A law of order estimation and leading-order terms for a family of averaged quantities on a multibaker chain system

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In this study a family of local quantities defined on each partition and its averaging on a macroscopic small region, site, are defined on a multibaker chain system. On its averaged quantities, a law of order estimation (LOE) in the bulk system is proved, making it possible to estimate the order of the quantities with respect to the representative partition scale parameter $\Delta$. Moreover, the form of the leading-order terms of the averaged quantities is obtained, and the form enables us to have the macroscopic quantity in the continuum limit, as $\Delta \to 0$, and to confirm its partitioning independency. These deliverables fully explain the numerical results obtained by Ishida, consistent with the irreversible thermodynamics.

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I. INTRODUCTION

It has been widely accepted that a coarse-grained entropy ($\varepsilon$-entropy) should be employed in order for expanding the nonequilibrium statistical mechanics based on the Gibbs entropy$^{1-18}$. The coarse graining or the partitioning of the phase space, often accomplished by a Markov partition, encounters the problem to find a local or coarse-grained form that recovers the nonequilibrium thermodynamics in the macroscopic or large system limit.

The change of the coarse-grained entropy on a macroscopic small region, e.g. site on a multibaker chain, has been decomposed into three terms: entropy flow, entropy flow due to a thermostat, and nonnegative entropy production$^1$, and its similar or extended formalism is introduced by Vollmer, Breymann, Mátyás, et al.$^{2-4,14-16,18}$ Although the entropy balance equation, based on the information theory, is confirmed to agree with the phenomenological one in the macroscopic limit, each term does not necessarily agree with a corresponding macroscopic one: the estimated entropy flow due to a thermostat is contaminated by a flux term and the contributions of convection and diffusion cannot be distinguished$^{19}$. The conventional decomposition inherently suffers from the need of an appropriate physical principle when finding a coarse-grained form corresponding to an arbitrary macroscopic quantity.

On the other hand, Ishida$^{6,19}$ introduced a completely different way to find local or coarse-grained terms. It can be interpreted as the decomposition, not of the quantity of the coarse-grained entropy change on a macroscopic small region, but of the time-evolution operator of local entropy density defined on each partition. In this approach, the decomposition is first performed on the level of a partition or a master equation, based on the symmetric properties of macroscopic entropy balance equation, and the local terms are identified. The properties are symmetry or anti-symmetry for the inversion of partition, density pairs and a given drift velocity. Next, a spatio-temporal averaging of the local quantities are performed on a macroscopic small region. The essence of the coarse graining of this method lies in the averaging, different from the conventional method putting emphasis on the partitioning of the phase space. In this approach the quantity defined on each partition is, therefore, appropriate to be called not coarse-grained quantity but local one. Finally, the macroscopic (continuum) limit, where the representative partition scale $\Delta$ goes to zero, are taken to compare with macroscopic quantities. On a volume-preserving$^6$ and a dissipative$^{19}$ multibaker chain systems, each term of the local entropy balance equation recovers the corresponding...
phenomenological term of irreversible thermodynamics. The mathematical procedure which
stands on the symmetry, in principle, is applicable to find a local, partition-level form of
an arbitrary macroscopic quantity, or to find the local balance equation corresponding to a
macroscopic one for an arbitrary quantity.

The procedure makes a family of local quantity which is symmetric or anti-symmetric for
some inversions and needs the property for the limiting behavior of its averaged quantity.
It is quantified by the order exponent $q$ of a given averaged quantity which behaves like
$O(\Delta^q)$ in the macroscopic limit. The order explains the macroscopic observability and the
scale separation of vanishing quantities: when $q$ equals zero, the averaged quantity of a
given local form has a limit value and it is macroscopically observable. In contrast, when
$q$ is positive the averaged quantity vanishes and it is not macroscopically observable. For
sufficiently large positive $q$, the quantity approaches more rapidly to zero in the limit, and
therefore the scale of such a quantity is well separated from the observable one. Moreover,
Ishida\textsuperscript{6} showed that the positivity of entropy production in the volume-preserving system
relies on the property of a local residual term and numerically confirmed that the order
exponent of its averaged quantity is positive. It is natural for us to consider that an isolated
system is governed by a volume-preserving dynamical system, and therefore, the behavior is
responsible for the law of increasing entropy. Thus, the exponent $q$ is physically essential.

In this study, a family of local quantities defined on each partition and its averaging
are first defined (Sec. II). Then, we shall prove the law of order estimation (LOE) for
the averaged values of the local quantities on the multibaker chain system (Sec. III). It
is applicable to a family of local quantity in the bulk system, and all of the important
orders mentioned above can be estimated. The estimation law is a slight extension originally
conjectured by Ishida\textsuperscript{19} from the numerical experiments on two multibaker chain systems\textsuperscript{6,19}.
Finally, we shall obtain the form of the leading-order terms of the averaged quantities in
Sec. IV. It is quite important because it is just the form of the macroscopic quantity when $q$
equals zero. In the coarse graining approach it is expected that these results are independent
of partitioning, and the leading form also enables us to discuss the independency that has
been numerically confirmed on the multibaker chain\textsuperscript{6,19}. 
II. AVERAGING OF A FAMILY OF LOCAL QUANTITIES

Ishida\textsuperscript{19} introduced a family of local quantities defined on each partition. They are found by the decomposition of the equations for local probability density and local entropy density, based on symmetric properties. When the local quantities is averaged on a macroscopic small region, Ishida numerically confirmed that the two decomposed equations recover the corresponding macroscopic balance equations of the irreversible thermodynamics.

Now let us consider a slightly extended family of local quantities, called $T$-type local quantities and its averaged value. In what follows, we utilize the notation of Ishida\textsuperscript{19}.

**Definition II.1** ($T$-type local quantity). Let $i,j$ be integers to identify the partition number of a Markov partition, and suppose that the transition volume $\tilde{W}_{ji}$ or $\tilde{W}_{ij}$ is nonzero, where $\tilde{W}_{ji} \equiv W_{ji} \Delta V_i$ by use of the transition probability $W_{ji}$ from $i$-th to $j$-th partition and the partition volume (Liouville measure) $\Delta V_i$ of the $i$-th partition. Let $e_i$, defined by $\Delta V_i / \sum_k \tilde{W}_{ik}$, denote the expansion rate on the $i$-th partition and $\rho^{(l)}_i$, defined by the probability measure $P^{(l)}_i$ on the $i$-th partition at the $l$-th step over the partition volume $\Delta V_i$, denote the local density. Suppose also that integers $\epsilon$ and $\delta$ are either 1 or -1 and that $f(x,y)$ is symmetric or antisymmetric for the interchange of $x$ and $y$. Then we define the following family $A^{(l)}_i$ of local quantity on the $i$-th (arbitrary) partition at the $l$-th time step, called a transitional type ($T$-type) local quantity:

\begin{align}
A^{(l)}_i & \equiv \sum_j A^{(l)}_{ij}, \quad \text{(1a)} \\
A^{(l)}_{ij} & \equiv \left[ (1 + \delta e_j) \tilde{W}_{ji} + \epsilon (1 + \delta e_i) \tilde{W}_{ij} \right] f(\rho^{(l)}_j, \rho^{(l)}_i). \quad \text{(1b)}
\end{align}

**Remark II.1.** The summand $A^{(l)}_{ij}$ is symmetric or antisymmetric for the interchange of partition pair $(i,j)$ and for the inversion of external parameter, such as the drift velocity $v$, for, at least, the multibaker chain system. The quantity is transitional because it is made by the decomposition of a transport equation associated with a master equation. As an arbitrary function $g(x,y)$ consists of symmetric and antisymmetric parts, the local quantity of the form $\sum_j W_{ij} g(\rho^{(l)}_j, \rho^{(l)}_i)$ can be expressed as the summation of $T$-type local quantities.

**Example II.2.** Set $\delta = \epsilon = -1$ and $f = (\rho^{(n)}_j + \rho^{(n)}_i)/4$. Then

\begin{align}
A^{(l)}_{ij} = \frac{\bar{U}_{j-i}^{[a]} \rho^{(l)}_j + \rho^{(l)}_i}{\Delta_{i,j}}, \quad \bar{U}_{j-i}^{[a]} & \equiv \frac{(1 - e_j) \tilde{W}_{ji} - (1 - e_i) \tilde{W}_{ij}}{2} \Delta_{i,j}, \quad \text{(2)}
\end{align}
where $\Delta_{i,j}$ is expected to reflect the distance between $i$-th and $j$-th partitions. This is a component of local probability density change\textsuperscript{19}.

