**Abstract**

In the paper we give a complete classification of all $\mathcal{R}$-cross-sections of the semigroup of order-preserving transformations on $n$-element set in terms of certain binary trees.

**Keywords:** Order-preserving transformation, Cross-section, Green’s relation, Binary tree

1 Introduction

Let $\rho$ be an equivalence relation on a semigroup $S$. A subsemigroup $S'$ of $S$ is called a $\rho$-cross-section in $S$ if $S'$ contains exactly one representative of each $\rho$-class. In general, given $S$ and $\rho$, a $\rho$-cross-section with respect to an equivalence $\rho$, chosen to be tightly enough connected with the multiplication in $S$, imposes a non-trivial restriction on $S$. Studying semigroups under such restrictions may shed some light on the structure of $S$ as the $\rho$-cross-section $S'$ may be thought of as a skeleton of $S$ to which the flesh of $\rho$-classes is attached.

As Howie [7] writes, “on encountering a new semigroup, almost the first question one asks is ‘What are the Green relations like?’” Therefore, it is not a surprise that studying $\rho$-cross-sections has started from $\rho$ being one of these all-pervading Green equivalencies $\mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{J}$. There are many publications devoted to $\rho$-cross-sections, with $\rho$ being one of the Green relations, in various transformation monoids and other semigroups with transparent Green structure; see Chapter 12 of the book [5] and references therein.

The present paper deals with $\mathcal{R}$- and $\mathcal{L}$-cross-sections in the semigroup $\mathcal{O}_n$ of all order-preserving transformations on the $n$-element chain. This is a very well-studied regular submonoid of the symmetric semigroup $\mathcal{T}_n$; see [5] Chapter 14 and references therein for various structural and combinatorial aspects of $\mathcal{O}_n$. However, to the best of our knowledge, the $\mathcal{R}$- and $\mathcal{L}$-cross-sections of $\mathcal{O}_n$ have not yet been classified. The present paper closes this gap.

Since $\mathcal{O}_n$ is a regular subsemigroup of $\mathcal{T}_n$, its Green equivalencies $\mathcal{R}$ and $\mathcal{L}$ are nothing but the restrictions of the corresponding equivalences on $\mathcal{T}_n$. One might therefore expect that the $\mathcal{R}$- and $\mathcal{L}$-cross-sections in $\mathcal{O}_n$ are sort of restrictions of the $\mathcal{R}$- and $\mathcal{L}$-cross-sections of $\mathcal{T}_n$ whose descriptions were found by respectively Pekhterev [8] and the present author [1,2]. We will see that this is true only for the $\mathcal{L}$-cross-sections. In contrast, for the $\mathcal{R}$-cross-sections, the situation turns out to very different, and their classification requires certain novel ingredients. Surprisingly

*Supported by the Ministry of Science and Higher Education of the Russian Federation (Ural Mathematical Center project No. 075-02-2022-877)
enough a classical data structure from the theory of combinatorial algorithms, namely binary search trees, plays a crucial role here.

The paper is organized as follows. Section 2 contains first observations and a few examples of $O$-cross-sections in $O_n$. In Sect. 3 we collect necessary definitions and notation from the theory of binary trees. In Sect. 4 we propose a construction in terms of certain binary trees which produces $O$-cross-sections of $O_n$. We show also that each $O$-cross-section of $O_n$ is given in this way. In Sect. 5 we discuss connections between $L$- and $O$-cross-sections of $O_n$. Section 6 is devoted to the classification of $O$-cross-sections of $O_n$ up to isomorphism.

2 $O$-cross-sections of $O_n$: first examples

We denote by $\pi$ the set \{1, 2, ..., $n$\} and by $T_n$ the semigroup of all transformations on $\pi$. Let $\pi$ be ordered in a natural way. Recall that a transformation $\alpha \in T_n$ is called order-preserving provided that $x \leq y$ implies $\alpha(x) \leq \alpha(y)$. The set of all order-preserving transformations is denoted by $O_n$. A subsemigroup $R$ of $O_n$ constitutes an $O$-cross-section of $O_n$ if it contains exactly one representative from each $R$-class of $O_n$. Dually, a subsemigroup $L$ of $O_n$ constitutes an $L$-cross-section of $O_n$ if it contains exactly one representative from each $L$-class of $O_n$. For brevity, we write $O_n$ and $L_n$ for the set of all $O$- and $L$-cross-sections of $O_n$ respectively. Analogously, we denote by $O(T_n)$ and $L(T_n)$ the $O$- and $L$-cross-sections of $T_n$.

We start with exhibiting two natural series of $O$-cross-sections of $O_n$. First example of “dense” cross-sections is inspired by the description of $O(T_n)$. Second example of “dual” cross-sections is due to the connection of $O_n$ with its dual $O_n^*$ and the description of $L(T_n)$. After that, we consider the third example of an $O$-cross-section of $O_n$, which is neither dense, nor dual.

Let further $\text{im}(\alpha)$ stand for the image of a transformation $\alpha \in T_n$; the kernel of $\alpha$ is denoted by $\ker(\alpha)$. The quotient set $\pi/\ker(\alpha)$ is said to be the partition assigned to $\alpha$. We write $\pi_{\alpha}$ for the identity transformation on $\pi$ and $c_{\alpha}$ for the constant transformation with a singleton image $\{x\}$, $x \in \pi$. We use the matrix notation of a transformation $\alpha$:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix},$$

where $A_1 \cup A_2 \cup \cdots \cup A_k = \pi/\ker(\alpha)$ with $A_1 \alpha = a_1, A_2 \alpha = a_2, \ldots, A_k \alpha = a_k$.

Recall that $O_n$ is a regular subsemigroup of $T_n$, so its Green equivalencies are just the restrictions of the corresponding equivalencies on $T_n$: for all $\alpha, \beta \in O_n$

- $\alpha \mathrel R \beta$ if and only if $\ker(\alpha) = \ker(\beta)$;
- $\alpha \mathrel L \beta$ if and only if $\text{im}(\alpha) = \text{im}(\beta)$;
- $\alpha \mathrel H \beta$ if and only if $\ker(\alpha) = \ker(\beta)$ and $\text{im}(\alpha) = \text{im}(\beta)$;
- $\alpha \mathrel D \beta$ if and only if $|\text{im}(\alpha)| = |\text{im}(\beta)|$;
- $\mathcal J = \mathcal D$.

We begin with a short lemma.

Lemma 2.1 Let $T$ be a regular subsemigroup of a semigroup $S$, $\mathcal R$ one of the Green’s relations and $C \subseteq T$ an $\mathcal R$-cross-section of $S$. Then $C$ is an $\mathcal R$-cross-section of $T$.

Proof It is known that $\mathcal R_T = \mathcal R_S \cap (T \times T)$. Hence $C$ is a subsemigroup of $T$ with exactly one element in each $\mathcal R$-class of $T$. □

Note that it can be seen from the description of $L(T_n)$ [1, 2] that each of $L$-cross-sections of $O_n$ respects a certain linear order on $\pi$. Therefore the description of $L(T_n)$ gives rise to an $L$-cross-section of $O_n$. We recall the necessary definitions and facts from [1, 2] in Section 3 of the paper.
2.1 Dense $R$-cross-sections

A description of $R$-cross-sections of $T_n$ is known [8]. For the reader’s convenience, we recall the description of $R(T_n)$ in what follows.

Fix a linear order $\prec$ on $\mathbb{P}$. Let $\mathbb{P} = \{u_1 \prec u_2 \prec \ldots \prec u_n\}$. For a nonempty set $A \subset \mathbb{P}$ denote by $\text{min}(A)$ the minimal element of $A$ with respect to $\prec$. If $A, B \subset \mathbb{P}$ are nonempty and disjoint, we will write $A \prec B$ provided that $\text{min}(A) \prec \text{min}(B)$. Let $R(\prec)$ denote the set of all elements of $T_n$, which have the form

$$\alpha = \left( \begin{array}{cccc} A_1 & A_2 & \cdots & A_k \\ u_1 & u_2 & \cdots & u_k \end{array} \right),$$

where $A_1 \prec A_2 \prec \cdots \prec A_k$ for all disjoint partitions $A_1 \cup A_2 \cup \ldots A_k$ of $\mathbb{P}$.

Pekhterev has proved the following result:

**Theorem 2.2 (8).** For every linear order $\prec$ on $\mathbb{P}$ the set $R(\prec)$ is an $R$-cross-section of $T_n$. Conversely, each $R$-cross-section of $T_n$ has the form $R(\prec)$ for some linear order $\prec$ on $\mathbb{P}$.

An $R$-cross-section of $T_n$, $n > 2$, however, does not belong to $O_n$. Indeed, the kernel classes of an order-preserving transformation are the convex subsets of the chain $\mathbb{P}$. That is if $x, y \in A$ for $A \in \mathbb{P}/\ker(\alpha)$ then $\{x \mid a < x < b\} \subset A$. But transformations of $T_n$ have non-convex partitions in general. Apparently, the following fact holds.

**Proposition 2.3** Let $(\mathbb{P}, \prec)$ be a naturally ordered, $A_1, A_2, \ldots, A_t$ be a partition of $\mathbb{P}$ into $t$ disjoint intervals, $1 \leq t \leq n$. Then each of the sets

$$R(\prec) \cap O_n = \left\{ \left( \begin{array}{ccc} A_1 & A_2 & \cdots \\ 1 & 2 & \cdots \\ n - (t - 1) & n - (t - 2) & \cdots & n - 1 \end{array} \right) \in O_n \mid 1 \leq t \leq n \right\}$$

and

$$R(\prec^{-1}) \cap O_n = \left\{ \left( \begin{array}{ccc} A_1 & A_2 & \cdots \\ 1 & 2 & \cdots \\ n - (t - 1) & n - (t - 2) & \cdots & n - 1 \end{array} \right) \in O_n \mid 1 \leq t \leq n \right\}$$

constitutes the $R$-cross-section of $O_n$.

We call such $R$-cross-sections to be dense. Hence for $n > 2$ there always exist at least two (dense) $R$-cross-sections of $O_n$. We will see also that the description of $R(O_n)$ dose not reduce to $R(T_n) \cap O_n$ also.
Example 2.4 The dense $R$-cross-sections of $O_4$:

$$
R_1 = \left\{ \begin{pmatrix} 12 & 3 & 4 \\ 1 & 2 & 3 \\ \end{pmatrix}, \begin{pmatrix} 1 & 23 & 4 \\ 1 & 2 & 3 \\ \end{pmatrix}, \begin{pmatrix} 123 & 4 \\ 1 & 2 \\ \end{pmatrix}, \begin{pmatrix} 123 & 4 \\ 1 & 2 \\ \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \\ \end{pmatrix} \right\},
$$

$$
R_2 = \left\{ \begin{pmatrix} 12 & 3 & 4 \\ 2 & 3 & 4 \\ \end{pmatrix}, \begin{pmatrix} 1 & 23 & 4 \\ 2 & 3 & 4 \\ \end{pmatrix}, \begin{pmatrix} 123 & 4 \\ 3 & 4 \\ \end{pmatrix}, \begin{pmatrix} 123 & 4 \\ 3 & 4 \\ \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \right\}.
$$

2.2 Dual $R$-cross-sections of $O_n$

However, there exist other examples of $R_n$-cross-sections in $O_n$. Higgins [6] has showed that there exists an injective homomorphism $*$ from $O_n$ to the dual $O_{n+1}$ on $(n+1)$-element set. The homomorphism maps $\alpha \in O_n$ to $\alpha^* \in O_{n+1}$, where $\alpha^*$ is defined as follows.

Denote by $K = \{k_1, k_2, \ldots, k_t\}$ the set of the maximal members of kernel classes of $\alpha$, written in ascending order. Let $\text{im}(\alpha) = \{r_1, r_2, \ldots, r_t\}$ with $k_i \alpha = r_i$ for all $1 \leq i \leq t$. For each $x \in [n+1]$ define

$$
\alpha^x = \begin{cases} 
1 & \text{if } x \leq r_1, \\
k_i + 1 & \text{if } r_i < x \leq r_{i+1}, 1 \leq i < t, \\
n+1 & \text{if } x > r_t.
\end{cases}
$$

The idea is: if we take an $L$-cross-section $L(O_n)$ of $O_n$ and consider its dual $L^*$, will we get an $R$-cross-section of $O_{n+1}$? Can we get all $R$-cross-section of $O_{n+1}$ in that way? For now we can claim the following:

Proposition 2.5 The duals $L(O_n)^* \cup \{c_1\}$ and $L(O_n)^* \cup \{c_{n+1}\}$ are $R$-cross-sections of $O_{n+1}$.

Proof By the definition for $\alpha \in L(O_n)$, $\text{im}(\alpha)$ goes through all non-empty subsets of $\pi$. The points of each subset of $\pi$ breaks $[n+1]$ into convex partition. Hence, the transformations from $L(O_n)$ induce $2^n - 1$ convex partitions on $[n+1]$ (except the 1-element partition $[n+1]$ itself). Therefore the dual of $L(O_n)$ contains exactly one representative from each $R$-class of $O_{n+1}$, except the constant one. Using the fact that $*$ is a homomorphism we get that $L(O_n)^*$ is closed under the multiplication. By the construction we have $1 \alpha^* = 1$ and $(n+1)\alpha^* = n+1$ for all $\alpha^* \in L(O_n)^*$. Thus $L(O_n)^* \cup \{c_1\}$ and $L(O_n)^* \cup \{c_{n+1}\}$ constitute the $R$-cross-sections of $O_{n+1}$.

Example 2.6 To illustrate Proposition 2.5 we give the present example: in Fig. 2 we depicted the transformations of $L$-cross-section $L$ of $O_3$ with solid arrows. It is an easy exercise to verify that the set $L$ constitutes an $L$-cross-section. The duals of the transformations are presented with dashed arrows.

Consider the the set $L^*$ of the dual transformations of $L$:

$$
L^* = \left\{ \begin{pmatrix} 1 & 2 & 34 \\ 1 & 2 & 3 \\ \end{pmatrix}, \begin{pmatrix} 1 & 23 & 4 \\ 1 & 2 & 3 \\ \end{pmatrix}, \begin{pmatrix} 12 & 3 & 4 \\ 1 & 2 & 3 \\ \end{pmatrix}, \begin{pmatrix} 1 & 234 \\ 1 & 2 \\ \end{pmatrix}, \begin{pmatrix} 12 & 34 \\ 1 & 2 \\ \end{pmatrix}, \begin{pmatrix} 123 & 4 \\ 1 & 2 \\ \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \right\}.
$$
It is easy to verify that each of the sets $L^* \cup \{c_1\}$ or $L^* \cup \{c_4\}$ forms an $R$-cross-section of $O_4$.

