Super-Ricci flows and improved gradient and transport estimates

Eva Kopfer∗

Abstract

We show that the heat flow on super-Ricci flows in the sense of Sturm in [22] satisfies transport estimates with respect to every $L^p$-Kantorovich distance, $p \in [1, \infty]$. As an application we construct Brownian motions on time-dependent metric measure spaces and present transport estimates for their trajectories.

The proof is inspired by Savaré and Bakry in [19] and [5] respectively and takes advantage of the self-improvement of the gradient estimates. For this we prove a refined version of Bochner’s inequality under strengthened assumptions on the metric.

1 Introduction and Statement of the main results

The heat flow on a metric measure space $(X,d,m)$ can be understood either as the gradient flow of the Cheeger energy $C_h$ on the Hilbert space $L^2(X,m)$ or as the gradient flow of the relative entropy $S$ on the space of probability measures $P_2(X)$ endowed with the $L^2$-Kantorovich distance $W_2$. It has been shown in [1] that these two notions coincide under the assumption that $(X,d,m)$ satisfies a lower Ricci curvature bound in the sense of Lott-Sturm-Villani.

A metric measure space is said to have a lower Ricci curvature bound $K$, in short $CD(K,\infty)$, if the relative entropy is $K$-convex along $L^2$-Kantorovich geodesics. This definition is consistent with the Riemannian case, i.e. a Riemannian manifold has Ricci curvature bounded from below by $K$ if and only if its relative entropy is $K$-convex. But even more holds true. Each of the following properties characterize a lower curvature bounds of the manifold: $L^p$-transport and gradient estimates of the heat flow for $p \in [1, \infty]$, and pathwise contraction for Brownian trajectories.

On general $CD(K,\infty)$-spaces these properties fail, and the heat flow does not even have to be linear. In order to obtain a more Riemannian-like behavior, Ambrosio, Gigli and Savaré introduced in [2] the notion of $RCD(K,\infty)$-spaces, i.e. $CD(K,\infty)$-spaces whose heat flow is linear. This notion is characterized by one single formula, namely the Evolution variational inequality of the heat flow with respect to the $L^2$-Kantorovich distance. Moreover one immediately

∗Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany (eva.kopfer@iam.uni-bonn.de)
reovers the $L^2$-transport and gradient estimates
\[ W_2(P_t \mu, P_t \nu) \leq e^{-Kt} W_2(\mu, \nu), \quad \Gamma(P_t u) \leq e^{-2Kt} P_t \Gamma(u), \]
where $\Gamma$ denotes the *Carré du champ* operator.
Savaré proved in [19] that the gradient estimate implies the stronger estimate
\[ (\Gamma(P_t u))^{\alpha} \leq e^{-2\alpha K t} P_t (\Gamma(u))^{\alpha}, \]
where $\alpha \in [1/2, 2]$. This has been first noticed by Bakry in [5] in the framework of Dirichlet forms. Crucial for this estimate is the self-improvement of Bochner’s inequality. By Kuwada’s duality [15] the stronger gradient estimate further implies stronger $L^p$-transport estimates
\[ W_p(P_t \mu, P_t \nu) \leq e^{-Kt} W_p(\mu, \nu) \quad p \in [1, \infty]. \]

In this paper we aim to show similar estimates for time-dependent metric measure spaces $(X, d_t, m_t)_{t \in I}$ which evolve under a super-Ricci flow in the sense of Sturm in [22]. Moreover we introduce Brownian motions and prove pathwise contraction estimates.

In [14] existence and uniqueness of the heat flow $(P_{t,s})_{t \geq s}$ have been shown, and moreover, that super-Ricci flows of RCD($K, N$)-spaces $(X, d_t, m_t)_{t \in I}$ are characterized by the time-dependent gradient estimate
\[ \Gamma_t(P_{t,s} u) \leq P_{t,s}(\Gamma_s(u)), \]
or equivalently, by the $L^2$-Kantorovich transport estimate
\[ W_{2,s}(\tilde{P}_{t,s} \mu, \tilde{P}_{t,s} \nu) \leq W_{2,\tau}(\mu, \nu). \]

What remained unsolved was the question whether there are stronger transport and gradient estimates in the sense of Savaré [19] and Bakry [5] respectively. The answer given in this paper is affirmative, but we have to strengthen our assumption regarding the time-dependence of the metric such that $t \mapsto \Gamma_t(u)$ is differentiable. This allows us to derive a dynamic version of Bochner’s inequality
\[ \frac{1}{2} \Delta_t (\Gamma_t(u)) - \Gamma_t(u, \Delta_t u) \geq \frac{1}{2} (\partial_t \Gamma_t)(u). \]

**Theorem 1.1.** Let $(X, d_t, m_t)_{t \in I}$ be a one-parameter family of geodesic Polish metric measure spaces satisfying [2], [3], [19] and [10] such that each $(X, d_t, m_t)$ is a RCD($K, N$) space. If the transport estimate [11] holds, then the dynamic Bochner inequality [12] holds at all $t \in I$.

In contrast to [14], where also a dynamic version of Bochner’s inequality has been derived, the function $u$ does not need to arise as a heat flow $P_{t,s}u_s$. This is an essential modification, since from this we obtain an improved gradient estimate.

**Theorem 1.2.** Let $(X, d_t, m_t)_{t \in I}$ be as in Theorem [17]. Then, if the dynamic Bochner inequality [12] and the regularity assumption [23] is satisfied, for every $\alpha \in [1/2, 1]$ we have for a.e. $\tau \leq t$ and $\sigma \geq s$ and every $u \in \text{Dom}(\text{Ch})$ and
\[ \Gamma_\tau(P_{\tau,\sigma} u)^\alpha \leq P_{\tau,\sigma}(\Gamma_\sigma(u)^\alpha) \quad m.a.e.. \]
As an application we introduce Brownian motions on time-dependent RCD($K,N$)-spaces and prove existence and uniqueness in law. On super-Ricci flows we construct couplings of Brownian motions satisfying a pathwise contraction estimate. Summarizing we obtain the following.

**Theorem 1.3.** Let $(X,d_t,m_t)_{t\in I}$ satisfy the assumptions in Theorem 1.1. Then, $(X,d_t,m_t)_{t\in I}$ is a super-Ricci flow if and only if one of the following equivalent properties holds

i) for all $s \leq t$, $\alpha \in [1/2,1]$ and $u \in \text{Dom}(Ch)$
\[ \Gamma_t(P_{t,s}u)^\alpha \leq P_{t,s}(\Gamma_s(u)^\alpha) \text{ m-a.e.}, \]

ii) for all $s \leq t$, $p \in [1,\infty]$ and $\mu,\nu \in \mathcal{P}(X)$
\[ W_{p,s}(\dot{P}_{t,s}\mu,\dot{P}_{t,s}\nu) \leq W_{p,t}(\mu,\nu), \]

iii) there exists a coupling of Brownian motions $(X_1^{s},X_2^{s})$ such that for all $s \leq t$
\[ d_s(X_1^{s},X_2^{s}) \leq d_t(X_1^{0},X_2^{0}) \text{ almost surely}. \]

A similar result as in Theorem 1.1 and Theorem 1.2 has been derived in [13] in the case of smooth Riemannian manifolds evolving as a super-Ricci flow. They give a characterization of super-Ricci flows in terms of a gradient estimate as in Theorem 1.2 with $\alpha = 1$ and $\alpha = 1/2$ and in terms of a Bochner’s formula. Arnaudon, Coulibaly and Thalmeier [4] showed existence of Brownian motions on a smooth time-dependent setting and apply their results to Ricci flows. Kuwada and Philipowski [16] construct couplings of Brownian motions such that the normalized Perelman’s $L$-distance of the coupling is a supermartingale, see also [23]. This construction is obtained on smooth Riemannian manifolds evolving as a super-Ricci flow.

**Example.** A possible example for the setting chosen in this paper is the super-Ricci flow on the spherical cone over the product of the 2-spheres with radius $1/\sqrt{3}$ constructed in [14]. This space is a RCD$(4,5)$-space, and the punctured cone is a 5-dimensional (non-complete) Riemannian manifold with constant curvature 4. A possible Ricci flow on the punctured cone is given by distances which shrink to one point homothetically in time. The completion of this flow is a super-Ricci flow which shrinks to a point homothetically in time. Hence, for time points smaller than the collapsing time the metrics satisfy the assumptions (9) and (10). The same argumentation can be used to obtain (2) for the measures.

## 2 Proof of the main results

In the sequel let $(X,d_t,m_t)_{t\in I}$, where $I = (0,T)$, be a one-parameter family of geodesic Polish metric measure spaces such that the following holds:
1. There exists a finite reference measure $m$ with full topological support such that $m_t = e^{-f_t}$ with Borel functions $(f_t)$ satisfying

$$|f_t(x)| \leq C, \quad |f_t(x) - f_t(y)| \leq C d_t(x, y), \quad |f_t(x) - f_s(x)| \leq L |t - s|,$$

with constants $C, L > 0$ independent of $x, y \in X$ and $s, t \in I$.

2. The distance is “log-Lipschitz” continuous, i.e.

$$|\log(d_t(x, y)/d_s(x, y))| \leq L |t - s|$$

for all $x, y \in X$ and all $s, t \in I$.

3. There exist constants $K, N \in \mathbb{R}$ such that for each $t \in I$ the space $(X, d_t, m_t)$ satisfies the Riemannian curvature-dimension bound $RCD(K, N)$ in the sense of [3], [11].

In the sequel let us introduce the time-dependent quantities which we are going to use.

Let $P(X)$ denote the space of all Borel probability measures. We set for each $p \in [1, \infty)$

$$W_{p,t}(\mu_1, \mu_2) = \min \left\{ \int_{X \times X} d_t^p(x, y) d\gamma(x, y) | \gamma \in \Pi(\mu_1, \mu_2) \right\}^{1/p},$$

where $\Pi(\mu_1, \mu_2)$ is the space of all measures in $P(X \times X)$ whose marginals $(e_i)_{#\mu}$ coincide with $\mu_i$. We also set

$$W_{\infty,t}(\mu_1, \mu_2) = \inf \left\{ ||d||_{L^\infty(\gamma)} | \gamma \in \Pi(\mu_1, \mu_2) \right\} = \lim_{p \to \infty} W_{p,t}(\mu_1, \mu_2),$$

with essential supremum $||d||_{L^\infty(\gamma)} = \inf \{ C \geq 0 | d(x, y) \leq C \gamma \text{-a.e. } x, y \}$. For the second equality see e.g. Lemma 3.2 in [15].

We recall that the Cheeger energy $\text{Ch}_t$ at time $t \in I$ is defined as the convex and lower-semicontinuous functional in $L^2(X, m_t)$

$$\text{Ch}_t(u) := \inf \left\{ \liminf_{n \to \infty} \frac{1}{2} \int_X \text{lip}_t(u_n)^2 dm_t \right\}$$

where the infimum is taken over all bounded Lipschitz functions $u_n \in \text{Lip}_t(X)$ such that $u_n \to u$ in $L^2(X, m_t)$ (cf. [1] [22]). Here, $\text{lip}_t u$ denotes the local Lipschitz constant w.r.t. the metric $d_t$

$$\text{lip}_t u(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_t(x, y)},$$

and $\text{Ch}_t$ admits the local representation formula

$$\text{Ch}_t(u) = \frac{1}{2} \int_X |
abla_t u|^2_{*} dm_t,$$

where $|\nabla_t u|_*$ is the minimal relaxed gradient [1]. Since $(X, d_t, m_t)$ satisfies a Riemannian curvature bound, (in particular $\text{Ch}_t$ is quadratic) $\mathcal{E}_t := 2\text{Ch}_t$ is a strongly local Dirichlet form with Carré du Champ

$$\Gamma_t(u) = |\nabla_t u|_*^2.$$
cf. [10, 11, 2], i.e.

\[ \mathcal{E}_t(u) = \int_X \Gamma_t(u) \, dm_t. \] (4)

Thanks to (4), \( \mathcal{E}(u,v) = \int_X \Gamma_t(u,v) \, dm_t \) where

\[ \Gamma_t(u,v) := \frac{1}{4} (\Gamma_t(u+v) - \Gamma_t(u - v)). \]

\( \Gamma(\cdot, \cdot) \) satisfies the chain rule and the Leibniz rule

\[ \Gamma_t(\theta(u), v) = \theta'(u) \Gamma_t(u,v), \quad \Gamma_t(uv, w) = u \Gamma_t(v, w) + v \Gamma_t(u, w), \]

where \( u, v, w \in \text{Dom}(\mathcal{E}_t) \) and \( \theta \in \text{Lip}(\mathbb{R}) \), \( \theta(0) = 0 \). We call the linear generator \( \Delta_t \) the Laplacian and

\[ -\int_X \Delta_t u \, v \, dm_t = \mathcal{E}_t(u,v) \quad \forall u \in \text{Dom}(\Delta_t), v \in \text{Dom}(\mathcal{E}_t), \]

with domain \( \text{Dom}(\Delta_t) \subset \text{Dom}(\mathcal{E}_t) \).

Due to our assumptions (2) and (3), the sets \( L^2(X, m_t) \) and \( W^{1,2}(X, d_t, m_t) := \mathcal{D}(\mathcal{E}_t) \) do not depend on \( t \) and the respective norms for varying \( t \) are equivalent to each other. We put \( \mathcal{H} = L^2(X, m) \) and \( \mathcal{F} = \text{Dom}(\mathcal{E}_t) \) for some fixed \( t_0 \) as well as

\[ \mathcal{F}_{(s, \tau)} = L^2((s, \tau) \to \mathcal{F}) \cap H^1((s, \tau) \to \mathcal{F}^*) \subset C([s, \tau] \to \mathcal{H}) \]

for each \( 0 \leq s < \tau \leq T \).

