The Classification of Time Invariants (First Integrals) of Multinominal Systems of O.D.E.s and the Surprising Link Between Algebraic and Logarithmic Time Invariants, Dictated by the Method of Arrays.

Lawrence Goldman *

Abstract

For a large class of systems of o.d.e.'s which have first integrals, the method of arrays yields the following results:

i) The first integrals $I$ can be found by solving systems of linear equations.

ii) How the first integral $I$ responds to changes in the system $S$.

iii) An easy way for finding the first integral for a special class of first integrals if they exist.

1 Introduction.

Let $S$ be an autonomous system of o.d.e.'s of the form:

$$S : \quad y_i' = y_i \sum_{j=1}^{r} c_{ij} y_1^{h_{j1}} \cdots y_n^{h_{jn}} \quad (i = 1, \ldots, n)$$ \hspace{1cm} 1.1

In the study of the problem: how a trajectory determined by $S$, responds to changes in the system $S$; it becomes apparent, that limiting our attention to trajectories that are given by first integrals of the form:

*lgoldman1@msn.com
\[ I = \sum_{k=1}^{q} e_k (y_1^{b_{k1}} \ldots y_n^{b_{kn}}), \]

the problem can be simplified. For, we can focus on the effect that changes in the exponent vectors \( H_j = (h_{j1}, \ldots, h_{jn}) \) and or the coefficient vectors \( C_j = (c_{1j}, \ldots, c_{nj})^t \) (\( j = 1, \ldots, r \)), of the system \( S \), have on the coefficients \( e_k \) and the exponent vectors \( B_k = (b_{k1}, \ldots, b_{kn}) \) (\( k = 1, \ldots, q \)) of the first integral \( I \).

This was done in my paper: Integrals of Multinomial Systems of Ordinary Differential Equations (Journal of Pure and Applied Algebra, 45 (1987) 225-240).

To facilitate the computation, a more efficient notation was employed. In this notation the system \( S \) of (1.1) becomes:

\[ S: \quad y' = y \sum_{j=1}^{r} C_j Y^H_j, \]

where \( y \) is the column vector \((y_1, \ldots, y_n)^t\),
\( C_j \) is the column vector \((c_{1j}, \ldots, c_{nj})^t\) and
\( H_j \) is the row vector \((h_{j1}, \ldots, h_{jn})\).

Similarly, (1.2) becomes:

\[ I = \sum_{k=1}^{q} e_k Y^{B_k}, \]

where \( B_k \) is the row vector \((b_{k1}, \ldots, b_{kn})\).

We will refer to (1.3) and (1.4) as the multinomial vector form of \( S \) and \( I \) (m.v.f. hereafter).

In this notation, the formula for the derivative of a monomial \( Y^B \), along the trajectory, is easily shown to be:

\[ (Y^B)' = Y^B \sum_{j=1}^{r} (B; C_j) Y^{H_j}, \]
where \((B; C_j) = \sum_{i=1}^n b_i c_{ij}\) (the inner product of the vectors \(B\) and \(C_j\)). Using (1.5), the derivative of \(I\) in (1.4) becomes:

\[
I' = \sum_{k=1}^{q} e_k \sum_{j=1}^{r} (B_k; C_j) Y^{B_k+H_j} = 0,
\]

neglecting all terms for which \((B_k; C_j) = 0\) and grouping the coefficients of equal monomials together, we get:

\[
I' = \sum_{i=1}^{p} \left( \sum_{H_a + B_k = E_i} e_k (B_k; C_\alpha) \right) Y^{E_i} = 0, \tag{1.6}
\]

from which the following was proven:

a) Each \(B_k\) \((k = 1, \ldots, q)\) satisfies a system of \(r\) linear equations.

b) The difference of any two exponent vectors of \(I\) is a linear combination of exponent vectors of \(S\). \tag{1.7}

c) \(e_1, \ldots, e_q\) is a solution of \(p\) linear homogeneous equations (the equation are the coefficients of \(Y^{E_i}\)) which must vanish, since \(y_1, \ldots, y_n\) are independent.

To make it possible to classify first integrals by the various relations between \(S\) and \(I\) implied by (1.7), the ‘method of arrays’ was developed. An integral array corresponding to a given first integral \(I\) of a system \(S\), is a pictorial representation of all the conditions required for \(I\) to be a first integral of \(S\).

In this paper we derive similar results for the cases where \(I\) is given by:

\[
I = \ln \left( Y^{B_1} \right) + \sum_{k=2}^{q} e_k Y^{B_k} \tag{1.8}
\]
\[ I = e_1 Y^{B_1} + \ln \left( 1 + \sum_{k=2}^{q} e_k Y^{B_k} \right), \]  

with some surprising results. In the case of (1.8), we find that the logarithmic integral is closely linked to an algebraic integral of the form (1.4), in the following sense. Let \( I, \) given by (1.4), be a first integral of a system \( S \) given by (1.3), where the coefficients and exponents of \( S \) are real or complex numbers. We will show the existence of a multinomial system \( S(\theta) \), depending on a set of parameters \( \theta = \theta_1, \ldots, \theta_m \) satisfying the following:

a) \( S(\theta) \) has a first integral \( I(\theta) \) such that \( e_k(\theta) \) and the components of \( B_k(\theta) \) \( (k = 1, \ldots, q) \) are rational functions of \( \theta_1, \ldots, \theta_m \).

