MULTIPLE POSITIVE SOLUTIONS FOR A SCHröDINGER LOGARITHMIC EQUATION

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Abstract. This article concerns with the existence of multiple positive solutions for the following logarithmic Schrödinger equation

\[
\begin{cases}
\epsilon^2 \Delta u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\]

where \(\epsilon > 0\), \(N \geq 1\) and \(V\) is a continuous function with a global minimum.

Using variational method, we prove that for small enough \(\epsilon > 0\), the “shape” of the graph of the function \(V\) affects the number of nontrivial solutions.

1. Introduction. Recently, the logarithmic Schrödinger equation given by

\[
i\epsilon \partial_t \Psi = -\epsilon^2 \Delta \Psi + (W(x) + w)\Psi - \Psi \log |\Psi|^2, \quad \Psi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, \quad N \geq 1,
\]

has received a special attention because it appears in a lot of physical applications, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation (see [14] and the references therein).

In its turn, standing waves solution, \(\Psi\), for this logarithmic Schrödinger equation is related to solutions of the equation

\[
-\epsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N.
\]

Besides the importance in applications, this last equation is very interesting in the mathematical point of view, because it arises a lot of difficulties to apply variational methods in order to find a solution for it. The natural candidate for the associated energy functional would formally be the functional

\[
\hat{I}_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^N} H(u)dx,
\]

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where

$$H(t) = \int_0^t s \log s^2 \, ds = -\frac{t^2}{2} + \frac{t^2 \log t^2}{2}, \quad \forall t \in \mathbb{R}$$

that is,

$$\hat{I}_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + (V(x) + 1)|u|^2) \, dx - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx.$$  

However, this functional is not well defined in $H^1(\mathbb{R}^N)$ because there is $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} u^2 \log u^2 \, dx = -\infty$. In order to overcome this technical difficulty some authors have used different techniques.

In [3], d’Avenia, Montefusco and Squassina have studied the existence of multiple solutions for a logarithmic elliptic equation of the type

$$\begin{cases}
-\Delta u + u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N).
\end{cases} \quad (P_1)$$

The authors obtained solutions for this equation by applying the non-smooth critical point theory, found in Degiovanii and Zani [5], to the energy functional defined on the space of radial functions $H^1_{rad}(\mathbb{R}^N)$. In [4], d’Avenia, Squassina and Zenari have used the same approach to show the existence of solution for a fractional logarithmic Schrödinger equation of the type

$$\begin{cases}
(-\Delta)^s u + u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^s(\mathbb{R}^N),
\end{cases} \quad (P_2)$$

for $s \in (0, 1)$ and $N > 2s$.

In [9], Squassina and Szulkin have showed the existence of multiple solutions for the following class of problem

$$\begin{cases}
-\Delta u + V(x)u = Q(x)u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases} \quad (P_3)$$

where $V, Q : \mathbb{R}^N \to \mathbb{R}$ are 1-periodic continuous functions verifying

$$\min_{x \in \mathbb{R}^N} Q(x) > 0 \quad \text{and} \quad \min_{x \in \mathbb{R}^N} (V + Q)(x) > 0.$$  

In that paper, the authors have used the minimax principles for lower semicontinuous functionals, developed by Szulkin [11], to prove the existence of geometrically distinct multiple solutions and the existence of a ground state solution. The multiple solutions follow by genus theory found in [11], while the existence of ground state follows by a specific deformation lemma, see [9, Lemma 2.14].

Later, Ji and Szulkin in [6] have established the existence of multiple solutions for a problem of the type

$$\begin{cases}
-\Delta u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases} \quad (P_4)$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function that satisfies

$$\lim_{|x| \to +\infty} V(x) = V_\infty \quad \text{where} \quad V_\infty + 1 \in (0, +\infty].$$

With the same approach explored in [9], the above authors showed that if $V_\infty = +\infty$, then $(P_4)$ has infinitely many solutions. If $V_\infty \in (-1, +\infty)$ and the spectrum $\sigma(-\Delta + V + 1) \subset (0, +\infty)$, then problem $(P_4)$ has a ground state solution. In [12], Tanaka and Zhang have studied the existence of solution for $(P_3)$. In that very nice paper the authors have observed that the positivity of $V$ is not essential.
Finally, in a recent work, Alves and de Morais Filho in [1] have used the minimax method found in [11] to show the existence and concentration of positive solution for the problem

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

(1.2)

where $\epsilon > 0$, $N \geq 1$ and $V$ is a continuous function with a global minimum.

Motivated by results found in the above mentioned papers, in the present paper we intend to study the existence of multiple solutions for problem (1.2) by supposing the following conditions on potential $V$:

$(V1)$ $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that

$$\lim_{|x| \to \infty} V(x) = V_\infty,$$

with $0 < V(x) < V_\infty$ for any $x \in \mathbb{R}^N$.

$(V2)$ There exist $l$ points $z_1, z_2, \ldots, z_l$ in $\mathbb{R}^N$ with $z_1 = 0$ such that

$$1 = V(z_i) = \min_{x \in \mathbb{R}^N} V(x), \text{ for } 1 \leq i \leq l.$$

By a change of variable, we know that problem (1.2) is equivalent to the problem

$$\begin{cases} -\Delta u + V(\epsilon x)u = u \log u^2, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

(1.3)

**Definition 1.1.** For us, a positive solution of (1.3) means a positive function $u \in H^1(\mathbb{R}^N)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(\epsilon x)u \cdot v)dx = \int_{\mathbb{R}^N} uv \log u^2dx, \text{ for all } v \in C_0^\infty(\mathbb{R}^N).$$

(1.4)

The main result to be proved is the theorem below.

**Theorem 1.1.** Suppose that $V$ satisfies $(V1)$ and $(V2)$. Then there is $\epsilon_* > 0$ such that problem (1.2) has at least $l$ positive solutions in $H^1(\mathbb{R}^N)$ for all $\epsilon \in (0, \epsilon_*)$.

