Dark matter distribution in galaxy U11454

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In this paper the dark matter distribution in the spiral galaxy U11454 is explored using only the well-known cored density profiles in the literature such as the pseudo-isothermal, Burkert, Einasto, exponential sphere, Beta and Brownstein profiles. It is assumed that the distribution of dark matter in the considered galaxy is spherically symmetric without taking into account its complex structural components. Instead, it is assumed that dark matter possesses non-vanishing pressure. The rotation curve data of the galaxy is fitted for each profile. The model free parameters are inferred, the total dark matter mass for each profile is calculated and our results are confronted and contrasted with the previous outcomes. Using Bayesian Information Criterion the best fit profile among the considered ones is selected. Furthermore, the hydrostatic equilibrium equation is solved and the pressure profiles for each above mentioned density profile are constructed in the weak gravitational field regime. Combining the pressure and corresponding density profiles one gains equations of state for the dark matter in the galaxy U11454. In addition, the speed of sound is estimated in the dark matter distribution and it is shown that its behavior is quite unusual especially for the Brownstein profile. Finally, the refracting index is calculated in order to assess gravitational lensing effects produced by the presence of dark matter and astrophysical implications of the obtained results are discussed.

Key words: dark matter, rotational curves, equation of state, speed of sound and refractive index.

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1 Introduction

Dark matter (DM) weakly interacts with ordinary matter via gravity. For this reason it is inaccessible to direct observations and measurements. Recent cosmological surveys have demonstrated that DM represents the 26.8 % of the total energy budget of the Universe [1]. So far, from a theoretical viewpoint it was assumed that DM consists of some experimentally undiscovered particles derived from the extension of the standard model of particle physics [2, 3]. Astronomers and astrophysicists indirectly receive all information about DM from rotation curves (RCs) of galaxies and gravitational lensing effects. RCs represent profiles of linear velocity $υ(r)$ of stars and gas in circular orbits as functions of radial distance $r$ from the galactic center to the halo. The RCs of galaxies have an amplitude that remains almost constant with increasing distance, even outside the stellar disk, which is not normally expected in Newtonian gravity (NG) [4, 5]. According to some studies it is presumed that the DM consists of nonrelativistic and collisionless particles, whose cross sections between the DM and baryons lie around $~10^{−26}$ cm$^2$ [6].

In this work to study the dark matter distribution in the chosen galaxy we mainly consider widely used density profiles such as: pseudo-isothermal (ISO), Burkert, Beta, Brownstein, Einasto, and exponential sphere in the framework of NG. By analyzing the RCs data points we derive the unknown model parameters of the DM halo for all profiles. In analogy to our previous work [7] we suppose that the DM equation of state (EoS) in a particular galaxy, by default, do not depend upon the density profiles. Hence, we intend to check whether this assumption is valid or not for the galaxy U11454 as well. Moreover, to avoid the cuspy halo problem we mainly draw our attention to the cored density profiles. This treatment is consistent with the fact that the DM density and pressure at the galactic center must be finite. Following Ref. [7] to find the pressure profiles, we solve the hydrostatic equilibrium equation within NG. The DM EoS for galaxy U11454
can be obtained by expressing pressure in terms of density for each individual model. For some profiles EoS is obtained analytically, but for others this can be done numerically only. Thus, the main objective of this work is to test whether the DM EoS depends on the adopted density profile or not. Using the DM EoS, we calculate the speed of sound and also the refractive index, which play a pivotal role in the formation of structures and in the gravitational lensing effect.

The paper is organized as follows. In section 2 the main features of our work are formulated. In section 3, the observational data of U11454 RCs are fitted by using the adopted density profiles and the EoS, the speed of sound and the refractive index are studied. In section 3.1, the basic results of the paper are presented. In section 4, we summarize conclusions and perspectives of our work.

2 Dark matter distribution in spiral galaxies

The distribution of DM in galaxies is not uniform, concentrating at their centers and dropping off at their periphery. Using the methods of numerical simulation of the dynamics of stars in galaxies, one can find the corresponding distribution function DM or its profile. In this paper, we have chosen the most commonly used cored DM density profiles, such as pseudo-isothermal (ISO), Beta, Burkert, Brownstein, Einasto and exponential sphere. All profiles that we use in this work are characterized by two parameters: the DM density at the centers of galaxies or the characteristic density $\rho_0$ and the scale radius $r_0$, except for the Einasto profile, which has one more free parameter.