b) There exist \( \tilde{\theta} \) such that \( S(\tilde{\theta}) = S, \ I(\tilde{\theta}) = I \)

c) If for some value \( \theta^* \), \( I(\theta^*) \) reduces to a non-zero constant, then there exists \( I(\theta^*) = \ln \left( Y^{B_1(\theta^*)} \right) + \sum_{k=2}^{q} e_k Y^{B_k(\theta^*)} \) which is a first integral of \( S(\theta^*) \). Thus \( S(\tilde{\theta}) \) which has an algebraic first integral given by (1.4) and \( S(\theta^*) \) which has a logarithmic first integral given by (1.8), both belong to the continuous system \( S(\theta) \)

In the case where the first integral of \( S \) is of the form (1.3), there exists \( S(\theta, \rho) \) where \( \theta \) is a set of continuous parameters as above, while \( \rho \) takes on positive integral values only. \( S(\theta, \rho) \) has a first integral given by

\[ I(\theta, \rho) = e_1(\theta, \rho) Y^{B_1(\theta, \rho)} + \ln \left( 1 + \sum_{k=2}^{\rho} e_k(\theta, \rho) Y^{B_k(\theta, \rho)} \right) \]

where \( e_k(\theta, \rho), B_k(\theta, \rho) \) are rational functions of \( \theta_1, \ldots, \theta_m, \rho \) and there exist \( \tilde{\theta}, \tilde{\rho} = q \) s.t. \( S(\tilde{\theta}, \tilde{\rho}) = S, I(\tilde{\theta}, \tilde{\rho}) = I \). Note that \( \rho \) is the number of monomials in \( I(\theta, \rho) \).

In addition, the method of arrays yields some curious results, such as:
i) Let $S$ be given by

$$y^{(n)} = f(y, y', \ldots, y^{(n-1)}) = \sum_{j=1}^{s} l_j Y^{M_j}$$

where $Y^{M_j} = \prod_{i=0}^{n-1} (y^{(i)})^{m_{ji}}, j = 1, \ldots, s$.

Let $S$ have a first integral

$$I = \sum_{k=1}^{q} e_k Y^{B_k}, Y^{B_k} = \prod_{i=0}^{n-1} (y^{(i)})^{b_{k,i} x_{hosti}}$$

then, the exponents of $I, B_K (k = 1, \ldots, q)$ are independent of $l_j (j = 1, \ldots, s)$ provided:

$$Y^{M_j} \neq (y^{(i)})^{-1} y^{(i+1)} y^{(n-1)}$$ \hspace{1cm} \text{(1.12)}$$

e.g. Let $S$ be given by

$$y'' = -l_1 y^{-1} (y')^2 + l_2 y + l_3 y^3$$

$Y^{M_1}$ violates (1.12), while $Y^{M_2}, Y^{M_3}$ do not. $S$ has a first integral

$$I = (l_1 + 1) y^{2l_1} (y')^2 - l_2 y^{2(l_1+1)} - \frac{l_3 (l_1 + 1) y^{2(l_1+2)}}{l_1 + 2}$$

$l_1$, the coefficient of $Y^{M_1}$ appears in every exponent of $I$, while $l_2, l_3$ appear in none.

ii) Let $S$ be given by

$$S: y' = y \sum_{j=1}^{r} C_j Y^{H_j}, \text{let } S \text{ have a first integral which is either:}$$

a) $I = \sum_{k=1}^{q} e_k Y^{B_k}$ or
b) \( I = \ln \left(Y^{B_i}\right) + \sum_{k=2}^{q} e_k Y^{B_k} \)

Let \( \sigma_{\alpha}(S) \) denote the system:

\[
y' = y \sum_{j=1}^{r} C_j Y^{\alpha H_j}, \quad \alpha \text{ any real or complex number.}
\]

Then \( \sigma_{\alpha}(S) \) has a first integral:

\[
\sigma_{\alpha}(I) = \sum_{k=1}^{q} \bar{e}_k Y^{\bar{B}_k},
\]

where \( \bar{B}_k = \alpha B_k, \quad \bar{e}_k = e_k \) (\( k = 1, \ldots, q \)).

# 2 The Derivative Formulas.

Throughout this paper we assume that \( y_1, \ldots, y_n \) are algebraically independent. Let \( y = (y_1, \ldots, y_n) \) be a solution of the system \( S \) given by (1.1) and let \( Y^{B} = y_1^{b_1} \ldots y_n^{b_n} \),

\[
(Y^{B})' = \sum_{i=1}^{n} b_i y_i^{-1} Y^{B} y_i' = Y^{B} \sum_{j=1}^{r} \sum_{i=1}^{n} (b_i c_{ij}) Y^{H_j}
\]

yielding

\[
\begin{cases}
  i) \quad (Y^{B})' = Y^{B} \sum_{j=1}^{r} (B; C_j) Y^{H_j} \\
  ii) \quad \left(\ln \left(Y^{B}\right)\right)' = Y^{-B} (Y^{B})' = \sum_{j=1}^{r} (B; C_j) Y^{H_j}
\end{cases}
\]

