Correspondence modules and persistence sheaves: a unifying perspective on one-parameter persistent homology

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Abstract
We develop a unifying framework for the treatment of various persistent homology architectures using the notion of correspondence modules. In this formulation, morphisms between vector spaces are given by partial linear relations, as opposed to linear mappings. In the one-dimensional case, among other things, this allows us to: (i) treat persistence modules and zigzag modules as algebraic objects of the same type; (ii) give a categorical formulation of zigzag structures over a continuous parameter; and (iii) construct barcodes associated with spaces and mappings that are richer in geometric information. A structural analysis of one-parameter persistence is carried out at the level of sections of correspondence modules that yield sheaf-like structures, termed persistence sheaves. Under some tameness hypotheses, we prove interval decomposition theorems for persistence sheaves and correspondence modules, as well as an isometry theorem for persistence diagrams obtained from interval decompositions. Applications include: (a) a Mayer-Vietoris sequence that relates the persistent homology of sublevelset filtrations and superlevelset filtrations to the levelset homology module of a real-valued function and (b) the construction of slices of 2-parameter persistence modules along negatively sloped lines.

Keywords Persistent homology · Correspondence modules · Persistence sheaves · Persistence diagrams

Mathematics Subject Classification MSC 55N31 · MSC 62R40 · MSC 18F20
1 Introduction

1.1 Context

Rooted in the works of Frosini [21] and Robins [31], over the years, persistent homology has experienced a vigorous development on many fronts, including theoretical foundations, computation, and applications. Through the assembly of homology across multiple scales into algebraic structures known as persistence modules, or $p$-modules, persistent homology provides a powerful technique for the study of structural properties of geometric objects and data using topology. As discussed below, there are many variants of persistence such as forward persistence, backward persistence, and zigzag persistence. One of the goals of this paper is to develop a categorical framework that allows us to view all of these variants as one. Not only does this novel formulation provide a unifying perspective, but it also reveals new forms of persistence and new relationships between different types of persistent structures. We refer to these generalized persistence modules as correspondence modules, or simply $c$-modules. In addition to developing this new framework, this paper carries out a structural analysis of persistent structures in this expanded setting, investigating interval decomposability of $c$-modules and stability properties that are crucial for theoretically sound applications.

A prototypical example of a forward $p$-module over $\mathbb{R}$ is the persistent homology of the sublevel set filtration of a space $X$ associated with a continuous function $f : X \to \mathbb{R}$. For each $t \in \mathbb{R}$, let $X_t := f^{-1}(-\infty, t]$. Clearly, $X_s \subseteq X_t$, for any $s \leq t$, so we obtain a filtration of $X$ indexed over $\mathbb{R}$. For an integer $i \geq 0$, applying the $i$th homology functor (with coefficients in a fixed field) to this filtration, we obtain a family of vector spaces $V_t := H_i(X_t)$ and vector space morphisms $v^i : V_s \to V_t$, for any $s \leq t$, induced by the inclusions $X_s \subseteq X_t$, satisfying $v^i_t = id$, $\forall t \in \mathbb{R}$, and $v^i_r = v^i_s \circ v^i_r$, if $r \leq s \leq t$. The family $(V_t, v^i_t)$ forms a forward $p$-module.

An analogous construction with superlevel sets $X_t := f^{-1}(t, \infty)$ yields a similar algebraic object, however, with a contravariant behavior. That is, if $s \leq t$, we have a morphism $v^i_s : V_t \to V_s$, where $V_t := H_i(X_t)$. The family $(V_t, v^i_s)$ is an example of a backward $p$-module.

Zigzag modules, introduced in [7], are parameterized over discrete subsets of $\mathbb{R}$. In a zigzag module, a morphism may point in either direction. An important example is the levelset persistence of a function $f : X \to \mathbb{R}$. For each $t \in \mathbb{R}$, let $X_t := f^{-1}(-\infty, t]$. Although there is no natural mapping from $X[s]$ to $X[t]$, for $s < t$, the interlevel set $X'_t := f^{-1}[s, t]$ can act as an interpolant, as there are inclusions $X[s] \hookrightarrow X'_t \hookrightarrow X[t]$. Thus, so long as we restrict the parameter values to a discrete set, with the aid of interlevel sets we get a sequence of spaces connected by morphisms alternating from forward to backward. Passing to homology, we obtain a zigzag module. This formulation is adequate to study the level sets of Morse-type functions, but insufficient for more general continuous functions, as there seems to be something inherently discrete about such structures. The question of how to give a category theory formulation of zigzag structures over $\mathbb{R}$ has been asked by
many and posed explicitly in [29]. Another characteristic of this zigzag formulation is that the homology of interlevel sets are introduced somewhat artificially in the sequence, as they are not the objects of interest. Correspondence modules provide a solution to both problems, as detailed in Sect. 7.1. Our approach via \( c \)-modules altogether removes the homology of interlevel sets from the sequence, recasting them as morphisms in the appropriate category. We note that levelset persistence over a continuous parameter also has been studied via a 2-parameter formulation of persistence employing interlevel sets [4, 15]. This and connections to our formulation are further discussed in Sect. 7.

Our constructs are based on two main concepts: \( c \)-modules and persistence sheaves, or \( p \)-sheaves. In a correspondence module, a morphism between two vectors spaces \( U \) and \( V \) is given by a partial linear relation; that is, a linear subspace of \( U \times V \). Letting \( \text{CVec} \) be the category whose objects are the vector spaces over a (fixed) field \( k \) and the morphisms from \( U \) to \( V \) are the partial linear relations from \( U \) to \( V \), a correspondence module over a poset \((P, \preceq)\) is a functor \( F : (P, \preceq) \to \text{CVec} \).

Formally, a \( p \)-module over \((P, \preceq)\) is a functor from \((P, \preceq)\) to \( \text{Vec} \), the category of vector spaces (over \( k \)) and linear mappings [14, 26]. A linear map \( T : U \to V \) may be viewed as a \( \text{CVec} \)-morphism by replacing it with its graph \( G_T \). Thus, via graphs, any \( p \)-module may be thought of as a \( c \)-module. Note that a “backward” mapping \( S : V \to U \) also may be viewed as a \( \text{CVec} \)-morphism from \( U \) to \( V \) by replacing \( S \) with \( G_S^* \), where the operation \( * \) swaps the coordinates of \( G_S \). From this viewpoint, persistence structures in which morphisms are given by linear mappings, regardless of whether the mappings are forward or backward, can all be formulated under the same category theory framework. Thus, zigzag modules [7, 23] also may be viewed as \( c \)-modules.

The focus of this paper is on \( c \)-modules over \((\mathbb{R}, \leq)\). Persistence sheaves are introduced mainly because it is difficult to directly analyze persistent structures in the \( \text{CVec} \) category, but they also are of interest in their own right. We note that a sheaf-theoretical approach to persistent homology was first investigated by Curry [19] and further studied by Kashiwara and Schapira [25], Berkouk and Ginot [1], and Berkouk et al. [2]. However, our formulation is different. Moreover, the \( p \)-sheaves we use to investigate the structure of \( c \)-modules satisfy a gluing property that is weaker than the standard gluing property for sheaves, as specified in Definition 14 and illustrated in Example 3. Persistence sheaves also encode finer information than sheaves over open sets of \( \mathbb{R} \), as shown in Example 4.

Under appropriate tameness hypotheses, a \( p \)-module admits a decomposition, unique up to isomorphism, as a direct sum of atomic units known as interval modules. This leads to a compact representation of persistence modules as persistence diagrams (\( p \)-diagrams) or barcodes. In a seminal piece, in which the expression persistent homology was introduced, Edelsbrunner et al. developed an algorithm to compute the persistent homology of a filtered simplicial complex and introduced an early form of \( p \)-diagrams [20]. Another landmark is the work of Zomorodian and Carlsson [33], where the persistent homology of a discrete simplicial filtration is formulated as a graded module over the polynomial ring \( k[x] \) from which one obtains interval decompositions under appropriate finiteness hypotheses. Barcodes were introduced in [11] as a representation of an interval decomposition. This work
also led to the insight that the fundamental structure underlying persistent homology is that of an inductive system of vector spaces; that is, a persistence module. It is in this algebraic setting that Crawley-Boevey showed that any pointwise finite-dimensional \( p \)-module over \((\mathbb{R}, \leq)\) admits an interval decomposition \([5, 18]\). As such, these \( p \)-modules may be represented by barcodes or \( p \)-diagrams, or alternatively, by Bubenik’s persistence landscapes \([6]\). For levelset persistence, using a series of zigzag structures over progressively denser discrete subsets of \( \mathbb{R} \), Carlsson et al. showed that one can define persistence diagrams associated with a continuous parameter \( t \in \mathbb{R} \) \([8]\). However, the question remained unanswered at the level of modules. This paper develops sheaf-theoretical analogues of the techniques of \([18]\) to prove interval decomposition theorems for \( c \)-modules and \( p \)-sheaves. The argument involves a whole hierarchy of decompositions that ultimately leads to an interval decomposition.

The stability of persistence diagrams is another theme of central interest, as it is important to ensure that persistent homology can be used reliably in applications. A breakthrough result in this direction is the celebrated stability theorem of Cohen-Steiner, Edelsbrunner and Harer \([16]\). If \( f, g : X \to \mathbb{R} \) satisfy some regularity conditions, then \( d_b(D_f, D_g) \leq \|f - g\|_\infty \), where \( D_f \) and \( D_g \) are the persistence diagrams associated with the sublevelset filtration of \( X \) induced by \( f \) and \( g \), respectively, and \( d_b \) denotes bottleneck distance. As one often is interested in comparing functional data defined on different domains, extensions to this setting have been studied in \([22, 24]\). For structural data (finite metric spaces), Chazal et al. showed that the persistence diagram of the Vietoris-Rips filtration is stable with respect to the Gromov-Hausdorff distance \([13]\). As the transition from functions, or other filtrations, to persistence diagrams goes through persistence modules, Chazal et al. \([12, 14]\) introduced an interleaving distance \( d_I \) between \( p \)-modules to analyze stability at an algebraic level. The Isometry Theorem, proven by Lesnick \([26]\) and Chazal et al. \([12, 14]\), states that \( d_b(D(\mathcal{U}), D(\mathcal{V})) = d_I(\mathcal{U}, \mathcal{V}) \), where \( \mathcal{U} \) and \( \mathcal{V} \) are \( p \)-modules over \( \mathbb{R} \) and \( D(\mathcal{U}) \) denotes the persistence diagram of \( \mathcal{U} \). We prove an isometry theorem for \( p \)-sheaves as a further extension of such stability results.

### 1.2 Main results

The sections of a \( c \)-module \( \mathcal{V} \) over any interval \( I \subseteq \mathbb{R} \) (see Definition 12) form a vector space \( F(I) \). Moreover, if \( I \subseteq J \), there is a restriction homomorphism \( F^J_I : F(J) \to F(I) \). Sections satisfy certain locality and gluing properties that yield a sheaf-like structure that we term persistence sheaf. Analysis of the structure of \( p \)-sheaves has the advantage of placing us back in the Vec category, at the expense of replacing the domain category \((\mathbb{R}, \leq)\) with the category \((\text{Int}, \subseteq)\), whose objects are the intervals of the real line with morphisms given by inclusions. It is in this framework that we establish the central results of the paper, which are as follows:

(I) Under appropriate tameness hypotheses, we prove interval decomposition theorems for \( c \)-modules and \( p \)-sheaves that lead to barcode or persistence diagram representations of their structures.
We prove an isometry theorem that states that, for any two interval decomposable \( p \)-sheaves, the bottleneck distance between their persistence diagrams is the same as an interleaving distance between their \( p \)-sheaves of sections. This distance extends the usual interleaving distance between \( p \)-modules (cf. [12, 14]).

We should point out that interval decompositions of tame \( p \)-sheaves may be approached via block decompositions for 2-D persistence modules [4, 15]. However, this is not sufficient to obtain an interval decomposition theorem for the class of pointwise finite-dimensional correspondence modules we are interested in because their \( p \)-sheaves of sections only satisfy a weaker form of tameness—see Examples 5 and 6, and Remark 5. For this reason we approach interval decompositions in a different way, developing a sheaf theoretical analogue of the arguments used by Crawley-Boevey in the proof of a decomposition theorem for pointwise finite-dimensional \( p \)-modules over \( \mathbb{R} \) [18]. This approach also provides a different perspective on interval decompositions of \( p \)-sheaves. We first prove the result for tame \( p \)-sheaves, as it is simpler to describe the arguments in this setting. The proof is then readily adapted to \( p \)-sheaves of sections of pointwise finite-dimensional \( c \)-modules, our primary objects of study. Similar remarks apply to our stability results.

One of the applications discussed in the paper illustrates particularly well the unifying quality of \( c \)-modules. For a function \( f : X \to \mathbb{R} \) and \( t \in \mathbb{R} \), the sublevel and superlevel sets \( X_t \) and \( X_t \), respectively, form a cover of \( X \) by two subspaces whose intersection is the level set \( X_t \). If \( X \) is a locally compact polyhedron, \( f \) is proper, and homology is Steenrod-Sitnikov [28], which satisfies a strong form of excision, there is a Mayer-Vietoris sequence for the cover \( X = X_t \cup X_t \). We construct a persistent Mayer-Vietoris sequence that ties together the persistent homology modules obtained from the sublevelset and superlevelset filtrations of \( X \) induced by \( f \), and the levelset homology \( c \)-module of \((X,f)\). Here we use in full force the fact that we can perform the direct sum of forward and backward \( p \)-modules and construct morphisms involving levelset \( c \)-modules, all as objects in the same category.

1.3 Organization

The rest of the paper is organized as follows. Section 2 introduces some basic terminology and the notion of correspondence modules. Section 3 is devoted to the basic properties of persistence sheaves. Tameness properties of \( c \)-modules and \( p \)-sheaves that imply interval decomposability are discussed in Sect. 4. The decomposition theorems for \( c \)-modules and \( p \)-sheaves are proven in Sect. 5 and the Isometry Theorem in Sect. 6. Section 7 discusses applications to: (i) levelset persistence; (ii) the construction of a persistent Mayer-Vietoris sequence; and (iii) 1-dimensional slices of 2-parameter persistence modules along lines of negative slope. We close the paper with some discussion in Sect. 8.
2 Correspondence modules

2.1 Preliminaries

We denote by Vec the category whose objects are the vector spaces over a fixed field $k$ with linear mappings as morphisms. A poset $(P, \leq)$ is treated as a category whose objects are the elements of $P$. For $s, t \in P$, there is a single morphism from $s$ to $t$ if $s \leq t$, and none otherwise. Abusing notation, we also denote the morphism by $s \leq t$.

A persistence module (p-module) over $P$ is a functor $\mathbb{V} : P \to \text{Vec}$. Correspondence modules, defined next, generalize p-modules by allowing more general morphisms between vector spaces.

Definition 1 Let $U$ and $V$ be vector spaces. A (linear, partial) correspondence from $U$ to $V$ is a linear subspace $C \subseteq U \times V$. We denote the set of all such linear correspondences by $C(U, V)$. If $C \in C(U, V)$, then:

(i) The domain of $C$ is defined as $\text{Dom}(C) = \pi_U(C)$ and the image of $C$ as $\text{Im}(C) = \pi_V(C)$, where $\pi_U$ and $\pi_V$ denote the projections onto $U$ and $V$, respectively. The kernel of $C$ is defined as $\ker(C) = \{u \in U \mid (u, 0) \in C\}$.

(ii) The reverse correspondence $C^* \in C(V, U)$ is defined as $C^* = \{(v, u) : (u, v) \in C\}$.

Example 1 Let $T : U \to V$ be a linear map and $G_T$ the graph of $T$. Then, $G_T \in C(U, V)$ and $G_T^* \in C(V, U)$. The graph of the identity map $I_V : V \to V$ gives the diagonal correspondence $\Delta_V := G_{I_V}$.

Definition 2 Let $C_1 \in C(U, V)$ and $C_2 \in C(V, W)$. The composition $C_2 \circ C_1 \in C(U, W)$ is defined as $C_2 \circ C_1 = \{(u, w) \in U \times W : \exists v \in V \text{ with } (u, v) \in C_1 \text{ and } (v, w) \in C_2\}$.

With this composition operation, we form a category CVec having vector spaces over $k$ as objects and correspondences $C \in C(U, V)$ as morphisms from $U$ to $V$. In this category, the identity morphism of an object $V$ is $\Delta_V$, the graph of the identity map.

Remark 1 The category CVec does not have zero morphisms and therefore is not an abelian category. Indeed, suppose $Z \in C(V, W)$ is a zero morphism. Let $U \neq 0$ be a non-trivial vector space and consider the relations $C_1, C_2 \in C(U, V)$ given by $C_1 = 0 \times 0$ and $C_2 = U \times 0$. Then, $\text{Dom}(Z \circ C_1) = 0$ and $\text{Dom}(Z \circ C_2) = U \neq 0$ so that $Z \circ C_1 \neq Z \circ C_2$, contradicting the fact that $Z$ is a zero morphism.

Lemma 1 Let $C \subseteq U \times V$ be a correspondence. If $\text{Dom}(C) = U$ and $\ker(C^*) = 0$, then $C$ is the graph of a linear mapping $T : U \to V$. In particular, isomorphisms in CVec are given by linear mappings. More precisely, let $C \subseteq U \times V$ and $D \subseteq V \times U$
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be correspondences such that \( D \circ C = \Delta_U \) and \( C \circ D = \Delta_V \). Then, there is an isomorphism \( T : U \to V \) such that \( G_T = C \) and \( G^*_T = D \).

**Proof** The assumptions on \( C \) imply that, for each \( u \in U \), there is a unique \( v \in V \) such that \((u, v) \in C\). Define \( T \) by \( T(u) = v \), which is linear with the desired properties. For the statement about isomorphisms, \( D \circ C = \Delta_U \) implies that \( \text{Dom}(C) = U \). Thus, to show that \( C \) is the graph of a linear mapping \( T : U \to V \), it suffices to verify that \( \ker(C^*) = 0 \). If \( \ker(C^*) \neq 0 \), there exists \( v \neq 0 \in V \) such that \((0, v) \in C \). Since \((0, 0) \in D\), we have \((0, v) \in C \circ D\), which contradicts with \( C \circ D = \Delta_V \). Hence \( \ker(C^*) = 0 \). Similarly, there is \( S : V \to U \) such that \( G_S = D \). Note that \( D \circ C = \Delta_U \) and \( C \circ D = \Delta_V \) imply that \( S \circ T = I_U \) and \( T \circ S = I_V \).

\( \square \)

**Definition 3** A correspondence module (abbreviated \( c \)-module) over a poset \((P, \leq)\) is a functor \( \mathbb{U} : P \to \text{CVec} \).

We adopt the following notation:

(i) \( V_t \) for the vector space associated with \( t \in P \); that is, \( V_t := \mathbb{U}(t) \);

(ii) \( v^t_s \) for the morphism associated with \( s \leq t \); that is, \( v^t_s := \mathbb{V}(s \leq t) \).

**Definition 4** Let \( \mathbb{U} \) and \( \mathbb{V} \) be \( c \)-modules over \( P \).

(i) A morphism \( F : \mathbb{U} \to \mathbb{V} \) is a natural transformation from the functor \( \mathbb{U} \) to the functor \( \mathbb{V} \). In other words, a collection of compatible morphisms \( f_t \in C(U_t, V_t) \), \( t \in P \), meaning that \( f_t \circ v^t_s = v^t_s \circ f_s \), for any \( s \leq t \). A morphism \( F \) is an isomorphism if \( f_t \) is a \text{CVec} isomorphism, \( \forall t \in P \).

(ii) \( \mathbb{U} \) is a submodule of \( \mathbb{V} \) if \( U_t \) is a subspace of \( V_t \), \( \forall t \in P \), and \( u^t_s = (U_s \times U_t) \cap v^t_s \), for any \( s \leq t \).

**Definition 5** The graph functor \( \mathbb{G} : \text{Vec} \to \text{CVec} \) is defined by \( \mathbb{G}(V) = V \), for any object \( V \), and \( \mathbb{G}(T) = G_T \), for any morphism \( T \).

**Example 2** Persistence Modules and Zigzag Modules as \( c \)-Modules

(i) Let \( \mathbb{U} : P \to \text{Vec} \) be a persistence module over \( P \). The composition \( \mathbb{V} = \mathbb{G} \circ \mathbb{U} \) yields a correspondence module. Thus, any persistence module may be viewed as a \( c \)-module via the graphs of its morphisms.

(ii) Consider the poset \( P_n = \{0, 1, 2, \ldots, n\} \) with the usual ordering \( \leq \). A zigzag module \( \mathbb{V} \) over \( P_n \) is a sequence

\[
V_0 \xleftarrow{p_1} V_1 \xleftarrow{p_2} V_2 \xleftarrow{} \ldots \xleftarrow{} V_{n-1} \xleftarrow{p_n} V_n ,
\]  

(1)
where each $V_i$ is a $k$-vector space and $\xrightarrow{p_i}$ denotes either a forward morphism $f_i : V_{i-1} \to V_i$ or a backward morphism $g_i : V_i \to V_{i-1}$ [7]. Let $C_i \subseteq V_{i-1} \times V_i$ be defined by:

(a) $C_i = G_{f_i}$, the graph of $f_i$, if $p_i$ is a forward homomorphism;
(b) $C_i = G_{g_i}$, the reverse of the graph of $g_i$, if $p_i$ is a backward homomorphism.

Set $C_{ii} = \Delta_{V_i} \subseteq V_i \times V_i$, for $0 \leq i \leq n$, and $C_{ij} = C_j \circ \ldots \circ C_{i+1} \subseteq V_i \times V_j$, for $0 \leq i < j \leq n$. These correspondences induce a $c$-module structure on $\mathcal{V}$. More precisely, $\mathcal{V}(i) := V_i$ and $\mathcal{V}(i \leq j) := C_{ij}$ define a functor $\mathcal{V} : (P_n, \leq) \to \text{CVec}

We denote by $\text{CMod}(P)$, or simply $\text{CMod}$, the category whose objects are the $c$-modules over $P$ with natural transformations as morphisms. Next, we show that morphisms in $\text{CMod}$ have images in the category theory sense. Zero morphisms, kernels and cokernels are not well defined in $\text{CMod}$. However, in some special situations, we can associate a kernel or a cokernel $c$-module to a morphism.

**Definition 6** Let $\mathbb{U}$ and $\mathcal{V}$ be $c$-modules over $P$ and $F : \mathbb{U} \to \mathcal{V}$ a $\text{CMod}$ morphism given by compatible $\text{CVec}$ morphisms $f_i : U_i \to V_i$. We adopt the notation $[v_i]$ for the element of cokernel of $f_i$ represented by $v_i \in V_i$.

