Numerical Method for Highly Non-linear Mean-reverting Asset Price Model with CEV-type Process

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Abstract

It is well documented from various empirical studies that the volatility process of an asset price dynamics is stochastic. This phenomenon called for a new approach to describing the random evolution of volatility through time with stochastic models. In this paper, we propose a mean-reverting theta-rho model for asset price dynamics where the volatility diffusion factor of this model follows a highly non-linear CEV-type process. Since this model lacks a closed-form formula, we construct a new truncated EM method to study it numerically under the Khasminskii-type condition. We justify that the truncated EM solutions can be used to evaluate a path-dependent financial product.

Key words: Asset price model, stochastic volatility, truncated EM scheme, strong convergence, financial product, Monte Carlo scheme.

1 Introduction

Several stochastic differential equations (SDEs) have been developed to describe random evolution of financial variables in time. The Black-Scholes model in [1] is widely used to describe time-series evolution of asset price dynamics under one of the assumptions that the asset price is log-normally distributed. However, as supported by many empirical evidence, the log-normality assumption does not hold exactly in reality. Various alternative stochastic models have since been proposed as modified versions of

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the Black-Scholes model. In 1977, Vasicek in [2] developed the well-known mean-reverting model as an alternative model for capturing short-term interest rate dynamics through time. This model is governed by

$$dx(t) = \alpha_1(\mu_1 - x(t))dt + \sigma_1 dB_1(t)$$

with initial data $x(0) = x_0$, where $\alpha_1, \mu_1, \sigma_1 > 0$ and $B_1(t)$ is a scalar Brownian motion. One main unique feature of SDE (1) is that the expectation of $x(t)$ converges to the long-term value $\mu_1$ with the speed $\alpha_1$. However, in practice, this model yields negative values. Cox, Ingersoll and Ross (CIR) in [3] addressed this drawback by extending SDE (1) to an alternative model often called the mean-reverting square root process which is driven by

$$dx(t) = \alpha_1(\mu_1 - x(t))dt + \sigma_1 \sqrt{x(t)} dB_1(t).$$

The square root diffusion factor avoids possible negative values. Later, Lewis in [4] generalised SDE (2) to the mean-reverting-theta process governed by

$$dx(t) = \alpha_1(\mu_1 - x(t))dt + \sigma_1 x(t)^\theta dB_1(t),$$

where $\theta \geq 1/2$. The SDE (3) has been found a useful tool for modelling interest rate, asset price and other financial variables. However, by applying $\chi^2$ tests to US treasury bill data, it has been shown that $\theta > 1$. For instance, Chan et al. in [5] applied the Generalised Moment method to the treasury bill data to estimate $\theta = 1.449$. Similarly, with the same data, Nowman in [6] also estimated $\theta = 1.361$ using the Gaussian Estimation method.

There are many literature where several classes of SDE (3) with parametric restrictions have been studied. For instance, Higham and Mao in [10] studied strong convergence of Monte Carlo simulations involving SDE (3) for $\theta = 1/2$. Mao in [20] studied strong convergence of EM method for SDE (3) when $\theta \in [1/2, 1]$. Wu et al. in [11] established weak convergence of EM method for $\theta > 1$. Dong-Hyun et al. documented a unique type of SDE (3) in [7] which admits closed-form solutions for bond prices and a concave relationship between interest rates and yields. Further discussions relating to SDE (3) could also be found in [8], [9], among others.

The original Black-Scholes model assumes constant volatility for asset price and even for options with different maturities and strikes over a trading period. This assumption makes the Black-Scholes model reproduce flat volatility surface in option pricing. However, in practice, volatility has been observed empirically to change as asset price changes. Essentially, this means that volatility is characterised by a smile or skew surface instead
of a flat surface. This characteristic is important for pricing and evaluating complex financial derivatives. As a result, several authors have proposed a variety of volatility models to explain the volatility surface curve adequately. For instance, Dupire (1994) developed the local volatility model in [12] to precisely match the observed smile or skew surface of market volatility data. Subsequently, stochastic volatility models have also been introduced as alternative models for modelling the random nature of volatility through time. One of the most notable stochastic volatility models is the diffusion class of Constant Elasticity of Variance (CEV) model driven by

$$d\varphi(t) = \mu_2\varphi(t)dt + \sigma_2\varphi(t)^\phi dB_2(t),$$

(4)

with initial data $\varphi(0) = \varphi_0$, $\mu_2, \sigma_2 > 0$, $\phi > 1$ and $B_2(t)$ is a scalar Brownian motion. SDE (4) is widely used by researchers and market practitioners for modelling volatility and other financial quantities (see, e.g., [13, 14]). In 2012, the authors in [21] established the weak convergence result of the Hull and White type model where the instantaneous volatility follows

$$d\varphi(t) = \alpha_2(\mu_2 - \varphi(t))dt + \sigma_2\varphi(t)^\phi dB_2(t),$$

(5)

for $\phi > 1$. The reader is referred, for example, to [15, 16, 17] for further coverage of stochastic volatility models in finance.

From the empirical viewpoint, it would be more desirable in modelling context to generalise SDE (3) as a highly non-linear SDE of the form

$$dx(t) = \alpha_1(\mu_1 - x(t)\rho)dt + \sigma_1\sqrt{\varphi(t)x(t)^\theta} dB_1(t)$$

(6)

for asset price dynamics, where $\rho > 1$. Here, the variance function $\varphi(t)$ is driven by a highly non-linear type of SDE (5) of the form

$$d\varphi(t) = \alpha_2(\mu_2 - \varphi(t)^r)dt + \sigma_2\varphi(t)^\phi dB_2(t),$$

(7)

where $r > 1$ and $B_1(t)$ is independent of $B_2(t)$.

The highly non-linear component of SDE (6) makes it well-suited for explaining non-linearity in asset price. On the other hand, the inherent super-linear CEV dynamics may capture extreme non-linearity in market volatility to reproduce volatility surface curve adequately. Obviously, SDE (6) is not analytically tractable. The drift and diffusion terms are of super-linear growth. In this case, we recognise the need to develop an implementable numerical method to estimate the exact solution. However, to the best of our knowledge, there exists no relevant literature devoted to the convergent approximation of the system of SDE (6) in the strong sense. In
this paper, we aim to close this gap by constructing several new numerical tools to study this model from viewpoint of financial applications.

The rest of the paper is organised as follows: In Section 2, we introduce some useful mathematical notations. In Section 3, we study the existence of a unique positive solution of SDE (6) and establish the finite moment of the solution. We construct a new truncated EM method to approximate SDE (6) in Section 4. In Section 5, we study numerical properties such as the finite moment and the finite time strong convergence of the numerical solutions. We implement numerical examples to validate the theoretical findings and conclude the paper with a financial application in Section 6.

2 Preliminaries

Throughout this paper unless specified otherwise, we employ the following notation. Let \( \{ \Omega, \mathcal{F}, \mathbb{P} \} \) be a complete probability space with a filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) satisfying the usual conditions (i.e., it is increasing and right continuous while \( \mathcal{F}_0 \) contains all \( \mathbb{P} \) null sets), and let \( \mathbb{E} \) denote the expectation corresponding to \( \mathbb{P} \). Let \( B_1(t) \) and \( B_2(t) \), \( t \geq 0 \), be scalar Brownian motions defined on the above probability space and are independent of each other. If \( x, y \) are real numbers, then we denote \( x \vee y \) as the maximum of \( x \) and \( y \), and \( x \wedge y \) as the minimum of \( x \) and \( y \). Let \( \mathbb{R} = (\mathbb{R}, \mathbb{R}^+) \), and for an empty set \( \emptyset \), we set \( \inf \emptyset = \mathbb{R}^+ \). For a set \( A \), we denote its indicator function by \( 1_A \). Moreover, we let \( T \) be an arbitrary positive number. Now consider the following scalar dynamics

\[
\begin{align*}
 dx(t) &= f_1(x(t))dt + \sqrt{\varphi(t)}g_1(x(t))dB_1(t) \quad (8) \\
 d\varphi(t) &= f_2(\varphi(t))dt + g_2(\varphi(t))dB_2(t), \quad (9)
\end{align*}
\]

as equations of SDEs (6) and (7), where \( f_1(x) = \alpha_1(\mu_1-x^\theta), g_1(x) = \sigma_1x^\theta, f_2(\varphi) = \alpha_2(\mu_2-\varphi^r) \) and \( g_2(\varphi) = \sigma_2\varphi^\phi \). Let \( H \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}) \), where \( C^{2,1}(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}) \) is the family of all real-valued functions \( H(\cdot, \cdot) \) defined on \( \mathbb{R} \times \mathbb{R}^+ \). Also let \( LH : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be the Itô diffusion operator such that

\[
\begin{align*}
 LH(x, \varphi, t) &= H_t(x, t) + H_x(x, t)f_1(x) + \varphi^2H_{xx}(x, t)g_1(x)^2 \quad (10) \\
 LH(\varphi, t) &= H_t(\varphi, t) + H_\varphi(\varphi, t)f_2(\varphi) + \varphi^2H_{\varphi\varphi}(\varphi, t)g_2(\varphi)^2, \quad (11)
\end{align*}
\]

where \( H_t(x, t), H_t(\varphi, t), H_x(x, t) \) and \( H_{\varphi}(\varphi, t) \) are first-order partial derivatives with respect to \( t, x \) and \( \varphi \), and, \( H_{xx}(x, t) \) and \( H_{\varphi\varphi}(\varphi, t) \) are second-order partial derivatives with respect to \( x \) and \( \varphi \) respectively. Given the
diffusion operator, we can now write the Itô formula as
\[
\begin{align*}
    dH(x(t),t) &= LH(x(t),t) dt + \sqrt{\varphi(t)} H_x(x(t),t) g_1(x(t)) dB_1(t) \quad (12) \\
    dH(\varphi(t),t) &= LH(\varphi(t),t) dt + H_\varphi(\varphi(t),t) g_2(\varphi(t)) dB_2(t) \quad \text{a.s.} \quad (13)
\end{align*}
\]

The reader may refer to [20] for further details about the Itô formula.

