A New Algorithm for Solving the Word Problem in Braid Groups

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Abstract

One of the most interesting questions about a group is if its word problem can be solved and how. The word problem in the braid group is of particular interest to topologists, algebraists and geometers, and is the target of intensive current research. We look at the braid group from a topological point of view (rather than a geometrical one). The braid group is defined by the action of diffeomorphisms on the fundamental group of a punctured disk. We exploit the topological definition of the braid group in order to give a new approach for solving its word problem. Our algorithm is faster, in comparison with known algorithms, for short braid words with respect to

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the number of generators combining the braid, and it is almost independent
of the number of strings in the braids. Moreover, the algorithm is based on
a new computer presentation of the elements of the fundamental group of a
punctured disk. This presentation can be used also for other algorithms.

Key Words: Fundamental group, Braid group, Word problem, Algorithm
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Introduction

Let $D$ be a closed disk, and $K = \{k_1, ..., k_n\}$ be $n$ points in $D$. Let $B$ be the group
of all diffeomorphisms $\beta$ of $D$ such that $\beta(K) = K$, $\beta|_{\partial D} = Id|_{\partial D}$. The braid group
is derived from $B$ by identifying two elements if their actions on $\pi_1(D \setminus K, u)$ are
equal. To simplify the algorithm, we choose a geometric base of $\pi_1(D \setminus K, u)$, and
we look at the action of $\beta \in B$ on the elements of this geometrical base.
Thus, in order to determine if two words in the braid group are identical, we check whether their actions on the different elements of the chosen geometrical base are identical. Accordingly, to make this checking procedure efficient, we produced a new computerized presentation, and two new algorithms:

1. A presentation of the geometrical base of $\pi_1(D \setminus K, u)$.
2. An algorithm to compute the action.
3. An algorithm for reducing the presentation into a unique form.

The composition of these components holds the solution for the word problem in the braid group.

In section 1 we will give a short presentation of the fundamental group, algebraic and topological definitions of the braid group, and finally we will present the word problem in the braid group and some of the known solutions for it (Garside [6], Dehornoy [3], Birman Ko and Lee [4]). In section 2, we will present our algorithms. Section 3 will be dedicated to the proof of the correctness of the algorithms. Section 4 will deal with some aspects of their complexity. Finally, in section 5 we give conclusions, future applications of the new presentation, and further plans.

1 Braid group and preliminaries

In this section, we will recall some definitions that we will use in the sequel. Some of them will concern the fundamental group, others will describe Artin’s braid group. We will give two equivalent definitions of the braid group. The first definition is Artin’s definition [8], and the second is based on the group of diffeomorphisms of the punctured disk. The latter will give us the tools needed for solving the word problem, which will be presented at the end of this section.

1.1 The fundamental group

$D$ is a closed oriented unit disk in $\mathbb{R}^2$, $K = \{k_1, ..., k_n\} \subset D$ is a finite set of points, and $u \in \partial D$. We look at the fundamental group of $D \setminus K$ denoted by $\pi_1(D \setminus K, u)$. It is known that the fundamental group of a punctured disk with $n$ holes is a free group on $n$ generators.

Let $q$ be a simple path connecting $u$ with one of the $k_i$, say $k_{i_0}$, such that $q$ does not meet any other point $k_j$ where $j \neq i_0$. To $q$ we will assign a loop $l(q)$ (which is an element of $\pi_1(D \setminus K, u)$) as follows:
Definition 1.1 $l(q)$
Let $c$ be a simple loop equal to the (oriented) boundary of a small neighborhood $V$ of $k_i$ chosen such that $q' = q \setminus (V \cap q)$ is a simple path. Then $l(q) = q' \cup c \cup q'^{-1}$. We will use the same notation for the element of $\pi_1(D \setminus K, u)$ corresponding to $l(q)$.

Definition 1.2 Let $(T_1, \ldots, T_n)$ be an ordered set of simple paths in $D$ which connect the $k_i$'s with $u$ such that:

1. $T_i \cap k_j = \emptyset$ if $i \neq j$ for all $i, j = 1, \ldots, n$.

2. $\bigcap_{i=1}^n T_i = \{u\}$.

3. for a small circle $c(u)$ around $u$, each $u'_i = T_i \cap c(u)$ is a single point and the order in $(u'_1, \ldots, u'_n)$ is consistent with the positive orientation of $c(u)$.

We say that two such sets $(T_1, \ldots, T_n)$ and $(T'_1, \ldots, T'_n)$ are equivalent if $l(T_i) = l(T_i')$ in $\pi_1(D \setminus K, u)$ for all $i = 1, \ldots, n$.

An equivalence class of such sets is called a bush in $D \setminus K$.

