MINIMAL PERIODS FOR ORDINARY DIFFERENTIAL EQUATIONS
IN STRICTLY CONVEX BANACH SPACES AND EXPLICIT BOUNDS
FOR SOME $\ell^p$-SPACES

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Abstract. Let $x(t)$ be a non-constant $T$-periodic solution to the ordinary
differential equation $\dot{x} = f(x)$ in a Banach space $X$, where $f$
is assumed to be Lipschitz continuous with constant $L$. Then there exists a
constant $c$ such that $TL \geq c$, with $c$ only depending on $X$. It is
known that $c \geq 6$ in any Banach space and that $c = 2\pi$ in any Hilbert
space, but whereas the bound of $c = 2\pi$ is sharp in any Hilbert space,
there exists only one known example of a Banach space such that $c = 6$
is optimal. In this paper, we show that the inequality $TL \geq 6$ is in fact
strict in any strictly convex Banach space. Moreover, we improve the lower
bound for $\ell^p(\mathbb{R}^n)$ and $L^p(M,\mu)$ for a range of $p$ close to $p = 2$
by using a form of Wirtinger’s inequality for functions in $W^{1,p}_{per}([0,T],L^p(M,\mu))$.

1. Introduction

Consider the ordinary differential equation $\dot{x} = f(x)$ in a Banach space $X$, where $f$
is Lipschitz continuous with constant $L$, that is for any $x,y \in X$
$$\|f(x) - f(y)\|_X \leq L\|x - y\|_X.$$ 
In this case one can relate the period $T$ of any non-constant periodic orbit to the Lipschitz
constant $L$ via the inequality $TL \geq c$. In 1969, Yorke [7] proved that $c = 2\pi$
when $X = \mathbb{R}^n$ with its usual norm. Lasota & Yorke [6] showed that the proof extends to arbitrary Hilbert
spaces and they proved the bound $c = 4$ for any Banach space. This was improved
to $c = 4.5$ by Busenberg & Martelli [1] and finally to $c = 6$ by Busenberg, Fisher & Martelli
[2] who also gave another proof for $c = 2\pi$ in any Hilbert space using Wirtinger’s inequality.
An obvious extension of the simple two-dimensional example
$$\dot{x} = Ly \quad \dot{y} = -Lx$$
shows that $c = 2\pi$ is sharp in any Hilbert space. Busenberg, Fisher & Martelli [3] also
constructed an example of an ODE on a periodic orbit of period 1, which when viewed as
a subset of $L^1([0,1]^2)$ has Lipschitz constant $L = 6$, showing that $c = 6$ is sharp for general
Banach spaces.

However, some interesting questions about minimal periods remain unanswered. Does
there exist an ODE in a finite-dimensional Banach space such that the lower bound of

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$TL = 6$ is obtained? Does $TL \geq 2\pi$ characterise Hilbert spaces? Is there a (non-Hilbert) Banach space for which $c > 6$?

The results in this paper address this last question. First we show that in strictly convex Banach spaces necessarily $TL > 6$. For these normed topological vector spaces the unit ball is a strictly convex set. It is easy to see that the unit balls in $\ell^1$ and $\ell^\infty$ contain a line segment and are therefore not strictly convex sets whereas the unit balls for all $1 < p < \infty$ are strictly convex. This result nicely complements the current theory because the only example for a Banach space with $c = 6$ is $L^1$.

However, we prove not only that the inequality is strict in any strictly convex Banach space but we are also able to push the bound a little further for the simplest family of interesting finite-dimensional Banach spaces, namely $\ell^p(R^n)$, that is $R^n$ equipped with the $\ell^p$-norm,

$$
\|(x_1, \ldots, x_n)\|_{\ell^p} = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}.
$$

It is remarkable that even for Euclidean spaces with the family of $\ell^p$-norms the optimal constant is not known for $p \neq 2$. Our second contribution in this paper is to point out that by using a generalised form of Wirtinger’s inequality, one can find explicit bounds on $c$ which are strictly larger than 6 in a range of $\ell^p$-spaces near $p = 2$ ($1.43 \lesssim p \lesssim 3.35$). A similar argument also works in the infinite-dimensional Lebesgue spaces $L^p(M, \mu)$.

