Deflection of light ray due to a charged body using Material Medium Approach

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The gravitational deflection of light ray is an important prediction of General Theory of Relativity. In this paper we develop an analytical expression of the deflection of light ray without any weak field approximation due to a charged gravitating body represented by Reissner-Nordström (RN) and Janis-Newman-Winicour (JNW) space time geometry, using material medium approach. It is concluded that although both the geometries represent the charged, non-rotating, spherically symmetric gravitating body, but the effect of charge on the gravitational deflection is just opposite to each other. The gravitational deflection decreases with charge in the RN geometry and increases with charge in the JNW geometry. The calculations obtained here are compared with other methods done by different authors. The formalism is applied to an arbitrary selected pulsar PSRB1937 + 21 as a gravitating body, as a test case.

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I. INTRODUCTION

One of the striking predictions of General Theory of Relativity is the deflection of light ray in the presence of a gravitating mass. The predictions of gravitational effect on light, began by Einstein in 1913 and was confirmed by Eddington in 1919, during the total solar eclipse.

In this paper we have used the Material Medium Approach to find the deflection of light ray due to a charged gravitating body represented by Reissner-Nordström (RN) space-time and Janis-Newman-Winicour (JNW) space time, without assuming any weak field approximation. The material medium approach is different from conventional null geodesic approach, through which the deflection of light ray can be calculated. The material medium approach has been used by several researchers (details in Section II), where the effect of gravity is viewed as a change in the refractive index of the medium through which light is travelling.

The Reissner-Nordström (RN) metric and Janis-Newman-Winicour (JNW) metric both are static solution of Einstein-Maxwell field equation for a charged, non-rotating, spherically symmetric gravitating body. The RN solution has an event horizon and a Cauchy horizon but JNW solution has a curvature singularity and a naked singularity. If there is no charge, both the solutions reduce to the Schwarzschild solution.

The concept of scalar field theory came before the general theory of relativity. In 1956 O. Bergmann discussed the scalar field theory as a theory of Gravitations and in 1957 O. Bergmann and R. Leipnik discussed the field equations of a static spherically symmetric scalar field. Many authors studied the field of charged particles in General Relativity.

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Role of scalar field in Gravitational lensing by JNW black hole was discussed by Virbhadra et al. in 1998. In 2002 Eiroa et al. discussed about the Gravitational lensing by RN black hole in strong field limit. The two groups of authors studied the null geodesics in a general static spherically symmetric spacetime and calculated the Einstein’s deflection angle. Further they discussed about the image formation and lensing effect. In 2000 Virbhadra and Ellis first defined the photon sphere which is the starting point of the strong field limit expansion and in the next paper Claudel, Virbhadra and Ellis discussed the geometry of photon surfaces and calculated the radius of the photon sphere in different space-time. In 2002, Virbhadra and Ellis discussed the gravitational lensing by naked singularities. In the same year, Bozza extended the analytical theory of strong lensing for a general class of static spherically symmetric matrices. In 2003 Bhadra also discussed the lensing effect in the strong gravity regime due to a charged black hole in case of strong field limit. Recently, Amore and Arceo in 2006 considered the gravitational lensing for different black holes including RN metric type and JNW metric type. In 2012 Chowdhury et al. studied the circular geodesics in the JNW and Gamma metric space-time. In 2014 Chakraborty and SenGupta estimated the perihelion precession and bending of light due to charged black hole using RN metric and showed that perihelion shift decreases with the increase of charge and bending of light is almost similar to that by Schwarzschild field at a large distance from the source.

The present paper is organized as follows: In section II A we describe the RN and JNW space-time. In section II B we obtain the velocity and refractive index due to RN and JNW geometry and explain the effects by plotting graphs. In a separate section II C we obtain the deflection angle and describe the results obtained by RN and JNW geometry. In section II D we compared the refractive index and bending angle due to a charged gravitating body for both the space time with other methods and by other authors. Finally in section II E we make discussion and draw conclusions on our result.

II. OPTICAL MEDIUM APPROACH

The deflection of light ray can be obtained by the most general method as null geodesic. As already outlined in section I, the Optical Medium Approach is an alternate method by which also we can calculate the gravitational deflection of light ray. In this method the gravitational effect is represented by an equivalent refractive index of the medium. Thus if a ray of light passes through a material medium, the light ray will deviate due to the variation of the refractive index of the corresponding media. This method was first introduced by Tamm in 1924. Balaz in 1958 considered that if an electromagnetic wave passed through the gravitational field of an rotating body, then the polarization vector gets rotated. In 1960 Plebanski studied the scattering of light ray by gravitational field. On the level of geometrical optics the author formulated the generalized form of the Einstein’s deflection angle and examined the direction of the plane of polarization. Felice in 1971 used the concept of equivalent material medium to deduce the refractive index and the deflection angle of a light ray in a static and spherically symmetric space time. B. Mashhoon obtained the scattering cross section and polarization of the scattered wave by Schwarzschild and Kerr gravitating body. Fischbach and Freeman in 1980 used the same approach and obtained the refractive index and deflection up to 2nd order in Schwarzschild geometry. Evans, Rosenquist, Nandi and Islam used the Fermat’s principle to calculate the effective refractive index to derive the deflection angle in Schwarzschild geometry for massless and massive particles. P.M. Alsing and Islam in 1998 extended the formalism of Evans, Nandi and Islam in case of Kerr field geometry. Sereno is also used the Fermat’s principle to discuss the gravitational lensing and Faraday rotation in the weak field limit. In 2004, Sereno discussed the gravitational lensing in weak field limit of RN metric and used Fermat’s principle to calculate the deflection angle. Ye and Lin in 2008 discussed the exterior and interior solution of the refractive index for a static spherically symmetric gravitational field and found the most general formula of refractive index in terms of potential in weak
field limit and also discussed the effect of lensing.
Very recently Material Medium Approach was used by Sen\textsuperscript{36} and Roy and Sen\textsuperscript{37} to calculate the light deflection angle due to a gravitating body represented by Schwarzschild and Kerr space-time geometry respectively.