3 Theoretical properties

In this section, we discuss pathwise existence of unique positive solutions and finite moments of the solutions to SDEs (8) and (9). The following assumption on the parameters is crucial to obtain the results.

**Assumption 3.1.** The parameters of SDEs (8) and (9) satisfy
\[
\begin{align*}
    1 + \rho &> 2\theta, \quad \text{(14)} \\
    1 + r &> 2\phi, \quad \text{(15)}
\end{align*}
\]

for \(\rho, \theta, \phi, r > 1\).

3.1 Existence and uniqueness of solution

**Lemma 3.2.** Let equation (15) hold. Then there exists a unique global solution \(\varphi(t)\) to SDE (9) on \(t \in [0, T]\) for any given initial data \(\varphi_0 > 0\) and \(\varphi(t) > 0\) a.s.

**Proof.** Apparently, the coefficients of SDE (9) are locally Lipschitz continuous in \(\mathbb{R}\). Hence there exists a unique positive maximal local solution \(\varphi(t)\) on \(t \in [0, \tau_e]\), where \(\tau_e\) is the explosion time (e.g., see [20, 21]). Let us extend the domain of SDE (9) from \(\mathbb{R}_+\) to \(\mathbb{R}\) by setting the coefficients to 0 for \(\varphi(t) < 0\). Then for every sufficiently large integer \(n > 0\), such that \(\varphi(0) \in \left(\frac{1}{n}, n\right)\), define the stopping time as
\[
\tau_n = \inf\{t \in [0, \tau_e) : |\varphi(t)| \notin (1/n, n)\} \quad (16)
\]

and set \(\tau_\infty = \lim_{n \to \infty} \tau_n\). To complete the proof, we need to show that \(\tau_\infty = \infty\) a.s. That is, it is enough to prove that \(\mathbb{P}(\tau_n \leq T) \to 0\) as \(n \to \infty\) for any given \(T > 0\) and hence, \(\mathbb{P}(\tau_\infty = \infty) = 1\). For \(\gamma \in (0, 1)\), define a \(C^2\)-function \(H : \mathbb{R}_+ \to \mathbb{R}_+\) by
\[
H(\varphi) = \varphi^\gamma - 1 - \gamma \log(\varphi). \quad (17)
\]
Apparently, $H(\varphi) \to \infty$ as $\varphi \to 0$ or $\varphi \to \infty$. By (11), we compute

$$LH(\varphi) = \gamma(\varphi^{\gamma - 1} - 1/\varphi)f_2(\varphi) + \frac{1}{2}(\gamma(\gamma - 1)\varphi^{\gamma - 2} + \gamma\varphi^{-2})g_2(\varphi)^2$$

$$= \alpha_2\mu_2\varphi^{\gamma - 1} - \alpha_2\gamma\varphi^{\gamma + r - 1} - \alpha_2\mu_2\varphi^{-1} + \alpha_2\gamma\varphi^{r - 1}$$

$$+ \frac{\sigma_2}{2}\gamma(\gamma - 1)\varphi^{\gamma + 2\phi - 2} + \frac{\sigma_2}{2}\gamma\varphi^{2\phi - 2}.$$ 

So clearly, for $\gamma \in (0, 1)$ and by (15), we infer $-\alpha_2\mu_2\gamma\varphi^{-1}$ leads and tends to $-\infty$ for small $\varphi$. Similarly, we infer $-\alpha_2\gamma\varphi^{\gamma + r - 1}$ leads and also tends to $-\infty$ for large $\varphi$. So there exists a constant $K_0$ such that

$$LH(\varphi) \leq K_0. \quad (18)$$

By the Itô formula, we have

$$\mathbb{E}[H(\varphi(T \wedge \tau_n))] \leq H(\varphi_0) + K_0T. \quad (19)$$

It then follows that,

$$\mathbb{P} (\tau_n \leq T) \leq \frac{H(\varphi_0) + K_0T}{H(1/n) \wedge H(n)}. \quad (20)$$

This implies that $\lim_{n \to \infty} \mathbb{P}(\tau_n \leq T) \to 0$ as required. \hfill \Box

**Lemma 3.3.** Let Assumption 3.1 hold. Then for any given initial data $x_0 > 0$ and $\varphi_0 > 0$, there exists a unique global solution $x(t)$ to SDE (8) on $t \in [0, T]$ and $x(t) > 0$ a.s.

**Proof.** Similarly, we treat SDE (8) as an SDE in $\mathbb{R}^2$ by setting its coefficients to 0 whenever $x(t) < 0$ or $\varphi(t) < 0$. Obviously, the coefficients are locally Lipschitzian. Thus there exists a unique positive maximal local solution $(x(t), \varphi(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time (e.g., see [21]). So for any sufficiently large integer $n > 0$, define the stopping times

$$\varsigma_n = \tau_e \wedge \inf \{t \in [0, \tau_e) : |x(t)| \notin (1/n, n)\},$$

$$\tau_m = \tau_e \wedge \inf \{t \in [0, \tau_e) : |\varphi(t)| \notin (1/m, m)\}.$$ 

Now let

$$\varsigma_{mn} = \varsigma_n \wedge \tau_m. \quad (21)$$

Set $\varsigma_\infty = \lim_{n \to \infty} \varsigma_n$ and $\tau_\infty = \lim_{m \to \infty} \tau_m$. Define a $C^2$-function by

$$H(x) = x^\gamma - 1 - \gamma\log(x). \quad (22)$$

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for $\gamma \in (0, 1)$. Then for $s \in [0, T \wedge \varrho_{mn}]$, we apply (10) to compute

$$LH(x(s), \varphi(s))$$

$$= \gamma(x(s)^{\gamma - 1} - 1/x(s))f_1(x(s)) + \frac{1}{2}(\gamma(\gamma - 1)x(s)^{\gamma - 2} + \gamma x(s)^{-2})\varphi(s)g_1(x(s))^2$$

$$\leq \alpha_1 \mu_1 x(s)^{\gamma - 1} - \alpha_1 \gamma x(s)^{\gamma + \rho - 1} - \alpha_1 \mu_1 x(s)^{-1} + \alpha_1 \gamma x(s)^{\rho - 1}$$

$$+ \frac{\sigma_1}{2} \gamma(\gamma - 1)x(s)^{\gamma + 2\theta - 2} + \frac{\sigma_1}{2} \gamma x(s)^{2\theta - 2};$$

Moreover, we can derive that

$$E[H(x(T \wedge \varrho_{mn}))] = E[H(x(T \wedge \varsigma_n \wedge \tau_m))] \geq E[H(x(\varsigma_n))|_{\varsigma_n \leq T \wedge \tau_m}]$$

$$\geq [H(1/n) \wedge H(n)]P(\varsigma_n \leq T \wedge \tau_m).$$

By the Itô formula, we derive that

$$E[H(x(T \wedge \varrho_{mn}))] \leq H(x_0) + E \int_0^{T \wedge \varrho_{mn}} LH(x(s), \varphi(s))ds.$$ 

This implies

$$[H(1/n) \wedge H(n)]P(\varsigma_n \leq T \wedge \tau_m) \leq H(x_0) + E \int_0^{T \wedge \varrho_{mn}} LH(x(s), \varphi(s))ds.$$ 

Meanwhile, for $\gamma \in (0, 1)$ and by (15), we can find a constant $K_1$ such that

$$[H(1/n) \wedge H(n)]P(\varsigma_n \leq T \wedge \tau_m) \leq H(x_0) + K_1 T. \quad (23)$$

This means we have

$$P(\varsigma_n \leq T \wedge \tau_m) \leq \frac{H(x_0) + K_1 T}{H(1/n) \wedge H(n)}. \quad (24)$$

So, by letting $n \to \infty$, we obtain $P(\varsigma_n \leq T \wedge \tau_m) \to 0$. By setting $m \to \infty$ and using Lemma 3.2, we have $P(\varsigma_\infty \leq T) = 0$. This implies $P(\varsigma_\infty > T) = 1$. 

### 3.2 Finite moments

In the sequel, we show that the moments of SDEs (8) and (9) are finite.

**Lemma 3.4.** Let equation (15) hold. Then for any $p \geq 2$, the solution $\varphi(t)$ to SDE (9) satisfies

$$\sup_{0 \leq t < \infty} (E|\varphi(t)|^p) \leq c_1,$$ 

where $c_1$ is a constant.
See [18] for the proof.