(2.1)
i) of (2.1) has two interesting consequences:
a) A necessary and sufficient condition for a monomial $Y^B$ to be a first integral of $S$ is that $(B; C_j) = 0 \ (j = 1, \ldots, n)$, this implies:

i) monomial first integrals of $S$ are independent of the exponent vectors $H_j$ of $S \ (j = 1, \ldots, r)$.

ii) Let the matrix $(c_{ij})$ of $S$ have rank $s \leq n$, then $S$ has exactly $n - s$ independent first integrals. For the system of homogeneous linear equations:

$$(B; C_j) = 0 \ (j = 1, \ldots, r)$$

has exactly $n - s$ independent solutions.

b) Let $r = n$ and let $(H_i, C_j) = 0$ for all $i \neq j$, then

$$(Y^{H_i})' = (H_i, C_i)Y^{2H_i} \ (i = 1, \ldots, n).$$

Setting $z_i = Y^{H_i}$ yields a complete separation of variables, in $z_1, \ldots, z_n$.

Remark: The sufficiency condition of a) applies to any system of the form:

$$y_i' = y_i \sum_{j=1}^{r} c_{ij}f_j(y_1, \ldots, y_n).$$

b) applies to any system of the form:

$$y_i' = y_i \sum_{j=1}^{r} c_{ij}f_j(Y^{H_j}).$$

3 The Method of Arrays.

Before we give a systematic treatment of the “method of arrays”, we give a few examples to show how to construct the continuous system $S(\theta)$ and its first integral $I(\theta)$, to which a given system $S$ and its first integral $I$ belong.

This can be done for any system $S$ of the form (1.1) whose first integral $I$ is such that

$$I' = \frac{A(y)}{B(y)}$$

where $A(y), B(y)$
are linear combinations of monomials.

Example 1.

\[
\begin{align*}
\left\{ \begin{array}{l}
y_1' = y_2 = y_1(y_1^{-1}y_2) \\
y_2' = -y_1 = y_2(-y_1y_2^{-1})
\end{array} \right. \\
\text{or } y' = y(C_1Y^{H_1} + C_2Y^{H_2})
\end{align*}
\]

where,

\[
H_1 = (-1, 1), \quad H_2 = (1, -1), \quad C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

(the latter is the m.v.f. of \( S \)). \( S \) has a first integral:

\[
I = y_2^2 + y_1^2 = e_1Y^{B_1} + e_2Y^{B_2},
\]

where \( B_1 = (0, 2), \ B_2 = (2, 0), \ e_1 = e_2 = 1 \)

Applying the derivative formula \( i) \) of (2.1) to \( I \), we get:

\[
\begin{align*}
I' &= e_1(Y_1^{B_1})' + e_2(Y_2^{B_2})' \\
&= e_1[(B_1; C_1)Y_1^{B_1} + (B_1; C_2)Y_2^{B_1}] + e_2[(B_2; C_1)Y_2^{B_2} + (B_2; C_2)Y_1^{B_2}] = 0.
\end{align*}
\]

Substituting for \( B_1, B_2, C_1, C_2, H_1, H_2 \) their values, given by (3.1), (3.2), in (3.3) we find:

\[
(B_1; C_1) = (B_2, C_2) = 0
\]

\( B_1 + H_2 = B_2 + H_1 = E_1 \), \quad \text{so that}

\[
I' = [e_1(B_1; C_2) + e_2(B_2; C_1)]Y^{E_1} = 0
\]

Thus, the relations between \( I \) and \( S \) are:

\[
\begin{align*}
\left\{ \begin{array}{l}
i) \quad (B_1; C_1) = (B_2; C_2) = 0 \\
ii) \quad B_1 + H_2 = B_2 + H_1 = E_1 \\
iii) \quad e_1(B_1; C_2) + e_2(B_2; C_1) = 0
\end{array} \right.
\]

\( 3.4 \)
These relations are summarized by the array:

\[
A = \begin{pmatrix} B_1 & B_2 \\ H_2 & H_1 \end{pmatrix} E_1
\]

Which obeys the following:

\[
\begin{align*}
&i) \quad H_\alpha \text{ appears in the } k^{th} \text{ column of a } p \times q \text{ array if and only if } (B_k, C_\alpha) \neq 0. \\
&ii) \quad H_\alpha \text{ appears in the } j^{th} \text{ row and the } k^{th} \text{ column of a } p \times q \\
&\quad \text{array if and only if: } B_k + H_\alpha = E_j, \quad 1 \leq j \leq p, \quad 1 \leq k \leq q. \\
\end{align*}
\]

In our case the array tells us that:

\[
l' = [e_1(B_1; C_2) + e_2(B_2; C_1)]Y^{E_1} = 0
\]

where

\[
E_1 = B_1 + H_2 = B_2 + H_1.
\]

Now, ii) of (3.4) and i) of (3.5) imply:

\[
\begin{align*}
(B_1; C_2) &= (B_1 - B_2; C_2) + (B_2; C_2) \\
&= (H_1 - H_2; C_2) + 0 = (H_1 - H_2; C_2) \neq 0 \\
(B_2; C_1) &= (B_2 - B_1; C_1) + (B_1; C_1) \\
&= (H_2 - H_1; C_1) + 0 = (H_2 - H_1; C_1) \neq 0
\end{align*}
\]

