ON CONFORMAL POWERS OF THE DIRAC OPERATOR ON EINSTEIN MANIFOLDS

MATTHIAS FISCHMANN\textsuperscript{1}, CHRISTIAN KRATTENTHALER\textsuperscript{2} AND PETR SOMBERG\textsuperscript{1}

Abstract. We determine the structure of conformal powers of the Dirac operator on Einstein $\text{Spin}$-manifolds in terms of the product formula for shifted Dirac operators. The result is based on the techniques of higher variations for the Dirac operator on Einstein manifolds and spectral analysis of the Dirac operator on the associated Poincaré-Einstein metric, and relies on combinatorial recurrence identities related to the dual Hahn polynomials.

1. Introduction

Conformally covariant operators like the Yamabe-Laplace or the Dirac operator are of central interest in geometric analysis on manifolds. The Yamabe-Laplace operator is a representative of conformally covariant operators termed GJMS operators, cf. [GJMS92, GZ03, GP03], and this is analogous for the Dirac operator, cf. [HS01, GMP12, Fis13]. Their original construction is based on the ambient metric, or, equivalently, on the associated Poincaré-Einstein metric introduced by Fefferman and Graham [FG85, FG11].

In the case of GJMS operators, it is shown in [GH04] that in even dimensions $n$ there exists in general no conformal modification of the $k$-th power of the Laplace operator for $k > n/2$. In the case of conformal powers of the Dirac operator on general $\text{Spin}$-manifolds, all known constructions break down for even dimensions $n$ if the order of the operator exceeds the dimension. The effect of non-existence for higher order conformal powers of the Laplace and Dirac operators does not apply for certain classes of manifolds, for example flat manifolds [Slo93] or Einstein manifolds [Gov06].

The proper understanding of the internal structure of conformal powers of the Laplace and Dirac operators is a difficult task, see the progress for GJMS operators in [Juh13]. In particular, there exists a sequence of second order differential operators such that all GJMS operators are polynomials in this collection of second order differential operators, and vice versa. Such a structure, in terms of first order differential operators, is also available for low order examples of conformal powers of the Dirac operator, cf. [Fis13, Chapter 6]. Explicit formulas are available on flat manifolds, where they are just powers of the Laplace or Dirac operators, on the spheres [Bra95, ES10], where they factor into a product of shifted Laplace or Dirac operators, or on Einstein manifolds, where the GJMS operators factor into a product of shifted Laplace operators [Gov06, FG11].
The main aim of our article is to complete these results for conformal powers of the Dirac operator on Einstein Spin-manifolds. The result is based on the proper understanding of higher variations of the Dirac operator on Einstein manifolds and the spectral analysis of the Dirac operator on the associated Poincaré-Einstein metric. The derivations of specific formulas rely on combinatorial recurrence identities related to dual Hahn polynomials.

The product structure, or factorization, of conformal powers of the Laplace and Dirac operators is applied in theoretical physics, cf. [Dow11, Dow13], to compute conformal and multiplicative anomalies of functional determinants in the context of the AdS/CFT correspondence.

The paper is organized as follows. In Section 3 we discuss variations to all orders of the Dirac operator on semi-Riemannian Einstein Spin-manifolds with respect to the 1-parameter family of metrics arising from the Poincaré-Einstein metric, cf. Theorem 3.5. In Section 4 we briefly recall the dual Hahn polynomials, which form a special class of generalized hypergeometric functions. The solution of certain recurrence relation, derived in Section 5, has an interpretation in terms of dual Hahn polynomials. Section 5 contains our main theorem, Theorem 5.2. Its proof is based on a recurrence relation, deduced from the construction of conformal powers on the Dirac operator of a semi-Riemannian Einstein Spin-manifold via the associated Poincaré-Einstein metric, cf. Proposition 4.1. The final section collects several statements and applications, aiming at the description of a still largely conjectural holographic deformation of the Dirac operator.

2. Semi-Riemannian Spin-geometry, Clifford algebras, and Poincaré-Einstein spaces

In the present section we review conventions and notation related to semi-Riemannian Spin-geometry and the Poincaré-Einstein metric construction used throughout the article.

Let $(M, h)$ be a semi-Riemannian Spin-manifold of signature $(p, q)$ and dimension $n = p + q$. Then any orthonormal frame $\{e_i\}_{i=1}^n$ fulfills $h(e_i, e_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = -1$ for $1 \leq i \leq p$ and $\varepsilon_i = 1$ for $p + 1 \leq i \leq n$.

The Clifford algebra of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{p,q})$, denoted by $Cl(\mathbb{R}^{p,q})$, is a quotient of the tensor algebra of $\mathbb{R}^n$ by the two sided ideal generated by the relations $x \otimes y + y \otimes x = -2 \langle x, y \rangle_{p,q}$ for all $x, y \in \mathbb{R}^n$. In the even case $n = 2m$, the complexified Clifford algebra $Cl_C(\mathbb{R}^{p,q})$ has a unique irreducible representation up to isomorphism, whereas in the odd case $n = 2m + 1$ it has two non-equivalent irreducible representations on $\Delta_n := \mathbb{C}^{2m}$, again unique up to isomorphism. The restriction of this representation to the spin group $Spin(p,q)$, regarded as a subgroup of the group of units $Cl^*(\mathbb{R}^{p,q})$, is denoted by $\kappa_n$.

The choice of a Spin-structure $(Q, f)$ on $(M, h)$ provides an associated spinor bundle $S(M, h) := Q \times_{(Spin_0(p,q), \kappa_n)} \Delta_n$, where $Spin_0(p,q)$ denotes the connected component of the spin group containing the identity element. (We could work with the full spin group as well, because we do not need the existence of a scalar product). Then the Levi-Civita connection $\nabla^h$ on $(M, h)$ lifts to a covariant derivative $\nabla^{h,S}$ on the spinor bundle. The associated Dirac operator is denoted by $\slashed{D}$. 
Let \( \hat{h} = e^{2\sigma}h \) be a metric conformally related to \( h, \sigma \in C^\infty(M) \). The spinor bundles for \( \hat{h} \) and \( h \) can be identified by a vector bundle isomorphism \( F_\sigma : S(M,h) \rightarrow S(M,\hat{h}) \), and the Dirac operator satisfies the conformal transformation law

\[
\hat{D}(e^{\frac{1-n}{2}\sigma}\psi) = e^{-\frac{1-n}{2}\sigma}\hat{D}\psi,
\]

for all smooth sections \( \psi \in \Gamma(S(M,h)) \), and \( \hat{\ } \) denotes the evaluation with respect to \( \hat{h} \). Conformal odd powers of the Dirac operator were constructed in [HS01, GMP12, Fis13], and are denoted by \( D_{2N+1} = D^{2N+1} + \text{LOT} \), for \( N \in \mathbb{N}_0 (N < \frac{n}{2} \text{ for even } n) \). Here, LOT stands for “lower order terms.” They satisfy

\[
\hat{D}_{2N+1}(e^{\frac{2N+1-n}{2}\sigma}\hat{\psi}) = e^{-\frac{2N+1-n}{2}\sigma}\hat{D}_{2N+1}\psi
\]

for all smooth function \( \sigma \in C^\infty(M) \) and sections \( \psi \in \Gamma(S(M,h)) \).

