INSTABILITY OF MULTI-SOLITONS FOR DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

PHAN VAN TIN

Abstract. In [19] and [26], the authors proved the stability of multi-solitons for derivative nonlinear Schrödinger equations. Roughly speaking, sum of finite stable solitons is stable. We predict that if there is one unstable solition then multi-soliton is unstable. This prediction is proved in [7] for classical nonlinear Schrödinger equations. In this paper, we proved this prediction for derivative nonlinear Schrödinger equations by using the method of Côte-Le Coz [7] with the help of Gauge transformation.

CONTENTS
1. Introduction
1.1. Instability of multi-solitons for (1.1) 2
1.2. Instability of multi-solitons for (1.2) 4
2. Proof of main results
2.1. Construction of approximation profiles 6
2.2. Proof of Theorems 1.2 and 1.4 9
2.3. Orbital instability of soliton and multi-solitons 13
3. Appendix 16
Acknowledgement 19
References 19

1. Introduction

We consider the following derivative nonlinear Schrödinger equations

\[ iu_t + u_{xx} + i|u|^2u_x + b|u|^4u = 0, \]

(1.1)

and

\[ iu_t + u_{xx} + i|u|^{2\sigma}u_x = 0, \]

(1.2)

where \( b \in \mathbb{R} \), \( \sigma \geq 1 \) and \( u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) is unknown function.

The local well posedness and the global well posedness of derivative nonlinear Schrödinger equations were studied in many works (see [1, 3, 5, 6, 11, 15, 16, 17, 21, 23, 25, 27, 28, 32, 31] for (1.1) and see e.g [14, 24] for (1.2)). The existence of blow up solutions for (1.1) and (1.2) is an open question.

The equation (1.1) and (1.2) have Hamilton structures and they do not possess the Galilean invariant. The family of solitons of the derivative nonlinear Schrödinger equations have two parameters. A soliton of (1.1) and (1.2) is a solution of form \( R_{\omega,c}(t, x) = e^{i\omega t} \phi_{\omega,c}(x-ct) \), where \( \omega > 0 \) and \( c^2 < 4\omega \). The stability and instability of solitons are proved in many works (see [22, 18, 13, 11, 4] for (1.1) and [20, 12] for (1.2)).

Multi-soliton is a solution of (1.1), (1.2) which behaves at large time like a sum of finite solitons. In [29, 30], Tin proved the existence of multi-solitons for (1.1) and (1.2) respectively. The author used fixed point method, Strichartz estimates and gauge transformations to obtain the desired results. The stability of multi-solitons was proved in [19] for (1.1) in the case \( b = 0 \) and for (1.2) in the case \( \sigma \in (1, 2) \) in [26] provided all solitons are stable. Roughly speaking, the multi-solitons...
behave at large time like a sum of stable solitons are stable. We predict that if there is one unstable soliton then multi-soliton is unstable in some sense. This prediction was proved in the case of classical nonlinear Schrödinger equation by the work of Côte-Le Coz [7]. In this paper, using the idea of Côte-Le Coz, we show that if soliton of (1.1) ((1.2)) is linearly unstable then it is orbitally unstable. Moreover, multi-soliton behaving like a sum of one unstable soliton and finite solitons is not unique and unstable.

1.1. Instability of multi-solitons for (1.1). The flow of (1.1) in $H^1(\mathbb{R})$ satisfies the following conservation laws.

\[
\begin{align*}
\text{Energy} & \quad E(u) := \frac{1}{2} \|u\|_{L^2}^2 + \frac{1}{4} \int_\mathbb{R} |u|^4 u_x \, dx - \frac{b}{6} \|u\|_{L^6}^6, \\
\text{Mass} & \quad Q(u) := \frac{1}{2} \|u\|_{L^2}^2, \\
\text{Momentum} & \quad P(u) := -\frac{1}{2} \int_\mathbb{R} u_x \bar{u} \, dx.
\end{align*}
\]

For each $\omega, c \in \mathbb{R}$ and $u \in H^1(\mathbb{R})$, we define

\[
S_{\omega,c}(u) = E(u) + \omega Q(u) + cP(u).
\]

Recall that a soliton of (1.1) is a solution of form $R_{\omega,c} = e^{i\omega t} \phi_{\omega,c}(x - ct)$, for $\phi_{\omega,c}$ is a critical point of $S_{\omega,c}$. Moreover, $\phi_{\omega,c}$ is (up to phase shift and translation) of form

\[
\phi_{\omega,c} = \Phi_{\omega,c} \exp \left( \frac{ic}{2} x - \frac{i}{4} \int_{-\infty}^x |\phi_{\omega,c}(y)|^2 \, dy \right),
\]

where $\Phi_{\omega,c}$ is given by if $\gamma := 1 + \frac{16}{9} b > 0$,

\[
\Phi_{\omega,c}^2(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2) \cosh(\sqrt{c^2 + \gamma}) - c}} \quad \text{if} \quad -2\sqrt{\omega} < c < 2\sqrt{\omega},
\]

\[
\Phi_{\omega,c}^2(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2) \cosh(\sqrt{4\omega - c^2}) - c}} \quad \text{if} \quad c = 2\sqrt{\omega},
\]

and if $\gamma \leq 0$ ($b \leq \frac{3}{16}$),

\[
\Phi_{\omega,c}^2(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2) \cosh(\sqrt{4\omega - c^2}) - c}} \quad \text{if} \quad 2\sqrt{\omega} < c < -2\sqrt{\omega},
\]

where $s_* = s_*(\gamma) = \sqrt{\frac{2}{\gamma}}$.

We note that the following condition on the parameters $\gamma$ and $(\omega, c)$ is a necessary and sufficient condition for the existence of non-trivial solutions of (1.1) vanishing at infinity (see [2]):

if $\gamma > 0$ ($b > \frac{-3}{16}$) then $-2\sqrt{\omega} < c \leq 2\sqrt{\omega}$,

if $\gamma \leq 0$ ($b \leq \frac{-3}{16}$) then $-2\sqrt{\omega} < c < -2s_*\sqrt{\omega}$.

Define $d(\omega, c) = S_{\omega,c}(\phi_{\omega,c})$ and

\[
H_{\omega,c}(v) = (E''(\phi_{\omega,c}) + \omega Q''(\phi_{\omega,c}) + cP''(\phi_{\omega,c}))(v) = -\partial_{xx} v + \omega v + i c \partial_x v - 2i \partial_t \phi_{\omega,c} \Re(\phi_{\omega,c} \bar{v}) - i [\phi_{\omega,c}] \partial_x v - b[|\phi_{\omega,c}|^4 v + 4|\phi_{\omega,c}|^2 \phi_{\omega,c} \Re(\phi_{\omega,c} \bar{v})].
\]

Let $n(H_{\omega,c})$ be the number of negative eigenvalue of $H_{\omega,c}$ and $p(d''(\omega, c))$ be the number of positive eigenvalue of the matrix $d''(\omega, c)$, which is defined by

\[
d''(\omega, c) = \begin{bmatrix}
\partial_x^2 d(\omega, c) & \partial_x \partial_t d(\omega, c) \\
\partial_t \partial_x d(\omega, c) & \partial_t^2 d(\omega, c)
\end{bmatrix} = \begin{bmatrix}
\partial_{\omega} Q(\phi_{\omega,c}) & \partial_{\omega} P(\phi_{\omega,c}) \\
\partial_c Q(\phi_{\omega,c}) & \partial_c P(\phi_{\omega,c})
\end{bmatrix}.
\]

The stability/instability of solitons $R_{\omega,c}$ can be given by the abstract theory of Grillakis-Shatah-Strauss [9, 10]. We have the following result.

**Theorem 1.1.**

\[
p(d''(\omega, c)) \leq n(H_{\omega,c}).
\]

Furthermore, under the condition that $d$ is non-degenerate at $(\omega, c)$:
(i) If \( p(d''(\omega, c)) = n(H_{\omega, c}) \) then \( R_{\omega, c} \) is orbitally stable;
(ii) If \( n(H_{\omega, c}) - p(d''(\omega, c)) \) is odd then \( R_{\omega, c} \) is orbitally unstable.

Let \( K \in \mathbb{N}, K > 1 \). For each \( 1 \leq j \leq K \), let \((\theta_j, x_j) \in \mathbb{R}^2\) and \((c_j, \omega_j)\) satisfy the condition of existence of soliton. For each \( j \in \{1, 2, \ldots, K\} \), we set
\[
R_j(t, x) = e^{i\theta_j} R_{\omega_j, c_j}(t, x - x_j).
\]
We define for each \( j \), \( h_j = \sqrt{4\omega_j - c_j^2} \). As in [29, Lemma 4.1],
\[
|R_j(t, x)| \lesssim e^{-\frac{h_j}{2} |x - c_j t|}.
\]
The profile of a multi-soliton is a sum of the form:
\[
R = \sum_{j=1}^{K} R_j.
\]

A solution of (1.1) is called a multi-soliton if
\[
\|u(t) - R(t)\|_{H^1} \to 0 \text{ as } t \to \infty.
\]

Since solutions of (1.1) are invariant by phase shift and translation, we may assume that \( \theta_1 = x_1 = 0 \) without loss of generality. For convenience, we denote \( \phi_j = \phi_{\omega_j, c_j} \) for all \( j \) and \( \phi = \phi_1 \). Then \( R_1(t, x) = e^{i\omega_1 t} \phi(x - c_1 t) \). We have
\[
- \phi_{xx} + \omega_1 \phi + ic_1 \phi_x - i|\phi|^2 \phi_x - b|\phi|^4 \phi = 0.
\]
Let \( u(t, x) \) be a solution of (1.1) and set \( u = e^{i\omega_1 t} (\phi(x - c_1 t) + v(t, x - c_1 t)) \). Using (1.7), we have
\[
0 = i u_t + u_{xx} + i|u|^2 u_x + b|u|^4 u = i(\omega_1 e^{i\omega_1 t} (\phi + v) + e^{i\omega_1 t} (-c_1 \phi_x + v_t - c_1 v_x)) + e^{i\omega_1 t} (\phi_{xx} + v_{xx}) + e^{i\omega_1 t} |\phi + v|^4 (\phi + v) + e^{i\omega_1 t} (-\omega_1 v + iv_t - ic_1 v_x + v_{xx}) + i(|\phi + v|^2 (\phi_x + v_x) - |\phi|^2 \phi_x) + b(|\phi + v|^4 (\phi + v) - |\phi|^4 \phi))
\]
where \( Lc \) is linearized operator around \( R_1 \) and is defined by
\[
Lc(v) = -iv_{xx} + i\omega_1 v - c_1 v_x + 2Re(\phi \overline{\phi}) \phi_x + |\phi|^2 v_x - ib(|\phi|^4 v + 4|\phi|^2 \phi Re(\phi \overline{\phi})),
\]
and the quadratic term in \( v, \mathcal{M}_c \) is defined by
\[
\mathcal{M}_c(v) = 2Re(\overline{\phi} \phi v_x) + |v|^2 \phi_x + |v|^2 v_x - ib(\phi + v)(4Re(\phi \overline{\phi})^2 + 2|v|^2 |\phi|^2 + 4|v|^2 Re(\phi \overline{\phi}) + |v|^4).
\]
We may check that \( Lc = iH_{\omega_1, c_1} \). We need the following assumption.
\[
Lc has an eigenvalue \( \lambda \in \mathbb{C} \) such that \( \rho := Re \lambda > 0 \). \quad (A1)
\]
Our first goal is to prove that if soliton is linearly unstable then it is orbitally unstable. To do this, we prove the following result.

**Theorem 1.2.** Assume that \( (A1) \) holds. Then there exists a function \( Y(t) \) such that \( \|Y(t)\|_{H^2} \leq Ce^{-\rho t} \) and \( e^{\rho t}\|Y(t)\|_{H^2} \) is non-zero and periodic (where \( \rho \) is given by \( (A1) \)) and \( Y(t) \) is a solution to the linearized flow around \( R_1 \). For all \( a \in \mathbb{R} \), there exist \( T_0 \in \mathbb{R} \) large enough, a constant \( C > 0 \) and a solution \( u_a \) to (1.1) defined on \([T_0, \infty)\) such that
\[
\|u_a(t) - R_1(t) - aY(t)\|_{H^2} \leq Ce^{-\rho t}, \quad \forall t \geq T_0.
\]
As a consequence of 1.2, we prove that under \( (A1) \), \( R_1 \) is orbitally unstable. We prove the following result.

