On the strict positivity and the spectrum of the two-dimensional Brown-Ravenhall operator with an attractive potential of the Bessel-Macdonald type

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Abstract

The Brown-Ravenhall operator was initially proposed as an alternative to describe the fermion-fermion interaction via Coulomb potential and subject to relativity. This operator is defined in terms of the associated Dirac operator and the projection onto the positive spectral subspace of the free Dirac operator. In this paper, we propose to analyze a modified version of the Brown-Ravenhall operator in two-dimensions. More specifically, we consider the Brown-Ravenhall operator with an attractive potential given by a Bessel-Macdonald function (also known as $K_0$-potential) using the Foldy-Wouthuysen unitary transformation. The $K_0$-potential is derived of the parity-preserving QED$_3$ model as a framework for evaluation of the fermion-fermion interaction potential. We prove that the two-dimensional Brown-Ravenhall operator with $K_0$-potential is bounded from below when the coupling constant is below a specified critical value (a property also referred to as stability). As by product, it is shown that the operator is in fact positive. We also investigate the location and nature of the spectrum of the Brown-Ravenhall operator with $K_0$-potential.

1 Introduction

The quantum electrodynamics in three space-time dimensions (QED$_3$) has been drawn attention, since the works by Schonfeld, Deser, Jackiw and Templeton [1, 2], as a potential theoretical framework to be applied to quasi-planar condensed matter systems [3], namely high-$T_c$ superconductors [4, 5], quantum Hall [6], topological insulators [7], topological superconductors [8] and graphene [9, 10, 11]. Thenceforth, planar quantum electrodynamics models have been studied in many physical configurations: small (perturbative) and large (non perturbative) gauge transformations, abelian and non-abelian gauge groups, fermions families, even or odd under parity, compact space-times, space-times with boundaries, curved space-times, discrete (lattice) space-times, external fields and finite temperatures. In condensed matter systems, quasiparticles usually stem from two-particle (Cooper pairs), particle-quasiparticle (excitons) or two-quasiparticle (bipolarons) non-relativistic bound states.

Regarding the present article and the physics of new “Dirac materials”[12], in particular due to the importance of the Dirac’s equation in the description of graphene, one fact deserves to be highlighted: while the electrons in graphene are essentially confined in $d = 2$, the electric field clearly still acts in all three spatial dimensions. This has been used to justify
the choice of the Coulomb potential in $d = 3$, given by the Riesz potential operator $-\Delta^{-1/2}$, to treat the interaction between electrons in graphene. Bearing in mind this issue together with the fact that there are QED$_3$ models in which, fermion-fermion, fermion-antifermion or antifermion-antifermion scattering potentials – mediated by massive\(^a\) scalars or vector mesons – can be attractive\(^b\) and of $K_0$-type (a Bessel-Macdonald function) \([5, 3, 11, 14, 15, 16, 17]\), here we propose to discuss quantum relativistic effects in the fermions scattering by a short-range potential to treat the interaction between these particles in $d = 2$. We deal with the potential theory in spaces of Bessel potentials. Let us recall that the operator $(-\Delta + 1)^{-s/2}$, for $s > 0$, is called Bessel potential operator, and it has an integral representation with the following (Bessel) convolution kernel:

$$G_s(x) = \frac{1}{2^{d-1} \pi^{d/2} \Gamma(s/2)} K_{d-s}(|x|)|x|^{d-2s},$$

where $K_{d-s}$ is the modified Bessel function of the second kind also called Bessel-Macdonald function and $\Gamma$ denotes the Gamma function (see more details in Appendix). More specifically, in this article we study the two-dimensional Brown-Ravenhall operator, i.e., the projection of the Dirac operator perturbed by a $K_0$-potential (the two-dimensional Brown-Ravenhall operator in $d = 2$ perturbed by a Coulomb potential has been analyzed by Bouzouina \([18]\) and Walter \([19]\)). We prove that the Brown-Ravenhall operator with $K_0$-potential in $d = 2$ is bounded from below when the coupling constant is below a specified critical value (a property also referred to as stability). As a consequence, the operator is then self-adjoint. Furthermore, we show that the operator is in fact positive. We conclude with an analysis of the essential spectrum. A similar analysis in the non-relativistic case, more specifically considering the Schrödinger operator in two-dimensions with an attractive potential given by a Bessel-Macdonald function, has been performed in Ref. \([20]\). Due to the purpose of this paper, for the convenience of the reader, in Appendix A we gather a few facts about Bessel’s potential class, of which the potential $K_0$ is part.

Remark 1. The following notations will be used consistently throughout the article: $x, y, z, \ldots$ will denote points of the $d$-dimensional euclidean space $\mathbb{R}^d$, $|x - y|$ the euclidean distance of the points $x, y$, $|x| = |x - 0|$, $p, q, \ldots$ points of the dual space, $p \cdot x$ the inner product of the vectors $p$ and $x$. The gradient $\nabla \Psi$ of a differentiable function $\Psi$ is $\nabla \Psi = (\partial \Psi / \partial x_1, \ldots, \partial \Psi / \partial x_d)$. If $\Psi, \Phi \in L^2(\mathbb{R}^d)$, then we set $\langle \Psi, \Phi \rangle = \int_{\mathbb{R}^d} \overline{\Psi}(x) \Phi(x) \, dx$. $\Psi * \Phi$ will denote the convolution of $\Psi$ and $\Phi$, $\widehat{\Psi}$ the Fourier transform of $\Psi$. We are adopting the following convention for the Fourier transform on $\mathbb{R}^d$:

$$[\mathcal{F} \Psi](p) = \widehat{\Psi}(p) = \int_{\mathbb{R}^d} e^{ih^{-1}p \cdot x} \Psi(x) \, dx,$$

$$[\mathcal{F}^{-1} \Psi](x) = \Psi(x) = \frac{1}{(2\pi \hbar)^d} \int_{\mathbb{R}^d} e^{-ih^{-1}p \cdot x} \widehat{\Psi}(p) \, dp,$$

with $\Psi \in \mathcal{S}(\mathbb{R}^d)$. Here, by $\mathcal{S}(\mathbb{R}^d)$ we mean the set of all rapidly decreasing functions on $\mathbb{R}^d$. The $h$ is introduced to keep the units consistent with the physical interpretation. Of course, the invariance of space $\mathcal{S}(\mathbb{R}^d)$ under the Fourier transform implies that $\widehat{\Psi} \in \mathcal{S}(\mathbb{R}^d)$.

