Abelian Theory

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This exposition begins with a systematic account of the theory of group schemes, ultimately specializing to algebraic tori. This done, I then take up two central objectives.

- First, I define the local Langlands correspondence in the case of a $\mathbb{K}$-torus ($\mathbb{K}$ a non-archimedean local field) and prove it when

$$T = \text{Res}_{L/\mathbb{K}}(G_{m,L}),$$

$L$ a finite Galois extension of $\mathbb{K}$ ($\text{Res}_{L/\mathbb{K}}$ the functor of restriction of the scalars). Historically, this was first done by Langlands in the late 1960’s and served to confirm his far reaching conjectures for reductive groups in the “simplest situation” ($GL_1$ “is” local class field theory . . . ).

- Second, working within the context of a $\mathbb{Q}$-torus $T$, I define precisely the terms “Tamagawa measure” and “Tamagawa number”, make some computations to illustrate the general picture, and set up Ono’s celebrated theorem connecting these notions with the Tate-Shafarevich group.
§1. GROUP SCHEMES

1: NOTATION SCH is the category of schemes, RNG is the category of commutative rings with unit.

Fix a scheme $S$ — then the category $\text{SCH}/S$ of schemes over $S$ (or of $S$-schemes) is the category whose objects are the morphisms $X \to S$ of schemes and whose morphisms

$$\text{Mor}(X \to S, Y \to S)$$

are the morphisms $X \to Y$ of schemes with the property that the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & \longleftarrow & S
\end{array}
$$

commutes.

[Note: Take $S = \text{Spec}(\mathbb{Z})$ — then $\text{SCH}/S = \text{SCH}$.]

2: N.B. If $S = \text{Spec}(A)$ ($A$ in RNG) is an affine scheme, then the terminology is “schemes over $A$” (or “$A$-schemes”) and one writes $\text{SCH}/A$ in place of $\text{SCH}/\text{Spec}(A)$.

3: NOTATION Abbreviate $\text{Mor}(X \to S, Y \to S)$ to $\text{Mor}_S(X, Y)$ (or to $\text{Mor}_A(X, Y)$ if $S = \text{Spec}(A)$).

4: REMARK The $S$-scheme $\text{id}_S : S \to S$ is a final object in $\text{SCH}/S$. 1-1
5: **THEOREM** \( \text{SCH}/S \) has pullbacks:

\[
\begin{align*}
X \times_S Y & \rightarrow Y \\
\downarrow & \downarrow \\
X & \rightarrow S
\end{align*}
\]

[Note: Every diagram]

\[
\begin{array}{c}
\xymatrix{Z \ar[rr]^u & & \quad X \times_S Y \ar[rr]^q & & Y \quad (f \circ u = g \circ v) \\
& \downarrow^v & \downarrow^p & \downarrow^q & \\
X \times_S Y & \ar[r]_p & X & \ar[r]_f & S
\end{array}
\]

admits a unique filler

\[
(u, v)_S : Z \rightarrow X \times_S Y
\]

such that

\[
\begin{cases}
p \circ (u, v)_S = u \\
q \circ (u, v)_S = v
\end{cases}
\]

6: **FORMALITIES** Let \( X, Y, Z \) be objects in \( \text{SCH}/S \) — then

\[
\begin{align*}
X \times_S S & \approx X, \\
X \times_S Y & \approx Y \times_S X,
\end{align*}
\]

and

\[
(X \times_S Y) \times_S Z \approx X \times_S (Y \times_S Z).
\]

7: **REMARK** If \( X, Y, X', Y' \) are objects in \( \text{SCH}/S \) and if \( u : X \rightarrow X', v : Y \rightarrow Y' \) are \( S \)-morphisms, then there is a unique morphism \( u \times_S v \) (or just \( u \times v \)) rendering the
diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{u} & X' & \xrightarrow{} & S \\
X \times_{S} Y & \xrightarrow{u \times_{S} v} & X' \times_{S} Y' & \xrightarrow{} & Y' \times_{S} Y \\
Y & \xrightarrow{v} & Y' & \xrightarrow{} & S \\
\end{array}
\]

commutative.

[Spelled out, \( u \times_{S} v = (u \circ p, v \circ q)_{S} \).]

8: BASE CHANGE Let \( u: S' \rightarrow S \) be a morphism in \( \text{SCH} \).

- If \( X \rightarrow S \) is an \( S \)-object, then \( X \times_{S} S' \) is an \( S' \)-object via the projection

\[
X \times_{S} S' \rightarrow S',
\]

denoted \( u^*(X) \) or \( X_{(S')} \) and called the base change of \( X \) by \( u \).

- If \( X \rightarrow S, Y \rightarrow S \) are \( S \)-objects and if \( f: (X \rightarrow S) \rightarrow (Y \rightarrow S) \) is an \( S \)-morphism, then

\[
\begin{array}{c}
X \times_{S} S' \xrightarrow{f \times_{S} \text{id}_{S'}} Y \times_{S} S'
\end{array}
\]

is a morphism of \( S' \)-objects, denoted \( u^*(f) \) or \( f_{(S')} \) and called the base change of \( f \) by \( u \).

These considerations thus lead to a functor

\[
u^*: \text{SCH}/S \rightarrow \text{SCH}/S'
\]
called the base change by \( u \).
9: N.B. If $u' : S'' \to S'$ is another morphism in $\text{SCH}$, then the functors $(u \circ u')^*$ and $(u')^* \circ u$ from $\text{SCH}/S$ to $\text{SCH}/S''$ are isomorphic.

10: LEMMA Let $u : S' \to S$ be a morphism in $\text{SCH}$. Suppose that $T' \to S'$ is an $S'$-object -- then $T'$ can be viewed as an $S$-object $T$ via postcomposition with $u$ and there are canonical mutually inverse bijections

$$\text{Mor}_{S'}(T', X_{(S')}) \cong \text{Mor}_{S}(T, X)$$

functorial in $T'$ and $X$.

11: NOTATION Each $S$-scheme $X \to S$ determines a functor

$$(\text{SCH}/S)^{\text{OP}} \to \text{SET},$$

viz. the assignment

$$T \to \text{Mor}_{S}(T, X) \equiv X_{S}(T),$$

the set of $T$-valued points of $X$.

[Note: In terms of category theory,

$$X_{S}(T) = h_{X \to S}(T \to S).]$$

12: LEMMA To give a morphism $(X \to S) \xrightarrow{f} (Y \to S)$ in $\text{SCH}/S$ is equivalent to giving for all $S$-schemes $T$ a map

$$f(T) : X_{S}(T) \to Y_{S}(T)$$
which is functorial in $T$, i.e., for all morphisms $u: T' \to T$ of $S$-schemes the diagram

$$
\begin{array}{ccc}
X_S(T) & \xrightarrow{f(T)} & Y_S(T) \\
X_S(u) & & \downarrow Y_S(u) \\
X_S(T') & \xrightarrow{f(T')} & Y_S(T')
\end{array}
$$

commutes.

13: DEFINITION  A group scheme over $S$ (or an $S$-group) is an object $G$ of $\text{SCH}/S$ and $S$-morphisms

$$
m : G \times_S G \to G \quad \text{("multiplication")}
$$

$$
e : S \to G \quad \text{("unit")}
$$

$$
i : G \to G \quad \text{("inversion")}
$$

such that the diagrams

$$
\begin{array}{ccc}
G \times_S G \times_S G & \xrightarrow{m \times \text{id}_G} & G \times_S G \\
\text{id}_G \times m & & m \\
G \times_S G & \xrightarrow{m} & G
\end{array}
$$

$$
\begin{array}{ccc}
G \times_S S & \xrightarrow{(\text{id}_G, e)_S} & G \times_S G \\
\quad & & m \\
G & \xrightarrow{\text{id}_G} & G
\end{array}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{(\text{id}_G, i)_S} & G \times_S G \\
\downarrow & & m \\
S & \xrightarrow{e} & G
\end{array}
$$

commute.
14: **REMARK** To say that \((G; m, e, i)\) is a group scheme over \(S\) amounts to saying that \(G\) is a group object in \(\text{SCH}/S\).

15: **LEMMA** Let \(G\) be an \(S\)-scheme — then \(G\) gives rise to a group scheme over \(S\) iff for all \(S\)-schemes \(T\), the set \(G_S(T)\) carries the structure of a group which is functorial in \(T\) (i.e., for all \(S\)-morphisms \(T' \to T\), the induced map \(G_S(T) \to G_S(T')\) is a homomorphism of groups).

16: **REMARK** It suffices to define functorial group structures on the \(G_S(A)\), where \(\text{Spec}(A) \to S\) is an affine \(S\)-scheme.

[This is because morphisms of schemes can be “glued”.

17: **LEMMA** Let \(u : S' \to S\) be a morphism in \(\text{SCH}\). Suppose that \((G; m, e, i)\) is a group scheme over \(S\) — then

\[
(G \times_S S'; m_{(S')}, e_{(S')}, i_{(S')})
\]

is a group scheme over \(S'\).

[Note: For every \(S'\)-object \(T' \to S'\),

\[
(G \times_S S')_{S'}(T') = G_S(T),
\]

where \(T\) is the \(S\)-object \(T' \to S' \xrightarrow{u} S\).]

18: **THEOREM** If \((X, \mathcal{O}_X)\) is a locally ringed space and if \(A\) is a commutative ring with unit, then there is a functorial set-theoretic bijection

\[
\text{Mor}(S, \text{Spec}(A)) \approx \text{Mor}(A, \Gamma(X, \mathcal{O}_X)).
\]
[Note: The “Mor” on the LHS is in the category of locally ringed spaces and the “Mor” on the RHS is in the category of commutative rings with unit.]

19: EXAMPLE Take $S = \text{Spec}(\mathbb{Z})$ and let

$$A^n = \text{Spec}(\mathbb{Z}[t_1, \ldots, t_n]).$$

Then for every scheme $X$,

$$\text{Mor}(X, A^n) \approx \text{Mor}(\mathbb{Z}[t_1, \ldots, t_n], \Gamma(X, \mathcal{O}_X)) \approx \Gamma(X, \mathcal{O}_X)^n (\phi \to (\phi(t_1), \ldots, \phi(t_n))).$$

Therefore $A^n$ is a group object in $\text{SCH}$ called affine $n$-space.

[Note: Here $\Gamma(X, \mathcal{O}_X)$ is being viewed as an additive group, hence the underlying multiplicative structure is being ignored.]

20: N.B. Given any scheme $S$,

$$A^n_S = A^n \times_{\mathbb{Z}} S \to S$$

is an $S$-scheme and for every morphism $S' \to S$,

$$A^n_S \times_S S' \approx A^n \times_{\mathbb{Z}} S \times_S S' \approx A^n_{S'}.$$  

21: NOTATION Write $G_a$ in place of $A^1$.

22: NOTATION Given $A$ in $\text{RNG}$, denote

$$G_a \times_{\mathbb{Z}} \text{Spec}(A)$$

by $G_a \otimes A$ or still, by $G_{a,A}$. 

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23: N.B. \[ G_{a,A} = \text{Spec}(\mathbb{Z}[t]) \times_{\mathbb{Z}} \text{Spec}(A) \]
\[ = \text{Spec}(\mathbb{Z}[t] \otimes A) \]
\[ = \text{Spec}(A[t]). \]

24: LEMMA \( G_{a,A} \) is a group object in \( \text{SCH}/A \).

There are two other “canonical” examples of group objects in \( \text{SCH}/A \).

- \( G_{m,A} = \text{Spec}(A[u,v]/(uv - 1)) \)
  which assigns to an \( A \)-scheme \( X \) the multiplicative group \( \Gamma(X, \mathcal{O}_X)^\times \) of invertible elements in the ring \( \Gamma(X, \mathcal{O}_X) \).

- \( \text{GL}_{n,A} = \text{Spec}(A[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}]) \)
  which assigns to an \( A \)-scheme \( X \) the group

\[ \text{GL}_n(\Gamma(X, \mathcal{O}_X)) \]

of invertible \( n \times n \)-matrices with entries in the ring \( \Gamma(X, \mathcal{O}_X) \).

25: DEFINITION If \( G \) and \( H \) are \( S \)-groups, then a homomorphism from \( G \) to \( H \) is a morphism \( f : G \to H \) of \( S \)-schemes such that for all \( S \)-schemes \( T \) the induced map \( f(T) : G_S(T) \to H_S(T) \) is a group homomorphism.

26: EXAMPLE Take \( S = \text{Spec}(A) \) — then

\[ \det_A : \text{GL}_{n,A} \to G_{m,A} \]

is a homomorphism.
**27: Definition** Let $G$ be a group scheme over $S$—then a subscheme (resp. an open subscheme, resp. a closed subscheme) $H \subset G$ is called an $S$-subgroup scheme (resp. an open $S$-subgroup scheme, resp. a closed $S$-subgroup scheme) if for every $S$-scheme $T$, $H_S(T)$ is a subgroup of $G_S(T)$.

**28: Example** Given a positive integer $n$, $\mu_{n,A}$ is the group object in $\text{SCH}/A$ which assigns to an $A$-scheme $X$ the multiplicative subgroup of $\Gamma(X, \mathcal{O}_X)^\times$ consisting of those $\phi$ such that $\phi^n = 1$, thus

$$\mu_{n,A} = \text{Spec}(A[t]/(t^n - 1))$$

and $\mu_{n,A}$ is a closed $A$-subgroup of $G_{m,A}$.

