Introduction

The main aim of the present article is to overview the applications of soliton equations to the geometry of Abelian varieties.

Novikov was the first to indicate a possibility of this by conjecturing that Jacobi varieties are exactly principally-polarised Abelian varieties such that the Kadomtsev–Petviashvili equation integrates in their theta functions.

This conjecture followed a stormy starting period of the development of finite-zone integration of soliton equations (the Korteweg–de Vries equation, the Toda lattice, the sine–Gordon equation, etc.) (see [31, 107]) which led to the method of Baker–Akhieser functions. This method was proposed by Krichever for constructing finite-zone solutions to the Kadomtsev–Petviashvili equation and describing commutative rings of ordinary differential operators of rank 1 ([61, 62]). The latter results inspired the Novikov conjecture. Its proof by Shiota ([98]) led to solution of one of the oldest and prominent problems of algebraic geometry, the Riemann–Schottky problem.

At the same time, Arbarello and De Concini ([2]) showed that the Novikov conjecture interweaves tightly with the ideas of Gunning who proposed to describe Jacobi varieties as admitting sufficiently many trisecants. The last condition amounts to validity of special theta function identities ([45]). In their research Arbarello and De Concini used the observation of Mumford that many soliton equations (the Korteweg–de Vries, Kadomtsev–Petviashvili, and sine–Gordon equations) are “hidden” in the Fay trisecant formula ([82]).

The present article gives a survey of these papers on the Riemann–Schottky problem as well as of the papers on the analogue of the Novikov conjecture for Prym varieties. The analogue is abstracted further because the Veselov–Novikov, Landau–Lifschitz and BKP equations integrated in Prym theta functions are also “hidden” in the Fay and Beauville–Debarre quadrisecant formulæ.

Chapter 1 contains a brief introduction to the analytic theory of theta functions (see also [11, 31]) which can be considered as self-contained if supplemented with information on cohomologies with coefficients in sheaves, for instance, in the amount of Chern’s book [17].

Chapter 2 provides necessary information on the theta functions of Jacobi and Prym (the detailed proofs are exposed, for instance, in [36, 11]) and the inference of the trisecant and quadrisecant formulæ.

Chapters 3 and 4 are devoted to application of soliton equations to the theory of Jacobi and Prym varieties. Therewith necessary information on finite-zone solutions to non-linear equations and on Baker–Akhieser functions is exposed in brief in §8. This information is given in detail in the survey articles on finite-zone theory [31, 32, 33, 27] (see also [9, 64]). We hope that the present survey completes them in the part related to application of finite-zone theory to the geometry of Abelian varieties.

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Chapter 1. Abelian varieties and theta functions

§1. A condition for a complex torus to be algebraic

1.1. Algebraic varieties.

A complex manifold is called a projective algebraic variety if it is embeddable into the complex projective space $\mathbb{C}P^n$ as the set of zeros of a system of homogeneous polynomials.

By definition, $\mathbb{C}P^n$ is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ for the following action of $\mathbb{C}^*$:

$$(z^0, z^1, \ldots, z^n) \rightarrow (\lambda z^0, \lambda z^1, \ldots, \lambda z^n), \lambda \in \mathbb{C}^*.$$ 

This space is endowed with the Fubini–Study metric written in terms of the homogeneous coordinates $(z^0 : \ldots : z^n)$ as

$$\frac{(\sum_j z^j \bar{z}^j) \cdot (\sum_k z^k \bar{d}^k) - (\sum_j z^j \bar{d}^j) \cdot (\sum_k \bar{z}^k dz^k)}{(\sum_k z^k \bar{z}^k)^2}.$$ 

Assign to this Hermitian metric the uniquely-determined form

$$\omega = \sqrt{-1} \frac{(\sum_j z^j \bar{z}^j) \cdot (\sum_k z^k dz^k) - (\sum_j z^j \bar{d}^j) \cdot (\sum_k \bar{z}^k d^k)}{(\sum_k z^k \bar{z}^k)^2}.$$ 

Straightforward computations show the validity of the following:

1) $\omega$ is a closed form;
2) $[\omega]$ is an integer cohomology class: $[\omega] \in H^2(\mathbb{C}P^n; \mathbb{Z})$.

Complex manifolds with the above properties fall into special classes:

1) a compact manifold $M$ is called a Kähler manifold if there exists a Hermitian metric $h_{ij} dz^i \wedge d\bar{z}^j$ on $M$ such that the form $\omega = \sqrt{-1} \frac{h_{ij} dz^i \wedge d\bar{z}^j}{2\pi}$ associated with this metric is closed. The metric $h_{ij}$ is said to be Kähler;
2) a Kähler manifold $M$ is called a Hodge manifold if the form $\omega$ associated with the Kähler metric on $M$ represents an integer cohomology class, $[\omega] \in H^2(M; \mathbb{Z})$. This form $\omega$ is said to be Hodge.

Unless stated otherwise, repeated upper and lower indices in the same expression imply summation in (1) and in the sequel.

Give an invariant definition of $\omega$. It suffices to do that locally, in the tangent space at a point. Decompose the Hermitian form

$$(v, w) = h_{ij} v^i \bar{w}^j$$

into the sum of its real and imaginary parts:

$$H(v, w) = H_R(v, w) + \sqrt{-1}H_I(v, w)$$

where the real forms $H_R$ and $H_I$ are R-bi-linear. Since $H$ is Hermitian, $H_R$ is symmetric and positive definite whereas $H_I$ is skew-symmetric. Since $H(\sqrt{-1}v, w) = \sqrt{-1}H(v, w)$, we have

$$H_R(v, w) = H_I(\sqrt{-1}v, w).$$
Hence the Hermitian metric is determined by the skew-symmetric form $H_I$ alone. Given $x \in M$, put
\[ \omega(x)(v,w) = -\frac{1}{\pi} H_I(x)(v,w) \quad (4) \]
where $v, w \in T_x M$.

The equivalence of (1) and (4) is clear for diagonal metrics. The general case reduces to this one by change of a linear basis.

1.2. Theorems of Kodaira and Chow.

Clearly the properties that a manifold is a Kähler or Hodge manifold are inherited by submanifolds. For instance, let $Y$ be a submanifold of a Kähler manifold $X$. Then the Kähler metric on $X$ induces a Kähler metric on $Y$. This also holds in the case of Hodge manifolds. Hence every projective algebraic variety is a Hodge manifold. Kodaira proved that the converse statement is also valid.

1.2.1. Kodaira Theorem. A complex manifold is projective algebraic if and only if it is a Hodge manifold.

Proof of this theorem consists in embedding a Hodge manifold into a projective space of sufficiently high dimension and applying the following

1.2.2. Chow Theorem. An analytic subset $X \subset \mathbb{CP}^n$ is distinguished in $\mathbb{CP}^n$ as the set of zeros of a system of homogeneous polynomials.

For complex tori the equivalence of the property of being a Hodge manifold and that of being an algebraic variety was established by Lefschetz (see §4).

1.3. A criterion for a complex torus to be a Hodge manifold.

A complex torus $\mathbb{C}^n/\Lambda$, with $\Lambda$ a lattice of rank $2n$ in $\mathbb{C}^n$, is called an Abelian variety (or an Abelian torus) if it is a projective algebraic variety. By the Lefschetz theorem, this is equivalent to the property of being a Hodge torus.

1.3.1. Riemann Criterion. A complex torus $M = \mathbb{C}^n/\Lambda$ is a Hodge manifold if and only if there exists a complex linear basis for $\mathbb{C}^n$ such that in this basis the lattice $\Lambda$ is written as
\[
\Lambda = \Delta \delta N_1 + \Omega N_2, \quad N_1, N_2 \in \mathbb{Z}^n, \quad (5)
\]
where
1) $\Delta$ stands for the diagonal $(n \times n)$-matrix $\text{diag}(\delta_1, \ldots, \delta_n)$ with integer entries, $\delta_j > 0$, and $\delta_k$ divides $\delta_{k+1}$ for every $k = 1, \ldots, (n - 1)$;
2) the matrix $\Omega$ is symmetric;
3) the matrix $\text{Im} \Omega$ is positive definite.

Proof of the Riemann criterion.

Suppose that the complex torus $M = \mathbb{C}^n/\Lambda$ is a Hodge manifold. Take a Hermitian metric $h_{ij} dz^i \overline{dz}^j$ on $\mathbb{C}^n$ invariant under translations by the vectors of $\Lambda$ and inducing a metric that defines a Hodge structure on $M$.

1.3.2. Let $x^1, \ldots, x^{2n}$ be $\mathbb{R}$-linear coordinates on $\mathbb{R}^{2n} = \mathbb{C}^n$ and let $\omega = \omega_{ij} dx^i \wedge dx^j$ be a closed 2-form on $\mathbb{C}^n/\Lambda$. Define an averaging operator as follows
\[
\omega_{ij} \rightarrow \tilde{\omega}_{ij} = \frac{\int_{\Pi} \omega_{ij}(z^1, \ldots, z^n) dz^1 \wedge \ldots \wedge dz^n}{\int_{\Pi} dz^1 \wedge \ldots \wedge dz^n}
\]
where $\Pi$ is the fundamental domain of $\Lambda$. Then the averaging operator transforms $\omega$ into a 2-form with constant coefficients which is cohomologous to $\omega$. Therewith Hodge forms are transformed into Hodge forms.

The proof is clear.
1.3.3. Every closed 2-form \( \omega \) on \( M = \mathbb{C}^n/\Lambda \) defines a skew-symmetric form \( Q \) on \( \Lambda \). The value of \( Q \) at a pair \((u, v)\) is equal to the integral of \( \omega \) over the 2-torus spanned by \( u \) and \( v \). If \( [\omega] \in H^2(M; \mathbb{Z}) \) then \( Q \) is integer-valued. If \( \omega \) is a Hodge form then \( Q \) is non-degenerate.

Let \( v, w \in \Lambda \) and let \( \Pi_{v, w} \) be the parallelogram in \( \mathbb{C}^n \) spanned by \( v \) and \( w \). This parallelogram projects onto the 2-cycle \( T_{v, w} \) in \( M \). Define \( Q \) as follows

\[
Q(v, w) = \int_{T_{v, w}} \omega.
\]

By (1), if \( \omega \) is a Hodge form then \( \omega^n \) is proportional to the volume form, and we infer that \( Q \) is non-degenerate. This proves Proposition 1.3.3.

1.3.4. Frobenius Criterion. Let \( M = \mathbb{C}^n/\Lambda \) be a complex torus with \( \Lambda \) a lattice of rank \( 2n \). The torus \( M \) is a Hodge manifold if and only if there exists a skew-symmetric real \( \mathbb{R} \)-bi-linear form \( Q \) on \( \mathbb{C}^n \times \mathbb{C}^n \) such that

1) the form \( \langle v, w \rangle = Q(\sqrt{-1}v, w) \) is symmetric and positive definite;

2) \( Q \) is integer-valued on \( \Lambda \times \Lambda \).

Proof of the Frobenius criterion.

Let \( M \) be a Hodge torus. Take a Hermitian metric with constant coefficients on \( M \) generating a Hodge form. By (3) and 1.3.3, the form \( \omega \) associated with this metric satisfies conditions 1 and 2.

Let \( Q \) satisfies the hypotheses of 1.3.4. Then the Hermitian metric \( \langle v, w \rangle = \pi(Q(\sqrt{-1}v, w) + \sqrt{-1}Q(v, w)) \) generates the associated Hodge form.

The Frobenius criterion is established.

1.3.5. Let \( Q \) be a non-degenerate integer skew-symmetric form on \( \Lambda = \mathbb{Z}^{2n} \). Then in a suitable basis \( \lambda_1, \ldots, \lambda_{2n} \) the form \( Q \) is written as

\[
Q = \begin{pmatrix}
0 & \Delta_\delta \\
-\Delta_\delta & 0
\end{pmatrix}
\]

where \( \Delta_\delta \) stands for the diagonal integer matrix \( \text{diag}(\delta_1, \ldots, \delta_n) \), \( \delta_k > 0 \), and \( \delta_j \) divides \( \delta_{j+1} \) for every \( j = 1, \ldots, n - 1 \). An \( n \)-tuple \( \delta_k \) satisfying these conditions is an invariant of \( Q \).

We prove this proposition.

Take the minimal value \( \delta_1 = Q(\lambda_1, \lambda_{n+1}) \) among all possible positive values of the form \( Q(\lambda, \lambda') \). Denote by \( \Lambda' \) the orthogonal complement to the sublattice \( \mathbb{Z}\{\lambda_1, \lambda_{n+1}\} \) in \( \Lambda \). We have

\[
\lambda + Q(\lambda_1, \lambda_{n+1}) \delta_1 \lambda_{n+1} - Q(\lambda, \lambda_{n+1}) \delta_1 \lambda_1 \in \Lambda'.
\]

for \( \lambda \in \Lambda \). Hence \( \Lambda' = \mathbb{Z}^{2n-2} \) and the restriction of \( Q \) onto \( \Lambda' \) satisfies the hypotheses of Proposition 1.3.5. Using this procedure successively we thus construct a basis \( \lambda_1, \ldots, \lambda_{2n} \) for \( \Lambda \) such that in this basis \( Q \) takes the shape (3).

Prove that \( \delta_1 \) divides \( \delta_j \) by way of contradiction. Let \( \delta \) be the greatest common divisor of \( \delta_1 \) and \( \delta_2 \) such that \( \delta < \delta_1 \). Then there exist integers \( k \) and \( n \) such that \( \delta = k\delta_1 + m\delta_2 \). However,

\[
Q(k\lambda_1 + m\lambda_2, \lambda_{n+1} + \lambda_{n+2}) = \delta < \delta_1
\]

which contradicts the choice of \( \delta_1 \). Hence \( \delta_1 \) divides \( \delta_2 \). We similarly prove that \( \delta_j \) divides \( \delta_{j+1} \) for every \( j, 1 \leq j \leq n - 1 \).
It follows from the construction of \( \{ \delta_j \} \) that this \( n \)-tuple is an invariant of \( Q \). This proves Proposition 1.3.5.

1.3.6. Let \( \omega \) be the Hodge form associated with a Hermitian metric with constant coefficients on a Hodge torus \( M = \mathbb{C}^n/\Lambda \) and let \( \lambda_1, \ldots, \lambda_{2n} \) be a basis for the lattice \( \Lambda \) in which \( Q \) takes the shape (6). Then the set of the vectors \( \lambda_1, \ldots, \lambda_n \) is a complex linear basis for \( \mathbb{C}^n \).

Proof of Proposition 1.3.6. Since the coefficients of a Hermitian form are constant, the form \( Q, \mathbb{R} \)-linearly-extended onto \( \mathbb{C}^n \), coincides with \( \omega \) and determines a Hermitian metric by (2–4).

Consider the succession of vectors
\[
v_1 = \lambda_1, \ v_2 = \lambda_2 - \frac{Q(\sqrt{-1}\lambda_2, v_1)}{Q(\sqrt{-1}v_1, v_1)}v_1, \ldots, \ v_k = \lambda_k - \sum_{j<k} \frac{Q(\sqrt{-1}\lambda_k, v_j)}{Q(\sqrt{-1}v_j, v_j)}v_j, \ldots
\]
By (7–9), all the expressions \( Q(\sqrt{-1}\lambda_k, v_j)/Q(\sqrt{-1}v_j, v_j) \) are real. It follows from (2) and (6) that \( (v_j, v_k) = 0 \) for \( j \neq k \). Since \( Q(v_k, \lambda_{n+k}) \neq 0 \), the vectors \( v_1, \ldots, v_n \) do not vanish and form a complex linear basis for \( \mathbb{C}^n \). Now Proposition 1.3.6 follows from the coincidence of the linear spans of \( (v_1, \ldots, v_n) \) and \( (\lambda_1, \ldots, \lambda_n) \) over \( \mathbb{C} \).

Now we turn directly to proving the Riemann criterion.

Let \( M = \mathbb{C}^n/\Lambda \) be a Hodge torus. By Propositions 1.3.2–1.3.6, there exist a basis \( V_0 = (\lambda_1, \ldots, \lambda_{2n}) \) for \( \Lambda \) and a Hodge form \( \omega \) with constant coefficients such that in this basis \( \omega \) is written as
\[
\omega = \sum_{k=1}^n \delta_k dy^k \wedge dy^{n+k}. \quad (7)
\]
Take a complex basis \( V_{\mathbb{C}} \) and a real basis \( V_{\mathbb{R}} \) for \( \mathbb{C}^n \) as follows
\[
V_{\mathbb{C}} = (\frac{1}{\delta_1}\lambda_1, \ldots, \frac{1}{\delta_n}\lambda_n),
V_{\mathbb{R}} = (\frac{1}{\delta_1}\lambda_1, \ldots, \frac{1}{\delta_n}\lambda_n, \lambda_{n+1}, \ldots, \lambda_{2n}). \quad (8)
\]
Then \( \Lambda \) takes the shape (6) and we have the formula relating \( V_{\mathbb{R}} \) and \( V_{\mathbb{C}} \)
\[
\begin{pmatrix}
x^1 \\
\vdots \\
x^n 
\end{pmatrix}
= (I_n, \Omega)
\begin{pmatrix}
x^1 \\
\vdots \\
x^{2n} 
\end{pmatrix}, \quad (9)
\]
where \( I_n \) is the identity \((n \times n)\)-matrix.

By (7–9), in the basis (8) \( \omega \) is written as
\[
\omega = \sum_{k=1}^n dx^k \wedge dx^{n+k}. \quad (10)
\]
Since \( \omega \) is associated with the Hodge metric \( h_{ij} \) by (1), we infer from (10) that
\[
\omega = \frac{\sqrt{-1}}{2\pi} h_{ij}(dx^i + \Omega^i_k dx^{n+k}) \wedge (dx^j + \Omega^j_l dx^{n+l}) = \sum_{m=1}^n dx^m \wedge dx^{n+m}. \quad (11)
\]
Since $h_{ij}$ is symmetric, we conclude that (11) is equivalent to

$$\frac{\sqrt{-1}}{2\pi} h_{kj}(\Omega_l^j - \Omega_l^j) dx^k \wedge dx^{n+l} = \sum_{m=1}^{n} dx^m \wedge dx^{n+m}$$  \hspace{1cm} (12)

and

$$T_{lm} = T_{ml} \text{ where } T_{lm} = h_{kj} \Omega_k^l \bar{\Omega}_j^l.$$  \hspace{1cm} (13)

It follows from (12) that

$$T_{lm} = h_{kj} \Omega_k^l \Omega_m^j + \frac{2\pi}{\sqrt{-1}} \Omega_l^m.$$  \hspace{1cm} (14)

Now we conclude that

1) the matrix $Im \Omega$ is positive definite (this follows from (12));
2) the matrix $\Omega$ is symmetric: $\Omega_{lm} = \Omega_{ml}$ (this follows from (13) and (14)).

Hence we prove that if $C^n/\Lambda$ is a Hodge manifold then it admits a form given in the statement of the Riemann criterion.

If $C^n/\Lambda$ takes this shape, then, by (12–14), the Hermitian form $h_{ij} dz^i \wedge d\bar{z}^j$ defined by (12) is a Hodge form and, consequently, this torus is a Hodge torus.

The Riemann criterion is established.

1.4. Polarizations and moduli of Abelian varieties.

In this subsection of §1 we identify Hodge and Abelian tori. The complete proof of the coincidence of these families is given in §4.

A Hodge torus, or an Abelian variety, $M = C^n/\Lambda$ is polarised with polarization $[\omega] \in H^2(M; \mathbb{Z})$ if the cohomology class of a Hodge form $\omega$ is fixed.

By 1.3.3 and 1.3.5, the polarization $[\omega]$ uniquely defines the non-degenerate integer skew-symmetric 2-form

$$Q : H^1(M) \times H^1(M) \to \mathbb{Z}$$

taking the shape (4) in a suitable basis for $\Lambda$. Moreover the $n$-tuple of positive integers $\delta_1, \ldots, \delta_n$ is uniquely determined by this form. This $n$-tuple $\delta_1, \ldots, \delta_n$ is called a polarization type. An Abelian variety is principally-polarised if $\delta_1 = \ldots = \delta_n = 1$.

Two polarised Abelian varieties $(M_1, [\omega_1])$ and $(M_2, [\omega_2])$ are equivalent if there exists a bi-holomorphic mapping $f : M_1 \to M_2$ such that $f^*([\omega_2]) = [\omega_1]$. Given pairs $(n, \delta)$, the sets of equivalence classes of $n$-dimensional Abelian varieties with polarization of type $\delta$ are complex manifolds $A^n_\delta$, the moduli spaces of Abelian varieties.

Denote by $A_n$ the moduli spaces of $n$-dimensional principally-polarised Abelian varieties.

The moduli spaces $A^n_\delta$ are as follows.

Denote by $H_n$ the Siegel upper half-plane formed by symmetric $(n \times n)$-matrices with positive definite imaginary parts. It follows from the proof of the Riemann criterion and Propositions 1.3.3 and 1.3.5 that every $n$-dimensional Abelian variety with polarization of type $\delta$ is equivalent to the torus $C^n/\Lambda$ with $\Lambda$ of the shape (6) and $\Omega \in H_n$. It is clear that two polarised Abelian varieties $M_1 = C^n/\{\Delta_1 \mathbb{Z}^n + \Omega_1 \mathbb{Z}^n\}$ and $M_2 = C^n/\{\Delta_2 \mathbb{Z}^n + \Omega_2 \mathbb{Z}^n\}$ are equivalent if and only if their lattices are adjoint by a transformation, in $GL(2n, \mathbb{Z})$, preserving (6).
Precise the statement above. The group $SL(\delta, \mathbb{Z})$ is a subgroup of $GL(2n, \mathbb{Z})$ formed by elements $g$ such that

\[
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}
\cdot
\begin{pmatrix}
  0 & \Delta_\delta \\
  -\Delta_\delta & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
  A^* & C^* \\
  B^* & D^*
\end{pmatrix}
= 
\begin{pmatrix}
  0 & \Delta_i \\
  -\Delta_i & 0
\end{pmatrix}
\] (15)

where

\[ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

and $A, B, C,$ and $D$ are integer $(n \times n)$-matrices.

The action of $Sp(\delta, \mathbb{Z})$ on $\mathcal{H}_n$ generated by automorphisms of lattices is as follows

\[ \Omega \rightarrow (A\Omega + B\Delta_\delta) \cdot (\Delta_\delta^{-1}C\Omega + \Delta_\delta^{-1}D\Delta_\delta)^{-1}. \] (16)

We obtain the usual modular action of $SL(2, \mathbb{Z})$ on the complex upper half-plane for $n = 1$ and $\delta = 1$.

1.4.1. The moduli space of $n$-dimensional Abelian varieties with polarization of type $\delta$ is the quotient space of the Siegel upper half-plane $\mathcal{H}_n$ for the action (16) of $Sp(\delta, \mathbb{Z})$:

\[ \mathcal{A}_n^\delta = \mathcal{H}_n / Sp(\delta, \mathbb{Z}). \]

An isogeny of Abelian varieties $M$ and $M'$ is a holomorphic finite-sheeted covering $\varphi : M \rightarrow M'$.

1.4.2. Let $M$ be a polarised Abelian variety. Then there exist a principally-polarised Abelian variety $M'$ and an isogeny

\[ \varphi : M \rightarrow M' \]

with degree $\Delta = \delta_1 \ldots \delta_n$ where $(\delta_1, \ldots, \delta_n)$ is the polarization type of $M$.

We prove Proposition 1.4.2. Let $M$ be an Abelian variety $\mathbb{C}^n/\Lambda$ of the shape (7) and let $(\lambda_1, \ldots, \lambda_{2n})$ be a basis for $\Lambda$. Assume that a Hodge form $\omega$ takes the shape (8) in this basis.

Consider the lattice $\Lambda'$ spanned by the vectors of the basis (8). This lattice contains $\Lambda$ as a sublattice with index $\Delta$. The projection

\[ \varphi : \mathbb{C}^n/\Lambda \rightarrow \mathbb{C}^n/\Lambda' \]

is an isogeny of degree $\Delta$. Choose a form $\omega'$ such that $\varphi^* (\omega') = \omega$. Then, by (10), it defines the principal polarization on $M'$. This proves Proposition 1.4.2.

§2. Line bundles on complex tori

2.1. Line bundles, Picard group, Chern classes, and positive bundles.

A line bundle $L$ on a complex manifold $M$ is a vector bundle with a fibre $\mathbb{C}$ and holomorphic coordinate transformation functions, i.e.,

1) a projection $p : L \rightarrow M$ is defined;

2) there exists an open cover $\mathcal{U} = \{U_\alpha\}$ of $M$ such that a bundle is trivial over each element of this family:

\[ U_\alpha \times \mathbb{C} \overset{\xi_\alpha}{\cong} p^{-1}(U_\alpha), \quad p \cdot \xi_\alpha(z, v) = z \in U_\alpha \subset M; \]
3) the coordinate transformations
\[ g_{\alpha\beta}(z) = \xi^{-1}_\alpha \xi_{\beta|p^{-1}(z)} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \]
are holomorphic functions.

By definition, the coordinate transformations satisfy the following conditions
\[ g_{\alpha\beta} \cdot g_{\beta\alpha} = 1, \quad g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1. \]

By these conditions, the family of functions \( g_{\alpha\beta} \) defining the line bundle uniquely is a 1-cocycle in \( C^1(U, O^*) \). The sheaf \( O^* \) is defined as follows. Let \( O^*(U) \) be a multiplicative group of non-vanishing holomorphic functions on \( U \subset M \). It is said that two cocycles \( g \) and \( g' \) are cohomologous if there exists a family of sections \( f_\alpha : U_\alpha \to O^*(U_\alpha) \) such that \( g'_{\alpha\beta} = (f_\alpha/f_\beta)g_{\alpha\beta} \).

Two line bundles are called equivalent if the cocycles \( g \) and \( g' \) corresponding to these bundles are cohomologous. Therewith the functions \( f_\alpha \) deform the trivializations as follows \( \xi_\alpha \to \xi_\alpha/f_\alpha \).

The set of equivalence classes of line bundles on \( M \) is a commutative group, the Picard group \( \text{Pic}(M) \), under the operations of tensor product
\[ L, L' \to L \otimes L', \quad g_{\alpha\beta}, g'_{\alpha\beta} \to g_{\alpha\beta} \cdot g'_{\alpha\beta} \]
and passing to the dual bundle
\[ L \to L^*, \quad g_{\alpha\beta} \to g_{\alpha\beta}^{-1}. \]

We thus obtain the following proposition.

2.1.1. Structure of the Picard group. The Picard group of line bundles on \( M \) is isomorphic to the first cohomology group of \( M \) with coefficients in the sheaf of germs of non-vanishing holomorphic functions: \( \text{Pic}(M) = H^1(M; O^*) \).

