dipole magnetic field (step 1 in Fig. 1). The rotational losses of the axisymmetrically rotating star ejecting relativistic plasma are comparable with the rotational losses of the oblique dipole in vacuum. The energy of rotation is carried out near the surface of the axisymmetric rotator by the Poynting flux. This is why this model can be used for study of the process of the Poynting flux transformation into the kinetic energy of the plasma.

However even with such simplification the structure of the axisymmetric flow of the plasma from the star with the initially dipole magnetic field appears too complicated. There were a lot of attempts to solve this problem in massless approximation (Michel 1973; Beskin, Gurevich & Istomin 1983; Lyubarskii 1990; Contopoulos, Kazanas & Fendt 1999). To simplify the problem Michel (1963) was the first who used the rotator with the prescribed split-monopole poloidal magnetic field for the investigation of the relativistic plasma flow in the magnetosphere of the axisymmetrical rotator (step 2). It follows from Fig. 1 that there are no closed field lines in the split-monopole model. All lines go to infinity. This allows one to remarkably simplify the analysis of the flow.

Michel (1963) considered the flow of the plasma in the prescribed split-monopole magnetic field. The affect of the moving plasma on the poloidal magnetic field was not taken into account. In the self-consistent solution the plasma and the electromagnetic field affect each other. The problem of the self-consistent plasma flow from the rotator with the initially split-monopole magnetic field was investigated in nonrelativistic and relativistic limits in the papers by Sakurai (1985), Bogovalov (1992), Bogovalov & Tsinganos (1999). The phrase “initially split-monopole magnetic field” means that the normal component of the magnetic field on the surface of the star is constant but changes sign on the magnetic equator. In this paper we firstly consider the model of the oblique rotator with the split-monopole magnetic field on the surface of the star (step 3). This model allows one to investigate the plasma flow in conditions more typical for real radio pulsars than it occurs in the axisymmetric model and to connect the model with the split-monopole magnetic field with the real pulsars (step 4).

Since the density of the relativistic plasma produced in the pulsar magnetosphere is high enough to screen the electric field parallel to the magnetic field, the magnetohydrodynamical approximation is used in this paper to describe the flow of the plasma. The plasma is considered as cold to a first approximation (Lyubarsky 1995).

3. Reduction of the oblique rotator problem to the axisymmetrical problem

System of time dependent equations defining the temporal evolution of the flow of the relativistic plasma (Akhiezer et al. 1975)

\[ m \pi q, \quad \int H \cdot \mathbf{v} = q \cdot \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{H}, \]  

\[ \frac{\partial \mathbf{H}}{\partial t} = \text{curl} \mathbf{E}, \]  

\[ \text{curl} \mathbf{H} = 4 \pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}, \]  

\[ \text{div} \mathbf{H} = 0, \]  

\[ \text{div} \mathbf{E} = 4 \pi q, \]  

(1)

(2)

(3)

(4)

(5)
with step 4 connecting this model with real pulsars. Step 1 is the transition from the oblique rotator with a dipole magnetic field to the axisymmetric rotator. Step 2 is the transition from the axisymmetric rotator with the split-monopole magnetic field to the axisymmetric rotator with the dipole magnetic field. In this paper the transition from the oblique rotator with the dipole magnetic field to the axisymmetric rotator. In Fig. 1. The sequence of models introduced to simplify the solution of the problem of the plasma outflows from radio pulsars.

\[ \frac{\partial E}{\partial t} + \nabla \cdot E = 0, \]

\[ \frac{\partial \gamma v_\theta}{\partial t} + (\mathbf{v} \cdot \nabla) v_\theta - \frac{\gamma (v_\theta^2 + v_\varphi^2)}{r} = \]

\[ q \cdot E_\varphi + \frac{1}{4\pi} \left\{ \frac{1}{r \sin \theta} \frac{\partial H_r}{\partial \varphi} - \frac{1}{r} \frac{\partial (r H_\varphi)}{\partial r} \right\} H_r - \]

\[ - \frac{1}{r} \frac{\partial (r H_\theta)}{\partial r} H_\varphi + \frac{1}{r \sin \theta} \frac{\partial H_\varphi}{\partial \theta} + \frac{1}{c} \left( H_\varphi \frac{\partial E_\varphi}{\partial t} - H_r \frac{\partial E_\theta}{\partial t} \right), \]

\[ \frac{\partial H_r}{\partial t} = - \frac{1}{r \sin \theta} \left( \frac{\partial (\sin \theta E_\varphi)}{\partial \varphi} - \frac{\partial E_\theta}{\partial \varphi} \right), \]

\[ \frac{\partial H_\theta}{\partial t} = - \left( \frac{\partial (r E_\varphi)}{\partial r} - \frac{\partial E_\varphi}{\partial \theta} \right), \]

\[ \frac{\partial H_\varphi}{\partial t} = - \left( \frac{\partial (r E_\theta)}{\partial r} - \frac{\partial E_\varphi}{\partial \theta} \right). \]

