GLOBAL EXISTENCE AND DECAY RATES FOR A GENERIC COMPRESSIBLE TWO–FLUID MODEL

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ABSTRACT. We investigate global existence and optimal decay rates of a generic non-conservative compressible two–fluid model with general constant viscosities and capillary coefficients. Bresch, et al. in the seminal work (Arch Rational Mech Anal 196:599–629, 2010) considered the compressible two–fluid model with a special type of density–dependent viscosities ($\mu^\pm (\rho^\pm) = \mu^\pm \rho^\pm$, $\lambda^\pm = 0$). However, as indicated by themselves, their methods cannot deal with the case of constant viscosity coefficients. Besides, Cui, et al. (SIAM J Math Anal 48:470–512, 2016) studied the same model with a more special type of density-dependent viscosity ($\mu^\pm (\rho^\pm) = \nu \rho^\pm$, $\lambda^\pm = 0$) and equal capillary coefficients ($\sigma^+ = \sigma^- = \sigma$). Since their analysis relies heavily this special choice for viscosities and capillary coefficients, the case of general constant viscosities and capillary coefficients cannot be handled in their settings. The main novelty of this work is three–fold: First, for any integer $\ell \geq 3$, we show that the densities and velocities converge to their corresponding equilibrium states at the $L^2$ rate $(1 + t)^{-\frac{3}{4}}$, and the $k$–($k \in [1, \ell]$)–order spatial derivatives of them converge to zero at the $L^2$ rate $(1 + t)^{-\frac{3}{4} - \frac{k}{2}}$, which are the same as ones of the compressible Navier–Stokes system, Navier–Stokes–Korteweg system and heat equation. Second, the linear combination of the fraction densities $(\beta^+ \alpha^+ \rho^+ + \beta^- \alpha^- \rho^-)$ converges to its corresponding equilibrium state at the $L^2$ rate $(1 + t)^{-\frac{3}{4}}$, and its $k$–($k \in [1, \ell]$)–order spatial derivative converges to zero at the $L^2$ rate $(1 + t)^{-\frac{3}{4} - \frac{k}{2}}$, but the fraction densities $(\alpha^\pm \rho^\pm)$ themselves converge to their corresponding equilibrium states at the $L^2$ rate $(1 + t)^{-\frac{3}{4}}$, and the $k$–($k \in [1, \ell]$)–order spatial derivatives of them converge to zero at the $L^2$ rate $(1 + t)^{-\frac{3}{4} - \frac{k}{2}}$, which are slower than ones of their linear combination $(\beta^+ \alpha^+ \rho^+ + \beta^- \alpha^- \rho^-)$ and the densities. We think that this phenomenon should owe to the special structure of the system. Finally, for well–chosen initial data, we also prove the lower bounds on the decay rates, which are the same as those of the upper decay rates. Therefore, these decay rates are optimal for the compressible two–fluid model.

1. Introduction.

1.1. Background and motivation. As is well–known, multi–fluid flows are very common in nature. Such a terminology includes the flows of non–miscible fluids such as air and water; gas, oil and water. For the flows of miscible fluids, they usually form a “new” single fluid possessing its own rheological properties. One interesting example is the stable emulsion between oil and water which is a non–Newtonian fluid, but oil and water are Newtonian ones.

One of the classic examples of multi–fluid flows is small amplitude waves propagating at the interface between air and water, which is called a separated flow. In view of modeling, each fluid obeys its own equation and couples with each other through the free surface in this case. Here, the motion of the fluid is governed by the pair of compressible Euler equations with free surface:

$$\partial_t \rho_i + \nabla \cdot (\rho_i v_i) = 0, \quad i = 1, 2,$$

(1.1)
\[ \partial_t (\rho_i v_i) + \nabla \cdot (\rho_i v_i \otimes v_i) + \nabla p_i = -g \rho_i e_3 \pm F_D. \quad (1.2) \]

In above equations, \( \rho_i \) and \( v_i \) represent the density and velocity of the upper fluid (air), and \( \rho_2 \) and \( v_2 \) denote the density and velocity of the lower fluid (water). \( p_i \) denotes the pressure. \(-g \rho_i e_3 \) is the gravitational force with the constant \( g > 0 \) the acceleration of gravity and \( e_3 \) the vertical unit vector, and \( F_D \) is the drag force. As mentioned before, the two fluids (air and water) are separated by the unknown free surface \( z = \eta(x,y,t) \), which is advected with the fluids according to the kinematic relation:

\[ \partial_t \eta = u_{1,z} - u_{1,x} \partial_x \eta - u_{1,y} \partial_y \eta \quad (1.3) \]
on two sides of the surface \( z = \eta \) and the pressure is continuous across this surface.

When the wave’s amplitude becomes large enough, wave breaking may happen. Then, in the region around the interface between air and water, small droplets of liquid appear in the gas, and bubbles of gas also appear in the liquid. These inclusions might be quite small. Due to the appearances of collapse and fragmentation, the topologies of the free surface become quite complicated and a wide range of length scales are involved. Therefore, we encounter the situation where two–fluid models become relevant if not inevitable. The classic approach to simplify the complexity of multi–phase flows and satisfy the engineer’s need of some modeling tools is the well–known volume–averaging method (see [15, 22] for details). Thus, by performing such a procedure, one can derive a model without surface: a two–fluid model. More precisely, we denote \( \alpha^\pm \) by the volume fraction of the liquid (water) and gas (air), respectively. Therefore, \( \alpha^+ + \alpha^- = 1 \). Applying the volume–averaging procedure to the equations (1.1) and (1.2) leads to the following generic compressible two–fluid model:

\[
\begin{cases}
\partial_t (\alpha^\pm \rho^\pm) + \text{div}(\alpha^\pm \rho^\pm u^\pm) = 0, \\
\partial_t (\alpha^\pm \rho^\pm u^\pm) + \text{div}(\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P = -g \alpha^\pm \rho^\pm e_3 \pm F_D, 
\end{cases} \quad (1.4)
\]

where the two fluids are assumed to share the common pressure \( P \).

We have already discussed the case of water waves, where a separated flow can lead to a two–fluid model from the viewpoint of practical modeling. As stated before, two–fluid flows are very common in nature, but also in various industry applications such as nuclear power, chemical processing, oil and gas manufacturing. In terms of the context, the models used for simulation may be very different. However, averaged models share the same structure as (1.4). By introducing viscosity and capillary effects, one can generalize the above system (1.4) to

\[
\begin{cases}
\partial_t (\alpha^\pm \rho^\pm) + \text{div}(\beta^\pm \rho^\pm u^\pm) = 0, \\
\partial_t (\alpha^\pm \rho^\pm u^\pm) + \text{div}(\beta^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P = \text{div}(\alpha^\pm \tau^\pm) + \sigma^\pm \alpha^\pm \rho^\pm \nabla \Delta(\alpha^\pm \rho^\pm), \\
P = p^\pm(\rho^\pm) = A^\pm(\rho^\pm)^\gamma, \quad (1.5)
\end{cases}
\]

where \( \rho^\pm(x,t) \geq 0, u^\pm(x,t) \) and \( p^\pm(\rho^\pm) = A^\pm(\rho^\pm)^\gamma \) denote the densities, velocities of each phase, and the two pressure functions, respectively. \( \gamma \geq 1, A^\pm > 0 \) are positive constants. In what follows, we set \( A^+ = A^- = 1 \) without loss of any generality. Moreover, \( \tau^\pm \) are the viscous stress tensors

\[ \tau^\pm := \mu^\pm \left( \nabla u^\pm + \nabla u^\pm^T \right) + \lambda^\pm \text{div} u^\pm \text{Id}, \quad (1.6) \]

where the constants \( \mu^\pm \) and \( \lambda^\pm \) are shear and bulk viscosity coefficients satisfying the physical condition: \( \mu^\pm > 0 \) and \( 2 \mu^\pm + 3 \lambda \pm \geq 0 \), which implies that \( \mu^\pm + \lambda^\pm > 0 \). For more information about this model, we refer to [1, 3–5, 9–13, 23, 25, 27–29] and references therein. However, it is well–known that as far as mathematical analysis of two–fluid model is concerned, there are many technical challenges. Some of them involve, for example:
• The corresponding linear system of the model has multiple eigenvalue, which makes mathematical analysis (well–posedness and stability) of the model become quite difficult and complicated;

• Transition to single–phase regions, i.e, regions where the mass \( \alpha^+ \rho^+ \) or \( \alpha^- \rho^- \) becomes zero, may occur when the volume fractions \( \alpha^\pm \) or the densities \( \rho^\pm \) become zero;

• The system is non–conservative, since the non–conservative terms \( \alpha^\pm \nabla \rho^\pm \) are involved in the momentum equations. This brings various mathematical difficulties for us to employ methods used for single phase models to the two–fluid model.

Bresch et al. in the seminal work [4] considered the generic two-fluid model (1.5) with the following special density-dependent viscosities:

\[
\mu^\pm(\rho^\pm) = \mu^\pm \rho^\pm, \lambda^\pm(\rho^\pm) = 0. \tag{1.7}
\]

They obtained the global weak solutions in periodic domain with \( 1 < \gamma^\pm < 6 \). However, as indicated by themselves, their methods rely heavily on the above special density-dependent viscosities, and particularly cannot handle the case of constant viscosity coefficients as in (1.6). Later, Bresch–Huang–Li [5] established the global existence of weak solutions in one space dimension without capillary effects (i.e., \( \sigma^\pm = 0 \)) when \( \gamma^\pm > 1 \) by taking advantage of the one space dimension. Recently, Cui–Wang–Yao–Zhu [7] obtained the time–decay rates of classical solutions for model (1.5) with the following special density-dependent viscosities with equal viscosity coefficients, and equal capillary coefficients:

\[
\mu^\pm(\rho^\pm) = \nu \rho^\pm, \lambda^\pm(\rho^\pm) = 0, \sigma^+ = \sigma^- = \sigma. \tag{1.8}
\]

Based on the above special choice for viscosities and capillary coefficients, they can take a linear combination of model (1.5) to reformulate it into two 4 × 4 systems whose linear parts are decoupled with each other and possess the same dissipation structure as that of the compressible Navier–Stokes-Korteweg system, and then employ the similar arguments as in [2, 26] to prove their main results. However, since this reformulation played a crucial role in their analysis, the case of constant viscosities, even if the equal constant viscosities (i.e., \( \mu^\pm(\rho^\pm) = \nu, \lambda^\pm(\rho^\pm) = \lambda \)), cannot be handled in their settings.

In conclusion, all the works [4, 7] depend essentially on the special density-dependent viscosities. Therefore, a natural and important problem is that what will happen when we consider the general constant viscosities as in (1.6). That is to say, what about the global well–posedness and large time behavior of Cauchy problem to the two–fluid model (1.5)–(1.6) in high dimensions. However, to our best knowledge, so far there is no result on mathematical theory of the two–fluid model (1.5)–(1.6) in high dimensions.

The main purpose of this work is to establish global well–posedness and large time behavior of classical solution to the two–fluid model (1.5)–(1.6). More precisely, for any integer \( \ell \geq 3 \), we show that the densities and velocities of model (1.5)–(1.6) converge to their corresponding equilibrium states at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2}} \), and the \( k(\in [1, \ell]) \) order spatial derivatives of them converge to zero at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2} - \frac{1}{2}} \), which are the same as ones of the compressible Navier–Stokes system [8, 14], Navier–Stokes–Korteweg system [2, 26] and heat equation. Moreover, the linear combination of the fraction densities \( \beta^+ \alpha^+ \rho^+ + \beta^- \alpha^- \rho^- \) converges to its corresponding equilibrium state at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2}} \), and its \( k(\in [1, \ell]) \) order spatial derivative converges to zero at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2} - \frac{1}{2}} \), but the fraction densities \( \alpha^\pm \rho^\pm \) themselves converge to their corresponding equilibrium states at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2}} \), and the \( k(\in [1, \ell]) \) order spatial derivatives of them converge to zero at the \( L^2 \) rate \( (1 + t)^{-\frac{k}{2} - \frac{1}{2}} \), which are slower than ones of their linear combination \( \beta^+ \alpha^+ \rho^+ + \beta^- \alpha^- \rho^- \) and the densities. We think that this phenomenon should owe to the special structure of the system. Finally, for well–chosen initial data,
we also prove the lower bounds on the decay rates, which are the same as those of the upper decay rates. Therefore, these decay rates are optimal for the compressible two–fluid model.

1.2. New formulation of system (1.5) and Main Results. In this subsection, we devote ourselves to reformulating the system (1.5) and stating the main results. The relations between the pressures of (1.5) implies

\[ dP = s_+^2 dp^+ = s_-^2 dp^-, \quad \text{where} \quad s_{\pm} := \sqrt{\frac{dP}{d\rho^\pm}(\rho^\pm)}. \]  

(1.9)

Here \( s_{\pm} \) represent the sound speed of each phase respectively. As in [4], we introduce the fraction densities

\[ R^\pm = \alpha^\pm \rho^\pm, \]  

(1.10)

which together with the relation \( \alpha^+ + \alpha^- = 1 \) leads to

\[ dp^+ = \frac{1}{\alpha^+}(dR^+ - \rho^+ d\alpha^+), \quad dp^- = \frac{1}{\alpha^-}(dR^- + \rho^- d\alpha^+). \]  

(1.11)

From (1.9)–(1.10), we finally get

\[ d\alpha^+ = \frac{\alpha^+ s_+^2}{\alpha^+ - \rho^+ s_+^2 + \alpha^- \rho^- s_-^2} dR^+ - \frac{\alpha^- s_-^2}{\alpha^- - \rho^+ s_+^2 + \alpha^- \rho^- s_-^2} dR^-. \]

Substituting the above equality into (1.11) yields

\[ dp^+ = \frac{s_+^2}{\alpha^+ - \rho^+ s_+^2 + \alpha^- \rho^- s_-^2} (\rho^- dR^+ + \rho^+ dR^-), \]

and

\[ dp^- = \frac{s_-^2}{\alpha^- - \rho^+ s_+^2 + \alpha^- \rho^- s_-^2} (\rho^- dR^+ + \rho^+ dR^-), \]

which combined with (1.9) imply for the pressure differential \( dP \)

\[ dP = \mathcal{E}^2 (\rho^- dR^+ + \rho^+ dR^-), \]  

(1.12)

where

\[ \mathcal{E}^2 := \frac{s_+^2 s_-^2}{\alpha^+ - \rho^+ s_+^2 + \alpha^- \rho^- s_-^2}, \quad \text{and} \quad s_{\pm}^2 = \frac{dP(\rho^\pm)}{d\rho^\pm} = \tilde{\gamma} \frac{P(\rho^\pm)}{\rho^\pm}. \]

Next, by using the relation \( \alpha^+ + \alpha^- = 1 \) again, we can get

\[ \frac{R^+}{\rho^+} + \frac{R^-}{\rho^-} = 1, \quad \text{and therefore} \quad \rho^+ = \frac{R^- \rho^+}{\rho^+ - R^+}. \]  

(1.13)

By virtue of (1.5), we have

\[ \varphi(\rho^+, R^+, R^-) := P(\rho^+) - P\left(\frac{R^- \rho^+}{\rho^+ - R^+}\right) = 0. \]

Consequently, for any given two positive constants \( \bar{R}^+, \bar{R}^- \), there exists \( \bar{\rho}^+ > \bar{R}^+ \) such that

\[ \varphi(\bar{\rho}^+, \bar{R}^+, \bar{R}^-) = 0. \]

Differentiating \( \varphi \) with respect to \( \rho^+ \), we get

\[ \frac{\partial \varphi}{\partial \rho^+}(\rho^+, R^+, R^-) = \tilde{s}_+^2 + \tilde{s}_-^2 \frac{R^- R^+}{(\rho^+ - R^+)^2}, \]

which implies

\[ \frac{\partial \varphi}{\partial \rho^+}(\bar{\rho}^+, \bar{R}^+, \bar{R}^-) > 0. \]
Thus, this together with Implicit Function Theorem and (1.10), (1.13) implies that the unknowns \( \rho^\pm, \alpha^\pm \) and \( \mathcal{C} \) can be given by

\[
\rho^\pm = \rho^\pm(R^+, R^-), \quad \alpha^\pm = \alpha^\pm(R^+, R^-), \quad \text{and therefore} \quad \mathcal{C} = \mathcal{C}(R^+, R^-).
\]

We refer to [4], pp. 614 for the details.

Therefore, we can rewrite system (1.5) into the following equivalent form:

\[
\begin{dcases}
\partial_t R^+ + \text{div}(R^+ u^+) = 0, \\
\partial_t (R^+ u^+) + \text{div}(R^+ u^+ u^+) + \alpha^+ \mathcal{C}^2 [\rho^- \nabla R^+ + \rho^+ \nabla R^-]
\end{dcases}
\]

where \( \mathcal{C} = \mathcal{C}(R^+, R^-) \).

In the present paper, we consider the Cauchy problem of (1.14) subject to the initial condition

\[
(R^+, u^+, R^-, u^-)(x, 0) = (R_0^+, u_0^+, R_0^-, u_0^-) \quad \text{as} \quad |x| \to \infty \in \mathbb{R}^3
\]

where two positive constants \( \tilde{R}^+ \) and \( \tilde{R}^- \) denote the background doping profile, and in the present paper are taken as 1 for simplicity.

