Search for dark energy potentials in quintessence theory

Yusuke Muromachi, Akira Okabayashi, Daiki Okada, Tetsuya Hara and Yutaka Itoh

Department of Physics, Kyoto Sangyo University, Kyoto 603-8555, Japan

Abstract

The time evolution of the equation of state $w$ for quintessence models with a scalar field as dark energy is studied up to the third derivative ($d^3w/da^3$) with respect to scale factor $a$, in order to predict the future observations and specify the scalar potential parameters with the observables. The third derivative of $w$ for general potential $V$ was derived and applied to several types of potential. They are the inverse power-law ($V = M^{4+\alpha}/Q^\alpha$), exponential ($V = M^4 \exp(\beta M/Q)$), mixed ($V = M^{4+\gamma} \exp(\beta M/Q)/Q^\gamma$), cosine ($V = M^4(\cos(Q/f) + 1)$) and the Gaussian types ($V = M^4 \exp(-Q^2/\sigma^2)$), which are prototypical potentials for the freezing and thawing models. If the parameter number for a potential form is $n$, it is necessary to find at least for $n+2$ independent observations to identify the potential form and the evolution of scalar field ($Q$ and $\dot{Q}$). Such observations would be the values of $\Omega_Q, w, dw/da, \cdots$ and $dw^n/da^n$. From these specific potentials, we could predict the $n+1$ and higher derivative of $w$, $dw^{n+1}/da^{n+1}, \cdots$. Since four of the above mentioned potentials have two parameters, it is necessary to calculate the third derivative of $w$ for them to estimate the predict values. If they are tested observationally, it will be understood whether the dark energy could be described by the scalar field with this potential. At least it will satisfy the necessary conditions.
I. INTRODUCTION

The acceleration in the expansion of the universe was discovered by the intensive observations of the cosmology almost ten and several years ago \[1, 2\]. Although the dark energy was introduced to cause the late-time accelerated universe, the physical mechanism and origin have been poorly understood \[3, 4\]. Two theoretical viewpoints have been proposed so far. One is associated with modification of gravity. The other is associated with matter field theories \[5\]. From the latter viewpoint, we explore the possibilities of the scalar fields in quintessence models and study how relevant to the dark energy.

In the quintessence models, the scalar fields cause the time evolution of the universe. Since the scalar field theories involve \(n\) independent parameters, we notice that in principle \(n\) time derivatives of the equation of state with observable \(\Omega_Q\) and \(w\) are enough to specify the scalar potentials and to predict the higher derivatives. In this paper, we have carried out the calculations of the third derivative of the equation of state for five scalar potentials to identify the models and to predict the future observations. The parameters in the potentials can be determined by the knowledge of the first, second, and the higher derivatives, with the observable \(\Omega_Q\) and \(w\). The first and second derivatives have been reported in the previous paper \[6\].

Usually, the variation of the equation of state \(w\) for the dark energy is described by \[7–10\]

\[
   w(a) = w_0 + w_a(1 - a),
\]

where \(a, w_0,\) and \(w_a\) are the scale factor \((a = 1\) at current\), the current value of \(w(a)\) and the first derivative of \(w(a)\) by \(w_a = -dw/da\), respectively.

We have extended the parameter space, in this paper,

\[
   w(a) = w_0 + w_a(1 - a) + \frac{1}{2}w_{a2}(1 - a)^2 + \frac{1}{3!}w_{a3}(1 - a)^3,
\]

where \(w_{a2} = -d^2w/da^2\) and \(w_{a3} = -d^3w/da^3\). One of the new ingredients of this work in comparison with past works is the inclusion of this third derivative for the parameter space.

We follow the single scalar field formalism of Steinhardt et al. (1999) \[11, 12\] and investigate three potentials for so-called freezing model \[13\], in which the field is rolling towards down its potential minimum, as \(V = M^{4+a}/Q^a\) \[14\], \(V = M^4 \exp(\beta M/Q)\), and \(V = M^{4+\gamma}/Q^\gamma \exp(\zeta Q^2/M_{pl}^2)\) \[15\]. Two of them are supported by observational data \[16\].
We study other two potentials for so-called thawing model, in which the field is nearly constant at first and then starts to evolve slowly down the potential; \( V = M^4 \cos(Q/f) + 1) \) \[17, 18\] and \( V = M^4 \exp(-Q^2/\sigma^2) \) \[18\].

The cosine type is called the pseudo Nambu-Goldstone boson potential \[17, 18\], which is the prototype potential of thawing model. The above mentioned potentials are motivated by particle physics. Investigation of those potentials with the method \[6\] is another main new ingredient of this work.

The goal of this paper is to explore the dark energy under the quintessence model in a single scalar field by assuming the potential. We have tried to increase the parameter space to examine the features of dark energy, by adding the third derivative. To determine the potential form we must observe the expansion history of the universe. If the parameter number is \( n \) for the potential form, it will be necessary for \( n + 2 \) independent observations to determine the potential form, \( Q \) and \( \dot{Q} \) at some time for the time variation of scalar field. Such observations would be values of \( \Omega_Q, w, dw/da, \cdots \), and \( dw^n/da^n \). From these specific potentials, we could predict the \( n + 1 \) and higher derivative of \( w : dw^{n+1}/da^{n+1}, \cdots \). Because four of the above mentioned potentials have two parameters, it is necessary to calculate the third derivative of \( w \) for them to estimate the predict values. If they are the predicted one, it will be understood that the dark energy could be described by the scalar field with this potential. At least it will satisfy the necessary conditions. One of the above mentioned potentials has three parameters, so it is necessary to calculate the fourth derivative of \( w \) to estimate the predict values, which is not calculated in this paper. However, the principle would be the same to calculate them.

In Sect. II, the equation of state for the scalar field are presented and the results of the first, second, and third derivatives of \( w_Q \) are summarized, where the detailed calculations are displayed in Appendix. Three potentials of freezing model are studied in Sect. III, and two potentials of thawing model are described in Sect. IV. The conclusions and discussion are considered in Sect. V.
II. FIRST, SECOND, AND THIRD DERIVATIVES OF $w_Q$

A. Scalar field

For the dark energy, we consider the scalar field $Q(x, t)$, where the action for this field in the gravitational field is described by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu Q \partial_\nu Q - V(Q) \right] + S_M,$$

(3)

where $S_M$ is the action of the matter field and $G$ is the gravitational constant, occasionally putting $G = 1 \ [4]$. Neglecting the coordinate dependence, the equation for $Q(t)$ becomes

$$\ddot{Q} + 3H \dot{Q} + V' = 0,$$

(4)

where $H$ is the Hubble parameter, overdot is the derivative with time, and $V'$ is the derivative with $Q$. Putting $\kappa = 8\pi/3$, $H$ satisfies the following equation

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \kappa (\rho_B + \rho_Q) = \kappa \rho_c,$$

(5)

where $\rho_B$, $\rho_Q$ and $\rho_c$ are the energy density of the background, the scalar field, and the critical density of the universe. The energy density and pressure for the scalar field are written by

$$\rho_Q = \frac{1}{2} \dot{Q}^2 + V,$$

(6)

and

$$p_Q = \frac{1}{2} \dot{Q}^2 - V,$$

(7)

respectively. Then the parameter $w_Q$ for the equation of state is described by

$$w_Q \equiv \frac{p_Q}{\rho_Q} = \frac{\frac{1}{2} \dot{Q}^2 - V}{\frac{1}{2} \dot{Q}^2 + V}.$$

(8)

B. Time variation of $w_Q$

It is assumed that the current value of $w_Q$ is slightly different from a negative unity by $\Delta(> 0)$

$$w_Q = -1 + \Delta.$$

(9)
By using Eq. (8), $\dot{Q}^2$ is written as

$$\dot{Q}^2 = \frac{2\Delta V}{2 - \Delta},$$

(10)

which becomes, using the density parameter $\Omega_Q = \rho_Q/\rho_c$, 

$$\dot{Q}^2 = 2(\rho_c \Omega_Q - V).$$

(11)

Combining Eqs. (10) and (11), $V$ is given by

$$V = \rho_c \Omega_Q \left(1 - \frac{\Delta}{2}\right).$$

(12)

From Eqs. (11) and (12), $\dot{Q}$ is expressed

$$\dot{Q} = \sqrt{\Delta(\rho_c \Omega_Q)}.$$  

(13)

Since $\rho_c$ is given by the observation through the Hubble parameter $H$, $\dot{Q}$ is determined by $\Omega_Q$ and $\Delta$, which also determine the value of $V$. If we adopt the form and parameters of each potential, the value of $V$ could be used to estimate the value of $Q$. Actually, the evolution of $H$ in Eq. (4) depends on the background densities which include radiation density. The effect of radiation density can be ignored in the near past ($z \leq 10^3$) and so is not considered in this work.

C. First derivative of $w_Q$

To investigate the variation of $w_Q$, we calculate $dw_Q/da$, using Eqs. (4), (6) and (7), after Ref. [6]

$$\frac{dw_Q}{da} = \frac{1}{\dot{a}} \frac{d}{dt} \left( \frac{p_Q}{\rho_Q} \right) = \frac{1}{\dot{a}} \left( \frac{\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q}{\rho_Q^2} \right)$$

$$= \frac{2V \dot{Q}}{a H \rho_Q^2} \left(-3H \dot{Q} - \frac{V'}{V} \rho_Q\right).$$

(14)

If the first derivative is observed, $\frac{V'}{V} M_{pl}$ is specified by,

$$\frac{V'}{V} M_{pl} = -\sqrt{\frac{8\pi}{3\Delta \Omega_Q (2 - \Delta)^2}} \left\{ a \frac{dw_Q}{da} + 3\Delta(2 - \Delta) \right\}.$$  

(15)

where $M_{pl}$ is the Planck mass. To investigate further, we must consider each potential form.
D. Second derivative of \( w_Q \)

From Eq. (14), the second derivative of \( w_Q \) is given by

\[
\frac{d^2 w_Q}{d a^2} = \frac{1}{\dot{a}^2 \rho_Q^2} [(\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q) \dot{\rho}_Q^2 - (\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\ddot{\rho}_Q + 2\dot{\rho}_Q \dot{p}_Q)]. \tag{16}
\]

After the calculation in the paper [6], \( d^2 w/da^2 \) becomes

\[
\frac{d^2 w_Q}{d a^2} = \frac{3 \Omega_Q}{4 \pi a^2} \left( 1 - \frac{\Delta}{2} \right) \times \left[ -\Delta M^2_{pl} \frac{V''}{V} + \sqrt{\frac{8 \pi \Delta}{\Omega_Q}} \left( (1 - \Delta) (6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right) \times M_{pl} \left( \frac{V'}{V} \right) + \left( 1 - \frac{\Delta}{2} \right) M^2_{pl} \left( \frac{V'}{V} \right)^2 + \frac{8 \pi \Delta}{\Omega_Q} (7 - 6 \Delta) \right]. \tag{17}
\]

In the limit \( \Delta \to 0 \), the signature of \( d^2 w_Q/da^2 \) is positive under the condition \( V'/V \neq 0 \).

