Coset Realization of Unifying $\mathcal{W}$-Algebras

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Abstract

We construct several quantum coset $\mathcal{W}$-algebras, e.g. $\hat{sl}(2,\mathbb{R})/\hat{U}(1)$ and $\hat{sl}(2,\mathbb{R}) \oplus \hat{sl}(2,\mathbb{R})/\hat{sl}(2,\mathbb{R})$, and argue that they are finitely nonfreely generated. Furthermore, we discuss in detail their rôle as unifying $\mathcal{W}$-algebras of Casimir $\mathcal{W}$-algebras. We show that it is possible to give coset realizations of various types of unifying $\mathcal{W}$-algebras, e.g. the diagonal cosets based on the symplectic Lie algebras $sp(2n)$ realize the unifying $\mathcal{W}$-algebras which have previously been introduced as ‘$\mathcal{WD}_{-n}$’. In addition, minimal models of $\mathcal{WD}_{-n}$ are studied. The coset realizations provide a generalization of level-rank-duality of dual coset pairs. As further examples of finitely nonfreely generated quantum $\mathcal{W}$-algebras we discuss orbifolding of $\mathcal{W}$-algebras which on the quantum level has different properties than in the classical case. We demonstrate in some examples that the classical limit according to Bowcock and Watts of these nonfreely finitely generated quantum $\mathcal{W}$-algebras probably yields infinitely nonfreely generated classical $\mathcal{W}$-algebras.

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Contents

1. Introduction 2
1.1. Notation 4
2. Non-freely generated $\mathcal{W}$-algebras in cosets and orbifolds 5
2.1. Coset $\mathcal{W}$-algebras 5
2.1.1. The cosets $sl(2,\mathbb{R})/U(1)$ and $SVIR(N = 2)/U(1)$ 5
2.1.2. The coset $\mathcal{W}_{2,1}^{sl(3)}/U(1)$ 12
2.1.3. The coset $sl(2,\mathbb{R})_{\kappa} \oplus sl(2,\mathbb{R})_{\mu}/sl(2,\mathbb{R})_{\kappa+\mu}$ 16
2.2. Orbifolds of quantum $\mathcal{W}$-algebras 23
2.2.1. General remarks and results 24
2.2.2. The orbifold of $\mathcal{W}(2,3)$ 28
2.2.3. Remarks on the orbifold of $\mathcal{WA}_{n-1}$ 30
3. General structures in cosets and orbifolds 32
3.1. Vacuum preserving algebras (VPA) and classical limits 32
3.2. Coset realization of unifying $\mathcal{W}$-algebras and level-rank-duality 38
3.2.1. Unifying $\mathcal{W}$-algebras for the $\mathcal{WA}_n$ Casimir algebras 38
3.2.2. Level-rank-duality for the cosets $so(n)_k \oplus so(n)_1/so(n)_{k+1}$ 42
3.2.3. Realization of $\mathcal{WD}_{-n}$ as diagonal $sp(2n)$ cosets 44
3.2.4. Minimal models of $\mathcal{WD}_{-m}$ 46
3.2.5. The coset $\hat{g}_k/g$ for a simple Lie algebra $g$ 48
4. Conclusion 49

Appendix A: The simple fields with spin 3, 4, 5 of $sl(2,\mathbb{R})/U(1)$ 52
Appendix B: Some structure constants of $\mathcal{W}^{sl(3)}_{2,1}/U(1)$ 52
Appendix C: The primary spin 4 generator of $sl(2,\mathbb{R})_{\kappa} \oplus sl(2,\mathbb{R})_{\mu}/sl(2,\mathbb{R})_{\kappa+\mu}$ 53
Appendix D: Minimal models of Casimir $\mathcal{W}$-algebras 54
Appendix E: The orbifold of the $N = 1$ Super Virasoro algebra 54
Appendix F: Generators and structure constants of the orbifold of $\mathcal{W}(2,3)$ 56
References 57
1. Introduction

One of the most interesting questions in two dimensional conformal invariant quantum field theory is the classification of rational conformal field theories (RCFT). An important tool in the investigation of this question is provided by extended conformal algebras also called $\mathcal{W}$-algebras (see e.g. [1 − 3]). Although $\mathcal{W}$-algebras have been the object of intense studies in the last few years, a complete satisfactory classification of $\mathcal{W}$-algebras and their representations has not been achieved yet. One has to distinguish between algebras which exist only for fixed values of the Virasoro central charge $c$, called nondeformable, and deformable (or generic) $\mathcal{W}$-algebras existing for generic central charge $c$. For the latter class the structure constants are continuous functions of $c$ apart from a finite set of singularities. Intense studies revealed that it is possible to explain most of the nondeformable $\mathcal{W}$-algebras as truncations or extensions of generically existing ones (see e.g. [4 − 8]).

It was noticed very recently that the class of deformable $\mathcal{W}$-algebras contains at least two subclasses which have completely different features of the classical counterparts [9]. The first class consists of deformable $\mathcal{W}$-algebras originating from quantum Drinfeld-Sokolov (DS) type hamiltonian reduction of affine Kac-Moody algebras (see e.g. [10 − 12]). These algebras are by far the best understood ones. The main property of this class of algebras is that the classical counterparts are Poisson bracket algebras based on finitely and freely generated rings of differential polynomials. It is believed that these algebras can be classified by $sl(2, \mathbb{R})$ embeddings into simple Lie algebras [13, 10, 14]. The so-called Casimir algebras [15] correspond to the principal embedding. In contrast hereto the second class of generically existing quantum $\mathcal{W}$-algebras have as classical counterparts infinitely generated rings of differential polynomials with infinitely many relations, i.e. the ring is nonfreely generated [9].

In this paper we discuss features of quantum $\mathcal{W}$-algebras belonging to this new class. The first observation is that although the classical Poisson bracket algebra is infinitely generated the quantum $\mathcal{W}$-algebra is generated by a finite number of simple fields in all cases studied up to now. This is due to the fact that on the quantum level normal ordered versions of the classical relations between the generators eliminate all but a finite number of generators [9]. A very intriguing point is that the unexplained solutions of $\mathcal{W}(2, 4, 6)$ [5, 16] and $\mathcal{W}(2, 3, 4, 5)$ [7] with generic null fields obtained by direct construction are of this type and can be explained in terms of coset constructions (see also [9]). In [9] it has been argued that all classical coset algebras are infinitely generated. In the present paper we argue for certain cases that the corresponding quantum $\mathcal{W}$-algebras are finitely generated. In the case of $sl(2, \mathbb{R})/U(1)$ we give arguments that the commutant of the $U(1)$-current yields the unexplained solution of $\mathcal{W}(2, 3, 4, 5)$. We argue that this coset algebra is isomorphic to the commutant of the $U(1)$-current in the $N = 2$ Super Virasoro algebra $SVIR(N = 2)$. We also present supporting arguments for the realization [9] of the previously unexplained solution of $\mathcal{W}(2, 4, 6)$ as the diagonal coset $sl(2, \mathbb{R})_\kappa \oplus sl(2, \mathbb{R})_{-\kappa} / sl(2, \mathbb{R})_{\kappa - 1/2}$ and study its representation theory.

In [17] it has been observed that $\mathcal{W}$-algebras of the second class have the interesting property of being unifying algebras for special series of minimal models of Casimir $\mathcal{W}$-algebras. In this paper we give explicit coset realizations for many of these unifying
algebras which generalize level-rank-duality of coset pairs [18, 19]. In special cases like the unitary models of the $\mathcal{W}A_{k-1}$ algebras the unifying algebras can be inferred directly from level-rank-duality. The coset $\widehat{sl}(2, \mathbb{R})_k/U(1)$ for example describes the first unitary model of the $\mathcal{W}A_{k-1}$ Casimir algebra, the so-called $\mathbb{Z}_k$ parafermions [20]. For the corresponding values of the central charge $c = \frac{2(k-1)}{k+2}$ the algebras $\mathcal{W}A_{k-1}$ truncate to $\mathcal{W}(2, 3, 4, 5)$ [17].

Even in the cases where level-rank-duality can be exploited our considerations go beyond the ‘$T$-equivalence’ of coset pairs [1, 18, 19]. We compute the spins of the finite generating set of the unifying algebras and check in some examples the isomorphism of the extended symmetry algebras.

The unifying algebras for $\mathcal{WC}_k$ minimal models are the new algebras $\mathcal{WD}_{-n}$ [8, 17]. We argue that these algebras $\mathcal{WD}_{-n}$ can be realized in terms of a diagonal $\widehat{sp}(2n)$ coset. The first member $\mathcal{WD}_{-1}$ is the formerly unexplained $\mathcal{W}(2, 4, 6)$. In fact, negative-dimensional groups have already been encountered in the representation theory of the classical groups (see e.g. [21]). The existence of $\mathcal{WD}_{-n}$ can be expected by a deep connection of the negative-dimensional orthogonal groups $SO(-2n)$ and the symplectic groups $Sp(2n)$.

Apart from the coset construction there is another construction which leads to $\mathcal{W}$-algebras with infinitely nonfreely generated classical counterparts: orbifolding of $\mathcal{W}$-algebras (see e.g. [22 – 24]). If a given $\mathcal{W}$-algebra possesses an outer automorphism one considers the projection onto the invariant subspace. A special case of this construction is the bosonic projection of a $\mathcal{W}$-algebra containing fermionic fields. We show that in contrast to the classical case [24] the orbifold of a quantum Casimir algebra does in general not contain a Casimir algebra as a subalgebra for generic central charge.

The plan of the paper is as follows. In the first part of section 2 we will work out explicitly the algebraic structure of some cosets. Our starting point are the cosets $\widehat{sl}(2, \mathbb{R})/U(1)$ and $SVIR(N = 2)/U(1)$ which we argue to lead to a finitely generated algebra of type $\mathcal{W}(2, 3, 4, 5)$. After some remarks concerning the unifying properties of these cosets we proceed with the coset $\mathcal{W}_{2,1}^{\widehat{sl}(3)}/U(1)$ leading to a $\mathcal{W}(2, 3, 4, 5, 6, 7)$ (different from $\mathcal{W}A_6$).

As a further important example we investigate the diagonal coset $\widehat{sl}(2, \mathbb{R})_\kappa \oplus \widehat{sl}(2, \mathbb{R})_\mu/\widehat{sl}(2, \mathbb{R})_{\kappa + \mu}$ which, in particular, for $\mu = -\frac{1}{2}$ explains the special solution of $\mathcal{W}(2, 4, 6)$. In the second part of section 2 we study orbifolds of quantum $\mathcal{W}$-algebras. We determine the spin content of several orbifolds, the explicit form of the invariant fields and compute the structure constants for some examples.

In section 3 we discuss first the vacuum preserving algebra (VPA) and classical limits of some prominent examples of deformable $\mathcal{W}$-algebras outside the Drinfeld-Sokolov class. It turns out that the VPA is actually infinite dimensional in contrast to algebras in the DS class. We continue with a detailed discussion of the realization of unifying algebras for Casimir algebras by coset and orbifold constructions. Using character techniques we compute the spin content of these algebras and discuss the relation to level-rank-duality.

The calculations presented in this paper have been performed in parts with a small special purpose computer algebra system discussed in [25] which we used for calculations with $\mathcal{W}$-algebra modules. For the more complicated OPE calculations we used the Mathematica-package of ref. [26].
1.1. Notation

We begin with a short account of the notations which will be used in the following. For a detailed description we refer to the review article [1].

Denote the extended symmetry algebra of a local chiral conformally invariant quantum field theory in two dimensions by $\mathcal{F}$. We assume that $\mathcal{F}$ is equipped with the following three important operations: The commutator, the normal ordered product and the derivative $\partial$ of local quantum fields. Alternatively, we can demand that $\mathcal{F}$ is closed with respect to the short-distance operator product expansion (OPE) of two local fields. The singular part of an OPE reproduces the commutator, whereas the constant term yields the normal ordered product (NOP). The ‘mode expansion’ of a field $\phi$ is defined by

$$\phi(z) = \sum_{n-d(\phi) \in \mathbb{Z}} \phi_n \, z^{n-d(\phi)}.$$  \hspace{1cm} (1.1.1)

$d(\phi)$ is the ‘conformal dimension’ of $\phi$ and $\phi_n$ are the ‘modes’ (we deviate here from the standard conventions because negative modes annihilate the vacuum state). The modes of the energy momentum tensor $L$ satisfy the Virasoro algebra:

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}.$$  \hspace{1cm} (1.1.2)

The Lie bracket structure in $\mathcal{F}$ is fixed by a general formula for the commutator of two chiral fields of half integral conformal dimension which involves universal polynomials and some structure constants given by two- and three point functions [4,14]. For the singular part of the OPE of two fields $\phi_i, \phi_j$ we will use the following shorthand notation:

$$\phi_i \star \phi_j = \sum_k C^k_{ij} \phi_k.$$  \hspace{1cm} (1.1.3)

The coefficients $C^k_{ij}$ describe the coupling of the conformal families of $\phi_i, \phi_j$ and $\phi_k$. In repeated normal ordered products we use the convention: $\phi_k \ldots \phi_2 \phi_1 = (\phi_k (\ldots (\phi_2 \phi_1)))$ unless stated explicitly. In the form $\chi \phi$ the NOP occurs as the constant term in the OPE of $\chi$ and $\phi$ but in general it is not a quasi-primary field. In explicit calculations it is sometimes useful to work with the quasi-primary projection [4] of $\partial^n \phi_i \phi_j$ defined by $^1$)

$$\mathcal{N}(\phi_j, \partial^n \phi_i) := \sum_{r=0}^{n} (-)^r \binom{n}{r} \binom{2(d(i)+d(j)+n-1)}{r}^{-1} \binom{2d(i)+n-1}{r} \partial^r ((\partial^{n-r} \phi_i) \phi_j)$$

$$- (-)^n \sum_{\{k: h(ijk)\geq 1\}} C^k_{ij} \binom{h(ijk)+n-1}{n} \binom{2(d(i)+d(j)+n-1)}{n}^{-1}$$

$$\times \binom{2d(i)+n-1}{h(ijk)+n} \binom{h(ijk)-1}{n}^{-1} \frac{\partial^{h(ijk)+n} \phi_k}{(\sigma(ijk)+n)(h(ijk)-1)!}.$$  \hspace{1cm} (1.1.4)

$^1$) This formula is the same as in [4], the only difference is the normal ordering convention, i.e. $\phi_i \phi_j = N(\phi_j, \phi_i)$. 

4
with \( \sigma(ijk) = d(i) + d(j) + d(k) - 1 \), \( h(ijk) = d(i) + d(j) - d(k) \).

From any finite set of fields the operations \( \partial \) and \( \mathcal{N} \) generate infinitely many fields. It is therefore convenient to define ‘simple’ fields which are noncomposite and nonderivative. To be more precise simple fields are defined to have a vanishing projection onto derivative and composite fields. An algebra generated by the simple fields \( \phi_1, \ldots, \phi_n \) is called a \( \mathcal{W}(d(\phi_1), \ldots, d(\phi_n)) \). Primary fields composed of two simple fields and their derivatives are sometimes abbreviated using their dimensions if they determine the simple fields uniquely.

For example we write \( P^{d(\phi_1)d(\phi_2)}_{d(\phi_1)+d(\phi_2)+2} \) for the primary field appearing in the OPE \( \phi_1(z)\phi_2(w) \) at the zero of order 2 with normalization \( C^{\hat{\phi}_1\hat{\phi}_2}_{\phi_1\phi_2} = 1 \).

We denote the Casimir algebra corresponding to a simple Lie algebra \( \mathcal{L}_n \) by \( \mathcal{W}\mathcal{L}_n \). Casimir algebras arise from quantum Drinfeld-Sokolov reduction with the principal \( sl(2) \) embedding into \( \mathcal{L}_n \). It should be clear to the reader that we reserve the notation \( \mathcal{W}\mathcal{B}_n \) for the purely bosonic algebras \( \mathcal{W}\mathcal{B}_n \cong \mathcal{W}(2,4,\ldots,2n) \). In earlier papers, the same notation was used for \( \mathcal{W} \)-algebras of type \( \mathcal{W}(2,4,\ldots,2n,n+\frac{1}{2}) \) that contain one fermionic field. We denote these algebras by \( \mathcal{W}\mathcal{B}(0,n) \) because they can be obtained from quantum DS reduction for the Super Lie algebras \( \mathcal{B}(0,n) \). For a \( \mathcal{W} \)-algebra coming from DS reduction of a nonprincipal \( sl(2) \) embedding into \( \mathcal{L}_n \) we use the notation \( \mathcal{W}\mathcal{S}_{\mathcal{L}_n}^S \) where \( S \) denotes the embedding. For example, for \( \mathcal{L}_n = \mathcal{A}_n \), \( S \) is the \( r \)-tuple of the dimensions of the irreducible \( sl(2) \) representations which appear in the defining representation of \( \mathcal{L}_n \) thus determining the embedding. The Polyakov-Bershadsky algebra \([27,28]\) which is obtained by a DS reduction of the nonprincipal \( sl(2) \) embedding into \( sl(3) \) is abbreviated in this notation by \( \mathcal{W}_{2,1}^{sl(3)} \).

2. Non-freely generated \( \mathcal{W} \)-algebras in cosets and orbifolds

2.1. Coset \( \mathcal{W} \)-algebras

In this section we present the results of explicit constructions of the symmetry algebras of several cosets. In order to do so we have to define the coset construction algebraically: The algebraic coset \( \mathcal{W}/\hat{g}_k \) of a \( \mathcal{W} \)-algebra \( \mathcal{W} \) with a Kac-Moody subalgebra \( \hat{g}_k \) is defined as the commutant of \( \hat{g}_k \) in \( \mathcal{W} \). Note that Kac-Moody algebras are also \( \mathcal{W} \)-algebras and therefore \( \mathcal{W} \) can also be a Kac-Moody algebra. Similarly, \( \mathcal{W}/g \) refers to those fields in \( \mathcal{W} \) that commute with the horizontal Lie subalgebra \( g \) of \( \hat{g}_k \), i.e. the \( g \)-singlets in \( \mathcal{W} \). If one is interested in rational models one may define the coset algebra slightly different namely as the maximal extended symmetry algebra which contains the algebraic coset algebra as a subalgebra at the given value of the central charge. Since we are mainly interested in cosets which exist for generic values of the central charge we will use only algebraic cosets and call them just cosets from now on.

2.1.1. The cosets \( \tilde{sl(2,\mathbb{R})}/\hat{U}(1) \) and \( SVIR(N = 2)/\hat{U}(1) \)

In this section we study the quantum versions of the cosets \( \tilde{sl(2,\mathbb{R})}/\hat{U}(1) \) and \( SVIR(N = 2)/\hat{U}(1) \). It has been shown in \([9]\) that these cosets are infinitely generated with infinitely many relations in the classical case. However, we will argue that in the quantum case these cosets have at least a finitely generated subalgebra \( \mathcal{W}(2,3,4,5) \) (most probably the cosets are equal to this algebra) what is already indicated by counting arguments of the invariants.
in the classical coset. By explicit calculation of the first generators in the commutant we find that no new generators with conformal dimension 6,7 or 8 appear. This is contrary to the claim of [29] \(^2\) that one needs a new generator at each integer conformal dimension greater than one. Furthermore, by computing structure constants explicitly we find agreement with the second solution for \(\mathcal{W}(2,3,4,5)\) [7]. In fact, a character argument indicates that the cosets \(\widehat{sl}(2,\mathbb{R})/U(1)\) and \(SVIR(N = 2)/U(1)\) are isomorphic to \(\mathcal{W}(2,3,4,5)\).

**The coset \(\widehat{sl}(2,\mathbb{R})/U(1)\)**

The equivalence \(\widehat{sl}(2,\mathbb{R})/U(1) \cong \mathcal{W}(2,3,4,5)\) can be checked explicitly and we present here the calculations, even if they agree in part with those performed by other authors [29,31]. We start from the \(sl(2,\mathbb{R})\) Kac-Moody algebra

\[
J^o \star J^o = 2k I \quad J^o \star J^\pm = \pm 2J^\pm \quad J^+ \star J^- = kI + J^o, 
\]

(2.1.1)

It is easily verified that the Virasoro-operator

\[
L = \frac{1}{2k(k+2)} (2k J^+ J^- - J^o J^o - k \partial J^o) 
\]

(2.1.2)

has central charge \(c = \frac{2(k-1)}{k+2}\) and commutes with \(J^o\), that is \(L \star J^o = 0\). We can now construct a spin-3 field \(W_3\) that is primary with respect to \(L\) and commutes with \(J^o\). \(W_3\) is a linear combination of operators with zero \(U(1)\)-charge and we find

\[
W_3 = -6k^2J^+ \partial J^- - 12kJ^o J^+ J^- + 4J^o J^o J^o + 6k^2 \partial J^+ J^- + 6k \partial J^o J^0 + k^2 \partial^2 J^o. \quad (2.1.3)
\]

Starting from \(W_3\) we find the complete algebra by successively calculating the OPEs

\[
W_3 \star W_3 = d_{3,3}I + W_4 \quad W_3 \star W_4 = \frac{d_{4,4}}{d_{3,3}} W_3 + W_5 \quad W_3 \star W_5 = \frac{d_{5,5}}{d_{4,4}} W_4 + \tilde{C}^6_{35} P^3_{5} 
\]

\[
W_4 \star W_4 = d_{4,4} I + \tilde{C}^4_{44} W_4 + \tilde{C}^6_{44} P^3_{6} 
\]

\[
W_4 \star W_5 = \frac{d_{5,5}}{d_{4,4}} W_3 + \tilde{C}^5_{45} W_5 + \tilde{C}^7_{45} P^3_{7} + \tilde{C}^8_{45} P^3_{8} 
\]

\[
W_5 \star W_5 = d_{5,5} I + \frac{d_{5,5}}{d_{5,5}} \tilde{C}^5_{45} W_4 + \tilde{C}^7_{55} P^3_{6} + \tilde{C}^8_{55} P^3_{7} + \tilde{C}^8_{55} P^3_{8} + \tilde{C}^8_{58} P^3_{5}. \quad (2.1.4)
\]

The primary spin-4 and spin-5 fields as well as some two- and three-point functions are given in appendix A. The tilde for the structure constants \(\tilde{C}^k_{ij}\) distinguishes them from the structure constants \(C^k_{ij}\) used when the fields are in standard normalization. We make the following two important observations: In eq. (2.1.4) new generators of dimensions \(>5\) do not appear so that the algebra closes within the fields \(L, W_3, W_4\) and \(W_5\). The algebra can be identified with the special \(\mathcal{W}(2,3,4,5)\) by comparing the structure constants with

\(^2\) It has been known to the authors of [29] shortly after its publication that there is no independent spin 6 field [30].
those given in [7] for this algebra. For example, we obtain for the structure constant \( C_{33}^4 \) in standard normalization
\[
(C_{33}^4)^2 = \frac{128(k-3)(2k+1)(2k+3)^2}{(k-2)(k+2)(3k+4)(16k+17)} = \frac{16(c+2)(c+10)^2(5c-4)}{3(c+7)(2c-1)(5c+22)}
\] (2.1.5)
in accordance with the \( \mathcal{W}(2,3,4,5) \) of [7] different from \( \mathcal{W}_{A_4} \). The corresponding structure constant in [31, 29] and in [32] (derived from a construction using \( SU(2) \) parafermions) agrees with eq. (2.1.5).

The identification of \( sl(\mathbf{2}, \mathbb{R})/U(1) \) with the special \( \mathcal{W}(2,3,4,5) \) is further supported by the following calculation of the vacuum character of \( sl(\mathbf{2}, \mathbb{R})/U(1) \). To this end one has to count the uncharged states in the module generated freely by \( J^\pm \):
\[
\frac{1}{\prod_{n \geq 1}(1 - z^2 q^n)(1 - z^{-2} q^n)} = \frac{(1-z^2)\sum_{m \in \mathbb{Z}} \phi_m(q) z^{2m}}{\prod_{n \geq 1}(1 - q^n)^2} = \frac{\sum_{m \in \mathbb{Z}} (\phi_m(q) - \phi_{m-1}(q)) z^{2m}}{\prod_{n \geq 1}(1 - q^n)^2}
\] (2.1.6)
where the first manipulation follows from the well-known identity
\[
\frac{1}{\prod_{n \geq 0}(1 - z^2 q^n)(1 - z^{-2} q^{n+1})} = \frac{\sum_{m \in \mathbb{Z}} \phi_m(q) z^{2m}}{\prod_{n \geq 1}(1 - q^n)^2}
\] (2.1.7a)
(see e.g. [33]) with
\[
\phi_m(q) = \sum_{r \geq 0} (-1)^r q^\frac{r(r+1)}{2} + m r, \quad \phi_{-m}(q) = q^m \phi_m(q).
\] (2.1.7b)

The vacuum character \( B_0^0(q) \) of \( sl(\mathbf{2}, \mathbb{R})/U(1) \) is given by the \( m = 0 \) term of the r.h.s. of eq. (2.1.6) which is the generating function of the \( U(1) \) singlets:
\[
B_0^0(q) = \frac{\phi_0(q) - q \phi_1(q)}{\prod_{n \geq 1}(1 - q^n)^2}.
\] (2.1.8)

This agrees with the corresponding formula of [29] after correcting the following misprint: the exponent of \( f(q) \) in formula (4.14) of [29] should read 2 not 3. For the difference of \( B_0^0(q) \) and the character \( \chi_{2,3,4,5} \) of the vacuum module freely generated by fields of dimension 2, 3, 4 and 5 we obtain
\[
B_0^0(q) - \chi_{2,3,4,5} = -2q^8 - 4q^9 - 9q^{10} + O(q^{11}).
\] (2.1.9)

This supports the identification of the coset algebra \( sl(\mathbf{2}, \mathbb{R})/U(1) \) with the special solution of \( \mathcal{W}(2,3,4,5) \) because both algebras have two generic null fields at conformal dimension 8 [7].

**Commuting charges**

One might wonder if the confusion in the literature about the existence of a finite set of generators for the quantum coset \( sl(\mathbf{2}, \mathbb{R})/U(1) \) has any impact on the associated set of
commuting charges [34]. Note that the computation of [35] need not necessarily reflect the general case because it strongly relies on \( k = -1 \), i.e. \( c = -4 \). For \( c = -4 \) the existence of one charge per integer dimension is due to the fact that \( c = -4 \) is exactly one of the two values where \( \mathcal{W}_\infty \) truncates to the \( \mathcal{W}(2,3,4,5) \) under consideration [17]. Therefore, the conserved charges might be specific to \( c = -4 \) and the general case could be different.

