Double-layer potentials for a generalized bi-axially symmetric Helmholtz equation II

A. S. Berdyshev\textsuperscript{a, b}, A. Hasanov\textsuperscript{c} and T. Ergashev\textsuperscript{d}

\textsuperscript{a}Kazakh National Pedagogical University, Almaty, Kazakhstan; \textsuperscript{b}Institute of Information and Computational Technologies, Almaty, Kazakhstan; \textsuperscript{c}Institute of Mathematics, Uzbek Academy of Sciences, Tashkent, Uzbekistan; \textsuperscript{d}Tashkent Institute of Engineers of Irrigation and Mechanization of Agriculture, Tashkent, Uzbekistan

\textbf{ABSTRACT}

In earlier papers, the double-layer potential has been successfully applied in solving boundary value problems for elliptic equations. All the fundamental solutions of the generalized bi-axially symmetric Helmholtz equation were known [Complex Var Elliptic Equ. 2007;52(8):673–683], while the potential theory was constructed only for the first one [Sohag J Math. 2015;2(1):1-10]. Here, in this paper, our goal is to construct theory of double-layer potentials corresponding to the next fundamental solution. We used some properties of one of Appell’s hypergeometric functions with respect to two variables to prove the limiting theorems, while integral equations concerning the denseness of double-layer potentials are derived.

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1. Introduction

Potential theory has been successfully implemented in analysis as well as in computation of boundary value problems for elliptic partial differential equations. Practical applications can be seen in numerous areas such as electrodynamics, fracture mechanics, fluid mechanics, electromagnetic, and acoustics. Thanks to latest potential theory results, boundary value problems can be represented in the form of integral equation. And quite powerful numerical approximation schemes can be derived from the integral equation formulation for problems with known Green’s functions.

The double-layer potential is appeared to be critical in solving boundary value problems of elliptic equations. The representation of the of the (first) boundary value problem solution is sought as a double-layer potential with unknown density and the function is
determined by applying certain property leading to a Fredholm equation of the second kind (see [1,2]).

Method of complex analysis (based upon analytic functions), has been applied by Gilbert [3] to construct an integral representation of solutions for the following generalized bi-axially symmetric Helmholtz equation:

\[ H^{\lambda}_{\alpha,\beta} (u) \equiv u_{xx} + u_{yy} + \frac{2\alpha}{x} u_x + \frac{2\beta}{y} u_y - \lambda^2 u = 0, \]

\[ \begin{pmatrix} 0 < \alpha < \frac{1}{2}; \ 0 < \beta < \frac{1}{2} \end{pmatrix}, \quad \left( H^{\lambda}_{\alpha,\beta} \right) \]

where \( \alpha, \beta \) and \( \lambda \) are constants. When \( \lambda = 0 \) this equation is known as the equation of the generalized axially symmetric potential theory whose name is due to Weinstein who first considered fractional dimensional space in potential theory (see [4]). The special case where \( \lambda = 0 \) was also investigated by (among others) Erdelyi (see [5,6]), Gilbert (see [7–11]), Gilbert and Howard [12], Ranger [13] and Henrici (see [14,15]). Various interesting problems associated with the equation \( (H^{\lambda}_{\alpha,\beta}) \) were studied by many authors (see, e.g. [16–26, 27, 28–32]).

Recently, fundamental solutions of the equation \( (H^{\lambda}_{\alpha,\beta}) \) were found (see [33]). In fact, Appell's hypergeometric function with respect to two variables of the second kind can be represented as the fundamental solutions of the equation \( (H^{\lambda}_{\alpha,\beta}) \) when \( \lambda = 0 \), that is, the Appell function \( F_2(a, b_1, b_2; c_1, c_2; x, y) \) defined by (see [34, p.224,Eq.5.7.1(7)]; see also [35, p.14, Eq.(12)] and [36, p.23, Eq.1.3(3)])

\[ F_2(a; b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n x^m y^n}{(c_1)_m (c_2)_n m!n!}, \quad (1) \]

where \( (a)_v \) denotes the general Pochhammer symbol defined (for \( a, \nu \in \mathbb{C}\setminus\{0\} \)), in terms of the familiar Gamma function, by

\[ (a)_v := \frac{\Gamma (a + v)}{\Gamma (a)} = \begin{cases} 1 & (\nu = 0; \ a \in \mathbb{C}\setminus\{0\}), \\ a(a + 1) \cdots (a + \nu - 1) & (\nu = n \in \mathbb{N}; \ a \in \mathbb{C}\setminus\{0\}), \end{cases} \]

it is conventionally understood that \( (0)_0 := 1 \) and assumed tacitly that the \( \Gamma \)-quotient exists. In case of \( \lambda = 0 \) fundamental solutions look like

\[ q_1 (x, y; x_0, y_0) = k_1 (r^2)^{-\alpha - \beta} F_2 (\alpha + \beta; \alpha, \beta; 2\alpha, 2\beta; \xi, \eta), \quad (2) \]

\[ q_2 (x, y; x_0, y_0) = k_2 (r^2)^{\alpha - \beta - 1} x^{1-2\alpha} y^{1-2\beta} F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta), \quad (3) \]

\[ q_3 (x, y; x_0, y_0) = k_3 (r^2)^{-\alpha + \beta - 1} y^{1-2\beta} y_0^{1-2\beta} F_2 (1 + \alpha - \beta; \alpha, 1 - \beta; 2\alpha, 2 - 2\beta; \xi, \eta). \quad (4) \]

and

\[ q_4 (x, y; x_0, y_0) = k_4 (r^2)^{\alpha + \beta - 2} x^{1-2\alpha} y^{1-2\beta} x_0^{1-2\alpha} y_0^{1-2\beta} \times F_2 (2 - \alpha - \beta; 1 - \alpha, 1 - \beta; 2 - 2\alpha, 2 - 2\beta; \xi, \eta), \quad (5) \]
where