A. Reissner-Nordström (RN) space-time and Janis-Newman-Winicour (JNW) space-time

The Reissner-Nordström space-time is the spherically symmetric solution of coupled equations of Einstein and of Maxwell. A non-rotating black hole with gravitational mass $m$ and a charge length $Q$ can be represented by a Reissner-Nordström(RN)\textsuperscript{1,2} line element. In Boyer-Lindquist co-ordinates ($ct, r, \theta, \phi$),\textsuperscript{38} the RN metric is in the form\textsuperscript{39}:

$$ds^2 = (1 - \frac{2m}{r} + \frac{Q^2}{r^2})c^2 dt^2 - (1 - \frac{2m}{r} + \frac{Q^2}{r^2})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (1)

where $Q^2 = \frac{Gc^2}{4\pi\epsilon_0}$ and $e$ is the scalar charge, $\frac{e}{\epsilon_0}$ is the coulomb force-constant, $G$ is the gravitational constant, $c$ is the speed of light. $Q$ has the dimension of length. Further, $2m = r_g$ and $r_g$ is the Schwarzschild radius $= \frac{2GM}{c^2}$.

This metric has an event horizon at $r_+ = m + \sqrt{m^2 - Q^2}$ and a Cauchy horizon at $r_- = m - \sqrt{m^2 - Q^2}$. For $Q^2 > m^2$, $r_+$ or $r_-$ has no real solution and hence $Q$ has a limit as $Q^2 \leq m^2$. Thus event horizon exists for $0 \leq Q^2 \leq m^2$ and Cauchy horizon exists for $0 < Q^2 \leq m^2$.

Photon sphere is a region where light travels in close orbits due to strong gravitational effect of the gravitating body. Thus it is the minimum radius of the stable orbit. The radius of the photon sphere of this metric is\textsuperscript{13}

$$r_{ps}^\pm = \frac{3m \pm \sqrt{9m^2 - 8Q^2}}{2}$$  \hspace{1cm} (2)

Thus the surface $S^+$ for radius $r_{ps}^+$ exists for $0 \leq Q^2 \leq 9/8m^2$. And the surface $S^-$ for radius $r_{ps}^-$ exists for $0 < Q^2 \leq 9/8m^2$. Both the surfaces $S^+$ and $S^-$ coincide for $Q^2 = 9/8m^2$.

Jenis-Newman-Winicour (JNW) space-time\textsuperscript{3} on the other hand also represents the most general spherically symmetric, static and asymptotically flat solution of Einstein’s field equation which is coupled to a massless scalar field. The JNW solution in the co-ordinates ($ct, r, \theta, \phi$) can be represented by the line element\textsuperscript{10}:

$$ds^2 = (1 - \frac{l}{r})\gamma c^2 dt^2 - (1 - \frac{l}{r})^{-\gamma} dr^2 - r^2 (1 - \frac{l}{r})^{1-\gamma} (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (3)

with scalar field

$$\Phi = \frac{Q}{l\sqrt{4\pi}} \ln(1 - \frac{l}{r})$$  \hspace{1cm} (4)

where

$$l = 2\sqrt{m^2 + Q^2}$$  \hspace{1cm} (5a)

and

$$\gamma = \frac{2m}{l} = \frac{m}{\sqrt{m^2 + Q^2}}$$  \hspace{1cm} (5b)
When $Q = 0$, $\gamma = 1$ and finally the JNW metric also goes to the Schwarzschild metric. The JNW metric has a curvature singularity at $r = l$ i.e. $r$ has a limit as $l < r < \infty$.

The radius of the photon sphere of this geometry is

$$r_{ps} = \frac{l(1 + 2\gamma)}{2}$$

$$= \sqrt{m^2 + Q^2(1 + \frac{2m}{\sqrt{m^2 + Q^2}})}$$

$$= 2m + \sqrt{m^2 + Q^2}$$

(6)

which exists only for $\frac{1}{2} < \gamma \leq 1$ i.e. for $0 \leq Q^2 < 3m^2$ which is mentioned as a weak naked singularity. If we consider the value of $\gamma \leq 1/2$ then we will get the strong naked singularity. Naked singularity may or may not be within photon sphere. If the singularity point is within photon sphere then it is called as weak naked singularity. If the singularity point is not covered by any photon sphere then it is called as strong naked singularity.$^{12}$

B. Refractive index as calculated in RN and JNW space-time

By following the same procedure as Sen$^{36}$ and Roy and Sen$^{37}$ we can get the isotropic form of the line element in the field of RN space-time and JNW space-time in terms of $(ct, \rho, \theta, \phi)$ as

$$ds^2 = \frac{(1 - \frac{m^2 - Q^2}{4\rho^2})^2}{(1 + \frac{m^2 - Q^2}{4\rho^2})^2} c^2 dt^2 - (1 + \frac{m^2 - Q^2}{4\rho^2})^2(\rho^2 + \rho^2(\sin^2\theta d\phi^2))$$

(7)

and

$$ds^2 = \frac{(1 - \frac{l^2}{4\rho^2})^2}{(1 + \frac{l^2}{4\rho^2})^2} c^2 dt^2 - (1 + \frac{l^2}{4\rho^2})^2(\rho^2 + \rho^2(\sin^2\theta d\phi^2))$$

(8)

respectively.