The heat equations

A function \( u \) is called solution to the heat equation

\[ \Delta_t u = \partial_t u \quad \text{on } (s, \tau) \times X \]

if \( u \in \mathcal{F}_{(s, \tau)} \) and if for all \( w \in \mathcal{F}_{(s, \tau)} \)

\[ -\int_s^\tau \mathcal{E}_t(u_t, w_t) \, dt = \int_s^\tau \langle \partial_t u_t, w_t e^{-f_t} \rangle_{\mathcal{F}^*, \mathcal{F}} \, dt \] (5)

where \( \langle \cdot, \cdot \rangle_{\mathcal{F}^*, \mathcal{F}} = \langle \cdot, \cdot \rangle \) denotes the dual pairing. Note that thanks to (2), \( w \in L^2((s, \tau) \to \mathcal{F}) \) if and only if \( we^{-f} \in L^2((s, \tau) \to \mathcal{F}) \).

Further a function \( v \) is called solution to the adjoint heat equation

\[ -\Delta_s v + \partial_s f \cdot v = \partial_s v \quad \text{on } (\sigma, t) \times X \]

if \( v \in \mathcal{F}_{(\sigma, t)} \) and if for all \( w \in \mathcal{F}_{(\sigma, t)} \)

\[ \int_\sigma^t \mathcal{E}_s(v_s, w_s) \, ds + \int_\sigma^t \int_X v_s \cdot w_s \cdot \partial_s f_s \, dm_s \, ds = \int_\sigma^t \langle \partial_s v_s, w_s e^{-f_s} \rangle_{\mathcal{F}^*, \mathcal{F}} \, ds. \]

We recall the following results from [14].
Theorem 2.1. (i) For each $0 \leq s < \tau \leq T$ and each $h \in \mathcal{H}$ there exists a unique solution $u \in \mathcal{F}(s,\tau)$ to the heat equation $\partial_t u_t = \Delta u_t$ on $(s,\tau) \times X$ with $u_s = h$.

(ii) The heat propagator $P_{t,s} : h \mapsto u_t$ admits a kernel $p_{t,s}(x,y)$ w.r.t. $m_s$, i.e.,

$$P_{t,s} h(x) = \int p_{t,s}(x,y) h(y) \, dm_s(y). \quad (6)$$

If $X$ is bounded, for each $(s',y) \in (s,T) \times X$ the function $(t,x) \mapsto p_{t,s}(x,y)$ is a solution to the heat equation on $(s',T) \times X$.

(iii) All solutions $u : (t,x) \mapsto u_t(x)$ to the heat equation on $(s',T) \times X$ are Hölder continuous in $t$ and $x$. All nonnegative solutions satisfy a scale invariant parabolic Harnack inequality of Moser type.

(iv) The heat kernel $p_{t,s}(x,y)$ is Hölder continuous in all variables, it is Markovian

$$\int p_{t,s}(x,y) \, dy := \int p_{t,s}(x,y) \, dm_s(y) = 1 \quad (\forall s < t, \forall x)$$

and has the propagator property

$$p_{t,r}(x,z) = \int p_{t,s}(x,y) p_{s,r}(y,z) \, dm_s(y) \quad (\forall r < s < t, \forall s, z).$$

Theorem 2.2. (i) For each $0 \leq \sigma < t \leq T$ and each $g \in \mathcal{H}$ there exists a unique solution $v \in \mathcal{F}(t,\sigma)$ to the adjoint heat equation $\partial_s v_s = -\Delta v_s + (\partial_s f_s)v_s$ on $(\sigma,t) \times X$ with $v_t = g$.

(ii) This solution is given as $v_s(y) = P_{t,s}^* g(y)$ in term of the adjoint heat propagator

$$P_{t,s}^* g(y) = \int p_{t,s}(x,y) g(x) \, dm_t(x). \quad (7)$$

If $X$ is bounded, for each $(t',x) \in (0,t) \times X$ the function $(s,y) \mapsto p_{t,s}(x,y)$ is a solution to the adjoint heat equation on $(0,t') \times X$.

(iii) All solutions $v : (s,y) \mapsto v_s(y)$ to the adjoint heat equation on $(\sigma,t) \times X$ are Hölder continuous in $s$ and $y$. All nonnegative solutions satisfy a scale invariant parabolic Harnack inequality of Moser type.

By duality, the propagator $(P_{t,s})_{s \leq t}$ acting on bounded continuous functions induces a dual propagator $(\hat{P}_{t,s})_{s \leq t}$ acting on probability measures as follows

$$\int u \, d(\hat{P}_{t,s} \mu) = \int (P_{t,s} u) \, dm \quad \forall u \in \mathcal{C}_b(X), \forall \mu \in \mathcal{P}(X).$$

The time-dependent function $v_t(x) = P_{t,s} u(x)$ is a solution to the heat equation, whereas the time-dependent measure $\nu_t(dy) = P_{t,s} \mu(dy)$ is a solution to the dual heat equation

$$-\partial_s \nu = \hat{\Delta}_s \nu.$$

Again $\hat{\Delta}_s$ is defined by duality: $\int u \, d(\hat{\Delta}_s \mu) = \int \Delta_s u \, dm \quad \forall u, \forall \mu$.

Lemma 2.3 (Lemma 3.8 in [14]). Let $u, g \in \mathcal{F}$ and $t \in I$ with $g \in L^1(X,m_t)$. Then,

$$\lim_{h \to 0} \frac{1}{h} \left( \int u g dm_t - \int u P_{t,t-h}^* g dm_{t-h} \right) = \int \Gamma_t(u,g) \, dm_t.$$
In this section we aggravate the regularity of the map

2.1 From transport estimates to Bochner’s inequality

such that for each $C$ assume that there exists a $h$ which proves the assertion.

More precisely,

$$e^{-3Lr}E_r(u_r) + 2\int_s^T e^{-3Lt} \int_X |\Delta_t u_t|^2 \, dm_t \, dt \leq e^{-3Ls} \cdot E_s(u_s).$$

(iii) For all solutions $v$ to the adjoint heat equation on $(\sigma, t) \times X$ and all $s \in (\sigma, t)$,

$$E_s(v_s) + \|v_s\|^2_{L^2(m_s)} \leq e^{3L(t-s)} \cdot \left[ E_s(v_t) + \|v_t\|^2_{L^2(m_t)} \right].$$

Moreover, $v_s \in \text{Dom}(\Delta_s)$ for a.e. $s \in (\sigma, t)$.

2.1 From transport estimates to Bochner’s inequality

In this section we aggravate the regularity of the map $r \mapsto \log d_r(x, y)$. We assume that there exists a $C^0$ map $r \mapsto h_r(x, y)$, uniformly bounded $|h_r(x, y)| \leq C$ such that for each $s, t \in I$ and $x, y \in X$

$$a_r(x, y) = d_s(x, y)e^{f \int_s^t h_r(x, y) \, dr}.$$ Consequence, for each $x, y \in X$, $r \mapsto \log d_r(x, y)$ is continuously differentiable with derivative $h_r(x, y) = \frac{d}{dr} \log d_r(x, y)$. Moreover we assume that

$$\forall x \in X, r \in I \text{ the limit } \lim_{y \to x} h_r(x, y) := H_r(x) \text{ exists, measurable in } x,$n and $r \mapsto H_r(x)$ is continuous $\forall x \in X$.

We obtain the following lemma.

Lemma 2.5. Let $u \in \text{Lip}(X)$. Then for all $s, t \in I$ and $x \in X$

$$\text{lip}_x u(x) = \text{lip}_x u(x)e^{-f \int_s^t H_r(x) \, dr}.$$

Proof. For $s < t$, we obtain from the very definition of the local slope

$$\text{lip}_x u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_s(x, y)} \leq \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_s(x, y)} e^{-\liminf_{y \to x} f \int_s^t h_r(x, y) \, dr}$$

where we applied dominated convergence. Changing the roles of $s$ and $t$ yields

$$\text{lip}_x u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_s(x, y)} \leq \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_t(x, y)} e^{-\liminf_{y \to x} f \int_s^t h_r(x, y) \, dr}$$

which proves the assertion.
We apply our observation to the minimal relaxed gradient. We say that 
\( G \in L^2(X,m_t) \) is a \textit{t-relaxed gradient} of \( u \in L^2(X,m_t) \) if there exists Lipschitz functions \( u_n \in L^2(X,m_t) \) such that 
\[
\lim_{n \to \infty} u_n \to u \quad \text{in} \quad L^2(X,m_t) \quad \text{and} \quad \lim_{n \to \infty} \operatorname{lip}_s u_n \to \tilde{G} \quad \text{in} \quad L^2(X,m_t), \quad \tilde{G} \leq G \text{ m.a.e. in } X.
\]
\( G \) is the \textit{minimal t-relaxed gradient} \( |\nabla_t u|_s \), if its \( L^2(X,m_t) \) norm is minimal among all relaxed gradients, see \[1\] Definition 4.2. The collection of all \( t \)-relaxed gradients is convex and closed in \( L^2(X,m_t) \) [1 Lemma 4.3].

**Proposition 2.6.** For m-a.e. \( x \in X \)
\[
|\nabla_t u|_s(x) = |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr}
\]
for each \( u \in \mathcal{F} \) and for all \( s,t \in I \). In particular for m-a.e. \( x \in X, \ t \to |\nabla_t u|_s(x) \) is continuously differentiable.

**Proof.** Assume \( s \leq t \). Let \( u_n \in L^2(X,m_s) \) be a sequence of Borel Lipschitz functions such that \( u_n \to u \) and \( \lim_{n \to \infty} \operatorname{lip}_s u_n \to |\nabla_s u|_s \) in \( L^2(X,m_s) \), see Lemma 4.3 in [1]. Then since \( H \) is uniformly bounded
\[
\lim_{n \to \infty} \operatorname{lip}_s u_n(x)e^{-\int_l^s H_r(x) \, dr} \to |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \quad \text{in} \quad L^2(X,m_s).
\]
This implies that \( |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \) is a relaxed gradient of \( u \) with respect to the \( d_t \) norm, and hence from Lemma 4.4 in [1]
\[
|\nabla_t u|_s(x) \leq |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \quad \text{m-a.e. in } X.
\]
Changing the roles of \( s \) and \( t \) yields that
\[
|\nabla_t u|_s(x) = |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \quad \text{m-a.e. in } X.
\]
Choosing \( s \) and \( t \) from a dense and countable set \( D \) in \( I \) the argument from above implies that m-a.e. in \( X \)
\[
|\nabla_t u|_s(x) = |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \quad \text{for each } s \text{ and } t \text{ in } D.
\]
Since the dependence of the left and the right side of the equality is continuous with respect to \( s \) and \( t \), we conclude that for m-a.e. \( x \in X, |\nabla_t u|_s(x) = |\nabla_s u|_s(x)e^{-\int_l^s H_r(x) \, dr} \) holds for every \( s \) and \( t \) in \( I \).

Similarly, we choose \( u \) in a dense and countable set \( C \) in \( \mathcal{F} \) (\[2\] Proposition 4.10)) and obtain that m-a.e. equation \( \text{(11)} \) holds for every \( s,t \in I \) and every \( u \in C \). Given \( u \in \mathcal{F} \) we approximate \( u \) by a sequence \( u_n \in C \), i.e. \( |\nabla_t u_n| \to |\nabla_t u| \) in \( L^2(X,m_t) \). Then there exists a subsequence \( u_{n_k} \) such that for m-a.e. \( x \in X, |\nabla_t u_{n_k}|(x) \to |\nabla_t u|(x) \). Equality \( \text{(11)} \) implies that for the same subsequence \( |\nabla_t u_{n_k}|(x) \to |\nabla_t u|(x) \) for m-a.e. \( x \). Hence we showed that for m-a.e. \( x \in X, \text{(11)} \) holds for every \( u \in \mathcal{F} \) and every \( s,t \in I \).

The last assertion follows directly from the fact that \( r \to H_r(x) \) is supposed to be continuous for all \( x \in X \).

Following \[14\] we give a weak dynamic version of Bochner’s inequality.
**Definition 2.7.** We say that the dynamic Bochner inequality holds at time $t$ if for all $u \in \text{Dom}(\Delta_t) \cap L^\infty(X, m_t)$ such that $\Gamma_t(u) \in L^\infty(X, m_t)$, and all $g \in \text{Dom}(\Delta_t) \cap L^\infty(X, m_t)$ with $g \geq 0$

$$\frac{1}{2} \int \Gamma_t(u) \Delta_t g \, dm_t + \int (\Delta_t u)^2 g + \Gamma_t(u, g) \Delta_t u \, dm_t \geq \frac{1}{2} \int (\partial_t \Gamma_t(u)) g \, dm_t.$$  \hspace{1cm} (12)

This is a “real” Bochner’s inequality in the sense that on the one hand $u$ and $g$ do not have to arise as a heat flow (see [14, Definition 5.5]), and on the other we employ the time-derivative $\partial_t \Gamma_t(u)$ in contrast to [14, Definition 5.5, Definition 5.6]). For the proof of Theorem 1.1 we follow the argumentation in the proof of Theorem 5.13 in [14]. The argumentation in [14] is inspired by [9], where the authors prove the equivalence between Wasserstein contraction estimates and Bochner’s inequality in the static setting.

**Proof of Theorem 1.1.** Define $u = h^1_t u_0$, where $u_0 \in L^\infty(X, m_t) \cap L^2(X, m_t)$ and $h^1_t$ the static semigroup mollification

$$h^1_t u_0 := -\frac{1}{\varepsilon^2} \int_0^\infty H^1_t u_0 \kappa(\varepsilon) \, dr.$$  

Here, $(H^1_t)_{t \geq 0}$ denotes the (static) semigroup associated to $E_t$ and $\kappa \in C_c^\infty((0, \infty))$ with $\kappa \geq 0$ and $\int_0^\infty \kappa_r \, dr = 1$. Recall that $u, \Delta_t u \in \text{Dom}(\Delta_t) \cap \text{Lip}_b(X)$.

