INTRINSICALLY WEIGHTED MEANS OF MARKED POINT PROCESSES
ALEXANDER MALINOWSKI,* University of Göttingen
MARTIN SCHLATHER,** University Mannheim
ZHENGJUN ZHANG,*** University of Wisconsin at Madison

Abstract
For a non-stationary or non-ergodic marked point process (MPP) on \( \mathbb{R}^d \), the definition of averages becomes ambiguous as the process might have a different stochastic behavior in different realizations (non-ergodicity) or in different areas of the observation window (non-stationarity). We investigate different definitions for the moments, including a new hierarchical definition for non-ergodic MPPs, and embed them into a family of weighted mean marks. We point out examples of application in which different weighted mean marks all have a sensible meaning. Further, asymptotic properties of the corresponding estimators are investigated as well as optimal weighting procedures.

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1. Introduction
Marked point processes (MPPs) provide an adequate framework for modeling irregularly scattered events in space or time in that they incorporate the joint distribution of the observed values and the point locations (e.g., [7, 8, 14, 19, 20, 22]). Due to the variety of possible forms of dependence between marks and locations in an MPP framework, already the notion of the mean, which is usually considered as being the simplest summary statistic, rises tantalizing and challenging questions.

An introductory example for the type of MPP averages being considered within this paper is the trading process in financial markets. Transactions of assets are typically characterized by the two quantities price and volume; a benchmark quantity that is of major interest especially for institutional investors is the so-called volume-weighted average price (VWAP) (e.g., [3, 15]). The VWAP of \( n \) transactions with prices \( p_i \) and traded volumes \( v_i, i = 1, \ldots, n \), is defined as \( p_{VWAP} = \frac{\sum (p_i v_i)}{\sum v_i} \).
We embed this example in the following general MPP framework: We consider stationary MPPs on $\mathbb{R}^d$ of the form

$$\Phi = \{(t_i, y_i, z_i) : i \in \mathbb{N}\},$$

where $t_i \in \mathbb{R}^d$ is the point location, $y_i \in \mathbb{R}$ is the first mark and $z_i \in [0, \infty)$ is a second mark of the $i$th point of $\Phi$. Let $\Phi_g = \{t : (t, y, z) \in \Phi\}$ denote the ground process of point locations of $\Phi$ and let us denote the marks at a location $t \in \Phi_g$ by $y(t)$ and $z(t)$. The non-negativity assumption on the $z$-component simplifies technical assumptions when employing this mark component as weights for averages of the first mark component $y(t)$ or $f(y(t))$ for some function $f : \mathbb{R} \to \mathbb{R}$. In intuitive notation we write the corresponding weighted mean as

$$\mu_f^{(1)} = \mathbb{E}[z(t)f(y(t)) | t \in \Phi_g],$$

(1)

where we assume that the $z$-component is normalized such that $\mathbb{E}[z(t) | t \in \Phi_g] = 1$. Here, the conditioning on “$t \in \Phi_g$” is understood in the sense of the Palm mark distribution. Since the weights $z(t)$ are provided by the MPP itself and may depend on both the marks $y(t)$ and the point locations $t \in \Phi_g$, we refer to $\mu_f^{(1)}$ as intrinsically weighted mean mark of $\Phi$. The formal definition of $\mu_f^{(1)}$ and related quantities will be given at the beginning of Section 2.

When a system of randomly distributed objects is modeled by means of MPPs, there can exist different sensible choices of intrinsic weights $z(t)$ leading to different weighted mean marks that are relevant for one and the same process, but for different statistical questions:

- **Average height of trees:** Consider $n$ forests of about equal size, each of which is sampled on an area with fixed size and shape. Then the unweighted average of the height of all trees provides a measure of the entire timber stand, which is relevant for forest inventory applications. This amounts to $z(t) = 1$ in (1). Additionally, the average height of a typical forest (as opposed to a typical tree) might be of interest, independently of how dense the trees occur in the different forests. Then, a nested definition of mean seems to be adequate where we first average within each forest and then between all forests. This is equivalent to using a weighted average over all trees with $z(t)$ being proportional to the inverse of the number of trees in the forest that location $t$ belongs to.

- **Density of insects on plants**, cf. [2]: Consider $n$ plants and a population of insects distributed over the plants. Let $k_i, i = 1, \ldots, n$, be the number of insects on the $i$th plant. In this set-up there are different well-established definitions of density referring to different ecological effects. The ordinary density of insects, also called resource-weighted density, is $(k_1 + \ldots + k_n)/n$ and quantifies the average availability of resources. In contrast, the organism-weighted density is the density that an average insect experiences. Each individual on plant $i$ experiences a density of $k_i$ insects per plant, i.e., the organism-weighted density is $(k_i^2 + \ldots + k_n^2)/(k_1 + \ldots + k_n)$. In MPP notation, each insect is represented by a point, marked by the total number of insects on the plant on which the insect is located. Then the organism-weighted density corresponds to the ordinary mean mark ($z(t) = 1$), whereas the resource-weighted density is the average of all plant-wise averages of the marks, i.e., $z(t) = (nk_i)^{-1} \sum_{i=1}^n k_i$ if $t$ belongs to plant $i$. 

Sampling of continuous-space processes: Measurements of continuous-space or continuous-time processes usually aim at estimating or predicting the underlying process and the mean of interest is therefore the spatial or temporal mean over the whole domain of the process. Since measurement locations are not necessarily independent of the underlying process, knowledge of the pattern of point locations might already provide information about the values of the process. Such a situation is commonly referred to as biased or preferential sampling and different weighting approaches exist to correct for this form of biases (e.g., [11]). Although most statistical methods only use stationarity, ergodicity is often implicitly assumed. In case of non-ergodicity, which means that different realizations can have a different stochastic behavior, we are faced with an additional dimension of biasedness: Within each ergodic subclass, the pattern of point locations can be independent of the underlying process, while there might be a strong dependence between the pattern of measurement locations and the process itself if multiple realizations are considered. For a simple example, consider a Gaussian random field with a random mean \( m \) combined with a Poisson point process of measurement locations whose intensity of points is a function of \( m \).

While ergodicity of MPPs is necessary for a straightforward interpretation of the mark distribution as the distribution of a typical point and, at least implicitly, is required by many applications for consistent estimation, in this paper, we investigate the behavior of moment-based summary statistics in case of non-ergodic MPPs and intend to point out problems of ambiguity in this context. When the different forests and plants in the above examples are perceived as a set of MPP realizations and exhibit different ecological characteristics, non-ergodicity has to be included. Examples for non-ergodic MPPs that evolve in time can easily be found in the financial world: For subsequent days of asset trading, the process of executed transactions can be considered as different realizations of a possibly non-ergodic MPP. To treat non-ergodic MPPs adequately, we propose intrinsically weighted mean marks as a special case of [11] in which the weights are constant within each ergodic class but allow for compensating for differences between the different ergodicity classes. A direct application of the theory developed within this paper is [17], in which interaction effects within high-frequency financial data are investigated via MPP methods.

The remainder of this article is organized as follows: In Section 2 we recall and generalize moment-based characteristics for MPPs which also form the central tool for the analysis of interactions in MPPs. We study their behavior and interpretation for non-ergodic processes and, following the idea of the above examples, propose alternative definitions of moment-based summary statistics in Section 3. Different estimators for the above characteristics and their asymptotic properties are discussed in Section 4. The paper closes with a comparison of the point process set-up with estimation of continuous-space processes, which typically occur within geostatistical applications. The appendix reviews basic results from ergodic theory and contains some of the proofs of Section 4.
2. MPP moment-measures and measurement of interaction effects

Throughout the paper \( \Phi = \{(t_i, y_i, z_i) : i \in \mathbb{N}\} \) is a stationary and simple marked point process on \( \mathbb{R}^d \) with marks \( (y(t_i), z(t_i)) = (y_i, z_i) \in \mathbb{R} \times [0, \infty) \), and \( \Phi_g = \{ t : (t, y, z) \in \Phi \} \) is its ground process of point locations. In particular, the point configuration \( \Phi_g \) is locally finite. For the general theory of point processes, the reader is referred to \cite{3, 7, 24}, for example. Let us remark that the following definitions of MPP statistics can directly be generalized to MPPs on Polish spaces whose marks are also in a Polish space.

One of the most basic mark summary statistic is the weighted mean mark \( \mu_f^{(1)} \), which we introduced in (1) as a conditional mean, conditional on the event \( \{ t \in \Phi_g \} \).

Since for fixed \( t \in \mathbb{R}^d \), this is a zero-probability event, the classical formal definition is

\[
\mu_f^{(1)} = \frac{\mathbb{E} \sum_{(t, y) \in \Phi} z f(y) 1_B(t)}{\mathbb{E} \sum_{(t, y) \in \Phi} 1_B(t)}
\]

(2)

for any Borel set \( B \subset \mathbb{R}^d \) with \(|B| > 0\). Here we implicitly exclude the degenerate case \( z(t) \equiv 0 \). Due to the stationarity of \( \Phi \), this definition does not depend on the choice of \( B \).

**Proposition 2.1.** Both definitions of \( \mu_f^{(1)} \), (1) and (2), coincide.

**Proof.** The assertion follows from standard arguments of MPP theory \cite[chap. 13]{7}.

The most relevant example of \( f \) in practical application is \( f(y) = y^n \) for \( n = 1, 2, \ldots \) Then, if \( z(t) = 1 \) for \( t \in \Phi_g \), \( \mu_f^{(1)} \) simply represents the \( n \)-th moment of the (Palm) mark distribution. Note that in case the MPP represents measurements of an underlying continuous process, the mean mark can substantially differ from the mean of the underlying process due to stochastic dependence between the sampling locations and the process itself.

While the above statistic \( \mu_f^{(1)} \) reflects (average) properties of single points, second-order characteristics (in intuitive notation \( \mathbb{E} [ f(y(t_1), y(t_2)) \mid t_1, t_2 \in \Phi_g, t_1 \neq t_2 ] \)) provide a framework to investigate dependency structures within MPPs. We use the superscripts \((1)\) and \((2)\) to indicate whether first- or second-order measures are meant.

**Definition 2.1.** For any non-negative function \( f \) on \( \mathbb{R} \times \mathbb{R} \), we define a \( \sigma \)-finite measure on \( \mathbb{R}^d \times \mathbb{R}^d \) by

\[
\alpha_f^{(2)}(C) = \mathbb{E} \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi} z_1 f(y_1, y_2) 1_C((t_1, t_2)), \quad C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d),
\]

(3)

which we call weighted second moment measure. Here, “\( \neq \)” indicates that the sum runs over all pairs of points with \( (t_1, y_1) \neq (t_2, y_2) \).