**Lemma 3.5.** Let Assumption 3.1 hold. Then for any $p \geq 2$, the solution $x(t)$ to SDE (9) obeys

$$
\sup_{0 \leq t < \infty} (\mathbb{E}|x(t)|^p 1_{(t \leq \tau^*_m)}) \leq c_2,
$$

where for any sufficiently large integer $m > 0$,

$$
\tau^*_m = \inf\{t \geq 0 : \varphi(t) \notin (1/m, m)\}
$$

and $c_2$ is a constant dependent on $m$.

**Proof.** For any sufficiently large integer $n > 0$, define the stopping times

$$
\varsigma^*_n = \inf\{t \geq 0 : x(t) \notin (1/n, n)\}.
$$

Then set $\varrho^*_m = \varsigma^*_n \wedge \tau^*_m$. For $s \in [0, t \wedge \varrho^*_m]$, we apply (11) to $H(x, t) = e^t x^p$ to compute

$$
LH(x(s), \varphi(s))
$$

$$
= e^s x(s)^p + pe^s x(s)^{p-1} f_1(x(s)) + \frac{1}{2} p(p-1)e^s x(s)^{p-2} \varphi(s) g_1(x(s))^2
$$

$$
= e^s(x(s)^p + \alpha_1 \mu_1 x(s)^{p-1} - \alpha_1 \rho x(s)^{p+1} - \frac{\sigma_1}{2} p(p-1) \varphi(s)x(s)^{2p-p-2})
$$

$$
\leq e^s(x(s)^p + \alpha_1 \mu_1 x(s)^{p-1} - \alpha_1 \rho x(s)^{p+1} - \frac{m \sigma_1}{2} p(p-1) x(s)^{2p-2}).
$$

By the Itô formula, we get

$$
\mathbb{E}[e^{t \wedge \varrho^*_m} | x(t \wedge \varrho^*_m)|^p] \leq x_0^p + \mathbb{E} \int_0^{t \wedge \varrho^*_m} LH(x(s), \varphi(s)) ds.
$$

Noting that

$$
\mathbb{E}[e^{t \wedge \varrho^*_m} | x(t \wedge \varrho^*_m)|^p] = \mathbb{E}[e^{t \wedge \varsigma^*_n \wedge \tau^*_m} | x(t \wedge \varsigma^*_n \wedge \tau^*_m)|^p]
$$

$$
\geq \mathbb{E}[e^{t \wedge \varsigma^*_n} | x(t \wedge \varsigma^*_n)|^p 1_{(t \wedge \varsigma^*_n \leq \tau^*_m)}],
$$

we obtain

$$
\mathbb{E}[e^{t \wedge \varsigma^*_n} | x(t \wedge \varsigma^*_n)|^p 1_{(t \wedge \varsigma^*_n \leq \tau^*_m)}] \leq x_0^p + \mathbb{E} \int_0^{t \wedge \varrho^*_m} LH(x(s), \varphi(s)) ds.
$$
So, by Assumption 3.1, we can find a constant $K_3$ such that

$$E[e^{t \wedge \zeta^*_n} | x(t \wedge \zeta^*_n)|^p 1_{(t \wedge \zeta^*_n) \leq \tau^*_n}] \leq x_0^p + e^t K_3$$

By letting $n \to \infty$, we can apply the Fatou lemma to have

$$E|x(t)|^p 1_{(t \leq \tau^*_n)} \leq x_0^p e^{-t} + K_3,$$

and consequently,

$$\sup_{0 \leq t < \infty} (E|x(t)|^p 1_{(t \leq \tau^*_n)}) \leq c_2$$

as the desired assertion. The proof is thus complete.

\section{Numerical method}

In this section, we construct the truncated EM method to approximate SDEs (8) and (9). But before then, we need to introduce the following lemmas which are needed to perform the convergence analysis (see [19]).

\begin{lemma}
For any $R > 0$, there exist positive constants $K_R$ and $L_R$ such that the coefficients of SDE (8) and SDE (9) satisfy

$$|f_1(x) - f_1(\bar{x})| \vee |g_1(x) - g_1(\bar{x})| \leq K_R |x - \bar{x}|,$$

$$|f_2(\varphi) - f_2(\bar{\varphi})| \vee |g_2(\varphi) - g_2(\bar{\varphi})| \leq L_R |\varphi - \bar{\varphi}|$$

for all $\varphi, \bar{\varphi} \in \mathbb{R}$ and $x, \bar{x} \in \mathbb{R}^2$ with $|x| \vee |\bar{x}| \vee |\varphi| \vee |\bar{\varphi}| \leq R$.
\end{lemma}

\begin{lemma}
Let Assumption 3.1 hold. Then for any $p \geq 2$, there exist $K_4 = K(p) > 0$ and $K_5 = K(p) > 0$ such that the coefficients terms of SDE (8) and (9) fulfil

$$xf_1(x) + p - \frac{1}{2} |\sqrt{\varphi}g_1(x)|^2 \leq K_4 (1 + \varphi |x|^2)$$

$$\varphi f_2(\varphi) + p - \frac{1}{2} |g_2(\varphi)|^2 \leq K_5 (1 + |\varphi|^2)$$

$\forall \varphi \in \mathbb{R}_+, \forall x \in \mathbb{R}^2_+$. See [19] for the proof.
\end{lemma}

\subsection{Numerical schemes}

To begin with, let us extend the domain of SDE (9) from $\mathbb{R}_+$ to $\mathbb{R}$ and SDE (8) from $\mathbb{R}^2_+$ to $\mathbb{R}^2$. We should mention that these extensions do not affect the positivity of the solutions and the local Lipschitz conditions. We
define the truncated scheme by first choosing a strictly increasing continuous
function \( \nu : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \nu(r) \to \infty \) as \( r \to \infty \) and
\[
\sup_{|x| \vee |\varphi| \leq r} \left( |f_1(x)| \vee |f_2(\varphi)| \vee |g_1(x) \vee g_2(\varphi) \right) \leq \nu(r), \quad \forall r \geq 0. \tag{31}
\]
Denote by \( \nu^{-1} \) the inverse function of \( \nu \) and we see that \( \nu^{-1} \) is strictly
ingcreasing continuous function from \( [\nu(0), \infty) \) to \( \mathbb{R}_+ \). We also choose a
number \( \Delta^* \in (0, 1] \) and a strictly decreasing function \( h : (0, \Delta^*] \to (0, \infty) \)
such that
\[
h(\Delta^*) \geq \nu(1), \lim_{\Delta \to 0} h(\Delta) = \infty \text{ and } \Delta^{1/4} h(\Delta) \leq 1 \quad \forall \Delta \in (0, 1]. \tag{32}
\]
For any given step size \( \Delta \in (0, 1) \), we define the truncated functions by
\[
f_1^\Delta(x) = \begin{cases} f_1(x \wedge \nu^{-1}(h(\Delta))), & \text{if } x \geq 0, \\ \alpha_1 \mu_1, & \text{if } x < 0, \end{cases}
\]
\[
g_1^\Delta(x) = \begin{cases} g_1(x \wedge \nu^{-1}(h(\Delta))), & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}
\]
\[
f_2^\Delta(\varphi) = \begin{cases} f_2(\varphi \wedge \nu^{-1}(h(\Delta))), & \text{if } \varphi \geq 0, \\ \alpha_2 \mu_2, & \text{if } \varphi < 0, \end{cases}
\]
and
\[
g_2^\Delta(\varphi) = \begin{cases} g_2(\varphi \wedge \nu^{-1}(h(\Delta))), & \text{if } \varphi \geq 0, \\ 0, & \text{if } \varphi < 0, \end{cases}
\]
for \( \varphi \in \mathbb{R} \) and \( x \in \mathbb{R}^2 \). Clearly, we observe
\[
|f_1^\Delta(x)| \vee |f_2^\Delta(\varphi)| \vee g_1^\Delta(x) \vee g_2^\Delta(\varphi) \leq \nu(\nu^{-1}(h(\Delta))) = h(\Delta) \tag{33}
\]
for \( \varphi \in \mathbb{R}, x \in \mathbb{R}^2 \). The truncated functions \( f_1^\Delta \) and \( g_1^\Delta(x) \), and \( f_2^\Delta \) and \( g_2^\Delta \)
maintain \([28]\) and \([29]\) respectively as shown in the following lemma.