From this equation, we estimate \( d^2 w_Q/da^2 \) for each potential in the following. If the first derivative is observed, \( \frac{V'}{V} M_{pl} \) is estimated. Then \( \frac{V''}{V} M_{pl} \) could be estimated, if \( \frac{d^2 w_Q}{d a^2} \) is observed, as

\[
\frac{V''}{V} M^2_{pl} = -\frac{1}{4 \Delta} \left[ \frac{d^2 w_Q}{da^2} \frac{16 \pi a^2}{3 \Omega_Q (1 - \frac{\Delta}{2})} - 6 \sqrt{\frac{8 \pi \Delta}{3 \Omega_Q}} \left( (1 - \Delta) (6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right) M_{pl} \frac{V'}{V} \right. \\
- \left. 4 \left( 1 - \frac{\Delta}{2} \right) \left( M_{pl} \frac{V'}{V} \right)^2 - \frac{32 \pi \Delta}{\Omega_Q} \Delta (7 - 6 \Delta) \right]. \tag{18}
\]

E. Third derivative of \( w_Q \)

From Eq. (16), the third derivative of \( w_Q \) is given by

\[
\frac{d^3 w_Q}{d a^3} = \frac{1}{(\dot{a} \rho_Q)^2} \left[ \frac{(p_Q^{(3)} \rho_Q + p_Q \dot{\rho}_Q - p_Q \dot{\rho}_Q)}{\dot{a} \rho_Q^2} - (\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(a^{(3)} \rho_Q^2 + 4 \dot{\rho}_Q \rho_Q^2 + 2 \dot{\rho}_Q^2) \dot{a} \rho_Q \right. \\
- \left. \left\{ (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q) \dot{\rho}_Q^2 - (\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\ddot{\rho}_Q + 2 \dot{\rho}_Q \dot{p}_Q) \right\} (3 \dot{a} \rho_Q + 4 \dot{\rho}_Q) \right] 
\]
\[
\begin{align*}
&= \frac{3^2 \left(1 - \frac{\Delta}{2}\right) \Omega_Q}{2a^3(8\pi)^2} \left[ -\sqrt{\frac{128\pi\Delta^3}{3}} \Omega_Q M_{\text{pl}}^3 \frac{V''''}{V} + 16\pi\Delta \left\{ 10 + 3\Omega_Q \left(\frac{2}{3} - \Delta\right) - 8\Delta \right\} M_{\text{pl}}^2 \frac{V''}{V} \\
&+ \left( \sqrt{\frac{(8\pi)^3}{3}} \Omega_Q \Delta \right) \left\{ -14\Delta + 9\Delta^2 + \frac{164\Delta - 104 + \Omega_Q \left(\frac{64}{3} - 46\Delta + 45\Delta^2\right)}{\Omega_Q} \right\} \\
&- 9 \left\{ \Omega_Q \left(\frac{2}{3} - \Delta\right) - \frac{8}{3}\Delta \right\} \sqrt{\frac{(8\pi)^3\Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3}\Omega_Q \right\} M_{\text{pl}} \frac{V'}{V} \\
&+ 16 \left(1 - \frac{\Delta}{2}\right) \sqrt{\frac{8\pi}{3}} \Omega_Q \Delta M_{\text{pl}}^3 \frac{V' V''}{V}
\right]
\end{align*}
\]

In the next section, we investigate the potential forms. Although potential parts such as \(V'/V, V''/V, \text{and} V''''/V\) are varying, the coefficients do not change in Eq. \((19)\). Thus it is

\[
\begin{align*}
\frac{V''''}{V} M_{\text{pl}}^3 &= -\sqrt{\frac{3}{128\pi\Delta^3\Omega_Q}} \left[ \frac{d^3w_Q}{da^3} - \frac{2a^3(8\pi)^2}{3^2 (1 - \frac{\Delta}{2}) \Omega_Q} - 16\pi\Delta \left\{ 10 + 3\Omega_Q \left(\frac{2}{3} - \Delta\right) - 8\Delta \right\} M_{\text{pl}}^2 \frac{V''}{V} \\
&- \left( \sqrt{\frac{(8\pi)^3}{3}} \Omega_Q \Delta \right) \left\{ -14\Delta + 9\Delta^2 + \frac{164\Delta - 104 + \Omega_Q \left(\frac{64}{3} - 46\Delta + 45\Delta^2\right)}{\Omega_Q} \right\} \\
&- 9 \left\{ \Omega_Q \left(\frac{2}{3} - \Delta\right) - \frac{8}{3}\Delta \right\} \sqrt{\frac{(8\pi)^3\Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3}\Omega_Q \right\} M_{\text{pl}} \frac{V'}{V} \\
&- 16 \left(1 - \frac{\Delta}{2}\right) \sqrt{\frac{8\pi}{3}} \Omega_Q \Delta M_{\text{pl}} M_{\text{pl}}^3 \frac{V' V''}{V} \\
&- 16\pi \left(1 - \frac{\Delta}{2}\right) (5\Omega_Q \Delta + 20\Delta - 2\Omega_Q - 6) \left( M_{\text{pl}} \frac{V'}{V} \right)^2 \\
&- \frac{128\pi^2\Delta}{3} \left( -99\Delta + 54\Delta^2 + \frac{42\Omega_Q - 112 + 36\Delta^2 + 84\Delta}{\Omega_Q} \right) \\
&+ \frac{384\pi^2\Delta}{\Omega_Q} (7 - 6\Delta) \left\{ \Omega_Q \left(\frac{2}{3} - \Delta\right) - \frac{8}{3}\Delta \right\}
\end{align*}
\]
convenient to define the following notations,

\[ A = \frac{3^2 \left(1 - \frac{\Delta}{2}\right) \Omega_Q}{2a^3(8\pi)^2}, \]

\[ B = -\sqrt[3]{\frac{128\pi\Delta^3}{3}} \Omega_Q, \]

\[ C = 16\pi \Delta \left\{ 10 + 3\Omega_Q \left(\frac{2}{3} - \Delta\right) - 8\Delta \right\}, \]

\[ D = \sqrt[3]{\frac{(8\pi)^3}{3}} \Omega_Q \Delta \left\{ -14\Delta + 9\Delta^2 + \frac{164\Delta - 104 + \Omega_Q \left(\frac{64}{3} - 46\Delta + 45\Delta^2\right)}{\Omega_Q} \right\} \]

\[-9 \left\{ \Omega_Q \left(\frac{2}{3} - \Delta\right) - \frac{8\Delta}{3} \right\} \sqrt[3]{\frac{(8\pi)^3 \Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\}, \]

\[ E = 16 \left(1 - \frac{\Delta}{2}\right) \sqrt[3]{\frac{8\pi}{3}} \Omega_Q \Delta, \]

\[ F = 16\pi \left(1 - \frac{\Delta}{2}\right) (5\Omega_Q \Delta + 20\Delta - 2\Omega_Q - 6), \]

\[ G = \frac{128\pi^2 \Delta}{3} \left( -99\Delta + 54\Delta^2 + \frac{42\Omega_Q - 112 + 36\Delta^2 + 84\Delta}{\Omega_Q} \right), \]

\[-384\pi^2 \Delta \left(7 - 6\Delta\right) \left\{ \Omega_Q \left(\frac{2}{3} - \Delta\right) - \frac{8\Delta}{3} \right\}. \]  

By using these notations, Eqs. (19) and (20) become

\[ \frac{d^3w_Q}{da^3} = A \left[ B \frac{V''}{V} M_{pl}^3 + C \frac{V''}{V} M_{pl}^2 + D \frac{V'}{V} M_{pl} + E \frac{V''}{V} M_{pl} V'' M_{pl} + F \left(\frac{V'}{V} M_{pl}\right)^2 + G \right], \]

\[ (22) \]

and

\[ \frac{V'''}{V} M_{pl}^3 = \frac{1}{B} \left[ \frac{d^3w_Q}{da^3} \frac{1}{A} - C \frac{V''}{V} M_{pl}^2 - D \frac{V'}{V} M_{pl} - E \frac{V'}{V} M_{pl} V'' M_{pl} - F \left(\frac{V'}{V} M_{pl}\right)^2 - G \right], \]

\[ (23) \]

respectively.

### III. FREEZING MODEL

In the freezing model, \( w_Q \) will approach to \(-1\). Then the first derivative of \( w_Q \) is expected not positive ( \( dw_Q/da \leq 0 \)). If it is necessary, we adopt the current scale factor is \( a = 1 \). In the following, we investigate the power inverse potential \( V = M^4(M/Q)^{\alpha}; (\alpha > 0) \),
the exponential potential \( V = M^4 \exp(\beta M/Q); (\beta > 0) \), and the mixed type potential \( V = \frac{M^{4+\gamma}}{Q^\gamma} \exp(\zeta Q^2/M^2_{\text{pl}}); (\gamma, \zeta > 0) \), respectively.

A. \( V = M^{4+\alpha}/Q^\alpha \)

The parameters of the potential are \( M \) and \( \alpha \). From Eq. (12), \( Q \) is given by

\[
Q = \left( \frac{M^{4+\alpha}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})} \right)^{1/\alpha}.
\]  
(24)

If we take \( Q = Q_0 M_{\text{pl}} \) at current, \( M \) becomes

\[
M = M_{\text{pl}} \left( Q_0^\alpha \frac{\rho_c}{M^4_{\text{pl}}} \Omega_Q (1 - \frac{\Delta}{2}) \right)^{1/(4+\alpha)}.
\]  
(25)

Then \( Q_0, \Omega_Q, \Delta, \) and \( \alpha \) determine the parameter \( M \), which means that parameters determining the accelerating behavior are \( Q_0, \Omega_Q, \Delta, \) and \( \alpha \). The problem is how to estimate \( Q_0 \) and \( \alpha \).

1. First derivative

Using \( V' / V = -\alpha / Q \), Eq. (14) becomes

\[
\frac{d w_Q}{d a} = \frac{2 V \dot{Q}}{a H Q \rho_Q} \left( \alpha - \frac{3 H \dot{Q} Q}{\rho_Q} \right).
\]  
(26)

From Eq. (26), \( Q \) is derived as

\[
Q = \frac{2 \alpha \rho_Q V \dot{Q}}{H (a \rho_Q^2 \frac{d w_Q}{d a} + 6 V \dot{Q}^2)},
\]  
(27)

then \( Q/M_{\text{pl}} \) is given by

\[
\frac{Q_0}{\alpha} = \sqrt{3 \Delta \Omega_Q} \frac{2 \pi}{2 \pi} \left( \frac{1 - \Delta/2}{(a \frac{d w_Q}{d a} + 6 \Delta (1 - \Delta/2))} \right).
\]  
(28)

If \( d w_Q / d a \) is observed, \( Q_0/\alpha \) will be determined by the observed values \( \Omega_Q, \Delta, \) and \( d w_Q / d a \).
2. Second derivative

Since the following relations are derived

\[ \frac{V''}{V} = \frac{\alpha(\alpha + 1)}{Q^2}, \quad \frac{V'}{V} = -\frac{\alpha}{Q}, \quad \left( \frac{V'}{V} \right)^2 = \frac{\alpha^2}{Q^2}, \]

we substitute them into Eq. (17) and obtain

\[ \frac{d^2w_Q}{da^2} = \frac{3}{4\pi} \frac{\Omega_Q}{a^2} \left( 1 - \frac{\Delta}{2} \right) \times \left[ -\Delta \frac{\alpha(\alpha + 1)}{(Q/M_{pl})^2} - \sqrt{\frac{6\pi\Delta}{\Omega_Q}} \left( 1 - \Delta \right) \left( 6 + \Omega_Q \right) - \frac{1}{3}\Omega_Q \right] \left( \frac{\alpha}{Q/M_{pl}} \right) \]

\[ + \left( 1 - \frac{\Delta}{2} \right) \left( \frac{\alpha}{Q/M_{pl}} \right)^2 + \frac{8\pi\Delta}{\Omega_Q} \left( 7 - 6\Delta \right) \]

(29)

If \( dw_Q/da \) is observed, \( (Q/M_{pl})/\alpha \) will be determined by Eq. (28). If \( d^2w_Q/da^2 \) is observed, one could estimate the value of \( \alpha \) from the above equation as

\[ \alpha = -\left[ \frac{4\pi}{3} \frac{a^2}{\Omega_Q} \left( 1 - \frac{\Delta}{2} \right) \right]^{-1} \frac{d^2w_Q}{da^2} \left( \frac{Q/M_{pl}}{\alpha} \right)^2 + \sqrt{\frac{6\pi\Delta}{\Omega_Q}} \left( 1 - \Delta \right) \left( 6 + \Omega_Q \right) - \frac{1}{3}\Omega_Q \left( \frac{Q/M_{pl}}{\alpha} \right) \]

\[ - \left( 1 - \frac{\Delta}{2} \right) - \frac{8\pi\Delta}{\Omega_Q} \left( 7 - 6\Delta \right) \left( \frac{Q/M_{pl}}{\alpha} \right)^2 \left( \frac{Q/M_{pl}}{\alpha} \right)^2 \left( \frac{Q/M_{pl}}{\alpha} \right)^2 \]

(30)

3. Third derivative

Since the following equations are derived

\[ \frac{V'''}{V} = -\frac{\alpha(\alpha + 1)(\alpha + 2)}{Q^3}, \quad \frac{V'V''}{V^2} = -\frac{\alpha^2(\alpha + 1)}{Q^3}, \]

we substitute them into Eq. (22), using Eq. (21), and obtain

\[ \frac{d^3w_Q}{da^3} = A \left[ -B\alpha(\alpha + 1)(\alpha + 2) \left( \frac{M_{pl}}{Q} \right)^3 + C\alpha(\alpha + 1) \left( \frac{M_{pl}}{Q} \right)^2 - D\alpha \left( \frac{M_{pl}}{Q} \right) \right. \]

\[ \left. - E\alpha^2(\alpha + 1) \left( \frac{M_{pl}}{Q} \right)^3 + F\alpha^2 \left( \frac{M_{pl}}{Q} \right)^2 + G \right]. \]

(32)

Because we get \( Q, \alpha \) through the observations of \( dw_Q/da, d^2w_Q/da^2 \), we could predict the third derivative for this potential.
B. \( V = M^4 \exp(\beta M/Q) \)

This potential has also two independent parameters of \( \beta \) and \( M \). From Eq. (12), the potential relates to the observables

\[
V = \rho_c \Omega_Q \left( 1 - \frac{\Delta}{2} \right) = M^4 \exp \left( \frac{\beta M}{Q} \right),
\]

which is written by

\[
\frac{\beta M}{Q} = \ln \left[ \frac{\rho_c \Omega_Q \left( 1 - \frac{\Delta}{2} \right)}{M^4} \right].
\]

1. First derivative

Since the first derivative of the potential is \( V' = M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{\beta M}{Q^2} \right) \), then

\[
\frac{V'}{V} = -\frac{\beta M}{Q^2}.
\]

Using Eqs. (26), (5), and (13), the first derivative of \( w_Q \) becomes

\[
\frac{dw_Q}{da} = \frac{2V}{a \rho_Q} \left( \frac{\beta M}{Q^2} M_{pl} \sqrt{\frac{3\Delta \Omega_Q}{8\pi}} - 3\Delta \right).
\]

We get \( \frac{\beta M}{Q} M_{pl} \) from the observables

\[
\frac{\beta M}{Q} M_{pl} \frac{M_{pl}}{Q} = \sqrt{\frac{8\pi}{3\Delta \Omega_Q(2 - \Delta)^2}} \left( a \frac{dw_Q}{da} + 3\Delta(2 - \Delta) \right).
\]

In the following Eq. (40), we could estimate \( M_{pl}/Q \) by the observables. After then we could estimate \( \beta M/Q \) by the observables through this equation.