**29: Example** Fix a prime number $p$ and suppose that $A$ has characteristic $p$.

Given a positive integer $n$, $\alpha_{n,A}$ is the group object in $\text{SCH}/A$ which assigns to an $A$-scheme $X$ the additive subgroup of $\Gamma(X, \mathcal{O}_X)$ consisting of those $\phi$ such that $\phi^{p^n} = 0$, thus

$$\alpha_{n,A} = \text{Spec}(A[t]/(t^{p^n}))$$

and $\alpha_{n,A}$ is a closed $A$-subgroup of $G_{a,A}$.

**30: Construction** Let $f : G \to H$ be a homomorphism of $S$-groups. Define $\text{Ker}(f)$ by the pullback square

$$\begin{array}{ccc}
\text{Ker}(f) = S \times_H G & \longrightarrow & G \\
\downarrow & & \downarrow f \\
S & \underset{e}{\longrightarrow} & H
\end{array}$$

Then for all $S$-schemes $T$,

$$\text{Mor}_S(T, \text{Ker}(f)) = \text{Ker}(G_S(T) \xrightarrow{f(T)} H_S(T)),$$
so Ker$(f)$ is an $S$-group.

31: EXAMPLE The kernel of $\det_A$ is $\text{SL}_{n,A}$.

32: N.B. Other kernels are $\mu_{n,A}$ and $\alpha_{n,A}$.

33: CONVENTION If $P$ is a property of morphisms of schemes, then an $S$-group $G$ has property $P$ if this is the case of its structural morphism $G \to S$.

E.g.: The property of morphisms of schemes being quasi-compact, locally of finite type, separated, étale etc.

34: LEMMA Let

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{\Delta_{Y/X}} & Y
\end{array}
$$

be a pullback square in $\text{SCH}$. Suppose that $f$ is a closed immersion—then the same holds for $f'$.

35: APPLICATION Let $g : Y \to X$ be a morphism of schemes that has a section $s : X \to Y$. Assume: $g$ is separated—then $s$ is a closed immersion.

[The commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow{s} & & \downarrow{\Delta_{Y/X}} \\
Y & \xrightarrow{(id_Y,s \circ g)_X} & Y \times_X Y
\end{array}
$$

is a pullback square in $\text{SCH}$. But $g$ is separated, hence the diagonal morphism $\Delta_{Y/X}$ is a closed immersion. Now quote the preceding lemma.]

1-10
If \( G \to S \) is a group scheme over \( S \), then the composition

\[
S \xrightarrow{e} G \to S
\]

is \( \text{id}_S \). Proof: \( e \) is an \( S \)-morphism and the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{e} & G \\
\downarrow{id_S} & & \downarrow \\
S & = & S
\end{array}
\]

commutes. Therefore \( e \) is a section for the structural morphism \( G \to S \):

\[
G \to S \xrightarrow{e} G.
\]

**36: LEMMA** Let \( G \to S \) be a group scheme over \( S \) – then the structural morphism \( G \to S \) is separated iff \( e : S \to G \) is a closed immersion.

[To see that "closed immersion" \( \implies \) "separated", consider the pullback square

\[
\begin{array}{ccc}
G & \xrightarrow{\Delta_{G/S}} & S \\
\downarrow{e} & & \downarrow{e} \\
G \times_S G & \xrightarrow{\text{mor}(\text{id}_G \times \text{id})} & G
\end{array}
\]

**37: LEMMA** If \( S \) is a discrete scheme, then every \( S \)-group is separated.

**38: APPLICATION** Take \( S = \text{Spec}(k) \), where \( k \) is a field – then the structural morphism \( X \to \text{Spec}(k) \) of a \( k \)-scheme \( X \) is separated.
Fix a field $k$.

1: **DEFINITION** A $k$-algebra is an object in $\text{RNG}$ and a ring homomorphism $k \to A$.

2: **NOTATION** $\text{ALG}/k$ is the category whose objects are the $k$-algebras $k \to A$ and whose morphisms

\[
(k \to A) \longrightarrow (k \to B)
\]

are the ring homomorphisms $A \to B$ with the property that the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & & \uparrow \\
k & \equiv & k
\end{array}
\]

commutes.

3: **DEFINITION** Let $A$ be a $k$-algebra —then $A$ is finitely generated if there exists a surjective homomorphism $k[t_1, \ldots, t_n] \to A$ of $k$-algebras.

4: **DEFINITION** Let $A$ be a $k$-algebra —then $A$ is finite if there exists a surjective homomorphism $k^n \to A$ of $k$-modules.

5: **N.B.** A finite $k$-algebra is finitely generated.

Recall now that $\text{SCH}/k$ stands for $\text{SCH}/\text{Spec}(k)$. 

2-1
6: **Lemma** The functor

\[ A \to \text{Spec}(A) \]

from \((\text{ALG}/k)^{\text{op}}\) to \(\text{SCH}/k\) is fully faithful.

7: **Definition** Let \(X \to \text{Spec}(k)\) be a \(k\)-scheme — then \(X\) is locally of finite type if there exists an affine open covering \(X = \bigcup_{i \in I} U_i\) such that for all \(i\), \(U_i = \text{Spec}(A_i)\), where \(A_i\) is a finitely generated \(k\)-algebra.

8: **Definition** Let \(X \to \text{Spec}(k)\) be a \(k\)-scheme — then \(X\) is of finite type if \(X\) is locally of finite type and quasi-compact.

9: **Lemma** If a \(k\)-scheme \(X \to \text{Spec}(k)\) is locally of finite type and if \(U \subset X\) is an open affine subset, then \(\Gamma(U, O_X)\) is a finitely generated \(k\)-algebra.

10: **Application** If \(A\) is a finitely generated \(k\)-algebra, then the \(k\)-scheme \(\text{Spec}(A) \to \text{Spec}(k)\) is of finite type.

11: **Lemma** If \(X \to \text{Spec}(k)\) is a \(k\)-scheme of finite type, then all subschemes of \(X\) are of finite type.

12: **Rappel** Let \((X, O_X)\) be a locally ringed space. Given \(x \in X\), denote the stalk of \(O_X\) at \(x\) by \(O_{X,x}\) — then \(O_{X,x}\) is a local ring. And:

- \(m_x\) is the maximal ideal in \(O_{X,x}\).
- \(\kappa(x) = O_{X,x}/m_x\) is the residue field of \(O_{X,x}\).
**13: CONSTRUCTION** Let \((X, \mathcal{O}_X)\) be a scheme. Given \(x \in X\), let \(U = \text{Spec}(A)\) be an affine open neighborhood of \(x\). Denote by \(p\) the prime ideal of \(A\) corresponding to \(x\), hence \(\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = A_p\) (the localization of \(A\) at \(p\)) and the canonical homomorphism \(A \to A_p\) leads to a morphism

\[
\text{Spec}(\mathcal{O}_{X,x}) = \text{Spec}(A_p) \to \text{Spec}(A) = U \subset X
\]

doing schemes (which is independent of the choice of \(U\)).

**14: N.B.** There is an arrow \(\mathcal{O}_{X,x} \to \kappa(x)\), thus an arrow \(\text{Spec}(\kappa(x)) \to \text{Spec}(\mathcal{O}_{X,x})\), thus an arrow

\[
i_x : \text{Spec}(\kappa(x)) \to X
\]

whose image is \(x\).

Let \(K\) be any field, let \(f : \text{Spec}(K) \to X\) be a morphism of schemes, and let \(x\) be the image of the unique point \(p\) of \(\text{Spec}(K)\). Since \(f\) is a morphism of locally ringed spaces, at the stalk level there is a homomorphism

\[
\mathcal{O}_{X,x} \to \mathcal{O}_{\text{Spec}(K),p} = K
\]

of local rings meaning that the image of the maximal ideal \(m_x \subset \mathcal{O}_{X,x}\) is contained in the maximal ideal \(\{1\}\) of \(K\), so there is an induced homomorphism

\[
i : \kappa(x) \to K.
\]

Consequently,

\[
f = i_x \circ \text{Spec}(i).
\]

**15: SCHOLIUM** There is a bijection

\[
\text{Mor}(\text{Spec}(K), X) \to \{(x, i) : x \in X, \ i : \kappa(x) \to K\}.
\]
If \( X \to \text{Spec}(k) \) is a \( k \)-scheme, then for any \( x \in X \), there is an arrow

\[
\text{Spec}(\kappa(x)) \to X,
\]

from which an arrow

\[
\text{Spec}(\kappa(x)) \to \text{Spec}(k),
\]

or still, an arrow \( k \to \kappa(x) \).

16: LEMMA  Let \( X \to \text{Spec}(k) \) be a \( k \)-scheme locally of finite type – then \( x \in X \) is closed iff the field extension \( \kappa(x)/k \) is finite.

17: APPLICATION  Let \( X \to \text{Spec}(k) \) be a \( k \)-scheme locally of finite type. Assume: \( k \) is algebraically closed – then

\[
\{ x \in X : x \text{ closed} \} = \{ x \in X : k = \kappa(x) \}
\]

\[
= \text{Mor}_k(\text{Spec}(k), X) \equiv X(k).
\]

18: DEFINITION  A subset \( Y \) of a topological space \( X \) is dense in \( X \) if \( \overline{Y} = X \).

19: DEFINITION  A subset \( Y \) of a topological space \( X \) is very dense in \( X \) if for every closed subset \( F \subset X \), \( \overline{F \cap Y} = F \).

20: N.B.  If \( Y \) is very dense in \( X \), then \( Y \) is dense in \( X \).

[Take \( F = X \) : \( X \cap Y = \overline{Y} = X \).]

21: LEMMA  Let \( X \to \text{Spec}(k) \) be a \( k \)-scheme locally of finite type – then

\[
\{ x \in X : x \text{ closed} \}
\]

is very dense in \( X \).
22: **DEFINITION** Let $X \rightarrow \text{Spec}(k)$ be a $k$-scheme — then a point $x \in X$ is called $k$-rational if the arrow $k \rightarrow \kappa(x)$ is an isomorphism.

23: **N.B.** Sending a $k$-morphism $\text{Spec}(k) \rightarrow X$ to its image sets up a bijection between the set

$$X(k) = \text{Mor}_k(\text{Spec}(k), X)$$

and the set of $k$-rational points of $X$.

24: **REMARK** $X(k)$ may very well be empty.

[Consider what happens if $k'/k$ is a proper field extension.]

Given a $k$-scheme $X \rightarrow \text{Spec}(k)$ and a field extension $K/k$, let

$$X(K) = \text{Mor}_k(\text{Spec}(K), X)$$

be the set of $K$-valued points of $X$. If $x : \text{Spec}(K) \rightarrow X$ is a $K$-valued point with image $x \in X$, then there are field extensions

$$k \rightarrow \kappa(x) \rightarrow K.$$

25: **N.B.** $\text{Spec}(K)$ is a $k$-scheme, the structural morphism $\text{Spec}(K) \rightarrow \text{Spec}(k)$ being derived from the arrow of inclusion $j : k \rightarrow K$.]

Let $G = \text{Gal}(K/k)$. Given $\sigma : K \rightarrow K$ in $G$,

$$\text{Spec}(\sigma) : \text{Spec}(K) \rightarrow \text{Spec}(K),$$

hence

$$\xymatrix{ \text{Spec}(K) \ar[r]^{\text{Spec}(\sigma)} & \text{Spec}(K) \ar[r]^{x} & X, }$$

2-5
and we put
\[ \sigma \cdot x = x \circ \text{Spec}(\sigma). \]

- \( \sigma \cdot x \) is a \( K \)-valued point.

[There is a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\sigma} & K \\
\downarrow{j} & & \downarrow{j} \\
k & \xrightarrow{id_k} & k
\end{array}
\]

so \( \sigma \circ j = j \circ id_k = j \), and if \( \pi : X \to \text{Spec}(k) \) is the structural morphism, there is a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{x} & X \\
\downarrow{\text{Spec}(j)} & & \downarrow{\pi} \\
\text{Spec}(k) & = & \text{Spec}(k)
\end{array}
\]

so \( \pi \circ x = \text{Spec}(j) \). The claim then is that the diagram

\[
\begin{array}{ccc}
\text{Spec}(K) & \xrightarrow{x \circ \text{Spec}(\sigma)} & X \\
\downarrow{\text{Spec}(j)} & & \downarrow{\pi} \\
\text{Spec}(k) & = & \text{Spec}(k)
\end{array}
\]

commutes. But

\[
\pi \circ x \circ \text{Spec}(\sigma) = \text{Spec}(j) \circ \text{Spec}(\sigma) = \text{Spec}(\sigma \circ j) = \text{Spec}(j).
\]
The operation
\[
\begin{cases}
G \times X(\mathbb{K}) \to X(\mathbb{K}) \\
(\sigma, x) \to \sigma \cdot x
\end{cases}
\]
is a left action of $G$ on $X(\mathbb{K})$.

[Given $\sigma, \tau \in G : \mathbb{K} \xrightarrow{\tau} \mathbb{K} \xrightarrow{\sigma} \mathbb{K}$, it is a question of checking that
\[
(\sigma \circ \tau) \cdot x = \sigma \cdot (\tau \cdot x).
\]
But the LHS equals
\[
x \circ \text{Spec}(\sigma \circ \tau) = x \circ \text{Spec}(\tau) \circ \text{Spec}(\sigma)
\]
while the RHS equals
\[
\tau \cdot x \circ \text{Spec}(\sigma) = x \circ \text{Spec}(\tau) \circ \text{Spec}(\sigma).
\]

26: NOTATION Let
\[
\mathbb{K}^G = \text{Inv}(G)
\]
be the invariant field associated with $G$.