**Corollary 1.3.** Under the hypothesis of Theorem 1.2, \( R_1 \) is orbitally unstable in the following sense: There exist \( \varepsilon > 0, (T_n) \subset \mathbb{R}^+, (u_{0,n}) \subset H^2(\mathbb{R}) \) and solution \( (u_n) \) of (1.1) defined on \([T_n, 0]\) with \( u_n(0) = u_{0,n} \) such that
\[
\lim_{n \to \infty} \|u_{0,n} - R_1(0)\|_{H^2} = 0 \quad \text{and} \quad \inf_{y \in \mathbb{R}, \theta \in \mathbb{R}} \|u_n(T_n) - e^{i\theta} \phi(\cdot - y)\|_{L^2} \geq \varepsilon \quad \forall n \in \mathbb{N}.
Under the assumption of Theorem 1.2, we prove the existence of a one parameter family of multi-solitons. This implies that multi-soliton is not unique. Moreover, we prove instability for high relative speed of multi-solitons.

**Theorem 1.4.** Let $K \in \mathbb{N}$, $K > 1$. For each $j = 1, \ldots, K$, let $(\theta_j, x_j) \in \mathbb{R}^2$ and $(c_j, \omega_j)$ satisfy the condition of existence of soliton and $R_j$ be defined by (1.4). Let $h_n = \inf_j \beta_j$, where $\beta_j$ is the given constant in Proposition 3.1 and $\omega_\ast = \frac{1}{2} \min \{|c_j - c_k| : j, k = 1, \ldots, K, j \neq k\}$. Assume that (A1) holds. There exists $v_\ast > 0$ such that if $v_n > v_\ast$ then the following holds. There exist $Y(t)$ such that $\|Y(t)\|_{L^2} \leq Ce^{-\rho t}$ and $e^{\omega t} \|Y(t)\|_{L^2}$ is non-zero and periodic, where $\rho$ is given by (A1) and $Y(t)$ is a solution to the linearized flow around $R_1$. For all $a \in \mathbb{R}$, there exist $T_0 \in \mathbb{R}$ large enough, a solution $u_a$ to (1.1), and a constant $C > 0$ such that

$$\left\| u_a(t) - \sum_{j=1}^K R_j(t) - aY(t) \right\|_{L^2} \leq Ce^{-\rho t}.$$  

**Corollary 1.5.** Let $R$ be the multi-soliton profile defined by (1.6). Under the hypotheses of Theorem 1.4, the multi-soliton around $R$ satisfies the following instability property. There exists $\varepsilon > 0$, such that for all $n \in \mathbb{N} \setminus \{0\}$ and for $T > 0$ large enough the following holds. There exist $I_n, J_n \in \mathbb{R}$, $J_n < I_n < -T$ and a solution $w_n \in C([I_n, J_n], H^2(\mathbb{R}))$ to (1.1) such that

$$\lim_{n \to \infty} \|w_n(I_n) - R(I_n)\|_{L^2} = 0, \quad \text{and} \quad \inf_{y_j \in \mathbb{R}, \theta_j \in \mathbb{R}} \left| w_n(J_n) - \sum_{j=1}^K \phi_j(-y_j)e^{i\theta_j} \right| \geq \varepsilon.$$  

**Remark 1.6.** Replacing $\phi$ by $\overline{\phi}$ in the definition of $L_C$, we obtain the new operator denoted by $L_C^T$. By similar argument in [7, Proof of Corollary 2], we may prove that if $L_C^T$ has a eigenvalue with positive real part then the soliton $R_1$ (in Corollary 1.3) and the multi-soliton $R$ (in Corollary 1.5) are unstable forward in time. However, not like in [7] for classical nonlinear Schrödinger equation, in our case (A1) does not imply that $L_C^T$ has an eigenvalue with positive real part.

From [10, Theorem 5.1], if $d''(\omega_1, c_1)$ is non-singular and $n(H_{\omega_1, c_1} - p(d''(\omega_1, c_1)))$ odd then $-iH_{\omega_1, c_1}$ has at least one pair of real non-zero eigenvalues $\pm \lambda$. In that case, (A1) holds. We have the following result.

**Theorem 1.7.** Assume that $d''(\omega_1, c_1)$ is non-singular and $n(H_{\omega_1, c_1} - p(d''(\omega_1, c_1)))$ odd. Then the conclusions of Corollary 1.3 and Corollary 1.5 hold.

**Remark 1.8.** From the works of Colin-Ohta [4], Ohta [22] and Hanashi [13], we see that $p(d''(\omega, c)) = 1$ if $b = 0$ or $b < 0$ or $b > 0$ and $-2\sqrt{2} < c < 2\sqrt{2}$, $p(d''(\omega, c)) = 0$ if $b = 0$ and $2\kappa \sqrt{2} < c < 2\sqrt{2}$ for some constant $\kappa = \kappa(b) \in (0, 1)$. We predict that $n(H_{\omega, c}) = 1$ for all $b$ and the condition $n(H_{\omega_1, c_1} - p(d''(\omega_1, c_1)))$ odd is replaced by $p(d''(\omega_1, c_1)) = 0$.

### 1.2. Instability of multi-solitons for (1.2)

In this section, for simplicity, we use the same notation in Section 1.1.

The flow of (1.2) in $H^1(\mathbb{R})$ satisfies the following conservation laws.

- **Energy** $E(u) := \frac{1}{2} \|u_x\|_{L^2}^2 + \frac{1}{2(\sigma + 1)} \int_{\mathbb{R}} |u|^{2\sigma} u_x \overline{u} dx$.
- **Mass** $Q(u) := \frac{1}{2} \|u\|_{L^2}^2$.
- **Momentum** $P(u) := -\frac{1}{2} \int_{\mathbb{R}} u_x \overline{u} dx$.

For each $\omega, c \in \mathbb{R}$ and $u \in H^1(\mathbb{R})$, we define

$$S_{\omega, c}(u) = E(u) + \omega Q(u) + c P(u).$$

A soliton of (1.2) is a solution of form $R_{\omega, c}(t, x) = e^{i\omega t} \phi_{\omega, c}(x - ct)$, for $\phi_{\omega, c}$ is a critical point of $S_{\omega, c}$. Moreover, $\phi_{\omega, c}$ is (up to phase shift and translation) of form

$$\phi_{\omega, c}(x) = \Phi_{\omega}(x) \exp \left( -\frac{i}{2} x - \frac{1}{2\sigma + 2} \int_{-\infty}^{x} \Phi_{\omega, c}^{\sigma}(y) dy \right).$$
where \( \omega > \frac{c^2}{4} \) and

\[
\Phi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma + 1)(4\omega - c^2)}{2\sqrt{\omega} \left( \cosh(\sqrt{4\omega - c^2}y) - \frac{c^2}{2\sqrt{\omega}} \right)}.
\]

For each \( \omega, c \in \mathbb{R} \), let \( d(\omega, c), H_{\omega,c}, n(H_{\omega,c}), p(d''(\omega, c)) \) be defined as in Section 1.1. Similar in the case (1.1), Stability/instability of solitons of \((A2)\) obeys Theorem 1.1. In [20], the authors proved that \( n(H_{\omega,c}) = 1 \) for all \( \sigma > 0 \). Thus, \( R_{\omega,c} \) is orbitally stable if \( p(d''(\omega, c)) = 1 \) and orbitally unstable if \( p(d''(\omega, c)) = 0 \).

Let \( K \in \mathbb{N} \), \( K > 1 \). For each \( 1 \leq j \leq K \), let \( (\theta_j, x_j) \in \mathbb{R}^2 \) and \( (c_j, \omega_j) \) satisfy the condition of existence of soliton. For each \( j \in \{1, 2, \ldots, K\} \), we set

\[
R_j(t, x) = e^{i\theta_j} R_{\omega_j,c_j}(t, x - x_j).
\]

We define for each \( j \), \( h_j = \sqrt{4\omega_j - c_j^2} \). As in [30, Lemma 3.1],

\[
|R_j(t, x)| \lesssim e^{-\frac{h_j}{2}|x - c_j t|}.
\]

The profile of a multi-soliton is a sum of the form:

\[
R = \sum_{j=1}^{K} R_j. \tag{1.9}
\]

Since (1.2) is invariant under phase shift and translation, we may assume that \( \theta_1 = x_1 = 0 \). For convenience, we denote \( \phi_j = \phi_{\omega_j,c_j} \) and \( \phi = \phi_1 \). By an elementary calculation, we see that the linearized operator around \( R_1 \) of (1.2) is the following.

\[
L_C(\nu) = -iv_{xx} + i\omega_1 v - c_1 v_x + v_y|\phi|^{2\sigma} + 2\sigma \phi_x \Re(\bar{\phi} \phi)|\phi|^{2(\sigma - 1)}.
\]

We may check that \( L_C = iH_{\omega_1,c_1} \). We need the following assumption.

\[
L_C \text{ has an eigenvalue } \lambda \in \mathbb{C} \text{ such that } \rho := \Re \lambda > 0. \tag{A2}
\]

We have the following result.

**Theorem 1.9.** Let \( \sigma = 1 \) or \( \sigma = 2 \) or \( \sigma \geq \frac{5}{2} \). Under \((A2)\), \( R_1 \) is orbitally unstable by the same sense as in Corollary 1.3.

Moreover, we have the following result.

**Theorem 1.10.** Let \( \sigma = 1 \) or \( \sigma = 2 \) or \( \sigma \geq \frac{5}{2} \). Let \( R \) be the multi-solitons profile defined by (1.9). Assume that \((A2)\) holds. Then the multi-soliton around \( R \) is unstable by the same sense as in Corollary 1.5.

Using [10, Theorem 5.1], we have if \( d''(\omega_1, c_1) \) is non-singular and \( p(d''(\omega, c)) = 0 \) then \( -iH_{\omega_1,c_1} \) has a pair of real non-zero eigenvalue \( \pm \lambda \). In that case, \((A2)\) holds. Thus, we have the following result.*

**Theorem 1.11.** Let \( \sigma = 1 \) or \( \sigma = 2 \) or \( \sigma \geq \frac{5}{2} \) and \( R_1 \) be such that \( d''(\omega_1, c_1) \) is non-singular and \( p(d''(\omega_1, c_1)) = 0 \) then the conclusions of Theorem 1.9 and Theorem 1.10 hold.

**Remark 1.12.** Define

\[
\varphi(t, x) = \exp \left( \frac{i}{2} \int_{-\infty}^{x} |u(t, y)|^2 \, dy \right) u(t, x),
\]

\[
\psi = \partial_x \varphi - \frac{i}{2} |\varphi|^{2\sigma} \varphi.
\]

From [30, page 6], if \( u \) solves (1.2) then \((\varphi, \psi)\) solves

\[
L\varphi = P(\varphi, \psi),
\]

\[
L\psi = Q(\varphi, \psi),
\]
where $P, Q$ are defined by

$$P(\varphi, \psi) = i\sigma|\varphi|^{2(\sigma-1)} \varphi \overline{\psi} - \sigma(\sigma - 1) \varphi \int_{-\infty}^{\infty} |\varphi|^{2(\sigma-2)} \text{Im}(\psi^2 \overline{\varphi}) \, dy,$$

$$Q(\varphi, \psi) = -i\sigma|\varphi|^{2(\sigma-1)} \psi \overline{\varphi} - \sigma(\sigma - 1) \psi \int_{-\infty}^{\infty} |\varphi|^{2(\sigma-2)} \text{Im}(\psi^2 \overline{\varphi}) \, dy.$$

Since [30, Remark 1.2], the conditions $\sigma = 1$ or $\sigma = 2$ or $\sigma \geq \frac{5}{2}$ ensure that $P(\varphi, \psi)$ and $Q(\varphi, \psi)$ are Lipschitz continuous on bounded set of $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. This is important point in the proof of Theorem 1.9, 1.10.

