Anomalous quantum mass flow of atoms in \(p\)-wave resonance

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I analyze an atomic Fermi gas with a planar \(p\)-wave interaction, motivated by the experimentally observed anisotropy in \(p\)-wave Feshbach resonances. An axial superfluid state is verified. A domain wall object is discovered to be a new topological defect of this superfluid and an explicit solution has been found. Gapless quasiparticles appear as bound states on the wall, dispersing in the continuum of reduced dimensions. Surprisingly, they are chiral, deeply related to fermion zero modes and anomalies in quantum chromodynamics. The chirality of the superfluid is manifested by a persistent anomalous mass current of atoms in the groundstate. This spectacular quantum phenomenon is a prediction for future experiments.

The \(p\)-wave Feshbach resonance has recently been demonstrated accessible and robust in atomic gases of \(^6\)Li and \(^{40}\)K \(\Box\). Pioneer theoretical studies \(\Box\) point to interesting new properties associated with the resonance. For a simple model, one may consider a system of spinless fermionic atoms interacting through a \(p\)-wave potential, isotropic in the 3D orbital space. Such an interaction would enjoy the usual SO(3) spherical symmetry in orbital space. However, the \(p\)-wave Feshbach resonances are split between the perpendicular and parallel orbitals with respect to the magnetic field axis, due to the (anisotropic) magnetic dipole-dipole interaction. This is known both experimentally and theoretically \(\Box\).

The energy splitting, in units of Bohr magneton, is significantly greater than both the free atom Fermi energy [for a typical gas density] and the resonance width [which is narrow]. Therefore, it is perhaps best to distinguish the \(p_{x,y}\) orbitals from the \(p_z\) orbital and model the two \(p\)-wave resonances separately.

The planar \(p\)-wave model Focus on the resonance involving the \(p_{x,y}\) orbitals. I postulate a planar \(p\)-wave model for the atomic Fermi gas with the interaction parametrized by a single coupling constant \(g\). Using \(a_k^\dagger\) and \(a_k\) as creation and annihilation operators for atoms with momentum \(k\), the Hamiltonian reads

\[
H = \sum_k \epsilon_k a_k^\dagger a_k + \frac{g}{2\gamma} \sum_{q,k,k'} \vec{k} \cdot \vec{k}' a_k^\dagger a_{-\vec{k}+\vec{k}'}^\dagger a_{\vec{k}-\vec{k}'} a_{-\vec{k}'+\vec{k}'} ,
\]

(1)

where \(\epsilon_k = \frac{k^2}{2m} - \mu\) is the energy spectrum of the free atom, measured from the chemical potential \(\mu\), and \(\gamma\) is the space volume \((\hbar = 1\) unless explicitly restored\). A convention is adopted: the boldfaced vector labels three components \(\vec{k} = (k_x, k_y, k_z)\) and the arrow vector collects the \(x, y\) components only, \(\vec{k} = (k_x, k_y)\). When expanded in terms of the spherical harmonics, the interaction term contains only \(p_x\) and \(p_y\) orbitals as indicated by the arrow vector; it explicitly breaks the 3D rotational invariance down to a lower symmetry of SO(2) that rotates about the \(z\)-axis. Only \(L_z\) of the three angular momenta is conserved.

To compare with a physical system, the coupling constant \(g\) is related to the \(p\)-wave scattering length \(a_1\) and effective range \(r_0\) by \(g \approx \frac{2\pi k^2 q_0^2}{m r_0}\) with \(a_1 < 0 \\Box\).

As I shall show below, this atomic gas model predicts remarkable properties, including a new domain wall soliton and an anomalous quantum mass flow of atoms persisting in one direction along the wall.