In the proof of Theorem 1.1, we adapt for our problem some ideas explored in Cao and Noussair [2], where the existence and multiplicity of solutions have considered for the following class of problem

$$\begin{cases} -\Delta u + u = A(\epsilon x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

(1.5)

Using Ekeland’s variational principle and concentration compactness principle of Lions [8], Cao and Noussair proved that if $A$ has $l$ equal maximum points, then problem (1.5) has at least $l$ positive solutions and $l$ nodal solutions if $\epsilon > 0$ is small enough. We would like to point out that different of [2], where the energy functional is $C^1$, we cannot work directly with the energy functional associated with (1.3) because it is not continuous, and so, it is not $C^1$. Have this in mind, for each $R > 0$, we first find a solution $u_{\epsilon,R} \in H^1_0(B_R(0))$, and after, taking the limit of $R \to +\infty$ we get a solution for the original problem.

The plan of the paper is as follows. In Section 2 we show some preliminary results which will be used later on. In Section 3 we prove the existence of multiple solutions for an auxiliary problem, while in Section 4 we prove Theorem 1.1.

**Notation:** From now on in this paper, otherwise mentioned, we use the following notations:
• $B_r(u)$ is an open ball centered at $u$ with radius $r > 0$, $B_r = B_r(0)$.
• If $g$ is a measurable function, the integral $\int_{\mathbb{R}^N} g(x) \, dx$ will be denoted by $\int g(x) \, dx$. Moreover, we denote by $g^+$ and $g^-$ the positive and negative part of $g$ given by $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \max\{-g(x), 0\}$.

$C$ denotes any positive constant, whose value is not relevant.
$| \cdot |_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^N)$, for $p \in [1, +\infty]$.
$H^1_0(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u \text{ has compact support} \}$.
$a_n(1)$ denotes a real sequence with $a_n(1) \to 0$ as $n \to +\infty$.
$2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$.

2. Preliminaries. Hereafter, we consider the problem
\begin{align*}
\begin{cases}
-\Delta u + V(0)u = u \log u^2, & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{cases}
\end{align*}
(2.1)

The corresponding energy functional associated to (2.1) will be denoted by $J_0 : H^1(\mathbb{R}^N) \to (-\infty, +\infty)$ and defined as
\[ J_0(u) = \frac{1}{2} \int (|\nabla u|^2 + (V(0) + 1)|u|^2) \, dx - \frac{1}{2} \int u^2 \log u^2 \, dx. \]

In [9] is proved that problem (2.1) has a positive solution attained at the infimum
\[ c_0 := \inf_{u \in N_0} J_0(u), \]
(2.2)
where
\[ N_0 = \left\{ u \in D(J_0) \setminus \{0\} : J_0(u) = \frac{1}{2} \int |u|^2 \, dx \right\} \]
and
\[ D(J_0) = \left\{ u \in H^1(\mathbb{R}^N) : J_0(u) < +\infty \right\}. \]

Mutatis mutandis the previous notations, we shall also use the energy level
\[ c_\infty = \inf_{u \in N_\infty} J_\infty(u), \]
corresponding to problem (2.1), replacing $V(0)$ by $V_\infty$. Using the definition of $c_0$ and $c_\infty$, it follows that
\[ c_0 < c_\infty. \]

Related to the numbers $c_0$ and $c_\infty$, we would like to point out that they are the mountain pass levels of the functionals $J_0$ and $J_\infty$ respectively.

Following the approach explored in [1, 6, 9], due to the lack of smoothness of $J_0$ and $J_\infty$, let us decompose them into a sum of a $C^1$ functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, let us define the following functions:
\[ F_1(s) = \begin{cases}
0, & s = 0 \\
-\frac{1}{2} s^2 \log s^2, & \frac{1}{2} s^2 < |s| < \delta \\
-\frac{1}{2} s^2 (\log \delta^2 + 3) + 2\delta |s| - \frac{1}{2} \delta^2, & |s| \geq \delta
\end{cases}, \]
and
\[ F_2(s) = \begin{cases}
0, & |s| < \delta \\
\frac{1}{2} s^2 \log(s^2/\delta^2) + 2\delta |s| - \frac{3}{2} s^2 - \frac{1}{2} \delta^2, & |s| \geq \delta
\end{cases}. \]
Therefore
\[ F_2(s) - F_1(s) = \frac{1}{2} s^2 \log s^2, \quad \forall s \in \mathbb{R}, \]
and the functionals \( J_0, J_\infty : H^1(\mathbb{R}^N) \to (\mathbb{R}, +\infty) \) may be rewritten as
\[ J_0(u) = \Phi_0(u) + \Psi(u) \quad \text{and} \quad J_\infty(u) = \Phi_\infty(u) + \Psi(u), \quad u \in H^1(\mathbb{R}^N) \]
where
\[ \Phi_0(u) = \frac{1}{2} \int (|\nabla u|^2 + (V(0) + 1)|u|^2) \, dx - \int F_2(u) \, dx, \]
\[ \Phi_\infty(u) = \frac{1}{2} \int (|\nabla u|^2 + (V_{\infty} + 1)|u|^2) \, dx - \int F_2(u) \, dx, \]
and
\[ \Psi(u) = \int F_1(u) \, dx. \]

It was proved in [6] and [9] that \( F_1 \) and \( F_2 \) verify the following properties:
\[ F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R}). \quad \tag{2.9} \]

If \( \delta > 0 \) is small enough, \( F_1 \) is convex, even, \( F_1(s) \geq 0 \) for all \( s \in \mathbb{R} \) and
\[ F'_1(s)s \geq 0, \quad s \in \mathbb{R}. \quad \tag{2.10} \]

For each fixed \( p \in (2, 2^*) \), there is \( C > 0 \) such that
\[ |F_2'(s)| \leq C|s|^{p-1}, \quad \forall s \in \mathbb{R}. \quad \tag{2.11} \]

3. An auxiliary functional. In what follows, let us fix \( R_0 > 0 \) such that \( z_i \in B_{R_0}(0) \) for all \( i \in \{0, 1, \ldots, l\} \). Moreover, for each \( R > R_0 \), we set the functional
\[ J_{\epsilon,R}(u) = \frac{1}{2} \int_{B_R(0)} (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2) \, dx - \frac{1}{2} \int_{B_R(0)} u^2 \log u^2 \, dx, \quad u \in H^1_0(B_R(0)). \quad \tag{3.1} \]

It is easy to check that \( J_{\epsilon,R} \in C^1(H^1_0(B_R(0)), \mathbb{R}) \) with
\[ J'_{\epsilon,R}(u)v = \int_{B_R(0)} (\nabla u \nabla v + V(\epsilon x)uv) \, dx - \int_{B_R(0)} vu^2 \log u \, dx, \quad \forall u, v \in H^1_0(B_R(0)). \]

Hereafter, \( H^1_0(B_R(0)) \) is endowed with the norm
\[ \|u\|_{\epsilon,R} = \left( \int (|\nabla u|^2 + (V(\epsilon x) + 1)|u|^2) \, dx \right)^\frac{1}{2}, \quad \tag{3.2} \]
which is also a norm in \( H^1(\mathbb{R}^N) \). Moreover, this norm is equivalent to the usual norms in \( H^1_0(B_R(0)) \) and \( H^1(\mathbb{R}^N) \) respectively.

