The double role of Einstein’s equations: as equations of motion and as vanishing energy-momentum tensor∗

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Diffeomorphism covariant theories with dynamical background metric, like gravity coupled to matter fields in the way expressed by Einstein-Hilbert’s action or relativistic strings described by Polyakov’s action, have ‘on-shell’ vanishing energy-momentum tensor $t_{\mu\nu}$ because $t_{\mu\nu}$ is, essentially, the Eulerian derivative associated with the dynamical background metric and thus $t_{\mu\nu}$ vanishes ‘on-shell.’ Therefore, the equations of motion for the dynamical background metric play a double role: as equations of motion themselves and as a reflection of the fact that $t_{\mu\nu} = 0$. Alternatively, the vanishing property of $t_{\mu\nu}$ can be seen as a reflection of the so-called ‘problem of time’ present in diffeomorphism covariant theories in the sense that $t_{\mu\nu}$ are written as linear combinations of first class constraints only.

I. INTRODUCTION

There are in literature several attempts to define the energy-momentum tensor for gravity coupled to matter fields. One can say that there is, currently, non-agreement about how it must be defined or even if its definition makes sense or not. The fact of having in the literature various definitions for it is usually regarded as the reflection of the fact that this task makes no sense:  

‘Anybody who looks for a magic formula for “local gravitational energy-momentum” is looking for the right answer to the wrong question’

is, for instance, a quotation found in page 467 of Ref. 1. In spite of the strongness of that statement, the issue of the energy-momentum tensor for gravity coupled to matter fields is investigated in this paper. Intuitively, it is expected that all dynamical fields of which an action depends on contribute to the full energy-momentum tensor for the full system of dynamical fields. So, from this perspective it is natural to expect a contribution of the gravitational field to the full energy-momentum tensor. The viewpoint adopted here about the definition of the energy-momentum tensor is, from the mathematical point of view, very simple. However, it will be more important for us to explore the conceptual aspects involved in such a definition. Specifically, the definition of the energy-momentum tensor $t_{\mu\nu}$ is taken as that coming from the variation of the action under consideration with respect to the dynamical metric $g_{\mu\nu}$ the action depends on

$$ t_{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta S[g_{\mu\nu}, \phi]}{\delta g^{\mu\nu}}. $$

(1)

From the definition of Eq. (1) it is evident that, for a dynamical background metric, the energy-momentum tensor $t_{\mu\nu}$ vanishes ‘on-shell’

$$ t_{\mu\nu} = 0. $$

(2)

In particular, in the case of gravity coupled to matter fields $\phi$, $t_{\mu\nu} = T_{\mu\nu} - \frac{c^4}{8\pi G} G_{\mu\nu}$, which vanishes because of Einstein’s equations. To ‘avoid this difficulty’ (i.e., the ‘on-shell’ vanishing property of $t_{\mu\nu}$) people usually say that this way of defining the energy-momentum tensor just gives the ‘right’ form for the energy-momentum tensor of the matter fields $\phi$ only, $T_{\mu\nu}$, by simply identifying in the expression for $t_{\mu\nu}$ the contribution of the matter fields $\phi$. We disagree with that point of view because it puts the matter fields $\phi$ and the geometry $g_{\mu\nu}$ on non the same footing.

∗ To Jerzy Plebański on the occasion of his 75th the birthday.
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It might be possible to consider the first order formalism if fermions want to be included. The ideas developed here can without any problem be applied to that case.
(i.e., the status of the matter fields $\phi$ is (from that perspective) different to the status of the geometry $g_{\mu\nu}$ even though both fields are dynamical ones). Here, on the other hand, we argue that there is nothing wrong either with the definition of the energy-momentum tensor $t_{\mu\nu}$ given in Eq. (1) nor with the fact that it vanishes ‘on shell.’ This way of interpreting things has several conceptual consequences:

1. all fields the action depends on are on the same conceptual footing and thus it is not a surprise that they all contribute (as dynamical fields) to the full energy-momentum tensor $t_{\mu\nu}$.

2. a natural definition for the energy-momentum tensor of the gravitational field arises.

3. the vanishing property of $t_{\mu\nu}$ is not interpreted as a ‘problem’ which must be corrected somehow but rather as a reflection of the double role that the equations of motion associated with the dynamical background play or, alternatively, as a reflection of diffeomorphism covariance.

