Anomalies in quantum field theory
Properties and characterization

E. Kraus

Physikalisches Institut, Universität Bonn,
Nußallee 12, D–53115 Bonn, Germany
E-mail: kraus@th.physik.uni-bonn.de

Abstract

We consider the Adler-Bardeen anomaly of the $U(1)$ axial current in abelian and non-abelian gauge theories and present its algebraic characterization as well as an explicit evaluation proving regularization scheme independence of the anomaly. By extending the gauge coupling to an external space-time dependent field we get a unique definition for the quantum corrections of the topological term. It also implies a simple proof of the non-renormalization theorem of the Adler-Bardeen anomaly. We consider local gauge couplings in supersymmetric theories and find that there the renormalization of the gauge coupling is determined by the topological term in all loop orders except for one loop. It is shown that in one-loop order the quantum corrections to the topological term induce an anomalous breaking of supersymmetry, which is characterized by similar properties as the Adler-Bardeen anomaly.

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1 Introduction

Anomalies appear as a breaking of classical symmetries in quantum field theory. They arise from divergent loop diagrams and they cannot be removed by adjusting counterterms to the classical action. As such they are seen to be true quantum effects appearing in the procedure of renormalization and quantization of the classical field theory.

In the first part of the paper we consider the Adler-Bardeen anomaly \[1, 2, 3\] of softly broken $U(1)$ axial symmetry in abelian and non-abelian gauge theories. The Adler–Bardeen anomaly is determined by the topological term $\text{Tr} G \tilde{G}$ and algebraically characterized as a non-variation under gauge and BRS symmetry, respectively. We prove explicitly that the coefficient of the anomaly is determined by convergent one-loop integrals, which are not subject of renormalization. As such its coefficient is independent from the regularization scheme used for the subtraction of UV divergences. Another characteristic property of the Adler-Bardeen anomaly is its non-renormalization theorem \[4\], which states, that the coefficient of the anomaly is not renormalized in higher orders. We give a new and simple proof of the non-renormalization theorem by extending the gauge coupling to an external field. Then the renormalization of the topological term is unambiguously determined by gauge invariance and implies the non-renormalization theorem.

In the second part we turn to supersymmetric gauge theories. It was recently shown that with local coupling supersymmetric Yang-Mills theories have an anomalous breaking of supersymmetry in one-loop order \[5\]. Contrary to the Adler-Bardeen anomaly the supersymmetry anomaly is a variation under BRS-symmetry, but the respective counterterm depends on the logarithm of the coupling. Thus its coefficient is scheme independent and is not introduced in the procedure of renormalization.

Evaluating the symmetry identities one can show, that the anomaly of supersymmetry is induced by the renormalization of the topological term \[6\]. In addition we demonstrate that in all orders except for one loop the renormalization of the topological term determines the renormalization of the gauge coupling constant. As an application we derive the closed expression of the gauge $\beta$ function in terms of its one-loop coefficient and the anomaly coefficient.

2 The Adler-Bardeen anomaly

We start the considerations with the axial anomaly appearing in QED in the axial current Ward identity \[7\]. The classical action of QED with one Dirac spinor is
given by:

$$\Gamma_{\text{cl}} = -\frac{1}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu} - \int d^4x \, \frac{1}{2\xi} (\partial^\mu A_\mu)^2 
+ \int d^4x \, \{ (i\psi_L \sigma^\mu D_\mu \psi_L + i\psi_R \sigma^\mu D_\mu \psi_R) - m (\psi_L \psi_R + \psi_L \psi_R) \}$$  \hspace{1cm} (1)

with

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu ,$$

$$D_\mu \psi_P = (\partial_\mu + ieQ_PA_\mu)\psi_P, \quad P = L, R, \quad \text{and} \quad Q_L = -1, Q_R = +1 ,$$

$$D_\mu \psi_P \equiv (D_\mu \psi_P)\dagger$$  \hspace{1cm} (2)

Here, $\psi_P^\alpha$ and $\psi_R^\alpha$ are two-component Weyl spinors and $\bar{\psi}_L^\dot{\alpha}$ and $\bar{\psi}_R^\dot{\alpha}$ are their complex conjugates

$$(\psi_L^\alpha)\dagger = \bar{\psi}_L^\dot{\alpha}.$$  \hspace{1cm} (3)

The left- and right-handed Weyl spinors compose the 4-component Dirac spinor according to

$$\Psi = \begin{pmatrix} \psi_L^\alpha \\ \bar{\psi}_R^\dot{\alpha} \end{pmatrix} .$$  \hspace{1cm} (4)

Up to the gauge fixing term $(\partial^\mu A_\mu)^2$ the classical action (1) is invariant under $U(1)$ gauge transformations:

$$\delta_{\text{gauge}} A_\mu = \frac{1}{e} \partial_\mu \omega ,$$

$$\delta_{\text{gauge}} \psi_P = -i\omega Q_P \psi_P .$$  \hspace{1cm} (5)

$U(1)$ gauge invariance is expressed by the gauge Ward identity

$$\left( w^{\text{gauge}} - \partial^\nu \frac{1}{e} \frac{\delta}{\delta A^\nu} \right) \Gamma_{\text{cl}} = -\frac{1}{\xi e} \Box A ,$$  \hspace{1cm} (6)

with the Ward operator

$$w^{\text{gauge}} = \sum_{P=L,R} \left( \delta_{\text{gauge}} \psi_P^\alpha \frac{\delta}{\delta \psi_P^\alpha} + \delta_{\text{gauge}} \bar{\psi}_P^\dot{\alpha} \frac{\delta}{\delta \bar{\psi}_P^\dot{\alpha}} \right) .$$  \hspace{1cm} (7)
QED is renormalizable which means that the Ward identity (8) can be fulfilled to all orders in perturbation theory in its classical form:

\[
(w_{\text{gauge}} - \partial^\nu \frac{1}{e} \frac{\delta}{\delta A^\nu}) \Gamma = -\frac{1}{e\xi} \square \partial A ,
\]

where \(\Gamma = \Gamma_{\text{cl}} + O(\hbar)\) is the generating functional of one-particle-irreducible (1PI) Green functions. The Ward identity together with a subtraction procedure for removing the ultraviolet divergences defines the 1PI Green functions of QED.