Example II.3. The residual entropy source $r_{s,i}^{(l)}$ (Eq. (20f) of Ref. 19) can be expressed by the summation of some $T$-type quantities.

Hereafter, we also assume that the function $f$ is of class $C^\infty$ and has a nonnegative integer $m$, called a characteristic exponent, and a characteristic function $\hat{C}(x)$, defined as follows.

Definition II.2 (Characteristic exponent and characteristic function). If there exists the following power index $m$ and nonzero bounded function $\hat{C}(x)$:

$$\forall s, t(\neq s) \in \mathbb{R}, \exists m \in \mathbb{R}, \lim_{h \to 0} \frac{f(x + sh, x + th)}{(\frac{s-t}{2}h)^m} = \hat{C}(x),$$

then the exponent $m$ and the function $\hat{C}(x)$ are called a characteristic exponent and a characteristic function, respectively.

Remark II.4. Assume that a function $f$ of the variable $X(\equiv (x-y)/2)$ and $Y(\equiv (x+y)/2)$ is of class $C^1$ in the neighborhood of $X = 0$. Then the characteristic exponent $m$ can be expressed as:

$$m = \lim_{X \to 0} \frac{X}{f} \frac{\partial f(X, Y)}{\partial X}.$$

Definition II.3 ($T^1$-type local quantity). Assume that a function $f(x, y)$ to define a $T$-type local quantity is of class $C^\infty$ and that $f$ has a constant, nonnegative integer characteristic exponent $m$ and a characteristic function $\hat{C}(x)$. Then this type of local quantity $A_{ij}^{(l)}$ is called $T^1$-type, expressed by $A^{(l)}(f, m, \hat{C})$.

Remark II.5. It imply that the function $f$ can be expressed as

$$f(x, y) = \tilde{f}(X, Y) \equiv \tilde{f}\left(\frac{x-y}{2}, \frac{x+y}{2}\right) = \hat{C}(Y)X^m + C^{(m+2)}(Y)X^{m+2} + C^{(m+4)}(Y)X^{m+4} + \cdots,$$

in a neighborhood of $X = 0$.

Remark II.6. When the absolute value of $X$ is sufficiently small, $f$ can be well approximated by $\hat{C}(Y)X^m$. Replacing $f$ in Eq. (1b) with $\hat{C}(Y)X^m$ and comparing the local quantity $A_{ij}^{(n)}$ with Table 1 in Ref. 19, we are easily convinced that the quantity of the present study is an extension in the sense that the symmetric part $\hat{C}(Y) (Y \equiv (\rho_j^{(l)} + \rho_i^{(l)}/2)$ is not restricted to the product of the power of two symmetric order components $Y$ and $\ln Y$. 

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For the above family of local quantity $A_i^{(l)}$, Eq. (1b), we can introduce its averaged quantity $\langle A \rangle_R^{(l)}$ on a macroscopic small region $R$.  

**Definition II.4** (Spatio-temporal averaging on $R$). Let $R$ denote a union of partitions of a Markov partition, $\Delta V_R$ the volume of the region $R$, $\tau$ timely increment (time step) of a given map. For a local quantity $A_i^{(l)}$, defined on the $i$-th partition at the $l$-th step, we can define its spatio-temporal averaging on $R$ as follows:

$$\langle A \rangle_R^{(l)} = \frac{1}{\tau \Delta V_R} \sum_{i \subset R} A_i^{(l)}, \quad (5)$$

where $i \subset R$ means that the quantity $A_i^{(l)}$ is summed with respect to all the partitions included in the region $R$.

In the next section we see that a law of order estimation (LOE) holds for the averaged quantity in the bulk of a multibaker chain system.

**III. LAW OF ORDER ESTIMATION (LOE)**

Hereafter we shall confine our discussion to a triadic multibaker chain, introduced by Vollmer et al.\textsuperscript{2-5} and Ishida.\textsuperscript{6,19}

**Definition III.1** (Multibaker-chain map). Let $R_m \equiv [0, N] \times [0, \Delta r] \times [0, h]$ be a domain defined on $\mathbb{Z} \times \mathbb{R}^2$. Consider the following map $T: R_m \to R_m$, called a multibaker-chain map. For the case of $0 < n < N$,

$$T(n, x, y) = \begin{cases} 
(n - 1, x / \eta_n(0), \nu_n(0)y), & 0 \leq x < \eta_n(0)\Delta r, \\
(n, (x - \eta_n(0)\Delta r) / \eta_n(1), \nu_n(0)h + \nu_n(1)y), & \eta_n(0)\Delta r \leq x < (\eta_n(0) + \eta_n(1))\Delta r, \\
(n + 1, [x - (\eta_n(0) + \eta_n(1))\Delta r] / \eta_n(2), (\eta_n(0) + \eta_n(1))\Delta r + \nu_n(2)y), & \eta_n(0) + \eta_n(1)\Delta r \leq x \leq \Delta r,
\end{cases} \quad (6)$$
and an appropriate map is given for the case of \( n = 0 \) or \( N \). Herein \((n,x,y)\) is regarded as a position \((x,y)\) on the \( n \)th site, \( A_n \subset \mathbb{R}^2 \), and \( A_0 \) and \( A_N \) is regarded as two boundary sites. The invertible condition\(^2\)^\(^3\)^\(^19\) yields \( \nu_n(\omega) = \eta_n(2 - \omega) \).

**Remark III.1.** By convention, the domain \( R_m \) is regarded as a one-dimensional chain of \((N+1)\) rectangular sites, and therefore, the map is called a multibaker chain system. Each site has the origin fixed at the lower left corner and the coordinates \( x \) and \( y \) are the horizontal and vertical direction, respectively. We shall take \( r \) direction in which two disjoint sites connect, and the width \( \Delta r \) is treated as the distance between the two sites.

In what follows, a restricted type of multibaker chain system is dealt with.

**Definition III.2** (UOFP-type multibaker-chain map). Consider the following transient probability \( \eta_n(\omega) \) on each site \( A_n \) that is uniform except the two boundary sites.

\[
\eta_n(0) = \frac{\beta}{2}(1 - \frac{Pe_g}{2}), \quad \eta_n(2) = \frac{\beta}{2}(1 + \frac{Pe_g}{2}), \quad \eta_n(1) = 1 - \beta,
\]

where \( \beta \equiv \frac{2 \tau D}{\Delta r^2} \), and \( D \) denotes a diffusion coefficient, \( \tau \) a time step. \( Pe_g(\equiv v \Delta r / D) \) is the so-called grid Péclet number\(^2\)^\(^0\). The condition that \( \eta(\omega) \) should be positive leads to the order of the time step \( \tau \) must be greater than or equal to the order of \( \Delta r^2 \), and Ishida\(^6\)^\(^19\) fixed the order at \( \Delta r^2 \). In other words, the order exponent of \( \beta \) with respect to \( \Delta r \) must be nonnegative. At the boundary sites \( n=0 \) and \( N \), other transient probabilities are given, determining the macroscopic boundary condition. Once all of the transition probabilities are given, a Markov partition \((n, \omega_k)\) on the \( n \)th site, the partition volume \( \Delta V(n, \omega_k) \) on the partition, and the probability measure \( P^{(l)}(n, \omega_k) \) at the \( l \)th step are defined, where \( \omega_k(\equiv \omega_0 \omega_1 \cdots \omega_{k-1}) \) is a \( k \)-digit trit number specifying a partition of width \( \Delta r \) by height \( h \times \nu_n(\omega_d) \), forming a line in its numerical order from bottom to top on each site\(^19\). The “bulk” of this system is defined by the sites \( A_n \) for \( 0 < n - k < n < n + k < N \) and time step \( l > k \).