As for the Poincaré-Einstein metric construction we refer to [FG11]. The Poincaré-Einstein metric associated with an \( n \)-dimensional semi-Riemannian manifold \((M,h)\), \( n \geq 3 \), is \( X := M \times (0,\varepsilon), \varepsilon \in \mathbb{R}_+ \), equipped with the metric

\[
g_+ = r^{-2}(dr^2 + h_r),
\]

for a 1-parameter family of metrics \( h_r \) on \( M, h_0 = h \). The requirement of the Einstein condition on \( g_+ \) for \( n \) odd,

\[
Ric(g_+) + ng_+ = O(r^\infty),
\]

uniquely determines the family \( h_r \), while for \( n \) even the conditions

\[
Ric(g_+) + ng_+ = O(r^{n-2}), \quad tr(Ric(g_+) + ng_+) = O(r^{n-1}),
\]

uniquely determine the coefficients \( h_{(2)}, \ldots, h_{(n-2)}, \tilde{h}_{(n)} \) and the trace of \( h_{(n)} \) in the formal power series

\[
h_r = h + r^2h_{(2)} + \cdots + r^{n-2}h_{(n-2)} + r^n(h_{(n)} + \tilde{h}_{(n)} \log r) + \cdots.
\]

For example, we have

\[
h_{(2)} = -P, \quad h_{(4)} = \frac{1}{4} \left( P^2 - \frac{B}{n-3} \right),
\]

where \( P \) is the Schouten tensor and \( B \) is the Bach tensor associated with \( h \).

All constructions in the present article, based on the Poincaré-Einstein metric, depend for even \( n \) on the coefficients \( h_{(2)}, \ldots, h_{(n-2)} \) and \( tr(h_{(n)}) \) only. Choosing different representatives \( h, \tilde{h} \in [\hat{h}] \) in the conformal class leads to Poincaré-Einstein metrics \( g^1_+ \) and \( g^2_+ \) related by a diffeomorphism \( \Phi : U_1 \subset X \rightarrow U_2 \subset X \), where both \( U_i, i = 1,2 \), contain \( M \times \{0\}, \Phi|_M = Id_M \), and \( g^1_+ = \Phi^*g^2_+ \) (up to a finite order in \( r \), for even \( n \)).

3. Variation of the Dirac operator induced by the Poincaré-Einstein metric

In this section we give a complete description of the variations of the Dirac operator, associated with the 1-parameter family \( h_r \) induced by the Poincaré-Einstein metric \( g_+ \), assuming that \((M,h_0 = h)\) is Einstein.

For a general 1-parameter family of metrics \( h_r \) on a Riemannian \( Spin \)-manifold, the first variation of the Dirac operator was discussed in [BG92], which we will adapt and
make explicit for the 1-parameter family of metrics \( h_r \) induced by the Poincaré-Einstein metric.

Motivated by a proof of the fundamental theorem of hypersurface theory and a new way to identify spinors for different metrics, [BGM05] introduced the technique of generalized cylinders to derive the first order variation formula for the Dirac operator with respect to a deformation of the underlying metric.

In general, the topic of higher metric variations for the Dirac operator was not discussed in the literature. In case \((M, h)\) is Einstein, the associated Poincaré-Einstein metric takes a very simple form, cf. Equation (3.1). This allows for a complete description of variation formulas of general order for the Dirac operator associated with \( h_r \). The higher variation formulas are used in Section 5 to make the construction of conformal powers of the Dirac operator very explicit, ending in a product structure for \( \mathcal{D}_{2N+1}, N \in \mathbb{N}_0 \).

Throughout the article, we use the standard notation in semi-Riemannian geometry, e.g., \( \text{Ric}, \tau \) are the Ricci tensor and its scalar curvature, respectively.

Let \((M, h)\) be a semi-Riemannian Einstein manifold of dimension \( n \) with normalized Einstein metric \( h \),

\[
\text{Ric}(h) = 2\lambda(n-1)h, \quad \lambda \in \mathbb{R}.
\]

This implies \( P = \frac{J}{n}h \), where \( J = \frac{\tau}{2(n-1)} \) is the normalized scalar curvature and \( P = \frac{1}{n-2}(\text{Ric} - Jh) \) is the Schouten tensor. The associated Poincaré-Einstein metric \( g_+ = r^{-2}(dr^2 + h_r) \) on \( X \) is determined by the 1-parameter family of metrics \( h_r \) on \( M \),

\[
h_r = h - r^2 \frac{J}{n}h + r^4 \left( \frac{J}{2n} \right)^2 h = \left( 1 - \frac{J}{2n}r^2 \right)^2 h, \quad h_0 = h.
\]

For \( r \in \mathbb{R}_+ \) small enough we consider a point-wise isomorphism \( f_r : T_xM \to T_xM \), \( x \in M \), relating \( h = h_0 \) and \( h_r \) via

\[
Y \mapsto f_rY := \left( 1 - \frac{J}{2n}r^2 \right)^{-1} Y, \quad \text{for all } Y \in \Gamma(TM),
\]

characterized by \( h(Y, U) = h_r(f_rY, f_rU) \) for all \( Y, U \in \Gamma(TM) \) and \( f_0 = \text{Id}_{TM} \).

Let us introduce the Levi-Civita covariant derivatives on \( TM \) corresponding to \( h \) and \( h_r \):

\[
\nabla^h, \nabla^{h_r} : \Gamma(TM) \to \Gamma(T^*M \otimes TM),
\]

and

\[
\nabla^{h,h_r} : \Gamma(TM) \to \Gamma(T^*M \otimes TM)
\]

\[
Y \mapsto \left\{ U \to \nabla^{h,h_r}_Y U := (f_r^{-1} \circ \nabla^{h_r}_Y \circ f_r)U \right\}. \quad (3.2)
\]

The covariant derivatives \( \nabla^h, \nabla^{h_r}, \nabla^{h,h_r} \) extend by the Leibniz rule and the spin representation to tensor-spinor fields. For example one has

\[
(\nabla^h_U f)(Y) = \nabla^h_U(fY) - f(\nabla^h_U Y),
\]

for all \( f \in \text{End}(TM) \), and \( U, Y \in \Gamma(TM) \).