In the sequel, \( \mathcal{N}_{\epsilon,R} \) denotes the Nehari manifold associated with \( J_{\epsilon,R} \), that is,
\[ \mathcal{N}_{\epsilon,R} = \left\{ u \in H^1_0(B_R(0)) \setminus \{0\} : J'_{\epsilon,R}(u)u = 0 \right\}, \]

or equivalently,
\[ \mathcal{N}_{\epsilon,R} = \left\{ u \in H^1_0(B_R(0)) \setminus \{0\} : J_{\epsilon,R}(u) = \frac{1}{2} \int |u|^2 \, dx \right\}. \]

The next three lemmas show that \( J_{\epsilon,R} \) verifies the mountain pass geometry and the well known \( (PS) \) condition.

**Lemma 3.1.** For all \( \epsilon > 0, R > R_0 \), the functional \( J_{\epsilon,R} \) has the mountain pass geometry.
Proof. 

(i): Note that \( J_{\varepsilon,R}(u) \geq \frac{1}{2} \|u\|_p^2 - \int_{B_R(0)} F_2(u) \, dx \). Hence, from (2.11), fixed \( p \in (2, 2^*) \), it follows that
\[
J_{\varepsilon,R}(u) \geq \frac{1}{2} \|u\|_p^2 - C\|u\|_p^p \geq C_1 > 0,
\]
for some \( C_1 > 0 \) and \( \|u\|_p > 0 \) small enough. Here the constant \( C_1 \) does not depend on \( \varepsilon \) and \( R > 0 \).

(ii): Let us fix \( u \in D(J_{\varepsilon,R}) \setminus \{0\} \) with \( \text{supp} u \subset B_{R_0}(0) \) and \( s > 0 \). Using (2.4) we get
\[
J_{\varepsilon,R}(su) = \frac{s^2}{2} \|u\|_p^2 - \frac{1}{2} \int_{B_{R_0}(0)} s^2 u^2 \log(\|su\|_p^2) \, dx
\]
= \( s^2 \left[ J_{\varepsilon,R}(u) - \log s \int_{B_{R_0}(0)} u^2 \, dx \right] \),
from where it follows that
\[
J_{\varepsilon,R}(su) \to -\infty \quad \text{as} \quad s \to +\infty.
\]
Thereby, there is \( s_0 > 0 \) independent of \( \varepsilon > 0 \) small enough and \( R > R_0 \) such that \( J_{\varepsilon,R}(s_0 su) < 0 \). \( \Box \)

Lemma 3.2. All \((PS)\) sequences of \( J_{\varepsilon,R} \) are bounded in \( H_0^1(B_R(0)) \).

Proof. Let \((u_n) \subset H_0^1(B_R(0))\) be a \((PS)_d\) sequence. Then,
\[
\int_{B_R(0)} |u_n|^2 \, dx = 2J_{\varepsilon,R}(u_n) - J'_{\varepsilon,R}(u_n)u_n = 2d + o_n(1) + o_n(1)\|u_n\|_e \leq C + o_n(1)\|u_n\|_e,
\]
for some \( C > 0 \). Consequently
\[
|u_n|_{L^2(B_R(0))}^2 \leq C + o_n(1)\|u_n\|_e. \tag{3.3}
\]
Now, let us employ the following logarithmic Sobolev inequality found in [7],
\[
\int u^2 \log u^2 \, dx \leq \frac{a^2}{\pi} |\nabla u|_{L^2(\mathbb{R}^N)}^2 + (\log |u|_{L^2(\mathbb{R}^N)}^2 - N(1 + \log a)) |u|_{L^2(\mathbb{R}^N)}^2 \tag{3.4}
\]
for all \( a > 0 \). Fixing \( \frac{a^2}{\pi} = \frac{1}{4} \) and \( \xi \in (0, 1) \), the inequalities (3.3) and (3.4) yield
\[
\int_{B_R(0)} u_n^2 \log u_n^2 \, dx \leq \frac{1}{4} |\nabla u_n|_{L^2(B_R(0))}^2 + C(\log |u_n|_{L^2(B_R(0))}^2 + 1) |u_n|_{L^2(B_R(0))}^2
\]
\leq \frac{1}{4} |\nabla u_n|^2 + C_1 (1 + \|u_n\|_e)^{1+\xi}. \tag{3.5}
\]
Since
\[
d + o_n(1) = J_{\varepsilon,R}(u_n) = \frac{1}{2} |\nabla u_n|^2 + \int_{B_R(0)} (V(\varepsilon x) + 1)|u_n|^2 \, dx - \frac{1}{2} \int_{B_R(0)} u_n^2 \log u_n^2 \, dx
\]
assertion (3.5) assures that
\[
d + o_n(1) \geq C[\|u_n\|_e^2 - (1 + \|u_n\|_e)^{1+\xi}], \tag{3.6}
\]
showing that the sequence \((u_n)\) is bounded in \( H_0^1(B_R(0)) \). \( \Box \)

Lemma 3.3. The mountain pass level \( c_{\varepsilon,R} \) of \( J_{\varepsilon,R} \) can be characterized by
\[
c_{\varepsilon,R} = \inf_{u \in \mathcal{N}_{\varepsilon,R}} J_{\varepsilon,R}(u).
\]
Proof. See [1, Lemma 3.3].