Definition 1.3 A g-base (geometrical base) of $\pi_1(D \setminus K, u)$ is an ordered free base of $\pi_1(D \setminus K, u)$ which has the form $(l(T_1), \ldots, l(T_n))$, where $(T_1, \ldots, T_n)$ is a bush in $D \setminus K$.

For convenience, we choose $D$ to be the unit disk and the set $\{k_1, \ldots, k_n\}$ on the $x$-axis ordered from left to right and $u = (0, -1)$ and hence $u \in \partial D$.

We would like to point out a particular g-base which will be used in the paper. Choose $T_i$ to be the straight line connecting $u$ with $k_i$, then we call $(l(T_1), \ldots, l(T_n))$ the standard g-base of $\pi_1(D \setminus K, u)$ and it is shown in the following figure:

![Figure 1: The standard g-base](image-url)
1.2 Artin’s braid group

In this subsection, we will give two equivalent definitions for the braid group. The first is algebraic and the second is topological, which will be used to present our algorithms in this paper.

1.2.1 The algebraic definition for the braid group

Here we will lay out Artin’s definition as used in most cases.

**Definition 1.4** Artin’s braid group \( B_n \) is the group generated by \( \{\sigma_1, ..., \sigma_{n-1}\} \) submitted to the relations

1. \( \sigma_i \sigma_j = \sigma_j \sigma_i \) where \( |i - j| \geq 2 \)
2. \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for all \( i = 1, ..., n-2 \)

One can look at this as a geometrical definition, since it can be interpreted to the set of ties of \( n \) strings going from top to bottom. This is done by assigning a positive switch between any adjacent pair of strings to one of the generators. This means that \( \sigma_i \) corresponds to the geometrical element described in the following figure:

![Figure 2: The geometrical braid associated with \( \sigma_i \)](image)

The operation for this group can be described as the concatenation of two geometrical sets of strings resulting in what is called a braid.

**Example 1.5** The geometrical braid that corresponds to \( \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_3 \) is presented in the following figure:

![Figure 3: The geometrical braid \( \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_3 \)](image)
1.2.2 The topological definition for the braid group

Let \( D, K, u \) be as above.

**Definition 1.6** Let \( B \) be the group of all diffeomorphisms \( \beta \) of \( D \) such that \( \beta(K) = K, \beta|_{\partial D} = \text{Id}|_{\partial D} \). For \( \beta_1, \beta_2 \in B \) we say that \( \beta_1 \) is equivalent to \( \beta_2 \) if \( \beta_1 \) and \( \beta_2 \) induce the same automorphism of \( \pi_1(D \setminus K, u) \). The quotient of \( B \) by this equivalence relation is called the braid group \( B_n[D, K] \) (\( n = \#K \)). The elements of \( B_n[D, K] \) are called braids.

**Remark 1.7** For the canonical homomorphism \( \psi : B \to \text{Aut}(\pi_1(D \setminus K, u)) \), we actually have \( B_n[D, K] \cong \text{Im}(\psi) \).

We recall two facts from [3] section III.

1. If \( K' \subset D' \), where \( D \) is another disk, and \( \#K = \#K' \) then \( B_n[D, K] \cong B_n[D', K'] \).

2. Any braid \( \beta \in B_n[D, K] \) transforms a g-base to a g-base. Moreover, for every two g-bases, there exists a unique braid which transforms one g-base to another.

We distinguish some elements in \( B_n[D, K] \) called half-twists.

Let \( a, b \in K \) be two points. We denote \( K_{a,b} = K \setminus \{a, b\} \). Let \( \sigma \) be a simple path in \( D \setminus (\partial D \cup K_{a,b}) \) connecting \( a \) with \( b \). Choose a small regular neighborhood \( U \) of \( \sigma \) and an orientation preserving diffeomorphism \( f : \mathbb{R}^2 \to \mathbb{C} \) such that \( f(\sigma) = [-1, 1], f(U) = \{z \in \mathbb{C} \mid |z| < 2\} \).

Let \( \alpha(x), 0 \leq x \) be a real smooth monotone function such that:

\[
\alpha(x) = \begin{cases} 
1 & 0 \leq x \leq \frac{3}{2} \\
0 & 2 \leq x
\end{cases}
\]

Define a diffeomorphism \( h : \mathbb{C} \to \mathbb{C} \) as follows: for \( z = re^{i\varphi} \in \mathbb{C} \) let \( h(z) = re^{i(\varphi + \alpha(r)\pi)} \).

For the set \( \{z \in \mathbb{C} \mid 2 \leq |z|\} \), \( h(z) = \text{Id} \), and for the set \( \{z \in \mathbb{C} \mid |z| \leq \frac{3}{2}\} \), \( h(z) \) a rotation by 180° in the positive direction.

