On surfaces of general type with $q = 5$

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Abstract. We prove that a complex surface $S$ with irregularity $q(S) = 5$ that has no irrational pencil of genus $> 1$ has geometric genus $p_g(S) \geq 8$. As a consequence, we are able to classify minimal surfaces $S$ of general type with $q(S) = 5$ and $p_g(S) < 8$. This result is a negative answer, for $q = 5$, to the question asked in [13] of the existence of surfaces of general type with irregularity $q$ that have no irrational pencil of genus $> 1$ and with the lowest possible geometric genus $p_g = 2q - 3$ (examples are known to exist only for $q = 3, 4$).

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1. Introduction

Let $S$ be a smooth complex projective surface with irregularity $q(S) := h^0(\Omega^1_{S}) \geq 3$. The existence of a fibration $f : S \to B$ with $B$ a smooth curve of genus $b > 1$ (“an irrational pencil of genus $b > 1$”) gives much geometrical information on $S$ (cf. the survey [14]). However, surfaces with an irrational pencil of genus $b > 1$ can hardly be regarded as “general” among the irregular surfaces of general type: for instance, for $b < q(S)$ the Albanese variety of such a surface $S$ is not simple.

By the classical Castelnuovo-De Franchis theorem (cf. [6, Proposition X.9]), if $S$ has no irrational pencil of genus $> 1$ then the inequality $p_g(S) \geq 2q(S) - 3$ holds, where $p_g(S) := h^0(K_S)$ is, as usual, the geometric genus. This fundamental inequality has been recently generalized in [17] to Kähler varieties of arbitrary dimension.

The surfaces of general type $S$ for which the equality $p_g(S) = 2q(S) - 3$ holds are studied in [13]. There those with an irrational pencil of genus $> 1$ are classified and the inequality $K_S^2 \geq 7\chi(S) - 1$ is proven for $S$ minimal. However, the question of the existence of surfaces with $p_g(S) = 2q(S) - 3$ having no irrational pencil of

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genus \( b > 1 \) is wide open. At present, the state of the art is as follows:

- for \( q = 3 \), the only such surfaces are (the minimal desingularization of) a theta divisor in a principally polarized Abelian threefold ([11, 18]);
- for \( q = 4 \), a family of examples is constructed in [19];
- for \( q \geq 5 \), no example is known.

One is led to conjecture that for \( q > 4 \) there are no surfaces with \( p_g = 2q - 3 \) that have no irrational pencil. In this note we settle the case \( q = 5 \):

**Theorem 1.1.** Let \( S \) be a smooth projective complex surface with \( q(S) = 5 \) that has no irrational pencils of genus \( > 1 \). Then:

\[ p_g(S) \geq 8. \]

As a consequence we obtain the following classification theorem:

**Theorem 1.2.** Let \( S \) be a minimal complex surface of general type with \( q(S) = 5 \) and \( p_g(S) \leq 7 \). Then either:

(i) \( p_g(S) = 6, K^2_S = 16 \) and \( S \) is the product of a curve of genus 2 and a curve of genus 3; or
(ii) \( p_g(S) = 7, K^2_S = 24 \) and \( S = (C \times F)/\mathbb{Z}_2 \), where \( C \) is a curve of genus 7 with a free \( \mathbb{Z}_2 \)-action, \( F \) is a curve of genus 2 with a \( \mathbb{Z}_2 \)-action such that \( F/\mathbb{Z}_2 \) has genus 1 and \( \mathbb{Z}_2 \) acts diagonally on \( C \times F \). The map \( f : S \to C/\mathbb{Z}_2 \) induced by the projection \( C \times F \to C \) is an irrational pencil of genus 4 with general fibre \( F \) of genus 2.

The idea of the proof of Theorem 1.1 is to obtain contradictory upper and lower bounds for \( K^2_S \) under the assumption that \( p_g(S) < 8 \) and \( S \) is minimal.