\textbf{Lemma 3.1.} \textit{The covariant derivative} \( \nabla^{h,h_r} \) \textit{is metric for} \( h \), \textit{and its torsion} \( T^r \) \textit{satisfies}

\[
T^r(U, Y) = f_r^{-1}((\nabla^{h_r}_U f_r)(Y) - (\nabla^{h_r}_Y f_r)(U)), \quad \text{for all } U, Y \in \Gamma(TM).
\]

\textit{(3.3)}
Let $Y, U, Z \in \Gamma(TM)$. First we show the $h$-metricity of $\nabla^{h,hr}$:

\[
(\nabla^{h,hr}_Y h)(U, Z) = Y(h(U, Z)) - h(\nabla^{h,hr}_Y U, Z) - h(\nabla^{h,hr}_Y Z, U)
\]

\[
= Y(h_r(f_r U, f_r Z)) - h(f^{-1}_r \nabla^{h,hr}_Y (f_r U), Z) - h(f^{-1}_r \nabla^{h,hr}_Y (f_r Z), U)
\]

\[
= Y(h_r(f_r U, f_r Z)) - h(h^{hr}_r (f_r U), f_r Z) - h(h^{hr}_r (f_r Z), f_r U)
\]

\[
= \nabla^{h,hr}_Y h_r(f_r U, f_r Z) = 0.
\]

The second statement follows from

\[
T^r(U, Y) = \nabla^{h,hr}_Y Y - \nabla^{h,hr}_Y U - [U, Y]
\]

\[
= f^{-1}_r(\nabla^{h,hr}_U f_r Y - \nabla^{h,hr}_Y f_r U) - \nabla^{h,hr}_U Y + \nabla^{h,hr}_Y U
\]

\[
= f^{-1}_r(\nabla^{h,hr}_U f_r Y - f_r(\nabla^{h,hr}_U Y) - \nabla^{h,hr}_Y f_r U + f_r(\nabla^{h,hr}_Y U))
\]

\[
= f^{-1}_r((\nabla^{h,hr}_U f_r)(Y) - (\nabla^{h,hr}_Y f_r)(U)).
\]

It is well known that two $h$-metric covariant derivatives on $(M, h)$ differ by their torsions.

**Lemma 3.2.** We have

\[
h(\nabla^{h,hr}_U Y, Z) = h(\nabla^{h,hr}_U Y, Z) + \frac{1}{2}[h(T^r(Z, Y), U) - h(T^r(Y, U), Z) - h(T^r(U, Z), Y)]
\]

for all $U, Y, Z \in \Gamma(TM)$.

**Proof.** Let $U, Y, Z \in \Gamma(TM)$. Any two covariant derivatives differ by a tensor field $\omega \in \Gamma(T^*M \otimes TM \otimes T^*M)$, i.e., $\nabla^{h,hr}_Y Y = \nabla^{h,hr}_U Y + \omega(U)Y$. Since both covariant derivatives are metric with respect to $h$, we get

\[
0 = \nabla^{h,hr}_U h(Y, Z) - \nabla^{h,hr}_Y h(Y, Z) = -[h(\omega(U)Y, Z) + h(\omega(U)Z, Y)].
\]

Since $\nabla^h$ is torsion-free, we have

\[
T^r(U, Y) = \nabla^{h,hr}_U Y - \nabla^{h,hr}_Y U - [U, Y]
\]

\[
= \nabla^{h,hr}_U Y - \nabla^{h,hr}_Y U - \nabla^h Y + \nabla^h U
\]

\[
= \omega(U)Y - \omega(Y)U.
\]

Using Equation (3.4), we see that this implies

\[
h(T^r(Z, Y), U) - h(T^r(Y, U), Z) - h(T^r(U, Z), Y)
\]

\[
= 2h(\omega(U)Y, Z) = 2h(\nabla^{h,hr}_U Y - \nabla^{h,hr}_Y Y, Z),
\]

and the proof is complete. $\square$

Any $h$-metric covariant derivative induces a covariant derivative on the spinor bundle $S(M, h)$, hence $\nabla^h$, $\nabla^{hr}$ and $\nabla^{h,hr}$ induce

\[
\nabla^{h,S} : \Gamma(S(M, h)) \to \Gamma(T^*M \otimes S(M, h)),
\]

\[
\nabla^{hr,S} : \Gamma(S(M, h_r)) \to \Gamma(T^*M \otimes S(M, h_r)),
\]

\[
\nabla^{r,S} : \Gamma(S(M, h)) \to \Gamma(T^*M \otimes S(M, h)).
\]
Note that the last of the above covariant derivatives equals $\nabla^{h,h_r,S}$, but we will use the abbreviation $\nabla^{r,S}$. It follows from Lemma 3.2 that locally
\[
\nabla^{r,S}_{\xi_i} \psi = s_i(\psi) + \frac{1}{2} \sum_{j<k} \varepsilon_j \varepsilon_k h(\nabla^{h,h_r}_{\xi_i} s_j, s_k) s_j \cdot s_k \cdot \psi
\]
\[
= \nabla^{h,S}_{\xi_i} \psi + \frac{1}{8} \sum_{j<k} \varepsilon_j \varepsilon_k r^{\tau}_{ij} s_j \cdot s_k \cdot \psi, \tag{3.6}
\]
where $\psi \in \Gamma(S(M,h))$, $\{s_i\}_{i=1}^n$ is an $h$-orthonormal frame, $r^{\tau}_{ij} := (1 - \sigma - \sigma^2) r^{\tau}_{ij}$, with $\sigma r^{\tau}_{ij} = T^{\tau}_{ijk}$ and $T^{\tau}(U,Y,Z) := h(T^{\tau}(U,Y), Z)$ for $U,Y,Z \in \Gamma(TM)$. The isometry $f_r : T_x M \to T_x M$, $x \in M$, pulls-back $h$ to $h_r$ and lifts to a spinor bundle isomorphism $\beta_r : S(M,h) \to S(M,h_r)$.

It preserves the base points on $M$ and satisfies $\beta_r(U \cdot \psi) = f_r(U) \cdot \beta_r(\psi)$ for all $U \in \Gamma(TM), \psi \in \Gamma(S(M,h))$.

**Lemma 3.3.** Let $\psi \in \Gamma(S(M,h)), Y \in \Gamma(TM)$. Then
\[
\nabla^r_{\psi} \psi = (\beta_r^{-1} \circ \nabla^{h_r,S} \circ \beta_r) \psi. \tag{3.7}
\]

**Proof.** For $\psi \in \Gamma(S(M,h))$, we have
\[
\nabla^r_{\psi} \psi = Y(\psi) + \frac{1}{2} \sum_{j<k} \varepsilon_j \varepsilon_k h(\nabla^{h_r}_{\psi} s_j, s_k) s_j \cdot s_k \cdot \psi
\]
\[
= Y(\psi) + \frac{1}{2} \sum_{j<k} \varepsilon_j \varepsilon_k h_r(\nabla^{h_r}_{Y} f_r s_j, f_r s_k) \beta_r^{-1}(f_r s_j \cdot f_r s_k \cdot \beta_r \psi)
\]
\[
= (\beta_r^{-1} \circ \nabla^{h_r,S} \circ \beta_r) \psi,
\]
which completes the proof. \qed

Let us introduce the notation
\[
f^{(l)} := \frac{d^l}{dr^l}(f_r)|_{r=0}, \quad T^{(l)}(U,Y) := \frac{d^l}{dr^l}(T^{r}(U,Y))|_{r=0}, \tag{3.8}
\]
for $l \in \mathbb{N}_0$.

