On volumes of classical supermanifolds

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Abstract. We consider the volumes of classical supermanifolds (such as the supersphere, complex projective superspace, Stiefel and Grassmann supermanifolds) with respect to natural metrics or symplectic structures. We show that the formulae for the volumes of these supermanifolds can be obtained from the formulae for the volumes of the corresponding ordinary manifolds (under some universal normalization of the volume) by analytic continuation with respect to parameters.

The volumes of nontrivial supermanifolds may be identically equal to zero. In the 1970s Berezin showed that the total Haar measure of the unitary supergroup $U(n|m)$ vanishes except in the cases $m = 0$ and $n = 0$, when the supergroup is the ordinary unitary group $U(n)$ or $U(m)$. Some time ago Witten conjectured that the Liouville volume of a compact even symplectic supermanifold is always equal to zero (except for ordinary manifolds). We give counterexamples to this conjecture, present a simple explanation of Berezin’s theorem, and generalize this theorem to the Stiefel supermanifold $V_{r|s}(C^{n|m})$. We mention a connection with recent work of Mkrtchyan and Veselov on universal formulae in Lie algebra theory.

Bibliography: 32 titles.

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§1. Introduction

The notion of volume can acquire unexpected features in the supercase. For example, the total volume of a nontrivial supermanifold may be equal to zero. This is because the Berezin integral, which underlies the notion of volume, is not related to sets and measures in their usual sense, thus making considerations based on the positivity of measure inapplicable. A striking example is the superanalogue of the unitary group: the total ‘Haar measure’ of the unitary supergroup $U(n|m)$ vanishes if $nm > 0$ (see [1], [2], as also [3], Pt. II, Ch. 5). (This fact has consequences for the theory of representations of $U(n|m)$. Since one cannot divide by zero, as well as by infinity, it becomes impossible to average, and this makes supergroups ‘noncompact’ in some sense. The theory of integration on supermanifolds also shows that the presence of odd variables is an analogue of noncompactness: see [4].)

In the autumn of 2012, Witten asked the author whether the Liouville volume of any compact symplectic supermanifold (with an even symplectic form)
always vanishes except in the case of ordinary manifolds. Despite the deep reasons that led Witten to his conjecture, the author quickly came up with a counterexample: the superanalogue of a complex projective space endowed with an analogue of the classical Fubini-Study form. The simplest case when the Liouville volume of the projective superspace $\mathbb{CP}^{n|m}$ does not vanish is the superspace $\mathbb{CP}^{1|1}$ with $\text{vol}(\mathbb{CP}^{1|1}) = 2\pi$ (the volume here has the noteworthy property of being scale-invariant)$^1$.

The study of volumes of complex projective superspaces led the author to observations that seem to be of independent interest. Namely, the explicit formula for the volume of $\mathbb{CP}^{n|m}$ turned out to be an analytic continuation of the corresponding formula for the ordinary complex projective space $\mathbb{CP}^n$ (up to a universal normalizing factor which depends only on the dimension). This encouraged us to study more examples. It turned out that this fact (on analytic continuation) holds for other classical manifolds such as superspheres and complex Stiefel and Grassmann supermanifolds. The volume formulae for these manifolds can be obtained using simple geometric arguments. In particular, in the course of our work we obtained a simple explanation of Berezin’s theorem (mentioned above) on the supergroup $\text{U}(n|m)$ and generalized this theorem to Stiefel supermanifolds (whose volume need not be identically equal to zero).

The possibility of expressing the volumes of classical manifolds and supermanifolds by analytic functions, and the structure of these functions, are quite remarkable. It is also interesting to see a contact with recent studies of Mkrtchyan and Veselov devoted to ‘universal formulae’ in Lie algebra theory: see [6]–[9]. (These papers develop, on the one hand, the physical idea of ‘$(N \rightarrow -N)$-symmetry’ (see, for example, [10]) and, on the other hand, the philosophy of the ‘universal Lie algebra’ going back to Vogel and Deligne (see, for example, [11]).) In particular, the papers cited contain a ‘universal formula’ for the volume of a compact Lie group in terms of the Vogel parameters $\alpha$, $\beta$, $\gamma$. Related topics are discussed in §4; see also [12] by Khudaverdian and Mkrtchyan.

The analytic functions expressing the ‘normalized volumes’ depend on complex variables having the geometric meaning of ‘indices’ (that is, numbers of the form $n - m$ in superdimension $n|m$). This suggests a possible nontrivial infinite-dimensional generalization, provided that the ‘indices’ are expressed as genuine indices of elliptic operators; then one can still make sense of these volumes.

While preparing this paper for publication, the author came across the paper [13], where the volumes of many compact manifolds are collected or newly calculated. (Concrete values of these volumes are important in various physical applications.) It would be interesting to compare the formulae in [13] with ours and perform their ‘superization’. (We note that formulae from different sources may differ by the choice of the initial normalization.)

The paper is structured as follows. In §2 we study the volumes of superspheres and complex projective superspaces. In §3 we consider the complex Stiefel supermanifolds, unitary supergroups and Grassmann supermanifolds. The results are

$^1$After a thorough search of literature, we found that the answer for the volume of $\mathbb{CP}^{n|m}$ (which differs by a factor from our formula) was already known (see [5]), but apparently not widely known, otherwise Witten’s question would not have been raised.
briefly discussed in § 4. In § 5 and § 6 we present some background information on supermanifolds, Berezin integrals, volume forms and Riemannian submersions.

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§ 2. Superspheres and complex projective superspaces

2.1. The supersphere and its volume. To define the supersphere $S^{n|2m}$, we consider the superspace $\mathbb{R}^{n+1|2m}$ as a vector supermanifold and endow it with a scalar product such that

$$ (x, x) = (x_1)^2 + \cdots + (x_{n+1})^2 + 2\xi_1\eta_1 + \cdots + 2\xi_m\eta_m $$

in the standard coordinates $x^1, \ldots, x^{n+1}, \xi^1, \eta^1, \ldots, \xi^m, \eta^m$. (We need an even number of odd coordinates to guarantee that the scalar product is nondegenerate.) Then, as usual, the supersphere of radius $R$ is the submanifold $S^{n|2m}_R \subset \mathbb{R}^{n+1|2m}$ of all vectors of length $R$. It is given by the equation

$$ (x_1)^2 + \cdots + (x_{n+1})^2 + 2\xi_1\eta_1 + \cdots + 2\xi_m\eta_m = R^2. \quad (2.1) $$

When $R = 1$, we do not indicate the radius.

The Euclidean metric on $\mathbb{R}^{n+1|2m}$,

$$ \delta s^2 = (\delta x^1)^2 + \cdots + (\delta x^{n+1})^2 + 2\delta\xi^1\delta\eta^1 + \cdots + 2\delta\xi^m\delta\eta^m, $$

induces a Riemannian metric on the submanifold $S^{n|2m}_R$. This gives rise to a volume element. When speaking about the volume of a supersphere, we mean its total volume with respect to this Riemannian volume element. (The tensor notation in the supercase, including the use of $\delta s$ instead of $ds$ in the formula for the metric, is explained in § 5.)

**Theorem 1.** The volume of the supersphere of radius $R$ is equal to

$$ \text{vol}(S^{n|2m}_R) = 2R^{n-2m-2} \frac{\pi^{(n+1)/2}2^m}{\Gamma\left(\frac{n+1}{2} - m\right)}. \quad (2.2) $$

The formula (2.2) can be obtained from (2.1) by calculating an integral with the delta function. Alternatively, it can be obtained using a Gaussian integral (as for the ordinary sphere).

The formula for the volume of a supersphere was probably established many times by different authors. In particular, it was deduced, among other things, in the Ph.D. thesis [14] of Haunch, who was then a Ph.D. student of Khudaverdian and the author. For $m = 0$ we obtain the familiar formula for the ordinary sphere. It is convenient to consider the cases $n = 2k$ and $n = 2k + 1$ separately (as for the ordinary sphere). We have the following corollary (in slightly changed notation).
Corollary 1. The following formulae hold:
\[
\text{vol}(S^2_R|^{2m}) = 2R^{2n-2m} \frac{\pi^{n+1/2} 2^m}{\Gamma(n - m + \frac{1}{2})}, \tag{2.3}
\]
\[
\text{vol}(S^{2n+1}_R|^{2m}) = 2R^{2n+1-2m} \frac{\pi^{n+1} 2^m}{\Gamma(n - m + 1)}. \tag{2.4}
\]

Recall that \(\Gamma(z)\) has poles at the nonpositive integers \(z = 0, -1, -2, \ldots\) and only there. It follows that the volume of the ‘odd-dimensional’ supersphere \(S^{2n+1}_R|^{2m}\) vanishes for \(m > n\), and when it is nonzero, it can be written in the form
\[
\text{vol}(S^{2n+1}_R|^{2m}) = 2R^{2n+1-2m} \frac{\pi^{n+1} 2^m}{(n - m)!}. \tag{2.5}
\]

On the contrary, the volume of the ‘even-dimensional’ supersphere \(S^{2n}_R|^{2m}\) never vanishes (and is expressed in terms of double factorials, as in the ordinary case).

2.2. Background about \(\mathbb{C}P^{n|m}\). Metric, symplectic structure and volume element. We recall the superanalogues of well-known constructions for the ordinary complex projective space. As in the ordinary case, the complex projective superspace \(\mathbb{C}P^{n|m}\) is defined as the quotient of the space of all nonzero (even) vectors in \(\mathbb{C}^{n+1}|{m}\) by the action of \(\mathbb{C}^* = \mathbb{C}\setminus\{0\}\).