The laws of conservation of the magnetic field and the matter flux, together with Coulomb law take a form

\[ \frac{1}{r^2} \frac{\partial (r^2 H_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \sin \theta H_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial H_\varphi}{\partial \varphi} = 0. \]

\[ E_r + \frac{1}{c} (v_\theta H_\varphi - v_\varphi H_\theta) = 0, \]

\[ E_\varphi + \frac{1}{c} (v_\varphi H_r - v_r H_\varphi) = 0, \]

\[ E_\varphi + \frac{1}{c} (v_\varphi H_\theta - v_\theta H_r) = 0. \]

In this system \( q \) is the induced space electric charge density, \( \theta \) is the polar angle and \( \varphi \) is the azimuthal angle.

Boundary conditions on the surface of the star with radius \( R_* \) for the system of equations above are:

1. The Lorenz-factor \( \gamma_0 \) is specified constant;
2. The normal component of the density of the matter flux is specified constant and uniform;
3. The normal component of the poloidal magnetic field \( H_0 \) does not depend on coordinates and time in every point of the surface of the star and changes sign on the magnetic equator.
4. The tangential component of the electric field is continuous on the star surface;
5. It is assumed that the velocity of the plasma exceeds fast magnetosonic velocity (the flow is super sonic) at the infinity. This system of equations together with the boundary conditions describes the stationary axisymmetric flow as well.

Let’s assume that the self-consistent solution for the problem of the plasma flow from the axisymmetric rotator with an initially monopole-like poloidal magnetic field is specified. The phrase “initially monopole-like magnetic
Let’s introduce the velocity sume that this perturbed self-consistent solution is known.

is symmetric flow (Bogovalov & Tsinganos 1999). We as-

field are perturbed by rotation in the self-consistent ax-

the case of the oblique rotator as well.

in the lower hemisphere. Similar operation can be done in

magnetic field by formal reverse sign of the magnetic fields

can be obtained from the solution for the monopole-like

sheet. It is obvious that the solution for the split-monopole

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in Fig. 2. There is no current sheet in this flow in con-

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model with the monopole-like magnetic field all field lines

like magnetic field differs from the split-monopole mag-

solutions for more realistic plasma flows. The monopole-

magnetic fields can not be created in reality, this rather

magnetic field on the surface of the star is constant and

field” means that the normal component of the poloidal

magnetic field on the surface of the star is constant and
does not change sign on the magnetic equator. Actually
this field has no magnetic equator at all. Although such
magnetic fields can not be created in reality, this rather
artificial mathematical model is convenient to construct
solutions for more realistic plasma flows. The monopole-
like magnetic field differs from the split-monopole mag-
netic field only by direction of the field lines in one of
hemispheres and was introduced by Michel [1966]. In the
model with the monopole-like magnetic field all field lines
are out coming from the surface of the star as it is shown in
Fig. 2. There is no current sheet in this flow in con-
trast to the model with the split-monopole magnetic field
which contains the current sheet in the equatorial plane.
The current sheet is a contact discontinuity (Landau &
Lifshitz 1963) in ideal MHD approximation. The magnetic
field changes direction at the passage through the current
sheet. It is obvious that the solution for the split-monopole
can be obtained from the solution for the monopole-like
magnetic filed by formal reverse sign of the magnetic fields
in the lower hemisphere. Similar operation can be done in
the case of the oblique rotator as well.

The flow and the monopole-like poloidal magnetic
field are perturbed by rotation in the self-consistent ax-

isymmetric flow (Bogovalov & Tsinganos 1999). We as-
sume that this perturbed self-consistent solution is known.
Let’s introduce the velocity \( \mathbf{V}(r) \), the poloidal \( \mathbf{B}_p(r) \) and
toroidal \( B_\varphi(r) \) magnetic fields and the density of the
plasma \( n(r) \) in the self-consistent axisymmetric flow with
the initially monopole-like magnetic field. The poloidal
electric field \( \mathbf{E} \) and the poloidal magnetic field are con-
nected in the axisymmetric flow \( \mathbf{E} = (r \sin \theta \Omega/c) \mathbf{B}_p \times \mathbf{e}_\varphi \)
(Weber & Davis 1967), where \( \mathbf{e}_\varphi \) is the unit vector di-
rected in the azimuthal direction. The toroidal electric
field is equal to 0. These variables depend only on coor-
dinates. Dependence on time is absent for the stationary
axisymmetric flow. This solution satisfies automatically
all the equations of the system \( \text{[1][8]} \) and corresponding
boundary conditions.