For fixed \( q \) and \( p_g \), by Noether’s formula giving an upper bound for \( K^2 \) is the same as giving a lower bound for the topological Euler characteristic \( c_2 \). More precisely, it is the same as giving a lower bound for \( h^{1,1} \), the only Hodge number which is not determined by \( p_g \) and \( q \). In our situation, the upper bound follows directly from the result of [9] that if \( S \) is a surface of general type with \( q = 5 \), having no irrational pencils, then \( h^{1,1} \geq 11 + t \), where \( t \) is bigger or equal to the number of curves contracted by the Albanese map.

If the canonical system \( |K_S| \) has no fixed components, one can apply the results of [2] to get a lower bound for \( K^2_S \) which is enough to rule out this possibility. Hence the bulk of the proof consists in obtaining a lower bound for \( K^2_S \) under the assumption that \( |K_S| \) has a fixed part \( Z > 0 \). This is done in Section 2, where we improve by 1 in the case \( Z > 0 \) a well known inequality for surfaces with birational bicanonical map due to Debarre (cf. Corollary 2.7). The proof is based on a subtle numerical analysis of the intersection properties of the fixed and moving part of \( |K_S| \) that is, we believe, of independent interest.
It would be possible to generalize Theorem 1.1 for $q \geq 6$, if a good lower bound for $h^{1,1}(S)$ could be established. Unfortunately it is very difficult to extend the methods of [9] for $q \geq 6$. Recently, a lower bound on $h^{1,1}$ has been obtained in [12] by completely different methods, but it is not strong enough for our purposes.

**Notation and conventions:** a *surface* is a smooth complex projective surface. We use the standard notation for the invariants of a surface $S$: $p_g(S) := h^0(\omega_S) = h^2(\mathcal{O}_S)$ is the geometric genus, $q(S) := h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$ is the irregularity and $\chi(S) := p_g(S) - q(S) + 1$ is the Euler–Poincaré characteristic.

An irrational pencil of genus $b$ of a surface $S$ is a fibration $f : S \to B$, where $B$ is a smooth curve of genus $b > 0$.

We use $\equiv$ to denote linear equivalence and $\sim$ to denote numerical equivalence of divisors.

An effective divisor $D$ on a smooth surface is $k$-connected if for every decomposition $D = A + B$, with $A, B > 0$ one has $AB \geq k$. (Recall that on a minimal surface of general type every $n$-canonical divisor is 1-connected and, unless $n = 2$ and $K^2_S = 1$, it is also 2-connected (cf. [3])).

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## 2. Reider divisors

Let $S$ be a surface and let $M$ be a nef and big divisor on $S$ such that $M^2 \geq 5$. By Reider’s theorem, if a point $P$ of $S$ is a base point of $|K_S + M|$, then there is an effective divisor $E$ passing through $P$ such that either:

- $E^2 = -1, ME = 0$ or
- $E^2 = 0, ME = 1$.

This suggests the following definition:

**Definition 2.1.** Let $M$ be a nef and big divisor on a surface $S$. An effective divisor $E$ such that $E^2 = k$ and $EM = s$ is called a $(k, s)$ divisor of $M$.

By [8, (0.13)], the $(-1, 0)$ divisors and the $(0, 1)$ divisors are 1-connected. In addition, if $E$ is a $(1, 0)$ divisor, using the index theorem one shows that the intersection form on the components of $E$ is negative definite. In particular, there exist only finitely many $(-1, 0)$ divisors of $M$ on $S$.

**Lemma 2.2.** Let $M$ be a nef divisor with $M^2 \geq 5$ on a surface $S$. Then:

1. if $E$ is a reducible $(0, 1)$ divisor $E$ of $M$, and $0 < C < E$ then $C^2 < 0$;
2. if $E_1, E_2$ are two distinct $(0, 1)$ divisors of $M$, then $E_1E_2 = 0$ and $E_1$ and $E_2$ are disjoint.
Proof. Let $E, C$ be as in (i). The index theorem gives $C^2 < 0$ if $MC = 0$ and $C^2 \leq 0$ if $MC = 1$. Assume that $C^2 = 0$. Then $EC = (E - C)C > 0$, since $E$ is 1-connected, and therefore $(E + C)^2 \geq 2$. Since $M^2 \geq 5$ and $M(C + E) = 2$ we have a contradiction to the index theorem. Hence $C^2 < 0$.