\[ k_1 = \frac{2^{2\alpha + 2\beta}}{4\pi} \frac{\Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta)}{\Gamma (2\alpha) \Gamma (2\beta)}, \quad (6) \]

\[ k_2 = \frac{2^{-2\alpha + 2\beta}}{4\pi} \frac{\Gamma (1 - \alpha) \Gamma (\beta) \Gamma (1 - \alpha + \beta)}{\Gamma (2 - 2\alpha) \Gamma (2\beta)}, \quad (7) \]

\[ k_3 = \frac{2^{2\alpha - 2\beta}}{4\pi} \frac{\Gamma (\alpha) \Gamma (1 - \beta) \Gamma (1 + \alpha - \beta)}{\Gamma (2\alpha) \Gamma (2 - 2\beta)}, \quad (8) \]

\[ k_4 = \frac{2^{4-2\alpha - 2\beta}}{4\pi} \frac{\Gamma (1 - \alpha) \Gamma (1 - \beta) \Gamma (2 - \alpha - \beta)}{\Gamma (2 - 2\alpha) \Gamma (2 - 2\beta)}, \quad (9) \]

\[ \left[ \begin{array}{c} r_2^2 \\ r_2^2 \\ r_2^2 \\ r_2^2 \\ r_2^2 \\ r_2^2 \end{array} \right] = \left( \begin{array}{c} -x \\ x + x_0 \\ - y \\ y - y_0 \\ - \xi \\ \eta \end{array} \right)^2, \quad \xi = \frac{r^2 - r_1^2}{r^2}, \quad \eta = \frac{r^2 - r_2^2}{r^2}. \quad (10) \]

The fundamental solutions (2)–(5) possess the following properties:

\[ x^{2\alpha} \frac{\partial q_1 (x, y; x_0, y_0)}{\partial x}\big|_{x=0} = 0, \quad y^{2\beta} \frac{\partial q_1 (x, y; x_0, y_0)}{\partial y}\big|_{y=0} = 0, \quad (11) \]

\[ q_2 (x, y; x_0, y_0)\big|_{x=0} = 0, \quad y^{2\beta} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial y}\big|_{y=0} = 0, \quad (12) \]

\[ x^{2\alpha} \frac{\partial q_3 (x, y; x_0, y_0)}{\partial x}\big|_{x=0} = 0, \quad q_3 (x, y; x_0, y_0)\big|_{y=0} = 0, \quad (13) \]

\[ q_4 (x, y; x_0, y_0)\big|_{x=0} = 0 \quad \text{and} \quad q_4 (x, y; x_0, y_0)\big|_{y=0} = 0. \quad (14) \]

In the paper [25] using fundamental solution \( q_1 (x, y; x_0, y_0) \) in the domain defined by

\[ \Omega \subset R^2_+ = \{(x, y) : x > 0, \ y > 0\}, \quad (15) \]

the double-layer potential theory for the equation \((H_{\alpha,\beta}^0)\) was investigated. Here, in this publication, our goal is to construct theory of double-layer potentials corresponding to the next fundamental solution \( q_2 (x, y; x_0, y_0) \). Thus, we are using some properties of one of Appell's hypergeometric functions with respect to two variables to prove limiting theorems and derive integral equations concerning a denseness of double-layer potentials.

## 2. Green’s formula

We consider the following identity to begin with:

\[ x^{2\alpha} y^{2\beta} \left[ u H_{\alpha,\beta}^0 (v) - v H_{\alpha,\beta}^0 (u) \right] = \frac{\partial}{\partial x} \left[ x^{2\alpha} y^{2\beta} (v_x u - v u_x) \right] \]

\[ + \frac{\partial}{\partial y} \left[ x^{2\alpha} y^{2\beta} (v_y u - v u_y) \right]. \quad (16) \]
By integrating both parts of the above identity (16) on a $\Omega$ domain in (15), and using Green's formula, we get the following equation:

$$
\iint_{\Omega} x^{2\alpha} y^{2\beta} \left[ u H_{\alpha,\beta}^0 (v) - v H_{\alpha,\beta}^0 (u) \right] \, dx \, dy \\
= \int_{S} x^{2\alpha} y^{2\beta} u (v_x \, dy - v_y \, dx) - x^{2\alpha} y^{2\beta} v (u_x \, dy - u_y \, dx),
$$

where $S = \partial \Omega$ is a boundary of the $\Omega$-domain. If functions $u(x, y)$ and $v(x, y)$ are solutions of the equation $(H_{\alpha,\beta}^0)$, we get from (17) that

$$
\int_{S} x^{2\alpha} y^{2\beta} \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, ds = 0,
$$

where

$$
\frac{\partial}{\partial n} = \frac{dy}{ds} \frac{\partial}{\partial x} - \frac{dx}{ds} \frac{\partial}{\partial y}, \quad \frac{dy}{ds} = \cos (n, x), \quad \frac{dx}{ds} = - \cos (n, y),
$$

$n$ being the exterior normal to the curve $S$. We also obtain the following identity:

$$
\iint_{\Omega} x^{2\alpha} y^{2\beta} \left[ u_x^2 + u_y^2 + \lambda^2 u^2 \right] \, dx \, dy = \int_{S} x^{2\alpha} y^{2\beta} u \frac{\partial u}{\partial n} \, ds,
$$

where function $u(x, y)$ is a solution of the equation $(H_{\alpha,\beta}^0)$. The special case of (18) when $v = 1$ reduces to the following form:

$$
\int_{S} x^{2\alpha} y^{2\beta} \frac{\partial u}{\partial n} \, ds = 0.
$$

We note from (21) that the integral of the of a equation's $(H_{\alpha,\beta}^0)$ solution normal derivative with a weight $x^{2\alpha} y^{2\beta}$ along the boundary $S$ of the $\Omega$-domain in (15) is equal to zero.