\(^a\)Otherwise, if the mediated quanta were massless, the interaction potential would be a logarithm-type (confining) potential \([13]\).

\(^b\)While the obtained scattering potentials for $p$-wave states showed up repulsives, the $s$-wave states (angular momentum state $\ell = 0$) show attractive.
2 The modified Brown-Ravenhall operator in $d = 2$

The Brown-Ravenhall operator [21] was initially proposed as an alternative to describe the fermion-fermion interaction via Coulomb potential and subject to relativity. This operator is defined in terms of the associated Dirac operator and the projection onto the positive spectral subspace of the free Dirac operator. In what follows, we consider a version of the two-dimensional Brown-Ravenhall operator with the $K_0$-potential

$$B(x) = \Lambda_+ (D(x) - \delta K_0(\beta|x|)) \Lambda_+ \quad \delta > 0 ,$$  

where $\delta$ is the coupling parameter taken to be, without the loss of generality, non-negative. The constants $\delta$ and $\beta$ shall depend on some model parameters, like coupling constants, characteristic lengths, mass parameters or vacuum expectation value of a scalar field. From expression $V(x) = -\delta K_0(\beta|x|)$ we see that $\delta$ has energy dimension and gives us an energy scale for the interaction among the two particles. In turn, the parameter $\beta$ has inverse length dimension, thus fixing a length scale, an interaction range, which is related to the mass of the boson-mediated quantum exchanged during the two particle scattering [3, 5, 11, 14, 15, 16, 17, 20].

In (2.1) the notation is as follows:

1. The operator $D$ is the free Dirac operator in $d = 2$; it is a first order operator acting on spinor-valued functions $\Psi(x) = (\psi_1(x), \psi_2(x))$, with 2 components, of the space variable $x = (x_1, x_2)$. We denote by $\mathbb{C}^2$ the 2-dimensional complex vector space in which the values of $\Psi(x)$ lie. $D$ has the form

$$D = -i\hbar \sigma \cdot \nabla + mc^2 \sigma_3 = -i\hbar c \left( \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} \right) + mc^2 \sigma_3 .$$

where $\hbar$ is the Planck constant, $m > 0$ is the mass of the fermionic particle under consideration, $c$ is the velocity of light and $\sigma = (\sigma_1; \sigma_2)$ and $\sigma_3$ are the Pauli $2 \times 2$-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The $\sigma_j$ matrices are introduced in view of making the Dirac operator a square root of the Laplace operator; they satisfy by construction the following anti-commutating relations:

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} \mathbb{I}_{2 \times 2} , \quad j, k = 1, 2 .$$

Remark 2. The free Dirac operator $D$ is essentially self-adjoint on the dense subspace $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ and self-adjoint on the Sobolev space $\mathfrak{Dom}(D) = H^1(\mathbb{R}^2; \mathbb{C}^2)$, its spectrum is given by

$$\sigma(D) = (-\infty, -mc^2] \cup [mc^2, +\infty) ,$$

and it has as form domain the space $\mathfrak{D}(D) = H^{1/2}(\mathbb{R}^2; \mathbb{C}^2)$ (see for example [22, Chapter 7] for more details on the spaces $H^1(\mathbb{R}^2; \mathbb{C}^2)$ and $H^{1/2}(\mathbb{R}^2; \mathbb{C}^2)$). Naturally, the negative spectrum is associated with antiparticles, in relativistic theories.

2. $\Lambda_+ \overset{\text{def}}{=} \chi_{(0, \infty)}(D)$, where $\chi_{(0, \infty)}$ is the characteristic function of the interval $(0, +\infty)$, is the projection of $L_2(\mathbb{R}^2; \mathbb{C}^2)$ onto the positive spectral subspace of $D$, i.e.,

$$\Lambda_+ = \frac{1}{2} \left( \mathbb{I}_{2 \times 2} + \frac{-i\hbar c \sigma \cdot \nabla + mc^2 \sigma_3}{\sqrt{-\hbar^2 c^2 \Delta + m^2 c^4}} \right) ,$$

where $\Delta$ is the laplacian operator on $\mathbb{R}^2$. The underlying Hilbert space in which $D$ acts is

$$\mathcal{H}_+ \overset{\text{def}}{=} \Lambda_+(L_2(\mathbb{R}^2) \otimes \mathbb{C}^2) = L_2(\mathbb{R}^2) \otimes \mathbb{C} .$$
3 The boundedness from below and positivity

In applications it is often very important to determine the lowest point of the spectrum of a self-adjoint operator. This problem makes sense only if the operator is bounded from below, since otherwise the spectrum extends to $-\infty$. In this section, the boundedness from below of the Brown-Ravenhall operator with $K_0$-potential is analyzed. We will first prove the following

**Theorem 3.1.** The operator \((2.1)\) is bounded from below if and only if
\[
\delta \leq \delta_c \overset{\text{def.}}{=} \frac{4c\hbar\beta^2}{\pi}.
\]

As a second result, we will show that the operator \((2.1)\) is in fact positive.

**Theorem 3.2.** Let \(E[\Psi] \overset{\text{def.}}{=} \langle \Psi, B\Psi \rangle\) be the energy associated with \(B\). Then, for \(\delta \leq \delta_c\)
\[
\inf_{\Psi \in \text{Dom}(B) \atop \|\Psi\|_2 = 1} E[\Psi] \geq mc^2 \left(1 - \frac{2\hbar\beta}{mc}\right),
\]
that is, the energy of the operator \(B\) is strictly positive.

3.1 Preamble: reduction of spinors

The first step in order to prove the Theorems 3.1 and 3.2 is a reduction of spinors. We will follow the same strategy as Zelati-Nolasco [23]: we now introduce the Foldy-Wouthuysen transformation (FW), given by a unitary transformation \(U_{\text{FW}}\) which transforms the free Dirac operator into the diagonal form (see details in [24, the case in \(d = 1 + 2\)] and [25, the case in \(d = 1 + 3\)])
\[
D_{\text{FW}} = U_{\text{FW}}DU_{\text{FW}}^{-1} = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix},
\]
where \(H = \sqrt{-\hbar^2c^4\Delta + m^2c^4}\) is the so-called quasi-relativistic operator (the relativistic (free) hamiltonian operator). This operator has been extensively studied for a long time (we refer to [26, 27, 28, 29, 30, 31, 32, 33]).