In this section we will present a few explicit results indicating that the misinterpretation of the generating set has no impact on the conserved charges. In particular, it seems to be true that there is one conserved charge for each integer dimension [34] – in contrast to the \( \mathcal{W}A_n \) Casimir algebras where some of these charges are missing. The generic null fields (or relations) play a crucial rôle because the commutator of two charges \( P_i \) and \( P_j \) does no longer need to be exactly zero but can be proportional to the integral over a null field, relaxing the constraints on the commuting charges. Using the explicit form of the abstract \( \mathcal{W}(2,3,4,5) \)-algebra [7] \(^3\), we can indeed construct the first few commuting charges:

\[
\begin{align*}
\mathcal{P}_1 &= \oint L \\
\mathcal{P}_2 &= \oint W_3 \\
\mathcal{P}_3 &= \oint \left( W_4 + \frac{3}{16(2+c)} C_{33}^4 LL \right) \\
\mathcal{P}_4 &= \oint \left( W_5 + \frac{64}{5(22+5c)} C_{33}^4 LW_3 \right) \\
\mathcal{P}_5 &= \oint \left( (4+c) W_3 W_3 + \frac{5c^3 - 100c^2 - 1100c - 32}{2(7+c)(2c-1)(22+5c)} LLL + \frac{(20320 + 10928c + 1032c^2 + 34c^3 + 5c^4)}{24(1-2c)(7+c)(22+5c)} \partial L \partial L - \frac{3}{2} C_{33}^4 LW_4 \right).
\end{align*}
\]

The various null fields with dimensions \( \geq 8 \) appear in the commutators of \( \mathcal{P}_3 \) with \( \mathcal{P}_4 \) and of \( \mathcal{P}_2, \mathcal{P}_3 \) and \( \mathcal{P}_4 \) with \( \mathcal{P}_5 \). The presence of the charge \( \mathcal{P}_5 \) is contrary to the case of the \( \mathcal{W}A_4 \) algebra with the same spin content, where all charges of dimension 0 (mod 5) are missing.

We have checked that specializing (2.1.10) to \( c = -4 \) (\( k = -1 \)) one finds agreement with eq. (27) of [35]. One can also check that the classical limit of (2.1.10) is consistent with the classical conserved charges of [36]. The computation of this classical limit which has to be performed on the construction in terms of \( \widehat{sl}(2,\mathbb{R}) \) is straightforward but tedious because leading orders in \( \hbar \) cancel and the classical conserved charges arise from (2.1.10) as subleading orders in \( \hbar \).

The coset \( SVIR(N = 2)/U(1) \)

In the following we report an explicit construction of the generators in the quantum coset \( SVIR(N = 2)/U(1) \). The classical version of this coset was presented in [9] where also some of the results presented below were already mentioned.

For the \( N = 2 \) Super Virasoro algebra we use the explicit form of [37] eq. (2.12). The coset we intend to consider is defined as the commutant of the \( U(1) \)-current \( J \). For explicit

\(^3\) Note that the prefactor of \( C_{33}^{\mathcal{P}_{33}} \) in the appendix of this reference has a misprint and must therefore be multiplied by a factor of \( \frac{4}{5} \)!
calculations it is convenient to use a Vertex-operator approach. First we note that
\[ [J_m, X_n] = 0 \quad \forall m, n \quad \iff \quad J_{-m} X_{d(X)} |0\rangle = 0 , \quad m = 0, 1, \ldots, d(X) \tag{2.1.11} \]
for any field \(X\) with fixed scale-dimension \(d(X)\). For later purposes we also note a similar result for primary fields. The equivalence we will use is
\[ [\hat{L}_m, X_n] = (n - (d(X) - 1) m) X_{m+n} \quad \forall m, n \quad \iff \quad \hat{L}_{-m} X_{d(X)} |0\rangle = 0 , \quad m = 1, \ldots, d(X) \tag{2.1.12} \]
for the primarity of a field \(X\) with scale-dimension \(d(X)\) with respect to some energy momentum tensor \(\hat{L}\).

One now proceeds as follows: First one makes the most general ansatz at a given scale dimension for an invariant field (it is better to use standard normal ordered products and derivatives). In our case we also have a \(J_0\)-grading in addition to the \(L_0\)-grading. Therefore, in our ansatz we may restrict to ‘uncharged’ composite fields (fields with \(J_0\)-grade 0). This automatically ensures that the condition \(J_0 X_{d(X)} |0\rangle = 0\) is satisfied. When extracting conditions for the ansatz from the vacuum module it is important to use a basis in the space of fields which can most conveniently be implemented by choosing a lexicographic ordering.

First we find a unique invariant field at scale dimension 2
\[ \hat{L} = L - \frac{3}{2c} JJ. \tag{2.1.13} \]
\(\hat{L}\) satisfies the Virasoro-algebra eq. (1.1.2) with shifted central charge \(\hat{c} = c - 1\). Exploiting the first condition eq. (2.1.11) we find two- respective four- and six-dimensional invariant spaces of fields at conformal dimensions 3, 4 and 5. Imposing additionally the primarity condition eq. (2.1.12) we find unique primary invariant fields at scale dimensions 3, 4 and 5. For brevity we omit the lengthy expressions for the primary fields \(W^{(4)}, W^{(5)}\) and present only \(W^{(3)}\):
\[ W^{(3)} = \nu \left( -\frac{6}{c^2} JJ + \frac{6}{c} LJ + \overline{G} G + \frac{c - 9}{3c} \partial^2 J - \partial L \right). \tag{2.1.14} \]
The normalization constant \(\nu\) is fixed to \(\nu^2 = \frac{3(\hat{c}+1)}{2(2\hat{c}-1)(\hat{c}+7)}\) by imposing the normalization condition \(d_{3,3} = \frac{\hat{c}}{3}\). Now a standard computation shows that the structure constants connecting the primary fields \(W^{(3)}, W^{(4)}, W^{(5)}\) are identical to those obtained for the special solution of \(W(2,3,4,5)\) involving null fields [7]. This suggests that the coset considered here coincides with this algebra.

Analogous to \(\widehat{sl(2, \mathbb{R})/U(1)}\) this identification can also be inferred from a character argument, i.e. from the vacuum character of \(SVIR(N=2)/U(1)\). According to the results of [9] we have to count the states in the vacuum representation of the \(N=2\) algebra which are charge neutral and do not contain \(J\).
We use the Jacobi-triple product identity to write the $N=2$ vacuum character as

$$
\frac{\prod_{n \geq 1} (1 + z^2 q^{n+\frac{1}{2}})(1 + z^{-2} q^{n+\frac{1}{2}})}{\prod_{n \geq 1} (1 - q^n)(1 - q^{n+1})} = \frac{(1 - q)}{\prod_{n \geq 1} (1 - q^n)^2 (1 + z^2 q^{\frac{1}{2}})(1 + z^{-2} q^{\frac{1}{2}})} \sum_{m \in \mathbb{Z}} q^m z^{2m}. 
$$

(2.1.15)

Expanding the denominator into the geometric series

$$
\frac{1}{(1 + z^2 q^{\frac{1}{2}})(1 + z^{-2} q^{\frac{1}{2}})} = \sum_{\alpha, \beta \geq 0} (-1)^{\alpha + \beta} q^{\frac{\alpha + \beta}{2}} z^{2(\alpha - \beta)}. 
$$

(2.1.16)

one obtains for the vacuum character $A_0^0(q)$ of the $\widehat{U}(1)$-commutant in $SVIR(N=2)$:

$$
A_0^0(q) = \frac{(1 - q)}{\prod_{n \geq 1} (1 - q^n)^2} \sum_{\alpha, \beta \geq 0} (-1)^{\alpha + \beta} q^{\frac{\alpha + \beta}{2}} q^{\frac{(\alpha - \beta)^2}{2}}. 
$$

(2.1.17)

This is compatible with three additional generators of dimensions 3, 4, 5 respectively and two generic null fields at scale dimension 8 – precisely what has been found for the special $W(2, 3, 4, 5)$ in [7]. This supports the above identification quite convincingly.

The question if the coset algebras $\mathfrak{sl}(2, \mathbb{R})/U(1)$ and $SVIR(N=2)/U(1)$ are isomorphic is a natural question since we claimed that both cosets are isomorphic to $W(2, 3, 4, 5)$. First we show that the vacuum characters of $\mathfrak{sl}(2, \mathbb{R})/U(1)$ eq. (2.1.8) and $SVIR(N=2)/U(1)$ eq. (2.1.17) are equal. To this end we define

$$
f(\alpha, \beta) = (-1)^{\alpha + \beta} q^{\frac{\alpha + \beta}{2}} q^{\frac{(\alpha - \beta)^2}{2}}. 
$$

(2.1.18)

Since $f(\alpha + 1, \beta + 1) = qf(\alpha, \beta)$ most terms in eq. (2.1.17) cancel so that only the two axes $\alpha = 0$ and $\beta = 0$ survive. Hence we obtain

$$
A_0^0(q) = \frac{1}{\prod_{n \geq 1} (1 - q^n)^2} \left( \sum_{\alpha \geq 0} (-1)^{\alpha} q^{\frac{(\alpha + 1)\alpha}{2}} + \sum_{\beta \geq 0} (-1)^{\beta + 1} q^{\frac{(\beta + 1)(\beta + 2)}{2}} \right). 
$$

(2.1.19)

Using the definition of $\phi_m(q)$ (eq. (2.1.7b)) this coincides obviously with the vacuum character eq. (2.1.8) of $\mathfrak{sl}(2, \mathbb{R})/U(1)$.

However, it is not difficult to show that the commutants are algebraically isomorphic. One uses the realizations of $\mathfrak{sl}(2, \mathbb{R})$ and the $SVIR(N=2)$ in terms of a free boson and $\mathbb{Z}_k$ parafermions (see e.g. [38, 39]). The $U(1)$-current $J$ is just given by the derivative of the free boson $\phi$. The fields $J^\pm$ resp. $G, \overline{G}$ are represented by two parafermionic currents $\psi^\pm$ with conformal dimension $1 - \frac{1}{k}$ dressed with a vertex operator $e^{\pm i\alpha(k)\phi}$ with a suitably chosen $\alpha(k)$ (compare [38, 39]). The parafermions $\psi^\pm$ themselves can be realized in terms of two free bosons [38]. The $U(1)$-commutant is generated by the invariants quadratic in $J^\pm$ resp. $G, \overline{G}$ implying that in both cases the commutant is built out of the two parafermions. Thus the two $\widehat{U}(1)$-commutants are isomorphic. Note that in the $k \to \infty$ limit the two
Parafermions have conformal dimension 1 and the character of the commutant is exactly the one calculated in eq. (2.1.8).

Until now we have not treated the cancellation of the classical generators and relations in the quantum case. Therefore, let us understand why in the quantum case we fail to get a new generator at scale dimension 6 in the coset $SVIR(N = 2)/U(1)$ and compare it with the classical situation. The field $((\bar{G}G)(\bar{G}G))$ does not belong to the set of quantum generators – classically it vanishes identically because of the Pauli principle. However, due to the nonassociativity of the normal ordered product it satisfies the following equality (quantum):

$$
((\bar{G}G)(\bar{G}G)) = (c + 9) \partial^3 \bar{G} G + 2 \partial^2 \bar{G} \partial G + \partial \bar{G} \partial^2 G - \frac{a}{9} \bar{G} \partial^3 G + \partial J (\partial \bar{G} G)
+ \partial J (\bar{G} \partial G) + 2 J (\partial^2 \bar{G} G) + 2 J (\bar{G} \partial^2 G) + 2 \partial L (\bar{G} G) + 2 L (\partial \bar{G} G) - 2 L (\bar{G} \partial G)
+ 2 \partial^2 L L - \frac{3}{2} \partial^3 L J - \partial^2 L \partial J - \frac{1}{3} L \partial^3 J + \frac{1}{6} \partial^4 J J + \frac{1}{6} \partial^3 J \partial J - \frac{c - 4}{60} \partial^4 J + \frac{c}{15} \partial^3 L.
$$

(2.1.20)

From this expression one can see explicitly that $((\bar{G}G)(\bar{G}G))$ tends to zero in the classical limit (compare section 3.1.). This equality implies that the square of the generator $W^{(3)}$ contains correction terms that cancel contributions which might give rise to new generators. This mechanism guarantees that the quantum coset $SVIR(N = 2)/\hat{U}(1)$ has at least a finitely generated subalgebra of type $W_{(2,3,4,5)}$. Such cancellations do not occur classically. Therefore, the coset under consideration is infinitely generated classically with a first relation at scale dimension 6.

It is also straightforward to derive the representations of the coset algebra from those of the $N = 2$ Super Virasoro algebra. Each highest weight representation of the Super Virasoro algebra satisfying

$$
L_0 \mid h, \tau \rangle = h \mid h, \tau \rangle \quad J_0 \mid h, \tau \rangle = \tau \mid h, \tau \rangle
$$

(2.1.21)

gives rise to one (in general reducible) representation of the coset algebra with the following eigenvalue equations for the highest weight vector:

$$
\hat{L}_0 \mid h, \tau \rangle = \hat{h} \mid h, \tau \rangle \quad W^{(i)}_0 \mid h, \tau \rangle = w_i \mid h, \tau \rangle \quad i = 3, 4, 5.
$$

(2.1.22)

Obviously, the central charge is shifted by one: $\hat{c} = c - 1$. $\hat{h}$ and $w_i$ can be expressed through $h$, $\tau$ and $c$ using the realization of the generators. One finds:

$$
\hat{h} = h - \frac{3\tau^2}{2c} \quad w_3 = \nu \left( h \frac{6\tau}{c} - \tau \frac{c^2 + 18\tau^2}{3c^2} \right).
$$

(2.1.23)

In particular, all minimal models of the coset algebra can be derived from those of the $N = 2$ Super Virasoro algebra. Note that the $N = 2$ Super Virasoro algebra presumably has only unitary minimal models.

We know that the $W(2, 3, 4, 5)$ is the unifying algebra for the first unitary model of $WA_{k-1}$ [17], i.e. that $WA_{k-1}$ truncates for $c(k) = \frac{2k-1}{k+2}$ to $W(2, 3, 4, 5)$. From the above coset realizations

$$
\frac{sl(2, \mathbb{R})}{U(1)} \cong \frac{SVIR(N = 2)}{U(1)} \cong W(2, 3, 4, 5)
$$

(2.1.24)
we expect that the first unitary model of $\mathcal{WA}_{k-1}$ agrees with the rational models of $SVIR(N = 2)/\widehat{U(1)}$ coming from the unitary minimal models of the $N = 2$ Super Virasoro algebra. Indeed, using eq. (2.1.23) one can compute the conformal dimensions and $w_i$ quantum numbers for $SVIR(N = 2)/\widehat{U(1)}$ and one finds perfect agreement with the first unitary model of $\mathcal{WA}_{k-1}$.

For $\mathcal{WA}_{k-1}$ we are in the fortunate position that at least some structure constants are known generally [40]:

$$\left( C_{3\,3}^4 \right)^2 = \frac{64(k - 3)(c + 2)(c(k + 3) + 2(4k + 3)(k - 1))}{(k - 2)(5c + 22)(c(k + 2) + (3k + 2)(k - 1))}$$

$$C_{3\,3}^4 C_{4\,4}^4 = \frac{48(c^2(k^2 - 19) + 3c(6k^3 - 25k^2 + 15) + 2(k - 1)(6k^2 - 41k - 41))}{(k - 2)(5c + 22)(c(k + 2) + (3k + 2)(k - 1))} \quad (2.1.25a)$$

We compute the so far unknown structure constants $(C_{3\,4}^5)^2$ and $C_{4\,5}^5$. This can be done in two different ways. Either one uses an ansatz involving a representation theoretic argument [8] or one evaluates special Jacobi identities for $\mathcal{WA}_{k-1}$. Both methods yield the same result:

$$\left( C_{3\,4}^5 \right)^2 = \frac{25(c k + 4 c + 15k^2 - 3k - 12)(5c + 22)(k - 4)}{(ck + 2c + 3k^2 - k - 2)(7c + 114)(k - 2)}$$

$$C_{4\,5}^5 = \frac{15}{8}((2 + c)(114 + 7c)(k - 3)(8k^2 + ck - 2k + 3c - 6))^{-1}C_{3\,3}^4$$

$$\times (6756 + 1420c - 483c^2 - 97c^3 - 6972k^2 - 5192ck^2 - 467c^2k^2 + 3c^3k^2 + 216k^3 + 856ck^3 + 94c^2k^3) \quad (2.1.25b)$$

where we have also presented the result for $C_{4\,5}^5$ that was obtained exclusively with the methods of [8]. Inserting $c(k) = \frac{2k - 1}{k^2 + 2}$ into eq. (2.1.25) reproduces eq. (2.1.5) verifying again the truncation of $\mathcal{WA}_{k-1}$ at $c(k)$ to $W(2, 3, 4, 5)$ given in [17]. For this particular value of the central charge all generators with scale dimension 6 or higher turn out to be null fields. This can be verified directly for $k = 6$ by inspecting the structure constants presented in [7] for $\mathcal{WA}_5 \cong W(2, 3, 4, 5, 6)$ (see also [17]).

This identification can be interpreted in a more general way if one inspects the “dual coset pairs” given in [1, 18, 19]. Here dual coset pairs possessing equivalent energy momentum tensors (“$T$-equivalence”) have been presented, e.g. $sl(\widehat{k}, \mathbb{R})_\kappa \oplus sl(\widehat{k}, \mathbb{R})_\mu / sl(\widehat{k}, \mathbb{R})_{\kappa + \mu} \cong sl(\widehat{\kappa + \mu}, \mathbb{R})_k / \{ sl(\widehat{\kappa}, \mathbb{R})_k \oplus \widehat{U(\mu)}_k \}$. Specializing to $\kappa = \mu = 1$ we know that the left hand side yields the first unitary model of $\mathcal{WA}_{k-1}$ at $c(k)$ [15]. The right hand side reduces to the coset $sl(2, \mathbb{R})_k / \widehat{U(1)}$ and we get back the equivalence considered above. Our investigations go beyond [18, 19]: we have verified that the maximally extended symmetry algebras are isomorphic and not just the energy momentum tensors. In section three we will discuss the relation of unifying $\mathcal{W}$-algebras to level-rank duality in more general situations.

### 2.1.2. The coset $\mathcal{W}^{sl(3)}/\widehat{U(1)}$

Like the $SVIR(N = 2)$-algebra, the $\mathcal{W}^{sl(3)}$-algebra of Polyakov and Bershadsky [27, 28] has a primary field of dimension 1 and two of dimension $\frac{3}{2}$ in addition to the energy momentum tensor. $\mathcal{W}^{sl(3)}$ is obtained by quantum hamiltonian reduction for the nonprincipal
embedding of $sl(2)$ in $sl(3)$. In contrast to the $SVIR(N = 2)$-algebra the spin $\frac{3}{2}$ fields $G^\pm$ obey bosonic statistics. One important consequence is that the OPE of the generators does not close linearly, but quadratically in $J^0$.

In the normalization of Bershadsky [28] the algebra reads

\begin{align}
J^0 \star J^0 &= \frac{1}{3} (2k + 3) I \\
J^0 \star G^\pm &= \pm G^\pm \\
G^+ \star G^- &= (k + 1) (2k + 3) I + 3 (k + 1) J^0 + 3 P^{11}_2 \\
\end{align}

(2.1.26)

where the central charge $c$ of this algebra is connected to $k$ by

\begin{equation}
c = -(2k + 3) (3k + 1)(k + 3)^{-1}
\end{equation}

(2.1.27)

and the field $P^{11}_2$ is the composite primary field:

\begin{equation}
P^{11}_2 = J^0 J^0 + \frac{2}{3} (k + 3)(3k + 1)^{-1} L.
\end{equation}

(2.1.28)

The classical version of the coset $W_{2,1}^{sl(3)/U(1)}$ has been treated in [41] and an infinite generating set of the commutant has been given. A nonredundant (infinite) set of generators and the full (infinite) generating set of relations between these generators have been presented in [9]. We discuss now the quantum version of this coset model.

As for $sl(2,\mathbb{R})/U(1)$, we construct $W_{2,1}^{sl(3)/U(1)}$ by computing a Virasoro-operator $\hat{L}$ and a primary $W_3$-field both commuting with $J^0$ 4), and recursive evaluation of the OPEs starting with $W_3 \star W_3$. The bosonic statistics of the fields $G^\pm$ implies that the classical limit of $P^{33}_6$ whose leading term is $((G^+ G^-) (G^+ G^-))$ does not vanish. Therefore, in the coset-algebra a $W_6$-field appears, contrary to the coset $SVIR(N = 2)/U(1)$. Analogously the only composite field with scale dimension 7, $P^{34}_7$, does not vanish in the classical limit. From counting the invariant fields (commuting with $J^0$) up to scale dimension 8 we expect that the quantum coset does not contain a spin 8 generator which is confirmed by our explicit calculations below.

First we have to construct the modified energy momentum tensor and the $W_3$-field both commuting with the $U(1)$-current $J^0$. These fields are given by: 5)

\begin{align}
\hat{L} &= L - \frac{3}{2}(3 + 2k)^{-1} J^0 J^0 \\
W_3 &= \frac{1}{2(3 + 2k)^2} \left( 2(3 + 2k)^2 G^+ G^- - 18(2 + k) J^0 J^0 + 6(3 + k)(3 + 2k) L J^0 - 6(3 + 2k)^2 \partial J^0 J^0 + (3 + k)(3 + 2k)^2 \partial L - 2(3 + 2k)(6 + 4k + k^2) \partial^2 J^0 \right)
\end{align}

(2.1.29, 2.1.30)

where the central charge is again shifted by one,

\begin{equation}
\hat{c} = c - 1 = -6(1 + k)^2 (3 + k)^{-1}.
\end{equation}

(2.1.31)

4) Since $W_3$ commutes with $J^0$ it is simultaneously primary under $L$ and $\hat{L}$.

5) The difference of $\hat{L}$ to the one of the $SVIR(N = 2)/U(1)$-coset is due to the different normalization of $J^0$. 

13
We verified the following OPEs:

\[ W_3 \star W_3 = d_{3,3} I + W_4 \quad W_3 \star W_4 = \frac{d_{4,4}}{d_{3,3}} W_3 + W_5 \quad W_3 \star W_5 = \frac{d_{5,5}}{d_{4,4}} W_4 + W_6 \]

\[ W_3 \star W_6 = \frac{d_6}{d_{3,3}} W_3 + \frac{d_{6,6}}{d_{5,5}} W_5 + W_7 + \tilde{C}_{3,6}^{P_{34}} P_{34} \]

\[ W_4 \star W_4 = d_{4,4} I + \tilde{C}_{4,4}^{W_4} W_4 + \tilde{C}_{4,4}^{P_{33}} W_6 + \tilde{C}_{4,4}^{P_{34}} P_{33} \]

\[ W_4 \star W_5 = \frac{d_{5,5}}{d_{3,3}} W_3 + \tilde{C}_{5,4}^{W_5} W_5 + \tilde{C}_{4,5}^{P_{34}} P_{34} + \tilde{C}_{4,5}^{P_{34}} P_{34} \]

\[ W_4 \star W_6 = \tilde{C}_{4,6}^{W_4} W_4 + \tilde{C}_{4,6}^{P_{33}} W_6 + \tilde{C}_{4,6}^{P_{34}} P_{33} + \tilde{C}_{4,6}^{P_{34}} P_{34} + \tilde{C}_{4,6}^{P_{34}} P_{34} + \tilde{C}_{4,6}^{P_{34}} P_{34} = \tilde{C}_{4,6}^{P_{34}} P_{34}. \]  

(2.1.32)

The fields \( W_4 \) to \( W_7 \) are again defined recursively by these OPEs. In order to save space we omit the lengthy expressions of these fields. For our purpose (i.e. to show that there is no new spin-8 field) it was not necessary to define the new fields \( W_6 \) and \( W_7 \) to be orthogonal to the composite fields \( P_{33}^{P} \) and \( P_{34}^{P} \), i.e. they are just defined by eq. (2.1.32).

The appearance of \( W_3 \) in the OPE of \( W_3 \) with \( W_6 \) reflects this fact, where \( d_6 \) is the off-diagonal element of the spin-6 fields: \( W_6 \star P_{33}^{P} = d_6 I + \ldots \). The OPE of \( W_4 \) with \( W_6 \) does not show a new spin-8 field. We expect therefore that the \( \mathcal{W}_{2,1}^{sl(3)}/\hat{U}(1) \)-coset has a subalgebra of type \( \mathcal{W}(2, 3, 4, 5, 6, 7) \) (being different from the \( \mathcal{W}A_6 \)-algebra). The OPEs eq. (2.1.32) do not exclude completely the existence of fields with dimension \( \geq 9 \) in the coset, but supporting counting arguments on the classical level [9] can be given. Moreover, these counting arguments indicate two generic null fields at dimension 10.

The structure constant \( C_{3,3}^{4} \) in standard normalization for this algebra is given by

\[
(C_{3,3}^{4})^2 = \frac{128k^2 (3 + k) (5 + 3k) (12 + 5k)}{(1 + 2k) (9 + 4k) (-18 + 19k + 15k^2)}.
\]

(2.1.33)

Rewriting it in terms of the new central charge \( \hat{c} \) would lead to square roots in the expression. Additional structure constants are given in appendix B.

The spin content of \( \mathcal{W}_{2,1}^{sl(3)}/\hat{U}(1) \) suggests that this algebra is a unifying algebra for a certain model of the \( \mathcal{W}A_{n-1} \) Casimir algebras. Indeed, with the relation \( k = \frac{n-3}{2} \), we find that the central charge of the \( \mathcal{W}_{2,1}^{sl(3)}/\hat{U}(1) \)-coset eq. (2.1.31) coincides with the central charge \( c_{A_{n-1}}(n + 1, n + 3) = -3(n-1)^2/n+3 \) of the nonunitary model \( p = n + 1, q = p + 2 \) of \( \mathcal{W}A_{n-1} \). Furthermore, we find agreement of the structure constants \( (C_{3,3}^{4})^2 \) of \( \mathcal{W}A_{n-1} \) eq. (2.1.25a) and the \( \mathcal{W}_{2,1}^{sl(3)}/\hat{U}(1) \)-coset eq. (2.1.33) for these particular choices of \( k \) and \( c \). Thus we expect that the generators of \( \mathcal{W}A_{n-1} \) with scale dimension \( \geq 8 \) are null fields for these values of the central charge so that the Casimir algebras \( \mathcal{W}A_{n-1} \) truncate to this \( \mathcal{W}(2, 3, 4, 5, 6, 7) \). This is also supported by the study of Kac-determinants in [17].