**Lemma 3.4.** The functional \( J_{\epsilon,R} \) satisfies the \((PS)\) condition.

**Proof.** Let \( (u_n) \subset H^1_0(B_R(0)) \) be a \((PS)_d\) sequence for \( J_{\epsilon,R} \), that is,
\[
J_{\epsilon,R}(u_n) \to d \quad \text{and} \quad J'_{\epsilon,R}(u_n) \to 0.
\]

By Lemma 3.2, we can assume that there is \( u \in H^1_0(B_R(0)) \) and a subsequence of \( (u_n) \), still denoted by itself, such that
\[
u_n \to u \quad \text{in} \quad H^1_0(B_R(0))
\]
\[
u_n \to u \quad \text{in} \quad L^q(B_R(0)), \quad \forall q \in [1,2^*)
\]
and
\[
u_n(x) \to u(x) \quad \text{a.e. in} \quad B_R(0).
\]

Setting \( f(t) = t \log t^2, F(t) = \int_0^t f(s) \, ds = \frac{1}{2} (t^2 \log t^2 - t^2) \) for all \( t \in \mathbb{R} \) and fixing \( p \in (2,2^*) \), there is \( C > 0 \) such that
\[
|f(t)| \leq C(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R}
\]
and
\[
|F(t)| \leq C(1 + |t|^p), \quad \forall t \in \mathbb{R}.
\]
Hence, by the Sobolev embeddings,
\[
\int_{B_R(0)} f(u_n) u_n \, dx \to \int_{B_R(0)} f(u) u \, dx
\]
and
\[
\int_{B_R(0)} f(u_n) v \, dx \to \int_{B_R(0)} f(u) v \, dx, \quad \forall v \in H^1_0(B_R(0)).
\]

Now, using the limits \( J'_{\epsilon,R}(u_n)u_n = J'_{\epsilon,R}(u_n)u + o_n(1) \), it is easy to see that
\[
\|u_n - u\|_2^2 = \int_{B_R(0)} f(u_n) u_n \, dx - \int_{B_R(0)} f(u_n) u \, dx + o_n(1) = o_n(1),
\]
showing the lemma. \(\square\)

In the following, let us fix \( \rho_0 > 0 \) satisfying \( \overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} = \emptyset \) for \( i \neq j \) and \( i, j \in \{1,...,l\} \), \( \bigcup_{i=1}^{l} B_{\rho_0}(z_i) \subset B_{R_0}(0) \) and \( K_{\epsilon} = \bigcup_{i=1}^{l} \overline{B_{2\epsilon}(z_i)} \). Moreover, we also set the function \( Q_{\epsilon}: H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N \) by
\[
Q_{\epsilon}(u) = \frac{\int \chi(\epsilon x)g(\epsilon x) |u|^2 \, dx}{\int g(\epsilon x) |u|^2 \, dx},
\]
where \( \chi: \mathbb{R}^N \to \mathbb{R}^N \) is given by
\[
\chi(x) = \begin{cases} 
  x, & \text{if } |x| \leq R_0, \\
  R_0 \frac{x}{|x|}, & \text{if } |x| > R_0,
\end{cases}
\]
and \( g: \mathbb{R}^N \to \mathbb{R} \) is a radial positive continuous function such that
\[
g(z_i) = 1, \quad \forall i \in \{1,...,l\} \quad \text{and} \quad g(x) \to 0 \quad \text{as} \quad |x| \to +\infty. \quad (3.7)
\]
The following lemma is very useful to obtain \((PS)_c\) sequences associated with \( J_{\epsilon,R} \).

**Lemma 3.5.** There exist \( \alpha_0 > 0, \epsilon_1 \in (0,\epsilon_0) \) small enough and \( R_1 > R_0 \) large enough such that if \( u \in N_{\epsilon,R} \) and \( J_{\epsilon,R}(u) \leq c_0 + \alpha_0 \), then \( Q_{\epsilon}(u) \in K_{\epsilon} \), \( \forall \epsilon \in (0,\epsilon_1) \) and \( R \geq R_1 \).
Proof. If the lemma is not true, then exist \( \alpha_n \to 0, \epsilon_n \to 0, R_n \to +\infty \) and \( u_n \in \mathcal{N}_{\epsilon_n,R_n} \) such that

\[ J_{\epsilon_n,R_n}(u_n) \leq c_0 + \alpha_n \]

and

\[ Q_{\epsilon_n}(u_n) \notin K_{\frac{\alpha_n}{2}}. \]

Since \( c_{\epsilon_n,R_n} \geq c_0 \), the above inequality gives

\[ c_0 \leq c_{\epsilon_n,R_n} \leq J_{\epsilon_n,R_n}(u_n) \leq c_0 + \alpha_n, \]

and so,

\[ u_n \in \mathcal{N}_{\epsilon_n,R_n} \quad \text{and} \quad J_{\epsilon_n,R_n}(u_n) = c_{\epsilon_n,R_n} + o_n(1). \]

Setting the functional \( \Psi_{\epsilon_n,R_n} : H_0^1(B_{R_n}) \to \mathbb{R} \) by

\[ \Psi_{\epsilon_n,R_n}(u) = J_{\epsilon_n,R_n}(u) - \frac{1}{2} \int_{B_{\epsilon_n,R_n}(0)} |u|^2 \, dx, \]

we derive

\[ \mathcal{N}_{\epsilon_n,R_n} = \{ u \in H_0^1(B_R(0)) \setminus \{ 0 \} : \Psi_{\epsilon_n,R_n}(u) = 0 \}. \]

A simple computation ensures that there is \( \beta > 0 \), which is independent of \( n \), such that

\[ \Psi'_{\epsilon_n,R_n}(u)u = - \int_{B_{\epsilon_n,R_n}(0)} |u|^2 \leq -\beta, \quad \forall n \in \mathbb{N}, \]

otherwise \( c_{\epsilon_n,R_n} \to 0 \), which is absurd. With this information in our hands, we can apply the Ekeland Variational principal found in [13, Theorem 8.5] to assume, without loss of generality, that \( \| J'_{\epsilon_n,B_{R_n}}(u_n) \| \to 0 \) as \( n \to +\infty \).

By [1, Section 6], we need to consider the following two cases:

(a) \( u_n \to u \neq 0 \) in \( L^2(\mathbb{R}^N) \),

or

(b) There exists \( \{ y_n \} \subset \mathbb{R}^N \) such that \( \vartheta_n = u_n(\cdot + y_n) \to \vartheta \neq 0 \) in \( L^2(\mathbb{R}^N) \).

