Fundamentals of bicomplex pseudoanalytic function theory:
Cauchy integral formulas, negative formal powers and
Schrödinger equations with complex coefficients

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Abstract

The study of the Dirac system and second-order elliptic equations with complex-valued coefficients on the plane naturally leads to bicomplex Vekua-type equations [8], [14], [6]. To the difference of complex pseudoanalytic (or generalized analytic) functions [3], [25], the theory of bicomplex pseudoanalytic functions has not been developed. Such basic facts as, e.g., the similarity principle or the Liouville theorem in general are no longer available due to the presence of zero divisors in the algebra of bicomplex numbers.

In the present work we develop a theory of bicomplex pseudoanalytic formal powers analogous to the developed by L. Bers [3] and especially that of negative formal powers. Combining the approaches of L. Bers and I. N. Vekua with some additional ideas we obtain the Cauchy integral formula in the bicomplex setting. In the classical complex situation this formula was obtained under the assumption that the involved Cauchy kernel is global, a very restrictive condition taking into account possible practical applications, especially when the equation itself is not defined on the whole plane. We show that the Cauchy integral formula remains valid with the Cauchy kernel from a wider class called here the reproducing Cauchy kernels. We give a complete characterization of this class.

To our best knowledge these results are new even for complex Vekua equations. We establish that reproducing Cauchy kernels can be used to obtain a full set of negative formal powers for the corresponding bicomplex Vekua equation and present an algorithm which allows one their construction.

Bicomplex Vekua equations of a special form called main Vekua equations are closely related to stationary Schrödinger equations with complex-valued potentials. We use this relation to establish useful connections between the reproducing Cauchy kernels and the fundamental solutions for the Schrödinger operators which allow one to construct the Cauchy kernel when the fundamental solution is known and vice versa. Moreover, using these results we construct the fundamental solutions for the Darboux transformed Schrödinger operators.

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1 Introduction

In the present work we study the bicomplex Vekua equations of the form

\[ \partial_z W = aW + b\overline{W} \quad (1) \]

where \( a, b \) and \( W \) are functions of the complex variable \( z = x + jy \) and take values in the algebra of bicomplex numbers. The conjugation \( \overline{W} \) is with respect to the imaginary unit \( j \) and \( \partial_z = \frac{1}{2}(\partial_x + j\partial_y) \). Every bicomplex Vekua equation (1) is equivalent to a first order system

\[ \begin{align*}
\partial_x u - \partial_y v &= \alpha u + \beta v \\
\partial_x v + \partial_y u &= \gamma u + \delta v
\end{align*} \quad (2) \]

where all the involved functions are complex, and vice versa, the system can be written in the form (1). As always, whenever it is possible, the introduction of an appropriate algebraic structure leads to a deeper understanding of the system of equations and this is why it is preferable to study (2), (3) in the form (1). Equation (1) arose in [8] in relation with the Dirac system with electromagnetic and scalar potentials in the two-dimensional case. In [14] it was shown that equation (1) when \( a \equiv 0 \) and \( b = \partial_z f/f \) where \( f \) is a scalar function (see Section 2) is closely related to the stationary Schrödinger equation

\[ \Delta u = qu \quad (4) \]

with \( q = \Delta f/f \). Vekua equations with coefficients \( a \) and \( b \) of this special form are called Vekua equations of the main type or main Vekua equations [16]. In the classical complex case they are closely related to so-called \( p\)-analytic functions (for the theory of \( p\)-analytic functions we refer to the book [20], for the relation to the main Vekua equation to [16, Chapter 5] and for their applications to [1], [10], [12], [16], [21], [27], [28]).

The relation between the main Vekua equation and the stationary Schrödinger equation is of the same nature as the relation between the Cauchy-Riemann system and the Laplace equation. The scalar part of the solution of the main Vekua equation is necessarily a solution of the corresponding Schrödinger equation and vice versa, for any solution \( u \) of the Schrödinger equation its “conjugate metaharmonic” counterpart \( v \) (see [16, Chapter 3]) can be constructed such that the obtained bicomplex function \( W = u + jv \) will be a solution of the main Vekua equation. The obtained counterpart in its turn is a solution of another Schrödinger equation the potential of which is a Darboux transformation of the initial potential \( q \). In [13] (see also [16]) the procedure for construction of \( v \) by \( u \) and vice versa was obtained in the explicit form. Recently Sh. Garuchava in [11] established a correspondence between the procedure from [13] and the two-dimensional Darboux transformation from [19].

The theory of bicomplex Vekua equations is far from being complete though publications attempting to extend the essential properties of pseudoanalytic (or generalized analytic) functions onto the solutions of more general systems on the plane are numerous. We refer to [26] for some first results in studying (1) and emphasize that such important facts as the similarity principle were obtained in that paper under the overrestrictive condition on the coefficients \( b \equiv 0 \). Under this condition the bicomplex Vekua equation (1) reduces to a pair of decoupled complex Vekua equations which obviously simplifies its study. In [22] relations...
between classes of bicomplex Vekua equations and complexified Schrödinger equations were studied. In the recent work \cite{2} several results on bicomplex pseudoanalytic functions from the point of view of Bauer-Peschl differential operators can be found.

The main difficulty in studying the bicomplex Vekua equation comes from the existence of zero divisors in the algebra of bicomplex numbers. For example, when the solution \( W \) of (1) does not have zero divisors (the values of the function do not coincide with a zero divisor at any point \( z \) of the domain of interest) for such \( W \) we prove a similarity principle (Theorem 14 below). Unfortunately in general this fact is unavailable which forces one to look for alternative ideas for developing the corresponding pseudoanalytic function theory.

In \cite{6}, \cite{8}, \cite{14}, \cite{16} it was noticed that several constructive results from Bers’ theory remain valid in the bicomplex case. For example, if a generating sequence corresponding to the bicomplex Vekua equation (1) is known, the construction of corresponding positive formal powers can be performed by means of Bers’ algorithm based on the concept of the \((F,G)\)-integration. However to the difference from the complex pseudoanalytic function theory it is not clear how to prove the expansion and Runge theorems in the bicomplex case, the results ensuring the completeness of the formal powers in the space of all solutions of the Vekua equation. Moreover, these results in the classical complex case were obtained for global formal powers only (see the definition in Subsection 3.3) meanwhile the only system of formal powers whose explicit form is known corresponds to the case of analytic functions, \( \{(z - z_0)^n\}_{0}^{\infty} \). Recently for a wide class of main bicomplex Vekua equations the expansion and the Runge theorems were obtained in \cite{6}, \cite{7} using the approach based on so-called transmutation (or transformation) operators. It is important to emphasize that in \cite{6} and \cite{7} the results concerning the completeness of systems of local formal powers were obtained which opened the way to apply them in practical solution of boundary value and eigenvalue problems for second-order elliptic equations with variable coefficients (see \cite{5} and \cite{9}).

The main subject of the present work is the study of negative formal powers for the bicomplex Vekua equation and their applications. Based on the results on the Cauchy kernels we obtain Cauchy integral formulas for bicomplex pseudoanalytic functions. In the classical theory of complex pseudoanalytic functions in fact there are two types of Cauchy integral formulas. One is based on Cauchy kernels for an adjoint Vekua equation and the other involves the Cauchy kernels for the initial Vekua equation (here we call these two Cauchy integral formulas the first and the second respectively). We obtain a relation between both Cauchy kernels and prove both kinds of Cauchy integral formulas. It is important to mention that the Cauchy kernels involved in the obtained Cauchy integral formulas are not required to be global but instead belong to a much more general class which we call reproducing Cauchy kernels. We give a complete characterization of this class. To our best knowledge these results are new even for complex Vekua equations. We establish that reproducing Cauchy kernels can be used to obtain a full set of negative formal powers for the corresponding bicomplex Vekua equation and present an algorithm which allows one their construction.

Bicomplex Vekua equations of a special form called main Vekua equations are closely related to stationary Schrödinger equations with complex-valued potentials. We use this relation to establish direct connections between the reproducing Cauchy kernels and the fundamental solutions for the Schrödinger operators which allow one to construct the Cauchy kernel when the fundamental solution is known and vice versa. Moreover, using these results we construct the fundamental solutions for the Darboux transformed Schrödinger operators.
These results are also new in the context of the classical complex Vekua equations and among other applications allow one to obtain in a closed form a reproducing Cauchy kernel and a set of negative formal powers for an important class of main Vekua equations.

The layout of the paper is as follows. In Section 2 we introduce the necessary formalism concerning the bicomplex numbers. Some of the facts presented here can be found in several sources, e.g., [17], [23], [24]. Nevertheless we needed to introduce a special though quite natural norm and hence prove several related properties which probably are first published. In Section 3 we introduce the necessary definitions from pseudoanalytic function theory, prove several results concerning bicomplex pseudoanalytic functions, like the mentioned above similarity principle (for functions without zero divisors). By analogy with the complex case we define formal powers and obtain their basic properties. We introduce the main Vekua equation in relation with the stationary Schrödinger equation and obtain some auxiliary results for its solutions. In Section 4 the Cauchy integral formulas for bicomplex pseudoanalytic functions are obtained and the characterization of the reproducing Cauchy kernels is given. In Section 5 we establish an important relation between the negative formal powers corresponding to different bicomplex Vekua equations which in fact leads to an algorithm for constructing the negative formal powers. We give several applications of this result in Section 6 using a valuable observation that for the main Vekua equation the adjoint and the successor coincide. This leads to the possibility to construct a reproducing Cauchy kernel for the Vekua equation from a known fundamental solution for a related Schrödinger equation and vice versa and also gives a method for constructing the fundamental solutions for a chain of Darboux transformed Schrödinger operators. Finally, several examples of explicitly calculated kernels and fundamental solutions are presented.

2 Bicomplex numbers

Together with the imaginary unit \( i \) we consider another imaginary unit \( j \), such that

\[
  j^2 = i^2 = -1 \quad \text{and} \quad ij = ji. \tag{5}
\]

We have then two copies of the field of complex numbers, \( \mathbb{C}_i := \{a + ib, \ \{a, b\} \subset \mathbb{R}\} \) and \( \mathbb{C}_j := \{a + jb, \ \{a, b\} \subset \mathbb{R}\} \). The expressions of the form \( W = u + jv \) where \( \{u, v\} \subset \mathbb{C}_i \) are called bicomplex numbers. The conjugation with respect to \( j \) we denote as \( \overline{W} = u - jv \). The components \( u \) and \( v \) will be called the scalar and the vector part of \( W \) respectively. We will use the notation \( u = \text{Sc} W \) and \( v = \text{Vec} W \).

The set of all bicomplex numbers with a natural operation of addition and with the multiplication defined by the laws (5) represents a commutative ring with a unit. We denote it by \( \mathbb{B} \). An element \( W \in \mathbb{B} \) is invertible if and only if \( WW \neq 0 \) and the inverse element is defined by the equality \( W^{-1} = \overline{W} / WW \).

Let \( \mathcal{R}(\mathbb{B}) \) be the set formed by the invertible elements of \( \mathbb{B} \) and \( \sigma(\mathbb{B}) \) denote the generalized zeros of \( \mathbb{B} \) (zero divisors), that is

\[
  \sigma(\mathbb{B}) = \{W \in \mathbb{B}: W \neq 0 \text{ and } WW = 0\}.
\]

It is convenient to introduce the pair of idempotents \( P^+ = \frac{1}{2}(1 + ij) \) and \( P^- = \frac{1}{2}(1 - ij) \) \((P^\pm)^2 = P^\pm\). As it can be verified directly, \( P^\pm \in \sigma(\mathbb{B}) \) and \( P^+ + P^- = 1 \).
Proposition 1  Let $W \in \mathbb{B}$. Then

(i) there exist the unique numbers $W^+, W^- \in \mathbb{C}_i$ such that $W = P^+W^+ + P^-W^-$ which can be computed from $W$ as follows

$$W^\pm = \text{Se} W \mp i \text{Vec} W,$$

(ii) a nonzero element $W$ belongs to $\sigma(\mathbb{B})$ iff $W = P^+W^+$ or $W = P^-W^-$. 