**Lemma 4.3.** Let Assumption \([3.1]\) hold. Then, for all \( \Delta \in (0, \Delta^*) \) and \( p \geq 2 \),
the truncated functions satisfy
\[
x f_1^\Delta(x) + \frac{p - 1}{2} |\sqrt{\varphi} g_1^\Delta(x)|^2 \leq K_6 (1 + \varphi |x|^2), \tag{34}
\]
\[
\varphi f_2^\Delta(\varphi) + \frac{p - 1}{2} |g_2^\Delta(\varphi)|^2 \leq K_7 (1 + |\varphi|^2) \tag{35}
\]
\( \forall \varphi \in \mathbb{R}_+, \forall x \in \mathbb{R}^2_+ \), where \( K_6 \) and \( K_7 \) are independent of \( \Delta \).
Proof. See [19] for the proof of (35). To prove (34), fix any $\Delta \in (0, \Delta^*)$. Then for $\varphi \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with $|x| \vee |\varphi| \leq \nu^{-1}(h(\Delta))$, by (30), we obtain
\[
x f_1^A(x) + \frac{p-1}{2} |\sqrt{\varphi} g_1^A(x)|^2 \leq x f_1(x) + \frac{p-1}{2} |\sqrt{\varphi} g_1(x)|^2 \leq K_4(1 + \varphi |x|^2)
\]
as required. For $\varphi \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with $|x| \vee |\varphi| > \nu^{-1}(h(\Delta))$, we get
\[
x f_1^A(x) + \frac{p-1}{2} |\sqrt{\varphi} g_1^A(x)|^2 \leq x f_1(\nu^{-1}(h(\Delta))) + \frac{p-1}{2} |\sqrt{\varphi} g_1(\nu^{-1}(h(\Delta)))|^2
\]
\[
\leq \nu^{-1}(h(\Delta)) f_1(\nu^{-1}(h(\Delta))) + \frac{p-1}{2} |\sqrt{\nu^{-1}(h(\Delta))} g_1(\nu^{-1}(h(\Delta)))|^2
\]
\[
+ \left( \frac{x}{\nu^{-1}(h(\Delta))} - 1 \right) \nu^{-1}(h(\Delta)) f_1(\nu^{-1}(h(\Delta)))
\]
\[
\leq K_4(1 + \nu^{-1}(h(\Delta))|\nu^{-1}(h(\Delta))|^2)
\]
\[
+ \left( \frac{x}{\nu^{-1}(h(\Delta))} - 1 \right) \nu^{-1}(h(\Delta)) f_1(\nu^{-1}(h(\Delta))).
\]
Again, we observe from [29] that $x f_1(x) \leq K_4(1 + \varphi |x|^2)$ for any $\varphi \in \mathbb{R}$ and $x \in \mathbb{R}^2$, we obtain
\[
x f_1^A(x) + \frac{p-1}{2} |\sqrt{\varphi} g_1^A(x)|^2 \leq K_4(1 + \nu^{-1}(h(\Delta))|\nu^{-1}(h(\Delta))|^2)
\]
\[
+ \left( \frac{x}{\nu^{-1}(h(\Delta))} - 1 \right) K_4(1 + \nu^{-1}(h(\Delta))|\nu^{-1}(h(\Delta))|^2)
\]
\[
\leq \frac{x}{\nu^{-1}(h(\Delta))} K_4(1 + \nu^{-1}(h(\Delta))|\nu^{-1}(h(\Delta))|^2)
\]
\[
\leq x \cdot K_5(1 + \nu^{-1}(h(\Delta))|\nu^{-1}(h(\Delta))|)
\]
\[
\leq x \cdot K_5(1 + \varphi \cdot x) \leq K_7(1 + \varphi |x|^2),
\]
where $K_7 = 2K_4$ as the required assertion in (34). We should mention that using these proofs, we could similarly establish the case when $\varphi \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with $|x| > \nu^{-1}(h(\Delta))$ and $|\varphi| \leq \nu^{-1}(h(\Delta))$ and the case when $\varphi \in \mathbb{R}$ and $x \in \mathbb{R}^2$ with $|x| \leq \nu^{-1}(h(\Delta))$ and $|\varphi| > \nu^{-1}(h(\Delta))$. \qed

Let us now form the discrete-time truncated EM solutions $Y_\Delta(t_k) \approx \varphi(t_k)$ and $X_\Delta(t_k) \approx x(t_k)$ to SDEs [8] and [9] for $t_k = k\Delta$ respectively, by setting $Y_\Delta(0) = \varphi_0$, $X_\Delta(0) = x_0$ and computing
\[
Y_\Delta(t_{k+1}) = Y_\Delta(t_k) + f_2^\Delta(Y_\Delta(t_k)) \Delta + g_2^\Delta(Y_\Delta(t_k)) \Delta B_{2k}
\]
(36)
for $k = 0, 1, 2, \cdots$, where $\Delta = t_{k+1} - t_k$, $\Delta B_{1k} = (B_1(t_{k+1}) - B_1(t_k))$ and $\Delta B_{2k} = (B_2(t_{k+1}) - B_2(t_k))$. Let us now form corresponding versions of the continuous-time truncated EM solutions. The first versions are defined by

$$
\bar{\phi}_\Delta(t) = \sum_{k=0}^{\infty} Y_\Delta(t_k) 1_{[t_k,t_{k+1})}(t) \quad \text{(38)}
$$

$$
\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) 1_{[t_k,t_{k+1})}(t). \quad \text{(39)}
$$
on $t \geq 0$. These are the continuous-time step processes. The other versions are the continuous-time continuous processes defined on $t \geq 0$ by

$$
\varphi_\Delta(t) = \varphi(0) + \int_0^t f_2^\Delta(\bar{\varphi}_\Delta(s)) ds + \int_0^t g_2^\Delta(\bar{\varphi}_\Delta(s)) dB_2(s) \quad \text{(40)}
$$

$$
x_\Delta(t) = x(0) + \int_0^t f_1^\Delta(\bar{x}_\Delta(s)) ds + \int_0^t \sqrt{\bar{\varphi}(s)} |g_1^\Delta(\bar{x}_\Delta(s))| dB_1(s). \quad \text{(41)}
$$

Obviously $\varphi_\Delta(t)$ and $x_\Delta(t)$ are Itô processes on $t \geq 0$ respectively satisfying Itô differentials

$$
d\varphi_\Delta(t) = f_2^\Delta(\bar{\varphi}_\Delta(t)) dt + g_2^\Delta(\bar{\varphi}_\Delta(t)) dB_2(t)
$$

$$
dx_\Delta(t) = f_1^\Delta(\bar{x}_\Delta(t)) dt + \sqrt{\bar{\varphi}(t)} |g_1^\Delta(\bar{x}_\Delta(t))| dB_1(t).
$$

For all $k \geq 0$, we clearly observe that $\varphi_\Delta(t_k) = \bar{\varphi}_\Delta(t_k) = Y_\Delta(t_k)$ and $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$.

5 Numerical properties

In this section, we establish the moment bounds and finite time strong convergence results for the truncated EM solutions.

5.1 Finite moments

In the sequel, let us recall the following useful lemmas. The proofs of these lemmas are in [19] and therefore omitted.

**Lemma 5.1.** Let equation (15) hold. Then for any $p \geq 2$, the solution of (40) satisfies

$$
\sup_{0 \leq \Delta \leq \Delta^*} \sup_{0 \leq t \leq T} (E|\varphi_\Delta(t)|^p) \leq c_4, \quad \text{(42)}
$$

$\forall T \geq 0$ where $c_4 := c_4(\varphi_0, p, T, K_7)$ may change between occurrences.
It is important to note that (42) also holds for $\bar{\varphi}_\Delta(t)$ because $\varphi_\Delta(t_k)$ and $\bar{\varphi}_\Delta(t_k)$ coincide at discrete time $t_k$ for all $k \geq 0$.

**Lemma 5.2.** For any $\Delta \in (0, \Delta^*)$ and $\forall t \geq 0$, we have
\[ E| \varphi_\Delta(t) - \bar{\varphi}_\Delta(t) |^p \leq c_p \Delta^{p/2} (h(\Delta))^p \] (43)
and consequently,
\[ \lim_{\Delta \to 0} E| \varphi_\Delta(t) - \bar{\varphi}_\Delta(t) |^p = 0, \] (44)
where $c_p$ is a positive constant which depends only on $p$.

In addition to the above lemmas, we also need the following lemmas.

**Lemma 5.3.** For any $\Delta \in (0, \Delta^*)$ and $\forall t \geq 0$, we have
\[ E| x_\Delta(t) - \bar{x}_\Delta(t) |^p \leq C_p \Delta^{p/2} (h(\Delta))^p \] (45)
and consequently,
\[ \lim_{\Delta \to 0} E| x_\Delta(t) - \bar{x}_\Delta(t) |^p = 0, \] (46)
where $C_p$ is a positive constant which depends only on $p$.

**Proof.** Fix any $\Delta \in (0, \Delta^*)$ and $t \geq 0$. Then there is a unique integer $k \geq 0$ such that $t_k \leq t \leq t_{k+1}$. By elementary inequality, we derive
\[
E| x_\Delta(t) - \bar{x}_\Delta(t) |^p = E| x_\Delta(t) - \bar{x}_\Delta(t_k) |^p \\
\leq c(p) \left( E \left| \int_{t_k}^t f_1^\Delta(\bar{x}_\Delta(s)) ds \right|^p + E \left| \int_{t_k}^t \sqrt{\varphi(s)} g_1^\Delta(\bar{x}_\Delta(s)) dB(s) \right|^p \right) \\
\leq c(p) \left( \Delta^{p-1} E \int_{t_k}^t |f_1^\Delta(\bar{x}_\Delta(s))|^p ds + \Delta^{p-2} E \int_{t_k}^t |\sqrt{\varphi(s)} g_1^\Delta(\bar{x}_\Delta(s))|^p ds \right).
\]
So by Lemma (5.1) and (33), we have
\[
E| x_\Delta(t) - \bar{x}_\Delta(t) |^p \leq c(p) \left( \Delta^{p-1} (h(\Delta))^p \Delta + |c_4|^{p/2} \Delta^{(p-2)/2} (h(\Delta))^p \right) \\
\leq c(p) \left( \Delta^p (h(\Delta))^p + |c_4|^{p/2} \Delta^{p/2} (h(\Delta))^p \right) \\
\leq C_p \Delta^{p/2} (h(\Delta))^p,
\]
where $C_p = c(p)(1 \lor |c_4|^{p/2})$. Nothing that $\Delta^{p/2} (h(\Delta))^p \leq \Delta^{p/4}$ from (32), we obtain (47) from (46) by letting $\Delta \to 0$. 