We should mention the interesting related result, due to Zevin [9], that if $X$ is a finite-dimensional Banach space and one considers the second order equation $\ddot{x} = f(x)$ with $f : X \to X$ Lipschitz with constant $L^2$, then $TL \geq 2\pi$ independent of the space $X$. (The paper [9] claims a similar result for the first order equation $\dot{x} = f(x)$, but there is a small error in the proof of equation (11). Nevertheless, Zevin’s argument readily yields the result we have stated for $\ddot{x} = f(x).$)

2. Minimal periods in strictly convex Banach spaces

Let us start this section by stating the main result of this paper:

**Theorem 2.1.** Let $X$ be a strictly convex Banach space. Then

$$TL > 6.$$  

In fact the proof of this statement is a refinement of an integral inequality originally introduced by Busenberg, Martelli & Fisher [2]. The revised version of the result is summarised in the following lemma.

\^[1]Unfortunately there appears to be an error in one of the integral calculations in the paper by Zevin [8] which claims to show that $c = 2\pi$ in $\ell^\infty(R^n)$.
Lemma 2.2. Let $X$ be a normed space and $y : \mathbb{R} \to X$ be a continuous, $T$-periodic function such that $\|\dot{y}(t)\|$ is integrable. Then
\[
\int_0^T \int_0^T \|y(t) - y(s)\| \, ds \, dt \leq \frac{T}{6} \int_0^T \int_0^T \|\dot{y}(t) - \dot{y}(s)\| \, ds \, dt.
\]

If $X$ is a strictly convex Banach space, then the above inequality is in fact strict.

Before we go into details of the proof, we show how Busenberg, Fisher & Martelli used it to establish $TL \geq 6$ for any Banach space.

Proof of Theorem 2.1: Applying Lemma 2.2 and using the Lipschitz continuity of $f$, it follows that
\[
\int_0^T \int_0^T \|x(t) - x(s)\| \, ds \, dt \leq \frac{T}{6} \int_0^T \int_0^T \|\dot{x}(t) - \dot{x}(s)\| \, ds \, dt
\]
\[
\leq \frac{LT}{6} \int_0^T \int_0^T \|x(t) - x(s)\| \, ds \, dt.
\]

Dividing both sides of the inequality by $\int_0^T \int_0^T \|x(t) - x(s)\| \, ds \, dt$ yields the result. \hfill \Box

We now turn to the main proof of this section.

Proof of Lemma 2.2: We know that $y$ is periodic with period $T$. Hence its integral over one period is shift invariant and thus
\[
\int_0^T \int_0^T \|y(t + s) - y(s)\| \, ds \, dt = \int_0^T \int_0^T \|y(t) - y(s)\| \, ds \, dt.
\]

Using the above observation, we can derive the following integral expression
\[
\int_0^T \int_0^T \|y(t) - y(s)\| \, ds \, dt = \int_0^T \int_0^T \|y(t + s) - y(s)\| \, ds \, dt
\]
\[
= \int_0^T \int_0^T \frac{(T-t)t}{T} \left\| \frac{y(t+s) - y(s)}{t} - \frac{y(s) - y(s+t-T)}{T-t} \right\| \, ds \, dt
\]
\[
= \int_0^T \int_0^T \frac{(T-t)t}{T^2} \left\| \int_0^T \dot{y} \left( s + \frac{tr}{T} \right) - \dot{y} \left( s + \frac{tr}{T} - r \right) \, dr \right\| \, ds \, dt
\]
\[
\leq \int_0^T \int_0^T \frac{(T-t)t}{T^2} \left\| \int_0^T \dot{y} \left( s + \frac{tr}{T} \right) - \dot{y} \left( s + \frac{tr}{T} - r \right) \right\| \, dr \, ds \, dt
\]
\[
= \int_0^T \int_0^T \frac{(T-t)t}{T^2} \int_0^T \left\| \frac{\dot{y}}{T} \left( s + \frac{tr}{T} \right) - \frac{\dot{y}}{T} \left( s + \frac{tr}{T} - r \right) \right\| \, ds \, dr \, dt
\]
(1)
The last inner integral has been taken over one period, so we may shift it by \( tr/T \) in order to obtain
\[
\int_0^T \int_0^T \|y(r) - y(s)\| \, dr \, ds \leq \int_0^T (T - t) \frac{t}{T^2} dt \int_0^T \int_0^T \|\dot{y}(s + r) - \dot{y}(s)\| \, ds \, dr
\]
\[
= \frac{T}{6} \int_0^T \int_0^T \|\dot{y}(r) - \dot{y}(s)\| \, ds \, dr
\]
giving us the desired inequality for arbitrary Banach spaces.