Proof. (i) Straightforward.  
(ii) It follows directly from the definition of $\sigma(\mathbb{B})$ and from the equality $W\overline{W} = W^+W^-$. 

For $W = P^+W^+ + P^-W^- \in \mathbb{B}$ we introduce the notation

$$|W| = \frac{1}{2} \left( |W^+|_{\mathbb{C}_i} + |W^-|_{\mathbb{C}_i} \right),$$

where $|.|_{\mathbb{C}_i}$ is the usual norm in $\mathbb{C}_i$.

Proposition 2 The function defined in (7) is a norm in $\mathbb{B}$ and possesses the following properties.

(i) If $W \in \mathbb{C}_i$ (that is, Vec$W = 0$) then $|W| = |W|_{\mathbb{C}_i}$.

(ii) If $W \in \mathbb{C}_j$ and $V \in \mathbb{B}$ then $|W| = |W|_{\mathbb{C}_j}$ and $|WV| = |W||V|$. 

(iii) If $W, V \in \mathbb{B}$ then $|WV| \leq 2|W||V|$ and

$$|\text{Se} W| \leq |W|, \quad |\text{Vec} W| \leq |W|, \quad |W| \leq |\text{Se} W| + |\text{Vec} W|.$$ 

Proof. The proof of (i), (ii) and of the last part of (iii) is straightforward. Let $W = P^+W^+ + P^-W^-$ and $V = P^+V^+ + P^-V^- \in \mathbb{B}$. Then

$$WV = P^+W^+V^+ + P^-W^-V^-$$

and

$$|WV| = \frac{1}{2} (|W^+V^+| + |W^-V^-|) \leq \frac{1}{2} (|W^+| + |W^-|) (|V^+| + |V^-|) = 2|W||V|.$$ 

Proposition 3 $\mathcal{R}(\mathbb{B})$ is open in $\mathbb{B}$. 

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Proof. Let \( W \in \mathcal{R}(\mathbb{B}) \). Then \( W = P^+W^+ + P^-W^- \) where \( W^+ \neq 0 \) and \( W^- \neq 0 \). Define \( r = \min \left\{ \frac{|W^+|}{2}, \frac{|W^-|}{2} \right\} \) and consider the open ball \( B(W, r) \). If \( V \in (\mathcal{R}(\mathbb{B}))^c = \sigma(\mathbb{B}) \cup \{0\} \) then \( V = P^+V^+ \) for which we have

\[
|W - V| = \frac{1}{2} (|W^+ - V^+| + |W^-|) \geq \frac{|W^-|}{2} \geq r
\]
or \( V = P^-V^- \) for which we have

\[
|W - V| = \frac{1}{2} (|W^+| + |W^- - V^-|) \geq \frac{|W^+|}{2} \geq r.
\]
Thus, \( V \notin B(W, r) \) and hence \( B(W, r) \subset \mathcal{R}(\mathbb{B}) \) which proves that \( \mathcal{R}(\mathbb{B}) \) is open. \( \blacksquare \)

An exponential function of a bicomplex variable is defined by the equality

\[
E[W] := P^+e^{W^+} + P^-e^{W^-}, \quad W \in \mathbb{B}.
\]

**Proposition 4**

(i) \( E[W+V] = E[W]E[V] \), \( \forall W, V \in \mathbb{B} \), in particular, \( E[W] \) is invertible with the inverse given by \( E[-W] \);

(ii) \( |E[W] - 1| \leq e^{2|W|} - 1 \) for all \( W \in \mathbb{B} \).

**Proof.** (i) Straightforward.

(ii) Follows from the equality

\[
E[W] - 1 = P^+ \left( e^{W^+} - 1 \right) + P^- \left( e^{W^-} - 1 \right)
\]
and from the fact that \( |e^{W^\pm} - 1| \leq e^{|W^\pm|} - 1 \). \( \blacksquare \)

## 3 Bicomplex pseudoanalytic functions

### 3.1 Generating pair and first properties of bicomplex pseudoanalytic functions

**Definition 5** A pair of \( \mathbb{B} \)-valued functions \( F \) and \( G \) possessing Hölder continuous partial derivatives in \( \Omega \subset \mathbb{C}_j \) with respect to the real variables \( x \) and \( y \) is said to be a generating pair if it satisfies the inequality

\[
\text{Vec}(FG) \neq 0 \quad \text{in } \Omega.
\]  

(8)

Condition (8) implies that every bicomplex function \( W \) defined in a subdomain of \( \Omega \) admits the unique representation \( W = \phi F + \psi G \) where the functions \( \phi \) and \( \psi \) are scalar (=\( \mathbb{C}_i \)-valued).

Assume that \((F,G)\) is a generating pair in a domain \( \Omega \).
Definition 6 Let the \( \mathbb{B} \)-valued function \( W \) be defined in a neighborhood of \( z_0 \in \Omega \). In a complete analogy with the complex case we say that at \( z_0 \) the function \( W \) possesses the \((F,G)\)-derivative \( W(z_0) \) if the (finite) limit
\[
\hat{W}(z_0) = \lim_{z \to z_0} \frac{W(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0}
\]
equals where \( \lambda_0 \) and \( \mu_0 \) are the unique scalar constants such that \( W(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0) \).

We will also use the notation
\[
\frac{d(F,G)W(z_0)}{dz} = \hat{W}(z_0).
\]

Let us introduce the bicomplex operators
\[
\partial_z = \frac{1}{2} (\partial_x + j \partial_y), \quad \partial_{\bar{z}} = \frac{1}{2} (\partial_x - j \partial_y)
\]
acting on \( \mathbb{B} \)-valued functions. Similarly to the complex case (see, e.g., [16, Chapter 2]) it is easy to show that if \( \hat{W}(z_0) \) exists then at \( z_0 \), \( \partial_z W \) and \( \partial_{\bar{z}} W \) exist and the equations
\[
\partial_z W = aW + b\overline{W} \tag{11}
\]
and
\[
\hat{W} = \partial_{\bar{z}} W - AW - B\overline{W} \tag{12}
\]
hold, where \( a, b, A \) and \( B \) are the characteristic coefficients of the pair \((F,G)\) defined by the formulas
\[
a = a_{(F,G)} = - \frac{F \partial_z G - G \partial_z F}{FG - \overline{FG}}, \quad b = b_{(F,G)} = \frac{F \partial_{\bar{z}} G - G \partial_{\bar{z}} F}{FG - \overline{FG}},
\]
\[
A = A_{(F,G)} = - \frac{F \partial_z G - G \partial_z F}{FG - \overline{FG}}, \quad B = B_{(F,G)} = \frac{F \partial_{\bar{z}} G - G \partial_{\bar{z}} F}{FG - \overline{FG}}.
\]
Notice that \( FG - \overline{FG} = -2j \text{Vec}(FG) \neq 0 \).

If \( \partial_z W \) and \( \partial_{\bar{z}} W \) exist and are continuous in some neighborhood of \( z_0 \), and if (11) holds at \( z_0 \), then \( W(z_0) \) exists, and (12) holds. Let us notice that \( F \) and \( G \) possess \((F,G)\)-derivatives, \( \hat{F} \equiv \hat{G} \equiv 0 \), and the following equalities are valid which determine the characteristic coefficients uniquely
\[
\partial_z F = aF + b\overline{F}, \quad \partial_{\bar{z}} G = aG + b\overline{G},
\]
\[
\partial_z F = AF + B\overline{F}, \quad \partial_{\bar{z}} G = AG + B\overline{G}.
\]

Definition 7 A function will be called \( \mathbb{B} \)-(\(F,G)\)-pseudoanalytic (or, simply, \( \mathbb{B} \)-pseudoanalytic, if there is no danger of confusion) in a domain \( \Omega \) if \( \hat{W}(z) \) exists everywhere in \( \Omega \).

Remark 8 When \( F \equiv 1 \) and \( G \equiv j \) the corresponding bicomplex Vekua equation is
\[
\partial_z W = 0, \tag{13}
\]
and its study in fact reduces to the complex analytic function theory. To see this notice that with the aid of the idempotents $P^\pm$ the operator $\partial z$ admits the following representation
\[ \partial z = P^+ dz + P^- d\bar{z} \] (14)

where
\[ dz = \frac{1}{2}(\partial_x - i\partial_y) \quad d\bar{z} = \frac{1}{2}(\partial_x + i\partial_y) \]

are the complex Cauchy-Riemann operators. Then the function $W = P^+ W^+ + P^- W^-$ satisfies (13) iff the scalar functions $W^+$ and $W^-$ (defined by (6)) are antiholomorphic and holomorphic respectively.

**Remark 9** The bicomplex Vekua equation (11) is equivalent to the following first order elliptic system
\[ \partial_x u - \partial_y v = (\alpha + \theta) u + (\gamma - \beta) v \\
\partial_x v + \partial_y u = (\beta + \gamma) u + (\alpha - \theta) v \]

where $u = \text{Sc} W$, $v = \text{Vec} W$, $\alpha = 2 \text{Sc} a$, $\beta = 2 \text{Vec} a$, $\theta = 2 \text{Sc} b$, $\gamma = 2 \text{Vec} b$ are $\mathbb{C}_1$-valued functions. We stress that if $a$ and $b$ are $\mathbb{C}_j$-valued functions then $\alpha$, $\beta$, $\theta$ and $\gamma$ are real valued functions and the bicomplex Vekua equation (11) can be decoupled into a pair of independent complex Vekua equations. For example, $W$ is a solution of (11) iff $H_1 := \text{Re} \text{Sc} W + j \text{Re} \text{Vec} W$ and $H_2 := \text{Im} \text{Sc} W + j \text{Im} \text{Vec} W$ satisfy
\[ \partial z H_1 = aH_1 + b\bar{H}_1, \quad \partial z H_2 = aH_2 + b\bar{H}_2. \] (15)

Notice that all the functions involved in (15) are $\mathbb{C}_j$-valued which means that the theory developed by L. Bers and I. Vekua on complex Vekua equations ([3], [25], [16]) can be applied to study (15) and therefore to the study of (11) in this special case. In general the reduction of a bicomplex Vekua equation to a pair of decoupled complex Vekua equations is not possible.

Let $\Omega \subset \mathbb{R}^2$ and $A$, $B$ be the operators acting on complex functions by the rules
\[ (A\Phi)(z) = \frac{1}{\pi} \int_{\Omega} \frac{\Phi(z)}{z - \zeta} d\Omega_\zeta \]
and
\[ B = CAC \]

where $C$ is the operator of complex conjugation (with respect to $i$). We recognize immediately that $A$ and $B$ are the right inverse operators of $d\bar{z}$ and $dz$ respectively. Consider the operator
\[ TzW = P^+ BW^+ + P^- AW^- \]
acting on bicomplex functions.

**Proposition 10** Let $W$ be a bounded measurable $\mathbb{B}$-valued function defined in some bounded domain $\Omega \subset \mathbb{C}_j$. 8
(i) If $|W(z)| \leq M$ in $\Omega$ then $|T_\tau W(z)| \leq k_1M$ for all $z \in \Omega$, $k_1$ depending only on the area of $\Omega$.

(ii) For every $z_1, z_2 \in C_j$,

$$|T_\tau W(z_1) - T_\tau W(z_2)| \leq k_2M \left|z_1 - z_2\right| \left\{1 + \log^+ \frac{1}{|z_1 - z_2|}\right\}$$

where $k_2$ depends only on the diameter of $\Omega$ and $\log^+ \alpha = \log \alpha$ if $\alpha > 1$, and $\log^+ \alpha = 0$ if $\alpha \leq 1$.

(iii) In every connected component of the complement of $\overline{\Omega}$, $T_\tau W(z)$ is a bicomplex analytic function.

(iv) If $W$ satisfies the Hölder condition at a point $z_0 \in \Omega$ then $\partial_\tau T_\tau W(z_0)$ and $\partial_z T_\tau W(z_0)$ exist, and $\partial_\tau^2 T_\tau W(z_0) = W(z_0)$.

(v) If $W$ is Hölder continuous in $\Omega$, then so are $\partial_\tau T_\tau W$ and $\partial_z T_\tau W$.

