Weakly almost periodic functionals on the measure algebra

Matthew Daws

Leeds

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Outline

1. Weakly almost periodic functionals
2. Hopf von Neumann algebras
3. Further directions
Let $G$ be a locally compact group;
Locally compact groups

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Let $M(G)$ be the collection of all finite Borel measures on $G$; again equipped with the convolution product. Then $L^1(G)$ is an (essential) ideal in $M(G)$. $M(G) = L^1(G)$ if and only if $G$ is discrete.
Weakly almost periodic functionals

For \( f \in C^b(G) \) and \( s \in G \), define the left translate by

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Matthew Daws (Leeds)
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Generalise: $f$ is almost periodic if $L_G(f)$ is (relatively) compact.

Generalise: $f$ is weakly almost periodic if $L_G(f)$ is (relatively) compact, in the weak topology on $C^b(G)$. 
A group compactification of $G$ is a pair $(H, \phi)$ of a compact group $H$ and a continuous homomorphism $\phi : G \to H$, which has dense range (but may not be injective).
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The Bohr (or almost periodic) compactification is the maximal group compactification of $G$, say $bG$. 

Replace “compact group” by “compact semitopological semigroup” (that is, separate continuity of the product) and we replace “almost periodic” by “weakly almost periodic.”
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Links with compactifications

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Replace “compact group” by “compact semitopological semigroup” (that is, separate continuity of the product) and we replace “almost periodic” by “weakly almost periodic”.
For a Banach algebra $\mathcal{A}$, a functional $\mu \in \mathcal{A}^*$ is (weakly) almost periodic if the orbit

$$\{ a \cdot \mu : a \in \mathcal{A}, \|a\| = 1 \}$$

is relatively (weakly) compact in $\mathcal{A}$. Here $\mathcal{A}$ acts on $\mathcal{A}^*$ in the usual way.
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A bounded approximate identity argument shows that

$$\text{ap}(L^1(G)) = \text{ap}(G), \quad \text{wap}(L^1(G)) = \text{wap}(G),$$

where $C^b(G) \subseteq L^\infty(G) = L^1(G)^*$. 

(See Ulger, 1986, or Wong, 1969, or Lau, 1977.)
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$wap(\mathcal{A})$ has interesting links with the Arens products on $\mathcal{A}^{**}$. 
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$\text{wap}(\mathcal{A})$ has interesting links with the Arens products on $\mathcal{A}^{**}$. In general, little can be said about $\text{wap}(\mathcal{A})$ and $\text{ap}(\mathcal{A})$. 
Measure algebras

What can we say about $\text{ap}(M(G))$ or $\text{wap}(M(G))$?

To be more precise: the history above was backwards. To show that $\text{wap}(L^1(G))$ is a subalgebra of $L^\infty(G)$ requires the result that $\text{wap}(L^1(G)) = \text{wap}(G)$, and then an application of Grothendieck's repeated limit criterion for weak compactness.
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A representation of $G$ is a group homomorphism $\pi : G \to \text{iso}(E)$, the isometry group of a Banach space $E$, which is weak operator topology continuous.

$\hat{\pi}(f) = \int_G f(s) \pi(s) \, ds$,

Bounded approximate identities allows you to build $\pi$ from $\hat{\pi}$.
A representation of $G$ is a group homomorphism $\pi : G \to \text{iso}(E)$, the isometry group of a Banach space $E$, which is weak operator topology continuous.

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“Multiplying” functionals

Given \( \pi : G \to \text{iso}(E) \), a coefficient functional of \( \pi \) is

\[
F \in C^b(G), \quad F(s) = \langle \mu, \pi(s)x \rangle \quad (s \in G),
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where \( \mu \in E^* \) and \( x \in E \). Write \( F = \omega_{\pi,\mu,x} \).
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$$\pi = \pi_1 \otimes \pi_2 : G \to \text{iso}(E_1 \otimes E_2), \quad s \mapsto \pi_1(s) \otimes \pi_2(s),$$
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This is exactly the proof that the Fourier-Stieltjes algebra is an algebra (all coefficient functionals of unitary representations).
Young, Kaiser and Interpolation

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**Theorem**

\[ \mu \in \text{wap}(A^*) \text{ if and only if there exists a reflexive Banach space } E, \text{ a representation } \pi : A \to \mathcal{B}(E), \text{ and } x \in E, \mu \in E^* \text{ with} \]

\[ \langle \mu, a \rangle = \langle \mu, \pi(a)(x) \rangle \quad (a \in A). \]
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So \( F \in \text{wap}(L^1(G)) \) if and only if \( F \) is the coefficient functional of a representation on a reflexive Banach space.
Reflexive tensor products

Let $E$ and $F$ be reflexive Banach spaces. There exists a norm on $E \otimes F$ such that:

1. $\|x \otimes y\| = \|x\|\|y\|$ for $x \in E, y \in F$;
2. Given $T \in \mathcal{B}(E)$ and $S \in \mathcal{B}(F)$, the map $T \otimes S$ is bounded, with norm $\|T\|\|S\|$;
3. the completion is reflexive.
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So:

- $\text{wap}(L^1(G))$ is the space of coefficient functionals on reflexive spaces;
- Multiplication is the same as tensoring;
- Reflexive spaces are stable under tensoring.
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So $\text{wap}(L^1(G))$ is a subalgebra of $C^b(G)$. 
The measure algebra

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Change categories!
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Change categories!