2. Second derivative

Since the second derivative of the potential is

\[
V'' = M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{\beta M}{Q^2} \right)^2 + M^4 \exp \left( \frac{\beta M}{Q} \right) \frac{2\beta M}{Q^3},
\]

(38)
\[ \frac{V''}{V} = \left( -\frac{\beta M}{Q^2} \right)^2 + \frac{2\beta M}{Q^3}. \]

The second derivative of \( w_Q \) is obtained by Eq. (17)
\[
\frac{d^2 w_Q}{da^2} = \frac{3}{16\pi} \frac{\Omega_Q}{a^2} \left[ -4\Delta \left\{ \left( \frac{\beta M}{Q} \right)^2 \left( \frac{M_{pl}}{Q} \right)^2 \right. \right.
\]
\[
- \left. \left. 2 \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^2 \right\} \right. \]
\[
- 6\sqrt{\frac{8\pi\Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)
\]
\[
+ 4 \left( 1 - \frac{\Delta}{2} \right) \left\{ \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^2 + \frac{32\pi}{\Omega_Q} \Delta(7 - 6\Delta) \right\}. \tag{39}
\]

From the observation of \( \frac{dw_Q}{da} \), it is derived the value \( \frac{4\beta\rho_{c}M_{pl}}{Q} \) in Eq. (37). Then we rewrite the above equation by
\[
\frac{M_{pl}}{Q} = -\frac{1}{2} \left( \frac{\beta M}{Q} \right)^{-1} \left( \frac{M_{pl}}{Q} \right)^2 + \frac{1}{4\Delta} \left\{ \frac{d^2 w_Q}{da^2} \frac{16\pi a^2}{3\Omega_Q (1 - \frac{\Delta}{2})} \right. \]
\[
+ \left. 6\sqrt{\frac{8\pi\Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^2 \right.
\]
\[
- 4 \left( 1 - \frac{\Delta}{2} \right) \left\{ \left( \frac{\beta M}{Q} \right)^2 \left( \frac{M_{pl}}{Q} \right) - \frac{32\pi}{\Omega_Q} \Delta(7 - 6\Delta) \right\} \right]. \tag{40}
\]

Then we find out \( \frac{M_{pl}}{Q} \). From Eq. (37), \( \frac{4\beta\rho_{c}M_{pl}}{Q} \) is estimated and \( M \) is derived through Eq. (34) by
\[
M = \left[ \rho_{c} \Omega_Q \left( 1 - \frac{\Delta}{2} \right) \exp \left( -\frac{\beta M}{Q} \right) \right]^\frac{1}{4}. \tag{41}
\]

The value of \( \beta \) is estimated through Eq. (34). As the two parameters of \( \beta \) and \( M \) are specified, it becomes possible to predict the third derivative of \( w \).

3. Third derivative

Since the third derivative of the potential is
\[
V''' = M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{\beta M}{Q^2} \right)^3 + 4M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{\beta^2 M^2}{Q^5} \right)
\]
\[
+ M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{\beta M}{Q^2} \right) \left( -\frac{2\beta M}{Q^3} \right) + M^4 \exp \left( \frac{\beta M}{Q} \right) \left( -\frac{6\beta M}{Q^4} \right), \tag{42}
\]
\[ V''/V \text{ leads to} \]
\[ \frac{V''}{V} = -\frac{\beta^3 M^3}{Q^6} - \frac{6\beta^2 M^2}{Q^5} - \frac{6\beta M}{Q^4}. \quad (43) \]

Then the third derivative of \( w \) is given through Eqs. (22) and (21) by
\[ \frac{d^3 w_Q}{d a^3} = A \left[ -B \left( \frac{\beta M M_{pl}}{Q} \right)^3 + 6 \left( \frac{\beta M}{Q} \right)^2 \left( \frac{M_{pl}}{Q} \right)^3 + 6 \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^3 \right] \]
\[ + C \left( \frac{\beta M M_{pl}}{Q} \right)^2 + 2 \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^2 \] 
\[ - D \left( \frac{\beta M M_{pl}}{Q} \right) - E \left( \frac{\beta M M_{pl}}{Q} \right)^3 + 2 \left( \frac{\beta M}{Q} \right) \left( \frac{M_{pl}}{Q} \right)^3 \] 
\[ + F \left( \frac{\beta M M_{pl}}{Q} \right) + G \]. \quad (44)

This is the predictive value for this potential.

**C.** \( V = \frac{M^{4+\gamma}}{Q^\gamma} \exp \left( \frac{\zeta Q^2}{M_{pl}^2} \right) \)

There are three parameters \( \zeta, M, \) and \( \gamma \) for this potential. If we use the relation of Eq. (12) for the potential with the observables, the parameter \( M \) is expressed by
\[ M = \left[ Q^\gamma \rho_c \Omega_Q \left( 1 - \frac{\Delta}{2} \right) \exp \left( -\frac{\zeta Q^2}{M_{pl}^2} \right) \right]^{1/\gamma}, \quad (45) \]
where there are three unspecified \( \gamma, \zeta \), and \( Q \) parameters.

**1. First derivative**

Since the following relations are derived,
\[ V' = -\frac{\gamma M^{4+\gamma}}{Q^{\gamma+1}} \exp \left( \frac{\zeta Q^2}{M_{pl}^2} \right) + \frac{M^{4+\gamma}}{Q^\gamma} \exp \left( \frac{\zeta Q^2}{M_{pl}^2} \right) \frac{2\zeta Q}{M_{pl}^2}, \]
\[ \frac{V'}{V} = -\frac{\gamma}{Q} + \frac{2\zeta Q}{M_{pl}^2}, \quad (46) \]
we substitute them into Eq. (??)
\[ -\gamma \left( \frac{M_{pl}}{Q} \right) + 2\zeta \left( \frac{Q}{M_{pl}} \right) = -\sqrt{\frac{8\pi}{3\Delta \Omega_Q (2-\Delta)^2}} \left( \frac{d w_Q}{d a} + 3\Delta (2-\Delta) \right). \quad (47) \]

From the observables, including \( \Delta, \Omega_Q \), and \( dw_Q/da \), we could estimate
\[ \frac{V'}{V} M_{pl} = -\gamma \left( \frac{M_{pl}}{Q} \right) + 2\zeta \left( \frac{Q}{M_{pl}} \right) = X, \quad (48) \]
where we put \( X = \frac{V'}{V} M_{pl} \).
2. Second derivative

Since the following equations are obtained

\[
V'' = \frac{M^{4+\gamma}}{Q^\gamma} \exp \left( \frac{\zeta Q^2}{M_{pl}^2} \right) \left[ \frac{\gamma(\gamma + 1)}{Q^2} - \frac{2\zeta(2\gamma - 1)}{M_{pl}^2} + \left( \frac{2\zeta Q}{M_{pl}^2} \right)^2 \right],
\]

\[
\frac{V''}{V} = \frac{\gamma(\gamma + 1)}{Q^2} - \frac{2\zeta(2\gamma - 1)}{M_{pl}^2} + \left( \frac{2\zeta Q}{M_{pl}^2} \right)^2,
\]

we substitute them into Eq. (17) and derive for \( M_{pl}^2 \frac{V''}{V} = Y \);

\[
Y = \gamma(\gamma + 1) \left( \frac{M_{pl}}{Q} \right)^2 - 2\zeta(2\gamma - 1) + \left( \frac{2\zeta}{M_{pl}} \right)^2 = -\frac{1}{4\Delta} \left[ \frac{16\pi a^2}{3\Omega Q \left( 1 - \frac{\Delta}{2} \right)} \frac{d^2 w_Q}{da^2} \right] - 6\sqrt{\frac{8\pi\Delta}{3\Omega Q}} \left\{ (1 - \Delta)(6 + \Omega Q) - \frac{1}{3} \Omega Q \right\} X - 4 \left( 1 - \frac{\Delta}{2} \right) X^2 - \frac{32\pi}{\Omega Q} \Delta(7 - 6\Delta).
\]

From the observables, including \( dw_Q^2/da^2 \), we could estimate \( Y \). If we make the square of \( X \)

\[
X^2 = \gamma^2 \left( \frac{M_{pl}}{Q} \right)^2 - 4\gamma \zeta + \left( \frac{2\zeta}{M_{pl}} \right)^2,
\]

\( Y \) is expressed by

\[
Y = \gamma(\gamma + 1) \left( \frac{M_{pl}}{Q} \right)^2 - 2\zeta(2\gamma - 1) + \left( \frac{2\zeta}{M_{pl}} \right)^2 = X^2 + \gamma \left( \frac{M_{pl}}{Q} \right)^2 + 2\zeta.
\]

Then we could estimate \( \gamma \left( \frac{M_{pl}}{Q} \right)^2 + 2\zeta \) from \( X \) and \( Y \).

3. Third derivative

There is still unspecified parameter, which is different from the potentials with two parameters. Checking the third derivative of \( w_Q \) in Eq. (19), there is still unknown term \( M_{pl}^3 \frac{V'''}{V} \), which must be investigated. The third derivative of the potential is

\[
V''' = \frac{M^{4+\gamma}}{Q^\gamma} \exp \left( \frac{\zeta Q^2}{M_{pl}^2} \right) \left[ -\frac{\gamma(\gamma + 1)(\gamma + 2)}{Q^3} + \frac{3\gamma^2 2\zeta}{Q M_{pl}^2} + 3Q(1 - \gamma) \left( \frac{2\zeta}{M_{pl}^2} \right)^2 + \left( \frac{2\zeta Q}{M_{pl}^2} \right)^3 \right],
\]

then \( V'''/V \) is given by

\[
\frac{V'''}{V} = -\frac{\gamma(\gamma + 1)(\gamma + 2)}{Q^3} + \frac{3\gamma^2 2\zeta}{Q M_{pl}^2} + 3Q(1 - \gamma) \left( \frac{2\zeta}{M_{pl}^2} \right)^2 + \left( \frac{2\zeta Q}{M_{pl}^2} \right)^3.
\]
If we use the third power of $X$, $Z = M_{pl}^3 \frac{V'''}{V}$ is expressed as

$$Z = M_{pl}^3 \frac{V'''}{V} = X^3 - 3\gamma^2 \left( \frac{M_{pl}}{Q} \right)^3 - 2\gamma \left( \frac{M_{pl}}{Q} \right)^3 + 12\zeta^2 \left( \frac{Q}{M_{pl}} \right)
$$

$$= -2X^3 + 3XY - 2\gamma \left( \frac{M_{pl}}{Q} \right)^3,$$

where we have used $XY = X^3 - \gamma^2 \left( \frac{M_{pl}}{Q} \right)^3 + 4\zeta^2 \left( \frac{Q}{M_{pl}} \right)$. If $dw^3_Q/da^3$ is observed, $Z$ could be estimated from Eq. (19). So it is possible to specify three parameters $\gamma, \zeta$, and $Q/M_{pl}$ from the observables $X, Y, \text{and } Z$.

From Eq. (51), we put $2\zeta$ into Eq. (48) and obtain

$$-2\gamma \left( \frac{M_{pl}}{Q} \right) = X - (Y - X^2) \left( \frac{Q}{M_{pl}} \right),$$

and put it into Eq. (51). Then we get

$$(2X^3 - 3XY + Z) \left( \frac{Q}{M_{pl}} \right)^2 + (Y - X^2) \left( \frac{Q}{M_{pl}} \right) - X = 0. \quad (55)$$

Because $Q/M_{pl}$ is derived from the above equation as

$$\frac{Q}{M_{pl}} = \frac{X^2 - Y + \sqrt{9X^4 - 14X^2Y + 4XZ + Y^2}}{2(2X^3 - 3XY + Z)},$$

$\gamma$ is estimated from Eq. (54), $\zeta$ is derived through Eq. (51), and $M$ is estimated by Eq. (45), respectively. For this potential, three parameters are specified through the observations $dw_Q/da, d^2w_Q/da^2$, and $d^3w_Q/da^3$. However, it is necessary to calculate the fourth derivative of the potential to predict $d^4w_Q/da^4$.