27: LEMMA The set $X(\mathbb{K})^G$ of fixed points in $X(\mathbb{K})$ for the left action of $G$ on $X(\mathbb{K})$ coincides with the set $X(\mathbb{K}^G)$.

28: APPLICATION If $\mathbb{K}$ is a Galois extension of $k$, then
\[
X(\mathbb{K})^G = X(\mathbb{k}).
\]
Take $\mathbb{K} = \mathbb{k}^{\text{sep}}$, thus now $G = \text{Gal}(\mathbb{k}^{\text{sep}}/\mathbb{k})$.

29: DEFINITION Suppose given a left action $G \times S \to S$ of $G$ on a set $S$ —then $S$ is called a $G$-set if $\forall \ s \in S$, the $G$-orbit $G \cdot s$ is finite or, equivalently, the stabilizer $G_S \subset G$ is an open subgroup of $G$. 2-7
**30: EXAMPLE** Let $X \to \text{Spec}(k)$ be a $k$-scheme locally of finite type—then $\forall x \in X(k^{\text{sep}})$, the $G$-orbit $G \cdot x$ of $x$ in $X(k^{\text{sep}})$ is finite, hence $X(k^{\text{sep}})$ is a $G$-set.

**31: DEFINITION** Let $X \to \text{Spec}(X)$ be a $k$-scheme—then $X$ is \'{e}tale if it is of the form

$$X = \bigsqcup_{i \in I} \text{Spec}(K_i),$$

where $I$ is some index set and where $K_i/k$ is a finite separable field extension.

There is a category $\text{\'{E}T}/k$ whose objects are the \'{e}tale $k$-schemes and there is a category $G\text{-SET}$ whose objects are the $G$-sets.

Define a functor

$$\Phi : \text{\'{E}T}/k \to G\text{-SET}$$

by associating with each $X$ in $\text{\'{E}T}/k$ the set $X(K^{\text{sep}})$ equipped with its left $G$-action.

**32: LEMMA** $\Phi$ is an equivalence of categories.

**PROOF** To construct a functor

$$\Psi : G\text{-SET} \to \text{\'{E}T}/k$$

such that

$$\Psi \circ \Phi \approx \text{id}_{\text{\'{E}T}/k} \quad \text{and} \quad \Phi \circ \Psi \approx \text{id}_{G\text{-SET}},$$

take a $G$-set $S$ and write it as a union of $G$-orbits, say

$$S \approx \bigsqcup_{i \in I} G \cdot s_i.$$ 

Let $K_i \supset k$ be the finite separable field extension inside $k^{\text{sep}}$ corresponding to the open subgroup $G_{s_i} \subset G$ and assign to $S$ the \'{e}tale $k$-scheme $\bigsqcup_{i \in I} \text{Spec}(K_i)$. Proceed . . . .

The foregoing equivalence of categories induces an equivalence between the correspond-
ing categories of group objects:

$$\text{étale group } k\text{-schemes } \approx G\text{-groups},$$

where a $G$-group is a group which is a $G$-set, the underlying left action being by group automorphisms.

**33: CONSTRUCTION**  Given a group $M$, let $M_k$ be the disjoint union

$$\coprod_M \text{Spec}(k),$$

the constant group $k$-scheme, thus for any $k$-scheme $X \to \text{Spec}(k)$,

$$\text{Mor}_k(X, M_k)$$

is the set of locally constant maps $X \to M$ whose group structure is multiplication of functions.

[The terminology is standard but not the best since if $M$ is nontrivial, then

$$\text{Mor}_k(X, M_k) \approx M$$

only if $X$ is connected.]

**34: EXAMPLE**  For any étale group $k$-scheme $X$,

$$X \times_k \text{Spec}(k^{\text{sep}}) \approx X(k^{\text{sep}})_k \times_k \text{Spec}(k^{\text{sep}}).$$

[Note: Here (and elsewhere),

$$\times_k = \times_{\text{Spec}(k)}.$$]
35: **RAPPEL** An $A$ in **RNG** is reduced if it has no nilpotent elements $\neq 0$ (i.e., $\exists a \neq 0: a^n = 0$ $(\exists n)$).

36: **DEFINITION** A scheme $X$ is reduced if for any nonempty open subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.

[Note: This is equivalent to the demand that all the local rings $\mathcal{O}_{X,x}$ $(x \in X)$ are reduced.]

37: **DEFINITION** Let $X$ be a $k$-scheme — then $X$ is geometrically reduced if for every field extension $K \supset k$, the $K$-scheme $X \times_k \text{Spec}(K)$ is reduced.

38: **LEMMA** If $X$ is a reduced $k$-scheme, then for every separable field extension $K/k$, the $K$-scheme $X \times_k \text{Spec}(K)$ is reduced.

39: **APPLICATION** Assume: $k$ is a perfect field — then every reduced $k$-scheme $X$ is geometrically reduced.

40: **THEOREM** Assume: $k$ is of characteristic zero. Suppose that $X$ is a group $k$-scheme which is locally of finite type — then $X$ is reduced, hence is geometrically reduced.
§3. AFFINE GROUP k-SCHEMES

Fix a perfect field $k$.

[Recall that a field $k$ is perfect if every field extension of $k$ is separable (equivalently, char($k$) = 0 or char($k$) = $p > 0$ and the arrow $x \to x^p$ is surjective).]

1: DEFINITION An affine group $k$-scheme is a group $k$-scheme of the form Spec($A$), where $A$ is a $k$-algebra.

2: EXAMPLE $G_{a,k} = \text{Spec}(k[t])$ is an affine group $k$-scheme.

3: EXAMPLE $G_{m,k} = \text{Spec}(k[t, t^{-1}])$ is an affine group $k$-scheme.

4: EXAMPLE $\mu_{n,k} = \text{Spec}(k[t]/(t^n - 1))$ $(n \in \mathbb{N})$ is an affine group $k$-scheme.

There is a category $\text{GRP}/k$ whose objects are the group $k$-schemes and whose morphisms are the morphisms $f : X \to Y$ of $k$-schemes such that for all $k$-schemes $T$ the induced map

$$f(T) : \text{Mor}_k(T, X) \to \text{Mor}_k(T, Y)$$
is a group homomorphism.

5: NOTATION

\[ \text{AFF} - \text{GRP}/k \]

is the full subcategory of \( \text{GRP}/k \) whose objects are the affine group \( k \)-schemes.

6: NOTATION

\[ \text{GRP} - \text{ALG}/k \]

is the category of group objects in \( \text{ALG}/k \) and

\[ \text{GRP} - (\text{ALG}/k)^{\text{OP}} \]

is the category of group objects in \((\text{ALG}/k)^{\text{OP}}\).

7: LEMMA  The functor

\[ A \to \text{Spec}(A) \]

from \((\text{ALG}/k)^{\text{OP}}\) to \( \text{SCH}/k \) is fully faithful and restricts to an equivalence

\[ \text{GRP} - (\text{ALG}/k)^{\text{OP}} \to \text{AFF} - \text{GRP}/k. \]

8: REMARK  An object in \( \text{GRP} - (\text{ALG}/k)^{\text{OP}} \) is a \( k \)-algebra \( A \) which carries the structure of a commutative Hopf algebra over \( k \): \( \exists \) \( k \)-algebra homomorphisms

\[ \Delta : A \to A \otimes_k A, \quad \varepsilon : A \to k, \quad S : A \to A \]

satisfying the “usual” conditions.
9: N.B. There is another way to view matters, viz. any functor \( \text{ALG}/k \to \text{GRP} \) which is representable by a \( k \)-algebra serves to determine an affine group \( k \)-scheme (and vice versa). From this perspective, a morphism \( G \to H \) of affine group \( k \)-schemes is a natural transformation of functors, i.e., a collection of group homomorphisms \( G(A) \to H(A) \) such that if \( A \to B \) is a \( k \)-algebra homomorphism, then the diagram

\[
\begin{array}{ccc}
G(A) & \rightarrow & H(A) \\
\downarrow & & \downarrow \\
G(B) & \rightarrow & H(B)
\end{array}
\]

commutes.

[Note: Suppose that

\[
\begin{cases}
G = h^X = \text{Mor}(X, -) \\
H = h^Y = \text{Mor}(Y, -)
\end{cases}
\]

Then from Yoneda theory,

\[\text{Mor}(G, H) \approx \text{Mor}(Y, X).] \]

10: EXAMPLE \( k[t, t^{-1}] \) represents \( G_{m,k} \) and

\[k[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}]\]

represents \( \text{GL}_{n,k} \). Given any \( k \)-algebra \( A \), the determinant is a group homomorphism

\[\text{GL}_{n,k}(A) \to G_{m,k}(A)\]

and

\[\det_k \in \text{Mor}(\text{GL}_{n,k}, G_{m,k}).\]
[Note: There is a homomorphism \( k[t, t^{-1}] \to k[t_{11}, \ldots, t_{nn}, \det(t_{ij})^{-1}] \)

of \( k \)-algebras that defines \( \det_k \). E.g.: If \( n = 2 \), then the homomorphism in question sends \( t \) to \( t_{11}t_{22} - t_{12}t_{21} \).]

11: PRODUCTS

Let

\[
\begin{align*}
G &= h^X \quad (X \text{ in } \text{ALG}/k) \\
H &= h^Y \quad (Y \text{ in } \text{ALG}/k)
\end{align*}
\]

be affine group \( k \)-schemes. Consider the functor

\[
G \times H : \text{ALG}/k \to \text{GRP}
\]

defined on objects by

\[
A \to G(A) \times H(A).
\]

Then this functor is represented by the \( k \)-algebra \( X \otimes_k Y \):

\[
\text{Mor}(X \otimes_k Y, A) \approx \text{Mor}(X, A) \times \text{Mor}(Y, A) \\
= G(A) \times H(A).
\]

12: EXAMPLE

Take

\[
\begin{align*}
G &= G_{m, \mathbb{R}} \\
H &= G_{m, \mathbb{R}}
\end{align*}
\]

Then

\[
(G_{m, \mathbb{R}} \times G_{m, \mathbb{R}})(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times = \mathbb{C}^\times
\]

and

\[
(G_{m, \mathbb{R}} \times G_{m, \mathbb{R}})(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times.
\]
Let $k'/k$ be a field extension — then for any $k$-algebra $A$, the tensor product $A \otimes_k k'$ is a $k'$-algebra, hence there is a functor

$$\text{ALG}/k \to \text{ALG}/k'$$

termed *extension of the scalars*. On the other hand, every $k'$-algebra $B'$ can be regarded as a $k$-algebra $B$, from which a functor

$$\text{ALG}/k' \to \text{ALG}/k$$

termed *restriction of the scalars*.

**13: LEMMA** For all $k$-algebras $A$ and for all $k'$-algebras $B'$,

$$\text{Mor}_{k'}(A \otimes_k k', B') \approx \text{Mor}_k(A, B).$$

**14: SCHOLIUM** The functor “extension of the scalars” is a left adjoint for the functor “restriction of the scalars”.

Let $G$ be an affine group $k$-scheme. Abusing the notation, denote still by $G$ the associated functor

$$\text{ALG}/k \to \text{GRP}.$$ 

Then there is a functor

$$G_{k'} : \text{ALG}/k' \to \text{GRP},$$

namely

$$G_{k'}(A') = G(A),$$

where $A$ is $A'$ viewed as a $k$-algebra.

**15: LEMMA** $G_{k'}$ is an affine group $k'$-scheme and the assignment $G \to G_{k'}$ is functorial:
\[ \text{AFF-GRP}/k \rightarrow \text{AFF-GRP}/k'. \]

[Note: Suppose that \( G = h^X \) — then

\[
\text{Mor}_{k'}(X \otimes_k k', A') \approx \text{Mor}_k(X, A) \\
= G(A) \\
= G_{k'}(A').
\]

Therefore \( G_{k'} \) is represented by \( X \otimes_k k' \):

\[
G_{k'} = h^X \otimes_k k'.
\]

Matters can also be interpreted “on the other side”:

\[
\begin{tikzcd}
G_{k'} = \text{Spec}(X \times_k k') = \text{Spec}(X) \times_k \text{Spec}(k') \ar[r] \ar[d] & \text{Spec}(k') \ar[d] \\
G = \text{Spec}(X) \ar[r] & \text{Spec}(k)
\end{tikzcd}
\]

**16: Definition** \( G_{k'} \) is said to have been obtained from \( G \) by extension of the scalars.

**17: Notation** Given an affine group \( k' \)-scheme \( G' \), let \( G_{k'}/k \) be the functor

\[
\text{ALG}/k \rightarrow \text{GRP}
\]

defined by the rule

\[
A \rightarrow G'(A \otimes_k k').
\]
[Note: If \( k' = k \), then \( G_{k'/k} = G \).]

18: **THEOREM** Assume that \( k'/k \) is a finite field extension —then \( G_{k'/k} \) is an affine group \( k \)-scheme and the assignment \( G' \to G_{k'/k} \) is functorial:

\[
\text{AFF-GRP}/k' \to \text{AFF-GRP}/k.
\]

19: **DEFINITION** \( G_{k'/k} \) is said to have been obtained from \( G' \) by restriction of the scalars.

20: **LEMMA** Assume that \( k'/k \) is a finite field extension —then for all affine group \( k \)-schemes \( H \),

\[
\text{Mor}_k(H, G_{k'/k}) \cong \text{Mor}_{k'}(H_{k'}, G').
\]

21: **SCHOLIUM** The functor “restriction of the scalars” is a right adjoint for the functor “extension of the scalars”.

