A NEW KIND OF UNIFORMLY ACCELERATED REFERENCE FRAMES

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A new kind of uniformly accelerated reference frames with a line-element different from the Møller and Rindler ones is presented, in which every observer at \( x, y, z = \text{consts.} \) has the same constant acceleration. The laws of mechanics are checked in the new kind of frames. Its thermal property is studied. The comparison with the Møller and Rindler uniform accelerated reference frames is also made.

Keywords: accelerated reference frame, uniform acceleration

1. Introduction

It is very well-known that an inertial reference frame in a flat spacetime equips with a Minkowski coordinates so that

\[
 ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{1}
\]

In the literature, a uniformly accelerated reference frame is defined as a set of observers who remain at rest with respect to a given observer Alice who is accelerating at a constant rate with respect to the instantaneously comoving inertial frames\(^1\). Two observers Alice and Bob are said to be at rest with respect to each other if the time elapses on the clock of Alice during the passage of a light signal from Alice to Bob and back again always has the same value. By definition, two observers who at each instant are moving with the same velocity \( v(t) \) with respect to an inertial frame will remain at rest only when \( v(t) \) is a constant. Therefore, a set of observers who has a single acceleration is not said to constitute a uniformly accelerated reference frame. Instead, a uniformly accelerated reference frame is usually described by Rindler coordinate system\(^2\)

\[
 ds^2 = \xi^2 d\eta^2 - d\xi^2 - dy^2 - dz^2, \tag{2}
\]
or Møller coordinate system\textsuperscript{3–5}

\[ ds^2 = (1 + g\xi)^2 d\eta^2 - d\xi^2 - dy^2 - dz^2, \]

in which static observers at different places in the spacetime, whose spatial coordinates are constants, have different accelerations.

It should be noted, however, that such a kind of uniformly accelerated reference frame is not suitable to describe the frame consisting of a group of electrons in a uniform electric field or the motion of phonon in an accelerated crystalloid.

On the other hand, when a clock experiences a uniform acceleration, its time elapses at different rate time to time. Thus, to define “at rest” with the help of such a kind of clock is not without questions. The concept ‘at rest’ of such a kind is quite different from that in the laboratory (inertial) frame. Obviously, an alternative definition of ‘at rest’ will lead to an alternative definition of a uniformly accelerated reference frame and thus an alternative, uniformly accelerated coordinate system.

In the present paper, we shall study a new kind of reference frame in which each observer moves with the same acceleration, which is thought to be unable to serve as an acceleration reference frame in Ref. 1. Since the reference frame have only constant parameter $a$ distinguishing from an inertial reference frame, we also refer to it as a uniformly accelerated reference frame.

The paper is organized as follows. In Sec. 2, the Rindler spacetime is reviewed briefly. In Sec. 3, uniformly accelerated reference frames in flat spacetime are introduced. Sec. 4 is a short discussion on the thermal property of the new uniformly accelerated reference frame. In the last section, the concluding remarks are given.

2. Rindler Spacetime

In Rindler spacetime (2), the 4-velocity of a particle standing at constant-$\tilde{\xi}, y, z$ is $U^{\mu} = \{\xi^{-1}, 0, 0, 0\}$ and its 4-acceleration is

\[ a^{\mu} := U^{\nu} \nabla_{\nu} U^{\mu} = \{0, \xi^{-1}, 0, 0\}. \]  

The magnitude of the acceleration is $a = (-g_{\mu\nu} a^{\mu} a^{\nu})^{1/2} = \xi^{-1}$, which depends on the position of particle in the spacetime. Let $\tilde{\eta} = g\eta$ and $\tilde{\xi} = (1 + g\xi)/g$, where $g$ is a constant. Then the line-element (2) is written in the Møller coordinates (3). The magnitude of the acceleration of a particle standing at constant-$\xi, y, z$ is

\[ a = \frac{g}{1 + g\xi}. \]

