KOLMOGOROV CONDITION NEAR HYPERBOLIC
SINGULARITIES OF INTEGRABLE HAMILTONIAN
SYSTEMS

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Abstract. In this paper we show that, if an integrable Hamiltonian
system admits a nondegenerate hyperbolic singularity then it will satisfy
the Kolmogorov condegeneracy condition near that singularity (under a
mild additional condition, which is trivial if the singularity contains a
fixed point).

Key words: integrable system, hyperbolic singularity, KAM theory,
Kolmogorov condition

AMS subject classification: 58F14, 58F07, 58F05, 70H05

1. Introduction

The celebrated Kolmogorov–Arnold–Moser theorem (e.g., [1, 8, 10]) says
that, under a small perturbation, most invariant tori of an integrable Hamiltonian
system persist. This theorem is stated under a non-degeneracy condition, called the
Kolmogorov condition, which says that the Hessian of the integrable Hamiltonian function \( H \) with respect to a family of action variables \( (I_i) \) does not vanish: \( \det(\partial^2 H / \partial I_i \partial I_j) \neq 0 \). There are many generalizations of this theorem, which require a weaker non-degeneracy condition
than the Kolmogorov condition (see, e.g., Rüssmann [14]). However, the
Kolmogorov condition is quite natural, and integrable Hamiltonian systems
which are not resonant are expected to satisfy this condition in general.
On the other hand, in practice, this condition is not easy to verify directly,
because the computation of the above determinant often involves Abelian
integrals and transcendental functions (see, e.g., Horozov [6] for the case of
spherical pendulum).

In this paper, we will show that if an integrable Hamiltonian system
admits a nondegenerate singularity of hyperbolic type, then the Hessian
\( \det(\partial^2 H / \partial I_i \partial I_j) \neq 0 \) is different from zero everywhere in (the regular part
of) a neighborhood of that hyperbolic singularity, provided that the integrable
subsystem on a corresponding center manifold satisfies the Kolmogorov condition (if the singularity has a fixed point, i.e. a point at which
the differential of the momentum map vanishes, then the center manifold
is just a point, and this last condition is empty). In fact, we will show
the following asymptotic formula for \( \det(\partial^2 H / \partial I_i \partial I_j) \) near a nondegenerate
hyperbolic singular fiber \( N \) of corank \( k \) of the system (a fiber means a con-
nected component of a level set of the momentum map; the corank \( k \) is the

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maximal corank of the differential of the momentum map on $N$):

$$\det(\partial^2 H/\partial I_i \partial I_j)(z) = \frac{g(z)}{\prod_{i=1}^k F_{n-k+i}(z) (\ln F_{n-k+i}(z))^3},$$

where $F_{n-k+1}, \ldots, F_n$ are a well-chosen set of smooth first integrals which vanish on $N$ (more precisely, these functions are chosen so that the local bifurcation diagram of the momentum map near the image of $N$ is a union of $k$ transversal hypersurfaces given by $\prod_{i=1}^k F_{n-k+i} = 0$), $z$ represents a regular fiber (i.e., a Liouville torus) near the singularity, and $g(z)$ is a first integral in a connected component of a neighborhood of $N$ minus the singular fibers, such that the limit $\lim_{z \to N} g(z)$ exists and is different from zero. Formula (1.1) implies immediately that $\det(\partial^2 H/\partial I_i \partial I_j)(z) \neq 0$ for $z$ close enough to the hyperbolic singular fiber $N$, and moreover $\det(\partial^2 H/\partial I_i \partial I_j)(z) \to \infty$ when $z$ tends to $N$.

Our result may be viewed as a significant improvement of a result of Knörrer [7], which says that the Kolmogorov condition $\det(\partial^2 H/\partial I_i \partial I_j) \neq 0$ is satisfied almost everywhere near a nondegenerate hyperbolic singular fiber of corank 1 or 2. Here we show that it is satisfied everywhere near the singular fiber, and in our result there is no restriction on the corank of the singularity.