Let \(z^a, \zeta^\mu\) be the standard coordinates in \(\mathbb{C}^{n+1}|{m}\). They are also homogeneous coordinates on \(\mathbb{C}P^{n|m} := \mathbb{C}^{n+1}|{m}\}/\mathbb{C}^*\). For each fixed \(k\) we can regard the variables \(w^a(k) := z^a/z^k\) and \(\theta^\mu(k) := \zeta^\mu/z^k\) as coordinates on the \(k\)th affine chart of the projective superspace \(\mathbb{C}P^{n|m}\). They are referred to as inhomogeneous coordinates in \(\mathbb{C}P^{n|m}\). (Altogether there are \(n+1\) charts. In what follows we do not indicate the index \(k\).)

One can identify \(\mathbb{C}^{n+1}|{m}\) with \(\mathbb{R}^{2n+2|m}\) by putting \(z^a = x^a + iy^a\) and \(\zeta^\mu = \xi^\mu + i\eta^\mu\), where the variables \(x^a, y^a, \xi^\mu, \eta^\mu\) are assumed to be real. The standard Euclidean metric on \(\mathbb{C}^{n+1}|{m}\),
\[
\delta s^2 = \sum \delta z^a \delta \bar{z}^a + i \sum \delta \zeta^\mu \delta \bar{\zeta}^\mu = \sum ((\delta x^a)^2 + (\delta y^a)^2) + 2 \sum \delta \xi^\mu \delta \eta^\mu,
\]
is the real part of the Hermitian form
\[
H = \sum \delta z^a \otimes \delta \bar{z}^a + i \sum \delta \zeta^\mu \otimes \delta \bar{\zeta}^\mu.
\]

Up to a factor, the imaginary part of \(H\) is the standard symplectic form on \(\mathbb{C}^{n+1}|{m}\):
\[
\omega_0 = -\frac{1}{2} \text{Im} H = \frac{i}{2} \left( \sum \delta z^a \delta \bar{z}^a - i \sum \delta \zeta^\mu \delta \bar{\zeta}^\mu \right)
= \sum dx^a dy^a + \frac{1}{2} \sum ((d\xi^\mu)^2 + (d\eta^\mu)^2).
\]

(As already mentioned, the difference in notation, for example, between \(dz^a\) and \(\delta z^a\), is explained in § 5.) Notice that
\[
\omega_0 = \frac{i}{2} \partial \bar{\partial} N_0,
\]
where
\[ N_0 = \sum z^a \overline{z}^a + i \sum \zeta^\mu \overline{\zeta}^\mu \]
is the scalar square of the radius vector. (The symbol \( \partial \) stands for the holomorphic part of the exterior differential: \( d = \partial + \overline{\partial} \).)

The symplectic form \( \omega_0 \) does not descend to \( \mathbb{CP}^{n|m} \), but its restriction to the supersphere \( S^{2n+1|2m}_R \subset \mathbb{CP}^{n|m} \) does. We have an analogue of the familiar diagram

\[
\begin{array}{c}
S^{2n+1|2m}_R \\
\downarrow \quad p \\
\mathbb{C}^{n+1|m} \setminus \{0\} \\
\downarrow q \\
\mathbb{CP}^{n|m}
\end{array}
\]

The diagonal arrow is a superanalogue of the Hopf fibration. The projection is the quotient over the action of \( U(1) \). The equation of \( S^{2n+1|2m}_R \) in the complex coordinates is
\[
\sum z^a \overline{z}^a + i \sum \zeta^\mu \overline{\zeta}^\mu = R^2.
\]

The following proposition gives an analogue of the Fubini-Study form for \( \mathbb{CP}^{n|m} \).

**Proposition 1.** The form \( i^* \omega_0 \) on the supersphere \( S^{2n+1|2m}_R \) is invariant under the action of the supergroup \( \mathbb{U}(1) = \Pi U(1) \). Hence there exists a unique form \( \omega \in \Omega^2(\mathbb{CP}^{n|m}) \) (the Fubini-Study form) such that
\[
i^* \omega_0 = p^* \omega.
\]

The form \( \omega \) determines a symplectic structure on \( \mathbb{CP}^{n|m} \). In local coordinates it is given by
\[
\omega = \frac{i}{2} R^2 \frac{(dw \cdot d\overline{w} - i d\theta \cdot d\overline{\theta})N - (dw \cdot \overline{w} + i d\theta \cdot \overline{\theta})(d\overline{w} \cdot w - i d\overline{\theta} \cdot \theta)}{N^2},
\]
where \( N = w \cdot \overline{w} + i \theta \cdot \overline{\theta} \). We sometimes abbreviate sums of the form \( \sum dw^a d\overline{w}^a \) to \( dw \cdot d\overline{w} \), that is, \( dw \cdot d\overline{w} = \sum dw^a d\overline{w}^a \). It is also convenient to include one ‘fake’ inhomogeneous coordinate which is identically equal to 1, so that one of the terms in \( N \) is equal to 1. The expression for \( \omega \) in the homogeneous coordinates (or, equivalently, the pullback \( q^* \omega \) to \( \mathbb{C}^{n+1|m} \setminus \{0\} \)) takes a similar form:
\[
q^* \omega = \frac{i}{2} R^2 \frac{(dz \cdot d\overline{z} - i d\zeta \cdot d\overline{\zeta})N_0 - (dz \cdot \overline{z} + i d\zeta \cdot \overline{\zeta})(d\overline{z} \cdot z - i d\overline{\zeta} \cdot \zeta)}{N_0^2},
\]
where \( N_0 = z \cdot \overline{z} + i \zeta \cdot \overline{\zeta} \). One can also express \( \omega \) in terms of the Kähler potential:
\[
\omega = \frac{i}{2} R^2 \bar{\partial} \bar{\partial} \ln N, \quad q^* \omega = \frac{i}{2} R^2 \bar{\partial} \bar{\partial} \ln N_0.
\]

By an abuse of notation we refer to the symplectic form \( \omega \) and the corresponding Riemannian metric on \( \mathbb{CP}^{n|m} \) as the Fubini-Study form. Thus the construction is
completely analogous to the ordinary case. (The definition of the Fubini-Study metric usually does not include the factor $R$, but it is convenient for us to have a scaling factor. When we wish to stress the presence of $R$, we write it as a subscript.)

The following volume element on $\mathbb{CP}^{n|m}$ is simultaneously the Riemannian volume element (generated by the metric) and the Liouville volume element (generated by the symplectic structure).

**Proposition 2.** In the local coordinates $w^a, w^\alpha, \theta^\mu, \bar{\theta}^\mu$, the volume element corresponding to the Fubini-Study form is

$$dV_{\mathbb{CP}^{n|m}} = R^{2(n-m)} \left( \frac{i}{2} \right)^{n-m} \frac{dw^1 d\bar{w}^1, \ldots, dw^n d\bar{w}^n | d\theta^1 d\bar{\theta}^1, \ldots, d\theta^m d\bar{\theta}^m}{N^{n-m+1}}.$$  \hspace{1cm} (2.9)

(The meaning of the bracket notation is explained in §5.)

To deduce (2.9) one must find the Berezinian of the Hermitian matrix corresponding to the Fubini-Study form. We omit the details of this calculation, but mention that it is convenient to use a shorthand notation where the odd and even variables are denoted by the same letter with odd or even indices respectively. We have

$$\omega = \frac{i}{2} R^2 \left( N^{-1} dw^a h_{\bar{a} b} d\bar{w}^b - N^{-2} dw^a h_{\bar{a} \bar{b}} w^b (1) \bar{w}^p h_{p \bar{q}} d\bar{w}^q \right)$$

$$= \frac{i}{2} R^2 dw^a N^{-1} h_{a \bar{a}} \left( \delta_{\bar{a}}^b - N^{-1} \bar{w}^q (1) \bar{w}^p h_{p b} \right) d\bar{w}^b,$$

where $h_{a \bar{a}}$ stands for the matrix of the standard Hermitian form in $\mathbb{C}^{n|m}$ and $N = 1 + w \bar{w}$. (Here we do not include the ‘fake’ coordinate.) Hence the desired Berezinians is the product of two Berezinians. Then we apply the useful identity

$$\text{Ber}(1 - uv) = 1 - vu,$$

where $u$ and $v$ are an even column vector and an even row vector respectively. In our case this yields

$$\text{Ber} \left( \delta_{\bar{a}}^b - N^{-1} \bar{w}^q (1) \bar{w}^p h_{p b} \right) = 1 - N^{-1} w \bar{w} = N^{-1} (N - w \bar{w}) = N^{-1}.$$

Together with $\text{Ber}(N^{-1} h_{a \bar{a}})$ (where $n|m$ is the size of the matrix), this gives the denominator $N^{n-m+1}$ in (2.9).

Our next task is to calculate the total volume of $\mathbb{CP}^{n|m}$ with respect to the volume element (2.9). To do this, we establish a correspondence between the volume elements on $\mathbb{CP}^{n|m}$ and on the supersphere.