Since \( J_{\epsilon_n,R_n}(u_n) = \frac{1}{2} \int_{B_{\epsilon_n,R_n}(0)} |u_n|^2 \geq c_0 > 0 \), we derive that \( \liminf_{n \to \infty} t_n > 0 \). Hence, the above conclusion ensures that

(a') \( u_n \to u \neq 0 \) in \( L^2(\mathbb{R}^N) \), for some \( u \in H^1(\mathbb{R}^N) \)

or

(b') There exists \( \{ y_n \} \subset \mathbb{R}^N \) such that \( v_n = u_n(\cdot + y_n) \to v \neq 0 \) in \( L^2(\mathbb{R}^N) \), for some \( v \in H^1(\mathbb{R}^N) \).

If (a') holds, we have that

\[ Q_{\epsilon_n}(u_n) = \frac{\int \chi_0(\epsilon_n x) |g(\epsilon_n x)| |u_n|^2 \, dx}{\int g(\epsilon_n x) |u_n|^2 \, dx} \to \frac{\int \chi(0) g(0) |u|^2 \, dx}{\int g(0) \int |u|^2 \, dx} = 0 \in K_{\frac{\alpha_n}{2}}. \]

From this, \( Q_{\epsilon_n}(u_n) \in K_{\frac{\alpha_n}{2}} \) for \( n \) large enough, which is a contradiction.

Now, if (b') holds, we must distinguish two cases:

(I) \( |\epsilon_n y_n| \to +\infty \)

or

(II) \( \epsilon_n y_n \to y \) for some \( y \in \mathbb{R}^N \), for some subsequence.
If (I) holds, we have that $J'_\infty(v)v \leq 0$. Thus, there is $s \in (0, 1]$ such that $sv \in \mathcal{N}_\infty$

\begin{align*}
2c_\infty & \leq 2J_\infty(sv) - J'_\infty(sv)sv \\
& = \int |sv|^2 \, dx \\
& \leq \int |v|^2 \, dx \leq \liminf_{n \to \infty} \int |v_n|^2 \, dx = \liminf_{n \to \infty} \int |u_n|^2 \, dx \\
& = \lim_{n \to +\infty} 2J_{\epsilon_n, R_n}(u_n) = 2c_0,
\end{align*}

which contradicts (2.3).

Now, if (II) holds, the previous argument yields

$c_{V(y)} \leq c_0$, \hfill (3.8)

where $c_{V(y)}$ is the mountain pass level of the functional $J_{V(y)} : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

\[ J_{V(y)}(u) = \frac{1}{2} \int (|\nabla u|^2 + (V(y) + 1)|u|^2) \, dx - \frac{1}{2} \int u^2 \log u^2 \, dx. \]

One can see that

\[ c_{V(y)} = \inf_{u \in \mathcal{M}_{V(y)}} J_{V(y)}(u), \]

where

\[ \mathcal{M}_{V(y)} = \left\{ u \in D(J_{V(y)}) \setminus \{0\} : J_{V(y)}(u) = \frac{1}{2} \int |u|^2 \, dx \right\}. \]

If $V(y) > 1$, as in [1], it is possible to prove that $c_{V(y)} > c_0$, which contradicts (3.8). Then $V(y) = 1$ and $y = z_i$ for some $i = 1, \ldots, l$. Hence

\[ Q_{\epsilon_n}(u_n) = \int \frac{\chi(\epsilon_n x) g(\epsilon_n x) u_n}{g(\epsilon_n x)} |u_n|^2 \, dx = \frac{\int \chi(\epsilon_n x + y_n) g(\epsilon_n x + \epsilon_n y_n) |v_n|^2 \, dx}{\int g(\epsilon_n x + \epsilon_n y_n) |v_n|^2 \, dx} \]

\[ \to \frac{\int \chi(z_i) g(z_i) |v|^2 \, dx}{\int g(z_i) |v|^2 \, dx} = z_i \in K_{c_0}^0 \]

from where it follows that $Q_{\epsilon_n}(u_n) \in K_{c_0}^0$ for $n$ large, which is absurd, because we are assuming that $Q_{\epsilon_n}(u_n) \notin K_{c_0}^0$. This finishes the proof. \hfill \Box

Next, we specify the following symbols.

\begin{align*}
\Omega_{\epsilon, R}^i & = \left\{ u \in \mathcal{N}_{\epsilon, R} : |Q_\epsilon(u) - z_i| < \rho_0 \right\}, \\
\partial \Omega_{\epsilon, R}^i & = \left\{ u \in \mathcal{N}_{\epsilon, R} : |Q_\epsilon(u) - z_i| = \rho_0 \right\}, \\
\alpha_{\epsilon, R}^i & = \inf_{u \in \Omega_{\epsilon, R}^i} J_{\epsilon, R}(u), \\
\tilde{\alpha}_{\epsilon, R}^i & = \inf_{u \in \partial \Omega_{\epsilon, R}^i} J_{\epsilon, R}(u).
\end{align*}

**Lemma 3.6.** Given $\gamma \in (0, (\epsilon_\infty - c_0)/2) > 0$, there exists $\epsilon_2 \in (0, \epsilon_1)$ small enough such that

\[ \alpha_{\epsilon, R}^i < c_0 + \gamma \text{ and } \alpha_{\epsilon, R}^i < \tilde{\alpha}_{\epsilon, R}^i, \]

for all $\epsilon \in (0, \epsilon_2)$ and $R \geq R_1 = R_1(\epsilon) > R_0$. 

Proof. Let \( u \in H^1(\mathbb{R}^N) \) be a ground state solution for \( J_0 \), i.e.,
\[
u \in \mathcal{N}_0, \quad J_0(u) = c_0 \quad \text{and} \quad J'_0(u) = 0.
\]
Hereafter, for each \( w \in D(J_0) \), \( J'_0(w) : H^1_0(\mathbb{R}^N) \to \mathbb{R} \) means the functional given by
\[
\langle J'_0(w), z \rangle = \langle \Phi'_0(w), z \rangle - \int F'_1(w)z \, dx, \quad \forall z \in H^1_0(\mathbb{R}^N)
\]
and
\[
\| J'_0(w) \| = \sup \left\{ \| J'_0(w), z \| : z \in H^1_0(\mathbb{R}^N) \quad \text{and} \quad \| z \|_\epsilon \leq 1 \right\}.
\]
If \( \| J'_0(w) \| \) is finite, then \( J'_0(w) \) may be extended to a bounded operator in \( H^1(\mathbb{R}^N) \),
and so, it can be seen as an element of \( (H^1(\mathbb{R}^N))' \).