### 3. A double layer potential $w^{(2)}(x_0, y_0)$

Let $\Omega$ in (15) be a domain bounded by intervals $(0, a)$ and $(0, b)$ of the $x$ and $y$-axes, respectively, and a curve $\Gamma$ with the extremities at points $A(a, 0)$ and $B(0, b)$. The parametrical equations of the curve $\Gamma$ are given by the functions $x = x(s)$ and $y = y(s)$ ($s \in [0, l]$), where $l$ denotes the length of $\Gamma$. We assume that $\Gamma$ curve possesses the following properties:

(i) The functions $x = x(s)$ and $y = y(s)$ have continuous derivatives $x'(s)$ and $y'(s)$ on a segment $[0, l]$, and do not vanish simultaneously;

(ii) The second derivatives $x''(s)$ and $y''(s)$ satisfy to Hoelder condition on $[0, l]$, where $l$ denotes the length of the curve $\Gamma$;

(iii) The following conditions are being satisfied in some neighborhoods of points $A(a, 0)$ and $B(0, b)$:
\[
\left| \frac{dx}{ds} \right| \leq cy^{1+\varepsilon} (s) \quad \text{and} \quad \left| \frac{dy}{ds} \right| \leq cx^{1+\varepsilon} (s), \quad 0 < \varepsilon < 1, \quad c = \text{a constant} \quad (22)
\]

\((x, y)\) being the coordinates of a variable point on the curve \(\Gamma\). Let's consider the following integral

\[
w^{(2)}(x_0, y_0) = \int_0^l x^{2\alpha} y^{2\beta} \mu_2 (s) \frac{\partial q_2 (x, y; x_0, y_0)}{\partial n} \, ds, \quad (23)
\]

where the density \(\mu_2(s) \in C[0, l]\) and \(q_2\) is given in (3). We call the integral (23) a double-layer potential with denseness \(\mu_2(s)\).

Furthermore, we investigate some properties of a double-layer potential \(w^{(2)}(x_0, y_0)\) with denseness \(\mu_2(s)\).

**Lemma 3.1:** The following formula holds true:

\[
w^{(2)}_1 (x_0, y_0) = \begin{cases} 
  i (x_0, y_0) - \frac{1}{2} & (x_0, y_0) \in \Omega, \\
  i (x_0, y_0) - \frac{1}{2} & (x_0, y_0) \in \Gamma, \\
  i (x_0, y_0) & (x_0, y_0) \notin \bar{\Omega},
\end{cases} \quad (24)
\]

where a \(\Omega\)-domain and the curve \(\Gamma\) are described as in this section and \(\bar{\Omega} := \Omega \cup \Gamma\);

\[
i (x_0, y_0) = k_2 (1 - 2\alpha) x_0^{1-2\alpha} \int_0^b y^{2\beta} (x_0^2 + (y - y_0)^2)^{\alpha-\beta-1} \\
\times F \left( 1 - \alpha + \beta, \beta; 2\beta; \frac{-4yy_0}{x_0^2 + (y - y_0)^2} \right) \, dy.
\]

**Proof:** Case 1. When \((x_0, y_0) \in \Omega\), we cut a circle centered at \((x_0, y_0)\) with a small radius \(\rho\) off the \(\Omega\)-domain and denote the remaining by \(\Omega^0\) and circuit of the cut-off-circle by \(C_\rho\). The function \(q_2(x, y; x_0, y_0)\) in (3) is a regular solution of the equation \((H^0_{\alpha,\beta})\) in the domain \(\Omega^0\). By using the following derivative formula of Appell’s hypergeometric function ([25], p. 19, (20)):

\[
\frac{\partial^{m+n} F_2 \left( a; b_1, b_2; c_1, c_2; x, y \right)}{\partial x^m \partial y^n} \\
= \frac{(a)_m (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} F_2 \left( a + m + n; b_1 + m, b_2 + n; c_1 + m, c_2 + n; x, y \right) \quad (25)
\]
we have

\[
\frac{\partial q_2 (x, y; x_0, y_0)}{\partial x} = (1 - 2\alpha)k_2 (r^2)^{\alpha - \beta - 1} x^{-2\alpha} x_0^{1-2\alpha} \\
\times F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \\
+ 2(\alpha - \beta - 1)k_2 (r^2)^{\alpha - \beta - 2} (x - x_0)x_0^{1-2\alpha} \\
\times F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \\
- 2k_2 (r^2)^{\alpha - \beta - 2} x_0^{1-2\alpha} (x - x_0) \\
\times \left[ \frac{(1 - \alpha + \beta)(1 - \alpha)}{2 - 2\alpha} \xi F_2 (2 - \alpha + \beta; 1 - \alpha, 1 + \beta; 2 - 2\alpha, 1 + 2\beta; \xi, \eta) \right] \\
+ \frac{(1 - \alpha + \beta)\beta}{2\beta} \eta F_2 (2 - \alpha + \beta; 1 - \alpha, 1 + \beta; 2 - 2\alpha, 1 + 2\beta; \xi, \eta) \right].
\]

Further we apply the following known contiguous relation (see [35, p.21]):

\[
\frac{b_1}{c_1} xF_2 (a + 1; b_1 + 1, b_2; c_1 + 1, c_2; x, y) + \frac{b_2}{c_2} yF_2 (a + 1; b_1, b_2 + 1; c_1, c_2 + 1; x, y) \\
= F_2 (a + 1; b_1, b_2; c_1, c_2; x, y) - F_2 (a; b_1, b_2; c_1, c_2; x, y),
\]

\[\text{(27)}\]

to (26), and obtain

\[
\frac{\partial q_2 (x, y; x_0, y_0)}{\partial x} = (1 - 2\alpha)k_2 (r^2)^{\alpha - \beta - 1} x^{-2\alpha} x_0^{1-2\alpha} \\
\times F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \\
- 2(1 - \alpha + \beta)k_2 (r^2)^{\alpha - \beta - 2} x_0^{2-2\alpha} \\
\times F_2 (2 - \alpha + \beta; 2 - \alpha, \beta; 3 - 2\alpha, 2\beta; \xi, \eta) \\
- 2(1 - \alpha + \beta)k_2 (r^2)^{\alpha - \beta - 2} x_0^{1-2\alpha} (x - x_0) \\
\times F_2 (2 - \alpha + \beta; 1 - \alpha, 1 + \beta; 2 - 2\alpha, 1 + 2\beta; \xi, \eta).
\]

\[\text{(28)}\]