Unfortunately, we are not able to verify that the highest weight representations of the corresponding models coincide since the highest weights of the minimal models of the \( \mathcal{W}_{2,1}^{sl(3)} \)-algebra are not known. We calculated explicitly the highest weights of the first few minimal models of \( \mathcal{W}_{2,1}^{sl(3)} \) in the sector with periodic boundary conditions for the fields
with half-integer spin. The induced representations of the coset lie all in the Kac-table of the corresponding $\mathcal{WA}_{n-1}$ models. Our explicit calculations show that the set of highest weights given in [28] is certainly too big.

Generalizations

Our aim is to generalize the result of the preceding section to unifying $\mathcal{W}$-algebras of the nonunitary models $c_{A_{n-1}}(n + 1, n + r)$ of $\mathcal{WA}_{n-1}$.

From the experience with the case $r = 3$ (the Polyakov-Bershadsky algebra) we expect that these unifying $\mathcal{W}$-algebras should arise as cosets of DS type $\mathcal{W}$-algebras $\mathcal{W}^{sl(r)}_{r-1,1}$ which have a $U(1)$ Kac-Moody subalgebra. Consider therefore the $r$-dimensional defining representation of $sl(r)$. One embeds $sl(r - 1) \oplus U(1)$ into $sl(r)$ by separating the last node from the Dynkin diagram of $sl(r)$ (the first $r - 2$ nodes realize $sl(r - 1)$). The defining representation of $sl(r)$ splits into the $r - 1$ dimensional defining representation of the $sl(r - 1)$ subalgebra and the trivial representation. Take now the principal $sl(2)$ embedding into the $sl(r - 1)$. Quantum hamiltonian reduction yields an algebra of type $\mathcal{W}(1, 2, \ldots, r - 1, \frac{r}{2}, \frac{r}{2})$ with all fields bosonic. The spin $\frac{r}{2}$ fields have $U(1)$-charge $\pm 1$ whereas all other simple fields are uncharged. In the notation introduced in section 1.1. this algebra is denoted by $\mathcal{W}^{sl(r)}_{r-1,1}$. Following [9] we obtain for the coset:

$$\frac{\mathcal{W}^{sl(r)}_{r-1,1}}{U(1)} \cong \mathcal{W}(2, 3, \ldots, 2r + 1).$$

(2.1.34)

Comparing with table 1 in [17] we see that this algebra has indeed the spin content of the unifying algebra for the models $c_{A_{n-1}}(n + 1, n + r)$.

To get a confirmation let us treat the case $r = 4$. We constructed explicitly the OPEs for the algebra $\mathcal{W}^{sl(4)}_{3,1} \cong \mathcal{W}(1, 2, 2, 3)$. To check our conjecture we calculated the structure constant $C_{3, 3}^4$ in the coset $\mathcal{W}(1, 2, 2, 3)/U(1) \cong \mathcal{W}(2, 3, 4, 5, 6, 7, 8, 9)$ yielding truncations of $\mathcal{WA}_{n-1}$ to $\mathcal{W}(2, 3, 4, 5, 6, 7, 8, 9)$ for:

$$c_1(n) = -\frac{4(n - 1)(2n - 3)}{(n - 3)(n - 4)}$$

$$c_2(n) = -\frac{4(n - 1)(2n - 1)}{(n + 4)}$$

$$c_3(n) = -\frac{(3n + 1)(5n + 3)}{(n + 3)}.$$  

(2.1.35)

The first truncation corresponds to the $\mathcal{W}_\infty$ truncation of section 3 in [17] with $r = 4$. The second truncation is exactly the truncation of the nonunitary models $c_{A_{n-1}}(n + 1, n + 4)$ of $\mathcal{WA}_{n-1}$. This confirms our claim from above.

For a final check we compute the central charge of the algebra $\mathcal{W}^{sl(r)}_{r-1,1}$ as a function of the level $k$ of the underlying Kac-Moody $sl(r)_k$. We use the formula $c = N_t - \frac{1}{2} N_s - 12 |k + h^\vee \delta - \frac{1}{\sqrt{k + h^\vee}} \rho|^2$ (see e.g. [12]). For the embedding under consideration one has $N_t = r - 1, N_s = 2$ for $r$ odd and $N_t = r + 1, N_s = 0$ for $r$ even. A matrix representation for $\rho$ and $\delta$ is $\rho = \text{diag}(\frac{r-1}{2}, \frac{r-3}{2}, \ldots, -\frac{r-1}{2}), \delta = \text{diag}(\frac{r}{2}, \ldots, \frac{r}{2} - \frac{|r|}{2}, 0, -\frac{r}{2} + \frac{|r|}{2}, \ldots, -\frac{r-2}{2})$. The scalar product of two matrices is the usual one: $a \cdot b = \text{tr}(ab)$. With this data it is straightforward to calculate $c$ as a function of $k$ for $\mathcal{W}^{sl(r)}_{r-1,1}$:

$$c = -\frac{(kr^2 - 2kr + r^3 - 3r^2 + 1)(kr - k + r^2 - 2r)}{k + r}.$$  

(2.1.36)
From the fact that the coset (2.1.34) is a unifying algebra for $\mathcal{W}A_{n-1}$ at $cA_{n-1}(n+1, n+r)$ one must have $c - 1 = cA_{n-1}(n+1, n+r)$. This is indeed satisfied for

$$k = \frac{n - r^2 + 2r}{r - 1}. \quad (2.1.37)$$

Note that the relation (2.1.37) between the level $k$ and the rank of $\mathcal{W}A_{n-1}$ is linear with a denominator $r - 1$. This simple relation increases our confidence that the cosets (2.1.34) play indeed the rôle of unifying $\mathcal{W}$-algebras.

2.1.3. The coset $sl(2, \mathbb{R})_{\kappa} \oplus sl(2, \mathbb{R})_{\mu}/sl(2, \mathbb{R})_{\kappa+\mu}$

In this section we study the diagonal $sl(2, \mathbb{R})$ coset. Noting from [42, 18, 9] that

$$\lim_{\mu \rightarrow \infty} \frac{sl(2, \mathbb{R})_{\kappa} \oplus sl(2, \mathbb{R})_{\mu}}{sl(2, \mathbb{R})_{\kappa+\mu}} = \frac{sl(2, \mathbb{R})_{\kappa}}{sl(2, \mathbb{R})} \quad (2.1.38)$$

and that the l.h.s. at generic $\mu$ is a deformation of the r.h.s. we conclude that much information about the coset we are interested in can already be obtained from the simpler coset $sl(2, \mathbb{R})_{\kappa}/sl(2, \mathbb{R})$. Therefore, we treat this coset first.

Upon a mode expansion of the OPE eq. (2.1.1) we obtain the following commutation relations for the Kac-Moody algebra $sl(2, \mathbb{R})_{\kappa}$:

$$[J_{m}^{(\pm)}, J_{n}^{(\pm)}] = 0, \quad [J_{m}^{(0)}, J_{n}^{(0)}] = 2\kappa \delta_{m,-n} n, \quad [J_{m}^{(\pm)}, J_{n}^{(\mp)}] = \pm 2J_{m+n}^{(\pm)}, \quad [J_{m}^{(+)}, J_{n}^{(-)}] = J_{m+n}^{(0)} + \kappa \delta_{m,-n} n. \quad (2.1.39)$$

We want to construct the invariant subspace under the natural action of $sl(2, \mathbb{R})$ on $sl(2, \mathbb{R})$. This means that we have to construct all fields commuting with $\{ J_{0}^{(\pm)}, J_{0}^{(0)} \}$. From the coefficient of $[J_{0}^{(0)}, X_{m}]$ in front of $X_{m}$ one concludes first that $X$ must be $J_{0}^{(0)}$-uncharged.

It is straightforward to make the most general ansatz in uncharged fields of scale dimension 2 and determine the field(s) that also commute(s) with $J_{0}^{(\pm)}$. It is no surprise to find a unique invariant field (up to normalization) at dimension 2, which is just given by the Sugawara construction:

$$2(\kappa + 2) \ L : = \mathcal{N}(J^{(+)}, J^{(-)}) + \frac{1}{2} \mathcal{N}(J^{(0)}, J^{(0)}) + \mathcal{N}(J^{(-)}, J^{(+)}) \quad (2.1.40)$$

with $\mathcal{N}(J^{(0)}, J^{(0)}) = J^{(0)} J^{(0)}, \mathcal{N}(J^{(\pm)}, J^{(\mp)}) = J^{(\mp)} J^{(\pm)} \pm \frac{1}{2} \partial J^{(0)}$. It is straightforward to check that $L$ satisfies the Virasoro algebra eq. (1.1.2) with central charge $c = 3\kappa(\kappa + 2)^{-1}$. Note that the original currents are primary with respect to $L$. Therefore, one can conveniently use quasi-primary normal ordered products in the calculations which we shall do below.
One can proceed along the same lines to find the next independent invariant field at scale dimension 4. However, this approach becomes unfeasible for higher dimensions. Therefore, we use group theoretic knowledge about the generators of this coset [9]. Let \( g^{ij} \) be the metric on the simple Lie algebra and \( f^k_{ij} \) be the structure constants. With the inverse of the metric \( g_{ij} \) we define \( \epsilon_{ijkl} := \sum_l g_{kl} f^l_{ij} \). Then, the \( SL(2, \mathbb{R}) \) invariant generators are given by [9]

\[
S_{m,n} := \sum_{i,j} g_{ij} \partial^n J^{(j)} \partial^m J^{(i)},
\]

(2.1.41a)

\[
S_{m,n,k} := \sum_{i,j,l} \epsilon_{ijkl} \partial^k J^{(l)} \partial^n J^{(j)} \partial^m J^{(i)}
\]

(2.1.41b)

\[
= \partial^k J^{(-)} (\partial^n J^{(0)} \partial^m J^{(+)}) - \partial^k J^{(-)} (\partial^n J^{(+)} \partial^m J^{(0)}) - \partial^k J^{(0)} (\partial^n J^{(-)} \partial^m J^{(+)}) + \partial^k J^{(0)} (\partial^n J^{(+)} \partial^m J^{(-)}) + \partial^k J^{(0)} (\partial^n J^{(-)} \partial^m J^{(0)}) - \partial^k J^{(0)} (\partial^n J^{(+)} \partial^m J^{(-)}).
\]

Instead of the second order invariants \( S_{m,n} \) we use their quasi-primary projections:

\[
W^{(n+2)} := \mathcal{N}(J^{(+)}, \partial^n J^{(-)}) + \frac{1}{2} \mathcal{N}(J^{(0)}, \partial^n J^{(0)}) + \mathcal{N}(J^{(-)}, \partial^n J^{(+)})
\]

(2.1.42)

for all even \( n \). For the third order invariants it is more complicated to obtain quasi-primary projections; we will come back to this problem below. Note that it can also be verified case by case that \( W^{(n)} \) and \( S_{m,n,k} \) indeed commute with \( J_0^{(\pm)} \) (they commute with \( J_0^{(0)} \) by construction) using the fact that \([ J_0^{(\pm)}, X] = 0 \) is equivalent to \( J_0^{(\pm)} X_{d(X)} |0 \) = 0 which is in the spirit of (2.1.11).

In order to obtain a generating set for the quasi-primary projections of the third order invariants we first note that the third order invariants are classically completely antisymmetric and therefore the quantization \( S_{m,n,k} \) can be expressed in terms of lower order invariants if any of the arguments coincide. Next, the derivative \( \partial \) maps third order invariants to third order invariants. Keeping all this in mind we may choose as independent generators

\[
S_{0,1,2}, S_{0,1,4}, S_{0,1,5}, S_{0,1,6}, S_{0,1,7}, S_{0,1,8}
\]

(2.1.43)

up to dimension 12. The quasi-primary projection of these fields can now be calculated by orthogonalization with respect to all derivatives – using again the vacuum representation. One obtains the following quasi-primary third order invariants:

\[
\hat{S}_{0,1,2} := S_{0,1,2} - \frac{1}{35} \partial^4 \hat{L} - \frac{1}{15} \partial^2 W^{(4)},
\]

\[
\hat{S}_{0,1,4} := S_{0,1,4} - \frac{1}{126} \partial^6 \hat{L} + \frac{19}{39} \partial^4 W^{(4)} - \frac{10}{13} \partial^2 \hat{S}_{0,1,2},
\]

\[
\hat{S}_{0,1,5} := S_{0,1,5} - \frac{1}{210} \partial^7 \hat{L} - \frac{1}{198} \partial^5 W^{(4)} + \frac{45}{91} \partial^3 W^{(6)} - \frac{50}{91} \partial^2 \hat{S}_{0,1,2} + \frac{1}{4} \partial W^{(8)} - \frac{15}{8} \partial \hat{S}_{0,1,4},
\]

\[
\hat{S}_{0,1,6} := S_{0,1,6} - \frac{1}{330} \partial^8 \hat{L} - \frac{7}{858} \partial^6 W^{(4)} + \frac{17}{39} \partial^4 W^{(6)} - \frac{5}{13} \partial^3 \hat{S}_{0,1,2} + \frac{41}{68} \partial^2 W^{(8)}
\]

\[- \frac{315}{136} \partial^2 \hat{S}_{0,1,4} - \frac{7}{3} \partial \hat{S}_{0,1,5},
\]

\[
\hat{S}_{0,1,7} := S_{0,1,7} - \frac{1}{495} \partial^9 \hat{L} - \frac{4}{429} \partial^7 W^{(4)} + \frac{14}{39} \partial^5 \hat{L} - \frac{7}{26} \partial^3 \hat{S}_{0,1,2} + \frac{140}{153} \partial^3 W^{(8)}
\]

\[- \frac{245}{102} \partial^2 \hat{S}_{0,1,4} - \frac{196}{57} \partial^2 \hat{S}_{0,1,5} + \frac{1}{5} \partial W^{(10)} - \frac{14}{5} \partial \hat{S}_{0,1,6},
\]

(2.1.44)
Using (2.1.44) we have computed determinants of all quasi-primary fields up to scale dimension 9. From the zeroes of the determinants one can read off those values of $\kappa$ where truncations of the coset algebra occur. All determinants have a singularity at $\kappa = -2$ which reflects the fact that the $c(\kappa) = 3\kappa(\kappa + 2)^{-1} \to \infty$ limit of this coset algebra is not well defined (see also section 3.1.). Furthermore, all fields are null fields at $\kappa = c = 0$. The remaining exceptional values of $\kappa$ are listed in table 1. For these values of $\kappa$ we have further determined the kernels of the $d$-matrices giving us precise information which generators drop out. The resulting generating sets are also presented in table 1.

| $\kappa$ | $c$ | algebra |
|----------|-----|---------|
| generic  | generic | $\mathcal{W}(2,4,6,6,8,8,9,10,10,12)$ |
| 1        | 1   | $\mathcal{W}(2)$ |
| 2        | $\frac{3}{2}$ | $\mathcal{W}(2,4,6)$ |
| 3        | $\frac{9}{5}$ | $\mathcal{W}(2,4,6,6,8,9)$ |
| 4        | 2   | $\mathcal{W}(2,4,6,6,8,8,9,10)$ |
| $\frac{15}{7}$ | 5 | $\mathcal{W}(2,4,6,6,8,8,9,10,10)$ |
| $\frac{-1}{2}$ | $\frac{1}{2}$ | $\mathcal{W}(2,4,6)$ |
| $\frac{-4}{3}$ | $\frac{-6}{3}$ | $\mathcal{W}(2,6,8,10,12)$ |
| $\frac{-8}{5}$ | $\frac{-12}{5}$ | $\mathcal{W}(2,4,6,8,9,10,12)$ |
| $\frac{-12}{7}$ | $\frac{-18}{7}$ | $\mathcal{W}(2,4,6,6,8,9,10)$ |

Table 1: Truncations of the $sl(2,\mathbb{R})_\kappa / sl(2,\mathbb{R})$-coset algebra

The information beyond scale dimension 9 for positive integer $\kappa$ in table 1 is taken from the character arguments presented in [43] (see also [1]). For the remaining cases information about dimensions higher than 9 is obtained from character arguments which will be discussed at the end of this section. The algebras of type $\mathcal{W}(2,4,6)$ appearing in table 1 are not identical; the case $\kappa = -\frac{1}{2}$ was discussed in detail in [9] and corresponds to the solution which was unexplained for some time [5, 16], the case $\kappa = 2$ is the bosonic projection of the $N=1$ Super Virasoro algebra.

It is now straightforward to obtain a primary set of generators by orthogonalization of (2.1.42) and (2.1.44) with respect to normal ordered products $N$. Up to scale dimension 6 one can choose for example the following primary generators in addition to $L$:

$$
\Phi^{(4)} = 24(\kappa + 2)^2N(L,L) - (37\kappa + 44)W^{(4)},
$$

$$
\Phi^{(6a)} = 1920(\kappa + 2)^3N(N(L,L),L) - 40(209\kappa + 328)(\kappa + 2)N(W^{(4)},L) + 18(31\kappa + 24)(9\kappa + 16)W^{(6)} + 45(9\kappa + 16)(\kappa + 8)\hat{S}_{0,1,2},
$$

$$
\Phi^{(6b)} = 3840(675\kappa^3 + 75\kappa^2 - 1403\kappa + 4328)(\kappa + 2)^3N(N(L,L),L) + M(\kappa + 2)^2N(L,\partial^2L) - 360(89\kappa + 136)(15\kappa - 8)(5\kappa + 13)(5\kappa - 2)(\kappa + 2)N(W^{(4)},L) + 5(3645\kappa^3 + 19947\kappa^2 - 200\kappa - 40192)(89\kappa + 136)(5\kappa - 2)W^{(6)} - 30(645\kappa^2 - 928\kappa - 3392)(89\kappa + 136)(5\kappa - 2)\hat{S}_{0,1,2},
$$

(2.1.45)
where we have used the abbreviation
\[ M := 36(18225\kappa^4 + 485475\kappa^3 + 637424\kappa^2 - 1738048\kappa - 2584576). \]
The two point functions (or central terms) read
\[ d_{4,4} = 120(37\kappa + 44)(3\kappa + 4)(\kappa - 1)^2, \quad d_{6a,6a} = 8M(9\kappa + 16)(\kappa - 1)^2, \quad d_{6b,6b} = 2520M(89\kappa + 136)(5\kappa + 8)(5\kappa - 2)(3\kappa + 4)(2\kappa + 1)(\kappa - 1)^2. \] (2.1.46)

Note that we have chosen \( d_{6a,6b} = d_{6b,6a} = 0 \). The normalization constants (2.1.46) indeed vanish for the corresponding values of \( \kappa \) of table 1. The additional zeroes belong to Virasoro-minimal values of \( c \) (where a correction term becomes a null field and thus the primary projection fails).

Finally, we present the first nontrivial structure constant of this coset algebra in standard normalization \( (\hat{d}_{4,4} = \frac{1}{4}) \):
\[ (C_{4,4}^4)^2 = \frac{2(99\kappa^2 - 25\kappa - 236)^2}{5(37\kappa + 44)(3\kappa + 4)(\kappa + 2)(\kappa - 1)} = \frac{(35c^2 + 211c - 354)^2}{15(5c + 22)(c + 6)(c - 1)}. \] (2.1.47)

Further structure constants would strongly depend on the choice of basis and therefore we omit them.

Next, we consider two commuting copies of the Kac-Moody algebra \( J^{(i,\pm,0)}(i = 1, 2) \) based on \( sl(2,\mathbb{R}) \). The \( J^{(i,\pm,0)} \) each satisfy eq. (2.1.39) independently, the mixed commutators being zero. We denote the first level by \( \kappa \) and the second level by \( \mu \). All the currents are primary spin 1 fields with respect to some energy momentum tensor. We are however not going to use any information about the nature of this energy momentum tensor or its central charge.

We want to construct the invariant subspace under the action of the diagonally embedded \( sl(2,\mathbb{R}) \). This means that we have to construct all fields commuting with
\[ J^{(\pm)} := J^{(1,\pm)} + J^{(2,\pm)}, \quad J^{(0)} := J^{(1,0)} + J^{(2,0)}. \] (2.1.48)

This task is simplified by the observation that this \( sl(2,\mathbb{R}) \) is generated by the horizontal \( sl(2,\mathbb{R}) \) subalgebra of (2.1.48) and all modes of the current \( J^{(0)} \). So the commutant consists of those \( sl(2,\mathbb{R}) \)-invariant polynomials which also commute with \( J^{(0)} \). The \( sl(2,\mathbb{R}) \)-invariant fields are generated by (2.1.41) where we have now the freedom to insert the currents \( J^{(1)} \) and \( J^{(2)} \) for any of the arguments. The second order invariants coming from (2.1.41a) are
\[ S_{m,n}(r,s) := \partial^n J^{(s,-)} \partial^m J^{(r,+)} + \frac{1}{2} \partial^n J^{(s,0)} \partial^m J^{(r,0)} + \partial^n J^{(s,+)} \partial^m J^{(r,-)}. \] (2.1.49)

The third order invariants \( S_{m,n,k}(r,s,t) \) are obtained from (2.1.41b) in the same manner.

For explicit calculations we use again the Vertex-operator approach of section 2.1.1. First, condition (2.1.11) is used to find the invariant fields. Then, the primary generators can be computed from (2.1.12).
The first invariant we find with the ansatz (2.1.49) and the condition (2.1.11) is the coset energy momentum tensor

\[
L = \frac{1}{\kappa + \mu + 2} \left( \frac{\mu}{2(\kappa + 2)} S_{0,0}(1,1) - S_{0,0}(1,2) + \frac{\kappa}{2(\mu + 2)} S_{0,0}(2,2) \right). \tag{2.1.50}
\]

The central charge \(c\) of the coset theory is found immediately to be

\[
c = \frac{3\mu}{\mu + 2} \left( 1 - \frac{\mu + 2}{(\kappa + \mu + 2)(\kappa + 2)} \right). \tag{2.1.51}
\]

Note that for the diagonal coset the original currents are not primary with respect to the coset energy momentum tensor (2.1.50). This is the reason why it is not advantageous to use quasi-primary normal ordered products. The next primary generator can be constructed at scale dimension 4 using eqs. (2.1.49), (2.1.11), (2.1.12). The explicit form can be found in appendix C. From this realization one obtains the central term (or normalization constant) which is omitted here. A tedious calculation yields the self-coupling constant of the dimension 4 generator. In standard normalization it reads

\[
(C_{14}^4)^2 = 18 \left( 99(\kappa^2 \mu^4 + \kappa^4 \mu^2) - 25(\kappa \mu^4 + \kappa^4 \mu) - 236(\mu^4 + \kappa^4) + 198 \kappa^3 \mu^3 + 742(\kappa^2 \mu^3 + \kappa^3 \mu^2) - 672(\kappa \mu^3 + \kappa^3 \mu) - 1888(\mu^3 + \kappa^3) + 576 \kappa^2 \mu^2 - 4088(\kappa \mu^2 + \kappa^2 \mu) - 5168(\mu^2 + \kappa^2) - 8592 \kappa \mu - 5568(\kappa + \mu - 1792)^2 (5(\kappa - 1)(\mu - 1)(2 + \kappa)(2 + \mu)(4 + 3\kappa)(4 + 3\mu)(2 + \kappa + \mu) (5 + \kappa + \mu)(8 + 3(\kappa + \mu))(176(1 + \kappa + \mu) + 44(\kappa^2 + \mu^2) + 192 \kappa \mu + 37(\kappa^2 \mu + \kappa \mu^2)) \right)^{-1}. \tag{2.1.52}
\]

A few remarks about the structure constant eq. (2.1.52) are in place. It is symmetric in the levels \(\kappa\) and \(\mu\) reflecting the symmetry of the construction. Furthermore, eq. (2.1.47) is recovered from eq. (2.1.52) in the limit \(\mu \to \infty\), as it should be. Also the singularities for one of the levels \(\kappa\) or \(\mu\) equal to 1 or \(-\frac{4}{3}\) are expected because in these cases the field \(\Phi^{(4)}\) should be a null field. For \(\mu = 2\) one recovers the structure constants of the bosonic projection of the \(N = 1\) Super Virasoro algebra presented in [5, 25] (see also appendix E). For \(\mu = -\frac{1}{2}\) the structure constant of the \(W(2,4,6)\) in [5, 16] is reproduced. This last identity confirms the identification in [9] of the last unexplained solution \(W(2,4,6)\) with the above \(sl(2, \mathbb{R})\) coset at \(\mu = -\frac{1}{2}\).

One should mention that one of the levels can be replaced by the central charge \(c\). The other level still occurs as a parameter in the structure constants. Thus, the resulting \(W\)-algebra can be regarded as a 1-parameter family of algebras of type \(W(2,4,6,8,9,10,12)\) for generic \(c\).

### Representation theory of \(sl(2, \mathbb{R})_\kappa \oplus \overset{\mu}{\otimes} / sl(2, \mathbb{R})_{\kappa - \frac{1}{2}}\)

We conclude this section with a discussion of the representation theory of the coset \(sl(2, \mathbb{R})_\kappa \oplus \overset{\mu}{\otimes} / sl(2, \mathbb{R})_{\kappa - \frac{1}{2}}\). For coset algebras of affine Kac-Moody algebras one has a natural approach to representation theory, i.e. to the set of irreducible highest weight
modules (see e.g. [1, 44]). One obtains a highest weight module $L_{\Lambda'}^{g'/g'}$ of the coset algebra $g'/g'$ by the decomposition of a highest weight module $L_{\Lambda}^{g}$ of $\hat{g}$ under the $g'$ action $L_{\Lambda}^{g} = \bigoplus_{\Lambda'} L_{\Lambda,g'}^{\Lambda} \otimes L_{\Lambda'}^{g'}$ where $\Lambda'$ runs over the weights of $g'$ ($k' = jk$, where $j$ is the Dynkin index of the embedding $g' \hookrightarrow g$). The corresponding formula for the characters is given by

$$\chi_{L_{\Lambda}}(q) = \sum_{\Lambda'} b_{\Lambda'}^{\Lambda}(q) \chi_{L_{\Lambda'}}(q)$$  \hspace{1cm} (2.1.53a)

with the so-called branching functions $b_{\Lambda'}^{\Lambda}(q)$. Similarly, for the cosets $\hat{g}/g'$ one obtains a decomposition

$$\chi_{L_{\Lambda}}(q) = \sum_{j} b_{j}^{\Lambda}(q) \chi_{j}$$  \hspace{1cm} (2.1.53b)

where the $\chi_{j}$ are now characters of irreducible representations of the Lie algebra $g'$.