Figure 2: $L^*$-cross-section of $T_3$ and its dual in $T_4$

The $R$-cross-sections of $O_n$ obtained in this way we will call dual for brevity.

The direct calculations show that the dual preimages of $R_1 \setminus c_1$ and $R_2 \setminus c_4$ of dense cross-sections $R_1$ and $R_2$ do not form the $L^*$-cross-sections of $O_3$.

**Definition 2.7** We say that an $R$-cross-section $R$ has a fixed point $x$, if $xf = x$ for all non-constant $f \in R$.

Note that each of dense $R$-cross-sections of $(\{n+1\}, \prec)$ always has the only fixed point: either 1, or $n+1$. On the other hand, by the construction each set $L(O_n)^*$ always has at least two fixed points, namely, 1 and $n+1$.

**Example 2.8** However, there exists an $R$-cross-section of $O_4$ which is not dense but has the only fixed point 1:

$R_5 = \left\{\begin{array}{c}
\{1\} & \{2\} & \{34\} \\
1 & 2 & 3
\end{array}\right\}, \left\{\begin{array}{c}
\{1\} & \{23\} & \{4\} \\
1 & 3 & 4
\end{array}\right\}, \left\{\begin{array}{c}
\{12\} & \{3\} & \{4\} \\
1 & 3 & 4
\end{array}\right\}.$

Thus, at the moment we have examples of two types of $R$-cross-sections of $O_n$ with completely different origins. We have also the evidence of the existing of other ones. In fact, we will show that all $R$-cross-sections of $O_n$ can be generated in some sense by a sequence of $L^*$-cross-sections of $O_{i_1}, O_{i_2}, \ldots, O_{i_k}$, with $i_1 + i_2 + \ldots + i_k = n$. The sequence in turn is tightly connected with a certain type of binary search trees and their duals, so-called inner trees. We will state the necessary types of binary trees, examples and interpretations trees as diagrams in the following section.
3 Trees and diagrams

Recall that a rooted binary tree is an acyclic connected graph in which one vertex is specified as a root, each vertex \( v \) has at most two children and a unique parent (except the root, which has no one). The trees occurring in this paper assumed to be rooted binary trees unless otherwise stated. We denote the root of a binary tree by \( r \), the parent of a vertex \( v \) by \( p(v) \). The children are referred to as the left child (the son of a vertex) and the right child (the daughter of a vertex). The son and the daughter of \( v \) are denoted by \( s(v) \) and \( d(v) \), respectively. A leaf is a vertex of the tree which has no children. Vertices which are not leaves are called internal nodes. Each internal vertex \( v \) of a tree has the left and right subtree. A subtree may be empty. The non-empty subtree consists of a vertex (a child of \( v \)) and the descendants of the vertex in the tree. A binary tree is said to be full if each its internal vertex has exactly two children.

There exists a unique path from the root to a vertex in the tree. We denote by \( \omega(v) \) the set \( \{v, p(v), p(p(v)), \ldots, r\} \) containing the path of a vertex \( v \). The level of a vertex is the length of its path. Hence, the level of the root equals 0; the level of a non-root vertex is the level of its parent plus 1.

Let \( T(n) \) denote a tree whose vertices are labeled with 1, 2, …, \( n \). We do not make a difference between a vertex and its label for convenience. So, if a vertex \( v \) is labeled with a number \( a \) with \( a < b \) for some \( b \in \pi \) we write \( v < b \).

3.1 Order-preserving binary tree

In computer science, a binary search tree (BST) \([4]\) is a rooted binary tree with the following property: for a vertices \( x, y \) of the tree if \( y \) is a vertex in the left subtree of \( x \) then \( y \leq x \). If \( y \) is a vertex in the right subtree of \( x \) then \( x \leq y \). We need to modify the notion of a BST for our purposes.

**Definition 3.1** An order-preserving tree is a rooted binary tree \( T(n) \) with the following property: for a vertices \( x, y \in T(n) \) if \( y \) is a vertex in the left subtree of \( x \) then \( y < x \). If \( y \) is a vertex in the right subtree of \( x \) then \( x < y \). We need to modify the notion of a BST for our purposes.

Thus, for a natural \( n \), an order-preserving binary tree is a kind of strict BST with the vertex set \( \{1, 2, \ldots, n\} \). It is also important for us to fix the bounds of each vertex in the tree:

**Definition 3.2** We define the canonical bounds of a vertex of an order-preserving binary tree \( T(n) \) by induction:

1) For the root the canonical bounds are \( 1 \leq r \leq n \).

2) Let \( v \in T(n) \) be a vertex with the canonical bounds \( a \leq v \leq b \) which has been already defined. If \( v \) has a son (a daughter) then a child has the following canonical bounds

\[
a \leq s(v) < v \quad \text{and} \quad v < d(v) \leq b \quad \text{respectively.}
\]

**Figure 3:** Examples of order-preserving binary trees for \( n = 4, 5 \).
The definition implies that each canonical bound of a vertex $v$ is either belongs to $\omega(v)$, or to the set $\{1, n\}$.

Note that an order-preserving binary tree on $T(n)$ can be regarded as a model $(\Pi, \leq, \prec)$, where $\prec$ is a strong partial order on $\Pi$, defined as follows: for all $x, y \in \Pi$ set $x \prec y$ if $x$ is a descendant of $y$. Since $\Pi$ assumed to be equipped with the natural order $\leq$ through the paper, we write $(\Pi, \prec, \leq)$ instead $(\Pi, \leq, \prec)$, or just $T(n)$.

It is convenient for our purposes to picture an order-preserving binary tree of $(\Pi, \prec)$ in an “unfolded” form (diagram) which is described below. The main advantage of the diagrams is that it “preserves the scale” of the tree and makes the “inner” trees (which we define further) more visual.

By $U_q$, $0 \leq q \leq n - 1$, we denote the set of all vertices in the $q$-th level of an order-preserving tree $T(n)$. To each vertex $x \in U_q$ of the tree we assign the point $(x, q)$ and the line segment with the endpoints $(x, q)$ and $(x, n - 1)$ of the coordinate plane (see Fig. 4).

Since $T(n)$ is order-preserving, the children of each $(x, q)$ are determined by the diagram of the tree in a unique way. The points of the same level are incomparable with respect to $\prec$.

![Figure 4](image.png)

Figure 4: "Unfolded" representation (diagram) of the order-preserving binary tree $//(\Pi, \leq, \prec)$.

Note that the diagram of an order-preserving binary tree completely determines the original tree. For example, the diagrams of the trees from Fig. 3 are presented in Fig. 5.

![Figure 5](image.png)

Figure 5: Diagrams of the order-preserving binary trees shown on Fig. 3.

Everywhere below the maximal element in an interval with respect to $\prec$ we call the highest, while the word “maximal” stands for the maximal element with respect to $\leq$. We also say the element is the lowest if it is minimal with respect to $\prec$, while the word “minimal” relates with usual order.

Our goal now to give the notion of a decreasing tree, which is a special case of order-preserving trees. This leads us to the notion of so-called "inner" binary tree of an order-preserving
tree, whose vertices are marked by intervals. Inner trees can be extremely well seen in the diagram of the order-preserving tree. We discuss this construction more detail in the next subsection.

3.2 Inner binary tree

Let \( i, j \in \mathbb{N} \) and \( i \leq j \). The interval \([i, j]\) is the set \( \{ k \in \mathbb{N} \mid i \leq k \leq j \} \). We write \([i, i]\) instead of \([i, i]\).

Consider the diagram of an order-preserving tree \( T(n) \). Label the cells in diagrams from left to right in following way: the cell which is next after vertex \( i \), \( 1 \leq i < n \), we label by \( i' \).

**Definition 3.3** We define the **inner tree** \( \Gamma \) of an order-preserving tree \( T(n) \) as a full binary tree labeled with intervals of \([1', (n-1)']\), defined by induction as follows:

1. The interval \([1', (n-1)']\) is the root of \( \Gamma \).
2. Let \([i', j']\) \( \in \Gamma \). If \( i' = j' \) then \( i' \) is a leaf. Let \( i' \neq j', x \in T(n) \) be the highest point with \( i < x < j + 1 \). Then \([i', (x-1)']\) and \([x', j']\) are the daughter and the son of \([i', j']\) respectively.

**Example 3.4** According to the previous definition the inner tree of \( T(5) \) has the form (see Fig 6):

![Figure 6: The inner tree \( \Gamma \) of the order-preserving tree \( T(5) \).](image)

Note that the inner tree is determined by an order-preserving tree in a unique way. Nevertheless the same inner tree could be associated with different order-preserving trees since an order-preserving tree is not full in general. For instance, \( \Gamma \) is also the inner tree of \( T_1(5) \) (see Fig 5).

Let \( x \in T(n) \) be a vertex with the canonical bounds \( a, b, a < b \). Then the **left inner tree** \( \Gamma_l(x) \) and the **right inner tree** \( \Gamma_r(x) \) of \( x \) are the subtrees of the inner tree \( \Gamma \) rooted at \([a', (x-1)']\) and \([x', (b-1)']\) respectively. We define \( \Gamma_l(x) = \emptyset \) if \( x = 1 \) and \( \Gamma_r(x) = \emptyset \) if \( x = n \).

**Example 3.5** Consider tree \( T(4) \) from Fig 3. For the vertex \( 2 \in T(4) \) the left inner tree \( \Gamma_l(2) \) is isomorphic to a point (1-element full binary tree), tree \( \Gamma_r(2) \) is isomorphic to a 2-element full binary tree (see Fig 7). The right inner tree \( \Gamma_r(4) \) of 4 is empty.

Exactly the interconnections between inner trees of the vertices play a crucial role in the definition of decreasing binary trees.

3.3 Decreasing binary tree

We will need “to compare” the trees frequently in the sense of following definitions.

**Definition 3.6** By a homomorphism between two trees \( T_1 \) and \( T_2 \) we mean a 1-1-map from the vertex set of \( T_1 \) into the vertex set of \( T_2 \) that sends the root of \( T_1 \) to the root of \( T_2 \) and preserves the parent–child relation and the genders of non-root vertices.
We denote the highest and the closest vertex to \( V \) on the right. Then

\[ \{ V \} = \{ r \} \]

**Example 3.11** Let \( T \) be a binary tree.

Given two trees \( T_1 \) and \( T_2 \), we say that \( T_1 \) subordinates \( T_2 \) (and write \( T_1 \hookrightarrow T_2 \)) if there exists a homomorphism \( T_1 \to T_2 \). The empty tree is assumed to subordinate every binary tree.

Recall that by \( \omega(1) \), \( \omega(n) \) we denote the paths from the vertex 1 and \( n \) respectively, to the root in \( \mathcal{T} \).

**Definition 3.8** We say an order-preserving tree \( \mathcal{T} \) to be **decreasing**, if for all \( x, y \in \mathcal{T} \) with \( x \prec y \) the following conditions hold:

1. if \( x, y \in \omega(1) \) then the right inner tree of \( x \) subordinates the right inner tree of \( y \);
2. if \( x, y \in \omega(n) \) then the left inner tree of \( x \) subordinates the left inner tree of \( y \);
3. if \( x, y \notin \omega(1) \cup \omega(n) \) then the inner trees of \( x \) subordinate the respective inner trees of \( y \).

**Example 3.9** The tree \( T(4) \) from Fig. 5 is not decreasing, since \( 4 < 2 \) and \( 2, 4 \in \omega(4) \), but it does not hold \( \Gamma_1(4) \hookrightarrow \Gamma_1(2) \). Trees \( T(5) \) and \( T_2(5) \) are decreasing.

### 3.4 Order-preserving tree of a convex partition of \( \mathcal{T} \)

It remains to define one more type of binary trees. The one is defined for a fixed order-preserving tree \( \mathcal{T} \) and a convex partition of \( \mathcal{T} \).

Let \( K = K_1 \cup K_2 \cup \ldots \cup K_m \) be a convex partition of \( \mathcal{T} \) into \( m \) intervals:

\[ K_i = [k_{i-1}, k_i] \]

For every \( a \in \mathcal{T} \), denote by \( K(a) \) the interval of \( K \) that contains \( a \).

**Definition 3.10** Define a partition order-preserving tree \( T(K) \) with respect to \( \mathcal{T} \) by induction as follows.

1. Let \( K(0) \in K \) be the root of \( T(K) \).
2. Assume \( V \) is a vertex of \( T(K) \). By \( K \in \mathcal{T} \) we denote the highest and the closest to \( V \) on the left vertex of \( \mathcal{T} \), which does not belong to any of already defined vertices of \( T(K) \). In dual way, by \( K \) denote the highest and the closest vertex to \( V \) on the right. Then

\[ s(V) := K(m_i) \text{ and } d(V) := K(m_d). \]

For brevity in what follows we refer to \( T(K) \) as to the partition tree. The vertices \( r, m_i, m_d \) are said to be the **leading** element of the interval.

**Example 3.11** Let \( T(5) \) be the decreasing tree (see Fig. 3). \( K = \{(1, 2), (3, 4), (5)\} \). Since \( r = 3 \), we have \( K(0) = (3, 4) \) is a root of \( T(K) \). The leading elements of \( \{1, 2\} \) and \( \{5\} \) are \( m_i = 1, m_d = 5 \) respectively. Thus \( s((3, 4)) = \{1, 2\}, d((3, 4)) = \{5\} \).
4 Description of \( \mathcal{R} \)-cross-sections of \( \mathcal{O}_n \)

In this section we introduce a semigroup \( \Phi_\prec \subset \mathcal{O}_n \). We will prove that the semigroup \( \Phi_\prec \) forms an \( \mathcal{R} \)-cross-section and conversely, each \( \mathcal{R} \)-cross-section of \( \mathcal{O}_n \) is given by \( \Phi_\prec \) for a certain \( \prec \).

4.1 Semigroup \( \Phi_\prec \)

Let \( (\mathcal{P}, \prec) \) be a decreasing tree. Fix a convex partition \( \mathcal{K} \) of \( \mathcal{P} \) into \( m \) intervals. Let \( T(\mathcal{K}) \) be the partition tree associated with \( (\mathcal{P}, \prec) \). First we show that there exists a unique homomorphism from \( T(\mathcal{K}) \) to \( (\mathcal{P}, \prec) \).