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Lemma 5.4. Let Assumption 3.1 hold. Then for any \( p \geq 2 \), the truncated EM solution of (42) satisfies
\[
\sup_{0 \leq t < \infty} \left( \mathbb{E}|x_{\Delta}(t)|^p 1_{\{t \leq h_{m}\}} \right) \leq c_5,
\]
(47)
where for any sufficiently large integer \( m > 0 \),
\[
h_m = \inf\{ t \geq 0 : \bar{\phi}(t) \notin (1/m, m) \}
\]
and \( c_5 := c_5(x_0, \phi_0, p, T, K_6, m) \) may change value between occurrences.

Proof. Fix any \( \Delta \in (0, \Delta^*) \) and for every sufficiently large integer \( n > 0 \),
define
\[
h_n^* = \inf\{ t \geq 0 : x_{\Delta}(t) \notin (1/n, n) \}.
\]
Now set \( \bar{\Delta}_{mn} = h_m \wedge h_n^* \). By the Itô formula, we derive from (41) that
\[
\mathbb{E}|x_{\Delta}(t \wedge \bar{\Delta}_{mn})|^p - |x_0|^p
\leq \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} p|x_{\Delta}(s)|^{p-2} (x_{\Delta}(s)f_1^\Delta(\bar{x}_{\Delta}(s)) + \frac{p-1}{2}|\sqrt{\Delta(s)}g_1^\Delta(\bar{x}_{\Delta}(s))|^2) \, ds
= J_1 + J_2,
\]
where
\[
J_1 = \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} p|x_{\Delta}(s)|^{p-2} (\bar{x}_{\Delta}(s)f_1^\Delta(\bar{x}_{\Delta}(s)) + \frac{p-1}{2}|\sqrt{\Delta(s)}g_1^\Delta(\bar{x}_{\Delta}(s))|^2) \, ds
J_2 = \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} p|x_{\Delta}(s)|^{p-2}(x_{\Delta}(s) - \bar{x}_{\Delta}(s))f_1^\Delta(\bar{x}_{\Delta}(s))ds.
\]
By the Young inequality, we have
\[
J_1 = K_6 \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} |x_{\Delta}(s)|^{p-2}(1 + |\bar{x}_{\Delta}(s)|^2) \, ds
\leq K_6 \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} \left( |x_{\Delta}(s)|^{(p-2)\frac{p}{p-2}} \left( (1 + |\bar{x}_{\Delta}(s)|^2) |\bar{x}_{\Delta}(s)|^p \right)^{\frac{2}{p}} \right) \, ds
\leq K_6 \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} \left( (p-2)|x_{\Delta}(s)|^p + 2(1 + |\bar{x}_{\Delta}(s)|^2 |\bar{x}_{\Delta}(s)|^p) \right) ds
\leq K_6 \mathbb{E} \int_0^{t \wedge \bar{\Delta}_{mn}} \left( (p-2)|x_{\Delta}(s)|^p + 2m^2 |\bar{x}_{\Delta}(s)|^p \right) ds
\]
where \( r_1 = K_6[2T + ((p - 2) \lor 2m^\frac{p}{2})] \). Also, by Lemma 5.1 we have
\[
\mathcal{J}_2 \leq \mathbb{E} \int_0^{t \wedge \bar{\alpha}_{mn}} p|x_\Delta(s)|^{p-2}(x_\Delta(s) - \bar{x}_\Delta(s)) f_1^\Delta(\bar{x}_\Delta(s)) ds
\]
\[
\leq p\mathbb{E}\left( \int_0^{t \wedge \bar{\alpha}_{mn}} |x_\Delta(s)|^{(p-2)} \frac{p}{(p-2)} ds \right)^\frac{(p-2)}{p} \left( \int_0^{t \wedge \bar{\alpha}_{mn}} (x_\Delta(s) - \bar{x}_\Delta(s))^\frac{1}{2} f_1^\Delta(\bar{x}_\Delta(s))^\frac{1}{2} ds \right)^\frac{2}{p}
\]
\[
\leq (p - 2)\mathbb{E} \int_0^{t \wedge \bar{\alpha}_{mn}} |x_\Delta(s)|^p ds + 2 \int_0^{T} \left( \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)||f_1^\Delta(\bar{x}_\Delta(s))| \right)^\frac{p}{2} ds
\]
\[
\leq (p - 2)\mathbb{E} \int_0^{t \wedge \bar{\alpha}_{mn}} |x_\Delta(s)|^p ds + 2c_1^{1/2}T^{p/4}(\Delta^p)^p.
\]
Noting from (32) that \([\Delta^{1/4}(h(\Delta))]^p \leq 1\), we have
\[
\mathcal{J}_2 \leq r_2\mathbb{E} \int_0^{t \wedge \bar{\alpha}_{mn}} |x_\Delta(s)|^p ds,
\]
where \( r_2 = (2c_1^{1/2})T \lor (p - 2) \). We now combine \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) to get
\[
\mathbb{E}|x_\Delta(t \wedge \bar{\alpha}_{mn})|^p \leq |x_0|^p + \mathbb{E} \int_0^{t \wedge \bar{\alpha}_{mn}} (r_1|x_\Delta(s)|^p + (r_1 + r_2)|\bar{x}_\Delta(s)|^p) ds
\]
\[
\leq |x_0|^p + (2r_1 + r_2) \int_0^t \sup_{0 \leq \bar{\alpha}_s \leq t} \left( \mathbb{E}|x_\Delta(t \wedge \bar{\alpha}_s)|^p \right) ds.
\]
The Gronwall inequality yields
\[
\sup_{0 \leq t < \infty} (\mathbb{E}|x_\Delta(t \wedge \bar{\alpha}_{mn})|^p) \leq c_5
\]
where \( c_5 = |x_0|^p e^{(2r_1 + r_2)} \) is independent of \( \Delta \). Noting that
\[
\sup_{0 \leq t < \infty} (\mathbb{E}|x_\Delta(t \wedge \bar{\alpha}_{mn})|^p) \geq \sup_{0 \leq t < \infty} (\mathbb{E}|x_\Delta(t \wedge h_n^*)|^p \mathbb{I}_{1_{t \wedge h_n^*} \leq k_n^*})
\]
we can set \( n \to \infty \) to obtain
\[
\sup_{0 \leq t < \infty} (\mathbb{E}|x_\Delta(t)|^p \mathbb{I}_{1_{t \leq k_n^*}}) \leq c_5
\]
as the desired result. The proof is now complete. \( \square \)
5.2 Strong convergence

Before we establish the main result in this section, we need the following lemmas. The proofs of these lemmas could be found in [18].

**Lemma 5.5.** Let equation (15) hold and \( T > 0 \) be fixed. Then for any \( \epsilon \in (0, 1) \), there exists a pair of positive constants \( n = n(\epsilon) \) and \( \Delta^1 = \Delta^1(\epsilon) \) such that for each \( \Delta \in (0, \Delta^1] \), we have

\[
P(\vartheta_n \leq T) \leq \epsilon,
\]

where

\[
\vartheta_n = \vartheta(\Delta, n) = \inf\{ t \in [0, T] : \varphi(\Delta(t)) \notin (1/n, n) \}. \tag{49}
\]

is a stopping time.

**Lemma 5.6.** Let equation (15) hold. Then for any \( p \geq 2, T > 0 \), we have

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |\varphi(\Delta \wedge \vartheta_n) - \varphi(t \wedge \vartheta_n)|^p \right) \leq K_1 \Delta^{p/4} \tag{50}
\]

for any sufficiently large \( n \) and any \( \Delta \in (0, \Delta^* \], \) where \( K_1 \) is a constant independent of \( \Delta \) and \( \vartheta_n \) is a stopping time. Consequently, we have

\[
\lim_{\Delta \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} |\varphi(\Delta \wedge \vartheta_n) - \varphi(t \wedge \vartheta_n)|^p \right) = 0. \tag{51}
\]

Let us proceed to establish the following useful lemmas.

