NOTES ON $H^{\log}$: STRUCTURAL PROPERTIES, DYADIC VARIANTS, AND BILINEAR $H^1$-BMO MAPPINGS

ODYSSEAS BAKAS, SANDRA POTT, SALVADOR RODRÍGUEZ-LÓPEZ, AND ALAN SOLA

Abstract. This article is devoted to a study of the Hardy space $H^{\log}(\mathbb{R}^d)$ introduced by Bonami, Grellier, and Ky. We present an alternative approach to their result relating the product of a function in the real Hardy space $H^1$ and a function in $BMO$ to distributions that belong to $H^{\log}$ based on dyadic para-products. We also point out analogues of classical results of Hardy-Littlewood, Zygmund, and Stein for $H^{\log}$ and related Musielak-Orlicz spaces.

1. Introduction

The Lebesgue spaces $L^p$ with $1 \leq p \leq \infty$ are fundamental in mathematical analysis. They are easy to define, they are Banach spaces with many useful properties, and they encode a natural notion of regularity in a measurable function. Nevertheless, there are many instances where $L^p$ spaces, especially with $p = 1$, do not capture finer properties of functions or operators acting on functions. In such instances, it may be necessary to consider substitutes for $L^1$, as is the case when studying endpoint bounds for operators on $L^p$ as $p \to 1^+$. For instance, the ubiquitous Hardy-Littlewood maximal function exhibits precisely this kind of behaviour near $p = 1$. The maximal function is defined for a locally integrable function $f : \mathbb{R}^d \to \mathbb{C}$ by setting

$$M(f)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy,$$

where $B(x, r)$ denotes the open ball in $\mathbb{R}^d$ centered at $x$ with radius $r > 0$, and $|A|$ denotes the Lebesgue measure of $A \subseteq \mathbb{R}^d$. It is a basic fact that the mapping $f \mapsto M(f)$ is bounded on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$. The maximal operator is also bounded from $L^1(\mathbb{R}^d)$ to weak-$L^1$, but does not map $L^1(\mathbb{R}^d)$ to itself (see, for instance, [27] for an in-depth discussion).

However, $M(f)$ is locally integrable provided $f$ is compactly supported and satisfies the $L \log L$ condition

$$\int_{\mathbb{R}^d} |f(x)| \log^+ |f(x)| \, dx < \infty,$$

where, as usual, $\log^+ |x| = \max\{\log |x|, 0\}$. In a 1969 paper, E. M. Stein [25] proved that this $L \log L$ condition is both sufficient and necessary for integrability of the

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Hardy-Littlewood maximal function, in the following sense: if $f$ is supported in some finite ball $B = B(r)$ of radius $0 < r < \infty$, then
\[ \int_B M(f)dx < \infty \quad \text{if, and only if,} \quad \int_B |f(x)| \log^+ |f(x)|dx < \infty. \]

Thus, $L \log L$ is a natural substitute for $L^1$ for the purposes of studying the boundedness of the Hardy-Littlewood maximal function in the scale of $L^p$ spaces.

Another classical result that involves the space $L \log L$ is due to Zygmund, and asserts that the periodic Hilbert transform $H$ maps $L \log L(\mathbb{T})$ to $L^1(\mathbb{T})$; see e.g. Theorem 2.8 in Chapter VII of [35]. Zygmund’s theorem implies that $L \log L(\mathbb{T})$ is contained in the real Hardy space $H^1(\mathbb{T})$ consisting of integrable functions on the torus whose Hilbert transforms are integrable. Moreover, as shown by Stein in [25], Zygmund’s theorem has a partial converse, namely if $f \in H^1(\mathbb{T})$ and $f$ is non-negative, then $f$ necessarily belongs to $L \log L(\mathbb{T})$. Therefore, in view of the aforementioned results of Zygmund and Stein, the Hardy space $H^1(\mathbb{T})$ is, in terms of magnitude, associated with the Orlicz space $L \log L(\mathbb{T})$.

In several problems in harmonic analysis it is natural to consider Hardy-Orlicz spaces, for instance when one studies certain problems related to endpoint mapping properties of operators; see e.g. [34, Theorem 8], Théorème 2 in Chapitre II and Théorème 1 (c) in Chapitre IV of [20] as well as [14, 16, 28, 29] or, even more generally, Musielak-Orlicz Hardy spaces [33]. In this paper we shall mainly focus on certain structural aspects of the space $H^{log}(\mathbb{R}^d)$ appearing in the work of A. Bonami, S. Grellier, and L. D. Ky [4].

Before we proceed with the outline of our paper, let us give a formal definition of the space $H^{log}(\mathbb{R}^d)$. Let $\Psi : \mathbb{R}^d \times [0, \infty) \to [0, \infty)$ denote the function given by
\[ \Psi(x,t) := \frac{t}{\log(e+t) + \log(e+|x|)}, \quad (x,t) \in \mathbb{R}^d \times [0, \infty). \]

If $B$ is a subset of $\mathbb{R}^d$, one defines $L_\Psi(B)$ to be the space of all locally integrable functions $f$ on $B$ satisfying
\[ \int_B \Psi(x,|f(x)|)dx < \infty. \]

We also fix a non-negative function $\phi \in C^\infty(\mathbb{R}^d)$, which is supported in the unit ball of $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \phi(y)dy = 1$ and $\phi(x) = c_d$ for all $|x| \leq 1/2$, where $c_d$ is a constant depending on the dimension $d$. Given an $\epsilon > 0$, we employ the notation $\phi_\epsilon(x) := e^{-\epsilon d} \phi(e^{-1}x)$, $x \in \mathbb{R}^d$.

**Definition** ($H^{log}$, see [4, 32]). Let $\phi$ be as above. If $f$ is a tempered distribution on $\mathbb{R}^d$, consider the maximal function
\[ M_\phi(f)(x) := \sup_{\epsilon > 0} |(f * \phi_\epsilon)(x)|, \quad x \in \mathbb{R}^d. \]

The Hardy space $H^{log}(\mathbb{R}^d)$ is defined to be the space of tempered distributions $f$ on $\mathbb{R}^d$ such that $M_\phi(f) \in L_\Psi(\mathbb{R}^d)$, that is, $M_\phi(f)$ satisfies
\[ \int_{\mathbb{R}^d} \Psi(x, |M_\phi(f)(x)|)dx < \infty. \]

The reason for defining $H^{log}(\mathbb{R}^d)$ comes from the study of products of functions in the real Hardy space $H^1(\mathbb{R}^d)$ and its dual space $BMO(\mathbb{R}^d)$. To be more specific, following earlier work by Bonami, T. Iwaniec, P. Jones, and M. Zinsmeister in [4], it was shown by Bonami, Grellier, and Ky [4] that the product $fg$, in the

\[ ^1 \text{Notice that the aforementioned endpoint bounds for } M \text{ and } H \text{ can be regarded as special cases of the fact that if } T \text{ is any sublinear operator that is bounded on } L^{p_0} \text{ for some } p_0 > 1 \text{ and maps } L^1 \text{ to weak-}L^1, \text{ then } T \text{ locally maps } L \log L \text{ to } L^1. \]
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sense of distributions, of a function $f \in H^1(\mathbb{R}^d)$ and a function $g$ of bounded mean oscillation in $\mathbb{R}^d$ can in fact be represented as a sum of a continuous bilinear mapping into $L^1(\mathbb{R}^d)$ and a continuous bilinear mapping into $H^{1,\log}(\mathbb{R}^d)$. Following \cite{4}, for a function $g$ of bounded mean oscillation in $\mathbb{R}^d$, we set

$$\|g\|_{BMO^+(\mathbb{R}^d)} := \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \left| \int_Q g(x) - \langle g \rangle_Q \right| dx + \left| \int_{[0,1]^d} g(x) dx \right|,$$

where $\langle g \rangle_Q := |Q|^{-1} \int_Q g(x) dx$. The aforementioned result of Bonami, Grellier, and Ky can be stated as follows.

**Theorem 1** \cite{4}. There exist two bilinear operators $S, T$ and a constant $C_d > 0$ such that

$$\|S(f,g)\|_{L^1(\mathbb{R}^d)} \leq C_d \|f\|_{H^1(\mathbb{R}^d)} \|g\|_{BMO^+(\mathbb{R}^d)}$$

and

$$\|T(f,g)\|_{H^{1,\log}(\mathbb{R}^d)} \leq C_d \|f\|_{H^1(\mathbb{R}^d)} \|g\|_{BMO^+(\mathbb{R}^d)}$$

with

$$f \cdot g = S(f,g) + T(f,g)$$

in the sense of distributions.

The operators $S$ and $T$ in the statement of Theorem \ref{t1} are not unique and they are given in \cite{4} in terms of paraproducts that are constructed by using continuous wavelets. See also \cite{15, 2}, and \cite{33} for further developments. For an introduction to the theory of wavelets, we refer the reader to Y. Meyer’s book \cite{19}.

Having seen why $H^{1,\log}(\mathbb{R}^d)$ is worthy of study, we wish to further elucidate its structure. Our paper consists of three parts.

**Part I: Sections 2 and 3** In the first part of this paper we present analogues of the aforementioned theorems of Zygmund and Stein for $H^{1,\log}(\mathbb{R}^d)$. Such results can be derived from more general results previously obtained in the setting of Orlicz spaces, see for instance \cite{5, 12}. (We are grateful that these facts were pointed out to us in connection with an earlier note on this subject.) We give a self-contained account here, including a discussion of sharpness, and indicate some minor modifications that need to be made to obtain results in the Musielak-Orlicz setting.

For an $H^{1,\log}$ version of Stein’s theorem, we need to identify the correct analogue of $L \log L$ in this context, which turns out to be $L \log \log L$: given a measurable subset $B$ of $\mathbb{R}^d$, $L \log \log L(B)$ denotes the class of all locally integrable functions $f$ with $\text{supp}(f) \subseteq B$ and

$$\int_B |f(x)| \log^+ \log^+ |f(x)| dx < \infty.$$

Here is our version of Stein’s lemma for $L_\Psi$.

**Theorem 2.** Let $f$ be a measurable function supported in a closed ball $B \subseteq \mathbb{R}^d$. Then $M(f) \in L_\Psi(B)$ if, and only if, $f \in L \log L(B)$.

Our proof in fact leads to a more general version of Theorem \ref{t2}. We discuss this, and give a proof of Theorem \ref{t3} in Section \ref{sec:proof}.

Next is the analogue of Zygmund’s result for $H^{1,\log}(\mathbb{R}^d)$.

**Theorem 3.** Let $B$ denote the closed unit ball in $\mathbb{R}^d$.

If $f$ is a measurable function satisfying $f \in L \log L(B)$ and $\int_B f(y) dy = 0$, then $f \in H^{1,\log}(\mathbb{R}^d)$. 

We remark that the mean-zero condition in the hypothesis is in fact necessary in order to place a compactly supported function in $H^{\log}(\mathbb{R}^d)$; see Lemma 8 below for a more general version of this fact valid for compactly supported distributions.

Part II: Sections 4, 5 and 6. The second part of this paper covers two different themes. In the first one, we show that one can simplify the argument in [4] that establishes Theorem 1 by reducing matters to appropriate dyadic counterparts. To be more specific, in Section 4 we introduce a dyadic version of $H^{\log}$ in the periodic setting and then, by establishing a characterisation of dyadic $H^{\log}$ in terms of atomic decompositions, we show that $H^{\log}$ coincides with an intersection of two translates of dyadic $H^{\log}$, a result of independent interest; see Section 5. In Section 6, we show that, in view of the aforementioned result of Section 5, one can obtain a simplified proof of Theorem 1 in the periodic setting in which only dyadic paraproducts are involved.

Part III: Section 7. The last part of this paper also features two different themes: in Section 7.1, we discuss some variants and further extensions of Theorems 2 and 3 to the periodic setting. In Section 7.2, we establish a version of a classical inequality of G. H. Hardy and J. E. Littlewood [10] that gives a description of the order of magnitude of Fourier coefficients of distributions in $H^{\log}(\mathbb{T})$.

2. Proof of the Stein-type Theorem for $L_\Psi$ and further extensions

We begin with an elementary observation that will be implicitly used several times in the sequel: if $\Phi : [0, \infty) \to [0, \infty)$ is an increasing function, then for every positive constant $\alpha_0$ one has
\[
\int_B \Phi(|g(x)|)\,dx \leq \Phi(\alpha_0)|B| + \int_{\{|g| > \alpha_0\}} \Phi(|g(x)|)\,dx
\]
for each measurable set $B$ in $\mathbb{R}^d$ with finite measure.

We now turn to the proof of our first theorem.

Proof of Theorem 2. Assume first that $f \in L^{\log \log}$. The main observation is that locally the space $L_\Psi$ essentially coincides with the Orlicz space defined in terms of the function $\Psi_0(t) := t \cdot \log(e + t)^{-1}, t \geq 0$ and so, one can employ the arguments of Stein [25]. In view of this observation, we remark that the fact that $f \in L^{\log \log}$ implies $M(f) \in L_{\Psi_0}(B)$ is well-known; see for instance [3, p.242], [12, Sections 4 and 7]. We shall also include the proof of this implication here for the convenience of the reader.