Let $g \in F \cap L^\infty(X, m_t)$ such that $g \geq 0$. Then, the transport estimate (11) together with Lemma 5.10 and Lemma 5.11 in [14] eventually yields

$$-\frac{1}{2} \int P_{t,s}(\Gamma_s(u)) g \, dm_t + \int \Gamma_t(P_{t,s} u, u) g \, dm_t \leq \frac{1}{2} \int \Gamma_t(u) g \, dm_t,$$

see [14] Section 5]. Then following the lines in [14], we subtract $\frac{1}{2} \int \Gamma_t(u) g \, dm_t$ on each side and divide by $t - s$ obtaining

$$\frac{1}{2(t - s)} \left[ \int \Gamma_t(u) g \, dm_t - \int P_{t,s}(\Gamma_s(u)) g \, dm_t \right] + \frac{1}{t - s} \left[ \int \Gamma_t(P_{t,s}u, u) g \, dm_t - \int \Gamma_t(u) g \, dm_t \right] \leq 0.$$  \hspace{1cm} (13)

We decompose the first term on the left-hand side into the following two terms

$$\frac{1}{2(t - s)} \left[ \int \Gamma_t(u) g \, dm_t - \int \Gamma_s(u) P_{t,s}^* g \, dm_s \right] = \frac{1}{2(t - s)} \left[ \int \Gamma_t(u) g \, dm_t - \int \Gamma_t(u) P_{t,s}^* g \, dm_s \right] + \frac{1}{2} \int \frac{\Gamma_t(u) - \Gamma_s(u)}{t - s} P_{t,s}^* g \, dm_s.$$ 

Recall that $\Gamma_t(u) \in F$ [19, Lemma 3.2] and thus we can apply Lemma 2.3, which gives us

$$\lim_{s \to t} \frac{1}{t - s} \left[ \int \Gamma_t(u) g \, dm_t - \int \Gamma_t(u) P_{t,s}^* g \, dm_s \right] = \int \Gamma_t(u, \cdot) g \, dm_t,$$  \hspace{1cm} (14)
while, since $\left| \frac{\Gamma_f(u) - \Gamma_\tau(u)}{t-s} \right| \leq 2L \Gamma_{t}(u) \in L^\infty(X, m_t)$,

$$\liminf_{s \nearrow t} \int \frac{\Gamma_t(u) - \Gamma_s(u)}{(t-s)} (P_{t,s}g) dm_s \geq \liminf_{s \nearrow t} \int \frac{\Gamma_t(u) - \Gamma_s(u)}{(t-s)} gdm_t + \liminf_{s \nearrow t} \int \frac{\Gamma_t(u) - \Gamma_s(u)}{(t-s)} (P_{t,s}ge^{-f_s} - ge^{-f_t}) dm$$

$$\geq \int (\partial_t \Gamma_t)(u) gdm_t - \limsup_{s \nearrow t} 2L \|\Gamma_t(u)\|_{L^\infty(X, m_s)} \|P_{t,s}^{*}ge^{-f_s} - ge^{-f_t}\|_{L^1(X, m_s)}$$

$$= \int (\partial_t \Gamma_t)(u) gdm_t,$$

where we used Proposition 2.4 in the last inequality and that $P_{t,s}^{*}ge^{-f_s} \to ge^{-f_t}$ in $L^1(X, m)$ as $s \to t$.

Regarding the second term on the left-hand side of (13), note that the Leibniz rule and the integration by parts formula is applicable and we get

$$\int \Gamma_t(P_{t,s}u, u) gdm_t = \int \Gamma_t(gP_{t,s}u, u) dm_t - \int \Gamma_t(g, u) P_{t,s}udm_t$$

$$= - \int \psi P_{t,s}^{*} (g \Delta_t u) dm_s - \int P_{t,s}^{*}(\Gamma_t(g, u)) u dm_s.$$

Subtracting $\int \Gamma_t(u) gdm_t$ and applying (16)

$$\frac{1}{t-s} \left( \int \Gamma_t(P_{t,s}u, u) gdm_t - \int \Gamma_t(u) gdm_t \right)$$

$$= \frac{1}{t-s} \left(- \int \psi P_{t,s}^{*} (g \Delta_t u) dm_s + \int \psi (g \Delta_t u) dm_t \right)$$

$$+ \frac{1}{t-s} \left(- \int P_{t,s}^{*}(\Gamma_t(g, u)) u dm_s + \int \Gamma_t(u, g) u dm_t \right).$$

Letting $s \nearrow t$ we have since $g \in \mathcal{F} \cap L^\infty(X, m_t)$ and $\Delta_t u \in \text{Lip}_b(X)$, $g\Delta_t u \in \mathcal{F} \cap L^1(X, m_t)$

$$\lim_{s \nearrow t} \frac{1}{t-s} \left(- \int u P_{t,s}^{*} (g \Delta_t u) dm_s + \int u (g \Delta_t u) dm_t \right) = \int \Gamma_t(u, g \Delta_t u) dm_t$$

by virtue of Lemma 2.3. In order to determine

$$\lim_{s \nearrow t} \frac{1}{(t-s)} \left(- \int P_{t,s}^{*}(\Gamma_t(g, u)) u dm_s + \int \Gamma_t(u, g) u dm_t, \right),$$

we need to argue whether $\Gamma_t(g, u) \in \mathcal{F}$. But this is the case, since, due to our static RCD$(K, \infty)$ assumption, we may apply Theorem 3.4 in [19] and obtain

$$\Gamma_t(\Gamma_t(g, u)) \leq 2(\gamma_2(u) - KT_t(g) \Gamma_t(g) + 2(\gamma_2(g) - KT_t(g)) \Gamma_t(g) u \quad m_t \text{-a.e.,}$$

where $\gamma_2(u), \gamma_2(g) \in L^1(X, m_t)$. Our regularity assumptions on $u$ and $g$ provide that the right hand side is in $L^1(X, m_t)$ and consequently Lemma 2.3 implies

$$\lim_{s \nearrow t} \frac{1}{t-s} \left(- \int P_{t,s}^{*}(\Gamma_t(g, u)) u dm_s + \int \Gamma_t(u, g) u dm_t \right) = \int \Gamma_t(\Gamma_t(g, u), u) dm_t.$$
Combining these observations we find
\[ \lim_{s \nearrow t} \frac{1}{t-s} \left( \int \Gamma_t(P_{t-s}u, u) \, dm_t - \int \Gamma_t(u) \, dm_t \right) = \int \Gamma_t(u, g \Delta_t u) \, dm_t + \int \Gamma_t(\Gamma_t(g, u), u) \, dm_t = -\int (\Delta_t u)^2 g + \Gamma_t(g, u) \Delta_t u \, dm_t. \]

Hence from (13), (14), (15) and (17)
\[ \frac{1}{2} \int \Gamma_t(h^2 \Delta_t u, \Delta_t u) \, dm_t + \int (\Delta_t(h^2 \Delta_t u))^2 g + \Gamma_t(g, h^2 \Delta_t u) \, dm_t \geq \frac{1}{2} \int (\partial \Gamma_t)(h^2 \Delta_t u) \, dm_t. \]

Since \((\partial \Gamma_t)(u)(x) = -2H_t(x)\Gamma_t(u)(x)\) for \(m\text{-a.e. } x \in X\) and \(|H_t(x)| \leq C\), we obtain the assertion by letting \(\varepsilon \to 0\) with taking into account that
\[ ||h^2 \Delta_t u - u||_X \to 0 \text{ as } \varepsilon \to 0 \text{ and } \Delta_t(h^2 \Delta_t u) = h^2 \Delta_t u. \]

\[ \square \]

2.2 Self-improvement of the gradient estimate

Quasi-regular Dirichlet forms

We follow the approach in [19] and briefly recall the notion of quasi-regular Dirichlet forms developed in [18] and [10]. We denote by \(F = \{ u \in L^2(X, m) | E(u) < \infty \}\) the domain of a Dirichlet form \(E: L^2(X, m) \to [0, \infty]\), where \(X\) is a Polish space and \(m\) is a \(\sigma\)-finite Borel measure. \(F\) is a Hilbert space with norm \(\|u\|_F^2 = \|u\|_{L^2(X, m)}^2 + E(u)\). If \(F\) is a closed set in \(X\) we denote
\[ F_F := \{ u \in F | u(x) = 0 \text{ for } m\text{-a.e. } x \in X \setminus F \}. \]

**Definition 2.8.** Given a Dirichlet form \(E\) on a Polish space \(X\), an \(E\)-nest is an increasing sequence of closed subsets \((F_k)_{k \in \mathbb{N}} \subset X\) such that \(\cup_{k \in \mathbb{N}} F_k\) is dense in \(F\).

A set \(N \subset X\) is \(E\)-polar if there is an \(E\)-nest \((F_k)_{k \in \mathbb{N}}\) such that \(N \subset X \setminus \cup_{k \in \mathbb{N}} F_k\). If a property holds in a complement of an \(E\)-polar set we say that it holds \(E\)-quasi-everywhere (\(E\)-q.e.).

A function \(u: X \to \mathbb{R}\) is said to be \(E\)-quasi-continuous if there exists an \(E\)-nest \((F_k)_{k \in \mathbb{N}}\) such that every restriction \(f|_{F_k}\) is continuous on \(F_k\).

The Dirichlet form \(E\) is said to be quasi-regular if the following three properties hold.

i) There exists an \(E\)-nest \((F_k)_{k \in \mathbb{N}}\) consisting of compact sets.
ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-quasi-continuous representatives.

iii) There exists an $\mathcal{E}$-polar set $N \subset X$ and a countable collection of $\mathcal{E}$-quasi-continuous functions $(f_k)_{k \in \mathbb{N}} \subset \mathcal{F}$ separating the points of $X \setminus N$.

For every $u \in \mathcal{F}$ the quasi-regularity implies that $u$ admits an $\mathcal{E}$-quasi-continuous representative $\tilde{u}$. The representative is unique q.e. and if $u \in \mathcal{F}$ with $|u| \leq C$ m-a.e., then $|\tilde{u}| \leq C$ q.e.. (18)

The following Lemma is taken from [19, Lemma 2.6].

**Lemma 2.9.** Let $\mathcal{E}$ be a strongly local, quasi-regular Dirichlet form with linear generator $\Delta$. Let $\psi \in L^1(X,m) \cap L^\infty(X,m)$ nonnegative and $\varphi \in L^1(X,m) \cap L^2(X,m)$ such that

$$\int_X \psi \Delta g \, dm \geq - \int_X \varphi g \, dm$$

for any nonnegative $g \in \mathcal{F} \cap L^\infty(X,m)$ with $\Delta g \in L^\infty(X,m)$. Then $\psi \in \mathcal{F}$ with

$$\mathcal{E}(\psi) \leq \int_X \psi \varphi \, dm, \quad \int \varphi \, dm \geq 0,$

and there exists a unique finite Borel measure $\mu := \mu^+ - \mu^-$ such that every $\mathcal{E}$-polar set is $|\mu|$-negligible, the q.e. representative of any function in $\mathcal{F}$ belongs to $L^1(\mathbb{R},|\mu|)$ and

$$-\mathcal{E}(\psi,g) = - \int \Gamma(\psi,g) \, dm = \int \tilde{g} \, d\mu \text{ for every } g \in \mathcal{F}.$$

We denote by $\Delta^* u$ the measured valued Laplacian, i.e. the signed measure $\mu = \mu^+ - \mu^-$ such that

$$\mathcal{E}(u,\varphi) = \int \varphi \, d\mu \text{ for every } \varphi \in \mathcal{F}. \quad (19)$$

**Contraction estimates for the heat flows $P_{t,s}$ and $\hat{P}_{t,s}$**

For each $t \in I$ we define the Hessian

$$H_t[u](g,h) := \frac{1}{2} \left( \Gamma_t(g,\Gamma_t(u,h)) + \Gamma_t(h,\Gamma_t(u,g)) - \Gamma_t(u,\Gamma_t(g,h)) \right).$$

Recall that on a family of closed Riemannian manifolds $(M,g_t)$ we obtain the equality

$$H_t[u](g,h) = \langle \nabla_t^2 u \nabla_t g, \nabla_t h \rangle_{g_t}.$$

Further note that $||\nabla_t^2 u \nabla_t g, \nabla_t h||_{g_t} \leq ||\nabla_t^2 u||_{HS} ||\nabla_t g|| ||\nabla_t h||$, where $|| \cdot ||_{HS}$ denotes the Hilbert-Schmidt norm. If the manifold has Ricci curvature bounded from below by some $K \in \mathbb{R}$ then with $|| \cdot ||_2 = || \cdot ||_{L^2}$ and $K_- = \max\{-K,0\}$

$$||\nabla_t^2 u||_{HS}^2 \leq (1 + K_-/2) (||\nabla_t u||_2^2 + ||u||_2^2).$$

We define the distribution valued $\Gamma_{2,K}$-operator

$$\Gamma_{2,K}(u) : \mathcal{F} \cap L^\infty \cap L^1 \to \mathbb{R}$$

as in [14].
Definition 2.10. For each $u \in \text{Dom}(\Delta_t)$ such that $u, \Gamma_t(u) \in L^\infty(X, m_t)$ we define
\[
\Gamma_{2,t}(u)(g) = \int -\frac{1}{2} \Gamma_t(\Gamma_t(u), g) \, dm_t + \int (g(\Delta_t u)^2 + \Gamma_t(g, u) \Delta_t u) \, dm_t,
\]
where $g \in \mathcal{F}$ such that $g \in L^1(X, m_t) \cap L^\infty(X, m_t)$.

Note that thanks to the static RCD($K, N$)-condition the domain of the Laplacian coincides with the domain of the Hessian, i.e. $\text{Dom}(\Delta_t) = W^{2,2}(X, d_t, m_t)$, and
\[
|\Gamma_{2,t}(u)(g)| \leq ||g||_\infty ||\Delta_t u||_2^2 + C||g||_\infty \sqrt{\Gamma_t(u)} ||\Delta_t u||_2 \sqrt{\|\Delta_t u\|^2_2 + ||u||_2^2},
\]
cf. Section 5 in [14]. Moreover, each $\mathcal{E}_t = 2\text{Ch}_t$ defines a quasi-regular Dirichlet form ([19, Theorem 4.1]).

