Traveling wave solutions for the generalized (2+1)-dimensional Kundu-Mukherjee-Naskar equation

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Abstract

In this paper, we consider two types of traveling wave systems of the generalized Kundu-Mukherjee-Naskar equation. Firstly, due to the integrity, we obtain their energy functions. Then, the dynamical system method is applied to study bifurcation behaviours of the two types of traveling wave systems to obtain corresponding global phase portraits in different parameter bifurcation sets. According to them, every bounded and unbounded orbits can be identified clearly and investigated carefully which correspond to various traveling wave solutions of the generalized Kundu-Mukherjee-Naskar equation exactly. Finally, by integrating along these orbits and calculating some complicated elliptic integral, we obtain all type I and type II traveling wave solutions of the generalized Kundu-Mukherjee-Naskar equation without loss.

Keywords: GKMN equation, traveling wave solution, bifurcation, dynamical system

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1. Introduction

In this paper, we consider the following generalized (2+1)-dimensional Kundu-Mukherjee-Naskar (GKMN) equation \[1\]

\[ iq_t + a q_{xy} + ibq(qq^*_x - q^*q_x) = 0, \quad i = \sqrt{-1}, \]  

(1)

where complex function \( q = q(x, y, t) \) represents the profile of soliton, \( q^* \) is the complex conjugation of \( q(x, y, t) \), \( x, y \) and \( t \) are the spatial and temporal variables respectively. Real parameters \( a \) and \( b \) are the dispersion coefficient and nonlinear coefficient respectively. This equation can be used to describe optical wave propagation through coherently excited resonant waveguides, especially in the phenomenon of the bending of light beams\[2\].

When the coefficients \( a = 1 \) and \( b = 2 \), the GKMN equation degenerates to the classical KMN equation

\[ iq_t + q_{xy} + 2i q(qq^*_x - q^*q_x) = 0, \]  

(2)

which was first proposed by Kundu, Mukherjee and Naskar in 2014\[3, 4, 5\]. Besides description of the dynamics of optical soliton propagation in optical fibers, Eq. (2) also can be used extensively to address the problems of oceanic rogue waves, hole waves and an ion acoustic wave in a magnetized plasma\[5, 6, 7\].

Since the KMN equation is completely integrable and possesses the dynamic characteristics similar to the standard nonlinear Schrödinger equation\[5, 7\], it can be regarded as an integrable generalization of the well-known nonlinear Schrödinger equation

\[ q_t + iq_{xx} + 2i|q|^2q = 0. \]  

(3)

Unlike the conventional Kerr type nonlinearity in the Eq. (3), the nonlinear terms of Eq. (2) can be considered as the current nonlinearity caused by chirality\[8\]. Hence, Eq. (3) only allows bright (dark) soliton for focusing (defocusing) nonlinearity or with anomalous (normal) dispersion, whereas Eq. (2) admits both bright and dark soliton solutions regardless of its positive spatial dispersive term\[9\].
Traveling wave solutions of the KMN equation have been always focused on by people. In 2015, Mukherjee showed the connection between Eq. (2) and Kadomtsev Petviashvili equation and obtained its one-soliton solution, two-soliton solution and static lump solution by using Hirota bilinear method[7]. In 2017, Wen[10] pointed out that the solutions, satisfying the focusing nonlinear Schrödinger equation

$$q_y + iq_{xx} + 2i|q|^2q = 0 \tag{4}$$

and the complex modified Korteweg-de Vries equation

$$q_t + q_{xxx} + 6|q|^2q_x = 0, \tag{5}$$

must satisfy Eq. (2). So, by generalizing the $n$-fold Darboux transformation of Eqs. (4) and (5) to the perturbation $(n; M)$-fold Darboux transformation[10], he obtained the higher order rogue wave solutions of Eq. (2). Recently, as a generalization model of Eq. (2), the GKMN equation has aroused people’s more extensive interests and attentions. In 2018, Peng applied ansatz method to obtain the bright soliton, dark soliton and power series solutions of Eq. (1) and constructed the complexitons through the tanh method[11]. In 2019, Yildilm got bright, dark, singular, combo bright-dark, combo singular and singular periodic solitons of Eq. (1) by using the modified simple equation method, Riccati function method and so on[12, 13, 14, 15, 16]. In the same year, Ekici explored the plane wave solutions of Eq. (1) via the extended trial function method[17]. Later, Kudryashov used the Jacobi elliptic functions[18] to construct the general solution of the Eq. (1). With the aid of extended rational sinh-Gordon equation expansion method[19], Sulaiman obtained the trigonometric functions solutions of Eq. (1). In addition, Jhangeer applied the direct extended algebraic approach to Eq. (1) to derive the complex waves[20]. In 2020, Rizvi[21] got dark, bright, periodic U-shaped and singular solitons through the generalized Kudryashov method. Subsequently, Kumar[22] discussed singular, dark, combined dark-singular solitons and other hyperbolic solutions by using the csch method, extended tanh-coth method and extended rational sinh-cosh method. Meanwhile, Talarposhti[23] and Ghanbari[24] derived some new solitary solutions by using
the Exp-function method. More recently, Rezazadeh constructed the analytical solutions of Eq. (1) by utilizing the functional variable method [25].

Although various concise and efficient methods have been put forward to obtain so many profound results about traveling wave solutions of the Eq. (1), there still exist some problems unsolved. Firstly, due to the limitations caused by both the ansatz equations and the assumption about solutions in these direct methods, some solutions of Eq. (1) could be lost. In addition, we note that unbounded traveling wave solutions of Eq. (1) have not been reported in previous work. So, in this paper, we will try to apply the bifurcation method of dynamical system [26, 27] to solve these problems. This method allows detailed analysis of phase space geometry of the traveling wave system of Eq. (1). Through exploring various orbits of traveling wave system of Eq. (1), we construct its traveling wave solutions uniformly, including the bounded traveling wave solutions and unbounded ones.

Our paper is organized as follows: In section 2, we derive two types of traveling wave systems of Eq. (1), including a singular traveling wave system. Then, by studying bifurcation of the two traveling wave systems (8) and (13) by dynamical system method.

2. Traveling wave systems and bifurcation analysis

In this section, inspired by previous work [12, 18], we firstly derive two types of traveling wave systems of Eq. (1). Then, we study bifurcation of the two traveling wave systems (8) and (13) by dynamical system method.

2.1. Two types of traveling wave systems of the GKMN equation

Firstly, we assume that the type I traveling wave solution has the form

$$q(x, y, t) = p(\xi)\exp(\psi(x, y, t)i),$$

(6)
where real function $p(\xi)$ is the portion of the amplitude with $\xi = x + my - ct$ and real function $\psi(x, y, t) = \kappa x + \omega y - rt + \theta$ is the phase component. Parameters $m$, $c$, $r$ and $\theta$ are reals and represent the inverse width of the soliton in the direction $y$, wave velocity, wave number and phase constant respectively. Real parameters $\kappa$ and $\omega$ denote the frequencies along the $x$ and $y$ directions respectively.

Substituting the solution (6) into Eq. (1), we obtain

\[
\begin{align*}
\text{real part:} & \quad amp'' + (r - a\kappa \omega)p + 2\kappa bp^3 = 0, \\
\text{imaginary part:} & \quad (-c + am\kappa + a\omega)p' = 0,
\end{align*}
\]

(7)

where $'$ denotes $d/d\xi$. From the imaginary part in (7), we have

\[c = am\kappa + a\omega.\]

It means that system (7) has the equivalent form

\[
\begin{align*}
p' &= y, \\
y' &= \frac{a\kappa \omega - r}{am}p - \frac{2kb}{am}p^3,
\end{align*}
\]

(8)

with constraint $c = am\kappa + a\omega$. Obviously, system (8) is a Hamiltonian system with the energy function

\[H(p, y) = \frac{1}{2}y^2 - \frac{a\kappa \omega - r}{2am}p^2 + \frac{kb}{2am}p^4.\]

(9)

Then, we consider the type II traveling wave solution

\[q(x, y, t) = \phi(\xi)\exp((\varphi(\xi) - \mu t)i), \quad \xi = x + my - ct,\]

(10)

where $\phi(\xi)$ is the amplitude function, $\varphi(\xi)$ is the phase function and real parameters $m$, $c$ and $\mu$ represent the inverse width of the soliton in the direction $y$, wave velocity and wave frequency respectively. Here, we suppose that the amplitude function $\phi(\xi) \neq 0$, otherwise the solution $q(x, y, t)$ of Eq. (1) degenerates to the trivial solution. Substituting (10) into (1) and separating the real and imaginary parts, we have

\[
\begin{align*}
\text{real part:} & \quad am\phi'' + n\phi\phi' + c\phi - am(\varphi')^2 + 2b\phi^3\varphi' = 0, \\
\text{imaginary part:} & \quad -c\phi' + 2am'\varphi' + am\phi' = 0,
\end{align*}
\]

(11)
where \( \prime \) denotes \( d/d\xi \). From the second equation in (11), we can solve

\[
\phi' = \frac{e}{am\phi^2} + \frac{c}{2am},
\]

where \( e \) is the integral constant. Plugging (12) into the real part in (11), we obtain

\[
\begin{align*}
\phi' &= y, \\
y' &= \frac{\alpha_1\phi^6 + \alpha_2\phi^4 + \alpha_3}{\phi^3},
\end{align*}
\]

(13)

where \( \alpha_1 = -\frac{cb}{a^2m^2} \), \( \alpha_2 = -\frac{\mu}{am} - \frac{c^2 + 8be}{4} \) and \( \alpha_3 = \frac{e^2}{a^2m^2} \neq 0 \). If \( \alpha_3 = 0 \), system (13) degenerates to system (8). System (13) is a singular traveling wave system. With the transformation \( d\xi = \phi^3d\zeta \), it can be converted to the associated regular system

\[
\begin{align*}
\phi' &= \phi^3y, \\
y' &= \alpha_1\phi^6 + \alpha_2\phi^4 + \alpha_3,
\end{align*}
\]

(14)

where \( \prime \) stands for \( d/d\zeta \), which has the energy function

\[
H(\phi, y) = \frac{1}{2}y^2 - \frac{\alpha_1}{4}\phi^4 - \frac{\alpha_2}{2}\phi^2 + \frac{\alpha_3}{2}\frac{1}{\phi^2}.
\]

(15)

When \( \phi \neq 0 \), systems (13) has the same vector field as system (14). So, the function (15) is also the energy function of system (13).

2.2. Bifurcation analysis

Firstly, we study the distribution and properties of equilibria of system (8) and (14).