Having specified the vertex function of QED one can consider the action of axial transformations on the 1PI Green functions of QED. Axial symmetry is classically broken by the mass term of matter fields and its local version implies partial conservation of the axial current. We introduce in the classical action (1) an external vector field \(V^\mu\) coupling to the axial current, i.e. we extend the covariant derivatives in (1) to,

\[
D^\mu \psi_P = (\partial^\mu + ieQ_P A^\mu + iV^\mu) \psi_P .
\]

Then softly broken axial symmetry can be expressed in form of a softly broken axial Ward identity

\[
\left(w_{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu}\right) \Gamma_{\text{cl}} = 2im(\bar{\psi}_L \psi_R + \bar{\psi}_L \psi_R)
\]

with the Ward operator of axial transformations

\[
w_{\text{axial}} \equiv \sum_{P=L,R} \left(-i\bar{\psi}_P \frac{\delta}{\delta \psi_P} + i\bar{\psi}_P \frac{\delta}{\delta \bar{\psi}_P}\right)
\]

In one-loop order the axial Ward identity is broken by the Adler-Bardeen anomaly:

\[
\left(w_{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu}\right) \Gamma^{(1)} = e^2 \epsilon^{\mu\nu\rho\sigma}[F_{\mu\nu} F_{\rho\sigma}] \cdot \Gamma + \text{soft terms} .
\]

The axial anomaly is characterized by the following properties:

1. If the gauge Ward identity is established, then \(\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\) cannot be expressed as the variation of a 4-dimensional field monomial. Indeed, the only possible counterterm with the \(\epsilon\)-tensor

\[
\Gamma_{\text{ct,noninv}} = 4r^{(1)} e^2 \int d^4x \epsilon^{\mu\nu\rho\sigma} V_\mu A_\nu \partial_\rho A_\sigma
\]
is not gauge invariant. Adding it to the classical action the anomaly is shifted to the gauge Ward identity, i.e.,

\[ (w^\text{gauge} - \frac{1}{e} \partial^\nu \frac{\delta}{\delta A^\nu}) \Gamma^{(1)} = r^{(1)} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} (A) F_{\rho \sigma} (V) , \]

\[ (w^\text{axial} - \partial^\nu \frac{\delta}{\delta V^\nu}) \Gamma^{(1)} = \text{soft terms} . \] (14)

From its algebraic characterization one can immediately derive that the coefficient of the anomaly \( r^{(1)} \) is scheme-independent and independent of the specific subtraction procedure used for renormalization of 1PI Green functions.

2. The anomaly is not renormalized \[4\], i.e., one has

\[ (w^\text{axial} - \partial^\nu \frac{\delta}{\delta V^\nu}) \Gamma = e^{2r^{(1)}} \epsilon^{\mu \nu \rho \sigma} [F_{\mu \nu} F_{\rho \sigma}]_4 \cdot \Gamma + 2i m [\bar{\psi}_L \psi_R + \bar{\psi}_L \psi_R]_4 \cdot \Gamma . \] (15)

It is obvious that for a precise definition of the non-renormalization theorem one first has to define the Green functions with anomaly insertions \([F_{\mu \nu} \bar{F}_{\mu \nu}]_4 \cdot \Gamma \). A well-suited procedure for this definition is the usage of local gauge couplings, which makes possible to define the renormalization of \( \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} \) unambiguously by gauge-invariance and to state the non-renormalization of the Adler-Bardeen anomaly in a scheme-independent framework.

In the following subsections we will have a closer look on the properties of the Adler-Bardeen anomaly.

2.1 Scheme independence of the anomaly

The algebraic characterization of the Adler-Bardeen anomaly implies, that it is induced by convergent one-loop integrals, which are not subject of renormalization \[4\]. (For a recent review see \[7\].) To work this property out more explicitly we test the axial Ward-identity with respect to two photon fields and obtain after Fourier transformation:

\[ -iq^\rho \Gamma_{V^\rho A^\mu A^\nu} (q, p_1, p_2) = r^{(1)} \delta^{\rho \mu \rho \sigma} p_1^\rho p_2^\sigma + \text{soft terms} . \] (16)

Gauge invariance implies:

\[ p_1^\mu \Gamma_{V^\rho A^\mu A^\nu} (q, p_1, p_2) = p_2^\nu \Gamma_{V^\rho A^\mu A^\nu} (q, p_1, p_2) = 0 \] (17)
We carry out a tensor decomposition of the Green function $\Gamma_{V^\rho A^\mu A^\nu}(q,p_1,p_2)$ and find:

$$
\Gamma_{V^\rho A^\mu A^\nu}(q,p_1,p_2) = i\epsilon^\rho\mu\lambda\Sigma_{\text{div}}(p_1,p_2) - i\epsilon^\rho\mu\lambda\lambda' p_2\lambda p_1\lambda' \sum_{i=1,2} p_i^\rho \Sigma_i(p_1,p_2) + (p_1 \leftrightarrow p_2, \mu \leftrightarrow \nu) .
$$

(18)

According to their momentum structure the scalar function $\Sigma_i(p_1,p_2)$ arise from convergent one-loop integrals. Using now transversality in the photon legs we find that the divergent part is uniquely determined by the convergent one-loop integrals:

$$
\Sigma_{\text{div}}(p_1,p_2) = p_2^2 \Sigma_2(p_1,p_2) + p_1 \cdot p_2 \Sigma_1(p_1,p_2) .
$$

(19)

Thus, in the end one finds that the anomaly coefficient is determined in terms of the convergent one-loop functions $\Sigma_i$. Evaluating (16) for asymptotic moment much larger than the fermion mass, where the soft mass contribution vanishes, one obtains:

$$
8r^{(1)} = \Sigma_{\text{div}}^{(1)}(p_1,p_2) + \Sigma_{\text{div}}^{(1)}(p_2,p_1)
$$

$$
= (p_2^2 \Sigma_2^{(1)}(p_1,p_2) + p_1^2 \Sigma_2^{(1)}(p_2,p_1)) + p_1 \cdot p_2 (\Sigma_1(p_1,p_2) + \Sigma_1(p_2,p_1)) ,
$$

(20)

which demonstrates scheme and regularization independence of $r^{(1)}$. The explicit evaluation of integrals can be found in the original paper [1].

2.2 The non-renormalization theorem and the local gauge coupling

For proving the non-renormalization theorem of the Adler-Bardeen anomaly [4] we define in a first step the Green functions $\epsilon^{\mu\nu\rho\sigma}[F_{\mu\nu}F_{\rho\sigma}].\Gamma$. For this purpose it is useful to extend the coupling constant to an external space-time dependent field, the local coupling $e(x)$ [8]. With local gauge coupling the gauge transformation of the photon field (5) is generalized to

$$
\delta_{\text{gauge}} A_\mu = \frac{1}{e(x)} \partial_\mu \omega .
$$

(21)

The kinetic action of the photon field

$$
\Gamma_{\text{cl,kin}} = \int d^4x \left( -\frac{1}{4e^2(x)} F_{\mu\nu}(eA) F_{\mu\nu}(eA) \right) 
$$

(22)
is gauge invariant under transformations with local coupling and is the unique extension of the ordinary gauge invariant action \( \Pi \). Renormalization of QED with local gauge coupling is performed in the same way as with constant coupling. We treat the local coupling as an external field and perform the limit to constant coupling for all explicit diagrams. In particular, the perturbative expansion is a power series in the coupling and the 1PI Green function satisfy the topological formula in loop order \( l \):

\[
N_e \Gamma^{(l)} = (N_A + N_\psi + 2(l - 1)) \Gamma^{(l)} .
\]

(23)

\( N_e, N_A, N_\psi \) are the usual leg counting differential operators, e.g.

\[
N_e = \int d^4x \, e(x) \frac{\delta}{\delta e(x)} ,
\]

(24)

which include for fermion fields the complex conjugate and a sum over left and right-handed fields.