To be more precise, we note that for $x \in B$ one has
\[
\log(e + M(f)(x)) \leq \log((e + |x|)(e + M(f)(x))) \leq c \log(e + M(f)(x)),
\]
for a constant $c$ that only depends on $B$. Next, an integration by parts yields
\[
\int_e^y \frac{1}{\log \alpha}\,d\alpha = \frac{y}{\log y} - e + \int_e^y \frac{1}{\log^2 \alpha}\,d\alpha,
\]
so that
\[
\frac{y}{\log y} \leq e + \int_e^y \frac{1}{\log \alpha}\,d\alpha, \quad \text{for } y > e.
\]
Together, these two observations imply that
\[
\int_B \Psi(x, M(f)(x))\,dx \lesssim_B 1 + \int_{B \cap \{|M(f)| > \alpha\}} \left( \int_e^{M(f)(x)} \frac{1}{\log \alpha}\,d\alpha \right)\,dx
\]
\[
= 1 + \int_e^\infty \frac{1}{\log \alpha} \cdot |\{x \in B : M(f)(x) > \alpha\}|\,d\alpha.
\]
To estimate the last integral, note that there exists an absolute constant $C_d > 0$ such that
\begin{equation}
|x \in \mathbb{R}^d : M(f)(x) > \alpha| \leq \frac{C_d}{\alpha} \int_{|f| > \alpha/2} |f(x)| \, dx
\end{equation}
for all $\alpha > 0$; see e.g. [23, (5)] or Section 5.2 (a) in Chapter I in [24]. We thus deduce from (2.2) that
\begin{align*}
\int_B \Psi(x,M(f)(x)) \, dx & \lesssim_B 1 + \int_B |f(x)| \left( \int_{e}^{2|f(x)|} \frac{1}{\alpha \log \alpha} \, d\alpha \right) \, dx \\
& \lesssim 1 + \int_B |f(x)| \log^+ \log^+ |f(x)| \, dx,
\end{align*}
which implies that $M(f) \in L_\Psi(B)$.

To prove the reverse implication, assume that for some $f$ supported in $B$ with $f \in L^1(B)$ we have $M(f) \in L_\Psi(B)$. Our task is to show that $f \in L \log L(B)$. In order to accomplish this, we shall make use of the fact that there exists a constant $\rho > 2$, depending only on $\|f\|_{L^1(B)}$ and $B$, such that we also have $M(f) \in L_\Psi(\rho B)$ and moreover, for every $\alpha \geq e^r$,
\begin{equation}
\|x \in \rho B : M(f)(x) > \alpha\| \geq \frac{c_2}{\alpha} \int_{B^c(\{f > \alpha\})} |f(x)| \, dx,
\end{equation}
where $c_1$, $c_2$ are positive constants that can be taken to be independent of $f$ and $\alpha$. Indeed, arguing as in the proof of [23, Lemma 1], note that for every $r > 1$ one has
\begin{equation}
M(f)(x) \lesssim \frac{1}{(r - 1)^d |B|} \|f\|_{L^1(B)} \quad \text{for all} \ x \in \mathbb{R}^d \setminus rB.
\end{equation}
Hence, if we choose $\rho > 2$ to be large enough, then $M(f)(x) < e^r \leq \alpha$ for all $x \in \mathbb{R}^d \setminus \rho B$ and so, (2.3) follows from [23, Inequality (6)].

Furthermore, one can check that $M(f) \in L_\Psi(\rho B)$. Indeed, if we write $B = B(x_0, r_0)$ then, as in [23], it follows from the definition of $M$ and the fact that $\text{supp}(f) \subseteq B$ that there exists a constant $c_0 > 0$, depending only on the dimension, such that for every $x \in 2B \setminus B$ one has
\begin{equation}
M(f)(x) \leq c_0 M(f) \left( x_0 + r_0^2, \frac{x - x_0}{|x - x_0|^2} \right)
\end{equation}
and so, $M(f) \in L_\Psi(2B)$. To show that (2.5) implies that $M(f) \in L_\Psi(B)$, observe first that the function $\Psi_0$ is increasing on $[0, +\infty)$, and for all $t \geq 1$ and all $s > 0$,
\begin{align*}
1 - \frac{\log(e + s)}{\log(e + ts)} &= \frac{\log(e + s)}{\log(e/t + s)} + \log t \\
& \geq \frac{\log(e + s)}{\log(e + s) + t} + \log t \\
& \geq \frac{1}{1 + \log t},
\end{align*}
so $\Psi_0$ satisfies
\begin{equation}
t(1 + \log t)^{-1} \Psi_0(s) \leq \Psi_0(st) \leq t \Psi_0(s),
\end{equation}
which implies that for all $c > 0$ and all $s > 0$
\[ \Psi_0(cs) \sim_c \Psi_0(s). \]
Observe that a change to polar coordinates, followed by another a change of variables and elementary estimates yield
\[
\int_{2B\setminus B} \Psi_0(Mf(x))dx \lesssim \int_{2B\setminus B} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0 \theta/s))d\sigma(\theta)ds \\
\sim r_0^{-1} \int_0^1 t^{-1-d} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0 t\theta))d\sigma(\theta)dt \\
\sim \int_0^1 t^{-1-d} \int_{S^{d-1}} \Psi_0(Mf(x_0 + r_0 t\theta))d\sigma(\theta)dt \\
\lesssim \int_B \Psi(x, Mf(x))dx.
\]

Moreover, we deduce from (2.4) that \(M(f)\) belongs to \(L_\varphi(\rho B \setminus 2B)\) and it thus follows that \(M(f) \in L_\varphi(\rho B)\), as desired.

Next, note that by the same reasoning as in the proof of sufficiency and by Fubini’s theorem,
\[
\int_{\rho B} \Psi(x, Mf(x))dx \gtrsim \int_{\rho B \cap \{M(f) > \max\{\alpha, |x_0|+r_0\}\}} \frac{M(f)(x)}{\log(M(f)(x))} dx \\
\gtrsim \int_{\rho B \cap \{M(f) > \max\{\alpha, |x_0|+r_0\}\}} \left(\int_{\max\{\alpha, |x_0|+r_0\}}^\infty \frac{1}{\log \alpha} d\alpha\right) dx \\
\gtrsim \int_{\max\{\alpha, |x_0|+r_0\}}^\infty \frac{1}{\log \alpha} |\{x \in \rho B : M(f)(x) > c_2 \alpha\}| d\alpha.
\]

By using (2.3), we now get
\[
\infty \gtrsim \int_{\rho B} \Psi(x, Mf(x))dx \gtrsim \int_B |f(x)| \left(\int_{\max\{\alpha, |x_0|+r_0\}}^\infty \frac{1}{\alpha \log \alpha} d\alpha\right) dx \\
\gtrsim 1 + \int_B |f(x)| \log^+ \log^+ |f(x)| dx
\]
and this completes the proof of Theorem 2.

\[\square\]

Remark 4. Let \(B_0\) denote the closed unit ball in \(\mathbb{R}^d\). Given a small \(\delta \in (0, e^{-c})\), if, as on pp. 58–59 in [2], one considers \(f := \delta^{-d} \chi_{\{|x| < \delta\}}\) then \(M(f)(x) \sim |x|^{-d}\) for all \(|x| > 2\delta\) and so,
\[
(2.7) \quad \int_{B_0} |f(x)| \log^+ \log^+ |f(x)| dx \sim \log(\log(\delta^{-1})) \sim \int_{B_0} \Psi(x, Mf(x))dx.
\]

This shows that given \(L_\varphi(B_0)\), the space \(L \log \log L(B_0)\) in the statement of Theorem 2 is best possible in general, in terms of size.

Indeed, the left-hand side of (2.7) follows by direct calculation. On the other hand, (2.4), (2.6), a change to polar coordinates, and further change of variables yield
\[
\int_{B_0} \Psi(x, Mf(x))dx \sim 1 + \int_{2\delta}^1 \frac{1}{\log(e + s^{-d})} \frac{ds}{s} \\
\sim 1 + \int_2^{(2\delta)^{-1}} \frac{1}{\log(e + u^d)} \frac{du}{u} \sim 1 + \int_e^{(2\delta)^{-1}} \frac{1}{\log(u)} \frac{du}{u},
\]
from where the right-hand side of (2.7) follows.
2.1. Further generalisations. Assume that $\Psi : \mathbb{R}^d \times [0, \infty)$ is a non-negative function satisfying the following properties:

1. For every $x \in \mathbb{R}^d$ fixed, $\Psi(x, t) = \Psi_x(t)$ is Orlicz in $t \in [0, \infty)$, namely $\Psi_x(0) = 0$, $\Psi_x$ is increasing on $[0, \infty)$ with $\Psi_x(t) > 0$ for all $t > 0$ and $\Psi_x(t) \to \infty$ as $t \to \infty$.

Moreover, assume that there exists an absolute constant $C_0 > 0$ such that $\Psi_x(2t) \leq C_0 \Psi_x(t)$ for all $x \in \mathbb{R}^d$ and every $t \in [0, \infty)$.

2. If $K$ is a compact set in $\mathbb{R}^d$, then there exist $x_1, x_2 \in K$ and a constant $C_K > 0$ such that $C_K^{-1} < \Psi(x_1, t) \leq \Psi(x, t) \leq \Psi(x_2, t) < C_K$

for every $x \in K$ and for all $t > 0$.

3. If we write $\Psi(x, t) = \Psi_x(t) = \int_0^t \psi_x(s)ds$, then for every $\alpha_0$, $\beta_0 > 0$, if $0 < \alpha_0 < \beta_0$, one has

$$\int_{\alpha_0}^{\beta_0} \frac{\psi_x(s)}{s} ds < \infty$$

for every $x \in \mathbb{R}^d$.

By carefully examining the proof of Theorem 2, one obtains the following result.

**Theorem 5.** Let $\Psi(x, t) = \int_0^t \psi_x(s)ds$, $(x, t) \in \mathbb{R}^d \times [0, \infty)$, be as above. Fix a closed ball $B$ with $B \subseteq \mathbb{R}^d$ and let $f$ be such that $\text{supp}(f) \subseteq B$. Then, $M(f) \in L_\Psi(B)$ if, and only if,

$$\int_{\{|f| > \alpha_0\}} |f(x)| \left( \int_{\alpha_0}^{\beta_0} \frac{\psi_x(s)}{s} ds \right) dx < \infty$$

for every $\alpha_0 > 0$.

Theorem 5 applies to certain Orlicz spaces considered in connection with convergence of Fourier series, see e.g. [1, 24], and the recent paper by V. Lie [17]; we give some sample applications in Subsection 7.1.1

3. Proof of the Zygmund-type Theorem for $H^{\log}(\mathbb{R}^d)$

We begin with the following elementary lemmas.

**Lemma 6.** Consider the function $g : [0, \infty)^2 \to [0, \infty)$ given by

$$g(s, t) := \frac{1}{\log(e + t) + \log(e + s)^2}, \quad (s, t) \in [0, \infty)^2.$$ 

Then one has

$$\Psi(x, t) \leq \int_0^t g(|x|, \tau) d\tau \leq 2\Psi(x, t)$$

for all $(x, t) \in \mathbb{R}^d \times [0, \infty)$.

**Proof.** The function $t \mapsto g(s, t) = [\log((e + t)(e + s))]^{-1}$ is decreasing, so clearly

$$\int_0^t g(|x|, s)ds \geq tg(|x|, t) = \Psi(x, t).$$

We now address the upper bound. A calculation yields that

$$\partial_t(t^e g(|x|, t)) = \frac{t^e}{\log(e + t) + \log(1 + |x|)} \left( \frac{\epsilon}{t} - \frac{1}{(e + t)[\log(e + t) + \log(e + |x|)]} \right),$$

and we observe that the term within the parenthesis is positive if, and only if,

$$\frac{\epsilon}{t} - \frac{1}{(e + t)[\log(e + t) + \log(e + |x|)]} > 0,$$
which for $\epsilon = 1/2$ is equivalent to the inequality
\[(e + t)\log(e + t) + \log(e + |x|)] > 2t.\]
But clearly
\[(e + t)\log(e + t) + \log(e + |x|)] \geq 2(e + t) > 2t.\]
Thus $s \mapsto s^t g(|x|, s)$ is increasing for $\epsilon = 1/2$, which implies that
\[
\int_0^t g(|x|, s)ds = \int_0^t s^t s^t g(|x|, s)ds \leq \frac{1}{1 - \epsilon} \Psi(x, t) = 2\Psi(x, t)
\]
and this completes the proof of the lemma. \qed

\textbf{Lemma 7.} Let $x_0 \in \mathbb{R}^d$ be fixed and for $u \in S(\mathbb{R}^d)$ define $\langle \tau_{x_0}f, u \rangle := \langle f, \tau_{-x_0}u \rangle$, where $\gamma_{-x_0}u(x) := u(x - x_0)$, $x \in \mathbb{R}^d$.

Then $f \in H^{\log}(\mathbb{R}^d)$ if, and only if, $\tau_{x_0}f \in H^{\log}(\mathbb{R}^d)$.

\textbf{Proof.} Note that it suffices to prove that for any $x_0 \in \mathbb{R}^d$ and $f \in H^{\log}(\mathbb{R}^d)$ one also has that $\tau_{x_0}f \in H^{\log}(\mathbb{R}^d)$.

Towards this aim, fix an $x_0 \in \mathbb{R}^d$ and an $f \in H^{\log}(\mathbb{R}^d)$. Observe that, by using a change of variables and the translation invariance of $M_\phi$, we may write
\[
I := \int_{\mathbb{R}^d} \frac{M_\phi(\tau_{x_0}f)(x)}{\log(e + |x|) + \log(e + M_\phi(\tau_{x_0}f)(x))} dx
\]
as
\[
I = \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx.
\]
To prove that $I < \infty$, we split
\[
I = I_1 + I_2,
\]
where
\[
I_1 := \int_{|x| > 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx
\]
and
\[
I_2 := \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x - x_0|) + \log(e + M_\phi(f)(x))} dx.
\]
To show that $I_1 < \infty$, observe that for $|x| > 4|x_0|$ one has
\[
\frac{4|x - x_0|}{5} < |x| < \frac{4|x - x_0|}{3}
\]
and so,
\[
I_1 \leq \int_{|x| > 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx \\
\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx.
\]
Since $f \in H^{\log}(\mathbb{R}^d)$, the last integral is finite and we thus deduce that $I_1 < \infty$. Next, to show that $I_2 < \infty$, we have
\[
I_2 \leq \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{1 + \log(e + M_\phi(f)(x))} dx \\
\leq \int_{|x| \leq 4|x_0|} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx \\
\leq \int_{\mathbb{R}^d} \frac{M_\phi(f)(x)}{\log(e + |x|) + \log(e + M_\phi(f)(x))} dx
\]
and so, $I_2 < \infty$, as $f \in H^{\log}(\mathbb{R}^d)$. Therefore, $I < \infty$ and it thus follows that $\tau_{x_0}f \in H^{\log}(\mathbb{R}^d)$. \qed
To obtain the desired variant of Zygmund’s theorem, we shall use the fact that functions in $H^{log}(\mathbb{R}^d)$ have mean zero; see Lemma 1.4 in [3]. One can actually establish the following more general fact.