Now we show how the self-consistent solution for the
plasma flow in the magnetosphere of the oblique rotator
with the initially split - monopole magnetic field can be
obtained from the known solution for the axisymmetric
problem. Let’s consider the following transformation of
the axisymmetric solution. The velocity and the density
are taken the same as for the axisymmetric rotator

\[ \mathbf{v}(r, t) = V(r), \quad n(r, t) = N(r). \]  \( \text{(19)} \)

The poloidal magnetic field \( \mathbf{H}_p(r, t) \) for the oblique
rotator is obtained from the poloidal magnetic field of
the axisymmetric rotator as follows

\[ \mathbf{H}_p(r, t) = \eta(r, t)\mathbf{B}_p(r). \]  \( \text{(20)} \)

The same procedure gives us the toroidal magnetic field

\[ H_\varphi(r, t) = \eta(r, t)B_\varphi(r), \]  \( \text{(21)} \)

where \( \eta(r, t) \) is unknown function to be specified. The
poloidal electric field \( E \) is defined as in the axisymmet-
ric case

\[ E_\theta = -\frac{r \sin \theta \Omega}{c} H_r, \]  \( \text{(22)} \)

\[ E_r = \frac{r \sin \theta \Omega}{c} H_\theta, \]  \( \text{(23)} \)

and the toroidal electric field \( E_\varphi = 0 \). It is easy to show
that this is indeed the solution of the problem for the
oblique rotator and to specify the function \( \eta \).

The frozen-in conditions \( \text{[16][18]} \) are fulfilled automatic-
ly for the solution \( \text{[1][12][23]} \). At the stationary rotation
the dependence of \( \eta \) on \( \varphi \) and \( t \) can be presented by one
variable \( \xi = \varphi - \Omega t \) in the spherical system of coordinates.
Thus, \( \eta \) depends on three variables \( r \), \( \theta \) and \( \xi \). Then the
induction equation \( \text{[13]} \) is reduced to the equation

\[ \frac{\partial}{\partial \xi} \left( \frac{E_\theta}{r \sin \theta} + \frac{\Omega H_r}{c} \right) = 0 \]  \( \text{(24)} \)

which is fulfilled due to condition \( \text{[22]} \). The same is valid
for the induction equation \( \text{[11]} \) which is reduced to the equa-
tion

\[ \frac{\partial}{\partial \xi} \left( \frac{E_r}{r \sin \theta} - \frac{\Omega H_\theta}{c} \right) = 0 \]  \( \text{(25)} \)

which is satisfied due to condition \( \text{[23]} \). Third induction
equation \( \text{[12]} \) after substitution of conditions \( \text{[22][23]} \) is re-
duced to the flux conservation condition \( \text{[14]} \) which takes a
form

\( (\mathbf{B} \cdot \nabla) \eta = 0. \)  \( \text{(26)} \)

since the total magnetic field \( \mathbf{B} \) of the axisymmetric flow
satisfies the equation \( \nabla \cdot \mathbf{B} = 0 \). It implies that function
\( \eta \) is constant on the field line of the magnetic filed of the
axisymmetric flow.

Fig. 2. The solution in the model with the split-monopole
magnetic field can be obtained from the solution for the
monopole-like magnetic field by change of sign of the mag-
netic field in lower hemisphere. The same is valid for re-
verse operation. There is a current sheet at the equatorial
plane in the split-monopole model which is a contact dis-
continuity in ideal MHD approximation.

\( \text{[42x-3425]} \text{let’s introduce the velocity sume that this perturbed self-consistent solution is known.} \)

\( \text{[42x-3306]} \text{is symmetric flow (Bogovalov & Tsinganos 1999). We as-} \)

\( \text{[42x-3186]} \text{field are perturbed by rotation in the self-consistent ax-} \)

\( \text{[42x-3068]} \text{the case of the oblique rotator as well.} \)

\( \text{[42x-2924]} \text{in the lower hemisphere. Similar operation can be done in} \)

\( \text{[42x-2805]} \text{magnetic field by formal reverse sign of the magnetic fields} \)

\( \text{[42x-2685]} \text{can be obtained from the solution for the monopole-like} \)

\( \text{[42x-2567]} \text{Lifshitz 1963) in ideal MHD approximation. The magnetic} \)

\( \text{[42x-2209]} \text{The current sheet is a contact discontinuity (Landau &} \)

\( \text{[42x-1971]} \text{contrast to the model with the split-monopole magnetic field} \)

\( \text{[42x-1852]} \text{in Fig. 2. There is no current sheet in this flow in con-} \)

\( \text{[42x-1732]} \text{are out coming from the surface of the star as it is shown} \)

\( \text{[42x-1614]} \text{model with the monopole-like magnetic field all field lines} \)

\( \text{[42x-1375]} \text{solutions for more realistic plasma flows. The monopole-} \)

\( \text{[42x-1256]} \text{like magnetic field differs from the split-monopole mag-} \)

\( \text{[42x-1137]} \text{magnetic fields can not be created in reality, this rather} \)