Proposition 2.11. Suppose that Bochner’s inequality holds at time $t \in I$. Then for every $u \in \text{Dom}(\Delta_t)$ with $u, \Gamma_t(u) \in L^\infty(X, m_t)$

i) $\Gamma_t(u) \in \mathcal{F}$ with
\[
\frac{1}{2} \mathcal{E}_t(\Gamma_t(u)) \leq L||\Gamma_t(u)||_\infty \mathcal{E}_t(u) + ||\Gamma_t(u)||_\infty ||\Delta_t u||_2^2
+ C||\Delta_t u||_2 \sqrt{||\Gamma_t(u)||_2^2} \sqrt{||\Delta_t u||_2^2 + ||u||_2^2}.
\]

ii) There exists a finite nonnegative Borel measure $\mu_+ = \mu_+(t)$ such that every $\mathcal{E}_t$-polar set is $\mu_+$-negligible and for each $g \in \mathcal{F}$ the $\mathcal{E}_t$-q.c. representative $\tilde{g} \in L^1(X, \mu_+)$ with
\[
2 \Gamma_{2,t}(u)(g) = \int g(\partial_t \Gamma_t)(u) \, dm_t + \int \tilde{g} \, d\mu_+.
\]

In particular $\Gamma_{2,t}(u)$ is a finite Borel measure with
\[
2 \Gamma_{2,t}(u) = (\partial_t \Gamma_t)(u)m + \mu_+.
\]

Proof. Let $u_{\varepsilon} = h^\varepsilon_t u$. Choosing $\psi = \Gamma_t(u_{\varepsilon})$ and $\varphi = -(\partial_t \Gamma_t)(u_{\varepsilon}) - 2\Gamma_t(u_{\varepsilon}, \Delta_t u_{\varepsilon})$ in Lemma 2.9 and applying Bochner’s inequality together with the Leibniz rule yields
\[
\mathcal{E}_t(\Gamma_t(u_{\varepsilon})) \leq - \int \Gamma_t(u_{\varepsilon}((\partial_t \Gamma_t)(u_{\varepsilon}) + 2\Gamma_t(u_{\varepsilon}, \Delta_t u_{\varepsilon})) \, dm_t.
\]

Applying the Leibniz rule once again we obtain
\[
\mathcal{E}_t(\Gamma_t(u_{\varepsilon})) \leq - \int (\Gamma_t(u_{\varepsilon})(\partial_t \Gamma_t)(u_{\varepsilon}) - 2(\Delta_t u_{\varepsilon})^2 \Gamma_t(u_{\varepsilon}) - 2\Gamma_t(u_{\varepsilon}, \Gamma_t(u_{\varepsilon})) \Delta_t u_{\varepsilon}) \, dm_t.
\]

Note that as $\varepsilon \to 0$, $\Gamma(u_{\varepsilon}) \to \Gamma(u)$ pointwise, in $L^1$ and in the weak* $L^\infty$ topology. The latter is due to the fact that $\Gamma(u_{\varepsilon} - u)$ is uniformly bounded and converges to 0 in $L^1$. Moreover by the uniform boundedness of $\Gamma(u_{\varepsilon})$ in $L^\infty$ we obtain that $\Gamma(u_{\varepsilon}) \to \Gamma(u)$ in $L^2$. Hence we find
\[
\mathcal{E}_t(\Gamma_t(u)) \leq \liminf_{\varepsilon \to 0} \mathcal{E}_t(\Gamma_t(u_{\varepsilon}))
\]
and
\[
\int \Gamma_t(u)(\partial_t \Gamma_t)(u) \, dm_t = \int \Gamma_t(u)^2 e^{H_t} \, dm_t
\]
\[
= \lim_{\varepsilon \to 0} \int \Gamma_t(u_\varepsilon)^2 e^{H_t} \, dm_t = \lim_{\varepsilon \to 0} \int \Gamma_t(u_\varepsilon)(\partial_t \Gamma_t)(u_\varepsilon) \, dm_t,
\]

while
\[
\int (\Delta_t u)^2 \Gamma_t(u) \, dm_t = \lim_{\varepsilon \to 0} \int (h_\varepsilon^t \Delta_t u)^2 \Gamma_t(u_\varepsilon) \, dm_t = \lim_{\varepsilon \to 0} \int (\Delta_t u_\varepsilon)^2 \Gamma_t(u_\varepsilon) \, dm_t.
\]

In order to show that
\[
\lim_{\varepsilon \to 0} \int \Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon)) \Delta_t u_\varepsilon \, dm_t = \int \Gamma_t(u, \Gamma_t(u)) \Delta_t u \, dm_t,
\]
we show that \(\Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon))\) weakly converges to \(\Gamma_t(u, \Gamma_t(u))\) in \(L^2\). Take a sufficiently smooth testfunction \(\varphi \in \mathcal{F} \cap L^\infty\), then we easily deduce
\[
\int \Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon)) \varphi \, dm_t = - \int \Delta_t u_\varepsilon \Gamma_t(u_\varepsilon) \varphi \, dm_t - \int \Gamma_t(u_\varepsilon, \varphi) \Gamma_t(u_\varepsilon) \, dm_t
\]
\[
\quad \to - \int \Delta_t u \Gamma_t(u) \varphi \, dm_t - \int \Gamma_t(u, \varphi) \Gamma_t(u) \, dm_t
\]
by the strong \(L^2\) convergence of \(\Delta_t u_\varepsilon\), the weak*-\(L^\infty\) convergence of \(\Gamma_t(u_\varepsilon)\) and the \(L^1\) convergence of \(\Gamma(u_\varepsilon, \varphi)\). Moreover \(||\Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon))||_2\) is uniformly bounded in \(\varepsilon\) since
\[
\int ||\Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon))||^2 dm_t \leq 4||\Gamma_t(u_\varepsilon)^2||_\infty C(||\Delta_t u_\varepsilon||_2^2 + ||u_\varepsilon||_2^2)
\]
\[
\leq C||\Gamma_t(u)^2||_\infty(||||\Delta_t u||_2^2 + ||u||_2^2||)
\]
since the domain of the Laplacian coincides with the domain of the Hessian, cf. [14, Section 3], [12]. Consequently we obtain that \(\Gamma_t(u_\varepsilon, \Gamma_t(u_\varepsilon))\) weakly converges to \(\Gamma_t(u, \Gamma_t(u))\) in \(L^2\) since \(\mathcal{F} \cap L^\infty\) is dense in \(L^2\) [11, Theorem 4.5].

We conclude
\[
\frac{1}{2} \mathcal{E}_t(\Gamma_t(u)) \leq - \int \frac{1}{2} \Gamma_t(u)(\partial_t \Gamma_t)(u) - \Gamma_t(u)(\Delta_t u)^2 - \Gamma_t(u, \Gamma_t(u)) \Delta_t u \, dm_t
\]
\[
\leq L||\Gamma_t(u)||_\infty \mathcal{E}_t(u) + ||\Gamma_t(u)||_\infty ||\Delta_t u||_2^2 + C||\Delta_t u||_2 ||\Gamma_t(u)||_\infty ||\Delta_t u||_2^2 + ||u||_2^2.
\]

We show the second claim again by using the semigroup mollification \(u_\varepsilon := h_\varepsilon^t u\). By Lemma [29] we deduce that
\[
\int g \, d\Delta_t^* \Gamma_t(u_\varepsilon) = \int \tilde{g} 2 \Gamma_t(u_\varepsilon, \Delta_t u_\varepsilon) \, dm_t
\]
\[
= \int \tilde{g} \, dm_{\mu_t}(u_\varepsilon) + \int \tilde{g}(\partial_t \Gamma_t)(u_\varepsilon) \, dm_t,
\]
where \(\Delta_t^*\) is the measure valued Laplacian, and \(\mu_t(u_\varepsilon)\) the nonnegative Borel measure with \(\mu_t(u_\varepsilon)(X) \leq \int (\Delta_t u_\varepsilon)^2 + \frac{1}{2}(\partial_t \Gamma_t)(u_\varepsilon) \, dm_t\). Hence, since \(g = \tilde{g}\)
\[
\int g \, d\mu_+(u_z) = \int -\Gamma_t(\Gamma_t(u_z), g) \, dm_t + \int 2g(\Delta_t u_z)^2 + 2\Gamma_t(g, u_z)(\Delta_t u_z) \, dm_t - \int g(\partial_t \Gamma_t)(u_z) \, dm_t.
\]

Note that the right hand side converges as \( \varepsilon \to 0 \) since \( \Gamma(u_z) \to \Gamma(u) \) weakly in \( F \). Indeed, take a test function \( \varphi \in \text{Dom}(\Delta_t) \). Then

\[
\lim_{\varepsilon \to 0} \int \Gamma_t(\Gamma_t(u_z), \varphi) \, dm_t = \lim_{\varepsilon \to 0} \int \Gamma_t(u_z) \Delta_t \varphi \, dm_t = \int \Gamma_t(\Gamma_t(u), \varphi) \, dm_t.
\]

Since \( E_t(\Gamma_t(u_z)) \) is uniformly bounded in \( \varepsilon \) by the first claim and \( \text{Dom}(\Delta_t) \) is dense in \( F \) we deduce that

\[
\lim_{\varepsilon \to 0} \int \Gamma_t(\Gamma_t(u_z), g) \, dm_t = \int \Gamma_t(\Gamma_t(u), g) \, dm_t \quad \forall g \in F.
\]

Define the linear functional \( \tilde{\mu}_+(u) : F \cap L^\infty \to \mathbb{R} \) by

\[
\tilde{\mu}_+(u)(g) := \lim_{\varepsilon \to 0} \int g \, d\mu_+(u_z).
\]

Note that if \( g \geq 0 \) we have \( \tilde{\mu}_+(u)(g) \geq 0 \) by the Bochner inequality. The Hahn-Banach theorem implies that there exists a linear functional \( M : F \to \mathbb{R} \) such that \( M(g) = \mu_+(u)(g) \) for all \( g \in F \cap L^\infty \) and \( M(g) \geq 0 \) for all \( g \in F \) such that \( g \geq 0 \) a.e.. Moreover, if \( g \in F \) with \( g \leq 1 \) m-a.e.

\[
M(g) = \mu_+(u)(g) = \lim_{\varepsilon \to 0} \int g \, d\mu_+(u_z) \leq \mu_+(u_z)(X) \leq \int (\Delta_t u)^2 + C\Gamma_t(u) \, dm_t.
\]

Thus by Proposition 2.5 in [19] there exists a unique finite and nonnegative Borel measure \( \mu_+ \) in \( X \) such that every \( E_t \)-polar set is \( \mu_+ \)-negligible and for each \( g \in F \) the \( E_t \)-q.e. representative \( \hat{g} \in L^1(X, \mu_+) \) with

\[
M(g) = \int \hat{g} \, d\mu_+.
\]

Consequently

\[
2\Gamma_{2,t}(u)(g) = \int g(\partial_t \Gamma_t)(u) \, dm_t + \int \hat{g} \, d\mu_+,
\]

and hence \( \Gamma_{2,t} \) is measure valued with \( 2\Gamma_{2,t}(u) = (\partial_t \Gamma_t)(u) m_t + \mu_+ \).

By virtue of Lebesgue’s decomposition theorem we denote by \( \gamma_{2,t}(u) \in L^1(X, m_t) \) the density wrt \( m_t \)

\[
\Gamma_{2,t}(u) = \gamma_{2,t}(u) m_t + \Gamma^\perp_{2,t}(u), \quad \Gamma^\perp_{2,t}(u) \perp m_t,
\]

and thus by the above Lemma

\[
\gamma_{2,t}(u) \geq \frac{1}{2}(\partial_t \Gamma_t)(u) \text{ m-a.e. and } \Gamma^\perp_{2,t}(u) \geq 0.
\]
Lemma 2.12. Let \( \bar{u} = (u_i)_{i=1}^n \) with \( u_i \in \text{Dom}(\Delta_t) \) such that \( u_i, \Gamma_t(u) \in L^\infty(X, m_t) \) and let \( \Psi \in C^2(\mathbb{R}^n) \) with \( \Psi(0) = 0 \). Then

\[
\Gamma_{2,t}(\Psi(\bar{u})) = \sum_{i,j} \Gamma_{2,t}(u_i, u_j)(\partial_i \Psi)(\bar{u})(\partial_j \Psi)(\bar{u}) + 2 \sum_{i,j,k} (\partial_i \Psi)(\bar{u})(\partial_j \partial_k \Psi)(\bar{u}) H_t[u_i](u_j, u_k) m_t + \sum_{i,j,k,h} (\partial_i \partial_k \Psi)(\bar{u})(\partial_j \partial_h \Psi)(\bar{u}) \Gamma_t(u_i, u_j) \Gamma_t(u_k, u_h) m_t.
\]

In particular \( m_t \)-a.e.

\[
\gamma_{2,t}(\Psi(\bar{u})) = \sum_{i,j} \gamma_{2,t}(u_i, u_j)(\partial_i \Psi)(\bar{u})(\partial_j \Psi)(\bar{u}) + 2 \sum_{i,j,k} (\partial_i \Psi)(\bar{u})(\partial_j \partial_k \Psi)(\bar{u}) H_t[u_i](u_j, u_k) + \sum_{i,j,k,h} (\partial_i \partial_k \Psi)(\bar{u})(\partial_j \partial_h \Psi)(\bar{u}) \Gamma_t(u_i, u_j) \Gamma_t(u_k, u_h).
\]

Proof. Note that \( \Psi(\bar{u}) \in \text{Dom}(\Delta_t) \) with \( \Gamma_t(u) \in L^\infty \) since

\[
\Gamma_t(\Psi(\bar{u})) = \sum_{i,j} \partial_i \Psi(\bar{u}) \partial_j \Psi(\bar{u}) \Gamma_t(u_i, u_j) \in L^1 \cap L^\infty,
\]

\[
\Delta_t(\Psi(\bar{u})) = \sum_{i} \partial_i \Psi(\bar{u}) \Delta_t u_i + \sum_{i,j} \partial_{ij} \Psi(\bar{u}) \Gamma_t(u_i, u_j) \in L^2.
\]