Let us focus on the diagonal cosets $\hat{g}_{k_{1}} \oplus \hat{g}_{k_{2}} / \hat{g}_{k_{1}+k_{2}}$ where the modules $L_{\Lambda,\Lambda'}$ are believed to be irreducible. In this case, the branching functions are equal to the characters of the coset up to some prefactor. The representations of $\hat{g}_{k_{1}} \oplus \hat{g}_{k_{2}}$ are now labeled by two weights $\Lambda_{1}$ and $\Lambda_{2}$ instead of a single weight $\Lambda$. One has the following formula for the branching functions for integrable weights $\Lambda_{i}$, $i = 1, 2$ at level $k_{i}$ (see e.g. [33]):

$$b_{\Lambda_{1},\Lambda_{2}}^{\Lambda_{3}} = \sum_{w \in \tilde{W}} \epsilon(w)c_{\Lambda_{3}-w\ast\Lambda_{2}}^{\Lambda_{1}}(q)q^{\frac{|w(\Lambda_{2}+\rho)(k_{1}+k_{2}+h')-(\Lambda_{1}+\rho)(k_{1}+k_{2}+h')|}{2k_{1}(k_{2}+h')(k_{1}+k_{2})}}$$  \hspace{1cm} (2.1.54)

where $\Lambda_{3}$ is an integrable weight at level $k_{1} + k_{2}$ such that $\Lambda_{1} + \Lambda_{2} - \Lambda_{3}$ belongs to the long root lattice of $\hat{g}$, $\tilde{W}$ denotes the Weyl group of $\hat{g}$ and $w \ast \cdot$ denotes the shifted action of the Weyl group element $w$. The $c_{\Lambda}^{\Lambda}(q)$ are the Kac-Peterson string functions [45] which are defined via the identity $b_{\Lambda}^{\Lambda}(q) = \eta(q)^{l}c_{\Lambda}^{\Lambda}(q)$ where $b_{\Lambda}^{\Lambda}(q)$ are the branching functions of the coset $\hat{g}/h$ with $h$ the Cartan subalgebra of $g$, $l$ the rank of $h$ and $\eta(q)$ is Dedekind’s eta-function.

The limit $k_{2} \rightarrow \infty$ of the coset $\hat{g}_{k} \oplus \hat{g}_{k_{2}} / \hat{g}_{k+k_{2}}$ is the coset $\hat{g}_{k}/g$. The vacuum character for $\hat{g}_{k}/g$ is obtained from the limit $k_{2} \rightarrow \infty$ of (2.1.54) with $\Lambda_{1} = k\Lambda_{0}$, $\Lambda_{2} = k_{2}\Lambda_{0}$, $\Lambda_{3} = (k+k_{2})\Lambda_{0}$ ($\Lambda_{0}$ is the first fundamental weight of $\hat{g}$):

$$b_{0}^{k\Lambda_{0}}(q) = \sum_{w \in W} \epsilon(w)c_{w\ast\rho+k\Lambda_{0}}^{k\Lambda_{0}}(q)q^{\frac{1}{2k_{1}}|w\ast\rho|^{2}}$$  \hspace{1cm} (2.1.55)

where $W$ is the Weyl group of $g$. With the explicit form of the Kac-Peterson string functions [45] it is relatively easy to calculate the vacuum characters for $g = sl(2, \mathbb{R})$ and the integer levels specified in table 1 above yielding the spins of the generating fields of the coset.

However, we are interested in the generalization of eq. (2.1.54) to the case where one of the levels is fractional so that the so-called admissible representations of the Kac-Moody algebra carrying a representation of the modular group enter the game [46]. In particular, we would like to calculate the branching functions of the coset $sl(2, \mathbb{R})_{k} \oplus$
\(
\text{sl}(2, \mathbb{R})_{\frac{1}{2}}/\text{sl}(2, \mathbb{R})_{k_{\frac{1}{2}}}
\). If the level \(k = k_1\) is an integer we are allowed to calculate the branching functions of the coset involving admissible representations:

\[
\chi_L(q) = \chi_{\Lambda_1}(q)\chi_{\Lambda_2}(q) = \sum_{\Lambda_3} b^{\Lambda_1,\Lambda_2}_{\Lambda_3}(q)\chi_{\Lambda_3}(q),
\]

(2.1.56)

where the sum runs over all admissible representations (we denoted the levels of \(\Lambda_i\) explicitly). Eq. (2.1.54) may still applied to one fractional and one integer level if one modifies the range of summation such that the powers of \(q\) remain integer spaced. Using the parametrization of [46] for the admissible representations \(k_2 = -\frac{1}{2}, p = 2k + 4, q = p - 1, l = \sqrt{2}\Lambda_1\) and \(j = \sqrt{2}\Lambda_2 + 1, j' = \sqrt{2}\Lambda_3 + 1\) one obtains from eq. (2.1.54):

\[
b^j_{j'}(q) = q^{-\frac{m}{2}} \sum_{m \in \mathbb{Z}} \left( c^j_{j'-mp}(q)q^{-(mpq+jq-j'p)^2-(p-q)^2} - c^j_{j'+mp}(q)q^{-(mpq+jq-j'p)^2-(p-q)^2} \right)
\]

(2.1.57)

where \(0 < j < \frac{p}{2}, 0 < j' < q, j - j' \leq 2\mathbb{Z}, l \in \{0,1\}\) and \(c = -1 + \frac{12(p-q)^2}{pq}\). By definition, the string functions satisfy

\[
c^j_m(q) = \frac{B^j_m(q)}{\eta(q)}
\]

(2.1.58)

where the \(B^j_m(q)\) are the branching functions of the coset \(\text{sl}(2, \mathbb{R})_{\frac{1}{2}}/U(1)\). One can read off these branching functions from (2.1.7a) observing that the modules of \(\text{sl}(2, \mathbb{R})_{\frac{1}{2}}\) are freely generated in terms of a \(\beta - \gamma\) system (see [9]). Substituting \(z^2 \rightarrow z^2q^{\frac{1}{4}}\) in (2.1.7a) leads to the character of \(\text{sl}(2, \mathbb{R})_{\frac{1}{2}}/U(1)\). Multiplication with the correct prefactor gives the result

\[
B^j_m = q^{\frac{m}{2}} q^{\frac{m(m+1)}{2}} \phi_m(q) \prod_{n \geq 1} (1 - q^n)
\]

(2.1.59)

where \(\phi_m(q)\) is defined in eq. (2.1.7b). Note that the following symmetry holds

\[
b^0_{j'} = b^1_{q-j'}.
\]

(2.1.60)

Thus we get the following Kac-table for the conformal dimensions of the branching functions:

\[
h(j, j') = \frac{|j - j'|(|j - j'| + 1)}{2} - \frac{(jq - j'p)^2 - (p-q)^2}{2pq}
\]

for \(j' < j < \frac{p}{2}\)

\[
h(j, j') = h(j, q - j')
\]

for \(j \leq j' < q = p - 1, j' < \frac{p}{2}\)

\[
|j - j'| \equiv 0 \pmod{2}.
\]

(2.1.61)

The second line of (2.1.61) does not reflect a symmetry but means that the conformal dimension for \(j' \geq j\) is obtained from the first line evaluated at \(h(j, q - j')\).

For the rational models of \(\mathcal{W}(2, 4, 6)\) calculated in [16] we get perfect agreement with the set of highest weights given by eq. (2.1.61). Taking the limit \(p \rightarrow \infty\) one obtains

\[
b^1_{01}(q) = c^0_0(q) - qc^0_2(q)
\]

from which one can read off the generators of the underlying
\(\mathcal{W}\)-algebra yielding a \(\mathcal{W}(2, 4, 6)\). For general fractional level the vacuum character can be computed using the results of [47] for string functions \(^6\).

Eq. (2.1.61) agrees with the conjecture of [8] for the minimal series of this \(\mathcal{W}(2, 4, 6)\) which was obtained by formal extrapolation to \(\mathcal{WD}_{-1}\). We will continue this formal extrapolation to the algebras \(\mathcal{WD}_{-m}\) in section 3.2.4.

Using formula (D.2) of appendix D it is easy to show that the conformal dimensions for the \(\mathcal{WC}_m\) minimal models with central charge \(c_{\mathcal{C}_m}(m + 2, 2m + 3)\) agree exactly with (2.1.61). This supports that \(\mathcal{WD}_{-1} \cong \mathcal{W}(2, 4, 6)\) is the unifying algebra for the minimal models of the Casimir algebras \(\mathcal{WC}_m\) at \(c_{\mathcal{C}_m}(m + 2, 2m + 3)\) as indicated in [17].

**2.2. Orbifolds of quantum \(\mathcal{W}\)-algebras**

In the previous section we discussed cosets. Coset constructions can be viewed as projections onto the subspace invariant under an *inner* symmetry (realized by a subalgebra) of a \(\mathcal{W}\)-algebra. Orbifolds are projections onto subspaces invariant under *outer* automorphisms (leaving the algebraic structure invariant) and behave therefore similarly to coset constructions. It was shown in [9] that these two constructions lead in general to nonfreely generated \(\mathcal{W}\)-algebras. We will further comment on the similar behaviour of these two constructions in section 3.1.

There are two further questions, both turning up in the general context of classification of \(\mathcal{W}\)-symmetries, which motivate the study of orbifold constructions of \(\mathcal{W}\)-algebras. The first motivation is that recently projections of the generators onto invariant subspaces were calculated for the classical Casimir algebras \(\mathcal{WA}_{n-1}\) in [24] and were shown to give rise to other Casimir algebras. In all known cases of quantum Casimir \(\mathcal{W}\)-algebras one can read off from the structure constants that such an identity is not true on the quantum level. In this section we will show that indeed such an identity is never true for the quantum Casimir algebras \(\mathcal{WA}_{n-1}\).

The second motivation is the observation that instead of considering symmetry algebras containing fermions – in particular Super \(\mathcal{W}\)-algebras – it might be simpler to use their bosonic projections [50]. In particular, a classification program might turn out to be simpler for the bosonic case. From this point of view it is certainly important to study orbifolds of \(\mathcal{W}\)-algebras.

Orbifolds also turn up in various applications of CFT. For example they occur as the chiral algebras of the GSO projected models used in superstring theory [51]. Orbifolds are also useful for applications to statistical mechanics. Spin models on the cylinder, torus or other higher genus surfaces are difficult to realize in experiments. From this point of view the most natural boundary conditions for spin systems are *free* boundary conditions. At conformally invariant second order phase transitions of such two dimensional statistical

---

\(^6\) Note that the example in table 3 of [47] arising from a coset with two fractional levels \(k_1, k_2\) cannot be a RCFT in contrast to the claim of [47]. Firstly, it is known that there is no RCFT with positive conformal dimensions and \(c = \frac{1}{5}\) (see e.g. [48]). Secondly, using the ideas of [49] one can check that there is no representation of the modular group inducing a ‘good’ fusion algebra with conformal dimensions as in [47].

23
systems the spectrum generating algebra should be the orbifold of the underlying $W$-algebra. This expectation comes from the observation in [23] that generally boundary conditions of $\mathbb{Z}_n$ quantum spin systems are in one to one correspondence with boundary conditions of their spectrum generating algebras $WA_{n-1}$.

In this section we will take an algebraic approach to orbifolds which is different from the usual orbifolds that deal with the complete CFT (see e.g. [22]). If a $W$-algebra has a nontrivial outer automorphism group one can consider the projection onto the invariant subspace. We will denote this projection by ‘orbifold’. Available results on outer automorphisms of $W$-algebras have been collected in [23]. In all known cases the group of outer automorphisms is discrete, in many cases the automorphism group is just $\mathbb{Z}_2$. For simplicity, we will restrict to the case of $\mathbb{Z}_2$ automorphisms. Note that this covers in particular the bosonic projections of fermionic $W$-algebras. Here the automorphism maps any fermion $\psi$ to $-\psi$ and the invariant subspace is precisely given by the space of the bosonic fields.

2.2.1. General remarks and results

By definition, the energy momentum tensor $L$ of any $W$-algebra is invariant under all outer automorphisms. Therefore, orbifold constructions never change the energy momentum tensor $L$ (in contrast to coset constructions). This observation will be exploited using a quasi-primary basis from the very beginning which simplifies the transition to primary generators. For $\mathbb{Z}_2$ automorphisms a minimal generating set of the classical orbifolds was presented in [9]. The normal ordered versions of these generators are also a (redundant) generating set for the quantum orbifolds. From this we obtain in the case of $\mathbb{Z}_2$ orbifolds that the nonzero quasi-primary normal ordered products

$$\mathcal{N}(\phi^{(1)}, \partial^n \phi^{(2)})$$

(2.2.1)

of any two generators $\phi^{(j)}$ transforming under the automorphism as $\phi^{(j)} \mapsto -\phi^{(j)}$ constitute a generating set for the orbifold together with the invariant generators. One can further restrict to those normal ordered products eq. (2.2.1) where the fields $\phi^{(j)}$ appear in a certain order.

The field content of an orbifold can be easily predicted using a character argument which also enables one (at least in principle) to determine the representations of the orbifold. Let $V$ be an irreducible highest weight module of the underlying $W$-algebra. Then we define a character $\chi(q, z)$ which also encodes the $\mathbb{Z}_2$ automorphism by

$$\chi(q, z) := \text{tr}_V \left( q^{(L_0 - \frac{c}{24})} z^P \right)$$

(2.2.2)

where $P$ denotes the parity of a state in $V$ with respect to the automorphism. The parity of invariant states is zero ($P = 0$). The states transforming with a sign under the automorphism group are defined to have parity $P = 1$. The subspace of $V$ invariant under the automorphism as well as the subspace of states transforming with a sign both provide irreducible representations of the orbifold $W$-algebra. From this we conclude that the decomposition

$$\chi(q, z) = \chi^{(0)}(q) + z \chi^{(1)}(q)$$

(2.2.3)
yields the two characters $\chi^{(i)}(q)$ for the two representations of the orbifold obtained from $V$. In particular, the constant part $\chi^{(0)}_0(q)$ in $z$ of the vacuum character $\chi_0(q, z)$ of the underlying $\mathcal{W}$-algebra is the vacuum character of the orbifold. It is straightforward to compute this character for a $\mathcal{W}$-algebra without null fields (which applies to Drinfeld-Sokolov $\mathcal{W}$-algebras at generic $c$) up to a finite order. The field content of the orbifold can now be read off by determining the minimal set of fields whose free vacuum module is at least as large.

If $\chi_i(q, z)$ are the characters of a rational model of the underlying $\mathcal{W}$-algebra, the characters obtained from (2.2.3) are the characters of the associated rational model of the orbifold. Note, however, that for this statement to be valid one has to take the representations of all sectors of the original algebra into account, in particular for bosonic algebras also the twisted sector [23]. Then the statement about rational models follows from the identity (3.6) in [23] for the partition functions.

Note that instead of determining the field content of the orbifold using this character argument it can also be computed by applying a basis algorithm to those simple fields that will drop out. The invariant normal ordered products must be contained in the orbifold. Either they can be considered as composite or must be added to the generating set. This approach is slightly more involved but has the small advantage that one can trace invariant fields and up to the cancellation of correction terms in relations and generators (see [9]) ensure closure of the orbifold $\mathcal{W}$-algebra.

The primary generators in the orbifold can be efficiently determined using the definition of simple fields (see section 1.1.). Instead of taking the quasi-primary normal ordered products (2.2.1) we orthogonalize them with respect to all quasi-primary normal ordered products in the orbifold because simple fields are orthogonal to derivatives and quasi-primary normal ordered products.

Finally, the structure constants are determined by a standard procedure: Two- and three-point functions of the simple fields are evaluated in the vacuum representation and the coupling constants arise as solutions of a linear system of equations (see section 1.1.).

Below we will use the following notations: $W^{(\delta)}$ denotes a simple field of dimension $\delta$ in the original algebra, $\Phi^{(n)}$ denotes a simple field of dimension $n$ in the orbifold. $d_{n,n}$ is the central term in the commutator of $\Phi^{(n)}$ with itself which equals its norm squared. The structure constants in the original algebra are denoted by $C^{(\mu)}_{W^{(\nu)} W^{(\lambda)}}$ whereas for the orbifold we just write the dimensions, i.e. $C^{m}_{k l}$. Note that the structure constants for the orbifold are given in the standard normalization $\hat{d}_{n,n} = \frac{2}{n} \hat{d}$ and not in the induced one given by $d_{n,n}$.

# Throwing out one field

Before presenting a collection of results for $\mathbb{Z}_2$ orbifolds we will first consider the simplest case where we project out a single field $W^{(\delta)}$ using an automorphism $W^{(\delta)} \mapsto -W^{(\delta)}$. In order to determine the field content of the orbifold let us look at all invariant fields that can be built out of $W^{(\delta)}$ alone. This argumentation will rely on the absence of null fields, i.e. it will apply to generic $c$ only.

First, we discuss the bosonic case where $\delta$ is an integer $\delta = n$. According to eq. (2.2.1) all fields $\mathcal{N}(W^{(n)}, \partial^{2k}W^{(n)})$ ($k \geq 0$) are invariant under the automorphism. Their primary projections $\Phi^{(2(n+k))}$ cannot be composite for $k < n$. At dimension $4n$ we have
invariant fields of the form \( \mathcal{N}(W^{(n)}, \partial^{2n}W^{(n)}) \) and \( \mathcal{N}(\mathcal{N}(W^{(n)}, W^{(n)}), W^{(n)}) \). These two fields are equivalent to \( \Phi^{(4n)} \) and \( \mathcal{N}(\Phi^{(2n)}, \Phi^{(2n)}) \). Clearly, there is precisely one additional generator \( \Phi^{(4n)} \) at scale dimension 4. At dimension 4 + 2 the invariant fields are \( \mathcal{N}(W^{(n)}, \partial^{2n+2}W^{(n)}) \) and \( \mathcal{N}(\mathcal{N}(W^{(n)}, W^{(n)}), \partial^2W^{(n)}) \). Assuming the cancellation procedure described in [9] to be valid in general, this space is also spanned by \( \mathcal{N}(\Phi^{(2n)}, \partial^2\Phi^{(2n)}) \) and \( \mathcal{N}(\Phi^{(2n+2)}, \Phi^{(2n)}) \). Finally, at dimension 4 + 4 there are two fourth order and one second order invariant fields (three altogether). At this dimension we have in the orbifold \( \mathcal{N}(\Phi^{(2n)}, \partial^4\Phi^{(2n)}), \mathcal{N}(\Phi^{(2n+2)}, \partial^2\Phi^{(2n)}), \mathcal{N}(\Phi^{(2n+2)}, \Phi^{(2n+2)}) \) and \( \mathcal{N}(\Phi^{(2n+4)}, \Phi^{(2n)}) \). Thus, there must be a generic null field at dimension 4 + 4. In summary, we have generators \( \Phi^{(2k)}, n \leq k \leq 2n \) in the orbifold and a first generic null field at dimension 4 + 4.

The same procedure can be applied to a fermionic field, i.e. \( \delta \) half-integer. Again, we obtain generators of the orbifold \( \Phi^{(2\delta+2k+1)} \) from the primary projection of \( \mathcal{N}(W^{(\delta)}, \partial^{2k+1}W^{(\delta)}) \) for \( 0 \leq k \leq \delta - \frac{1}{2} \) and the first generic null field at dimension \( 4\delta + 4 \).

Note that the Casimir algebras \( \mathcal{W}_D \) and \( \mathcal{W}_B(0,n) \) have this property. \( \mathcal{W}_D \) (\( n > 4 \)) contains precisely one simple bosonic field which can be projected out [23]. This field has dimension \( n \). \( \mathcal{W}_B(0,n) \) has precisely one simple fermionic field. The bosonic subalgebra can therefore be obtained by projecting out the fermion with dimension \( n + \frac{1}{2} \). It was noted in [1] that this bosonic subspace can be realized in terms of the diagonal coset \( (\mathcal{B}_n)_k \oplus (\mathcal{B}_n)_1/(\mathcal{B}_n)_{k+1} \). Eq. (7.19) of [1] is a closed formula for the vacuum character of this coset.

### Results

From this result and the character argument explained above one can predict the generating set for many orbifolds. Table 2 contains a collection of results 7. Strictly speaking, one would have to check in each case separately that the cancellation mechanism of [9] indeed takes place. To determine the orbifold of the cosets \( \text{sl}(2, \mathbb{R})_\kappa \oplus \text{sl}(2, \mathbb{R})_\mu / \text{sl}(2, \mathbb{R})_{\kappa+\mu} \) and \( \text{sl}(2, \mathbb{R})_\kappa / \mathcal{U}(1) \) one has to use the explicit knowledge of the invariants (and relations) generating the coset because of the presence of generic null fields. In the case of \( \text{sl}(2, \mathbb{R})_\kappa / \mathcal{U}(1) \) the outer automorphism of the coset comes from the inner automorphism of the \( \text{sl}(2, \mathbb{R}) \) Kac-Moody algebra (2.1.1) that maps \( J^+ \leftrightarrow J^- \), \( J^o \leftrightarrow -J^o \). This automorphism leaves the even dimensional generators of the coset invariant and changes the sign of the odd dimensional ones.

The automorphism of the other coset \( \text{sl}(2, \mathbb{R})_\kappa \oplus \text{sl}(2, \mathbb{R})_\mu / \text{sl}(2, \mathbb{R})_{\kappa+\mu} \) is induced by the map \( J^{(1 \cdot \cdot)} \leftrightarrow J^{(2 \cdot \cdot)} \). The effect is that for \( \kappa = \mu \) the quadratic fields (2.1.41a) are left invariant and the third order invariants (2.1.41b) change their sign. With this information the field content of the orbifold can easily be inferred from [9] by dropping the third order invariants from the generating set and the relations.

7) For the field content of the orbifolds of \( \mathcal{W}_A_2 \) and \( \mathcal{W}_A_3 \) see also [1].
| algebra       | projection       | dimension of first generic null field |
|--------------|------------------|---------------------------------------|
| $W(2, 1)$    | $W(2, 2, 4)$     | 8                                     |
| $W(2, \frac{3}{2})$ | $W(2, 4, 6)$     | 10                                    |
| $W(2, 2)$    | $W(2, 4, 6, 8)$  | 12                                    |
| $WA_2 \cong W(2, 3)$ | $W(2, 6, 8, 10, 12)$ | 16                                    |
| $WA_3 \cong W(2, 3, 4)$ | $W(2, 4, 6, 8, 10, 12)$ | 16                                    |
| $WA_4 \cong W(2, 3, 4, 5)$ | $W(2, 4, 6, 8, 10)$  | 14                                    |
| $WA_5 \cong W(2, 3, 4, 5, 6)$ | $W(2, 4, 6, 8, 10, 12)$ | 16                                    |
| $sl(2, \mathbb{R})_k/\tilde{U}(1) \cong W(2, 3, 4, 5)$ | $W(2, 4, 6, 8, 10)$  | 14                                    |
| $sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_k/\tilde{sl}(2, \mathbb{R})_{2k}$ | $W(2, 4, 6, 8, 10, 12, 14, 16, 18)$ | 22                                    |
| $WD_n \cong W(2, 4, \ldots, 2n-2, n)$ | $W(2, 4, \ldots, 4n)$  | $4n + 4$                              |
| $WB(0,n) \cong W(2,4,\ldots,2n,n+\frac{1}{2})$ | $W(2, 4, \ldots, 4n + 2)$ | $4n + 6$                              |
| $SW(\frac{3}{2}, 2)$ | $W(2^2, 4^2, 5, 6^2, 7)$  | 9                                     |

Table 2: Field content of some orbifolds of $W$-algebras

In the simplest cases one can also determine some primary generators of the orbifold and calculate the corresponding structure constants. We briefly summarize results for $W(2, 1)$ and $W(2, 2)$. The more interesting cases of the $W(2, 3)$ and $WA_{n-1}$ will be treated in later subsections and the orbifold of the $N = 1$ Super Virasoro algebra $W(2, \frac{3}{2})$ can be found in appendix E. Below, we present structure constants connecting additional simple fields.

**$W(2, 1)$:** The (Lie) algebra $W(2, 1)$ is the extension of the Virasoro algebra $L$ by a primary $U(1)$ current $J$. The map $J \mapsto -J$ is the unique nontrivial automorphism of the algebra, providing us with one of the simplest examples of an orbifold construction. The orbifold contains two additional primary fields of dimensions 2 and 4. Both of them are null fields at $c = 1$. At $c = -\frac{17}{5}$ the dimension 4 generator vanishes. The vanishing of the additional dimension 2 generator at $c = 1$ is to be expected because the Sugawara energy momentum tensor of the current is the unique energy momentum tensor for $c = 1$, i.e. $L = \frac{1}{2}N(J,J)$ at $c = 1$. The structure constants connecting these two fields can be determined as:

\[
\begin{align*}
(C_{22}^2)^2 &= 4(c-2)^2(c-1)^{-1} \\
(C_{22}^4)^2 &= 24(5c+17)c^3((25c^2+180c+383)(c-1))^{-1} \\
C_{22}^2C_{44}^2 &= 4(25c^3+95c^2-61c-383)(c-2)(25c^2+180c+383)(c-1))^{-1} \\
C_{44}^4C_{22}^2 &= 36(375c^4+2400c^3+2090c^2-9864c-11801)c^2((25c^2+180c+383)^2(c-1))^{-1}.
\end{align*}
\]

**$W(2, 2)$:** The algebra $W(2, 2)$ admits a nontrivial outer automorphism iff the self-coupling constant vanishes. In this case, it can be realized in terms of two commuting copies of the Virasoro algebra $(L_1$ and $L_2$) with equal central charge. $W := L_1 - L_2$ is primary with respect to $L := L_1 + L_2$. Furthermore, the map $L \mapsto -L$, $W \mapsto -W$ is an automorphism of this algebra. The generating set of the orbifold was discussed in [9]. It was also verified...
in [9] that there is indeed no dimension 10 generator in the orbifold which supports the character argument predicting a \( \mathcal{W}(2, 4, 6, 8) \).