Next we prove (ii). We have:

$M^2 \geq 5, \quad M(E_1 + E_2) = 2, \quad M(E_1 - E_2) = 0,$

hence by the index theorem we obtain:

$2E_1E_2 = (E_1 + E_2)^2 \leq 0, \quad -2E_1E_2 = (E_1 - E_2)^2 \leq 0.$

So $E_1E_2 = 0$. By 1-connectedness of $E_1, E_2$ we conclude that neither divisor is contained in the other. Then we can write $E_1 = A + B, E_2 = A + C$ where $A \geq 0, B, C > 0$ and $B$ and $C$ have no common components.

Since $M$ is nef and $ME_1 = 1$, we have $1 \geq MB(=MC)$ and so $B^2 \leq 0, C^2 \leq 0$. Then, since $0 = (E_1 - E_2)^2 = (B - C)^2$, we conclude that $B^2 = C^2 = BC = 0$. Hence by (i) $B = E_1$ and $C = E_2$, namely $A = 0$ and $E_1$ and $E_2$ are disjoint. □

Lemma 2.3. Let $S$ be a surface and let $M$ be a nef and big divisor such that the linear system $|M|$ has no fixed components. Let $E$ be a $(0, 1)$ divisor of $M$ and let $C$ be the only irreducible component of $E$ such that $MC = 1$. Then either $|M|$ has a base point on $C$ or $C$ is a smooth rational curve.

Proof. Suppose $|M|$ has no base points on $C$. Then, since $MC = 1$ the restriction map $H^0(M) \to H^0(C, M|_C)$ has image of dimension at least 2. It follows that $C$ is a smooth rational curve. □

Proposition 2.4. Let $X$ be a non ruled surface and let $M$ be a divisor of $X$ such that:

- $M^2 \geq 5$,
- the linear system $|M|$ has no fixed components and maps $X$ onto a surface.

Let $C$ be an irreducible curve contained in the fixed locus of $|K_X + M|$. Then either:

(i) $C$ is contained in a $(-1, 0)$ divisor of $M$, $MC = 0$ and $C^2 < 0$;

or

(ii) $C$ is contained in a $(0, 1)$ divisor of $M$, $MC \leq 1$ and $C^2 \leq 0$.

Proof. Let $P \in C$ be a point. By Reider’s theorem, there is a $(-1, 0)$ divisor or a $(0, 1)$ divisor of $M$ passing through $P$.

Assume for contradiction that $C$ is not a component of any $(-1, 0)$ or $(0, 1)$ divisor of $M$. Since there are only finitely many distinct $(-1, 0)$ divisors of $M$ in $S$, we can assume that there is a $(0, 1)$ divisor passing through a general point $P$ of $C$. It follows that there are infinitely many $(0, 1)$ divisors on $S$. Recall that two distinct
(0, 1) divisors are disjoint by Lemma 2.2. Thus, since $|M|$ has a finite number of base points, by Lemma 2.3 $X$ is ruled, against the assumptions. So $C$ is contained in a $(−1, 0)$ divisor or a $(0, 1)$ divisor $E$ of $M$. In the first case, $M$ being nef implies that $MC = 0$ and so $C^2 < 0$ by the index theorem. In the second case, again by nefness $MC ≤ 1$ and again by the index theorem $C^2 ≤ 0$.

**Lemma 2.5.** Let $S$ be a surface and let $M$ be a nef and big divisor of $S$ and let $E$ be a $(0, 1)$ divisor of $M$. If $L$ is a divisor such that $(M − L)^2 > 0$ and $M(M − L) > 0$, then $EL ≤ 0$.

**Proof.** Write $γ := M(M − L)$. Then $M(γE − (M − L)) = 0$. Since $(M − L)^2 > 0$ and $E^2 = 0$, $γ(M − L) / E$. Thus, by the index theorem $0 > (γE − (M − L))^2 = −2γE(M − L) + (M − L)^2$.