We can, now, use i) of (3.4) and (3.6) to solve for \(B_1, B_2, e_1, e_2\). For, \(B_1, B_2\) are solutions of the linear systems:

\[
\begin{align*}
&i) \quad \begin{cases} (B_1; C_1) = 0 \\
(B_1; C_2) = (H_1 - H_2; C_2) \end{cases} \\
&ii) \quad \begin{cases} (B_2; C_2) = 0 \\
(B_2; C_1) = (H_2 - H_1; C_1) \end{cases}
\end{align*}
\]

These systems have unique non-zero solutions provided:

\[
\begin{align*}
&i) \quad d = \det \begin{pmatrix} c_{11} & c_{12} \\
&c_{21} & c_{22} \end{pmatrix} \neq 0 \\
&ii) \quad (H_1 - H_2; C_1) \neq 0 \\
&iii) \quad (H_1 - H_2; C_2) \neq 0.
\end{align*}
\]

9
We can now use \( \text{iii} \) of (3.4) to find \( e_1, e_2 \).
Since the only conditions on \( S \) are the inequalities (3.8), we may take for \( S(\theta) \), the continuous system to which \( S \), given by (3.1), belongs, the full 8 parameter system:
\[
S(\theta) : \quad y' = y\left[ \begin{array}{c} c_{11} \\ c_{21} \end{array} \right] Y^{(h_{11}, h_{12})} + \left( \begin{array}{c} c_{12} \\ c_{22} \end{array} \right) Y^{(h_{21}, h_{22})} \]
\]
3.9

subject only to the inequalities (3.8).
We can, now, solve the linear systems (3.7) for \( B_1(\theta), B_2(\theta) \), and obtain:
\[
\begin{align*}
&\text{i)} \quad B_1(\theta) = \frac{(H_2 - H_1; C_2)}{d} (c_{21}, c_{11}) \\
&\text{ii)} \quad B_2(\theta) = \frac{(H_2 - H_1; C_1)}{d} (c_{22}, -c_{12})
\end{align*}
\]
3.10

To check that (3.10) are solutions to (3.7), note that \((-c_{21}, c_{11}), (c_{22}, -c_{12})\) are normal to \( C_1, C_2 \) respectively and that
\[( (-c_{21}, c_{11}); C_2) = d \quad ((c_{22}, -c_{12}); C_1) = d. \]

\( B_2(\theta) \) may also be obtained, when \( B_1(\theta) \) is known, by using \( \text{ii} \) of (3.4) yielding:
\[
B_2(\theta) = B_1(\theta) + H_2 - H_1
\]
3.11

To show that (3.11) agrees with \( \text{ii} \) of (3.10), we show that (3.11) is a solution of the linear system:
\[
\begin{align*}
(B_2(\theta); C_2) &= 0 \\
(B_2(\theta); C_1) &= (H_2 - H_1; C_1)
\end{align*}
\]

For
\[
\begin{align*}
(B_2(\theta); C_2) &= (B_1(\theta); C_2) + (H_2 - H_1; C_2) \\
&= (H_1 - H_2; C_2) + (H_2 - H_1; C_2) = 0 \\
(B_2(\theta), C_1) &= (B_1(\theta), C_1) + (H_2 - H_1, C_1) = (H_2 - H_1, C_1),
\end{align*}
\]

10
Thus (3.11) and $ii$ of (3.10) are both solutions of the linear system $ii$ of (3.7) which has a unique solution when the determinant $d$ is not equal to zero.

To find $e_1(\theta), e_2(\theta)$ we use $iii$ of (3.4)

$$e_1(H_1 - H_2; C_2) + e_2(H_2 - H_1, C_1) = 0,$$
yielding

$$e_1 = (H_2 - H_1; C_1), \quad e_2 = (H_2 - H_1, C_2)$$

Thus

$$I(\theta) = (H_2 - H_1, C_1)(y_1^{-c_{12}}y_2^{c_{11}})^{(H_1-H_2+C_2)}d$$

$$+ (H_2 - H_1, C_2)(y_1^{c_{22}}y_2^{-c_{12}})^{(H_2-H_1+C_1)}d$$

setting $c_{12} = c_{21} = 0, \quad c_{11} = 1, \quad c_{22} = -1$ we get $\bar{d} = -1$ and $H_1 = (-1, 1), \quad H_2 = (1, -1)$ in (3.8) and (3.12) we get:

$S(\bar{\theta}) = S$ of (3.1) and $I(\bar{\theta}) = 2I$ of (3.2)

We now show that the system given by (3.9) has a first integral even when the inequalities are violated (one at a time).

Let $d = 0$ then $C_2 = lC_1$ and by $i$ of (2.2) $S$ has the monomial first integral $Y^B$ where $(B; C_1) = (B; C_2) = 0$. Now, let $d \neq 0$ and let $\bar{\theta}$ be such that $(H_1 - H_2, C_2) = 0$ while $(H_1 - H_2, C_1) \neq 0$, then

$I(\bar{\theta}) = (H_2 - H_1; C_1)$ and fails to define a first integral of $S(\bar{\theta})$. Fortunately, $(H_1 - H_2, C_2) = 0$ is the very condition required for $S(\bar{\theta})$ to have a first integral of the form:

$$I^*(\bar{\theta}) = \ln \left( Y^{B_1} \right) + Y^{B_2}$$

For applying the derivative formula $ii$ of (2.1) to $I^*(\bar{\theta})$ we get

$$(I^*(\bar{\theta}))' = (B_1; C_2)Y^{H_2} + (B_2; C_1)Y^{B_2+H_1} = 0$$

Thus the relations between $S(\bar{\theta})$ and $I^*(\bar{\theta})$ are:
i) \( (B_1; C_1) = (B_2; C_2) = 0 \)

ii) \( H_2 = B_2 + H_1 = E_1 \).