We have further determined a basis of primary fields and calculated the structure constants. Omitting those involving the complicated dimension 8 generator one obtains the following list:

\[
(C^4_{44})^2 = 2(5c^2 + 66c - 176)^2((5c + 44)(5c + 22)c)^{-1}
\]
\[
(C^4_{66})^2 = 8(7c + 136)(5c + 22)^2(c + 4)^2(c - 1)(3(7c + 68)(5c + 44)(2c - 1)(c + 24)c)^{-1}
\]
\[
C^4_{44}C^4_{66} = 4(5c^2 + 66c - 176)(7c + 68)(5c + 88)(2c - 1)(9(5c + 44)(5c + 22)(c + 24)c)^{-1}
\]
\[
C^6_{44}C^6_{66} = 20(1106c^5 + 50845c^4 + 705182c^3 + 2270104c^2 - 5361664c - 1192448)\times
(5c + 22)(c + 4)(27(7c + 68)(5c + 44)(2c - 1)(c + 24)c)^{-1}.
\]

It is interesting to notice that for \( c \in \{-4, 1, -\frac{136}{7}\} \) the coupling constant \( C^6_{44} \) vanishes whereas \( \Phi^{(4)} \) is not a null field (for \( c \in \{1, -\frac{136}{7}\} \) the field \( \Phi^{(6)} \) is a null field). Thus, for these values of the central charge the orbifold of \( \mathcal{W}(2, 2) \) reduces to \( \mathcal{W}(2, 4) \) or has at least a \( \mathcal{W}(2, 4) \) subalgebra.

\section*{2.2.2. The orbifold of \( \mathcal{W}(2, 3) \)}

Zamolodchikov’s \( \mathcal{W}(2, 3) \) [3] is not only one of the first \( \mathcal{W} \)-algebras which appeared in the literature but also one of the most frequently used ones. Therefore we also discuss it in detail here. The computations which will be reported below were carried out with the \( \mathcal{W}(2, 3) \) as it was presented e.g. in [23]. The field content of the orbifold for generic \( c \) can be found e.g. in [1].

First, we have computed determinants of the invariant quasi-primary fields up to scale dimension 13. The zeroes of the determinants tell us where null fields occur. For these values of the central charge we have further calculated the dimension of the space of nonnull invariant fields. Define a counting function

\[
\Pi(q) := \sum_{n=1}^{\infty} q^n (\# \text{ quasi-primary fields with dimension } n). \tag{2.2.6}
\]

\( \Pi(q) \) and the vacuum character are related by \( q^{\frac{c}{24}} \chi_0(q) = (1 - q)^{-1} \Pi(q) + 1 \). These counting functions are presented up to order 13 for generic value of \( c \) and all exceptional values of the central charge in table 3. Table 3 also contains the field content of a \( \mathcal{W} \)-algebra which would give rise precisely to these counting functions.
| $c$ | $\Pi(q)$ | $\text{orbifold}$ | first null field |
|-----|----------|-------------------|-----------------|
| generic | $q^2+q^4+3q^6+5q^8+2q^9+8q^{10}+5q^{11}+16q^{12}+10q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10, 12)$ | 16 |
| $-2$ | $q^2+q^4+2q^6+3q^8+q^9+5q^{10}+2q^{11}+8q^{12}+4q^{13}+O(q^{14})$ | $\mathcal{W}(2, 10)$ | 20 |
| $-23$ | $q^2+q^4+2q^6+3q^8+q^9+5q^{10}+3q^{11}+9q^{12}+5q^{13}+O(q^{14})$ | $\mathcal{W}(2, 8)$ | 16 |
| $\frac{6}{5}$ | $q^2+q^4+2q^6+5q^8+2q^9+8q^{10}+5q^{11}+15q^{12}+10q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10)$ | $>13$ |
| $\frac{4}{5}$ | $q^2+q^4+2q^6+3q^8+q^9+4q^{10}+2q^{11}+7q^{12}+3q^{13}+O(q^{14})$ | $\mathcal{W}(2)$ | 20 |
| $-\frac{98}{5}$ | $q^2+q^4+3q^6+5q^8+2q^9+8q^{10}+5q^{11}+15q^{12}+10q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10)$ | $>13$ |
| $-\frac{186}{5}$ | $q^2+q^4+3q^6+4q^8+2q^9+7q^{10}+4q^{11}+13q^{12}+8q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 10)$ | $>13$ |
| $-\frac{40}{7}$ | $q^2+q^4+3q^6+5q^8+2q^9+8q^{10}+5q^{11}+14q^{12}+9q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10)$ | 12 |
| $-\frac{114}{7}$ | $q^2+q^4+2q^6+3q^8+q^9+4q^{10}+2q^{11}+7q^{12}+3q^{13}+O(q^{14})$ | $\mathcal{W}(2)$ | 26 |
| $-\frac{470}{7}$ | $q^2+q^4+3q^6+5q^8+2q^9+8q^{10}+5q^{11}+15q^{12}+10q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10)$ | $>13$ |
| $-\frac{490}{11}$ | $q^2+q^4+3q^6+5q^8+2q^9+7q^{10}+5q^{11}+14q^{12}+9q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8)$ | $>13$ |
| $-\frac{774}{13}$ | $q^2+q^4+3q^6+5q^8+2q^9+8q^{10}+5q^{11}+15q^{12}+10q^{13}+O(q^{14})$ | $\mathcal{W}(2, 6, 8, 10)$ | $>13$ |

Table 3: Orbifold of $\mathcal{W}(2, 3)$ where generators become null fields

Using the procedure described in section 2.2.1. we calculated the composite primary fields of dimension 6, 8 and 10 in the orbifold. The dimension 6 generator is given by

$$\Phi^{(6)} = 9(43c - 844)(5c + 22)\mathcal{N}(L, \partial^2 L) + 480(191c + 22)\mathcal{N}(\mathcal{N}(L, L), L) - 90(7c + 68)(5c + 22)(2c - 1)\mathcal{N}(\mathcal{W}^{(3)}, \mathcal{W}^{(3)}).$$

(2.2.7)

The two point function turns out to be

$$d_{6,6} = 3600(7c + 114)(7c + 68)(5c + 22)(5c - 4)(2c - 1)(c + 23) = 2(c + 2).$$

(2.2.8)

After rescaling to standard normalization one obtains the following structure constant:

$$\left(C_{6,6}^6\right)^2 = \frac{50(14c^3 + 915c^2 + 14758c - 22344)^2(c + 2)}{3(7c + 2144)(7c + 68)(5c + 22)(5c - 4)(2c - 1)(c + 23)}.$$  

(2.2.9)

The explicit form of the spin 8 and 10 generators, other two-point functions and additional structure constants can be found in appendix F.

At $c = -23$ the field $\Phi^{(6)}$ is a null field whereas $\Phi^{(8)}$ is nonzero. This is to be expected because one knows that for this value of the central charge $\mathcal{W}(2, 3)$ has a $\mathcal{W}(2, 8)$ subalgebra [52]. At $c = -2$ both fields $\Phi^{(6)}$ and $\Phi^{(8)}$ turn out to be null fields. However $\Phi^{(10)}$ is nonzero and $\mathcal{W}(2, 3)$ has a $\mathcal{W}(2, 10)$ subalgebra at $c = -2$. This agrees with the results of [16] and in particular confirms that the field of dimension 10 is quadratic in $\mathcal{W}^{(3)}$. Furthermore, $c = \frac{4}{5}$ and $c = -\frac{114}{7}$ are Virasoro minimal and therefore the orbifold must be just the Virasoro algebra. These statements about $c = -2, -23, \frac{4}{5}, -\frac{114}{7}$ are confirmed by the dimensional arguments in table 3 and we can use known facts (see e.g. [52, 23, 16]) about representations of these algebras to predict the dimension of the first null field. For the
remaining cases one has to be more careful because we have not checked all structure constants. For some values of the central charge it might turn out that null fields actually make the orbifold inconsistent. Note, however, that the induced normalizations eq. (2.2.8), (F.2) for the generators of the orbifold are consistent with the field content predicted in table 3.

2.2.3. Remarks on the orbifold of \( \mathcal{WA}_{n-1} \)

It has been shown in [24] that the orbifolds of the classical \( \mathcal{WA}_{n-1} \) possess other classical Casimir algebras as subalgebras. In this subsection we will show that such a relation does not hold true for the quantum orbifolds.

For \( \mathcal{WA}_{n-1} \cong \mathcal{W}(2, \ldots, n) \) some structure constants (2.1.25) are known generally. The first primary composite field in the \( \mathbb{Z}_2 \) orbifold (the \( \mathbb{Z}_2 \) automorphism changes the sign of the odd dimensional simple fields) can be calculated for all \( n \geq 4 \):

\[
\Phi^{(6)} = 27(43c - 844)(5c + 22)(c + 24)\mathcal{N}(L, \partial^2 L) + 1440(191c + 22)(c + 24)\mathcal{N}(\mathcal{N}(L, L), L) \\
+ 1980(7c + 68)(5c + 22)(2c - 1)C_{W(4)}^{W(4)} W_{(3)} W^{(4)}, L) \\
- 270(7c + 68)(5c + 22)(2c - 1)(c + 24)\mathcal{N}(W^{(3)}, W^{(3)}).
\]

Using (2.1.25a) this induces the following normalization

\[
d_{6,6} = \left(32400c(7(n + 2)(n - 2)c^4 + (21n^3 + 380n^2 - 1800)c^3 + (1399n^3 + 1585n^2 \\
- 7700)c^2 + 4(1179n^3 + 4375n^2 - 1770)c - 32(473n^2 - 81n - 81)(n - 1)) \\
(7c + 68)(5c + 22)^2(2c - 1)(c + 24)(c + 2)\right) \left((cn + 2c + 3n^2 - n - 2)(n - 2)\right)^{-1}.
\]

The simplest nontrivial structure constant of the orbifold reads in the standard normalization \( (\phi_{6,6} = \frac{c}{6}) \)

\[
(C_{44}^{6})^2 = \left(29c^3n^2 - 284c^3 + 255c^2n^3 - 427c^2n^2 - 368c^2 + 540cn^3 + 3016cn^2 - 1636c \\
- 1920n^3 + 2032n^2 - 112\right)^2 (5c + 22)^2 \\
= \left(7c^4n^2 - 28c^4 + 21c^3n^3 + 380c^3n^2 - 1800c^3 + 1399c^2n^3 + 1585c^2n^2 - 7700c^2 \\
+ 4716cn^3 + 17500cn^2 - 7080c - 15136n^3 + 17728n^2 - 2592\right) \\
(cn + 2c + 3n^2 - n - 2)(7c + 68)(2c - 1)(c + 24)(c + 2)(n - 2) \right)^{-1}
\]

where we have used both (2.1.25a) and (2.1.25b). Fortunately (2.1.25b) vanishes for \( n = 4 \) such that we can apply it to \( \mathcal{W}(2, 3, 4) \) as well. Note that the structure constant (2.2.12) is nonzero for any \( n \) and the central charge \( c \) generic. This means that the invariant original generators do not close among themselves and one is forced to include the dimension 6
Specializing (2.2.11) to \( n = 4 \) we obtain for the induced normalization of the first composite field in the orbifold of \( W(2,3,4) \):

\[
d_{6,6} = 10800(7c+114)(7c+68)(5c+22)^2(3c+116)(2c-1)(c+24)(c+13)(c+2)(c-1)c(c+7)^{-1}.
\]

(2.2.13)

Of course, it is also straightforward to specialize (2.1.25) and (2.2.12) to \( n = 4 \) to obtain the first structure constants. From (2.2.13) we read off some interesting values of the central charge \( c \) for the orbifold of \( W(2,3,4) \). For \( c \in \{ -13, 1, -\frac{116}{3} \} \) the field \( \Phi^{(6)} \) is a null field but does not make \( W(2,3,4) \) inconsistent (like it happens e.g. for \( c = -24 \)). Thus, for these values of the central charge the orbifold of \( W(2,3,4) \) is a \( W(2,4) \) or at least has a \( W(2,4) \) subalgebra. In particular, \( W(2,3,4) \) itself has a \( W(2,4) \) subalgebra for \( c \in \{ -13, 1, -\frac{116}{3} \} \). For \( W(2,3,4) \) there are 8 quasi-primary invariant fields at scale dimension 8. Their determinant reads (up to a nonzero constant of proportionality which depends on the choice of basis):

\[
\det_8 \sim (11c+702)(7c+114)(7c+27)(5c+22)^4(3c+116)^2(c+51)(c+13)^2(c+2)(c-1)^3c^8(c+7)^{-3}.
\]

(2.2.14)

From this we observe that the additional scale dimension 8 generator in the orbifold vanishes for \( c \in \{ -13, 1, -\frac{116}{3}, -51, -\frac{27}{7}, -\frac{702}{11} \} \) which includes in particular the three values of \( c \) where already the scale dimension 6 composite generator vanishes. Whereas \( c = -51 \) and \( c = -\frac{27}{7} \) do not belong to the minimal series of the \( WB/C \) Casimir algebras the value \( c = -\frac{702}{11} \) lies in the minimal series of \( WB_3 \) (\( p = 11, q = 7 \)). Comparison of the set of highest weights of the minimal models and structure constants indicates that the orbifold of \( W(2,3,4) \) is a \( WB_3 \) for \( c = -\frac{702}{11} \) (compare appendix D).

Coupling constants connecting two simple fields with primary normal ordered products have been determined for \( W(2,3,4,5) \) and \( W(2,3,4,5,6) \) before in [7]. However, in this work the coefficient of \( N(W^{(3)},W^{(3)}) \) was chosen independent of \( c \) such that one cannot read off the values of the central charge where \( \Phi^{(6)} \) becomes a null field.

For the orbifold of \( W(2,3,4,5) \) we obtain from the specialization of (2.2.11) to \( n = 5 \):

\[
d_{6,6} = 10800(7c+68)(7c-8)(5c+22)^2(3c+116)(2c-1)(c+24)(c+23)(c+2)c.
\]

(2.2.15)

It is remarkable that for \( c = \frac{8}{7} \) the field \( \Phi^{(6)} \) turns out to be a null field. Thus, the orbifold of \( W(2,3,4,5) \) has a \( W(2,4) \) subalgebra at \( c = \frac{8}{7} \) or probably even reduces to \( W(2,4) \). On the one hand this is the first unitary minimal model of \( W(2,3,4,5) \) – the \( \mathbb{Z}_5 \) parafermions [20]. On the other hand this is presumably the only nontrivial unitary minimal model of \( W(2,4) \) [53]. The orbifold construction explains why precisely half of the representations of \( W(2,4) \) at \( c = \frac{8}{7} \) are parafermionic representations [53] because each representation of the original algebra splits into two representations when a \( \mathbb{Z}_2 \) orbifolding procedure is
applied. We also re-encounter the value $c = -\frac{116}{3}$ which is the only other rational value of $c$ for which the dimension 6 generator vanishes and $C_{44}^6 = 0$.

For $n = 6$, i.e. the orbifold of $\mathcal{W}(2,3,4,5,6)$ one finds no rational value of the central charge $c$ where the additional (composite) dimension 6 generator could drop out.

3. General structures in cosets and orbifolds

3.1. Vacuum preserving algebras (VPA) and classical limits

Once a construction of a $\mathcal{W}$-algebra as a reduction of a Kac-Moody algebra or a similar linear system is known, the $\mathcal{W}$-algebra is usually well under control. In particular, one can easily discuss its classical counterpart, i.e. the analogous reduction of the corresponding classical linear system. Note, however, that in general several constructions of the same quantum $\mathcal{W}$-algebra are possible and can lead to different classical counterparts. For a classification one needs more general methods which do not refer to any particular construction. Two closely related ideas in this direction have been put forward in [13] for deformable $\mathcal{W}$-algebras: The vacuum preserving algebra (VPA) as well as a particular classical limit of a $\mathcal{W}$-algebra. These methods work nicely for $\mathcal{W}$-algebras obtained by Drinfeld-Sokolov reduction [13, 14]. The $\mathcal{W}$-algebras discussed in this paper are not in the DS class and therefore it is interesting to see to what extent these methods work for them.

We start with a discussion of a general approach to the VPA. First, one introduces the ‘vacuum preserving modes’ of all quasi-primary fields. The space spanned by them carries a Lie algebra structure. Next, one considers the limit $c \to \infty$ of this algebraic structure. The VPA is the smallest subalgebra of this algebra containing the vacuum preserving modes of the simple fields. To be more precise, the vacuum preserving modes of a quasi-primary field $\Phi$ are given by

$$\{ \Phi_n \mid |n| < d(\Phi) \}. \quad (3.1.1)$$

The vacuum preserving modes of all quasi-primary fields have the important property [13, 14] that the commutator closes among them and does not have any central term. Note that the vacuum preserving modes $L_{\pm 1}, L_0$ of the energy momentum tensor $L$ form an $sl(2)$ subalgebra of all vacuum preserving modes. The space spanned by all vacuum preserving modes (3.1.1) is in general infinite dimensional and the commutator still depends continuously on $c$. In order to cure the second property one takes the limit $c \to \infty$. In general, one will have to rescale the generators $W^{(i)}$ of the finitely generated quantum $\mathcal{W}$-algebra in order to make sense of the limit $c \to \infty$:

$$\hat{W}^{(i)} := c^{-\alpha_i} W^{(i)}. \quad (3.1.2)$$

The exponents $\alpha_i$ have to be adapted in order to make all structure constants connecting the fields $\hat{W}^{(i)}$ bounded and nontrivial for $c \to \infty$. Even in this limit, the algebra of all vacuum preserving modes is a very complicated object. Therefore, the VPA is defined as the smallest subalgebra of the limit $c \to \infty$ of the algebra of all vacuum preserving modes that contains the vacuum preserving modes of the simple fields.

It was shown in [13] that this works nicely for the algebras in the DS class: One can associate to them a finite dimensional Lie algebra with an $sl(2)$ embedding that encodes
the spin content of these algebras. For the $W$-algebras in the DS class one can set all $\alpha_i := 0$ and then the VPA is defined by

$$V := \text{span}\{W_n^{(i)} \mid |n| < d(W^{(i)})\}$$

(3.1.3)

where $W^{(i)}$ are the simple fields of the $W$-algebra. On this space a Lie bracket is induced by taking the limit $c \to \infty$ of the commutator. The crucial point is that the commutator linearizes, i.e. that the induced Lie bracket closes in the space (3.1.3). The data (3.1.3) together with the Lie bracket and the $sl(2)$ embedding is equivalent to the original data used for the DS reduction [13, 14].

The situation is much less clear for $W$-algebras outside the DS class, i.e. those $W$-algebras which do not have ‘nice’ asymptotic properties of the structure constants. We will first discuss one example in detail: The bosonic projection of the $N = 1$ Super Virasoro algebra. Let us for the moment forget about the construction of this $W(2,4,6)$ and look what can be said about the VPA solely from inspection of the structure constants (Set 1 in section 6.2 of [5]). Denote the generators of scale dimension 2, 4 and 6 by $L, W^{(i)}$ ($i = 4, 6$). We observe from the structure constants in [5] that we have to rescale

$$\hat{L} := L, \quad \hat{W}^{(i)} := \frac{1}{\sqrt{c}} W^{(i)}.$$  

(3.1.4)

Then, all structure constants connecting these three simple fields are bounded and nonzero for $c \to \infty$. Unlike for $W$-algebras in the DS class the commutator does not linearize for this $W(2,4,6)$, i.e. the commutator does not close in the space (3.1.3). The structure constants $C_{W^{(4)}W^{(4)}}^{W^{(4)}W^{(6)}}$ and $C_{W^{(6)}W^{(6)}}^{W^{(4)}W^{(4)}}$ are invariant under the rescaling (3.1.4). Furthermore, these structure constants tend to a nonzero constant in the limit $c \to \infty$. This means that the vacuum preserving modes $\hat{W}^{(8)}_n, |n| < 8$ of the primary projection $\hat{W}^{(8)}$ of $(\hat{W}^{(4)}\hat{W}^{(4)})$ have to be included into the VPA. We have checked that the same happens at scale dimension 10. At scale dimension 10 one finds the quadratic fields $(\partial^2 \hat{W}^{(4)}\hat{W}^{(4)})$ and $(\hat{W}^{(4)}\hat{W}^{(6)})$. Due to the presence of a generic null field at scale dimension 10 [44] these two fields give rise to precisely one primary field at scale dimension 10. Since both $(\partial^2 \hat{W}^{(4)}\hat{W}^{(4)})$ and $(\hat{W}^{(4)}\hat{W}^{(6)})$ do not decouple in the limit $c \to \infty$, the vacuum preserving modes of the primary field with dimension 10 have to be included into the VPA. Thus, the VPA does definitely not close on the vacuum preserving modes of the simple fields only but one has to include the vacuum preserving modes of further primary fields (at least up to scale dimension 10). We expect that one must indeed include the vacuum preserving modes of infinitely many primary fields because there is no reason to expect any of the crucial structure constants to vanish. Therefore, we actually expect the VPA of this $W(2,4,6)$ to be infinite dimensional.

Similar reasoning applies to the second $W(2,4,6)$ outside the DS class (Set 2 in section 6.2 of [5] and eq. (6) in [16]). The situation is slightly different at scale dimension 10. Here, no null field is present and therefore we do indeed have two primary fields which turn up in the commutator of $\hat{W}^{(6)}$ with itself. However, up to that stage only a particular linear combination of these fields appears in the commutators and therefore it might be sufficient to include the vacuum preserving modes of only one field at scale dimension 10.
into the VPA. As before the realization of this $\mathcal{W}(2,4,6)$ in terms of the coset $sl(\widehat{2},\mathbb{R})_k \oplus sl(\widehat{2},\mathbb{R})_{-1/2}/sl(\widehat{2},\mathbb{R})_{k-1/2}$ [9] (see also section 2.1.3.) is not needed for the discussion of the VPA.

Let us now turn to the $\mathcal{W}(2,3,4,5)$ which we discussed in section 2.1.1. (see also [9]). The structure constants for this algebra were first calculated in [7] checking associativity of the OPE without reference to the cosets $sl(\widehat{2},\mathbb{R})_k/U(1)$ or $SVIR(N=2)/U(1)$. Denote the generators of this $\mathcal{W}(2,3,4,5)$ by $L$ and $W^{(i)}$ ($i = 3, 4, 5$). Then we see from the structure constants in table 2 of ref. [7] that we have to rescale as in eq. (3.1.4). Up to scale dimension 8 there are 6 further quadratic fields in terms of the $W^{(i)}$, two of them at scale dimension 8 give rise to null fields (see eq. (2.11) and (2.12) of ref. [7]) leaving us with 4 primary fields: One each at scale dimensions 6 and 7, two at scale dimension 8. As in the previous cases, the structure constants connecting the simple primary fields with these composite primary fields are invariant under the rescaling (3.1.4) and tend to a nonzero constant for $c \to \infty$ (compare table 2 of ref. [7]). This means that the vacuum preserving modes of the fields at scale dimensions 6 and 7 have to be included into the VPA. Only a linear combination of the two fields at scale dimension 8 appears in the OPEs of the simple fields. Therefore, so far we have to include the vacuum preserving modes of only one primary field at scale dimension 8 into the VPA. As before, we have not calculated structure constants involving fields of higher dimension, but there is no reason to expect those structure constants involving primary fields to vanish. Thus, we expect also the VPA of this algebra to be infinite dimensional.

The above discussion can further be supported by looking at the classical counterparts of these algebras that correspond to the above constructions [9]. These classical counterparts are infinitely generated. From [13,14] we know that in the classical case the VPA consists of the vacuum preserving modes of all generators. In particular, for the examples under consideration it is definitely infinite dimensional. Even more, the composite primary fields which had to be added precisely correspond to the additional generators of the classical counterparts. This means that the VPA of a $\mathcal{W}$-algebra encodes some information of possibly underlying constructions.

We have seen in three examples outside the DS class that their VPA is an infinite dimensional Lie algebra with $sl(2)$ embedding which decomposes into finite dimensional representations under this $sl(2)$. This indicates that probably all $\mathcal{W}$-algebras outside the DS class have infinite dimensional VPAs which makes the VPA as a tool for classification unhandy. At least, for any explicitly known algebra the construction of the VPA is purely algorithmic but does unfortunately not necessarily stop after finitely many steps. It should be noted that the impact of the realization of composite fields contributing to the VPA in terms of finitely many simple ones is not completely clear to us, neither is the impact of the relations satisfied by the generators of these $\mathcal{W}$-algebras [9] on the VPA.

Next we consider classical limits of $\mathcal{W}$-algebras. A classical Kac-Moody algebra is a Lie algebra and can therefore be quantized according to Dirac’s rule. In particular, the r.h.s. of the commutator is multiplied by $\hbar$, and the canonical classical limit of the quantum Kac-Moody algebra is the limit $\hbar \to 0$. For a $\mathcal{W}$-algebra that arises as some reduction of a
Kac-Moody algebra, this classical limit of the Kac-Moody algebra induces a classical limit of the $\mathcal{W}$-algebra. If we have a particular construction in mind we refer to the induced classical limit as ‘the classical limit’. In contrast hereto, the procedure of [13] is a set of rules to re-institute $\hbar$’s in a $\mathcal{W}$-algebra without referring to any particular reduction and afterwards define a classical limit $\hbar \to 0$. We will refer to this classical limit as the ‘BW classical limit’.

To be more explicit, the BW classical limit is defined by the following set of rules

$$W^{(i)} \mapsto \frac{\hat{W}^{(i)}_{\hbar}}{\hbar^{1+\alpha_i}}, \quad c \mapsto \frac{\hat{c}}{\hbar} \quad (3.1.5)$$

where $W^{(i)}$ are the simple generators of the $\mathcal{W}$-algebra. Then, the quantum fields $W^{(i)}(z)$ are replaced by classical fields $w^{(i)}(z) := \lim_{\hbar \to 0} \hat{W}^{(i)}_{\hbar}(z)$ and Poisson brackets as well as the ring structure are defined by the following identifications

$$\{w^{(i)}(z), w^{(j)}(w)\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [\hat{W}^{(i)}_{\hbar}(z), \hat{W}^{(j)}_{\hbar}(w)], \quad w^{(i)}(z)w^{(j)}(z) = \lim_{\hbar \to 0}(\hat{W}^{(i)}_{\hbar}\hat{W}^{(j)}_{\hbar})(z) \quad (3.1.6)$$

where the choice of normal ordering prescription on the quantum level is actually irrelevant. The crucial point is to choose the exponents $\alpha_i$ in (3.1.5) such that the limits in (3.1.6) exist and are nontrivial. The reader should note that in general the existence of such a set of exponents $\alpha_i$ is not guaranteed. For $\mathcal{W}$-algebras in the DS class one can choose all $\alpha_i = 0$ [13] and then (3.1.5) and (3.1.6) are one-to-one maps between the one-parameter families of $\mathcal{W}$-algebras on the quantum level and on the classical level where the parameter is the central charge. It should be noted that the BW classical limit according to (3.1.5) and (3.1.6) is not necessarily the same as the one induced by a reduction. Close inspection shows that even in the DS class the central charges of these two classical limits are indeed different, and therefore the classical limits are equivalent only when looking at one-parameter families of $\mathcal{W}$-algebras. Note that also nondeformable $\mathcal{W}$-algebras can have a classical limit (compare e.g. the $\beta - \gamma$ system of [9]). Thus, deformability and existence of a classical limit should not be confused. However, only for deformable $\mathcal{W}$-algebras the rules (3.1.5) can be applied.