\( \text{[42x-899]} \text{magnetic field on the surface of the star is constant and} \)

\( \text{[42x-541]} \text{field” means that the normal component of the poloidal} \)

\( \text{[42x-422]} \text{continuity in ideal MHD approximation} \)

\( \text{[42x102]} \text{plane in the split-monopole model which is a contact dis-} \)

\( \text{[42x221]} \text{verse operation. There is a current sheet at the equatorial} \)

\( \text{[42x341]} \text{netic field in lower hemisphere. The same is valid for re-} \)

\( \text{[42x459]} \text{monopole-like magnetic field by change of sign of the mag-} \)

\( \text{[42x579]} \text{magnetic field can be obtained from the solution for the} \)

\( \text{[42x591]} \text{Fig. 2.} \)

\( \text{[42x718]} \text{S.V.Bogovalov: On the physics of cold MHD winds from oblique rotators.} \)

\( \text{[42x3437]} \)
is convenient to introduce function

\[ \eta \]

the points where the field lines come into the surface. It goes out from the surface of the star and is equal to the points on the surface of the star where the field line boundary conditions that the function \( \eta \) is.

The angle \( \chi \) with the sign of the product \( e \cdot B \) is the magnetic moment \( M \) used in equation (29).

Therefore the general solution is

\[ \eta(r, \theta, \varphi, t) = D \left( \sin(\chi) \sin(\theta - \int_{R_1}^{R_2} \frac{B_\theta dr}{r B_r}) \times \sin(\varphi - \int_{R_1}^{R_2} \frac{B_\varphi dr}{r B_r} - \Omega t) + \cos(\theta - \int_{R_1}^{R_2} \frac{B_\theta dr}{r B_r}) \cos \chi \right). \]

(34)

It follows from this solution that \( \eta^2 = 1 \) and \( \eta \) changes sign when the magnetic field changes direction. It is easy to show now that equations of motion (23-25) are also satisfied for the solution \( \eta \) defined by (34). Notice that on the left hand side of these equations there is no function \( \eta \). In the right hand side of the equations of motion function \( \eta \) comes in combination \( A_\eta \frac{\partial B_k}{\partial x_i} \),

where \( A_\eta \) and \( B_k \) are arbitrary components of fields of the axisymmetric solution, \( x_i \) spatial or time coordinates in 4-space. This relationship can be presented as

\[ A_\eta \frac{\partial \eta B_k}{\partial x_i} = \eta^2 A_\eta \frac{\partial B_k}{\partial x_i} + A_\eta B_k \frac{1}{2} \frac{\partial \eta^2}{\partial x_i} = A_\eta \frac{\partial B_k}{\partial x_i}. \]

(35)

Therefore, the function \( \eta \) disappears in the equations of motion. Here we ignore the difference in the dynamics of the current sheet and the surrounding plasma assuming that the current sheet is the mathematic discontinuity as usual in ideal MHD. This assumption can be violated for the oblique rotators at large distance from the star. But at large distances the dynamics of the current sheet can be considered particularly in WKB approximation (Coroniti & Tsinganos 1999).

Thus, we obtain the self-consistent solution for the oblique rotator from the known self-consistent solution for the axisymmetric rotator. The sketch demonstrating the structure of the cold wind from the oblique rotator is presented in Fig. 3. The structure of the plasma flow is symmetric in relation to the equator. The form of the poloidal field lines is the same as for the axisymmetric rotator. In general there is collimation of the plasma flow to the axis of rotation, although the effect of the collimation depends on the parameters of the problem (Bogovalov & Tsinganos 1999). In the axisymmetric flow the current sheet dividing the magnetic fluxes of opposite directions is located on the equator. In the wind from the oblique rotator the current sheet takes a form of a wave. In the poloidal plane the poloidal magnetic field lines change direction on the current sheet. At first glance it seems that this behavior contradicts to the magnetic field freezing.

The bottom panel of Fig. 3 shows the structure of the field lines in the equatorial plane. It is seen that there is no contradiction with the magnetic flux freezing since the total magnetic field depends on the azimuthal angle \( \varphi \). The velocity and the density of the plasma do not depend on \( \chi \) and \( \varphi \) and are the same as for the axisymmetric rotator.

Fig. 3. The initial position of the star and the geometry used in equation (29).

Fig. 3 shows the initial position of the star with the magnetic moment \( \mathbf{M} \) inclined to the axis of rotation at the angle \( \chi \) at \( t = 0 \). It follows from equation (24) and the boundary conditions that the function \( \eta \) is.

The sign of the function \( \eta \) on the surface of the star varies with the sign of the product \( (e \cdot e_M) \), where \( e \) is the unit vector directed to the point on the surface of the star, \( e_M \) is the unit vector directed along the magnetic moment. This product can be presented as

\[ (e \cdot e_M) = \sin \chi \sin \theta \sin \varphi + \cos \chi \cos \theta. \]

(28)

Then, on the surface of the star, the function \( \eta \) is

\[ \eta(\theta, \varphi - \Omega t) = D(\sin \chi \sin \theta \sin(\varphi - \Omega t) + \cos \chi \cos \theta). \]

(29)

Actually this is the boundary condition to equation (26).