The content of the paper is organized as follows. In Sect. 2, the issue of the energy-momentum tensor for Polyakov’s action is studied where point (3) is explicitly displayed. Next, in Sect. 3, the system of gravity coupled to matter fields is analyzed. Finally, our conclusions are collected in Sect. 4.

II. POLYAKOV’S ACTION

Before considering general relativity coupled to matter fields, and as a warming up, let us study the dynamics of relativistic bosonic strings propagating in an arbitrary $D$-dimensional fixed (i.e., non dynamical) background spacetime with metric $g = g_{\mu\nu}(X)dX^\mu dX^\nu$; $\mu, \nu = 0, 1, \ldots, D − 1$. This system can be described, for instance, with the action

$$S[\gamma^{ab}, X^\mu] = \alpha \int_{\mathcal{M}} \sqrt{-\gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)}.$$  

(3)

The variation of the action (3) with respect to the background coordinates $X^\mu$ and the inverse metric $\gamma^{ab}$ yields the equations of motion

$$\nabla^a \nabla_a X^\mu + \Gamma^\mu_{\alpha\beta} \gamma^{bc} \partial_b X^\alpha \partial_c X^\beta = 0,$$

(4)

$$\frac{\alpha}{2} \gamma^{cd} h_{cd} - \alpha h_{ab} = 0,$$

(5)

with

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X),$$

(6)

the induced metric on the world sheet $\mathcal{M}$.

Energy-momentum tensor. The energy-momentum tensor $t_{ab}$ for the full system of fields is defined by the variational variation of the action (3) with respect to the inverse metric $\gamma^{ab}$

$$t_{ab} := \frac{\sqrt{-\gamma}}{\alpha} \delta S[\gamma^{ab}, X^\mu] \delta \gamma^{ab}$$

$$= \frac{\alpha}{2} \gamma^{cd} h_{cd} - \alpha h_{ab}.$$  

(7)

Thus, $t_{ab}$ is defined with respect to any ‘local’ observer on the world sheet $\mathcal{M}$. Note that the trace of $t_{ab}$ vanishes ‘off-shell,’ $t_{ab} \gamma^{ab} = 0$. Moreover, by using Eq. (5)

$$t_{ab} = 0,$$

(8)

i.e., the energy-momentum tensor vanishes ‘on-shell’ or, what is the same, $t_{ab} = 0$ is just a reflection of having a dynamical background metric on the world sheet $\mathcal{M}$. In this sense, Eq. (5) plays a double role. The first role of Eq. (5) is that it is the equation of motion associated with the dynamical field $\gamma^{ab}$. The second role of Eq. (5) is that it is the reflection of the fact that the energy-momentum tensor of the full system vanishes, $t_{ab} = 0$, which in fact implies that $\gamma_{ab}$ is proportional to the induced metric $h_{ab}$, $\gamma_{ab} = e^{\Omega} h_{ab}$. Alternatively, the fact that the energy-momentum tensor $t_{ab}$ vanishes from the viewpoint of an observer sitting on the world sheet establishes a balance between the intrinsic metric $\gamma_{ab}$ and the induced metric $h_{ab}$ in the precise form given by $\gamma_{ab} = e^{\Omega} h_{ab}$.
Diffeomorphism covariance (active diffeomorphism invariance). Diffeomorphism covariance or general covariance from the active point of view means the following. Let \((X^\mu(\xi), \gamma_{ab}(\xi))\) be any solution of the equations of motion \(\text{(11)}\) and \(\text{(12)}\) with respect to the coordinate system associated with the local coordinates \(\xi\) and let \(f: M \to M\) be any diffeomorphism of the world sheet \(M\) onto itself then, the new configuration

\[
X'^\mu(\xi) = X^\mu(f(\xi)),
\]

\[
\gamma'_{ab}(\xi) = \frac{\partial f^c}{\partial \xi^a} \frac{\partial f^d}{\partial \xi^b} \gamma_{cd}(f(\xi)),
\]

is also a (mathematically different) solution to the equations of motion \(\text{(11)}\) and \(\text{(12)}\) with respect to the same observer. Even though, \((X^\mu(\xi), \gamma_{ab}(\xi))\) and \((X'^\mu(\xi), \gamma'_{ab}(\xi))\) are mathematically distinct configurations they represent the same physical solution with respect to the local observer, i.e., any diffeomorphism of the world sheet \(M\) induces a gauge transformation on the fields living on \(M\) which is given in Eq. \(\text{(11)}\). As it is well-known, gauge symmetries are, in the canonical formalism, associated with first class constraints.\(\footnote{1}\)