In passing we mention that one obtains from (3.1.6) the classical quasi-primary projection of the product of two classical quasi-primary fields:

$$\mathcal{Q}(w^{(i)}(z)\partial^n w^{(j)}(z)) = \lim_{\hbar \to 0} \mathcal{N}(\hat{W}^{(i)}_{\hbar}, \partial^n \hat{W}^{(j)}_{\hbar})(z). \quad (3.1.7)$$

Applying (3.1.5) and (3.1.6) to the formula for the quasi-primary normal ordered product $\mathcal{N}$ (eq. (1.1.4)) we immediately obtain an explicit formula for $\mathcal{Q}(w^{(i)}(z)\partial^n w^{(j)}(z))$ (under assumptions such as $\alpha_k < 1 + \alpha_i + \alpha_j$ for all $k$). Let $w^{(i)}(z)$ and $w^{(j)}(z)$ be two classical quasi-primary fields. Then

$$\mathcal{Q}(w^{(i)}(z)\partial^n w^{(j)}(z)) := \sum_{r=0}^{n} (-1)^r \left( \begin{array}{c} n \cr r \end{array} \right) \frac{(2d(w^{(j)})+n-1)}{\left(2d(w^{(j)})+d(w^{(i)})+n-1\right)} \partial^r w^{(i)}(z)\partial^{n-r} w^{(j)}(z) \quad (3.1.8)$$
is quasi-primary and has dimension \( d(w^{(i)}) + d(w^{(j)}) + n \). Let us now apply these ideas to the algebras outside the DS class. Requiring the linear term (in terms of the generators) on the r.h.s. of the commutator to be bounded and nonzero for \( \hbar \to 0 \) we conclude that the exponents \( \alpha_i \) in (3.1.5) and (3.1.2) are actually identical. In particular, for the three examples already discussed above the exponent for the Virasoro field \( L \) is \( \alpha_0 = 0 \) and all other \( \alpha_i = \frac{1}{2} \). In order to have a well-defined limit of the commutator in (3.1.6) all structure constants connecting the additional simple fields with quadratic fields not containing \( L \) must be at most of order \( O(c^{-1}) \) as \( c \to \infty \).

However, from the discussion of the VPA we conclude that the coupling constants to the quadratic fields in terms of the generators tend to a nonzero constant for \( c \to \infty \). This means that one must decouple the quadratic fields from the ring and rescale them independently, i.e. one must introduce further relations and further generators in order to make sense of the BW classical limit. This is similar to the VPA, in particular those fields have to be introduced as new generators whose modes had to be included into the VPA. This indicates that the BW classical limits of these three algebras are probably infinitely generated and satisfy infinitely many constraints.

At least for the bosonic projection of the \( N = 1 \) Super Virasoro algebra we can verify in terms of the underlying realization that one field each at scale dimensions 8 and 10 decouples in the classical limit and gives rise to a new generator. Recall that the field \( G \) in appendix E has to be rescaled with \( \alpha_1 = 0 \) (the exponent \( \alpha_0 \) for \( L \) is also zero). Denote the classical limits corresponding to \( L(z), G(z) \) by \( l(z), g(z) \). After these substitutions, the coefficient of \( g\partial^5 g \) in (E.3) vanishes in this classical limit and one has to add its primary projection \( \mathcal{P}(g\partial^5 g) \) to the generating set. This primary projection can be obtained by applying the limiting procedure to the corresponding primary quantum field and is explicitly given by

\[
\mathcal{P}(g\partial^5 g) = Q(g\partial^5 g) - \frac{1380}{13c} Q(Q(g\partial^3 g)l) - \frac{182}{11c} Q(Q(g\partial g)\partial^2 l) + \frac{16524}{11c^2} Q(Q(Q(g\partial g)l)l).
\]

A similar phenomenon happens at scale dimension 10 where the primary projection of \( g\partial^7 g \) has to be added to the generating set (compare also the discussion in [9], in particular appendix A loc. cit.).

For these three examples (the \( W(2,3,4,5) \) and the two algebras of type \( W(2,4,6) \)) the realization in terms of a reduction is known both on the quantum as well as on the classical level. Therefore, we can compare the result of the limiting procedure (3.1.5) and (3.1.6) to the corresponding classical algebras [9]. First, we remark that the fields which we had

\[8)\] This statement can be proven directly (without classical limit) by taking Poisson brackets with the modes \( l_{\pm 1} \) and \( l_0 \) of the classical Virasoro generator on the right hand side of (3.1.8) and verifying that it does indeed transform like a quasi-primary field.

Note that finding the projection of the \( n \)th derivative of the product of two modular forms onto a modular form (see e.g. [54]) is analogous to determining the quasi-primary projection. In fact, the formula (1) of [54] (called the ‘\( n \)th Rankin-Cohen bracket’) is more compact but equivalent to (3.1.8).
to introduce as generators in the BW classical limit indeed turn up as generators in the classical reduction and that the classical counterparts of these algebras are indeed infinitely generated [9]. Rescaling of the generators with exponents $\alpha_i > 0$ implies a vanishing central term in the limit $\hbar \to 0$. On the classical level, all generators (with exception of the Virasoro field $l(z)$) are at least second order in the fields of the underlying algebra and the Poisson brackets of such fields do indeed not contain any central term. For any orbifold (including in particular the bosonic projection of the $N = 1$ Super Virasoro algebra), the energy momentum tensor $L$ in the projection is noncomposite and the central charge $c$ is a free parameter on the classical level. Thus, the BW classical limit of this $\mathcal{W}(2, 4, 6)$ has a chance to be identical to the $\mathcal{W}$-algebra obtained from the classical orbifold. The situation is different for the other $\mathcal{W}(2, 4, 6)$ and the $\mathcal{W}(2, 3, 4, 5)$ which we realized in terms of cosets. Note that the classical coset energy momentum tensor $l(z)$ is composite and therefore has no central term, i.e. $c = 0$ on the classical level. However, the BW limiting procedure (3.1.5) and (3.1.6) gives rise to nonzero Virasoro centre $c$ showing that the BW classical limit is not identical to the classical coset. Even more, close inspection of the structure constants of the $\mathcal{W}(2, 4, 6)$ arising in the diagonal $sl(2, \mathbb{R})$ coset shows that some structure constants still are proportional to $\frac{1}{c}$ after taking the BW classical limit. Because this $c$-dependence cannot be completely scaled away, one cannot simply set $c = 0$ in the BW classical limit. This is not very surprising because for cosets of Kac-Moody algebras the energy momentum tensor $L$ satisfies $[L_m, L_n] = \hbar(n - m)L_{m+n} + \hbar^2 c(n^3 - n)\delta_{m+n, 0}$. This means that one should modify the BW procedure by substituting $c \mapsto \hat{c}$ instead of (3.1.5) in order to obtain at least the correct classical form of the Virasoro algebra. However, this substitution does not introduce any $\hbar$'s in the structure constants and would therefore leave us with $c$-dependent structure constants in this modified BW classical limit – something we do not want either. Furthermore, in the case of the $\mathcal{W}(2, 3, 4, 5)$, the classical counterpart of the realization in terms of the coset $SVIR(N = 2)/\hat{U}(1)$ has a non-vanishing Virasoro centre in contrast to the classical coset $sl(2, \mathbb{R})/\hat{U}(1)$. So, the limiting procedure (3.1.5), (3.1.6) might correspond to the classical coset $SVIR(N = 2)/\hat{U}(1)$ but it is not clear how to obtain the classical counterpart of the other coset realization by a limiting procedure of this type. Even more, one can check that the classical coset $SVIR(N = 2)/\hat{U}(1)$ admits a primary generating set whereas the classical coset $sl(2, \mathbb{R})/\hat{U}(1)$ does not. This shows that the Poisson brackets carried by these two classical cosets are not at all related to each other. These ambiguities for the classical counterpart might be related to the possibility that the BW procedure (3.1.5), (3.1.6) does not always give rise to a classical $\mathcal{W}$-algebra.

In summary, we have seen that one can introduce the VPA and study the BW classical limit of any quantum $\mathcal{W}$-algebra in an algorithmic manner. In doing so one recovers many features of a corresponding classical counterpart without using any knowledge about the underlying construction (although the attempt to construct the BW classical limit could fail). This gives rise to the hope that all $\mathcal{W}$-algebras outside the DS class which are finitely generated on the quantum level belong to the same class of $\mathcal{W}$-algebras with infinitely generated classical counterparts [9]. From this point of view coset constructions and orbifolds behave in a very similar manner.
3.2. Coset realization of unifying \( \mathcal{W} \)-algebras and level-rank-duality

In section 2.1.3. we were able to show that the special \( \mathcal{W}(2,4,6) \) is realized by the coset \( \widehat{sl(2,\mathbb{R})}_k \oplus \widehat{sl(2,\mathbb{R})}_{-k} / \widehat{sl(2,\mathbb{R})}_{k-\frac{1}{2}} \) and we have studied also its minimal models. Below we will show that this algebra is the first member of a new series of unifying algebras - denoted as \( WD_{-m} \) - which are unifying objects for some \( WC \) minimal models [17]. In the spirit of [8] one can write down the minimal series of these algebras. Using character arguments it is possible to give an explicit coset realization based on the symplectic Lie algebras \( sp(2m) \). Furthermore, we present coset realizations of unifying algebras of some series of minimal models of the (Casimir) algebras \( \mathcal{WA}, \mathcal{WB} \) and \( \text{Orb}(WD) \) proposed in [17]. These relationships generalize level-rank-duality of coset pairs. Furthermore, we study the diagonal cosets \( \hat{g}_\kappa \oplus \hat{g}_\mu / \hat{g}_{\kappa+\mu} \) for \( g = A_n, B_n, C_n, D_n \) and some special values of \( \kappa, \mu \) on the level of characters.

3.2.1. Unifying \( \mathcal{W} \)-algebras for the \( \mathcal{WA}_n \) Casimir algebras

Due to the level-rank duality [18, 19]

\[
\frac{\widehat{sl(n)}_k \oplus \widehat{sl(n)}_1}{\widehat{sl(n)}_{k+1}} \cong \frac{\widehat{sl(k+1)}_n}{\widehat{sl(k)}_n \oplus U(1)} = \mathcal{CP}(k)
\]

one expects that the symmetry algebra of the \( \mathcal{CP}(k) \) model is a unifying \( \mathcal{W} \)-algebra for the \( k \)th unitary model of \( \mathcal{WA}_{n-1} \). Note that the l.h.s. of (3.2.1) is defined for integer \( n \) and arbitrary \( k \), whereas the r.h.s. is defined for integer \( k \) and general \( n \). The isomorphism in (3.2.1) is valid iff \( k \) and \( n \) are both positive integers.

We will calculate the spin content of \( \mathcal{CP}(k) \) using character techniques \(^9\). According to [9] we have to count the states in the complement of \( \widehat{sl(k)}_n \oplus U(1) \) in \( \widehat{sl(k+1)}_n \) that are invariant under \( sl(k) \oplus U(1) \). This will be carried out for generic level \( n \) using the character argument described in section 2.1.3. Let \( \Delta^k \) be the root system of \( sl(k+1) \) and let \( \Theta \) be in \( h^* \) (\( h \) is a Cartan subalgebra of the horizontal subalgebra \( sl(k+1) \) of \( sl(k+1)_n \)). We will write \( \Theta = (\bar{\theta}, \Theta_k) \) where \( \bar{\theta} \) corresponds to \( sl(k) \) and \( \Theta_k \) to \( U(1) \). The counting function for the subspace of the vacuum module of \( \widehat{sl(k+1)}_n \) that does not contain any states generated by \( \widehat{sl(k)}_n \oplus U(1) \) is given by:

\[
\prod_{\bar{\alpha} \in \Delta^k \setminus \Delta^{k-1}} \prod_{n \geq 1}(1 - e^{2\pi i \bar{\theta} \cdot \bar{\alpha} q^n}) \sum_{\lambda, m} C_{\lambda, m}^0(q) e^{2\pi i (\bar{\theta} \cdot \bar{\lambda} + \Theta_k m)}.
\]

(3.2.2)

Since this module carries a representation of \( sl(k) \oplus U(1) \) it can be decomposed into representations of \( sl(k) \oplus U(1) \) which is indicated by the r.h.s. of (3.2.2). The \( C_{\lambda, m}^0(q) \) are the string functions where \( \bar{\lambda} \) and \( m \) are the weights of \( sl(k) \) and \( U(1) \) respectively that label the representation. Invariance under \( U(1) \) is equivalent to \( m = 0 \). The part invariant under \( sl(k) \) can be obtained by summation over its Weyl group. Thus, the vacuum character of

\(^9\) The (unique) simple field with spin 3 has been calculated in [55].
the coset under consideration is given by the branching function \( B_{0,0}^0(q) \) which can be written as the following sum over string functions (compare (2.1.55)):

\[
B_{0,0}^0(q) = \sum_{w \in W} \epsilon(w) C_{w(\rho)+\rho,0}^0(q)
\]

(3.2.3)

where \( W \) is the Weyl group of \( sl(k) \). The string functions can be obtained as follows. Inserting \( z^2 = e^{2\pi i \vec{\phi} \cdot \vec{\alpha}} \) with \( \vec{\alpha} \) a positive root into (2.1.6), the left hand side of (3.2.2) can be written in the form

\[
\frac{q^{-\frac{2k}{2\pi}}}{\prod_{\vec{\alpha} \in \Delta^k \backslash \Delta_{k-1}} \prod_{n \geq 1} (1 - e^{2\pi i \vec{\phi} \cdot \vec{\alpha} q^n})} = \sum_{\{n_1, ..., n_k\} \in \mathbb{Z}^k} \prod_{i=1}^k (\phi_{n_i}(q) - \phi_{n_i-1}(q)) \eta^{2k}(q) e^{2\pi i \vec{\phi} \cdot (\sum_{i=1}^k n_i \vec{\alpha}_i)}.
\]

(3.2.4)

Using an embedding of the roots of \( sl(k+1) \) into \( \mathbb{R}^k \) one obtains a map \((\vec{X}, m) \rightarrow (n_1, ..., n_k)\). Now, comparison of the r.h.s. of eqs. (3.2.2) and (3.2.4) gives an explicit representation of the string functions in terms of the \( \phi \)'s. It is now not difficult to calculate the vacuum character of the \( CP(k) \) model for the first few values of \( k \). In table 4 we present the results for \( 1 \leq k \leq 4 \) and give the spin content of the coset as well as a conjecture for general \( k \) (for \( k = 1 \) compare section 2.1.1.).

| \( k \) | vacuum character |
|-------|------------------|
| 1     | \( B_{0,0}^0(q) = \eta(q)^{-2}(\phi_0(q) - q \phi_1(q)) \) |
|       | \( B_{0,0}^0(q) - \chi_{2,3,4,5}(q) = -2q^8 - 4q^9 - 9q^{10} + O(q^{11}) \) |
| 2     | \( B_{0,0}^0(q) = \eta(q)^{-4}((\phi_0(q) - q \phi_1(q))^2 - (\phi_1(q) - \phi_0(q))^2) \) |
|       | \( B_{0,0}^0(q) - \chi_{2, ..., 11}(q) = -2q^{14} - 6q^{15} - 15q^{16} + O(q^{17}) \) |
| 3     | \( B_{0,0}^0(q) - \chi_{2, ..., 19}(q) = -2q^{22} - 6q^{23} - 17q^{24} + O(q^{25}) \) |
| 4     | \( B_{0,0}^0(q) - \chi_{2, ..., 29}(q) = -2q^{32} - 6q^{33} + O(q^{34}) \) |
| \( k \) | \( B_{0,0}^0(q) - \chi_{2, ..., k^2+3k+1}(q) = -2q^{k^2+3k+4} - 6q^{k^2+3k+5} + O(q^{k^2+3k+6}) \) |

Table 4: Vacuum character for the \( CP(k) \)-model

From the vacuum characters given in table 4 and the arguments presented above we conjecture that

\[
CP(k) = \frac{sl(k+1)}{sl(k) \oplus U(1)} \cong W(2, 3, \ldots, k^2 + 3k + 1).
\]

(3.2.5)

Note that (3.2.5) is compatible with the truncations predicted in [17] for the l.h.s. of (3.2.1).

**Relations to linear \( W_\infty \) algebras**

At this point a few remarks on the relation of our results and the various linear \( W_\infty \) algebras are in place. First, we note that

\[
W_\infty \cong \lim_{n \to \infty} W_{\mathcal{A}_{n-1}},
\]

\[
W_{1,\infty} \cong \lim_{k \to \infty} U(1) \oplus CP(k)
\]

(3.2.6)
where at least the first equality is well-known (see e.g. [56]). Note that the limit in the second line of (3.2.6) is defined via the coset realization of \( CP(k) \), i.e. the limit is taken for fixed level \( n \). Furthermore, the algebras \( W_{n}^{gl(n)} \) which were used in [57] for the classification of quasifinite representations of \( W_{1+∞} \) are related to \( WA_{n−1} \) by [57]:

\[
W_{n}^{gl(n)} \cong \widehat{U}(1) \oplus WA_{n−1}. \tag{3.2.7}
\]

From (3.2.6) and (3.2.7) one can immediately derive further identities. An important one is that \( W_{1+∞}/\widehat{U}(1) \cong \lim_{k \to ∞} \widehat{CP}(k) \) where the latter non-linear infinitely generated algebra can formally be identified with \( WA_{−1} \), i.e. its structure constants can be obtained from those of \( WA_{n−1} \) by setting \( n = 0 \). In particular, all truncations presented in [17] for \( WA_{n−1} \) can immediately applied to \( W_{1+∞} \) setting \( n = 0 \) and shifting the central charge by one. Since the unifying algebras for the unitary models of \( W_{∞} \) are related to algebras of type \( WA_{1} \), all known truncations of the linear \( W_{∞} \) algebras arise as accumulation points of the minimal series of some unifying \( W \)-algebra. These truncations in turn seem to be in one-to-one correspondence with the quasifinite representations of the linear \( W_{∞} \) algebras (see e.g. [57] for a proof in the case of \( W_{1+∞} \) and \( c ∈ N \)).

For example, one can take the limit \( n \to ∞ \) of the level \( n \) in the \( CP(k) \) models in order to obtain unitary representations of \( W_{∞} \) at \( c = 2k \) which has been done already some time ago [58]. Our previous computations show that the identity \( W_{∞} \cong \widehat{CP}(k) \) at \( c = 2k \) implies a truncation of \( W_{∞} \) to an algebra of type \( W(2, 3, . . . , k^2 + 3k + 1) \) at \( c = 2k \). Similarly, the truncations of \( W_{∞} \) to an algebra of type \( W(2, 3, . . . , 2r + 1) \) at \( c = −2r \) can be understood as accumulation points of the unifying algebras \( W_{r−1,1}^{sl(r)}/\widehat{U}(1) \). Above, we already explained the relation of the unifying property of \( W_{n}^{gl(n)} \) to the truncations of \( W_{1+∞} \) to algebras of type \( W(1, 2, . . . , n) \) at \( c = n \). The truncations of \( W_{1+∞} \) to algebras of type \( W(1, 2, . . . , n^2 − 1) \) at \( c = −n + 1 \) [59] also arise from unifying \( W \)-algebras: They are related to algebras of type \( W(2, 3, . . . , n^2 − 1) \cong WA_{−n} \) that can be considered as continuations of the \( WA_{l} \) series to negative rank. The algebras \( WA_{−n−1} \) can be realized in terms of the coset \( sl(n)_k \oplus sl(n)_{−1}/sl(n)_{k−1} \) [60].

The general unifying algebra for \( WA_{n} \)-minimal models

From the preceding discussion it is natural to expect that the unifying \( W \)-algebras for all \( WA_{n−1} \) minimal models can be obtained by cosets of some Drinfeld-Sokolov reductions based on \( sl(r) \) that have a \( \widehat{sl(k)} \oplus \widehat{U}(1) \) Kac-Moody subalgebra \(^{10}\). Therefore, let us consider the algebras

\[
W_{r−k,1}^{sl(r)} = W(1^{k^2}, 2, 3, . . . , r−k, (\frac{r−k+1}{2})^{2k}). \tag{3.2.8}
\]

The \( k^2 \) currents form a \( \widehat{sl(k)} \oplus \widehat{U}(1) \) Kac-Moody algebra, the fields of dimension \( 2, . . . , r−k \) are singlets with respect to this Kac-Moody and the \( 2k \) fields of dimension \( \frac{r−k+1}{2} \)

\(^{10}\) We would like to thank E. Ragoucy for pointing this out to us.
are a Kac-Moody multiplet. More precisely, these $2k$ fields transform as two $U(1)$-charge conjugate defining representations of $sl(k)$. The conjecture is that

$$\frac{\mathcal{W}^{sl(r)}}{sl(k) \oplus U(1)} \cong \mathcal{W}A_{n-1} \quad \text{at} \quad c_{A_{n-1}}(n + k, n + r). \quad (3.2.9)$$

Using the truncations of [17] the conjecture (3.2.9) implies in particular

$$\frac{\mathcal{W}^{sl(r)}}{sl(k) \oplus U(1)} \cong \mathcal{W}(2, 3, \ldots, (k + 1)r + k) \quad (3.2.10)$$

with two generic null fields at dimension $(k + 1)r + k + 3$. The case $k = 1$ was already discussed at the end of section 2.1.2. and the conjectures (3.2.9) and (3.2.10) were confirmed. Identifying $\hat{\mathcal{W}}^{sl(r)}$ with the unconstrained $sl(r + 1)$ Kac-Moody algebra, the case $r = k + 1$ is identical to the $\mathbb{CP}(k)$ cosets which we have just discussed. The results for the $\mathbb{CP}(k)$ cosets also confirm the conjectured identities (3.2.9) and (3.2.10). One can also apply (3.2.9) to the case $k = 0$. In this case, the $sl(2)$ embedding is the principal one and there is no Kac-Moody subalgebra. Thus, the l.h.s. of (3.2.9) is just $\mathcal{W}A_{r-1}$ and we recover eq. (2.4) of ref. [17] (note that this equality was originally observed in [61]).

As a first check we compute the relation of the level $l$ of the underlying $\hat{\mathcal{W}}^{(r)}$ as a function of $r$, $k$ and $n$ which generalizes (2.1.37):

$$l = \frac{n + r}{r - k} - r. \quad (3.2.11)$$

Like in (2.1.37) the level $l$ is linear in the rank $n$ of $\mathcal{W}A_{n-1}$ and becomes rational for the identifications in (3.2.9) ($r$, $k$ and $n$ positive integers). Note that the level $l'$ of the Kac-Moody subalgebra $\hat{sl}(k)_r$ satisfies $l' = l + r - k - 1$.

In order to perform a further check let us consider the case $k = 2$. In this case we have according to (3.2.8)

$$\mathcal{W}^{sl(r)}_{r-2,1} = \mathcal{W}(1^4, 2, 3, \ldots, r-2, \left(\frac{r-1}{2}\right)^4).$$

Denote the four fields of scale dimension $\frac{r-1}{2}$ by $W^\pm_1$ and $W^\pm_2$. The upper index of these fields refers to the $U(1)$-charge. $W^+_1$ and $W^+_2$ are two $sl(2)$-doublets. In the spirit of [9] one can see that the classical invariants under $sl(2) \oplus U(1)$ are generated by

$$S_{m,n} := \partial^m W^+_1 \partial^n W^-_1 + \partial^m W^-_2 \partial^n W^+_2 \quad (3.12a)$$

and the relations are generated by

$$\begin{vmatrix}
S_{m_1,n_1} & S_{m_1,n_2} & S_{m_1,n_3} \\
S_{m_2,n_1} & S_{m_2,n_2} & S_{m_2,n_3} \\
S_{m_3,n_1} & S_{m_3,n_2} & S_{m_3,n_3}
\end{vmatrix} = 0 \quad (3.12b)$$

\(^{11)}\) We would like to acknowledge help for these computations by J. de Boer.
with two sets of pairwise distinct integers \( \{m_1, m_2, m_3\} \) and \( \{n_1, n_2, n_3\} \). Following the argumentation of [9] it is now simple to determine the field content of the coset (3.2.10) for \( k = 2 \):

\[
\mathcal{W}_{r-2,12}^{sl(2)} \cong \mathcal{W}(2, 3, \ldots, 3r + 2)
\]

(3.2.13)

with two generic null fields at dimension \( 3r + 5 \).

In order to perform further checks of (3.2.10) on the level of the characters we need a generalization of (2.1.7):

\[
\frac{1}{\prod_{m \geq \Delta - 1}(1 - e^{2\pi i \theta q^m}) \prod_{m \geq \Delta}(1 - e^{-2\pi i \theta q^m})} = \sum_{n \in \mathbb{Z}} \frac{\psi_n^\Delta(q)}{\prod_{m > 0}(1 - q^m)^2} e^{2\pi i \theta n}
\]

(3.2.14a)

with

\[
\psi_n^\Delta(q) = \sum_{m \geq \Delta - 1} (-1)^{m-\Delta+1} q^{(m-\Delta+1)(m-\Delta+2)} \prod_{\nu = m-\Delta+2}^{m-1} (1 - q^n).
\]

(3.2.14b)

Now one can compute the vacuum character of the coset (3.2.9) in complete analogy to the \(\mathbb{C}P^1\) models using (3.2.14), i.e. the only modification is that one has to substitute \(\psi_n^\Delta(q)\) with \(\Delta = r-k+1\) for \(\phi_n(q)\). We have checked for \(k = 3, 4 \leq r \leq 8\) and \(k = 4, 5 \leq r \leq 8\) that this character argument is in agreement with (3.2.10).

### 3.2.2. Level-rank-duality for the cosets \(so(n)_k \oplus so(n)_1 / so(n)_{k+1}\)

We consider now the impact of level-rank-duality on the coset \(so(n)_k \oplus so(n)_1 / so(n)_{k+1}\). From known results in the literature [18, 19] and arguments to be presented below, we conjecture that

\[
(\text{Orb}) \left( \frac{so(n)_k \oplus so(n)_1}{so(n)_{k+1}} \right) \cong \text{Orb} \left( \frac{so(k+1)_n}{so(k)_n} \right).
\]

(3.2.15)

For \(n\) even, the coset on the l.h.s. realizes \(WD_n\) and one has to take an orbifold (see section 2.2) whereas for \(n\) odd, the coset already corresponds to the orbifold of \(WB(0, \frac{n-1}{2})\) and no additional projection has to be taken. Eq. (3.2.15) implies that the orbifold of the coset \(WD\)-algebra \(so(k+1)_n / so(k)_n\) is a unifying \(WD\)-algebra for the orbifolds of the \(k\)th unitary minimal models of \(WD_n\) and \(WB(0, \frac{n-1}{2})\) for \(n\) even or odd respectively. It should be clear to the reader that the methods used in [18, 19] to derive level-rank-duality are insensitive to orbifolds and therefore they originally did not turn up in the equality (3.2.15). Because of the orbifolding procedure the coset \(so(k+1)_n / so(k)_n\) is more difficult to deal with in full generality than those we discussed before. Therefore we will just look at the first three examples.