Before stating our main result, let us first introduce the notations and conventions used throughout this paper. We use \( H^k(\mathbb{R}^3) \) to denote the usual Sobolev space with norm \( \| \cdot \|_{H^k} \) and \( L^p, 1 \leq p \leq \infty \) to denote the usual \( L^p(\mathbb{R}^3) \) space with norm \( \| \cdot \|_{L^p} \). For the sake of conciseness, we do not precise in functional space names when they are concerned with scalar-valued or vector-valued functions, \( \| f \|_X \) denotes \( \| f \|_X + \| g \|_X \). We will employ the notation \( a \leq b \) to mean that \( a \leq Cb \) for a universal constant \( C > 0 \) that only depends on the parameters coming from the problem. We denote \( \nabla = \partial_x = (\partial_1, \partial_2, \partial_3) \), where \( \partial_i = \partial_{x_i}, \forall i = 1 \) and put \( \partial_1 f = \nabla f = \nabla(\nabla^{-1} f) \). Let \( \Lambda^s \) be the pseudo differential operator defined by

\[
\Lambda^s f = \hat{\phi}^{-1}(|\xi|^s \hat{f}) \quad \text{for} \quad s \in \mathbb{R},
\]

where \( \hat{f} \) and \( \hat{\phi}(f) \) are the Fourier transform of \( f \). The homogenous Sobolev space \( H^s(\mathbb{R}^3) \) with norm given by \( \| f \|_{H^s} \overset{\Delta}{=} \| \Lambda^s f \|_{L^2} \). For a radial function \( \phi \in C_0^\infty(\mathbb{R}^3_+) \) such that \( \phi(\xi) = 1 \) when \( |\xi| \leq \frac{\eta}{2} \) and \( \phi(\xi) = 0 \) when \( |\xi| \geq \eta \), where \( \eta \) is defined in Lemma 2.1 we define the low–frequency part of \( f \) by

\[
f^l = \hat{\phi}^{-1}[\phi(\xi)\hat{f}]
\]

and the high–frequency part of \( f \) by

\[
f^h = \hat{\phi}^{-1}[(1 - \phi(\xi))\hat{f}].
\]

It is direct to check that \( f = f^l + f^h \) if Fourier transform of \( f \) exists.

Now, we are in a position to state our main result.

**Theorem 1.1.** Assume that \( R_0^+ - 1, R_0^- - 1 \in H^{\ell+1}(\mathbb{R}^3) \) and \( u_0^+, u_0^- \in H^\ell(\mathbb{R}^3) \) for an integer \( \ell \geq 3 \). Then there exists a constant \( \delta_0 \) such that if

\[
K_0 := \| (R_0^+ - 1, R_0^- - 1, u_0^+, u_0^-) \|_{H^{\ell+1} \cap L^1} + \| (u_0^+, u_0^-) \|_{H^\ell \cap L^1} \leq \delta_0,
\]

then the Cauchy problem (1.14)–(1.15) admits a unique solution \( (R^+, u^+, R^-, u^-) \) globally in time in the sense that

\[
R^+ - 1, R^- - 1 \in C^{0}([0, \infty); H^{\ell+1}(\mathbb{R}^3)) \cap C^{1}([0, \infty); H^{\ell}(\mathbb{R}^3)), \quad u^+, u^- \in C^{0}([0, \infty); H^{\ell}(\mathbb{R}^3)) \cap C^{1}([0, \infty); H^{\ell-2}(\mathbb{R}^3)).
\]
and satisfies
\[
\|(R^+ - 1, R^- - 1)(t)\|^2_{H^{k+1}} + \|(u^+, u^-)(t)\|^2_{H^k} \\
+ \|\beta^+(R^+ - 1) + \beta^-(R^- - 1)\|^2_{H^k} + \int_0^t \|(\nabla(R^+ - 1, R^- - 1)(\tau))\|^2_{H^k} + \|(u^+, u^-)(\tau)\|^2_{H^k} \, d\tau \leq CK_0^2.
\] (1.17)
Moreover, the following convergence rates hold true.

- **Upper bounds.** For any \( t \geq 0 \), and \( 0 \leq k \leq \ell \), it holds that
  \[
  \left\| \nabla^k (\rho^+ - \hat{\rho}^+, \rho^- - \hat{\rho}^-)(t) \right\|_{H^{k-\ell}} \leq C_K(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.18)
  \[
  \left\| \nabla^k (u^+, u^-)(t) \right\|_{H^{k-\ell}} \leq C_K(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.19)
  \[
  \left\| \nabla^k [\beta^+(R^+ - 1) + \beta^-(R^- - 1)](t) \right\|_{H^{k-\ell}} \leq C_K(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.20)
  \[
  \left\| \nabla^k (R^+ - 1, R^- - 1, \ldots)(t) \right\|_{H^{k+1}} \leq C_K(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.21)
where \( \hat{\rho}^+ = \rho^+(1, 1) \) denote equilibrium states of \( \rho^+ \) respectively, and \( \beta^\pm = \sqrt{\frac{R^+}{\rho^+}} \).

- **Lower bounds.** Let \((\bar{n}^0_\ell, u^0_\ell, R^0_\ell, u^0_\ell) = (R^0_\ell - 1, u^0_\ell, R^0_\ell - 1, u^0_\ell)\) and assume that Fourier transform of functions \((\bar{n}^0_\ell, u^0_\ell, \bar{n}^0_\ell, u^0_\ell)\) satisfy
  \[
  \bar{n}^0_\ell(\xi) = 0, \quad \Lambda^{-1} \text{div } \bar{n}^0_\ell(\xi) = \bar{\rho}^0_\ell(\xi) = 0, \quad \Lambda^{-1} \text{div } \bar{u}^0_\ell(\xi) - K^0_\ell \sim |\xi|^s,
  \] (1.22)
for any \(|\xi| \leq \eta\), where \( \theta < 2 \) and \( s > 0 \) are two given constants. Then there is a positive constant \( C_0 \) independent of \( t \) such that for any large enough \( t \) and \( 0 \leq k \leq \ell \), it holds that
  \[
  \min \left\{ \left\| \nabla^k (\rho^+ - \hat{\rho}^+)(t) \right\|_{H^{k-\ell}}, \left\| \nabla^k (\rho^- - \hat{\rho}^-)(t) \right\|_{H^{k-\ell}} \right\} \geq C_0(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.23)
  \[
  \min \left\{ \left\| \nabla^k u^+(t) \right\|_{H^{k-\ell}}, \left\| \nabla^k u^-(t) \right\|_{H^{k-\ell}} \right\} \geq C_0(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.24)
  \[
  \min \left\{ \left\| \nabla^k [\beta^+(R^+ - 1) + \beta^-(R^- - 1)](t) \right\|_{H^{k-\ell}} \right\} \geq C_0(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.25)
  \[
  \min \left\{ \left\| \nabla^k (R^+ - 1, R^- - 1, \ldots)(t) \right\|_{H^{k+1}} \right\} \geq C_0(1 + t)^{-\frac{s}{2} - \frac{\ell}{2}},
  \] (1.26)

**Remark 1.2.** Compared to Cui–Wang–Yao–Zhu [7], where the model (1.5) with (1.8) was considered, the main new contribution of Theorem 1.1 is four-fold: First, as mentioned before, (1.8) played an essential role in their analysis. Therefore, we need to develop new thoughts to overcome the difficulties arising from the general constant viscosities as in (1.6), which will be explained later. Second, (1.13) and (1.20) give the optimal decay rates on the densities and linear combination of the fraction densities: \( \beta^+(R^+ - 1) + \beta^-(R^- - 1) \), which are totally new as compared to Cui–Wang–Yao–Zhu [7]. Third, noticing that in Cui–Wang–Yao–Zhu [7], the \((\ell - 1)\)–th and \( \ell \)–th spatial derivatives of the velocities decay at the same \( L^2 \) rate \( (1 + t)^{-\frac{s}{2} - \frac{\ell}{2}} \), and the \( \ell \)–th and \((\ell + 1)\)–th spatial derivatives of the fraction densities decay at the same \( L^2 \) rate \( (1 + t)^{-\frac{s}{2} - \frac{\ell}{2}} \), which are slower than ones in (1.19) and (1.21). Finally, for well–chosen initial data, (1.23)–(1.26) show the lower bounds on the decay rates, which are the same as those of the upper decay rates, and completely new as compared to Cui–Wang–Yao–Zhu [7]. Therefore, our decay rates are optimal in this sense.
Now, let us sketch the strategy of proving Theorem 1.1 and explain some of the main difficulties and techniques involved in the process. Different from Cui–Wang–Yao–Zhu [7] where the model (1.5) with (1.6) was considered, we need to develop new ideas to tackle with the difficulties from general constant viscosities and capillary coefficients. To see this, by taking \( n^\pm = R^\pm - 1 \), one can write the corresponding linear system of model (1.5) in terms of the variables \((n^+, u^+, n^-, u^-)\):

\[
\begin{cases}
\partial_t n^+ + \text{div } u^+ = 0, \\
\partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - v_1^+ \Delta u^+ - v_2^+ \nabla \text{div } u^+ - \sigma^+ \nabla \Delta n^+ = 0, \\
\partial_t n^- + \text{div } u^- = 0, \\
\partial_t u^- + \beta_1 \nabla n^+ + \beta_4 \nabla n^- - v_1^- \Delta u^- - v_2^- \nabla \text{div } u^- - \sigma^- \nabla \Delta n^- = 0,
\end{cases}
\]

where \( v_1^\pm = \frac{\mu^\pm}{\rho^\pm}, \ v_2^\pm = \frac{\mu^\pm + \lambda^\pm}{\rho^\pm} > 0, \ \beta_1 = \frac{\nu^2(1,1) \rho^-}{\rho^+}, \ \beta_2 = \beta_3 = \nu^2(1,1), \ \beta_4 = \frac{\nu^2(1,1) \rho^-}{\rho^+} \). In view of (1.8), the system (1.27) can be reduced to

\[
\begin{cases}
\partial_t n^+ + \text{div } u^+ = 0, \\
\partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - \nu \Delta u^+ - \nu \nabla \text{div } u^+ - \sigma \nabla \Delta n^+ = 0, \\
\partial_t n^- + \text{div } u^- = 0, \\
\partial_t u^- + \beta_3 \nabla n^+ + \beta_4 \nabla n^- - \nu \Delta u^- - \nu \nabla \text{div } u^- - \sigma \nabla \Delta n^- = 0.
\end{cases}
\]  

Based on the above special linear system, the main observation of Cui–Wang–Yao–Zhu [7] is to introduce four linear combinations:

\[
N^+ := \beta_3 n^+ + \beta_4 n^-, \quad N^- := \beta_3 n^+ - \beta_4 n^-,
\]

\[
U^+ := \beta_3 u^+ + \beta_4 u^-, \quad U^- := \beta_3 u^+ - \beta_4 u^-.
\]

Then, the system (1.28) can be divided into two new linear system:

\[
\begin{cases}
\partial_t N^+ + \text{div } U^+ = 0, \\
\partial_t U^+ + (\beta_1 + \beta_4) \nabla N^+ - \nu \Delta U^+ - \nu \nabla \text{div } U^+ - \sigma \nabla \Delta N^+ = 0,
\end{cases}
\]  

and

\[
\begin{cases}
\partial_t N^- + \text{div } U^- = 0, \\
\partial_t U^- + (\beta_1 - \beta_4) \nabla N^- - \nu \Delta U^- - \nu \nabla \text{div } U^- - \sigma \nabla \Delta N^- = 0
\end{cases}
\]

It is worth mentioning that (1.29) are decoupled from \((N^-, U^-)\) and possesses the same dissipative structure as that of the compressible Navier–Stokes–Korteweg system [2, 26], while (1.30) are coupled with \(\nabla N^+\) and possesses the similar dissipative structure as that of the compressible Navier–Stokes–Korteweg system [2, 26]. Thus, by making full use of these good properties of (1.29) and (1.30), Cui–Wang–Yao–Zhu [7] can modify the methods of [2, 26] to prove their main results. However, since their analysis relies heavily on this reformulation, the case of general constant viscosities as in (1.6), even if the equal constant viscosities, cannot be handled in their settings. Indeed, for the case of the equal constant viscosities (i.e. \( \mu^\pm(\rho^\pm) = \nu, \ \lambda^\pm(\rho^\pm) = \lambda \)), we have

\[
v_1^\pm = \frac{\nu}{\rho^\pm}, \text{ and } v_2^\pm = \frac{\nu + \lambda}{\rho^\pm}.
\]

This particularly implies that the system (1.27) cannot be reduced to system (1.28), since \( v_1^+ \neq v_1^- \) and \( v_2^+ \neq v_2^- \) when \( \rho^+ \neq \rho^- \). The key idea here is that, instead of using this reformulation, we will work on the model (1.5) with (1.6) directly, which makes the problem become quite difficult and complicated. In what follows, we will give a brief interpretation for the main idea of the proof.
To begin with, we give a heuristic description of our strategy. Multiplying \((1.27)_1, (1.27)_2, (1.27)_3\) and \((1.27)_4\) by \(\frac{\beta_1}{\beta_2} n^+, \frac{1}{\beta_2} u^+, \frac{\beta_1}{\beta_3} u^-\) and \(\frac{1}{\beta_3} u^-\), one can easily get the nature energy equation of the linear system \((1.27)\):

\[
\partial_t \mathcal{E}_0(t) + \mathcal{D}_0(t) := \partial_t \int_{\mathbb{R}^3} \left( \frac{\beta_1}{\beta_2} |n^+|^2 + \frac{\beta_1}{\beta_3} |n^-|^2 + \frac{\sigma_-}{\beta_2} |\nabla n^+|^2 + \frac{\sigma_-}{\beta_3} |\nabla n^-|^2 + \frac{1}{\beta_2} |u^+|^2 + \frac{1}{\beta_3} |u^-|^2 \right) \, dx \\
+ \int_{\mathbb{R}^3} \left( \frac{1}{\beta_2} (v_1^+ |\nabla u^+|^2 + v_2^+ |\text{div} u^+|^2) + \frac{1}{\beta_3} (v_1^- |\nabla u^-|^2 + v_2^- |\text{div} u^-|^2) \right) \, dx = 0, \\
(1.31)
\]

where \(\mathcal{E}_0(t)\) and \(\mathcal{D}_0(t)\) denote the nature energy and dissipation, respectively. Noticing the fact that \(\beta_1 \beta_4 = \beta_2 \beta_3 = \beta_3^2\), it is clear that

\[
\mathcal{E}_0(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left( (\beta^+ n^+ + \beta^- n^-)^2 + \frac{\sigma_+}{\beta_2} |\nabla n^+|^2 + \frac{\sigma_-}{\beta_3} |\nabla n^-|^2 + \frac{1}{\beta_2} |u^+|^2 + \frac{1}{\beta_3} |u^-|^2 \right) \, dx. \\
(1.32)
\]

This together with the energy equation \((1.31)\) makes it impossible for us to get the uniform energy estimates of \(n^\pm\) simultaneously, even though in the linear level, but possibly the uniform energy estimates of their linear combination: \(\beta^+ n^+ + \beta^- n^-\), and \(\nabla n^\pm\). Therefore, the linear combination: \(\beta^+ n^+ + \beta^- n^-\) may be a good dissipative variable. On the other hand, by virtue of Mean Value Theorem, we have \(\rho^+ - \rho^- \sim \frac{\sqrt{2(1.1)}}{\sqrt{x^2}} (\beta^+ n^+ + \beta^- n^-)\). In the spirit of these heuristic observations, it is natural to conjecture that \(\rho^+ - \rho^-\) have the same decay rate in time as the density of the compressible Navier–Stokes–Korteweg system \((2.1)\). As a matter of fact, this key observation plays a vital role in our analysis. Roughly speaking, our proofs mainly involves the following four steps.

First, we deduce spectral analysis and linear \(L^2\) estimates on \((n^+, u^+, n^-, u^-)\) to the solution to the linear system \((2.1)\). To derive time–decay estimates of the linear system \((2.1)\), it requires us to make a detailed analysis on the properties of the semigroup. We therefore encounter a fundamental obstacle that the matrix \(\mathcal{A}(\xi)\) in \((2.13)\) is a 8–order matrix and is not self–adjoint. Particularly, it is easy to check that the matrix \(\mathcal{A}(\xi)\) cannot be diagonalizable (see \([24]\) pp.807 for example). Thus, it seems impossible to apply the usual time decay investigation through spectral analysis. To get around this difficulty, we will employ the Hodge decomposition technique firstly introduced by Danchin \([6]\) to split the linear system into three systems. One is a \(4 \times 4\) system and the other two are classic heat equations. Unfortunately, Green Matrix \(\mathcal{A}_1(\xi)\) for Fourier transform of the \(4 \times 4\) system may have multiple eigenvalue, and particularly cannot be diagonalizable. To overcome this difficulty, we first deal with the case of no multiple eigenvalue. Then, in the spirit of the case of no multiple eigenvalue, we handle the case of multiple eigenvalue. The key idea here is that we first assume that a similar expression of the semigroup as in \((2.30)\) holds, which however is crucial for the derivation of time–decay estimates. Then, by employ a clever decomposition and careful analysis, we can get the explicit expressions of \(P_i(\xi)\) for \(i = 1, 2, 3, 4\). We refer to the proof of \((2.30)\) for details.