From now on we consider the case when \( X \) is in fact a strictly convex Banach space. The only actual inequality in the above argument occurs in line (1) where we use the triangle inequality for the Banach space \( X \). Note that in doing so, we have a weight \( (T - t)t \) in front of the inner integral which vanishes at \( t = 0, T \). In particular, if we show that this inequality actually has to be strict for some \( s \) and some \( 0 < t < T \), our statement follows. Additionally, because of the weight, these conditions are tight as the triangle inequality could fail to be strict at \( t = 0, T \) without causing the chain of inequalities to become strict.

Note that from the continuity of \( \dot{y}(t) \) we obtain that the functions
\[
(s, t) \rightarrow \left\| \int_0^T \dot{y} \left(s + \frac{tr}{T} \right) - \dot{y} \left(s + \frac{tr}{T} - r \right) \, dr \right\|
\]
are continuous as well. Fix \( s \) and \( 0 < t < T \), fix an arbitrarily fine decomposition \( 0 = a_0 < a_1 < \ldots < a_n = T \) and abbreviate
\[
b_i := \int_{a_i}^{a_{i+1}} \dot{y} \left(s + \frac{tr}{T} \right) \quad \text{and} \quad c_i := \int_{a_i}^{a_{i+1}} \dot{y} \left(s + \frac{tr}{T} - r \right) .
\]

If there is in fact equality in (1), then we need to have equality in every step of iteratively applying the triangle inequality and thus
\[
\left\| \sum_{i=0}^{n-1} b_i - c_i \right\| = \left\| b_0 - c_0 \right\| + \left\| \sum_{i=1}^{n-1} b_i - c_i \right\|
\]
\[
= \left\| b_0 - c_0 \right\| + \left\| b_1 - c_1 \right\| + \left\| \sum_{i=2}^{n-1} b_i - c_i \right\|
\]
\[
= \ldots
\]
\[
= \sum_{i=0}^{n-1} \left\| b_i - c_i \right\| .
\]
W.l.o.g. we assume that all the terms satisfy \( b_i - c_i \neq 0 \). Strict convexity implies in the last line of this argument that \( b_{n-2} - c_{n-2} \) and \( b_{n-1} - c_{n-1} \) are collinear. By the same reasoning
\( b_{n-3} - c_{n-3} \) and \( (b_{n-2} - c_{n-2}) + (b_{n-1} - c_{n-1}) \) are collinear, however, the last expression itself is collinear to \( b_{n-2} - c_{n-2} \) as well as \( b_{n-1} - c_{n-1} \). Iterating this argument shows that all \( b_i - c_i \) are necessarily collinear. Using the continuity of \( \dot{y}(t) \), making the partition sufficiently small and applying the fundamental theorem of calculus, we can deduce that for every fixed \( s \) and \( 0 < t < T \) there exists a vector \( \mathbf{v} \in X \) and a function \( g : [0, T] \to \mathbb{R} \) such that for all \( 0 \leq r \leq T \)

\[
\dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) = g(r)\mathbf{v}.
\]

(2) \( \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) = g(r)\mathbf{v} \).

Note, however, that both \( g \) and \( \mathbf{v} \) depend on the previously fixed \( s, t \). Since \( y \) is not constant, it is possible to find and fix an \( s \) such that \( \dot{y}(s) \neq 0 \).

We now claim that this already implies that for all \( 0 \leq r \leq T \)

\[
\dot{y}(s + r) = \tilde{g}(r)\mathbf{v} + \dot{y}(s).
\]

Suppose this was false, then there is an \( r \) such that

\[
\dot{y}(s + r) \notin \{\dot{y}(s) + \lambda \mathbf{v} | \lambda \in \mathbb{R}\}.
\]

In particular,

\[
\min_{\lambda \in \mathbb{R}} \|\dot{y}(s + r) - \dot{y}(s) + \lambda \mathbf{v}\| > 0.
\]

This, however, can be seen to contradict (2) by taking \( t \) sufficiently small.