In this system, the limit of the partition scale \( \Delta \equiv \Delta r \rightarrow 0 \) with the total length \( L \equiv N \Delta r \) and trit number \( k \) fixed defines a continuum limit. In this limit the averaged equation for \( P^{(l)}(n, \omega_k) \) on a bulk site \( A_n \) recovers the one-dimensional Fokker-Planck equation for macroscopic (averaged) measure density \( \rho^{19} \). In addition, the distance between the bulk domain composed by all bulk sites and the boundary sites approaches to zero. Hereafter, the multibaker-chain map (system) mentioned above is called a uniform, one-dimensional Fokker-Planck (UOFP) type.
Remark III.2. The type is independent of the dynamics on the boundary sites.

The definition of the bulk system for the UOFP-type multibaker chain allows us to express the local density on each partition by

\[ \rho_{l}(n, \omega_{k}) \equiv \frac{P_{l}(n, \omega_{k})}{\Delta V(n, \omega_{k})} = \frac{\eta_{n+1-\omega_{0}}(\omega_{0})P_{l-1}(n+1-\omega_{0}, \omega_{k-1} \leftarrow \omega_{0})}{\nu_{n}(\omega_{0})\Delta V(n+1-\omega_{0}, \omega_{k-1} \leftarrow \omega_{0})} = \frac{\eta(\omega_{0})}{\nu(\omega_{0})}\rho_{l-1}(n+1-\omega_{0}, \omega_{k-1} \leftarrow \omega_{0}), \quad (8) \]

where \(\omega_{k-1} \leftarrow \omega_{0} \cdots \omega_{1} \leftarrow \omega_{0}\) and this is the so-called left shift of symbolic dynamics. Please note herein that the volume expansion rate \(e(n, \omega_{k})\), i.e. the reciprocal of volume contraction rate, at a partition \((n, \omega_{k})\) can be written as

\[ e(n, \omega_{k}) = \frac{\Delta V(n, \omega_{k})}{\eta_{n+1-\omega_{0}}(\omega_{0})\Delta V(n+1-\omega_{0}, \omega_{k-1} \leftarrow \omega_{0})} = \frac{\nu_{n}(\omega_{0})}{\eta_{n+1-\omega_{0}}(\omega_{0})} \quad (9) \]

from the definition of expansion rate (Def. II.1). It is essential that the relation (8) holds recursively and the product of the contraction rates is independent of the site number \(n\) while the \(n\)th site considered is in the bulk. Conversely, if the \(n\)th site is outside the bulk, the product must be contaminated by the discontinuous contraction rate given at the boundary sites, and Ishida\(^{19}\) has numerically confirmed that the LOE, proved below, is not fulfilled for such non-bulk sites near the boundaries. This is the onset of boundary effects\(^{14,19}\). In what follows, the subscripts of \(\eta\) and \(\nu\) are omitted because these transient probabilities are independent of the site number \(n\) in the bulk system.

Substituting Eqs. (8) and (9) into Eq. (1b), we obtain the following form of the averaged quantity \(\langle A \rangle_{l}^{(f)}\), Eq. (5), on a bulk site \(R = A_{n}\) for a \(T^{1}\)-type local quantity \(A^{(f)}(f, m, \hat{C})\) as follows:

\[ \langle A \rangle_{l}^{(f)}(n) \equiv K/(\tau \Delta r), \quad (10a) \]
where

\[
K = \sum_{\omega} \sum_{\omega'} \sum_{\omega_k} \left[ (\eta(\omega') + \delta \nu(\omega')) \nu(\omega_k) \Delta r \right. \\
\times f \left( \rho^{(l)}(n + \omega' - 1, \omega_k), \rho^{(l)}(n, \omega_k) \right) \\
+ \epsilon (\eta(\omega_0) + \delta \nu(\omega_0)) \nu(\omega_{k-1}) \Delta r \\
\times f \left( \rho^{(l)}(n + 1 - \omega_0, \omega_k), \rho^{(l)}(n, \omega_k) \right) \\
= \sum_{\omega} \sum_{\omega_k} (\eta(\omega) + \delta \nu(\omega)) \nu(\omega_k) F_n^{(l)}(\epsilon, m, \omega, \omega_k) \Delta r, \\
\]

and

\[
F_n^{(l)}(\epsilon, m, \omega, \omega_k) \equiv F_n^{(l)}(\epsilon, \omega, \omega_k) + \epsilon (-1)^m F_{n+1-\omega}(\omega, \omega_k), \\
F_n^{(l)}(\omega, \omega_k) \equiv f \left( \frac{\eta(\omega)}{\nu(\omega)} \rho^{(l-1)}(n, \omega_{k-1}) \\
+ \frac{\eta(\omega_0)}{\nu(\omega_0)} \rho^{(l-1)}(n + 1 - \omega_0, \omega_{k-1}) \right). \\
\]

The following lemma plays essential role for estimating the order of \( K \) or \( \langle A \rangle_{A_n}^{(l)} \).

**Lemma III.3.** The order exponent of \( K/(\beta \Delta r) \) for \( m \neq 0 \) or \( K/\Delta r \) for \( m = 0 \) with respect to \( \Delta r \) is even number, determined by the property of the summation with respect to only two trinary bits \( \omega \) and \( \omega_0 \).

**Proof.** Firstly, we define the following function of an integer \( a \):

\[
S_{\nu}(a) \equiv \begin{cases} 
1, & a = 0, \\
\frac{3 Pe_{a}}{2}, & a : odd, \\
\beta, & a(\neq 0) : even,
\end{cases}
\]

and it is originated from the quantity \( \sum_{\omega} \nu(\omega) I_{\omega}^{a} \), where \( I_{\omega} \equiv 1-\omega \). For \( a \neq 0 \), the definition of bulk transient probability \( (7) \) and the invertible condition \( (\text{Def. II.1}) \) have the quantity agree with \( S_{\nu}(a) \). The definition at \( a = 0 \) is useful to express the summation with respect to \( \omega \) when the summand does not include the form \( I_{\omega}^{a} \) of \( a \neq 0 \). Herein, it is important to note that the function increases its order by one with respect to \( \Delta \) when \( a \) changes from an even number to an odd one because the order of \( Pe_{a} \) is \( O(\Delta) \). In addition, we define the following function related to the above function

\[
S_{\nu\nu}(\delta, a) \equiv (\delta + (-1)^a)S_{\nu}(a) \sim \sum_{\omega} (\eta(\omega) + \delta \nu(\omega)) I_{\omega}^{a}, \]

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and, therefore, the function inherit the order-increase property.

On the other hand, when the partition scale $\Delta r$ is sufficiently small we have the following relations:

$$\frac{\eta(\omega)}{\nu(\omega)} = 1 + \sum_{n=1}^{\infty} \left(-1\right)^n \frac{(Pe_g I_\omega)^n}{2^{n-1}},$$

(13a)

$$\rho^{(l-1)}(n + 1 - \omega_0, \tilde{w}_{k-1}) = \rho^{(l-1)}(n, \tilde{w}_{k-1}) + \left(\frac{\partial \rho}{\partial r}\right)^{(l-1)}(n, \tilde{w}_{k-1})(I_{\omega_0} \Delta r)^1 + O^{(l-1)}((I_{\omega_0} \Delta r)^2),$$

(13b)

$$\sum_{\tilde{w}_{k-1}} \nu(\tilde{w}_{k-1}) g\left(A_n^{(l-1)}(\tilde{w}_{k-1})\right) = g(A_n^{(l)}) + O(\Delta r),$$

(13c)

where $A_n^{(l)}(\tilde{w}_{k})$ is a family of quantities that fulfills recursively the following relation

$$A_n^{(l)}(\tilde{w}_{k}) = \frac{\eta(\omega_0)}{\nu(\omega_0)} A_n^{(l-1)}_{n+1-\omega_0}(\tilde{w}_{k-1}),$$

(14)

such as $\rho^{(l-1)}(n, \tilde{w}_{k-1})$ and $(\partial \rho/\partial r)^{(l-1)}(n, \tilde{w}_{k-1})$. Herein, $A_n^{(l)}$, defined by $A_n^{(l)}(\omega_0)$, is a family of quantities given on the $n$th site.