With the notation

\[
C(B, I) = \begin{cases} \{ (t_1, t_2) : t_1 \in B, t_2 \in t_1 + I \}, & d = 1, \\ \{ (t_1, t_2) : t_1 \in B, t_2 \in t_1 + \{ x \in \mathbb{R}^d : \| x \| \in I \} \}, & d > 1, \end{cases}
\]

\[
C(t, I) = C([0, t], I),
\]

\[
C(I) = C([0, 1], I),
\]
for \(B \in \mathcal{B}(\mathbb{R}^d), t \in \mathbb{R}^d, I \in \mathcal{B}(\mathbb{R})\),
\[
\alpha_f^{(2)}(C(I)) = \frac{\alpha_f^{(2)}(C(I))}{\alpha_f^{(2)}(A(I))}, \quad I \in \mathcal{B}(\mathbb{R}),
\]
defines a \(\sigma\)-finite measure on \(\mathbb{R}\). Well-known examples of second-order mark characteristics for stationary and isotropic MPPs are Cressie’s mark variogram and covariance function \([5]\), Stoyan’s \(k_{mm}\)-function \([20]\), and Isham’s mark correlation function \([12]\), which can all be expressed in terms of \([13]\) or \([11]\) with a constant \(z\)-component. \([21]\) provides a unifying notation for the above characteristics and further introduces new functions, \(E\) and \(V\), where \(E(r)\) and \(V(r)\) represent the mean and variance of a mark, respectively, given that there exists a further point at distance \(r > 0\). For the one-dimensional case, e.g., for temporal processes, \([16]\) extend those characteristics to the non-isotropic set-up, where a negative value of \(r\) means that the point that is conditioned on is in the past. The above second-order characteristics only involve the three functions \(f(y_1, y_2) = y_1 y_2, f(y_1, y_2) = y_1\) and \(f(y_1, y_2) = y_1^2\).

**Definition 2.2.** (cf. \([21]\)) For a general non-negative function \(f\) on \(\mathbb{R} \times \mathbb{R}\), we define
\[
\mu_f^{(2)}(I) = \frac{\alpha_f^{(2)}(C(I))}{\alpha_f^{(2)}(C(I))}, \quad I \in \mathcal{B}(\mathbb{R}),
\]
if \(\alpha_f^{(2)}(C(I)) > 0\). Here, \(\alpha_f^{(2)}\) is short notation for \(\alpha_f^{(2)}\) with \(f \equiv 1\). We call \(\mu_f^{(2)}\) the (weighted) second-order mean mark.

In the following, we always assume that \(I\) is chosen such that \(\alpha_1^{(2)}(C(I)) > 0\). Note that the distinction between \(d = 1\) and \(d > 1\) in the definition of the set \(C(B, I)\) allows to capture a possibly anisotropic behavior of \(\mu_f^{(2)}\) in the one-dimensional case. In particular,
\[
\alpha_f^{(2)}(C(I)) = \begin{cases} 
\mathbb{E}_\Phi \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi, t_1 \in [0, 1]} z_1 f(y_1, y_2) 1_{t_2 - t_1 \in I}, & d = 1 \\
\mathbb{E}_\Phi \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi, t_1 \in [0, 1]} z_1 f(y_1, y_2) 1_{\|t_2 - t_1\| \in I}, & d > 1.
\end{cases}
\]

For higher dimensions, it is also possible to assign different directions of isotropy, but the technical burden increases considerably as \(\mu_f^{(2)}\) will not be a function of a scalar argument anymore. For further notational convenience, we assume that the derivative of \(\alpha_f^{(2)}\) w.r.t. the Lebesgue measure exists, which is then referred to as *product density* and denoted by \(\rho_f^{(2)}\).

Due to the stationarity of \(\Phi\), we have \(\rho_f^{(2)}(t_1, t_2) = \rho_f^{(2)}(0, t_2 - t_1)\) for almost all \((t_1, t_2) \in \mathbb{R}^d\) and hence \(\alpha_f^{(2)}(C) = \int_C \rho_f^{(2)}(0, h_2 - h_1) d(h_1 \times h_2), \quad C \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)\).

Let \(\rho_f^{(2)}(r), \quad r \in \mathbb{R}\), denote the derivative of \(\alpha_f^{(2)}(C(\cdot))\) w.r.t. the one-dimensional Lebesgue measure. Obviously, \(\alpha_f^{(2)}(C(\cdot))\) is dominated by \(\alpha_f^{(2)}(C(\cdot))\), which ensures that the limit of \(\mu_f^{(2)}(I)\) for \(|I| \to 0\) exists and can be expressed in terms of Radon-Nikodym derivatives. For \(r \neq 0\) we define
\[
\mu_f^{(2)}(r) = \left. \frac{\partial \alpha_f^{(2)}(C(\cdot))}{\partial \alpha_f^{(2)}(C(\cdot))} \right|_{r=0} = \rho_f^{(2)}(r) / \rho_f^{(2)}(r).
\]
Note that for \( d = 1 \), we have \( \rho_f^{C,(2)}(r) = \rho_f^{(2)}(0, r) \). With a slight abuse of notation, we refer to both definitions (5) and (6) as \( \mu_f^{(2)} \). For \( r \neq 0 \) and \( f \) only depending on its first argument, \( \mu_f^{(2)}(r) \) can be interpreted as the (weighted) expectation of a mark at location \( t \) subject to the conditioning that \( \Phi \) has a point at location \( t \) and at location \( t + re_1 \), i.e., \( \mu_f^{(2)}(r) = E[z(t)f(y(t)) \mid t, t + re_1 \in \Phi_g] \), where \( e_1 \) denotes the vector \((1, 0, \ldots, 0)^T \in \mathbb{R}^d \). For \( \mu_f^{(2)}(I) \), this interpretation becomes slightly ambiguous: Considering an event at time \( t \), there may be multiple other points located within the set \( t + I \) and in case that interactions of higher order are present, these will be reflected by the second-order statistic \( \mu_f^{(2)}(I) \) as well. More precisely, by the definitions in (5) and (6),

\[
\mu_f^{(2)}(I) = \alpha^{(2)}(C(I))^{-1} \int_I \mu_f^{(2)}(r) \, dr,
\]

where \( \mu_f^{(2)}(r) \) is a weighted average of conditional expectations \( \mu_f^{(2)}(r) \) with weights being proportional to the expected number of pairs of points with distance \( dr \).

**Remark 2.1.** (a) The extension to moment measures of higher order is straightforward and allows to condition on arbitrary point constellations. In practice, however, mostly first- and second-order statistics are considered.

(b) The non-negativity condition on \( f \) can be weakened by considering the restriction of \( \mu_f^{(2)}(\cdot) \) to some bounded set \( J \in \mathcal{B} \mathbb{R} \). Then it is sufficient for \( f \) that \( \alpha_h^{(2)}(C(J)) < \infty \) is satisfied for \( h = f_+ = \max\{f, 0\} \) or for \( h = f_- = -\min\{f, 0\} \).

(c) Another generalization allows to include further conditioning on the marks. For \( f_{\text{cond}} \) a non-negative function on \( \mathbb{R} \times \mathbb{R} \) we consider

\[
\mu_{f, f_{\text{cond}}}^{(2)}(I) = \frac{\alpha_{f, f_{\text{cond}}}^{(2)}(C(I))}{\alpha_{f_{\text{cond}}}^{(2)}(C(I))} = \frac{\mu_{f, f_{\text{cond}}}^{(2)}(I)}{\mu_{f_{\text{cond}}}^{(2)}(I)}.
\]

Choosing \( f_{\text{cond}} \) to be an indicator function \( f_{\text{cond}}(y_1, y_2) = 1_A(y_1)1_B(y_2) \) conditions the marks on the events \( A \) and \( B \), respectively.

**Remark 2.2.** For \( d > 1 \), \( \mu_f^{(2)} \) is a function of the Euclidean distance between two points, whereas for \( d = 1 \), \( \mu_f^{(2)} \) is a function of the signed distance. In the latter case, \( \mu_f^{(2)}(\cdot) \) is in general not symmetric: Consider a temporal process consisting of pairs of points \((t_1, t_2)\) with \( t_1 < t_2 \) and with small intra- but large inter-pair distances. Assume that the marks of different pairs are stochastically independent and that for each pair of points, \( f(y_2, y_1) > f(y_1, y_2) \) holds. Then \( \mu_f^{(2)}(r) > \mu_f^{(2)}(-r) \) holds for all \( r > 0 \) that are small enough and that can occur as intra-pair distances.

For notational convenience, we will write \( \mu_f^{(i)} \) to indicate that a statement is valid for \( \mu_f^{(1)} \) and \( \mu_f^{(2)} \).
3. New moment measures for non-ergodic MPPs

Ergodicity makes spatial averages over suitably increasing observation windows of a single realization converge to the corresponding expectation over the state space:

\[ |W|^{-1} \int_W X(T_x \Phi) \, dx \overset{\text{a.s.}}{\to} E(X(\Phi)), \quad \text{for } |W| \to \infty \text{ suitably}, \]

for any integrable function \( X \) on the space of all locally finite counting measures. Here, \( T_x \) denotes the shift of the whole random point pattern \( \Phi \) by \( x \in \mathbb{R}^d \). In essence, ergodicity enables consistent estimation of MPP moment measures by observing a single realization on a suitably increasing domain. In this section, though, we consider the opposite situation, namely where \( \Phi \) is a non-ergodic process.

The following proposition directly relates to the fact that a non-ergodic MPP can be seen as hierarchical model, which, in a first step, draws an ergodic source of randomness out of which the final realization is drawn in a second step.

**Proposition 3.1.** Let \( \Phi \) be a non-ergodic MPP with probability law \( P \). By \( \mathcal{M}_0 \) and \( \mathcal{M}_0 \) we denote the space of all locally finite counting measures on \( \mathbb{R}^d \times \mathbb{R} \times [0, \infty) \) and the usual \( \sigma \)-algebra, respectively. (See Appendix A for more details.) Then

\[
\mu_f^{(1)} = \frac{E_Q \left[ \mu_{f, \Phi \mid Q}^{(1)} \cdot \alpha_{\Phi \mid Q}^{(1)}(B) \right]}{\alpha^{(1)}(B)}, \quad \mu_f^{(2)}(\cdot) = \frac{E_Q \left[ \mu_{f, \Phi \mid Q}^{(2)}(\cdot) \cdot \alpha_{\Phi \mid Q}^{(2)}(C(\cdot)) \right]}{\alpha^{(2)}(C(\cdot))},
\]

where \( Q \sim \lambda \) is a random variable with values in the space \( \mathcal{P}_{\text{erg}} \) of all ergodic MPP probability laws, distributed according to some probability measure \( \lambda \), such that \( P(M) = \int_{\mathcal{P}_{\text{erg}}} Q^*(M) \lambda(dQ^*) \), \( M \in \mathcal{M}_0 \). If \( \mu_f^{(2)} \) is evaluated for a fixed distance \( r \in \mathbb{R} \), \( \alpha^{(2)}(C(r)) \) has to be replaced by \( \rho_r^{C,(2)}(r) \) in (9).

**Proof.** The ergodic decomposition theorem (cf. Theorem A.2) guarantees the existence and uniqueness of a decomposition \( P(\cdot) = \int_{\mathcal{P}_{\text{erg}}} Q^*(\cdot) \lambda(dQ^*) \) and a corresponding mixing random variable \( Q \sim \lambda \). Conditioning \( \Phi \) on \( Q \), we can decompose the moment measures \( \alpha^{(j)} \) and obtain

\[
\mu_f^{(2)}(r) = \frac{\partial E_Q \alpha^{(2)}_{f, \Phi \mid Q}(C(\cdot))}{\partial \alpha^{(2)}(C(\cdot))} \Bigg|_{=r} = \frac{\partial E_Q \alpha^{(2)}_{f, \Phi \mid Q}(C(\cdot))/\partial \nu(\cdot)}{\partial \alpha^{(2)}(C(\cdot))/\partial \nu(\cdot)} \Bigg|_{=r}
\]

\[
= \frac{E_Q \rho_r^{C,(2)}(0, r)}{\rho_1^{C,(2)}(0, r)} = \frac{E_Q \left[ \mu_{f, \Phi \mid Q}^{(2)}(r) \cdot \rho_r^{C,(2)}(0, r) \right]}{\rho_1^{C,(2)}(0, r)},
\]

where \( \nu \) denotes the Lebesgue measure. For \( \mu_f^{(2)}(I) \) and \( \mu_f^{(1)} \), the decomposition is analogous.