**Remark 1.13.** From the work of Liu-Simpson-Sulem [20], we have if $\sigma \geq 2$ or $\sigma \in (1, 2)$ and $2\omega < c < 2\sqrt{\omega}$ then $p(d''(\omega, c)) = 0$ and if $\sigma \in (0, 1)$ or $\sigma \in (1, 2)$ and $-2\sqrt{\omega} < c < 2\omega\sqrt{\omega}$ then $p(d''(\omega, c)) = 1$.

The proofs of Theorem 1.9 and 1.10 are similar the proofs of Corollary 1.3 and Corollary 1.5 respectively. In this paper, we admit this and we only focus on the proofs of the results in Section 1.1.

## 2. Proof of main results

As said above, we only prove the results in Section 1.1. The results in Section 1.2 are proved by similar argument.

### 2.1. Construction of approximation profiles.

For convenience, we use the same notation as in [7]. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and use the notation $a + ib = \left( \begin{array}{c} a \\ b \end{array} \right)$ ($a, b \in \mathbb{R}$). Given $v \in \mathbb{C}$, we denote $v^+$ is its real part and $v^-$ is its imaginary part. To avoid confusion, we denote with an index whether we consider the operator with $\mathbb{C}$, $\mathbb{R}^2$, or $\mathbb{C}^2$-valued functions.

Let $L_C$ be defined by (1.8). We define

$$L_C(v) = -iv_{xx} + 2R \text{Re}(\overline{R_1}) R_{1x} + |R_1|^2 v_x$$

$$- ib(|R_1|^4 v + 4|R|^2 RR \text{Re}(\overline{R_1})),$$

and the nonlinear operators

$$\mathcal{N}_C(v) = 2R \text{Re}(\overline{R_1}) v_x + |v|^2 R_{1x} + |v|^2 v_x$$

$$- ib \left( (R_1 + v)(4R \text{Re}(\overline{R_1}) + 2|v|^2 |R_1|^2) + 4|v|^2 R \text{Re}(\overline{R_1} + |v|^4) - 4|R_1|^2 v R \text{Re}(\overline{R_1}) \right),$$

$$\mathcal{M}_C(v) = e^{-\omega t} \mathcal{N}_C(e^{\omega t} v) = 2R \text{Re}(\phi \overline{\sigma}) v_x + |v|^2 \phi_x + |v|^2 v_x$$

$$- ib(\phi + v)(4R \text{Re}(\overline{\sigma} v) + 2|v|^2 |\phi|^2 + 4|\phi|^2 R \text{Re}(\overline{\sigma} v) + |v|^4) - 4ib|\phi|^2 v R \text{Re}(\overline{\sigma} v).$$

We have

$$L_{R_2} \left( \begin{array}{c} v^+ \\ v^- \end{array} \right) = \left( \begin{array}{c} \text{Re}(L_C(v)) \\ \text{Im}(L_C(v)) \end{array} \right)$$

$$= \left( \begin{array}{c} v_{xx} - \omega_1 v_x - c_1 v_x + 2(v^+ \phi^+ + v^- \phi^-) \phi_x^+ + |\phi|^2 v_x^+ + b|\phi|^4 v^+ + 4b|\phi|^2 \phi^-(\phi^+ v^+ + \phi^- v^-) \\ -v_{xx}^+ + \omega_1 v^+ - c_1 v_x^+ + 2(v^+ \phi^+ v^- \phi^-) \phi_x^+ + |\phi|^2 v_x^+ - b|\phi|^4 v^+ + 4b|\phi|^2 \phi^-(\phi^+ v^+ + \phi^- v^-) \end{array} \right)$$

$$= \left( \begin{array}{c} -c_1 \partial_x + 2\phi^+ \phi^- + b|\phi|^2 \phi x + 2b|\phi|^2 \phi^+ \phi^- \partial_x + \omega_1 + 2\partial x + b|\phi|^4 + 4b|\phi|^2 \phi^2 \phi^2 \partial_x \\ -c_1 \partial x + 2\phi^+ \phi^- + b|\phi|^2 \phi x - 4b|\phi|^2 \phi^2 \phi^2 \partial_x \end{array} \right) \left( \begin{array}{c} v^+ \\ v^- \end{array} \right)$$

We see that $L_{R_2}$ is an $R$-linear operator on $H^2(\mathbb{R}, \mathbb{R}^2) \to L^2(\mathbb{R}, \mathbb{R}^2)$. To have some eigenfunctions, we extend $L_{R_2}$ to $L_{C^2} : H^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C}^2)$, which is a $C$-linear operator.

Define $v = e^{i\omega t} \bar{u}$ and $\phi = e^{i\omega t} \bar{\phi}$. By an elementary calculation, we have

$$L_C(v) = L_C(e^{i\omega t} \bar{u})$$

$$= e^{i\omega t} L_C^0(\bar{v}),$$
where

\[ L_\varphi^c(\tilde{\nu}) = -i\tilde{\nu}_{xx} + i \left( \omega_1 - \frac{c_1^2}{4} \right) \tilde{\nu} + 2\text{Re}(\phi \tilde{\nu}) \left( \frac{ic_1}{2} \phi - \phi_x - 2ib|\phi|^2 \phi \right) \]

+ $|\phi|^2 \left( \frac{ic_1}{2} + \tilde{v} \right) - ib|\phi|^4 \tilde{\nu}$.

Thus, $L_\varphi^c$ equals to

\[
\begin{pmatrix}
-\partial_{xx} + (\omega_1 - \frac{c_1^2}{4}) & W_3 \\
W_1,1 + W_1,2 \partial_x & \partial_{xx} - (\omega_1 - \frac{c_1^2}{4}) + W_2
\end{pmatrix}
\]

where

\[ W_{1,1} = \tilde{\phi}^+ \left( -\frac{c_1}{2} \tilde{\phi}^- + \phi_x + 2b|\phi|^2 \tilde{\phi}^- \right), \]

\[ W_{1,2} = |\phi|^2, \]

\[ W_2 = \tilde{\phi}^- \left( -\frac{c_1}{2} \tilde{\phi}^- + \phi_x + 2b|\phi|^2 \tilde{\phi}^- \right) - \frac{c_1}{2} |\phi|^2 + b|\phi|^4 \]

\[ W_3 = \tilde{\phi}^+ \left( \frac{c_1}{2} \tilde{\phi}^+ + \phi_x - 2b|\phi|^2 \tilde{\phi}^+ \right) + \frac{c_1}{2} |\phi|^2 - b|\phi|^4 \]

\[ W_{4,1} = \tilde{\phi}^- \left( \frac{c_1}{2} \tilde{\phi}^+ + \phi_x - 2b|\phi|^2 \tilde{\phi}^+ \right) \]

\[ W_{4,2} = |\phi|^2. \]

Thus, $W_{1,1}, W_{1,2}, W_2, W_3, W_{4,1}, W_{4,2}$ are exponentially decaying at infinity. Moreover,

\[ L_\varphi^c \left( \frac{\tilde{v}^+}{\tilde{v}^-} \right) = \begin{pmatrix}
\text{Re}(L^c_L(\tilde{\nu})) \\
\text{Im}(L^c_L(\tilde{\nu}))
\end{pmatrix} = \begin{pmatrix}
\text{Re}(e^{-i\frac{c_1}{4}x} L^c_L(v)) \\
\text{Im}(e^{-i\frac{c_1}{4}x} L^c_L(v))
\end{pmatrix} \]

\[ = \begin{pmatrix}
\cos \left( \frac{c_1}{4} x \right) & \sin \left( \frac{c_1}{4} x \right) \\
-\sin \left( \frac{c_1}{4} x \right) & \cos \left( \frac{c_1}{4} x \right)
\end{pmatrix} \begin{pmatrix}
\text{Re}(L^c_L(v)) \\
\text{Im}(L^c_L(v))
\end{pmatrix} \]

\[ = \begin{pmatrix}
\cos \left( \frac{c_1}{4} x \right) & \sin \left( \frac{c_1}{4} x \right) \\
-\sin \left( \frac{c_1}{4} x \right) & \cos \left( \frac{c_1}{4} x \right)
\end{pmatrix} L^c_{R^2} \begin{pmatrix}
\cos \left( \frac{c_1}{4} x \right) & \sin \left( \frac{c_1}{4} x \right) \\
-\sin \left( \frac{c_1}{4} x \right) & \cos \left( \frac{c_1}{4} x \right)
\end{pmatrix}^{-1} \begin{pmatrix}
\tilde{v}^+ \\
\tilde{v}^-
\end{pmatrix}. \]

This implies that the spectrum set and the resolvent set of $L^c_L$ (or $L^c_{R^2}$) are the same to the spectrum set and the resolvent set of $L^c_\varphi = (L^c_{R^2})^*$.

Let $\alpha > 0$ be the decay rate given by Proposition 3.1 for eigenfunctions of $L$ with eigenvalue $\lambda$ (see (A1)). Taking a small value of $\alpha$, we assume that $\alpha \in (0, \frac{h_1}{2})$, where $h_1 = \sqrt{4\omega_1 - c_1^2}$. For $K = \mathbb{R}, \mathbb{R}^2, \mathbb{C}$ or $\mathbb{C}^2$, denote

\[ \mathcal{H}(K) = \{ v \in H^\infty(\mathbb{R}, K)|e^{\alpha |x|} |\partial_x^2 v| \in L^\infty(\mathbb{R}) \text{ for any } \alpha \in \mathbb{N} \}. \]

We have the following properties of $L^c_L$.

**Proposition 2.1.**

(i) The eigenvalue $\lambda = \rho + i\theta$ can be chosen with maximal real part. We denote $Z(x) = \begin{pmatrix} Z^+(x) \\ Z^-(x) \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C})$, an associated eigenfunction.

(ii) $\phi \in \mathcal{H}(\mathbb{R}^2)$ and $Z \in \mathcal{H}(\mathbb{C}^2)$.

(iii) Let $\mu \notin \text{Sp}(L_{R^2})$ and $A \in \mathcal{H}(\mathbb{C}^2)$. There exists a solution $X \in \mathcal{H}(\mathbb{C}^2)$ to $(L - \mu I)X = A$ and $(L - \mu I)^{-1}$ is a continuous operator on $\mathcal{H}(\mathbb{C}^2)$.

Since $L_{R^2}$ and $L^c_{R^2}$ are conjugates of each other, we only need to prove Proposition 2.1 for $L^c_{R^2}$.

**Proof.** (i) We see that if $\lambda$ is an eigenvalue of $L^c_L$ with eigenfunction $v$ then $\lambda$ is an eigenvalue of $L_{R^2}$ with eigenfunction $\begin{pmatrix} v \\ -iv \end{pmatrix}$. Thus, from (A1), there exists an eigenvalue of $L^c_{R^2}$ with positive
real part. Since $L_2^\Delta$ is a compact perturbation of 
$$
\begin{pmatrix}
0 & \Delta_x - \frac{\lambda^2}{4} \\
-\Delta_x + \frac{\lambda^2}{4} & 0
\end{pmatrix},
$$
the essential spectrum of $L_2^\Delta$ is the set \( \{ iy : y \in \mathbb{R}, |y| \geq \frac{\lambda^2}{4} \} \) and there exists an eigenvalue $\lambda$ with maximal real part.

(ii) It is well known that $\phi$ and its derivative are exponentially decay with decay rate $\frac{\lambda}{2}$. Combining with the fact that $\phi$ solves an elliptic equation, we have $\phi \in H(\mathbb{R}^2)$. Since $L_2^\Delta Z = \lambda Z$, using Proposition 3.1 (i), we have $Z \in H(\mathbb{C}^2)$.

(iii) This part follows from Proposition 3.1 (ii). \qed

We need the following definition.

