SEMISEMNORMALITY, CANONICAL MODULES, AND REGULARITY OF CUT POLYTOPES

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ABSTRACT. Motivated by a conjecture of Sturmfels and Sullivant we study normal cut polytopes. After a brief survey of known results for normal cut polytopes it is in particular observed that for simplicial and simple cut polytopes their cut algebras are normal and hence Cohen–Macaulay. Moreover, seminormality is considered. It is shown that the cut algebra of $K_5$ is not seminormal which implies again the known fact that it is not normal. For normal Gorenstein cut algebras and other cases of interest we determine their canonical modules. The Castelnuovo–Mumford regularity of a cut algebra is computed for various types of graphs and bounds for it are provided if normality is assumed. As an application we classify all graphs for which the cut algebra has regularity less than or equal to 4.

1. INTRODUCTION

The MAXCUT problem in combinatorial optimization is one of the 21 NP-complete problems of Karp [24] with interesting applications. For an overview of these we refer to [13, 14]. Closely related to this is the polyhedral point of view of the story, i.e. the study of the cut polytope $\text{Cut}^\square(G)$ of a graph $G$ and its geometric properties (see Section 2 for definitions). Facts based on cases of graphs where polynomial time algorithms for MAXCUT are known or other special knowledge exists (see, e.g., [1, 4, 23]) led to a larger number of beautiful results where all or at least many facet defining inequalities of cut polytopes and related objects can be explicitly described. See [2, 9, 12, 29], the book of Deza–Laurent [15] and also [10, 11] for statements on enumerations of facets.

The connection to algebraic geometry and commutative algebra was initiated by Sturmfels and Sullivant, who introduced the cut algebra $\mathbb{K}[G]$ over a field $\mathbb{K}$ and its defining cut ideal $I_G$ for a graph $G$ (see [39]). Observe that $\mathbb{K}[G]$ is the toric algebra associated to the 0/1-polytope $\text{Cut}^\square(G)$. See Section 2 for these notations and [6] as a general reference on toric algebra and geometry. Note that in [39] also applications of cut objects to algebraic statistics are discussed. From purely algebraic prospects several conjectures of that paper are of great interest, from which we summarize the following combined one:

Conjecture 1.1. ([39]) The following statements are equivalent:

(i) $G$ is $K_5$-minor-free;
(ii) $I_G$ is generated in degrees less than or equal to 4;
(iii) $\mathbb{K}[G]$ is Cohen–Macaulay;
(iv) $\mathbb{K}[G]$ is normal.

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In [39] it was already observed that for (ii), (iv) it is necessary that (i) holds. By a theorem of Hochster [20] one knows that (iv) implies (iii). Further partial results are discussed in [39] as well as in [17, 28, 32, 35]. The characterizations of \(\mathbb{K}[G]\) being a complete intersection in [35, Theorem 6.9] and of \(\mathbb{K}[G]\) being a normal Gorenstein algebra in [33, Theorem 3.4] yield further support for the conjecture. Beside these facts Conjecture 1.1 is open and remains a challenging task. Additionally to the mentioned work, further properties of \(\mathbb{K}[G]\) and \(I_G\) were studied in particular in [34, 36, 37].

The main goal of this paper is to study \(\mathbb{K}[G]\) in the normal situation and to consider algebraic properties of interest. We start in Section 3 with a brief survey of known facts related to Conjecture 1.1, which deepens the previous discussion. Our first contribution is in Corollary 3.3 the observation that using results of [32] one can give a new short proof that for a ring graph \(G\) the algebra \(\mathbb{K}[G]\) is normal. This is interesting since the first proof of this in [28] contains an error as was pointed out in [36]. We conclude Section 3 by discussing examples where \(G\) is an outerplanar graph and where \(\text{Cut}^\square(G)\) is a simplicial or simple polytope, respectively. We expect that further polytopal properties of \(\text{Cut}^\square(G)\) should provide further support for Conjecture 1.1.

A natural approach to relax the property normality is to study seminormal algebras. See [3, 6, 8, 19, 21, 30, 31, 41, 42, 43] for results related to seminormality in commutative algebra and related areas. In Section 4, we present in Theorem 4.3 a proof that \(\mathbb{K}[K_5]\) is not seminormal, which complements the discussion in [35, Example 7.2] and reproves also some (computer based) facts in [39, Table 1]. If a cut algebra is seminormal, then the underlying graph has to be \(K_5\)-minor-free (see Corollary 4.4). Remarkably, using this fact and other clever arguments related to very ampleness, Lason and Michałek [27] were recently able to prove that \(\mathbb{K}[G]\) is seminormal if and only if it is normal. An interesting consequence of this fact is that results of Ohsugi [32] hold in the seminormal situation like that seminormality is a minor-closed property of the underlying graphs.

For a Cohen–Macaulay algebra there is always an associated canonical module, which is of great importance for the algebra itself (see [6, 7] for details). The canonical module of cut algebras has not been studied before. We determine this module for normal Gorenstein cut algebras of graphs in Theorem 5.2 and other cases of interest. The remaining part of this work is devoted in Section 6 to study the Castelnuovo–Mumford regularity of cut algebras. For arbitrary standard graded algebras this is one of its key homological invariants. First results related to this invariants were obtained in [28] and [35]. See also [34] for relevant facts. We compute the Castelnuovo–Mumford regularity for various types of graphs including ring graphs (see Theorem 6.6) and provide upper as well as lower bounds of it for all normal cut algebras. Extending the classification in [35] of graphs for which the cut algebra has regularity 0 or 1, we classify in Theorem 6.10 all graphs for which their cut algebra has regularity less than or equal to 4.

Finally, we formulate some research problems in Section 7.

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2. Preliminaries

In this section we recall basic facts and notation used in the following. For further details to graph theory we refer to the book of Diestel [16]. A general reference for monoids, polytopes, their algebras and properties of them is the book of Bruns and Gubeladze [6].

2.1. Graphs. Let \( G = (V, E) \) be a graph with vertex set \( V(G) = V \neq \emptyset \) and edge set \( E(G) = E \). We always consider undirected and simple graphs (i.e. without multiple edges and without loops). An edge \((v, w) \in E\) is also denoted by \( vw \).

A graph \( H \) is a subgraph of a graph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). The subgraph is called induced if \( E(H) \) contains all possible edges \( vw \in E(G) \) with \( v, w \in V(H) \). In this case we also denote \( H \) by \( G_W \) where \( \emptyset \neq W = V(H) \subseteq V(G) \).

A cycle \( C = C_n \) of length \( n \) is a graph with \( n \) vertices \( \{v_1, \ldots, v_n\} \) and edges \( \{v_i, v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n, v_1\} \). A triangle is a cycle of length 3. A cycle of a graph \( G \) is a cycle \( C \) of some length which is a subgraph of \( G \). A chord of such a cycle is an edge \( vw \in E(G) \setminus E(C) \) with \( v, w \in V(C) \). A cycle of a graph is induced if and only if it is a cycle without chords. A graph is said to be chordal if all its induced cycles are triangles. An edge \( e \) of a graph is called a bridge if there does not exist a cycle which contains \( e \).

A complete graph with \( n \) vertices is a graph \( G \) with \( |V(G)| = n \) and \( E(G) = \{vw : v, w \in V(G), v \neq w\} \). We denote this graph by \( K_n \). An induced complete subgraph of a graph \( G \) is also called a clique of it. A graph \( G \) is a bipartite graph if there exists a partition \( V(G) = V_1 \cup V_2 \) with \( V_i \cap V_j = \emptyset \) such that any edge of \( G \) is of the type \( vw \) with \( v \in V_i, w \in V_j \). Recall that \( G \) is bipartite if and only if \( G \) has no cycle of odd length. A complete bipartite graph \( G \) is a bipartite graph \( G \) as above where any \( v \in V_1, w \in V_2 \) yield an edge \( vw \in E(G) \). If \( |V_1| = m, |V_2| = n \), then this type of graph is also denoted by \( K_{m,n} \).

Edge deletion and edge contraction of a graph \( G \) at an edge \( e \in E(G) \) are denoted by \( G \setminus e \) and \( G/e \). A graph is said to be a minor of \( G \) if it can be obtained from it by a sequence of edge deletions and edge contractions. Recall that if \( K_n \) is a minor of \( G \), then \( K_n \) can be obtained from \( G \) by using only edge contractions. By Kuratowski’s Theorem a graph is planar if and only if it is \( K_5 \)- and \( K_{3,3} \)-minor-free (after removing isolated vertices). Maximal planar graphs are called triangulations. An outerplanar graph is a graph which has a planar drawing such that all vertices belong to the outer face of that drawing.

Let \( G_1 \) and \( G_2 \) be two graphs such that \( H = (G_1) \cap (G_2) = (G_2) \cap (G_1) \). The new graph \( G_1 \# G_2 = G_1 \# H G_2 \) with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \) is called the \( H \)-sum of \( G_1 \) and \( G_2 \). If \( H = K_{n+1} \), then \( G_1 \# G_2 = G_1 \# K_{3,n+1} \) is called an \( n \)-clique-sum or simply an \( n \)-sum of graphs.