\(1+1\) canonical viewpoint of the energy-momentum tensor. The vanishing of \(t_{ab}\) is just a reflection of the gauge symmetry \(\text{(3)}\). To see this, \(M = R \times \Sigma\) and the metric \(\gamma_{ab}\) is put in the ADM form

\[
(\gamma_{ab}) = \left( \begin{array}{cc} -N^2 + \lambda^2 & \chi \lambda \\ \chi \lambda & \chi \end{array} \right),
\]

\[
(\gamma^{ab}) = \left( \begin{array}{cc} -\frac{N}{\chi} & \frac{\lambda}{\chi} \\ \frac{\lambda}{\chi} & \frac{1}{\chi} - \frac{\chi}{N^2} \end{array} \right),
\]

and so \(\sqrt{-\gamma} := \sqrt{-\det \gamma_{ab}} = \epsilon N \sqrt{\chi}\) with \(\epsilon = +1\) if \(N > 0\) and \(\epsilon = -1\) if \(N < 0\). Due to the fact \(\partial \sigma^\alpha\) is time-like and \(\partial \sigma^\beta\) is space-like then \(-N^2 + \lambda^2 \chi < 0\) and \(\chi > 0\). Taking into account Eq. \(\text{(10)}\), the action of Eq. \(\text{(3)}\) acquires the form

\[
S[X^\mu, P_\mu, M, \lambda] = \int_R d\tau \int_\Sigma d\sigma \left[ X^\mu P_\mu - (MH + \lambda D) \right],
\]

with

\[
H := P_\mu P_\nu g^{\mu\nu} + 4\alpha^2 X'^\mu X'^\nu g_{\mu\nu},
\]

\[
D := X'^\mu P_\mu. \tag{11}
\]

The dependence of the phase space variables and Lagrange multipliers in terms of the Lagrangian variables is

\[
P_\mu := -\frac{2\alpha \sqrt{\chi}}{N} X'^\nu g_{\mu\nu} + \frac{2\alpha \lambda \sqrt{\chi}}{N} X'^\nu g_{\mu\nu},
\]

\[
M := -\frac{N}{4\alpha \sqrt{\chi}}, \tag{13}
\]

where \(X'^\mu = \frac{\partial X^\nu}{\partial \sigma}\).

Hamilton’s principle applied to the action \(\text{(11)}\) yields the dynamical equations

\[
\dot{X}^\mu = 2MP_\nu g^{\mu\nu} + \lambda X'^\mu, \tag{14}
\]

\[
\dot{P}_\mu = MY^{\theta\phi} \frac{\partial g_{\theta\phi}}{\partial X^\mu} + (8\alpha^2 M X'^\mu g_{\mu\nu} + \lambda P_\mu)', \tag{15}
\]

and

\[
H \approx 0, \quad D \approx 0, \tag{16}
\]

which are the Hamiltonian and diffeomorphism first class constraints; respectively. Here, \(Y^{\theta\phi} = P_\mu P_\nu g^{\theta\mu} g^{\phi\nu} - 4\alpha^2 X^{\theta} X^{\phi} g_{\theta\phi}\).

Next, the induced metric \(h_{ab}\) in terms of the phase space variables and Lagrange multipliers is written down

\[
h_{\tau\tau} = X^\mu X^\nu g_{\mu\nu},
\]

\[
= 4M^2 P_\mu P_\nu g^{\mu\nu} + 4\lambda P_\mu X'^\mu + \lambda^2 X'^\mu X'^\nu g_{\mu\nu},
\]

\[
h_{\tau\sigma} = X^\mu X^\nu g_{\mu\nu},
\]

\[
= 2MP_\mu X'^\mu + \lambda X'^\mu X'^\nu g_{\mu\nu},
\]

\[
h_{\sigma\sigma} = X'^\mu X'^\nu g_{\mu\nu}, \tag{17}
\]
where the dynamical equation (14) was used. Therefore, by using Eqs. (7), (10), (17), and the definition of the constraints (12), the components of the energy-momentum tensor (7) in terms of the phase space variables and the Lagrange multipliers acquire the form:

\[
\begin{align*}
 t_{\tau\tau} &= -2\alpha M^2 \left( 1 + \frac{\lambda^2}{16\alpha^2 M^2} \right) H - (4\alpha M\lambda) D, \\
 t_{\tau\sigma} &= -\frac{\lambda}{8\alpha} H - (2\alpha M) D, \\
 t_{\sigma\sigma} &= -\frac{1}{8\alpha} H.
\end{align*}
\]