Similarly, we observe that

\[
\frac{\partial q_2 (x, y; x_0, y_0)}{\partial y} = -2(1 - \alpha + \beta)k_2 (r^2)^{\alpha - \beta - 2} x_0^{1-2\alpha} y_0 \\
\times F_2 (2 - \alpha + \beta; 1 - \alpha, 1 + \beta; 2 - 2\alpha, 1 + 2\beta; \xi, \eta) \\
- 2(1 - \alpha + \beta)k_2 (r^2)^{\alpha - \beta - 2} x_0^{1-2\alpha} (y - y_0) \\
\times F_2 (2 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta).
\]

\[\text{(29)}\]
Thus, with the help of (28) and (29), it follows from (3) and (19) that

\[
\frac{\partial q_2(x, y; x_0, y_0)}{\partial n} = - (1 - \alpha + \beta) k_2 \left( r^2 \right)^{\alpha - \beta - 1} x^{1 - 2\alpha} x_0^{1 - 2\alpha}
\times F_2 (2 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \frac{\partial}{\partial n} \left[ \ln r^2 \right] + \frac{\partial q_2(x, y; x_0, y_0)}{\partial n} \bigg|_{x=0}
\times F_2 (2 - \alpha + \beta; 2 - \alpha, \beta; 3 - 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds}
\times F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 1 + \beta; 1 + 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds}
\\times F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds}
\]

By applying (21) and considering identity (12), we get the following formula:

\[
w_2^{(2)} (x_0, y_0) = k_2 x_0^{1 - 2\alpha} \lim_{\rho \to 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} \frac{\partial q_2(x, y; x_0, y_0)}{\partial n} \bigg|_{x=0} \frac{dy}{ds} ds
\]

Substituting from (30) into (31), we find that

\[
w_2^{(2)} (x_0, y_0) = k_2 x_0^{1 - 2\alpha} \lim_{\rho \to 0} \left\{ (1 - \alpha + \beta) \left[ - J_1 - 2x_0J_2 + 2y_0J_3 \right] + (1 - 2\alpha)J_4 \right\} + J_5,
\]

where

\[
\begin{align*}
J_1(x_0, y_0) &= \int_{C_\rho} x y^{2\beta} \left( r^2 \right)^{\alpha - \beta - 1} F_2 (2 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \frac{\partial}{\partial n} \left[ \ln r^2 \right] ds, \\
J_2(x_0, y_0) &= \int_{C_\rho} x y^{2\beta} \left( r^2 \right)^{\alpha - \beta - 2} F_2 (2 - \alpha + \beta; 2 - \alpha, \beta; 3 - 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds} ds, \\
J_3(x_0, y_0) &= \int_{C_\rho} x y^{2\beta} \left( r^2 \right)^{\alpha - \beta - 2} F_2 (2 - \alpha + \beta; 1 - \alpha, 1 + \beta; 1 + 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds} ds, \\
J_4(x_0, y_0) &= \int_{C_\rho} y^{2\beta} \left( r^2 \right)^{\alpha - \beta - 1} F_2 (1 - \alpha + \beta; 1 - \alpha, \beta; 1 + \beta; 2 - 2\alpha, 2\beta; \xi, \eta) \frac{dy}{ds} ds, \\
J_5(x_0, y_0) &= \int_{0}^{b} y^{2\beta} \left[ x^{2\alpha} \frac{\partial q_2(x, y; x_0, y_0)}{\partial n} \right] \bigg|_{x=0} ds.
\end{align*}
\]
Then, by introducing the polar coordinates: \( x = x_0 + \rho \cos \phi \) and \( y = y_0 + \rho \sin \phi \), we get
\[
J_1 (x_0, y_0) = \int_0^{2\pi} (x_0 + \rho \cos \phi)(y_0 + \rho \sin \phi)^{2\beta} (\rho^2)^{\alpha-\beta-1} \times F_2 (2 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi; \eta) \, d\phi.
\] (33)

We use the following known formulas (see [37], p. 253, (26), [34], p. 113, (4)):
\[
F_2 (a; b_1, b_2; c_1, c_2; x, y) = \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{(c_1)_i (c_2)_i i!} x^i y^i \times F (a+i, b_1+i; c_1+i; x) F (a+i, b_2+i; c_2+i; y),
\] (34)

and
\[
F (a, b; c, x) = (1 - x)^{-b} F \left( c-a, b; c, \frac{x}{x-1} \right),
\] (35)

then, we obtain
\[
F_2 (a; b_1, b_2; c_1, c_2; x, y)
= (1-x)^{-b_1} (1-y)^{-b_2} \sum_{i=0}^{\infty} \frac{(a)_i (b_1)_i (b_2)_i}{(c_1)_i (c_2)_i i!} \left( \frac{x}{1-x} \right)^i \left( \frac{y}{1-y} \right)^i \times F \left( c_1-a, b_1+i; c_1+i; \frac{x}{x-1} \right) F \left( c_2-a, b_2+i; c_2+i; \frac{y}{y-1} \right),
\] (36)

where \( F(a; b; c; x) \) is hypergeometric function of Gauss ([34], p. 69, (2)). Hence we have
\[
F_2 (2 - \alpha + \beta; 1 - \alpha, \beta; 2 - 2\alpha, 2\beta; \xi, \eta)
= (\rho^2)^{1-\alpha+\beta} \left( \rho^2 + 4x_0^2 + 4x_0 \rho \cos \phi \right)^{\alpha-1} \left( \rho^2 + 4y_0^2 + 4y_0 \rho \sin \phi \right)^{-\beta} P_{11},
\] (37)