The equation can be presented in the form

\[ B_r \frac{\partial \eta}{\partial r} + B_\theta \frac{\partial \eta}{\partial \theta} + B_\varphi \frac{\partial \eta}{\partial \varphi} = 0. \]

(30)

Equations of characteristics for this equation are

\[ \frac{dr}{B_r} = \frac{r \sin \theta d\varphi}{B_\varphi} \]

(31)

and

\[ \frac{dr}{B_r} = \frac{r d\theta}{B_\theta}. \]

(32)

Therefore the general solution is

\[ \eta(r, \theta, \varphi, t) = f(\theta - \int_{R_1}^{R_2} \frac{B_\theta dr}{r B_r}, \varphi - \int_{R_1}^{R_2} \frac{B_\varphi dr}{r \sin \theta B_r}), \]

(33)
Istomin (1983). The evolution of the inclination angle as it happens in the solution by Beskin, Gurevich & Goldwire 1970, Davis & Goldstein 1970) or disalignment (Ostriker & Gunn 1969, Pacini 1968, Michel along the axis of rotation as it happens in vacuum approximation (Ostriker & Gunn 1969, Pacini 1968, Michel & Goldwire 1970) Davis & Goldstein 1970) or disalignment as it happens in the solution by Beskin, Gurevich & Istomin (1983). The evolution of the inclination angle \( \chi \) is defined by torques on the surface of the star (Michel & Goldwire 1970)

\[
L_y = \int \left[ T_{r\theta} \cos \varphi - T_{r\varphi} \cos \theta \sin \varphi \right] d^2 S, \tag{36}
\]

\[
L_x = -\int \left[ T_{r\theta} \cos \varphi - T_{r\varphi} \cos \theta \sin \varphi \right] d^2 S, \tag{37}
\]

and

\[
L_z = \int T_{r\varphi} \sin \theta \ d^2 S, \tag{38}
\]

where \( T_{r\theta} \) and \( T_{r\varphi} \) are the components of the energy-momentum tensor \( T_{r\theta} \) and integration in (37) is performed over the surface of the star. The energy-momentum tensor is also biquadratic on the components of the electromagnetic field (Landau & Lifshitz 1975). Therefore it does not depend on the function \( \eta \) and the angles \( \chi \) and \( \varphi \). The torques for the oblique rotator are the same as for the axisymmetric rotator. Integration in (37) over the surface of the star gives \( L_x = L_y = 0 \). There are no torques which can change the inclination angle of the oblique rotator with the initially split-monopole magnetic field. This is not totally unexpected conclusion. The torques on the star rotating in vacuum differs from the torque on the start ejecting ideal plasma. The torque on the star rotating in vacuum always align the rotational and magnetic axes (Soper 1972). The temporal evolution of the inclination angle \( \chi \) of the star ejecting plasma depends on the distribution of the magnetic field on the surface of the star. It was shown by perturbation theory in first order approximation on \( \Omega \) that magnetic and rotation axes tend to align when the magnetic flux is more concentrated to the magnetic poles and evolve to \( \chi = 90^\circ \) when there is less flux density at the poles than at the equator (Mestel & Selley 1970). Thus, the result obtained here for arbitrary angular velocity of the rotator with the split-monopole magnetic field is due to combination of two conditions: by the ejection of the plasma simultaneously with the uniform magnetic flux distribution on the surface of the star.

4. The flow of the plasma at \( \frac{v_0^3}{U_0} \ll 1 \).

The solution of the problem of the cold plasma flow from the rotator with the initially split-monopole magnetic field is defined by the following parameters and variables: \( H_0 \), \( mn_0 \), \( v_0 \), \( \Omega \), c, r, \( \theta \), \( \varphi \), \( R_\ast \), \( \chi \) and \( t \). Lower index “0” denotes the values on the surface of the star. Since the dependence on \( \chi \), \( \varphi \) and \( t \) is known, we can consider only the dependence of the solution for the axisymmetric flow on other 8 parameters. There are 3 parameters in this set with independent dimension. Therefore, according to the theory of dimensions (Barenblatt 1979) the solution in dimensionless variables depends only on 5 parameters. Let us write this dependence as

\[
A = f(R_{\ast}/a, \alpha_1, \alpha_2, r/a, \theta). \tag{39}
\]
Here \( \mathbf{A} \) is the vector of the state of the plasma including density, velocity and magnetic field, \( \mathbf{f} \) is the unknown vector-function and \( \alpha \) is some parameter with dimension of length. It is reasonable to consider limiting case when the dimensionless radius of the star goes to zero. It allows one to decrease the number of variables in the solution on one. Now the solution crucially depends only on two parameters and two variables in the dimensionless presentation.