To get the isotropic form of the line element in terms of $(ct, \rho, \theta, \phi)$ we have introduced a new co-ordinate as

$$\rho = \frac{1}{2}(r - m \pm \sqrt{r^2 - 2mr + Q^2})$$

(9a)

or

$$r = \rho(1 + \frac{m}{\rho} + \frac{m^2 - Q^2}{4\rho^2})$$

(9b)

for RN space-time and

$$\rho = \frac{1}{2}[r - \frac{l}{2} + r^{\frac{3}{2}}(r - l)^{\frac{1}{2}}]$$

(10a)

or

$$r = \rho(1 + \frac{l}{4\rho})^2$$

(10b)

for JNW space-time.
In a spherical coordinate system $ds^2 = f(\rho)dt^2 - d\vec{\rho}^2$ where the quantity $d\vec{\rho}^2 = \{d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\}$ has the dimension of the square of the infinitesimal length vector $\vec{\rho}$. Thus, by setting $ds = 0$, the velocity of light can be identified as $v(\rho) = \sqrt{f(\rho)}$ (Sen\cite{36}).

Now from the above isotropic expression the velocity of light ray in terms of $\rho$ can be written as:

$$v(\rho) = \frac{(1 - \frac{m^2 - Q^2}{4\rho^2})}{(1 + \frac{m}{\rho} + \frac{m^2 - Q^2}{4\rho^2})^2}c$$

for RN space-time.

Again

$$v(\rho) = \frac{(1 - \frac{L}{4\rho})^{2\gamma - 1}}{(1 + \frac{L}{4\rho})^{2\gamma + 1}}c$$

for JNW space-time.

So with the value of $\rho$ as in Eqn. \cite{9a} and \cite{10a}, the velocity $v(r, Q)$ in terms of $r$ and $Q$ becomes

$$v(r, Q) = \frac{r^2 - rr_g + Q^2}{r^2}c$$

$$= (1 - \frac{r_g}{r} + \frac{Q^2}{r^2})c$$

$$= (1 - \frac{r_g}{r})c + \frac{Q^2}{r^2}c$$

The first term of the above expression refers to the velocity of the light ray due to Schwarzschild geometry\cite{36} and the second term is due to the charge under RN geometry.

and

$$v(r, Q) = \left(\frac{r - l}{r}\right)\gamma c$$

$$= r - \sqrt{r_g^2 + 4Q^2} \frac{r_g}{\sqrt{r_g^2 + 4Q^2}}c$$

$$= (1 - \frac{r_g}{r} \sqrt{1 + \frac{4Q^2}{r_g^2}})\gamma c$$

where we have substituted the value of $l = 2\sqrt{m^2 + Q^2}$, $\gamma = \frac{2m}{r}$ and $m = \frac{r_g}{r}$.

Therefore the refractive index of light ray, $n(r, Q)$ can be expressed by the relation

$$n(r, Q) = \frac{1}{(1 - \frac{r_g}{r}) + \frac{Q^2}{r^2}}$$

for RN space-time.

And

$$n(r, Q) = \left(\frac{1}{1 - \frac{r_g}{r} \sqrt{1 + \frac{4Q^2}{r_g^2}}}\right)^{\gamma}$$

(16)
for JNW space-time.
Replacing $r/r_g$ by $x$ and $Q/r_g$ by $q$, the above expression of refractive indices become:

$$n(x, q) = \frac{1}{1 - \frac{1}{x} + \frac{q^2}{x^2}}$$

$$= \frac{1}{1 - \frac{1}{x}} [1 + \frac{q^2}{x^2(1 - \frac{1}{x})}]^{-1}$$

$$= \frac{x}{x - 1} [1 + \frac{q^2}{x(x - 1)}]^{-1}$$

$$= n_0(x)[1 + C_x]^{-1} \quad (17)$$

for RN space-time. In the above, we introduced the parameters $n_0(x) = \frac{x}{x - 1}$ and $C_x = \frac{q^2}{x(x - 1)}$. We can show for all $r >> r_g$ and $r >> Q$, we must have $x >> 1$ and $x >> q$. Now as $x >> 1$, we can approximate $(x - 1) \sim x$ and then we can finally show that $C_x << 1$.

And

$$n(x, q) = \left( \frac{1}{1 - \frac{1}{x} \sqrt{1 + 4q^2}} \right)^{\frac{1}{\sqrt{1 + 4q^2}}} \quad (18)$$

for JNW space-time.

Now at $Q = 0$ (or $q = 0$), the Reissner-Nordström space-time and Janis-Newman-Winicour space-time, both become the Schwarzschild space-time, so that the velocity and refractive index become

$$v(x) = (1 - \frac{1}{x})c = \frac{r - r_g}{r}c \quad (19)$$

and

$$n(x) = \frac{x}{x - 1} = \frac{r}{r - r_g} \quad (20)$$

which are exactly same as the velocity and refractive index calculated by Sen\textsuperscript{36} for Schwarzschild space-time.