**Remark 3.** Let us recall that to the operator \(H\) can be defined for all \(\Psi \in H^1(\mathbb{R}^2; \mathbb{C}^2)\) as the inverse Fourier transform of the \(L_2\)-function \(\sqrt{c^2p^2 + m^2c^4} \hat{\Psi}(p)\) (where \(\hat{\Psi}\) denotes the Fourier transform of \(\Psi\) and \(p = |p|\)). To \(H\) we can associate the following quadratic form
\[
q_H(\Phi, \Psi) \overset{\text{def.}}{=} \langle \Phi, H\Psi \rangle = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \sqrt{c^2p^2 + m^2c^4} \overline{\Phi(p)}\hat{\Psi}(p) \, dp,
\]
which can be extended to all functions \(\Phi, \Psi \in \mathfrak{H}(H) = H^{1/2}(\mathbb{R}^2; \mathbb{C}^2)\), where
\[
H^{1/2}(\mathbb{R}^2; \mathbb{C}^2) = \left\{ \Psi \in L_2(\mathbb{R}^2; \mathbb{C}^2) \mid \int_{\mathbb{R}^2} (1 + p^2) |\hat{\Psi}(p)|^2 \, dp < \infty \right\}.
\]
It is known that \(H\) restricted on \(C_0^\infty(\mathbb{R}^d)\) is essentially self-adjoint, and that \(\sigma(H) = \sigma_{\text{ess}}(H) = [mc^2, \infty)\). An excellent mathematical, comprehensive and self-contained analysis of the spectral properties of the operators \(B, D\) and \(H\) (perturbed by the Coulomb potential) can be found in [34].
Therefore the positive energy subspace for \( D_{B} \) bounded from below. In that case, Eq.(3.2) will define a self-adjoint operator bounded from below. Of course, \( \langle B \rangle \) form (3.2) defines a self-adjoint operator the introduction of the operator \( B \) in the state \( \Psi \) in the FW-representation is associated to the quadratic form \( \psi \) \( \mathcal{U} \)

\[
\mathcal{H}_{+} = \left\{ \psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \in L_{2}(\mathbb{R}^{2}) \otimes \mathbb{C}^{2} \mid \psi \in L_{2}(\mathbb{R}^{2}) \otimes \mathbb{C} \right\} .
\]

In the FW-representation the associated quadratic form acting on \( \mathcal{H}_{+} \) is defined by

\[
\langle \varphi, B_{FW} \psi \rangle = \langle \varphi, H \psi \rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C})} + \langle \varphi, V_{FW} \psi \rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C})} ,
\]

for any \( \varphi, \psi \in H^{1/2}(\mathbb{R}^{2}; \mathbb{C}) \), where \( V_{FW} \psi = Q^{*} U_{FW} V U_{FW}^{-1} Q \psi \), with \( V(x) = -\delta K_{0}(\beta|x|) \) and \( Q : \mathbb{C} \to \mathbb{C}^{2} \), \( Q(z_{1}) = (z_{1}, 0) \), \( Q^{*} : \mathbb{C}^{2} \to \mathbb{C} \), \( Q^{*}(z_{1}, z_{2}) = z_{1} \), so that

\[
\langle \varphi, H \psi \rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C})} = \left\langle \Lambda_{+} U_{FW}^{-1} Q \varphi, D \Lambda_{+} U_{FW}^{-1} Q \psi \right\rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C}^{2})} = \left\langle \Lambda_{+} U_{FW}^{-1} \left( \begin{array}{c} \varphi \\ 0 \end{array} \right), D \Lambda_{+} U_{FW}^{-1} \left( \begin{array}{c} \psi \\ 0 \end{array} \right) \right\rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C}^{2})} ,
\]

and

\[
\langle \varphi, V_{FW} \psi \rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C})} = \left\langle \Lambda_{+} U_{FW}^{-1} Q \varphi, V \Lambda_{+} U_{FW}^{-1} Q \psi \right\rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C}^{2})} = \left\langle \Lambda_{+} U_{FW}^{-1} \left( \begin{array}{c} \varphi \\ 0 \end{array} \right), V \Lambda_{+} U_{FW}^{-1} \left( \begin{array}{c} \psi \\ 0 \end{array} \right) \right\rangle_{L_{2}(\mathbb{R}^{2}; \mathbb{C}^{2})} .
\]

Note that \( U_{FW}^{-1} Q \varphi = \Lambda_{+} U_{FW}^{-1} Q \varphi \in \Lambda_{+} L_{2}(\mathbb{R}^{2}) \otimes \mathbb{C}^{2} \) for any \( \varphi \in L_{2}(\mathbb{R}^{2}; \mathbb{C}) \). Therefore, if we define \( \psi = U_{FW}^{-1} Q \varphi \), then \( \Lambda_{+} \psi = \psi \) and

\[
\langle \varphi, B_{FW} \varphi \rangle = -\delta \left\langle \Lambda_{+} \psi, K_{0}(\beta|x|) \Lambda_{+} \psi \right\rangle = -\delta \left\langle \psi, K_{0}(\beta|x|) \psi \right\rangle .
\]

Using the above description, for any \( \psi \) in the positive spectral subspace, the expectation of \( B \) in the state \( \Psi \) in the FW-representation is associated to the quadratic form

\[
\langle \psi, B_{FW} \psi \rangle = \langle \psi, H \psi \rangle - \delta \langle \psi, K_{0}(\beta|x|) \psi \rangle .
\]

Hence, the transition from \( \Psi \in L_{2}(\mathbb{R}^{2}; \mathbb{C}^{2}) \) to the reduced spinor \( \psi \in L_{2}(\mathbb{R}^{2}; \mathbb{C}) \) through the introduction of the operator \( B_{FW} \) is possible because we are working in \( \mathcal{H}_{+} \). The quadratic form (3.2) defines a self-adjoint operator \( B_{FW} \) if we can show that the form \( \langle \psi, B_{FW} \psi \rangle \) is bounded from below. Of course, \( \langle \Psi, B \Psi \rangle \) is bounded from below if and only if \( \langle \psi, B_{FW} \psi \rangle \) is bounded from below. In that case, Eq.(3.2) will define a self-adjoint operator \( B_{FW} \).
3.2 Proof of Theorem 3.1