IV. THAWING MODEL

The definition of the thawing model is that equation of state is $w = -1$ at early times and then it increases from $-1$, so it is expected $\frac{dw_Q}{da} \geq 0$.

A. $V = M^4 \left[ \cos \left( \frac{Q}{f} \right) + 1 \right]$.

Two parameters are $M$ and $f$, where $f$ ($>0$) is the energy scale of spontaneous symmetry break down. The potential is related to the observation by Eq. (12) as

$$\rho_c \Omega_Q \left( 1 - \frac{\Delta}{2} \right) = M^4 \left[ \cos \left( \frac{Q}{f} \right) + 1 \right] \left( = 2M^4 \cos^2 \left( \frac{Q}{2f} \right) \right). \quad (56)$$
1. First derivative

Since the first derivative of the potential is \( V' = -\frac{M^4}{f} \sin \left( \frac{Q}{f} \right) \), then

\[
X = \frac{V'}{V} = -\frac{\sin \left( \frac{Q}{f} \right)}{f \left[ \cos \left( \frac{Q}{f} \right) + 1 \right]} \ = -\frac{1}{f} \tan \left( \frac{Q}{2f} \right),
\]

where we put \( X = V'/V \). From Eq. (14), the first derivative of \( w_Q \) becomes

\[
\frac{dw_Q}{da} = \frac{2 - \Delta}{a} \left( -3\Delta - X \sqrt{\frac{3\Delta \Omega_Q}{8\pi G}} \right).
\]

If \( dW/da \) is observed, \( X \) is estimated from

\[
X = -\sqrt{\frac{8\pi G}{3\Delta \Omega_Q}} \left( \frac{a}{2 - \Delta} \frac{dW}{da} + 3\Delta \right).
\]

If \( dw_Q/da \geq 0 \), then \( X < 0 \). It means

\[
\frac{1}{f} \tan \left( \frac{Q}{2f} \right) > 0.
\]

2. Second derivative

Since the second derivative of the potential is \( V'' = -\frac{M^4}{f^2} \cos \left( \frac{Q}{f} \right) \), \( Y = V''/V \) is given by

\[
Y = \frac{V''}{V} = -\frac{\cos \left( \frac{Q}{f} \right)}{f^2 \left[ \cos \left( \frac{Q}{f} \right) + 1 \right]}. \tag{58}
\]

The second derivative of \( w_Q \) is derived by Eq. (17) as

\[
\frac{d^2 w_Q}{da^2} = \frac{3}{16\pi G} \frac{\Omega_Q}{a^2} \left( 1 - \frac{\Delta}{2} \right) \left[ -4\Delta Y + 6 \sqrt{\frac{8\pi G \Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} X \right. \tag{59}
\]

\[
+ 4 \left( 1 - \frac{\Delta}{2} \right) X^2 + \frac{32\pi G}{\Omega_Q} \Delta(7 - 6\Delta) \right].
\]

If \( d^2 w_Q/da^2 \) is observed, it becomes possible to estimate \( Y \) as

\[
Y = -\frac{1}{4\Delta} \left[ \frac{d^2 w_Q}{da^2} \frac{16\pi a^2}{3\Omega_Q (1 - \frac{\Delta}{2})} + 6 \sqrt{\frac{8\pi G \Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} X \right.
\]

\[
- 4 \left( 1 - \frac{\Delta}{2} \right) X^2 - \frac{32\pi}{\Omega_Q} \Delta(7 - 6\Delta) \right]. \tag{60}
\]
From Eqs. (57) and (58), \(f\) is estimated by

\[
f = \frac{1}{\sqrt{X^2 - 2Y}}. \tag{61}
\]

From Eq. (57), \(Q\) is estimated from \(X/Y = f \tan(Q/f)\) as

\[
Q = \frac{1}{\sqrt{X^2 - 2Y}} \tan^{-1} \left( \frac{X}{Y} \sqrt{X^2 - 2Y} \right), \tag{62}
\]

which is equivalent, from \(X = -\frac{1}{f} \tan^{-1}(Q/2f)\), to

\[
Q = \frac{2}{\sqrt{X^2 - 2Y}} \tan^{-1} \left( -\frac{X}{\sqrt{X^2 - 2Y}} \right). \tag{63}
\]

From Eq. (56), \(M\) is also determined. Then it becomes possible to predict the third derivative.

3. Third derivative

Since the following relations are derived

\[
V''' = \frac{M^4}{f^3} \sin \left( \frac{Q}{f} \right),
\]

\[
\frac{V''}{V} = \frac{2}{f^3} \left[ \cos \left( \frac{Q}{f} \right) + 1 \right] = -\frac{X}{f^2} = -X(X^2 - 2Y),
\]

the third derivative of \(w_Q\) is given through Eq. (22) by

\[
\frac{d^3 w_Q}{da^3} = A \left[ BX(2Y - X^2)M_{pl}^3 + CYM_{pl}^2 + DXM_{pl} + EXYM_{pl}^3 + FX^2M_{pl}^2 + G \right]. \tag{66}
\]

This is the predictable value for this potential.

B. \(V = M^4 \exp \left( -\frac{Q^2}{\sigma^2} \right)\)

Two parameters are \(M\) and \(\sigma\).

1. First Derivative of \(w_Q\)

Since the first derivative of the potential is \(V' = -\frac{2Q}{\sigma^2}M^4 \exp \left( -\frac{Q^2}{\sigma^2} \right)\), \(V'/V\) becomes

\[
\frac{V'}{V} = -\frac{2Q}{\sigma^2}. \tag{67}
\]
From Eq. (??),
\[
\frac{2Q}{\sigma^2} = \sqrt{\frac{8\pi G}{3\Delta \Omega_Q (2 - \Delta)^2}} \left( a \frac{dw_Q}{da} + 3\Delta (2 - \Delta) \right). \tag{68}
\]

If \(dw_Q/da\) is observed, \(\frac{2Q}{\sigma^2}\) could be estimated.

2. Second derivative

Since the second derivative of the potential is
\[
V'' = \left( -\frac{2}{\sigma^2} + \frac{4Q^2}{\sigma^4} \right) M^4 \exp \left( -\frac{Q^2}{\sigma^2} \right), \quad \frac{V''}{V}
\]
becomes
\[
\frac{V''}{V} = -\frac{2}{\sigma^2} + \frac{4Q^2}{\sigma^4}. \tag{69}
\]

Because \(\frac{2Q}{\sigma^2}\) can be derived when \(dw_Q/da\) is observed, \(\left( \frac{2Q}{\sigma^2} \right)^2 = \frac{4Q^2}{\sigma^4}\) is estimated. From Eq. (17), the second derivative is given by
\[
\frac{d^2w_Q}{da^2} = \frac{3}{16\pi G a^2} \frac{\Omega_Q}{a} \left( 1 - \frac{\Delta}{2} \right) \left[ -4\Delta \left( -\frac{2}{\sigma^2} + \frac{4Q^2}{\sigma^4} \right) \\
+ 6\sqrt{\frac{8\pi G \Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} \left( -\frac{2Q}{\sigma^2} \right) \\
+ 4 \left( 1 - \frac{\Delta}{2} \right) \left( -\frac{2Q}{\sigma^2} \right)^2 + \frac{32\pi G \Delta}{\Omega_Q} (7 - 6\Delta) \right]. \tag{70}
\]

If \(d^2w_Q/da^2\) is observed, \(\sigma\) is specified by
\[
\sigma^2 = \left[ 2 \left( \left( \frac{2Q}{\sigma^2} \right)^2 + \frac{1}{4\Delta} \left\{ \frac{d^2w_Q}{da^2} \frac{16\pi a^2}{3\Omega_Q (1 - \frac{\Delta}{2})} - 6\sqrt{\frac{8\pi \Delta}{3\Omega_Q}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} \left( -\frac{2Q}{\sigma^2} \right) \\
- 4 \left( 1 - \frac{\Delta}{2} \right) \left( -\frac{2Q}{\sigma^2} \right)^2 - \frac{32\pi G \Delta}{\Omega_Q} (7 - 6\Delta) \right) \right]^{-1}. \tag{71}
\]

The value \(Q\) and \(M\) can be also specified by Eq. (68) and Eq. (12), respectively.

3. Third derivative

Since the third derivative of the potential is
\[
V''' = \left( \frac{12Q}{\sigma^4} - \frac{8Q^3}{\sigma^6} \right) M^4 \exp \left( -\frac{Q^2}{\sigma^2} \right), \quad \frac{V'''}{V}
\]
given by
\[
\frac{V'''}{V} = \frac{12Q}{\sigma^4} - \frac{8Q^3}{\sigma^6}. \tag{72}
\]
If the parameters are specified when \( \frac{dw_Q}{da} \) and \( \frac{d^2w_Q}{da^2} \) are observed, it will become possible to predict the third derivative which is given by

\[
\frac{d^3w_Q}{da^3} = A \left[ B \left( \frac{12Q}{\sigma^4} - \frac{8Q^3}{\sigma^6} \right) M_{pl}^3 + C \left( -\frac{2}{\sigma^2} + \frac{4Q^2}{\sigma^4} \right) M_{pl}^2 \right. \\
- D \frac{2Q}{\sigma^2} M_{pl} + E \left( \frac{4Q}{\sigma^4} - \frac{8Q^3}{\sigma^6} \right) M_{pl}^3 + F \left( -\frac{2Q}{\sigma^2} \right)^2 M_{pl}^2 + G \right].
\]

(V. CONCLUSIONS AND DISCUSSION)

If more details of the accelerating universe is observed, it will be important to find out the time variation of the equation of state to understand the so-called dark energy. Many potentials are proposed to explain the acceleration in the context of quintessence with a single scalar field. It is necessary to distinguish which type of the potential will be the theory to explain the expansion. To differentiate the potential, it is necessary to specify the parameters of the potential. In this paper we have studied the method to find out the potential and calculated the third derivative of the equation of state for several potentials for this purpose.

At present, backward observation of large-scale structure of the universe has been undertaken to estimate \( w_Q \) at the age \( (1 + z) \) \[^{10}\]. Actually, the values of \( w(a = a_0), \frac{dw}{da} \) \[^{7–9}\] and \( \frac{d^2w}{da^2} \) \[^{6}\] have been pursued to be determined from the observation:

\[
w(a) = w(a = a_0) + \frac{dw}{da} da + \frac{1}{2} \frac{d^2w}{da^2} (da)^2.
\]

Since observations of the third and higher derivatives of \( w \) could be expected in the future

\[
w(a) = w(a_0 = 1) + \frac{dw}{da} da + \frac{1}{2} \frac{d^2w}{da^2} (da)^2 + \frac{1}{3!} \frac{d^3w}{da^3} (da)^3 + \cdots,
\]

we have calculated the third derivative of \( w \) for general potential \( V \) and applied to five typical potentials. Three are the freezing model; the inverse power type \( (V = M^{4+\alpha}/Q^\alpha) \), the exponential type \( (V = M^4 \exp(\beta M/Q)) \), and the mixed type \( (V = M^4/Q^7 \exp(\zeta Q^2/M_{pl}^2)) \), and two are the thawing model; the PNGB type \( (V = M^4 (\cos(Q/f) + 1)) \), and the Gaussian type \( (V = M^4 \exp(Q^2/\sigma^2)) \). The four of them have two parameters and one has three parameters to identify the form.

The common points of these potentials are that it is necessary to observe the \( n \) derivatives of \( w_Q \) \( (\frac{dw_Q}{da}, \frac{d^2w_Q}{da^2}, \cdots, \frac{d^n w_Q}{da^n}) \) to specify the \( n \) parameters of the potential. It
becomes possible to predict the $n + 1$ derivative of $w_Q \left( d^{n+1}w_Q/da^{n+1} \right)$ from the specific potential.

For example about the inverse power-law potential ($V = M^{4+\alpha}/Q^\alpha$), the observed first and second derivatives ($dw_Q/da$, $d^2w_Q/da^2$) with $H, w_Q$, and $\Omega_Q$ could determine the two parameters of the potential $M$ and $\alpha$. The point is that the third derivative ($d^3w_Q/da^3$) is described by the current values of parameters $\Omega_Q, w_Q$, and its time derivatives, $dw_Q/da$, and $d^2w_Q/da^2$. If it is predicted value, it could be understood that the dark energy would be described by the quintessence with a single scalar field of this potential. At least it will satisfy the necessary condition. It seems to be difficult to define the sufficient condition for the model of the dark energy. However, in principle, the higher derivatives $d^n w_Q/da^n$ ($n \geq 3$) could be predicted from the specific potentials.

The evolution of forward and/or backward time variation could be analyzed at some fixed time point. If the potential is known, the evolution will be estimated from values $Q$ and $\dot{Q}$ at this point, because the equation for scalar field is the second derivative equation as in Eq. 4.