However, Fischbach and Freeman\textsuperscript{26} have also calculated the refractive index in terms of PPN parameters where the photon is propagating in a Minkowskian space-time but with a local index of refraction. But in our process, we converted the metric into isotropic form (in terms of $\rho$) by co-ordinate transformation to get a general expression of index of refraction.

Fischbach and Freeman\textsuperscript{26} defined the index of refraction as an infinite convergent series as

$$n(r) = 1 + A/r + B/r^2 + ......$$

with $A = r_g$ and $B = f(r_g)$ where as the expression for Schwarzschild co-ordinate by Sen\textsuperscript{36} can also be represented by the infinite convergent series as

$$n(r) = 1 + (r_g/r) + (r_g/r)^2 + (r_g/r)^3 + .......$$

Thus both the expressions are same in the weak field limit.

Now we will study the variation of refractive index ($n(x, q)$) as a function of $q$. As a test case, we consider a pulsar PSRB 1937+21\textsuperscript{41} as the charged gravitating body. This pulsar is of mass $1.35M_\odot$, it has time period $1.557$ ms and physical radius $20.2$ Km (Nunez et al\textsuperscript{42}).
Fig. 1 shows the variation of refractive index \( n(x, q) \) with normalized charge radius \( q \) in RN geometry and JNW geometry. RN geometry explains the decrease of refractive index with the increase of \( q \) and at \( q = 0 \) the value of refractive index is maximum. According to RN geometry, it has two horizons and accordingly has some limitation to choose the value of \( Q \) as \( Q^2 \leq m^2 \) or \( Q^2 \leq 0.25r_g^2 \) so that \( q^2 \leq 0.25 \). Here we have chosen the value of \( Q \) accordingly for the given pulsar, so that the maximum value is \( q = 0.5 \) or \( Q = 1.9949 \) km as the value of Schwarzschild radius, \( r_g \) of that pulsar is approximately 3.9898 km.

In JNW geometry with the increase of \( q \), refractive index increases and at \( q = 0 \) the value of refractive index is minimum. JNW metric has a curvature singularity at \( r = l \) and a naked singularity at \( 1/2 < \gamma < 1 \) which gives \( 0 \leq Q^2 < 3m^2 \) or \( 0 \leq Q^2 < 0.75r_g^2 \), so that \( q^2 < 0.75 \) (expression for \( l \) and \( \gamma \) are given in Eqn. (5a) and (5b)). We have also considered the maximum value of \( Q \) as 3.4552 km or \( q = 0.8660 \) remembering such conditions.

C. Calculation of deflection of light ray from refractive index

Using the expression of refractive index the trajectory of light ray can be written as\(^{35,37,40}\):

\[
\Delta \psi = 2 \int_b^\infty \frac{dr}{r \sqrt{\left( \frac{n(r)}{n(b)} \right)^2 - 1} - \pi}
\]

As reported earlier by Sen\(^ {36} \) and Roy and Sen\(^ {37} \), in the present paper also we are considering that the light is approaching from asymptotic infinity \( (r = -\infty) \) towards the gravitating body and then it goes to \( r = +\infty \) after undergoing certain amount of deflection \( (\Delta \psi) \). Here, the gravitating body is characterized by the Schwarzschild radius \( r_g \) and the charge length \( Q \). Here the closest distance of approach or the impact parameter \((b)\) is considered as the physical radius of the gravitating body. When the light ray passes through the closest distance of approach, the tangent to the trajectory becomes perpendicular to the vector \( \vec{r} \) (which is \( \frac{\vec{b}}{b} \)).
Now we change the variable from $r$ to $x = \frac{r}{r_g}$ according as the Roy and Sen, so that $dr = r_g \, dx$ and the corresponding limit changes to $x = \frac{r}{r_g} = v$ and $x = \infty$, as the limit of $r$ changes from $r = b$ and $r = \infty$.

Therefore, the value of deflection can be written as:

$$\Delta \psi = 2n(v, q) v \int_{v}^{\infty} \frac{dx}{x \sqrt{(n(x, q) x)^2 - (n(v, q) v)^2}} - \pi$$

$$= 2I - \pi$$

(22)

where

$$I = n(v, q) v \int_{v}^{\infty} \frac{dx}{x \sqrt{(n(x, q) x)^2 - (n(v, q) v)^2}}$$

$$= D \int_{v}^{\infty} \frac{dx}{x \sqrt{(n(x, q) x)^2 - D^2}}$$

(23)

where we have substituted $n(v, q) v = D$.