**Lemma 5.7.** Let Assumption 3.1 hold and \( T > 0 \) be fixed. Define a stopping time by

\[
\vartheta_n^* = \vartheta^*(\Delta, n) = \inf\{ t \in [0, T] : x(\Delta(t)) \notin (1/n, n) \}. \tag{52}
\]

Then for any \( \epsilon \in (0, 1) \), there exists a pair of positive constants \( n = n(\epsilon) \) and \( \Delta^1 = \Delta^1(\epsilon) \) such that for each \( \Delta \in (0, \Delta^1] \), we have

\[
P(\vartheta_n^* \leq T) \leq \epsilon. \tag{53}
\]

**Proof.** We apply the Itô formula to (22) to compute

\[
\mathbb{E}(H(x(\vartheta_n^*))) - H(x(0)) = \mathbb{E} \int_0^{\vartheta_n^*} \left( H_x(x(s))f^\Delta_x(\bar{x}(s)) + \frac{1}{2} H_{xx}(x(s))\bar{\varphi}(s)g^\Delta\bar{\varphi}(s)^2 \right) ds
\]

\[
\leq J_3 + J_4 + J_5
\]

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where,
\[ J_3 = E \int_0^{t \wedge \vartheta_n} \left( H_x(x_\Delta(s)) f_1^\Delta(x_\Delta(s)) + \frac{1}{2} H_{xx}(x_\Delta(s)) \varphi_\Delta(s) g_1^\Delta(x_\Delta(s))^2 \right) ds \]
\[ J_4 = E \int_0^{t \wedge \vartheta_n} H_x(x_\Delta(s)) \left( f_1^\Delta(\bar{x}_\Delta(s)) - f_1^\Delta(x_\Delta(s)) \right) ds \]
\[ J_5 = E \int_0^{t \wedge \vartheta_n} \frac{1}{2} H_{xx}(x_\Delta(s)) \left( \varphi_\Delta(s) g_1^\Delta(\bar{x}_\Delta(s))^2 - \varphi_\Delta(s) g_1^\Delta(x_\Delta(s))^2 \right) ds. \]

So, by (10) and (14), we can find a constant \( K_9 \) such that
\[ J_3 \leq E \int_0^{t \wedge \vartheta_n} L H(x_\Delta(s), \varphi_\Delta(s)) ds \leq K_9 T. \]

By the definition of the truncated functions, we note for \( s \in [0, t \wedge \vartheta_n] \),
\[ f_1^\Delta(x_\Delta(s)) = f_1(x_\Delta(s)) \quad \text{and} \quad g_1^\Delta(x_\Delta(s)) = g_1(x_\Delta(s)). \]

So by Lemma 4.1, we have where,
\[ J_4 \leq E \int_0^{t \wedge \vartheta_n} H_x(x_\Delta(s)) |f_1(\bar{x}_\Delta(s)) - f_1(x_\Delta(s))| ds \leq E \int_0^{t \wedge \vartheta_n} K_n H_x(x_\Delta(s)) |\bar{x}_\Delta(s) - x_\Delta(s)| ds. \]

Similarly,
\[ J_5 = E \int_0^{t \wedge \vartheta_n} \frac{1}{2} H_{xx}(x_\Delta(s)) \left( \varphi_\Delta(s) g_1(\bar{x}_\Delta(s))^2 - \varphi_\Delta(s) g_1(x_\Delta(s))^2 \right) ds \]
\[ = E \int_0^{t \wedge \vartheta_n} \frac{1}{2} H_{xx}(x_\Delta(s)) \left( \varphi_\Delta(s) g_1(\bar{x}_\Delta(s))^2 - \varphi_\Delta(s) g_1(x(s))^2 \right) ds \]
\[ + E \int_0^{t \wedge \vartheta_n} \frac{1}{2} H_{xx}(x_\Delta(s)) \left( \varphi_\Delta(s) g_1(x(s))^2 - \varphi_\Delta(s) g_1(x_\Delta(s))^2 \right) ds \]
\[ \leq E \int_0^{t \wedge \vartheta_n} \frac{\varphi_\Delta(s)}{2} H_{xx}(x_\Delta(s)) |g_1(\bar{x}_\Delta(s)) - g_1(x_\Delta(s))| |g_1(\bar{x}_\Delta(s)) + g_1(x_\Delta(s))| ds \]
\[ + E \int_0^{t \wedge \vartheta_n} \frac{1}{2} H_{xx}(x_\Delta(s)) g_1(x(s))^2 |\varphi_\Delta(s) - \varphi(s)| ds. \]

Noting from (31) that \( x_\Delta(s), \bar{x}_\Delta(s) \in [1/n, n] \) for \( s \in [0, t \wedge \vartheta_n] \), we have \( g_1(\bar{x}_\Delta(s)) \supset g_1(x_\Delta(s)) \leq \nu(n) \). So by Lemma 4.1, we obtain
\[ J_5 \leq E \int_0^{t \wedge \vartheta_n} \varphi_\Delta(s) H_{xx}(x_\Delta(s)) |g_1(\bar{x}_\Delta(s)) - g_1(x_\Delta(s))| ds \]

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where

So by the Young inequality and Lemmas 5.1, 5.2 and 5.3, we now have

\[ \mathbb{E}(H(x_{\Delta}(t \wedge \vartheta^*_n))) \leq H(x(0)) + K_9 T + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} K_n H_x(x_{\Delta}(s))|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|ds \]

\[ + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} \frac{\nu(n)^2}{2} H_{xx}(x_{\Delta}(s))|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} K_n \bar{\varphi}_{\Delta}(s) H_{xx}(x_{\Delta}(s))|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 \int_0^T \mathbb{E}|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \tau_3 \int_0^T \mathbb{E}(|\bar{\varphi}_{\Delta}(s)|^2(|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^{2} \frac{1}{2}) \right) ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 \int_0^T \mathbb{E}|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \tau_3 \int_0^T \mathbb{E}(|\bar{\varphi}_{\Delta}(s)|^2) \left( \frac{1}{2} (|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^2) \right) \right) ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 \int_0^T \mathbb{E}|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \tau_3 \int_0^T \mathbb{E}(|\bar{\varphi}_{\Delta}(s)|^2) \left( \frac{1}{2} (|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^2) \right) \right) ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 c_p T \Delta^{p/2}(h(\Delta))^p \]

Combining $J_3$, $J_4$ and $J_5$, we then have

\[ \mathbb{E}(H(x_{\Delta}(t \wedge \vartheta^*_n))) \]

\[ \leq H(x(0)) + K_9 T + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} K_n H_x(x_{\Delta}(s))|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|ds \]

\[ + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} \frac{\nu(n)^2}{2} H_{xx}(x_{\Delta}(s))|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \mathbb{E} \int_0^{t \wedge \vartheta^*_n} K_n \bar{\varphi}_{\Delta}(s) H_{xx}(x_{\Delta}(s))|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 \int_0^T \mathbb{E}|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \tau_3 \int_0^T \mathbb{E}(|\bar{\varphi}_{\Delta}(s)|^2(|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^{2} \frac{1}{2}) \right) ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 \int_0^T \mathbb{E}|\bar{\varphi}_{\Delta}(s) - \varphi(s)| ds \]

\[ + \tau_3 \int_0^T \mathbb{E}(|\bar{\varphi}_{\Delta}(s)|^2) \left( \frac{1}{2} (|\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^2) \right) \right) ds \]

\[ \leq H(x(0)) + K_9 T + \tau_2 c_p T \Delta^{p/2}(h(\Delta))^p \]

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This implies
\[
\mathbb{P}(\vartheta_n^* \leq T) \leq \frac{H(x(0)) + K_T + \nu_2 c_p T \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} C_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p T}{H(1/n) \wedge H(n)}. \tag{55}
\]
For any \( \epsilon \in (0, 1) \), we may select sufficiently large \( n \) such that
\[
\frac{H(x(0)) + K_T + \frac{\nu_3}{2} c_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p T}{H(1/n) \wedge H(n)} \leq \epsilon \tag{56}
\]
and sufficiently small of each step size \( \Delta \in (0, \Delta^1) \) such that
\[
\frac{\nu_2 c_p T \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} C_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p \Delta^{p/2}(h(\Delta))^p + \frac{\nu_3}{2} c_p T}{H(1/n) \wedge H(n)} \leq \epsilon. \tag{57}
\]
We now combine (56) and (57) to get the required assertion. \( \square \)

**Lemma 5.8.** Let Assumption 3.1 hold. Set
\[
\vartheta_n^* = \varrho_{mn} \wedge \vartheta_n \wedge \vartheta_n^*,
\]
where \( \varrho_{mn} \), \( \vartheta_n \) and \( \vartheta_n^* \) are (21), (49) and (52) respectively. Then for any \( p \geq 2 \), \( T > 0 \), we have
\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |x_{\Delta}(t \wedge \vartheta_n^*) - x(t \wedge \vartheta_n^*)|^p \right) \leq K_2 \Delta^{p(1/2\wedge1/4\wedge1/8)}(h(\Delta))^{p(1/2\wedge1)} \tag{58}
\]
for any sufficiently large \( n \) and any \( \Delta \in (0, \Delta^* \wedge 1] \), where \( K_2 \) is a constant independent of \( \Delta \). Consequently, we have
\[
\lim_{\Delta \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} |x_{\Delta}(t \wedge \vartheta_n^*) - x(t \wedge \vartheta_n^*)|^p \right) = 0. \tag{59}
\]
Proof. It follows from \((8)\) and \((41)\) that

\[
\mathbb{E}\left( \sup_{0 \leq t \leq t_1} |x_{\Delta}(t \land v_n^*) - x(t \land v_n^*)|^p \right) \leq J_6 + J_7,
\]
where

\[
J_6 = 2^{p-1}\left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} (f_{i \Delta}(\bar{x}_{\Delta}(s)) - f_1(x(s))) ds \right|^p \right)
\]

\[
J_7 = 2^{p-1}\left( \mathbb{E} \left( \sup_{0 \leq t \leq t_1} \left| \int_0^{t_1 \land v_n^*} (\sqrt{|\varphi_{\Delta}(s)|} g_1^\Delta(\bar{x}_{\Delta}(s)) - \sqrt{|\varphi(s)|} g_1(x(s))) dB(s) \right|^p \right) \right).
\]