To a line bundle \( L \) one assigns its Chern class \( c_1(L) \). Define it in terms of the coordinate transformations \( g_{\alpha\beta} \). Let \( \mathcal{U} \) be an open cover of \( M \) by simply-connected domains in \( M \) such that \( L \) is trivial over each domain of this cover and let \( \{g_{\alpha\beta}\} \) be the corresponding coordinate transformations. Define functions
\[ h_{\alpha\beta} = \frac{1}{2\pi \sqrt{-1}} \log (g_{\alpha\beta}), \quad \log (g_{\alpha\beta}) = -\log (g_{\beta\alpha}). \]

and construct a cocycle \( z_{\alpha\beta\gamma} \in C^2(\mathcal{U}; \mathbb{Z}) \) from these functions as follows
\[ z_{\alpha\beta\gamma} = h_{\alpha\beta} + h_{\beta\gamma} - h_{\gamma\alpha}. \]

The cohomology class \( c_1(L) \in H^2(M; \mathbb{Z}) \) realised by this cocycle is called the (first) Chern class of \( L \).

Since the choice of other branches of the logarithm in the definition of \( h_{\alpha\beta} \)
\( (\log (g_{\alpha\beta}) \to \log (g_{\alpha\beta}) + 2\pi \sqrt{-1}k_{\alpha\beta}, k_{\alpha\beta} \in \mathbb{Z}) \) is equivalent to the change of the cocycle \( z_{\alpha\beta\gamma} \) to the cohomologous cocycle \( z \to z + dk \); this definition is correct.

By (22), the Chern class \( c_1 \) is a homomorphism of the Picard group into \( H^2(M; \mathbb{Z}) \):
\[ c_1(L \otimes L') = c_1(L) + c_1(L'), \quad c_1(L^*) = -c_1(L). \]

Recall the invariant definition of \( c_1 \). Consider the exact sequence of sheaves
\[ 0 \to \mathbb{Z} \to O \xrightarrow{\exp} O^* \to 0 \]
where \( Z \) is the constant sheaf, \( \mathcal{O} \) is the sheaf of germs of holomorphic functions, and \( \exp \) is the exponential homomorphism \( f \to \exp(2\pi\sqrt{-1}f) \). Assign to this sequence the following exact cohomology sequence

\[
\cdots \to H^k(M;\mathbb{Z}) \to H^k(M;\mathcal{O}) \to H^k(M;\mathcal{O}^*) \to H^{k+1}(M;\mathbb{Z}) \to \cdots
\] (22)

Consider the boundary homomorphism (22) for \( k = 1 \):

\[
\delta : H^1(M;\mathcal{O}^*) = \text{Pic}(M) \to H^2(M;\mathbb{Z}).
\]

The first Chern class is the image of \( L \) under this homomorphism:

\[
\delta(L) = c_1(L).
\]

A line bundle \( L \) is called positive if its first Chern class is realized by a Hodge form \( \omega \), namely, \( c_1(L) = [\omega] \).

2.2. Chern classes of line bundles on tori.

Let \( M = \mathbb{C}^n/\Lambda \) be a complex torus and let \( \pi : \mathbb{C}^n \to M \) be the universal covering of \( M \). For every line bundle \( L \) on \( M \) there exists a pull-back \( \pi^*(L) \), a line bundle on \( \mathbb{C}^n \). By the \( \bar{\partial} \)-lemma of Poincare, \( H^1(\mathbb{C}^n;\mathcal{O}^*) = 0 \) (see, for instance, [41]). Hence the bundle \( \pi^*(L) \) is trivial, \( \pi^*(L) = \mathbb{C}^n \times \mathbb{C} \).

Fix a trivialization of \( \pi^*(L) \). Construct the automorphisms of a fibre \( e_\lambda(z) : \mathbb{C} \to \mathbb{C} \), \( e_\lambda(z) \in \mathbb{C}^* \)

by comparing the trivializations of \( \pi^*(L) \) at the points \( z \) and \( z + \lambda \) with \( \lambda \in \Lambda \). The set of all such automorphisms is a set of multiplicators \( \{e_\lambda \in \mathcal{O}^*(U)\}_{\lambda \in \Lambda} \) such that

\[
e_\lambda(z + \lambda)e_\lambda(z) = e_\lambda(z + \lambda')e_{\lambda'}(z) = e_{\lambda + \lambda'}(z).
\] (23)

Every set of functions \( \{e_\lambda \in \mathcal{O}^*(U)\}_{\lambda \in \Lambda} \) satisfying (23) defines a line bundle which is obtained from \( \mathbb{C}^n \times \mathbb{C} \) by factorization under the action of the group \( \Lambda \)

\[
(z,v) \to (z + \lambda, e_\lambda(z) \cdot v), \quad z \in \mathbb{C}^n, v \in \mathbb{C}.
\]

Two sets of multiplicators \( e_\lambda \) and \( \tilde{e}_\lambda \) define the same bundle if and only if there exists a holomorphic function \( \varphi : \mathbb{C} \to \mathbb{C}^* \) such that

\[
\tilde{e}_\lambda(z) = e_\lambda(z)\varphi(z + \lambda)\varphi^{-1}(z).
\] (24)

The function \( \varphi \) changes a trivialization of \( \pi^*(L) \).

We find a formula which expresses the Chern class in terms of multiplicators of a bundle. Fix a basis \( \lambda_1, \ldots, \lambda_{2n} \) for the lattice \( \Lambda \) and denote by \( x^1, \ldots, x^{2n} \) the real coordinates on \( \mathbb{C}^n \) referred to this basis. Consider the cover of \( M \) by the sets

\[
U_\lambda = \{|x^k - \lambda^k| \leq \frac{3}{4}\}
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \Lambda \). By the definition of multiplicators, the coordinate transformations are written as

\[
g_{\alpha,\beta}(z) = e_{\alpha - \beta}(z + \beta).
\] (25)
Define functions $f_\alpha(z)$ as follows

$$e_\alpha(z) = e^{2\pi \sqrt{-1} f_\alpha(z)}. \tag{26}$$

We show how to construct from the cocycle (20) a 2-form realising it. The nerve $N(\mathcal{U})$ of the minimal subcover of the cover $U_\alpha$ is homeomorphic to $M$ and its cohomologies with coefficients in $\mathbb{Z}$ coincide with $H^*(M; \mathbb{Z})$. The formula (20) gives the value of the cocycle $z$ at a two-dimensional simplex $(\alpha, \beta, \gamma) \in N(\mathcal{U})$.

Take vectors $\tau, \sigma \in \Lambda$ and span over them the parallelogram $\Pi_{\tau,\sigma}$ which is projected onto the torus $T_{\tau,\sigma} \subset M$. Given values of $z$ at two-dimensional simplexes, we define the value of $c_1$ at this torus.

By (20) and (24),

$$c_1([T_{\tau,\sigma}]) = f_\tau(z) + f_\sigma(z + \tau) - f_\sigma(z) - f_\sigma(z + \tau). \tag{27}$$

2.2.1. The formula (27) expresses the Chern class $c_1$ of a line bundle on $M = \mathbb{C}^n / \Lambda$. The Chern class defines an integer skew-symmetric 2-form $Q$ on $\Lambda \times \Lambda$ by $Q(\tau, \sigma) = c_1([T_{\tau,\sigma}])$. The form $\omega$ realising $c_1(L)$, i.e., $[\omega] = c_1(L)$, is as follows

$$\omega = \frac{1}{2} Q(\lambda_k, \lambda_l) dx^k \wedge dx^l.$$ 

The following proposition interprets the Chern class as a class realised by the curvature form of a line bundle (17).

A $(p,q)$-form is a linear combination of differential forms of the type $f\, dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}$. For a compact complex manifold $M$ we distinguish $H^2_{1,1}(M; \mathbb{Z})$, a subgroup of $H^2(M; \mathbb{Z})$, formed by elements realised by $(1,1)$-forms.

2.2.2. The first Chern class of a line bundle $L$ on a compact complex manifold $M$ lies in $H^2_{1,1}(M; \mathbb{Z})$, $c_1(L) \in H^2_{1,1}(M; \mathbb{Z})$.

We prove Proposition 2.2.2 by applying the de Rham theorem of comparison between the sheaf cohomologies and de Rham’s cohomologies. The equivalence of (27) and (29) for tori is checked by straightforward computations.

Take a Hermitian metric on the line bundle. This means that we choose positive functions $a_\alpha : U_\alpha \to \mathbb{R}$, the metric tensor, such that

$$a_\alpha \cdot |g_{\alpha\beta}|^2 = a_\beta \text{ in } U_\alpha \cap U_\beta. \tag{28}$$

Let $d = d^1 + d^2$ be the decomposition of the operator of inner derivative $d$ into its holomorphic and anti-holomorphic parts where $d^1$ and $d^2$ transform a $(p,q)$-form into a $(p+1,q)$-form and a $(p,q+1)$-form. It is clear that $d^2 = d^2 = 0$. By applying the operator $d^1$ to the logarithms of both sides of (28) and considering that the coordinate transformations are holomorphic, obtain

$$d^1 \log (a_\alpha) + d^1 \log (g_{\alpha\beta}) = d^1 \log (a_\beta).$$

Since $dd^1 \log (g_{\alpha\beta}) = 0$, the form $dd^1 \log (a_\alpha)$ is correctly defined everywhere on $M$.

By comparing the de Rham cohomologies and spectral cohomologies and (20), obtain

$$c_1(L) = \left[ \frac{1}{2\pi \sqrt{-1}} d^1 d^1 \log (a_\alpha) \right]. \tag{29}$$

This proves Proposition 2.2.2.
2.3. Construction of line bundles with prescribed Chern classes on tori.

2.3.1. Structure of the group $H^2_{1,1}(M; \mathbb{Z})$. Let $M = \mathbb{C}^n/\Lambda$ be a complex torus. Then the group $H^2_{1,1}(M; \mathbb{Z})$ is isomorphic to the group of skew-symmetric $\mathbf{R}$-bi-linear 2-forms $\omega$ on $\mathbb{C}^n \times \mathbb{C}^n$ integer-valued on $\Lambda \times \Lambda$ and satisfying the following condition

$$Q(u, v) = Q(\sqrt{-1}u, \sqrt{-1}v). \quad (30)$$

By Propositions 1.3.2 and 1.3.3, it suffices to consider 2-forms $\omega \in H^2(M; \mathbb{Z})$ with constant coefficients. Assign to every 2-form $\omega$ an integer skew-symmetric bi-linear form $Q$ on $\Lambda \times \Lambda$. Extend $Q$ to a $\mathbf{R}$-bi-linear form on $\mathbb{C}^n \times \mathbb{C}^n$, preserving the previous notation. This form coincides with $\omega$.

Put $Q = \alpha_{ij}dz^i \wedge d\bar{z}^j + \beta_{ij}dz^i \wedge dz^j + \gamma_{ij}d\bar{z}^i \wedge d\bar{z}^j$. It is clear that $Q$ satisfies (30) if and only if $\alpha_{ij} = \gamma_{ij} = 0$ which is equivalent to $\omega \in H^2_{1,1}(M; \mathbb{Z})$. This proves Proposition 2.3.1.

Let $\omega$ be a 2-form on $M = \mathbb{C}^n/\Lambda$ realising an element of $H^2_{1,1}(M; \mathbb{Z})$ and let $Q$ be associated with $\omega$ (see Proposition 1.3.3). By (30), the $\mathbf{R}$-bi-linear form

$$H(u, v) = -\pi(Q(\sqrt{-1}u, v) + \sqrt{-1}Q(u, v))$$

is Hermitian,

$$H(u, v) = \overline{H(v, u)}, \quad H(\alpha u, v) = \alpha H(u, v), \alpha \in \mathbb{C}. \quad \text{If } \omega \text{ is a Hodge form with constant coefficients then } H \text{ is the Kähler metric associated with } \omega.$$ 

Define functions $e_{\lambda}(z), \lambda \in \Lambda, z \in \mathbb{C}^n$ as follows:

$$e_{\lambda}(z) = \alpha(\lambda) \exp \left( H(z, \lambda) + \frac{1}{2}H(\lambda, \lambda) \right). \quad (31)$$

2.3.2. The set of functions (31) is a set of multiplicators of a line bundle $L$ on $M = \mathbb{C}^n/\Lambda$ if and only if the equality

$$\frac{\alpha(\lambda + \lambda')}{\alpha(\lambda)\alpha(\lambda')} = \exp (\pi \sqrt{-1}Q(\lambda, \lambda')) \quad \text{for every } \lambda, \lambda' \in \Lambda \quad (32)$$

holds. If (32) is valid, then $\omega$ is the first Chern class of $L$.

Proposition 2.3.2 is derived from (23) and (24) by straightforward computations. The following proposition is a corollary of 2.3.2.

2.3.3. For every element $[\omega] \in H^2_{1,1}(M; \mathbb{Z})$ there exists a line bundle $L$ on $M = \mathbb{C}^n/\Lambda$ defined by multiplicators (31) such that $c_1(L) = [\omega]$.

In order to prove this proposition it suffices to show that (32) are compatible for every skew-symmetric form $Q$ integer-valued on $\Lambda \times \Lambda$.

As in 1.3.5, we construct a basis $\lambda_1, \ldots, \lambda_{2n}$ for $\Lambda$ such that $Q$ takes the shape (1) in this basis. For every subspace $V_k = \mathbf{R}\{\lambda_k, \lambda_{n+k}\}$ we define a function $\alpha_k$ as follows: $\alpha_k(x\lambda_k + y\lambda_{n+k}) = \exp (\pi \sqrt{-1}\delta_kxy)$. Now define a function $\alpha(v) = \alpha_1(v_1) \ldots \alpha_n(v_n)$ where $v = v_1 + \ldots v_n, v_k \in V_k$. We are left to notice that $\alpha$ satisfies (32). The proof of Proposition 2.3.3 is complete.

2.4. Line bundles with $c_1 = 0$: the group $Pic^0(M)$.
2.4.1. If the Chern class of a line bundle on a complex torus $M$ vanishes, then this bundle can be defined by constant multiplicators.

We prove this proposition. Assume $c_1(L) = 0$. By the definition of $c_1$ and \([22]\), \([L] \in H^1(M; \mathcal{O}^*)\) lies in the image of the homomorphism $H^1(M; \mathcal{O}) \rightarrow H^1(M; \mathcal{O}^*)$. Indeed, the cocycle $\zeta \in H^1(M; \mathcal{O})$ is defined by the sections $\log (g_{\alpha \beta})$ and is transformed into the cocycle $g_{\alpha \beta}$ defining $L$.

For every compact complex manifold

1) the group $H^1(M; \mathcal{O})$ is isomorphic to the subgroup of $H^1(M; C)$ formed by the cohomology classes of closed $(0,1)$-forms;

2) the natural homomorphism $H^1(M; C) \rightarrow H^1(M; \mathcal{O})$ generated by the embedding of sheaves $C \rightarrow \mathcal{O}$ is onto. In terms of the de Rham cohomologies, this homomorphism is a projection

$$\xi = \xi^{1,0} + \xi^{0,1} \rightarrow \xi^{0,1}$$

where $\xi = \xi^{1,0} + \xi^{0,1}$ is the decomposition of a closed form into the sum of a $(1,0)$-form and a $(0,1)$-form (\([13]\)).

Therefore $\zeta$ is cohomologous to the cocycle $\log (g_{\alpha \beta})$ with constant coefficients. This means that the line bundle is defined by constant multiplicators. This proves Proposition 2.4.1.

Consider line bundles with constant multiplicators. By Propositions 2.3.2 and 2.4.1, these are exactly line bundles with $c_1 = 0$.

2.4.2. The equivalence classes of line bundles with $c_1 = 0$ on a torus $M$ are in a one-one correspondence with points of the dual torus $\hat{M} = \text{Pic}^0(M)$.

Let $L$ be a line bundle defined by the constant multiplicators $e_{\lambda}$. It is easy to construct a function $\varphi(z)$, of the shape $\varphi(z) = \exp (a_k z^k)$, such that $|\varphi(\lambda)| = |e_{\lambda}|^{-1}$. By \([24]\), the constant multiplicators $\tilde{e}_{\lambda}(z) = e_{\lambda} \varphi(z + \lambda) \varphi(z)^{-1}$ define the same bundle $L$ and $|\tilde{e}_{\lambda}| = 1$. Such sets of multiplicators are in a one-one correspondence with the homomorphisms

$$\psi : \Lambda \rightarrow \mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}.$$  (33)

This correspondence is defined by

$$\psi(\lambda) = e_{\lambda}.  \quad (34)$$

The set of homomorphisms \([33]\) is in a one-one correspondence with the torus $\hat{M} = \text{Hom}(\Lambda, \mathbb{C}_1)$.

Two distinct homomorphisms defines two distinct bundles. Prove this by way of contradiction. Assume that $\psi_1$ and $\psi_2$ define equivalent bundles. Then, by \([24]\), there exists an entire function $\varphi(z)$ such that $\psi_1(\lambda) = \psi_2(\lambda) \varphi(z + \lambda) \varphi^{-1}(z)$ for any $z \in \mathbb{C}^n$ and $\lambda \in \Lambda$. Since $|\psi_1(\lambda)| = |\psi_2(\lambda)| = 1$, the function $\varphi(z)$ is bounded and thus, by the Liouville theorem, it is constant. Hence $\psi_1 = \psi_2$. The proof of Proposition 2.4.2 is complete.

By Proposition 2.4.2 and \([34]\), the subgroup $\text{Pic}^0(M)$, formed by equivalence classes of bundles with $c_1 = 0$, coincides with $\hat{M} = \text{Hom}(\Lambda, \mathbb{C}_1)$ as a group also. Since $H^0(M; \mathcal{O}^*) = 0$, the other definition of $\text{Pic}^0(M)$ follows from the exact sequence \([24]\).

2.4.3. $\text{Pic}^0(M) = H^1(M; \mathcal{O})/H^1(M; \mathbb{Z})$.

The following theorem is derived from Propositions 2.3.2, 2.3.3, and 2.4.2.
2.4.4. **Appel–Humbert Theorem.** 1) Every line bundle on the complex torus $M$ can be defined by multiplicators (31); 2) The sequence

$$0 \to \hat{M} = \text{Pic}^0(M) \to \text{Pic}(M) \to H^2_{L,1}(M; \mathbb{Z}) \to 0,$$

where the first homomorphism is an embedding and the second assigns to a bundle its Chern class, is exact.

2.5. **Positive line bundles on tori and their sections.**

Let $M = \mathbb{C}^n/\Lambda$ be a torus endowed with a Hodge form $\omega$. By Propositions 1.3.5 and 1.3.6, there exists a basis $\mathcal{V} = (\lambda_1, \ldots, \lambda_{2n})$ for the lattice $\Lambda$ such that

1) in this basis $\omega$ takes the shape (31), with $\delta_j \in \mathbb{Z}, \delta_j > 0$;

2) the set of vectors $\lambda_1/\delta_1, \ldots, \lambda_n/\delta_n$ is a complex basis for $\mathbb{C}^n$ (denote by $z^1, \ldots, z^n$ the coordinates referred to this basis);

3) in terms of the coordinates $(z^1, \ldots, z^n)$ the lattice $\Lambda$ is written as $\Delta_\delta N_1 + \Omega N_2, N_i \in \mathbb{Z}^n$ where the matrix $\Omega$ is symmetric and positive definite.

By the Appel–Humbert theorem, the moduli space $\mathcal{L}(M, [\omega])$ of line bundles on $M$ with $c_1 = [\omega]$ is isomorphic to $\text{Pic}^0(M)$. Indeed, if $L, L' \in \mathcal{L}(M, [\omega])$ there exists a unique bundle $L'' \in \text{Pic}^0(M)$ such that $L' = L \otimes L''$.

However, $M$ acts on $\mathcal{L}(M, [\omega])$ by translations as follows. Put $L \in \mathcal{L}(M, [\omega])$. Let $\tau_\mu : M \to M$ be the translation $z \to z + \mu$. Then the bundle $\tau_\mu^*(L)$ is defined. In terms of multiplicators this action is simply as follows $e_\lambda(z) \to e_\lambda(z + \mu)$.

Let $L \in \mathcal{L}(M, [\omega])$ be defined by the following multiplicators

$$e_{\lambda_k} = 1, \quad e_{\lambda_n + k} = \exp(-2\pi \sqrt{-1} z^k), \quad k = 1, \ldots, n. \quad (36)$$

Thus this bundle lies in $\mathcal{L}(M, [\omega])$. Then the bundle $\tau_\mu^*(L) \otimes L^* \in \mathcal{L}(M, 0)$ is defined by the constant multiplicators

$$\tilde{e}_{\lambda_k}^{(\mu)} = 1, \quad \tilde{e}_{\lambda_n + k}^{(\mu)} = \exp(-2\pi \sqrt{-1} \mu^k), \quad k = 1, \ldots, n, \quad (37)$$

and by straightforward computations we derive from (37) the following proposition.

2.5.1. Assume that $L \in \mathcal{L}(M, [\omega])$. Then the mapping

$$\tau : M \to \text{Pic}^0(M), \quad \mu \mapsto \tau_\mu^*(L) \otimes L^*, \quad (38)$$

is a homomorphism onto the group $\text{Pic}^0(M)$. The kernel of this homomorphism is isomorphic to the sublattice spanned by $\{\delta_k^{-1} \lambda_k, \delta_k^{-1} \lambda_{n+k}\}$. Every bundle $L \in \mathcal{L}(M, 0)$ is written as $\tau_0^*(L) \otimes L^*$.

Denote by $\mathcal{O}(L)$ the sheaf of germs of holomorphic sections of $L$. Global sections are in one-one correspondence with entire functions $f(z)$ satisfying the periodicity conditions

$$f(z + \lambda) = e_\lambda(z)f(z) \quad (39)$$

where $e_\lambda$ are the multiplicators defining $L$. The set of all global sections is the vector space $H^0(M; \mathcal{O}(L))$.

2.5.2. Assume that a bundle $L$ is positive and $c_1 = [\omega]$. Then

$$h^0(L) = \dim_\mathbb{C} H^0(M; \mathcal{O}(L)) = \delta_1 \cdot \ldots \cdot \delta_n$$

where $(\delta_1, \ldots, \delta_n)$ is the polarization type of $\omega$.

Proof of Proposition 2.5.2.

Since the translation by $\mu$ is homotopic to the identity and the bundles in $\mathcal{L}(M, [\omega])$ are translates of one another, it suffices to find $h^0(L)$ for an arbitrary
bundle $L \in \mathcal{L}(M, [\omega])$. Consider the bundle with multiplicators \((36)\). Since $e_{\lambda_k} \equiv 1$, every section $f$ expands into the Fourier series in $\exp (2\pi \sqrt{-1} \omega_k \delta_k^i z^k)$:

$$f(z) = \sum_{l \in \mathbb{Z}^n} a_l \exp (2\pi \sqrt{-1} \sum_k k^k \delta_k^{-1} z^k). \quad (40)$$

By comparing the Fourier series expansion of $f(z + \lambda_{n+k})$ and the Fourier series expansion of $f(z)$ and considering \((39)\) we derive

$$a_{l+d_k} = \exp (2\pi \sqrt{-1} (l, \Delta^{-1} \lambda_{n+k})) a_l \quad (41)$$

where $(u, v) = \sum_k u^k v^k$ and $d_k \in \mathbb{Z}^n$, $d_k^k = \delta_k, d_k^j = 0$ for $j \neq k$. Thus we derive $h^0(L) \leq \delta_1 \cdot \ldots \cdot \delta_n$.

Consider functions $\theta_{\bar{l}}$, $\bar{l} \in \Pi_{\delta}$ of the shape \((40–41)\) such that

$$a_l = \begin{cases} 0, & \text{for } l \in \Pi_{\delta} \ \text{,} \\ 1, & \text{for } l = \bar{l}. \end{cases} \quad (42)$$

By \((40–41)\), the series \((40)\) is uniquely defined for every $\bar{l} \in \Pi_{\delta}$. All these series define entire functions. We explain this in §3 for the theta function of a principally-polarised Hodge torus.

It is clear that the functions $\theta_{\bar{l}}$ are linearly independent as sections of $O(L)$. Hence, $h^0(L) = \delta_1 \cdot \ldots \cdot \delta_n$. This proves Proposition 2.5.2.

The following proposition follows from Propositions 2.5.1 and 2.5.2.

2.5.3. Let $M$ be a Hodge torus with polarization $[\omega]$. Then the following statements are equivalent:

1) $[\omega]$ is the principal polarization;
2) the homomorphism \((38)\) $\tau : M \to \hat{M}$ is an isomorphism;
3) $h^0(L) = 1$ where $c_1(L) = [\omega]$.

§3. Theta functions

3.1. The theta function of a principally-polarised Hodge torus.

Every Hodge torus $M$ with the principal polarization $[\omega]$ is represented as follows

1) $M = \mathbb{C}^n / \Lambda$ and $\omega = \sum_k dx^k \wedge dx^{n+k}$ in the basis $(\lambda_1, \ldots, \lambda_2n)$ for $\Lambda$;
2) the lattice $\Lambda$ is written as $\{ \mathbb{Z}^n + \Omega \mathbb{Z}^n \}$ in the basis $(\lambda_1, \ldots, \lambda_n)$ for $\mathbb{C}^n$;
3) the $(n \times n)$-matrix $\Omega$ is symmetric and its imaginary part $Im \Omega$ is positive definite.

This follows from Propositions 1.3.5 and 1.3.6 and the Riemann criterion. In the sequel we mean by a principally-polarised Hodge torus a complex torus represented in this shape.

Let $L$ be the bundle defined by the following multiplicators

$$e_{\lambda_k} \equiv 1, \quad e_{\lambda_{n+k}} = \exp (-\pi \sqrt{-1} \Omega_{kk} + 2\pi \sqrt{-1} z^k) \quad (43)$$

where $\Omega_{ij} = \Omega_{ij}^k$ and $\lambda_{n+k} = \Omega_{kk}, \ k = 1, \ldots, n$.

By \((27)\), $c_1(L) = [\omega]$ and every bundle with the same Chern class is a translate of $L$ (Proposition 2.5.1). It thus suffices to construct the global sections of this bundle only. By Proposition 2.5.2, the space of these sections is one-dimensional.
Consider the formal series

\[ \theta(z, \Omega) = \sum_{m \in \mathbb{Z}^n} \exp(\pi\sqrt{-1}(\Omega m, m) + 2\pi\sqrt{-1}(m, z)). \] (44)

Since \( \text{Im} \ Omega \) is positive definite, for every compact subset \( U \subset \mathbb{C}^n \) there exists a constant \( c(U) \) such that

\[ |\exp(\pi\sqrt{-1}(\Omega m, m) + 2\pi\sqrt{-1}(m, z))| < c(U) \exp(-d|m|^2) \]

where \( d \) is a positive constant which depends on \( \Omega \). This implies that the series (44) converges uniformly on every compact set \( U \subset \mathbb{C}^n \) and defines an entire function.

**Definition.** The theta function \( \theta(z, \Omega) \) is that defined by (44).

In the sequel we mean by \( \theta(z) \) the function \( \theta(z, \Omega) \).

It follows from (44) that the theta function satisfies the following periodicity conditions

\[ \theta(z + m, \Omega) = \theta(z, \Omega), \]
\[ \theta(z + \Omega m, \Omega) = \exp(-\pi\sqrt{-1}(\Omega m, m) - 2\pi\sqrt{-1}(m, z)) \cdot \theta(z, \Omega), \quad m \in \mathbb{Z}^n. \] (45)

This implies Proposition 3.1.1.

**3.1.1.** \( \theta(z, \Omega) \) is a global section of \( L \) unique up to a constant multiple.

Since \( \mu \rightarrow M \) is an isomorphism, every line bundle on \( M \) with \( c_1 = [\omega] \) is uniquely expressed in the shape \( \tau_{\mu}^*(L) \) and the space of its global sections is generated by \( \theta(z + \mu, \Omega) \).