**Lemma 8.** If $f \in H^{log}(\mathbb{R}^d)$ is a compactly supported distribution, then $\tilde{f}(0) = 0$.

**Proof.** Let $f$ be a compactly supported distribution in $H^{log}(\mathbb{R}^d)$. In light of Lemma 7 we may assume, without loss of generality, that $f$ is supported in a closed ball $B_r$ centered at 0 with radius $r > 0$, i.e. $\text{supp}(f) \subseteq B_r := \{ x \in \mathbb{R}^d : |x| < r \}$.

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be supported inside $B_r$ and be equal to 1 on the support of $f$.

To prove the lemma, take an $x \in \mathbb{R}^d$ with $|x| > 2r$ and observe that, by the definition of $\phi$, if we take $\epsilon = 4|x|$ we have that

$$|f * \phi(x)| = \frac{1}{\epsilon^d} |\langle f, \chi\phi(\epsilon^{-1}(\cdot - x)) \rangle| \gtrsim \frac{1}{|x|^d} |\langle f, \chi \rangle|$$

as we then have $\phi(\epsilon^{-1}(x - y)) = c_d$ for $y \in B_r$. Notice that $|\langle f, 1 - \chi \rangle| = 0$. Therefore, for all $|x| > 2r$ and $\epsilon = 4|x|$, we have

$$M_\phi(f)(x) \gtrsim \frac{1}{|x|^d} |\langle f, 1 \rangle|,$$

and so, we deduce from Lemma 3 that

$$\Psi(x, M_\phi(f)(x)) \gtrsim \frac{1}{|x|^d \log(\epsilon + |x|)} |\tilde{f}(0)|$$

for $|x|$ large enough.

Hence, if $\tilde{f}(0) \neq 0$, then the function $\Psi(x, M_\phi(f)(x))$ does not belong to $L^1(\mathbb{R}^d)$, which is a contradiction. 

We are now ready to prove Theorem 3.

**Proof of Theorem 3.** Let $B$ denote the closed unit ball in $\mathbb{R}^d$. Fix a function $f$ with $\text{supp}(f) \subseteq B$, $\int_B f(y)dy = 0$ and $f \in L \log \log L(B)$. First of all, observe that

$$M_\phi(f)(x) \lesssim M(f)(x) \quad \text{for all } x \in \mathbb{R}^d,$$

where $M(f)$ denotes the Hardy-Littlewood maximal function of $f$; see e.g. Theorem 2 on pp. 62–63 in [26]. We thus deduce from Lemma 3 that

$$\Psi(x, M_\phi(f)(x)) \lesssim \Psi(x, M(f)(x)) \quad \text{for all } x \in \mathbb{R}^d$$

and hence, by using Theorem 2 we obtain

$$\int_{2B} \Psi(x, M_\phi(f)(x))dx \lesssim 1 + \int_{2B} |f(x)| \log^+ \log^+ |f(x)|dx,$$

where $2B := \{ x \in \mathbb{R}^d : |x| \leq 2 \}$.

To estimate the integral of $\Psi(x, M_\phi(f)(x))$ for $x \in \mathbb{R}^d \setminus 2B$, we shall make use of the cancellation of $f$. To be more specific, observe that if $|x| > 2$ then for every $\epsilon < |x|/2$, one has that

$$f * \phi \epsilon(x) = \frac{1}{\epsilon^d} \int_B f(y) \phi \left( \frac{x - y}{\epsilon} \right)dy = 0$$

since $|x - y|/\epsilon > 1$ whenever $y \in B$. Therefore, we may restrict ourselves to $\epsilon \geq |x|/2$ when $|x| > 2$. Hence, for $\epsilon \geq |x|/2$, by exploiting the cancellation of $f$ and using a
Lipschitz estimate on \( \phi_\epsilon \), we obtain
\[
|f \ast \phi_\epsilon(x)| = \frac{1}{e^\epsilon} \left| \int_B f(y) \phi_\epsilon \left( \frac{x-y}{e} \right) dy \right| = \frac{1}{e^\epsilon} \left| \int_B f(y) \left[ \phi_\epsilon \left( \frac{x-y}{e} \right) - \phi_\epsilon \left( \frac{x}{e} \right) \right] dy \right| \\
\lesssim_{\phi} \frac{1}{e^{d+1}} \int_B |y \cdot f(y)| dy \lesssim \frac{1}{|x|^{d+1}} \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right].
\]

We thus deduce that, for every \( x \in \mathbb{R}^d \setminus 2B \),
\[
|M_\phi(f)(x)| \lesssim \frac{1}{|x|^{d+1}} \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right]
\]
and so,
\[
\int_{\mathbb{R}^d \setminus 2B} \Psi(x, M_\phi(x)) dx \leq \left[ 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy \right] \int_{\mathbb{R}^d \setminus 2B} \frac{1}{|x|^{d+1}} \log(e + |x|) dx \leq 1 + \int_B |f(y)| \log^+ \log^+ |f(y)| dy,
\]
as desired. Therefore, Theorem 3 is now established by using the last estimate combined with (3.1).

3.1. A partial converse. As in the classical setting of the real Hardy space \( H^1 \), see [25], Theorem 3 has a partial converse. To be more precise, if a function \( f \) is positive on an open set \( U \) and \( f \) belongs to \( H^{\log}(\mathbb{R}^d) \), then the function \( \lambda \in L \log \log L(K) \) for every compact set \( K \subset U \).

Indeed, to see this, note that if \( f \) is as above then
\[
M_\phi(f)(x) \gtrsim M(f \cdot \eta_K)(x) \quad \text{for all } x \in K,
\]
where \( \eta_K \) is an appropriate Schwartz function with \( \eta_K \sim 1 \) on \( K \); see e.g. Section 5.3 in Chapter III in [27]. Hence, by using Lemma 6 and Theorem 2, we get
\[
\int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x)) dx \geq \int_{\mathbb{R}^d} \Psi(x, M_\phi(f)(x)) dx \gtrsim \int_K \Psi(x, M_\phi(f)(x)) dx \gtrsim 1 + \int_K |f(x)| \log^+ \log^+ |f(x)| dx.
\]

4. Dyadic \( H^{\log} \) on \( T \)

In this section we introduce a dyadic variant of \( H^{\log}(\mathbb{T}) \) and in the next section we shall prove that it admits a characterisation in terms of atomic decompositions. Here, we adopt the convention that \( \mathbb{T} \cong \mathbb{R}/2\pi \mathbb{Z} \).

Before we proceed, let us recall that, following [3], \( H^{\log}(\mathbb{T}) \) is defined as the class of all \( f \in \mathcal{D}' \) whose non-tangential maximal function \( f^{\ast} \) satisfies
\[
\int_0^{2\pi} \Psi_0(|f^{\ast}(t)|) dt < \infty,
\]
where \( \Psi_0 \) is as above; \( \Psi_0(t) := t \cdot |\log(e + t)|^{-1} \), \( t \geq 0 \). Here, \( \mathcal{D}' \) denotes the class of all distributions on \( \mathbb{T} \). For \( f \in H^{\log}(\mathbb{T}) \), one sets
\[
\|f\|_{H^{\log}(\mathbb{T})} := \inf \left\{ \lambda > 0 : (2\pi)^{-1} \int_0^{2\pi} \Psi_0(\lambda^{-1}|f^{\ast}(t)|) dt \leq 1 \right\}.
\]
In what follows, for \( f \in \mathcal{D}' \) and \( n \in \mathbb{Z} \), we write \( \hat{f}(n) := (f,e_n) \) where \( e_n(\theta) := e^{in\theta}, \ \theta \in [0,2\pi) \).

**Remark 9.** Recall that for \( p > 0 \), the real Hardy space \( H^p(\mathbb{T}) \) is defined as the class of all \( f \in \mathcal{D}' \) such that \( f^* \in L^p(\mathbb{T}) \).

It is well-known that for \( p \geq 1 \), elements in \( H^p(\mathbb{T}) \) are functions and moreover, \( H^p(\mathbb{T}) \equiv L^p(\mathbb{T}) \) for \( p > 1 \). Clearly,

\[
H^1(\mathbb{T}) \subset H^{log}(\mathbb{T}) \subset H^p(\mathbb{T}) \quad \text{for all } 0 < p < 1.
\]

### 4.1. Definition of dyadic \( H^{log} \) on \( \mathbb{T} \)

Let \( \mathcal{I} \) be a given system of dyadic arcs in \( \mathbb{T} \). In particular, following the formulation of [18], we assume that \( \mathcal{I} \) is of the form

\[
\mathcal{I}^k := \{ [2\pi(2^{-k}m + \delta), 2\pi(2^{-k}(m + 1) + \delta)) : k \in \mathbb{N}_0, m = 0, \ldots, 2^k - 1 \}
\]

for some \( \delta \in (0,1) \).

If \( I \in \mathcal{I} \), then \( h_I \) denotes the cancellative Haar function associated to \( I \) that is,

\[
h_I := |I|^{-1/2}(\chi_L - \chi_R),
\]

where \( L_- \) and \( L_+ \) denote the left and right halves of \( I \), respectively.

Here, we shall adopt the following convention: if \( f = \{f_I\}_{I \in \mathcal{I}} \cup \{ f_0 \} \) is a collection of complex numbers, we consider the associated sequence of functions \( (f_N)_{N \in \mathbb{N}} \) given by

\[
f_N(\theta) := f_0 + \sum_{|I| \geq 2^{-N}} f_I h_I(\theta), \quad \theta \in \mathbb{T}.
\]

As usual, the dyadic square function \( S_\mathcal{I}[f_N] \) of \( f_N \) is given by

\[
S_\mathcal{I}[f_N](\theta) := |f_0| + \left( \sum_{|I| \geq 2^{-N}} |f_I|^2 \chi_I(\theta) / |I| \right)^{1/2}, \quad \theta \in \mathbb{T}.
\]

One then defines the dyadic square function \( S_\mathcal{I}[f] \) of \( f \) as the pointwise limit

\[
S_\mathcal{I}[f] := \lim_{N \to \infty} S_\mathcal{I}[f_N] = |f_0| + \left( \sum_{I \in \mathcal{I}} |f_I|^2 \chi_I / |I| \right)^{1/2}.
\]

**Definition.** We define \( h^{log}_2(\mathbb{T}) \) as the class of all collections \( f = \{ f_I \}_{I \in \mathcal{I}} \cup \{ f_0 \} \) of complex numbers satisfying

\[
\int_{\mathbb{T}} \Psi_0(S_\mathcal{I}[f](\theta)) d\theta < \infty.
\]

If \( f \in h^{log}_2(\mathbb{T}) \), we set

\[
\| f \|_{h^{log}_2(\mathbb{T})} := \inf \left\{ \lambda > 0 : (2\pi)^{-1} \int_{\mathbb{T}} \Psi_0(\lambda^{-1} S_\mathcal{I}[f](\theta)) d\theta \leq 1 \right\}.
\]

**4.1.1. Some remarks.** It can easily be seen that there exists an absolute constant \( C_0 > 1 \) such that for all \( f, g \in h^{log}_2(\mathbb{T}) \) and \( \mu \in \mathbb{C} \) one has

\[
\| f + g \|_{h^{log}_2(\mathbb{T})} \leq C_0(\| f \|_{h^{log}_2(\mathbb{T})} + \| g \|_{h^{log}_2(\mathbb{T})})
\]

and

\[
\| \mu f \|_{h^{log}_2(\mathbb{T})} = |\mu| \| f \|_{h^{log}_2(\mathbb{T})}.
\]

Moreover, \( f = 0 \) if, and only if, \( \| f \|_{h^{log}_2(\mathbb{T})} = 0 \). In particular, \( \| \cdot \|_{h^{log}_2(\mathbb{T})} \) is a quasi-norm on the linear space \( h^{log}_2(\mathbb{T}) \) and one can show that \( (h^{log}_2(\mathbb{T}), \| \cdot \|_{h^{log}_2(\mathbb{T})}) \) is complete.

Let \( \mathcal{F}_\mathcal{I}(\mathbb{T}) \) denote the class of all functions \( f \in L^1(\mathbb{T}) \) such that the collection \( \{(f,h_I)\}_{I \in \mathcal{I}} \cup \{ f(0) \} \) consists of finitely many non-zero terms. Note that if
$f \in F_T(\mathbb{T})$ one can write $f = \tilde{f}(0) + \sum_{I \in \mathcal{I}} (f, h_I) h_I$ and moreover, by identifying functions in $F_T(\mathbb{T})$ with the corresponding collections of their Haar coefficients, we may regard $F_T(\mathbb{T})$ as a dense subspace of $h_T^0(\mathbb{T})$.

4.2. $h_T^0(\mathbb{T})$ and $H_T^0(\mathbb{T})$. Our goal in this section is to show that every collection in $h_T^0(\mathbb{T})$ can be regarded in a ‘canonical’ way as an element of $\mathcal{D}'$. More specifically, we shall prove that if $f \in h_T^0(\mathbb{T})$, then the corresponding sequence of functions $(f_N)_{N \in \mathbb{N}}$ in $F_T(\mathbb{T})$ converges in the sense of distributions to some $f \in \mathcal{D}'$. If $\nu$ denotes the associated map from $h_T(\mathbb{T})$ to $\mathcal{D}'$, then one defines $H_T^0(\mathbb{T}) := \nu[h_T^0(\mathbb{T})]$ and $\|f\|_{H_T^0(\mathbb{T})} := \|f\|_{h_T^0(\mathbb{T})}$ for $f \in H_T^0(\mathbb{T})$ with $f = \nu(f)$.

