Two-Loop Euler-Heisenberg QED Pair-Production Rate

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We study the divergence of large-order perturbation theory in the worldline expression for the two-loop Euler-Heisenberg QED effective Lagrangian in a constant magnetic field. The leading rate of divergence is identical, up to an overall factor, to that of the one-loop case. From this we deduce, using Borel summation techniques, that the leading behaviour of the imaginary part of the two-loop effective Lagrangian for a constant E field, giving the pair-production rate, is proportional to the one-loop result. This also serves as a test of the mass renormalization, and confirms the earlier analysis by Ritus.

1. INTRODUCTION

Euler and Heisenberg [1], and many others since [2,3,4,5,6], computed the exact renormalized one-loop QED effective action for electrons in a uniform electromagnetic field background. When the background is purely that of a static magnetic field, the effective action is minus the effective energy of the electrons in that background. When the background is purely that of a uniform electric field, the effective action has an imaginary part which determines the pair-production rate of electron-positron pairs from vacuum [1,2,3,4,5,6].

In this paper we consider the two-loop Euler-Heisenberg effective action, and we show how the divergence of the perturbative expression for the effective action with a uniform magnetic background is related to the non-perturbative imaginary part of the effective action with a uniform electric background. The two-loop Euler-Heisenberg effective Lagrangian, describing the effect of a single photon exchange in the electron loop, was first calculated by Ritus [7], and later recalculated by Dittrich and Reuter [8] for the magnetic field case. In both cases the proper-time method and the exact Dirac propagator in the uniform field were used. More recently the magnetic field computation was repeated in [9] using the more convenient ‘worldline’ formalism [10,11]. This calculation revealed that the previous results by Ritus and Dittrich/Reuter were actually incompatible, and differed precisely by a finite electron mass renormalization. This prompted yet another recalculation of this quantity in the worldline formalism [12], now using dimensional regularisation instead of a proper-time cutoff as had been used in the previous calculations. That calculation confirmed the correctness of Ritus’s result, and conversely showed that the final result given by [8] was not expressed in terms of the physical electron mass. As part of our analysis here, we show how this finite difference in the mass renormalization affects the large-order behaviour of perturbation theory, and how this affects the leading contribution to the imaginary part of the effective action in the electric field case.

For a uniform magnetic background, of strength \(B\), the one-loop effective Lagrangian has a simple “proper-time” integral representation [16,17,18,19]:

\[
\mathcal{L}^{(1)} = -\frac{m^4}{8\pi^2} \left( \frac{eB}{m^2} \right)^2 \int_0^\infty \frac{ds}{s^2} \left( \coth \frac{s}{2m^2} - \frac{s}{3} \right) e^{-m^2s/(eB)}
\]

The \(\frac{1}{s}\) term is a subtraction of the zero field (\(B = 0\)) effective action, while the \(\frac{s}{3}\) subtraction corresponds to a logarithmically divergent charge renormalization. For a given strength \(B\), this integral can be evaluated numerically [16]. Alternatively, we can make contact with a perturbative evaluation of the one-loop effective action by making an asymptotic expansion of the integral in the weak field limit – i.e., for small values of the dimensionless parameter \(\frac{eB}{m^2}\).
Here the $B_{2n}$ are Bernoulli numbers [13]. Each term in this expansion of $\mathcal{L}^{(1)}$ is associated with a one-fermion-loop Feynman diagram. Note that only even powers of $eB$ appear, as expected due to charge conjugation invariance (Furry’s theorem). The divergent $O(e^{2})$ self-energy term is not included as it contributes to the bare Lagrangian by charge renormalization.

The expansion (2) is the prototypical “effective field theory” effective Lagrangian [14], where the low energy effective Lagrangian, for energies well below the fermion mass scale $m$, is expanded as

$$\mathcal{L} = m^4 \sum_n a_n \frac{O(n)}{m^n}$$  

with $O(n)$ being an operator of dimension $n$. For QED in a uniform background, the higher dimensional operators $O(n)$ are formed from powers of Lorentz invariant combinations of the uniform field strength $F_{\mu\nu}$. For a uniform magnetic background this simply means even powers of $B$, as in (2). Note that the ‘low energy’ condition here means that the cyclotron energy $eBm$ is well below the fermion mass scale $m$; in other words, $\frac{eB}{m} \ll 1$.