**3.2. Theta functions with characteristics.**

Given a pair of vectors \( a, b \in \mathbb{R}^n; \theta[a, b](z, \Omega) \), the theta function with characteristics \( a \) and \( b \) is defined by

\[ \theta[a, b](z, \Omega) = \theta[a, b](z, \Omega) = \exp(\pi\sqrt{-1}(\Omega a, a) + 2\pi\sqrt{-1}(a, z + b)) \cdot \theta(z + \Omega a + b, \Omega) = \sum_{m \in \mathbb{Z}^n} \exp(\pi\sqrt{-1}(\Omega(m + a), (m + a)) + 2\pi\sqrt{-1}(m + a, z + b)). \] (46)

The analogues of the periodicity conditions (45) for theta functions with characteristics are written as

\[ \theta[a, b](z + m, \Omega) = \exp(2\pi\sqrt{-1}(a, m)) \cdot \theta[a, b](z, \Omega), \]
\[ \theta[a, b](z + \Omega m, \Omega) = \exp(-\pi\sqrt{-1}(\Omega m, m) - 2\pi\sqrt{-1}(m, z)) \times \exp(-2\pi\sqrt{-1}(b, m)) \cdot \theta[a, b](z, \Omega), \quad m \in \mathbb{Z}^n. \] (47)

If the characteristics \( a \) and \( b \) are rational, then the functions \( \theta[a, b](z) \) determine sections of line bundles on \( M \).

Consider the tensor products of \( L \). Every bundle \( L^d \) is defined by the multipli-
cators \( e^d_{\lambda i} \) where \( e^d_{\lambda i} \) define \( L \) and take the shape (43).

**3.2.1. Each of the families of functions**

1) \( \theta[a/d, 0](d \cdot z, d \cdot \Omega), \quad 0 \leq a^i < d \),
2) \( \theta[0, b/d](z, d^{-1} \cdot \Omega), \quad 0 \leq b^i < d \)

is a basis for global sections of \( L^d \), i.e., a basis for \( H^0(M; \mathcal{O}(L^d)) \).

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These bases are related by the identity
\[ \theta(0, b/d)(z, d^{-1} \cdot \Omega) = \sum_a \exp\left(2\pi \sqrt{-1}(a, b)\right) \cdot \theta[a/d, 0](d \cdot z, d \cdot \Omega). \tag{48} \]

It follows from \([47]\) that these functions are sections of \(L^d\). For proving that these families are bases it suffices to compare them with a basis \(\{ \theta_l \}\) of the shape \([12]\) for sections of \(L^d\). Identity \([48]\) is proved by comparing the Fourier series expansions of the left-hand and right-hand sides of \((48)\).

3.3. Modular transformations.

In §1.4 the action of \(Sp(\Delta, \mathbb{Z})\) on the Siegel upper-half plane \(\mathcal{H}_n\) is described. This action consists in changing the matrix \(\Omega\) for \(\Omega'\) for sections of \(L^3\). These families are bases it suffices to compare them with a basis by \((46)\). In this case \((51)\) is written as
\[ \begin{align*}
\theta[a', b'][z', \Omega'] &= C_0 \cdot \exp\left(\pi \sqrt{-1}(z, Tz)\right) \cdot \theta[a, b](z, \Omega) \\
\text{with } C_0 \text{ a constant independent of } z,
\end{align*} \tag{50} \]

The explicit formula for \(C_0\) and the proof of \((50)\) are given in \([53]\).

3.4. Addition theorems for theta functions.

The most important theta function identities are the Riemann addition theorems \(3.4.1\) and \(3.4.2\).

3.4.1. Binary addition theorem of Riemann.

\[ \theta[a, c](z_1 + z_2, \Omega) \cdot \theta[b, d](z_1 - z_2, \Omega) = \sum_{e \in \mathbb{Z}^n/2\mathbb{Z}^n} \theta[a + b + c + d, (2z_1, 2\Omega)] \cdot \theta[a - b + c - d, (2z_2, 2\Omega)], \tag{51} \]

for \(\theta : \mathbb{C}^n \to \mathbb{C}, z_1, z_2 \in \mathbb{C}^n, \text{ and } a, b, c, d \in \mathbb{R}^n\).

It suffices to prove \((51)\) for \(a = b = c = d = 0\) to which the general case reduces by \((40)\). In this case \((51)\) is written as
\[ \theta(z_1 + z_2, \Omega) \cdot \theta(z_1 - z_2, \Omega) = \sum_{e \in \mathbb{Z}^n/2\mathbb{Z}^n} \theta\left(\frac{e}{2}, 0\right)(2z_1, 2\Omega) \cdot \theta\left(\frac{e}{2}, 0\right)(2z_2, 2\Omega). \tag{52} \]

The inverse of \((52)\) is the following addition theorem:
\[ \theta(z_1 + z_2, \Omega) \cdot \theta(z_1 - z_2, \Omega) = \sum_{e \in \mathbb{Z}^n/2\mathbb{Z}^n} \theta\left(0, \frac{e}{2}\right)(z_1, \frac{\Omega}{2}) \cdot \theta\left(0, \frac{e}{2}\right)(z_2, \frac{\Omega}{2}). \tag{53} \]
The ternary addition theorem is a generalization of the binary addition theorem of Riemann for the case of products of four theta functions.

3.4.2. Ternary addition theorem of Riemann.

\[ \theta[a_1, b_1](z_1) \cdot \ldots \cdot \theta[a_4, b_4](z_4) = \]
\[
\frac{1}{2^n} \cdot \sum_{c,d \in \mathbb{Z}^n} \exp \left(-4\pi i (d, a_1) + \ldots + (d, a_4) + c, b_4 + d) \right) \] (54)

for \( \theta : \mathbb{C}^n \to \mathbb{C}^n \) and \( a_i, b_j \in \mathbb{R}^n \) where \( \mathbb{a} = aT, \mathbb{b} = bT, \mathbb{z} = zT, \) and

\[
T = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\]

Here the units in the formula for the matrix \( T \) mean the identity \( (n \times n) \)-matrices.

The proofs of the addition theorems are exposed, for instance, in [82]. For \( n = 1 \) the proofs are obtained by straightforward comparing the Fourier series expansions of the left-hand and right-hand sides of (52–54). Generally, this requires repeating these arguments by coordinates (27).

3.5. The theta divisor.

A divisor on a variety \( M \) is a formal integer linear combination \( D = \sum_k a_k V_k (a_k \in \mathbb{Z}) \) of finitely many analytic hypersurfaces in \( M \). The set of divisors is the divisor group \( \text{Div}(M) \) under the formal addition. A divisor is called effective if \( a_k \geq 0 \) for every \( k \).

A divisor \( (f) \) of a function \( f \) is a linear combination of the sets of its zeros and poles counted with multiplicities. Therewith the sets of zeros and poles are taken with positive and negative signs respectively.

The group \( \text{Div}(M) \) is described in terms of sections of sheaves as follows. Let \( \mathcal{M}^* \) be the multiplicative sheaf of germs of meromorphic functions, on \( M \), non-vanishing identically. The sheaf \( \mathcal{O}^* \) is a subsheaf of \( \mathcal{M}^* \). Consider the space of sections of the sheaf \( \mathcal{M}^*/\mathcal{O}^* \).

3.5.1. \( \text{Div}(M) = H^0(M; \mathcal{M}^*/\mathcal{O}^*) \).

Indeed, a global section of \( \mathcal{M}^*/\mathcal{O}^* \) is defined by a family of local sections \( f_\alpha \in \mathcal{M}^*(U_\alpha) \) such that \( f_\alpha / f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta) \). Thus local divisors \( (f_\alpha) \) are glued into a global divisor which is identified with the section \( \{ f_\alpha \} \) of \( \mathcal{M}^*/\mathcal{O}^* \).

Consider the exact sequence of sheaves

\[
0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0.
\] (55)

Assign for this sequence the exact cohomology sequence which we need only extracting the following fragment

\[
H^0(M; \mathcal{M}^*) \to H^0(M; \mathcal{M}^*/\mathcal{O}^*) \overset{\delta}{\to} H^1(M; \mathcal{O}^*) = \text{Pic}(M).
\] (56)

The homomorphism

\[
\delta : \text{Div}(M) \to \text{Pic}(M)
\] (57)

assigns to a divisor \( D \) an associated line bundle as follows. Let \( D \) is defined by a section \( \{ f_\alpha \} \) of \( \mathcal{M}^*/\mathcal{O}^* \). Then the family of functions \( g_{\alpha \beta} = f_\alpha / f_\beta \) defines the coordinate transformations for the bundle \( \delta(D) = [D] \).
Two divisors $D_1$ and $D_2$ are called linearly equivalent, $D_1 \approx D_2$, if $D_1 - D_2 \in \text{Ker} \delta$. By (54), $D_1 \approx D_2$ implies that $D_1 - D_2$ is the divisor of zeros of a meromorphic function on $M$.

The theta divisor $\Theta = (\theta)$ is the zero set of the theta function (44).

The following proposition is clear.

3.5.2. 1) The line bundle $L = |\Theta|$ is defined by multiplicators (44) and the theta function generates $H^0(M; \mathcal{O}(L))$. 2) $|\Theta + \mu| = \tau^*_{\mu}(L)$.

To make our exposition complete, we state the following fact which is proved by applying the Stokes theorem. The proof is exposed, for instance, in [41].

3.5.3. Let $M$ be a compact complex $n$-dimensional manifold and let $L$ be a line bundle on $M$ such that $L = |D|$. Then $c_1(L)$ is dual to the cycle $D \in H_{2n-2}(M; \mathbb{Z})$ by the Poincare duality.

§4. Theta functions and mappings of tori into projective spaces.

Secants of Abelian varieties

4.1. Linear systems.

Let $L = |D|$ be a line bundle, on a compact complex manifold, associated with an effective divisor $D$. The divisor $D$ is defined locally by the functions $f_\alpha \in \mathcal{M}(U_\alpha)$ where $\mathcal{M}$ is the sheaf of germs of meromorphic sections of $L$. It is seen that these functions define a meromorphic section $f_0$ of $L$.

Denote by $\mathcal{L}(D)$ the space of all meromorphic functions $f$ on $M$ such that the divisor $(f) + D$ is effective, i.e., $(f) + D \geq 0$, and denote by $|D|$ the set of effective divisors linearly equivalent to $D$. There exists a one-to-one correspondence between the points of the projective space associated with $\mathcal{L}(D)$ and the divisors of $|D|$. It is as follows:

$$D' \in |D| \leftrightarrow [f] \in P\mathcal{L}(D) : (f) = D' - D. \quad (58)$$

A section $f \cdot f_0$ of $\mathcal{M}$ is holomorphic if and only if $f \in \mathcal{L}(D)$. Therefore the correspondence (58) is extended as follows

$$|D| \leftrightarrow PH^0(M; \mathcal{O}(|D|)). \quad (59)$$

A subset of effective divisors corresponding to a linear subspace of $H^0(M; \mathcal{O}(|D|))$ by (59) is called a linear system of divisors. The simplest example is the complete linear system $|D|$.

The set of base points $B(X)$ of a linear system $X$ is the intersection of all divisors of this system. Assume that $E(X) \subset H^0(M; \mathcal{O}(|D|))$ is a subspace corresponding to $X$ by (24). Then $B(X)$ comprises the points of $M$ at which all sections in $E(X)$ vanish.

Given a linear system $X$, the following mapping is defined up to projective transformations of $\mathbb{CP}^k$:

$$\varphi : M \setminus B(X) \to \mathbb{CP}^k : \varphi(x) = (f_1(x) : \ldots : f_{k+1}(x)) \quad (60)$$

where $\{f_\alpha\}$ is a basis for $E(X)$.

4.2. The linear systems $|d\Theta|$. The Lefschetz theorem.

Consider the mappings (20) for the linear systems $|d\Theta|$ where $d \geq 2$.

4.2.1. The sets of base points of $|d\Theta|$ are empty for $d \geq 2$.

To prove this proposition it suffices to find for every point $z \in M = \mathbb{C}^n/\{\mathbb{Z}^n + \Omega \mathbb{Z}^n\}$ sections $f_d \in H^0(M; \mathcal{O}(L^d))$ such that $f_d(z) \neq 0$. 

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By \eqref{eq:15}, the function $f_d^\mu(z) = \theta(z + \mu_1) \cdots \theta(z + \mu_d)$ defines a global section of $\mathcal{O}(L^d)$ for $\mu_1 + \cdots + \mu_d = 0$. Since the theta divisor is an analytic hypersurface in $M$, for every point $z \in M$ there exists a vector $\nu = (\nu_1, \ldots, \nu_d)$ such that $\theta(z + \nu_k) \neq 0$ for any $k$. Now it suffices to take $f_d^\mu$ for $f_d$. The proof of Proposition 4.2.1 is complete.

Denote by
\[ \varphi_d : M = \mathbb{C}^n / \{Z^n + \Omega Z^n\} \to \mathbb{C}P^{d-1} \] the mapping for the linear system $|d\Theta|$. By Proposition 3.2.1, $\varphi_d$ can be defined by the following formula
\[ \varphi_d(z) = (\theta(z_1^\mu, 0) | (d \cdot z, d \cdot \Omega) : \ldots : \theta(z_n^\mu, 0) | (d \cdot z, d \cdot \Omega)) \in \mathbb{C}P^{d-1} \] where $\{a_i\} = Z^n/dZ^n$.

**4.2.2. Lefschetz Theorem.** The mapping $\varphi_d$ is an embedding for $d \geq 3$.

For brevity we prove this only for $d = 3$. It is easily seen how from the proof of Proposition 4.2.1 to obtain the proof in the general case.

1) First, prove that the rank of $\varphi_d$ is maximal.

Take an arbitrary point $z_* \in M$. Choose a basis $(\theta_0, \theta_1, \ldots, \theta_N)$ for $H^0(M; \mathcal{O}(L^2))$ such that $\theta_0(z_*) = 1$ and $\theta_1(z_*) = \ldots = \theta_N(z_*) = 0$. Then the rank of $\varphi_2$ at $z_*$ coincides with the rank of the following matrix
\[ J(z_*) = \begin{pmatrix} \frac{\partial \theta_1(z_*)}{\partial z_1} & \cdots & \frac{\partial \theta_1(z_*)}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \theta_N(z_*)}{\partial z_1} & \cdots & \frac{\partial \theta_N(z_*)}{\partial z_n} \end{pmatrix}. \]

Assume that $\text{rank} J(z_*) < n$. This means that there exist non-trivial vectors $a_1, \ldots, a_n$ such that
\[ a_1 \frac{\partial \theta_k(z_*)}{\partial z_1} + \cdots + a_n \frac{\partial \theta_k(z_*)}{\partial z_n} = 0 \] for any $k$. Choose $\mu$ and $\nu$ such that
\[ \theta(z_* + \nu) \cdot \theta(z_* - \mu - \nu) \neq 0. \] (63)

By \eqref{eq:15}, the function
\[ F(z, \mu, \nu) = \theta(z + \mu) \cdot \theta(z + \nu) \cdot \theta(z - \mu - \nu) \] defines a section of $L^d (d = 3)$ for any $\mu, \nu$. Hence this function is represented by a linear combination of functions of the basis $\{\theta_k\}$ and the identity
\[ a_1 \frac{\partial F(z_*, \mu, \nu)}{\partial z_1} + \cdots + a_n \frac{\partial F(z_*, \mu, \nu)}{\partial z_n} = 0 \] (65)
holds. Define the function
\[ \xi(z) = a_1 \frac{\partial \log \theta(z)}{\partial z_1} + \cdots + a_n \frac{\partial \log \theta(z)}{\partial z_n}. \]

It is seen that \eqref{eq:15} is equivalent to the following equality
\[ \xi(z + \mu) + \xi(z + \nu) + \xi(z - \mu - \nu) = 0. \] (66)
By (53) and (54), the function \( \tilde{\xi}(z) = \xi(z + \mu) \) has no poles and thus it is an entire function. It follows from (63) that \( \tilde{\xi}(z) \) satisfies the following periodicity conditions

\[
\hat{\xi}(z + \lambda_k) = \tilde{\xi}(z), \quad \hat{\xi}(z + \lambda_{n+k}) = \tilde{\xi}(z) - 2\pi \sqrt{-1}a_k. \tag{67}
\]

By (67), the derivatives \( \partial \hat{\xi}/\partial z^k \) of \( \hat{\xi} \) are \( \Lambda \)-periodic and thus they are bounded. Since these derivatives are entire functions, they are constant and \( \hat{\xi}(z) = \sum_k b_k z^k + c \). However, it follows from (67) that \( b_k \equiv 0 \) and, therefore, \( \xi \) is constant. Since \( \hat{\xi}(z + \lambda_{n+k}) - \hat{\xi}(z) = -2\pi \sqrt{-1}a_k \), we have \( a_k \equiv 0 \).

We conclude that the family of constants \( \{a_k\} \) is trivial and \( J(z_*) \) has maximal rank. Since \( z_* \) is arbitrary, the rank of \( \varphi_d \) equals \( n \) everywhere.

2) Prove that \( \varphi_d(z_1) = \varphi_d(z_2) \) implies \( z_1 - z_2 \in \Lambda \). This amounts to the injectivity of \( \varphi_d \).

Assume that \( \varphi_d(z_1) = \varphi_d(z_2) \). Hence there exists a non-zero constant \( C \) such that \( \psi(z_1) = C\psi(z_2) \) for every section \( \psi \) of \( L^2 \). Consider a function \( F(z, \mu, \nu) \) of the shape (64). Since this function defines a section of \( L^2 \), the following identity

\[
F(z_1, \mu, \nu) = \frac{\theta(z_1 + \mu) \cdot \theta(z_1 + \nu) \cdot \theta(z_1 + \nu - \mu)}{\theta(z_2 + \mu) \cdot \theta(z_2 + \nu) \cdot \theta(z_2 + \nu - \mu)} = C \tag{68}
\]

holds. For arbitrary \( \mu \) it is possible to find \( \nu \) such that

\[
\theta(z_1 + \nu) \cdot \theta(z_1 + \nu - \mu) \cdot \theta(z_2 + \nu) \cdot \theta(z_2 + \nu - \mu) \neq 0. \tag{69}
\]

By (49), the function \( \hat{\xi} = \log(\theta(z_1 + z)/\theta(z_2 + z)) \) is holomorphic. As for \( \hat{\xi} \) it is shown that the derivatives \( \hat{\xi}(z) \) are constant. Therefore, this function is linear: \( \hat{\xi}(z) = 2\pi \sqrt{-1} \sum b_k z^k + c \). It follows from (63) that

\[
\hat{\xi}(z + \lambda_k) = \hat{\xi}(z) + 2\pi \sqrt{-1} \beta_k,
\]

\[
\hat{\xi}(z + \lambda_{n+k}) = \hat{\xi}(z) + 2\pi \sqrt{-1}(\bar{z}_2^k - z_1^k) + 2\pi \sqrt{-1} \alpha_k, \quad \alpha_i, \beta_j \in \mathbb{Z}.
\]

Since \( \hat{\xi} \) is linear, we have \( b_k = \beta_k \) and

\[
z_2^k - z_1^k = -\alpha_k + \sum_i \Omega_{kt} \beta_t.
\]

The last equality is equivalent to \( z_2 - z_1 \in \Lambda \).

The proof of the Lefschetz theorem is complete. The following theorem is a corollary of the Lefschetz and Chow theorems.

**4.2.3. A principally-polarised Hodge torus is an Abelian variety.**

In the case of a non-principally-polarised Hodge torus \( (M, [\omega]) \) the space of sections of a line bundle \( L \to M \) with \( c_1(L) = [\omega] \) is spanned by the functions \( \theta_l \) (10) (32).

Substituting any of the functions \( \theta_l \) instead of the theta function \( \theta(z, \Omega) \) in the proofs of Proposition 4.2.1 and the Lefschetz theorem we may repeat all arguments and prove the following theorem.

**4.2.4. Lefschetz Theorem (the general case).** Let \( (M, [\omega]) \) be a Hodge torus, let \( L \) be a line bundle with \( c_1(L) = [\omega] \), and let \( X_d \) be the linear system corresponding to \( H^0(M; \mathcal{O}(L^d)) \). Then the set of base points of \( X_d \) is empty for \( d \geq 2 \) and this system defines by (64) an embedding of \( M \) into a complex projective space for \( d \geq 3 \).
This theorem and the theorem of Chow imply the following criterion.

**4.2.5. Criterion for a complex torus to be algebraic.** A complex torus is an Abelian variety if and only if it is a Hodge torus.

In the general case the Kodaira theorem is proved by construction of a line bundle $L$ with $c_1(L) = [\omega]$ from a Hodge form $\omega$ and embedding $M$ into $\mathbb{C}P^N$ by a mapping of the shape \( [\theta(\omega, L)] \) for a line system corresponding to $H^0(M; \mathcal{O}(L^d))$ for sufficiently large $d$ \( (\ref{eq:linebundle}) \).

**4.3. The mapping $\varphi_2$ and Kummer varieties.**

Let $M = \mathbb{C}^n/\Lambda$ be an Abelian variety with the principal polarization $[\omega]$. The reflection acts on $M$ as follows

\[ \sigma : M \to M : \sigma(z) = -z. \]

This action explains why the claim of the Lefschetz theorem fails for $d = 2$. The mapping $\varphi_2$ is defined by the section $\theta(e, 0)(2z, 2\Omega)$ of the bundle $[2\Omega]$ where $2e \in \mathbb{Z}^n/2\mathbb{Z}^n$. All these functions are even and the rank of $\varphi_2$ vanishes at zero. This also means that $\varphi_2$ splits as follows

\[ M \xrightarrow{\pi} M/\sigma \xrightarrow{\Phi} \mathbb{C}P^{2^n-1}, \quad \varphi_2 = \Phi \circ \pi \]

with $\pi$ the natural projection.

The variety $K(M, \Theta) = \Phi(M/\sigma)$ is called a **Kummer variety** and the mapping $\Phi$ is called a **Kummer map**.

A polarised Abelian variety $(M, [\omega])$ is called reducible if it is a direct product of two Abelian varieties $(M_1, [\omega_1])$ and $(M_2, [\omega_2])$ of positive dimensions and $[\omega] = [\omega_1] + [\omega_2]$.

**4.3.1. Theorem on injectivity of the Kummer map.** Let the principally-polarised Abelian variety $M = \mathbb{C}^g/\{Z^g + \Omega Z^g\}$ be irreducible. Then the Kummer map

\[ \Phi : z \to (\theta(e, 0)(2z, 2\Omega) : \ldots : \theta(n_r, 0)(2z, 2\Omega)), \]

where $\{n_j\} = \mathbb{Z}^g/2\mathbb{Z}^g$ and $r = 2^g$, is an embedding of the variety with singularities $M/\sigma$ into $\mathbb{C}P^{2^n-1}$.

We explain the proof of this theorem assuming Proposition 4.3.2 to be valid.

**4.3.2. Let an Abelian variety be principally-polarised and irreducible. Then its theta divisor $\Theta$ is irreducible and non-invariant under translations, i.e., $\Theta + \mu = \Theta$ implies $\mu \in \Lambda$.**

For brevity we omit the proof of this proposition. Notice that its proof is based on the equivalence of reducibility of a principally-polarised Abelian variety to reducibility of its theta divisor and the non-compatibility of existence of non-trivial translations transforming variety into itself with a principal polarization.

Assume that $\varphi_2(z_1) = \varphi_2(z_2)$. Then there exists a non-zero constant $C$ such that

\[ \frac{\theta(e, 0)(2z_1, 2\Omega)}{\theta(e, 0)(2z_2, 2\Omega)} = C \]

for any $e \in \frac{1}{2}\mathbb{Z}^n/2\mathbb{Z}^n$. It follows from \( (\ref{eq:theta}) \) that

\[ \frac{\theta(z + z_1) \cdot \theta(z - z_1)}{\theta(z + z_2) \cdot \theta(z - z_2)} = C \]

(71)

for any $z \in \mathbb{C}^n$. 

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Let \((z + z_1)\) be a non-singular point of the theta divisor and let \(\theta(z - z_1)\) does not vanish. Then, by (71), \(\theta(z + z_2) \cdot \theta(z - z_2) = 0\).

Suppose that \(\theta(z - z_2) = 0\). Then in a neighbourhood of \(z\) the function \(\theta(z + z_1)/\theta(z - z_2)\) is holomorphic and, since the theta divisor is irreducible, \(\Theta + (z_1 + z_2) = \Theta\). Since the theta divisor is non-invariant under translations, \(z_1 + z_2 \in \Lambda\). The case of \(\theta(z + z_2) = 0\) is considered similarly. This completes the proof.

An important example of a Kummer variety is the quotient space of a two-dimensional torus \(\mathbb{C}^2/\Lambda\) for the reflection. It contains 16 singular points corresponding to the sublattice \(\Lambda/2\). By resolution of singularities we obtain a non-singular algebraic surface known as a \textit{Kummer surface}, or a K3 surface.

4.4. \textbf{Secants of Abelian varieties.}

Let \(M = \mathbb{C}^n/\Lambda\) be an irreducible principally-polarised Abelian variety and \(K(M, \Theta) \subset \mathbb{C}P^{2n-1}\) be its Kummer variety.

\textbf{Definition.} \textit{N-secant of an Abelian variety} \(M\) is a \((N - 2)\)-dimensional plane \(V (= \mathbb{C}P^{N-2}) \subset \mathbb{C}P^{2n-1}\) which intersects the Kummer variety \(K(M, \Theta)\) at \(N\) distinct points.

Let \(V\) be a \(N\)-secant passing through points \(\Phi(z_1), \ldots, \Phi(z_N)\). Then there exist non-zero constants \(c_1, \ldots, c_N\) such that
\[
   c_1 \cdot \theta(e, 0)(2z_1, 2\Omega) + \ldots + c_N \cdot \theta(e, 0)(2z_N, 2\Omega) = 0
\]
for any \(e \in \frac{1}{2}\mathbb{Z}^n/2\mathbb{Z}^n\). The addition theorem (72) and (72) imply Proposition 4.4.1.

4.4.1. The following statements (i) and (ii) are equivalent:
(i) points \(\Phi(z_1), \ldots, \Phi(z_N)\) of \(K(M, \Theta)\) lie on the same \(N\)-plane and satisfy the relation \(c_1 \Phi(z_1) + \ldots + c_N \Phi(z_N) = 0\),

(ii) the \textit{theta functional identity}
\[
   c_1 \theta(z + z_1) \theta(z - z_1) + \ldots + c_N \theta(z + z_N) \theta(z - z_N) = 0, \quad z \in \mathbb{C}^n
\]
holds.

Consider conditions on the theta divisor providing the existence of \(N\)-secants. It seems that they were first considered in [111]. Denote by \(\Theta\), the translate of the theta divisor by \(\mu\), i.e., \(\Theta + \mu\).