where
\[
P_{11} = \sum_{i=0}^{\infty} \frac{(2 - \alpha + \beta)_i (1 - \alpha)_i (\beta)_i}{(2 - 2\alpha)_i (2\beta)_i i!} \left( \frac{4x_0^2 + 4x_0 \rho \cos \phi}{\rho^2 + 4x_0^2 + 4x_0 \rho \cos \phi} \right)^i \times \left( \frac{4y_0^2 + 4y_0 \rho \sin \phi}{\rho^2 + 4y_0^2 + 4y_0 \rho \sin \phi} \right)^i \times F \left( -\alpha - \beta, 1 - \alpha + i; 2 - 2\alpha + i; \frac{4x_0^2 + 4x_0 \rho \cos \phi}{\rho^2 + 4x_0^2 + 4x_0 \rho \cos \phi} \right) \times F \left( \alpha + \beta - 2, \beta + i; 2\beta + i; \frac{4y_0^2 + 4y_0 \rho \sin \phi}{\rho^2 + 4y_0^2 + 4y_0 \rho \sin \phi} \right).
\]

Using the well-known Gauss’s summation formula for \( F(a; b; c; 1) \) ([34], p. 112, (46))
\[
F (a, b; c; 1) = \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)}, c \neq 0, -1, -2, \ldots, \text{Re} (c-a-b) > 0,
\]
we obtain
\[
\lim_{\rho \to 0} P_{11} = \frac{\Gamma (2 - 2\alpha) \Gamma (2\beta)}{\Gamma (2 - \alpha + \beta) \Gamma (\beta) \Gamma (1 - \alpha)}.
\] (38)

Thus, by virtue of the identities (33), (37), and (38), we get
\[
- (1 - \alpha + \beta) k_2 x_0^{1-2\alpha} \lim_{\rho \to 0} J_1 (x_0, y_0) = -1.
\] (39)

Similarly, by considering the corresponding identities and the fact that
\[
\lim_{\rho \to 0} \rho \ln \rho = 0,
\] (40)

we find that
\[
\lim_{\rho \to 0} J_2 (x_0, y_0) = \lim_{\rho \to 0} J_3 (x_0, y_0) = \lim_{\rho \to 0} J_4 (x_0, y_0) = 0.
\] (41)

Now we consider the integral \( J_5 (x_0, y_0) \), which, taking into account formula (30), takes the form
\[
J_5 (x_0, y_0) = i (x_0, y_0) (42)
\]

Hence, by virtue of (39)–(42), from (32) at \((x_0, y_0) \in \Omega\) follows
\[
w_1^{(2)} (x_0, y_0) = i (x_0, y_0) - 1.
\] (43)

**Case 2.** Let \((x_0, y_0) \in \Gamma\), we cut a circle \(C_\rho\) centered at \((x_0, y_0)\) with a small radius \(\rho\) off the domain \(\Omega\) and denote the remaining part of the curve by \(\Gamma - C_\rho\). Let \(C'_\rho\) denote a part of the circle \(C_\rho\) lying inside the domain \(\Omega\). We consider the domain \(\Omega_\rho\) which is bounded by a curve \(\Gamma - C_\rho, C'_\rho\) and segments \([0, a]\) and \([0, b]\) along the x- and y-axes, respectively. Then we have
\[
w_1^{(2)} (x_0, y_0) \equiv \int_0^1 x^{2\alpha} y^{2\beta} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial n} \, ds
\]
\[
= \lim_{\rho \to 0} \int_{\Gamma - C_\rho} x^{2\alpha} y^{2\beta} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial n} \, ds. \] (44)

When the point \((x_0, y_0)\) lies outside the domain \(\Omega_\rho\), it is found that, in this domain \(q_2 (x, y; x_0, y_0)\) is a regular solution of the equation \(H_{\alpha, \beta}^0\). Therefore, by virtue of (21), we have
\[
\int_{\Gamma - C_\rho} x^{2\alpha} y^{2\beta} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial n} \, ds
\]
\[
= \int_0^b y^{2\beta} \left[ x^{2\alpha} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial x} \right]_{x=0} \, dy + \int_{C'_\rho} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \left\{ q_2 (x, y; x_0, y_0) \right\} \, ds. \] (45)

Substituting from (45) into (44), we get
\[
w_1^{(2)} (x_0, y_0) = i (x_0, y_0) + \lim_{\rho \to 0} \int_{C_\rho} x^{2\alpha} y^{2\beta} \frac{\partial q_2 (x, y; x_0, y_0)}{\partial n} \, ds. \] (46)
Now, again by introducing the polar coordinates in the second summand and calculating the limit at \( \rho \to 0 \), we obtain

\[
w^{(2)}_1(x_0, y_0) = i(x_0, y_0) - \frac{1}{2}.
\]

**Case 3.** When \((x_0, y_0) \notin \tilde{\Omega}\), it is noted that the function \(q_2(x, y; x_0, y_0)\) is a regular solution of the equation \((H^0_{\alpha, \beta})\). Hence, in view of the formula (21), we have

\[
w^{(2)}_1(x_0, y_0) = \int_{d}^{b} x^{2\alpha} y^{2\beta} \frac{\partial}{\partial n} \{q_2(x, y; x_0, y_0)\} \, ds
\]

\[
= \int_{0}^{b} y^{2\beta} \left[ x^{2\alpha} \frac{\partial q_2(x, y; x_0, y_0)}{\partial x} \right]_{x=0} \, dy = i(x_0, y_0).
\]

The proof of Lemma 3.1 is thus completed.