Thus, Eq. (18) clearly expresses the conceptual reason of the vanishing property of \( t_{ab} \), i.e., \( t_{ab} \) vanish because they are (modulo the dynamical Eq. (14)) linear combinations of the first class constraints (16) for the system, which are the 1 + 1 canonical version of the gauge symmetry (9). In conclusion, a vanishing energy-momentum tensor \( t_{\mu\nu} = 0 \) is a reflection of the fact that the hamiltonian of the theory is just a linear combination of first class constraints. One could say that \( t_{\mu\nu} = 0 \) is equivalent to the definition of the ‘constraint surface’, however, this is not so because Eq. (18) was written by using the dynamical equation (14).

### III. GRAVITY COUPLED TO MATTER FIELDS

Now, let us study the Einstein-Hilbert action coupled to matter fields

\[
S[g_{\mu\nu}, \phi] = \frac{c^3}{16\pi G} \int_M \sqrt{-g} \ R \ d^4 x + \int_M \sqrt{-g} \ L_{\text{matter fields}}(\phi) \ d^4 x,
\]

where \( R \) is the scalar curvature and \( L_{\text{matter fields}}(\phi) \) denotes the contribution of the matter fields dynamically coupled to gravity and denoted generically by \( \phi \). Hamilton’s principle yields the equations of motion for the system

\[
\frac{\delta S}{\delta \phi} = 0, \\
G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \) is Einstein’s tensor and \( T_{\mu\nu} \) is the contribution of the matter fields \( \phi \) to the energy-momentum tensor.\(^2\)

**Energy-momentum tensor.** Again, the energy-momentum tensor \( t_{\mu\nu} \) is obtained by the variation of the full action (19) with respect to the inverse metric \( g^{\mu\nu} \). Therefore, the energy-momentum tensor for the full system of fields is

\[
t_{\mu\nu} = T_{\mu\nu} - \frac{c^4}{8\pi G} G_{\mu\nu}.
\]

Some remarks follow:

1) from this perspective, \( T_{\mu\nu} \) is the contribution of the matter fields \( \phi \) while \( -\frac{c^4}{8\pi G} G_{\mu\nu} \) is the contribution of the gravitational field \( g_{\mu\nu} \) to \( t_{\mu\nu} \). Therefore, the matter fields \( \phi \) and the gravitational field \( g_{\mu\nu} \) are put on the same ontological status in the sense that both of them contribute (as dynamical fields) to the full energy-momentum tensor \( t_{\mu\nu} \). In addition, and in contrast to the dynamical system described by Polyakov’s action, note that the full energy-momentum tensor of gravity and matter fields given in Eq. (22) is composed of two additive parts each one being associated to each field, i.e., there is a splitting of the contributions of the fields (\( g_{\mu\nu} \) and \( \phi \)) to \( t_{\mu\nu} ).^3

2) for observers which detect a gravitational field the energy-momentum tensor identically vanishes, \( t_{\mu\nu} = 0 \), because of Einstein’s equations (21). This means that for this type of observers, there is a balance between the ‘content’ of energy and momentum densities and stress associated with the matter fields \( \phi \) (which is characterized in \( T_{\mu\nu} \)) and the

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\(^2\) Strictly speaking, the field \( g_{\mu\nu} \) also contributes to \( T_{\mu\nu} \).