About the derivative of $w_Q$, if $w_Q(z_i)$ at redshift $z_i$ is observed, the derivative of $w_Q$ is given by $dw_Q/da \simeq (w_Q(z_i) - w_Q(z_{i+1}))/a(z_i) - a(z_{i+1}))$, for $a(z_i) > a(z_{i+1})$. The $n$-th derivative of $w_Q$ could be derived through the observation $w_Q(z_i)$, where $i$ takes $1, 2, \cdots, n+1$, respectively. The differences of $w_Q(z_i)$ could give the higher derivative of $w_Q(z_i)$. If we get the form of the potential, we could predict any higher derivative of $w_Q$ by the observables through the method developed in the paper [6] and this paper.

If $\Delta < 0$, we must consider fully different models such as phantoms [19], quintom [20] or k-essence [21]. There are other models which are proposed to explain dark energy such as chameleon field [22], tachyon field [23], dilaton field [24], holographic dark energy [25], modified gravity theory [26], and so on. These models should be considered to parameterize the characteristic features in relation with the high derivatives of the accelerated expansion velocity and observable quantities.

Appendix
Appendix A: Derivation of the third derivative of $w_Q$ in Eq. (19)

Here, we describe the calculation to derive the third derivative of $w_Q$. It should be noted that the third derivative is denoted by the superscript by (3) and the differences should be noted by parentheses ( ), curly brackets { }, and brackets [ ].

From Eq. (16), the third derivative is given as

$$
\frac{d^3 w_Q}{da^3} = \frac{dt}{da} \frac{d}{dt} \left[ \frac{(\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)\dot{\rho}_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)}{\dot{\rho}_Q^4} \right]
$$

$$
= \frac{1}{\dot{\rho}_Q} \left[ \{(\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)\dot{\rho}_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)\} \dot{\rho}_Q^4 
\right.
\left. - \{(\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)\dot{\rho}_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)\} \dot{\rho}_Q^4 \right],
$$

(A1)

where the dot symbol · means the derivative with time.

The derivative of the first term in the above equation becomes

$$
\{(\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)\dot{\rho}_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)\} \cdot \dot{\rho}_Q
$$

$$
= (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q) \dot{\rho}_Q^2 + (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2) 
\left. \right. 
- (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q) \dot{\rho}_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)
$$

$$
= (p_Q^{(3)} \rho_Q + \ddot{p}_Q \rho_Q - \ddot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)}) \dot{\rho}_Q^2 + (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)
\left. \right. 
- (\ddot{p}_Q \rho_Q + \ddot{p}_Q \rho_Q - \ddot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q)
\left. \right. 
- (p_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q + 2\ddot{\rho}_Q \ddot{\rho}_Q + 2\ddot{\rho}_Q \ddot{\rho}_Q)
\left. \right. 
- (p_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q + 2\ddot{\rho}_Q \ddot{\rho}_Q + 2\ddot{\rho}_Q \ddot{\rho}_Q)
\left. \right. 
= (p_Q^{(3)} \rho_Q + \ddot{p}_Q \rho_Q - \ddot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)}) \dot{\rho}_Q^2 + (\ddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\ddot{\rho}_Q^2 + 4\ddot{\rho}_Q \ddot{\rho}_Q + 2\ddot{\rho}_Q^2 + 2\ddot{\rho}_Q \ddot{\rho}_Q),
$$

(A2)
then

\[
\frac{d^3 w_Q}{da^3} = \frac{1}{\dot{a}^7 \rho_Q^8} \left\{ (p_Q^{(3)} \rho_Q + \dot{p}_Q \dot{\rho}_Q - \ddot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)}) \dot{a} \rho_Q^2 \\
- (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(a^{(3)} \dot{\rho}_Q^2 + 4a \dot{\rho}_Q \ddot{\rho}_Q + 2 \dot{\rho}_Q \dot{\rho}_Q - 2 \dot{a} \rho_Q \dot{\rho}_Q) \dot{a} \rho_Q^4 \\
- \{(\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q) \dot{a} \rho_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\ddot{\rho}_Q^2 + 2 \dot{a} \rho_Q \dot{\rho}_Q)\}(3a^2 \dot{a} \rho_Q^4 + 4 \dot{a} \rho_Q^4) \right\}
\]

\[
= \frac{1}{(\dot{a} \rho_Q)^5} \left\{ (p_Q^{(3)} \rho_Q + \ddot{p}_Q \dot{\rho}_Q - \ddot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)}) \dot{a} \rho_Q^2 \\
- (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(a^{(3)} \dot{\rho}_Q^2 + 4a \dot{\rho}_Q \ddot{\rho}_Q + 2 \dot{\rho}_Q \dot{\rho}_Q - 2 \dot{a} \rho_Q \dot{\rho}_Q) \dot{a} \rho_Q \\
- \{(\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q) \dot{a} \rho_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\ddot{\rho}_Q^2 + 2 \dot{a} \rho_Q \dot{\rho}_Q)\}(3a^2 \dot{a} \rho_Q^4 + 4 \dot{a} \rho_Q^4) \right\}.
\]

(A3)

Since the second derivatives are described in the paper [6], the third derivatives of \( p_Q \), and \( \rho_Q \) are shown here as,

\[
p_Q^{(3)} = \left( -3 \frac{a^{(3)} a - \ddot{a} \dddot{a}}{a^2} + 42H \dddot{H} - 2Q \dddot{V}'' \right) \dot{Q}^2 + \left( -3 \frac{\dddot{a}}{a} + 21H^2 - 2V'' \right) 2\dot{Q}(-3H \dddot{Q} - V') \]

\[
+ 12(\dddot{H} \dot{Q} V' + H \dddot{Q} V' + H \dot{Q}^2 V'') + 4V' \dot{Q} V''
\]

\[
= \left( -3 \frac{a^{(3)} a + \dddot{a}}{a^2} + 42H \frac{\dddot{a} a - \dddot{a} a}{a^2} - 2Q \dddot{V}'' \right) \dot{Q}^2 + \left( -3 \frac{\dddot{a}}{a} + 21H^2 - 2V'' \right) (-6H \dot{Q}^2 - 2Q V')
\]

\[
+ 12 \left\{ \frac{\dddot{a} a - \dddot{a} a}{a^2} \dot{Q} V' + H(-3H \dddot{Q} - V')V' + H \dot{Q}^2 V'' \right\} + 4V' \dot{Q} V''
\]

\[
= \left( -3 \frac{a^{(3)} a + \dddot{a} a - 42H^3 - 2Q \dddot{V}''}{a} \right) \dot{Q}^2
\]

\[
+ 18 \frac{\dddot{a} a}{a} \dot{Q}^2 + 6 \frac{\dddot{a} a}{a} \dot{Q} V' - 12H^3 Q^2 - 42H^2 \dot{Q} V' + 12H \dot{Q}^2 V'' + 4Q V' V''
\]

\[
+ 12 \frac{\dddot{a} a}{a} \dot{Q} V' - 12H^2 \dddot{Q} V' - 36H^2 \dot{Q} V' - 12H V'^2 + 12H \dot{Q}^2 V'' + 4V' \dot{Q} V''
\]

\[
= \left( -3 \frac{a^{(3)} a}{a} + 63H \frac{\dddot{a} a}{a} - 168H^3 + 24H V'' - 2Q \dddot{V}'' \right) \dot{Q}^2
\]

\[
+ \left( 18 \frac{\dddot{a} a}{a} - 90H^2 + 8V'' \right) V' \dot{Q} - 12HV'^2.
\]

(A4)
Hereafter we calculate each term of Eq. (A3) consequently. Beforehand we show the necessary pieces as  

\[
\frac{\ddot{a}}{a} = 4\pi G\rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right],
\]

\[
\dot{Q}^2 = \rho_c \Omega_Q \Delta = \rho_Q \Delta,
\]

\[
\dot{\rho}_Q = -3H\rho_Q \Delta - 2V'\dot{Q},
\]

\[
\dot{\rho}_Q = -3H\rho_Q \Delta,
\]

\[
\rho_c = \frac{3H^2}{8\pi G} \Rightarrow H^2 = \frac{8\pi G}{3}\rho_c.
\]

(A5)
Next we calculate the elements appeared in this section as

\[
\frac{d}{dt} \left( \frac{\ddot{a}}{a} \right) = \frac{a^{(3)}a - \ddot{a}a}{a^2} = a^{(3)}a - \frac{\ddot{a}a}{a},
\]

\[
a^{(3)}a = \frac{d}{dt} \left( \frac{\ddot{a}}{a} \right) + \frac{\ddot{a} \dot{a}}{a}.
\]

\[
= \frac{d}{dt} \left\{ 4\pi G \left( -\frac{\rho Q}{3} - \rhoQ \right) \right\} + 4\pi GH \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right]
\]

\[
= 4\pi G \left( -\frac{\rho Q}{3} - \rhoQ \right) + 4\pi GH \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right]
\]

\[
= 4\pi G \left\{ H\rho_Q \Delta + 3H \rho_Q \Delta + 2V' \dot{Q} + H \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \right\}
\]

\[
= 4\pi G \left\{ 4H \rho_Q \Delta + 2V' \sqrt{\rho_Q \Delta} + H \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \right\}.
\]

(A6)

\[
\ddot{p}_Q = \left\{ -12\pi G \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] + 21H^2 - 2V'' \right\} \rho_Q \Delta + 12HV' \sqrt{\rho_Q \Delta} + 2V'^2,
\]

(A7)

\[
\ddot{\rho}_Q = \left\{ -12\pi G \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] + 21H^2 \right\} \rho_Q \Delta + 6HV' \sqrt{\rho_Q \Delta}.
\]

(A8)

Hereafter we calculate each term in the bracket [ ] of Eq. (A3) separately.

At first, the second term becomes

\[
\left\{ (\ddot{p}_Q \rho_Q - p_Q \ddot{p}_Q) \dot{a} \rho_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{p}_Q)(\ddot{a} \rho_Q^2 + 2\dot{a} \rho_Q \dot{p}_Q) \right\} (3\dot{a} \rho_Q + 4\dot{a} \dot{p}_Q).
\]

Since the left curly bracket { } part is calculated in the paper [6], the right parenthesis ( ) part is calculated as

\[
3\dot{a} \rho_Q + 4\dot{a} \dot{p}_Q = a \left( 3\frac{\ddot{a}}{a} \rho_Q + 4\frac{\dot{a}}{a} \rho_Q \right)
\]

\[
= a \left( 12\pi G \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \rho_Q - 12H^2 \rho_Q \Delta \right)
\]

\[
= 12\pi G \rho_c \rho_Q a \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right].
\]

(A9)

Then the second term within bracket [ ] of Eq. (A3) becomes

\[
\left\{ (\ddot{p}_Q \rho_Q - p_Q \ddot{p}_Q) \dot{a} \rho_Q^2 - (\ddot{p}_Q \rho_Q - p_Q \ddot{p}_Q)(\ddot{a} \rho_Q^2 + 2\dot{a} \rho_Q \dot{p}_Q) \right\} (3\dot{a} \rho_Q + 4\dot{a} \dot{p}_Q)
\]

24
\[ 12\pi G \rho_c \rho_Q a \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right] \times \frac{1}{2} a \rho_Q^5 \Delta^2 \left[ 1 - \frac{\Delta}{2} \right] \sqrt{\frac{8\pi G}{3}} \rho_c \left[ - \frac{4 V''}{\Delta V} \right] + 6 \sqrt{\frac{8\pi G}{3\Omega_Q \Delta^2}} \left\{ (1 - \Delta)(6 + \Omega_Q) - \frac{1}{3} \Omega_Q \right\} \frac{V'}{V} + \left( \frac{2}{\Delta} \right)^2 \left( 1 - \frac{\Delta}{2} \right) \left( \frac{V'}{V} \right)^2 + \frac{32\pi G}{\Delta \Omega_Q} (7 - 6\Delta) \right]. \]

(A10)

Next we consider the first term:

\[ \{(p_Q^{(3)} \rho_Q + \ddot{p}_Q \rho_Q - \dot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)}) \dot{a} \rho_Q^2 - (\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(a^{(3)} \rho_Q^2 + 4\dot{a} \rho_Q \dot{\rho}_Q + 2\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \ddot{\rho}_Q) \} \dot{\rho}_Q. \]

Two elements of the above equation are following,

\[ \dot{a} \rho_Q^2 = a \sqrt{\frac{8\pi G}{3}} \rho_c \rho_Q^2, \]

\[ \ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q = -2V \rho_Q \sqrt{\rho_c} \left( \sqrt{24\pi G \Delta} + \frac{V'}{V} \sqrt{\Delta \Omega_Q} \right), \]

then we need to calculate the following two parts

(i) \( p_Q^{(3)} \rho_Q + \ddot{p}_Q \rho_Q - \dot{p}_Q \ddot{\rho}_Q - p_Q \rho_Q^{(3)} \),

(ii) \( a^{(3)} \rho_Q^2 + 4\dot{a} \rho_Q \dot{\rho}_Q + 2\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \ddot{\rho}_Q \).