**\(k = 1\):** One observes that \(so(2)_n / so(1)_n \cong U(1)\) and that the first unitary minimal models of \(WD_n\) and \(WB(0, \frac{n-1}{2})\) all have \(c = 1\). Therefore, the RCFTs related to \(k = 1\) boil down to the classification of \(c = 1\) theories [62, 48]. The r.h.s. of (3.2.15) is given by the
orbifold branch of the $c = 1$ RCFTs [62] leading to a symmetry algebra of type $\mathcal{W}(2, 4, \frac{d}{2})$ with $d \in \mathbb{N}$. Notice that at least for the first members of the series $\mathcal{WD}_{\pm}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ truncate to an algebra of type $\mathcal{W}(2, 4, \frac{n}{2})$ for $c = 1$ [16]. The $\mathbb{Z}_2$ orbifold of these algebras is again a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, \frac{d}{2})$. However, the spin content of this algebra depends on the level $n$ in (3.2.15), i.e. $d = d(n)$. In this respect the case $k = 1$ is different from the cases $k > 1$.

$k = 2$: For the unifying $\mathcal{W}$-algebra of the second unitary minimal models we can make use of the fact that $\widehat{so}(3)_n/so(2)_n \cong sl(2, \mathbb{R})_{2n}/U(1)$. This coset has been extensively discussed in section 2.1.1. In particular, we know that the set of spins of the generators is not a subset of those of the Casimir algebra $\mathcal{WD}_{\pm}$. Therefore, we definitely have to take the $\mathbb{Z}_2$ orbifold on the r.h.s. of (3.2.15). This orbifold has been argued in section 2.2.1. to lead to a $\mathcal{W}(2, 4, 6, 8, 10)$. Note that the structure constants $C_{14}$ in appendix A and that of $\mathcal{WD}_{\pm}$ are indeed equal for $k = 2n$, i.e.

$$c = \frac{2n - 1}{n + 1}$$  \hspace{1cm} (3.2.16)

(compare [17]). Since for $k = 2$ the minimal models on the r.h.s. of (3.2.15) are orbifolds of $\mathbb{Z}_{2n}$ parafermions, one can easily look at the first few minimal models on both sides of (3.2.15) (for the relation between $\mathbb{Z}_{2n}$ parafermions and $\mathcal{WD}_{\pm}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ see also [2]). One observes that already the $\mathbb{Z}_{2n}$ parafermionic models contain more fields than the second unitary minimal models of $\mathcal{WD}_{\pm}$. This is a first argument that we also have to take a $\mathbb{Z}_2$ orbifold of the l.h.s. of (3.2.15). Furthermore, one can also examine the structure constants of the $\mathcal{W}(2, 4, 6, 8)$ that corresponds to $\mathcal{WD}_{2}$. From their explicit expressions [63] one concludes that for no positive value of the central charge $c$ any of the two fields with conformal dimension 4 drops out. This is a further strong argument to take the $\mathbb{Z}_2$ orbifold also on the l.h.s. of (3.2.15).

$k = 3$: Finally we look at the third unitary minimal models. Observing that $\widehat{so}(4)_n/so(3)_n \cong sl(2, \mathbb{R})_{n} \oplus sl(2, \mathbb{R})_{n}/sl(2, \mathbb{R})_{2n}$ leads us to a coset which we have already discussed in section 2.1.3. Like for $k = 2$ we know that the set of spins of the generators is not a subset of those of $\mathcal{WD}_{\pm}$ showing once again that a $\mathbb{Z}_2$ orbifold has to be taken on the r.h.s. of (3.2.15). In section 2.2.1. we have argued that this orbifold leads to an algebra of type $\mathcal{W}(2, 4, 6, 8, 10, 12, 14, 16, 18)$. Again, we can check equality of the structure constants $C_{14}$ of $\mathcal{WD}_{\pm}$ with (2.1.52) at $\kappa = \mu = n$, i.e.

$$c = \frac{3n^2}{(n + 2)(n + 1)}. \hspace{1cm} (3.2.17)$$

From these examples we conjecture for the spin content of $\text{Orb}(\widehat{so}(k+1)_n/\widehat{so}(k)_n)$ the following:

$$\text{Orb} \left( \frac{\widehat{so}(k+1)_n}{\widehat{so}(k)_n} \right) \cong \mathcal{W}(2, 4, \ldots, k(k + 3)). \hspace{1cm} (3.2.18)$$

This spin content of the coset can also be obtained by looking at the Kac-determinant [17].
3.2.3. Realization of $\mathcal{WD}_{-n}$ as diagonal $sp(2n)$ cosets

In [17] we proposed unifying $\mathcal{W}$-algebras for the minimal models of $\mathcal{WC}_{n}$ with central charge $c_{\mathcal{C}_{n}}(n + k + 1, 2n + 2k + 1)$. From the study of the Kac-determinant we conjectured the following spin content for the unifying algebras

$$\mathcal{WD}_{-k} \cong \mathcal{W}(2, 4, \ldots, 2k(k + 2)).$$  \hfill (3.2.19)

Our aim is to give an explicit coset realization of these algebras. Therefore, one has to pose the question if one can make sense of $\mathcal{D}_{-n} = so(-2n)$. Indeed negative-dimensional groups $SU(-n), SO(-2n), Sp(-2n)$ have been introduced in the context of representation theory of the classical groups $SU(n), SO(2n), Sp(2n)$ (see e.g. [21]). There exist striking relations in representation theory which can be explained in a more natural way by ‘analytic continuations’ in $n$. For example the dimension formula of an irreducible representation of $SO(2n)$ equals up to a sign the dimension formula of $Sp(2n)$ for the transposed Young tableau upon the substitution $n \rightarrow -n$. Furthermore, the $p$th order Casimir of $SO(2n)$ in the totally antisymmetric rank-$r$ tensor representation equals up to a sign the $p$th order Casimir of $Sp(2n)$ in the totally symmetric rank-$r$ tensor representation upon the substitution $n \rightarrow -n$ [21]. These relations arise naturally if one defines the negative-dimensional groups via $SO(-2n) \cong Sp(2n)$ and $Sp(-2n) \cong SO(2n)$. The overbar means the interchange of symmetrization and antisymmetrization.

We conclude that $\mathcal{D}_{-n} = so(-2n)$ is related to $\mathcal{C}_{n} = sp(2n)$. From the Sugawara central charge one can establish the identification: $so(-2n)_{\kappa} \leftrightarrow sp(2n)_{-\kappa}$. This leads us to the study of the general coset $sp(2n)_{\kappa} \oplus sp(2n)_{\mu}/sp(2n)_{\kappa + \mu}$. The central charges for the minimal models read

$$c_{\kappa, \mu}(n) = \frac{n\mu\kappa(2n + 1)(2n + 2 + \mu + \kappa)}{(\mu + \kappa + n + 1)(\mu + n + 1)(\kappa + n + 1)}.$$  \hfill (3.2.20)

Specification to $\mu = -\frac{1}{2}$ yields the formula

$$c(n, \kappa) = -\frac{\kappa n(2\kappa + 4n + 3)}{(\kappa + n + 1)(2\kappa + 2n + 1)}.$$  \hfill (3.2.21)

This coincides with the $c_{\mathcal{C}_{n}}(\kappa + n + 1, 2\kappa + 2n + 1)$ minimal models of the $\mathcal{WC}_{n}$ Casimir algebras. Furthermore, looking at formula eq. (2.15) of [17] (this equation describes the truncation of $\mathcal{WC}_{n}$ to $\mathcal{WD}_{m}$) and substituting $m$ by $-m$ we recover eq. (3.2.21). These facts strongly indicate that the $\mathcal{WD}_{-n}$ algebras can be realized as the diagonal $sp(2n)$ coset. Further confirmation for this relationship comes from the comparison of the coupling constant $C_{44}^{4}$ of $\mathcal{WD}_{-n}$ (obtained by analytic continuation of the coupling constant of $\mathcal{WD}_{n}$) with $C_{44}^{4}$ of the diagonal coset (see section 3.2.5.).

The next step to be carried out is the determination of the vacuum character of the diagonal $sp(2n)$ coset. As before, it is useful to study the coset $sp(2n)_{-\frac{1}{2}}/sp(2n)$ first, since the former one can be viewed as a deformation of it. To determine the vacuum character of $sp(2n)_{-\frac{1}{2}}$ one uses the realization of $sp(2n)_{-\frac{1}{2}}$ by $n$ commuting $(\beta, \gamma)$ systems. This is due
to the fact that the vacuum module of $\hat{sp}(2n)\frac{1}{2}$ is freely generated in terms of the $(\beta, \gamma)$ systems which is not true for the canonical choice in terms of the currents themselves. We start with $n$ bosonic ghost-antighost fields $(\beta_i, \gamma_i)$

$$\beta_i(z) \gamma_i(w) = \frac{1}{z - w} + \text{reg.} \quad \text{(3.2.22)}$$

Define the currents $H_i = \beta_i \gamma_i$ satisfying the OPEs

$$H_i(z) H_j(w) = -\frac{\delta_{ij}}{(z - w)^2} + \text{reg.} \quad \text{(3.2.23)}$$

The ghost fields $\beta_i, \gamma_i$ have charge $\pm 1$ with respect to the currents $H_i$:

$$H_i(z) \gamma_j(w) = \delta_{ij} \frac{1}{z - w} + \text{reg.} \quad H_i(z) \beta_j(w) = -\delta_{ij} \frac{1}{z - w} + \text{reg.} \quad \text{(3.2.24)}$$

The realization of $\hat{sp}(2n)\frac{1}{2}$ in the Cartan-Weyl basis $\{H_i(z), E_{\alpha}(z)\}$ is given by

| root vector $\alpha$ | current $E_{\alpha}(z)$ |
|----------------------|-------------------------|
| $2e_i$               | $\gamma_i \gamma_i$    |
| $-2e_i$              | $-\beta_i \beta_i$     |
| $e_i + e_j$          | $\gamma_i \gamma_j$    |
| $e_i - e_j$          | $-\beta_i \beta_j$     |
| $-e_i + e_j$         | $\beta_i \gamma_j$     |

The branching functions of the coset $\hat{sp}(2n)\frac{1}{2}/sp(2n)$ are determined by the decomposition of the highest weight modules $\chi_{\bar{\Lambda}}(q, \bar{\theta})$ of $\hat{sp}(2n)\frac{1}{2}$ into highest weight modules $\chi_{\bar{\Lambda}}(\bar{\theta})$ of $sp(2n)$:

$$\chi_{\bar{\Lambda}}(q, \bar{\theta}) = \sum_{\lambda} b_{\bar{\Lambda}}^{\lambda}(q) \chi_{\bar{\Lambda}}(\bar{\theta}). \quad \text{(3.2.25)}$$

For the vacuum character one has the identity ($\kappa = -\frac{1}{2}$) (see eq. (2.1.55)):

$$b_{0}^{\kappa \Lambda_0}(q) = \sum_{w \in W} \epsilon(w) c_{w* \rho + \kappa \Lambda_0}(q) q^{\frac{1}{2} |w*\rho|^2} \quad \text{(3.2.26)}$$

with $\rho = \sum_{i=1}^{n} (n - i + 1) e_i$. We have to determine the string functions $c_{w* \rho + \kappa \Lambda_0}(q)$. One obtains from (2.1.7a) with $z^2 = e^{2\pi i \theta} q^\frac{1}{2}$ the following identity

$$\frac{1}{\prod_{n \geq 0}(1 - e^{2\pi i \theta} q^{n+\frac{1}{2}})(1 - e^{-2\pi i \theta} q^{n+\frac{1}{2}})} = \sum_{m \in \mathbb{Z}} q^{\frac{m}{2} \phi_m(q)} e^{2\pi i \theta m} \prod_{n \geq 1} (1 - q^n)^2. \quad \text{(3.2.27)}$$

$^{12}$ If $\{e_1, \ldots, e_n\}$ are orthonormal unit vectors in the standard euclidean $\mathbb{R}^n$, the root system of $C_n = sp(2n)$ is realized by the vectors $\pm 2e_i$ and $\pm(e_i \pm e_j)$.  

45
Using the realization of $\widehat{sp(2n)}_{-\frac{1}{2}}$ in terms of the $(\beta, \gamma)$ systems one obtains for the vacuum character of $\widehat{sp(2n)}_{-\frac{1}{2}}$:

$$
\prod_{i=1}^{n} \prod_{k \geq 0} \left( 1 - e^{2\pi i \theta_i q^k} \right) \left( 1 - e^{-2\pi i \theta_i q^{k+\frac{1}{2}}} \right) = \sum_{n \in \mathbb{Z}^n} q^{\frac{3n}{4\pi}} \chi_{\frac{1}{2} \Lambda_0}^{\frac{1}{2}}(q) q^{\frac{\lambda_1 + \ldots + \lambda_n}{2} + \lambda_n} \phi_{\lambda_1}(q) \ldots \phi_{\lambda_n}(q) e^{2\pi i \bar{\theta} \bar{X}}.
$$

Thus the string functions of $\widehat{sp(2n)}_{-\frac{1}{2}}$ are given by

$$
c_{\bar{X}}^{\frac{1}{2} \Lambda_0}(q) = q^{-|\bar{X}|^2} = \frac{q^{\frac{3n}{4\pi}}}{\eta^{2n}(q)} q^{\frac{\lambda_1 + \ldots + \lambda_n}{2} + \lambda_n} \phi_{\lambda_1}(q) \ldots \phi_{\lambda_n}(q).
$$

Inserting this into eq. (3.2.26) yields the explicit form of the vacuum character of the $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$ coset. In table 5 below we present the first few examples and give a conjecture for the spin content in the general case. The spin content of $\mathcal{WD}_m$ is indeed compatible with the truncations of $\mathcal{WC}_m$ Casimir algebras [17].

| n  | vacuum character |
|-----|------------------|
| 1   | $b^0_1(q) = \eta(q)^2 (\phi_0(q) - q \phi_2(q))$ |
|     | $b^0_0(q) - \chi_{2,4,6}(q) = -q^{11} - 2q^{12} - 3q^{13} + O(q^{14})$ |
| 2   | $b^0_0(q) = \eta(q)^4 (\phi_0 \phi_2 - q \phi_1^2 + 2q^2 \phi_3 \phi_1 - q^2 \phi_4 \phi_0 + q^3 \phi_4 \phi_2 - q^3 \phi_3^2)$ |
|     | $b^0_0(q) - \chi_{2,4,6,..,16}(q) = -q^{21} - 2q^{22} + O(q^{23})$ |
| 3   | $b^0_0(q) - \chi_{2,4,6,..,30}(q) = -q^{35} - 2q^{36} + O(q^{38})$ |
| 4   | $b^0_0(q) - \chi_{2,4,6,..,48}(q) = -q^{53} - 2q^{54} + O(q^{55})$ |
| n   | $b^0_0(q) - \chi_{2,..,2n(n+2)}(q) = -q^{2n(n+2)+5} - 2q^{2n(n+2)+6} + O(q^{2n(n+2)+7})$ |

Table 5: Vacuum character for $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$

3.2.4. Minimal models of $\mathcal{WD}_m$

It has been argued in [8] that the coset algebra $sl(2, \mathbb{R})_\kappa \oplus sl(2, \mathbb{R})_{-\frac{1}{2}} / sl(2, \mathbb{R})_{\kappa-\frac{1}{2}} \cong \mathcal{W}(2,4,6)$ [9] can be regarded in a formal way as an algebra $\mathcal{WD}_m$ in the following sense: Its structure constants are given by those of the orbifold of $\mathcal{WD}_m$ by setting $m = -1$. Even its minimal models could be deduced with the help of some known examples [16] from those of the $\mathcal{WD}_m$-algebras.

We will now try to continue the minimal models of $\mathcal{WD}_m$ to negative values of $m$ beyond $m = -1$. The central charges of the minimal models of the $\mathcal{WD}_m$-algebras are given by

$$
c_{\mathcal{D}_m}(p,q) = m \left( 1 - 2(m-1)(2m-1) \frac{(p-q)^2}{pq} \right)
$$

and following [8] we have to make certain assumptions for the allowed values of $\mathcal{WD}_m$: $q = p + 1; p = 2n + 2x + 1$ odd. The value $n = 0$ corresponds to the trivial model at $c = 0$.  

46
implying \( x = m \). This leads to the following ansatz for the central charge of the minimal models of \( \mathcal{WD}_{-m} \)

\[
c_{\mathcal{D}_{-m}}(n) = -\frac{mn(3+4m+2n)}{(1+m+n)(1+2m+2n)}.
\] (3.2.31)

This central charge equals the one of the \( \hat{sp}(2n) \) cosets eq. (3.2.21). The dimensions of the highest weights can be obtained as follows: Starting from the dimensions for the \( \mathcal{WD}_m \)-algebra \[64\]

\[
h(\vec{l}, \vec{l}') (n) = \frac{\sum_{r=1}^{m} x_r + 2}{2p(p+1)} \sum_{r<s} x_r x_s.
\] (3.2.32)

where \( x_r = l_r(p+1) - l'_r(p-1) \) with positive integers \( l_r \) and \( l'_r \). The fundamental weights obey

\[
\omega_r \cdot \omega_s = r \quad 1 \leq r \leq s \leq m-2
\]
\[
\omega_r \cdot \omega_{m-1} = \omega_r \cdot \omega_m = \frac{r}{2} \quad 1 \leq r \leq m-2
\]
\[
\omega_{m-1} \cdot \omega_{m} = \frac{m-2}{4} \quad \omega_{m-1}^2 = \omega_{m}^2 = \frac{m}{4}.
\] (3.2.33)

The experience with the algebra \( \mathcal{W}(2,4,6) \) tells us that we have to extend the summation in (3.2.32) to infinity and take \( l_r = l'_r = 1 \) (i.e. \( x_r = 0 \)) for all \( r > n \) and therefore we obtain by inserting the fundamental weights eq. (3.2.33) into eq. (3.2.32) and replacing \( m \rightarrow -m \), \( p \rightarrow 2n+2m+1 \)

\[
h_{\vec{l}, \vec{l}'} (n) = \frac{\sum_{r=1}^{n} r(x_r - 2m - 1 - r)x_r + 2 \sum_{r<s} r x_r x_s}{4(n+m+1)(2n+2m+1)}.
\] (3.2.34)

On the \( l_r, l'_r \) we have to impose the additional constraints

\[
\sum_{r=1}^{n} (l_r + l'_r - 2) \leq m + 1, \quad 1 \leq l_r, l'_r \leq m + 1.
\] (3.2.35)

Using eq. (D.2) for the minimal models of the Casimir algebras we have checked in a few cases that the minimal models of \( \mathcal{WD}_{-n} \) coincide with those minimal models of \( \mathcal{WC}_m \) that one expects from eq. (2.15) of \[17\].

From the realization of \( \mathcal{WD}_{-m} \) in terms of the diagonal \( \hat{sp}(2n) \) coset it should be possible to check these formulae with representation theory of Kac-Moody algebras in the way outlined in section 2.1.3.

\[13\) The formula in [64] contains a misprint.

47
3.2.5. The coset $\hat{g}_k/g$ for a simple Lie algebra $g$

In order to confirm the above identifications we compute a general structure constant for the cosets $\hat{g}_k/g$. For general simple $g$ we have the following invariants at dimension 2 and 4:

\[ \text{dim} = 2: \quad L = \sum_{i,j} \frac{g_{ij}}{(k + h^\vee)} J^i J^j \]

\[ \text{dim} = 4: \quad LL, \quad \partial^2 L, \quad V_4 = \sum_{i,j} \frac{g_{ij}}{(k + h^\vee)} J^i \partial^2 J^j \]  \hspace{1cm} (3.2.36)

where $g_{ij}$ is the metric of $g$ and $h^\vee$ the dual Coxeter number and the central charge is given by $c = (k \dim g)/(k + h^\vee)$. These are all independent invariants up to conformal dimension 4 for $sl(2)$ with arbitrary level $k$, for $sl(n), so(2n), so(2n+1)$ with level $k = 1$ and for $sp(2)$ with level $k = -\frac{1}{2}$ whereas one has to take into account additional invariants in the general case. In these special cases where $V_4$ is the only new invariant at scale dimension 4 one can construct a primary field $W_4$ out of $V_4$:

\[ W_4 = V_4 - \frac{3}{5} \partial^2 L - \frac{24}{5c + 22} LL. \]  \hspace{1cm} (3.2.37)

It is now relatively easy to compute the structure constant $C_{44}^4$ for this coset $\mathcal{W}$-algebra. For this purpose it is sufficient to consider the 4th-order pole of $W_4 \ast W_4$. Here we find

4th-order pole of $\quad V_4 \ast V_4 = 12(k + h^\vee) h^\vee \partial^2 L + 4 \left(18k + 11h^\vee\right) (k + h^\vee) V_4$  \hspace{1cm} (3.2.38)

and from eq. (3.2.37)

\[ \text{4th-order pole of} \quad W_4 \ast W_4 = 4 \left(\frac{-118h^\vee + 19ch^\vee + 36k + 54ck}{(22 + 5c)(k + h^\vee)}\right) W_4 + \right. \]

\[ 672 \left(\frac{-10h^\vee + ch^\vee + 12k + 6ck}{(22 + 5c)^2(k + h^\vee)}\right) LL + 24 \left(\frac{-10h^\vee + ch^\vee + 12k + 6ck}{(22 + 5c)(k + h^\vee)}\right) \partial^2 L. \]  \hspace{1cm} (3.2.39)

Since for a general spin-4 field with OPE $W_4 \ast W_4 = d_{4,4} I + \tilde{C}_{44}^4 W_4 + ...$ the 4th-order pole has the form

\[ \tilde{C}_{44}^4 W_4 + \frac{168 d_{4,4}}{c(22 + 5c)} LL + \frac{6 d_{4,4}}{5c} \partial^2 L \]  \hspace{1cm} (3.2.40)

we can read off

\[ d_{4,4} = \frac{4 c (-10h^\vee + ch^\vee + 12k + 6ck)}{(22 + 5c)(k + h^\vee)} \]  \hspace{1cm} (3.2.41)

\[ \tilde{C}_{44}^4 = 4 \frac{-118h^\vee + 19ch^\vee + 36k + 54ck}{(22 + 5c)(k + h^\vee)} \]

and for the normalized structure constant

\[ (C_{44}^4)^2 = \frac{(\tilde{C}_{44}^4)^2 c}{4d_{4,4}} = \frac{(-118h^\vee + 19ch^\vee + 36k + 54ck)^2}{(22 + 5c)(k + h^\vee)(-10h^\vee + ch^\vee + 12k + 6ck)}. \]  \hspace{1cm} (3.2.42)
Table 6: Structure constant $C_{44}^4$ for some $\hat{g}/g$

For $g = so(2n)$ and $k = 1$ we recover the algebra $\mathcal{WD}_n$ at $c = n$. For $g = so(2n + 1)$ and $k = 1$ we find the structure constant for the bosonic projection of $\mathcal{WB}(0, n)$ at $c = n + \frac{1}{2}$.

Both solutions are in agreement with the general structure constant for $\mathcal{WD}_n$ and the bosonic projection of $\mathcal{WB}(0, n)$ given in [8]. We should mention that for $g = sl(2)$ and arbitrary level $k$ we recover eq. (2.1.47).

For the coset $sp(2n)_k \oplus sp(2n)_{-\frac{1}{2}}/sp(2n)$ we obtain an identical expression for $(C_{44}^4)^2$ as for $\mathcal{WD}_n$ if we replace $n$ by $-n$. As in [9] one can deform this algebra to generic central charges giving rise to the coset $sp(2n)_k \oplus sp(2n)_{-\frac{1}{2}}/sp(2n)_{-\frac{1}{2}}$ which we formally identified with the algebra $\mathcal{WD}_{-n}$.

4. Conclusion

In this paper we studied various examples of quantum $\mathcal{W}$-algebras belonging to a new class of deformable $\mathcal{W}$-algebras with infinitely nonfreely generated classical limits which showed up recently [9]. By explicit calculation of the operator product expansions we provided evidence that the coset algebras $sl(2, \mathbb{R})/\hat{U}(1)$ and $\mathcal{W}_{2,1}^{sl(3)}/\hat{U}(1)$ are finitely generated on the quantum level as one can infer e.g. from character arguments. The mechanism which prohibits the additional generators in the quantum case is a cancellation due to normal ordering of the classical relations. From comparison of structure constants we obtained that $sl(2, \mathbb{R})/\hat{U}(1)$ as well as the isomorphic coset $SV I R(N = 2)/\hat{U}(1)$ provide a realization of the special nonfreely generated $\mathcal{W}(2, 3, 4, 5)$ found earlier [7]. We performed calculations showing that the fourth special solution of $\mathcal{W}(2, 4, 6)$ [5] is realized as the diagonal coset $sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_{-\frac{1}{2}}/sl(2, \mathbb{R})_{-\frac{1}{2}}$. We presented also the self-coupling constant of the unique primary spin 4 field in the general coset $sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\mu /sl(2, \mathbb{R})_{k+\mu}$.