(i) Let $\text{Im}_t = \text{Im}(f_t)$ and $\text{im}^t_s := v_t \cap (\text{Im}_s \times \text{Im}_t)$. We refer to the pair $\text{Im}(F) := ([\text{Im}_t : t \in P], \{\text{im}^t_s : s \leq t\})$ as the image of $F$.

(ii) Similarly, letting $K_t = \ker(f_t)$ and $k_t^s := u_t \cap (K_s \times K_t)$, define the kernel of $F$ as the pair $\ker(F) := ([K_t : t \in P], \{k_t^s : s \leq t\})$.

(iii) Let $Q_t = \text{coker}(f_t)$ and $q_t^s := Q_s \times Q_t$ be the subspace given by $([v_s], [v_t]) \in q_t^s$ if and only if there exist $a_s \in \text{Im}(f_s)$ and $a_t \in \text{Im}(f_t)$ such that $(v_s + a_s, v_t) \in f_t$. Define the cokernel of $F$ as the pair $\text{coker}(F) := ([Q_t : t \in P], \{q_t^s : s \leq t\})$.

**Proposition 1** *(Images and Kernels)* If $F : \mathbb{U} \to \mathcal{V}$ is a $\text{CMod}$ morphism, then

(i) $\text{Im}(F)$ is a submodule of $\mathcal{V}$ and the inclusion $\text{Im}(F) \hookrightarrow \mathcal{V}$ is an image of the morphism $F$ in the $\text{CMod}$ category;

(ii) If $G : \mathcal{W} \to \mathbb{U}$ is another $\text{CMod}$ morphism and the sequence

\[
W_t \xrightarrow{g_t} U_t \xrightarrow{f_t} V_t
\]

is exact in $\text{CVec}$, $\forall t \in P$, then $\ker(F)$ is a submodule of $\mathbb{U}$;

(iii) If $\mathcal{V}$ is a persistence module, then $\text{coker}(F)$ is a $c$-module.

**Proof** (i) To verify that $\text{Im}(F)$ is a submodule of $\mathcal{V}$, it suffices to check the composition rule $\text{im}^t_s = \text{im}^t_r \circ \text{im}^s_r$, for any $r \leq s \leq t$. The inclusion $\text{im}^t_r \supseteq \text{im}^t_s \circ \text{im}^s_r$ is straightforward. For the reverse inclusion, let $(v_r, v_s) \in \text{im}^t_s$. Then, there exist $v_s \in V_s$, $u_r \in U_r$ and $u_t \in U_t$ such that $(v_r, v_s) \in v_t^r$, $(v_s, v_t) \in v_t^s$, $(u_r, v_r) \in f_t$ and $(u_t, v_t) \in f_t$.
Our goal is to show that $v_s \in \text{Im}(f_s)$, as this implies that $(v_r, v_t) \in \text{im}^r \circ \text{im}^t$. Since $(u_s, v_s) \in v_s^* \sigma f_s = f_s^* \sigma u_s^*$, there exists $u_s \in U_s$ such that $(u_r, u_t) \in u_s^*$ and $(u_s, v_s) \in f_s^*$, showing that $v_s \in \text{Im} f_s$, as desired. The universal property for images in CMod is easily verified for $\text{Im}(F)$.

(ii) The assumption that the sequences are exact implies that $\ker(F) = \text{Im}(G)$. Hence, (i) implies that $\ker(F)$ is a submodule of $U$.

(iii) Since $V$ is a $p$-module, the correspondences $v^r_\mathcal{A}$ are given by graphs of linear mappings; that is, $v^r_\mathcal{A} = G \phi^r_\mathcal{A}$, where $\phi^r_\mathcal{A} : V_s \to V^r_\mathcal{A}$ are linear mappings. Let $\text{coker}(F) := (Q^r_\mathcal{A}, q^r_\mathcal{A})$. For $r \leq s \leq t$, we first show that $q^r_\mathcal{A} \subseteq q^s_\mathcal{A} \circ q^r_\mathcal{A}$. Let $([v_r], [v_t]) \in q^r_\mathcal{A}$. Then, there exist $a_r \in \text{Im}(f_s)$ and $a_t \in \text{Im}(f_t)$ such that $(v_r + a_r, v_r + a_t) \in v^r_\mathcal{A}$, which means that $\phi^r_\mathcal{A}(v_r + a_r) = v_t + a_t$. Let $v_s = \phi^r_\mathcal{A}(v_r)$ and $a_s = \phi^r_\mathcal{A}(a_r)$. A diagram chase shows that $a_s \in \text{Im}(f_s)$. Clearly, $\phi^r_\mathcal{A}(v_r + a_r) = v_s + a_s$ and $\phi^r_\mathcal{A}(v_s + a_s) = v_t + a_t$, showing that $([v_r], [v_t]) \in q^r_\mathcal{A}$ and $([v_r], [v_t]) \in q^r_\mathcal{A}$; that is, $([v_r], [v_t]) \in q^r_\mathcal{A} \circ q^r_\mathcal{A}$. For the converse inclusion, suppose that $([v_r], [v_t]) \in q^r_\mathcal{A}$ and $([v_r], [v_t]) \in q^r_\mathcal{A}$.

We close this section with a discussion of direct sums of c-modules.

**Definition 7** Let $\{\mathcal{V}_\lambda, \lambda \in \Lambda\}$, be an indexed collection of c-modules. Define the **direct sum** $\mathcal{V} = \bigoplus_{\lambda \in \Lambda} \mathcal{V}_\lambda$ by $V_t = \bigoplus_{\lambda \in \Lambda} V^r_{\lambda}$, with correspondences $v^r_s = \bigoplus_{\lambda \in \Lambda} \mathcal{V}^r_{\lambda} (s \leq t) \subseteq \mathcal{V}_s \times V_t$, for any $s \leq t$.

**Proposition 2** If $\{\mathcal{V}_\lambda, \lambda \in \Lambda\}$ is an indexed collection of c-modules, then the direct sum $\mathcal{V} = \bigoplus_{\lambda \in \Lambda} \mathcal{V}^r_{\lambda}$ also is a c-module.

**Proof** We only need to show that $v^r_s \circ v^s_r = v^r_s$, for any $r \leq s \leq t$. We first verify that $v^r_s \circ v^s_r \supseteq v^r_s$. Given $(v_r, v_t) \in v^r_s$, write $(v_r, v_t) = \sum_{\lambda \in \Lambda} c^r_{\lambda}(v^r_s, v^r_s)$, where $(v^r_s, v^r_s) \in \mathcal{V}^r_{\lambda}(r \leq t)$ and all but finitely many coefficients $c^r_{\lambda} \in k$ vanish. For each $\lambda \in \Lambda$, there exists $v^r_s \in \mathcal{V}^r_{\lambda}$ such that $(v^r_s, v^r_s) \in \mathcal{V}^r_{\lambda}(r \leq s)$ and $(v^r_s, v^r_s) \in \mathcal{V}^r_{\lambda}(s \leq t)$. Letting $v_s = \bigoplus_{\lambda \in \Lambda} c^r_{\lambda} v^r_s$, it follows that $(v_r, v_s) \in v^r_s$ and $(v_s, v_t) \in v^s_r$. Thus, $(v_r, v_t) \in v^r_s \circ v^s_r$, as claimed.

Conversely, let $(v_r, v_t) \in v^r_s \circ v^s_r$. Then, there exists $v_s \in \bigoplus_{\lambda \in \Lambda} V^r_s$ such that $(v_r, v_s) \in v^r_s$ and $(v_s, v_t) \in v^s_r$. Write

$$(v_r, v_s) = \sum_{\lambda \in \Lambda} c^r_{\lambda}(v^r_s, v^r_s) \quad \text{and} \quad (v_s, v_t) = \sum_{\lambda \in \Lambda} d^s_{\lambda}(v^s_r, v^s_r).$$

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where $(v^d_r, v^d_s) \in \mathbb{V}^d (r \leq s)$, $(v^s_r, v^s_s) \in \mathbb{V}^s (s \leq t)$ and all but finitely many scalars $c^d_i, d^d_i \in k$ vanish. Then, $v_s = \sum_{\lambda \in \Lambda} c^s_{\lambda} v^s_{\lambda}$ and $v_r = \sum_{\lambda \in \Lambda} d^s_{\lambda} v^s_{\lambda}$, which implies $c^d_i = d^d_i$, $\forall \lambda \in \Lambda$. Hence, $(v_r, v_s) = \sum_{\lambda \in \Lambda} c^d_i (v^d_r, v^d_s) \in \mathbb{V}^d$.

**Definition 8** A correspondence module $\mathbb{V}$ is indecomposable if $\mathbb{V} \cong \mathbb{V}_1 \oplus \mathbb{V}_2$ implies that either $\mathbb{V}_1 = 0$ or $\mathbb{V}_2 = 0$.

### 2.2 Interval correspondence modules

We now specialize to correspondence modules over $(\mathbb{R}, \leq)$. We introduce interval $c$-modules associated with each interval $I \subseteq \mathbb{R}$. Unlike interval $p$-modules (cf. [14]), there may be up to four non-isomorphic interval $c$-modules associated with $I$ (cf. [4, 8, 15]).

Let $\mathbb{E}$ be the set of extended, decorated real numbers defined as

$$\mathbb{E} = \mathbb{R} \times \{+, -\} \cup \{-\infty, +\infty\}. \quad \text{(4)}$$

For $t \in \mathbb{R}$, we use the abbreviations $t^+ := (t, +)$, $t^- := (t, -)$, and $t^*$ for either $t^+$ or $t^-$. Throughout the paper, $\mathbb{E}$ is equipped with the total ordering $\leq$ given by:

(i) $t^*_1 \leq t^*_2$, for any $t_1, t_2 \in \mathbb{R}$ satisfying $t_1 < t_2$;

(ii) $t^- \leq t^+$, for any $t \in \mathbb{R}$;

(iii) $-\infty \leq t^* \leq +\infty$, $\forall t \in \mathbb{R}$, and $-\infty \leq +\infty$;

(iv) $p \leq p$, $\forall p \in \mathbb{E}$.

For $p, q \in \mathbb{E}$, we write $p < q$ to mean that $p \leq q$ and $p \neq q$. Decorated numbers give a uniform notation for intervals in $\mathbb{R}$, whether open, closed, or half-open. We adopt the following identification between objects in Int and elements of $\{ (p, q) \in \mathbb{E}^2 \mid p < q \}$ (cf. [14]):

(a) For $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, $(t_1, t_2) \leftrightarrow (t^+_1, t^-_2), [t_1, t_2) \leftrightarrow (t^+_1, t^-_2), (t_1, t_2] \leftrightarrow (t^+_1, t^+_2)$, and $[t_1, t_2] \leftrightarrow (t^-_1, t^+_2)$;

(b) For $t \in \mathbb{R}$, the one-point interval $[t, t)$ corresponds to $(t^-, t^+)$ $\in \mathbb{E}^2$;

(c) For $t \in \mathbb{R}$, $(-\infty, t) \leftrightarrow (-\infty, t^-), (-\infty, t) \leftrightarrow (-\infty, t^+), (t, +\infty) \leftrightarrow (t^+, +\infty)$ and $[t, +\infty) \leftrightarrow (t^-, +\infty)$;

(d) The entire real line $\mathbb{R}$ corresponds to $(-\infty, +\infty) \in \mathbb{E}^2$.

**Definition 9** Let $(p, q) \in \mathbb{E}^2$, $p < q$, represent an interval in $\mathbb{R}$. We define $c$-modules $I \{p, q\}, I \{p, q\}, I(p, q) \text{ and } I(p, q)$ associated with $(p, q)$, referred to as interval $c$-modules, as follows (see Fig. 1):
(i) Let \( u_1 \) denote any of the above \( c \)-modules and \( t \in \mathbb{R} \). Define \( u_1(t) = k \), \( \forall t \in (p, q) \), and \( u_1(t) = 0 \), otherwise. For \( s, t \in \mathbb{R} \) and \( s \leq t \), set \( u_1(s \leq t) = \Delta_k \) if \( s, t \in (p, q) \), and \( u_1(s \leq t) = 0 \times 0 \) if \( s, t \not\in (p, q) \); 

(ii) \( \llbracket p, q \rrbracket(s \leq t) = 0 \times 0 \) if \( s \not\in (p, q) \) or \( t \not\in (p, q) \); 

(iii) \( \llbracket p, q \rrbracket(s \leq t) = k \times 0 \) if \( s \in (p, q) \) and \( t \in (q, +\infty) \); 

(iv) \( \llangle p, q \rrangle(s \leq t) = 0 \times k \) if \( s \in (-\infty, p) \) and \( t \in (p, q) \); 

(v) \( \llangle p, q \rrangle(s \leq t) = k \times 0 \) if \( s \in (p, q) \) and \( t \in (q, +\infty) \). 

**Remark 2** 

(a) If \( (p, q) \in \mathbb{E}^2 \) is a finite interval, the four modules in Definition 9 fall in different isomorphism classes. If \( p = -\infty \), then \( \llbracket p, q \rrbracket = \llangle p, q \rrangle \) and \( \llbracket p, q \rrbracket = \llangle p, q \rrangle \). Similarly, if \( q = +\infty \), \( \llbracket p, q \rrbracket = \llangle p, q \rrangle \) and \( \llbracket p, q \rrbracket = \llangle p, q \rrangle \). If \( (p, q) = (-\infty, +\infty) \), all four \( c \)-modules coincide. 

(b) We adopt the pictorial representation of these four types of interval modules indicated in Fig. 1. 

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**Fig. 1** Interval \( c \)-modules associated with the interval \((p, q)\)
Lemma 2 (Indecomposability) Interval c-modules are indecomposable.

Proof The argument is standard. Let \( I \) be an interval c-module of any of the types described in Definition 9. The corresponding interval in \( \mathbb{R} \) is denoted \( I \). Suppose \( \eta \) is a c-module isomorphism from \( I \) to \( U \oplus V \). By Lemma 1, \( \forall t \in I, \eta_t \) is the graph of a linear isomorphism \( \phi_t : k \to U_t \oplus V_t \). Let \( \pi : U \oplus V \to U \oplus V \) denote projection onto \( U \) followed by inclusion into \( U \oplus V \). Then, \( \psi_t = \phi_t^{-1} \circ \pi \circ \phi_t : k \to k \) is an idempotent of \( k \). Thus, \( \psi_t \) is given by multiplication by 0 or 1. Since \( \phi_t \) induce a c-module morphism, one can verify that this scalar is independent of \( t \in I \). Hence, either \( U = 0 \) or \( V = 0 \). \( \square \)

3 Persistence sheaves

Henceforth, all c-modules will be over \((\mathbb{R}, \leq)\). In this section, we develop a framework for the study of the structure of c-modules over \( \mathbb{R} \) based on the concept of persistence sheaves.

Let \( 2^{\mathbb{R}} \) be the category whose objects are the subsets of \( \mathbb{R} \) with inclusion of sets as morphisms. We may think of the objects of \( 2^{\mathbb{R}} \) as the open sets of \( \mathbb{R} \) in the discrete topology. We denote by Int the subcategory of \( 2^{\mathbb{R}} \) whose objects are all intervals \( I \subseteq \mathbb{R} \), open, closed or half-open (half-closed).

Definition 10 (Presheaves)

(i) A discrete presheaf is a contravariant functor from \( 2^{\mathbb{R}} \) to Vec; that is, a functor \( G : (2^{\mathbb{R}})^{\text{op}} \to \text{Vec} \).

(ii) A persistence presheaf is a functor \( F : \text{Int}^{\text{op}} \to \text{Vec} \).

Here, the superscript op denotes the opposite category. Clearly, any discrete presheaf defines a persistence presheaf via restriction to Int. For a persistence presheaf \( F \), we adopt the following terminology:

(a) We refer to an element of the vector space \( F(I) \) as a section of \( F \) over \( I \).

(b) For \( I \subseteq J \), we refer to the linear map \( F^J_I := F(I \subseteq J) : F(J) \to F(I) \) as a restriction homomorphism. If \( s \in F(J) \), we sometimes use the notation \( s|_I \) for \( F^J_I(s) \).

(c) If \( I = \{a\} \) is a singleton, we simplify the notation for \( F(I), F^J_I \) and \( s|_I \) to \( F_a, F^J_a \) and \( s|_a \), respectively.

(d) If \( s \in F(J) \), the support of \( s \) is defined as \( \text{supp}[s] := \{a \in I : F^J_a(s) \neq 0\} \).

(e) If \( s \) and \( s' \) are sections of \( F \), we write \( s \leq s' \) to indicate that \( s \) is a restriction of \( s' \).

Similar notation and terminology are adopted for discrete presheaves.

Remark 3 Let \( F \) be the restriction of a discrete presheaf \( G \) to Int. If \( J \subseteq \mathbb{R} \) is an interval, \( A \subseteq J \), and it is clear from the context what the presheaf \( G \) is, we abuse
notation and write \( F^I_A \) for the restriction homomorphism “inherited” from \( G \). Similarly, if \( s \in F(J) \), we write \( s|_A \) for \( G^J_A(s) \). More generally, if \( V \) is a subspace of \( F(J) \), we set \( V|_A = \{ s|_A : s \in V \} \).

**Definition 11 (Morphisms)**

(i) Let \( F \) and \( G \) be persistence presheaves. A morphism from \( F \) to \( G \) is a natural transformation \( \Phi : F \to G \); that is, a collection \( \Phi := \{ \phi_I : F(I) \to G(I) : I \in \text{Int} \} \) of linear mappings such that \( G^I_I \Phi = \Phi F^I_I \), for any \( I \subseteq J \).

(ii) \( \Phi : F \to G \) is an isomorphism if there is a morphism \( \Psi : G \to F \) such that \( \Psi \phi \Phi \) and \( \Phi \psi \Psi \) are the identity morphisms of \( F \) and \( G \), respectively.

**Definition 12** Let \( V \) be a \( c \)-module and \( A \subseteq \mathbb{R} \).

(i) A section \( s \) over \( A \) is an indexed family \( s = (v_t) \), \( t \in A \), such that \( v_t \in V_t \) and \( (v_r, v_t) \in v_{t+r} \), for any \( r, t \in A \) with \( r \leq t \). The domain of \( s \) is the set \( A \) and we denote it \( \text{Dom}(s) \). The support of \( s \) is defined as \( \text{supp}[s] = \{ t \in \text{Dom}(s) : v_t \neq 0 \} \). Pointwise addition and scalar multiplication induce a \( k \)-vector space structure on the collection of all sections over \( A \). If \( s \) is a section over \( B \) and \( A \subseteq B \), the restriction of \( s \) to \( A \) is denoted \( s|_A \).

(ii) The discrete presheaf of sections of \( V \), denoted \( D_V \), is the contravariant functor that associates to each \( A \subseteq \mathbb{R} \) the vector space \( D_V(A) \) of all sections of \( V \) over \( A \). If \( A \subseteq B \), then \( D_V(A \subseteq B) \) is the linear mapping given by \( s \mapsto s|_A \), for any section \( s \) over \( B \).

(iii) The persistence presheaf of sections of \( V \) is the restriction of \( D_V \) to \( \text{Int} \).

To define persistence sheaves, we introduce the notion of connected covering.

**Definition 13** Let \( X \) be a set and \( X = \bigcup A \), \( \lambda \in \Lambda \), a covering of \( X \) by subsets \( A \). The covering is connected if for any non-trivial partition \( \Lambda = \Lambda_0 \sqcup \Lambda_1 \), the intersection of the sets \( X_0 = \bigcup_{\lambda \in \Lambda_0} A \lambda \) and \( X_1 = \bigcup_{\lambda \in \Lambda_1} A \lambda \) is non-empty.

**Lemma 3** Let \( X = \bigcup A \), \( \lambda \in \Lambda \), be a covering of \( X \) with \( A \lambda \neq \emptyset \), \( \forall \lambda \in \Lambda \). The covering is connected if and only if, for any \( \lambda, \mu \in \Lambda \), there is a finite sequence \( \lambda_0, \ldots, \lambda_n \in \Lambda \) such that \( \lambda_0 = \lambda, \lambda_n = \mu \) and \( X_{\lambda_{i-1}} \cap X_{\lambda_i} \neq \emptyset \), for \( 1 \leq i \leq n \).

**Proof** Consider the equivalence relation on \( \Lambda \) generated by \( \lambda \sim \mu \) if \( X_{\lambda} \cap X_{\mu} \neq \emptyset \). Then, \( \lambda \sim \mu \) if and only if a sequence as above exists. It is simple to verify that the covering is connected if and only if \([ \lambda ] = \Lambda, \forall \lambda \in \Lambda \). \( \square \)

**Lemma 4** Let \( \{ A \lambda, \lambda \in \Lambda \} \) be a connected covering of a set \( X \). If \( V \subseteq X \) is such that, for each \( \lambda \in \Lambda, X_{\lambda} \subseteq V \) or \( X_{\lambda} \cap V = \emptyset \), then \( V = X \) or \( V = \emptyset \).
Proof Let $\Lambda_0 = \{ \lambda \in \Lambda \mid X_\lambda \subseteq V \}$ and $\Lambda_1 = \{ \lambda \in \Lambda \mid X_\lambda \cap V = \emptyset \}$. This gives a partition of $\Lambda$ with the property that $X_0 \cap X_1 = \emptyset$, with $X_0$ and $X_1$ as in Definition 13. Since the covering is connected, either $\Lambda_0 = \emptyset$ or $\Lambda_1 = \emptyset$. This implies that $V = X$ or $V = \emptyset$. \hfill \Box

Definition 14 Let $F$ be a persistence presheaf. $F$ is a persistence sheaf (abbreviated $p$-sheaf) if the following conditions are satisfied:

(i) (Locality) For any covering $I = \cup I_\lambda$, $\lambda \in \Lambda$, of $I$ by non-empty intervals $I_\lambda$, if $s \in F(I)$ is such that $s|_{I_\lambda} = 0$, $\forall \lambda \in \Lambda$, then $s = 0$;

(ii) (Connective Gluing) For any connected covering $I = \cup I_\lambda$, $\lambda \in \Lambda$, of $I$ by intervals $I_\lambda$, if $s_\lambda \in F(I_\lambda)$, $\lambda \in \Lambda$, are sections of $F$ such that $s_\lambda|_{I_\lambda \cap I_{\lambda'}} = s_{\lambda'}|_{I_\lambda \cap I_{\lambda'}}$, $\forall \lambda, \lambda' \in \Lambda$, then there is a section $s \in F(I)$ such that $s|_{I_\lambda} = s_\lambda$, $\forall \lambda \in \Lambda$.

Note that property (ii) differs from the usual gluing property for sheaves because of the connectivity condition on the coverings. The next example shows that the sections of a $c$-module in general do not satisfy the usual gluing property.