**Definition 2.2.** Let $\xi \in C^\infty(\mathbb{R}^+, H^\infty(\mathbb{R}))$ and $\chi : \mathbb{R}^+ \to (0, \infty)$. Then we denote $\xi(t) = O(\chi(t))$ as $t \to \infty$.

if, for all $s \geq 0$, there exists $C(s) > 0$ such that

$$
\forall t \geq 0, \quad \|\xi(t)\|_{H^s} \leq C(s)\chi(t).
$$

Define $Y_1 := \left( \Re(Z) = \left( \Re(Z^+) \right) \right)$ and $Y_2 := \left( \Im(Z) = \left( \Im(Z^+) \right) \right)$. Then $Y_1, Y_2 \in H(\mathbb{R}^2)$, and

$$
\begin{aligned}
L_{\mathbb{R}^2}Y_1 &= \rho Y_1 - \theta Y_2, \\
L_{\mathbb{R}^2}Y_2 &= \theta Y_1 + \rho Y_2.
\end{aligned}
$$

(2.2)

Denote

$$
Y(t) = e^{-\rho t}(\cos(\theta t)Y_1 + \sin(\theta t)Y_2).
$$

(2.3)

**Lemma 2.3.** The function $Y(t)$ solves the following equation.

$$
\partial_t Y + L_{\mathbb{R}^2}Y = 0.
$$

**Proof.** The desired result follows from (2.2) and the definition of $Y$ (2.3). For detail proof, we refer reader to [7, Lemma 21]. \qed

**Proposition 2.4.** Let $N_0 \in \mathbb{N}$ and $a \in \mathbb{R}$. Then there exists a profile $W^{N_0} \in C^\infty([0, \infty), H(\mathbb{R}^2))$, such that

$$
\partial_t W^{N_0} + L_{\mathbb{R}^2}W^{N_0} = M_{\mathbb{R}^2}(W^{N_0}) + O(e^{-(N_0+1)t}),
$$

as $t \to \infty$ and $W^{N_0}(t) = aY(t) + O(e^{-2\rho t})$.

For simplicity, in the proof of this proposition, we write $W$ for $W^a$. We look for $W$ in the following form

$$
W(t, x) = \sum_{k=1}^{N_0} e^{-\rho k t} \left( A_{j,k}(x) \cos(j\theta t) + B_{j,k}(x) \sin(j\theta t) \right),
$$

where $A_{j,k} = \left( A^+_j, A^-_{j,k} \right)$ and $B_{j,k} = \left( B^+_j, B^-_{j,k} \right)$ are some functions in $H(\mathbb{R}^2)$ which are determined later.

We have the following expression of $M_{\mathbb{R}^2}(W)$.

**Lemma 2.5.** We have

$$
M_{\mathbb{R}^2}(W) = \sum_{j=0}^{\infty} \sum_{k=1}^{N_0} e^{-\rho k t} \left( A_{j,k}(x) \cos(j\theta t) + B_{j,k}(x) \sin(j\theta t) \right) + O(e^{-N_0+1}t),
$$

where $A_{j,k}, B_{j,k}$ depend on $A_{1,n}, B_{1,n}$ and $\partial_x A_{1,n}, \partial_x B_{1,n}$ only for $l \leq n \leq k - 1$.

**Proof.** Remark that there exists a polynomial $P_{N_0} \in H(\mathbb{R}^2)[X, Y, Z, T]$ with coefficients in $H(\mathbb{R}^2)$ and valuation at least 2, such that

$$
M_{\mathbb{R}^2}(W) = P_{N_0}(v^+, v^-, v^+_x, v^-_x) + O(|v|^{N_0+1})
$$

$$
= \sum_{m=2}^{N_0} \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} \sum_{j=0}^{m-p_1-p_2} \left( \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} Q_{j,p_1,p_2,m}(x) v^+_j \partial_x v^+_1 v^-_1 v^-_x v^+_x v^-_x v^-_x v^-_x v^-_x v^-_x + O(v^{N_0+1}) \right).
$$

The rest of the proof follows from [7, Claim 24]. \qed
Proof of Proposition 2.4. The desired result is proved by similar argument in [7, Proof of Proposition 22]. □

Define

\[ V_1^{N_0}(t, x) := e^{i\omega_1 t} W^{N_0}(t, x - c_1 t), \quad U_1^{N_0}(t, x) := R_1(t, x) + V_1^{N_0}(t, x). \]

Then we define

\[ Err_1^{N_0}(t, x) := i\partial_t U_1^{N_0} + \partial_{xx} U_1^{N_0} + i|U_1^{N_0}|^2 \partial_x U_1^{N_0} + b|U_1^{N_0}| U_1^{N_0} \]

\[ = i\partial_t V_1^{N_0} + \partial_{xx} V_1^{N_0} + i(|R_1(t)| + V_1^{N_0})^2 \partial_x (R_1(t) + V_1^{N_0}) - |R_1(t)|^2 \partial_x R_1(t) \]

\[ + b(|R_1(t)| + V_1^{N_0})^4 (R_1(t) + V_1^{N_0}) - |R_1(t)|^4 R_1(t) \]

\[ = i(\partial_t V_1^{N_0} + L_c V_1^{N_0} + N_c(V_1^{N_0})). \]

Remark: that \( V_1^{N_0}(t, x) = e^{i\omega_1 t} W^{N_0}(t, x - c_1 t) \) and \( R_1(t, x) = e^{i\omega_1 t} \phi(x - c_1 t) \), we have

\[ \partial_t V_1^{N_0} = e^{i\omega_1 t} (i\omega_1 W^{N_0} + \partial_t W^{N_0} - c\partial_x W^{N_0}) \]

\[ L_c V_1^{N_0} = e^{i\omega_1 t} \left( 2\mathcal{R}(\partial \phi W^{N_0}) \partial_x \phi + |\phi|^2 \partial_x W^{N_0} - i\partial_{xx} W^{N_0} - i\mathcal{R}(\partial \phi W^{N_0}) \right) \]

\[ N_c(V_1^{N_0}) = e^{i\omega_1 t} M_c(W^{N_0}). \]

Thus,

\[ Err_1^{N_0}(t, x) = i(\partial_t V_1^{N_0} + L_c V_1^{N_0} + N_c(V_1^{N_0})) \]

\[ = i e^{i\omega_1 t}(\partial_t W^{N_0} + L_c W^{N_0} + M_c(W^{N_0})). \]

By Proposition 2.4, \( Err_1^{N_0}(t, x) = O(e^{-\rho |N_0| + t}) \). Moreover, \( W^{N_0}(t) = a Y(t) + O(e^{-2\rho t}) \) and then \( V_1^{N_0}(t, x) = ae^{i\omega_1 t} Y(t, x - c_1 t) + O(e^{-2\rho t}) \), where \( Y(t) \) is defined by (2.3). This implies that, for all \( s \geq 0 \), there exists \( C(N_0, s) \) such that

\[ \forall t \geq 0, \quad \| V_1^{N_0} \|_{H^s} \leq C(N_0, s) e^{-\rho t}. \]

2.2. Proof of Theorems 1.2 and 1.4.

Proof of Theorem 1.2. Let \( N_0 \) to be determined later. Define

\[ \varphi(t, x) = \exp \left( \frac{i}{2} \int_{-\infty}^{x} |u(t, y)|^2 \, dy \right) u(t, x) \]

\[ \psi = \exp \left( \frac{i}{2} \int_{-\infty}^{x} |u(t, y)|^2 \, dy \right) \partial_y u(t, x) = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi, \]

\[ h(t, x) = \exp \left( \frac{i}{2} \int_{-\infty}^{x} |U_1^{N_0}(y, t)| \, dy \right) U_1^{N_0}(t, x), \]

\[ k = \exp \left( \frac{i}{2} \int_{-\infty}^{x} |U_1^{N_0}(y, t)| \, dy \right) \partial_x U_1^{N_0}(t, x) = \partial_x h - \frac{i}{2} |h|^2 h. \]

From [29, page 8], we see that if \( u \) solves (1.1) then \((\varphi, \psi)\) solves the following system

\[ \begin{cases} L \varphi = P(\varphi, \psi), \\ L \psi = Q(\varphi, \psi), \end{cases} \]

where \( L = i\partial_t + \partial_{xx} \) and

\[ P(\varphi, \psi) = i\varphi \bar{\psi} - b|\varphi|^4 \varphi, \]

\[ Q(\varphi, \psi) = -i\varphi^2 \bar{\psi} - 3b|\varphi|^4 \psi - 2b|\varphi|^2 \varphi \bar{\psi}. \]

From (2.4), by similar arguments in [29, page 9], we have \( h, k \) solves the following system

\[ \begin{cases} L h = P(h, k) + Err_1^{N_0}(1), \\ L k = Q(h, k) + Err_1^{N_0}(2), \end{cases} \]
where

\[ \text{Err}^{N_0}(1) = \text{Err}_1^{N_0} \exp \left( \frac{i}{2} \int_{-\infty}^{\infty} |U_1^{N_0}|^2 \, dy \right) - h \int_{-\infty}^{\infty} \text{Im} \left( \text{Err}_1^{N_0} \bar{U}_1^{N_0} \right) \, dy \]

\[ \text{Err}^{N_0}(2) = \partial_x \text{Err}_1^{N_0}(1) - i |h|^2 \text{Err}_1^{N_0}(1) + \frac{i}{2} h^2 \text{Err}_1^{N_0}(1). \]

Since \( \text{Err}_1^{N_0} = O(e^{-\rho(N_0+1)t}) \), we have \( (\text{Err}_1^{N_0}(1), \text{Err}_1^{N_0}(2)) = O(e^{-\rho(N_0+1)t}) \). We do a fixed point around \( q := (h, k) \) of (2.6). Set \( \tilde{w} := (\tilde{\phi}, \tilde{\psi}) = (\phi, \psi) - (h, k) \), \( F(\phi, \psi) = (P(\phi, \psi), Q(\phi, \psi)) \) and \( \text{Err}_1^{N_0} = (\text{Err}_1^{N_0}(1), \text{Err}_1^{N_0}(2)) = O(e^{-\rho(N_0+1)t}) \). We have

\[ \tilde{\psi} = \partial_x \tilde{\phi} - \frac{i}{2} (|\tilde{\phi} + h|^2 \tilde{\phi} + h) - |h|^2 h. \] (2.8)

Moreover, \( \tilde{w} \) solves the following system

\[ L\tilde{w} = F(\tilde{w} + q) - F(q) - \text{Err}_1^{N_0}. \] (2.9)

In Duhamel form, \( \tilde{w} \) satisfies, for \( t \leq s \)

\[ \tilde{w}(s) = S(s-t)w(t) - i \int_t^s S(s-\tau)(F(\tilde{w} + q) - F(q) - \text{Err}_1^{N_0})(\tau) \, d\tau. \]

Thus,

\[ S(-s)w(s) = S(-t)w(t) - i \int_t^\infty S(-\tau)(F(\tilde{w} + q) - F(q) - \text{Err}_1^{N_0})(\tau) \, d\tau. \]

We find \( \tilde{w} \) such that \( \tilde{w}(t) \to 0 \) as \( t \to \infty \). Letting \( s \to \infty \) as \( \tilde{w}(s) \to 0 \), we need to find \( \tilde{w} \) satisfying the fixed point equation

\[ \tilde{w}(t) = i \int_t^\infty S(t-\tau)(F(\tilde{w} + q) - F(q) - \text{Err}_1^{N_0})(\tau) \, d\tau. \]

We define the map

\[ \Phi : v \mapsto \Phi(v) = i \int_t^\infty S(t-\tau)(F(v + q) - F(q) - \text{Err}_1^{N_0})(\tau) \, d\tau. \]

Let \( B, T_0 \) to be determined later. For \( \tilde{w} \in C([T_0, \infty), H^2(\mathbb{R}) \times H^2(\mathbb{R})) \), define

\[ \|\tilde{w}\|_{X_{T_0, N_0}} = \sup_{t \geq T_0} e^{\rho(N_0+1)t} \|\tilde{w}(t)\|_{H^2 \times H^2}, \quad \text{for} \quad (\|w(t)\|_{H^2 \times H^2} = \|\tilde{\phi}\|_{H^2} + \|\tilde{\psi}\|_{H^2}) \]

to be norm of the Banach space

\[ X_{T_0, N_0} := \{ \tilde{w} \in C([T_0, \infty), H^2(\mathbb{R}) \times H^2(\mathbb{R})) \|\tilde{w}\|_{X_{T_0, N_0}} < \infty \}. \]