For any \( 1 \leq i \leq l \), there is \( \epsilon_1 > 0 \) such that
\[
|Q_\epsilon(u(x - z_i)) - z_i| < \rho, \quad \forall \epsilon \in (0, \epsilon_1).
\]
Now, we fix \( R > R_1 = R_1(\epsilon) \) and \( t_{\epsilon,R} > 0 \) such that function
\[
 u(t_{\epsilon,R}u(x)) = t_{\epsilon,R}\varphi_R(x)
\]
\[ u(x - \frac{z_i}{\epsilon}) \in \mathcal{N}_{\epsilon,R}, \]
\[
|Q_\epsilon(u(x_{\epsilon,R}) - z_i| < \rho, \quad \forall \epsilon \in (0, \epsilon_1) \quad \text{and} \quad R > R_1,
\]
and
\[
J_{\epsilon,R}(u_{\epsilon,R}^i) \leq c_0 + \frac{\alpha_0}{8}, \quad \forall \epsilon \in (0, \epsilon_1) \quad \text{and} \quad R > R_1.
\]
Here, \( \varphi_R(x) = \varphi(\frac{x}{R}) \) with \( \varphi \in C_0^\infty(\mathbb{R}^N), 0 \leq \varphi(x) \leq 1 \) for all \( x \in \mathbb{R}^N, \varphi(x) = 1 \text{ for } x \in B_{1/2}(0) \text{ and } \varphi(x) = 0 \text{ for } x \in B_1^c(0) \). Therefore,
\[
 u_{\epsilon,R}^i \in \Omega_{\epsilon,R}^i, \quad \forall \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1,
\]
from it follows that
\[
\alpha_{\epsilon,R}^i < c_0 + \frac{\alpha_0}{4}, \quad \forall \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1. \quad (3.9)
\]
Then, decreasing \( \alpha_0 \) if necessary,
\[
\alpha_{\epsilon,R}^i < c_0 + \gamma, \quad \forall \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1,
\]
which is the first inequality. To obtain the second one, note that if \( u \in \partial \Omega_{\epsilon,R}^i \), then
\[
u \in \mathcal{N}_{\epsilon,R} \quad \text{and} \quad |Q_\epsilon(Ru) - z_i| = \rho_0 > \frac{\rho_0}{2},
\]
that is, \( Q_{\epsilon,R}(u) \not\in K_{2\epsilon} \). Thus, from Lemma 3.5,
\[
J_{\epsilon,R}(u) > c_0 + \alpha_0, \quad \text{for all} \quad u \in \partial \Omega_{\epsilon,R}^i \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1,
\]
and so
\[
\tilde{\alpha}_{\epsilon,R}^i = \inf_{u \in \partial \Omega_{\epsilon,R}^i} J_{\epsilon,R}(u) \geq c_0 + \alpha_0, \quad \forall \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1. \quad (3.10)
\]
Consequently, from (3.9)-(3.10),
\[
\alpha_{\epsilon,R}^i < \tilde{\alpha}_{\epsilon,R}^i, \quad \text{for all} \quad \epsilon \in (0, \epsilon_2) \quad \text{and} \quad R \geq R_1,
\]
and the results are derived by fixing \( \epsilon_2 \in (0, \epsilon_1) \).

\[\square\]

**Theorem 3.1.** There are \( \epsilon_+ \in (0, \epsilon_2) \) small enough and \( R_1 = R_1(\epsilon) > R_0 \) large enough such that \( J_{\epsilon,R} \) has at least 1 nontrivial critical points for \( \epsilon \in (0, \epsilon_+) \) and \( R \geq R_1 \). Moreover, all of the solutions are positive.
Proof. From Lemma 3.6, there exist $0 < \epsilon_* < \epsilon_2$ small enough and $R_1 > R_0$ large enough such that

$$\alpha^i_{\epsilon,R} < \tilde{\alpha}^i_{\epsilon,R}, \quad \text{for all } \epsilon \in (0, \epsilon_*) \text{ and } R \geq R_1.$$  

Arguing as in [2, Proof of Theorem 2.1], the above inequality permits to use the Ekeland’s variational principle to get a $(PS)_{\alpha^i_{\epsilon,R}}$ sequence $(u^i_n) \subset \Omega^i_{\epsilon,R}$ for $J_{\epsilon,R}$. Noting that $\alpha^i_{\epsilon,R} < c_0 + \gamma$, from Lemma 3.4 there exists $u^i$ such that $u^i_n \rightarrow u^i$ in $H^1_0(B_R(0))$. So

$$u^i \in \Omega^i_{\epsilon,R}, \quad J_{\epsilon,R}(u^i) = \alpha^i_{\epsilon,R} \text{ and } J'_{\epsilon,R}(u^i) = 0.$$  

Since

$$Q_{\epsilon}(u^i) \in B_{\rho_0}(z_i), \quad Q_{\epsilon}(u^j) \in B_{\rho_0}(z_j),$$

$$B_{\rho_0}(z_i) \cap B_{\rho_0}(z_j) = \emptyset \quad \text{for } i \neq j.$$  

We deduce that $u^i \neq u^j$ for $i \neq j$ for $1 \leq i, j \leq l$. Hence $J_{\epsilon,R}$ possesses at least $l$ nontrivial critical points for all $\epsilon \in (0, \epsilon_*)$ and $R \geq R_1$. Finally, decreasing $\gamma$ and increasing $R_1$ if necessary, we can assume that

$$2c_{\epsilon,R} < c_0 + \gamma, \quad \text{for } \epsilon \in (0, \epsilon^*) \text{ and } R \geq R_1.$$  

The above inequality permits to conclude that all of the solutions do not change sign, and as $f(t) = t \log t^2$ is an odd function, we can assume that they are nonnegative. Now, the positivity of the solutions in $B_R(0)$ follows by maximum principle. \hfill \Box