[Accordingly, there are arrows of adjunction \[
\begin{align*}
G &\to (G_{k'})_{k'/k} \\
(G_{k'/k})_{k'} &\to G'
\end{align*}
\].]

22: **NOTATION**

\[
\text{Res}_{k'/k} : \text{AFF-GRP}/k' \to \text{AFF-GRP}/k
\]

is the functor defined by setting

\[
\text{Res}_{k'/k}(G') = G_{k'/k}.
\]

So, by definition,

\[
\text{Res}_{k'/k}(G')(A) = G'(A \otimes_k k').
\]

3-7
And in particular:
\[
\text{Res}_{k'/k}(G')(k) = G'(k \otimes_k k') = G'(k').
\]

**23: EXAMPLE** Take \( G' = A^n_k \) – then
\[
\text{Res}_{k'/k}(A^n_k) \approx A^{nd}_k \quad (d = [k' : k]).
\]

**24: EXAMPLE** Take \( k = \mathbb{R}, k' = \mathbb{C}, G' = G_{m,\mathbb{C}}, \) and consider
\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}).
\]
Then
\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}})(\mathbb{R}) = \mathbb{C}^x
\]
and
\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}})(\mathbb{C}) = \mathbb{C}^x \times \mathbb{C}^x.
\]

[Note:
\[
\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}})
\]
is not isomorphic to \( G_{m,\mathbb{R}} \) (its group of real points is \( \mathbb{R}^\times \)).]

**25: LEMMA** Let \( k' \) be a finite Galois extension of \( k \) – then
\[
(\text{Res}_{k'/k}(G'))_{k'} \approx \prod_{\sigma \in \text{Gal}(k'/k)} \sigma G'.
\]

3-8
∀ σ ∈ Gal(k'/k), there is a pullback square

\[
\begin{array}{ccc}
\sigma G' & \longrightarrow & \text{Spec}(k') \\
\downarrow & & \downarrow \text{Spec(σ)} \cdot \\
G' & \longrightarrow & \text{Spec}(k')
\end{array}
\]

26: EXAMPLE Take \( k = \mathbb{R}, k' = \mathbb{C}, G' = G_{m,\mathbb{C}} \) — then

\[
(\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}))_{\mathbb{C}} \approx G_{m,\mathbb{C}} \times \sigma G_{m,\mathbb{C}} \\
\approx G_{m,\mathbb{C}} \times G_{m,\mathbb{C}}.
\]

Let \( G \) be an affine group \( k \)-scheme.

27: DEFINITION A character of \( G \) is an element of

\[
X(G) = \text{Mor}_k(G, G_{m,k}).
\]

Given \( \chi \in X(G) \), for every \( k \)-algebra \( A \), there is a homomorphism

\[
\chi(A) : G(A) \to G_{m,k}(A) = A^\times.
\]

Given \( \chi_1, \chi_2 \in X(G) \), define

\[
(\chi_1 + \chi_2)(A) : G(A) \to G_{m,k}(A) = A^\times
\]

by the stipulation

\[
(\chi_1 + \chi_2)(A)(t) = \chi_1(A)(t)\chi_2(A)(t),
\]

from which a character \( \chi_1 + \chi_2 \) of \( G \), hence \( X(G) \) is an abelian group.
**28: EXAMPLE** Take $G = G_{m,k}$ — then the characters of $G$ are the morphisms $G \to G_{m,k}$ of the form

$$t \to t^n \quad (n \in \mathbb{Z}),$$

i.e.,

$$X(G) \approx \mathbb{Z}.$$

**29: EXAMPLE** Take $G = G_{m,k} \times \cdots \times G_{m,k}$ (d factors) — then the characters of $G$ are the morphisms $G \to G_{m,k}$ of the form

$$(t_1, \ldots, t_d) \to t_1^{n_1} \cdots t_d^{n_d} \quad (n_1, \ldots, n_d \in \mathbb{Z}),$$

i.e.,

$$X(G) \approx \mathbb{Z}^d.$$

**30: EXAMPLE** Given an abelian group $M$, its group algebra $k[M]$ is canonically a $k$-algebra. Consider the functor $D(M) : \text{ALG}/k \to \text{GRP}$ defined on objects by the rule

$$A \to \text{Mor}(M, A^\times).$$

Then $\forall A$,

$$\text{Mor}(M, A^\times) \approx \text{Mor}(k[M], A),$$

so $k[M]$ represents $D(M)$ which is therefore an affine group $k$-scheme. And

$$X(D(M)) \approx M,$$

the character of $D(M)$ corresponding to $m \in M$ being the assignment

$$D(M)(A) = \text{Mor}(M, A^\times)$$

$$\xrightarrow{f \mapsto f(m)} A^\times = G_{m,k}(A).$$
**31: NOTATION** Given $\chi' \in X(G')$, let $N_{k'/k}(\chi')$ stand for the rule that assigns to each $k$-algebra $A$ the homomorphism

$$G_{k'/k} \to G_{m,k}(A) = A^\times$$

defined by the composition

$$G_{k'/k}(A) \to G'(A \otimes_k k')$$

$$G'(A \otimes_k k') \to G_{m,k}(A \otimes_k k')^\times = (A \otimes_k k')^\times$$

$$(A \otimes_k k')^\times \to A^\times.$$

Here the first arrow is the canonical isomorphism, the second arrow is $\chi'(A \otimes_k k')$, and the third arrow is the norm map.

**32: LEMMA** The arrow

$$\chi' \to N_{k'/k}(\chi')$$

is a homomorphism

$$X(G') \to X(G_{k'/k})$$

of abelian groups.

**33: THEOREM** The arrow

$$\chi' \to N_{k'/k}(\chi')$$

is bijective, hence defines an isomorphism

$$X(G') \to X(G_{k'/k})$$

of abelian groups.
Consider

\[ \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}). \]

Then its character group is isomorphic to the character group of \( G_{m,\mathbb{C}} \), i.e., to \( \mathbb{Z} \). Therefore

\[ \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}) \]

is not isomorphic to \( G_{m,\mathbb{R}} \times G_{m,\mathbb{R}} \).
§4. ALGEBRAIC TORI

Fix a field \( k \) of characteristic zero.

1: DEFINITION Let \( G \) be an affine group \( k \)-scheme —then \( G \) is algebraic if its associated representing \( k \)-algebra \( A \) is finitely generated.

2: REMARK It can be shown that every algebraic affine group \( k \)-scheme is isomorphic to a closed subgroup of some \( \text{GL}_n, k \) \((\exists n)\).

3: CONVENTION The term algebraic \( k \)-group means “algebraic affine group \( k \)-scheme”.

4: N.B. It is automatic that an algebraic \( k \)-group is reduced (cf. §2, #40), hence is geometrically reduced (cf. §2, #39).

5: LEMMA Assume that \( k'/k \) is a finite field extension —then the functor

\[
\text{Res}_{k'/k} : \text{AFF-GRP}/k' \to \text{AFF-GRP}/k
\]

sends algebraic \( k' \)-groups to algebraic \( k \)-groups.

Given a finite field extension \( k'/k \), let \( \Sigma \) be the set of \( k \)-embeddings of \( k' \) into \( k^{\text{sep}} \) and identify \( k' \otimes_k k^{\text{sep}} \) with \((k^{\text{sep}})^\Sigma\) via the bijection which takes \( x \otimes y \) to the string \((\sigma(x)y)_{\sigma \in \Sigma}\).

6: LEMMA Let \( G' \) be an algebraic \( k' \)-group —then

\[
(G_{k'/k}) \times_k \text{Spec}(k^{\text{sep}}) \cong \prod_{\sigma \in \Sigma} \sigma G',
\]
where $\sigma G'$ is the algebraic $k^{\text{sep}}$-group defined by the pullback square

$$
\begin{array}{cc}
\sigma G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \text{Spec}(\sigma) \\
G' & \rightarrow & \text{Spec}(k')
\end{array}
$$

[Note: To review, the LHS is

$$(\text{Res}_{k'/k})(G')_{k^{\text{sep}}}$$

and the Galois group $\text{Gal}(k^{\text{sep}}/k)$ operates on it through the second factor. On the other hand, to each pair $(\tau, \sigma) \in \text{Gal}(k^{\text{sep}}/k) \times \Sigma$, there corresponds a bijection $\sigma G' \rightarrow (\tau \circ \sigma)G'$ leading thereby to an action of $\text{Gal}(k^{\text{sep}}/k)$ on

$$\prod_{\sigma \in \Sigma} \sigma G'.$$

The point then is that the identification

$$(\text{Res}_{k'/k})(G')_{k^{\text{sep}}} \approx \prod_{\sigma \in \Sigma} \sigma G'$$

respects the actions, i.e., is $\text{Gal}(k^{\text{sep}}/k)$-equivariant.]

**7: N.B.** Consider the commutative diagram

$$
\begin{array}{cc}
(\tau \circ \sigma)G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \text{Spec}(\tau) \\
\sigma G' & \rightarrow & \text{Spec}(k^{\text{sep}}) \\
\downarrow & & \downarrow \text{Spec}(\sigma) \\
G' & \rightarrow & \text{Spec}(k')
\end{array}
$$

4-2
Then the “big” square is a pullback. Since this is also the case of the “small” bottom square, it follows that the “small” upper square is a pullback.

8: Definition A split $k$-torus is an algebraic $k$-group $T$ which is isomorphic to a finite product of copies of $G_{m,k}$.

9: Example The algebraic $\mathbb{R}$-group

$$\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}})$$

is not a split $\mathbb{R}$-torus (cf. §3, #24 and #34).

10: Lemma If $T$ is a split $k$-torus, then $X(T)$ is a finitely generated free abelian group.

11: Theorem The functor

$$T \rightarrow X(T)$$

from the category of split $k$-tori to the category of finitely generated free abelian groups is a contravariant equivalence of categories.

12: N.B. \(\forall k\)-algebra $A$,

$$T(A) \approx \text{Mor}(X(T), A^\times).$$

[Note: Explicated,

$$T \approx \text{Spec}(k[X(T)]) \quad \text{(cf. §3, #30)}.$$

Therefore

$$T(A) \approx \text{Mor}(\text{Spec}(A), T)$$

4-3
\[
\approx \text{Mor}(\text{Spec}(A), \text{Spec}(k[X(T)])) \\
\approx \text{Mor}(k[X(T)], A) \\
\approx \text{Mor}(X(T), A^\times).]
\]

13: **DEFINITION** A \(k\)-torus is an algebraic \(k\)-group \(T\) such that
\[
T_{k^{\text{sep}}} = T \times_k \text{Spec}(k^{\text{sep}})
\]
is a split \(k^{\text{sep}}\)-torus.

14: **N.B.** A split \(k\)-torus is a \(k\)-torus.

15: **EXAMPLE** Let \(k'/k\) be a finite field extension and take \(G' = G_{m,k'}\) — then the algebraic \(k\)-group \(G_{k'/k}\) is a \(k\)-torus (cf. #6).

16: **DEFINITION** Let \(T\) be a \(k\)-torus — then a splitting field for \(T\) is a finite field extension \(K/k\) such that \(T_K\) is a split \(K\)-torus.

17: **THEOREM** Every \(k\)-torus \(T\) admits a splitting field which is minimal (i.e., contained in any other splitting field) and Galois.

18: **NOTATION** Given a \(k\)-scheme \(X\) and a Galois extension \(K/k\), the Galois group \(\text{Gal}(K/k)\) operates on
\[
X_K = X \times_k \text{Spec}(K)
\]
via the second term, hence \(\sigma \to 1 \otimes \sigma\).

[Note: \(1 \otimes \sigma\) is a \(k\)-automorphism of \(X_K\).]
19: NOTATION Given $k$-schemes $X, Y$ and a Galois extension $K/k$, the Galois group $\text{Gal}(K/k)$ operates on $\text{Mor}_K(X_K, Y_K)$ by the prescription

$$\sigma f = (1 \otimes \sigma)f(1 \otimes \sigma)^{-1}.$$  

[Note: If $f \in \text{Mor}_K(X_K, Y_K)$, then the condition $\sigma f = f$ for all $\sigma \in \text{Gal}(K/k)$ is equivalent to the condition that $f$ is the lift of a $k$-automorphism $\phi : X \to Y$, i.e., $f = \phi \otimes 1$.]

20: LEMA Let $K/k$ be a Galois extension and let $G = \text{Gal}(K/k)$ –then for any $k$-algebra $A$ and for any $k$-scheme $X$,

$$X(A \otimes_k K)^G = X(A).$$

[Note: This generalizes §2, #28 to which it reduces if $A = k$.]

21: DEFINITION Let $G$ be a finite group –then a $G$-module is an abelian group $M$ supplied with a homomorphism $G \to \text{Aut}(M)$.

22: N.B. A $G$-module is the same thing as a $\mathbb{Z}[G]$-module (in the usual sense when $\mathbb{Z}[G]$ is viewed as a ring).

23: DEFINITION Let $G$ be a finite group –then a $G$-lattice is a $\mathbb{Z}$-free $G$-module $M$ of finite rank.

24: LEMA If $T$ is a $k$-torus split by a finite Galois extension $K/k$, then

$$X(T_K) = \text{Mor}_K(T_K, G_{m,K})$$

is a $\text{Gal}(K/k)$-lattice.
25: THEOREM Fix a finite Galois extension \( \mathbb{K}/k \) – then the functor

\[ T \to X(T_k) \]

from the category of \( k \)-tori split by \( \mathbb{K}/k \) to the category of \( \text{Gal}(\mathbb{K}/k) \)-lattices is a contravariant equivalence of categories.