So $E(M − L) > 0$, and therefore $EL ≤ 0$. □

**Proposition 2.6.** Let $S$ be a smooth minimal surface of general type and let $M$ be a divisor such that

- $Z := K_S − M > 0$;
- the linear system $|M|$ has no fixed components and maps $S$ onto a surface.

Then the following hold:

(i) if $M^2 ≥ 5 + KZ$, then $h^0(2M) < h^0(K_S + M)$;
(ii) if $M^2 ≥ 5$, $(M − Z)^2 > 0$ and $M(M − Z) > 0$, then there are no $(0, 1)$ divisors of $M$. Furthermore $h^0(2M) < h^0(K_S + M)$ and every irreducible fixed component $C$ of $|K_S + M|$ satisfies $MC = 0$.

**Proof.** We observe first of all that $h^0(2M) = h^0(K_S + M)$ if and only if $Z$ is the fixed part of $|K_S + M|$.

(i) Assume for contradiction that $h^0(2M) = h^0(K_S + M)$. Let $C$ be an irreducible component of $Z$. By Proposition 2.4, $C^2 ≤ 0$ and $MC ≤ 1$. Now

$$−2 ≤ C^2 − KC ≤ C^2 + KZ,$$

and hence $C^2 ≥ −2 − KZ$. It follows

$$(M − C)^2 = M^2 − 2MC + C^2 ≥ M^2 − 2 − 2 − KZ = M^2 − 4 − KZ > 0.$$ In addition, we have:

$$M(M − C) = (M − C)^2 + C(M − C) ≥ (M − C)^2 − C^2 ≥ (M − C)^2 > 0.$$ Since $MZ ≥ 2$ by the 2-connectedness of canonical divisors, there is at least a component $D$ of $Z$ such that $MD > 0$. By Proposition 2.4, we have $MD = 1$ and $D$ is contained in a $(0, 1)$ divisor $E$ of $M$. Then Lemma 2.5 gives $EC ≤ 0$ for all the components of $Z$, and so $EZ ≤ 0$. □
But now since $ME = 1$ and $E^2 = 0$ we obtain that $KE = 1 + EZ \leq 1$. On
the other hand, $K_S E$ is $> 0$ by the index theorem and it is even by the adjunction
formula, hence we have a contradiction.

(ii) Let $E$ be a $(0, 1)$ divisor of $M$. Then we have $EZ \leq 0$ by Lemma 2.5 and
we get a contradiction as above. So there are no $(0, 1) \text{ divisors of } M \text{ on } S$. Hence by
Proposition 2.4 every irreducible fixed curve of $|K_S + M|$ satisfies $MC = 0$. Since
$MZ \geq 2$ by the 2-connectedness of the canonical divisors, not every component of
$Z$ can be a fixed component of $|K_S + M|$ and therefore $h^0(K_S + M) > h^0(2M)$. □

As a consequence, we obtain the following refinement of [10, Theorem 3.2 and
Remark 3.3]:

**Corollary 2.7.** Let $S$ be a minimal surface of general type whose canonical map is
not composed with a pencil. Denote by $M$ the moving part and by $Z$ the fixed part
of $|K_S|$. If $Z > 0$ and $M^2 \geq 5 + K_SZ$, then

$$K_S^2 + \chi(S) = h^0(K_S + M) + K_SZ + MZ/2 \geq h^0(2M) + K_SZ + MZ/2 + 1.$$  

Furthermore, if $h^0(K_S + M) = h^0(2M) + 1$ then $|K_S + M|$ has base points and
there is a $(-1, 0)$ divisor or a $(0, 1)$ divisor $E$ of $M$ such that $EZ \geq 1$.  

**Proof.** Since $M$ is nef and big, by Kawamata-Viehweg vanishing $h^0(K_S + M) = \chi(K_S + M)$, hence the equality follows by the Riemann-Roch theorem whilst the
inequality is Proposition 2.6, (i).