Starting from the representation theory of the cosets realizing $\mathcal{W}(2, 3, 4, 5)$ and $\mathcal{W}(2, 4, 6)$ we collected further evidence that they are unifying objects of special series of minimal models of Casimir algebras, e.g. $\mathcal{W}(2, 3, 4, 5)$ ‘unifies’ the first unitary model of $\mathcal{WA}_n$. These aspects were already discussed in an earlier work [17] where we put the emphasis on the truncations of the Casimir algebras which they suffer at the particular values of the central charge $c$. From these truncations infinitely many unifying algebras can be proposed. For some of them we were able to present coset realizations thereby generalizing the level-rank-duality of coset pairs (see e.g. [1, 18, 19]). For the Casimir algebras $\mathcal{WA}_{n-1}$ we conjectured coset realizations for all unifying $\mathcal{W}$-algebras predicted in table 1 of [17]. Further important examples are the $\mathcal{WD}_{-n}$ algebras which arise from the symplectic cosets $sp(2n)_k \oplus sp(2n)_{-\frac{1}{2}}/sp(2n)_{-\frac{1}{2}}$ and $sp(2n)_{-\frac{1}{2}}$. They are unifying algebras for $\mathcal{WC}$ minimal models. The coset realizations of unifying algebras known to us are collected in table 7.
Finally, we investigated orbifolds of quantum \( W \)-algebras in a purely algebraic manner. They behave similarly to the coset models discussed above, i.e. they belong to the class of finitely nonfreely generated \( W \)-algebras. They occur also naturally in the context of unifying \( W \)-algebras. We stress the fact that in contrast to the orbifold of a classical Casimir \( W \)-algebras the orbifold of a quantum Casimir \( W \)-algebra does not contain a Casimir subalgebra. In examples we discussed some properties of the vacuum preserving algebra (VPA) and the BW classical limit of nonfreely generated quantum \( W \)-algebras. In contrast to algebras of the Drinfeld-Sokolov class the VPA does not yield a finitely nonfreely generated \( W \)-algebra. Analogously, the classical limits are not finitely generated any more.

The coset realization of unifying \( W \)-algebras gives a so far unknown coset realization for some of the minimal models of the non-simply laced Casimir \( W \)-algebras \( W_B n \) and \( W_C n \). A further interesting observation concerning minimal models of diagonal cosets is the following conjecture: At least one of the levels \( \kappa, \mu \) has to be an integer if \( \hat{g}_{\kappa} \oplus \hat{g}_{\mu}/\hat{g}_{\kappa+\mu} \) has a minimal model. Note that the only counterexample to this conjecture we know of which was presented in [47] is incorrect (see end of section 2.1.3.).

There are several interesting open question to answer in the future. It has been conjectured in [17] that all minimal models of Casimir algebras related to the classical Lie algebras can also be obtained as minimal models of unifying \( W \)-algebras. However, the existence of all these unifying \( W \)-algebras is not yet firmly established. Secondly, one could address the question whether all these algebras can be realized as cosets of quantum DS reductions. Finally, the representation theory of unifying \( W \)-algebras as well as the representation theory of quantum DS reductions related to nonprincipal \( sl(2) \) embeddings has not been worked out so far. However, in the case of \( sl(\hat{2}, \mathbb{R})_{\kappa}/U(1) \) and \( sl(\hat{2}, \mathbb{R})_{\kappa} \oplus sl(\hat{2}, \mathbb{R})_{-\kappa}/sl(\hat{2}, \mathbb{R})_{-\frac{1}{2}} \) one can see from the considerations in this paper that all their minimal models also arise from identifications with Casimir algebras. It would be interesting to know if all minimal models of all unifying \( W \)-algebras are also minimal models of the corresponding Casimir algebras. If this should be true, one could hope to reconstruct the representation theory of \( W \)-algebras obtained by DS reduction for a nonprincipal embedding using the fact that the coset by the Kac-Moody subalgebra that survives the reduction is a unifying \( W \)-algebra for Casimir \( W \)-algebras.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Casimir algebra} & \text{central charge} & \text{coset realization} & \text{dimensions of} & \text{dimension of} \\
& & \text{of unifying algebra} & \text{simple fields} & \text{first null field} \\
\hline
W_{A_{n-1}} & c_{A_{n-1}}(n+k, n+r) & \frac{W_{r-1,k}^{(r)}}{W_{r-1,k}} & 2, 3, \ldots, kr + r + k & kr + r + k + 3 \\
WB_n & c_{B_n}(2n+k-1, 2n+1) & \text{(Orb)} \left( \frac{so(k) \oplus so(k)}{so(k)_{n+1}} \right) & 2, 4, \ldots, 2k & 2k + 4 \\
WC_n & c_{C_n}(n+k+1, 2n+2k+1) & \frac{sp(2k)_{n} \oplus sp(2k)_{n+1}}{sp(2k)_{n}} & 2, 4, \ldots, 2k^2 + 4k & 2k^2 + 4k + 5 \\
\text{Orb } (WD_n) & c_{D_n}(2n+k-2, 2n+k-1) & \text{Orb } \frac{so(k+1)_{2n}}{so(k)_{2n}} & 2, 4, \ldots, k^2 + 3k & k^2 + 3k + 4 \\
\hline
\end{array}
\]

Table 7: Coset realizations of unifying \( W \)-algebras
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Appendix A: The simple fields with spin 3, 4, 5 of $sl(2,\mathbb{R})/U(1)$

\[ W_3 = -6k^2 J^+ \partial J^- - 12k J^o J^+ J^- + 4J^o J^o J^o + 6k^2 \partial J^+ J^- + 6k \partial J^o J^o + k^2 \partial^2 J^o \]

\[ W_4 = 48k (3 + 2k) (17 + 16k)^{-1} (6 (5 - 6k) k^2 J^+ J^+ J^- J^- + 12k^2 (1 + k + k^2) J^+ \partial^2 J^- + 12k^2 (11 + 5k) J^o J^+ \partial J^- + 12k (6 + 11k) J^o J^o J^+ J^- - 3 (6 + 11k) J^o J^o J^o J^o - 12k^2 (11 + 5k) J^o J^+ \partial J^- + 12k^2 (-8 + 3k - 3k^2) \partial J^+ \partial J^- + 12k^2 (-5 + 6k) \partial J^o J^+ J^- - 6k (6 + 11k) J^o J^o J^o J^o + 3k (6 - 5k^2) \partial J^o \partial J^o + 12 (-3 + k) (-2 + k) k^2 \partial^2 J^+ J^- - 12k (1 + k + k^2) \partial^2 J^o J^o - k^2 (1 + k + k^2) \partial^3 J^o \]

\[ W_5 = -288k^2 (3 + 2k) (4 + 3k) (107 + 64k)^{-1} (60 (7 - 10k) k^3 J^+ J^+ \partial J^- J^- + 20k^3 (5 + 3k + k^2) J^+ \partial^3 J^- + 120 (7 - 10k) k^2 J^o J^+ J^+ J^- J^- + 60k^2 (8 + 19k + 3k^2) J^o J^o \partial^2 J^- + 60k^2 (64 + 17k) J^o J^+ J^+ J^- + 120k^2 (12 + 19k) J^o J^o J^+ J^- - 24 (12 + 19k) J^o J^o J^o J^o - 60k^2 (64 + 17k) J^o J^o J^+ J^- - 480k^2 (5 + 3k + k^2) J^o \partial J^+ J^+ J^- + 180 (k - 4) (k - 3) k^2 J^o \partial^2 J^+ J^- + 60k^3 (10k - 7) \partial J^+ J^+ J^- J^- + 60k^3 (4k - 2k^2 - 17) \partial J^+ \partial^2 J^- + 180k^2 (3k^2 - 4) \partial J^o J^+ \partial J^- + 240k^2 (10k - 7) \partial J^o J^o J^+ J^- - 60k^2 (12 + 19k) \partial J^o J^o J^o J^o + 60k^2 (14k - 11k^2 - 12) \partial J^o \partial J^+ J^- + 180k^2 (4 - 3k^2) \partial J^o J^o J^o J^o + 120 (k - 4) (k - 3) k^3 \partial^2 J^+ \partial J^- + 60k^2 (8 - 7k + 4k^2) \partial^2 J^o J^+ J^- - 30k (16 + 12k + 7k^2) \partial^2 J^o J^o J^o J^o + 30k^2 (4 - 3k^2) \partial^2 J^o \partial J^o J^o + 20 (3 - k) (k - 4) k^3 \partial^3 J^+ J^- - 20k^2 (5 + 3k + k^2) \partial^3 J^o J^o - k^3 (5 + 3k + k^2) \partial^4 J^o \]

\[ d_{3,3} = 48 (k - 2) (k - 1) k^3 (4 + 3k) \]

\[ d_{4,4} = 331776 (k - 3) (k - 2) (k - 1) k^6 (1 + 2k) (3 + 2k)^2 (4 + 3k) (17 + 16k)^{-1} \]

\[ d_{5,5} = 477757440(k - 4)(k - 3)(k - 2)(k - 1) k^9 (1 + 2k)(3 + 2k)^2 (4 + 3k)^2 (8 + 5k)(107 + 64k)^{-1} \]

\[ \tilde{C}_{4,4}^4 = 10368 k^3 (3 + 2k) (4k^3 - 15k^2 - 33k - 4) (17 + 16k)^{-1} \]

\[ \tilde{C}_{4,5}^5 = 25920 k^3 (3 + 2k) (4 + 3k) (32k^3 - 236k^2 - 535k - 125) ((17 + 16k) (107 + 64k))^{-1} \]

Appendix B: Some structure constants of $W_{2,1}^{sl(3)}/U(1)$

\[ d_{3,3} = -(1 + k)^2 (3 + k) (1 + 2k) (9 + 4k) (3 + 2k)^{-1} \]

\[ d_{4,4} = 48k^2 (1 + k)^2 (3 + k)^4 (1 + 2k)(5 + 3k)(9 + 4k)(12 + 5k)(3 + 2k)^{-2} (18 - 19k - 15k^2)^{-1} \]

\[ d_{5,5} = 480k^2 (1 + k)^2 (3 + k)^6 (2k - 1)(1 + 2k)(5 + 2k)^2 (5 + 3k)(9 + 4k)(12 + 5k)(3 + 2k)^{-3} (50 + 5k - 7k^2)^{-1} \]

\[ d_{6,6} = \frac{1600k^2 (1 + k)^2 (3 + k)^8 (-1 + 2k)(1 + 2k)(5 + 2k)^2 (5 + 3k)(9 + 4k)(12 + 5k)}{(3 + 2k)^4 (11 + 2k - k^2) (-50 + 5k - 7k^2)^2} \times \]

\[ (-57600 + 25260k + 67829k^2 - 3738k^3 - 19182k^4 + 2540k^5 + 4809k^6 + 882k^7) \]
\[ d_0 = \frac{80k^2(1 + k)^2(3 + k)^6(-1 + 2k)(1 + 2k)(5 + 2k)^2(5 + 3k)(9 + 4k)(12 + 5k)}{3(3 + 2k)^3(11 + 2k - k^2)(7k^2 - 5k - 50)(33 + 46k + 21k^2)^{-1}} \]

\[ C_{44}^4 = 24(3 + k)^2(216 + 186k + 823k^2 + 1184k^3 + 573k^4 + 90k^5)((3 + 2k)(-18 + 19k + 15k^2))^{-1} \]

\[ C_{44}^6 = \frac{4}{5} \]

\[ C_{45}^5 = 60(3 + k)^2(-10800 - 1740k - 2408k^2 - 14301k^3 - 5041k^4 + 4947k^5 + 3233k^6 + 510k^7) \]

\[ (3 + 2k)(-50 - 5k + 7k^2)(-18 + 19k + 15k^2) \]

\[ C_{45}^7 = \frac{2}{3} \]

\[ C_{46}^4 = \frac{160}{3}(3 + k)^4(2k - 1)(5 + 2k)^2(15k^2 + 19k - 18)(3 + 2k)^{-2}((k^2 - 2k - 11)(7k^2 - 5k - 50))^{-1} \]

\[ \times (576 - 132k - 143k^2 + 485k^3 + 351k^4 + 63k^5) \]

\[ C_{46}^6 = \frac{4}{3}(3 + k)^2((3 + 2k)(k^2 - 2k - 11)(7k^2 - 5k - 50)(15k^2 + 9k - 18))^{-1} \]

\[ (10254600 + 3254580k - 4772094k^2 + 5168399k^3 + 6444501k^4 - 267576k^5 - 1247560k^6 + 114411k^7 + 233577k^8 + 39690k^9). \]

**Appendix C: The primary spin 4 generator of \( sl(\hat{2}, \mathbb{R})_\kappa \oplus sl(\hat{2}, \mathbb{R})_\mu / sl(\hat{2}, \mathbb{R})_{\kappa + \mu} \)**

\[ \Phi^{(4)} := (3\mu + 11)\mu S_{0,0}(1,1) S_{0,0}(1,1) - 4 (3\mu + 11)(\kappa + 2) S_{0,0}(1,2) S_{0,0}(1,1) \]

\[ + (2\mu \kappa - 11\mu - 11\kappa - 22)(3\kappa + 4)(\mu - 1)^{-1} S_{0,0}(2,2) S_{0,0}(1,1) \]

\[ + (4\mu \kappa + 23\mu + 23\kappa + 76)(3\kappa + 4)(\mu - 1)^{-1} S_{0,0}(1,2) S_{0,0}(1,2) \]

\[ - 4(\mu + 2)(3\kappa + 11)(3\kappa + 4)(\kappa - 1)((3\mu + 4)(\mu - 1))^{-1} S_{0,0}(2,2) S_{0,0}(1,2) \]

\[ + (3\kappa + 11)(3\kappa + 4)(\kappa - 1)\kappa((3\mu + 4)(\mu - 1))^{-1} S_{0,0}(2,2) S_{0,0}(2,2) \]

\[ + (37\mu^2 k + 44\mu^2 + 37\mu k^2 + 192\mu k + 176\mu + 44k^2 + 176k + 176)(3\kappa + 4)((3\mu + 4)(\mu - 1))^{-1} S_{0,0,1}(1,2,2) \]

\[ + (37\mu^2 k + 44\mu^2 + 37\mu k^2 + 192\mu k + 176\mu + 44k^2 + 176k + 176)(\mu - 1)^{-1} S_{0,0,1}(2,1,1) \]

\[ + \frac{3}{2}(5\mu k + 4\mu + 5k^2 + 20\kappa + 8) \mu S_{1,1}(1,1) \]

\[ - (22\mu^2 k + 32\mu^2 + 22\mu k^2 + 147\mu k + 164\mu + 59k^2 + 236\mu + 200) \mu (2(\mu - 1))^{-1} S_{0,2}(1,1) \]

\[ + (5\mu k^2 - 23\mu k - 36\mu + 5k^3 + 5k^2 - 96\kappa - 112) S_{2,0}(1,2) \]

\[ - (5\mu^2 k - 5\mu^2 + 5\mu k^2 + 6\mu k - 38\mu - 5k^2 - 38\kappa - 56)(3\kappa + 4)(\mu - 1)^{-1} S_{1,1}(1,2) \]

\[ + (5(3\mu + 4)(\mu - 1))^{-1}((4\mu k + 23\mu + 23k + 76)(3\mu + 4)(3\mu k^2 + \mu k + 16\mu - 9k^2 - 23\kappa + 12) \]

\[ + (\mu + 2)(3\kappa + 11)(\kappa - 1)(-12\mu^2 k + 44\mu^2 + 60\mu k + 280\mu + 168\kappa + 384)) S_{0,2}(1,2) \]

\[ - (22\mu^2 k + 59\mu^2 + 22\mu k^2 + 147\mu k + 236\mu + 32k^2 + 164\mu + 200)(3\kappa + 4)(2(3\mu + 4)(\mu - 1))^{-1} S_{0,2}(2,2) \]

\[ + 3(5\mu^2 + 5\mu k + 20\mu + 4k + 8)(3\kappa + 4)(\kappa - 1)\kappa(2(3\mu + 4)(\mu - 1))^{-1} S_{1,1}(2,2). \]
Appendix D: Minimal models of Casimir $\mathcal{W}$-algebras

Let $\mathcal{K}$ be a simple Lie algebra of rank $l$ over $\mathbb{C}$. The rational models of the Casimir $\mathcal{W}$-algebra related to this Lie algebra have central charge

$$c_\mathcal{K}(p, q) = l - \frac{12}{pq} (q \rho - p \rho^\vee)^2 \quad p, q \text{ coprime, } \quad h^\vee \leq p \quad h \leq q \quad (D.1)$$

where $p$ and $q$ have to be chosen minimal, $h$ ($h^\vee$) denotes the (dual) Coxeter number of $\mathcal{K}$ and $\rho$ ($\rho^\vee$) denotes the sum of its (dual) fundamental weights $\lambda_i$ ($\lambda_i^\vee$). The conformal dimensions of the minimal model are given by [65]:

$$h_{\lambda, \nu^\vee} = \frac{1}{2pq} ((q \lambda - p \nu^\vee)^2 - (q \rho - p \rho^\vee)^2) \quad (D.2)$$

where $\lambda$ ($\nu^\vee$) lies in the (dual) weight lattice so that $\lambda = \sum_{i=1}^l l_i \lambda_i$, $\nu^\vee = \sum_{i=1}^l l_i^\vee \lambda_i^\vee$. $\lambda$ and $\nu^\vee$ have to satisfy $\sum_{i=1}^l l_i m_i \leq p - 1$, $\sum_{i=1}^l l_i^\vee m_i^\vee \leq q - 1$ where $m_i$ are the normalized components of the highest root $\psi$ in the directions of the simple roots $\alpha_i$, i.e. $\psi = \sum_{i=1}^l m_i \frac{\alpha_i}{\alpha_i^+}$. $m_i^\vee$ is given by $m_i^\vee = \frac{2}{\alpha_i} m_i$. Note that the set of conformal dimensions given by this condition has a symmetry so that all conformal dimensions of the minimal model occur with the same multiplicity in it (in the non-simply laced cases the multiplicity is just 2). For more details see [65, 44, 2, 66].

| Lie algebra | $(m_i)$ | $(m_i^\vee)$ |
|-------------|--------|-------------|
| $A_l$       | $(1, \ldots, 1)$ | $(1, \ldots, 1)$ |
| $B_l$       | $(1, 2, \ldots, 2, 1)$ | $(1, 2, \ldots, 2)$ |
| $C_l$       | $(1, \ldots, 1)$ | $(2, \ldots, 2, 1)$ |
| $D_l$       | $(1, 2, \ldots, 2, 1, 1)$ | $(1, 2, \ldots, 2, 1, 1)$ |
| $E_6$       | $(1, 2, 2, 3, 2, 1)$ | $(1, 2, 2, 3, 2, 1)$ |
| $E_7$       | $(2, 2, 3, 4, 3, 2, 1)$ | $(2, 2, 3, 4, 3, 2, 1)$ |
| $E_8$       | $(2, 3, 4, 6, 5, 4, 3, 2)$ | $(2, 3, 4, 6, 5, 4, 3, 2)$ |
| $F_4$       | $(1, 2, 3, 2)$ | $(2, 4, 3, 2)$ |
| $G_2$       | $(2, 1)$ | $(2, 3)$ |

Table 8: Values of $m_i$, $m_i^\vee$ for all simple Lie algebras [66].

Appendix E: The orbifold of the $N=1$ Super Virasoro algebra

The orbifold of this algebra has been proposed by P. Bouwknegt who also determined the field content [44]. Explicit constructions were carried out before in [67] and [25] and the classical version of this orbifold was discussed in [9]. In [9] it was also shown how two normal ordered analogues of classical relations ensure that the orbifold contains at least a closed $\mathcal{W}(2, 4, 6)$ as subalgebra and how a third relation gives rise to a first generic null field at scale dimension 10. However, neither a primary basis of generating fields nor the corresponding structure constants were computed in [9]. These calculations will be presented in this appendix.

For our calculations we adopt the (noncovariant) conventions for the extension of the Virasoro algebra by a spin $\frac{3}{2}$ fermion $G$ used e.g. in [68]. Orthogonalizing the $\mathbb{Z}_2$ invariant
normal ordered products $\mathcal{N}(G, \partial G)$ and $\mathcal{N}(G, \partial^3 G)$ with respect to other normal ordered products one obtains the composite primary fields in the projection as:

$$
\Phi^{(4)} = (5c + 22)\mathcal{N}(G, \partial G) - 17\mathcal{N}(L, L)
$$

$$
\Phi^{(6)} = 5(7c + 68)(2c - 1)(c + 24)\mathcal{N}(G, \partial^3 G) - 130(7c + 68)(2c - 1)\mathcal{N}(\mathcal{N}(G, \partial G), L) + 20(218c - 293)\mathcal{N}(\mathcal{N}(L, L), L) - 6(11c - 86)(c + 24)\mathcal{N}(L, \partial^2 L).
$$

(E.1)

The two point functions of these fields can easily computed to be

$$
d_{4,4} = \frac{1}{6} c(10c - 7)(5c + 22)(4c + 21)
$$

$$
d_{6,6} = 50c(14c + 11)(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)(c + 11).
$$

(E.2)

For the structure constants one obtains exactly those of the first solution (Set 1) in [5].

In order to demonstrate the notational advantages of quasi-primary fields we also present a quasi-primary analogue of the relations presented in appendix A of [9]. The quasi-primary version of (A.2) in [9] is:

$$
(192 - 31c)\mathcal{N}(G, \partial^5 G) + 90\mathcal{N}(\mathcal{N}(G, \partial G), \mathcal{N}(G, \partial G)) + \frac{81}{10}\mathcal{N}(L, \partial^4 L) + 154\mathcal{N}(\mathcal{N}(G, \partial G), \partial^2 L) - 420\mathcal{N}(\mathcal{N}(G, \partial^3 G), L) = 0.
$$

(E.3)

Clearly, $\mathcal{N}(G, \partial^5 G)$ can be expressed in terms of normal ordered products of invariant fields with lower dimension.

The commutators of the simple fields of an algebra of type $\mathcal{W}(2, 4, 6)$ involve fields up to dimension 10. Therefore, closure of the algebra is ensured if in addition to eq. (E.3) also $\mathcal{N}(G, \partial^7 G)$ can be expressed in terms of $L, \mathcal{N}(G, \partial G)$ and $\mathcal{N}(G, \partial^3 G)$. A suitable relation in the spirit of (A.4) in [9] can easily be established but is omitted.

Finally, we recall the discussion of singularities for special $c$-values that has been carried out in [25]. For $c \in \{-\frac{11}{14}, -\frac{68}{7}, \frac{1}{2}, -24, -11\}$ the field $\Phi^{(6)}$ is a null field before normalization and one obtains a $\mathcal{W}(2, 4)$. For $c \in \{\frac{7}{10}, -\frac{21}{4}\}$ also the field $\Phi^{(4)}$ is a null field before normalization and the bosonic sector of the Super Virasoro algebra coincides with the Virasoro algebra. It might seem that for $c = -\frac{22}{5}$ we obtain a $\mathcal{W}(2, 6)$ which is not the case. Here, singularities in the structure constants $C^X_{66}$ forces one to normalize $\Phi^{(6)}$ to zero. This shows that there is no consistent $\mathcal{W}$-algebra in the bosonic sector at $c = -\frac{22}{5}$ and one is left with the Virasoro algebra again.

Note that it is straightforward to derive the representations of this orbifold $\mathcal{W}$-algebra from the well-known representations of the $N = 1$ Super Virasoro algebra. In particular, it has been observed in [6] that the representation theory of $\mathcal{W}(2, 4)$ and the $N = 1$ Super Virasoro algebra at $c = -11$ is much the same which is clear keeping the above truncations of the orbifold in mind (see also [50]).
Appendix F: Generators and structure constants of the orbifold of $W(2,3)$

Using the procedure described in section 2.2.1, one calculates the composite primary fields of dimension 8 and 10 in the orbifold of $W(2,3)$ to be:

\[
\Phi^{(8)} = 262080(1919c - 642)\mathcal{N}(\mathcal{N}(L, L), L) \\
+ 1512(3965c^2 - 168232c + 1940316)\mathcal{N}(\mathcal{N}(L, \partial^2 L), L) \\
- 205(13c^2 - 1096c + 14556)(13c + 516)\mathcal{N}(L, \partial^4 L) \\
- 589680(5c + 22)(5c + 3)(3c + 46)\mathcal{N}(\mathcal{N}(W^{(3)}, W^{(3)}), L) \\
+ 2730(13c + 516)(5c + 22)(5c + 3)(3c + 46)\mathcal{N}(W^{(3)}, \partial^2 W^{(3)})
\]

\[
\Phi^{(10)} = 22522500(17c + 944)(11c + 232)(5c + 22)(c + 47)(c + 2)\mathcal{N}(W^{(3)}, \partial^4 W^{(3)}) \\
- 42702660000(11c + 232)(5c + 22)(c + 47)(c + 2)\mathcal{N}(\mathcal{N}(W^{(3)}, \partial^2 W^{(3)}), L) \\
- 187110000(17c + 944)(11c + 232)(7c - 130)(5c + 22)\mathcal{N}(\mathcal{N}(W^{(3)}, W^{(3)}), \partial^2 L) \\
+ 11675640000(32c + 25)(11c + 232)(5c + 22)\mathcal{N}(\mathcal{N}(W^{(3)}, W^{(3)}), L, L) \\
- 2431(3325c^3 - 642870c^2 + 39648336c - 320267008)(5c + 22)\mathcal{N}(L, \partial^6 L) \\
+ 26520(522225c^3 - 42458420c^2 + 1123770804c - 8347445152)\mathcal{N}(\mathcal{N}(L, L), \partial^4 L) \\
+ 1108800(708305c^2 + 132859254c - 2814883952)\mathcal{N}(\mathcal{N}(L, L), \partial^2 L) \\
- 20756736000(9421c - 13918)\mathcal{N}(\mathcal{N}(\mathcal{N}(L, L), L), L). 
\]

The two point functions of these fields turn out to be

\[
d_{8,8} = 178869600(13c + 516)(7c + 114)(5c + 186)(5c + 22)
\]

\[
(5c + 3)(5c - 4)(3c + 46)(c + 2)^2c 
\]

\[
d_{10,10} = 1168739664000000(17c + 944)(11c + 490)(11c + 232)(7c + 114)
\]

\[
(7c + 40)(5c + 22)(5c - 4)(c + 47)(c + 23)(c + 2)^2c . 
\]

After rescaling to standard normalization one obtains the following structure constants:

\[
(C_{66}^8)^2 = \frac{300(7c + 68)^2(5c + 186)(2c - 1)^2(c + 30)^2}{(13c + 516)(7c + 114)(5c + 22)(5c + 3)(5c - 4)(3c + 46)} 
\]

\[
C_{66}^6 C_{88}^6 = \frac{60(14c^2 + 915c^2 + 14758c - 22344)(21c^2 + 754c - 1176)(5c + 3)(3c + 46)}{(13c + 516)(7c + 114)(7c + 68)(5c + 22)(5c - 4)(2c - 1)} 
\]

\[
C_{66}^8 C_{88}^8 = \frac{70Q(7c + 68)(2c - 1)(c + 30)}{(13c + 516)^2(7c + 114)(5c + 22)(5c + 3)(5c - 4)(3c + 46)(c + 2)} 
\]

with

\[
Q := 4525c^6 + 5031245c^5 + 148843726c^4 + 1411010708c^3 + 1061946744c^2 - 1038009888c - 12099262464. 
\]

We omit the structure constants involving fields of scale dimension 10 or higher because they are complicated but not very illuminating.
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