To this end, let $f = \{f_I\}_{I \in \mathcal{I}} \cup \{f_0\}$ be a given collection in $H_T^0(\mathbb{T})$. It can easily be seen that for every $p \in (1/2, 1)$ one has

$$\psi_0(t) \geq (1-p)t^p \quad \text{for all } t \geq 0.\tag{4.1}$$

Hence, by using (4.1), we deduce that

$$\|S_2[f]\|_{L^p(\mathbb{T})} \leq D(p, f),\tag{4.2}$$

where $D(p, f)$ is a (finite) positive constant given by

$$D(p, f) := (p-1)^{-1/p} \left( \int_\mathbb{T} \psi_0(S_2[f](\theta))d\theta \right)^{1/p}.\tag{4.3}$$

Since

$$S_2[f] \geq \left( \sum_{I \in \mathcal{I}: |I| = 2^{-k}} |f_I|^2 \frac{\log N}{|I|} \right)^{1/2} \quad \text{for all } k \in \mathbb{N}_0,$$

we deduce from (4.2) that

$$D(p, f) \geq \left( \int_\mathbb{T} \left( \sum_{I \in \mathcal{I}: |I| = 2^{-k}} |f_I|^2 \frac{\log N}{|I|} \right)^{p/2} d\theta \right)^{1/p} \quad \text{for all } k \in \mathbb{N}_0,$$

We thus have

$$D(p, f) \geq \left( \sum_{I \in \mathcal{I}: |I| = 2^{-k}} |f_I|^p |I|^{1-p/2} \right)^{1/p} \quad \text{for all } k \in \mathbb{N}_0,\tag{4.4}$$

where we used the fact that the arcs $I \in \mathcal{I}$ with $|I| = 2^{-k}$ are mutually disjoint. We shall combine (4.3) with the following standard estimate.

**Lemma 10.** There exists an absolute constant $C_0 > 0$ such that

$$|\{\phi, h_I\}| \leq C_0 \|\phi\|_{L^\infty(\mathbb{T})} |I|^{3/2}$$

for all $\phi \in C^1(\mathbb{T})$ and for all $I \in \mathcal{I}$.

**Proof.** We may assume without loss of generality that $I \subset \mathbb{T}$ can be regarded as an interval in $[0, 2\pi)$. By using a change of variables and the mean value theorem, we have

$$|\{\phi, h_I\}| = |I|^{-1/2} \int_{I_+} \phi(\theta)d\theta - \int_{I_-} \phi(\theta)d\theta$$

$$= |I|^{-1/2} \int_{I_+} [\phi(\theta) - \phi(\theta + |I|/2)]d\theta$$

$$\leq \|\phi\|_{L^\infty(\mathbb{T})} |I|^{3/2},$$

as desired. \qed
As mentioned above, we shall prove that the sequence of functions \((f_N)_{N \in \mathbb{N}}\) associated to \(f\) converges in the sense of distributions. Towards this aim, by using Lemma \([10]\) for \(N, M \in \mathbb{N}\) with \(N > M\), we have

\[
|\langle f_N - f_M, \phi \rangle| = \left| \sum_{k \leq M} f_I \langle h_I, \phi \rangle \right| \leq \sum_{k=M+1}^{N} \left| \sum_{I \in \mathcal{I}; |I| = 2^{-k}} |f_I||\langle \phi, h_I \rangle| \right| \\
\lesssim \|\phi\|_{L^\infty(T)} \sum_{k=M+1}^{N} \sum_{|I| = 2^{-k}} |f_I||I|^{3/2}. 
\]

Fix a \(p \in (1/2, 1)\) and note that the previous estimate implies that

\[
|\langle f_N - f_M, \phi \rangle| \lesssim \|\phi\|_{L^\infty(T)} \sum_{k=M+1}^{N} \left( \sum_{|I| = 2^{-k}} |f_I|^p |I|^{3p/2} \right)^{1/p}.
\]

Since

\[
\sum_{k=M+1}^{N} \left( \sum_{|I| = 2^{-k}} |f_I|^p |I|^{3p/2} \right)^{1/p} = \sum_{k=M+1}^{N} \left( \sum_{|I| = 2^{-k}} |f_I|^p |I|^{1-p/2} |I|^{2p-1} \right)^{1/p} = \sum_{k=M+1}^{N} 2^{-(2-1/p)k} \left( \sum_{|I| = 2^{-k}} |f_I|^p |I|^{1-p/2} \right)^{1/p},
\]

it follows from \([14]\) that

\[
\sum_{k=M+1}^{N} \left( \sum_{|I| = 2^{-k}} |f_I|^p |I|^{3p/2} \right)^{1/p} \lesssim \left( \sum_{k=M+1}^{N} 2^{-(2-1/p)k} \right) D(p, f).
\]

Therefore, \((1.5)\) and \((1.6)\) imply that \((f_N)_{N \in \mathbb{N}}\) is Cauchy in \(\mathcal{D}'\) and so, it converges to some \(f \in \mathcal{D}'\). A completely analogous argument shows that

\[
\sup_{N \in \mathbb{N}} |\langle f_N, \phi \rangle| \leq |f_0|\|\phi\|_{L^\infty(T)} + c_p D(p, f)|\phi'|_{L^\infty(T)} \quad \text{for all } \phi \in C^1(T),
\]

where \(c_p > 0\) is a constant that depends only on \(p\). Hence,

\[
|\langle f, \phi \rangle| \leq |f_0|\|\phi\|_{L^\infty(T)} + c_p D(p, f)|\phi'|_{L^\infty(T)} \quad \text{for all } \phi \in C^1(T).
\]

5. Atomic decomposition of \(H^{10^6}(\mathbb{T})\)

5.1. Atomic decomposition of \(H^{10^6}(\mathbb{T})\). In \([15]\), Ky showed that \(H^{10^6}(\mathbb{R}^d)\) admits a decomposition in terms of \(H^{10^6}(\mathbb{R}^d)\)-atoms, see also Chapter 1 in \([33]\). An adaptation of Ky’s argument to the periodic setting establishes an analogous result in the periodic setting.

In order to state the characterisation of \(H^{10^6}(\mathbb{T})\) in terms of atomic decompositions, we need the following definitions.

**Definition** \((13)\). Let \(I \subset \mathbb{T}\) be an arc. A measurable function \(a_I\) on \(\mathbb{T}\) is said to be an \(H^{10^6}(\mathbb{T})\)-atom associated to \(I\) whenever

- \(\text{supp}(a_I) \subseteq I\),
- \(\int a_I(\theta)d\theta = 0\), and
- \(\|a_I\|_{L^\infty(\mathbb{T})} \leq \|x_I\|_{L^\infty(\mathbb{T})}\).

Following \((13, 33)\), see also \([29]\), we give the definition of atomic \(H^{10^6}(\mathbb{T})\).
Definition. The atomic Hardy-Orlicz space $H_{\text{at},T}^{\log}(T)$ is defined as the space of all $f \in \mathcal{D}'$ that can be written as

$$f - \hat{f}(0) = \sum_{k \in \mathbb{N}} b_{I_k} \text{ in } \mathcal{D}',$$

where $b_{I_k} = \mu_k a_{I_k}$ with $a_{I_k}$ being an atom in $H^{\log}(T)$ associated to some arc $I_k$ and $\{\mu_k\}_{k \in \mathbb{N}}$ is a collection of complex scalars such that

$$\sum_{k \in \mathbb{N}} |I_k| \Psi_0(\|b_{I_k}\|_{L^\infty(T)}) < \infty.$$

If $\{b_{I_k}\}_{k \in \mathbb{N}}$ is as above, let

$$\Lambda_\infty(f, \{b_{I_k}\}_{k \in \mathbb{N}}) := \inf \left\{ \lambda > 0 : \Psi_0(\lambda^{-1} \|\hat{f}(0)\|) + \sum_{k \in \mathbb{N}} |I_k| \Psi_0(\lambda^{-1} \|b_{I_k}\|_{L^\infty(T)}) \leq 1 \right\}$$

and define

$$\|f\|_{H_{\text{at},T}^{\log}(T)} := \inf \left\{ \Lambda_\infty(f, \{b_{I_k}\}_{k \in \mathbb{N}}) : f - \hat{f}(0) = \sum_{k \in \mathbb{N}} b_{I_k} \text{ with } \{b_{I_k}\}_{k \in \mathbb{N}} \text{ being as above} \right\}.$$

By arguing as in [15] one can show that

$$H_{\text{at}}^{\log}(T) \cong H^{\log}(T).$$

5.2. Atomic decomposition of $H^{\log}_T(T)$.

Definition. Let $I$ be an arc in $\mathcal{I}$. A measurable function $a_I$ on $T$ is said to be an $H^{\log}_T$-atom associated to $I \in \mathcal{I}$ whenever

- $\text{supp}(a_I) \subset I$,
- $\int_T a_I(\theta)d\theta = 0$, and
- $\|a_I\|_{L^\infty(T)} \leq \|\chi_I\|_{L^\infty(T)}$.

In analogy with the non-dyadic case, define $H^{\log}_{\text{at},T}(T)$ to be the class of all $f \in \mathcal{D}'$ for which there exists a sequence $\{\beta_{I_k}\}_{k \in \mathbb{N}}$ of scalar multiples of $H^{\log}_T$-atoms such that

$$f - \hat{f}(0) = \sum_{k \in \mathbb{N}} \beta_{I_k} \text{ in } \mathcal{D}'$$

and

$$\sum_{k \in \mathbb{N}} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^\infty(T)}) < \infty.$$

For $f \in H^{\log}_{\text{at},T}(T)$, we set

$$\|f\|_{H^{\log}_{\text{at},T}(T)} := \inf \left\{ \Lambda_\infty(f, \{\beta_{I_k}\}_{k \in \mathbb{N}}) : f - \hat{f}(0) = \sum_{k \in \mathbb{N}} \beta_{I_k} \text{ in } \mathcal{D}' \right\},$$

where

$$\Lambda_\infty(f, \{\beta_{I_k}\}_{k \in \mathbb{N}}) := \inf \left\{ \lambda > 0 : \Psi_0(\lambda^{-1} \|\hat{f}(0)\|) + \sum_{k \in \mathbb{N}} |I_k| \Psi_0(\lambda^{-1} \|\beta_{I_k}\|_{L^\infty(T)}) \leq 1 \right\}.$$

We shall prove that $H^{\log}_T(T)$ is contained in $H^{\log}_{\text{at},T}(T)$. To this end, we shall show first that every function $f \in F_T(T)$ admits a decomposition in terms of $H^{\log}_T$-atoms.

Proposition 11. For every $f \in F_T(T)$ there exists a finite collection of multiples of $H^{\log}_T$-atoms $\{\beta_{I_k}\}_{k=1}^N$ such that:

- $f - \hat{f}(0) = \sum_{k=1}^N \beta_{I_k}$ and
• \( \Lambda_\infty(f, [\beta_k]_{k=1}^N) \leq C_0 \|f\|_{H^{2N}_{\infty}(\mathbb{T})} \),
where \( C_0 > 0 \) is an absolute constant that is independent of \( f \).

**Proof.** The proof is a variant of the non-dyadic case presented in Chapter 1 of [33].

Fix an \( f \in F_\mathcal{T}(\mathbb{T}) \). Without loss of generality, we may assume that \( \tilde{f}(0) = 0 \).

Notice that, since \( \Psi_0 \) satisfies the conditions of [33], Section 3], one has

\[
(5.2) \quad c_1 \int_{\mathbb{T}} \Psi_0(M_{\mathcal{T}}[f](\theta))d\theta \leq \int_{\mathbb{T}} \Psi_0(S_{\mathcal{T}}[f](\theta))d\theta \leq c_2 \int_{\mathbb{T}} \Psi_0(M_{\mathcal{T}}[f](\theta))d\theta
\]

where \( c_1, c_2 > 0 \) are absolute constants. Here, \( M_{\mathcal{T}}[f] \) is the dyadic maximal function of \( f \), namely

\[
M_{\mathcal{T}}[f](\theta) := \sup_{N \in \mathbb{N}_0} |\mathbb{E}_{\mathcal{T}, N}[f](\theta)|,
\]

where

\[
\mathbb{E}_{\mathcal{T}, N}[f] := \sum_{I \in \mathcal{N}, |I| = 2^{-N}} (f)_{I} \chi_{I}.
\]

Notice that, as \( f \in F_\mathcal{T}(\mathbb{T}) \), there exists an \( N_0 \in \mathbb{N} \) such that

\[
(5.3) \quad \mathbb{E}_{\mathcal{T}, N}[f] \equiv f \quad \text{for all } N \geq N_0.
\]

For \( \lambda > 0 \), by arguing as on pp. 33–34 in [33], one can show that the set

\[
\Omega_\lambda := \{ \theta \in T : M_{\mathcal{T}}[f](\theta) > \lambda \}
\]

can be written as a finite union of mutually disjoint arcs in \( \mathcal{T} \). We may thus write \( \Omega_\lambda = \bigcup_k I(\lambda, k) \), where the union is finite and the arcs \( \{ I(\lambda, k) \}_{k} \) are mutually disjoint and in \( \mathcal{T} \). We then define

\[
g_\lambda(\theta) := \begin{cases} f(\theta) & \text{if } x \in \mathbb{T} \setminus \Omega_\lambda, \\ (f)_I & \text{if } \theta \in I(\lambda, k) \end{cases}
\]

and \( b_\lambda := f - g_\lambda = \sum_k b_{\lambda,k} \), where \( b_{\lambda,k} := \lambda I(\lambda, k) b \). By using [33], one can easily check that

\[
(5.4) \quad \|g_\lambda\|_{L^\infty(\mathbb{T})} \leq \lambda.
\]