iii) \( (B_1; C_2) + (B_2; C_1) = 0 \) \( \text{3.15} \)

ii) of (3.15) implies \( B_2 = H_2 - H_1 \) and i) and iii) of (3.15) yield the system of linear equations:

\[
\begin{align*}
(B_1, C_1) &= 0 \\
(B_1, C_2) &= -(B_2, C_1) = (H_1 - H_2; C_1)
\end{align*}
\]

\( \text{3.16} \)

and \( B_1 = \frac{(H_1 - H_2; C_1)}{d} (-c_{21}, c_{11}) \)

\[ I^*(\theta^*) = \frac{(H_1 - H_2; C_1)}{d} \ln \left( y_1^{-c_{21}} y_2^{c_{22}} \right) + y_1^{h_{21}-h_{11}} y_2^{h_{22}-h_{12}} \]

\( \text{3.17} \)

The integral array of \( I^*(\theta^*) \) is \( A^* = (H_2 \circ H_1) \) (the \( \circ \) above \( H_2 \) indicates that \( B_1 \) is not added to \( H_2 \) to get ii) of (3.15)). Similarly, if \( S(\theta^{**}) \) is such that \( (H_1 - H_2, C_2) \neq 0, d \neq 0 \) but, \( (H_1 - H_2; C_1) = 0 \) then the relations between \( I^*(\theta^{**}) \) and \( S(\theta^{**}) \) are:

i) \( (B_1, C_1) = (B_2, C_2) = 0 \)

ii) \( B_1 + H_2 = H_1 = E_1 \)

iii) \( (B_1; C_2) + (B_2; C_1) = 0 \) \( \text{3.18} \)

the integral array of \( I^*(\theta^{**}) \) is

\[ A^{**} = (H_2 \circ H_1) \] and

\[ I^*(\theta^{**}) = y_1^{(h_{11}-h_{21})} y_2^{(h_{12}-h_{22})} + \frac{(H_2 - H_1, C_2)}{d} \ln \left( y_1^{c_{22}} y_2^{-c_{12}} \right) \]

\( \text{3.19} \)
Summarizing the above:

The system $S$ of (3.1) and its first integral $I$ of (3.2) belong to the continuous 8 parameter system $S(\theta)$ of (3.9) and its first integral $I(\theta)$ of (3.12) which exists provided $S(\theta)$ satisfies the 3 inequalities:

i) $d \neq 0$

ii) $(H_1 - H_2; C_1) \neq 0$

iii) $(H_1 - H_2; C_2) \neq 0$

If $d = 0$, $S(\theta)$ has the monomial integral $I = Y^B$, where $(B; C_1) = (B; C_2) = 0$.

If $S(\theta^*)$ is such that $d \neq 0$, $(H_1 - H_2, C_1) \neq 0$ but $(H_1 - H_2; C_2) = 0$, then $S(\theta^*)$ has the logarithmic integral $I^*(\theta^*)$ given by (3.17).

If $d \neq 0$, $(H_1 - H_2; C_2) \neq 0$ but $(H_1 - H_2, C_1) = 0$, when $\theta = \theta^{**}$. Then $S(\theta^{**})$ has the logarithmic integral $I^*(\theta^{**})$ given by (3.19).

4

Let $S_3(\theta)$ be the system given by:

$$y' = y[C_1 Y^{H_1} + C_2 Y^{H_2} + C_3 Y^{H_3}],$$

where $C_1, H_1, C_2, H_2$ are as in (3.9), subject only to the three inequalities (3.8). We shall refer to the system $S(\theta)$ of (3.9) as $S_2(\theta)$ and write

$$S_3(\theta) = S_2(\theta) + C_3 Y^{H_3}$$

Since $n = 2$ and $C_1, C_2$ are linearly independent, we may write

$$C_3 = l_1 C_1 + l_2 C_2$$  \hspace{1cm} 4.1$$

Now, $S_2(\theta)$ has a first integral $I_2(\theta)$ given by (3.12). We are going to show that $S_3(\theta)$ has a first integral

$$I_3(\theta) = I_2(\theta) + e_3 Y^{B_3}$$

subject only to the following:
\[(H_3; C_3) = l_1(H_1; C_1) + l_2(H_2; C_2)\]  \hspace{1cm} 4.2

where \(l_1, l_2\) are as in (4.1), and \(S_3(\theta)\) satisfies the following additional inequalities:

\[
\begin{align*}
  i) & \quad (H_1 - H_3; C_1, C_3) \neq 0 \\
  ii) & \quad (H_2 - H_3; C_2, C_3) \neq 0
\end{align*}
\]  \hspace{1cm} 4.3

There are, now, 2 cases to consider.