4. Existence of solution for original problem. In the following, for each $i \in \{1, \ldots, l\}$ and $\epsilon \in (0, \epsilon_*)$, we set $R_n = +\infty$ and $u^i_n = u^i_{\epsilon,R_n}$ be a solution obtained in Theorem 3.1. Then,

$$\int_{B_{\rho_0}(z_i)} (\nabla u^i_n \cdot \nabla v + V(\epsilon x)u^i_n v) \, dx = \int_{B_{\rho_0}(z_i)} u^i_n \log |u^i_n|^2 v \, dx, \quad \forall v \in H^1(B_{R_n}(0))$$  

and

$$J_{\epsilon,R_n}(u^i_n) = \alpha^i_{\epsilon,R_n}, \quad \forall n \in \mathbb{N}.$$  

Proposition 4.1. There exists $u^i \in H^1(\mathbb{R}^N)$ such that $u^i_n \rightarrow u^i$ in $H^1(\mathbb{R}^N)$ and $u^i \neq 0$ for all $i \in \{1, \ldots, l\}$.

Proof. Since $(\alpha^i_{\epsilon,R_n})$ is a bounded sequence, it is easy to check that $(u^i_n)$ is a bounded sequence. Hence, we may assume that $u^i_n \rightarrow u^i$ for some $u^i \in H^1(\mathbb{R}^N)$. Arguing by contradiction, we assume that there is $i_0 \in \{1, \ldots, l\}$ such that $u^{i_0} = 0$. In the sequel $(u_n)$ and $(\alpha_n)$ denote $(u^i_n)$ and $(\alpha^i_{\epsilon,R_n})$ respectively.

To proceed further we need to use the Concentration Compactness Principle, due to Lions [8], employed to the following sequence

$$\rho_n(x) := \frac{|u_n(x)|^2}{|u_n|^2}, \quad \forall x \in \mathbb{R}^N.$$  

This principle assures that one and only one of the following statements holds for a subsequence of $(\rho_n)$, still denoted by itself:

- (Vanishing):
  $$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_K(y)} \rho_n \, dx = 0, \quad \forall K > 0; \quad (4.1)$$
(Compactness): There exists a sequence of points \((y_n) \subset \mathbb{R}^N\) such that for all \(\eta > 0\), there exists \(K > 0\) such that
\[
\int_{B_K(y_n)} \rho_n \, dx \geq 1 - \eta, \quad \forall n \in \mathbb{N}; \tag{4.2}
\]

(Dichotomy): There exist \((y_n) \subset \mathbb{R}^N\), \(\alpha \in (0,1)\), \(K_1 > 0\), \(K_n \to +\infty\) such that the functions \(\rho_{1,n}(x) := \chi_{B_{K_1}(y_n)}(x)\rho_n(x)\) and \(\rho_{2,n}(x) := \chi_{B_{K_n}(y_n)}(x)\rho_n(x)\) satisfy
\[
\int \rho_{1,n} \, dx \to \alpha \quad \text{and} \quad \int \rho_{2,n} \, dx \to 1 - \alpha. \tag{4.3}
\]

Our objective is to show that \((\rho_n)\) verifies the Compactness condition and in order to do so we act by excluding the others two possibilities. But this fact will lead to a contradiction, showing the proposition.

The vanishing case (4.1) can not occur, otherwise we conclude that \(|u_n|_p \to 0\), and so, \(F'_2(u_n)u_n \to 0\) in \(L^1(\mathbb{R}^N)\). Arguing as in the previous section, it is possible to prove that \(u_n \to 0\) in \(H^1(\mathbb{R}^N)\). However, this convergence contradicts the fact that \(\alpha_n \geq C_1\) for all \(n \in \mathbb{N}\), see Lemma 3.1.

Let us show that Dichotomy also does not hold. Suppose that this is not the case. Under this assumption, we claim that \((y_n)\) is unbounded, because otherwise, in this case, using the fact that \(|u_n|_{L^2(\mathbb{R}^N)} \not\to 0\), the first convergence in (4.3) leads to
\[
\int_{B_{K_1}(y_n)} |u_n|^2 \, dx = |u_n|^2_2 \int_{\mathbb{R}^N} \rho_{1,n} \, dx \geq \delta
\]
for some \(\delta > 0\) and for sufficiently large \(n\). Then, picking \(R' > 0\) such that \(B_{K_1}(y_n) \subset B_{R'}(0)\), for all \(n \in \mathbb{N}\), it follows that
\[
\int_{B_{R'}(0)} |u_n|^2 \, dx \geq \delta, \quad \text{for all } n \text{ sufficiently large.}
\]

Since \(u_n \to 0\) in \(L^2(B_{R'}(0))\), the above inequality is impossible. Thereby \((y_n)\) is an unbounded sequence. In what follows, we set
\[
v_n(x) := u_n(x + y_n), \quad x \in \mathbb{R}^N. \tag{4.4}\]

Hence \((v_n) \subset H^1(\mathbb{R}^N)\) is bounded and, up to subsequence, we may assume that \(v_n \rightharpoonup v\) and by the first part of (4.3) we have \(v \not= 0\).

**Claim 4.1.** \(F'_1(v)v \in L^1(\mathbb{R}^N)\) and \(J'_\infty(v)v \leq 0\).

Note that, if \(\varphi \in C_c^\infty(\mathbb{R}^N)\), \(0 \leq \varphi \leq 1\), \(\varphi \equiv 1\) in \(B_1(0)\) and \(\varphi \equiv 0\) in \(B_2(0)^c\), defining \(\varphi_R := \varphi(\cdot/y)\) and \(v = \varphi_R(\cdot - y_n)u_n\), the following equality holds
\[
\int \left( \nabla v_n \cdot \nabla (\varphi_Rv_n) + (V(\epsilon(x + y_n)) + 1)\varphi_R(v_n)^2 \right) \, dx = \int F'_1(v_n)v_n\varphi_R \, dx
\]
\[
= \int F'_2(v_n)v_n\varphi_R \, dx + o_n(1). \tag{4.5}
\]

Fixing \(R\) and passing to the limit in the above equality when \(n \to \infty\) we get
\[
\int (\varphi_R|\nabla v|^2 + v\varphi_R \cdot \nabla v) + (V_{\infty} + 1)|\varphi_Rv|^2 \, dx \leq \int F'_2(v)v\varphi_R \, dx.
\]

Now, the claim follows, using that \(F'_1(t)t \geq 0\) for all \(t \in \mathbb{R}\), and applying Fatou’s lemma in the last inequality, as \(R \to +\infty\).