So by the H"older inequality, \((27)\) and \((54)\), we have

\[
J_6 \leq 2^{p-1} T^{p-1} \left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} f_{i \Delta}(\bar{x}_{\Delta}(s)) - f_1(x(s)) ds \right|^p \right)
\]

\[
\leq 2^{p-1} T^{p-1} \left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} f_1(\bar{x}_{\Delta}(s)) - f_1(x(s)) ds \right|^p \right)
\]

\[
\leq 2^{p-1} T^{p-1} K_n \mathbb{E} \int_0^{t_1 \land v_n^*} |\bar{x}_{\Delta}(s) - x(s)|^p ds
\]

\[
\leq 2^{2(p-1)} T^{p-1} K_n \mathbb{E} \int_0^{t_1 \land v_n^*} |\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^p ds
\]

\[
+ 2^{2(p-1)} T^{p-1} K_n \mathbb{E} \int_0^{t_1 \land v_n^*} |x_{\Delta}(s) - x(s)|^p ds
\]

\[
\leq 2^{2(p-1)} T^{p-1} K_n \mathbb{E} \int_0^T |\bar{x}_{\Delta}(s) - x_{\Delta}(s)|^p ds
\]

\[
+ 2^{2(p-1)} T^{p-1} K_n \mathbb{E} \sup_{0 \leq t \leq s} \mathbb{E}|x_{\Delta}(t \land v_n^*) - x(t \land v_n^*)|^p ds.
\]

By the Burkholder-Davis-Gundy inequality and \((54)\), we also have

\[
J_7 \leq 2^{p-1} T^{p-2} C_p \left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} |\sqrt{|\varphi_{\Delta}(s)|} g_1^\Delta(\bar{x}_{\Delta}(s)) - \sqrt{|\varphi(s)|} g_1(x(s))|^p ds \right) \right)
\]

\[
\leq 2^{p-1} T^{p-2} C_p \left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} |\sqrt{|\varphi_{\Delta}(s)|} g_1(\bar{x}_{\Delta}(s)) - \sqrt{|\varphi(s)|} g_1(x(s))|^p ds \right) \right),
\]

where \(C_p\) is a positive constant. By elementary inequality, we now have

\[
J_7 \leq 2^{2(p-1)} T^{p-2} C_p \left( \mathbb{E} \left| \int_0^{t_1 \land v_n^*} |\sqrt{|\varphi_{\Delta}(s)|} g_1(\bar{x}_{\Delta}(s)) - \sqrt{|\varphi(s)|} g_1(x_{\Delta}(s))|^p ds \right) \right)
\]
\begin{align*}
&+ 2(\ell - 1) T^{\nu^*_n} C_\ell \left( \mathbb{E} \int_0^{\bar{t}_1} |\sqrt{\varphi_\Delta(s)} g_1(x_\Delta(s)) - \sqrt{\varphi(s)} g_1(x(s))|^p ds \right) \\
&\leq 2(\ell - 1) T^{\nu^*_n} C_\ell \left( \mathbb{E} \int_0^{\bar{t}_1} \left| \sqrt{\varphi_\Delta(s)} g_1(x_\Delta(s)) - \sqrt{\varphi(s)} g_1(x(s)) \right|^p ds \right) \\
&+ 2(\ell - 1) T^{\nu^*_n} C_\ell \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi(s)|^p g_1(x_\Delta(s)) \left| \sqrt{\varphi_\Delta(s)} - \sqrt{\varphi(s)} \right|^p ds \right) \\
&\leq 2(\ell - 1) T^{\nu^*_n} C_\ell \left( \mathbb{E} \int_0^{\bar{t}_1} g_1(x_\Delta(s)) \left| \sqrt{\varphi_\Delta(s)} - \sqrt{\varphi(s)} \right|^p ds \right) \\
&+ 2(\ell - 1) T^{\nu^*_n} C_\ell \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi(s)|^{\frac{p}{2}} g_1(x_\Delta(s)) - g_1(x(s)) |^p ds \right).
\end{align*}

It follows from \([16]\) and \([31]\) that

\begin{align*}
\mathcal{J}_7 &\leq 2(\ell - 1) T^{\nu^*_n} C_\ell \nu(n)^p \left( \mathbb{E} \int_0^{\bar{t}_1} |\sqrt{\varphi_\Delta(s)} - \sqrt{\varphi(s)}|^p ds \right) \\
&+ 2(\ell - 1) T^{\nu^*_n} C_\ell \nu(n) \left( \mathbb{E} \int_0^{\bar{t}_1} |g_1(x_\Delta(s)) - g_1(x(s))|^p ds \right) \\
&\leq 2(\ell - 1) T^{\nu^*_n} C_\ell \nu(n) \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi(s) - \varphi_\Delta(s)|^p ds \right) \\
&+ 2(\ell - 1) T^{\nu^*_n} C_\ell \nu(n) \left( \mathbb{E} \int_0^{\bar{t}_1} |g_1(x_\Delta(s)) - g_1(x(s))|^p ds \right).
\end{align*}

Then, by elementary inequality and \([27]\), we have

\begin{align*}
\mathcal{J}_7 &\leq 2(\ell - 1) 2^{\frac{\ell}{2} - 1} T^{\nu^*_n} C_\ell \nu(n)^p \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi(s) - \varphi_\Delta(s)|^\frac{p}{2} ds \right) \\
&+ 2(\ell - 1) 2^{\frac{\ell}{2} - 1} T^{\nu^*_n} C_\ell \nu(n)^p \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi(s) - \varphi_\Delta(s)|^\frac{p}{2} ds \right) \\
&+ 2^{\ell - 1} T^{\nu^*_n} C_\ell \nu(n)^p K_n \left( \mathbb{E} \int_0^{\bar{t}_1} |x_\Delta(s) - x(s)|^p ds \right) \\
&+ 2^{\ell - 1} T^{\nu^*_n} C_\ell \nu(n)^p K_n \left( \mathbb{E} \int_0^{\bar{t}_1} |x_\Delta(s) - x(s)|^p ds \right) \\
&\leq 2(\ell - 1) 2^{\frac{\ell}{2} - 1} T^{\nu^*_n} C_\ell \nu(n)^p \left( \mathbb{E} \int_0^{\bar{t}_1} |\varphi_\Delta(s) - \varphi_\Delta(s)|^p ds \right)^\frac{1}{2} \\
&+ 2^{\ell - 1} 2^{\frac{\ell}{2} - 1} T^{\nu^*_n} C_\ell \nu(n)^p \int_0^{\bar{t}_1} \left( \sup_{0 \leq t \leq s} \mathbb{E} |\varphi_\Delta(t \wedge \nu_n) - \varphi(t \wedge \nu_n)|^p \right)^\frac{1}{2} ds \\
&+ 2^{\ell - 1} 2^{\frac{\ell}{2} - 1} T^{\nu^*_n} C_\ell \nu(n)^p K_n \int_0^{T} \mathbb{E} |\varphi_\Delta(s) - x(s)|^p ds
\end{align*}
\[ + 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p K_n \int_0^{t_1} \sup_{0 \leq t \leq s} \mathbb{E}|x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p ds. \]

So by combining \( J_6 \) and \( J_7 \), we now get

\[
\mathbb{E}\left( \sup_{0 \leq t \leq t_1} |x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p \right) \leq 2^{p-1} T^{p-1} K_n \int_0^T \mathbb{E} |x_\Delta(s) - x_\Delta(s)|^p ds \\
+ 2^{p-1} T^{p-1} K_n \int_0^{t_1} \sup_{0 \leq t \leq s} \mathbb{E}|x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p ds \\
+ 2^{p-1} T^{p-1} \bar{C}_{p}\mu(n)^p \int_0^T \mathbb{E} |\bar{\varphi}_\Delta(s) - \varphi_\Delta(s)| \frac{1}{2} ds \\
+ 2^{p-1} T^{p-1} \bar{C}_{p}\mu(n)^p \int_0^{t_1} \left( \sup_{0 \leq t \leq s} \mathbb{E}|\bar{\varphi}_\Delta(t \wedge \nu_n^*) - \varphi_\Delta(t \wedge \nu_n^*)|^p \right) \frac{1}{2} ds \\
+ 2^{p-1} T^{p-1} \bar{C}_{p}\mu(n)^p \int_0^T \mathbb{E} |x_\Delta(s) - x_\Delta(s)|^p ds \\
+ 2^{p-1} T^{p-1} \bar{C}_{p}\mu(n)^p K_n \int_0^{t_1} \sup_{0 \leq t \leq s} \mathbb{E}|x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p ds. \]

So, by Lemmas 5.2, 5.7 and 5.7, we hence have

\[
\mathbb{E}\left( \sup_{0 \leq t \leq t_1} |x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p \right) \\
\leq 2^{p-1} 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p T(c_p \Delta^{p/2} (h(\Delta))^p) \frac{1}{2} \\
+ 2^{p-1} 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p T(K_1 \Delta^{p/4}) \frac{1}{2} \\
+ (2^{p-1} T^{p-1} K_n + 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p K_n) C_p \Delta^{p/2} (h(\Delta))^p \\
+ (2^{p-1} T^{p-1} K_n + 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p K_n) \int_0^{t_1} \sup_{0 \leq t \leq s} \mathbb{E}|x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p ds. \]