**Example 3.1.** The so-called log-Gaussian Cox process \([18]\) is ergodic if and only if the underlying stationary Gaussian random field \( Z \) is ergodic. A sufficient condition for \( Z \) being ergodic is that the covariance function decays to zero. Amongst others, \([8]\) and \([20]\) use log-Gaussian Cox processes, combined with an intensity-dependent marking, as parametric models for preferential sampling applications.
Proposition 3.1 shows that in case of non-ergodicity, \( \mu_f^{(i)} \) is an average of its ergodic subclasses counterparts, in which each class \( Q^* \) is implicitly weighted by the respective intensity \( \alpha_{Q=Q^*}^{(i)} \). If all ergodic subprocesses \( \Phi|Q = Q^* \) have the same intensity measure, the weights cancel out and we have \( \mu_f^{(i)} = E_Q \mu_f^{(i)|Q} \). Since in the general case, a single ergodicity class with low probability may exhibit a large value of \( \alpha_f^{(i)} \), and thus drive the value of \( \mu_f^{(i)} \), the demand for a new characteristic \( \tilde{\mu}_f^{(i)} \) arises naturally, that summarizes the properties of all ergodicity classes irrespectively of how the processes of point locations differ between the different ergodicity classes. We meet these requirements by a definition that excludes the implicit weighting proportional to the \( i \)th order intensities:

**Definition 3.1.** Let \( \lambda \) and \( Q \) be the ergodic decomposition mixture measure and mixture variable, respectively, of \( \Phi \), and let \( E_Q|\mu_f^{(i)|\Phi} < \infty \). Then we call

\[
\tilde{\mu}_f^{(i)} = E_Q \mu_f^{(i)|\Phi} = \int_{P_{\text{erg}}} \mu_f^{(i)|\Phi=Q^*} \lambda(\text{d}Q^*).
\]

(10)

the (equally-weighted) average \( i \)-th order mean mark of \( \Phi \).

Relating to the introductory forest example, the classical definition of the mean mark in (2) corresponds to the average height of all trees, irrespectively of differences w.r.t. the tree densities between the different forests, while the new definition in (10) refers to the average height of a typical forest.

**Remark 3.1.** Comparing the new definition with (3) yields that \( \tilde{\mu}_f^{(i)} \) coincides with \( \mu_f^{(i)} \) if \( \alpha_{\Phi=Q^*}^{(i)} \) is \( \lambda \)-a.s. constant. This is particularly the case if \( \Phi \) is ergodic.

**Lemma 3.1.** For any \( I \in B(\mathbb{R}) \) we have

\[
\tilde{\mu}_f^{(2)}(I) = E_Q \left[ \alpha^{(2)}_{\Phi=Q^*}(C(I))^{-1} \int_I \mu_f^{(2)|\Phi=Q^*}(r) \, d\alpha^{(2)}_{\Phi=Q^*}(C(r)) \right].
\]

If, for \( \lambda \)-almost all measures \( Q^* \), \( \mu_f^{(2)|\Phi=Q^*}(r) \) is uniformly bounded by some positive constant \( c(Q^*) \) and \( E_Q c(Q) < \infty \), for \( I \in B(\mathbb{R}) \) and \( r \in \mathbb{R} \), we have

\[
\lim_{I \rightarrow \{r\}} \tilde{\mu}_f^{(2)}(I) = \tilde{\mu}_f^{(2)}(r).
\]

**Proof.** The first assertion follows directly from applying the representation (7) to the ergodic subprocesses \( \Phi|Q = Q^* \). Since \( \lim_{I \rightarrow \{r\}} \tilde{\mu}_f^{(2)}(I) = \tilde{\mu}_f^{(2)}(r) \) by construction, the second assertion is merely an application of Lebesgue’s dominated convergence theorem.

From Lemma 3.1 we see that the nested conditional mean \( \tilde{\mu}_f^{(2)}(r) \) is a Radon-Nikodym derivative of \( \alpha_f^{(2)}(C(\cdot)) \) w.r.t. \( \alpha^{(2)}(C(\cdot)) \) if and only if the expectation of \( \alpha^{(2)}_{\Phi=Q^*}(C(\cdot)) \mu_f^{(2)|\Phi=Q^*}(\cdot) \) factorizes. This contrasts the ordinary conditional mean \( \mu_f^{(2)}(r) \), which is already defined as a Radon-Nikodym derivative of \( \alpha_f^{(2)}(C(\cdot)) \) w.r.t. \( \alpha^{(2)}(C(\cdot)) \).
The ergodic decomposition and an analog to Definition 3.1 can be applied to any expectation-based functional of an MPP including the Palm mark distribution itself. While the classical definition of the mean mark represents a typical point, irrespectively of the different ergodicity classes, the two-stage-expectation \( \hat{\mu}_f^{(1)} \) refers to the mean of a typical realization. We provide more details on the meaning of the differences between \( \mu_f^{(2)} \) and \( \hat{\mu}_f^{(1)} \) and between different estimators in the next section.

4. Estimation principles for the new MPP moment-measures

4.1. The ergodic case

For ergodic processes \( \Phi \), the pointwise ergodic theorem for MPPs (Proposition A.1 in the Appendix) yields that

\[
\mathbb{E} \left[ \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi} z_1 f(y_1, y_2) \mathbf{1}_{(t_1, t_2) \in C(I)} \right] = \lim_{n \to \infty} \left[ n^{-d} \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \varphi} z_1 f(y_1, y_2) \mathbf{1}_{(t_1, t_2) \in C(n1, I)} \right]
\]

for almost all realizations \( \varphi \) of \( \Phi \), which builds the basis for the estimators being discussed in this section. For readability reasons, and since we will be only dealing with second-order statistics from now on, we drop the superscript \((2)\) in all the estimators of \( \mu_f^{(2)} \).

Applying the standard estimator for MPP moment measures to a realization of \( \Phi \) observed on the set \([0, T], T \in (0, \infty)^d\), we obtain

\[
\hat{\mu}_f(I, \Phi, T) = \frac{\hat{\alpha}_f(I, \Phi, T)}{\hat{\alpha}_1(I, \Phi, T)}, \tag{11}
\]

where \( \hat{\alpha}_f(I, \Phi, T) = \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi} z_1 f(y_1, y_2) \mathbf{1}_{(t_1, t_2) \in C(T, I)} \).

Lemma 4.1. If \( \Phi \) is ergodic, \( \hat{\mu}_f(I, \Phi, T) \) is consistent for \( \mu_f^{(2)}(I) \). Here, “\( T \to \infty \)” is understood componentwise. If \( \Phi \) is non-ergodic, \( \hat{\mu}_f(I, \Phi, T) \) is consistent if and only if \( \mu_f^{(2)}(I) \) is constant w.r.t. \( Q^* \).

Proof. By Proposition A.1, the tuple consisting of the numerator and the denominator of (11), each normalized by the volume of \([0, T]\), converges a.s. to the vector \((\alpha_f^{(2)}(C(I)), \alpha^{(2)}(C(I)))\) if \( \Phi \) is ergodic. The first assertion thus follows from the continuous mapping theorem. In the non-ergodic case, clearly only \( \mu_f^{(2)}(I) \) can be estimated consistently for \( Q^* \) being the respective ergodicity class. Though, if \( \mu_f^{(2)}(I) \) is constant w.r.t. \( Q^* \) we have \( \mu_f^{(2)}(I) = \mu_f^{(2)}(I) \) for any \( Q^* \in \mathcal{P}_{\text{erg}} \).

To establish asymptotic normality of \( \hat{\mu}_f(I, \Phi, T) \), we introduce some idealized assumptions. In particular, we assume stochastic independence between the point locations and the marks of the MPP. For simplicity, we restrict to the case where \( f \) only depends on its first argument and the MPP is a process on \( \mathbb{R} \).
\textbf{Condition 4.1.} (\textit{m}-dependent Random Field Model.) Let $\hat{\Phi}$ be a stationary unmarked point process on $\mathbb{R}$, for which neighboring points have some minimum distance $d_0 > 0$. Let $\{Y(t) : t \in \mathbb{R}\}$ be an independent stationary process with finite second moments and a covariance function $C$ that has finite range, i.e., $C(h) = 0$ for all $|h| > h_0$ for some $h_0 > 0$. Then, with $m = [d_0/h_0]$, we say that an MPP $\Phi$ is an \textit{m}-dependent Random Field Model, if $\Phi = \{(t_i, Y(t_i), 1) \mid t_i \in \hat{\Phi}\}$.

The following theorem transfers a central limit theorem (CLT) for arrays of \textit{m}-dependent random variables to the MPP context. It also covers a thinning of the MPP in which the threshold increases with the observation window. The result allows to derive asymptotically exact confidence intervals for the estimator of $\mu^{(2)}(I)$ and is applied in [17] in the context of extreme value analysis for MPPs.

\textbf{Theorem 4.1.} (CLT for \textit{m}-dependent Random Field Models.) Let $\Phi$ be an ergodic MPP that satisfies Condition 4.1. For $f : \mathbb{R} \to [0, \infty)$ and $u \geq 0$, let $f_u, f_{\text{cond}, u} : \mathbb{R} \to [0, \infty)$ be given by $f_u(y) = (f(y) - u)_+ = (f(y) - u)1_{f(y) > u}$ and $f_{\text{cond}, u}(y) = 1_{f(y) > u}$.

Let

$$\hat{\alpha}^*_{f_u}(I, \Phi, T) = \sum_{(t_1, y_1), (t_2, y_2) \in \Phi} \left( f_u(y_1) - \mu^{(2)}_{f_u, f_{\text{cond}, u}}(I) \right) \cdot f_{\text{cond}, u}(y_1) \cdot 1_{(t_1, t_2) \in C(T, I)}$$

be a centered version of $\hat{\alpha}_{f_u}(I, \Phi, T)$, where $\mu^{(2)}_{f_u, f_{\text{cond}, u}}(I)$ is defined as in (8). Let $(u_T)_{T \geq 0}$ be a family of non-negative, non-decreasing numbers such that the following conditions are satisfied:

$$u_\infty = \lim_{T \to \infty} u_T \in [0, \infty] \quad \text{exists},$$

$$\lim_{T \to \infty} \mathbb{E}\left[ f_{u_T}(Y(0))^i \mid f(Y(0)) > u_T \right] < \infty \quad (i = 1, \ldots, 4),$$

$$T^{-1} \hat{\alpha}_1(I, \Phi, T) - \lambda \rightarrow a.s. \quad (T \to \infty).$$

Then, for $I \in \mathcal{B}(\mathbb{R})$ and $T \to \infty$, we have

$$\frac{\hat{\alpha}^*_{f_{u_T}}(I, \Phi, T)}{\sqrt{\hat{\alpha}_{f_{\text{cond}, u_T}}(I, \Phi, T)}} \overset{\text{d}}{\to} \mathcal{N}(0, s_{u_\infty}),$$

where

$$s_{u_\infty} = \lim_{T \to \infty} \left\{ (\lambda_{u_T} T)^{-1} \mathbb{E}\left[ \hat{\alpha}^*_{f_{u_T}}(I, \Phi, T) \right] \right\},$$

$$\lambda_u = \mathbb{E}_\Phi \left[ \hat{\alpha}_{f_{\text{cond}, u}}(I, \Phi, 1) \right], \quad u \geq 0.$$
4.2. The non-ergodic case

If \( \Phi \) is non-ergodic, consistent estimation of summary statistics generally requires multiple realizations of the process. Let \( P \) and \( \lambda \) denote the probability law and the ergodic mixture measure of \( \Phi \), respectively. Then, drawing iid realizations of \( \Phi \) corresponds to drawing ergodicity classes according to the mixture measure \( \lambda \). Though, a finite collection of realizations merely approximates the mixing measure \( \lambda \) and we can only expect consistency if both \( n \) and \( T \) tend to infinity simultaneously. To see why \( n \to \infty \) is not sufficient, consider an MPP with infinitely many ergodicity classes \( Q_1, Q_2, \ldots \) and with \( \mathbb{E}_{\Phi|Q=Q_i}(\Phi([0,1])) = 2^{-i} \). Then, for fixed \( T \), the probability of observing at least one point in a realization that belongs to class \( i \) tends to zero as \( i \to \infty \). Hence, the classes \( Q_i \), for \( i \) large, are only captured by the estimator if \( T \) also tends to infinity.