A similar asymptotic formula for $\det(\partial^2 H/\partial I_i \partial I_j)$ near simple focus-focus singularities (with just 1 singular point on the singular fiber) of an integrable Hamiltonian system with 2 degrees of freedom was obtained by Rink [13] and Dullin and Vu-Ngoc [4]. We suspect that similar results hold for any nondegenerate singularity without elliptic components, and for many degenerate singularities as well, although the question remains open even for the case of a focus-focus singularity with several singular points on the singular fiber in an integrable system with 2 degrees of freedom, to my knowledge. The reason is that asymptotic formulas for the action functions near a generic focus-focus singularity with more than one singular points can be quite more complicated than the case with just one singular point or the hyperbolic case.

Most singularities of finite-dimensional integrable Hamiltonian systems are nondegenerate in a natural sense, and a large part of these nondegenerate singularities are of hyperbolic type. For example, most integrable cases of rigid body problems (see, e.g., Chapter 14 of [2]), geodesic flows on multi-dimensional ellipsoids, finite-dimensional subsystems of the integrable focusing cubic non-linear Schrödinger equations or the sine-Gordon equation, etc., admit hyperbolic singularities of various rank and corank, and our result can be applied to them.

Remark that, if the system is analytic and if $\det(\partial^2 H/\partial I_i \partial I_j) \neq 0$ somewhere then it is different from zero almost everywhere, at least in a connected component of the set of Liouville tori. So even though our result has a local character, it can be applied to show that the Kolmogorov condition is satisfied almost everywhere globally. On the other hand, when an integrable Hamiltonian system is perturbed, then Liouville tori which are too near unstable singularities are destroyed due to phenomena like separatrix splitting (looking at it another way, a small global perturbation will look
big in action-angle coordinates in a neighborhood of a Liouville torus which is too close to an unstable singularity, and so KAM theory does not apply there). So the applicability of our result to KAM theory for Liouville tori which are very close to hyperbolic singularities is quite limited.

The rest of this paper is organized as follows: in Section 2 we will recall some known facts about the structure of nondegenerate singularities of integrable Hamiltonian systems and give a more precise statement of our main result, in Section 3 we will recall an asymptotic formula for the action functions near nondegenerate hyperbolic singularities, and in Section 4 we will prove the asymptotic formula.

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2. Hyperbolic singularities

In order to state our result more precisely, let us recall here some facts and definitions (see, e.g., [16, 17, 2]). Denote by $F = (F_1, \ldots, F_n) : (M, \omega) \to \mathbb{R}^n$ a smooth momentum map of an integrable Hamiltonian function $H$ on a $2n$-dimensional symplectic manifold $M$ with the symplectic form $\omega$. We will always assume that the map $F$ is proper. Then, according to the classical Liouville-Mineur theorem [11, 12], each connected component $T$ of a regular level set of the momentum map $F$ is an $n$-dimensional torus, called a Liouville torus, and in a neighborhood $U(T)$ of $T$ there is a so-called action-angle coordinate system $(I_1, q_1, \ldots, I_2, q_2)$, where $q_i$ are cyclic coordinates (defined modulo 1), such that the symplectic form is $\omega = \sum_{i=1}^{n} dI_i \wedge dq_i$, and the first integrals $F_i$ depend only on the action variables $I_1, \ldots, I_n$.

A point $y \in M$ is called a singular point of the system if $\text{rank } dF(p) < n$. The number $n - \text{rank } dF(p)$ is called the corank. If $dF(y) = 0$ then we say that $y$ is a fixed point. When $y$ is a fixed point, then it makes sense to talk about the quadratic part $F^{(2)} = (F_1^{(2)}, \ldots, F_n^{(2)})$ of the momentum map at $y$. The functions $F_1^{(2)}, \ldots, F_n^{(2)}$ are quadratic functions on the tangent space $T_y M$, which Poisson-commute with respect to $\omega(y)$. The space of quadratic functions on $T_y M$ together with the Poisson bracket is naturally isomorphic to the Lie algebra $Sp(2n, \mathbb{R})$ of infinitesimal symplectic linear transformations, and $(F_1^{(2)}, \ldots, F_n^{(2)})$ span an Abelian subalgebra of this Lie algebra. $y$ is called nondegenerate if this Abelian subalgebra is a Cartan subalgebra of $Sp(2n, \mathbb{R})$. More generally, a singular point $y$ of corank $k$ is called nondegenerate if it becomes a fixed nondegenerate singular point after a local reduction with respect to a local free Poisson $\mathbb{R}^{n-k}$-action generated by $(n - k)$ components of the momentum map near $y$. 
According to the linearization theorem for nondegenerate singular points, due to Vey [15] in the analytic case and Eliasson [5] in the smooth case, near a nondegenerate singular point $y$ of corank $k$ there is a local smooth symplectic coordinate system $(p_1, q_1, \ldots, p_n, q_n)$, such that the functions $f_1, \ldots, f_k, p_{k+1}, \ldots, p_n$ are local first integrals of the system, where each $f_i$ is of one of the following three types:

1. $f_i = p_i^2 + q_i^2$ (elliptic type)
2. $f_i = p_iq_i$ (hyperbolic type)
3. \[
\begin{align*}
    f_i &= p_i q_{i+1} - p_{i+1} q_i \\
    f_{i+1} &= p_i q_i + p_{i+1} q_{i+1}
\end{align*}
\] (focus-focus type)

We say that $y$ is a hyperbolic singular point if all of its components are of hyperbolic type, i.e., $f_i = p_iq_i$ for all $i = 1, \ldots, k$ in the above local normal form.

The momentum map $\mathbf{F}$ gives rise to a singular torus fibration: by definition, each fiber is a connected component of a level set (i.e., the preimage of a point in $\mathbb{R}^n$) of $\mathbf{F}$. Regular fibers of this fibration are Liouville tori, and singular fibers are those which contain at least one singular point of the system. Denote by $\mathcal{B}$ the base space of this singular fibration, with the induced topology from $M$. In general, $\mathcal{B}$ is a stratified $n$-dimensional space with an integral affine structure, and singular points of $\mathcal{B}$ correspond to the singularities of $\mathbf{F}$ (see [17]). We may consider the Hamiltonian function $H$ as a function on $\mathcal{B}$. For each point $z \in \mathcal{B}$, denote by $N_z$ the corresponding fiber of the system. If $z$ is a regular point, i.e. $N_z$ is a Liouville torus, then we will say that $H$ satisfies the Kolmogorov condition at $z$ if there is a local integral affine coordinate system $(I_1, \ldots, I_n)$ near $z$ on $\mathcal{B}$ (i.e., a local system of action variables) such that $\det(\partial^2 H/\partial I_i \partial I_j)(z) \neq 0$.

Consider now a singular fiber $N = N_x$ of the system. We say that $N$ is a nondegenerate hyperbolic singularity of corank $k$, if the following two conditions are satisfied (see [16] and Chapter 9 of [2]):

1) Each point of $N$ is either regular, or nondegenerate hyperbolic singular of corank smaller or equal to $k$, and there is at least one nondegenerate hyperbolic singular point of corank $k$ on $N$.

2) The non-splitting condition (which was called the “topological stability condition” in [16]), there is a neighborhood of $N$ in $M$, such that when we restrict $\mathbf{F}$ to this neighborhood, then the set of its singular values in $\mathbb{R}^n$ (i.e., the local bifurcation diagram) is a union of $k$ local transversal (i.e., in generic position) smooth hypersurfaces intersecting at $\mathbf{F}(N)$

Consider now such a hyperbolic singularity $N_x$ of corank $k$ (where $x$ denotes the corresponding singular point on the base space $\mathcal{B}$). Denote by $y$ a hyperbolic singular point $y$ of corank $k$ in $N$. By Vey-Eliasson theorem, there is a local symplectic coordinate system $(p_1, q_1, \ldots, p_n, q_n)$ in which the $n$ functions $f_1 = p_1, \ldots, f_{n-k} = p_{n-k}, f_{n-k+1} = p_{n-k+1} q_{n-k+1}, \ldots, f_n = p_n q_n$ are local first integrals of the system. We will make the following assumption about $H$:

3) $H$ is really nondegenerate hyperbolic at $N$, in the sense that when writing $H$ as a function of $n$ variables $f_1, \ldots, f_n$, we have $\frac{\partial H}{\partial f_i}(y) \neq 0$ for all $i = 1, \ldots, k$. (In other words, the eigenvalues of the reduced linearized
Hamiltonian system of $H$ are all non-zero real numbers. Remark that this condition does not depend on the choice of the corank $k$ point $y$ in $N$.

According to the topological decomposition theorem for nondegenerate singularities [16], there is a neighborhood $(U(N_x), \mathcal{L})$ of $N_x$ in $M$ together with the singular torus foliation $\mathcal{L}$ of the system, which is diffeomorphic to an almost direct product of corank 1 hyperbolic singularities. In other words, we may write

$$(2.1) \quad (U(N_x), \mathcal{L}) \xrightarrow{\text{diffeo}} \left( \mathbb{T}^{n-k} \times D^{n-k} \times (U_1, \mathcal{L}_1) \times \ldots \times (U_k, \mathcal{L}_k) \right)/\Gamma,$$

where $\mathbb{T}^{n-k} \times D^{n-k}$ denotes a trivial fibration by $(n-k)$-dimensional tori over an $(n-k)$-dimensional disk, each $(U_i, \mathcal{L}_i)$ is a 2-dimensional surface together with a singular circle fibration given by the level sets of a Morse function with one hyperbolic singular level set (there may be many singular points on the singular level set), $\Gamma$ is a finite group which acts freely and component-wise on the product (its action on $D^{n-k}$ is trivial).

Note that the above direct decomposition is not symplectic, i.e. the symplectic form $\omega$ on $U(N_x)$ cannot be written as a direct sum of the symplectic forms on the components in general. However, according to [16], $(U(N_x), \mathcal{L})$ admits a partial system of action-angle variables. In particular, there is a system of $(n-k)$ action functions $(I_1, \ldots, I_{n-k})$ defined in $(U(N_x), \mathcal{L})$ which gives rise to a locally free Hamiltonian $\mathbb{T}^{n-k}$-action which preserves the system.

The singular point $y$ of corank $k$ in $N$ projects to a singular point $\hat{y}$ of corank $k$ in $(U_1, \mathcal{L}_1) \times \ldots \times (U_k, \mathcal{L}_k)$ modulo $\Gamma$. The set $P = (\mathbb{T}^{n-k} \times D^{n-k} \times \{\hat{y}\})/\Gamma$ is a symplectic submanifold in $M$, called the center manifold of the system through $y$. The restriction of our integrable Hamiltonian system to this center manifold $P$ is a regular integrable Hamiltonian system with action functions $I_1, \ldots, I_{n-k}$. Our last condition on $H$ is the following:

4) If $k < n$ then the restriction $H_P$ of $H$ to the center manifold $P = (\mathbb{T}^{n-k} \times D^{n-k} \times \{\hat{y}\})/\Gamma$ satisfies the Kolmogorov condition at the $(n-k)$-dimensional torus containing $y$ on $P$: $\det(\partial^2 H_P/\partial I_i \partial I_j)_{i,j \leq n-k}(y) \neq 0$.

Remark that the above condition does not depend on the choice of the corank $k$ point $y$ in $N$, and can also be rephrased as follows: $x$ lies on a $(n-k)$-dimensional stratum $S$ in $\mathcal{B}$ with a local system of affine coordinates $I_1, \ldots, I_{n-k}$, and we require that the restriction $H_S$ of $H$ to this stratum $S$ satisfy the condition $\det(\partial^2 H_S/\partial I_i \partial I_j)_{i,j \leq n-k}(x) \neq 0$.

Finally, changing the momentum map in $U(N_x)$ without changing the associated singular torus fibration of the system, we can assume that the following condition on the momentum map is satisfied:

5) $F_1 = I_1, \ldots, F_{n-k} = I_{n-k}$ are action functions, and for each $i = 1, \ldots, k$, $F_{n-k+i}$ is a Morse function on the component $(U_i, \mathcal{L}_i)$ in the decomposition (2.1) which gives rise to the singular fibration $\mathcal{L}_i$, is equal to zero on the singular fiber of $(U_i, \mathcal{L}_i)$, and is invariant under the action of $\Gamma$.

Theorem 2.1. Consider a smooth integrable Hamiltonian system with Hamiltonian function $H$ and with a proper momentum map $F = (F_1, \ldots, F_n)$ on a $2n$-dimensional symplectic manifold $(M, \omega)$, which admits a nondegenerate hyperbolic singularity $N_x$ of corank $k$ ($1 \leq k \leq n$). Assume that $H$ satisfies the conditions 3) and 4) above, i.e., $H$ is really nondegenerate hyperbolic at
$N_z$, and satisfies the Kolmogorov condition on a corresponding local $2(n-k)$-dimensional center manifold. Assume moreover that the momentum map $\mathbf{F}$ has been chosen in such a way that it satisfies the above condition 5). Then we have the following asymptotic formula:

\begin{equation}
\det(\partial^2 H/\partial I_i \partial I_j)_{i,j \leq n}(z) = \frac{g(z)}{\prod_{i=1}^{k} F_{n-k+i}(z)(\ln F_{n-k+i}(z))^3},
\end{equation}

where $z$ denotes a regular point on the base space $\mathcal{B}$ of the system near $x$, $g(z)$ is a smooth first integral in a connected component of a neighborhood of $x$ in $\mathcal{B}$ minus the singular part, such that the limit $\lim_{z \to x} g(z)$ exists and is different from zero, and $I_1, \ldots, I_n$ is a system of action functions (in the regular connected component which contains $z$). In particular, $\det(\partial^2 H/\partial I_i \partial I_j)_{i,j \leq n}(z) \neq 0$ for any regular point $z$ which lies in a sufficiently small neighborhood of $x$ in $\mathcal{B}$.

Recall that a particular (and maybe most practical) case of the above theorem is when the hyperbolic singularity $N_x$ is of corank $n$, i.e. when it contains a fixed point. In that case, the only additional condition (Condition 3) on $H$ is that the eigenvalues of the linear part of the Hamiltonian vector field of $H$ at a fixed point on $N_x$ are all different from zero.

3. ASYMPTOTIC FORMULA FOR ACTION FUNCTIONS

We will keep the notations of the previous section. Consider a hyperbolic singularity $N_x$ of corank $k$. Recall that in a neighborhood of $N_x$ there are $n-k$ regular actions functions $I_1, \ldots, I_{n-k}$. In this section we will write down an asymptotic formula for the remaining (singular) $k$ action functions in a complete system of action functions.

Remark that the actions functions change by an affine transformation, and the determinant $\det(\partial^2 H/\partial I_i \partial I_j)$ changes by a non-zero multiplicative constant, when we replace $\mathcal{U}(N_x)$ by a finite covering of it and lift the system to that finite covering. So without loss of generality, and for convenience, from now on we will assume that our singularity $N_x$ is of direct product type, i.e. the finite group $\Gamma$ in the decomposition (2.1) is trivial:

\begin{equation}
(\mathcal{U}(N_x), \mathcal{L}) \overset{\text{diff}}{=} \mathbb{T}^{n-k} \times D^{n-k} \times (\mathcal{U}_1, \mathcal{L}_1) \times \cdots \times (\mathcal{U}_k, \mathcal{L}_k)
\end{equation}

We will assume that the momentum map has been chosen in such a way that it satisfies condition 5) of the previous section, i.e. $F_i = I_i$ for $1 \leq i \leq n-k$ and $F_{n-k+i}$ is a Morse function on $(\mathcal{U}_i, \mathcal{L}_i)$ which gives rise to the fibration $\mathcal{L}_i$ for $1 \leq i \leq k$, and such that $F_{n-k+i} = 0$ on the singular fiber of $(\mathcal{U}_i, \mathcal{L}_i)$.

Consider a regular point $z$ near $x$ in the base space $\mathcal{B}$, so that the Liouville torus $N_z$ lies in the neighborhood $\mathcal{U}(N_x)$ of $N_x$. We can view the momentum map as a map from $\mathcal{B}$ to $\mathbb{R}^n$. Without loss of generality, we can assume that $F_1(x) = \ldots = F_n(x) = 0$ and $F_{n-k+1}(z) > 0, \ldots, F_n(z) > 0$.

Under the direct decomposition (3.1), we have

\begin{equation}
N_z = \mathbb{T}^{n-k}(z) \times S_1(z) \times \cdots \times S_k(z),
\end{equation}
where $\mathbb{T}^{n-k}(z)$ is a fiber in $\mathbb{T}^{n-k} \times D^{n-k}$ and each $S_i(z)$ is a regular circle fiber in $(U_i, L_i)$ on which $F_{n-k+i}$ is constant and positive.

Denote by $C$ the closure of intersection of the local regular stratum which contains $z$ in $\mathcal{B}$ with the base of $(\mathcal{U}(N_z), \mathcal{L})$ (i.e. the image of the projection of $\mathcal{U}(N_z)$ to $\mathcal{B}$). The set $C$ may be identified with a neighborhood of 0 of the “corner” set $\{(F_1, \ldots, F_n) \in \mathbb{R}^n | F_{n-k+i} \geq 0 \forall i = 1, \ldots, k\}$, with local coordinates $(F_1, \ldots, F_n)$.

On the interior of $C$ we have two different coordinate systems: the momentum coordinate system $(F_1, \ldots, F_n)$, and an action coordinate system $(I_1, \ldots, I_n)$, where $I_1, \ldots, I_{n-k}$ are action variables mentioned above (recall that $F_1 = I_1, \ldots, F_{n-k} = I_{n-k}$), and each $I_{n-k+i}$ ($i = 1, \ldots, k$) is an action variable defined as follows:

$$I_{n-k+i}(z) = \int_{\gamma_i(z)} \theta,$$

where $\theta$ is a primitive of the symplectic form $\omega$ in $\mathcal{U}(N_z)$ (i.e., $d\theta = \omega$), and $\gamma_i(z)$ is the 1-cycle on $N_z$ which is represented by $S_i(z)$. On $C$, the Hamiltonian $H$ is a smooth function of the variables $F_1, \ldots, F_n$, but $I_{n-k+i}$ ($i = 1, \ldots, k$) are not. The following proposition about the asymptotic behavior of the $k$ singular action functions $I_{n-k+i}$, viewed as functions of $n$ variables $(F_1, \ldots, F_n)$ on $C$ near the origin, will be the main ingredient in the proof of Theorem 2.1.

**Proposition 3.1.** With the above notations and assumptions, we have, for $i = 1, \ldots, k$,

$$I_{n-k+i} = \psi_i F_{n-k+i} \ln F_{n-k+i} + \phi_i,$$

on $C$, where $\psi_i = \psi_i(F_1, \ldots, F_n)$ and $\phi_i = \phi_i(F_1, \ldots, F_n)$ are smooth functions of $n$ variables $(F_1, \ldots, F_n)$, and $\psi_i(0, \ldots, 0) \neq 0$.

In particular, the action functions $I_{n-k+i}$ admit a continuous extension on the boundary of $C$ (because $F_{n-k+i} \ln F_{n-k+i}$ tends to 0 when $F_{n-k+i}$ tends to 0).

The above proposition is not a new result: it has been known for some time to people (e.g., Alexey Bolsinov and Vu Ngoc San [3]) who work on symplectic invariants of integrable Hamiltonian systems, and is a direct consequence of the theorems of Eliasson [5] and Miranda and myself [9] on the local canonical normal form of an integrable Hamiltonian system near a nondegenerate singular point or orbit. Let us sketch here its proof:

For simplicity, first consider the case with $n = k = 1$. In this case, we have just one first integral $F$, one singular action function $I$, and up to a constant and a sign, $I(z)$ is equal to the symplectic area of the region $R(z)$ between the singular fiber $F^{-1}(0)$ and the regular fiber which contains $z$. Near each singular point $y_i$ ($i = 1, \ldots, m$, where $m$ is the number of hyperbolic singular points on the singular fiber) we have a local symplectic coordinate system $(p_i, q_i)$ in which the local fibration of the system is given by $p_i q_i = constant$. Denote by $D_i = \{ -\epsilon < p_i < \epsilon, -\epsilon < q_i < \epsilon \}$ charts around $y_i$ chosen small enough so that they don’t intersect. The region $R(z)$ can be cut into “singular pieces” $R_i(z) = R_z \cap D_i$ and the rest $\hat{R}(z) = R(z) \setminus \cup_i R_i(z)$ (at least one of the singular pieces $R_i(z)$ is non-empty). The symplectic area of $\hat{R}(z)$ is a smooth function with respect to $F$, while the symplectic area
of each non-empty singular piece $R_i(z)$ is of the type $\psi F \ln F + \phi$ where $\psi$ and $\phi$ are smooth with respect to $F$, with $\psi(0) < 0$. Summing up these symplectic area gives us the desired formula for $I(z)$.