The diffeomorphism \( h \) defined above induces an automorphism on \( \pi_1(D \setminus K, u) \), that switches the position of two generators of \( \pi_1(D \setminus K, u) \), as can be seen in the figure:
Figure 4: The action of the diffeomorphism $h$

Considering $(f \circ h \circ f^{-1})|_D$ (we will compose from left to right), we get a diffeomorphism of $D$ which switches $a$ and $b$ and is the identity on $D \setminus U$. Thus it defines an element of $B_n[D, K]$.

**Definition 1.8** Let $H(\sigma)$ be the braid defined by $(f \circ h \circ f^{-1})|_D$. We call $H(\sigma)$ the positive half-twist defined by $\sigma$.

The half-twists generate $B_n$. In fact, one can choose $n - 1$ half-twists that generates $B_n$ (see below):

**Definition 1.9** Let $K = \{k_1, ..., k_n\}$, and $\sigma_1, ..., \sigma_{n-1}$ be a system of simple paths in $D \setminus \partial D$ such that each $\sigma_i$ connects $k_i$ with $k_{i+1}$ and

for all $i, j \in \{1, ..., n - 1\}$, $i < j$,

\[
\begin{align*}
\sigma_i \cap \sigma_j &= \emptyset & 2 \leq |i - j| \\
\sigma_i \cap \sigma_{i+1} &= \{k_{i+1}\} & i = 1, ..., n - 2.
\end{align*}
\]

Let $H_i = H(\sigma_i)$. The ordered system of (positive) half twists $(H_1, ..., H_{n-1})$ are called a frame of $B_n[D, K]$.

**Theorem 1.10** If $(H_1, ..., H_{n-1})$ is a frame of $B_n[D, K]$, then $B_n[D, K]$ is generated by $\{H_i\}_{i=1}^{n-1}$. Moreover, if $(H_1, ..., H_{n-1})$ is a frame of $B_n[D, K]$, then the set $\{H_i\}_{i=1}^{n-1}$ with the two relations $H_i H_j = H_j H_i$ if $2 \leq |i - j|$ and $H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}$ for any $i = 1, ..., n - 2$ are sufficient enough to present $B_n[D, K]$ and therefore this definition and Artin’s definition for the braid group are equivalent.

**Proof:** See [9].

As the standard frame we will use a frame which its paths are the straight segments connecting the point $k_i$ to $k_{i+1}$ $i = 1, ..., n - 1$.

1.2.3 The word problem

First we define what is called a braid word.
Definition 1.11 Let $b \in B_n$ be a braid. Then it is clear that $b = \sigma_{i_1}^{e_1} \cdots \sigma_{i_l}^{e_l}$ for some sequence of generators, where $i_1, \ldots, i_l \in \{1, \ldots, n-1\}$ and $e_1, \ldots, e_l \in \{1,-1\}$. We will call such a presentation of $b$ a braid word, and $\sigma_{i_k}^{e_k}$ will be called the $k^{th}$ letter of the word $b$. $l$ is the length of the braid word.

We will distinguish between two relations on the braid words.

Definition 1.12 Let $w_1$ and $w_2$ be two braid words. We will say that $w_1 = w_2$ if they represent the same element of the braid group.

Definition 1.13 Let $w_1$ and $w_2$ be two braid words. We will say that $w_1 \equiv w_2$ if $w_1$ and $w_2$ are identical letter by letter.

Now, we can introduce the word problem: Given two braid words $w_1$ and $w_2$, decide whether $w_1 = w_2$ or not.

1.3 Two known algorithms for the word problem

There are several known algorithms for solving the word problem for the braid group. In this section, we will summarize some of them. The complexity of different algorithms varies, but to our knowledge, the best known solution is of complexity of $O(l^2)$, where $l$ is the length of the longer braid word.

1.3.1 Garside’s solution

Garside gave a solution for the braid word problem in 1969. His solution is based on the definition of positive words, which contain only generators with positive power. Then, he stated that the fundamental word of the braid group $\Delta_n$ has a property that enables to replace all the generators with a negative power. This can be done simply by noticing the fact that for any $i$, there exists a positive braid word $w_i$ for which $\sigma_i^{-1} = \Delta_n^{-1}w_i$.

Another property of the $\Delta_n$ is that for any $i$ we have that $\sigma_i \Delta_n = \Delta_n \sigma_{n-1}$. This gives a method for writing a given braid word $w$ in such a way that $w = w_1w_2$ where $w_1 = \Delta_n^r$, $r \leq 0$ which is a negative braid word and $w_2$ is a positive braid word.