Note that for \( N \in \mathbb{N} \) large enough, one has \( \Omega_{2^n} = \emptyset \) for all \( n \geq N \) and so, \( g_{2^n} \equiv f \) for all \( n \geq N \). We thus have

\[
f = \sum_{n=0}^{N-1} (g_{2^{n+1}} - g_{2^n}) = \sum_{n=0}^{N-1} (b_{2^n} - b_{2^{n+1}}).
\]

We write \( \Omega_{2^n} = \bigcup_k I(2^n, k) \) and define \( \beta_{n,k} := (b_{2^n} - b_{2^{n+1}}) \chi_{I(2^n, k)} \). It can easily be seen that \( \int_{I(2^n, k)} \beta_{n,k}(x')dx' = 0 \). Moreover, it follows from [5.4] that

\[
\|\beta_{n,k}\|_{L^\infty(\mathbb{T})} \leq \|g_{2^n}\|_{L^\infty(\mathbb{T})} + \|g_{2^{n+1}}\|_{L^\infty(\mathbb{T})} \leq 3 \cdot 2^n.
\]

Hence, \( \beta_{n,k} \) are multiples of \( H^1_{\mathcal{T}} \)-atoms and moreover,

\[
\sum_{n=0}^{N-1} \sum_{k} |I(2^n, k)| \Psi_0(|\beta_{n,k}|_{L^\infty(\mathbb{T})}/\lambda) \leq \sum_{n=0}^{N-1} \sum_{k} |I(2^n, k)| \Psi_0(2^n/\lambda)
\]

\[
= \sum_{n=0}^{N-1} \Psi_0(2^n/\lambda) \sum_{k} |I(2^n, k)|
\]

\[
= \sum_{n=0}^{N-1} \Psi_0(2^n/\lambda) \sum_{l \in \mathbb{N}_0} |\{ 2^{n+l} < M_{\mathcal{T}}[f] \leq 2^{n+l+1} \}|.
\]
Note that there exists a $c_0 > 0$ such that $\Psi_0(2^{-l}t) \leq 2^{-cd_0}t$ for all $t \geq 0$ and $l \in \mathbb{N}_0$. Hence,

$$\sum_{n=0}^{N-1} \Psi_0(2^n/\lambda) \sum_{l \in \mathbb{N}_0} \{ \{2^n < M_T[f] \leq 2^{n+l+1}\} \}$$

$$= \sum_{l \in \mathbb{N}_0} \sum_{n=0}^{N-1} \Psi_0(2^n/\lambda) \{ \{2^n < M_T[f] \leq 2^{n+l+1}\} \}$$

$$\leq \sum_{l \in \mathbb{N}_0} \sum_{m \geq l} \Psi_0(2^m/\lambda) \{ \{2^m < M_T[f] \leq 2^{m+1}\} \}$$

$$\leq \sum_{l \in \mathbb{N}_0} \sum_{m \geq l} 2^{-cd_0} \Psi_0(2^m/\lambda) \{ \{2^m < M_T[f] \leq 2^{m+1}\} \} \lesssim \int_T \Psi_0(\lambda^{-1} M_T[f](\theta)) d\theta,$$

which combined with (5.22), completes the proof of the proposition. 

By arguing as on p. 109 in [27], the density of $F_T(\mathbb{T})$ in $H_T^\log(\mathbb{T})$ and Proposition 11 imply the following result.

**Proposition 12.** One has $H_T^\log(\mathbb{T}) \subseteq H_{at,T}^\log(\mathbb{T})$.

### 5.3. Proof of the reverse inclusion

The main result of this section is that the converse of Proposition 12 also holds.

**Proposition 13.** One has $H_{at,T}^\log(\mathbb{T}) \subseteq H_T^\log(\mathbb{T})$.

In order to prove Proposition 13 we shall establish first the following dyadic variant of [33, Lemma 1.3.5].

**Lemma 14.** Let $I \in \mathcal{I}$ be a given arc. For every $L^\infty$-function $\beta_I$ supported in $I$ one has

$$\int_I \Psi_0(S_T[\beta_I](\theta)) d\theta \leq (1 + (2\pi)^{1/2} |I|) \Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})}).$$

**Proof.** We argue as in the proof of [33, Lemma 1.3.5]. More specifically, since $\Psi$ is increasing and $\Psi_0(ct) \leq c\Psi_0(t)$ for all $c \geq 1$ and $t \geq 0$, we have

$$\int_I \Psi_0(S_T[\beta_I](\theta)) d\theta = \int_I \Psi_0\left(\frac{S_T[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\right) d\theta$$

$$\leq \int_I \Psi_0\left(1 + \frac{S_T[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\right) d\theta$$

$$\leq \Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})}) \int_I \left(1 + \frac{S_T[\beta_I](\theta)}{\|\beta_I\|_{L^\infty(\mathbb{T})}}\right) d\theta$$

$$= \Psi_0(\|\beta_I\|_{L^\infty(\mathbb{T})}) \left(|I| + \|\beta_I\|_{L^\infty(\mathbb{T})}^{-1} \int_I S_T[\beta_I](\theta) d\theta\right).$$

To complete the proof of the lemma, observe that by using the Cauchy-Schwarz inequality and the fact that $S_T$ is an isometry on $L^2(\mathbb{T})$, one has

$$\int_I S_T[\beta_I](\theta) d\theta \leq (2\pi |I|)^{1/2} \|S_T[\beta_I]\|_{L^2(\mathbb{T})} = (2\pi |I|)^{1/2} \|\beta_I\|_{L^2(\mathbb{T})}$$

$$\leq (2\pi)^{1/2} |I| \|\beta_I\|_{L^\infty(\mathbb{T})},$$

as desired. \qed
5.4. Proof of Proposition 15. Let \( f \) be a given distribution in \( H^{\log}_{\text{at}, T}(T) \). Without loss of generality, we may assume that \( \tilde{f}(0) = 0 \).

By definition, there exists a sequence of multiples of \( H^{\log}_{\text{at}, T} \)-atoms \( \{\beta_{I_k}\}_{k \in \mathbb{N}} \) such that

\[
f = \lim_{N \to \infty} \sum_{k=1}^{N} \beta_{I_k} \quad \text{in } \mathcal{D}' \quad \text{and} \quad \sum_{k=1}^{\infty} |\beta_{I_k}|_{L^{\infty}(T)} < \infty.
\]

For \( N \in \mathbb{N} \), we set \( b_N := \sum_{k=1}^{N} \beta_{I_k} \). Note that since \( \Psi_0 \in L^{\infty}(T) \) one has \( b_N = \sum_{I \in \mathcal{I}} (b_{N,I}) h_I \) a.e. on \( T \) and in \( L^2(T) \).

In what follows, we shall use several times the fact that

\[
\Psi_0 \left( \sum_{j=1}^{L} t_j \right) \leq \sum_{j=1}^{L} \Psi_0(t_j)
\]

for any finite collection of non-negative numbers \( \{t_j\}_{j=1}^{L} \); see the proof of Lemma 1.1.6 (i).

**Lemma 15.** Let \( I \in \mathcal{I} \) be given. If \( \{b_{N,I}\}_{N \in \mathbb{N}} \) as above, then the sequence of complex numbers \( \{b_{N,I}\}_{N \in \mathbb{N}} \) converges.

**Proof.** It follows from Lemma 14 and (5.5) that for \( N > M \) one has

\[
\int_{T} \Psi_0(S_{I}[b_N - b_M](\theta)) d\theta \leq \sum_{k=M+1}^{N} \int_{T} \Psi_0(S_{I}[\beta_{I_k}](\theta)) d\theta
\]

\[
\leq (1 + (2\pi)^{1/2}) \sum_{k=M+1}^{N} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^{\infty}(T)})
\]

and so,

\[
\lim_{M,N \to \infty} \int_{T} \Psi_0(S_{I}[b_N - b_M](\theta)) d\theta = 0.
\]

Observe that, since \( \Psi_0 \) is increasing, one has

\[
\int_{T} \Psi_0(S_{I}[b_N - b_M](\theta)) d\theta \geq \int_{T} \Psi_0(|(b_N, h_I) - (b_M, h_I)||I|^{-1/2}) d\theta
\]

\[
= |I| \Psi_0(|(b_N, h_I) - (b_M, h_I)||I|^{-1/2})
\]

for all \( I \in \mathcal{I} \). Since \( \Psi_0 \) is continuous and \( \Psi_0(t) = 0 \) if, and only if, \( t = 0 \), we deduce from the previous inequality and (5.6) that

\[
\lim_{M,N \to \infty} |(b_N, h_I) - (b_M, h_I)| = 0.
\]

Hence, \( \{b_{N,I}\}_{N \in \mathbb{N}} \) is Cauchy in \( C \) and so, it converges. \( \square \)

In view of Lemma 15, we may define \( b := \{b_I\}_{I \in \mathcal{I}} \) with \( b_I := \lim_{N \to \infty} \langle b_N, h_I \rangle \).

We claim that \( b \in H^{\log}_{\text{at}, T}(T) \) with

\[
\int_{T} \Psi_0(S_{I}[b](\theta)) d\theta \leq (1 + (2\pi)^{1/2}) \sum_{k=1}^{\infty} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^{\infty}(T)}).
\]

Indeed, by Lemma 14 and the definition of \( \{b_{N,I}\}_{N \in \mathbb{N}} \), one has

\[
\int_{T} \Psi_0(S_{I}[b](\theta)) d\theta \leq (1 + (2\pi)^{1/2}) \sum_{k=1}^{\infty} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^{\infty}(T)}) \quad \text{for all } N \in \mathbb{N}.
\]
Fix an $M \in \mathbb{N}$ and note that, by combining (5.8) with Fatou’s lemma, one gets
\[
\int_T \liminf_{N \to \infty} \Psi_0 \left( \left\{ \sum_{I \in \mathcal{I}} |b_I|^2 |I|^{-1} \chi_I(\theta) \right\}^{1/2} \right) d\theta 
\leq \liminf_{N \to \infty} \int_T \Psi_0(S_T[b_N](\theta)) d\theta 
\leq (1 + (2\pi)^{1/2}) \sum_{k=1}^{\infty} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^\infty(T)}).
\]

Since $\Psi_0$ is continuous, we deduce that
\[
\text{(5.9)} \quad \int_T \Psi_0 \left( \left\{ \sum_{I \in \mathcal{I}} |b_I|^2 |I|^{-1} \chi_I(\theta) \right\}^{1/2} \right) d\theta \leq (1 + (2\pi)^{1/2}) \sum_{k=1}^{\infty} |I_k| \Psi_0(\|\beta_{I_k}\|_{L^\infty(T)})
\]
for all $M \in \mathbb{N}$. Hence, (5.7) is obtained by using (5.9), the monotone convergence theorem, and the continuity of $\Psi_0$.

If we now define the sequence of functions $\{B_M\}_{M \in \mathbb{N}}$ with finite wavelet expansions given by
\[
B_M(\theta) := \sum_{I \in \mathcal{I}, |I| \geq 2^{-M}} b_I h_I(\theta) \quad (\theta \in T),
\]
then, as explained in Section 4.2, $B_M$ converges in $D'$ to some $b \in D'$. It thus suffices to prove that $b \equiv f$. To this end, it is enough to show that, in view of the definition of $\{b_N\}_{N \in \mathbb{N}}$, one has
\[
\text{(5.10)} \quad b = \lim_{N \to \infty} b_N \text{ in } D'.
\]

For $M, N \in \mathbb{N}$, consider the function $\delta_{M,N}$ given by
\[
\delta_{M,N}(\theta) := B_M(\theta) - (b_N)_M(\theta) \quad (\theta \in T),
\]
where $(b_N)_M$ is the ‘truncation’ of the Haar series representation of $b_N$ allowing Haar projections corresponding to arcs $I \in \mathcal{I}$ with $|I| \geq 2^{-M}$ that is,
\[
(b_N)_M(\theta) := \sum_{I \in \mathcal{I}, |I| \geq 2^{-M}} \langle b_N, h_I \rangle h_I(\theta) \quad (\theta \in T).
\]

We claim that for any fixed $\phi \in C^\infty(T)$ one has
\[
\text{(5.11)} \quad \lim_{M \to \infty} \langle \delta_{M,N}, \phi \rangle = \langle b - b_N, \phi \rangle \text{ uniformly in } N \in \mathbb{N}.
\]

Indeed, fix a $\phi \in C^\infty(T)$ and write
\[
|\langle \delta_{M,N} - (b - b_N), \phi \rangle| \leq |\langle B_M - b, \phi \rangle| + |\langle (b_N)_M - b_N, \phi \rangle|.
\]
Let $p \in (1/2, 1)$ be fixed. By arguing as in Section 4.2 one deduces that there exists an absolute constant $c_0 > 0$ such that
\[
|\langle B_M - b, \phi \rangle| \leq c_0 \|\phi\|_{L^\infty(T)} D(p, b) \sum_{k=M+1}^{\infty} 2^{-(2-1/p)k}.
\]
where $D(p, b)$ is as in Section 4.2 and is finite, in view of (5.7). Similarly, one has

\[(5.14) \quad \|((b_N)_M - b_N, \phi)\| \leq c_0 \|\phi'\|_{L^\infty(\mathbb{T})} D(p, b_N) \sum_{k=M+1}^{N} 2^{-2(-1/p)k},\]

where

\[D(p, b_N) = (p - 1)^{-1/p} \left( \int_{\mathbb{T}} \Psi_0(S_T[b_N](\theta))d\theta \right)^{1/p}.\]

By using Lemma 14 and the definition of $b_N$, one deduces that

\[D(p, b_N) \leq (p - 1)^{-1/p} \left( 1 + (2\pi)^{1/2} \sum_{k=1}^{N} I_k |\Psi_0(\|\beta_k\|_{L^\infty(\mathbb{T})})| \right)^{1/p} \leq (1 + (2\pi)^{1/2})(p - 1)^{-1/p} \left( \sum_{k=1}^{\infty} I_k |\Psi_0(\|\beta_k\|_{L^\infty(\mathbb{T})})| \right)^{1/p},\]

that is, $D(p, b_N)$ is bounded by a finite constant that is independent of $N \in \mathbb{N}$. Hence, (5.14) implies that

\[(5.15) \quad \|((b_N)_M - b_N, \phi)\| \leq c_p \|\phi'\|_{L^\infty(\mathbb{T})} \left( \sum_{k=1}^{\infty} I_k |\Psi_0(\|\beta_k\|_{L^\infty(\mathbb{T})})| \right)^{1/p} \sum_{k=M+1}^{N} 2^{-2(-1/p)k},\]

where $c_p > 0$ is a constant depending only on $p$. Therefore, by combining (5.13) with (5.15), we deduce that (5.11) holds.