2.2. Polytopes and Cones. A hyperplane \( H = H_{a,b} = \{x \in \mathbb{R}^d : a \cdot x - b = 0\} \), where \( a \in \mathbb{R}^d, b \in \mathbb{R} \), induces two (closed) half-spaces \( H^+ \) and \( H^- \) defined by \( a \cdot x - b \geq 0 \) and \( a \cdot x - b \leq 0 \). Polyhedra are intersections of finitely many half-spaces. Thus, such a polyhedron \( P \) is the set of solutions of a system of inequalities, which we write as \( A x \preceq b \) where \( A \) is a real-valued \( n \times d \)-matrix for some \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \) and \( b \in \mathbb{R}^n \). By Weyl–Minkowski, bounded polyhedra are exactly the polytopes, i.e. convex hulls \( \text{conv}(x_1, \ldots, x_m) \) of finitely many points in \( x_1, \ldots, x_m \in \mathbb{R}^d \). If a polyhedron \( P \) is defined by linear inequalities, i.e. \( b = 0 \), then we obtain all finitely generated (convex) cones of the form \( \text{cone}(y_1, \ldots, y_m) = \mathbb{R}_{\geq 0} y_1 + \cdots + \mathbb{R}_{\geq 0} y_m \) for \( y_1, \ldots, y_m \in \mathbb{R}^d \).
Let $P$ be a polyhedron. Its \textit{dimension} $\dim P$ is the dimension of the affine hull of $P$. A hyperplane $H = H_{a,b}$ is said to be a \textit{support hyperplane} of $P$ if all $x \in P$ satisfy $a \cdot x \leq b$ and $H \cap P \neq \emptyset$. The intersection $F = H \cap P$ is called a \textit{face} of $P$ and $H$ is also said to be a \textit{support hyperplane} associated with $F$. Observe that here $P$ is also a face of itself. Note that faces of polytopes are again polytopes and faces of cones are cones. Faces of dimension 0 and 1 are called \textit{vertices} and \textit{edges}, respectively. A face of dimension $\dim P - 1$ is a \textit{facet} of $P$. Then the inequality $a \cdot x \leq b$ of the support hyperplane is also said to be \textit{facet defining}.

We introduce further notation of polytopes. A polytope of dimension $d$ is a \textit{simplex} if it is the convex hull of $d + 1$ affinely independent points. It is \textit{simple} if each vertex of it is contained in exactly $d$ facets. We say that a polytope is \textit{simplicial} if each facet is a simplex. A polytope $P$ is a \textit{lattice polytope} if its vertices are lattice points of the integral lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$. The set of all lattice points in $P \cap \mathbb{Z}^d$ of $P$ is written as $L_P$.

### 2.3. Monoids and their algebras

All \textit{monoids} in this paper are commutative, written additively and contain a neutral element. Let $\text{gp}(M)$ be the abelian group induced by $M$. A subset $I$ of $M$ is called an \textit{ideal} of $M$ if $M + I \subseteq I$.

A monoid $M$ is \textit{affine} if it is finitely generated and isomorphic to a submonoid of some $\mathbb{Z}^n$. Let $M \subseteq \mathbb{Z}^n$ be an affine monoid. The (relative) \textit{interior} of $M$ is defined by $\text{int}(M) = M \cap \text{int} (\text{cone}(M))$, which is an ideal of $M$ and where $\text{int}(\text{cone}(M))$ denotes the (relative) interior of the conical hull $\text{cone}(M)$ of $M$. For a face $F$ of $\text{cone}(M)$ the set $F \cap M$ is a submonoid of $M$, which is called a \textit{face} of $M$.

Given a field $\mathbb{K}$ one can associate to a monoid $M$ a $\mathbb{K}$-algebra $\mathbb{K}[M]$, called its \textit{monoid algebra}, as follows. As a vector space $\mathbb{K}[M]$ is free with a basis consisting of the symbols $X^a$, $a \in M$. The elements $X^a$ are called the \textit{monomials} of $\mathbb{K}[M]$. The multiplication on $\mathbb{K}[M]$ is defined on the monomials by $X^a X^b = X^{a+b}$ and this extends linearly to all element of $\mathbb{K}[M]$. The \textit{normalization} of a monoid $M$, denoted by $\overline{M}$, is the set $\overline{M} = \{ x \in \text{gp}(M) : nx \in M \text{ for some nonzero } n \in \mathbb{N} \}$. Note that $\overline{M}$ is again a monoid. An $x \in \overline{M}$ is also said to be \textit{integral} over $M$. We say that $M$ is \textit{normal} if $M = \overline{M}$. The algebra $\mathbb{K}[M]$ is normal if and only if $M$ is a normal monoid.

Let $P \subseteq \mathbb{R}^d$ be a lattice polytope. The \textit{polytopal monoid} $M_P$ associated with $P$ is the submonoid of $\mathbb{R}^{d+1}$ generated by $(x, 1)$ for $x \in L_P$. As $L_P$ is finite, $M_P$ is affine. Observe also that for a lattice polytope $P$, the monoid algebra $\mathbb{K}[M_P]$, associated to the polytopal monoid $M_P$, equals to its \textit{polytopal algebra} $\mathbb{K}[P]$ defined as

$$\mathbb{K}[P] = \mathbb{K}[y^a z : a \in P \cap \mathbb{Z}^d].$$

Here $y^a = y_1^{a_1} \cdots y_d^{a_d}$ for $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ and $z$ is another variable. If $P \subseteq \mathbb{R}^d_{\geq 0}$, then this algebra is a $\mathbb{K}$-subalgebra of $\mathbb{K}[y_1, \ldots, y_d, z]$. It is standard graded induced by setting $\deg(z) = 1$ and $\deg(y_i) = 0$ for $i = 1, \ldots, d$. Observe that $\mathbb{K}[P]$ is a toric algebra.

### 2.4. Cut polytopes and their algebras

Let $G = (V, E)$ be a graph. Vectors in $\mathbb{R}^E$ are written as $x = (x_e)_{e \in E}$. For any subset $A \subseteq V$ we define its \textit{cut set} as $\text{Cut}(A) = \{ e \in E : |A \cap e| = 1 \}$ and its \textit{cut vector} $\delta_A = (\delta_{A,e})_{e \in E} \in \mathbb{R}^E$ by

$$\delta_{A,e} = \begin{cases} 1 & \text{if } e \in \text{Cut}(A), \\ 0 & \text{otherwise}. \end{cases}$$
We see that cut sets correspond one-to-one to cut vectors. Note that $\delta_A = \delta_{A^c}$ where $A^c = V \setminus A$. The cut polytope of $G$, denoted by $\text{Cut}^\square(G)$, is the convex hull of all $\delta_A$ for $A \subseteq V$. This polytope is full-dimensional and has at most $2^{|V|-1}$ many vertices with equality if $G$ is connected. Setting $A = \emptyset$ yields that the zero vector is always a vertex.

In general a description of all facet defining inequalities of cut polytopes is not known and one of the main open problems of the field is to get insights to these (see, e.g., [15]). However, for some interesting special cases one can describe all facets. For later use we recall the following consequence of results of Barahona [2] (see also [4, Theorem 2.3] and [32, Proposition 2.1]).

**Proposition 2.1.** ([2, Section 3]) Let $G$ be a $K_5$-minor-free graph. Then $\text{Cut}^\square(G)$ is the solution set of the following system of (valid) inequalities:

$$0 \leq x_e \leq 1, \quad \text{where } e \in E(G) \text{ does not belong to a triangle},$$

$$\sum_{f \in F} x_f - \sum_{e \in E(C) \setminus F} x_e \leq |F| - 1,$$

where $C$ ranges over all induced cycles of $G$ and $F \subseteq E(C)$ with $|F|$ odd. Moreover, these inequalities define all facets of $\text{Cut}^\square(G)$.

The affine monoid $M_G$ associated to $\text{Cut}^\square(G)$ is generated by $(\delta_A, 1)$ for $A \subseteq V$. As noted in [32, (1)] it follows from [26, Page 258] that for $x \in \mathbb{Z}^E$ and $\alpha \in \mathbb{Z}$ we have that

$$(x, \alpha) \in \text{gp}(M_G) \text{ if and only if } \sum_{e \in C} x_e \equiv 0 \text{ mod } 2,$$

where $C$ ranges over all cycles of $G$. The following consequence of Proposition 2.1 was observed in [32, Corollary 2.2]: let $G$ be a $K_5$-minor-free graph, $x \in \mathbb{Q}^E$ and $\alpha \in \mathbb{Z}_{\geq 0}$. Then $(x, \alpha) \in \mathbb{Q}_{\geq 0}M_G$ if and only if

$$(2.1) \quad (x, \alpha) \in \text{gp}(M_G) \text{ if and only if } \sum_{e \in C} x_e \equiv 0 \text{ mod } 2,$$

where $C$ ranges over all cycles of $G$ and $F \subseteq E(C)$ with $|F|$ odd. Note that $(x, \alpha)$ is an element in the interior of cone$(M_G)$ if and only if all inequalities are strictly satisfied.

The polytopal algebra associated to $\text{Cut}^\square(G)$ is called the cut algebra of $G$ and denoted in the following by $\mathbb{K}[G]$ for a given field $\mathbb{K}$. Observe that $\text{Cut}^\square(G)$ has dimension $|E|$ and thus $\dim \mathbb{K}[G] = |E| + 1$. Note that the definition of this algebra in [39] is equivalent to the one given here by [35, Lemma 4.2]. For a presentation choose a polynomial ring

$$S_G = \mathbb{K}[q_A : A \subseteq V]$$

where we set $q_A = q_{A^c}$. Thus, this ring is generated only by $2^{|V|-1}$ many variables. Consider the following surjective $\mathbb{K}$-algebra homomorphism:

$$\varphi_G : S_G \rightarrow \mathbb{K}[G], \quad q_A \mapsto y^{\delta_A} z.$$

The kernel $I_G$ of $\varphi_G$ is a graded ideal which is called the cut ideal of $G$. It is a toric ideal and thus generated by (pure) binomials. Recall that many algebraic properties of interest of $\mathbb{K}[G]$ can be characterized by corresponding monoidal properties of $M_G$ and vice versa.
3. Normality of Cut Polytopes

In this section we present a brief survey of known results related to Conjecture 1.1. Also some new contributions to this conjecture are discussed. In the following we say that a cut polytope satisfies some algebraic property if its cut algebra has this property. For a graph $G$ let $\mu(I_G)$ be the maximal degree of a minimal (binomial) generator of $I_G$.

It is known that the cut polytope of $K_5$ is not normal, not Cohen–Macaulay and $\mu(I_{K_5}) = 6$ (see [39, Table 1] for a computational approach and, e.g., [35, Example 7.2] for a proof). In [39, Pages 699–700] it was observed that if $K[G]$ is normal, Cohen–Macaulay or $\mu(I_G) \leq 4$, then the given graph $G$ has to be $K_5$-minor-free (see also [35, Sections 4–5] for additional remarks). Here we concentrate on the Cohen–Macaulay part of the conjecture.