The Euler-Heisenberg Lagrangian encodes the information on the low-energy limit of the one-loop $N$-photon amplitudes in a way which is highly convenient for the derivation of various nonlinear QED effects such as vacuum birefringence (see, e. g., [15] and refs. therein) or photon splitting [16,17]. The experimental observation of vacuum birefringence is presently attempted by laser experiments [18,19]. There is also recent experimental evidence for vacuum effects in pair production with strong laser electric fields [20].

The one-loop Euler-Heisenberg perturbative effective action (2) is not a convergent series. The one-loop expansion coefficients in (2), alternate in sign [since $\text{sign}(B_{2n}) = (-1)^{n+1}$], but grow factorially in magnitude (see also Table 1):

$$a_n^{(1)} = - \frac{2^{2n}B_{2n+4}}{(2n + 4)(2n + 3)(2n + 2)} \sim (-1)^n \frac{\Gamma(2n + 2)}{8\pi^4} \left( 1 + \frac{1}{2^{2n+4}} + \frac{1}{3^{2n+4}} + \ldots \right)$$

So the perturbative expansion (2) is a divergent series. This divergent behaviour is not a bad thing; it is completely analogous to generic behaviour that is well known in perturbation theory in both quantum field theory and quantum mechanics [21]. For example, Dyson [22] argued physically that QED perturbation theory is non-analytic, and therefore presumably divergent, as an expansion in the fine structure constant $\alpha$, because the theory is unstable when $\alpha$ is negative. As is well known, the divergence of high orders of perturbation theory can be used to extract information about non-perturbative decay and tunneling rates, thereby providing a bridge between perturbative and non-perturbative physics [24]. It has been argued [23], based on the behaviour of the one-loop Euler-Heisenberg effective Lagrangian (3), that the effective field theory expansion (3) is generically divergent. Here we consider this question at the two-loop level.

We stress that for energies well below the scale set by the fermion mass $m$, the divergent nature of the effective Lagrangian is not important, as the first few terms in the series (2) provide an accurate approximation. However, the divergence properties do become important when the external energy scale approaches the fermion mass scale $m$. The divergence is also the key to understanding how non-perturbative imaginary contributions to the effective action arise from real perturbation theory.

II. BOREL ANALYSIS OF THE ONE-LOOP EULER-HEISENBERG EFFECTIVE LAGRANGIAN

To begin, we review very briefly some basics of Borel summation [24,25,26,27]. Consider an asymptotic series expansion of some function $f(g)$
where \( g \to 0^+ \) is a small dimensionless perturbation expansion parameter. In an extremely broad range of physics applications \([21]\) one finds that perturbation theory leads not to a convergent series but to a divergent series in which the expansion coefficients \( a_n \) have leading large-order behaviour

\[
a_n \sim (-1)^n \rho^n \Gamma(n+\nu) \quad (n \to \infty)
\]

for some real constants \( \rho, \mu > 0 \), and \( \nu \). When \( \rho > 0 \), the perturbative expansion coefficients \( a_n \) alternate in sign and their magnitude grows factorially, just as in the Euler-Heisenberg case \((5)\). Borel summation is a useful approach to this case of a divergent, but alternating series. Non-alternating series must be treated somewhat differently.

To motivate the Borel approach, consider the classic example: \( a_n = (-1)^n \rho^n n! \), and \( \rho > 0 \). The series \((5)\) is clearly divergent for any value of the expansion parameter \( g \). Write

\[
f(g) \sim \sum_{n=0}^{\infty} a_n g^n (\rho g)^n \int_0^\infty ds s^ne^{-s} \\
\sim \frac{1}{\rho g} \int_0^\infty ds \left( \frac{1}{1+s} \right) \exp \left[ -\frac{s}{\rho g} \right]
\]

where we have formally interchanged the order of summation and integration. The final integral, which is convergent for all \( g > 0 \), is defined to be the sum of the divergent series. To be more precise \([24,25]\), the formula \((7)\) should be read backwards: for \( g \to 0^+ \), we can use Laplace’s method to make an asymptotic expansion of the integral, and we obtain the asymptotic series in \((5)\) with expansion coefficients \( a_n = (-1)^n \rho^n n! \).