**Lemma 3.4.**
\begin{enumerate}
\item Let $l \in \mathbb{N}_0$. Then
\[
f^{(2l+1)} U = 0, \quad f^{(2l)} U = \frac{(2l)!}{2^n} \left( \frac{d}{d r} \right)^l U, \quad U \in \Gamma(TM). \tag{3.9}
\]
\item Let $l \in \mathbb{N}$. Then the torsion $T^r$ of $\nabla^{h,h_r}$ fulfills
\[
T^{(l)}(U,Y) = 0 \tag{3.10}
\]
for all $U,Y \in \Gamma(TM)$.
\end{enumerate}

**Proof.** Expansion of $f_r$ into a formal power series
\[
f_r = (1 - \frac{r^2}{2})^{-1} Id_{TM} = \sum_{l \geq 0} \frac{r^{2l}}{(2l)!} (\frac{d}{d r})^l Id_{TM}
\]
gives the first statements. It follows from Lemma 3.1 that the $l$-th derivative of $T^r$ at $r = 0$ is given by a sum, where each contribution contains derivatives of $f_r$, $f_r^{-1}$ and $\nabla^{h_r}$ at $r = 0$. Using $f^{(2l+1)} = 0$ and $f^{(2l)} = \frac{(2l)!}{2^n} (\frac{d}{d r})^l Id_{TM}$, for all $l \in \mathbb{N}_0$, we just have to show that $\frac{d^l}{dr^l}(\nabla^{h_r})|_{r=0}$ acts as a covariant derivative, hence annihilating the identity
map. But this is obvious, since \( \frac{d}{dr} \) does not effect the property of \( \nabla^{hr} \) being a covariant derivative.

In what follows, we use two Dirac operators induced by \( \nabla^{h,S} \) and \( \nabla^{hr,S} \):

\[
\begin{align*}
\mathcal{D}^{h} & : \Gamma(S(M,h)) \to \Gamma(S(M,h)), \\
\mathcal{D}^{hr} & : \Gamma(S(M,h_r)) \to \Gamma(S(M,h_r)).
\end{align*}
\] (3.11)

Furthermore, we define

\[
\mathcal{D}^{h,hr} : \Gamma(S(M,h)) \to \Gamma(S(M,h))
\]

\[
\psi \mapsto \beta_{r}^{-1} \circ \mathcal{D}^{hr} \circ \beta_{r}(\psi).
\] (3.12)

Lemma 3.3 and \( \beta_{r}(U \cdot \psi) = f_{r}(U) \cdot \beta_{r}(\psi) \) for \( U \in \Gamma(TM) \), \( \psi \in \Gamma(S(M,h)) \) imply

\[
\mathcal{D}^{h,hr}_{r=0} \psi = \sum_{i=1}^{n} \varepsilon_{i}s_{i} \cdot \nabla^{r,S}_{f_r(s_i)} \psi,
\] (3.13)

and the \( r \)-derivatives of \( \mathcal{D}^{h,hr} \) at \( r = 0 \) yield the variation formulas for \( \mathcal{D}^{h} \) with respect to the 1-parameter deformation \( h_r \) of \( h \). Note that \( \mathcal{D}^{h,hr} \) is not the Dirac operator induced by \( \nabla^{r,S} \).

**Theorem 3.5.** Let \( (M, h) \) be a semi-Riemannian Einstein Spin-manifold with \( \dim(M) = n \), i.e., \( \text{Ric} = 2(n-1)\lambda h \). Let \( g_{+} \) be the associated Poincaré-Einstein metric on \( X \) with \( g_{+} = r^{-2}(dr^2 + h_r) \), \( h_r = (1 - \frac{f}{2n} r^2)^2 h \). Then, for all \( l \in \mathbb{N}_0 \), \( \psi \in \Gamma(S(M,h)) \), we have

\[
\begin{align*}
\frac{d^{(2l)}}{dr^{(2l)}}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} & = (2l)! \left( \frac{f}{2n} \right)^l \mathcal{D}^{h} \psi, \\
\frac{d^{(2l+1)}}{dr^{(2l+1)}}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} & = 0.
\end{align*}
\] (3.14)

**Proof.** The \( r \)-derivatives of \( \mathcal{D}^{h,hr}_{r=0} \) at \( r = 0 \) are

\[
\frac{d}{dr}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} = \sum_{k=0}^{l} \frac{\binom{l}{k}}{k!} \sum_{i=1}^{n} \varepsilon_{i}s_{i} \cdot \nabla^{r,S}_{f_r(s_i)}^{(l-k)s_{i}} \psi,
\]

where \( \nabla^{r,S}_{f_r(s_i)}^{(k)} := \frac{d^{k}}{dr^{k}}(\nabla^{r,S}) |_{r=0} \), for \( k \in \mathbb{N}_0 \). From Equation (3.6), we obtain

\[
\frac{d}{dr}(\nabla^{r,S}_{s_{i}} \psi) = \frac{1}{8} \sum_{j \neq k} \varepsilon_{j}\varepsilon_{k} \frac{d^{l}}{dr^{l}}(T^{r,\sigma}_{jik}) s_{j} \cdot s_{k} \cdot \psi,
\]

which vanishes at \( r = 0 \) due to Lemma 3.4 and the linearity of \( \frac{d}{dr} \). Thus we get

\[
\frac{d}{dr}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} = \sum_{i=1}^{n} \varepsilon_{i}s_{i} \cdot \nabla^{0,S}_{f_r(s_i)} \psi.
\]

Since \( \nabla^{0,S} \) agrees with the spinor covariant derivative \( \nabla^{h,S} \) on \( S(M,h) \), we may conclude by Lemma 3.4 that

\[
\frac{d^{(2l)}}{dr^{(2l)}}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} = (2l)! \left( \frac{f}{2n} \right)^l \mathcal{D}^{h} \psi, \quad \frac{d^{(2l+1)}}{dr^{(2l+1)}}(\mathcal{D}^{h,hr}_{r=0} \psi) |_{r=0} = 0,
\]

hence completing the proof. \( \Box \)
4. Generalized hypergeometric functions and dual Hahn polynomials

The aim of the present section is to introduce a certain class of polynomials, to prove some of their combinatorial properties, and to give their interpretation in terms of dual Hahn polynomials. These polynomials will be responsible for the product structure of conformal powers of the Dirac operator.

The Pochhammer symbol of a complex number \( a \in \mathbb{C} \) is denoted by \( (a)_l \), and it is defined by \( (a)_l := a(a+1) \cdots (a+l-1) \) for \( l \in \mathbb{N} \), and \( (a)_0 := 1 \). The generalized hypergeometric function \( {}_pF_q \), for \( p, q \in \mathbb{N} \), with \( p \) upper parameters, \( q \) lower parameters, and argument \( z \), is defined by

\[
{}_pF_q \left[ \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right] := \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_p)_l}{(b_1)_l \cdots (b_q)_l} \frac{z^l}{l!},
\]

(4.1) for \( a_i \in \mathbb{C} \) \((1 \leq i \leq q)\), \( b_j \in \mathbb{C} \setminus \{-\mathbb{N}_0\} \) \((1 \leq j \leq q)\), and \( z \in \mathbb{C} \).

For later purposes, we introduce the polynomials

\[
\tilde{q}_m(y) := \sum_{l=0}^{m} (-1)^{m-l}(\frac{m}{2} + l)_{m-l}(k-m)_{m-l} \left( \frac{m}{l} \right) \prod_{j=1}^{l}(y-j^2),
\]

(4.2) for \( m \in \mathbb{N}_0 \), \( k, n \in \mathbb{N} \) and an abstract variable \( y \).

**Proposition 4.1.** The polynomials \( \tilde{q}_m(y) \), \( m \in \mathbb{N}_0 \), satisfy the recurrence relation

\[
\tilde{q}_{m+1}(y) = (y - 2m + 1)_{m=l}(k-m)_{m-l+1} \left( \frac{m}{l} \right) \prod_{j=1}^{l}(y-j^2),
\]

(4.3) with \( \tilde{q}_{-1}(y) := 0 \) and \( \tilde{q}_0(y) := 1 \).