Thus by definition for each \( g \in \mathcal{F} \cap L^\infty \)

\[
2 \Gamma_{2,t}(\Psi(\bar{u}))(g) = \int -\Gamma_t(g, \Gamma_t(\Psi(\bar{u}))) + 2g(\Delta_t \Psi(\bar{u}))^2 + 2g(\Psi(\bar{u})) \Delta_t \Psi(\bar{u}) \ dm_t.
\]

We calculate using the notation \( \psi = \Psi(\bar{u}), \psi_i = \partial_i \Psi(\bar{u}) \) and \( \psi_{ij} = \partial_{ij} \Psi(\bar{u}) \) for
the first term
\[ \int -\Gamma_t(g, \Gamma_t(\Psi(\bar{u}))) \, dm_t \]
\[ = \sum_{i,j} \left\{ \int -\Gamma_t(g\psi_i\psi_j, \Gamma_t(u_i, u_j)) \, dm \right. 
\left. + \int g\left( \Gamma_t(u_i, u_j)\Delta_t(\psi_i\psi_j) + 2\Gamma_t(\psi_i\psi_j, \Gamma_t(u_i, u_j)) \right) \, dm_t \right\} \]
\[ = \sum_{i,j} \int -\Gamma_t(g\psi_i\psi_j, \Gamma_t(u_i, u_j)) \, dm + \int 2g(I + II) \, dm_t, \]
where
\[ I = \sum_{i,j,k,h} \Gamma_t(u_i, u_j) \left( \psi_i(\psi_jk\Delta_tu_k + \psi_jkh\Gamma_t(u_k, u_h)) + \psi_ik\psi_jh\Gamma_t(u_k, u_h) \right) \]
and
\[ II = \sum_{i,j,k} \psi_i\psi_jk \left( \Gamma_t(u_k, \Gamma_t(u_j, u_i)) + \Gamma_t(u_j, \Gamma_t(u_i, u_k)) \right). \]

On the other hand
\[ \int 2g(\Delta_t\Psi(\bar{u}))^2 + 2\Gamma_t(g, \Psi(\bar{u})) \, dm_t \]
\[ = \sum_{i,j} 2 \int \left( \Delta_tu_i\Delta_tu_jg\psi_i\psi_j + \Gamma_t(u_i, g\psi_i\psi_j) \Delta_tu_j \right) \, dm_t \]
\[ - \sum_{i,j,k,h} 2g\left( \psi_i\Delta_tu_k\psi_jk\Gamma_t(u_i, u_j) + \psi_i\Gamma_t(u_k, u_h)\psi_jh\Gamma_t(u_i, u_j) \right. 
\left. + \psi_ik\psi_jh\Gamma_t(u_i, u_k) \right) \, dm_t. \]

Adding up and collecting terms yields
\[ 2\Gamma_{2,t}(\Psi(\bar{u}))(g) \]
\[ = \sum_{i,j} \int \left( -\Gamma_t(g\psi_i\psi_j, \Gamma_t(u_i, u_j)) + 2g\psi_i\psi_j\Delta_tu_i\Delta_tu_j + 2\Gamma_t(u_i, g\psi_i\psi_j)\Delta_tu_j \right) \, dm_t 
\left. + \sum_{i,j,k} 2g\psi_i\psi_jk \left( \Gamma_t(u_k, \Gamma_t(u_i, u_j)) + \Gamma_t(u_j, \Gamma_t(u_i, u_k)) \right) \right) \, dm_t 
\left. + \sum_{i,j,k,h} 2g\psi_ik\psi_jh\Gamma_t(u_k, u_h) \right) \, dm_t \]
\[ = 2 \sum_{i,j} \Gamma_{2,t}(u_i, u_j)(g\psi_i\psi_j) + \sum_{i,j,k} 4g\psi_i\psi_jk(H_t[u_i](u_k, u_j)) \, dm_t 
\left. + \sum_{i,j,k,h} 2g\psi_ik\psi_jh\Gamma_t(u_k, u_h) \right) \, dm_t \]
for each \( g \in \mathcal{F} \cap L^\infty. \)
For arbitrary $g \in F$, set $g^n := g \wedge n$. Then, by dominated convergence (recall that $\tilde{g} \in L^1(X,\mu_+)$)

$$\lim_{n \to \infty} \int g^n \, d\Gamma_{2,t}(\Psi(\tilde{u})) = \lim_{n \to \infty} \left( \int g^n (\partial_i \Gamma_t)(\Psi(\tilde{u})) \, dm_t + \int g^n \, d\mu_+ \right)$$

$$= \int g \, d\Gamma_{2,t}(\Psi(\tilde{u})).$$

Similarly we can pass to the limit for the other integrals and obtain for all $g \in F$

$$2\Gamma_{2,t}(\Psi(\tilde{u}))(g) = 2 \sum_{i,j} \Gamma_{2,t}(u_i, u_j)(g \psi_i \psi_j) + \sum_{i,j,k} \int 4g_i \psi_j \psi_k (H_t[u_i](u_k, u_j)) \, dm_t$$

$$+ \sum_{i,j,k,h} \int 2g_i \psi_j \psi_k \Gamma_t(u_k, u_h) \Gamma_t(u_i, u_j) \, dm_t,$$

and hence the result.

**Proposition 2.13.** Suppose that Bochner’s inequality holds at time $t$. Then for every $u \in \text{Dom}(\Delta_t) \cap L^\infty(X,\mu_t)$ such that $\Gamma_t(u) \in L^\infty(X,\mu_t)$

$$\Gamma_t(\Gamma_t(u)) \leq 4(\gamma_{2,t}(u) - \frac{1}{2} \partial_t \Gamma_t(u)) \Gamma_t(u).$$

**Proof.** We choose the same polynomial $\Psi : \mathbb{R}^3 \to \mathbb{R}$ as in [19] by

$$\Psi(\tilde{u}) := \lambda u_1 + (u_2 - a)(u_3 - b) - ab, \quad \lambda, a, b \in \mathbb{R},$$

where $\tilde{u} = (u_1, u_2, u_3)$, where each $u_i \in \text{Dom}(\Delta_t) \cap L^\infty(X,\mu_t)$ with $\Gamma_t(u_i) \in L^\infty(X,\mu_t)$. We apply Lemma 2.11 and obtain

$$\gamma_{2,t}(\Psi(\tilde{u})) \geq \frac{1}{2}(\partial_t \Gamma_t)(\Psi(\tilde{u})) \quad \text{m-a.e. in } X,$$  \hspace{1cm} (21)

where both sides of the inequality depend on $\lambda, a, b \in \mathbb{R}$. Choosing $\lambda, a, b$ in a dense and countable subset $D$ of $\mathbb{R}$ yields that (21) holds m-a.e. for all $\lambda, a, b$ in $D$. Since

$$(\partial_t \Gamma_t)(\Psi(\tilde{u})) = \sum_{i,j} \partial_i \Psi(\tilde{u}) \partial_j \Psi(\tilde{u}) (\partial_i \Gamma_t)(u_i, u_j),$$

and

$$\gamma_{2,t}(\Psi(\tilde{u})) = \sum_{i,j} \partial_i \Psi(\tilde{u}) \partial_j \Psi(\tilde{u}) \gamma_{2,t}(u_i, u_j) + 2 \sum_{i,j,k} \partial_i \Psi(\tilde{u}) \partial_j \Psi(\tilde{u}) H_t[u_i](u_j, u_k)$$

$$+ \sum_{i,j,k,h} \partial_i \Psi(\tilde{u}) \partial_j \Psi(\tilde{u}) \Gamma_t(u_k, u_h) \Gamma_t(u_i, u_j) \Gamma_t(u_k, u_h),$$

cf. [19] Lemma 3.3], both sides are continuous in $\lambda, a, b$, and hence we conclude that (21) holds for all $\lambda, a, b$ in $\mathbb{R}$.

Thus, for m-a.e. $x \in X$ we may set $a := u_2(x), b := u_3(x)$ so that

$$\partial_1 \Psi(\tilde{u})(x) = \lambda, \quad \partial_2 \Psi(\tilde{u})(x) = 0 = \partial_3 \Psi(\tilde{u})(x)$$

$$\partial_{23} \Psi(\tilde{u})(x) = 1 = \partial_{22} \Psi(\tilde{u})(x), \quad \partial_j \Psi(\tilde{u})(x) = 0 \text{ else},$$

18
proof.\footnote{\label{A1}Lemma~3.3.6} yields
\[
\lambda^2 \gamma_2(t(u_1) + 4\lambda H_t[u_1](u_2, u_3) + 2(\Gamma_t(u_2, u_3)^2 + \Gamma_t(u_2)\Gamma_t(u_3)) \geq \frac{1}{2}\lambda^2 (\partial_t \Gamma_t)(u_1).
\]
Using Cauchy-Schwartz inequality \(\Gamma_t(u_2, u_3)^2 \leq \Gamma_t(u_2)\Gamma_t(u_3)\) this can be transformed into
\[
\lambda^2 \left(\gamma_2(t(u_1) - \frac{1}{2}(\partial_t \Gamma_t)(u_1)) + 4\lambda H_t[u_1](u_2, u_3) + 4\Gamma_t(u_2)\Gamma_t(u_3) \geq 0,
\]
and since \(\lambda\) is arbitrary \footnote{\label{A2}Lemma~3.3.6} we obtain
\[
(H_t[u_1](u_2, u_3))^2 \leq \left(\gamma_2(t(u_1) - \frac{1}{2}(\partial_t \Gamma_t)(u_1)) \right) \Gamma_t(u_2)\Gamma_t(u_3).
\]
From the definition of the Hessian we deduce that
\[
H_t[u_1](u_2, u_3) + H_t[u_2](u_1, u_3) = \Gamma_t(\Gamma_t(u_1, u_2), u_3)
\]
and consequently
\[
|\Gamma_t(\Gamma_t(u_1, u_2), u_3)| \leq \sqrt{\Gamma_t(u_3)} \left(\sqrt{\gamma_2(t(u_1) - \frac{1}{2}(\partial_t \Gamma_t)(u_1)) \sqrt{\Gamma_t(u_2)}}
\right.
\]
\[
+ \sqrt{\gamma_2(t(u_2) - \frac{1}{2}(\partial_t \Gamma_t)(u_2)) \sqrt{\Gamma_t(u_1)}}
\). \tag{22}
\]
We obtain \(22\) for arbitrary \(u_3 \in \mathcal{F} \cap L^\infty(X, m_t)\) by approximating \(u_3\) by a sequence \(u_3^j\) converging in energy with
\[
\Gamma_t(u_3^j) \rightarrow \Gamma_t(u), \quad \Gamma_t(u_3^j, \Gamma_t(u_1, u_2)) \rightarrow \Gamma_t(u_3, \Gamma_t(u_1, u_2))
\]
pointwise and in \(L^1(X, m_t)\), cf. Theorem 3.4 in \footnote{\label{B1}Theorem 3.4.} Hence we may choose \(u_3 = \Gamma_t(u_1, u_2)\), and obtain the result choosing \(u_1 = u_2\).

Now we are ready to prove Theorem \footnote{\label{C2}Theorem~1.2.} We will assume that
\[
u_r \in \text{Lip}(X) \text{ for all } r \in (s, t) \text{ with } \operatorname{supmise}_{r,s} u_r(x) < \infty. \tag{23}
\]

\textbf{Proof of Theorem~1.2.} Define for each \(\varepsilon > 0\) the concave and smooth function \(\omega_\varepsilon(\cdot) := (\varepsilon + \cdot)^\alpha - \varepsilon^\alpha\). Note that this function satisfies
\[
2\omega_\varepsilon(r) + 4r\omega''_\varepsilon(r) \geq 0. \tag{24}
\]
For each \(s, t \in (0, T)\) under consideration as well as \(u \in \text{Lip}(X)\) and \(g \in \mathcal{F} \cap L^\infty\) with \(g \geq 0\), we set \(u_r = P_{\tau, s} u, g_r = P_{\tau, s} g\) for \(r \in [s, t]\). Note that for a.e. \(r \in [s, t]\) \(u_r \in \text{Dom}(\Delta_r)\) and \(u, \Gamma_r(u) \in L^\infty(X, m_r)\).

We consider the function
\[
h_r^\varepsilon := \int g_r \omega_\varepsilon(\Gamma_r(u_r)) dm_r.
\]
Choose \(s \leq \sigma < \tau \leq t\) and \(\delta > 0\) sufficiently small that \(\sigma \leq \tau - \delta\) such that
\[
h_r^\varepsilon \leq \liminf_{\delta \to 0} \frac{1}{\delta} \int_{\tau - \delta}^\tau h_r dr \quad \text{and} \quad h_\sigma^\varepsilon \geq \limsup_{\delta \to 0} \frac{1}{\delta} \int_\sigma^\sigma + \delta h_r dr. \tag{25}
\]

19
Note that by Lebesgue’s density theorem, this is true at least for a.e. $\sigma \geq s$ and for a.e. $\tau \leq t$. Then from
\[ \int_{\tau - \delta}^{\tau} h_r \, dr - \int_{\sigma}^{\sigma + \delta} h_r \, dr = \int_{\sigma}^{\tau - \delta} (h_{r+\delta} - h_r) \, dr, \]
and the concavity of $\omega_\delta$ we deduce
\[ h^\varepsilon_r - h^\varepsilon_\sigma \leq \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} [h_{r+\delta} - h_r] \, dr \]
\[ \leq \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \omega_\varepsilon(\Gamma_{r+\delta}(u_{r+\delta})) d(\mu_{r+\delta} - \mu_r) \, dr \]
\[ + \liminf_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X g_r \omega'_\varepsilon(\Gamma_r(u_r)) [\Gamma_{r+\delta}(u_{r+\delta}) - \Gamma_r(u_r)] \, dm_r \, dr \]
\[ + \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X g_r \omega'_\varepsilon(\Gamma_r(u_r)) \Gamma_{r+\delta}(u_{r+\delta}, u_{r+\delta} - u_r) \, dm_r \]
\[ + \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X g_r \omega'_\varepsilon(\Gamma_r(u_r)) \Gamma_{r+\delta}(u_{r+\delta}, u_r) \, dm_r \, dr \]
\[ =: (I) + (II) + (III) + (III'). \]

Let us denote with a slight abuse of notation $\hat{g}_r = g_r \omega'_\varepsilon(\Gamma_r(u_r))$. Note that
\[ \hat{g}_r \in L^1 \cap L^\infty(X) \text{ and } \hat{g}_r \in \mathcal{F}. \]
Each of the four terms will be considered separately. Since $r \mapsto \mu_r$ is a solution to the dual heat equation, we obtain
\[ (I) = \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \omega_\varepsilon(\Gamma_{r+\delta}(u_{r+\delta})) \cdot \left(- \int_r^{r+\delta} \Delta q g_q \, dq \right) \, dm_r \, dr \]
\[ = - \liminf_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \omega_\varepsilon(\Gamma_r(u_r)) \left(\frac{1}{\delta} \int_r^{r+\delta} \Delta q g_q e^{-f_q} \, dq \right) \, dm_r \, dr \]
\[ = - \int_{\sigma}^{\tau} \int_X \omega_\varepsilon(\Gamma_r(u_r)) \cdot \Delta_r g_r \, dm_r \, dr \]
due Lebesgue’s density theorem applied to $r \mapsto \Delta_r g_r e^{-f_r}$. Note that the latter function is in $L^2$ (Theorem 2.34) and the function $r \mapsto \omega_\varepsilon(\Gamma_r(u_r))$ is in $L^\infty$ thanks to (20).