Without loss of generality we assume \( \mathcal{K} \). Without loss of generality we consider \( \omega(1) \). We induct on the level of a vertex \( K \in \omega(K^{(1)}) \). The root of \( \omega(K^{(1)}) \) always contains \( r \in \omega(1) \).

First we show that there exists a unique homomorphism \( f \) of \( \mathcal{K} \) into \( (\mathcal{P}, \prec) \).

**Lemma 4.1** For all \( K \in \omega(K^{(1)}) \) the leading element of \( K \) belongs to \( \omega(1) \). Dually, for all \( K \in \omega(K^{(n)}) \) the leading element of \( K \) belongs to \( \omega(n) \).

**Proof** Without loss of generality we consider \( \omega(K^{(1)}) \). We induct on the level of a vertex \( K \in \omega(K^{(1)}) \). The root of \( \omega(K^{(1)}) \) always contains \( r \in \omega(1) \). Therefore, the induction base holds.

The proof for \( \omega(K^{(n)}) \) is dual.

**Lemma 4.2** If there exists a homomorphism \( f: T(\mathcal{K}) \to (\mathcal{P}, \prec) \) then \( \omega(K^{(1)}) \prec \omega(1) \) and \( \omega(K^{(n)}) \prec \omega(n) \). Moreover, if \( K \notin \omega(K^{(1)}) \) then \( Kf \notin \omega(1) \); if \( K \notin \omega(K^{(n)}) \) then \( Kf \notin \omega(n) \).

**Proof** Without loss of generality we assume \( r \neq 1 \) and consider the left subtrees \( \omega(1) \) and \( \omega(K^{(1)}) \). It is easy to see that \( |\omega(K^{(1)})| \leq |\omega(1)| \) by Definition 3.10. We induct on the level of a vertex \( K \in \omega(K^{(1)}) \).

If \( f \) is a homomorphism then we have \( K^{(r)}f = r \). Let the assumption holds for first \( q + 1 \) levels of \( \omega(K^{(1)}) \) and \( Kf = x \) with \( x \in \omega(K^{(1)}) \).

Since \( f \) preserves the genders and parent-child relations, we have \( s(K)f = s(x) \) and first statement of the lemma for the left subtrees is proved.

Assume the converse: let \( K \notin \omega(K^{(1)}) \) and \( Kf = y \) for \( y \in \omega(1) \). Then \( K \) is a male vertex. Let \( P \in \omega(K^{(1)}) \) be the lowest ancestor of \( K \). Thus, there exists female ancestor \( F \in T(\mathcal{K}) \) of \( K \) and \( K \prec F \prec P \). Therefore, we get that the female vertex \( Ff \) is an ancestor of \( Kf \in \omega(1) \) which is impossible.

If \( r = n \) then it remains nothing to prove. Otherwise the proof is dual.
Lemma 4.3 For each decreasing tree $\langle \mathcal{T}, \prec \rangle$, a convex partition $\bar{K}$ and an interval tree $T(\bar{K})$ associated with $\langle \pi, \prec \rangle$, there exists a unique homomorphism $f : T(\bar{K}) \rightarrow \langle \pi, \prec \rangle$.

Proof We construct a homomorphism $f : T(\bar{K}) \rightarrow \langle \pi, \prec \rangle$ by induction on the level of $K_q \in T(\bar{K})$. Since $K^{(\ell)}$ is always defined, the base clearly holds: $K^{(\ell)}f = r$.

Let $k$ be the depth of $T(\bar{K})$. Suppose $f$ is well-defined for all vertices of $T(\bar{K})$ whose level is less than $q + 1$. Let $K_f = x$ and $K \in U_q$. For the concreteness assume that $x < r$.

Denote by $y$ the leading element of the interval $K$. Note that by the construction of the partition tree we have $\text{lev}(y) \geq \text{lev}(x)$. There are following cases to consider.

Case 1. Suppose $x \in K$. Since the level of the leading element of the interval is the least one of the elements in $K$ and $\text{lev}(y) \geq \text{lev}(x)$, we get that $x$ is the leading element of $K$. Thus, $K$ has the son (the daughter) in $T(\bar{K})$ whenever there exists the son (the daughter) of $x$. Therefore if $s(K_q), d(K_q)$ exist then $s(K_q)f = s(x), d(K_q)f = d(x)$ as required.

Case 2. $x \notin K$. Thus we have $x \neq r$ and $\text{lev}(y) > \text{lev}(x)$.

a) If $K \in \omega(K^{(1)})$ then by Lemma 4.2 $x \in \omega(1)$. Since $K \in \omega(K^{(1)})$, then by Lemma 4.1 we have $y \in \omega(1)$. The tree is decreasing, therefore right inner tree of $y$ subordinates the right inner tree of $x$. Thus $K$ has a child (children) whenever $y$ has a child (children), which in turn holds whenever $x$ has a child (children). Thus $s(K)f = s(x), d(K)f = d(x)$ are defined and unique.

b) Suppose $K \notin \omega(K^{(1)})$, whence $y \notin \omega(1)$. By Lemma 4.2 it holds also $x \notin \omega(1)$.

1. Let $y \prec x$. Since $\langle \mathcal{T}, \prec \rangle$ is decreasing, the inner trees of $y$ subordinate the respective inner trees of $x$. Hence, $s(K_q)f, d(K_q)f$ are well-defined and unique.

2. Assume $y \not\prec x$. Denote by $a, b \in \omega(1), a \neq b$ the lowest ancestors of $y$ and $x$ respectively. We have $a \prec b$ and $\Gamma_i(K^{(o)}) \rightarrow \Gamma_i(a) \rightarrow \Gamma_i(b)$. Thus $s(K)f = s(x), d(K)f = d(x)$ are well-defined.

3. Let $y \not\prec x$ and $x, y$ have the same lowest ancestor, say $a \in \omega(1)$. Denote by $t$ the lowest common ancestor of $x, y$ with $t \neq a$. Note that there always exists at least one ancestor $s(a)$ of $x, y$ which satisfies these conditions.

Again, without loss of generality for concreteness we assume $y < t < x$. Furthermore, by the inductive assumption the pre-images of $\{p(x), \ldots, t \in \omega(1)\}$ have been already defined. Let $K^t = t$. Since $f$ is a homomorphism of order-preserving trees, $f$ is order-preserving itself. Thus, $K^t f = t < K_f = x$ implies $t' < y$ for the leading element $t'$ of $K'$. The level of $t'$ cannot be less than $t$, therefore $t' \prec t$. Then we have $\Gamma_i(t') \rightarrow \Gamma_i(t)$, whence $s(K)f = s(x), d(K)f = d(x)$ are well-defined.

Definition 4.4 Let $\langle \pi, \prec \rangle$ be a decreasing tree. Fix a convex partition $\bar{K}$ of $\pi$ into $m$ blocks, $1 \leq m \leq n$. Let $f : T(\bar{K}) \rightarrow \langle \pi, \prec \rangle$ be a homomorphism. Denote by $\psi^\bar{K}$ the transformation of $\pi$ with the partition $\bar{K}$ such that $K, \phi^\bar{K} = \{K' \in K \} = K_f$ for all $K_f \in K$.

Example 4.5 Let $T_3(5)$, $\bar{K}$ be as in Example 4.4 For $\{3, 4\} = 3, \{1, 2\} = s(3) = 1, \{5\} = d(3) = 4, we have $\psi^\bar{K} = \left( \begin{array}{ccc} 1 \{12\} \{34\} \{35\} \begin{array}{c} 1 \begin{array}{c} 3 \begin{array}{c} 4 \end{array} \end{array} \end{array} \end{array} \right)$.

We denote by $\Phi_\prec$ the set of transformations $\phi^\bar{K}$ for all possible convex partitions $\bar{K}$ of $\pi$. The set $\Phi_\prec$ for $T_3(5)$ is presented in Table II.

Lemma 4.6 The set $\Phi_\prec$ constitutes an $\mathcal{R}$-cross-section of semigroup $\mathcal{O}_n$.

Proof Let $\langle \pi, \prec \rangle$ be a fixed order-preserving tree on $\pi$, $T(\bar{K})$ be the order-preserving tree of a fixed partition $\bar{K}$ of $\pi$. Since no other decreasing tree appears in the proof, an element of $\Phi_\prec$ is completely determined by the partition $\bar{K}$ of $\pi$. Thus we will write $\psi^\bar{K}$ for brevity. Let $K \in \bar{K}$, and $K \phi^\bar{K} = v$. Then according to Definition 4.4

$s(K) = K' \iff K' \phi^\bar{K} = s(v)$,

d$(K) = K'' \iff K'' \phi^\bar{K} = d(v)$,
First we need several auxiliary results. Consider an \( R \)-cross-section of \( O_\infty \). We begin with simple properties of \( R \).
Lemma 4.7 The following statements hold:

(i) Each $\mathcal{R}$-cross-section of $\mathcal{O}_n$ has a fixed point.

(ii) An $\mathcal{R}$-cross-section of $\mathcal{O}_n$ has at most two fixed points: 1 and $n$.

Proof (i) Let $c_r \in R$ be a constant transformation. Since for all $\alpha \in R$ it holds $c_r \alpha = c_r \alpha \in R$, and thus $c_r \alpha = c_r \alpha$, we get $r \alpha = r$.

(ii) Note that the points 1 and $n$ are the only ones that always belong to different classes of each 2-element convex partition of $\pi$. Therefore, if $R$ has two fixed points then the only possible ones are 1 and $n$.

Definition 4.8 Given an $\mathcal{R}$-cross-section $R$ of $\mathcal{O}_n$ and its fixed point $r$, construct a strictly increasing chain $W_0 = \{r\} \subset W_1 \subset W_2 \subset \ldots \subset W_t = \pi$ of subsets of $\pi$ by the rule:

$$W_i = W_{i-1} \cup \{x \in \text{im}(\alpha) \mid \alpha \in R, |\text{im}(\alpha)| = i + 1, x \notin W_{i-1}\}, 1 \leq i < n.$$  

In virtue of Lemma 4.7 the chain always has at least two components.

By $W'_i$ denote the set $W_0$ if $i = 0$ and $W_i \setminus W_{i-1}$ if $1 \leq i < t$.

Recall that a transversal of a partition is called a set that contains exactly one representative from each block of the partition.

Definition 4.9 Let $v$ be a vertex of an order-preserving tree $(\pi, \prec)$ with $|\omega(v)| = k$. Denote by $\Theta^v \subset \Phi_\pi$ the set of all transformations $\Theta^v$ such that $\omega(v)$ is a transversal of $\pi/\ker(\Theta^v)$.

We will write also just $\Theta^v$ if the length of the path does not matter for us. Note that by the definition of $\Phi_\pi$, we get immediately $\text{im}(\Theta^v) = \omega(v)$ for all $\Theta^v \in \Theta^v$. Thus, $\Theta^v \subset \Phi_\pi$ is a set of full idempotents whose image is $\omega(v)$. Furthermore, it is easy to see that $\Theta^v$ is a semigroup of left zeroes for each $v \in (\pi, \prec)$.

Lemma 4.10 Let $R$ be an $\mathcal{R}$-cross-section of $\mathcal{O}_n$. Then the sets $W_0, W'_1, \ldots, W'_t$ form the levels of an order-preserving tree such that for every $v \in W_t$ it holds $\Theta^v \subset R$.

Proof We induct on the number of levels $t$.

The base of induction holds. Indeed, if $c_r \in R$ then $W_0 = \{r\}$. Thus the root of required binary tree $r = r$, $1 \leq r \leq n$. In this case $\Theta^v = c_r \in R$.

Suppose the lemma holds for a natural $s$, $1 \leq s < t$: there defined a unique order-preserving tree with the vertex set $W_s$. The tree’s levels as sets are exactly $W_0, W'_1, \ldots, W'_s$. Also we have $\Theta^v \subset R$ for all $v \in W_s$. Let $v \in W'_s$ and $a, b \in \pi$ be the canonical bounds of $v$ as a vertex of the tree, $a < b$.

Without loss of generality we consider the left interval $[a, v - 1]$. Observe that if $v = 1$ or $a \notin \omega(v)$ and $v = a + 1$ then there is nothing to prove. Otherwise we will construct an interval $K \subseteq [a, v - 1]$ with $K \cap W_s = \emptyset$; and $\alpha \in R$ such that $K \subseteq \pi/\ker(\alpha)$ and $K \alpha \in W'_{s+1} \cap K$. Then we show that $|W'_{s+1} \cap K| = 1$.

Case 1. Let $a \notin \omega(v)$, therefore $a = 1$. Let $K := [1, v - 1]$. By the definition of order-preserving binary tree $K \cap W_s = \emptyset$. Let $\alpha \in R$ be such that $K \in \pi/\ker(\alpha)$ and the set $\{1\} \cup \omega(v)$ is a transversal of $\pi/\ker(\alpha)$. Since $|\omega(v)| = s + 1$ we get $|\text{im}(\alpha)| = s + 2$. Hence $K \alpha \in W'_{s+1}$. By the assumption $\Theta^v \subset R$. Let $\Theta^v \in \Theta^v$. We have $\ker(\Theta^v \alpha) = \ker(\Theta^v)$, therefore $\Theta^v \alpha = \Theta^v$, whence $\alpha|_{\omega(v)} = \text{id}_{\omega(v)}$. Since $\alpha$ is order-preserving, we get $K \alpha < v \alpha = v$. Thus, $K \alpha \in W'_{s+1} \cap K$.

Case 2. Let $a \in \omega(v)$. Then denote $K := [a + 1, v - 1]$. Since $a$ is a canonical bound of $v$, we get $K \cap W_s = \emptyset$. Consider $\alpha \in R$ such that $\{a + 1\} \cup \omega(v)$ is a transversal of $\pi/\ker(\alpha)$, $K \subset \pi/\ker(\alpha)$. Again, we get $\ker(\Theta^v \alpha) = \ker(\Theta^v)$, whence $a = a \alpha < K \alpha < v \alpha = v$. 

\[ a = a \alpha < K \alpha < v \alpha = v. \]
Thus \((a + 1) \alpha \in W_{t+1} \cap K\).