It is also worth noting that the scale parameter $\Delta r$ or $Pe_g$ in Eq. (13) is accompanied by $I_\omega$ or $I_{\omega_0}$. It follows that odd-ordered terms with respect to $\Delta r$ in the summand of $F_n^{(l)}$, Eq. (10c), increases their order by one through the operations (11) and (12). Consequently, the order exponent of $K/(\beta \Delta r)$ for the case of $m \neq 0$ or $K/\Delta r$ for $m = 0$ must be even number. Note herein that the coefficient $\beta$ does not appear for the case of $m = 0$ because $S_\nu(0) = 1$. In addition, Eq. (13c) shows that the summation with respect to $\tilde{w}_{k-1}$ is order one ($O(\Delta^0)$) and does not affect the order of $K$. That’s why we can estimate the order of $K$ or $(A_n^{(l)})_{A_n}$ from the properties of the summation with respect to only two trinary bits $\omega$ and $\omega_0$. 

**Remark III.4.** Eqs. (3), (13) and (14) show that the leading-order terms of $F_n^{(l)}(\omega, \tilde{w}_{k})$, Eq. (10a), is $O(\Delta^m)$ and that the leading terms are identical to that of $F_{n+1-\omega}(\omega, \tilde{w}_{k})$. Thus the leading-order term of $F_n^{(l)}$, Eq. (10a), has the coefficient of the quantity $1 + \epsilon(-1)^m$.

**Remark III.5.** In Eq. (13b), the term $(\partial \rho/\partial r)^{(l-1)}(n, \tilde{w}_{k-1})$ is merely a formal expression based on the Taylor expansion. From the recursive definition (5), it is defined if and only if the derivative of $\rho_n^{(l)}(\omega_0)$ with respect to $r$ at $r = n \Delta r$ is defined in the continuum limit, $\Delta r \to 0$. The property fully depends on the appropriateness of both the macroscopic small region $A_n$ and the transient probabilities.
Now we are in a position to prove the following law of order estimation that enable us to estimate the order of the averaged quantity $\langle A\rangle^{(l)}_{A_n}$.

**Theorem III.6 (LOE: Law of order estimation).** Consider the averaged value $\langle A\rangle^{(l)}_{A_n}$ of a $T^1$-type local quantity $A^{(l)}(f, m, \hat{C})$ for a UOFP-type multibaker chain system partitioned by a Markov partition $(n, \omega_k)$. The order of $\langle A\rangle^{(l)}_{A_n}$ can be expressed as follows:

(Case I): If the characteristic exponent $m$ is an even number, then

$$\langle A\rangle^{(l)}_{A_n} = \frac{K}{\tau \Delta r} \sim \begin{cases} O(\tau^{-1}), & m = 0 \text{ and } \epsilon = \delta = 1, \\ O(\Delta r^{m-2}), & m > 0 \text{ and } \epsilon = \delta = 1, \\ O(\Delta r^m), & \text{elsewhere.} \end{cases} \quad (15a)$$

(Case II): If $m$ is an odd number, then

$$\langle A\rangle^{(l)}_{A_n} = \frac{K}{\tau \Delta r} \sim \begin{cases} O(\Delta r^{m-1}), & \epsilon = -1 \text{ or } \delta = 1, \\ O(\Delta r^{m+1}), & \text{elsewhere.} \end{cases} \quad (15b)$$

**Proof. (Case I)**

If $f$ is symmetric for the interchange of $x$ and $y$, i.e. $m$ is an even number, the order of $K$ depends on $\epsilon$ and $\delta$, refined two cases: (I-1) $\epsilon = \delta = 1$ and (I-2) elsewhere. For the former case, $1 + \epsilon(-1)^m \neq 0$ and $\delta + 1 \neq 0$. From Lemma III.3, therefore, $m$th order terms in $F^{(l)}_{n}(\omega, \omega_k)$ with the coefficient of $I^{a}_{\omega} I^{b}_{\omega_0}$, such that $2(a + b) = m$ for two integers $a$ and $b$, determines the leading order of $K$. Considering the operations (11) and (12), we obtain

$$K \sim \begin{cases} \Delta r, & m = 0, \\ \Delta r^{\beta} \Delta r^m, & m > 0. \end{cases}$$

For the case of $m \neq 0$, $K$ can have some terms with the coefficient of $\beta^2$ for the case of $a \neq 0$ and $b \neq 0$. Note that, however, such terms cannot be, in general, the leading-order terms because $\beta$ has the nonnegative order exponent (cf. Def. III.2). They are the leading terms if and only if $\beta$ is order one. That is why $K$ can be expressed as above mentioned.

For the latter case (I-2), the $m$th order terms vanish. The next $(m+1)$th order terms have the coefficient of $I^{a}_{\omega} I^{b}_{\omega_0}$ and one of the integer power indices $a$ and $b$ is positive odd number,
occurring the order increase with the coefficient $\beta$ (Lemma III.3). This results in the same order as the $(m+2)$th order terms with the coefficient of $I^2\omega I^2\omega$, such that $2(a + b) = m + 2$. Therefore,

$$K \sim \Delta r \beta \Delta r^{m+2}.$$ 

The order of averaged quantity $\langle A \rangle_{A_n}$, Eq. (10a), for the Case I, Eq. (15a), is estimated from these results.

(Case II)

If $f$ is antisymmetric, i.e. $m$ is an odd number, the order of $K$ also depends on $\epsilon$ and $\delta$, refined two cases: (II-1) $\epsilon = -1$ or $\delta = 1$ and (II-2) elsewhere. For the former case, $1 + \epsilon(-1)^m \neq 0$ or $\delta + 1 \neq 0$. When $1 + \epsilon(-1)^m \neq 0$, the $m$th order terms in $F_n(l, \omega, \omega_k)$ remains. But the order increases by one through the operations (11) and (12). The leading terms have the coefficient of $\beta$ because $m$ is positive. When $1 + \epsilon(-1)^m = 0$ and $\delta + 1 \neq 0$, the $m$th order terms vanish. But the next $(m+1)$th order terms have the coefficient of $I^2\omega I^2\omega$, such that $2(a + b) = m + 1$, remaining as the $(m+1)$th order terms with the coefficient $\beta(\delta + 1)$. As a result,

$$K \sim \Delta r \beta \Delta r^{m+1}.$$ 

For the latter case (II-2), the $(m+1)$th order terms vanish. The next $(m+2)$th order terms have the coefficient of $I^2\omega I^2\omega$ and one of the integer power indices $a$ and $b$ is positive odd number, occurring again the order increase with the coefficient $\beta$. This results in the same order as the $(m+3)$th order terms with the coefficient of $I^2\omega I^2\omega$, such that $2(a + b) = m + 3$. For the case, therefore,

$$K \sim \Delta r \beta \Delta r^{m+3}.$$ 

As a result, the order of averaged quantity $\langle A \rangle_{A_n}$ for the Case II is estimated as Eq. (15b).

Corollary III.7. If the order of $\tau$ is $O(\Delta r^2)$ and the symmetric part $\hat{C}(Y)$ ($Y \equiv (\rho_j^{(l)} + \rho_i^{(l)})/2$) is restricted to the product of the power of two symmetric order components $Y$ and $\ln Y$, the estimated orders (15) is the statement of the LOE conjectured by Ishido. The LOE proved here is an extension of the empirical LOE.
Remark III.8. The resultant order exponent is always even number with an exception for the case that $m=0$ and $\epsilon=\delta=1$. Therefore, the vanishing-order terms are more than or equal to the second order, and the property plays a role in well separating convergent or macroscopic $O(1)$ terms from them. Such a behavior has been numerically confirmed in residual terms, such as residual entropy source (see below), residual entropy flux\textsuperscript{6,19}.

Example III.9. For the case of the quantity (2), characteristic exponent $m=0$ but $\epsilon=\delta=-1$ and, therefore, its averaged quantity is $O(1)$. This is a component of the macroscopic density change\textsuperscript{19}.

Example III.10. The averaged quantity of the residual entropy source (Example II.3) vanishes in the continuum limit because its order is estimated to be $O(\Delta r^2)$. This is responsible for the positive entropy production in the volume-preserving system, i.e. for the law of increasing entropy\textsuperscript{6,19}.

IV. LEADING-ORDER TERMS OF AN AVERAGED QUANTITY

The proof of the LOE, mentioned above, can be regarded as the introduction for deriving the form of the leading-order terms in the bulk system. Successively, we shall obtain the specific form of the leading terms of the averaged value $\langle A \rangle^{(l)}_{A_n}$ of a $T^1$-type local quantity $A^{(l)}(f, m, \hat{C})$ for a UOFP-type multibaker chain.