4.4.2. Assuming \(\dim M \geq (N - 1)\), the statements (i) and (ii) are equivalent:

(i) there exist pairwise distinct points \(\mu_1, \ldots, \mu_{N-1}, x, y \in M\) such that the analytic subset \(\Theta_{\mu_1} \cap \ldots \cap \Theta_{\mu_{N-1}}\) is of codimension \(N - 1\) and

\[
   \Theta_{\mu_1} \cap \ldots \cap \Theta_{\mu_{N-1}} \subset \Theta_x \cup \Theta_y;
\]

(ii) the Kummer variety \(K(M, \Theta)\) admits an \(N\)-secant and (72) holds with \(c_N \neq 0\) for \(z_1 = (x + y)/2 - \mu_1, \ldots, z_{N-1} = (x + y)/2 - \mu_{N-1}\), and \(z_N = (x - y)/2\).

By Proposition 4.4.1, (ii) implies (i). While proving that (ii) follows from (i) we apply the Koszul complex of the intersection \(\Theta_{\mu_1} \cap \ldots \cap \Theta_{\mu_{N-1}}\) (19). We consider only the case of \(N = 4\) (8) which illustrates how the proof becomes more complicated for \(N > 4\) and how it simplifies for \(N = 3\) (32).

Consider the bundle \(L = [2\Theta + x + y]\) and its sections \(s_0 = \theta(z - x)\theta(y - y), s_1 = \theta(z - x)\theta(z - x + y - \mu_1), \ldots, s_3 = \theta(z - x)\theta(z - x + y - \mu_3)\). Denote by \(I\) the subsheaf \(\mathcal{O}_M\) generated by the sections vanishing on \(\Theta_{\mu_1} \cap \ldots \cap \Theta_{\mu_{N-1}}\).

Consider the tensor product of the Koszul complex \((K_i, d_i)\) and \(\mathcal{O}(L)\):
\[
   0 \rightarrow \mathcal{O}(L^{-2}) \xrightarrow{d_1} \mathcal{O}(L^{-1}) \oplus \mathcal{O}(L^{-1}) \oplus \mathcal{O}(L^{-1}) \xrightarrow{d_2}
\]
\[ \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{d_3} I \otimes \mathcal{O}(L) \to 0 \]

where \( d_1(f) = (s_1 f, s_2 f, s_3 f) \), \( d_2(f, g, h) = (c_2 h - c_3 g, c_3 f - c_1 h, c_1 g - c_2 f) \), and \( d_3(f, g, h) = (s_1 f + s_2 g + s_3 h) \).

Since \( \dim \Theta_{\mu_1} \cap \Theta_{\mu_2} \cap \Theta_{\mu_3} = \dim M - 3 \), this sequence is exact.

By the Kodaira vanishing theorem applied to the positive bundle \( L \) ([17, 41]), \( H^k(M; \mathcal{O}(L^{-m})) = 0 \) for \( m > 0 \) and \( k < \dim M \). Successively applying the exact cohomology sequence to the triples \( 0 \to A = K_i/d_i-1(K_{i-1}) \to K_{i+1} \to K_{i+1}/d_{i+1}(A) \to 0 \) we arrive at the exact sequence

\[ 0 \to \tilde{A} \xrightarrow{d_{N-2}} \mathcal{O} \oplus \ldots \oplus \mathcal{O} \xrightarrow{d_{N-1}} I \cdot \mathcal{O}(L) \]  

(75)

where \( H^1(M; \tilde{A}) = 0 \). It follows from the exact cohomology sequence for (76) that \( H^0(M; I \otimes \mathcal{O}(L)) = d_{N-1}(H^0(M; \mathcal{O} \oplus \ldots \oplus \mathcal{O})) = d_{N-1}(\mathbb{C}^{N-1}) \). By (74), the function \( s_0 \) defines a section of \( I \otimes \mathcal{O}(L) \) and, therefore, is a linear combination of the sections \( s_1, \ldots, s_{N-1} \). By (73), this is equivalent to existence of a secant. This proves Proposition 4.4.2.

Chapter 2. Theta functions of Riemann surfaces

§5. Theta functions of Jacobi varieties

5.1. Riemann surfaces and their period matrices.

A Riemann surface is a complex manifold of complex dimension one. A compact surface \( \Gamma \) is homeomorphic to a sphere with handles. The number of handles is called the genus of \( \Gamma \).

A basis \( a_1, \ldots, a_g, b_1, \ldots, b_g \) for 1-cycles on a compact surface \( \Gamma \) of genus \( g \) is called canonical if its intersection form is as follows

\[ a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_j = \delta_{ij} \]

where \( \delta_{ij} \) is the Kronecker symbol. The canonical basis is realised by contours such that by cutting \( \Gamma \) along these contours we obtain a \( 4g \)-gon \( \tilde{\Gamma} \) with sides \( a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1} \) taken in order of traversal.

In the sequel we suppose that Riemann surfaces are compact and the bases of 1-cycles on surfaces are canonical.

Since a Hermitian metric on a one-dimensional complex manifold is proportional to a Hodge metric, every Riemann surface is projective algebraic, or an algebraic curve.

A differential is a 1-form \( \omega \) on \( \Gamma \). A differential is called Abelian (or holomorphic) if it is represented in a neighbourhood of every point by the form \( \omega = f(z)dz \) where \( f(z) \) is a holomorphic function. In the same manner meromorphic differentials are defined. It is clear that every Abelian differential as well as the adjoint differential \( \bar{\omega} = f(z)d\bar{z} \) is closed.

5.1.1. Reciprocity Law (for smooth differentials). Let \( \omega \) and \( \omega' \) be closed differentials on a Riemann surface of genus \( g \). Then the identity

\[ \int_{\Gamma} \omega \wedge \omega' = \sum_{k=1}^{g} (\int_{a_k} \omega \cdot \int_{b_k} \omega' - \int_{b_k} \omega \cdot \int_{a_k} \omega') = 0 \]  

(76)

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holds.

In order to prove this proposition we apply the Stokes theorem to a domain $\tilde{\Gamma}$ which is projected onto $\Gamma$ by the universal covering $\{ z \in \mathbb{C} : \text{Im} z > 0 \} \rightarrow \Gamma$. By applying this proposition to an Abelian differential $\omega$ and its adjoint differential $\bar{\omega}$ we infer the following proposition.

**5.1.2.** Let $\omega$ be a non-zero holomorphic differential on a Riemann surface of genus $g$. Then the identity

$$\text{Im} \sum_{k=1}^{g} \int_{a_k} \omega \cdot \int_{b_k} \omega < 0 \quad (77)$$

holds.

The conjugation $\omega \rightarrow \bar{\omega}$ induces an involution on the group $H^1(M; \mathbb{C})$ decomposed into the direct sum of the subgroups $H^{1,0}(M; \mathbb{C})$ and $H^{0,1}(M; \mathbb{C})$ generated by holomorphic and anti-holomorphic differentials. These subgroups permute under the conjugation and each of them is isomorphic to $\mathbb{C}^g$. Considering (77) we obtain

**5.1.3.** The space of holomorphic differentials on a Riemann surface of genus $g$ is isomorphic to $\mathbb{C}^g$ and every holomorphic differential is determined uniquely by its $a$-periods.

Hence to every canonical basis for 1-cycles there is assigned the canonical basis for Abelian differentials $\omega_1, \ldots, \omega_g$ which is defined uniquely by the formula

$$\int_{a_j} \omega_k = \delta_{jk}. \quad (78)$$

The period matrix, of Riemann surface $\Gamma$, associated with this basis is constructed as follows

$$B(\Gamma)_{jk} = \int_{b_j} \omega_k. \quad (79)$$

As one can see the period matrix is determined from a canonical basis for $H_1(\Gamma)$. Passage to a new canonical basis is achieved by a transformation in $Sp(2g, \mathbb{Z})$. Therewith the period matrix is transformed by (79). By applying (77) to the differentials $\omega = \sum k c_k \omega_k$, where $c_k \in \mathbb{R}$, we obtain the following proposition.

**5.1.4.** The period matrix of a Riemann surface lies in the Siegel upper half-plane, $B(\Gamma) \in \mathcal{H}_g$.

Riemann surfaces of genus $g$ are in a one-one correspondence with points of the complex manifold $\mathcal{M}_g$, the moduli space of Riemann surfaces of genus $g$. It is known that

$$\text{dim}_\mathbb{C} \mathcal{M}_g = \begin{cases} 0, & g = 0; \\ 1, & g = 1; \\ 3g - 3, & g \geq 2. \end{cases} \quad (80)$$

Hence the mapping

$$B : \mathcal{M}_g \rightarrow \mathcal{A}_g = \mathcal{H}_g / Sp(2g, \mathbb{Z}) \quad (81)$$

is defined. For a suitable compactification of $\mathcal{M}_g$, this mapping $B$ is algebraic. The same statement is valid for the Prym mappings (99) and (116) which are defined below (19).

**5.2.** Jacobi variety and Abel map. Theta function of a Riemann surface.

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It follows from (51) that to every Riemann surface \( \Gamma \) we may canonically assign a principally-polarised Abelian variety, its \textit{Jacobi variety}:

\[
J(\Gamma) = \mathbb{C}^g / \{ \mathbb{Z}^g + B(\Gamma) \mathbb{Z}^g \}.
\]  

(82)

Given a point \( p_0 \in \Gamma \), we consider the mapping

\[
A : \Gamma \to J(\Gamma) : A(p) = (\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g)
\]

(83)

which is called an \textit{Abel mapping}. Its definition does not depend on the choice of the path of integration \( \gamma \). Indeed, let us change \( \gamma \) for \( \gamma' \). The contour \( \gamma^{-1} \cdot \gamma' \) is closed and realises the 1-cycle \( \zeta \). Therewith \( A(p) \) is changed for the sum of \( A(p) \) and the vector \( (\int_{\gamma} \omega_1, \ldots, \int_{\gamma} \omega_g) \) which belongs to the lattice \( \{ \mathbb{Z}^g + B(\Gamma) \mathbb{Z}^g \} \).

An Abel mapping extends linearly onto the divisor group of \( \Gamma \).

Let \( \theta(z) = \theta(z, B(\Gamma)) \) be the \textit{theta function} of a Riemann surface \( \Gamma \). Construct the functions \( f(p) = \theta(A(p) - e) \). Since their values depend on paths of integrations in (84), these functions are many-valued on \( \Gamma \). The differences between branches are given by (85). However, by (84), if \( f \) does not vanish identically then the set of zeros of \( f(p) \) is a correctly defined divisor on \( \Gamma \).

\textbf{5.2.1.} There is a unique point \( K_{\Gamma} \in J(\Gamma) \) called the “vector of Riemann constants” such that one of the two possibilities takes place:

1) \( A(\Gamma) \subset \Theta_e = \Theta + e \) and \( \theta(A(p) - e) = 0 \);
2) if \( f(p) = \theta(A(p) - e) = 0 \) then \( p \in \{p_1, \ldots, p_g\} \) and

\[
A(p_1) + \ldots + A(p_g) + K_{\Gamma} = e.
\]

(84)

In the second case the divisor \( D = p_1 + \ldots + p_g \) is uniquely determined by (84).

Since the second case holds for generic \( e \in J(\Gamma) \), (84) determines almost everywhere the inverse of

\[
A : S^g \Gamma \to J(\Gamma), \quad A(p_1, \ldots, p_g) = A(p_1) + \ldots + A(p_g)
\]

where \( S^g \Gamma \) is the \( g \)-th symmetric power of \( \Gamma \).

It is clear that if an effective divisor \( D \) of degree \( g \) is the divisor of poles of a non-trivial meromorphic function then \( A \) is non-invertible at \( A(D) \). Such divisors are called \textit{special}.

\textbf{5.2.2.} Let \( \Sigma \subset S^g \Gamma \) be the subset formed by special divisors. Then the mapping \( A \) is inverted by (84) on \( S^g \Gamma \setminus \Sigma \).

The theta divisor \( \Theta \) is described by the following theorem.

\textbf{5.2.3. Riemann Theorem on the theta divisor of a Jacobi variety.}

\[
\Theta = A(S^g \Gamma) + K_{\Gamma}.
\]

(85)

\textbf{5.3. Abel and Riemann–Roch theorems.}

A divisor \( D \) on a surface is a linear combination of finite number of points of the shape \( D = a_1 p_1 + \ldots + a_n p_n, a_k \in \mathbb{Z}, p_k \in \Gamma \). The degree of \( D \) is the following sum of coefficients \( \text{deg} \ D = a_1 + \ldots + a_n \in \mathbb{Z} \).

Recall that \( [D] = \delta(D) \) is a line bundle constructed from \( D \) (see §3.5, (77)). The proof of the following theorem is given in (95).

\textbf{5.3.1. Let \( M \) be a projective algebraic variety. Then every line bundle on \( M \) has a meromorphic section and hence the homomorphism (77) \( \delta : \text{Div}(M) \to \text{Pic}(M) \) is a mapping onto.}
The Abel theorem enables us to linearise $\delta$.

5.3.2. Abel Theorem. Assume that $\deg D_1 = \deg D_2$. Then $D_1 - D_2$ is the divisor of a meromorphic function on $\Gamma$ if and only if $A(D_1) = A(D_2)$.

By the exact sequence (56), Proposition 5.3.1, and the Abel theorem, we have the following proposition.

5.3.3. Structure of the group $\text{Pic}(\Gamma)$. The sequence

$$0 \to J(\Gamma) \to \text{Pic}(\Gamma) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

is exact.

The first homomorphism in (86) is an embedding of a subgroup generated by line bundles $[D]$ with deg $D = 0$. This sequence is an analogue of (35). Indeed, $H^2(\Gamma; \mathbb{Z}) = \mathbb{Z}$ and it is easy to infer from (19–20) that the homomorphism $\deg$ assigns to a line bundle its first Chern class.

The divisor of a meromorphic differential $\omega$ is the formal sum $\omega$ of its zeros and poles taken with positive and negative signs counted with multiplicities. By the Abel theorem, all the divisors of meromorphic differentials are linearly equivalent. This equivalence class is called the canonical class of a surface and is denoted by $C(\Gamma)$. It is described by the following proposition.

5.3.4. 1) $\deg C(\Gamma) = 2g - 2$; 2) $A(C(\Gamma)) = -2K_{\Gamma}$.

The dimensions $l(D)$ of the spaces $H^0(\Gamma; \mathcal{O}([D])) = L(D)$ (see §4.1) are given by the Riemann–Roch theorem.

5.3.5. Riemann–Roch Theorem.

$$l(D) = \deg D - g + 1 + l(C(\Gamma) - D).$$

(87)

The Riemann–Roch formula for non-special effective divisors of degree $g$ degenerates into $l(D) = 1$.

5.4. Trisecants of Jacobi varieties.

5.4.1. Fay Trisecant Formula ([36]). For every quadruple $p_1, p_2, p_3, \text{ and } p_4$ of points of a Riemann surface $\Gamma$ there exist constants $c_1(\bar{p}), c_2(\bar{p}), \text{ and } c_3(\bar{p})$ such that

$$c_1 \cdot \theta(z + p_1 - p_3) \cdot \theta(z + p_2 - p_4) + c_2 \cdot \theta(z + p_1 - p_4) \cdot \theta(z + p_2 - p_3) +$$

$$c_3 \cdot \theta(z + p_1 + p_2 - p_3 - p_4) \cdot \theta(z) = 0.$$  

(88)

This means that the following points

$$\Phi(\frac{p_1 + p_2 - p_3 - p_4}{2}), \Phi(\frac{p_1 + p_3 - p_2 - p_4}{2}), \Phi(\frac{p_1 + p_4 - p_2 - p_3}{2})$$

(89)

of the Kummer variety lie on the same line $\mathbb{C}P^1 \subset \mathbb{C}P^{2g-1}$ called a trisecant.

For brevity here and below we denote $A(p_k)$ by $p_k$.

The trisecant formula for a generic quadruple $\bar{p}$ follows from Propositions 4.4.2 and 5.2.1 and the Riemann theorem on the theta divisor of a Jacobi variety.

In a generic case we can assume

$$\dim \Theta_{-(p_1 - p_3)} \cap \Theta_{-(p_1 - p_4)} = g - 2.$$

In fact, this is always valid ([82]). We are left to prove that

$$\Theta_{-(p_1 - p_3)} \cap \Theta_{-(p_1 - p_4)} \subset \Theta_{-(p_1 + p_2 - p_3 - p_4)} \cup \Theta.$$  

(90)
Assume that \( \theta(u - p_3) = \theta(u - p_4) = 0 \) and \( u = z + p_1 \). If \( \theta(u - p) = 0 \) for every \( p \in \Gamma \) then \( \theta(u) = 0 \). Otherwise \( \theta(u - p) = 0 \) for \( p = p_3, p_4, q_1, \ldots, q_{g-2} \) where \( q_j \in \Gamma \) and \( u = p_1 + p_2 + q_1 + \ldots + q_{g-2} + K \). However in this case \( u' = z + p_1 + p_2 - p_3 - p_4 = p_2 + q_1 + \ldots + q_{g-2} + K \) and, by the Riemann theorem, \( \theta(u') = 0 \). The formula (71) is established and it suffices to apply Proposition 4.4.2.

For an arbitrary quadruple \( \tilde{p} \) the existence of a trisecant which passes through points (69) is obtained from a generic case by passing to a limit. Explicit formula for \( c_k(\tilde{p}) \) can be derived from (68) (see also [34]).

§6. Theta functions of Prym varieties of double coverings with two branch points

6.1. Prym varieties of ramified coverings.

Let \( \Gamma_0 \) be a Riemann surface of genus \( g \) and \( q_0 \) and \( q_\infty \) be a pair of distinct points on \( \Gamma_0 \). Denote by \( \gamma_0 \) and \( \gamma_\infty \) the contours bounding small disks with centres in \( q_0 \) and \( q_\infty \).

There is a one-to-one correspondence between ramified double coverings \( \Gamma \to \Gamma_0 \) with branch points at \( q_0 \) and \( q_\infty \) and homomorphisms

\[
\rho : H_1(\Gamma \setminus \{q_0, q_\infty\}; \mathbb{Z}) \to \mathbb{Z}_2
\]

such that \( \rho([\gamma_0]) = \rho([\gamma_\infty]) = 1 \). This correspondence is as follows. Let \( \gamma \) be a contour on \( \Gamma_0 \). Put \( \rho([\gamma]) = 1 \) if the pull-back of \( \gamma \) on the covering is non-closed. Otherwise, put \( \rho([\gamma]) = 0 \). It follows from this correspondence that a covering of \( \Gamma_0 \) which is ramified at a pair of fixed points is determined from these data in up to \( 2^g \) possible ways.

Denote by \( \mathcal{M}_{g,2} \) the moduli space of Riemann surfaces of genus \( g \) with pair of marked points and denote by \( \mathcal{D} \mathcal{R}_g \) the moduli space of ramified double coverings of surfaces of genus \( g \) with two branch points. The mapping \( \Gamma \to (\Gamma_0, q_0, q_\infty) \) is the \( 2^g \)-sheeted covering \( \mathcal{D} \mathcal{R}_g \to \mathcal{M}_{g,2} \). It follows from (68) that \( \dim \mathcal{D} \mathcal{R}_g ( = \dim \mathcal{M}_{g,2} = \dim \mathcal{M}_g + 2) = 3g - 1 \) for \( g \geq 2 \).

Let \( \pi : \Gamma \to \Gamma_0 \) be a ramified covering with branch points at \( q_0 \) and \( q_\infty \). Denote by \( \sigma : \Gamma \to \Gamma \) the involution permuting the branches of the covering. It is clear that \( \Gamma_0 = \Gamma/\sigma \) and the involution \( \Gamma \to \Gamma \) uniquely determines the covering.

There exists a canonical basis for 1-cycles \( a_1, \ldots, a_{2g}, b_1, \ldots, b_{2g} \) on the surface \( \Gamma \) of genus \( 2g \) such that

\[
\sigma(a_k) + a_{g+k} = \sigma(b_k) + b_{g+k} = 0, \quad 1 \leq k \leq g,
\]

and the cycles \( a_1, \ldots, a_g, b_1, \ldots, b_g \) are projected into the canonical basis for cycles \( \{a_k, b_k\} \) on \( \Gamma_0 \) where \( a_k - \pi_k(a_k) = b_k - \pi_k(b_k) = 0 \).

Let \( \omega_1, \ldots, \omega_{2g} \) be a basis for holomorphic differentials on \( \Gamma \) and let \( \tilde{\omega}_1, \ldots, \tilde{\omega}_g \) be a basis for holomorphic differentials on \( \Gamma_0 \) which are related to the canonical bases for cycles by (71). It follows from (71) that

\[
\sigma^*(\omega_k) = -\omega_{g+k} \quad \text{and} \quad \omega_k - \omega_{g+k} = \pi^*(\tilde{\omega}_k) \quad \text{for } 1 \leq k \leq g.
\]

A differential \( \omega \) is called a Prym differential if \( \sigma^*(\omega) = -\omega \). The set of the forms \( u_1 = \omega_1 + \omega_{g+1}, \ldots, u_g = \omega_g + \omega_{2g} \) is a basis for holomorphic Prym differentials. Consider the matrix of their \( b \)-periods

\[
\Pi(\Gamma, \sigma)_{jk} (= \Pi_{jk}) = \int_{b_k} u_j, \quad 1 \leq j, k \leq g.
\]
This matrix is symmetric and, by (91–92), relates to the period matrices of $\Gamma$ and $\Gamma_0$ as follows

$$B(\Gamma) = \frac{1}{2} \left( \begin{array}{cc} \Pi + B_0 & \Pi - B_0 \\ \Pi - B_0 & \Pi + B_0 \end{array} \right)$$

(94)

where

$$B(\Gamma)_{jk} = \int_{b_k} \omega_j, \quad B(\Gamma_0)_{jk} = \int_{\pi^* (b_k)} \tilde{\omega}_j.$$ 

The principally-polarised Abelian variety

$$Pr(\Gamma, \sigma) = C^g / \{ Z^g + \Pi Z^g \}$$

is called the Prym variety of the covering $\Gamma \to \Gamma / \sigma$.

The Prym variety is defined for any involution $\sigma$ on $\Gamma$ in an invariant manner as follows. Let $A : \Gamma \to J(\Gamma)$ be the Abel mapping with initial point $\tilde{q}$ and let $\sigma : \Gamma \to \Gamma$ be an involution. Then $J(\Gamma) = A(S^\sigma \Gamma)$ where $n$ is the genus of $\Gamma$ and the involution $\sigma$ induces the following involution $\sigma : J(\Gamma) \to J(\Gamma)$:

$$\sigma(\int_{\tilde{q}} \omega + \ldots + \int_{\tilde{q}} \omega) = \int_{\sigma(\tilde{q})} \omega + \ldots + \int_{\sigma(\tilde{q})} \omega.$$ (95)

The Prym variety is an anti-invariant subvariety

$$Pr(\Gamma, \sigma) = \{ z \in J(\Gamma) : \sigma(z) = -z \}. \quad (96)$$

In the present case it follows from (92) that

$$\sigma(z_1, \ldots, z_g, z_{g+1}, \ldots, z_{2g}) = (-z_{g+1}, \ldots, -z_{2g}, -z_1, \ldots, -z_g) \quad (97)$$

and the embedding

$$\varphi : Pr(\Gamma, \sigma) \to J(\Gamma), \quad \varphi(z_1, \ldots, z_g) = (z_1, \ldots, z_g, z_1, \ldots, z_g) \quad (98)$$

is defined. By (95), derive the following proposition.

6.1.1. Let $[\mu]$ be the principal polarization of $J(\Gamma)$. Then the polarization $\varphi^*([\mu])/2$ is principal.

Hence the Prym map

$$Pr : D\mathcal{R}_g \to A_g, \quad (\Gamma, \sigma) \overset{Pr}{\to} (Pr(\Gamma, \sigma), \varphi^*([\mu])/2) \quad (99)$$

is correctly defined.

Analogues of Proposition 6.1.1 are also valid for two classes of double coverings, for unramified and hyperelliptic coverings. In the first case the polarization, induced by an embedding, is a multiple of the principal one. In the second case a Prym variety coincides with the Jacobi variety of a hyperelliptic Riemann surface which is a ramified double covering of the 2-sphere.

6.2. Theta function of Prym variety.

The Jacobi variety $J(\Gamma_0)$ is isomorphic to the subvariety invariant under $\sigma$, $J(\Gamma_0) = \{ z \in J(\Gamma) : \sigma(z) = z \}$, and it is embedded into $J(\Gamma)$ as follows

$$\pi^* : J(\Gamma_0) \to J(\Gamma), \quad \pi^*(z_1, \ldots, z_g) = (z_1, \ldots, z_g, -z_1, \ldots, -z_g). \quad (100)$$

The projections

$$\pi_1 : J(\Gamma) \to Pr(\Gamma, \sigma), \quad \pi_2 : J(\Gamma) \to J(\Gamma_0)$$
written in coordinates as
\[ \pi_1(u, v) = u + v, \quad \pi_2(u, v) = u - v, \quad u, v \in C^g \] (101)
are also given. The composition \( Pr(\Gamma, \sigma) \times J(\Gamma_0) \xrightarrow{\varphi \pi^*} J(\Gamma) \) of the embeddings \( \varphi \) and \( \pi^* \) is an isogeny of degree \( 2^{2g} \). Notice that
\[ 2e = \varphi \cdot \pi_1(e) + \pi^* \cdot \pi_2(e). \]

By (94) and (101), we have the following proposition.

6.2.1. 1) \( \theta\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right](z, B) = (102) \)
\[ \sum_{2e \in \mathbb{Z}^g/2\mathbb{Z}^g} \theta\left[ \begin{array}{c} (a+b)/2 + e \\ c + d \end{array} \right](\pi_1(z), 2\Pi) \cdot \theta\left[ \begin{array}{c} (a-b)/2 + e \\ c - d \end{array} \right](\pi_2(z), 2B_0) \]
for all characteristics \( a, b, c, d \in \mathbb{R}^g \).

2) The formula
\[ \frac{\theta(z, B)^2}{\theta(\pi_1(z), \Pi) \cdot \theta(\pi_2(z), B_0)} \]
defines a meromorphic function on \( J(\Gamma) \).

The Abel–Prym mapping with initial point at \( q_0 \) is defined as follows
\[ A_{Pr} : \Gamma \rightarrow Pr(\Gamma, \sigma), \quad A_{Pr}(q) = \int_{q_0}^q u_j. \] (104)

Denote by \( A_0 \) and \( A \) the Abel mappings for \( \Gamma_0 \) and \( \Gamma \) with initial points \( \pi(q_0) \) and \( q_0 \). By the definitions of these mappings and (101), we have
\[ A_{Pr} = \pi_1 \cdot A, \quad A_0 \cdot \pi = \pi_2 \cdot A. \] (105)

Infer also that
\[ C(\Gamma) = \pi^{-1}(C(\Gamma_0)) + q_0 + q_\infty. \] (106)

By virtue of Proposition 5.2.1 applied to (103), Proposition 6.2.1, and (106), the following proposition is valid.

6.2.2. Assume that \( f(p) = \theta(A_{Pr}(p) - e, \Pi) \). Then, for every \( e \in Pr(\Gamma, \sigma) \), either of the two possibilities holds
1) \( f(p) = 0 \);
2) if \( D \) is a divisor of zeros of the function \( f(p) \) then
\[ \varphi(e) = A(D) - A(q_0) - A(q_\infty) + \pi^*(K_{\Gamma_0}), \] (107)
and this implies
\[ D + \sigma(D) - q_0 - q_\infty \approx C(\Gamma). \] (108)
Moreover a divisor \( D \) satisfying (107) is unique.