Second, we make energy estimates of the nonlinear system \((2.1)\). To begin with, similar to the proof of \((1.31)\), for \(0 \leq k \leq \ell\), we have

\[
\frac{d}{dt} \left\{ \|\nabla (\beta^+ n^+ + \beta^- n^-)\|_{L^2}^2 + \frac{\sigma_+}{\beta_2} \|\nabla^{k+1} n^+\|_{L^2}^2 + \frac{\sigma_-}{\beta_3} \|\nabla^{k+1} n^-\|_{L^2}^2 + \frac{1}{\beta_2} \|\nabla^{k+1} u^+\|_{L^2}^2 + \frac{1}{\beta_3} \|\nabla^{k+1} u^-\|_{L^2}^2 \right\} \\
+ C \left( v_1^+ \|\nabla^{k+1} u^+\|_{L^2}^2 + v_2^+ \|\nabla \text{div} u^+\|_{L^2}^2 + v_1^- \|\nabla^{k+1} u^-\|_{L^2}^2 + v_2^- \|\nabla \text{div} u^-\|_{L^2}^2 \right) \\
(1.33)
\]
\[ \leq \delta \left( \| \nabla^{k+1}(n^+, n^-) \|^2_{H^1} + \| \nabla^k(u^+, u^-) \|^2_{L^2} \right). \]

Noticing that (1.33) only involves the dissipation of \( u^\pm \), we need to derive the dissipation estimates of \( n^\pm \).

In fact, for \( 0 \leq k \leq \ell \), it holds that
\[
\frac{d}{dt} \left\{ \left( \nabla^{k+1} u^+, \frac{1}{\beta_2} \nabla \nabla^{k} n^+ \right) + \left( \nabla^{k} u^-, \frac{1}{\beta_3} \nabla \nabla^{k} n^- \right) \right\} 
+ C \| \nabla^{k+1} (\beta^+ n^+ + \beta^- n^-) \|^2_{L^2} + \| \nabla^{k+2} n^+ \|^2_{L^2} + \| \nabla^{k+2} n^- \|^2_{L^2} \leq \left( \delta \| \nabla^{k+1}(n^+, n^-) \|^2_{H^1} + \| \nabla^{k+1}(u^+, u^-) \|^2_{L^2} \right). \tag{1.34} \]

It should be mentioned that different from Lemma 2.2 and Lemma 2.3 of [7], where \( \| \nabla^{k}(n^+, n^-) \|^2_{H^1} \) are involved in the right-hand side of the corresponding energy inequality, (1.33) and (1.34) are new and different since their right-hand side only includes \( \| \nabla^{k+1}(n^+, n^-) \|^2_{H^1} \), and particularly excludes the term \( \| \nabla^{k}(n^+, n^-) \|^2_{L^2} \). These new types of energy inequality (1.33) and (1.34) are crucial for us to close energy estimates at each \( k \)-th level.

Third, we close energy estimates and prove the upper bounds on optimal decay rates. As mentioned before, the nature energy equation (1.31) implies that it seems impossible to get the energy estimates of \( n^\pm \), but possibly the linear combination \( \beta^+ n^+ + \beta^- n^- \) and the derivatives of \( n^\pm \). Inspired by this key observation, the main idea here is to introduce two new time-weighted energy functionals and then estimate them separately. To begin with, we define the following two time-weighted energy functionals
\[
E_k(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{3}{2}} \left( \| \nabla^{k}(\beta^+ n^++\beta^- n^-) \|^2_{H^1} + \| \nabla^{k+1}(n^+, n^-)(\tau) \|^2_{H^1} \right) \right\}, \tag{1.35} \]
for \( 0 \leq k \leq \ell \), and
\[
E_0(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{1}{2}} \| n^+ - n^- \|_{L^2} \right\}. \tag{1.36} \]

Then, by virtue of (1.33) and (1.34), we have
\[
\frac{d}{dt} E_0(t) + C E_0(t) \leq C \left( \| (\beta^+ n^+ + \beta^- n^-)(t) \|^2_{L^2} + \| \nabla(n^+, n^-)(t) \|^2_{L^2} + \| (u^+, u^-)(t) \|^2_{L^2} \right), \tag{1.37} \]
where \( E_0(t) \) is equivalent to \( \| (\beta^+ n^+ + \beta^- n^-)(t) \|^2_{H^1} + \| \nabla(n^+, n^-)(t) \|^2_{H^1} + \| (u^+, u^-)(t) \|^2_{H^1} \). Next, we estimate the terms in the right-hand side of (1.37). To illustrate our idea, we use \( \| (u^+, u^-)(t) \|^2_{L^2} \) as an example. By virtue of Duhamel’s principle, integration by parts, linear estimates obtained in Step 1, we have
\[
\| (u^+, u^-)(t) \|^2_{L^2} \leq K_0 (1 + t)^{-\frac{3}{4}} \| U(0) \|_{L^1} + \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (n^+, n^-)(\tau) \|_{L^1} d\tau \]
\[
+ \int_0^t \| e^{(t-\tau)\beta}(F_2, F_3)(\tau) \|_{L^2} d\tau \leq (1 + t)^{-\frac{3}{4}} \left( K_0 + E_0(t) E_0(t) \right) + \int_0^t \| e^{(t-\tau)\beta}(F_2, F_3)(\tau) \|_{L^2} d\tau. \tag{1.38} \]

On the other hand, it is clear that the strongly coupling terms like \( g_+ (n^+, n^-) \partial n^+ + \bar{g}_+ (n^+, n^-) \partial n^- \) in (2.3) and \( g_- (n^+, n^-) \partial n^+ - \bar{g}_- (n^+, n^-) \partial n^- \) in (2.5) devote the slowest time-decay rates to the third term on the right-hand side of (1.38). Therefore, this together with (1.38) implies that we can only obtain the following estimate:
\[
\| (u^+, u^-)(t) \|^2_{L^2} \leq (1 + t)^{-\frac{3}{4}} \ln(1 + t) \left( K_0 + E_0(t) E_0(t) \right), \]
which however is not quickly enough for us to close energy estimates. To overcome this essential difficulty, it is crucial to develop new thoughts to deal with the trouble terms: \( g_+ (n^+, n^-) \partial n^+ + \bar{g}_+ (n^+, n^-) \partial n^- \)
and \(g_-(n^+, n^-) \partial n^- + \tilde{g}_-(n^+, n^-) \partial n^+\). The main idea here is that in view of the linear combination \(\beta^+ n^+ + \beta^- n^-\), we rewrite them in a clever way. More specifically, we surprisingly find that

\[
\begin{align*}
g_+(n^+, n^-) \partial n^+ + \tilde{g}_+(n^+, n^-) \partial n^- &= \partial t G_2 + \text{good terms}, \\
g_-(n^+, n^-) \partial n^- + \tilde{g}_-(n^+, n^-) \partial n^+ &= \partial t G_4 + \text{good terms}.
\end{align*}
\]

With this key observation, one can shift the derivatives of \(G_2\) and \(G_4\) onto the solution semigroup to derive the desired decay estimates of \(\|(u^{+, \ell}, u^{-, \ell})\|_{L^2}\). It is worth mentioning that good dissipation properties of \(\rho^\pm - \bar{\rho}^\pm\) play an important role in this process. Therefore, we finally deduce that

\[
E_0^\prime(t) \leq \left[ K_0 + E_0(t) E_0^\prime(t) + \left( E_0^\prime(t) \right)^2 \right].
\]  

(1.39)

Next, we deal with \(E_0(t)\). As mentioned before, due to special structure of the natural energy equation \(1.31\), we cannot employ the nonlinear energy estimates obtained in Step 2. Instead, we will make full use of the benefit of low-frequency and high-frequency decomposition to get

\[
\|(n^+, n^-)\|_{L^2} \leq \left( \|(n^{+, \ell}, n^{-, \ell})\|_{L^2} + \|(n^{+, h}, n^{-, h})\|_{L^2} \right)
\]

\[
\leq \left( \|(n^{+, \ell}, n^{-, \ell})\|_{L^2} + \|\nabla (n^+, n^-)\|_{L^2} \right)
\]

\[
\leq (1 + t)^{-\frac{2}{3}} \left[ K_0 + E_0(t) E_0^\prime(t) + \left( E_0^\prime(t) \right)^2 \right],
\]

which leads to

\[
E_0(t) \leq \left[ K_0 + E_0(t) E_0^\prime(t) + \left( E_0^\prime(t) \right)^2 \right].
\]  

(1.40)

Finally, combining (1.39) with (1.40), we can close energy estimates (see the proof of (4.19) for details). For \(1 \leq k \leq \ell\), the new difficulty is how to control the terms involving \(\nabla^{2+2} u^\pm\) which however don’t belong to the solution space. To get around this difficulty, we separate the time interval into two parts and make full use of the benefit of low-frequency and high-frequency decomposition (see the proof of (4.31) for details). Then, we can employ similar arguments to prove \(E_0^\prime(t) \leq CK_0\) by induction. This in turn proves the optimal decay rates in (1.19)–(1.21). Finally, we can make full use of the expression of the pressure differential in (1.12), and (1.19)–(1.21) to prove (1.18). We refer to the proof of (4.25) for details.

In the last step, we show the low-bounds on the convergence rates of solutions. First, we deal with \((n^+, n^-, u^+, u^-, \beta^+ n^+ + \beta^- n^-)\). To begin with, we will employ Plancherel theorem and careful analysis on the solution semigroup to derive the lower convergence rate in \(H^{-1}(\mathbb{R}^3)\) and \(L^2(\mathbb{R}^3)\). Then, we can prove the lower bound on the convergence rates in \(L^2\)-norm for the higher-order spatial derivatives by an interpolation trick. Second, we tackle with \((\rho^\pm - \bar{\rho}^\pm)\). By fully using the expression of the pressure differential \(dP\) and embedding estimate of Riesz potential, we can get the lower convergence rate in \(L^2(\mathbb{R}^3)\). For \(1 \leq k \leq \ell\), we can prove the lower bound on the \(L^2\) convergence rates for the \(k\)-th spatial derivatives by an interpolation trick.

2. **Spectral analysis and linear \(L^2\) estimates.**

2.1. **Reformulation.** In this subsection, we first reformulate the system. Setting

\[
n^\pm = R^\pm - 1,
\]
the Cauchy problem \((1.4)-(1.5)\) can be reformulated as

\[
\begin{aligned}
\frac{\partial u_1 + \nabla + F_1,}{} \\
\frac{\partial u_1 + \beta_1 \nabla + \beta_2 \nabla - u_1^+ \Delta u_1 - u_2^+ \nabla \nabla u_1 - \sigma \nabla \Delta u_1 = F_2,}{} \\
\frac{\partial u_1 + \beta_1 \nabla + \beta_2 \nabla - u_1^- \Delta u_1 - u_2^- \nabla \nabla u_1 - \sigma \nabla \Delta u_1 = F_3,}{}
\end{aligned}
\]

\[
\begin{aligned}
(n_1^+, u_1^+, n_1^-) & = (0, 0, 0) & (n_1^+, u_1^+, n_1^-) & \to (0, 0, 0), & as & |x| \to +\infty,
\end{aligned}
\]

where \(u_1^+ = \frac{u_1^+}{\rho_1^+}, u_2^+ = \frac{u_2^+}{\rho_1^+} > 0\), \(\beta_1 = \frac{\rho_1^2 (1, 1)}{\rho_1^+ (1, 1)}\), \(\beta_2 = \beta_3 = \rho_2^2 (1, 1)\), \(\beta_4 = \frac{\rho_1^2 (1, 1) \rho_2^2}{\rho_2^2 (1, 1)}\) (which imply \(\beta_1 \beta_4 = \beta_2 \beta_3 = \beta_2^2\)), and the nonlinear terms are given by

\[
\begin{aligned}
F_1 &= - \nabla (n^+ u^+), \\
F_2 &= - \nabla (n^+ n^-) \partial n^+ - \nabla (n^+ n^-) \partial n^- - (u^+ \cdot \nabla) u^+ \\
&+ \mu^+ h_+ (n^+ n^-) \partial n^+ \partial u^+ + \mu^+ k_+ (n^+ n^-) \partial n^- \partial u^+, \\
F_3 &= - \nabla (n^- u^-), \\
F_4 &= - \nabla (n^- n^-) \partial n^- - \nabla (n^- n^-) \partial n^- - (u^- \cdot \nabla) u^+ \\
&+ \mu^- h_- (n^- n^-) \partial n^- \partial u^- + \mu^- k_- (n^- n^-) \partial n^- \partial u^+, \\
\end{aligned}
\]

where

\[
\begin{aligned}
g_+(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)} - \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
g_-(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)} - \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
g_+(n^+, n^-) &= \rho^2 (n^+ + 1, n^- + 1) - \rho^2 (1, 1), \\
g_-(n^+, n^-) &= \rho^2 (n^+ + 1, n^- + 1) - \rho^2 (1, 1), \\
h_+(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
h_-(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
k_+(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
k_-(n^+, n^-) &= \frac{\rho^2 (1, 1)}{\rho^2 (1, 1)}, \\
l_+ (n^+, n^-) &= \frac{1}{\rho_+ (n^+ + 1, n^- + 1)} - \frac{1}{\rho_+ (1, 1)},
\end{aligned}
\]
Define $\tilde{U} = (\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-)$. In terms of the semigroup theory for evolutionary equation, we will investigate the following initial value problem for the corresponding linear system of (2.1):

\[
\begin{aligned}
\begin{cases}
\tilde{U}_t &= \mathcal{B}\tilde{U}, \\
\tilde{U}|_{t=0} &= U_0,
\end{cases}
\end{aligned}
\] (2.11)

where the operator $\mathcal{B}$ is given by

\[
\mathcal{B} = \begin{pmatrix}
0 & -\text{div} & 0 & 0 \\
-\beta_1 \nabla + \sigma^+ \nabla\Delta & \nu_1^+ \nabla + v_1^+ \nabla \otimes \nabla & -\beta_2 \nabla & 0 \\
0 & 0 & 0 & -\text{div} \\
-\beta_4 \nabla & 0 & -\beta_4 \nabla + \sigma^- \nabla\Delta & \nu_2^- \nabla + v_2^- \nabla \otimes \nabla
\end{pmatrix}.
\]

Applying Fourier transform to the system (2.11), one has

\[
\begin{aligned}
\begin{cases}
\hat{\tilde{U}}_t &= \mathcal{A}(\xi)\hat{\tilde{U}}, \\
\hat{\tilde{U}}|_{t=0} &= \hat{U}_0 = (\tilde{n}_0^+, \tilde{u}_0^+, \tilde{n}_0^-, \tilde{u}_0^-),
\end{cases}
\end{aligned}
\] (2.12)

where $\hat{\tilde{U}}(\xi, t) = \mathcal{F}(\tilde{U}(x, t))$, $\xi = (\xi_1, \xi_2, \xi_3)^T$ and $\mathcal{A}(\xi)$ is defined by

\[
\mathcal{A}(\xi) = \begin{pmatrix}
0 & -i\xi_1 & 0 & 0 \\
-i\beta_1 \xi_1 - i\sigma^+ |\xi|^2 \xi & -\nu_1^+ |\xi|^2 I_{3\times 3} - v_2^+ \xi \otimes \xi & -i\beta_3 \xi & 0 \\
0 & 0 & 0 & -i\xi_2^T \\
-i\beta_1 \xi_2 & 0 & -i\beta_3 - i\sigma^- |\xi|^2 \xi & -\nu_1^- |\xi|^2 I_{3\times 3} - v_2^- \xi \otimes \xi
\end{pmatrix}.
\]

To derive the linear time-decay estimates, by using a real method as in [17, 18], one need to make a detailed analysis on the properties of the semigroup. Unfortunately, it seems untractable, since the system (2.12) has eight equations and the matrix $\mathcal{A}(\xi)$ cannot be diagonalizable (see [24] pp.807 for example). To overcome this difficulty, we take Hodge decomposition to system (2.11) such that it can be decoupled into three systems. One has four equations, and the other two are classic heat equations. This key observation allows us to derive the optimal linear convergence rates.