Example 3 Using the notation introduced in Definition 9, let $\mathbb{V} = \llbracket -1^- , 0^- \rrbracket \oplus \llbracket 0^- , 1^+ \rrbracket$ be the direct sum of the interval $c$-modules of type $\llbracket . . \rrbracket$ associated with the intervals $[-1, 0)$ and $[0, 1]$, respectively. Then, the constant sections $s_1$ over $[-1, 0)$ and $s_2$ over $[0, 1]$ defined by $s_1 \equiv (1, 0)$ and $s_2 \equiv (0, 1)$ have disjoint domains. However, there is no section $s$ over the interval $[-1, 1]$ that coincides with $s_1$ over $[-1, 0)$ and with $s_2$ over $[0, 1]$.

Example 4 Here we describe a $c$-module $\mathbb{V}$ whose $p$-sheaf $F$ of sections is non-trivial, but the space of sections over any open interval is trivial. Thus, $p$-sheaves contain finer structural information than sheaves over open sets of $\mathbb{R}$ equipped with the standard topology. Let $\mathbb{V} := \llbracket 0^- , 0^+ \rrbracket$ be the interval module over the singleton $I = \{ 0 \}$ introduced in Definition 9. For any open interval $J \subseteq \mathbb{R}$, we have $F(J) = 0$ because the relation $\nu_s = 0$, for any $s < t$. On the other hand, the space of sections $F(0)$ is a 1-dimensional vector space.

Proposition 3 The persistence presheaf of sections of a $c$-module $\mathbb{V}$ is a $p$-sheaf.

Proof Locality is clearly satisfied, so we verify the connective gluing property. Let $I = \cup I_\lambda$, $\lambda \in \Lambda$, be a connected covering of an interval $I$ by intervals $I_\lambda$, and let $s_\lambda \in V(I_\lambda)$, $\lambda \in \Lambda$, be sections of $\mathbb{V}$ that agree on overlaps. There is a well-defined family $s = (v_t)$, $t \in I$, such that $s|_{I_\lambda} = s_\lambda$, $\forall \lambda \in \Lambda$. We need to show that $s$ is a section. Let $S$ be the collection of all sections $r$ of $\mathbb{V}$ satisfying $r \leq s$; that is, sections $r$ that coincide with the restriction of $s$ to some subinterval of $I$. $S$ is non-empty because the restriction of $s$ to any $I_\lambda$ gives a section. Moreover, $(S, \leq)$ is a partially ordered set such that each chain in $S$ has an upper bound in $S$. By Zorn’s lemma, $S$ contains a maximal element $\hat{s}$ defined on an interval $J \subseteq I$. By construction, for each
\( \lambda \in \Lambda, I_\lambda \cap J = \emptyset \) or \( I_\lambda \subseteq J \), for otherwise, we would be able to extend \( \hat{s} \) to a larger interval. Since \( J \neq \emptyset \), Lemma 4 ensures that \( J = I \), proving that \( s = \hat{s} \).

**Definition 15** Let \( F_\lambda, \lambda \in \Lambda \), be \( p \)-sheaves. The direct sum \( \bigoplus F_\lambda \) is the \( p \)-sheaf defined by:

(i) \((\bigoplus F_\lambda)(I) = \bigoplus F_\lambda(I)\), for any interval \( I \);
(ii) \((\bigoplus F_\lambda)(I \subseteq J) = \bigoplus F_\lambda(I \subseteq J)\).

**Definition 16** Let \( F \) and \( G \) be \( p \)-sheaves. \( F \) is a subsheaf of \( G \) if (i) for any interval \( I \), \( F(I) \subseteq G(I) \) and (ii) for any \( I \subseteq J \) and \( s \in F(J) \), \( F_I^J(s) = G_I^J(s) \).

It is easy to verify that the intersection of a collection of subsheaves of a \( p \)-sheaf \( F \) is also a subsheaf of \( F \).

**Definition 17** Let \( S \) be a set of sections of a \( p \)-sheaf \( F \). The \( p \)-sheaf generated by \( S \) is the intersection of all subsheaves \( G \) of \( F \) with the property that if \( s \in S \), then \( s \) is a section of \( G \). This subsheaf of \( F \) is denoted \( F(S) \).

**Definition 18** Let \((p, q) \in \mathbb{E}^2\) be an interval. We define the interval \( p \)-sheaves \( k[p, q], k[p, q], k(p, q) \) and \( k(p, q) \) associated with \((p, q)\), as follows:

(i) \( k[p, q](I) = k \) if \( I \subseteq (p, q) \) and \( k[p, q](I) = 0 \) otherwise. The element \( 1 \in k[p, q](p, q) \) is called the unit section of \( k[p, q] \).
(ii) \( k[p, q](I) = k \) if \( I \subseteq (p, +\infty) \) and \( I \cap (p, q) \neq \emptyset \), and \( k[p, q](I) = 0 \) otherwise. The element \( 1 \in k[p, q](p, +\infty) \) is called the unit section of \( k[p, q] \).
(iii) \( k[p, q](I) = k \) if \( I \subseteq (-\infty, q) \) and \( I \cap (p, q) \neq \emptyset \), and \( k[p, q](I) = 0 \) otherwise. The element \( 1 \in k[p, q](-\infty, q) \) is called the unit section of \( k[p, q] \).
(iv) \( k[p, q](I) = k \) if \( I \cap (p, q) \neq \emptyset \) and \( k[p, q](I) = 0 \) otherwise. We refer to the element \( 1 \in k[p, q](-\infty, +\infty) \) as the unit section of \( k[p, q] \).
(v) If \( F \) is any of the above \( p \)-sheaves and \( I \subseteq J \), then \( F_I^J \) is the identity map if \( F(J) = F(I) = k \), and \( F_I^J \) is the trivial map, otherwise.

We refer to each of these four possibilities as the type of the interval \( p \)-sheaf associated with \((p, q)\).

**Proposition 4** For any interval \((p, q) \in \mathbb{E}^2, p < q\), the following holds:

(i) \( k[p, q] \) is isomorphic to the \( p \)-sheaf of sections of \( \llbracket p, q \rrbracket \);
(ii) \( k[p, q] \) is isomorphic to the \( p \)-sheaf of sections of \( \llbracket p, q \rrbracket \);
(iii) \( k[p, q] \) is isomorphic to the \( p \)-sheaf of sections of \( \llbracket p, q \rrbracket \);
(iv) \( k(p, q) \) is isomorphic to the \( p \)-sheaf of sections of \( \llbracket p, q \rrbracket \).
Proof The proof is straightforward. \qed

Proposition 5 Let \( s \) be a section of a \( p \)-sheaf \( F \) with \( \text{Dom}(s) = (x, y) \in \mathbb{R}^2, \) \( x < y, \) and let \( F(s) \) be the subsheaf generated by \( s. \) Suppose that \( \text{supp}(s) \) is an interval \((p, q) \in \mathbb{R}^2, \) where \( x \leq p < q \leq y. \) Then, the following statements hold:

\begin{enumerate}[(i)]
  \item If \( x = p \) and \( q < y, \) then \( F(s) \) is isomorphic to \( k[p, q]. \)
  \item If \( x < p \) and \( q = y, \) then \( F(s) \) is isomorphic to \( k(p, q]. \)
  \item If \( x < p \) and \( q < y, \) then \( F(s) \) is isomorphic to \( k(p, q). \)
  \item If \( x = p \) and \( q = y, \) then \( F(s) \) is isomorphic to \( k[p, q]. \)
\end{enumerate}

Proof For statement (i), by the connective gluing property, there is a section \( \tilde{s} \) over \((p, +\infty)\) such that \( \tilde{s}|_{(p, y)} = s \) and \( \tilde{s}|_{(q, +\infty)} = 0. \) Note that \( \tilde{s} \) must be a section of any subsheaf of \( F \) having \( s \) as a section. For an interval \( I \subseteq (p, +\infty), \) we denote by \( \langle \tilde{s}|_I \rangle \) the subspace of \( F(I) \) spanned by \( \tilde{s}|_I. \) Let \( \text{Int}_{p,q} \) be the collection of all intervals satisfying \( I \subseteq (p, +\infty) \) and \( I \cap (p, q) \neq \emptyset. \) Define a persistence presheaf \( G \subseteq F \) by \( G(I) = \langle \tilde{s}|_I \rangle \) if \( I \in \text{Int}_{p,q}, \) and \( G(I) = 0 \) otherwise. Note that \( \langle \tilde{s}|_I \rangle \neq 0, \forall I \in \text{Int}_{p,q}, \) and \( \langle \tilde{s}|_I \rangle = 0 \) if \( I \notin \text{Int}_{p,q}. \) By construction, \( G \) is a sub-presheaf of \( F(s). \)

Let \( \Phi : G \to k[p, q] \) be the homomorphism defined as follows: \( \phi_I(\langle \tilde{s}|_I \rangle) = 1 \) if \( I \in \text{Int}_{p,q}, \) and \( \phi_I = 0, \) otherwise. Similarly, define \( \Psi : k[p, q] \to G \) by \( \psi_I(1) = \langle \tilde{s}|_I \rangle \) if \( I \in \text{Int}_{p,q} \) and \( \psi_I = 0, \) otherwise. Then, for any interval \( I, \phi_I \) and \( \psi_I \) are mutual inverses, so \( \Phi \) is a presheaf isomorphism. Since \( k[p, q] \) is a sheaf, \( G \) is not just a presheaf, but a subsheaf of \( F. \) Thus, \( G = F(s), \) showing that \( F(s) \) is isomorphic to \( k[p, q]. \) The proofs for the other cases are similar. \qed

4 Tameness

This section discusses tameness conditions for \( p \)-sheaves and \( c \)-modules under which we prove interval decomposition theorems in Sect. 5.

Definition 19 Let \( F \) be a \( p \)-sheaf.

\begin{enumerate}[(i)]
  \item \( F \) satisfies the descending chain condition (DCC) on images if for any ascending sequence \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) of intervals and any interval \( I \subseteq \cap I_n, \) the chain \( \text{Im}(F(I_1)) \supseteq \text{Im}(F(I_2)) \supseteq \ldots \) is stable, that is, it eventually becomes constant;
  \item \( F \) satisfies the descending chain condition (DCC) on kernels if for any ascending sequence \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) of intervals and any interval \( I \supseteq \cup I_n, \) the chain \( \text{ker}(F(I_1)) \supseteq \text{ker}(F(I_2)) \supseteq \ldots \) is stable.
\end{enumerate}
Definition 20 Let $V$ be a $c$-module and $t \in \mathbb{R}$.

(i) $V$ satisfies the descending chain condition (DCC) on images at $t$ if for any sequence $\cdots \leq \ell_2 \leq \ell_1 \leq t$, the chain $\text{Im}(v_{\ell_1}) \supseteq \text{Im}(v_{\ell_2}) \supseteq \cdots$ stabilizes in finitely many steps, and for any sequence $t \leq u_1 \leq u_2 \leq \cdots$, the chain $\text{Dom}(v_{u_1}) \supseteq \text{Dom}(v_{u_2}) \supseteq \cdots$ is stable.

(ii) $V$ satisfies the descending chain condition (DCC) on kernels at $t$ if for any sequence $t \leq \cdots \leq u_2 \leq u_1$, the chain $\text{ker}(v_{u_1}) \supseteq \text{ker}(v_{u_2}) \supseteq \cdots$ is stable, and for any sequence $\ell_2 \leq \ell_1 \leq \cdots \leq t$, the chain $\text{ker}(v_{\ell_1})^* \supseteq \text{ker}(v_{\ell_2})^* \supseteq \cdots$ is stable. Here, $^*$ denotes the operation of reversing correspondences (see Definition 1).

Definition 21 (Tameness)

(i) A $p$-sheaf $F$ is tame if it satisfies the DCC on both images and kernels.

(ii) A $c$-module is virtually tame if it satisfies the DCC on both images and kernels at each $t \in \mathbb{R}$.

(iii) A $c$-module is tame if its $p$-sheaf of sections is tame.

Remark 4 If a $c$-module $V$ has the property that all correspondences $v_s^t$, $s < t$, are finite-dimensional, then $V$ is virtually tame. In particular, a pointwise finite-dimensional $c$-module is virtually tame.

The next examples show that the sheaf of sections of a virtually tame $c$-module is not necessarily tame.

Example 5 Let $k$ be a field. Consider the $c$-module $V$ given by $V_t = k$, $\forall t \in \mathbb{R}$, with connecting morphisms $v_s^t = k \times k$, for any $s < t$, and $v_t^t = \Delta_k$, the graph of the identity map, $\forall t \in \mathbb{R}$. $V$ is virtually tame because each $V_t$ is 1-dimensional. Furthermore, it admits the interval decomposition $V \cong \bigoplus_{x \in \mathbb{R}} \langle x \rangle$, where $\langle x \rangle$ denotes the interval module of type $\langle \rangle$ supported on the singleton $\{x\}$. Note, however, that $F \not\cong \bigoplus_{x \in \mathbb{R}} F_x$, where $F_x$ is the $p$-sheaf of sections of $\langle x \rangle$. Indeed, for any interval $I \subseteq \mathbb{R}$, $F(I)$ comprise all sequences $(v_t)$ with $v_t \in k$ and $t \in I$. On the other hand, the sections of the direct sum $p$-sheaf are the sections over $I$ whose supports are finite sets. Although $F$ satisfies the DCC on images, $F$ does not satisfy the DCC on kernels and therefore is not tame. Indeed, consider the chain $I_n = [0, n]$ and let $I = [0, +\infty)$. Then, the chain $\text{ker}(F_{I_n}^I)$ is not stable.

Example 6 This example shows that the virtual tameness of $V$ does not imply the tameness of $F$ even for persistence modules. Let $I_n = (0, 1/n]$, $I_n^+ := [0^+, 1/n^+]$ be the associated interval $p$-module of type $\langle \rangle$, and $V = \bigoplus I_n$. $V$ is pointwise finite dimensional and thus virtually tame. However, its sheaf of sections is not tame, as the DCC on kernels is not satisfied. Indeed, let $I = (0, +\infty)$ and consider the chain $J_n = (1/n, +\infty)$. Then, the chain $\text{ker}(F_{J_n}^I)$ is strictly decreasing, thus not stable.
Remark 5 In spite of the above examples, suppose the c-module \( \mathcal{V} \) satisfies the constructibility hypothesis that it is pointwise finite-dimensional and there is a finite set \( T = \{ t_1, \ldots, t_n \} \subseteq \mathbb{R}, t_i < t_{i+1}, 1 \leq i < n \), such that for any \( s, t \in \mathbb{R} \) satisfying \( s \leq t < t_1 \), \( t_n < s \leq t \), or \( t_i < s \leq t < t_{i+1} \), for some \( i \), the relation \( v'_s \) is an isomorphism. Then, \( \dim F(I) < \infty \), for any interval \( I \), implying that \( F \) is tame.

Proposition 6 Let \( F \) be a p-sheaf and \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) an ascending sequence of intervals. Then, the following holds:

(i) If \( F \) satisfies the DCC on images and \( I \subseteq \cap I_n \), then \( \text{Im}(F^{I_n^{mf}}) = \text{Im}(F^{I_n^{fm}}) \) for \( m \) large enough;

(ii) If \( F \) satisfies the DCC on kernels and \( \cup I_n \subseteq I \), then \( \ker(F^{I_n^{fm}}) = \ker(F^{I_m^{fm}}) \) for \( m \) large enough.

Proof (i) Set \( I_0 = I \). For any \( n \geq 0 \), the DCC on images, applied to the interval \( I_n \) and the chain \( I_{n+1}^I \subseteq I_{n+2}^I \subseteq \ldots \), ensures that there exists \( N(n) > n \) such that \( \text{Im}(F^{I_n^{fm}}(s)) = \text{Im}(F^{I_0^{fm}}(s)), \forall m \geq N(n) \). Set \( V_n = \text{Im}(F^{I_0^{fm}}(s)) \subseteq F(I_n) \). Then, \( F^{I_n^{fm}}(V_m) = V_n \), for any \( m \geq n \). By construction, for any \( s_0 \in V_0 \), there is a sequence of sections \( s_n \in V_n, n \geq 1 \), such that \( F^{I_n}(s_n) = s_n \), for \( m \geq n \). By the connective gluing property, there exists \( s \in F(\cup I_n) \) such that \( F^{I_n}(s) = s_n, \forall n \geq 0 \). This implies that \( \text{Im}(F^{I_n}(s)) \supseteq V_0 = \text{Im}(F^{I_n^{fm}}(s)), \forall m \geq N(0) \). The inclusion \( \text{Im}(F^{I_n}(s)) \subseteq \text{Im}(F^{I_n^{fm}}(s)) \) is clearly satisfied \( \forall m \geq N(0) \), so this proves the claim.

(ii) By the DCC on kernels, we can choose \( n_0 > 0 \) such that \( \ker(F^{I_0^{fm}}(s)) = \ker(F^{I_n^{fm}}(s)) \), for any \( n \geq n_0 \). Clearly, \( \ker(F^{I_{n_0}^{fm}}(s)) \subseteq \ker(F^{I_n^{fm}}(s)) \). For the opposite inclusion, let \( s \in \ker(F^{I_{n_0}^{fm}}(s)) \), so that \( s_{|_{n_0}} = 0, \forall n > n_0 \). By locality, \( s_{|_{I_n}} = 0 \), which implies \( s \in \ker(F^{I_{n_0}^{fm}}(s)) \). Hence, \( \ker(F^{I_n^{fm}}(s)) \subseteq \ker(F^{I_n^{fm}}(s)) \), concluding the proof.

Our next goal is to prove a version of Proposition 6 for virtually tame c-modules. We begin with an extension result for sections of a c-module.

Proposition 7 Let \( \mathcal{V} \) be a c-module and \( A \subseteq \mathbb{R} \). If \( \mathcal{V} \) is virtually tame, then any section of \( \mathcal{V} \) over \( A \) can be extended to a section over an interval containing \( A \).

Proof Let \( f \) be a section over \( A \) and let \( S \) be the set of all sections that extend \( f \) over some set containing \( A \). For \( g, h \in S \), write \( g \leq h \) to mean that \( h \) extends \( g \). Note that \( (S, \leq) \) is a poset in which each ascending chain has an upper bound. By Zorn’s lemma, there exists a maximal element \( \bar{f} \in S \). We claim that the domain of \( \bar{f} \) is an interval. Suppose not, then there exists \( r \notin \text{Dom}(\bar{f}) \) such that \( L = \{ t \in \text{Dom}(\bar{f}) \mid t < r \} \) and \( U = \{ t \in \text{Dom}(\bar{f}) \mid r < t \} \) are not empty. Choose an increasing sequence \( \{ \ell'_n \} \subseteq L \) with \( \lim \ell'_n = \sup L \) and a decreasing sequence \( \{ u_n \} \subseteq U \) with \( \lim u_n = \inf U \). If \( \sup L \in L \), we assume that \( \ell'_n = \sup L, \forall n \). Similarly, \( u_n = \inf U \), for every \( n \), if \( \inf U \in U \). For \( n \geq 1 \), set

\[
\ell'_n = \sup L, \forall n.
\]

Similarly, \( u_n = \inf U \), for every \( n \), if \( \inf U \in U \). For \( n \geq 1 \), set...
\[ \text{Im}^r_{\mathcal{E}, u_n} = \{ v \in V_r \mid (\tilde{f}(\mathcal{E}_n), v) \in v^r_{\mathcal{E}_n} \text{ and } (v, \tilde{f}(u_n)) \in v_{u_n}^r \}, \quad (5) \]
a non-empty affine subspace of \( V_r \). Note that these affine subspaces form a nested sequence
\[ \text{Im}^r_{\mathcal{E}, u_1} \supseteq \text{Im}^r_{\mathcal{E}, u_2} \supseteq \text{Im}^r_{\mathcal{E}, u_3} \supseteq \ldots \]
We show that this sequence is stable. To this end, let
\[ \ker^r_{\mathcal{E}, u_n} = \{ v \in V_r \mid (0, v) \in v^r_{\mathcal{E}_n} \text{ and } (v, 0) \in v_{u_n}^r \}, \quad (7) \]
which also form a nested sequence
\[ \ker^r_{\mathcal{E}, u_1} \supseteq \ker^r_{\mathcal{E}, u_2} \supseteq \ker^r_{\mathcal{E}, u_3} \supseteq \ldots \]
Note that the stability of the sequence \( \ker^r_{\mathcal{E}, u_n} \) implies the stability of \( \text{Im}^r_{\mathcal{E}, u_n} \). Indeed, suppose \( \exists N \) such that \( \ker^r_{\mathcal{E}, u_n} = \ker^r_{\mathcal{E}, u_N} \forall n \geq N \). For any \( v_n \in \text{Im}^r_{\mathcal{E}, u_n} \), we may write
\[ \text{Im}^r_{\mathcal{E}, u_N} = v_n + \ker^r_{\mathcal{E}, u_n}, \quad (9) \]
Since the choice of \( v_N \in \text{Im}^r_{\mathcal{E}, u_N} \) in (9) is arbitrary, using (6) and the stability of kernels, we have
\[ \forall n \geq N, \text{ as claimed. The stability of (8) follows from } \ker^r_{\mathcal{E}, u_n} = \ker(v^r_{\mathcal{E}_n})^\ast \cap \ker(v_{u_n}^r) \text{ and the fact that the right-hand side of this equation stabilizes by virtual tameness. To conclude, pick } v \in \cap \text{Im}^r_{\mathcal{E}, u_n} \text{ and extend } \tilde{f} \text{ to } r \text{ via the assignment } \tilde{f}(r) = v. \text{ This contradicts the maximality of } \tilde{f}. \]

**Proposition 8** If \( F \) is the \( p \)-sheaf of sections of a virtually tame \( c \)-module \( \mathcal{V} \), then for any ascending sequence of intervals \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) the following holds:

(i) If \( A \subseteq \cap I_n \) is finite, then \( \text{Im}(F^m_{I_n}) = \text{Im}(F^m_A) \) for \( m \) large enough;
(ii) If \( A \subseteq I \supseteq \cup I_n \), where \( I \) is an interval and \( A \) is a finite set, then \( \ker(F^m_{\cup I_n})|_A = \ker(F^m_{I_n})|_A \) for \( m \) large enough.

**Proof** (i) Since \( F^m_{I_n} = F^m_{I} \circ F^m_{I_n} \), it follows that \( \text{Im}(F^m_{I_n}) \subseteq \text{Im}(F^m_A), \forall m \). For the reverse inclusion, let \( f \in \text{Im}(F^m_A) \). We show that, for \( m \) sufficiently large, there is a section \( g \) defined over \( \cup I_n \) such that \( g|_A = f \). Let \( s_0 = \min A \) and \( t_0 = \max A \). We construct \( g \) by gluing sections over the following three intervals: \( I_0 = [s_0, t_0] \), \( I_- = \{ x \in \cup I_n : x \leq s_0 \} \), and \( I_+ = \{ x \in \cup I_n : x \geq t_0 \} \).