Now we proceed for both cases. We will show that \(|W_{t+1} \cap K| = 1\). Denote \(K \alpha = x\) for \(x \in K\) and suppose there exists \(y \in K \cap W_{t+1}\) such that \(y \neq x\). That is, there exists \(y \in R\) with \(y \in \operatorname{im}(\gamma)\) and \(|\operatorname{im}(\gamma)| = s + 2\). Consider the composition \(\gamma \alpha\). Clearly, \(x \in \operatorname{im}(\gamma \alpha)\). By the definition of a chain \(W\) the condition \(x \in W_{t+1}\) implies \(|\operatorname{im}(\gamma \alpha)| = s + 2\). Consequently, \(\ker(\gamma \alpha) = \ker(\gamma)\), whence \(\gamma \alpha = \gamma\), which contradicts \(x \neq y\). Therefore, \(x \in W_{t+1}\) is unique. Thus, \(x\) is the son of \(v\) and it is well-defined.

In particular, the above proof imply that for all \(\beta \in R\) such that \(\{x\} \cup \omega(v)\) is a transversal of \(\pi/\ker(\beta)\), it holds \(\omega(v) \beta = \omega(v)\) and \(x \beta = x\). Therefore, by Definition 4.13 \(\beta \in \Theta^r\). Hence, \(\Theta^r \subseteq R\).

\(\Box\)

**Corollary 4.11** Let \(R\) be an \(\mathcal{R}\)-cross-section of \(O_n\). \(W_0, W_1, \ldots, W_t\) the levels of the order-preserving tree \((\pi, \prec)\) associated with \(R\). Then the following statements hold.

(a) It holds \(|W_i| \leq 2^k\) for all \(i, 0 \leq i \leq t\). For all \(\alpha \in R\) with \(\operatorname{im}(\alpha) \supseteq W_i\) it holds \(|\operatorname{im}(\alpha)| \leq \sum_{k=0}^{i} 2^k\).

(b) Let \(\alpha \in R, v \in \operatorname{im}(\alpha)\). Then \(\omega(v) \subseteq \operatorname{im}(\alpha)\).

**Proof**

(a) Since \(W_t = \emptyset\), the proof follows directly from the definition of a binary tree.

(b) Suppose \(\omega(v) \subseteq U_{k+1} \cap \operatorname{im}(\alpha)\), \(1 \leq k \leq t + 1\). The condition \(v \in U_{k+1} \cap \operatorname{im}(\alpha)\) implies \(|\operatorname{im}(\alpha)| \geq k\). Let \(\theta^r \in \Theta^r_k\) be a fixed transformation. Then we have \(v \in \operatorname{im}(\alpha \theta^r)\), hence \(|\operatorname{im}(\alpha \theta^r)| \geq k\) for all \(\theta^r \in \Theta^r_k\). Since \(\theta^r = \theta^r_{\omega(v)}\), \(\beta^r\) contains a transversal of a partition of \(\theta^r\). In virtue of arbitrariness \(\theta^r \in \Theta^r_k\), we have

\[\operatorname{im}(\alpha) \supseteq \{p_0, p_1, \ldots, p_{k-1}\}\]

\(\Box\)

We denote by \((\pi, \prec_R)\) the order-preserving binary tree associated with \(R\).

**Lemma 4.12** The order-preserving tree \((\pi, \prec_R)\) is decreasing.

**Proof** Case when \(n = 1, 2\) is trivial. Assume \(n > 2\). Let \(x, y \in \pi\) be such that \(x \prec y\). Let \(x \in W_k\), \(y \in W_m\), \(k < m\). To prove that the tree is order-preserving we need to verify conditions 1)-3) of Definition [3].

1. Let \(x, y \in \omega(1)\). We denote by \(T_i(x)\) and \(T_i(y)\) the right-side subtrees of \((\pi, \prec_R)\) rooted at \(s(x)\) and \(s(y)\) respectively (the respective tree is empty if the vertex has no son). To prove that condition 1) of Definition [3] holds we will show that there exists a homomorphism between \(T_i(x)\) and \(T_i(y)\). Apparently, the existence of the homomorphism implies that the right inner tree of \(x\) subordinates the right inner tree of \(y\). We will construct the transformation \(\alpha \in R\) such that if \(T_i(x) \neq \emptyset\) then \(T_i(x) \alpha \subseteq T_i(y)\). We will show that \(\alpha |_{T_i(x)}\) is a homomorphism.

Let \(\alpha \in R\) be such that \([x, p(x) - 1] \cup \omega(p(y))\) are the leading elements of the partition \(\pi/\ker(\alpha)\). Since \(p(x), y\) are higher than \(x\), we get \(p(x) \alpha = y \alpha = p(y) \alpha\).

We claim that \(x \alpha = y\). \(p(x) \alpha = p(y)\). Indeed, let \(\theta^r \in \Theta^r_{k+1}\), \(\theta^r \theta^r_{\omega(1)} \in \Theta^r_{\omega(1)}\). Note that \(|\operatorname{im}(\theta^r)| = m + 1\). Thus, \(x \alpha \in W_m\). Further, we have \(\ker(\theta^r) \alpha = \ker(\theta^r)\). Consequently, \(p(x) \alpha = p(y) \alpha = p(y)\). By the definition of an order-preserving tree and since \(\alpha \in O_n, y\) is the only element in \(W_m\) which is less than \(p(x) \alpha = p(y)\). Therefore, \(x \alpha = y\).

If \(x\) has no daughter then \(T_i(x)\) is empty, whence \(T_i(x)\) subordinates \(T_i(y)\) by the definition. Suppose \(|T_i(x)| \geq 1\). We will show that \(T_i(x) \rightarrow T_i(y)\). We induct on the level of a vertex of \(T_i(x)\). Consider the root \(d(x)\) of the tree. We claim that \(d(x) \alpha \in W_{m+1}\). Indeed, the condition \(|\operatorname{im}(\theta^r)\alpha)| = |\operatorname{im}(\theta^r)| + 1 = m + 2\), implies the required. Moreover, since \(\alpha \in O_n\), we get

\[x \alpha = y < d(x) \alpha < d(y) = p(y) = p(x) \alpha\]

Whence the only possibility is \(d(x) \alpha = d(y)\) and the induction base holds.

Suppose \(\alpha\) induces a homomorphism \(T_i(x) \rightarrow T_i(y)\) for the first \(l\) levels of \(T_i(x)\). Note that as was shown above \(f\) preserves the canonical bounds of the vertices of the subtree. Let
\(v \in T \cap W_{k+1}\) be a vertex, \(v_1, v_2\) be its canonical bounds, \(v_1 < v_2\). By the assumption we have \(v \alpha \in T \cap W_{m+1}\). Without loss of generality suppose \(v\) has a son. Then we have \(|\text{im} (\theta (v) \alpha)| = |\text{im} (\theta (v) \alpha)| + 1\). Therefore, \(s(v) \alpha \in W_{m+1}\) and \(v_1 \alpha < s(v) \alpha < v \alpha\). Hence, \(s(v) \alpha\) is the son of \(v \alpha\) by the definition of an order-preserving tree. Consequently, \(T \rightarrow T \omega\).

If \(y > r\) and \(x, y \in \omega(n)\) then in dual way \(T \rightarrow T \omega\), where \(T \omega\) is the left-side subtrees of \((\pi, \omega)\) rooted at \(s(x)\) and \(s(y)\) respectively. Hence condition 2) of Definition 4.3 also holds.

2. Let \(x, y \notin \omega(1) \cup \omega(n)\). We denote by \(T \omega\) the subtrees of \((\pi, \omega)\) rooted at \(x\) and \(y\) respectively. Note that to prove condition 3) of Definition 4.3 it is enough to show that there exists a homomorphism between \(T \omega\) and \(T \omega\). As an the previous item we will construct the transformation \(\beta \in R\) such that \(T \omega \beta \subseteq T \omega\). \(\beta\) is a homomorphism. Without loss of generality suppose \(x < y < r\). The condition \(x < y\) implies that there exists a minimal ancestor \(c \in \omega(1)\) for both \(x, y\). Furthermore, there are no \(t \in \omega(y)\) with \(x < t < y\). Thus, we have \(c < x \leq y < p(e)\). Moreover, let \(a, b\) be the canonical bounds of \(x, a < b\); \(q, h \in \omega(y)\) be the canonical bounds of \(y\). Then it holds \(a < b < h \leq r\). Let \(\beta \in R\) be a transformation such that \(\omega(p(y)) \cup T \omega\) are the leading elements of partition \(\pi/\ker(\beta)\). Then it holds \(q \beta = a \beta\) and \(b \beta = h \beta\), since \(a, b, y, h\) are higher than \(x\).

Let \(\theta(p(y)) = \theta(p(y))\). Then \(\theta(p(y)) \beta = \theta(p(y))\). Thus, the set \(T \omega \beta\) is bounded by the canonical bounds of \(y\), i.e. \(T \omega \beta \subseteq T \omega\). Then \(|\text{im} (\theta (v) \beta)| = |\text{im} (\theta (v) \alpha)| + 1\), where \(x \beta \in W_{m+1}\). Thus, the only possibility is \(x \beta = y\).

If \(T \omega\) is 1-element then the required is proved. Suppose \(\beta\) induces a homomorphism of the first \(l\) levels of \(T \omega\) to \(T \omega\). Let \(v \in T \cap W_{m+1}\); \(v_1, v_2\) be the canonical bounds of the vertex, \(v_1 < v_2\). Without loss of generality suppose \(v\) has the son. Just as in the previous item one can show that \(s(v) \beta\) defines the son of \(v \beta\). Dually, \(d(v) \beta\) defines the daughter of \(v \beta\). Therefore, \(T \omega\) subordinates \(T \omega\) as required.

If \(y < x < r\) then it holds \(q \beta = a \beta\) and \(b \beta = h \beta\). The further proof is exactly the same. If \(r > y\) we also get the same cases.

Now we are ready to prove

Lemma 4.13. For every \(\pi\)-cross-section \(R\) of \(\omega\) coincides with \(\Phi^\omega\) for a fixed decreasing binary tree \((\pi, <)\).

Proof. Consider a \((t + 1)\)-level decreasing binary tree \((\pi, <)\) that arises with an \(\pi\)-cross-section \(R\) of \(\omega\). We claim that a transformation \(\alpha \in R\) with a partition \(K\) has the form \(\Phi^K\). The proof is by induction on the level \(U_t\) of the partition tree \(T(K)\).

Suppose \(t = 0\). Then \(K^{\hat{\alpha}} = \alpha\) by Lemma 4.7. So the base of induction holds. Assume that there exists \(k, 0 < k < t\), such that \(\alpha|_{U_0 \cup \ldots \cup U_{k-1}} = \Phi^{K|_{U_0 \cup \ldots \cup U_{k-1}}}\).

Let \(X \in U_{k-1}\). By the assumption \(X \alpha = X \Phi^K\). Let \(X \alpha = x, a, b\) be the canonical bounds of \(x, a < b\). By the construction of \(\Phi^K\) we have \(x \in W_{m}, m < k - 1\). If \(X\) has no children then there is nothing to prove. Without loss of generality suppose \(X\) has the son \(S \in T(K)\). Let \(S \alpha = y\). Since \(\alpha \in \Omega_n\) we get \(x, a, x\) are the canonical bounds of \(y\). In other words, \(y\) is a vertex of the subtree of \((\pi, <)\) rooted at \(s(x)\). Therefore, \(\omega(x) \subseteq \omega(y)\). Note that for all \(v \in S\) we have \(\text{im} (\theta (v) \alpha)\). On one hand, we have \(|\text{im} (\theta (v) \alpha)| = |\text{im} (\theta (v) \alpha)| + 1\). On the other hand, by Corollary 4.11 we get \(\omega(y) \subseteq \text{im}(\theta (v) \alpha)\). Therefore \(\gamma = s(x) = S \Phi^K\) as required.

Thus, Lemmas 4.6 and 4.13 imply immediately the following fact.

Theorem 4.14. Let \((\pi, <)\) be a decreasing binary tree. Then semigroup \(\Phi^\omega\) is an \(\pi\)-cross-section of \(\omega\). Conversely, each \(\pi\)-cross-section of \(\omega\) is given by the semigroup \(\Phi^\omega\) for a decreasing binary tree.
5 On $L$-cross-sections of $\mathcal{O}_n$ and $R$-cross-sections of $\mathcal{O}_{n+1}$

We have pointed already that there is a connection between $L$-cross-sections of $T_n$ and $R$-cross-sections of $\mathcal{O}_{n+1}$. In this section we discuss the connection in more detail. A key notion in the description of $L$-cross-sections of $T_n$ plays a notion of a respectful tree (the definition will be stated below). It turns out, that the structure of an $R$-cross-section of $\mathcal{O}_n$ has an alternative interpretation in terms of respectful trees.

5.1 Respectful trees

Recall the notion of a respectful tree as it was proposed in [3]. If $u$ and $v$ are two vertices of the same tree $T$, we say that $u$ subordinates $v$ if the subtree rooted at $u$ subordinates the subtree rooted at $v$.

**Definition 5.1** A full binary tree is said to be respectful if it satisfies two conditions:

(S1) if a male vertex has a nephew, the nephew subordinates his uncle;
(S2) if a female vertex has a niece, the niece subordinates her aunt.

For an illustration, the tree shown in Fig. 9 satisfies (S1) but fails to satisfy (S2): the daughter of the root has a niece but this niece does not subordinates her aunt. On the other hand, the tree shown in Fig. 10 is respectful. (In order to ease the inspiration of this claim, we have shown the uncle–nephew and the aunt–niece relations in this tree with dotted and dashed arrows respectively.)

![Figure 9: An example of a non-respectful tree](image)

To preserve the notations of previous papers concerning respectful trees we will denote the respectful tree by $\Gamma$. There are also equivalent definition of respectful tree (see, [2]). Using current terminology the one can be stated as follows:

**Proposition 5.2** ([2], Proposition 1) A full binary tree $\Gamma$ is respectful if and only if every non-root vertex subordinates his (her) parent.