Case 1.

\[l_1 \times l_2 \neq 0\]  \hspace{1cm} 4.4

This inequality and the inequality \(d = d_{12} \neq 0\) imply that any two of \(C_1, C_2, C_3\) are linearly independent. To find \(I_3(\theta)\), let

\[B_3 = B_1 + H_3 - H_1\]  \hspace{1cm} 4.5

where \(B_1\) is as in \(i)\) of (3.10). We show that (4.2) implies \((B_3; C_3) = 0\). For, using \(i)\) of (3.7) we get:
\[
\begin{aligned}
  i) & \quad (B_3; C_3) = (B_3 + H_3 - H_1; C_3) \\
       & \quad = (B_3; C_3) + (H_3, C_3) - (H_1, C_3) \\
       & \quad = (B_3; C_3) + l_1(C_1 + l_2C_2) + l_1(H_1, C_1) \\
       & \quad + l_2(H_3, C_2) - l_1(H_1, C_1) - l_2(H_1, C_2) \\
       & \quad = l_2(B_1, C_2) + l_2(H_2 - H_1; C_2) \\
       & \quad = l_2[(H_1 - H_2, C_2) + (H_2 - H_1, C_2)] = 0 \\

  Also

  ii) & \quad (B_3, C_1) = (B_1 + H_3 - H_1; C_1) \\
       & \quad = (B_1; C_1) + (H_3 - H_1; C_1) \\
       & \quad = 0 + (H_3 - H_1; C_1) \neq 0 \\

  iii) & \quad (B_3; C_2) = (B_1 + H_3 - H_1; C_2) \\
       & \quad = (B_1; C_2) + (H_3 - H_1; C_2) \\
       & \quad = (H_1 - H_2; C_2) + (H_3 - H_1; C_2) \\
       & \quad = (H_3 - H_2; C_2) \neq 0 \\

  iv) & \quad (B_1, C_3) = (B_3 + H_1 - H_3; C_3) \\
       & \quad = (B_3; C_3) + (H_1 - H_3; C_3) = (H_1 - H_3; C_3) \neq 0 \\

  v) & \quad B_2 + H_3 = B_1 + H_2 - H_1 + H_3 \\
       & \quad = B_1 + H_3 - H_1 + H_2 = B_3 + H_2 \\

  vi) & \quad (B_2; C_3) = (B_2 - B_3; C_3) + (B_3; C_3) \\
       & \quad = (H_2 - H_3, C_3) + 0 \neq 0
\end{aligned}
\]

To get the results of (4.6) we used (3.4), (1.2), (1.3), (4.2) and (4.3).
The relations (4.6) are summarized by the array:

\[
A = \begin{pmatrix}
  B_1 & B_2 & B_3 \\
  H_2 & H_1 & 0 \\
  H_3 & 0 & H_1 \\
  0 & H_3 & H_2 \\
\end{pmatrix} \begin{pmatrix}
  E_1 \\
  E_2 \\
  E_3 \\
\end{pmatrix}
\]

Columns \( k \) says \((B_k; C_j) \neq 0\) if and only if \( k \neq j \) \((k = 1, 2, 3)\) Row 1 gives the relations (3.4).
Row 2 implies \( B_1 + H_3 = B_3 + H_1 = E_2 \).
Row 3 implies \( B_2 + H_3 = B_3 + H_2 = E_3 \).
Thus,

\[
I' = [e_1(B_1, C_2) + e_2(B_2; C_1)]Y^{E_1}
\]
\[ + \quad [e_1(B_1; C_3) + e_3(B_3; C_1)] Y^{E_2} \\
+ \quad [e_2(B_2; C_3) + e_3(B_3, C_2)] Y^{E_3} = 0 \]

and \((B_k; C_j)\) are given by (4.6) \((j, k = 1, 2, 3)\).

To find \(I_3(\theta)\) of \(S_3(\theta)\), we have \(I_2(\theta)\) is as in (3.12) and we solve for \(B_3\) by using the linear system:

\[
\begin{align*}
(B_3, C_3) &= 0 \\
(B_3, C_1) &= (H_3 - H_1, C_1)
\end{align*}
\]

Yielding: \(B_3\) in the same form as \(B_1, B_2\) e.g.:

\[ B_3 = \frac{(H_3 - H_1, C_1)}{d_{13}} (c_{23}, -c_{13}), \quad d_{13} = \begin{vmatrix} c_{11} & c_{13} \\ c_{21} & c_{23} \end{vmatrix} \]

\(e_1, e_2\) are as in \(I_2\) given by (3.12).

Setting the coefficient of \(Y^{E_2}\) to zero we get

\[ e_3 = -e_1 \frac{(B_1; C_3)}{(B_3; C_1)} = -e_1 \frac{(H_1 - H_3; C_3)}{(H_3 - H_1; C_1)} \]

We now must show that

\[
\begin{vmatrix} (B_1; C_2) & (B_2; C_1) & 0 \\ (B_1; C_3) & 0 & (B_3; C_1) \\ 0 & (B_2; C_3) & (B_3; C_2) \end{vmatrix} = 0
\]

Now,

\(B_1, C_3\) = \(l_1(B_1, C_1) + l_2(B_1, C_2) = l_2(B_1, C_2)\)

\(B_2, C_3\) = \(l_1(B_2, C_1) + l_2(B_2, C_2) = l_1(B_2, C_1)\)