**Lemma 3.2:** The following formula holds true:

\[
w^{(2)}_1(x_0, 0) = \begin{cases} 
  i(x_0, 0) - 1 & (x_0 \in (0, a)) \\
  i(x_0, 0) - \frac{1}{2} & (x_0 = 0 \text{ or } x_0 = a) \\
  i(x_0, 0) & (a < x_0),
\end{cases}
\]

(47)

where

\[
i(x_0, 0) = \frac{1 - 2\alpha}{1 + 2\beta} k_2 b^{2\beta+1} x_0^{2\alpha - 2\beta} \left( x_0^2 + b^2 \right)^{-1+\alpha-\beta} F \left( 1, \beta + \frac{3}{2}; \frac{1}{2}; \frac{b^2}{x_0^2 + b^2} \right).
\]

**Proof:** For considering the first case when \(x_0 \in (0, a)\), we introduce a straight line \(y = h\) for a sufficiently small positive real number \(h\) and consider a domain \(\Omega_h\) which is the part of the domain \(\Omega\) lying above the straight line \(y = h\). Applying the formula (21), we obtain

\[
w^{(2)}_1(x_0, 0) = \int_{0}^{b} x^{2\alpha} y^{2\beta} \frac{\partial q_2(x, y; x_0, 0)}{\partial x} \bigg|_{x=0} \, dy
\]

\[
+ \lim_{h \to 0} \int_{0}^{x_1} x^{2\alpha} y^{2\beta} \frac{\partial q_2(x, y; x_0, 0)}{\partial y} \bigg|_{y=h} \, dx,
\]

(48)

where \(x_1(\varepsilon)\) is an abscissa of a point at which the straight line \(y = h\) intersects the curve \(\Gamma\).

It follows from (42), (29) and (48) that

\[
w^{(2)}_1(x_0, 0) = i(x_0, 0) - 2 (1 - \alpha + \beta) k_2 x_0^{1-2\alpha} \lim_{h \to 0} h^{1+2\beta}
\]

\[
\times \int_{0}^{x_1} x \left[ \frac{F \left( 2 - \alpha + \beta, 1 - \alpha; 2 - 2\alpha, \frac{-4x x_0}{(x-x_0)^2 + h^2} \right)}{(x-x_0)^2 + h^2} \right]^{2-\alpha+\beta} \, dx.
\]

(49)
Now, by using the hypergeometric transformation formula (35) inside the integrand (49), we have

\[ w_1^{(2)}(x_0, 0) = i(x_0, 0) - 2(1 - \alpha + \beta) k_2 x_0^{1-2\alpha} \lim_{h \to 0} h^{1+2\beta} \]
\[ \times \int_{0}^{x_1} x \left[ \frac{F(-\alpha - \beta, 1 - \alpha; 2 - 2\alpha, \frac{4x_0}{(x-x_0)^2 + h^2})}{(x-x_0)^2 + h^2} \right]^{1+\beta} \left[ (x+x_0)^2 + h^2 \right]^{-\alpha-\beta} dx, \] (50)

which, upon setting \( x = x_0 + ht \) inside the integrand, yields

\[ w_1^{(2)}(x_0, 0) = i(x_0, 0) - 2(1 - \alpha + \beta) k_2 x_0^{1-2\alpha} \]
\[ \times \lim_{h \to 0} \int_{l_1}^{l_2} (x_0 + ht) \left[ \frac{F(-\alpha - \beta, 1 - \alpha; 2 - 2\alpha, \frac{4x_0(x_0+ht)}{(2x_0+ht)^2 + h^2})}{(1 + t^2)^{\beta+1} \left[ (2x_0 + ht)^2 + h^2 \right]^{1-\alpha}} \right] dt, \] (51)

where

\[ l_1 = -\frac{x_0}{h}, \quad l_2 = \frac{x_1 - x_0}{h}. \]

Considering

\[ \lim_{h \to 0} F(-\alpha - \beta, 1 - \alpha; 2 - 2\alpha, \frac{4x_0(x_0+ht)}{(2x_0+ht)^2 + h^2}) = F(-\alpha - \beta, 1 - \alpha; 2 - 2\alpha, 1) = \frac{\Gamma (2 - 2\alpha) \Gamma (1 + \beta)}{\Gamma (2 - \alpha + \beta) \Gamma (1 - \alpha)}, \]

and

\[ \int_{-\infty}^{+\infty} \frac{dt}{(1 + t^2)^{\beta+1}} = \frac{\pi \Gamma (2\beta)}{2^{2\beta-1} \beta \Gamma^2 (\beta)}, \]

we find from (51) that

\[ w_1^{(2)}(x_0, 0) = i(x_0, 0) - 1. \] (52)

The other three cases when \( x_0 = 0, x_0 = a \) and \( x_0 > a \) can be proved by using arguments similar to those detailed above in the first case. This evidently completes our proof of Lemma 3.2.

**Lemma 3.3:** The following identities are true

\[ w_1^{(2)}(0, y_0) = \begin{cases} 
-1 & (y_0 \in (0, b)), \\
-\frac{1}{2} & (y_0 = 0 \text{ or } y_0 = b), \\
0 & (b < y_0). 
\end{cases} \] (53)

**Proof:** The proof of Lemma 3.3 would run parallel to that of Lemma 3.2. ■
Theorem 3.1: For any points \((x, y)\) and \((x_0, y_0)\) \(\in \mathbb{R}^2_+\) and \(x \neq x_0\) and \(y \neq y_0\), the following inequality is true:

\[
|q_2(x, y; x_0, y_0)| \leq C x^{1-2\alpha} x_0^{1-2\alpha} (r_1^2)^{\alpha-1} (r_2^2)^{-\beta} \ln \left( \frac{r_1^2}{r_2^2} + \frac{r_2^2}{r_1^2} \cdot \frac{r_1^2}{r_2^2} \right),
\]

(54)

where \(C\) is constant, \(\alpha\) and \(\beta\) are real parameters with \((0 < \alpha, \beta < \frac{1}{2})\) as in the equation \((H_{\alpha, \beta}^\lambda)\) (with \(\lambda = 0\), and \(r, r_1\) and \(r_2\) are as in \((10)\).