In the following we will show that due to the extension to local gauge coupling the insertions \( [F^{\mu\nu} \tilde{F}_{\mu\nu}] \) into photon self energies are uniquely defined by gauge invariance with local gauge coupling. For this purpose it is useful to couple the anomaly to an external field \( \Theta(x) \) in the classical action \( \Pi \), i.e.,

\[
\Gamma_{cl} \to \Gamma_{cl} - \frac{1}{4} \int d^4x \, \Theta(x) F^{\mu\nu}(eA) \tilde{F}_{\mu\nu}(eA) , \quad \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} .
\]

(25)

Differentiation with respect to \( \Theta \) yields the Green functions with insertions \( F \tilde{F} \):

\[
\frac{\delta}{\delta \Theta(x)} \Gamma = -\frac{1}{4} [F^{\rho\sigma}(eA) \tilde{F}_{\rho\sigma}(eA)] \cdot \Gamma .
\]

(26)

An additional characterization for the renormalization is provided by the property, that \( F \tilde{F} \) couples is a total derivative and vanishes upon integration. Thus one has

\[
\int d^4x \, \frac{\delta}{\delta \Theta(x)} \Gamma_{cl} = -\frac{1}{4} \int d^4x \, F^{\mu\nu}(eA) \tilde{F}_{\mu\nu}(eA) = 0 .
\]

(27)

This property can be trivially maintained in the procedure of renormalization, i.e.,

\[
\int d^4x \, \frac{\delta}{\delta \Theta(x)} \Gamma = -\frac{1}{4} \int d^4x \, [F^{\rho\sigma}(eA) \tilde{F}_{\rho\sigma}(eA)] \cdot \Gamma = 0 ,
\]

(28)

and implies that the UV divergences corresponding to the respective diagrams are total derivatives.
Considering now the possible invariant counterterms to the field $\Theta$ in the classical action, we find as the only gauge invariant terms satisfying the constraint (28)

$$\Gamma^{(l)} = \int d^4x \, z^{(l)}_{V \Theta} \Theta(x) \partial^\mu (e^{2l} j_{\mu}^{\text{axial}}).$$  

(29)

It corresponds to a field redefinition of the external axial vector:

$$V^\mu \rightarrow V^\mu - z^{(l)}_{V \Theta} e^{2l} \partial^\mu \Theta.$$  

(30)

Higher-order counterterms to $F \tilde{F}$ itself vanish, since they have the general form

$$\int d^4x \, \Theta(x) e^{2l}(x) F_{\mu\nu}(eA) \tilde{F}_{\mu\nu}(eA),$$  

(31)

and violate the property (28) for local gauge coupling. Hence, higher-order corrections to $\Gamma_{\Theta A \nu A^\nu}$ are non-local and are determined by gauge invariance and by the identity (28).

For the explicit construction we consider the Green functions once differentiated with respect to $e(x)$:

$$\lim_{e \rightarrow \text{const.}} \frac{\delta}{\delta \Theta(z)} \frac{\delta}{\delta e(y)} \frac{\delta}{\delta A^\alpha(x_1)} \frac{\delta}{\delta A^\nu(x_2)} \Gamma \bigg|_{A, \psi = 0} \equiv \Gamma_{\Theta A \nu A^\nu}(z, y, x_1, x_2).$$  

(32)

The gauge Ward identity (8) with local coupling yields the identities ($q + p + p_1 + p_2 = 0$):

$$p^\mu_1 \Gamma_{\Theta A \nu A^\nu}(q, p, p_1, p_2) = 0,$$

$$ep^\mu_1 \Gamma_{e A \nu A^\nu}(q, p, p_1, p_2) - p^\mu_1 \Gamma_{e A \nu A^\nu}(q, p_1 + p, p_2) = 0.$$  

(33)

Due to the construction the Green functions once differentiated with respect to the local coupling are related to the Green functions with constant coupling. From the topological formula (23) one gets

$$2(l + 1) \Gamma^{(l)}_{\Theta A \nu A^\nu}(q, p_1, p_2) = e \Gamma^{(l)}_{e A \nu A^\nu}(q, 0, p_1, p_2).$$  

(34)

Using that Green functions with $F \tilde{F}$ insertions vanish for vanishing incoming momenta at the insertion (see (28)) we find the following tensor decomposition for general $\Gamma_{e A \nu A^\nu}$:

$$\Gamma_{e A \nu A^\nu}(q, p, p_1, p_2)$$

$$= e^{\mu \nu \rho \sigma} q_\rho p_1_\sigma \Sigma_2 (p_1, p_2, p)$$

$$+ e^{\mu \lambda \rho \sigma} q_\lambda p_1_\rho p_2_\sigma \left( p_1^\nu \Sigma_1 (p_1, p_2, p) + p_2^\nu \Sigma_2 (p_1, p_2, p) + p^\nu \Sigma_3 (p_1, p_2, p) \right)$$

$$+ (\mu \leftrightarrow \nu, p_1 \leftrightarrow p_2).$$  

(35)
From (34) we obtain for $\Gamma_{\Theta,AA}$ at constant coupling

$$\Gamma^{(l)}_{\Theta,AA}(q, p_1, p_2) \equiv \epsilon^{\mu \nu \rho \sigma} p_1 \rho p_2 \sigma \Sigma^{(l)}_{\Theta}(p_1, p_2)$$

$$= \frac{e}{2(l + 1)} \epsilon^{\mu \nu \rho \sigma} p_1 \rho p_2 \sigma \left( \Sigma^{(l)}_{\text{div}}(p_1, p_2, 0) + \Sigma^{(l)}_{\text{div}}(p_2, p_1, 0) \right) .$$

(36)

Applying the gauge Ward identity (33) to (35) the local part $\Sigma_{\text{div}}$ is determined by the convergent tensor integrals for all loop orders $l \geq 1$ and according to (36) one has at the same time determined $\Gamma_{\Theta,AA}$:

$$\Sigma^{(l)}_{\Theta}(p_1, p_2) = \frac{e}{2(l + 1)} \left( \Sigma^{(l)}_{\text{div}}(p_1, p_2, 0) + \Sigma^{(l)}_{\text{div}}(p_2, p_1, 0) \right)$$

$$= - \frac{e}{2l} \left( p_1 \cdot p_2 \left( \Sigma^{(l)}_1(p_1, p_2, 0) + \Sigma^{(l)}_1(p_2, p_1, 0) \right) 

+ p_2^2 \Sigma^{(l)}_2(p_1, p_2, 0) + p_1^2 \Sigma^{(l)}_2(p_2, p_1, 0) \right) .$$

(37)

Thus, one has constructed the Green functions with insertions of $F \tilde{F}$ uniquely from extension to local gauge coupling independent from the existence of an anomalous Ward identity.