Now, one can write $w_2 = \Delta_n^q w_3$, where $q$ is maximal. By doing this, he can increase $r$ resulting in the minimal way of writing $w = \Delta_n^{r-q} w_3$. By organizing $w_3$ in a lexicographic order, we obtain what is called Garside’s normal form of the braid word $w$. 
Garside proved, that two braid words $w$ and $w'$ are equal if and only if their normal forms are the same.

There are some implementations for solving the braid word using this solution, and variations of it as can be found, for example, in [3], [8], [2] and [7]. For achieving the best complexity by this method, one has to expand the size of the set of generators of the braid group, resulting in the complexity of $O(l^2)$ where $l$ is the length of the longer of the two braid words.

1.3.2 Dehornoy’s solution

Dehornoy ([3], [4]) used a different approach for solving the problem. His approach is based on a definition of a $\sigma$-reduced braid word, which is a braid word that for any integer $i$, any occurrence of the letter $\sigma_i$ is separated from any occurrence of the letter $\sigma_i^{-1}$ by at least one occurrence of a letter $\sigma_j^\pm$ with $j < i$.

Dehornoy presented an algorithm for transforming any braid word to its reduced form. He proved that the reduced form of a braid word $w$ is $\text{Id}$ (i.e. the null braid word) if and only if $w$ is the identity word. This gives a simple way of checking whether two braid words $w$ and $w'$ are equal, simply by writing $w'' = w(w')^{-1}$ and reducing $w''$. If the reduced form of $w''$ is $\text{Id}$, it means that $w = w'$.

The reduction process is actually a type of an unknotting process that unties the twisted strings in a braid, by adding proper sequences and transforming locally twisted strings into an untwisted state as shown in the following figure:

Dehornoy conjectured that the complexity of his algorithm is bounded by $O(l^2)$ where $l$ is the length of the longer braid word.

In the next section we will present our algorithm, which is based on a completely different approach.
2 The presentation of the new algorithm for solving the word problem

The algorithm that we are going to present in order to solve the word problem in the braid group is based on the interplay between its two definitions. We will fix the standard frame and the standard g-base that will be used as a starting position. We associate the generator $\sigma_i$ to the half-twist $H_i$ in the standard frame for every $i = 1, ..., n - 1$. By using our two algorithms and encoding the g-bases in a unique way, and by using an algorithmic way to explore the changes that happen to the standard g-base while the braid word acts on it, we produce a practical algorithm for the word problem. Mathematically, we compare two braid words by taking one braid word and compute the result of its action on the standard g-base of the fundamental group. Then, we take the other braid word and compute the same result. The two braid words are equal if and only if the two resulted g-bases are identical.

2.1 The computerized implementation of the g-base

In this subsection, we will describe the way we encode the g-base. It involves some conventions.

Recall that $D$ is the closed unit disk, the point $u$ is the point $(0, -1)$ and the points in $K$ are on the $x$-axis.

In order to encode the path in $D$, which is an element of the g-base, we will distinguish some positions in $D$.

Notation 2.1 We will denote by $(i, 1)$ a point close to $k_i$ but above it, $(i, -1)$ a point close to $k_i$ but below it, and $(i, 0)$ the point $k_i$ itself.

We will also denote the point $u$ by $(-1, 0)$ (which is not its position in $D$, rather only a notation).

To represent a path in $D$, we will use a linked list which its links are based on the notations above, which represents the position of the path in relation to the points $u$ and $k_i$, $i = 1, ..., n$.

Each link of the list holds the two numbers as described above. We will call them (point,position).

Example 2.2 The list $(1, 0) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (4, -1) \rightarrow (5, 0)$ represents the following path:
As a rule, we will never connect the point \(u\) to any point \((i, -1)\). This will be done in order to obtain a unique way of representation, and to make the automatic computation of the twists easier.

We will be able to tell whether a path \((-1, 0) \rightarrow (i, 1)\) is passing to the left or to the right of the point \(i\) simply by checking its continuation. If the path is turning to the left \((-1, 0) \rightarrow (i, 1) \rightarrow (i - 1, e)\), then it is passing to the right of the point \(i\), and if the path is turning to the right \((-1, 0) \rightarrow (i, 1) \rightarrow (i + 1, e)\), then it is passing to the left of the point \(i\) (where \(e \in \{-1, 1, 0\}\)).