5.2 The faithful interval marking of a respectful tree

We have already used the intervals as labels of binary tree vertices when were spoken about the inner trees and the partition trees. In our construction for $L$-cross-sections of $T_n$, we also use certain markings of respectful trees by intervals of the set $\pi$ considered as a chain under a fixed linear order:

$$u_1 \prec u_2 \prec \ldots \prec u_n.$$
Let $\Gamma$ be a fixed respectful tree. If $i, j \in \pi$ and $i \leq j$, the interval $[u_i, u_j]$ is the set $\{k \in \pi \mid u_i \preceq k \preceq u_j\}$. We write $[u_i]$ instead of $[u_i, u_i]$. Now, a faithful interval marking of a tree $\Gamma$ is a map $\mu_\preceq$ from the vertex set of $\Gamma$ into the set of all intervals in $\pi$ such that for each vertex $v$,

- the number of leaves in the subtree $\Gamma(v)$ rooted at $v$ is equal to the number of elements in the interval $v\mu_\preceq$;
- if $v\mu_\preceq = [u_p, u_q]$ for some $u_p, u_q \in \pi$ with $u_p \preceq u_q$, and $s$ and $d$ are respectively the son and the daughter of $v$, then $s\mu_\preceq = [u_p, u_t]$ and $d\mu_\preceq = [u_t, u_q]$ for some $t$ such that $p \leq t < q$.

It easy to see that given a fixed linear order $\preceq$ every respectful tree $\Gamma$ admits a unique faithful interval marking $\mu_\preceq$.

Fig. 11 demonstrates a faithful interval marking $\mu_\preceq$ for usual order $<$ of the tree from Fig. 10.

For the respectful tree $\Gamma$ with its faithful interval marking $\mu_\preceq$ we will not make a difference between a vertex of the tree and the interval associated with it.

### 5.3 $L$-cross-sections of $T_n$

First we recall the description of $L$-cross-sections of $T_n$ for convenience. Let $\Gamma$ be a respectful tree with its faithful interval marking. For each $M \subset \pi$ we construct inductively the transformation with the image $M$, using partial transformations, whose domains go through vertices of $\Gamma$ bottom up.
For functions $f$ and $g$ with disjoint domains, we denote by $f \cup g$ the union of $f$ and $g$ (viewed as sets of pairs). In other words, if $h = f \cup g$, then $\text{dom}(h) = \text{dom}(f) \cup \text{dom}(g)$ and for all $x \in \text{dom}(h)$, $xh = xf$ if $x \in \text{dom}(f)$, and $xh = xg$ if $x \in \text{dom}(g)$.

Let $M \subset \pi$, by $\langle M \rangle$ we denote the intersection of all vertices of $\Gamma$ which contains $M$. Given $\langle M \rangle = B$ and $A \in \Gamma$, we define the mapping $\alpha_M^A$ inductively as follows:

(a) if $A = \emptyset$ then $\alpha_M^\emptyset = \emptyset$ (empty mapping);
(b) if $M = \{m'\}$ and $A \neq \emptyset$, then $\text{dom}(\alpha_M^A) = A$ and $x'\alpha_M^A = m'$ for every $x' \in A$;
(c) if $|M| > 1$ and $A \neq \emptyset$, then $\alpha_M^A = \alpha_{M\setminus\{m\}}^A \cup \alpha_{M\setminus\{m'\}}^A$.

We denote by $\alpha_M$ the full transformation $\alpha_M^n$. The description of $\mathcal{L}(\mathcal{T}_n)$ gives the following:

**Theorem 5.3 (1, Theorem 1)** For every respectful tree $\Gamma$ with $n$ leaves and its faithful interval marking $\mu_{\prec}$, the set $L^\Gamma = \{\alpha_M | M \subset \pi\}$ forms an $\mathcal{L}$-cross-section of $\mathcal{T}_n$. Conversely, every $\mathcal{L}$-cross-section of $\mathcal{T}_n$ is given by $L^\Gamma$ for a respectful tree $\Gamma$ with a faithful interval marking.

It remains to recall the notion of similar respectful trees (see 2). We have already defined a homomorphism of the binary trees. It is natural to define an isomorphism between two trees as a bijective homomorphism between the ones.

By an anti-isomorphism between two trees $T_1$ and $T_2$ we mean the bijective mapping that sends root of $T_1$ to the root of $T_2$, preserves the parent-child relations and inverses the genders of the vertices.

**Definition 5.4** Recall that respectful trees $\Gamma_1$, $\Gamma_2$ are called similar (write $\Gamma_1 \sim \Gamma_2$) if there exists either isomorphism or anti-isomorphism between the trees.

In other words, $\Gamma_1 \sim \Gamma_2$ if $\Gamma_1$ is a mirror reflection of $\Gamma_2$.

**Example 5.5** Consider an example of similar respectful trees.

![Figure 12: A pair of similar respectful trees](image)

We need to recall also the following lemma.

**Lemma 5.6 (2, Lemma 2)** Let $\Gamma_1$, $\Gamma_2$ be the respectful trees with $n$ leaves and the faithful interval markings $\mu_{\prec_1}$, $\mu_{\prec_2}$ respectively. Then the $\mathcal{L}$-cross-sections associated with $\Gamma_1$, $\Gamma_2$ are coincide if and only if one of the following conditions holds:

(i) $\Gamma_1 = \Gamma_2$;
(ii) there exists an anti-isomorphism from $\Gamma_1$ to $\Gamma_2$ and $\prec_2 = \prec_1^{-1}$.

Now we are ready to describe $\mathcal{L}(\mathcal{T}_n)$.

**Theorem 5.7** Let $\Gamma$ be respectful tree with $n$ leaves and a faithful interval marking $\mu_{\prec}$, where $<$ is the usual order on $\pi$. The set $L^\Gamma = \{\alpha_M | M \subset \pi\}$ forms an $\mathcal{L}$-cross-section of $\mathcal{T}_n$. Conversely, every $\mathcal{L}$-cross-section of $\mathcal{T}_n$ is given by $L^\Gamma$ for a respectful tree $\Gamma$ with a faithful interval marking $\mu_{\prec}$.
5.4 Elementary trees

We define now a special type of decreasing trees, called elementary, which, we will see, generate in a sense all decreasing trees.

Definition 5.8 Say a subtree \( T^k \) of the decreasing tree to be elementary, if \( U \subseteq U_1 = \{ x, p(x) \} \) and for all \( v \in T^k \), \( |T^k| > 2 \) it holds either \( x < v < p(x) \) or \( p(x) < v < x \).

Looking at the diagram of an order-preserving tree (decreasing, in particular) it is easy to see that the tree can be considered as a union of elementary trees. However, at the moment we are interested in a special case when the decreasing tree is elementary itself. That is, it holds either \( (\pi, <) = T^k \) or \( (\pi, <) = T^n \). In this case, clearly, the elementary trees correspond to \( \mathcal{S} \)-cross-sections of \( O_n \) with two fixed points. Again, if an \( \mathcal{S} \)-cross-section has two fixed points then by Definition 5.4 we get \( U_0 \cup U_1 = \{ 1, n \} \). Thus the cross-section corresponds to one of the elementary trees \( T^1, T^n \).

On the other hand, by Proposition 5.5 each dual to an \( \mathcal{S} \)-cross-section of \( O_n \) gives us the pair of \( \mathcal{S} \)-cross-sections of \( O_{n+1} \) with two fixed points. In turn, each \( \mathcal{S} \)-cross-section is determined by a respectful tree. We will see now that a respectful tree determines the pair of elementary trees on \( [n, 1] \).

Lemma 5.9 Given a respectful tree \( \Gamma(n) \) with \( n \)-leaves, there exists a unique (up to the choice of the root) diagram of \( (n + 1) \)-element elementary trees \( T^1, T^{n+1} \) such that \( \Gamma \) is the common inner tree of the trees.

Proof We induct on the number \( n \) of leaves of \( \Gamma(n) \). Let \( n = 1 \). Obviously \( T^1, T^2 \) are the elementary trees with the common inner tree \( \Gamma(1) \).

Let \( k \) be a natural number. Suppose the claim holds for all natural \( l < k \). Consider a respectful tree \( \Gamma(k) \). Without loss of generality denote the root of \( \Gamma(k) \) by \( r' := [l', k'] \).

Consider the binary trees \( T^1, T^{k+1} \) such that \( U_0 \cup U_1 = \{ 1, k + 1 \} \) and \( \Gamma(k) \) is the inner tree for both of trees. For concreteness consider \( T^{k+1} \). Let \( d(1) := b \). Let there exist \( a \in T^{k+1} \) with \( y < x \). Without loss of generality assume that \( a < b \) (see Fig. 13). Then the left inner tree of \( a \) subordinates the left inner tree of \( b \) by Proposition 5.2. The right-hand inner tree of \( a \) is the niece of \( [b', k'] \). Hence, the right-hand inner tree of \( a \) subordinates the right-hand inner tree of \( b \).

The converse statement also holds: given an elementary tree determines the respectful tree.

Lemma 5.10 Let \( (\pi, <) \) be a decreasing binary tree, \( x, p(x) \in \pi \). The inner tree of elementary tree \( T^n \) is respectful.
Without loss of generality assume $p(x) < x$. Consider the inner tree $\Gamma$ of $T^1$. We induct on the number leaves of $\Gamma$. If $|\Gamma| \leq 2$ then there is nothing to prove. Let $\Gamma$ have $k$ leaves, $k > 2$. Denote by $b$ the son of $x$. Without loss of generality assume that $b$ has the son $a$ (again, one may use for the illustration Fig. 13 having in mind that $\tilde{\Gamma}$). By the definition of a decreasing tree, we have that the inner trees of $a$ subordinate the respective inner trees of $b$, which implies directly that the niece $[a',(b-1)'] \in \Gamma$ subordinates her aunt $[b',(x-1)'] \in \Gamma$. In the same way if there exists the nephew of $p(x)$, then the nephew subordinates his uncle. Consider the tree $\tilde{T}$, which implies directly that the niece $\tilde{\Gamma}$ is respectful. Consider the tree $T$ whose first levels $U_0 \cup U_1 = \{p(x), b\}$ and the inner tree coincides with $\Gamma(b)$. By the condition the diagram determines an order-preserving tree. Thus, $T$ is a respectful tree. Therefore, by the induction assumption the inner tree is respectful which completes the proof.

We need now to generalize the notion of canonical bounds for the vertices of a partition tree with respect to an elementary tree. Let $T^i$ be an elementary tree, $\tilde{K}$ be a convex $(i + 1)$-element partition of $[p(x), x]$. Consider the partition tree $T(\tilde{K})$. Let $k$ be the depth of the tree. Since $T^i$ is elementary, the vertices of first two levels of $T(\tilde{K})$ always contains the end-points of the interval $[p(x), x]$. Therefore, we define the canonical bounds for the vertices $X \in U_i$, $2 \leq i \leq k$. We induct on the level $i$ of a vertex $X \in \tilde{K}$. Note that in the case we have always $|U_2| = 1$. Without loss of generality assume $p(x) < x$. Let $i = 2$, then define $K'_i(p(x)) < X < K'_i(b)$. Let $X \in U_i$, $2 \leq q < k$. Assume $A, B \in T(\tilde{K})$ be the canonical bounds of $X$, $A < X < B$. If $X$ has the son $s(X)$ then $A < s(X) < X$. If $X$ has the daughter $d(X)$ then $X < d(X) < B$.

Now we will see that due to the connection between respectful and elementary trees, having a respectful tree one can get immediately the diagram of the dual $(L^1)^\ast$.

**Lemma 5.11** Let $\Gamma$ be the inner tree of an elementary tree $([n+1], \prec)$. Then

$$(L^1)^\ast = \Phi_\prec \setminus \{e_r\}. $$

**Proof** Consider the $\mathcal{L}$-cross-section $L^1$, associated with $\Gamma$ and the $\mathcal{R}$-cross-section $\Phi_\prec$, associated with $([n+1], \prec)$. Let $m \in L^1$ with $M = \{r_1, r_2, \ldots, r_t\}$. Then $M$ induces the $(t+1)$-element convex partition $K_M$ of $[n+1]$: $K_M = [1,r_1] \cup [r_1 + 1, r_2] \cup \ldots \cup [r_t + 1,n+1]$.

![Figure 13: A connection between respectful and decreasing trees](image-url)
Similarly, let $w \in (n+1, \infty)$ be the vertex induced by $s(B)$ and $d(B)$. Therefore, $K^{(w+1)}$ is the daughter of $K^{(w)}$ and $K^{(w+1)} \rho^K_M = z + 1$.

Let
\[ X \cap s(B) = [u', p'], \quad X \cap d(B) = [q', (v - 1)']. \]

By the definition $\alpha^A_M = \alpha^A_M \cap s(B) \cup d(B)$, whence $\alpha^A_M = p'$, $(z + 1)' \alpha^A_M = q'$. Consequently $p + 1, q \in K^{(w+1)}$ and $K^{(w+1)} \alpha^A_M = z + 1$ as required. \hfill $\Box$

Therefore, we get the following fact.

**Theorem 5.12** The $\mathcal{R}$-cross-section of $O_{n+1}$ has two fixed points if and only if it is produced by an $\mathcal{L}$-cross-section of $O_n$.
Proof. Necessity. Suppose \( R \) has two fixed points and let \( ([n+1],[<]) \) be the decreasing tree associated with \( R \). As we observed the tree \( ([n+1],[<]) \) is elementary. Thus, on one hand \( c_0 \in \{c_1,c_{n+1}\} \). On the other hand by Lemma 5.10 the inner tree \( \Gamma \) is respectful. Therefore, by previous lemma \( \Phi = (L^{\Gamma})^* \cup \{c_0\} \).

Proposition 5.5 implies immediately the sufficiency. \( \square \)

5.5 Decomposition of the decreasing trees

Clearly, every order-preserving tree can be presented as a union of elementary trees in many ways. Consider the set \( \omega(n,r) = \omega(1) \cup \omega(n) \) ordered with respect to \( < \). We will denote it also by \( \omega(R) \). The set is defined by the tree in a unique way and always has at least two elements 1 and \( n \). We can present the original tree as a union of the following elementary subtrees (see Fig. 15):

\[
T_1, T_p(1), T_d(n), T_{p(n)}, T_{n-1}, \ldots
\]

(2)

Figure 15: A minimal decomposition of an order-preserving tree

On one hand, it is easy to see that any other decomposition of \( T(n) \) into elementary trees, contains more components. On the other hand, for all \( a \in \omega(R) \) there is no elementary tree \( T \subseteq (\pi, \preceq) \) such that \( T \subseteq T \), \( x \in \pi \). We say the decomposition (2) is minimal.