Moreover, by arguing again as in Section 4.2 one shows that for any fixed $p \in (1/2, 1)$ there exists a constant $c'_p > 0$, depending only on $p$, such that

\[(5.16) \quad \|\delta_{M,N}, \phi\| \leq c'_p \|\phi'\|_{L^\infty(\mathbb{T})} \left( \int_{\mathbb{T}} \Psi_0 \left( \mathbb{E} \left[ \left\{ \sum_{l \in \mathbb{Z}} |b_l - \langle b_N, h_l \rangle|^2 |q|^{-1} \chi_{l}(\theta) \right\}^{1/2} \right] \right)^{1/p} d\theta \]

for all $M, N \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{T})$. We shall prove that the right-hand side of (5.16) tends to 0 as $N \to \infty$ for all $M \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{T})$. To this end, note that for any $L \in \mathbb{N}$ and for every collection \(\{t_j\}_{j=1}^{L}\) of non-negative numbers, one has

\[(5.17) \quad \Psi_0 \left( \mathbb{E} \left[ \sum_{j=1}^{L} t_j \right] \right)^{1/2} \leq \sum_{j=1}^{L} \Psi_0(t_j^{1/2}).\]

Indeed, (5.17) is obtained by combining (5.9) with

\[\left( \sum_{j=1}^{L} t_j \right)^{1/2} \leq \sum_{j=1}^{L} t_j^{1/2}.\]

Observe that by using (5.17) one has

\[\int_{\mathbb{T}} \Psi_0 \left( \mathbb{E} \left[ \sum_{l \in \mathbb{Z}} |b_l - \langle b_N, h_l \rangle|^2 |q|^{-1} \chi_{l}(\theta) \right] \right)^{1/2} d\theta \leq \sum_{l \in \mathbb{Z}} \int_{\mathbb{T}} \Psi_0 \left( |b_l - \langle b_N, h_l \rangle|^2 |q|^{-1} \chi_{l}(\theta) \right) d\theta = \sum_{l \in \mathbb{Z}} \int_{\mathbb{T}} |b_l - \langle b_N, h_l \rangle| |q|^{-1/2}\]
and hence, (5.16) implies that

\[
(5.18) \quad \|\delta_{M,N}, \phi\|_{L^\infty(T)} \leq c_p' \|\phi\|_{L^\infty(T)} \left( \sum_{t \in \mathbb{Z}, |t| \geq 2^m} |I| \Psi_0(|b_I - \langle b_N, h_I \rangle||I|^{-1/2}) \right)^{1/p}.
\]

Since the sum on the right-hand side of (5.18) is finite and \(\Phi\) is continuous, it follows from the definition of \(b\) that

\[
(5.19) \quad \lim_{N \to \infty} \langle \delta_{M,N}, \phi\rangle = 0 \quad \text{for all } M \in \mathbb{N}.
\]

Therefore, by combining (5.11) and (5.19), it follows that

\[
\lim_{N \to \infty} \langle b - b_N, \phi\rangle = \lim_{N \to \infty} \langle \delta_{M,N}, \phi\rangle = \lim_{M \to \infty} \lim_{N \to \infty} \langle \delta_{M,N}, \phi\rangle = 0
\]

for all \(\phi \in C^\infty(T)\). Hence, (5.10) holds and so, the proof of Proposition 13 is complete.

5.5. Concluding remarks. By combining Propositions 12 and 13 one obtains the following theorem.

**Theorem 16.** One has \(H_{log}^1(T) \cong H_{log}^1(T)\).

By using [18, Proposition 2.1] and the atomic decomposition of \(H_{log}^1(T)\); see [5.11], one shows that \(H_{log}^1(T) + H_{log}^1(J1/3(T)) \cong H_{log}^1(T)\). We thus deduce from Theorem 16 the following variant of T. Mei’s theorem 15 for \(H_{log}^1(T)\).

**Theorem 17.** One has

\[
H_{log}^1(T) + H_{log}^1(T) \approx H_{log}^1(T).
\]

**Remark 18.** Let \(I\) be a given system of dyadic arcs in \(T\). For \(p \in (0, \infty)\), define the dyadic Hardy space \(h^p_I(T)\) as the class of all collections of complex numbers \(\mathcal{F} = \{f_I\}_{I \in \mathcal{I}} \cup \{f_0\}\) such that \(S_I[f] \in L^p(T)\).

By arguing as in Sections 7 and 8, one can show that \(h^p_I(T)\) can be identified with a dyadic \(H^p\) space \(H^p_I(T)\) of distributions on \(T\) and moreover, the following extension of T. Mei’s theorem 15 holds

\[
H^p_I(T) + H^p_{J1/3}(T) \cong H^p(T) \quad \text{for all } p \in (1/2, 1].
\]

We remark that dyadic Hardy spaces for \(p < 1\) have also been considered in [31] and [21], but the definitions there are different than ours.

6. Proof of the Bonami-Grellier-Ky theorem in the periodic setting

For a function \(b \in L^1(T)\), we set

\[
\|b\|_{BMO_T^+} := \left( \sup_{I \in \mathcal{I}} \frac{1}{|I|^2} \int_I |b(\theta) - \langle b, I \rangle|^2 d\theta \right)^{1/2} + \left| \int_T b(\theta) d\theta \right|.
\]

We then define \(BMO_T^+\) as the class of all functions \(b \in L^1(T)\) such that \(\|b\|_{BMO_T^+} < \infty\). One defines \(BMO_T^+\) similarly.

Recall the following standard consequence of the John-Nirenberg type result in the dyadic case.

**Lemma 19.** There exists an absolute constant \(C_0 > 0\) such that for every function \(b \in BMO_T^+(T)\) one has

\[
\|b\|_{L^1(T)} \leq C_0 \|b\|_{BMO_T^+(T)},
\]

where \(\|b\|_{L^1(T)} := \inf \{\lambda > 0 : \int_T \psi(|b(x)|/\lambda) dx \leq 1\} \) and \(\psi(t) := e^t - t - 1, t \geq 0\).
The following variant of \cite[Proposition 2.1]{3} is obtained by combining Lemma \cite[Lemma 2.1]{3}.

**Proposition 20.** For all functions $f$, $b$ such that $f \in L^1(\mathbb{T})$ and $b \in BMO^+_T(\mathbb{T})$, one has

$$\|f \cdot b\|_{L^{1,\infty}(\mathbb{T})} \lesssim \|f\|_{L^1(\mathbb{T})} \|b\|_{BMO^+_T(\mathbb{T})}.$$ 

In this section, we present the following dyadic version of \cite[Theorem 1.1]{4}.

**Theorem 21.** There exist two bilinear operators $S$ and $T$ on the product space $H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T})$ such that

$$f \cdot b = S_T(f, b) + T_T(f, b) \text{ in } \mathcal{D}'$$

with $S_T : H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T}) \rightarrow L^1(\mathbb{T})$ and $T_T : H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T}) \rightarrow H^{\log}_T(\mathbb{T})$.

The proof of Theorem 21 that we present here is a variant of the corresponding one given by Bonami, Grelle, and Ky in \cite{4} (that establishes \cite[Theorem 1.1]{4}). To be more specific, let $f \in H^1_T(\mathbb{T})$ be a function with finite wavelet expansion. If $b$ is a function in $BMO^+_T(\mathbb{T})$ that also has a finite wavelet expansion, then we may write

$$f \cdot b = \Pi_1(f, b) + \Pi_2(f, b) + \Pi_3(f, b),$$

where

$$\Pi_1(f, b)(\theta) := \sum_{I,J \in \mathcal{I}, J \supseteq I} f_I g_J h_I(\theta) h_J(\theta),$$

$$\Pi_2(f, b)(\theta) := \sum_{I,J \in \mathcal{I}, J \supseteq I \setminus I} f_I g_J h_I(\theta) h_J(\theta),$$

and

$$\Pi_3(f, b)(\theta) := \sum_{I,J \in \mathcal{I}, I \supseteq J} f_I g_J h_I(\theta) h_J(\theta).$$

We shall prove that:

- $\Pi_1$ can be extended as a bounded bilinear operator from $H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T})$ to $H^{\log}_T(\mathbb{T})$,
- $\Pi_2$ can be extended as a bounded bilinear operator from $H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T})$ to $H^1_T(\mathbb{T})$, and
- $\Pi_3$ can be extended as a bounded bilinear operator from $H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T})$ to $L^1(\mathbb{T})$.

One can thus conclude that Theorem 21 holds by taking $S_T := \Pi_1$ and $T_T := \Pi_2 + \Pi_3$.

**Proposition 22.** The bilinear operator $\Pi_3$ extends into a bounded bilinear operator from $H^1_T(\mathbb{T}) \times BMO^+_T(\mathbb{T})$ to $L^1(\mathbb{T})$.

**Proof.** It is well-known that $H^1_T(\mathbb{T})$ admits a characterisation in terms of atoms. More specifically, recall that a measurable function $a$ is said to be an $H^1_T(\mathbb{T})$-atom if it is either the constant function or there exists an $\Omega \in \mathcal{I}$ such that supp$(a) \subseteq \Omega$, $\int_{\mathbb{T}} a(\theta) d\theta = 0$, and $\|a\|_{L^1(\mathbb{T})} \lesssim \|\Omega\|^{-1/2}$. Then, $f \in H^1_T(\mathbb{T})$ if, and only if, there exist a sequence $(\lambda_k)_k$ of non-negative scalars and a sequence $(a_k)_k$ of $H^1_T(\mathbb{T})$-atoms such that

$$f = \sum_{k \in \mathbb{N}} \lambda_k a_k.$$
in the $H^1_T$-norm and moreover, one has

$$\|f\|_{H^1_T(\mathbb{T})} \sim \inf \left\{ \sum_{k \in \mathbb{N}} |\lambda_k| : f = \sum_k \lambda_k a_k \right\}.$$  

Hence, to show that $\Pi_1$ maps $H^1_T(\mathbb{T}) \times BMO^*_T(\mathbb{T})$ to $L^1(\mathbb{T})$, it is enough to prove that there exists a constant $C > 0$ such that

$$\|\Pi_3(a, b)\|_{L^1(\mathbb{T})} \leq C\|b\|_{BMO^*_T(\mathbb{T})}$$

for all non-constant $H^1_T(\mathbb{T})$-atoms $a$ and every $b \in BMO^*_T(\mathbb{T})$ with $\int_\mathbb{T} b(\theta)d\theta = 0$ such that $a$ and $b$ have finite wavelet expansions. To this end, assume that $a = \sum_{I \subseteq \Omega} a_I h_I$ has finite wavelet expansion and is associated to some $\Omega \in \mathcal{I}$. Then,

$$\Pi_3(a, b)(\theta) = \sum_{I \subseteq \Omega : J \ni I} a_I b_J h_I(\theta) h_J(\theta) = \sum_{I \subseteq \Omega} a_I b_I \frac{\chi_I(\theta)}{|I|}$$

and hence, by using the Cauchy-Schwarz inequality, one gets the pointwise estimate

$$\|\Pi_3(a, b)(\theta)\| \leq S_T[a] \cdot S_T[b],$$

where $P_I b(\theta) := \sum_{I \subseteq \Omega} b_I h_I(\theta)$. We thus have by using the Cauchy-Schwarz inequality and the $L^2$-boundedness of $S_T$,

$$\|\Pi_3(a, b)\|_{L^1(\mathbb{T})} \leq \|a\|_{L^2(\mathbb{T})} \|P_I b\|_{L^2(\mathbb{T})}.$$  

Since $\|a\|_{L^2(\mathbb{T})} \leq |\Omega|^{-1/2}$ and $\|P_I b\|_{L^2(\mathbb{T})} \leq |\Omega|^{1/2}\|b\|_{BMO^*_T(\mathbb{T})}$, (6.1) follows from (6.2).

**Proposition 23.** The bilinear operator $\Pi_2$ extends into a bounded bilinear operator from $H^1_T(\mathbb{T}) \times BMO^*_T(\mathbb{T})$ to $H^1_T(\mathbb{T})$.