**Proof.** We prove the statement by comparing the coefficients of \( \prod_{j=1}^{l}(y-j^2) \) on both sides of (4.3). The only detail to observe is that one must replace \( y \) in the coefficient of \( \tilde{q}_m(y) \) on the right-hand side of (4.3) by \( (y-(l+1)^2)+(l+1)^2 \), with \( l \) being the summation index of the sum in the definition of \( \tilde{q}_m(y) \). The term \((y-(l+1)^2)\) then combines with the product \( \prod_{j=1}^{l+1}(y-j^2) \) to become \( \prod_{j=1}^{l+1}(y-j^2) \). The verification that the coefficients of \( \prod_{j=1}^{l}(y-j^2) \) do indeed agree is then a routine matter. \( \square \)

**Remark.** Our considerations are motivated by [FG11], where the analogous recurrence relation

\[
q_{m+1}(y) = (y - 2m + 1)_{m=l}(k-m)_{m-l+1} \left( \frac{m}{l} \right) q_m(y)
\]

\[
- m(m-k)(m+\frac{n}{2})(m+1+k)q_{m-1}(y),
\]

for \( k, n \in \mathbb{N} \), \( m \in \mathbb{N}_0 \), and \( q_{-1}(y) := 0 \), \( q_0(y) := 1 \), appears. Its solution is given by

\[
q_m(y) := \sum_{l=0}^{m} (-1)^{m-l}(\frac{m}{2} + l)_{m-l}(k-m)_{m-l} \left( \frac{m}{l} \right) \prod_{j=1}^{l}(y-j(j-1)).
\]

In the rest of the section, we discuss interpretations of \( \tilde{q}_m(y) \) and \( q_m(y) \) in terms of dual Hahn polynomials, cf. [KM61], [KLS10]. The Hahn polynomial \( Q_n(x) := Q_n(x; \alpha, \beta, \gamma, \delta) \),
for all \( m \in \mathbb{N}_0, k, n \in \mathbb{N} \), we have

\[
\tilde{q}_m(\lambda(y - 1)) = (-1)^m \binom{n}{\frac{n}{2} + 1} m(k - m)_m \binom{y}{\frac{n}{2} + 1, 1 - k} 3F_2 \left[ \frac{-(y - 1), -m, 1 + y}{\frac{n}{2} + 1, 1 - k}; 1 \right]
\]

\[
= (-1)^m \binom{n}{\frac{n}{2} + 1} m(k - m)_m Q_{y - 1}(m; \frac{n}{2}, \frac{n}{2}, \frac{n}{2} - k, \frac{n}{2})
\]

\[
= (-1)^m \binom{n}{\frac{n}{2} + 1} m(k - m)_m R_m(\lambda(y - 1); \frac{n}{2}, 1 - \frac{n}{2}, k)
\]

(4.7)

and

\[
q_m(\lambda(y)) = \binom{n}{\frac{n}{2}} m(k - m)_m (-1)^m 3F_2 \left[ \frac{-y, -m, 1 + y}{\frac{n}{2}, 1 - k}; 1 \right]
\]

\[
= \binom{n}{\frac{n}{2}} m(k - m)_m (-1)^m Q_y(m; \frac{n}{2}, \frac{n}{2} - 1, 1 - \frac{n}{2}, k)
\]

\[
= \binom{n}{\frac{n}{2}} m(k - m)_m (-1)^m R_m(\lambda(y); \frac{n}{2} - 1, 1 - \frac{n}{2}, k).
\]

(4.8)

Hence, up to a multiplicative factor, both \( \tilde{q}_m(y) \) and \( q_m(y) \) can be realized as dual Hahn polynomials.
5. Product structure (factorization) of conformal powers of the Dirac operator

In the present section, we show that conformal powers of the Dirac operator on Einstein manifolds obey a product structure, in the sense that they factor into linear factors based on shifted Dirac operators. This result is parallel to the case of conformal powers of the Laplace operator on Einstein manifolds, cf. [Gov06, Theorem 1.2].

Let us denote the Dirac operator on \((M, h)\) by \(\mathcal{D}\). (Notice that in Section \(3\) we used \(\mathcal{D}^h\) instead of \(\mathcal{D}\).) The proof of our main result, Theorem 5.2, relies on the construction of conformal powers of the Dirac operator.

**Theorem 5.1** ([GMP12]). Let \((M, h)\) be a semi-Riemannian Spin-manifold of dimension \(n\). For every \(N \in \mathbb{N}_0\) (\(N \leq n/2\) for even \(n\)) there exists a linear differential operator, called conformal power of the Dirac operator,

\[
\mathcal{D}_{2N+1} : \Gamma(S(M, h)) \rightarrow \Gamma(S(M, h)), \tag{5.1}
\]

satisfying

1. \(\mathcal{D}_{2N+1}\) is of order \(2N + 1\) and \(\mathcal{D}_{2N+1} = \mathcal{D}^{2N+1} + \text{LOT}\), where, as before, LOT denotes lower order terms;
2. \(\mathcal{D}_{2N+1}\) is conformally covariant, that is,

\[
\mathcal{D}_{2N+1} \left( e^{\frac{2N+1}{2} \sigma} \psi \right) = e^{-\frac{2N+1}{2} \sigma} \mathcal{D}_{2N+1} \psi \tag{5.2}
\]

for every \(\psi \in \Gamma(S(M, h))\), \(\sigma \in \mathcal{C}^\infty(M)\).

We briefly outline the main point of the proof, which will be then analyzed in detail on Einstein manifolds. Let \(g_+\) be the associated Poincaré-Einstein metric on \(X\) with conformal infinity \((M, [h])\). The conformal compactification of \((X = M \times (0, \varepsilon), g_+)\) is

\[
(M \times [0, \varepsilon], \bar{g} := r^2 g_+ = dr^2 + h_r),
\]

where \(\bar{g}\) smoothly extends to \(r = 0\). Corresponding spinor bundles are denoted by

\[
S(M, h), \quad S(X, g_+), \quad S(X, \bar{g}),
\]

respectively. The spinor bundle \(S(X, \bar{g})|_{r=0}\) is isomorphic to \(S(M, h)\) if \(n\) is even, and it is isomorphic to \(S(M, h) \oplus S(M, h)\) if \(n\) is odd. The proof of Theorem 5.1 is based on the extension of a boundary spinor \(\psi \in \Gamma(S(X, \bar{g})|_{r=0})\) to the interior \(\theta \in \Gamma(S(X, \bar{g}))\): one requires \(\theta\) to be a formal solution of

\[
D(\bar{g})\theta = i\lambda \theta, \quad \lambda \in \mathbb{C}. \tag{5.3}
\]

Here, \(D(\bar{g})\) arises by applying the vector bundle isomorphism \(F_r : S(X, g_+) \rightarrow S(X, \bar{g})\), which exists since \(g_+\) and \(\bar{g}\) are conformally equivalent, to the equation \(\mathcal{D}^h \varphi = i\lambda \varphi\), \(\lambda \in \mathbb{C}\) and \(\varphi \in \Gamma(S(X, g_+))\). The solution of Equation (5.3) is obstructed for \(\lambda = -\frac{2N+1}{2}\), and the obstruction induces a conformally covariant linear differential operator \(\mathcal{D}_{2N+1} = \mathcal{D}^{2N+1} + \text{LOT}\).