Considering iid realizations \( \Phi_1, \ldots, \Phi_n \) of \( \Phi \), different possibilities arise of how to put together the respective estimators. Let \( w = (w_1, \ldots, w_n) \) denote a vector of weight functions \( w_i : M_0 \times [0, \infty)^d \to [0, \infty) \). We assume that for \( \lambda \)-almost all ergodic MPP laws \( Q^* \) there exist constants \( w_i^*(Q^*) \geq 0 \) with \( w^*(Q^*) = \sum_{i=1}^n w_i^*(Q^*) > 0 \) to which the weights converge stochastically within the respective ergodicity class, i.e.,

\[
P_{\Phi|Q=Q_i}(|w_i(\Phi, T) - w_i^*(Q^*)| > \varepsilon) \to 0 \quad (T \to \infty)
\]  

for all \( \varepsilon > 0 \). Then we consider estimators of the form

\[
\hat{\mu}_f^w(I, w) = \hat{\mu}_f^w(I, w, (\Phi_1, \ldots, \Phi_n), T)
\]

\[
= \left( \sum w_i(\Phi_i, T) \right)^{-1} \sum_{i=1}^n w_i(\Phi_i, T) \hat{\mu}_f(I, \Phi_i, T),
\]

Note that the functions \( w_i \) might also depend on \( I \). With \( w_1 = \ldots = w_n = n^{-1} \), we obtain as a special case

\[
\hat{\mu}_f^n(I) = \hat{\mu}_f^n(I, (\Phi_1, \ldots, \Phi_n), T) = n^{-1} \sum_{i=1}^n \hat{\mu}_f(I, \Phi_i, T).
\]

In order to estimate \( \mu_f^{(2)}(I) \) consistently, according to the decomposition in [3], the weights have essentially to be chosen as

\[
w_i(\Phi_i, T) = \hat{\alpha}_f(C(T, I), \Phi_i)/v_T = \sum_{t_1, t_2 \in \Phi_i} 1_{(t_1, t_2) \in C(T, I)}/v_T,
\]

where \( v_T \) is the volume of the cube \([0, T] \). By Proposition 1.1, \( \hat{\alpha}_f^{(2)}(C(T, I), \Phi_i)/v_T \) converges to \( \alpha_\Phi^{(2)}(C(I)) \) a.s. as \( T \to \infty \), where \( Q_i \) is the realized ergodicity class of \( \Phi_i \). With \( w \) being the vector of weights from [15], we define

\[
\hat{\mu}_f^w(I, (\Phi_1, \ldots, \Phi_n), T) = \hat{\mu}_f^w(I, w, (\Phi_1, \ldots, \Phi_n), T),
\]

which, in a sense, represents the family of all pairs of points with a distance contained in \( I \) from all realizations. This choice of weights satisfies the above stochastic convergence condition [12] and is sufficient but not necessary for consistency. The following theorem gives a weaker set of conditions that is still sufficient for consistency.
Theorem 4.2. Let \( \Phi_i, i \in \mathbb{N} \), be iid copies of a possibly non-ergodic MPP \( \Phi \) and let \( Q_{j_i} \) denote the respective ergodicity classes. For weight functions \( \tilde{w}_i : \mathbb{R}^d \to [0, \infty) \) and iid random factors \( W_i \), with \( \mathbb{E}[|W_i|] < \infty \), \( i \in \mathbb{N} \), let \( w_i(\Phi_i, \mathbf{T}) = W_i \cdot \tilde{w}_i(\Phi_i, \mathbf{T}) \) and \( w = (w_1(\Phi_1, \mathbf{T}), \ldots, w_n(\Phi_n, \mathbf{T})) \). Then, \( \hat{\mu}_n^{w, \text{weight}}(I, w) \) is consistent for \( \mu_f^{(2)}(I) \) if the following conditions hold:

\[
 W_i > 0 \quad \text{a.s.,} \\
 \Var \tilde{w}_i(\Phi_i, \mathbf{T}) \leq c_1 \quad \text{for some } c_1 > 0, \\
 n^{-1} \mathbb{E} \sum_{i=1}^n \tilde{w}_i \geq c_2 > 0 \quad \forall n \geq n_0 \text{ for some } n_0 \in \mathbb{N}, \\
 E[W_i \tilde{w}_i(\Phi_i, \mathbf{T})] = E[W_i] \cdot E[\tilde{w}_i(\Phi_i, \mathbf{T})] \\
 \mathbb{E} \left[ W_i \cdot \hat{\alpha}^{(2)}(C(\mathbf{T}, I), \Phi_i) \mu_f^{(2)}(\Phi_i, \Phi_i Q=Q_{j_i}, I) \right] = \mathbb{E}[W_i] \cdot \mathbb{E}\left[ \hat{\alpha}^{(2)}(C(\mathbf{T}, I), \Phi_i) \mu_f^{(2)}(\Phi_i, \Phi_i Q=Q_{j_i}, I) \right] \\
 P \left\{ \max_{i=1}^n \left| \frac{\sum_{j=1}^n \tilde{w}_j(\Phi_j, \mathbf{T})}{\sum_{j=1}^n \tilde{w}_j(\Phi_j, \mathbf{T})} - \mu_f^{(2)}(I) \right| \right\} \to 0 \quad (n, \mathbf{T} \to \infty)
\]

Proof. We consider

\[
 \left| \frac{\sum_{i=1}^n w_i(\Phi_i, \mathbf{T}) \hat{\mu}_f(I, \Phi_i, \mathbf{T})}{\sum_{i=1}^n w_i(\Phi_i, \mathbf{T})} - \mu_f^{(2)}(I) \right| \\
 \leq \left| \sum_{i=1}^n w_i(\Phi_i, \mathbf{T}) \left[ \hat{\mu}_f(I, \Phi_i, \mathbf{T}) - \mu_f^{(2)}(\Phi_i, \Phi_i Q=Q_{j_i}, I) \right] \right| \\
 + \left| \sum_{i=1}^n W_i \tilde{w}_i(\Phi_i, \mathbf{T}) \mu_f^{(2)}(\Phi_i, \Phi_i Q=Q_{j_i}, I) \right| \\
 - \mu_f^{(2)}(I) \right| 
\]

By Lemma 4.11, \( \hat{\mu}_f(I, \Phi_i, \mathbf{T}) \) is consistent (for \( \mathbf{T} \to \infty \)) within the respective ergodicity class. Thus, \( \hat{\mu}_f(I, \Phi_i, \mathbf{T}) \to 0 \) in probability if \( \mathbf{T} \to \infty \). Using the short notation \( \alpha_i = \hat{\alpha}^{(2)}(C(\mathbf{T}, I), \Phi_i) \) and \( \tilde{w}_i = \tilde{w}_i(\Phi_i, \mathbf{T}) \), we have

\[
 (24) = \left| \sum_{i=1}^n W_i \alpha_i \left[ \mu_f(\Phi_i Q=Q_{j_i}, I) - \mu_f^{(2)}(I) \right] \right| \\
 \leq \max_{i=1}^n \left\{ \frac{\sum_{j=1}^n \tilde{w}_j(\alpha_j)}{\alpha_i \sum_{j=1}^n \tilde{w}_j} \cdot \frac{\sum_{j=1}^n W_j \alpha_j}{\sum_{j=1}^n W_j \tilde{w}_j} \cdot \frac{\sum_{j=1}^n \tilde{w}_j}{\sum_{j=1}^n W_j \tilde{w}_j} \cdot \frac{\sum_{i=1}^n W_i \alpha_i \left[ \mu_f(\Phi_i Q=Q_{j_i}, I) - \mu_f^{(2)}(I) \right]}{\sum_{i=1}^n W_i \alpha_i} \right\}
\]

Since by assumption, \( n^{-1} \mathbb{E} \left( \sum \tilde{w}_i \right) \in \mathbb{N} \) is eventually bounded away from 0 and the variance of the \( \tilde{w}_i \) is uniformly bounded, the law of large numbers yields that \( \sum_{j=1}^n \tilde{w}_j / \sum_{j=1}^n \tilde{w}_j \) and \( \sum_{j=1}^n W_j \tilde{w}_j / \sum_{j=1}^n W_j \tilde{w}_j \) converge to 1 in probability. Additionally using that \( \mathbb{E}[W_j \tilde{w}_j] = \mathbb{E}[W_j] \mathbb{E}\tilde{w}_j \), for \( n \to \infty \), we get the convergence

\[
 \frac{\sum_{j=1}^n \tilde{w}_j}{\sum_{j=1}^n W_j \tilde{w}_j} \mathbb{E}\tilde{w}_j / \mathbb{E}[W_j] \mathbb{E}\tilde{w}_j \to \frac{1}{\mathbb{E}[W_1]}
\]
as \( n \to \infty \). Similarly, for \( n \to \infty \) and \( n, T \to \infty \), we have

\[
\frac{\sum_{j=1}^{n} W_j \alpha_j}{\sum_{j=1}^{n} \alpha_j} \xrightarrow{p} \frac{E[W_1 \alpha_1]}{E\alpha_1},
\]

\[
\frac{\sum_{i=1}^{n} W_i \alpha_i \mu_j(I)}{\sum_{i=1}^{n} W_i \alpha_i} \xrightarrow{p} \frac{E\left[\alpha_i(\Phi) \mu_j(I) \cdot \mu_j(I)\right]}{E\left[\alpha_i(\Phi) \mu_j(I)\right]} = \mu_j(I),
\]

respectively. Together with (22) we obtain that (24) converges to 0 in probability, which completes the proof.

Note that if \( \tilde{w} \equiv \hat{w} \) for all \( i \in \mathbb{N} \) for some weight function \( \tilde{w} \) with \( E|\tilde{w}(\Phi, T)| < \infty \), the \( \tilde{w}(\Phi_i, T) \) are iid and conditions (13), (19) and (20) become obsolete.