The proof below is inspired by the work of Cotsiolis-Tavoularis [35]. Firstly, let us estimate the term \( \langle \psi, K_0(\beta|x|)\psi \rangle \). According to Appendix A, if we combine (A.2) and (A.3), it follows that

\[
\langle \psi, K_0(\beta|x|)\psi \rangle \leq |\langle \psi, K_0(\beta|x|)\psi \rangle| = \left| \int_{\mathbb{R}^2} \overline{\psi}(x)(K_0 + \psi)(x) \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\psi}(x)K_0(\beta|x-y|)\psi(y) \, dy \, dx \right|
\]

\[
= \frac{1}{2} \int_{0}^{\infty} e^{-\frac{\pi}{4\beta} \eta^{-1}} \left[ \int_{\mathbb{R}^2} \overline{\psi}(x)e^{-\frac{\pi^2|x-y|^2}{\eta}}\psi(y) \, dy \right] \, d\eta .
\]

Applying the Young inequality (cf. [22, Theorem 4.2]) for \( p = r = 2 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \), we have

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\psi}(x)e^{-\frac{\pi^2|x-y|^2}{\eta}}\psi(y) \, dy \, dx \leq C_{2,1,2,2} \| \psi \|_{2}^2 \left\| e^{-\frac{\pi^2|y|^2}{\eta}} \right\|_{1} .
\]

We can calculate \( C_{2,1,2,2} \) with the help of Theorem 4.2 from Ref. [22]. According to this theorem, the sharp constant \( C_{p,q,r,m} \) equals \( (C_pC_qC_r)^m \), where

\[
C_p^m = p^{1/p} \left( \frac{1}{p'} \right)^{1/p'} , \quad \text{with} \quad \frac{1}{p} + \frac{1}{p'} = 1 .
\]

Hence, for \( p = r = 2 \) and \( q = 1 \), we obtain that \( C_{2,1,2,2} = 1 \). Moreover, if we set \( r = |x-y| \), it follows that (cf. the Table of Integrals of Gradshteyn-Ryzhik [36, 3.321, 3., p.336])

\[
\left\| e^{-\frac{\pi^2|y|^2}{\eta}} \right\|_{1} = \frac{1}{2} \beta .
\]

Therefore, using the Table of Integrals of Gradshteyn-Ryzhik [36, 3.361, 2.8, p.344], we obtain

\[
\langle \psi, K_0(\beta|x|)\psi \rangle \leq \frac{1}{4\beta} \left( \int_{0}^{\infty} e^{-\frac{\pi}{4\beta} \eta^{-1/2}} \, d\eta \right) \| \psi \|_{2}^2 = \frac{\pi}{2\beta} \| \psi \|_{2}^2 .
\]

Let us now estimate the term \( \langle \psi, H\psi \rangle \). Applying the Hölder inequality (cf. [22, Theorem 2.3]), we have

\[
\langle \psi, \sqrt{-h^2c^2\Delta + m^2c^4} \psi \rangle \leq \left\| \psi, \sqrt{-h^2c^2\Delta + m^2c^4} \psi \right\|_{2} \leq \| \psi \|_{2} \left\| \sqrt{-h^2c^2\Delta + m^2c^4} \psi \right\|_{2} .
\]

Next, let \( \psi, \varphi \in \mathcal{S}(\mathbb{R}^2; \mathbb{C}) \) – remember that \( \mathcal{S}(\mathbb{R}^2; \mathbb{C}) \subset H^1(\mathbb{R}^2; \mathbb{C}) \subset H^{1/2}(\mathbb{R}^2; \mathbb{C}) \) [22, Chapter 7] and this subset is dense (since \( \mathcal{S}(\mathbb{R}^2; \mathbb{C}) \) is dense in \( L_2(\mathbb{R}^n; \mathbb{C}) \). Then, by Parseval’s formula, we have

\[
\langle \psi, \varphi \rangle = \langle \hat{\psi}, \hat{\varphi} \rangle = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \left( c^2\varphi^2 + m^2c^4 \right)^{1/2} \hat{\psi}(p) \left( c^2\varphi^2 + m^2c^4 \right)^{-1/2} \hat{\varphi}(p) \, dp ,
\]

\[\text{6}\]
with \( p = |\mathbf{p}| \). Thus,

\[
|\langle \psi, \varphi \rangle|^2 \leq \left( \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} (c^2 p^2 + m^2 c^4) |\overline{\psi}(\mathbf{p})|^2 d\mathbf{p} \right) \left( \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} \frac{1}{c^2 p^2 + m^2 c^4} |\varphi(\mathbf{p})|^2 d\mathbf{p} \right)
\]

\[
= \left\| \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \right\|^2 \left( \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} \frac{1}{c^2 p^2 + m^2 c^4} |\varphi(\mathbf{p})|^2 d\mathbf{p} \right)
\]

\[
\leq \left\| \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \varphi \right\|^2 \left\| \frac{1}{(2\pi \hbar)^2} \int_{\mathbb{R}^2} \frac{1}{c^2 p^2 + m^2 c^4} d\mathbf{p} \right\|
\]

\[
= \left\| \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \varphi \right\|^2 \left\| \frac{1}{(2\pi \hbar)^2} \int_{0}^{\infty} \frac{1}{c^2 p^2 + m^2 c^4} \, d\mathbf{p} \right\| .
\]

Note that the constant factor \((2\pi \hbar \beta)^{-2}\) has been introduced in order to keep the units consistent with the physical interpretation. Applying the change of variable \( p/mc \to k \) the integral in the above expression takes the following form:

\[
\left| \frac{1}{2\pi c^2 \hbar^2 \beta^2} \int_{0}^{\infty} \frac{1}{k^2 + 1} \, dk \right| = \left| \frac{1}{4\pi c^2 \hbar^2 \beta^2} \int_{-\infty}^{\infty} \frac{1}{(k - i)(k + i)} \, dk \right| = \frac{1}{4\pi \hbar^2 \beta^2},
\]

by Cauchy’s Residue Theorem.