**Theorem 1.** When \( kb(ak\omega - r) > 0 \), system (8) has three equilibria \( P_1(-\sqrt{\frac{ak\omega - r}{2kb}}, 0) \), \( P_2(0, 0) \) and \( P_3(\sqrt{\frac{ak\omega - r}{2kb}}, 0) \). If \( \frac{kb}{am} < 0(> 0) \), \( P_1 \) and \( P_3 \) are saddles(centers), while \( P_2 \) is a center(saddle). When \( kb(ak\omega - r) < 0 \), system (8) has only one simple equilibrium \( P_2(0, 0) \). If \( \frac{kb}{am} < 0(> 0) \), \( P_2 \) is a saddle(center). When \( ak\omega - r = 0 \), system (8) has a unique degenerate equilibrium \( P_2(0, 0) \). If \( \frac{kb}{am} < 0(> 0) \), \( P_2 \) is still a saddle(center).
Proof. When $κb(akω − r) \neq 0$, by the theory of dynamical system and the properties of Hamiltonian system [26, 27, 28], it is easy to verify the corresponding results above.

Especially, when $akω−r = 0$, we see that system (8) has a unique equilibrium $P_2(0, 0)$ with the degenerate Jacobian matrix

$$M(P_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$  

In this case, system (8) has the associated normal form

$$\begin{cases} 
  p' = y, \\
  y' = akp^k[1 + f_1(p)] + bn p^n y[1 + f_2(p)] + y^2 f_3(p, y) = -\frac{2kb}{am}p^3, 
\end{cases}$$

where $k = 3, a_k = \frac{2kb}{am}, b_n = 0, f_1(p) = 0, f_2(p) = 0$ and $f_3(p, y) = 0$. From the fact that $k$ is an odd number, the degenerate equilibrium $P_2$ is a saddle when $a_k > 0$, whereas $P_2$ is a center when $a_k < 0$ and $b_n = 0$ according to the qualitative theory of differential equation [28, Theorem 7.2, Chapter 2].

Furthermore, we note that when $κb(akω − r) > 0$, the energy of three equilibria has the following relationship

$$h(P_2) = 0,$$
$$h(P_1) ≡ h(P_3) = -\frac{(akω−r)^2}{8abkm},$$

which means that the energy of saddles $P_1$ and $P_3$ is always equivalent. So, according to the properties of Hamilton system [26], we have the following results:

**Case 1.** When $κb(akω−r) > 0$ and $\frac{kb}{am} < 0$, there exist two heteroclinic orbits $Γ_0$ and $Γ_0$ connecting the saddles $P_1$ and $P_3$. Center $P_2$ is surrounded by a family of periodic orbits

$$γ(h) = \{H(p, y) = h, h \in (0, -\frac{(akω−r)^2}{8abkm})\}.$$  

$γ(h)$ tends to $P_2$ as $h → 0$, and tends to $Γ_0$ and $Γ_0$ as $h → -\frac{(akω−r)^2}{8abkm}$. Except the periodic orbit and the heteroclinic orbits, other orbits of system (8) are unbounded. Please see Fig. 1(a).
Case 2. When $\kappa b(\alpha \kappa \omega - r) > 0$ and $\frac{\kappa b}{am} > 0$, all orbits are bounded. There exist two homoclinic orbits $\Upsilon_1$ and $\Upsilon_2$ connecting the saddle $P_2$. Centers $P_1$ and $P_3$ are surrounded by two families of periodic orbits

$$\gamma_R(h) = \{H(p,y) = h, h \in \left(-\frac{(\alpha \kappa \omega - r)^2}{8abkm}, 0\right)\},$$

$$\gamma_L(h) = \{H(p,y) = h, h \in \left(-\frac{(\alpha \kappa \omega - r)^2}{8abkm}, 0\right)\}.$$

$\gamma_R(h)$ and $\gamma_L(h)$ tend to $P_1$ and $P_3$ respectively as $h \to -\frac{(\alpha \kappa \omega - r)^2}{8abkm}$, and tend to $\Upsilon_1$ and $\Upsilon_2$ respectively as $h \to 0$ shown in Fig. 1(b).

Case 3. When $\kappa b(\alpha \kappa \omega - r) < 0$ or $\alpha \kappa \omega - r = 0$, system (8) has only one equilibrium. If $\frac{\kappa b}{am} < 0 (> 0)$, all orbits of system (8) are unbounded (bounded) shown in Figs. 2 and 3.

![Phase portraits of system (8) for $\kappa b(\alpha \kappa \omega - r) > 0$.](image)

(a) $\frac{\kappa b}{am} = -\frac{1}{2}$, $\frac{\alpha \kappa \omega - r}{am} = -4$. (b) $\frac{\kappa b}{am} = \frac{1}{2}$, $\frac{\alpha \kappa \omega - r}{am} = 4$.

Figure 1: Phase portraits of system (8) for $\kappa b(\alpha \kappa \omega - r) > 0$.

**Theorem 2.** When $\alpha_1 > 0$, $\alpha_2 < 0$ and $0 < \alpha_3 < -\frac{4\alpha_2^3}{27\alpha_1^2}$, system (14) has two pairs of equilibria $\tilde{P}_{1,2}(\pm u_1, 0)$ and $\tilde{P}_{3,4}(\pm u_2, 0)$, where $P_{1,2}$ are centers and $P_{3,4}$ are saddles. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = -\frac{4\alpha_2^3}{27\alpha_1^2}$, system (14) has a pair of cusps $\tilde{P}_{5,6}(\pm u_3, 0)$. When either $\alpha_1 > 0$, $\alpha_2 \geq 0$ or $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 < 0$, or $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 = 0$, system (14) has a saddle-node bifurcation at $\tilde{P}_{4,5}(\pm u_4, 0)$. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = 0$, system (14) has a pair of saddles $\tilde{P}_{7,8}(\pm u_5, 0)$, and when $\alpha_1 > 0$, $\alpha_2 \geq 0$ or $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 > 0$, system (14) has a pair of nodes $\tilde{P}_{9,10}(\pm u_6, 0)$. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = 0$, or $\alpha_1 > 0$, $\alpha_2 \geq 0$ or $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 > 0$, system (14) has a saddle-node bifurcation at $\tilde{P}_{11,12}(\pm u_7, 0)$. When $\alpha_1 > 0$, $\alpha_2 < 0$ and $\alpha_3 = 0$, or $\alpha_1 > 0$, $\alpha_2 \geq 0$ or $\alpha_1 > 0$, $\alpha_2 < 0$, $\alpha_3 > 0$, system (14) has a pair of saddles $\tilde{P}_{13,14}(\pm u_8, 0)$.
\( \frac{\kappa b}{am} = -2, \frac{a\kappa\omega - r}{am} = 4. \)

\( \frac{\kappa b}{am} = 2, \frac{a\kappa\omega - r}{am} = -4. \)

Figure 2: Phase portraits of system (8) for \( \kappa b(a\kappa\omega - r) < 0. \)

\( \frac{\kappa b}{am} = \frac{1}{2}, a\kappa\omega - r = 0. \)

\( \frac{\kappa b}{am} = \frac{1}{2}, a\kappa\omega - r = 0. \)

Figure 3: Phase portraits of system (8) for \( a\kappa\omega - r = 0. \)

\( \alpha_3 > -\frac{4\alpha_2^2}{27\alpha_1^2}, \) system (14) has no equilibrium. When \( \alpha_1 < 0, \) system (14) has a pair of centers \( \bar{P}_{7,8}(\pm u_4, 0). \)

Proof. Similar to the proof of Theorem 1 we only give the detail proof for degenerate equilibria for simplicity. When \( \alpha_1 > 0, \alpha_2 < 0 \) and \( \alpha_3 = -\frac{4\alpha_2^2}{27\alpha_1^2}, \) a
direct computation shows that system (14) has a pair of degenerate equilibria \( \tilde{P}_{5,6}(\pm u_3,0) \) with the degenerate Jacobian matrix
\[
M(\tilde{P}_{5,6}) = \begin{bmatrix} 0 & \pm u_3^3 \\ 0 & 0 \end{bmatrix}.
\]

In order to judge the type of the equilibrium \( \tilde{P}_{5}(u_3,0) \), we make the homeomorphic transformation
\[
u = \phi - u_3, \quad v = y + \frac{1}{u_3^3}(3u_3^2\alpha y + 3u_3\alpha^2 y + \alpha^3 y),
\]
which transforms system (14) to the normal form
\[
\begin{cases}
u' = v, \\
v' = a_k u^k[1 + g_1(u)] + b_n u^n [1 + g_2(u)] + v^2 g_3(u,v),
\end{cases}
\]
where \( k = 2, \quad M = \frac{\alpha_2 + 3u_3^2\alpha_1}{\alpha_1}, \quad a_k = \frac{4\alpha_1 M}{u_3^3}, \quad g_1(u) = \frac{1}{4u_3^2 M}((8u_3^5 M + 16u_3^4 M)u + (12u_3^3 + 24u_3^3 + 25u_3^3 M)u^2 + (42u_3^3 + 24u_3^3 + 19u_3^3 M) u^3 + (54u_3^3 + 9u_3^3 + 7u_3 M) u^4 + (33u_3^2 + M) u^5 + 9u_3 u^6 + u^7),
\]
\( b_n = 0 \) and \( g_3(u,v) = \frac{3u_3^2 + 6u_3 u + 3a^2}{u_3^3 + 3u_3^2 u + 3u_3 u^2 + u^3}. \)

From the fact that \( k = 2 \) is an even number and \( b_n = 0 \), we come to the conclusion that equilibrium \( \tilde{P}_{5}(u_3,0) \) is a cusp according to the qualitative theory of differential equation [28, Theorem 7.3, Chapter 2].

Similarly, applying another homeomorphic transformation
\[
u = \phi + u_3, \quad v = y - \frac{1}{u_3^3}(3u_3^2\alpha y - 3u_3\alpha^2 y + \alpha^3 y),
\]
to system (14), one can check equilibrium \( \tilde{P}_{6}(-u_3,0) \) is also a cusp.

Based on the properties of the system (14), the bifurcation results of system (13) are given as follows.

**Case I.** When \( \alpha_1 > 0, \alpha_2 < 0 \) and \( 0 < \alpha_3 \leq \frac{4\alpha_3}{27\alpha_1^2} \), there exist two homoclinic orbits \( \Pi^+_0 \) and \( \Pi^-_0 \) connecting saddles \( \tilde{P}_3 \) and \( \tilde{P}_4 \) respectively. Centers \( \tilde{P}_1 \) and \( \tilde{P}_2 \) are surrounded by two families of periodic orbits, respectively
\[
\gamma^+_1(h) = \{H(\phi, y) = h, h \in (h(u_1,0), h(u_2,0))\},
\]
\[
\gamma^-_1(h) = \{H(\phi, y) = h, h \in (h(-u_1,0), h(-u_2,0))\}.
\]
\( \gamma^+_1(h) \) and \( \gamma^-_1(h) \) respectively tend to \( \tilde{P}_{1,2}(\pm u_1,0) \) as \( h \to h(\pm u_1,0) \), and tend to \( \Pi^+_0 \) and \( \Pi^-_0 \) as \( h \to h(\pm u_2,0) \). All orbits are unbounded except for the periodic orbits and the homoclinic orbits shown in Fig. 4(a).