1. **RN space time**

The general expression of refractive index due to charged body in Reissner Nordström space time is represented by equation (12) or (17). Now, by substituting the value of $n(x, q)$ in the above equation (23) we get the deflection of light ray in the field of Reissner-Nordström space-time as

$$\Delta \psi = \left( \frac{D_r}{D_0} - 1 \right) \pi + 2D_r \left\{ \int_{0}^{a} \frac{dz}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \right\}$$

$$- \frac{1}{2} \int_{0}^{a} \frac{(1-z)}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \left[ -\frac{2q^2 z^2 (1-z) + q^4 z^4}{((1-z) + q^2 z^2)^2} + D_0^2 z^2 (1-z)^2 \right] \frac{dz}{1 - D_0^2 z^2(1 - z)^2}$$

$$+ \frac{3}{8} \int_{0}^{a} \frac{(1-z)}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \left[ -\frac{2q^2 z^2 (1-z) + q^4 z^4}{((1-z) + q^2 z^2)^2} + D_0^2 z^2 (1-z)^2 \right] \frac{dz}{1 - D_0^2 z^2(1 - z)^2}$$

$$+ \frac{5}{16} \int_{0}^{a} \frac{(1-z)}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \left[ -\frac{2q^2 z^2 (1-z) + q^4 z^4}{((1-z) + q^2 z^2)^2} + D_0^2 z^2 (1-z)^2 \right] \frac{dz}{1 - D_0^2 z^2(1 - z)^2}$$

$$+ \frac{35}{64} \int_{0}^{a} \frac{(1-z)}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \left[ -\frac{2q^2 z^2 (1-z) + q^4 z^4}{((1-z) + q^2 z^2)^2} + D_0^2 z^2 (1-z)^2 \right] \frac{dz}{1 - D_0^2 z^2(1 - z)^2}$$

$$+ \frac{63}{256} \int_{0}^{a} \frac{(1-z)}{\sqrt{1 - D_0^2 z^2(1 - z)^2}} \left[ -\frac{2q^2 z^2 (1-z) + q^4 z^4}{((1-z) + q^2 z^2)^2} + D_0^2 z^2 (1-z)^2 \right] \frac{dz}{1 - D_0^2 z^2(1 - z)^2}$$

(24)

To get the above equation (24) from equation (23) we have done some algebraic calculations and the details of the calculation are given in Appendix A.

2. **JNW space time**

The refractive index in JNW space time is expressed by the equation (10) or (18). Thus with the value of refractive index $(n(x, q))$ and equation (23) the value of deflection of light ray in the field of JNW space time as
\[ \Delta \psi = 2D_j \int_0^a \frac{z(1-z\sqrt{1+4q^2})^{\sqrt{1+4q^2}-1}}{\sqrt{1-D_j^2z^2(1-z\sqrt{1+4q^2})^2/\sqrt{1+4q^2}}} \, dz \tag{25} \]

Here also the details of the calculation are given in Appendix B.

Here also the details of the calculation are given in Appendix B.

The above expressions (24) and (25) have been obtained without applying any weak field approximation at any stage. Thus, we may consider, these are the exact expressions of deflection of light ray in RN and JNW space-time using an equivalent material medium approach.

Here also if we apply the boundary condition as \( q = 0 \), both the expressions (24) and (25) exactly match with the bending angle due to Schwarzschild metric as given by Sen.\(^{36}\)

Considering the impact parameter as the physical radius of the gravitating body the variations of deflection (\( \Delta \psi \)) as a function of charge radius \( q \), for RN and JNW geometry are shown in Fig. 2. It also shows that as the value of the \( q \) increases, the value of deflection decreases in RN geometry and increase in JNW geometry.

The variations of deflection (\( \Delta \psi \)) as a function of impact parameter (b) for different values of \( Q \) are shown in Fig. 3. Figure shows the variation in RN geometry and JNW geometry in lower and upper panel respectively. For RN geometry we have considered the value of \( Q \) as 0, 0.5m, 1.0m (as \( Q^2 \leq m^2 \)) or 0, 0.25rg, 0.5rg. And for JNW geometry the value of \( Q \) as 0, 0.5m, 1.0m, 1.5m (as \( 0 \leq Q^2 < 3m^2 \)) or 0, 0.25rg, 0.5rg, 0.75rg. As the value of \( b \) increases i.e. as we move towards asymptotically flat space, these curves merge into each other. The deflection with \( Q = 0 \) indicates the deflection with respect to the Schwarzschild geometry.

**D. Comparison with other recent work**

In our present work, we obtained the refractive index and angle of deflection of light ray due to RN space-time and JNW space-time following *Material Medium Approach*. We can
FIG. 3. Deflection ($\Delta \psi$) as a function of impact parameter ($b$) for an arbitrarily selected pulsar PSRB 1937+21.

compare our obtained results with others recent work.

FIG. 4. Deflection ($\Delta \psi$) as a function of charge length ($q$) in RN geometry for an arbitrarily selected pulsar PSRB 1937+21.
FIG. 5. Deflection ($\Delta \psi$) as a function of impact parameter in RN geometry for an arbitrarily selected pulsar PSRB 1937+21.

FIG. 6. Deflection ($\Delta \psi$) as a function of $q$ in JNW geometry for an arbitrarily selected pulsar PSRB 1937+21.
1. RN geometry:

Sereno in 2004\cite{sereno2004} studied the gravitational lensing by RN black hole in the weak field limit in quasi-Minkovskian co-ordinate. The author followed the Fermat’s principle and obtained the approximated value of refractive index in quasi-Minkovskian co-ordinate (eqn. 34). But here we have calculated the exact value of refractive index in Boyer and Lindquist co-ordinate due to RN black hole. In 2002 Eiroa et al.\cite{eiroa2002} also studied the strong gravitational lensing by RN black hole using null geodesic method in Boyer-Lindquist co-ordinate upto 2nd order of charge (eqn. 55). Very recently S. Chakraborty and A. K. Sen\cite{chakrabortysen} also studied the deflection of light ray by Kerr-Newman geometry, using null geodesic method in Boyer-Lindquist co-ordinate upto 4th order of charge and mass. If we put rotation parameter equal to zero, we get the the deflection angle due to RN geometry (eqn. 37). In fig.\ref{fig:sereno} and fig.\ref{fig:eiroa} we have plotted the deflection angle ($\Delta \psi$) as a function of charge radius ($q$) and impact parameter respectively using the expression of different authors. In both the figures the set of curves follow similar pattern, but the difference lies in the magnitude of deflection values. The reason could be due to the fact that, Eiroa et al.\cite{eiroa2002} calculated upto 2nd order of charge and Chakraborty and Sen\cite{chakrabortysen} calculated upto 4th order of charge and mass. But in the present work we have not used any approximation to calculate the deflection angle. So our calculated values are claimed to be most exact so far.