Let us be more specific. Let \((M, h)\) be a semi-Riemannian Einstein Spin-manifold, normalized by \(Ric(h) = \frac{2(n-1)J}{n}h\) for constant normalized scalar curvature \(J \in \mathbb{R}\). Consider the embedding \(\iota_r : M \rightarrow X\) given by \(\iota_r(m) := (r, m)\). Then \((M, \iota_r^*(\bar{g}) = h_r)\) is
a hypersurface in \((X, \bar{g})\) with trivial space-like normal bundle. It follows from [BGM05] that the Dirac operator \(\bar{D}^g\) of \((X, \bar{g})\) and the leaf-wise (or, hypersurface) Dirac operator

\[
\tilde{\bar{D}}^h := \partial_r \cdot \sum_{i=1}^n \epsilon_i s_i \cdot \tilde{\nabla}_{s_i}^{h_r} : \Gamma(S(X, \bar{g})) \to \Gamma(S(X, \bar{g}))
\]

for an \(h_r\)-orthonormal frame \(\{s_i\}\) on \(M\) are related by

\[
\iota_r^* \partial_r \cdot \bar{D}^g = \tilde{\bar{D}}^h \quad \text{if } n \text{ is even,}
\]

\[
\iota_r^* \partial_r \cdot \bar{D}^g = 0 \quad \text{if } n \text{ is odd.}
\]

The equation \(\bar{D}^g \varphi = i \lambda \varphi, \lambda \in \mathbb{C} \text{ and } \varphi \in \Gamma(S(X, g_+))\), is equivalent to Equation \((5.3)\) by combination of conformal covariance, Equation \((5.4)\), and the isomorphism \(F_r\), where the linear differential operator \(D(\bar{g}) : \Gamma(S(X, \bar{g})) \to \Gamma(S(X, \bar{g}))\) is given by

\[
D(\bar{g}) \theta = -r \partial_r \cdot \tilde{\bar{D}}^h \theta - \frac{n}{2} r H_r \partial_r \theta + r \partial_r \cdot \nabla_{\partial_r}^{g, S} \theta - \frac{n}{2} \partial_r \cdot \theta
\]

for \(\theta = F_r(\varphi)\). Using Theorem \(3.5\), we find the explicit formulas

\[
\tilde{\bar{D}}^h = \left(1 - \frac{J}{2n} r^2\right)^{-1} D_r, \quad H_r = \frac{2n}{n} \left(1 - \frac{J}{2n} r^2\right)^{-1}.
\]

This is a consequence of the Einstein assumption on \(M\). In general, there is no explicit formula analogous to Equation \((5.3)\). We decompose the spinor bundle \(S(X, \bar{g})\) into the \(\pm i\)-eigenspaces \(S^{\pm \iota_r}(X, \bar{g})\) with respect to the linear map \(\partial_r : S(X, \bar{g}) \to S(X, \bar{g})\) satisfying \(\partial_r^2 = -1\). The formal solution of Equation \((5.3)\) is constructed inside

\[
\mathcal{A} := \{ \theta = \sum_{j \geq 0} r^j \theta_j \mid \theta_j \in \Gamma(S(X, \bar{g})), \ \nabla_{\partial_r}^{g, S} \theta_j = 0 \},
\]

and

\[
\tilde{\theta} := r^{\frac{\lambda}{2} + \lambda} \theta = \sum_{j \geq 0} r^{\frac{\lambda}{2} + \lambda + j} (\theta_j^+ + \theta_j^-) \in \mathcal{A}
\]

for \(\theta_j^\pm \in \Gamma(S^{\pm \iota_r}(X, \bar{g})), j \in \mathbb{N}_0\), is a solution of Equation \((5.3)\) provided the coupled system of recurrence relations

\[
\begin{align*}
\lambda \theta_j^+ & = \tilde{\bar{D}} \theta_{j-1}^+ + \frac{n+j-2}{2n} \theta_{j-2}^+, \\
(2\lambda + j) \theta_j^- & = \tilde{\bar{D}} \theta_{j-1}^- + \frac{2n+j-2}{2n} \theta_{j-2}^-,
\end{align*}
\]

holds for all \(j \in \mathbb{N}_0\). Note that we only consider restrictions to \(r = 0\) and then extend \(\theta_j^\pm, j \geq 0\), by parallel transport with respect to \(\nabla^{g, S}\) along the geodesic induced by the \(r\)-coordinate. The initial data are given by \(\theta_0^+ := \psi^+\) for some \(\psi^+ \in \Gamma(S^{+\iota_r}(X, \bar{g})|_{r=0})\), and \(\theta_0^- = 0\). Assuming \(\lambda \notin -\mathbb{N} + \frac{1}{2}\), the system can be solved uniquely for all \(j \in \mathbb{N}\) if
\( n \) is odd, and for all \( j \in \mathbb{N} \) such that \( j \leq n \) if \( n \) is even. The obstruction at \( \lambda = -\frac{2N+1}{2} \), for \( N \in \mathbb{N}_0 \) (\( N \leq \frac{n}{2} \) for even \( n \)), is given by \( \mathcal{D}_{2N+1} \) for \( N \in \mathbb{N} \).

The application of \( \tilde{\nabla} \) to the system (5.6) together with the shift of \( j \) to \( j - 1 \), respectively \( j - 3 \), implies

\[
(j - 1)\tilde{\nabla}\theta_{j-1}^+ = \tilde{\nabla}^2 \theta_{j-2}^+ + \frac{n+j-3}{2n}J\tilde{\nabla}\theta_{j-3}^+, \\
(2\lambda + j - 1)\tilde{\nabla}\theta_{j-1}^- = \tilde{\nabla}^2 \theta_{j-2}^- + \frac{2\lambda+n+j-3}{2n}J\tilde{\nabla}\theta_{j-3}^-,
\]

\[
\tilde{\nabla}\theta_{j-3}^- = (j - 2)\theta_{j-2}^+ - \frac{n+j-4}{2n}J\theta_{j-4}^+, \\
\tilde{\nabla}\theta_{j-3}^- = (2\lambda + j - 2)\theta_{j-2}^- - \frac{2\lambda+n+j-4}{2n}J\theta_{j-4}^-.
\]

These formulas can be used to decouple the system (5.6) into

\[
j\theta_j^+ = \left( \frac{1}{2\lambda+1-j}\tilde{\nabla}^2 + \frac{(2\lambda+n+j-3)(j-2)}{2n(2\lambda+j-1)}J + \frac{(2\lambda+j-1)(n+j-2)}{2n(2\lambda+j-1)}J \right) \theta_{j-2}^+ \\
- \frac{(2\lambda+n+j-3)(n+j-4)}{4n^2(2\lambda+j-1)}J^2 \theta_{j-4}^+,
\]

\[
(2\lambda + j)\theta_j^- = \left( \frac{1}{j-1}\tilde{\nabla}^2 + \frac{(2\lambda+j-2)(n+j-3)}{2n(j-1)}J + \frac{(2\lambda+n+j-2)(j-1)}{2n(j-1)}J \right) \theta_{j-2}^- \\
- \frac{(2\lambda+n+j-4)(n+j-3)}{4n^2(j-1)}J^2 \theta_{j-4}^-.
\] (5.7)

for all \( j \geq 2 \), with the initial data \( \theta_0^+ = \psi^+ \), \( \theta_0^- = 0 \), \( \theta_1^+ = 0 \), and \( \theta_1^- = \frac{1}{2\lambda+1}\tilde{\nabla}\psi^+ \).