In particular, we have

\[
\mathbb{E}\left( \sup_{0 \leq t \leq t_1} |x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p \right) \leq (\varpi_3 + \varpi_4 + \varpi_5) \Delta^{p(1/2\wedge 1/4\wedge 1/8)} (h(\Delta))^{p(1/2\wedge 1)} \\
+ \varpi_6 \int_0^{t_1} \sup_{0 \leq t \leq s} \mathbb{E}|x_\Delta(t \wedge \nu_n^*) - x(t \wedge \nu_n^*)|^p ds \]

where

\[
\varpi_3 = 2^{p-1} 2^{p-1} T^{p-2} \bar{C}_{p}\mu(n)^p T c_p^{\frac{1}{2}} \]
\[ \varpi_4 = 2^{2(p-1)}2^{\frac{p-2}{2}}C_p\nu(n)^pTK_1^\frac{1}{2} \]
\[ \varpi_5 = (2^{2(p-1)}T^{p-1}K_n + 2^{3(p-1)}T^{\frac{p-2}{2}}C_p\nu^2K_n)C_p\Delta^{p/2}(h(\Delta))^p \]
\[ \varpi_6 = 2^{2(p-1)}T^{p-1}K_n + 2^{3(p-1)}T^{\frac{p-2}{2}}C_p\nu^2K_n. \]

The Gronwall inequality shows
\[ \mathbb{E}\left( \sup_{0 \leq t \leq t_1} |x(\tau_{\nu_n}^*) - x(t \wedge \nu_n^*)|^p \right) \leq \mathcal{K}_2 \Delta^{p(1/2 \wedge 1/4 \wedge 1/8)}(h(\Delta))^{p(1/2 \wedge 1)} \]

as the required assertion, where
\[ \mathcal{K}_2 = (\varpi_1 + \varpi_2 + \varpi_3)e^{\varpi_4}. \]

\[ \square \]

The following lemma shows that the truncated EM solutions converge strongly to the exact solution without the stopping time.

**Theorem 5.9.** Let Assumptions 3.1 hold. Then for any \( p \geq 2 \), we have
\[ \lim_{\Delta \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} |x_{\Delta}(t) - x(t)|^p \right) = 0 \quad (60) \]

and consequently

\[ \lim_{\Delta \to 0} \mathbb{E}\left( \sup_{0 \leq t \leq T} |\bar{x}_{\Delta}(t) - x(t)|^p \right) = 0. \quad (61) \]

**Proof.** Let \( \varrho_{mn}, \vartheta_n^* \) and \( v_n^* \) be the same as before. Now set
\[ e_{\Delta}(t) = x_{\Delta}(t) - x(t). \]

For any arbitrarily \( \delta > 0 \), we derive from the Young inequality that
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} |e_{\Delta}(t)|^p \right) = \mathbb{E}\left( \sup_{0 \leq t \leq T} |e_{\Delta}(t)|^p 1_{\{\varrho_{mn} > T \text{ and } \vartheta_n^* > T\}} \right) \]
\[ + \mathbb{E}\left( \sup_{0 \leq t \leq T} |e_{\Delta}(t)|^p 1_{\{\varrho_{mn} \leq T \text{ or } \vartheta_n^* \leq T\}} \right) \]
\[ \leq \mathbb{E}\left( \sup_{0 \leq t \leq T} |e_{\Delta}(t)|^p 1_{\{v_n^* > T\}} \right) + \frac{\delta}{2} \mathbb{E}\left( \sup_{0 \leq t \leq T} |e_{\Delta}(t)|^{2p} \right) \]
\[ + \frac{1}{2\delta} \mathbb{E}(\varrho_{mn} \leq T \text{ or } \vartheta_n^* \leq T). \quad (63) \]
Then for $p \geq 2$, Lemmas 3.5 and 5.4 give us

\[ E\left( \sup_{0 \leq t \leq T} |e(t)|^{2p} \right) \leq 2^{2p} E\left( \sup_{0 \leq t \leq T} |x(t)|^p \vee \sup_{0 \leq t \leq T} |x(t)|^{p} \right)^2 \leq 2^{2p}(c_2 \vee c_3)^2. \]  

(64)

By Lemmas 3.3 and 5.7, we have

\[ P(\nu_n^* \leq T) \leq P(\vartheta_{mn} \leq T) + P(\vartheta_n^* \leq T). \]  

(65)

Also, by Lemma 5.8, we get

\[ E\left( \sup_{0 \leq t \leq T} |e(t)|^{1/2 \wedge 1/4 \wedge 1/8} \right) \leq K_2 \Delta^{p(1/2 \wedge 1/4 \wedge 1/8)}(h(\Delta))^{p(1/2 \wedge 1)}. \]  

(66)

Therefore, we substitute (64), (65) and (66) into (62) to have

\[ E\left( \sup_{0 \leq t \leq T} |e(t)|^p \right) \leq 2^{2p}(c_2 \vee c_3)^2 \delta + K_2 \Delta^{p(1/2 \wedge 1/4 \wedge 1/8)}(h(\Delta))^{p(1/2 \wedge 1)} \]  

+ \[ P(\vartheta_{mn} \leq T) + P(\vartheta_n^* \leq T). \]

Given $\epsilon \in (0, 1)$, we can choose $\delta$ so that

\[ \frac{2^{2p}(c_2 \vee c_3)^2 \delta}{2} \leq \frac{\epsilon}{4}. \]  

(67)

Furthermore, for any given $\epsilon \in (0, 1)$, there exists $n_o$ such that for $n \geq n_o$, we may choose $\delta$ to obtain

\[ \frac{1}{2\delta} P(\vartheta_{mn} \leq T) \leq \frac{\epsilon}{4}. \]  

(68)

and then choose $n(\epsilon) \leq n_o$ such that for $\Delta \in (0, \Delta^1]$, we have

\[ \frac{1}{2\delta} P(\vartheta_n^* \leq T) \leq \frac{\epsilon}{4}. \]  

(69)

Lastly, we may choose $\Delta \in (0, \Delta^1]$ sufficiently small for $\epsilon \in (0, 1)$ such that

\[ K_2 \Delta^{p(1/2 \wedge 1/4 \wedge 1/8)}(h(\Delta))^{p(1/2 \wedge 1)} \leq \frac{\epsilon}{4}. \]  

(70)

We then (67), (68), (69) and (70), to have

\[ E\left( \sup_{0 \leq t \leq T} |x(t) - x(t)|^p \right) \leq \epsilon. \]

as desired. By Lemma 5.3, we also obtain (61) by letting $\Delta \to 0$. \qed
6 Numerical application

We now provide numerical demonstrations to support the theoretical result.

6.1 Simulation

In what follows, let us consider the following form of SDE (6)

\[ dx(t) = 2(1 - x(t)^5)dt + 3\sqrt{|\varphi(t)|}x(t)^{5/4}dB_1(t), \] (71)

with initial data \( x_0 = 0.2 \), where \( \varphi(t) \) is driven by SDE (7) of the form

\[ d\varphi(t) = 2(2 - \varphi(t)^2)dt + 0.5\varphi(t)^{3/2}dB_2(t) \] (72)

with initial data \( \varphi_0 = 2 \). Apparently, the coefficient terms \( f_1(x) = 2(1 - x^5), g_1(x) = 3x^{5/4}, f_2(x) = 2(2 - x^2) \) and \( g_2(\varphi) = 0.5\varphi^{3/2} \) of SDE (71) and SDE (72) are locally Lipschitz continuous. Moreover, we observe that

\[ \sup_{|x| \vee |\varphi| \leq u} \left( |f_1(x)| \vee |f_2(\varphi)| \vee g_1(x) \vee g_2(\varphi) \right) \leq 10.5\nu^5, \quad \nu \geq 0, \]

If we choose \( h(\Delta) = \Delta^{-1/2} \), then \( \nu^{-1}(h(\Delta)) = (\Delta/10.5)^{-1/10} \). Using a step size of \( 10^{-3} \), we get Monte Carlo simulated sample trajectories of SDE (72) and SDE (71) in Figure 1 and Figure 2 respectively.

6.2 Evaluation

In this session, we justify that the truncated EM solutions can be used to compute a barrier option with a European payoff \( P \). Let the asset price be the exact solution \( x(T) \) to SDE (6), \( B \) be a fixed barrier, \( T \) be an expiry date and \( \Lambda \) a strike price. Then the exact payoff of a barrier option is

\[ P(T) = \mathbb{E}\left[ (x(T) - \Lambda)^+ 1_{\sup_{0 \leq t \leq T} x(t) < B} \right]. \]

Using the step process (39), we could compute the approximate payoff by

\[ P^\Delta(T) = \mathbb{E}\left[ (\bar{x}_\Delta(T) - \Lambda)^+ 1_{\sup_{0 \leq t \leq T} \bar{x}_\Delta(t) < B} \right]. \]

So, from Theorem 5.9 we have

\[ \lim_{\Delta \to 0} |P(T) - P^\Delta(T)| = 0. \]

See [10] [20] for the detailed account.
Figure 1: Simulated sample path of \( \varphi(t) \) using \( \Delta = 0.001 \)
Figure 2: Simulated sample path of $x(t)$ using $\Delta = 0.001$
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