Look at Hopf von Neumann algebras and corepresentations.
Hopf von Neumann algebras

A (commutative) Hopf von Neumann algebra is a pair \((L^\infty(X), \Gamma)\) where \(\Gamma : L^\infty(X) \to L^\infty(X \times X)\) is a unital, normal, \(*\)-homomorphism which is co-associative:
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\begin{array}{ccc}
L^\infty(X) & \xrightarrow{\Gamma} & L^\infty(X \times X) \\
\downarrow \Gamma & & \downarrow \text{id} \otimes \Gamma \\
L^\infty(X \times X) & \xrightarrow{\Gamma \otimes \text{id}} & L^\infty(X \times X \times X)
\end{array}
\]

As \(\Gamma\) is normal, it drops to give a contraction \(L^1(X) \times L^1(X) \to L^1(X \times X)\). Then \(\Gamma\) is co-associative if and only if this product is associative.
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As \(\Gamma\) is normal, it drops to give a contraction

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L^1(X) \times L^1(X) \xrightarrow{\Gamma^*} L^1(X \times X) \xrightarrow{\Gamma^*} L^1(X).
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As \(\Gamma\) is normal, it drops to give a contraction

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L^1(X) \times L^1(X) \longrightarrow L^1(X \times X) \overset{\Gamma^*}{\longrightarrow} L^1(X).
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Then \(\Gamma\) is co-associative if and only if this product is associative.
The motivating example is $L^\infty(G)$ with the map

$$
\Gamma : L^\infty(G) \rightarrow L^\infty(G \times G);
\Gamma(F)(s, t) = F(st) \quad (F \in L^\infty(G), s, t \in G).
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Examples

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As $M(G) = C_0(G)^*$, we can lift the product from $C_0(G)$ to $M(G)^* = C_0(G)^{**}$, so $M(G)^*$ becomes a commutative von Neumann algebra.
The motivating example is $L^\infty(G)$ with the map

$$\Gamma : L^\infty(G) \rightarrow L^\infty(G \times G);$$

$$\Gamma(F)(s, t) = F(st) \quad (F \in L^\infty(G), s, t \in G).$$

Then $\Gamma_*$ induces the usual convolution product on $L^1(G)$.

As $M(G) = C_0(G)^*$, we can lift the product from $C_0(G)$ to $M(G)^* = C_0(G)^{**}$, so $M(G)^*$ becomes a commutative von Neumann algebra.

We can lift the product from $M(G)$ to a co-associative map on $M(G)^*$, turning $M(G)^*$ into a Hopf von Neumann algebra.
A suitable generalisation of a representation is a \textit{co-representation} of $(L^\infty(X), \Gamma)$. 

The von Neumann algebra $L^\infty(X) \otimes B(H)$ has predual $L^1(X)^{\hat{\otimes} T}(H)$, the projective tensor product of $L^1(X)$ and the trace-class operators on $H$. 
Representations?

A suitable generalisation of a representation is a co-representation of \((L^\infty(X), \Gamma)\).

A co-representation of \(L^\infty(X)\) on a Hilbert space \(H\) is an element \(W \in L^\infty(X) \overline{\otimes} B(H)\) (von Neumann tensor product);
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\[ L^\infty(X) \overline{\otimes} B(H) = (L^1(X) \widehat{\otimes} \mathcal{T}(H))^* = B(L^1(X), B(H)), \]
Co-representations

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via the dual pairing

\[ \langle T, f \otimes \tau \rangle = \langle T(f), \tau \rangle \quad (T \in \mathcal{B}(L^1(X), \mathcal{B}(H)), f \in L^1(X), \tau \in \mathcal{T}(H)) \]
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So \( W \in L^\infty(X) \bar{\otimes} \mathcal{B}(H) \) induces \( \pi : L^1(X) \rightarrow \mathcal{B}(H); \)
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So \( W \in L^\infty(X) \widehat{\otimes} \mathcal{B}(H) \) induces \( \pi : L^1(X) \to \mathcal{B}(H) \); \( W \) is a corepresentation if and only if \( \pi \) is a (Banach algebra) representation.
Tensoring co-representations