Let \( [s_n, t_n] \subseteq I_n \), \( n \geq 1 \), be a sequence of intervals such that \( [s_n, t_n] \subseteq [s_{n+1}, t_{n+1}] \), \( A \subseteq \cap [s_n, t_n] \), and \( \cup [s_n, t_n] = \cup I_n \), \( \forall n \geq 1 \). Note that we also have
A \subseteq [s_0, t_0] \subseteq [s_1, t_1]. For each \( n \geq 0 \), the virtual tameness of \( \forall \) implies that the descending chains

\[
\text{Im}(v_{s_n}^{n}) \supseteq \text{Im}(v_{s_n}^{n+1}) \supseteq \ldots \quad \text{and} \quad \text{Dom}(v_{s_n}^{n+1}) \supseteq \text{Dom}(v_{s_n}^{n+2}) \supseteq \ldots
\]

are stable. Thus, \( \exists N_n \geq n \) such that \( \text{Im}(v_{s_n}^{n}) \) and \( \text{Dom}(v_{s_n}^{n}) \) are constant for \( m \geq N_n \). Set \( L_n := \text{Im}(v_{s_n}^{n}) \) and \( U_n := \text{Dom}(v_{s_n}^{n}) \), for \( m \geq N_n \). By construction, for any \( x_n \in L_n \) and \( y_n \in U_n \), \( n \geq 0 \), there exist \( x_{n+1} \in L_{n+1} \) and \( y_{n+1} \in U_{n+1} \) such that

\[
(x_{n+1}, x_n) \in v_{s_{n+1}}^{n} \quad \text{and} \quad (y_{n}, y_{n+1}) \in v_{s_{n+1}}^{n+1}.
\]

If \( f \in \text{Im}(F^{I}_{A_n}) \) and \( m \geq N_0 \), then \( x_0 := f_{s_0} \in L_0 \) and \( y_0 := f_{t_0} \in U_0 \). Iteratively, as described above, construct \( x_n \in L_n \) and \( y_n \in U_n \) satisfying (11). The sequences \( \{x_n\} \) and \( \{y_n\} \), \( n \geq 0 \), together with \( f \), yield a section of \( \forall \) over the set \( \{s_n : n \geq 0\} \cup A \cup \{t_n : n \geq 0\} \). By Proposition 7, this section can be extended to a section over \( \cup I_n \) with the desired properties.

(ii) \( F^{I}_{I_m} = F^{I}_{I_m} \circ F^{I}_{\cup I_m} \) implies that \( \ker(F^{I}_{\cup I_m}) \subseteq \ker(F^{I}_{I_m}) \). Thus, \( \ker(F^{I}_{\cup I_m}) \mid A \subseteq \ker(F^{I}_{I_m}) \mid A \), for every \( m \). For the reverse inclusion, let \( A_0 = A \cap (\cup I_n), A_- = \{a \in A : a < t, \forall t \in \cup I_n\} \) and \( A_+ = \{a \in A : a > t, \forall t \in \cup I_n\} \). Set \( r_- = \max A_-, r_+ = \min A_+ \),

\[
I_- = I \setminus (r_-, +\infty) \quad \text{and} \quad I_+ = I \setminus (-\infty, r_+).
\]

Let \( f \in \ker(F^{I}_{I_m}) \mid A \). Since \( A \) is finite, \( \exists N_0 > 0 \) such that \( A_0 \subseteq I_m \), for \( m \geq N_0 \). Hence, \( f \mid A_0 \equiv 0 \) if \( m \geq N_0 \). Pick a section \( \hat{f} \in \ker(F^{I}_{I_m}) \) such that \( \hat{f} \mid A = f \) and set \( g = \hat{f} \mid \cup I_n \).

Next, we show that we can extend \( g \) to a section \( h \) over \( I_- \cup I_+ \cup \cup I_n \), such that \( h \mid \cup I_n \equiv 0 \), provided that \( m \) is sufficiently large. Note that any such \( h \) will have the property that \( h \mid A = f \).

Let \( [s_n, t_n], n \geq 1 \), be as in the proof of (i). By construction, \( r_- < \ldots \leq s_2 \leq s_1 \) and \( t_1 \leq t_2 \leq \ldots < r_+ \). By virtual tameness, there exists \( n_0 \geq N_0 \) such that the chains

\[
\ker(v_{s_n}^{r_-}) \supseteq \ker(v_{s_n}^{r_+}) \supseteq \ldots \quad \text{and} \quad \ker(v_{t_1}^{r_-}) \supseteq \ker(v_{t_1}^{r_+}) \supseteq \ldots
\]

are stable at \( s_n \) and \( t_n \), \( n \geq n_0 \), respectively. Hence, if \( f \in \ker(F^{I}_{I_m}) \mid A, m \geq n_0 \), we have that \( g_{r_-} \in \ker(v_{s_n}^{r_-}) \) and \( g_{r_+} \in \ker(v_{t_1}^{r_+}) \), for any \( n \geq n_0 \), which implies that the section \( g \) can be extended, as claimed. By Proposition 7, we can further extend \( g \) to a section over \( I \). This concludes the proof.

5 Decomposition Theorems

In this section, we prove interval decomposition theorems for tame \( p \)-sheaves and virtually tame \( c \)-modules that lead to representations of their structures by barcodes or persistence diagrams. We develop a sheaf-theoretical analogue of the techniques
employed by Crawley-Boevey to obtain such decompositions for pointwise finite-dimensional persistence modules [18].

5.1 Coverings

The arguments and constructions in [18] use the notion of sections of a vector space whose definition we recall next. To avoid confusion with sections of $c$-modules and $p$-sheaves, we rename them splittings.

**Definition 22** (*Splittings of Vector Spaces*, cf. [18])

(i) A splitting of a vector space $V$ is a pair $(F^-, F^+)$ of subspaces $F^- \subseteq F^+ \subseteq V$.

(ii) A collection $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ of splittings of $V$ is disjoint if for all $\lambda \neq \mu$, either $F^-_\lambda \subseteq F^-_\mu$ or $F^+_\mu \subseteq F^+_\lambda$.

(iii) A collection of splittings $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ covers $V$ if for any subspace $U \subseteq V$, with $U \neq V$, $\exists \lambda \in \Lambda$ such that $U + F^-_\lambda \neq U + F^+_\lambda$; and it strongly covers $V$ provided that for all subspaces $U, W \subseteq V$ with $W \nsubseteq U$, $\exists \lambda \in \Lambda$ such that $U + (F^-_\lambda \cap W) \neq U + (F^+_\lambda \cap W)$.

**Proposition 9** (Crawley-Boevey [18]) Let $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ be a set of splittings that is disjoint and covers $V$.

(i) If $W_\lambda$ is a complement of $F^-_\lambda$ in $F^+_\lambda$, $\forall \lambda \in \Lambda$, then the inclusions $W_\lambda \subseteq V$ induce a direct sum decomposition $V = \bigoplus_{\lambda \in \Lambda} W_\lambda$.

(ii) If $\{(G^-_\sigma, G^+_\sigma) : \sigma \in \Sigma\}$ is another set of splittings that is disjoint and strongly covers $V$, then the family

$$\{(F^-_\lambda + G^-_\sigma \cap F^+_\lambda, F^+_\lambda + G^+_\sigma \cap F^+_\lambda) : (\lambda, \sigma) \in \Lambda \times \Sigma\}$$

of splittings is disjoint and covers $V$.

**Corollary 1** Suppose that $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ is disjoint and covers $V$ and $\{(G^-_\sigma, G^+_\sigma) : \sigma \in \Sigma\}$ is disjoint and strongly covers $V$. If $W_{\sigma, \lambda}$ is a complement of $(F^-_\lambda \cap G^+_\sigma) + (G^-_\sigma \cap F^+_\lambda)$ in $G^+_\sigma \cap F^+_\lambda$, then:

(i) The inclusions $W_{\sigma, \lambda} \subseteq V$ induce a decomposition $V = \bigoplus_{\sigma, \lambda} W_{\sigma, \lambda}$.

(ii) For any $\lambda \in \Lambda$, the inclusions $F^-_\lambda, W_{\sigma, \lambda} \subseteq F^+_\lambda$ induce a direct sum decomposition $F^+_\lambda = F^-_\lambda \bigoplus (\bigoplus_{\sigma} W_{\sigma, \lambda})$.

**Proof** (i) Note that

$$\frac{G^+_\sigma \cap F^+_\lambda}{F^-_\lambda \cap G^+_\sigma + G^-_\sigma \cap F^+_\lambda} \cong \frac{F^-_\lambda + G^+_\sigma \cap F^+_\lambda}{F^-_\lambda + G^-_\sigma \cap F^+_\lambda}. \quad (13)$$
Since the isomorphism in (13) is induced by inclusion, a complement of \((F^-_\lambda \cap G^+_{\sigma}) + (G^-_{\sigma} \cap F^+_\lambda)\) in \(G^+_{\sigma} \cap F^+_\lambda\) is also a complement of \(F^-_\lambda + G^-_{\sigma} \cap \mathcal{F}^+_\lambda\). Thus, the claim follows from Proposition 9.

(ii) Since \(W_{\sigma, \lambda}\) is a complement of \(F^-_\lambda + G^-_{\sigma} \cap \mathcal{F}^+_\lambda\) in \(F^-_\lambda + G^-_{\sigma} \cap \mathcal{F}^+_\lambda\), we have \(W_{\sigma, \lambda} \cap \mathcal{F}^-_\lambda = 0\) and \(W_{\sigma, \lambda} \subseteq \mathcal{F}^+_\lambda\), \(\forall \sigma \in \Sigma\). For a fixed \(\lambda\), to simplify notation, set \(H^-_{\sigma} := F^-_\lambda + G^-_{\sigma} \cap \mathcal{F}^+_\lambda\) and \(H^+_{\sigma} := F^+_\lambda + G^+_{\sigma} \cap \mathcal{F}^+_\lambda\). Now, we show that \(\bigoplus_{\sigma} W_{\sigma, \lambda} \cap \mathcal{F}^-_\lambda = 0\). Let \(v_1 + v_\sigma + \cdots + v_n = 0\) be a relation with \(v_\lambda \in \mathcal{F}^-_\lambda\) and \(v_\sigma \in W_{\sigma, \lambda}\). Since the splittings \(\{(G^-_{\sigma}, G^+_{\sigma}) : \sigma \in \Sigma\}\) of \(\mathcal{F}^+_\lambda\) are disjoint, so are the splittings \(\{(H^-_{\sigma}, H^+_{\sigma}) : \sigma \in \Sigma\}\) of \(\mathcal{F}^+_\lambda\). By disjointness, we may assume that \(H^+_{\sigma_1} \subseteq H^-_{\sigma_{i+1}}\) for all \(i < n\). Thus, \(v_{\sigma_1} = -v_{\lambda} - v_{\sigma_1} - \cdots - v_{\sigma_n} \in H^-_{\sigma_n}\) because \(v_{\lambda} \in \mathcal{F}^-_\lambda \subseteq H^-_{\sigma_n}\) and \(v_{\sigma_i} \in H^+_{\sigma_i} \subseteq H^-_{\sigma_n}\). Hence, \(v_{\sigma_n} \in H^-_{\sigma_n} \cap W_{\sigma_n, \lambda} = 0\). Similarly, we show that all other terms in the relation vanish. Therefore, \(\bigoplus_{\sigma} W_{\sigma, \lambda} \cap \mathcal{F}^-_\lambda = 0\) and \(\bigoplus_{\sigma} W_{\sigma, \lambda} \subseteq \mathcal{F}^+_\lambda\).

Choose \(W_{\lambda} \subseteq \mathcal{F}^+_\lambda\) such that \(W_{\lambda} \oplus \left( \bigoplus_{\sigma} W_{\sigma, \lambda} \right)\) is a complement of \(\mathcal{F}^-_\lambda\) in \(\mathcal{F}^+_\lambda\). Since the set of splitting \(\{(\mathcal{F}^-_\lambda, \mathcal{F}^+_\lambda) : \lambda \in \Lambda\}\) is disjoint and covers \(V\), it follows that

\[
\left( \bigoplus_{\lambda} W_{\lambda} \right) \oplus \left( \bigoplus_{\sigma, \lambda} W_{\sigma, \lambda} \right) = V. \quad (14)
\]

On the other hand, by (i), \(\bigoplus_{\sigma, \lambda} W_{\sigma, \lambda} = V\). Thus, \(W_{\lambda} = 0\), \(\forall \lambda \in \Lambda\). This proves that \(\mathcal{F}^+_\lambda = \mathcal{F}^-_\lambda \oplus \left( \bigoplus_{\sigma} W_{\sigma, \lambda} \right)\).

Let \(F\) be a \(p\)-sheaf and \(x, y, p, q \in \mathbb{E}\) satisfy \(x \leq p < q \leq y\). We use the following abbreviations:

\[
\begin{align*}
\text{Im}^{(x, y)}_{(p, q)} &= \text{Im}(F^{(x, y)}_{(p, q)}) \\
\text{ker}^{(x, y)}_{(p, q)} &= \text{ker}(F^{(x, y)}_{(p, q)}) \\
\text{Im}^{(x, y)}_{(p, q)} &= \bigcup_{z \leq x} \text{Im}^{(z, y)}_{(p, q)} \\
\text{Im}^{(x, y)}_{(p, q)} &= \bigcup_{y < z} \text{Im}^{(x, z)}_{(p, q)} \\
\text{ker}^{(x, y)}_{(p, q)} &= \bigcup_{x \leq z < p} \text{ker}^{(z, y)}_{(p, q)} \\
\text{ker}^{(x, y)}_{(p, q)} &= \bigcup_{q < z \leq y} \text{ker}^{(x, z)}_{(p, q)},
\end{align*}
\]

with the convention that \(\text{Im}^{(x, y)}_{(p, q)} = 0\) if \(x = -\infty\), and \(\text{Im}^{(x, y)}_{(p, q)} = 0\) if \(y = +\infty\). Similarly, for kernels, we make the convention that \(\text{ker}^{(x, y)}_{\emptyset} = F(x, y)\), for any \(x < y\).

\[
\begin{align*}
\text{ker}^{(x, y)}_{(p, q)} &= \text{ker}^{(x, y)}_{(x, y)} \cap \text{ker}^{(x, y)}_{(p, q)} \\
\text{ker}^{(x, y)}_{(p, q)} &= \text{ker}^{(x, y)}_{(p, q)} \cap \text{ker}^{(x, y)}_{(p, q)} \\
\text{ker}^{(x, y)}_{(p, q)} &= \text{ker}^{(x, y)}_{(x, p)} \cap \text{ker}^{(x, y)}_{(p, q)} \\
\text{ker}^{(x, y)}_{(p, q)} &= \text{ker}^{(x, y)}_{(x, p)} \cap \text{ker}^{(x, y)}_{(q, y)}.
\end{align*}
\]

\textbf{Lemma 5} \textbf{(Covering Lemma for Sheaves)} Let \(F\) be a \(p\)-sheaf and \(p, q \in \mathbb{E}\) with \(p < q\). Then,

(i) \(\{(\text{Im}^{(x, y)}_{(p, q)}), x \leq p\}\) is a disjoint set of splittings of \(F(p, q)\);

(ii) \(\{(\text{Im}^{(p, y)}_{(p, q)}), y \geq q\}\) is a disjoint set of splittings of \(F(p, q)\);
Furthermore, if \( F \) satisfies the DCC on images, the splittings in (i) and (ii) form strong coverings. Similarly, if \( F \) satisfies the DCC on kernels, the splittings in (iii) and (iv) form strong coverings. In particular, if \( F \) is tame, each of the above sets of splittings strongly covers the corresponding vector space of sections.

**Proof** (i) For \( x \leq p \), we use the abbreviations \( F_x^- = \text{Im}\,(x,q)_{(p,q)} \) and \( F_x^+ = \text{Im}\,(x,q)_{(p,q)} \). It is simple to check that \( \{ (F_x^- , F_x^+) : x \leq p \} \) is a disjoint set of splittings. To verify the strong covering property under the assumption that \( F \) satisfies the DCC on images, let \( U, W \subseteq F(p,q) \) be subspaces with \( W \not\subseteq U \). Set

\[
S = \{ (z,q) : z \leq p \text{ and } \text{Im}\,(z,q)_{(p,q)} \cap W \not\subseteq U \}
\]  

(15)

Note that \( S \neq \emptyset \) because \( (p,q) \in S \). Let \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \) be a sequence of intervals from \( S \) such that \( \bigcup_{I \in S} I = \bigcup_{n \geq 1} I_n \). Write the interval \( I_n \) as \( I_n = (x_0,q) \), with \( x_0 \leq p \). By Proposition 6,

\[
F_{x_0}^+ \cap W = \text{Im}\,(x_0,q)_{(p,q)} \cap W = \text{Im}\,m_{(p,q)} \cap W \not\subseteq U,
\]

(16)

for \( m \) sufficiently large. On the other hand, for any \( z < x_0 \), we have that \( (z,q) \notin S \). This implies that \( \text{Im}\,(z,q)_{(p,q)} \cap W \subseteq U \). Therefore,

\[
F_{x_0}^- \cap W = \text{Im}\,(x_0,q)_{(p,q)} \cap W = \bigcup_{z < x_0} \text{Im}\,(z,q)_{(p,q)} \cap W \subseteq U.
\]

(17)

It follows from (16) and (17) that

\[
U + F_{x_0}^- \cap W \neq U + F_{x_0}^+ \cap W,
\]

(18)

concluding the argument. The proofs of the other statements are similar. \( \square \)

**Lemma 6** *(Covering Lemma for Modules)* Let \( F \) be the \( p \)-sheaf of sections of a \( c \)-module \( \forall \) and \( p,q \in \mathbb{E} \) with \( p < q \). Then,

(i) For any \( A \subseteq (p,q) \), \( \{ (\text{Im}\,(x,q)_{(p,q)} | A), \text{Im}\,(x,q)_{(p,q)} | A) : x \leq p \} \) is a disjoint set of splittings of \( F(p,q) | A \);

(ii) For any \( A \subseteq (p,q) \), \( \{ (\text{Im}\,(y,q)_{(p,q)} | A), \text{Im}\,(y,q)_{(p,q)} | A) : y \geq q \} \) is a disjoint set of splittings of \( F(p,q) | A \);

(iii) For any \( A \subseteq (\infty,q) \), \( \{ (\ker\,(x,q)_{(p,q)} | A), \ker\,(x,q)_{(p,q)} | A) : x < q \} \) is a disjoint set of splittings of \( F(\infty,q) | A \);

(iv) For any \( A \subseteq (p,\infty) \), \( \{ (\ker\,(y,q)_{(p,q)} | A), \ker\,(y,q)_{(p,q)} | A) : y > p \} \) is a disjoint set of splittings of \( F(p,\infty) | A \).
Moreover, if \( \mathcal{V} \) is virtually tame and \( A \) is finite, then each of the above set of splittings is a strong cover.

**Proof** The proof is nearly identical to that of Lemma 5. The only changes needed are to restrict the relevant vector spaces of sections to \( A \) and to use Proposition 8 in lieu of Proposition 6 in the tameness argument. \( \square \)

### 5.2 Decomposition of Tame \( p \)-Sheaves

Our next goal is to decompose a tame \( p \)-sheaf \( F \) as a direct sum of “atomic” subsheaves that are sheaf-theoretical analogues of interval \( c \)-modules. The building blocks of this decomposition are described next.

**Definition 23** For any \( p \)-sheaf \( F \) and \( x, y \in \mathbb{E} \) with \( x < y \), we let:

(i) \( F[x, y] \) be a complement of \( \text{Im}^{x,y} + \text{Im}^{x,y} \) in \( \text{Im}^{x,y} = F(x, y); \)

(ii) \( F[x, y) \) be a complement of \( \ker^{x, +\infty} + \ker^{y, +\infty} \) in \( \ker^{x, +\infty}, \)

(iii) \( F(x, y) \) be a complement of \( \ker^{(-\infty, y)} + \ker^{(-\infty, y)} \) in \( \ker^{(-\infty, y)} \)

(iv) \( F(x, y) \) be a complement of \( \ker^{x, y} + \ker^{x, y} \) in \( \ker^{x, y} \).

**Proposition 10** For any \( p \)-sheaf \( F \), the following statements hold:

(i) If \( x, y \in \mathbb{E}, x < y, \) and \( I \subseteq (x, y) \) is an interval, then

\[
\text{Im}^{x,y} = F[x, y]_I \oplus \left( \text{Im}^{x,y} + \text{Im}^{x,y} \right)_I;
\]

(ii) If \( x, y \in \mathbb{E}, x < y, \) and \( I \subseteq (x, +\infty) \) is an interval, then

\[
\ker^{x, +\infty} = F[x, y]_I \oplus \left( \ker^{x, +\infty} + \ker^{x, +\infty} \right)_I;
\]

(iii) If \( x, y \in \mathbb{E}, x < y, \) and \( I \subseteq (-\infty, y) \) is an interval, then

\[
\ker^{(-\infty, y)} = F[x, y]_I \oplus \left( \ker^{(-\infty, y)} + \ker^{(-\infty, y)} \right)_I;
\]

(iv) If \( x, y \in \mathbb{E}, x < y, \) and \( I \) is an interval, then

\[
\ker^{x, y} = F[x, y]_I \oplus \left( \ker^{x, y} + \ker^{x, y} \right)_I.
\]

**Proof** Here we just prove (ii), the proofs of the other statements being similar. Since, by definition, \( \ker^{x, +\infty} = F[x, y] \oplus \left( \ker^{x, +\infty} + \ker^{x, +\infty} \right) \), we just need to show that
the two summands in this direct sum decomposition remain independent after restriction to $I$; that is,
\[ F[x,y]_I \cap \left( \ker_{(y,\infty)}^{(x,\infty)} I + \ker_{(\infty,y)}^{(x,\infty)} I \right) = 0. \] (19)

Given $s_0 \in F[x,y]$ and $s_1 \in \left( \ker_{(y,\infty)}^{(x,\infty)} + \ker_{(\infty,y)}^{(x,\infty)} \right)$ with $s_0|_I = s_1|_I$, let $s = s_0 - s_1$. Clearly, $s|_I = 0$. If $I \cap (x, y) = \emptyset$, the left-hand side of (19) equals 0, so the proof is trivial. If $I \cap (x, y) \neq \emptyset$, by the gluing property, $s$ is the sum of two sections from $\ker_{(y,\infty)}^{(x,\infty)} + \ker_{(\infty,y)}^{(x,\infty)}$. Hence, $s \in \ker_{(y,\infty)}^{(x,\infty)} + \ker_{(\infty,y)}^{(x,\infty)}$, which implies that $s_0 = s_1 + s \in \ker_{(y,\infty)}^{(x,\infty)} + \ker_{(\infty,y)}^{(x,\infty)}$, so that $s_0 = s_1 = 0$. The result follows.