Thus, we employ the following models:

1. The ISO profile [8]:
   \[ \rho_{\text{ISO}}(r) = \frac{\rho_0}{1 + x^2}, \]  \hspace{1cm} (1)

   where $x(r) = r/r_0$, $r$ is the radial coordinate / distance. The model depends upon the two constants $r_0$ and $\rho_0$, so do the following profiles.

2. Exponential sphere [9]:
   \[ \rho_{\text{Exp}}(r) = \rho_0 e^{-x}. \] \hspace{1cm} (2)

3. Burkert profile [10]:
   \[ \rho_{\text{Bur}}(r) = \frac{\rho_0}{(1 + x)(1 + x^2)}. \] \hspace{1cm} (3)

4. The Beta profile with $\beta = 1$ [11, 12]:
   \[ \rho_{\text{Beta}}(r) = \frac{\rho_0}{(1 + x^2)^{3/2}}. \] \hspace{1cm} (4)

5. Brownstein profile [12, 13]:
   \[ \rho_{\text{Bro}}(r) = \frac{\rho_0}{1 + x^3}. \] \hspace{1cm} (5)

6. Einasto profile [14]:
   \[ \rho_{\text{Ein}}(r) = \rho_0 \exp\left[2\alpha\left(1 - x^{1/\alpha}\right)\right], \] \hspace{1cm} (6)

   where $\alpha$ is the Einasto free parameter, which $\alpha$ determines the slope of the profile [15].

In Figure 1 (left panel) we illustrate the dependence of $\rho/\rho_0$ on $r/r_0$ for different DM halo profiles and in the right panel the gray thick points show the observational data points with their error bars for galaxy U11454; solid curves show ISO (black), exponential sphere (red), Einasto (green), Burkert (blue), Beta (orange) and Brownstein (purple) profiles.

As stated above, our computations are performed in the NG regime to fulfill the simplest assumptions over star motions in the disk and halo of galaxies. In the next subsection, we consider the adopted strategy to fit galactic data of U11454 [16, 17].
2.1 Galaxy U11454 within Newtonian gravity

The main purpose of this paper is to calculate the equation of state of DM in the NG regime. To do so we start with the standard Newtonian hydrostatic equilibrium equations, given by \[18, 19\]

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \tag{7}
\]

\[
\frac{dP(r)}{dr} = -\rho(r) \frac{G M(r)}{r^2}, \tag{8}
\]

\[
\frac{d\Phi(r)}{dr} = \frac{G M(r)}{r^2}, \tag{9}
\]

where \(M(r)\) is the mass profile enclosed inside the sphere of radius \(r\), \(P(r)\) is the DM pressure, \(G\) is the Newtonian gravitational constant and \(\Phi(r)\) is the gravitational potential. To obtain expressions for the pressure the density profiles Eqs. (1) – (6) are plugged in Eqs. (7) – (8). We integrate the expression for pressure, using the fact that \(P\) tends to zero as \(r\) approaches infinity.

All calculations for the ISO, Beta and exponential sphere profiles were performed analytically, whereas for the Einasto, Brownstein and Burkert profiles were performed numerically. The pressure and EoS formulas for the ISO and exponential sphere profiles in Eqs. (1) and (2) are given in Ref. [7] for Beta profile we obtain the following equation

\[
P(r) = 4\pi G n_0^2 \rho_0^2 \left\{ \frac{1}{2(1+x^2)} + \frac{1+2x^2}{x\sqrt{1+x^2}} \arcsinh(x) - 2 \ln \left[ \frac{2\sqrt{1+x^2}}{1+x^2} \right] \right\}, \tag{10}
\]
Combining Eq. (10) together with Eq. (4), we obtain the EoS for the Beta profile, respectively

\[ P(\rho) = 4\pi G r_0^2 \rho_0^2 \times \left( \frac{\xi}{2} + \frac{2 - \xi}{\sqrt{\xi - 1}} \right) \text{arsinh} \left( \frac{\sqrt{\xi - 1}}{\xi} \right) - 2 \ln \left( \frac{2^{\frac{1}{2}}}{\sqrt{\xi}} \right) \] (11)

where \( \xi = (\rho/\rho_0)^{2/3} \). Eq. (11) for the Beta profile has been obtained here for the first time.