Now we turn to the axial Ward identity. For local gauge coupling the anomalous axial Ward identity takes the following general form:

$$\left( w^{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu} \right) \Gamma = \sum_{l \geq 1} r^{(l)} e^{2(l-1)} [F^\mu \nu(eA) \tilde{F}_\mu \nu(eA)] \cdot \Gamma + \text{soft terms} .$$

(38)

Up to matter part contributions (30) the insertions on the right-hand-side are uniquely defined by gauge invariance.

When the coupling is local the first order is distinguished from higher orders: Only in this order the expression of the anomaly is a total derivative. Hence, integrating the local Ward identity we find

$$W^{\text{axial}} \Gamma = \sum_{l \geq 2} \int d^4 x \ r^{(l)} e^{2(l-1)} [F^\mu \nu(eA) \tilde{F}_\mu \nu(eA)] \cdot \Gamma + \text{soft terms} .$$

(39)

Differentiating this identity twice with respect to $A^\mu$ and once with respect to the local coupling, the construction yields immediately the non-renormalization theorem of the Adler-Bardeen anomaly: Since the right-hand-side is vanishing for a test with respect to photon fields

$$\frac{\delta}{\delta A^\mu(x_1)} \frac{\delta}{\delta A^\nu(x_2)} \frac{\delta}{\delta e(y)} W^{\text{axial}} \Gamma = 0$$

we find order by order

$$r^{(l)} = 0 \quad \text{for} \quad l \geq 2 .$$

(40)
3 The axial anomaly in non-abelian gauge theories

The construction of the previous section can be immediately applied to non-abelian gauge theories. We consider non-abelian gauge theories with a simple gauge group as $SU(N)$ and left- and right-handed fermionic matter fields transforming under an irreducible representation and its complex conjugate representation respectively. Thus one has parity conservation and the theory is free of gauge anomalies.

Extending the gauge coupling to an external field and including a space-time dependent $\Theta$ angle in the same way as it was done for QED we find the following gauge invariant action:

$$\Gamma_{YM} = -\frac{1}{4} \text{Tr} \int d^4x \left( \frac{1}{g^2} G^{\mu\nu}(gA)G_{\mu\nu}(gA) + \Theta \epsilon^{\mu
u\rho\sigma} G_{\mu\nu}(gA)G_{\rho\sigma}(gA) \right) + \int d^4x \left( (i\bar{\psi}_L \gamma^\mu D_{\mu} \psi_L + i\bar{\psi}_R \sigma^\mu D_{\mu} \psi_R) - m(\psi_L^T \psi_R + \bar{\psi}_L^T \bar{\psi}_R) \right).$$

$$A^\mu = A^\mu_a \tau_a$$ are the gauge fields and $\tau_a$ are the generators of the gauge group:

$$[\tau_a, \tau_b] = i f_{abc} \tau_c , \quad (43)$$

with the normalization

$$\text{Tr}(\tau_a \tau_b) = \delta_{ab} . \quad (44)$$

The non-abelian field strength tensor and the covariant derivative defined by

$$G^{\mu\nu}(A) = \partial^\mu A^\nu - \partial^\nu A^\mu + i[A^\mu, A^\nu] ,$$

$$D^\mu \psi_L = (\partial^\mu - g(x) iT_a A^\mu + iV^\mu) \psi_L ,$$

$$D^\mu \psi_R = (\partial^\mu - g(x) iT^*_a A^\mu + iV^\mu) \psi_R . \quad (45)$$

In the definition of covariant derivatives we have already introduced the abelian external vector field $V^\mu$ whose interaction with matter fields is defined by the $U(1)$ axial Ward-identity (see (10) and (11)):

$$\left( w^{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu} \right) \Gamma_{YM} = 2im(\psi_L^T \psi_R - \bar{\psi}_L^T \bar{\psi}_R) . \quad (46)$$
Quantization of non-abelian gauge theories cannot be performed by a gauge Ward
identity but one has to use the corresponding BRS transformations. For this pur-
pose we rewrite the gauge transformations into BRS transformations by replacing
the gauge transformation parameter $\omega$ with the ghost field $c$

$$s\phi = \delta_{c}^{\text{gauge}}\phi, \quad \phi \equiv A^{\mu}, \psi, \bar{\psi},
$$

$$s\Theta(x) = sg(x) = sV^{\mu} = 0 ,$$

(47)

and introduce an anti-ghost and an auxiliary $B$-field with

$$s\bar{c} = B , \quad sB = 0 .$$

(48)

Then the gauge fixing term can be extended to a BRS-invariant action:

$$\Gamma_{\text{g.f.}} + \Gamma_{\phi\pi} = s \int d^{4}x \ Tr(\bar{c}\partial A) .$$

(49)

Finally adding to the classical action the external field part,

$$\Gamma_{\text{cl}} = \Gamma_{\text{YM}} + \Gamma_{\text{g.f.}} + \Gamma_{\phi\pi} + \Gamma_{\text{ext.f.}} ,$$

(50)

with

$$\Gamma_{\text{ext.f.}} = \int d^{4}x \left( \rho^{\mu}sA_{\mu} + (Y_{L}s\psi_{L} + Y_{R}s\psi_{R} + c.c.) \right)$$

(51)

we encode BRS transformations in the Slavnov–Taylor identity

$$S(\Gamma) = 0 .$$

(52)

The Slavnov–Taylor identity is valid to all orders also if we include the local
coupling and the $\Theta$ angle.

As for QED (27) the renormalization of the $\Theta$ angle is restricted by the equation

$$\int d^{4}x \ \frac{\delta}{\delta\Theta}\Gamma = 0 ,$$

(53)

which is a non-trivial restriction with local couplings.

For the renormalization of the $\Theta$ angle in non-abelian gauge theories we find the
same results as for QED: The only invariant counterterms to the $\Theta$ angle are the
redefinitions of the axial vector field into the $\Theta$ angle (see (29) and (30)) and
invariant counterterms to the $\Theta$ angle itself are excluded by the identity (53) (see
(31)). However, contrary to QED the topological term itself is renormalized by
the wave-function renormalization of vector fields $A^{\mu} \rightarrow z_{A}A^{\mu}$. 

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Therefrom we conclude that the insertions of the topological term into gluon self energies are unambiguously determined by the symmetries of the theory, i.e.,

\[ s_\Gamma \left[ \text{Tr}(G^{\mu\nu} \tilde{G}_{\mu\nu}) \right] \cdot \Gamma = 0 \quad \text{and} \quad \int d^4x \left[ \text{Tr} G^{\mu\nu} \tilde{G}_{\mu\nu} \right] \cdot \Gamma = 0 . \quad \text{(54)} \]

The explicit evaluation gives rise to similar identities as (33) which involve in addition the wave function renormalization of gluons. In order to get rid of the wave function renormalizations one can use background gauge fields which satisfy a non-abelian gauge Ward identity. Insertions of the topological term into background self energies satisfy the same identities of transversality as the photon fields (33).