Proposition 6.2.2 describes the inverse \( (84) \) of the Abel mapping restricted to the Prym variety \( \varphi(Pr(\Gamma, \sigma)) \subset J(\Gamma) \).

6.3. Quadrisecants of Prym varieties.
6.3.1. Fay Quadrisecant Formula (§7). For every quadruple \( p_1, p_2, p_3, p_4 \in \Gamma \) there exist constants \( c_1(\bar{p}), c_2(\bar{p}), c_3(\bar{p}), \) and \( c_4(\bar{p}) \) such that

\[
c_1 \cdot \theta(z - p_1 - p_2) \cdot \theta(z - p_3 - p_4) + c_2 \cdot \theta(z - p_1 - p_3) \cdot \theta(z - p_2 - p_4) +
\]

\[
c_3 \cdot \theta(z - p_1 - p_4) \cdot \theta(z - p_2 - p_3) + c_4 \cdot \theta(z - p_1 - p_2 - p_3 - p_4) \cdot \theta(z) = 0. \quad (109)
\]

This means that the following points \( \Phi((p_1 + p_2 - p_3 - p_4)/2), \Phi((p_1 + p_3 - p_2 - p_4)/2), \Phi((p_1 + p_4 - p_2 - p_3)/2), \) and \( \Phi((p_1 + p_2 + p_3 + p_4)/2) \) of the Kummer variety \( K(P_T(\Gamma, \sigma), \Theta) \) lie on the same plane \( \mathbb{C}P^2 \subset \mathbb{C}P^{2g - 1} \) called a quadrisecant.

Here, \( p_i \) stands for \( A_{p_i}(p_i) \) and \( \theta(z) = \theta(z, \Pi(\Gamma, \sigma)) \). Here, as in §5.4, it suffices to prove that for a generic quadruple of points the inclusion

\[
X = \theta_{p_1+p_2} \cap \theta_{p_1+p_3} \cap \theta_{p_1+p_4} \subset \theta \cup \theta_{p_1+p_2+p_3+p_4}
\] (110)

holds and that \( \dim X = g - 3 \). Since the condition on the dimension of \( X \) is not strong, we sketch the proof of (110). We may assume that all the points \( p_1, \ldots, p_4 \) are distinct.

Put \( u = z - p_1 \). It follows from Proposition 6.2.2 and (110) that one of the two possibilities holds: 1) \( \theta(u - p) \equiv 0 \), or 2) \( p_2, p_3, p_4 \in D \) where \( D \) is the divisor of zeros of the function \( f(p) = \theta(z - p) \).

By (105), if \( \theta(u - p) \equiv 0 \) then \( z - p_1 - \sigma(p_1) = z \) and \( \theta(z) = \theta(u - \sigma(p_1)) = 0 \).

We prove that if \( \theta(u - p) \) does not vanish identically then \( \theta(z - (p_1 + \ldots + p_4)) = 0 \).

Put \( D_1 = D_0 - (p_2 + p_3 + p_4) \).

The set of divisors satisfying \( (108) \) is a reducible subvariety \( M \subset S^{2g} \Gamma \). The Abel mapping \( \hat{A} : D \to A(D - q_0 - q_{\infty}) + K_\Gamma \) is inverted on a component \( M_1 \subset M \) by \( (107) \). The set of divisors \( \pi(D) \) where \( D \in M_1 \) is a subvariety \( M' \subset S^{2g} \Gamma_0 \). The covering \( M \to M' \) is ramified over the subvariety \( M'_0 \) formed by divisors of zeros of Abelian differentials. It was proved \( (105) \) that the end-points of the pull-back to \( M_1 \) of a closed path in \( M' \setminus M'_0 \) differ on a divisor of the shape \( D - \sigma(D) \) where \( \deg D \) is even. Hence, conclude that \( D_2 = \sigma(p_2 + p_3 + p_4) + D_1 \) does not lie in \( M_1 \) and, therefore, the Abel mapping is non-invertible at \( \hat{A}(D_2) \).

By Proposition 6.2.2 and the non-invertibility of the Abel mapping at \( \hat{A}(D_2) \), \( \theta(u' - p) \equiv 0 \) where \( u' = \hat{A}(D_2) = u - p_2 - p_3 - p_4 \). Thus derive \( \theta(z - (p_1 + \ldots + p_4)) = 0 \).

The inclusion \( (110) \) is established.

We thus outlined the proof of the quadrisecant formula which is given in \( (109) \). This proof, although similar to the proof of the Fay trisecant formula given above, shows the difference of the formulae from the geometric point of view.

The formula \( (109) \) was first obtained by Fay \( (37) \) who applied the theory of the Schottky–Jung relations.

§7. Theta functions of Prym varieties of unramified coverings

7.1. Prym varieties of unramified coverings.

Let \( \Gamma_0 \) be a Riemann surface of genus \( g \). There is a one-to-one correspondence between unramified double coverings \( \Gamma \to \Gamma_0 \) and homomorphisms

\[
\rho : H_1(\Gamma; \mathbb{Z}) \to \mathbb{Z}_2.
\] (111)

This correspondence is as follows. Let \( \gamma \) be a contour on \( \Gamma_0 \). Put \( \rho([\gamma]) = 1 \) if its pull-back on the covering is non-closed. Otherwise, put \( \rho([\gamma]) = 0 \). It follows
from this correspondence that an unramified double covering is determined by the
surface \( \Gamma_0 \) up to \( 2^{2g} \) possibilities. Moreover, for one of them, \( \rho \equiv 0 \), the covering
space has two components.

Denote by \( DU_g \) the moduli space of unramified double coverings of surfaces of
genus \( g \). The mapping \( \Gamma \to \Gamma_0 \) is a \( 2^{2g} \)-sheeted covering \( DU_g \to M_{g+1} \). It follows
from (80) that \( \dim DU_g (= \dim M_g) = 3g - 3 \) for \( g \geq 2 \).

Let \( \pi : \Gamma \to \Gamma_0 \) be an unramified double covering. Denote by \( \sigma : \Gamma \to \Gamma \) the
involution permuting the branches of the covering. It is clear that \( \Gamma_0 = \Gamma/\sigma \) and the
involution \( \Gamma \to \Gamma \) uniquely determines the covering.

There exists a canonical basis \( a_0, b_0, a_1, \ldots, a_{2(g-1)}, b_1, \ldots, b_{2(g-1)} \) for \( 1 \)-cycles
on the surface \( \Gamma \) of genus \( 2g - 1 \) such that

\[
\sigma(a_0) - a_0 = \sigma(b_0) - b_0 = \sigma(a_k) - a_{g-1+k} = \sigma(b_k) - b_{g-1+k} = 0
\quad (112)
\]

for \( 1 \leq k \leq g - 1 \) and the cycles \( a_0, b_0, a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1} \) are projected into
a canonical basis \( \{ \tilde{a}_k, \tilde{b}_k \} \) for cycles on \( \Gamma_0 \) where \( \tilde{a}_0 = \pi_*(a_0), \tilde{b}_0 = \pi_*(b_0) \) and
\( \tilde{a}_k - \pi_*(a_k) = \tilde{b}_k - \pi_*(b_k) = 0 \) for \( 1 \leq k \leq g - 1 \).

In this basis the homomorphism \( \rho \) is simply written as

\[
\rho(a_j) = 0, \quad \rho(b_j) = 0 \quad \text{for } j \neq 0, \quad \rho(b_0) = 1.
\]

Let \( \omega_0, \omega_1, \ldots, \omega_{2(g-1)} \) be a basis for holomorphic differentials on \( \Gamma \) and let \( \tilde{\omega}_0, \tilde{\omega}_1, \ldots, \tilde{\omega}(g-1) \) be a basis for holomorphic differentials on \( \Gamma_0 \) which are related
to the canonical bases for cycles by (78). It follows from (112) that \( \sigma^*(\omega_0) = \omega_0 \), \( \sigma^*(\omega_k) = \omega_{g-1+k} \) and \( \omega_k + \omega_{g-1+k} = \pi^*(\tilde{\omega}_k) \) for \( 1 \leq k \leq g - 1 \).

Similarly as in the case of ramified coverings a differential \( \omega \) is called a Prym
differential if \( \sigma^*(\omega) = -\omega \). The set of the forms \( \psi_k = \omega_k + \omega_{g-1+k} \) where \( 1 \leq k \leq g - 1 \) is a basis for holomorphic Prym differentials.

The Prym variety of the covering \( \Gamma \to \Gamma_0 \) is defined by (74) as in the case of
ramified coverings and is as follows

\[
Pr(\Gamma, \sigma) = C^g - 1 / \{ Z^g - 1 + \Pi(\Gamma, \sigma)Z^g - 1 \}
\]

where \( \Pi(\Gamma, \sigma) \) is given by (73). However, the relation between the Jacobi and Prym
varieties is different:

\[
B(\Gamma) = \begin{pmatrix}
T_0 & T_1 & T_1 \\
\Pi + T_2 / 2 & (\Pi - T_2) / 2 & (\Pi + T_2) / 2 \\
\Pi - T_2 / 2 & (\Pi + T_2) / 2 & (\Pi - T_2) / 2
\end{pmatrix}
\]

where

\[
B(\Gamma_0) = \begin{pmatrix}
T_0 / 2 & T_1 & T_1 \\
T_1 & T_2 & T_2 \\
(\Pi + T_2) / 2 & (\Pi - T_2) / 2 & (\Pi + T_2) / 2
\end{pmatrix}, \quad T_0 = \int_{b_0} \omega_0, \quad T_1 = \int_{b_1} \omega_0.
\]

The involution \( \sigma : J(\Gamma) \to J(\Gamma) \) is written as

\[
\sigma(z_0, z_1, \ldots, z_{g-1}, z_g, \ldots, z_{2(g-1)}) = (z_0, z_g, \ldots, z_{2(g-1)}, z_1, \ldots, z_{g-1}),
\quad (114)
\]

and the embedding of the Prym variety into \( J(\Gamma) \) is given by

\[
\varphi : Pr(\Gamma, \sigma) \to J(\Gamma), \quad \varphi(z_1, \ldots, z_g) = (0, z_1, \ldots, z_{g-1}, -z_1, \ldots, -z_{g-1}).
\quad (115)
\]

By (113), derive the following proposition.
Let $[\mu]$ be the principal polarization of $J(\Gamma)$. Then $\varphi^*([\mu])/2$ is the principal polarization of the Prym variety $Pr(\Gamma, \sigma)$.

Thus as in the case of ramified coverings the Prym map is defined by

$$Pr : DU_g \to A_{g-1}, \quad (\Gamma, \sigma) \mapsto (Pr(\Gamma, \sigma), \varphi^*([\mu])/2).$$

### 7.2. Theta function of a Prym variety and the quadrisecant formula for unramified coverings.

The Jacobi variety $J(\Gamma_0)$ is the $\sigma$-invariant subvariety of $J(\Gamma)$ embedded as follows

$$\pi^*: J(\Gamma_0) \to J(\Gamma), \quad \pi^*(z_0, z_1, \ldots, z_{g-1}) = (2z_0, z_1, \ldots, z_{g-1}).$$

The projections

$$\pi_1: J(\Gamma) \to Pr(\Gamma, \sigma), \quad \pi_2: J(\Gamma) \to J(\Gamma_0),$$

are defined by

$$\pi_1(u, v, w) = v - w, \quad \pi_2(u, v, w) = (u, v + w), \quad u \in \mathbb{C}, v, w \in \mathbb{C}^g.$$ 

The composition $Pr(\Gamma, \sigma) \times J(\Gamma_0)$ is an isogeny of degree $2^{g-1}$ and

$$2e = \varphi \cdot \pi_1(e) + \pi^* \cdot \pi_2(e).$$

State the analogue of Proposition 6.2.1.

#### 7.2.1. Beauville–Debarre Quadrisecant Formula.

For every quadruple $p_1, p_2, p_3, p_4 \in \Gamma$ the following points $\Phi((p_1 + p_2 - p_3 - p_4)/2), \Phi((p_1 + p_3 - p_2 - p_4)/2), \Phi((p_1 + p_4 - p_2 - p_3)/2)$ and $\Phi((p_1 + p_2 + p_3 + p_4)/2)$ of the Kummer variety $K(Pr(\Gamma, \sigma), \Theta)$ lie on the same complex projective plane in $\mathbb{C}P^{2g-1}$.
Chapter 3. Jacobi varieties and soliton equations (the Riemann–Schottky problem, the Novikov conjecture, and trisecants)

§8. Baker–Akhieser functions and rings of commuting differential operators

8.1. Finite-zone operators and Baker–Akhieser functions.

A Baker–Akhieser function, on a surface $\Gamma$ of genus $g$, corresponding to a point $q \in \Gamma$, a local parameter $k^{-1}$ in a neighbourhood of $q$ ($k(q) = \infty$), a polynomial $Q(k)$, and a divisor $D = p_1 + \ldots + p_g$ is a function $\psi(p)$ such that

1) $\psi$ is meromorphic on $\Gamma \setminus q$ and $(\psi) \geq -D$ (see §4.1);
2) the function $\psi(p) \cdot \exp(-Q(k))$ is analytic in a neighbourhood of $q$.

Denote by $S = \{\Gamma, q, k^{-1}, Q(k), D\}$ the set of “spectral data” and denote by $\Lambda(S)$ the space of all Baker–Akhieser functions corresponding to these data. The following proposition is a corollary of the Riemann–Roch theorem.

8.1.1. Let the divisor of poles of $\psi$ be non-special. Then for a generic polynomial $Q$ the divisor of zeros of $\psi$ is also non-special and $\dim \Lambda(S) = 1$.

This definition is naturally generalised for $n$-point or vector functions. Baker–Akhieser functions were introduced by Krichever ([61]) and now it is the main tool of the method of finite-zone integration of soliton equations.

In the simplest case when $\Gamma$ is a hyperelliptic surface $w^2 = E^{2g+1} + \ldots$, $q$ is the point at infinity, $Q(k) = kx$, and $k = \sqrt{E}$, the Baker–Akhieser function is the Bloch eigenfunction of the following operator

$$L = \frac{d^2}{dx^2} + u(x). \quad (122)$$

To be precise, under the assumption that the operator (122) is periodic, i.e., $u(x + T) = u(x)$, we call $\psi(x, p)$ a Bloch function if

$$L \psi = E(p) \cdot \psi;$$
$$\psi(x + T, p) = \mu(p) \cdot \psi(x, p).$$

Here $p$ belongs to a double-sheeted covering of the $E$-plane which, thus, parametrises the set of the common eigenfunctions of the operator (122) and of the translation operator $f(x) \to f(x + T)$.

If a covering $(E, \mu) \to E$ has finitely many branch points, then it is shown that this number is odd and the covering space is a hyperelliptic surface

$$w^2 = E^{2g+1} + \ldots$$
on which the Bloch function $\psi(x, p)$ is defined. The Bloch function has the following asymptotic expansion $\psi(x, p) \sim \exp(\sqrt{Ex}) \cdot (1 + O(E^{-1/2}))$ as $E \to \infty$ ([31]).

Definition. The operator (122) is called finite-zone ($g$-zone) if its Bloch function is defined on a Riemann surface of finite genus (of genus $g$).

Generally, an operator is called finite-zone if it has an eigenfunction being a Baker–Akhieser function which can be multi-point and vector-valued.

8.2. Finite-zone solutions to the Korteweg–de Vries and Kadomtsev–Petviashvili equations.

The Korteweg–de Vries equation (KdV)

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}) \quad (123)$$

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is represented by an \( L, A \)-pair, the condition of commutation of two differential operators:
\[
[L, - \frac{\partial}{\partial t} + A] = L_t + [L, A] = 0 \tag{124}
\]
where
\[
L = \frac{\partial^2}{\partial x^2} + u(x, t), \quad A = \frac{\partial^3}{\partial x^3} - \frac{3}{2} u(x, t) \frac{\partial}{\partial x} - \frac{3}{4} \frac{\partial u(x, t)}{\partial x}. \tag{125}
\]

There exists an infinite family of differential operators \( A_{2n+1} \) of the shape
\[
A_{2n+1} = \frac{\partial^{2n+1}}{\partial x^{2n+1}} + \ldots,
\]
such that
1) it contains the operator \( A = A_3 \);  
2) the commutation conditions \( \{L, A\} \) on \( A = A_{2n+1} \) generate an infinite family of commuting flows on the space of potentials \( u(x) \). These flows are described by the non-linear equations of the KdV type, the KdV hierarchy of non-linear equations \( \{67\} \).

The stationary solutions have the following prominent properties.

**8.2.1. (Novikov \( \{91\} \), Dubrovin and Novikov \( \{33\} \) The potential \( u(x) \) of the operator \( \{122\} \) is finite-zone if and only if it is a stationary solution to an equation of the KdV hierarchy**
\[
[L, A_{2n+1} + c_{2n-1} \cdot A_{2n-1} + \ldots + c_1 \cdot A_1] = 0. \tag{126}
\]

The operator \( L \) has \( g \) zones if \( \{126\} \) holds for \( n = 2g + 1 \) and fails for less values of \( n \). The finite-zone operators transform into finite-zone ones under the KdV flows.

In 1974 in the articles by Novikov, Dubrovin, Matveev, Its, and Lax the class of finite-zone operators was introduced and studied \( \{90, 33, 55, 68\} \). In 1975 some of these results were obtained also in \( \{76\} \). Another approach to the periodic problem for KdV was considered by Marchenko \( \{74\} \). A detailed survey of these papers was done in \( \{31\} \). Recently Shabat and Veselov have found a very interesting characterization of finite-zone operators \( \{122\} \) in terms of the Darboux transformations \( \{110\} \). For brevity we just explain the procedure of constructing finite-zone solutions to the KdV equation (Theorems 8.2.2 and 8.2.3).

Let \( \Gamma = \{w^2 = E^{2g+1} + \ldots\} \) be a hyperelliptic surface, let \( q = (E = \infty) \) be the point at infinity, and let \( k = \sqrt{E} \). Put \( Q(k) = kx \). For a non-special divisor \( D = p_1 + \ldots + p_g \) of degree \( g \) the space \( \Lambda(S) \) is one-dimensional where \( S = \{\Gamma, q, k^{-1}, Q, D\} \).

A set \( \Sigma = \{\Gamma, q, k^{-1}, D\} \) is called the “spectral data” of the Schrödinger operator \( \{122\} \).

**8.2.2. Generally, when \( \dim \Lambda(S) = 1 \), there exists a unique operator \( L \) of the shape \( \{122\} \) such that \( L\psi = E\psi \) where \( \psi \in \Lambda(S) \) and \( \psi(x, k) = \exp(kx)(1 + O(k^{-1})) \).**

Prove this proposition. Put
\[
\psi(x, k) = \exp(kx) \cdot (1 + \frac{\xi(x)}{k} + O(\frac{1}{k^2})). \tag{127}
\]

Then for \( u = -2\xi_x \) the function \( (L - k^2)\psi \) does not belong to \( \Lambda(S) \) and \( \dim \Lambda(S) = 1 \) implies \( (L - k^2)\psi = \lambda \cdot \psi \) with \( \lambda \) a constant. However, \( \exp(-kx) \cdot (L - k^2)\psi \to 0 \) as \( E \to \infty \). This implies that \( \lambda = 0 \) and \( L\psi = E\psi \) for \( u = -2\xi_x \).
In fact, we obtain the solution to the inverse problem, i.e., the procedure of reconstructing \( L \) from its “spectral data” \( S \):

\[
S \rightarrow u(x) = -2\xi_x
\]  

(128)

Finite-zone solutions to the KdV equation are obtained as follows. Change the spectral data \( S \) by substituting \( Q_1(k) = kx + k^3t \) for \( Q(k) = kx \) and construct from the new spectral data \( S' \) the function \( \psi(x, t, k) \) such that

\[
\psi(x, k) = \exp \left(kx + k^3t \cdot \left(1 + \frac{\xi(x, t)}{k} + O\left(\frac{1}{k^2}\right)\right)\right).
\]

(129)

Put

\[
u(x, t) = -2\xi_x.
\]

(130)

The function \( f = (-\partial_t + A)\psi \) belongs to \( \Lambda(S') \) and, since \( \exp(-kx - k^3t) \cdot (-\partial_t + A)\psi \rightarrow 0 \) as \( E \rightarrow \infty \), this function vanishes identically. Hence \([L, -\partial_t + A] \psi = 0\) but the operator \([L, -\partial_t + A] \) is the operator of multiplication by the scalar function \( f(x, t) = (u_x - (6uu_x + u_{xxx})/4) \) and \( \psi \) is non-trivial. We conclude that \( f \) vanishes identically and thus \( u(x, t) = -2\xi_x \) satisfies the KdV equation (123). This proves the following proposition.

**8.2.3.** Given a set of spectral data of generic position, the finite-zone Schrödinger operator \([\ref{125}, \ref{128}]\) and the finite-zone solution to the KdV equation \([\ref{125}, \ref{130}]\) are constructed uniquely.

It was shown in \([\ref{55}]\) that this solution is expressed by the theta function formula

\[
u(x, t) = 2\partial_x^2 \log \theta(Ux + Wt + Z, \Omega)
\]

(131)

where \( U \) and \( W \) are constant vectors, \( \Omega = B(\Gamma) \), and \( Z = -A(D) - K_\Gamma \). The inference of this formula is now routine and is explained, for instance, in the surveys \([\ref{71}, \ref{72}, \ref{27}]\). The function \( \psi \) is the Bloch eigenfunction of \( L \) and together with the “spectral data” is explicitly constructed from it \([\ref{31}]\).

The Kadomtsev–Petviashvili equation (KP)

\[
\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left( u_t - \frac{1}{4}(6uu_x + u_{xxx}) \right) = 0
\]

(132)

is the commutation condition

\[
[-\frac{\partial}{\partial y} + L, -\frac{\partial}{\partial t} + A] = 0
\]

where

\[
L = \frac{\partial^2}{\partial x^2} + u(x, y, t), \quad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + w.
\]

This equation is a two-dimensional generalization of the KdV equation and degenerates into it if \( u(x, y) \) does not depend on \( y \).

**8.2.4.** (Krichever \([\ref{71}, \ref{27}]\)) Let an arbitrary Riemann surface \( \Gamma \) of genus \( g \), a point \( q \in \Gamma \), a local parameter \( k^{-1} \) in a neighbourhood of \( q \), a non-special effective divisor \( D = p_1 + \ldots + p_g \) of degree \( g \), and a polynomial \( Q(k) = kx + k^3y + k^3t \) form the set of spectral data \( S = \{\Gamma, q, k^{-1}, Q(k), D\} \). Generally, when \( \dim \Lambda(S) = 1 \), the function \( \psi \in \Lambda(S) \) is defined uniquely by the condition \( \exp(-Q(k)) \cdot \psi(x, y, t, k) \rightarrow 1 \) as \( k \rightarrow \infty \). The operators \( L \) and \( A \) of the shape \([\ref{125}]\) satisfying

\[
(-\frac{\partial}{\partial y} + L)\psi = (-\frac{\partial}{\partial t} + A)\psi = 0
\]

(133)
are reconstructed uniquely from $\psi$. The reconstruction formulae are written as

$$
\begin{align*}
  u &= -2\frac{\partial \xi_1}{\partial x}, \\
  w &= 3\xi_1 \frac{\partial \xi_1}{\partial x} + 3\frac{\partial^3 \xi_1}{\partial x^3} - 3\frac{\partial \xi_2}{\partial x}.
\end{align*}
$$

(134)

where

$$
\psi(x, k) = \exp (kx + k^2 y + k^3 t) \cdot (1 + \frac{\xi_1(x, t)}{k} + \frac{\xi_2(x, t)}{k^2} + O(\frac{1}{k^3})).
$$

(135)

Moreover the function $u(x, y, t)$ is a solution to the KP equation written as

$$
u(x, y, t) = 2\partial_x^2 \log \theta(Ux + V y + W t + Z, \Omega)
$$

(136)

where $U, V,$ and $W$ are constant vectors, $\Omega = B(\Gamma)$, and $Z = -A(D) - K\Gamma$.

The proof of this theorem is similar to the proofs of Theorems 8.2.2 and 8.2.3 given above. It is essential that for the operator (122) the direct problem of constructing its Bloch function does not possess an effective solution. The direct problem is solved effectively for the matrix Dirac operator and in the middle of the 70s this enables construction of finite-zone solutions to the non-linear Schrödinger equation (Its and Kotlyarov) and the sine–Gordon equation (Kozel and Kotlyarov). These solutions are expressed in terms of theta functions of hyperelliptic surfaces (see the survey [31]). An axiomatization of the analytic properties of Bloch functions in the definition of Baker–Akhieser functions enables construction of a wide class of solutions without solving the direct problem. The KP equation was the first one to which this scheme, the Krichever method, was applied. Moreover, its finite-zone solutions are constructed from all Riemann surfaces.

8.3. **Commutative rings of ordinary linear differential operators.**

By Theorem 8.2.1, the problem of describing commutative rings of ordinary linear differential operators is related to that for finite-zone operators and was posed by Novikov. Its effective solution for commuting operators of rank 1 was obtained by Krichever ([61, 63]) who repeat in this case some results of Burchnall and Chaundy ([14]).

Here we mean by operators ordinary linear differential operators.

A pair of commuting operators $L_1$ and $L_2$ has rank $l$ if for a generic pair of eigenvalues $(\lambda_1, \lambda_2)$ the space of common eigenfunctions corresponding to these eigenvalues is $l$-dimensional. In this case $l$ is the greatest common divisor of the orders of $L_1$ and $L_2$.

As before we mean by “spectral data” $\Sigma$ the set formed by a Riemann surface $\Gamma$ of genus $g$, a point $q \in \Gamma$, a local parameter $k^{-1}$ in a neighbourhood of $q$, and a generic effective divisor $D$ of degree $g$. Let $\psi(x, p)$ be the Baker–Akhieser function corresponding to $\Sigma$ and normalised uniquely by the following condition: $\exp(-kx) \cdot \psi(x, p) \to 1$ for $p \to q$.

**8.3.1.** Let $f$ be a meromorphic function on $\Gamma$ with a single pole $q$ and let $Q(k)$ be a polynomial such that $f(k) = Q(k) + O(1)$. Then an operator $L_f$ such that $L_f \psi(x, p) = f(p) \psi(x, p)$ is constructed uniquely from $f$.

The proof of Theorem 8.3.1 is similar to the proofs of Theorems 8.2.2 and 8.2.4. For the function $f$ with a pole of multiplicity 2 at $q$ this is exactly Theorem 8.2.2. The operator $L_f$ is reconstructed effectively by formulae analogous to (130) and (134).

Denote by $M(q) = \cup_d L(dq)$ the algebra of meromorphic functions on $\Gamma$ having a single pole at $q$. 

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8.3.2. The procedure \( f \rightarrow L_f \) sets an isomorphism of the algebra \( M(q) \) to a commutative algebra of ordinary differential operators. Generic pairs, of operators from this algebra, are of rank 1.

The converse statement is also valid.

8.3.3. Every maximal commutative algebra \( R \) of rank 1 of ordinary differential operators is isomorphic to the algebra \( \{ L_f \} \approx M(q) \) for suitable spectral data \( \Sigma \).