Proof: It is implied from \((36)\) that

\[
q_2(x, y; x_0, y_0) = \frac{k_2 x^{1-2\alpha} x_0^{1-2\alpha}}{(r_1^2)^{1-\alpha} (r_2^2)^{\beta}} \quad \times \quad \sum_{i=0}^{\infty} \frac{(1-\alpha+\beta)_i (1-\alpha)_i (\beta)_i}{(2-2\alpha)_i (2\beta)_i i!} \left( 1 - \frac{r_2^2}{r_1^2} \right)^i \left( 1 - \frac{r_1^2}{r_2^2} \right)^i 
\]

\[
\times F \left( \begin{array}{l} 1 - \alpha - \beta, 1 - \alpha + i; 2 - 2\alpha + i; 1 - \frac{r_2^2}{r_1^2} \\ \alpha + \beta - 1, \beta + i; 2\beta + i; 1 - \frac{r_2^2}{r_2^2} \end{array} \right). 
\]

(55)

Now, in view of the following inequality:

\[
F \left( \begin{array}{l} 1 - \alpha - \beta, 1 - \alpha + i; 2 - 2\alpha + i; 1 - \frac{r_2^2}{r_1^2} \\ \alpha + \beta - 1, \beta + i; 2\beta + i; 1 - \frac{r_2^2}{r_2^2} \end{array} \right) \leq C_1,
\]

where \(C_1 > 0\) is constant, we get

\[
|q_2(x, y; x_0, y_0)| \leq C \cdot k_2 x^{1-2\alpha} x_0^{1-2\alpha} (r_1^2)^{\alpha-1} (r_2^2)^{-\beta} 
\]

\[
\times 3F_2 \left[ \begin{array}{c} 1 - \alpha + \beta, 1 - \alpha, \beta; \\ 2 - 2\alpha, 2\beta; \end{array} \right]_2 \left( 1 - \frac{r_2^2}{r_1^2} \right)^i \left( 1 - \frac{r_2^2}{r_2^2} \right)^i,
\]

where \(3F_2\) \([a, b, c; d, e; z]\) - generalized hypergeometric function \([13]\).

By virtue of the following known formula \([13], \text{p.} 446, (81)\):

\[
3F_2 \left[ \begin{array}{c} a_1, a_2, a_3; \\ b_1, b_2, c; \end{array} \right]_2 z = \frac{\Gamma(b_1) \Gamma(b_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3)} 
\]

\[
\times \left\{ - \sum_{k=0}^{\infty} \frac{c_k}{k!} (1-z)^k \ln(1-z) + \sum_{k=0}^{\infty} d_k^+ (1-z)^k \right\},
\]

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
where $b_1 + b_2 - a_1 - a_2 - a_3 = 0; |1 - z| < 1; |\arg(1 - z)| < \pi; \text{Re} a_j > 0; a_j \neq 0, -1, -2, \ldots; j = 1, 2, 3; c_k$ and $d_k^k$ are known constants, we observe from (54) that $q_2(x, y, x_0, y_0)$ has a logarithmic singularity at $r = 0$. Hence Theorem 3.1 is proved.

**Theorem 3.2:** If the curve $\Gamma$ satisfies to conditions (22) the inequality takes place

$$\int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial q_2(x, y; x_0, y_0)}{\partial n} \right| \, ds \leq C_1,$$

where $C_1$ is a constant.

**Proof:** Case 1. Let $x_0 > 0$ and $y_0 > 0$. Using the result of the Lemma 3.1 and by virtue of formula (35) we obtain

$$i(x_0, y_0) = k_2 (1 - 2\alpha) x_0^{1-2\alpha} \int_0^b y^{2\beta} F \left( \alpha + \beta - 1, \beta; 2\beta; \frac{-4yy_0}{x_0^2 + (y + y_0)^2} \right) \, dy.$$

In this case, taking into account the following estimates:

$$\left| F \left( \alpha + \beta - 1, \beta; 2\beta; \frac{-4yy_0}{x_0^2 + (y + y_0)^2} \right) \right| < A = \text{const},$$

$$0 < \frac{1}{(x_0^2 + (y - y_0)^2)^{1-\alpha}} < d_1 = \text{const}, \quad 1 - \alpha > 0,$$

$$0 < \frac{1}{(x_0^2 + (y + y_0)^2)^\beta} < d_2 = \text{const}, \quad \beta > 0,$$

$$0 < \frac{1}{(x_0^2 + (y - y_0)^2)^{1-\alpha} \left( x_0^2 + (y + y_0)^2 \right)^\beta} < d_1 d_2 = \text{const},$$

we obtain

$$|i(x_0, y_0)| < k_2 (1 - 2\alpha) d_1 d_2 A x_0^{1-2\alpha} \int_0^b y^{2\beta} \, dy = k_2 (1 - 2\alpha) d_1 d_2 A x_0^{1-2\alpha} \frac{b_1^{1+2\beta}}{1 + 2\beta}.$$

From this we conclude that

$$\int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial q_2(x, y; x_0, y_0)}{\partial n} \right| \, ds < \infty, x_0 > 0, y_0 > 0.$$

Case 2. Let $x_0 \geq 0$ and $y_0 = 0$. If $x_0 > 0$, then $0 < \frac{b_1^{2}}{x_0^2 + b_2^2} < 1$, therefore from Lemma 3.2 follows that

$$F \left( 1, \beta + \frac{1}{2}; \beta + \frac{3}{2}; \frac{b_1^2}{x_0^2 + b_2^2} \right) < B < \infty.$$
and

\[ |i(x_0, 0)| = B |k_2| \frac{1 - 2\alpha}{1 + 2\beta} \frac{x_0^{1-2\alpha} b^{2\beta+1}}{(x_0^2 + b^2)^{1-\alpha + \beta}}, x_0 > 0. \]