For $x, y \in \mathbb{E}$, $x < y$, we may view $F[x,y]$ as a persistence presheaf with $F[x,y](I) = F[x,y]|_I$ if $I \subseteq (x, +\infty)$, and $F[x,y](I) = 0$, otherwise. Morphisms are induced by restriction of sections of $F$. Similarly, we may treat $F(x,y)$, $F(x,y)$ and $F[x,y]$ as persistence presheaves, where $F(x,y)(I) = 0$ if $I \not\subseteq (-\infty, y)$ and $F[x,y](I) = 0$ if $I \not\subseteq (x, y)$. Henceforth, we refer to these interchangeably as $p$-sheaves or spaces of sections, the meaning determined by the context.

Let $m[x,y] = \dim(F[x,y])$, $m[x,y] = \dim(F(x,y))$, $m(x,y) = \dim(F(x,y))$, and $m(x,y) = \dim(F(x,y))$.

**Proposition 11** If $F$ is a $p$-sheaf and $x, y \in \mathbb{E}$, then:

1. $F[x,y] \cong m[x,y]k[x,y]$, if $-\infty < x < y < +\infty$;
2. $F[x,y] \cong m[x,y]k[x,y]$, if $-\infty < x < y \leq +\infty$;
3. $F(x,y) \cong m[x,y]k[x,y]$, if $-\infty \leq x < y < +\infty$;
4. $F(x,y) \cong m(x,y)k(x,y)$, if $-\infty \leq x < y \leq +\infty$.

**Proof** We prove (ii), the proofs of the other statements being similar. Choose a basis $B = \{ s_\lambda : \lambda \in \Lambda \}$ of the space of sections $F(x,y)$. By definition, for each $\lambda \in \Lambda$, $\text{Dom}(s_\lambda) = (x, +\infty)$. Note that $\text{supp}(s_\lambda) = (x, y)$. Indeed, suppose $\exists t \in (x, y)$ such that $s_\lambda(t) = 0$. Then, we may write $s_\lambda$ as the sum of sections in $\ker_{(y,\infty)}^{(x,\infty)}$ and $\ker_{(\infty,y)}^{(x,\infty)}$, which implies that $s_\lambda = 0$, a contradiction. By Proposition 5, the $p$-sheaf generated by $s_\lambda$ satisfies $F(s_\lambda) \cong k(x,y)$. Since $B$ is a basis, $F(x,y) \cong m[x,y]k[x,y]$.

**Lemma 7** (Decomposition Lemma) Let $F$ be a tame $p$-sheaf. If $p, q \in \mathbb{E}$, $p < q$, then the space of sections $F(p,q)$ may be decomposed as
\[ F(p,q) = \bigoplus_{-\infty < x \leq y \leq p < q < +\infty} F[x,y]|_I \bigoplus_{-\infty < x < y \leq p < q \leq +\infty} F[x,y]|_I \bigoplus_{-\infty < x \leq y < p < q < +\infty} F(x,y)|_I \bigoplus_{-\infty < x < y \leq p < q \leq +\infty} F(x,y)|_I, \]
where $I = (p,q)$. 

\[ \text{Springer} \]
Proof The proof of the lemma is in three steps, each providing a decomposition of $F(p, q)$ that is gradually refined to the target decomposition.

Step 1. Since $F$ satisfies the DCC on images, by Lemma 5 the families of splittings

$$(F_x^-, F_x^+), \quad -\infty \leq x \leq p,$$

and

$$(G_y^-, G_y^+), \quad q \leq y \leq +\infty,$$

are disjoint and strongly cover $F(p, q)$. By the connective gluing property for sections, we have:

$$F_x^+ \cap G_y^+ = \text{Im}(x, y) \cap \text{Im}(p, q) = \text{Im}(x, y),$$

$$F_x^- \cap G_y^+ = \text{Im}(x, y) \cap \text{Im}(p, q) = \text{Im}(x, y),$$

$$F_x^+ \cap G_y^- = \text{Im}(x, y) \cap \text{Im}(p, q) = \text{Im}(x, y).$$

Therefore,

$$\frac{F_x^+ \cap G_y^+}{(F_x^- \cap G_y^+) + (F_x^+ \cap G_y^-)} = \frac{\text{Im}(x, y)_{(p, q)}}{\text{Im}(x, y)_{(p, q)} + \text{Im}(x, y)_{(p, q)}}$$

$$= \frac{\text{Im}(x, y)_{(x, y)}}{\text{Im}(x, y)_{(x, y)} + \text{Im}(x, y)_{(x, y)}}$$

(23)

where the last equality follows from the fact that $I = (p, q) \subseteq (x, y)$. By Proposition 10(i) and (23), $F[x, y]|_I$ is a complement of $(F_x^- \cap G_y^+) + (F_x^+ \cap G_y^-)$ in $F_x^+ \cap G_y^+$.

Thus, Corollary 1(i) implies that

$$F(p, q) = \bigoplus_{-\infty < x \leq p < y \leq +\infty} F[x, y]|_I \bigoplus_{-\infty < x \leq p} F[x, +\infty]|_I$$

$$+ \bigoplus_{q \leq y < +\infty} F[-\infty, y]|_I \bigoplus F[-\infty, +\infty]|_I.$$ (24)

Note that, up to this point in the proof, we only have used the DCC on images, not the full tameness of $F$. Before proceeding to the next step recall that, according to Definition 23(i),

(a) $F[x, +\infty]|_I$ is a complement of $\text{Im}(x, +\infty)_{(x, +\infty)}|_I$ in $\text{Im}(x, +\infty)_{(x, +\infty)}|_I$;

(b) $F[-\infty, y]|_I$ is a complement of $\text{Im}(-\infty, y)_{(-\infty, y)}|_I$ in $\text{Im}(-\infty, y)_{(-\infty, y)}|_I$;

(c) $F[-\infty, +\infty]|_I = F(-\infty, +\infty)|_I$. 

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Step 2. Now we show that, in (24), the summands $F[x, +\infty]|_I$, $-\infty < x \leq p$, may be replaced with $\bigoplus_{p < x \leq +\infty} F(x,y)|_I$. Consider the families of splittings

$$ (F^-_x, F^+_x) = \left( \text{Im}^{(x, +\infty)}_{(p, +\infty)}|_I, \text{Im}^{(x, +\infty)}_{(p, +\infty)}|_I \right), $$

\(-\infty \leq x \leq p\), and

$$ (G^-_y, G^+_y) = \left( \ker^{(p, +\infty)}_{(y, +\infty)}|_I, \ker^{(p, +\infty)}_{(y, +\infty)}|_I \right), $$

\(p < y \leq +\infty\), that are disjoint and strongly cover $F(p, +\infty)|_I$. Using the gluing property, one may verify that

$$ \frac{F^+_x \cap G^+_y}{(F^-_x \cap G^+_y) + (F^+_x \cap G^-_y)} = \frac{\ker^{(x, +\infty)}_{(y, +\infty)}|_I}{\ker^{(x, +\infty)}_{(y, +\infty)}|_I + \ker^{(x, +\infty)}_{(y, +\infty)}|_I}. $$

It follows from Proposition 10(ii) and (27) that $F(x,y)|_I$ is a complement of $(F^-_x \cap G^+_y) + (F^+_x \cap G^-_y)$ in $F^+_x \cap G^+_y$. Corollary 1(ii), applied to $x = p$, gives

$$ \text{Im}^{(x, +\infty)}_{(x, +\infty)}|_I = \text{Im}^{(x, +\infty)}_{(x, +\infty)}|_I \bigoplus_{p < y \leq +\infty} F(x,y)|_I. $$

Thus, we may choose $F[x, +\infty]|_I$ to be $\bigoplus_{p < y \leq +\infty} F(x,y)|_I$.

Similarly, we may choose $F[-\infty, y]|_I$ to be $\bigoplus_{p < y \leq +\infty} F(x,y)|_I$. Therefore, we may rewrite (24) as

$$ F(p,q) = \bigoplus_{-\infty < q < p < y < +\infty} F[x,y]|_I \bigoplus_{-\infty < x < p < y < +\infty} F(x,y)|_I \bigoplus_{-\infty < x < q < y < +\infty} F(x,y)|_I. $$

Step 3. To complete the proof, we decompose the last summand in (29). Consider the families of splittings

$$ (F^-_x, F^+_x) = \left( \ker^{(-\infty, +\infty)}_{(-\infty, x)}|_I, \ker^{(-\infty, +\infty)}_{(-\infty, x)}|_I \right), $$

\(-\infty \leq x < +\infty\), and

$$ (G^-_y, G^+_y) = \left( \ker^{(-\infty, +\infty)}_{(y, +\infty)}|_I, \ker^{(-\infty, +\infty)}_{(y, +\infty)}|_I \right), $$

\(-\infty < y \leq +\infty\), that are disjoint and strongly cover $F(-\infty, +\infty)|_I$. Arguing as in Step 2 and using the fact that

$$ \frac{F^+_x \cap G^+_y}{(F^-_x \cap G^+_y) + (F^+_x \cap G^-_y)} = \frac{\ker^{(-\infty, +\infty)}_{x,y}|_I}{\ker^{(-\infty, +\infty)}_{x,y}|_I + \ker^{(-\infty, +\infty)}_{x,y}|_I}, $$

we obtain the decomposition
Combining (29) and (33), we obtain the desired decomposition. □

Remark 6 (Tameness Conditions)

(i) As pointed in the proof of Lemma 7, to obtain the decomposition in (24), we do not need the full tameness hypothesis on \( F \), it suffices to assume that \( F \) satisfies the DCC on images. The DCC on kernels only is needed in Steps 2 and 3 in the proof of the Decomposition Lemma.

(ii) Similarly, to obtain (29), in addition to the DCC on images for \( F \), we only need the DCC on kernels for \( F[x, y] \) and \( F[x, y] \), not the tameness of \( F \).

Theorem 1 (Interval Decomposition of p-řmeaves) If \( F \) is a tame p-sheaf, then

\[
F = \bigoplus_{-\infty < x < y < +\infty} F[x, y] \quad \bigoplus_{-\infty < x < y < +\infty} F[x, y] \\
\bigoplus_{-\infty < x < y < +\infty} F[x, y] \quad \bigoplus_{-\infty < x < y < +\infty} F[x, y].
\]

Moreover,

\[
F \cong \bigoplus_{-\infty < x < y < +\infty} m[x, y] \ k[x, y] \quad \bigoplus_{-\infty < x < y < +\infty} m[x, y] \ k[x, y] \\
\bigoplus_{-\infty < x < y < +\infty} m[x, y] \ k[x, y] \quad \bigoplus_{-\infty < x < y < +\infty} m[x, y] \ k[x, y],
\]

where \( m[x, y] \), \( m[x, y] \), \( m[x, y] \), and \( m[x, y] \) denote the multiplicities of the corresponding interval p-sheaves.

Proof To prove (34), since all summands are subsheaves of \( F \), it suffices to verify that the direct sum decomposition is satisfied by the space of sections of \( F \) over each interval \( I \). This is an immediate consequence of Lemma 7 because \( F[x, y](I) = 0 \) if \( I \not\subseteq (x, y) \), \( F[x, y](I) = 0 \) if \( I \not\subseteq (-\infty, y) \), and \( F[x, y](I) = 0 \) if \( I \not\subseteq (x, +\infty) \).

The isomorphism in (35) follows from (34) and Proposition 11. □

Remark 7 Suppose that \( F \) is decomposable into interval p-sheaves, where \( F \) is not necessarily tame. Then, \( m[x, y] = \dim F[x, y] \). Similarly, \( m[x, y] = \dim F[x, y] \), \( m[x, y] = \dim F[x, y] \) and \( m[x, y] = \dim F[x, y] \). Hence, for any interval decomposable p-sheaf, the decomposition is unique.
5.3 Decomposition of virtually tame c-modules

The main goal of this section is to prove an interval decomposition theorem for c-modules for which all correspondences \( v_s' \) are finite dimensional; in particular, for pointwise finite dimensional c-modules. The result is discussed in the more general setting of virtually tame correspondence modules. We begin with an analogue of Lemma 7 for c-modules.

**Lemma 8** Let \( \mathbb{V} \) be a virtually tame c-module and \( F \) its \( p \)-sheaf of sections. If \( p, q \in \mathbb{E}, p < q \), then the space of sections \( F(p, q) \) satisfies

\[
F(p, q)|_A = \bigoplus_{-\infty < s < p \leq y < +\infty} F[x, y]|_A \bigoplus_{-\infty < s \leq p < y \leq +\infty} F[x, y]|_A \bigoplus_{-\infty < s < y \leq +\infty} F(x, y)|_A,
\]

for any finite set \( A \subseteq (p, q) \).

**Proof** The proof is identical to that of Lemma 7. The only changes needed are to restrict the relevant vector spaces of sections to \( A \) and to replace the use of Lemma 5 with the Covering Lemma for Modules (Lemma 6). \( \square \)

**Theorem 2** (Interval Decomposition of c-Modules) If \( \mathbb{V} \) is a virtually tame c-module, then

\[
\mathbb{V} \cong \bigoplus_{-\infty < s < y < +\infty} m[x, y] \mathbb{V}[x, y] \bigoplus_{-\infty < s \leq y \leq +\infty} m[x, y] \bigoplus_{-\infty < y < +\infty} m(x, y),
\]

where \( m[x, y], m(x, y), m[x, y], \) and \( m(x, y) \) denote the multiplicities of the corresponding interval c-modules.

**Proof** We first show that each vector space \( V_t, t \in \mathbb{R} \), decomposes as claimed. Letting \( p = t^-, q = t^+ \) and \( A = \{t\} \), Lemma 8 and Proposition 11 imply that

\[
V_t \cong \bigoplus_{-\infty < s < y < +\infty} m[x, y] \mathbb{V}[x, y] \bigoplus_{-\infty < s \leq y \leq +\infty} m[x, y] \bigoplus_{-\infty < y < +\infty} m(x, y).
\]

To verify the decomposition for correspondences, let \( s, t \in \mathbb{R}, \) with \( s < t \). Set \( p = t^- \) and \( q = t^+ \), and \( A = \{s, t\} \). Note that \((p, q)\) is the closed interval \([s, t]\). Proposition 7 implies that \( v_s' = F(p, q)|_A \). The desired decomposition for correspondences now follows from Lemma 8 and Proposition 11. \( \square \)
Remark 8 Suppose that a \( c \)-module \( \mathbb{V} \) is virtually tame and let \( F \) be its \( p \)-sheaf of sections. As in Remark 7, the multiplicity \( m[x, y] \) of the interval component \([x, y]\) equals \( \dim F[x, y] \). Similarly, \( m[x, y] = \dim F[x, y] \) and \( m(x, y) = \dim F(x, y) \). Hence, the decomposition of a virtually tame \( c \)-module is unique. Moreover, if \( F \) is interval decomposable (not necessarily tame), then the multiplicity of each interval component obtained from the decomposition of \( \mathbb{V} \) is the same as the multiplicity of the corresponding interval summand of \( F \).

6 The Isometry Theorem

In this section, we prove one of the main results of this paper, the stability of persistence diagrams associated with interval decomposable \( p \)-sheaves. We define the interleaving distance \( d_I(F, G) \) between any two \( p \)-sheaves \( F \) and \( G \) and, assuming that the sheaves are decomposable, we also define the bottleneck distance \( d_b(dgm(F), dgm(G)) \) between their persistence diagrams. The Isometry Theorem states that

\[
d_b(dgm(F), dgm(G)) = d_I(F, G),
\]

for any decomposable \( p \)-sheaves \( F \) and \( G \). The inequality

\[
d_b(dgm(F), dgm(G)) \geq d_I(F, G)
\]

follows directly from the definition of \( d_I \) and \( d_b \). However, the proof of the algebraic stability statement

\[
d_b(dgm(F), dgm(G)) \leq d_I(F, G)
\]

involves rather delicate arguments.

6.1 Interleavings

This section introduces the notions of interleaving and interleaving distance for \( p \)-sheaves, extending the corresponding concepts for persistent modules [12, 14] to our setting, as needed in the formulation of the Isometry Theorem.

Given a decorated number \( p = t^* \in \mathbb{E}, t \in \mathbb{R}, \) and \( \epsilon \in \mathbb{R} \), let \( p + \epsilon := (t + \epsilon)^* \), with the additional convention that \( -\infty + \epsilon = -\infty \) and \( +\infty + \epsilon = +\infty \).

Definition 24 (Dilations and Erosions)

(i) If \( I = (p, q) \in \mathbb{E}^2 \) is an interval, the \( \epsilon \)-dilation of \( I, \epsilon \geq 0 \), is defined as \( I^\epsilon := (p - \epsilon, q + \epsilon) \).

(ii) The \( \epsilon \)-erosion of \( I = (p, q), \epsilon > 0 \), is defined as \( I^{-\epsilon} := (p + \epsilon, q - \epsilon) \), if \( p + \epsilon < q - \epsilon \). Otherwise, \( I^{-\epsilon} = \emptyset \).

(iii) For \( \epsilon > 0 \) and \( Z \subseteq \mathbb{R} \), if \( Z = \bigcup_{\lambda \in \Lambda} I_\lambda \) is its representation as the disjoint union of its connected components, define \( Z^{-\epsilon} = \bigcup_{\lambda \in \Lambda} I_\lambda^{-\epsilon} \).
Correspondence modules and persistence sheaves

Definition 25  Let $F$ and $G$ be $p$-sheaves and $\epsilon \geq 0$.

(i) An $\epsilon$-homomorphism $\Phi : F \to G$ is a collection

$$\{ \phi^I_{I-\epsilon} : F(I) \to G(I^{-\epsilon}) : I \in \text{Int} \}$$

of linear maps such that $G^I_{I-\epsilon} \circ \phi^I_{I-\epsilon} = \phi^J_{J-\epsilon} \circ F^I_{I-\epsilon}$, for any $I \subseteq J$, with the convention that $G(I^{-\epsilon}) = 0$ if $I^{-\epsilon} = \emptyset$. We refer to a $0$-homomorphism simply as a $\mathbb{R}$-homomorphism.

(ii) The $\epsilon$-erosion of $F$ is the $\epsilon$-homomorphism $\epsilon^e_F : F \to F$ given by

$$\{ F^I_{I-\epsilon} : I \in \text{Int} \}. (\text{Note that } \epsilon^0_F \text{ is the identity.})$$

To motivate the definition of interleaving and explain how it generalizes the notion of interleaving of persistence modules (cf. [12, 14]), let $\forall$ be a $p$-module and $F$ be its sheaf of sections. Note that vectors in $V_s$, $s \in \mathbb{R}$, are in one-to-one correspondence with sections of $F$ over $[s, +\infty)$, where $v_s \in V_s$ corresponds to the section $(v_i)_{i \in \mathbb{R}}$, given by $v_i = v^s_i(v_s)$. Under this correspondence, the transition map $v^s_i$ becomes a $(t - s)$-erosion map. The difference for more general $p$-sheaves is that we need to consider sections over all intervals, not just those of the form $[s, +\infty)$.

Definition 26  (Interleaving persistence sheaves) Let $F$ and $G$ be $p$-sheaves.

(i) An $\epsilon$-interleaving between $F$ and $G$, $\epsilon \geq 0$, is a pair of $\epsilon$-homomorphisms $\Phi^\epsilon : F \to G$ and $\Psi^\epsilon : G \to F$ such that $\Psi^\epsilon \circ \Phi^\epsilon = \epsilon^2_F$ and $\Phi^\epsilon \circ \Psi^\epsilon = \epsilon^2_G$. (Note that a 0-interleaving is an isomorphism.)

(ii) Given $\epsilon \geq 0$, $F$ and $G$ are $\epsilon^+\text{-interleaved}$ if they are $(\epsilon + \delta)$-interleaved for every $\delta > 0$.

(iii) The interleaving distance between $F$ and $G$ is defined as

$$d_I(F, G) = \inf\{ \epsilon > 0 : F \text{ and } G \text{ are } \epsilon\text{-interleaved} \}$$

$$= \min\{ \epsilon \geq 0 : F \text{ and } G \text{ are } \epsilon^+\text{-interleaved} \},$$

with the convention that $d_I(F, G) = \infty$ if no interleaving exists.

Remark 9  It is simple to verify that the two forms of Definition 26(iii) are equivalent. Indeed, let $a := \inf\{ \epsilon > 0 : F, G \text{ are } \epsilon\text{-interleaved} \}$ and $b := \min\{ \epsilon \geq 0 : F, G \text{ are } \epsilon^+\text{-interleaved} \}$. By the definition of $a$, for any $\delta > 0$ $F$, $G$ are $a + \delta$-interleaved which implies $a \geq b$. Similarly, by the definition of $b$, for any $\delta > 0$, $F$, $G$ are $b + \delta$-interleaved, which implies $a \leq b$. Hence $a = b$. We also note that $d_I$ is an extended pseudo-metric on the space of isomorphism classes of $p$-sheaves.

Remark 10  Given $s, t \in \overline{\mathbb{R}}$, $s \leq t$, there are up to four intervals $(x, y) \in \mathbb{E}^2$, defined by $s$ and $t$, given by the different combinations of decorations for $s$ and $t$. More precisely, let
\[ A_{s,t} = \{(x, y) \in \{s^-, s^+\} \times \{t^-, t^+\} : x < y\}, \]

with the convention that \(-\infty^* = -\infty\) and \(+\infty^* = +\infty\). Then, the intervals associated with \(s\) and \(t\) are \((x, y) \in A_{s,t}\). For a fixed interval-sheaf type, say \(k[\ , \ ]\), any two \(p\)-sheaves in the collection \(\{k[x, y] : (x, y) \in A_{s,t}\}\) are \(0^+\)-interleaved. The same applies to the other three types: \(k(x, y)\), \(k[x, y]\), and \(k(x, y)\). Hence, the interleaving distance is blind to changes in decorations of the endpoints of an interval.

As it is unclear how to formulate the interleaving distance between \(c\)-modules intrinsically, we do so by mapping \(c\)-modules to their corresponding \(p\)-sheaves of sections.