The second term can estimated according to Proposition 2.6.
\[ (II) = \liminf_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \hat{g}_r \left[ \Gamma_{r+\delta}(u_r) - \Gamma_r(u_r) \right] \, dm_r \, dr \]
\[ = \int_{\sigma}^{\tau} \int_X \hat{g}_r \left( \partial_r \Gamma_r(u_r) \right) \, dm_r \, dr. \]

The term $(III')$ is transformed as follows
\[ (III') \]
\[ = \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \hat{g}_{r+\delta} \Gamma_{r+\delta}(u_{r+\delta}, u_{r+\delta} - u_r) \, dm_{r+\delta} \, dr \]
\[ = - \liminf_{\delta \searrow 0} \frac{1}{\delta} \int_{\sigma}^{\tau - \delta} \int_X \left( \Gamma_{r+\delta}(\hat{g}_{r+\delta}, u_{r+\delta}) + \hat{g}_{r+\delta} \Delta_r u_{r+\delta} \right) \left(\frac{1}{\delta} \int_r^{r+\delta} \Delta q g_q \, dq \right) \, dm_{r+\delta} \, dr \]
\[ = - \int_{\sigma}^{\tau} \int_X \left( \hat{g}_r(u_r) + \hat{g}_r \Delta_r u_r \right) \cdot \Delta_r u_r \, dm_r \, dr. \]
Here again we used Lebesgue’s density theorem (applied to $r \mapsto \varrho_r$) and the ‘nearly continuity’ of $r \mapsto \tilde{g}_r$ as map from $(s, t)$ into $L^2(X, m)$ and as map into $\mathcal{F}$ (Lusin’s theorem). Moreover, we used the boundedness (uniformly in $r$ and $x$) of $g_r$ and of $\Gamma_r(u_r)$ as well as the square integrability of $\Delta_r u_r$.

Similarly, the term $(III')$ will be transformed:

$$(III') = \limsup_{\delta \searrow 0} \frac{1}{\delta} \int_X \int_{\tau - \delta}^{\tau + \delta} \tilde{g}_r \Gamma_r(u_{r+\delta} - u_r) \, dm_r \, dr$$

$$= -\liminf_{\delta \searrow 0} \frac{1}{\delta} \int_X \int_{\tau - \delta}^{\tau + \delta} \left( \Gamma_r(\tilde{g}_r, u_r) + \tilde{g}_r \Delta_r u_r \right) \cdot \left( \int_r^{\tau + \delta} \Delta_g u_q \, dq \right) \, dm_r \, dr$$

$$= -\int_X \int_{\tau - \delta}^{\tau + \delta} \left( \Gamma_r(\tilde{g}_r, u_r) + \tilde{g}_r \Delta_r u_r \right) \cdot \left( \Delta_r u_r \right) \, dm_r \, dr.$$

We therefore obtain

$$h^\varepsilon - h^\varepsilon = (I) + (II) + (III') + (III'')$$

$$\leq \int_X \int_{\tau - \delta}^{\tau + \delta} \left[ -\omega_\varepsilon(\Gamma_r(u_r)) \cdot \Delta_r g_r + \tilde{g}_r \left( \partial_r \Gamma_r \right)(u_r) - 2 \left( \Gamma_r(\tilde{g}_r, u_r) + \tilde{g}_r \Delta_r u_r \right) \Delta_r u_r \right] \, dm_r \, dr$$

$$= \int_X \int_{\tau - \delta}^{\tau + \delta} \left[ \Gamma_r(\Gamma_r(u_r), \tilde{g}_r) - \Gamma_r(\Gamma_r(u_r)) \omega_\varepsilon''(\Gamma_r(u_r)) g_r + \tilde{g}_r \left( \partial_r \Gamma_r \right)(u_r) \right] \, dm_r \, dr$$

$$- 2 \left( \Gamma_r(\tilde{g}_r, u_r) + \tilde{g}_r \Delta_r u_r \right) \Delta_r u_r \, dm_r \, dr$$

$$= \int_X \int_{\tau - \delta}^{\tau + \delta} \left[ \Gamma_r(\Gamma_r(u_r)) \omega_\varepsilon''(\Gamma_r(u_r)) g_r + \tilde{g}_r \left( \partial_r \Gamma_r \right)(u_r) \right] \, dm_r \, dr.$$

Applying (20), Proposition 2.13 (24) and taking into account the concavity of $\omega_\varepsilon$ we further deduce for a.e. $r \in [s, t]$,

$$h^\varepsilon - h^\varepsilon \leq \int_X \int_{\tau - \delta}^{\tau + \delta} \left[ -2 \gamma_{2,r}(u_r) \tilde{g}_r + \tilde{g}_r \left( \partial_r \Gamma_r \right)(u_r) \right] \, dm_r \, dr$$

$$\leq \int_X \int_{\tau - \delta}^{\tau + \delta} \left[ - \gamma_{2,r}(u_r) - \frac{1}{2} \left( \partial_r \Gamma_r \right)(u_r) \right] \left( 2 \omega_\varepsilon''(\Gamma_r(u_r)) + 4 \omega_\varepsilon''(\Gamma_r(u_r)) \Gamma_r(u_r) \right) \, dm_r \, dr \leq 0.$$

Hence we showed that, given $u$ and $g$, there exists exceptional sets (which are null sets) for $\tau$ and $\sigma$ outside of these sets

$$\int_X \omega_\varepsilon(\Gamma_r(\tau_{\sigma} u)) \, g \, dm_{\tau} - \int_X \tau_{\sigma} \omega_\varepsilon(\Gamma_r(u)) \, g \, dm_{\tau} \leq 0 \quad (26)$$

holds. Choosing $g$’s from a dense countable set one may achieve that the exceptional sets for $\sigma$ and $\tau$ in (26) do not depend on $g$. Next we may assume that $\sigma, \tau \in [s, t]$ with $\sigma < \tau$ is chosen such that (26) simultaneously holds for all $u$ from a dense countable set $C_1$ in $\text{Lip}(X)$. We approximate arbitrary $u \in \text{Lip}(X)$ by $u_n \in C_1$ in energy and in $L^2$ such that $\sqrt{\Gamma_r(\tau_{\sigma} u_n)} \to G$ in $L^2$, for some $G \in L^2(X)$. This is possible since $\|\sqrt{\Gamma_r(\tau_{\sigma} u_n)}\|_{L^2(X)}$ is uniformly bounded. Then we have on the one hand

$$\limsup_{n \to \infty} \int_X \tau_{\sigma} \omega_\varepsilon(\Gamma_r(u_n)) \, g \, dm_{\tau} \leq \int_X \tau_{\sigma} \omega_\varepsilon(\Gamma_r(u)) \, g \, dm_{\tau} \quad (27)$$
since
\[ \int_X P_{\tau,\sigma} \omega_\varepsilon(\Gamma_\sigma(u_n)) \, g \, dm_\tau - \int_X P_{\tau,\sigma} \omega_\varepsilon(\Gamma_\sigma(u)) \, g \, dm_\tau \leq \int_X P^*_\tau,\sigma g \omega'_\varepsilon(\Gamma_\sigma(u)) \, (\Gamma_\sigma(u_n) - \Gamma_\sigma(u)) \, dm_\sigma \]
\[ \leq ||P^*_\tau,\sigma g \omega'_\varepsilon(\Gamma_\sigma(u))||_{L^\infty(X)} \left| \int_X \Gamma_\sigma(u_n) - \Gamma_\sigma(u) \, dm_\sigma \right|. \]

On the other hand we find
\[ \liminf_{n \to \infty} \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) \, g \, dm_\tau \geq \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u)) \, g \, dm_\tau. \] (28)

Indeed, since \( P_{\tau,\sigma} u_n \to P_{\tau,\sigma} u \) and \( \Gamma(P_{\tau,\sigma} u_n) \to G \) in \( L^2(X) \) we know \( \Gamma(P_{\tau,\sigma} u) \leq G^2 \) \( m \text{-a.e.} \) and hence
\[ \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) \, g \, dm_\tau - \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u)) \, g \, dm_\tau 
= \int_X \tilde{\omega}_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) \, g \, dm_\tau - \int_X \tilde{\omega}_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u)) \, g \, dm_\tau 
\geq \int_X \tilde{\omega}_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) (\Gamma_\tau(P_{\tau,\sigma} u_n) - \Gamma_\tau(P_{\tau,\sigma} u)) \, g \, dm_\tau 
\geq \int_X \tilde{\omega}_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) (\Gamma_\tau(P_{\tau,\sigma} u_n) - G) \, g \, dm_\tau, \]

where \( \tilde{\omega}(r) = \omega(r^2) \), which is convex and monotone. Combining (20), (27) and (28) yields
\[ \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u)) \, g \, dm_\tau - \int_X P_{\tau,\sigma} \omega_\varepsilon(\Gamma_\sigma(u)) \, g \, dm_\tau \]
\[ \leq \liminf_n \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) \, g \, dm_\tau - \limsup_n \int_X P_{\tau,\sigma} \omega_\varepsilon(\Gamma_\sigma(u_n)) \, g \, dm_\tau \]
\[ \leq \liminf_n \left( \int_X \omega_\varepsilon(\Gamma_\tau(P_{\tau,\sigma} u_n)) \, g \, dm_\tau - \int_X P_{\tau,\sigma} \omega_\varepsilon(\Gamma_\sigma(u_n)) \, g \, dm_\tau \right) \leq 0. \]

Letting \( \varepsilon \to 0 \) we showed that
\[ \int_X (\Gamma_\tau(P_{\tau,\sigma} u))^a \, g \, dm_\tau \leq \int_X P_{\tau,\sigma} (\Gamma_\sigma(u))^a \, g \, dm_\tau. \] (29)

Since \( \text{Lip}(X) \) is dense in \( F \) we can extend (29) to arbitrary \( u \in F \). Since \( g \) is arbitrary we obtain the result. \( \square \)

3 Application to super-Ricci flows and couplings of Brownian motions

In this section we apply the previous results to super-Ricci flows as defined in [24]. The defining property is the relative entropy \( S: I \times P(X) \to (-\infty, \infty] \) given by
\[ S_t(\mu) = \int \rho \log \rho \, dm_t \]
whenever \( \mu = \rho m_t \), and \( S_t(\mu) = \infty \) otherwise.
Definition 3.1 (Theorem 1.3 in [14]). We say that \((X, d_t, m_t)\) is a super-Ricci flow if one of the following equivalent assertions holds

i) For a.e. \(t \in (0, T)\) and every \(W_t\)-geodesic \((\mu^a)_{a \in [0, 1]}\) in \(\mathcal{P}(X)\) with \(\mu^0, \mu^1 \in \text{Dom}(S)\)

\[
\partial_+ S_t(\mu^a)|_{a=1} - \partial_- S_t(\mu^a)|_{a=0} \geq -\frac{1}{2} \partial_+ W_t^2(\mu^0, \mu^1) \tag{30}
\]

('dynamic convexity').

ii) For all \(0 \leq s < t \leq T\) and \(\mu, \nu \in \mathcal{P}(X)\)

\[
W_s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_t(\mu, \nu) \tag{31}
\]

('transport estimate').

iii) For all \(u \in \text{Dom}(E)\) and all \(0 < s < t < T\)

\[
|\nabla_t(P_{t,s}u)|^2 \leq P_{t,s}(|\nabla_s u|^2) \tag{32}
\]

('gradient estimate').

iv) For all \(0 < s < t < T\) and for all \(u_s, g_t \in F\) with \(g_t \geq 0, g_t \in L^\infty, u_s \in \text{Lip}(X)\) and for a.e. \(r \in (s, t)\)

\[
\Gamma_{2,r}(u_r)(g_r) \geq \frac{1}{2} \int \Gamma_r(u_r)g_r dm_r \tag{33}
\]

('dynamic Bochner inequality' or 'dynamic Bakry-Emery condition') where \(u_r = P_{r,s}u_s\) and \(g_r = P^\ast_{t,r}g_t\). Moreover, the regularity assumption \(\text{(23)}\) is satisfied.