**Proof.** The proof follows directly from the well known properties of the operator $A$ [3, pag 7].

With the help of the last proposition the following useful results concerning bicomplex pseudoanalytic functions are obtained.

**Proposition 11** Let $W$ be a bounded measurable $\mathbb{B}$-valued function in some domain $\Omega$. Set

$$h = W - T_\tau(aW + b\overline{W}).$$

Then, $W$ is $\mathbb{B}$-pseudoanalytic iff $h$ is $\mathbb{B}$-analytic.

**Proof.** Suppose that $h$ is $\mathbb{B}$-analytic. Then as $W$ is bounded, by Proposition 10 the function $T_\tau(aW + b\overline{W})$ is Hölder continuous. As $h$ is Hölder continuous (being $\mathbb{B}$-analytic) the function $W$ is Hölder continuous as well and by Proposition 10 the function $T_\tau(aW + b\overline{W})$ is continuously differentiable. As $\partial_\tau h = 0$, application of $\partial_\tau$ to $W$ gives

$$\partial_\tau W = \partial_\tau h + \partial_\tau T_\tau(aW + b\overline{W}) = aW + b\overline{W}.$$ 

The other direction of this statement can be proved using similar techniques as in [3, p. 8].

**Proposition 12** A $\mathbb{B}$-pseudoanalytic function in $\Omega$ has Hölder continuous partial derivatives in every compact subdomain of $\Omega$.

**Proof.** The proof follows from the above proposition and from the properties of the operator $T_\tau$ established in Proposition 10.

**Theorem 13** The limit of a uniformly convergent sequence of $\mathbb{B}$-pseudoanalytic functions is $\mathbb{B}$-pseudoanalytic.
Proof. Let $W_n(z)$ be a sequence of bounded $\mathbb{B}$-pseudoanalytic functions in a bounded domain $\Omega$ such that $W_n(z) \to W(z)$ uniformly in $\Omega$. It follows from Proposition 11 that the functions $h_n$ defined by

$$h_n = W_n - T_\zeta(aW_n + bW_n)$$

are $\mathbb{B}$-analytic in $\Omega$. Then the uniform limit $h = W - T_\zeta(aW + bW)$ is also a $\mathbb{B}$-analytic function in $\Omega$. The last equality together with Proposition 11 allows one to conclude that $W$ is $\mathbb{B}$-pseudoanalytic in $\Omega$.

Theorem 14 Let $W$ be a $\mathbb{B}$-pseudoanalytic function in $\Omega$ except perhaps a finite number of points $\{z_1, \ldots, z_p\}$ where $W$ is allowed to be unbounded. Then if $W^{-1}W$ is a bounded measurable function in $\Omega$, there exists a $\mathbb{B}$-analytic function $\Psi$ in $\Omega \setminus \{z_1, \ldots, z_p\}$ such that

$$W = \Psi E[S], \quad \text{in } \Omega,$$

where $S = T_\zeta(a + \frac{W}{W}b)$.

Proof. Application of $\partial_\zeta$ to $\Psi = WE[-S]$ gives

$$\partial_\zeta \Psi = (\partial_\zeta W) E[-S] + W \partial_\zeta E[-S]$$

$$= (aW + bW) E[-S] + W \left( -a + \frac{W}{W}b \right) E[-S] = 0.$$  

Then $\Psi$ is $\mathbb{B}$-analytic. ■

3.2 Vekua’s equation for $(F,G)$-derivatives and the antiderivative

Definition 15 Let $(F,G)$ and $(F_1,G_1)$ be two generating pairs in $\Omega$. $(F_1,G_1)$ is called a successor of $(F,G)$ and $(F,G)$ is called a predecessor of $(F_1,G_1)$ if

$$a_{(F_1,G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)}.$$  

By analogy with the complex case [3] (see also [16]) we have the following statements

Theorem 16 Let $W$ be a bicomplex $(F,G)$-pseudoanalytic function and let $(F_1,G_1)$ be a successor of $(F,G)$. Then $W$ is an $(F_1,G_1)$-pseudoanalytic function.

Definition 17 A sequence of generating pairs $\{(F_m,G_m)\}$, $m = 0, \pm 1, \pm 2, \ldots$ is called a generating sequence if $(F_{m+1},G_{m+1})$ is a successor of $(F_m,G_m)$. If $(F_0,G_0) = (F,G)$, we say that $(F,G)$ is embedded in $\{(F_m,G_m)\}$.

Let $W$ be a bicomplex $(F,G)$-pseudoanalytic function. Using a generating sequence in which $(F,G)$ is embedded we can define the higher derivatives of $W$ by the recursion formula

$$W^{[0]} = W; \quad W^{[m+1]} = \frac{d(F_m,G_m)W^{[m]}}{dz}, \quad m = 0, 1, \ldots.$$  

10
Definition 18 Let \((F, G)\) be a generating pair. Its adjoint generating pair \((F^*, G^*)\) is defined by the formulas
\[
F^* = -\frac{2F}{FG - FG}, \quad G^* = \frac{2G}{FG - FG}.
\]

Proposition 19 \((F, G)^{**} = (F, G)\) and
\[
a_{(F^*, G^*)} = -a_{(F, G)} \quad b_{(F^*, G^*)} = -b_{(F, G)}
\]
\[A_{(F^*, G^*)} = -A_{(F, G)} \quad B_{(F^*, G^*)} = -B_{(F, G)}\]

Proposition 20 If \((F^{-1}, G^{-1})\) is a predecessor of \((F, G)\) then \((F^{-1}, G^{-1})^*\) is a successor of \((F, G)^*\).

Definition 21 Let \(\Gamma\) be a rectifiable curve leading from \(z_0\) to \(z_1\) and \(W\) a continuous function on \(\Gamma\). The \((F, G)^*\)-integral of \(W\) along \(\Gamma\) is defined by
\[
* \int_{\Gamma} W d_{(F, G)} z = \text{Sc} \int_{\Gamma} G^* W dz + j \text{Sc} \int_{\Gamma} F^* W dz
\]
and the \((F, G)\)-integral is defined as
\[
\int_{\Gamma} W d_{(F, G)} z = F(z_1) \text{Sc} \int_{\Gamma} G^* W dz + G(z_1) \text{Sc} \int_{\Gamma} F^* W dz.
\]

Definition 22 A continuous \(\mathbb{B}\)-valued function \(W\) defined in \(\Omega\) is called \((F, G)\)-integrable if for every closed curve \(\Gamma\) lying in a simply connected subdomain of \(\Omega\),
\[
* \int_{\Gamma} W d_{(F, G)} z = 0.
\]

Proposition 23 Let \((F, G)\) be a predecessor of \((F_1, G_1)\) and \(W\) be a continuous function defined in a simply connected domain \(\Omega\). Then

(i) \(W\) is \((F_1, G_1)\)-pseudoanalytic in \(\Omega\) iff \(W\) is \((F, G)\)-integrable, that is iff
\[
\text{Sc} \int_{\Gamma} G^* W dz = \text{Sc} \int_{\Gamma} F^* W dz = 0
\]
along every closed path \(\Gamma \subset \Omega\).

(ii) If \(W\) is \((F_1, G_1)\)-pseudoanalytic in \(\Omega\) and \(z_0 \in \Omega\) then \(w := \int_{\tau_0}^{\tau} W(\tau) d_{(F, G)} \tau\) is \((F, G)\) pseudoanalytic and \(W = \frac{d_{(F, G)} w}{dz}\).

Remark 24 The above proposition remains true for multiply-connected domains, except that in (ii) \(w\) may be multiple valued.
3.3 Formal powers

Let \((F, G)\) be a generating pair corresponding to \(\Box\) in some domain \(\Omega \subset \mathbb{R}^2\), \(n \in \mathbb{Z}\) and \(z_0 = x_0 + jy_0 \in \Omega\). We call formal powers of order \(n\) and center \(z_0\) to a pair of solutions \(Z^{(n)}(1, z_0, z)\), \(Z^{(n)}(j, z_0, z)\) of \(\Box\) in \(\Omega \setminus \{z_0\}\) such that
\[
Z^{(n)}(1, z_0, z) \sim (z - z_0)^n, \quad Z^{(n)}(j, z_0, z) \sim j(z - z_0)^n, \quad \text{as } z \to z_0, \tag{16}
\]
where \((z - z_0)^n = [(x - x_0) + j(y - y_0)]^n\) are the usual powers of the \(\mathbb{B}\)-analytic functions. For \(\alpha \in \mathbb{B}\), the following definition will be useful
\[
Z^{(n)}(\alpha, z_0, z) := \text{Sc } \alpha \ Z^{(n)}(1, z_0, z) + \text{Vec } \alpha \ Z^{(n)}(j, z_0, z). \tag{17}
\]
The function defined by \(\Box\) is called formal power of the order \(n\) with the coefficient \(\alpha\) and the center \(z_0\). In this work we are mainly interested in negative formal powers, that is, when \(n < 0\). We emphasize that the formal powers are not uniquely defined. For example, if a regular solution of \(\Box\) is added to a negative formal power the resulting solution will be again a negative formal power of the same order, center and coefficient as the initial one.

The special case when \(n = -1\) is distinguished: the formal power \(Z^{(-1)}(\alpha, z_0, z)\) playing an important role in the study of pseudoanalytic functions is called Cauchy kernel. Next we obtain some asymptotic formulas for the Cauchy kernel that will be important in order to establish the Cauchy integral formula in the subsequent section. First we introduce the following definition.

**Definition 25** Let \(W, g\) be a \(\mathbb{B}\)-valued and an \(\mathbb{R}\)-valued functions respectively. We agree that the notation \(W = \mathcal{O}(g)\), as \(z \to z_0\), means that the function \(\frac{W(z)}{g(z)}\) is bounded in some neighborhood of \(z_0\).

**Proposition 26** For \(\alpha = 1\) or \(\alpha = j\) the asymptotic formulas
\[
\lim_{z \to z_0} \frac{Z^{(-1)}(\alpha, z_0, z)}{Z^{(-1)}(\alpha, z_0, z_0)} = 1 \tag{18}
\]
and
\[
Z^{(-1)}(\alpha, z_0, z) = \frac{\alpha}{z - z_0} + \mathcal{O}(\log |z - z_0|), \quad \text{as } z \to z_0 \tag{19}
\]
hold.

**Proof.** We prove the theorem for \(\alpha = 1\), the case when \(\alpha = j\) can be treated by analogy.

(i) By the definition of \(Z^{(-1)}(1, z_0, z)\) we obtain
\[
\lim_{z \to z_0} (z - z_0)Z^{(-1)}(1, z_0, z) = 1. \tag{20}
\]
From the fact that \(\mathcal{R}(\mathbb{B})\) is open (Proposition 3) and \(1 \in \mathcal{R}(\mathbb{B})\) we conclude that there exists some neighborhood of \(z_0\) denoted by \(N_{z_0}\) in which \((z - z_0)Z^{(-1)}(1, z_0, z) \in \mathcal{R}(\mathbb{B})\). Particularly \(Z^{(-1)}(1, z_0, z) \in \mathcal{R}(\mathbb{B})\) in this neighborhood because \((z - z_0) \in \mathcal{R}(\mathbb{B})\) for all \(z, z_0 \in \mathbb{C}_j\). Hence
Theorem 14 guarantees the existence of a generating pair, that is, \((F,G)\). From Propositions 4 and 10 (in both part \((ii)\)), we have that \(F\) and \(G\) are \(\mathbb{B}\)-analytic functions defined everywhere and that \((F,G)\) is a complete generating pair. However, as we discuss next, under certain conditions regarding the point of infinity they are uniquely determined. Until the end of this subsection we suppose that \(F\) and \(G\) are \(\mathbb{C}_j\)-valued functions defined everywhere and that \((F,G)\) is a complete generating pair, that is, \((F,G)\) is a generating pair in \(\mathbb{R}^2\) such that \(F(\infty), G(\infty)\) exist, \(\text{Vec}(F(\infty))G(\infty) \neq 0\) and the functions \(F(\frac{1}{z}), G(\frac{1}{z})\) are Hölder continuous.

The results of L. Bers \([4]\) together with Remark \([9]\) allow us to obtain the following statements.