For a non-alternating series, such as \( a_n = \rho^n n! \), we need \( f(\rho) \). The Borel integral \((7)\) is \([24,25]\) an analytic function of \( g \) in the cut \( g \) plane: \( |\arg(g)| < \pi \). So a dispersion relation (using the discontinuity across the cut along the negative \( g \) axis) can be used to define the imaginary part of \( f(g) \) for negative values of the expansion parameter:

\[
\text{Im} f(-g) \sim \frac{\pi}{\rho g} \exp\left[ -\frac{1}{\rho g} \right]
\]

The imaginary contribution \((8)\) is non-perturbative (it clearly does not have an expansion in positive powers of \( g \)) and has important physical consequences. Note that \((8)\) is consistent with a principal parts prescription for the pole that appears on the \( s > 0 \) axis if we make the formal manipulations as in \((7)\):

\[
\sum_{n=0}^{\infty} \rho^n n! g^n \sim \frac{1}{\rho g} \int_0^\infty ds \left( \frac{1}{1-s} \right) \exp \left[ -\frac{s}{\rho g} \right]
\]

Similar formal arguments can be applied to the case when the expansion coefficients have leading behaviour \((6)\). Then the leading Borel approximation is

\[
f(g) \sim \frac{1}{\mu} \int_0^\infty ds \left( \frac{1}{1+s} \right) (s/\rho g)^{\nu/\mu} \exp \left[ -\left( \frac{s}{\rho g} \right)^{1/\mu} \right]
\]

For the corresponding non-alternating case, when \( g \) is negative, the leading imaginary contribution is

\[
\text{Im} f(-g) \sim \frac{\pi}{\mu} \left( \frac{1}{\rho g} \right)^{\nu/\mu} \exp \left[ -\left( \frac{1}{\rho g} \right)^{1/\mu} \right]
\]

Note the separate meanings of the parameters \( \rho, \mu \) and \( \nu \) that appear in the formula \((10)\) for the leading large-order growth of the expansion coefficients. The constant \( \rho \) clearly combines with \( g \) as an effective expansion.
parameter. The power of the exponent in (11) is determined by $\mu$, while the power of the prefactor in (11) is determined by the ratio $\frac{\mu}{\nu}$.

It must be stressed that these formulas (11) and (11) are formal, being based on assumed analyticity properties of the function $f(g)$. The Borel dispersion relations could be complicated by the appearance of additional poles and/or cuts in the complex $g$ plane, signalling new physics [27]. In certain special cases these analyticity assumptions can be tested rigorously, but we have in mind the situation in which one is confronted with the expansion coefficients $a_n$ of a perturbative expansion, without corresponding information about the function that this series is supposed to represent. This is a common circumstance in physical applications of perturbation theory. For example, Borel techniques have recently been used to study the divergence of the derivative expansion for QED effective actions in inhomogeneous backgrounds [28].

Returning to the Euler-Heisenberg effective Lagrangian, the question of whether the perturbative expansion is alternating or non-alternating is directly relevant. For a uniform magnetic background, the one-loop Euler-Heisenberg series (2) is precisely of the form (5) with

$$g = \left(\frac{eB}{m}\right)^2.$$  

Moreover, from (4) the expansion coefficients $a_n^{(1)}$ have leading large-order behaviour of the form (5), with $\rho = \frac{1}{\pi}$, and $\mu = \nu = 2$. In fact, taking into account the sub-leading corrections indicated in (5), the proper-time integral representation (5) of the divergent series (2) [28]. For a uniform electric background, the only difference perturbatively is that $B^2$ is replaced by $-E^2$, that is, $g = \left(\frac{eE}{m}\right)^2$ is replaced by $-g = -(\frac{eE}{m})^2$. So the perturbative one-loop Euler-Heisenberg series (2) becomes non-alternating. Then from (11), with $\rho = \frac{1}{\pi}$ and $\mu = \nu = 2$, we immediately deduce the leading behaviour of the imaginary part of the one-loop Euler-Heisenberg effective Lagrangian:

$$\text{Im}\mathcal{L}^{(1)} \sim m^4 \frac{eE}{m^2} \frac{eE}{m^2} \exp \left[ -\frac{m^2\pi}{eE} \right]$$  

(12)

This imaginary part has direct physical significance - it gives half the electron-positron pair production rate in the uniform electric field $E$. Actually, since we also know the sub-leading corrections (4) to the leading large-order behaviour of the expansion coefficients $a_n^{(1)}$, we can apply (11) successively to go beyond the leading behaviour in (12):

$$\text{Im}\mathcal{L}^{(1)} \sim m^4 \frac{eE}{m^2} \frac{eE}{m^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left[ -\frac{m^2\pi k}{eE} \right]$$  

(13)