Now, setting \( \varphi = \psi \) in (3.3), we obtain

\[
4c^2 \hbar^2 \beta^2 \| \psi \|_2^4 \leq \| \psi \|_2 \left\| \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \right\|_2^2,
\]

i.e.,

\[
2c \hbar \beta \| \psi \|_2^2 \leq \| \psi \|_2 \left\| \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \right\|_2.
\]

Hence, the operator \( B_{FW} \) is bounded from below if

\[
\left\langle \psi, \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \right\rangle - \delta \langle \psi, K_0(\beta |\mathbf{x}|) \psi \rangle \geq 0,
\]

i.e., if and only if

\[
\delta \geq \frac{2c \hbar \beta}{\hbar} \left\| \psi \right\|_2^2 \geq 0,
\]

where \( \psi \in L_2(\mathbb{R}^2; \mathbb{C}) \). In other words, \( \langle \psi, B_{FW} \psi \rangle \) is lower bounded if

\[
\delta \leq \delta_c = \frac{4c \beta^2}{\pi}.
\]

Hence, the proof of Theorem 3.1 is concluded.

**Remark 4.** If \( \delta \leq \delta_c \), then the quadratic form \( \langle \psi, B_{FW} \psi \rangle \) defines (according to Friedrichs [37]) a unique self-adjoint operator \( B_{FW} \). Hence, since the quadratic form \( \langle \psi, B \psi \rangle \) is bounded from below once \( \langle \psi, B_{FW} \psi \rangle \) is bounded from below, the operators \( B \) and \( B_{FW} \) can be defined as the corresponding Friedrichs extensions. Thus, the critical coupling constant as occurring in Theorem 3.1 can be mathematically thought of as that coupling constant were a natural definition of self-adjointness ceases to exist.
3.3 Positivity of the operator $B$

To prove the positivity of operator (2.1), we make the following observation: in view of Theorem 3.1 the kinetic energy of $B$, after a reduction of spinors, it is simply equal to $\langle \psi, \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \rangle$, for all $\psi \in H^{1/2}(\mathbb{R}^2; \mathbb{C})$. Hence, it suffices to prove the equivalent of Theorem 3.2 for the form $\langle \psi, B_{FW} \psi \rangle$.

**Proposition 3.3.** Let $E[\psi] \overset{\text{def}}{=} \langle \psi, B_{FW} \psi \rangle$ be the energy associated with $B_{FW}$. Then, for $\delta \leq \delta_c$

$$\inf_{\psi \in H^{1/2}(\mathbb{R}^2; \mathbb{C})/\|\psi\|_2 = 1} E[\psi] \geq mc^2 \left( 1 - \frac{2\delta \beta}{mc} \right), \quad (3.4)$$

that is, the energy of the operator $B_{FW}$ is strictly positive.

**Proof.** As $\delta \leq \delta_c$ it will be sufficient to prove the strict positivity for $\delta = \delta_c$.

$$\langle \psi, B_{FW} \psi \rangle = \langle \psi, \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} \psi \rangle - \delta \langle \psi, K_0(\beta|x|) \psi \rangle$$

$$\geq mc^2 \|\psi\|^2_2 - \frac{\delta \pi}{2\beta} \|\psi\|^2_2$$

$$= mc^2 \left( 1 - \frac{\delta \pi}{2\beta mc^2} \right) \|\psi\|^2_2$$

$$= mc^2 \left( 1 - \frac{2\delta \beta}{\delta_c mc} \right) \|\psi\|^2_2.$$

Therefore, for $\psi \in H^{1/2}(\mathbb{R}^2; \mathbb{C})$ (with $\|\psi\|_2 = 1$) and $\delta = \delta_c$, we obtain

$$\langle \psi, B_{FW} \psi \rangle \geq mc^2 \left( 1 - \frac{2\delta \beta}{mc} \right).$$

The result implies that $B_{FW}$ is non-negative and has no eigenvalue at 0 when the coupling constant $\delta$ does not exceed or equal the specified critical value $\delta_c$.

**Remark 5.** The positivity of the two-dimensional Brown-Ravenhall operator with Coulomb potential has been proven by Walter [19].

4 Locating the essential spectrum of the operator $B$

In this section we shall determine $\sigma_{\text{ess}}(B)$. First, we note that the map $\langle \Psi, B \Psi \rangle \rightarrow \langle \psi, B_{FW} \psi \rangle$, where $\Psi \in L_2(\mathbb{R}^2; \mathbb{C}^2)$ and $\psi \in L_2(\mathbb{R}^2; \mathbb{C})$, determines a unitary equivalence between the operators $B$ and $B_{FW}$, hence, they have the same spectral properties. This leads us to the following result:

**Theorem 4.1.** Assume that $0 < \delta \leq \delta_c$. Then for the essential spectrum of the Brown-Ravenhall operator (2.1) one has $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(B_{FW}) = [mc^2, \infty)$.

The proof of Theorem 4.1 depends fundamentally on the following (see [38, p.143]).

**Definition 4.2.** A potential function $V(x)$ is called a Kato potential if $V$ is real and $V \in L_2(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, where the $\varepsilon$ indicates that for any $\varepsilon > 0$, we can decompose $V = V_1 + V_2$ with $V_1 \in L_2(\mathbb{R}^d)$ and $V_2 \in L_\infty(\mathbb{R}^d)$, with $\|V_2\|_\infty < \varepsilon$. 

8
Lemma 4.3. The potential \( V(x) = -\delta K_0(\beta |x|) \) is a Kato potential.

Proof. For any \( \varepsilon > 0 \), let \( \chi_\varepsilon(|x|) \) be the function that is 1 on \( \{ x \mid |x| \leq (\delta \varepsilon)^{-1} \} \) and that vanishes outside \( \{ x \mid |x| < 2(\delta \varepsilon)^{-1} \} \). Then, we decompose the potential \( V(x) \) as

\[
V(x) = -\delta \chi_\varepsilon(|x|) K_0(\beta |x|) - \delta (1 - \chi_\varepsilon(|x|)) K_0(\beta |x|) = V_1(x) + V_2(x).
\]