Define

\[ X_{T_0, N_0}(B) := \{ \tilde{w} \in X_{T_0, N_0} \|\tilde{w}\|_{X_{T_0, N_0}} \leq B \}. \]

We will find a fixed point of \( \Phi \) in \( X_{T_0, N_0}(B) \). By (2.5), we can assume \( T_0 \) is large enough such that

\[ B e^{-\rho(N_0+1)T_0} \leq 1, \quad \text{and} \quad \|V_1^{N_0}\|_{H^3} \leq 1. \] (10.20)

We see that

\[ \|q\|_{H^2 \times H^2} = \|h\|_{H^2} + \|k\|_{H^2} \leq C(\|U_1^{N_0}\|_{H^3} + \|U_1^{N_0}\|_{H^3}^3) \leq C(\|V_1^{N_0}\|_{H^3} + \|R_1\|_{H^3} + \|R_1\|_{H^3}) \leq C(2 + \|\phi\|_{H^3} + \|\phi\|_{H^3}^3). \]

Define \( r = C(2 + \|\phi\|_{H^3} + \|\phi\|_{H^3}^3) + 1 \). Due to smoothness of \( F \), there exists a constant \( K \) such that

\[ \forall a, b \in B_{H^2 \times H^2}(r), \quad \|F(a) - F(b)\|_{H^2 \times H^2} \leq K \|a - b\|_{H^2 \times H^2}. \]

In particular,

\[ \|F(q + v) - F(q)\|_{H^2 \times H^2} \leq K \|v\|_{H^2 \times H^2}. \]
For any $v \in X_{T_0,N_0}(B)$, we have
\[
\|\Phi(v)\|_{H^2 \times H^2} = \left\| \int_0^\infty S(t-\tau)(F(v + q) - F(q) - E \overline{\psi}_{\tau}^{N_0}) d\tau \right\|_{H^2 \times H^2} \\
\leq \int_0^\infty \left( \|F(v + q) - F(q)\|_{H^2 \times H^2} + \|E \overline{\psi}_{\tau}^{N_0}\|_{H^2 \times H^2} \right) d\tau \\
\leq \int_0^\infty (K \|v\|_{H^2 \times H^2} + C(N_0) e^{-\rho(N_0 + 1) \tau}) d\tau \\
\leq KB + C(N_0) e^{-\rho(N_0 + 1) t}.
\]
Choose $N_0$ large enough such that $\frac{K}{(N_0 + 1) \rho} \leq \frac{1}{2}$ and choose $B = \frac{2C(N_0)}{(N_0 + 1) \rho}$. Finally, choose $T_0$ large enough such that (2.10) holds. Hence, we have
\[
\|\Phi(v)(t)\|_{H^2 \times H^2} \leq Be^{-\rho(N_0 + 1) t}.
\]
This implies that $\Phi$ maps $X_{T_0,N_0}(B)$ to itself. Now, we prove that $\Phi$ is a contraction in $X_{T_0,N_0}(B)$. Let $v_1, v_2 \in X_{T_0,N_0}(B)$, we have
\[
\Phi(v_1)(t) - \Phi(v_2)(t) = i \int_t^\infty S(t-s)(F(v_1 + q) - F(v_2 + q))(s) ds.
\]
Thus,
\[
e^{\rho(N_0 + 1) t} \|\Phi(v_1)(t) - \Phi(v_2)(t)\|_{H^2 \times H^2} \\
e^{\rho(N_0 + 1) t} \left\| \int_t^\infty S(t-s)(F(v_1 + q) - F(v_2 + q))(s) ds \right\|_{H^2 \times H^2} \\
\leq e^{\rho(N_0 + 1) t} \int_t^\infty \|F(v_1 + q) - F(v_2 + q)(s)\|_{H^2 \times H^2} ds \\
\leq e^{\rho(N_0 + 1) t} \int_t^\infty K \|v_1 - v_2\|_{H^2 \times H^2} ds \\
\leq Ke^{\rho(N_0 + 1) t} \int_t^\infty e^{-\rho(N_0 + 1) \tau} \|v_1 - v_2\|_{X_{T_0,N_0}} d\tau \\
\leq Ke^{\rho(N_0 + 1) t} \|v_1 - v_2\|_{X_{T_0,N_0}} \frac{e^{-\rho(N_0 + 1) t}}{(N_0 + 1) \rho} \\
\leq \frac{K}{(N_0 + 1) \rho} \|v_1 - v_2\|_{X_{T_0,N_0}}.
\]
Taking supremum over $t \geq T_0$, we have
\[
\|\Phi(v_1) - \Phi(v_2)\|_{X_{T_0,N_0}} \leq \frac{K}{(N_0 + 1) \rho} \|v_1 - v_2\|_{X_{T_0,N_0}} \leq \frac{1}{2} \|v_1 - v_2\|_{X_{T_0,N_0}}.
\]
Hence, $\Phi$ is a contraction on $X_{T_0,N_0}(B)$ and $\Phi$ has a fixed point $\tilde{w}$.

Next, we prove that the solution $w = (\tilde{\varphi}, \tilde{\psi})$ of (2.9) satisfies the relation (2.8) if $N_0$ is large enough. Define $v = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi$ and $\tilde{v} = v - k = \partial_x \tilde{\varphi} - \frac{i}{2} (|\varphi + h|^2 \varphi - |\varphi|^2 h)$, we need to prove that $\tilde{v} = \tilde{\psi}$. By similar argument as in [29],
\[
L \tilde{\psi} - L \tilde{v} = (\tilde{\psi} - \tilde{\psi}) A(\tilde{\psi}, \tilde{\varphi}, \tilde{\varphi}, h, k) + \tilde{\psi} - \tilde{\psi} B(\tilde{\psi}, \tilde{\varphi}, \tilde{\varphi}, h, k) - i (\tilde{\varphi} + h)^2 \partial_x (\tilde{\varphi} - \tilde{\psi}),
\]
where
\[
A = -i (\tilde{\psi} + \tilde{\varphi} + 2k)(|\tilde{\varphi} + h|^2 - 3b |\tilde{\varphi} + h|^4 - \frac{1}{2} |\tilde{\varphi} + h|^4) \\
B = -2b |\tilde{\varphi} + h|^2 (|\tilde{\varphi} + h|^2 - 2i (\tilde{\varphi} + h) \left( \tilde{\varphi} + k + \frac{i}{2} |\tilde{\varphi} + h|^2 (\tilde{\varphi} + h) \right) - |\tilde{\varphi} + h|^2 (\tilde{\varphi} + h)^2).
\]
Thus, \[
\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \\
\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left( \int_N^{\infty} (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial_x (\tilde{\varphi} + h)\|_{L^\infty}^2) \, ds \right),
\]
\[
\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left( ((N-t)(\|A\|_{L^\infty} + \|B\|_{L^\infty} + ...) \\
+ 2(\|\tilde{\psi}\|_{L^\infty} + \|h\|_{L^\infty}) (\|\partial_x \tilde{\varphi}\|_{L^\infty} + \|\partial_x h\|_{L^\infty})) \right)
\]
\[
\lesssim e^{-2(\rho(N_0+1)N - (N-t)C_*)}, \text{ for } N \gg t,
\]
where \(C_*\) depends on \(R_1\) (by using the bounded of \(\|\varphi\|_{H^2} + \|\tilde{\psi}\|_{H^2} + \|h\|_{H^2} + \|k\|_{H^1}\)). Choosing \(N_0\) large enough and letting \(N \to \infty\) we obtain \(\tilde{\psi} \equiv \tilde{v}\) and hence (2.8) holds. Thus, we prove that there exists a solution \((\tilde{\varphi}, \tilde{\psi})\) of (2.9) such that \(\tilde{\psi} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h)\). Define \(\varphi = \tilde{\varphi} + h, \psi = \tilde{\psi} + k\). Hence, \((\varphi, \psi)\) solves (2.6) and \(\psi = \partial_x \varphi - \frac{i}{2}|\varphi|^2 \varphi\). Setting \(u(t, x) = \exp\left( \frac{i}{2} \int_{-\infty}^x |\varphi(t, y)|^2 \, dy \right) \varphi(t, x)\), we have \(u\) solves (1.1). Moreover,
\[
\|u - U_1^{N_0}\|_{H^2} = \left\| \exp\left( \frac{-i}{2} \int_{-\infty}^x |\varphi(y)|^2 \, dy \right) \varphi - \exp\left( \frac{-i}{2} \int_{-\infty}^x |h(y)|^2 \, dy \right) h \right\|_{H^2} \\
\lesssim \|\varphi - h\|_{H^2} = \|\tilde{\varphi}\|_{H^2} \leq Ce^{-\rho(N_0+1)t}, \text{ for } t \geq T_0.
\]
Thus, \(u(t) = R_1(t) + U_1^{N_0}(t) + O(e^{-2at})\), for \(t\) large enough. This completes the proof of Theorem 1.2. \(\square\)

Proof of Theorem 1.4. Let \(v_2\) to be fixed later and assume that \(v_* > v_2\). Let \(N_0\) to be defined later and \(a \in \mathbb{R}\). Let \(V_1^{N_0}(t), U_1^{N_0}(t)\) and error term \(Err_1^{N_0}(t)\) associated to \(R_1(t)\) and an eigenvalue \(\lambda = \rho + i\theta\) of \(L_C\). We look for a solution to (1.1) of the form \(u(t) = U_1^{N_0}(t) + \sum_{j \geq 2} R_j(t) + w(t)\). We use similar argument in the proof of Theorem 1.2. We define
\[
\varphi(t, x) = \exp\left( \frac{i}{2} \int_{-\infty}^x |u(t, y)|^2 \, dy \right) u(t, x),
\]
\[
\psi = \partial_x \varphi - \frac{i}{2}|\varphi|^2 \varphi,
\]
and
\[
h(t, x) = \exp\left( \frac{i}{2} \int_{-\infty}^x |U_1^{N_0}(t, y) + \sum_{j \geq 2} R_j(t, y)|^2 \, dy \right) (U_1^{N_0}(t, x) + \sum_{j \geq 2} R_j(t, x)),
\]
\[
k = \partial_x h - \frac{i}{2}|\varphi|^2 h.
\]
We see that if \(u\) solves (1.1) then \((\varphi, \psi)\) solves (2.6).

Let \(f(u) = i|u|^2 u_x + b|u|^4 u\) and \(L\) be the Schrödinger operator defined as in the proof of Theorem 1.2. Define
\[
Err_2^{N_0} = L(U_1^{N_0} + \sum_{j \geq 2} R_j) + f(U_1^{N_0} + \sum_{j \geq 2} R_j)
\]
Thus, by choosing \(v_2 \gg (N_0 + 1)\rho\) and Lemma 3.6, we have
\[
Err_2^{N_0} = LU_1^{N_0} + f(U_1^{N_0}) + \sum_{j \geq 2} (LR_j + f(R_j)) + (f(U_1^{N_0} + \sum_{j \geq 2} R_j) - f(U_1^{N_0}) - \sum_{j \geq 2} f(R_j))
\]
\[
= Err_1^{N_0} + (f(U_1^{N_0} + \sum_{j \geq 2} R_j) - f(U_1^{N_0}) - \sum_{j \geq 2} f(R_j))
\]
\[
= O(e^{-\rho(N_0+1)t}) + O(e^{-h_*v_*t}) = O(e^{-\rho(N_0+1)t}),
\]
Thus, by an elementary calculation, we have $q = (h, k)$ solve

$$Lq = F(q) + Er_{r_2}^{N_0},$$

where $F = (P, Q)$ is given as in the proof of Theorem 1.2 and $Er_{r_2}^{N_0} = O(e^{-\rho(N_0+1) t})$.