This is Schwinger’s classic result [3] for the imaginary part of the one-loop effective Lagrangian in a uniform electric field $E$. To elucidate the physical meaning of the individual terms of this series it is useful to employ the following alternative representation due to Nikishov [4],

$$\frac{2}{\hbar} \text{Im}\mathcal{L}^{(1)} VT = -\sum_{r} \int \frac{d^3pV}{(2\pi\hbar)^3} \ln(1 - \bar{n}_p),$$  

$$\bar{n}_p = \exp\left(-\pi \frac{m^2 + p^2}{eE}\right)$$  

(14)

Here $\bar{n}_p$ is the mean number of pairs produced by the field in the state with given momentum $p$ and spin projection $r$. An expansion of the logarithm in $\bar{n}_p$ and term-by-term integration leads back to Schwinger’s formula (13). Thus the leading term in this formula can be interpreted as the mean number $\bar{n}_p$ of pairs in the unit 4-volume $VT$, while the higher ($k \geq 2$) terms describe the coherent creation of $k$ pairs.

Pair creation can occur for any value of the electric field strength, though due to the exponential suppression factors one is presently still far away from being able to observe spontaneous pair creation by macroscopic fields in the laboratory. However, it can be arranged for electrons traversing the focus of a terawatt laser to see a critical field in their rest frame. This has recently led to the first observation of pair creation in a process involving only real photons [20].

For the one-loop Euler-Heisenberg QED effective Lagrangian, this large-order perturbation theory analysis is greatly simplified by the fact that we know the exact formula (5) for the expansion coefficients $a_n^{(1)}$. This
will not be the case below, when we discuss the two-loop Euler-Heisenberg effective Lagrangian. So, for the sake of numerical comparison, we compare the exact one-loop coefficients $a_n^{(1)}$ with their leading large-order behaviour. The coefficients are listed in Table 1 up to $n = 15$. Since the growth is fast, it is convenient to compare the logarithms, as is done in Figure 1. With 16 terms it is straightforward to fit the the values of $\rho$, $\mu$ and $\nu$ appearing in (4); moreover, there is sufficient accuracy to fit the overall coefficient $\frac{1}{8\pi^4}$. In Figure 1 we plot $A_n^{(1)} = \log |a_n^{(1)}|$, and $C_n^{(1)} = \log[\Gamma(2n+2)/(8\pi^{2n+4})]$. The agreement is spectacular, already for $n = 0$. Indeed, on this scale the two plots are indistinguishable. To go beyond the leading large-order behaviour, we plot the difference $A_n^{(1)} - C_n^{(1)}$. This can be fitted to the correct form $\log(1 + \frac{1}{2n+4}) \approx \frac{1}{2n+4}$, with remarkable accuracy for $n \geq 2$, as illustrated in Figure 2.

TABLE I. The one-loop Euler-Heisenberg coefficients $a_n^{(1)}$ from (4) and their magnitudes $|a_n^{(1)}|$. The last two columns list the calculated two-loop Euler-Heisenberg coefficients $a_n^{(2)}$ in (21), and their magnitudes $|a_n^{(2)}|$. Note that both $a_n^{(1)}$ and $a_n^{(2)}$ alternate in sign and grow factorially in magnitude.