The second part (ii) becomes

\[ a^{(3)} \rho_Q^2 + 4\dot{a} \rho_Q \dot{\rho}_Q + 2\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \ddot{\rho}_Q = a \left( a^{(3)} \rho_Q^2 + 4\dot{a} \rho_Q \dot{\rho}_Q + 2\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \ddot{\rho}_Q \right) \]
\[ a \left[ 4\pi G \left\{ 4H\rho_Q \Delta + 2V' \sqrt{\rho_Q \Delta} + H\rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \right\} \rho_Q^2 \right.
\]
\[ + 16\pi G\rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \rho_Q (-3H\rho_Q \Delta) + 2H (-3H\rho_Q \Delta)^2 \]
\[ + 2H\rho_Q \left( \left\{-12\pi G\rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] + 21H^2 \right\} \rho_Q \Delta + 6H V' \sqrt{\rho_Q \Delta} \right) \]
\[ = a \left[ 16\pi G \rho_Q^3 H \Delta + 8\pi G \rho_Q^2 V' \sqrt{\rho_Q \Delta} + 4\pi G H \rho_Q^2 \rho_c \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] \right. \]
\[ - 48\pi G H \rho_c \rho_Q^2 \Delta \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] + 18H^3 \rho_Q^2 \Delta^2 \]
\[ - 24\pi G \rho_c \rho_Q^2 H \Delta \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \right] + 42H^3 \rho_Q^2 \Delta + 12H^2 \rho_Q V' \sqrt{\rho_Q \Delta} \]
\[ = a \left[ \Omega_Q \left( \frac{2}{3} - \Delta \right) \left( 4\pi G H \rho_Q^2 \rho_c - 72\pi G \rho_c \rho_Q H \Delta \right) \right. \]
\[ + 16\pi G \rho_Q^3 H \Delta + 8\pi G \rho_Q^2 V' \sqrt{\rho_Q \Delta} + 42H^3 \rho_Q^2 \Delta + 12H^2 \rho_Q V' \sqrt{\rho_Q \Delta} \]
\[ = 4\pi G \rho_c \rho_Q^2 a \left[ H \Omega_Q \left( \frac{2}{3} - \Delta \right) (1 - 18\Delta) + \frac{4H \rho_Q \Delta}{\rho_c} + \frac{2V' \sqrt{\rho_Q \Delta}}{\rho_c} \right. \]
\[ + 12H \Delta^2 + 28H \Delta + \frac{8V' \sqrt{\rho_Q \Delta}}{\rho_Q} \]
\[ = 4\pi G \rho_c \rho_Q^2 a \left[ H \Omega_Q \left( \frac{2}{3} - \Delta \right) (1 - 18\Delta) + 2V' \sqrt{\frac{\Delta \Omega_Q}{\rho_c}} + 4H \Delta (\Omega_Q + 3\Delta + 7) + 8V' \sqrt{\frac{\Delta}{\rho_Q}} \right. \]
\[ = 4\pi G \rho_c \rho_Q^2 a \left[ H \Omega_Q \left( \frac{2}{3} - \Delta \right) (1 - 18\Delta) + 4H \Delta (\Omega_Q + 3\Delta + 7) + 2 \frac{V'}{V} \left( 1 - \frac{\Delta}{2} \right) \sqrt{\Delta \rho_Q (\Omega_Q + 4)} \right]. \]

The right hand part of the first term becomes
\[ (p_Q \rho_Q - p_Q \rho_Q \dot{\rho}_Q)(a^{(3)} \rho_Q^3 + 4a \rho_Q \rho_Q + 2a \dot{\rho}_Q^2 + 2a \rho_Q \ddot{\rho}_Q) \]
\[ = - \frac{8\pi G \rho_Q^3 \rho_c^2}{\Omega_Q} \left( \frac{1 - \frac{\Delta}{2}}{2} \right) \left[ \sqrt{24\pi G \Delta + \frac{V'}{V} \sqrt{\Delta \Omega_Q}} \right] \left[ H \Omega_Q \left( \frac{2}{3} - \Delta \right) (1 - 18\Delta) \right. \]
\[ + 4H \Delta (\Omega_Q + 3\Delta + 7) + 2 \frac{V'}{V} \left( 1 - \frac{\Delta}{2} \right) \sqrt{\Delta \rho_Q (\Omega_Q + 4)} \right]. \]

At last we calculate the first term as
\[ p_Q^{(3)} \rho_Q + \ddot{p}_Q \rho_Q - \ddot{p}_Q \dot{Q} - p_Q \rho_Q^{(3)} \]
\[
\begin{align*}
&= \left[ \left( -3\frac{a^{(3)}}{a} + 63H\ddot{a} - 168H^3 + 24HV'' - 2\dot{Q}V''\right) \right] \dot{Q}^2 \\
&+ \left( 18\frac{\ddot{a}}{a} - 90H^2 + 8V'' \right) V'\dot{Q} - 12HV'^2 \left( \frac{1}{2} \dot{Q}^2 + V \right) \\
&+ \left[ \left( -3\frac{\ddot{a}}{a} + 21H^2 - 2V'' \right) \dot{Q}^2 + 12H\dot{Q}V' + 2V'^2 \right] (-3H\dot{Q}^2) \\
&- (-3H\dot{Q}^2 - 2V'\dot{Q} \left[ \left( -3\frac{a^{(3)}}{a} + 63H\ddot{a} - 168H^3 + 6HV'' \right) \dot{Q}^2 \\
&+ \frac{1}{2} \dot{Q}^2 - V \right] \left[ \left( -3\frac{a^{(3)}}{a} + 63H\ddot{a} - 168H^3 + 6HV'' \right) \dot{Q}^2 \right]
\end{align*}
\]

We concentrate the first and fourth terms and sort out as

\[
\begin{align*}
p_Q^{(3)}p_Q - p_Qp_Q^{(3)} &= \left[ \left( 18HV'' - 2\dot{Q}V''\right) \dot{Q}^2 + \left( \frac{\ddot{a}}{a} - 24H^2V' + 8V'V'' \right) \dot{Q} - 6HV'^2 \right] \frac{1}{2} \dot{Q}^2 \\
&+ \left[ \left( -6\frac{a^{(3)}}{a} + 126H\ddot{a} - 336H^3 + 30HV'' - 2\dot{Q}V'' \right) \dot{Q}^2 \\
&+ \left( 30\frac{\ddot{a}}{a} - 156H^2V' + 8V'V'' \right) \dot{Q} - 18HV'^2 \right] V
\end{align*}
\]

\[
\begin{align*}
&= \left[ \left( \frac{18H}{V}V'' - 2\dot{Q}V''\right) \dot{Q}^2 \right] V + \left( \frac{\ddot{a}}{a} - 24H^2V' + 8V'V'' \right) \dot{Q} - 6H \left( \frac{V'}{V} \right)^2 \frac{1}{2} \dot{Q}^2 V^2 \\
&+ \left[ \left( -6\frac{a^{(3)}}{a} + 126H\ddot{a} - 336H^3 + 30HV'' - 2\dot{Q}V'' \right) \dot{Q}^2 \right] V \\
&+ \left( 30\frac{\ddot{a}}{a} - 156H^2V' + 8V'V'' \right) \dot{Q} - 18H \left( \frac{V'}{V} \right)^2 V^3
\end{align*}
\]

\[
\begin{align*}
&= \left[ \frac{18H}{V}\dot{Q}^2 V'' - 2\dot{Q}V'' \right] V + \left( 6\frac{\ddot{a}}{a} - 24H^2V' + 8V'V'' \right) \dot{Q} - 6H \left( \frac{V'}{V} \right)^2 \frac{1}{2} \dot{Q}^2 V^2 \\
&+ \left[ -6\frac{a^{(3)}}{a} \dot{Q}^2 + 126H\ddot{a} \dot{Q}^2 - 336H^3\dot{Q}^2 + 30H\dot{Q}^2 V'' - 2\dot{Q}^3 V'' \right] V \\
&+ \frac{30}{a} \dddot{V} - 156H^2\dddot{V} + 8\dddot{V} V'' V' - 18H \left( \frac{V'}{V} \right)^2 V^3
\end{align*}
\]

(A13)
Here we show the necessary parts as

\[
\frac{H \dot{Q}^2}{V} = \frac{\rho_Q \Delta \sqrt{8\pi G \rho_c}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})} = \frac{\Delta \sqrt{8\pi G \rho_c}}{1 - \frac{\Delta}{2}}, \tag{A15}
\]

\[
\frac{\dot{Q}^3}{V} = \frac{\rho_Q \Delta \sqrt{\rho_Q \Delta}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})} = \frac{\Delta \sqrt{\rho_Q \Delta}}{1 - \frac{\Delta}{2}}, \tag{A16}
\]

\[
\frac{\ddot{a} \dot{Q}}{a V} = \frac{4\pi G \rho_c \Omega_Q \left(\frac{2}{3} - \Delta\right) \sqrt{\rho_Q \Delta}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})} = \frac{4\pi G \left(\frac{2}{3} - \Delta\right) \sqrt{\rho_Q \Delta}}{1 - \frac{\Delta}{2}}, \tag{A17}
\]

\[
\frac{H^2 \dot{Q}}{V} = \frac{8\pi G \rho_c \sqrt{\rho_Q \Delta}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})} = \frac{8\pi G \sqrt{\rho_Q \Delta}}{3 \Omega_Q (1 - \frac{\Delta}{2})}, \tag{A18}
\]

\[
\frac{a^{(3)} \dot{Q}^2}{a V^2} = \frac{4\pi G \left\{4H \rho_Q \Delta + 2V' \sqrt{\rho_Q \Delta} + H \rho_c \left[\Omega_Q \left(\frac{2}{3} - \Delta\right)\right]\right\}}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})^2} \frac{\rho_Q \Delta}{\rho_c^2 \Omega_Q^2 (1 - \frac{\Delta}{2})^2} = \frac{4\pi G \Delta}{\rho_c \Omega_Q (1 - \frac{\Delta}{2})^2} \left\{4H \rho_Q \Delta + 2V' \sqrt{\rho_Q \Delta} + H \rho_c \left[\Omega_Q \left(\frac{2}{3} - \Delta\right)\right]\right\}
\]

\[
= \frac{16\pi G \sqrt{8\pi G \rho_c \Delta^2}}{(1 - \frac{\Delta}{2})^2} + \frac{8\pi G \Delta \sqrt{\rho_Q \Delta}}{1 - \frac{\Delta}{2}} \frac{V'}{V} + \frac{4\pi G \Delta \sqrt{8\pi G \rho_c \Delta}}{(1 - \frac{\Delta}{2})^2},
\]

\[
\frac{\ddot{a} H \dot{Q}^2}{a V^2} = \frac{4\pi G \rho_c \Omega_Q \left(\frac{2}{3} - \Delta\right) \sqrt{8\pi G \rho_c \rho_Q \Delta}}{\rho_c^2 \Omega_Q^2 (1 - \frac{\Delta}{2})^2}
\]

\[
= \frac{4\pi G \left(\frac{2}{3} - \Delta\right) \sqrt{8\pi G \rho_c \Delta}}{(1 - \frac{\Delta}{2})^2}, \tag{A19}
\]

\[
\frac{H^3 \dot{Q}^2}{V^2} = \frac{8\pi G \rho_c \sqrt{8\pi G \rho_c \rho_Q \Delta}}{\rho_c^2 \Omega_Q^2 (1 - \frac{\Delta}{2})^2} = \frac{8\pi G \sqrt{8\pi G \rho_c \Delta}}{3 \rho_c \Omega_Q (1 - \frac{\Delta}{2})^2}. \tag{A20}
\]
Then $p_Q^{(3)} \rho_Q - p_Q^{(3)} \rho_Q$ becomes as

$$
\begin{align*}
&= \left[ 18 \sqrt{\frac{8\pi G}{3} \rho_c} V'' - \frac{2\Delta \sqrt{\rho_Q \Delta V''}}{1 - \frac{3}{2}} \right] \frac{1}{2} \Delta \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right)^2 + 24\pi G \frac{\rho_Q \Delta V'}{1 - \frac{3}{2}} \frac{V'}{V} - \frac{64\pi G \sqrt{\rho_Q \Delta V'}}{\Omega_Q} \left( 1 - \frac{3}{2} \right) \frac{V'}{V} \\
&\quad + 8\sqrt{\rho_Q \Delta V'} V'' - \sqrt{96\pi G \rho_c} \left( \frac{V'}{V} \right)^2 \frac{1}{2} \Delta \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right)^2 \\
&\quad + \frac{96\pi G \sqrt{\frac{8\pi G}{3} \rho_c} \Delta^2}{(1 - \frac{3}{2})^2} - \frac{48\pi G \Delta \sqrt{\rho_Q \Delta V'}}{1 - \frac{3}{2}} \frac{V'}{V} - \frac{24\pi G \Delta \sqrt{\frac{8\pi G}{3} \rho_c} \left( \frac{3}{2} - \Delta \right)}{(1 - \frac{3}{2})^2} \\
&\quad + \frac{504\pi G \left( \frac{3}{2} - \Delta \right) \Delta \sqrt{\frac{8\pi G}{3} \rho_c}}{(1 - \frac{3}{2})^2} - \frac{896\pi G \sqrt{\frac{8\pi G}{3} \rho_c} \Delta}{\Omega_Q} \left( 1 - \frac{3}{2} \right) \frac{V''}{V} - \frac{2\Delta \sqrt{\rho_Q \Delta V''}}{1 - \frac{3}{2}} \frac{V''}{V} \\
&\quad + \frac{120\pi G \left( \frac{3}{2} - \Delta \right) \sqrt{\rho_Q \Delta V'}}{(1 - \frac{3}{2})^2} - \frac{416\pi G \sqrt{\rho_Q \Delta V'}}{\Omega_Q} \left( 1 - \frac{3}{2} \right) \frac{V'}{V} + 8\sqrt{\rho_Q \Delta V'} V'' \\
&\quad - 18\sqrt{\frac{8\pi G}{3} \rho_c} \left( \frac{V'}{V} \right)^2 \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right)^3. \tag{A21}
\end{align*}
$$

To combine the first and second term, we must multiply the first term by $\times \frac{1 - \frac{3}{2}}{\Delta^2}$ and the second term by $\times \frac{2(1 - \frac{3}{2})^2}{\Delta^2}$ and arrange as

$$
\begin{align*}
&= \frac{1}{2} \Delta^2 \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right) \times \left[ -\frac{4\Delta \sqrt{\rho_Q \Delta V''}}{\Delta} + \frac{12\sqrt{\frac{8\pi G}{3} \rho_c} (-\Delta + 5) V''}{\Delta} \right] \frac{V'}{V} \\
&\quad + \frac{8\pi G \sqrt{\rho_Q \Delta}}{\Delta} \left\{ -10 + 3\Delta + \frac{44\Delta - 104 + 30\Omega_Q \left( \frac{3}{2} - \Delta \right) \left( 1 - \frac{3}{2} \right)}{\Delta \Omega_Q} \right\} \frac{V'}{V} \\
&\quad + \frac{16\sqrt{\rho_Q \Delta \left( 1 - \frac{3}{2} \right) V' V''}}{\Delta^2} - \frac{12\sqrt{\frac{8\pi G \rho_c}{3}} \left( 1 - \frac{3}{2} \right)}{\Delta} \left( -1 + \frac{3}{\Delta} \right) \left( \frac{V'}{V} \right)^2 \\
&\quad + 16\pi G \sqrt{\frac{8\pi G}{3} \rho_c} \left[ -72 + \frac{1}{\Delta \Omega_Q}(40\Omega_Q - 112) \right]. \tag{A22}
\end{align*}
$$

Subsequently we concentrate the second and third terms, and sort out as

$$
\bar{p}_Q \dot{\rho}_Q - \dot{p}_Q \bar{\rho}_Q
$$
= \left[ \left( -3 \frac{\ddot{a}}{a} + 21 H^2 - 2 V'' \right) Q^2 + 12 H \dot{Q} V' + 2 V''^2 \right] (-3 \dot{H} \dot{Q}^2) + (3 \dot{H} \dot{Q}^2 + 2 V' \dot{Q}) \left[ \left( -3 \frac{\ddot{a}}{a} + 21 H^2 \right) \dot{Q}^2 + 6 H \dot{Q} V' \right] \\
= -3 \dot{H} \dot{Q}^2 (2 V'' \dot{Q}^2 + 6 H \dot{Q} V' + 2 V''^2) + 2 V' \dot{Q} \left[ \left( -3 \frac{\ddot{a}}{a} + 21 H^2 \right) \dot{Q}^2 + 6 H \dot{Q} V' \right] \\
= -3 \dot{H} \dot{Q}^2 V^2 \left[ -2 \frac{V'' \dot{Q}^2}{V} + 6 \frac{H \dot{Q} V'}{V} + 2 \left( \frac{V'}{V} \right)^2 \right] + 2 V' \dot{Q} \left[ \left( -3 \frac{\ddot{a}}{a} + 21 H^2 \right) \frac{\dot{Q}^2}{V} + 6 \frac{H \dot{Q} V'}{V} \right] \\
= 6 H \dot{Q}^4 V \frac{V''}{V} - 18 H^2 \dot{Q}^3 V \frac{V'}{V} + 6 \dot{Q}^2 V^2 \left( \frac{V'}{V} \right)^2 \\
- 6 \frac{\ddot{a}}{a} \dot{Q}^3 V \frac{V''}{V} + 42 H^2 \dot{Q}^3 V \frac{V'}{V} + 12 H \dot{Q}^2 V^2 \left( \frac{V'}{V} \right)^2 \\
= 6 H \dot{Q}^4 V \frac{V''}{V} + \left( 24 H^2 \dot{Q}^3 V - 6 \frac{\ddot{a}}{a} \dot{Q}^3 \right) \frac{V'}{V} + 6 H \dot{Q}^2 V^2 \left( \frac{V'}{V} \right)^2. \quad (A23)

B The necessary elements are displayed:

\[ H \dot{Q}^4 V = \sqrt{\frac{8 \pi G}{3}} \rho_c \rho_Q^3 \Delta^2 \left( 1 - \frac{\Delta}{2} \right) \quad (A24) \]
\[ H^2 \dot{Q}^3 V = \frac{8 \pi G}{3} \rho_c \rho_Q^2 \Delta \sqrt{\rho_Q \Delta} \left( 1 - \frac{\Delta}{2} \right) \quad (A25) \]
\[ \frac{\ddot{a}}{a} \dot{Q}^3 = 4 \pi G \rho_Q^3 \rho_Q \Delta \left( \frac{2}{3} - \Delta \right) \left( 1 - \frac{\Delta}{2} \right) \Delta \quad (A26) \]
\[ H \dot{Q}^2 V^2 = \sqrt{\frac{8 \pi G}{3}} \rho_c \rho_Q^3 \Delta \left( 1 - \frac{\Delta}{2} \right)^2 \quad (A27) \]

Then Eq. \[(A23)\] becomes

\[ = 6 \sqrt{\frac{8 \pi G}{3}} \rho_c \rho_Q^3 \Delta^2 \left( 1 - \frac{\Delta}{2} \right) \frac{V''}{V} + \left\{ 64 \pi G \rho_c \rho_Q^2 \Delta \sqrt{\rho_Q \Delta} \left( 1 - \frac{\Delta}{2} \right) \right\} \frac{V'}{V} + 6 \sqrt{\frac{8 \pi G}{3}} \rho_c \rho_Q^3 \Delta \left( 1 - \frac{\Delta}{2} \right)^2 \left( \frac{V'}{V} \right)^2. \]

So we must multiply the whole equation by \( \sqrt{\frac{\Delta^2 \rho_Q^2}{1 - \frac{\Delta}{2}}} \) and arrange as

\[ \frac{1}{2} \Delta^2 \rho_Q^2 \left( 1 - \frac{\Delta}{2} \right) \times \left[ 12 \sqrt{\frac{8 \pi G}{3}} \rho_c \frac{V''}{V} + \left\{ 128 \pi G \sqrt{\Delta} \rho_c \rho_Q \Delta \sqrt{\rho_Q \Delta} \left( \frac{\Delta}{3} - \Delta \right) \frac{V'}{V} - 48 \pi G \left( \frac{2}{3} - \Delta \right) \sqrt{\rho_Q \Delta} \frac{V'}{V} \right\} \frac{V'}{V} \right. \]

\[ \left. + \frac{12}{\Delta} \sqrt{\frac{8 \pi G}{3}} \rho_c \left( 1 - \frac{\Delta}{2} \right) \left( \frac{V'}{V} \right)^2 \right]. \quad (A28) \]
Then the first Equation of \( p_Q^{(3)} \rho_Q + \tilde{p}_Q \tilde{\rho}_Q - \hat{p}_Q \hat{\rho}_Q - p_Q \rho_Q^{(3)} \) becomes as

\[
\begin{align*}
&= \frac{1}{2} \Delta^2 \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right) \left[ -\frac{4}{\Delta} \sqrt{\rho_Q \Delta} \frac{V'''}{V} + \frac{60}{\Delta} \sqrt{\frac{8\pi G}{3}} \rho_c \frac{V''}{V} \\
&+ \frac{8\pi G \sqrt{\rho_Q \Delta}}{\Delta} \left\{ -14 + 9\Delta + \frac{60\Delta - 104 + 30\Omega_Q \left( \frac{\Delta}{3} - \Delta \right) \left( 1 - \frac{\Delta}{2} \right)}{\Delta \Omega_Q} \right\} \frac{V'}{V} \\
&+ \frac{16 \sqrt{\rho_Q \Delta}}{\Delta^2} \left( 1 - \frac{\Delta}{2} \right) \frac{V'}{V} \frac{V''}{V} + \frac{12 \sqrt{\frac{8\pi G}{3}} \rho_c \left( 1 - \frac{\Delta}{2} \right)}{\Delta} \left( 2 - \frac{3}{\Delta} \right) \left( \frac{V'}{V} \right)^2 \\
&+ 16\pi G \sqrt{\frac{8\pi G}{3}} \rho_c \left\{ -72 + \frac{1}{\Delta \Omega_Q} \left( 40\Omega_Q - 112 \right) \right\} \right]. \\
\end{align*}
\]

(A29)

To put together the above whole calculations, the first term of Eq. (A3) becomes

\[
\begin{align*}
\{ (p_Q^{(3)} \rho_Q + \tilde{p}_Q \tilde{\rho}_Q - \hat{p}_Q \hat{\rho}_Q - p_Q \rho_Q^{(3)}) \hat{a} \rho_Q^2 - (\tilde{p}_Q \rho_Q - p_Q \rho_Q^{(3)}) (a^{(3)} \rho_Q^2 + 4\hat{a} \rho_Q \hat{\rho}_Q + 2\hat{a} \rho_Q^2 + 2\hat{a} \rho_Q \hat{\rho}_Q) \} \hat{a} \rho_Q \\
= \left\{ \frac{1}{2} \Delta^2 \rho_Q^3 \left( 1 - \frac{\Delta}{2} \right) \times a \sqrt{\frac{8\pi G}{3}} \rho_c \rho_Q^2 \left[ -\frac{4}{\Delta} \sqrt{\rho_Q \Delta} \frac{V'''}{V} + \frac{60}{\Delta} \sqrt{\frac{8\pi G}{3}} \rho_c \frac{V''}{V} \\
+ \frac{8\pi G \sqrt{\rho_Q \Delta}}{\Delta} \left\{ -14 + 9\Delta + \frac{60\Delta - 104 + 30\Omega_Q \left( \frac{\Delta}{3} - \Delta \right) \left( 1 - \frac{\Delta}{2} \right)}{\Delta \Omega_Q} \right\} \frac{V'}{V} \\
+ \frac{16 \sqrt{\rho_Q \Delta}}{\Delta^2} \left( 1 - \frac{\Delta}{2} \right) \frac{V'}{V} \frac{V''}{V} + \frac{12 \sqrt{\frac{8\pi G}{3}} \rho_c \left( 1 - \frac{\Delta}{2} \right)}{\Delta} \left( 2 - \frac{3}{\Delta} \right) \left( \frac{V'}{V} \right)^2 \\
+ 16\pi G \sqrt{\frac{8\pi G}{3}} \rho_c \left\{ -72 + \frac{1}{\Delta \Omega_Q} \left( 40\Omega_Q - 112 \right) \right\} \right] \\
+ \frac{8\pi G a^{(3)} \rho_Q^2 \rho_c^{3/2}}{\Omega_Q} \left( 1 - \frac{\Delta}{2} \right) \left[ \sqrt{24\pi G \Delta} + \frac{V'}{V} \sqrt{\Delta \Omega_Q} \right] \left( 2 - \frac{3}{\Delta} \right) \left( \frac{V'}{V} \right)^2 \\
+ 4\hat{a} \rho_Q \hat{\rho}_Q \left( 1 - \frac{\Delta}{2} \right) \sqrt{\Delta \rho_Q \Omega_Q} \left( 2 - \frac{3}{\Delta} \right) \left( \frac{V'}{V} \right)^2 \right\} \times a \sqrt{\frac{8\pi G}{3}} \rho_c \rho_Q.
\end{align*}
\]