As mentioned in the introduction, by Hochster’s result on affine monoid rings normality implies Cohen–Macaulayness for cut algebras. Hence, one key part of the conjecture is to show that cut algebras of $K_5$-minor-free graphs are normal. Note that in [39] Conjecture 1.1 is verified computationally for graphs with up to 6 vertices. In [40, Theorem 3.2] Sullivant proved that Cut□($G$) is a compressed polytope if and only if $G$ is $K_5$-minor-free and has no induced cycle of length greater than 4. One should note that Sullivant’s notion of compressed refers to general affine lattices and not just to $\mathbb{Z}^r$. This differs from the definition of compressed polytopes that has also appeared in the literature. It is known that compressed polytopes are normal, which supports further Conjecture 1.1 in this case. In particular, this can be applied to $K_5$-minor-free chordal graphs. Ohsugi proved in [32, Corollary 2.4 and Theorem 3.2] the following two important results with respect to normality of cut polytopes.

**Theorem 3.1 ([32]).** Let $G$ be a graph.

(i) Let $H$ be a minor of $G$. If Cut□($G$) is normal, then Cut□($H$) is normal. Thus, normality is a minor-closed property.

(ii) Let $G = G_1 \# G_2$ be a 0-, 1- or 2-sum of $G_1$ and $G_2$. Then the cut polytope Cut□($G$) is normal if and only if the cut polytopes Cut□($G_1$) and Cut□($G_2$) are normal.

In [32, Theorem 3.2] the proof of “the only if” direction of (ii) relies on [39, Lemma 3.2(1)] which is not correct as was pointed out in [35]. Note that all other results in [32] use only the “if part” of (ii). But this can easily be repaired. In (ii) the graphs $G_1$, $G_2$ are minors of $G_1 \# G_2$, which can be obtained by using only edge contractions (and removing possible multiple edges, loops and isolated vertices, which all do not change cut polytopes). Then one can apply [39, Lemma 3.2(2)] to conclude the proof of (ii). The following proposition summarizes further related and important results:

**Proposition 3.2.** Let $G$ be a graph.

(i) ([32, Theorem 3.8]) If $G$ is $K_5 \setminus e$-minor-free, then Cut□($G$) is normal.

(ii) ([17, Corollary 2.8]) The cut ideal $I_G$ is generated in degrees less than or equal to 2 if and only if $G$ is $K_4$-minor-free.

(iii) ([25, Section 4]) Suppose that $G$ has a universal vertex. Then $I_G$ is generated in degrees less than or equal to 4 if and only if $G$ is $K_5$-minor-free.

An interesting class of graphs are ring graphs defined as follows.
Recall that a cut vertex \( v \in V(G) \) of a graph \( G \) is a vertex such that induced subgraph on \( V(G) \setminus v \) has more connected components than \( G \). A block of \( G \) is a maximal connected subgraph of \( G \) without cut vertices. Then \( G \) is called a ring graph if any block of \( G \) which is not a bridge or a vertex can be constructed from a cycle by adding cycles using the operation of taking 1-sums. One can see that ring graphs are exactly those graphs which one can obtain (up to isolated vertices) from trees and cycles using 0- or 1-sums. Moreover, ring graphs are \( K_4 \)-minor-free. See [28] for some further details and their study related to cut objects. In the latter paper it is stated that cut algebras of ring graphs are Cohen–Macaulay (see [28, Theorem 6.2]). As was pointed out in [36] this proof contains a mistake which makes it interesting to give an alternative argument. Using Theorem 3.1 this is an easy task:

**Corollary 3.3.** Let \( G \) be a ring graph. Then \( \text{Cut} \square(G) \) is normal.

**Proof.** There are at least two arguments using Ohsugi’s results. For the first one observe that cut polytopes of trees and cycles are normal as one might check as an exercise. Then one concludes the proof using Theorem 3.1(ii) and the definition of ring graphs. There exists also an even quicker second proof by applying Proposition 3.2(i) (which was proved by Ohsugi using Theorem 3.1(i)), since ring graphs are \( K_4 \)-minor-free graphs.

Studying this conjecture further and having Kuratowski’s Theorem in mind, one could try to prove it in the case of planar graphs. But proving the case of planar triangulations is equivalent to showing that any \( K_5 \)-minor-free graph yields a normal cut polytope (see [32, Conjecture 4.3]).

**Example 3.4.** A partial related result to the latter discussion is that for example cut polytopes of outerplanar graphs are normal as one can prove using the results mentioned in this section. Details are left to the reader.

Further evidence to Conjecture 1.1 are given by the characterizations of \( \mathbb{K}[G] \) being a complete intersection in [35] and being a normal Gorenstein algebra in [33]. Special knowledge about polytopes can also help to support 1.1.

**Example 3.5.** Let \( G \) be a graph such that the cut polytope \( \text{Cut} \square(G) \) is either simplicial or simple, then Conjecture 1.1 is true for \( \text{Cut} \square(G) \). Indeed, the simplicial case is obtained by [9, Theorem 4.5], where all possible polytopes are listed, and then using [32, Examples 3.7] as well as [35, Table 1] to verify all parts of the conjecture. For simple polytopes one either applies [22, Theorem 1] or [9, Lemma 4.2 and Theorem 4.3] to obtain a list of possible polytopes, which are \( k \)-sums of normal cut polytopes with \( k \leq 2 \). Then normality follows from Theorem 3.1(ii) and \( \mu(I_G) \leq 4 \) by applying [39, Theorem 2.1]. Note that in both cases normality alone could also be deduced using Proposition 3.2(i).

### 4. Seminormality of Cut Polytopes

A natural approach to relax the normality property in Conjecture 1.1 is to study seminormal cut algebras. In particular, see [6, 8, 21] for references of related results to affine monoid rings, which are of relevance in the following. Compared to normality, the definition of seminormality seems to be less known and we recall it here in the context of monoids (see, e.g., [6, Definition 2.39]):
**Definition 4.1.** A monoid $M$ is called *seminormal* if $x \in \operatorname{gp}(M)$ with $2x, 3x \in M$ implies that $x \in M$. The *seminormalization* $^+M$ of $M$ is the intersection of all seminormal submonoids of $\operatorname{gp}(M)$ which contain $M$.

One can see that $^+M$ is a monoid which is affine if $M$ is affine. We have $M \subseteq ^+M$ with equality if and only if $M$ is seminormal. Note that every normal monoid is seminormal and in general we have $M \subseteq ^+M \subseteq \overline{M}$. Hochster and Roberts proved in [21, Proposition 5.32] that an affine monoid is seminormal if and only if $\mathbb{K}[M]$ is seminormal in the algebraic sense. For an affine monoid $M$, set $M_* = \text{int}(M) \cup \{0\}$. Recall the following result:

**Proposition 4.2.** ([6, Proposition 2.20]) An affine monoid $M$ is seminormal if and only if $(M \cap F)_*$ is a normal monoid for every face $F$ of $\operatorname{cone}(M)$. In particular, if $M$ is seminormal, then $M_* = \overline{M}_*$, i.e. $M_*$ is normal.

In this section we study (non-) seminormal cut algebras and related questions. One of our results is that the cut polytope of $K_5$ is not seminormal. In particular, this complements the discussion in [35, Example 7.2] and reproves also some (computer based) facts in [39, Table 1] related to $K_5$. For sets $\{1, \ldots, n\}$ we write also $[n]$.

**Theorem 4.3.** The polytope $\operatorname{Cut}(K_5)$ is not seminormal. In particular, it is not normal.

**Proof.** The cut polytope for $K_5$, $\operatorname{Cut}(K_5)$ is the convex hull of the following cut vectors:

- $a_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
- $a_1 = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0)$
- $a_2 = (1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0)$
- $a_3 = (0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0)$
- $a_4 = (0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1)$
- $a_5 = (0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1)$
- $a_6 = (0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0)$
- $a_7 = (1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0)$
- $a_8 = (1, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1)$
- $a_9 = (1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$
- $a_{10} = (1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1)$
- $a_{11} = (0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1)$
- $a_{12} = (0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1)$
- $a_{13} = (0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1)$
- $a_{14} = (0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1)$
- $a_{15} = (0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1)$

The cut vector $a_0$ corresponds to the choice $0 \subseteq V$. The cut vectors $a_1, \ldots, a_5$ are induced by $\{1\}, \ldots, \{5\}$. These cut vectors have exactly 4 nonzero entries as each vertex belongs to 4 edges. The other cut vectors correspond to the cardinality 2 subsets of $[5]$ and they have 6 nonzero entries. Set

$$M = \left\{ \sum_{i=0}^{15} z_i (a_i, 1) : z_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Observe that $\operatorname{Cut}(K_5)$ is seminormal if and only if the $M$ is a seminormal affine monoid. Consider the element

$$x = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4) \in \mathbb{Z}^{11}.$$

We claim that $x \in \overline{M}_* \setminus M_*$, where $\overline{M}_*$ is the normalization of $M_*$. Then $M_*$ is not normal and Proposition 4.2 yields that $M$ cannot be seminormal.
At first one sees that $x \in \text{gp}(M)$, since, e.g., $x = \sum_{i=1}^{5}(a_{i}, 1) - (a_{0}, 1)$. Thus, $x \in \text{gp}(M_\ast)$ by [6, Corollary 2.25]. Note that

$$4x = \sum_{i=0}^{15}(a_{i}, 1) \in \text{int}(M) \subseteq M_\ast.$$ 

That $4x \in \text{int}(M)$ can, e.g., be deduced from the fact that it is a multiple of a lifting of the barycenter of the vertices of $\text{Cut}^\square(K_5)$ to $M$. Alternatively, one checks the facet description of $\text{Cut}^\square(K_5)$ in [15, Chapter 30.6] to deduce that statement. Hence, $x \in M_\ast$.