Since \( y \) is periodic with period \( T \),

\[
\int_0^T \dot{y}(s + r)dr = \mathbf{0} = \left(\int_0^T \tilde{g}(r)dr\right)\mathbf{v} + T\dot{y}(s).
\]

This implies that \( \dot{y}(s) \) is a scalar multiple of \( \mathbf{v} \), in which case

\[
\dot{y}(s + r) = \left(\tilde{g}(r) - \frac{1}{T} \int_0^T \tilde{g}(r)dr\right)\mathbf{v}.
\]

This establishes that \( \dot{y}(t) \) is one-dimensional, that is

\[
\dot{y}(t) = h(t)\mathbf{v}
\]

for some \( \mathbf{v} \neq \mathbf{0} \) and a continuous, \( T \)-periodic function \( h : [0, T] \to \mathbb{R} \).

Going back to an earlier stage of the argument, we had that for any fixed \( s \) and \( 0 < t < T \) the application of the triangle inequality needs to be strict, that is

\[
\left\|\int_0^T \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) dr\right\| = \int_0^T \left\|\dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right)\right\| dr.
\]

Plugging in the relation \( \dot{y}(t) = h(t)\mathbf{v} \), we require that for any fixed \( s, t \) with \( 0 < t < T \)

\[
\left|\int_0^T h\left(s + \frac{tr}{T}\right) - h\left(s + \frac{tr}{T} - r\right) dr\right| = \int_0^T \left|h\left(s + \frac{tr}{T}\right) - h\left(s + \frac{tr}{T} - r\right)\right| dr.
\]
However, since \( h \) is continuous and
\[
\int_0^T h(z) \, dz = 0,
\]
h has to vanish in a point, say \( h(s) = 0 \). For \( t \) very small, we have
\[
\lim_{t \to 0} \left| \int_0^T h \left( s + \frac{tr}{T} \right) - h \left( s + \frac{tr}{T} - r \right) \, dr \right| = \left| \int_0^T h(s - r) \, dr \right| = 0
\]
while
\[
\lim_{t \to 0} \int_0^T \left| h \left( s + \frac{tr}{T} \right) - h \left( s + \frac{tr}{T} - r \right) \right| \, dr = \int_0^T |h(s - r)| \, dr,
\]
proving that \( h \equiv 0 \).  

3. A generalised form of Wirtinger’s inequality

The second part of this paper is devoted to establishing explicit bounds for a certain class of \( \ell^p \)-spaces. The idea of our approach goes back to the proof that \( TL \geq 2\pi \) in any Hilbert space which is based on an analogue of Wirtinger’s inequality for Hilbert spaces. In the following we adapt this idea by using the work of Croce & Dacorogna which found the optimal constant in a generalised set of Wirtinger inequalities, including the case of interest to us here. They showed that for
\[
 u \in \left\{ W^{1,p}_{\text{per}}(0, 1) \text{ with } \int_0^1 u(t) \, dt = 0 \text{ and } u(0) = u(1) \right\},
\]
where \( W^{1,p}_{\text{per}} \) is the space of \( L^p \)-functions \( u \) whose weak first derivative lies in \( L^p \), one has
\[
\left( \int_0^1 |u(t)|^p \right)^{1/p} \leq C_p \left( \int_0^1 |\dot{u}(t)|^p \, dt \right)^{1/p},
\]
where
\[
C_p = \frac{p}{4(p - 1)^{1/p} \int_0^1 t^{-\frac{1}{p}} (1 - t)^{\frac{1}{p} - 1} \, dt}
\]
is sharp. (Note that the integral appearing in the denominator is in fact the beta function \( B(1/p', 1/p) \) where \( p' \) is the Hölder conjugate of \( p \). Croce and Dacorogna consider functions defined on \( (-1, 1) \) but the form of the inequality here is more suitable for us in what follows.)

**Corollary 3.1.** Let \( u \in W^{1,p}_{\text{per}}([0, T], X) \) where \( X \) is either \( \ell^p(\mathbb{R}^n) \) or \( L^p(M, \mu) \) and assume that \( \int_0^T u(t) \, dt = 0 \). Then
\[
\int_0^T \|u(t)\|_{X}^p \, dt \leq C_p T p \int_0^T \|\dot{u}(t)\|_{X}^p \, dt,
\]
where \( C_p \) is given in (3) and is optimal.
Proof. By a simple change of variables it suffices to prove the result for \( T = 1 \). When \( X = L^p(\mathbb{R}^n) \) we have
\[
\int_0^1 \sum_{j=1}^n |u_j(t)|^p \, dt = \sum_{j=1}^n \int_0^1 |u_j(t)|^p \, dt \leq C_p^p \sum_{j=1}^n \int_0^1 |\dot{u}_j(t)|^p \, dt,
\]
from which (4) is immediate. One can see that the constant is optimal by considering \( u = (u_1, \ldots, u_n) \) with \( u_j \in W^{1,p}_{\text{per}}(0,1) \) and \( u_j = 0 \) for \( j = 2, \ldots, n \).