Define $\tilde{w} = (\tilde{\varphi}, \psi) = (\varphi, \psi) - (h, k)$. Then $\tilde{w}$ solves

$$L\tilde{w} = F(\tilde{w} + q) - F(q) - Er_{r_2}^{N_0}. \tag{2.11}$$

By similar argument in the proof of Theorem 1.2, there exists a solution $\tilde{w}$ of (2.11) such that

$$\sup_{t \geq T_0} e^{\rho(N_0+1) t} \|w(t)\|_{H^2 \times H^2} \leq B,$$

for some $T_0, N_0, B$. From this and the Grönwall inequality, we may prove that $\tilde{\varphi} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h)$. Hence, we obtain a solution $u$ of (1.1) such that

$$\|w(t)\|_{H^2} = \|u - U_1^{N_0} - \sum_{j \geq 2} R_j \|_{H^2} \leq \|\varphi - h\|_{H^2} = \|\tilde{\varphi}\|_{H^2} \leq e^{-\rho(N_0+1) t},$$

as $t$ large enough. Thus, $u(t) = U_1^{N_0}(t) + \sum_{j \geq 2} R_j(t) + w(t)$ satisfies the desired property. \qed

2.3. Orbital instability of soliton and multi-solitons. In this section, we prove Corollary 1.3 and Corollary 1.5.

Let $u \in C([T_0, \infty), H^2(\mathbb{R}))$ be the solution constructed in Theorem 1.2. Thus,

$$\forall t \geq T_0, \quad \|u(t) - R(t) - Y(t)\|_{H^2} \leq C e^{-2\rho t}.$$

We have the following lemma.

**Lemma 2.6.** There exist $\varepsilon > 0$, $t_0 \geq T_0$ and $M > 0$ such that

$$\inf_{y \in \mathbb{R}, \theta \in \mathbb{R}} \|u(t_0) - \phi(x - y) e^{i\theta}\|_{L^2(B(0, M))} = \varepsilon > 0.$$

**Proof.** The proof of this lemma is similar to the proof of [7, Lemma 31]. \qed

**Proof of Corollary 1.3.** Take a sequence $(S_n)$ such that $S_n \to \infty$ as $n \to \infty$, and define $T_n = t_0 - S_n$ and

$$u_n(t, x) = u(t + S_n, x + c_1 S_n) e^{-i\omega_1 S_n}.$$

Then $u_n \in C([T_n, 0], H^2(\mathbb{R}))$ is a solution of (1.1). Since $u(t) \approx R(t)$ as $t \geq T_0$, we have

$$u_n(t, x) \approx R(t + S_n, x + c_1 S_n) e^{-i\omega_1 S_n} = e^{i\omega_1 t} \phi(x - c_1 t).$$

Thus,

$$u_n(0, x) = \phi(x) + O(e^{-\rho S_n})$$

and hence

$$\|u_n(0) - R(0)\|_{H^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover,

$$u_n(T_n, x) = u(t_0, x + c_1 S_n)e^{-i\omega_1 S_n}.$$ 

Due to Lemma 2.6, we deduce that for all $n \in \mathbb{N}$, we have

$$\inf_{y \in \mathbb{R}, \theta \in \mathbb{R}} \|u_n(T_n) - e^{i\theta} \phi(-y)\|_{L^2} \geq \inf_{y \in \mathbb{R}, \theta \in \mathbb{R}} \|u(t_0) - e^{i\theta} \phi(-y)\|_{L^2} \geq \varepsilon,$$

which is the desired result. \qed

**Proof of Corollary 1.5.** Let $T > 0$, $M$ be given by Lemma 2.6 and $\varepsilon$, $(u_n)$, $(T_n)$ be given by Corollary 1.3. Given $I < -T$, define $\tilde{u}_n \in C([I + T_n, I], H^2(\mathbb{R}))$ by

$$\tilde{u}_n(t, x) = u_n(t - I, x - c_1 t).$$

By decreasing $I$ if possible, we assume that $\omega_1 I = 0(2\pi)$. We have $\|\tilde{u}_n(I) - R(I)\|_{H^2} = \|u_n(0) - R(0)\|_{H^2} \to 0$, as $n \to \infty$ and $\tilde{u}_n(I + T_n)$ is $\varepsilon$-away from the $\phi$-soliton family. Consider the backward solution $w_n \in C([T^*, I], H^2(\mathbb{R}))$ of (1.1) with the initial data at time $I$

$$w_n(I, x) = \tilde{u}_n(I, x) + \sum_{j=2}^{K} R_j(I, x).$$
If \( T^* > -\infty \) then \( w_n \) is a blow up solution. Consider the case \( T^* = -\infty \). Note that \( u_n \in C([T_n, 0], H^2(\mathbb{R})) \) and \([0, T_n]\) is compact, the set \( \{u_n(t) | t \in [0, T_n]\} \) is compact in \( H^2(\mathbb{R}) \). Thus, \( \sup_{t \in [0, T_n]}||\tilde{u}_n(t)||_{H^2(|x| \geq R)} \to 0 \) as \( R \to \infty \). Hence, by the localized of \( R_j \), there exists a function \( \eta(I) \) such that \( \eta(I) \to 0 \) as \( I \to -\infty \) and

\[
\forall t \in [I + T_n, I] \quad \sum_{j \geq 2} ||\tilde{u}_n(t)R_j(t)||_{H^2} \leq \eta(I).
\]

Define \( x_j(t) = c_j t + x_j \). Recall that \( R_j(t, x) = \exp(\i \omega t e^{i\theta}) \phi_j(x - x_j(t). \) For \( t < 0 \) small enough, \( x_j(t) \) is far away from \( x_1(t) \) for each \( j \geq 2 \).

Denote \( J = I + T_n \) and

\[
z(t) = w_n(t) - \left( \tilde{u}_n(t) + \sum_{j = 2}^{K} R_j(t) \right).
\]

Let \( F = (P, Q) \) be given as in the proof of Theorem 1.2. Define

\[
\varphi_n(t, x) = w_n(t, x) \exp \left( \frac{i}{2} \int_{-\infty}^{x} |w_n(t, y)|^2 dy \right),
\]

\[
\psi_n = \partial_x \varphi_n - \frac{i}{2} |\varphi_n|^2 \varphi_n,
\]

\[
h_n(t, x) = (\tilde{u}_n(t, x) + \sum_{j = 2}^{K} R_j(t, x)) \exp \left( \frac{i}{2} \int_{-\infty}^{x} |u_n + \sum_{j = 2}^{K} R_j|^2 dy \right),
\]

\[
k_n = \partial_x h_n - \frac{i}{2} |h_n|^2 h_n,
\]

\[
\tilde{w}_n = (\varphi_n, \psi_n) - (h_n, k_n),
\]

\[
q = (h_n, k_n).
\]

Recall that \( f(u) = i |u|^2 u + b |u|^4 u \). We have for \( t \in [I + T_n, I] \)

\[
L(\tilde{u}_n + \sum_{j = 2}^{K} R_j) + f(\tilde{u}_n + \sum_{j = 2}^{K} R_j) = f(\tilde{u}_n + \sum_{j = 2}^{K} R_j) - f(\tilde{u}_n) - \sum_{j = 2}^{K} f(R_j)
\]

\[
= \sum_{j \geq 2} O(\tilde{u}_n R_j) + \sum_{j \neq k \neq 1} O(R_j R_k) \leq C \eta(I), \text{ as } I \to -\infty.
\]

We see that \( \tilde{w}_n(I) = 0 \). As in the proof of Theorem 1.2, we deduce that \( \tilde{w}_n \) solves

\[
\tilde{w}_n = i \int_{t}^{t} S(t - s)(F(\tilde{w}_n) + q) - F(q) + Err(s) ds,
\]

where \( ||Err(s)||_{H^1 \times H^1} \leq C \eta(I) \). Since \( F \) is lipschitz continuous on bounded set of \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \), we have

\[
||\tilde{w}_n(t)||_{H^1 \times H^1} \leq C \int_{t}^{T} (||\tilde{w}_n(s)||_{H^1 \times H^1} + R(I)) ds
\]

\[
\leq C \int_{t}^{T} ||\tilde{w}_n(s)||_{H^1 \times H^1} ds + C \eta(t - I).
\]

Hence, by Grönwall inequality, we have

\[
||\tilde{w}_n(t)||_{H^1 \times H^1} \leq C \eta(I)(t - I)e^{C(t - I)} \leq C \eta(I), \quad \forall t \in [J, I]
\]

Thus, for \( t \in [J, I] \)

\[
||\varphi_n - h_n||_{H^2} \leq ||\tilde{w}_n(t)||_{H^1 \times H^1} \leq C \eta(I)
\]

Remark that \( \tilde{u}_n(J) = u_n(T_n) \). This implies that for all \( n \), we have

\[
||w_n(J) - u_n(T_n) - \sum_{j = 2}^{K} R_j(J)||_{H^2} \leq ||\varphi_n - h_n||_{H^2} \leq C \eta(I).
\]
Choose $I_n$ such that $C_n\eta(I_n) < \frac{\varepsilon}{3}$, $J_n = I_n + T_n$. We have
\[\|z(J_n)\|_{H^2} \leq \frac{\varepsilon}{3}.\]

Given $y_j, \gamma_j, c_j(t) = \omega_j t + \theta_j$ we have
\[
\left\| w(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} \\
\geq \left\| u_n(T_n) + \sum_{j=2}^{K} R_j(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} - \left\| w_n(J_n) - u_n(T_n) - \sum_{j=2}^{K} R_j(J_n) \right\|_{L^2} \\
\geq \left\| u_n(T_n) - \phi(\cdot - y_1)e^{i\gamma_1} + \sum_{j=2}^{K} \phi_j(\cdot - x_j(J_n)e^{ic_j(J_n)} - \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} - \frac{\varepsilon}{3}.
\]

If $\inf_{\omega, \gamma} \left\| w(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} > \varepsilon$ for infinite many $n$ then we obtain the desired result. We assume that for $n$ large enough,
\[
\inf_{y_j, \gamma_j} \left\| w(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} \leq \varepsilon.
\]

Choosing $y_j, \gamma_j$ near minimizer such that
\[
\left\| w_n(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} \leq 2\varepsilon.
\]

Consider $L^2$-norm on balls $B(x_j(J_n), R)$ around each $R_j, \ j \geq 2$. By localized of each $\phi_j$ and $u_n(T_n) = \tilde{u}_n(J_n)$, for $J_n$ small enough, we have
\[
2\varepsilon + \varepsilon \geq \left\| u_n(T_n) - \phi(\cdot - y_1)e^{i\gamma_1} + \phi_j(\cdot - x_j(J_n))e^{ic_j(J_n)} - \sum_{j=2}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2(B(x_j(J_n), R))} \\
\geq \left\| \phi_j(\cdot - x_j(J_n)) e^{ic_j(J_n)} - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2(B(x_j(J_n), R))}, \ \forall j \geq 2.
\]

Thus, each $j \geq 2$ there exists $y_{k(j)} \neq 1$ near $x_j(J_n)$. Hence, each $j \geq 2$, there exists only one $y_{k(j)}$ near $x_j(J_n)$. Since $\phi_j \neq \phi_k$, for $j \neq k$ we have $k(j) = j$ for all $j \geq 2$ i.e. $y_k - x_k(J_n) = O(1)$, for all $j \geq 2$ uniformly in $n$. This implies that
\[
\left\| \sum_{j=2}^{K} \phi_j(\cdot - x_j(J_n)) e^{ic_j(J_n)} - \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2(B(0,M))} = O_{n \to \infty}(1) \leq \frac{\varepsilon}{3}.
\]

Thus,
\[
\inf_{y_j, \gamma_j} \left\| w_n(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2} \geq \left\| w_n(J_n) - \sum_{j=1}^{K} \phi_j(\cdot - y_j)e^{i\gamma_j} \right\|_{L^2(B(0,M))} \\
\geq \left\| u_n(T_n) - \phi(\cdot - y_1)e^{i\gamma_1} \right\|_{L^2(B(0,M))} - \frac{2\varepsilon}{3} \\
\geq \varepsilon - \frac{2\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.
\]

where we use Corollary 1.3. Moreover,
\[
\left\| w_n(I_n) - \sum_{j=1}^{K} R_j(I_n) \right\|_{H^2} \to 0,
\]
as \( n \to \infty \). Thus, we obtain the desired result. \( \square \)

3. Appendix

In this section, we consider an operator \( L : H^2(\mathbb{R}, \mathbb{C}^2) \subset L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2) \) of the form

\[
L = \begin{pmatrix}
W_{1,1} + W_{1,2} \partial_x & \partial_x - \frac{h_2}{\rho} + W_2 \\
-\partial_x + \frac{h_1}{\rho} + W_3 & W_{4,1} + W_{4,2} \partial_x
\end{pmatrix},
\]

where \( h_1 \in \mathbb{R} \) and \( W_{1,1}, W_{1,2}, W_2, W_3, W_{4,1}, W_{4,2} \) belong to \( \mathcal{H}(\mathbb{C}) \), where \( \mathcal{H}(\mathbb{C}) \) is defined by (2.1). We prove the following result.