Assume that $x \in M_\ast$. Then $x \in M$ and one gets

$$x = (a_{i_1}, 1) + (a_{i_2}, 1) + (a_{i_3}, 1) + (a_{i_4}, 1) \text{ for some } i_j \in [15].$$

Observe that the sum of the first 10 entries of $x$ is 20 and sum of the entries of $a_j$’s are either 4 or 6. The only possibility is then that exactly two of the involved $a_j$ have the property that the sum of their entries is equal 4 and for the other two it is 6. Without loss of generality we may assume that $i_1, i_2 \in [5]$ and $i_3, i_4 \in [15] \setminus [5]$. Next we see that $a_{i_1} + a_{i_2}$ has exactly three entries equal to zero. Indeed, they correspond to the edges of the triangle with vertices $[5] \setminus \{i_1, i_2\}$. If one compares $a_{i_3}, a_{i_4}$ at those entries, then the possibilities are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$. Note that these correspond to all cut vectors of a triangle. Anyhow, adding two of such vectors one cannot get $(2, 2, 2)$. Thus, $x$ cannot exist, which concludes the proof.

Recall that for two (graded) $\mathbb{K}$-algebras $A, B$ and an injective (homogeneous) homomorphism $t: A \to B$ one calls $A$ an algebra retract of $B$, if there exists a (homogeneous) homomorphism $\pi: B \to A$ such that $\pi \circ t = \text{id}_A$. Note that given a face $F$ of a polytope $P$, then it follows from [6, Corollary 4.34] that $\mathbb{K}[P]$ is a graded retract of $\mathbb{K}[F]$, which we call a face retract. This is a useful concept, since, e.g., if in the retract situation above $B$ is seminormal or normal, then $A$ has also this property (see, e.g., [18, Proposition 3.1]).

**Corollary 4.4.** Let $G$ be a graph such that $\text{Cut}^\square(G)$ is seminormal. Then $G$ is $K_5$-minor-free.

**Proof.** Assume that $G$ has a $K_5$-minor. Recall that this minor can be obtained by taking edge contractions only (plus possible deleting isolated vertices). Hence, the cut polytope corresponding to that $K_5$-minor $\text{Cut}^\square(K_5)$ is a face of $\text{Cut}^\square(G)$ by [39, Lemma 3.2(2)]. Since $\text{Cut}^\square(G)$ is seminormal, then so is $\text{Cut}^\square(K_5)$ by the retract situation described above. This is a contradiction to Theorem 4.3 and hence $G$ has to be $K_5$-minor-free.

Note that if Conjecture 1.1 is true, then normality and seminormality of cut polytopes are equivalent. Remarkably, Lasoń and Michałek [27] were able to prove the latter fact. Their proof is based on Corollary 4.4 and very interesting additional facts related to the question in which situations cut polytopes are very ample.

**Theorem 4.5 ([27]).** Let $G$ be a graph. Then the following statements are equivalent:

(i) $\text{Cut}^\square(G)$ is normal;

(ii) $\text{Cut}^\square(G)$ is seminormal.

In particular, Theorem 3.1 is true by replacing the word normal with seminormal everywhere.
5. Canonical Modules of Cut Polytopes

Given a (graded) Cohen–Macaulay algebra there exists its (graded) canonical module which captures many important properties of the algebra. For its algebraic definition and the theory itself we refer to [6, 7]. Let $M$ be a normal and thus a Cohen–Macaulay affine monoid (i.e. its algebra $\mathbb{K}[M]$ has the property). The canonical module of $\mathbb{K}[M]$ has in this case a nice description due to Danilov and Stanley (see [38, Theorem 6.7] or [6, Theorem 6.31]). Indeed, it is the ideal generated by $\text{int}(M)$ inside $\mathbb{K}[M]$. The main goal of this section is to determine this module for certain normal cut polytopes.

A special class of Cohen–Macaulay algebras are the Gorenstein ones, which are of great interest in commutative algebra. For a normal affine monoid $M$ it is well-known that $\mathbb{K}[M]$ is Gorenstein if and only if $\text{int}(M) = x + M$ for some $x \in \text{int}(M)$ (see, e.g., [6, Theorem 6.33]). Then we call $x$ also a generator for $\text{int}(M)$.

Observe that Ohsugi classified in [33, Theorem 3.4] all normal Gorenstein cut polytopes:

**Theorem 5.1 ([33]).** The cut polytope $\text{Cut}^{\Box}(G)$ of a graph $G$ is normal and Gorenstein if and only if $G$ is $K_5$-minor-free and $G$ satisfies one of the following conditions:

(i) $G$ is a bipartite graph without induced cycle of length greater or equal to 6.
(ii) $G$ is a bridgeless chordal graph.

Using this result we can show:

**Theorem 5.2.** Let $G$ be a graph such that $\text{Cut}^{\Box}(G)$ is normal and Gorenstein. Then the generator $x$ of the canonical module $\text{int}(M_G)$ is either

(i) $x = (1, 1, \ldots, 1, 2)$ if $G$ is a bipartite graph without induced cycle of length greater or equal to 6, or
(ii) $x = (2, 2, \ldots, 2, 4)$ if $G$ is a bridgeless chordal graph.

The above theorem is also stated in [33, Remark 5.2] but without a proof. For the convenience of the reader we show it in the following. We subdivide the proof of Theorem 5.2 and discuss at first the following two lemmas.

**Lemma 5.3.** With the assumptions of Theorem 5.2(i) and $E(G) = \{e_1, \ldots, e_m\}$ the generator $x$ of the canonical module $\text{int}(M_G)$ is

$$x = (1, 1, \ldots, 1, 2) \in \mathbb{Z}^{m+1}.$$

**Proof.** Since $G$ is bipartite there is a partition of vertices $V$ of $G$ say $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$. All edges of $G$ are then of the form $v_1v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. Note that $(1, 1, \ldots, 1, 1) \in \mathbb{Z}^m$ is a cut vector corresponding to $V_1$ and $(0, 0, \ldots, 0, 0) \in \mathbb{Z}^m$ is a cut vector corresponding to $\emptyset$, which yields

$$x = (1, 1, \ldots, 1, 2) = (1, 1, \ldots, 1, 1) + (0, 0, \ldots, 0, 1) \in M_G \subseteq \mathbb{Z}^{m+1}.$$

$\text{int}(M_G)$ contains no vectors of the form $(a, 1) \in \mathbb{Z}^{m+1}$, which follows, e.g., from (2.2) since $G$ does not contain triangles. It remains to observe that $x$ is an element of $\text{int}(M_G)$ to conclude that it is the generator of $\text{int}(M_G)$, since we know by assumption that there can be at most one element in $\text{int}(M_G)$ with last coordinate 2.
For this we recall again that $G$ contains no triangles and more generally it contains only induced cycles of even length, because it is bipartite. By assumption on the length of induced cycles, the only possible length for such a cycle is 4. It follows from (2.2) and (2.3) that $x \in \text{int}(M_G)$ if and only if $0 < x_e < 2$ where $e \in E(G)$ and
\[
\begin{align*}
x_a < x_b + x_c + x_d, \\
x_b < x_a + x_c + x_d, \\
x_c < x_a + x_b + x_d, \\
x_d < x_a + x_b + x_c,
\end{align*}
\]
for each induced cycle $C$ of $G$ of length 4 with edges $\{a, b, c, d\}$ corresponding to some of the first $m$ coordinates in $\mathbb{Z}^{m+1}$. This is easily verified for $x$ and concludes the proof. □

**Lemma 5.4.** With the assumptions of Theorem 5.2(ii) and $E(G) = \{e_1, \ldots, e_m\}$ the generator $x$ of the canonical module $\text{int}(M_G)$ is
\[
x = (2, 2, \ldots, 2, 4) \in \mathbb{Z}^{m+1}.
\]

**Proof.** Set $x = (2, 2, \ldots, 2, 4) \in \mathbb{Z}^{m+1}$. Since $G$ is bridgeless and chordal, each edge belongs to a cycle, triangles are the only induced cycles of $G$ and thus each edge belongs to a triangle. At first observe that for any cycle $C$ of $G$ the element $x$ satisfies trivially $\sum_{e \in C} x_e \equiv 0 \mod 2$ and thus $x \in \text{gp}(M_G)$ by (2.1). By (2.2) and (2.3) we have $x \in \mathbb{Q}_{\geq 0}M_G$ if and only if
\[
\sum_{f \in F} x_f - \sum_{e \in E(C) \setminus F} x_e \leq 4(|F| - 1),
\]
where $C$ ranges over all induced cycles of $G$ and $F \subseteq E(C)$ with $|F|$ odd. As mentioned above, $C$ has to have length 3 with edges $\{a, b, c\}$.

Hence, the inequalities look like $x_a + x_b + x_c \leq 8$ and $x_a \leq x_b + x_c$, and others by permuting the indices. We see that $x$ satisfies all desired inequalities. In particular,
\[
x \in \mathbb{Q}_{\geq 0}M_G \cap \text{gp}(M_G) = M_G,
\]
because $M_G$ is normal by assumption. Since all inequalities are even strictly satisfied, one obtains that $x \in \text{int}(M_G)$. Assume that $\text{int}(M_G)$ contains a vector $y = (a, k)$ for some integer $1 \leq k \leq 3$ and $a \in \mathbb{Z}^m$. It is easy to see that $k \geq 2$, e.g., by reducing this to the connected case and using the structure of cut polytopes. Choose a triangle $C$ of $G$ with edges $\{a, b, c\}$. Assume that $k = 2$. Then by (2.2) and (2.3) we have
\[
y_a + y_b + y_c \leq 4, \ y_a < y_b + y_c, \ y_b < y_a + y_c, \ y_c < y_a + y_b.
\]
On the other hand by comparing the possible cut vectors and their coordinates at $a, b, c$, we see that $a$ has to be a sum of exactly two of the vectors
\[
(0, 0, 0), \ (1, 1, 0), \ (1, 0, 1), \ (0, 1, 1)
\]
and can verify that this is not possible. Hence, $k \geq 3$. A similar argument with 4 replaced by 6 and considering sums of three of the latter vectors also rules out the possibility $k = 3$. This concludes the proof. □

*(Proof of Theorem 5.2).* This follows from Theorem 5.1, Lemma 5.3 and Lemma 5.4. □

Observe that alternatively one can use also [6, Theorem 6.33] to show Theorem 5.2.
Example 5.5.