To begin with, let $\phi^\pm = \Lambda^{-1} \text{div} \tilde{u}^\pm$ be the “compressible part” of the velocities $\tilde{u}^\pm$, and denote $\phi^\pm = \Lambda^{-1} \text{curl} \tilde{u}^\pm$ (with $\text{curl} z = (\partial_z \tilde{z}_3 - \partial_{\tilde{z}_1}, \partial_z \tilde{z}_1 - \partial_{\tilde{z}_3}, \partial_z \tilde{z}_2 - \partial_{\tilde{z}_2})^T$) by the “incompressible part” of the velocities $\tilde{u}^\pm$. Then, we can rewrite the system (2.11) as follows:

\[
\begin{aligned}
\begin{cases}
\partial_t \tilde{n}^+ + \Lambda \phi^+ = 0, \\
\partial_t \phi^+ - \beta_1 \Lambda \tilde{n}^+ - \beta_2 \Lambda \tilde{n}^- + \nu_1^+ \phi^+ - \sigma^+ \Lambda^3 \tilde{n}^+ = 0, \\
\partial_t \tilde{n}^- + \Lambda \phi^- = 0, \\
\partial_t \phi^- - \beta_3 \Lambda \tilde{n}^+ - \beta_4 \Lambda \tilde{n}^- + \nu_2^- \phi^- - \sigma^- \Lambda^3 \tilde{n}^- = 0,
\end{cases}
\end{aligned}
\] (2.13)

and

\[
\begin{aligned}
\begin{cases}
\partial_t \phi^+ + v_1^+ \Lambda^2 \phi^+ = 0, \\
\partial_t \phi^- + v_1^- \Lambda^2 \phi^- = 0, \\
(\phi^+, \phi^-)|_{t=0} = (n_0^+, \Lambda^{-1} \text{div} \tilde{u}_0^+, n_0^-, \Lambda^{-1} \text{div} \tilde{u}_0^-)(x),
\end{cases}
\end{aligned}
\] (2.14)

where $\nu^\pm = v_1^\pm + v_2^\pm$. 
2.2. Spectral analysis and linear $L^2$ estimates. In view of the semigroup theory, we may represent the IVP \((2.13)\) for \(W = (\bar{u}^+, \varphi^+, \bar{u}^-, \varphi^-)\) as
\[
\begin{cases}
\dot{W}_t = B_1 W, \\
W|_{t=0} = W_0,
\end{cases}
\]
where the operator \(B_1\) is defined by
\[
B_1 = \begin{pmatrix}
0 & -\Lambda & 0 & 0 \\
\beta_1 \Lambda + \sigma^+ \Lambda^3 & -v^+ \Lambda^2 & \beta_2 \Lambda & 0 \\
0 & 0 & 0 & -\Lambda \\
\beta_3 \Lambda & 0 & \beta_4 \Lambda + \sigma^- \Lambda^3 & -v^- \Lambda^2
\end{pmatrix}.
\]
Taking Fourier transform to system \((2.15)\), we obtain
\[
\begin{cases}
\hat{W} = \mathcal{A}_1(\xi) \hat{W}, \\
\hat{W}|_{t=0} = \hat{W}_0,
\end{cases}
\]
where \(\hat{W}(\xi, t) = \hat{W}(W(x, t))\) and \(\mathcal{A}_1(\xi)\) is given by
\[
\mathcal{A}_1(\xi) = \begin{pmatrix}
0 & -|\xi| & 0 & 0 \\
\beta_1 |\xi| + \sigma^+ |\xi|^3 & -v^+ |\xi|^2 & \beta_2 |\xi| & 0 \\
0 & 0 & 0 & -|\xi| \\
\beta_3 |\xi| & 0 & \beta_4 |\xi| + \sigma^- |\xi|^3 & -v^- |\xi|^2
\end{pmatrix}.
\]
We compute the eigenvalues of matrix \(\mathcal{A}_1(\xi)\) from the determinant
\[
\det(\lambda I - \mathcal{A}_1(\xi)) = \lambda^4 + (v^+ |\xi|^2 + v^- |\xi|^2) \lambda^3 + (|\beta_1| \lambda^2 + |\beta_4| \lambda^2 + \sigma^+ |\xi|^4 + \sigma^- |\xi|^4 + v^+ v^- |\xi|^4) \lambda^2 \\
+ (|\beta_1 v^- |\xi|^4 + |\beta_4 v^+ |\xi|^4 + v^+ \sigma^- |\xi|^6 + v^- \sigma^+ |\xi|^6) \lambda \\
+ \beta_1 \sigma^- |\xi|^6 + \beta_4 \sigma^+ |\xi|^6 + \sigma^+ \sigma^- |\xi|^8 = 0,
\]
which implies that matrix \(\mathcal{A}_1(\xi)\) possesses four eigenvalues:
\[
\begin{cases}
\lambda_1 = -\frac{\beta_1 v^+ + \beta_4 v^-}{2(\beta_1 + \beta_4)} |\xi|^2 + i \sqrt{\beta_1 + \beta_4} |\xi| + O(|\xi|^3), \\
\lambda_2 = -\frac{\beta_1 v^- + \beta_4 v^+}{2(\beta_1 + \beta_4)} |\xi|^2 - i \sqrt{\beta_1 + \beta_4} |\xi| + O(|\xi|^3), \\
\lambda_3 = -\frac{(\beta_1 v^- + \beta_4 v^+)^2 - 4(\beta_1 + \beta_4) (\beta_1 \sigma^+ + \beta_4 \sigma^-)}{2(\beta_1 + \beta_4)} |\xi|^2 + O(|\xi|^4), \\
\lambda_4 = -\frac{(\beta_1 v^- + \beta_4 v^+)^2 - 4(\beta_1 + \beta_4) (\beta_1 \sigma^- + \beta_4 \sigma^+)}{2(\beta_1 + \beta_4)} |\xi|^2 + O(|\xi|^4),
\end{cases}
\]
Set \(\mathcal{D} = \sqrt{\frac{(\beta_1 v^- + \beta_4 v^+)^2 - 4(\beta_1 + \beta_4) (\beta_1 \sigma^+ + \beta_4 \sigma^-)}{2(\beta_1 + \beta_4)}}, \lambda_3 = -\frac{(\beta_1 v^- + \beta_4 v^+)^2 + \mathcal{D}}{2(\beta_1 + \beta_4)}, \lambda_4 = -\frac{(\beta_1 v^- + \beta_4 v^+)^2 - \mathcal{D}}{2(\beta_1 + \beta_4)}.
\]
In what follows, we will divide the issue into two cases and discuss them respectively.

Case 1: \(\lambda_3 \neq \lambda_4\). In this case, we have \(P^{-1} \mathcal{A}_1(\xi) P = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}\). Next, we derive the expressions of the project operators \(P_i\) for \(i = 1, 2, 3, 4\). By making tedious calculations, it is clear that the
semigroup $e^{\xi t}$ can be expressed as

$$e^{\xi t} = e^{\lambda_1 t} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} + e^{\lambda_2 t} P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1}$$

$$+ e^{\lambda_3 t} P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} + e^{\lambda_4 t} P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} \quad (2.19)$$

where the projectors $P_i(\xi)$ can be computed as

$$P_1(\xi) = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} P(\lambda_2 I - J)(\lambda_3 I - J)(\lambda_4 I - J) P^{-1} \quad (2.20)$$

$$P_2(\xi) = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} P(\lambda_1 I - J)(\lambda_3 I - J)(\lambda_4 I - J) P^{-1} \quad (2.21)$$

$$P_3(\xi) = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} = \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} P(\lambda_1 I - J)(\lambda_2 I - J)(\lambda_4 I - J) P^{-1}$$

$$= \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} P(\lambda_1 I - J)(\lambda_2 I - J)(\lambda_4 I - J) P^{-1} \quad (2.22)$$

$$= \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_3)} P(\lambda_1 I - J)(\lambda_2 I - J)(\lambda_4 I - J) P^{-1} \quad (2.22)$$
Consequently, we can represent the solution of IVP (2.16) as

$$\hat{w}(\xi, t) = e^{i\phi_1(\xi)}\hat{\psi}_0(\xi) = \left( \sum_{j=1}^{4} e^{i\phi_j} P_j(\xi) \right) \hat{\psi}_0(\xi),$$

(2.24)

where

$$P_1(\xi) = \begin{pmatrix} \frac{1}{2}(1 + \xi^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

$$P_2(\xi) = \begin{pmatrix} \frac{1}{2}(1 + \xi^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{-1} + O(|\xi|),$$

(2.25)

$$P_3(\xi) = \begin{pmatrix} \frac{1}{2}(1 + \xi^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{-1} + O(|\xi|),$$

(2.26)

$$P_4(\xi) = \begin{pmatrix} \frac{1}{2}(1 + \xi^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{-1} + O(|\xi|).$$

(2.27)
In this case, we have $P^{-1} \mathcal{A} P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = J$. Compared to Case 1, where the matrix $\mathcal{A} (\xi)$ can be diagonalizable, it is rather difficult to get the expression of the solution in terms of the solution semigroup as in (2.24) due to the appearance of the multiple roots. The key idea here is that we first conjecture that a similar formula as (2.24) holds, and then derive the explicit expressions of $P_i(\xi)$ for $i = 1, 2, 3, 4$. To see this, we first notice that

\[
\begin{cases}
\hat{\mathcal{W}}(\xi, t) = PJ P^{-1} \mathcal{W}(\xi, t), \\
\hat{\mathcal{W}}(\xi, 0) = \hat{\mathcal{W}}_0(\xi),
\end{cases}
\]

which implies

\[
\hat{\mathcal{W}}(\xi, t) = P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & t e^{\lambda_3 t} \\ 0 & 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1} \mathcal{W}_0(\xi).
\]

(2.29)

Inspired by Case 1, we obtain the following decomposition holds:

\[
\hat{\mathcal{W}}(\xi, t) = P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & t e^{\lambda_3 t} \\ 0 & 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1} \mathcal{W}_0(\xi)
\]

(2.30)

Then, after careful and sophisticated calculations, we find the explicit expressions of $P_i(\xi)$ for $i = 1, 2, 3, 4$, which are given by

\[
P_i(\xi) = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1}
\]

\[
= \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)^2} \lambda_2 I - J)^2 P^{-1}
\]

\[
= \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)^2} \lambda_2 I - J)^P(\lambda_3 I - J)^2 P^{-1}
\]

\[
= \frac{(r_2 I - A)(r_3 I - A)^2}{(r_2 - r_1)(r_3 - r_1)^2}
\]

\[
= \begin{pmatrix} \frac{\beta_1}{2(\beta_1 + \beta_4)} & \frac{\beta_1}{2(\beta_1 + \beta_4)} - i & \frac{\beta_2}{2(\beta_1 + \beta_4)} & \frac{\beta_2}{2(\beta_1 + \beta_4)} - i \\ \frac{\beta_1}{2(\beta_1 + \beta_4)} - i & \frac{2(\beta_1 + \beta_4)}{\beta_2} & \frac{\beta_2}{2(\beta_1 + \beta_4)} - i & \frac{2(\beta_1 + \beta_4)}{\beta_2} \\ \frac{\beta_2}{2(\beta_1 + \beta_4)} & \frac{\beta_2}{2(\beta_1 + \beta_4)} - i & \frac{\beta_3}{2(\beta_1 + \beta_4)} & \frac{\beta_3}{2(\beta_1 + \beta_4)} - i \\ \frac{\beta_3}{2(\beta_1 + \beta_4)} & \frac{\beta_3}{2(\beta_1 + \beta_4)} - i & \frac{\beta_4}{2(\beta_1 + \beta_4)} & \frac{\beta_4}{2(\beta_1 + \beta_4)} - i \end{pmatrix} + \mathcal{O}(1) ,
\]
\[
P_2(\xi) = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)^2} P(\lambda_1 I - J)(\lambda_3 I - J)^2 P^{-1}
\]

\[
P_3(\xi) = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1} = \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} P(\lambda_1 I - J)(\lambda_2 I - J) + (\lambda_1 + \lambda_2 - 2\lambda_3) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P^{-1}
\]

\[
P_4(\xi) = P \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1} = -\frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} P(\lambda_1 I - J)(\lambda_2 I - J)(\lambda_3 I - J) P^{-1}
\]

\[
P_4(\xi) = P \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{-1} = -\frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} P(\lambda_1 I - J)(\lambda_2 I - J)(\lambda_3 I - J) P^{-1}
\]

\[
(2.32)
\]

\[
(2.33)
\]

\[
(2.34)
\]
By virtue of (2.24)-(2.26) and (2.30)-(2.34), we can establish the following estimates on low-frequency part of the solution \( \tilde{\mathcal{U}}(\xi, t) \) to the IVP (2.16).

**Lemma 2.1.** Let \( \bar{V} = \min \{ \frac{\beta_1 \nu^+ + \beta_2 \nu^-}{2(\beta_1 + \beta_2)}, -\lambda_3 \} \) if \( \mathcal{R} \) is a real number, and \( \bar{V} = \min \{ \frac{\beta_1 \nu^+ + \beta_2 \nu^-}{2(\beta_1 + \beta_2)}, \frac{\beta_1 \nu^+ - \beta_2 \nu^-}{2(\beta_1 + \beta_2)} \} \) if \( \mathcal{R} \) is an imaginary number, then there exists a sufficiently small positive constant \( \eta \), such that the following estimates hold

\[
\left| \hat{n}^+ \right|, \left| \hat{n}^- \right| \lesssim \frac{e^{-\bar{V} \frac{|\xi|^2}{4}}}{|\xi|} \left( \left| \hat{n}_0^+ \right| + \left| \hat{\phi}_0^+ \right| + \left| \hat{n}_0^- \right| + \left| \hat{\phi}_0^- \right| \right) + te^{-\bar{V} \frac{|\xi|^2}{4}} \left( \left| \hat{n}_0^+ \right| + \left| \hat{\phi}_0^+ \right| + \left| \hat{n}_0^- \right| + \left| \hat{\phi}_0^- \right| \right),
\]

(2.35)

\[
\left| \bar{\phi}^+ \right|, \left| \bar{\phi}^- \right|, \left| \beta_1 \hat{n}^+ + \beta_2 \hat{n}^- \right| \lesssim e^{-\bar{V} \frac{|\xi|^2}{4}} \left( \left| \hat{n}_0^+ \right| + \left| \hat{\phi}_0^+ \right| + \left| \hat{n}_0^- \right| + \left| \hat{\phi}_0^- \right| \right),
\]

(2.36)

for any \( |\xi| \leq \eta \).

The key estimates in (2.35) and (2.36) enable us to establish the optimal \( L^2 \)-convergence rate on the low-frequency part of the solution, which is stated in the following proposition.

**Proposition 2.2** (\( L^2 \)-theory). For any \( k > -\frac{3}{4} \), there exists a positive constant \( C \) independent of \( t \), such that

\[
\left\| \nabla^k \left( \hat{n}^+ + \hat{n}^- \right)(t) \right\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}} \left\| \hat{\mathcal{U}}(0) \right\|_{L^\infty},
\]

(2.37)

\[
\left\| \nabla^k \left( \bar{\phi}^+ + \bar{\phi}^- + \beta_1 \hat{n}^+ + \beta_2 \hat{n}^- \right)(t) \right\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}} \left\| \hat{\mathcal{U}}(0) \right\|_{L^\infty},
\]

(2.38)

for any \( t \geq 0 \).

It should be mentioned that the above \( L^2 \)-convergence rates are optimal. As a matter of fact, we also have the lower bounds on the convergence rates which are stated in the following proposition.

**Proposition 2.3.** Assume that \( \left( \hat{n}_0^+, \hat{\phi}_0^+, \hat{n}_0^-, \hat{\phi}_0^- \right) \in L^1 \) satisfies

\[
\hat{n}_0^+(\xi) = \hat{\phi}_0^+(\xi) = \hat{n}_0^-(\xi) = 0, \quad \text{and} \quad \hat{\phi}_0^-(\xi) - c_0 \sim |\xi|^s
\]

(2.39)

for any \( |\xi| \leq \eta \), where \( c_0 \) and \( s \) are given two positive constants. Then, there exists a positive constant \( C_1 \) independent of \( t \), such that it holds that

\[
\min \left\{ \| \hat{n}^+ \|(t) \|_{L^2}, \| \hat{n}^- \|(t) \|_{L^2} \right\} \geq C_1 c_0 (1 + t)^{-\frac{s}{2}},
\]

(2.40)

\[
\min \left\{ \| \bar{\phi}^+ \|(t) \|_{L^2}, \| \bar{\phi}^- \|(t) \|_{L^2}, \| \beta_1 \hat{n}^+ + \beta_2 \hat{n}^- \|(t) \|_{L^2} \right\} \geq C_1 c_0 (1 + t)^{-\frac{s}{2}},
\]

(2.41)

for sufficiently large \( t \).