**Case II.** When \( \alpha_1 > 0, \alpha_2 < 0 \) and \( \alpha_3 = -\frac{4\alpha_2^3}{27\alpha_1^2} \), all orbits of system (13) are unbounded. In the positive \( \phi \)-axis, the orbit \( \Omega^1 \) is different from orbit \( \Omega_1 \). More precisely, the \( \omega \)-limit set of \( \Omega_1 \) and the \( \alpha \)-limit set of \( \Omega \) correspond to the same equilibrium \( \tilde{P}_5(u_3,0) \) shown in Fig. 4(b).

**Case III.** When \( \alpha_1 > 0, \alpha_2 \geq 0 \) or \( \alpha_1 > 0, \alpha_2 < 0 \) and \( \alpha_3 > -\frac{4\alpha_2^3}{27\alpha_1^2} \), there only exist unbounded orbits of system (13) shown in Fig. 4(c).

**Case IV.** When \( \alpha_1 < 0 \), there exist two families of periodic orbits shown in Fig. 4(d).

### 3. Exact solutions of system (8) and (13)

In this section, we seek the explicit expressions of bounded and unbounded solutions of systems (8) and (13).

#### 3.1. Bounded solutions of system (8)

According to the bifurcation results in Theorem 1, to seek bounded solutions of system (8), there are three cases need to be discussed.

1. When \( k\beta(a\kappa\omega - r) > 0 \) and \( \frac{k\beta}{am} < 0 \), we consider two subcases as follows.
   1. Consider the periodic orbits shown in Fig. 1(a), whose energy is lower than energy of the saddle \( P_1 \), but higher than energy of center \( P_2 \). Any one of them can be expressed by
      
      \[
      y = \pm \sqrt{-\frac{k\beta}{am}} \sqrt{(p - p_1)(p - p_2)(p_3 - p)(p_4 - p)},
      \]

      where \( p_1, p_2, p_3 \) and \( p_4 \) satisfy the constraint condition 
      \( p_1 < -\sqrt{\frac{a\kappa\omega - r}{2k\beta}} < p_2 < p < p_3 < \sqrt{\frac{a\kappa\omega - r}{2k\beta}} < p_4 \). Assuming that the period is \( 2T_0 \) and choosing initial value \( p(0) = p_2 \), we have
      
      \[
      \int_{p_2}^{p} \frac{-am}{k\beta} \sqrt{(p - p_1)(p - p_2)(p_3 - p)(p_4 - p)} \, dp = \int_{0}^{\xi} d\xi, \quad 0 < \xi < T_0,
      \]
(a) $\alpha_1 = 1, \alpha_2 = -4, \alpha_3 = 0.1$.

(b) $\alpha_1 = 1, \alpha_2 = -4, \alpha_3 = \frac{256}{27}$.

(c) $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 0.1$.

(d) $\alpha_1 = -1, \alpha_2 = 0, \alpha_3 = 0.1$.

Figure 4: Phase portraits of system (13).

$$
- \int_p^{p_2} \frac{am}{k\beta} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = \int_\xi^0 d\xi, \quad -T_0 < \xi < 0,
$$

which can be rewritten as

$$
\int_p^{p_2} \frac{am}{k\beta} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = |\xi|, \quad -T_0 < \xi < T_0.
$$

By calculating the elliptic integral

$$
\int_p^{p_2} \frac{dp}{\sqrt{(p-p_1)(p-p_2)(p_3-p)(p_4-p)}} = g \cdot sn^{-1} \left( \sqrt{\frac{(p_3-p_1)(p-p_2)}{(p_3-p_2)(p-p_1)}}, k \right),
$$
where \( g = \frac{2}{\sqrt{(p_3 - p_1)(p_4 - p_2)}} \), \( k^2 = \left(\frac{p_3 - p_2}{p_4 - p_2}\right)\left(\frac{p_4 - p_1}{p_3 - p_1}\right) \), we get the first type of periodic solution of system (8).

\[ p_{b_1}(\xi) = p_1 + \frac{(p_2 - p_1)(p_3 - p_1)}{(p_3 - p_1) - (p_3 - p_2)} \sin^2\left(\sqrt{-\frac{\kappa b(p_3 - p_2)(p_3 - p_1)}{4am}} \xi\right), \quad -T_0 < \xi < T_0. \]

(2) Consider the heteroclinic orbit \( \Gamma^0 \) shown in Fig. 1(a) whose energy is equal to energy of the saddle \( P_1 \). It can be expressed by

\[ y = \sqrt{-\frac{\kappa b}{am}(p - p_5)^2(p_6 - p)^2}, \]

where \( p_5 \) and \( p_6 \) satisfy the constraint condition \( -\sqrt{\frac{ak\omega - r}{2kb}} = p_5 < p < p_6 = \sqrt{\frac{ak\omega - r}{2kb}} \). Choosing initial value \( p(0) = \frac{p_5 + p_6}{2} = 0 \), we have

\[ \int_{\xi}^{p} \frac{-am}{\kappa b (p - p_5)(p_6 - p)} \, dp = \int_{-\infty}^{\infty} d\xi, \quad -\infty < \xi < +\infty, \]

Noting that

\[ \int_{0}^{p} \frac{dp}{(p - p_5)(p_6 - p)} = \frac{2}{p_6 - p_5} \tanh^{-1} \frac{2p - (p_5 + p_6)}{p_6 - p_5}, \]

we obtain the expression of kink wave solution of system (8)

\[ p_{b_2}(\xi) = \frac{p_6 - p_5}{2} \tanh\left(\frac{p_6 - p_5}{2} \sqrt{-\frac{\kappa b}{am}} \xi\right), \quad -\infty < \xi < +\infty. \]

Applying similar calculation to another heteroclinic orbit \( \Gamma_0 \) shown in Fig. 1(a) we can get the corresponding kink wave solution of system (8) as follows

\[ p_{b_2}(\xi) = \frac{p_6 - p_5}{2} \tanh\left(\frac{p_6 - p_5}{2} \sqrt{-\frac{\kappa b}{am}} \xi\right), \quad -\infty < \xi < +\infty. \]

2. When \( \kappa b(ak\omega - r) > 0 \) and \( \frac{\kappa b}{am} > 0 \), we need to consider three subcases in this case.

(1) Consider the family of periodic orbits inside the homoclinic orbit \( \gamma_2 \) shown in Fig. 1(b) whose energy is lower than energy of the saddle \( P_2 \), but higher than energy of center \( P_1 \). Any one of them can be expressed by

\[ y = \pm \sqrt{\frac{\kappa b}{am}(p - p_7)(p_8 - p)(p_9 - p)(p_{10} - p)}, \]

13
where \( p_7, p_8, p_9 \) and \( p_{10} \) satisfy the constraint condition \( p_7 < p < p_8 < 0 < p_9 < \sqrt{\frac{a\omega r}{2kb}} < p_{10} \). Assuming that the period is \( 2T_1 \), similar to the calculation of solution \( p_{b_1}(\xi) \), we get the second type of periodic solution of system (8)

\[
p_{b_2}(\xi) = p_{10} - \frac{(p_{10} - p_8)(p_{10} - p_7)}{(p_{10} - p_8) + (p_8 - p_7) \text{sn}^2(\sqrt{\frac{kb}{am}} \sqrt{(p_{10} - p_8)(p_9 - p_7)} \xi)}, \quad -T_1 < \xi < T_1.
\]

Similarly, we can give the periodic solutions of system (8) corresponding to the family of periodic orbits inside the homoclinic orbit \( \Upsilon_1 \) shown in Fig. 1(b)

\[
p_{b_3}(\xi) = p_8 + \frac{(p_9 - p_8)(p_{10} - p_8)}{(p_{10} - p_8) - (p_{10} - p_9) \text{sn}^2(\sqrt{\frac{kb}{am}} \sqrt{(p_{10} - p_8)(p_9 - p_7)} \xi)}, \quad -T_1 < \xi < T_1,
\]

where \( p_7 < -\sqrt{\frac{a\omega r}{2kb}} < p_8 < 0 < p_9 < p < p_{10} \).

(2) Consider the homoclinic orbits shown in Fig. 1(b) whose energy is equal to energy of the saddle \( P_2 \). The homoclinic orbit \( \Upsilon_2 \) can be expressed by

\[
y = \pm \sqrt{\frac{kb}{am}} \sqrt{(p + p_{11})(p_2 - p)},
\]

where \( p_{11} \) and \( -p_{11} \) satisfy the constraint condition \( -p_{11} < p < 0 < \sqrt{\frac{a\omega r}{2kb}} < p_{11} \). Choosing initial value \( p(0) = -p_{11} \), we have

\[
\int_{-p_{11}}^{p} \sqrt{\frac{am}{kb}} \frac{dp}{\sqrt{(p + p_{11})(p_2 - p)}} = \int_{0}^{\xi} d\xi, \quad \xi > 0,
\]

\[
-\int_{p}^{-p_{11}} \sqrt{\frac{am}{kb}} \frac{dp}{\sqrt{(p + p_{11})(p_2 - p)}} = \int_{\xi}^{0} d\xi, \quad \xi < 0,
\]

which can be rewritten as

\[
\int_{-p_{11}}^{p} \sqrt{\frac{am}{kb}} \frac{dp}{\sqrt{(p + p_{11})(p_2 - p)}} = |\xi|.
\]

Noting that

\[
\int_{-p_{11}}^{p} \frac{dp}{p \sqrt{(p + p_{11})(p_2 - p)}} = \ln\left(\frac{-p_{11} - \sqrt{p_{11}^2 - p^2}}{-p_{11}}\right),
\]

we obtain the expression of solitary wave solution of system (8)

\[
p_{b_4}(\xi) = \frac{-2p_{11}\exp\left(\sqrt{\frac{kb}{am}} p_{11} |\xi|\right)}{\exp(2\sqrt{\frac{kb}{am}} p_{11} |\xi| + 1)}, \quad -\infty < \xi < +\infty.
\]
Similarly, we can get another solitary wave solution of system (8) corresponding to the homoclinic orbit \( Y_p \)

\[
p_{b_p}(\xi) = \frac{2p_{11}\exp(\sqrt{\frac{kb}{am}p_{11}} \mid \xi \mid)}{\exp(2\sqrt{\frac{2b}{am}p_{11}} \mid \xi \mid) + 1}, \quad -\infty < \xi < +\infty,
\]

where \(-p_{11} < -\sqrt{\frac{a\nu - r}{2kb}} < 0 < p < p_{11}\).

(3) Consider the family of large amplitude periodic orbits shown in Fig. 1(b), whose energy is higher than energy of the saddle \( P_2 \). Any one of them can be expressed by

\[
y = \pm \sqrt{\frac{kb}{am} \int \frac{dp}{(p + p_{12})(p_{12} - p)(p^2 + p_{12}^2 - \frac{a\nu - r}{kb})}},
\]

where \( p_{12} \) and \(-p_{12} \) satisfy the constraint condition \( p_{12} > 0, p_{12}^2 - \frac{a\nu - r}{kb} > 0 \) and \(-p_{12} < p < p_{12} \). Assuming that the period is \( 2T_2 \) and choosing initial value \( p(0) = \frac{-p_{12} + p_{12}}{2} = 0 \), we have

\[
\int_0^p \sqrt{\frac{am}{kb}} \frac{dp}{(p + p_{12})(p_{12} - p)(p^2 + p_{12}^2 - \frac{a\nu - r}{kb})} = \int_0^\xi d\xi, \quad 0 < \xi < T_2,
\]

\[
-\int_p^0 \sqrt{\frac{am}{kb}} \frac{dp}{(p + p_{12})(p_{12} - p)(p^2 + p_{12}^2 - \frac{a\nu - r}{kb})} = \int_\xi^0 d\xi, \quad -T_2 < \xi < 0,
\]

which can be rewritten as

\[
\int_0^p \sqrt{\frac{am}{kb}} \frac{dp}{(p + p_{12})(p_{12} - p)(p^2 + p_{12}^2 - \frac{a\nu - r}{kb})} = |\xi|, \quad -T_2 < \xi < T_2.
\]

Noting that

\[
\int_0^p \frac{dp}{(p + p_{12})(p_{12} - p)(p^2 + p_{12}^2 - \frac{a\nu - r}{kb})} = g \cdot \text{sn}^{-1}\left( \frac{p^2(2kbp_{12}^2 - a\nu + r)}{p_{12}^2(kbp_{12}^2 + kbp_{12}^2 - a\nu + r)} k \right),
\]

where \( g = \sqrt{\frac{kb}{2kbp_{12}^2 - a\nu + r}}, \quad k^2 = \frac{kbp_{12}^2}{2kbp_{12}^2 - a\nu + r} \), we obtain the third type of periodic solution of system (8)

\[
p_{b_3}(\xi) = \frac{(kbp_{12}^2 - a\nu + r)p_{12}^2sn^2(\sqrt{\frac{2kbp_{12}^2 - a\nu + r}{am}} \xi)}{2kbp_{12}^2 - a\nu + r - kbp_{12}^2sn^2(\sqrt{\frac{2kbp_{12}^2 - a\nu + r}{am}} \xi)}, \quad -T_2 < \xi < T_2.
\]
When $\kappa b(\ak \omega - r) \leq 0$ and $\frac{\kappa b}{am} > 0$, one can check that the periodic orbits shown in Figs. 2(b) and 3(b) have the same form as solution $p_{b_0}(\xi)$. We ignore them here for simplicity.

3.2. Unbounded solutions of system (8)

In this subsection, we need to consider two cases to get unbounded solutions of system (8).

1. When $\kappa b(\ak \omega - r) > 0$ and $\frac{\kappa b}{am} < 0$, we have five subcases to discuss.

(1) Consider the first type of unbounded orbits, for example $\Gamma^1$ and $\Gamma_1$, shown in Fig. 1(a) whose energy $h_0$ is higher than energy of saddles $P_1$ and $P_3$. They can be expressed respectively by

$$y = \pm \sqrt{-\frac{\kappa b}{am} p^4 + \frac{\ak \omega - r}{am} p^2 + 2h_0},$$

where $0 < p < +\infty$. For the sake of simplicity, we take $\Gamma_1$ for example to calculate its corresponding solution. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p} \frac{dp}{p^4 + \frac{\ak \omega - r}{\kappa b} p^2 - \frac{2amh_0}{\kappa b}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{p}^{+\infty} \frac{dp}{p^4 + \frac{\ak \omega - r}{\kappa b} p^2 - \frac{2amh_0}{\kappa b}} = g \cdot cn^{-1}\left(\frac{p^2 - \frac{2amh_0}{\kappa b}}{p^2 + \frac{2amh_0}{\kappa b}}, k\right),$$

where $g = \frac{1}{2} \sqrt{-\frac{2amh_0}{\kappa b}}$, $k^2 = \frac{2\sqrt{-2am\kappa bh_0} + \ak \omega - r}{4\sqrt{-2am\kappa bh_0}}$, we obtain the first type of unbounded solution of system (8)

$$p_{u_0}(\xi) = \sqrt{-\frac{2amh_0}{\kappa b}} \cdot \sqrt{-1 + \frac{2}{1 - cn(2\sqrt{\frac{2amh_0}{\kappa b}} \xi)}}, \quad 0 < \xi < \xi_0,$$

where $\xi_0 = 2\sqrt{-\frac{am}{2\kappa bh_0}} \cdot \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{1 - \frac{2\sqrt{-2am\kappa bh_0} + \ak \omega - r}{4\sqrt{-2am\kappa bh_0}} \cdot \sin^2 \theta}$.

It is not difficult to check that the corresponding unbounded solution of $\Gamma^1$ has the same form as $p_{u_0}(\xi)$. 

16
Consider the second type of unbounded orbits $\Gamma^2$, $\Gamma_2$, $\Gamma^3$ and $\Gamma_3$ shown in Fig. 1(a) whose energy is equal to energy of saddles $P_1$ and $P_3$. The orbits $\Gamma^2$ and $\Gamma_2$ can be expressed respectively by

$$y = \pm \sqrt{-\frac{kb}{am}} \sqrt{(p - \sqrt{\frac{a\kappa \omega - r}{2kb}})^2(p + \sqrt{\frac{a\kappa \omega - r}{2kb}})^2},$$

where $0 < \frac{a\kappa \omega - r}{2kb} < p < +\infty$. Similar to the discussion above, we only need to discuss the orbit $\Gamma_2$. Choosing initial value $p(0) = +\infty$, we have

$$-\int_{+\infty}^{p} \frac{-am}{kb} \frac{dp}{(p - \sqrt{\frac{a\kappa \omega - r}{2kb}})(p + \sqrt{\frac{a\kappa \omega - r}{2kb}})} = \int_{0}^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{p}^{+\infty} \frac{dp}{(p - \sqrt{\frac{a\kappa \omega - r}{2kb}})(p + \sqrt{\frac{a\kappa \omega - r}{2kb}})} = \sqrt{\frac{kb}{2(a\kappa \omega - r)}} \ln \left( \frac{p + \sqrt{\frac{a\kappa \omega - r}{2kb}}}{p - \sqrt{\frac{a\kappa \omega - r}{2kb}}} \right),$$

we obtain the second type of unbounded solution of system (8)

$$p_{u_1}(\xi) = \sqrt{\frac{a\kappa \omega - r}{2kb}} \cdot \left( 1 + \frac{2}{\exp(\frac{2(r-a\kappa \omega)}{am} \xi) - 1} \right), \quad \xi > 0.$$

One can check that the corresponding unbounded solution of $\Gamma^2$ has the same form as $p_{u_1}(\xi)$. If choosing initial value $p(0) = -\infty$ and applying similar calculation to the orbit $\Gamma^3$, we can get the corresponding unbounded solution of system (8)

$$p_{u_1'}(\xi) = -\sqrt{\frac{a\kappa \omega - r}{2kb}} \cdot \left( 1 + \frac{2}{\exp(\frac{2(r-a\kappa \omega)}{am} \xi) - 1} \right), \quad \xi > 0,$$

where $-\infty < p < -\sqrt{\frac{a\kappa \omega - r}{2kb}} < 0$. And it is not difficult to conclude that the corresponding unbounded solution of the orbit $\Gamma_3$ has the same form as $p_{u_1'}(\xi)$.

(3) Consider the third type of unbounded orbits, for example $\Gamma^+_4$, shown in Fig. 1(a) whose energy is lower than energy of the saddle $P_1$, but higher than energy of center $P_2$. It can be expressed by

$$y = \pm \sqrt{-\frac{kb}{am}} \sqrt{(p - p_1')(p - p_2')(p - p_3')(p - p_4')}.$$
where \( p_1', p_2', p_3' \) and \( p_4' \) satisfy the constraint condition 
\[
p_1' < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_2' < 0 < p_3' < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_4' < p < +\infty.
\]
Similarly, we only need to consider the lower branch of \( \Gamma_4^+ \). Choosing initial value \( p(0) = +\infty \), we have
\[
-\int_{+\infty}^{p} \sqrt{\frac{am}{kb}} \frac{dp}{(p - p_1')(p - p_2')(p - p_3')(p - p_4')} = \int_{0}^{\xi} d\xi, \quad \xi > 0.
\]
Noting that
\[
\int_{+\infty}^{p} \sqrt{\frac{am}{kb}} \frac{dp}{(p - p_1')(p - p_2')(p - p_3')(p - p_4')} = g \cdot \text{sn}^{-1}\left(\frac{p_4'}{p}, k\right),
\]
where \( g = \frac{1}{p_4'}, k^2 = \frac{p_2^2}{p_4'} \), we get the third type of unbounded solution of system \( (8) \)
\[
p_{u_2}(\xi) = \frac{p_4'}{\text{sn}(p_4' \sqrt{-\frac{kb}{am\xi}})}, \quad 0 < \xi < \xi_1,
\]
where \( \xi_1 = \frac{4}{p_4'} \sqrt{-\frac{am}{kb}} \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{p_4'^2}{p_4'^2} \sin^2 \theta}} \).
If choosing initial value \( p(0) = -\infty \) and adopting the similar calculation to the upper branch of unbounded orbit \( \Gamma_4^- \), we can get the corresponding unbounded solution of system \( (8) \)
\[
p_{u_1}(\xi) = \frac{p_1'}{\text{sn}(p_1' \sqrt{-\frac{kb}{am\xi}})}, \quad 0 < \xi < \xi_1',
\]
where \( -\infty < p < p_1' < -\sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_2' < 0 < p_3' < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_4', \xi_1' = -\frac{4}{p_1'} \sqrt{-\frac{am}{kb}} \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{p_1'^2}{p_1'^2} \sin^2 \theta}} \).

(4) Consider the fourth type of unbounded orbits, for example \( \Gamma_5^+ \) and \( \Gamma_5^- \), shown in Fig. 1(a), whose energy is equal to energy of the center \( P_2 \). The unbounded orbit \( \Gamma_5^+ \) can be expressed by
\[
y = \pm \sqrt{-\frac{kb}{am}} \sqrt{(p + p_5')(p^2 - p_5')},
\]
where \( p_5' \) satisfies the constraint condition \( 0 < \sqrt{\frac{a\kappa\omega - r}{2\kappa b}} < p_5' < p < +\infty \).
Choosing initial value \( p(0) = +\infty \), we have

\[
- \int_{+\infty}^{p} \sqrt{\frac{am}{kb}} \frac{dp}{\sqrt{(p + p_5')(p^2(p - p_5'))}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{p}^{+\infty} \frac{dp}{p \sqrt{(p + p_5')(p - p_5')}} = \frac{1}{p_5'} \arcsin \frac{p_5'}{p},
\]

we obtain the fourth type of unbounded solution of system (8)

\[
p_{u3}(\xi) = p_5' \csc(p_5' \sqrt{-\frac{kb}{am}}), \quad 0 < \xi < \xi_2,
\]

where \( \xi_2 = \frac{2\pi}{p_5'} \sqrt{-\frac{am}{kb}} \).

If choosing initial value \( p(0) = -\infty \), we can get another unbounded solution of system (8) corresponding to the upper branch of unbounded orbit \( \Gamma_{-}^5 \)

\[
p_{u3}(\xi) = p_5' \csc(-p_5' \sqrt{-\frac{kb}{am}}), \quad 0 < \xi < \xi_2',
\]

where \(-\infty < p < p_5' < -\sqrt{\frac{a\kappa \omega - r}{2kb}} < 0, \ \xi_2' = -\frac{2\pi}{p_5'} \sqrt{-\frac{am}{kb}}\).

(5) Consider the fifth type of unbounded orbit, for example \( \Gamma_{6}^+ \), shown in Fig. 1(a), whose energy is lower than energy of the center \( P_2 \). It can be expressed by

\[
y = \pm \sqrt{-\frac{kb}{am}} \sqrt{(p + p_6')(p^2(p - p_6') + \frac{a\kappa \omega - r}{kb})},
\]

where \( p_6' \) satisfies the constraint condition \( 0 < \sqrt{\frac{a\kappa \omega - r}{kb}} < p_6' < p < +\infty \).

Choosing initial value \( p(0) = +\infty \), we have

\[
- \int_{+\infty}^{p} \sqrt{-\frac{am}{kb}} \frac{dp}{\sqrt{(p + p_6')(p^2 + p_6^2 - \frac{a\kappa \omega - r}{kb})}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{p}^{+\infty} \frac{dp}{\sqrt{(p + p_6')(p - p_6')(p^2 + p_6^2 - \frac{a\kappa \omega - r}{kb})}} = g \cdot sn^{-1}\left( \sqrt{\frac{2kbp_6^2 - a\kappa \omega + r}{kbp_6^2 + kbp_6^2 - a\kappa \omega + r}}, k \right),
\]
where \( g = \sqrt{\frac{\kappa b}{2kbp_g' - a\kappa \omega + r}} \), \( k^2 = \frac{kbp_g'^2 - a\kappa \omega + r}{2kbp_g'^2 - a\kappa \omega + r} \), we obtain the fifth type of unbounded solution of system (8)

\[
p_{u_4}(\xi) = \sqrt{-\frac{kbp_g'^2 - a\kappa \omega + r}{\kappa b} + \frac{2kbp_g'^2 - a\kappa \omega + r}{\kappa b sn^2(\sqrt{-\frac{2kbp_g'^2 - a\kappa \omega + r}{am}}\xi)}}\quad 0 < \xi < \xi_3,
\]

where \( \xi_3 = \sqrt{-\frac{4am}{2kbp_g'^2 - a\kappa \omega + r}} \cdot \int_0^\frac{\pi}{2} \frac{d\theta}{\sqrt{1 - \frac{\kbp_g'^2 - a\kappa \omega + r}{2kbp_g'^2 - a\kappa \omega + r} \sin^2 \theta}}. \)

If choosing initial value \( p(0) = -\infty \) and applying similar calculation to the upper branch of unbounded orbit \( \Gamma_6^- \), it is obvious to conclude that the corresponding unbounded solution has the same form as unbounded solution \( p_{u_4}(\xi) \).

2. When \( \kbp(a\kappa \omega - r) \leq 0 \) and \( \frac{\kbp}{am} < 0 \), we need to consider three subcases in this case.

(1) Consider the first type of unbounded orbits, for example \( \Upsilon^3 \), \( \Upsilon_3 \), \( Y^1 \) and \( Y_1 \), shown in Figs. 2(a) and 3(a) whose energy \( h_0' \) is higher than energy of the saddle \( P_2 \). One can check that the corresponding unbounded solution of \( \Upsilon_3 \) has the same form as solution \( p_{u_0}(\xi) \). We ignore it here for simplicity.

In particular, when \( \kbp(a\kappa \omega - r) = 0 \), the unbounded orbits \( Y^1 \) and \( Y_1 \) shown in Fig. 3(a) can be expressed respectively by

\[
y = \pm \sqrt{-\frac{\kbp}{am} p^4 + 2h_0'},\]

where \( 0 < p < +\infty \). Choosing initial value \( p(0) = +\infty \), we have

\[
- \int_{+\infty}^p \frac{\kbp}{am} \frac{dp}{\sqrt{p^4 - 2amh_0'}} = \int_0^\xi d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{-p}^{+\infty} \frac{dp}{\sqrt{p^4 - 2amh_0'}} = \frac{1}{2} \sqrt{-\frac{\kbp}{2amh_0'}} \cdot cn^{-1}\left(\frac{p^2 - 1}{p^2 + 1}, k\right),
\]

where \( k^2 = \frac{1}{2} \) we obtain the sixth type of unbounded solution of system (8)

\[
p_{u_0'}(\xi) = \sqrt{\frac{2}{1 - cn(8h_0'\xi)} - 1}, \quad 0 < \xi < \xi_4',
\]

20
where \( \xi_4' = \frac{1}{2h_0'} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{1}{2} \sin^2 \theta}} \). 

(2) Consider the second type of unbounded orbits shown in Figs. 2(a) and 3(a), whose energy is equal to energy of the saddle \( P_2 \). The unbounded orbits \( \Upsilon_4 \) and \( \Upsilon_4' \) can be expressed by

\[
y = \pm \sqrt{-\frac{kb}{am}} \sqrt{p^4 + \frac{r - a\kappa \omega}{kb} p^2},
\]

where \( 0 < p < +\infty \). Choosing initial value \( p(0) = +\infty \), we have

\[
- \int_0^p \frac{am}{kb} \frac{dp}{p \sqrt{p^2 + \frac{r - a\kappa \omega}{kb}}} = \int_0^\xi d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{-\infty}^{+\infty} \frac{dp}{p \sqrt{p^2 + \frac{r - a\kappa \omega}{kb}}} = \sqrt{\frac{kb}{r - a\kappa \omega}} \cdot \ln \left( \frac{\sqrt{p^2 + \frac{r - a\kappa \omega}{kb}} + \sqrt{\frac{r - a\kappa \omega}{kb}}}{p} \right),
\]

we obtain the seventh type of unbounded solution of system (8)

\[
p_{u_{6}} (\xi) = 2 \sqrt{-\frac{r - a\kappa \omega}{kb}} \cdot \frac{\exp \left( \sqrt{-\frac{a\kappa \omega - r}{am}} \xi \right)}{1 - \exp \left( 2 \sqrt{-\frac{a\kappa \omega - r}{am}} \xi \right)}, \quad \xi > 0.
\]

Similar calculation can be applied to the unbounded orbit \( \Upsilon_5 \) shown in Fig. 2(a). If choosing \( p(0) = -\infty \) and \( -\infty < p < 0 \), we obtain another unbounded solution of system (8) as follows

\[
p_{u_{6}'} (\xi) = 2 \sqrt{-\frac{r - a\kappa \omega}{kb}} \cdot \frac{\exp \left( \sqrt{-\frac{a\kappa \omega - r}{am}} \xi \right)}{1 - \exp \left( 2 \sqrt{-\frac{a\kappa \omega - r}{am}} \xi \right)}, \quad \xi > 0.
\]

In particular, when \( \kappa b (a\kappa \omega - r) = 0 \), the unbounded orbits \( Y^2 \) and \( Y_2 \) shown in Fig. 3(a) can be expressed respectively by

\[
y = \pm \sqrt{-\frac{kb}{am} p^2},
\]

where \( 0 < p < +\infty \). By a direct calculation, we obtain the corresponding unbounded solution of system (8)

\[
p_{u_{7}} (\xi) = \sqrt{-\frac{am}{kb}} \cdot \frac{1}{\xi}, \quad \xi > 0.
\]
If choosing \( p(0) = -\infty \) and adopting the similar calculation to the unbounded orbit \( Y^3 \) shown in Fig. 3(a) we have
\[
p_{u_i}(\xi) = -\sqrt{-\frac{am}{kb}} \cdot \frac{1}{\xi}, \quad \xi > 0,
\]
where \(-\infty < p < 0\).

(3) Consider the third type of unbounded orbits, for example \( Y_6^+, Y_6^-, Y_4^+ \) and \( Y_4^- \), shown in Figs. 2(a) and 3(a), whose energy is lower than energy of the saddle \( P_2 \). It is not difficult to conclude that the corresponding unbounded solutions have the same form as solution \( p_{u_3}(\xi) \). Due to the tedious expressions of the solutions, we ignore them here for simplicity.

3.3. Bounded solutions of system (13)

From the energy function \( H(\phi, y) = \frac{1}{2} y^2 - \frac{\alpha_1}{4} \phi^4 - \frac{\alpha_2}{2} \phi^2 + \frac{\alpha_3}{2} \phi^2 = h \) and the first equation of system (13), we can obtain the following expression
\[
\xi = \int_{\phi_0}^{\phi} \phi d\phi \frac{\sqrt{2\frac{\alpha_2}{2} \phi^2 + 2h\phi^2 - \alpha_3}}{\sqrt{\frac{\alpha_2}{2} \phi^2 + 2h\phi^2 - \alpha_3}},
\]
where \( \phi > 0 \) and \( \psi = \phi^2 \). Due to the symmetry of system (13), it is easy to check that the bounded solutions of system (13) where \( \phi < 0 \) have the same expressions of bounded solutions of system (13) where \( \phi > 0 \) except for the difference of signs.

I. When \( \alpha_1 > 0, \alpha_2 < 0 \) and \( 0 < \alpha_3 < -\frac{4\alpha_2^3}{27\alpha_1^2} \), we have two subcases in this case.

(i) Consider the periodic orbits shown in Fig. 4(a) whose energy is lower than energy of the saddle \( \tilde{P}_3 \), but higher than energy of center \( \tilde{P}_1 \). Assuming that \( r_1, r_2 \) and \( r_3 \) satisfy the constraint condition \( 0 < r_1 < \psi < r_2 < r_3 \), the period is \( 2T_{0'} \) and choosing \( \psi_0 = \psi(0) = r_1 \), we have
\[
\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}}} = \int_0^\xi d\xi, \quad 0 < \xi < T_{0'},
\]
\[
- \int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}}} = \int_\xi^0 d\xi, \quad -T_{0'} < \xi < 0,
\]
which can be rewritten as
\[
\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}}} = |\xi|, \quad -T_0' < \xi < T_0'.
\]
Noting that
\[
\int_{r_1}^{\psi} \frac{d\psi}{\sqrt{(\psi - r_1)(r_2 - \psi)(r_3 - \psi)}} = g \cdot sn^{-1}(\sqrt{\frac{\psi - r_1}{r_2 - r_1}}k),
\]
where \( g = \frac{2}{\sqrt{r_3 - r_1}}, \ k^2 = \frac{r_2 - r_1}{r_3 - r_1} \), we get the first type of periodic solution of system (13)
\[
\psi_b_1(\xi) = \sqrt{r_1 + (r_2 - r_1) sn^2(\frac{\alpha_1 (r_3 - r_1)}{2}\xi)}, \quad -T_0' < \xi < T_0'.
\]
(ii) Consider the homoclinic orbit \( \Pi_0^+ \) shown in Fig. 4(a) whose energy is equal to energy of the saddle \( \tilde{P}_3 \). Assuming that \( r_4 \) and \( r_5 \) satisfy the constraint condition \( 0 < r_4 < \psi < r_5 \) and choosing \( \psi_0 = \psi(0) = r_4 \), we have
\[
\int_{r_4}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_4)(r_5 - \psi)}}} = \int_{0}^{\xi} d\xi, \quad \xi > 0,
\]
\[
-\int_{\psi}^{r_4} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_4)(r_5 - \psi)}}} = \int_{\xi}^{0} d\xi, \quad \xi < 0,
\]
which can be rewritten as
\[
\int_{r_4}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 (r_5 - \psi)\sqrt{\psi - r_4}}} = |\xi|, \quad -\infty < \xi < +\infty.
\]
Noting that
\[
\int_{r_4}^{\psi} \frac{d\psi}{(r_5 - \psi)\sqrt{\psi - r_4}} = -\frac{1}{\sqrt{r_5 - r_4}} \ln \frac{\sqrt{r_5 - r_4} - \sqrt{\psi - r_4}}{\sqrt{r_5 - r_4} + \sqrt{\psi - r_4}},
\]
we obtain the expression of solitary wave solution of system (13)
\[
\psi_b_2(\xi) = \sqrt{r_4 + (r_5 - r_4)(1 - \exp(\sqrt{2\alpha_1 (r_5 - r_4)\xi}))^2} - (1 + \exp(\sqrt{2\alpha_1 (r_5 - r_4)\xi}))^2, \quad -\infty < \xi < +\infty.
\]
II. When \( \alpha_1 < 0 \), there only exist two families of periodic orbits shown in Fig. 4(d) Assuming that \( r_6 \) and \( r_7 \) satisfy the constraint conditions \( 0 < r_6 < \psi < r_7 \),
\( r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1} > 0 \), the period is \( 2T_1' \) and choosing \( \psi_0 = \psi(0) = r_6 \), we have

\[
\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{-2 \alpha_1 \sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1})}}} = \int_{0}^{\xi} d\xi, \quad 0 < \xi < T_1',
\]

\[
- \int_{\psi}^{r_6} \frac{d\psi}{\sqrt{-2 \alpha_1 \sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1})}}} = \int_{\xi}^{0} d\xi, \quad -T_1' < \xi < 0,
\]

which can be rewritten as

\[
\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{-2 \alpha_1 \sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1})}}} = |\xi|, \quad -T_1' < \xi < T_1'.
\]

Noting that

\[
\int_{r_6}^{\psi} \frac{d\psi}{\sqrt{(\psi - r_6)(r_7 - \psi)(\psi + r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1})}} = g \cdot sn^{-1}(\frac{(2r_7 + r_6 + \frac{2 \alpha_2}{\alpha_1})(\psi - r_6)}{(r_7 - r_6)(\psi + r_6 + r_7 + \frac{2 \alpha_2}{\alpha_1})} k),
\]

where \( g = \frac{2}{\sqrt{2r_7 + r_6 + \frac{2 \alpha_2}{\alpha_1}}}, \quad k^2 = \frac{r_7 - r_6}{2r_7 + r_6 + \frac{2 \alpha_2}{\alpha_1}} \), we obtain the second type of periodic solution of system (13)

\[
\phi(\xi) = \sqrt{r_6 + \frac{(2 \alpha_1 r_6 + \alpha_1 r_7 + 2 \alpha_2)(r_7 - r_6) \sqrt{\frac{2 \alpha_1 r_7 + \alpha_1 r_6 + 2 \alpha_2}{2}} \xi}{\alpha_1 (r_7 - r_6) \sqrt{\frac{-2 \alpha_1 r_7 + \alpha_1 r_6 + 2 \alpha_2}{2}} - (2 \alpha_1 r_7 + \alpha_1 r_6 + 2 \alpha_2)}} \cdot -T_1' < \xi < T_1'.
\]

3.4. Unbounded solutions of system (13)

In this subsection, to seek the unbounded solutions of system (13) where \( \psi > 0 \) and \( \phi = \sqrt{\psi} > 0 \), we need to discuss three cases. If applying similar calculation to the unbounded orbits of system (13) where \( \phi < 0 \), one can check the corresponding unbounded solutions have the same expressions as unbounded solutions where \( \phi > 0 \) except for the difference of signs.

**I.** When \( \alpha_1 > 0, \alpha_2 < 0 \) and \( 0 < \alpha_3 < -\frac{4 \alpha_2^3}{27 \alpha_1^2} \), we have five subcases in this case.

(i) Consider the first type of unbounded orbits, for example \( \Pi_1 \), shown in Fig. 4(a), whose energy is higher than energy of the saddle \( \tilde{P}_3 \). We consider to discuss the lower branch of orbit \( \Pi_1 \) for simplicity. Assuming that \( r_1' \) satisfies
the constraint condition $0 < r'_1 < \psi < +\infty$, $h_1 \in (h(\tilde{P}_3), +\infty)$ and choosing $\psi_0 = \psi(0) = +\infty$, we have

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r'_1) \left[(\psi + (2\alpha_2 + r'_1) \psi + r'_1 + \frac{4h_1}{\alpha_1})\right]}}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.$$  

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{\sqrt{(\psi - r'_1) \left[(\psi + (2\alpha_2 + r'_1) \psi + r'_1 + \frac{4h_1}{\alpha_1})\right]}} = g \cdot cn^{-1} \left(\frac{\psi - r'_1 - A}{\psi - r'_1 + A}k\right),$$

where $A^2 = \frac{2\alpha_1 r'_1^2 + (\alpha_1 + 2\alpha_2) r'_1 + 4h_1}{\alpha_1}$, 

and $k^2 = \frac{\sqrt{8\alpha_1^2 r'_1^2 + 4\alpha_1(\alpha_1 + 2\alpha_2) r'_1 + 16\alpha_1 h_1 - (2\alpha_2 + 3\alpha_1 r'_1)}}{\sqrt{32\alpha_1^2 r'_1^2 + 16\alpha_1(\alpha_1 + \alpha_2) r'_1 + 64\alpha_1 h_1}}$, we get the first type of unbounded solution of system \[13\]

$$\phi_{\alpha_1}(\xi) = \sqrt{r'_1 - \frac{2\alpha_1 r'_1^2 + (\alpha_1 + 2\alpha_2) r'_1 + 4h_1}{\alpha_1}} + \frac{2\sqrt{\frac{2\alpha_1 r'_1^2 + (\alpha_1 + 2\alpha_2) r'_1 + 4h_1}{\alpha_1}}}{1 - cn \left(\sqrt{2\alpha_1^2 + 4(\alpha_1 + 2\alpha_2) r'_1 + 4h_1} \xi\right)},$$

where $0 < \xi < \xi_{0''}$ and

$$\xi_{0''} = \frac{4}{\sqrt{2\alpha_1^2 + 4(\alpha_1 + 2\alpha_2) r'_1 + 4h_1}} \int_{0}^{\frac{\pi}{2}} d\theta \cdot \sqrt{1 - \frac{\sqrt{8\alpha_1^2 r'_1^2 + 4\alpha_1(\alpha_1 + 2\alpha_2) r'_1 + 16\alpha_1 h_1 - (2\alpha_2 + 3\alpha_1 r'_1)}}{\sqrt{32\alpha_1^2 r'_1^2 + 16\alpha_1(\alpha_1 + \alpha_2) r'_1 + 64\alpha_1 h_1}} \cdot \sin^2 \theta}.$$  

Similar calculation can be applied to the upper branch of the orbit $\Pi_1$. And one can check the corresponding solution has the same form as solution $\phi_{\alpha_1}(\xi)$.

(ii) Consider the second type of unbounded orbit $\Pi_2$ shown in Fig. 4(a) whose energy is equal to energy of the saddle $\tilde{P}_3$. Similar to the discussion above, we only need to discuss the lower branch of orbit $\Pi_2$. Assuming that $r'_2$ and $r'_3$ satisfy the constraint condition $0 < r'_2 < r'_3 < \psi < +\infty$ and choosing $\psi_0 = \psi(0) = +\infty$, we have

$$-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r'_2) \left[(\psi + (2\alpha_2 + r'_2) \psi + r'_2 + \frac{4h_1}{\alpha_1})\right]}}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.$$  

Noting that

$$\int_{\psi}^{+\infty} \frac{d\psi}{(\psi - r'_2) \sqrt{\psi - r'_2}} = -\frac{1}{\sqrt{r'_3 - r'_2}} \ln \frac{\sqrt{\psi - r'_2} - \sqrt{r'_3 - r'_2}}{\sqrt{\psi - r'_2} + \sqrt{r'_3 - r'_2}},$$

25
we get the second type of unbounded solution of system \[ \{13\}

\[
\phi_{u2}(\xi) = \sqrt{r_{2}' + \frac{(r_{3}' - r_{2}') (1 + \exp(\sqrt{2\alpha_1 (r_{3}' - r_{2}') \xi}))^2}{(1 - \exp(\sqrt{2\alpha_1 (r_{3}' - r_{2}') \xi}))^2}}, \quad \xi > 0.
\]

(iii) Consider the third type of unbounded orbits, for example unbounded orbit \(\Pi_3\), shown in Fig. 4(a) whose energy is higher than energy of the center \(\tilde{P}_1\), but lower than energy of saddle \(\tilde{P}_3\). Assuming that \(r_{4'}, r_{5'}\) and \(r_{6'}\) satisfy the constraint condition \(0 < r_{4'} < r_{5'} < r_{6'} < \psi < +\infty\) and choosing \(\psi_0 = \psi(0) = +\infty\), it can be expressed by

\[
-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{\psi - r_{4'}} (\psi - r_{5'}) (\psi - r_{6'})}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{\psi}^{+\infty} \frac{d\psi}{\sqrt{\psi - r_{4'}} (\psi - r_{5'}) (\psi - r_{6'})} = g \cdot sn^{-1}\left(\sqrt{r_{6'} - r_{4'} \psi - r_{4'}}, k\right),
\]

where \(g = \frac{2}{\sqrt{r_{6'} - r_{4'}}}\), \(k^2 = \frac{r_{5'} - r_{4'}}{r_{6'} - r_{4'}}\), we get the third type of unbounded solution of system \[ \{13\}\]

\[
\phi_{u3}(\xi) = \sqrt{r_{4'} + \frac{r_{6'} - r_{4'}}{sn^2(\sqrt{\alpha_1 (r_{6'} - r_{4'}) \xi})}}, \quad 0 < \xi < \xi_1'',
\]

where \(\xi_1'' = \sqrt{\frac{8}{\alpha_1 (r_{6'} - r_{4'})}} \cdot \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{r_{5'} - r_{4'}}{r_{6'} - r_{4'}} \cdot \sin^2 \theta}}\).

(iv) Consider the fourth type of unbounded orbits, for example \(\Pi_4\), shown in Fig. 4(a) whose energy is equal to energy of the center \(\tilde{P}_2\). Assuming that \(r_{7'}\) and \(r_{8'}\) satisfy the constraint condition \(0 < r_{7'} < r_{8'} < \psi < +\infty\) and choosing \(\psi_0 = \psi(0) = +\infty\), we have

\[
-\int_{+\infty}^{\psi} \frac{d\psi}{\sqrt{2\alpha_1 \sqrt{(\psi - r_{7'})^2 (\psi - r_{8'})}}} = \int_{0}^{\xi} d\xi, \quad \xi > 0.
\]

Noting that

\[
\int_{\psi}^{+\infty} \frac{d\psi}{(\psi - r_{7'}) \sqrt{\psi - r_{8'}}} = \frac{1}{\sqrt{r_{8'} - r_{7'}}} (\pi - 2 \arctan \sqrt{\frac{\psi - r_{8'}}{r_{8'} - r_{7'}}}),
\]

26
we obtain the fourth type of unbounded solution of system \[13\]

\[
\phi_{u_4}(\xi) = \sqrt{r_{q'} + (r_{q'} - r_{q''}) \cdot \cot^2 \left( \frac{\alpha_1 (r_{q'} - r_{q''})}{2} \right) \xi}}, \quad 0 < \xi < \xi_{q''},
\]

where \(\xi_{q''} = \sqrt{\frac{2}{\alpha_1 (r_{q'} - r_{q''}) \cdot \pi}}\).

(v) Consider the fifth type of unbounded orbits, for example \(\Pi_5\), shown in Fig. 4(a), whose energy is lower than energy of the center \(\tilde{P}_2\). It is not difficult to obtain the fifth type of unbounded solution as follows

\[
\phi_{u_5}(\xi) = \sqrt{r_{q'} - \sqrt{\frac{2\alpha_1 r_{q'}^2 + (\alpha_1 + 2\alpha_2) r_{q'} + 4h_2}{\alpha_1} + \frac{2\sqrt{2\alpha_1 r_{q'}^2 + (\alpha_1 + 2\alpha_2) r_{q'} + 4h_2}}{\alpha_1}}} + \frac{\pi}{1 - \alpha_n(\sqrt{\frac{2\alpha_1 r_{q'}^2 + (\alpha_1 + 2\alpha_2) r_{q'} + 4h_2}{\alpha_1}} \xi)},
\]

where \(0 < r_{q'} < \psi < +\infty, h_2 \in (0, h(\tilde{P}_2)), 0 < \xi < \xi_{q''}\) and

\[
\xi_{q''} = \frac{4}{\sqrt{2\alpha_1}} \sqrt{\frac{\alpha_1 r_{q'}^2 + (\alpha_1 + 2\alpha_2) r_{q'} + 4h_2}{\alpha_1}} \int_0^\pi \frac{d\theta}{\sqrt{1 - \frac{\sqrt{8\alpha_1^2 r_{q'}^2 + 4\alpha_1 (\alpha_1 + 2\alpha_2) r_{q'} + 16\alpha_1 h_2 - (\alpha_2 + 3\alpha_1) r_{q'}}}{\alpha_1}}} \cdot \sin^2 \theta.
\]

II. When \(\alpha_1 > 0, \alpha_2 < 0\) and \(\alpha_3 = -\frac{4\alpha_2^3}{27\alpha_1^2}\), we consider two subcases in this case.

(i) Consider the first type of unbounded orbits \(\Omega^1\) and \(\Omega_1\) shown in Fig. 4(b) whose energy is equal to energy of the cusp \(\tilde{P}_5\). Choosing \(\psi_0 = \psi(0) = +\infty\), we have

\[-\int_+^{\psi} \frac{d\psi}{\sqrt{2\alpha_1} \sqrt{\left(\psi + \frac{2\alpha_2}{3\alpha_1}\right)(\psi + \frac{2\alpha_2}{3\alpha_1})}} = \int_0^\xi d\xi, \quad \xi > 0.\]

Noting that

\[
\int_\psi^{+\infty} \frac{d\psi}{\sqrt{\psi + \frac{2\alpha_2}{3\alpha_1}}} = \frac{2}{\sqrt{\psi + \frac{2\alpha_2}{3\alpha_1}}},
\]

we get the sixth type of unbounded solution of system \[13\]

\[
\phi_{u_6}(\xi) = \sqrt{-\frac{2\alpha_2}{3\alpha_1} + \frac{2}{\alpha_1 \xi^2}}, \quad \xi > 0.
\]

(ii) Consider other unbounded orbits, for example \(\Omega_2\) and \(\Omega_3\), shown in Fig. 4(b) they can be expressed uniformly. Assuming that \(r_{10'}\) satisfies the constraint condition \(0 < r_{10'} < \psi < +\infty, h_3 \neq h(\tilde{P}_5)\) and choosing \(\psi_0 = \psi(0) = \cdots\)
+∞, we obtain

\[ \phi_{u7}(\xi) = \sqrt{r_{10}' - \sqrt{\frac{2\alpha_1 r_{11}^2 + (\alpha_1 + 2\alpha_2) r_{11}' + 4h_3}{\alpha_1}}} + \frac{2\sqrt{2\alpha_1 r_{11}^2 + (\alpha_1 + 2\alpha_2) r_{11}'} + 4h_3}{\alpha_1}, \]

where \( 0 < \xi < \xi_{4''} \) and

\[ \xi_{4''} = \frac{4}{\sqrt{2\alpha_1}} \frac{d\theta}{\sqrt{32\alpha_1^2 r_{11}^2 + 16\alpha_1 (\alpha_1 + 2\alpha_2) r_{11}' + 64\alpha_1 h_3}} \cdot \sin^2 \theta. \]

III. Similarly, when \( \alpha_1 > 0, \alpha_2 \geq 0, \) one can check that the unbounded solution \( \phi_{u8}(\xi) \) has the same form as solution \( \phi_{u4}(\xi) \). We get the corresponding unbounded solution of system \([13]\)

\[ \phi_{u8}(\xi) = \sqrt{r_{11}' - \sqrt{\frac{2\alpha_1 r_{11}^2 + (\alpha_1 + 2\alpha_2) r_{11}' + 4h_4}{\alpha_1}}} + \frac{2\sqrt{2\alpha_1 r_{11}^2 + (\alpha_1 + 2\alpha_2) r_{11}'} + 4h_4}{\alpha_1}, \]

where \( 0 < r_{11}' < \psi < +\infty, h_4 \) is a real number, \( 0 < \xi < \xi_{5''} \) and

\[ \xi_{5''} = \frac{4}{\sqrt{2\alpha_1}} \frac{d\theta}{\sqrt{32\alpha_1^2 r_{11}^2 + 16\alpha_1 (\alpha_1 + 2\alpha_2) r_{11}' + 64\alpha_1 h_4}} \cdot \sin^2 \theta. \]

4. Exact traveling wave solutions of Eq. \([1]\)

In order to get the type I traveling wave solution of the GKMN equation, it only needs to substitute the solution \( p(\xi) \) of system \([8]\) into the formula \([6]\). Here we list them in the Appendix for the sake of simplicity.

But, it is not so easy to get the type II traveling wave solution of the GKMN equation. It needs us to substitute the solutions \( \phi(\xi) \) of system \([13]\) into the ODE \([12]\) to solve \( \varphi(\xi) \), and then plug \( \phi(\xi) \) and \( \varphi(\xi) \) into the formula \([10]\). (S1) Noting that

\[ \phi_{b1}(\xi) = \sqrt{r_1 + (r_2 - r_1) \sin^2 \left( \frac{\alpha_1 (r_3 - r_1)}{2} \xi \right)}, \quad T_0' < \xi < T_0', \]
and
\[\int \frac{du}{1 + k \cdot \text{sn}^2(u)} = \frac{u}{2} + \frac{1}{2(1 + k)} \tan^{-1}[(1 + k)\text{tn}(u) \cdot nd(u)],\]
we have
\[\varphi_{b_1}(\xi) = \int \left( \frac{e}{\text{am} \varphi_{b_1}^2(\xi)} + \frac{c}{2 \text{am}} \right) d\xi = \frac{e + cr_1}{2 \text{am} r_1} \xi + \frac{r_1}{2 \text{am} r_1} \cdot \tan^{-1} \left( \frac{r_2}{r_1} \text{tn} \left( \sqrt{\frac{\alpha_1(r_3 - r_1)}{2}} \xi \right) \cdot nd \left( \sqrt{\frac{\alpha_1(r_3 - r_1)}{2}} \xi \right) \right) + C_1,\]
where \(C_1\) is a constant.

Thus we obtain the final solution \(q_1(x, y, t) = \phi_{b_1}(\xi) \exp(\varphi_{b_1}(\xi) - \mu t)i\).

(S2) Noting that
\[\varphi_{b_2}(\xi) = \sqrt{r_4 + \frac{(r_5 - r_4)(1 - \exp(\sqrt{2\alpha_1(r_5 - r_4)\xi}))^2}{1 + \exp(\sqrt{2\alpha_1(r_5 - r_4)\xi})^2}}, \quad -\infty < \xi < +\infty,\]
we have
\[\varphi_{b_2}(\xi) = \int \left( \frac{e}{\text{am} \varphi_{b_2}^2(\xi)} + \frac{c}{2 \text{am}} \right) d\xi = \frac{2e \sqrt{2\alpha_1(r_5 - r_4) + cr_5} \xi + (r_5 - r_4) e}{2 \text{am} r_5 \sqrt{r_4(r_5 - r_4)}} \cdot \arctan \left( \frac{r_5 - r_4}{2 \sqrt{r_4(r_5 - r_4)}} \exp(\sqrt{2\alpha_1(r_5 - r_4)\xi}) + \frac{r_5 - r_4}{2 \sqrt{r_4(r_5 - r_4)}} - \frac{r_5}{2 r_4} \right) + C_2,\]
where \(C_2\) is a constant.

Thus we obtain the final solution \(q_2(x, y, t) = \phi_{b_2}(\xi) \exp(\varphi_{b_2}(\xi) - \mu t)i\).

(S3) Noting that
\[\phi_{b_3}(\xi) = \sqrt{r_6 + \frac{(2\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)(r_7 - r_6) \text{sn}^2(\sqrt{\frac{-2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2} \xi})}{\alpha_1(r_7 - r_6) \text{sn}^2(\sqrt{\frac{-2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2} \xi}) - (2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2)}, \quad -T_{1'} < \xi < T_{1'},\]
and
\[\int \frac{du}{1 \pm k \cdot \text{sn}(u)} = \frac{1}{k^2} \left[ E(u) + k(1 \pm k \cdot \text{sn}(u)) \cdot \text{cd}(u) \right],\]
where $k' = \sqrt{1 - k^2}$, we have

$$\phi_{b_3}(\xi) = \int \left( \frac{e}{am\phi_{b_3}'(\xi)} + \frac{c}{2am} \right) d\xi$$

$$= \left( \frac{e\alpha_1}{am(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)} \right) \sqrt{\frac{2}{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2} + \frac{c}{2am}} \xi$$

$$+ \frac{e(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2 - \alpha_1)}{am(3\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)(2\alpha_1 r_6 + \alpha_1 r_7 + 2\alpha_2)} \cdot \left( E\left( \sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}} \right) \right)$$

$$+ \left( \frac{\alpha_1 r_7 - \alpha_1 r_6}{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2} \cdot cd\left( \sqrt{-\frac{2\alpha_1 r_7 + \alpha_1 r_6 + 2\alpha_2}{2}} \xi \right) \right) + C_3,$$

where $C_3$ is a constant.

Thus we obtain the final solution $q_3(x, y, t) = \phi_{b_3}(\xi)\exp(\varphi_{b_3}(\xi) - \mu t)i$.

(S4) Noting that

$$\phi_{u_1}(\xi) = \sqrt{r_1' - \sqrt{\frac{2a_1 r_1'^2 + (a_1 + 2a_2)r_1' + 4h_1}{a_1}} + \frac{2\sqrt{2a_1 r_1'^2 + (a_1 + 2a_2)r_1' + 4h_1}}{a_1}}$$

$$1 - cn\left( \sqrt{\frac{2a_1 r_1'^2 + (a_1 + 2a_2)r_1' + 4h_1}{a_1}} \xi \right),$$

and

$$m \int \frac{1 - cn(u)}{1 + \alpha \cdot cn(u)} du = \frac{1}{\alpha} \left[ -u + \frac{1}{1 - \alpha} \left( \Pi(\varphi, \frac{\alpha^2}{\alpha^2 - 1}, k) - \alpha \cdot f_1 \right) \right],$$

where $0 < \xi < \xi_0''$, $\varphi = am(u)$ and $f_1 = \sqrt{\frac{1 - a^2}{k^2 + k'^2 \alpha^2}} \cdot \tan^{-1} \sqrt{\frac{k^2 + k'^2 \alpha^2}{1 - a^2}}$.
Similar calculation can be applied to the solutions where $k = \sqrt{1 - k^2}$, we have

$$
\varphi_{u_1}(\xi) = \int \left( \frac{e}{\text{am} \phi_{u_1}(\xi)} + \frac{c}{2am} \right) d\xi
= \left( \frac{e}{\text{am}(r_{1'} - \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1})} + \frac{c}{2am} \right) \xi
+ e\left( \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1} + r_{1'} \right)
+ \frac{2am r_{1'} \sqrt{8a_1^2 r_{1'}^2 + 4a_1(a_1 + 2a_2) r_{1'} + 16a_1 h_1}}{\sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1}}
\cdot \Pi[\text{am}(\sqrt{8a_1^2 r_{1'}^2 + 4a_1(a_1 + 2a_2) r_{1'} + 16a_1 h_1 - u}), - \frac{(\sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1} - r_{1'})^2}{4 \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1}} k]$$

$$
- \frac{c}{am r_{1'} \sqrt{8a_1^2 r_{1'}^2 + 4a_1(a_1 + 2a_2) r_{1'} + 16a_1 h_1}}
\cdot \left[ \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1} + r_{1'} \right]
\cdot \tan^{-1}\left( \left( \frac{4 \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1} - r_{1'})^2}{4 \sqrt{2a_1 r_{1'}^2 + (a_1 + 2a_2) r_{1'} + 4h_1} r_{1'}} \right)
\cdot \sqrt{2a_1 r_{1'}^2 + 16a_1 h_1} \right) + C_4,
$$

where $k^2 = \frac{\sqrt{8a_1^2 r_{1'}^2 + 4a_1(a_1 + 2a_2) r_{1'} + 16a_1 h_1} - (2a_2 + 3a_1 r_{1'})}{\sqrt{32a_1^2 r_{1'}^2 + 16a_1(a_1 + a_2) r_{1'} + 64a_1 h_1}}$, $C_4$ is a constant.

Thus we obtain the final solution $q_4(x,y,t) = \varphi_{u_1}(\xi)\exp(\varphi_{u_1}(\xi) - \mu t)i$.

Similar calculation can be applied to the solutions $\phi_{u_3}(\xi)$, $\phi_{u_7}(\xi)$ and $\phi_{u_9}(\xi)$, we ignore them here for simplicity.

(S5) Noting that

$$
\phi_{u_2}(\xi) = \sqrt{r_{1'} + (r_{2'} - r_{2'})} \left( 1 + \exp(\sqrt{2a_1(r_{3'} - r_{2'})}) \right)^2,
\quad \xi > 0,
$$

31
we have
\[
\varphi_{u_2}(\xi) = \int \left( \frac{e}{am\varphi_{u_2}^2(\xi)} + \frac{c}{2am} \right) d\xi
\]
\[
= \frac{4e(r_{2'} - r_{3'})}{amr_{3'}\sqrt{2\alpha_1(r_{3'} - r_{2'})}} \ln(r_{3'}\sqrt{2\alpha_1(r_{3'} - r_{2'})}\xi + r_{3'} - 2r_{2'})
\]
\[
+ \frac{4e(r_{2'} - r_{3'})^2}{amr_{3'}\sqrt{2\alpha_1(r_{3'} - r_{2'})}(r_{3'}\exp(\sqrt{2\alpha_1(r_{3'} - r_{2'})}\xi) + r_{3'} - 2r_{2'})} + \frac{c}{2am}\xi + C_5,
\]
where \(C_5\) is a constant.

Thus we obtain the final solution \(q_5(x, y, t) = \varphi_{u_2}(\xi)\exp(\varphi_{u_2}(\xi) - \mu t)i\).

(S6) Noting that
\[
\varphi_{u_3}(\xi) = \sqrt{r_{4'} + \frac{r_{6'} - r_{4'}}{\sqrt{\frac{\alpha_1(r_{6'} - r_{4'})}{2}}\xi}}, \quad 0 < \xi < \xi_{1''},
\]
and
\[
\int \frac{du}{1 + k \cdot sn^2(u)} = \frac{u}{2} + \frac{1}{2(1 + k)} \cdot \tan^{-1}[(1 + k) \cdot tn(u) \cdot nd(u)],
\]
we have
\[
\varphi_{u_3}(\xi) = \int \left( \frac{e}{am\varphi_{u_2}^2(\xi)} + \frac{c}{2am} \right) d\xi
\]
\[
= \frac{e + cr_{4'}\xi - e(r_{6'} - r_{4'})}{2amr_{4'}} - \frac{e(r_{6'} - r_{4'})}{4amr_{4'}^2r_{6'}} \sqrt{\frac{2}{\alpha_1(r_{6'} - r_{4'})}}
\]
\[
\cdot \tan^{-1}\left( \frac{r_{6'}}{r_{6'} - r_{4'}} \cdot tn\left( \frac{\alpha_1(r_{6'} - r_{4'})}{2}\xi \right) \cdot nd\left( \frac{\alpha_1(r_{6'} - r_{4'})}{2}\xi \right) \right) + C_6,
\]
where \(C_6\) is a constant.

Thus we obtain the final solution \(q_6(x, y, t) = \varphi_{u_3}(\xi)\exp(\varphi_{u_3}(\xi) - \mu t)i\).

(S7) Noting that
\[
\varphi_{u_4}(\xi) = \sqrt{r_{5'} + (r_{5'} - r_{7'}) \cdot \cot^2\left( \frac{\alpha_1(r_{5'} - r_{7'})}{2}\xi \right)}, \quad 0 < \xi < \xi_{2''},
\]
we have
\[
\varphi_{u_4}(\xi) = \int \left( \frac{e}{am\varphi_{u_2}^2(\xi)} + \frac{c}{2am} \right) d\xi
\]
\[
= \frac{2e + cr_{5'}\xi - e}{2amr_{7'}} \cdot \frac{2}{\alpha_1 r_{5'}} \cdot \arctan\left( \frac{r_{5'}}{r_{5'} - r_{7'}} \cdot \tan\left( \frac{\alpha_1(r_{5'} - r_{7'})}{2}\xi \right) \right) + C_7,
\]

32
where $C_7$ is a constant.

Thus we obtain the final solution $q_7(x, y, t) = \phi_{u_4}(\xi)\exp(\varphi_{u_4}(\xi) - \mu t)i$.

(S8) Noting that

$$\phi_{u_4}(\xi) = \sqrt{-\frac{2\alpha_2}{3\alpha_1} + \frac{2}{\alpha_1\xi^2}}, \quad \xi > 0,$$

we have

$$\varphi_{u_4}(\xi) = \int \left( \frac{e}{am\phi_{u_4}^2(\xi)} + \frac{c}{2am} \right) d\xi$$

$$= \frac{c\alpha_2 - 3e\alpha_1}{2am\alpha_2} \xi - \frac{9ec\alpha_1^2}{4am\alpha_2^2} \sqrt{-\frac{3}{\alpha_2}} \cdot \arctan\left( \sqrt{-\frac{\alpha_2}{3\xi}} \right) + C_8,$$

where $C_8$ is a constant.

Thus we obtain the final solution $q_8(x, y, t) = \phi_{u_6}(\xi)\exp(\varphi_{u_6}(\xi) - \mu t)i$.

5. Discussion and conclusion

In this paper, by using the dynamical system method, we study two kinds of traveling wave systems of the GKMN equation and obtain all type I and type II traveling wave solutions of it. Especially, some new solutions $q_{b_7}$, $q_{u_9}(\iota = 2, 2', 4..8, 4'.8')$ and $q_j(j = 1..8)$ have not been reported before, which not only help ones to understand the complicated physical phenomena described by the model further, but also can be used to verify the correctness of the numerical solutions. In particular, we can generate more solutions of Eq. (1) by using of these new solutions. For example, when $a = -1$ and $b = -2$, one can consider these new solutions as "seeds" and apply the perturbation $(n;M)$-fold Darboux transformation $\tilde{\varphi} = T(\lambda)\varphi$ and $\tilde{q}_{n-1} = q_0 - 2B^{(n-1)}$ mentioned in [10] to construct more new solutions. In addition, this method is an effective way to deal with traveling waves of a PDE and can be used in other PDE models.

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## Appendix

### Table 1: Type I traveling wave solutions of Eq. \( (1) \)

| \( \lambda \) | Traveling wave solution | \( \xi > 0 \) | \( \xi < 0 \) |
|---|---|---|---|
| \( \lambda < 0 \) | \( a_1 \) | \( r \) | \( \sqrt{\omega} \) | \( T \) |
| \( \lambda > 0 \) | \( a_2 \) | \( r \) | \( \sqrt{\omega} \) | \( T \) |
| \( \lambda < 0 \) | \( a_3 \) | \( r \) | \( \sqrt{\omega} \) | \( T \) |
| \( \lambda > 0 \) | \( a_4 \) | \( r \) | \( \sqrt{\omega} \) | \( T \) |

### Notes:
- \( a_1, a_2, a_3, a_4 \) are parameters determined by the specific conditions of the problem.
- \( \lambda \) represents the wave speed.
- \( \xi \) is the spatial coordinate.
- \( \omega \) is the frequency.
- \( T \) is the period of the wave.

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38