**Definition 27** (Interleaving Correspondence Modules) Let \(\mathcal{V}\) and \(\mathcal{W}\) be \(c\)-modules and \(F\) and \(G\) be their \(p\)-sheaves of sections, respectively. The *interleaving distance* between \(\mathcal{V}\) and \(\mathcal{W}\) is defined by

\[ d_I(\mathcal{V}, \mathcal{W}) := d_I(F, G). \]

### 6.2 Persistence diagrams

To motivate the definition of persistence diagrams, suppose that a \(p\)-sheaf \(F\) is interval decomposable; that is,

\[
F \cong \bigoplus_{-\infty < x < y < +\infty} m[x, y] k[x, y] \bigoplus_{-\infty < x < y \leq +\infty} m[x, y] k[x, y] \\
\bigoplus_{-\infty \leq x < y < +\infty} m[x, y] k(x, y) \bigoplus_{-\infty \leq x < y \leq +\infty} m(x, y) k(x, y),
\]

where \(m[x, y], m(x, y), m(x, y)\), and \(m(x, y)\) denote the multiplicities of the various interval summands. Then, \(m[x, y]\) coincides with the dimension of the space of sections \(F[x, y]\), defined in Sect. 5, subsequently reinterpreted as a \(p\)-sheaf. Similarly, for \(m(x, y), m(x, y), \) and \(m(x, y)\). Hence, the decomposition of \(F\), if it exists, is uniquely determined by \(m[x, y], m(x, y), m(x, y)\), and \(m(x, y)\). Although the construction of the spaces of sections \(F[x, y], F(x, y), F[x, y]\), and \(F(x, y)\) involve choices of complementary subspaces to certain spaces of sections, the dimensions of these complements are independent of the choices made. Therefore, the following definition of persistence diagrams is well posed.

**Definition 28** Given any \(p\)-sheaf \(F\), define its *decorated persistence diagram* \(\text{Dgm}(F)\) as the quadruple of functions \(([m], \langle m \rangle, [m], \langle m \rangle)\) with domains

(i) \(\text{Dom}([m]) = \{(x, y) \in \mathbb{E}^2 : -\infty < x < y < +\infty\}\),
(ii) \(\text{Dom}(\langle m \rangle) = \{(x, y) \in \mathbb{E}^2 : -\infty < x < y \leq +\infty\}\),
(iii) \(\text{Dom}(\langle m \rangle) = \{(x, y) \in \mathbb{E}^2 : -\infty \leq x < y < +\infty\}\),
(iv) \(\text{Dom}(\langle m \rangle) = \{(x, y) \in \mathbb{E}^2 : -\infty \leq x < y \leq +\infty\}\),
given by
\[ [m](x, y) = \dim F[x, y], \quad [m](x, y) = \dim F[x, y], \quad \langle m \rangle(x, y) = \dim F(x, y), \quad \langle m \rangle(x, y) = \dim F(x, y). \]

As pointed out in Remark 10, for a given interval \( p \)-sheaf type, the interleaving distance is not sensitive to changes in the decoration of the endpoints of an interval. Thus, to obtain an isometry theorem, we should not distinguish persistence diagrams associated with those intervals. To this end, as in [14], we introduce “undecorated” versions of persistence diagrams.

**Definition 29** Given any \( p \)-sheaf \( F \), we define its (undecorated) persistence diagram \( \text{dgm}(F) \) as the quadruple of functions \([\overline{m}], [\overline{m}], \langle \overline{m} \rangle, \langle \overline{m} \rangle\) with domains

\[
\begin{align*}
(i) & \quad \text{Dom}(\overline{m}) = \{(s, t) \in \mathbb{R} \times \mathbb{R} : s < t\}, \\
(ii) & \quad \text{Dom}(\overline{m}) = \{(s, t) \in \mathbb{R} \times \mathbb{R} : s < t\}, \\
(iii) & \quad \text{Dom}(\langle \overline{m} \rangle) = \{(s, t) \in \mathbb{R} \times \mathbb{R} : s < t\}, \\
(iv) & \quad \text{Dom}(\langle \overline{m} \rangle) = \{(s, t) \in \mathbb{R} \times \mathbb{R} : s \leq t\},
\end{align*}
\]

given by \([\overline{m}](s, t) = \sum_{(x,y) \in A_{s,t}} [m](x, y)\), with \( A_{s,t} \) as in (40). The function values \([\overline{m}](s, t), \langle \overline{m} \rangle(s, t), \) and \( \langle \overline{m} \rangle(s, t) \) are defined similarly.

**Remark 11** We often refer to the domain of each of the four functions in a \( p \)-diagram as a multiset and to the value of the function at \((s, t)\) as the multiplicity of \((s, t)\). We also frequently treat the multiplicity functions as defined on \( \mathbb{R} \times \mathbb{R} \) by extending them to be zero outside their original domains.

Notice that for singletons \((s^-, s^+)\subset 2^\mathbb{R}, s \in \mathbb{R}\), forgetting decorations returns points on the diagonal \( \Delta = \{(s, s) : s \in \mathbb{R}\}. \) Nonetheless, in Definition 29(i)-(iii), \( \Delta \) is not included in the domain of the multiplicity function because \( k[s^-, s^+] \) and \( k[s^-, s^+] \) are all 0+- interleaved with the trivial \( p \)-sheaf. This is not the case, however, for \( k(s^-, s^+) \). Indeed, the \( \delta \)-erosion map for global sections of this \( p \)-sheaf is the identity, for any \( \delta > 0 \), implying that \( k(s^-, s^+) \) is not \( \delta \)-interleaved with the trivial \( p \)-sheaf, for any \( \delta > 0 \).

Henceforth, we refer to undecorated persistence diagrams simply as persistence diagrams, or \( p \)-diagrams. In order to define the bottleneck distance between \( p \)-diagrams, we first discuss a variant of the notion of matching of multisets that is suited to our goals. Abusing terminology, we often refer to a multiset \((A, f)\), with multiplicity function \( f \), simply as \( A \).

A partial matching between the multisets \((A, f)\) and \((B, g)\) is a multiset injection from a subset of \(A\) to \(B\). More formally, a resolution of \(A\) is a mapping \( \pi_A : \hat{A} \to A \) such that \( |\pi_A^{-1}(a)| = f(a) \), \( \forall a \in A \). A partial matching between \(A\) and \(B\) is an injection \( \sigma : \hat{A} \to \hat{B} \), where \( \hat{A} \subseteq \hat{A} \). The partial matching \( \sigma \) is surjective if \( \text{Im}(\sigma) = \hat{B} \). We abuse terminology and refer to elements of \( \hat{A} \) as elements of the multiset \( A \), and to \( \sigma(a) \) as an element of the multiset \( B \).
As usual, the bottleneck distance will be based on the $\ell_\infty$-distance on the extended plane, denoted $d_\infty : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and given by

$$d_\infty((s_1, t_1), (s_2, t_2)) = \max\{|s_1 - s_2|, |t_1 - t_2|\},$$

with the convention that $|\infty - \infty| = 0$. Note that $d_\infty((s, t), \Delta) = |s - t|/2$.

**Definition 30** Let $A$ and $B$ be multisets with domain $\mathbb{R} \times \mathbb{R}$, $\epsilon > 0$, and $L > 0$. A partial matching $\sigma$ between $A$ and $B$ is an $(\epsilon, L)$-matching provided that:

(i) if $\sigma(a) = b$, then $d_\infty(a, b) \leq \epsilon$;
(ii) if $a \in \text{supp}(A) \setminus \text{Dom}(\sigma)$, then $d_\infty(a, \Delta) \leq L\epsilon$;
(iii) if $b \in \text{supp}(B) \setminus \text{Im}(\sigma)$, then $d_\infty(b, \Delta) \leq L\epsilon$.

A partial matching $\sigma$ is a full $\epsilon$-matching if it is a multiset bijection between $A$ and $B$ that satisfies condition (i) above.

**Definition 31** The persistence diagrams $dgm_i = (f_i, f_i^*, f_i^1, f_i^2)$, $i \in \{1, 2\}$, are $\epsilon$-matched if the following holds:

(i) $f_1^1$ and $f_2^2$ are $(\epsilon, 2)$-matched;
(ii) $f_1^1$ and $f_3^2$ are $(\epsilon, 1)$-matched;
(iii) $f_2^1$ and $f_4^2$ are $(\epsilon, 1)$-matched;
(iv) $f_4^1$ and $f_3^2$ are fully $\epsilon$-matched.

**Remark 12** The different choices of $L$ in (i)–(iv) above is to appropriately match unpaired sections to trivial sections. More precisely,

(a) $k[s^*, t^*]$ is $\epsilon$-interleaved with the trivial section if and only if $|s - t| \leq 4\epsilon$ or $d_\infty((s, t), \Delta) \leq 2\epsilon$;
(b) $k[s^*, t^*]$ is $\epsilon$-interleaved with the trivial section if and only if $|s - t| \leq 2\epsilon$ or $d_\infty((s, t), \Delta) \leq \epsilon$;
(c) $k(s^*, t^*)$ is $\epsilon$-interleaved with the trivial section if and only if $|s - t| \leq 2\epsilon$ or $d_\infty((s, t), \Delta) \leq \epsilon$;
(d) $k(s^*, t^*)$ is not $\epsilon$-interleaved with the trivial section for any $\epsilon > 0$, hence we need a full $\epsilon$-match in Definition 31(iv).

**Definition 32** The bottleneck distance between persistence diagrams is the extended pseudo-metric defined as

$$d_b(dgm_1, dgm_2) = \inf\{\epsilon > 0 : dgm_1 \text{ and } dgm_2 \text{ are } \epsilon \text{-matched}\}.$$

The next proposition shows that a converse to the stability of persistence diagrams holds.

**Proposition 12** Let $F$ and $G$ be decomposable $p$-sheaves. If $dgm(F)$ and $dgm(G)$ are $\epsilon$-matched, $\epsilon > 0$, then $F$ and $G$ are $\epsilon^+$-interleaved. Thus,
\[ d_I(F, G) \leq d_p(dgm(F), dgm(G)). \]

**Proof** Let \( \epsilon \geq 0 \). If \( dgm(F) = (f_1, f_2, f_3, f_4) \) and \( dgm(G) = (g_1, g_2, g_3, g_4) \) are \( \epsilon \)-matched, there exist:

(a) an \((\epsilon, 2)\)-matching between \( f_1 \) and \( g_1 \);
(b) an \((\epsilon, 1)\)-matching between \( f_2 \) and \( g_2 \);
(c) an \((\epsilon, 1)\)-matching between \( f_3 \) and \( g_3 \);
(d) a full \( \epsilon \)-matching between \( f_4 \) and \( g_4 \).

We show that the subsheaves \([F]\) and \([G]\) are \( \epsilon^+\)-interleaved. The argument for the components of type \((\), [ ), and ( ) of \( F \) and \( G \) are similar. As the diagrams disregard decorations of the endpoints of an interval, suppose that for \( s_i, t_i \in \mathbb{R}, s_i < t_i, i = 1, 2 \), the interval modules \( k[s_i^+, t_i^-] \) and \( k[s_2^+, t_2^-] \) are \( (\epsilon, 2)\)-matched under the given \( \epsilon \)-matching of diagrams. Then, \( d^{\infty}(s_1, t_1), (s_2, t_2)) \leq \epsilon \), which implies that \( k[s_1^+, t_1^-] \) and \( k[s_2^+, t_2^-] \) are \( (\epsilon + \delta)\)-interleaved for any \( \delta > 0 \). Assembling all of these pairwise interleavings, yields the desired \( (\epsilon + \delta)\)-interleaving. \( \square \)

### 6.3 Algebraic stability

This section is devoted to the proof of the algebraic stability of persistence diagrams; that is, the statement that

\[ d_p(dgm(F), dgm(G)) \leq d_I(F, G). \]

The general strategy for the proof of stability, particularly in Lemma 14 and Theorem 4 below, is similar to that used by Bjerkevik [3] to study algebraic stability in the context of multi-parameter persistence modules. The arguments needed for \( p \)-sheaves, however, are quite distinct.

**Definition 33** Let \( F \) be a \( p \)-sheaf. A set \( B_F \) of non-trivial sections of \( F \) is a **basis of \( F \)** if for any interval \( I \subseteq \mathbb{R} \), the set

\[ B_F(I) := \{ f|_I : f \in B_F, I \subseteq \text{Dom}(f) \text{ and } f|_I \neq 0 \} \]

is a basis of \( F(I) \).

**Lemma 9** If \( B_F \) is a basis of a \( p \)-sheaf \( F \), then each section in \( B_F \) has connected support. Moreover, if \( f \in B_F \), then the subsheaf \( F(f) \) spanned by \( f \) (see Definition 17) is isomorphic to an interval \( p \)-sheaf.

**Proof** Suppose that \( f \in B_F \) has disconnected support. Then, there exist \( r < s < t \) such that \( f|_r \neq 0, f|_s = 0 \) and \( f|_t \neq 0 \). By the gluing property, \( f \) may be decomposed as \( f = a + b \), where \( a_x = f_x \), for \( x \in L := \text{Dom}(f) \cap (-\infty, s] \), and \( a_x = 0 \), for \( x \in R := \text{Dom}(f) \cap [s, +\infty) \). This implies that \( b_x = f_x \), \( \forall x \in R \) and \( b|_L \equiv 0 \). Note
that both \(a\) and \(b\) are non-trivial sections. By the definition of basis, \(a\) and \(b\) can be uniquely expressed as linear combinations

\[
a = c_0f + \sum_{f_s \in B_F \setminus \{f\}} c_sf_s \quad \text{and} \quad b = d_0f + \sum_{f_s \in B_F \setminus \{f\}} d_sf_s.
\]  

(43)

Since \(f = a + b\), it follows that \(c_0 + d_0 = 1\). On the other hand,

\[
a|_L = c_0f|_L + \sum c_sf_s|_L \quad \text{and} \quad b|_R = d_0f|_R + \sum d_sf_s|_R.
\]  

(44)

Since both \(a|_L\) and \(b|_R\) are non-trivial and coincide with \(f|_L\) and \(f|_R\), respectively, it follows that \(c_0 = d_0 = 1\). This contradicts the fact that \(c_0 + d_0 = 1\).

The fact that, for any \(f \in B_F\), the subsheaf \(F(f)\) is isomorphic to an interval sheaf now follows from Proposition 5.

Let \(F\) be interval decomposable. If \(\Phi : \sum_{\lambda \in \Lambda} F_\lambda \rightarrow F\) is a \(p\)-sheaf isomorphism, where each \(F_\lambda\) is an interval \(p\)-sheaf, then the images under \(\Phi\) of the unit sections \(s_\lambda\) of \(F_\lambda\) (see Definition 18) form a basis of \(F\). The following proposition provides a converse statement. Hence, we may view an interval decomposition of a \(p\)-sheaf as a choice of basis.

**Proposition 13** If \(B_F\) is a basis of \(F\), then there are interval modules \(F_\lambda, \lambda \in \Lambda\), and an isomorphism \(\Phi : \sum_{\lambda \in \Lambda} F_\lambda \rightarrow F\) that maps the unit sections \(s_\lambda\) of \(F_\lambda\) bijectively onto \(B_F\).

**Proof** Set \(F_\lambda := F(f_\lambda)\). Lemma 9 implies that each \(F_\lambda\) is isomorphic to an interval \(p\)-sheaf. Moreover, the isomorphism may be chosen to map the unit section \(s_\lambda\) of \(F_\lambda\) to \(f_\lambda\). The fact that \(B_F\) is a basis implies that these induce an isomorphism \(\Phi : \sum_{\lambda \in \Lambda} F_\lambda \rightarrow F\) with the desired properties.

Let \(f\) and \(g\) be sections of \(F\) and \(G\), respectively, with connected support. By Proposition 5, \(F(f)\) and \(G(g)\) are (isomorphic to) interval \(p\)-sheaves.

**Definition 34** Under the above assumptions, define \(f\) and \(g\) to be \(\epsilon\)-matched if:

(i) \(F(f)\) and \(G(g)\) are interval \(p\)-sheaves of the same type;

(ii) \(d_H(\text{supp}(f), \text{supp}(g)) \leq \epsilon\), where \(d_H\) denotes (extended) Hausdorff distance.

**Lemma 10** Under the assumptions of Definition 34, if the sections \(f\) and \(g\) are \(\epsilon\)-matched, then \(F(f)\) and \(G(g)\) are \(\epsilon\)-interleaved.

**Proof** Define \(\epsilon\)-homomorphisms \(\Phi^\epsilon : F(f) \rightarrow G(g)\) and \(\Psi^\epsilon : G(g) \rightarrow F(f)\) by \(\Phi^\epsilon(f) = g|_{\text{Dom}(f)\cap \text{Dom}(g)\cap \epsilon}\) and \(\Psi^\epsilon(g) = f|_{\text{Dom}(g)\cap \epsilon}\). This completes characterize these homomorphisms because the \(p\)-sheaves are spanned by \(f\) and \(g\). Then, \((\Phi^\epsilon, \Psi^\epsilon)\) is an \(\epsilon\)-interleaving.  

\(\Box\)
Let \( \sigma \) be an \( \varepsilon \)-mapping if \( f \) and \( \sigma(f) \) are \( \varepsilon \)-matched, \( \forall f \in \text{Dom}(\sigma) \).

\( \mathbf{BG} \) and \( \mathbf{BF} \) are said to be \( \varepsilon \)-matched if there exists an injective \( \varepsilon \)-map \( \sigma : \text{Dom}(\sigma) \subseteq B_F \to B_G \) such that \( X_F^\varepsilon \subseteq \text{Dom}(\sigma) \) and \( X_G^\varepsilon \subseteq \text{Im}(\sigma) \).

**Proposition 14** If \( B_F \) and \( B_G \) are \( \varepsilon \)-matched, then \( \text{dgm}(F) \) and \( \text{dgm}(G) \) are \( \varepsilon \)-matched.

**Proof** This follows directly from the definitions of \( \varepsilon \)-matchings for sections and for persistence diagrams.

By Proposition 14, we can translate a diagram-matching problem into a question of matching bases and this is how we approach the proof of the stability theorem.

**Lemma 11** Let \( \varepsilon > 0 \). If there exist injective \( \varepsilon \)-mappings \( \sigma : X_F^\varepsilon \to B_G \) and \( \tau : X_G^\varepsilon \to B_F \), then there is an \( \varepsilon \)-matching between \( B_F \) and \( B_G \).

**Proof** Let \( G \) be the bipartite graph with vertex set partitioned as \( B_F \sqcup B_G \) and whose edge set is the union of the graphs of the maps \( \sigma \) and \( \tau \). We denote an edge \( \{u, v\} \), where \( \sigma(u) = v \), by \( u \xrightarrow{\sigma} v \). Similarly, \( u \xrightarrow{\tau} v \) if \( v = \tau(u) \). Since \( \sigma \) and \( \tau \) are injections, each vertex of \( G \) has degree \( \leq 2 \).

We construct an \( \varepsilon \)-matching over each connected component of \( G \) and take the union of these to obtain the desired matching. Let \( C \) be any connected component of \( G \) and \( V_C \) its vertex set. Let \( C_F^\varepsilon := X_F^\varepsilon \cap V_C \) and \( C_G^\varepsilon := X_G^\varepsilon \cap V_C \), the sets of vertices in \( C \) comprised of \( \varepsilon \)-significant sections of \( F \) and \( G \), respectively. Then, at least one of the following properties is satisfied: (a) \( C_F^\varepsilon \subseteq \text{Im } \sigma|_{C_F^\varepsilon} \) or (b) \( C_G^\varepsilon \subseteq \text{Im } \tau|_{C_G^\varepsilon} \). Indeed, there is a sequence

\[
\ldots \xrightarrow{\tau} v_{-1} \xrightarrow{\sigma} v_0 \xrightarrow{\tau} v_1 \xrightarrow{\sigma} \ldots
\]

(45)

such that \( V_C = \cup_i \{v_i\} \). This sequence may be infinite or finite on either end. Note that if it is finite on the right, then the last element is a section that is \( 2\varepsilon \)-trivial. If the sequence is infinite on the left or finite starting with an edge of type \( \xrightarrow{\tau} \), then the construction implies that \( C_F^\varepsilon \subseteq \text{Im } \sigma|_{C_F^\varepsilon} \), regardless of the behavior of the sequence on the right end. Similarly, if it is finite on the left, starting with a type \( \xrightarrow{\sigma} \) edge, then \( C_G^\varepsilon \subseteq \text{Im } \tau|_{C_G^\varepsilon} \).

If property (a) above is satisfied, then \( \sigma|_{C_F^\varepsilon} : C_F^\varepsilon \to B_G \cap C \) is an \( \varepsilon \)-matching between \( B_F \cap V_C \) and \( B_G \cap V_C \). Let \( \zeta := \tau|_{C_G^\varepsilon} : C_G^\varepsilon \to B_F \cap V_C \). If property (b) is
satisfied, then \( \tau^{-1}|_{\text{Im}(\zeta)} : \text{Im}(\zeta) \supseteq C^e \to C^e \) is an \( \epsilon \)-matching between \( B_F \cap V_C \) and \( B_G \cap V_C \). Assembling the matchings over all connected components of \( G \) yields an \( \epsilon \)-matching between \( B_F \) and \( B_G \).

We now proceed to the core argument in the proof of the algebraic stability of persistence diagrams, namely, the construction of the injective mappings called for in the hypothesis of Lemma 11. In the remainder of this section, we index the bases of \( F \) and \( G \) as \( B_F = \{ f_\lambda : \lambda \in \Lambda_F \} \) and \( B_G = \{ g_\mu : \mu \in \Lambda_G \} \), respectively. We also adopt the following abbreviations:

\[
D_\lambda := \text{Dom}(f_\lambda), \quad D_\mu := \text{Dom}(g_\mu),
\]

\[
Z_\lambda := D_\lambda \setminus \text{supp}(f_\lambda), \quad Z_\mu := D_\mu \setminus \text{supp}(g_\mu).
\]

For \( \epsilon \geq 0 \), \( \Phi^\epsilon : F \to G \) and \( \Psi^\epsilon : G \to F \) denote \( \epsilon \)-homomorphisms. Using \( B_F \) and \( B_G \), an \( \epsilon \)-morphism \( \Phi^\epsilon \) may be represented by a “matrix” \((\phi^\mu_\lambda)\), where

\[
\Phi^\epsilon(f_\lambda) = \sum_{\mu \in \Lambda_G} \phi^\mu_\lambda g_\mu|_{D^\epsilon},
\]

for any \( \lambda \in \Lambda_F \). Here, we make the convention that \( \phi^\mu_\lambda = 0 \) if \( g_\mu|_{D^\epsilon} = 0 \); that is, \( D^\epsilon \subseteq Z_\mu \). Thus, for a fixed \( \lambda \), only finitely many entries \( \phi^\mu_\lambda \) can be non-zero. Similarly, \( \Psi^\epsilon \) may be represented by a matrix \((\psi_\mu^\lambda)\) whose entries, for a fixed \( \mu \), vanish for all but finitely many values of \( \lambda \).

**Lemma 12** If \( \phi^\mu_\lambda \neq 0 \), then \( D^\epsilon_\lambda \subseteq D_\mu \) and \( Z^\epsilon_\lambda \subseteq Z_\mu \). Moreover, the interval \( p \)-sheaves \( F(f_\lambda) \) and \( G(g_\mu) \) are of the same type.