2. JNW geometry:

In 1998 Virbhadra et al.\cite{virbhadra1998} calculated the Einstein deflection angle (up to second order) (eqn. 24) with JNW space-time using null geodesic method. In Fig.\ref{fig:sereno} and Fig.\ref{fig:eiroa} we have plotted deflection angle as a function of charge radius and impact parameter calculated by Virbhadra et al.\cite{virbhadra1998} and present work. Here also all the curves follow similar pattern, but the values of deflection angle differ between the curves. Virbhadra et al.\cite{virbhadra1998} calculated only up

FIG. 7. Deflection($\Delta \psi$) as a function of impact parameter in JNW geometry for an arbitrarily selected pulsar PSRB 1937+21.
to second order and we have not considered any approximation in our present calculation. So here again, our calculated values are claimed to be most exact.

From the above discussions, we may notice that the RN space-time and JNW space-time both are not similar kind of space-time although both represents the static solution of Einstein-Maxwell field equation for a charged, non-rotating and spherically symmetric gravitating body. With the increase of scalar charge the deflection angle decreases due to RN geometry and increases due to JNW geometry.

III. DISCUSSION AND CONCLUSIONS

In this paper we have presented the light deflection angle due to a charged gravitating body in Reissner-Nordström space time and Janis, Newman and Winicour space time, following Material Medium Approach. We have calculated the refractive index and deflection angle for both the space time, without using any weak field approximation. The plots of deflection angle against charge and impact parameter, show the same pattern as obtained by the others using most conventional method of Null geodesic. From the results obtained in the present work, it is concluded that the bending angle decreases with the increase of charge in RN space time and increases with charge in JNW space time.

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Appendix A: Bending angle due to RN geometry

We rewrite the Eqn.(22) as

$$\triangle \psi = 2I - \pi$$

(A1)

where

$$I = n(v,q)v\int_{v}^{\infty} \frac{dx}{x\sqrt{(n(x,q)x)^2 - (n(v,q)v)^2}}$$

$$= D \int_{v}^{\infty} \frac{dx}{x\sqrt{(n(x,q)x)^2 - D^2}}$$

(A2)

with $D = n(v,q)v$.

To evaluate the above integral, we follow a procedure similar to what was done by Sen and Roy and Sen. Thus with the value of refractive index from Eqn.[17] and $D = D_r$ (where $D_r = n(v,q)v$, $n(v,q)$ is the refractive index due to RN geometry at the limit of impact parameter) we have
\[ I = D_r \int_0^\infty \frac{dx}{\sqrt{\{n_0(x)(1 + C_x)^{-1}\cdot x\}^2 - D_r^2}} \]

\[ = D_r \int_0^\infty \frac{dx}{\sqrt{n_0^2(x)x^2 - D_0^2 + n_0^2(x)x^2(1 + C_x)^{-2} - n_0^2(x)x^2 + D_0^2 - D_r^2}} \]

\[ = D_r \int_0^\infty \frac{dx}{x\sqrt{n_0^2(x)x^2 - D_0^2}} \left[ 1 + \frac{n_0^2(x)x^2((1 + C_x)^{-2} - 1)}{n_0^2(x)x^2 - D_0^2} \right]^{\frac{1}{2}} \]

\[ = D_r \int_0^\infty \frac{dx}{x\sqrt{n_0^2(x)x^2 - D_0^2}} \left[ 1 + K(x) \right]^{-\frac{1}{2}} \tag{A3} \]

where \( D_0 = n_0(v).v \) (corresponding to Schwarzschild deflection). And we have also denoted

\[ K(x) = \frac{n_0^2(x)x^2((1 + C_x)^{-2} - 1) + D_0^2 - D_r^2}{n_0^2(x)x^2 - D_0^2} \tag{A4} \]

Here, we can show that \( K(x) \ll 1 \). To evaluate the value of \( K(x) \) of Eqn.\( \text{(A4)} \), the value of \(((1 + C_x)^{-2} - 1)\) is as follows:

\[ (1 + C_x)^{-2} - 1 = \frac{1}{(1 + C_x)^2} - 1 \]

\[ = \frac{1}{1 + \frac{q^2}{x(x-1)^2}} - 1 \]

\[ = \frac{2q^2x(x-1) + q^4}{x^2(x-1)^2} \]

\[ = -\frac{2q^2(x(x-1) + q^4)}{(x(x-1) + q^2)^2} \tag{A5} \]