Let’s measure the velocity of the plasma in units of the initial velocity \( v_0 \) and all geometrical variables measure in the initial radius of the fast mode surface (FMS) \( r_f \). The FMS is the surface where the poloidal velocity of the plasma equals the local velocity of the fast mode MHD perturbations. In the comoving coordinate system where the electric field is equal to zero, this condition takes a perturbations. The presentation is justified for arbitrary parameter \( \epsilon \neq 0 \). It was shown that at the limit under consideration the function \( f_1^{(0)}(\epsilon, \mathbf{X}) \) follows in the ordinary dimensionless variables (Bogovalov, 1997)

\[
U_r = U_0, \quad U_\varphi = 0, \quad U_\theta = 0, \quad n(r) = n_0 \left( \frac{r_0}{r} \right)^2. \tag{43}
\]

\[
H_\varphi = -\left( \frac{r \Omega \sin \theta}{c} \right) H_p, \quad E = \left( \frac{r \Omega \sin \theta}{c} \right) H_p, \tag{44}
\]

and

\[
H_r = \begin{cases} 
H_0 \left( \frac{\Omega}{r} \right)^2, & \text{if } \theta \leq \pi/2 \\
-H_0 \left( \frac{\Omega}{r} \right)^2, & \text{if } \theta > \pi/2
\end{cases} \quad \text{when } \theta = 0. \tag{45}
\]

This is exactly the solution obtained by Michel (1973a) for the massless plasma. It is easy to show that the full system of the equations of motion is satisfied for this solution except the frozen-in condition \( \mathbf{E} \times \mathbf{B} = 0 \). The residual in this condition on the FMS is

\[
\delta v_\varphi = \epsilon \left( \frac{\alpha/\epsilon}{1 + v_0/c} \right). \tag{46}
\]

The residual shows that the corrections to the solution are indeed proportional to powers of \( \alpha \) and can be neglected in the region limited by the FMS provided that \( \alpha \ll 1 \). This residual also shows that expansion \( \mathbf{f}_1^{(0)}(\epsilon, \mathbf{X}) \) contains terms \( \left( \epsilon/\alpha \right)^k \), where \( k \) is integer number. Therefore the first term in expansion \( \mathbf{f}_1^{(0)}(\epsilon, \mathbf{X}) \) gives incorrect result in the limit \( \alpha \to 0, \epsilon \to 0 \) provided that \( \alpha/\epsilon = \text{const} \) which corresponds to the slow rotation of the star at arbitrary velocity \( v_0 \). Mathematically it means that the point \( \left( \alpha = 0, \epsilon = 0 \right) \) is a particular point of the expansion. The terms \( \left( \epsilon/\alpha \right)^k \) can be eliminated from expansion \( \mathbf{f}_1^{(0)}(\epsilon, \mathbf{X}) \) if to define the function

\[
g_1(\alpha/\epsilon, \mathbf{X}) = \lim_{\alpha \to 0} \frac{A_1(\alpha, \epsilon, \mathbf{X})}{f_1^{(0)}(\epsilon, \mathbf{X})}. \tag{47}
\]

Then the expansion can be presented as

\[
A_1 = f_1^{(0)}(\epsilon, \mathbf{X}) \left( g(\alpha/\epsilon, \mathbf{X}) + \sum_{k=1} \alpha^k g_1^{(k)}(\epsilon, \mathbf{X}) \right). \tag{48}
\]

The expansion \( \sum_{k=1} \alpha^k g_1^{(k)}(\epsilon, \mathbf{X}) \) already does not contain terms \( \left( \epsilon/\alpha \right)^k \) and function \( F_{\alpha} = f_1^{(0)}(\epsilon, \mathbf{X}) \left( g(\alpha/\epsilon, \mathbf{X}) \right) \) gives correct first term at \( \alpha \to 0 \) for arbitrary parameter \( \epsilon \) including point \( \epsilon = 0 \). This function was calculated analytically by Bogovalov (1992). The corrected solution of the problem in the limit \( \alpha \to 0 \) at arbitrary \( \epsilon \) coincides
with the solution with exception of the toroidal magnetic field. It is replaced by
\[ H_\varphi = -\left(\frac{r\Omega \sin \theta}{v_0}\right)H_r. \]  \tag{49}

According to this solution the plasma moves radially in the split-monopole poloidal magnetic field with constant poloidal velocity and without any toroidal motion. At large distances the collimation of the plasma to the axis of rotation of the slow rotators is located below the line at the increase of \( \sigma \). All the pulsars are located in the region of slow rotators \( \sigma < 1 \) as the fast rotators eject the Poynting flux dominated wind provided that \( U_0 > 200 \).