(i) Note that for a complete bipartite and $K_5$-minor-free graph every induced cycle is of length 4. Hence, Theorem 5.1 yields that the cut polytope is normal and Gorenstein. By Theorem 5.2(i) the generator of $\text{int}(M_G)$ is

$$(1, 1, \ldots, 1, 2).$$

Note that complete bipartite graphs are in particular Ferrers graphs (see, e.g., [35, Example 5.10(3)]) and one can see that the statements from here also hold in this more general case.

(ii) Let $G = G_1 \# G_2$ be a 0- or 1-sum of two graphs $G_1$ and $G_2$, which are both $K_5$-minor-free and bipartite graphs without induced cycle of length $\geq 6$. Then $G$ has the same properties as the $G_k$'s and thus the generator $x$ of the canonical module $\text{int}(M_G)$ is by Theorem 5.2(i)

$$x = (1, 1, \ldots, 1, 2).$$

(iii) Let $G = G_1 \# G_2$ be a $i$-sum for $i = 0, 1, 2, 3$ of two graphs $G_1$ and $G_2$, which are both $K_5$-minor-free and bridgeless chordal graphs. Then $G$ has the same properties as the $G_k$'s and thus the generator $x$ of the canonical module $\text{int}(M_G)$ is by Theorem 5.2(ii)

$$x = (2, 2, \ldots, 2, 4)$$

For certain clique-sums one can determine $\text{int}(M_G)$ for a given graph $G$ explicitly like in the previous example. Another case of this type is:

Lemma 5.6. Let $G_1$ and $G_2$ be two $K_5$-minor-free graphs, $G = G_1 \#_0 G_2$ and $W_1, W_2 \subseteq \mathbb{N}$ be two finite sets. Suppose that a system of generators of $\text{int}(M_{G_1})$ has last coordinates $\gamma_1$ with $\gamma_1 \in W_1$ and a system of generators of $\text{int}(M_{G_2})$ has last coordinates $\gamma_2$ with $\gamma_2 \in W_2$. Then $\text{int}(M_G)$ has a system of generators with elements of the form $(x, y, \beta)$ with

$$\beta \in W_1 \cup W_2, \beta \geq \max(\min W_1, \min W_2), (x, \beta) \in \text{int}(M_{G_1}) \text{ and } (y, \beta) \in \text{int}(M_{G_2}),$$

where for any choice $\beta \in W_1 \cap W_2$, any generator $(x, \beta) \in \text{int}(M_{G_1})$ and any generator $(y, \beta) \in \text{int}(M_{G_2})$ a vector $(x, y, \beta)$ is part of that system.

Proof. Let $G = G_1 \#_0 G_2$ be a 0-sum of two graphs $G_1$ and $G_2$. The cut vectors of $G$ are exactly of the form $(\delta_{A_1}, \delta_{A_2})$ for $A_1 \subseteq V(G_1)$ and $A_2 \subseteq V(G_2)$. Thus,

$$M_G = \{(x, y, \alpha) : (x, \alpha) \in M_{G_1} \text{ and } (y, \alpha) \in M_{G_2}\}.$$

Suppose that $(x, y, \alpha) \in \text{int}(M_G)$. Using Proposition 2.1 one sees that $(x, \alpha) \in \text{int}(M_{G_1})$ and $(y, \alpha) \in \text{int}(M_{G_2})$. There exists a generator $(\tilde{x}, \beta_1)$ of $\text{int}(M_{G_1})$ and a generator $(\tilde{y}, \beta_2)$ of $\text{int}(M_{G_2})$ such that

$$(x, \alpha - \beta_1) = (x, \alpha) - (\tilde{x}, \beta_1) \in M_{G_1} \text{ and } (y, \alpha - \beta_2) = (y, \alpha) - (\tilde{y}, \beta_2) \in M_{G_2},$$

where $\beta_1 \in W_1$ and $\beta_2 \in W_2$. Assume that $\beta_1 = \beta_2 = \beta$. Then

$$(x, y, \alpha) = (\tilde{x}, \tilde{y}, \beta) + (v, w, \alpha - \beta) \in (\tilde{x}, \tilde{y}, \beta) + M_G$$

with $(\tilde{x}, \tilde{y}, \beta) \in \text{int}(M_G)$ and $(v, w, \alpha - \beta) \in M_G.$
Next suppose that $\beta_1 < \beta_2 \leq \alpha$. Note that $M_{G_1}$ is generated by certain elements of the form $(a_1, 1), \ldots, (a_n, 1)$. There exists a presentation $(v, \alpha - \beta_1) = \sum_{k=1}^{n} a_k (a_k, 1)$ with $i_k \in [n]$. Set 

$$(\tilde{z}, \beta_2) = (\tilde{x}, \beta_1) + \sum_{k=1}^{n} (a_k, 1) \in \text{int}(M_{G_1}).$$

Then $(x, \alpha) \in (\tilde{z}, \beta_2) + M_{G_1}$ and thus, as above, $(x, y, \alpha) \in (\tilde{z}, \tilde{y}, \beta_2) + M_{G}$. The case $\beta_1 > \beta_2$ is treated in the same way and this concludes the proof. 

This rather technical lemma can for example be used in the following situation:

**Example 5.7.** Assume that $G$ is a $K_5$-minor-free and chordal graph. The property chordal is by [16, Proposition 5.5.1] equivalent to the fact that $G$ is a clique-sum of complete graphs. Using the $K_5$-minor-freeness this means that $G$ has to be a clique-sum of certain $K_2$, $K_3$ and $K_4$. Reordering this we see that $G$ is a 0-sum of certain $K_5$-minor-free bridgeless chordal graphs (where no $K_2$ is allowed to use) and some trees. Thus,

$$G = G_1 \# T_1 \# G_2 \# G_3 \# \ldots \# G_n \# T_n,$$

where $G_i$ are $K_5$-minor-free bridgeless chordal graphs and $T_i$ are trees.

For a tree $T$ it follows from (2.1), (2.2) and (2.3) that its elements $(x, \alpha) \in \text{int}(M_T)$ have to satisfy $1 \leq x = \alpha - 1$ for $e \in E(T)$ and $x_e \in \mathbb{Z}$. As a (not minimal) system of generators we can choose 

$$(x, 4) \text{ with } 1 \leq x = \alpha - 1 	ext{ and } x_e \in \mathbb{Z} \text{ together with all elements in } \text{int}(M_T) \text{ with } 2 \leq \alpha \leq 3.$$ 

Let $2_k = (2, \ldots, 2) \in \mathbb{Z}^{E(G_k)}$ for $k \in [n]$. The observation for trees, Lemma 5.4 and Lemma 5.6 imply that $\text{int}(M_G)$ has the following system of generators:

$$\left\{ (2, 1, x_1, \ldots, x_n, 4) : 1 \leq (x_k)_e \leq 3 \right\} \text{ for all } e \in E(T_k).$$

**Corollary 5.8.** Let $G_1$ be a graph which is both $K_5$-minor-free and bridgeless chordal and $G_2$ be a graph which is both $K_5$-minor-free and bipartite without induced cycle of length $\geq 6$. Let $G = G_1 \# G_2$. Then $\text{int}(M_G)$ has a system of generators of the form

$$\{ (2, y, 4) : (2, 4) \in \text{int}(M_G_1) \text{ and } (y, 4) \in \text{int}(M_G_2) \}.$$ 

**Proof.** The corollary follows from Theorem 5.2 and Lemma 5.6. 

**Lemma 5.9.** Let $G_1$ be a graph which is both $K_5$-minor-free and bridgeless chordal and $G_2$ be a graph which is both $K_5$-minor-free and bipartite without induced cycle of length $\geq 6$. Let $G = G_1 \# G_2$. Then $\text{int}(M_G)$ has a system of generators with elements of the form

$$(2, 2, y, 4) \text{ such that } (2, 2, 4) \in \text{int}(M_{G_1}) \text{ and } (2, y, 4) \in \text{int}(M_{G_2}).$$

**Proof.** Assume that with respect to a suitable ordering the common edge $e$ of $G_1$ and $G_2$, at which the 1-sum is performed, is the first edge of $E(G)$, then the remaining edges from $E(G_1)$ follow and finally we see the edges from $E(G_2)$. Observe that

$$M_G = \{ (\gamma, x, y, \alpha) : (\gamma, x, \alpha) \in M_{G_1} \text{ and } (\gamma, y, \alpha) \in M_{G_2} \}.$$ 

Suppose that $(\gamma, x, y, \alpha) \in \text{int}(M_G)$. It follows from Proposition 2.1 that $(\gamma, x, \alpha) \in \text{int}(M_{G_1})$ and $(\gamma, y, \alpha) \in \text{int}(M_{G_2})$. Hence, using Theorem 5.2,

$$(\gamma, x, \alpha) = (2, 2, 4) + (\gamma - 2, y, \alpha - 4),$$

where $(\gamma - 2, y, \alpha - 4) \in M_{G_1}$. 


In particular, one obtains $\gamma - 2 \leq \alpha - 4$ by (2.3) and thus $\gamma \leq \alpha - 2$. We see also that $(\gamma, y, \alpha) = (1, 1, 2) + (\gamma - 1, w, \alpha - 2)$, where $(\gamma - 1, w, \alpha - 2) \in M_{G_2}$. Recall that the affine monoid $M_{G_1}$ is generated by elements of the form 
\[(\delta_1^1(e), a_1, 1), \ldots, (\delta_{n_1}^1(e), a_{n_1}, 1),\]
and the affine monoid $M_{G_2}$ is generated by elements of the form 
\[(\delta_1^2(e), b_1, 1), \ldots, (\delta_{n_2}^2(e), b_{n_2}, 1).\]
There exists a presentation, $(\gamma - 1, w, \alpha - 2) = \sum_{k=1}^{\alpha - 2}(\delta_i^2(e), b_i, 1)$ with $i_k \in [n]$. As noted above $\gamma \leq \alpha - 2$. Hence, $\gamma - 1 < \alpha - 2$ and there must exist $j$ and $j'$ such that $\delta_{i_j}^2(e) = 0$ and $\delta_{i_{j'}}^2(e) = 1$. Set
\[(2, z, 4) = (1, 1, 2) + (0, b_{i_j}, 1) + (1, b_{i_{j'}}, 1) \in \text{int}(M_{G_2}).\]
Then $(\gamma, y, \alpha) \in (2, z, 4) + M_{G_2}$ and thus, $(\gamma, x, y, \alpha) \in (2, 2, z, 4) + M_G$. $\square$

**Example 5.10.** Let $G$ be a ring graph without induced cycle of length $\geq 5$. Since it is a 0- or 1-sum of cycles and trees, $G$ can be obtained from a sequence of 0- or 1-sum of triangles, squares, and trees. Recall that $\text{int}(M_C)$ is generated by $(2, 2, 2, 4)$ for a triangle $C$ and $\text{int}(M_D)$ is generated by $(1, 1, 1, 1, 2)$ for a square $D$. Then by Corollary 5.8 and Lemma 5.9, the canonical module $\text{int}(M_G)$ has generators whose last coordinates are 4.