Substituting the value of \(((1 + C_x)^{-2} - 1)\) from Eqn.\( \text{(A5)} \) and \( n_0(x) = x/(x-1) \) we can write the value of \( K(x) \) as

\[ K(x) = \frac{n_0^2(x)x^2((1 + C_x)^{-2} - 1) + D_0^2 - D_r^2}{n_0^2(x)x^2 - D_0^2} \]

\[ = \frac{n_0^2(x)x^2((1 + C_x)^{-2} - 1) + D_0^2}{n_0^2(x)x^2 - D_0^2(1 + C_x)^{-2}} \]

\[ = \frac{x^4}{(x-1)^2} \left\{ \frac{-2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right\} + D_0^2 \left( \frac{2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right) \]

\[ = \frac{x^4}{(x-1)^2} \left\{ \frac{-2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right\} + D_0^2 \left( \frac{2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right) \]

\[ = \frac{x^4}{(x-1)^2} \left\{ \frac{-2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right\} + D_0^2 \left( \frac{2q^2x(x-1) + q^4}{(x(x-1) + q^2)^2} \right) \]

\[ \tag{A6} \]

At this stage we can show that \( K(x) \ll 1 \). As \( K(x) \) is discontinuous at \( x = v \), we can remove its discontinuity and evaluate its value by applying L'Hospital's rule.

Therefore, from Eqs.\( \text{(A1)} \) and \( \text{(A3)} \) one can write:

\[ \Delta \psi = 2D_r \int_0^\infty \frac{dx}{x\sqrt{n_0^2(x)x^2 - D_0^2}} \left[ 1 - \frac{1}{2} K(x) + \frac{q}{2} K^2(x) - \frac{5}{16} K^3(x) + \frac{35}{288} K^4(x) - \frac{63}{284} K^5(x) + \ldots \right] - \pi \]

\[ = 2[I_0 + I_1 + I_2 + I_3 + \ldots] - \pi \tag{A7} \]
where, we have introduced some other notations:

\[ I_0 = D_r \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} \]  
(A8a)

\[ I_1 = D_r \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} (-1/2K(x)) \]  
(A8b)

\[ I_2 = D_r \int_v^\infty \frac{dx}{x \sqrt{n_0^2(x)x^2 - D_0^2}} (3/8K^2(x)) \]  
(A8c)

and so on.

Now \( I_0 \) can be evaluate by following the same procedure as Sen\(^{36}\) and Roy and Sen\(^{37}\). According to Roy and Sen\(^{37}\), \( I_0 \) can be split into two integrals as \( I_{01} \) and \( I_{02} \). Here, \( D_r \) is replaced by \( D_r \). Thus the value of \( I_{01} \) and \( I_{02} \) will be:

\[ I_{01} = \frac{D_r}{D_0} \cdot \frac{\pi}{2} \]  
(A9)

and

\[ I_{02} = D_r \int_0^a \frac{zdz}{\sqrt{1 - D_0^2z^2(1 - z)^2}} \]  
(A10)

where we change the variable as \( z = \frac{1}{v} \), so that the limits of this integration changes from \( z = \frac{1}{v} = a \) (say) to \( z = 0 \). The integral \( (A10) \) can be evaluated in terms of Elliptical function as expressed by Eqn.(18) of Sen\(^{36}\) and finally for a given value of \( a \), its numerical value can be obtained as Roy and Sen\(^{37}\).

Now substituting the value of \( K(x) \) from Eqn.\(^{(A6)}\), \( n_0(x) = \frac{x}{\pi} \) and applying the change of variable as \( z = \frac{1}{x} \) the integral \( I_1 \) becomes

\[ I_1 = -\frac{1}{2} D_r \int_v^\infty \frac{K(x)}{x \sqrt{n_0^2(x)x^2 - D_0^2}} dx \]

\[ = -\frac{1}{2} D_r \int_v^\infty \frac{1}{x \sqrt{\frac{x^4}{(x-1)^2} - D_0^2}} - x^4 \left( \frac{2q^2x(x-1)+q^4}{x(x-1)q^2} \right) + D_0^2(x-1)^2 \left( \frac{2q^2(x-1)+q^4}{x(x-1)q^2} \right) dx \]

\[ = -\frac{1}{2} D_r \int_v^\infty \frac{(x-1)}{\sqrt{x^6 - D_0^2x^2(x-1)^2}} - x^4 \left( \frac{2q^2x(x-1)+q^4}{x(x-1)q^2} \right) + D_0^2(x-1)^2 \left( \frac{2q^2(x-1)+q^4}{x(x-1)q^2} \right) dx \]

\[ = -\frac{1}{2} D_r \int_0^a \frac{(1-z)}{\sqrt{1 - D_0^2z^2(1 - z)^2}} - x^4 \left( \frac{2q^2z^2(1-z)+q^4z^4}{((1-z)+q^2z^2)^2} \right) + D_0^2z^2(1-z)^2 \left( \frac{2q^2z^2(1-z)+q^4z^4}{((1-z)+q^2z^2)^2} \right) dx \]  
(A11)

Applying the same procedure, the other integrals \( I_2, I_3 \) etc. are as follows:

\[ I_2 = \frac{3}{8} D_r \int_0^a \frac{(1-z)}{\sqrt{1 - D_0^2z^2(1 - z)^2}} - x^4 \left( \frac{2q^2z^2(1-z)+q^4z^4}{((1-z)+q^2z^2)^2} \right) + D_0^2z^2(1-z)^2 \left( \frac{2q^2z^2(1-z)+q^4z^4}{((1-z)+q^2z^2)^2} \right) dx \]  
(A12)
\[I_3 = -\frac{5}{16} D_r \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^3 dz \]