**Proposition 27** Under the above conditions for any integer \(n\) and for every pair of numbers \(\alpha \in \mathbb{B}, \alpha \neq 0\) and \(z_0 \in \mathbb{C}_j\) there exists one and only one \((F,G)\)-pseudoanalytic function \(Z^{(n)}(\alpha, z_0, z)\) defined in \(\mathbb{C}_j \setminus \{z_0\}\) such that

\[
Z^{(n)}(\alpha, z_0, z) \sim \alpha (z - z_0)^n, \quad z \to z_0
\]
and
\[ Z^{(n)}(\alpha, z_0, z) = \mathcal{O}(|z|^n), \quad z \to \infty \] (22)
and having no other zeros or poles except at \( z = \infty \).

The functions described in the above proposition are called global formal powers. The negative global formal powers enjoy the following interesting relations.

**Theorem 28** Let \( \{ (F_m, G_m) \} \), \( m = 0, \pm 1, \pm 2, \ldots \) be a generating sequence of complete generating pairs in which \((F, G)\) is embedded and \( Z_m^{(-n)}(\alpha, z_0, z) \), \( Z_m^*^{(-n)}(\alpha, z_0, z) \) denote the \((F_m, G_m)\)- and the \((F_m, G_m)^*\)- negative formal powers, respectively. Then for any positive integer \( n \) we have
\[
\begin{align*}
\text{Sc} Z_m^{(-n)}(j, z_0, z) + j \text{Sc} Z_m^{(-n)}(1, z_0, z) &= (-1)^n Z_m^{*(-n)}(j, z, z_0) \\
\text{Vec} Z_m^{(-n)}(j, z_0, z) + j \text{Vec} Z_m^{(-n)}(1, z_0, z) &= (-1)^n Z_m^{*(-n)}(1, z, z_0).
\end{align*}
\]

Due to the above theorem for fixed values of \( z \) the global formal powers \( Z_m^{(-n)}(j, z_0, z) \) are continuously differentiable in the variable \( z_0 \).

There are currently no results regarding the existence or the uniqueness for the global formal powers of a general bicomplex Vekua equation though in Subsection 4.1 we establish an existence result under certain additional conditions. We end this section by mentioning that even for complex Vekua equations the construction of the negative formal powers (global or not) is a very difficult task. In Section 6 we construct the negative formal powers in the explicit form for some classes of bicomplex main Vekua equations.

### 3.4 A relation between the Schrödinger equation and the main Vekua equation

Let \( q \) be a complex \((C_i\text{-valued})\) continuous function in \( \Omega \subseteq \mathbb{R}^2 \). Consider the two-dimensional stationary Schrödinger equation
\[ \Delta u - qu = 0 \quad \text{in } \Omega \] (23)
and assume that it possesses a \( C_i \)-valued particular solution \( f \in C^2(\Omega) \) such that \( f(z) \neq 0 \) for all \( z \in \Omega \). We will need a result from [13] (see also [16]) relating the Schrödinger equation with a Vekua equation of a special kind.

**Theorem 29** Let \( W = u + jv \) with \( u = \text{Sc} W, \ v = \text{Vec} W \) be a solution of the main bicomplex Vekua equation
\[ \partial_t W = \frac{\partial f}{f} W \quad \text{in } \Omega. \] (24)
Then \( u \) is a solution of (23) and \( v \) is a solution of
\[ \Delta v - q_1 v = 0 \quad \text{in } \Omega \] (25)
where \( q_1 = 2 \frac{(\partial_t f)^2 + (\partial_s f)^2}{f^2} - q. \)
Remark 30 A generating pair corresponding to (24) can be chosen in the form

\[(F,G) = (f, \frac{j}{f})\]  \(\text{(26)}\)

The corresponding characteristic coefficients are

\[a(f, \frac{j}{f}) = A(f, \frac{j}{f}) = 0, \quad b(f, \frac{j}{f}) = \frac{\partial_z f}{f}, \quad B(f, \frac{j}{f}) = \frac{\partial_y f}{f}\]

and the successor equation of (24) is

\[\partial_z W = -\frac{\partial_z f}{f}W\]  \(\text{(27)}\)

We need the following notation. Let \(W\) be a \(B\)-valued function defined on a simply connected domain \(\Omega\) with \(u = \text{Sc} W\) and \(v = \text{Vec} W\) such that

\[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \quad \text{in } \Omega\]  \(\text{(28)}\)

and let \(\Gamma \subset \Omega\) be a simple rectifiable curve leading from \(z_0 = x_0 + jy_0\) to \(z = x + jy\). Then the integral

\[\overline{AW}(z) := 2 \left( \int_{\Gamma} u dx + v dy \right)\]

is path-independent, and all \(C_i\)-valued solutions \(\varphi\) of the equation \(\partial_{\varphi} \varphi = W\) in \(\Omega\) have the form \(\varphi(z) = \overline{AW}(z) + c\) where \(c\) is an arbitrary \(C_i\)-constant.

Remark 31 It is easy to check that if \(u\) is a solution of the Schrödinger equation (23) in a simply connected domain \(\Omega\) then the \(B\)-valued function \(jf^2 \partial_{\varphi} (f^{-1}u)\) satisfies (28) and therefore the function constructed by the rule

\[T_f(u) := f^{-1} \overline{A}(jf^2 \partial_{\varphi} (f^{-1}u))\]  \(\text{(29)}\)

is well defined in \(\Omega\). Application of \(T_f\) to a more general class of functions will be also considered but as in this case the result of such operation may depend on the choice of the curve \(\Gamma\) leading from \(z_0\) to \(z\), the notation \(T_f, \Gamma(u)\) will be used.

Theorem 32 If \(u\) is a \(C_i\)-valued solution of the Schrödinger equation (23) in a simply connected domain \(\Omega\) then \(v = T_f(u)\) is a solution of (25) and \(W = u + jv\) is a solution of (24).

Corollary 33 Let \(u_n \in C^2(\Omega)\) be a sequence of solutions of (25) and \(u\) be such that \(u_n \rightarrow u\), \(\partial_x u_n \rightarrow \partial_x u\) and \(\partial_y u_n \rightarrow \partial_y u\) uniformly in \(\Omega\). Then \(u\) is a solution of (24).

Proof. The assumption together with the above theorem allows one to conclude that the sequence of solutions of (24) given by \(W_n := u_n + jT_f(u_n)\) converges uniformly to \(W := u + jT_f(u)\). Then by Theorem 13 \(W\) is a solution of (24) and from Theorem 29 its scalar part \(u\) is a solution of (23).
4 Cauchy integral formulas for bicomplex pseudoanalytic functions

Consider the bicomplex Vekua equation
\[ \partial_z W = aW + b\overline{W} \] (30)
in a simply connected domain \( \Sigma \subset \mathbb{C}_j \) where the \( \mathbb{B} \)-valued functions \( a \) and \( b \) are Hölder continuous. Throughout this section \( \Omega \) is a bounded domain such that \( \overline{\Omega} \subset \Sigma \) and regular in the following sense [3, p. 2] \( \partial \Omega \) consists of a finite number of piecewise differentiable simple closed Jordan curves.

4.1 First Cauchy’s integral formula

For complex pseudoanalytic functions the integral representation called here first Cauchy’s integral formula was obtained in [25]. In that representation the Cauchy kernel corresponding to an adjoint Vekua equation is used. In the present section we obtain this Cauchy integral formula for \( \mathbb{B} \)-pseudoanalytic functions.

The adjoint equation of (30) has the form
\[ \partial_{\overline{z}} V = -aV - b\overline{V}. \] (31)

In the next two propositions a useful characterization of solutions of (31) is obtained.

**Proposition 34** Let \( W \) and \( V \) be solutions of (30) and (31) respectively in \( \Omega \) and continuous in \( \Omega \). Then
\[ \text{Vec} \int_{\partial \Omega} W(\tau)V(\tau)d\tau = 0. \] (32)

**Proof.** The proof is analogous to the proof given in the complex situation [25, p. 169].

The converse of this statement can be obtained implementing Bers’ result on a generalization of Morera’s theorem for solutions of equation (31).

**Proposition 35** Let \( (F,G) \) be a generating pair corresponding to (30) and \( V \) be a continuous function in \( \Omega \). Then \( V \) is a solution of (31) in \( \Omega \) iff the following equalities hold
\[ \text{Vec} \int_{\Gamma} G(\tau)V(\tau)d\tau = \text{Vec} \int_{\Gamma} F(\tau)V(\tau)d\tau = 0 \] (33)
along every closed path \( \Gamma \) situated in a simply connected subdomain of \( \Omega \).

**Proof.** If \( V \) is a solution of (31) then (33) is a consequence of the previous proposition. To prove the statement in the opposite direction notice that if \( (F,G) \) is a generating pair corresponding to (30) then from Proposition 19 one has that \( (jF^*, jG^*) \) is a generating pair corresponding to the equation
\[ W_\tau = -aW + \overline{\mathcal{O}(F,G)}W, \]
and its successor is equation (31). This, together with Proposition 28 implies that the solutions of (31) are \((jF^*, jG^*)\)-integrable functions. Notice that \((jF^*, jG^*)^* = (-jF, -jG)\) and then equality (33) tells us that \( V \) is \((jF^*, jG^*)\)-integrable. Thus, \( V \) is a solution of (31).
Proposition 36 (First Cauchy’s integral formula) Let \( W \) be a solution of (30) in \( \Omega \) continuous in \( \Omega \) and \( \hat{Z}^{(-1)}(\alpha, z_0, z) \) be a Cauchy kernel of (31) in \( \Sigma \). Then,

\[
\text{Vec} \int_{\partial \Omega} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau - j \text{Vec} \int_{\partial \Omega} W(\tau) \hat{Z}^{(-1)}(j, z_0, \tau) d\tau = \begin{cases} 
2\pi W(z_0), & \text{if } z_0 \in \Omega \\
0, & \text{if } z_0 \in \Sigma \setminus \{\bar{\Omega}\}.
\end{cases}
\]

Proof. When \( z_0 \in \Sigma \setminus \{\bar{\Omega}\} \) the equality follows from Proposition 34. For \( z_0 \in \Omega \) let us prove the scalar part of the equality,

\[
\text{Vec} \int_{\partial \Omega} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau = 2\pi \text{Sc} W(z_0).
\]

Let \( D_\varepsilon(z_0) \) be a disk with the center \( z_0 \) and radius \( \varepsilon \) and \( \Omega_\varepsilon = \Omega \setminus \{D_\varepsilon(z_0)\} \). Then for a sufficiently small \( \varepsilon \) from Proposition 34 we obtain

\[
\text{Vec} \int_{\partial \Omega_\varepsilon} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau = 0,
\]

that is

\[
\text{Vec} \int_{\partial \Omega} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau = \text{Vec} \int_{\partial D_\varepsilon} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau.
\]

By (19) we have that

\[
\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \frac{W(\tau)}{\tau - z_0} d\tau = 2\pi j W(z_0).
\]

Thus,

\[
\text{Vec} \int_{\partial \Omega} W(\tau) \hat{Z}^{(-1)}(1, z_0, \tau) d\tau = \text{Vec}(2\pi j W(z_0)) = 2\pi \text{Sc} W(z_0).
\]

The vector part of the Cauchy integral formula is proved in the same way.

Remark 37 As the adjoint of equation of (31) is (30) a similar Cauchy’s integral formula holds for solutions of (31) in terms of Cauchy kernels corresponding to (30).

Following [25, p. 174] we use the first Cauchy integral formula to obtain relations between Cauchy kernels corresponding to mutually adjoint bicomplex Vekua equations.