In particular, $a = g$ at $\xi = 0$. The horizon for (3) is at $\xi_H = -1/g$. By definition, the “surface gravity” on the horizon is\textsuperscript{6}

\[ \kappa := \lim_{\xi \to \xi_H} (Va) = g, \]

where $V = \sqrt{g_{00}} = 1 + g\xi$ is the redshift factor, and the Hawking temperature is

\[ T_H := \frac{\kappa}{2\pi} = \frac{g}{2\pi}. \]
All above equations are rewritten in natural units. In SI units which is used in the following discussion, the line-elements (2) and (3), acceleration (5), and Hawking temperature (7) may be written as

\[ ds^2 = \frac{c^5}{G_N \hbar} \xi^2 d\tilde{\eta}^2 - d\tilde{\xi}^2 - dy^2 - dz^2 \] (8)

and

\[ ds^2 = \left(1 + \frac{g \xi}{c^2}\right)^2 c^2 d\eta^2 - d\xi^2 - dy^2 - dz^2, \] (9)

\[ a = \frac{g}{1 + g \xi/c^2}, \] (10)

and

\[ T_H = \frac{g \hbar}{2\pi c k_B}, \] (11)

respectively. It is obvious that the line-element (9) is more natural to describe an accelerated observer than Eq.(8) because Eq.(9) only contains the speed of light \( c \) while Eq.(8) also contains the Newtonian gravitational constant \( G_N \) and Planck constant \( \hbar \) which are not related to accelerated motion directly.

For metrics (2) and (3), the congruence of trajectories of accelerated observers has vanishing expansion, spatial components of shear and twist. The non-zero components of shear and twist are only \( \sigma_{01} = \sigma_{10} = -\frac{1}{2} \partial_1 U^0 \) and \( \omega_{01} = -\omega_{10} = \frac{3}{2} \partial_1 U^0 \). Therefore, the distance between two static observers, Alice and Bob, measured in the comoving inertial frame always has the same value.

3. New Uniformly Accelerated Reference Frame and Its Line-element in Flat Spacetime

Now, we consider a uniformly accelerated reference frame consisting of a set of accelerated observers with proper acceleration \( a \) in a given direction, say, in \( x \)-direction, as shown in Fig. 1. At \( t = 0 \), the observers are at \( (x = X, y, z) \), where \( t, x, y, z \) are Minkowski coordinates of a flat spacetime. The trajectory of each observer is a hyperbola of same eccentricity but different asymptotes in the Minkowski spacetime. Namely, different observers have different horizons.

The trajectory of an accelerated observer with acceleration \( a \) passing \((0, X, y, z)\) at \( t = 0 \) is (see, e.g., 2)

\[ x(t) = X + \frac{c^2}{a} \left[ \left(1 + \frac{a^2 t^2}{c^2}\right)^{1/2} - 1 \right]. \] (12)
Fig. 1. Trajectories (real) of observers in an accelerated reference frame. Different observers in the same accelerated reference frame have different horizons (dash).

Since

\[
\frac{dt}{ds} = \frac{1}{c} \frac{x(t) - X + \frac{c^2}{a}}{\left[ (x(t) - X + \frac{c^2}{a})^2 - c^2 t^2 \right]^{1/2}}
\]

\[
\frac{dx}{ds} = \frac{ct}{\left[ (x(t) - X + \frac{c^2}{a})^2 - c^2 t^2 \right]^{1/2}}
\]

\[
\frac{dy}{ds} = \frac{dz}{ds} = 0,
\]

on the trajectory

\[
\frac{dt}{ds} = \frac{1}{c} \left[ 1 + \frac{a^2}{c^2 t^2} \right]^{1/2}.
\]

Its integration under the initial condition that \( t = 0 \) when \( s = 0 \) is

\[
t = \frac{c}{a} \sinh \frac{as}{c^2}.
\]

Substituting it in Eq.(12), we obtain

\[
x(t) = X + \frac{c^2}{a} \left( \cosh \frac{as}{c^2} - 1 \right).
\]
In order to get a line-element for the uniformly accelerated reference frame in the flat spacetime, we choose new coordinates \((s = cT, X, y, z)\) so that
\[
dt = \cosh \frac{aT}{c} dT
\]
\[
dx = dX + c \sinh \frac{aT}{c} dT.
\]
The line-element of the flat spacetime can be written as
\[
ds^2 = c^2 dT^2 - 2c \sinh \frac{aT}{c} dTdX - dX^2 - dy^2 - dz^2
\]
in terms of the new coordinates. This is the line-element for the uniformly accelerated reference frame with the acceleration \(a\).