### 2.2 Speed of sound and optical properties

Using the speed of sound \( c_s \), we can account for perturbations caused by the DM fluid. Its definition in case of adiabatic perturbations is [20]

\[ c_s^2 = \left( \frac{\partial P}{\partial \rho} \right)_S. \] (12)

The speed of sound for the ISO and exponential sphere profiles are given in Ref. [7] and for the Beta profile we find it from Eq. (11)

\[ c_{s(Beta)}^2 = \frac{4\pi G r_0^2 \rho_0}{3} \sqrt{\frac{\xi}{1 - \xi}} \left\{ \sqrt{1 - \xi} \text{arsinh} \left( \frac{1}{\sqrt{\xi}} \right) - 1 \right\}. \] (13)

An unusual feature of DM is the fact that with increasing density, the speed of sound decreases for all profiles apart from the Brownstein profile where the opposite is true. It is known from cosmology that if the speed of sound in the central part of the DM distribution is less than in its outer parts, then it allows the formation of large-scale structures in the Universe [21]. Thus, we can state that the Brownstein profile does not allow forming galaxies. Hence, we can rule it out.

Furthermore, we calculate and show that DM gives a refractive index which is slightly larger than pure vacuum. This can be seen when studying the optical properties of the galaxy under consideration. In this fashion, we here examine the refractive index of DM halo of the galaxy U11454 for the ISO, exponential sphere, Burkert, Beta, Brownstein and Einasto profiles. The refractive index in the field of DM in NG is given by [22]

\[ n(r) = 1 - \frac{\Phi(r)}{c^2} - \int G M (r) \frac{c^2 \rho^2}{c^2 r^2} dr, \] (14)

where \( \Phi(r) \) is now the internal gravitational potential for a given DM density profile, \( c \) is the speed of light in vacuum. The integration is carried out within the range of the radial coordinate taken from the RC data points.

For the profiles considered above in the galaxy U11454, we have shown the dependence of the refractive index \( n \) on the radial coordinate in the DM distribution in Figure 2. The value of the refractive index is small in the halo, as we expected. In the core region the refractive index grows slowly, but remains extremely small, growing less than 1 part in \( 10^6 \). In Figure 2 we plotted the refractive index DM for different profiles. The refractive index in vacuum is slightly less than in DM for all the profiles that we used at a distance within (0.4 - 11.9) kpc which corresponds to RCs data points. In general, it can be used even to study the light propagation process in the epoch of DM dominance [23].
2.3 The role of dark matter equation of state

The EoS of a given liquid characterizes the physical properties of the liquid itself and shows how it develops when it is not in equilibrium. Consequently, the thermodynamic properties of the liquid are hidden inside the state parameter defined by [24]

$$\omega = \frac{P}{\rho c^2},$$

(15)

which is also known as a state or barotropic parameter. In our case, the pressure does not vanish inside the DM distribution and the corresponding expressions for the EoS are complicated for some profiles. Eq. (15) can be written either as a function of the radial coordinate \( r \) or density \( \rho \) as shown in Fig. 3.

It is interesting to study the general behavior of the EoS. It follows that \( \omega \) tends to zero for large distances or low densities (except for the isothermal profile, where it becomes constant). At small distances or high densities, \( \omega \) tends to a constant value for the cored profiles, at least in the considered range of distances provided by the RC data points.

In our calculations, we can take into account the corrections of general theory of relativity. At large distances when \( \rho \ll \rho_0 \), the corrections can be neglected. However, near the central part of the galaxy, its deviations from NG would be significantly noticeable, since the corrections of general relativity in pressure are clearly much larger than in NG [25]. This is simply a consequence of the Tolman-Oppenheimer-Volkov system of equations.
3 Methods and analyses

We aim to show that as a result of our analysis, by testing different profiles for such a galaxy, DM halo density law can be accounted for the kinematics of the whole family of disk (spiral) galaxies like U11454. The internal structure of U11454 is not known and so the strategy is to get the rotational velocity.

For completeness, it is important to stress that the contributions of gas in the bulge and disk of the galaxy are absolutely essential to characterize the RCs. In turn, the RC allows one to determine the distribution of the galaxy's mass along the radial distance. In the standard approach, all the constituents of the galaxy must be accounted for in analogy with Ref. [9]. However their contributions with respect to the DM halo are small and one can neglect them for simplicity. Therefore, we adopt

$$v^2_{tot} = v^2_{profile}.$$  \hspace{1cm} (16)

where \(v_{profile}\) is the contributions of the DM halo profile velocity. So, \(v_{profile}\) is defined by

$$v(r) = \sqrt{\frac{GM(r)}{r}}$$ and

$$M(r) = \int_0^r 4\pi r^2 \rho(r)dr.$$  \hspace{1cm} (17)
where the DM mass $M(r)$ has been obtained from Eq. (17), by integrating Eq. (7) and making use of $\rho(r)$, i.e. the DM density profile taken from Eqs. (1) through (6).