Axial superfluid For \(g < 0\), the interaction term in Hamiltonian \(\Box\) can be decoupled by introducing a planar vector field \(\Phi\) via the Hubbard-Stratonovich transformation, standard in the path integral formalism of many-body theory. The field operator \(\Phi_q\) is identified by the quantum equation of motion with a composite, \(p\)-wave atom pair,

\[
\Phi_q = -\frac{g}{\gamma} \sum_k \vec{k} a_\vec{k}^\dagger a_{-\vec{k}+\vec{k}'} + \frac{1}{2} \Sigma_q \vec{k} a_\vec{k}^\dagger a_{-\vec{k}+\vec{k}'} .
\]

(2)

\(\Phi\) is the superfluid order parameter. This order parameter is a complex, two-component vector in orbital space. We parametrize it with four independent real variables as follows,

\[
\Phi_q = \left(\begin{array}{c} \Phi_x \\ \Phi_y \\ \Phi_z \\ \Phi_0 \end{array}\right) = \rho e^{-i\varphi/2} e^{i\sigma_2 (\cos \chi)} i \sin \chi ,
\]

(3)

where \(\mathbb{I}\) is a \(2 \times 2\) identity matrix and \(\sigma_2\) the second Pauli matrix. In literature, one encounters other popular parameterizations, such as writing the order parameter as a rank-2 tensor or a sum of real and imaginary vectors. Whatever other representations are can be mapped, in a one-to-one correspondence manner, to the above form.

An advantage of our parametrization is that its symmetry transformation property is made transparent. The matrices \(\mathbb{I}\) and \(\sigma_2\) are the generators of the U(1) phase and SO(2) orbital rotation transformations, respectively. So, \(\vartheta\) represents the phase of the atom-pair wavefunction and \(\varphi\) is the azimuthal angle (about the \(z\)-axis) of the order parameter as a planar vector in orbital space. For any finite modulus \(\rho\), both symmetries are spontaneously broken down. The third angle variable, \(\chi\), transforms \(\chi \rightarrow -\chi\) under the time reversal symmetry.

For a homogeneous state of (spatially) uniform \(\Phi\), the effective potential (or free energy if extended to finite temperature) density can be exactly calculated. I found

\[
V_{\text{eff}} = \frac{\rho^2}{2|g|} - \int \frac{d^3k}{2(2\pi)^3} \left[ \sqrt{\epsilon_k^2 + |\vec{k} \cdot \langle \Phi_0 \rangle|^2} - |\epsilon_k| \right] .
\]

(4)
Notice that the potential does not depend on the phase $\vartheta$ and the orbital angle $\varphi$ because of the symmetries. (Set $\vartheta = \varphi = 0$ hereafter). Given a constant $g$, the effective potential is minimized at a finite value $\rho = \rho_0$ and at $\chi = \pm \frac{\pi}{4}, \pi \pm \frac{\pi}{4}$ (see Fig. 1 and Eq. (5)). $\rho_0$ is not universal, depending on the Fermi and ultra-violet cutoff momenta. The four minima of $\chi$ are degenerate. A $\pi$ phase identification, due to the U(1) phase symmetry of the order parameter, reduces the four into two identical groups, each of two minima. So the angle variable $\chi$ should be limited to half the scope of $2\pi$, say in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, to avoid the parametrization redundancy. I emphasize that the two minima $\chi = \pm \frac{\pi}{4}$ are discrete due to time reversal symmetry. This will be of important topological implications later. Then, the (mean-field) states are the (axial) $p_x \pm ip_y$ superfluid: $\Phi_0 = e^{\Phi_0} (\frac{1}{\sqrt{2}})$ for $\chi = \pm \frac{\pi}{4}$, respectively. One of these two states will become the groundstate, breaking the symmetry spontaneously. My finding agrees to the results of other different $p$-wave models.\footnote{See [4, 5].}

The degeneracy of the $\chi = \pm \frac{\pi}{4}$ states motivates us to look for a spatially nonuniform, soliton solution that intervenes the two minima and varies smoothly from one to another in real space. Such a configuration is topologically stable since the two minima are related by the discrete time reversal $Z_2$ symmetry, not by any continuous symmetries such as orbital rotation and phase transformation.