Therefore, there is \(t_\infty \in (0,1]\) such that \(t_\infty v \in \mathcal{N}_\infty\), and so,
\[ c_\infty \leq J_\infty(t_\infty v) = \frac{t_\infty^2}{2} \int |v|^2 dx \leq \liminf_{n \to \infty} \frac{1}{2} \int |v_n|^2 dx \leq \limsup_{n \to \infty} \frac{1}{2} \int |u_n|^2 dx = \limsup_{n \to \infty} J_{c_n, R_n}(u_n) = \limsup_{n \to \infty} \alpha_n \leq c_0 + \gamma, \]

which contradicts the fact that \( \gamma < c_\infty - c_0 \). Thus, in any case, Dichotomy does not occur and, actually, Compactness must hold. To reach our goal let us state the last claim.

**Claim 4.2.** The sequence of points \((y_n) \subset \mathbb{R}^N\) in (4.2) is bounded.

The proof of this claim consists in assuming by contradiction that the sequence of points \((y_n)\) is unbounded. Then, up to subsequence, \(\|y_n\| \to +\infty\), and we proceed as in the case of Dichotomy, where \((y_n)\) was unbounded, reaching that \(c_0 + \gamma \geq c_\infty\).

In view of Claim 4.2, for a given \(\eta > 0\), there exists \(R > 0\) such that, by (4.2),

\[ \int_{B_R(0)} |\rho_n|^2 dx < \eta, \quad \forall n \in \mathbb{N}, \]

or equivalent to

\[ \int_{B_R(0)} |u_n|^2 dx \leq \eta |u_n|^2 \leq b\eta, \quad \forall n \in \mathbb{N}, \]  

where \(b = \sup_{n \in \mathbb{N}} |u_n|^2\). Then, for \(R_1 \geq \max\{R, R_0\}\), due to the convergence \(u_n \to 0\) in \(L^2(B_{R_1}(0))\), there exists \(n_0 \in \mathbb{N}\) such that

\[ \int_{B_{R_1}(0)} |u_n|^2 dx \leq \eta, \quad \forall n \geq n_0. \]  

Then, by (4.6) and (4.7), it follows that if \(n \geq n_0\),

\[ \int |u_n|^2 dx \leq \eta + \int_{B_{R_1}(0)} |u_n|^2 dx \leq \eta + b\eta \leq C\eta \]

for some \(C\) that does not depend on \(\eta\). As \(\eta\) is arbitrary, we can conclude that \(u_n \to 0\) in \(L^2(\mathbb{R}^N)\). Since \((u_n)\) is bounded in \(H^1(\mathbb{R}^N)\), by interpolation on the Lebesgue spaces, it follows that

\[ u_n \to 0 \text{ in } L^p(\mathbb{R}^N), \text{ for all } 2 \leq p < 2^*. \]

However, this limit implies that \(J_{\epsilon, R_n}(u_n) = \alpha_n \to 0\), which is impossible, because \(\alpha_n \geq c_\epsilon > 0\) for all \(n \in \mathbb{N}\).

As an immediate consequence of Proposition 4.1, we have the corollary.

**Corollary 4.1.** For each sequence \((u_n^i) \subset H^1(\mathbb{R}^N)\) given in Proposition 4.1 and for small \(\epsilon \in (0, \epsilon_*)\), we have that \(u^i \neq 0\) and \(J'_\epsilon(u^i)v = 0\) for all \(v \in C_0^\infty(\mathbb{R}^N)\). Moreover, the following limits hold

\[ Q_\epsilon(u_n^i) \to Q_\epsilon(u^i), \quad i = 1, 2, \ldots, l. \]  

Since

\[ Q_\epsilon(u_n^i) \in \overline{B_{\rho_0}(z_i)}, \quad \forall n \in \mathbb{N}, \]

we have that

\[ Q_\epsilon(u^i) \in \overline{B_{\rho_0}(z_i)}. \]  

\[ (4.9) \]
Proof. By Proposition 4.1, we know that $u^i \not\equiv 0$ for all $i \in \{1, \ldots, l\}$. The limit $u^i_n \to u^i$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [2, 2^*)$ ensures that
\[
\int u^i_n \log |u^i_n|^2 v dx \to \int u^i \log |u^i|^2 v dx, \quad \forall v \in C_0^\infty(\mathbb{R}^N).
\]
Since
\[
\int (\nabla u^i_n \nabla v + (V(\epsilon x) + 1) u^i_n v) dx \to \int (\nabla u^i \nabla v + (V(\epsilon x) + 1) u^i v) dx, \quad \forall v \in C_0^\infty(\mathbb{R}^N),
\]
we can conclude that $J'_\epsilon(u^i)v = 0$ for all $v \in C_0^\infty(\mathbb{R}^N)$. Using the fact that $g(x) \to 0$ as $|x| \to +\infty$, it is easy to show that
\[
\int \chi(\epsilon x) g(\epsilon x) |u^i_n|^2 dx \to \int \chi(\epsilon x) g(\epsilon x) |u^i|^2 dx
\]
and
\[
\int g(\epsilon x) |u^i_n|^2 dx \to \int g(\epsilon x) |u^i|^2 dx.
\]
The above limits ensure that (4.8) and (4.9) hold. \hfill \Box

4.1. Proof of Theorem 1.1. By Corollary 4.1, for each $i \in \{1, \ldots, l\}$ and $\epsilon \in (0, \epsilon_*)$, there is a solution $u^i \in H^1(\mathbb{R}^N) \setminus \{0\}$ for problem (1.2) such that
\[
Q_\epsilon(u^i) \in B_{r_\epsilon}(z_i).
\]
Since
\[
B_{r_\epsilon}(z_i) \cap B_{r_\epsilon}(z_j) = \emptyset \quad \text{and} \quad i \neq j,
\]
it follows that $u^i \neq u^j$ for $i \neq j$. Due to a change of variable, the functions $v^i(x) = u^i(x/\epsilon), \forall x \in \mathbb{R}^N, i \in \{1, \ldots, l\}$ are $l$ positive solutions of problem (1.2).

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