| $n$ | $a_n^{(1)}$ | $|a_n^{(1)}|$ | $a_n^{(2)}$ | $|a_n^{(2)}|$ |
|-----|-------------|-------------|-------------|-------------|
| 0   | $\frac{1}{20}$ | 0.00138889  | $\frac{1}{8}$ | 0.790123    |
| 1   | $\frac{-1}{1260}$ | 0.000793651 | $\frac{1}{2025}$ | 0.601975    |
| 2   | $\frac{1}{30}$ | 0.0015873  | $\frac{135308}{99229}$ | 1.36365    |
| 3   | $\frac{-2}{257}$ | 0.00673401 | $\frac{791384}{125775}$ | 6.20328    |
| 4   | $\frac{11056}{225229}$ | 0.0490887 | $\frac{8519287552}{180093175}$ | 47.3048    |
| 5   | $\frac{-64}{111}$ | 0.547009  | $\frac{-167313653536}{89432124}$ | 544.23     |
| 6   | $\frac{231488}{20775}$ | 8.64568   | $\frac{132488118784}{11036625}$ | 8811.21    |
| 7   | $\frac{-11229952}{21154062270081664}$ | 183.956   | $\frac{-21154062270081664}{11036625}$ | 191062     |
| 8   | $\frac{21520656}{61847}$ | 5069.69   | $\frac{34572799881766503424}{89432124}$ | 5.34718 10^6 |
| 9   | $\frac{-127278777}{1056035}$ | 175674   | $\frac{-6386544346643762224}{34029543873125}$ | 1.87676 10^8 |
| 10  | $\frac{7765178531888}{1056035}$ | 7.47579 10^6 | $\frac{188413511360959841129676288}{25632171540080664}$ | 8.07369 10^9 |
| 11  | $\frac{-662363120222}{225}$ | 3.83273 10^8 | $\frac{-173667069737807846129664}{426449669696864}$ | 4.17809 10^11 |
| 12  | $\frac{35575875420672}{522855}$ | 2.33002 10^10 | $\frac{298448156828080479680470208}{116548600343557375}$ | 2.5607310^13 |
| 13  | $\frac{-72279050546275104}{430104}$ | 1.65728 10^12 | $\frac{-261690819380374253044755653922816}{14265124909993875}$ | 1.83448 10^15 |
| 14  | $\frac{1616768040230331984}{118575}$ | 1.36349 10^14 | $\frac{412837705885275820951997292842967552}{2257119099206430875}$ | 1.51892 10^17 |
| 15  | $\frac{-127193705305899008}{99}$ | 1.2840 10^16 | $\frac{-401755841596726235272214096383644249088}{2791061222991831125}$ | 1.43944 10^19 |
FIG. 1. Plots of $A_n^{(1)} = \log(|a_n^{(1)}|)$ and $C_n^{(1)} = \log[2\Gamma(2n + 2)/(8\pi^{2n+4})]$ up to $n = 15$. The two plots are indistinguishable on this scale.

FIG. 2. Plot (solid line) of the difference $A_n^{(1)} - C_n^{(1)}$ between $A_n^{(1)} = \log(|a_n^{(1)}|)$ and the leading large order fit $C_n^{(1)} = \log[\Gamma(2n + 2)/(8\pi^{2n+4})]$ from (1), up to $n = 15$. The dashed line shows the fit of this difference to the function $\log(1 + \frac{1}{2\beta^2 n^2})$, which is excellent already for $n \geq 2$.

III. THE TWO-LOOP EULER-HEISENBERG EFFECTIVE LAGRANGIAN

We now turn to the two-loop Euler-Heisenberg effective Lagrangian, describing the effect of a single photon exchange in the electron loop. This quantity was first studied by Ritus [7]. Using the exact electron propagator in a constant field found by Fock [29] and Schwinger [3], and a proper-time cutoff as the UV regulator, he obtained the on-shell renormalized $L^{(2)}$ in terms of a certain two-parameter integral. From this integral the imaginary part of the Lagrangian was then extracted by a painstaking analysis of its analyticity properties, yielding a representation analogous to Schwinger’s one-loop formula (13) [30]. Adding up the one-loop and the two-loop contributions to the imaginary part the result reads

$$\text{Im}L^{(1)} + \text{Im}L^{(2)} \sim \frac{m^4}{8\pi^3} \beta^2 \sum_{k=1}^{\infty} \left[ \frac{1}{k^2} + \alpha \pi K_k(\beta) \right] \exp \left[ -\frac{\pi k}{\beta} \right]$$

where $\beta = \frac{eE}{m^2}$. For the function $K_k(\beta)$ the following small $\beta$-expansion was obtained in [30],

$$K_k(\beta) = -\frac{c_k}{\sqrt{\beta}} + 1 + O(\sqrt{\beta})$$

$c_1 = 0$, \hspace{1cm} $c_k = \frac{1}{2\sqrt{k}} \sum_{l=1}^{k-1} \frac{1}{\sqrt{l(k-l)}}, \hspace{1cm} k \geq 2$
According to [7] the physical interpretation of the individual terms in the series in terms of coherent multipair creation can be carried over to the two-loop level. This requires one to absorb the term linear in $\alpha$ into the exponential factor, rewriting

$$
\left[ \frac{1}{k^2} + \alpha \pi K_0 \left( \frac{eE}{m^2} \right) \right] \exp \left[ - \frac{k \pi m^2}{eE} \right] \sim \frac{1}{k^2} \exp \left[ - \frac{k \pi m^2}{eE} \right]
$$

(17)

The individual terms in the expansion (15) of $K_0(\beta)$ are then to be absorbed into the mass shift $m_\ast - m$. For the lowest order terms in this expansion, those given in (15), the physical interpretation of the corresponding mass shifts in terms of the coherent pair production picture is discussed in [7]. For example, the leading "1" in the expansion of $K_1(\beta)$ after exponentiation yields a mass shift that can be identified as the classical change in mass caused for one particle in a created pair by the acceleration due to its partner.