Proposition 38 Let \( \Sigma = \mathbb{C}_j \) and suppose that \( Z^{(-1)}(\alpha, z_0, z) \), \( \hat{Z}^{(-1)}(\alpha, z_0, z) \) are Cauchy kernels in \( \mathbb{C}_j \) corresponding to (30) and (31) respectively, both behaving like \( O(|z|^{-1}) \) as \( z \to \infty \). Then

\[
\hat{Z}^{(-1)}(1, z_0, z) = -\text{Sc} Z^{(-1)}(1, z_0, z) + j \text{Sc} Z^{(-1)}(j, z, z_0)
\]

and

\[
\hat{Z}^{(-1)}(j, z_0, z) = \text{Vec} Z^{(-1)}(1, z, z_0) - j \text{Vec} Z^{(-1)}(j, z, z_0).
\]
Proof. Let $z_0, z \in \mathbb{C}_j$, $z_0 \neq z$ and $\varepsilon > 0$ be sufficiently small such that $z \in \Omega_\varepsilon = D_\varepsilon(z_0) \setminus D_\varepsilon(z_0)$ where $D_\varepsilon(z_0)$ is a disc with the center $z_0$ and radius $\varepsilon$. Then from the first Cauchy integral formula we have
\[
2\pi \hat{Z}^{(-1)}(1, z_0, z) = \text{Vec} \int_{\partial D_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(1, z, \tau) d\tau - j \text{Vec} \int_{\partial \Omega_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(j, z, \tau) d\tau
\]
or equivalently,
\[
2\pi \hat{Z}^{(-1)}(1, z_0, z) = \text{Vec} \int_{\partial D_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(1, z, \tau) d\tau - j \text{Vec} \int_{\partial D_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(j, z, \tau) d\tau - \text{Vec} \int_{\partial D_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(1, z, \tau) d\tau + j \text{Vec} \int_{\partial D_\varepsilon} \hat{Z}^{(-1)}(1, z_0, \tau) Z^{(-1)}(j, z, \tau) d\tau.
\]
Taking into account the asymptotic behaviour of the kernels at infinity it is easy to verify that the first and the second integrals on the right-hand side of the equality tends to zero when $\varepsilon \to 0$. Then (34) is obtained from the last equality applying the first Cauchy integral formula to the last two integrals and considering $\varepsilon \to 0$. A similar reasoning proves (35). $
$
Remark 39 Under the assumptions of Proposition 27 (which in particular imply that $a$ and $b$ are $\mathbb{C}_j$-valued functions) equations (30) and (31) possess unique Cauchy kernels (called global Cauchy kernels) satisfying the conditions of Proposition 28 and therefore they can be computed one from another by means of (34) and (35). For a general bicomplex Vekua equation we were not able to obtain comparable definitive results regarding the existence or uniqueness of the kernels possessing such properties. However from the previous proposition we immediately obtain the following uniqueness result.

Corollary 40 If both equations (30) and (31) possess Cauchy kernels fulfilling the assumptions of Proposition 28 then they are unique.

4.2 Second Cauchy’s integral formula and reproducing Cauchy kernels

Definition 41 Let $\Omega \subseteq \Sigma$ be a regular domain, $W$ be a continuous function up to $\overline{\Omega}$ and $Z^{(n)}(\alpha, z_0, z)$ be a formal power corresponding to (30) in $\Sigma$, continuous in the variable $z_0$ for fixed values of $z$. Following [3] we define
\[
\int_{\partial \Omega} Z^{(n)}(jW(\tau)) d\tau, \tau, z_0 = \int_a^b Z^{(n)}(jW(\gamma(t))) \gamma'(t) dt, \gamma(t), z_0
\]
where $\gamma(t) : [a, b] \to \mathbb{C}_j$ is a parametrization of $\partial \Omega$.

Theorem 42 (Second Cauchy’s integral formula) [27] Let $Z^{(-1)}(\alpha, z_0, z)$, $\hat{Z}^{(-1)}(\alpha, z_0, z)$ be Cauchy kernels corresponding to (30) and (31) respectively satisfying (34), (35) in $\Sigma$ and $W$ be a solution of (30) in $\Omega$ continuous in $\overline{\Omega}$. Then
\[
\int_{\partial \Omega} Z^{(-1)}(jW(\tau)) d\tau, \tau, z_0 = 2\pi W(z_0), \quad (36)
\]
for \( z_0 \in \Omega \) and
\[
\int_{\partial \Omega} Z^{(-1)}(jW(\tau)d\tau, \tau, z_0) = 0
\]
(37)

for \( z_0 \in \Sigma \setminus \{\Omega\} \).

**Proof.** Direct calculation gives us the equality
\[
\int_{\partial \Omega} Z^{(-1)}(jW(\tau)d\tau, \tau, z_0) =
\]
\[
= \text{Vec} \int_{\partial \Omega} W(\tau) \left\{ -\text{Sc} Z^{(-1)}(1, \tau, z_0) + j \text{Sc} Z^{(-1)}(j, \tau, z_0) \right\} d\tau
\]
\[
- j \text{Vec} \int_{\partial \Omega} W(\tau) \left\{ \text{Vec} Z^{(-1)}(1, \tau, z_0) - j \text{Vec} Z^{(-1)}(j, \tau, z_0) \right\} d\tau. \tag{38}
\]

From (34), (35), (38) and from the first Cauchy integral formula we obtain the result. □

**Remark 43** Formulas (36) and (37) were obtained independently in [3] and [25] using global Cauchy kernels. Our proof which follows that from [25] reveals that the second Cauchy integral formula is equivalent to the first when an appropriate Cauchy kernel corresponding to (30) is used. The Cauchy kernels involved, \( Z^{(-1)}(\alpha, z_0, z) \) and \( \tilde{Z}^{(-1)}(\alpha, z_0, z) \) should satisfy equalities (34), (35) in the domain of interest but not necessarily be global. It is worth noticing that equalities (36) and (37) are not valid for an arbitrary Cauchy kernel of (30).

For example, consider the following Cauchy kernels
\[
Z^{(-1)}(1, \zeta, z) = \frac{1}{z - \zeta} + \xi, \quad Z^{(-1)}(j, \zeta, z) = \frac{j}{z - \zeta}.
\]
of the equation \( \partial_z W = 0 \), where \( z = x + jy \) and \( \zeta = \xi + j\eta \). If \( \Omega \) is the unitary disk, \( z_0 = 0 \) and \( W \equiv 1 \) then the left-hand side of (30) is
\[
\int_0^{2\pi} Z^{(-1)}(j^2 e^{j\theta} d\theta, e^{j\theta}, 0) = - \int_0^{2\pi} \left[ (\cos \theta) Z^{(-1)}(1, e^{j\theta}, 0) + (\sin \theta) Z^{(-1)}(j, e^{j\theta}, 0) \right] d\theta
\]
\[
= \int_0^{2\pi} (1 - \cos^2 \theta) d\theta = \pi
\]
which is different from \( 2\pi W(0) \).

**Definition 44** We say that a Cauchy kernel \( Z^{(-1)}(\alpha, z_0, z) \) defined in \( \Sigma \) reproduces a solution \( W \) of (30) if for every domain \( \Omega \) where \( W \) is pseudoanalytic in \( \Omega \) and continuous in \( \Omega \) the formula (36) holds. A local Cauchy kernel is called reproducing if it reproduces all solutions of (30).

Combining results of the preceding sections we obtain the following criteria describing the reproducing Cauchy kernels.
Theorem 45  Let \((F, G)\) be a generating pair and \(Z^{(-1)}(\alpha, z_0, z)\) be a Cauchy kernel, both corresponding to (31) in \(\Sigma\) and such that for each fixed \(z\) the kernel \(Z^{(-1)}(\alpha, z_0, z)\) is a continuous function in the variable \(z_0\) and
\[
\lim_{z_0 \to z} (z - z_0)Z^{(-1)}(\alpha, z_0, z) = \alpha, \quad \alpha = 1, j.
\] (39)

Then the following statements are equivalent:

(i) \(Z^{(-1)}(\alpha, z_0, z)\) reproduces both solutions \(F\) and \(G\);

(ii) There exists a Cauchy kernel \(\hat{Z}^{(-1)}(\alpha, z_0, z)\) corresponding to (31) such that (34) and (35) hold;

(iii) Second Cauchy’s integral formula holds;

(iv) \(Z^{(-1)}(\alpha, z_0, z)\) is a reproducing Cauchy kernel.

Proof.

(i) \(\Rightarrow\) (ii) Let \(z_0 \in \Sigma\) and \(\Omega\) be a domain such that \(z_0 \notin \overline{\Omega} \subset \Sigma\). Then from (i) we have
\[
\int_{\partial \Omega} Z^{(-1)}(jF(\tau)d\tau, \tau, z_0) = \int_{\partial \Omega} Z^{(-1)}(jG(\tau)d\tau, \tau, z_0) = 0.
\]

From this equality and from (38) we obtain
\[
\text{Vec} \int_{\partial \Omega} F(\tau)w_1(z_0, \tau)d\tau = \text{Vec} \int_{\partial \Omega} G(\tau)w_2(z_0, \tau)d\tau = 0 \tag{40}
\]
where
\[
w_1(z_0, z) := -\text{Sc} Z^{(-1)}(1, z, z_0) + j \text{ Sc} Z^{(-1)}(j, z, z_0),
\]
\[
w_2(z_0, z) := \text{Vec} Z^{(-1)}(1, z, z_0) - j \text{ Vec} Z^{(-1)}(j, z, z_0).
\]

From (40) and Proposition 40 we conclude that \(w_1, w_2\) are solutions of (31) in \(\Omega \setminus \{z_0\}\). On the other hand using (39) we see that the left-hand side of these equalities defines Cauchy kernels corresponding to (31). Thus, (31) and (35) hold.

(ii) \(\Rightarrow\) (iii) follows from Theorem 42.

(iii) \(\Rightarrow\) (iv) and (iv) \(\Rightarrow\) (i) is straightforward.

\[\blacksquare\]

In the following example we slightly change the Cauchy kernel from Remark 43 and obtain a reproducing but not global Cauchy kernel.

Example 46  Consider the Cauchy kernel
\[
Z^{(-1)}(1, \zeta, z) = \frac{1}{z - \zeta} + \xi, \quad Z^{(-1)}(j, \zeta, z) = \frac{j}{z - \zeta} + \eta
\]
corresponding to the equation \( \partial_{\bar{z}} W = 0 \) with \( \zeta = \xi + j\eta \). Since

\[
- \text{Sc} Z^{(-1)}(1, z, z_0) + j \text{Sc} Z^{(-1)}(j, z, z_0) = \frac{1}{z - z_0} + z
\]  

(41)

and

\[
\text{Vec} Z^{(-1)}(1, z, z_0) - j \text{Vec} Z^{(-1)}(j, z, z_0) = \frac{1}{z - z_0}.
\]  

(42)

due to Theorem 45, \( Z^{(-1)}(\alpha, \zeta, z) \) is a reproducing Cauchy kernel in any bounded domain.

Let us prove that this kernel is not a restriction of any global Cauchy kernel in any simply connected domain containing \((0, 0)\) and \((1, 0)\) as interior points. Indeed, take \( \zeta = 1 + \frac{1}{n} \) and \( z = \zeta - \frac{1}{n} = 1 + \frac{1}{n} - \frac{1}{n+1} \). Then \( Z^{(-1)}(1, \zeta, z) = 0 \). On the other hand as we know a global Cauchy kernel can not vanish except at \( z = \infty \).

5 Construction of negative formal powers

Let \( \{ (\hat{F}_n, \hat{G}_n) \} \), \( n = 0, 1, 2 \ldots \) be a generating sequence in a domain \( \Sigma \subset \mathbb{C}_j \) embedding a generating pair corresponding to \((31)\). In this section we show how a set of negative formal powers corresponding to \((30)\) can be constructed following a simple algorithm when the sequence \( \{ (\hat{F}_n, \hat{G}_n) \} \) and a reproducing Cauchy kernel for \((30)\) are known.

Let \( Z^{(-1)}(\alpha, z_0, z) \) be a reproducing Cauchy kernel of \((30)\) enjoying the properties from the hypothesis of Theorem 45. Then, as was established in Theorem 45, \( Z^{(-1)}(\alpha, z_0, z) \) constructed by means of \((34)\) and \((35)\) is a reproducing Cauchy kernel for \((30)\). For an integer \( n \geq 2 \) define

\[
\hat{Z}^{(-n)}_{n-1}(\alpha, z_0, z) := \frac{(-1)^n-1}{(n-1)!} \frac{d}{dz} \frac{d}{dz} \cdots \frac{d}{dz} \frac{d}{dz} \frac{d}{dz} \frac{d}{dz} \hat{G}_{n-2}(\alpha, z_0, z).
\]

By construction, \( \hat{Z}^{(-n)}_{n-1}(\alpha, z_0, z) \) is an \( (\hat{F}_{n-1}, \hat{G}_{n-1}) \)-formal power of the order \(-n\). These are related with the formal powers for equation \((30)\) in the following way.