**Proof.** We only deal with **Case 1**, since the proof of **Case 2** is similar. Because the proofs are similar, we will focus on the proofs concerning the terms \( \varphi^{+, -} \) and \( \tilde{n}^{+, -} \) for simplicity. To begin with, by virtue of (2.24) and (2.30), we have

\[
\tilde{\varphi}^{+, -} = \left( \frac{\beta_2}{2(\beta_1 + \beta_4)} + O(|\xi|) \right) e^{\lambda_1(|\xi|)} \hat{n}_0^{+, -} \left( \frac{\beta_2}{2(\beta_1 + \beta_4)} + O(|\xi|) \right) e^{\lambda_2(|\xi|)} \hat{\phi}_0^{+, -} + \left( \frac{\beta_1}{(\beta_1 + \beta_4)} + O(|\xi|) \right) e^{\lambda_3(|\xi|)} \hat{n}_0^{+, -} \left( \frac{\beta_1}{(\beta_1 + \beta_4)} + O(|\xi|) \right) e^{\lambda_4(|\xi|)} \hat{\phi}_0^{+, -}
\]

\[
+ \left( \frac{\beta_3 (\beta_4 \nu^+ + \beta_1 \nu^-)}{(\beta_1 + \beta_4) \mathcal{R}} + O(|\xi|) \right) e^{\lambda_5(|\xi|)} \hat{n}_0^{+, -} \left( \frac{\beta_3 (\beta_4 \nu^+ + \beta_1 \nu^-)}{(\beta_1 + \beta_4) \mathcal{R}} + O(|\xi|) \right) e^{\lambda_6(|\xi|)} \hat{\phi}_0^{+, -}
\]
\[
\sim \frac{\beta_2}{\beta_1 + \beta_4} e^{-\frac{\beta_1 v^+ + \beta_4 v^-}{2(\beta_1 + \beta_4)} |\xi|^2} \cos \left[ \left( \sqrt{\beta_1 + \beta_4} |\xi| + O(|\xi|^3) \right)t \right] \tilde{q}_0^{i - j} + \left( \frac{\beta_2 (\beta_4 v^+ + \beta_1 v^-)}{(\beta_1 + \beta_4) R} + \frac{\beta_2 \lambda_4}{R} \right) e^{\lambda_4 |\xi|^2} \tilde{q}_0^{j - i},
\]

which together with Plancherel theorem and the double angle formula implies

\[
\| \phi^{i - j} \|^2_{L^2} = \| \phi^{j - i} \|^2_{L^2}
\]

\[
\geq \frac{\beta_2^2}{2(\beta_1 + \beta_4)^2} \int_{|\xi| \leq \frac{1}{2}} e^{-\frac{\beta_1 v^+ + \beta_4 v^-}{2(\beta_1 + \beta_4)} |\xi|^2} \cos \left[ \left( \sqrt{\beta_1 + \beta_4} |\xi| + O(|\xi|^3) \right)t \right] d\xi
\]

\[
+ 2\beta_2 \beta_4 \int_{|\xi| \leq \eta} e^{-\frac{\beta_1 v^+ + \beta_4 v^-}{2(\beta_1 + \beta_4)} |\xi|^2} \cos \left[ \left( \sqrt{\beta_1 + \beta_4} |\xi| + O(|\xi|^3) \right)t \right] d\xi
\]

\[
\times \left( \frac{\beta_2 (\beta_4 v^+ + \beta_1 v^-)}{(\beta_1 + \beta_4) R} + \frac{\beta_2 \lambda_4}{R} \right) e^{\lambda_4 |\xi|^2} \tilde{q}_0^{j - i} + \left( \frac{\beta_2 (\beta_4 v^+ + \beta_1 v^-)}{(\beta_1 + \beta_4) R} + \frac{\beta_2 \lambda_4}{R} \right) e^{\lambda_4 |\xi|^2} \tilde{q}_0^{j - i} d\xi
\]

\[
- C_1 \int_{|\xi| \leq \eta} e^{-\frac{\beta_1 v^+ + \beta_4 v^-}{2(\beta_1 + \beta_4)} |\xi|^2} |\xi|^2 d\xi
\]

\[
\geq C_2 c_0 (1 + t)^{-\frac{3}{4}} - C_3 \left( (1 + t)^{-\frac{3}{4}} + (1 + t)^{-\frac{3}{2} - 2\delta} \right)
\]

\[
\geq C_4 c_0 (1 + t)^{-\frac{3}{4}}
\]

if \( t \) large enough. Similarly, for the term \( \tilde{n}^{i - j} \), we have

\[
\tilde{n}^{i - j} = \left( \frac{\beta_2}{2(\beta_1 + \beta_4)^2} + O'(|\xi|) \right) \left( e^{4\lambda_4 |\xi|^2} - e^{4\lambda_4 |\xi|^2} \right) \tilde{q}_0^{j - i}
\]

\[
+ \left( \frac{\beta_2}{2(\beta_1 + \beta_4)^2} + O'(|\xi|) \right) \left( e^{\lambda_4 |\xi|^2} - e^{\lambda_4 |\xi|^2} \right) \tilde{q}_0^{j - i}
\]

\[
\sim \left( \frac{\beta_2}{2(\beta_1 + \beta_4)^2} + O'(|\xi|) \right) \left( e^{\lambda_4 |\xi|^2} - e^{\lambda_4 |\xi|^2} \right) \tilde{q}_0^{j - i}
\]

\[
+ \left( \frac{\beta_2}{2(\beta_1 + \beta_4)^2} + O'(|\xi|) \right) \left( \frac{\beta_1 v^+ + \beta_4 v^-}{(\beta_1 + \beta_4) R} + \frac{\beta_2 \lambda_4}{R} \right) e^{\lambda_4 |\xi|^2} \tilde{q}_0^{j - i},
\]

which together with Plancherel theorem leads to

\[
\| \tilde{n}^{i - j} \|^2_{L^2} = \| \tilde{n}^{j - i} \|^2_{L^2} \geq C_5 c_0 (1 + t)^{-\frac{3}{4}}.
\]
Therefore, we have completed the proof of Proposition 2.3.

In terms of the classic theory of the heat equation, the solutions $\phi^{\pm}$ to the IVP (2.14) possess the following decay estimates.

**Proposition 2.4 (L$^2$-theory).** For any $k > -\frac{3}{2}$, there exists a positive constant $C$ independent of $t$ such that

$$
\left\| \nabla^k \phi^{\pm}(t) \right\|_{L^2} \leq C (1 + t)^{-\frac{3}{2} - \frac{k}{2}} \left\| \phi^{\pm}(0) \right\|_{L^2},
$$

(2.44)

for any $t \geq 0$.

Finally, noticing the definition of $\varphi^{\pm}$ and $\phi^{\pm}$, and the fact that the relations

$$
\tilde{u}^{\pm} = -\nabla \varphi^{\pm} - \nabla \text{div} \phi^{\pm}
$$

involves pseudo-differential operators of degree zero, the estimates in space $H^k(\mathbb{R}^3)$ for the original function $\tilde{u}^{\pm}$ will be the same as for $(\varphi^{\pm}, \phi^{\pm})$. Combining Proposition 2.2, 2.3 and 2.4, we finally deduce that the solution $\tilde{U}$ to the IVP (2.11) has the following decay rates in time.

**Proposition 2.5.** Let $k > -\frac{3}{2}$ and assume that the initial data $U_0 \in L^1(\mathbb{R}^3)$, then for any $t \geq 0$, the global solution $\tilde{U} = (\tilde{n}^+, \tilde{u}^+, \tilde{n}^-, \tilde{u}^-)'$ of the IVP (2.11) satisfies

$$
\left\| \nabla^k (\tilde{n}^+, \tilde{n}^-) (t) \right\|_{L^2} \leq C (1 + t)^{-\frac{3}{2} - \frac{k}{2}} \left\| U(0) \right\|_{L^1},
$$

(2.45)

$$
\left\| \nabla^k (\tilde{u}^+, \tilde{n}^+, \tilde{n}^-, \tilde{u}^-) \right\|_{L^2} \leq C (1 + t)^{-\frac{3}{2} - \frac{k}{2}} \left\| U(0) \right\|_{L^1}.
$$

(2.46)

If in addition, the initial data satisfies (2.39), the following lower-bounds on convergence rate hold

$$
\min \left\{ \left\| \tilde{n}^{+\pm}(t) \right\|_{L^2}, \left\| \tilde{n}^{-\pm}(t) \right\|_{L^2} \right\} \geq C_1 c_0 (1 + t)^{-\frac{3}{2},}
$$

(2.47)

$$
\min \left\{ \left\| \tilde{u}^{+\pm}(t) \right\|_{L^2}, \left\| \tilde{u}^{-\pm}(t) \right\|_{L^2}, \left\| \beta_1 \tilde{n}^{+\pm} + \beta_2 \tilde{n}^{-\pm}(t) \right\|_{L^2} \right\} \geq C_1 c_0 (1 + t)^{-\frac{3}{2},}
$$

(2.48)

if $t$ large enough.

3. **Energy estimates for the nonlinear system.**

In this section, we devote ourselves to deriving the a priori energy estimates for the nonlinear system (2.1). To see this, we assume a priori that for sufficiently small $\delta > 0$,

$$
\| (n^+, n^-) \|_{H^{\ell+1}} + \| (u^+, u^-) \|_{H^\ell} \leq \delta.
$$

(3.1)

In what follows, a series of lemmas on the energy estimates are given. First, we deduce energy estimate on $(n^+, u^+, n^-, u^-)$, which is stated in the following lemma.

**Lemma 3.1.** Assume that the notations and hypotheses of Theorem 2.1 and (3.1) are in force. Then, for $0 \leq k \leq \ell$, it holds that

$$
\frac{1}{2} \frac{d}{dt} \left\{ \left\| \nabla^k (\beta^+ n^+ + \beta^- n^-) \right\|_{L^2}^2 + \frac{\sigma^+}{\beta_2} \left\| \nabla^k u^+ \right\|_{L^2}^2 + \frac{\sigma^-}{\beta_3} \left\| \nabla^k u^- \right\|_{L^2}^2 + \frac{1}{\beta_2} \left\| \nabla^k u^+ \right\|_{L^2}^2 + \frac{1}{\beta_3} \left\| \nabla^k u^- \right\|_{L^2}^2 \right\}
$$

$$
+ \frac{3}{4 \beta_2} \left( \left\| \nabla^k u^+ \right\|_{L^2}^2 + \left\| \nabla^k \text{div} u^+ \right\|_{L^2}^2 \right) + \frac{3}{4 \beta_3} \left( \left\| \nabla^k u^- \right\|_{L^2}^2 + \left\| \nabla^k \text{div} u^- \right\|_{L^2}^2 \right)
$$

$$
\leq C \delta \left( \left\| \nabla^k (n^+, n^-) \right\|_{H^\ell}^2 + \left\| \nabla^k (u^+, u^-) \right\|_{H^\ell}^2 \right),
$$

(3.2)
for some constant $C > 0$ independent of $\delta$, where $\beta^+ = \sqrt{\beta_1 \beta_2}$ and $\beta^- = \sqrt{\beta_1 \beta_3}$.

**Remark 3.2.** Compared to Lemma 2.2 and Lemma 2.3 of [7], where $\|\nabla^k(n^+,n^-)\|^2_{L^2}$ are involved in the right-hand side of the corresponding energy inequality, (3.3) is new and different since its right-hand side only includes $\|\nabla^{k+1}(n^+,n^-)\|^2_{L^2}$, and particularly excludes the term $\|\nabla^k(n^+,n^-)\|^2_{L^2}$. We remark that this new type of energy inequality (3.3) enables us to close our energy estimates at each kth level and plays an essential role in our analysis.

**Proof.** For $0 \leq k \leq \ell$, multiplying $\nabla^k(\Omega_1), \nabla^k(\Omega_2), \nabla^k(\Omega_3)$ and $\nabla^k(\Omega_4)$ by $\frac{\beta_1}{\beta_2} \nabla^k n^+, \frac{1}{\beta_2} \nabla^k u^+, \frac{\beta_1}{\beta_3} \nabla^k n^-$ and $\frac{1}{\beta_3} \nabla^k u^-$ respectively, summing up and then integrating the resultant equation over $\mathbb{R}^3$ by parts, we have

$$
\frac{1}{2} \frac{d}{dt} \left\{ \|\beta^+ \nabla^k n^+\|^2_{L^2} + \|\beta^- \nabla^k n^-\|^2_{L^2} + \frac{\sigma^+}{\beta_2} \|\nabla^{k+1} n^+\|^2_{L^2} \right\}
+ \frac{\sigma^-}{\beta_3} \|\nabla^{k+1} n^-\|^2_{L^2} + \frac{1}{\beta_2} \|\nabla^k u^+\|^2_{L^2} + \frac{1}{\beta_3} \|\nabla^k u^-\|^2_{L^2} + \frac{1}{\beta_2} \|\nabla^k u^+\|^2_{L^2} \right\}
+ \frac{1}{\beta_2} \left( v_1^+ \|\nabla^{k+1} u^+\|^2_{L^2} + v_2^+ \|\nabla^k \text{div} u^+\|^2_{L^2} \right) + \frac{1}{\beta_3} \left( v_1^- \|\nabla^{k+1} u^-\|^2_{L^2} + v_2^- \|\nabla^k \text{div} u^-\|^2_{L^2} \right)
\begin{aligned}
= & \left\langle \nabla^k F_1, \frac{\beta_1}{\beta_2} \nabla^k n^+ \right\rangle + \left\langle \nabla^k F_2, \frac{1}{\beta_2} \nabla^k u^+ \right\rangle + \left\langle \nabla^k F_3, \frac{\beta_1}{\beta_3} \nabla^k n^- \right\rangle + \left\langle \nabla^k F_4, \frac{1}{\beta_3} \nabla^k u^- \right\rangle \\
- & \left\langle \Delta^{k+1} n^+, \frac{\beta_1 \sigma^+}{\beta_2} \nabla^k F_1 \right\rangle - \left\langle \Delta^{k+1} n^-, \frac{\beta_1 \sigma^-}{\beta_3} \nabla^k F_3 \right\rangle - \left\langle \nabla^{k+1} n^-, \nabla^k u^- \right\rangle - \left\langle \nabla^{k+1} n^+, \nabla^k u^+ \right\rangle
\end{aligned}

:= I_1^k + I_2^k + I_3^k + I_4^k + I_5^k + I_6^k + I_7^k + I_8^k,

where $I_j^k, j = 1, 2, \cdots, 8$, denote the nonlinear terms in the above equation, which will be estimated as follows. Notice that the nonlinear source terms $F_k (k = 1, 2, 3, 4)$ possess the following equivalent properties:

$$
F_1 \sim n^+ \partial_k u_k^+ + u_k^+ \partial_k n^+, \quad (3.4)
$$

$$
F_2 \sim (n^+ + n^-) \partial_j n^+ + (u^+ \cdot \nabla) u_j^+ + \partial_j u_j^+ \partial_j n^+ + \partial_j u_j^+ \partial_j n^- + (n^+ + n^-) \partial_j n^- - \partial_j u_j^-, \quad (3.5)
$$

$$
F_3 \sim n^- \partial_k u_k^- + u_k^- \partial_k n^-, \quad (3.6)
$$

$$
F_4 \sim (n^+ + n^-) \partial_j n^- + (u^- \cdot \nabla) u_j^- + \partial_j u_j^- \partial_j n^+ + \partial_j u_j^- \partial_j n^- + (n^+ + n^-) \partial_j n^+. \quad (3.7)
$$

Firstly, for $I_1^k$, it follows from integration by parts, Lemma A.1 Lemma A.2 Young inequality and (3.1) that

$$
|I_1^k| \leq C \|\nabla^k (n^+ u^+)^+\|_{L^2} \|\nabla^{k+1} n^+\|_{L^2}
\leq C \left( \|\nabla^k n^+\|_{L^6} \|u^+\|_{L^3} + \|n^+\|_{L^3} \|\nabla^k u^+\|_{L^6} \right) \|\nabla^{k+1} n^+\|_{L^2}
\leq C \delta \left( \|\nabla^{k+1} n^+\|^2_{L^2} + \|\nabla^{k+1} u^+\|^2_{L^2} \right). \quad (3.8)
$$

Similarly, for $I_4^k$, we have

$$
|I_4^k| \leq C \delta \left( \|\nabla^{k+1} n^-\|_{L^2}^2 + \|\nabla^{k+1} u^-\|_{L^2}^2 \right). \quad (3.9)
$$
Noting (3.5) and employing similar arguments used in (3.8), one has

\[
|I^n_k| \leq C \left( \langle \nabla^k [(n^+ + n^-) \nabla n^+], \nabla^k u^+ \rangle + C \left( \langle \nabla^k [(u^+ \cdot \nabla) u^+], \nabla^k u^+ \rangle + C \langle \nabla^k (\nabla n^+ \cdot \nabla u^+), \nabla^k u^+ \rangle + C \langle \nabla^k (\nabla n^+ (\nabla u^+)^T), \nabla^k u^+ \rangle + C \langle \nabla^k (n^+ + n^-) \nabla n^-, \nabla^k u^+ \rangle \right) \right) \\
\leq C \|\nabla^k u^+\|_{L^2} \left( \|[(n^+ + n^-)]_t\|_{L^2} \|\nabla^{k+1} n^+\|_{L^2} + \|\nabla^{k+1} (n^+ + n^-)\|_{L^2} \right) + C \|\nabla^k u^+\|_{L^2} \left( \|u^+\|_{L^2} \|\nabla^{k+1} u^+\|_{L^2} + \|\nabla^{k+1} u^+\|_{L^2} \right) + C \|\nabla^k u^+\|_{L^2} \left( \|\nabla^{k+1} u^+\|_{L^2} \|\nabla n^-\|_{L^2} + \|\nabla^{k+1} n^-\|_{L^2} \right) + C \|\nabla^k u^+\|_{L^2} \left( \|\nabla^{k+1} n^-\|_{L^2} + \|\nabla^{k+1} u^+\|_{H^1} \right). \tag{3.10}
\]