By using the Table of Integrals of Gradshteyn-Ryzhik [36, 6.521, 6.4, p.665], in polar coordinates, we obtain

\[
\|V_1\|_2 = \left( \delta^2 \int_{\mathbb{R}^2} |\chi_\varepsilon(|x|) K_0(\beta |x|)|^2 d^2x \right)^{1/2} = \left( 2\pi \delta^2 \int_0^\infty \chi_\varepsilon^2(\rho) K_0^2(\beta \rho) \rho d\rho \right)^{1/2} \lesssim \frac{\pi^{1/2} \delta}{\beta}.
\]

Therefore, \( V_1 \in L_2(\mathbb{R}^2) \). On the other hand, considering that at the limit when \( |x| \to \infty \) we have

\[
K_0(\beta |x|) \simeq \sqrt{\frac{\pi}{2\beta |x|}} e^{-\beta |x|},
\]

i.e., \( K_0(\beta |x|) \) is a bounded function vanishing at \( \infty \), then \( V_2 \in L_\infty(\mathbb{R}^2) \), with

\[
\sup_{x \in \mathbb{R}^2} |\delta (1 - \chi_\varepsilon(|x|)) K_0(\beta |x|)| \leq \varepsilon.
\]

Hence, the potential \( V(x) = -\delta K_0(\beta |x|) \) is a Kato potential. \( \square \)

Proof of Theorem 4.1. Let us start by defining \( B_{FW0} = H = \sqrt{-\hbar^2 c^2 \Delta + mc^2} \). From Remark 3, we know that \( \sigma_{ess}(B_{FW0}) = [mc^2, \infty) \). On the other hand, in order to locate \( \sigma_{ess}(B_{FW}) \), where \( B_{FW} = B_{FW0} + V \) (with \( V(x) = -\delta K_0(\beta |x|) \)), we study the resolvent operator \( (B_{FW} - \lambda \mathbb{I})^{-1} \) for a some \( \lambda \notin \sigma(B_{FW}) \).

By the second resolvent equation, for any value of \( \lambda \in \rho(B_{FW}) \cap \rho(B_{FW0}) \), where \( \rho(B_{FW}) \) and \( \rho(B_{FW0}) \) are the resolvent sets of \( B_{FW} \) and \( B_{FW0} \), respectively, we have

\[
(B_{FW} - \lambda \mathbb{I})^{-1} - (B_{FW0} - \lambda \mathbb{I})^{-1} = (B_{FW} - \lambda \mathbb{I})^{-1} V(B_{FW0} - \lambda \mathbb{I})^{-1}, \tag{4.1}
\]

(we recall that since \( B_{FW} \) is a positive self-adjoint operator, once \( \langle \psi, B_{FW} \psi \rangle \) is bounded from below if \( \delta \leq \delta_0 \), \( \lambda \in \rho(B_{FW}) \) if and only if \( (B_{FW} - \lambda \mathbb{I}) : \text{Dom}(B_{FW}) \to \mathbb{H}_+ \) is bijective and its inverse is bounded). We will show that \( (B_{FW} - \lambda \mathbb{I})^{-1} - (B_{FW0} - \lambda \mathbb{I})^{-1} \) is compact as an operator on \( L_2(\mathbb{R}^2; \mathbb{C}) \) and therefore \( \sigma_{ess}(B_{FW}) = \sigma_{ess}(B_{FW0}) = [mc^2, \infty) \) by Weyl’s criterion [39, Theorem XIII.14].

In view of the self-adjointness of \( B_{FW} \) follows that \( (B_{FW} - \lambda \mathbb{I})^{-1} \) is bounded by

\[
\|B_{FW} - \lambda \mathbb{I}\|^{-1} \lesssim |\text{Im} \lambda|^{-1},
\]

(cf. Ref.[38, Corollary 5.7]). Thus, it remains for us to show that \( V(B_{FW0} - \lambda \mathbb{I})^{-1} \) is compact.

Taking into account the basic fact of inclusion [22, Chapter 7],

\[
H^1(\mathbb{R}^2; \mathbb{C}) \subset H^{1/2}(\mathbb{R}^2; \mathbb{C}),
\]

i.e., since the range of \( (B_{FW0} - \lambda \mathbb{I})^{-1} \), namely \( \text{Dom}(H) = H^1(\mathbb{R}^2; \mathbb{C}) \), lies in the form domain \( \mathcal{D}(H) = H^{1/2}(\mathbb{R}^2; \mathbb{C}) \), we just show that \( V(x) = -\delta K_0(\beta |x|) \) is a compact operator from \( H^1(\mathbb{R}^2; \mathbb{C}) \) to \( L_2(\mathbb{R}^2; \mathbb{C}) \). This is enough to guarantee that the perturbation \( V \) does not modify the essential spectrum of the operator \( H \). For this we will need the following compactness theorem, the proof of which can be found in [40, Theorem 1.10, p.355].
Theorem 4.4 (Rellich compactness criterion). Let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open and bounded set. Then for any bounded sequence $(\psi_k)_{k \in \mathbb{N}}$ in $H_0^1(\Omega)$ there exists a subsequence $(\psi_{k_\ell})_{\ell \in \mathbb{N}}$ in $H_0^1(\Omega)$ that converges strongly in $L_2(\Omega)$.

Remark 6. $H_0^1(\Omega)$, with the scalar product $(\cdot, \cdot)_{H^1(\Omega)}$ defined by

$$
(\psi, \varphi)_{H^1(\Omega)} = (\psi, \varphi)_{L_2(\Omega)} + \sum_{k=1}^d (D_k \psi, D_k \varphi)_{L_2(\Omega)},
$$

is a Hilbert space by construction, being a closed subspace of the Hilbert space $H^1(\Omega)$ (cf. [41, Lemma 22.1, p.112]).

Returning to the proof of Theorem 4.1, select a sequence $(\varphi_{\ell})_{\ell \in \mathbb{N}} \subset L_2(\mathbb{R}^2; \mathbb{C})$ such that $\varphi_{\ell} \xrightarrow{\infty} 0$. Assume that

$$
\psi_{\ell} = (B_{FW0} - \lambda \mathbb{1})^{-1} \varphi_{\ell},
$$

where $\psi_{\ell} \in \mathcal{D}(B_{FW0})$. The fact that $(\varphi_{\ell})_{\ell \in \mathbb{N}}$ converges weakly implies that $(\varphi_{\ell})_{\ell \in \mathbb{N}}$ is bounded in $L_2(\mathbb{R}^2; \mathbb{C})$. As $(B_{FW0} - \lambda \mathbb{1})^{-1}$ is bounded, the sequence $(\psi_{\ell})_{\ell \in \mathbb{N}}$ is also bounded in $\mathcal{D}(B_{FW0})$ and converges weakly to zero in $\mathcal{D}(B_{FW0}) \subset L_2(\mathbb{R}^2; \mathbb{C})$.

