Conditions for the confirmation of three-particle non-locality

Peter Mitchell¹, Sandu Popescu¹,² and David Roberts¹

¹H.H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, UK
²Hewlett-Packard Laboratories, Stoke Gifford, Bristol BS12 6QZ, UK

(19 Nov 2001)

The notion of genuine three-particle non-locality introduced by Svetlichny ¹ is discussed. Svetlichny’s inequality which can distinguish between genuine three-particle non-locality and two-particle non-locality is analyzed by reinterpreting it as a frustrated network of correlations. Its quantum mechanical maximum violation is derived and a situation is presented that produces the maximum violation. It is shown that the measurements performed in recent experiments to demonstrate GHZ entanglement ², ³ do not allow this inequality to be violated, and hence can not be taken as confirmation of genuine three-particle non-locality. Modifications to the experiments that would make such a confirmation possible are discussed.

I. INTRODUCTION

Generalized Bell inequalities have been reported for N-particle systems which show that quantum mechanics violates local realism in these situations ⁴ ⁵ ⁶. However such results are insufficient to show that all of the particles in a system are acting non-locally - it is possible to imagine a non-local many-particle system as consisting of a finite number of non-local sub-systems, but with only local correlations present between these sub-systems. For example a state of three particles |Ψ⟩₁₂₃ which can be decomposed as |ψ⟩₁₂ |φ⟩₃ only exhibits non-local correlations between particles 2 and 3. Similarly, a density matrix ρ₁₂₃ which is a mixture of states of the form |ψ⟩₁₂ |φ⟩₃, |η⟩₂ |ξ⟩₁₃ and |χ⟩₁ |θ⟩₁₂ contains only two particle non-locality (though it might be very difficult to show this if only the density matrix is given but not the explicit decomposition). Suppose however that |Ψ⟩₁₂₃ can not be decomposed - does this necessarily imply that it has three particle non-locality? This question was first raised by G. Svetlichny ¹. More precisely, Svetlichny asked the following: We know that the correlations between the results of measurements performed on triplets of particles in the state |Ψ⟩₁₂₃ cannot be described by local hidden variables. Could they however be described by a hybrid local - nonlocal system, in which non-local correlations are present only between two particles (which two particles are nonlocally correlated can change in different runs of the experiment) while they are only locally correlated with the third? If “yes” than although |Ψ⟩₁₂₃ can not be decomposed as a direct product of one particle versus a (possible entangled) state of the other two, the nonlocality exhibited by this state is still only two particle nonlocality.

Formally Svetlichny’s model is the following. Let \( P(A = a, B = b, C = c) \) be the probability for obtaining a results \( A = a, B = b \) and \( C = c \) when observable \( A \) is measured on the first particle, \( B \) on the second and \( C \) on the third. In a local hidden variables model each particle in the triplet is endowed at source with the same hidden variable \( λ \) and later, when subjected to measurements, each particle behaves independently of the others, taking into account only the value of the hidden variable and the measurement to which itself is subjected, but not to what measurements the other particles were subjected and/or the results they yield. Hence, \( P(A = a, B = b, C = c) \) can be expressed as

\[
P(A = a, B = b, C = c)_{\text{local}} = \int \rho(λ) dλ P_1(A = a|λ) P_2(B = b|λ) P_3(C = c|λ),
\]

where \( ρ(λ) \) describes the probability that the hidden variable has a particular value \( λ \). As it is well-known no such local hidden variables model can account for the correlations generated by entangled states.

In the hybrid local-nonlocal hidden variables model considered by Svetlichny, \( P(A = a, B = b, C = c)_{SV} \) is given by:

\[
P(A = a, B = b, C = c)_{SV} = q_{12} \int ρ_{12}(λ) dλ P_{1,2}(A = a, B = b|λ) P_3(C = c|λ)
+ q_{23} \int ρ_{23}(λ) dλ P_{2,3}(B = b, C = c|λ) P_1(A = a|λ)
+ q_{13} \int ρ_{13}(λ) dλ P_{1,3}(A = a, C = c|λ) P_2(B = b|λ),
\]
subject to $q_{12} + q_{23} + q_{13} = 1$ and $\int \rho_{i,j}(\lambda) d\lambda = 1$.

Thus when repeated measurements are performed on an ensemble, the three terms in equation 2 correspond to the three possible factorizations of two particle non-locality between the three particles, (1,2)-3, (2,3)-1 and (1,3)-2, with $q_{12}, q_{23}$ and $q_{13}$ the probabilities of each particular factorization being present.

Svetlichny derived an inequality which is obeyed by all such hybrid local - two-particle nonlocal models, and showed that some quantum states violate the inequality, hence they are genuinely three-particle nonlocal.

In this paper Svetlichny’s inequality is first given a novel interpretation as a frustrated network of correlations. It is hoped that such an interpretation will give physical insight into the subsequent discussions. We then derive the maximal possible violation of Svetlichny’s inequality and a quantum state is then presented which violates it maximally. Finally we discuss the experimental status of the verification of genuine 3-particle non-locality, and suggest simple modifications to the recent experiments by D. Bouwmeester et al. [2] and Pan et al. [3] which may make such a verification possible.

II. INTERPRETING GENERALIZED BELL INEQUALITIES AS FRUSTRATED NETWORKS

Bell-type inequalities are generally expressed in terms of the expectation values of observables. In this section it is shown how it is possible to interpret generalized Bell inequalities as frustrated networks of correlations, and accordingly we present an alternative derivation of Svetlichny’s inequality based upon such an interpretation. (In fact all presently known Bell type inequalities can be described in such a way \[7\], and this leads to a better understanding of their physical meaning.)

Consider a situation of three spatially separated two dimensional systems. System A is subject to one of the measurements $A$ or $A'$, and similarly for systems B and C. The result of any measurement is labeled $\pm 1$. Suppose $A$, $B$ and $C$ have been measured. Since the outcomes $a$, $b$ and $c$ can only be equal to $\pm 1$, we have only two possibilities: either $a = bc$ or $a = -bc$; we refer to the two cases as A being correlated to BC or anti-correlated to BC. Furthermore, when $a = bc$ it is also the case that $b = ac$ and $c = ab$ thus we can talk about correlation without mentioning explicitly between which partitions; similarly for anti-correlation. Define the probability of correlation, $P_c(ABC)$ as the probability that $A$, $B$ and $C$ are correlated, and $P_a(ABC)$ as the probability that they are anti-correlated. Now consider the expression

$$S = P_a(ABC) + P_a(ABC') + P_a(A'BC) + P_a(A'B'C') + P_a(AB'C') + P_a(A'B'C') + P_a(A'B'C')$$  \(3\)

Suppose initially that limited non-locality takes the form that particles A and B form a non-local subsystem AB and that this subsystem is locally correlated with particle C. The other possible factorizations of the system, A-BC and B-AC, will be considered later. Then equation (3) corresponds to the network shown below:

Figure 1

Recall that in our interpretation of Svetlichny’s inequality non-locality between A and B means these particles are regarded as a composite system. Hence the outcomes for the paired measurements $AB, AB', A'B$ and $A'B'$ are completely unconstrained from each other. Furthermore, locality of C versus AB means that for any local hidden variable model the choice of which of the paired measurements $AB, AB', A'B$ and $A'B'$ to make is independent of whether $C$ or $C'$ is measured.
In the most general hidden variable model that can be considered, for each value of the hidden variable $\lambda$, the measurements can yield different outcomes, according to the associated probabilities, such as $P(A = a|\lambda)$. However, it can be easily shown that any such model can be re-cast into a deterministic model in which for each value of $\lambda$ the outcomes are completely determined, i.e. the probabilities of obtaining each of the possible measurements is either 0 or 1. In particular, for each value of $\lambda$ we have a given, well-defined assignment of $\pm 1$ values for $ab, ab', a'b, a'b', c$ and $c'$, and the probabilities of correlation and anticorrelation are either 0 or 1.

Referring to equation (3), one can easily check that for no assignment of $\pm 1$ values for the results of measurements can all the eight probabilities be equal to 1, nor can all of them be equal to 0. In fact at least two of the bonds in figure 1 must be satisfied by any combination of $\pm 1$ at the vertices, and only a maximum of six out of the total of eight bonds may ever be satisfied. Hence the network is frustrated (in other words not all links can be simultaneously satisfied) and for every value of $\lambda$, $2 \leq S \leq 6$. Furthermore, since the the inequality holds for every value of $\lambda$, it also holds for the average.

As a last step, due to the symmetry under permutation of particles, the same inequality holds for all 2 versus 1 partitions, and thus for the grand average over all possible partitions and all assignments of the hidden variable. While the values of $S$ for given different partitions or particular values of the hidden variable are not accessible experimentally - indeed, when performing the measurements on an ensemble of triplets of particles we don’t know what is the hidden variable or the partition of a particular triplet, the grand average is experimentally observable. This is Svetlichny’s inequality, in a slightly different form than originally proposed.

The original inequality reads

$$S_v = |E(ABC) + E(ABC') + E(A'BC) - E(A'BC') + E(AB'C) - E(AB'C') - E(A'B'C) - E(A'B'C')| \leq 4,$$  

(4)

where $E(ABC)$ represents the expectation value of the product $ABC$. This form of the inequality can be easily deduced from eq.(3) and the above established bounds for $S$, by noting that the expectation values are related to the correlation/anti-correlation probabilities by

$$E = 2P_e - 1 = 1 - 2P_a$$  

(5)

### III. THE PREDICTIONS OF QUANTUM MECHANICS FOR THREE-BODY SYSTEMS

We now derive the maximum possible quantum mechanical violation of Svetlichny’s inequality and show a particular case in which the inequality is maximally violated.

It is possible to show that $S_v = 4\sqrt{2}$ is the maximum possible quantum mechanical violation of Svetlichny’s inequality; this is the equivalent of Cire’son’s bound for the CHSH inequality. For a state $|\psi\rangle$, $S_v$ can be written as:

$$S_v = |\langle\psi| AB(C + C') |\psi\rangle + \langle\psi| AB'(C - C') |\psi\rangle + \langle\psi| A'B(C - C') |\psi\rangle + \langle\psi| A'B'(-C - C') |\psi\rangle|,$$  

(6)

by replacing in (3) the expectation values by their quantum expression and grouping the terms. Using Schwarz’ inequality we can bound the magnitude of each term:

$$|\langle\psi| AB(C + C') |\psi\rangle| \leq \sqrt{|\langle\psi| ABAB |\psi\rangle||\langle\psi| (C + C')(C + C') |\psi\rangle|}$$  

(7)

$$\leq \sqrt{2 + \langle\psi| CC' + C'C |\psi\rangle},$$  

(8)

where the last inequality obtains since $\langle\psi| ABAB |\psi\rangle = \langle\psi| CC |\psi\rangle = \langle\psi| C'C' |\psi\rangle = 1$. Similar results are found for the other three terms. If we now let $x = \langle\psi| CC' + C'C |\psi\rangle$, then

$$|S_v| \leq 2(\sqrt{2 + x}) + 2(\sqrt{2 - x})$$  

(9)

Thus $|S_v| \leq 4\sqrt{2}$ with the maximum absolute value being attained at $x = 0$.

For a GHZ state of three spin 1/2 particles $|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle)$, where $\uparrow$ and $\downarrow$ represent spins polarized “up” or “down” along the z axis, Svetlichny’s inequality is violated if, for example, measurements are made in the xy plane along some appropriate directions. In this case $E(ABC) = \langle\psi| \overset{\rightarrow}{\sigma} \otimes \overset{\rightarrow}{\sigma} \otimes \overset{\rightarrow}{\sigma} |\psi\rangle = -\cos(\alpha + \beta - \gamma)$, where we labeled the angles from the $x$ axis. The inequality will be violated by choosing $\alpha = 0, \alpha' = \frac{\pi}{2}, \beta = \frac{\pi}{4}, \beta' = \frac{\pi}{4}, \gamma = 0, \gamma' = \frac{\pi}{2}$. Then $S_v = 4\sqrt{2}$.  

3
IV. EXPERIMENTS

Experiments to produce and analyze 3-particle entangled states are far more difficult than those on 2-particle entangled states which are now routinely performed. In fact the very first such experiments have only very recently been performed. Unfortunately although the beautiful work of Svetlichny is now more than a decade old, the notion of genuine 3-particle nonlocality which it introduced has not been widely known and the experiments on 3-particle entanglement have not been specifically designed to verify the existence of such correlations. In this section we revisit the experiments of Bouwmeester et al. [2] and Pan et al. [3], two of the first experiments to test 3-particle entanglement. We show that the particular measurements performed in these experiments are such that they do not produce (according to quantum mechanics) any violations of Svetlichny’s inequality, so that they cannot be used for the verification of the existence of genuine three-particle non-locality, (although they prove 3-party entanglement). Ironically, history repeats itself. The pre 1964 measurements performed in order to establish the existence of entanglement, though able to confirm entanglement, turned out to be precisely those that were not appropriate for testing Bell’s inequalities!

The two experiments described in [2] and [3] use essentially the same experimental set-up to produce the three-photon entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|HHV\rangle - |VVH\rangle)$. Here H represents horizontal polarization and V vertical polarization. To verify that indeed such a GHZ state had been produced, different tests were made.

It is simpler to represent the state in the z basis writing $|H\rangle = |\uparrow\rangle$ and $|V\rangle = |\downarrow\rangle$. Then $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle)$. In [2] measurements (of the optical equivalent) of spin in the z and x directions were performed. In the subsequent experiment [3] measurements along z, x and y were performed. Unfortunately, as it is straightforward to check, measurements along x,y and z do not lead to Svetlichny inequality violations for the GHZ state, so the analysis of the data already obtained cannot prove the existence of genuine 3-party correlations. On the other hand, it is easy to modify the experiments so as to produce a maximum violation of Svetlichny’s inequality. It is sufficient to make measurements in the xy plane using the angles listed above in section three. Where to measure an angle $\theta$ in this plane it is necessary to perform a measurement which has eigenvectors $\frac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle)$ and $\frac{1}{\sqrt{2}}(|\uparrow\rangle - e^{i\theta}|\downarrow\rangle)$, that is $\frac{1}{\sqrt{2}}(|H\rangle + e^{i\theta}|V\rangle)$ and $\frac{1}{\sqrt{2}}(|H\rangle - e^{i\theta}|V\rangle)$. It should then be possible to confirm that the state produced demonstrates genuine three-particle non-locality.

Acknowledgments We would like to thank D. Bouwmeester, D. Collins, N. Gisin, A. Kent and N. Linden for very useful discussions.

[1] G. Svetlichny, Phys Rev D 35, 10, 3066 (1987).
[2] D. Bouwmeester, J.-W. Pan, M. Daniell, H. Weinfurther, and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999).
[3] J.-W. Pan, D.Bouwmeester, M. Daniell, H. Weinfurther, and A. Zeilinger, Nature 403, 515 (2000).
[4] S. Popescu, D. Rohrlich, Phys. Lett. A, 166, 293 (1992).
[5] N. Gisin, H. Bechmann-Pasquinucci, Phys.Lett. A246, 1 (1998).
[6] N D. Mermin. Phys Rev Lett 65, 1838 (1990).
[7] S. Popescu et al., in preparation.

More precisely, the measurements in [2] and [3] do not allow testing of genuine 3-particle nonlocality via Svetlichny’s original inequality. We don’t know whether these particular measurements could violate some other similar inequality, or they can be modeled by a limited 2-particle nonlocal model, and hence they are useless.