Introducing \( \phi_l := \theta_{2l+1}^- \) for \( l \in \mathbb{N}_0 \), Equation (5.7) is equivalent to

\[
2l(2\lambda + 2l + 1)\phi_l = \left( \tilde{\nabla}^2 + \frac{(2\lambda+2l-1)(n+2l-2)}{2n}J + \frac{2l(2\lambda+n+2l-1)}{2n}J \right) \phi_{l-1} \\
- \frac{(2\lambda+n+2l-3)(n+2l-2)}{4n^2}J^2 \phi_{l-2},
\] (5.8)

for \( l \in \mathbb{N} \) and \( \phi_0 := \frac{1}{2\lambda+1}\tilde{\nabla}\psi^+ \). We define the solution operators \( \tilde{q}_l(y) \) by

\[
4^l!(\frac{n}{2\lambda})^l(\lambda + \frac{3}{2})_l \phi_l = \tilde{q}_l(y)\phi_0,
\] (5.9)

where \( y := \frac{\tilde{\nabla}^2}{\nabla^2} \). Then Equation (5.8) yields a recurrence relation for \( \tilde{q}_l(y) \), namely

\[
\tilde{q}_l(y) = (y + (\lambda + l - \frac{1}{2})(l + \frac{n}{2} - 1) + (\lambda + l + \frac{n}{2} - \frac{3}{2}))\tilde{q}_{l-1}(y) \\
- (l - 1)(l + \frac{n}{2} - 1)(l + \lambda - \frac{1}{2})(l + \lambda + \frac{n}{2} - \frac{3}{2})\tilde{q}_{l-2}(y),
\]

\( l \in \mathbb{N}, \tilde{q}_{-1}(y) := 0 \) and \( \tilde{q}_0(y) := 1 \). Changing \( l \) to \( (m+1) \) and substituting \( \lambda = -\frac{2N+1}{2} \) for \( N \in \mathbb{N}_0 \), we obtain

\[
\tilde{q}_{m+1}(y) = (y - 2m(N - m - \frac{n}{2} - \frac{1}{2}) - \frac{n}{2}(N - 1) - N)\tilde{q}_m(y) \\
- m(m - N)(m + \frac{n}{2})(m - N + \frac{n}{2} - 1)\tilde{q}_{m-1}(y).
\]

The unique solution of the recurrence relation (5.10) is given by

\[
\tilde{q}_m(y) := \sum_{l=0}^{m} (-1)^{m-l}(\frac{n}{2} + 1 + l)_{m-l} (N - m)_{m-l} \binom{m}{l} \prod_{j=1}^{l} (y - j^2),
\] (5.10)
cf. Proposition 4.1 and it specializes for \( m = N \)

\[
\tilde{q}_N(y) = \prod_{j=1}^{N}(y-j^2) = \prod_{j=1}^{N}\left(\sqrt{\frac{n}{2j}}\tilde{\nabla} - j\right)\left(\sqrt{\frac{n}{2j}}\tilde{\nabla} + j\right).
\]

The solution \( \phi_l \), cf. Equation (5.9), multiplied by \((2\lambda + 1)\) is obstructed at \( \lambda = -\frac{2N+1}{2} \), \( N \in \mathbb{N}_0 \), and we get

\[
4^N N!(-N)N\phi_N = \left(\frac{n}{2j}\right)^{-N}\tilde{q}_N(y)\tilde{\nabla}\psi^+.
\]

Repeating all the previous steps with eigen-equation (5.3) for the eigenvalue \( -\lambda \) and initial data \( \psi^- \in \Gamma(S^{-\tilde{\delta}_k}(X,\bar{g})|_{r=0}) \), the obstruction at \( \lambda = -\frac{2N+1}{2} \), for \( N \in \mathbb{N}_0 \), induces

\[
D_{2N+1} = \left(\frac{n}{2j}\right)^{-N}\prod_{j=1}^{N}\left(\sqrt{\frac{n}{2j}}\tilde{\nabla} - j\right)\left(\sqrt{\frac{n}{2j}}\tilde{\nabla} + j\right)
\]

\[
= \tilde{\nabla}\prod_{j=1}^{N}\left(\tilde{\nabla} - j\sqrt{\frac{2j}{n}}\right)\left(\tilde{\nabla} + j\sqrt{\frac{2j}{n}}\right),
\]

the conformal power of the Dirac operator in the factorized form. Note that there is no restriction on \( N \in \mathbb{N}_0 \) in the case of even \( n \). Thus we have the following result.

**Theorem 5.2.** Let \((M,h)\) be a semi-Riemannian Einstein Spin-manifold of dimension \( n \), normalized by \( \text{Ric}(h) = \frac{2(n-1)}{n} J h \) for constant normalized scalar curvature \( J \in \mathbb{R} \).

The \((2N+1)\)-th conformal power of the Dirac operator, \( N \in \mathbb{N}_0 \), satisfies

\[
D_{2N+1}\psi = \prod_{j=1}^{2N+1}\left(\tilde{\nabla} - (N - j + 1)\sqrt{\frac{2j}{n}}\right)\psi
\]

\[
= \tilde{\nabla}\prod_{j=1}^{N}\left(\tilde{\nabla}^2 - j^2\left(\frac{2j}{n}\right)\right)\psi
\]

for all \( \psi \in \Gamma(S(M,h)) \). The empty product is regarded as 1.

In particular, Theorem 5.2 applies to the standard round sphere \((S^n, h)\) of radius 1. We get

\[
D_{2N+1} = (\tilde{\nabla} - N) \cdots (\tilde{\nabla} - 1)\tilde{\nabla}(\tilde{\nabla} + 1) \cdots (\tilde{\nabla} + N)
\]

for all \( N \in \mathbb{N}_0 \), since the scalar curvature is \( \tau = n(n-1) \) and so \( J = \frac{n}{2} \). This agrees with the results in [Bra95, ES10].

**6. Application: Holographic deformation of the Dirac operator on Einstein manifolds**

The inversion formula for GJMS operators \( P_{2N}(g), N \in \mathbb{N} (N \leq \frac{n}{2} \text{ for even } n) \), cf. [Juh13], implies the existence of a sequence of second order linear differential operators \( \mathcal{M}_{2N} \) acting on functions and fulfilling

\[
P_{2N}(g) \in \mathbb{N}[\mathcal{M}_2, \ldots, \mathcal{M}_{2N}],
\]

\[
\mathcal{M}_{2N} \in \mathbb{Z}[P_2(g), \ldots, P_{2N}(g)].
\]
Let us define a sequence of first order differential operators acting on the spinor bundle $S(M, h)$:

$$M_{2N+1} := (-1)^N(N!)^2 \left( \frac{2l}{n} \right)^N \mathcal{D}, \ N \in \mathbb{N}_0. \quad (6.2)$$

Notice that the operators $M_{2N+1}$ are deformations of the Dirac operator.