With these ingredients the non-renormalization theorem of the axial anomaly can be proven as in QED. With local coupling the axial Ward identity takes the general form:

\[ \left( w_{\text{axial}} - \partial^\nu \frac{\delta}{\delta V^\nu} \right) \Gamma = \sum_{l \geq 1} r^{(l)} g^{2(l-1)} \left[ \text{Tr}(G^{\mu\nu}(gA) \tilde{G}_{\mu\nu}(gA)) \right] \cdot \Gamma + \text{soft terms} . \quad \text{(55)} \]

All insertions on the right-hand-side are unambiguously defined by the identities (54). Integration of the local Ward identity (55) yields then

\[ r^{(l)} = 0 \quad \text{for} \quad l \geq 2 , \quad \text{(56)} \]

i.e. the non-renormalization theorem of the Adler-Bardeen anomaly.

### 4 Supersymmetric Yang-Mills theories

Now we extend the Yang-Mills theories without matter to supersymmetric theories: We introduce in addition to the gauge fields the gaugino fields \( \lambda^\alpha = \lambda^\alpha_a \tau_a \) and their complex conjugate fields \( \bar{\lambda}^\dot{\alpha} \) and the auxiliary fields \( D = D_a \tau_a \). For constant coupling the action

\[ \Gamma_{\text{SYM}} = \text{Tr} \int d^4x \left( -\frac{1}{4g^2} G^{\mu\nu}(gA)G_{\mu\nu}(gA) + i\lambda^\alpha \sigma_\mu \lambda^\dot{\alpha} + \frac{1}{8} D^2 \right) , \quad \text{(57)} \]

with

\[ D^\mu \lambda = \partial^\mu \lambda - ig[A^\mu, \lambda] , \quad \text{(58)} \]
is invariant under non-abelian gauge transformations and supersymmetry transformations:

\[\delta \alpha A^\mu = i\sigma^\mu_{\dot{\alpha}\alpha} \bar{\chi}_\alpha, \quad \bar{\delta} \dot{\alpha} A^\mu = -i\lambda^\alpha \sigma^\mu_{\alpha\dot{\alpha}}, \]
\[\delta \alpha \lambda_\beta = -\frac{i}{2} (\epsilon_{\alpha\beta} \partial - \sigma_{\alpha\beta} G_{\mu\nu}) , \quad \bar{\delta} \dot{\alpha} \lambda_\beta = -\frac{i}{2} (\epsilon_{\dot{\alpha}\dot{\beta}} \partial - \bar{\sigma}^{\mu\nu} G_{\mu\nu}) , \]
\[\delta \alpha \bar{\lambda}_\dot{\alpha} = 0 , \quad \bar{\delta} \dot{\alpha} \bar{\lambda}_\dot{\alpha} = 0 , \quad \delta \alpha D = 2\sigma^\mu_{\alpha\dot{\alpha}} D_\mu \bar{\chi}_\alpha, \quad \bar{\delta} \dot{\alpha} D = 2D_\mu \lambda^\alpha \sigma^\mu_{\alpha\dot{\alpha}} . \tag{59}\]

In the Wess-Zumino gauge \cite{9, 10} that we have used here the algebra of supersymmetry transformation does not close on translations but involves an additional field dependent gauge transformation:

\[\{\delta \alpha, \bar{\delta} \dot{\alpha}\} = 2i\sigma^\mu_{\alpha\dot{\alpha}} (\partial^\mu + \delta^\mu_{\text{gauge}} ) , \]
\[\{\delta \alpha, \delta \beta\} = \{\bar{\delta} \dot{\alpha}, \bar{\delta} \dot{\beta}\} = 0 . \tag{60}\]

Thus, on gauge invariant field monomials the supersymmetry algebra takes their usual form, and it is possible to classify the Lagrangians into the usual \(N = 1\) multiplets.

The Lagrangians of supersymmetric field theories are the \(F\) or \(D\) components of a \(N = 1\) multiplet. In particular one finds that the Lagrangian of the super-Yang-Mills action is the highest component of a chiral and an antichiral multiplet. Using a superfield notation in the chiral representation the chiral Lagrangian multiplet \(L_{SYM}\) is given by

\[L_{SYM} = -\frac{1}{2} g^2 \text{Tr} \lambda^\alpha \lambda_\alpha + \Lambda^\alpha \theta_\alpha + \theta^2 L_{SYM} , \tag{61}\]

with the chiral super-Yang-Mills Lagrangian \(L_{SYM}\)

\[L_{SYM} = \text{Tr} \left( -\frac{1}{4} G^{\mu\nu} (gA)G_{\mu\nu}(gA) + ig\lambda^\alpha \sigma^\mu_{\alpha\dot{\alpha}} D_\mu (g \bar{\chi}^\dot{\alpha}) + \frac{1}{8} g^2 D^2 \right. \]
\[\left. -\frac{i}{8} e^{\mu\nu\rho\sigma} G_{\mu\nu}(gA)G_{\rho\sigma}(gA) \right) . \tag{62}\]

By complex conjugation one obtains the the respective antichiral multiplet \(\bar{L}_{SYM}\).

The crucial point for the present considerations is the fact that the topological term \(\text{Tr} \ G \bar{G}\) appears in the supersymmetric Lagrangians \cite{12} and is related to the kinetic term \(-\frac{1}{4} \text{Tr} (GG)\) via supersymmetry. In previous sections we have
shown that the higher order corrections to $\text{Tr}G\tilde{G}$ are unambiguously determined from gauge invariance, and therefore one expects the same effect for the kinetic term, which describes the renormalization of the gauge coupling. In ordinary perturbation theory these relations are not found, since the topological term disappears from the action by integration. However, with a local gauge coupling one is able to include the complete Lagrangian with the topological term into the action and the improved renormalization properties become apparent.