Similarly, for \( X = L^p(M, \mu) \) we have
\[
\int_0^1 \int_U |u(x,t)|^p \, d\mu \, dt = \int_U \int_0^1 |u(x,t)|^p \, dt \, d\mu \leq C_p^p \int_U \int_0^1 |\dot{u}(x,t)|^p \, dt \, d\mu = C_p^p \int_U \int_0^1 |\ddot{u}(x,t)|^p \, dt \, d\mu \, dt,
\]
and (4) follows once more. Optimality of the constant follows by taking \( f(t,x) = f(t)1_A \) for some \( f \in W^{1,p}_{\text{per}}(0,1) \) and \( A \subset U \) with \( \mu(A) > 0 \). \( \square \)

4. Improved lower bounds in \( L^p(\mathbb{R}^n) \) and \( L^p(M, \mu) \)

Having established Wirtinger’s inequality for \( W^{1,p}_{\text{per}}([0,T], X) \) where \( X \) is either \( L^p(\mathbb{R}^n) \) or \( L^p(M, \mu) \), we can now prove the second contribution of this paper. The simple proof is essentially that for \( p = 2 \) due to [2] which is a particular case of this result if one notes that \( C_2^{-1} = 2\pi \).

**Theorem 4.1.** Let \( x \) be a non-constant \( T \)-periodic solution to \( \dot{x} = f(x) \) in either \( X = L^p(\mathbb{R}^n) \) or \( X = L^p(M, \mu) \). Further, suppose that \( f \) is Lipschitz continuous from \( X \) into \( X \) with Lipschitz constant \( L \). Then
\[
TL \geq C_p^{-1}.
\]

**Proof.** As the function \( x \) is a solution to the ODE, it is differentiable by definition. Moreover, a simple calculation shows that
\[
\int_0^T x(t+h) - x(t) \, dt = 0.
\]
Hence Wirtinger’s inequality for \( W^{1,p}_{\text{per}}((0,T), X) \) is applicable to \( x(t+h) - x(t) \) and thus
\[
\int_0^T \|x(t+h) - x(t)\|_X^p \, dt \leq C_p^p T^p \int_0^T \|\dot{x}(t+h) - \dot{x}(t)\|_X^p \, dt
\]
\[
= C_p^p T^p \int_0^T \|f(x(t+h)) - f(x(t))\|_X^p \, dt
\]
\[
\leq L^p C_p^p T^p \int_0^T \|x(t+h) - x(t)\|_X^p \, dt.
\]
Dividing both sides by \( \int_0^T \|x(t+h) - x(t)\|_X^p \, dt \), which is non-zero as \( x \) is non-constant, yields (5). \( \square \)
Theorem 4.1 gives an improved lower bound on the product of Lipschitz constant $L$ and period $T$ for the spaces $\ell^p(\mathbb{R}^n)$ and $L^p(M, \mu)$ for a range of $p$ around $p = 2$. Figure 1 plots $C_p^{-1}$ against $p$ for $1 \leq p \leq 4$, and shows that $C_p^{-1} > 6$ for $1.43 \leq p \leq 3.35$.

Figure 1. Improved lower bound near $p = 2$ using Wirtinger’s inequality

Remark 1. For $1 \leq p < \infty$ one can construct an example of an ODE in $L^p(M, \mu)$ satisfying Lipschitz conditions on its derivative with period $2\pi$. Suppose there are two sets $A \cap B = \emptyset$ such that

$$0 < \mu(A) = \mu(B).$$

and consider the ODE

$$\dot{z} = f(z)$$

with $f : L^p(M, \mu) \to L^p(M, \mu)$ given by

$$f(z) = -\frac{\chi_B}{\mu(A)} \int_A z d\mu + \frac{\chi_A}{\mu(B)} \int_B z d\mu.$$