**Theorem 47** Under the above conditions suppose that the functions \( \hat{Z}^{(-n)}_{n-1}(\alpha, z_0, z), \alpha = 1, j \) are continuous in the variable \( z_0 \) for fixed values of \( z \) and that

\[
\lim_{z_0 \to z} (z - z_0)^n \hat{Z}^{(-n)}_{n-1}(\alpha, z_0, z) = \alpha, \quad \alpha = 1, j.
\]

Then

\[
Z^{(-n)}(1, z_0, z) := (-1)^n \text{Sc} \hat{Z}^{(-n)}_{n-1}(1, z, z_0) + j(-1)^{n+1} \text{Sc} \hat{Z}^{(-n)}_{n-1}(j, z, z_0),
\]

(44)

\[
Z^{(-n)}(j, z_0, z) := (-1)^{n+1} \text{Vec} \hat{Z}^{(-n)}_{n-1}(1, z, z_0) + j(-1)^n \text{Vec} \hat{Z}^{(-n)}_{n-1}(j, z, z_0)
\]

(45)

are negative formal powers corresponding to \((30)\).

**Proof.** Let \( z_0 \in \Sigma \) and \( \Omega \) be a domain such that \( z_0 \notin \overline{\Omega} \subset \Sigma \). As mentioned above \( Z^{(-1)}(\alpha, z_0, z) \) constructed by means of \((34)\) and \((35)\) is a reproducing Cauchy kernel corresponding to \((31)\). This in particular implies that

\[
\int_{\partial \Omega} \hat{Z}^{(-1)}(j\hat{F}(\tau)d\tau, \tau, z_0) = \int_{\partial \Omega} \hat{Z}^{(-1)}(j\hat{G}(\tau)d\tau, \tau, z_0) = 0.
\]
Taking in the above equalities \( n - 1 \) Bers’ derivatives with respect to the variable \( z_0 \) we obtain
\[
\int_{\partial \Omega} \hat{Z}^{(-n)}_{n-1}(j \hat{F}(\tau) d\tau, \tau, z_0) = \int_{\partial \Omega} \hat{Z}^{(-n)}_{n-1}(j \hat{G}(\tau) d\tau, \tau, z_0) = 0
\]
which equivalently can be written as follows
\[
\text{Vec} \int_{\partial \Omega} \hat{F}(\tau) w_{1,2}(z_0, \tau) d\tau = \text{Vec} \int_{\partial \Omega} \hat{G}(\tau) w_{1,2}(z_0, \tau) d\tau = 0,
\]
where
\[
w_1(z_0, z) := - \text{Sc} \hat{Z}^{(-n)}_{n-1}(1, z, z_0) + j \text{Sc} \hat{Z}^{(-n)}_{n-1}(j, z, z_0),
\]
\[
w_2(z_0, z) := \text{Vec} \hat{Z}^{(-n)}_{n-1}(1, z, z_0) - j \text{Vec} \hat{Z}^{(-n)}_{n-1}(j, z, z_0).
\]

Equalities (46) together with Proposition 35 allow one to conclude that \( w_{1,2}(z_0, z) \) are solutions of \((30)\) in \( \Omega \setminus \{z_0\} \). On the other hand, using (43) we see that the left-hand side of (41) and (45) defines formal powers corresponding to \((30)\). \( \blacksquare \)

**Remark 48** The generating sequence \( \left\{ \left( \hat{F}_n, \hat{G}_n \right) \right\} \) required in the above construction can be obtained as follows. Let \( \{(F_n, G_n)\}, \ n = 0, 1, 2 \ldots \) be a generating sequence embedding a generating pair corresponding to \((30)\). Then by Propositions 14 and 20, \( \left( \hat{F}, \hat{G} \right) := (j F^*_1, j G^*_1) \) is a generating pair for \((31)\) embedded in the generating sequence \( \left\{ \left( \hat{F}_n, \hat{G}_n \right) \right\} \), \( n = 0, 1, 2 \ldots \) where \( \left( \hat{F}_n, \hat{G}_n \right) := (j (F_{n-1}^*)^*, j (G_{n-1}^*)^*) \).

**Remark 49** Let us mention that if \( \{(F_n, G_n)\}, \ n = 0, 1, 2 \ldots \) is a complete generating sequence of \( \mathbb{C}_j \)-valued generating pairs then by Theorem 28, the corresponding \((F,G)\)- and \( \left( \hat{F}_n, \hat{G}_n \right) \)- negative global formal powers satisfy (24) and (25).

### 6 Applications to the main Vekua and Schrödinger equations

Let us suppose that the Schrödinger equation (23) has a nonvanishing particular \( \mathbb{C}_i \)-valued solution \( f \in C^2(\Omega) \). Then as it was explained in Section 3.3 the main Vekua equation (24) is closely related to the Schrödinger equation (23). In this section we use this relation for obtaining the following results. Starting from a fundamental solution of (23) we construct explicitly a reproducing kernel and corresponding negative formal powers for (24) as well as a fundamental solution for the Darboux transformed equation (25).

**Definition 50** We say that a function \( M(\zeta, z) \) satisfies condition I in \( \Omega \times \Omega \) if \( M(\zeta, z) \) together with its first partial derivatives corresponding to \( \zeta \) are continuous functions in \( \Omega \times \Omega \setminus \{\text{diag}\} \), \( M(\zeta, z) \) is bounded in every neighborhood of every point \((z_0, z_0)\) and the same partial derivatives behave like \( O(\log |z - \zeta|) \) as \((\zeta, z) \to (z_0, z_0)\).

We are mainly interested in fundamental solutions for (23)
\[
S(\zeta, z) = \log |z - \zeta| + R(\zeta, z)
\]
fulfilling the following conditions:
(C1) \( S(z, \zeta) \) is a solution of (23) in both variables \( z \) and \( \zeta \);

(C2) \( R(\zeta, z) \in C^1(\Omega \times \Omega \setminus \{ \text{diag} \}) \) and \( R \) is bounded in some neighborhood of every point \((z_0, z_0), z_0 \in \Omega\).

(C3) The functions \( \partial_x R(\zeta, z), \partial_y R(\zeta, z) \) satisfy the condition I in \( \Omega \times \Omega \).

6.1 Construction of a reproducing Cauchy kernel

The purpose of this subsection is to construct a reproducing kernel for the main Vekua equation (24) from a fundamental solution of (23). First we notice the following fact.

Proposition 51 Under the choice of the generating pair corresponding to (24) in the form (26) the successor and the adjoint equation for (24) coincide.

Proof. This is a direct consequence of Remark 30. ■

This observation together with Theorem 45 allow us to state that \( Z^{(-1)}(\alpha, \zeta, z) \) is a reproducing kernel for (24) when the formulas

\[
Z^{(-1)}_1(1, \zeta, z) = - \text{Sc} Z^{(-1)}(1, z, \zeta) + j \text{Sc} Z^{(-1)}(j, z, \zeta),
\]

\[
Z^{(-1)}_1(j, \zeta, z) = \text{Vec} Z^{(-1)}(1, z, \zeta) - j \text{Vec} Z^{(-1)}(j, z, \zeta).
\]

hold where \( Z^{(-1)}(\alpha, \zeta, z) \) is some Cauchy kernel of (27). We will start by constructing \( Z^{(-1)}_1(1, \zeta, z) \) and then the above formulas can be used to obtain \( Z^{(-1)}_1(j, \zeta, z) \) and \( Z^{(-1)}(\alpha, \zeta, z) \).

Proposition 52 Let \( S(\zeta, z) \) be a fundamental solution of (23) satisfying (C1) – (C3). Then the function defined as follows

\[
Z^{(-1)}_1(1, \zeta, z) := 2(\partial_z S(\zeta, z) - \frac{\partial_z f(z)}{f(z)}) S(\zeta, z)
\]

satisfies the following properties:

(i) \( Z^{(-1)}_1(1, \zeta, z) = \frac{1}{z - \zeta} - 2 \frac{\partial_z f(z)}{f(z)} \log |z - \zeta| + M(\zeta, z) \)

where \( M(\zeta, z) \) satisfies condition I.

(ii) \( Z^{(-1)}_1(1, \zeta, z) \) is a Cauchy kernel of (24);

(iii) The scalar and the vector parts of \( Z^{(-1)}_1(1, \zeta, z) \) are solutions of (26) in the variable \( \zeta \).

Proof. (i) Equality (51) and the properties of \( M \) follow directly from (49) and from the conditions (C1) – (C3) defining the fundamental solution \( S \).

(ii) By construction \( Z^{(-1)}_1(1, \zeta, z) \) is a solution of (27) and from (50) we have

\[
\lim_{z \to \zeta} (z - \zeta) Z^{(-1)}_1(1, \zeta, z) = 1.
\]
from which we conclude that this function is a Cauchy kernel.

(iii) We prove that the scalar part of (49)

$$\text{Sc} Z_1^{-1}(1, \zeta, z) = \partial_x S(\zeta, z) - \frac{f_z(z)}{f(z)} S(\zeta, z)$$

is a solution of (23) in the variable $\zeta$. As $S(\zeta, z)$ is already a solution of (23) with respect to $\zeta$, it is sufficient to prove that $\partial_x S(\zeta, z)$ is a solution of the same equation in $\zeta$. We give a rigorous proof of this fact. Let $z_0 = x_0 + jy_0 \in \Omega$ and define $\varphi(\zeta) = \partial_x S(\zeta, z_0)$,

$$\varphi_n(\zeta) = S(\zeta, z_0 + \frac{1}{n}) - S(\zeta, z_0), \quad n \in \mathbb{N}.$$ 

Each function $\varphi_n(\zeta)$ is a solution of (23) and $\varphi_n(\zeta) \to \varphi(\zeta)$ pointwise. Let $\zeta_0 \in \Omega, \zeta_0 \neq z_0$. We will prove that

$$\varphi_n(\zeta) \to \varphi(\zeta), \quad \partial_\xi \varphi_n(\zeta) \to \partial_\xi \varphi(\zeta) \text{ and } \partial_\eta \varphi_n(\zeta) \to \partial_\eta \varphi(\zeta)$$

uniformly is some neighborhood of $\zeta_0$. This together with Corollary 33 allows us to conclude that $\varphi(\zeta)$ is a solution of (23) in such neighborhood of $\zeta_0$, and also in $\Omega \setminus \{z_0\}$ (because $\zeta_0$ is an arbitrary point).

By the mean value theorem there exists some point $x_n \in [x_0, x_0 + \frac{1}{n}]$ such that

$$\varphi_n(\zeta) = \partial_x S(\zeta, x_n + jy_0)$$

Let $\varepsilon > 0$. Then by the continuity of $\partial_x S$ in $\Omega \times \Omega \setminus \{\text{diag}\}$ there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\varphi(\zeta) - \varphi_n(\zeta)| = |\partial_x S(\zeta, z_0) - \partial_x S(\zeta, x_n + jy_0)| < \varepsilon$$

for all $\zeta$ belonging to the disk $D_\varepsilon(\zeta_0)$ of radius $r$ and center $\zeta_0$ and $n \geq n_0$. This shows that $\varphi_n(\zeta) \to \varphi(\zeta)$ uniformly in $D_\varepsilon(\zeta_0)$. Let us now prove $\partial_\xi \varphi_n(\zeta) \to \partial_\xi \varphi(\zeta)$. From the continuity of $\partial_\xi \partial_x S$ in $\Omega \times \Omega \setminus \{\text{diag}\}$ and using the mean value theorem we have that there exists $x_n \in [x_0, x_0 + \frac{1}{n}]$ such that

$$\partial_\xi \varphi_n(\zeta) = \frac{\partial_\xi S(\zeta, z_0 + \frac{1}{n}) - \partial_\xi S(\zeta, z_0)}{\frac{1}{n}}$$

$$= \partial_x \partial_\xi S(\zeta, x_n + iy_0)$$

$$= \partial_\xi \partial_x S(\zeta, x_n + iy_0).$$

As above, given $\varepsilon > 0$ we can find some $r$ such that

$$|\partial_\xi \varphi_n(\zeta) - \partial_\xi \varphi(\zeta)| = |\partial_\xi \partial_x S(\zeta, x_n + iy_0) - \partial_\xi \partial_x S(\zeta, z_0)| < \varepsilon$$

for all $\zeta \in D_r(\zeta_0)$ which implies that $\partial_\xi \varphi_n(\zeta) \to \partial_\xi \varphi(\zeta)$ uniformly in $D_r(\zeta_0)$. The corresponding convergence for the partial derivatives with respect to $\eta$ is proved analogously.