**Proof.** As in the proof of the previous proposition, it suffices to prove that there exists an absolute constant $C > 0$ such that

$$\|\Pi_2(a, b)\|_{H^1_T(\mathbb{T})} \leq C\|b\|_{BMO^*_T(\mathbb{T})}$$

for each non-constant $H^1_T(\mathbb{T})$-atom $a$ and for each $b \in BMO^*_T(\mathbb{T})$ with $\int_\mathbb{T} b(\theta)d\theta = 0$ such that $a$ and $b$ have finite wavelet expansions. Towards this aim, write $a = \sum_{I \subseteq \Omega} a_I h_I$ for some $\Omega \in \mathcal{I}$ and notice that

$$\Pi_2(a, b)(\theta) = \sum_{I, J \subseteq \Omega : J \ni I} a_I b_J h_I(\theta) h_J(\theta) = \sum_{I, J \subseteq \Omega : J \ni I} a_I (P_I b_J) h_I(\theta) h_J(\theta)$$

$$= \sum_{I, J \subseteq \Omega : J \ni I} a_I (P_I b_J) h_I(c_J) h_J(\theta),$$

where $c_J$ denotes the centre of $J$. Moreover, observe that since

$$\langle a \rangle_J = |J|^{-1} \int_J a = \sum_{I \subseteq \Omega : J \ni I} a_I |J|^{-1} \int_J h_I(\theta)d\theta = \sum_{I \subseteq \Omega : J \ni I} a_I h_I(c_J),$$

one may rewrite $\Pi_2(a, b)$ as

$$\Pi_2(a, b)(\theta) = \sum_{J \subseteq \Omega} \langle a \rangle_J (P_I b_J) h_J(\theta).$$

Hence,

$$S_T[\Pi_2(a, b)](\theta) = \left( \sum_{J \subseteq \Omega} \langle a \rangle_J^2 |(P_I b_J)^2 \frac{\chi_J(\theta)}{|J|} \right)^{1/2} \leq M(a)(\theta) \cdot S_T[P_I b](\theta),$$

where $M(a)(\theta) := \left( \sum_{J \subseteq \Omega} \langle a \rangle_J^2 \right)^{1/2}$.
where $M$ denotes the Hardy-Littlewood maximal operator acting on functions defined over $\mathbb{T}$. Hence, by using the Cauchy-Schwarz inequality and the $L^2$-boundedness of $M$ and $S_T$, one gets
\[
\|\Pi_1(a, b)\|_{H^1_1} = \|S_T(\Pi_2(a, b))\|_{L^1_1} \lesssim \|a\|_{L^2_1} \|P_3 b\|_{L^2_1}.
\]
As in the proof of the previous lemma, note that one has $\|a\|_{L^2_1} \leq |\Omega|^{-1/2}$ and $\|P_3 b\|_{L^2_1} \leq |\Omega|^{1/2}\|b\|_{BMO_2^1}$ and so, (6.3) follows from the last estimate.

It follows from Propositions 22 and 23 that if we define
\[
T(f, b)(\theta) := \Pi_2(f, b)(\theta) + \Pi_3(f, b)(\theta)
\]
then $T$ is a bilinear operator that maps $H^1_1(\mathbb{T}) \times BMO_2^1(\mathbb{T})$ to $L^1_1(\mathbb{T})$. Therefore, to complete the proof of Theorem 21, it remains to handle $\Pi_4$.

**Proposition 24.** The bilinear operator $\Pi_4$ extends into a bounded bilinear operator from $H^1_1(\mathbb{T}) \times BMO_2^1(\mathbb{T})$ to $H^1_1(\mathbb{T})$.

**Proof.** Fix an $f \in H^1_1(\mathbb{T})$ and $b \in BMO_2^1(\mathbb{T})$ with finite wavelet expansions and moreover, assume that $\int_{\mathbb{T}} b(\theta) d\theta = 0$.

First of all, arguing as above, one may write
\[
\Pi_4(f, b)(\theta) = \sum_{J \in I} f_J(b) \theta h_J(\theta).
\]
Let $a$ be a non-constant $H^1_1(\mathbb{T})$-atom with finite wavelet expansion that is supported in some $\Omega \in I$ so that $\|a\|_{L^2_1} \leq |\Omega|^{-1/2}$. We claim that
\[
\Pi_4(a, b)(\theta) = \Pi_4(a, P_\Omega b)(\theta) + \langle b \rangle_{\Omega} \cdot a(\theta).
\]
Indeed, to see this, write
\[
\Pi_4(a, b)(\theta) = \sum_{J \in I} a_J(b) \theta h_J(\theta) = \Pi_4(a, P_\Omega b)(\theta) + \sum_{J \in I} a_J(b - P_\Omega b) \theta h_J(\theta)
\]
and observe that
\[
\sum_{J \in I} a_J(b - P_\Omega b) \theta h_J(\theta) = \sum_{J \in I} a_J \left( |J|^{-1} \int_{J \supseteq \Omega} b_J h_J(\theta') d\theta' \right) h_J(\theta)
\]
\[
= \sum_{J \in I} a_J \left( \sum_{J \supseteq \Omega, J \subseteq \Omega} b_J h_J(c_{\Omega}) \right) h_J(\theta)
\]
\[
= \left( \sum_{J \in I, J \subseteq \Omega} b_J h_J(c_{\Omega}) \right) a(\theta)
\]
\[
= \langle b \rangle_{\Omega} \cdot a(\theta),
\]
where $c_{\Omega}$ denotes the centre of $\Omega$. Hence, the proof of (4.4) is complete.

We may assume that $f = \sum_{k=1}^N \lambda_k a_k$, where each atom $a_k$ has a finite wavelet expansion. By using (6.4), one may write
\[
\Pi_4(f, b) = \beta_1 + \beta_2,
\]
where
\[
\beta_1(\theta) := \sum_{k=1}^N \lambda_k \Pi_4(a_k, P_{\Omega_k} b)(\theta)
\]
and
\[
\beta_2(\theta) := \sum_{k=1}^N \lambda_k \langle b \rangle_{\Omega_k} a_k(\theta).
\]
For the first term we have
\[
\|\Pi_1(a_k, P\Omega b)\|_{H^1_T} = \|S_2[\Pi_1(a_k, P\Omega b)]\|_{L^1_T}
\]
\[
= \left\| \left( \sum_{j \in J} |m_j(\Omega)|^2 |(a_k)_j|^2 \frac{\chi_{\Omega}}{|T|} \right)^{1/2} \right\|_{L^1_T}
\leq \|M(P\Omega b)S_T[a_k]\|_{L^1_T}
\leq \|M(P\Omega b)\|_{L^2(T)} \|S_T[a_k]\|_{L^2(T)}
\leq C\|P\Omega b\|_{L^2(T)} \|a_k\|_{L^2(T)}
\leq C|\Omega_k|\|b\|_{BMO^+_T(T)}\|\Omega_k|^{-1/2} = C\|b\|_{BMO^+_T(T)}
\]
for all \(k = 1, \ldots, N\). Hence,
\[
\|\beta_1\|_{H^1_T} \lesssim \|f\|_{H^1_T} \|b\|_{BMO^+_T(T)}
\]
and so, one deduces that
\[
\|\beta_1\|_{H^1_T} \lesssim \|f\|_{H^1_T} \|b\|_{BMO^+_T(T)}
\]
It remains to treat \(\beta_2\). The goal is to prove that
\[
\|S_2[\beta_2]\|_{L^{\infty}(T)} \lesssim \|b\|_{BMO^+_T(T)} \sum_{k=1}^N |\lambda_k|
\]
where the implied constant is independent of \(b, f\) (and \(N\)). To this end, observe that
\[
S_2[\beta_2](\theta) \leq \sum_{k=1}^N |\lambda_k| \langle b \rangle_{\Omega_k} |S_2[a_k]|(\theta)
\]
\[
\leq \sum_{k=1}^N |\lambda_k| |b(\theta) - \langle b \rangle_{\Omega_k} |S_2[a_k]|(\theta) + |b(\theta)| \sum_{k=1}^N |\lambda_k| |S_2[a_k]|(\theta)
\]
\[
= \sum_{k=1}^N |\lambda_k| |P\Omega b(s)| |S_2[a_k]|(\theta) + |b(\theta)| \sum_{k=1}^N |\lambda_k| |S_2[a_k]|(\theta).
\]
Hence,
\[
\|S_2[\beta_2]\|_{L^{\infty}(T)} \lesssim \left\| \sum_{k=1}^N |\lambda_k| |P\Omega b|S_2[a_k]| \right\|_{L^{\infty}(T)} + \left\| |b| \sum_{k=1}^N |\lambda_k| |S_2[a_k]| \right\|_{L^{\infty}(T)}
\]
\[
\leq \left\| \sum_{k=1}^N |\lambda_k| |P\Omega b|S_2[a_k]| \right\|_{L^1(T)} + \left\| |b| \sum_{k=1}^N |\lambda_k| |S_2[a_k]| \right\|_{L^{\infty}(T)}.
\]
By arguing as above, it can easily be seen that
\[
\left\| \sum_{k=1}^N |\lambda_k| |P\Omega b|S_2[a_k]| \right\|_{L^1(T)} \lesssim \|b\|_{BMO^+_T(T)} \sum_{k=1}^N |\lambda_k|.
\]
Therefore, the proof of (6.5) is reduced to showing that
\[
\left\| |b| \sum_{k=1}^N |\lambda_k| |S_2[a_k]| \right\|_{L^{\infty}(T)} \lesssim \|b\|_{BMO^+_T(T)} \sum_{k=1}^N |\lambda_k|.
\]
To this end, note that by using Proposition 20 one gets
\[ \left\| \sum_{k=1}^{N} |\lambda_k| S \right\|_{L_1}(T) \lesssim \sum_{k=1}^{N} |\lambda_k| \left\| S \right\|_{L_1}(T) \]
and since for each \( H^1 \)-atom one has \( \| S \|_{L_1}(T) \leq 1 \), (6.6) follows from the last estimate. This completes the proof of (6.6). \( \square \)

6.1. Passing from dyadic to non-dyadic decompositions. Assume that \( f \in H^1(T) \) and \( b \in BMO^+(\mathbb{T}) \). Then, as shown in 13, one has
\[ \text{BMO}(T) = \text{BMO}_{T^0}(T) \cap \text{BMO}_{T^1/3}(T) \]
and there exist \( f_1 \in H^1_{T^0}(\mathbb{R}) \) and \( f_2 \in H^1_{T^1/3}(\mathbb{R}) \) such that \( f = f_1 + f_2 \). Having fixed such a decomposition of \( f \), write
\[ f \cdot b = f_1 \cdot b + f_2 \cdot g = S_T(f_1, b) + T_T(f_1, b) + S_{T^1/3}(f_2, b) + T_{T^1/3}(f_2, b) \]
in \( \mathcal{D}' \).
Hence, a periodic version of Theorem 11 is obtained by taking
\[ S(f, b) := S_T(f_1, b) + S_{T^1/3}(f_2, b) \]
and
\[ T(f, b) := T_T(f_1, b) + T_{T^1/3}(f_2, b). \]

7. Some further remarks in the periodic setting

7.1. Variants of Theorems 2 and 3 on \( T \). There is a periodic version of Theorem 2 namely \( M(f) \in L_{\Psi_0}(\mathbb{T}) \) if, and only if, \( f \in L \log \log L(T) \). Combining this with Lemma 6 one obtains the following result.

Proposition 25. If \( f \in L \log \log L(T) \), then \( f \in H^{\log}(T) \).

Moreover, arguing as in the Section 3 and using the necessity in (an appropriate periodic version of) Theorem 2 as well as Proposition 25 and Lemma 6 one can show that if \( f \) is a non-negative function in \( H^{\log}(\mathbb{T}) \), then \( f \in L \log \log L(T) \).

Proposition 26. One has
\[ \{ f \in L \log \log L(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T} \} = \{ f \in H^{\log}(T) : f \geq 0 \text{ a.e. on } \mathbb{T} \}. \]

Proof. Note that Proposition 25 implies that
\[ \{ f \in L \log \log L(T) : f \geq 0 \text{ a.e. on } \mathbb{T} \} \subseteq \{ f \in H^{\log}(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T} \}. \]

To prove the reverse inclusion, take a non-negative function \( f \) in \( H^{\log}(T) \) and notice that it follows from the work of Stein 25 that
\[ \| \{ \theta \in \mathbb{T} : M(f)(\theta) > c_1 \alpha \} \| \geq \frac{C}{\alpha} \int_{|f| > \alpha} |f(\theta)| d\theta, \]
where \( c_1, c_2 > 0 \) are absolute constants. Hence, by arguing as in the proof of Theorem 2, it follows from (7.2) (noting that the periodic case is easier as one does not need to consider the contribution away from the support of \( f \)) that
\[ \int_{\mathbb{T}} \Psi_0(M(f))(\theta) d\theta \gtrsim 1 + \int_{\mathbb{T}} |f(x)| \log^+ |f(\theta)| d\theta. \]
Since \( f \geq 0 \) a.e. on \( \mathbb{T} \), as in the Euclidean case, one has
\[ f^*(\theta) \geq \sup_{0 < r < 1} |(P_r \ast f)(\theta)| \gtrsim M(f)(\theta) \text{ for a.e. } \theta \in \mathbb{T}, \]
where \( P_r \) denotes the Poisson kernel in the periodic setting. Hence, by using (7.3), (7.4), and Lemma 6 we deduce that \( f \in L \log \log L(T) \) and so,
\[ \{ f \in H^{\log}(T) : f \geq 0 \text{ a.e. on } \mathbb{T} \} \subseteq \{ f \in L \log \log L(\mathbb{T}) : f \geq 0 \text{ a.e. on } \mathbb{T} \}. \]
The desired fact is a consequence of (7.1) and (7.5). \( \square \)
7.1.1. Some further applications. We conclude with some applications of Theorem 5 in the periodic setting. The function
\[ \Psi(x, t) = \Psi(t) = t \log^+ t \log^+ \log^+ t \]
appearing in [24] satisfies the hypotheses of Theorem 5 and we now determine which space maps into \( L\Phi \) via the maximal function. With the associated \( \psi \) defined as before, an integration by parts yields
\[ \int \frac{\psi(s)}{s} ds = \frac{1}{2} (\log^+ s)^2 \log^+ \log^+ s + \log^+ s \log^+ s - \frac{1}{4} (\log^+ s)^2. \]
This allows us to conclude that, for this choice of \( \Psi \),
\[ M(f) \in L\Phi(\mathbb{T}) \quad \text{if, and only if,} \quad f \in L \log^2 L \log\log L(\mathbb{T}). \]