**Example 2.3** The list \((-1, 0) \rightarrow (3, 1) \rightarrow (2, 0)\) represents the following path:

![Figure 7](image)

The list \((-1, 0) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (3, -1) \rightarrow (2, 0)\) represents the following path:

![Figure 8](image)

In order to unify our treatment of all the paths of the g-base, we will concatenate all of them into one list, which means that after we arrive at the end of one path (i.e. a link \((i, 0)\)), the following link will be \((-1, 0)\) marking the beginning of the next path. For convenience, and not for mathematical reasons, we add the link \((-1, 0)\) at the end of the list.
Example 2.4 The list $(-1,0) \rightarrow (1,1) \rightarrow (2,0) \rightarrow (-1,0) \rightarrow (1,0) \rightarrow (-1,0) \rightarrow (4,0) \rightarrow (-1,0) \rightarrow (4,1) \rightarrow (3,0) \rightarrow (-1,0)$ represents the g-base in the following figure (the small circles around the points are omitted):

![Figure 9: The g-base represented by the list](image)

2.2 The algorithm

Now, we are ready to present the algorithm:

**Algorithm 2.5** ProcessWord($w$)

**input**: $w$ - a braid word.

**output**: a list which represents the g-base resulted after the action of the word's letters on the standard g-base.

ProcessWord($w$)

g ← list that represents the standard g-base.

For every letter $\sigma_i$ in $w$ do

1. act on $g$ using $\sigma_i$ by applying $(Positive/Negative)HalfTwist(\sigma_i, g)$ function.

2. reduce $g$ to its unique form using Reduce($g$) function.

return $g$.

Now, we will present the $PositiveHalfTwist(\sigma_i, g)$ function.

**Algorithm 2.6** PositiveHalfTwist($\sigma_i, g$)

**input**: 
$\sigma_i$ - the generator of the braid group acting on the g-base.
g - the list representing the g-base.

**output**: a list representing the g-base after the action of $\sigma_i$ on $g$

PositiveHalfTwist($\sigma_i, g$)

for each sequence of links in $g$ of the type $(i,e)$ or $(i+1,e)$ ($e \in \{-1,1,0\}$) do

BeforeSection ← The link just before the first link in the sequence
AfterSection ← The link after the last link in the sequence
FirstLink ← The first link in the sequence
SecondLink ← the second link in the sequence
if BeforeSection = (−1, 0) then
   act upon one of the following cases:
   if FirstLink.Point = i and SecondLink.Point = 0 then
      add the link (i − 1, −1) after BeforeSection
      BeforeSection ← the new link
   if FirstLink.Point = i + 1 and FirstLink.Position = 0 then
      add the link (i + 2, −1) after BeforeSection
      BeforeSection ← the new link
   if FirstLink.Point = i and SecondLink.Point = i + 1 then
      add the link (i − 1, −1) after BeforeSection
      BeforeSection ← the new link
   if FirstLink.Point = i and SecondLink.Point = i − 1 then
      add the links (i − 1, −1) → (i, −1) after BeforeSection
      BeforeSection ← the first new link
   if FirstLink.Point = i + 1 and SecondLink.Point = i + 2 then
      add the links (i + 2, −1) → (i + 1, −1) after BeforeSection
      BeforeSection ← the first new link
   if FirstLink.Point = i + 1 and SecondLink.Point = i then
      add the link (i + 2, −1) after BeforeSection
      BeforeSection ← the new link
for any link L between BeforeLink and AfterLink do
   L.Position ← −L.Position
   L.Point ← 2i + 1 − L.Point
if BeforeSection.Point = i − 1 then
   add the links (i, −1) → (i + 1, −1) after BeforeSection
else
   add the links (i + 1, 1) → (i, 1) after BeforeSection
   if AfterSection.Point = i − 1 then
      add the links (i + 1, −1) → (i, −1) after BeforeSection
   else
      add the links (i, 1) → (i + 1, 1) after BeforeSection

In order to obtain the $NegativeHalfTwist(\sigma_i, g)$ function, one has to use the $PositiveHalfTwist(\sigma_i, g)$ function while replacing the last two 'if statements' with the following:
   if BeforeSection.Point = i − 1 then
      add the links (i, 1) → (i + 1, 1) after BeforeSection
   else
      add the links (i + 1, −1) → (i, −1) after BeforeSection
if AfterSection.Point = $i - 1$ then
    add the links $(i + 1, 1) \rightarrow (i, 1)$ after BeforeSection
else
    add the links $(i, -1) \rightarrow (i + 1, -1)$ after BeforeSection

Now, we will present the algorithm for the function $Reduce(g)$. This function reduces the list that represents the g-base to a unique form without changing its homotopy type. This is done by applying several reduction rules that are induced from homotopic equivalences. The full proof of the validity of the rules will be given in the next section.