In Example 5.7, for a tree $T$, we have discussed the structure of elements $(x, 4) \in \text{int}(M_T)$. For a square $D$, by analyzing its cut vectors, it can be seen that $(x, 4) \in \text{int}(M_D)$ if and only if for each $e \in E(D)$ we have $1 \leq x_e \leq 3$ with $x_e \in \mathbb{Z}$, and either all entries of $x$ are equal or they are pairwise equal. Hence, again using Corollary 5.8 and Lemma 5.9, one gets a complete description of a set of generators of $\text{int}(M_G)$.

6. **Castelnuovo–Mumford Regularity of Cut Polytopes**

In this section we study the Castelnuovo–Mumford regularity of cut polytopes, i.e. the Castelnuovo–Mumford regularity of their cut algebras. Let us recall the definition. For a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ and a finitely generated graded $R$-module $M \neq 0$ its graded Betti numbers are 
\[\beta_{i,j}^R(M) := \dim_{\mathbb{K}} \text{Tor}_i^R(\mathbb{K}, M)_j \text{ for } i = 0, \ldots, n \text{ and } j \in \mathbb{Z}.\]
The Castelnuovo–Mumford regularity of $M$ is defined as 
\[\text{reg}_R M = \max \{ j - i : \beta_{i,j}^R(M) \neq 0 \}.\]
For a graph $G$ with cut algebra $\mathbb{K}[G]$ and cut ideal $I_G$ we determine their regularities always with respect to the polynomial ring $S_G$ as defined in Section 2. To simplify the notation we omit $S_G$ as an index and simply write $\text{reg} \mathbb{K}[G]$ and $\text{reg} I_G$ for the regularities. Observe that by [35, Proposition 3.2] we have $(I_G)_1 = 0$ if and only if $G$ is connected. So in this case $\text{reg} I_G \geq 2$ or $I_G = 0$. Recall that $\dim \mathbb{K}[G] = |E(G)| + 1$.

In the following we need the following useful fact. Let $T = S/J$ be a standard graded algebra where $S$ is a polynomial ring and $J$ is a graded ideal. If $J \neq 0$, then it follows from the definition that 
\[\text{reg}_S T = \text{reg} J - 1.\]
If $T$ is Cohen–Macaulay with graded canonical module $\omega_T$ and (Krull-)dimension $d$, then (e.g., by [6, Equation (6.6)])

\[(6.1) \quad \text{reg}_S T = d - \min\{i \in \mathbb{Z} : (\omega_T)_i \neq 0\}.
\]

At first we discuss a general lower bound for the regularity in the normal case.

**Proposition 6.1.** Let $G = (V, E)$ be a graph such that Cut$^\square(G)$ is normal. Then
\[
\text{reg} \mathbb{K}[G] \geq |E| - 3.
\]

**Proof.** Set
\[
v = (x, \alpha) = (2, 2, \ldots, 2, 4) \in \mathbb{Z}^{|E| + 1}.
\]
Note that $v \in \text{gp}(M_G)$ by (2.1). Trivially $v$ satisfies strictly the inequalities in (2.2). For those in (2.3) let us consider an induced cycle $C$ of $G$ of length $n \geq 3$ and let $F \subseteq E(C)$ be an odd subset. For $v$ we compute
\[
\sum_{f \in F} x_f - \sum_{e \in E(C) \setminus F} x_e = 2|F| - 2(n - |F|) < 4(|F| - 1).
\]
Thus, also the inequalities in (2.3) are strictly satisfied and this implies
\[
v \in \mathbb{Q}_{\geq 0}(M_G) \cap \text{int}(\text{cone}(M_G)).
\]
By assumption Cut$^\square(G)$ is normal. Hence, $\mathbb{Q}_{\geq 0}(M_G) \cap \text{gp}(M_G) = M_G$ and we get
\[
v \in M_G \cap \text{int}(\text{cone}(M_G)) = \text{int}(M_G).
\]
Let $\omega$ be the canonical module of $\mathbb{K}[G]$. Then $v \in \text{int}(M_G)$ implies $\omega_4 \neq 0$. Using (6.1) this yields
\[
\text{reg} \mathbb{K}(G) = \dim \mathbb{K}[G] - \min\{i \in \mathbb{Z} : \omega_i \neq 0\} \geq |E| + 1 - 4 = |E| - 3.
\]
\[\square\]
For our next main result we observe before:

**Lemma 6.2.** Let $G = (V, E)$ be a bipartite graph with $|E| \geq 1$ such that Cut$^\square(G)$ is normal. Then
\[
\text{reg} \mathbb{K}[G] \geq |E| - 1.
\]

**Proof.** Let $\omega$ be the canonical module of $\mathbb{K}[G]$. As in the beginning of the proof of Lemma 5.3 we see that $x = (1, \ldots, 1, 2) \in M_G$. We claim that $x \in \text{int}(M_G)$. Let $C$ be an induced cycles of $G$ of even length $n \geq 4$ and $F \subseteq E(C)$ with $|F|$ odd. Note that $|F| - (n - |F|) < 2(|F| - 1)$ and thus
\[
\sum_{f \in F} x_f - \sum_{e \in E(C)} x_e < 2(|F| - 1).
\]
Moreover, $0 < x_e < 2$ where $e \in E(G)$. Hence, it follows from (2.2) and (2.3) that $x \in \text{int}(M_G)$ and so $\omega_2 \neq 0$. Using Equation (6.1) this yields
\[
\text{reg} \mathbb{K}[G] \geq \dim \mathbb{K}[G] - 2 = |E| + 1 - 2 = |E| - 1,
\]
which concludes the proof. \[\square\]
We can also provide upper bounds for normal cut polytopes, which often are equalities.
Theorem 6.3. Let $G = (V, E)$ be a graph such that $\text{Cut} \square (G)$ is normal. Then
\[
\text{reg } \mathbb{K}[G] = \begin{cases} 
  |E| - 1 & \text{if any induced cycles of } G \text{ is of even length (i.e., it is bipartite),} \\
  |E| - 2 & \text{if } G \text{ has no triangles and has an induced cycle of odd length,} \\
  |E| - 3 & \text{if } G \text{ contains a triangle.}
\end{cases}
\]

Proof. Let $v = (x, \alpha) \in \text{int}(M_G)$ with $\alpha \in \mathbb{N}$ chosen in a way that (using (6.1))
\[
\text{reg } \mathbb{K}(G) = |E| + 1 - \alpha.
\]

Case 1: Assume at first that any induced cycles of $G$ (if existing) is of even length and in particular, $G$ contains no triangles. It follows from (2.2) that in this case $\alpha \geq 2$, since the facet defining inequalities $0 \leq x_e \leq 1$ have no chance to be strictly satisfied by integers. This implies already $\text{reg } \mathbb{K}[G] \leq |E| - 1$. The equality follows from this and Lemma 6.2.

Case 2: Next we consider the case that $G$ contains no triangles, but an induced cycle $C$ of odd length. As in Case 1 we see immediately $\alpha \geq 2$. In particular, for $e \in E(G)$ and the coordinate $x_e$ of $v$ we get by (2.2) that
\[
1 \leq x_e \leq \alpha - 1.
\]

Assume that $\alpha = 2$. Then $x_e = 1$ for $e \in E(G)$. But for $e \in E(C)$ and the coordinates $x_e$ of $v$, the congruences in (2.1) imply the contradiction $|E(C)| = \sum_{e \in C} x_e \equiv 0 \mod 2$. Hence, $\alpha \geq 3$ and $\text{reg } \mathbb{K}[G] \leq |E| - 2$.

Case 3: Finally, we consider the case that $G$ contains (at least) one triangle $C$ with edges $\{e, f, g\}$. Then $v = (x, \alpha) \in \text{int}(M_G)$. Using (2.1), (2.2) and (2.3) applied to the cycle $C$ yield that $(x_e, x_f, x_g, \alpha) \in \text{int}(M_C)$. Then by Theorem 5.2(ii) we get that $\alpha \geq 4$. Thus, $\text{reg } \mathbb{K}[G] \leq |E| - 3$. This together with the lower bound of Proposition 6.1 yields the equality in this case. \hfill \Box

Corollary 6.4. Let $G = (V, E)$ be a cycle of even length. Then
\[
\text{reg } \mathbb{K}[G] = |E| - 1.
\]

Proof. The proof follows from Corollary 3.3 and from the first case of Theorem 6.3. \hfill \Box

Having Theorem 6.3 in mind, it is an interesting question to find a class of graphs where where always equality occurs instead of inequalities in the second case. We will see that ring graphs have this property. The study of the regularity of cut algebras for ring graphs was initiated in [28, Proposition 4.4 and Corollary 6.5]. Using a different approach we give in Theorem 6.6 a complete description of the regularity in that case. To prepare the proof of this result, we discuss at first a lemma.

Lemma 6.5. Let $G = (V, E)$ be a cycle of odd length $2n + 1 > 3$ for $n \in \mathbb{N}$. Then
\[
\text{reg } \mathbb{K}[G] = |E| - 2.
\]

Proof. Note that $\text{Cut} \square (G)$ is normal by Corollary 3.3. Let $\omega$ be its canonical module. By the second case of Theorem 6.3 we have $\text{reg } \mathbb{K}[G] \leq |E| - 2$.