**Proof** \( D^\epsilon_\lambda \subseteq D_\mu \) follows directly from the definition of \( \phi^\mu_\lambda \). Suppose that \( Z_\lambda \neq \emptyset \) and let \( I \subseteq \mathbb{R} \) be a connected component of \( Z_\lambda \). Then,

\[
0 = \Phi^\epsilon(f_\lambda|_I) = \sum_{\mu \in \Lambda_G} \phi^\mu_\lambda g_\mu|_{I^\epsilon}.
\]

Since non-trivial sections in \( \{g_\mu|_{I^\epsilon}\} \) are independent, we have \( \phi^\mu_\lambda g_\mu|_{I^\epsilon} = 0 \) for any \( \mu \). Hence, \( \phi^\mu_\lambda \neq 0 \) implies that \( g_\mu|_{I^\epsilon} = 0 \). Thus, \( Z^\epsilon_\lambda \subseteq Z_\mu \).

Lemma 9 shows that both \( F(f_\lambda) \) and \( G(g_\mu) \) are isomorphic to interval \( p \)-sheaves. The fact that these interval \( p \)-sheaves are of the same type follows directly from \( D^\epsilon_\lambda \subseteq D_\mu \) and \( Z^\epsilon_\lambda \subseteq Z_\mu \). \( \square \)

To state the next lemma, we introduce some terminology. Let \( \alpha : \mathbb{E} \to \mathbb{R} \times \mathbb{N} \) be given by \( \alpha(t^+) = (t, 1) \) and \( \alpha(t^-) = (t, -1) \), \( \forall t \in \mathbb{R} \), \( \alpha(+\infty) = (+\infty, 1) \), and \( \alpha(-\infty) = (-\infty, -1) \). Define “+” and “−” operations in \( \mathbb{R} \times \mathbb{N} \) that perform coordinate-wise addition or subtraction. If we order the elements of \( \mathbb{R} \times \mathbb{N} \) by \( (a, n) < (b, m) \) if \( a < b \), or if \( a = b \) and \( n < m \), then \( \alpha \) is order preserving.

Springer
Lemma 13 Let \( \lambda, \lambda' \in \Lambda_F \) and suppose that \( \phi_{\lambda}^{\mu} \psi_{\lambda'}^{\mu} \neq 0 \) for some \( \mu \in \Lambda_G \). If any one of the conditions

(i) \( F(f_{\lambda}) \approx k[p, q], F(f_{\lambda'}) \approx k[p', q'] \) and \( \alpha(q) - \alpha(p) \geq \alpha(q') - \alpha(p') \),

(ii) \( F(f_{\lambda}) \approx k[p, q], F(f_{\lambda'}) \approx k[p', q'] \) and \( \alpha(p) + \alpha(q) \leq \alpha(p') + \alpha(q') \),

(iii) \( F(f_{\lambda}) \approx k[p, q], F(f_{\lambda'}) \approx k[p', q'] \) and \( \alpha(p) + \alpha(q) \geq \alpha(p') + \alpha(q') \),

(iv) \( F(f_{\lambda}) \approx k[p, q], F(f_{\lambda'}) \approx k[p', q'] \) and \( \alpha(q) - \alpha(p) \leq \alpha(q') - \alpha(p') \).

is satisfied, then \( g_{\mu} \) is \( \epsilon \)-matched with either \( f_{\lambda} \) or \( f_{\lambda'} \).

Proof By Lemma 12, the interval \( p \)-sheaf \( G(g_{\mu}) \) has the same type as the intervals \( p \)-sheaves \( F(f_{\lambda}) \) and \( F(f_{\lambda'}) \) and

1. \( D_{\lambda}^e \subseteq D_{\mu}^e \) and \( Z_{\lambda}^e \subseteq Z_{\mu}^e \);
2. \( D_{\lambda'}^e \subseteq D_{\mu}^e \) and \( Z_{\mu}^e \subseteq Z_{\lambda'}^e \).

Thus, to prove the lemma, it suffices to show that the assumptions that

3. \( d_H(\text{supp}(f_{\lambda}), \text{supp}(g_{\mu})) > \epsilon \)
4. \( d_H(\text{supp}(f_{\lambda'}), \text{supp}(g_{\mu})) > \epsilon \)

lead to a contradiction. We divide the argument into four cases.

Case 1. Suppose that (i) is satisfied, so that we may write \( G(g_{\mu}) \approx k[x, y] \). By (1), we have \( x \leq p + \epsilon \) and \( y \geq q - \epsilon \), whereas (3) implies that \( x < p - \epsilon \) or \( y > q + \epsilon \). If \( x < p - \epsilon \), the inequality \( y \geq q - \epsilon \) ensures that

\[
\alpha(y) - \alpha(x) > \alpha(q) - \alpha(p). \tag{50}
\]

If \( y > q + \epsilon \), the inequality \( x < p + \epsilon \) also implies (50). Similarly, (2) and (4) yield

\[
\alpha(q') - \alpha(p') > \alpha(y) - \alpha(x). \tag{51}
\]

Then, (50) and (51) imply that \( \alpha(q) - \alpha(p) < \alpha(q') - \alpha(p') \), which contradicts the hypothesis.

Case 2. Suppose that (ii) is satisfied and write \( g_{\mu} \approx k[x, y] \). By (1), we have \( x \leq p + \epsilon \) and \( y \leq q + \epsilon \), whereas (3) implies that \( x < p - \epsilon \) or \( y < q - \epsilon \). Either possibility yields \( \alpha(x) + \alpha(y) < \alpha(p) + \alpha(q) \). Similarly, by (2) and (4), we have \( \alpha(p') + \alpha(q') < \alpha(x) + \alpha(y) \). Then, \( \alpha(p') + \alpha(q') < \alpha(p) + \alpha(q) \), which is a contradiction.

Case 3. Suppose that (iii) is satisfied and let \( g_{\mu} \approx k[x, y] \). By (1) we have \( x \geq p - \epsilon \) and \( y \geq q - \epsilon \), whereas (3) implies that \( x > p + \epsilon \) or \( y > q + \epsilon \), either giving \( \alpha(x) + \alpha(y) > \alpha(p) + \alpha(q) \). Similarly, by (2) and (4), we have \( \alpha(p') + \alpha(q') > \alpha(x) + \alpha(y) \). Then, \( \alpha(p') + \alpha(q') > \alpha(p) + \alpha(q) \), which is a contradiction.

Case 4. Suppose that (iv) holds and write \( g_{\mu} \approx k[x, y] \). By (1) we have \( x \geq p - \epsilon \) and \( y \leq q + \epsilon \), whereas (3) implies \( x > p + \epsilon \) or \( y < q - \epsilon \), either yielding \( \alpha(y) - \alpha(x) < \alpha(q) - \alpha(p) \). Similarly, by (2) and (4), we have
\[ α(q') - α(p') < α(y) - α(x). \] Hence, \( α(q') - α(p') < α(q) - α(p) \), contradicting the hypothesis. 

**Definition 37** Given \( f \in B_F \), define \( m^e_G(f) \subseteq B_G \) as the the subset of all sections \( g \in B_G \) such that \( f \) and \( g \) are \( e \)-matched.

Let \( A = \{f_{\lambda_1}, \ldots, f_{\lambda_m}\} \subseteq B_F \) be a finite, indexed sub-collection of basis elements with the property that all interval subsheaves \( F(f_{\lambda_i}), 1 \leq i \leq m \), are of the same type and the inequalities specified in the hypotheses of Lemma 13 are satisfied for any \( i \leq j \).

More precisely, if \( F(f_{\lambda_i}) \cong k[p_i, q_i] \) and \( F(f_{\lambda_j}) \cong k[p_j, q_j], \ i \leq j, \) then \( α(q_i) - α(p_i) \geq α(q_j) - α(p_j) \), and similarly for other interval sheaf types. Define

\[
v(A) := \{ g_\mu \in B_G : \phi_{\lambda_i}^\mu \psi_{\lambda_j} \neq 0 \text{ for some pair } i \leq j \}. \tag{52}
\]

Note that \( |v(A)| < \infty \) because, for each fixed pair \( i \leq j \), only finitely many entries \( \phi_{\lambda_i}^\mu \) and \( \phi_{\lambda_j}^\mu \) can be non-zero. Moreover, Lemma 13 implies that each section in \( v(A) \) is \( e \)-matched with at least one section in \( A \). Thus, we have

\[
v(A) \subseteq \cup_{f \in A} m^e_G(f). \tag{53}
\]

We remark that the special ordering of the elements of \( A \) is essential in the argument that shows that (53) holds.

**Proposition 15** Let \( A \) be as above. If \((\Phi^e, \Psi^e)\) is an \( e \)-interleaving and \( A \subseteq X^e_F \) (that is, all sections in \( A \) are \( 2e \)-significant), then

\[
|A| \leq |v(A)| < \infty.
\]

**Proof** Let \( v(A) = \{g_{\mu_1}, \ldots, g_{\mu_n}\} \). Since \( \Psi^e \circ \Phi^e = e^2_F \) and all sections in \( A \) are \( 2e \)-significant, we have that \( \sum_{i=1}^n \phi_{\lambda_i}^\mu \psi_{\lambda_i}^\mu = \delta_{ij} \), for any \( i \leq j \), where \( \delta_{ij} \) is Kronecker’s delta. Thus, the product of the matrices \( K := (\phi_{\lambda_i}^\mu)_{n \times m} \) and \( L := (\psi_{\lambda_j}^\mu)_{n \times m} \) is a triangular matrix with diagonal entries all equal to 1. Hence, the product \( KL \) has rank \( m \). This implies that \( |v(A)| = n \geq \text{rank}(K) \geq \text{rank}(KL) = m = |A| \). \( \square \)

As pointed out earlier, the general strategy for the proof of stability is similar to that used by Bjerkevik in the study of multi-parameter persistence [3], although the arguments for \( p \)-sheaves differ substantially. A key matching result needed in the proof of the stability theorem, we employ the following classic result in combinatorics and graph theory.

**Theorem 3** (Hall’s Matching Theorem) Let \( \{Z_x : x \in X\} \) be a collection of non-empty subsets of a set \( Z \) such that \( |A| \leq |\cup_{x \in A} Z_x| < \infty \), for any finite set \( A \subseteq X \). Then, there exists an injective map \( \zeta : X \rightarrow Z \) such that \( \zeta(x) \in Z_x, \forall x \in X \).

**Lemma 14** (The Matching Lemma) Suppose that \( F \) and \( G \) are decomposable \( p \)-sheaves with bases \( B_F \) and \( B_G \), respectively, and let

\[ \zeta \] Springer
The fact that 
then 
\( f \) \( \in \mathbb{Z} \)
we construct
straightforward consequence of Definition 26 that if 
\( \frac{\mathbb{Z}}{\mathbb{Z}} \)
countable sets. Let

\[
A = [A] \cup [A] \cup \langle A \rangle \cup \langle A \rangle,
\]
where \([A]\) only contains sections \( f \) such that \( F(f) \cong k[p, q] \), the subset \([A]\) only contains sections \( f \) such that \( F(f) \cong k[p, q] \), and similarly for the other two terms. If any of these subsets is empty, we simply discard it. We can index the elements of \([A]\) appropriately, so that \([A]\) satisfies the assumptions of Proposition 15. Therefore, for this labeling of \([A]\), we have \(|[A]| \leq | \cup f \in [A] m_G^e(f) | \). Similar inequalities can be obtained for the other three subsets of \( A \) after appropriate labeling of their elements. Note that the sets \( \cup f \in [A] m_G^e(f) \), \( \cup f \in [A] m_G^e(f) \), \( \cup f \in [A] m_G^e(f) \), and \( \cup f \in [A] m_G^e(f) \) are pairwise disjoint because they contain sections that span interval sheaves of different types. Therefore,

\[
|A| = |[A]| + |+[A]| + |\langle A \rangle| + |\langle A \rangle| \leq | \cup f \in [A] m_G^e(f) |.
\]

By Theorem 3, there is an injection \( \zeta : X \rightarrow B_G \) such that \( \zeta(f) \in m_G^e(f) \), \( \forall f \in X \). In other words, \( \zeta \) is an injective \( e \)-map.

**Theorem 4 (The Algebraic Stability Theorem)** Let \( F \) and \( G \) be decomposable \( p \)-sheaves and \( e > 0 \). If \( F \) and \( G \) are \( e \)-interleaved, then \( B_F \) and \( B_G \) are \((e + \delta)\) -matched, for any \( \delta > 0 \).

\[
d_b(dgm(F), dgm(G)) \leq d_I(F, G).
\]

**Proof** We begin by reducing the argument to the case in which both \( B_F \) and \( B_G \) are countable sets. Let \((\Phi^x, \Psi^x)\) be an \( \epsilon \)-interleaving between \( F \) and \( G \) represented in the bases \( B_F \) and \( B_G \) by the matrices \((\phi^x)\) and \((\psi^x)\), respectively. Let \( G \) be the bipartite graph with vertex set partitioned as \( B_F \cup B_G \), and with an edge between \( f \) and \( g \) if \((\phi_x) \neq 0 \) or \((\psi_x) \neq 0 \). Then, any connected component \( C \) of \( G \) is a bipartite subgraph whose vertex set \( V \) is countable. This can be seen as follows. If we fix \( v_0 \in V \), then for any integer \( n \geq 0 \), the set \( B_n := \{ v \in C : d(v_0, v) \leq n \} \) is finite, where \( d(v_0, v) \) is the distance in the graph \( C \). Hence, \( V = \cup_{n \geq 0} B_n \) is countable. Moreover, it is a straightforward consequence of Definition 26 that if \( F \) and \( G \) are \( e \)-interleaved, so are \( F(C_1) \) and \( G(C_2) \), the subsheaves of \( F \) and \( G \) spanned by \( C_1 \) and \( C_2 \), respectively.

If there is an \((e + \delta)\)-matching between \( C_1 \) and \( C_2 \), for any connected component \( C \) of \( G \), then assembling the matchings over all connected components of \( G \), we obtain an \((e + \delta)\)-matching between \( B_F \) and \( B_G \). Thus, without loss of generality, we assume that \( B_F \) and \( B_G \) are countable.
By Lemma 11, to construct an \((\epsilon + \delta)\)-matching between \(B_F\) and \(B_G\), it suffices to construct injective \((\epsilon + \delta)\)-maps \(\sigma : X^c_F \rightarrow B_G\) and \(\tau : X^c_G \rightarrow B_F\). To construct \(\sigma\), we filter the countable set \(X^c_F \subseteq B_F\) and define it inductively over the filtration. Let
\[
X := \{f \in X^c_F : |m^c_G(f)| < \infty\}
\]
and \(\zeta : X \rightarrow B_G\) be an injective \(\epsilon\)-map whose existence is guaranteed by the Matching Lemma (Lemma 14).

Before proceeding with the construction, we recall a standard fact: if \(N_1, N_2, \ldots \subseteq Z\) are infinite subsets of a set \(Z\), there exist infinite subsets \(N'_i \subseteq N_i\), \(i \geq 1\), such that \(N'_i \cap N'_j = \emptyset\), for any \(i \neq j\).

List the sections in \(X^c_F \setminus X\) as \(\{f_i : i \geq 1\}\). For any fixed \(\delta > 0\), since \(|m^c_G(f_i)| = \infty\), by Lemma 15 below, there exists a sequence
\[
S_i := \{g^n_i : n \geq 1\} \subseteq m^c_G(f_i)
\]
such that any two sections in \(S_i\) are \(\delta\)-matched. Furthermore, by the remark in the previous paragraph, we can assume that \(S_i \cap S_j = \emptyset\), for any \(i \neq j\). If \(X = \emptyset\), we simply define \(\sigma(f_i), i \geq 1\), to be any element of \(S_i\). This concludes the construction of \(\sigma\) because it is an injective \(\epsilon\)-mapping, in particular, an injective \((\epsilon + \delta)\)-map. If \(X \neq \emptyset\), let
\[
X_0 := X \setminus \cup_{i \geq 1} \zeta^{-1}(S_i)
\]
and define \(\sigma_0 : X_0 \rightarrow B_G\) as \(\sigma_0 = \zeta|_{X_0}\), which is an injective \(\epsilon\)-map. For \(i \geq 1\), inductively define
\[
X_i := X_{i-1} \cup \zeta^{-1}(S_i) \cup \{f_i\}.
\]
Assuming that an injective \((\epsilon + \delta)\)-map \(\sigma_{i-1} : X_{i-1} \rightarrow B_G\) has been constructed, we extend it to \(\sigma_i : X_i \rightarrow B_G\), as follows:

(i) \(\sigma_i(f_i) = g^1_i\);
(ii) if \(f \in \zeta^{-1}(S_i) \subseteq X\) and \(\zeta(f) = g^j_i\), set \(\sigma_i(f) = g^{j+1}_i\).

Since \(f\) and \(g^j_i\) are \(\epsilon\)-matched and \(g^j_i\) and \(g^{j+1}_i\) are \(\delta\)-matched, we have that \(f\) and \(\sigma_i(f) = g^{j+1}_i\) are \((\epsilon + \delta)\)-matched. Thus, \(\sigma_i\) is an injective \((\epsilon + \delta)\)-map. As \(X^c_F = \cup_{i \geq 1} X_i\), the mapping \(\sigma : X^c_F \rightarrow B_G\) given by \(\sigma|_{X_i} = \sigma_i\) has the desired properties.

Similarly, we construct an injective \((\epsilon + \delta)\)-map \(\tau : X^c_G \rightarrow B_F\). By Lemma 11, there exists an \((\epsilon + \delta)\)-matching between \(B_F\) and \(B_G\).

For the stability statement, let \(\epsilon > d_F(F, G)\). Then, for any \(\delta > 0\), there is an \((\epsilon + \delta)\)-matching between \(B_F\) and \(B_G\). Taking the infimum over \(\delta > 0\), it follows that \(d_\delta(dgm(F), dgm(G)) \leq \epsilon\). Taking the infimum over \(\epsilon > 0\), we obtain
\[
d_\delta(dgm(F), dgm(G)) \leq d_F(F, G).
\]
\[\square\]
Lemma 15  Let \( f \in B_F \) be a 2\( e \)-significant section and \( S \subseteq B_G \) an infinite subset with the property that each section \( g \in S \) is \( e \)-interleaved with \( f \). Then, for any \( \delta > 0 \), there exists an infinite sub-collection \( S_\delta \subseteq S \) such that any two sections in \( S_\delta \) are \( \delta \)-interleaved.

**Proof** Without loss of generality, we may assume that \( S \) is countable. Suppose that \( \text{supp}(f) = (s^*, t^*) \) and let \( S = \{ g_n : n \geq 1 \} \) with \( \text{supp}(g_n) = (s_n^*, t_n^*) \), \( \forall n \geq 1 \). Then, we have \( s - e \leq s_n \leq s + e \) and \( t - e \leq t_n \leq t + e \) for all \( n \). Hence, \( \{(s_n, t_n)\} \) is a bounded set under the \( d^\infty \) metric and therefore has at least one accumulation point, say, \( (s_0, t_0) \). This implies that, for any \( \delta > 0 \), we can choose a sub-collection of intervals \( \{(s_{n_i}^*, t_{n_i}^*)\}_{i=1}^\infty \), each contained in the \( \delta/2 \)-neighborhood of \( (s_0^*, t_0^*) \), with \( s_{n_i} \to s_0 \) and \( t_{n_i} \to t_0 \). By construction, any pair of sections in \( S_\delta = \{ g_{n_i} : i \geq 1 \} \) are \( \delta \)-interleaved.  

\[ \square \]

Theorem 5 *(The Isometry Theorem)* If \( F \) and \( G \) are decomposable \( p \)-sheaves, then

\[ d_p(\text{dgm}(F), \text{dgm}(G)) = d_I(F, G). \]

**Proof** This follows from Proposition 12 and Theorem 4.  

\[ \square \]

Corollary 2  Let \( \mathbb{V} \) and \( \mathbb{W} \) be decomposable \( c \)-modules. Then,

\[ d_p(\text{dgm}(\mathbb{V}), \text{dgm}(\mathbb{W})) = d_I(\mathbb{V}, \mathbb{W}). \]

**Proof** This follows from Theorem 5, the definition of interleaving distance for \( c \)-modules (Definition 27) and the fact that the barcode of a decomposable \( c \)-module coincides with the barcode of its \( p \)-sheaf of sections (see Remark 8).  

\[ \square \]

7 Applications

The formulation of persistent structures developed in this paper largely has been motivated by applications. This section explores some of these applications, explaining how correspondence modules relate to levelset zigzag persistence, to slices of 2-D persistence modules, as well as how to obtain homological barcodes richer in geometric information than those obtained from sublevel (or superlevel) set filtrations or extended persistence [17]. Among other things, we establish a Mayer-Vietoris sequence relating sublevel and superlevel set homology modules to levelset homology modules.

7.1 Levelset persistence

Let \( f : X \to \mathbb{R} \) be a continuous function defined on a topological space \( X \). It is of great interest to summarize the topological changes in the level sets of \( f \) across function values, as such summaries can provide valuable insights on \( f \). However,
unlike sublevel sets, there are no natural mappings relating different level sets, so a common practice is to use interlevel sets to interpolate level sets in a zigzag structure [7–9]. To be more precise, for \( s \leq t \), denote the interlevel set between \( s \) and \( t \) by 
\[
X^t_s := f^{-1}([s, t])
\]
To further simplify notation, write \( X[t] \) for the level sets \( X^t_s \). At the topological space level, we have inclusions \( X[s] \hookrightarrow X^t_s \hookrightarrow X[t] \) that induce homomorphisms
\[
H_*(X[s]) \xrightarrow{\phi^t_s} H_*(X^t_s) \xleftarrow{\psi^t_s} H_*(X[t])
\] (60)
on homology (with field coefficients). Thus, for an increasing sequence \((t_n)\), we obtain a zigzag module
\[
\ldots \hookrightarrow H_*(X[t_{n-1}]) \xrightarrow{\phi_{n-1}^t} H_*(X^t_{t_{n-1}}) \xleftarrow{\psi_{n-1}^t} H_*(X[t_n]) \hookrightarrow \ldots
\] (61)
Using correspondences, we eliminate the homology of interlevel sets from (61), treating the triple \((H_*(X^t_s), \phi^n_{n-1}, \psi^n_{n-1})\) as a CVec-morphism from \( H_*(X[t_{n-1}]) \) to \( H_*(X[t_n]) \). This has the virtue of leaving only the homology of the level sets as objects in the sequence, also leading to a categorical formulation of level set persistence that easily extends to a continuous parameter \( t \in \mathbb{R} \).

To state the next theorem, recall that we denote the graph of a mapping \( T \) by \( G_T \) and the operator that reverses correspondences by \( \ast \). For \( s, t \in \mathbb{R}, s \leq t \), define a correspondence \( h^t_s \subseteq H_*(X[s]) \times H_*(X[t]) \) by \( h^t_s = G^t_s \circ G^s_t \), and let \( \mathbb{H}^-_*(f) := (H_*(X[t]), h^t_s), s, t \in \mathbb{R}, s \leq t \), which we refer to as the levelset c-module associated with \( f \). As in [8], the homology theory used is Steenrod-Sitnikov homology [28].