(A30)
Here we arrange the above equation by adjusting the factor

\[
= \frac{1}{2} a \rho_Q^5 \Delta^2 \left(1 - \frac{\Delta}{2}\right) \sqrt{\frac{8 \pi G}{3} \rho_c} \left[-\frac{4}{\Delta} \sqrt{\rho_Q \Delta} \frac{V'''}{V} + \frac{60}{\Delta} \sqrt{\frac{8 \pi G}{3} \rho_c} \frac{V''}{V}\right]
+ \frac{8 \pi G \sqrt{\rho_Q \Delta}}{\Delta} \left\{-14 + 9 \Delta + \frac{60 \Delta - 104 + 30 \Omega_Q \left(\frac{2}{3} - \Delta\right) \left(1 - \frac{\Delta}{2}\right)}{\Delta \Omega_Q} \right\} \frac{V'}{V}
+ \frac{16 \sqrt{\rho_Q \Delta}}{\Delta^2} \frac{(1 - \frac{\Delta}{2}) V'' V'''}{V V} + \frac{12 \sqrt{\frac{8 \pi G}{3} \rho_c} \left(1 - \frac{\Delta}{2}\right)}{\Delta} \left(2 - \frac{3}{\Delta}\right) \left(\frac{V'}{V}\right)^2
+ 16 \pi G \sqrt{\frac{8 \pi G}{3} \rho_c} \left\{-72 + \frac{1}{\Delta \Omega_Q} \left(40 \Omega_Q - 112\right)\right\}
+ \frac{2 \sqrt{24 \pi G}}{\Delta^2 \Omega_Q} \left\{\sqrt{24 \pi G} \Delta H \Omega_Q \left(\frac{2}{3} - \Delta\right) \left(1 - 18 \Delta\right) + \sqrt{24 \pi G} 4 \Delta^2 (\Omega_Q + 3 \Delta + 7)\right\} \frac{V'}{V}
+ \left\{2 \sqrt{24 \pi G} \Delta \left(1 - \frac{\Delta}{2}\right) \sqrt{\Delta \rho_Q (\Omega_Q + 4)} + H \Omega_Q \left(\frac{2}{3} - \Delta\right) \left(1 - 18 \Delta\right) \sqrt{\Delta \Omega_Q}\right\} \frac{V'}{V}
+ \frac{4 H \Delta (\Omega_Q + 3 \Delta + 7) \sqrt{\Delta \Omega_Q}}{\Delta} \frac{V'}{V}
+ 2 \sqrt{\Delta \Omega_Q} \left(1 - \frac{\Delta}{2}\right) \sqrt{\Delta \rho_Q (\Omega_Q + 4)} \left(\frac{V'}{V}\right)^2 \right\} \times a \sqrt{\frac{8 \pi G}{3} \rho_c \rho_Q}.
\]  

(A31)

Then we put together the above equation by Eq. (A10) times by - and factortize by \(\frac{1}{2} a \rho_Q^5 \Delta^2 \left(1 - \frac{\Delta}{2}\right) \sqrt{\frac{8 \pi G}{3} \rho_c}\). Then Eq. (A3) becomes
\[
\frac{d^3w_Q}{da^3} = \frac{1}{(a \rho_Q)^5} \left\{ \left( p_Q^{(3)} \rho_Q + \dot{p}_Q \rho_Q^2 - p_Q \rho_Q^{(3)} \right) \dot{\alpha} \rho_Q^2 \\
- (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\dot{a} \rho_Q^2 + 4 \dot{\alpha} \rho_Q \dot{\rho}_Q + 2 \ddot{\alpha} \rho_Q \dot{\rho}_Q + 2 \dot{\alpha} \dot{\rho}_Q \dot{\rho}_Q) \right\} \dot{\alpha} \rho_Q \\
- \left\{ (\ddot{p}_Q \rho_Q - p_Q \dot{\rho}_Q) \dot{\alpha} \rho_Q^2 - (\dddot{p}_Q \rho_Q - p_Q \ddot{\rho}_Q)(\dddot{\alpha} \rho_Q^2 + 2 \dddot{\alpha} \rho_Q \dot{\rho}_Q) \right\} (3 \dot{\alpha} \rho_Q + 4 \dddot{\alpha} \rho_Q)
\]

= \frac{1}{a^5 \left( \frac{8 \pi G}{3} \rho_c \right)^2} \sqrt{\frac{8 \pi G}{3} \rho_c^5} \frac{1}{2} a^5 \rho_Q^2 \Delta^2 \left( 1 - \frac{\Delta}{2} \right) \sqrt{\frac{8 \pi G}{3} \rho_c \times \rho_Q a} \\
\times \left( \sqrt{\frac{8 \pi G}{3} \rho_c} \rho_Q \Delta V'' \frac{4}{\Delta} \sqrt{\frac{8 \pi G}{3} \rho_c} \rho_Q \Delta V'' + 6 \Delta \sqrt{\frac{8 \pi G}{3} \rho_c} \rho_Q \Delta V'' \right) \\
+ \frac{8 \pi G \rho_Q \Delta}{\Delta} \left\{ -14 + 9 \Delta + \frac{164 \Delta - 104 + \Omega_Q \left( \frac{64}{3} - 46 \Delta + 45 \Delta^2 \right)}{\Delta \Omega_Q} \right\} V'' \\
+ \frac{16 \rho_Q \Delta \left( 1 - \frac{\Delta}{2} \right) V' V''}{\Delta^2} + \frac{12 \left( 1 - \frac{\Delta}{2} \right)}{\Delta} \sqrt{\frac{8 \pi G}{3} \rho_c} (\Omega_Q - \frac{3}{\Delta} + 6) \left( V'' \right)^2 \\
+ 16 \pi G \sqrt{\frac{8 \pi G}{3} \rho_c} \left\{ -99 + 54 \Delta + \frac{42 \Omega_Q - 112 + 36 \Delta^2 + 84 \Delta}{\Delta \Omega_Q} \right\} \\
- 12 \pi G \rho_c \left\{ \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right\} \left\{ - \frac{4}{\Delta} V'' \frac{6}{\Delta} \sqrt{\frac{8 \pi G}{3} \Omega_Q \Delta^2} \left( 1 - \Delta \right) \left( 6 + \Omega_Q \right) - \frac{1}{3} \Omega_Q \right\} V'' \\
+ \left( \frac{2}{\Delta} \right)^2 \left( 1 - \frac{\Delta}{2} \right) \left( V'' \right)^2 + \frac{32 \pi G}{\Delta \Omega_Q} (7 - 6 \Delta) \right\}\]

= \frac{3^2 \Delta^2 \left( 1 - \frac{\Delta}{2} \right) \Omega_Q}{2a^3 (8 \pi G)^2 \rho_c} \left\{ - \sqrt{\frac{128 \pi G}{3 \Delta} \Omega_Q \rho_c} \frac{V''}{V} + \frac{16 \pi G \rho_c}{\Delta} \left\{ 10 + 3 \Omega_Q \left( \frac{2}{3} - \Delta \right) - 8 \Delta \right\} \frac{V''}{V} \\
+ \left( \sqrt{\frac{8 \pi G}{3 \Delta} \Omega_Q \rho_c} \left\{ -14 + 9 \Delta + \frac{164 \Delta - 104 + \Omega_Q \left( \frac{64}{3} - 46 \Delta + 45 \Delta^2 \right)}{\Delta \Omega_Q} \right\} \\
- 9 \left( \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right) \sqrt{\frac{8 \pi G}{3 \Omega_Q \Delta^2} \rho_c} \left\{ (1 - \Delta) \left( 6 + \Omega_Q \right) - \frac{1}{3} \Omega_Q \right\} \right\} \frac{V''}{V} \\
+ \frac{16 \left( 1 - \frac{\Delta}{2} \right)}{\Delta^2} \sqrt{\frac{8 \pi G}{3 \Omega_Q \Delta \rho_c} \frac{V'}{V}} \\
+ \left\{ \frac{32 \pi G \rho_c}{\Delta} \left( \Omega_Q - \frac{3}{\Delta} + 6 \right) - 12 \pi G \rho_c \left\{ \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right\} \left( \frac{2}{\Delta} \right)^2 \left( 1 - \frac{\Delta}{2} \right) \right\} \left( \frac{V''}{V} \right)^2 \\
+ \frac{128 \pi G^2 \rho_c}{3 \Delta \Omega_Q} \left\{ -99 + 54 \Delta + \frac{42 \Omega_Q - 112 + 36 \Delta^2 + 84 \Delta}{\Delta \Omega_Q} \right\} \\
- \frac{384 \pi G^2 \rho_c}{\Delta \Omega_Q} (7 - 6 \Delta) \left\{ \Omega_Q \left( \frac{2}{3} - \Delta \right) - \frac{8}{3} \Delta \right\} \right\} \right\} \\
\right\} (A32)\]
References

[1] A. G. Riess et al., Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, Astron. J. 116, 1009 (1998).

[2] S. Perlmutter et al. Measurements of $\Omega$ and $\Lambda$ from 42 High-Redshift Supernovae, Astrophys. J. 517, 565 (1999).

[3] S. Weinberg, The Cosmological Constant Problem, Rev. Mod. Phys. 61 (1989), 1.

[4] L. Amendola and S. Tsujikawa, Dark Energy, (Cambridge University Press, Cambridge, 2010).

[5] S. Tsujikawa, Quintessence: A Review, Class. Quant. Gravi. 30 (2013), 214003. [arXiv:1304.1961v2 [gr-qc]].

[6] T. Hara, R. Sakata, Y. Muromachi, and Y. Itoh, Time variation of Equation of State for Dark Energy , Prog. Theor. Exp. Phys., 2014, 113E01 (2014), [arXiv:1409.2726 [astro-ph]].

[7] E. V. Linder, Exploring the expansion history of the universe, Phys. Rev. Lett. 90 (2003), 091301, [arXiv:0208512v1 [astro-ph]].

[8] M. Chevallier and D. Polarski, Accelerating universes with scaling dark matter, Int. J. Mod. Phys. D10 (2001), 213, [arXiv:0009008v2 [gr-qc]].

[9] S. Ray, M.Yu.Khlopov, P. P. Ghosh and Utpal Mukhopadhyay, Phenomenology of $\Lambda$-CDM model: a possibility of accelerating Universe with positive pressure, Int. J. Theor. Phys. 50 (2011), 939, [arXiv:0711.0686 [gr-qc]].

[10] N. Suzuki et al., The Hubble Space Telescope Cluster Supernova Survey. V., Astrophys. J. 746 (2012) 85.

[11] P. J. Steinhardt, L. Wang and I. Zlatev, Cosmological tracking solutions, Phys. Rev. D 59 (1999), 123504.

[12] I. Zlatev, L. Wang, P. J. Steinhardt, Quintessence, Cosmic Coincidence, and the Cosmological Constant, Phys. Rev. Lett. 82 (1999), 896.

[13] R. R. Caldwell and E. V. Linder, Limits of Quintessence, Phys. Rev. Lett. 95 (2005), 141301.

[14] P. J. E. Peebles and B. Ratra, The Cosmology with a time-variable cosmological “constant”, Astrop. J. 325 (1988), L17.

[15] P. Brax and J. Martin, Quintessence and Supergravity, Phys. Lett. B468 (1999), 40, [arXiv:9905040v2 [astro-ph]].
[16] P. Wang, C. Chen and P. Chen, *Confronting tracker field quintessence with data*, JCAP. **02** (2012), 016, [arXiv:1108.1424 [astro-ph]].

[17] J. A. Frieman, C. T. Hill, A. Stebbins, and I. Waga, *Cosmology With Ultralight Pseudo Nambu-Goldstone Bosons*, Phys. Rev. Lett. **75** (1995), 2077.

[18] S. Dutta and R. J. Scherrer, *Hilltop Quintessence*, Phys. Rev. D **78** (2008), 123525, [arXiv:0809.4441 [astro-ph]].

[19] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, *Phantom Energy and Cosmic Doomsday*, Phys. Rev. Lett. **D 91** (2003), 071301.

[20] Y. F. Cai, E. N. Saridakis, M. R. Setare and J. Q. Xia, *Quintom Cosmology: Theoretical implications and observations*, Phys. Rept. **493** (2010), 1, [arXiv:0909.2776 [hep-th]].

[21] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, *Essentials of k-Essence*, Phys. Rev. D **63** (2001), 103510, [arXiv:0006373v1 [astro-ph]].

[22] J. Khoury, *Chameleon Field Theories*, [arXiv:1306.4326 [astro-ph. CO]].

[23] B. Novosyadlyj, *Tachyonic fields in Cosmology*, [arXiv:1311.0227v3 [asrr-ph. CO]].

[24] Y. Fujii, *Mass of the dilaton and the cosmological constant*, Prog. Theor. Phys. **110**, 433, [arXiv:021203[gr-qc]].

[25] M. Li, *A Model of Holographic Dark Energy*, Phys. Lett. B **603** (2004), 1.

[26] T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, *Modified Gravity and Cosmology*, Physics Reports **513**, 1 (2012), 1.