The rotators with the initially relativistic plasma flow firstly cross the line \( \sigma_0 = 1 \) at the increase of \( \sigma_0 \) at constant \( U_0 \) and became the slow rotators ejecting the Poynting flux dominated wind. At the increase of \( \sigma \) they also cross the line \( \sigma = U_0^2 \) and become the fast rotators. The relativistic stationary plasma flow from the fast rotators was investigated recently by Beskin, Kuznetsova & Rafikov \( \text{[1998]} \). Surprisingly, the authors found that the perturbation of the poloidal magnetic field go to zero as \( \sigma_0/U_0^2 \to 0 \) and acceleration of the plasma remains very non effective like in the model by Michel \( \text{[1963]} \) with the prescribed monopole-like poloidal magnetic field. The Lorentz-factor of the plasma reaches the value \( \gamma_\infty \sim (\sigma_0 \gamma_0)^{1/3} \).

The plasma flow from the slow oblique rotator becomes especially simple. In particular, the function \( \eta \) for the slow rotators \( \alpha \ll 1 \) is
\[ \eta(r, \theta, \varphi, t) = D(\sin \chi \sin \theta \sin(\varphi - \Omega(t - r/v_0)) + \cos \theta \cos \chi). \tag{52} \]

Rotational losses of the slow rotators with the initially split-monopole magnetic field are defined by the total Poynting flux through the surface of the star
\[ \dot{E}_{\text{rot}} = \frac{2}{3} \frac{H_\varphi^2 R_\odot^4 \Omega^2}{v_0}, \tag{53} \]
and they do not depend on the angle $\chi$. It is useful also to express the rotational losses through the total magnetic flux $\psi$ of the open field lines of one direction (for example, outgoing from the star)

$$E_{\text{rot}} = \frac{1}{6\pi^2} \frac{\psi^2 \Omega^2}{v_0},$$

where $\psi = 2\pi R_0 R^2 \psi$ for the split-monopole magnetic field.

5. Implications to radio pulsars.

How the model considered above is connected with the real radio pulsars? It is reasonable to assume that the rotator with the split-monopole magnetic field is equivalent to real radio pulsar if it provides the plasma flow similar to the plasma flow from the real pulsar at distances larger than the light cylinder. Therefore the equivalent split-monopole rotator should have the matter flux, the poloidal magnetic field flux from one of the poles, $\gamma_0$, $v_0$, inclination angle $\chi$ and $\Omega$ the same as the real pulsar has.

The magnetic flux from one of the poles of the split-monopole rotator is $\psi = 2\pi R^2 \psi$. The magnetic flux of the open field lines from the radio pulsar with dipole magnetic field is defined by the radius of the polar cap $R_p = R_0 \sqrt{\frac{R_0 \Omega}{c}}$ (Goldreich & Julian 1969). Therefore the flux of the open field lines from one of the polar caps of the radio pulsar is estimated as

$$\psi = \pi R_0 R^2 \frac{R_0 \Omega}{c}.$$  

The ratio of the poloidal magnetic field to the matter flux density $B_0 = \rho_0 c$ is constant on the field line. Therefore this ratio for the split-monopole rotator and for the radio pulsar should be the same. These conditions allow one to define all parameters. The split-monopole rotator equivalent to the pulsar with the magnetic field on the polar cap $H_0$, angular velocity $\Omega$ and ejecting plasma with initial density $n_0$ and $\gamma_0$ has the magnetization parameter

$$\sigma_r = \frac{H^2_0}{8n_0 \rho c^2 \gamma_0} \left( \frac{R_0 \Omega}{c} \right)^3.$$  

Below we consider the relativistic plasma with $v_0 = c$. The rotational losses of this object are

$$E_{\text{rot}} = \frac{1}{6} \frac{H^2_0 R^2 \psi \Omega^4}{c^3}.$$  

These rotational losses equal the losses of the dipole rotating in vacuum at the angle of inclination $\chi = 30^\circ$.

The initial density of the plasma $n_0$ is defined by the processes of multiplication of plasma in the electromagnetic cascades initiated by primary particles with the Lorentz-factor $\gamma_{\text{gap}} \sim 2 \times 10^7$ in the magnetosphere of pulsar. In the inner gap theories (Ruderman & Sutherland 1973; Arons 1981) the primary beam has Goldreich-Julian density

$$n_{\text{GJ}} \sim \frac{Q H_0}{2\pi c^2},$$

We neglect here the dependence of $n_{\text{GJ}}$ on the inclination angle. The primary particles produce $\lambda$ secondary electrons and positrons. Therefore the initial density of the plasma is $n_0 = \lambda n_{\text{GJ}}$. Calculations of cascades performed by Daugherty & Harding (1982) and by Gurevich & Isitomin (1983) show that $\lambda \sim 10^4$. Outer gap model (Cheng & Ruderman 1986; Romani 1990) gives similar values of $n_0$. After substitution of (58) in (56), $\sigma_r$ takes a form

$$\sigma_r = \frac{e H_0}{4 \lambda mc^2 \gamma_0} \left( \frac{R_0 \Omega c}{c} \right)^2.$$  