To make sure that Eq.(19) describes an accelerated reference frame, let us consider an arbitrary static observer in the new coordinates such that
\[
U^\mu = c(1, 0, 0, 0).
\]
The 4-acceleration
\[
a^\mu = U^\nu \nabla_\nu U^\mu = a \tanh \frac{aT}{c} \delta^\mu_0 + \frac{a}{\cosh \frac{aT}{c}} \delta^\mu_1
\]
has the magnitude
\[
(-g_{\mu\nu} a^\mu a^\nu)^{1/2} = a,
\]
which is independent of the position of a static observer in the spacetime. This is unlike the Rindler spacetime in which the static observer at different place has different acceleration. Therefore, we say that \(a\) here is the acceleration of the accelerated reference frame and that \((T, X, y, z)\) constitutes an accelerated coordinate system.

It is remarkable that the accelerated coordinate system differs from an inertial coordinate system only by a non-diagonal term. At the time origin, the concepts of space and time in an accelerated reference frame are identical to those of inertial observers.

It can be shown that such a kind of accelerating system is locally synchronizable at \(T = 0\) hyperplane.

In an inertial reference frame, the second law of mechanics reads
\[
m_0 c \frac{dU^\mu}{ds} = F^\mu.
\]
Since the 4-velocity in an inertial reference frame, \(U^\mu_{\text{iner}}\), and that in the accelerated reference frame whose origin is at rest with respect to the inertial reference frame at \(t = 0\), \(U^\mu_{\text{acc}}\), are related by
\[
U^\mu_{\text{iner}} = \frac{\partial x^\nu}{\partial X^\nu} U^\nu_{\text{acc}},
\]
which reads

\[ U_0^{\text{iner}} = U_0^{\text{acc}} \cosh \frac{aT}{c} \]
\[ U_1^{\text{iner}} = U_0^{\text{acc}} \sinh \frac{aT}{c} + U_1^{\text{acc}} \]  \tag{25}
\[ U_2^{\text{iner}} = U_2^{\text{acc}} \]
\[ U_3^{\text{iner}} = U_3^{\text{acc}}. \]

\[
\frac{dU_0^{\text{iner}}}{ds} = \frac{dU_0^{\text{acc}}}{ds} \cosh \frac{aT}{c} + \frac{a}{c^3} (U_0^{\text{acc}})^2 \sinh \frac{aT}{c} \frac{dU_0^{\text{acc}}}{ds} + \frac{a}{c^3} (U_0^{\text{acc}})^2 \frac{dU_1^{\text{acc}}}{ds} \\
\frac{dU_1^{\text{iner}}}{ds} = \frac{dU_0^{\text{acc}}}{ds} \sinh \frac{aT}{c} + \frac{a}{c^3} (U_0^{\text{acc}})^2 \cosh \frac{aT}{c} + \frac{dU_1^{\text{acc}}}{ds} \\
\frac{dU_2^{\text{iner}}}{ds} = \frac{dU_2^{\text{acc}}}{ds} \\
\frac{dU_3^{\text{iner}}}{ds} = \frac{dU_3^{\text{acc}}}{ds}. \]  \tag{26}

Similarly,

\[ F_0^{\text{iner}} = F_0^{\text{acc}} \cosh \frac{aT}{c} \]
\[ F_1^{\text{iner}} = F_0^{\text{acc}} \sinh \frac{aT}{c} + F_1^{\text{acc}} \]  \tag{27}
\[ F_2^{\text{iner}} = F_2^{\text{acc}} \]
\[ F_3^{\text{iner}} = F_3^{\text{acc}}. \]