**Algorithm 2.7** $Reduce(g)$

**input:** $g$ - a list representing a g-base.

**output:** a list which represents a g-base homotopic to $g$. Its representation is unique.

$Reduce(g)$

*for each link $L$ in the list do*

  *FirstLink $\leftarrow$ the link right after $L$*

  *SecondLink $\leftarrow$ the link right after FirstLink*

  *if FirstLink = SecondLink then*

    *delete FirstLink and SecondLink from the list*

  *if FirstLink = $(i, 1)$ or $(i, -1)$ and SecondLink = $(i, 0)$ then*

    *delete FirstLink from the list*

  *if FirstLink = $(i, 0)$ then*

    *delete all links between FirstLink and the first appearance of $(-1, 0)$*

  *if FirstLink = $(-1, 0)$ then*

    *delete all links of the type $(i, -1)$ after it*

  *$L \leftarrow$ the next or previous link as necessary*


3 Verification of the new algorithm (correctness)

In this section, we will lay out the proof for the correctness of the two algorithms.
3.1 Correctness of the \((Positive/Negative)\)Half\(Twist(\sigma_i, g)\) algorithm

We will begin our proof of the correctness of the algorithm by proving that the algorithm works on parts of the paths that are not directly connected to \(u\) (i.e. \((-1,0)\) is not BeforeSection).

**Proposition 3.1** Let \(\sigma_i\) be the generator acting on the g-base. Then any part of the path which does not contain the points \(i\) or \(i+1\) is not affected by the twist.

\textbf{Proof:} Since the action of the twist is defined locally, any part of the path out of the twisted region (that contains only the points \(i\) and \(i+1\)) is not affected. \(\square\)

We need to check the behavior of the path locally in the twisted region. By \textit{local behavior} we mean the behavior of the links of the type \((i,e)\) or \((i+1,e)\), where \(\sigma_i\) is the generator of the specified letter in the braid word, and \(e \in \{-1,1,0\}\)

**Proposition 3.2** Let \(\sigma_i\) be the generator acting on the g-base. The local behavior of the path is given by the following changes:

1. The link’s position changes to \(-\text{-position}\)
2. The link’s point changes from \(i\) to \(i+1\) and vice versa.

\textbf{Proof:} From its definition, the actual local action of the braid is a rotation of \(180^\circ\). Therefore, a part of the path of the g-base’s element, which was beneath a point before the rotation, will now be above a point, and the part of the path that was above a point before the rotation will now be beneath a point. Hence, if the position was equal to \(-1\) before the twist, it will be equal to 1 after the twist, and vice versa.

Moreover, if the point in the path was equal to \(i\), then the point in the path will be \(i+1\) after the twist, and if the point in the path was equal to \(i+1\), then the point in the path will be \(i\) after the twist. \(\square\)

After we have rotated the path locally, we will have to connect it to the global path. This should be done by adding proper prefix and postfix sequences before and after the part that has been twisted.
Proposition 3.3 Let $\sigma_i$ be the positive half-twist acting on the $g$-base. Then, the prefix sequence we have to add is as follows:

1. $(i, -1) \rightarrow (i + 1, -1)$ if the local section of the path is connected to a point to the left of the point $i$.

2. $(i + 1, 1) \rightarrow (i, 1)$ if the local section of the path is connected to a point to the right of the point $i + 1$.

Proof: If the point just before the local section of the path is to the left of the twist, then the connecting path should be beneath the twisted region. On the contrary, if the point just before the local section is to the right of the twist, then the connecting path should be above the twisted region. So, all we need to add is the two links above the twisted region or beneath it as necessary, as shown in the following figure:

![Figure 10: Prefix added after the twist](image)

Proposition 3.4 Let $\sigma_i$ be the positive half-twist acting on the $g$-base. Then, the postfix sequence we have to add is as follows:

1. $(i + 1, -1) \rightarrow (i, -1)$ if the local section of the path is connected to a point left to the point $i$.

2. $(i, 1) \rightarrow (i + 1, 1)$ if the local section of the path is connected to a point to the right of the point $i + 1$.

Proof: The proof is similar to the proof of proposition 3.3, see the following figure:
Figure 11: Postfix added after the twist

The local action of the braid generator $\sigma_i^{-1}$ is computable in the same way as the action of the generator $\sigma_i$. The prefix and the postfix sequences that we have to add are not the same sequences due to the direction of the twist. Therefore, we have the following proposition:

**Proposition 3.5** Let $\sigma_i$ be the negative half-twist acting on the $g$-base. Then, the prefix sequence we have to add is as follows:

1. $(i, 1) \to (i+1, 1)$ if the local section of the path is connected to a point to the left of the point $i$.
2. $(i+1, -1) \to (i, -1)$ if the local section of the path is connected to a point to the right of the point $i+1$.

The postfix sequence we have to add is as follows:

1. $(i + 1, 1) \to (i, 1)$ if the local section of the path is connected to a point left of the point $i$.
2. $(i, -1) \to (i + 1, -1)$ if the local section of the path is connected to a point right of the point $i+1$.

Now, we will consider the case where the link $(-1, 0)$ is followed immediately by the local section of the path. In this case, we alter the path homotopically so that the preceding link to the local section of the path will not be $(-1, 0)$. By doing this, we will reduce the problem to the one already proved by the above
propositions, hence, we will be able to use the same algorithmic methods in these cases.