To uncover the soliton granted by the topology of broken time-reversal symmetry, we consider temporally stationary, spatially nonuniform configurations of $\chi(r)$ with $r$ the space coordinate while keeping the modulus constant. The order parameter then becomes

$$\tilde{\Phi}(r) \equiv \left( \frac{\Phi^x(r)}{\Phi^0(r)} \right) = \rho_0 \left( \frac{\cos \chi(r)}{i \sin \chi(r)} \right)$$

where $\tilde{\Phi}(r) = \sum \phi e^{i q \cdot r} \phi_q$.

Our goal is to find an effective theory for the angle field $\chi(r)$ that retains both spatial derivative terms and potential terms. Such an effective theory is obtained perturbatively through the Wilsonian renormalization approach, by integrating out fermionic atoms at high energies (measured from the Fermi level) while keeping low energy fermions within a thin shell of thickness, say $\kappa$, along the Fermi sphere in the momentum space. I have found the effective free energy functional of $\chi(r)$.

$$F = \frac{\rho_0^2 m k_F}{2 C_F} \int d^3 r \left[ (\nabla \chi)^2 + \frac{1}{2 \xi^2} \left( 1 + \cos(4 \chi) \right) \right].$$

The characteristic length $\xi$, emergent in the low energy effective theory, is the coherence scale on which the order parameter varies. A perturbative calculation determines $\xi = \frac{k_{\Delta_0}}{m \Delta_0} \times C_1 (\frac{k_F}{\xi}, \frac{\kappa}{\xi})$ where $\Delta_0$ is the maximum energy gap of quasiparticle excitations. $C_\Phi$ and $C_1$ are dimensionless positive constants depending on the infrared and ultraviolet cutoffs ($\kappa$ and $\Lambda$), and $k_F$ is the Fermi wavevector $k_F^2/(2m) = \mu$. [Detailed derivations, including coefficients $C_\Phi$ and $C_1$, will be given elsewhere.] From either this perturbative calculation or the general Ginzburg-Landau phenomenology, the order parameter should vary slowly compared with the Fermi wavelength, i.e., $1/(\xi k_F) \ll 1$. The effective theory is invariant under a $Z_2$ time-reversal symmetry, $\chi \rightarrow -\chi$.

The effective theory happens to be a variant of the well-known sine-Gordon Lagrangian in the 3D Euclidean space. It is known to possess (nonuniform) soliton solutions\footnote{See [4, 5].}. Applying the textbook method, we find soliton and anti-soliton configurations (say, constant in $y, z$ directions),

$$\chi_{\pm}(r) = \pm \arctan \left( \tanh \left( x/\xi \right) \right),$$

where $\pm$ are the two topological charges (or winding numbers). For either configuration, $\chi(r)$ changes sign when $x$ varies from $-\infty$ to $+\infty$, developing a wall that separates two domains of opposite angular momentum. The domain wall is centered at $x = 0$, a position however set arbitrarily for simplicity. The thickness of the wall is characterized by $\xi$. Its energy per unit area is calculated to be $\rho_0^2/(C_\Phi \xi)$, proportional to $\Delta_0/\xi^2$ (recall $\Delta_0$ the maximum energy gap).

\textbf{Gapless chiral fermion bound states} What happens to the quasiparticle states when the order parameter develops a soliton defect? Consider an order parameter configuration in real space depicted in Fig. 2c. $\chi$
varies in a characteristic distance of \( \xi \), far greater than the Fermi wavelength \( k_F^{-1} \). The fermionic atoms view the soliton defect as a fairly flat, “classical” off-diagonal potential that hybridizes the particle and hole states of atoms. So we can loosely speak of three different regions along the \( x \)-axis—left (\( p_x - ip_y \)), domain wall (\( p_x \) state), and right (\( p_x + ip_y \)) boxes. Now imagine a momentum space for each ‘macroscopic box’, each beginning with free fermionic atoms labeled by 3-component definite momentum \( \mathbf{k} \). An important observation is that the off-diagonal pairing potential seen by the atomic states in the \( k_y \)-axis direction must be vanishingly small in the domain wall box (exact zero at the center of the box) but maximizes in infinites in the left and right boxes. Therefore, like the Caroli-de Gennes-Matricon state \( | \Phi \rangle \) in the vortex core of a superconductor where the order parameter vanishes, we expect to see zero energy states peaked at the domain wall and bound by the rising \( p_y \)-wave component of the gap potential at far away from the wall. Unlike the vortex core states, the domain wall fermions are only bound in the direction perpendicular to the wall (\( x \)-direction) and disperses like in continuum in the parallel directions (Fig. 2).

![FIG. 2: Heuristic illustration of gapless chiral quasiparticle bound states: (a) momentum-space energy gap in the axial \( p_x + ip_y \) superfluid state; (b) gap in the \( p_x \) state; (c) the soliton defect in the real space with the angular momentum \( \tilde{L} = -i\Phi^* \times \Phi = \tilde{\epsilon}_z p_\theta \sin \chi \); (d) the gapless bound state profile with \( \langle \Phi^0 \rangle \) as the “off-diagonal” potential (\( \cdots \)). The arrow in (c) indicates the direction of anomalous mass flow.](image)

Such heuristic argument can be made rigorous by analyzing the Hamiltonian \( \hat{H} \) in the axial state with the domain wall defect. The (mean-field) Hamiltonian can be diagonalized by using the Bogoliubov transformation

\[
a \hat{\mathbf{r}} = \sum_{n\mathbf{k}_||} \left[ u_{n\mathbf{k}_||} \hat{c}_{n\mathbf{k}_||} + v_{n\mathbf{k}_||}^* \hat{c}_{n\mathbf{k}_||}^\dagger \right]
\]

where \( \mathbf{k}_|| = (0, k_y, k_z) \) is a momentum parallel to the domain wall, and \( n \) a quantum number from the quantization of \( x \)-direction. The unitarity of the transformation requires that \( \int d^3 \mathbf{r} [u_{n\mathbf{k}_||} \hat{c}_{n\mathbf{k}_||}(\mathbf{r})u_{n\mathbf{k}_||}^* \hat{c}_{n\mathbf{k}_||}^\dagger(\mathbf{r}) + v_{n\mathbf{k}_||} \hat{c}_{n\mathbf{k}_||}(\mathbf{r})v_{n\mathbf{k}_||}^* \hat{c}_{n\mathbf{k}_||}^\dagger(\mathbf{r})] = \delta_{n\mathbf{k}_||} \delta_{\mathbf{k}_||} \). The unbroken symmetries of translation parallel to the domain wall simplify the eigenstates,

\[
\left( u_{n\mathbf{k}_||}(\mathbf{r}), v_{n\mathbf{k}_||}(\mathbf{r}) \right) = \left( \tilde{u}_{n\mathbf{k}_||}(x), \tilde{v}_{n\mathbf{k}_||}(x) \right) \frac{e^{i\mathbf{k}_|| \cdot \mathbf{r}}}{2\pi}.
\]

We are primarily interested in finding the low energy quasiparticle states that have a parallel momentum in order of \( |\mathbf{k}_|| \sim k_F \) (Fermi wavevector) and vary slowly in \( x \) direction in real space. For \( \xi k_F \gg 1 \), the eigenstates are dictated by the following Bogoliubov-de Gennes equations

\[
- i\rho_0 \left( \cos \chi \partial_x - k_y \sin \chi \right) \tilde{v}_{n\mathbf{k}_||} = \left( E_{n\mathbf{k}_||} - \epsilon_k \right) \tilde{u}_{n\mathbf{k}_||},
- i\rho_0 \left( \cos \chi \partial_x + k_y \sin \chi \right) \tilde{u}_{n\mathbf{k}_||} = \left( E_{n\mathbf{k}_||} + \epsilon_k \right) \tilde{v}_{n\mathbf{k}_||},
\]

where \( \chi = \chi_+(\mathbf{r}) \) [Eq. (7)] and \( \rho_0 \) is a constant related to the maximum gap by \( \rho_0 k_F = \Delta_0 \).

The above equations have symmetries. While reversing the sign of both \( \chi \) and \( k_y \) simultaneously or reversing the sign of \( k_z \) independently, the eigenvalues are invariant. One can also prove that given any eigenstate \( \left( \tilde{u}_{n\mathbf{k}_||}, \tilde{v}_{n\mathbf{k}_||} \right) \) with eigenvalue \( E_{n\mathbf{k}_||} \), there always exists another eigenstate, \( \left( -\tilde{v}_{n\mathbf{k}_||}, \tilde{u}_{n\mathbf{k}_||} \right) \), with an eigenvalue of opposite sign, \( -E_{n\mathbf{k}_||} \), where \( \mathbf{k}_|| = (0, -k_y, k_z) \). Such a doubling of eigenstates are not surprising in a superfluid system that had originally enjoyed time-reversal and parity symmetries before undergoing superfluid. However, as I shall show later, the phenomenon of doubling eigenstates breaks down when chiral fermions appear. There, the physical eigenstates are no longer symmetrically available upon reversing the sign of momentum, say \( k_y \rightarrow -k_y \).

The reduced 1D eigenvalue problem resembles, but differs in fundamental ways from, the Dirac equation of one spatial dimension analyzed by Jackiew and Rebbi \( \mathbb{[1]} \) who found the chiral zero-energy bound state (zero fermion mode). In our case, we would expect that their zero-energy mode translates into gapless chiral quasiparticle bound states (chiral fermions). Indeed, focusing on the gapless branch \( (n = 0) \), I found two branches of such states. They are

\[
E_{0\mathbf{k}_||}^+ = + \epsilon_{k_z}, \quad \tilde{v}_{0\mathbf{k}_||}^+ \sim [\cosh(\xi/\xi)]^{-k_y}, \quad \tilde{v}_{0\mathbf{k}_||}^+ = 0,
E_{0\mathbf{k}_||}^- = - \epsilon_{k_z}, \quad \tilde{v}_{0\mathbf{k}_||}^- = 0, \quad \tilde{v}_{0\mathbf{k}_||}^- \sim [\cosh(\xi/\xi)]^{k_y},
\]

with ‘\( \sim \)’ meaning a proper normalization to be achieved yet. Clearly, the \( E^+ \) and \( E^- \) branches are only normalizable (so become physical bound states) for \( k_y \geq 0 \) and \( k_z \geq 0 \), respectively. The superscript ‘\( \pm \)’ is best interpreted as a sign of positive or negative chirality for the existing gapless fermion modes. For either chirality the energy \( E_{n\mathbf{k}_||} \) can be positive or negative, depending on
the value of $\epsilon_{k_z} = k_z^2/(2m) - \mu$. The doubling of eigenvalues is removed by the presence of chirality, with only one mode being physical bound state, supporting the claim made early. For gapped branches (denoted with $n > 0$), I would speculate, based on the study of domain wall quasiparticle excitations in $^3$He [12], that bound states exist for both positive and negative momentum $k_y$ (hence not chiral) and the doubling should restore.

**Anomalous quantum mass flow** With the eigenfunctions diagonalizing the superfluid Hamiltonian, we can directly calculate the atom occupation number in the state vector space, specified by the quantum numbers $|n k_z\rangle$. I found that the atom occupation number per state is $N_{n k_z} = \int dx|\tilde{\psi}_{n k_z}(x)|^2$, derived from the total atom number $N = \int d^3r|\langle a^\dagger(r) a(r)\rangle| = \sum_{n k_z} N_{n k_z}$. Clearly, the discrete quantum number $n$ takes the role of $k_z$ for a homogeneous superfluid state (where the domain wall defect is absent). Finite temperature effects, which are not included in this paper, will modify the above result through extra terms and factors dependent on the Fermi-Dirac distribution function. The occupation weight is essentially given by the eigenfunction $\tilde{\psi}$ alone, consistent with the result established for the homogeneous superconductivity. The physical normalizability constrains that non-zero $\tilde{\psi}_{0k_z}$ exist only for negative chirality of $k_y \leq 0$ (Eq. [11]). We then must conclude that the atoms of quantum number $n = 0$ only occupy half the reduced 2D $k_y$-$k_z$ momentum plane (parallel to the domain wall), with a spectral weight $|\tilde{\psi}_{0k_z}|^2$. The another half space is exactly empty for the gapless states in the superfluid groundstate (zero temperature). Bear in mind that the $n > 0$ states are different. This is the origin of an anomalous chiral mass flow of atoms.

A direct calculation of the mass current of atoms verifies the above intuitive speculation. I found that the chiral gapless bound states give rise to a spectacular anomalous current of atoms equal to

$$j = \frac{h k_z^3}{6\pi^2} \hat{e}_y,$$

per unit length in $z$-axis (perpendicular to the flow). The current persists without an additional external field. Its direction is set by the topological charge of the soliton defect. For an anti-soliton, the anomalous mass flow reverses direction. In real experiments that the magnetic field sets the $z$-axis ($\mathbf{H} \parallel \hat{e}_z$), the prediction is that the anomalous mass current flows parallel to the domain wall in the direction of $\nabla L_z \times \hat{e}_z$ where $\nabla L_z$ is taken as the gradient of the angular momentum $L_z$ at the wall (Fig. 2c). The mass flow returns at the boundaries of the atomic trap which constitute an anti-soliton (anti-domain wall). The total mass current, therefore, does not violate conservation law.

At finite temperature, excitations with positive energy will proliferate those unoccupied states in the empty half space and/or deplete the occupied states. The finite temperature effect is then to create a counter mass flow in opposite direction. I conjecture that the anomalous mass flow should decrease with increasing temperature and should be strongest in magnitude for temperatures (times the Boltzmann constant) below the level splitting of quasiparticle bound states, characteristically given by $\sim \Delta^2/\mu$.

**Discussion** The anomalous current is reminiscent of that in the $^3$He A-phase [12,13,14] and in the PbTe semiconductor [17], but differs in nature. First, the domain wall defect I discovered is a new kind different than the “twist” texture of Ref. [12,13,14] for $^3$He-A. In the latter, the angular momentum $L$ sweeps its direction in real space with a unit magnitude. This domain wall is however that $\hat{L}$ is fixed in the $\hat{z}$-axis direction but changes sign and magnitude across the wall (Fig. 2b). Second, treating the pairing potential semi-classically, the quasiparticles see a gapless line (in the $k_y$-$k_z$ plane of Fig. 2b) in the domain wall region in this case as opposed to nodal points everywhere in the $^3$He-A case. These features, among others, are new for the atomic gas of anisotropic $p$-wave Feshbach resonances. The $^3$He-A results [12,13,14] do not directly apply.

This domain wall defect seems interesting and new for the experiment of atomic Fermi gases, currently developing rapidly. I speculate that superfluid domains of opposite angular momentum may be easier to realize than a homogeneous (axial) superfluid, since the latter requires rotating the whole gas (or other means) for a net input of macroscopic angular momentum. I am not aware whether the anomalous current predicted for the $^3$He-A has been observed experimentally. It would be first if the $p$-wave resonant atomic gas shows this spectacular quantum anomaly. That would be also of interest to the study of lattice quantum chromodynamics attempting to simulate chiral quark fields by fermion zero-energy mode bound at a domain wall [17].

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