First of all, we shall define some variables and abridged notations as follows:

$$x_1 \equiv \frac{\eta(\omega)}{\nu(\omega)} \rho^{(l-1)}(n, \omega_{k-1})$$

$$x_2 \equiv \frac{\eta(\omega)}{\nu(\omega)} \rho^{(l-1)}(n+1-\omega, \omega_{k-1})$$

$$y_1 \equiv \frac{\eta(\omega_0)}{\nu(\omega_0)} \rho^{(l-1)}(n+1-\omega_0, \omega_{k-1})$$

$$y_2 \equiv \frac{\eta(\omega_0)}{\nu(\omega_0)} \rho^{(l-1)}(n+1-\omega_0+1-\omega, \omega_{k-1})$$

and

$$X_i \equiv \frac{x_i - y_i}{2}, \quad Y_i \equiv \frac{x_i + y_i}{2}, \quad (i = 1 \text{ or } 2).$$

In this section, we shall also utilize the abridged notation for the quantity $\hat{A}_n^{(l)} \equiv A_n^{(l-1)}(\omega_{k-1})$, defined in Eq. (14). For example, $\rho^{(l-1)}(n, \omega_{k-1})$, $(\partial \rho / \partial r)^{(l-1)}(n, \omega_{k-1})$, and $(\partial^2 \rho / \partial r^2)^{(l-1)}(n, \omega_{k-1})$. 

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are abridged to $\rho_n^{(l)}$, $\rho_r^{(l)}$ and $(\rho_{rr})_n^{(l)}$, respectively. Then $X_i$ and $Y_i$ can be written in the following power series:

\[
X_i = \sum_{p,q\geq 0} (-1)^{p+q} \mathcal{X}_{p,q}^{(i)} \left( \frac{I_\omega \Delta r}{2} \right)^p \left( \frac{I_{\omega_0} \Delta r}{2} \right)^q \\
= \sum_{p\geq 0} \mathcal{X}_p^{(i)}(\omega, \omega_0)(\Delta r/2)^p, \tag{16a}
\]

\[
Y_i = \sum_{p,q\geq 0} (-1)^{p+q} \mathcal{Y}_{p,q}^{(i)} \left( \frac{I_\omega \Delta r}{2} \right)^p \left( \frac{I_{\omega_0} \Delta r}{2} \right)^q \\
= \sum_{p\geq 0} \mathcal{Y}_p^{(i)}(\omega, \omega_0)(\Delta r/2)^p, \tag{16b}
\]

where

\[
\mathcal{X}_p^{(i)}(\omega, \omega_0) \equiv (-1)^p \sum_{q=0}^p \mathcal{X}_{p-q,q}^{(i)} I_\omega^{-q} I_{\omega_0}^q,
\]

and $\mathcal{Y}_p^{(i)}(\omega, \omega_0)$ is similarly defined. We can determine the coefficients $\mathcal{X}_{p,q}^{(i)}$ and $\mathcal{Y}_{p,q}^{(i)}$ from Eqs. (13a) and (13b), and typical ones are shown in Table. I. From the table, we can find that $\mathcal{X}_1^{(i)}(\omega, \omega_0)$ is independent of $i$, and hereafter the superscript ($i$) is omitted. Furthermore, $\hat{C}$ is regarded as the function of $\rho$ because $Y_i$ converges to $\rho_n^{(l)}$ in the continuum limit, $\Delta r \to 0$, and, for example, we shall denote $(\partial \hat{C}/\partial \rho)|_{\rho=\rho_n^{(l)}}$ by $\hat{C}_\rho(\rho_n^{(l)})$.

The following lemma is crucial for obtaining the leading-order terms of $\langle A \rangle_{A_n}^{(l)}$.

\textbf{Lemma IV.1.} $F_n^{(l)}$, Eq. (10a), has the following leading terms with respect to $\Delta r$. This is the necessary and sufficient expansion form in order to find the leading-order terms of $\langle A \rangle_{A_n}^{(l)}$.
| $(p, q)$ | $\mathcal{X}^{(1)}_{p,q}$ | $\mathcal{X}^{(2)}_{p,q}$ | $\mathcal{Y}^{(1)}_{p,q}$ | $\mathcal{Y}^{(2)}_{p,q}$ |
|----------|----------------|----------------|----------------|----------------|
| $(0, 0)$  | 0               | 0              | $\frac{v}{p} \rho^{(l)}_n$ | $\frac{v}{p} \rho^{(l)}_n$ |
| $(1, 0)$  | $\frac{v}{p} \rho^{(l)}_n$ | $\mathcal{X}^{(1)}_{1,0}$ | $\mathcal{X}^{(1)}_{1,0}$ | $\frac{v}{p} \rho^{(l)}_n$ |
| $(0, 1)$  | $\frac{v}{p} \rho^{(l)}_n$ | $\mathcal{X}^{(1)}_{0,1}$ | $\mathcal{X}^{(1)}_{0,1}$ | $\mathcal{Y}^{(1)}_{0,1}$ |
| $(2, 0)$  | $(\frac{v}{D})^2 \rho^{(l)}_n$ | $(\frac{v}{D})^2 \rho^{(l)}_n$ | $\mathcal{X}^{(1)}_{2,0}$ | $\mathcal{X}^{(1)}_{2,0}$ |
| $(0, 2)$  | $\frac{2v}{D} (\rho_r)_n$ | $\mathcal{X}^{(1)}_{0,2}$ | $\mathcal{X}^{(1)}_{0,2}$ | $\mathcal{Y}^{(1)}_{0,2}$ |
| $(3, 0)$  | $(\frac{v}{D})^3 \rho^{(l)}_n$ | $(\frac{v}{D})^3 \rho^{(l)}_n$ | $\mathcal{X}^{(1)}_{3,0}$ | $\mathcal{X}^{(1)}_{3,0}$ |
| $(2, 1)$  | $-2 (\frac{v}{D})^2 (\rho_r)_n$ | $\mathcal{X}^{(1)}_{2,1}$ | $\mathcal{X}^{(1)}_{2,1}$ | $\mathcal{Y}^{(1)}_{2,1}$ |
| $(1, 2)$  | $2 \frac{v}{D} (\rho_r)_n$ | $\mathcal{X}^{(1)}_{1,2}$ | $\mathcal{X}^{(1)}_{1,2}$ | $\mathcal{Y}^{(1)}_{1,2}$ |
| $(0, 3)$  | $\frac{v}{D}^3 \rho^{(l)}_n$ | $\mathcal{X}^{(1)}_{0,3}$ | $\mathcal{X}^{(1)}_{0,3}$ | $\mathcal{Y}^{(1)}_{0,3}$ |
\[ \mathcal{F}_n^{(l)}(\epsilon, m, \omega, \omega_k) = (1 + \epsilon(-1)^m) \Delta X_1(\omega, \omega_0)^m \hat{C}'(\hat{\rho}_n^{(l)}) \left( \frac{\Delta r}{2} \right)^m + \left[ \Delta X_1(\omega, \omega_0)^m \hat{C}_\rho(\hat{\rho}_n^{(l)}) \left( \Delta Y_1^{(l)}(\omega, \omega_0) + \epsilon(-1)^m \Delta Y_1^{(2)}(\omega, \omega_0) \right) \right] \left( \frac{\Delta r}{2} \right)^{m+1} + m \Delta X_1(\omega, \omega_0)^{m-1} \hat{C}_\rho(\hat{\rho}_n^{(l)}) \left( \Delta Y_1^{(l)}(\omega, \omega_0) + \epsilon(-1)^m \Delta Y_1^{(2)}(\omega, \omega_0) \right) \right] \left( \frac{\Delta r}{2} \right)^{m+1} + \left( \frac{m}{2} \right) \Delta X_1(\omega, \omega_0)^{m-2} \hat{C}_\rho(\hat{\rho}_n^{(l)}) \left( \Delta Y_1^{(l)}(\omega, \omega_0) \right)^2 + \epsilon(-1)^m \Delta X_2^{(2)}(\omega, \omega_0) \left( \Delta Y_1^{(2)}(\omega, \omega_0) \right)^2 \right) \left( \frac{\Delta r}{2} \right)^{m+2} + O(\Delta r^{m+3}). \tag{17} \]