We have 6 possible different cases:

1. If we have the sequence $(-1,0) \rightarrow (i,0)$, then we add a link just below the point to the left of the local section which is $(i-1,-1)$. As a result, this point is the one preceding the local section (see figure (a)).

2. If we have the sequence $(-1,0) \rightarrow (i+1,0)$, then we add a link just below the point to the right of the local section which is $(i+2,-1)$. As a result, this point is the one preceding the local section (see figure (b)).

3. If we have the sequence $(-1,0) \rightarrow (i,1) \rightarrow (i+1,e) \ (e \in \{-1,1,0\})$, then we add a link just below the point to the left of the local section which is $(i-1,-1)$. As a result, this point is the one preceding the local section (see figure (c)).

4. If we have the sequence $(-1,0) \rightarrow (i,1) \rightarrow (i-1,e) \ (e \in \{-1,1,0\})$, then we add two links. The first is just below the point to the left of the local section which is $(i-1,-1)$ and therefore will be the preceding of the local section sequence, and the second will be just below the point $i$ which is $(i,-1)$ (see figure (d)).

5. If we have the sequence $(-1,0) \rightarrow (i+1,1) \rightarrow (i+2,e) \ (e \in \{-1,1,0\})$, then we add two links. The first is just below the point to the right of the local section which is $(i+2,-1)$ and therefore will be the preceding of the local section sequence, and the second will be just below the point $i+1$ which is $(i+1,-1)$ (see figure (e)).

6. If we have the sequence $(-1,0) \rightarrow (i+1,1) \rightarrow (i,e) \ (e \in \{-1,1,0\})$, then we add a link just below the point to the right of the local section which is $(i+2,-1)$. As a result, this point is the one preceding the local section (see figure (f)).
Figure 12: Homotopical modifications of the elements of the g-base

This concludes the proof of the correctness of the PositiveHalfTwist(σ₁, g) and the NegativeHalfTwist(σ₁, g) functions. We still have to prove the correctness of the Reduce(g) function, and that it does not change the homotopy type of the
elements of the g-base. These proofs will make it possible to derive the uniqueness of the presentation.

### 3.2 Correctness of the Reduce\((g)\) algorithm

Here we will lay out the proof of the correctness of the `Reduce\((g)\)` algorithm. We will prove that the algorithm does not change the homotopy type of the elements of the g-base, and that it returns a list which represents the g-base in a unique form.

**Proposition 3.6** Let \(g\) be a list representing a g-base. Then, the list returned by the function `Reduce\((g)\)` represents a g-base which is homotopically equivalent to \(g\).

**Proof:** The `Reduce\((g)\)` algorithm is based on four reduction rules. We will present the rules and we will prove that each one of them preserves the homotopy type of \(g\).

1. If we have two consecutive equal links, we can omit them both.
2. If we have a sequence of \((i, \pm 1) \rightarrow (i, 0)\), we can omit the first link.
3. If we have links between \((i, 0)\) and \((-1, 0)\), we can omit them all.
4. If we have a sequence that starts with \((-1, 0)\) and continues to \((i, -1)\), we can omit the latter link.

Concerning the first rule, the meaning of the situation of two consecutive equal links, is that the path is moving above (or beneath) a point and immediately retracing back. Homotopically, this is equivalent to a point. Hence, we can omit the two links.

Concerning the second rule, the link \((i, \pm 1)\) represents a point which is directly above or below the point \((i, 0)\) and very close to it. Therefore, we result in a homotopic path after omitting the link \((i, \pm 1)\).

The third rule is trivial since any link that was added between the end point of one path \((i, 0)\) and the beginning point of the next path \((-1, 0)\) is not even a part of the g-base presentation and therefore has to be erased.
The fourth rule is based on the fact that the shortest path between two points is a straight line. Therefore, any point that we add can be omitted without changing the homotopy type of the g-base. We should remember that in our presentation of the paths of the g-base, we use the convention that the start point \((-1, 0)\) will always be connected directly to the point \((i, 0)\) or to the point above \((i, 1)\) but never to the point \((i, -1)\).

\section*{Lemma 3.7}

Let \(g\) be a representation of a g-base. Then \(g\) does not contain any sequence of the following type \((i - 1, e) \to (i, \pm 1) \to (i, \mp 1) \to (i + 1, e), e \in \{-1, 1, 0\}\).

\begin{proof}

We will prove it by induction. The initial g-base we have is the standard g-base, which does not contain any such sequence.