Now, according to Lemma 4.3, for any $\varepsilon > 0$, we can decompose $V = V_1 + V_2$, where $V_1 \in L_2(\mathbb{R}^2)$ and $V_2 \in L_\infty(\mathbb{R}^2)$. Then, applying the Hölder inequality,

$$
\|V_2 \psi_{\ell}\|_2 \leq \|V_2\|_\infty \|(B_{FW0} - \lambda \mathbb{1})^{-1} \varphi_{\ell}\|_2 \leq \varepsilon \|\varphi_{\ell}\|_2,
$$

and it follows that $\|V_2(B_{FW0} - \lambda \mathbb{1})^{-1}\| \leq \varepsilon$. Next, observe that $V_1$ is bounded so that there is $K > 0$ such that $|V_1(x)| \leq K$, for all $x \in \mathbb{R}^2$. Let $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$ be the open ball with radius $R > 0$ around zero in $\mathbb{R}^2$ and $\theta_R \in C_0^\infty(B_{2R})$ be such that $0 \leq \theta_R \leq 1$, $\theta_R \circ B_R = 1$. Then, $(\theta_R \psi_{\ell})_{\ell \in \mathbb{N}}$ is bounded in $H^1(B_{2R}; \mathbb{C})$. We recall that Rellich’s Compactness Theorem gives us that the inclusion

$$
H^1(B_{2R}; \mathbb{C}) \hookrightarrow L_2(B_{2R}; \mathbb{C}),
$$

is compact. This means that there exists a subsequence $(\theta_R \psi_{k_\ell})_{k \in \mathbb{N}} \subset (\theta_R \psi_{\ell})_{\ell \in \mathbb{N}}$ such that $(\theta_R \psi_{k_\ell})_{k \in \mathbb{N}}$ converges strongly in $L_2(B_{2R}; \mathbb{C})$. We shall show that $\|V_1 \psi_{\ell}\|$ converges to 0. With the help of the function $\theta_R$ defined above, it follows that

$$
\|V_1 \psi_{\ell}\|_2 \leq \|V_1 \theta_R \psi_{\ell}\|_2 + \|V_1(1 - \theta_R) \psi_{\ell}\|_2 \leq \|V_1 \theta_R \psi_{\ell}\|_2 + \|V_1(1 - \theta_R)\|_\infty \|\psi_{\ell}\|_2.
$$

The first term can be made smaller than $\varepsilon$ by choosing $\ell$ large since $V_1$ is bounded, i.e., there is $\ell_0 \in \mathbb{N}$ such that $\|\theta_R \psi_{\ell}\| \leq \varepsilon/K$ for all $\ell \geq \ell_0$. As $\psi_{\ell}$ converges weakly to zero, there is a positive constant $M$ such that $\|\psi_{\ell}\|_2 \leq M$. Hence, by assumption, the second term can be made smaller than $\varepsilon$ times a positive constant by choosing $R$ large. Consequently, for $\ell$ large enough, the right-hand side of (4.3) is smaller than $\varepsilon$ times a positive constant. This implies that $V_1 \psi_{\ell} \rightarrow 0$ in $L_2(\mathbb{R}^2; \mathbb{C})$ as $\ell \rightarrow \infty$, since $\varepsilon$ is arbitrary, and hence $V_1(B_{FW0} - \lambda \mathbb{1})^{-1}$ is compact.

Finally, by (4.2) we have

$$
\|V(B_{FW0} - \lambda \mathbb{1})^{-1} - V_1(B_{FW0} - \lambda \mathbb{1})^{-1}\| < \varepsilon,
$$

so $V(B_{FW0} - \lambda \mathbb{1})^{-1}$ is approximated by compact operators and is itself compact. Thus, we have that $(B_{FW} - \lambda \mathbb{1})^{-1} - (B_{FW0} - \lambda \mathbb{1})^{-1}$ is equal to the product of a compact operator and a bounded operator, and since the product of a compact operator with a bounded operator is compact, this proves the required compactness on $L_2(\mathbb{R}^2; \mathbb{C})$. This means that $B_{FW}$ and $B_{FW0}$ have the same essential spectrum by Weyl’s criterion, namely $\sigma_{\text{ess}}(B_{FW}) = \sigma_{\text{ess}}(B_{FW0}) = [mc^2, \infty)$ [39, Theorem XIII.14].
In many applications, a self-adjoint operator has a number of eigenvalues below the bottom of the essential spectrum. Thus, under the assumptions of Theorem 4.1, a natural question to ask is whether it is possible that $B_{FW}$ has discrete eigenvalues below $mc^2$. We recall that if this is the case, the eigenvalues of $B_{FW}$ are characterized by the min-max principle. This theorem establishes that since $B_{FW}$ is self-adjoint, and if $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ are eigenvalues of $B_{FW}$ below the essential spectrum, respectively, the infimum of the essential spectrum, once there are no more eigenvalues left, then

$$\lambda_n = \inf_{\psi, \ldots, \psi} \sup_{\psi \in \Omega(\psi_1, ..., \psi_n)} \langle \psi, B_{FW} \psi \rangle,$$

where $\Omega(\psi_1, ..., \psi_n) = \{ \psi \in \text{Dom}(B_{FW}) \mid \|\psi\| = 1, \psi \in \text{span}\{\psi_1, ..., \psi_n\} \}$. Hence, if there exists $\psi \in \text{Dom}(B_{FW})$ such that $\langle \psi, B_{FW} \psi \rangle < mc^2$, then $B_{FW}$ has at least one eigenvalue below the bottom of the essential spectrum, $\sigma_{ess}(B_{FW})$. Indeed, if this were not true then $\sigma(B_{FW}) \cap (0, mc^2) = \varnothing$ meaning that $\sigma(B_{FW}) \subset [mc^2, \infty)$. By the spectral theorem, this would imply that $B_{FW} \geq 0$, i.e., $\langle \varphi, B_{FW} \varphi \rangle \geq 0$ for all $\varphi \in \text{Dom}(B_{FW})$ in contradiction to the assumption $\langle \psi, B_{FW} \psi \rangle < mc^2$. Moreover, in view of Proposition 3.3, we know also that the lower bound of the operator $B_{FW}$ is positive and hence all discrete eigenvalues belong to the interval $(0, mc^2)$. Since each eigenvalue $\lambda_j$ has finite multiplicity, there must be an infinite number of eigenvalues which accumulate at $mc^2$. This proves the following

**Proposition 4.5.** Consider the operator $B_{FW}$. If there exists $\psi \in \text{Dom}(B_{FW})$ such that $\langle \psi, B_{FW} \psi \rangle < mc^2$, then $B_{FW}$ has at least one eigenvalue in $(0, mc^2)$.