Then Hölder’s inequality gives that for $L = 1$ the quantity

$$I = \|f(z) - f(w)\|_{L^p(M, \mu)}^p$$
satisfies
\[ I = \left\| -\chi_B \frac{1}{\mu(A)} \int_A z - wd\mu + \chi_A \frac{1}{\mu(A)} \int_B z - wd\mu \right\|_{L^p}^p \]
\[ = \left( \frac{1}{\mu(A)} \int_A z - wd\mu \right)^p \mu(B) + \left( \frac{1}{\mu(B)} \int_B z - wd\mu \right)^p \mu(A) \]
\[ \leq \frac{1}{\mu(A)^p} \left( \int_A |z - w|^p d\mu \right) \mu(A)^{p-1} \mu(B) + \frac{1}{\mu(B)^p} \left( \int_B |z - w|^p d\mu \right) \mu(B)^{p-1} \mu(A) \]
\[ \leq \|z - w\|_{L^p}^p \]

and one explicit \(2\pi\)-periodic solution is given by
\[ z(t) = -(\cos t) \chi_A + (\sin t) \chi_B. \]

Notice that this example can be generalised further to the case when \(0 < \mu(A) \neq \mu(B)\).

**Remark 2.** Let \(X\) be a Banach space which obeys ‘almost’ a Hilbert space structure in the sense of the norm, that is there exists an \(\varepsilon > 0\) such that
\[ (1 - \varepsilon)\|x\|_H \leq \|x\|_X \leq (1 + \varepsilon)\|x\|_H. \]
Let \(x : \mathbb{R} \to X\) be a \(T\)-periodic solution to the ODE \(\dot{x} = f(x)\) with \(f\) being Lipschitz continuous with Lipschitz constant \(L\). Since
\[ \|f(x) - f(y)\|_H \leq \frac{1}{1 - \varepsilon} \|f(x) - f(y)\|_X \leq \frac{1}{1 - \varepsilon} L \|x - y\|_X \leq \frac{1 + \varepsilon}{1 - \varepsilon} L \|x - y\|_H, \]
it follows that \(f\) is also Lipschitz continuous with respect to the Euclidean norm with Lipschitz constant \(L' = \frac{1 + \varepsilon}{1 - \varepsilon} L\). At the same time, the length of the curve \(x\) as measured in the Hilbert space is smaller than \((1 + \varepsilon)T\) and using the fact that \(c = 2\pi\) in any Hilbert space we may conclude that
\[ TL \geq 2\pi \frac{1 - \varepsilon}{(1 + \varepsilon)^2}. \]

However, this approximation lags behind the numerical results for \(\ell^p\) obtained at the beginning of this section, especially for high dimensions.

**Remark 3.** Dvoretzky’s theorem in [5] guarantees that for any \(\varepsilon > 0\) there exists \(n \in \mathbb{N}\) sufficiently large such that any Banach space with \(\dim X \geq n\) contains a two-dimensional subspace with Banach-Mazur distance to \(\ell^2_2\) at most \(1 + \varepsilon\). The example of a simple circle in \(\ell^2_2\) realizes \(TL = 2\pi\). This means that in any Banach space \(X\) it is possible to construct an ODE satisfying \(TL \leq 2\pi + \varepsilon\), where \(\varepsilon\) depends only on the dimension of \(X\). We do not know whether there is always an ODE for which \(TL \leq 2\pi\).

5. **Conclusion**

As discussed in the introduction, the key question is what intrinsic property of a space \(X\) determines the largest (and hence best) constant \(C_X\) such that \(LT \geq C_X\). One of these intrinsic properties is strict convexity for which we have shown that the constant must be
strictly larger than 6. A natural question that arises is whether there exists a Banach space in which the optimal constant is neither 6 nor $2\pi$.

However, explicit bounds are difficult to obtain. Even in the simple case $X = \ell^p(\mathbb{R}^n)$ this is not known, although our simple argument gives an explicit lower bound for $p$ around $p = 2$. It is interesting that a simple calculation shows that $C_p = C_{p'}$ when $p$ and $p'$ are conjugates; but it is not known whether the optimal constants in $\ell^p$ and $\ell^{p'}$ do in fact coincide (this interesting question was suggested to one of us in a personal communication from Mario Martelli).

While the use of an $L^p$-based Wirtinger inequality suits the $\ell^p$-spaces, there is no reason why these exponents should match. Given a Banach space $X$ it would be interesting to determine the optimal constants in the family of inequalities

$$\left( \int_0^1 \| u(t) \|^p_X \, dt \right)^{1/p} \leq C_p(X) \left( \int_0^T \| \dot{u}(t) \|^p_X \, dt \right)^{1/p},$$

noting that as a consequence of such a family of inequalities and the argument of Theorem 4.1 one would obtain

$$TL \geq \sup_p C_p(X)^{-1}.$$

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