Now, at each step, when we add new links we connect them to links to the left or right of the local twisted section. This means that we will always connect the local section of the path to the other ends by a sequence that does not contain both \((i, -1)\) and \((i, 1)\), and therefore we will not create at any step a sequence that was forbidden by the lemma. Consider the fact that the change in the local section of the path is only a twist of 180°, then this twist will not add any forbidden sequences. Moreover, using the \textit{Reduce}(g) function, we eliminate any unnecessary links, resulting in the fact that each connection is the shortest possible (with regard to the convention that we will never connect the point \((-1, 0)\) to any \((i, -1)\)). Therefore, the \textit{Reduce}(g) function will not add any forbidden sequences. So by induction we proved the lemma.
\end{proof}

\section*{Corollary 3.8}

Let \(g\) be a representation of a g-base. Then the representation of the g-base obtained by \textit{Reduce}(g) is unique.

\begin{proof}

As stated above, by the correctness of the algorithms, and since \textit{Reduce}(g) will always connect two points by the shortest path without changing the homotopic type of the g-base, we obtain a unique representation of the g-base.
\end{proof}

This concludes the proof of the correctness of the algorithm.

\section*{Theorem 3.9}

The \textit{ProcessWord}(w) algorithm will result in a unique representation of the g-base after the action of the braid word \(w\) on the standard g-base.
4 Complexity

In this section, we will compute the complexity of the two functions that we use: 
(Positive/Negative)HalfTwist(σ_i, g) and Reduce(g).

Proposition 4.1 Let l_g be the length of the list representing a g-base. Then, 
the complexity of the function (Positive/Negative)HalfTwist(σ_i, g) is bounded 
by O(l_g).

Proof: The algorithm goes through all the links in the list, acting at most once 
on every link. As an upper bound, the algorithm might add two links for every 
link that was in the list, resulting in a list with length 3l_g. Therefore at most the 
algorithm will perform 4l_g operations, which yields in the complexity of O(l_g). □

Remark 4.2 In practice the number of links actually added is much smaller than 
the upper bound given above. This is the reason we have a practical and an efficient 
algorithm for short words.

Proposition 4.3 Let l_g be the length of the list representing a g-base. Then, the 
complexity of the function Reduce(g) is bounded by O(l_g).

Proof: This result is a consequence of the fact that the algorithm will check every 
link at most twice, and that each link that was inserted in the list can be extracted 
only once. We have to notice that there cannot be a situation of the following type 
(i, e) → (i + 1, e) → (i + 2, e) → (i + 2, e) → (i + 1, e) → (i, e), since after each 
step of inserting links we delete the unnecessary ones, and the fact that the local 
section is always two points wide. This fact is what makes it possible to make sure 
that in the worst case while going through the list, in order to delete unnecessary 
links, we will need to retrace only one step. That means that keeping in memory 
the link before the one we are currently checking is sufficient, and that each link 
can be passed at most twice; therefore, the complexity bound is O(l_g). □

Note that even if one wants to use a general half-twist which allows the local 
section to be larger than two points (with proper modification of the prefix and 
postfix sequences), obtaining the complexity of O(l_g) is still possible by using a 
doubly-connected list.
5 Conclusions

We would like to state here that although for very long braid words this algorithm’s running time is long, since the complexity of the g-base presentation grows with the length of the braid word, for short braid words we have obtained a quick algorithm in comparison with other methods. This is true because of the fact that Garside’s algorithm involves the replacement of the generators in a negative power by a subword of size $n(n-1)$, where $n$ is the number of strings in the braid group (although, for variations of his algorithm the size of the fundamental word $\Delta_n$ reduces [2]), and because each step of reduction in Dehornoy’s algorithm involves the insertion of at least two subwords of $O(n)$ length. In our algorithm for short braid words over a large number of strings we obtain a very short description of the g-base, that yields a very fast algorithm. This means that we have presented a very useful and practical algorithm. We also would like to point out that we have an implementation of the algorithm on a computer.

Instead of the restriction on the length of the braid word, one might consider a restriction on the size of the presentation of the g-base. Therefore by excluding braid word where the number of twists of the g-base’s paths grows dramatically, we still have an efficient algorithm even for longer braid words.

We would like to point out some of the future applications in which we believe that the new approach may help. We think that there has to be a connection between presentations of two conjugated braid words, what might bring a practical fast algorithm for solving the conjugacy problem in the braid group.

We believe that we can do the unprocess of the algorithm, which means to compute the braid word from a given g-base. Another thing is obvious when looking at the braid monodromy as a homomorphism from one fundamental group of a punctured disk to another. We believe that using this new method will make it easier to compute the braid monodromy automatically, at least for some of the cases.

Another implication of the algorithm and the new method for presenting the g-base is a similar implementation for the Moishezon-Teicher algorithm for computing the braid monodromy of real line arrangements and plane curves. For this implementation we have an efficient computer program.
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