26: N.B. Suppose that \( T \) is a \( k \)-torus split by a finite Galois extension \( \mathbb{K}/k \). Form \( \mathbb{K}[X(T_k)] \), thus operationally, \( \forall \sigma \in \text{Gal}(\mathbb{K}/k) \),

\[ \sigma \left( \sum_i a_i \chi_i \right) = \sum_i \sigma(a_i) \sigma(\chi_i) \quad (a_i \in \mathbb{K}, \chi_i \in X(T_k)). \]

Pass now to the invariants

\[ \mathbb{K}[X(T_k)] \quad (G = \text{Gal}(\mathbb{K}/k)). \]

Then

\[ T \approx \text{Spec}(\mathbb{K}[X(T_k)]^G). \]

And

\[ T(A \otimes_k \mathbb{K})^G = T(A) \]

\[ \approx \text{Mor}(\text{Spec}(A), T) \]

\[ \approx \text{Mor}(\text{Spec}(A), \text{Spec}(\mathbb{K}[X(T_k)])^G) \]

\[ \approx \text{Mor}_k(\mathbb{K}[X(T_k)]^G, A) \]

\[ \approx \text{Mor}_k(\mathbb{K}[X(T_k)], A \otimes_k \mathbb{K})^G \]

\[ \approx \text{Mor}_\mathbb{Z}(X(T_k), (A \otimes_k \mathbb{K})^\times)^G \]

\[ \approx \text{Mor}_\mathbb{Z}[G](X(T_k), (A \otimes_k \mathbb{K})^\times). \]

[Note: Let \( T = \text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}}) \) – then on the one hand,

\[ \text{Mor}_{\mathbb{Z}[G]}(\mathbb{Z}[G], (A \otimes_k \mathbb{K})^\times) \approx (A \otimes_k \mathbb{K})^\times, \]

4-6]
while on the other,

\[ \text{Res}_{K/k}(G_{m,K})(A) = (A \otimes_k K)^\times \]

\[ \approx \text{Mor}_{Z[G]}(X(T_K), (A \otimes_k K)^\times). \]

Therefore

\[ X(T_K) \approx Z[G]. \]

Take \( k = \mathbb{R}, K = \mathbb{C} \), and let \( \sigma \) be the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) — then every \( \mathbb{R} \)-torus \( T \) gives rise to a free \( \mathbb{Z} \)-module of finite rank supplied with an involution corresponding to \( \sigma \). And conversely . . . .

There are three “basic” \( \mathbb{R} \)-tori.

1. \( T = G_{m,\mathbb{R}} \). In this case,

\[ X(T_\mathbb{C}) = X(G_{m,\mathbb{C}}) \approx \mathbb{Z} \]

and the Galois action is trivial.

2. \( T = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}) \). In this case,

\[ X(T_\mathbb{C}) \approx X(G_{m,\mathbb{C}} \times G_{m,\mathbb{C}}) \quad (\text{cf. } \S 3, \#26) \]

\[ \approx \mathbb{Z} \times \mathbb{Z} \]

and the Galois action swaps coordinates.

3. \( T = \text{SO}_2 \). In this case,

\[ X((\text{SO}_2)_\mathbb{C}) \approx X(G_{m,\mathbb{C}}) \]

\[ \approx \mathbb{Z} \]

and the Galois action is multiplication by \(-1\).
[Note:

\[ \text{SO}_2 : \text{ALG}/\mathbb{R} \to \text{GRP} \]

is the functor defined by the rule

\[ \text{SO}_2(A) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in A \& a^2 + b^2 = 1 \right\}. \]

Then \( \text{SO}_2 \) is an algebraic \( \mathbb{R} \)-group such that

\[ (\text{SO}_2)_\mathbb{C} \approx G_{m, \mathbb{C}}, \]

so \( \text{SO}_2 \) is an \( \mathbb{R} \)-torus and \( \text{SO}_2(\mathbb{R}) \) can be identified with \( S (= \{ z \in \mathbb{C} : z\overline{z} = 1 \}) \).

27: THEOREM Every \( \mathbb{R} \)-torus is isomorphic to a finite product of copies of the three basic tori described above.

Here is the procedure. Fix a \( \mathbb{Z} \)-free module \( M \) of finite rank and an involution \( \iota : M \to M \) — then \( M \) can be decomposed as a direct sum

\[ M_+ \oplus M_{sw} \oplus M_- , \]

where \( \iota = 1 \) on \( M_+ \), \( \iota \) is a sum of 2-dimensional swaps on \( M_{sw} \) (or still, \( M_{sw} = \oplus \mathbb{Z}[G_{al}(\mathbb{C}/\mathbb{R})] \)), and \( \iota = -1 \) on \( M_- \).

28: SCHOLIUM If \( T \) is an \( \mathbb{R} \)-torus, then there exist unique nonnegative integers \( a, b, c \) such that

\[ T(\mathbb{R}) \approx (\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b \times S^c. \]

29: REMARK The classification of \( \mathbb{C} \)-tori is trivial: Any such is a finite product of the \( G_{m, \mathbb{C}} \).
30: **RAPPEL**  Let $\mathbb{K}/k$ be a finite Galois extension and let $A$ be a $k$-algebra—then there is a norm map

\[(A \otimes_k \mathbb{K})^\times \to A^\times (\approx (A \otimes_k k)^\times)\].

31: **CONSTRUCTION**  Let $\mathbb{K}/k$ be a finite Galois extension—then there is a norm map

\[N_{\mathbb{K}/k} : \text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}}) \to G_{m,k}\].

[For any $k$-algebra $A$,]

\[\text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}})(A) = G_{m,\mathbb{K}}(A \otimes_k \mathbb{K}) = (A \otimes_k \mathbb{K})^\times \to A^\times = G_{m,k}(A)\].

[Note: $N_{\mathbb{K}/k}$ is not to be confused with the arrow of adjunction
\[G_{m,k} \to \text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}}).\]]

32: **N.B.**

\[N_{\mathbb{K}/k} \in X(\text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}})).\]

33: **NOTATION**  Let $\text{Res}_{\mathbb{K}/k}^{(1)}(G_{m,\mathbb{K}})$ be the kernel of $N_{\mathbb{K}/k}$.

34: **LEMMA**  $\text{Res}_{\mathbb{K}/k}^{(1)}(G_{m,\mathbb{K}})$ is a $k$-torus and there is a short exact sequence

\[1 \to \text{Res}_{\mathbb{K}/k}^{(1)}(G_{m,\mathbb{K}}) \to \text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}}) \to G_{m,k} \to 1.\]
35: EXAMPLE Take \( \mathbb{k} = \mathbb{R}, \mathbb{K} = \mathbb{C} \) then
\[
\text{Res}_{\mathbb{C}/\mathbb{R}}^{(1)}(G_{m,\mathbb{C}}) \approx \text{SO}_2
\]
and there is a short exact sequence
\[
1 \to \text{SO}_2 \to \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}) \to G_{m,\mathbb{R}} \to 1.
\]
[Note: On \( \mathbb{R} \)-points, this becomes
\[
1 \to S \to \mathbb{C}^\times \to \mathbb{R}^\times \to 1.
\]
]

36: DEFINITION Let \( T \) be a \( \mathbb{k} \)-torus – then \( T \) is \( \mathbb{k} \)-anisotropic if \( X(T) = \{0\} \).

37: EXAMPLE \( \text{SO}_2 \) is \( \mathbb{R} \)-anisotropic.

38: THEOREM Every \( \mathbb{k} \)-torus \( T \) has a unique maximal \( \mathbb{k} \)-split subtorus \( T_s \) and a unique maximal \( \mathbb{k} \)-anisotropic subtorus \( T_a \). The intersection \( T_s \cap T_a \) is finite and \( T_s \cdot T_a = T \).

39: LEMMA \( \text{Res}_{\mathbb{K}/\mathbb{k}}^{(1)}(G_{m,\mathbb{K}}) \) is \( \mathbb{k} \)-anisotropic.

PROOF Setting \( G = \text{Gal}(\mathbb{K}/\mathbb{k}) \), under the functoriality of #25, the norm map
\[
\text{N}_{\mathbb{K}/\mathbb{k}} : \text{Res}_{\mathbb{K}/\mathbb{k}}(G_{m,\mathbb{K}}) \to G_{m,\mathbb{k}}
\]
corresponds to the homomorphism \( \mathbb{Z} \to \mathbb{Z}[G] \) of \( G \)-modules that sends \( n \) to \( n (\sum_{G} \sigma) \), the quotient \( \mathbb{Z}[G]/\mathbb{Z}(\sum_{G} \sigma) \) being \( X(T_{\mathbb{K}}) \), where
\[
T = \text{Res}_{\mathbb{K}/\mathbb{k}}^{(1)}(G_{m,\mathbb{K}}).
\]
And

\[ \mathbb{Z}[G]^G = \mathbb{Z}(\sum_G \sigma). \]

**40: N.B.** \( \text{Res}_{\mathbb{K}/k}^{(1)}(G_{m,\mathbb{K}}) \) is the maximal \( k \)-anisotropic subtorus of \( \text{Res}_{\mathbb{K}/k}(G_{m,\mathbb{K}}) \).

**41: DEFINITION** Let \( G, H \) be algebraic \( k \)-groups — then a homomorphism \( \phi : G \to H \) is an isogeny if it is surjective with a finite kernel.

**42: DEFINITION** Let \( G, H \) be algebraic \( k \)-groups — then \( G, H \) are said to be isogeneous if there is an isogeny between them.

**43: THEOREM** Two \( k \)-tori \( T', T'' \) per #25 are isogeneous iff the \( \mathbb{Q}[\text{Gal}(\mathbb{K}/k)] \)-modules

\[
\begin{align*}
\bigg\{ \quad X(T'_G) \otimes_{\mathbb{Z}} \mathbb{Q} \\
X(T''_G) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{align*}
\]

are isomorphic.
§5. THE LLC

1: N.B. The term “LLC” means “local Langlands correspondence” (cf. #26).

Let \( \mathbb{K} \) be a non-archimedean local field — then the image of \( \text{rec}_K : \mathbb{K}^\times \to G_K^{ab} \) is \( W_K^{ab} \) and the induced map \( \mathbb{K}^\times \to W_K^{ab} \) is an isomorphism of topological groups.

2: SCHOLIUM There is a bijective correspondence between the characters of \( W_K \) and the characters of \( \mathbb{K}^\times \):

\[
\text{Mor}(W_K, \mathbb{C}^\times) \approx \text{Mor}(\mathbb{K}^\times, \mathbb{C}^\times).
\]

[Note: “Character” means continuous homomorphism. So, if \( \chi : W_K \to \mathbb{C}^\times \) is a character, then \( \chi \) must be trivial on \( W_K^\times \) (\( \mathbb{C}^\times \) being abelian), hence by continuity, trivial on \( W_K^\times \), thus \( \chi \) factors through \( W_K/W_K^\times = W_K^{ab} \).]

Let \( T \) be a \( \mathbb{K} \)-torus — then \( T \) is isomorphic to a closed subgroup of some \( \text{GL}_{n, \mathbb{K}} \) (\( \exists n \)). But \( \text{GL}_{n, \mathbb{K}}(\mathbb{K}) \) is a locally compact topological group, thus \( T(\mathbb{K}) \) is a locally compact topological group (which, moreover, is abelian).

3: N.B. For the record,

\[
G_{m, \mathbb{K}}(\mathbb{K}) = \mathbb{K}^\times = \text{GL}_{1, \mathbb{K}}(\mathbb{K}).
\]

4: EXAMPLE Let \( \mathbb{L}/\mathbb{K} \) be a finite extension and consider \( T = \text{Res}_{\mathbb{L}/\mathbb{K}}(G_{m, \mathbb{L}}) \) — then \( T(\mathbb{K}) = \mathbb{L}^\times \).

Roughly speaking, the objective now is to describe \( \text{Mor}(T(\mathbb{K}), \mathbb{C}^\times) \) in terms of data attached to \( W_K \) but to even state the result requires some preparation.
5: **N.B.** The case when $T = G_{m, K}$ is local class field theory . . . .