Turning to the space \( L \log\log L \log\log\log\log L \) appearing in Lie’s paper [17], we can check where the maximal operator maps this space. Performing the appropriate computations, we obtain that
\[ \int_{T} M(f) \log(M(f) + e) \log^+ \log^+ \log^+ M(f) d\theta < \infty \]
if, and only if,
\[ f \in L \log\log L \log\log\log\log L(\mathbb{T}). \]

Roughly speaking, the contents of Theorem 5 and the computations presented above can be summarised as follows. Let \( \Phi_0 \) be a given Orlicz function, namely \( \Phi_0 : [0, \infty) \to [0, \infty) \) is an increasing function with \( \Phi_0(0) = 0 \) and \( \Phi_0(t) \to \infty \) as \( t \to \infty \). Suppose that one can find non-negative, increasing functions \( M, S \) with
\[ \Phi_0(t) = M(t) \cdot S(t) \quad (t > 0) \]
and such that, for \( 0 < \alpha < t \), one can easily compute
\[ F_\alpha(t) := \int_{\alpha}^{t} \frac{M'(s)}{s} ds \]
in closed form and, moreover, that there exists an \( \alpha_0 > 0 \) with the property that for every \( \alpha \geq \alpha_0 \) one has
\[ F_\alpha(t) \cdot S(t) \gtrsim \int_{\alpha}^{t} \left( \frac{M(s)}{s} + F(s) \right) \cdot S'(s) ds \quad \text{for all } t \geq \alpha. \]
Then, by arguing as in Section 2 one deduces the 'concrete' relation
\[ f \in L\Phi_0(\mathbb{T}) \quad \text{if, and only if,} \quad M(f) \in L_{F_\alpha S}(\mathbb{T}), \]
for any \( \alpha \geq \alpha_0. \)

7.2. A variant of an inequality of Hardy and Littlewood for \( H^{\log}(\mathbb{T}) \). A classical result due to Hardy and Littlewood asserts that for every \( p \in (0, 1] \) there exists a constant \( C_p > 0 \) such that
\[ (7.6) \quad \left( \sum_{n=1}^{\infty} \frac{|f_n|^p}{|n|^2-p} \right)^{1/p} \leq C_p \|F\|_{H^p(\mathbb{D})} \]
for all analytic functions \( F(z) = \sum_{n=0}^{\infty} f_n z^n \) in the Hardy space \( H^p(\mathbb{D}) \) on the unit disc \( \mathbb{D} \); [11, Theorem 16]. It follows from (7.6) that for every \( p \in (0, 1] \) there exists a constant \( B_p > 0 \)
\[ (7.7) \quad \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|\hat{f}(n)|^p}{|n|^2-p} \right)^{1/p} \leq B_p \|f\|_{H^p(\mathbb{T})}. \]
Theorem 27. There exists a constant $C > 0$ such that
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\Psi_0(|n\hat{f}(n)|)}{n^2} \leq C \int_{\mathbb{T}} \Psi_0(f^*(\theta))d\theta,
\]
where $\Psi_0(t) := t \cdot [\log(e + t)]^{-1}$, $t \geq 0$.

Proof. We shall prove that there exists an absolute constant $C_0 > 0$ such that
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\Psi_0(|n\hat{a}I(n)|)}{n^2} \leq C_0 |I| \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})})
\]
for any $L^\infty$-function $a_I$ supported in some arc $I$ in $\mathbb{T}$ with $\int_I a_I(\theta)d\theta = 0$.

To this end, we fix such a function $a_I$ (and an arc $I$) and write
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\Psi_0(|n\hat{a}I(n)|)}{n^2} = A + B,
\]
where
\[
A := \sum_{|n| \leq |I|^{-1}} \frac{\Psi_0(|n\hat{a}I(n)|)}{n^2}
\]
and
\[
B := \sum_{|n| > |I|^{-1}} \frac{\Psi_0(|n\hat{a}I(n)|)}{n^2}.
\]

We shall prove that
\[
A \lesssim |I| \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})})
\]
and
\[
B \lesssim |I| \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})}).
\]

To prove (7.9), by using the cancellation of $a_I$ and the fact that $|e^{-inx} - e^{-iny}| \leq |n||x - y| \leq |n||I|$ for all $n \in \mathbb{Z}$ and $x, y \in I$, one has
\[
|\hat{a}_I(n)| \leq |n||I|^2 \|a_I\|_{L^\infty(\mathbb{T})} \quad \text{for all } n \in \mathbb{Z}.
\]

Since $\Psi_0$ is increasing and there exists an absolute constant $A_0 > 0$ such that $\Psi_0(st) \leq A_0 s^{2/3} \Psi_0(t)$ for all $t > 0$ and $s \in (0, 1)$; see 33. Example 1.1.5 (i), it follows from (7.11) that
\[
A \leq \sum_{1 \leq |n| \leq |I|^{-1}} \frac{\Psi_0(n^2|I|^2 \|a_I\|_{L^\infty(\mathbb{T})})}{n^2} \lesssim \sum_{1 \leq |n| \leq |I|^{-1}} \frac{(|I|^2 n^2)^{2/3} \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})})}{n^2} = |I|^{1/3} \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})}) \sum_{1 \leq |n| \leq |I|^{-1}} n^{-2/3}
\]
\[
\lesssim |I| \Psi_0(\|a_I\|_{L^\infty(\mathbb{T})}).
\]

Hence, (7.9) holds. To establish (7.10), note that by using Hölder’s inequality for $p = 4$ and $p' = 4/3$ and Parseval’s identity, one obtains
\[
B \leq B_1 \cdot B_2,
\]
where
\[
B_1 := \|a_I\|_{L^p(\mathbb{T})}^{1/2} \leq |I|^{1/4} \|a_I\|_{L^\infty(\mathbb{T})}^{1/2}
\]
and

\[ B_2 := \left( \sum_{|n| > |I|^{-1}} \frac{\hat{\Psi}_0(|n\tilde{a}_I(n)|)}{n^2} \right)^{3/4} \]

with \( \hat{\Psi}_0(t) := t^{2/3} |\log(e + t)|^{-4/3}, t > 0. \) Since \( |\tilde{a}_I(n)| \leq |I||a_I||L^\infty(\mathbb{T}) \) and \( \hat{\Psi}_0 \) is increasing on \([0, \infty)\), we have

\[ B_2 \leq \left( \sum_{|n| > |I|^{-1}} \frac{\hat{\Psi}_0(|n||a_I||L^\infty(\mathbb{T})|)}{n^2} \right)^{3/4} \]

\[ \leq |I|^{1/2} \frac{\|a_I||L^\infty(\mathbb{T})\|}{\log(e + \|a_I||L^\infty(\mathbb{T})\|)} \left( \sum_{|n| > |I|^{-1}} n^{-4/3} \right)^{3/4} \]

\[ \lesssim |I|^{3/4} \frac{\|a_I||L^\infty(\mathbb{T})\|}{\log(e + \|a_I||L^\infty(\mathbb{T})\|)} \]

and so, (7.10) holds as

\[ B \leq B_1 \cdot B_2 \lesssim |I|\hat{\Psi}_0(\|a_I||L^\infty(\mathbb{T})\|). \]

Therefore, in view of (7.9) and (7.10), (7.8) holds.

To complete the proof of the theorem, take an \( f \in H^{\log}(\mathbb{T}) \) and note that there exists a sequence \( \{b_{I_k}\}_{k \in \mathbb{N}} \) of multiples of atoms in \( H^{\log}(\mathbb{T}) \), supported in arcs \( I_k \), such that

\[ f - \hat{f}(0) = \sum_{k \in \mathbb{N}} b_{I_k} \text{ in } \mathcal{D}' \]

and

\[ \sum_{k \in \mathbb{N}} |I_k|\hat{\Psi}_0(\|b_{I_k}\||L^\infty(\mathbb{T})) \leq A \int_\mathbb{T} \hat{\Psi}_0(f^*(\theta))d\theta, \]

where \( A > 0 \) is an absolute constant. Hence, by using (7.5) and (7.8) we get

\[ \sum_{n \in \mathbb{Z}\setminus\{0\}} \Psi_0(|n\tilde{f}(n)|) = \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{Z}\setminus\{0\}} \Psi_0(|nb_{I_k}(n)|) \]

\[ \lesssim \sum_{k \in \mathbb{N}} |I_k|\Psi_0(\|b_{I_k}\||L^\infty(\mathbb{T})) \]

\[ \lesssim \int_\mathbb{T} \hat{\Psi}_0(f^*(\theta))d\theta \]

and this completes the proof of our theorem. \( \square \)

**Remark 28.** One deduces from Theorem 27 that for any \( f \in H^{\log}(\mathbb{T}), \)

\[ (7.12) \]

\[ \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{|	ilde{f}(n)|}{|n| \log(e + |n|)} < \infty. \]

Indeed, observe that

\[ (7.13) \]

\[ |	ilde{f}(n)| \lesssim_f 1 + |n| \text{ for all } n \in \mathbb{Z}. \]

To see this, note that \( (H^p(\mathbb{T}))^* \cong \Lambda_{p-1}^{-1}(\mathbb{T}) \) for \( p < 1; \) see §7.4 in [3] and so, for \( f \in H^{\log}(\mathbb{T}) \subset H^{2/3}(\mathbb{T}) \) one has

\[ \|\langle f, e_n \rangle\|_{L^2(\mathbb{T})} = \|e_n\|_{L^\infty(\mathbb{T})} + \sup_{x,y \in [0,2\pi]} \frac{|e^{inx} - e^{iny}|}{|x-y|^{1/2}} \lesssim 1 + |n|^{1/2} \]

\[ \leq 1 + |n| \]
for all \( n \in \mathbb{Z} \), where \( e_n(x) := e^{inx}, x \in \mathbb{T} \). Therefore, in view of Theorem 27 and (7.12), (7.13) holds.

Theorem 27 can be used to exhibit distributions in \( H^p(\mathbb{T}) \setminus H^{10}([0,2\pi]) \) for \( p \in (0,1) \). For instance, it follows from Theorem 27 that the Dirac distribution \( \delta_0 \) does not belong to \( H^{10}([0,2\pi]) \).

Furthermore, Theorem 27 is sharp in the following sense: if \( \tilde{\Psi} : [0,\infty) \to [0,\infty) \) is any increasing function with \( \lim_{t \to \infty} \tilde{\Psi}(t)/\Psi(t) = \infty \), then there is no constant \( C > 0 \) such that

\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\tilde{\Psi}(|n f(n)|)}{n^2} \leq C \int_\mathbb{T} \Psi_0(f^*(\theta)) d\theta
\]

for all \( f \in H^{10}([0,2\pi]) \). Indeed, take a function \( \tilde{\Psi} \) as above and suppose that (7.14) holds true. Let \( N \) be a large positive integer that will eventually be sent to infinity. Consider the function

\[
a_N(\theta) := N2^N e^{i2^n\theta} \chi_{[0,2\pi\mathbb{N})}(\theta), \quad \theta \in [0,2\pi).
\]

One can easily check that

\[
\|a_N\|_{H^{10}(\mathbb{T})} \lesssim 1,
\]

where the implied constant is independent of \( N \).

Consider the interval \( I_N := [2^{N-2},2^{N-1}) \) and observe that there exists an absolute constant \( c_0 > 0 \) such that for every natural number \( n \) in \( I_N \) one has

\[
|\hat{a}_N(n)| = N2^N \left| \frac{e^{-i2^n(2^Nn-1)}}{2\pi|n|} \right| = N2^N \left| \frac{\sin [\pi(n2^{-N}-1)]}{2\pi(2^n-n)} \right| \geq c_0 N,
\]

where we used the identity \(|e^{is} - e^{it}| = |\sin((s-t)/2)|\) for \( s = -2\pi(2^{-N}n-1) \) and \( t = 0 \) as well as the fact that for \( n \in I_N \) one has \( 2^{N-n} \sim 2^N \) and \( |\sin[\pi(2^{-N}n-1)]| \sim 1 \).

Hence, (7.14) and (7.15) imply that

\[
1 \geq \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\tilde{\Psi}(|n \hat{a}_N(n)|)}{n^2} \geq \sum_{n=2^{N-2}}^{2^{N-1}} \tilde{\Psi}(|n \hat{a}_N(n)|) \geq \tilde{\Psi}(c_02^{N-2}N) \sum_{n=2^{N-2}}^{2^{N-1}} n^{-2}
\]

\[
\approx \tilde{\Psi}(c_02^{N-2}N) \frac{2^N}{2^N} = \frac{\Psi_0(c_02^{N-2}N)}{\Psi_0(c_02^{N-2}N)} \approx \frac{\Psi_0(c_02^{N-2}N)}{\Psi_0(c_02^{N-2}N)}.
\]

which yields a contradiction by taking \( N \in \mathbb{N} \) ‘large enough’.

**Remark 29.** As a consequence of sharpness of Theorem 27 discussed above, one deduces that

\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|\hat{f}(n)|}{n \log(e + |n \hat{f}(n)|)} \lesssim \int_\mathbb{T} \Psi_0(f^*(\theta)) d\theta
\]

is false when \( 0 < s < 1 \).

It follows from Theorem 27 and [5, (8.3)] that there exists a constant \( C > 0 \) such that

\[
\sum_{n=1}^{\infty} \frac{\Psi_0(|n f(n)|)}{n^2} \leq C \sup_{0 < r < 1} \int_0^{2\pi} \Psi_0(|F(r e^{i\theta})|) d\theta
\]
for all analytic functions \( F(z) = \sum_{n=0}^{\infty} f_n z^n \) in the unit disc \( D \) for which the quantity on the right-hand side of (7.16) is finite.

We remark that variants of (7.6) and (7.7) for certain classes of Hardy-Orlicz spaces have been obtained in [22] and [32] (see also [13, 23, 30]), which do not include the case of \( H^{log}(\mathbb{T}) \) treated above. Moreover, our methods are completely different from those in the aforementioned references.

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