Formulas (47), (48) together with (iii) of the above proposition suggest that a Cauchy kernel of (27) with the coefficient $\alpha = j$ can be constructed from (49) according to the formula

$$Z_1^{-1}(j, \zeta, z) := T_{f(\zeta)}(-Z_1^{-1}(1, \zeta, z)). \quad (51)$$
where the integral operator \( T_{f(\zeta)} \) (defined by (29)) acts with respect to the variable \( \zeta \) along some path \( \Gamma \subset \Omega \) joining \( \zeta_0 \) with \( \zeta \) and not passing through the point \( z \). We prove this fact in the following proposition.

**Proposition 53** Let \( Z_1^{(-1)}(1, \zeta, z) \) be the Cauchy kernel of \( (27) \) given by (49), \( \zeta_0 \in \Omega \) and \( \Omega_0 \) be a simply connected subdomain of \( \Omega \) such that \( \zeta_0 \notin \Omega_0 \). Then

(i) the function (51) is a Cauchy kernel with the coefficient \( j \) for equation (49) in \( \Omega_0 \)

(ii) the scalar and the vector parts of \( Z_1^{(-1)}(j, \zeta, z) \) are solutions of (29) in the variable \( \zeta \)

(iii) the equality holds

\[
Z_1^{(-1)}(j, \zeta, z) = \frac{j}{z - \zeta} + 2j \frac{\partial_x f(z)}{f(z)} \log |z - \zeta| + N(\zeta, z) \tag{52}
\]

where \( N(\zeta, z) \) is a function satisfying condition I in \( \Omega_0 \).

**Proof.** It follows from (iii) of Proposition 52 and from Remark 31 that (51) is a well-defined function when \( \zeta \) belongs to any simply connected subdomain of \( \Omega_0 \) not containing \( z \). Since the function \( M(\zeta, z) \) in (50) satisfies condition I, it is easy to see that

\[
\lim_{\varepsilon \to 0} T_{f(\zeta)}|_{\Gamma_\varepsilon}(-Z_1^{(-1)}(1, \zeta, z)) = 0
\]

where \( \Gamma_\varepsilon \) is the boundary of a disk with the center \( z \) and radius \( \varepsilon \). This implies that (51) is an univalent function in \( \Omega_0 \setminus \{z\} \). Then (ii) of this proposition follows from Theorem 32.

Let us prove (iii). It is clear that the function \( N(\zeta, z) \) in (52) together with its first partial derivatives corresponding to \( \zeta \) are continuous functions in \( \Omega_0 \times \Omega_0 \setminus \{\text{diag}\} \). It remains to study the behavior of \( N(\zeta, z) \) and of its partial derivatives corresponding to \( \zeta \) when \( (\zeta, z) \to (z_0, z_0) \in \Omega_0 \times \Omega_0 \). Consider the scalar part of \( N(\zeta, z) \),

\[
\text{Sc} \ N(\zeta, z) = T_{f(\zeta)}(-\text{Sc} \ Z_1^{(-1)}(1, \zeta, z)) - \frac{y - \eta}{|z - \xi|^2} - \frac{\partial_y f(z)}{f(z)} \log |z - \xi| . \tag{53}
\]

Let \( z_0 \in \Omega_0, \varepsilon_0 > 0 \) such that \( D_{2\varepsilon_0}(z_0) \subset \Omega_0 \) and \( \zeta = \xi + j\eta, z = x + jy \in D_{\varepsilon_0}(z_0) \). Consider the path \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \) where \( \Gamma_1 \) is some rectifiable curve joining \( \zeta_0 \) with \( z + \varepsilon_0 \) and not passing through the point \( z \), and

\[
\Gamma_2(t) = t + jy, \quad t \in [x + \varepsilon_0, x + |\zeta - z|] ,
\]

\[
\Gamma_3(t) = z + |\zeta - z| e^{jt}, \quad t \in [0, \arg(\zeta - z)] .
\]

It is clear that \( \Gamma \) leads from \( \zeta_0 \) to \( \zeta \) and \( \Gamma_2 \cup \Gamma_3 \subset \Omega_0 \). We have

\[
T_{f(\zeta),\Gamma}(-\text{Sc} \ Z_1^{(-1)}(1, \zeta, z)) = \sum_{m=1}^{3} T_{f(\zeta),\Gamma_m}(-\text{Sc} \ Z_1^{(-1)}(1, \zeta, z)), \tag{54}
\]

where

\[
\text{Sc} \ Z_1^{(-1)}(1, \zeta, z) = \frac{x - \xi}{|z - \xi|^2} - \frac{f_x(z)}{f(z)} \log |z - \xi| + \text{Sc} \ M(\zeta, z).
\]

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The function \( T_{f(\zeta),\Gamma_1}(-\text{Sc} Z_1^{(1)}(1, \zeta, z)) \) is continuous (since \( z \notin \Gamma_1 \)) then and as \( \text{Sc} M(\zeta, z) \) satisfies condition I, the function \( T_{f(\zeta),\Gamma_1}\text{Sc} Z_1^{(1)}(\zeta, z) \) is bounded in some neighborhood of \((z_0, \zeta_0)\). The remaining terms of (54) are

\[
T_{f(\zeta),\Gamma_2} \left( \frac{\xi - x}{|z - \xi|^2} + \frac{f_x(z)}{f(z)} \log |z - \xi| \right) = \frac{f_y(z + |\zeta - \xi|)}{f(\zeta)} \log |z - \xi| - \frac{f_y(z + \varepsilon_0)}{f(\zeta)} \log \varepsilon_0
\]

\[
+ \frac{1}{f(\zeta)} \int_{x + \varepsilon_0}^{x + |z - \xi|} \left( \frac{f_x(z)}{f(z)} f_y(t + iy) - f_x(t + iy) \right) \log |t - x| dt,
\]

\[
T_{f(\zeta),\Gamma_3} \left( \frac{\xi - x}{|z - \xi|^2} \right) = \frac{y - \eta}{|z - \xi|^2} - \frac{1}{f(\zeta)} \int_0^{\arg(\xi - z)} f_{\xi}(\Gamma_3(t)) dt
\]

and

\[
T_{f(\zeta),\Gamma_3} \left( \log |z - \xi| \right) = \frac{1}{f(\zeta)} \int_0^{\arg(\xi - z)} \left( f_{\eta}(\Gamma_3(t)) \sin t + f_{\xi}(\Gamma_3(t) \cos t) \right) dt.
\]

Substituting these relations on the right-hand side of (53) we conclude that the function \( \text{Sc} N(\zeta, z) \) is bounded when \( (\zeta, z) \to (z_0, \zeta_0) \). Differentiating (53) with respect to \( \xi \) and using the relation

\[
\partial_\xi T_{f(\zeta),\Gamma_1}(-\text{Sc} Z_1^{(1)}(1, \zeta, z)) = -\frac{f_x(\zeta)}{f(\zeta)} T_{f(\zeta),\Gamma_1}(-\text{Sc} Z_1^{(1)}(1, \zeta, z))
\]

\[
- \frac{f_y(\zeta)}{f(\zeta)} \text{Sc} Z_1^{(1)}(1, \zeta, z) + \partial_\eta \text{Sc} Z_1^{(1)}(1, \zeta, z)
\]

we obtain that \( \partial_\xi \text{Sc} N(\zeta, z) = O(|\log |z - \xi||) \) as \( (\zeta, z) \to (z_0, \zeta_0) \). A similar reasoning can be used to establish that \( \partial_\eta \text{Sc} N(\zeta, z) = O(|\log |z - \xi||) \) as \( (\zeta, z) \to (z_0, \zeta_0) \). This shows that \( \text{Sc} N(\zeta, z) \) satisfies condition I in \( \Omega_0 \). An analogous reasoning is applicable to \( \text{Vec} N(\zeta, z) \) that finishes the proof of part (iii).

Finally, we prove that (51) is a solution of (27) in the variable \( z \). This together with (52) will imply (i). Using the fact that \( Z_1^{(1)}(1, \zeta, z) \) is a solution of (27) and with the help of Theorem 13 we can prove, reasoning as in Proposition 52 that \( \partial_\xi Z_1^{(1)}(1, \zeta, z), \partial_\eta Z_1^{(1)}(1, \zeta, z) \) are also solutions of (27) in \( z \). Then the fact that (51) is a solution of (27) is obtained from the integration with respect to \( \zeta \) of solutions of (27) in the variable \( z \). The proposition is proved.

**Example 54** Let \( f = x \). Consider the corresponding main Vekua equation \( \partial_\xi W = \frac{1}{2x} \overline{W} \) in some domain \( \Omega \) such that \( \overline{\Omega} \) has no common point with the axis \( x = 0 \). The successor equation has the form

\[
\partial_\xi W = -\frac{1}{2x} \overline{W}
\]
and the related Schrödinger equations are $\Delta u = 0$ and $\Delta v = \frac{2}{x^2} v$. A fundamental solution for the Laplace equation satisfying (C1) – (C3) can be chosen as $S(\zeta, z) = \log|z - \zeta|$. A reproducing Cauchy kernel for (52) is obtained by means of the procedure described above and has the following form

$$Z_1^{(-1)}(1, \zeta, z) = \frac{1}{z - \zeta} - \frac{1}{x} \log|z - \zeta|, \quad (56)$$

where $z = x + iy$, $\zeta = \xi + j\eta$ and $\zeta_0 = \xi_0 + j\eta_0 \in \Omega$ is some fixed point. Here the function inside the brackets is a regular solution of (55) for $z$ belonging to a domain not containing $\zeta_0$. Therefore

$$Z_1^{(-1)}(j, \zeta, z) = \frac{j}{z - \zeta} + \frac{y - \eta}{x\xi}(\log|z - \zeta| - 1) + \frac{j}{\xi} \log|z - \zeta| \quad (57)$$

is a Cauchy kernel of (55) as well. Using Theorem 45 it is easy to verify that the expressions (56) and (57) also represent a reproducing Cauchy kernel for (55). The corresponding reproducing Cauchy kernel for the main Vekua equation such that (47) and (48) hold has the form

$$Z^{(-1)}(1, \zeta, z) = \frac{1}{z - \zeta} + \frac{1}{\xi} \log|z - \zeta| - j \frac{y - \eta}{x\xi}(\log|z - \zeta| - 1), \quad (58)$$

$$Z^{(-1)}(j, \zeta, z) = \frac{j}{z - \zeta} - \frac{1}{x} \log|z - \zeta|. \quad (59)$$

### 6.2 Construction of negative formal powers for main Vekua equations

In this subsection we explain how a set of negative formal powers for equation (24) can be constructed from a fundamental solution of (23) satisfying properties (C1) – (C2). In the previous section a reproducing Cauchy kernel for equation (24) was constructed. Then, as was shown in Section 5, using such Cauchy kernel and by means of formulas (44), (45) a set of negative formal powers for (24) can be obtained whenever a generating sequence embedding $(\hat{F}, \hat{G})$ is known, where $(\hat{F}, \hat{G})$ is a generating pair corresponding to the adjoint equation of (24). Notice that $(f, \frac{1}{f})$ is a generating pair for (24) and by Proposition 51 $(\hat{F}, \hat{G})$ is a successor of $(f, \frac{1}{f})$. Hence it is sufficient to know a generating sequence embedding $(f, \frac{1}{f})$. As was shown in [3] (see also [16]) when $f$ has a separable form $f = \phi(x)\psi(y)$ where $\phi$ and $\psi$ are arbitrary twice continuously differentiable functions, there exists a periodic generating sequence with a period two in which $(f, \frac{1}{f})$ is embedded,

$$(F,G) = \left( \phi \psi, \frac{j}{\phi \psi} \right), \quad (F_1,G_1) = \left( \frac{\psi}{\phi}, \frac{j \phi}{\psi} \right), \quad (F_2,G_2) = (F,G), \quad (F_3,G_3) = (F_1,G_1), \ldots$$