Thus (4.10) becomes:

\[
(B_1, C_2)(B_2, C_1) \begin{vmatrix} 1 & 1 & 0 \\ l_2 & 0 & (B_3; C_1) \\ 0 & l_1 & (B_3; C_2) \end{vmatrix} = 0
\]

16
\[= (B_1, C_2)(B_2, C_1)[-l_1(B_3, C_1) - l_2(B_3, C_2)]\]

\[= (B_1, C_2)(B_2, C_1)[-\langle B_3; C_3 \rangle] = 0 \text{ by } i \text{ of } (4.6).\]

Setting \(e_1 = 1 \text{ (in } I_2(\theta))\), we get:

\[e_2 = \frac{(H_1 - H_2; C_2)}{H_1 - H_2; C_1}, \quad e_3 = \frac{(H_1 - H_3; C_3)}{(H_1 - H_3; C_1)} \text{ by } (4.9)\]

Thus:

\[I_3(\theta) = I_2(\theta) + \frac{(H_1 - H_3, C_3)}{(H_1 - H_3, C_1)} \frac{(y_{c^{23}} y_{c^{13}})}{4^{13}}\]

where \(I_2(\theta)\) is the first integral given by (3.12), multiplied by \(\frac{1}{(H_1 - H_2; C_1)}\).

Let \(S_2(\bar{\theta}), I_2(\bar{\theta})\) be as in (3.1), (3.2) respectively. Then

\[C_3 = \left( \begin{array}{c} l_1 \\ -l_2 \end{array} \right) \quad \text{and} \]

\[(H_3; C_3) = -l_1 + l_2 \quad \text{which implies}\]

\[l_1(h_{31} + 1) = l_2(h_{32} + 1) \]

and \(B_3 = B_1 + H_3 - H_1 = (h_{31} + 1, h_{32} + 1) \text{ subject to } (4.12).\)

\[e_3 = \frac{(H_3 - H_1; C_3)}{(H_3 - H_1; C_1)} = \frac{2l_2}{h_{31} + 1}\]

Thus

\[I_3(\bar{\theta}) = I_2(\bar{\theta}) + e_3 Y^{B_3} = y_2^2 + y_1^2 + \frac{2l_2}{h_{31} + 1} y_1^{h_{31} + 1} y_2^{h_{32} + 1} \]

\[\text{is a first integral of}\]

\[S_3(\bar{\theta}) = S_2(\bar{\theta}) + C_3 Y^{H_3}\]
\[ S_3(\bar{\theta}) : \quad y'_1 = y_2 + l_1 y^{h_{31}+1}_1 y^{h_{32}}_2 \]
\[ y'_2 = -y_1 - l_2 y^{h_{31}}_1 y^{h_{32}+1}_2 \]

subject to (4.12).

Case 2. One of \( l_1, l_2 \) is zero, say \( l_1 \), then:
\[
C_3 = l_2 C_2
\]

Let \((H_3, C_3)\) be subject to the condition:
\[
(H_3, C_3) = l_2(H_2, C_2)
\]
which implies:
\[
(H_3 - H_2, C_2) = 0
\]

In addition, let \( S_3(\theta) \) satisfy the following inequalities:
\[
(H_3 - H_1; C_1, C_3) \neq 0
\]

Let \( B_1, B_2 \) be as in (3.12) then:
\[
(B_1, C_2) \neq 0 \quad \text{implies} \quad (B_1, C_3) \neq 0
\]
\[
(B_2, C_2) = 0 \quad \text{implies} \quad (B_2, C_3) = 0
\]

Let \( B_3 = B_1 + H_3 - H_1 \) then
\[
(B_3; C_1) = (B_1 + H_3 - H_1; C_1)
\]
\[
= (B_1; C_1) + (H_3 - H_1; C_1) = 0 + (H_3 - H_1, C_1) \neq 0
\]
\[
(B_3, C_2) = (B_1 + H_3 - H_1, C_2)
\]
\[
= (B_1, C_2) + (H_3 - H_1; C_2)
\]
\[
= (H_1 - H_2; C_2) + (H_3 - H_1; C_2)
\]
\[
= (H_3 - H_2; C_2) = 0
\]
by (4.15) and
\[
(B_3; C_3) = (B_3, C_2) = 0
\]

Thus
\[ I'_3(\theta) = I'_2(\theta) + e_3(\text{Y}B_3)' \]
\[ = [e_1(B_1; C_2) + e_2(B_2; C_1)]\text{Y}_E^1 \]
\[ + [e_1(B_1; C_3) + e_3(B_3; C_1)]\text{Y}_E^2 \]
\[ = 0 \]

All the relation between \( B_1, B_2, B_3, C_1, C_2, C_3, H_1, H_2, H_3 \) are summarized by the 2×3 array:
\[
A = \begin{pmatrix}
B_1 & B_2 & B_3 \\
H_2 & H_1 & 0 \\
H_3 & 0 & H_1
\end{pmatrix}
\]
\( E_1 \)
\( E_2 \)

\( B_1, B_2 \) are as in \( I_2(\theta) \) and \( B_3 = B_1 + H_3 - H_1 \) may, also, be obtained by solving the linear system:
\( (B_3, C_3) = 0 = l_2(B_3, C_2) \)
\( (B_3, C_1) = (H_3 - H_1; C_1) \)

which yields
\[ B_3 = \frac{(H_3 - H_1; C_1)}{d_{12}}(c_{22} - c_{12}) \]