**2.3. The volume elements for $S^{2n+1|2m}_R$ and $\mathbb{CP}^{n|m}_R$.** To find the desired relation we endow $\mathbb{C}^{n+1|m} \setminus \{0\}$ with an atlas consisting of the same number of local charts as for the projective superspace. The coordinates in each chart are the polar radius $r$, the angular coordinate $\phi$ in the fibres of the Hopf fibration and the inhomogeneous coordinates on $\mathbb{CP}^{n|m}$. For each fixed $k$ (the chart index) we have

$$z^a = r e^{i \phi} N^{-1/2} w^a, \quad \zeta^\mu = r e^{i \phi} N^{-1/2} \theta^\mu.$$  \hspace{1cm} (2.10)
Here $\alpha = \text{Arg} z^k$, $r^2 = z \cdot \bar{z} + i\zeta \cdot \bar{\zeta}$, $w^a = z^a / z^k$ and $N = w \cdot \bar{w} + i\theta \cdot \bar{\theta}$, as above. (To avoid cumbersome notation, we indicate only one angular coordinate on the circle. Using more detailed notation, the number of charts increases at least by a factor of 2.)

**Proposition 3.** In the local coordinates $r, \alpha, w^a, \bar{w}^a, \theta^\mu, \bar{\theta}^\mu$ the Euclidean volume element for $\mathbb{C}^{n+1|m}$ is given by

$$dV_{\mathbb{C}^{n+1|m}} = \left(\frac{i}{2}\right)^{n-m} r^{2n+1-2m} \frac{[dr, d\alpha, dw^1, d\bar{w}^1, \ldots, dw^n, d\bar{w}^n, d\theta^1, d\bar{\theta}^1, \ldots, d\theta^m, d\bar{\theta}^m]}{N^{n-m+1}}.$$

(2.11)

The formula (2.11) is proved by substituting the explicit expressions for the differentials of the $z^a$ and $\zeta^\mu$ obtained from (2.10) into $[dz, d\bar{z}; d\zeta, d\bar{\zeta}]$ and then simplifying the result using the rules of the symbolic bracket notation.

**Corollary 2.** For the supersphere $S^{2n+1|2m}_R$ we have the following formula in the local coordinates $\alpha, w^a, \bar{w}^a, \theta^\mu, \bar{\theta}^\mu$ adjusted to the Hopf fibration:

$$dV_{S^{2n+1|2m}_R} = R^{2n+1-2m} \left(\frac{i}{2}\right)^{n-m} \frac{[d\alpha, dw^1, d\bar{w}^1, \ldots, dw^n, d\bar{w}^n, d\theta^1, d\bar{\theta}^1, \ldots, d\theta^m, d\bar{\theta}^m]}{N^{n-m+1}} = R d\alpha \cdot dV_{\mathbb{C}^{n|m}_R}.$$

This means that the Hopf fibration $S^{2n+1|2m}_R \to \mathbb{C}P^{n|m}_R$ determines a factorization of the volume element of the supersphere. It follows that the total volume (already known) of the supersphere is the product of the volume of $\mathbb{C}P^{n|m}_R$ (which we want to find) and the length of the circle $S^1_R$.

### 2.4. The formula for the volume of $\mathbb{C}P^{n|m}_R$.

**Theorem 2.** The volume of the complex projective superspace is given by

$$\text{vol}(\mathbb{C}P^{n|m}_R) = R^{2(n-m)} \frac{\pi^{2m}}{\Gamma(n-m+1)}.$$  

(2.12)

**Example 1.** In the simplest nontrivial case we obtain $\text{vol}(\mathbb{C}P^{1|1}_R) = 2\pi$ (independent of the radius!). In particular, the volume is nonzero, thus giving the simplest example of a compact symplectic supermanifold with nonzero volume.

**Example 2.** For $m > n$ we obtain $\text{vol}(\mathbb{C}P^{n|m}_R) = 0$ (since the gamma function has poles at all nonpositive integers).

**Remark 1.** Using the well-known homeomorphism $\mathbb{C}P^n \approx (\mathbb{C}P^1)^n / S_n$ (the symmetric power), one can write the formula for the volume of the ordinary complex projective space in the form

$$\text{vol}(\mathbb{C}P^n_R) = R^{2n} \frac{\pi^n}{n!} = \left(\frac{\text{area}(\mathbb{C}P^1_R)}{n!}\right)^n.$$
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which is, of course, not a proof of this formula). Note that area($\mathbf{CP}_R^1$) = $\pi R^2$
in the Fubini-Study metric (which differs by a factor of 1/4 from the metric of the 2-sphere: $\delta s_{S^2}^2 = 4 \delta s_{\mathbf{CP}_R^1}^2$). It would be interesting to know whether there is
a similar interpretation for the superspace $\mathbf{CP}_R^{n|m}$.

(Remark 2.

The formula for the volume of $\mathbf{CP}_R^{n|m}$ in the form $1/(n - m)!$ and a formula for the volume element close to our (2.9) were given in [5] without detailed
proof, in particular, without explicit calculation of the integral. The Fubini-Study
form in [5] differs from ours by the coefficient $\pi^{-1}$, but this does not explain the
absence of a factor in the expression for the volume (most probably, the formula
in [5] contains a misprint, or a common factor is simply ignored).

2.5. A useful restatement: volumes as analytic functions. Our formulae
for volumes may be restated as follows. Recall that the (super)dimensions take
values in the ring $\hat{\mathbb{Z}} = \mathbb{Z}[\Pi]/(\Pi^2 - 1)$, so that $n|m = n + m\Pi$. Define the index
of a supermanifold $M^{n|m}$ by the formula

$$\text{ind} M^{n|m} := n - m$$

(2.13)

if $\dim M^{n|m} = n|m$. Equivalently, $\text{ind} M = \chi(\dim M)$, where $\chi: \hat{\mathbb{Z}} \rightarrow \mathbb{Z}$ is the ring
homomorphism $n|m \mapsto n - m$. For a complex supermanifold $M^{2n|2m}$ we can also
speak of the dimension and index over $\mathbb{C}$:

$$\dim_{\mathbb{C}} M^{2n|2m} = n|m, \quad \text{ind}_{\mathbb{C}} M^{2n|2m} := n - m.$$ 

We also define the Gaussian factor $g_D$ for a given (super)dimension $D = n|m \in \hat{\mathbb{Z}}$
by the formula

$$g_{n|m} := (\sqrt{\pi})^n (\sqrt{2})^m.$$ 

(2.14)

(If $D = n|2m$, then $g_{n|2m}$ is the value of the standard Gaussian integral over $\mathbb{R}^{n|2m}$;
see §5. For aesthetic reasons we introduce $g_D$ for an arbitrary $D$.)

Define the following functions of the complex variable $z$:

$$\mathcal{V}(S; R, z) := R^2 \frac{2\sqrt{\pi}}{\Gamma\left(\frac{z+1}{2}\right)},$$

(2.15)

$$\mathcal{V}(\mathbf{CP}; R, z) := R^{2z} \frac{1}{\Gamma(z + 1)}.$$ 

(2.16)

Here $R$ is a real parameter. The formulae for the volumes of the superspheres and
complex projective superspaces can then be rewritten as

$$\text{vol}(S^{n|2m}_R) = g_D \cdot \mathcal{V}(S; R, z),$$

(2.17)

where $D = \dim S^{n|2m} = n|2m$, $z = \text{ind} S^{n|2m} = n - 2m$ and

$$\text{vol}(\mathbf{CP}^{n|m}_R) = g_{2D} \cdot \mathcal{V}(\mathbf{CP}; R, z),$$

(2.18)

where $2D = \dim \mathbf{CP}^{n|m} = 2n|2m$ and $z = \text{ind}_{\mathbb{C}} \mathbf{CP}^{n|m} = n - m$. It follows that
if we normalize the volumes of supermanifolds by dividing by the Gaussian factor,
then the normalized volumes are expressed by analytic functions of ‘index’-type variables:

\[
\frac{\text{vol}(S^D)}{g^D} = \mathcal{V}(S; R, z), \quad \text{where } z = \text{ind} S^D, \tag{2.19}
\]

\[
\frac{\text{vol}(\mathbb{CP}^D)}{g^{2D}} = \mathcal{V}(\mathbb{CP}; R, z), \quad \text{where } z = \text{ind} \mathbb{CP}^D, \tag{2.20}
\]

for arbitrary (super)dimensions \(D\).

The analytic functions \(\mathcal{V}(S; R, z)\) and \(\mathcal{V}(\mathbb{CP}; R, z)\) of the variable \(z\) may be regarded as the volumes of spheres and complex projective spaces after normalization (as above) and analytic continuation to all complex values of the parameter. Since the index of an ordinary manifold coincides with its dimension, the normalized volumes of supermanifolds (considered above) are obtained by analytic continuation of the normalized volumes of their classical counterparts: first replace the dimensions (such as \(n\)) by the complex variable \(z\) and then substitute \(z = n - m\).

**Remark 3.** The relation between the volume of a complex projective (super)space, the volume of an odd-dimensional (super)sphere, and the length of a circle remains valid for all complex values of \(z\):

\[
\mathcal{V}(S; R, 2z + 1) = \mathcal{V}(\mathbb{CP}; R, z) \cdot \mathcal{V}(S; R, 1).
\]

§ 3. The unitary supergroup. Stiefel and Grassmann supermanifolds

**3.1. The unitary supergroup.** Recall that the unitary supergroup \(U(n|m)\) is defined by the matrix equation

\[
gHg^* = H \tag{3.1}
\]

for the matrix \(g \in \text{Mat}(n|m; \mathbb{C})\), where

\[
\begin{pmatrix}
  g_{00} & g_{01} \\
  g_{10} & g_{11}
\end{pmatrix}^* := \begin{pmatrix}
  \bar{g}_{00}^T & \bar{g}_{10}^T \\
  \bar{g}_{01} & \bar{g}_{11}
\end{pmatrix}, \tag{3.2}
\]

\[
H = \begin{pmatrix}
  1 & 0 \\
  0 & i
\end{pmatrix}. \tag{3.3}
\]

More geometrically, \(U(n|m)\) can be defined as the supergroup of all linear transformations of the superspace \(\mathbb{C}^{n|m}\) that preserve the Hermitian metric. In complete analogy with the ordinary case we obtain that

\[
\dim U(n|m) = (n|m)^2, \tag{3.4}
\]

\[
\text{ind} U(n|m) = (n - m)^2. \tag{3.5}
\]

Notice also that \(U(n|m) \cong U(m|n)\).

The following remarkable fact was discovered by Berezin in the 1970s.

**Theorem 3** (Berezin, see [1] and [2]). The total Haar measure of the semigroup \(U(n|m)\) vanishes when \(n > 0\) and \(m > 0\) (that is, when the supergroup does not reduce to an ordinary unitary group).
Remark 4. Berezin obtained his theorem probably around 1976. He announced it in a short paper [1] and set it out in detail in the preprint [2], which concludes a series of five ITEP preprints where Berezin presented his results on the theory of supergroups and Lie superalgebras and their representations. These preprints circulated among experts. They were included as part II in the English version [3] of his posthumous book [17] (compiled by Berezin’s friends using his manuscripts). In particular, the preprint [2] is published as Chapter 5 of Part II in [3]. Berezin’s method of proof is a direct calculation of the invariant volume form on $U(n|\mathfrak{m})$ in concrete coordinates. He shows that the answer does not contain the maximal product of odd coordinates and, therefore, the integral of $1$ is equal to zero. Berezin also performed a similar calculation for the supergroup $OSp(n|2m)$, but there one cannot make the same conclusion about the total volume since the underlying topological space $O(n) \times Sp(2m)$ is noncompact.

Example 3. Theorem 3 admits a direct verification in the simplest case of the supergroup $U(1|1)$. One can find the following explicit parametrization for the matrices $g = g(\alpha, \beta|\theta) \in U(1|1)$:

$$g(\alpha, \beta|\theta) = \begin{pmatrix} e^{i\alpha} \left( 1 + \frac{i}{2} \theta \theta \right) & \theta \\ \bar{\theta} e^{i\beta} & e^{-i\alpha} \left( 1 - \frac{i}{2} \theta \bar{\theta} \right) e^{i\beta} \end{pmatrix}.$$ 

Here the ‘angular’ parameters $\alpha$ and $\beta$ are even real variables and $\theta$ is an odd complex variable. A direct calculation yields that

$$[dg \cdot g^{-1}] = -2i[d\alpha, d\beta|d\theta, d\bar{\theta}]$$

with respect to the basis

$$e_1 = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}, \quad \varepsilon_1 = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$$

of the corresponding Lie superalgebra $\mathfrak{u}(1|1)$. (Here $dg \cdot g^{-1}$ is the canonical right-invariant 1-form on the Lie supergroup $G$ with values in its Lie superalgebra $\mathfrak{g}$. Expanding the $\mathfrak{g}$-valued form $dg \cdot g^{-1}$ with respect to the chosen basis of $\mathfrak{g}$, we obtain a basis of right-invariant 1-forms on $G$ as coefficients. Their ‘bracket’ operation (see §5), which is denoted by $[dg \cdot g^{-1}]$, provides the desired volume element on $G$.) Up to a constant factor, this is the invariant volume element on $U(1|1)$. Integrating, we have

$$\text{vol}(U(1|1)) = 0$$

by the definition of the Berezin integral (since the integrand contains no odd variables $\theta, \bar{\theta}$).

In the next subsection we calculate the volumes of complex Stiefel supermanifolds, which include unitary supergroups as special cases. In particular, we shall obtain a simple geometric explanation of Berezin’s theorem.

\footnote{Unfortunately, the book [3] is not easily available, and Berezin’s pioneering results contained in it still have not received due appreciation.}
3.2. Stiefel supermanifolds. Consider the superspace $\mathbb{C}^{n|m}$ with a standard Hermitian form. We denote the complex Stiefel supermanifold by $V_{r|s}(\mathbb{C}^{n|m}) = \mathbb{C}V(n|m, r|s)$. It is defined as the closed submanifold

$$V_{r|s}(\mathbb{C}^{n|m}) \subset \underbrace{\mathbb{C}^{n|m} \times \cdots \times \mathbb{C}^{n|m}}_{r} \times \underbrace{\Pi \mathbb{C}^{n|m} \times \cdots \times \Pi \mathbb{C}^{n|m}}_{s},$$

of all orthonormal $r|s$-frames in $\mathbb{C}^{n|m}$. (More precisely, the vectors are pairwise orthogonal, the first $r$ vectors are even with scalar square 1, and the last $s$ vectors are odd with scalar square $i$.)

As in the ordinary case,

$$V_{r|s}(\mathbb{C}^{n|m}) = U(n|m)/U(n - r|m - s).$$

One can also establish that

$$\dim V_{r|s}(\mathbb{C}^{n|m}) = (r|s) \cdot (2(n|m) - (r|s)) = r(2n - r) + s(2m - s) - 2(ns + mr - rs),$$

$$\text{ind} V_{r|s}(\mathbb{C}^{n|m}) = (r - s)(2(n - m) - r + s),$$

(3.6) (3.7)

(this generalizes the familiar formula $\dim V_r(\mathbb{C}^n) = r(2n - r)$ for the dimension of an ordinary Stiefel manifold).

Notice that if $V$ is a Hermitian vector superspace, then the superspace $\Pi V$ with reversed parity also has a Hermitian structure naturally induced by the formula

$$\langle \Pi u, \Pi v \rangle := i(-1)^\bar{u} \langle u, v \rangle.$$

(3.8)

If the initial Hermitian structure of $V$ is positive definite, then so is the induced structure on $\Pi V$. By definition, the Riemannian metric on $V_{r|s}(\mathbb{C}^{n|m})$ is induced by the Euclidean metric on $\mathbb{C}^{n|m}$ (the real part of the Hermitian form). This yields the following useful assertion.

**Proposition 4.** The parity reversion functor $\Pi$ induces an isometry of Riemannian supermanifolds

$$V_{r|s}(\mathbb{C}^{n|m}) \approx V_{s|r}(\mathbb{C}^{m|n}).$$

(3.9)

The following assertion is crucial for our calculation of the volume of a Stiefel supermanifold.

**Theorem 4.** For $r > 0$ there is a locally trivial bundle

$$V_{r-1|s}(\mathbb{C}^{n-1|m}) \longrightarrow V_{r|s}(\mathbb{C}^{n|m}) \quad \text{(3.10)}$$

$$\quad \downarrow$$

$$\mathbb{S}^{2n-1|2m}$$

(thus $V_{r-1|s}(\mathbb{C}^{n-1|m})$ is the standard fibre). This bundle is a Riemannian submersion.
We recall that a fibration of Riemannian (super)manifolds is called a *Riemannian submersion* if its projection induces an isometry of each horizontal subspace onto the tangent space to the base at the corresponding point. Then there is a factorization of the volume element of the total space and, in homogeneous situations, the volume of the total space is the product of volumes of the base and the standard fibre (see §6.2).

We omit the details, but notice that our verification of the fact that the bundle \( V_{r|s}(C^n|m) \to S^{2n-1|2m} \) is a Riemannian submersion follows from the commutative diagram

\[
\begin{align*}
V_{r|s}(C^n|m) & \longrightarrow (C^n|m)^r \times (\I C^n|m)^s \\
S^{2n-1|2m} & \longrightarrow C^n|m
\end{align*}
\]

since the metrics on the Stiefel supermanifold and the supersphere are after all induced from the standard metric of \( C^n|m \). It follows also that the induced metric on the fibres of \( V_{r|s}(C^n|m) \to S^{2n-1|2m} \) is compatible with the metric of \( V_{r-1|s}(C^{n-1}|m) \).

The bundle \( V_{r|s}(C^n|m) \to S^{2n-1|2m} \) is nontrivial, but it can be written as a direct product on a dense domain. Iterating, we obtain a ‘diffeomorphism’ (on a dense domain) for \( r > 0 \):

\[
V_{r|s}(C^n|m) \approx S^{2n-1|2m} \times S^{2n-3|2m} \times \cdots \times S^{2(n-r)+1|2m} \times V_{0|s}(C^{n-r}|m) \\
\approx S^{2n-1|2m} \times S^{2n-3|2m} \times \cdots \times S^{2(n-r)+1|2m} \times V_{s}(C^{n-m|r}) \\
\approx S^{2n-1|2m} \times S^{2n-3|2m} \times \cdots \times S^{2(n-r)+1|2m} \times S^{2(m-1)+1|2(n-r)} \\
\times \cdots \times S^{2(m-s)+1|2(n-r)}
\]

(the last equality holds for \( s > 0 \)). This idea of ‘decomposition into a product of superspheres’ helps to visualize the situation. In any case, the corresponding factorization of volumes holds by Theorem 4. Recall that the volume of an odd-dimensional supersphere \( S^{2k+1|2m} \) vanishes when \( k - m < 0 \). Hence the volume

\[
\text{vol}(V_{r|s}(C^n|m)) = \text{vol}(S^{2n-1|2m}) \text{vol}(S^{2n-3|2m}) \cdots \text{vol}(S^{2(n-r)+1|2m}) \text{vol}(V_{s}(C^{n-m|r}))
\]

can be nonzero only when \( n - r - m + 1 > 0 \), that is, \( r < n - m + 1 \). At the same time, if \( s > 0 \), then

\[
\text{vol}(V_{s}(C^{n-m|r})) = \text{vol}(S^{2(m-1)+1|2(n-r)}) \cdots \text{vol}(S^{2(m-s)+1|2(n-r)}),
\]

which for the same reason can be nonzero only when \( m - 1 - n + r + 1 > 0 \), that is, \( r > n - m \). This is impossible for integer \( r \). Thus we arrive at the following theorem, whose part 1 generalizes Berezin’s theorem on the volume of \( U(n|m) \).

**Theorem 5.** 1. The volume of the complex Stiefel supermanifold \( V_{r|s}(C^n|m) \) vanishes unless \( r = 0 \) or \( s = 0 \).
2. For \( s = 0 \) the volume is given by

\[
\text{vol}(V_r(C^n|^{m})) = \text{vol}(S^{2(n-1)+1|2m}) \cdots \text{vol}(S^{2(n-r)+1|2m}) = g_D R^N(2\sqrt{\pi})^r \frac{1}{\Gamma(n-m)\Gamma(n-m-1) \cdots \Gamma(n-m-r+1)}
\]

(3.11)

(where we have re-introduced the scaling factor \( R \) in the metric). Here

\[ D = \dim V_r(C^n|^{m}) \quad \text{and} \quad \chi = \chi(D) = \text{ind} V_r(C^n|^{m}). \]

(In Theorem 5 we use the multiplicativity of the Gaussian factor \( g_{\dim} \) and the scaling factor \( R^{\text{ind.}} \).

Rewriting the fraction as

\[
\frac{\Gamma(n-m-r)\Gamma(n-m-r-1) \cdots \Gamma(1)}{\Gamma(n-m)\Gamma(n-m-1) \cdots \Gamma(1)},
\]

we obtain the following corollary.

**Corollary 3** (‘final formula’). The volume of a complex Stiefel supermanifold, when distinct from zero, is given by

\[
\text{vol}(V_r(C^n|^{m})) = g_D \cdot R^{\chi(D)}(2\sqrt{\pi})^r \frac{G(n-m-r+1)}{G(n-m+1)},
\]

(3.12)

where

\[ D = \dim V_r(C^n|^{m}) = r(2n-r)|mr, \quad \chi(D) = \text{ind} V_r(C^n|^{m}) = r(2(n-m) - r). \]

Here \( G(z) \) is the Barnes function (the ‘double gamma function’). It is defined as a certain infinite product, but essentially it is determined by the fundamental recursive functional relation

\[
G(z+1) = G(z)\Gamma(z).
\]

Moreover, \( G(1) = G(2) = G(3) = 1 \) (and all nonpositive integers are zeros of \( G(z) \)). See [18] for a modern exposition of the theory and new results about the function \( G(z) \).

**3.3. Analytic formula for the volume.** As in the cases of superspheres and complex projective superspaces, there is an analytic function expressing the volume of a Stiefel supermanifold. For complex variables \( z \) and \( w \) we put

\[
\mathcal{Y}(CV; R, z, w) := R^{w(2z-w)}(2\sqrt{\pi})^w \frac{G(z-w+1)}{G(z+1)}.
\]

(3.13)

Then the normalized volume \( \text{vol}(V_r(C^n|^{m})) \) is given by

\[
\frac{\text{vol}(V_r(C^n|^{m}))}{g_D} = \mathcal{Y}(CV; R, z, w),
\]

(3.14)

where \( D = \dim V_r(C^n|^{m}) = r(2n-r)|mr \) and \( z = n-m, w = r \).
3.4. The case of a Grassmann supermanifold. For the complex Grassmann supermanifold $G_{r\mid s}(C^{n\mid m}) = CG(n\mid m, r\mid s)$ we have

$$G_{r\mid s}(C^{n\mid m}) \cong V_{r\mid s}(C^{n\mid m})/U(r\mid s) \cong U(n\mid m)/U(n - r\mid m - s) \times U(r\mid s).$$

As in the purely even case, the Grassmann supermanifold $G_{r\mid s}(C^{n\mid m})$ has an analogue of the Fubini-Study metric. The dimension and index are

$$\dim G_{r\mid s}(C^{n\mid m}) = 2(r\mid s) \cdot (n\mid m),$$

$$\text{ind } G_{r\mid s}(C^{n\mid m}) = 2(r - s)(n - m - r + s).$$

For complex variables $z$ and $w$ we set

$$\mathcal{V}(CG; R, z, w) := R^{2w(z-w)} \frac{G(w+1)G(z-w+1)}{G(z+1)},$$

where $G(z)$ is the Barnes function. Note the symmetry under $w \to z - w$.

We expect that the volume of a complex Grassmann supermanifold is expressed by the following analytic formula:

$$\frac{\text{vol}(G_{r\mid s}(C^{n\mid m}))}{g_D} = \mathcal{V}(CG; R, z, w),$$

where $D = \dim G_{r\mid s}(C^{n\mid m})$, $z = n - m$, $w = r - s$. (For example, when $w = 1$, we obtain again the formula for $CP^D_R$. On the other hand, assuming that the projections $U(n\mid m) \to G_{r\mid s}(C^{n\mid m})$ and $V_{r\mid s}(C^{n\mid m}) \to G_{r\mid s}(C^{n\mid m})$ are Riemannian submersions, which is beyond reasonable doubt, we arrive at this formula at least when the volume of the total space is nonzero. However, (3.18) must be valid for all values of the parameters, including those cases when the volumes of the Stiefel supermanifold and the unitary supergroup vanish.)

§ 4. Conclusions. Discussion

4.1. We have considered the volumes of classical supermanifolds: the supersphere, the complex projective superspace and complex Stiefel and Grassmann supermanifolds. We normalized the volumes by factoring out the Gaussian factor and showed that the normalized volumes of these supermanifolds result from analytic continuation of their values for the corresponding ordinary manifolds. We gave a geometric proof of the formula for the volume of complex Stiefel supermanifolds, which covers Berezin’s theorem on the vanishing of the volume of the unitary supergroup. On the other hand, our calculations provide examples of compact symplectic supermanifolds with nonzero phase volume (counterexamples to Witten’s conjecture).

Volumes of supermanifolds exhibit scale invariance with scaling factor $R^{\text{ind}}$ when the metric is multiplied by $R^2$. This could be expected because of the properties of Berezinians (superdeterminants). In particular, we see that the volume is scale-invariant when $\text{ind } M = 0$. On the other hand, the complex variables in our analytic formulae for the volumes are of ‘index’ type. This suggests the possibility of extending the theory of volumes to appropriate infinite-dimensional cases.
4.2. One of the questions that must be clarified is how to agree the analytic expressions with those special cases when the volume vanishes identically for a whole range of parameters. It is characteristic that distinguishing such ranges always requires not just the knowledge of ‘index-type variables’ occurring in the analytic formulae, but also of additional parameters such as the full dimension, that is, the knowledge of the number of even and odd variables separately. Then the analytic formula (such as (3.13)) may well exist and give a nonzero answer if we forget about the additional restrictions. What does it describe then?

A similar question arises in the context of ‘universal formulae’ for Lie algebras (see [7]–[9], [12]). In particular, there is an analytic formula for the volume of a compact Lie group, such as $U(n)$, in terms of the parameters of the Vogel plane. When expressed in terms of $n$, this formula coincides (up to normalization) with our expression that uses the Barnes function. Rather surprisingly, the history of explicit formulae for the volumes of compact Lie groups is quite recent: Hua [19], §§3.1, 3.7 (a pioneering monograph, where concrete volumes were calculated ad hoc); Marinov [20], [21]; Fegan [22], Macdonald [23], Hashimoto [24]; Kac and Peterson [25], equation 4.32.1. One must clearly understand that these formulae by definition deal with a discrete set of objects. Their extension to various continuous parameters is never unique and always includes additional considerations to fix the choice of the extension. These additional considerations may be more or less natural. (One could recall the example of the gamma function that analytically extends the factorial from a discrete to a continuous argument. The uniqueness of such an extension is fixed by the additional condition of convexity. Similar things appear in the theory of the Barnes function.) From this point of view, the ‘universal formula’ for the volume of a group, first obtained in [7] as a result of calculations in quantum field theory, has an advantage: it is analytic by construction instead of being an interpolation of a discrete formula. A problem in connection with this ‘universal formula’ is its seemingly anomalous behaviour under the expected symmetry transformations. For example, the formula for the volume of the unitary group in terms of the Barnes function,

$$\frac{\text{vol}(U(n))}{g_{n^2}} = R^n (2\pi)^{n/2} \frac{1}{G(n+1)}$$

(see our formula (3.12) with $m = 0$ and $r = n$), does not possess the expected symmetry under $n \to -n$. (The Barnes function satisfies a more complicated Kinkelin identity: see Adamchik [18].) How can we explain this fact? A physicist could call it an ‘anomaly in the volume formula’. The most complete explanation was given in [12]: the ‘correct’ universal volume formula (which was originally obtained in [7]) is an integral depending on parameters, and it is essential that this integral is not a single-valued analytic function of the parameters. Careless calculations of the integral give only one branch of this multi-valued function (an example is the formula above in terms of the Barnes function). The symmetry under which the formula is ‘anomalous’ actually interchanges branches.

The $(n \to -n)$-symmetry problem in the volume formula for the unitary group is a particular case of the analogous question for our formulae. For example, we know that the Stiefel supermanifolds $V_{r|s}(C^n|m)$ and $V_{s|r}(C^m|n)$ are isometric. This should correspond to a simultaneous change of variables $z \to -z$ and $w \to -w$ in
the volume formula (3.13). How can we then perform an argument based on integral formulae? (We have no such formulae yet.) This is the way to an explanation of the agreement between analytic formulae and volumes which vanish identically in whole ranges of the values of the parameters.

It would be very interesting to clarify all these questions.

4.3. The following observation is curious although not directly related to our main theme (the volumes of supermanifolds). For ordinary symplectic manifolds, a Hamiltonian action of a Lie group enables one to define a momentum map from the manifold to the dual of the Lie algebra. A classical theorem due to Atiyah (see [26]) and Guillemin-Sternberg (see [27]) states that the image of the momentum map for a Hamiltonian action of the torus on a compact symplectic manifold is a convex polytope. For example, the standard ‘coordinate’ action of \( T^{n+1} \) on \( \mathbb{CP}^n \) yields an \( n \)-simplex. Buchstaber queried the author for an analogue of this in the supercase, say, for the complex projective superspace. The answer for \( \mathbb{CP}^{n|m} \) was unexpected: the image of the momentum map in \( \mathbb{R}^{n+m+1} \) is an infinitesimal neighbourhood of an \( n \)-simplex inside an \( (n+m) \)-simplex. (This follows from the explicit formulae for the Hamiltonians that generate the action of \( T^{n+m+1} \) on \( \mathbb{CP}^{n|m} \): even nilpotent elements appear.) It is still unknown what the corresponding general theory could be if it exists.

§ 5. Background on supermanifolds, volumes and integrals

We refer the reader to the following sources on the theory of supermanifolds: [17] (its extended English translation was published as [3]), [28], [29], Chs. 3 and 4, [4], [30] and [31]. We assume it to be known that all objects (modules, algebras) in supergeometry carry a \( \mathbb{Z}_2 \)-grading called parity. The parity of a homogeneous element is denoted by a tilde over the corresponding letter. The objects may also carry a \( \mathbb{Z} \)-grading (the ‘degree’ or ‘weight’). In general, the parity is not assumed to be equal to the degree modulo 2, and it is parity, rather than degree, that is used in the sign rule.

5.1. Tensors on supermanifolds. One difference from the ordinary case is the necessity to distinguish between even and odd differentials and, more generally, to consider not only the tangent bundle \( TM \) and the cotangent bundle \( T^*M \), but also the antitangent bundle \( \Pi TM \) and the anticotangent bundle \( \Pi T^*M \) obtained by parity reversion in the fibres.

If the differential \( \delta f \) of a function \( f \) is understood as the principal linear part of the increment\(^3\), then the map \( f \mapsto \delta f \) preserves parity, \( \delta f = \tilde{f} \). (Here we intentionally write \( \delta \) instead of the customary \( d \).) This is the even differential: \( \delta f \) takes values in \( T^*M \). We have \( \delta f(X) = (-1)^\tilde{f} X \partial_X f \) for every tangent vector \( X \). The differentials \( \delta x^a \) of the local coordinates provide a basis of sections of \( T^*M \). We notice that they form a right dual basis of the basis \( \partial_a = \partial/\partial x^a \) of partial derivatives in \( TM \),

\[
\delta x^a(\partial_b) = \langle \delta x^a, \partial_b \rangle = (-1)^\tilde{a} \delta^a_b, \quad \langle \partial_a, \delta x^b \rangle = (-1)^\tilde{a} \delta^b_a, \quad \langle \partial_a, \partial_x \rangle = \delta^b_a.
\] (5.1)

\(^3\)That is, \( \delta f = f - f(x_0) \ (\mod m^2_{x_0}) \), where the ideal \( m_{x_0} \) consists of all functions vanishing at \( x_0 \).
The odd differential $d$ is defined as the composite $d := \Pi \circ \delta$ (where $\Pi$ is understood as an odd formal symbol whose product with an object reverses the object’s parity, with $\Pi^2 = 1$), whence the $dx^a$ form a basis in $\Pi T^* M$. We have $df = \tilde{f} + 1$ for all $f$ and $\tilde{dx}^a = \tilde{a} + 1$. There is an odd pairing $\langle df, X \rangle := \delta f(X)$. Thus,

$$\langle dx^a, \partial_b \rangle = \langle \partial_b, dx^a \rangle = (-1)^{\tilde{a} \tilde{b}}. \quad (5.2)$$

Every covariant tensor on a supermanifold expands with respect to the tensor products of both $\delta x^a$ and $dx^a$. But products of the $dx^a$ can be converted to products of the $\delta x^a$ and vice versa using the sign rule, with a possible appearance of the common factor $\Pi$. This corresponds algebraically to the natural isomorphism $(\Pi V)^{\otimes k} \cong \Pi^k V^{\otimes k}$ for any module $V$. We also mention a natural isomorphism $\Pi V^* \cong (\Pi V)^*$.  

Example 4. Suppose that $T(\partial_{a_1}, \ldots, \partial_{a_k}) = T_{a_1 \ldots a_k}$. Then

$$T = \delta x^{a_1} \otimes \cdots \otimes \delta x^{a_k} T_{a_1 \ldots a_k} (\widetilde{(-1)^{\bar{a}_1 (\bar{a}_2 + \cdots + \bar{a}_k + T) + \bar{a}_2 (\bar{a}_3 + \cdots + \bar{a}_k + T) + \cdots + \bar{a}_k T}). \quad (5.3)$$

Example 5. For covariant tensors of rank 2 we have

$$dx^a \otimes dx^b = (\Pi \delta x^a) \otimes (\Pi \delta x^b) = (-1)^{\bar{a} \bar{b}} \Pi^2 \delta x^a \otimes \delta x^b = (-1)^{\bar{a}} \delta x^a \otimes \delta x^b \quad (5.4)$$

(recall that $\Pi^2 = 1$).

The symmetric algebra and the exterior algebra are defined in the usual way, taking the sign rule into account. The same can be said about symmetrization, alternation, symmetric products and wedge products. As in the ordinary case, symmetric (resp. antisymmetric) tensors are expressed in terms of symmetric (resp. wedge) products of basis vectors or covectors.

Example 6. Suppose that the covariant tensor in Example 4 is symmetric. Then

$$T = \delta x^{a_1} \otimes \cdots \otimes \delta x^{a_k} T_{a_k \ldots a_1} (\widetilde{(-1)^{\bar{a}_1 + \cdots + \bar{a}_k}) T}$$

$$= \delta x^{a_1} \ldots \delta x^{a_k} T_{a_k \ldots a_1} (-1)^{\bar{a}_1 + \cdots + \bar{a}_k) \widetilde{T},$$

where the second line contains the symmetric product of differentials.

For each module $V$, the natural isomorphism $(\Pi V)^{\otimes k} \cong \Pi^k V^{\otimes k}$ induces a natural isomorphism $S^k (\Pi V) \cong \Pi^k \Lambda^k (V)$. Thus a $k$-form on a supermanifold may equivalently be regarded either as a section of the exterior power $\Lambda^k (T^* M)$ or as a section of the symmetric power $\Pi^k S^k (\Pi T^* M)$ (with possible parity shift).

Example 7. Using the identification $dx^a \otimes dx^b = (-1)^{\bar{a}} \delta x^a \otimes \delta x^b$ we obtain

$$dx^a \otimes dx^b = \frac{1}{2} (dx^a \otimes dx^b + (-1)^{\bar{a} + \bar{b} + 1} dx^b \otimes dx^a)$$

$$= (-1)^{\bar{a} + 1} \frac{1}{2} (\delta x^a \otimes \delta x^b - (-1)^{\bar{a} \bar{b}} \delta x^b \otimes \delta x^a) = (-1)^{\bar{a}} \delta x^a \wedge \delta x^b. \quad (5.5)$$

Writing a 2-form $\omega$ in terms of symmetric products of odd differentials,

$$\omega = dx^a dx^b \omega_{ba}, \quad (5.6)$$
where $\omega_{ab} = (-1)^{(\bar{a}+1)(\bar{b}+1)} \omega_{ba}$, we can also express it in terms of wedge products of even differentials,

$$\omega = \delta x^a \wedge \delta x^b \omega_{ba}',$$

(5.7)

where $\omega'_{ab} := (-1)^{\bar{b}} \omega_{ab}$. Conversely, wedge products of even differentials can be expressed in terms of symmetric products of odd differentials. We mention the symmetry condition for the coefficients: $\omega'_{ab} = -(-1)^{\bar{a} \bar{b}} \omega_{ba}$.

An advantage of using the symmetric powers $S^k(\Pi T^*M)$ is that $k$-forms may be regarded in this language as fibrewise polynomial functions on the supermanifold $\Pi T M$. By definition arbitrary functions on $\Pi T M$ are pseudodifferential forms on $M$.

5.2. The Berezin integral and volume elements. Consider the superspace $R^{n|m}$ with standard coordinates $x^a$ (even) and $\xi^\mu$ (odd). Here it is convenient to use distinct letters for even and odd coordinates. Consider a function $f \in C^\infty(R^{n|m})$ admitting the following coordinate expansion with respect to odd variables:

$$f(x, \xi) = f_0(x) + \xi^\mu f_\mu(x) + \frac{1}{2} \xi^\nu \xi^\mu f_{\mu \nu}(x) + \cdots + \frac{1}{m!} \xi^{\mu_1} \cdots \xi^{\mu_m} f_{\mu_1 \cdots \mu_m}(x),$$

where the coefficients $f_{\mu_1 \cdots \mu_k}$ are assumed to be symmetric in their subscripts (taking the sign rule into account, that is, antisymmetric in the naive sense). Assume that the functions $f_{\mu_1 \cdots \mu_k}$ are either compactly supported or rapidly decaying on $R^n$.

**Definition 1** (the Berezin integral). The integral of $f$ with respect to the variables $x^1, \ldots, x^n, \xi^1, \ldots, \xi^m$ is defined as

$$\int_{R^{n|m}} D(x, \xi) f(x, \xi) := \int_{R^n} f_{1 \ldots m}(x) d^n x.$$

(5.8)

(There are various conventions on the common sign in the definition of the integral. Our choice is one of these possibilities.)

We emphasize that unlike the ordinary integral over even variables, the Berezin integral is defined formally (but not arbitrarily, the definition is designed to make the integral of a derivative equal to zero). It is not related to measure theory in the ordinary sense. Just the opposite, in the supercase, measure and volume are defined in terms of the Berezin integral and may therefore possess unexpected properties.

**Example 8.** Consider a purely odd superspace $R^{0|m}$. By the definition of the integral we have

$$\int_{R^{0|m}} D\xi = 0,$$

which may be interpreted as the vanishing of the volume of $R^{0|m}$ (with respect to the ‘coordinate measure’ element $D\xi$). One can compare this with the infinite volume of $R^n$ with respect to the Lebesgue measure element $d^n x$.

We recall the change of variables formula for the Berezin integral.
Theorem 6. For an invertible change of variables \( x = x(x', \xi') \), \( \xi = \xi(x', \xi') \) on \( \mathbb{R}^{n|m} \)

\[
\int_{\mathbb{R}^{n|m}} D(x, \xi) f(x, \xi) = \pm \int_{\mathbb{R}^{n|m}} D(x', \xi') \frac{D(x, \xi)}{D(x', \xi')} (x', \xi') f(x(x', \xi'), \xi(x', \xi')) ,
\]

where \( \pm = \text{sign det } \frac{\partial x}{\partial x'}(x', 0) \),

\[
\frac{D(x, \xi)}{D(x', \xi')} = \text{Ber } \frac{\partial(x, \xi)}{\partial(x', \xi')} .
\]

Here the superdeterminant, or Berezinian, \( \text{Ber } J \) of an even block matrix is defined by

\[
\text{Ber} \begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix} := \frac{\det(J_{00} - J_{01}J_{11}^{-1}J_{10})}{\det J_{11}} = \frac{\det J_{00}}{\det(J_{11} - J_{10}J_{00}^{-1}J_{01})} .
\]

Remark 5. The rapid decay of all coefficients in the expansion of \( f \) with respect to odd variables is important for the change of variables formula to hold (although the definition of the integral involves only one coefficient \( f_{1...m} \)).

On invertible even matrices the Berezinian is a multiplicative function, and it is essentially uniquely determined by this property. The fact that it is a rational expression, not a polynomial, is fundamental. The Berezinian is invariant under the following superanalogue of transpose:

\[
\text{Ber } J^T = \text{Ber } J
\]

for

\[
\begin{pmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{pmatrix}^T = \begin{pmatrix} J_{00}^T & J_{10}^T \\ -J_{01}^T & J_{11}^T \end{pmatrix}
\]

(where \( ^T \) for the blocks of the matrix is the ordinary transpose).

The symbol

\[
D(x, \xi) = D(x^1, \ldots, x^n, \xi^1, \ldots, \xi^m)
\]

is the coordinate volume element. By definition, this symbol is defined in every system of coordinates and is multiplied by the Berezinian under a change of coordinates:

\[
D(x, \xi) = D(x', \xi') \frac{D(x, \xi)}{D(x', \xi')} .
\]

The symbol \( D(x, \xi) \) is analogous to \( d^n x \) in ordinary multiple integrals (over even variables). In the ordinary case it is very helpful to express the coordinate volume element \( d^n x \) as the complete wedge product \( dx^1 \wedge \cdots \wedge dx^n \) of the differentials of all coordinates. This is impossible in the supercase since the Berezinian is a fraction and cannot arise from multilinear operations. As a substitute for the maximal wedge product, one can introduce the symbolic bracket

\[
[dx^1, \ldots, dx^n|d\xi^1, \ldots, d\xi^m]
\]

as an alternative notation for \( D(x^1, \ldots, x^n, \xi^1, \ldots, \xi^m) \).
More generally, one can define the \textit{bracket symbol} on an arbitrary free module over a commutative superalgebra as a function of bases that satisfies the following axioms.

- \textit{(Homogeneity)} If we multiply an element of the basis by an invertible factor, then the bracket is multiplied by the same factor raised to power $+1$ for a basis element in an even position and to power $-1$ for a basis element in an odd position$^4$.

- \textit{(Invariance under elementary transformations)} The symbol remains unchanged when a basis vector is replaced by its sum with another basis vector multiplied by a coefficient of appropriate parity.

These axioms model the characteristic properties of the Berezinian. The square bracket of a basis of a free module $V$ is a basis element of the one-dimensional module $\text{Ber}_V$ (the Berezinian of a free module). Calculations with the symbol $[dx_1, \ldots, dx_n, d\xi_1, \ldots, d\xi_m]$ on a supermanifold are easily seen to be as efficient as those with the wedge product $dx_1 \wedge \cdots \wedge dx_n$ on an ordinary manifold. In what follows we use an obvious shorthand notation such as $[dx|d\xi]$.

\textbf{Example 9.} Consider the superspace $\mathbb{C}^{n|m}$ with complex coordinates $z^a = x^a + iy^a$, $\zeta^\mu = \xi^\mu + i\eta^\mu$. Using elementary transformations we obtain

\begin{align*}
[dz, \overline{dz}|d\zeta, \overline{d\zeta}] &= [dx + i dy, dx - i dy|d\xi + i d\eta, d\xi - i d\eta] \\
&= [2dx, dx - i dy|2d\xi, d\xi - i d\eta] = [2dx, -i dy|2d\xi, -i d\eta] \\
\end{align*}

By homogeneity we deduce that

$$[dz, d\overline{z}|d\zeta, d\overline{d\zeta}] = (-2i)^{n-m}[dx, dy|d\xi, d\eta],$$

whence

$$[dx, dy|d\xi, d\eta] = \left(\frac{i}{2}\right)^{n-m} [dz, d\overline{z}|d\zeta, d\overline{d\zeta}].$$

\textbf{Remark 6.} Till now we have been dealing with coordinate superspaces such as $\mathbb{R}^{n|m}$. To integrate over supermanifolds, one must first of all be able to integrate in one chart independently of the choice of coordinates. As in the ordinary case, this is achieved by either integrating Berezin volume forms, that is, objects of the form $f(x) Dx$, or integrating functions against a fixed volume element. The passage to several charts for compact supermanifolds is performed using partitions of unity as in the ordinary case.

\subsection*{5.3. Gaussian integrals. \textit{‘Gaussian factor’}.}

Consider a nondegenerate even quadratic function $Q(x)$ on $\mathbb{R}^{n|2m}$. (Nondegeneracy implies that the odd part of the dimension must be $2m$.) It corresponds to an even symmetric bilinear form, which we denote by the same letter, so that $Q(x) = Q(x,x)$. In coordinates,

$$Q(x) = x^a x^b Q_{ab} = x^a Q_{ab} (-1)^b x^b,$$

\begin{equation}
(5.15)
\end{equation}

$^4$Here we intentionally allow some ambiguity, assuming that either all even basis vectors are in even positions and all odd basis vectors are in odd positions, or vice versa. This enables us to suppress the difference between $dx^a$, $d\xi^\mu$ and $\delta x^a$, $\delta\xi^\mu$ in the notation for the bracket.
where $Q_{ab} = Q_{ba}(-1)^{\tilde{a} \tilde{b}}$. Consider the Gaussian integral defined by $Q$. One can see that

$$\int_{\mathbb{R}^{n|2m}} e^{-Q(x)} \, Dx = \frac{1}{(\text{Ber} \, Q)^{1/2} (\sqrt{\pi})^{n2m}}, \quad (5.16)$$

where Ber $Q$ is the Berezinian of the matrix $(Q_{ab})$. Here $(\sqrt{\pi})^{n2m}$ is the value of the ‘standard’ Gaussian integral that corresponds to the matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & J_{2m} \end{pmatrix},$$

where $I_n$ is the identity matrix and $J_{2m} = \text{diag}((0_{1 \times 1}, \ldots, 0_{1 \times 1}))$, so that

$$Q(x) = (x^1)^2 + \cdots + (x^n)^2 + 2\xi^1\xi^2 + \cdots + 2\xi^{2m-1}\xi^{2m}$$

(here we use separate notation for even and odd variables in the right-hand side while $x$ in the left-hand side stands for the tuple of all variables, odd and even).

The number $(\sqrt{\pi})^{n2m}$, which is a function of dimension, plays an important role in our formulae. We denote this quantity by

$$g_{n|2m} := (\sqrt{\pi})^{n2m} \quad (5.17)$$

and call it the Gaussian factor. (Formally one can use the notation $g_{n|m}$ for arbitrary $n|m$.)

§ 6. Volumes and Riemannian submersions

6.1. Volume elements arising from Riemannian and symplectic structures. An even Riemannian metric or an even symplectic form on a supermanifold generates a volume element, like on an ordinary manifold\(^5\).

Consider an odd covariant tensor $T$ of rank two (without any conditions of symmetry). In local coordinates,

$$T = \delta x^a \otimes \delta x^b T_{ab}(-1)^{\tilde{a} \tilde{b}}, \quad (6.1)$$

where $T_{ab} = T(e_a, e_b)$ and $\widetilde{T}_{ab} = \tilde{a} + \tilde{b}$. Under a change of coordinates we have

$$T_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} (-1)^{\tilde{a}(\tilde{b} + 1)} T_{ab} = \frac{\partial x^a}{\partial x^{a'}} T_{ab} \frac{\partial x^b}{\partial x^{b'}} (-1)^{\tilde{b}(\tilde{b} + 1)}$$

(the last factor is the supertranspose of the Jacobi matrix). It follows that

$$\text{Ber}(T_{a'b'}) = \text{Ber}(T_{ab}) \cdot \left(\text{Ber}\left(\frac{\partial x^a}{\partial x^{a'}}\right)\right)^2.$$

Thus the expression

$$dV := \sqrt{\text{Ber}(T_{ab})} \, Dx \quad (6.2)$$

\(^5\)Odd Riemannian metrics or symplectic structures behave quite differently from their classical prototypes. In particular, there are no volume elements associated with them, which has important consequences. Odd symplectic geometry underlies the Batalin-Vilkovisky quantization method. Odd Riemannian geometry has so far attracted little attention (see [32] for details).
determines an invariant volume element independently of the symmetry assumptions on $T$.

Suppose that we are given an even Riemannian metric

$$
\delta s^2 = \delta x^a \delta x^b g_{ab} = \delta x^a g_{ab}(-1)^{\bar{b}} \delta x^b.
$$

Here $g_{ab} = g_{ab}(x)$, $\tilde{\omega}_{ab} = \tilde{a} + \tilde{b}$ and $g_{ab} = (-1)^{\tilde{a}\tilde{b}} g_{ba}$. In particular,

$$
\delta s^2 = \delta x^a \otimes \delta x^b g_{ab}(-1)^{\tilde{a}\tilde{b}}.
$$

Hence the formula

$$
dV := \sqrt{\text{Ber}(g_{ab})} Dx
$$

determines an invariant Riemannian volume element, just as in the classical case.

A similar assertion holds for even symplectic forms $\omega$. If

$$
\omega = dx^a dx^b \omega_{ba},
$$

where $\omega_{ab} = (-1)^{\tilde{a}+1}(\tilde{b}+1) \omega_{ba}$, then

$$
\omega = \delta x^a \otimes \delta x^b (-1)^{\tilde{a}\tilde{b}} \omega_{ab}(-1)^{\tilde{b}+1}.
$$

Put $T_{ab} = \omega_{ab}(-1)^{\tilde{b}+1}$. One can see that

$$
\text{Ber}(T_{ab}) = \text{Ber}(\omega_{ab}(-1)^{\tilde{b}+1}) = \text{Ber}(\omega_{ab}).
$$

Thus,

$$
dV := \sqrt{\text{Ber}(\omega_{ab})} Dx
$$

is an invariant volume element corresponding to the even symplectic structure (the Liouville volume element).

**Remark 7.** For an ordinary symplectic manifold $M^{2n}$, the Liouville volume element can be written as a top-degree differential form $\omega^n/n!$. An analogue of this expression in the supercase is the pseudodifferential form $e^{-\omega}$ (see [4], for example). By definition the integral of a pseudodifferential form is taken over the totality of variables including the coordinates $x^a$ and their differentials $dx^a$ (regarded as independent commuting variables whose parity is opposite to that of $x^a$). In particular, integration over the variables $dx^a$ in the expression $e^{-\omega}$ (the Gaussian integral!) yields a square root of the Berezinian of the matrix $\omega_{ab}$, whence we arrive at the Liouville volume element as written above. The difference from the purely even case is that this square root cannot be expressed as a polynomial in the coefficients $\omega_{ab}$.

It is known that the Riemannian and symplectic structures are combined in the notion of a Kähler structure. This extends fully to the supercase. We have a tensor $H$ defining an even Hermitian metric on a complex supermanifold $M^{2n|2m}$ with holomorphic local coordinates $z^a$,

$$
H = \delta z^a \otimes \delta \overline{z}^b h_{ab}(-1)^{\bar{a}\bar{b}},
$$

where $h_{ab} = h_{ab}(x)$, $\tilde{\omega}_{ab} = \tilde{a} + \tilde{b}$ and $h_{ab} = (-1)^{\tilde{a}\tilde{b}} h_{ba}$. In particular,
where \( h_{\alpha\bar{\beta}} = (-1)^{\alpha\bar{\beta}} h_{\bar{\beta}\alpha} \). Then \( \delta s^2 = \text{Re} H \) determines a Riemannian metric and \( \omega = -\frac{1}{2} \text{Im} H \) determines a symplectic structure. The corresponding Riemannian and Liouville volume elements coincide. We easily see that the square root can now be extracted and the volume element is given by the formula

\[
dV = \left(\frac{i}{2}\right)^{n-m} \text{Ber}(h_{\alpha\bar{\beta}}) D(z, \bar{z}).
\]

(6.10)

### 6.2. Relation between volumes and Riemannian submersions.

We recall the following classical notion. Consider a locally trivial bundle \( p: E \to M \) whose total space \( E \) and base \( M \) are endowed with Riemannian metrics. The tangent bundle \( TE \) is decomposed into a direct sum \( VE \oplus HE \), where \( VE = \text{Ker} Tp \) (the vertical subbundle) and the horizontal subbundle \( HE = (VE)^\perp \) is defined as the orthogonal complement with respect to the Riemannian metric on \( E \). The restriction of the tangent map \( Tp \) to the horizontal subspace \( HE_z \) maps this subspace isomorphically onto the tangent space \( T_{p(z)}M \). The bundle \( p: E \to M \) is called a Riemannian submersion if the map \( Tp(z)|_{HE_z}: HE_z \to T_{p(z)}M \) is an isometry for all \( z \in E \). This notion transfers directly to supermanifolds.

Consider the direct product coordinates \( x^a, y^i \) on the bundle \( E \), so that the \( x^a \) are local coordinates on \( M \) and the projection sends \( (x^a, y^i) \) to \( (x^a) \). The Riemannian metrics on the supermanifolds \( E \) and \( M \) can be written as

\[
\delta s^2_E = \delta x^a \delta x^b g_{ba}(x, y) + 2 \delta x^a \delta y^i g_{ia}(x, y) + \delta y^i \delta y^j g_{ji}(x, y),
\]

(6.11)

\[
\delta s^2_M = \delta x^a \delta x^b g^0_{ba}(x).
\]

(6.12)

The following proposition is crucial.

**Proposition 5.** The bundle \( p: E \to M \) is a Riemannian submersion if and only if the components of the metrics on the total space and the base satisfy the following relation in direct product coordinates:

\[
g_{ab}(x, y) = g^0_{ab}(x) + g_{ak}(x, y)g^{kl}(x, y)g_{lb}(x, y).
\]

(6.13)

To prove (6.13), notice first of all that the horizontal subspace \( HE_z \) has a basis consisting of vectors of the form

\[
e_a = \frac{\partial}{\partial x^a} - g_{ai}(x, y)g^{ij}(x, y) \frac{\partial}{\partial y^j},
\]

which are projected onto the tangent vectors \( \partial/\partial x^a \) on the base. Then the desired formula (6.13) follows from the definition of a Riemannian submersion,

\[
(e_a, e_b) = \left( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right)(x),
\]

where the right-hand side is the scalar product of tangent vectors on \( M \).

**Corollary 4.** In the Riemannian submersion case, the Berezinian of the metric tensor satisfies

\[
\text{Ber} \begin{pmatrix} g_{ab} & g_{aj} \\ g_{aj} & g_{ij} \end{pmatrix} = \text{Ber} \begin{pmatrix} g^0_{ab} & 0 \\ g_{aj} & g_{ij} \end{pmatrix}.
\]
This is proved by an elementary transformation of rows.

**Corollary 5** (factorization of the volume element). Let \( p: E \to M \) be a locally trivial bundle and a Riemannian submersion. Then

\[
dV_E(z) = dV_{E_z}(z) \cdot dV_M(x),
\]

(6.14)

where \( dV_E(z) \) is the Riemannian volume element for \( E \), \( dV_{E_z}(z) \) is the Riemannian volume element on the fibre \( E_z \) (with respect to the induced metric) and \( dV_M(x) \) is the Riemannian volume element of the base \( M \) at the point \( x = p(z) \).

This yields a ‘Cavalieri principle’ for Riemannian submersions: the volume of the total space is the integral of the volume of the fibres over the base (with respect to the Riemannian volume element of the base). If all fibres have the same volume (which is the case in any homogeneous situation), then the volume of the total space is the product of the volumes of the base and the fibre.

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