### 3.1 Numerical results

We apply the Levenberg-Marquardt nonlinear least squares method [26, 27] to find the minimum of the $\chi^2$ function. The Levenberg-Marquardt algorithm is an iterative technique that locates the minimum of a function which is expressed as the sum of squares of nonlinear functions. It consists in a combination of the Gauss–Newton algorithm and the method of the steepest descent gradient. So, the $\chi^2$ function is defined by

$$
\chi^2 = \sum_{i=1}^{N} \left[ \frac{\nu_i^{\text{obs}} - \nu(\rho_0, r_0, r)}{\sigma_{\nu, i}^{\text{obs}}} \right]^2,
$$

where $\nu_i^{\text{obs}}$ and $\sigma_{\nu, i}^{\text{obs}}$ are the $N$ data points of U11454 RC and their corresponding errors, respectively (see Fig. 1, right panel), while $\nu(\rho_0, r_0, r)$ is given by Eq. (17) and describes the RCs for each DM profile. The best fit parameters, minimizing the $\chi^2$ for each DM profile are listed in Table 1 and shown in Fig. 1 (right panel). The $\rho_0$ and $r_0$ parameters listed in Table 1 are in agreement with the results of Ref. [28]. The amount of DM mass in the galaxy, computed by using different profiles, is also shown in Table 1 and it is shown to be consistent with the results of Ref. [28]. For the Einasto profile we obtained $\alpha = 1.8 \pm 0.3$. The $\chi^2$ values are also shown in the last column of Table 1.

To compare the selected 6 profiles, which have different number of parameters and are not nested into each other, we employ the Bayesian Information Criterion (BIC) [29]. BIC is a selection criterion among a finite set of models, conceived to solve the overfitting issue when increasing the number of parameters in the fitting function. For completeness, please notice that other selection criteria could also be used, e.g. the Akaike information and/or DIC criteria [30, 31]. Starting from the $\chi^2$ definition, the BIC is defined as

$$
\text{BIC} = \chi^2 + k \ln N,
$$

where $k$ is the number of model parameters. For the Einasto profile $k = 3$, while for all the other profiles $k = 2$. A profile with a minimum BIC value is favored, according to Ref. [32]. As one can see from Table 1 for galaxy U11454 the BIC value is minimum for the Isothermal profile and is maximum for the Brownstein profile, though the difference between the values is not large.

### Table 1 – Model parameters for the analyzed galaxy U11454. We reported for each profile the density $\rho_0$, $r_0$ and the masses expressed in terms of solar mass $M_\odot$. For every column we report the error bars. The last two columns report the BIC statistical outputs and the corresponding chi squares used for computing the BIC values. a DM mass calculated using the last data point in the halo for $r$. b DM mass calculated using the scale radius $r_0$. The values BIC = [84, 91, 79, 76, 85, 67] are for Beta, Brownstein, Burkert, Einasto, exponential sphere and ISO, respectively; we define $\Delta \text{BIC} = \text{BIC} - \text{BIC}_0$, with $\text{BIC}_0 = 67$ the Isothermal reference value. For the Einasto profile $\alpha = 1.8 \pm 0.3$.