Assuming this exponentiation to work one can, of course, obtain some partial information on the higher-loop corrections to the imaginary part. More remarkably, since the above physical interpretation requires the mass appearing in the exponent to be the physical one, it allows one to determine the physical renormalized loop corrections to the imaginary part. Moreover, since the above physical interpretation requires the mass appearing in the exponent to be the physical one, it allows one to determine the physical renormalized mass from an inspection of the renormalized Lagrange function alone, rather than by a calculation of the (lower order) electron self energy.

Following the pioneering work by Ritus and his collaborators, a first recalculation of the Euler-Heisenberg Lagrangian was done by Dittrich and Reuter [9] for the magnetic field case. The more recent recalculation in [9] showed that the two previous results were actually incompatible, and differed precisely by a finite electron mass renormalisation. All three calculations had been done using a proper-time cutoff rather than dimensional regularisation. This cutoff leads to relatively simple integrals, but due to its non-universality makes it, at the two-loop level, already non-trivial to determine the physical renormalized electron mass. A fourth calculation of this quantity [12], now using dimensional regularisation, yielded complete agreement with Ritus’s result after a perturbative expansion of both results in powers of the $B$ field had been performed. In the following we will push the same calculation to $O(B^3)$, and analyze the rate of growth of the expansion coefficients.

IV. BOREL ANALYSIS OF THE TWO-LOOP EULER-HEISENBERG EFFECTIVE LAGRANGIAN

The world-line expression for the two-loop on-shell renormalized Euler-Heisenberg effective Lagrangian is [9,12]:

$$
L^{(2)} = \alpha \frac{m^4}{(4\pi)^3} \left( \frac{eB}{m^2} \right)^2 \int_0^\infty \frac{ds}{s^3} e^{-m^2 s/(eB)} \int_0^1 du \left[ \frac{L(s, u)}{u(1 - u)} - 2s^2 + \frac{6}{u(1 - u)} \left( \frac{s^2}{\sinh^2 s} + s \coth s \right) \right]
$$

$$
-12\alpha \frac{m^4}{(4\pi)^3} eB \int_0^\infty \frac{ds}{s} e^{-m^2 s/(eB)} \left[ \coth s - \frac{1}{s} \frac{s}{3} \right] \left[ \frac{3}{2} - \gamma - \log \left( \frac{m^2 s}{eB} \right) \right]
$$

(18)

Here $\alpha \approx \frac{1}{137}$ is the fine-structure constant, and $\gamma = 0.5772...$ is Euler’s constant. The function $L(s, u)$ appearing in the integrand of the first term in (18) is defined by the following relations:

$$
L(s, u) = s \coth s \left[ \frac{\log \left( \frac{u(1 - u)}{G(u, s)} \right)}{u(1 - u) - G(u, s)} \right]^2 F_1 + \frac{F_2}{G(u, s)[u(1 - u) - G(u, s)]} + \frac{F_3}{u(1 - u)[u(1 - u) - G(u, s)]}
$$

$$
G(u, s) = \frac{\cosh s - \cosh((1 - 2u)s)}{2s \sinh s}
$$

$$
F_1 = 4s(\coth s - \tanh s)G(u, s) - 4u(1 - u)
$$

$$
F_2 = 2(1 - 2u) \frac{\sinh((1 - 2u)s)}{\sinh s} + s(8\tanh s - 4\coth s)G(u, s) - 2
$$

$$
F_3 = 4u(1 - u) - 2(1 - 2u) \frac{\sinh((1 - 2u)s)}{\sinh s} - 4sG(u, s)\tanh s + 2
$$

(19)
The second term in the two-loop expression \((18)\) is generated by the one-loop electron mass renormalisation, which at the two-loop level becomes necessary in addition to the photon wave function renormalisation. For this mass renormalization term we have found the following exact expansion:

\[
\frac{eB}{m^2} \int_0^\infty \frac{ds}{s} e^{-m^2 s/(eB)} \left[ \coth s - 1 - \frac{s}{3} \right] \left[ \frac{3}{2} - \gamma - \log \left( \frac{m^2 s}{eB} \right) \right] + \frac{eB}{m^2 s} \] = \left( \frac{eB}{m^2} \right)^2 \sum_{n=0}^{\infty} \frac{2^{2n+4}B_{2n+4}}{(2n+4)(2n+3)} \left( \frac{3}{2} - \gamma - \psi(2n+2) \right) \left( \frac{eB}{m^2} \right)^{2n} \quad (20)
\]