We will now extend the coupling constant $g$ to an external field $g(x)$ \[8, 3\]. For maintaining at the same time supersymmetry the coupling has to be extended to a supermultiplet. Therefore we introduce a chiral and an antichiral field multiplet $\eta$ and $\bar{\eta}$:

$$
\eta = \eta + \theta \chi + \theta^2 f,
\bar{\eta} = \bar{\eta} + \bar{\theta} \chi + \bar{\theta}^2 \bar{f},
$$

which couple to the Lagrangian multiplet of the Super-Yang-Mills action:

$$
\Gamma_{\text{SYM}} = -\frac{1}{4} \int dS \eta L_{\text{SYM}} - \frac{1}{4} \int d\bar{S} \bar{\eta} L_{\text{SYM}}
= \int d^4x \left( \eta L_{\text{SYM}} - \frac{1}{2} \chi^\alpha \Lambda_\alpha - \frac{1}{2} f g^2 \text{Tr} \chi^\alpha \lambda_\alpha + \text{c.c.} \right).
\tag{63}
$$

We identify the real part of $\eta$ with the inverse of the square of the local coupling:

$$
\eta + \bar{\eta} = \frac{1}{g^2(x)},
\tag{64}
$$

and observe that the imaginary part of $\eta$ couples to the topological term. Thus, it takes the role of a space time dependent $\Theta$ angle

$$
\eta - \bar{\eta} = 2i \Theta.
\tag{65}
$$

Hence, dependence on the superfield $\eta$ and $\bar{\eta}$ is governed by a topological formula (cf. (23)) and by the identity (53):

$$
\int d^4x \left( \frac{\delta}{\delta \eta} - \frac{\delta}{\delta \bar{\eta}} \right) \Gamma = -i \int d^4x \frac{\delta}{\delta \Theta} \Gamma = 0.
\tag{66}
$$

Constructing with these ingredients the invariant counterterms to $\Gamma_{\text{SYM}}$ we find from gauge invariance and supersymmetry:

$$
\Gamma_{\text{ct,phys}}^{(l)} = z_{\text{YM}}^{(l)} \left( -\frac{1}{8} \int dS \eta^{1-l} L_{\text{SYM}} - \frac{1}{8} \int d\bar{S} \bar{\eta}^{1-l} \bar{L}_{\text{SYM}} \right)
= z_{\text{YM}}^{(l)} \int d^4x \left( -\frac{1}{4} (2g^2)^{l-1} G^{\mu\nu}(gA) G_{\mu\nu}(gA)
- \frac{1}{8} (2g^2)^{l}(l - 1) \Theta G^{\mu\nu}(gA) \tilde{G}_{\mu\nu}(gA) + \ldots \right).
\tag{67}
$$
These counterterms as well as the respective UV divergences are excluded in loop order $l \geq 2$ by the identity (66), which governs the renormalization of the $\Theta$ angle:

$$z^{(l)}_{YM} = 0 \quad \text{for} \quad l \geq 2 .$$

Thus, in loop orders $l \geq 2$ the topological term $\text{Tr} \, G \tilde{G}$ determines the renormalization of the coupling. It is obvious from eq. (67) that the one-loop order is special, and we will show in the next section that there the loop corrections to $\text{Tr} \, G \tilde{G}$ induce an anomaly of supersymmetry.

## 5 The anomalous breaking of supersymmetry

To quantize supersymmetric field theories in the Wess-Zumino gauge one includes the gauge transformations, supersymmetry transformations and translations into the nilpotent BRS operator [11, 12]:

$$s\phi = \delta^{\text{gauge}}_{c} + \epsilon^{\alpha} \delta_{\alpha} + \bar{\epsilon}^{\dot{\alpha}} \delta^{\dot{\alpha}} - i \omega^{\mu} \delta_{\mu}^{T}. $$

(69)

The fields $c(x)$ are the usual Faddeev-Popov ghosts, $\epsilon^{\alpha}$, $\bar{\epsilon}^{\dot{\alpha}}$ and $\omega^{\mu}$ are constant ghosts of supersymmetry and translations, respectively.

As for usual gauge theories BRS transformations are encoded in the Slavnov-Taylor identity:

$$S(\Gamma_{\text{cl}}) = 0$$

(70)

with

$$\Gamma_{\text{cl}} = \Gamma_{\text{SYM}} + \Gamma_{g.f.} + \Gamma_{\phi \pi} + \Gamma_{\text{ext.f.}} .$$

(71)

Details of the construction are not relevant for the following considerations, but it is only important to keep in mind that the Slavnov-Taylor identity for supersymmetric actions expresses gauge invariance as well as supersymmetry.

In addition to the Slavnov-Taylor identity the dependence on the external fields $\eta$ and $\bar{\eta}$ is restricted to all orders by the identity (66). [3]

In the course of renormalization the Slavnov-Taylor identity (70) has to be established for the 1PI Green functions to all orders of perturbation theory. From the quantum action principle one finds that the possible breaking terms are local in one-loop order:

$$S(\Gamma) = \Delta_{\text{brs}} + \mathcal{O}(h^{2}) .$$

(72)
Algebraic consistency yields the constraint

$$s_{\Gamma_{cl}} \Delta_{\text{brs}} = 0.$$  \hspace{1cm} (73)

Gauge invariance can be established as usually, i.e. one has

$$\mathcal{S}(\Gamma) \bigg|_{\epsilon, \bar{\epsilon} = 0} = \mathcal{O}(\hbar^2),$$  \hspace{1cm} (74)

and the remaining breaking terms depend on the supersymmetry ghosts $\epsilon$ and $\bar{\epsilon}$, and represent as such a breaking of supersymmetry. Having gauge invariance established, the supersymmetry algebra closes on translations. Using the supersymmetry algebra one obtains that the remaining breaking terms of supersymmetry are variations of field monomials with the quantum numbers of the action:

$$\Delta_{\text{brs}} = s_{\Gamma_{cl}} \hat{\Gamma}_{ct}.$$  \hspace{1cm} (75)

However, not all of the field monomials in $\hat{\Gamma}_{ct}$ represent scheme-dependent counterterms of the usual form. There is one field monomial in $\hat{\Gamma}_{ct}$, which depends on the logarithm of the gauge coupling, but whose BRS variation is free of logarithms:

$$\Delta_{\text{anomaly}} = s \int d^4x \ln g(x) (L_{\text{SYM}} + \bar{L}_{\text{SYM}})$$

$$= (\epsilon^\alpha \delta_\alpha + \bar{\epsilon}^i \delta_{\bar{\alpha}}) \int d^4x \ln g(x) (L_{\text{SYM}} + \bar{L}_{\text{SYM}})$$

$$= \int d^4x \left( i \ln g(x) \left( \partial_\mu \Lambda^\alpha \sigma^\mu_{\alpha\bar{\alpha}} \bar{\epsilon}^{\bar{\alpha}} - \epsilon^\alpha \sigma^\mu_{\bar{\alpha} \bar{\alpha}} \partial_\mu \bar{\Lambda}^{\bar{\alpha}} \right) \right.$$  

$$\left. - \frac{1}{2} g^2(x) (\epsilon \chi + \bar{\epsilon} \bar{\chi}) (L_{\text{SYM}} + \bar{L}_{\text{SYM}}) \right).$$

Indeed, due to the total derivative in the first line the breaking $\Delta_{\text{anomaly}}$ is free of logarithms for constant coupling and for any test with respect to the local coupling. Moreover one immediately verifies, that $\Delta_{\text{anomaly}}$ satisfies the topological formula in one-loop order. Therefore $\Delta_{\text{anomaly}}$ satisfies all constraints on the breakings and can appear as a breaking of the Slavnov-Taylor identity in the first order of perturbation theory.