Consequently,

\[ \int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial q_2(x, y, 0; x_0, 0)}{\partial n} \right| \, ds < \infty, x_0 > 0. \]

If \( x_0 \to 0 \), then by virtue of equality \( \lim_{x_0 \to 0} x_0^{1-2\alpha} \ln x_0 = 0 \) we get

\[ \int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial q_2(x, y, 0; 0, 0)}{\partial n} \right| \, ds < \infty. \]

**Case 3.** Let \( y_0 \geq 0 \) and \( x_0 = 0 \). In this case by virtue of Lemma 3.3 a result is a clear:

\[ \int_{\Gamma} x^{2\alpha} y^{2\beta} \left| \frac{\partial q_2(x, y, 0, y_0)}{\partial n} \right| \, ds < \infty, y_0 \geq 0. \]

Thus, the Theorem 3.2 is completely proved. \( \blacksquare \)

**Theorem 3.3:** The following limiting formulas hold true for a double-layer potential (23):

\[ w_i^{(2)}(t) = -\frac{1}{2} \mu_2(t) + \int_0^I \mu_2(s) K_2(s, t) \, ds \quad (56) \]

and

\[ w_e^{(2)}(t) = \frac{1}{2} \mu_2(t) + \int_0^I \mu_2(s) K_2(s, t) \, ds, \quad (57) \]

where, as usual, \( \mu_2(t) \in C[0, l] \),

\[ K_2(s, t) = [x(s)]^{2\alpha} [y(s)]^{2\beta} \frac{\partial}{\partial n} \left\{ q_2 \left[ x(s), y(s); x_0(t), y_0(t) \right] \right\} \]

\[ ((x(s), y(s)) \in \Gamma; (x_0(t), y_0(t)) \in \Gamma), \]

\( w_i^{(2)}(t) \) and \( w_e^{(2)}(t) \) are limiting values of the double-layer potential (23) at \( (x_0(t), y_0(t)) \to \Gamma \) from the inside and the outside, respectively.
Proof: We consider the function

$$w^{(2)}(x_0, y_0) = \int_0^1 Q_2(s; x_0, y_0) \ ds,$$

where

$$Q_2(s; x_0, y_0) := x^{2\alpha(s)} y^{2\beta(s)} \frac{\partial q_2(x(s), y(s); x_0, y_0)}{\partial n}.$$

We denote by $w_i^{(2)}(\xi_0, \eta_0)$ (respectively, by $w^{(2)}_e(\xi_0, \eta_0)$) a limit of the function $w^{(2)}(x_0, y_0)$ when $(x_0, y_0) \to (\xi_0, \eta_0)$ from the inside (respectively from the outside). A point $(\xi_0, \eta_0)$ is a internal point of the curve $\Gamma$. To prove the equalities in Theorem 3.3, we consider the difference

$$w^{(2)}(x_0, y_0) - w^{(2)}(\xi_0, \eta_0) = \int_0^1 \left[ Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0) \right] \mu_2(s) \ ds$$

$$= \int_0^1 \left[ Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0) - (\mu_2(s) - \mu_2(t)) \right] \ ds$$

$$+ \mu_2(t) \int_0^t \left[ Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0) \right] \ ds = \Delta_1 + \Delta_2.$$

For any real number $\varepsilon > 0$ we find the number $\delta > 0$ such that $|\mu_2(s) - \mu_2(t)| < \varepsilon$ when $|s - s_0| < \delta$. Let be $|\mu_2(s)| < M, s \in \Gamma$. We obtain

$$|\Delta_1| \leq 2M \int_{t_1}^{t-\delta} \left| Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0) \right| \ ds$$

$$+ 2M \int_{t+\delta}^{t_2} \left| Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0) \right| \ ds$$

$$+ \varepsilon \left( \int_0^1 |Q_2(s; x_0, y_0)| \ ds + \int_0^1 |Q_2(s; \xi_0, \eta_0)| \ ds \right).$$

Since $\int_0^1 |Q_2(s; x, y)| \ ds$ is bounded, and the difference $Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0)$ uniformly tends to $0$ from outside the interval $t - \delta < s < t + \delta$, then for $(x_0, y_0)$ sufficiently close to $(\xi_0, \eta_0)$ the inequality $|\Delta_1| < \varepsilon N$ is true.

Now we consider $\Delta_2$. When $(x_0, y_0) \in \Omega$, we draw a circle $\gamma$ centered at $(x_0, y_0)$ with a small radius $\varepsilon$. The circular domain bounded by a circle $\gamma$ we denote by $\delta_0$.

We draw straight lines $x = \delta_1$ and $y = \delta_2$, $0 < \delta_1 < x_0 - \varepsilon, 0 < \delta_2 < y_0 - \varepsilon$ and consider the domain $\Omega_\delta_1 \delta_2 \varepsilon$ which is bounded by the curve $\Gamma$, the straight lines $x = \delta_1, y = \delta_2$, and a circle $\gamma$. Applying Green’s formula to the domain $\Omega_\delta_1 \delta_2 \varepsilon$, after several transformations we obtain $\lim_{(x_0, y_0) \to (\xi_0, \eta_0)} \int_0^1 |Q_2(s; x_0, y_0) - Q_2(s; \xi_0, \eta_0)| ds = -\frac{1}{2}$, when a point $(x_0, y_0)$ is lying inside the domain $\Omega$. 
If the point \((x_0, y_0)\) is lying outside of the domain \(\Omega\), then it is not necessary to build a circle \(\gamma\), the remaining considerations remain the same

\[
\lim_{(x_0, y_0) \to (\xi_0, \eta_0)} \int_0^l \left[ Q_2 \left( s; x_0, y_0 \right) - Q_2 \left( s; \xi_0, \eta_0 \right) \right] ds = \frac{1}{2}.
\]

Thus the Theorem 3.3 is proved.

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