Similarly, for \(I^n_4\), it holds that

\[
|I^n_4| \leq C \delta \left( \|\nabla^{k+1} (n^+, n^-)\|_{L^2}^2 + \|\nabla^k u^-\|_{H^1}^2 \right). \tag{3.11}
\]

For the term \(I^n_5\), it holds that

\[
|I^n_5| \leq C \left( \langle \nabla^k [n^+ \text{div} u^+ + u^+ \cdot \nabla n^+], \Delta \nabla^k n^+ \rangle \right) \\
\leq C \left( \|n^+\|_{L^2} \|\nabla^{k+1} u^+\|_{L^2} + \|u^+\|_{L^2} \|\nabla^{k+1} n^+\|_{L^2} \right) \|\nabla^{k+2} n^+\|_{L^2} \\
+ C \left( \|u^+\|_{L^2} \|\nabla^{k+1} n^+\|_{L^2} + \|n^+\|_{L^2} \|\nabla^{k+1} u^+\|_{L^2} \right) \|\nabla^{k+2} n^+\|_{L^2} \\
\leq C \delta \left( \|\nabla^{k+1} n^+\|_{H^1}^2 + \|\nabla^{k+1} u^+\|_{L^2}^2 \right). \tag{3.12}
\]

Similarly, for \(I^n_6\), we have

\[
|I^n_6| \leq C \delta \left( \|\nabla^{k+1} u^-\|_{H^1}^2 + \|\nabla^{k+1} u^+\|_{L^2}^2 \right). \tag{3.13}
\]

For \(I^n_7\) and \(I^n_8\), it follows from integration by parts, (2.1), and (2.1) that

\[
I^n_7 + I^n_8 = - \left( \langle \nabla^k n^-, \nabla^k u^+ \rangle - \langle \nabla^{k+1} n^-, \nabla^k u^- \rangle \right) \\
= \langle \nabla^k n^-, \nabla^k \text{div} u^+ \rangle + \langle \nabla^k n^-, \nabla^k \text{div} u^- \rangle \tag{3.14}
\]

Furthermore, similar to the proofs of \(I^n_2\) and \(I^n_5\), the last two terms on the right-hand side of (3.14) can be estimated as follows:

\[
\left| \langle \nabla^k n^-, \nabla^k \text{F}^1 \rangle \right| + \left| \langle \nabla^k n^+, \nabla^k \text{F}^3 \rangle \right| \leq C \delta \left( \|\nabla^{k+1} (n^+, n^-)\|_{L^2}^2 + \|\nabla^{k+1} (u^+, u^-)\|_{L^2}^2 \right). \tag{3.15}
\]

Finally, substituting (3.8)-(3.15) into (3.1) and using the smallness of \(\delta\), we get (3.2) and thus complete the proof of Lemma 3.1. \(\Box\)

Notice that (3.2) only involves the dissipative estimates of \(u^\pm\). Next, we deduce the dissipative estimates for \(n^\pm\), which is stated in the following lemma.
Lemma 3.3. Assume that the notations and hypotheses of Theorem 4.7 and (3.1) are in force. Then, for $0 \leq k \leq \ell$, it holds that

$$
\begin{align*}
\frac{d}{dt} \left\{ \langle \nabla^k u^+ \frac{1}{\beta_2} \nabla \nabla^k n^+ \rangle + \langle \nabla^k u^- \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle \right\} \\
+ \| \nabla^{k+1} (\beta^+ \nabla n^+ + \beta^- \nabla n^-) \|^2_{L^2} + \frac{3\sigma^+}{4\beta_2} \| \nabla^{k+2} n^+ \|^2_{L^2} + \frac{3\sigma^-}{4\beta_3} \| \nabla^{k+2} n^- \|^2_{L^2}
\leq C \left( \delta \| \nabla^{k+1} (n^+ \nabla n^-) \|^2_{L^2} + \| \nabla^{k+1} (u^+ \nabla u^-) \|^2_{L^2} \right),
\end{align*}
$$

(3.16)

for some constant $C > 0$ independent of $\delta$.

Proof. For $0 \leq k \leq \ell$, applying $\nabla^k$ to (2.1)2, (2.1)4 and then multiplying the resultant equations by $\frac{1}{\beta_2} \nabla^k \nabla n^+$ and $\frac{1}{\beta_3} \nabla^k \nabla n^-$ respectively, summing up and then integrating over $\mathbb{R}^3$, we obtain

$$
\begin{align*}
\frac{d}{dt} \left\{ \langle \nabla^k u^+ \frac{1}{\beta_2} \nabla \nabla^k n^+ \rangle + \langle \nabla^k u^- \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle \right\} \\
+ \| \nabla^{k+1} (\beta^+ \nabla n^+ + \beta^- \nabla n^-) \|^2_{L^2} + \frac{\sigma^+}{\beta_2} \| \nabla^{k+2} n^+ \|^2_{L^2} + \frac{\sigma^-}{\beta_3} \| \nabla^{k+2} n^- \|^2_{L^2}
= \langle \nabla^k u^+ \frac{1}{\beta_2} \partial_t \nabla \nabla^k n^+ \rangle + \langle \nabla^k \div u^+ \frac{1}{\beta_2} \nabla \nabla^k n^+ \rangle + \langle \nabla^k \div u^- \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle \\
+ \langle \nabla^k F_2 \frac{1}{\beta_2} \nabla \nabla^k n^+ \rangle + \langle \nabla^k u^- \frac{1}{\beta_3} \partial_t \nabla \nabla^k n^- \rangle + \langle \nabla^k u^- \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle \\
+ \langle \nabla^k \div u^- \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle + \langle \nabla^k F_4 \frac{1}{\beta_3} \nabla \nabla^k n^- \rangle
:= J_1^k + J_2^k + J_3^k + J_4^k + J_5^k + J_6^k + J_7^k + J_8^k.
\end{align*}
$$

(3.17)

We shall estimate each term in the right hand side of (3.17). First, it follows from integration by parts, (2.1)1 and (2.1)3 that

$$
\begin{align*}
J_1^k + J_3^k &= \langle \nabla^k u^+ \frac{1}{\beta_2} \partial_t \nabla \nabla^k n^+ \rangle + \langle \nabla^k u^- \frac{1}{\beta_3} \partial_t \nabla \nabla^k n^- \rangle \\
= \langle \nabla^k \div u^+ \frac{1}{\beta_2} \partial_t \nabla \nabla^k n^+ \rangle + \langle \nabla^k \div u^- \frac{1}{\beta_3} \partial_t \nabla \nabla^k n^- \rangle \\
= \frac{1}{\beta_2} \| \nabla^k \div u^+ \|^2_{L^2} + \frac{1}{\beta_2} \| \nabla^k \div u^- \|^2_{L^2}
- \left( \langle \nabla^k \div u^+ \frac{1}{\beta_2} \nabla F_1 \rangle - \langle \nabla^k \div u^- \frac{1}{\beta_3} \nabla F_3 \rangle \right).
\end{align*}
$$

(3.18)

Moreover, the last two terms in the right-hand-hand side of (3.18) can be bounded by

$$
\begin{align*}
\left| \langle \nabla^k \div u^+ \frac{1}{\beta_2} \nabla F_1 \rangle \right| &\leq C \left( \| n^+ \|_{L^\infty} \| \nabla^{k+1} u^+ \|_{L^2} + \| \nabla^k n^+ \|_{L^2} \| \nabla u^+ \|_{L^2} \right) \| \nabla^{k+1} u^+ \|_{L^2} \\
+ C \left( \| u^+ \|_{L^\infty} \| \nabla^{k+1} n^+ \|_{L^2} + \| \nabla^k u^+ \|_{L^2} \| \nabla n^+ \|_{L^2} \right) \| \nabla^{k+1} n^+ \|_{L^2}
\leq C \delta \left( \| \nabla^{k+1} n^+ \|_{L^2}^2 + \| \nabla^{k+1} u^+ \|_{L^2}^2 \right),
\end{align*}
$$

(3.19)

and similarly,

$$
\left| \langle \nabla^k \div u^- \frac{1}{\beta_3} \nabla F_3 \rangle \right| \leq C \delta \left( \| \nabla^{k+1} n^- \|_{L^2}^2 + \| \nabla^{k+1} u^- \|_{L^2}^2 \right). \tag{3.20}
$$
By virtue of integration by parts and Young inequality, we have
\[ |J_9^k| + |J_9^2| + |J_9^3| + |J_9| \leq \varepsilon \|\nabla^{k+2}(n^+, n^-)\|_{L^2}^2 + C_\varepsilon \|\nabla^{k+1}(u^+, u^-)\|_{L^2}^2, \]  \[(3.21)\]
where \(\varepsilon\) is a sufficiently small positive constant which will be determined later. Similar to the proof of \(I_2\), for \(J_9\), it holds that
\[ |J_9| \leq C \left( \|\nabla^{k+1}(n^+, n^-)\|_{L^2}^2 + \|\nabla^{k+1}(u^+, u^-)\|_{L^2}^2 + \|\nabla^{k+1}(n^+, n^-)\|_{L^2}^2 \right) \]  \[(3.22)\]

Finally, putting \((3.18)\)–\((3.23)\) into \((3.17)\), using the smallness of \(\delta\) and choosing \(\varepsilon\) small enough, we get \((3.16)\), and thus complete the proof of Lemma 3.3.

\[ |J_9| \leq C \delta \left( \|\nabla^{k+1}(n^+, n^-)\|_{H^k}^2 + \|\nabla^{k+1}u^-\|_{H^k}^2 \right). \]  \[(3.23)\]

4. Proof of Theorem 1.1

4.1. Proof of global existence and uniqueness. In this subsection, we shall show global existence and uniqueness of solutions stated in Theorem 1.1. By virtue of the classic local existence results in [19, 20] and the continuation in time of the local solution, we see that to prove the global existence result of Theorem 1.1, it suffices to close the a priori assumption \((3.1)\) and prove the energy estimate \((1.17)\). To begin with, we define the following two time-weighted energy functionals
\[ E_k^0(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{k}{2} + \frac{1}{2}} \|\nabla^{k}(\beta^{n^+} + \beta^{-n^-}, u^+, u^-)(\tau)\|_{H^{k-1}} + \|\nabla^{k+1}(n^+, n^-)(\tau)\|_{H^{k-1}} \right\}, \]  \[(4.1)\]
for \(0 \leq k \leq \ell\), and
\[ E_0(t) = \sup_{0 \leq \tau \leq t} \left\{ (1 + \tau)^{\frac{1}{2}} \|n^+, n^-(\tau)\|_{L^2}^2 \right\}. \]  \[(4.2)\]
Choosing a sufficiently large positive constant \(D_1\), and computing \((3.2)\) \& \((3.16)\), and then summing up the resultant inequality from \(k = 0\) to \(\ell\), we have from the smallness of \(\delta\) that
\[ \frac{d}{dt} E_0^0(t) + C (\|\nabla(u^+, u^-)(t)\|_{H^k}^2 + \|\nabla^2(n^+, n^-)(t)\|_{H^{k-1}}^2 + \|\nabla^2(n^+, n^-)(t)\|_{H^{k-1}}^2) \leq C \delta \left( \|\nabla(n^+, n^-)(t)\|_{L^2}^2 + \|(u^+, u^-)(t)\|_{L^2}^2 \right), \]  \[(4.3)\]
where
\[ E_0^0(t) = \frac{D_1}{2} \frac{d}{dt} \left\{ \|\nabla^l (\beta^{n^+} + \beta^{-n^-})\|_{H^{k-1}}^2 + \frac{\sigma^+}{\beta^2} \|\nabla n^+\|_{H^{k-1}}^2 + \frac{\sigma^-}{\beta^3} \|\nabla n^+\|_{H^{k-1}}^2 + \frac{1}{\beta^2} \|u^+\|_{H^{k-1}}^2 + \frac{1}{\beta^3} \|u^-\|_{H^{k-1}}^2 \right\}. \]
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\[ \frac{d}{dt} \mathcal{E}_0(t) + D_2 \mathcal{E}_0(t) \leq C \left( \| (\beta^+ n^+ + \beta^- n^-) \|_{L^2}^2 + \| \nabla (n^+ + n^-) \|_{L^2}^2 + \| (u^+, u^-) \|_{H^s}^2 \right), \quad (4.4) \]

where \( D_2 \) is a positive constant independent of \( \delta \).

Defining \( U = (n^+, u^+, n^-, u^-)^t \) and \( \mathcal{F} = (F^1, F^2, F^3, F^4)^t \), it follows from Duhamel’s principle that

\[ U(t) = e^{\mathcal{F}t} U(0) + \int_0^t e^{(t-\tau)\mathcal{F}} \mathcal{F}(\tau) d\tau, \quad (4.5) \]

which together with Plancherel theorem, integration by parts, Proposition 2.3, Lemma A.3, (4.1) and (4.2) implies

\[ \| (u^+, u^-) \|_{L^2} \leq C(1+ t)^{-\frac{3}{2}} \| U(0) \|_{L^1} + \int_0^t (1+ t - \tau)^{-\frac{3}{2}} \| (n^+ u^+, n^- u^-) \|_{L^2} d\tau \]

\[ + \int_0^t \| e^{(t-\tau)\mathcal{F}} (F_2, F_4)(\tau) \|_{L^2} d\tau \]

\[ \leq CK_0 (1+ t)^{-\frac{3}{2}} + \int_0^t (1+ t - \tau)^{-\frac{3}{2}} (1+ \tau)^{-1} E_0(t) E_0(t) d\tau \]

\[ + \int_0^t \| e^{(t-\tau)\mathcal{F}} (F_2, F_4)(\tau) \|_{L^2} d\tau \]

\[ \leq (1+ t)^{-\frac{3}{2}} \left( CK_0 (1+ t) E_0(t) E_0(t) \right) + \int_0^t \| e^{(t-\tau)\mathcal{F}} (F_2, F_4)(\tau) \|_{L^2} d\tau. \]

As mentioned before, the strongly coupling terms like \( g_+ (n^+, n^-) \partial n^+ + \tilde{g}_+ (n^+, n^-) \partial n^- \) in (2.3) and \( g_- (n^+, n^-) \partial n^- + \tilde{g}_- (n^+, n^-) \partial n^+ \) in (2.5) devote the slowest time–decay rates to the third term on the right–side of (4.6), which prevents us from deriving the desired decay rates of \( \| (u^+, u^-) \|_{L^2} \). To overcome this difficulty, the key idea here is to make full use of good properties of \( \beta^+ n^+ + \beta^- n^- \), and cleverly rewrite them as follows:

\[ g_+ (n^+, n^-) \partial n^+ + \tilde{g}_+ (n^+, n^-) \partial n^- \]

\[ = \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} \right) \partial n^+ + \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \beta_1 \partial n^- \]

\[ \leq \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \left( \beta_1 \partial n^+ + \beta_2 \partial n^- \right) \]

\[ = \partial_1 \left[ \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \left( \beta_1 n^+ + \beta_2 n^- \right) \right] \]

\[ - \partial_1 \left[ \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \left( \beta_1 n^+ + \beta_2 n^- \right) \right] \]

\[ - \partial_1 \left[ \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \left( \beta_1 n^+ + \beta_2 n^- \right) \right] \]

\[ = -\partial_1 G_1 \left[ \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \right] (\beta_1 n^+ + \beta_2 n^-) \]

\[ = -\partial_1 G_1 \left[ \frac{1}{\beta_1} \left( \frac{\rho^+ (n^+ + 1, n^- + 1)}{\beta_1} - \beta_1 \right) \right] (\beta_1 n^+ + \beta_2 n^-) \]
which together with the smallness of $\delta$
This together with Hölder inequality, Lemma A.1, (3.1), (4.1) and (4.2) implies

$$-\mathcal{G}^2(n^+ + 1, n^- + 1) \left( \frac{\bar{\rho}^+ \rho^-(n^+ + 1, n^- + 1)}{\bar{\rho}^+ \rho^-(n^+ + 1, n^- + 1)} - 1 \right) \partial_t n^-,$$

and

$$g_- (n^+, n^-) \partial_t n^- + \bar{g}_- (n^+, n^-) \partial_t n^+$$

$$= \left( \mathcal{G}^2 \rho^+(n^+ + 1, n^- + 1) - \beta_d \right) \partial_t n^- + \left( \mathcal{G}^2 (n^+ + 1, n^- + 1) - \beta_1 \right) \partial_t n^+$$

$$= \frac{1}{\beta_d} \left( \mathcal{G}^2 \rho^+(n^+ + 1, n^- + 1) - \beta_d \right) \left( \beta_3 \partial_t n^+ + \beta_4 \partial_t n^- \right)$$

$$-\mathcal{G}^2(n^+ + 1, n^- + 1) \left( \frac{\bar{\rho}^- \rho^+(n^+ + 1, n^- + 1)}{\bar{\rho}^- \rho^+(n^+ + 1, n^- + 1)} - 1 \right) \partial_t n^+$$

$$= \partial_t \left[ \frac{1}{\beta_d} \left( \mathcal{G}^2 \rho^+(n^+ + 1, n^- + 1) - \beta_d \right) \left( \beta_3 n^+ + \beta_4 n^- \right) \right] (4.8)$$