**Theorem 6** Let \( X \) be a locally compact polyhedron. If \( f : X \to \mathbb{R} \) is a proper continuous function and \( H_* \) is Steenrod-Sitnikov homology with field coefficients, then \( \mathbb{H}^-_*(f) \) is a c-module.

**Proof** To prove that \( \mathbb{H}^-_*(f) \) is a c-module, it suffices to verify the validity of the composition rule \( h^t_s = h^t_r \circ h^r_s \), for any \( r \leq s \leq t \). For the argument we present, it will be useful to consider the inclusions of interlevel sets \( X^r_s \hookrightarrow X^t_r \hookrightarrow X^t_s \), for \( r \leq s \leq t \), and the induced homomorphisms
\[
H_*(X^r_s) \xrightarrow{\rho^r_{s,t}} H_*(X^t_r) \xleftarrow{\sigma^r_{s,t}} H_*(X^t_s)
\] (62)
The proof amounts to a chase in the commutative diagram
Example 7 Let \( f : X \to \mathbb{R} \) be as indicated in Fig. 2(i). The interval decomposition of the 0-dimensional levelset \( c \)-module \( \mathbb{H}_0^- (f) \) contains all four types of interval modules, as indicated in the barcode. The bar types follow the convention made in Remark 2. Panel (ii) of the figure shows a similar calculation for the 1-dimensional homology module \( \mathbb{H}_1^- (g) \).

We note that although the homology module \( \mathbb{H}_1^- (f) \) is trivial, the persistent Mayer-Vietoris sequence, discussed in the next section, shows how the 1-dimensional homology of \( X \) relates to the levelset \( c \)-module of the mapping \( f \). An alternative approach to 1-dimensional topology via level sets is to consider the Reeb graph of \( f \), which in this case is homeomorphic to \( X \); however, this leaves the realm of homology theory. Similarly, the global longitudinal 1-cycles of \( Y \) are not captured by \( \mathbb{H}_1^- (g) \), but they relate to \( \mathbb{H}_1^- (g) \) through the persistent Mayer-Vietoris sequence.
Remark 14 There is a connection between the present formulation of levelset homology using $c$-modules and 2-D persistence modules. Consider $\mathbb{R}^2$ with the partial ordering in which $(s_0, t_0) \leq (s_1, t_1)$ if $s_0 \geq s_1$ and $t_0 \leq t_1$. As a category, this ordering corresponds to $\mathbb{R}^{\text{op}} \times \mathbb{R}$. Let $\Delta^+ := \{(s, t) \in \mathbb{R}^2 : s \leq t\}$ and $\Delta^- := \{(s, t) \in \mathbb{R}^2 : s \geq t\}$. Similar to the extension of zigzag modules to an exact 2-D persistence module [4, 15], it is possible to extend a $c$-module to a persistent module over $\mathbb{R}^2$. Place the $c$-module $H^*(f)$ along the diagonal $\Delta \subseteq \mathbb{R}^2$. Under the assumptions of Theorem 6, one can define an exact $p$-module over $\Delta^+$, extending $H^*(f)$, whose vector space at $(s, t) \in \Delta^+$ is $V_{(s,t)} := H_s(X^t)$ and whose morphisms $v_{(s_0,s_1)}$ are the mappings on homology induced by the inclusions $X_{s_0} \hookrightarrow X_{s_1}$. Letting $F$ denote the $p$-sheaf of sections of the $c$-module $\mathbb{H}^-(f)$, define a persistence module over $\Delta^-$ by $V_{(s,t)} := F([t, s])$ and $v_{(s_0,s_1)} = F^{[t_0,s_1]}$. One can verify that these two structures combine to yield a single exact 2-D persistence module $\mathcal{V}$ over $\mathbb{R}^{\text{op}} \times \mathbb{R}$.

### 7.2 A persistent mayer-vietoris sequence

Let $f : X \rightarrow \mathbb{R}$ be a continuous function. Under the assumptions of Theorem 6, in this section, we construct a Mayer-Vietoris (M-V) sequence of $c$-modules for covers of $X$ given by sublevel and superlevel sets of $f$. The level set $c$-modules $\mathbb{H}^+_i(f)$, $i \geq 0$, constructed in Section 7.1, represents the homology of the intersections of the elements of these covers. Recall that CMod is the category whose objects are the $c$-modules (over $\mathbb{R}$) with natural transformations as morphisms. In general, morphisms in CMod do not have kernels. However, in this particular case, there is sufficient structure to formulate an exact M-V sequence. We begin with the construction of the objects in the sequence.

Denote the sublevel sets of $f$ by $X^t := f^{-1}((-\infty, t])$ and the superlevel sets of $f$ by $X_t := f^{-1}((t, +\infty))$. For $s \leq t$ and $i \geq 0$, let $t_i^s : H_i(X^s) \rightarrow H_i(X^t)$ and $j_i^s : H_i(X_t) \rightarrow H_i(X_s)$ be the morphisms induced on homology by the inclusions $X^t \subseteq X^s$ and $X_t \subseteq X_s$, respectively. We denote the graphs of $t_i^s$ and $j_i^s$ by $I_i^s$ and $J_i^s$, respectively. Then, $\mathbb{H}^+_i(f) := (H_i(X^s), I_i^s)$ and $\mathbb{H}^-_i(f) := (H_i(X_t), J_i^s)$ are the homology $c$-modules associated to the sublevel and superlevel filtrations of $X$, respectively. The direct sum $\mathbb{H}^+_i(f) \oplus \mathbb{H}^-_i(f)$ is the homology $c$-module associated with the covers $X = X^t \cup X_t$, $t \in \mathbb{R}$. Note that, for $s \leq t$, the elements $(a_s, b_s) \in H_i(X^s) \oplus H_i(X_t)$ and $(a_t, b_t) \in H_i(X^t) \oplus H_i(X_s)$ are in correspondence in $\mathbb{H}^+_i(f) \oplus \mathbb{H}^-_i(f)$ if and only if $t_i^s(a_s) = a_t$ and $j_i^s(b_s) = b_t$. We also define the “constant” $c$-module $\mathbb{H}_i(X)$ in which the vector space over any $t \in \mathbb{R}$ is $H_i(X)$ with the diagonal subspace $\Delta_{H_i(X)}$ as correspondence, for any $s \leq t$.

Now we define the relevant $c$-module morphisms for the M-V sequence. Let $p_i^s : H_i(X^s) \rightarrow H_s(X)$ and $q_i^s : H_i(X_t) \rightarrow H_s(X)$ be the mappings induced on homology by the inclusions $X^t \subseteq X$ and $X_t \subseteq X$. The commutativity of the diagrams
for any \( s \leq t \), implies that the mappings \( p_i^t - q_i^t : H_i(X^t) \oplus H_i(X^s) \to H_i(X) \), given by \((a_t, b_t) \mapsto p_i^t(a_t) - q_i^t(b_t)\), induce a \( c \)-module morphism

\[
p_i - q_i : \mathbb{H}^+_i(f) \oplus \mathbb{H}^-_i(f) \to \mathbb{H}_i(X).
\]

Similarly, the inclusions \( X[t] \subseteq X^t \) and \( X[t] \subseteq X_s \), \( t \in \mathbb{R} \), induce mappings \( \psi_i^t : H_i(X[t]) \to H_i(X^t) \) and \( \phi_i^t : H_i(X[t]) \to H_i(X_t) \) on homology, which in turn induce a \( c \)-module morphism

\[
\psi_i \oplus \phi_i : \mathbb{H}^{-}_i(f) \to \mathbb{H}^+_i(f) \oplus \mathbb{H}^-_i(f).
\]

To define the connecting \( c \)-module morphisms, we work under the assumptions of Theorem 6. For each \( t \in \mathbb{R} \), consider the cover \( X = X^t \cup X_s \), as well as the coarser cover \( X = X[t] \cup X_s \), for \( s \leq t \). Naturality of the Mayer-Vietoris sequence implies that inclusions yield a commutative diagram

\[
\begin{array}{ccc}
\cdots & \to & H_i(X) \xrightarrow{\partial_i} H_{i-1}(X[s]) \to H_{i-1}(X^s) \oplus H_{i-1}(X_s) \cdots \to H_0(X) \\
\downarrow{id} & & \downarrow{\phi_i^t} \\
\cdots & \to & H_i(X) \xrightarrow{\partial_i} H_{i-1}(X^t) \to H_{i-1}(X^t) \oplus H_{i-1}(X_t) \cdots \to H_0(X) \\
\downarrow{id} & & \downarrow{\psi_i^s} \\
\cdots & \to & H_i(X) \xrightarrow{\partial_i} H_{i-1}(X[t]) \to H_{i-1}(X^t) \oplus H_{i-1}(X_t) \cdots \to H_0(X).
\end{array}
\]

In particular, for any \( i \geq 1 \) and \( s \leq t \), the diagram

Fig. 2 Persistence diagrams for levelset homology: (i) \( \mathbb{H}_i^{-}(f) \) and (ii) \( \mathbb{H}_i^{-}(g) \)
commutes, showing that the mappings \( \partial_i : H_i(X) \to H_{i-1}(X[t]) \), \( t \in \mathbb{R} \), induce a \( c \)-module connecting morphism

\[
\Delta_i : \mathbb{H}_i(X) \to \mathbb{H}_{i-1}^{-}(X).
\] (70)

**Theorem 7** If \( f : X \to \mathbb{R} \) is a proper, continuous function defined on a locally compact polyhedron \( X \) and \( H_* \) is Steenrod-Sitnikov homology, then the Mayer-Vietoris sequence

\[
\ldots \longrightarrow \mathbb{H}_{i+1}(X) \xrightarrow{\Delta_{i+1}} \mathbb{H}_i^{-1}(f) \xrightarrow{\psi_i \oplus \phi_i} \mathbb{H}_i^Y(f) \oplus \mathbb{H}_i^A(f) \xrightarrow{p_i - q_i} \ldots
\]

is exact in the CMod category.

**Proof** If \( W_t \xrightarrow{E} W_s \xrightarrow{P} \mathbb{Y} \) are any two consecutive morphisms in the sequence, by construction, \( W_t \xrightarrow{g_t} U_t \xrightarrow{f_t} \) is exact, \( \forall t \in \mathbb{R} \). By Proposition 1(i) and (ii), all morphisms in the sequence have well-defined image and kernel \( c \)-modules. CMod exactness follows from CVec exactness at each \( t \in \mathbb{R} \). \( \square \)

**Example 8** Let \( f : X \to \mathbb{R} \) be projection of the space depicted in Fig. 3 to a horizontal axis. By Proposition 1(iii), \( \text{coker}(p_1 - q_1) \) is a \( c \)-module. The barcode for \( \text{coker}(p_1 - q_1) \), also shown in the figure, exactly captures the horizontal spread of the 1-dimensional cycles of \( X \).

In this example, the barcode for extended persistence [17] also encodes the same geometric properties, but \( c \)-modules present the information in a categorical framework that naturally integrates various different types of persistence architectures.

### 7.3 Slicing 2-D Persistence Modules

Multidimensional persistent homology is of great practical interest, as topological analysis of complex data frequently gives rise to topological or simplicial filtrations that depend on multiple parameters. However, unlike the one-dimensional case in which the barcode of a sufficiently tame \( p \)-module yields a complete invariant, it is impossible to obtain a discrete, complete representation of the structure of multidimensional \( p \)-modules [10]. As such, it is of interest to define, albeit incomplete, computable and informative invariants for these modules, such
as rank invariants [10, 30] and some numeric invariants [32], to summarize their structural properties. Lesnick and Wright slice 2-D persistence modules along affine lines of non-negative slopes to obtain a family of one-dimensional \( p \)-modules whose structures may be described by persistence diagrams [27]. Here, we show how to define a \( c \)-module structure along affine lines of negative slope. If the original 2-D persistence module is pointwise finite dimensional, then these negatively sloped slices are virtually tame thus admitting an interval decomposition.

We view \( (\mathbb{R}^2, \leq) \) as a poset, where \((x_1, y_1) \leq (x_2, y_2)\) if and only if \(x_1 \leq x_2\) and \(y_1 \leq y_2\). Let \( \ell' \subseteq \mathbb{R}^2 \) be an affine line of negative slope. We fix an orientation for \( \ell' \) via the unit vector \( u = e^{i\theta}, -\pi/2 < \theta < 0 \), parallel to \( \ell' \). This induces a linear ordering on \( \ell' \) given by \( s \leq t \) if and only if \((t - s) \cdot u \geq 0 \). Given a 2-D persistence module \( \mathbb{U} : \mathbb{R}^2 \to \text{Vec} \), we define a \( c \)-module \( \mathbb{V} : \mathbb{R} \to \text{CVec} \), termed the slice of \( \mathbb{U} \) along \( \ell' \), as follows. The vector space at \( t \in \ell' \) is \( V_t := U_t \). To define the correspondences, we introduce some notation. For \( a, b \in \mathbb{R}^2 \), let \( a \lor b \in \mathbb{R}^2 \) be the (unique) element that is initial with respect to the property that \( a \leq a \lor b \) and \( b \leq a \lor b \). If \( a = (x_1, y_1) \) and \( b = (x_2, y_2) \) satisfy \( x_1 \leq x_2 \) and \( y_1 \geq y_2 \), then \( a \lor b := (x_2, y_1) \). For \( s, t \in \ell' \) with \( s \leq t \), let \( P(s, t) \) be collection of all finite sequences \( T = (t_i)_{i=0}^{n} \) of points in \( \ell' \) satisfying \( t_0 \leq \cdots \leq t_n \), \( t_0 = s \), and \( t_n = t \). Letting \( r_i = t_{i-1} \lor t_i \), define the staircase \( \Gamma_T \) associated to \( T \in P(s, t) \) as the sequence in \( \mathbb{R}^2 \) given by

\[
t_0 \leq r_1 \leq t_1 \leq \cdots \leq r_{n-1} \leq t_n,
\]

as depicted in Fig. 4(i). Using staircases, define a correspondence \( v_i' \), as follows.

**Definition 38** A pair \((v_s, v_i) \in v_i' \) if and only if for any staircase \( \Gamma_T, T \in P(s, t) \), there are vectors that interpolate \( v_s \) and \( v_i \) along \( \Gamma_T \), as illustrated in Fig. 4(ii). More precisely, there are vectors \( w_i \in U_{t_i}, 0 \leq i \leq n \), and \( z_i \in U_{r_i}, 1 \leq i \leq n \), such that

(i) \( w_0 = v_s \) and \( w_n = v_i \);
(ii) \( u_{i+1}^r(w_{i-1}) = z_i \) and \( u_i^r(w_i) = z_i \), for \( 1 \leq i \leq n \).

**Lemma 16** Let \( \mathbb{V} \) be a slice of a 2-D persistence module \( \mathbb{U} \) along a line of negative slope, and let \( S, T \in P(s, t) \) with \( S \subseteq T \). If \( v_s \in V_s \) and \( v_i \in V_i \), may be interpolated along \( \Gamma_T \), then \( v_s \) and \( v_i \) also may be interpolated along the coarser staircase \( \Gamma_S \).

**Proof** Using an iterative argument, it suffices to consider the case where \( S \) and \( T \) differ by a single element, say,

\[
T = \{t_0, \ldots, t_j, \ldots, t_n\} \text{ and } S = \{t_0, \ldots, t_{j-1}, \hat{t}_j, t_{j+1}, \ldots, t_n\},
\]

where \( \hat{t}_j \) indicates deletion of \( t_j \). Using the notation of Definition 38, suppose \( w_i \in U_{t_i} \) and \( z_i \in U_{r_i} \) interpolate \( v_s \) and \( v_i \) along \( \Gamma_T \). Let \( \bar{r}_j = t_{j-1} \lor t_{j+1} \) and \( \bar{z}_j = u_{j+1}^r(w_{j-1}) \). Then,
\[ \bar{z}_j = u_{j+1}^r (w_{j-1}) = u_{j+1}^r \circ u_{j}^p (w_j) = u_{j+1}^r \circ u_{j+1}^{r+1} (w_j) \]
\[ = u_{j+1}^r (z_{j+1}) = u_{j+1}^r \circ u_{j+1}^{r+1} (w_{j+1}) = u_{j+1}^r (w_{j+1}). \]  
(73)

Thus, \( \bar{z}_j = u_{j+1}^r (w_{j-1}) = u_{j+1}^r (w_{j+1}) \), showing that the vectors

\[ w_0, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n \quad \text{and} \quad r_1, \ldots, r_{j-1}, \bar{r}_j, r_{j+2}, \ldots, r_n \]  
(74)

interpolate \( v_s \) and \( v_t \) along the staircase \( \Gamma_S \).

**Theorem 8** Let \( U : \mathbb{R}^2 \to \text{Vec} \) be a pointwise finite-dimensional persistence module. If \( \ell' \subseteq \mathbb{R}^2 \) is a negatively sloped line, then the slice of \( U \) along \( \ell' \) is a virtually tame \( c \)-module.

**Proof** Let \( V \) be the slice of \( U \) along \( \ell' \). To show that \( V \) is a \( c \)-module, it suffices to verify the composition rule \( v'_r = v'_r \circ v'_s \) for morphisms. We begin with the inclusion \( v'_s \circ v'_s \subseteq v'_r \). Let \((v_r, v_s) \in v'_r \) and \((v_s, v_t) \in v'_s \). Given \( T \in P(r, t) \), let \( \bar{T} = T \cup \{s\} \).

Write \( T \) as a union \( \bar{T} = T_1 \cup T_2 \), \( T_1 \in P(r, s) \), and \( T_2 \in P(s, t) \). By assumption, we may interpolate \( v_r \) and \( v_s \) along \( \Gamma_{T_1} \), as well as \( v_s \) and \( v_t \) along \( \Gamma_{T_2} \). Concatenating these interpolations, we obtain an interpolation of \( v_r \) and \( v_t \) along \( \Gamma_{\bar{T}} \). By Lemma 16, \( v_r \) and \( v_t \) also can be interpolated along the coarser staircase \( \Gamma_{\bar{T}} \). This proves that \((v_r, v_t) \in v'_r \).

For the reverse inclusion, let \((v_r, v_s) \in v'_r \) and \( s \in \ell' \) be such that \( r \leq s \leq t \). Our goal is to show that there exists \( v_s \in V_s = U_s \) such that \((v_r, v_s) \in v'_s \) and \((v_s, v_t) \in v'_t \).

Given \( T_1 \in P(r, s) \) and \( T_2 \in P(s, t) \), let \( V_{T_1, T_2} \) be the affine subspace of \( V_s \) comprising those vectors \( v_s \in V_s \) such that:

(i) \( v_r \) and \( v_s \) may be interpolated along \( \Gamma_{T_1} \);
(ii) \( v_s \) and \( v_t \) may be interpolated along \( \Gamma_{T_2} \).

**Fig. 3** Barcode for the cokernel of \( p_1 - q_1 : \mathbb{H}^\bullet_p (f) \oplus \mathbb{H}^\bullet_s (f) \to H_1(X) \).
Note that $V_{T_1,T_2}$ is non-empty because $(v_r,v_t) \in v_s^i$ implies that $v_r$ and $v_t$ may be interpolated along the staircase $\Gamma_{T_1 \cup T_2}$. To conclude the argument, we show that as this implies that any $v_s \in W_s$ satisfies $(v_r,v_s) \in v_r^i$ and $(v_s,v_t) \in v_t^i$, as desired.

Let $A = \{ V_{T_1,T_2} : T_1 \in P(r,s) \text{ and } T_2 \in P(s,t) \}$, partially ordered via inclusion. Since $V_s$ is finite dimensional, each descending chain in $A$ stabilizes in finitely many steps, thus having a lower bound. By Zorn’s Lemma, $A$ has a minimal element $V_{R_1,R_2}$. We show that $V_{R_1,R_2} \subseteq V_{T_1,T_2}$, for any $T_1 \in P(r,s)$ and $T_2 \in P(s,t)$. Let $S_1 = T_1 \cup R_1$ and $S_2 = T_2 \cup R_2$. By Lemma 16, $V_{S_1,S_2} \subseteq V_{T_1,T_2}$ and $V_{S_1,S_2} \subseteq V_{R_1,R_2}$. By minimality, $V_{R_1,R_2} = V_{S_1,S_2} \subseteq V_{T_1,T_2}$. Hence, $W_s = V_{R_1,R_2} \neq \emptyset$, as claimed.

The virtual tameness of $\mathcal{V}$ follows from the assumption that $\mathcal{U}$ is pointwise finite dimensional. 

8 Closing Remarks

This paper introduced and developed two main concepts: (i) correspondence modules that generalize structures such as persistence modules and zigzag modules and (ii) persistence sheaves that provide a pathway to the structural analysis of $c$-modules. Using sheaf-theoretical arguments, we proved interval decomposition theorems for sufficiently tame $c$-modules and $p$-sheaves parameterized over $\mathbb{R}$, as well as a stability theorem for persistence-diagram representations of $p$-sheaves. Applications discussed in the paper include: (1) a new formulation of continuously parameterized levelset persistence in a category theory framework; (2) a Mayer-Vietoris sequence that brings together levelset, sublevelset and superlevelset homology modules of a real-valued function; and (3) 1-dimensional slices of 2-dimensional persistence modules along lines of negative slope.

Fig. 4 Interpolating $v_s \in V_s$ and $v_t \in V_t$ along a staircase

\[ W_s := \bigcap_{T_1 \in P(r,s)} \bigcap_{T_2 \in P(s,t)} V_{T_1,T_2} \neq \emptyset, \]
This study of persistent homology from the viewpoint of $c$-modules and $p$-sheaves opens up avenues for further investigation. We conclude with a discussion of some of the questions raised by the results of this paper.

(a) The isometry statement $d_I(dgm(\mathcal{V}), dgm(\mathcal{W})) = d_I(\mathcal{V}, \mathcal{W})$ for decomposable $c$-modules was established using an interleaving distance defined by mapping a $c$-module to its $p$-sheaf of sections. How to define the interleaving distance between $c$-modules more intrinsically so that the isometry theorem holds?

(b) This paper has focused primarily on structural and stability questions associated with $c$-modules. However, the practical relevance of $c$-modules depends heavily on computability of interval decompositions. Thus, a basic problem is that of developing and implementing an algorithm to calculate the persistence diagram of a sufficiently tame $c$-module.

(c) There are algorithms to calculate the persistence diagrams for the sublevelset and superlevelset homology modules of a function $f : X \to \mathbb{R}$. To what extent can the persistence diagrams for the levelset $c$-module of $f$ be inferred from these and the homology groups $H_n(X)$ via the persistent Mayer-Vietoris sequence of Section 7.2? More generally, given an exact sequence of $c$-modules in which the barcodes for every other term is known, can we infer the other barcodes?

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