(methods for construction of generating sequences in more general situations are discussed in [15], [16] and [18]).
Proposition 53. Then a fundamental solution of (25) is constructed as follows.

\[ Z^{(-2)}(1, \zeta, z) = \frac{1}{(z - \zeta)^2} + \frac{j}{x} \text{Vec} \frac{1}{z - \zeta}, \]

\[ Z^{(-2)}(j, \zeta, z) = \frac{j}{(z - \zeta)^2} - \frac{1}{\xi z - \zeta} + \frac{j}{x} \left( \text{Vec} \frac{1}{z - \zeta} + \frac{1}{\xi} \log |z - \zeta| \right), \]

for an odd \( n \geq 3 \)

\[ Z^{(-n)}(1, \zeta, z) = \frac{1}{(z - \zeta)^n} - \frac{1}{(n - 1)\xi (z - \zeta)^{n-1}} \frac{j}{(n - 1)x} \text{Sc} \left( \frac{j}{(z - \zeta)^{n-1}} - \frac{1}{(n - 2)\xi (z - \zeta)^{n-2}} \right), \]

\[ Z^{(-n)}(j, \zeta, z) = \frac{j}{(z - \zeta)^n} - \frac{j}{(n - 1)x} \text{Sc} \frac{1}{(z - \zeta)^{n-1}}, \]

and for an even \( n \geq 4 \)

\[ Z^{(-n)}(1, \zeta, z) = \frac{1}{(z - \zeta)^n} + \frac{j}{(n - 1)\xi (z - \zeta)^{n-1}} \frac{j}{(n - 1)x} \text{Sc} \left( \frac{j}{(z - \zeta)^{n-1}} - \frac{1}{(n - 2)\xi (z - \zeta)^{n-2}} \right), \]

\[ Z^{(-n)}(j, \zeta, z) = \frac{j}{(z - \zeta)^n} + \frac{j}{(n - 1)\xi (z - \zeta)^{n-1}} \frac{j}{(n - 1)x} \text{Vec} \left( \frac{j}{(z - \zeta)^{n-1}} - \frac{1}{(n - 2)\xi (z - \zeta)^{n-2}} \right). \]

6.3 Construction of fundamental solutions using the Darboux-type transformation

Let \( f \in C^2(\Omega) \) be a nonvanishing particular solution of (23). Consider the operator \( T_f \) defined by (29) that transforms regular solutions of (23) into regular solutions of the Darboux transformed equation (25). We stress that direct application of \( T_f \) to a fundamental solution of (23) does not lead to a fundamental solution of (25). The result rather should be a multivalued solution of (25) with the behaviour similar to that of \( \text{arg}(z - \zeta) \) (the imaginary part of the function \( \ln(z - \zeta) \)). In this subsection we describe a procedure to construct a fundamental solution of (25) from a known fundamental solution of (23).

Let \( S(\zeta, z) \) be a fundamental solution of (23) satisfying the conditions (C1)–(C3). Using formula (49) one can construct \( Z_1^{(-1)}(1, \zeta, z) \) and use it to obtain \( Z_1^{(-1)}(j, \zeta, z) \) by means of (51). Then a fundamental solution of (25) is constructed as follows.

Proposition 56. Let \( Z_1^{(-1)}(j, \zeta, z) \) be the Cauchy kernel given by (51) and characterized by Proposition 55. Then

\[ S_1(z, \zeta) := \text{Vec} \int_{z_0}^{z} Z_1^{(-1)}(j, \zeta, \tau)d_j(f, \hat{\pi}) \tau \]

\[ = \frac{1}{j(z)} \text{Vec} \int_{z_0}^{z} f(\tau) Z_1^{(-1)}(j, \zeta, \tau) d\tau \]

(60)

is a fundamental solution of (23) in \( \Omega_0 \) enjoying properties (C1)–(C3). Here \( z_0 \) is some fixed point different from \( \zeta_0 \) and such that \( z_0 \in \Omega \setminus \overline{\Omega_0} \).
Proof. By construction the integral on the right-hand side of (60) does not depend on the path joining $z_0$ with $z$ (obviously not passing through the point $\zeta$). Substituting into (60) the representation (52) of $Z_1^{(-1)}(j,\zeta,\tau)$ we obtain

$$S_1(z,\zeta) = 1 \frac{f(z)}{f(z_0)} \text{Vec} \int_{z_0}^{z} f(\tau) \left( \frac{j}{\tau - \zeta} + 2 \frac{\partial_\tau f(\tau)}{f(\tau)} \log |\tau - \zeta| + N(\zeta, \tau) \right) d\tau$$

$$= \log |z - \zeta| - \frac{f(z_0)}{f(z)} \log |z_0 - \zeta| + 1 \frac{f(z)}{f(z_0)} \text{Vec} \int_{z_0}^{z} f(\tau) N(\zeta, \tau) d\tau.$$  

From the last equality and using the fact that $N(\zeta, \tau)$ satisfies condition I we conclude that $S_1(z, \zeta)$ is indeed a fundamental solution of (25). The remaining properties follow directly from the properties of $Z_1^{(-1)}(j, \zeta, \tau)$ established in Proposition 53.

Example 57 Let us consider the case described in Example 54. Using the kernel (57) and formula (60) with $z_0 = \zeta + 1$ we obtain the following fundamental solution for the operator $\Delta - \frac{j^2}{\pi^2} I$

$$S_1(z, \zeta) = \log |z - \zeta| + \frac{|z - \zeta|^2}{2x\xi_}\log |z - \zeta| - \frac{|z - \zeta|^2 + 2(y - \eta)^2 - 1}{4x\xi}.$$  

Remark 58 Since the fundamental solution (60) possesses the properties (C1) – (C3) one can apply to it the above procedure and construct fundamental solutions of new Schrödinger equations obtained from (25) by applying further Darboux transformations. The procedure can be repeated a finite number of times. In this way it is possible to obtain fundamental solutions of several Schrödinger equations in a closed form as well as the negative formal powers for the corresponding main Vekua equations.

References

[1] Astala K and Päivärinta L 2006 Calderón’s inverse conductivity problem in the plane. Annals of Mathematics, 163, No. 1, 265-299.

[2] Berglez P 2010 On some classes of bicomplex pseudoanalytic functions. In Progress in Analysis and its Applications, M. Ruzhansky and J. Wirth eds., World Scientific ISBN-13 978-981-4313-16-2, pp. 81-88.

[3] Bers L 1952 Theory of pseudo-analytic functions. New York University.

[4] Bers L 1956 Formal powers and power series. Communications on Pure and Applied Mathematics 9, 693–711.

[5] Campos H M, Castillo R, Kravchenko V V Construction and application of Bergman-type reproducing kernels for boundary and eigenvalue problems in the plane. Complex Variables and Elliptic Equations, Published on-line.
[6] Campos H M, Kravchenko V V, Mendez L M Complete families of solutions for the Dirac equation: an application of bicomplex pseudoanalytic function theory and transmutation operators. Advances in Applied Clifford Algebras, to appear.

[7] Campos H M, Kravchenko V V, Torba S M 2012 Transmutations, L-bases and complete families of solutions of the stationary Schrödinger equation in the plane. Journal of Mathematical Analysis and Applications, v.389, issue 2, 1222–1238.

[8] Castaño A and Kravchenko V V 2005 New applications of pseudoanalytic function theory to the Dirac equation. J. of Physics A: Mathematical and General, v. 38, 9207-9219.

[9] Castillo R, Kravchenko V V and Reséndiz R 2011 Solution of boundary value and eigenvalue problems for second order elliptic operators in the plane using pseudoanalytic formal powers. Mathematical Methods in the Applied Sciences, v. 34, issue 4, 455-468.

[10] Fischer Ya 2011 Approximation dans des classes de fonctions analytiques generalisees et resolution de problemes inverses pour les tokamaks. These de doctorat presentee pour obtenir le grade de docteur de l’Universite Nice-Sophia Antipolis Specialite : Mathematiques Appliquees, 275 pp.

[11] Garuchava Sh. 2011 On the Darboux transformation for Carleman-Bers-Vekua system. In: Recent Developments in Generalized Analytic Functions and Their Applications. Proceedings of the International Conference on Generalized Analytic Functions and Their Applications Tbilisi, Georgia, 12 – 14 September 2011 Edited by G.Giorgadze, ISBN 978-9941-0-3687-3, Tbilisi State University, 51-55.

[12] Kravchenko V V 2005 On the relationship between p-analytic functions and the Schrödinger equation. Zeitschrift für Analysis und ihre Anwendungen, 24, No. 3, 487-496.

[13] Kravchenko V V 2005 On a relation of pseudoanalytic function theory to the two-dimensional stationary Schrödinger equation and Taylor series in formal powers for its solutions. Journal of Physics A: Mathematical and General, v. 38, No. 18, 3947-3964.

[14] Kravchenko V V 2006 On a factorization of second order elliptic operators and applications. Journal of Physics A: Mathematical and General, v. 39, 12407-12425.

[15] Kravchenko V V 2008 Recent developments in applied pseudoanalytic function theory. Beijing: Science Press “Some topics on value distribution and differentiability in complex and p-adic analysis”, eds. A. Escassut, W. Tutschke and C. C. Yang, 267-300.

[16] Kravchenko V V 2009 Applied pseudoanalytic function theory. Basel: Birkhäuser, Series: Frontiers in Mathematics

[17] Kravchenko V V and Shapiro M V 1996 Integral representations for spatial models of mathematical physics. Harlow: Addison Wesley Longman Ltd., Pitman Res. Notes in Math. Series, v. 351.

[18] Kravchenko V V, Tremblay S 2010 Explicit solutions of generalized Cauchy-Riemann systems using the transplant operator. Journal of Mathematical Analysis and Applications, v. 370, issue 1, 242-257.
[19] Matveev V and Salle M 1991 Darboux transformations and solitons. N.Y. Springer.

[20] Polozhy G N 1965 Generalization of the theory of analytic functions of complex variables: $p$-analytic and $(p,q)$-analytic functions and some applications. Kiev University Publishers (in Russian).

[21] Ramirez M P, Gutierrez A, Sanchez V D, Rodriguez O 2010 Study of the General Solution for the Two-Dimensional Electrical Impedance Equation. In: Lecture Notes in Electrical Engineering, Electronic Engineering and Computing Technology, S.-I. Ao and L. Gelman (eds.), Springer, v. 60, 563-574.

[22] Rochon D 2008 On a relation of bicomplex pseudoanalytic function theory to the complexified stationary Schrödinger equation. Complex Variables and Elliptic Equations, v. 53, No. 6, 501-521.

[23] Rochon D and Shapiro M 2004 On algebraic properties of bicomplex and hyperbolic numbers. An. Univ. Oradea Fasc. Mat. 11, 71–110.

[24] Rochon D and Tremblay S 2004 Bicomplex quantum mechanics: I. The generalized Schrödinger equation. Advances in Applied Clifford Algebras 14, No. 2, 231-248.

[25] Vekua I N 1959 Generalized analytic functions. Moscow: Nauka (in Russian); English translation Oxford: Pergamon Press 1962.

[26] Youvaraj G P, Jain R K 1990 On pseudo-analytic matrix functions. Complex Variables, Theory and Application, v. 15, issue 4, 259-278.

[27] Zabarankin M and Krokhmal P 2007 Generalized Analytic Functions in 3D Stokes Flows. The Quarterly Journal of Mechanics and Applied Mathematics, v. 60, no. 2, 99–123.

[28] Zabarankin M and Ulitko A F 2006 Hilbert formulas for $r$-analytic functions in the domain exterior to spindle. SIAM Journal of Applied Mathematics 66, No. 4, 1270-1300.