Now the definition of decreasing tree and Lemma 5.10 imply immediately the following

**Theorem 5.13** Let \( a_1 = 1 < a_2 < \ldots < a_{t-1} < a_k = n \) be a sequence of natural numbers with a marked point \( a_t \), \( 1 \leq t \leq k \leq n \). Let \( n_i = a_{i+1} - a_i, \Gamma(n_i) \) denote a respectful tree on \( n_i \)-element set such that

\[
\Gamma(n_1) \hookrightarrow \Gamma(n_2) \hookrightarrow \ldots \hookrightarrow \Gamma(n_k),
\]

\[
\Gamma(n_{k-1}) \hookrightarrow \Gamma(n_{k-2}) \hookrightarrow \ldots \hookrightarrow \Gamma(n_1).
\]

The sequence with a marked point \( a_t \) and the trees define a unique decreasing tree.

Since every decreasing tree determines the \( \mathcal{R} \)-cross-section, we can regard the last theorem as the alternative description of \( \mathcal{R} \)-cross-sections of \( \mathcal{O}_n \) in terms of respectful trees.

In particular, the sequence \( \omega(\pi,r) = \{1\} \) with a respectful tree, define the dual \( \mathcal{R} \)-cross-section up to the choice of the root. The other special case is when \( \omega(\pi,r) = \{1,2,\ldots,n\} \) with \( r = 1 \) or \( r = n \). We get in these cases the dense cross-sections.
6 Classification of $\mathcal{R}$-cross-sections of $\mathcal{O}_n$ up to an isomorphism

6.1 Necessary conditions

Let $R$ and $\hat{R}$ be the $\mathcal{R}$-cross-sections of $\mathcal{O}_n$ associated with diagrams $(\pi, \prec_1)$ and $(\pi, \prec_2)$ respectively.

First we will look for necessary conditions two $\mathcal{R}$-cross-sections to be isomorphic. We will see in the following lemma that the sets of idempotents $\Theta^\xi$ play an important role in the structure of $\mathcal{R}$-cross-sections. To avoid the ambiguous we write $\hat{U}_i$ for the $i$-th level of $\hat{R}$.

Lemma 6.1 Let $R \cong \hat{R}$. Then for each $x \in U_k$ with $0 \leq k < n$ there exists $y \in \hat{U}_k$ such that

$$\Theta^x \xi = \Theta^y.$$ 

Moreover, the mapping $\xi : (\pi, \prec_1) \to (\pi, \prec_2) : x \mapsto y$ if and only if $\Theta^x \xi = \Theta^y$, is well-defined, bijective and preserves the levels of the vertices.

Proof We proceed by induction on the level $i$ of a vertex $v \in R$. Denote by $r_1$, $r_2$ the roots of $R$ and $\hat{R}$ respectively. Since $c_{\xi} \xi = c_v$, we have $\Theta^x \xi = \Theta^y$, $r_1^x = r_2$, so the induction base holds.

Suppose the statement is true for all $U_1$, $U_i$ with $1 \leq i < k$.

Let $a \in U_{k-1}$. First note that $\omega(a^2) = \omega(a)^2$. Indeed, for all $b \in \omega(a)$, $\theta^b \in \Theta^b$ and only for them it holds $\theta^b \theta^a = \theta^b$. Therefore, $\text{im}(\theta^d) = \{a^2, \, \alpha(a)^2, \ldots, \alpha(1)^2\} = \omega(a^2)$. On the other hand, $\text{im}(\theta^d) = \omega(a^2)$. Therefore,

$$\omega(a^2) = \omega(a)^2 = \text{im}(\theta^d).$$

Let $x \in U_k$ be the son (the daughter) of $a$. Consider the transformation $\theta^x \xi$. For all $c \in \omega(x)$ and only for them we have $\theta^x \theta^c = \theta^c$, whence by the induction assumption we get $\omega(a^2) \subset \text{im}(\theta^x \xi)$. Let $y \in \text{im}(\theta^x \xi)$, $y \notin \omega(a^2)$. Then by Corollary 4.11 it holds $\omega(y) \subset \text{im}(\theta^y \xi)$. Since for all $z \in \omega(y)$ it holds $\theta^x(\theta^y \xi) = \theta^z$, the only possibility is when

$$\text{im}(\theta^x \xi) = \omega(a^2) \cup \{y\} \text{ with } \alpha(y) \in \omega(a^2).$$

In fact, we will show that $y$ is either $s(a^2)$, or $d(a^2)$. Suppose the converse. Then $\theta^x \xi \notin \Theta^y$ for all $i \in \pi$. Since $p(y) \in \omega(a^2)$, we get $y \in U_{m+k}$, $m \leq k - 1$. Therefore, by the induction assumption $y = \nu^2$ for some $\nu \in U_m$. Then, on one hand, we have $\theta^x \theta^y \in \Theta^{\nu^2}$. Therefore, by the induction assumption we get $(\theta^x \theta^y)^2 \xi \in \Theta^{\nu^2}$. On the other hand, it holds $(\theta^x \theta^y)^2 \xi = (\theta^y \xi) \theta^x = \Theta^x \xi$, which contradicts the assumption. Hence, we have that $y$ is either $s(a^2)$, or $d(a^2)$.

Further, assume $\alpha, \beta \in \Theta^x$ are such that $\alpha \xi = \theta^x(a^2)$ and $\beta \xi = \theta^x(d(a^2))$. Then we have $\alpha \beta \in \Theta^x$ and $(\alpha \beta)^2 \xi = (\alpha^2 \xi)(\beta^2 \xi) = \theta^x(a^2) \theta^x(d(a^2)) \in \Theta^x$, we get a contradiction. Thus, we get either $\Theta^x \xi \subset \Theta^x(a^2)$, or $\Theta^x \xi \subset \Theta^x(d(a^2))$.

Let $\alpha \in \Theta^x$ be such that $\alpha \xi \in \Theta^x(a^2)$. Then $\alpha$ is idempotent and for all $\theta^x$ it holds $(\alpha \theta^x)^2 \xi = \alpha \xi$. Therefore $\alpha \theta^x = \alpha$, whence $\text{im}(\alpha) = \omega(x)$. Consequently, $\alpha \in \Theta^x$ as required and $x^2$ is well-defined. Since $\xi$ is isomorphic we get $\xi$ is bijective.

Consider the mapping $\xi$ from the proof of Lemma 6.1 in more detail.

Corollary 6.2 Let $R \cong \hat{R}$. Then for all $a, b \in (\pi, \prec_1)$, $\alpha \in R$ the following holds:

1. $\omega(a^2) = \omega(a)^2$;
2. the condition $a < b$ holds if and only if $a^2 < b^2$;
3. $\text{im}(\alpha \xi) = \text{im}(\alpha)^2$. 

Proof The first claim follows immediately from the proof ofLemma 6.1. It remains to verify items (2)-(3).

(2) Indeed, it holds

\[ \theta^A \theta^B = \theta^B \theta^A = \theta^B \theta^\circ = \theta^\circ \theta^B = \alpha^B < B^B. \]

(3) Let \( x \in \pi, \beta \in R \) such that im(\beta) = \omega(x). Then for all \( \theta^B \in \Theta^B \) it holds \( \beta \theta^B \xi = \beta^B \xi = (\beta^B \xi) | \theta^B \].

To verify this, we note that there are \( 1 \in \{ a \} \) such that \( x \in \im \alpha \). By Corollary 4.11 (b), we get \( \omega(x) \subset \im \alpha \). Then it holds im(\alpha \theta^B) = \omega(x). Therefore for all \( \theta^B \in \Theta^B \) im(\alpha \theta^B) = \omega(x) = im(\alpha^B)(\theta^B) \], whence \( \omega(x^2) \subset im(\alpha^B) \). In view of arbitrariness \( x \), we get \( \im(\alpha^B) = \im(\alpha^B) \].

We also need the following lemma in the sequel.

Lemma 6.3 Given a pair of isomorphic \( \mathcal{R} \)-cross-sections \( R, \hat{R} \) of \( \partial \omega, x \in \omega(R) \), the following statements hold:

1. \( |\Theta| = 1, |\Theta^B| = |\Theta^B| \cdot |p(x) - x| \), where \( |p(x) - x| \) is the absolute difference of \( p(x) \) and \( x \).
2. \( |\xi^2(x) - x^2| = |p(x)^2 - x^2| \).

Proof (1) Clearly, \( |\Theta^B| = 1 \). Assume the statement holds for the first \( k \) levels of the tree, \( k < n \).

Let \( x \in U_{k+1} \cap \omega(R) \). Without loss of generality assume \( x < r \). Denote by \( p \) the set \( \{ x \} \). For all \( \theta^B \in \Theta^B \), we have \( 1^{\theta^B} = \ldots = x^{\theta^B} = x \) the interval \( K^B(x) \) is totally determined by its right-hand end-point. The condition \( p(x)^\theta^B = \theta^B \) implies that there are \( |p(x)| \) possibilities to choose the interval \( K^B(x) \). Hence \( |\Theta^B| = |\Theta^B| \cdot (p(x) - x) \) as required.

(2) Let \( x^2 \in \omega(\hat{R}) \). By Corollary 6.3 it holds \( p(x)^2 = p(x^2) \). Therefore \( p(x)^2 \in \omega(\hat{R}) \). According to the previous item we have

\[ |\Theta| = |\Theta^B| \cdot |p(x) - x| = |\Theta^B| \cdot |p(x)^2 - x^2| = |\Theta^2| \]

Since \( |\Theta^B| = |\Theta^B| \), we get \( |p(x) - x| = |p(x)^2 - x^2| \).

Now we are ready to investigate the connection between the sets \( \omega(R) \) and \( \omega(\hat{R}) \) of isomorphic cross-sections.

Lemma 6.4 Let \( R \cong \hat{R} \). Then it holds \( \omega(R)^2 = \omega(\hat{R}) \) and for all \( a_i \in \omega(R) \), we have either \( a_i = a_i^2 \) or \( a_i + a_i^2 = n + 1 \).

Proof The proof is by induction on the level of a vertex.

It is clear that \( r_i = r_i \). If \( r_i \in \{ 1, n \} \) then we get \( \{ r_i \} = 1 \). By Corollary 6.3 then \( \{ r_i \} = 1 \), therefore \( r_i \in \{ 1, n \} \). Let \( r_i \notin \{ 1, n \} \), \( d := d(r_i) \). Consider the set \( D \subset R \) with \( \im \alpha \). By Corollary 6.3 (4), we have

\[ D^2 = \{ \alpha \in \hat{R} \mid \im \alpha = \{ r_2, d^2 \} \}. \]

We use the fact that \( |D| = |D^2| \). Clearly, \( |D| = n - r_i \). For \( D^2 \) we have two possibilities. If \( 1 \leq d^2 < r_i \), we get \( |D^2| = r_2 - 1 \). If \( r_i < d^2 \leq n \), we get \( |D^2| = n - r_2 \). Hence, we have either \( r_1 = r_2 \) or \( r_1 + r_2 = n + 1 \).

Suppose the statement holds for \( \omega(R) \cap U_p \) with \( m \leq p < s \). Let \( a_i \in \omega(R) \cap U_p \). Without loss of generality assume \( r_2 = r_2 \) and \( r_1 < a_i \). By the assumption then we have \( a_i^2 \in \omega(\hat{R}) \) and \( a_i^2 \in \omega(\hat{R}) \). By Lemma 6.3 (2), it suffices to show that \( a_i^2 \in \omega(\hat{R}) \). By Corollary 6.3 (2), it suffices to show that \( a_i^2 \) is a child of \( a_i^2 \). If \( a_i = a_{i-1} = 1 \) then \( a_i^2 = a_{i-1}^2 = 1 \), so \( a_i^2 \) has a unique child. Thus, in this case the statement holds.
Let \(|a_i - a_{i-1}| = \middle|a_i^t - a_{i-1}^t\middle| > 1\). Suppose \(a_i^t \notin \omega(R)\). That is, \(a_i^t\) is the son of \(a_i^t\). Hence, we have \(a_i^t < a_{i-1}^t < a_i^t\).

Suppose the left inner tree of \(a_{i-1}^t\) is defined on \(x\)-element set (i.e., \(a_{i-1} - a_i = x\)) and the right tree of \(a_{i-1}^t\) is defined \(y\)-element set (i.e., \(n - a_{i+1} = y\)). By Corollary 6.2 (3), the inner trees associated with \(a_i^t\) are defined on \(x\)- and \(y\)-element sets respectively:

\[
\omega(|a_i^t - a_{i-1}^t|)(a_i^t - a_{i-1}^t) = xy,
\]
\[
a_i^t - a_{i-1}^t = x + y.
\]

Now, on the one hand, note that

\[
|\Theta^t_{a_i^t}| = |\Theta^t_{a_{i-1}^t}| = |\Theta^a_{a_i^t} | = |\Theta^a_{a_{i-1}^t}|.
\]

On the other hand, by the assumption it holds \(a_i^t - a_{i-1}^t = a_i - a_{i-1} = x + y\), therefore we have

\[
|\Theta^a_{a_i^t}| = |\Theta^a_{a_{i-1}^t}|(a_i - a_{i-1})(a_{i+1} - a_i) = |\Theta^a_{a_i^t}|(x + y)x,
\]

which contradicts the condition \(|\Theta^a_{a_i^t}| = |\Theta^a_{a_{i-1}^t}|\). Hence, \(a_i^t \in \omega(\hat{R})\). The second part of the statement of the lemma follows immediately from Lemma 6.3 (2).

Thus, if \(R \cong \hat{R}\) then “the skeletons” \(\omega(R), \omega(\hat{R})\) of the respective order-preserving trees are the same up to a mirror reflection. Furthermore, the elementary trees \(T^a\) are also very “close” to \(T^{\hat{a}}\), \(a \in \omega(R)\):

**Corollary 6.5** Let \(R \cong \hat{R}\), then for all \(a \in \omega(R)\) the mapping \(\hat{a}: T^a \rightarrow T^{\hat{a}}\) is bijective and preserves the parent-child relations.

**Proof** Let \(a \in \omega(R)\). By Lemma 5.4, it holds \(a^t \in \omega(\hat{R})\). We have \((T^a)^t \cap \omega(\hat{R}) = \{a^t, p(a)^t = p(a^t) \} \) with \(|p(a) - a| = |p(a^t) - a^t|\). By Corollary 6.2 (2), we get now the required.

