Vector model in higher dimensions

Mikhail Goykhman and Michael Smolkin

The Racah Institute of Physics, The Hebrew University of Jerusalem,
Jerusalem 91904, Israel
E-mail: michael.goykhman@mail.huji.ac.il,
michael.smolkin@mail.huji.ac.il

Abstract: We study UV behaviour of the $O(N)$ vector model with quartic interaction in $4 < d < 6$ dimensions to the next-to-leading order in the large-$N$ expansion. We derive and perform consistency checks that provide evidence for the existence of a non-trivial UV fixed point and explore the corresponding CFT. In particular, we calculate a CFT data associated with the three-point functions of the fundamental scalar and Hubbard-Stratonovich fields. We compare our findings with their counterparts obtained within a proposed alternative description of the model in terms of $N+1$ massless scalars with cubic interactions.
1 Introduction

In the past elementary particle theory was based on the assumption that nature must be described by a renormalizable quantum field theory. However, a dramatic progress in the realm of critical phenomena revealed that any non-renormalizable theory will look as if it were renormalizable at sufficiently low energies. From this perspective renormalizable theories correspond to a subset of RG flow trajectories for which all but a few of the couplings vanish. In recent years it has become increasingly apparent that renormalizability is not a fundamental physical requirement, and any realistic quantum field theory may contain non-renormalizable as well as renormalizable interactions. In fact, the experimental success of The Standard Model merely imposes a constraint on the characteristic energy scale of any non-renormalizable interaction. In particular, it
remains unclear what is the fundamental principle behind the choice of the trajectory corresponding to the real world from the infinite number of possible theories.

One of the admissible ways to address this problem is to demand that the Hamiltonian of the theory belongs to a trajectory which terminates at a fixed point in UV. Theories with this property are called asymptotically safe. This concept was originally introduced by Weinberg [1, 2] as an approach to select physically consistent quantum field theories (QFT). It provided an alternative to the standard requirement of renormalizability. The original argument was motivated by an attempt to avoid hitting the Landau pole along the RG flow by imposing a demand that the flow of a QFT terminates at the UV fixed point. Asymptotically safe theories are not necessarily renormalizable in the usual sense. However, they are interacting QFTs with no unphysical singularities at high energies.

Just as the requirement of renormalizability leaves only few possible interaction terms which one is allowed to include in the Lagrangian, so does the requirement of asymptotic safety imposes an infinite number of constraints, limiting the number of physically acceptable theories. Indeed, a UV fixed point in general has only finite number of relevant deformations, therefore any asymptotically safe field theory is entirely specified by a finite number of parameters. Some of such theories are represented by a renormalizable field theory at low energies, while others might be non-renormalizable.

Thus, for instance, a scalar field theory respecting $\phi \to -\phi$ symmetry rules out the possibility of any renormalizable interaction in five dimensions, yet it exhibits a non-trivial UV fixed point with just two relevant deformations in higher dimensions, and therefore it provides an example of asymptotically safe non-renormalizable quantum field theory [1]. An immediate application of the asymptotic safety is related to the problem of quantum gravity [1–4] which is ongoing (see, e.g., [5] and references therein for recent work).

Several techniques for studying UV properties of QFTs in particular and their RG flows in general have been developed. One of the approaches is to perform a dimensional continuation, and subsequently apply a perturbative expansion around the desired integer-valued physical space-time dimension. This is done with the hope that the result will have sufficiently accurate convergence behaviour to be applied to physically meaningful space-time dimensions, such as $d = 3, 4, 5$. In this spirit the Wilson-Fisher fixed point has originally been derived in $4 - \epsilon$ dimensions, in which case at the fixed point the coupling constant is perturbative in $\epsilon$ [6]. In the context of asymptotic safety, gravity has been studied in $2 + \epsilon$ dimensions, see [2–4] for some of the early works.

Recently the $O(N)$ vector model of the scalar fields $\phi^a$, $a = 1, \ldots, N$, and $\sigma$ with the relevant cubic interactions $\phi^a \phi^a \phi$ and $\sigma^3$ in $d = 6 - \epsilon$ dimensions has been extensively
studied perturbatively in $\epsilon$, see e.g., [7, 8]. This model exhibits an IR fixed point, and it was suggested as an alternative description of the critical scalar $O(N)$ model with quartic interaction $(\phi^a \phi^a)^2$. Several consistency checks for this proposal were given [7, 8]. Some of them, such as matching of the coefficients of the three-point functions $\langle \phi^a \phi^b \sigma \rangle$ and $\langle \sigma \sigma \sigma \rangle$ in both theories, fall within a universal class of results obtained for a generic large-$N$ CFT by the methods of conformal bootstrap [9, 10]. However, some other checks, such as matching of the anomalous dimensions of certain operators in these theories to high order both in $\epsilon$ and $1/N$ expansion, are rather non-trivial [7, 8]. Recent work [11] is also dedicated to calculation of non-perturbative imaginary-valued contributions to the scaling dimensions as a result of fluctuations around instanton background.

Large-$N$ expansion suggests another useful and widely applied tool to study non-perturbative aspects of QFTs. A significant part of the large $N$ lore is due to the original observation made by 't Hooft [12] that the $SU(3)$ QCD becomes more tractable when generalized to the $SU(N)$ gauge theory with matter, and considered in the large-$N$ limit (see also [13] and references therein for some of the earlier arguments regarding the accuracy of such an approximation). In the context of the AdS/CFT correspondence the large-$N$ limit of the $SU(N)$ gauge theories has been subsequently related to the weak Newton/string coupling limit of the dual bulk theory in the AdS space [14–16]. Other recent applications of the large-$N$ formalism include studies of the large-$N$ three-dimensional $SU(N)$ gauge theories with the Chern-Simons interaction, and generalizations of these models to study fundamental vector matter at finite temperature and chemical potential (see, e.g., [17–22] and references therein).

Some of the earlier ideas related to applications of the large-$N$ methods are due to the work by Parisi (see [23] and references therein), who in particular developed a systematic proof of renormalizability of the $O(N)$ vector model in $4 < d < 6$ dimensions to each order in the $1/N$ expansion. In particular, one can study the $O(N)$ vector model in a physically interesting dimension $d = 5$. Therefore, this model provides a useful testing ground for the formalism of asymptotic safety [1].

Furthermore, it was shown that the large-$N$ approach can meet the dimensional continuation method (see, e.g., [7] for recent developments in this direction), thanks to the simple observation that the IR Wilson-Fisher fixed point in $d = 4 - \epsilon$ dimensions can be analytically continued to obtain perturbative UV fixed point in $d = 4 + \epsilon$ dimensions, albeit the coupling constant at that fixed point is negative-valued, and therefore the theory is unstable [1]. In fact, the work of [7] was partially motivated by an attempt to design a UV completion of the critical $O(N)$ vector model in $4 < d < 6$ dimensions.\footnote{It should be noted that the cubic $\sigma^3$ theory considered in [7] is still expected to run into instabilities}
In this work we continue studying the UV properties of the $O(N)$ vector model with quartic interaction $(\phi^a \phi^a)^2$ in $4 < d < 6$ dimensions. We scrutinize renormalization in the $d = 5$ case and provide additional evidence for the asymptotic safety of the model. The calculations are done to the next-to-leading order in the $1/N$ expansion, and we successfully recover the known anomalous scaling dimensions associated with the UV CFT [9, 10, 24, 25]. The calculations can be readily extended to general $d$, and therefore similar conclusions hold for the IR CFT in $2 < d < 4$ as well as for the UV CFT in $4 < d < 6$.

For a CFT in general dimension our goal is to derive a CFT data associated with the three-point functions $\langle \phi^a \phi^b \sigma \rangle$ and $\langle \sigma \sigma \sigma \rangle$ to the next-to-leading order in the $1/N$ expansion. The results for the $\langle \phi^a \phi^b \sigma \rangle$ agree with literature [9, 10], while the $O(1/N^{3/2})$ result for $\langle \sigma \sigma \sigma \rangle$ is new. Setting $d = 6 - \epsilon$ in our expression we compare our findings with their counterparts calculated for the IR fixed point of the cubic model studied in [7, 8]. We notice that the $O(1/N^{3/2})$ coefficients of the $\langle \sigma \sigma \sigma \rangle$ three-point functions of these CFTs do not quite match.

The rest of this paper is organized as follows. In section 2 we define the model studied in this work and set our notation. Next we assume existence of the fixed point and derive a general relation between the counterterms at the fixed point. In what follows this relation is used as a consistency check for the existence of the fixed point. In section 3 we focus on the model in 5D. We calculate various counterterms and provide additional evidence in favor of asymptotic safety of the model. In section 4 we review diagrammatic and computational techniques for a CFT in general $d$, and use them to evaluate various anomalous dimensions and two-point functions coefficients. In section 5 we calculate the three-point functions associated with a CFT that emerges at the UV fixed point of the model. We discuss our results in section 6.

## Setup

In this paper we focus on a $d$-dimensional Euclidean vector model governed by the bare action

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} g_2 \phi^2 + \frac{g_4}{N} (\phi^2)^2 \right),$$  

(2.1)

where the field $\phi$ has $N$ components, but we suppress the vector index for brevity. We keep the dimension $d$ general most of the time, having in mind the region $4 < d < 6$, and performing some of the specific calculations in $d = 5$ and $d = 6 - \epsilon$. The UV

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because of negative mode associated with fluctuations around instanton background. See [11] for the recent detailed account of the stability issues in this model.
behaviour of the model is our main objective, but for now we keep the mass parameter $g_2$ to preserve generality.

The straightforward and well-known approach (sometimes referred to as the Hubbard-Stratonovich transformation) to study the model such as (2.1) is to introduce the auxiliary field $s$ which has no impact on the original path integral,

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} g_2 \phi^2 + \frac{g_4}{N} (\phi^2)^2 \right) - \frac{1}{4g_4} \left( s - \frac{2g_4}{\sqrt{N}} \phi^2 \right)^2. \quad (2.2)$$

After a straightforward simplification this action takes the form

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} g_2 \phi^2 - \frac{1}{4g_4} s^2 + \frac{1}{\sqrt{N}} s \phi^2 \right). \quad (2.3)$$

As usual, the model (2.3) has to be renormalized.