6: **EXAMPLE** Suppose that $T$ is $K$-split:

$$T \approx G_{m, K} \times \cdots \times G_{m, K} \quad (d \text{ factors}).$$

Then

$$
\prod_{i=1}^{d} \text{Mor}(W_{K}, \mathbb{C}^{\times}) \approx \prod_{i=1}^{d} \text{Mor}(K_{i}^{\times}, \mathbb{C}^{\times}) \\
\approx \text{Mor}(\prod_{i=1}^{d} K_{i}^{\times}, \mathbb{C}^{\times}) \\
\approx \text{Mor}(T(K), \mathbb{C}^{\times}).
$$

Given a $K$-torus $T$, put

$$
\begin{cases}
X^{*}(T) = \text{Mor}_{K_{sep}}(T_{K_{sep}}, G_{m, K_{sep}}) \\
X_{*}(T) = \text{Morg}_{K_{sep}}(G_{m, K_{sep}}, T_{K_{sep}})
\end{cases}.
$$

7: **LEMMA** Canonically,

$$X_{*}(T) \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \approx \text{Mor}(X^{*}(T), \mathbb{C}^{\times}).$$

**PROOF** Bearing in mind that

$$\text{Mor}_{K_{sep}}(G_{m, K_{sep}}, G_{m, K_{sep}}) \approx \mathbb{Z},$$

define a pairing

$$X^{*}(T) \times X_{*}(T) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{Z}$$

by sending $(\chi^{*}, \chi_{*})$ to $\chi^{*} \circ \chi_{*} \in \mathbb{Z}$. This done, given $\chi_{*} \otimes z$, assign to it the homomorphism

$$\chi^{*} \rightarrow z^{\langle \chi^{*}, \chi_{*} \rangle}.$$
8: NOTATION Given a \( \mathbb{K} \)-torus \( T \), put
\[
\hat{T} = \text{Spec}(\mathbb{C}[X_*(T)]).
\]

9: LEMMA \( \hat{T} \) is a split \( \mathbb{C} \)-torus such that
\[
\begin{align*}
X^*(\hat{T}) &\equiv \text{Mor}_\mathbb{C}(\hat{T}, G_{m,\mathbb{C}}) \approx X_*(T) \\
X_*(\hat{T}) &\equiv \text{Mor}_\mathbb{C}(G_{m,\mathbb{C}}, \hat{T}) \approx X^*(T)
\end{align*}
\]

Therefore
\[
\text{Mor}(X_*(T), \mathbb{C}^\times) \approx \text{Mor}(X^*(\hat{T}), \mathbb{C}^\times) \\
\approx X_*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times \\
\approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]

10: LEMMA
\[
\hat{T}(\mathbb{C}) \approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]

PROOF In fact,
\[
\hat{T}(\mathbb{C}) \approx \text{Mor}(X^*(\hat{T}), \mathbb{C}^\times) \quad (\text{cf. } \S 4, \#12) \\
\approx \text{Mor}(X_*(T), \mathbb{C}^\times) \\
\approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.
\]

11: DEFINITION \( \hat{T} \) is the complex dual torus of \( T \).

12: EXAMPLE Under the assumptions of \#6,
\[
\hat{T}(\mathbb{C}) \approx X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \\
\approx \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{C} \\
\approx (\mathbb{C}^\times)^d.
\]
Therefore
\[ \text{Mor}(W\hat{\mathbb{K}}, \hat{T}(\mathbb{C})) \approx \text{Mor}(W\hat{\mathbb{K}}, (\mathbb{C}^\times)^d) \approx \prod_{i=1}^{d} \text{Mor}(W\hat{\mathbb{K}}, \mathbb{C}^\times) \approx \text{Mor}(T(\mathbb{K}), \mathbb{C}^\times). \]

13: RAPPEL If \( G \) is a group and if \( A \) is a \( G \)-module, then
\[ H^1(G, A) = \frac{Z^1(G, A)}{B^1(G, A)}. \]

- \( Z^1(G, A) \) (the 1-cocycles) consists of those maps \( f : G \to A \) such that
\[ f(\sigma \tau) = f(\sigma) + \sigma(f(\tau)). \]

- \( B^1(G, A) \) (the 1-coboundaries) consists of those maps \( f : G \to A \) for which \( \exists \) an \( a \in A \) such that \( \forall \sigma \in G \),
\[ f(\sigma) = \sigma a - a. \]

[Note: \( H^1(G, A) = \text{Mor}(G, A) \) if the action is trivial.]

14: NOTATION If \( G \) is a topological group and if \( A \) is a topological \( G \)-module, then
\[ \text{Mor}_c(G, A) \]
is the group of continuous group homomorphisms from \( G \) to \( A \). Analogously,
\[ \begin{cases} 
Z^1_c(G, A) = \text{“continuous 1-cocycles”} \\
B^1_c(G, A) = \text{“continuous 1-coboundaries”}
\end{cases} \]
and
\[ H_1^c(G, A) = \frac{Z_1^c(G, A)}{B_1^c(G, A)}. \]

Let \( T \) be a \( \mathbb{K} \)-torus — then \( G_\mathbb{K} (= \text{Gal}(\mathbb{K}_{\text{sep}}/\mathbb{K})) \) operates on \( X^*(G) \), thus \( W_\mathbb{K} \subset G_\mathbb{K} \) operates on \( X^*(G) \) by restriction. Therefore \( \hat{T}(\mathbb{C}) \) is a \( W_\mathbb{K} \)-module, so it makes sense to form
\[ H_1^c(W_\mathbb{K}, \hat{T}(\mathbb{C})). \]

**15: NOTATION** \( \text{TOR}_\mathbb{K} \) is the category of \( \mathbb{K} \)-tori.

**16: LEMMA** The assignment
\[ T \to H_1^c(W_\mathbb{K}, \hat{T}(\mathbb{C})) \]
defines a functor
\[ \text{TOR}_\mathbb{K}^{\text{OP}} \to \text{AB}. \]

[Note: Suppose that \( T_1 \to T_2 \) — then
\[ (T_1)_{\mathbb{K}_{\text{sep}}} \to (T_2)_{\mathbb{K}_{\text{sep}}} \]
\[ \Rightarrow \]
\[ X^*(T_2) \to X^*(T_1) \]
\[ \Rightarrow \]
\[ \hat{T}_2(\mathbb{C}) \to \hat{T}_1(\mathbb{C}) \]
\[ \Rightarrow \]
\[ H_1^c(W_\mathbb{K}, \hat{T}_2(\mathbb{C})) \to H_1^c(W_\mathbb{K}, \hat{T}_1(\mathbb{C})). \]

**17: LEMMA** The assignment
\[ T \to \text{Mor}_c(T(\mathbb{K}), \mathbb{C}^\times) \]
defines a functor
\[ \text{TOR}_K^\text{op} \to \text{AB}. \]

**18: Theorem** The functors
\[ T \to H_1^c(W_K, \tilde{T}(\mathbb{C})) \]
and
\[ T \to \text{Mor}_c(T(K), \mathbb{C}^\times) \]
are naturally isomorphic.

**19: Scholium** There exist isomorphisms
\[ \iota_T : H_1^c(W_K, \tilde{T}(\mathbb{C})) \to \text{Mor}_c(T(K), \mathbb{C}^\times) \]
such that if \( T_1 \to T_2 \), then the diagram
\[
\begin{array}{ccc}
H_1^c(W_K, \tilde{T}_1(\mathbb{C})) & \xrightarrow{\iota_{T_1}} & \text{Mor}_c(T_1(K), \mathbb{C}^\times) \\
\uparrow & & \uparrow \\
H_1^c(W_K, \tilde{T}_2(\mathbb{C})) & \xrightarrow{\iota_{T_2}} & \text{Mor}_c(T_2(K), \mathbb{C}^\times)
\end{array}
\]
commutes.

**20: Example** Under the assumptions of #12, the action of \( G_K \) is trivial, hence the action of \( W_K \) is trivial. Therefore
\[ H_1^c(W_K, \tilde{T}(\mathbb{C})) = \text{Mor}_c(W_K, \tilde{T}(\mathbb{C})) \]
\[ \approx \text{Mor}_c(T(K), \mathbb{C}^\times). \]

[Note: The earlier use of the symbol Mor tacitly incorporated “continuity”.]
There is a special case that can be dealt with directly, viz. when $L/K$ is a finite Galois extension and

$$T = \text{Res}_{L/K}(G_{m,L}).$$

The discussion requires some elementary cohomological generalities which have been collected in the Appendix below.

21: **RAPPEL** $W_L$ is a normal subgroup of $W_K$ of finite index:

$$W_K/W_L \approx G_K/G_L \approx \text{Gal}(L/K).$$

Proceeding,

$$T_{\text{grp}} \approx \prod_{\sigma \in \text{Gal}(L/K)} \sigma G_{m,L} \quad \text{(cf. #6)},$$

so

$$X^*(T) \approx Z[W_K/W_L],$$

where

$$Z[W_K/W_L] \approx \text{Ind}^{W_K}_{W_L}$$

$$\equiv Z[W_K] \otimes_{Z[W_L]} Z.$$

It therefore follows that

$$\hat{T}(\mathbb{C}) \approx X^*(T) \otimes_{Z} \mathbb{C}^\times$$

$$\approx Z[W_K] \otimes_{Z[W_L]} Z \otimes_{Z} \mathbb{C}^\times$$

$$\approx Z[W_K] \otimes_{Z[W_L]} \mathbb{C}^\times$$

$$\equiv \text{Ind}^{W_K}_{W_L}.$$

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Consequently

\[ H^1(W_K, \hat{T}(\mathbb{C})) \approx H^1(W_K, \text{Ind}_{W_L}^{W_K} W_{C^\times}) \]

\[ \approx H^1(W_L, C^\times) \quad \text{(Shapiro's lemma)} \]

\[ \approx \text{Mor}(W_L, C^\times) \]

\[ \approx \text{Mor}(L^\times, C^\times) \]

\[ \approx \text{Mor}(T(K), C^\times), \]

which completes the proof modulo “continuity details” that we shall not stop to sort out.

22: DEFINITION The L-group of T is the semidirect product

\[ L_T = \hat{T}(\mathbb{C}) \rtimes W_K. \]

Because of this, it will be best to first recall “semidirect product theory”.

23: RAPPEL If G is a group and if A is a G-module, then there is a canonical extension of G by A, namely

\[ 0 \to A \overset{i}{\to} A \rtimes G \overset{\pi}{\to} G \to 1, \]

where \( A \rtimes G \) is the semidirect product.

24: DEFINITION A splitting of the extension

\[ 0 \to A \overset{i}{\to} A \rtimes G \overset{\pi}{\to} G \to 1 \]

is a homomorphism \( s : G \to A \rtimes G \) such that \( \pi \circ s = \text{id}_G \).
25: **FACT** The splittings of the extension

\[ 0 \to A \xrightarrow{i} A \times G \xrightarrow{\pi} G \to 1 \]

determine and are determined by the elements of \( \mathbb{Z}^1(G, A) \).

Two splittings \( s_1, s_2 \) are said to be equivalent if there is an element \( a \in A \) such that

\[ s_1(\sigma) = i(a)s_2(\sigma)i(a)^{-1} \quad (\sigma \in G). \]

If

\[
\begin{cases}
  f_1 \leftrightarrow s_1 \\
  f_2 \leftrightarrow s_2
\end{cases}
\]

are the 1-cocycles corresponding to \( \begin{cases} s_1 \\ s_2 \end{cases} \), then their difference \( f_2 - f_1 \) is a 1-coboundary.

26: **SCHOLIUM** The equivalence classes of splittings of the extension

\[ 0 \to A \xrightarrow{i} A \times G \xrightarrow{\pi} G \to 1 \]

are in a bijective correspondence with the elements of \( H^1(G, A) \).

Return now to the extension

\[
0 \to \hat{T}(\mathbb{C}) \to \hat{T}(\mathbb{C}) \rtimes W_\kappa \to W_\kappa \to 1
\]

but to reflect the underlying topologies, work with continuous splittings and call them **admissible homomorphisms**. Introducing the obvious notion of equivalence, denote by \( \Phi_\kappa(T) \) the set of equivalence classes of admissible homomorphisms, hence

\[ \Phi_\kappa(T) \approx H^1_c(W_\kappa, \hat{T}(\mathbb{C})). \]
On the other hand, denote by $A_K(T)$ the group of characters of $T(\mathbb{K})$, i.e.,

$$A_K(T) \approx \text{Mor}_c(T(\mathbb{K}), \mathbb{C}^\times).$$

**27: THEOREM** There is a canonical isomorphism

$$\Phi_K(T) \rightarrow A_K(T).$$

[This statement is just a rephrasing of #18 and is the LLC for tori.]

**28: HEURISTIC** To each admissible homomorphism of $W_K$ into $LT$, it is possible to associate an irreducible automorphic representation of $T(\mathbb{K})$ (a.k.a. a character of $T(\mathbb{K})$) and all such arise in this fashion.

It remains to consider the archimedean case: $\mathbb{C}$ or $\mathbb{R}$.

- If $T$ is a $\mathbb{C}$-torus, then $T$ is isomorphic to a finite product

$$G_{m,\mathbb{C}} \times \cdots \times G_{m,\mathbb{C}}$$

and

$$T(\mathbb{C}) \approx \text{Mor}(X^*(T), \mathbb{C}^\times)$$

$$\approx X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.$$  

Furthermore, $W_\mathbb{C} = \mathbb{C}^\times$ and the claim is that

$$H^1_c(W_\mathbb{C}, \hat{T}(\mathbb{C})) \equiv \text{Mor}_c(\mathbb{C}^\times, \hat{T}(\mathbb{C}))$$

is isomorphic to

$$\text{Mor}_c(T(\mathbb{C}), \mathbb{C}^\times).$$
But

\[ \text{Mor}_c(\mathbb{C}^\times, \hat{T}(\mathbb{C})) \approx \text{Mor}_c(\mathbb{C}^\times, X^*(T) \otimes \mathbb{C}^\times) \]
\[ \approx \text{Mor}_c(\mathbb{C}^\times, \text{Mor}(X^*(T), \mathbb{C}^\times)) \]
\[ \approx \text{Mor}_c(X^*(T) \otimes \mathbb{C}^\times, \mathbb{C}^\times) \]
\[ \approx \text{Mor}_c(T(\mathbb{C}), \mathbb{C}^\times). \]

- If \( T \) is an \( \mathbb{R} \)-torus, then \( T \) is isomorphic to a finite product

\[ (G_{m,\mathbb{R}})^a \times (\text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}))^b \times (\text{SO}_2)^c \]

and it is enough to look at the three irreducible possibilities.

1. \( T = G_{m,\mathbb{R}} \). The point here is that \( W_{ab}^{\mathbb{R}} \approx \mathbb{R}^\times \approx T(\mathbb{R}) \).