**Proof.** First of all, we should notice that higher-order terms in Eq. (4), i.e. \( C^{(m+2)}(Y)X^{m+2} \), \( C^{(m+4)}(Y)X^{m+4} \), \( \cdots \), does not affect the leading-order terms of \( \langle A \rangle_\Delta^{(l)} \). From the term \( C^{(m+2)}(Y)X^{m+2} \), for example, \( (m+2) \)th order terms appears. If the leading \( m \)th order terms, originated from the term \( \hat{C}(Y)X^m \), vanish, the \( (m+2) \)th order terms also do because

\[ 1 + \epsilon(-1)^m = 1 + \epsilon(-1)^{m+2} (= 1 + \epsilon(-1)^{m+4} = \cdots). \]

Similarly, when the leading-order terms of \( \hat{C}(Y)X^m \) increase its order through the operations (II) and (II), those of \( C^{(m+2)}(Y)X^{m+2} \) also do because the same parameters of \( \epsilon \) and \( \delta \) are shared. Therefore, the difference of order exponents between the two leading-order terms originated from \( \hat{C}(Y)X^m \) and \( C^{(m+2)}(Y)X^{m+2} \) is always kept constant of two. Similar discussion can be applied to the leading terms originated from the order-disjoint two terms \( C^{(m+2a)}(Y)X^{m+2a} \) and \( C^{(m+2a+2)}(Y)X^{m+2a+2} \). Consequently, the order exponents of these leading terms are equally spaced, with an interval of two. That is why the function \( f \) can be replaced by \( \hat{C}(Y)X^m \) in order to obtain the specific leading terms of \( \langle A \rangle_\Delta^{(l)} \). Replacing \( f \) with \( \hat{C}(Y)X^m \) and substituting Eq. (16) into \( \mathcal{F}_n^{(l)} \), Eq. (11c), yields Eq. (17).

As described above we should treat \( (m+3) \)th order terms for the case (II-2), which is the same order as the higher-order term \( O(\Delta r^{m+3}) \) in Eq. (17). In this case, however, the
operations (11) and (12) on the higher-order term make the same order terms vanish because
of the coefficient \((δ + 1)\). That is why the expansion (17) is necessary and sufficient to find
the leading-order terms of \(K\) or \(\langle A \rangle_{An}\).

Corollary IV.2. The LOE (Theorem III.6 or Eq. (15)) immediately follows from the
Lemma IV.1 (or Eq. (17)) and properties (11), (12) and (13c).

Now we are in a position to have the leading-order terms of the avera ged quantity \(\langle A \rangle_{An}\).

In what follows, some variables are defined as follows

\[
X^{(1)}_{1,0} = X^{(2)}_{1,0} \equiv \frac{v}{D} \rho^{(l)}_n \equiv X_{1,0},
\]

\[
X^{(1)}_{0,1} = X^{(2)}_{0,1} \equiv -\frac{v}{D} \rho^{(l)}_n + (\rho_r)^{(l)}_n \equiv X_{0,1},
\]

and this is the replacement of \(\frac{v}{\rho^{(l)}_n}\) and \(\frac{v}{(\rho_r)^{(l)}_n}\) in \(\frac{X^{(1)}}{X^{(2)}_{0,1}}\) or \(\frac{X^{(1)}}{X^{(2)}_{1,0}}\) (cf. Table. I) with \(\rho^{(l)}_n\)
and \((\rho_r)^{(l)}_n\), respectively. Other quantities \(X^{(i)}_{p,q}\) and \(Y^{(i)}_{p,q}\) are similarly defined. These operations are originated from the property (13c). \(\rho^{(l)}_n\) and \((\rho_r)^{(l)}_n\) are identified with \(\rho(r,t)\) and \((\partial \rho / \partial r)(r,t)\), respectively, when we regard these quantities as those defined at a position \(r = n Δr\) and time \(t = lτ\).

In addition, we assume that \(\sum_{p=a}^b A_p = 0\) when \(b < a\). The leading terms often include
the form of \([((A)^{(1)} + \epsilon(-1)^m (A)^{(2)})]\) (see Case (II-1) below). In what follows, therefore, only
the first part \((A)^{(1)}\) is described and the second counterpart is often abridged to \((A)^{(2)}\).

Theorem IV.3 (Leading-order form). Consider the averaged value \(\langle A \rangle_{An}\) of a \(T_1\)-type
local quantity \(A^{(l)}(f,m,\hat{C})\) for a UOFP type multibaker chain system partitioned by a Markov
partition \((n,ω_k)\). Its leading-order terms have the following form:

Case (I-1): (m-2)th order term

For the case of \(m=0\)

\[
\frac{K}{τΔr} \cong \frac{4 \hat{C}(\rho^{(l)}_n)}{τ},
\]

and for the case \(m = 2m' (m' > 0)\)

\[
\frac{K}{τΔr} \cong \frac{D \hat{C}(\rho^{(l)}_n)}{2^{m-3}} \left( X_{0,1}^m + X_{1,0}^m + \beta \sum_{p=1}^{m'-1} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{2(m'-p)} \right) Δr^{m-2}.
\]

(18a)
Case (II-1): (m-1)th order term

For the case of \( m = 2m' + 1 \) \((m' \geq 0)\) we obtain

\[
\frac{K}{\tau \Delta r} \approx -(1 + \epsilon(-1)^m) \frac{v \hat{C}(\rho_n^{(l)})}{2m} \left[ (\delta + 1) \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} \rho_{0,1}^{2(m'-p)+1} S_{\nu}(2p) \right. \\
+ (\delta - 1) \sum_{p=0}^{m'} \left( \frac{m}{2p+1} \right) X_{1,0}^{2p+1} \rho_{0,1}^{2(m'-p)} S_{\nu}(2(m'-p)) \right] \Delta r^{m-1} \\
+ (\delta + 1) \frac{D}{2m} \hat{C}(\rho_n^{(l)}) \left[ \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{2(m'-p)+1} \left( r_{0,1}^{(1)} - \epsilon Y_{0,1}^{(2)} \right) S_{\nu}(2m'-p) \right. \\
\left. + \sum_{p=0}^{m'} \left( \frac{m}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)} \left( r_{0,1}^{(1)} - \epsilon Y_{0,1}^{(2)} \right) S_{\nu}(2m'-p) \right] \Delta r^{m-1} \\
+ (\delta + 1) \frac{mD}{2m} \hat{C}(\rho_n^{(l)}) \left\{ \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{2(m'-p)} \left[ X_{2,0}^{(1)} S_{\nu}(2m'-p) + X_{0,2}^{(1)} S_{\nu}(2p) \right. \right. \\
\left. - \epsilon \left( X_{2,0}^{(2)} S_{\nu}(2m'-p) + X_{0,2}^{(2)} S_{\nu}(2p) \right) \right] \\
\left. + \beta \sum_{p=0}^{m'-1} \left( \frac{2m'}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \left( 1 - \epsilon X_{1,1}^{(2)} \right) \right\} \Delta r^{m-1}. \tag{18b} \]

Case (I-2): mth order term

For \( m = 2m' \) \((m' \geq 0)\) the leading terms are as follows:

\[
\frac{K}{\tau \Delta r} \approx \beta \frac{(1 + \epsilon)(\delta - 1)}{2^{m+1}} \frac{v^2 \hat{C}(\rho_n^{(l)})}{D} \sum_{p=0}^{m'-1} \left( \frac{m}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \Delta r^m \\
- \frac{v}{2^{m+1}} \hat{C}(\rho_n^{(l)}) \left\{ \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{2(m'-p)} \left[ (\delta + 1) Y_{0,1}^{(1)} S_{\nu}(2p) \right. \right. \\
\left. \left. + (\delta - 1) Y_{0,1}^{(1)} S_{\nu}(2m'-p) \right] + \epsilon(3) \right\} \\
+ \beta \sum_{p=0}^{m'-1} \left( \frac{m}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \\
\times \left[ (\delta - 1) Y_{0,1}^{(1)} + (\delta + 1) Y_{0,1}^{(1)} + \epsilon(2) \right] \right\} \Delta r^m
\]
\[- \frac{m}{2^{m+1} + 1} \nu \hat{C}(\rho_n^{(l)}) \left\{ \sum_{p=0}^{m'-1} \binom{m-1}{2p} X_{1,0}^{2p} X_{0,1}^{2(m'-p)-1} \left[ (\delta + 1) X_{0,2}^{(1)} S_\nu(2p) + \beta (\delta - 1) X_{1,1}^{(1)} + \beta (\delta + 1) X_{2,0}^{(1)} + \epsilon^{(2)} \right] \right. \\
+ \sum_{p=0}^{m'-1} \binom{m-1}{2p+1} X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \left[ \beta (\delta - 1) X_{0,2}^{(1)} + \beta (\delta + 1) X_{1,1}^{(1)} + (\delta - 1) X_{2,0}^{(1)} S_\nu(2(m' - p - 1)) + \epsilon^{(2)} \right] \right\} \Delta r^m \\
+ (\delta + 1) \frac{D}{2^{m+1}} \left\{ \sum_{p=0}^{m'} \binom{m}{2p} X_{1,0}^{2p} X_{0,1}^{2(m'-p)} \left[ \hat{C}_p(\rho_n^{(l)}) \left( Y_{0,2}^{(1)} S_\nu(2p) + \epsilon^{(2)} \right) \right] \\
+ \frac{1}{2} \hat{C}_{pp}(\rho_n^{(l)}) \left( Y_{0,1}^{(2)} S_\nu(2p) + Y_{1,0}^{(2)} S_\nu(2(m' - p)) + \epsilon^{(2)} \right) \right\} \Delta r^m \\
+ (\delta + 1) \frac{mD}{2^{m+1}} \left\{ \sum_{p=0}^{m'-1} \binom{m-1}{2p} X_{1,0}^{2p} X_{0,1}^{2(m'-p)-1} \left[ \hat{C}_p(\rho_n^{(l)}) \left( X_{0,2}^{(1)} Y_{0,1}^{(1)} S_\nu(2p) + \beta X_{1,1}^{(1)} Y_{1,0}^{(1)} + \beta X_{2,0}^{(1)} Y_{0,1}^{(1)} + \epsilon^{(2)} \right) + \hat{C}(\rho_n^{(l)}) \left( X_{0,2}^{(1)} S_\nu(2p) + \beta X_{2,1}^{(1)} + \epsilon^{(2)} \right) \right] \right. \\
\left. + \sum_{p=0}^{m'-1} \binom{m-1}{2p+1} X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \left[ \hat{C}_p(\rho_n^{(l)}) \left( X_{0,2}^{(1)} Y_{0,1}^{(1)} + X_{1,1}^{(1)} Y_{1,0}^{(1)} S_\nu(2(m' - p - 1)) + \epsilon^{(2)} \right) \right] \right\} \Delta r^m \\
+ (\delta + 1) \frac{m(m-1)}{2^{m+2}} D \hat{C}(\rho_n^{(l)}) \left\{ \sum_{p=0}^{m'-1} \binom{m-2}{2p} X_{1,0}^{2p} X_{0,1}^{2(m'-p)-1} \left[ X_{0,2}^{(1)} S_\nu(2p) + 2\beta X_{0,2}^{(1)} X_{2,0}^{(1)} + \beta X_{1,1}^{(1)} + X_{2,0}^{(1)} S_\nu(2(m' - p - 1)) + \epsilon^{(2)} \right] \right. \\
\left. + \sum_{p=0}^{m'-2} \binom{m-2}{2p+1} X_{1,0}^{2p+1} X_{0,1}^{2(m'-p)-1} \left[ 2X_{0,2}^{(1)} X_{1,1}^{(1)} + 2X_{1,1}^{(1)} X_{2,0}^{(1)} + \epsilon^{(2)} \right] \right\} \Delta r^m. \]
Case (II-2): (m+1)th order term

For the case of $m = 2m' + 1$ ($m' > 0$)

$$\frac{K}{\tau \Delta r} \equiv - \frac{\beta}{2m+1} \frac{v^2 \dot{C}_\rho (\rho_n^{(l)})}{D} \left[ \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{(2m'-p)+1} (Y_{1,0}^{(1)} - Y_{1,0}^{(2)}) \right. \\
+ \left. \sum_{p=0}^{m'} \left( \frac{m-1}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{(2m'-p)} (Y_{0,1}^{(1)} - Y_{0,1}^{(2)}) \right] \Delta r^{m+1}$$

$$- \frac{m \beta}{2m+1} \frac{v^2 \dot{C}_\rho (\rho_n^{(l)})}{D} \left[ \sum_{p=0}^{m'} \left( \frac{m-1}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{(2m'-p)-1} \left( X_{0,2}^{(1)} + X_{2,0}^{(1)} - (2) \right) \right] \Delta r^{m+1}$$

$$+ \frac{v}{2m+1} \left\{ \beta \sum_{p=0}^{m'} \left( \frac{m}{2p} \right) X_{1,0}^{2p} X_{0,1}^{(2m'-p)+1} \left[ \dot{C}_\rho (\rho_n^{(l)}) (Y_{1,1}^{(1)} - Y_{1,1}^{(2)}) \right. \\
+ \left. \dot{C}_\rho (\rho_n^{(l)}) (2m'-p)(\beta Y_{0,2}^{(1)} + Y_{2,0}^{(1)} S_\nu (2(m' - p))) \right. \\
- (2)) \right) + \left. \frac{1}{2} \dot{C}_\rho (\rho_n^{(l)}) (\beta Y_{0,1}^{(1)} + Y_{1,0}^{(1)} S_\nu (2(m' - p)) - (2)) \right] \right\} \Delta r^{m+1}$$

$$+ \frac{mv}{2m+1} \left\{ \sum_{p=0}^{m'} \left( \frac{m-1}{2p} \right) X_{1,0}^{2p} X_{0,1}^{(2m'-p)} \left[ \dot{C}_\rho (\rho_n^{(l)}) \left( \beta X_{0,2}^{(1)} Y_{1,0}^{(1)} ight. \\
+ \left. \beta X_{1,1}^{(1)} Y_{1,0}^{(1)} + X_{2,0}^{(1)} Y_{1,0}^{(1)} S_\nu (2(m' - p)) - (2)) \right) \\
+ \dot{C}_\rho (\rho_n^{(l)}) \left( \beta X_{1,2}^{(1)} + X_{2,0}^{(1)} S_\nu (2(m' - p)) - (2)) \right) \right] \right. \\
+ \left. \beta \sum_{p=0}^{m'-1} \left( \frac{m-1}{2p+1} \right) X_{1,0}^{2p+1} X_{0,1}^{(2m'-p)-1} \left[ \dot{C}_\rho (\rho_n^{(l)}) \left( X_{0,2}^{(1)} Y_{0,1}^{(1)} + X_{1,1}^{(1)} Y_{1,0}^{(1)} ight. \\
+ X_{2,0}^{(1)} Y_{0,1}^{(1)} - (2)) \right) + \dot{C}_\rho (\rho_n^{(l)}) \left( X_{2,1}^{(1)} + X_{0,3}^{(1)} - (2) \right) \right] \right\} \Delta r^{m+1}$$
\[
+ \frac{m(m - 1)}{2^{m+2}} \nu \tilde{C}(\rho_n^{(l)}) \left[ 2 \beta \sum_{p=0}^{m'-1} \left( \frac{m - 2}{2p} \right) X_{1,0}^{2} X_{0,1}^{2(m'-p)-1} \left( X_{0,2}^{(1)} X_{1,1}^{(1)} \right) \\
+ X_{1,1}^{(1)} X_{2,0}^{(1)} - \left( \frac{m - 2}{2p + 1} \right) X_{1,0}^{2} X_{0,1}^{2(m' - p - 1)} \right] \\
\times \left( \beta X_{0,2}^{(1)^2} + 2 \beta X_{0,2}^{(1)} X_{2,0}^{(1)} + \beta X_{1,1}^{(1)} + X_{2,0}^{(1)^2} S_{\nu}(2(m' - p - 1)) \right) \\
- \left( \frac{m - 2}{2p + 1} \right) \Delta r^{m+1}. \quad (18d)
\]

**Proof.** follows immediately from Lemma IV.1, i.e. the operations (11), (12) and (13c) on the operand (summand) \( \mathcal{F}_n^{(l)} \), Eq. (17). \( \square \)

**Remark IV.4.** We can find that the leading terms involves \( \beta \) and, therefore, its quantity depends on a coarse-graining, partitioning parameter: it depends on how the time step \( \tau \) approaches to zero in the continuum limit, \( \Delta r \to 0 \). It is easy for us to convince or show that the second leading-order terms are also dependent on \( \beta \). We can numerically confirm the dependency when \( \Delta r \) approaches to zero. The parameter \( \beta \), however, is nothing to do with the leading order, estimated from the LOE. Thus the order itself is independent of \( \beta \). Moreover, we can find that the leading terms are independent of \( \beta \) if and only if their order is \( O(1) \). Such an “observable” order is achieved for an appropriate characteristic exponent \( m \) except the case (II-2) because \( m \) cannot be negative integer. It follows that its limit value in the continuum limit, i.e. macroscopic quantity, is also independent of the parameter \( \beta \).