**Proposition 3.1.** Let \( \lambda \in \mathbb{C} \setminus \{iy, y \in \mathbb{R}, |y| \geq \frac{h_1^2}{4}\} \), and \( U = \begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C}^2) \) such that \( LU = \lambda U \). We have the following results.

(i) There exist \( C > 0 \) and \( \alpha > 0 \) such that for all \( x \in \mathbb{R} \) we have

\[
|u(x)| + |v(x)| + |u'(x)| + |v'(x)| \leq Ce^{-\alpha |x|}.
\]

Moreover, \( u, v \in H(\mathbb{C}) \).

(ii) Let \( \lambda \notin \text{Sp}(L) \) and take \( A \in H(\mathbb{C}^2) \). Then there exists \( \eta \in H(\mathbb{C}^2) \) such that \( (L - \lambda I)X = A \).

To prove Proposition 3.1, we study the fundamental solutions to Helmholtz equations. For a given \( \mu \in \mathbb{C} \), a fundamental solution to Helmholtz equation in \( \mathbb{R} \) is a solution of

\[
(-\partial^2_x - \mu)g_{\mu} = \delta_0.
\]

For \( \mu = \rho e^{i\theta} \) with \( \rho \geq 0 \) and \( \theta \in (0, 2\pi) \), we define \( \sqrt{\mu} = \rho \tau \frac{e^{i\theta}}{\tau} \). We have the following result (see [7, Lemma 26]).

**Lemma 3.2.** Let \( \mu \in \mathbb{C} \setminus \{\mathbb{R}^+\} \). Then there exist \( \tau > 0 \) and \( C > 0 \) such that

\[
|g_{\mu}(x)| \leq Cg_{-\tau}(x) \quad \forall x \in \mathbb{R} \setminus \{0\}.
\]

In particular, \( g_{\mu} \) is exponentially decaying at infinity with decay rate \( \sqrt{\tau} \) i.e. \( |g_{\mu}(x)| \leq Ce^{-\sqrt{\tau}|x|} \) for \( |x| \) large enough.

**Proof.** We have \( \sqrt{\mu} = \rho \frac{\tau}{2} e^{i\phi} \). It is well known that \( g_{\mu} = \frac{i}{2\sqrt{\mu}} e^{i\sqrt{\mu}|x|} \). Thus, choosing \( \tau > 0 \) such that \( \sqrt{\tau} = \rho \frac{\tau}{2} \sin \left( \frac{\phi}{2} \right) \), we have

\[
|g_{\mu}(x)| = \frac{1}{2|\sqrt{\mu}|} |e^{i\frac{\tau}{2} e^{i\phi}} e^{i\sqrt{\mu}|x|}| \leq \frac{1}{2\sqrt{\rho}} e^{-\rho \frac{\tau}{2} \sin \left( \frac{\phi}{2} \right)|x|}.
\]

Since

\[
g_{-\tau}(x) = \frac{1}{2\sqrt{\rho} \sin \left( \frac{\phi}{2} \right)} e^{-\rho \frac{\tau}{2} \sin \left( \frac{\phi}{2} \right)|x|},
\]

we obtain the desired result. \( \square \)

The following regularity result on eigenfunctions is trivial.

**Lemma 3.3.** Under the assumptions of Proposition 3.1, the functions \( u, v \in H^{\infty}(\mathbb{R}, \mathbb{C}) \) and \( \lim_{|x| \to \infty} (|u(x)| + |v(x)| + |\partial_x u(x)| + |\partial_x v(x)|) = 0 \).

For the rest of the proof, we work with the following operator

\[
L' = iPLP^{-1} = \begin{pmatrix}
\partial_{xx} - \frac{h_2}{\rho^2} + W_{1,1} \partial_x + W_{1,2} & W_{1,2} \partial_x + W_{2,2} \\
W_{3,1} \partial_x + W_{3,2} & -\partial_{xx} + \frac{h_1}{\rho^2} + W_{4,1} \partial_x + W_{4,2}
\end{pmatrix},
\]
where $P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ and
\[
\begin{align*}
W_{1,1} &= \frac{i}{2}W_{1,2} + \frac{i}{2}W_{4,2} \\
W_{1,2} &= \frac{i}{2}W_{1,1} + \frac{1}{2}W_2 - \frac{1}{2}W_3 + \frac{i}{2}W_4 \\
W_{2,1} &= \frac{i}{2}W_{2,1} - \frac{i}{2}W_{4,2} \\
W_{2,2} &= \frac{i}{2}W_{1,1} - \frac{i}{2}W_2 - \frac{1}{2}W_3 - \frac{i}{2}W_{4,1} \\
W_{3,1} &= \frac{i}{2}W_{1,2} - \frac{i}{2}W_{4,2} \\
W_{3,2} &= \frac{i}{2}W_{1,2} + \frac{1}{2}W_2 + \frac{1}{2}W_3 - \frac{i}{2}W_{4,1} \\
W_{4,1} &= \frac{i}{2}W_{1,2} - \frac{i}{2}W_2 \\
W_{4,2} &= \frac{i}{2}W_{1,2} - \frac{1}{2}W_2 + \frac{i}{2}W_3 + \frac{i}{2}W_{4,1}.
\end{align*}
\]
Thus, $W_{i,j} \in \mathcal{H}(\mathbb{C})$ for each $i = 1, \ldots, 4$ and $j = 1, 2$. Then the spectrum of $L'$ is $\text{Sp}(L') = i\text{Sp}(L)$. We see that if $\lambda$ is an eigenvalue of $L$ with eigenvector $U$ then $\lambda' = i\lambda$ is an eigenvalue of $L'$ with eigenvector $U' = \begin{pmatrix} u' \\ v' \end{pmatrix} = PU$.

Write $L' - \lambda'I = H + K$, where
\[
H := \begin{pmatrix} \partial_{xx} - \frac{\mu^2}{4} - \lambda' & 0 \\
0 & -\partial_{xx} + \frac{\mu^2}{4} - \lambda' \end{pmatrix} \quad \text{and} \quad K := \begin{pmatrix} W_{1,1}\partial_x + W_{1,2} & W_{1,2}\partial_x + W_{2,1} \\
W_{3,1}\partial_x + W_{3,2} & W_{4,1}\partial_x + W_{4,2} \end{pmatrix}.
\]

Define
\[
F := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := KU' = \begin{pmatrix} (W_{1,1}\partial_x + W_{1,2})u' + (W_{2,1}\partial_x + W_{2,2})v' \\
(W_{3,1}\partial_x + W_{3,2})u' + (W_{4,1}\partial_x + W_{4,2})v' \end{pmatrix}.
\]

We have
\[
u' = g \begin{pmatrix} \frac{\mu^2}{4} & -\lambda' \\ \lambda' & -\frac{\mu^2}{4} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \partial_{f_1} \\ \partial_{f_2} \end{pmatrix} = \begin{pmatrix} (\partial_x f_1) \\ (\partial_x f_2) \end{pmatrix}.
\]

Let $\mu_1 = -\frac{\mu^2}{4} - \lambda'$ and $\mu_2 = \lambda' - \frac{\mu^2}{4}$. Since $\lambda \notin \{iy, y \in \mathbb{R}, |y| \geq \frac{\mu^2}{4}\}$, we have $\mu_1, \mu_2$ satisfy the assumption of Lemma 3.2. Let $\tau_1, \tau_2$ be given as in Lemma 3.2 and set $\tau := \min\{\tau_1, \tau_2\}$. Define
\[
\hat{F} := \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \quad \hat{G} := \begin{pmatrix} \hat{g}_1 \\ \hat{g}_2 \end{pmatrix} = \begin{pmatrix} \partial_{f_1} \\ \partial_{f_2} \end{pmatrix}.
\]

$\hat{u} := g_{-\tau} \ast \hat{f}_1$ and $\hat{v} := g_{-\tau} \ast \hat{f}_2$.

$\hat{u}^1 := g_{-\tau} \ast \hat{g}_1$ and $\hat{v}^1 := g_{-\tau} \ast \hat{g}_2$.

**Lemma 3.4.** There exists $C > 0$ such that
\[
|u'| \leq Cu \quad \text{and} \quad |v'| \leq C\hat{u}, |\partial_x u'| \leq C\hat{u}^1 \quad \text{and} \quad |\partial_x v'| \leq C\hat{u}^2.
\]

**Proof.** From Lemma 3.2, $|g_{\mu_1}| \leq Cg_{-\tau_1} \leq Cg_{-\tau}$ for some $C > 0$. Thus,
\[
|u'| = |g_{\mu_1} \ast (-f_1)| \leq Cg_{-\tau} \ast \hat{f}_1 = Cu,
\]
\[
|\partial_x u'| = |g_{\mu_1} \ast \partial_x (-f_1)| \leq Cg_{-\tau} \ast \hat{g}_1 = C\hat{u}^1.
\]

Similarly, we have $|v'| \leq C\hat{u}$ and $|\partial_x v'| \leq C\hat{u}^1$ for some $C > 0$. This completes the proof. \hfill \Box

**Lemma 3.5.** Set $w := \hat{u} + \hat{v} + \hat{u}^1 + \hat{v}^1$. There exist $C > 0$ and $\alpha > 0$ such that
\[
w(x) \leq Ce^{-\alpha|x|}, \quad \forall x \in \mathbb{R}.
\]

The proof of Lemma 3.5 follows closely the proof of [8, Theorem 1.1] or [7, Lemma 29].
Proof. Set \( f := \bar{f}_1 + \bar{f}_2 + \bar{g}_1 + \bar{g}_2 \). We have \( w \in C^0(R) \). Indeed, \( w \) solves
\[
- \partial_{xx} w + \tau w = f, \tag{3.2}
\]
and from \( f \in L^2(\mathbb{R}) \), this implies \( w \in H^2(\mathbb{R}) \) and then \( w \in C^0(\mathbb{R}) \).
Now, we prove that there exists \( R > 0 \) such that for all \( x \in \mathbb{R} \) with \( |x| > R \) we have
\[
\frac{\tau w(x) - f(x)}{w(x)} \geq \frac{\tau}{2}. \tag{3.3}
\]
Indeed, setting \( T(x) := \sum_{i=1}^4 \sum_{j=1}^2 |W_{i,j}| + |\partial_2 W_{i,j}| \). Since \( u' \) solves \((- \partial_{xx} - \mu_1)u' = f_1 \), we have \(|\partial_{xx}u'| \leq C(|u'| + |f_1|) \leq C(|u'| + |\partial_x u'| + |\partial_x u'| + |\partial_{xx}u'|)\), for some \( C > 0 \). Similarly, \(|\partial_{xx}u'| \leq C(|u'| + |f_2|) \leq C(|u'| + |\partial_x u'| + |\partial_x u'| + |\partial_{xx}u'|)\). Combining Lemma 3.4, we have
\[
f = \bar{f}_1 + \bar{f}_2 + \bar{g}_1 + \bar{g}_2
\leq T(x)(|u'| + |v'| + |\partial_x u'| + |\partial_x v'| + |\partial_{xx}u'| + |\partial_{xx}v'|)
\leq CT(x)(|u'| + |v'| + |\partial_x u'| + |\partial_x v'|) = CT(x)w.
\]
Thus,
\[
\frac{\tau w(x) - f(x)}{w(x)} \geq \tau - CT(x) \geq \frac{\tau}{2},
\]
for \(|x| > R\) large enough, by decaying of the function \( T \). This proves (3.3).
Note that \( w \geq 0 \). Since \( w \in C^0(\mathbb{R}) \cap H^2(\mathbb{R}) \), there exists \( C_R \) such that for all \( x \in \mathbb{R} \) with \(|x| < R\), we have
\[
0 \leq w \leq C_R.
\]
Define \( \psi(x) := C_R e^{-\frac{\tau}{2}(|x|-R)} \). We have
\[
- \partial_{xx} \psi + \frac{\tau}{2} \psi \geq 0 \quad \text{on } \mathbb{R} \setminus \{0\},
\]
\[
w(x) - \psi(x) \leq 0 \quad \text{on } \{x \in \mathbb{R}, |x| < R\}. \tag{3.4}
\]
Thus, we only need to prove that \( w(x) \leq \psi(x) \) for \(|x| > R\). We prove by contradiction. Assume that \( w(x_0) > \psi(x_0) \) for some \(|x_0| > R\). Define
\[
\Omega := \{x \in \mathbb{R}, w(x) > \psi(x)\}.
\]
Then \( \Omega \) is not empty and for all \( x \in \Omega \), we have \(|x| > R\) and for all \( x \in \partial \Omega \) we have \( w(x) = \psi(x) \). Moreover, by (3.2), (3.3) and (3.4), we have
\[
\partial_{xx}(w - \psi) = \partial_{xx}w - \partial_{xx}\psi = \tau w - f - \partial_{xx}\psi
\geq \frac{\tau}{2}(w - \psi) > 0.
\]
By maximal principle, this implies that \( w - \psi \leq 0 \) on \( \Omega \), a contradiction. Thus, for all \( x \in \mathbb{R} \) we have
\[
w(x) \leq \psi(x) = C_R e^{-\frac{\tau}{2}(|x|-R)} = C e^{-\sqrt{\tau} |x|}.
\]
This implies the desired result. \( \square \)