To begin with, we introduce the renormalized fields $\tilde{\phi}$, $\tilde{s}$

$$\phi = \sqrt{Z_\phi} \tilde{\phi}, \quad s = \sqrt{Z_s} \tilde{s}, \quad (2.4)$$

where the field strength renormalization constants $Z_{\phi,s}$ are expressed in terms of the counterterms $\delta_{\phi,s}$ as

$$Z_\phi = 1 + \delta_\phi, \quad Z_s = 1 + \delta_s. \quad (2.5)$$

Similarly, the renormalized mass $m$ and coupling $\tilde{g}_4$ are defined by

$$g_2 Z_{\phi} = m^2 + \delta_m, \quad (2.6)$$
$$g_4 Z_{\phi}^2 = \tilde{g}_4 + \delta_4, \quad (2.7)$$

where $\delta_m$ and $\delta_4$ are mass and quartic coupling counterterms respectively. \(^2\)

It turns out that all counterterms vanish to leading order in the $1/N$ expansion. Therefore to carry out calculations to the next-to-leading order, it is sufficient to linearize the full action with respect to various counterterms. The action (2.3) in terms of renormalized parameters is thus given by

$$S = \int d^d x \left( \frac{1}{2} (\partial_\mu \tilde{\phi})^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{2} \delta_\phi (\partial_\mu \tilde{\phi})^2 + \frac{1}{2} \delta_m \tilde{\phi}^2 \right.$$  
$$\left. - \frac{1}{4\tilde{g}_4} \left( 1 + 2\delta_\phi + \delta_s - \frac{\delta_4}{\tilde{g}_4} \right) \tilde{s}^2 + \frac{1}{\sqrt{N}} \left( 1 + \frac{1}{2} \delta_s + \delta_\phi \right) \tilde{s} \tilde{\phi}^2 \right). \quad (2.9)$$

\(^2\)It follows from the definition (2.7) that the interaction term in the original bare action (2.1) can be expressed as the following sum

$$g_4 (\phi^2)^2 = \tilde{g}_4 (\tilde{\phi}^2)^2 + \delta_4 (\tilde{\phi}^2)^2. \quad (2.8)$$
It is convenient to define the following combinations of the counterterms,
\[
\hat{\delta}_s = 2\delta_\phi + \delta_s - \frac{\delta_4}{\sqrt{g_4}},
\]
\[
\hat{\delta}_4 = \frac{1}{2}\delta_s + \delta_\phi,
\]
in terms of which the action can be rewritten as follows
\[
S = \int d^d x \left( \frac{1}{2} (\partial\phi)\partial\phi + \frac{1}{2} m^2 \phi^2 - \frac{1}{4g_4} \bar{s}^2 + \frac{1}{\sqrt{N}} \bar{s} \bar{\phi}^2 
+ \frac{1}{2} \delta_\phi (\partial\phi)\partial\phi - \frac{1}{2} \delta_m \phi^2 - \frac{1}{4g_4} \hat{\delta}_4 \bar{s}^2 + \frac{1}{\sqrt{N}} \frac{\hat{\delta}_4}{\sqrt{g_4}} \bar{s} \bar{\phi}^2 \right).
\]
(2.12)

Loop corrections along with renormalization conditions fix the counterterms \(\delta_\phi, \delta_m, \hat{\delta}_s, \) \text{and} \(\hat{\delta}_4\). Inverting (2.10), (2.11) one then obtains
\[
\delta_s = 2 \left( \frac{\hat{\delta}_4}{\sqrt{g_4}} - \delta_\phi \right),
\]
\[
\frac{\delta_4}{g_4} = 2 \frac{\hat{\delta}_4}{\sqrt{g_4}} - \hat{\delta}_s.
\]
(2.13)\hspace{1cm} (2.14)

Finally, renormalization of the quartic coupling constant can be read off (2.5), (2.7) and (2.14)
\[
g_4 = \bar{g}_4 \left( 1 + 2 \frac{\hat{\delta}_4}{\sqrt{g_4}} - 2\delta_\phi - \hat{\delta}_s \right) = \bar{g}_4 \left( 1 + \delta_s - \hat{\delta}_s \right),
\]
(2.15)
where in the last equality (2.13) was used.

The Callan-Symanzik equation can be used to impose various relations between the counterterms of the theory. In particular, using it for the three-point function \(\langle \phi \phi s \rangle\), we argue in Appendix A that at the UV (IR) fixed point in \(4 < d < 6\) \((2 < d < 4)\) dimensions one obtains
\[
\mu \frac{\partial}{\partial \mu} \left( \delta_s - \hat{\delta}_s \right) = 0 + \mathcal{O}(1/N^2),
\]
(2.16)
where \(\mu\) is an arbitrary renormalization scale. In section 3 we explicitly confirm this relation in \(d = 5\) at the next-to-leading order in the \(1/N\) expansion. We interpret it as a consistency check of the assumption that the theory is asymptotically safe.
3 UV fixed point in $d = 5$

A vast amount of literature, starting from the earlier works [1, 23], is dedicated to studying properties of the UV conformal fixed point of the $O(N)$ vector models with quartic interaction in $4 < d < 6$. While the quartic interaction is non-renormalizable in $d > 4$, one can take advantage of the fact that this model is renormalizable at each order in the $1/N$ expansion [23]. In this section we consider (2.12) in $d = 5$ dimensions. The main goal is to systematically derive various anomalous dimensions [9, 10, 24, 25] and to verify explicitly that they satisfy certain constraints which are expected to hold at the UV fixed point. The calculations are carried out to the next-to-leading order in the $1/N$ expansion.

To begin with, we list all necessary Feynman rules for the model (2.12). A solid line will be used to denote the propagators of various components of the scalar field $\phi$, whereas a dashed line is associated with the propagator of the Hubbard-Stratonovich auxiliary field $s$. Obviously, the matrix of propagators is diagonal, whereas each non-trivial interaction vertex carries two identical vector indices. Hence, for simplicity we suppress the Kronecker delta which explicitly emphasizes these facts. We also omit the momentum conservation delta function at each vertex. Vertices are denoted by a solid blob, whereas propagators are enclosed by black dots (absence of dots indicates an amputated external leg).

Note that we introduced a counterterm $\delta_0$. It is completely determined by the renormalization condition $\langle \tilde{s} \rangle = 0$. In what follows we assume this condition has been satisfied without elaborating the details.

The full propagator of the Hubbard-Stratonovich field $\tilde{s}$ to leading order in the $1/N$ expansion is thus given by an infinite sum of bubble diagrams.
This is a geometric series which can be readily written in a closed form.\footnote{One should take into account that each bubble comes with the symmetry factor 1/2.} We denote it by $G_s(p)$ and represent it diagrammatically by a wavy line

$$
p_{\ldots} = G_s(p) = -2\tilde{g}_4 \sum_{n=0}^{\infty} (-4\tilde{g}_4 B)^n = \frac{-2\tilde{g}_4}{1+4\tilde{g}_4 B},
$$

where each bubble in the infinite series is associated with a UV divergent loop integral

$$B(p) = \int \frac{d^5q}{(2\pi)^5} \frac{1}{(q^2 + m^2)((p + q)^2 + m^2)}.$$  \hspace{1cm} (3.1)

To regularize the UV divergence we introduce a spherically symmetric sharp cutoff $\Lambda$

$$B(p) = \Lambda \frac{1}{12\pi^3} - \frac{1}{64\pi^2 p} \left((4m^2 + p^2) \tan^{-1} \left( \frac{p}{2|m|} \right) + 2|m|p \right).$$  \hspace{1cm} (3.2)

The power law divergence can be eliminated by adjusting the counterterm $\hat{\delta}_s$. Unlike logarithmic divergences, the power law divergences depend on the details of regularization scheme. For instance, they are absent in dimensional regularization. Hence, we simply ignore such divergences in what follows to reduce clutter in the equations. In particular, in the large momentum limit which we are interested in for the purpose of finding the UV fixed point, we obtain

$$B \big|_{p \gg m} = -\frac{p}{128\pi}.$$  \hspace{1cm} (3.3)

Using the asymptotic behaviour (3.3) rather than the full expression (3.2), and thus focusing on the UV regime, one avoids passing through the pole of the propagator $G_s(p)$ at some finite momentum \cite{23}. Therefore in all of our calculations we take the limit $\tilde{\gamma}_4 \rho \gg 1$ to simplify the propagator of the Hubbard-Stratonovich field (see, \textit{e.g.}, \cite{7}),

$$G_s(p) = -\frac{1}{2B(p)} = \frac{64\pi}{p}.$$  \hspace{1cm} (3.4)

By assumption, the beta function for $\tilde{\gamma}_4$ is expected to exhibit a UV fixed point. As pointed out in Appendix A, the counterterms must therefore satisfy equation (2.16). In particular, (2.16) serves as a non-trivial consistency check of the assumption about the UV behaviour of the beta function. In order to explicitly verify this identity, we proceed to calculation of the counterterms to $\mathcal{O}\left(\frac{1}{N}\right)$ order. Of course, it is natural to set $m = 0$ since we are ultimately interested in the UV behaviour of the loop integrals.
3.1 Scalar field two-point function

To leading order in the $1/N$ expansion only $G_s(p)$ exhibits loop corrections. As argued above, there are no non-trivial UV divergences associated with loop diagrams at this order, and all counterterms thus vanish. However, a non-trivial renormalization is induced at the next-to-leading order in $1/N$.