\(^3\) See footnote b.
‘content’ of energy and momentum densities and stress associated with the gravitational field (which is characterized in $-\frac{c^4}{8\pi G} G_{\mu\nu}$)

\[
\begin{align*}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow
\end{align*}
\]

in a precise form such that both fluxes cancel, and thus leading to a vanishing ‘flux’, i.e., $t_{\mu\nu} = 0$. Once again, the vanishing property of $t_{\mu\nu}$ for the system of gravity coupled to matter fields is just a reflection of the fact that the background metric is dynamical. More precisely, $t_{\mu\nu} = 0$ tells us that the ‘reaction’ of the dynamical background metric is such that it just cancels the effect of ‘flux’ associated with the matter fields. It is impossible (and makes no sense) to have a locally non-vanishing ‘flux’ in this situation. If this were the case, there would be no explanation for the origin of that non-vanishing ‘flux’. Moreover, that hypothetic non-vanishing ‘flux’ would define privileged observers associated with it (the ether would come back!). It is important to emphasize that, in the case of having a dynamical background metric, the vanishing property of $t_{\mu\nu}$ is not interpreted here as a ‘problem’ that must be corrected somehow but exactly the other way around. In our opinion, there is nothing wrong with that property because it just reflects the double role that the equations of motion associated with the dynamical background play.

3) Connection with special relativity. In the conceptual framework of the special theory of the relativity of motion the background metric is fixed (i.e., non-dynamical\(^4\)), the only dynamical entities are the matter fields $\phi$ and thus any Lorentz observer can associate a non-vanishing ‘content’ of energy and momentum densities and stress

\[
\begin{align*}
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow
\end{align*}
\]

associated to them (the fields) and represented in $T_{\mu\nu}$. How does this non-vanishing $t_{\mu\nu} = T_{\mu\nu}$ in the special theory of the relativity of motion come out from general relativity where $t_{\mu\nu} = 0$? If one goes from the general to the special theory of the relativity of motion by using ‘locally’ freely-falling observers one finds that there is a contradiction between the fact explained above and the point 2) which states that $t_{\mu\nu} = 0$ must hold for any observer and, in particular, for ‘locally’ freely-falling observers which find ‘no evidence of gravity’ (Einstein’s equivalence principle). However, the contradiction disappears by noting that in the point 2) the background field metric is dynamical while in special relativity is not, i.e., the contradiction arises from the comparison of two conceptually different scenarios.

More precisely, the fact of ‘locally’ having the special theory of the relativity of motion and therefore a non-vanishing energy-momentum tensor (whose contribution comes only from the matter fields) is just a reflection of neglecting (by means of Einstein’s equivalence principle) the contribution of the dynamical background $g_{\mu\nu}$ to the full tensor, i.e., ‘locally’ freely-falling observers can not use Einstein’s equations simply because for them the background metric is non dynamical but it is fixed to be the Minkowski metric. What these observers do is simply to neglect the second term in the right-hand side of Eq. (22) under the pretext of Einstein’s equivalence principle. Note, however, that from the mathematical point of view it is not possible to do that because it is impossible to choose local coordinates (and thus a particular reference frame attached to it) such that with respect to these coordinates (with respect to this reference frame) the Riemann tensor $R_{\alpha\beta\gamma\delta}$ vanishes in a certain point (in whose neighborhood one could define the concept of an ‘inertial reference frame’ in the sense of the conceptual framework of the special theory of the relativity of motion). This mathematical impossibility is other way of saying that particular reference frames where the gravitational field (represented by the Riemann tensor) completely vanishes do not exist. This fact implies that it is impossible to cancel the effects of the gravitational field even for freely-falling observers because of the presence of tidal forces. Therefore, it is conceptually not possible to neglect gravity effects and thus all observers must conclude that the background metric is always dynamical and that its effects can not be neglected. Thus, conceptually, $t_{\mu\nu} = 0$ always. If, by hand (Einstein’s equivalence principle) the dynamics of the background metric is neglected then this fact leads to the arising of non-vanishing energy-momentum tensor associated with matter fields only.

IV. CONCLUDING REMARKS

The lesson from Polyakov’s action and from gravity coupled to matter fields leaves no room for speculations. It is completely clear the relationship between diffeomorphism covariance and a vanishing energy-momentum tensor $t_{\mu\nu}$ in both theories. Alternatively, one could say that the vanishing property of $t_{\mu\nu}$ is another manifestation of the so-called

