DETERMINISTIC HOMOGENIZATION OF UNSTEADY NAVIER-STOKES TYPE EQUATIONS

LAZARUS SIGNING

Abstract. In this paper we study the deterministic homogenization problems for unsteady Navier-Stokes type equations, on one hand in an open set $\Omega$ of $\mathbb{R}^N$, on the other hand in porous media $\Omega^\varepsilon$. In the second case, the equations are classical unsteady Navier-Stokes one, and the porous media are periodic.

1. Introduction

We study the homogenization of unsteady Navier-Stokes type equations in two distinct settings. In the first setting, the equations are considered in a fixed bounded open set in the $N$-dimensional numerical space and moreover the usual Laplace operator involved in the classical Navier-Stokes equations is replaced by an elliptic linear differential operator of order two, in divergence form, with spatially varying coefficients. In the second setting, the equations are the classical unsteady Navier-Stokes one and are considered in periodic porous media. Precisely, we investigate the following problems:

Problem I. Let $\Omega$ be a smooth bounded open set of $\mathbb{R}^N_x$ (the $N$-dimensional numerical space of variable $x = (x_1, ..., x_N)$), and let $T$ and $\varepsilon$ be real numbers with $T > 0$ and $0 < \varepsilon < 1$. We consider the partial differential operator

$$P^\varepsilon = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a^\varepsilon_{ij} \frac{\partial}{\partial x_j} \right)$$

in $\Omega \times ]0,T[$, where $a^\varepsilon_{ij}(x) = a_{ij}(x/\varepsilon)$ ($x \in \Omega$), $a_{ij} \in L^\infty(\mathbb{R}^N_y;\mathbb{R})$ (1 $\leq i, j \leq N$) with

$$(1.1) \quad a_{ij} = a_{ji},$$

and the assumption that there is a constant $\alpha > 0$ such that

$$(1.2) \quad \sum_{i,j=1}^{N} a_{ij}(y) \zeta_j \zeta_i \geq \alpha |\zeta|^2 \text{ for all } \zeta = (\zeta_j) \in \mathbb{R}^N$$

for almost all $y \in \mathbb{R}^N$, $\mathbb{R}^N_y$ being the $N$-dimensional numerical space $\mathbb{R}^N$ of variables $y = (y_1, ..., y_N)$ and $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^N$. The operator $P^\varepsilon$ acts on scalar functions, say $\varphi \in L^2(0,T;H^1(\Omega))$. However, we may as well view $P^\varepsilon$ as acting on vector functions $u = (u^i) \in L^2(0,T;H^1(\Omega)^N)$ in a diagonal way, i.e.,

$$(P^\varepsilon u)^i = P^\varepsilon u^i \quad (i = 1, ..., N).$$

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For any Roman character such as \( i, j \) (with \( 1 \leq i, j \leq N \)), \( u^i \) (resp. \( u^j \)) denotes the \( i \)-th (resp. \( j \)-th) component of a vector function \( u \) in \( L^1_{\text{loc}}(\Omega \times [0,T]) \) or in \( L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}_T) \) where \( \mathbb{R}_T \) is the numerical space \( \mathbb{R} \) of variables \( \tau \). Further, for any real \( 0 < \varepsilon < 1 \), we define \( u^\varepsilon \) as
\[
u^\varepsilon (x,t) = u \left( \frac{x}{\varepsilon} \varepsilon, \frac{t}{\varepsilon} \right) \quad ((x,t) \in \Omega \times [0,T])
\]
for \( u \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}_T) \), as is customary in homogenization theory. More generally, for \( u \in L^1_{\text{loc}}(Q \times \mathbb{R}^N \times \mathbb{R}_T) \) with \( Q = \Omega \times [0,T] \), it is customary to put
\[
u^\varepsilon (x,t) = u \left( x, t, \frac{x}{\varepsilon} \varepsilon, \frac{t}{\varepsilon} \right) \quad ((x,t) \in \Omega \times [0,T])
\]
whenever the right-hand side makes sense (see, e.g., [15]).

After these preliminaries, let \( f = (f^i) \in L^2 \left( 0, T; H^{-1}(\Omega; \mathbb{R}) \right) \). For any fixed \( 0 < \varepsilon < 1 \), we consider the initial boundary value problem
\[egin{align*}
\frac{\partial u^\varepsilon}{\partial t} + P^\varepsilon u^\varepsilon + \sum_{j=1}^N u^\varepsilon_j \frac{\partial u^\varepsilon}{\partial x_j} + \text{grad} p^\varepsilon &= f \quad \text{in } \Omega \times [0,T], \\
div u^\varepsilon &= 0 \quad \text{in } \Omega \times [0,T], \\
u^\varepsilon = 0 \quad &\text{on } \partial \Omega \times [0,T], \\
u^\varepsilon (0) &= 0 \quad \text{in } \Omega
\end{align*}
\]
where \( \frac{\partial u^\varepsilon}{\partial x_j} = \left( \frac{\partial u^1}{\partial x_j}, ..., \frac{\partial u^N}{\partial x_j} \right) \). As in [31], for \( N = 2 \) ([13]–[16]) uniquely define \((u^\varepsilon, p^\varepsilon)\) with \( u^\varepsilon \in W(0,T) \) and \( p^\varepsilon \in L^2 \left( 0, T; L^2(\Omega; \mathbb{R}) \right) \), where
\[
W(0,T) = \left\{ u \in L^2 \left( 0, T; V \right) : u' \in L^2 \left( 0, T; V' \right) \right\}
\]
\( V \) being the space of functions \( u \) in \( H^1_0(\Omega; \mathbb{R}) \) with \( \text{div} u = 0 \) (\( V' \) is the topological dual of \( V \)), and where
\[
L^2(\Omega; \mathbb{R}) \rightarrow \mathbb{R} = \left\{ v \in L^2(\Omega; \mathbb{R}) : \int_\Omega v dx = 0 \right\}.
\]
Let us recall that \( W(0,T) \) is provided with the norm
\[
\|u\|_{W(0,T)} = \left( \|u\|_{L^2(0,T; V)}^2 + \|u'\|_{L^2(0,T; V')}^2 \right)^{\frac{1}{2}} \quad (u \in W(0,T)),
\]
which makes it a Hilbert space with the following properties (see [31]): \( W(0,T) \) is continuously embedded in \( C \left([0,T]; L^2(\Omega)^N\right) \) and is compactly embedded in \( L^2 \left( 0, T; L^2(\Omega)^N \right) \).

Our aim here is to investigate the asymptotic behavior, as \( \varepsilon \to 0 \), of \((u^\varepsilon, p^\varepsilon)\) under an abstract assumption on \( a_{ij} \) (\( 1 \leq i, j \leq N \)) covering a wide range of concrete behaviours beyond the classical periodicity hypothesis. The latter states that \( a_{ij} (y + k) = a_{ij} (y) \) for almost all \( y \in \mathbb{R}^N \) and for all \( k \in \mathbb{Z}^N \) (\( \mathbb{Z} \) denotes the integers). The study of this problem turns out to be of benefit to the modelling of heterogeneous fluid flows, in particular multi-phase flows, fluids with spatially
varying viscosities, and others. The linear version of this problem (i.e., the homogenization of (1.3)-(1.6) without the term $\sum_{j=1}^{N} u_j^\varepsilon \partial u_j^\varepsilon / \partial x_j$) has been studied by the author [27] under the periodicity hypothesis on the coefficients $a_{ij}$ via the two-scale convergence techniques. We mention also the paper by Choe and Kim [5] dealing with that linear version by the well known asymptotic expansion combined with Tartar’s energy method. Further, the steady version was first investigated in [22] by the sigma-convergence method. This paper deals with a more complicated situation where the equations are non-stationary and non-linear, and the estimates of the pressure and the acceleration become a laborious issue as it is shown in the proof of Proposition 2.1. As far as i know, this topic has not yet been seriously investigated.

The main result of this part of the work can be stated as follows: Let $(u_\varepsilon, p_\varepsilon) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R}))$ be the unique solution of (1.3)-(1.6). As $\varepsilon$ goes to zero, $(u_\varepsilon, p_\varepsilon)$ converges in some topology to some $(u_0, p_0) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R}))$, where $(u_0, p_0)$ is the unique solution of the initial boundary value problem (2.53)-(2.56). The macroscopic homogenized equations (2.53)-(2.56) is of the incompressible Navier-Stokes type. This result is proved in the periodic setting by Theorem 2.3 and Lemma 2.5, and in general deterministic setting by Theorem 3.1 and Lemma 3.2.

Our approach is the sigma-convergence method derived from two-scale convergence ideas [1], [14] by means of so-called homogenization algebras [17], [18].

Problem II. Let us put

$$Y = \left( -\frac{1}{2}, \frac{1}{2} \right)^N$$

with $N \geq 2$, $Y$ being viewed as a subset of $\mathbb{R}^N_y$ (the space $\mathbb{R}^N$ of variables $y = (y_1, ..., y_N)$). Let $Y_s$ be a connected open set in $\mathbb{R}^N$ with $\overline{Y}_s \subset Y$ ($\overline{Y}_s$ the closure of $Y_s$ in $\mathbb{R}^N$) and with smooth boundary $\Gamma = \partial \overline{Y}_s$. We put

$$Y_f = Y \setminus \overline{Y}_s$$

and

$$\Theta = \bigcup_{k \in \mathbb{Z}^N} (k + \overline{Y}_s).$$

In view of the compactness of $\overline{Y}_s$, it is an easy exercise to verify that $\Theta$ is closed in $\mathbb{R}^N$. Now, let $\Omega$ be a connected smooth bounded open set in $\mathbb{R}^N_x$. Let $0 < \varepsilon < 1$. We define

$$\Omega_\varepsilon = \Omega \setminus \varepsilon \Theta.$$ 

This is a Lipschitz bounded open set in $\mathbb{R}^N_x$.

We are now in a position to state the problem under consideration in the present setting. Let $T > 0$. Given $f = (f_i) \in L^2(0, T; L^2(\Omega; \mathbb{R})^N)$, we consider the initial boundary value problem in $\Omega_\varepsilon \times [0, T]$ for the Navier-Stokes equations:

(1.7) \quad $\frac{\partial u_\varepsilon}{\partial \tau} - \nu \Delta u_\varepsilon + \sum_{j=1}^{N} u_j^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} + \text{grad} p_\varepsilon = f$ in $\Omega_\varepsilon \times [0, T]$,

(1.8) \quad $\text{div} u_\varepsilon = 0$ in $\Omega_\varepsilon \times [0, T]$,

(1.9) \quad $u_\varepsilon = 0$ on $\partial \Omega_\varepsilon \times [0, T]$. 
where $\nu > 0$ is the kinematic viscosity coefficient. For $N = 2$ the problem (1.7)-(1.10) admits a unique solution $(u_\varepsilon, p_\varepsilon)$ with $u_\varepsilon \in L^2(0, T; H^1_0(\Omega_\varepsilon)^N)$ and $p_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon)/\mathbb{R})$ (see, e.g., [10], [31]).

The aim is to study the limiting behavior of $(u_\varepsilon, p_\varepsilon)$ as $\varepsilon \to 0$. In other words, our purpose here is to discuss the homogenization of the initial boundary value problem, (1.7)-(1.10), the non-stationary Navier-Stokes equations governing an incompressible fluid flow in the domain $\Omega_\varepsilon$.

Many authors have studied the homogenization, in porous media, of fluid flows governed by the Stokes as well as the Navier-Stokes equations in various physical contexts. We refer for example to [2], [3], [6], [8], [12] and [13]. Those authors derive mostly the Darcy’s law without the proof of a global convergence result as it is stated and proved in Theorem 4.1. Our topic here is concerned with the compressible fluid flow in the domain $\Omega_\varepsilon$.

By means of the sigma-convergence, we derive the homogenized problem for (1.7)-(1.10) which is given by (1.26)-(1.27). Equation (1.27) is the Darcy’s law with a time parameter. A similar result as been established for the stationary case in [23, Section 4], but in view of the difficulties encountered in the proof of estimates for the pressure and the acceleration in the non-stationary case, Theorem 4.1 seems not to have been published before in the literature.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notation. If $X$ and $F$ denote a locally compact space and a Banach space, respectively, then we write $C(X; F)$ for continuous mappings of $X$ into $F$, and $B(X; F)$ for those mappings in $C(X; F)$ that are bounded. We denote by $\mathcal{K}(X; F)$ the mappings in $C(X; F)$ having compact supports. We shall assume $B(X; F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x \in X} \|u(x)\|$ ($\|\cdot\|$ denotes the norm in $F$). For shortness we will write $C(X) = C(X; \mathbb{C})$, $B(X) = B(X; \mathbb{C})$ and $\mathcal{K}(X) = \mathcal{K}(X; \mathbb{C})$. Likewise in the case when $F = \mathbb{C}$, the usual spaces $L^p(X; F)$ and $L^p_{\text{loc}}(X; F)$ ($X$ provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{\text{loc}}(X)$, respectively. Finally, the numerical space $\mathbb{R}^N$ and its open sets are each provided with Lebesgue measure denoted by $dx = dx_1...dx_N$.

The rest of the paper is organized as follows. Section 2 is devoted to the homogenization of (1.3)-(1.6) under the periodicity assumption on the coefficients $a_{ij}$. In Section 3 we reconsider the homogenization of (1.3)-(1.6) in a more general setting. The periodicity hypothesis on the coefficients $a_{ij}$ is here replaced by an abstract assumption covering a variety of concrete behaviours, the periodicity being a particular case. Finally, in Section 4 we discuss the homogenization of problem (1.7)-(1.10).

2. Periodic homogenization of unsteady Navier-Stokes type equations

2.1. Preliminaries. Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^N$. For fixed $0 < \varepsilon < 1$, we introduce the bilinear form $a^\varepsilon$ on $H^1_0(\Omega; \mathbb{R})^N \times H^1_0(\Omega; \mathbb{R})^N$ defined by

$$a^\varepsilon(u, v) = \sum_{k=1}^N \sum_{i,j=1}^N \int_\Omega a_{ij}^{\varepsilon} \frac{\partial u^k}{\partial x_j} \frac{\partial v^k}{\partial x_i} dx$$
for \( \mathbf{u} = (u^k) \) and \( \mathbf{v} = (v^k) \) in \( H^1_0(\Omega; \mathbb{R})^N \). By virtue of (1.1), the form \( a^\varepsilon \) is symmetric. Further, in view of (1.2),

\[ a^\varepsilon (\mathbf{v}, \mathbf{v}) \geq \alpha \| \mathbf{v} \|^2_{H^1_0(\Omega)^N} \]

for every \( \mathbf{v} = (v^k) \in H^1_0(\Omega; \mathbb{R})^N \) and \( 0 < \varepsilon < 1 \), where

\[ \| \mathbf{v} \|^2_{H^1_0(\Omega)^N} = \left( \sum_{k=1}^N \int_{\Omega} |\nabla v^k|^2 \, dx \right)^\frac{1}{2} \]

with \( \nabla v^k = \left( \frac{\partial v^k}{\partial x_1}, \ldots, \frac{\partial v^k}{\partial x_N} \right) \). On the other hand, it is clear that a constant \( c_0 > 0 \) exists such that

\[ |a^\varepsilon (\mathbf{u}, \mathbf{v})| \leq c_0 \| \mathbf{u} \|_{H^1_0(\Omega)^N} \| \mathbf{v} \|_{H^1_0(\Omega)^N} \]

for every \( \mathbf{u}, \mathbf{v} \in H^1_0(\Omega; \mathbb{R})^N \) and \( 0 < \varepsilon < 1 \). We introduce also the trilinear form \( b \) on \( H^1_0(\Omega; \mathbb{R})^N \times H^1_0(\Omega; \mathbb{R})^N \times H^1_0(\Omega; \mathbb{R})^N \) defined by

\[ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{k=1}^N \sum_{j=1}^N \int_{\Omega} \bar{u}^j \frac{\partial v^k}{\partial x_j} \, w^k \, dx \]

for \( \mathbf{u} = (u^k), \mathbf{v} = (v^k) \) et \( \mathbf{w} = (w^k) \in H^1_0(\Omega; \mathbb{R})^N \). The form \( b \) has the following properties \cite{31} pp.162-163:

\[ |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c(N) \| \mathbf{u} \|_{H^1_0(\Omega)^N} \| \mathbf{v} \|_{H^1_0(\Omega)^N} \| \mathbf{w} \|_{H^1_0(\Omega)^N} \]

for all \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \in H^1_0(\Omega; \mathbb{R})^N \), where the positive constant \( c(N) \) depends on \( N \) and \( \Omega \);

\[ b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \quad (\mathbf{u} \in V, \mathbf{v} \in H^1_0(\Omega; \mathbb{R})^N) \]

and

\[ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad (\mathbf{u} \in V, \mathbf{v} \in H^1_0(\Omega; \mathbb{R})^N). \]

For \( \mathbf{u} \) and \( \mathbf{v} \in H^1_0(\Omega; \mathbb{R})^N \), let us consider the linear form \( B(\mathbf{u}, \mathbf{v}) \) on \( H^1_0(\Omega; \mathbb{R})^N \) defined by

\[ (B(\mathbf{u}, \mathbf{v}), \mathbf{w}) = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad (\mathbf{w} \in H^1_0(\Omega; \mathbb{R})^N). \]

Let us set for \( \mathbf{u} \in H^1_0(\Omega; \mathbb{R})^N \)

\[ \overline{B}(\mathbf{u}) = B(\mathbf{u}, \mathbf{u}). \]

In view of (2.3), we have \( \overline{B}(\mathbf{u}) \in H^{-1}(\Omega; \mathbb{R})^N \) and

\[ \left\| \overline{B}(\mathbf{u}) \right\|_{H^{-1}(\Omega)^N} \leq c(N) \| \mathbf{u} \|^2_{H^1_0(\Omega)^N}. \]

Before we can establish some estimates on the velocity \( \mathbf{u}_t \), the acceleration \( \frac{\partial \mathbf{u}}{\partial t} \), and the pressure \( p_\varepsilon \), let us recall the following results.

**Lemma 2.1.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^2 \). We have the following inequalities:

\[ \| \mathbf{v} \|_{L^4(\Omega)} \leq 2^{\frac{1}{4}} \| \mathbf{v} \|_{L^2(\Omega)} \frac{1}{2} \| \text{grad} \mathbf{v} \|_{L^2(\Omega)} \quad (\mathbf{v} \in H^1_0(\Omega; \mathbb{R})), \]

\[ |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq 2^{\frac{1}{4}} |\mathbf{u}|_2 \| \mathbf{v} \|_2 \| \mathbf{w} \|_2 \quad (\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1_0(\Omega; \mathbb{R})^2), \]
where $|\cdot|$ and $\|\cdot\|$ are respectively the norms in $L^2(\Omega)^N$ and $H^1_0(\Omega)^N$. Moreover, if $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $H$ being the closure of $V = \{u \in D(\Omega; \mathbb{R})^2 : \text{div} u = 0\}$ in $L^2(\Omega; \mathbb{R})^2$, then $\tilde{B}(u) \in L^2(0, T; V')$ and
\begin{equation}
(2.8) \quad \|\tilde{B}(u)\|_{L^2(0, T; V')} \leq 2\pi \|u\|_{L^\infty(0, T; H)} \|u\|_{L^2(0, T; V')}.
\end{equation}

The proof of the above lemma can be found in [31, pp.291-293]. The following regularity result is fundamental for the estimates of the solution $(u, p_e)$ of (1.3)-(1.6).

**Lemma 2.2.** Suppose in (1.3)-(1.6) that $N = 2$ and
\begin{equation}
(2.9) \quad f, f' \in L^2(0, T; V') \quad \text{and} \quad f(0) \in L^2(\Omega; \mathbb{R})^N.
\end{equation}
Then the solution $u_e$ verifies:
\begin{equation}
(2.10) \quad u_e \in L^2(0, T; V) \cap L^\infty(0, T; H).
\end{equation}

The proof of the above lemma follows by the same line of argument as in the proof of [31, p.299, Theorem 3.5]. So we omit it. We are now able to prove the result on the estimates.

**Proposition 2.1.** Under the hypotheses of Lemma 2.2, there exists a constant $c > 0$ (independent of $\varepsilon$) such that the pair $(u_e, p_e)$ solution of (1.3)-(1.6) in $W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})^N)$ satisfies:
\begin{equation}
(2.11) \quad \|u_e\|_{W(0, T)} \leq c
\end{equation}
\begin{equation}
(2.12) \quad \left\|\frac{\partial u_e}{\partial t}\right\|_{L^2(0, T; H^{-1}(\Omega)^N)} \leq c
\end{equation}
and
\begin{equation}
(2.13) \quad \|p_e\|_{L^3(0, T; L^2(\Omega))} \leq c.
\end{equation}

**Proof.** Let $(u_e, p_e)$ be the solution of (1.3)-(1.6). We have
\begin{equation}
(2.14) \quad (u_e'(t), v) + a^\varepsilon(u_e(t), v) + b(u_e(t), u_e(t), v) = (f(t), v) \quad (v \in V)
\end{equation}
for almost all $t \in [0, T]$, where $(\cdot, \cdot)$ denotes the duality pairing between $V'$ and $V$ as well as between $H^{-1}(\Omega; \mathbb{R})^N$ and $H^1_0(\Omega; \mathbb{R})^N$. By taking in particular $v = u_e(t)$ in (2.14), we have for almost all $t \in [0, T]$
\begin{equation}
\frac{d}{dt} |u_e(t)|^2 + 2\alpha \|u_e(t)\|^2 \leq \frac{1}{\alpha} \|f(t)\|_{V'}^2 + \alpha \|u_e(t)\|^2
\end{equation}
since $b(u_e(t), u_e(t), u_e(t)) = 0$ in view of (2.4) (where $|\cdot|$ and $\|\cdot\|$ are respectively the norms in $L^2(\Omega)^N$ and $H^1_0(\Omega)^N$). Hence, for every $s \in [0, T]$
\begin{equation}
|u_e(s)|^2 + \alpha \int_0^s \|u_e(t)\|^2 dt \leq \frac{1}{\alpha} \int_0^s \|f(t)\|_{V'}^2 dt
\end{equation}
and
\begin{equation}
|u_e(s)|^2 \leq \frac{1}{\alpha} \int_0^T \|f(t)\|_{V'}^2 dt
\end{equation}
since $u_x(0) = 0$. We have also

\begin{equation}
(2.16) \quad \alpha \int_0^T \|u_x(t)\|^2 \, dt \leq \frac{1}{\alpha} \int_0^T \|f(t)\|^2_{V'}, \, dt.
\end{equation}

On the other hand, the abstract parabolic problem for (1.3)-(1.6) gives

\[ u'_x = f - A_x u_x - \tilde{B}(u_x), \]

where $A_x$ is the linear operator of $V$ into $V'$ defined by

\[ (A_x u, v) = a^\ast(u, v) \quad (u, v \in V). \]

Hence, in view of (2.2)

\begin{equation}
(2.17) \quad \|u'_x\|_{L^2(0,T;V')} \leq \|f\|_{L^2(0,T;V')} + c_0 \|u_x\|_{L^2(0,T;V)} + \|\tilde{B}(u_x)\|_{L^2(0,T;V')},
\end{equation}

since $\tilde{B}(u_x) \in L^2(0,T;V')$ (see (2.8)). Thus, by (2.16), (2.17) and (2.8) one quickly arrives at (2.11). Let us show (2.12). By virtue of Lemma 2.2 we have $u'_x \in L^2(0,T;V) \cap L^\infty(0,T;H)$. On the other hand, we are allowed to differentiate (2.14) in the distribution sense on $]0,T[$. We get

\[ \frac{d}{dt}(u'_x, v) + a^\ast(u'_x, v) + b(u'_x, u_x, v) + b(u_x, u'_x, v) = (f', v) \quad (v \in V), \]

i.e.,

\[ \frac{d}{dt}(u'_x, v) = (f' - A_x u'_x - B(u'_x, u_x) - B(u_x, u'_x), v). \]

But the function $f' - A_x u'_x - B(u'_x, u_x) - B(u_x, u'_x) \in L^2(0,T;V')$: indeed, $f' \in L^2(0,T;V')$ by hypothesis, $A_x u'_x \in L^2(0,T;V')$, and further we have

\[ \int_0^T \|B(u'_x(t), u_x(t))\|_{V'}^2 \, dt \leq 2 \|u'_x\|_{L^\infty(0,T;H)} \|u_x\|_{L^\infty(0,T;H)} \int_0^T \|u'_x(t)\| \|u_x(t)\| \, dt \]

and

\[ \int_0^T \|B(u_x(t), u'_x(t))\|_{V'}^2 \, dt \leq 2 \|u'_x\|_{L^\infty(0,T;H)} \|u_x\|_{L^\infty(0,T;H)} \int_0^T \|u'_x(t)\| \|u_x(t)\| \, dt \]

by virtue of part (2.7) of Lemma 2.1. Thus by [31], p.250, Lemma 1.1, $u'_x \in L^2(0,T;V')$ and

\begin{equation}
(2.18) \quad (u'_x, v) + a^\ast(u'_x, v) + b(u'_x, u_x, v) + b(u_x, u'_x, v) = (f', v)
\end{equation}

for all $v \in V$. Furthermore, $u'_x \in L^2(0,T;V)$ and $u''_x \in L^2(0,T;V')$ imply that $u'_x \in W(0,T)$. But $W(0,T)$ is continuously embedded in $C([0,T];H)$ (see [31]), thus $u'_x \in C([0,T];H)$. Moreover, replacing $v$ by $u'_x(t)$ in (2.14), we obtain

\[ |u'_x(t)|^2 + a^\ast(u_x(t), u'_x(t)) + b(u_x(t), u_x(t), u'_x(t)) = (f(t), u'_x(t)) \]

As $u'_x \in C(0,T;H)$, one has in particular for $t = 0$,

\[ |u'_x(0)|^2 = (f(0), u'_x(0)) - a^\ast(u_x(0), u'_x(0)) - b(u_x(0), u_x(0), u'_x(0)) \]

where $(,)$ denotes also the scalar product in $H$. But $u_x(0) = 0$, thus by the preceding equality we obtain

\begin{equation}
(2.19) \quad |u'_x(0)| \leq |f(0)|.
\end{equation}
The inequality (2.19) shows that $u'_t(0)$ lies in a bounded subset of $H$. On the other hand, by taking in particular $v = u'_t(t)$ in (2.19), we get

$$\frac{d}{dt} |u'_t(t)|^2 + 2 \alpha ||u'_t(t)||^2 + 2b(u'_t(t), u_t(t), u'_t(t)) \leq 2 (f'(t), u'_t(t)),$$

since $b(u_t(t), u'_t(t), u'_t(t)) = 0$. But, by virtue of Lemma 2.1

$$2 |b(u'_t(t), u_t(t), u'_t(t))| \leq \frac{2}{\alpha} ||u'_t(t)|| ||u'_t(t)|| ||u_t(t)|| \leq \alpha ||u'_t(t)||^2 + \frac{2}{\alpha} ||u_t(t)||^2 |u'_t(t)|^2.$$

Hence, we deduce from (2.20) that

$$\frac{d}{dt} |u'_t(t)|^2 + \frac{\alpha}{2} ||u'_t(t)||^2 \leq \frac{2}{\alpha} ||f'(t)||_{L^2}^2 + \frac{2}{\alpha} ||u_t(t)||^2 |u'_t(t)|^2.$$

By (2.21) we have

$$\frac{d}{dt} |u'_t(t)|^2 - \frac{2}{\alpha} ||u_t(t)||^2 |u'_t(t)|^2 \leq \frac{2}{\alpha} ||f'(t)||_{L^2}^2,$$

As

$$\exp\left(-\int_0^t \frac{2}{\alpha} ||u_t(s)||^2 ds\right) \leq 1,$$

multiplying (2.22) by $\exp\left(-\int_0^t \frac{2}{\alpha} ||u_t(s)||^2 ds\right)$ yields

$$\left(\frac{d}{dt} |u'_t(t)|^2 - \frac{2}{\alpha} ||u_t(t)||^2 |u'_t(t)|^2\right) \exp\left(-\int_0^t \frac{2}{\alpha} ||u_t(s)||^2 ds\right) \leq \frac{2}{\alpha} ||f'(t)||_{L^2}^2,$$

i.e.,

$$\frac{d}{dt} \left(|u'_t(t)|^2 \exp\left(-\int_0^t \frac{2}{\alpha} ||u_t(s)||^2 ds\right)\right) \leq \frac{2}{\alpha} ||f'(t)||_{L^2}^2.$$

Thus, integrating (2.23) on $(0, t)$, $t \in (0, T)$, we have

$$|u'_t(t)|^2 \leq \left\{ |u'_t(0)|^2 + \frac{2}{\alpha} \int_0^T ||f'(t)||_{L^2}^2 dt \right\} \exp\left(\int_0^T \frac{2}{\alpha} ||u_t(s)||^2 ds\right)$$

for all $t \in (0, T)$. It follows from (2.16), (2.19) and (2.24) that the sequence $(u'_t)_{t \geq 0}$ is bounded in $L^2 \left(0, T; L^2(\Omega)^N\right)$. Hence, the sequence $(\frac{\partial u}{\partial t})_{t \geq 0}$ is bounded in $L^2 \left(0, T; H^{-1}(\Omega)^N\right)$ and (2.12) is proved. Further, by part (2.7) of Lemma 2.1 we have

$$|b(u, v, w)| \leq 2^\frac{1}{2} |u|^{\frac{1}{2}} |v|^{\frac{1}{2}} |w|^{\frac{1}{2}} \quad \left(u, v, w \in H^1_0(\Omega; \mathbb{R})^N\right).$$

Thus, if $u \in V$ then $b(u, v, w) = -b(u, w, v)$ and

$$|b(u, v, w)| \leq 2^\frac{1}{2} |u|^{\frac{1}{2}} |v|^{\frac{1}{2}} |w|^{\frac{1}{2}} \quad \left(v, w \in H^1_0(\Omega; \mathbb{R})^N\right).$$

In particular,

$$|b(u, u, v)| \leq 2^\frac{1}{2} |u| ||u|| |v| \quad \text{for } u \in V \text{ and } v \in H^1_0(\Omega; \mathbb{R})^N,$$

thus

$$\|\tilde{B}(u)\|_{H^{-1}(\Omega)^N} \leq 2^\frac{1}{2} |u| ||u|| \quad \left(u \in V\right).$$
It follows from the preceding inequality that \( \bar{B} (u_\varepsilon) \in L^2 \left( 0, T; H^{-1}(\Omega)^N \right) \) and
\[
(2.25) \quad \int_0^T \| \bar{B} (u_\varepsilon) \|^2_{H^{-1}(\Omega)^N} \, dt \leq 2 \| u_\varepsilon \|^2_{L^\infty(0,T;H)} \| u_\varepsilon \|^2_{L^2(0,T;V)}.
\]

On the other hand, \( p_\varepsilon (t) \in L^2 (\Omega; \mathbb{R}) / \mathbb{R} \) for almost all \( t \in (0, T) \). Consequently, by virtue of [30, p. 30] there exists some \( v_\varepsilon (t) \in H^1_b (\Omega; \mathbb{R})^N \) such that
\[
(2.26) \quad \text{div} v_\varepsilon (t) = p_\varepsilon (t)
\]
where the constant \( c_1 \) depends solely on \( \Omega \). Multiplying (1.4) by \( v_\varepsilon (t) \), we have for almost all \( t \in (0, T) \)
\[
(2.28) \quad (u_\varepsilon', v_\varepsilon (t)) + a^\varepsilon (u_\varepsilon (t), v_\varepsilon (t)) + b (u_\varepsilon (t), v_\varepsilon (t), v_\varepsilon (t)) - \int \Omega p_\varepsilon (t) \, \text{div} v_\varepsilon (t) \, dx = (f(t), v_\varepsilon (t)).
\]
Integrating (2.28) on \((0, T)\), and using (2.25)-(2.27) yield
\[
(2.29) \quad \| p_\varepsilon \|^2_{L^2(Q)} \leq c_1 \| u_\varepsilon' \|^2_{L^2(0,T;L^2(\Omega)^N)} \| p_\varepsilon \|^2_{L^2(Q)} + \sqrt{2} c_1 \| u_\varepsilon \|^2_{L^\infty(0,T;H)} \| u_\varepsilon \|^2_{L^2(0,T;V)} \| p_\varepsilon \|^2_{L^2(Q)}
+ c_1 \| f \|^2_{L^2(0,T;H^{-1}(\Omega))} \| p_\varepsilon \|^2_{L^2(Q)} + c_0 c_1 \| u_\varepsilon \|^2_{L^2(0,T;V)} \| p_\varepsilon \|^2_{L^2(Q)},
\]
where \( c \) is the constant in the Poincaré inequality, \( c_0 \) and \( c_1 \) are the constants in (2.2) and (2.24) respectively. It follows from (2.24) that
\[
(2.30) \quad \| p_\varepsilon \|^2_{L^2(Q)} \leq c_1 \| u_\varepsilon' \|^2_{L^2(0,T;L^2(\Omega)^N)} + \sqrt{2} c_1 \| u_\varepsilon \|^2_{L^\infty(0,T;H)} \| u_\varepsilon \|^2_{L^2(0,T;V)}
+ c_1 \| f \|^2_{L^2(0,T;H^{-1}(\Omega))} + c_0 c_1 \| u_\varepsilon \|^2_{L^2(0,T;V)}.
\]
Using (2.11), (2.12) and (2.16) already proved, one quickly arrives a (2.13) by (2.30).
The proof of the proposition is complete. \(\square\)

2.2. A convergence result for (1.3)-(1.6). We set \( Y = \left( -\frac{1}{2}, \frac{1}{2} \right)^N \), \( Y \) considered as a subset of \( \mathbb{R}^N \) (the space \( \mathbb{R}^N \) of variables \( y = (y_1, ..., y_N) \)). We set also \( Z = \left( -\frac{1}{2}, \frac{1}{2} \right) \), \( Z \) considered as a subset of \( \mathbb{R} \) (the space \( \mathbb{R} \) of variables \( \tau \)). Our purpose is to study the homogenization of (1.3)-(1.6) under the periodicity hypothesis on \( a_{ij} \).

2.2.1. Preliminaries. Let us first recall that a function \( u \in L^1_{\text{loc}}(\mathbb{R}^N_y \times \mathbb{R}_\tau) \) is said to be \( Y \times Z \)-periodic if for each \((k, l) \in \mathbb{Z}^N \times \mathbb{Z} \) (\( \mathbb{Z} \) denotes the integers), we have \( u(y + k, \tau + l) = u(y, \tau) \) almost everywhere (a.e.) in \((y, \tau) \in \mathbb{R}^N \times \mathbb{R} \). If in addition \( u \) is continuous, then the preceding equality holds for every \((y, \tau) \in \mathbb{R}^N \times \mathbb{R} \). The space of all \( Y \times Z \)-periodic continuous complex functions on \( \mathbb{R}^N_y \times \mathbb{R}_\tau \) is denoted by \( \mathcal{C}_{\text{per}}(Y \times Z) \); that of all \( Y \times Z \)-periodic functions in \( L^1_{\text{loc}}(\mathbb{R}^N_y \times \mathbb{R}_\tau) \) \((1 \leq p < \infty)\) is denoted by \( L^p_{\text{per}}(Y \times Z) \). \( \mathcal{C}_{\text{per}}(Y \times Z) \) is a Banach space under the supremum norm on \( \mathbb{R}^N_y \times \mathbb{R}_\tau \), whereas \( L^p_{\text{per}}(Y \times Z) \) is a Banach space under the norm
\[
\| u \|_{L^p(Y \times Z)} = \left( \int_Y \left( \int_Z |u(y, \tau)|^p \, d\tau \right)^\frac{1}{p} \right)^{\frac{1}{p}} \quad (u \in L^p_{\text{per}}(Y \times Z)).
\]
We will need the space $H^1_{\#}(Y)$ of $Y$-periodic functions $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ such that $\int_Y u(y) \, dy = 0$. Provided with the gradient norm,

$$\|u\|_{H^1_{\#}(Y)} = \left(\int_Y |\nabla_y u|^2 \, dy\right)^{\frac{1}{2}} \quad (u \in H^1_{\#}(Y)),$$

where $\nabla_y u = \left(\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_N}\right)$, $H^1_{\#}(Y)$ is a Hilbert space. We will also need the space $L^2_{\text{per}}(Z; H^1_{\#}(Y))$ with the norm

$$\|u\|_{L^2_{\text{per}}(Z; H^1_{\#}(Y))} = \left(\int_Z \int_Y \|\nabla_y u(y, \tau)\|^2 \, dy \, d\tau\right)^{\frac{1}{2}} \quad (u \in L^2_{\text{per}}(Z; H^1_{\#}(Y)))$$

which is a Hilbert space.

Before we can recall the concept of $\Sigma$-convergence, let us introduce one further notation. The letter $E$ throughout will denote a family of real numbers $0 < \varepsilon < 1$ admitting $0$ as an accumulation point. For example, $E$ may be the whole interval $(0, 1)$; $E$ may also be an ordinary sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n < 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In the latter case $E$ will be referred to as a fundamental sequence.

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ and $Q = \Omega \times [0, T]$ with $T \in \mathbb{R}^*_+$, and let $1 \leq p < \infty$.

**Definition 2.1.** A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(Q)$ is said to:

(i) weakly $\Sigma$-converge in $L^p(Q)$ to some $u_0 \in L^p(Q; L^p_{\text{per}}(Y \times Z))$ if as $E \ni \varepsilon \to 0$

$$\int_Q u_{\varepsilon}(x, t) \, \psi^\varepsilon(x, t) \, dx \, dt \to \int \int_{Q \times Y \times Z} u_0(x, t, y, \tau) \, \psi(x, t, y, \tau) \, dx \, dy \, d\tau$$

for all $\psi \in L^{p'}(Q; C_{\text{per}}(Y \times Z)) \quad \left(\frac{1}{p'} = 1 - \frac{1}{p}\right)$, where $\psi^\varepsilon(x, t) = \psi(x, t, \frac{\varepsilon}{\varepsilon}, \varepsilon)$, $((x, t) \in Q)$;

(ii) strongly $\Sigma$-converge in $L^p(Q)$ to some $u_0 \in L^p(Q; L^p_{\text{per}}(Y \times Z))$ if the following property is verified:

$$\|u_0 - v\|_{L^p(Q; C_{\text{per}}(Y \times Z))}$$

with

$$\|u_0 - v\|_{L^p(Q; C_{\text{per}}(Y \times Z))} \leq \frac{\eta}{2}, \text{ there is some } \alpha > 0 \text{ such that } \|u_\varepsilon - v\|_{L^p(Q)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha.$$

We will briefly express weak and strong two-scale convergence by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak $\Sigma$ and $u_\varepsilon \to u_0$ in $L^p(Q)$-strong $\Sigma$, respectively.

**Remark 2.1.** It is of interest to know that if $u_\varepsilon \to u_0$ in $L^p(Q)$-weak $\Sigma$, then (2.31) holds for $\psi \in C\left(\Omega; L^\infty_{\text{per}}(Y \times Z)\right)$. See [10 Proposition 10] for the proof.

Instead of repeating here the main results underlying two-scale convergence or $\Sigma$-convergence theory for periodic structures, we find it more convenient to draw the reader’s attention to a few references, see, e.g., [1], [11], [16] and [32].

However, we recall below two fundamental results. First of all, let

$$\mathcal{Y}(0, T) = \left\{ v \in L^2(0, T; H^1_{\#}(\Omega, \mathbb{R})) : v' \in L^2(0, T; H^{-1}(\Omega, \mathbb{R})) \right\}.$$

$\mathcal{Y}(0, T)$ is provided with the norm

$$\|v\|_{\mathcal{Y}(0, T)} = \left(\|v\|_{L^2(0, T; H^1_{\#}(\Omega))}^2 + \|v'\|_{L^2(0, T; H^{-1}(\Omega))}^2\right)^{\frac{1}{2}} \quad (v \in \mathcal{Y}(0, T))$$. 

which makes it a Hilbert space.

**Theorem 2.1.** Assume that $1 < p < \infty$ and further $E$ is a fundamental sequence. Let a sequence $(u_\varepsilon)_{\varepsilon \in \varepsilon}$ be bounded in $L^p(Q)$. Then, a subsequence $E'$ can be extracted from $E$ such that $(u_\varepsilon)_{\varepsilon \in \varepsilon'}$ weakly $\Sigma$-converges in $L^p(Q)$.

**Theorem 2.2.** Let $E$ be a fundamental sequence. Suppose a sequence $(u_\varepsilon)_{\varepsilon \in \varepsilon}$ is bounded in $Y'(0,T)$. Then, a subsequence $E'$ can be extracted from $E$ such that, as $E' \ni \varepsilon \to 0$,

\[
\begin{align*}
&u_\varepsilon \rightharpoonup u_0 \text{ in } Y'(0,T) \text{-weak,} \\
&\frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q) \text{-weak } (1 \leq j \leq N),
\end{align*}
\]

where $u_0 \in Y(0,T)$, $u_1 \in L^2(Q;L^2_{\operatorname{per}}(Z;H^1_\#(Y)))$.

The proof of Theorem 2.1 can be found in, e.g., [1], [11], whereas Theorem 2.2 has its proof in, e.g., [10] and [21].

2.2.2. **A global homogenization theorem.** Before we can establish a so-called global homogenization theorem for (1.3)–(1.6), we require a few basic notation and results. To begin, let

\[
\begin{align*}
\mathcal{V}_Y &= \left\{ \psi \in C^\infty_{\operatorname{per}}(Y;\mathbb{R})^N : \int_Y \psi(y) dy = 0, \operatorname{div}_y \psi = 0 \right\}, \\
\mathcal{V}_Y &= \left\{ w \in H^1_\#(Y;\mathbb{R})^N : \delta w = 0 \right\},
\end{align*}
\]

where: $C^\infty_{\operatorname{per}}(Y;\mathbb{R}) = C^\infty(\mathbb{R}^N;\mathbb{R}) \cap C_{\operatorname{per}}(Y)$, $\delta w$ denotes the divergence operator in $\mathbb{R}^N$. We provide $\mathcal{V}_Y$ with the $H^1_\#(Y)^N$-norm, which makes it a Hilbert space. There is no difficulty in verifying that $\mathcal{V}_Y$ is dense in $\mathcal{V}_Y$ (proceed as in [21] Proposition 3.2]). With this in mind, set

\[
F_0^1 = L^2(0,T;V) \times L^2(Q;L^2_{\operatorname{per}}(Z;\mathcal{V}_Y)).
\]

This is a Hilbert space with norm

\[
\|v\|_{F_0^1} = \left( \|v_0\|_{L^2(0,T;V)}^2 + \|v_1\|_{L^2(Q;L^2_{\operatorname{per}}(Z;\mathcal{V}_Y))}^2 \right)^{1/2},
\]

$v = (v_0, v_1) \in F_0^1$.

On the other hand, put

\[
F_0^\infty = \mathcal{D}(0,T;V) \times \left[ \mathcal{D}(Q;\mathbb{R}) \otimes \left[ C^\infty_{\operatorname{per}}(Z;\mathbb{R}) \otimes \mathcal{V}_Y \right] \right],
\]

where $C^\infty_{\operatorname{per}}(Z;\mathbb{R}) = C^\infty(\mathbb{R};\mathbb{R}) \cap C_{\operatorname{per}}(Z)$, $C^\infty_{\operatorname{per}}(Z;\mathbb{R}) \otimes \mathcal{V}_Y$ stands for the space of vector functions $w$ on $\mathbb{R}^N \times \mathbb{R}_t$ of the form

\[
w(y,\tau) = \sum_{i \text{ finite}} \chi_i(\tau) v_i(y) \quad (\tau \in \mathbb{R}, y \in \mathbb{R}^N)
\]

with $\chi_i \in C^\infty_{\operatorname{per}}(Z;\mathbb{R})$, $v_i \in \mathcal{V}_Y$, and where $\mathcal{D}(Q;\mathbb{R}) \otimes \left( C^\infty_{\operatorname{per}}(Z;\mathbb{R}) \otimes \mathcal{V}_Y \right)$ is the space of vector functions on $Q \times \mathbb{R}^N \times \mathbb{R}$ of the form

\[
\psi(x,t,y,\tau) = \sum_{i \text{ finite}} \varphi_i(x,t) w_i(y,\tau) \quad ((x,t) \in Q, (y,\tau) \in \mathbb{R}^N \times \mathbb{R})
\]
with \( \varphi_i \in \mathcal{D}(Q; \mathbb{R}) \), \( w_i \in C_0^\infty(\mathbb{R}; \mathbb{R}) \otimes \mathcal{V}_Y \). Since \( \mathcal{V} \) is dense in \( V \) (see [31] p.18), it is clear that \( \mathcal{F}_0^\infty \) is dense in \( \mathcal{F}_0 \).

Now, let
\[
U = V \times L^2(\Omega; L^2_{per}(Z; V_Y))
\]
Provided with the norm
\[
\|v\|_U = \left( \|v_0\|^2 + \|v_1\|^2_{L^2(\Omega; L^2_{per}(Z; V_Y))} \right)^{\frac{1}{2}} \quad (v = (v_0, v_1) \in U),
\]
\( U \) is a Hilbert space. Let us set
\[
\hat{a}_\Omega(u, v) = \sum_{i,j,k=1}^N \int \int_{\Omega \times Y \times Z} a_{ij} \left( \frac{\partial u_i^k}{\partial x_j} + \frac{\partial u_j^k}{\partial y_i} \right) \left( \frac{\partial v_i^k}{\partial x_j} + \frac{\partial v_j^k}{\partial y_i} \right) \, dxdydt
\]
for \( u = (u_0, u_1) \) and \( v = (v_0, v_1) \) in \( U \). This defines a symmetric continuous bilinear form \( \hat{a}_\Omega \) on \( U \times U \). Furthermore, \( \hat{a}_\Omega \) is \( \Omega \)-elliptic. Specifically,
\[
\hat{a}_\Omega(u, u) \geq \alpha \|u\|^2_U \quad (u \in U),
\]
as is easily checked by using [12] and the fact that \( \int_Y \frac{\partial u_i^k}{\partial y_i} (x, y, \tau) \, dy = 0 \).

Here is one fundamental lemma.

**Lemma 2.3.** Suppose \( N = 2 \). Suppose also that there exists a function \( u = (u_0, u_1) \in \mathcal{F}_0^\infty \) verifying
\[
(2.34) \quad u_0 \in W(0, T) \quad \text{with} \quad u_0(0) = 0,
\]
and the variational equation
\[
(2.35) \quad \left\{ \begin{array}{l}
\int_0^T (u_0'(t), v_0(t)) \, dt + \int_0^T \hat{a}_\Omega(u(t), v(t)) \, dt + \int_0^T b(u_0(t), u_0(t), v_0(t)) \, dt \\
\quad = \int_0^T (f(t), v_0(t)) \, dt \quad \text{for all} \quad v = (v_0, v_1) \in \mathcal{F}_0^\infty.
\end{array} \right.
\]
Then \( u \) is unique.

**Proof.** Let \( v_* = (v_0, v_1) \in U \) and \( \varphi \in \mathcal{D}([0, T]) \). By taking \( v = \varphi \otimes v_* \) in \( (2.35) \), we arrive at
\[
(2.36) \quad (u_0'(t), v_0(t)) + \hat{a}_\Omega(u(t), v_*) + b(u_0(t), u_0(t), v_*) = (f(t), v_0) \quad (v_* \in U)
\]
for almost all \( t \in (0, T) \). Suppose that \( u_* \) and \( u_{**} \) are two solutions of \( (2.34)-(2.35) \) with \( u_* = (u_{0,0}, u_{1,1}) \) and \( u_{**} = (u_{0,**}, u_{1,**}) \). Let \( u = u_* - u_{**} = (u_0, u_1) \) with \( u_0 = u_{0,0} - u_{0,**} \) and \( u_1 = u_{1,1} - u_{1,**} \). Let us show that \( u = 0 \). Using \( (2.36) \), we see that \( u \) verifies:
\[
(2.37) \quad (u_0'(t), v_0(t)) + \hat{a}_\Omega(u(t), v_*) + b(u_0(t), u_0(t), v_0(t)) + b(u_0(t), u_{**}(t), v_0(t)) = 0
\]
for all \( v_* \in U \) and for almost all \( t \in (0, T) \). But, by virtue of [31] p. 261
\[
(2.38) \quad \frac{d}{dt} \|u_0(t)\|^2 = 2 \langle u_0'(t), u_0(t) \rangle
\]
for almost all \( t \in (0, T) \). Hence, taking \( v_* = u(t) \) in \( (2.37) \), we obtain by \( (2.4), (2.5) \) and \( (2.38) \)
\[
(2.39) \quad \frac{d}{dt} \|u_0(t)\|^2 + 2\alpha \|u(t)\|^2_U \leq -2b(u_0(t), u_{**}(t), u_0(t))
\]
for almost all \( t \in (0, T) \). Furthermore, using part \( (2.7) \) of Lemma \( 2.1 \) yields
\[
2 \|b(u_0(t), u_{**}(t), u_0(t))\| \leq 2\frac{\|u_0(t)\|}{\|u_0(t)\|} \|u_{**}(t)\| \|u_0(t)\|.
\]
Thus, by the inequality
\[ 2^p |u_0(t)| \|u_0(t)\| \|u_{*0}(t)\| \leq 2\alpha \|u_0(t)\|^2 + \frac{1}{\alpha} |u_0(t)|^2 \|u_{*0}(t)\|^2 \]
we have:
\[ \frac{d}{dt} |u_0(t)|^2 + 2\alpha \|u(t)\|^2 \leq 2\alpha \|u(t)\|_0^2 + \frac{1}{\alpha} |u_0(t)|^2 \|u_{*0}(t)\|^2 \]
for almost all \( t \in (0, T) \), and then
\[ \frac{d}{dt} |u_0(t)|^2 - \frac{1}{\alpha} |u_0(t)|^2 \|u_{*0}(t)\|^2 \leq 0 \]
for almost all \( t \in (0, T) \). Multiplying the preceding inequality by \( \exp \left(-\frac{1}{\alpha} \int_0^t \|u_{*0}(s)\|^2 ds\right) \), we obtain
\[ \frac{d}{dt} \left[ |u_0(t)|^2 \exp \left(-\frac{1}{\alpha} \int_0^t \|u_{*0}(s)\|^2 ds\right) \right] \leq 0 \]
for almost all \( t \in (0, T) \). As \( u_0(0) = 0 \), integrating on \([0, t], (0 \leq t \leq T)\) the preceding inequality yields: \( |u_0(t)|^2 \leq 0 \) for all \( t \in (0, T) \), thus \( u_0(t) = 0 \) \( (t \in (0, T)) \).
Finally, the inequality (2.39) gives \( u = 0 \), and the lemma follows. \( \square \)

We are now able to prove the desired theorem. Throughout the remainder of the present section, it is assumed that \( a_{ij} \) is \( Y \)-periodic for any \( 1 \leq i, j \leq N \).

**Theorem 2.3.** Suppose that the hypotheses of Lemma 2.2 are satisfied. For \( 0 < \varepsilon < 1 \), let \( u_\varepsilon \) be defined by (2.33)-(1.4). Then, as \( \varepsilon \to 0 \) we have
\[(2.40) \quad u_\varepsilon \to u_0 \text{ in } W(0, T) \text{-weak},
(2.41) \quad \frac{\partial u_\varepsilon^k}{\partial x_j} = \frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j} \text{ in } L^2(Q) \text{-weak } \Sigma \quad (1 \leq j, k \leq N)
where \( u = (u_0, u_1) \) (with \( u_0 = (u_0^k) \) and \( u_1 = (u_1^k) \)) is the unique solution of
(2.38)-(2.39).

**Proof.** By Proposition 2.1 we see that the sequences \((p_\varepsilon)_{0<\varepsilon<1}\) and \((u_\varepsilon)_{0<\varepsilon<1} = (u_\varepsilon^1, \ldots, u_\varepsilon^N)_{0<\varepsilon<1}\) are bounded respectively in \( L^2(Q) \) and \( W(0, T) \). Further, it follows from (2.41) and (2.42) that for \( 1 \leq k \leq N \), the sequence \((u_\varepsilon^k)_{0<\varepsilon<1}\) is bounded in \( Y(0, T) \). Let \( E \) be a fundamental sequence. Then, by Theorems 2.1
(2.2) and the fact that \( W(0, T) \) is compactly embedded in \( L^2(Q)^N \), there exist a subsequence \( E' \) extracted from \( E \) and functions \( u_0 = (u_0^k)_{1 \leq k \leq N} \in W(0, T), u_1 = (u_1^k)_{1 \leq k \leq N} \in L^2(Q; L^2_{\text{per}}(Z; H_0^1(Y; R)^N)) \), and \( p \in L^2(Q; L^2_{\text{per}}(Y \times Z; R)) \) such that as \( \varepsilon \to 0 \), we have
(2.40)-(2.41) and
(2.42) \quad \|u_\varepsilon - u_0\|_{L^2(Q)^N} \to 0 \text{-strong},
(2.43) \quad p_\varepsilon \to p \text{ in } L^2(Q) \text{-weak } \Sigma.
But, by virtue of Lemma 2.3 the theorem will be entirely proved if we show that \( u = (u_0, u_1) \) verifies (2.35). Indeed, according to (1.4), we have \( \text{div } u_0 = 0 \) and \( \text{div } u_1 = 0 \). Therefore \( u = (u_0, u_1) \in H^1_0 \). Let us recall that \( u_0 \) can be considered as a continuous function of \([0, T]\) into \( H \) since \( W(0, T) \) is continuously embedded.
in $\mathcal{C}([0, T]; H)$. Let us show that $u_0(0) = 0$. For $v \in V$ and $\varphi \in \mathcal{C}^1([0, T])$ with $\varphi(T) = 0$ and $\varphi(0) = 1$, we have by an integration by part

$$
\int_0^T (u'_v(t), v) \varphi(t) dt + \int_0^T (u_\varepsilon(t), v) \varphi'(t) dt = - (u_\varepsilon(0), v).
$$

According to (1.6), we have by passing to the limit in the preceding equality as $E' \ni \varepsilon \to 0$

$$
\int_0^T (u'_v(t), v) \varphi(t) dt + \int_0^T (u_0(t), v) \varphi'(t) dt = 0.
$$

Hence $(u_0(0), v) = 0$ for all $v \in V$, and as $V$ is dense in $H$ we conclude that $u_0(0) = 0$. Now, let us check that $u = (u_0, u_1)$ verifies the variational equation of (2.30). For $0 < \varepsilon < 1$, let

$$
\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^0 \text{ with } \psi_0 \in \mathcal{D}(Q; \mathbb{R})^N \text{ and } \psi_1 \in \mathcal{D}(Q; \mathbb{R}) \otimes \left[ C^\infty_{per}(Z; \mathbb{R}) \otimes V_Y \right],
$$

i.e., $\Phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1(x, t, \varepsilon, \frac{t}{\varepsilon})$ for $(x, t) \in Q$. We have $\Phi_\varepsilon \in \mathcal{D}(Q; \mathbb{R})^N$. Thus, multiplying (1.3) by $\Phi_\varepsilon$ yields

$$
\int_0^T (u'_v(t), \Phi_\varepsilon(t)) dt + \int_0^T a^\varepsilon(u_\varepsilon(t), \Phi_\varepsilon(t)) dt + \int_0^T b(u_\varepsilon(t), u_\varepsilon(t), \Phi_\varepsilon(t)) dt

- \int_Q p_\varepsilon \text{div}(\Phi_\varepsilon) dx dt = \int_0^T (f(t), \Phi_\varepsilon(t)) dt.
$$

Let us note at once that

$$
\int_0^T (u'_v(t), \Phi_\varepsilon(t)) dt = - \sum_{i=1}^N \int_Q u^i [\frac{\partial \psi^i_0}{\partial t} + \varepsilon \left( \frac{\partial \psi^i_1}{\partial t} \right)^\varepsilon + \left( \frac{\partial \psi^i_1}{\partial \tau} \right)^\varepsilon] dx dt.
$$

Then by virtue of (2.42) we have

$$
\int_0^T (u'_v(t), \Phi_\varepsilon(t)) dt \to - \sum_{i=1}^N \int_Q u^i_0 \frac{\partial \psi^i_0}{\partial t} dx dt = \int_0^T (u'_v(t), \psi_0(t)) dt
$$

as $E' \ni \varepsilon \to 0$. In fact, on one hand

$$
\sum_{i=1}^N \left[ \int_Q u^i_0 \frac{\partial \psi^i_0}{\partial t} \right] dx dt
$$

and on the other hand

$$
\int \int_{Q \times Y \times Z} u^i_0 \frac{\partial \psi^i_0}{\partial \tau} dx dt d\tau = \int_Q u^i_0 \left( \int \int_{Y \times Z} \frac{\partial \psi^i_1}{\partial \tau} dx dt d\tau \right) dx dt = 0 \text{ by virtue of the } Y \times Z\text{-periodicity. The next point is to pass to the limit in (2.45) as } E' \ni \varepsilon \to 0.
$$

To this end, we note that as $E' \ni \varepsilon \to 0$,

$$
\int_0^T a^\varepsilon(u_\varepsilon(t), \Phi_\varepsilon(t)) dt \to \int_0^T \tilde{a}_0(u(t), \Phi(t)) dt,
$$

where $\Phi = (\psi_0, \psi_1)$ (proceed as in the proof of the analogous result in [19] p.179). Further, in view of (2.42) and (2.41), it follows from [17] Proposition 4.7 (see also
that
\[ \int_0^T b(u_\varepsilon(t), u_\varepsilon(t), \Phi_\varepsilon(t)) \, dt \to \int_0^T b(u_0(t), u_0(t), \psi_0(t)) \, dt \]
as \( E' \ni \varepsilon \to 0 \). Now, based on (2.43), there is no difficulty in showing that as \( E' \ni \varepsilon \to 0 \),
\[ \int_Q p \, \text{div} \Phi_\varepsilon \, dx \, dt \to \int \int \int_{Q \times Y} p \, \text{div} \psi_0 \, dx \, dt \, dy \, d\tau. \]
On the other hand, let us check that as \( \varepsilon \to 0 \)
(2.47)
\[ \int_0^T (f(t), \Phi_\varepsilon(t)) \, dt \to \int_0^T (f(t), \psi_0(t)) \, dt. \]
Indeed, if \( f \in L^2 \left( 0, T; L^2(\Omega; \mathbb{R})^N \right) \), (2.47) is immediate by using the classical fact that \( \Phi_\varepsilon \to \psi_0 \) in \( L^2(Q)^N \)-weak and \( \partial \Phi_\varepsilon/\partial x \to \partial \psi_0/\partial x \) in \( L^2(Q)^N \)-weak \((1 \leq j \leq N)\) as \( \varepsilon \to 0 \). In the general case, (2.47) follows by the density of \( L^2 \left( 0, T; L^2(\Omega; \mathbb{R})^N \right) \) in \( L^2 \left( 0, T; H^{-1}(\Omega; \mathbb{R})^N \right) \).

Having made this point, we can pass to the limit in (2.45) when \( E' \ni \varepsilon \to 0 \), and the result is that
(2.48)
\[ \int_0^T (u'_0(t), \psi_0(t)) \, dt + \int_0^T \hat{a}_0(u(t), \Phi(t)) \, dt + \int_0^T b(u_0(t), u_0(t), \psi_0(t)) \, dt \\
- \int_Q p_0 \, \text{div} \psi_0 \, dx \, dt = \int_0^T (f(t), \psi_0(t)) \, dt, \]
where \( p_0 \) denotes the mean of \( p \), i.e., \( p_0 \in L^2 \left( 0, T; L^2(\Omega; \mathbb{R}) \right) \) and \( p_0(x,t) = \int \int_{Q \times Z} p(x,t,y,\tau) \, dy \, d\tau \) a.e. in \( (x,t) \in Q \), and where \( \Phi = (\psi_0, \psi_1) \), \( \psi_0 \) ranging over \( D(Q; \mathbb{R})^N \) and \( \psi_1 \) ranging over \( D(Q; \mathbb{R}) \otimes \left[ C_\infty^0(Z; \mathbb{R}) \otimes V_Y \right] \). Taking in particular \( \psi_0 \) in \( D(0, T; V) \) and using the density of \( \Phi_0^N \) in \( F_0^1 \), one quickly arrives at (2.35). The unicity of \( u = (u_0, u_1) \) follows by Lemma 2.3. Consequently, (2.40) and (2.41) still hold when \( E' \ni \varepsilon \to 0 \). Hence when \( 0 < \varepsilon \to 0 \), by virtue of the arbitrariness of \( E \). The theorem is proved. \( \square \)

Now, we wish to give a simple representation of the vector function \( u_1 \) in Theorem 2.3 for further uses. For this purpose we introduce the bilinear form \( \hat{a} \) on \( L^2_{per}(Z; V_Y) \times L^2_{per}(Z; V_Y) \) defined by
\[ \hat{a}(u, v) = \sum_{i,j,k=1}^N \int_{Y \times Z} a_{ij} \frac{\partial u^k}{\partial y_j} \frac{\partial v^k}{\partial y_i} \, dy \, d\tau \]
for \( u = (u^k) \) and \( v = (v^k) \in L^2_{per}(Z; V_Y) \). Next, for each pair of indices \( 1 \leq i, k \leq N \), we consider the variational problem
(2.49)
\[ \begin{cases} 
X_{ik} \in L^2_{per}(Z; V_Y) : \\
\hat{a}(X_{ik}, w) = \sum_{i=1}^N \int_{Y \times Z} a_{i1} \frac{\partial u^k}{\partial y_i} \, dy \, d\tau \\
\text{for all } w = (w^j) \in L^2_{per}(Z; V_Y),
\end{cases} \]
which determines \( X_{ik} \) in a unique manner.
Lemma 2.4. Under the hypotheses and notation of Theorem 2.3, we have

\begin{equation}
\mathbf{u}_1(x, t, y, \tau) = - \sum_{i,k=1}^{N} \frac{\partial q_{ik}}{\partial x_i}(x, t) \chi_{ik}(y, \tau)
\end{equation}

almost everywhere in \((x, t, y, \tau) \in Q \times Y \times Z\).

Proof. In (2.50), we choose the test functions \(\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1)\) such that \(\mathbf{v}_0 = 0\) and \(\mathbf{v}_1(x, t, y, \tau) = \varphi(x, t) \mathbf{w}(y, \tau)\) for \((x, t, y, \tau) \in Q \times Y \times Z\), where \(\varphi \in \mathcal{D}(Q; \mathbb{R})\) and \(\mathbf{w} \in L^2_{\text{per}}(Z; V_Y)\). Then for almost every \((x, t)\) in \(Q\), we have

\begin{equation}
\hat{a}(\mathbf{u}_1(x, t), \mathbf{w}) = - \sum_{i,j,k=1}^{N} \frac{\partial q_{ik}}{\partial x_i}(x, t) \int_{Y \times Z} a_{ij}(y) \frac{\partial \chi_{jk}}{\partial y_j}(y, \tau) \, dyd\tau
\end{equation}

for all \(\mathbf{w} \in L^2_{\text{per}}(Z; V_Y)\).

But it is clear that \(\mathbf{u}_1(x, t)\) (for fixed \((x, t) \in Q\)) is the unique function in \(L^2_{\text{per}}(Z; V_Y)\) solving the variational equation (2.51). On the other hand, it is an easy exercise to verify that \(z(x, t) = - \sum_{i,k=1}^{N} \frac{\partial q_{ik}}{\partial x_i}(x, t) \chi_{ik}\) solves also (2.51). Hence the lemma follows immediately. \(\square\)

2.3. Macroscopic homogenized equations. Our aim here is to derive a well-posed initial boundary value problem for \((\mathbf{u}_0, p_0)\). To begin, for \(1 \leq i,j,k,h \leq N\), let

\[ q_{ijkh} = \delta_{kh} \int_Y a_{ij}(y) \, dy - \sum_{l=1}^{N} \int_{Y \times Z} a_{il}(y) \frac{\partial \chi_{jk}}{\partial y_l}(y, \tau) \, dyd\tau, \]

where \(\delta_{kh}\) is the Kronecker symbol and \(\chi_{jk} = \left(\chi_{jk}^k\right)\) is defined by (2.39). To the coefficients \(q_{ijkh}\) we associate the differential operator \(Q\) on \(Q\) mapping \(\mathcal{D}'(Q)^N\) into \(\mathcal{D}'(Q)^N\) \((\mathcal{D}'(Q)\) being the usual space of complex distributions on \(Q)\) as

\begin{equation}
(Qz)^k = - \sum_{i,j,h=1}^{N} q_{ijkh} \frac{\partial^2 z^h}{\partial x_i \partial x_j} \quad (1 \leq k \leq N) \quad \text{for } z = (z^k), \quad z^h \in \mathcal{D}'(Q).
\end{equation}

\(Q\) is the so-called homogenized operator associated to \(P^\varepsilon\) \((0 < \varepsilon < 1)\).

Now, let us consider the initial boundary value problem

\begin{equation}
\frac{\partial \mathbf{u}_0}{\partial t} + Q \mathbf{u}_0 + \sum_{j=1}^{N} \mathbf{u}_j^0 \frac{\partial \mathbf{u}_0}{\partial x_j} + \text{grad}p_0 = \mathbf{f} \quad \text{in } Q = \Omega \times ]0, T[,
\end{equation}

\begin{equation}
div \mathbf{u}_0 = 0 \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\mathbf{u}_0 = 0 \quad \text{on } \partial \Omega \times ]0, T[,
\end{equation}

\begin{equation}
\mathbf{u}_0(0) = 0 \quad \text{in } \Omega.
\end{equation}

Lemma 2.5. Suppose \(N = 2\). The initial boundary value problem (2.53)-(2.56) admits at most one weak solution \((\mathbf{u}_0, p_0)\) with \(\mathbf{u}_0 \in W(0, T)\) and \(p_0 \in L^2(0, T; L^2(\Omega; \mathbb{R}))\).

Proof. If \((\mathbf{u}_0, p_0) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R}))\) verifies (2.53)-(2.56), then we have

\[ \int_0^T \left( \left( \mathbf{u}_0(t) , \mathbf{v}_0(t) \right) dt + \sum_{i,j,k,h=1}^{N} \int_0^T q_{ijkh} \frac{\partial q_{ik}}{\partial x_i} \frac{\partial q_{jk}}{\partial x_j} \, dx \, dt \right) + \int_0^T b(\mathbf{u}_0(t), \mathbf{u}_0(t), \mathbf{u}_0(t)) \, dt = \int_0^T \left( \mathbf{f}(t), \mathbf{v}_0(t) \right) dt \]
for all \( v_0 \in L^2(0, T; V) \). From the previous equality, one quickly arrives at
\[
\int_0^T (u_0'(t), v_0(t)) dt + \sum_{i,j,k=1}^N \iint_{Q \times Y \times Z} a_{ij} \left( \frac{\partial u_k^0}{\partial x_j} + \frac{\partial u_k^0}{\partial y_j} \right) \partial v_k^0 \, dx \, dt \, d\tau
\]
where \( u_k^0(x, t, y, \tau) = -\sum_{i,h=1}^N \frac{\partial u^0}{\partial x_i} (x, t) \chi^h_k(y, \tau) \) for \((x, t, y, \tau) \in Q \times Y \times Z\). Let us check that \( u = (u_0, u_1) \) (with \( u_1(x, t, y, \tau) = -\sum_{i,k=1}^N \frac{\partial u_k^0}{\partial x_i} (x, t) \chi^i_k(y, \tau) \) for \((x, t, y, \tau) \in Q \times Y \times Z\)) satisfies (2.35). Indeed, by following a classical line of argument (see, e.g., [4]), we can give a suitable expression of \( \Phi \)'s such that \( \Phi = (\psi_0, \psi_1) \in \mathcal{D}(Q; \mathbb{R})^N \times \mathcal{D}(Q; \mathbb{R}) \otimes C_{per}^\infty (Z; \mathbb{R}) \otimes \mathcal{Y}_V \). Then, substituting (2.40) in (2.48) and choosing therein the \( \Phi \)'s such that \( \psi_0 = 0 \), a simple computation leads to (2.48) with evidently (2.51)-(2.56). Hence, the theorem follows by Lemma 2.5 since \( E \) is arbitrarily chosen.

This leads us to the following theorem.

**Theorem 2.4.** Suppose that the hypotheses of Theorem 2.2 are satisfied. For each \( 0 < \varepsilon \ll 1 \), let \( (u_\varepsilon, p_\varepsilon) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R}) \) be defined by (2.54)-(2.55). Then, as \( \varepsilon \to 0 \), we have \( u_\varepsilon \to u_0 \) in \( W(0, T) \)-weak and \( p_\varepsilon \to p_0 \) in \( L^2(0, T; L^2(\Omega)) \)-weak, where the pair \((u_0, p_0)\) lies in \( W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R}) \) and is the unique solution of (2.35)-(2.36).

**Proof.** Let \( E \) be a fundamental sequence. As in the proof of Theorem 2.2, there exists a subsequence \( E' \) extracted from \( E \) such that as \( E' \ni \varepsilon \to 0 \), we have (2.40)-(2.41) and (2.48) with \( u = (u_0, u_1) \in F_0^1 \) and \( u_0(0) = 0 \). Then, from (2.42) we have \( p_\varepsilon \to p_0 \) in \( L^2(0, T; L^2(\Omega)) \)-weak when \( E' \ni \varepsilon \to 0 \), where \( p_0 \) is the mean of \( p \). Hence, it follows that \( p_0 \in L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R}) \). Further, (2.48) holds for all \( \Phi = (\psi_0, \psi_1) \in \mathcal{D}(Q; \mathbb{R})^N \times \mathcal{D}(Q; \mathbb{R}) \otimes C_{per}^\infty (Z; \mathbb{R}) \otimes \mathcal{Y}_V \). Then, substituting (2.50) in (2.48) and choosing therein the \( \Phi \)'s such that \( \psi_0 = 0 \), a simple computation leads to (2.48) with evidently (2.51)-(2.56). Hence, the theorem follows by Lemma 2.5 since \( E \) is arbitrarily chosen.

**Remark 2.2.** The operator \( Q \) is elliptic, i.e., there is some \( \alpha_0 > 0 \) such that
\[
\sum_{i,j,k=1}^N q_{ijkh} \xi_{ik} \xi_{jh} \geq \alpha_0 \sum_{k,h=1}^N |\xi_{kh}|^2
\]
for all \( \xi = (\xi_{ij}) \) with \( \xi_{ij} \in \mathbb{R} \). Indeed, by following a classical line of argument (see, e.g., [4]), we can give a suitable expression of \( q_{ijkh} \), viz.
\[
q_{ijkh} = \tilde{a} \left( \chi_{ik} - \pi_{ik}, \chi_{jh} - \pi_{jh} \right),
\]
where, for each pair of indices \( 1 \leq i, k \leq N \), the vector function \( \pi_{ik} = (\pi_{ik}^1, ..., \pi_{ik}^N) : \mathbb{R}^N \to \mathbb{R} \) is given by \( \pi_{ik}^r(y) = y_i \delta_{kr} (r = 1, ..., N) \) for \( y = (y_1, ..., y_N) \in \mathbb{R}^N \). Hence, the above ellipticity property follows in a classical fashion.
3. GENERAL DETERMINISTIC HOMOGENIZATION OF UNSTEADY NAVIER-STOKES TYPE EQUATIONS

Our goal here is to extend the results of Section 2 to a more general setting beyond the periodic framework. The basic notation and hypotheses (except the periodicity assumption) stated before are still valid.

3.1. Preliminaries and statement of the homogenization problem. We recall that \( \mathcal{B}(\mathbb{R}^N_+) \), \( \mathcal{B}(\mathbb{R}_+) \) and \( \mathcal{B}(\mathbb{R}^N_+ \times \mathbb{R}_+) \) denote respectively the spaces of bounded continuous complex functions on \( \mathbb{R}^N_+ \), \( \mathbb{R}_+ \) and \( \mathbb{R}^N_+ \times \mathbb{R}_+ \). It is well known that the above spaces with the supremum norm and the usual algebra operations are commutative \( C^* \)-algebras with identity (the involution is here the usual complex conjugation).

Throughout the present Section 3, \( A_y \) and \( A_\tau \) denote respectively the separable closed subalgebras of the Banach algebras \( \mathcal{B}(\mathbb{R}^N_+) \) and \( \mathcal{B}(\mathbb{R}_+) \), \( A \) denotes the closure of \( A_y \otimes A_\tau \) in \( \mathcal{B}(\mathbb{R}^N_+ \times \mathbb{R}_+) \), which is also a separable closed subalgebra of \( \mathcal{B}(\mathbb{R}^N_+ \times \mathbb{R}_+) \). Further, we assume that \( A_y \) and \( A_\tau \) contain the constants, \( A_y \) and \( A_\tau \) are stable under complex conjugation, and finally, \( A_y \) and \( A_\tau \) have the following properties: For all \( u \in A_y \) and \( v \in A_\tau \), we have \( u^\varepsilon \rightarrow M(u) \) in \( L^\infty(\mathbb{R}^N_+)-\)weak * and \( v^\varepsilon \rightarrow M(v) \) in \( L^\infty(\mathbb{R}_+)-\)weak * as \( \varepsilon \rightarrow 0 \) (\( \varepsilon > 0 \)), where:

\[
u^\varepsilon(t) = v\left(\frac{t}{\varepsilon}\right) \quad (t \in \mathbb{R}),
\]

the mapping \( u \rightarrow M(u) \) of \( A_y \) (resp. \( A_\tau \)) into \( \mathbb{C} \), denoted by \( M \), is a positive continuous linear form on \( A_y \) (resp. \( A_\tau \)) with \( M(\mathbb{1}) = 1 \) (see [17]). Then, under those assumptions on \( A_y \) and \( A_\tau \), \( A \) contains the constants, \( A \) is stable under complex conjugation and for any \( w \in A \), we have \( w^\varepsilon \rightarrow M(w) \) in \( L^\infty(\mathbb{R}^{N+1}_+(x,t)) \)-weak * as \( \varepsilon \rightarrow 0 \) (\( \varepsilon > 0 \)) where

\[
w^\varepsilon(x,t) = w\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right) \quad ((x,t) \in \mathbb{R}^N \times \mathbb{R}).
\]

For the details, see [17].

\( A_y \), \( A_\tau \) and \( A \) are called \( H \)-algebras. \( A \) is the \( H \)-algebra product of \( A_y \) and \( A_\tau \). It is clear that \( A_y \), \( A_\tau \) and \( A \) are the commutative \( C^* \)-algebras with identity. We denote by \( \Delta(A_y) \), \( \Delta(A_\tau) \) and \( \Delta(A) \) the spectra of \( A_y \), \( A_\tau \) and \( A \) respectively, and by \( \mathcal{G} \) the Gelfand transformation on \( A_y \), \( A_\tau \) and \( A \). We recall that if \( B \) is a commutative \( C^* \)-algebras with identity, \( \Delta(B) \) is the set of all nonzero multiplicative linear forms on \( B \), and \( \mathcal{G} \) is the mapping of \( B \) into \( \mathcal{C}(\Delta(B)) \) such that \( \mathcal{G}(u)(s) = \langle s, u \rangle \) (\( s \in \Delta(B) \)), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( B' \) (the topological dual of \( B \)) and \( B \). The appropriate topology on \( \Delta(B) \) is the relative weak * topology on \( B' \). So topologized, \( \Delta(B) \) is a metrizable compact space, and the Gelfand transformation is an isometric isomorphism of the \( C^* \)-algebra \( B \) onto the \( C^* \)-algebra \( \mathcal{C}(\Delta(B)) \). See, e.g., [9] for further details concerning the Banach algebras theory.

The appropriate measures on \( \Delta(A_y) \), \( \Delta(A_\tau) \) and \( \Delta(A) \) are the so-called \( M \)-measures, namely the positive Radon measures \( \beta_y \), \( \beta_\tau \) and \( \beta \) (of total mass 1) on \( \Delta(A_y) \), \( \Delta(A_\tau) \) and \( \Delta(A) \) respectively, such that \( M(u) = \int_{\Delta(A_y)} \mathcal{G}(u) \, d\beta_y \) for \( u \in A_y \), \( M(v) = \int_{\Delta(A_\tau)} \mathcal{G}(v) \, d\beta_\tau \) for \( v \in A_\tau \) and \( M(w) = \int_{\Delta(A)} \mathcal{G}(w) \, d\beta \) for
is a Fréchet space and further, $G$ for every multi-index $\alpha$ isomorphism of $A$ equipped with a suitable locally convex topology (see [17]), (and in particular Remark 2.4 and Proposition 2.6 there) for more details.

We introduce the separated completion, $H$ and the canonical mapping $\psi$ into $C^\infty(\mathbb{R}^N)$ such that

$$D^\alpha \psi = \frac{\partial^{\alpha}}{\partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}} \in A_y$$

for every multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$, and let

$$D(\Delta (A_y)) = \{ \varphi \in C(\Delta (A_y)) : G^{-1}(\varphi) \in A_y \}$$

Endowed with a suitable locally convex topology (see [17]), $A_y^\infty$ (respectively $D(\Delta (A_y))$) is a Fréchet space and further, $G$ viewed as defined on $A_y^\infty$ is a topological isomorphism of $A_y^\infty$ onto $D(\Delta (A_y))$.

By a distribution on $\Delta (A_y)$ is understood any continuous linear form on $D(\Delta (A_y))$. The space of all distributions on $\Delta (A_y)$ is then the dual, $D'(\Delta (A_y))$, of $D(\Delta (A_y))$.

We endow $D'(\Delta (A_y))$ with the strong dual topology. In the sequel it is assumed that $A_y^\infty$ is dense in $A_y$, which amounts to assuming that $D(\Delta (A_y))$ is dense in $C(\Delta (A_y))$. Then $L^p(\Delta (A_y)) \subset D'(\Delta (A_y))$ $(1 \leq p \leq \infty)$ with continuous embedding (see [17] for more details). Hence we may define

$$H^1(\Delta (A_y)) = \{ u \in L^2(\Delta (A_y)) : \partial_i u \in L^2(\Delta (A_y)) \ (1 \leq i \leq N) \}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta (A_y)$ (exactly as the Schwartz derivative is defined in the classical case). This is a Hilbert space with norm

$$\| u \|_{H^1(\Delta (A_y))} = \left( \| u \|^2_{L^2(\Delta (A_y))} + \sum_{i=1}^{N} \| \partial_i u \|^2_{L^2(\Delta (A_y))} \right)^{\frac{1}{2}} (u \in H^1(\Delta (A_y))).$$

However, in practice the appropriate space is not $H^1(\Delta (A_y))$ but its closed subspace

$$H^1(\Delta (A_y))/C = \left\{ u \in H^1(\Delta (A_y)) : \int_{\Delta (A_y)} u(s) \ d\beta(s) = 0 \right\}$$

equipped with the seminorm

$$\| u \|_{H^1(\Delta (A_y))/C} = \left( \sum_{i=1}^{N} \| \partial_i u \|^2_{L^2(\Delta (A_y))} \right)^{\frac{1}{2}} (u \in H^1(\Delta (A_y))/C).$$

Unfortunately, the pre-Hilbert space $H^1(\Delta (A_y))/C$ is in general nonseparated and noncomplete. We introduce the separated completion, $H^1_{\text{sep}}(\Delta (A_y))$, of $H^1(\Delta (A_y))/C$, and the canonical mapping $J_y$ of $H^1(\Delta (A_y))/C$ into its separated completion. See [17] (and in particular Remark 2.4 and Proposition 2.6 there) for more details.
In the sequel, we assume also $A_2^\infty$ to be dense in $A_r$, where $A_r^\infty$ is the space of $w \in C^\infty (\mathbb{R}_r)$ such that
\[
\frac{d^{\alpha} w}{d\tau^{\alpha}} \in A_r \quad (\alpha \in \mathbb{N}).
\]

We will now recall the notion of $\Sigma$-convergence in the present context. Let $1 \leq p < \infty$, and let $E$ be as in Section 2.

**Definition 3.1.** A sequence $(u_\varepsilon)_{\varepsilon \in E} \subset L^p (Q)$ is said to:

(i) weakly $\Sigma$-converge in $L^p (Q)$ to some $u_0 \in L^p (Q \times \Delta (A)) = L^p (Q; L^p (\Delta (A)))$ if as $E \ni \varepsilon \to 0$,
\[
\int_Q u_\varepsilon (x) \psi^\varepsilon (x, t) \, dxdt \to \int \int_{Q \times \Delta (A)} u_0 (x, t, s, s_0) \hat{\psi} (x, s, s_0) \, dxdsds_0
\]
for all $\psi \in L^p (Q; \mathbb{R}) \left( \frac{1}{p} = 1 - \frac{1}{p} \right)$, where $\psi^\varepsilon$ is as in Definition 2.1, and where $\hat{\psi} (x, t, \ldots) = \mathcal{G} (\psi (x, t, \ldots))$ a.e. in $(x, t) \in Q.$

(ii) strongly $\Sigma$-converge in $L^p (Q)$ to some $u_0 \in L^p (Q \times \Delta (A))$ if the following property is verified:
\[
\left\{ \begin{array}{l}
\text{Given } \eta > 0 \text{ and } v \in L^p (Q; A) \text{ with } \\
\|u_0 - \hat{v}\|_{L^p (Q \times \Delta (A))} \leq \frac{\eta}{2}, \text{ there is some } \alpha > 0 \\
\text{such that } \|u_\varepsilon - v\|_{L^p (Q)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha.
\end{array} \right.
\]

**Remark 3.1.** The existence of such $v$'s as in (ii) results from the density of $L^p (Q; C (\Delta (A)))$ in $L^p (Q; L^p (\Delta (A)))$.

We will use the same notation as in Section 2 to briefly express weak and strong $\Sigma$-convergence.

Theorem 2.1 (together with its proof) carries over to the present setting. Instead of Theorem 2.2, we have here the following notion.

**Definition 3.2.** The $H$-algebra $A$ is said to be quasi-proper if the following conditions are fulfilled.

(QPR) 1. $D (\Delta (A_y))$ is dense in $H^1 (\Delta (A_y)).$

(QPR) 2. Given a fundamental sequence $E$, and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $Y (0, T)$, one can extract a subsequence $E'$ from $E$ such that as $E' \ni \varepsilon \to 0$, we have $u_\varepsilon \to u_0 \in Y (0, T)$-weak and $\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1$ in $L^2 (Q)$-weak $1 \leq j \leq N$, where $u_0 \in Y (0, T), u_1 \in L^2 \left( Q; \frac{1}{2} (\Delta (A_r) ; \cdot H^1 (\Delta (A_y))) \right)$.

The $H$-algebra $A = C_{\text{per}} (Y \times Z)$ (see Section 2) is quasi-proper. Other examples of quasi-proper $H$-algebras can be found in [24].

Having made the above preliminaries, let us turn now to the statement of the general deterministic homogenization problem for (1.3)-(1.4). For this purpose, let $\mathbb{E}^2 (\mathbb{R}^N)$ be the space of functions $u \in L^2_{\text{loc}} (\mathbb{R}^N)$ such that
\[
\|u\|_{\mathbb{E}^2} = \sup_{0 < \varepsilon \leq 1} \left( \int_{B_N} \left| \frac{u (x, \varepsilon)}{\varepsilon} \right|^2 \, dx \right)^{\frac{1}{2}} < \infty,
\]
where $B_N$ denotes the open unit ball in $\mathbb{R}^N$. $\mathbb{E}^2$ is a complex vector space, and the mapping $u \to \|u\|_{\mathbb{E}^2}$, denoted by $\|\cdot\|_{\mathbb{E}^2}$, is a norm on $\mathbb{E}^2$ which makes it a Banach space (this is a simple exercise left to the reader). We define $\mathbb{X}^1_y$ and $\mathbb{X}^2$ to be the closure of $A_y$ and $A$ in $\mathbb{E}^2 (\mathbb{R}^N)$ and $\mathbb{E}^2 (\mathbb{R}^{N+1})$ respectively. We provide $\mathbb{X}^2_y$ (resp.
\(X^2\) with the \(\Xi^2(\mathbb{R}^N)\)-norm (resp. \(\Xi^2(\mathbb{R}^{N+1})\)-norm), which makes it a Banach space.

**Remark 3.2.** Any function \(u \in X^2_y\) can be considered as a function in \(X^2\) which is independent of the variable \(\tau\). Indeed, let \(u \in X^p_y\) and \(\eta > 0\). There exists a function \(v \in A_y\) such that

\[
\|u - v\|_{\Xi^p(\mathbb{R}^N)} \leq \frac{\eta}{2}, \text{ i.e., } \sup_{0 < r \leq 1} \left( \int_{B_r} |u^r - v^r|^p \, dy \right)^{\frac{1}{p}} \leq \frac{\eta}{2}.
\]

but \(v = v \otimes 1 \in A_y \otimes A_\tau \subset A\) and

\[
\int_{B_{N+1}} |u^r - v^r|^p \, dy \, d\tau \leq 2 \int_{B_N} |u^r - v^r|^p \, dy \leq 2 \|u - v\|_{\Xi^p(\mathbb{R}^N)}.
\]

It follows from the preceding inequalities that \(u \in \Xi^p(\mathbb{R}^{N+1})\) and \(\|u - v\|_{\Xi^p(\mathbb{R}^{N+1})} \leq \eta\).

Our main purpose in the present section is to discuss the homogenization of \((1.3) - (1.6)\) under the assumption

\[(3.1)\]

As is pointed out in [17, 18, and 19], assumption \((3.1)\) covers a great variety of concrete behaviors. In particular, \((3.1)\) generalizes the usual periodicity hypothesis (see Section 2). Indeed, for \(A = \mathcal{C}_{\text{per}}(Y \times Z)\), we have \(X^2 = L_\text{per}^2(Y \times Z)\) (use Lemma 1 of [16]).

The approach we follow here is analogous to the one which was presented in Section 2. Throughout the rest of the section, it is assumed that \((3.1)\) is satisfied, and \(A\), the closure of \(A_y \otimes A_\tau\) in \(\mathcal{B}(\mathbb{R}^N_\mathbb{R})\) is quasi-proper.

### 3.2. A global homogenization theorem.

We need a few preliminaries. To begin, we set

\[\mathcal{G}(\psi) = \left( \mathcal{G}(\psi^i) \right)_{1 \leq i \leq N}\]

for any \(\psi = (\psi^i)\) with \(\psi^i \in A\) \((1 \leq i \leq N)\). We have \(\mathcal{G}(\psi) \in C(\Delta(A))^N\) and the transformation \(\psi \rightarrow \mathcal{G}(\psi)\) of \(A^N\) into \(C(\Delta(A))^N\) maps in particular \((A^\infty_\mathbb{R})^N\) isomorphically onto \(D(\Delta(A);R)^N\), where we denote

\[A^\infty_\mathbb{R} = A^\infty \cap C(\mathbb{R}^N;\mathbb{R})\]

Likewise, letting \(J_y(u) = (J_y(u^i))_{1 \leq i \leq N}\) for \(u = (u^i)\) with \(u^i \in H^1(\Delta(A_y)) / \mathbb{C}\) \((1 \leq i \leq N)\), we have \(J_y(u) \in H^1_\#(\Delta(A_y))^N\) and the transformation \(u \rightarrow J_y(u)\) of \(H^1(\Delta(A_y)) / \mathbb{C}\) \((1 \leq i \leq N)\) isometrically into \(H^1_\#(\Delta(A_y))^N\), where we denote

\[H^1_\#(\Delta(A_y);\mathbb{R}) = \{ u \in H^1_\#(\Delta(A_y)) : \partial_i u \in L^2(\Delta(A_y);\mathbb{R}) \ (1 \leq i \leq N) \}\].

For any \(u \in L^2(\Delta(A_\tau);H^1(\Delta(A_y)) / \mathbb{C})\), we put

\[J(u)(s_0) = J_y(u(s_0)) \quad (s_0 \in \Delta(A_\tau))\].

This defines a continuous linear mapping \(J\) of \(L^2(\Delta(A_\tau);H^1(\Delta(A_y)) / \mathbb{C})\) into \(L^2(\Delta(A_\tau);H^1_\#(\Delta(A_y)))\) with the equality

\[J(\chi \otimes v) = \chi \otimes J_y(v)\] for \(\chi \in L^2(\Delta(A_\tau))\) and \(v \in H^1(\Delta(A_y)) / \mathbb{C}\).
Furthermore, letting \( J(u) = (J(u^i)) \) for \( u = (u^i) \) with \( u^i \in L^2(\Delta(A_r); H^1(\Delta(A_y))/\mathbb{C}) \) (\( 1 \leq i \leq N \)), we have \( J(u) \in L^2(\Delta(A_r); H^1(\Delta(A_y))/\mathbb{C}) \) and the transformation \( u \to J(u) \) of \( L^2(\Delta(A_r); [H^1(\Delta(A_y))/\mathbb{C}]^N) \) into \( L^2(\Delta(A_r); [H^1(\Delta(A_y))/\mathbb{C}]^N) \) maps in particular \( L^2(\Delta(A_r); [H^1(\Delta(A_y))/\mathbb{C}]^N) \) isometrically into \( L^2(\Delta(A_r); H^1(\Delta(A_y)); \mathbb{C})^N) \).

We will set
\[
E_0^1 = L^2(0, T; H^1_0(\Omega; \mathbb{R}^N) \times L^2(\Omega; L^2(\Delta(A_r); H^1(\Delta(A_y)); \mathbb{R}))
\]
\[
\mathcal{E}_0^\infty = \mathcal{D}(Q; \mathbb{R}^N) \times \left( \mathcal{D}(Q; \mathbb{R}) \otimes \mathcal{D}(\Delta(A_r); \mathbb{R}) \otimes \mathcal{J}_Q \left[ (\mathcal{D}(\Delta(A_r); \mathbb{R})/\mathbb{C})^N \right] \right)
\]
where \( \mathcal{D}(\Delta(A_y); \mathbb{R})/\mathbb{C} = \mathcal{D}(\Delta(A_y); \mathbb{R}) \cap [H^1(\Delta(A_y))/\mathbb{C}] \). \( E_0^1 \) is topologized in an obvious way and \( \mathcal{E}_0^\infty \) is considered without topology. It is clear that \( \mathcal{E}_0^\infty \) is dense in \( E_0^1 \).

At the present time, let us consider the vector space
\[
U_0^1 = H^1_0(\Omega; \mathbb{R})^N \times L^2(\Omega; L^2(\Delta(A_r); H^1(\Delta(A_y)); \mathbb{R}))
\]
topologized in an obvious way. We put
\[
\widehat{\partial}_\Omega(u, v) = \sum_{i,j,k=1}^N \int_{\Omega} \int_{\Delta(A)} \widehat{a}_{ij} \left( \frac{\partial u^k}{\partial x_j} + \partial_j u^k \right) \left( \frac{\partial v^0}{\partial x_i} + \partial_i v^0 \right) dx d\beta
\]
for \( u = (u_0, u_1) \) and \( v = (v_0, v_1) \) in \( U_0^1 \) with, of course, \( u_0 = (u^0_k), u_1 = (u^1_k) \) (and analogous expressions for \( v_0 \) and \( v_1 \)), where \( \widehat{a}_{ij} = \mathcal{G}(a_{ij}) \). This gives a bilinear form \( \widehat{\partial}_\Omega \) on \( U_0^1 \times U_0^1 \), which is symmetric, continuous, and coercive (see \[\text{[17]}\]).

Now, let
\[
V_\Delta = \left\{ u = (u^i) \in H^1_0(\Delta(A_y); \mathbb{R})^N : \widehat{\partial}_\Omega u = 0 \right\}
\]
where
\[
\widehat{\partial}_\Omega u = \sum_{i=1}^N \partial_i u^i.
\]
Equipped with the \( H^1(\Delta(A_y))/\mathbb{C} \)-norm, \( V_\Delta \) is a Hilbert space. We next put
\[
F_0^1 = L^2(0, T; V) \times L^2(\Omega; L^2(\Delta(A_r); V))
\]
provided with an obvious norm. It is an easy exercise to check that Lemma \[\text{[2.3]}\] together with its proof can be carried over mutatis mutandis to the present setting. This leads us to the analogue of Theorem \[\text{[2.3]}\].

**Theorem 3.1.** Suppose \( \text{[3.1]} \) holds and further \( A \) is quasi-proper. On the other hand, suppose that the hypotheses of Lemma \[\text{[2.3]}\] are satisfied. For each real \( 0 < \varepsilon < 1 \), let \( u_{\varepsilon} = (u^\varepsilon_k) \) be defined by \[\text{[1.3]}\)-\[\text{[1.4]}\]. Then, as \( \varepsilon \to 0 \),
\[
(3.2) \quad u_{\varepsilon} \to u_0 \text{ in } W(0, T) - \text{weak},
\]
\[
(3.3) \quad \frac{\partial u^k_{\varepsilon}}{\partial x_j} \to \frac{\partial u^k_0}{\partial x_j} + \partial_j u^1_k \text{ in } L^2(Q) - \text{weak} \quad (1 \leq j, k \leq N),
\]
where \( u = (u_0, u_1) \) with \( u_0 = (u^k_0) \) and \( u_1 = (u^k_1) \) is the unique solution of \[\text{[2.3]}\]-\[\text{[2.3]}\].
Proof. This is an adaptation of the proof of Theorem 2.3 and we will not go too deeply into details. Starting from (2.11)-(2.13), we see that the generalized sequences \((u_n^ε)_{0 < \varepsilon < 1}\) and \((p_n^ε)_{0 < \varepsilon < 1}\) are bounded in \(W(0,T)\) and \(L^2(Q)\), respectively. Moreover, for \(1 \leq k \leq N\), the sequence \((\dddot{u}_k^n)_{0 < \varepsilon < 1}\) is bounded in \(\mathcal{V}(0,T)\). Hence, from any given fundamental sequence \(E\), one can extract a subsequence \(E'\) such that as \(E' \ni \varepsilon \to 0\), we have (2.43), (2.44) and (2.45), where \(p\) lies in \(L^2(Q;L^2(\Delta(A);\mathbb{R}))\) and \(u = (u_0,u_1)\) lies in \(\mathcal{F}_0^1\) with (2.34).

Now, for each real \(0 < \varepsilon < 1\), let

\[
\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon \quad \text{with}
\begin{align*}
\psi_0 &\in D(Q;\mathbb{R})^N, \psi_1 \in D(Q;\mathbb{R}) \otimes [\mathbb{R}A_y^\infty \otimes (\mathbb{R}A_y^\infty / \mathbb{C})^N] \\
\Phi &= (\psi_0, J(\psi_1))
\end{align*}
\]

where: \(\mathbb{R}A_y^\infty / \mathbb{C} = \{\psi \in A_y^\infty : M(\psi) = 0\}\), \(\mathbb{R}A_y^\infty = \{w \in A_y^\infty : M(w) = 0\}\), \(\hat{\psi}_1\) stands for the function \((x,t) \to \mathcal{G}(\psi_1(x,t,:))\) of \(Q\) into \([D(\Delta(A);\mathbb{R})/\mathbb{C}]^N\) (\(\hat{\psi}_1\) being viewed as a function in \(\mathcal{G}(Q;A^N)\)), \(J(\hat{\psi}_1)\) stands for the function \((x,t) \to J(\hat{\psi}_1(x,t,:))\) of \(Q\) into \(L^2(\Delta(A);H_y^1(\Delta(A_y);\mathbb{R})^N)\). It is clear that \(\Phi \in \mathcal{E}_0^\infty\). With this in mind, we can pass to the limit in (2.34) (with \(\Phi_\varepsilon\) given by (3.4)) as \(E' \ni \varepsilon \to 0\), and we obtain

\[
\int_0^T (u_0^\varepsilon(t), \psi_0(t)) dt + \int_0^T \dddot{a}_0(u(t), \Phi(t)) dt + \int_0^T b(u_0(t), u(t), \psi_0(t)) dt \\
- \int_0^T \int_{Q \times \Delta(A)} p \left(div\psi_0 + \hat{div}\hat{\psi}_1 \right) dx dt = \int_0^T (f(t), \psi_0(t)) dt.
\]

Therefore, thanks to the density of \(\mathcal{E}_0^\infty\) in \(\mathcal{F}_0^1\),

\[
\int_0^T (u_0^\varepsilon(t), v_0(t)) dt + \int_0^T \dddot{a}_0(u(t), v(t)) dt + \int_0^T b(u_0(t), u(t), v_0(t)) dt \\
- \int_0^T \int_{Q \times \Delta(A)} p \left(divv_0 + \hat{div}v_1 \right) dx dt = \int_0^T (f(t), v_0(t)) dt,
\]

and that for all \(v = (v_0,v_1) \in \mathcal{F}_0^1\). Taking in particular \(v \in \mathcal{F}_0^1\) leads us immediately to (2.35). Hence the theorem follows by the same argument as used in the proof of Theorem 2.3. \(\Box\)

As was pointed out in Section 2, it is of interest to give a suitable representation of \(u_1\) (in Theorem 5.1). To this end, let

\[
\dddot{a}(v,w) = \sum_{i,j,k=1}^N \int_{\Delta(A)} \dddot{a}_{ijk}v^i w^j w^k d\beta
\]

for \(v = (v^k)\) and \(w = (w^k)\) in \(L^2(\Delta(A);H_y^1(\Delta(A_y);\mathbb{R})^N)\). This defines a bilinear form \(\dddot{a}\) on \(L^2(\Delta(A);H_y^1(\Delta(A_y);\mathbb{R})^N) \times L^2(\Delta(A);H_y^1(\Delta(A_y);\mathbb{R})^N)\), which is symmetric, continuous and coercive. For each couple of indices \(1 \leq i, k \leq N\), we consider the variational problem

\[
\begin{align*}
\chi_{ik} &\in L^2(\Delta(A);V_{A_y}) : \\
\dddot{a}(\chi_{ik},w) &= \sum_{l=1}^N \int_{\Delta(A)} \dddot{a}_{ikl}w^k d\beta
\end{align*}
\]

for all \(w = (w^k) \in L^2(\Delta(A);V_{A_y})\).
which uniquely determines \( \chi_{ik} \).

**Lemma 3.1.** Under the assumptions and notation of Theorem 3.1, we have

\[
(3.7) \quad u_1(x, t, s, s_0) = -\sum_{i,k=1}^{N} \frac{\partial u_k^i}{\partial x_i}(x, t) \chi_{ik}(s, s_0)
\]

almost everywhere in \((x, t, s, s_0) \in Q \times \Delta (A) = \Omega \times ]0, T[ \times \Delta (A_y) \times \Delta (A_z)\).

**Proof.** This is a simple adaptation of the proof of Lemma 2.4; the verification is left to the reader. \( \square \)

### 3.3. Macroscopic homogenized equations

The aim here is to derive from (3.2) a well-posed initial boundary value problem for the couple \((u_0, p_0)\), where \(u_0\) is the weak limit in (3.2) and \(p_0\) is the mean of \(p\) in (3.3), i.e., \(p_0(x, t) = \int_{\Delta(A)} p(x, t, s, s_0)\,d\beta(s, s_0)\) for \((x, t) \in Q\). We will proceed exactly as in Subsection 2.3.

First, for \(1 \leq i, j, k, h \leq N\), let

\[
q_{ijkh} = \delta_{kh} \int_{\Delta(A_y)} \hat{a}_{ij}(s)\,d\beta_y(s) - \sum_{l=1}^{N} \int_{\Delta(A)} \hat{a}_{il}(s) \partial_l \chi_{jh}(s, s_0)\,d\beta(s, s_0),
\]

where \(\chi_{jh} = (\chi_{jh}^k)\) is defined as in (3.6). To these coefficients we associate the differential operator \(Q\) on \(Q\) given by (2.52). Finally, we consider the boundary value problem (2.53)-(2.56).

**Lemma 3.2.** Under the hypotheses of Theorem 3.1, the boundary value problem (2.53)-(2.56) admits at most one weak solution \((u_0, p_0)\) with \(u_0 \in W(0, T), p_0 \in L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R})\).

**Proof.** It is an easy exercise to show that if a couple \((u_0, p_0) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R})\) is a solution of (2.53)-(2.56), then the couple \(u = (u_0, u_1)\) in which \(u_1\) is given by (3.7) satisfies (2.34)-(2.35) and is therefore unique. Hence Lemma 3.2 follows at once. \( \square \)

We are now in a position to state and prove

**Theorem 3.2.** Let the hypotheses of Theorem 3.1 be satisfied. For each real \(0 < \varepsilon < 1\), let \((u_\varepsilon, p_\varepsilon) \in W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R})\) be defined by (1.3)-(1.6). Then, as \(\varepsilon \to 0\), we have \(u_\varepsilon \to u_0\) in \(W(0, T)\)-weak and \(p_\varepsilon \to p_0\) in \(L^2(Q)\)-weak, where the couple \((u_0, p_0)\) lies in \(W(0, T) \times L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R})\) and is the unique weak solution of (2.53)-(2.56).

**Proof.** As was pointed out above, from any arbitrarily given fundamental sequence \(E\) one can extract a subsequence \(E'\) such that as \(E' \ni \varepsilon \to 0\), we have (3.2)-(3.3) and (2.43) hence \(p_\varepsilon \to p_0\) in \(L^2(Q)\)-weak, where \(p_0\) is the mean of \(p\) and thus \(p_0 \in L^2(0, T; L^2(\Omega; \mathbb{R})/\mathbb{R})\), and where \(u = (u_0, u_1) \in \mathcal{F}_1\). Furthermore, (3.6) holds for all \(v = (v_0, v_1) \in E_1\). Substituting (3.7) in (3.5) and then choosing therein the particular test functions \(v = (v_0, v_1) \in E_1\) with \(v_1 = 0\) leads to Theorem 3.2 thanks to Lemma 3.2. \( \square \)

We can present \(q_{ijkh}\) in a suitable form as in Remark 2.2. For this purpose, we introduce the space \(\mathcal{M}\) of all \(N \times N\) matrix functions with entries in \(L^2(\Delta(A); \mathbb{R})\).
Specifically $\mathcal{M}$ denotes the space of $F = (F^{ij})_{1 \leq i,j \leq N}$ with $F^{ij} \in L^2(\Delta(A) ; \mathbb{R})$. Provided with the norm
\[
\|F\|_\mathcal{M} = \left( \sum_{i,j=1}^{N} \|F^{ij}\|_{L^2(\Delta(A))}^2 \right)^{\frac{1}{2}}, \quad F = (F^{ij}) \in \mathcal{M},
\]
$\mathcal{M}$ is a Hilbert space. Now, let
\[
\mathcal{A}(F, G) = \sum_{i,j,k=1}^{N} \int_{\Delta(A)} \hat{a}_{ij}(s) F^{jk}(s, s_0) G^{ik}(s, s_0) \, d\beta(s, s_0)
\]
for $F = (F^{jk})$ and $G = (G^{ik})$ in $\mathcal{M}$. This gives a bilinear form $\mathcal{A}$ on $\mathcal{M} \times \mathcal{M}$, which is symmetric, continuous and coercive. Furthermore
\[
\hat{a}(u, v) = \mathcal{A} \left( \nabla u, \nabla v \right), \quad u, v \in L^2 \left( \Delta(A) ; H^1_{loc}(\Delta(A) ; \mathbb{R}) \right)
\]
where $\nabla u = (\partial_i u^k)$ for any $u = (u^k) \in L^2 \left( \Delta(A) ; H^1_{loc}(\Delta(A) ; \mathbb{R}) \right)$. Now, by the same line of proceeding as followed in [4] (see also [15]) one can quickly show that
\[
q_{ijkh} = \mathcal{A} \left( \nabla X_{ik} - \theta_{ik}, \nabla X_{jh} - \theta_{jh} \right),
\]
where, for any couple of indices $1 \leq i, k \leq N$, $X_{ik}$ is defined by (3.6), and $\theta_{ik} = (\theta_{ik}^{lm}) \in \mathcal{M}$ with $\theta_{ik}^{lm} = \delta_{ij}\delta_{km}$. Having made this point, Remark 2.2 can then be carried over to the present setting.

### 3.4. Some Concrete Examples

In the present subsection we consider a few examples of homogenization problems for (1.3)-(1.6) in a concrete setting (as opposed to the abstract assumption (3.1)) and we show how their study leads to the abstract setting of Subsection 3.1 and so we may conclude by merely applying Theorems 3.1 and 3.2.

**Example 1.** (Almost periodic homogenization). We study here the homogenization of (1.3)-(1.6) under the concrete hypothesis that the family $(a_{ij})_{1 \leq i,j \leq N}$ verifies:

\[
(3.8) \quad a_{ij} \in L^2_{AP}(\mathbb{R}_y^N) \quad (1 \leq i, j \leq N),
\]

where $L^2_{AP}(\mathbb{R}_y^N)$ denotes the space of all functions $w \in L^2_{loc}(\mathbb{R}_y^N)$ that are almost periodic in the sense of Stepanoff (see, e.g., [23, Section 4]). According to [24, Proposition 4.1], the hypothesis (3.8) yields a countable subgroup $\mathcal{R}_y$ of $\mathbb{R}_y^N$ such that $a_{ij} \in L^2_{AP,\mathcal{R}_y}(\mathbb{R}_y^N)$ for $1 \leq i, j \leq N$, where

\[
L^2_{AP,\mathcal{R}_y}(\mathbb{R}_y^N) = \{ u \in L^2_{AP}(\mathbb{R}_y^N) : Sp(u) \subset \mathcal{R}_y \}, \quad Sp(u) \text{ being the spectrum of } u,
\]

i.e., $Sp(u) = \{ k \in \mathbb{R}^N : M(w_{\gamma_k}) \neq 0 \}$ with $\gamma_k(y) = \exp(2i\pi k.y) (y \in \mathbb{R}^N)$. Let us consider the $H$-algebra $A_y = AP_{\mathcal{R}_y}(\mathbb{R}_y^N) = \{ u \in AP(\mathbb{R}_y^N) : Sp(u) \subset \mathcal{R}_y \}$, where $AP(\mathbb{R}_y^N)$ denotes the space of almost periodic continuous complex functions on $\mathbb{R}_y^N$ (see, e.g., [7, Chapter 5] and [11, Chapter 10]). We have

\[
a_{ij} \in L^2_{AP,\mathcal{R}_y}(\mathbb{R}_y^N) \subset \mathcal{X}_y^2 \quad (1 \leq i, j \leq N)
\]
as it can be seen by using [10, Lemma 1]. In view of Remark 2.2 we consider the $a_{ij}$ as functions in $\mathcal{X}_y^2$ which are independent of the variable $\tau$. We see that for any countable subgroup $\mathcal{R}_\tau$ of $\mathbb{R}_\tau$, we have (3.7) with $A_y = AP_{\mathcal{R}_y}(\mathbb{R}_y^N)$ and
Example 2. Let $Y' = (-\frac{1}{2}, \frac{1}{2})^{N-1}$. We study the homogenization of \((1.9)-(1.10)\) under the following hypothesis:

\[(3.9)\]

where $L^2_{per}(Y')$ is the space of functions in $L^2_{loc}(\mathbb{R}^{N-1})$ that are $Y'$-periodic. Let $C_{per}(Y')$ denotes the space of $Y'$-periodic complex continuous functions on $\mathbb{R}^{N-1}$. Let us consider the $H$-algebra $A_y = B_\infty(\mathbb{R}; C_{per}(Y'))$. We recall that $B_\infty(\mathbb{R}; C_{per}(Y'))$ is the space of continuous functions $\psi : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{C}$ such that the mapping $y_N \to \psi(., y_N)$ send continuously $\mathbb{R}$ into $C_{per}(Y')$ and $\psi(., y_N)$ has a limit in $C_{per}(Y')$ (with the norm $\| \bullet \|_\infty$ as $|y_N| \to +\infty$). Under the hypothesis \((3.9)\), the condition \((3.1)\) is satisfied with $A = B_\infty(\mathbb{R}; A_y)$, where $B_\infty(\mathbb{R}; A_y)$ is the space of functions $\varphi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ such that the mapping $\tau \to \varphi(., \tau)$ send continuously $\mathbb{R}$ into $A_y$ and $\varphi(., \tau)$ has a limit in $A_y$ (with the norm $\| \bullet \|_\infty$ as $|\tau| \to +\infty$). Indeed, on one hand the space $K(\mathbb{R}; C_{per}(Y'))$ is contained in $A_y = B_\infty(\mathbb{R}; C_{per}(Y'))$, and $K(\mathbb{R}; C_{per}(Y'))$ is dense in $L^2(\mathbb{R}; L^2_{per}(Y'))$ as it’s easily seen by using the fact that $K(\mathbb{R})$ and $C_{per}(Y')$ are dense in $L^2(\mathbb{R})$ and $L^2_{per}(Y')$ respectively. On the other hand, let $(L^2, \ell^\infty)$ be the space of all $u \in L^2_{loc}(\mathbb{R}^N)$ such that

\[\| u \|_{2, \infty} = \sup_{k \in \mathbb{Z}^N} \left( \int_{k+y} |u(y)|^2 \, dy \right)^{\frac{1}{2}} < \infty,\]

where $Y = (-\frac{1}{2}, \frac{1}{2})^N$. This is a Banach space under the norm $\| \bullet \|_{2, \infty}$. $L^2(\mathbb{R}; L^2_{per}(Y'))$ is continuously embedded in $(L^2, \ell^\infty)(\mathbb{R}^N)$ and the later is continuously embedded in $\mathbb{Z}^2(\mathbb{R}^N)$, thus $L^2(\mathbb{R}; L^2_{per}(Y'))$ is continuously embedded $\mathbb{Z}^2(\mathbb{R}^N)$. It follows that $L^2(\mathbb{R}; L^2_{per}(Y')) \subset X_{\infty}$. Thus, by Remark 3.2, we see that \((3.6)\) implies \((3.1)\) for the $H$-algebra $A$, closure of $A_{y} \otimes B_\infty(\mathbb{R}; A_y)$ in $B(\mathbb{R}^N \times \mathbb{R})$ ($A_y$ is here $B_\infty(\mathbb{R})$, the space of continuous functions $w : \mathbb{R} \to \mathbb{C}$ such that $w(\tau)$ has a limit in $\mathbb{C}$ as $|\tau| \to +\infty$). Further, $B_\infty(\mathbb{R}; A_y)$ coincides with the closure of $A_{y} \otimes B_\infty(\mathbb{R}; A_y)$ in $B(\mathbb{R}^N \times \mathbb{R})$, hence $A = B_\infty(\mathbb{R}; A_y)$. Moreover if we denote by $A_{2y}$ the closure of $C_{per}(Y') \otimes C$ in $B(\mathbb{R}^N)$, then the pair $\{ A_y, A_{2y} \}$ satisfies the hypothesis $(H)$ of [25] Subsection 4.1 by virtue of the proof in [17] Corollary 4.4. But, $A_{2y} = AP_{\mathbb{R}}(\mathbb{R}^N)$ with $\mathbb{R} = \mathbb{Z}^{N-1} \times \{0\}$, and $A_y = B_\infty(\mathbb{R}; AP_{\mathbb{R}}(\mathbb{R}^N))$ is quasi-proper (see [24] Example 3.1). Therefore, by [25] Proposition 4.1 we see that $A$ is quasi-proper. Hence, the homogenization of \((1.9)-(1.10)\) follows by Theorems 3.1 and 3.2.

Example 3. Let us suppose that

\[(3.10)\]

where $L^2_{\infty, AP}(\mathbb{R}^N)$ denotes the closure in $(L^2, \ell^\infty)(\mathbb{R}^N)$ of the space of finite sums $\sum_{finite} \varphi_i u_i$ with $\varphi_i \in B_\infty(\mathbb{R}^N)$ and $u_i \in AP(\mathbb{R}^N)$. We have the continuous embedding of $L^\infty(\mathbb{R}^N)$ in $(L^2, \ell^\infty)(\mathbb{R}^N)$, thus

\[a_{ij} \in (L^2, \ell^\infty)(\mathbb{R}^N) \quad (1 \leq i, j \leq N).\]
One can also see that $L_{∞,AP}^2(\mathbb{R}^N)$ coincides with the closure of $B_∞(\mathbb{R}^N)+AP(\mathbb{R}^N)$ in $(L^2, l^∞)(\mathbb{R}^N)$. Let $1 ≤ i, j ≤ N$. By density there exists a sequence $(a_{ij}^n)_{n∈N}$ defined by $a_{ij}^n = b_{ij}^n + c_{ij}^n$ with $b_{ij}^n \in B_∞(\mathbb{R}^N)$ and $c_{ij}^n \in AP(\mathbb{R}^N)$ such that

\begin{equation}
(3.11) \quad a_{ij}^n → a_{ij} \text{ in } (L^2, l^∞)(\mathbb{R}^N) \text{ as } n → ∞.
\end{equation}

The family $\{c_{ij}^n : 1 ≤ i, j ≤ N \text{ and } n∈N\}$ is a countable set in $AP(\mathbb{R}^N)$, thus by [15, Proposition 5.1] there exists a countable subgroup $R$ of $\mathbb{R}^N$ such that

\begin{equation}
\{c_{ij}^n : 1 ≤ i, j ≤ N \text{ and } n∈N\} \subset AP_R(\mathbb{R}^N).
\end{equation}

Then the sequence $(a_{ij}^n)_{n∈N}$ lies in $B_∞,R(\mathbb{R}^N)$, where $B_∞,R(\mathbb{R}^N)$ denotes the closure in $B(\mathbb{R}^N)$ of $B_∞(\mathbb{R}^N)+AP_R(\mathbb{R}^N)$. We denote by $L_{∞,R}^2(\mathbb{R}^N)$ the closure in $(L^2, l^∞)(\mathbb{R}^N)$ of $B_∞,R(\mathbb{R}^N)$. It follows from (3.11) that

\begin{equation}
(3.12) \quad a_{ij} ∈ L_{∞,R}^2(\mathbb{R}^N) \quad (1 ≤ i, j ≤ N).
\end{equation}

Let $A_y = B_∞,R(\mathbb{R}^N)$ and $A_τ = B_∞(\mathbb{R}_τ)$. We denote by $A$ the closure of $A_y ⊗ A_τ$ in $B(\mathbb{R}_y^N × \mathbb{R}_τ)$. As $(L^2, l^∞)(\mathbb{R}_y^N)$ is continuously embedded in $Ξ^2(\mathbb{R}^N)$, we have $L_{∞,R}^2(\mathbb{R}_y^N) ⊂ \mathcal{X}_y^2$. It follows from (3.12) and Remark 4.1 that (3.1) is satisfied for $A$. Moreover, the pair $(A_y = AP_R(\mathbb{R}^N), A_τ = B_∞,R(\mathbb{R}_τ))$ satisfies the hypothesis (H) of [25, Subsection 4.1] (see the proof of [15, Corollary 4.2]). Furthermore, the $H$-algebra $A_y$ closure of $A_y ⊗ A_τ$ in $B(\mathbb{R}_y^N × \mathbb{R}_τ)$ is quasi-proper (see Example 4). Thus, by virtue of [25, Proposition 4.1] $A$ is quasi-proper. Therefore, the homogenization of (1.3)–(1.0) follows by Theorems 3.1 and 3.2.

4. Homogenization of unsteady Navier-Stokes equations in periodic porous media

The basic notation and hypotheses are those which are stated in Section 1, especially in Problem II. Throughout the present section, vector spaces are considered over $\mathbb{R}$ and scalar functions are assumed to take real values. Thus, for the sake of convenience, we will put $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{R})$, $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{R})$, $L^p(X) = L^p(X; \mathbb{R})$, $H^1(X) = H^1(X; \mathbb{R})$, etc., $X$ being an open set in $\mathbb{R}^N$.

The purpose here is to investigate the asymptotic behaviour, as $ε → 0$, of the solution, $(u_ε, p_ε)$, of (1.7)–(1.10) with $N = 2$. As was mentioned earlier, the hypothesis $N = 2$ guarantees the unicity in (1.7)–(1.10).

4.1. Preliminaries. Before we can study the asymptotic behavior of $u_ε$ and $p_ε$ as $ε → 0$, we require a few basic results.

Lemma 4.1. (Friedrichs inequality). There is a constant $c = c(Y_f) > 0$ such that

\begin{equation}
(4.1) \quad ∫_{Ω_ε} |u|^2 dx ≤ cε² ∫_{Ω_ε} |∇u|^2 dx
\end{equation}

for all $u ∈ H^1_0(Ω_ε)$ and all real $0 < ε < 1$.

Proof. See [25].

Now, if $w = (w^k)_{1≤k≤N}$ with $w^k \in L^p(Ω)$, or if $w = (w^{ij})_{1≤i,j≤N}$ with $w^{ij} \in L^p(Ω)$, where $Ω$ is an open set in $\mathbb{R}^N$, we will sometimes write $\|w\|_{L^p(Ω)}$ for $\|w\|_{L^p(Ω)^N}$ or for $\|w\|_{L^p(Ω)^N×N}$. This abuse is convenient and in common use.

The next lemma will allow us study the behaviour of the pressure $p_ε$. 

**Lemma 4.2.** There exists a linear operator $R_\varepsilon : H_0^1(\Omega)^N \to H_0^1(\Omega)^N$ with the following properties:

(P1) If $w \in H_0^1(\Omega)^N$ and $w$ is zero on $\Omega \setminus \Omega_\varepsilon$, then $R_\varepsilon w = w|_{\Omega_\varepsilon}$.

(P2) If $w \in H_0^1(\Omega)^N$ and $\text{div}w = 0$, then $\text{div}R_\varepsilon w = 0$.

(P3) There is a constant $c > 0$ (independent of $w$ and $\varepsilon$, as well) such that

$$
\|R_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \leq c \|w\|_{L^2(\Omega)} + c\varepsilon \|\nabla w\|_{L^2(\Omega)}
$$

and

$$
\|\nabla R_\varepsilon w\|_{L^2(\Omega_\varepsilon)} \leq \frac{c}{\varepsilon} \|w\|_{L^2(\Omega)} + c \|\nabla w\|_{L^2(\Omega)}
$$

for all $w \in H_0^1(\Omega)^N$ and all $0 < \varepsilon < 1$, where $\nabla w = \left(\frac{\partial w_i}{\partial x_i}\right)_{1 \leq i \leq N}$.

**Proof.** See [23].

**Lemma 4.3.** Assume that $f$, $f' \in L^2\left(0, T; L^2(\Omega)^N\right)$ and $f(0) \in L^2(\Omega)^N$. Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (1.7)-(1.10). Let $u_\varepsilon$ be identified with its extension by zero in $(\Omega \setminus \Omega_\varepsilon) \times ]0, T[$, which lies in $L^2\left(0, T; H_0^1(\Omega)^N\right)$. The following assertions are true.

There is a constant $C > 0$ such that

$$
\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon, \quad \|u_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^2 \quad \text{and} \quad \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2(\Omega)} \leq C \quad (0 < \varepsilon < 1).
$$

For each $0 < \varepsilon < 1$, there is a unique $\mathbf{p}_\varepsilon \in L^2\left(0, T; L^2(\Omega) / \mathbb{R}\right)$ such that

$$
\int_{\Omega_\varepsilon} \mathbf{p}_\varepsilon \cdot \nabla w \, dx = \int_{\Omega_\varepsilon} p_\varepsilon \text{div}(R_\varepsilon w) \, dx \quad (w \in H_0^1(\Omega)^N)
$$

for almost all $t \in [0, T]$.

Furthermore,

$$
\frac{\partial \mathbf{p}_\varepsilon}{\partial x_i}|_{\varepsilon \times ]0, T[} = \frac{\partial p_\varepsilon}{\partial x_i} \quad (1 \leq i \leq N)
$$

in the sense of distributions on $\Omega_\varepsilon \times ]0, T[$,

$$
\|\nabla \mathbf{p}_\varepsilon\|_{L^2(0, T; H^{-1}(\Omega)^N)} \leq C \quad \text{and} \quad \|\mathbf{p}_\varepsilon\|_{L^2(\Omega)} \leq C \quad (0 < \varepsilon < 1)
$$

where the constant $C > 0$ is independent of $\varepsilon$.

**Proof.** Let us prove the inequalities in (4.3). By (1.7) we have

$$
(u'_\varepsilon(t), v) + \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon(t) \cdot \nabla v \, dx + b(u_\varepsilon(t), u_\varepsilon(t), v) = \int_{\Omega_\varepsilon} f(t) \cdot v \, dx
$$

for all $v \in H_0^1(\Omega)^N$ with div$v = 0$, where the dot stands for the usual Euclidean inner product. Choosing the particular test function $v = u_\varepsilon(t)$ and noting that

$$
b(u_\varepsilon(t), u_\varepsilon(t), u_\varepsilon(t)) = 0 \quad \text{(see, e.g., [31] p.163)},
$$

we get immediately

$$
\frac{d}{dt} \|u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 + 2\nu \|\nabla u_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 = 2 \int_{\Omega_\varepsilon} f(t) \cdot u_\varepsilon(t) \, dx \text{ in } [0, T[.
$$
Integrating on \([0, t]\) (with \(t \in [0, T]\)) the preceding equality, we arrive at

\begin{equation}
\|u_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + 2\nu \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}^2 \leq 2 \|f\|_{L^2(Q)} \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \quad \text{in } [0, T].
\end{equation}

From (4.10) we get

\begin{equation}
\nu \|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}^2 \leq \|f\|_{L^2(Q)} \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.
\end{equation}

Therefore, using Lemma (4.11) in the preceding inequality leads to

\begin{equation}
\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq \frac{c^2\varepsilon^2}{\nu} \|f\|_{L^2(Q)} \quad \text{and} \quad \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq \frac{c^2\varepsilon^2}{\nu} \|f\|_{L^2(Q)},
\end{equation}

c being the constant in (4.11). On the other hand, let us differentiate (4.9) in the distribution sense on \([0, T]\). We have

\begin{equation}
(u''_\varepsilon (t), v) + \nu \int_{\Omega_\varepsilon} \nabla u'_\varepsilon (t) \cdot \nabla v dx + b(u'_\varepsilon (t), u_\varepsilon (t), v) + b(u_\varepsilon (t), u'_\varepsilon (t), v) = \int_{\Omega_\varepsilon} f'(t) \cdot v dx
\end{equation}

for all \(v \in H^1_0(\Omega_\varepsilon)^N\) with \(\text{div} v = 0\) and in particular for \(v = u'_\varepsilon (t)\),

\begin{equation}
(u''_\varepsilon (t), u'_\varepsilon (t)) + \nu \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + b(u'_\varepsilon (t), u_\varepsilon (t), u'_\varepsilon (t)) = \int_{\Omega_\varepsilon} f'(t) \cdot u'_\varepsilon (t) dx.
\end{equation}

In fact, since \(f, f' \in L^2(0,T;L^2(\Omega)^N)\) and \(f(0) \in L^2(\Omega)^N\), \(u'_\varepsilon\) belongs to \(L^2(0,T;V) \cap L^\infty(0,T;H)\) by virtue of Lemma (2.4) and further we recall that \(b(u_\varepsilon (t), u'_\varepsilon (t), u'_\varepsilon (t)) = 0\). Moreover, by (4.8) we get

\begin{equation}
\|u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + \nu \int_{\Omega_\varepsilon} \nabla u_\varepsilon (t) \cdot \nabla u'_\varepsilon (t) dx + b(u_\varepsilon (t), u_\varepsilon (t), u'_\varepsilon (t)) = \int_{\Omega_\varepsilon} f(t) \cdot u'_\varepsilon (t) dx
\end{equation}

and in particular

\begin{equation}
\|u'_\varepsilon (0)\|_{L^2(\Omega_\varepsilon)}^2 = \int_{\Omega_\varepsilon} f(0) \cdot u'_\varepsilon (0) dx.
\end{equation}

This implies

\begin{equation}
\|u'_\varepsilon (0)\|_{L^2(\Omega_\varepsilon)} \leq \|f(0)\|_{L^2(\Omega)}.
\end{equation}

On the other hand, by (4.11) we have

\begin{equation}
\frac{d}{dt} \|u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + 2\nu \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + 2b(u'_\varepsilon (t), u_\varepsilon (t), u'_\varepsilon (t)) = 2 \int_{\Omega_\varepsilon} f'(t) \cdot u'_\varepsilon (t) dx.
\end{equation}

Further, by virtue of (2.7) of Lemma (2.1) we have

\begin{align*}
|2b(u'_\varepsilon (t), u_\varepsilon (t), u'_\varepsilon (t))| & \leq \frac{2\varepsilon}{\nu} \|u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)} \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)} \|\nabla u_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)} \\
& \leq \nu \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + \frac{2}{\nu} \|u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 \|\nabla u_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2.
\end{align*}

Moreover,

\begin{align*}
2 \left| \int_{\Omega_\varepsilon} f'(t) \cdot u'_\varepsilon (t) dx \right| & \leq 2 \|f'(t)\|_{L^2(\Omega)} \|u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)} \\
& \leq 2 \varepsilon \|f'(t)\|_{L^2(\Omega)} \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)} \\
& \leq \frac{c^2\varepsilon^2}{\nu} \|f'(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla u'_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2.
\end{align*}
where \( c > 0 \) is the constant in (4.1). Therefore, by (4.13) we get
\[
\frac{d}{dt} \| u_\varepsilon ' (t) \|_{L^2(\Omega_\varepsilon)}^2 - \frac{2}{\nu} \| u_\varepsilon ' (t) \|_{L^2(\Omega_\varepsilon)}^2 \| \nabla u_\varepsilon (t) \|_{L^2(\Omega_\varepsilon)}^2 \leq \frac{c^2\varepsilon^2}{\nu} \| f' (t) \|_{L^2(\Omega)}^2.
\]
Hence,
\[
\frac{d}{dt} \left\{ \| u_\varepsilon ' (t) \|_{L^2(\Omega_\varepsilon)}^2 \exp \left( - \int_0^t \frac{2}{\nu} \| \nabla u_\varepsilon (s) \|_{L^2(\Omega_\varepsilon)}^2 \, ds \right) \right\} \leq \frac{c^2\varepsilon^2}{\nu} \| f' (t) \|_{L^2(\Omega)}^2.
\]
In view of (4.12), integrating the preceding inequality on \([0, t]\) (with \( t \in [0, T] \)) and using the first inequality of (4.10), one quickly arrives at
\[
\| u_\varepsilon ' (t) \|_{L^2(\Omega_\varepsilon)}^2 \leq \exp \left( \frac{2Tc^2\varepsilon^2}{\nu^3} \| f \|_{L^2(\Omega)}^2 \right). \tag{4.14}
\]
It follows from (4.10), (4.12) and (4.14) that (4.3) holds for all \( 0 < \varepsilon < 1 \) with an appropriate constant \( C > 0 \).

Now, let us prove (4.15). By (17) it is clear that
\[
\int_{\Omega_\varepsilon} u_\varepsilon ' (t) \cdot \nu dx + \int_{\Omega_\varepsilon} \nabla u_\varepsilon (t) \cdot \nabla v dx + b (u_\varepsilon (t), u_\varepsilon (t), v) - \int_{\Omega_\varepsilon} p_c \text{div} v dx = \int_{\Omega_\varepsilon} f (t) \cdot v dx \tag{4.15}
\]
for almost all \( t \in [0, T] \) and for all \( v \in H^1_0 (\Omega) \). For fixed \( 0 < \varepsilon < 1 \), let us consider the mapping \( F_\varepsilon \) of \( L^2 \left( 0, T; H^1_0 (\Omega) \right) \) into \( \mathbb{R} \) defined by
\[
F_\varepsilon (w) = - \int_0^T \int_{\Omega_\varepsilon} p_c \text{div} (R_\varepsilon w) \, dx \text{ for all } w \in L^2 \left( 0, T; H^1_0 (\Omega) \right), \tag{4.16}
\]
where \( R_\varepsilon \) is the restriction operator of Lemma 4.2. It is straightforward that \( F_\varepsilon \) is a continuous linear form on \( L^2 \left( 0, T; H^1_0 (\Omega) \right) \) and
\[
\langle F_\varepsilon, w \rangle = \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon ' R_\varepsilon w \, dx \, dt + \nu \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla R_\varepsilon w \, dx \, dt + \int_0^T \int_{\Omega_\varepsilon} b (u_\varepsilon (t), u_\varepsilon (t), R_\varepsilon w (t)) \, dt - \int_0^T \int_{\Omega_\varepsilon} f \cdot R_\varepsilon w \, dx \, dt
\]
for all \( w \in L^2 \left( 0, T; H^1_0 (\Omega) \right) \), in view of (4.15). The aim is to estimate each of the integrals on the right of the preceding equality. By (4.12) and (4.14), we see that
\[
\left| \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon ' R_\varepsilon w \, dx \, dt \right| \leq \sqrt{2} C \left( \| w \|_{L^2(\Omega)} + \varepsilon \| \nabla w \|_{L^2(\Omega)} \right).
\]
Next, combining (4.3) with (4.1) we get
\[
\nu \left| \int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla R_\varepsilon w \, dx \, dt \right| \leq \sqrt{2} \nu C \left( \| w \|_{L^2(\Omega)} + \varepsilon \| \nabla w \|_{L^2(\Omega)} \right).
\]
Further, by (4.2) we get immediately
\[
\left| \int_0^T \int_{\Omega_\varepsilon} f \cdot R_\varepsilon w \, dx \, dt \right| \leq \sqrt{2} \| f \|_{L^2(\Omega)} \left( \| w \|_{L^2(\Omega)} + \varepsilon \| \nabla w \|_{L^2(\Omega)} \right).
\]
Finally, recalling that
\[
b (u_\varepsilon (t), u_\varepsilon (t), R_\varepsilon w (t)) = -b (u_\varepsilon (t), R_\varepsilon w (t), u_\varepsilon (t)),
\]
it follows from (2.7) of Lemma 2.1 that
\[ |b(u_\varepsilon(t), u_\varepsilon(t), R_\varepsilon w(t))| \leq \sqrt{2} \|u_\varepsilon(t)\|_{L^2(\Omega)} \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)} \|\nabla R_\varepsilon w(t)\|_{L^2(\Omega)} .\]
Thus,
\[ \left| \int_0^T b(u_\varepsilon(t), u_\varepsilon(t), R_\varepsilon w(t)) \, dt \right| \leq \sqrt{2} \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \int_0^T \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)} \|\nabla R_\varepsilon w(t)\|_{L^2(\Omega)} \, dt.\]
Using (4.3) and (4.4), we arrive at
\[ \left| \int_0^T b(u_\varepsilon(t), u_\varepsilon(t), R_\varepsilon w(t)) \, dt \right| \leq 2Cc \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \left( \|w\|_{L^2(Q)} + \varepsilon \|\nabla w\|_{L^2(Q)} \right) .\]
Furthermore, by (4.9)
\[ \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq 2 \|f\|_{L^2(Q)} \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} .\]
Thus, combining (4.4) and the preceding inequality, we get
\[ \left| \int_0^T b(u_\varepsilon(t), u_\varepsilon(t), R_\varepsilon w(t)) \, dt \right| \leq (2C)^{\frac{3}{2}} c\varepsilon \|f\|_{L^2(Q)} \left( \|w\|_{L^2(Q)} + \varepsilon \|\nabla w\|_{L^2(Q)} \right) .\]
On the other hand, in view of (4.15) and property (P2) of Lemma 4.2 we have
\[ (F_\varepsilon, w) = 0 \text{ for all } w \in L^2(0,T;V) .\]
Further, (4.18) is satisfied in particular for all w \in D(0,T;\mathcal{V}). Thus, by a classical argument (see, e.g., [31 pp.14-15]) we obtain
\[ F_\varepsilon = \nabla \varphi, \]
where \( \varphi \in L^2(0,T;L^2(\Omega)/\mathbb{R} ) \) with
\[ \|\varphi\|_{L^2(Q)} \leq c \|\nabla \varphi\|_{L^2(0,T;H^{-1}(\Omega)^N)} .\]
Moreover, taking in particular \( w = \varphi \in D \left( 0, T; D(\Omega)^N \right) \) in (4.16) leads to
\[ \langle \nabla \varphi, \varphi \rangle = \langle \nabla \varphi, \varphi \rangle .\]
Hence (4.16) follows by the arbitrariness of \( \varphi \). In view of the estimates for the right hand of (4.16),
\[ |\langle \nabla \varphi, w \rangle| \leq c_0 \left( \|w\|_{L^2(Q)} + \varepsilon \|\nabla w\|_{L^2(Q)} \right) \]
for all \( 0 < \varepsilon < 1 \) and for all \( w \in L^2 \left( 0, T; H^1_0(\Omega)^N \right) \), where the constant \( c_0 > 0 \) is independent of \( \varepsilon \). We deduce that
\[ |\langle \nabla \varphi, w \rangle| \leq c_0 \|w\|_{L^2(0,T;H^1_0(\Omega)^N)} \left( w \in L^2 \left( 0, T; H^1_0(\Omega)^N \right) \right) \]
for all \( 0 < \varepsilon < 1 \). Hence (4.7) follows (use (4.19)) with an appropriate constant \( C > 0 .\) \qed
4.2. Homogenization results. Let

\[ W(Y) = \left\{ w \in H^1_{\text{per}}(Y)^N : w = 0 \text{ on } Y_s, \text{div}_y w = 0 \right\}. \]

This is a Hilbert space with the norm

\[ \|w\|_{W(Y)} = \left( \int_Y |\nabla_y w|^2 \, dy \right)^{\frac{1}{2}} \quad (w \in W(Y)) \]

equivalent to the \( H^1_{\text{per}}(Y)^N \)-norm. Now, we define

\[ \mathcal{H}_0(Q; L^2_{\text{per}}(Z; W(Y))) = \left\{ u \in L^2(Q; L^2_{\text{per}}(Z; W(Y))) : \text{div}_y u = 0, \, \tilde{u} \cdot n = 0 \text{ on } \partial \Omega \times [0,T] \right\} \]

where:

\begin{align*}
\mathbf{n} &= \left(n^i\right)_{1 \leq i \leq N} \text{ is the outward unit normal to } \partial \Omega, \\
\tilde{u}(x,t) &= \int \int_{Y \times Z} u(x,t,y,\tau) \, dy \, d\tau \quad ((x,t) \in Q = \Omega \times [0,T]).
\end{align*}

Provided with the \( L^2(Q; L^2_{\text{per}}(Z; W(Y))) \)-norm, \( \mathcal{H}_0(Q; L^2_{\text{per}}(Z; W(Y))) \) is a Hilbert space.

We will need the family of vector functions \( (\mathbf{x}_j)_{1 \leq j \leq N} \) defined, for each fixed \( 1 \leq j \leq N \), by the variational problem

\[ \int \int_{Y \times Z} \chi_j \cdot \nabla_y w \, dy \, d\tau = \int \int_{Y \times Z} w \cdot \mathbf{n} \, dy \, d\tau \quad \text{for all } w = \left(w^i\right) \in L^2_{\text{per}}(Z; W(Y)). \]

Let \( (K_{ij})_{1 \leq i, j \leq N} \) be the matrix defined by

\[ K_{ij} = \int \int_{Y \times Z} \chi_j^i (y, \tau) \, dy \, d\tau. \]

Lemma 4.4. The matrix \( (K_{ij}) \) is symmetric and positive definite.

Proof. It follows immediately from (4.21) that

\[ K_{ij} = \int \int_{Y \times Z} \nabla_y \chi_j \cdot \nabla_y \chi_i \, dy \, d\tau = \int_Z (\chi_j, \chi_i)_{W(Y)} \, d\tau, \]

where \( (,)_W(Y) \) denotes the inner product associated with the norm \( \| \cdot \|_{W(Y)} \) on \( W(Y) \). The symmetry property of \( (K_{ij}) \) follows at once. Now, we have

\[ \sum_{j=1}^N K_{ij} \xi_j = \int_Z (u(\xi), u(\xi))_{W(Y)} \, d\tau \quad \text{for all } u = (\xi) \in \mathbb{R}^N, \quad \text{where } u(\xi) = \sum_{i=1}^N \xi_i \mathbf{x}_i. \]

The positivity follows. Let us check the nondegeneracy. Suppose we are given some \( \xi = (\xi_j) \) such that \( \sum_{j=1}^N K_{ij} \xi_j = 0 \). Then, \( u(\xi) = 0 \). We deduce by (4.21) that

\[ \sum_{i=1}^N \xi_i \int \int_{Y \times Z} w^i \, dy \, d\tau = 0 \quad \text{for all } w = \left(w^i\right) \in L^2_{\text{per}}(Z; W(Y)), \]

and in particular (4.22) holds true for all \( w = 1 \otimes v \) with \( v = \left(v^j\right) \in W(Y) \). Choosing in (4.22) \( w = 1 \otimes v \) with \( v \in W(Y) \) such that \( \int_Y v \, dy = \xi \) (see [23, Remark 4.3]) leads to \( \xi_j = 0 \) (\( j = 1, \ldots, N \)) and so the lemma is proved.

We are now able to prove the following homogenization theorem.
Theorem 4.1. Assume that the hypotheses of Lemma 4.3 are satisfied and \( f \in L^2(0, T; H^1(\Omega)^N) \). Let \((u_\varepsilon, p_\varepsilon)\) be the solution of (4.4) and (4.10). Let \( u_\varepsilon \) be identified with its extension by zero in \((\Omega \setminus \{\varepsilon\}) \times [0, T]\), and let \( p_\varepsilon \) be defined in Lemma 4.3. Then, as \( \varepsilon \to 0 \),

\[
\frac{1}{\varepsilon} u^i_\varepsilon \to u^i_0 \text{ in } L^2Q-(\text{weak } \Sigma \quad (1 \leq i \leq N)),
\]

\[
\frac{1}{\varepsilon} \frac{\partial u^i_\varepsilon}{\partial x_j} \to \frac{\partial u^i_0}{\partial y_j} \quad \text{in } L^2Q-(\text{weak } \Sigma \quad (1 \leq i, j \leq N)),
\]

\[
\frac{\partial \bar{u}^i_\varepsilon}{\partial x_j} \to \bar{p}_0 \quad \text{in } L^2Q,
\]

where \( u_0 = (u^i_0) \) is uniquely defined by the variational problem

\[
\begin{cases}
\nu \int_Q \int_{Y \times Z} \nabla^2 \Phi u_0 \nabla \nu v \, dxdydt = \int_Q f \cdot \nu dv & \text{for all } v \in H_0(Q; L^2_{\text{per}}(Z; W(\nu))), \\
\text{and } \nu \text{ is the unique function in } L^2(0, T; L^2(\Omega) / \mathbb{R}) \text{ such that}
\end{cases}
\]

\[
\bar{u}^i_0 = \sum_{j=1}^{N} \frac{K_{ij}}{\nu} \left( f^j - \frac{\partial p_0}{\partial x_j} \right) \quad (1 \leq i \leq N).
\]

Proof. Let us first observe that the existence and uniqueness of \( u_0 \) in (4.26) is trivial (use, e.g., the Lax-Milgram lemma). On the other hand, taking account of \( u_0 \cdot n = 0 \) on \( \partial \Omega \times [0, T] \) and \( \text{div} u_0 = 0 \), we see that if \( p_0 \) lies in \( L^2(0, T; L^2(\Omega) / \mathbb{R}) \) and verifies (4.24), then \( p_0 \) satisfies

\[
\begin{cases}
- \sum_{i,j=1}^{N} \frac{K_{ij}}{\nu} \frac{\partial^2 p_0}{\partial x_i \partial x_j} = f & \text{in } \Omega \times [0, T], \\
\sum_{i,j=1}^{N} \frac{K_{ij}}{\nu} \frac{\partial p_0}{\partial x_i} n^i = g & \text{on } \partial \Omega \times [0, T],
\end{cases}
\]

where:

\[
f = - \sum_{i,j=1}^{N} \frac{K_{ij}}{\nu} \frac{\partial f^j}{\partial x_i},
\]

\[
g = \sum_{i,j=1}^{N} \frac{K_{ij}}{\nu} \left( f^j \left|_{\partial \Omega \times [0, T]} \right. \right) n^i.
\]

But (4.28) is a Neumann type problem admitting one, and only one, solution \( p_0 \) in \( L^2(0, T; H^1(\Omega) / \mathbb{R}) \).

Now, as seen earlier, the sequences \((\overline{u}_\varepsilon)_{0<\varepsilon<1}, (\overline{p}_\varepsilon)_{0<\varepsilon<1}, (\nabla \overline{u}_\varepsilon)_{0<\varepsilon<1}\) and \((\frac{\partial \overline{u}_\varepsilon}{\partial t})_{0<\varepsilon<1}\) are bounded in the \( L^2Q \) norm. Thus, given a fundamental sequence \( E \) (i.e., \( E \) is an ordinary sequence of reals \( 0 < \varepsilon_n < 1 \) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \)), by well known compactness results (see in particular [1], [11]) we can extract a subsequence \( E' \) from \( E \) such that as \( E' \ni \varepsilon \to 0 \), we have (4.28), \( \overline{p}_\varepsilon \to p_0 \) in \( L^2Q-(\text{weak } \Sigma \quad (1 \leq i, j \leq N)),
\]

\[
\frac{1}{\varepsilon} \frac{\partial \overline{u}^i_\varepsilon}{\partial x_j} \to z_{ij} \quad \text{in } L^2Q-(\text{weak } \Sigma \quad (1 \leq i, j \leq N)),
\]

\[
\frac{1}{\varepsilon} \frac{\partial \overline{u}^i_\varepsilon}{\partial x_j} \to \frac{\partial \overline{u}^i_0}{\partial y_j} \quad \text{in } L^2Q-(\text{weak } \Sigma \quad (1 \leq i, j \leq N)).
\]
where \( u_0^i, \) \( z_{ij} \in L^2 \left( Q; L^2_{\text{per}} \left( Z; L^2_{\text{per}} (Y) \right) \right) \) and \( p_0 \in L^2 \left( 0, T; L^2 (\Omega) \right) \). But based on (4.20), if \((w_\varepsilon)_{\varepsilon \in E'}\) is a sequence in \(W(0, T)\) such that \( w_\varepsilon \to w \) in \(W(0, T)\)-weak as \( E' \ni \varepsilon \to 0 \), then

\[
\left| \langle \nabla \varphi, w \rangle - \langle \nabla p_0, w \rangle \right| \leq c \left( \| w_\varepsilon - w \|_{L^2(Q)} + \varepsilon \| \nabla w_\varepsilon - \nabla w \|_{L^2(Q)} \right) + \left| \langle \nabla \varphi, w \rangle - \langle \nabla p_0, w \rangle \right|,
\]

and we see that the right hand of (4.30) tends to zero as \( E' \ni \varepsilon \to 0 \), since \( \nabla \varphi \to \nabla p_0 \) in \(L^2 \left( 0, T; H^{-1}(\Omega)^N \right)\)-weak and \( w_\varepsilon \to w \) in \(L^2(0, T)\)-strong as \( E' \ni \varepsilon \to 0 \). Thus by (4.30) we see that the sequence \((\varphi_\varepsilon)_{\varepsilon \in E'}\) actually strongly converges in \(L^2(Q)\) to \( p_0 \), so that (4.25) holds when \( E' \ni \varepsilon \to 0 \).

The next point is to check that \( u_0 \in L^2 \left( Q; L^2_{\text{per}} \left( Z; H^1_{\text{per}} (Y)^N \right) \right) \) with

\[
\frac{\partial u_0^i}{\partial y_j} = z_{ij} \quad (1 \leq i, j \leq N).
\]

To do this, consider a function \( \psi : Q \times \mathbb{R}_y^N \times \mathbb{R}_\tau \to \mathbb{R} \) of the form

\[
\psi (x, t, y, \tau) = \varphi (x, t) \Phi (y, \tau) \quad ((x, t) \in Q, (y, \tau) \in \mathbb{R}_y^N \times \mathbb{R})
\]

with \( \varphi \in \mathcal{D} (Q), \Phi \in \mathcal{C}_\text{per}^\infty (Y \times Z) = \mathcal{C}_\text{per} (Y \times Z) \cap \mathcal{C}_\text{per}^\infty (\mathbb{R}_y^N \times \mathbb{R}_\tau) \).

We have

\[
\frac{1}{\varepsilon} \int_Q \frac{\partial u_0^i}{\partial x_j} \psi^\varepsilon dxdt = -\frac{1}{\varepsilon} \int_Q u_0^i \frac{\partial \psi^\varepsilon}{\partial x_j} dxdt = -\int_Q u_0^i \left( \frac{\partial \psi}{\partial x_j} \right)^\varepsilon dxdt - \int_Q \frac{\partial u_0^i}{\partial y_j} \left( \frac{\partial \psi}{\partial x_j} \right)^\varepsilon dxdt.
\]

Letting \( E' \ni \varepsilon \to 0 \) and recalling (4.23) and (4.29), we are quickly led to

\[
\frac{\partial u_0^i (x, t, \ldots)}{\partial y_j} = z_{ij} (x, t, \ldots) \text{ a.e. in } (x, t) \in Q \quad (1 \leq i, j \leq N),
\]

which shows that \( u_0 \) belongs to \( L^2 \left( Q; L^2_{\text{per}} \left( Z; H^1_{\text{per}} (Y)^N \right) \right) \) with (4.31) hence (4.25) (as \( E' \ni \varepsilon \to 0 \)).

Now, let us check that \( u_0 (x, t, \ldots) = 0 \) in \( Y_s \times Z \) for almost all \((x, t) \in Q\). We consider \( \psi \) as in (4.32) such that \( \Phi = 0 \) in \( Y_f \times Z \). As \( E' \ni \varepsilon \to 0 \), (4.23) yields

\[
0 = \frac{1}{\varepsilon^2} \int_Q u_0^i \psi^\varepsilon dxdt \to \int_Q \int_{Y_s \times Z} u_0^i (x, t, y, \tau) \varphi (x, t) \Phi (y, \tau) dyd\tau.
\]

Consequently

\[
\int_{Y_s \times Z} u_0^i (x, t, y, \tau) \Phi (y, \tau) dyd\tau \text{ a.e. in } (x, t) \in Q \quad (1 \leq i \leq N)
\]

(since \( \varphi \) is arbitrary) and for all \( \Phi \in \mathcal{C}_\text{per}^\infty (Y \times Z) \) verifying \( \Phi = 0 \) in \( Y_f \times Z \). Thus, \( u_0 (x, t, \ldots) = 0 \) in \( Y_s \times Z \) a.e. in \((x, t) \in Q\). Furthermore, let \( \varphi \in \mathcal{D} (Q) \) and
$w \in C^\infty_{\text{per}}(Y \times Z)$. According to (4.23), we have as $E' \ni \varepsilon \to 0$,
\[
\sum_{i=1}^{N} \int_{Q} \frac{u_i^\varepsilon (x, t, \varepsilon^2) \varphi (x, t)}{\varepsilon} \frac{\partial w}{\partial y_i} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) dxdt \to \sum_{i=1}^{N} \int_{Q} \int_{Y \times Z} \frac{u_i^\varepsilon}{\varepsilon} \varphi \frac{\partial w}{\partial y_i} dxdtdydr.
\]
But the left-hand side reduces to the term
\[
- \sum_{i=1}^{N} \int_{Q} \frac{u_i^\varepsilon (x, t, \varepsilon^2) \varphi (x, t)}{\varepsilon} \frac{\partial w}{\partial x_i} \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) dxdt
\]
which goes to zero as $\varepsilon \to 0$. Hence, by the arbitrariness of $\varphi$,
\[
\int \int_{Y \times Z} \text{div}_{y} u_0 (x, .) w dyd\tau = 0 \quad \text{a.e. in} \ (x, t) \in Q,
\]
and that for any $w \in C^\infty_{\text{per}}(Y \times Z)$. Therefore $\text{div}_{y} u_0 (x, t, .) = 0$ a.e. in $(x, t) \in Q$.
Hence $u_0 \in L^2 (Q; L^2_{\text{per}} (Z; W (Y)))$.

Let us verify that $u_0$ belongs to $H_0 (Q; L^2_{\text{per}} (Z; W (Y)))$. Clearly as $E' \ni \varepsilon \to 0$,
\[
\frac{u_i}{\varepsilon} \to \tilde{u}_0 \quad \text{in} \quad L^2 (Q)^N \text{-weak}.
\]
Since the operator $u \to \text{div} u$ sends continuously $L^2 (0, T; L^2 (\Omega)^N)$ into $L^2 (0, T; H^{-1} (\Omega))$, it follows that
\[
\frac{\text{div} u_0}{\varepsilon^2} \to \text{div} \tilde{u}_0 \quad \text{in} \quad L^2 (0, T; H^{-1} (\Omega)) \text{-weak}.
\]
Hence $\text{div} \tilde{u}_0 = 0$. On the other hand, by the Stokes formula we have
\[
\sum_{i=1}^{N} \int_{Q} \frac{u_i^\varepsilon}{\varepsilon^2} \varphi \frac{\partial \varphi}{\partial x_i} dxdt = 0
\]
for all $\theta \in \mathcal{D} ([0, T])$ and all $\varphi \in \mathcal{D} (\overline{\Omega})$. Hence, on letting $E' \ni \varepsilon \to 0$, it follows
\[
\int_{\Omega} \tilde{u}_0 (x, t) \cdot \nabla \varphi (x) dx = 0 \quad \text{a.e. in} \ t \in [0, T]
\]
(since $\theta$ is arbitrary) for all $\varphi \in \mathcal{D} (\overline{\Omega})$. This shows that $\tilde{u}_0 \cdot n = 0$ on $\partial \Omega \times [0, T]$ (use the Stokes formula) and so $u_0 \in H_0 (Q; L^2_{\text{per}} (Z; W (Y)))$.

Finally, if we prove that $u_0$ satisfies the variational equation in (4.20) and $p_0$ satisfies (4.27) then, by virtue of the unicity in (4.20) and (4.27), it will follow that (4.23) hold not only as $E' \ni \varepsilon \to 0$ but also as $E \ni \varepsilon \to 0$ and further as $0 < \varepsilon \to 0$, owing to the arbitrariness of $E$; and so the proof will be complete.

For this purpose, we introduce the space
\[
\mathcal{H} (Q; L^2_{\text{per}} (Z; W (Y))) = \{ u \in L^2 (Q; L^2_{\text{per}} (Z; W (Y))) : \text{div} \tilde{u} \in L^2 (Q), \tilde{u} \cdot n = 0 \text{ on} \ \partial \Omega \times [0, T]\},
\]
which is a Hilbert space with the norm
\[
\| u \|_{\mathcal{H} (Q; L^2_{\text{per}} (Z; W (Y)))} = \left( \| u \|_{L^2 (Q; L^2_{\text{per}} (Z; W (Y)))}^2 + \| \text{div} \tilde{u} \|_{L^2 (Q)}^2 \right)^{\frac{1}{2}}.
\]
Next, let $\Phi \in \mathcal{D} (Q) \otimes C^\infty_{\text{per}} (Z) \otimes W (Y)$. We recall the standard notation
\[
\Phi^\varepsilon (x, t) = \Phi \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon} \right) \quad \text{for} \ (x, t) \in Q,
\]
and the formulas
\[ \nabla \Phi^\varepsilon = (\nabla_x \Phi)^\varepsilon + \frac{1}{\varepsilon} (\nabla_y \Phi)^\varepsilon \]
\[ \text{div} \Phi^\varepsilon = (\text{div}_x \Phi)^\varepsilon + \frac{1}{\varepsilon} (\text{div}_y \Phi)^\varepsilon . \]

Having made this point, take now in (4.15) the particular test function \( \mathbf{v} = \Phi^\varepsilon(t) \).

This yields
\[ \int_Q \frac{\partial \mathbf{u}_\varepsilon}{\partial t} \Phi^\varepsilon \, dx \, dt + \nu \int_Q \nabla \mathbf{u}_\varepsilon \cdot (\nabla_x \Phi)^\varepsilon \, dx \, dt + \nu \int_Q \frac{1}{\varepsilon} \nabla \mathbf{u}_\varepsilon \cdot (\nabla_y \Phi)^\varepsilon \, dx \, dt + \int_0^T b(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t)) \, dt - \int_Q \mathbf{p}_\varepsilon \cdot (\text{div}_x \Phi)^\varepsilon \, dx \, dt = \int_Q \mathbf{f} \cdot \Phi^\varepsilon \, dx \, dt. \]

The aim is to pass to the limit as \( E' \ni \varepsilon \to 0 \). In view of the inequalities in (4.4), we clearly have
\[ \int_Q \nabla \mathbf{u}_\varepsilon \cdot (\nabla_x \Phi)^\varepsilon \, dx \, dt \to 0 \]
and \( \int_Q \frac{\partial \mathbf{u}_\varepsilon}{\partial t} \Phi^\varepsilon \, dx \, dt \to 0 \)
as \( 0 < \varepsilon \to 0 \). On the other hand, using the equality
\[ b(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t)) = -b(\mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t), \mathbf{u}_\varepsilon(t)) \]
and (2.7) (of Lemma 2.1), we have
\[ \left| \int_0^T b(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t)) \, dt \right| \leq 2 \varepsilon \| \mathbf{u}_\varepsilon \|_{L^\infty(0,T; L^2(\Omega))} \int_0^T \| \nabla \mathbf{u}_\varepsilon(t) \|_{L^2(\Omega)} \| \nabla \Phi^\varepsilon(t) \|_{L^2(\Omega)} \, dt \]
\[ \leq 2 \varepsilon \| \mathbf{u}_\varepsilon \|_{L^\infty(0,T; L^2(\Omega))} \| \nabla \mathbf{u}_\varepsilon \|_{L^2(\Omega)} \| \nabla \Phi^\varepsilon \|_{L^2(\Omega)} . \]

Hence, by (4.4) and (4.17) we get
\[ \left| \int_0^T b(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t)) \, dt \right| \leq K \varepsilon^2 \| \nabla \Phi^\varepsilon \|_{L^2(\Omega)} \]
where \( K > 0 \) is a constant independent of \( \varepsilon \). But
\[ \sup_{0 \leq \varepsilon \leq 1} \varepsilon \| \nabla \Phi^\varepsilon \|_{L^2(\Omega)} < \infty . \]

Therefore, as \( 0 < \varepsilon \to 0 \)
\[ \int_0^T b(\mathbf{u}_\varepsilon(t), \mathbf{u}_\varepsilon(t), \Phi^\varepsilon(t)) \, dt \to 0 . \]

Finally, by (4.23)-(4.25), one quickly arrives at
\[ (4.33) \quad \nu \int_Q \int_{Y \times Z} \nabla_y \mathbf{u}_0 \cdot \nabla_y \Phi^\varepsilon \, dx \, dy \, dt - \int_Q p_0 \text{div} \Phi^\varepsilon \, dx \, dt = \int_Q \mathbf{f} \cdot \Phi^\varepsilon \, dx \, dt \]
and that for any \( \Phi \in \mathcal{D}(Q) \otimes C^\infty_{\text{per}}(Z) \otimes W(Y) \). Recalling that \( \mathcal{D}(Q) \otimes C^\infty_{\text{per}}(Z) \otimes W(Y) \) is dense in \( \mathcal{H}(Q; L^2_{\text{per}}(Z; W(Y))) \) (see [20], (1.33)) holds for all \( \Phi \in \mathcal{H}(Q; L^2_{\text{per}}(Z; W(Y))) \). Therefore, the variational equation in (4.23) follows at once by taking in particular \( \Phi = \mathbf{v} \) with \( \mathbf{v} \in \mathcal{H}_0(Q; L^2_{\text{per}}(Z; W(Y))) \). Thus, the proof is complete once (4.27) is established. To achieve this, let \( 1 \leq j \leq N \) be
arbitrary fixed. Take in (4.33) $\Phi = \varphi \otimes \chi_j$, where $\chi_j$ is defined by (4.21) and $\varphi$ is freely fixed in $\mathcal{D}(Q)$. Then it is an easy matter to arrive at

$$\nu \int \int_{Y \times Z} \nabla_y u_0(x,t) \cdot \nabla_y \chi_j dy + \sum_{i=1}^{N} K_{ij} \left( \frac{\partial p_0}{\partial x_i}(x,t) - f_i(x,t) \right) = 0$$

\[ a.e. \text{ in } (x,t) \in Q. \]

But

$$\int \int_{Y \times Z} \nabla_y u_0(x,t) \cdot \nabla_y \chi_j dy = \tilde{u}_0(x,t)$$

as is immediate by taking in (4.21) $w = u_0(x,t)$ for fixed $(x,t) \in Q$. Hence (4.27) follows. The theorem is proved. $\square$

Conclusion. In our study, we have been limited in spatial dimension $N = 2$. It would be interesting to investigate the case $N = 3$, customary used in physics. Unfortunately, we come up against the lack of uniqueness for Non-stationary Navier-Stokes equations in dimension $N \geq 3$. Moreover, for flows in porous media, we have been interested uniquely for the periodic case, the problem beyond the periodic setting being not only to be formulated mathematically, but to be justified by physics.

However, one convergence theorem has been proved for each problem, and we have derived the macroscopic homogenized model.

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University of Ngaoundere, Department of Mathematics and Computer Science, P.O.Box 454 Ngaoundere (Cameroon)
E-mail address: lsigning@uy1.uninet.cm