The $\mathcal{O}(1/N)$ contribution to the counterterms $\delta_{\phi,m}$ are derived from the requirement that the sum of the loop diagram

$$
\delta_{\phi,m} = -\frac{1}{N} \frac{64}{15\pi^2} \log \left( \frac{\Lambda}{\mu} \right),
$$

where $\mu$ is an arbitrary renormalization scale. The anomalous dimension of the scalar field $\phi$ at the UV fixed point can be readily evaluated using the Callan-Symanzik equation for the two-point function of $\tilde{\phi}$, see Appendix A

$$
\gamma_{\phi} = \frac{1}{2} \frac{\partial}{\partial \log \mu} \delta_{\phi} = \frac{1}{N} \frac{32}{15\pi^2}.
$$

This result is in full agreement with $[9, 10, 24, 25]$.

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\textsuperscript{4}If we keep the mass in the action (2.12), then there is an additional counterterm of the form

$$
\delta_m = -\frac{1}{N} \frac{320m^2}{3\pi^2} \log \left( \frac{\Lambda}{\mu} \right).
$$

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3.2 Renormalization of the interaction vertex

There are two loop diagrams that contribute to renormalization of the interaction vertex at the next to leading order in the $1/N$ expansion. To calculate the corresponding counterterm $\delta_4$ it is enough to set all external momenta to zero. The first diagram takes the form

\[
\left(-\frac{2}{\sqrt{N}}\right)^3 \int \frac{d^5 q}{(2\pi)^5} \left(\frac{-1}{2B(q)}\right)^2 \frac{1}{(q^2)^2} = -\frac{1}{N^{3/2}} \frac{128}{3\pi^2} \log \Lambda
\]

whereas the second diagram is given by

\[
\left(-\frac{2}{\sqrt{N}}\right)^5 N \int \frac{d^5 p}{(2\pi)^5} \left(\frac{-1}{2B(q)}\right)^2 \frac{1}{q^2} \left(-\frac{1}{2}\right) \frac{\partial B(q)}{\partial m^2} = -\frac{1}{N^{3/2}} \frac{512}{3\pi^2} \log \Lambda,
\]

where we used (3.1) to get the following simple relation

\[
-\frac{1}{2} \frac{\partial B(q)}{\partial m^2} = \int \frac{d^5 p}{(2\pi)^5} \frac{1}{(p^2 + m^2)^2((p + q)^2 + m^2)}.
\]
In particular, the loop integral can be calculated by taking derivative of (3.2) with respect to the mass parameter.

Finally, the tree level counterterm contribution can be read off the Feynman rules we listed in the beginning of this section. Combining everything together and requiring cancellation of the divergences, we obtain

\[
\hat{\delta}_4 \frac{\hat{g}_4}{\sqrt{g}_4} = -\frac{1}{N} \frac{320}{3\pi^2} \log \left( \frac{\Lambda}{\mu} \right).
\]  

(3.9)

3.3 Auxiliary field propagator

Finally, we proceed to calculate the counterterm \( \hat{\delta}_s \). To this end, we perform renormalization of the Hubbard-Stratonovich propagator. There are five loop diagrams which contribute at the next to leading order in the \( 1/N \) expansion. We tag these diagrams with \( C_i, i = 1, \ldots, 5 \) and denote their external momentum by \( u \). For simplicity, all the diagrams are amputated, i.e., the two external \( s \) propagators are factored out. Thus, for instance

\[
C_1 = \begin{array}{cccc}
- & - & u & - \\
p + u & r & p + r + u & p + r \\
p & r + u & - & -
\end{array}
\]

is given by

\[
C_1 = \frac{1}{2} \left( -\frac{2}{\sqrt{N}} \right)^4 N \int \frac{d^5 r}{(2\pi)^5} \frac{64\pi}{r} \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p^2(p + u)^2(p + r)^2(p + r + u)^2}.
\]  

(3.10)

This integral is somewhat laborious to evaluate in full generality. However, we are only interested in its behaviour at large momenta

\[
C_1 = \frac{1024\pi}{N} \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p^2(p + u)^2} \int \frac{d^5 r}{(2\pi)^5} \frac{1}{r^5} = \frac{1}{N} \frac{256}{3\pi^2} B(u) \log \Lambda + \ldots.
\]  

(3.11)

Similarly,
is given by

\[ C_2 = \frac{1}{2} \left( -\frac{2}{\sqrt{N}} \right)^6 N^2 \int \frac{d^5 q}{(2\pi)^5} \frac{64\pi}{q} \frac{64\pi}{|q + u|} \left( \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p^2(p + u)^2(p + q + u)^2} \right)^2. \]  

(3.12)

Expanding it in the region of large momenta, gives

\[ C_2 = \frac{2^{18}\pi^2}{N} \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p^2(p + u)^2} \int \frac{d^5 r}{(2\pi)^5} \frac{1}{(r^2)^2} \int \frac{d^5 q}{(2\pi)^5} \frac{1}{q^2(q + u)^2} \]

\[ = \frac{1}{N} \frac{1024}{3\pi^2} B(u) \log \Lambda + \ldots. \]  

(3.13)

Next, let us evaluate

\[ C_3 = - - u - p - q - p - u - \]

Using Feynman rules results in the following integral expression

\[ C_3 = \left( -\frac{2}{\sqrt{N}} \right)^4 N \int \frac{d^5 p}{(2\pi)^5} \frac{1}{(p^2)^2(p + u)^2} \int \frac{d^5 q}{(2\pi)^5} \frac{64\pi}{q(p + q)^2}. \]  

(3.14)

Performing integration over \( q \) and focussing on the logarithmic divergence\(^5\), yields

\[ C_3 = -\frac{1}{N} \frac{256}{15\pi^2} B(u) \log \Lambda + \ldots. \]  

(3.15)

The two remaining diagrams are

\[ C_4 = 2 \times - - u - \]

and

\(^5\)Recall that we ignore scheme dependent power law divergences.
An extra factor of two in $C_4$ comes from the interchange of the ordinary vertex with the counterterm. More specifically, the integral expressions for these diagrams read

$$C_4 = 2 \times \left( -\frac{2}{\sqrt{N}} \right) \left( -\frac{2}{\sqrt{N}} \sqrt{g_4} \right) N \int \frac{d^5 p}{(2\pi)^5} \frac{1}{p^2 (p + u)^2} = 4 \frac{\hat{\delta}_4}{\sqrt{g_4}} B(u), \quad (3.16)$$

$$C_5 = \left( -\frac{2}{\sqrt{N}} \right)^2 N \int \frac{d^5 p}{(2\pi)^5} \frac{-p^2 \delta_\phi}{(p^2)^2 (p + u)^2} = -4 \delta_\phi B(u). \quad (3.17)$$

Summing up the loop contributions to the Hubbard-Stratonovich propagator we obtain the full expression to the $O(1/N)$ order,

$$\bullet \bullet \bullet \bullet \bullet + \bullet \bullet \bullet \bullet \bullet = G_s^{(1/N)} = -\frac{1}{2B} + \left( -\frac{1}{2B} \right)^2 C,$$

where we denoted the total of loop diagrams $C_i, i = 1, \ldots, 5$ as

$$C = \sum_{i=1}^{5} C_i, \quad (3.18)$$

and depicted it symbolically with a blob. On the other hand, the auxiliary field propagator is given by

$$G_s^{(1/N)} = -\frac{1}{2B} (1 - \hat{\delta}_s). \quad (3.19)$$

Comparing these relations, and using expressions (3.11), (3.13), (3.15), (3.16), (3.17) for $C_i, i = 1, \ldots, 5$, and expressions (3.6), (3.9) for $\delta_\phi, \hat{\delta}_4$, we then obtain

$$\mu \frac{\partial}{\partial \mu} \left( 2 \left( \frac{\hat{\delta}_4}{\sqrt{g_4}} - \delta_\phi \right) - \hat{\delta}_s \right) = 0 + O(1/N^2). \quad (3.20)$$

Using the relation (2.13) we observe that the $d = 5$ result (3.20) agrees with the general expression (2.16). As discussed in section 2 this gives a non-trivial consistency check that the theory reaches a UV fixed point, thereby achieving asymptotic safety.
Finally, using the Callan-Symanzik equation for the two-point function of $\tilde{s}$, as discussed in appendix A, we derive the anomalous dimension of the auxiliary field $\tilde{s}$ at the order $O(1/N)$,

\[ \gamma_s = \frac{1}{2} \frac{\partial}{\partial \log \mu} \delta_s = \frac{1}{N} \frac{512}{5\pi^2}, \]  

which agrees with [9, 10, 24, 25].

4 Position space calculation in general dimension

Our goal in this section is to derive the next-to-leading $O\left(\frac{1}{N}\right)$ order corrections to the CFT propagators of the fundamental scalar field $\phi$ and the auxiliary Hubbard-Stratonovich field $s$, working in position space at the UV fixed point. Space-time dimension $d$ will be assumed to be general $4 < d < 6$ in this section (the results of this section are also applicable to the IR conformal fixed point in $2 < d < 4$ dimensions). To the leading order in the large-$N$ expansion the propagators are given by (below we will continue to skip writing down the $O(N)$ vector indices and the explicit factors of the Kronecker delta)

\[ \langle \phi(x_1)\phi(x_2) \rangle = C_\phi \frac{1}{|x_{12}|^{2\Delta_\phi}} , \]  

\[ \langle s(x_1)s(x_2) \rangle = C_s \frac{1}{|x_{12}|^{2\Delta_s}} , \]

where the amplitudes $C_{\phi,s}$ will be written down below, and $x_{12} = x_1 - x_2$. To calculate sub-leading $1/N$ corrections to these propagators we will use the technique developed in [24, 25], following the recent treatment in [26]. We are looking for the final expressions at the $O(1/N)$ order of the form

\[ \langle \phi(x_1)\phi(x_2) \rangle = C_\phi (1 + A_\phi) \frac{\mu^{-2\gamma_\phi}}{|x_{12}|^{2(\Delta_\phi + \gamma_\phi)}} , \]  

\[ \langle s(x_1)s(x_2) \rangle = C_s (1 + A_s) \frac{\mu^{-2\gamma_s}}{|x_{12}|^{2(\Delta_s + \gamma_s)}} , \]

where $\mu$ is an arbitrary renormalization scale. Everywhere in this paper we reserve the symbols $\Delta_{\phi,s}$ for the engineering scaling dimensions of the fields $\phi, s$,

\[ \Delta_\phi = \frac{d}{2} - 1 , \quad \Delta_s = 2 , \]  

satisfying the simple relation

\[ 2\Delta_\phi + \Delta_s = d , \]
and prefer to separate the anomalous dimensions $\gamma_{\phi,s}$ into explicit additional terms. While we have already derived the anomalous dimensions $\gamma_{\phi,s}$ in section 2 (focusing on the $d = 5$ dimensional model), in this section we are interested in deriving the amplitude correction coefficients $A_{\phi,s}$. Our results for these coefficients will agree with the literature (see, e.g., [27]).

### 4.1 Preliminary remarks

Performing the Fourier transform

$$\int \frac{d^d k}{(2\pi)^d} e^{i k \cdot x_1} \frac{1}{(k^2)^{\frac{d}{2} - \Delta}} = \frac{2^{2\Delta - d}}{\pi^{\frac{d}{2}}} \frac{\Gamma(\Delta)}{\Gamma\left(\frac{d}{2} - \Delta\right)} \frac{1}{|x|^2\Delta}, \quad (4.7)$$

we obtain that the coefficient $C_{\phi}$ in (4.3) is given by

$$C_{\phi} = \frac{2^{2\Delta_{\phi} - d}}{\pi^{\frac{d}{2}}} \frac{\Gamma(\Delta_{\phi})}{\Gamma\left(\frac{d}{2} - \Delta_{\phi}\right)}. \quad (4.8)$$

Using expression (3.4) for the $s$-propagator in momentum space, as well as the result for the bubble loop integral $B$ in general $d$ [7]

$$B(p) = \int \frac{d^d q}{(2\pi)^d} q^2 (p + q)^2 = -\frac{(p^2)^{\frac{d}{2} - 2}}{2^d(4\pi)^{\frac{d+1}{2}} \Gamma\left(\frac{d-1}{2}\right) \sin\left(\frac{\pi d}{2}\right)} \quad (4.9)$$

we obtain $^6$

$$C_s = \frac{2^d \Gamma\left(\frac{d-1}{2}\right) \sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2} - 1\right)}. \quad (4.10)$$

We proceed to reviewing the Feynman rules for CFT in position space [24, 25]. Working at the (UV) conformal fixed point, we do not need to keep separate notations (solid or dashed) for the $\phi$ and $s$ propagator lines. Instead, we put the power index $2a$ (determined by the scaling dimension $a$ of the corresponding field) on top of the line. We will have to start calculation of each diagram by collecting and writing down explicitly all the factors of $C_{\phi,s}$. The line itself will therefore be assumed to be normalized to unity.

$$x_1 \quad \begin{array}{c} 2a \end{array} \quad x_2 \quad = \frac{1}{|x_{12}|^{2a}}$$

This way a loop diagram simply amounts to adding two powers together.

$^6$This agrees with [7] after we reconcile conventions for the normalization of the Hubbard-Stratonovich field.
We will also need the propagator splitting/merging relation \([24, 25]\)
\[
\int d^d x_3 \frac{1}{(x_3^2)^a((x_3 - x_{12})^2)^b} = U(a, b, d - a - b) \frac{1}{(x_{12}^2)^{a+b-d+2}},
\]
where we denoted
\[
U(a, b, c) = \pi^{d/2} \frac{\Gamma\left(\frac{d}{2} - a\right) \Gamma\left(\frac{d}{2} - b\right) \Gamma\left(\frac{d}{2} - c\right)}{\Gamma(a) \Gamma(b) \Gamma(c)}.
\]
Expression (4.11) can be graphically represented as \([24, 25]\)

\[
2a \quad 2b = 2(a + b) - d \times U(a, b, d - a - b)
\]

where the middle point in the l.h.s. is assumed to be integrated over. Additionally, we will make use of the following identity for \(a_1 + a_2 + a_3 = d\), also known as the uniqueness relation \([24, 25, 28, 29]\)
\[
\int d^d x \frac{1}{|x_1 - x_2|^{2a_1}|x_2 - x_3|^{2a_2}|x_3 - x_1|^{2a_3}} = \frac{U(a_1, a_2, a_3)}{|x_{12}|^{d-2a_3}|x_{13}|^{d-2a_2}|x_{23}|^{d-2a_1}},
\]
where the function \(U\) was defined in (4.12). It can be graphically represented as \([24, 25]\)

\[
2a_1 \quad 2a_2 = \alpha \times \left(\frac{2}{\sqrt{N}}\right) U(a_1, a_2, a_3)
\]

where the middle vertex in the l.h.s. is assumed to be integrated over, and we have denoted \(\alpha = d - 2a_3\), \(\beta = d - 2a_2\), \(\gamma = d - 2a_1\). Here we have also taken into account the Feynman rule for the vertex

\[
= \frac{2}{\sqrt{N}}
\]
4.2 \( \phi \) propagator

The simplest application of the rules formulated above is given by calculation of the \( \phi \) propagator. We are considering the diagram

\[
P = x_1 \xrightarrow{2\Delta_\phi} x_3 \xrightarrow{2\Delta_\phi} x_4 \xrightarrow{2\Delta_\phi} x_2
\]

This diagram is divergent, and we regularize it by adding a small correction \( \delta \) to the power of the \( s \) line \([24, 25]\). Applying the rules formulated above, we calculate it as

\[
P(x_1, x_2) = \left( -\frac{2}{\sqrt{N}} \right)^2 C_\phi^3 C_s \mu^{-\delta} \int d^d x_3, 4 |x_{13}|^{2\Delta_\phi} |x_{24}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi + 2\Delta_s + \delta} \frac{1}{|x_{12}|^{2\Delta_\phi + \delta}}
\]

\[
= \frac{4}{N} C_\phi^3 C_s \mu^{-\delta} U \left( \Delta_\phi, \Delta_\phi + \Delta_s + \delta, \Delta_\phi + \Delta_s + \delta \right) \int d^d x_3 \frac{1}{|x_{13}|^{2\Delta_\phi} |x_{23}|^{d+\delta}} (4.14)
\]

\[
= \frac{4}{N} C_\phi^3 C_s \mu^{-\delta} U \left( \Delta_\phi, \Delta_\phi + \Delta_s + \delta, \Delta_\phi + \Delta_s + \delta \right) \frac{1}{|x_{12}|^{2\Delta_\phi + \delta}}.
\]

Combining this with the free propagator (4.1) and expanding around \( \delta = 0 \) we obtain

\[
\langle \phi(x_1)\phi(x_2) \rangle = \frac{C_\phi}{|x_{12}|^{2\Delta_\phi}} \left( 1 + \left( \frac{2\gamma_\phi}{\delta} + A_\phi + \mathcal{O}(\delta) \right) \frac{1}{(|x_{12}|\mu)^\delta} \right). \quad (4.15)
\]

Here

\[
\gamma_\phi = \frac{1}{N} \frac{2^d \sin \left( \frac{\pi d}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\pi^{3/2}(d-2)d\Gamma \left( \frac{d-2}{2} \right)}, \quad (4.16)
\]

\[
A_\phi = -\frac{1}{N} \frac{2^d (d^2 + 2d - 4) \sin \left( \frac{\pi d}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\pi^{3/2}(d-2)^2d\Gamma \left( \frac{d-2}{2} \right)}. \quad (4.17)
\]

Taking further the limit \( \delta \to 0 \) and subtracting the divergent \( \mathcal{O}(1/\delta) \) contribution, we obtain (4.3) to the next-to-leading order of the \( 1/N \) expansion, with \( \gamma_\phi, A_\phi \) given by (4.16), (4.17). In particular in \( d = 6 - \epsilon \) we obtain

\[
\gamma_\phi = \frac{\epsilon}{N} + \mathcal{O}(\epsilon^2), \quad (4.18)
\]

\[
A_\phi = -\frac{11\epsilon}{6N} + \mathcal{O}(\epsilon^2), \quad (4.19)
\]

while in \( d = 5 \) the \( \gamma_\phi \) expression reproduces (3.7). In general \( d \) our expressions agree with [9, 10, 24, 25, 27].
4.3 Hubbard-Stratonovich propagator

We proceed to derive the $1/N$ corrections to propagator of the auxiliary field $s$. Similarly to our calculation in section 3, we have three contributing diagrams $C_{1,2,3}$ (in this section we are using regularization by additional powers in propagators, and skip writing down the explicit counterterm diagrams corresponding to $C_{4,5}$ of section 3). As prescribed above, for each diagram we begin by collecting the amplitudes $C_{\phi,s}$ in explicit prefactors. To this end, denote

$$C_1(x_1, x_2) = C_3^2 C_4^4 \tilde{C}_1(x_1, x_2) \mu^{-\delta}, \quad (4.20)$$

where $\tilde{C}_1(x_1, x_2)$ is given by

$$\tilde{C}_1 = x_1 \rightarrow \frac{2\Delta_\phi - \eta}{2\Delta_\phi + \delta} \times \int \frac{d^d x_4}{|x_{34}|^{2\Delta_\phi - \frac{2\delta}{2}} |x_{45}|^{2\Delta_\phi + \delta} |x_{46}|^{2\Delta_\phi - \frac{2\delta}{2}}}.$$ 

Here we added an extra power $\eta$ to the $\phi$ propagator lines [24–26]. Since we are interested in sending $\delta \rightarrow 0$ at the end, this can be done as long as $\eta = O(\delta)$. Indeed, the resulting diagram is symmetric w.r.t. $\eta \rightarrow -\eta$, and therefore its series expansion around $\eta = 0$ takes the form

$$\tilde{C}_1 = f_0 + f_2 \eta^2 + f_4 \eta^4 + \ldots \quad (4.21)$$

Further expanding it around $\delta = 0$ and keeping in mind that the coefficients $f_a$ have at most simple poles at $\delta = 0$, we conclude that $\tilde{C}_1 = f_0$ in the limit $\delta \rightarrow 0$, so far as $\eta = O(\delta)$. It is convenient to choose $\eta = \delta/2$ [26] which gives

$$\tilde{C}_1 = \frac{1}{2} \left( -\frac{2}{\sqrt{N}} \right)^4 \frac{1}{U_\phi} \int d^d x_4 \frac{1}{|x_{34}|^{2\Delta_\phi - \frac{2\delta}{2}} |x_{45}|^{2\Delta_\phi + \delta} |x_{46}|^{2\Delta_\phi - \frac{2\delta}{2}}}.$$ 

This makes it possible to apply the uniqueness equation (4.13) to the integral over $x_4$. Applying subsequently the propagator merging relation (4.11) to the integral over $x_5$, we obtain

$$\tilde{C}_1 = \frac{8}{N} U \left( \Delta_\phi - \frac{\delta}{4}, \Delta_\phi - \frac{\delta}{4}, \Delta_\phi + \frac{\delta}{2} \right) U \left( \frac{d + \delta}{2}, \frac{d + \delta}{2}, -\delta \right) \times \int d^d x_{3,6} \frac{1}{|x_{13}|^{2\Delta_\phi + \delta} |x_{26}|^{2\Delta_\phi + \delta} |x_{30}|^{2\Delta_\phi + \delta}}.$$ 

(4.23)
To avoid repeating the same calculation in derivation of $C_{1,2,3}$, we postpone taking the integral over two edge points (the integral over $x_{3,6}$ in case of $C_1$) until after we have assembled all the terms $C_{1,2,3}$ together.

For the next diagram contributing to the $s$ propagator at the $\mathcal{O}(1/N)$ order we denote

$$C_2(x_1, x_2) = C_s^4 C_\phi^6 \tilde{C}_2(x_1, x_2) \mu^{-\delta}, \quad (4.24)$$

where

$$\tilde{C}_2 = x_1 \quad 2\Delta_s - \eta \quad x_7 \quad 2\Delta_s + \delta/2 \quad x_4 \quad 2\Delta_s + \eta \quad x_5 \quad 2\Delta_s + \delta/2 \quad x_6 \quad 2\Delta_s - \eta \quad x_2$$

Notice that here we regularized two internal $s$ propagators with the additional power $\delta/2$ rather than $\delta$. This is done for the consistency of regularization of $C_{1,2,3}$, which demands that the total compensating power of $\mu$ in the r.h.s. of (4.24) is $\mu^{-\delta}$, just like in the other diagrams $C_{1,3}$ [26]. We have also added extra exponents $\eta$ to some of the $\phi$ lines, relying on the same technique as we used in evaluating the diagram $\tilde{C}_1$. Setting $\eta = \frac{\delta}{2}$ [26] we will be able to calculate integrals over $x_{3,6}$ using the uniqueness relation (4.13),

$$\tilde{C}_2 = \frac{1}{2} \left(- \frac{2}{\sqrt{N}}\right)^6 N^2 \int d^d x_{7,8} \frac{1}{|x_{17}| \cdot |x_{28}|} \frac{1}{|x_{48}|^{2\Delta_s + \delta}} \int d^d x_{4,5} \frac{1}{|x_{48}|^{2\Delta_s + \frac{\delta}{2}} |x_{57}|^{2\Delta_\phi + \frac{\delta}{2}}} \times \int d^d x_3 \frac{1}{|x_{37}|^{2\Delta_\phi - \frac{\delta}{4}} |x_{34}|^{2\Delta_s + \frac{\delta}{4}} |x_{35}|^{2\Delta_\phi}} \int d^d x_6 \frac{1}{|x_{68}|^{2\Delta_\phi - \frac{\delta}{4}} |x_{56}|^{2\Delta_s + \frac{\delta}{4}} |x_{46}|^{2\Delta_\phi}}. \quad (4.25)$$

We can then write it down as

$$\tilde{C}_2 = \frac{32}{N} U \left(\Delta_\phi - \frac{\delta}{4}, \Delta_s + \frac{\delta}{4}, \Delta_\phi\right)^2 \int d^d x_{7,8} \frac{1}{|x_{17}| \cdot |x_{28}|^{2\Delta_s + \delta}} \tilde{C}_2(x_7, x_8; 0), \quad (4.26)$$

where we denoted

$$\tilde{C}_2(x_7; x_8; \eta') = \int d^d x_{4,5} \frac{1}{|x_{48}|^{2d - 3\Delta_s - \eta'} |x_{57}|^{2d - 3\Delta_s + \eta'} |x_{45}|^{2\Delta_s + \delta} |x_{47}|^{\Delta_s - \eta'} |x_{58}|^{\Delta_s + \eta'}. \quad (4.27)$$

Here we have re-introduced an additional exponent $\eta'$ [24–26]. By performing change of variables of integration

$$x_4 \rightarrow x_7 + x_8 - x_5, \quad x_5 \rightarrow x_7 + x_8 - x_4, \quad (4.28)$$
one can demonstrate that \( \tilde{c}_2(x_7, x_8; \eta') = \tilde{c}_2(x_7, x_8; -\eta') \). Therefore any choice of \( \eta' = \mathcal{O}(\delta) \) will not change the value of the integral in the \( \delta \to 0 \) limit, similarly to the calculation of \( \tilde{C}_1 \). Specifically choosing \( \eta' = \frac{\delta}{2} \) will make it possible to apply the uniqueness relation (4.13) to the integral over \( x_4 \) [26]. Applying subsequently the propagator merging relation (4.11) to the integral over \( x_5 \) gives

\[
\tilde{C}_2 = \frac{32}{N} U \left( \Delta_\phi - \frac{\delta}{2}, \Delta_s + \frac{\delta}{4}, -\Delta_\phi \right)^2 U \left( d - \frac{3\Delta_s}{2} - \frac{\delta}{4}, \Delta_s + \frac{\Delta_s}{2} - \frac{\delta}{4} \right) \times \frac{1}{|x_{17}|^{2\Delta_s + \delta} |x_{28}|^{2\Delta_s + \delta} |x_{78}|^{2d - 2\Delta_s + \delta}}. (4.29)
\]

Finally, for the third contributing diagram denote

\[
C_3(x_1, x_2) = C_s^3 C_\phi^4 \tilde{C}_3(x_1, x_2) \mu^{-\delta}, \quad (4.30)
\]

where

\[
\tilde{C}_3 = x_1 \quad 2\Delta_s \quad x_5 \quad 2\Delta_s \quad x_6 \quad 2\Delta_s \quad x_4 \quad 2\Delta_s \quad x_2
\]

This diagram contains the sub-diagram which we have already calculated, namely, the one-loop correction to the \( \phi \) propagator. Using that, we obtain

\[
\tilde{C}_3 = \frac{16}{N} U \left( \Delta_\phi, \Delta_\phi + \Delta_s + \frac{\delta}{2}, -\Delta_\phi \right) U \left( d + \frac{\delta}{2}, d - \frac{\delta}{2} - \Delta_\phi \right) \times \frac{1}{|x_{13}|^{2\Delta_s + \delta} |x_{24}|^{2\Delta_s + \delta} |x_{34}|^{2d - 2\Delta_s + \delta}}. (4.31)
\]

We now proceed to adding up the obtained expressions for \( C_{1,2,3} \). First of all notice that, as expected, expressions (4.23), (4.29), (4.31) for \( \tilde{C}_{1,2,3} \) have similar structure,

\[
\tilde{C}_i = \tilde{C}_i \int d^d x_{3,4} \frac{1}{|x_{13}|^{2\Delta_s + \delta} |x_{24}|^{2\Delta_s + \delta} |x_{34}|^{2d - 2\Delta_s + \delta}}. (4.32)
\]

Performing the remaining two integrals over \( x_{3,4} \) using the relation (4.11) we obtain

\[
\tilde{C}_i = U \left( \Delta_s + \frac{\delta}{2}, d - \Delta_s + \frac{\delta}{2}, -\delta \right) U \left( \Delta_s + \frac{\delta}{2}, \Delta_s + \frac{\Delta_s}{2} + \delta, \frac{d - \Delta_s - 3\delta}{2} \right) \tilde{C}_i \frac{1}{|x_{12}|^{2\Delta_s + \delta}}. (4.33)
\]
Summing up all three diagrams and expanding around \( \delta = 0 \) we obtain

\[
\langle s(x_1)s(x_2) \rangle = \frac{C_s}{|x_{12}|^{2\Delta_s}} \left( 1 + \left( \frac{2\gamma_s}{\delta} + A_s + \mathcal{O}(\delta) \right) \frac{1}{(|x_{12}|)^\delta} \right),
\]

where

\[
\gamma_s = \frac{1}{N} \frac{4 \sin \left( \frac{\pi d}{4} \right) \Gamma(d)}{\pi \Gamma \left( \frac{d}{2} + 1 \right) \Gamma \left( \frac{d}{2} - 1 \right)},
\]

\[
A_s = \frac{1}{N} \frac{1}{\pi^4} 4^{d-2} \sin \left( \frac{\pi d}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-1}{2} \right) ^2
\]

\[
\times \frac{\pi^2 \left( \pi \cot \left( \frac{\pi d}{2} \right) - \psi(0) \left( \frac{d}{2} - 2 \right) + \psi(0) \left( \frac{d}{2} + 1 \right) + \psi(0) (d-2) + \gamma \right)}{\Gamma \left( \frac{d}{2} - 2 \right) \Gamma \left( \frac{d}{2} + 1 \right) \Gamma(d-2)}
\]

\[
+ \frac{\pi \sin \left( \frac{\pi d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) \left( \psi(0) \left( \frac{d}{2} - 2 \right) - \psi(0) \left( \frac{d}{2} - 1 \right) + \psi(0) (d-2) + \gamma \right)}{\Gamma(d-2) \Gamma \left( \frac{d}{2} \right)}
\]

\[
+ \frac{2 \Gamma \left( 2 - \frac{d}{2} \right) ^2 \Gamma \left( \frac{d-2}{2} \right) ^2 \sin^2 \left( \frac{\pi d}{2} \right) \Gamma \left( 2 - \frac{d}{2} \right) \Gamma(d-3) \Gamma \left( \frac{d}{2} \right)}{(d-4) \left( H_{1-\frac{d}{2}} - H_{d-6} + H_{d+1} \right)},
\]

and

\[
\langle ss s \rangle = \mathcal{O}(1/\delta).
\]

5 Conformal three-point functions

In this section we will calculate the next-to-leading \( 1/N \) corrections to coefficients of the conformal three-point functions \( \langle \phi \phi s \rangle \) and \( \langle s s s \rangle \) of the fundamental scalar field \( \phi \) and the auxiliary Hubbard-Stratonovich field \( s \). In a general \( d \)-dimensional \( O(N) \) symmetric CFT the results for \( \langle \phi \phi s \rangle \) to the next-to-leading \( \mathcal{O} \left( \frac{1}{N^{3/2}} \right) \) order and for \( \langle s s s \rangle \) to the leading \( \mathcal{O} \left( \frac{1}{N^{1/2}} \right) \) order are known from general conformal bootstrap calculations [9, 10].
(see also [30, 31] for discussion in the context of the $O(N)$ sigma model). We begin by reproducing these general results in our model, and then proceed to deriving the new result for the coefficient of the three-point function $\langle sss \rangle$ to the order $O\left(\frac{1}{N^{3/2}}\right)$.

5.1 $\langle \phi \phi s \rangle$

The leading order result for the $\langle \phi \phi s \rangle$ three-point function follows directly from the tree-level diagram,

$$\langle \phi(x_1)\phi(x_2)s(x_3) \rangle = -\frac{2}{\sqrt{N}} C^2 C_s \int d^d x \frac{1}{|x_1 - x|^{2\Delta_\phi} |x_2 - x|^{2\Delta_\phi} |x_3 - x|^{2\Delta_s}}. \quad (5.1)$$

Using the uniqueness relation (4.13) we obtain the conformal three-point function

$$\langle \phi(x_1)\phi(x_2)s(x_3) \rangle = \frac{C_{\phi\phi s}}{|x_{12}|^{2\Delta_\phi - \Delta_s} |x_{13}|^{\Delta_\phi} |x_{23}|^{\Delta_s}}, \quad (5.2)$$

where the leading order coefficient is given by

$$C_{\phi\phi s} = -\frac{2}{\sqrt{N}} C^2 C_s U(\Delta_\phi, \Delta_\phi, \Delta_s). \quad (5.3)$$

It is convenient to write down three-point functions for the normalized fields

$$\phi \to \phi/C_\phi^{1/2}, \quad s \to s/C_s^{1/2},$$

i.e., for the fields whose two-point functions are normalized to unity. The corresponding three-point function coefficient (which we will be denoting with tilde) is given by

$$\tilde{C}_{\phi\phi s} = \frac{C_{\phi\phi s}}{C_\phi C_s^{1/2}}, \quad (5.5)$$

simplifying to

$$\tilde{C}_{\phi\phi s} = -\frac{2}{\sqrt{N}} C_\phi C_s^{1/2} U(\Delta_\phi, \Delta_\phi, \Delta_s), \quad (5.6)$$

which agrees with the general result of [9, 10]. In particular in $d = 6 - \epsilon$ dimensions we obtain

$$\tilde{C}_{\phi\phi s} = -\sqrt{\frac{6\epsilon}{N}} + O(\epsilon). \quad (5.7)$$

We now proceed to calculation of the next-to-leading $O\left(\frac{1}{N^{3/2}}\right)$ contribution to the three-point coefficient $C_{\phi\phi s}$. The corresponding contributing diagrams are comprised of corrections to the propagators, and corrections to the $\phi\phi s$ interaction vertex, all of which we have discussed above. We draw the $O(1/N)$ order propagators (4.3), (4.4) of $\phi$ and $s$ with a line and a solid blob.
\[ x_1 \xrightarrow{2\Delta} x_2 = C(1 + A) \frac{\mu^{-2\gamma}}{|x_{12}|^{2(\Delta + \gamma)}} \]

Here the set of parameters \((C, A, \Delta, \gamma)\) stands either for \((C_\phi, A_\phi, \Delta_\phi, \gamma_\phi)\) when we draw the propagator for \(\phi\), or for \((C_s, A_s, \Delta_s, \gamma_s)\), when we draw the propagator for \(s\).

We will denote the vertex \(\phi \phi s\) at the next-to-leading order \(O(1/N^3/2)\) by \(-\frac{2}{\sqrt{N}}(1 + \delta V)\), and draw it as the vertex blob diagram.

\[
\int d^d x \quad \begin{array}{c}
\stackrel{2\Delta_s}{x_1} \\
\bullet \quad x \\
\stackrel{2\Delta \phi}{x_2}
\end{array} = \frac{-2}{\sqrt{N} |x_{13}|^{\Delta_s} |x_{23}|^{\Delta \phi}} \frac{1 + \delta V}{|x_{12}|^{\Delta \phi + \Delta_s}}
\]

Here \(\delta V\) corresponds to the sum of two contributing loop diagrams, which we considered in momentum representation in \(d = 5\) in section 3. We do not calculate \(\delta V\) explicitly in coordinate representation in general \(d\). One can find the result for it in [27], which we use here to write down the answer for \(d = 6 - \epsilon\),

\[
\delta V = -\frac{1}{N} 29\epsilon^2 + O(\epsilon^3) . \tag{5.8}
\]

The three-point function coefficient can then be obtained by calculating the diagram symbolically drawn as

\[
\int d^d x_4 \quad \begin{array}{c}
\stackrel{2\Delta_s}{x_1} \\
\bullet \quad x_4 \\
\stackrel{2\Delta \phi}{x_2}
\end{array} \quad \begin{array}{c}
\bullet \quad x_3 \\
\stackrel{2\Delta}{x_4}
\end{array}
\]

This diagram, after normalization of its external legs, has the form

\[
\langle \phi(x_1) \phi(x_2) s(x_3) \rangle = \tilde{C}_{\phi \phi s} \frac{(1 + W_{\phi \phi s}) \mu^{-2\gamma_\phi - \gamma_s}}{|x_{12}|^{2(\Delta_\phi + \gamma_\phi) - (\Delta_s + \gamma_s)} |x_{13}|^{\Delta_\phi + \gamma_s} |x_{23}|^{\Delta_s + \gamma_s}} , \tag{5.9}
\]
and we want to calculate $W_{\phi \phi s}$ to the order $O\left(\frac{1}{N}\right)$. We can re-write (5.9) as regularized $\delta \to 0$ limit of

\[
\langle \phi(x_1)\phi(x_2)s(x_3) \rangle = \frac{\tilde{C}_{\phi \phi s}}{|x_{12}|^{2\Delta_\phi - \Delta_s}|x_{13}|^{\Delta_s}|x_{23}|^{\Delta_s}} (1 + W_{\phi \phi s}) + \frac{1}{\delta \mu^\beta} \left( \frac{2\gamma_\phi - \gamma_s}{|x_{12}|^\beta} + \frac{\gamma_s}{|x_{13}|^\beta} + \frac{\gamma_s}{|x_{23}|^\beta} \right).
\]

On the other hand, calculating the diagram directly we obtain

\[
\langle \phi(x_1)\phi(x_2)s(x_3) \rangle = -\frac{2}{\sqrt{N}} \frac{C_\phi^2(1 + A_\phi)^2C_s(1 + A_s)}{\sqrt{C_\phi^2(1 + A_\phi)^2C_s(1 + A_s)}} \left( \frac{\delta V}{|x_{12}|^{2\Delta_\phi - \Delta_s}|x_{13}|^{\Delta_s}|x_{23}|^{\Delta_s}} \right) + \int d^d x_4 \frac{1}{|x_{14}|^{2\Delta_\phi}|x_{24}|^{2\Delta_\phi}|x_{34}|^{2\Delta_s}} \left( 1 + \frac{2}{\delta \mu^\delta} \left( \frac{\gamma_\phi}{|x_{14}|^\delta} + \frac{\gamma_s}{|x_{24}|^\delta} + \frac{\gamma_s}{|x_{34}|^\delta} \right) \right).
\]

Comparing (5.10) and (5.11), while taking into account (5.6), we conclude

\[
W_{\phi \phi s} = A_\phi + \frac{A_s}{2} + \delta V.
\]

In $d = 6 - \epsilon$ from (5.12) using (4.19), (4.38), (5.8) we obtain

\[
W_{\phi \phi s} = \frac{22}{N} + O(\epsilon),
\]

in agreement with the result of [9, 10] for general $O(N)$ symmetric CFT in $d = 6 - \epsilon$ dimensions. In particular it reproduces the three-point function coefficient for the IR fixed point CFT of the $O(N)$ vector model with cubic interactions considered in [7] in $d = 6 - \epsilon$ dimensions.

5.2 \langle sss \rangle

In this subsection we will calculate the three-point function $\langle sss \rangle$ to the order $O\left(\frac{1}{N^{3/2}}\right)$. For the normalized field (5.4) this three-point function has the form

\[
\langle s(x_1)s(x_2)s(x_3) \rangle = \tilde{C}_s^3 \frac{(1 + W_{s^3})\mu^{-3\gamma_s}}{|x_{12}|^{\Delta_s + \gamma_s}|x_{13}|^{\Delta_s + \gamma_s}|x_{23}|^{\Delta_s + \gamma_s}},
\]

and our goal is to find the leading coefficient $\tilde{C}_s^3$ and the $1/N$ correction $W_{s^3}$.

The leading order contribution to the $\langle sss \rangle$ three-point function originates from the one-loop triangle diagram
Normalizing external legs of this diagram we obtain

\[ \langle s(x_1)s(x_2)s(x_3) \rangle \bigg|_{\text{leading}} = N \left( -\frac{2}{\sqrt{N}} \right)^3 C_\phi^3 C_s^3 \int d^4x_{4,5} \frac{1}{|x_{35}|^{2\Delta_s} |x_{45}|^{2\Delta_\phi} |x_{24}|^{2\Delta_s}} \times \int d^4x_6 \frac{1}{|x_{16}|^{2\Delta_s} |x_{46}|^{2\Delta_\phi} |x_{56}|^{2\Delta_\phi}} . \]  

(5.15)

Applying repeatedly the uniqueness formula (4.13) to the integrals over \( x_{4,5,6} \) we arrive at

\[ \langle s(x_1)s(x_2)s(x_3) \rangle \bigg|_{\text{leading}} = \tilde{C}_s^3 \bigg|_{x_{12}^{\Delta_s} x_{23}^{\Delta_s} x_{13}^{\Delta_s}} . \]  

(5.16)

where

\[ \tilde{C}_s^3 = -\frac{8}{\sqrt{N}} C_\phi^3 C_s^3 U(\Delta_\phi, \Delta_\phi, \Delta_s)^2 U \left( \frac{\Delta_s}{2}, \Delta_s, d - \frac{3\Delta_s}{2} \right) . \]  

(5.17)

One can demonstrate that for general \( d \)

\[ C_\phi^2 C_s U(\Delta_\phi, \Delta_\phi, \Delta_s) U \left( \frac{\Delta_s}{2}, \Delta_s, d - \frac{3\Delta_s}{2} \right) = \frac{d - 3}{2} , \]  

(5.18)

and consequently using (5.6) we obtain

\[ \tilde{C}_s^3 = 2(d - 3) C_{\phi\phi s} , \]  

(5.19)

in agreement with the general result of \([9, 10]\).

We now proceed to calculation of the sub-leading term \( W_{s3} \) in (5.14). Contributions to this term originate from the vertex corrections and corrections to the propagators of \( \phi \) and \( s \).
There are three kinds of vertex corrections diagrams. The first two have the structure of the $\mathcal{O}(1/N^{3/2})$ diagrams renormalizing the $\phi\phi s$ interaction vertex, discussed in section 3. Summing contributions of these diagrams to each of the three vertices of the leading $s^3$ triangle diagram, we get the contribution $3(w_1 + w_2)$. The third kind of vertex correction diagram, whose contribution we will denote by $w_3$, is given by the triangle diagram where each vertex is the leading $s^3$ triangle.

The propagators contribute $\frac{3}{2} A_s + 3A_\phi$, similarly to the calculation of $W_{\phi\phi s}$, where we have subtracted the term $-\frac{3}{2} A_s$, due to normalization of the external legs of the diagrams. In total we get

$$W_{s^3} = \frac{1}{C_{s^3}} \left( 3(w_1 + w_2) + w_3 \right) + \frac{3}{2} A_s + 3A_\phi,$$

where $\frac{1}{C_{s^3}}$ has been factored out for subsequent notation convenience.

The first vertex correction diagram has the form
Normalizing external legs of this diagram we obtain

$$W_1(x_1, x_2, x_3) = \left( \frac{2}{\sqrt{N}} \right)^5 N \frac{C_5^5 C_4^4}{\sqrt{C_s^3}} R_1(x_1, x_2, x_3)$$  \hspace{1cm} (5.21)

where

$$R_1 = \int d^d x_{4,5,6,7,8} \frac{1}{(|x_{45}| |x_{46}| |x_{57}| |x_{68}| |x_{78}|)^{2\Delta_\phi}} \frac{1}{(|x_{24}| |x_{35}| |x_{67}| |x_{18}|)^{2\Delta_s}}.$$  \hspace{1cm} (5.22)

The integral (5.22) has the form

$$R_1(x_1, x_2, x_3) = \frac{r_1}{|x_{12}|^{\Delta_s} |x_{13}|^{\Delta_s} |x_{23}|^{\Delta_s}},$$  \hspace{1cm} (5.23)

and we are looking for the coefficient $r_1$. Then

$$W_1(x_1, x_2, x_3) = \frac{w_1}{|x_{12}|^{\Delta_s} |x_{13}|^{\Delta_s} |x_{23}|^{\Delta_s}}, \quad w_1 = \left( \frac{2}{\sqrt{N}} \right)^5 N \frac{C_5^5 C_4^4}{\sqrt{C_s^3}} r_1.$$  \hspace{1cm} (5.24)

Integrating both sides of (5.23) w.r.t. $x_{1,3}$ we obtain

$$\int d^d x_{1,3} R_1(x_1, x_2, x_3) = r_1 U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) \int d^d x_1 \frac{1}{|x_{12}|^{3\Delta_s - d}}.$$  \hspace{1cm} (5.25)

The l.h.s. of (5.25) for $R_1$ given by (5.22) can be calculated by repeatedly applying the uniqueness relation (4.13) and the propagator merging relation (4.11). Reader can easily follow this sequence of integration, where we mark which relation ($u$ for uniqueness and $p$ for propagator merging) is applied

$$x_8(u) \rightarrow x_1(p) \rightarrow x_6(p) \rightarrow x_5(u) \rightarrow x_3(p) \rightarrow x_4(p).$$  \hspace{1cm} (5.26)

As a result we obtain

$$w_1 = -\frac{32}{N^{3/2}} C_5^5 C_4^4 U(\Delta_\phi, \Delta_\phi, \Delta_s)^3 U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) U \left( \Delta_s, \frac{\Delta_s}{2}, d - \frac{3\Delta_s}{2} \right).$$  \hspace{1cm} (5.27)

Using (5.17) we obtain

$$w_1 = \frac{4}{N} C_5^5 C_4^4 U(\Delta_\phi, \Delta_\phi, \Delta_s) U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right).$$  \hspace{1cm} (5.28)

---

7One can check that this conformal three-point function form is valid for this diagram individually, as well as for each separate diagram $R_{2,3}$ introduced below, by verifying explicitly that the scaling and inversion transformation properties of these integrals coincide with the scaling and inversion covariance transformation properties of the conformal three-point function.
One can demonstrate that for general $d$

$$C_\phi^2 C_s U(\Delta_\phi, \Delta_\phi, \Delta_s) U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) = -\frac{1}{2},$$  \hspace{0.5cm} (5.29)

and therefore

$$w_1 = -\frac{2}{N} \tilde{C}_s^3. $$  \hspace{0.5cm} (5.30)

The second vertex correction diagram has the form

Normalizing external legs of this diagram we obtain

$$W_2(x_1, x_2, x_3) = \left( -\frac{2}{\sqrt{N}} \right)^7 N^2 \frac{C_s^5 C_s^5 \sqrt{C_s^3}}{C_s^3} R_2(x_1, x_2, x_3) $$  \hspace{0.5cm} (5.31)
\[ R_2 = \int d^d x_{1-10} \frac{1}{(|x_{45}| |x_{46}| |x_{57}| |x_{67}| |x_{89}| |x_{910}|)^2} \Delta_\phi \frac{1}{(|x_{24}| |x_{35}| |x_{68}| |x_{79}| |x_{110}|)^2}. \]  

(5.32)

The integral (5.32) has the form

\[ R_2(x_1, x_2, x_3) = \frac{r_2}{|x_{12}| \Delta_s |x_{13}| |x_{23}| \Delta_s}, \]  

(5.33)

where we want to calculate the coefficient \( r_2 \). Then

\[ W_2(x_1, x_2, x_3) = \frac{w_2}{|x_{12}| \Delta_s |x_{13}| |x_{23}| \Delta_s}, \quad w_2 = \left( -\frac{2}{\sqrt{N}} \right)^7 N^2 \frac{C_\phi^7 C_s^5}{\sqrt{C_s^3}} r_2. \]  

(5.34)

Integrating both sides of (5.33) w.r.t. \( x_{1,3} \) we obtain

\[ \int d^d x_{1,3} R_2(x_1, x_2, x_3) = r_2 U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) \int d^d x_1 \frac{1}{|x_{12}|^{3 \Delta_s - d}}. \]  

(5.35)

We calculate the l.h.s. of (5.35) by repeatedly applying the uniqueness relation (4.13) and the propagator merging relation (4.11), following the chain of integration

\[ x_{10}(u) \rightarrow x_1(p) \rightarrow x_8(p) \rightarrow x_9(p) \rightarrow x_6(p) \rightarrow x_5(u) \rightarrow x_3(p) \rightarrow x_4(p). \]  

(5.36)

At the end we obtain

\[ w_2 = -\frac{128}{N^{3/2}} C_\phi^7 C_s^7 U(\Delta_\phi, \Delta_\phi, \Delta_s)^4 U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) U \left( \Delta_s, \frac{\Delta_s}{2}, d - \frac{3\Delta_s}{2} \right)^2. \]  

(5.37)

Using (5.17) we obtain

\[ w_2 = \frac{16}{\tilde{N}} C_{s3}^4 C_\phi^4 C_s^2 U(\Delta_\phi, \Delta_\phi, \Delta_s)^2 U \left( \frac{\Delta_s}{2}, \frac{\Delta_s}{2}, d - \Delta_s \right) U \left( \Delta_s, \frac{\Delta_s}{2}, d - \frac{3\Delta_s}{2} \right). \]  

(5.38)

Using (5.18), (5.29) this expression can be further simplified as

\[ w_2 = -\frac{4(d-3)}{\bar{N}} \tilde{C}_{s3}. \]  

(5.39)

The third vertex correction diagram looks as follows.
Normalizing external legs of this diagram we obtain

\[ W_3(x_1, x_2, x_3) = \tilde{C}_s^3 \left( -\frac{2}{\sqrt{N}} \right)^6 N^2 C_\phi^6 C_s^3 R_3(x_1, x_2, x_3) \]  (5.40)

where we have separated some of the factors and defined

\[ R_3 = \frac{1}{C_s^3} \left( -\frac{2}{\sqrt{N}} \right)^3 N \sqrt{C_\phi^3 C_s^3} \int d^d x_{4-12} \frac{1}{(|x_{45}| |x_{56}| |x_{78}| |x_{79}| |x_{89}| |x_{1011}| |x_{1012}| |x_{1112}|)^{2\Delta_s}} \times \frac{1}{(|x_{24}| |x_{510}| |x_{58}| |x_{711}| |x_{112}| |x_{39}|)^{2\Delta_s}}. \]  (5.41)

Here we recognize that the integral over \( x_{10,11,12} \) gives the leading \( \langle sss \rangle \) coefficient, resulting in

\[ R_3 = \int d^d x_{4-9} \frac{1}{(|x_{45}| |x_{46}| |x_{56}| |x_{78}| |x_{79}| |x_{89}|)^{2\Delta_s}} \times \frac{1}{(|x_{24}| |x_{58}| |x_{39}|)^{2\Delta_s}}. \]  (5.42)

The integral (5.42) has the form

\[ R_3(x_1, x_2, x_3) = \frac{r_3}{|x_{12}| \Delta_s |x_{13}| \Delta_s |x_{23}| \Delta_s}, \]  (5.43)

where we are interested in finding the coefficient \( r_3 \). Then

\[ W_3(x_1, x_2, x_3) = \frac{w_3}{|x_{12}| \Delta_s |x_{13}| \Delta_s |x_{23}| \Delta_s}, \quad w_3 = \tilde{C}_s^3 \left( -\frac{2}{\sqrt{N}} \right)^6 N^2 C_\phi^6 C_s^3 r_3. \]  (5.44)
Again, integrating both sides of (5.43) w.r.t. $x_{1,3}$ we get
\[ \int d^d x_{1,3} R_3(x_1, x_2, x_3) = r_3 \int d^d x_{1,3} \frac{1}{|x_{12}|^\Delta_3 |x_{13}|^\Delta_3 |x_{23}|^\Delta_3}. \]  
(5.45)

As before, we proceed to calculate the l.h.s. of (5.45) by repeatedly applying the uniqueness relation (4.13) and the propagator merging relation (4.11), following the chain of integration
\[ x_4(u) \rightarrow x_9(u) \rightarrow x_8(u) \rightarrow x_1(p) \rightarrow x_3(p) \rightarrow x_7(p). \]  
(5.46)

At the end we arrive at
\[ w_3 = \frac{64}{N} \tilde{C}_s^3 C_\phi^6 C_\Delta^3 U(\Delta_\phi, \Delta_\phi, \Delta_s) U \left( \frac{\Delta_s}{2}, \Delta_s, d - \frac{3\Delta_\phi}{2} \right) U \left( \frac{\Delta_s}{2}, \frac{\Delta_\phi}{2}, d - \Delta_s \right) \]
\[ \times U \left( \Delta_\phi, \frac{3\Delta_s - d}{2}, d - \Delta_s \right) U \left( \frac{\Delta_s}{2}, \frac{3\Delta_s - d}{2}, \frac{3d}{2} - 2\Delta_s \right). \]  
(5.47)

Using (5.18), (5.29) and
\[ C_\phi^2 C_s U \left( \Delta_\phi, \frac{3\Delta_s - d}{2}, d - \Delta_s \right) U \left( \frac{\Delta_s}{2}, \frac{3\Delta_s - d}{2}, \frac{3d}{2} - 2\Delta_s \right) = \frac{\sec \left( \frac{\pi d}{2} \right) \Gamma(d - 3)}{4 \Gamma \left( 3 - \frac{d}{2} \right) \Gamma \left( \frac{3d}{2} - 4 \right)} \]  
(5.48)

we obtain
\[ w_3 = -\frac{1}{N} \frac{4 \sec \left( \frac{\pi d}{2} \right) \Gamma(d - 2)}{\Gamma \left( 3 - \frac{d}{2} \right) \Gamma \left( \frac{3d}{2} - 4 \right)} \tilde{C}_s^3. \]  
(5.49)

To summarize, substituting (4.17), (4.36) (5.30), (5.39), (5.49), into (5.20) we obtain the general $d$ formula for the next-to-leading contribution to the $\langle sss \rangle$ three-point coefficient $W_{s^3}$. To save space we avoid writing this expression in general $d$, but write down the answer in $d = 6 - \epsilon$,
\[ W_{s^3} = \frac{24}{N} + O(\epsilon). \]  
(5.50)

Notice that this term is different from the result $W_{s^3} = \frac{162}{N}$ obtained in [7] for the IR fixed point CFT of the $d = 6 - \epsilon$ dimensional $O(N)$ vector model with cubic interaction in $d = 6 - \epsilon$ dimensions. This is at odds with the conjecture that the model of [7] represents an alternative description of the UV fixed point of the $O(N)$ vector model with quartic interaction in $d = 6 - \epsilon$ dimensions.
6 Discussion

In this work we explore the $O(N)$ critical vector model with quartic interaction in $4 < d < 6$ dimensions. This model provides a nice illustration of the asymptotically safe quantum field theory. While it is difficult to prove existence of the UV fixed point in full generality, perturbative approach, $\epsilon$- and large-$N$ expansions confirm the hypothesis of asymptotic safety in this model. Our calculations encompass the next-to-leading order analysis in the $1/N$ expansion and extend previously known results in a few ways.

We derive and perform consistency checks that provide an additional evidence for the existence of a non-trivial UV fixed point. Moreover, we continue non-perturbative studies of the emergent conformal field theory and calculate a CFT data associated with the three-point functions of the fundamental scalar and Hubbard-Stratonovich fields.

In [7] an alternative description of the critical $O(N)$ model in terms of $N+1$ massless scalars with cubic interactions was proposed. It was explicitly shown that the scaling dimensions of various operators within the alternative description match the known results for the critical $O(N)$ theory. While our findings confirm all the observations made in [7], we find partial discrepancy in a CFT data associated with the 3-point functions calculated in the current work. In the future, we plan to understand better the origin of this mismatch and explore its consequences.

In fact, the critical $O(N)$ model in higher dimensions does have certain unphysical features. The well-known $\epsilon$-expansion shows that the coupling constant at the Wilson-Fisher fixed point is negative in $4 + \epsilon$ dimensions [1]. This means that the potential is unbounded from below for large values of the field. However, perturbative calculations in $1/N$ do not reveal any sign of instability at the level of the correlation functions. Thus, for instance, unitarity bounds are satisfied. Of course, this argument only shows that the vacuum state of the theory is metastable if the instability observed within $\epsilon$-expansion persists in the $\epsilon \to 1$ limit.

Indeed, in the recent work [11] the authors provide a non-perturbative argument in favour of instability of the model. Their analysis rests on the observation that the path integral in the large-$N$ limit is entirely dominated by the saddle point. The corresponding saddle point equation is Weyl invariant at the fixed point, and therefore admits a family of solutions parametrized by size and location. This type of large-$N$ instantons was previously observed in the critical $\phi^6$ model in three dimensions [32]. In particular, it was argued that the instantons and associated instability of the critical $\phi^6$ model can be used as a toy model towards holographic resolution of the singularity and conformal factor problems in quantum cosmology. It would be interesting to explore
these aspects in the context of critical $\phi^4$ model.

Remarkably, the critical $O(N)$ model in higher dimensions exhibits interesting behaviour when coupled to a thermal bath. This behaviour might be closely related to the non-perturbative instabilities discussed above. Progress in this direction will be reported elsewhere [33].

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A Callan-Symanzik equations

In this appendix we use the Callan-Symanzik equation to derive various relations between the counterterms $\delta_\phi$, $\hat{\delta}_s$, $\hat{\delta}_4$, and the anomalous dimensions $\gamma_\phi$, $\gamma_s$. In particular we establish the identity (2.16) satisfied by the counterterms $\delta_s$ and $\hat{\delta}_s$. The calculation is carried out for the model (2.12) with mass set to zero $m = 0$. Depending on the dimension $d$ the model is assumed to sit either at the UV or IR fixed point.

Let us start with the two-point function for the renormalized field $\phi$,

$$
\langle \tilde{\phi}(p)\tilde{\phi}(q) \rangle = (2\pi)^d \delta(p + q) \left( \frac{1}{p^2} + \text{loop corrections} + \frac{-p^2}{(p^2)^2} \delta_\phi \right).
$$

It satisfies the Callan-Symanzik equation

$$
\left( \mu \frac{\partial}{\partial \mu} + 2\gamma_\phi \right) \langle \tilde{\phi}(p)\tilde{\phi}(q) \rangle = 0,
$$

which leads to

$$
\gamma_\phi = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \delta_\phi + O(1/N^2).
$$

Similarly, the Callan-Symanzik equation for the Hubbard-Stratonovich field reads

$$
\left( \mu \frac{\partial}{\partial \mu} + 2\gamma_s \right) \langle \tilde{s}(p)\tilde{s}(q) \rangle = 0,
$$

Substituting

$$
\langle \tilde{s}(p)\tilde{s}(q) \rangle = (2\pi)^d \delta(p + q) \left( -\frac{1}{2B(p)} + \text{loop corrections} + \frac{\hat{\delta}_s}{2B(p)} \right),
$$
we obtain
\[ \gamma_s = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \hat{\delta}_s + \mathcal{O}(1/N^2). \]  

Finally, for the three-point function
\[
\langle \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{s}(q) \rangle = (2\pi)^d \delta(p_1 + p_2 + q) \frac{1}{p_1^2} \frac{1}{p_2^2} 2B(q) 
\times \left( -\frac{2}{\sqrt{N}} + \text{loop corrections} - \frac{2}{\sqrt{N} \sqrt{g_4}} \hat{\delta}_4 + (-2\hat{\delta}_\phi - \hat{\delta}_s) \left( -\frac{2}{\sqrt{N}} \right) \right),
\]
the Callan-Symanzik equation at the fixed point takes the form
\[
\left( \mu \frac{\partial}{\partial \mu} + 2\gamma_\phi + \gamma_s \right) \langle \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{s}(q) \rangle = 0.
\]

As a result, we obtain
\[
\mu \frac{\partial}{\partial \mu} \left( \delta_s - \hat{\delta}_s \right) = 0 + \mathcal{O}(1/N^2),
\]
where we substituted (A.3), (A.6) and used (2.13).
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