Now we turn to the estimation of \( \hat{\mu}_j(I) \). By construction (cf. Definition 3.1), \( \hat{\mu}_j(I) \) consistently estimates \( \hat{\mu}_j(I) \); in contrast to \( \hat{\mu}_j(I) \), it reflects a random pair of points with distance \( I \) within a randomly chosen ergodicity class. Again, also other choices of weights are feasible for consistent estimation of \( \hat{\mu}_j(I) \), apart from the choice \( w_i(\Phi_i, T) = 1 \). By replacing \( \hat{\mu}_j(C(T, I), \Phi_i) \) by the constant 1 in Theorem 4.2 we get the following corollary.

**Corollary 4.1.** Under the assumptions of Theorem 4.2 with \( \hat{\mu}_j(C(T, I), \Phi_i) \) being replaced by the constant 1, \( \hat{\mu}_j^{wght}(I, w) \) is a consistent estimator for \( \hat{\mu}_j(I) \).

**Remark 4.1.** If \( \Phi \) is ergodic, \( \hat{\mu}_j^{wght}(I, w) \) is consistent for \( \hat{\mu}_j(I) \) (as \( T \to \infty \)) for any choice of weights \( w \) that satisfies (12). Note that in this case, consistency is independent of \( n \), which can be fixed to any finite value.

**Proof.** If \( \Phi \) is ergodic, the mixing measure \( \lambda \) is the one-point distribution \( \delta_P \) and condition (12) simply means stochastic convergence of the weights w.r.t. \( P \). The assertion directly follows from the continuous mapping theorem.

### 4.3. Variance minimization

In what follows, we seek for an optimal consistent estimator for \( \hat{\mu}_j(I) \) in the sense of minimal variance. We introduce some additional assumptions on the mark-location dependence for analytical tractability. For simplicity, we set \( \tilde{w}(\Phi_i, T) = W_i \). Let \( A_n^* \) denote the \( \sigma \)-algebra generated by the unmarked ground processes \( \Phi_{1,g}, \ldots, \Phi_{n,g} \), i.e., \( A_n^* = \sigma(\{\omega : \Phi_{1,g}(\omega)(B) = k \ : \ k \in \mathbb{N}, B \in B, i = 1, \ldots, n\}) \). We assume that \( E[\mu I, \Phi_i, T] | A_n^* \) is a.s. constant. We further assume that \( A_n^* \) is maximal w.r.t. this property and that \( \text{Var} [\hat{\mu}_j(I, \Phi_i, T) | A_n^*] \) is independent of the random ergodicity class \( Q \).

**Proposition 4.1.** With the above notation and assumptions, the variance minimizing weights for \( \hat{\mu}_j^{wght}(I, w, (\Phi_1, \ldots, \Phi_n), T) \) that satisfy (17)–(22) with \( \hat{\mu}_j^{(2)}(C(T, I), \Phi_i) \) being replaced by 1 are given by

\[
w_i(\Phi_i, T) = W_i = \text{Var} [\hat{\mu}_j(I, \Phi_i, T) | A_n^*]^{-1}.
\]

Note that an analog variance minimizing procedure via random factors \( W_i \) could also be included into the estimator \( \hat{\mu}_j^* \) of \( \hat{\mu}_j(I) \).
Proof of Proposition [4.1] For general \( A_n^* \)-measurable weights \( w_i(\Phi_i, T) \), \( i = 1, \ldots, n \), we have

\[
\text{Var} \left[ \hat{\mu}_f^{\text{wght}}(I, \mathbf{w}, (\Phi_1, \ldots, \Phi_n), T) \right] \\
= \mathbb{E} \left[ \frac{1}{(\sum w_i(\Phi_i, T))^2} \sum_{i=1}^n w_i(\Phi_i, T)^2 \text{Var} \left[ \hat{\mu}_f(I, \Phi_i, T) \mid A_n^* \right] \right] \\
+ \text{Var} \left[ \frac{1}{\sum w_i(\Phi_i, T)} \sum_{i=1}^n w_i(\Phi_i, T) \mathbb{E} \left[ \hat{\mu}_f(I, \Phi_i, T) \mid A_n^* \right] \right] \\
= \mathbb{E} \left[ \sum_{i=1}^n w_i^{\text{rel}}(\Phi_i, T)^2 \text{Var} \left[ \hat{\mu}_f(I, \Phi_i, T) \mid A_n^* \right] \right] + 0 \quad (25)
\]

with \( w_i^{\text{rel}}(\Phi_i, T) = w_i(\Phi_i, T) / \sum_{i=1}^n w_i(\Phi_i, T) \). Since any weighted average \( \sum_i v_i x_i \) with \( x_i > 0 \) and \( \sum v_i = 1 \) is minimized by \( v_i = x_i^{-1} / \sum x_i^{-1} \) (Lagrange method), the unconditional variance (25) is minimized by choosing

\[
w_i(\Phi_i, T) = W_i = \text{Var} \left[ \hat{\mu}_f(I, \Phi_i, T) \mid A_n^* \right]^{-1}.
\]

The \( W_i \) are \( A_n^* \)-measurable by definition of the conditional variance and satisfy (17)–(22) with \( \hat{\alpha}^{(2)}(C(T, I), \Phi_i) \) being replaced by 1. Maximality of \( A_n^* \) ensures optimality of the weights.

If there exist interaction effects in the MPP that are of higher than second order, the assumption on \( \mathbb{E}[\hat{\mu}_f(I, \Phi_i, T) \mid A_n^*] \) might not be satisfied anymore and weighting according to the above conditional variances should be handled with care. Clusters of point locations which tend to increase the conditional variance of \( \hat{\mu}_f \) given the ground process, can additionally influence the mean of other marks in excess of the bivariate interaction measured by \( \mu_f^{(2)}(I) \). Then, a bias will be introduced by using the above random weights. More generally, the more is known about the relation between \( \hat{\mu}_f(I, \Phi, T) \) and the ground process \( \Phi_g \), the more can be gained from using different (random) weights while preserving consistency of the estimator. Without any assumption, only deterministic or independent weights are feasible and then \( w_i(\Phi_i, T) = 1 \) is naturally the best choice, i.e., the use of \( \hat{\mu}_f^I(I) \).

We consider two simple examples of optimal weighting in the following. Here we assume that the \( z \)-components of the marks are 1 for all points. Recall that \( \hat{\mu}_f(I, \Phi, T) = \hat{\alpha}_f(I, \Phi, T) / \hat{\alpha}_1(I, \Phi, T) \), that the denominator is \( A_n^* \)-measurable, and that \( \hat{\alpha}_f(I, \Phi, T) \) is a sum consisting of \( \hat{\alpha}_1(I, \Phi, T) \) random summands.

Remark 4.2. In general, the summands of \( \hat{\alpha}_f(I, \Phi, T) \) are not iid. However, if conditionally on \( A_n^* \), the summands were iid with variance \( v \), the conditional variance \( \text{Var}[\hat{\mu}_f(I, \Phi, T) \mid A_n^*] \) would be \( v / \hat{\alpha}_1(I, \Phi, T) \).

In the following scenarios, we assume \( f \) to depend on its first argument, only. The proofs are given in Appendix C.

Example 4.1. Let \( \Phi \) have marks that are stochastically independent of the process of point locations and let these point locations be fully regularly spaced in every realization. Let \( \nu_T \) and \( N = N(T) \) denote the volume of \([0, T]\) and the random
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number of points in \([0, T]\), respectively, and assume that the \(f(y_i), i \in \mathbb{Z}\), are iid with variance \(v\). Then, asymptotically, \(\text{Var}[\hat{\mu}_f(I, \Phi, T)|A_n^s] \sim v/N\) and the resulting weights are \(w_i(\Phi_t) = N_i/v\), where \(N_i\) denotes the number of points within the \(i\)-th realization.

Since \(N(T)\) is usually much smaller than \(\hat{\alpha}_1(I, \Phi, T)\), the variance \(\text{Var}[\mu_f(I, \Phi, T)|A_n^s]\) in Proposition 4.1 is larger than the one in the hypothetical example in Remark 4.2.

In the following example, we consider arbitrary point locations but still assume independence between marks and locations.

Example 4.2. Let \(\tilde{\Phi}\) be a one-dimensional, stationary unmarked point process and \(Y\) a stationary continuous-time process which is independent of \(\Phi\) and such that \(f(Y)\) has finite second moments. We consider the MPP \(\Phi = \{(t, Y(t), 1): t \in \tilde{\Phi}\}\). Then

\[
\text{Var}[\hat{\mu}_f(I, \Phi, T)|A_n^s] = \sum_{t_1 \in \Phi_g \cap [0, T]} \sum_{s_1 \in \Phi_g \cap [0, T]} \frac{\text{Cov}[f(Y(t_1)), f(Y(s_1))] n(t_1, \Phi_g, I) n(s_1, \Phi_g, I)}{\left[\sum_{t_1 \in \Phi_g \cap [0, T]} n(t_1, \Phi_g, I)\right]^2},
\]

where \(n(t_1, \Phi_g, I) = \sum_{t_2 \in \Phi_g \setminus \{t_1\}} 1_{t_2 - t_1 \in I}\).

4.4. Remarks

Remark 4.3. The weighting of multiple realizations and the intrinsically weighted means coincide in the following sense: Let \(\Phi_1, \ldots, \Phi_n\) be iid copies of an MPP \(\Phi = \{(t_i, y_i, 1): i \in \mathbb{N}\}\), for which the second mark component equals 1 for all points. Then the weighting of realizations via \(w_i(\Phi_t, T)\) in the estimator (13) can alternatively be captured by the second mark component. For \(i = 1, \ldots, n\), let \(\Phi_i = \{(t_i, y_i) \in \Phi_i\}\), where \(w^{\text{rel}}(\Phi_i, T) = w_i(\Phi_i, T)/\sum_{k=1}^n w_k(\Phi_k, T)\). Let \(\Psi_n\) be the concatenation of the processes \(\Phi_1, \ldots, \Phi_n\), each restricted to the observation window \([0, T]\) and concatenated with a buffer of \(\max(I)\) and such that all points of \(\Psi_n\) are contained in \([0, T_n]\) for some \(T_n \in \mathbb{R}^d\). Then, with \(w = (w_i(\Phi_i, T))_{i=1}^n\), we have

\[
\hat{\mu}_f(I, \Psi_n, T_n) = \hat{\mu}_f^{\text{wght}}(I, w, (\Phi_1, \ldots, \Phi_n), T).
\]

We close this section with a note on the estimation of \(\hat{\mu}_f^{(2)}(r)\) and \(\hat{\mu}_f^{(2)}(r)\), \(r \in \mathbb{R}\).

Remark 4.4. For most MPPs used in applications, finding two points of an MPP with a fixed distance \(r\) within a bounded observation window, has probability zero. Then the simplest approach is to apply any of the estimators (11), (13), (14) or (16), with \(I\) being a small interval containing \(r\), e.g., \([r - \delta, r + \delta]\) for some \(\delta > 0\). This is equivalent to use (Nadaraya-Watson) kernel regression with the rectangular kernel, applied to the tuples \(\{(z_1 f(y_1), \text{dist}(t_2 - t_1)): (t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi\}\), where \(\text{dist}(x) = x\) if \(x \in \mathbb{R}^1\) and \(\text{dist}(x) = ||x||\) if \(x \in \mathbb{R}^d\) with \(d > 1\).

An obvious generalization is to replace the rectangular kernel by a general kernel \(K_h\) with bandwidth \(h\). For the basic estimator (11), this yields

\[
\hat{\mu}_f(r, \Phi, T) = \frac{\sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi, t_1 \in [0, T]} z_1 f(y_1) K_h(r - \text{dist}(t_2 - t_1))}{\sum_{(t_1, y_1), (t_2, y_2) \in \Phi, t_1 \in [0, T]} K_h(r - \text{dist}(t_2 - t_1))},
\]

likewise for the other estimators. If the support of \(K_h\) covers the whole real line, the denominator is always strictly larger than zero, which simplifies implementation, but
Also allows \( \hat{\mu}_f(r, \Phi, T) \) to be driven by pairs of points whose distance differs largely from \( r \).

5. Application to continuous-space processes

Picking up the introductory example on continuous-space processes, taking measurements from such a process with measurement locations that are possibly irregularly spaced but independent of the underlying process, leads to a subclass of MPPs. At the same time, particularly developed in the geostatistical context, there exist numerous methods of inference for continuous-space processes, including methods to account for biased and preferential sampling. We compare the concept of intrinsically weighted means of MPPs to statistical methods for continuous-space processes in the following.

One of the classical problems in geostatistical applications (e.g., [4]) is prediction of averages from measurements \( \{(t_i, Y(t_i)) : i = 1, \ldots, n\} \), where \( \{Y_t : t \in T\}, T \subset \mathbb{R}^d \), is a latent second-order stationary random field. When predicting global moments of \( Y \), redundancies in the data can be excluded via the spatial correlation structure, e.g., the best linear unbiased estimator (BLUE) for \( \mathbb{E}Y \) is well-known to be \( (\mathbf{1}'\Sigma^{-1}\mathbf{1})^{-1}\mathbf{1}'\Sigma^{-1}Y \), where \( \mathbf{1} = (1, \ldots, 1)' \), \( Y = (Y(t_1), \ldots, Y(t_n))' \) and \( \Sigma = \text{Cov}(Y(t_i), Y(t_j))_{i,j=1}^n \) (e.g., [4, p.179]). More generally, any estimator that is linear in a transformation \( \Phi \) of the data allows for assigning a different weight to each data point; then the estimator takes the form \( \sum_{i=1}^n z_i g(Y(t_i)) \) or \( \sum_{i,j=1}^n z_{ij} g(Y(t_i), Y(t_j)) \) (similarly for higher-order moments).

The weights \( z_i \) and \( z_{ij} \) are supposed to capture the spatial or temporal pattern of measurement locations when statistical inference from irregularly spaced data is carried out. Similar weighting procedures are used for declustering and debiasing methods, cf. [11].

**Assertion 5.1.** Identifying the geostatistical weights \( z_i \) with the \( z \)-component of the marked point process \( \Phi = \{(t_i, y_i, z_i) : i \in \mathbb{N}\} \), the estimator \( \sum_{i=1}^n z_i g(Y(t_i)) \) of \( \mathbb{E}g(Y) \) coincides with the canonical estimator for the weighted mean mark \( \mu_f^{(1)} \), defined by (2).

The geostatistical guiding principle of choosing optimal weights for aggregation of measurements adheres to the idea that a) there exists an underlying random field and b) that this field can be measured at any location without causally influencing the other measurements. It is important to note that this is far from being satisfied for processes in which the measurements reflect physical objects that interact with each other. Trees in a forest, for example, compete for resources and if another tree had been added at some point, the measured characteristics of the surrounding trees would have likely changed. Though, with increasing distance, interaction effects between single objects of an MPP may become negligible and the random field assumption might be sensible on a larger scale. This perspective motivates combining classical mean mark estimators for MPPs of the form \( \Phi = \{(t_i, y_i, 1) : i \in \mathbb{N}\} \) with a geostatistical weighting. Partitioning the observation window in smaller parts, we assign a \( z \)-component to \( \Phi \) such that \( z_i = z_j \) whenever \( t_i \) and \( t_j \) belong to the same cell of the partition. This leads to a classical unweighted average within each cell and therewith maintains the information contained in the small-scale pattern of the point locations. Between the different cells, we allow for a weighting in the geostatistical sense and therewith allow to smooth out large-scale irregularities in the distribution of point locations. We denote the resulting estimator by \( \hat{\mu}_f^{(1), \text{geo}} \).
**Assertion 5.2.** Considering a realization of $\Phi$ as a collection of realizations of a possibly non-ergodic MPP on smaller observation windows corresponding to the above partition, the form of $\hat{\mu}_f^{(1),\text{geo}}$ coincides with that of $\hat{\mu}_f^n$ and $\hat{\mu}_f^n,\text{wght}$, which estimate the average mean mark $\hat{\mu}_f^{(1)}$ (see Definition 3.1) instead of the classical mean mark $\mu_f^{(1)}$.

The application of such a weighting scheme is particularly of interest when the underlying process jumps between different regimes that differ substantially from each other, e.g., w.r.t. the intensity of point locations. In summary, applying the geostatistical idea of declustering in the MPP context in a sense corresponds to the concept of non-ergodic modeling.

To avoid possible confusion, we conclude this section with a final remark.

**Remark 5.1.** For certain choices of $f$, the random field counterpart of $\mu_f^{(2)}$ is well-defined. For $f(y_1, y_2) = y_1 y_2$, for instance, the counterpart is the ordinary (non-centered) covariance function. If $f$ only depends on one of the two marks of a pair of points, $\mu_f^{(2)}$ implicitly conditions on the existence of other points and there is no sensible way of interpreting suchlike statistic in a random field context, where there exist values at all points of the index space. Nevertheless, the geostatistical idea of variance-minimizing weights can be applied to $\mu_f^{(2)}$ by a simple mean squared error approach.

### 6. Discussion

The MPP summary statistics considered in this paper are (weighted) mean marks. In practice, the choice of weights is not always clear, for example when data from different stochastic sources are combined. In Section 5 we point out that, if there was an underlying continuous-time process from which the data were generated by a random sampling procedure, then the mean of interest would rather be the temporal average over the whole index space instead of the average over all sampling locations. The weights might then be chosen to compensate for the irregular distribution of point locations. Though, the assumption of a continuous-time background process is problematic if the points represent physical objects that influence each other. Then, the mean of interest might include the randomness of the point pattern, as it is reflected by the MPP moment measures $\alpha_f^{(2)}$.

Related questions arise when multiple realizations of a non-ergodic MPP are considered: Should the definition of mean include possibly different intensities of points between different ergodicity classes or not? A non-ergodic MPP can be seen as a hierarchical model and expectation functionals w.r.t. the point process can naturally be replaced by two-step expectations by averaging within each ergodicity class first and then aggregating the different classes (cf. Section 3). This alternative definition filters out the differences w.r.t. the point location patterns between different ergodicity classes. Which definition of mean should be chosen eventually depends on the purpose of the characteristic at hand and on the intended interpretation.

### Appendix A. Ergodic theory

Ergodicity is a mixing property that can be defined in the very general context of dynamical systems. A MPP on $\mathbb{R}^d$ together with the group of $\mathbb{R}^d$-indexed shift
operators is a special case of a dynamical system.

We denote by $M_0$ the set of all locally finite counting measures on $\mathbb{R}^d \times \mathbb{R}$, and by $M_0$ the smallest $\sigma$-algebra on $M_0$ that makes all mappings $M_0 \to \mathbb{N}_0 \cup \infty$, $\varphi \mapsto \varphi(S)$, measurable. Formally, a MPP $\Phi$ is a measurable mapping from some probability space $(\Omega, A, P)$ into $(M_0, M_0)$ and we can identify $(\Omega, A)$ with $(M_0, M_0)$ in the usual way. Let $T = \{T_x : x \in \mathbb{R}^d\}$ with $$ (T_x \varphi)(B \times L) = \varphi((B + x), L), \quad B \in B^d, L \in \mathbb{R}. \quad (26) $$

Recall that $\Phi$ is said to be stationary if the induced probability measure $P^\Phi$ is $T$-invariant. Further, a stationary MPP $\Phi$ is called ergodic if $P^\Phi(A)$ is either zero or one for all $T$-invariant sets $A \in M_0$. Let $A_0 \subset M_0$ be the sub-$\sigma$-algebra of all $T$-invariant sets in $M_0$, i.e., $A = T^{-1}A$ for all $A \in A_0$ and $T \in T$. The following theorem is commonly termed pointwise or individual ergodic theorem in literature and establishes almost sure convergence of a certain average of values of a random variable $X$.

**Definition A.1.** (Def. 12.2.I in [7].) An increasing sequence of bounded convex Borel sets $W_n \subset \mathbb{R}^d$ is called convex averaging sequence in $\mathbb{R}^d$ if the maximal radius of a ball contained in $W_n$ goes to infinity if $n$ increases.

**Theorem A.1.** (Prop. 12.2.II [7].) Let $(\Omega, A, P)$ be a probability space and $T = \{T_x : x \in \mathbb{R}^d\}$ a group of measure-preserving transformations acting on $(\Omega, A, P)$ such that the mapping $(T_x, \omega) \mapsto T_x \omega$ is jointly measurable, i.e., $(B(T) \otimes A, A)$-measurable. (Multiplication in $T$ is given by $T_x T_y = T_{x+y}$.) Let $\{W_n\}_{n \in \mathbb{N}}$ be a convex averaging sequence in $\mathbb{R}^d$ and $A_0$ the $\sigma$-algebra of $T$-invariant events. Then for all real-valued integrable functions $X$ on $(\Omega, A, P)$

$$ \bar{X}_n = \frac{1}{\nu(W_n)} \int_{W_n} X(T_x \omega) \nu(dx) \xrightarrow{a.s.} \mathbb{E}(X | A_0), \quad n \to \infty. $$

If $X$ is additionally $L_p$-integrable, then $\mathbb{E}(X | A_0)$ is also the $L_p$-limit of $\bar{X}_n$.

**Remark A.1.** If $P$ is ergodic (i.e., $P(A) \in \{0, 1\} \forall A \in A_0$) then $\mathbb{E}(X | A_0)$ reduces to the constant $\mathbb{E}X$. Loosely speaking, this means that a suitable average over transformations of a single realization converges to the expectation over the state space $\Omega$.

While Theorem A.1 refers to a general probability space with a general group of transformations action on it, the following Proposition relates this result to the context of MPPs on $\mathbb{R}^d$, in which the transformations $T_x$, $x \in \mathbb{R}^d$, are given by shifts of the whole point pattern by the vector $x$. Here, the point is that the index $x \in \mathbb{R}^d$ has a direct geometric meaning when $T_x$ is applied to a realization $\varphi$ of $\Phi$. This yields convergence of spatial averages within a single realization of the MPP to the state space mean.

The proof of the following Proposition is based on a simple sandwich argument, which can also be used for other consistency statements. We include the proof here, because to our knowledge, it is not available in this form in pertinent literature. A similar assertion can be found in [7, Thm. 12.2.IV].