However, being the variation of a field monomial depending on the logarithm of the coupling $\Delta_{\text{anomaly}}$ cannot be induced by divergent one-loop diagrams, which are all power series in the coupling. Thus, the corresponding counterterm is not related to a naive contribution induced in the procedure of subtraction and does not represent a naive redefinition of time-ordered Green functions. Therefore,
\[ \Delta_{\text{brs}}^{\text{anomaly}} \] is an anomalous breaking of supersymmetry in perturbation theory and we remain with
\[ S(\Gamma) = r_\eta^{(1)} \Delta_{\text{brs}}^{\text{anomaly}} + O(h^2) . \] (77)

From its characterization it is straightforward to prove with algebraic methods that the coefficient of the anomaly is gauge and scheme independent \[5\]. Furthermore, by evaluating the Slavnov–Taylor identity one can find an expression for \[r_\eta^{(1)}\] in terms of convergent loop integrals \[3\]. Using background gauge fields \[\hat{A}^\mu\] and Feynman gauge \[\xi = 1\] the anomaly coefficient is explicitly related to insertions of the topological term and the axial current of gluinos into self energies of background fields:
\[ g^2 r_\eta^{(1)} = -\frac{1}{2} \Sigma_{\eta-\pi}(p_1, -p_1) \bigg|_{\xi=1} , \] (78)

where \[\Sigma_{\eta-\pi}\] is defined by
\[
\Gamma_{\eta-\pi\hat{A}_a^\mu \hat{A}_b^\nu}(q, p_1, p_2) = \left( i \text{Tr} \left( \partial (g^2 \lambda \sigma \lambda) - \frac{1}{4} G^{\mu\nu} \tilde{G}_{\mu\nu}(gA + \hat{A}) \right) \right) \cdot \hat{A}_a^\mu \hat{A}_b^\nu(q, p_1, p_2) = i \epsilon^{\mu\rho\sigma} p_1\rho p_2\sigma \delta_{ab} (-2 + \Sigma_{\eta-\pi}(p_1, p_2)) . \] (79)

In section 2 and 3 these Green functions have been constructed in abelian as well as non-abelian gauge theories by extension to local gauge coupling in terms of convergent integrals (see \[37\] and \[54\]). Due to the Ward identity for the background gauge fields,
\[
(w_{\text{gauge}}^a - \partial^\nu \frac{\delta}{\delta \hat{A}_a^\nu}) \Gamma = 0 \] (80)

the 1PI Green functions \[\Gamma_{\eta-\pi\hat{A}_a^\mu \hat{A}_b^\nu}\] can be even constructed in the same way as it was demonstrated for \[\Gamma_{\Theta A^\mu A^\nu}\] in QED \[37\].

Explicit evaluation of the respective one-loop diagrams yields
\[ r_\eta^{(1)} = (-1 + 2) \frac{C(G)}{8\pi^2} = \frac{C(G)}{8\pi^2} , \] (81)

where the first term comes from the axial anomaly of gauginos and the second term from the insertion of the topological term.

We want to mention that the anomaly coefficient vanishes in SQED, since one-loop diagrams to \[\Gamma_{\eta-\pi\hat{A}_a^\mu \hat{A}_b^\nu}\] do not exist. The anomaly coefficient also vanishes in \[N = 2\] theories, since there the two fermionic fields just cancel the contribution arising from the topological term \[\text{Tr} G^{\mu\nu} \tilde{G}_{\mu\nu}\].
6 Implications of the anomaly

In the framework of perturbation theory the anomaly of supersymmetry cannot be removed by a local counterterm. However, one is able to proceed with algebraic renormalization nevertheless by rewriting the anomalous breaking in the form of a differential operator

\[ \Delta_{\text{anomaly}} = \int d^4x \left( g^6 r_{r}^{(1)}(\epsilon\chi + \chi\epsilon) \frac{\delta}{\delta g^2} \right. \]

\[ \left. - i r_{\eta}^{(1)} \partial_\mu \ln g^2 \left( \left( \sigma^\mu \tau \right)^\alpha \frac{\delta}{\delta \chi^\alpha} + (\epsilon \sigma^\mu)^\alpha \delta \frac{\delta}{\delta \chi^\alpha} \right) \right) \Gamma_{\text{cl}} \]  

(82)

Then one has

\[ (S + r_{\eta}^{(1)} \delta S) \Gamma = \mathcal{O}(\hbar^2) . \]  

(83)

For algebraic consistency one has to require the nilpotency properties of the classical Slavnov-Taylor operator also for the extended operator. Nilpotency determines an algebraic consistent continuation of (82). This continuation is not unique but contains at the same time all redefinitions of the coupling compatible with the formal power series expansion of perturbation theory.

As a result one finds an algebraic consistent Slavnov-Taylor identity in presence of the anomaly:

\[ S^{\text{an}}(\Gamma) = 0 \quad \text{and} \quad \int d^4x \left( \frac{\delta}{\delta \eta} - \frac{\delta}{\delta \eta} \right) \Gamma = 0 , \]  

(84)

where the anomalous part is of the form:

\[ S^{\text{an}}(\Gamma) = S(\Gamma) - \int d^4x \left( g^4 \delta F(g^2)(\epsilon\chi + \chi\epsilon) \frac{\delta}{\delta g^2} \right. \]

\[ \left. + i \frac{\delta F}{1 + \delta F} \partial_\mu g^{-2} \left( \left( \sigma^\mu \tau \right)^\alpha \frac{\delta}{\delta \chi^\alpha} + (\epsilon \sigma^\mu)^\alpha \delta \frac{\delta}{\delta \chi^\alpha} \right) \right) \Gamma , \]  

(85)

with

\[ \delta F(g^2) = r_{r}^{(1)} g^2 + \mathcal{O}(g^4) . \]  

(86)

The lowest order term is uniquely fixed by the anomaly, whereas the higher orders in \( \delta F(g^2) \) correspond to the scheme-dependent finite redefinitions of the coupling.

The simplest choice for \( \delta F \) is given by

\[ \delta F = r_{\eta}^{(1)} g^2 , \]  

(87)
and another choice is provided by

$$\frac{\delta F}{1 + \delta F} = r_{\eta}^{(1)} g^2.$$  \hspace{1cm} (88)

As seen below, the latter choice gives the NSZV expression of the gauge $\beta$ function \cite{14, 13}.