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**Appendix A: Bessel potential**

In this appendix we introduce spaces of Bessel potentials. Let us recall that the operator $(-\Delta + \mathbb{1}^{-s/2})$, for $s > 0$, is called Bessel potential operator. Thus, given any $s > 0$, the Bessel potential $G_s$ is defined to be that function whose Fourier transform $\hat{G}_s$ is given by

$$\hat{G}_s(p) = \frac{1}{(1 + p^2)^{s/2}} \quad \text{with} \quad p \in \mathbb{R}^d \quad \text{and} \quad p = |p|.$$

The following is a simple proof of this result for the $K_0$-potential in $\mathbb{R}^2$. The proof is based on the spherical symmetry of the $K_0$-function.

**Proposition A.6.** Given that $K_0$-potential is spherically symmetric, then

$$[\mathcal{F} K_0](p) = \hat{K}_0(p) = \frac{2\pi h^2}{p^2 + h^2 \beta^2} \quad \text{with} \quad p = |p|.$$

**Proof.** We started by writing

$$\hat{K}_0(p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_0(\beta \sqrt{x_1^2 + x_2^2}) e^{ih^{-1}(p_1 x_1 + p_2 x_2)} dx_1 dx_2.$$

For a potential depending only upon $r$ (central force field) it is expedient to introduce polar coordinates by the formulae $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, with $r = |x|$, and similarly in the
momentum domain by the formulae $p_1 = p\cos \varphi, \ p_2 = p\sin \varphi$, with $p = |p|$. It then follows that the Fourier transform in $d = 2$ can be written as

$$\hat{K}_0(p) = \int_0^\infty \int_{-\pi}^\pi K_0(\beta r)e^{i\beta r \cos(\varphi-\theta)}r dr d\theta$$

$$= \int_0^\infty K_0(\beta r)r dr \int_{-\pi}^\pi e^{i\beta r \cos(\varphi-\theta)}d\theta .$$

(A.1)

Using the integral definition of the zeroth-order Bessel function,

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{ix\cos(\varphi-\theta)}d\theta = \frac{1}{2\pi} \int_{-\pi}^\pi e^{ix\cos \eta}d\eta ,$$

Eq.(A.1) can then be written as

$$\hat{K}_0(p) = 2\pi \int_0^\infty K_0(\beta r)J_0(h^{-1}pr) r dr .$$

The conclusion follows from the Table of Integrals of Gradshtein-Ryzhik [36, 6.521, 2.10, p.665].

We quote below without proof the basic properties of $G_s$ relevant to our development (more details can be found in Refs. [43, 44, 45, 46]):

(i) if $s > 0$, then $G_s$ is a positive function in $L_1(\mathbb{R}^d)$ which is analytic except at 0 and is given by

$$G_s(x) = \frac{1}{2^{d-s/2} \pi^{d/2} \Gamma(s/2)} K_{d-s}(|x|)|x|^{d-s} ,$$

(A.2)

where $K_{d-s}$ is the modified Bessel function of the second kind also called Bessel-MacDonald function and $\Gamma$ denotes the Gamma function. The Bessel kernel can also be represented for $x \in \mathbb{R}^d \setminus \{0\}$ by the integral formula

$$G_s(x) = \frac{1}{(4\pi)^{s/2} \Gamma(s/2)} \int_0^\infty e^{-\frac{(\eta - |x|)^2}{4\eta}} e^{-\eta} \eta^{-(1+s/2)} d\eta .$$

(A.3)

(ii) $G_s * G_\tau = G_{s+\tau}$ if $\tau > 0$.

(iii) as $|x| \to 0$,

$$G_s(x) \simeq \begin{cases} 
\frac{\Gamma(d-s)}{2^{d-\frac{s}{2}} \pi^{d/2} \Gamma(s/2)} |x|^{s-d} & \text{if } 0 < s < d , \\
\frac{1}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \ln \frac{1}{|x|} & \text{if } s = d , \\
\frac{\Gamma(s-d)}{2^{d-\frac{s}{2}} \pi^{d/2} \Gamma(s/2)} & \text{if } s > d .
\end{cases}$$

(iv) as $|x| \to \infty$,

$$G_s(x) \simeq e^{-|x|} \frac{2^{d-s-1} \pi^{-\frac{d-s}{2}} \Gamma(s/2)|x|^{-\frac{d+s}{2}}}{2^{d-\frac{s}{2}} \pi^{d/2} \Gamma(s/2)} .$$
(v) there exists $c > 0$ such that for all $x \in \mathbb{R}^d$ and all $s \in (0, d)$

$$G_s(x) \simeq \frac{1}{|x|^{d-s}} e^{-c|x|} .$$

Closely related to the operator $(-\Delta + I)^{-s/2}$ is the Riesz potential operator, $(-\Delta)^{-s/2}$, which has an integral convolution kernel of the form

$$G_s(x) = \frac{\Gamma((d-s)/2)}{2^s \pi^{d/2} \Gamma(s/2)} \frac{1}{|x|^{d-s}} , \quad \text{if } s < d , \ 0 < s < 2 .$$

The Bessel potential is a potential similar to the Riesz potential but with better decay properties at infinity. Comparatively, the Yukawa potential is a particular case of a Bessel potential for $s = 2$ in $d = 3$, while the Coulomb potential is an example of a Riesz potential also in $d = 3$. Note that according to properties (iii) and (iv), for $s = d = 2$, the $K_0$-potential behaves as if it were the Yukawa potential in $d = 2$ (cf. [47, p.1006, Eq.(2.21a)]).

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