Then, the second law in the accelerated reference frame becomes

\[
m_0 c \frac{dU_0^{\text{acc}}}{ds} = F_0^{\text{acc}} - m_0 \frac{a}{c^2} (U_0^{\text{acc}})^2 \tanh \frac{aT}{c} \]
\[
m_0 c \frac{dU_1^{\text{acc}}}{ds} = F_1^{\text{acc}} - m_0 \frac{a}{c^2} (U_0^{\text{acc}})^2 \sech \frac{aT}{c} \]  \tag{28}
\[
m_0 c \frac{dU_2^{\text{acc}}}{ds} = F_2^{\text{acc}} \]
\[
m_0 c \frac{dU_3^{\text{acc}}}{ds} = F_3^{\text{acc}}. \]

It can also be obtained directly from

\[ F_\mu^{\text{acc}} = m_0 c \frac{DU_\mu^{\text{acc}}}{ds}, \]  \tag{29}

since

\[ m_0 c \frac{dU_\mu^{\text{acc}}}{ds} = m_0 c \left( \frac{DU_\mu^{\text{acc}}}{ds} - \frac{1}{c} \Gamma_\mu^{\nu\lambda} U_\nu^{\text{acc}} U_\lambda^{\text{acc}} \right), \]
\[ \Gamma_\mu^{\nu\lambda} U_\nu^{\text{acc}} U_\lambda^{\text{acc}} = \frac{a}{c^2} (U_0^{\text{acc}})^2 \left( \tanh \frac{aT}{c} \delta_0^{\mu} + \sech \frac{aT}{c} \delta_1^{\mu} \right). \]
In the Newtonian approximation,
\[ U_{\text{iner}}^0 \approx U_{\text{acc}}^0 \approx c, \quad cdU_{\text{iner}}^0/(ds) \approx cdU_{\text{acc}}^0/(ds) \approx 0, \quad F_{\text{iner}}^0 \approx F_{\text{acc}}^0 \approx 0, \quad \text{and} \quad aT/c \to 0, \]
the second law reduces to
\[ m_0c \frac{dU_{\text{acc}}^i}{ds} = F_{\text{acc}}^i - m_0a_1. \] (30)
It takes the standard form of the Newtonian mechanics in an accelerated reference frame. In addition, in the Newtonian approximation, Eqs.(15) and (12) reduce to
\[ t = T, \quad x = X + \frac{1}{2}aT^2. \] (31)
They are the standard relations between coordinates in an inertial and a uniformly accelerated system.

In contrast, the second law of mechanics in the Rindler metric (8),
\[ m_0c \frac{dU_R^\mu}{ds} = F_R^\mu - m_0\Gamma_\nu^\mu U_R^\nu U_R^\lambda \]
\[ = F_R^\mu - m_0U_R^0[\xi^{-1}U_R^1\delta_1^\mu + \frac{c^3}{G_N\hbar}\xi U_R^0\delta_1^\mu]. \]
reduces to
\[ m_0c \frac{dU_R^i}{ds} = F_R^i - m_0c^2\xi \delta_1^i = F_R^i - m_0a(\xi)\delta_1^i. \]
under the Newtonian approximation, where the approximation \( g_{00}(U_R^0)^2 \approx c^2 \) has been used. It is not the standard form of the second law Newtonian mechanics in an accelerated reference frame though the acceleration of constant-\( \xi \) trajectory is \( c^2/\xi \). The second law of mechanics in the metric (9) does not reduce to the standard Newtonian form in an accelerated reference frame either.