Proof of Proposition 3.1. (i) By using Lemma 3.3, 3.4 and 3.5, it is easy to imply that (3.1) holds. Since \( u, v \) solves a system of elliptic equations and (3.1), \( u, v \) and their derivative are exponentially decaying at rate \( \alpha \). This implies the desired result.
(ii) Since \( \lambda \notin \text{Sp}(L) \), there exists \( X \in H^2(\mathbb{R}, \mathbb{C}^2) \) such that \((L - \lambda Id)X = A\). Define \( L' = iPLP^{-1} \), \( X' = PX \), \( \lambda' = i\lambda \) and \( A' = iPA \) then
\[
(L' - \lambda' Id)X' = A'.
\]
Recall that \( L' - \lambda' = H + K \). Set \( Y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \) := \( KPX \), \( A' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) and \( X' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). We have
\[
x_1 = g_{-\frac{\sqrt{\tau}}{\tau} - \lambda'}(y_1 - a_1) \quad \text{and} \quad x_2 = g_{-\frac{\sqrt{\tau}}{\tau} - \lambda'}(a_2 - y_2).
\]
The terms \( g_{-\frac{\sqrt{\tau}}{\tau} - \lambda'}(a_1) \) and \( g_{-\frac{\sqrt{\tau}}{\tau} - \lambda'}(a_2) \) are exponentially decaying, with decay rate \( \alpha \). Since \( W_{i,j} \in \mathcal{H}(\mathbb{C}) \) for each \( i = 1, ..., 4 \) and \( j = 1, 2 \), we have \( y_1, y_2 \in \mathcal{H}(\mathbb{C}) \). Hence, \( g_{-\frac{\sqrt{\tau}}{\tau} - \lambda'}(y_1) \) and
as $N$ large enough. By similar argument, we obtain the similar estimates of the interaction of the derivatives of $X$. This completes the proof of Proposition 3.1. \hfill \Box

\begin{lem}
Let $U_j^{N_0}$, $(R_j)$ $(j = 1, \ldots, K)$ be profiles given as in the proof of Theorem 1.4 and \(f(u) = i|u|^2u_x + b|u|^4u\). Then
\[
 f(U_1^{N_0} + \sum_{j \geq 2} R_j) - f(U_1^{N_0}) - \sum_{j \geq 2} f(R_j) = O(e^{-h_*|v_*|t}),
\]
where $h_*$ and $v_*$ are defined as in Theorem 1.4.
\end{lem}

\begin{proof}
For $j \neq k$, since (1.5), we have
\[
 |R_j(t,x)R_k(t,x)| \lesssim e^{-\frac{\mathcal{R}_j}{4}|x-c_jt|}e^{-\frac{\mathcal{R}_k}{4}|x-c_kt|} \lesssim e^{-\frac{\mathcal{R}_j}{4}(|x-c_jt| + |x-c_kt|)} \lesssim e^{-\frac{\mathcal{R}_j}{4}|c_j-c_k||t|} \lesssim e^{-3h_*v_*|t|}.
\]
Thus,
\[
 ||R_j(t)R_k(t)||_{L^2} \lesssim \|\sqrt{|R_j(t)R_k(t)|}_{L^\infty}\| \sqrt{|R_j(t)||R_k(t)|}_{L^2} \lesssim e^{-h_*v_*|t|},
\]
for $t$ large enough. By similar argument, we obtain the similar estimates of the interaction of the derivatives of $R_j$ and $R_k$. Recall that $U_1^{N_0} = R_1 + V_1^{N_0}$, where $V_1^{N_0}(t,x) = e^{i\omega t}W^{N_0}(t,x - ct)$. Since, $W^{N_0} \in H(C)$, we have
\[
 |R_j(t,x)V_1^{N_0}(t,x)| \lesssim e^{-\frac{\mathcal{R}_j}{4}|x-c_jt|}e^{-\alpha|x-c_jt|} \lesssim e^{-\frac{\mathcal{R}_j}{4}|x-c_jt|}e^{-\frac{\mathcal{R}_j}{4}|x-c_kt|} \lesssim e^{-3h_*v_*|t|}.
\]
Thus, we deduce that
\[
 ||R_j(t)V_1^{N_0}(t)||_{L^2} \lesssim e^{-h_*v_*|t|},
\]
for $t$ large enough. Similarly, we obtain the similar estimates of the interaction of the derivatives of $R_j$ $(j \geq 2)$ and $V_1^{N_0}$. Moreover, we have
\[
 f(U_1^{N_0} + \sum_{j \geq 2} R_j) - f(U_1^{N_0}) - \sum_{j \geq 2} f(R_j) = \sum_{j \geq 2} O(R_jR_k) + \sum_{j \geq 2} O(V_1^{N_0}R_j).
\]
This implies the desired result. \hfill \Box

\section*{Acknowledgement}
I would like to thank Prof. Stefan Le Coz for his guidance and encouragement. I am supported by scholarship MESR in my Ph.D. This work is also supported by the ANR LabEx CIMI (grant ANR-11-LABX-0040) within the French State Programme "Investissements d’Avenir.

\section*{References}
\begin{thebibliography}{99}
\bibitem{1} H. Bahouri and G. Perelman. Global well-posedness for the derivative nonlinear Schrödinger equation, 2020.
\bibitem{2} H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. \textit{Arch. Rational Mech. Anal.}, 82(4):313–345, 1983.
\bibitem{3} H. A. Biagioni and F. Linares. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations. \textit{Trans. Amer. Math. Soc.}, 353(9):3649–3659, 2001.
\bibitem{4} M. Colin and M. Ohta. Stability of solitary waves for derivative nonlinear Schrödinger equation. \textit{Ann. Inst. H. Poincaré Anal. Non Linéaire}, 23(5):753–764, 2006.
\bibitem{5} J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for Schrödinger equations with derivative. \textit{SIAM J. Math. Anal.}, 33(3):649–669, 2001.
\bibitem{6} J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. \textit{SIAM J. Math. Anal.}, 34(1):64–86, 2002.
\bibitem{7} R. Côte and S. Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. \textit{J. Math. Pures Appl. (9)}, 96(2):135–160, 2011.
\bibitem{8} Y. Deng and Y. Li. Exponential decay of the solutions for nonlinear biharmonic equations. \textit{Commun. Contemp. Math.}, 9(5):753–768, 2007.
\bibitem{9} M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. \textit{J. Funct. Anal.}, 74(1):160–197, 1987.
\bibitem{10} M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. II. \textit{J. Funct. Anal.}, 94(2):308–348, 1990.
\end{thebibliography}
[11] B. L. Guo and Y. P. Wu. Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation. *J. Differential Equations*, 123(1):35–55, 1995.

[12] Z. Guo, C. Ning, and Y. Wu. Instability of the solitary wave solutions for the generalized derivative nonlinear Schrödinger equation in the critical frequency case. *Math. Res. Lett.*, 27(2):339–375, 2020.

[13] M. Hayashi. Potential well theory for the derivative nonlinear Schrödinger equation. *Anal. PDE*, 14(3):909–944, 2021.

[14] M. Hayashi and T. Ozawa. Well-posedness for a generalized derivative nonlinear Schrödinger equation. *J. Differential Equations*, 261(10):5424–5445, 2016.

[15] N. Hayashi and T. Ozawa. On the derivative nonlinear Schrödinger equation. *Phys. D*, 55(1-2):14–36, 1992.

[16] M. Hayashi and T. Ozawa. Finite energy solutions of nonlinear Schrödinger equations of derivative type. *SIAM J. Math. Anal.*, 25(6):1488–1503, 1994.

[17] R. Jenkins, J. Liu, P. Perry, and C. Sulem. Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities. *Anal. PDE*, 13(5):1539–1578, 2020.

[18] S. Kwon and Y. Wu. Orbital stability of solitary waves for derivative nonlinear Schrödinger equation. *J. Anal. Math.*, 135(2):473–486, 2018.

[19] S. Le Coz and Y. Wu. Stability of multisolitons for the derivative nonlinear Schrödinger equation. *Int. Math. Res. Not. IMRN*, (13):4120–4170, 2018.

[20] X. Liu, G. Simpson, and C. Sulem. Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation. *J. Nonlinear Sci.*, 23(4):557–583, 2013.

[21] C. Miao, Y. Wu, and G. Xu. Global well-posedness for Schrödinger equation with derivative in $H^\frac{1}{2} (\mathbb{R})$. *J. Differential Equations*, 251(8):2164–2195, 2011.

[22] M. Ohta. Instability of solitary waves for nonlinear Schrödinger equations of derivative type. *SUT J. Math.*, 50(2):399–415, 2014.

[23] T. Ozawa. On the nonlinear Schrödinger equations of derivative type. *Indiana Univ. Math. J.*, 45(1):137–163, 1996.

[24] G. d. N. Santos. Existence and uniqueness of solution for a generalized nonlinear derivative Schrödinger equation. *J. Differential Equations*, 259(5):2030–2060, 2015.

[25] H. Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Adv. Differential Equations*, 4(4):561–580, 1999.

[26] X. Tang and G. Xu. Stability of the sum of two solitary waves for (gDNLS) in the energy space. *J. Differential Equations*, 264(6):4094–4135, 2018.

[27] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem. *Funkcial. Ekvac.*, 23(3):259–277, 1980.

[28] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. II. *Funkcial. Ekvac.*, 24(1):85–94, 1981.

[29] T. Van Phan. Construction of multi-solitons and multi kink-solitons of derivative nonlinear Schrödinger equations. to appear in *Nonlinear Analysis*, 2022, arXiv:2102.00744v4.

[30] T. Van Phan. Construction of multi-solitons for a generalized derivative nonlinear Schrödinger equation, submitted, 2022, arXiv:2105.10173v3.

[31] Y. Wu. Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space. *Anal. PDE*, 6(8):1989–2002, 2013.

[32] Y. Wu. Global well-posedness on the derivative nonlinear Schrödinger equation. *Anal. PDE*, 8(5):1101–1112, 2015.

(Phan Van Tin) **INSTITUT DE MATHÉMATIQUES DE TOULOUSE** ; UMR5219, **UNIVERSITÉ DE TOULOUSE** ; **CNRS**, **UPS IMT**, F-31062 TOULOUSE CEDEX 9, **FRANCE**

*Email address*, Phan Van Tin: van-tin.phan@univ-tlse3.fr