For the second assertion it suffices to notice that $h^0(K_S + M) = h^0(2M) + 1$
means that the image of the restriction map $H^0(K_S + M) \to H^0(Z, (K_S + M)|_Z)$
is 1-dimensional. Since $(K_S + M)Z \geq 2$, the system $|K_S + M|$ has necessarily base
points. Thus there is a $(-1, 0)$ divisor or a $(0, 1)$ divisor $E$ of $M$. By adjunction
$K_SE - E^2$ is even and so necessarily $EZ \geq 1$. □

3. Proofs of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1.** Let $a : S \to A$ be the Albanese map of $S$. Notice that by
the classification of surfaces the assumptions that $q(S) = 5$ and $S$ has no irrational
pencil of genus $> 1$ imply that $S$ is of general type and $a$ is generically finite onto
its image. Without loss of generality we may assume that $S$ is minimal. By [5], an
irregular surface of general type having no irrational pencils of genus $> 1$ satisfies
$p_g \geq 2q - 3$. We assume for contradiction that $p_g(S) = 7 = 2q(S) - 3$, so that
$\chi(S) = 3$. We denote by $\varphi_K : S \to \mathbb{P}^6$ the canonical map and by $\Sigma$ the canonical
image. Since $q(S) > 2$, $\Sigma$ is a surface by [20].

We denote by $t$ the rank of the cokernel of the map $a^* : \text{NS}(A) \to \text{NS}(S)$. Note that $t$ is bigger than or equal to the number of irreducible curves contracted by
the Albanese map.
Denote as usual by $b_i(S)$ the $i$-th Betti number and by $c_2(S)$ the second Chern class of $S$. By [9, Theorem 1.3(i)], we have $b_2(S) \geq 31 + t$, namely $c_2(S) \geq 13 + t$. By Noether’s formula this is equivalent to:

$$K_S^2 \leq 23 - t. \quad (3.1)$$

Denote by $\mathbb{G}$ the Grassmannian of 2-planes of $H^0(\Omega^1_S)^\vee$ and by $\mathbb{G}^\vee$ the Grassmannian of 2-planes in $H^0(\Omega^1_S)$. By the Castelnuovo–De Franchis theorem, the kernel of the map $\rho \: \wedge^2 H^0(\Omega^1_S) \to H^0(K_S)$ does not contain any nonzero simple tensor. Hence $\rho$ induces a morphism $\mathbb{G}^\vee \to \mathbb{P}(H^0(K_S))$ which is finite onto its image. Since $\dim \mathbb{G}^\vee = 6$, it follows that $\ker \rho$ has dimension 3, $\rho$ is surjective and it induces a finite map $\mathbb{G}^\vee \to \mathbb{P}(H^0(K_S))$. As a consequence, we have the following facts:

(a) the surface $S$ is generalized Lagrangian, namely there exist independent 1-forms $\eta_1, \ldots, \eta_4 \in H^0(\Omega^1_S)$ such that $\eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4 = 0$. In addition, we may assume that $\eta_1 \wedge \eta_2$ is a general 2-form of $S$. In that case, the fixed part of the linear system $\mathbb{P}(\wedge^2 V)$, where $V = \langle \eta_1, \ldots, \eta_4 \rangle$, coincides with the fixed part of the canonical divisor (cf. [15, Section 3]).

(b) the canonical image $\Sigma$ is contained in the intersection of $\mathbb{G}$ with the codimension 3 subspace $T = \mathbb{P}(\text{Im} \rho^\vee) \subset \mathbb{P}^9 = \mathbb{P}(\wedge^2 H^0(\Omega^1_S))$ (where $\rho^\vee$ is the transpose of $\rho$).

(c) since $\mathbb{G}^\vee$ is the dual variety of $\mathbb{G}$, the space $T$ is not contained in an hyperplane tangent to $\mathbb{G}$, hence $Y := \mathbb{G} \cap T$ is a smooth threefold.