**Remark IV.5.** We also notice that the leading terms are also independent of the other partitioning parameter, digit number \( k \) for \( k \geq 1 \). As the results, the LOE and the macroscopic quantities are independent of partitioning parameters. This is essential for the coarse-grained dynamics to be consistent with irreversible thermodynamics. The least number of \( k = 1 \) corresponds to the approach of level-0 partitioning, utilized by Vollmer, Tél, Mátyás, et al.\(^2\)\(^5\)\(^14\)\(^16\)\(^18\). The property comes from the fact that the leading terms, as well as the LOE, are determined by the properties related to only two trinary bits \( \omega \) and \( \omega_0 \) (Lemma III.3). For the case of \( k=0 \), one of the trinary bit \( \omega_0 \) vanishes and, therefore, another LOE or leading form appears. However, Vollmer, Tél, and Breymann\(^2\) showed that such a “projected” dynamics is not physically relevant in the sense that it cannot explain the effects of thermostat.

**Example IV.6.** Ishida\(^19\) showed that the balance equation for local entropy density \( s^{(l)}_i \) (\( \equiv \) ...
\(-\rho_i^{(t)} \ln(\rho_i^{(t)}/\rho_r)\) can be expressed as follows:

\[
\Delta V_i \Delta s_i^{(t)} \equiv \Delta V_i (s_i^{(t+1)} - s_i^{(t)}) = \dot{S}_{c,i}^{(t)} + \dot{S}_{d,i}^{(t)} + \dot{S}_{p,i}^{(t)} + \dot{S}_{th,i}^{(t)} + r_{T,i}^{(t)}, \tag{19}
\]

where

\[
\dot{S}_{c,i}^{(t)} = \sum_j \frac{J_{\text{th},i}^{(t)}}{\Delta v_{i,j}} \left( 1 + \frac{\ln(\rho_j^{(t)}/\rho_r) + \ln(\rho_i^{(t)}/\rho_r)}{2} \right) + \frac{\tilde{D}_{\text{th},i}^{(t)}}{\Delta v_{i,j}} (\rho_j^{(t)} - \rho_i^{(t)}) \right),
\]

\[
\dot{S}_{d,i}^{(t)} = \sum_j \frac{J_{\text{th},i}^{(t)}}{\Delta v_{i,j}} \left( 1 + \frac{\ln(\rho_j^{(t)}/\rho_r) + \ln(\rho_i^{(t)}/\rho_r)}{2} \right) - \frac{\tilde{D}_{\text{th},i}^{(t)}}{\Delta v_{i,j}} (\rho_j^{(t)} + \rho_i^{(t)}) \right),
\]

\[
\dot{S}_{p,i}^{(t)} = \sum_j \frac{1}{\Delta v_{i,j}} \left( \tilde{U}_{\text{th},i}^{(t)} \frac{\rho_j^{(t)} - \rho_i^{(t)}}{2} - \frac{\tilde{D}_{\text{th},i}^{(t)}}{\Delta v_{i,j}} (\rho_j^{(t)} + \rho_i^{(t)}) \right),
\]

\[
r_{T,i}^{(t)} = -2\Delta V_i \rho_i^{(t+1)} \left( \frac{\ln(\rho_i^{(t+1)}/\rho_r) - \ln(\rho_i^{(t)})}{2} - \frac{\Delta \rho_i^{(t)}}{2\rho_i^{(t+1)}} \right).
\]

By use of the leading terms \(18\), and LOE \(13\), we can now derive the averaged form of each component of the above local entropy change in the bulk system as follows:

\[
\langle \dot{S}_c \rangle_A^{(t)} = -\frac{\partial}{\partial r} (sv - \rho v) + O(\Delta r^2),
\]

\[
\langle \dot{S}_d \rangle_A^{(t)} = D \frac{\partial^2 s}{\partial r^2} + O(\Delta r^2),
\]

\[
\langle \dot{S}_p \rangle_A^{(t)} = \frac{j^2}{\rho D} + O(\Delta r^2),
\]

\[
\langle \dot{S}_{th} \rangle_A^{(t)} = -\frac{ju}{D} + O(\Delta r^2),
\]

where \(s \equiv -\rho \ln(\rho/\rho_r)\), \(j \equiv \rho v - D \partial \rho/\partial r\). Herein \(\langle A \rangle_A^{(t)}\) denotes the averaged quantity on \(A_n\) at the \(lth\) time step, interpreted as the quantity at \(r = n \Delta r\) and \(t = l \tau\) on its right side. Considering that the averaging of \(\Delta V_i \Delta s_i^{(t)}\) is the time derivative of the averaged entropy density on \(A_n\) and that \(\langle r_T \rangle_A^{(t)}\) is \(O(\tau)^{19}\), the averaging of Eq. \(19\) recovers the macroscopic entropy balance equation of irreversible thermodynamics in the continuum limit, \(\Delta r \to 0\). It has been numerically confirmed by Ishida\(^{19}\).
V. CONCLUDING REMARKS

For $T^1$-type local quantities on the UOFP multibaker chain system, a law of order estimation (LOE) to estimate the order exponent of the averaged local quantities, originally conjectured by Ishida, becomes a theorem. Furthermore, the form of the leading-order terms for the quantities is derived, and we can confirm the order and the finite limit value in the continuum limit are independent of partitioning parameters, such as $\beta$ and trit number $k$. The results fully explain the numerical results in the bulk system, obtained by Ishida, and they are consistent with the irreversible thermodynamics.

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REFERENCES

1. T. Gilbert and J. Dorfman, J. Stat. Phys. 96, 225 (1999).
2. J. Vollmer, T. Tél, and W. Breymann, Phys. Rev. E 58, 1672 (1998).
3. W. Breymann, T. Tél, and J. Vollmer, Chaos 8, 396 (1998).
4. J. Vollmer, T. Tél, and W. Breymann, Phys. Rev. Lett. 79, 2759 (1997).
5. T. Tél, J. Vollmer, and W. Breymann, Europhys. Lett. 35, 659 (1996).
6. H. Ishida, Physica A 388, 332 (2009).
7. S. Tasaki and P. Gaspard, J. Stat. Phys. 81, 935 (1995).
8. T. Gilbert and J. Dorfman, Physica A 282, 427 (2000).
9. T. Gilbert, J. Dorfman, and P. Gaspard, Phys. Rev. Lett. 85, 1606 (2000).
10. P. Gaspard, J. Stat. Phys. 88, 1215 (1997).
11. J. Dorfman, P. Gaspard, and T. Gilbert, Phys. Rev. E 66, 026110 (2002).
12. F. Barra, P. Gaspard, and T. Gilbert, Phys. Rev. E 80, 021126 (2009).
13. F. Barra, P. Gaspard, and T. Gilbert, Phys. Rev. E 80, 021127 (2009).
14. J. Vollmer, T. Tél, and L. Mátyás, J. Stat. Phys. 101, 79 (2000).
15. L. Mátyás, T. Tél, and J. Vollmer, Phys. Rev. E 61, R3295 (2000).

16. L. Mátyás, T. Tél, and J. Vollmer, Phys. Rev. E 62, 349 (2000).

17. P. Gaspard, *Chaos, scattering and statistical mechanics*, Vol. 9 (Cambridge University Press, New York, 2005).

18. J. Vollmer, Phys. Rep. 372, 131 (2002).

19. H. Ishida, Entropy 15, 4345 (2013).

20. S. V. Patankar, *Numerical heat transfer and fluid flow* (Taylor & Francis, 1980).