**Theorem 6.1.** Let $(M, h)$ be a semi-Riemannian Einstein Spin-manifold of dimension $n$. For all $N \in \mathbb{N}_0$, we have

$$D_{2N+1} \in \mathbb{N}[M_1, \ldots, M_{2N+1}],$$

$$M_{2N+1} \in \mathbb{Z}[D_1, \ldots, D_{2N+1}]. \quad (6.3)$$

**Proof.** We prove, by induction, that for all $N \in \mathbb{N}_0$ we have $D_{2N+1} \in \mathbb{N}[M_1, M_3, \ldots, M_{2N+1}]$, and the coefficient of $M_{2N+1}$ equals 1. The case $N = 0$ is obvious by definition. Let us assume that $D_{2N-1} \in \mathbb{N}[M_1, \ldots, M_{2N-1}]$ such that the coefficient of $M_{2N-1}$ equals 1. By Theorem 5.2 we have

$$D_{2N+1} = D_{2N-1} \left( \mathcal{D}^2 - N^2 \left( \frac{2l}{n} \right) \right) = D_{2N-1} M_1^2 - N^2 \left( \frac{2l}{n} \right) D_{2N-1}. \quad (6.4)$$

Since $D_{2N-1} = B + M_{2N-1}$ for some $B \in \mathbb{N}[M_1, \ldots, M_{2N-3}]$, and $B$ contains in each contribution at least one $M_1$, we can absorb the factor $-\frac{2l}{n}$ into $M_1$ to get a contribution of $M_3$. To finish the proof, we note that $M_{2N+1} = -N^2 \left( \frac{2l}{n} \right) M_{2N-1}$, hence $D_{2N+1} \in \mathbb{N}[M_1, M_3, \ldots, M_{2N+1}]$ with the coefficient of $M_{2N+1}$ being 1. Notice that we can not expect a unique polynomial expression for $D_{2N+1}$. \hfill \square

Closely related to the sequence $[6.2]$ is the following generating function termed holographic deformation of the Dirac operator,

$$\mathcal{H}(r) := \sum_{N \geq 0} \frac{(-1)^N}{(N!)^2} \left( \frac{r}{2} \right)^{2N} M_{2N+1}, \quad (6.5)$$

a deformation of the Dirac operator on $M$ in the sense that $\mathcal{H}(0) = \mathcal{D}$. It has the holographic description

$$\mathcal{H}(r) \psi = \sum_{i=1}^{n} \varepsilon_i \sqrt{h_{r^{-1}}} (s_i) \cdot \nabla_{s_i}^{h,S} \psi, \quad (6.6)$$

where $\psi \in \Gamma(S(M, h))$, $\{s_i\}_{i=1}^{n}$ is an $h$-orthonormal frame and $\sqrt{h_{r^{-1}}}$ is a formal power series in $r$,

$$\sqrt{h_{r^{-1}}} = (1 - \frac{r^2}{2n})^{-1} h = \sum_{N \geq 0} \left( \frac{2l}{2n} r^2 \right)^N h. \quad (6.6)$$

**Remark.** The existence of $M_1$, $M_3$ and $M_5$ is shown for general curved manifolds in [Fis13, Chapter 6], and reduces in the case of Einstein manifolds to sequence $[6.2]$. Notice that the holographic description $[6.6]$ reproduces the first order contributions of $M_k$, $k = 1, 3, 5$. The structure of constant (zeroth order) terms of $M_k$, $k = 3, 5$, remains unclear.
References

[BG92] J.P. Bourguignon and P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, Communications in Mathematical Physics 144 (1992), no. 3, 581–599.

[BGM05] C. Bär, P. Gauduchon, and A. Moroianu, Generalized cylinders in semi-Riemannian and spin geometry, Mathematische Zeitschrift 249 (2005), no. 3, 545–580.

[Bra95] T. Branson, Sharp inequalities, the functional determinant, and the complementary series, Transaction of the American Mathematical Society 347 (1995), 3671–3742.

[Dow11] J.S. Dowker, Determinants and conformal anomalies of GJMS operators on spheres, Journal of Physics A: Mathematical and Theoretical 44 (2011), no. 11, 1–14.

[Dow13] ———. Spherical Dirac GJMS operator determinants, ArXiv e-prints (2013), http://arxiv.org/abs/1310.5563.

[ES10] D. Eelbode and V. Soucek, Conformally invariant powers of the Dirac operator in Clifford analysis, Mathematical Methods in Applied Science 33 (2010), no. 13, 1558–1570.

[FG85] C. Fefferman and C.R. Graham, Conformal invariants, Élie Cartan et les mathématiques d’aujourd’hui, Astérisque (hors série) (1985), 95–116.

[FG11] C. Fefferman and C. R. Graham, The ambient metric (AM-178), no. 178, Princeton University Press, 2011.

[Fis13] M. Fischmann, Conformally covariant differential operators acting on spinor bundles and related conformal covariants, Ph.D. thesis, Humboldt Universität zu Berlin, 2013, http://edoc.hu-berlin.de/dissertationen/fischmann-matthias-2013-03-04/PDF/fischmann.pdf.

[GH04] A.R. Gover and K. Hirachi, Conformally invariant powers of the Laplacian - A complete non-existence theorem, Journal of the American Mathematical Society 17 (2004), no. 2, 389–405.

[GJMS92] C.R. Graham, R.W. Jenne, L. Mason, and G. Sparling, Conformally invariant powers of the Laplacian, I: Existence, Journal of the London Mathematical Society 2 (1992), no. 3, 557–565.

[GMP12] C. Guillarmou, S. Moroianu, and J. Park, Bergman and Calderón projectors for Dirac operators, Journal of Geometric Analysis (2012), 1–39, http://dx.doi.org/10.1007/s12220-012-9338-9.

[Gov06] A.R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, Mathematische Annalen 336 (2006), no. 2, 311–334.

[GP03] A.R. Gover and L.J. Peterson, Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus, Communications in Mathematical Physics 235 (2003), no. 2, 339–378.

[GZ03] C.R. Graham and M. Zworski, Scattering matrix in conformal geometry, Inventiones mathematicae 152 (2003), 89–118.

[HS01] J. Holland and G. Sparling, Conformally invariant powers of the ambient Dirac operator, ArXiv e-prints (2001), http://arxiv.org/abs/math/0112033.

[Juh13] A. Juhl, Explicit formulas for GJMS-operators and Q-curvatures, Geometric and Functional Analysis 23 (2013), 1278–1370, http://arxiv.org/abs/1108.0273.

[KLS10] R. Koekoek, P.A. Lesky, and R.F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[KM61] S. Karlin and J.L. McGregor, The Hahn polynomials, formulas and an application, Scripta Mathematica 26 (1961), 33–46.

[Slo93] J. Slovák, Natural operators on conformal manifolds, Ph.D. thesis, Masaryk University, 1993.

1 E. Čech Institute, Mathematical Institute of Charles University, Sokolovská 83, Praha 8 - Karlín, Czech Republic.

2 Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria.