AN IMMERSED INTERFACE METHOD FOR PENNES BIOHEAT TRANSFER EQUATION

CHAMPIKE ATTANAYAKE
Department of Mathematics, Miami University
Middletown OH, 45042, USA

SO-HSIANG CHOU
Department of Mathematics and Statistics, Bowling Green State University
Bowling Green, OH 43402-0221, USA

(Communicated by Zhimin Zhang)

Abstract. We consider an immersed finite element method for solving one dimensional Pennes bioheat transfer equation with discontinuous coefficients and nonhomogenous flux jump condition. Convergence properties of the semidiscrete and fully discrete schemes are investigated in the $L^2$ and energy norms. By using the computed solution from the immersed finite element method, an inexpensive and effective flux recovery technique is employed to approximate flux over the whole domain. Optimal order convergence is proved for the immersed finite element approximation and its flux. Results of the simulation confirm the convergence analysis.

1. Introduction. In this paper we consider a bioheat transfer equation of the parabolic type

$$\begin{cases}
\gamma T_t - \beta T_{xx} = T_a - T, & (x,t) \in I \times J, \\
T(x,0) = T_0, & x \in I, \\
T(a,t) = T(b,t) = 0, & t > 0, \\
[T]_\alpha = 0, & [\beta T_x]_\alpha = Q,
\end{cases}$$

(1.1)

where $T$ is the temperature distribution over $I = [a,b]$ during a time period $J = [0,\bar{t}]$. The material parameters $\beta$ and $\gamma$ are piecewise constants, reflecting the interface nature of the problem. The jump of the quantity $q$ across the interface $\alpha$ is denoted by $[q]_\alpha$ and $Q = [\beta T_x]_\alpha$, the interface flux jump is considered given. $T_a$ is a fixed given temperature. The homogeneous temperature boundary conditions are for the simplicity of the presentation in the theoretical part. The case of mixed temperature and flux boundary conditions are considered in the numerical examples of the final section.

The heat transfer analysis of biological bodies during surface heating or cooling has been studied by many researchers [5, 7, 12, 13]. In many diagnostic and therapeutic applications of heat transfer, evaluation of transient and spatial distribution of temperature is required both on the skin surface and inside the living tissues. Investigation of transient temperature distribution requires solving a bioheat transfer

2010 Mathematics Subject Classification. Primary: 65N15, 65N30; Secondary: 35J60.

Key words and phrases. Flux recovery, immersed finite element method, immersed interface method, superconvergence, error analysis.
equation with relevant boundary and initial conditions. Several numerical methods including finite element (FEM) and finite difference (FDM), have been studied to simulate Pennes bioheat transfer equation\cite{3, 15}. Finite difference schemes were more developed to simulate Pennes equation in 1-D and 3-D in a triple layered skin structure allowing variable thermal properties across the layers. We refer the reader to \cite{3, 4} and the references therein for more details.

Obtaining the analytical solution of Pennes bioheat transfer equation in a multiple layered structure is difficult due to the variable thermal parameters and the interface conditions. In order to successfully approximate the transient temperature distribution in the tissues and at the skin surface of the biological body, it is necessary to consider the variation of thermal properties that appear in the bioheat transfer equation. Therefore, one of the goals of this article is to simulate Pennes bioheat transfer equation using the immersed finite element method (IFEM), which is suitable for handling variable thermal properties when heat transfer through the different layers of the biological body.

An IFEM requires construction of special basis functions for the interface elements while keeping the standard basis functions for non-interface elements. The interface basis functions are constructed to satisfy the interface jump conditions. Detailed explanation of this IFEM can be found in \cite{8, 9, 10, 11}, and references therein. In this paper due to the one dimensional nature of the problem, we used a special function to reduce the nonhomogeneous jump condition to a homogeneous one, making it easier to prove optimal convergence. In higher dimensions it is much harder to find such functions. See He et al. \cite{6} for related issues.

In many applications it is also important to accurately approximate the heat flux $u = -\beta T_x$ as well as the solution $T$ of the initial boundary value problem. Even though the finite element method provides a symmetric positive definite system to solve for the variable $T$, it does not approximate flux automatically. Chou and Tang \cite{1} introduced a technique to approximate flux of the solution $T$, when numerical solution is approximated accurately by the standard finite element method. Chou \cite{2} extended this method to IFE spaces (both cases are for elliptic problems). It has the special merit of reproducing the exact fluxes at the nodes and interface. In Section 3, we extend this methodology to parabolic problems and show that the approximate flux has optimal order convergence.

The rest of the paper is organized as follows. The error analysis for the semidiscrete problem and the fully discrete problem is given in Section 2. Section 3 is devoted to the convergence of the flux approximations. Finally, numerical results are presented in Section 4.

2. IFE method. In this section we study an immersed finite element method for the bioheat interface problem (1.1). In the error analysis, we shall use the standard notation for the Sobolev spaces $H^k(I)$ endowed with Sobolev norm $\| \cdot \|_{k, I}$. $H^k_0(I)$ is the usual space with homogenous boundary conditions. We use the shorthand $H^k$ for $H^k(I)$ when $I = (a, b)$. Let $\alpha$ be the interface point in $(a, b)$. For the ensuing analysis we need the function spaces

$$H^2_\alpha := H^2(a, \alpha) \cap H^2(\alpha, b) \cap H^1$$

equipped with the norm

$$\| u \|^2_{2, \alpha} := \| u \|^2_{2, (a, \alpha)} + \| u \|^2_{2, (\alpha, b)}.$$
To derive a natural weak formulation of the problem (1.1), we integrate it against \( q \in H_0^1 \) and use integration by parts on each subinterval. The resulting formulation is

\[
(\gamma T_t, q) + a(T, q) = (T_0, q) - Qq(\alpha), \quad \forall q \in H_0^1, \ t \in J,
\]

where

\[
a(T, q) = (\beta T', q') + (T, q),
\]

using the standard \( L^2 \) inner product \( (f, g) = \int_a^b f(x)g(x)dx \).

In order to handle the nonhomogeneous flux jump condition, let \( \tilde{p} \) be any boundary vanishing function in \( H_0^1 \) such that \( [\tilde{p}]_\alpha = 0 \) and \( [\beta \tilde{p}_x]_\alpha = Q \). Among all possible \( \tilde{p} \), we can choose those suitable for our numerical computation as well. For example we can define \( \tilde{p} \) as

\[
\tilde{p}(x) = \begin{cases} 
0, & a \leq x < \alpha, \\
-\frac{Q}{\beta(b-a)}(x-\alpha)^2 + \frac{Q}{\beta}(x-\alpha), & \alpha \leq x \leq b.
\end{cases}
\]

Rewriting \( T \) as \( T = p + \tilde{p}, p \in H_0^1 \), we see that it is sufficient to find \( p : J \mapsto H_0^1 \) such that

\[
(\gamma p_t, q) + a(p, q) = l(q), \quad \forall q \in H_0^1, \ t \in J,
\]

\[
p(0) = p_0,
\]

where the right hand side

\[
l(q) := (T_a, q) - a(\tilde{p}, q) - Qq(\alpha)
\]

is a bounded linear functional on \( H_0^1 \), using the Sobolev imbedding theorem on the last term. Note that by using smooth testing functions it can be shown that

\[
[\tilde{p}]_\alpha = 0, \quad \text{and} \quad [\beta \tilde{p}_x]_\alpha = 0.
\]

It is also easy to see that the bilinear form \( a(\cdot, \cdot) \) is bounded and coercive on \( H_0^1 \).

To approximate the weak problem (2.2), we consider the one dimensional linear IFE space introduced in [8, 10]. Let \( a = x_0 < x_1, \ldots, x_N = b \) be a partition of \( I = [a, b] \) and let the interface point \( \alpha \) be located in \( (x_\kappa, x_{\kappa+1}) \) for some \( \kappa \). We term \( (x_\kappa, x_{\kappa+1}) \) an interface element and the remaining elements non-interface elements. Let \( h_i = x_i - x_{i-1}, \ (i = 1, 2, \ldots, N) \). In the IFE space, for non-interface elements the local shape function is the usual linear function. For the interface element a local shape function takes the form

\[
\hat{\phi}(x) = \begin{cases} 
\hat{\phi}^-(x) = a^-x + b^- \quad & x \in [x_\kappa, \alpha], \\
\hat{\phi}^+(x) = a^+x + b^+ \quad & x \in [\alpha, x_{\kappa+1}], \\
[\hat{\phi}]_\alpha = [\beta \hat{\phi}']_\alpha = 0.
\end{cases}
\]

It is easy to see that function \( \hat{\phi} \) is uniquely determined by the above conditions [8]. The linear IFE space is defined as \( S_h = \text{span}\{\hat{\phi}_i\}_{i=1}^N \), where \( \hat{\phi}_i(x_k) = \delta_{ik} \) and \( \hat{\phi}_i \) is piecewise linear and glued together by the local shape basis functions. It is clear that \( S_h \subset H_0^1 \). Moreover, Lin et al. [11] proved that the interpolation error in this space has optimal order. That is, for any \( v \in H_0^1 \) with \( [v]_\alpha = [\beta v']_\alpha = 0 \) there exists a \( \hat{\pi}_hv \in S_h \) such that

\[
\|v - \hat{\pi}_hv\|_0 + h|v - \hat{\pi}_hv|_1 \leq Ch^2\|v\|_{2,\alpha},
\]
where the interpolation $\hat{\pi}_h v(x) = \sum_{i=1}^{N-1} v(x_i)\hat{\phi}_i(x)$. The semidiscrete immersed interface finite element problem based on the above weak formulation is: Find $p_h$ where $p_h : J \rightarrow \hat{S}_h$ satisfies
\begin{equation}
(\gamma p_h, \hat{\phi}) + a(p_h, \hat{\phi}) = l(\hat{\phi}), \quad \forall \hat{\phi} \in \hat{S}_h, \ t \in J
\end{equation}
with the initial condition $p_h(0) = \hat{\pi}_h p_0$. Of course, the approximate $T_h$ to $T$ can be obtained through $T_h = p_h + \hat{p}$. In the following we will concentrate on the error $p - p_h$, instead of $T - T_h$.

2.1. Semidiscrete error. In this section we study the convergence properties of the semidiscrete IFE method. We shall show that the error $p - p_h$ between the solutions of the continuous and spatially discrete problems are of optimal order. Let us first define the elliptic projection of $w \in H^0_0$ to be $R_tw \in \hat{S}_h$ such that
\begin{equation}
 a(w - R_tw, \hat{\phi}) = 0, \quad \forall \hat{\phi} \in \hat{S}_h.
\end{equation}
Note that $R_tw$ is well defined by the coercivity of $a$ on $\hat{S}_h$ due to the conformingness of $\hat{S}_h$. Furthermore, if $w$ is smoother in the sense that $w \in H^2_\alpha$ with $[w]_\alpha = [\beta w']_\alpha = 0$ then it is not hard to prove that
\begin{align*}
||w - R_tw||_1 &\leq Ch||w||_{2,\alpha} \quad \text{and} \quad ||w - R_tw||_0 \leq Ch^2||w||_{2,\alpha}.
\end{align*}
In fact, their proofs are similar to the standard linear elements, provided that we make sure the constants $C$ are independent of the interface parameter $\alpha$. For example, in the proof of the first inequality we have to use the coercivity constant $\min\{\beta^-, \beta^+\}$, which are all independent of $\alpha$. The second inequality can be proved using the first one and the usual duality argument while checking that none of the constants involved depend on $\alpha$. Unlike in 2-D IFE (nonconforming) methods, here we are having a conforming method and proving the above type of two inequalities is much simpler.

Now we study the parabolic equation. As in the study of standard parabolic equations [14], we introduce the elliptic projection $p^* = R_hp$. Moreover, substituting $w$ by $p$ and differentiating (2.6) with respect to $t$ leads to
\begin{equation}
 a(p^*_t - p_t, \hat{\phi}) = 0,
\end{equation}
and hence
\begin{equation}
||p^*_t - p_t||_0 + h||p^*_t - p_t||_1 \leq Ch^2||p_t||_{2,\alpha}.
\end{equation}
Now the total error is written as
\begin{equation}
p_h(t) - p(t) = \theta(t) + \rho(t), \quad \theta = p_h - p^*, \quad \rho = p^* - p,
\end{equation}
and the elliptic projection error satisfies
\begin{align*}
||\rho||_0 &= ||p^* - p||_0 \leq Ch^2||p||_{2,\alpha},
||\rho_t||_0 &= ||p^*_t - p_t||_0 \leq Ch^2||p_t||_{2,\alpha}.
\end{align*}

The next theorem shows that the error between the semidiscrete solution and the continuous solution is of optimal order in the $L^2$ norm.

Theorem 2.1. Let $p_h$ and $p$ be the solutions of the semidiscrete problem (2.5) and the parabolic problem (2.2), respectively. Then there exists a positive constant $C$ independent of $h$ such that
\begin{equation}
||p_h - p||_0 \leq Ch^2 \left( ||p_0||_{2,\alpha} + ||p||_{2,\alpha} + \int_0^t ||p_t||_{2,\alpha}ds \right).
\end{equation}
Proof. We only have to find error bound for \( \theta \). It follows from (2.2), (2.5) and (2.6) that
\[
(\gamma \theta_t, \hat{\phi}) + a(\theta, \hat{\phi}) = -(\gamma \rho_t, \hat{\phi}), \quad \forall \hat{\phi} \in \tilde{S}_h. \tag{2.9}
\]
If \( \hat{\phi} = \theta \) we have
\[
(\gamma \theta_t, \theta) + a(\theta, \theta) = -(\gamma \rho_t, \theta). \tag{2.10}
\]
Using the fact \( a(\theta, \theta) \geq 0 \), we get
\[
\frac{1}{2} \frac{d}{dt} \| \gamma^{1/2} \theta \|_0^2 = \frac{1}{2} \frac{d}{dt} \left( \| \gamma^{1/2} \theta \|_0^2 + \epsilon^2 \right) \leq \| \gamma^{1/2} \rho_t \|_0 \| \gamma^{1/2} \theta \|_0 \leq \| \gamma^{1/2} \rho_t \|_0 \left( \| \gamma^{1/2} \theta \|_0^2 + \epsilon^2 \right)^{1/2}, \quad \epsilon > 0,
\]
where the \( \epsilon^2 \) term was added since \( \| \gamma^{1/2} \theta \|_0^2 \) may not differentiable when \( \theta = 0 \). Since the second term is less than the fourth term, we have
\[
\left( \| \gamma^{1/2} \theta \|_0^2 + \epsilon^2 \right)^{1/2} \frac{d}{dt} \left( \| \gamma^{1/2} \theta \|_0^2 + \epsilon^2 \right)^{1/2} \leq \| \gamma^{1/2} \rho_t \|_0 \left( \| \gamma^{1/2} \theta \|_0^2 + \epsilon^2 \right)^{1/2}.
\]
Canceling the common factor and then integrating both sides with respect to \( t \) and let \( \epsilon \to 0 \), we get
\[
\| \gamma^{1/2} \theta(t) \|_0 \leq \| \gamma^{1/2} \theta(0) \|_0 + \int_0^t \| \gamma^{1/2} \rho_t \|_0 ds.
\]
Using the equivalence of the norms we have
\[
\| \theta(t) \|_0 \leq C \| \theta(0) \|_0 + C \int_0^t \| \rho_t \|_0 ds. \tag{2.10}
\]
Letting \( p_h(0) = \hat{p}_h p_0 \) we can see that,
\[
\| \theta(0) \|_0 \leq \| p_h(0) - p_0 \|_0 + \| p_0 - p^*(0) \|_0 \leq C h^2 \| p_0 \|_{2,\alpha} + C h^2 \| p_0 \|_{2,\alpha} \leq C h^2 \| p_0 \|_{2,\alpha}.
\]
Therefore, substituting the estimates for \( \| \theta(0) \|_0 \) and \( \| \rho_t \|_0 \) in (2.10) gives
\[
\| \theta(t) \|_0 \leq C h^2 \left( \| p_0 \|_{2,\alpha} + \int_0^t \| p_t \|_{2,\alpha} ds \right).
\]
Finally the total error
\[
\| p_h - p \|_0 \leq \| \theta \|_0 + \| \rho \|_0 \leq C h^2 \left( \| p_0 \|_{2,\alpha} + \int_0^t \| p_t \|_{2,\alpha} ds \right) + C h^2 \| p \|_{2,\alpha}.
\]
This completes the proof. \( \Box \)

In a similar fashion, we can obtain the error estimate for the semidiscrete solution in the energy norm.

**Theorem 2.2.** Under the assumptions of Theorem 2.1 we have
\[
\| p_h - p \|_1 \leq C h \left( \| p_0 \|_{2,\alpha} + \| p \|_{2,\alpha} + \left( \int_0^t \| p_t \|_{2,\alpha}^2 ds \right)^{1/2} \right).
\]
Proof. By setting $\dot{\phi} = \theta_t$ in (2.9) we obtain

$$(\gamma \theta_t, \theta_t) + a(\theta, \theta_t) = -(\gamma \rho_t, \theta_t),$$

or, explicitly

$$\|\gamma^{1/2} \theta_t\|_0^2 + \frac{1}{2} \frac{d}{dt} \left( \|\beta^{1/2} \theta_x\|_0^2 + \|\theta\|_0^2 \right) = -(\gamma \rho_t, \theta_t).$$

Let $\gamma_{\text{max}} = \max\{\gamma-, \gamma_+\}$ and $\gamma_{\text{min}} = \min\{\gamma-, \gamma_+\}$. Then

$$\gamma_{\text{min}}\|\theta_t\|^2_0 + \frac{1}{2} \frac{d}{dt} \left( \|\beta^{1/2} \theta_x\|_0^2 + \|\theta\|_0^2 \right) \leq \gamma_{\text{max}} \left( \|\theta_t\|_0 \|\rho_t\|_0 \right).$$

Integrating with respect to $t$ and then using the equivalence of the norms, it follows

$$\|\theta_x(t)\|_0^2 + \|\theta(t)\|_0^2 \leq C\|\theta_x(0)\|_0^2 + C\|\theta(0)\|_0^2 + C \int_0^t \|\rho_t\|_0^2 ds,$$

where $C$ depends on $\beta$ and $\gamma$. For $p_h(0) = \hat{\pi}p_0$, $\|\theta(0)\|_1 \leq Ch\|p_0\|_{2,\alpha}$. Substituting the estimates for $\|\rho_t\|_0$ and $\|\theta(0)\|_1$ in the above inequality we have

$$\|\theta\|_1 \leq Ch \left( \|p_0\|_{2,\alpha} + \left( \int_0^t \|p_t\|_{2,\alpha}^2 ds \right)^{1/2} \right).$$

Thus

$$\|p - p_h\|_1 \leq \|\theta\|_1 + \|\rho\|_1 \leq Ch \left( \|p_0\|_{2,\alpha} + \|p\|_{2,\alpha} + \left( \int_0^t \|p_t\|_{2,\alpha}^2 ds \right)^{1/2} \right),$$

which completes the proof. \qed

2.2. Completely discrete error. In this section we analyze a fully discrete scheme based on the backward Euler finite difference approximation. Optimal order convergence in the $L^2$ norm is established. We first partition the time interval $[0, T]$ into $M$ subintervals by $0 = t_0 < t_1 < \ldots < t_M = t$ points, with $t_n = nk$ and $k = t/M$, the uniform time stepsize. Then the completely discrete backward Euler Galerkin approximation $p^n_h \in \tilde{S}_h$ to $p^n = p(t_n)$ of problem (1.1) is defined as the solution of

$$(\gamma \tilde{\partial} p^n_h, \hat{\phi}) + a(p^n_h, \hat{\phi}) = l(\hat{\phi}), \quad \forall \hat{\phi} \in \tilde{S}_h,$$

where $l(q) = (T_a, q) - a(\hat{p}, q) - Qq(\alpha)$ and

$$\tilde{\partial} p^n_h = \frac{p^n_h - p^{n-1}_h}{k}.$$

Under appropriate regularity assumptions for $p$, the following theorem shows the fully discrete error has convergence rate of order $O(h^2 + k)$. 

Theorem 2.3. Let $p^n_N$ for $n = 1, \ldots, M$, and $p$ be the solutions of the fully discrete problem (2.12) and the parabolic problem (2.2), respectively. Then there exists a positive constant $C$ independent of the mesh parameters $h$ and $k$ such that for $n = 1, \ldots, M$

$$
\|p^n_N - p(t_n)\|_0 \leq Ch^2 \left( \|p_0\|_{2,\alpha} + \|p\|_{2,\alpha} + \int_0^{t_n} \|p_t\|_{2,\alpha} \right) + kC \int_0^{t_n} \|p_{tt}\|_0 ds.
$$

Proof. As in the previous theorem, we split the error into two terms as

$$
p^N_j - p(t_j) = \theta^j + \rho^j, \quad \text{where} \ \theta^j = \theta(t_j) = p^j_N - p^{\ast j}, \quad \rho^j = \rho(t_j) = p^{\ast j} - p(t_j),
$$

where $p^{\ast j} = p^* (t_j) = R_h p(t_j)$, the elliptic projection defined in (2.6). Since the estimates for $\rho^j$ are known, we only have to find a bound for $\theta^j$. It follows from (2.12), that

$$
(\gamma \tilde{\theta} \partial^j, \tilde{\phi}) + a(\theta^j, \phi) = (\gamma \tilde{\theta} (p^j_N - p^{\ast j}), \phi) + a(p^j_N - p^{\ast j}, \phi) = - (\partial \rho^j, \gamma \tilde{\phi}) - (\partial \rho^j, p^j_\eta, \gamma \tilde{\phi}). \quad (2.13)
$$

Now define \[14\]

$$
\omega^j := \tilde{\partial} \rho^j + (\partial \rho^j - p^j_\eta) := \omega^j_1 + \omega^j_2. \quad (2.14)
$$

For notational convenience, let us define $\tilde{\theta} = \gamma^{1/2} \theta$, $\tilde{\omega} = \gamma^{1/2} \omega$ and so on. Taking $\tilde{\phi} = \theta^j$,

$$
(\partial \tilde{\omega} \tilde{\theta} \partial^j, \tilde{\phi}) = (\partial \tilde{\omega} \tilde{\theta} \theta^j, \tilde{\phi}) = - (\tilde{\omega} \tilde{\theta} \theta^j, \tilde{\phi}).
$$

Since $(\partial \tilde{\omega} \tilde{\theta} \partial^j, \tilde{\phi}) \leq \|\tilde{\omega} \tilde{\theta} \|_0 \|\tilde{\theta} \|_0$, or

$$
\|\tilde{\omega} \tilde{\theta} \|_0^2 - (\tilde{\theta} \tilde{\theta}^{-1}, \tilde{\theta} \tilde{\theta}^{-1}) \leq k \|\tilde{\omega} \tilde{\theta} \|_0 \|\tilde{\theta} \|_0
$$

we have

$$
\|\tilde{\theta} \|_0^2 - (\tilde{\theta} \tilde{\theta}^{-1}, \tilde{\theta} \tilde{\theta}^{-1}) \leq k \|\tilde{\omega} \tilde{\theta} \|_0^2.
$$

Summing over $j$ from 1 to $n$ leads to

$$
\|\tilde{\theta} \|_0^2 \leq k \|\tilde{\theta} \|_0 + k \sum_{j=1}^n \|\tilde{\omega} \tilde{\theta} \|_0 \leq \|\tilde{\theta} \|_0 + k \sum_{j=1}^n \|\tilde{\omega} \tilde{\theta} \|_0 + k \sum_{j=1}^n \|\tilde{\omega} \tilde{\theta} \|_0.
$$

Using the norm equivalences we have

$$
\|\theta^j \|_0 \leq \|\tilde{\theta} \|_0 + kC \sum_{j=1}^n \|\omega^j_1 \|_0 + \|\omega^j_2 \|_0.
$$

Now we estimate the last two terms. For $\omega_1$ terms we have

$$
k \sum_{j=1}^n \|\omega^j_1 \|_0 = k \sum_{j=1}^n \|\tilde{\partial} \partial^j \|_0 = \sum_{j=1}^n \|\int_{t_{j-1}}^{t_j} \rho_t ds \|_0 \leq \sum_{j=1}^n Ch^2 \int_{t_{j-1}}^{t_j} \|p_t\|_{2,\alpha} ds \leq Ch^2 \int_0^{t_n} \|p_t\|_{2,\alpha} ds.
$$

As for the $\omega_2$ term we have

$$
k (\partial \rho^j - p^j_\eta) = p(t_j) - p(t_{j-1}) - kp_t(t_j) = - \int_{t_{j-1}}^{t_j} (s - t_{j-1}) p_{tt}(s) ds.
$$
Thus
\[
k \sum_{j=1}^{n} \|\omega_2^{j}\|_0 \leq \sum_{j=1}^{n} \left( \int_{t_{j-1}}^{t_j} (s - t_{j-1}) \|p_{tt}(s)\|_0 ds \right) \leq k \int_0^{t_n} \|p_{tt}\|_0 ds.
\]
Combining all the estimates completes the proof.

**Theorem 2.4.** Under the assumptions of Theorem 2.1 we have
\[
\|p_N^n - p(t_\ast)\|_1 \leq Ch \left( \|p_0\|_{2,0} + \|p\|_{2,0} + \left( \int_0^{t_n} \|p_t\|_{2,0}^2 ds \right)^{1/2} \right) + kC(\int_0^{t_n} \|p_{tt}\|_0^2 ds)^{1/2}.
\]

Proof. By setting \(\bar{\theta} = \partial \theta^j\) in (2.13) we obtain,
\[
(\gamma \bar{\partial} \theta^j, \partial \theta^i) + a(\theta^j, \partial \theta^i) = - (\gamma \bar{\partial} \rho^j, \partial \theta^i) - (\gamma \bar{\partial} \nu^j, \partial \theta^i).
\]
Note that for any sequence \(\eta^j, \|\partial \eta^j\|_0^2 \geq 0\) implies
\[
(\bar{\partial} \eta^j, \eta^j) \geq \frac{1}{2} \bar{\partial} \|\eta^j\|_0^2.
\]
Applying (2.16) with \(\eta^j = \beta^{1/2} \theta^j_0, \theta^j\), we have
\[
a(\theta^j, \partial \theta^i) = (\beta^{1/2} \theta^j_0, \bar{\partial} \beta^{1/2} \theta^j_0) + (\theta^j, \partial \theta^i) \geq \frac{1}{2} \bar{\partial} \|\beta^{1/2} \theta^j_0\|_0^2 + \frac{1}{2} \bar{\partial} \|\theta^j\|_0^2.
\]
Therefore, just as in the proof of Theorem 2.2, we can use the \(\epsilon\)-inequality \(ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, a, b > 0\) for \(\epsilon > 0\) small enough to absorb the \(\|\bar{\partial} \theta^j\|_0\) term on the right side into the first term on the left side, when estimating (2.15). Thus using this technique, we again derive
\[
\bar{\partial} \|\beta^{1/2} \theta^j_0\|_0^2 + \bar{\partial} \|\theta^j\|_0^2 \leq C\|\gamma \omega^j\|_0^2,
\]
where \(\omega^j\) is as in the proof of the last theorem. Summing the above inequality over \(j\) from \(j = 1\) to \(n\) and use equivalence of norms, we have
\[
\|\theta^n\|_1^2 \leq C\|\theta(0)\|_1^2 + Ck \sum_{j=1}^{n} (\|\omega_1^j\|_0^2 + \|\omega_2^j\|_0^2).
\]
The remaining part is just like in the last theorem.

3. **Flux recovery.** In this section we will construct a method to approximate the exact flux \(-\beta T_x\) in (1.1). Since we have concentrated on computing \(p_h\) to approximate \(p\) in \(T = p + \bar{p}\). We will first transform (1.1) accordingly. The transformed problem is
\[
\begin{aligned}
\gamma p_h - \beta p_{xx} + p = f(x, t) := T_a + \beta \bar{p}_{xx} - \bar{p}, \quad (x, t) \in I \times J, \\
p(x, 0) = p_0, \quad x \in I, \\
p(a, t) = p(b, t) = 0, \quad t > 0, \\
[p]_0 = 0; \ [\beta p_x]_a = 0.
\end{aligned}
\]
Thus it suffices to approximate \(u = -\beta p_x\), which will be done by a flux recovery process in which once \(p_N \approx p\) has been computed, the approximate flux \(u_N\) is obtained by using a simple formula. The \(u_N\) is globally continuous and linear elementwise except on the interface element. On the interface element, \(u_N\) is piecewise linear.
We now describe how to construct the approximate flux on non-interface elements first. Multiply (3.1) by \( \phi_j \) and integrate by parts over \( I_j = [x_{j-1}, x_j], j \neq \kappa \) (recall \( I_{\kappa+1} \) is the interface element) to get
\[
 u(x_j^-, t) = -\beta p_x(x_j, t)
 = \int_{x_j}^{x_{j-1}} -\beta p_x \phi_j' - p \phi_j + \int_{x_j}^{x_{j-1}} \gamma p_t \phi_j + \int_{x_j}^{x_{j-1}} f(x, t) \phi_j dx.
\]

Do the same over \( I_{j+1} = [x_j, x_{j+1}] \) to get
\[
 u(x_j^+, t) = -\beta p_x(x_j, t)
 = \int_{x_j}^{x_{j+1}} \beta p_x \phi_j' + p \phi_j dx + \int_{x_j}^{x_{j+1}} \gamma p_t \phi_j dx - \int_{x_j}^{x_{j+1}} f(x, t) \phi_j dx.
\]

Since \( p_N \) is a good approximate of \( p \) for a fixed time \( t_n \), it is natural to define the discrete fluxes \( u_N(x_j^-, t_n) \) and \( u_N(x_j^+, t_n) \) on \( I_j \) and \( I_{j+1} \), respectively as,
\[
 u_N(x_j^-, t_n) = \int_{x_j}^{x_{j-1}} -\beta p_N^{n,x} \phi_j' - p_N^n \phi_j dx - \int_{x_j}^{x_{j-1}} \gamma \partial p_N^n \phi_j dx + \int_{x_j}^{x_{j-1}} f(x, t_n) \phi_j dx,
\]
\[
 u_N(x_j^+, t_n) = \int_{x_j}^{x_{j+1}} \beta p_N^{n,x} \phi_j' + p_N^n \phi_j dx + \int_{x_j}^{x_{j+1}} \gamma \partial p_N^n \phi_j dx - \int_{x_j}^{x_{j+1}} f(x, t_n) \phi_j dx.
\]

Setting \( \hat{\phi} = \phi_j \) in (2.12), we have
\[
 \int_{x_j}^{x_{j-1}} \gamma \partial p_N^n \phi_j dx + \int_{x_j}^{x_{j+1}} \beta p_N^{n,x} \phi_j' + p_N^n \phi_j dx = \int_{x_j}^{x_{j+1}} f(x, t_n) \phi_j dx
\]
or
\[
 -\int_{x_j}^{x_{j-1}} \gamma \partial p_N^n \phi_j dx - \int_{x_j}^{x_{j+1}} (\beta p_N^{n,x} \phi_j' + p_N^n \phi_j) dx + \int_{x_j}^{x_{j+1}} f(x, t_n) \phi_j dx = \int_{x_j}^{x_{j+1}} \gamma \partial p_N^n \phi_j dx + \int_{x_j}^{x_{j+1}} \beta p_N^{n,x} \phi_j' + p_N^n \phi_j dx - \int_{x_j}^{x_{j+1}} f(x, t_n) \phi_j dx.
\]

Thus, from (3.3) and (3.4) we see that \( u_N(x_j^-, t_n) = u_N(x_j^+, t_n) \) and \( u_N(x_j, t_n) \) is well defined.

In a similar fashion by making \( \hat{\phi} = \phi_{j+1} \) in (2.12), it is easy to see that \( u_N(x_{j+1}^-, t_n) = u_N(x_{j+1}^+, t_n) = u_N(x_{j+1}, t_n) \). Therefore, following the flux approximation method in [1], for a given \( t_n \in J \) and for any \( x \in [x_j, x_{j+1}], j \neq k \), we define the approximate flux \( u_N(x, t_n) \) as
\[
 u_N(x, t_n) = \hat{\phi}_{j+1}(x) u_N(x_{j+1}, t_n) + \hat{\phi}_j(x) u_N(x_j, t_n) \text{ for } x \in I_j. \tag{3.5}
\]

To approximate flux on the interface element, \( I_\kappa \), we multiply (3.1) by the interface basis function \( \hat{\phi}_\kappa \), integrate by parts over \([x_\kappa, \alpha]\) and \([\alpha, x_{\kappa+1}]\), and use the interface conditions to get
\[
 u(x_{\kappa}^-, t) = -\beta p_x(x_\kappa, t)
 = \int_{x_\kappa}^{x_{\kappa+1}} \beta p_x \phi_\kappa' + p \phi_\kappa dx + \int_{x_\kappa}^{x_{\kappa+1}} \gamma p_t \phi_\kappa dx - \int_{x_\kappa}^{x_{\kappa+1}} f(x, t) \phi_\kappa dx, \tag{3.6}
\]
which has the same form as for a noninterface element. For the adjacent non-
interface element \( I_{k-1} = [x_{k-1}, x_k] \), we get
\[
    u(x_k, t) = -\beta p_x(x_k, t)
    = \int_{x_{k-1}}^{x_k} -\beta p_x \phi'_k - p_x \phi_k dx - \int_{x_{k-1}}^{x_k} \gamma p_t \phi_k dx + \int_{x_{k-1}}^{x_k} f(x, t) \phi_k dx.
\] (3.7)

Accordingly we define
\[
    u_N(x_k, t_n) = \int_{x_{k-1}}^{x_k} \beta p_{N,x} \phi'_k + p_{N} \phi_k dx + \int_{x_{k-1}}^{x_k+1} \beta p_{N,x} \phi'_k + p_{N} \phi_k dx
    + \int_{x_{k-1}}^{x_k} \gamma \tilde{p}_N \phi_k dx - \int_{x_{k-1}}^{x_k} f(x, t_n) \phi_k dx,
\] (3.8)

\[
    u_N(x_{k+1}, t_n) = \int_{x_{k+1}}^{x_k} -\beta p_{N,x} \phi_{k+1} - p_{N} \phi_k dx
    - \int_{x_{k+1}}^{x_k} \gamma \tilde{p}_N \phi_k dx - \int_{x_{k-1}}^{x_k} f(x, t_n) \phi_k dx.
\] (3.9)

Again the fact that \( u_N(x_{k}, t_n) = u_N(x_{k+1}, t_n) \) can be checked by setting \( \phi = \phi_k \) in (2.12). It is then not hard to see that all we need to check is
\[
    l(\phi_k) = \int_{x_{k-1}}^{x_k} f(x, t_n) \phi_k dx,
\]
which upon using integration by parts on \([x_k, \alpha]\) and \([\alpha, x_{k+1}]\) reduces to checking the validity of
\[
    (\beta \tilde{p}_x \phi_k)(\alpha^-) - (\beta \tilde{p}_x \phi_k)(x_{k-1}) + (\beta \tilde{p}_x \phi_k)(x_{k+1}) - (\beta \tilde{p}_x \phi_k)(\alpha^+) = -Q \phi_k(\alpha).
\]

Hence \( u_N(x_{k}, t_n) = u_N(x_{k+1}, t_n) = u_N(x_{k}, t_n) \). Similarly, it is easy to see that \( u_N(x_{k+1}, t_n) = u_N(x_{k}, t_n) \). Now we need to define \( u_N(\alpha, t_n) \), the approximate flux at the interface point. For this we take clue from (3.1) again. Integrating it over \([x_k, \alpha]\) and using \( u = -\beta p_x \), we get
\[
    u(\alpha, t) = u(x_k, t) + \int_{x_k}^{\alpha} (f - p - \gamma p_t) dx.
\] (3.10)

Similarly, integrating over \([\alpha, x_{k+1}]\) gives
\[
    u(\alpha, t) = u(x_{k+1}, t) + \int_{x_{k+1}}^{\alpha} (f - p - \gamma p_t) dx.
\]

Thus accordingly we define
\[
    u_N(\alpha^-, t_n) = u_N(x_k, t_n) + \int_{x_k}^{\alpha} (f - p_{N} - \gamma \tilde{p}_N) dx,
\] (3.11)
\[
    u_N(\alpha^+, t_n) = u_N(x_{k+1}, t_n) + \int_{x_{k+1}}^{\alpha} (f - p_{N} - \gamma \tilde{p}_N) dx.
\] (3.12)

Now we claim \( u_N(\alpha^-, t_n) = u_N(\alpha^+, t_n) \). Recall (3.8) and
\[
    u_N(x_{k+1}, t_n) = \int_{x_k}^{x_{k+1}} (\beta p_{N,x} \phi'_{k+1} + p_{N} \phi_{k+1}) dx
    - \int_{x_k}^{x_{k+1}} \gamma \tilde{p}_N \phi_{k+1} dx - \int_{x_k}^{x_{k+1}} f \phi_{k+1} dx.
\]
Substituting these relations into (3.11)-(3.12) and using \( \hat{\phi}_k + \hat{\phi}_{k+1} = 1 \) give the desired result. So we can define \( u_N(\alpha, t_n) = u_N(\alpha^+, t_n) \). Finally we define the global approximate flux by interpolation at the nodes and the interface point. In particular, \( u_N(x, t_n) \) on the interface element \([x_k, x_{k+1}]\) has the formula

\[
u_N(x, t_n) = \begin{cases} 
  u_N(x_k, t_n)\lambda_k + u_N(\alpha^-, t_n)\lambda_{\alpha^-}, & \forall x \in [x_k, \alpha], \\
  u_N(\alpha^+, t_n)\lambda_\alpha + u_N(x_{k+1}, t_n)\lambda_{\alpha^+}, & \forall x \in [\alpha, x_{k+1}], 
\end{cases}
\]

(3.13)

where \( \lambda_k, \lambda_{\alpha^-}, \) and \( \lambda_{\alpha^+}, \lambda_{\alpha^-} \) are the linear Lagrange nodal basis functions defined in \([x_k, \alpha]\) and \([\alpha, x_{k+1}]\), respectively. We summarize the above findings in the following theorem.

**Theorem 3.1.** The global approximate flux function \( u_N(x, t_n) \) is continuous at all nodes \( x_i, i = 1, \ldots, N - 1 \) and \( \alpha \).

Now we show the flux approximation converges to the exact flux in the order of \( O(h + k) \).

**Theorem 3.2.** Let \( u \) be the exact flux and \( u_N \) be the approximate flux. Then there exists a positive constant \( C \) independent of \( h \) and \( k \) such that

\[ \| u(t_n) - u_N^n \|_0 \leq C(h + k) \]

for \( n = 1, \ldots, M \).

**Proof.** We give a proof for the interface element \( I_\kappa = [x_k, x_{k+1}] \). The one for non-interface elements follows similarly. Consider the subinterval \( I_{\kappa^-} = [x_k, \alpha] \). Since \( \lambda_k + \lambda_{\alpha^-} = 1 \), let \( u(x, t) = u(x, t)\lambda_k + u(x, t)\lambda_{\alpha^-} \). Then using (3.13) we can write the error in \( I_{\kappa^-} \) as

\[
u(x, t_n) - u_N^N(x) = (u(x, t_n) - u_N^N(x_k))\lambda_k + (u(x, t_n) - u_N^N(x_k))\lambda_{\alpha^-} \\
= (u(x_k, t_n) - u_N^N(x_k))\lambda_k + (u(-, t_n) - u_N^N(\alpha^-))\lambda_{\alpha^-} \\
+ (u(x, t_n) - u(x_k, t_n))\lambda_k + (u(x, t_n) - u(\alpha^-, t_n))\lambda_{\alpha^-}
\]

(3.14)

From (3.6) and (3.8),

\[
\| (u(x_k, t_n) - u_N^N(x_k))\lambda_k \|_{0, I_{\kappa^-}} \\
\leq \| (\int_{x_k}^{\alpha^-} \beta^-(pN, x) - p\hat{\phi}_k dx + \int_{x_k}^{\alpha} (pN - p)\hat{\phi}_k dx + \int_{x_k}^{\alpha} \gamma^- (\hat{\partial}_{pN} - p)\hat{\phi}_k dx)\lambda_k \|_{0, I_{\kappa^-}} \\
\leq \beta^- \| pN - p \|_{0, I_{\kappa^-}} \| \hat{\phi}_k \|_{0, I_{\kappa^-}} \| \lambda_k \|_{0, I_{\kappa^-}} \\
+ (\gamma^- \| \hat{\partial}_{pN} - p \|_{0, I_{\kappa^-}} + \| pN - p \|_{0, I_{\kappa^-}}) \| \hat{\phi}_k \|_{0, I_{\kappa^-}} \| \lambda_k \|_{0, I_{\kappa^-}}.
\]

Now using the facts \( \| \lambda_k \|_{0, I_{\kappa^-}} \leq Ch^{1/2}, \| \hat{\phi}_k \|_{0, I_{\kappa^-}} \leq Ch^{-1/2}, \) and \( \| \hat{\phi}_k \|_{0, I_{\kappa^-}} \leq Ch^{1/2} \), we have

\[
\| (u(x_k, t_n) - u_N^N(x_k))\lambda_k \|_{0, I_{\kappa^-}} \leq \beta^- \| pN - p \|_{0, I_{\kappa^-}} \\
+ h (\gamma^- \| \hat{\partial}_{pN} - p \|_{0, I_{\kappa^-}} + \| pN - p \|_{0, I_{\kappa^-}}).
\]

(3.15)
Using the fact that \( \|\lambda_\alpha\|_{0,T_-} \leq C h^{1/2} \) and from (3.10) and (3.11)
\[
\| (u(\alpha^-, t_n) - u_n^\alpha (\alpha^-)) \lambda_\alpha \|_{0,T_-} \\
\leq \| (u(x_n, t_n) - u_n^\alpha (x_n)) \lambda_\alpha \|_{0,T_-} + \| \int_{x_n}^\alpha (p - p_N) ds \lambda_\alpha \|_{0,T_-} \\
+ \| \gamma^- \int_{x_n}^\alpha (p - \bar{p}_N) ds \lambda_\alpha \|_{0,T_-} \\
\leq \| (u(x_n, t_n) - u_n^\alpha (x_n)) \lambda_\alpha \|_{0,T_-} \\
+ \frac{h}{n} \left[ \| \gamma^- \bar{p}_N - p_1 \|_{0,T_-} + \| p_N - p \|_{0,T_-} \right],
\]
Note that the first term on the right can be estimated as (3.15). In addition,
\[
\| (u(x, t_n) - u(x, t_n)) \lambda_\alpha \|_{0,T_-} \leq \| \int_{x_n}^\alpha u'(s) ds \lambda_\alpha \|_{0,T_-} \\
\leq Ch^{1/2} \| p \|_{2,T_-} \| \lambda_\alpha \|_{0,T_-} \\
\leq Ch \| p \|_{2,T_-}.
\]
Similarly,
\[
\| (u(x, t_n) - u(\alpha^-, t_n)) \lambda_\alpha \|_{0,T_-} \leq Ch \| p \|_{2,T_-}.
\]
Applying (3.15), (3.16), (3.17) and (3.18) to (3.14), we can estimate \( \| u(x, t_n) - u_n^\alpha(x) \|_{0,T_-} \) by the typical local norms. Similarly on the interval \([a, x_{n+1}]\), the same pattern holds for \( \| u(x, t_n) - u_n^\alpha(x) \|_{0,T_-} \) as well as similar estimates for noninterface elements. Thus squaring and summing over all elements and using Theorems 2.3
and 2.4, we have
\[
\| u(x, t_n) - u_n^\alpha \|_0 \leq C(h + k).
\]

\[\square\]

4. Numerical examples. In this section we give two numerical examples to confirm our theory.

Example 1.
\[
\gamma T_t - \beta T_{xx} = f(x, t), \quad (x, t) \in [0, 1] \times [0, 2], \\
T(x, 0) = T_0, \quad x \in I = [0, 1], \\
T(0, t) = T(1, t) = 0, \quad t > 0, \\
[T]_\alpha = 0, \quad [\beta T]_\alpha = 1, \quad \alpha = \frac{1}{3}.
\]

Here,
\[
f(x, t) = \begin{cases} \\
\frac{x^2}{\beta^2} + \frac{t^2 x}{\beta^2} + x^2 + 2t, & x \leq \alpha, \\
-\frac{x^2}{\beta^2} + \frac{t^2 x}{\beta^2} - \frac{t^2}{\beta^2} + \frac{1}{\beta^2} + x^2 + 2t, & \alpha \leq x.
\end{cases}
\]
\[
T_0 = \begin{cases} \\
\frac{x^2}{12\beta^2} + \frac{t^2 x}{12\beta^2}, & x \leq \alpha, \\
\frac{x^2}{12\beta^2} + \frac{t^2 x}{12\beta^2} - \frac{t^2}{\beta^2} + \frac{1}{12\beta^2}, & \alpha \leq x.
\end{cases}
\]

where
\[
t_1^- = \left( 1 - \frac{\alpha^4}{12\beta^2} - Q \frac{1 - \alpha}{\beta^2} + \frac{\alpha^4}{12\beta^2} \right) \frac{1}{((1 - \alpha)/\beta^2 + \alpha/\beta^-)}, \\
t_1^+ = t_1^- + Q, \\
t_2 = \left( \frac{-\alpha^2}{\beta^-} + \frac{\alpha^2}{\beta^+} - \frac{1}{\beta^+} \right) \frac{1}{((\alpha - 1)/\beta^+ - \alpha/\beta^-)}.
\]
The exact solution is
\[ T(x, t) = \begin{cases} -\frac{x^4}{12\beta} + \frac{t}{\beta} x + t(-\frac{x^2}{\beta} + \frac{2x}{\beta}), & x \leq \alpha, \\ \frac{x^4}{12\beta} + \frac{t}{\beta} x - \frac{x^2}{\beta} + \frac{1}{12\beta} + t(-\frac{x^2}{\beta} + \frac{2x}{\beta} - \frac{1}{\beta}), & \alpha \leq x. \end{cases} \]

The tables below contain \( L^2 \) error of the solution \( \|T - T_h\|_0 \) and flux error \( \|u - u_h\|_\infty \) in the maximum norm for \( \beta^- = 1, \beta^+ = 100 \) and \( \beta^- = 100, \beta^+ = 1 \). Error values in Tables 1 and 2 clearly indicate that solution \( T_h \) converges to \( T \) with optimal rate \( O(h^2) \) and flux \( u_h \) has optimal convergence rate \( O(h) \).

| \( h \)  | \( \|T - T_h\|_0 \)  | Order  | \( \|u - u_h\|_\infty \)  | Order  |
|--------|------------------|-------|------------------|-------|
| 80     | \( 4.8644 \times 10^{-3} \) | -     | \( 1.9474 \times 10^{-4} \) | -     |
| 160    | \( 1.2524 \times 10^{-3} \) | 1.9575 | \( 9.7360 \times 10^{-5} \) | 1.00031 |
| 320    | \( 3.1238 \times 10^{-4} \) | 2.0033 | \( 4.8670 \times 10^{-5} \) | 1.00029 |
| 640    | \( 7.8674 \times 10^{-5} \) | 1.9893 | \( 2.4332 \times 10^{-5} \) | 1.00017 |
| 1280   | \( 1.9657 \times 10^{-5} \) | 2.008  | \( 1.2165 \times 10^{-5} \) | 1.00007 |