The construction of \( \{ L_f \} \) is straightforward. First, observe that the following proposition is valid.

8.3.4. Let two ordinary differential operators \( L_1 \) and \( L_2 \) commute. Then there exists a polynomial \( Q(x,y) \) such that \( Q(L_1, L_2) = 0 \).

Now the bundle of common eigenfunctions of \( L_1 \) and \( L_2 \) is constructed on the Riemann surface \( \Gamma = \{ Q(x,y) = 0 \} \) in terms of formal series. This bundle is one-dimensional and it is a Baker–Akhieser function \( \psi(x,p) \). For completeness, we need to consider Riemann surfaces with simplest singularities but with non-singular marked points \( q \). This situation was analysed separately in [14.III]. The surface \( \Gamma \setminus q \) is defined in invariant form as the spectrum of the ring \( R \setminus q = \text{Spec} R \).

The detailed proofs of these results are exposed in ([61, 63]). Algebraizations of this construction and, in particular, its generalization to the case of an arbitrary field of coefficients were given in [26, 84]. An effective description for pairs of commuting operators of rank \( l > 1 \) is still not available (see the surveys [13, 61, 16]). Krichever and Novikov found the generic pair of commuting operators of rank 2 corresponding to a surface of genus 1 ([26, 60], see also [12, 92]) and using their methods Mokhov described pairs of rank 3 corresponding to a surface of genus 1 ([77, 78]). The case of matrix ordinary differential operators was examined by Grinevich ([43]).

§ 9. The Novikov conjecture in the Riemann–Schottky problem and trisecants of Jacobi varieties

9.1. Effectivization of theta function formulae for finite-zone solutions.

The vectors \( U, V, \) and \( W \) and the constant \( C \) entering into the theta functional formulae (131) and (136) for solutions to the KdV and KP equations are determined by spectral data via transcendental equations. Novikov proposed an effective procedure for writing out these formulae. Namely, he proposed to find the relations for a Riemann matrix \( \Omega = B(\Gamma) \in \mathcal{A}_g \) and the vectors \( U, V, \) and \( W \) by straightforward substitution of the theta functional formulae into the KdV and KP equations. Therewith a system of algebraic equations for \( U, V, \) and \( W \) is obtained. The compatibility conditions on this system gives relations for the Riemann matrix.

For \( g = 2, 3 \) there are no relations for the Riemann matrix. This is a corollary of the following theorem.

9.1.1. Torelli Theorem. The mapping \( B : \mathcal{M} \rightarrow \mathcal{A}_g \) assigning the Jacobi variety to a Riemann surface is an embedding.

A geometric proof of this theorem is exposed, for instance, in [41], and a soliton proof is given in [27]. This theorem implies that \( \dim A_g = \dim \mathcal{M}_g \) for \( g = 2, 3 \) and thus a generic matrix \( \Omega \in \mathcal{H}_g \) is the period matrix of a Riemann surface. Indicate a single constraint on Jacobi varieties.

9.1.2. Mertens Theorem. The Jacobi variety of a Riemann surface is irreducible, i.e., it is not a direct product of Abelian varieties of positive dimension.

The irreducibility condition is expressed effectively in terms of theta constants,
the values of the theta function and its derivatives at zero.

9.1.3. ([97]) A principally-polarised Abelian variety \( C^g/(Z^g + \Omega Z^g) \) is irreducible if and only if the rank of the matrix

\[
\begin{pmatrix}
\hat{\theta}_{11}[n_1, 0] & \ldots & \hat{\theta}_{jk}[n_1, 0] & \hat{\theta}[n_1, 0] \\
\vdots & \ddots & \vdots & \vdots \\
\hat{\theta}_{11}[n_r, 0] & \ldots & \hat{\theta}_{jk}[n_r, 0] & \hat{\theta}[n_r, 0]
\end{pmatrix}
\] (137)

is maximal, i.e., equals \( g(g + 1)/2 + 1 \). Here

\[
\hat{\theta}[\alpha, \beta] = \theta[\alpha, \beta](0, 2\Omega), \quad \hat{\theta}_{jk}[\alpha, \beta] = \frac{\partial^2 \hat{\theta}[\alpha, \beta](0, 2\Omega)}{\partial z^j \partial z^k},
\] (138)

and \( \{2n_j\} = Z^g/2Z^g, r = 2^g \).

The programme of effective construction of two- and three-zone solutions to the KdV and KP equations was realised by Dubrovin. This programme is based on finding relations for \( U, V, \) and \( W \).

9.1.4. (Dubrovin ([27])) Let an Abelian variety \( C^g/(Z^g + \Omega Z^g) \) be irreducible.

1) The formula (131) defines for every \( Z \) a solution to the Korteweg–de Vries equation if and only if the relations

\[
\partial^4 U \hat{\theta}[n, 0] - \partial_U \partial_W \hat{\theta}[n, 0] + d\hat{\theta}[n, 0] = 0,
\] (139)

hold for every \( n \in \frac{1}{2}Z^g/2Z^g \) with \( d \) a constant;

2) The formula (136) defines for every \( Z \) a solution to the Kadomtsev–Petviashvili equation if and only if the relations

\[
\partial^4 U \hat{\theta}[n, 0] - \partial_U \partial_W \hat{\theta}[n, 0] + \partial^2 V \hat{\theta}[n, 0] + d\hat{\theta}[n, 0] = 0,
\] (140)

hold for every \( n \in \frac{1}{2}Z^g/2Z^g \) with \( d \) a constant.

Dubrovin’s proof is exposed in detail in the survey ([27]) and we just notice that in order to obtain (139) and (140) it suffices to substitute the theta formulae into the non-linear equations and to apply the binary addition theorem of Riemann and Theorem 3.2.1. As it was noticed later this substitution leads to the Hirota equations (196), for the theta function, reducing to (139) and (140) by the theorems mentioned above. Convenience of the formulae (139) and (140) consists in possibility of applying both to construction of explicit solutions and problems of algebraic geometry.

By using (139) and (140) Dubrovin obtained a direct procedure of construction of two- and three-zone solutions to the KP equation from an arbitrary irreducible principally-polarised Abelian variety of dimension \( g = 2 \) or \( g = 3 \). Since all Riemann surface of genus 2 are hyperelliptic, in the case of the KdV equation this procedure works for \( g = 2 \) only. For \( g = 3 \) it suffices to complete this procedure by an effective distinguishing the Jacobi varieties of hyperelliptic surfaces. This was also done in [27].

9.2. The Novikov conjecture in the Riemann–Schottky problem.

For \( g \geq 4 \) the subvariety \( J_g = B(\mathcal{M}_g) \) has a positive codimension, in \( \mathcal{A}_g \), equal to \( (g - 2)(g - 3)/2 \). Thus the procedure of construction of solutions to the KP equation by using the Dubrovin effectivization formulae (140) needs at the first step an effective description of the subvariety \( B(\mathcal{M}_g) \). Until the early 80s this was one of the most prominent problems of algebraic geometry.
9.2.1. Riemann–Schottky Problem. Find equations distinguishing the closure of the set of Jacobi varieties, \( \overline{J_g} = \overline{B(M_g)} \), in the moduli space \( A_g \) of principally-polarised Abelian varieties.

Before the Novikov conjecture a non-trivial relation was known only for \( g = 4 \). By the end of the 19th century it was found by Schottky who showed that this relation distinguishes the subvariety, of codimension 1 in \( A_4 \), possibly reducible and containing \( \overline{J_g} \) (§10.1). In 1981 Igusa proved that this subvariety is irreducible and thus the Schottky relation solves the Riemann–Schottky problem for \( g = 4 \) (§5).

By the end of the 70s Novikov made the following conjecture initiated by the Krichever construction of finite-zone solutions to the KP equation from an arbitrary Riemann surface (see §8.2).

9.2.2. Novikov Conjecture. An irreducible principally-polarised Abelian variety \( M = \mathbb{C}^g / \{ Z^g + \Omega \} \) is the Jacobi variety of a Riemann surface if and only if there exist vectors \( U \in \mathbb{C}^g \setminus \{ 0 \}, V, W \in \mathbb{C}^g \) such that the function

\[
U_Z(x, y, t) = 2 \frac{\partial^2 \log \theta(Ux + Vy + Wt + Z)}{\partial x^2}
\]

is a solution to the KP equation for every \( Z \in \mathbb{C}^g \).

According to this conjecture the relations solving the Riemann–Schottky problem are exactly the relations distinguishing the matrices \( \Omega \in H_g \) for which (140) are compatible. Rather soon a partial proof of this conjecture was obtained.

9.2.3. (Dubrovin (28)) The compatibility condition on (140) distinguishes the subvariety (probably reducible) of dimension \( 3g - 3 = \dim M_g \) in \( A_g \).

The Dubrovin theorem as well as the Schottky theorem for \( g = 4 \) solves the Riemann–Schottky problem locally but for all \( g \geq 4 \). In order to obtain a global solution it suffices to prove that this subvariety is irreducible.

The complete proof of the Novikov conjecture as well as the solution to the Riemann–Schottky problem was obtained by Shiota (§5). The important papers of Arbarello and De Concini (2) and Mulase (24) preceded his work. We explain all these results in §10.

9.3. The Fay trisecant formula and finite-zone solutions to soliton equations.

Before proceeding to discussion of the Riemann–Schottky problem, we expose the method of constructing finite-zone solutions to the KP equation by using the Fay trisecant formula. This method was introduced by Mumford (82) in the early 80s. It shows how the KP equation arises as a degeneration of a purely algebro-geometric identity. We expose this method following the paper (3) where it was applied to the Riemann–Schottky problem.

Denote by \( \tilde{\theta}(z) \) the vector \( [\theta[n_1, 0](z, 2\Omega), \ldots, \theta[n_r, 0](z, 2\Omega)] \) and denote by \( u \wedge v \) the product of vectors \( u, v \in \mathbb{C}^{2g} \) in the Grassmann algebra generated by \( \mathbb{C}^{2g} \).

Put \( \Omega = B(\Gamma) \) and take \( p_1, p_2, p_3, p_4 \in \Gamma \) where \( \Gamma \) is a Riemann surface of genus \( g \). By the Fay trisecant formula,

\[
\tilde{\theta}(p_1 + p_2 - p_3 - p_4) \wedge \tilde{\theta}(p_1 + p_3 - p_2 - p_4) \wedge \tilde{\theta}(p_1 + p_4 - p_2 - p_3) = 0 \tag{141}
\]

where \( p_k \) stands for \( A(p_k) \) and \( A \) is the Abel mapping (33).

We may assume that \( p_1 \) is the initial point of the Abel mapping, \( A(p_1) = 0 \).

Take a local parameter \( k^{-1} \) in a neighbourhood of \( p_1 \) such that \( k(p_1) = \infty \) and
expand the Abel mapping in a neighbourhood of $p_1$ into the series in the powers of $k^{-1}$:

$$A : k^{-1} \to \sum_{m=1}^{\infty} \frac{U_m}{k^m}$$

where $U_m \in \mathbb{C}^g$. Denote by $D_m$ the derivative in the direction $U_m$

$$D_m = U_m^1 \frac{\partial}{\partial z^1} + \ldots + U_m^g \frac{\partial}{\partial z^g}.$$

Introduce the formal operator $T(k)$

$$T(k) = \exp \left( \sum_{m=1}^{\infty} \frac{D_m}{k^m} \right) = \sum_{j \geq 0} \frac{S_j(D_1,\ldots)}{k^j}. \quad (142)$$

Consider the limit of (141) as $p_4 \to p_1$: $k^{-1}(p_4) = \alpha \to 0$. Expand the left-hand side of (141) into the series in the powers of $\alpha$. The first non-trivial term corresponds to $\alpha$:

$$\bar{\theta}(p_2 - p_3) \wedge D_1 \bar{\theta}(p_2 - p_5) \wedge \bar{\theta}(p_2 + p_3) = 0. \quad (143)$$

Consider now the limit of (143) as $p_3 \to p_1$: $k^{-1}(p_3) = \beta \to 0$ and expand the left-hand side of (143) into the series in the powers of $\beta$. The first non-trivial term corresponds to $\beta^2$:

$$\bar{\theta}(p_2) \wedge D_1 \bar{\theta}(p_2) \wedge (D_1^2 + D_2) \bar{\theta}(p_2 + p_3) = 0. \quad (144)$$

Mumford indicates (23) that the identity (144) first appeared in the book of Fay (36) (Corollary 2.13).

Consider the resulting limit of $p_2 \to p_1$, $k^{-1}(p_2) = \varepsilon \to 0$, and expand the left-hand side of (144) into the series in the powers of $\varepsilon$. Every term of this series defines a linear relation for values of the functions $\bar{\theta}[n,0](z,2\Omega)$ and their partial derivatives at zero. Moreover, the coefficients of this linear relation do not depend on the characteristic $n$. Symbolically, these equations are written as

$$T(\varepsilon) \circ (\bar{\theta}(z) \wedge D_1 \bar{\theta}(z) \wedge (D_1^2 + D_2) \bar{\theta}(z))|_{z=0} \equiv 0. \quad (145)$$

Notice that we implicitly used the facts that $\bar{\theta}(u) = \bar{\theta}(-u)$ and that the derivatives of odd order of $\bar{\theta}[n,0](z,2\Omega)$ vanish at $z = 0$.

As it was noticed in (33), (145) reduces to the Dubrovin effectivevization equations of the shape (139–140). Renormalising $D_j$, if need be, rewrite the system

$$\Delta_j(\bar{\theta}(z) \wedge D_1 \bar{\theta}(z) \wedge (D_1^2 + D_2) \bar{\theta}(z))|_{z=0} = 0 \quad j \leq n \quad (146)$$

where $\Delta_j = S_j(D_1,\ldots)$, as

$$(\Delta_j \cdot D_1 - \Delta_{j-1} \cdot (D_1^2 + D_2) + \sum_{m=3}^{j} d_{m+1} \Delta_{j-m}) \bar{\theta}(z)|_{z=0} = 0, \quad j \leq n \quad (147)$$

with $d_1, d_2, \ldots$ constants. Assign to every derivative $D_j$ its degree as follows:

$$\deg D_j = j. \quad (148)$$
This agrees with the definition of $D_j$ as the derivative with respect to the $j$-th term of the expansion of the Abel mapping. Then the principal term of $\theta$ is written as

\[(\Delta_j \cdot D_1 - \Delta_{j-1} \cdot (D_1^2 + D_2))\theta(z)|_{z=0} = 0. \quad (149)\]

In particular, since the principal terms are invariant under changes of the local parameter $k^{-1}$, these terms prevail.

9.3.1. Equations (143) and (144) obtained from the Fay trisecant formula by the degeneration $p_2, p_3, p_4 \to p_1$ are the effectivization equations for the theta functional solutions to non-linear equations in the KP hierarchy. The solutions to these equations are given by

\[
U_2(x, t_1, t_2, \ldots) = 2 \frac{\partial^2}{\partial x^2} \log \theta(U_1x + U_2t_1 + U_3t_2 + \ldots + Z, B(\Gamma)). \quad (150)
\]

The simplest equation of this family coincides with the KP equation and the finite-zone solutions to it are described by

\[
2 \frac{\partial^2}{\partial x^2} \log \theta(U_1x + \sqrt{3}U_2y + 3U_3t + Z, B(\Gamma)). \quad (151)
\]

If $\Gamma$ is a hyperelliptic surface, $p_1$ is a fixed point of the hyperelliptic involution $\sigma$, and a local parameter $k^{-1}$ is inverted by the involution, $\sigma(k) = -k$, then $U_2 = 0$ and the hierarchy (151) is the KdV hierarchy.

The hierarchy of sine–Gordon equations is also obtained by degenerating the trisecant formula. The finite-zone solutions to the sine-Gordon equation

\[
\frac{\partial^2 u}{\partial x \partial y} = \sin u \quad (152)
\]

are constructed from hyperelliptic Riemann surfaces. Let $p_1$ and $p_2$ be different branch points of the hyperelliptic covering $\Gamma \to \mathbb{C}P^1$. The sine–Gordon equation and its finite-zone solutions are obtained from the trisecant formula as $p_3 \to p_1$ and $p_4 \to p_2$.

In neighbourhoods of $p_1$ and $p_2$ we introduce local parameters $k_1^{-1}$ and $k_2^{-1}$ which are inverted by the hyperelliptic involution, $\sigma(k_j) = -k_j$. Denote by $U_j$ the derivative of the Abel mapping with respect to $k_j^{-1}$ at the point $p_j$ and denote by $D_j$ the derivative in the direction $U_j$.

Consider now the limit of (144) as $p_4 \to p_2$: $k_2^{-1}(p_3) = \alpha \to 0$, and expand the left-hand side of (141) into the series in the powers of $\alpha$. The term corresponding to $\alpha$ is

\[
\bar{\theta}(p_1 - p_3) \land D_2\bar{\theta}(p_1 - p_3) \land \bar{\theta}(p_1 + p_3 - 2p_2) = 0. \quad (153)
\]

Put $k_1^{-1}(p_3) = \beta \to 0$ and expand the left-hand side of (153) into the series in the powers of $\beta$. We obtain the series which starts with the linear term with the coefficient

\[
\bar{\theta}(0) \land D_1D_2\bar{\theta}(0) \land \bar{\theta}(2\delta) = 0 \quad (154)
\]

where $\delta = p_1 - p_2$. By (154), there exist constants $c_1, c_2,$ and $c_3$ such that

\[
c_1 \cdot \theta[n, 0](0, 2\Omega) \land \theta[n, 0](2z, 2\Omega) + c_2 \cdot \theta[n, 0](2\delta, 2\Omega) \land \theta[n, 0](2z, 2\Omega) + c_3 \cdot D_1D_2\theta[n, 0](0, 2\Omega) \land \theta[n, 0](2z, 2\Omega) = 0 \quad (155)
\]
where $\Omega = B(\Gamma)$. Inspecting the Fay identity more thoroughly, we may obtain that all the constants $c_j$ do not vanish. Apply the Riemann addition theorem (155) to (156) and derive

$$c_1 + c_2 \frac{\theta(z - \delta) \cdot \theta(z + \delta)}{\theta(z)^2} + \frac{c_3}{2} D_1 D_2 \log \theta(z) = 0$$

where $\theta(z) = \theta(z, B(\Gamma))$.

Since $p_1$ and $p_2$ are branch points of the hyperelliptic covering, it is easy to show that $\delta$ is a half-period:

$$\delta = \left( m_1 + B(\Gamma) m_2 \right) / 2, m_1, m_2 \in \mathbb{Z}^g$$

where $g$ is the genus of $\Gamma$. Take (156) for $z = u$ and $z = u + \delta$ and subtract one from another. After renormalization of constants we obtain

$$D_1 D_2 \log \frac{\theta(u + \delta)^2}{\theta(u)^2} = C' \cdot \left( \frac{\theta(u + 2\delta) \theta(u)}{\theta(u + \delta)^2} - \frac{\theta(u + \delta) \theta(u - \delta)}{\theta(u)^2} \right).$$

Considering that $\delta$ is a half-period we arrive at the following conclusion.

9.3.2. (156) The function

$$\mathcal{U}_Z = 2 \sqrt{-1} \log \frac{\theta(U_1 x + U_2 y + \delta + Z, B(\Gamma))}{\theta(U_1 x + U_2 y + Z, B(\Gamma))} = 2\pi (m, U_1 x + U_2 y + Z + \delta)$$

satisfies the sine–Gordon equation

$$\frac{\partial^2 \mathcal{U}}{\partial x \partial y} = C \cdot \sin \mathcal{U},$$

where $C$ is a non-vanishing constant, for any $Z \in \mathbb{Z}^g$.

The solution (158) was found in (158). The identity (156) which itself “contains” both the sine–Gordon equation and its finite-zone solutions was obtained by Fay in (156) (Proposition 2.10). The equations effectivising the theta functional formulae for solutions to the sine–Gordon equation were obtained in (156, 22).

Notice that, since we first, we considered not all possible degenerations and, second, we just considered only first non-trivial terms of vanishing identically series as while obtaining (143) and (144), we gave evidently not all equations which are contained in the trisecant formula.

§10. The Riemann–Schottky problem

10.1. The Schottky–Jung relations.

Let $\Gamma_0$ be a Riemann surface of genus $g$. Consider an unramified double covering $\Gamma \rightarrow \Gamma_0 = \Gamma / \sigma$. As in §7, assign to this covering a canonical basis for cycles such that the homomorphism (111) is written as

$$\rho(a_0) = 0, \rho(b_0) = 0, \rho(b_j) = 1$$

For brevity denote $B(\Gamma)$ by $\tau$ and denote $Pr(\Gamma, \sigma)$ by $\pi$.

We just consider the case when all characteristics have order 2, i.e., if $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^{2n})$ is a characteristic then $2\varepsilon^j \in \{0, 1\}$. In this case a characteristic $\varepsilon_j \in \mathbb{R}^n$, of a theta function of $n$ variables is called even if $4 \sum_j \varepsilon^j_1 \varepsilon^j_2$ is even and is called odd otherwise.
10.1.1. Schottky–Jung Proportionality Relations. For even characteristics \((\varepsilon_1, \varepsilon_2)\) the ratio

\[
\frac{\theta\left[\begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \end{array}\right](0, \pi)^2}{\theta\left[\begin{array}{c} 0 \\ \varepsilon_2 \end{array}\right](0, \tau) \cdot \theta\left[\begin{array}{c} 1 \\ \varepsilon_1 \end{array}\right](0, \tau)} = k^2(\pi, \tau)
\] (159)

is a constant independent of the characteristic.

This theorem was proved by Schottky for \(g = 4\) in 1888 but in general the relations were introduced in 1909 in the joint paper of Schottky and Jung ([34], see also the survey [34]) where they were conjectured. The proof of these relations was obtained by Farkas and Rauch ([35]).

The scheme of applying them to the Riemann–Schottky problem is as follows:

1) for a Riemann surface \(\Gamma_0\) construct an unramified double covering as it was shown above;

2) take a homogeneous identity met by all theta functions, for instance, the ternary addition theorem of Riemann, (54), and apply it to the Prym theta functions \(\theta(z, \pi)\);

3) using (159), substitute into this identity for the values of \(\theta[\alpha, \beta](z, \pi)\) at zero the values of theta functions of \(\Gamma\) with characteristics proportional to the first ones.

The resulting identity is not valid for all theta functions since it is geometric rather than analytic due to its origin.

The first non-trivial identity was found by Schottky for \(g = 4\). Its modern inference is given in [18]. Igusa proved that this identity distinguishes \(\bar{J}_4\) exactly ([54]). Subsequently, it was proved that the Schottky–Jung relations are sufficient for obtaining a local solution to the Riemann–Schottky problem.

10.1.2. (van Geemen ([39])) For any \(g \geq 4\) the Schottky–Jung relations define in \(A_g\) a subvariety, of dimension \(3g - 3\), containing \(\bar{J}_g\) as one of irreducible components.

As Donagi showed, these subvarieties are generally reducible and contain another but \(\bar{J}_g\) components even for \(g = 5\) ([23]).

10.2. Geometry of the theta divisor of a Jacobi variety and the Riemann–Schottky problem.

All approaches to the Riemann–Schottky problem preceding the Novikov conjecture were based on the description for the theta divisor of a Jacobi variety given by Riemann. In two of these approaches, this reveals rather brightly.

The first is based on investigation of \(\text{Sing}\Theta\), the set of singular points of the theta divisor, which was also described by Riemann for Jacobi varieties. By his theorem on singularities (see [41]), this set is rather big. Its examination leads to the following result.

10.2.1. (Andreotti and Mayer ([1])) The condition \(\dim \text{Sing}\Theta \geq g - 4\) defines in \(A_g\) a subvariety, of dimension \(3g - 3\), possibly reducible and containing \(\bar{J}_g\) as one of irreducible components.

In much the same ways as in the case of the Schottky–Jung relations, these subvarieties are reducible in general: just for \(g = 4\) such subvariety contains two components (Beauville, ([3])). Notice that the condition \(\dim \text{Sing}\Theta = g - 3\) defines in \(J_g\) exactly the Jacobi varieties of hyperelliptic surfaces.

Another approach, going back to the papers of Lie and Wirtinger, uses the representation of the theta divisor in the shape of translated sum of \(g - 1\) examples
of the embedded surface $\Gamma$:

$$\Theta = C + \ldots + C + \kappa$$

(160)

where $C = A(\Gamma)$ is a surface embedded into $J(\Gamma)$ and $\kappa = K_\Gamma$. A subvariety, of an Abelian variety, of the shape (160), is called a translation hypersurface. The investigation of such subvarieties was started by Lie. It follows from the theorem of Lie and Wirtinger that

10.2.2. A principally-polarised Abelian variety is the Jacobi variety of a non-hyperelliptic surface if and only if in a neighbourhood of one of its non-singular points it has two distinct representations of the shape (160).

The problem how to describe the hypothesis of Theorem 10.2.2 by equations was discussed by Poincare and Tchebotarev but for the moment the complete solution to it is not obtained (see the survey [72]).

10.3. Trisecants of Jacobi varieties.

One more approach, introduced by Gunning ([45, 46]), is based on the Fay trisecant formula.

Before exposing the Gunning theorem, we mention the Matsusaka–Hoyt criterion ([32, 72]) which is, roughly speaking, as follows. Let $M$ be an Abelian variety of complex dimension $g$, let $X \subset M$ be an effective divisor, and let $C$ be a 2-cycle in $M$. Denote by $D$ the Poincare duality operator $D : H_{2g-k}(M; \mathbb{Z}) \to H^k(M; \mathbb{Z})$. Put $\omega = D(X)$. Then the following statements are equivalent:

a) $(\omega^g, [M]) = g!$ and $\omega^{g-1} = (g-1)!D(C)$;

b) $M$ is a direct product of the Jacobi varieties of Riemann surfaces $\Gamma_1, \ldots, \Gamma_n$, and

$$X = \cup_j (J(\Gamma_1) \times \ldots \times J(\Gamma_j-1) \times \Theta_j \times J(\Gamma_{j+1}) \times \ldots \times J(\Gamma_n)),$$

$$C = \cup_j (0 \times \ldots \times 0 \times \Gamma_j \times 0 \times \ldots \times 0)$$

where $\Theta_j$ is the theta divisor of $J(\Gamma_j)$.

The Matsusaka–Hoyt criterion also describes the closure $\tilde{J}_g$ as the set of direct products of Jacobi varieties.

Gunning’s reasoning is based on the following rigidity of trisecants of Jacobi varieties. Let $M = \mathbb{C}^g / \{Z^g + \Omega Z^g\}$ be a principally-polarised Abelian variety. For every triple of points $\alpha, \beta, \gamma \in \mathbb{C}^g$ denote by $V_{2\alpha, 2\beta, 2\gamma}$ a set of vectors $t \in \mathbb{C}^g$ such that the vectors $\Phi(t/2 + \alpha), \Phi(t/2 + \beta)$, and $\Phi(t/2 + \gamma)$ are coplanar, the Kummer map is defined by (2) and (3).