The general (higher dimensional and higher corank) case is the same. The main idea is to cut a loop on $N_z$ which represents the 1-cycle $\gamma_i$ into several pieces; the integral of the primitive form $\theta$ over those pieces which pass nearby singular points will contribute singular terms of the type $\psi_i F_{n-k+i} \ln F_{n-k+i}$.

4. Proof of Theorem 2.1

We will work under the assumptions of Theorem 2.1 and with the notations introduced in the previous sections. Denote by $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ the frequency map, where $\Gamma_i = \partial H / \partial I_i$. We will first view $(\Gamma_i)$ as a map of $n$ variables $(F_1, \ldots, F_n)$ and find an asymptotic formula for $\det(\partial \Gamma_i / \partial F_j)$, and then deduce from that asymptotic formula the desired asymptotic formula for $\det(\partial^2 H / \partial I_i \partial I_j)$.

To simplify the formulas, we will use the following notations: by $(smooth)$ we mean a function on $C$ which is smooth with respect to the variables $(F_1, \ldots, F_n)$ (they must be smooth also on the boundary of $C$), by $(smooth*)$ we mean a smooth function which moreover does not vanish at the origin, by $(small)$ a continuous function of the variables $(F_1, \ldots, F_n)$ which vanishes at the origin, by $(smoothsmall)$ a function which is both $(smooth)$ and $(small)$, by $(continuous)$ a continuous function on $C$, and by $(continuous*)$ a continuous function which does not vanish at the origin.

It follows from Proposition 3.1 that we have:

$$\frac{\partial I_i}{\partial F_j} = (\text{smooth}*) \cdot \ln F_{n-k+i} + (\text{smooth})$$

(for $i \leq k$, and)

$$\frac{\partial I_i}{\partial F_j} = (\text{smooth}) \cdot F_{n-k+i} \ln F_{n-k+i} + (\text{smooth})$$

(for $j \neq n-k+i; \ i \leq k; \ j \leq n$)

Since $I_i = F_i \ \forall i \leq n-k$, we obviously have

$$\frac{\partial I_i}{\partial F_i} = 1$$

and

$$\frac{\partial I_i}{\partial F_j} = 0$$

for all $i \leq n-k, j \neq i, j \leq n$.

The asymptotic behavior (near the origin) of the entries of the matrix $\left(\frac{\partial I_i}{\partial F_j}\right)_{i=1,\ldots,n}$ are given by the above formulas. Let us now write down the asymptotic formulas for the entries of the inverse matrix $\left(\frac{\partial F_i}{\partial I_j}\right)_{i=1,\ldots,n}$. Direct computations show that:
\[(4.5) \quad \det \left( \frac{\partial I_i}{\partial F_j} \right)_{i=1,\ldots,n}^{j=1,\ldots,n} = (\text{smooth}^*). \prod_{j=1}^{k} \ln F_{n-k+j} + (\text{l.o.t.})\]

where (l.o.t.) (lower order terms) means terms of the following types:

\[(smoothsmall). \prod_{j=1}^{k} \ln F_{n-k+j}\] for some \(i\), and \((smooth). \prod_{j \in \Delta} \ln F_{n-k+j}\) where \(\Delta\) is a proper subset of \(\{1, \ldots, k\}\);

\[(4.6) \quad \frac{\partial F_{n-k+i}}{\partial I_{n-k+i}} = (\text{smooth}^*). \prod_{j \neq i} \ln F_{n-k+j} + (\text{l.o.t.})\]

\[(4.7) \quad \frac{\partial F_{n-k+i}}{\partial I_{n-k+s}} = (\text{smooth}). F_{n-k+i} \prod_{j \neq s} \ln F_{n-k+j} + (\text{smooth}). \prod_{j \neq i, s} \ln F_{n-k+j} + (\text{l.o.t.})\]

\[(4.8) \quad \frac{\partial F_{n-k+i}}{\partial I_t} = (\text{smooth}). F_{n-k+i} \prod_{j} \ln F_{n-k+j} + (\text{smooth}). \prod_{j \neq i} \ln F_{n-k+j} + (\text{l.o.t.})\]

\[(for \ s \neq i)\), and

\[(4.9) \quad \Gamma_t = \sum_{j=1}^{n} (\text{smooth}). F_{n-k+i} \prod_{j} \ln F_{n-k+j} + (\text{smooth}). \prod_{j \neq i} \ln F_{n-k+j} + (\text{l.o.t.})\]

\[(for \ t \leq n-k)\) The reader may have noticed that our (partial) ordering of the terms is generated by

\[\ln F_{n-k+i} \succ (\text{smooth}^*) \succ (\text{smoothsmall}).\]