(A13)

\[I_4 = \frac{35}{64} D_r \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^4 dz \]

(A14)

\[I_5 = -\frac{63}{256} D_r \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^5 dz \]

(A15)

Therefore, substituting all the values of \(I_0, I_1, I_2, I_3\) etc. the Eqn.\ref{eq:A7} becomes

\[
\Delta \psi = 2\left[ \frac{D_r \pi}{D_0} \right] + 2D_r \left\{ \int_0^a \frac{z dz}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \right. \]

\[
- \frac{1}{2} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right] dz \]

\[
+ \frac{3}{8} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^2 dz \]

\[
- \frac{5}{16} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^3 dz \]

\[
+ \frac{35}{64} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^4 dz \]

\[
- \frac{63}{256} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^5 dz \]

\[+ \ldots \ldots \ldots \ldots \ldots \]

\[
= \left( \frac{D_r}{D_0} - 1 \right) \pi + 2D_r \left\{ \int_0^a \frac{z dz}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \right. \]

\[
- \frac{1}{2} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right] dz \]

\[
+ \frac{3}{8} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^2 dz \]

\[
- \frac{5}{16} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^3 dz \]

\[
+ \frac{35}{64} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^4 dz \]

\[
- \frac{63}{256} \int_0^a \frac{(1 - z)}{\sqrt{1 - D_0^2 z^2 (1 - z)^2}} \left[ -\frac{2q^2 z^2 (1 - z) + q^4 z^4}{(1 - z) + q^2 z^2} + \frac{D_0^2 z^2 (1 - z)^2}{1 - D_0^2 z^2 (1 - z)^2} \right]^5 dz \]

\[+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

(A16)
The above expression represents the light deflection angle due to charged gravitating mass in RN space time.

Appendix B: Bending angle due to JNW geometry

Here also we will follow the same procedure as Appendix A. Now with the value of refractive index from Eqn. (18) and $D = D_j$ (where $D_j = n(v,q)v$, $n(v,q)$ is the refractive index due to JNW geometry at the limit of impact parameter), we have

$$I_j = D_j \int_{v}^{\infty} \frac{dx}{x(1 + \sqrt{1 + q^2})^{2/\sqrt{1 + 4q^2} - D_j^2}}$$

$$= D_j \int_{v}^{\infty} \frac{dx}{x(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}} - D_j^2}$$

$$= D_j \int_{v}^{\infty} \frac{(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}} - \frac{1}{x^2}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}})}{x^2 \sqrt{1 - D_j^2/(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{2/\sqrt{1 + 4q^2}})}} dx$$

$$+ D_j \int_{v}^{\infty} \frac{D_j}{x^2(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{2/\sqrt{1 + 4q^2}})} dx$$

$$= I_{j1} + I_{j2}$$

Now, let

$$y = \frac{D_j}{x}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}})$$

so that

$$dy = -\frac{D_j}{x^2}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}}) dx - \frac{D_j}{x^3}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}}) dx$$

Thus the limit changes to $y = 0$ and $y = \frac{D_j}{x}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}} = D_j \frac{1}{D_j} = 1$ as $x$ changes to $v$ and $\infty$. So,

$$I_{j1} = D_j \int_{v}^{\infty} \frac{(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}} - \frac{1}{x^2}(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{1/\sqrt{1 + 4q^2}})}{x^2 \sqrt{1 - D_j^2/(1 - \frac{1}{x^2}(1 + \sqrt{1 + q^2})^{2/\sqrt{1 + 4q^2}})}} dx$$

$$= \int_{0}^{1} \frac{dy}{\sqrt{1 - y^2}}$$

$$= \frac{\pi}{2}$$

Now by applying the change of variable as $z = \frac{1}{x}$ like Appendix A, we may write the above integral $I_{j2}$ as
\[ I_{J2} = D_j \int_0^\infty \frac{\frac{1}{2}(1 - \frac{1}{2}\sqrt{1 + 4q^2})^{\frac{1}{4} + \frac{1}{2}q^2} - 1}{x^2 \sqrt{1 - \frac{E^2}{2}(1 - \frac{1}{2}\sqrt{1 + 4q^2}^2)/\sqrt{1 + 4q^2}}} \, dx \]

\[ = D_j \int_0^a \frac{z(1 - z\sqrt{1 + 4q^2})^{\frac{1}{4} + \frac{1}{2}q^2} - 1}{\sqrt{1 - D_j^2 z^2(1 - z\sqrt{1 + 4q^2}^2)/\sqrt{1 + 4q^2}}} \, dz \]

(B3)

Thus from expression (A11) and (B1) the light deflection angle due to JNW space time can be written as

\[ \Delta \psi = 2[\frac{\pi}{2} + D_j \int_0^a \frac{z(1 - z\sqrt{1 + 4q^2})^{\frac{1}{4} + \frac{1}{2}q^2} - 1}{\sqrt{1 - D_j^2 z^2(1 - z\sqrt{1 + 4q^2}^2)/\sqrt{1 + 4q^2}}} \, dz] - \pi \]

\[ = 2D_j \int_0^a \frac{z(1 - z\sqrt{1 + 4q^2})^{\frac{1}{4} + \frac{1}{2}q^2} - 1}{\sqrt{1 - D_j^2 z^2(1 - z\sqrt{1 + 4q^2}^2)/\sqrt{1 + 4q^2}}} \, dz \]

(B4)

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