The following corollary is a consequence of Theorem [14] and Theorem 1.2. In particular, choosing \(\mu = \delta_x\) and \(\nu = \delta_y\) for some arbitrary \(x, y \in X\), Corollary 3.2 implies for \(p = \infty\)

\[
W_{\infty,s}(\hat{P}_{t,s}\delta_x, \hat{P}_{t,s}\delta_y) \leq d_t(x, y). \tag{34}
\]

Corollary 3.2. Suppose that \((X, d_t, m_t)_{t \in I}\) is a super-Ricci flow satisfying the assumptions in Theorem 1.4. Then

i) for every \(u \in F \cap L^\infty(X, m_s)\) and every \(\beta \in [1, 2]\)

\[
|\nabla_t(P_{t,s}u)|^2 \leq P_{t,s}(|\nabla_s u|^2). \tag{35}
\]

ii) for every \(\mu, \nu \in \mathcal{P}(X)\) and every \(p \in [1, \infty]\)

\[
W_p,s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \leq W_{p,t}(\mu, \nu). \tag{36}
\]

Proof. Note that, taking into account \(\Gamma(u) = |\nabla u|^2\) due to our static Riemannian curvature bound, \(\text{(35)}\) holds at least for a.e. \(s \leq t\) by Definition 3.1. Theorem 1.4 and Theorem 1.2. Then applying Kuwada’s duality [15] Theorem 2.2 implies that \(\text{(36)}\) holds at all these time instances. Indeed, \(\text{(35)}\) implies that
for all $u \in \text{Lip}_b(X)$, $|\nabla_t P_{s,t} u| \leq P_{s,t}(|\nabla_s u|^2)^{1/\beta}$ and thus by Proposition 3.11 in [3] $\text{lip}_t P_{s,t} u \leq P_{s,t}(|\nabla_s u|^2)^{1/\beta}$. We obtain
\[
\text{lip}_t P_{s,t} u \leq P_{t,s} (\text{lip}_s u)^{2/\beta}
\]
by virtue of $|\nabla u| \leq \text{lip} u$ (Lemma 4.4 in [1]) and the monotonicity of the functions $P_{s,t}$, $r^\beta$ and $r^{1/\beta}$. We deduce from Theorem 2.2 in [15] for a.e. $s \leq t$
\[
W_{p,s}(\hat{P}_{s,t} \mu, \hat{P}_{t,s} \nu) \leq W_{p,t}(\mu, \nu),
\]
where $p$ is the Hölder conjugate of $\beta$; $1/p + 1/\beta = 1$. Since both sides of (36) are continuous in $s$ and $t$ (see Lemma 3.3 below), we obtain that (36) holds for all times $s \leq t$ and thus also (35) holds for all times by Theorem 2.2 in [15]. The same applies to $p = 1$ in (35) by noting that $\text{lip}_t (P_{s,t} u) \leq P_{s,t} (\text{lip}_s u)^{2/\beta}$ for all $\beta \geq 2$ by virtue of Jensen’s inequality.

We obtain the following continuity estimate for the heat flow $\mu_s = \hat{P}_{t,s} \mu$, where $\mu \in \mathcal{P}(X)$.

**Lemma 3.3.** Let $\mu_s = \hat{P}_{t,s} \mu$. Then there exist constants $c, c' > 0$ depending only on $K, N$ and $L$ such that
\[
W_{p,t}((\mu_s, \mu_{s'}))^p \leq c|s - s'|^{p/2} e^{c|s - s'|/2}.
\]

**Proof.** Assume $0 < s < s' < t$. Then by $\mu_s = \hat{P}_{s',s} \mu_{s'}$ we estimate
\[
W_{p,t}((\mu_s, \mu_{s'}))^p \leq \int \int d^p_t(x,y)p_{s',s}(x,y) dm_s(y) d\mu_{s'}(x).
\]
By virtue of the Gaussian upper bounds ([17] Section 4) and the Bishop Gromov volume comparison in $\text{RCD}(K,N)$ spaces ([20] Theorem 2.3) we obtain for $\sigma = s' - s$ and $B_t(r,x)$ denoting the ball of radius $r$ around $x$ in the metric space $(X,d_X)$
\[
p_{s',s}(x,y) \leq \frac{C}{m_t(B_t(\sqrt{\sigma},x))} \cdot \exp\left(-\frac{d^2_t(x,y)}{C\sigma}\right)
\]
\[
A(R,x) \leq \left(\frac{R}{r}\right)^N \cdot e^{R\sqrt{K(N-1)}} \cdot A(r,x)
\]
for $R \geq r$ where $A(r,x) = \partial_t m_t(B_t(r,x))$ and thus (by integrating from 0 to $\sqrt{\sigma}$)
\[
A(R,x) \leq N \frac{R^{N-1}}{\sigma^{N/2}} \cdot e^{R\sqrt{K(N-1)}} \cdot m_t(B_t(\sqrt{\sigma},x))
\]
for $R \geq \sqrt{\sigma}$. Then estimating further yields (with varying constants)
\[
\int \int d^p_t(x,y)p_{s',s}(x,y) dm_s(y) d\mu_{s'}(x)
\]
\[
\leq \int_X \left[\frac{C}{m_t(B_t(\sqrt{\sigma},x))}\right] \cdot \int_X d^p_t(x,y) \cdot \exp\left(-\frac{d^2_t(x,y)}{C\sigma}\right) dm_t(y) \cdot d\mu_{s'}(x)
\]
\[
\leq C\sigma^{p/2} + C \cdot \int_X \int_0^\infty R^p \cdot \exp\left(-\frac{R^2}{C\sigma}\right) N \frac{R^{N-1}}{\sigma^{N/2}} \cdot e^{R\sqrt{K(N-1)}} dR \cdot d\mu_{s'}(x)
\]
\[
\leq C\sigma^{p/2} + c'' \sigma^{p/2} e^{c'\sigma/2} \leq c\sigma^{p/2} e^{c'\sigma/2}.
\]
(37)
Brownian motions

In the remainder of this section we follow the approach in [21] and construct couplings of two Brownian motions \((X^1_s)_{s \leq t}, (X^2_s)_{s \leq t}\) on X such that the distance \(d_s\) between \(X^1_s\) and \(X^2_s\) does not increase.

**Definition 3.4.** Let \(\mu \in \mathcal{P}(X)\) and \(t \in I\). We call a stochastic process \((X_s)_{s \leq t}\) on a probability space \((\Omega, \Sigma, \mathbb{P})\) with values in X a Brownian motion on X with initial distribution \(\mu\) if the process is sample-continuous and if for all \(s \leq t\)

\[
\mathbb{P}[X_s \in A] = \hat{P}_{t,s}(\mu)(A) = \int_X \int_A p_{t,s}(x, y) \, dm_s(y) \, d\mu(x).
\]

**Remark.** Let us remark that the Brownian motion defined here has time-dependent generator \(\Delta_s\) instead of \(\frac{1}{2} \Delta_s\). This is only for convenience and the stochastic process with generators \(\frac{1}{2} \Delta_s\) is given by \((\hat{X}_s)_{s \leq t/2}\), where \(\hat{X}_s := X_{2s}\).

In order to prove existence of a Brownian motion we consider for fixed \(t \in I\) the finite subset \(J = \{t_1, \ldots, t_r\}\) of \((0, t]\) and the finite dimensional distribution \(P_{J}^{\mu}\), where \(\mu \in \mathcal{P}(X)\), defined by

\[
P_{J}^{\mu}(B_r \times \ldots \times B_1) := \int_X \int_{B_r} \cdots \int_{B_1} p_{t_{2r}, t_1}(x_{2r}, x_{t_1}) \, dm_{t_1}(x_{t_1}) \cdots p_{t_r, t_1}(x, x_{t_1}) \, dm_{t_1}(x_{t_1}) \, d\mu(x).
\]

The family of probability measures \(\{P_{J}^{\mu}\mid J \text{ finite } \subset (0, t]\}\) defines a projective family, hence the Kolmogorov extension theorem [7, Theorem 35.5] implies that there exists a unique probability measure \(P_{[0,t]}^{\mu}\) on \((X^{[0,t]}, \mathcal{B}(X)^{[0,t]})\) such that \((\pi_J)_\ast P_{[0,t]}^{\mu} = P_{J}^{\mu}\). Here, \(\pi_J\) denotes the projection \(\omega \mapsto (\omega(t_1), \ldots, \omega(t_r))\) from \(X^{[0,t]}\) to \(X^r\).

For every \(s \in (0, t]\) the map \(\pi_s: \omega \mapsto \omega(s)\) from \(X^{(0,t]}\) to X is a stochastic process with finite-dimensional distributions \((P_{J}^{\mu})_{J}\) from Proposition yields existence of a continuous modification \((X_s)_{s \leq t}\), and hence a Brownian motion.

**Proposition 3.5.** For each \(t \in I\) and each \(\mu \in \mathcal{P}(X)\) there exists a Brownian motion on X with initial distribution \(\mu\), which is unique in law.

**Proof.** We need to show that there exists positive constants \(\alpha, \beta, c > 0\) such that the above mentioned process \(\pi_s\) satisfies

\[
E[d(\pi_{s'}, \pi_s)^\alpha] \leq c|s - s'|^{1+\beta}
\]  (38)

for all \(s', s \in (0, t]\). Then the Kolmogorov continuity theorem [7, Theorem 39.3] implies that there exists a modification \((X_s)_{s \leq t}\) such that the map \(s \mapsto X_s(\omega)\) is continuous for \(P_{[0,t]}^{\mu}\)-a.e. \(\omega\). Hence the process \((X_s)_{s \leq t}\) on the probability space \((X^{(0,t]}, \mathcal{B}(X)^{(0,t]}, P_{(0,t]}^{\mu})\) yields the desired properties. For \(\alpha > 2\) (38) follows from (37) in the proof of Lemma 3.3.

Since all finite-dimensional distributions are uniquely determined this process is unique in law.
Couplings of Brownian motions

We introduce the σ-field $B^\nu(X^2) := \bigcap_{\nu \in \mathcal{P}(X^2)} B^\nu(X^2)$ of universally measurable subsets of $X^2$, i.e. the intersection of all $B^\nu(X^2)$, where $\nu$ runs through the set $\mathcal{P}(X^2)$ and where $B^\nu(X^2)$ denotes the completion of the Borel σ-field on $X^2$ w.r.t. $\nu \in \mathcal{P}(X^2)$. Let $D := \{k2^{-n}|k, n \in \mathbb{N}\} \cap (0, t]$ denote the set of nonnegative dyadic numbers $s$ in $(0, t]$ and $D_n := \{k2^{-n}|k \in \mathbb{N}\} \cap (0, t]$ for fixed $n \in \mathbb{N}$.

In the remainder we will assume that the transport estimate \[(\ref{eq:transport_estimate})\] holds for all $p \in [1, \infty]$.

**Lemma 3.6.** For each $s \leq t$ there exists a Markov kernel $q^s_{t,s}$ on $(X^2, B^\nu(X^2))$ with the following properties:

1. For each $(x,y) \in X^2$ the probability measure $q^s_{t,s}((x,y), \cdot)$ is a coupling of the probability measures $p_{t,s}(x, \cdot)$ and $p_{t,s}(y, \cdot)$.
2. For each $(x,y) \in X^2$ and $q^s_{t,s}((x,y), \cdot)$-a.e. $(x', y') \in X^2$

$$d_s(x', y') \leq d_s(x, y).$$

**Proof.** By virtue of the transport estimate \[(\ref{eq:transport_estimate})\] there exists at least one probability measures with properties i) and ii). Indeed, define $\mu_s = P_{t,s} \delta_x$, $\nu_s = P_{t,s} \delta_y$ and let $\gamma_p \in \Pi(\mu_s, \nu_s)$ such that $W_{P,s}(\mu_s, \nu_s) = \|d_s\|_{L^p(\gamma_p)}$. Since $\gamma_p \in \Pi(\mu_s, \nu_s)$, $(\gamma_p)_{p \in \mathbb{N}}$ is tight \[(\ref{lem:finite_dim})\] and hence there exists a subsequence $p_k$ and a probability measure $\gamma$ such that $\gamma_{p_k}$ weakly converges to $\gamma$. Since $\Pi(\mu_s, \nu_s)$ is closed we obtain that $\gamma \in \Pi(\mu_s, \nu_s)$. Moreover, since $d_s \land R \in C_b(X \times X)$

$$\|d_s \land R\|_{L^p(\gamma)} = \lim_{k \to \infty} \|d_s \land R\|_{L^p(\gamma_{p_k})} \leq \lim_{k \to \infty} \|d_s\|_{L^p(\gamma_{p_k})} \leq d_s(x, y),$$

where the second inequality follows from the Hölder inequality and the last from Corollary 2.15. Letting $R \to \infty$ and $p \to \infty$, we obtain

$$\|d_s\|_{L^\infty(\gamma)} \leq d_s(x, y).$$

Hence the set of all these couplings $\gamma$ is non-empty and satisfies i) and ii).

Moreover, for given $x, y \in X$ this set is closed w.r.t. weak convergence in $\mathcal{P}(X^2)$. According to a measurable selection theorem \cite[Theorem 6.9.2]{Mayer} we may choose a coupling $q^s_{t,s}((x,y), \cdot)$ such that the map

$$(x,y) \mapsto q^s_{t,s}((x,y), \cdot), \quad (X^2, B^\nu(X^2)) \to (\mathcal{P}(X^2), B(\mathcal{P}(X^2)))$$

is measurable. \hfill \Box

**Lemma 3.7.** For each $n \in \mathbb{N}$ and $s, s' \in D_n$ there exists a Markov kernel $q^{(n)}_{s,s'}$ on $(X^2, B^\nu(X^2))$ with the following properties:

1. For each $(x,y) \in X^2$ the probability measure $q^{(n)}_{s,s'}((x,y), \cdot)$ is a coupling of $p_{s,s'}(x, \cdot)$ and $p_{s,s'}(y, \cdot)$.
2. For each $(x,y) \in X^2$

$$d_s(x', y') \leq d_s(x, y)$$

for $q^{(n)}_{s,s'}((x,y), \cdot)$-a.e. $(x', y')$. 

26
Proof. For $s = l2^{-n}$ and $s' = k2^{-n}$ with $l \geq k$ we put
\[ q_{s,s'}^{(n)} := q_{(k+1)2^{-n},s'} \circ \cdots \circ q_{s,(l-1)2^{-n}}. \]

Obviously we have for $r \in D_n$ such that $s' \leq r \leq s$,
\[ q_{r,s}^{(n)} \circ q_{s,r}^{(n)} = q_{s,s}^{(n)}, \tag{39} \]
and the properties $ij$ and $ii$ hold by iteration, cf. Lemma 2.3 in [21].