4. Wick Rotation of Uniformly Accelerated Coordinate System and Temperature

Now, let us study the thermal property of the line-element (19) and consider its Euclidean section. Under the Wick rotation
\[ T \to \tau = iT, \quad s \to is_E \]
the line-element (19) becomes
\[ ds_E^2 = d\tau^2 - 2c\sin \frac{a\tau}{c}d\tau dX + dX^2 + dy^2 + dz^2. \] (33)
Clearly, the metric spacetime has a period in \( \tau \) direction. The period is \( 2\pi c/a \). Namely, the time variable in line-element (19) has the imaginary period \( -i2\pi c/a \). On the same reasoning as that in Ref.8, the Green function, as a function of coordinates, is expected to acquire the same period automatically. Therefore, one may expect, according to the Green’s function theory at finite temperature\(^9\), that the observers in the accelerated reference frame at uniform acceleration \( a \) will observe the finite Hawking temperature \( a\hbar/(2\pi c k_B) \).
To confirm that the observer will observe the finite Hawking temperature, more systematic studies on the Green’s function are needed. But it is beyond the scope of this paper. Instead, it is recalled that the vacuum Green function for the detector with a uniform (proper) acceleration \( a \) is the same as the thermal Green function for an inertial detector with the temperature\(^{10} \)

\[
T_H = \frac{a\hbar}{2\pi c k_B},
\]

which leads to the conclusion that the uniformly accelerated detector in vacuum will detect a thermal bath of radiation at temperature \( T_H \). Thus, the particle detectors moving along Eq.(12) will all detect a thermal bath of radiation at the same temperature \( T_H \).

5. Concluding remarks

A new kind of uniformly accelerated reference frame and the corresponding coordinate system is constructed. Unlike the Møller and Rindler uniformly accelerated reference frames, the uniformly accelerated reference frame has a single acceleration parameter \( a \). Although the distance between two observers standing in the reference frame may change, the reference frame has the following advantages:

First of all, all observers with \( X = \text{const.} \) in the reference frame are on equal footing. In particular, each observer in the reference frame has the same acceleration, and the reference frame has spatial translation invariance. The frame is the relativistic version of a uniformly accelerated reference in the sense of Newtonian mechanics. So, it is easy to be understood. In contrast, in the Møller and Rindler uniformly accelerated reference frames there is no spatial translation invariance in the direction of acceleration. Instead, it possesses the boost invariance. The observers in the frame are not equivalent to each other. Therefore, they may be regarded as the uniformly accelerated reference frame in a small nearby region of a given observer.

Secondly, each observer in the new uniformly accelerated reference frame has its own horizon (as shown in Fig.1). It is something like the comoving observers in Friedmann-Robertson-Walker universe when horizon exists. On the other hand, all standing observers have the same horizon. It is similar to de Sitter spacetime in the static coordinate system.

Thirdly, each particle detector detects the same temperature in the new uniformly accelerated reference frame. It is again something like the comoving observers in Friedmann-Robertson-Walker universe, each of whom detects the same temperature of cosmic microwave background.

Fourthly, the reference frame may pick up a particular inertial reference frame, the laboratory reference frame, in which the distance of two standing observers keeps always unchanged and observers (or particles) have the same instant velocity at any time (The same temperature of different observers in a reference frame may also be used to define a special, simultaneous hypersurface, because the thermal equilibrium is closely related to the transitivity of simultaneity\(^{11} \)). It becomes clearer
if a piecewise uniformly accelerated reference frame is considered, which is first at rest in the laboratory, inertial frame, and begins to be uniformly accelerated at \( t = 0 \), and finally settles down an inertial frame at \( t = t_f \) with the final velocity

\[
v_f = \frac{dx}{dt} \bigg|_{t_f} = \frac{c^2 t_f}{x(t_f) - X + \frac{c^2}{a}} = at_f \left( 1 + \frac{a^2 t_f^2}{c^2} \right)^{-1/2}
\]  

(35)

with respect to the laboratory frame, as shown in Fig.2. The reference frame is more suitable than the Møller or Rindler uniformly reference frame to apply the particles in a linear accelerator, in which particles undergo the same acceleration, or the phonon in a uniformly accelerated crystalloid. The new frame can be used to study the "complicated fractional differences" due to acceleration, which have been neglected in a small-enough accelerated frame\(^4\). The inertial coordinates in the final inertial reference frame are related to the coordinates for the laboratory reference
It is easy to see from Fig.2 that the distance between the two observers with the same acceleration in the laboratory frame keeps unchanged, while the distance in comoving frame varies from time to time.

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