Using the Lefschetz hyperplane section theorem we see that $\text{Pic}(Y)$ is generated by the class of a hyperplane. Then $\Sigma$ is the scheme theoretic intersection of $Y$ with a hypersurface of degree $m \geq 2$ of $\mathbb{P}^6$. Thus, since $\mathbb{G}$ has degree 5 (cf. [16, Corollary 1.11]), it follows that $\deg \Sigma = 5m$ and by [16, Proposition 1.9] we have $\omega_\Sigma = \mathcal{O}_\Sigma(m - 2)$. By [13, Theorem 1.2], the degree $d$ of $\varphi_K$ is different from 2. Since $K_S^2 \leq 23$ by (3.1), the inequality $K_S^2 \geq d \deg \Sigma = 5dm$ gives $d = 1$, namely $\varphi_K$ is birational onto its image. So we have $m \geq 3$, since $\omega_\mathbb{G} = \mathcal{O}_\mathbb{G}(-5)$ (cf. [16, Proposition 1.9]) and $\Sigma$ is of general type.

Write $|K_S| = |M| + Z$, where $Z$ is the fixed part and $M$ is the moving part. If $Z = 0$, then in view of (a) we have $K_S^2 \geq 8\chi = 24$ by [2, Theorem 1.2]. This would contradict (3.1), hence $Z > 0$.

Since $m > 2$, every quadric that contains $\Sigma$ must contain $Y$. Recall that $Y$ is obtained from $\mathbb{G}$ by intersecting with 3 independent linear sections. Denote by $R$ the homogeneous coordinate ring of $\mathbb{G}$. Since $R$ is Cohen–Macaulay and $Y$ has codimension 3 in $\mathbb{G}$, these 3 linear sections form an $R$-regular sequence. As a consequence (cf. [7, Proposition 1.1.5]) the (vector) dimension of the space of quadrics of $\mathbb{P}^6$ containing $Y$ is the same as the (vector) dimension of the space of quadrics of $\mathbb{P}^9$ containing $\mathbb{G}$. Since the latter dimension is 5 (cf. [16, Proposition 1.2]), it follows that:

$$h^0(2M) \geq h^0(\mathcal{O}_{\mathbb{P}^6}(2)) - 5 = 23.$$
Then by (3.1) and Corollary 2.7 we have:

\[ 26-t \geq K_S^2 + \chi(S) = h^0(K_S+M) + K_SZ + MZ/2 \geq 23 + K_SZ + MZ/2 + 1. \]  

(3.2)

So \( K_SZ + MZ/2 \leq 2-t \). Recall that \( MZ \geq 2 \) by the 2-connectedness of canonical divisors.

Assume \( K_SZ = 0 \). Then every component of \( Z \) is an irreducible smooth rational curve with self-intersection \(-2\) and as such it is contracted by the Albanese map. Since \( K_SZ + MZ/2 \leq 2-t \), the only possibility is \( t = 1 \) and \( MZ = 2 \). Hence \( Z = rA \), where \( A \) is a \(-2\)-curve. Since \( MZ = 2 \) and \( K_SZ = 0 \), we have \( Z^2 = -2 \) and so \( r = 1 \). Hence \( Z \) is a \(-2\)-cycle of type \( A_1 \); in particular it is reduced and, in the terminology of [2], it is contracted by any subspace \( V \subseteq H^0(\Omega^1_S) \). Then, again by (a) and [2, Theorem 1.2], we get \( K^2 \geq 2\chi = 24 \), a contradiction.

So \( K_SZ > 0 \). Then by (3.2) necessarily \( K_SZ = 1 \), \( MZ = 2 \) (yielding \( Z^2 = -1 \)) and \( h^0(K_S + M) = 24 \leq h^0(2M) + 1 \). As we have already remarked, the canonical image \( \Sigma \) has degree \( \geq 15 \). Therefore \( M^2 \geq 15 > 5 + K_SZ = 6 \) and, by Corollary 2.7, there is a \((-1, 0)\) or a \((0, 1)\) divisor \( E \) of \( M \). Since the hypotheses of Proposition 2.6, (ii) are satisfied, \( E \) must be a \((-1, 0)\) divisor of \( M \).