2. \( T = \text{Res}_{\mathbb{C}/\mathbb{R}}(G_{m,\mathbb{C}}) \). One can imitate the argument used above for its non-archimedean analog.

3. \( T = \text{SO}_2 \). The initial observation is that \( X(T) = \mathbb{Z} \) with action \( n \rightarrow -n \), so \( \hat{T}(\mathbb{C}) = \mathbb{C}^\times \) with action \( z \rightarrow \frac{1}{z} \). And . . . .

**APPENDIX**

Let \( G \) be a group (written multiplicatively).

**1: DEFINITION** A left (right) \( G \)-module is an abelian group \( A \) equipped with a left (right) action of \( G \), i.e., with a homomorphism \( G \rightarrow \text{Aut}(A) \).

**2: N.B.** Spelled out, to say that \( A \) is a left \( G \)-module means that there is a map

\[
\begin{cases}
G \times A & \rightarrow A \\
(\sigma, a) & \rightarrow \sigma a
\end{cases}
\]

5-11
such that
\[ \tau(\sigma a) = (\tau \sigma) a, \quad 1a = a, \]
thus \( A \) is first of all a left \( G \)-set. To say that \( A \) is a left \( G \)-module then means in addition that
\[ \sigma(a + b) = \sigma a + \sigma b. \]

[Note: For the most part, the formalities are worked out from the left, the agreement being that
“left \( G \)-module” = “\( G \)-module”.]

3: NOTATION The group ring \( \mathbb{Z}[G] \) is the ring whose additive group is the free abelian group with basis \( G \) and whose multiplication is determined by the multiplication in \( G \) and the distributive law.

A typical element of \( \mathbb{Z}[G] \) is
\[ \sum_{\sigma \in G} m_\sigma \sigma, \]
where \( m_\sigma \in \mathbb{Z} \) and \( m_\sigma = 0 \) for all but finitely many \( \sigma \).

4: N.B. A \( G \)-module is the same thing as a \( \mathbb{Z}[G] \)-module.

5: LEMMA Given a ring \( R \), there is a canonical bijection
\[ \text{Mor}(\mathbb{Z}[G], R) \cong \text{Mor}(G, R^\times). \]

6: CONSTRUCTION Given a \( G \)-set \( X \), form the free abelian group \( \mathbb{Z}[X] \) generated by \( X \) and extend the action of \( G \) on \( X \) to a \( \mathbb{Z} \)-linear action of \( G \) on \( \mathbb{Z}[X] \) —then the resulting \( G \)-module is called a **permutation module**.

7: EXAMPLE Let \( H \) be a subgroup of \( G \) and take \( X = G/H \) (here \( G \) operates on \( G/H \) by left translation), from which \( \mathbb{Z}[G/H] \).
8: **DEFINITION** A $G$-module homomorphism is a $\mathbb{Z}[G]$-module homomorphism.

9: **NOTATION** $\text{MOD}_G$ is the category of $G$-modules.

10: **NOTATION** Given $A, B$ in $\text{MOD}_G$, write $\text{Hom}_G(A, B)$ in place of $\text{Mor}(A, B)$.

11: **LEMMA** Let $A, B \in \text{MOD}_G$ – then $A \otimes_{\mathbb{Z}} B$ carries the $G$-module structure defined by $\sigma(a \otimes a') = \sigma a \otimes \sigma a'$ and $\text{Hom}_\mathbb{Z}(A, B)$ carries the $G$-module structure defined by $(\sigma \phi)(a) = \sigma \phi(\sigma^{-1} a)$.

12: **LEMMA** If $G'$ is a subgroup of $G$, then there is a homomorphism $\mathbb{Z}[G'] \rightarrow \mathbb{Z}[G]$ of rings and a functor

$$\text{Res}^G_{G'} : \text{MOD}_G \rightarrow \text{MOD}_{G'}$$

of restriction.

13: **DEFINITION** Let $G'$ be a subgroup of $G$, then the functor of induction

$$\text{Ind}^G_{G'} : \text{MOD}_{G'} \rightarrow \text{MOD}_G$$

sends $A'$ to

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} A'.$$

[Note: $\mathbb{Z}[G]$ is a right $\mathbb{Z}[G']$-module and $A'$ is a left $\mathbb{Z}[G']$-module. Therefore the tensor product

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} A'$$

is an abelian group. And it becomes a left $G$-module under the operation $\sigma(r \otimes a') = \sigma r \otimes a'$.]
14: **EXAMPLE** Let $H$ be a subgroup of $G$. Suppose that $H$ operates trivially on $\mathbb{Z}$—then

$$\mathbb{Z}[G/H] \approx \text{Ind}_H^G \mathbb{Z}.$$

15: **FROBENIUS RECIPROCITY** $\forall A \text{ in } \text{MOD}_G, \forall A' \text{ in } \text{MOD}_{G'}$,

$$\text{Hom}_{G'}(A', \text{Res}_G^{G'} A) \approx \text{Hom}_G(\text{Ind}_G^{G'} A', A).$$

16: **REMARK** $\forall A \text{ in } \text{MOD}_G$,

$$\text{Ind}_G^{G'} \circ \text{Res}_G^{G'} A \approx \mathbb{Z}[G/G'] \otimes_{\mathbb{Z}[G]} A.$$

$[G \text{ operates on the right hand side diagonally: } \sigma(r \otimes a) = \sigma r \otimes \sigma a.]$

17: **LEMMA** There is an arrow of inclusion

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A' \to \text{Hom}_{G'}(\mathbb{Z}[G], A')$$

which is an isomorphism if $[G : G'] < \infty$.

18: **NOTATION** Given a $G$-module $A$, put

$$A^G = \{a \in A : \sigma a = a \ \forall \ \sigma \in G\}.$$ 

[Note: $A^G$ is a subgroup of $A$, termed the **invariants** in $A$.]

19: **LEMMA** $A^G = \text{Hom}_G(\mathbb{Z}, A)$ (trivial $G$-action on $\mathbb{Z}$).

[Note: By comparison, 

$$A = \text{Hom}_G(\mathbb{Z}[G], A).$$]

20: **LEMMA** $\text{Hom}_\mathbb{Z}(A, B)^G = \text{Hom}_G(A, B)$. 

5-14
\( \text{MOD}_G \) is an abelian category. As such, it has enough injectives (i.e., every \( G \)-module can be embedded in an injective \( G \)-module).

**21: Definition** The group cohomology functor \( H^q(G, -) : \text{MOD}_G \to \text{AB} \) is the right derived functor of \( (-)^G \).

[Note: Recall the procedure: To compute \( H^q(G, A) \), choose an injective resolution

\[
0 \to A \to I^0 \to I^1 \to \cdots.
\]

Then \( H^* (G, A) \) is the cohomology of the complex \( (I)^G \). In particular: \( H^0(G, A) = A^G \).]

**22: Lemma** \( H^q(G, A) \) is independent of the choice of injective resolutions.

**23: Lemma** \( H^q(G, A) \) is a covariant functor of \( A \).

**24: Lemma** If

\[
0 \to A \to B \to C \to 0
\]

is a short exact sequence of \( G \)-modules, then there is a functorial long exact sequence

\[
0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \\
\to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to H^2(G, A) \to \cdots \\
\cdots \to H^q(G, A) \to H^q(G, B) \to H^q(G, C) \to H^{q+1}(G, A) \to \cdots
\]

in cohomology.

**25: N.B.** If \( G = \{1\} \) is the trivial group, then

\[
H^0(G, A) = A, \quad H^q(G, A) = 0 \quad (q > 0).
\]

[Note: Another point is that for any \( G \), every injective \( G \)-module \( A \) is cohomologically acyclic:

\[
\forall q > 0, \ H^q(G, A) = 0.
\]

5-15]
**26: THEOREM (SHAPIRO’S LEMMA)** If \([G : G'] < \infty\), then \(\forall q\),

\[ H^q(G', A') \approx H^q(G, \text{Ind}_{G'}^G A'). \]

**27: EXAMPLE** Take \(A' = \mathbb{Z}[G']\) then

\[ H^q(G', \mathbb{Z}[G']) \approx H^q(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} \mathbb{Z}[G']) \approx H^q(G, \mathbb{Z}[G]). \]

**28: EXAMPLE** Take \(G' = \{1\}\) (so \(G\) is finite) then \(\mathbb{Z}[G'] = \mathbb{Z}\) and

\[ H^q(\{1\}, \mathbb{Z}) \approx H^q(G, \mathbb{Z}[G]). \]

But the LHS vanishes if \(q > 0\), thus the same is true of the RHS. However, this fails if \(G\) is infinite. E.g.: Take for \(G\) the infinite cyclic group: \(H^1(G, \mathbb{Z}[G]) \approx \mathbb{Z}\).

[Note: If \(G\) is finite, then \(H^0(G, \mathbb{Z}[G]) \approx \mathbb{Z}\) while if \(G\) is infinite, then \(H^0(G, \mathbb{Z}[G]) = 0\).]

**29: EXAMPLE** Take \(A' = \mathbb{Z}\) then

\[ H^q(G', \mathbb{Z}) \approx H^q(G, \text{Ind}_{G'}^G \mathbb{Z}) \approx H^q(G, \mathbb{Z}[G/G']). \]
§6. TAMAGAWA MEASURES

Suppose given a $\mathbb{Q}$-torus $T$ of dimension $d$ — then one can introduce

$$T(\mathbb{Q}) \subset T(\mathbb{R}), \quad T(\mathbb{Q}) \subset T(\mathbb{Q}_p)$$

$$\cup$$

$$T(\mathbb{Z}_p)$$

and

$$T(\mathbb{Q}) \subset T(\mathbb{A}).$$

1: **EXAMPLE** Take $T = G_{m,\mathbb{Q}}$ — then the above data becomes

$$\mathbb{Q}^\times \subset \mathbb{R}^\times, \quad \mathbb{Q}^\times \subset \mathbb{Q}_p^\times$$

$$\cup$$

$$\mathbb{Z}_p^\times$$

and

$$\mathbb{Q}^\times \subset \mathbb{A}^\times = \mathbb{I}.$$

2: **LEMMA** $T(\mathbb{Q})$ is a discrete subgroup of $T(\mathbb{A}).$

3: **RAPPEL** $\mathbb{I}^1 = \text{Ker } |\cdot|_\mathbb{A}$, where for $x \in \mathbb{I}$,

$$|x|_\mathbb{A} = \prod_{p \leq \infty} |x_p|_p.$$
And the quotient $\mathbb{I}^1/Q^\times$ is a compact Hausdorff space.

Each $\chi \in X(T)$ generates continuous homomorphisms

$$
\begin{cases}
\chi_p : T(Q_p) \to Q_p^\times \xrightarrow{|\cdot|_p} \mathbb{R}_{>0}^\times \\
\chi_\infty : T(\mathbb{R}) \to \mathbb{R}^\times \xrightarrow{|\cdot|_\infty} \mathbb{R}_{>0}^\times 
\end{cases}
$$

from which an arrow

$$
\begin{cases}
\chi_A : T(A) \to \mathbb{R}_{>0}^\times \\
x \to \prod_{p \leq \infty} \chi_p(x_p)
\end{cases}
$$

4: NOTATION

$$T^1(A) = \bigcap_{\chi \in X(T)} \ker \chi_A.$$  

5: N.B. The infinite intersection can be replaced by a finite intersection since if $\chi_1, \ldots, \chi_d$ is a basis for $X(T)$, then

$$T^1(A) = \bigcap_{i=1}^d \ker (\chi_i)_A.$$  

6: THEOREM The quotient $T^1(A)/T(\mathbb{Q})$ is a compact Hausdorff space.

7: CONSTRUCTION Let $\Omega_T$ denote the collection of all left invariant $d$-forms on $T$, thus $\Omega_T$ is a 1-dimensional vector space over $\mathbb{Q}$. Choose a nonzero element $\omega \in \Omega_T$—then $\omega$ determines a left invariant differential form of top degree on the $T(Q_p)$ and $T(\mathbb{R})$, which in turn determines a Haar measure $\mu_{Q_p, \omega}$ on the $T(Q_p)$ and a Haar measure $\mu_{\mathbb{R}, \omega}$ on $T(\mathbb{R})$.

The product

$$\prod_p \mu_{Q_p, \omega}(T(\mathbb{Z}_p))$$

may or may not converge.
**8: DEFINITION** A sequence Λ = \{Λₚ\} of positive real numbers is said to be a system of convergence coefficients if the product

\[
\prod_{p} Λₚµ_{Qₚ,ω}(T(Zₚ))
\]

is convergent.

**9: N.B.** Convergence coefficients always exist, e.g.,

\[
Λₚ = \frac{1}{µ_{Qₚ,ω}(T(Zₚ))}.
\]

**10: LEMMA** If the sequence Λ = \{Λₚ\} is a system of convergence coefficients, then

\[
µ_{ω,Λ} ≡ \prod_{p} Λₚµ_{Qₚ,ω} × µ_{R,ω}
\]

is a Haar measure on \(T(𝒜)\).

**11: N.B.** Let λ be a nonzero rational number – then

\[
µ_{Qₚ,λω} = |λ|ₚµ_{Qₚ,ω}, \quad µ_{R,λω} = |λ|_{∞}µ_{R,ω}.
\]

Therefore

\[
µ_{ω,Λ} ≡ \prod_{p} Λₚµ_{Qₚ,λω} × µ_{R,λω}
\]

\[
= (\prod_{p} |λ|ₚ) \prod_{p} Λₚµ_{Qₚ,ω} × |λ|_{∞}µ_{R,ω}
\]

\[
= \prod_{p ≤ ∞} |λ|ₚ \prod_{p} Λₚµ_{Qₚ,ω} × µ_{R,ω}
\]

\[
= µ_{ω,Λ}.
\]
And this means that the Haar measure $\mu_{\omega, \lambda}$ is independent of the choice of the rational density $\omega$.