**Proposition A.1.** Let $\Phi$ be stationary and ergodic and $T$ as in Theorem A.1. Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a non-negative function that satisfies $f(t-x, y, T_x \varphi) = f(t, y, \varphi)$ for all $x, y \in \mathbb{R}^d$ and $\varphi \in \Phi$. Then

$$ f(x, y, \varphi) \xrightarrow{a.s.} \mathbb{E}(f(x, y, \varphi) | A_0), \quad n \to \infty. $$

If $f$ is additionally $L_p$-integrable, then $\mathbb{E}(f(x, y, \varphi) | A_0)$ is also the $L_p$-limit of $f(x, y, \varphi)$.
for all \( t, x \in \mathbb{R}^d, y \in \mathbb{R} \), and that is integrable w.r.t. to the marked Campbell measure \( C(B \times L \times M) = \mathbb{E}_\Phi(B \cap [0, 1]^d) \times L \mathbb{1}_M(\Phi) \), \( B \in \mathcal{B}, L \in \mathcal{L}, M \in \mathcal{M}_0 \). We define random variables \( X, X_n : \mathcal{M}_0 \to \mathbb{R} \) by

\[
X(\varphi) = \sum_{(t, y) \in \varphi, t \in [0, 1]^d} f(t, y, \varphi)
\]

\[
X_n(\varphi) = \frac{1}{n^d} \sum_{(t, y) \in \varphi, t \in [0, n]^d} f(t, y, \varphi).
\]

Then \( X_n \) converges to \( \mathbb{E}^X \) almost surely if \( n \to \infty \).

**Proof.** An extension of the classical Campbell theorem (e.g., Lem. 13.1.II in [7]) guarantees that \( \mathbb{E}|X| < \infty \) if \( f \) is integrable w.r.t. the Campbell measure. The \( W_n = [0, n]^d \) obviously form an averaging sequence and

\[
X_n(\varphi) = \frac{1}{\nu(W_n)} \sum_{(t, y) \in \varphi, t \in W_n} f(t, y, \varphi) \int_{\mathbb{R}^d} 1_{[t, t+1]}(x) \nu(dx)
\]

\[
= \frac{1}{\nu(W_n)} \int_{\mathbb{R}^d} \sum_{(t, y) \in \varphi, t \in W_n \cap [x-1, x]} f(t, y, \varphi) \nu(dx),
\]

where \( x \pm 1 \) for \( x \in \mathbb{R}^d \) is defined component-wise. Note that the integrand on the RHS equals 0 whenever \( W_n \cap [x-1, x] = \emptyset \), which means that \( x \) is not contained in \( W_n \oplus [0, 1]^d \), which is, on its part, a subset of \( W_n+1 \). Thus, we can shrink the region of integration to \( W_n+1 \) without changing the integral. If we then drop the condition \( t \in W_n \) under the summation sign, we enlarge the whole expression since \( f \) is non-negative, i.e.

\[
X_n(\varphi) \leq \frac{1}{\nu(W_n)} \int_{W_n+1} \sum_{(t, y) \in \varphi, t \in [x-1, x]} f(t, y, \varphi) \nu(dx)
\]

\[
= \frac{1}{\nu(W_n)} \int_{W_n+1} \sum_{(t, y) \in \varphi, t \in [0, 1]^d} f(t, y, T_x \varphi) \nu(dx)
\]

\[
= \frac{\nu(W_{n+1})}{\nu(W_n)} \int_{W_n+1} X(T_x \varphi) \nu(dx),
\]

where the second equation uses that \( f(t-x, y, T_x \varphi) = f(t, y, \varphi) \) and the last equation uses that \( \nu \) is shift-invariant. Since the ratio \( \nu(W_{n+1})/\nu(W_n) \) converges to 1, Theorem A.1 yields that the RHS of (28) converges to \( \mathbb{E}^X | A_0 \) for almost all \( \varphi \in \mathcal{M}_0 \). Since \( \Phi \) was assumed to be ergodic, this conditional expectation equals \( \mathbb{E}^\Phi X \).

Similarly, if we restrict integration in (27) to the set \( W_{n-1} \), we reduce the value of the integral. Since \( W_{n-1} \oplus [-1, 0]^d \subset W_n \), we can again drop the condition \( t \in W \) under the summation sign and by the same argument as before, we have

\[
X_n(\varphi) \geq \frac{1}{\nu(W_n)} \int_{W_{n-1}} \sum_{(t, y) \in \varphi, t \in [x, x+1]} f(t, y, \varphi) \nu(dx) \xrightarrow{n \to \infty} \mathbb{E}^\Phi X
\]

for almost all \( \varphi \in \mathcal{M}_0 \). Thus, we have a sandwich relation for \( X_n(\varphi) \) and can conclude that \( X_n \to \mathbb{E}^\Phi X \) a.s.
Note that the convex averaging sequence \( \{[0, n^d]_{n \in \mathbb{N}} \} \) in Proposition A.1 can be replaced by any sequence \( \{W \oplus nV\}_{n \in \mathbb{N}} \) with \( W \) a bounded Borel set and \( V \subset \mathbb{R}^d \) a convex and bounded set with \( \nu(V) > 0 \) and \( 0 \in V \).

In case that \( \Phi \) is not ergodic, the following results provide a representation of \( \Phi \) as a mixture of a set of ergodic MPPs. To this end, let \( \mathcal{P} (\mathcal{P}_{\text{erg}} \text{ resp.}) \) denote the set of all probability measures on \((M_0, M_0)\) induced by stationary (and ergodic) MPPs and let \( \Pi_{\text{erg}} \) be the smallest \( \sigma \)-algebra making all mappings \( \mathcal{P}_{\text{erg}} \to [0, 1], P \mapsto P(A) \), measurable. We say that \( T \) fulfills the condition (LocCompGrp) if \( T \) is a locally compact, second-countable Hausdorff group of jointly measurable, surjective transformations.

From [9] we can extract the very general result

**Theorem A.2.** Let \((\Omega, A)\) be a measurable space with \( \Omega \) a complete separable metric space and \( A \) its Borel-\( \sigma \)-algebra. Let \( T \) be a set of measurable transformations of \( \Omega \) satisfying the condition (LocCompGrp) and let \( \mathcal{P} \) (\( \mathcal{P}_{\text{erg}} \) resp.) be the set of all \( T \)-invariant (and ergodic) probability measures on \((\Omega, A)\). Then there is a unique probability measure \( \lambda_P \) on \((\mathcal{P}_{\text{erg}}, \Pi_{\text{erg}})\) and a \( \mathcal{P}_{\text{erg}} \)-valued random variable \( Q_P \) s.t.

\[
P(A) = \int_{\mathcal{P}_{\text{erg}}} Q(A) \lambda_P(dQ) = \int_{\Omega} Q_P(\omega)(A) P(d\omega) \quad \forall A \in \mathcal{A},
\]

i.e., \( \lambda_P \) is the distribution of \( Q_P \).

In the context of MPPs on \( \mathbb{R}^d \), the group \( T \) of shifts, as defined in (26), obviously fulfills the condition (LocCompGrp), and since \( M_0 \) is a complete separable metric space and \( M_0 \) its Borel-\( \sigma \)-algebra (e.g., [13]), Theorem A.2 can directly be applied, which yields a decomposition of the non-ergodic MPP \( \Phi \sim P \):

\[
P(M) = \int_{\mathcal{P}_{\text{erg}}} Q(M) \lambda(dQ) \quad \forall M \in M_0.
\]

Note that each \( Q \) induces a new ergodic MPP \( \Phi_Q : \Omega \to M_0 \) which is given implicitly by \( P(\Phi_Q \in M) = Q(M), M \in M_0 \). By the second representation in Theorem A.2 we can also consider \( Q \) as a random variable on \((M_0, M_0, P)\) with distribution \( \lambda = \lambda_P \). Thus, \( \Phi \) and \( Q^\Phi \) have a joint distribution and the conditional distribution of \( \Phi \) given \( Q \) is well-defined:

\[
P(\cdot | Q = Q^*) = Q^*(\cdot).
\]

**Appendix B. Proof of Theorem 4.1**

The following lemma generalizes the classical individual ergodic theorem [7, Prop. 12.2.II] to a situation in which the thinning of the point process depends on the size of the observation window.

**Lemma B.1.** Let \( \Phi \) be a stationary and ergodic MPP on \( \mathbb{R} \) with real-valued marks and let \((u_T)_{T \geq 0}\) be a family of non-negative non-decreasing numbers such that

\[
\frac{T^{-1} \hat{\delta}_1(I, \Phi, T) - \lambda}{\mathbb{E}_\Phi \hat{\alpha}_{\text{cond}, u_T}(I, \Phi, 1)} \to 0 \quad \text{a.s. (} T \to \infty \text{)}.
\]
Then, for $T \to \infty$, we have the almost sure convergence
\[
\frac{\hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, T)}{TE \phi \hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, 1)} \to 1.
\]
Note that the almost sure convergence $(\lambda T)^{-1} \hat{\alpha}_1(I, \Phi, T) \to 1$ as $T \to \infty$ follows from the classical individual ergodic theorem (e.g., [7, Prop. 12.2.II]).

Proof of Lemma [B.3] With $g_u(y) = 1 - f_{\text{cond},u}(y)$, $y \in \mathbb{R}$, we obtain the almost sure convergence
\[
\frac{\hat{\alpha}_{g_{u,T}}(I, \Phi, T)}{TE \phi \hat{\alpha}_{g_{u,T}}(I, \Phi, 1)} \to 1
\]
from [7, Prop. 12.2.VII] and the subsequent remarks. Further, $\lambda = E \phi \hat{\alpha}_1(I, \Phi, 1) = E \phi \hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, 1) + E \phi \hat{\alpha}_{g_{u,T}}(I, \Phi, 1)$. Hence,
\[
\frac{\hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, T)}{TE \phi \hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, 1)} = \frac{\hat{\alpha}_1(I, \Phi, T) - \hat{\alpha}_{g_{u,T}}(I, \Phi, T)}{TE \phi \hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, 1)} = \frac{\lambda \hat{\alpha}_1(I, \Phi, T) - E \phi \hat{\alpha}_{g_{u,T}}(I, \Phi, 1) \hat{\alpha}_{g_{u,T}}(I, \Phi, T)}{E \phi \hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, 1)}
\]
and the RHS converges to 1 as long as $E \phi \hat{\alpha}_1, f_{\text{cond},u,T}(I, \Phi, 1)$ converges to 0 at a slower rate (in the sense of (29)) than $\frac{\hat{\alpha}_1(I, \Phi, T)}{\lambda}$ and $\frac{\hat{\alpha}_{g_{u,T}}(I, \Phi, T)}{E \phi \hat{\alpha}_{g_{u,T}}(I, \Phi, 1)}$ approach 1.