Table 1. Error analysis of the solution and the flux for the case with \( \beta^- = 1 \) and \( \beta^+ = 100 \).

| \( h \)  | \( \|T - T_h\|_0 \)  | Order  | \( \|u - u_h\|_\infty \)  | Order  |
|--------|------------------|-------|------------------|-------|
| 80     | \( 2.9367 \times 10^{-3} \) | -     | \( 7.1587 \times 10^{-4} \) | -     |
| 160    | \( 7.3474 \times 10^{-4} \) | 1.9988 | \( 3.5519 \times 10^{-5} \) | 1.0011 |
| 320    | \( 1.8409 \times 10^{-5} \) | 1.9967 | \( 1.7690 \times 10^{-5} \) | 1.0056 |
| 640    | \( 4.6033 \times 10^{-6} \) | 1.9997 | \( 8.8279 \times 10^{-6} \) | 1.0028 |
| 1280   | \( 1.1151 \times 10^{-6} \) | 1.9991 | \( 4.4096 \times 10^{-6} \) | 1.0014 |

Table 2. Error analysis of the solution and the flux for the case with \( \beta^- = 100 \) and \( \beta^+ = 1 \).

**Example 2.** To illustrate our method applied to Pennes bioheat equation we consider a double layered tissue structure in the domain of 1 mm cross section. The first and the second layers have a thickness of 0.33 mm and 0.67 mm, respectively. More specifically, consider the following initial boundary value problem in \([0, 1]\) with an interface at 0.13:
\[
\rho c \frac{\partial T}{\partial t} - \beta \frac{\partial^2 T}{\partial x^2} = W_0 c_0 (T_a - T), \quad x \in [0, 1].
\]

The elevation of the temperature at the left end of the tissue is kept constant, and the heat flux approaches 0 at the other end \([3]\). This is realistic for a situation such as a skin burn in a biological body. Thus, the initial and boundary conditions are
\[
T(x, 0) = \begin{cases} -\frac{x}{\beta} + 12 & \text{for } 0 \leq x \leq \alpha, \\ 12 - \frac{Q_0}{\beta} & \text{for } \alpha \leq x \leq 1. \end{cases}
\]
\[
T(0, t) = 12°C \quad \text{for } t > 0,
\]
\[
T_x(1, t) = 0 \quad \text{for } t > 0.
\]

At the interface, we assume the temperature \( T \) satisfies the jump conditions
\[
[T]_\alpha = 0, \quad [\beta T_x]_\alpha = Q = 1.
\]
| Parameters                  | Layer I  | Layer II |
|-----------------------------|----------|----------|
| $L (mm)$                    | 0.13     | 0.87     |
| $\rho (g/mm^3)$             | $1.2 \times 10^{-3}$ | $1.0 \times 10^{-3}$ |
| $C (J/g^\circ C)$          | 3.4      | 3.06     |
| $C_b (J/g^\circ C)$        | 4.2      | 4.2      |
| $\beta (W/mm^2^\circ C)$   | $5.2 \times 10^{-4}$ | $2.1 \times 10^{-4}$ |
| $W_b (g/mm^3s)$             | $5.0 \times 10^{-7}$ | $5.0 \times 10^{-7}$ |

Table 3. Parameters for the two layers in the tissue.

The thermal properties for the two layer are given in Table 3 [3]. Temperature distribution and flux distribution for our example are shown in Figure 1 and 2, respectively.

Figure 1. Temperature profile along the tissue depth at $t = 0.1, 10, 100, 1000$ and $10000$, with interface at 0.33mm.

Figure 2. Temperature flux along the tissue depth at $t = 0.1, 10, 100, 1000$ and $10000$, with interface at 0.33mm.
REFERENCES

[1] S. H. Chou and S. Tang, Conservative P1 conforming and nonconforming Galerkin FEMs: effective flux evaluation via a nonmixed method approach, *SIAM J. Numer. Anal.*, **38** (2000), 660–680.

[2] S. H. Chou, An immersed linear finite element method with interface flux capturing recovery, *Discrete and Continuous Dynamical Systems-Series-B*, **17** (2012), 2343–2357.

[3] W. Dai, H. Yu and R. Nassar, A forth order compact finite-difference scheme for solving a 1-D Pennes bioheat transfer equation in a tripple layered skin structure, *Numerical Heat Transfer*, **46** (2004), 447–461.

[4] W. Dai, H. Yu and R. Nassar, Optimal temperature distribution in a three dimensional tripple layered skin structure embedded with artery and vein vasculature, *Num. Heat Transfer*, **50** (2006), 809–834.

[5] Z. S. Deng and J. Liu, Mathematical modeling of temperature mapping over skin surface and its implementation in thermal disease diagnostics, *Comput. Biol. Med.*, **34** (2004), 495–521.

[6] X. He, T. Lin and Y. Lin, Immersed finite element methods for elliptic interface problems with non-Homogeneous jump conditions, *Inter. J. Numerical Analysis and Modeling*, **8** (2011), 284–301.

[7] S. C. Jiang, N. Ma and H. J. Li, Effects of thermal properties and geometrical dimensions on skin burn injuries, *Burns*, **28** (2002), 713–717.

[8] Z. Li, The immersed interface method using a finite element formulation, *Applied Numerical Mathematics*, **27** (1998), 253–267.

[9] Z. Li, T. Lin, Y. Lin and R. C. Rogers, An immersed finite element space and its approximation capability, *Numer. Methods. Partial Differential Equations*, **20** (2004), 338–367.

[10] Z. Li, T. Lin and X. Wu, New Cartesian grid methods for interface problems using the finite element formulation, *Numer. Math.*, **96** (2003), 61–98.

[11] T. Lin, Y. Lin and W. Sun, Error estimation of a class quadratic immersed finite element methods for elliptic interface problems, *Discrete and Continuous Dynamical Systems Series-B*, **7** (2007), 807–823.

[12] E. H. Liu, G. M. Saidel and H. Harasaki, Model analysis of tissue responses to transient and chronic heating, *Ann. Biomed. Eng.*, **31** (2003), 1007–1048.

[13] H. H. Pennes, Analysis of tissue and arterial blood temperature in the resting forearm, *J. Appl. Physiol.*, **1** (1948), 93–122.

[14] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer, 2006.

[15] D. A. Tori and J. D. Dale, A finite element model of skin subjected to a flash fire, *J. Biomed Eng.*, **116** (1994), 250–255.

Received October 2013; revised May 2014.

E-mail address: attanac@miamic.edu
E-mail address: chou@bgsu.edu