Let $V = \{v_1, \ldots, v_{2n+1}\}$ and $E = \{e_1, \ldots, e_{2n+1}\}$, where $e_i = v_iv_{i+1}$ for $1 \leq i < 2n+1$ and $e_{2n+1} = v_{2n+1}v_1$. 
Consider the cut vectors $\mathbf{a}$ corresponding to $\{v_1, v_3, \ldots, v_{2n+1}\} \subseteq V$ and $\mathbf{b}$ corresponding to $\{v_1\} \subseteq V$. Thus, $a_i = 1$ for $i \neq 2n+1$ and $a_{2n+1} = 0$ as well as $b_{e_i} = 1$ for $i = 1, 2n+1$ and $b_{e_i} = 0$ for $1 < i < 2n+1$. It follows that
\[
\mathbf{v} = (2, 1, 1, \ldots, 1, 3) = (\mathbf{a}, 1) + (\mathbf{b}, 1) + (0, 1) \in M_G.
\]
Let $F \subseteq E$ be an odd subset. If $e_1 \in F$, then
\[
|F| + 1 < 3(|F| - 1) + (2n + 1 - |F|),
\]
since $2n + 1 + |F| - 4 > 0$. If $e_1 \not\in F$, then similarly
\[
|F| < 3(|F| - 1) + (2n + 1 - |F| + 1).
\]
Using (2.2) and (2.3) one gets that $\mathbf{v} \in \text{int}(M_G)$ and then $\omega_3 \neq 0$. Hence, using Equation (6.1) this yields
\[
\text{reg} \mathbb{K}[G] \geq \dim \mathbb{K}[G] - 3 = |E| + 1 - 3 = |E| - 2.
\]
Thus, $\text{reg} \mathbb{K}[G] = |E| - 2$. \qed

For the proof of the next theorem we observe the following. A ring graph $G$ may have isolated vertices, which are not relevant for us since they have no effect on cut polytopes. So we always may assume without loss of generality that $G$ is connected for such a graph. $G$ can be a cycle or a tree. If this is not the case, then there exists always a decomposition of one of the following types (see, e.g., [35, Example 5.10(4)]):

(i) $G = \tilde{G} \times_k T$ for $k \in \{0, 1\}$ where $\tilde{G}$ is a ring graph and $T$ is a tree. One even can reduce the case $k = 1$ to (at most) two decompositions of type $\tilde{G} \times_0 T$.

(ii) $G = \tilde{G} \times_k C$ for $k \in \{0, 1\}$ where $\tilde{G}$ is a ring graph and $C$ is a cycle.

Note also that induced cycles of $G$ are the ones of $\tilde{G}$ in both cases, or in (ii) additionally the cycle $C$. With this preparation, we are ready to prove:

**Theorem 6.6.** Let $G = (V, E)$ be a ring graph with $|E| \geq 1$. Then:

\[
\text{reg} \mathbb{K}[G] = \begin{cases} 
|E| - 1 & \text{if any induced cycles of } G \text{ is of even length,} \\
|E| - 2 & \text{if } G \text{ has no triangles and has an induced cycle of odd length,} \\
|E| - 3 & \text{if } G \text{ contains a triangle.}
\end{cases}
\]

**Proof.** The polytope $\text{Cut}^\square(G)$ is normal by Corollary 3.3. Let $\omega$ be the associated canonical module generated as an ideal by $\text{int}(M_G)$ which trivially satisfies $\omega_0 = 0$. Without loss of generality we may assume that $G$ is connected.

**Cases 1, 3:** These cases follow directly from Theorem 6.3.

**Case 2:** Next we consider the case of a ring graph $G$, which does not contain triangles, but has induced cycles of odd length. It follows from the second case of Theorem 6.3 that $\text{reg} \mathbb{K}[G] \leq |E| - 2$. We claim that there exists an $\mathbf{x} \in \mathbb{Z}^E(G)$ such that
\[
(\mathbf{x}, 3) \in \text{int}(M_G).
\]
Note that by (2.2) necessarily $1 \leq x_e \leq 2$ for any $e \in E(G)$. Then (6.1) yields that $\text{reg} \mathbb{K}[G] \geq |E| - 2$ and it remains to prove the claim for Case 2.

If $G$ is equal to just such an odd cycle, then Lemma 6.5 and its proof yield the assertion of the theorem as well as the claim.
We show the claim by an induction on the number of 0- and 1-sums at most needed to construct $G$ as a ring graph using trees and cycles as discussed above, where we already verified the base case and consider for trees only 0-sums.

Assume that $G = G_1 \#_1 G_2$ is a 1-sum of two ring graphs $G_1, G_2$ which both do not contain triangles and $G_2 = C$ is a cycle. Let the edge $e = vw$, at which the 1-sum is performed, be the first edge of $E(G)$, then we assume that the remaining edges from $E(G_1)$ follow and finally we see the edges from $E(G_2)$ with respect to a suitable ordering.

If $G_1$ contains (if existing) only cycles of even length, then $G_2$ has to be an odd cycle. In $\text{int}(M_{G_1})$ there exists by the proof of Lemma 6.2 an element $(1, 1, \ldots, 1, 2)$ and then also

$$(1, v_1, 3) = (1, 1, \ldots, 1, 3) = (1, 1, \ldots, 1, 2) + (0, 0, \ldots, 0, 1) \in \text{int}(M_{G_1}),$$

since $(0, 0, \ldots, 0, 1) \in M_{G_1}$, as the vector induced by the cut vector of the empty set. It follows from Lemma 6.5 that $\text{int}(M_{G_2})$ contains $(1, v_2, 3) = (1, 2, 1, \ldots, 1, 3)$, since for such vectors the 2 might be at any position (except the last one). One gets that

$$(1, v_1, v_2, 3) \in \text{int}(M_G),$$

which follows from the fact that induced cycles of $G$ are those from $G_1, G_2$, the monoid $M_G$ is normal (thus $\mathbb{Q}_{\geq 0}M_G \cap \text{gp}(M_G) = M_G$), and then applying all this using (2.1), (2.2) and (2.3).

Next we assume that $G_1$ has induced cycles of odd length and $G_2$ might be an arbitrary cycle not equal to a triangle. If by induction hypothesis

$$(1, v_1, 3) \in \text{int}(M_{G_1}),$$

then one uses facts in the same manner as before by observing that $(1, 1, \ldots, 1, 3) \in \text{int}(M_{G_2})$ if $G_2$ is an even cycle, or $(1, 2, 1, \ldots, 1, 3) \in \text{int}(M_{G_2})$ if $G_2$ is an odd cycle, to conclude that $(1, \ldots, 3) \in \text{int}(M_G)$.

It remains to consider the subcase

$$(2, v_1, 3) \in \text{int}(M_{G_1}).$$

If $G_2$ is an odd cycle, then Lemma 6.5 implies that $\text{int}(M_{G_2})$ contains

$$(2, v_2, 3) = (2, 1, \ldots, 1, 3) \in \text{int}(M_{G_2}).$$

Since all monoids are normal, that induced cycles of $G$ are those from $G_1, G_2$, and then applying again (2.1), (2.2) and (2.3) yields

$$(2, v_1, v_2, 3) \in \text{int}(M_G).$$

Finally, assume that $G_2$ is an even cycle and thus

$$(1, 1, 1, \ldots, 1, 2) \in \text{int}(M_{G_2})$$

by the proof of Lemma 6.2. Since $G_2$ is bipartite, one can see analogously to the proof of Lemma 5.3 that

$$(1, 1, 1, \ldots, 1, 1) \in M_{G_2}$$

and thus

$$(2, v_2, 3) = (1, 1, 1, \ldots, 1, 2) + (1, 1, 1, \ldots, 1, 1) \in \text{int}(M_{G_2}).$$
In this case we get by our “standard” argument that $$\langle 2, v_1, v_2, 3 \rangle \in \text{int}(M_G)$$. Similarly, one proves the case that $$G = G_1 \#_5 G_2$$ is a 0-sum using graphs $$G_1, G_2$$ as above. Here the case distinction is much simpler, since one does not have to make sure that one particular chosen coordinate is equal for vectors produced in int($$M_{G_1}$$) and int($$M_{G_2}$$).

Before we discuss our next main result, we need to determine the regularity for one more special case. In the following $$P_n$$ denotes always a path of (edge-)length $$n$$, i.e. it has exactly $$n$$ edges.

**Lemma 6.7.** We have $$\text{reg } \mathbb{K}[C_5 \#_2 C_4] = 5$$.

**Proof.** Let $$G = C_5 \#_2 C_4$$ and $$E = E(G)$$ as well as $$V = V(G)$$. Since $$G$$ is $$K_5 \setminus e$$-minor-free, it follows from Proposition 3.2 (i) that $$\text{Cut}_G^\square$$ is normal. Let $$\omega$$ be its canonical module. By the second case of Theorem 6.3 we know already that $$\text{reg } \mathbb{K}[G] \leq 7 - 2 = 5$$. Choose an ordering of the vertices and edges. Let $$V = \{v_1, \ldots, v_6\}$$ and $$E = \{e_1, \ldots, e_7\}$$, where $$e_i = v_i v_{i+1}$$ for $$1 \leq i < 5$$, $$e_5 = v_5 v_1$$, $$e_6 = v_1 v_6$$ and $$e_7 = v_3 v_6$$.

That is, the cycle $$C_5$$ with vertices $$\{v_1, \ldots, v_5\}$$ is attached via the $$P_2$$-sum performed at the edges $$\{e_1, e_2\}$$ with $$C_4$$ with vertices $$\{v_1, v_2, v_3, v_6\}$$.

Consider the cut vectors $$a$$ corresponding to $$\{v_1, v_3, v_5\} \subseteq V$$ and $$b$$ corresponding to $$\{v_5\} \subseteq V$$. Thus, $$a_{e_i} = 1$$ for $$i \neq 5$$ and $$a_{e_5} = 0$$ as well as $$b_{e_i} = 1$$ for $$i = 4, 5$$ and $$b_{e_i} = 0$$ for any other $$i$$. It follows that $$v = (1, 1, 1, 2, 1, 1, 1, 3) = (a, 1) + (b, 1) + (0, 1) \in M_G$$.