On the other hand, by noticing the pressure differential $dP$, we have

$$\nabla P = \mathcal{G}^2(n^+ + 1, n^- + 1) \left[ \rho^- (n^+ + 1, n^- + 1) \nabla n^+ + \rho^+ (n^+ + 1, n^- + 1) \nabla n^- \right]$$

$$= \mathcal{G}^2(n^+ + 1, n^- + 1) \left[ \rho^- (n^+ + 1, n^- + 1) - \bar{\rho}^- \right] \nabla n^+$$

$$+ \mathcal{G}^2(n^+ + 1, n^- + 1) \left[ \rho^+ (n^+ + 1, n^- + 1) - \bar{\rho}^+ \right] \nabla n^-$$

$$+ \left[ \mathcal{G}^2(n^+ + 1, n^- + 1) - \mathcal{G}^2(1, 1) \right] \left[ \bar{\rho}^- \nabla n^+ + \rho^+ \nabla n^- \right]$$

$$+ \mathcal{G}^2(1, 1) \left( \bar{\rho}^- \nabla n^+ + \rho^+ \nabla n^- \right).$$

This together with Hölder inequality, Lemma A.1, (3.1), (4.1) and (4.2) implies

$$\| \nabla P \|_{L^2} \leq C \left( \| \nabla (n^+, n^-) \|_{L^2} \left( \| \rho^+ - \bar{\rho}^+ \|_{L^6} + \| \rho^- - \bar{\rho}^- \|_{L^6} \right) + \| \bar{\rho}^- \nabla n^+ + \rho^+ \nabla n^- \|_{L^2} \right)$$

$$\leq C \left( \delta \| \nabla P \|_{L^2} \left( \| \nabla (n^+, n^-) \|_{H^1} \| \nabla P \|_{L^2} + (1 + t)^{-\frac{1}{2}} E_0(t) \right) \right)$$

$$\leq C \left( \delta \| \nabla P \|_{L^2} + (1 + t)^{-\frac{1}{2}} E_0(t) \right),$$

which together with the smallness of $\delta$ gives

$$\| \nabla (P, \rho^+, \rho^-) \|_{L^2} \leq C (1 + t)^{-\frac{1}{2}} E_0(t). (4.10)$$
Therefore, denoting \( \tilde{p} = P(1, 1) \) by the equilibrium state of pressure \( P \), we have from embedding estimate of Riesz potential that

\[
\| P - \tilde{p} \|_{L^2} \\
\lesssim \| \Lambda^{-1} \nabla P \|_{L^2} \\
\lesssim (\| \mathcal{E}^2(\rho - \tilde{\rho}) \nabla n^+ \|_{L^\infty} + \| \mathcal{E}^2(\rho^+ - \tilde{\rho}^+) \nabla n^- \|_{L^\infty} \\
+ \| (\mathcal{E}^2 - \mathcal{E}^2(1, 1)) (\tilde{\rho} - \nabla n^+ + \tilde{\rho}^+ \nabla n^-) \|_{L^\infty} \\
+ (\| \mathcal{E}^2(1, 1)(\tilde{\rho} - n^+ + \tilde{\rho}^+ n^-) \|_{L^2} \\
\lesssim \| (\rho^+ - \tilde{\rho}^+ + \tilde{\rho}^-) \|_{L^2} \| \nabla n^+ \|_{L^2} + \| (\rho^- - \tilde{\rho}^-) \|_{L^2} \| \nabla n^- \|_{L^2} \\
+ \| (\mathcal{E}^2 - \mathcal{E}^2(1, 1)) \|_{L^2} \| \rho^- \nabla n^+ + \rho^+ \nabla n^- \|_{L^2} + \| \rho^+ n^+ + \rho^- n^- \|_{L^2} \\
\lesssim (1 + t)^{-\frac{\delta}{2}} E_0^0(t),
\]

which particularly gives

\[
\| (\rho^+ - \tilde{\rho}^+, \rho^- - \tilde{\rho}^-) \|_{L^2} \lesssim (1 + t)^{-\frac{\delta}{2}} E_0^0(t).
\]

Consequently, combining (4.11), (4.7), (4.8), (4.12) and using the fact that \( \frac{\delta}{\tilde{\rho}^- \tilde{\rho}^+} - 1 \sim \rho^+ - \tilde{\rho}^+ + \rho^- - \tilde{\rho}^- \), we conclude that

\[
\int_0^t \| e^{(t - \tau)\mathcal{M}_2} (F_2, F_4) (\tau) \|_{L^2} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} \| (G_2, G_4)(\tau) \|_{L^2} d\tau + C \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} \| (F_2 - \text{div} G_2, F_4 - \text{div} G_4)(\tau) \|_{L^2} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} \| (n^+, n^-)(\tau) \|_{L^2} \| (\beta^+ n^+ + \beta^- n^-)(\tau) \|_{L^2} d\tau \\
+ C \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} \| (n^+, n^-)(\tau) \|_{L^2} \| (\nabla^2(n^+, n^-)(\tau)) \|_{L^2} \| (\nabla(n^+, n^-)(\tau)) \|_{L^2} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-\frac{\delta}{2}} \| (1 + t - \tau)^{-\frac{\delta}{2}} (E_0(t)) \|_{L^2} d\tau + (1 + t - \tau)^{-\frac{\delta}{2}} \| (E_0(t)) \|^2 \leq C(1 + t)^{-\frac{\delta}{2}} \left[ K_0 + E_0(t) E_0^0(t) + \left( E_0^0(t) \right)^2 \right],
\]

where \( G_2 = (G_2^1, G_2^2, G_2^3) \) and \( G_4 = (G_4^1, G_4^2, G_4^3) \). Substituting (4.13) into (4.6) yields

\[
\| (u^{+, \bot}) \|_{L^2} \lesssim (1 + t)^{-\frac{\delta}{2}} \left[ K_0 + E_0(t) E_0^0(t) + \left( E_0^0(t) \right)^2 \right].
\]

Similarly, we also have

\[
\| (\beta^+ n^{+, \bot} + \beta^- n^{-, \bot})(t) \|_{L^2} + \| (n^{+, \bot}, n^{-, \bot})(t) \|_{L^2} \lesssim (1 + t)^{-\frac{\delta}{2}} \left[ K_0 + E_0(t) E_0^0(t) + \left( E_0^0(t) \right)^2 \right],
\]

and

\[
\| (n^{+, \bot}, n^{-, \bot})(t) \|_{L^2} \lesssim (1 + t)^{-\frac{\delta}{2}} \left[ K_0 + E_0(t) E_0^0(t) + \left( E_0^0(t) \right)^2 \right].
\]

Substituting (4.14)–(4.16) into (4.4) yields

\[
\frac{d}{dt} \phi_0^0(t) + D_2 \phi_0^0(t) \leq C(1 + t)^{-\frac{\delta}{2}} \left[ K_0 + E_0(t) E_0^0(t) + \left( E_0^0(t) \right)^2 \right].
\]
Applying Gronwall’s inequality to the above inequality, we can infer that

\[ \mathcal{A}_\ell(t) \leq e^{-Dt} \mathcal{A}_\ell(0) + C \int_0^t e^{-D(t-\tau)} (1 + \tau)^{-\frac{1}{2}} \left[ K_0 + E_0(t)E_0(t) + \left( E_0(t) \right)^2 \right] d\tau \]

\[ \leq C(1 + t)^{-\frac{1}{2}} \left[ K_0 + E_0(t)E_0(t) + \left( E_0(t) \right)^2 \right], \]

which together with (4.1) implies that

\[ E_0(t) \leq C \left[ K_0 + E_0(t)E_0(t) + \left( E_0(t) \right)^2 \right]. \quad (4.17) \]

Next, we deal with \( E_0(t) \). By virtue of Lemma [A.4] (4.1), (4.16) and (4.17), we have

\[ \|(n^+, n^-)\|_{L^2} \leq C \left( \|(n^{+, j}, n^{-, j})\|_{L^2} + \|(n^{+, h}, n^{-, h})\|_{L^2} \right) \]

\[ \leq C \left( \|(n^{+, j}, n^{-, j})\|_{L^2} + \|\nabla(n^+, n^-)\|_{L^2} \right) \]

\[ \leq C(1 + t)^{-\frac{1}{2}} \left[ K_0 + E_0(t)E_0(t) + \left( E_0(t) \right)^2 \right], \]

which leads to

\[ E_0(t) \leq C \left[ K_0 + E_0(t)E_0(t) + \left( E_0(t) \right)^2 \right]. \quad (4.18) \]

Finally, combining (4.17) with (4.18) and using (1.16), we conclude that

\[ E_0(t) + E_0(t) \leq C K_0. \quad (4.19) \]

By a standard continuity argument, this closes the a priori estimates (3.1) immediately since \( \mathcal{A}_0 \) is sufficiently small. This in turn allows us to integrate (3.4) directly in time, to obtain

\[ \|\nabla(n^+, n^-)(t)\|_{H^1}^2 + \|(u^+, u^-)(t)\|_{H^1}^2 + \|\beta^+ n^+ + \beta^- n^-)(t)\|_{H^1}^2 \]

\[ + \int_0^t \left( \|\nabla(n^+, n^-)(\tau)\|_{H^1}^2 + \|(u^+, u^-)(\tau)\|_{H^1}^2 + \|\beta^+ n^+ + \beta^- n^-)(\tau)\|_{H^1}^2 \right) d\tau \leq C K_0^2, \]

which together with (4.19) implies (1.17) immediately. Therefore, we have completed the global existence result of Theorem 1.1.

4.2. **Proof of upper bounds on decay rates.** In this subsection, we devote ourselves to proving the upper optimal convergence rate of the solution stated in (1.18)–(1.21) of Theorem 1.1. We first show (1.19)–(1.21). Noticing (4.1) and (4.19), it suffices to prove that \( E_j(t) \leq C K_0 \), for any \( 1 \leq j \leq \ell \). We will make full use of the low–frequency and high–frequency decomposition, and employ key linear convergence estimates obtained in Section 2 and uniform nonlinear energy estimates obtained in Section 3 to achieve this goal by induction.

**Theorem 4.1.** Assume that the hypotheses of Theorem 1.1 and (1.20) are in force. Then there exists a positive constant \( C \) independent of \( t \), such that

\[ E_j(t) \leq C K_0, \]

for \( 1 \leq j \leq \ell \).

**Proof.** We will employ mathematical inductive method to prove Theorem 4.1. Therefore, by noticing (4.1) and (4.19), it suffices to prove the following Lemma 4.2. Thus, the proof Theorem 4.1 is completed. \( \square \)
Lemma 4.2. Assume that the hypotheses of Theorem 4.1 and (4.20). If additionally
\[ E_{k-1}^\ell(t) \leq CK_0, \tag{4.20} \]
then it holds that
\[ E_k^\ell(t) \leq CK_0, \tag{4.21} \]
for \(1 \leq k \leq \ell\).

Proof. We will combine the key linear estimates with delicate nonlinear energy estimates based on good properties of the low–frequency and high–frequency decomposition to prove Theorem 4.1 and the process involves the following three steps.

Step 1. Energy estimates on \(\|\nabla^j(\beta_t n^+ + \beta^- n^-)\|_{H_{k=\ell}}^2 + \|\nabla^{j+1}(n^+, n^-)\|_{H_{k=\ell}}^2 + \|\nabla^j(u^+, u^-)\|_{H_{k=\ell}}^2\). Choosing a sufficiently large positive constant \(D_3\), and computing \(2 \times D_3 + (3.16)\), and then summing up the resultant inequality from \(k = j\) to \(\ell\), we have from the smallness of \(\delta\) that
\[
\frac{d}{dt} \delta_j^\ell(t) + C \left( \|\nabla^{j+1}(u^+, u^-)(t)\|_{H_{k=\ell}}^2 + \|\nabla^{j+2}(n^+, n^-)(t)\|_{H_{k=\ell}}^2 + \|\nabla^{j+1}(\beta_t n^+ + \beta^- n^-)(t)\|_{H_{k=\ell}}^2 \right)
\leq C\delta \left( \|\nabla^{j+1}(n^+, n^-)(t)\|_{L^2}^2 + \|\nabla^j(u^+, u^-)(t)\|_{L^2}^2 \right),
\tag{4.22}
\]
where
\[
\delta_j^\ell = \frac{D_3}{2} \frac{d}{dt} \left\{ \|\nabla^j(\beta_t n^+ + \beta^- n^-)\|_{H_{k=\ell}}^2 + \frac{\sigma^+}{p_2} \|\nabla^{j+1}n^+\|_{H_{k=\ell}}^2 + \frac{\sigma^-}{p_3} \|\nabla^{j+1}n^-\|_{H_{k=\ell}}^2 \right. \\
+ \frac{1}{p_2} \|\nabla^j u^+\|_{H_{k=\ell}}^2 + \frac{1}{p_3} \|\nabla^j u^-\|_{H_{k=\ell}}^2 \left. \right\} + \sum_{k=j}^{\ell} \left\{ \left\langle \nabla^{k} u^+, \frac{1}{p_2} \nabla^k n^+ \right\rangle + \left\langle \nabla^{k} u^-, \frac{1}{p_3} \nabla^k n^- \right\rangle \right\},
\]
which is equivalent to \(\|\nabla^j(\beta_t n^+ + \beta^- n^-)(t)\|_{H_{k=\ell}}^2 + \|\nabla^{j+1}(n^+, n^-)(t)\|_{H_{k=\ell}}^2 \leq \|\nabla^j(u^+, u^-)(t)\|_{H_{k=\ell}}^2\), since \(D_3\) is large enough. Then, (4.22) together with Lemma [4.4] yields
\[
\frac{d}{dt} \delta_j^\ell(t) + D_4 \delta_j^\ell(t) \leq C \left( \|\nabla^j(\beta_t n^+ + \beta^- n^-)\|_{H_{k=\ell}}^2 + \|\nabla^{j+1}(n^+, n^-)\|_{H_{k=\ell}}^2 + \|\nabla^j(u^+, u^-)\|_{H_{k=\ell}}^2 \right),
\tag{4.23}
\]
where \(D_4\) is a positive constant independent of \(\delta\).

Step 2. Decay estimates on \(\|\nabla^j(\beta_t n^+ + \beta^- n^-)\|_{H_{k=\ell}}^2 + \|\nabla^{j+1}(n^+, n^-)\|_{H_{k=\ell}}^2 + \|\nabla^j(u^+, u^-)\|_{H_{k=\ell}}^2\). To begin with, we deal with \(\|\nabla^j(u^+, u^-)\|_{H_{k=\ell}}^2\). To do this, using (4.1) and the inequality above (4.10), we have
\[
\|\nabla^j P\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}} E_j^\ell(t).
\tag{4.24}
\]
For \(2 \leq j \leq \ell\), by employing (4.9), Hölder inequality, Lemma [4.2] (4.1), (4.19) and (4.20), we have
\[
\|\nabla^j P\|_{L^2} \leq \|\nabla^{j-1}(\epsilon^2 (\rho^- - \rho^+) \nabla n^+)\|_{L^2} + \|\nabla^{j-1}(\epsilon^2 (\rho^+ - \rho^-) \nabla n^-)\|_{L^2} \\
+ \|\nabla^{j-1}(\epsilon^2 (\rho^+ \nabla n^+ + \rho^- \nabla n^-))\|_{L^2} \\
\leq \left( \|\nabla^{j-1}(\rho^+ + \rho^-)\|_{L^5} \epsilon^2 \|\nabla^j n^+\|_{L^2} + \|\rho^+ - \rho^-\| \|\nabla^{j-1} \epsilon^2 \|_{L^6} \|\nabla(n^+, n^-)\|_{L^2} \\
+ \|\nabla^j \rho^- \|_{L^2} \|\nabla^j(n^+, n^-)\|_{L^2} \right) + \left( \|\rho^+ \|_{L^2} \|\nabla^j(n^+, n^-)\|_{L^2} \right) \\
\leq (K_0 \|\nabla^j P\|_{L^2} + \|\rho^+ - \rho^-\| \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \\
+ \|\rho^- \nabla n^+ + \nabla^j(n^+, n^-)\|_{L^2} + \|\nabla^j \rho^+ \|_{L^2} \|\nabla(n^+, n^-)\|_{H^2}) \\
+ \left( \frac{1}{p_2} \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \right) \\
\approx (K_0 \|\nabla^j P\|_{L^2} + \|\rho^+ - \rho^-\| \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \\
+ \|\rho^- \nabla n^+ + \nabla^j(n^+, n^-)\|_{L^2} + \|\nabla^j \rho^+ \|_{L^2} \|\nabla(n^+, n^-)\|_{H^2}) \\
+ \left( \frac{1}{p_2} \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \right) \\
\approx (K_0 \|\nabla^j P\|_{L^2} + \|\rho^+ - \rho^-\| \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \\
+ \|\rho^- \nabla n^+ + \nabla^j(n^+, n^-)\|_{L^2} + \|\nabla^j \rho^+ \|_{L^2} \|\nabla(n^+, n^-)\|_{H^2}) \\
+ \left( \frac{1}{p_2} \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \right) \\
\approx (K_0 \|\nabla^j P\|_{L^2} + \|\rho^+ - \rho^-\| \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \\
+ \|\rho^- \nabla n^+ + \nabla^j(n^+, n^-)\|_{L^2} + \|\nabla^j \rho^+ \|_{L^2} \|\nabla(n^+, n^-)\|_{H^2}) \\
+ \left( \frac{1}{p_2} \|\nabla^j(n^+, n^-)\|_{H^2} \|\nabla(n^+, n^-)\|_{H^2} \right),
\]
\[ \leq K_0 \left\| \nabla^j P \right\|_{L^2} + K_0 (1 + t)^{-1 - \frac{j}{2} + (1 + t)^{-\frac{3}{2} - \frac{j}{4}} E_j^f(t), \]

which together with the smallness of \( K_0 \) implies

\[ \left\| \nabla^j (P, \rho^+, \rho^-) \right\|_{L^2} \leq C(1 + t)^{-\frac{j}{2} - \frac{3}{2}} \left[ K_0 + E_j^f(t) \right]. \quad (4.25) \]