10.3.1. ([29, 44]) Assuming $M$ to be the Jacobi variety of $\Gamma$, we have

$$V_{2\alpha, 2\beta, 2\gamma} = \{t \in \mathbb{C}^g : t + \alpha + \beta + \gamma \in \tilde{\Gamma}\}$$

(161)

where $\tilde{\Gamma}$ is a pre-image of $\Gamma = A(\Gamma) \subset M$ under the projection $\mathbb{C}^g \to M$.

The trisecant identity ([88, 89]) corresponds exactly to the case of $\alpha = p_2, \beta = p_3, \gamma = p_4$, and $t = (p_1 - p_2 - p_3 - p_4)$. This implies the following nice procedure of reconstructing the Riemann surface $\Gamma \subset J(\Gamma)$ from a triple of generic points of $\Gamma \subset J(\Gamma)$.

Gunning proved that if for an irreducible principally-polarised Abelian variety $M$ and points $\alpha, \beta, \gamma \in M$ the set $V_{2\alpha, 2\beta, 2\gamma}$ is not empty, then under some additional conditions we have that $V_{2\alpha, 2\beta, 2\gamma}$ contains a component $\Gamma$ such that $M = J(\Gamma)$. The following theorem was proved by using the Matsusaka–Hoyt criterion.

10.3.2. Theorem of Gunning. ([43]) An irreducible principally-polarised Abelian variety $M$ of dimension $g$ is the Jacobi variety of a Riemann surface if and only if there exist points $\alpha, \beta, \gamma \in \mathbb{C}^g$ such that
1) they represent distinct points of $M = \mathbb{C}^g/\{\mathbb{Z}^g + \Omega\mathbb{Z}^g\}$;
2) there are no complex multiplications $F : M \to M$ satisfying $F(\alpha - \gamma) = F(\beta - \gamma) = 0$;
3) $\dim_{(-\alpha - \beta)} V_{2\alpha,2\beta,2\gamma} > 0$.

A complex multiplication is an endomorphism of $M$ induced by a linear mapping $t \to ct + d$ where $t, d \in \mathbb{C}^g$ and $c \in \mathbb{C}$. The validity of the hypothesis 3 implies existence of a surface $\Gamma$ as a component of $V_{2\alpha,2\beta,2\gamma}$ passing through $(-\alpha - \beta)$.

The Gunning criterion was subsequently strengthened by Welters ([112, 113, 114]) and Debarre ([22]). Welters also derived its infinitesimal analogue. To avoid plunging into the scheme theory we formulate it roughly: if $M$ is an irreducible principally-polarised Abelian variety and for some point $\alpha$ “a limit subvariety” $\lim_{\delta, \gamma \to \alpha} V_{2\alpha,2\beta,2\gamma}$ is of positive dimension somewhere, then $M$ is the Jacobi variety of a Riemann surface. This criterion was translated into the language of equations by Arbarello and De Concini ([8]):

10.3.3. An irreducible principally-polarised Abelian variety $M$ of dimension $g$ is the Jacobi variety of a Riemann surface if and only if there exist vectors $U_1 \neq 0, U_2, \ldots \in \mathbb{C}^g$ such that these vectors and the theta function of $g$ satisfying the first $N(g)(< \infty)$ equations of the KP hierarchy for any $Z \in \mathbb{C}^g$. Moreover just it suffices to take the equations

$$ T(z) \circ (\bar{\theta}(z) \wedge D_t \bar{\theta}(z) \wedge (D_t^2 + D_2) \bar{\theta}(z))|_{z=0} \equiv 0 \mod \varepsilon^{N+1} \quad (162) $$

for $N = [(3/2)g \cdot g!]$.

This and Theorem 9.3.1 implies

10.3.4. (Arbarello and De Concini ([8], and Mulase ([79])) An irreducible principally-polarised Abelian variety $M$ of dimension $g$ is the Jacobi variety of a Riemann surface if and only if the functions (150) satisfy the first $N(g)(< \infty)$ equations of the KP hierarchy for any $Z \in \mathbb{C}^g$.

We avoid exposing the formula for $N(g)$ because the Arbarello-De Concini upper bound has already been given in Theorem 10.3.3 and Mulase ([79]), the author of the “soliton” proof of Theorem 10.3.4, pointed out merely that such bound exists. If the upper bound for $N(g)$ is decreased to $N(g) = 2$, then the Novikov conjecture is proved. This was done by Shiota (see §10.5).

10.4. On the soliton proof of Theorem 10.3.4.

The higher KP equations are the equations met by the function $u(x,t_1,\ldots) = -2\varepsilon x \left(25\right)$ with the spectral data for the KP equation with $Q(k) = kx + k^2t_1 + k^3t_2 + \ldots$ (here we treat $t$ as a temporal variable $t_1$).

Give a strong definition of this hierarchy. Let $R$ be the ring of smooth functions of $x,t_1,\ldots$. Introduce the algebra $E$ of formal pseudo-differential operators

$$ A = \sum_{n=0}^{N<\infty} p_n \left( \frac{d}{dx} \right)^n, \quad p_n \in R \quad (163) $$

with multiplication defined by the Leibniz rule

$$ \left( \frac{d}{dx} \right)^k \cdot f = \sum_{j \geq 0} \binom{k}{j} \frac{d^j f}{dx^k} \cdot \left( \frac{d}{dx} \right)^{k-j}, \quad k \in \mathbb{Z}, \ f \in R. \quad (164) $$

The order of the operator (163), $ordA$, equals $d$ if $p_d \neq 0$ and $p_j = 0$ for $j \geq d$. Denote by $E^-$ the subalgebra formed by operators of order $< 0$. For every operator $A$ a decomposition $A = A^+ + A^-$ into the sum of the differential operator $A^+$ and the operator $A^- \in E^-$ is defined.

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Let \( H \) be the affine subspace, of \( E \), formed by the operators \( L = d/dx + \psi \) where \( \psi \in E^- \). The *Kadomtsev–Petviashvili hierarchy* is the following system of evolution equations for \( H \) with respect to \( t_1, \ldots \):

\[
\frac{\partial}{\partial t_n} L = [B_n, L], \quad B_n = (L^n)^+.
\] (165)

The following proposition is derived from the definition of these flows by simple reasonings.

**10.4.1.** The KP hierarchy is equivalent to the following system of the Zakharov–Shabat equations:

\[
\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m].
\] (166)

The vector fields on \( H \) defined by the higher KP equations commute.

For the finite-zone solutions to the KP hierarchy the action of the KP flows (168) on potentials \( u(x, t_1, \ldots) \) of the operators \( L \) is linearised by (150). Here we treat the spatial coordinate \( y \) in (150) as the first among the temporal, \( t_1 \).

Put

\[
L = \frac{d}{dx} + U \left( \frac{d}{dx} \right)^{-1} + \sum_{k=2} u_k \left( \frac{d}{dx} \right)^{-k}.
\] (167)

If \( L \) satisfies (165), then all the functions \( u_k \) are expressed as polynomials in the function \( U \) and its derivatives. Thus it is natural to call \( U \) a solution to the KP hierarchy. Precisely this is meant in the statement of the Novikov conjecture, which also implies

\[
\frac{\partial L}{\partial \tau} = 0 \quad \text{for} \quad \frac{\partial U}{\partial \tau} = 0
\] (168)

where \( \tau = c_1 t_1 + \ldots + c_n t_n, c_j \in \mathbb{C} \).

Now it becomes clear how to derive Theorem 10.3.4 from the Burchnall–Chaundy–Krichever theory (§8.3). Let \( \theta(z, \Omega) \) be a theta function of \( g \) variables and the function

\[
U_Z(x, t_1, t_2, \ldots) = 2 \frac{\partial^2}{\partial x^2} \log \theta(Ux + V_1 t_1 + V_2 t_2 + \ldots + Z, \Omega)
\] (169)

be a solution to the first \( N(g) = 2g + 1 \) equations of the KP hierarchy for any \( Z \in \mathbb{C}^g \). Then there exist the following linear combinations

\[
c_1 V_1 + \ldots + c_{2m} V_{2m} + V_{2m+1} = d_1 V_1 + \ldots + d_{2n-1} V_{2n-1} + V_{2n} = 0
\] (170)

such that \( 2m + 1 \) is relatively prime with \( 2n \). Take the linear operators

\[
B' = c_1 B_1 + \ldots + c_{2m} B_{2m} + B_{2m+1}, \quad B'' = d_1 B_1 + \ldots + d_{2n-1} B_{2n-1} + B_{2n}.
\] (171)

Put \( \tau' = c_1 t_1 + \ldots + c_{2m} t_{2m} + t_{2m+1} \) and \( \tau'' = d_1 t_1 + \ldots + d_{2n-1} t_{2n-1} + t_{2n} \). By (165) and (168), we have

\[
\frac{\partial L}{\partial \tau'} - \frac{\partial L}{\partial \tau''} = [B', B''] = 0.
\] (172)

We conclude from (172) that, since the operators \( B' \) and \( B'' \) commute and are of co-prime orders, they are included into a maximal commutative algebra of differential operators \( A \) of rank 1. It follows from Theorem 8.3.3 that this algebra is obtained
by the Burchall–Chaundy–Krichever construction from a Riemann surface $\Gamma$ and additional spectral data. It is left to prove that $\Omega = B(\Gamma)$.

From the fact that the algebra $\mathcal{A}$ is maximal it is inferred that the vectors $U, V_1, V_2, \ldots$ span the tangent space to $J(\Gamma)$. This was strictly proved in [98].

Now, since the formal solution (169) is induced from the finite-dimensional torus $\mathbb{C}^g/\{\mathbb{Z} + \Omega \mathbb{Z}\}$, we conclude that $\Gamma$ is smooth. Since at the same time due to the Krichever construction the finite-zone solutions (136) and (150) to the KP hierarchy are constructed from the “spectral data” of $\mathcal{A}$, it is easy to conclude that these solutions are given exactly by (169).

We omit the rigorous proofs ([79, 98]) because we have explained the most important thing, i.e., how a Riemann surface arises in the “soliton” proof.

10.5. The Shiota theorem (the proof of the Novikov conjecture).

Shiota derived from Theorem 10.3.4 the proof of the Novikov conjecture showing that if the function

$$U_2(x, t_1, t_2) = 2 \frac{\partial^2}{\partial x^2} \log \theta(Ux + V_1 t_1 + V_2 t_2 + Z, \Omega)$$

satisfies the first two equations of the KP hierarchy (after the substitution $t_1 \to y, t_2 \to t$ this function satisfies the KP equation (132) then we may successively construct vectors $V_3, V_4, \ldots$ such that the function (169) satisfies the higher KP equations (150). The proof of Shiota is analytical. In (3) Arbarello and De Concini simplified his proof.

10.5.1. Shiota Theorem. Let $M = \mathbb{C}^g/\Lambda$ be an irreducible principally-polarised Abelian variety and $\theta$ be its theta function. Then (140) are solvable for $U, V, W \in \mathbb{C}^g$ with $U \neq 0$ if and only if $M$ is the Jacobi variety of a Riemann surface.

10.6. On new approaches to describing $\bar{J}_g$. The Welters trisecant conjecture.

Although the Riemann–Schottky problem was completely solved by Shiota, it still attracts attention of researchers. We indicate above five approaches to it: the Schottky–Jung relations, the geometry of $\text{Sing} \Theta$, translation surfaces, trisecants, the Novikov conjecture, and only the latter one led to a success. Theorem 10.3.3, of Welters, Arbarello, and De Concini, points at the deep relation between trisecants and soliton equations and is a forcible argument for the following conjecture.

10.6.1. Welters Conjecture. An irreducible principally-polarised Abelian variety is a Jacobi variety if and only if it, or, more precisely, its Kummer variety, admits a trisecant.

Although the statement of the conjecture is exceptionally strong it has already gained serious confirmations.

10.6.2. (Beauville and Debarre ([23])) If an irreducible principally-polarised Abelian variety admits a trisecant, then $\text{Sing} \Theta \geq g - 4$.

In view of Theorem 10.2.1 (of Andreotti and Mayer) this means that the hypothesis of the Welters conjecture solves the Riemann–Schottky problem locally, it distinguishes a subvariety, in $A_g$, which has $\bar{J}_g$ as one of its components.

Leaning upon Theorem 10.6.2 Debarre showed that

10.6.3. (Debarre ([23])) If an irreducible principally-polarised Abelian variety is the Prym variety of an unramified double covering and admits a trisecant, then it is the Jacobi variety of a Riemann surface.

The set of such Prym varieties is a $3g$-dimensional family and the Prym map $\text{Pr} : \mathcal{D}_g^+ \to A_g$ is of maximal rank at a generic point (see §12). Hence $\text{Pr}(\mathcal{D}_g^+) = \bar{J}_g$. 

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10.6.4. (Debarre ([2])) The Welters conjecture is valid for four- and five-
dimensional varieties.

Note two more approaches to describing $\overline{J}_g$:

1) A nice geometric property of $\overline{J}_g$ has been found recently by Buser and Sarnak: for large $g$ the subvariety $\overline{J}_g$ lies in a small neighbourhood of the boundary of $A_g$ ([13]). Methods used in its proof are far from ones discussed in this survey;

2) Recently Buchstaber and Krichever have introduced a programme of soliton description for $\overline{J}_g$ based on functional equations, distinguishing Baker–Akhieser functions, and addition theorems for theta functions. For the present, this programme is still not completed ([12, 13]).

Other geometric approaches to the Riemann–Schottky problem, related the previous and based on study of Kummer varieties, were considered in [86, 40].

Chapter 4. Prym varieties and soliton equations

§11. Soliton equations and Prym theta functions

11.1. Finite-zone two-dimensional Schrödinger operators and the Veselov–Novikov equation.

A direct generalization, of the scheme of finite-zone integration of the KdV equation, for $2 + 1$-equations is hindered by some circumstances.

The KdV equation is integrated in terms of the theta function of a surface $\Gamma$ parametrising the Bloch functions of one-dimensional Schrödinger operator. For $n$-dimensional Schrödinger operator $\Delta + u$ the global Bloch function is defined on an $n$-dimensional manifold $M^n$ as follows. Assume for simplicity that a potential $u(x^1, \ldots, x^n)$ is $\mathbb{Z}^n$-periodic. Let $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ and let $E \in \mathbb{C}$. The condition of existence of a Bloch function $\psi(x^1, \ldots, x^n)$ with multiplicators $\mu_j$ and eigenvalue $E$ is written as the condition for the kernel of the operator

$$L_{\nu, E} = \Delta + \sum_j \nu_j \partial_{x^j} + (u + \sum_j \nu_j^2 - E)$$

to be non-zero in the space $W^2_2(\mathbb{R}^n/\mathbb{Z}^n)$ where $\mu_j = \exp \eta_j$. This is equivalent to a property of the kernel of the operator

$$L_{\nu, E} \circ \Delta^{-1} : L^2_2(\mathbb{R}^n/\mathbb{Z}^n) \rightarrow L^2_2(\mathbb{R}^n/\mathbb{Z}^n).$$

(174)

to be non-zero. The operator (174) is of the shape $1 + P(\nu, E)$ where $P$ is a compact operator being a polynomial in $\nu$ and $E$. By the Keldysh theorem, non-invertibility of such operator is equivalent to vanishing of its determinant defined for such operators and this determinant is an entire function of $\nu$ and $E$. The manifold $M^n$ is the set of zeros of this determinant.

A natural analogue of the finite-zone condition is the condition for $M^n$ to be projective algebraic. However, the scheme of integration whereby a non-linear equation determines a deformation of a Bloch eigenfunction does not work. A Bloch function on a Riemann surface defines a line bundle from which an operator is re-constructed and which is deformed by the equation (see §8.3). This scheme works in the case of Riemann surfaces because their Picard groups are non-discrete (see
Theorem 5.3.3) and does not work for \( n = 2 \) because the Picard groups of “almost all” two-dimensional algebraic varieties are discrete.

We may obviate this difficulty for \( n = 2 \), for instance, by fixing an energy level \( E = \text{const} \). In this event \( M^2 \) is changed for a one-dimensional complex surface, the “spectrum” of \( L \) on the energy level \( E \). This works for the KP equation with \( L = -\partial_y + \partial^2_x + u(x,y) \) (134). The fact that the derivative with respect to \( y \) enters the operator \( L \) linearly is not accidental. Manakov showed that if two differential operators of several variables form a non-trivial \( L, A \)-pair, then derivatives with respect to all variables but of single one enter linearly into each of the operators (173).

Manakov introduced a generalization of \( L, A \)-pairs, the \( L, A, B \)-triples:

\[
\frac{\partial L}{\partial t} = [L, A] + B L. \tag{175}
\]

Such deformations preserve the “spectrum” of \( L \) on the zero energy level, \( E = 0 \).

A natural candidate for a deformed operator \( L \) is the two-dimensional Schrödinger operator

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + V(x,y). \tag{176}
\]

The development of the theory of two-dimensional Schrödinger operators finite-zone on one energy level was started in 1976 by Dubrovin, Krichever, and Novikov (29) by solving the inverse problem for these operators.

11.1.1. 1) (Dubrovin, Krichever, and Novikov (29)) Let \( \Gamma \) be a Riemann surface of genus \( g \), let \( q_1 \) and \( q_2 \) be a pair of distinct points of \( \Gamma \), let \( k_1^{-1} \) and \( k_2^{-1} \) be local parameters defined in some neighbourhoods of \( q_1 \) and \( q_2 \), and let \( D \) be an effective non-special divisor of degree \( g \). The two-point Baker–Akhieser function \( \psi(z, \bar{z}, p) \) corresponding to these spectral data and with asymptotic expansions

\[
\psi(z, \bar{z}, p) = \exp \left( k_1 z \right) \left( 1 + \frac{1}{k_1} + O \left( \frac{1}{k_1^2} \right) \right), \quad p \to q_1, \tag{177}
\]

\[
\psi(z, \bar{z}, p) = c(z, \bar{z}) \cdot \exp \left( k_2 \bar{z} \right) \left( 1 + O \left( \frac{1}{k_1} \right) \right), \quad p \to q_2
\]

is unique and presents an eigenfunction of the operator

\[
L = \partial \bar{\partial} + A(z, \bar{z}) \cdot \bar{\partial} + V(z, \bar{z}) \tag{178}
\]

with the zero eigenvalue

\[
L \psi = 0. \tag{179}
\]

The coefficients of the operator are reconstructed by the formulae

\[
A(z, \bar{z}) = -\partial \log c(z, \bar{z}), \quad V(z, \bar{z}) = -\bar{\partial} \xi_1^+. \tag{180}
\]

The potential \( V(z, \bar{z}) \) is written as

\[
V(z, \bar{z}) = 2 \partial \bar{\partial} \log \theta (\hat{U}^+ z + \hat{U}^- \bar{z} + \hat{Z}(D), B(\Gamma)). \tag{181}
\]

2) (Cherednik (14)) Moreover, if an anti-holomorphic involution \( \tau : \Gamma \to \Gamma \) acts such that \( \tau(q_1) = q_2, \tau(k_1) = -k_2 \), and \( \tau(D) + D - q_1 - q_2 \approx C(\Gamma) \), then the operator \( L \) is self-adjoint.
3) (Veselov and Novikov ([108])) If there exists a holomorphic involution \( \sigma : \Gamma \to \Gamma \) such that \( \sigma(q_j) = q_j, \sigma(k_j) = -k_j \), and
\[
\sigma(D) + D - q_1 - q_2 \approx C(\Gamma), \tag{182}
\]
then the operator \( L \) is potential, \( A \equiv 0 \), and the potential is expressed in terms of the Prym theta function of the ramified covering \( \Gamma \to \Gamma/\sigma \) as
\[
V(z, \bar{z}) = 2\partial\bar{\partial}\log \theta(U^+ z + U^- \bar{z} + Z(D), Pr(\Gamma, \sigma)). \tag{183}
\]
These conditions for the operator to be real and to be potential are compatible for \( \sigma \tau = \tau \sigma \) and \( \tau(D) = D \). The equation (182) is solvable for smooth Riemann surfaces only if the covering \( \Gamma \to \Gamma/\sigma \) has just two branch points.

Here \( \partial = \partial/\partial z \) and \( \bar{\partial} = \partial/\partial \bar{z} \) where \( z = x + \sqrt{-1} y \in \mathbb{C} \). The inference of (181) and (183) is analogous to that of (131).

For smooth periodic operators the conditions for operator to be potential found by Veselov and Novikov are necessary. Different proofs of this were given in [64] and in the paper of the author (Functional. Anal. Appl. 24:1 (1990), 76–77).

The set of the equations describing “isospectral” deformations of potential operators is the Veselov–Novikov hierarchy of equations.

11.1.2. (Veselov and Novikov ([109])) The Baker–Akhieser function corresponding to the same spectral data as in 11.1.1.3 and with the asymptotic expansions
\[
\psi(z, \bar{z}, t', t'', p) = \exp(k_1 z + \sum k_1^{2m+1} t'_m) \cdot (1 + O(1/k_1)), \quad p \to q_1, \tag{184}
\]
\[
\psi(z, \bar{z}, t', t'', p) = c(z, \bar{z}) \cdot \exp(k_2 z + \sum k_2^{2n+1} t'_n) \cdot (1 + O(1/k_2)), \quad p \to q_2
\]
is unique and satisfies the equations
\[
(\frac{\partial}{\partial t'_m} - A'_m)\psi = (\frac{\partial}{\partial t''_n} - A''_n)\psi = 0, \tag{185}
\]
where
\[
A'_m = \partial^{2m+1} + a'_{2m-1} \partial^{2m-1} + \ldots, \quad A''_n = \bar{\partial}^{2n+1} + a''_{2n-1} \bar{\partial}^{2n-1} + \ldots.
\]
The potentials \( V(z, \bar{z}, t', t'') \) reconstructed by (180) satisfy the non-linear equations represented by \( L, A, B \)-triples
\[
\frac{\partial L}{\partial t'_m} = [L, A'_m] + B'_m L, \quad \frac{\partial L}{\partial t''_n} = [L, A''_n] + B''_n L. \tag{186}
\]
The simplest of them are
\[
\frac{\partial V}{\partial t'_1} = \partial^3 V + \partial(uV), \quad \bar{\partial} u = 3\partial V, \tag{187}
\]
and
\[
\frac{\partial V}{\partial t''_1} = \bar{\partial}^3 V + \bar{\partial}(wV), \quad \partial w = 3\bar{\partial} V. \tag{188}
\]
The coordinated deformation (187) and (188) with respect to \( t = t'_1 = t''_1 \)
\[
\frac{\partial V}{\partial t} = \partial^3 V + \bar{\partial}^3 V + \partial(uV) + \bar{\partial}(wV), \tag{189}
\]
preserves the class of real operators. Solutions to these equations are given by

\[
V(z, \bar{z}, t', t'') = \nonumber 2\bar{\partial}\partial \log \theta (U^+z + U^-\bar{z} + W_{1}^+t'_1 + \ldots + W_{1}^-t''_1 + \ldots + Z(D), Pr(\Gamma, \sigma)).
\]  

The inference of (190) is analogous to that of (134).

The Veselov–Novikov equation (190) reduces to the KdV equation in the case when \( V \) does not depend on \( y \) and thus it is a generalization of the KdV equation different from the KP equation.

11.2. Infinite-dimensional Lie algebras in soliton theory and the hierarchies BKP and DKP.

Another definition of the KP hierarchy and its generalizations different from given §10.5 was introduced in the series of papers of Date, Jimbo, Kashiwara, and Miwa basing on the ideas of Sato (19). We explain it in the form close to the methods discussed above.

Finite-zone and soliton solutions to the KP equation are written as

\[ u = 2\partial^2_x \log \tau(x_1, x_2, \ldots). \]  

In the finite-zone case the function \( \tau \) is in fact a theta function which in terms of the physical variables \( x = x_1, y = x_2, t = x_3 \) is written as

\[ \tau(x_1, x_2, \ldots) = \theta(U_1x_1 + U_2x_2 + \ldots + Z, \Omega) \]  

and the solution \( \tau \) to the whole hierarchy belongs to the space of formal series \( R = \mathbf{C}[[x_1, x_2, \ldots]] \) in infinite number of variables.

Consider the general case. Let a group \( G \) act on a vector space \( V \), let \( v_0 \in V \), and let an operator \( A \) commuting with the group action be defined. Assume that \( A(v_0) = 0 \). Then

\[ A(g \cdot v_0) = 0 \quad \text{for any} \quad g \in G. \]  

Consider the group \( G = GL(\infty) \) acting diagonally on \( V = R \otimes R \). Let \( v_0 = 1 \otimes 1 \) and let the operator \( A \) be of the shape

\[ A \cdot \tau(x')\tau(x'') = \text{Res}_{z=0} \left( \exp \sum_{j \geq 1} z^j (x'_j - x''_j) \right) \times \]  

\[ \left( \exp - \sum_{j \geq 1} z^j \left( \frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} \right) \right) \cdot \tau(x')\tau(x''). \]  

The operator \( A \) is an example of vertex operators found by physicists in string theory. In terms of quantum field theory this system is expressed simpler.

Let \( F \) be the \( \mathbf{Z} \)-graded fermion Fock space. The Clifford algebra \( Cl \) generated by \( \psi_j \) and \( \psi^*_j \) and satisfying the relations \( \psi_j \psi^*_k + \psi^*_k \psi_j = \delta_{jk} \) and \( \psi_j^* \psi_k^* + \psi_k \psi_j = 0 \) acts on \( F \). The inclusion \( gl(\infty) \rightarrow Cl: E_{jk} \rightarrow \psi_j^* \psi_k \), where \( E_{jk} \) is a matrix with a single non-zero element, the unit at the intersection of the \( j \)-th row with \( k \)-th column, is defined. The operator \( A \) on \( F \otimes F \) is of the shape \( \sum_{j} \psi_j \otimes \psi^*_j \) and \( v_0 = |0\rangle \otimes |0\rangle \). It is easy to check that \( A \) and the action of the algebra \( gl(\infty) \) commute. \( F^{(0)} \) is transformed into \( R \) by bosonization and we obtain the system described above.

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11.2.1. \((19)\) The \(GL(\infty)\)-orbit of \(v_0\) is the set of all non-zero solutions to the equation \(Av = 0\).

The detailed description for the Fock space \(F\) and bosonization is given in [57]. Write out (194) in detail. Introduce the variables \(x_j = (x_j' + x_j'')/2, y_k = (x_j' - x_j'')/2\) in terms of which (194) is written as

\[
\sum_{j \geq 0} S_j(2y)S_{j+1}(-\tilde{\partial}_y)\tau(x + y)\tau(x - y) = 0
\]  
(195)

where \(\tilde{\partial}_y = \begin{pmatrix} \frac{\partial}{\partial y_1}, & \frac{1}{2} \frac{\partial}{\partial y_2}, & \frac{1}{3} \frac{\partial}{\partial y_3}, & \ldots \end{pmatrix}\), and \(S_j\) are the elementary Schur polynomials (see (142)).

The equations (195) are expressed by Hirota bi-linear equations. Let \(P(x_1, x_2, \ldots)\) be a polynomial in finitely many variables, then the Hirota bi-linear equation ([50]) is as follows

\[
P(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots)f(x_1 + y_1, \ldots)g(x_1 - y_1, \ldots)|_{y = 0} = 0.
\]  
(196)

Expand (195) into the Taylor series in \(y_1, y_2, \ldots\). Every coefficient of this expansion vanishes and this condition is of the shape of a Hirota equation. For instance, the vanishing of the coefficient at \(y_3\) implies

\[
(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0.
\]  
(197)

Renormalising \(x_1, x_2,\) and \(x_3\) in (197), changing \(\tau\) for \(\theta\) and applying the binary addition theorem of Riemann we obtain the Dubrovin effectivization equation for the KP equation (191) with \(d = 0\).

From a realization of representation of infinite-dimensional Lie algebras by using Clifford algebras, Date, Jimbo, Kashiwara, and Miwa showed how in this situation the KdV hierarchy corresponding to the algebra \(A_1^{(1)}\) appears and derived new hierarchies, natural analogues to the KP hierarchy.

In [19.IV] they introduced the BKP hierarchy corresponding to the central extension of the algebra \(so(\infty)\), the algebra \(b_\infty\). The BKP equations have the shape ([166]) for

\[
B_{2n+1} = \frac{\partial^{2n+1}}{\partial x^{2n+1}} + \sum_{k=1}^{2n-1} b_{nk} \frac{\partial^k}{\partial x^k}, \quad B_{2n} \equiv 0.
\]  
(198)

Existence of a commutation representation made it possible to find finite-zone solutions by using Baker–Akhieser functions [19.V]. These solutions are of the shape ([191] [192]). Like the solutions to the Veselov–Novikov equations, they are expressed in the Prym theta functions of double coverings with two branch points. The first equations of the BKP hierarchy in the Hirota shape are as follows

\[
(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau \cdot \tau = 0,
\]  
(199)

\[
(D_1^8 + 7D_1^5D_3 - 35D_1^3D_2^2 - 21D_3^2D_5 - 42D_3D_5 + 90D_1D_7)\tau \cdot \tau = 0.
\]

The CKP hierarchy relates to the algebra \(sp(\infty)\) [19.VI] and also admits a commutation representation. Its finite-zone solutions are constructed from double coverings of Riemann surfaces with \(2n > 2\) branch points. The Prym varieties of such coverings are not principally-polarised and for the moment explicit theta functional formulae for finite-zone solutions are not written.
In [19.IV] the $m$-component BKP hierarchies related to central extensions of the algebra $so(m \cdot \infty)$ were also introduced. Solutions to equations of this hierarchy are functions of infinitely many variables divided into $m$ classes of the shape $(x_1^{(1)}, x_2^{(1)}) \ldots, (x_1^{(m)}, x_2^{(m)}) \ldots$. The simplest example is the hierarchy corresponding to $m = 2$. In this case the first non-trivial equations are

\[ \hat{D}(D_3^2 - D_3) \tau \cdot \tau = 0, \]  
\[ \hat{D}(D_1^3 + 5D_3D_1^2 - 6D_3) \tau \cdot \tau = 0 \]

where $D_j = D_j^{(1)}$ and $\hat{D} = D_1^{(2)}$. In [19.VII] this hierarchy is also called BKP-II. In the sequel they became to call it the DKP hierarchy ([56]). No commutation representation for the DKP equations was found and thus its solutions were not discussed.

Notice that another approach to constructing hierarchies of non-linear equations related to Prym theta functions was considered in [89].

The ideas of papers [19] in the sequel were developed in different directions among which we mention two ones. Theorem 11.2.1 going back to Sato may be treated simply as follows: tau-functions of the KP hierarchy are parametrised by points of infinite-dimensional Grassmann space, $\mathbb{C}[[x_1, x_2, \ldots]]/GL(\infty)$. Investigation of cell decomposition, of this space, related to decomposition of all possible solutions into different classes (finite-zone, rational, soliton, etc.) was undertaken in [97]. The other interesting problem on canonical correspondence to every loop group a hierarchy of soliton equations was investigated in [58].

11.3. The Landau–Lifschitz equation.

In [20] the methods of [19] were applied to the Landau–Lifschitz equation (the LL equation) known in solid state physics:

\[ S_t = [S \times S_{xx}] + [S \times JS] \]  

where $S$ is a three-dimensional vector, $S_1^2 + S_2^2 + S_3^2 = 1$, and $J = diag(J_1, J_2, J_3)$.

Its integrability by methods of soliton theory was established by Borovik and Sklyanin by the end of the 70s. Hirota equations for (201) were found in [51] on base of the ansatz introduced by the physicists Bogdan and Kovalev:

\[ S_1 = (f g^* + g f^*)/(f f^* + g g^*), \]  
\[ S_2 = -i(g f^* - f g^*)/(f f^* + g g^*), \]  
\[ S_3 = (f f^* - g g^*)/(f f^* + g g^*). \]

These equations are

\[ D_1(f \cdot f^* + g \cdot g^*) = 0, \]  
\[ (D_2 - D_1^2)(f \cdot f^* - g \cdot g^*) = 0, \]  
\[ (D_2 - D_1^2 + \lambda)f \cdot g^* + \mu g \cdot f^* = 0, \]

and

\[ (D_2 - D_1^2 + \lambda)g \cdot f^* + \mu f \cdot g^* = 0 \]

where

\[ \lambda + \mu = J_3 - J_1, \lambda - \mu = J_3 - J_2. \]
As it was pointed out in [20] finite-zone solutions to the LL equation are expressed in terms of theta functions of the Prym varieties of double coverings of a special type. We describe it in §12.3.

§12. Methods of the finite-zone integration in the theory of Prym map

12.1. Analogues of the Torelli theorem for Prym maps.

We considered above two Prym maps corresponding to double coverings unramified (116) and ramified at two branch points (99).

Generally, analogues of the Torelli theorem are not valid for either of them.

For unramified coverings the situation is rather clear.

12.1.1. 1) The Prym map $\text{Pr} : D\mathcal{U}_g \to \mathcal{A}_{g-1}$ has maximal rank at a generic point. For $g \leq 5$ the rank equals $g(g-1)/2 < \text{dim } D\mathcal{U}_g$ and $\text{Pr}(D\mathcal{U}_g) = \mathcal{A}_{g-1}$, and for $g \geq 6$ the rank equals $3g - 3 = \text{dim } D\mathcal{U}_g$.

2) (Donagi and Smith (25)) For $g = 6$ the Prym map is a 27-sheeted covering at a generic point and $\text{Pr}(D\mathcal{U}_6) = \mathcal{A}_5$.

3) (Kanev (59), Friedman and Smith (38)) For $g \geq 7$ the Prym map is injective at a generic point.

The Donagi tetragonal construction shows that for any $g$ there exists principally-polarised Abelian varieties being the Prym varieties of several non-equivalent coverings.

The case of coverings ramified at a pair of points was not practically examined. By soliton methods it was proved that

12.1.2. (104) The Prym map $\text{Pr} : D\mathcal{R}_g \to \mathcal{A}_g$ has maximal rank at a generic point.

This proposition was proved as an auxiliary one while studying an analogue of the Novikov conjecture in the Riemann–Schottky problem (see §12.4).

12.2. Effectivization of theta functional formulae for solutions to the Veselov–Novikov equation.

Effectivization of theta functional formulae for finite-zone solutions to the Veselov–Novikov equation was developed in (104) by using the methods of (27). The results were announced by the author in Soviet Math. Dokl. 32 (1985), 843–846.

12.2.1. (104) Let $\theta$ be a theta function of an irreducible principally-polarised Abelian variety of dimension $g$. If for every $Z \in \mathbb{C}^g$ the formula

$$U_Z = 2\partial \bar{\partial} \log \theta(Uz + V\bar{z} + Wt + Z) + C$$

(206)

gives a solution to the “holomorphic” half-part of the Veselov–Novikov equation

$$\frac{\partial U_Z}{\partial t} = \partial^3 U_Z + \partial(vU_Z),$$

(207)

$v = 6\partial^2 \log \theta(Uz + V\bar{z} + Wt + Z) + d$,

then the relations

$$(D_2D_3 - 4D_1^2D_2 - dD_1D_2 - 3CD_1^2 - a)\theta[u,0] = 0$$

(208)

hold, with $a, d, C$ constants and $D_1 = \partial U, D_2 = \partial V, D_3 = \partial W$. For the “anti-holomorphic” half-part the relations have the same shape with $U$ and $V$ permuted.

Notice that for $a = d = C = 0$ (208) transforms into the Hirota equation for the first equation of the BKP hierarchy (200).
From every double ramified covering with two branch points a solution to the Veselov–Novikov equation is constructed. Hence, for every Prym variety there exist vectors \( U, V \in \mathbb{C}^g \setminus \{0\} \), \( W \in \mathbb{C}^g \) and constants \( a, d, C \) such that (208) hold. This led Novikov to an analogue of his conjecture for Prym varieties which we introduce here and discuss in §12.4.

**12.2.2. Analogue of the Novikov conjecture.** An irreducible principally-polarised Abelian variety is the Prym variety of a double covering with two branch points if and only if (208) are solvable with \( U, V \neq 0 \).

The relations (208) have the following property.

**12.2.3.** (104) If an Abelian variety is irreducible and vectors \( U \) and \( V \) are linearly independent, then by (208) the vector \( W \) and the constants \( a, c, d \) are reconstructed uniquely up to the transformations

\[
U \rightarrow \lambda U, \ V \rightarrow \mu V, \ W \rightarrow \lambda^3 W + \alpha U, \quad (209)
\]

\[a \rightarrow \lambda^3 \mu a, \ C \rightarrow \lambda \mu C, \ d \rightarrow \lambda^2 d + \alpha.\]

In particular, for \( g = 2 \) we obtain a direct procedure for constructing solutions to the Veselov–Novikov equation from an arbitrary irreducible principally-polarised Abelian variety and an arbitrary pair of linearly independent vectors. The constant \( C \) is reconstructed by two different ways, i.e., by the effectivization equations for “holomorphic” and “anti-holomorphic” half-parts of the Veselov–Novikov equation. The condition of coincidence of solutions implies non-trivial relations for theta constants in the case \( g = 2 \).

**12.2.4.** (104) Let \( \theta \) be a theta function of two variables and let

\[
(a_n^{11}, a_n^{12}, a_n^{22}, a_n) \text{ be the inverse matrix of } (\hat{\theta}_{111}[n,0] \hat{\theta}_{12}[n,0] \hat{\theta}_{22}[n,0] \hat{\theta}[n,0]):
\]

\[a_n^{pa} \hat{\theta}_{kl}[n,0] = \delta_k^p \delta_l^a, a_n \hat{\theta}[n,0] = 0, \quad a_n \hat{\theta}[n,0] = 1.\]

Then the relations

\[
\sum_n (a_n^{km} \hat{\theta}_{mmmm}[n,0] + 2a_n^{kk} \hat{\theta}_{kmkm}[n,0]) = 0,
\]

\[
\sum_n (a_n^{km} \hat{\theta}_{kmkm}[n,0] - a_n^{mm} \hat{\theta}_{mmmm}[n,0]) + 3a_n^{kk} \hat{\theta}_{kkkm}[n,0]) = 0,
\]

\[
\sum_n (a_n^{11} \hat{\theta}_{1112}[n,0] - a_n^{22} \hat{\theta}_{1222}[n,0]) = 0
\]

hold where \( 1 \leq k, m \leq 2, \ n \in 1/2(\mathbb{Z}^2/\mathbb{Z}^2) \).

**12.3. Quadrisecant formulae and the Veselov–Novikov, BKP, and Landau–Lifschitz equations.**

Denote by \( \hat{\theta}(z) \) the vector \( \{\theta[n_1,0](z,2\Omega), \ldots, \theta[n_r,0](z,2\Omega)\} \) and denote by \( u \wedge v \) the product of vectors \( u, v \in \mathbb{C}^{2g} \) in the Grassmann algebra.

Let \( \Gamma \rightarrow \Gamma/\sigma \) be a double covering with branch points at \( q_1 \) and \( q_2 \). By the Fay quadrisecant formula 6.3.1 for an arbitrary quadruple \( p_1, \ldots, p_4 \in \Gamma \) the identity

\[
\hat{\theta}(p_1 + p_2 + p_3 + p_4) \wedge \hat{\theta}(p_1 + p_2 - p_3 - p_4) \wedge
\]

\[
\hat{\theta}(p_1 + p_3 - p_2 - p_4) \wedge \hat{\theta}(p_1 + p_4 - p_2 - p_3) = 0
\]

holds where as in §6 we mean \( A_{p_r}(p_j) \) by \( p_j \).
Introduce local coordinates $k_j^{-1}$ in neighbourhoods of points $q_j$ and consider the expansions of the Abel–Prym mapping into series

$$k_1^{-1} \to A_{Pr}(q_1) + \sum_{m} \frac{U_m}{k_1^m}, \quad k_2^{-1} \to A_{Pr}(q_2) + \sum_{n} \frac{V_n}{k_2^n}. \quad (212)$$

If the involution $\sigma$ inverts parameters then $U_{2m} = V_{2n} = 0$. Denote by $D_j$ the derivative in the direction $U_j$ and denote by $\hat{D}_k$ the derivative in the direction $V_k$. Also introduce the operator $T(k)$

$$T(k_1^{-1}) = \exp\left(\sum_{m=1}^{\infty} \frac{D_m}{k_1^m}\right) = 1 + \sum_{n \geq 1} \frac{\Delta_n}{k_1^n}. \quad (213)$$

Take $q_1$ as the initial point of the Abel–Prym mapping, $A_{Pr}(q_1) = 0$. Since $q_2$ is a fixed point of the involution $\sigma: \Gamma \to \Gamma$ then we may assume that $A_{Pr}(q_2) = 0$.

Consider the degeneration (211) corresponding to the case $p_4 = q_1$ and $p_2, p_3, p_4 \to q_1$. We omit details and just give the final result.

12.3.1. (103) The equations

$$T(k_1^{-1}) \circ (\bar{\theta}(z) \wedge \Delta_1 \bar{\theta}(z) \wedge \Delta_1^2 \bar{\theta}(z) \wedge (\Delta_1 \Delta_2 - \Delta_3)\bar{\theta}(z))|_{z=0} \equiv 0 \quad (214)$$

are obtained from the Fay quadrisecant formula (211) by the degeneration $p_1, p_2, p_3, p_4 \to q_1$. By expanding the left-hand side of (214) into the series in the powers of $k_1^{-1}$ we obtain a hierarchy of equations in the Hirota shape. The principal members of this hierarchy, possibly after change of a local parameter, are written as

$$(\Delta_n \Delta_1 - \Delta_{n-1} \Delta_1^2 + \Delta_{n-2}(\Delta_1 \Delta_2 - \Delta_3))\theta \cdot \theta = 0. \quad (215)$$

Its members for $n = 5, 7$ coincide with (199) and the hierarchy (214) coincides with the BKP hierarchy.

12.3.2. (103) The equations

$$T(k_1^{-1}) \circ (\bar{\theta}(z) \wedge D_1 \bar{\theta}(z) \wedge \hat{D}_1 \bar{\theta}(z) \wedge D_1 \hat{D}_1 \bar{\theta}(z))|_{z=0} \equiv 0, \quad (216)$$

are obtained from the Fay quadrisecant formula (211) by the degeneration $p_1, p_2, p_3 \to q_1, p_4 \to q_2$. By expanding the left-hand side of (216) into the series in the powers of $k_1^{-1}$ we obtain a hierarchy of equations in the Hirota shape. The principal members of this hierarchy, possibly after change of local parameters, are written as

$$(\Delta_n \hat{D}_1 - \Delta_{n-1} \Delta_1 \hat{D}_1)\theta \cdot \theta = 0. \quad (217)$$

Its members for $n = 3, 5$ are of the shape (200) and the hierarchy (216) coincides with the hierarchy of “holomorphic” half-parts of the Veselov–Novikov equations or with the DKP hierarchy.

These propositions explain from the algebro-geometric point of view the existence of a pair of hierarchies related to double coverings of algebraic curves with two branch points and close to the Kadomtsev–Petviashvili hierarchy due to their properties.

The equivalence of a part of the Veselov–Novikov hierarchy and the DKP hierarchy indicated above explains why there is no commutation representation for the DKP equations by scalar differential operators.
We do not give explicit formula for finite-zone solutions to the DKP hierarchy. Notice only that they can be easily obtained from (212,214,215).

Now consider the Beauville–Debarre quadrisecant identity 7.2.2. We apply it to unramified coverings of the class related to the Landau–Lifschitz equation.

Let \( \Gamma \to E \) be a double covering with 4\(g \) branch points of a Riemann surface of genus 1 and let the branch set be invariant under the translation by a half-period: \( u \to u + \alpha \). Denote by \( \sigma \) the involution of \( \Gamma \) generated by this translation and denote by \( \omega \) the involution permuting the branches of the covering \( \Gamma \to E \). Evidently, \( \Gamma \to \Gamma/\sigma \) is an unramified double covering and the involutions \( \sigma \) and \( \omega \) commute: \( \sigma \omega = \omega \sigma \). We call the covering \( \Gamma \to \Gamma/\sigma \) an LL-covering.

As it was first indicated in [20] finite-zone solutions to the LL equations are constructed from such coverings. Explicit formulae in terms of Prym theta functions we derived in [10]. Choose a basis for Prym differentials \( u_k \) such that \( \sigma^*(u_k) = -u_k \). Fix a point \( p_0 \), a branch point of the covering \( \Gamma \to \Gamma/\omega \), and introduce the vector

\[
\mu_k = \int_{p_0}^{p_0} \sigma(p_0) u_k.
\]

It is evident that \( 2\mu \equiv 0 \) in \( \text{Pr}(\Gamma, \sigma) \). Then we have

\[
A_{\text{Pr}}(p) = \frac{1}{2} \int_{\sigma(p)}^{p} u = \frac{1}{2} \left( \int_{p_0}^{p} u + \mu + \int_{\sigma(p)}^{p_0} \sigma(p) \right) = \int_{p_0}^{p} u + \frac{\mu}{2}.
\]

Consider a quadruple \( p_1, p_2, p_3, p_4 \in \Gamma \) and denote by \( \delta \) the vector

\[
\delta_j = \int_{\omega(p)}^{p} u_j.
\]

Introduce a local parameter \( \epsilon \) in a neighbourhood of \( p = p_1 \) and expand the Abel–Prym mapping into the series in the powers of \( \epsilon \)

\[
A_{\text{Pr}}(\epsilon) = A_{\text{Pr}}(p) + U_1 \epsilon + U_2 \epsilon^2 + \ldots.
\]

Denote by \( D_j \) the derivative in the direction \( U_j \).

The Beauville–Debarre quadrisecant formula is of the shape (211) where for brevity we denote \( A_{\text{Pr}}(p) \) by \( p \) in arguments of theta functions.

Consider the degeneration of the quadrisecant formula as \( p_2, p_3 \to \sigma \omega(p), p_4 \to \omega(p) \). In this event

\[
p_2, p_3 \to p_1 + \mu, \quad p_4 \to -p_1 - \mu,
\]

\[
(p_1 + \ldots + p_1, p_1 + p_2 - p_3 - p_4, p_1 + p_2 - p_2 - p_4) \to \delta,
\]

in the Prym variety.

The simplest relations followed from this degeneration are

\[
(D_2 - D_1^2 + \alpha) \bar{\theta}(\delta) + \beta \bar{\theta}(2\mu + \delta) = 0.
\]

Transforming (218) into the Hirota equations by using the binary addition theorem of Riemann we obtain (204) and (205) for \( f = \theta(z + \mu + \delta), f^*(z), g = -i\theta(z + \delta), \) and \( g^* = -i\theta(z + \mu) \) where \( \theta \) is the Prym theta function of the covering \( \Gamma \to \Gamma/\sigma \) (204) is transformed into (205) by the translation \( z \to z + \mu \).

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For this choice of $f, f^*, g, g^*$ (202) is equivalent to

$$D_1 \left( \bar{\theta}(\mu + \delta) + \bar{\theta}(\mu - \delta) \right) = 0$$

and (203) is equivalent to

$$(D_2 - D_1^2) \left( \bar{\theta}(\mu + \delta) - \bar{\theta}(\mu - \delta) \right) = 0,$$

i.e., (202) and (203) are trivial consequences of the symmetry of the function $\bar{\theta}$ ($\bar{\theta}(z) = \bar{\theta}(-z)$). As a result we obtain the following.

12.3.3. (104) The equations of the LL hierarchy are obtained from the quadrisecant formula for LL-coverings in the limit as $p_2, p_3 \rightarrow \sigma \omega(p_1), p_4 \rightarrow \omega(p_1)$.

Notice that the class of LL-coverings is rather small comparatively with all unramified double coverings. Indeed, LL-coverings with $g$-dimensional Prym variety are defined locally by $2g$ parameters; one parameter defines the conformal class of a torus $E$ and $2g - 1$ parameters define branch points taken up to translation on the torus. All coverings with $g$-dimensional Prym variety are defined by $3g = \dim DU_{g+1}$ parameters.

Note that degenerations of quadrisecant formulae were considered in [47]. So far, possible relation of results obtained in [47] to soliton equations were not discussed.

12.4. Analogue of the Riemann–Schottky problem for Prym varieties.

In the spirit of the Novikov conjecture the Veselov–Novikov, BKP, and Landau–Lifschitz equations can be considered as characterizing Abelian varieties in which theta functions these equations are integrated. Of a special interest are the first two hierarchies with their solutions constructed from all Prym varieties of coverings with two branch points. In this connection we mentioned above an analogue of the Novikov conjecture 12.2.2.

An analogue of the Dubrovin theorem 9.2.3 was obtained in [103, 104]:

12.4.1. The solvability condition on (208), i.e., on the effectivization equations for finite-zone solutions to the Veselov–Novikov equation, distinguishes a subvariety in $A_g$ containing $Pr(DR_g)$ as one of components.

That is, the Veselov–Novikov equation locally solves the Riemann–Schottky problem for the Prym varieties of coverings with two branch points. In view of 12.1.2 the proof of 12.4.1 reduces to computing the dimension of this subvariety. In proofs of Theorems 9.2.3, 10.1.2, and 10.2.1, solving the Riemann–Schottky problem locally, this part was played by the Torelli theorem.

As it turns out using of higher equations of the Veselov–Novikov hierarchy as well as using of the BKP hierarchy for rejecting other components met considerable difficulties.

First, potential two-dimensional Schrödinger operators, finite-zone on the zero energy level, are constructed also by singular surfaces:

1) let the spectral surface $\Gamma$ of a Schrödinger operator also has double points $q_1, \ldots, q_k$;

2) all singular points $q_1, \ldots, q_n$ are fixed points of the involution $\sigma$ which does not permute the branches in their neighbourhoods (103).

The quotient surface $\Gamma/\sigma$ also has $k$ double points. The Jacobi varieties of the surfaces $\Gamma$ and $\Gamma/\sigma$ are non-compact but the Prym variety can be treated up to isogeny as the quotient space $J(\Gamma)/J(\Gamma/\sigma)$ and it is compact and principally-polarised.

Second, at the soliton proof of Theorem 10.3.4 we told that the KP-flows span the tangent space to the Jacobi variety of the surface $\Gamma$. For the BKP hierarchy

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this does not valid and theta functional solution can be concentrated on the non-full-dimensional winding of a principally-polarised Abelian variety.

This explains the following statement of the theorem of Shiota (99,100).

12.4.2. If for any \( Z \in \mathbb{C}^g \) the function

\[ U_Z = 2 \partial^2 \log \theta(Ux + V_1 t_1 + \ldots + Z, \Omega) \]

is a solution to the BKP hierarchy and \( \mathbb{C}^g \) is the linear span of the vectors \( U, V_1, \ldots \), then the principally-polarised Abelian variety \( \mathbb{C}^g/(\mathbb{Z}^g + \Omega \mathbb{Z}^g) \) is the Prym variety of a double covering \( \Gamma \to \Gamma/\sigma \) with two non-singular branch points and finitely many double points at which the involution \( \sigma \) does not permute the branches of \( \Gamma \).

An analogous result can be obtained for the Veselov–Novikov hierarchy (100).

The analogue of the Riemann–Schottky problem for coverings with two branch points was not discussed from a geometric point of view in contrast with the case of unramified coverings.

The most close in spirit to the soliton theory is the following result of Beauville and Debarre (7), an analogue of their Theorem 10.6.2.

12.4.3. The condition of existence of quadrisecant distinguishes a subvariety in \( \mathcal{A}_g \) containing \( \text{Pr}(\mathcal{D}U_{g+1}) \), a locus of Prym varieties of double unramified coverings, as one of components.

So far, the relation of this result to the theory of soliton equations is not clear. It seems that the Landau–Lifschitz equation can be used for characterizing Prym varieties of LL-coverings.

Recalling other approaches to the Riemann–Schottky problem we note the investigation of \( \text{Sing}_\Theta \) carried out in [83,101]. In particular, it enables us to understand and describe the difference between Prym and Jacobi varieties. Other geometric approaches were also considered ([94]) but we do not dwell on them. Notice only that the Donagi conjecture on the type of non-single-valuedness of the Prym map (see the survey [102]) and the analogue of the Riemann–Schottky problem for Prym varieties are still in question.

Final remarks

In conclusion we indicate some problems which are left beyond the present survey.

1. It seems that the first application of soliton theory to algebraic geometry was the following result of Dubrovin and Novikov ([33]). Let \( V_g \) be the moduli space of hyperelliptic surfaces of genus \( g \). Consider the bundle \( M_g \to V_g \) with a fibre \( J(\Gamma), \Gamma \in V_g \). Construct a \( 2g + 2 \)-sheeted covering over \( M_g \) corresponding to fixing one of the \( 2g + 2 \) branch points of the hyperelliptic covering \( \Gamma \to \mathbb{C}P^1 \). The space \( \tilde{M}_g \) is called the complete moduli space of hyperellitic Jacobi varieties. In ([33]) it was proved that \( \tilde{M}_g \) is a rational variety.

The finite-zone theory of equations related to hyperelliptic surfaces consequently led to many profound results (see their exposition in [82]).

2. Recently Nakayashiki, developing the ideas of Sato, has shown that there exists a natural definition of integrable non-linear equations expressed as commutation conditions on matrix differential operators ([87,88]). These equations define deformations of (Baker–Akhieser) vector bundles on arbitrary principally-polarised Abelian varieties and thus are not related to the geometry of Riemann surfaces. For the moment explicit formulae for equations and their solutions are neglected.
3. The trisecant identity is a far-reaching generalization of the addition theorem for elliptic functions (this is discussed in [34]). There exist another possibilities of generalizing this theorem, i.e., the addition theorems for hyperelliptic functions of large genus ([1]). Their relation to soliton equations has been observed recently and now it is under investigation ([11]).

4. The papers of Mulase [80] and Li and Mulase [70, 71] are devoted to category generalizations of the Burchnall–Chaundy–Krichever correspondence. In [80] on the language of category theory the part of the paper of Krichever and Novikov [66] where the KP hierarchy was realised as deformations of framed semi-stable vector bundles of rank $l \geq 1$ with $c_1 = lg$ over non-singular Riemann surfaces of genus $g$ is exposed. Theta functional identities and formulae are not discussed in [80, 70, 71]. It seems interesting to relate with theta functional formulae the results of [70] where the Prym varieties of $n$-sheeted coverings were ineffectively characterised in terms of the $n$-component KP hierarchies.

5. We did not address at all the relation of Riemann surfaces with finite-dimensional integrable systems having commutation representations (an up-to-date survey is given in [24]) and with difference soliton equations (a survey of these papers is contained in [30]).
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