Using the proofs of Lemma 6.2 and Lemma 6.5 as well as that $$C_4, C_5$$ are the only induced cycles of $$G$$ one gets that $$v \in \text{int}(M_G)$$ and then $$\omega_3 \neq 0$$. Hence, Equation (6.1) yields

$$\text{reg } \mathbb{K}[G] \geq \dim \mathbb{K}[G] - 3 = 7 + 1 - 3 = 5,$$

and thus $$\text{reg } \mathbb{K}[G] = 5$$. \hfill \Box

The strategy of the previous proof can also be used to show that $$\text{reg } \mathbb{K}[C_{2n+1} \#_2 C_{2k}] = 2n + 2k - 3$$ for $$n \geq 1$$ and $$k \geq 2$$. Note that cut polytopes with small regularity are normal.

**Lemma 6.8.** Let $$G$$ be a graph such that $$\text{reg } \mathbb{K}[G] \leq 4$$. Then $$\text{Cut}_G^\square$$ is normal.

**Proof.** For this proof we use the fact that $$I_{K_5}$$ has a minimal generator in degree 6 (see, e.g., [39, Table 1]), which implies that $$\text{reg } I_{K_5} \geq 6$$ and thus $$\text{reg } \mathbb{K}[K_5] \geq 5$$. Assume that $$G$$ has $$K_5$$ as a minor. Recall that this minor can be obtained by edge contractions only (plus possible deletions of isolated vertices, etc.) and that $$\mathbb{K}[K_5]$$ is an algebra retract of $$\mathbb{K}[G]$$ (see [35, Corollary 4.6]). Then, e.g., [35, Proposition 2.2] implies that

$$\text{reg } \mathbb{K}[G] \geq \text{reg } \mathbb{K}[K_5] \geq 5.$$

This is a contradiction. Hence, $$G$$ is $$K_5$$-minor-free.

By Theorem 6.3 or Theorem 5.2 (ii) we know that $$\text{reg } \mathbb{K}[K_5 \setminus e] = 6$$.

Assume that $$H = K_5 \setminus e$$ is a minor of $$G$$. Then we can get it from $$G$$ by certain edge deletions and edge contractions. Observe that the deletions do not decrease the number of vertices, but contractions have this property and also that operations deletions/contractions commute with each other.
Just by using only the edge contractions with respect to $G$ (plus possible deletions of isolated vertices, etc.) yields a graph $\tilde{H}$ on five vertices which contains $H$ as a subgraph. The only possibilities of such $\tilde{H}$ are $H$ and $K_5$. Since $G$ is $K_5$-minor-free, we obtain that $\tilde{H} = K_5 \setminus e$. Thus, the minor $K_5 \setminus e$ can be obtained from $G$ using only edge contractions. But then [35, Corollary 4.6] implies again the contradiction

$$\text{reg } \mathbb{K}[G] \geq \text{reg } \mathbb{K}[K_5 \setminus e] = 6.$$ 

Hence, $G$ is also $K_5 \setminus e$-minor-free and Proposition 3.2 (i) implies that Cut$^\square (G)$ is normal.

We recall the following known results from [35, Section 6] that classify connected graphs with very small regularities.

**Proposition 6.9.** Let $G = (V, E)$ be a connected graph with $|E| \geq 1$ and $r = \text{reg } \mathbb{K}[G]$. Then:

(i) $r = 0$ if and only if $G = C_3 = K_3$ or $G = P_1 = K_2$.
(ii) $r = 1$ if and only if $G = C_3 \# 0P_1 = K_3 \# 0K_2$ or $G = P_2$.

(One should note that the notation for $P_n$ differs by one in the index compared to [35]).

**Proof.** The case $r = 0$ follows from [35, Proposition 3.1, Proposition 3.2] and the case $r = 1$ is obtained from [35, Proposition 3.2, Corollary 6.11]. Alternatively, the proof follows also from Lemma 6.2 and Theorem 6.3.

We are ready to state our second main result of this section. Extending the results of Proposition 6.9 we describe all connected graphs whose cut algebras have regularities less than or equal to 4. Recall that in this case $(I_G)_1 = 0$. Thus, either $I_G = 0$ or the ideal is generated in degrees $\geq 2$.

**Theorem 6.10.** Let $G = (V, E)$ be a connected graph with $|E| \geq 1$ and $r = \text{reg } \mathbb{K}[G]$. Then:

(i) $r = 2$ if and only if $G$ is a tree with $|E| = 3$ or $G$ contains a $C_3$ with $|E| = 5$.
(ii) $r = 3$ if and only if $G$ is a tree with $|E| = 4$ or $G = C_4$ or $G = C_5$ or $G$ contains a $C_3$ with $|E| = 6$.
(iii) $r = 4$ if and only if $G$ is bipartite with $|E| = 5$ or $G = C_5 \# 0P_1$ or $G$ contains a $C_3$ with $|E| = 7$.

**Proof.** First note that by Lemma 6.8 any cut algebra with regularity $\leq 4$ is normal. It follows from Lemma 6.1 that $|E| \leq r + 3 \leq 7$. Conversely, any connected graph with $|E| \leq 7$ is $K_5 \setminus e$-minor free and thus by Proposition 3.2 (i) has a normal cut polytope.

In conclusion, we have to consider all possible graphs with $|E| \leq 7$ and have to determine their regularity, where we can use the fact that the cut polytopes are normal. We proceed on a case by case basis with respect to $|E| \geq 1$.

*Case $|E| = 1$: Here $G$ has to equal $P_1$ with $r = 0$ by Lemma 6.2. This case is included in Proposition 6.9(i) and not relevant here.*

*Case $|E| = 2$: There exists only the possibility $G = P_2$ with $r = 1$ by Lemma 6.2. This case is included in Proposition 6.9(ii) and not relevant here.*
Case $|E| = 3$: If $G$ is a tree, then $r = 2$ by Lemma 6.2. The only other possibility is $G = C_3 = K_3$ where $r = 0$ by Theorem 6.3, which is again included in Proposition 6.9(i) and not relevant here.

Case $|E| = 4$: If $G$ contains a triangle $C_3$, then $G = C_3\# P_1$ and $r = 1$ by Theorem 6.3, which is included in Proposition 6.9(ii) and not relevant here. Otherwise $G = C_4$ or $G$ is a tree and in both cases $r = 3$ by Lemma 6.2.

Case $|E| = 5$: If $G$ is a tree, then $r = 4$ by Lemma 6.2. If $G$ contains a triangle $C_3$, then $r = 2$ by Theorem 6.3. Next assume that $G$ has cycles, but does not contain a $C_3$. Then either $G = C_5$ with $r = 3$ by Lemma 6.5, or the bipartite graph $G = C_4\# P_1$ with $r = 4$ by Lemma 6.2.

Case $|E| = 6$: If $G$ is a tree, then $r = 5$ by Lemma 6.2 and this case is not listed in the theorem. If $G$ contains a triangle $C_3$, then $r = 3$ by Theorem 6.3.

Next assume that $G$ has cycles, but does not contain a $C_3$. The first case is that $G = C_6$ with $r = 5$ by Lemma 6.2, which has to be excluded. If $G \neq C_6$ contains a $C_5$, then this cycle cannot have a chord since otherwise there would exist also a triangle. Hence, in this case $G = C_5\# P_1$ with $r = 4$ by Theorem 6.6.

The remaining case is that $G$ contains no $C_3, C_5$ and $C_6$, but a $C_4$. Then $G$ is bipartite, because it has no cycles of odd length and thus $r = 5$ by Lemma 6.2; a case which has again to be excluded from our desired list.

Case $|E| = 7$: If $G$ is a tree, then $r = 6$ by Lemma 6.2. Again a subcase we have to exclude from our list. If $G$ contains a triangle then $r = 4$ by Theorem 6.3. Next assume that $G$ has cycles, but does not contain a $C_3$. The first case is that $G = C_7$ with $r = 5$ by Theorem 6.6, which has again to be excluded.

If $G \neq C_7$ contains a $C_6$, there exist two subcases. The graph $G$ could contain a chord of $C_6$ and then $G = C_4\# C_4$ with $r = 6$ by Lemma 6.2. Or $C_6$ is an induced cycle of $G$. Then $G = C_6\# P_1$ with $r = 6$ by Lemma 6.2. Both subcases are not relevant for us since $r > 4$.

The next subcase is that $G$ contains no $C_3, C_6$ and $C_7$, but a $C_5$. There cannot exist chords, since otherwise we would obtain triangles inside $G$. Then the choices for $G$ are $C_5\# P_2 C_4$ or $C_5\# P_1\# P_1$ or $C_5\# P_2$. In all these cases $r = 5$ by Lemma 6.7 and Theorem 6.6, which all have to be excluded.

The remaining case is that $G$ contains no $C_3, C_5, C_6$ and $C_7$, but a $C_4$. Here $G$ is bipartite with $r = 6$ by Lemma 6.2, which is again too large to be included in the desired list. □

7. PROBLEMS

In this short section we discuss some research questions, which are of interest for future activities. Related to Section 3 and Section 4 the main challenge is of course Conjecture 1.1.

In Section 5 we determined in many cases the canonical module of cut polytopes like in Theorem 5.2 in the normal Gorenstein case. Since the case of trees is here included in Case (i) of that theorem, it is a natural question to consider the case of cycles $C_n$ for $n \geq 5$. Cut polytopes of cycles are always normal by Corollary 3.3, but almost never Gorenstein (see Theorem 5.1), since only $C_3$ and $C_4$ induce this property for their cut polytopes.

**Problem 7.1.** Determine the canonical module of $\text{Cut}^\square (C_n)$ for any $n \geq 5$. 
Computational evidences using Normaliz [5] suggest that in the case of cycles \( C_n, n \geq 5 \) the canonical module is not generated in one degree. However, note that by Theorem 6.6 and using Equation (6.1) we know the smallest degree of generators for any \( C_n \).

In Section 6 many cases are studied where one knows the Castelnuovo–Mumford regularity. In particular, having Theorem 6.10 in mind, the following would be interesting:

**Problem 7.2.** Continue the classification of cut polytopes with small regularities.

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