\(^4\) This is also true for any field theory defined on a curved fixed background, however, this is not relevant for the present purposes.
‘the problem of time’ which, of course, is not a problem but a property of generally covariant theories. Moreover, the interplay between $t_{\mu\nu}$ and the Euler-Lagrange derivative associated with the dynamical background metric in the way expressed in Eq. (22) leaves no room for attempts of modifying the expression for the energy-momentum tensor adding, for instance, divergences because if this were done, say, that a hypothetic ‘right’ energy-momentum tensor $T_{\mu\nu}$ were built

$$T_{\mu\nu} = t_{\mu\nu} + \nabla_\gamma \chi_{\mu\nu\gamma}$$

(25)

by this procedure then, $T_{\mu\nu} = 0$ would imply

$$G_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} + \nabla_\gamma \chi_{\mu\nu\gamma})$$

(26)

thus modifying the original Einstein’s equations we start with which is a contradiction, i.e., any attempt to ‘improve’ the energy-momentum by adding divergence terms, $t_{\mu\nu} \rightarrow T_{\mu\nu}$, would modify the field equations associated with the background metric and there is currently no experimental reason to do that. So, ‘improvements’ for the energy-momentum tensor of the kind introduced by Belinfante are not allowed in diffeomorphism covariant theories.

As a final comment let us consider the theory of a massless scalar field defined on a flat background expressed in a generally covariant form[7]

$$S[g_{\mu\nu}, \phi, \lambda^{\mu\nu\gamma\delta}] = -\frac{1}{2} \int_{\mathcal{M}} \sqrt{-g} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{1}{4} \int_{\mathcal{M}} \sqrt{-g} \lambda^{\mu\nu\gamma\delta} R_{\mu\nu\gamma\delta}.$$  

(27)

Hamilton’s principle yields the equations of motion

$$\nabla^\gamma \nabla^\delta \lambda_{\mu\nu\gamma\delta} = T_{\mu\nu},$$

(28)

$$g^{\mu\nu} \nabla_{\mu} \phi = 0,$$

(29)

$$R_{\mu\nu\gamma\delta} = 0.$$  

(30)

Note that the first equation plays the role of Einstein’s equations, i.e., it is the equation associated with the dynamical background metric.\(^5\) Again, the energy momentum tensor for the system is

$$t_{\mu\nu} = T_{\mu\nu} - \nabla_\gamma \nabla^\delta \lambda_{\mu\nu\gamma\delta},$$  

(31)

and it vanishes because of Eq. (28). Therefore, one finds the same phenomenon found in gravity coupled to matter fields in the sense that if ‘locally’ freely-falling observers neglected the reaction of the background (i.e., neglecting $\nabla^\gamma \nabla^\delta \lambda_{\mu\nu\gamma\delta}$) they would observe just a non-vanishing $t_{\mu\nu} = T_{\mu\nu}$, as expected. However, note that conceptually (i.e., from the mathematical point of view) it is impossible to neglect $\nabla^\gamma \nabla^\delta \lambda_{\mu\nu\gamma\delta}$ because it is not possible to find a coordinate system and a point in which this term vanishes. Note also that the theory defined by the action of Eq. (27) is completely different to the theory defined by

$$S[\phi] = -\frac{1}{2} \int_{\mathcal{M}} \sqrt{-g} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi.$$  

(32)

(assuming that the background metric $g_{\mu\nu}$ is flat) in the following sense: in the field theory defined in Eq. (27), $t_{\mu\nu} = 0$ because of the dynamical equation for $g_{\mu\nu}$ while in the field theory defined in Eq. (32) the background metric $g_{\mu\nu}$ is non-dynamical and thus the theory has a non-vanishing energy-momentum tensor $t_{\mu\nu} = T_{\mu\nu}$. Of course, what defines a theory is its equations of motion, so one could say whether or no both theories are the same by simply looking at their equations of motion. However, from the present analysis, they have different full energy-momentum tensors and this fact indicates that each of these theories describe physically distinct scenarios. Moreover, the theory defined by Eq. (32) has, from the canonical point of view, a non-vanishing Hamiltonian while the Hamiltonian for the theory defined by Eq. (27) must involve first class constraints because of diffeomorphism covariance. In summary:

|                      | Theory of Eq. (27) | Theory of Eq. (32) |
|----------------------|-------------------|-------------------|
| general covariance (passive diff. inv.) | Yes               | Yes               |
| dynamical background metric                  | Yes               | Not               |
| diff. covariance (active diff. inv.)          | Yes               | Not               |
| vanishing $t_{\mu\nu}$                      | Yes               | Not               |

\(^5\) The background metric $g_{\mu\nu}$ is dynamical in the sense that the action (27) depends functionally on it.
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