We fix a distribution $\nu \in \mathcal{P}(X^2)$ with marginals $\nu_1$ and $\nu_2$. Similarly as before for any finite subset $J = \{t_1, \ldots, t_r\}$ of $D_n$ we consider the finite-dimensional distribution $Q_J^{(n)}$ on $(X^2)^{|J|}$
\[
Q_J^{(n)}(A_r \times \ldots \times A_1) = \int_{X^2} \int_{A_r} \ldots \int_{A_1} q_{t_r,t_1}^{(n)}((x_2,y_2),d(x_1,y_1)) \cdots q_{t_1,t_1}^{(n)}((x,y),d(x,y)) \nu(d(x,y)),
\]
where $q_{t_r,t_1}^{(n)} = q_{t_2,t_1}^{(n)} \circ q_{t_1,t_1}^{*}$ whenever $l2^{-n} < t < (l+1)2^{-n}$.

Lemma 3.8. For fixed finite $J \subset D_n$ the family $\{Q_J^{(n)}| n \in \mathbb{R}, n \geq m \}$ is a tight family of probability measures on $(X^2)^{|J|}$.

Proof. Let $J = \{t_1, \ldots, t_r\}$ with each $t_i \in D_m$. The families $\{\hat{P}_{t_i,t_1}(\nu_1)| i = 1, \ldots, r\}$ and $\{\hat{P}_{t_i,t_1}(\nu_2)| i = 1, \ldots, r\}$ are tight by virtue of Prokhorov’s theorem, see e.g. [8]. This means that given $\varepsilon > 0$ there exist compact sets $B_1, B_2 \subset X$ such that for all $i = 1, \ldots, r$
\[
\hat{P}_{t_i,t_1}(\nu_1)(X \setminus B_1) < \varepsilon, \quad \hat{P}_{t_i,t_1}(\nu_2)(X \setminus B_2) < \varepsilon.
\]

Applying $A_1 \times A_2 \subset X \times A_2 \cup A_1 \times X$ and (39) yields for the compact set $\tilde{B} = (B_1 \times B_2)^r$ and $n \in \mathbb{N}$
\[
Q_J^{(n)}((X^2)^r \setminus \tilde{B}) \leq \sum_{i=1}^r Q_{t_i,t_1}^{(n)}((X^2 \setminus B_1) \times B_2)
\leq \sum_{i=1}^r \bigg[ Q_{t_i,t_1}^{(n)}((X \setminus B_1) \times X) + Q_{t_i,t_1}^{(n)}((X \times (X \setminus B_2)) \bigg]
= \sum_{i=1}^r \bigg[ \hat{P}_{t_i,t_1}(\nu_1)(X \setminus B_1) + \hat{P}_{t_i,t_1}(\nu_2)(X \setminus B_2) \bigg]
\leq 2r \varepsilon,
\]
where the last two inequalities follow from $i)$ of Lemma 3.7 and the tightness of $\{\hat{P}_{t_i,t_1}(\nu_j)\}$, respectively. Hence the family $\{Q_J^{(n)}| n \in \mathbb{R}, n \geq m \}$ is tight.

For $J = \{t_1, \ldots, t_r\}$ as above we set
\[ e_1: (X^2)^r \to X^r, \quad ((x_1,y_1), \ldots, (x_r,y_r)) \mapsto (x_1, \ldots, x_r), \]
and similarly for $e_2$.
Proposition 3.9. There exists a projective family \( \{ Q_J^n \}_{J \text{ finite } \subset D} \) of probability measures and a subsequence \( (n_l)_{l \in \mathbb{N}} \) such that for each finite \( J \subset D \)

i) \( Q_J^{(n_l)} \to Q_J^\nu \) weakly in \( \mathcal{P}(\mathbb{R}^2_{\text{c}}) \) as \( l \to \infty \),

ii) and \( (e_1)_{\#}Q_J^{(n_l)} = P_{D}^\nu \), \( (e_2)_{\#}Q_J^{(n_l)} = P_{J}^\nu \).

In particular there exists a probability measure \( Q_D^\nu \in \mathcal{P}(\mathbb{R}^2) \) such that for all finite \( J \subset D \)

\[
(\pi_J)_{\#}Q_D^\nu = Q_J^\nu
\]

and

\[
(e_1)_{\#}Q_D^\nu = P_D^\nu, \quad (e_2)_{\#}Q_D^\nu = P_J^\nu.
\]

Proof. Lemma 3.8 yields for each fixed \( J \) the existence of a weakly converging subsequence \( Q_J^{(n_l)} \) by virtue of Prokhorov’s theorem. By a diagonal argument we may choose a subsequence such that \( Q_J^{(n_l)} \) weakly converges for all finite \( J \subset D \). Note that

\[
(\pi_J)_{\#}Q_J^{(n_l)} = P_{J}^\nu, \quad (\pi_J)_{\#}Q_J^{(n_l)} = P_{J}^\nu
\]

and hence the same holds true for the limit. We obtain the last assertion by applying Kolmogorov’s extension theorem.

The next theorem is in particular true for super-Ricci flows satisfying additionally (9) and (10). This is summarized in Theorem 1.3 which we prove at the end of this section.

Theorem 3.10. Let \( (X, d_t, m_t)_{t \in \mathbb{T}} \) be a family of \( \operatorname{RCD}(K,N) \) spaces such that (2) and (3) hold. Moreover we assume that the transport estimate (28) holds for every \( p \in [1, \infty] \). Then, for each \( x, y \in X \) there exists a continuous stochastic process \( (X_s)_{s \leq t} \) such that \( (X_s)_{s \leq t} \) is a coupling of the Brownian motions \( (X_1^1)_{s \leq t} \) and \( (X_1^2)_{s \leq t} \) with values in \( X \) and initial distributions \( \delta_x \) and \( \delta_y \) respectively and it satisfies for \( Q_D^{(\delta_x, \delta_y)\cdot \text{a.e. path}} \)

\[
d_s(X_s^1, X_s^2) \leq d_t(x, y),
\]

for each \( s \leq t \).

Proof. Set \( \nu = (\nu_1, \nu_2) = (\delta_x, \delta_y) \). Consider the coordinate process \( \pi_s = (\pi_s^1, \pi_s^2): (X^2)^{D} \to X^2 \). Under \( Q_D^\nu \) the process \( (\pi_s^1)_{s \in D} \) has distribution \( P_D^\nu \) and satisfies the continuity property (35). The corresponding statement holds true for the process \( (\pi_s^2)_{s \in D} \). Hence, the process \( \pi_t = (\pi_1^1, \pi_2^2) \) satisfies the Kolmogorov continuity theorem for \( \alpha > 2 \) since

\[
E[d_t(\pi_s, \pi_{s'})^\alpha] \leq 2^\alpha/2 \left( E[d_t(\pi_s^1, \pi_{s'}^1)^\alpha] + E[d_t(\pi_s^2, \pi_{s'}^2)^\alpha] \right)
\]

\[
\leq c 2^{\alpha/2} |s - s'|^{\alpha/2},
\]

with product metric \( \tilde{d}^2((x^1, y^1), (x^2, y^2)) = d^2(x^1, x^2) + d^2(y^1, y^2) \). Consequently there exists a continuous modification \( (X_s)_{s \leq t} = (X_s^1, X_s^2)_{s \leq t} \) defined by \( X_s = \ldots \)
Since the process \( (X_s^i)_{s \leq t}, i = 1, 2 \) is a Brownian motion by continuity of \( s \mapsto p_{t,s}(x,dy) \).

We need to justify that for \( Q_D^n \)-a.e. path

\[
d_s(X_s^1, X_s^2) \leq d_t(x, y).
\]

For each \( n \in \mathbb{N} \) let \( Q_D^n \) be the projective limit of the family \( (Q_j^n)_{j \in D_n} \), which exists thanks to the Kolmogorov extension theorem. Consider the coordinate process \( (\pi_s^n)_{s \in D_n} = (\pi_s^{1(n)}, \pi_s^{2(n)})_{s \in D_n} \) from \( (X_s^2)_D \to X^2 \). Then \( Q_D^n \)-a.e. we have \( d_s(\pi_s^{1(n)}, \pi_s^{2(n)}) \leq d(x, y) \) by virtue of Lemma 3.7. Applying Proposition 5.3 and ii) of Lemma 6.7 we obtain for a subsequence

\[
E[(d_s(\pi_s^{1}, \pi_s^{2}) \wedge R)^{1/p}] = \lim_{l \to \infty} E[(d_s(\pi_s^{1}, \pi_s^{2}) \wedge R)^{1/p}] \leq \lim_{l \to \infty} E[(d_t(x, y) \wedge R)^{1/p}] = d_t(x, y) \wedge R,
\]

for each \( s \in D \). Letting \( R \) and \( p \) tend to \( \infty \) we find for each \( s \in D \)

\[
d_s(\pi_s^{1}, \pi_s^{2}) \leq d_t(x, y).
\]

Since the process \( (X_s)_{s \in D} \) is a modification we get for each \( s \in D \) and \( Q_D^n \)-a.e.

\[
d_s(X_s^1, X_s^2) \leq d_t(x, y). \quad \text{Since } D \subset (0, t) \text{ is a dense and countable subset we}
\]

obtain the result by continuity of \( s \mapsto X_s(\omega) \).

\[\square\]

**Proof of Theorem 1.3.** By virtue of Corollary 3.2, Theorem 3.10 and Theorem 1.7 in [14] the only implication left to show is iii) implies one of the other assertions. We show that iii) implies ii). This can be seen by choosing an \( W_{p,t} \)-optimal transport plan \( \gamma \) between \( \mu, \nu \). Let \( q_{t,s} \) be the transition semigroup of the coupled process. Then \( q_{t,s}(z_1, dx, z_2, dy)\gamma(dz_1, dz_2) \) is a coupling of \( \tilde{P}_{t,s}\mu \) and \( \tilde{P}_{t,s}\nu \).

Hence

\[
W_{p,s}(\tilde{P}_{t,s}\mu, \tilde{P}_{t,s}\nu)^p \leq \int \int d_s^p(x, y)q_{t,s}(z_1, dx, z_2, dy)\gamma(dz_1, dz_2)
\]

\[
= \int E[d_s^p(X_s^1, X_s^2)|X_1^1 = z_1, X_1^2 = z_2] \gamma(dz_1, dz_2) \leq \int d_t^p(z_1, z_2) \gamma(dz_1, dz_2)
\]

\[
= W_{p,t}(\mu, \nu)^p,
\]

where we used iii) in the last inequality.

\[\square\]
References

[1] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, 195(2):289–391, 2013.

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.*, 163(7):1405–1490, 2014.

[3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. *Ann. Probab.*, 43(1):339–404, 2015.

[4] Marc Arnaudon, Koléhè Abdoulaye Coulibaly, and Anton Thalmaier. Brownian motion with respect to a metric depending on time; definition, existence and applications to Ricci flow. *Comptes Rendus Mathematique*, 346(13):773–778, 2008.

[5] D. Bakry. Transformations de Riesz pour les semi-groupes symétriques. II. étude sous la condition $\Gamma_2 \geq 0$. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 145–174. Springer, Berlin, 1985.

[6] Heinz Bauer. *Probability theory and elements of measure theory*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1981. Second edition of the translation by R. B. Burckel from the third German edition, Probability and Mathematical Statistics.

[7] Heinz Bauer. *Wahrscheinlichkeitstheorie*. de Gruyter Lehrbuch. [de Gruyter Textbook]. Walter de Gruyter & Co., Berlin, fifth edition, 2002.

[8] V. Bogachev. *Measure Theory*, volume 1. Springer-Verlag, Berlin, Heidelberg, 2007.

[9] François Bolley, Ivan Gentil, Arnaud Guillin, and Kazumasa Kuwada. Equivalence between dimensional contractions in Wasserstein distance and the curvature-dimension condition. *arXiv:1510.07793*, 2015.

[10] Zhen-Qing Chen and Masatoshi Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2012.

[11] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces. *Invent. Math.*, 201:993–1071, 2015.

[12] Nicola Gigli. Nonsmooth differential geometry-An approach tailored for spaces with Ricci curvature bounded from below. *arXiv:1407.0809*, 2014.

[13] Robert Haslhofer and Aaron Naber. Weak solutions for the Ricci flow I. *arXiv:1504.00911*, 2015.

[14] Eva Kopfer and Karl-Theodor Sturm. Heat flows on Time-dependent Metric Measure Spaces and Super-Ricci Flows. *arXiv:1611.02570*, 2017.
[15] Kazumasa Kuwada. Duality on gradient estimates and Wasserstein controls. *J. Funct. Anal.*, 258(11):3758–3774, 2010.

[16] Kazumasa Kuwada and Robert Philippowski. Coupling of Brownian motions and Perelman’s L-functional. *Journal of Functional Analysis*, 260(9):2742–2766, 2011.

[17] Janna Lierl and Laurent Saloff-Coste. Parabolic Harnack inequality for time-dependent non-symmetric Dirichlet forms. *arXiv:1205.6493*, 2012.

[18] Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Springer-Verlag, Berlin, 1992.

[19] Giuseppe Savaré. Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in RCD(K,∞) metric measure spaces. *Discr. Cont. Dyn. Syst. A*, 34(4):1641–1661, 2014.

[20] Karl-Theodor Sturm. On the geometry of metric measure spaces. I and II. *Acta Math.*, 169(1):65–131, 2006.

[21] Karl-Theodor Sturm. Metric measure spaces with variable Ricci bounds and couplings of Brownian motions. In *Festschrift Masatoshi Fukushima*, volume 17 of *Interdiscip. Math. Sci.*, pages 553–575. World Sci. Publ., Hackensack, NJ, 2015.

[22] Karl-Theodor Sturm. Super Ricci flows for metric measure spaces. I. *arXiv:1603.02193*, 2016.

[23] Peter Topping. Ricci flow: The foundations via optimal transportation. 2013.

[24] C. Villani. *Optimal transport, old and new*. Springer-Verlag, Berlin, Heidelberg, 2009.