Let $\mathbb{K}/\mathbb{Q}$ be a Galois extension relative to which $T$ splits — then

$$X(T_{\mathbb{K}}) = \text{Mor}_{\mathbb{K}}(T_{\mathbb{K}}, G_{m, \mathbb{K}})$$

is a $\text{Gal}(\mathbb{K}, \mathbb{Q})$ lattice. Call $\Pi$ the representation thereby determined and denote its character by $\chi_\Pi$. Let

$$L(s, \chi_\Pi, \mathbb{K}/\mathbb{Q}) = \prod_p L_p(s, \chi_\Pi, \mathbb{K}/\mathbb{Q})$$

be the associated Artin L-function and denote by $S$ the set of primes that ramify in $\mathbb{K}$ plus the “prime at infinity”.

12: **Lemma** $\forall p \notin S$,  

$$\mu_{\mathbb{Q}_p, \omega}(T(\mathbb{Z}_p)) = L_p(1, \chi_\Pi, \mathbb{K}/\mathbb{Q})^{-1}.$$  

13: **Scholium** The sequence $\Lambda = \{\Lambda_p\}$ defined by the prescription

$$\Lambda_p = L_p(1, \chi_\Pi, \mathbb{K}/\mathbb{Q}) \quad \text{if } p \notin S$$

and

$$\Lambda_p = 1 \quad \text{if } p \in S$$

is a system of convergence coefficients termed **canonical**.

14: **Lemma** The Haar measure $\mu_{\omega, \lambda}$ on $T(\mathbb{A})$ corresponding to a canonical system of convergence coefficients is independent of the choice of $\mathbb{K}$, denote it by $\mu_T$.

15: **Definition** $\mu_T$ is the **Tamagawa measure** on $T(\mathbb{A})$.  

6-4
Owing to Brauer theory, there is a decomposition of the character $\chi_\Pi$ of $\Pi$ as a finite sum

$$
\chi_\Pi = d\chi_0 + \sum_{j=1}^{M} m_j \chi_j,
$$

where $\chi_0$ is the principal character of $\text{Gal}(\mathbb{K}/\mathbb{Q})$ ($\chi_0(\sigma) = 1$ for all $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$), the $m_j$ are positive integers, and the $\chi_j$ are irreducible characters of $\text{Gal}(\mathbb{K}/\mathbb{Q})$. Standard properties of Artin L-functions then imply that

$$
L(s, \chi_\Pi, \mathbb{K}/\mathbb{Q}) = \zeta(s)^d \prod_{j=1}^{M} L(s, \chi_j, \mathbb{K}/\mathbb{Q})^{m_j}.
$$

16: FACT

$$
L(1, \chi_j, \mathbb{K}/\mathbb{Q})^{m_j} \neq 0 \quad (1 \leq j \leq M).
$$

Therefore

$$
\lim_{s \to 1} (s - 1)^d L(s, \chi_\Pi, \mathbb{K}/\mathbb{Q}) = \prod_{j=1}^{M} L(1, \chi_j, \mathbb{K}/\mathbb{Q})^{m_j}
$$

$$
\neq 0.
$$

17: LEMMA The limit on the left is positive and independent of the choice of $\mathbb{K}$, denote it by $\rho_T$.

18: DEFINITION $\rho_T$ is the residue of $T$.

Define a map

$$
T : T(\mathbb{A}) \to (\mathbb{R}_{>0})^d
$$

by the rule

$$
T(x) = ((\chi_1)_A(x), \ldots, (\chi_d)_A(x)).
$$
Then the kernel of $T$ is $T^1(\mathbb{A})$, hence $T$ drops to an isomorphism

$$T^1: T(\mathbb{A})/T^1(\mathbb{A}) \rightarrow (\mathbb{R}^\times_{>0})^d.$$ 

19: **DEFINITION** The **standard measure** on $T(\mathbb{A})/T^1(\mathbb{A})$ is the pullback via $T^1$ of the product measure

$$\prod_{i=1}^d \frac{d\lambda_i}{t_i}$$

on $(\mathbb{R}^\times_{>0})^d$.

Consider now the formalism

$$d(T(\mathbb{A})) = d(T(\mathbb{A})/T^1(\mathbb{A}))d(T^1(\mathbb{A})/T(\mathbb{Q}))d(T(\mathbb{Q}))$$

in which:

- $d(T(\mathbb{A}))$ is the Tamagawa measure on $T(\mathbb{A})$ multiplied by $\frac{1}{\rho_T}$.
- $d(T(\mathbb{A})/T^1(\mathbb{A}))$ is the standard measure on $T(\mathbb{A})/T^1(\mathbb{A})$.
- $d(T(\mathbb{Q}))$ is the counting measure on $T(\mathbb{Q})$.

20: **DEFINITION** The **Tamagawa number** $\tau(T)$ is the volume

$$\tau(T) = \int_{T^1(\mathbb{A})/T(\mathbb{Q})} 1$$

of the compact Hausdorff space $T^1(\mathbb{A})/T(\mathbb{Q})$ per the invariant measure

$$d(T^1(\mathbb{A})/T(\mathbb{Q}))$$

such that

$$\frac{\mu_T}{\rho_T} = d(T(\mathbb{A})/T^1(\mathbb{A}))d(T^1(\mathbb{A})/T(\mathbb{Q}))d(T(\mathbb{Q})).$$
To be completely precise, the integral formula

\[ \int_{T(\mathbb{A})} = \int_{T(\mathbb{A})/T^1(\mathbb{A})} \int_{T^1(\mathbb{A})} \]

fixes the invariant measure on \( T^1(\mathbb{A}) \) and from there the integral formula

\[ \int_{T^1(\mathbb{A})} = \int_{T^1(\mathbb{A})/T(\mathbb{Q})} \int_{T(\mathbb{Q})} \]

fixes the invariant measure on \( T^1(\mathbb{A})/T(\mathbb{Q}) \), its volume then being the Tamagawa number \( \tau(T) \).

[Note: If \( T \) is \( \mathbb{Q} \)-anisotropic, then \( T(\mathbb{A}) = T^1(\mathbb{A}) \).]

**EXAMPLE**

Take \( T = G_{m,\mathbb{Q}} \) and \( \omega = \frac{dx}{x} \) —then

\[ \text{vol}_{\frac{\mathbb{Z}}{p\mathbb{Z}}} (\mathbb{Z}^\times_p) = \frac{p - 1}{p} = 1 - \frac{1}{p} \]

and the canonical coverage coefficients are the

\[ (1 - \frac{1}{p})^{-1}. \]

Here \( d = 1 \) and

\[ \lim_{s \to 1} (s - 1)\zeta(s) = 1 \implies \rho_T = 1. \]

Working through the definitions, one concludes that \( \tau(T) = 1 \) or still,

\[ \text{vol}(\mathbb{I}^1/\mathbb{Q}^\times) = 1. \]

**REMARK**

Take \( T = \text{Res}_{\mathbb{K}/\mathbb{Q}}(G_{m,\mathbb{K}}) \) —then it turns out that \( \tau(T) \) is the Tamagawa number of \( G_{m,\mathbb{K}} \) computed relative to \( \mathbb{K} \) (and not relative to \( \mathbb{Q} \ldots \)). From this,
it follows that \( \tau(T) = 1 \), matters hinging on the “famous formula”

\[
\lim_{s \to 1} (s - 1)\zeta_K(s) = \frac{2^{\gamma_1}(2\pi)^{\gamma_2}}{w_K |d_K|^{1/2}}R_K R. 
\]

**24: LEMMA** Let \( F \) be an integrable function on \((\mathbb{R}_0^\times)^d\) – then

\[
\tau(T) = \frac{1}{\rho_T} \int_{T(\mathbb{A})/T(\mathbb{Q})} F(T(x))d\mu_T(x) 
\]

\[
\int_{(\mathbb{R}_0^\times)^d} F(t_1, \ldots, t_d) \frac{dt_1}{t_1} \ldots \frac{dt_d}{t_d}
\]

**25: EXAMPLE** Let \( T = G_{m, \mathbb{Q}} \) – then

\[
\tau(T) = \frac{1}{\rho_T} \int_{\mathbb{Q}^\times} F(|x|_K)d\mu_T(x) 
\]

\[
\int_0^\infty \frac{F(t)}{t} dt
\]

\( \rho_T \) being 1 in this case. To see that \( \tau(T) = 1 \), make the calculation by choosing

\[
F(t) = 2te^{-\pi t^2}.
\]

[Note: Recall that

\[
\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_0^\times
\]

is a fundamental domain for \( \mathbb{I}/\mathbb{Q}^\times \).]

**26: NOTATION** Put

\[
H^1(\mathbb{Q}, T) = H^1(\text{Gal}(\mathbb{Q}^{\text{sep}}/\mathbb{Q}), T(\mathbb{Q}^{\text{sep}}))
\]

6-8
and for $p \leq \infty$,

$$H^1(\mathbb{Q}_p, T) = H^1(\text{Gal}(\mathbb{Q}_p^{\text{sep}}/\mathbb{Q}_p), T(\mathbb{Q}_p^{\text{sep}})).$$

**27: LEMMA** There is a canonical arrow

$$H^1(\mathbb{Q}, T) \rightarrow H^1(\mathbb{Q}_p, T).$$

**PROOF** Put

$$G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \quad (\overline{\mathbb{Q}} = \mathbb{Q}^{\text{sep}})$$

and

$$G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \quad (\overline{\mathbb{Q}_p} = \mathbb{Q}_p^{\text{sep}}).$$

Then schematically

1. There is an arrow of restriction

$$\rho : G_p \rightarrow G$$

and a morphism $T(\mathbb{Q}) \rightarrow T(\overline{\mathbb{Q}_p})$ of $G_p$-modules, $T(\mathbb{Q})$ being viewed as a $G_p$-module via $\rho$.

2. The canonical arrow

$$H^1(\mathbb{Q}, T) \rightarrow H^1(\mathbb{Q}_p, T)$$
is then the result of composing the map

$$H^1(G, T(\mathbb{Q})) \rightarrow H^1(G_p, T(\mathbb{Q}))$$

with the map

$$H^1(G_p, T(\mathbb{Q})) \rightarrow H^1(G_p, T(\mathbb{Q}_p)).$$

28: NOTATION  Put

$$\text{III}(T) = \text{Ker}(H^1(\mathbb{Q}, T) \rightarrow \prod_{p \leq \infty} H^1(\mathbb{Q}_p, T)).$$

29: DEFINITION  \(\text{III}(T)\) is the Tate-Shafarevich group of \(T\).

30: THEOREM  \(\text{III}(T)\) is a finite group.

31: EXAMPLE  If \(K\) is a finite extension of \(\mathbb{Q}\), then

$$H^1(\mathbb{Q}, \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m, K)) = 1.$$ 

Therefore in this case

$$\#(\text{III}(T)) = 1.$$ 

32: REMARK  By comparison,

$$H^1(\mathbb{Q}, \text{Res}_{K/\mathbb{Q}}^{(1)}(\mathbb{G}_m, K)) \approx \mathbb{Q}^\times / N_{K/\mathbb{Q}}(K^\times).$$

[Consider the short exact sequence

$$1 \rightarrow \text{Res}_{K/\mathbb{Q}}^{(1)}(\mathbb{G}_m, K) \rightarrow \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m, K) \rightarrow N_{K/\mathbb{Q}} \rightarrow G_{m, \mathbb{Q}} \rightarrow 1.$$ ]
NOTATION

Put

\[ \eta(T) = \text{CoKer}\left( H^1(\mathbb{Q}, T) \to \prod_{p \leq \infty} H^1(\mathbb{Q}_p, T) \right). \]

THEOREM

\[ \eta(T) \] is a finite group.

MAIN THEOREM

The Tamagawa number \( \tau(T) \) is given by the formula

\[ \tau(T) = \frac{\#(\eta(T))}{\#(\text{III}(T))}. \]

EXAMPLE

If \( K \) is a finite extension of \( \mathbb{Q} \), then

\[ H^1(\mathbb{Q}_p, \text{Res}_{K/\mathbb{Q}}(G_{m, K})) = 1. \]

Therefore in this case

\[ \#(\eta(T)) = 1. \]

It follows from the main theorem that \( \tau(T) \) is a positive rational number. Still, there are examples of finite abelian extensions \( K/\mathbb{Q} \) such that

\[ \tau(\text{Res}_{K/\mathbb{Q}}^{(1)}(G_{m, K})) \]

is not a positive integer.
REFERENCES

Görtz, U. et al.
[1] Algebraic Geometry I., VieWeg + Teubner Verlag, 2010.

Langlands, R. P.
[2] The Representation Theory of Abelian Algebraic Groups, Pacific Journal of Mathematics, vol. 181, Issue 3, 1997, pp. 231-250.

Milne, J. S.
[3] Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field, Cambridge University Press, 2017.

Ono, T.
[4-(a)] Arithmetic of Algebraic Tori, Annals of Mathematics, vol. 74, 1961, pp. 101-139.
[4-(b)] On the Tamagawa Number of Algebraic Tori, Annals of Mathematics, vol. 78, 1963, pp. 47-73.