| Profiles | $\rho_0 \pm \sigma_{\rho_0}$ | $r_0 \pm \sigma_{r_0}$ | $M \pm \sigma_M^a$ | $M \pm \sigma_M^b$ | $\Delta \text{BIC}^c$ | $\chi^2$ |
|----------|-----------------|------------------|----------------|----------------|----------------|--------|
| Beta     | 104.0 $\pm$ 10.8 | 3.7 $\pm$ 0.2    | 6.0 $\pm$ 1.0  | 1.1 $\pm$ 0.3  | 17             | 1.5    |
| Brownstein | 81.0 $\pm$ 10.0 | 3.6 $\pm$ 0.3    | 5.7 $\pm$ 1.2  | 1.1 $\pm$ 0.4  | 24             | 2.8    |
| Burkert  | 139.0 $\pm$ 12.9 | 3.7 $\pm$ 0.2    | 6.1 $\pm$ 0.9  | 1.1 $\pm$ 0.3  | 12             | 1.0    |
| Einasto  | 10.5 $\pm$ 2.6   | 7.2 $\pm$ 0.9    | 6.4 $\pm$ 2.2  | 3.4 $\pm$ 2.0  | 9              | 0      |
| Exp. sphere | 130.9 $\pm$ 12.9 | 2.8 $\pm$ 0.2    | 5.9 $\pm$ 1.0  | 0.6 $\pm$ 0.2  | 18             | 1.7    |
| ISO      | 151.0 $\pm$ 11.7 | 1.9 $\pm$ 0.1    | 6.5 $\pm$ 0.8  | 0.3 $\pm$ 0.1  | 0              | 0.4    |
In Figure 4 we plotted $\rho(r)$ (left panel) and $P(r)$ (right panel) for galaxy U11454, using Eqs. (1) – (6) and (10), respectively.

![Figure 4 - Color online. Left panel: Logarithmic density profiles of DM in the halo. Right panel: Logarithmic pressure profiles of DM in the halo.](image)

Solving Eqs. (7) and (8) we plotted the EoS for all the profiles in Figure 5 (left panel). Besides, in Fig. 5 (right panel) we plotted the speed of sound according to Eq. (12) for all the profiles.

In Figs. 4 (right panel) and 5 (left panel) one can adopt different units by using the following conversion factors:

$$M_\odot/\text{pc}^3 = 6.77 \times 10^{-23} \text{ g/cm}^3 = 38.05 \text{ GeV/cm}^3,$$

$$M_\odot/(\text{pc s}^2) = 6.44 \times 10^{14} \text{ g/(cm s}^2) = 3.62 \times 10^{38} \text{ GeV/(cm s}^2).$$

It should be recalled that for ISO, exponential sphere, Burkert, Beta and Brownstein profiles $\rho_0$ is the central density. However for the Einasto profile $\rho_0$ is the characteristic density, i.e. one can choose arbitrary large central density. As can be seen from Fig. 5 (right panel), the behaviors are very different, some of them change drastically as a function of the density. Our results are similar to the ones of Ref. [16], but for another galaxy.
4 Conclusion

By analyzing the RC data points of the galaxy U11454, we inferred the free parameters of the considered profiles with the help of the least squares method. We used the well-known cored DM density profiles from the literature and, in addition, we considered the exponential sphere, Beta and Brownstein profiles for comparison. The exponential sphere profile is usually applied to study the inner parts of galaxies [9] and, in fact, for galaxy U11454 it showed a good result. This can be seen from Table 1.

Out of all the considered profiles, the Brownstein profile has the highest BIC value, and the ISO profile has the lowest value. The same is true also for the $\chi^2$. The difference in BIC between ISO profile (with the smallest value) and Brownstein profile (with the highest value) is $\Delta\text{BIC} = 24$ and exhibits very strong evidence against the Brownstein profile. Among all the other profiles, the only one with the least evidence against it, though still strong, is the Einasto profile with $\Delta\text{BIC} = 9$. So, for our purposes ISO and Einasto profiles are the most convenient profiles to use for studying the DM EoS of the galaxy U11454.

Inspecting the curves for $\rho(r)$, $P(r)$ and $P(\rho)$, we see that the behavior of the profiles is similar to each other, but the degeneracy among the profiles is completely broken by looking at the diagram of the speed of sound. The results show that the Burkert profile is similar to the Einasto profile, and the Beta profile is similar to the ISO profile. As can be seen in Fig. 5 (right panel), the speed of sound for the Brownstein profile behaves differently and does allow to form structures.

Figure 5 - Color online. Left panel: The equation of state of DM halo. Right panel: The speed of sound for the DM fluid for different DM profiles.
In conclusion our analyzes exhibit that for the considered galaxy the Isothermal, Einasto and Burkert profiles are the best models as they have the smaller BIC than the exponential sphere, Beta and Brownstein profiles. Unfortunately, according to our recent findings [7, 33, 34] the assumption that the EoS of DM do not depend on the model turned out to be not valid. All features of DM such as the state parameter, refractive index and sound speed depend on the adopted models. Therefore if it is possible one should search for an alternative model independent approach to explore the nature of DM. It would be also interesting to continue studies in the direction including the contributions from gas and disk, taking into account a composite structure of a galaxy. This issue will be addressed in our future works.

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