Here \(B_{2n}\) are the Bernoulli numbers, and \(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\) is the digamma function \([13]\). We have not succeeded in finding a closed-form expression for the expansion coefficients arising from the expansion of the double integral in \((18)\), although we suspect that one may exist. Instead, we have made an algebraic expansion of this integral, using MATHEMATICA and MAPLE. When combined with the exact expansion \((20)\) of the mass renormalization term we obtain an expansion of the form:

\[
\mathcal{L}^{(2)} = \alpha \frac{m^4}{(4\pi)^3} \left( \frac{eB}{m^2} \right)^4 \sum_{n=0}^{\infty} a_n^{(2)} \left( \frac{eB}{m^2} \right)^{2n} \quad (21)
\]

The expansion coefficients \(a_n^{(2)}\) are listed in Table 1, up to \(n = 15\). Note that those coefficients are in some sense less universal than their one-loop counterparts, since they depend on the one-loop normalization condition imposed on the renormalized electron mass.

Several comments are in order. First, the two-loop expansion coefficients \(a_n^{(2)}\) alternate in sign, just as in the one-loop magnetic background case \([4]\). Second, the magnitude \(|a_n^{(2)}|\) is clearly growing factorially fast with \(n\). Thus, the two-loop Euler-Heisenberg series \((21)\) is a divergent series, as is the one-loop Euler-Heisenberg series \([3]\). Note also that for each series the smallest magnitude coefficient is reached already for \(n = 1\), after which the coefficients begin to increase rapidly in magnitude.

To extract the leading large-\(n\) growth of \(|a_n^{(2)}|\) we fit \(a_n^{(2)}\) to the form in \((4)\). Once again, it is convenient to work with the logarithm \(D_n = \log(|a_n|)\) since the growth is so rapid. It is relatively straightforward to find that \(\mu = \nu = 2\) and \(\rho = \frac{1}{\pi}\). It is more difficult to fit the overall coefficient, but if we assume this is a simple power of \(\pi\) then our best fit for the leading large-order growth of the two-loop expansion coefficients in \((21)\) is:

\[
a_n^{(2)} \sim (-1)^n \frac{16}{\pi^2} \frac{\Gamma(2n+2)}{\pi^{2n}} \quad (22)
\]

This leading fit is displayed in Figure 3, in terms of \(A_n^{(2)} = \log(|a_n^{(2)}|)\). The fit is not as good as the one-loop fit shown in Figure 1, but it is still very good.

Note the remarkable similarity of the leading large-order growth \((22)\) of the two-loop expansion coefficients to the leading large-order growth of the one-loop expansion coefficients in \((4)\). The only difference is the overall coefficient. The parameters \(\rho, \mu\) and \(\nu\) in the general form \((1)\) are identical. Using the Borel technique to relate this leading growth rate to the leading non-perturbative imaginary part of the effective Lagrangian in a uniform electric field \(E\), we deduce that the two-loop leading imaginary part is proportional to the one-loop leading imaginary part \((12)\). In fact, from \((11)\) and \((22)\), we find the leading contribution

\[
\text{Im} \mathcal{L}^{(2)} \sim \alpha \frac{m^4}{8\pi^3} \left( \frac{eE}{m^2} \right)^2 \exp \left[ -\frac{m^2 \pi}{eE} \right] \quad (23)
\]
FIG. 3. Plots of $A_n^{(2)} = \log(|a_n^{(2)}|)$ (solid line), and $C_n^{(2)} = \log[16\Gamma(2n + 2)/\pi^{2n+2}]$ (dashed line), up to $n = 15$. While the agreement is not as good as for the one-loop case shown in Figure 1, the fit is still very good, even at low orders in $n$.

This has exactly the same dependence on the electric field $E$ as the one-loop case. So to two-loop order the leading non-perturbative behaviour of the imaginary part of the effective Lagrangian is:

$$\text{Im} \left( L^{(1)} + L^{(2)} \right) \sim (1 + \alpha \pi) \frac{m^4}{8\pi^2} \left( \frac{eE}{m^2} \right)^2 \exp \left[ -\frac{m^2\pi}{eE} \right]$$

(24)

This agrees with the leading term of Ritus’s formula (15).