Then \( M(E + Z) = 2 \) and so by the algebraic index theorem \( M^2(E + Z)^2 - 4 \leq 0 \), yielding \((E + Z)^2 \leq 0 \). Since \((E + Z)^2 = -2 + 2EZ \) and, by Corollary 2.7, \( EZ \geq 1 \), the only possibility is \( EZ = 1 \) and \((E + Z)^2 = 0 \). In this case \( K_S(E + Z) = 2 \) and this is impossible by the proof of [2, Proposition 8.2], which shows that a minimal irregular surface with \( q \geq 4 \), having no irrational pencils of genus \( > 1 \), cannot have effective divisors of arithmetic genus 2 and self-intersection 0.

Proof of Theorem 1.2. By [5], a surface of general type \( S \) with \( q(S) = 5 \) has \( p_g(S) \geq 6 \) and, in addition, if \( p_g(S) = 6 \) then \( S \) is the product of a curve of genus \( C \) and a curve of genus 3. Now statement (ii) is a consequence of Theorem 1.1 and [13, Theorem 1.1].

References

[1] M. A. Barja, Numerical bounds of canonical varieties, Osaka J. Math. 37 (2000), 701–718.
[2] M. A. Barja, J. C. Naranjo and G. P. Pirola, On the topological index of irregular surfaces, J. Algebraic Geom. 16 (2007), 435–458.
[3] W. Barth, C. Peters and A. Van de Ven, “Compact Complex Surfaces”, Ergebnisse der Mathematik, 3. Folge, Band 4, Springer, Berlin, 1984.
[4] A. Beauville, L’application canonique pour les surfaces de type général, Invent. Math. 55 (1979), 121–140.
[5] A. Beauville, L’inégalité \( p_g \geq 2q - 4 \) pour les surfaces de type général, appendix to [10].
[6] A. Beauville, “Complex Algebraic Surfaces”, second edition, L.M.S Student Texts 34, Cambridge University Press, Cambridge, 1996.
[7] W. Bruns and J. Herzog, “Cohen-Macaulay Rings”, revised edition, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1998.
[8] F. CATANESE, C. CILIBERTO and M. MENDES LOPES, On the classification of irregular surfaces of general type with non birational bicanonical map, Trans. Amer. Math. Soc. 350 (1998), 275–308.

[9] A. CAUSIN and G. P. PIROLA, Hermitian matrices and cohomology of Kaehler varieties, Manuscripta Math. 121 (2006), 157–168.

[10] O. DEBARRE, Inégalités numériques pour les surfaces de type général, with an appendix by A. Beauville, Bull. Soc. Math. France 110 (1982), 319–346.

[11] C. D. HACON and R. PARDINI, Surfaces with \( p_g = q = 3 \), Trans. Amer. Math. Soc. 354 (2002), 2631–2638.

[12] R. LAZARSFELD and M. POPA, Derivative complex BGG correspondence and numerical inequalities for compact Kähler manifolds, Invent. Math. 182 (2010), 605–633.

[13] M. MENDES LOPES and R. PARDINI, On surfaces with \( p_g = 2q – 3 \), Adv. Geom. 10 (2010), 549–555.

[14] M. MENDES LOPES and R. PARDINI, The geography of irregular surfaces, In: “Current Developments in Algebraic Geometry”, Math. Sci. Res. Inst. Publ., Cambridge Univ. Press 59 (2012), 349–378.

[15] M. MENDES LOPES and R. PARDINI, Severi type inequalities for surfaces with ample canonical class, Comment. Math. Helv. 86 (2011), 401–414.

[16] S. MUKAI, “Curves and Grassmannians”, Algebraic geometry and related topics (Inchon, 1992), 19–40, Conf. Proc. Lecture Notes Algebraic Geom., I, Int. Press, Cambridge, MA, 1993.

[17] G. PARESCHI and M. POPA, Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality, Duke Math. J. 150 (2009), 269–28.

[18] G. P. PIROLA, Algebraic surfaces with \( p_g = q = 3 \) and no irrational pencils, Manuscripta Math. 108 (2002), 163–170.

[19] C. SCHOEN, A family of surfaces constructed from genus 2 curves, Internat. J. Math. 18 (2007), 585–612.

[20] G. XIAO, Irregularity of surfaces with a linear pencil, Duke Math. J. 55 (1987), 596–602.