In fact, the stronger statement holds: the inner trees of \(T^a\) and \(T^{\hat{a}}\) are similar, for all \(a \in \omega(R)\) (see Proposition 6.7). To prove this we use the connection between the dual cross-sections with \(\mathcal{L}\)-cross-sections (see Sec. 5). Recall the following result from [2].
Proof. Let $\mathcal{R} = \mathcal{R}_n$ and consider below only the right-hand sided diagrams, unless otherwise stated.

Proposition 6.7. If $R \cong \bar{R}$ then the inner trees of $T^a$, $T^b$ are pairwise similar for all $a \in \omega(R)$.

Proof. Let $|\omega(R)| = k$, $a \in \omega(R)$ with $a \neq r$. Consider the elementary tree $T^a$. We denote the inner trees of $T^a$, $T^b$ by $\Gamma^a$, $\Gamma^b$ respectively. If $|T^a| \leq 3$ then Corollary 6.3 implies directly that $\Gamma^a \cong \Gamma^b$. Suppose $|T^a| = m > 3$.

Let $\Phi_{a1}, \Phi_{a2}$ be $\mathcal{R}$-cross-sections associated with the elementary trees $T^a$, $T^b$ respectively. We will construct now a subsemigroup $R(a) \subseteq R$ such that $R(a) \cong \Phi_{a2}$.

Recall that the subsets of $[p(a)], (a-1)''$ induces an interval partition of the set $[p(a), a]$ (see Subsection 2.2). If $p(a) = r$ then we denote by $R(a)$ the set of transformations $\gamma \in R$ whose partition induced by a subset $V \subseteq \Gamma^a$, for each $V \subseteq \Gamma^a$. If $p(a) \neq r$ then let $b \in \omega(a)$ be the parent of $p(a)$. In this case we denote by $R(a)$ the set of transformations $\gamma \in R$ whose partition induced by $\{x \mid x \in \omega(b)\} \cup V$, for each $V \subseteq \Gamma^a$. According to the description of $\mathcal{R}(O_n)$ we have $\gamma = x$ for all $x \in \omega(b)$, $\gamma \subseteq \Theta^b$. Clearly, it holds

$$R(a) \cong \{\gamma \mid \gamma \in R(a)\} \cong \Phi_{a2}.$$ 

In the same manner we define the subset $R(a^2) \subseteq R$ with $R(a^2) \cong \Phi_{a2}$. By Corollary 6.2 (4), it holds $\operatorname{im}(\gamma^2) = \operatorname{im}(\gamma^2)$. Since $\omega(v)^2 = \omega(v)^2$ for all $v \in (\mathcal{R}, \prec)$ and Corollary 6.3 we get $R(a)^2 = R(a^2)$.

Trees $T^a$, $T^b$ are elementary. Consequently, their inner trees are respectful. Let $L$, $\tilde{L}$ be order-preserving $\mathcal{R}$-cross-sections associated with the inner trees. Using the fact that $\hat{R}(a)^2 = R(a^2)$ we will show that $L \cong \tilde{L}$, whence by Theorem 6.3 we get the required immediately.

For $a \in R(a)$ denote by $\alpha_A \in L$, $A \in \Gamma^a$, the transformation such that $\alpha_A^\prime = \alpha|_{\Gamma^a}$. By $\alpha_{A} \in \tilde{L}$ we denote the transformation such that $\alpha_A^\prime = \alpha_{A}\mid_{\Gamma^a}$.

We claim that a mapping

$$\kappa : L \rightarrow \tilde{L} : \alpha_A \mapsto \alpha_{A}$$

is an isomorphism. Clearly, by the construction of $\ast$ and since $R(a)^2 = R(a^2)$, we get that $\kappa$ is bijective. Let $\beta \in R(a)$ and $\beta^\ast_B = \beta|_{\Gamma^b}$, for $B \in \Gamma^a$. Let $\alpha_C \in L$, $C \in \Gamma^a$, stand for $\alpha_{A}\beta_{B}$. It remains to show that $\alpha_C = \alpha_{A}\beta_{B}$. Hence that

$$\alpha_C = (\alpha_A\beta^\ast_B)^\prime = (\beta|_{\Gamma^b})(\alpha|_{\Gamma^a}).$$

Since $\alpha, \beta \in R(a)$ we get $(\beta|_{\Gamma^b})(\alpha|_{\Gamma^a}) = (\beta\alpha)|_{\Gamma^b}$. Consequently, $\beta_B^\ast = (\beta\alpha)|_{\Gamma^b}$. Then

$$\alpha_C^\ast = (\beta\alpha|_{\Gamma^b})^\prime = (\beta^\ast_B)(\alpha^\ast_C).$$

Since $\alpha, \beta \in R(a)$, we get $(\beta|_{\Gamma^b})(\alpha^\ast_C) = (\beta\alpha|_{\Gamma^b})(\alpha^\ast_C)$. Therefore, $\alpha_C = \alpha_A\beta_B$. Hence, $\kappa$ is isomorphism.

Furthermore, let $T^a = T^b$. We claim that $\psi : T^a \rightarrow T^b$ directly depends on the isomorphism/the anti-isomorphism between $\Gamma^a$ and $\Gamma^b$. To prove this we need to find a way how to "synchronize" the vertices of the elementary trees according to the isomorphism of their inner trees.

Definition 6.8. We denote by $\psi$ an isomorphism/an anti-isomorphism from $\Gamma^a$ to $\Gamma^b$. For every $x \in T^a$ define a mapping $\psi^{(a,b)} : T^a \rightarrow T^b$ induced by $\psi$ as follows:
Case 1: $\Gamma^a \cong \Gamma^b$, $\psi$ is an isomorphism

$$x_{\pi^{(a,b)}} = \begin{cases} y & \text{such that } x' \psi = y' \text{ if } x' \in \Gamma^a, \\ z & \text{such that } z \in T^b, z' \notin \Gamma^b, \text{ if } x' \notin \Gamma^a. \end{cases}$$

Case 2: $\Gamma^a \not\cong \Gamma^b$, $\psi$ is an anti-isomorphism

$$x_{\pi^{(a,b)}} = \begin{cases} y & \text{such that } x' \psi = (y-1)' \text{ if } x' \in \Gamma^a, \\ z & \text{such that } z \in T^b, (z-1)' \notin \Gamma^b, \text{ if } x' \notin \Gamma^a. \end{cases}$$

As the following example shows the mapping $\pi^{(a,b)}$ is not an isomorphism of $T^a$ to $T^b$: it does not preserve the root in general.

**Example 6.9** Consider order-preserving trees pictured in Fig. 17 Then $\pi^{(5,1)} : T^5 \to T^1$ and $\pi^{(5,9)} : T^5 \to T^9$ have the form

$$\pi^{(5,1)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \quad \pi^{(5,9)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 9 & 8 & 7 & 6 & 5 \end{bmatrix}$$

![Figure 17: Illustration of Example 6.9](image)

Despite $\pi^{(a,n)}$ is not an isomorphism, it has the following property:

**Lemma 6.10** For all $x \in T^a, p(a) \neq x \neq a$, it holds $\omega(x\pi^{(a,b)}) = \omega(x)p^{(a,b)}$.

**Proof** Let $T^a$ have $t$ levels. We induct on the level $i$ of $x \in T^a$, $2 \leq i \leq t - 1$. Clearly,

$$\{a, p(a)\} \pi^{(a,b)} = \{b, p(b)\}.$$ 

Therefore, if $x \in U_2$ then

$$\omega(x)(\pi^{(a,b)} = \{x, a, p(a)\} \pi^{(a,b)} = \{x, b, p(b)\} = \omega(x\pi^{(a,b)}).$$

Thus, the induction base holds. Suppose the statement holds for all $i < k$ for a natural $k, k < t$. Let $x \in U_k$ with the canonical bounds $y < x < z$. Without loss of generality assume $y$ is the parent of $x$. Let $[y', (z-1)'] \psi = [p', (m-1)']$ Then $\gamma\pi^{(a,b)} \in \{p, m\}$ and by the assumption $\omega(y)(\pi^{(a,b)} = \omega(y)p^{(a,b)}).$ Suppose $[p', (l-1)'], [l', (m-1)']$ the children of $[p', (m-1)'].$ Then on the one hand, by the construction of the inner tree $l$ is a child of $\gamma\pi^{(a,b)}.$ On the other hand, if $\psi$ is an isomorphism then $x' = l'.$ If $\psi$ is an anti-isomorphism then $x' = (l-1)'$. In both cases we get $\gamma\pi^{(a,b)} = l.$ Thus,

$$\omega(x)(\pi^{(a,b)} = \{x\} \pi^{(a,b)} \cup \omega(y)(\pi^{(a,b)} = \{l\} \cup \omega(y)p^{(a,b)}) = \omega(l) = \omega(x\pi^{(a,b)}).$$

We also need to recall the following fact.
Lemma 6.11 (2, Lemma 5) Let \( L, \hat{L} \) be \( \mathcal{L} \)-cross-sections associated to similar respectful trees \( \Gamma^a, \Gamma^b \), respectively. Let \( \tau : L \to \hat{L} \) be an isomorphism. Then for all \( A \in \Gamma^a \) and \( \beta \in L \), it holds \( (A\beta)\psi = A\psi(\beta\tau) \).

Remark 6.12 It is convenient to interpret this result graphically, picturing the transformations like it was in Fig. 2. If \( \psi \) is an isomorphism then we get the same depiction of transformation \( \beta\tau \in \hat{L} \) as the \( \beta \)'s one (up to the labels of the vertices). If \( \psi \) is an anti-isomorphism then \( \Gamma^b \) is a mirror reflection of \( \Gamma^a \). Therefore, to get the depiction of \( \beta\tau \) it is enough to flip the \( \beta \)'s one upside-down. Consider an example.

Example 6.13 Let \( \Gamma_1, \Gamma_2 \) be the trees from Example 5.5. Apparently, \( \psi \) is an anti-isomorphism, \( \psi = (1234) \). Without loss of generality consider \( \beta \in \Gamma_1 \) with \( \text{im} (\beta) = \{ 1, 2, 4 \} \). By the definition of \( \alpha_M \) we have \( \beta = (\{12\} 3 4) \). Then by Lemma 6.11 

\[
\{12\}\psi(\beta\tau) = \{43\}\beta\psi; \{12\}\beta\psi = 1\psi = 4, \text{ whence } \{43\}\beta\psi = 4.
\]

Analogously 

\[
\{3\}\psi(\beta\tau) = \{2\}\beta\psi; \{3\}\beta\psi = 2\psi = 3, \\
\{4\}\psi(\beta\tau) = \{1\}\beta\psi; \{4\}\beta\psi = 4\psi = 1.
\]

Therefore, \( \beta\tau = (1 2 3 4) \) and its diagram is the turned upside down \( \beta \)'s one (see Fig. 18).

Figure 18: Illustration of Example 6.13

Now we are ready to prove the following lemma.

Lemma 6.14 Let \( (\pi, \prec_1) = T^n \) and \( (\pi, \prec_2) \) be an elementary tree, \( R \leftrightarrow \hat{R} \). Then for all \( x, 1 < x < n \), it holds 

\[
x^i = x^i_{\pi(\pi, \prec_1)}.
\]

Proof Let \( \Gamma, \hat{\Gamma} \) be the inner trees of \( R, \hat{R} \) respectively; \( L, \hat{L} \) be \( \mathcal{L} \)-cross-sections of \( \mathcal{O}_{\pi-1} \) associated with \( \Gamma, \hat{\Gamma} \). Let \( \kappa : L \to \hat{L} : \alpha_M \mapsto \hat{\alpha}_M \) be the isomorphism defined in the proof of Proposition 6.7. Since \( (\pi, \prec_1) = T^n \) it holds \( \alpha_M^* \in R \) for all \( \alpha_M \in L \). Then for \( \hat{\alpha}_M \in \hat{L} \) we have 

\[
\hat{\alpha}_M = \hat{\alpha}_M^* \cdot \xi.
\]

Let \( x \in \pi \), \( 1 < x < n \). Let \( \alpha_M \in L \) be such that \( \alpha_M^* \in \Theta^i \). Then \( \alpha_M^* \xi \in \Theta^i \). Thus, if we show that \( \hat{\alpha}_M^* \xi \in \Theta^i \pi(x^i) \) we get the required.
Since \( \kappa \) is an isomorphism we can apply Lemma 6.1 to \( \alpha_\mathcal{M}, \hat{\alpha}_\mathcal{M} \). According to Proposition 6.2 we have to consider two possibilities:

**Case 1.** \( \Gamma^\eta \sim \hat{\Gamma} \), \( \psi \) is an isomorphism. As we have observed in Remark 6.12 in this case \( \hat{\alpha}_\mathcal{M}, \hat{\alpha}_\mathcal{M} \) has the same depiction. Since \( x\pi^{(n,n')} = x \) we get immediately \( \omega(x\pi^{(n,n')}) = \omega(x) \) for all \( x \in \pi, 1 \neq x \neq n \). Thus, we have \( \text{im}(\hat{\alpha}_\mathcal{M}) = \omega(x) = \omega(x\pi^{(n,n')}) \) as required.

**Case 2.** \( \Gamma^\eta \not\sim \hat{\Gamma} \), \( \psi \) is an anti-isomorphism. Then the depiction of \( \hat{\alpha}_\mathcal{M} \) is obtained from the \( \alpha_\mathcal{M} \)'s one if we consider the depiction upside down. Then clearly, \( \hat{\alpha}_\mathcal{M} \) and \( (\hat{\alpha}_\mathcal{M})^* \) also have the symmetric diagrams with \( \ker(\hat{\alpha}_\mathcal{M}) = (\ker \alpha_\mathcal{M})\pi^{(n,n')} \) and \( \text{im}(\hat{\alpha}_\mathcal{M}) = (\text{im} \alpha_\mathcal{M})\pi^{(n,n')} \). Thus, \( \text{im}(\hat{\alpha}_\mathcal{M}) = \omega(x)\pi^{(n,n')} \). By Lemma 6.10 it holds \( \omega(x)\pi^{(n,n')} = \omega(x\pi^{(n,n')}) \) for all \( x \in \pi, 1 \neq x \neq n \). Therefore, \( \hat{\alpha}_\mathcal{M} \in \Theta^{\mathcal{M}(x\pi^{(n,n)})} \), whence \( x^\sharp = x\pi^{(n,n')} \) as required.

At last, we can state the final form of the necessary condition of two \( \mathcal{R} \)-cross-sections being isomorphic.

### Lemma 6.15

Let \( \mathcal{R} \equiv \hat{\mathcal{R}} \); and the minimal fragmentations of \( R, \hat{R} \) consist of \( k \) elementary trees. Then for all \( a_i \in \omega(R), 1 \leq i \leq k \), and the inner trees \( \Gamma^\alpha, \Gamma^\hat{\alpha} \) of \( T^\alpha, T^{\hat{\alpha}} \) respectively, one of the following statements hold true:

(a) \( \Gamma^\alpha \Rightarrow \Gamma^\hat{\alpha}, \psi_1 \) is an isomorphism.

(b) \( \Gamma^\alpha \sim \Gamma^\hat{\alpha}, \psi_1 \) is an anti-isomorphism.

**Proof**

We induct on \( i, a_i \in \omega(R), 1 \leq i \leq k \).

Let \( i = 1 \) and \( a_1 \) be the daughter of \( r \). Then according to the proof of Proposition 6.4 we have \( \mathcal{R}(a_1) \equiv \mathcal{R}(a'_1) \) and \( \Gamma^\alpha \sim \Gamma^{\hat{\alpha}} \) (see the proof of Proposition 6.7). Assume \( \psi_1 \) is an isomorphism for definiteness.

Step. Let \( a_i \in \omega(R, r) \), \( 1 \leq i \leq k \). By Proposition 6.7 it holds \( \Gamma^{\psi_1} \Rightarrow \Gamma^{\psi_2} \). If \( |\Gamma^{\psi_1}| \leq 3 \) then clearly, the statement holds. Let \( |\Gamma^{\psi_1}| > 3 \), \( a \in T^{\psi_1} \) with \( a \notin \{ p(a_i), a_i, d(a_i) \} \).

Consider a transformation \( \alpha \in R \) such that \( T^{\psi_1} \) is a transversal of \( \psi_1 \ker(\alpha) \). Note that according to Definition 4.3 \( \alpha/\pi_\alpha \) defines the homomorphism from \( T^{\psi_1} \) to \( T^{\psi_2} \). Let \( \alpha = q \).

We will show that there exists the homomorphism \( T^{\psi_1} \rightarrow T^{\psi_2} \) that maps \( u\pi^{(a,a')} \) to \( q\pi^{(a,a')} \).

The condition clearly holds if and only if either of \( \psi_1 \) and \( \psi_2 \) are the isomorphisms, or both of \( \psi_1 \) and \( \psi_2 \) are the anti- isomorphisms.

Note that \( \text{im}(\theta^\alpha) = \omega(q) \). By Corollary 6.3 (3), we have \( \text{im}(\theta^\alpha \alpha) = \omega(q^\sharp) \). Therefore, \( u^\sharp(\alpha^\sharp z^\sharp) = q^\sharp \). By the proof of Proposition 6.7 we have \( \mathcal{R}(a_k) \equiv \mathcal{R}(a'_k) \), thus by Lemma 6.14 we have \( u^\sharp = u\pi^{(a_i,a')} \), \( q^\sharp = q\pi^{(a_i,a')} \).

Furthermore, it holds \( \text{im}(\theta^\alpha(\alpha z^\sharp)) = \text{im}(\theta^\alpha(\alpha^\sharp z^\sharp)) = r^\sharp \) for all \( x \in \omega(p(a_i)) \). Therefore by Definition 4.3 we have that the restriction \( \alpha^\sharp z^\sharp|_{\Gamma^\alpha} \) is the homomorphism from \( T^{\psi_1} \) to \( T^{\psi_2} \) and \( u\pi^{(a_i,a')}(\alpha^\sharp z^\sharp) = q\pi^{(a_i,a')} \) as required.  

### 6.2 The sufficient condition

In fact, the statement converse to Lemma 6.15 gives us the sufficient condition of isomorphic \( \mathcal{R} \)-cross-sections of \( \mathcal{O}_\mathcal{M} \):

**Lemma 6.16**

Let \( \mathcal{R}, \hat{\mathcal{R}} \) be the \( \mathcal{R} \)-cross-sections of \( \pi \) such that there exists an isomorphism (anti-isomorphism) \( \eta : \omega(R) \rightarrow \omega(\hat{R}) \) and for all \( a_i \in \omega(R) \) the inner trees \( \Gamma^\alpha, \Gamma^\hat{\alpha} \) of elementary trees \( T^\alpha, T^{\hat{\alpha}} \) are similar. Moreover, if \( \Gamma^\alpha \Rightarrow \Gamma^{\hat{\alpha}} \) it holds either \( \psi_1 \) is isomorphism for all \( a_i \in \omega(R) \), or \( \psi_2 \) is anti-isomorphism for all \( a_i \in \omega(R) \). Then \( \mathcal{R} \equiv \hat{\mathcal{R}} \).

We divide the proof into few steps. First, we assign with each \( \varphi \in \mathcal{R} \) the transformation \( \hat{\varphi} \). Then we investigate properties of \( \hat{\varphi} \). After that we prove that \( \xi : \mathcal{R} \rightarrow \hat{\mathcal{R}} : \varphi \mapsto \hat{\varphi} \) is isomorphism.
Each transformation of an $\mathcal{A}$-section is determined by its partition. We construct the partition $\phi$ basing on the partition $\varphi$. In outlines, we will do the following: cut the partition of $\varphi$, change the resulting parts with $\psi$, and then glue the new ones back together.

Let $K$ be the partition of $\varphi$. For a fixed $a_i \in \omega(R)$ by $\Lambda^{a_i}$ denote the kernel classes of the restriction $\varphi$ to the vertex set of the tree $T^{a_i}$: 

$$\Lambda^{a_i} = \{ X \cap T^{a_i} \mid X \in K, X \cap T^{a_i} \neq \emptyset \}.$$ 

Now we define a partition $\Lambda^{a,\eta}$ of a vertex set of $T^{a,\eta}$ as follows:

$$\Lambda^{a,\eta} = \{ X \pi^{(a,\eta)} \mid X \in \Lambda^{a_i} \},$$

where $X \pi^{(a,\eta)} = \{ x \pi^{(a,\eta)} \mid x \in X \}$.

Let $\Lambda_\eta = \cup_{a \in \omega(R)} \Lambda^{a,\eta}$. Note that for all $A, B \in \Lambda_\eta$ it holds either $A \cap B = \emptyset$ or $A \cap B \in \omega(R)$.

We define $\ker(\varphi)$ as follows: for all $x \in A, y \in B$, let

$$x \tilde{\varphi} = y \tilde{\varphi} \text{ if and only if either } A = B, \text{ or } A \cap B \neq \emptyset.$$ 

Further, let $x, y \in \omega(R)$ and $y$ have the child $z \in \omega(R)$. If $T^x \varphi = \{ y \}$, $y \neq \pi$ then $T^y \varphi \subseteq T^y$ and $T^x \varphi \subseteq T^y$. To avoid the ambiguity we will write $T^y \varphi \subseteq T^y$ for the highest non-root vertex $y$ with this property. If $y = \pi$ we will write formally $T^y \varphi \subseteq T^y$ to be consistent.

**Lemma 6.17** Let $R, \hat{R}$ satisfy the condition of Lemma 6.16, $x \in \omega(R)$ with $x \neq \pi$. If $T^x \varphi \subseteq T^y$ for $y \in \omega(R)$ then $T^{x,\eta} \varphi \subseteq T^{y,\eta}$ for all $x \in \omega(R)$.

**Proof** Let $|\omega(R)| = r + 1$. The proof is by induction on the level $i$ of $x \in \omega(R)$, $1 \leq i \leq t$.

Let $x$ be the daughter of $r \in \omega(R)$. By the construction of the $\mathcal{A}$-cross-sections, there is only two possibilities for $T^x$: it holds either $T^x \varphi = r$, or $T^x \varphi \subseteq T^3$. If $T^x \varphi = r$ then by the construction of $\ker(\varphi)$, it holds also $|\Lambda^{a,\eta}| = |\Lambda^a| = 1$ therefore, $(\pi) \varphi = (x \pi) \varphi$, whence $T^{x,\eta} \varphi = \pi$. If $T^x \varphi \subseteq T^3$ then $|\Lambda^{a,\eta}| > 1$. Consequently, $T^{x,\eta} \varphi \subseteq T^{x,\eta}$.

Let $x \in \omega(R)$ with level $k$, $0 \leq k < t$. Suppose $T^x \varphi \subseteq T^3$ implies $T^{x,\eta} \varphi \subseteq T^{x,\eta}$. Let $d \in \omega(R)$ be the daughter of $x$. If $T^d \varphi = 1$ then $x \varphi = d \varphi$, whence $T^d \varphi = \{ y \}$. Moreover, since $|\Lambda^{d,\eta}| = |\Lambda^d| = 1$, we have $(d \pi) \varphi = (x \pi) \varphi$. By the assumption $(x \pi) \varphi = y \eta$. Therefore $T^{d,\eta} \varphi \subseteq T^{x,\eta}$ as required.

Let $|T^d \varphi| > 1$. Then $T^d \varphi \subseteq T^d$ and $|\Lambda^{d,\eta}| = |\Lambda^d| > 1$. Since $(x \pi) \varphi = y \eta$ and $\eta : \omega(R) \rightarrow \omega(R)$ is isomorphism (anti-isomorphism), the definition of $\phi$ implies $(d \pi) \varphi = (d \pi) \eta$. Therefore, we get $T^{d,\eta} \varphi \subseteq T^{x,\eta}$ as required.

**Lemma 6.18** For all $x \in \omega(R)$ the following statements hold:

(i) If $T^x \varphi = \{ x \pi \}$ then $T^{x,\eta} \varphi = \{ x \pi \eta \}$.

(ii) If $|\Lambda^x| > 1$ then for all $X \in \Lambda^x$ it holds $X \pi^{(x,\eta)} \varphi = (X \varphi) \pi^{(x,\eta)}$.

**Proof** Let $x \in \omega(R), x \neq \pi$. Item (i) follows directly from the previous lemma.

Let $|T(A^x)| > 1$ and $T^x \varphi \subseteq T^3$ with $x \varphi = y, y \in \omega(R)$. By the description of $\mathcal{A}(\Omega_n)$ the mapping $T(A^x) \rightarrow T^3 : X \mapsto X \varphi$ defines a homomorphism of the trees. By Definition 6.8 and since $R, \hat{R}$ satisfy the condition of Lemma 6.16 the mapping $T(A^x) \rightarrow T^3 : X \pi^{(x,\eta)} \mapsto (X \varphi) \pi^{(x,\eta)}$, $X \in A^x$, is also homomorphism. Therefore, by the description of $\mathcal{A}(\Omega_n)$ we get $X \pi^{(x,\eta)} \varphi = (X \varphi) \pi^{(x,\eta)}$ as required.

Finally, we are ready to finish the proof of the sufficient condition.

**Lemma 6.19** The mapping $\xi : R \rightarrow \hat{R} : \varphi \mapsto \tilde{\varphi}$ is an isomorphism.
Proof. It is clear that ξ is a bijective mapping. Let \( \hat{\alpha}, \hat{\beta} \in \hat{R} \). Then for all \( x \in \omega(R) \) and if \( |\Lambda| = 1 \) according to Lemma 6.18, we have
\[
(x)\hat{\alpha}\hat{\beta} = x(\alpha\beta)\eta = (x\alpha)\beta\eta = x\hat{\alpha}\hat{\beta}.
\]
Let \( |\Lambda| > 1, A \in \Lambda^\alpha \). Denote by \( a' \) an element of \( \Lambda^{(x, x\eta)} \). Then
\[
a'\hat{\alpha}\hat{\beta} = A\pi^{(x, x\eta)}(\alpha\beta)\eta = A(\alpha\beta)\pi^{(x, (x\alpha)\beta)\eta} = (A\alpha)\beta\pi^{(x, \alpha\beta)\eta} = (A\alpha)\pi^{(x, x\alpha\eta)}\hat{\beta} = A\pi^{(x, x\eta)}\alpha\hat{\beta} = a'\hat{\alpha}\hat{\beta}.
\]
Therefore, ξ is an isomorphism as required. □

Lemmas 6.15 and 6.16 now give the classification of \( \mathcal{R} \)-cross-sections of \( O_n \) up to an isomorphism:

**Theorem 6.20** The \( \mathcal{R} \)-cross-sections \( R, \hat{R} \) of \( O_n \) are isomorphic if and only if there exists an isomorphism (anti-isomorphism) \( \eta \) of the trees \( \omega(R) \) and \( \omega(\hat{R}) \); and for all \( a_i \in \omega(R) \) with \( a_i > r_1 \) (\( a_i < r_1 \) respectively) and the inner trees \( \Gamma^a_i, \Gamma^\hat{a}_i \) of \( T^a_i, T^\hat{a}_i \) respectively, one of the following statements hold true:

(a) \( \Gamma^a_i \psi_i \sim \Gamma^\hat{a}_i, \psi_i \) is an isomorphism.

(b) \( \Gamma^a_i \psi_i \sim \Gamma^\hat{a}_i, \psi_i \) is an anti-isomorphism.

**References**

[1] Bondar, E.: \( \mathcal{L} \)-cross-sections of the finite symmetric semigroup. Algebra and Discrete Math., 18(1), 27–41 (2014).

[2] Bondar, E.: Classification of \( \mathcal{L} \)-cross-sections of the finite symmetric semigroup. Algebra and Discrete Mathematics, 21(1), 1–17 (2016).

[3] Bondar, E. A., Volkov, M. V.: Completely reachable automata. Describtional Complexity of Formal Systems. DCFS 2016 [Lect. Notes Comp. Sci. 9777], 1–17 (2016).

[4] Cormen, T. H.; Leiserson, C. E.; Rivest, R. L.; Stein, C. Introduction to Algorithms (3rd ed.). The MIT Press, Cambridge, Massachusetts; London, England (2009)

[5] Ganyushkin, O., Mazorchuk, V.: Classical Finite Transformation Semigroups: An Introduction. Algebra and Applications. Springer-Verlag, London (2009).

[6] Higgins, P.: Divisors of semigroups of order-preserving mappings on a finite chain. International Journal of Algebra and Computation, 5(6), 725–742 (1995).

[7] Howie, J. M.: Semigroups, past, present and future, Proc. Internat. Conference on Algebra and its Applications, 6–20 (2002).

[8] Pekhterev, V.: \( \mathcal{H} \)- and \( \mathcal{S} \)-cross-sections of the full finite semigroup \( S_n \), Algebra and Discrete Math., 2(3), 82–88 (2003).