With (4.24) and (4.25) in hand, we can borrow similar arguments used in (4.6) to get

\[ \left\| \nabla^j (u^+, u^-) \right\|_{L^2} \leq K_0 (1 + t)^{-\frac{j}{2} - \frac{3}{2}} + \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2} - \frac{j}{4}} \left\| (n^+, n^-) \right\|_{L^2} \left\| (u^+, u^-) \right\|_{L^2} d\tau 
\]
\[ + \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2}} \left\| (G_2, G_4) \right\|_{L^2} \left\| (\nabla^j (G_2, G_4)) \right\|_{L^2} d\tau 
\]
\[ + \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2}} \left\| (F_2 - \text{div} G_2, F_4 - \text{div} G_4) \right\|_{L^2} \left\| (\nabla^j (F_2 - \text{div} G_2, F_4 - \text{div} G_4)) \right\|_{L^2} d\tau 
\]
\[ := K_0 (1 + t)^{-\frac{j}{2} - \frac{3}{2} + K_1^j + K_2^j + K_3^j + K_4^j + K_5^j + K_6^j. \]

We shall estimate \( K_1^j \) with \( i = 1, 2, \ldots, 6 \). Firstly, it follows from Hölder inequality and (4.19) that

\[ \left| K_1^j \right| \leq \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2} - \frac{j}{4}} \left\| (n^+, n^-) \right\|_{L^2} \left\| (u^+, u^-) \right\|_{L^2} d\tau \]
\[ \leq K_0^2 \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2} - \frac{j}{4}} (1 + \tau)^{-1} d\tau \]
\[ \leq K_0^2 (1 + t)^{-\frac{j}{2} - \frac{3}{2}}. \quad (4.27) \]

Next, for \( K_2^j \), by using Lemma A.1, Lemma A.2, (4.1), (4.19), (4.20), we have

\[ \left| K_2^j \right| \leq \int_0^t (1 + t - \tau)^{-\frac{j}{2}} \left\| \nabla^j (n^+, n^-) \right\|_{L^2} d\tau \]
\[ \leq \int_0^t (1 + t - \tau)^{-\frac{j}{2}} \left( \left\| \nabla^j (n^+, n^-) \right\|_{L^2} \left\| (u^+, u^-) \right\|_{L^2} + \left\| (n^+, n^-) \right\|_{L^2} \left\| \nabla^j (u^+, u^-) \right\|_{L^2} \right) d\tau \]
\[ \leq K_0 \int_0^t (1 + t - \tau)^{-\frac{j}{2} - \frac{3}{2} - \frac{j}{4}} (1 + \tau)^{-1} \left( 1 + E_j^f(t) \right) d\tau \]
\[ \leq K_0 (1 + t)^{-1 - \frac{j}{2}} \left( 1 + E_j^f(t) \right). \quad (4.28) \]

Similar to the proofs of (4.27) and (4.28), for \( K_3^j \) and \( K_4^j \), it holds that

\[ \left| K_3^j \right| + \left| K_4^j \right| \leq K_0^2 (1 + t)^{-\frac{j}{2} - \frac{3}{2}} + K_0 (1 + t)^{-1 - \frac{j}{2}} \left( 1 + E_j^f(t) \right). \quad (4.29) \]

Employing similar arguments in (4.13), we have

\[ \left| K_5^j \right| \leq K_0^2 (1 + t)^{-\frac{j}{2} - \frac{3}{2}}, \quad (4.30) \]

where we have used (4.12) and (4.19). The last term \( K_6^j \) is much more complicated. The main idea of our approach is to make full use of the benefit of the low-frequency and high-frequency decomposition. To
see this, by virtue of (4.7), (4.8), (4.19), (4.20), (4.25), Lemma A.1, Lemma A.2, Lemma A.3 and Lemma A.4 we can bound the term $\left\| \xi \left( F_2^j - \text{div} G_2^j, F_4^j - \text{div} G_4^j \right) (\tau) \right\|_{L^\infty}$ by

$$
\leq \left\| \nabla^{-j} \left( u^+ \cdot \nabla u^+, u^- \cdot \nabla u^-, (\nabla n^+ + \nabla n^-) \nabla u^+, (\nabla n^+ + \nabla n^-) \nabla u^+ \right) (t) \right\|_{L^1} + \left\| \nabla^{-j} \left( (\rho^+ - \bar{\rho}^+ + \rho^- - \bar{\rho}^-) \nabla n^+, (\rho^+ - \bar{\rho}^+ + \rho^- - \bar{\rho}^-) \nabla n^- \right) (t) \right\|_{L^1} + \left\| \nabla^{\max(0, j-2)} \left( (\rho^+ - \bar{\rho}^+ + \rho^- - \bar{\rho}^-) \nabla^2 u^+, (\rho^+ - \bar{\rho}^+ + \rho^- - \bar{\rho}^-) \nabla^2 u^- \right) (t) \right\|_{L^1} 
\leq \left\| (u^+, u^-) (t) \right\|_{L^2} \left\| \nabla^j (u^+, u^-) (t) \right\|_{L^2} + \left\| \nabla (u^+, u^-) (t) \right\|_{L^2} \left\| \nabla^{j-1} (u^+, u^-) (t) \right\|_{L^2} \left( 4.31 \right)
$$

which implies

$$
\left| K_j^0 \right| \leq K_0^2 (1 + t)^{-\frac{3}{2} + \frac{1}{2}} + K_0 (1 + t)^{-\frac{1}{2} - \frac{1}{2}} E_j^0 (t). \quad (4.32)
$$

Substituting (4.27)–(4.30) and (4.32) into (4.26) gives

$$
\left\| \nabla^j (u^{+,,}, u^{-,}) \right\|_{L^2} \leq K_0^0 (1 + t)^{-\frac{j}{2} - \frac{1}{2}} + K_0 (1 + t)^{-\frac{j}{2} - \frac{1}{2}} \left( 1 + E_j^0 (t) \right). \quad (4.33)
$$

Similarly, we also have

$$
\left\| \nabla^j (\beta^+ n^{+,,} + \beta^- n^{-,}) (\tau) \right\|_{L^2} + \left\| \nabla^{j+1} (n^{+,,}, n^{-,}) (\tau) \right\|_{L^2} 
\leq K_0^2 (1 + t)^{-\frac{1}{2} - \frac{1}{2}} + K_0 (1 + t)^{-\frac{1}{2} - \frac{1}{2}} \left( 1 + E_j^0 (t) \right). \quad (4.34)
$$

Putting (4.33)–(4.34) into (4.24) and then integrating the resultant inequality over $[0, t]$, we have

$$
\delta_j^0 (t) \leq e^{-D_0 t} \delta_j^0 (0) + C \int_{0}^{t} e^{-D_3 (t - \tau)} (1 + \tau)^{-\frac{3}{2} - j} \left[ K_0^2 + K_0 \left( 1 + E_j^0 (t) \right) \right]^2 d\tau 
\leq CK_0^2 (1 + t)^{-\frac{3}{2} - j} \left( 1 + E_j^0 (t) \right)^2,
$$

which together with the smallness of $K_0$ gives

$$
E_j^0 (t) \leq CK_0. \quad (4.35)
$$

This gives (1.19)–(1.21) directly. Finally, (1.18) follows from (4.25) immediately.

Therefore, we complete the proof of the upper bounds on decay rates stated in Theorem 1.1. $\square$
4.3. Proof of lower bounds on decay rates. In this subsection, we shall prove the lower bounds on decay rates in (4.23)-(4.26), and thus complete the proof of Theorem 1.1.

Theorem 4.3. Assume that the hypotheses of Theorem 1.7 and (1.22) are in force. Then there is a positive constant $c_1$ independent of $t$ such that for any large enough $t$, it holds that

$$
\min \left\{ \| \nabla^k \left( \rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^- \right) \|_{L^2} \right\} 
\geq c_1 (1 + t)^{-\frac{3}{2} - \frac{k}{2}},
$$

for $0 \leq k \leq \ell$.

$$
\min \left\{ \| \nabla^k (u^+, u^-) \|_{L^2} + \| \nabla^k (\beta^+ n^+ + \beta^- n^-) \|_{L^2} \right\}
\geq c_1 (1 + t)^{-\frac{3}{2} - \frac{k}{2}},
$$

for $0 \leq k \leq \ell$, and

$$
\min \left\{ \| \nabla^k (n^+, n^-) \|_{L^2} \right\} \geq c_1 (1 + t)^{-\frac{3}{2} - \frac{k}{2}},
$$

for $0 \leq k \leq \ell + 1$.

Proof. If $t$ is large enough, it follows from (4.3), Proposition 2.3 (1.18)-(1.21), Lemma A.4 that

$$
\| \Lambda^{-1} (u^+, u^-, \beta^+ n^+ + \beta^- n^-) \|_{L^2} 
\leq \| \Lambda^{-1} (u^{+, 1}, u^{-, 1}, \beta^+ n^{+, 1} + \beta^- n^{-, 1}) \|_{L^2} + \| \Lambda^{-1} (u^{+, h}, u^{-, h}, \beta^+ n^{+, h} + \beta^- n^{-, h}) \|_{L^2}
\leq CK_0 (1 + t)^{-\frac{3}{4} + \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (F^1, F^3)(\tau) \|_{L^2} d\tau + \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (G_2, G_4)(\tau) \|_{L^2} d\tau + C \| (u^+, u^-, \beta^+ n^+ + \beta^- n^-) \|_{L^2}
\leq CK_0 \left( (1 + t)^{-\frac{3}{4} + \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-\frac{3}{4}} d\tau + \int_0^t (1 + t - \tau)^{-\frac{3}{4}} (1 + \tau)^{-1} d\tau \right)
\leq CK_0 (1 + t)^{-\frac{3}{4}},
$$

and

$$
\min \left\{ \| (u^+, u^-, \beta^+ n^+ + \beta^- n^-) \|_{L^2} \right\}
\geq \min \left\{ \| (u^{+, 1}, u^{-, 1}, \beta^+ n^{+, 1} + \beta^- n^{-, 1}) \|_{L^2} \right\}
\geq C_1 K_0^2 (1 + t)^{-\frac{3}{4} - \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (F^1, F^3)(\tau) \|_{L^2} d\tau - \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (G_2, G_4)(\tau) \|_{L^2} d\tau
- C \int_0^t (1 + t - \tau)^{-\frac{3}{4}} \| (F_2 - \text{div} G_2, F_4 - \text{div} G_4)(\tau) \|_{L^2} d\tau
\geq C_1 K_0^2 (1 + t)^{-\frac{3}{4} - CK_0^2 (1 + t)^{-\frac{3}{4}}}
\geq c_2 (1 + t)^{-\frac{3}{4}},
$$

since $\theta < 2$ and $K_0$ is sufficiently small. These together with the interpolation inequality

$$
\| f \|_{L^2} \leq C \| \Lambda^{-1} f \|_{L^2}^{\frac{1}{2}} \| \nabla^k f \|_{L^2}^{\frac{1}{2}}
$$

imply (4.37) immediately. Similarly, we can prove (4.38). Here, we omit the details for simplicity. Finally, we turn to prove (4.36). To begin with, by noting (4.9) and employing similar arguments in (4.11), we...
have
\[ \| P - \bar{P} \|_{L^2} \geq \| \Lambda^{-1} \nabla P \|_{L^2} \geq (\| \mathcal{E}^2(1,1)(\bar{\rho}^+ n^+ + \bar{\rho}^- n^-) \|_{L^2} - \| (\mathcal{E}^2(\rho^- - \bar{\rho}^-)\nabla n^+) \|_{L^p} + \| \mathcal{E}^2(\rho^+ - \bar{\rho}^+)\nabla n^- \|_{L^q}) \geq K_0(1 + t)^{1/4}\]
which leads to
\[ \| (\rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^-)(t) \|_{L^2} \geq K_0(1 + t)^{-\frac{1}{4}}. \tag{4.39} \]
Similarly, we have
\[ \| \nabla (\rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^-)(t) \|_{L^2} \geq K_0(1 + t)^{-\frac{1}{4}}. \tag{4.40} \]
For \(2 \leq k \leq \ell\), by employing an interpolation technique, (1.18) and (4.40), we obtain
\[ \| \nabla^k (\rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^-)(t) \|_{L^2} \geq C \| \nabla (\rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^-)(t) \|_{L^2}^{(1-k)} \| (\rho^+ - \bar{\rho}^+, \rho^- - \bar{\rho}^-)(t) \|_{L^2}^{(k-1)} \geq C_3(1 + t)^{-\frac{3}{4} + \frac{1}{4}}. \tag{4.41} \]
Consequently, (4.36) follows from (4.39)–(4.41) immediately, and thus the proof of Theorem 4.3 is completed.

Therefore, we have completed the proof of Theorem 1.1. \( \square \)

APPENDIX A. ANALYTIC TOOLS

We recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

**Lemma A.1.** Let \(0 \leq i, j \leq k\), then we have
\[ \| \nabla^a f \|_{L^p} \leq \| \nabla^i f \|_{L^q}^{1-a} \| \nabla^j f \|_{L^r}^a \]
where \(a\) satisfies
\[ \frac{i}{3} - \frac{1}{p} = \left( \frac{j}{3} - \frac{1}{q} \right)(1-a) + \left( \frac{k}{3} - \frac{1}{r} \right)a. \]
Especially, while \(p = q = r = 2\), we have
\[ \| \nabla^a f \|_{L^2} \leq \| \nabla^i f \|_{L^2}^{\frac{k-i}{2}} \| \nabla^j f \|_{L^2}^{\frac{k-j}{2}}. \]

**Proof.** This is a special case of [21], pp. 125, THEOREM. \( \square \)

Next, to estimate the \(L^p\)-norm of the spatial derivatives of the product of two functions, we shall recall the following estimate:
Lemma A.2. For any integer $k \geq 1$, we have
\[ \left\| \nabla^k (fg) \right\|_{L^p} \leq \left\| f \right\|_{L^{p_1}} \left\| \nabla^k g \right\|_{L^{p_2}} + \left\| \nabla^k f \right\|_{L^{p_3}} \left\| g \right\|_{L^{p_4}}, \]
and
\[ \left\| \nabla^k (fg) - f \nabla^k g \right\|_{L^p} \leq \left\| \nabla f \right\|_{L^{p_1}} \left\| \nabla^{k-1} g \right\|_{L^{p_2}} + \left\| \nabla^k f \right\|_{L^{p_3}} \left\| g \right\|_{L^{p_4}}, \]
where $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ and
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]

Proof. See [16]. \hfill \Box

Finally, the following two lemmas concern the estimate for the low–frequency part and the high–frequency part of $f$.

Lemma A.3. If $f \in L^r(\mathbb{R}^3)$ for any $2 \leq r \leq \infty$, then we have
\[ \left\| f^{l} \right\|_{L^r} + \left\| f^{h} \right\|_{L^r} \leq \left\| f \right\|_{L^r}. \]

Proof. For $2 \leq r \leq \infty$, by virtue of Young’s inequality for convolutions, for the low frequency, it holds that
\[ \left\| f^{l} \right\|_{L^r} \leq \left\| \phi^{-1} \right\|_{L^1} \left\| f \right\|_{L^r} \leq \left\| f \right\|_{L^r}, \]
and hence
\[ \left\| f^{h} \right\|_{L^r} \leq \left\| f \right\|_{L^r} + \left\| f^{l} \right\|_{L^r} \leq \left\| f \right\|_{L^r}. \]
\hfill \Box

Lemma A.4. Let $f \in H^k(\mathbb{R}^3)$ for any integer $k \geq 2$. Then there exists a positive constant $C_0$ such that
\[ \left\| \nabla^{j+1} f^{h} \right\|_{L^2} \leq C_0 \left\| \nabla^{j+1} f \right\|_{L^2}, \]
and
\[ \left\| \nabla^{j+1} f^{l} \right\|_{L^2} \leq C_0 \left\| \nabla^{j} f \right\|_{L^2}, \]
for any $0 \leq j \leq k - 1$.

Proof. This lemma can be shown directly by the definitions of the low–frequency and high–frequency of $f$ and the Plancherel theorem, and thus we omit the details. \hfill \Box

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