FIG. 4. Plot (solid line) of the difference $A_n^{(2)} - C_n^{(2)}$ between $A_n^{(2)} = \log(|a_n^{(2)}|)$ and the leading large order fit from (22), $C_n^{(2)} = \log[16\Gamma(2n + 2)/\pi^{2n+2}]$, up to $n = 15$. The dashed line represents the fit in (25).

To go beyond this leading term we need to look at corrections to the leading behaviour in (23). In Figure 4 we plot the difference of the logarithms, and we see that the $n$ dependence is much more gentle than the rapid fall-off found in the one-loop case, which was plotted in Figure 2. In fact, from the terms up to $n = 15$, we obtain the fit

$$a_n^{(2)} \sim (-1)^n \frac{16}{\pi^2} \frac{\Gamma(2n + 2)}{\pi^{2n}} \left[ 1 - \frac{0.44}{\sqrt{n}} + \ldots \right]$$

(25)

This is a considerably weaker $n$ dependence than is found for the first correction in the one-loop case (4). This means that in the two-loop case the dominant corrections are to the prefactor in the leading behaviour (23). This is in contrast to the one-loop case (13), where the first correction to the leading behaviour is exponentially suppressed. Indeed, applying the Borel relations, the correction term (25) leads to

$$\text{Im} \left( L^{(1)} + L^{(2)} \right) \sim \left( 1 + \alpha \pi \left[ 1 - (0.44)\sqrt{\frac{2eE}{\pi m^2}} + \ldots \right] \right) \frac{m^4}{8\pi^3} \left( \frac{eE}{m^2} \right)^2 \exp \left[ -\frac{m^2\pi}{eE} \right]$$

(26)
We emphasize that the fit in (25) is based on a simple fit to the first 16 two-loop coefficients. Nevertheless, the structure of (26) conforms already to the form of Ritus’s expansion in eq.(15). It would be interesting to probe this correction term in more detail by a further study of the analyticity properties of the integral representations [7,9,12] of the two-loop Euler-Heisenberg effective Lagrangian, or by looking at still higher orders in perturbation theory.

V. CONCLUDING REMARKS

Our analysis also permits us to study the dependence of (23), the leading non-perturbative imaginary contribution to the effective Lagrangian, on the electron mass renormalisation. In the world-line expression (13) for the two-loop Euler-Heisenberg effective Lagrangian, a finite change of the renormalised electron mass would amount to an arbitrary change of the constant $\frac{3}{2} - \gamma$ appearing in the second bracket of the second term. For example, in [8] it had been shown that the renormalised Lagrangian obtained by [8] differs from (13) precisely by a replacement of $\frac{3}{2}$ by $\frac{5}{6}$. A separate study of the contributions of the first and second term in (18) shows that, due to cancellations between both terms, the leading large-\(n\) growth of their sum is smaller than for each term separately. However this property holds true only if the renormalised electron mass is the physical one.

For definiteness, in Figure 5 we compare the correct leading growth (22) of the two-loop coefficients with the coefficients obtained by expanding out the two-loop result of [8]. The difference in the leading large-\(n\) growths is obvious. Thus, in agreement with the analysis of [8], we find that if and only if expressed in terms of the true electron mass will the imaginary part of the renormalised two-loop Lagrangian show the same exponential suppression factor \(\exp[-\pi m^2/(eE)]\) as for the one-loop Lagrangian.

To summarize, we have constructed the imaginary part of the two-loop QED Euler-Heisenberg Lagrangian in a constant electric field by a computer based calculation of its weak field expansion together with Borel summation techniques. The knowledge of the first 16 coefficients has turned out to be sufficient to verify the structure of the leading \((k = 1)\) term in Ritus’s eq.(15), and to obtain a numerical value for the first \(O(\sqrt{eE/m^2})\) correction contained in that formula. The method used here is significantly simpler than the one in [7,30], where the imaginary part was obtained by an analysis of the analyticity properties of the two-loop parameter integrals.

In particular, we have seen that the large order behaviour of the two-loop coefficients is the same (up to an overall constant factor) as the one-loop case. This means that the leading contribution to the imaginary part of the two-loop effective Lagrangian has the same form as in the one-loop case. This gives a new perspective to Ritus’s arguments [7] that the true renormalized electron mass \(m\) is such that the leading exponential factor in the pair production rate is \(\exp[-\pi m^2/(eE)]\). Since those arguments pertain to arbitrary loop orders,
and the leading exponential factor is directly related to the leading growth rate in the weak field expansion, they also lead one to expect that the Euler-Heisenberg Lagrangian may be amenable to this type of Borel analysis at any fixed loop order in perturbation theory.

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