Given $\pi_i : L^1(X) \to \mathcal{B}(H_i)$ representations, the tensored representation

$$\pi = \pi_1 \otimes \pi_2 : L^1(X) \to \mathcal{B}(H_1 \otimes H_2),$$

is associated to

$$W_{12}^{(1)} W_{13}^{(2)} \in L^\infty(X) \overline{\otimes} \mathcal{B}(H_1) \overline{\otimes} \mathcal{B}(H_2).$$
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A coefficient functional associated to \( \pi \) is

\[
\langle F, a \rangle = \langle \mu, \pi(a)(x) \rangle = \langle (\text{id} \otimes \omega_{\mu,x}) W, a \rangle \quad (a \in L^1(X)),
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where \( \omega_{\mu,x} \in \mathcal{I}(H) \) is the normal functional

\[
B(H) \to \mathbb{C}; \quad T \mapsto \langle \mu, T(x) \rangle.
\]
For reflexive spaces?

So multiplying coefficient functionals is equivalent to “multiplying” co-representations.
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So we need a co-representation theory for reflexive Banach spaces!
Weak*-tensor products

Fix a reflexive space $E$. We define $L^\infty(X) \bar{\otimes} B(E)$ to be the weak*-closure of $L^\infty(X) \otimes B(E)$ inside $B(L^2(X, E))$. That is, the closure of $L^\infty(X) \otimes E$ for some norm. Using the approximation property for $L^1(X)$, we can show that $B(L^1(X)) \approx B(E) = L^\infty(X) \otimes B(E)$. Then co-representations all still work, and are compatible with our way of tensoring reflexive spaces.
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Theorem

Let \((L^\infty(X), \Gamma)\) be a commutative Hopf von Neumann algebra. The \(\text{wap}(L^1(X))\) is a \(C^*\)-subalgebra of \(L^\infty(X)\).
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The analogous result for \(ap(L^1(X))\) is easy, once you think in terms of \(\Gamma\) (and not just look at \(L^1(X)\)).
But what is $\text{wap}(M(G))$?

For $L^1(G)$, we have that $\text{wap}(L^1(G)) = \text{wap}(G) = C(K)$ where $K$ is some compact semigroup, which we can characterise in terms of $G$. 
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But we only expect separate continuity, so we cannot expect something simple, like $\Gamma$ restricting to a map $C(K) \to C(K \times K)$.
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But we only expect \textit{separate} continuity, so we cannot expect something simple, like \( \Gamma \) restricting to a map \( C(K) \rightarrow C(K \times K) \).

Not clear that co-representations give much insight.
Weakly compact operators

We have that

\[ L^\infty(X \times X) = L^\infty(X) \varprojlim L^\infty(X) = (L^1(X) \hat{\otimes} L^1(X))^* = B(L^1(X), L^\infty(X)). \]
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Let \( \mathcal{W}(L^1(X), L^\infty(X)) \) be the collection of all weakly-compact operators \( L^1(X) \to L^\infty(X) \).
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Let \( \mathcal{W}(\ell^1(X), \ell^\infty(X)) \) be the collection of all weakly-compact operators \( \ell^1(X) \to \ell^\infty(X) \).

Again using factorisation results, it is possible to show:

**Theorem**

Identify \( \mathcal{B}(\ell^1(X), \ell^\infty(X)) \) with \( \ell^\infty(X \times X) \). Then \( \mathcal{W}(\ell^1(X), \ell^\infty(X)) \) is a subalgebra of \( \ell^\infty(X \times X) \).
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This immediately implies that \( \text{wap}(L^1(X)) \) is a subalgebra!
Semitopological semigroups

Recall that a topological semigroup $K$ is *semitopological* if the product is separately continuous.
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**Theorem**

Let $(L^\infty(X), \Gamma)$ be a commutative Hopf von Neumann algebra. Let $K$ be the character space of $wap(L^1(X))$. Then $\Gamma$ naturally induces a semigroup product on $K$ turning $K$ into a compact semitopological semigroup.
For the measure algebra

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We now know that \( K \) is, naturally, a compact semitopological semigroup.
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We now know that $K$ is, naturally, a compact semitopological semigroup.

But what can we say about $K$? It would be good to have an abstract characterisation of $K$ in terms of $G$. 
Non-commutative issues

I initially thought about these problems for *non-commutative* Hopf von Neumann algebras,
Non-commutative issues

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Let \((M, \Gamma)\) be a Hopf von Neumann algebra; let \(M_*\) be the predual of \(M\); let \(E\) be a reflexive (operator) space.

1. What is a good replacement for \(L_2(X, E)\)? Maybe Pisier's notion of vector-valued non-commutative \(L^p\) spaces? But does \(M\) act nicely on these?

2. Lacking the approximation property, can we show that \(\text{CB}(M_*, \text{CB}(E))\) is equal to \(M_* \otimes \text{CB}(E)\)? (True if \(E\) is a Hilbert space).

3. How to tensor two reflexive operator spaces?
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