The above formulas together with the formula \(\Gamma_i = \sum_{j=1}^{n} \frac{\partial H}{\partial I_i} \frac{\partial F_j}{\partial I_i}\) (for \(i = 1, \ldots, n\)) give rise to:

\[(4.10) \quad \Gamma_{n-k+i} = (\text{smooth}^*). \prod_{j \neq i} \ln F_{n-k+j} + (\text{l.o.t.})\]

\[(for \ i \leq k)\). The above asymptotic formulas for the frequency map lead directly to the following formulas:

\[(4.11) \quad \frac{\partial \Gamma_t}{\partial F_s} = \frac{\partial^2 H}{\partial F_t \partial F_s} + (\text{small})\]

\[(for \ t, s \leq n-k)\);
(4.12) \[ \frac{\partial \Gamma_{n-k+i}}{\partial F_s} = (\text{small}) \]

(for \( i \leq k, s \leq n - k \));

(4.13) \[ \frac{\partial \Gamma_{n-k+j}}{\partial F_{n-k+j}} = \left( \text{continuous}^* \right) \frac{1}{F_{n-k+j}(\ln F_{n-k+j})^2} \]

(for \( j \leq k \));

(4.14) \[ \frac{\partial \Gamma_{n-k+i}}{\partial F_{n-k+j}} = \left( \text{continuous} \right) \frac{1}{F_{n-k+j}(\ln F_{n-k+j})^2(\ln F_{n-k+i})} = \left( \text{small} \right) \frac{1}{F_{n-k+j}(\ln F_{n-k+j})^2} \]

(for \( i, j \leq k, i \neq j \) \); and

(4.15) \[ \frac{\partial \Gamma_i}{\partial F_{n-k+j}} = \left( \text{continuous} \right) \frac{1}{F_{n-k+j}(\ln F_{n-k+j})^2} \]

(for \( j \leq k, t \leq n - k \)).

In turn, the above asymptotic formulas for the entries of the matrix \( \left( \frac{\partial \Gamma_i}{\partial F_j} \right)_{i,j \leq n} \) imply that the leading term in the asymptotic expansion of the determinant \( \det \left( \frac{\partial \Gamma_i}{\partial F_j} \right) \) is of the form

(4.16) \[ \det \left( \frac{\partial^2 H(x)}{\partial F_s \partial F_t} \right)_{s,t \leq n-k} \prod_{j \leq k} \left( \frac{\text{continuous}^*}{F_{n-k+j}(\ln F_{n-k+j})^2} \right), \]

where \( \det \left( \frac{\partial^2 H(x)}{\partial F_s \partial F_t} \right)_{s,t \leq n-k} \neq 0 \) by our hypothesis, so we can write

(4.17) \[ \det \left( \frac{\partial \Gamma_i}{\partial F_j} \right) = \prod_{j \leq k} F_{n-k+j}(\ln F_{n-k+j})^2 \]

It follows from the asymptotic formula for the matrix \( \left( \partial I_i / \partial F_j \right) \) shown earlier in this section that we have

(4.18) \[ \det \left( \frac{\partial I_i}{\partial F_j} \right) = \left( \text{continuous}^* \right) \prod_{j \leq k} \ln F_{n-k+j} \]

The last two formulas, together with the fact that

\[ \det(\partial^2 H / \partial I_i \partial I_j) = \det(\partial \Gamma_i / \partial I_j) = \det(\partial \Gamma_i / \partial F_s) / \det(\partial I_j / \partial F_s) \]

give us the asymptotic formula

(4.19) \[ \det(\partial^2 H / \partial I_i \partial I_j) = \frac{\left( \text{continuous}^* \right)}{\prod_{i=1}^k F_{n-k+i}(\ln F_{n-k+i})^3} \]

on \( C \). The theorem is proved.
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