Algebraic renormalization with the symmetry operator (84) is performed in the conventional way. In particular one can derive the $\beta$ functions from an algebraic construction of the renormalization group equation in presence of the local coupling. Starting from the classical expression of the RG equation

$$\kappa \partial_{\kappa} \Gamma_{cl} = 0$$  \hspace{1cm} (89)

we construct the higher orders of the RG equation by constructing the general basis of symmetric differential operator with the quantum numbers of the action:

$$\mathcal{R} = \kappa \partial_{\kappa} + O(h),$$  \hspace{1cm} (90)

with

$$s_{\Gamma}^{\eta} \mathcal{R} \Gamma - \mathcal{R} S_{\eta}^{\eta}(\Gamma) = 0,$$  \hspace{1cm} (91)

$$\left[ \int d^4x \left( \frac{\delta}{\delta \eta} - \frac{\delta}{\delta \bar{\eta}} \right), \mathcal{R} \right] = 0.$$  \hspace{1cm} (92)

The general basis for the symmetric differential operators consists of the differential operator of the supercoupling $\eta$ and $\bar{\eta}$ and several field redefinition operators. The differential operator of the coupling determines the $\beta$ function of the coupling, whereas the field redefinition operators correspond to the anomalous dimensions of fields.

For the present paper we focus on the operator of the $\beta$ function and neglect the anomalous dimensions. For proceeding we construct first the RG operator, which is symmetric with respect to the classical Slavnov-Taylor operator, i.e. we set $\delta F = 0$. Using a superspace notation we find:

$$\mathcal{R}_{cl} = - \sum_{l \geq 1} \hat{\beta}^{(l)} g \left( \int dS \eta^{-l+1} \frac{\delta}{\delta \eta} + \int dS \bar{\eta}^{-l+1} \frac{\delta}{\delta \bar{\eta}} \right) + \ldots,$$  \hspace{1cm} (93)

and

$$s_{\Gamma} \mathcal{R}_{cl} \Gamma - \mathcal{R}_{cl} S(\Gamma) = 0.$$  \hspace{1cm} (94)
Evaluating then the consistency equation (92) we obtain

\[ \hat{\beta}_g^{(l)} = 0 \quad \text{for} \quad l \geq 2 . \]  

(95)

These restrictions are the same restrictions as we have found for the invariant counterterms of the super-Yang-Mills action (see (67) with (68)) and represent the fact, that for super-Yang-Mills the renormalization of the coupling is governed by the renormalization of the \( \Theta \) angle in loop orders greater than one.

Finally we extend the one-loop operator \( \mathcal{R}_{ct} \) to a symmetric operator with respect to the anomalous ST identity (91) and find by a straightforward calculation

\[
\mathcal{R} = \hat{\beta}_g^{(1)} \int d^4x \, g^3 \left( 1 + \delta F(g^2) \right) \frac{\delta}{\delta g} + \ldots \\
= \hat{\beta}_g^{(1)} \int d^4x \, g^3 \left( 1 + r_\eta^{(1)} g^2 + \mathcal{O}(h^2) \right) \frac{\delta}{\delta g} + \ldots . 
\]

(96)

For constant coupling we find from (96) the closed expression of the gauge \( \beta \) function

\[ \beta_g = \hat{\beta}_g^{(1)} g^3 \left( 1 + \delta F(g^2) \right) = \hat{\beta}_g^{(1)} g^3 \left( 1 + r_\eta^{(1)} g^2 + \mathcal{O}(h^2) \right) . \]  

(97)

Thus, the two-loop order is uniquely determined by the anomaly and the one-loop coefficient, whereas higher orders depend on the specific form one has chosen for the function \( \delta F(g^2) \). In particular one has for the minimal choice (87) a pure two-loop \( \beta \)-function and for the NSZV-choice (88) one obtains the NSZV expression [14] of the gauge \( \beta \) function of pure Super-Yang-Mills theories.

7 Conclusions

The extension of the gauge coupling to an external field has been shown to be a crucial step for determining the renormalization of the topological term. As its definition is a necessary prerequisite for the proof of the Adler-Bardeen non-renormalization its quantum corrections have been considered in the past with different methods [13, 17]. In a strict sense, however, these corrections are determined in perturbation theory with constant coupling only in combination with an anomalous chiral symmetry [18, 19, 24].

With local coupling, however, the situation is drastically changed: Using gauge invariance with local coupling it turns out that renormalization of the topological term is uniquely fixed by convergent one-loop integrals in the same way as it is for the triangle diagrams which induce the Adler-Bardeen anomaly.
In the present paper we have given two implications of the construction: First, a proof of the non-renormalization of the Adler–Bardeen anomaly, and second, we have shown that the quantum corrections to the topological term induce an anomaly of supersymmetry in supersymmetric Yang-Mills theories in one loop order and determine the renormalization of the coupling in all loop orders greater than one.

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References

[1] S.L. Adler, *Phys. Rev.* **177** (1969) 2426.

[2] W.A. Bardeen, *Phys. Rev.* **184** (1969) 1848.

[3] J.S. Bell and R. Jackiw, *Nuovo Cim.* **60A** (1969) 47.

[4] S.L. Adler and W.A. Bardeen, *Phys. Rev.* **182** (1969) 1517.

[5] E. Kraus, *Nucl. Phys.* **B620** (2002) 55; [hep-th/0107239](https://arxiv.org/abs/hep-th/0107239).

[6] E. Kraus *Phys.Rev.D* **65** (2002) 105003; [hep-ph/0110323](https://arxiv.org/abs/hep-ph/0110323).

[7] F. Jegerlehner, *Eur. Phys. J.* **C18** (2001) 673; [hep-th/0005253](https://arxiv.org/abs/hep-th/0005253).

[8] E. Kraus and D. Stöckinger, *Nucl.Phys.* **626** (2002) 73, [hep-th/0105028](https://arxiv.org/abs/hep-th/0105028).

[9] J. Wess and B. Zumino, *Nucl. Phys.* **B78** (1974) 1.

[10] B. de Wit and D. Freedman, *Phys. Rev.* **D12** (1975) 2286.

[11] P.L. White, *Class. Quantum Grav.* **9** (1992) 1663.

[12] N. Maggiore, O. Piguet and S. Wolf, *Nucl. Phys.* **B458** (1996) 403.

[13] W. Hollik, E. Kraus and D. Stöckinger, *Eur. Phys. J.* **C11** (1999) 365.

[14] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Nucl. Phys.* **B229** (1983) 381; *Phys. Lett.* **B166** (1986) 329.

[15] M.A. Shifman and A.I. Vainshtein, *Nucl. Phys.* **B277** (1986) 456.
[16] V.I. Zakharov *Phys. Rev.* **D42** (1990) 1208;  
A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B324** (1989) 495;  
M.A. Shifman and A.I. Vainshtein, *Nucl. Phys.* **B365** (1991) 312.

[17] M. Reuter *Mod.Phys.Lett.* **A12** (1997) 2777; hep-th/9604124.

[18] P. Breitenlohner, D. Maison and K. Stelle, *Phys. Lett.* **B134** (1984) 63.

[19] C. Lucchesi, O. Piguet and K. Sibold, *Int. Jour. Mod. Phys.* **A2** (1987) 385.

[20] A.A. Ansel’m and A.A. Iogansen, JETP **69** (1989) 670;  
M. Bos, *Nucl. Phys.* **B404** (1993) 215; hep-ph/9211319.