Proof of Theorem [4.1] We have
\[
\frac{\hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, T)}{\sqrt{\hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, T)}} = \frac{\hat{\alpha}_{f_{\text{cond},u,T}}^*(I, \Phi, T)}{\sqrt{\lambda_{u,T}} T} \frac{\sqrt{\lambda_{u,T} T}}{\sqrt{\hat{\alpha}_{f_{\text{cond},u,T}}(I, \Phi, T)}}
\]
and by Lemma [B.1] the last factor converges to 1. (Here, for $a \geq 0$, $[a]$ denotes the smallest integer $\geq a$.) Hence, for convergence of the LHS it is sufficient to show that $\hat{\alpha}_{f_{\text{cond},u,T}}^*(I, \Phi, T) / \sqrt{\lambda_{u,T} T}$ converges to a Gaussian variable. According to [3, Lemma 2.3], we can write $\Phi$ as a sum of Dirac measures $\delta_{(T_i,Y_i)}$, $i \in \mathbb{N}$, with random vectors $(T_i, Y_i)$ and $T_1 \leq T_2 \leq \ldots$. If only a finite observation window $[0,T]$ is considered, the number of summands $N(T)$ is also finite but random. Then we introduce a modified version of $\hat{\alpha}_{f_{\text{cond},u,T}}^*(I, \Phi, T)$, in which the sum is cut after a fixed number $N_{\text{max}} \in \mathbb{N}$ of terms:
\[
\hat{\alpha}_{f_{\text{cond},u,T}}^* N_{\text{max}}(I, \Phi, T) = \sum_{i=1}^{N(T)} \sum_{j=1}^{N(T)} \left[ f_u(Y_i) - \mu_{j,u,f_{\text{cond},u}}(I) \right] \cdot f_{\text{cond},u}(Y_i) \cdot 1_{T_j - T_i} \leq i \leq N_{\text{max}}.
\]
Then we have
\[
\frac{\hat{\alpha}_{f_{\text{cond},u,T}}^*(I, \Phi, T)}{\sqrt{\lambda_{u,T}}} = \frac{\hat{\alpha}_{f_{\text{cond},u,T}}^*(\lambda_{u,T} T)}{\sqrt{\lambda_{u,T}}} + \frac{\hat{\alpha}_{f_{\text{cond},u,T}}^*(\lambda_{u,T} T) - \hat{\alpha}_{f_{\text{cond},u,T}}^*(I, \Phi, \infty)}{\sqrt{\lambda_{u,T}}} (30)
\]
and the first summand of the RHS contains a non-random number of summands (namely $|\lambda_{uT}|$). By the minimum distance assumption in condition (m-dependent Random Field Model), each mark $Y_i$ occurs at most $|I|/d_0$ times in $\hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(I, \Phi, \infty)$. By the finite-range assumption on the covariance function of the underlying random field, the sequence $(Y_i)_{i \in \mathbb{N}}$ is $[h_0/d_0]$-dependent. Hence, the sequence of summands in $\hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(\Phi, I, \infty)$ is $|[I]/d_0\mathbb{R}]$-dependent. By assumption, the first four moments of the excesses $Z_i = [f_{uT}(Y_i) | f(Y_i) > uT]$ exist and converge to some constant in $(0, \infty)$ as $T \to \infty$. Then the sequence of summands in $\hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(\Phi, I, \infty)$ satisfies the assumptions of Berk’s CLT for triangular arrays of m-dependent random variables and thus, for $T \to \infty$, $\hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(I, \Phi, \infty)/\sqrt{\lambda_{uT}T}$ approaches a Gaussian distribution with zero mean and variance

$$u_\infty = \lim_{T \to \infty} \text{Var} \left[ \hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(I, \Phi, \infty) \right] /[\lambda_{uT}T].$$

Next, we show that the second summand in (30) converges to 0 in probability. We use the notation $\Delta \alpha_{f_{uT}} = \hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(I, \Phi, \Phi) - \hat{\alpha}_{f_{uT}}^{*[\lambda_{uT}]T}(I, \Phi, \Phi)$ and $\Delta \alpha_1 = \hat{\alpha}_{f_{\text{cond},u}}^{*[\lambda_{uT}]T}(I, \Phi, \Phi) - \hat{\alpha}_{f_{\text{cond},u}}^{*[\lambda_{uT}]T}(I, \Phi, \Phi)$ and consider

$$P(\{ |\Delta \alpha_{f_{uT}}| \geq \varepsilon \sqrt{\lambda_{uT}T} \} | |\Delta \alpha_1| \geq \varepsilon \lambda_{uT}|) \cdot P(\{ |\Delta \alpha_1| \geq \varepsilon \lambda_{uT} \})$$

$$+ P(\{ |\Delta \alpha_{f_{uT}}| \geq \varepsilon \sqrt{\lambda_{uT}T} \} | |\Delta \alpha_1| < \varepsilon \lambda_{uT}|) \cdot P(\{ |\Delta \alpha_1| < \varepsilon \lambda_{uT} \})$$

$$\leq P(\{ |\Delta \alpha_1| \geq \varepsilon \lambda_{uT} \}) + P(\{ |\Delta \alpha_{f_{uT}}| \geq \varepsilon \sqrt{\lambda_{uT}T} \} | |\Delta \alpha_1| < \varepsilon \lambda_{uT} \})$$

(31)

Note that $\hat{\alpha}_{f_{\text{cond},u}}^{*[\lambda_{uT}]T}(I, \Phi, \Phi) = \lambda_{uT}$ and hence

$$P(\{ |\Delta \alpha_1| \geq \varepsilon \lambda_{uT} \} = P \left( \{ |\hat{\alpha}_{f_{\text{cond},u}}(I, \Phi, \Phi) - \lambda_{uT}| - 1 \geq \varepsilon \} \right) \to 0 \text{ for } T \to \infty.$$ 

(32)

To estimate the last summand in (31), we use again that the sequence $(Y_i)_{i \in \mathbb{N}}$ is $[h_0/d_0]$-dependent and that the number of points in any interval of length $|I|$ is bounded by $c = |I|/d_0$. This means that each term $f_{uT}(Y_i)$ occurs at most $c$ times in the sum $\Delta \alpha_{f_{uT}}$. Obviously, the variance of $\Delta \alpha_{f_{uT}}$, or more generally all even centered moments of $\Delta \alpha_{f_{uT}}$, become maximal, if this boundary is bailed, i.e., if for a given total number $\Delta \alpha_1$ of summands, only $|\Delta \alpha_1|/c$ different $Y_i$ are involved. With $Z_i = \sum_{i=1}^{c}[f_{uT}(Y_i) | f(Y_i) > uT] - c(uT)$, where $c(u) = \mathbb{E} \left[ f_u(Y(0)) | f(Y(0)) > u \right]$, we get

$$P(\{ |\Delta \alpha_{f_{uT}}| \geq \varepsilon \sqrt{\lambda_{uT}T} \} | |\Delta \alpha_1| < \varepsilon \lambda_{uT}|)$$

$$= P \left( \{ |\Delta \alpha_{f_{uT}}| \geq \varepsilon \lambda_{uT}^2 \} | |\Delta \alpha_1| < \varepsilon \lambda_{uT} \} \right)$$

$$\leq P \left( \sum_{i=1}^{c} [|\lambda_{uT}|^{-1} \varepsilon] Z_i^4 \geq \varepsilon \lambda_{uT} \right)$$

$$\leq P \left( \sum_{i=1}^{c} [\lambda_{uT}]^{-1} \varepsilon Z_i^4 \geq \varepsilon \lambda_{uT} \right)$$

$$\leq P \left( \sum_{i=1}^{c} [\lambda_{uT}]^{-1} \varepsilon Z_i^4 \geq \varepsilon \lambda_{uT} \right)$$

(33)
Plugging this and (32) into (31) yields that \( \Delta \alpha_{f_{w,T}} \) \( \cdot [\lambda_{u,T}T]^{-1} \) \( \cdot (\varepsilon^4[\lambda_{u,T}T]^2)^{-1} \) \( \leq c^4 \sum_{i,j,k,l=1} E[Z^*_i Z^*_j Z^*_k Z^*_l] \cdot (\varepsilon^4[\lambda_{u,T}T]^2)^{-1} \) \( \leq c^4 \cdot [\varepsilon[\lambda_{u,T}T]c^{-1}] \cdot \left( \frac{h_0}{d_0} \right)^3 E[(Z^*_i)^4] \cdot (\varepsilon^4[\lambda_{u,T}T]^2)^{-1} \) \( = (\lambda_{u,T}T)^{-1} \varepsilon^{-3} \left( \frac{h_0}{d_0} \right)^3 E[(Z^*_i)^4] (1 + o(1)) \rightarrow 0, \quad (T \to \infty). \)

Appendix C. Proofs of Examples in Section 4

Proof of Example 4.1. For \( |I| \) and \( T \) large, we have \( \hat{\alpha}_1(I, \Phi, T) \sim N \cdot |I|/\nu_T \) and each distinct summand in \( \hat{\alpha}_f(I, \Phi, T) \) occurs \( N |I|/\nu_T \sim \hat{\alpha}_1(I, \Phi, T)/N \) times. Thus, \( \hat{\alpha}_f(I, \Phi, T) \sim \hat{\alpha}_1(I, \Phi, T) \sum_{i=1} f(y_i)/N \) and Var[\( \hat{\alpha}_f(I, \Phi, T) | A_n^* \) \( \sim \hat{\alpha}_1(I, \Phi, T)^2 v/N \)

Proof of Example 4.2. We have

\[
E[\hat{\alpha}_f(I, \Phi, T)]/\hat{\alpha}_1(I, \Phi, T) | A_n^* \]
\[
= \hat{\alpha}_1(I, \Phi, T)^{-1} \cdot E \left[ \sum_{(t_1, y_1, z_1), (t_2, y_2, z_2) \in \Phi, t_1 \in [0, T]} z_1 f(y_1) \cdot 1_{t_2 - t_1 \in I} A_n^* \right] \]
\[
= \hat{\alpha}_1(I, \Phi, T)^{-1} \cdot \sum_{t_1 \in \Psi \cap [0, T]} \cdot \# \{ t_2 \in \Phi : t_2 - t_1 \in I \} \cdot E[f(Y(t_1)) | A_n^*] \]
\[
= E[f(Y(0))].\]

and

\[
E[\hat{\alpha}_f(I, \Phi, T)^2 | A_n^*] \]
\[
= E \left[ \sum_{t_1, s_1 \in \Psi \cap [0, T]} f(Y(t_1)) f(Y(s_1)) \cdot \# \{ t_2 \in \Phi : t_2 - t_1 \in I \} \cdot \# \{ s_2 \in \Phi : s_2 - s_1 \in I \} | A_n^* \right] \]
\[
= \sum_{t_1, s_1 \in \Psi \cap [0, T]} n(t_1, \Phi_g, I) n(s_1, \Phi_g, I) \cdot E[f(Y(t_1)) f(Y(s_1)) | A_n^*] \]
\[
= \sum_{t_1, s_1 \in \Psi \cap [0, T]} n(t_1, \Phi_g, I) n(s_1, \Phi_g, I) \cdot \left[ E[f(Y(0)) | A_n^*]^2 + \text{Cov}[f(Y(t_1), f(Y(s_1)) | A_n^*] \right] \]
\[
= \sum_{t_1, s_1 \in \Psi \cap [0, T]} n(t_1, \Phi_g, I) n(s_1, \Phi_g, I) \cdot \text{Cov}[f(Y(t_1), f(Y(s_1))]
\]
\[
+ (E[f(Y(0))])^2 \hat{\alpha}_1(I, \Phi, T)^2.\]
Hence,

\[
\begin{align*}
\text{Var}[\hat{\alpha}_f(I, \Phi, T) / \hat{\alpha}_1(I, \Phi, T) | A_n^*] &= \mathbb{E}[(\hat{\alpha}_f(I, \Phi, T) / \hat{\alpha}_1(I, \Phi, T) | A_n^*)^2] - (\mathbb{E}[\hat{\alpha}_f(I, \Phi, T) / \hat{\alpha}_1(I, \Phi, T) | A_n^*])^2 \\
&= \hat{\alpha}_1(I, \Phi, T)^{-2} \cdot \mathbb{E}[\hat{\alpha}_f(I, \Phi, T)^2 | A_n^*] - (\mathbb{E}[f(Y(0))]^2 \\
&= \hat{\alpha}_1(I, \Phi, T)^{-2} \sum_{t_1, s_1 \in \Phi_\text{g} \cap [0, T]} n(t_1, \Phi_\text{g}, I) n(s_1, \Phi_\text{g}, I) \cdot \text{Cov}[f(Y(t_1), f(Y(s_1))].
\end{align*}
\]

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