The estimates of $\gamma_0$ are less definite. The flow of the plasma formed in the electromagnetic cascade consists of two components. One of them is the beam of the primary particles which can lose remarkable amount of energy. Another component is the plasma of the secondary particles. It is reasonable to take the average Lorentz-factor of the secondary particles as $\gamma_0$ because it is this plasma which screens the electric field and ensures the frozen-in condition. The average Lorentz-factor of the secondary particles strongly depends on the model of the gap. This parameter lies in the range $20 < \gamma_0 < \gamma_{\text{gap}}/\lambda$. To be definite we assume here that

$$\gamma_0 = 0.1 \frac{\gamma_{\text{gap}}}{\lambda} \sim 200.$$

This value well agrees with the results of cascade simulations performed by Daugherty & Harding (1982). For other $\gamma_0$ the magnetization parameter is scaled as $\sigma_r = \sigma_{200}(200/\gamma_0)$. The location of all radio pulsar observed by EGRET in gamma-rays above 100 MeV (Crab, Vela, Geminga, PSR 1706-44, PSR 1509-58, PSR 1951+32 and PSR 1055-55; Thompson 1998) in the parameter space $\sigma_r$, $U_0$ for all possible $U_0$ is shown in Fig. 6 by dashed lines. For the estimates of the magnetic field $H_0$ equation (57) was used. It was assumed also that the momentum of inertia of all pulsars is $10^{45} \text{g} \cdot \text{cm}^2$. It is seen that all the pulsars appears the slow rotators provided that $U_0 \gtrsim 200$. This implies that they do not accelerate plasma and do not collimate it considerably to the axis of rotation beyond the light cylinder if the flow of the plasma is dissipativeless (Bogovalov 1998; Bogovalov & Tsinganos 1999).

6. Conclusion

This paper clearly demonstrates that the oblique rotators do not differ strongly from the axisymmetric rotators. In the particular case considered in this paper the dynamics of the plasma ejected by the oblique rotator is exactly the same as the plasma dynamics of the wind from the axisymmetric rotator. It allows us to apply the all results obtained for the cold axisymmetric plasma flows to the oblique rotators. But these results show that for parameters typical for radio pulsars the acceleration of the relativistic plasma is not effective. Radio pulsars apparently are basically slow rotators. The obliqueness of the rotators does not change this conclusion.
Our results indicate that only violation of the ideal MHD can provide effective acceleration of the winds from radio pulsars since neither magneto-centrifugal mechanism or the acceleration by the gradient of the toroidal magnetic field at the collimation of the plasma to the axis of rotation are effective for the pulsars. The role of the dissipative processes in the relativistic winds is under discussion for many years (Coroniti [1990], Michel [1994], Mestel & Shibata [1994]). Recently Melatos & Melrose [1996] tried to argue that the violation of the frozen-in condition takes place in the wind from an oblique rotator and it can provide the acceleration of the wind. Our self-consistent solution of the problem for the particular case of the oblique rotator with the initially split-monopole magnetic field shows no evidence of the violation of the ideal MHD in the wind.

Mestel & Shibata [1994] argued that the frozen-in condition can be violated right after the light cylinder due to the formation of singularities in the stationary axisymmetric flow. This assumption is not confirmed by last results by Bogovalov [1997] and by Contopoulos, Kazanas & Fendt [1991].

It follows from all results obtained recently in the physics of relativistic plasma outflows that only dissipative processes can ensure plasma acceleration under pulsar conditions. One of these mechanisms was proposed by Coroniti [1990] and Michel [1994]. They argued that due to the geometrical reasons the dissipative processes can ensure plasma acceleration under pulsed conditions. Another possibility is the Poynting flux transformation to the geometrical flow. This assumption is not confirmed by last results by Bogovalov [1997] and by Contopoulos, Kazanas & Fendt [1991].

Another possibility is the Poynting flux transformation in the volume of the flow. This can happen if the Poynting flux dominated plasma flow is unstable in relation to MHD perturbations. The anomalously low conductivity due to the MHD turbulence would provide the violation of the ideality in the flow with corresponding transformation of the Poynting flux into the energy of the plasma. The advantage of this approach is that it can provide effective acceration even at the axisymmetric rotation. But the question about stability of the Poynting flux dominated plasma flow is still open and should be investigated in details separately.

Acknowledgements. Author acknowledge discussions with colleagues from Astrophysics group of the MPI für Kernphysik and Dr. Yu.E.Lyubarsky. Author thanks Dr. F.A. Aharonian for reading of the manuscript and useful comments as well as MPI für Kernphysik for warm hospitality and support during the work on the paper. Author is also grateful to unknown referee for useful remarks on the paper.

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