The vanishing of the coefficient of \( \text{Y}E_2 \) yields:
\[ e_3 = -e_1(B_1, C_3) - \frac{e_1 l_2(B_1; C_2)}{(B_3, C_1)} \]

setting \( e_1 = 1 \) we get
\[ e_3 = -l_2 \frac{(H_3 - H_2, C_2)}{(H_3 - H_1, C_1)} = \frac{l_2(H_2 - H_1, C_2)}{l_2(H_3 - H_1, C_1)} \]

Thus, if \( S_3(\theta) = S_2(\theta) + C_3\text{Y}H_3 \), where \( S_2(\theta) \) is as in (3.9), \( C_3 = l_2C_2 \) and \( H_3 \) is subject to the condition:
\( (H_3 - H_2, C_2) = 0 \)

and the inequalities \( (H_3 - H_1; C_1, C_3) \neq 0 \), then \( S_3(\theta) \) has the first integral:
\[ I_3(\theta) = I_2(\theta) + e_3\text{Y}B_3 = I_2(\theta) + l_2 \frac{(H_2 - H_1, C_2)}{(H_3 - H_1, C_1)} \left( y_{1}^{c_{22}}y_{2}^{c_{12}} \frac{(H_3 - H_1; C_1)}{d_{12}} \right) \]

where \( I_2(\theta) \) is given by (3.12).
Example 3.

\[ y'' - 2y'^2 + 3y^2 = 0 \]

has a first integral

\[ I = -4y + \ln \left( 1 - \frac{16}{3}y'^2 + 8y^2 + 4y \right) \]

Let \( y = y_1 \), \( y' = y_2 \), then the multinomial vector form for (5.1) is:

\[ y' = y(C_1Y^{H_1} + C_2Y^{H_2} + C_3Y^{H_3}) \]

where:

\[ H_1 = (-1, 1), \quad H_2 = (0, 1), \quad H_3 = (2, -1) \]

\[ C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}. \]

\[ I = e_1Y^{B_1} + \ln \left( 1 + e_2Y^{B_2} + e_3Y^{B_3} + e_4Y^{B_4} \right) \]

where:

\[ B_1 = (1, 0), \quad B_2 = (0, 2), \quad B_3 = (2, 0), \quad B_4 = (1, 0) \]

\[ e_1 = -4, \quad e_2 = -\frac{16}{3}, \quad e_3 = 8, \quad e_4 = 4. \]

Now,

\[ I' = e_1(Y^{B_1})' + \frac{e_2(Y^{B_2})' + e_3(Y^{B_3})' + e_4(Y^{B_4})'}{1 + e_2Y^{B_2} + e_3Y^{B_3} + e_4Y^{B_4}} = 0 \]

Clearing of fractions and using the relations:

\( (B_2; C_1) = 0, \quad (B_1, B_3, B_4; C_j) = 0 \) if \( j \neq 1 \),

yields:
\[ I' = e_1(B_1; C_1)Y^{B_1 + H_1}[1 + \sum_{k=2}^4 e_k Y^{B_k}] \\
+ e_2[(B_2; C_2)Y^{B_2 + H_2} + (B_2; C_3)Y^{B_2 + H_3}] \\
+ e_3(B_3; C_1)Y^{B_3 + H_1} + e_4(B_4; C_1)Y^{B_4 + H_1} = 0 \]

Grouping all the coefficients of the same vector together, we get:

\[ I' = [e_1(B_1; C_1) + e_4(B_4; C_1)]Y^{E_1} \\
+ [e_2(B_2; C_2) + e_1e_2(B_1; C_1)]Y^{E_2} \\
+ [e_2(B_2; C_3) + e_1e_3(B_1; C_1)]Y^{E_3} \\
+ [e_3(B_3; C_1) + e_1e_4(B_1; C_1)]Y^{E_4} = 0, \]

where:

i) \( E_1 = B_1 + H_1 = B_4 + H_1 = (0, 1) \)

ii) \( E_2 = B - 2 + H_2 = B_1 + B_2 + H_1 = (0, 3) \)

iii) \( E_3 = B_2 + H_3 = B_1 + B_3 + H_1 = (2, 1) \)

iv) \( E_4 = B_3 + H_1 = B_1 + B_4 + H_1 = (1, 1) \)

The integral array \( A \) of \( I \) is:

\[
A = \begin{pmatrix}
B_1 & B_2 & B_3 & B_4 & B_1 + \bar{B}_2 & B_1 + \bar{B}_3 & B_1 + \bar{B}_4 \\
H_1 & \emptyset & \emptyset & H_1 & \emptyset & \emptyset & \emptyset \\
\emptyset & H_2 & \emptyset & \emptyset & H_1 & \emptyset & \emptyset \\
\emptyset & \emptyset & H_3 & \emptyset & \emptyset & H_1 & \emptyset \\
\emptyset & \emptyset & \emptyset & H_1 & \emptyset & \emptyset & H_1 \\
\hline
\end{pmatrix}
\]

\( M(A) = \begin{pmatrix}
(B_1; C_1) & 0 & 0 & (B_4; C_1) & 0 & 0 & 0 \\
0 & (B_2; C_2) & 0 & 0 & (B_1; C_1) & 0 & 0 \\
0 & (B_2; C_3) & 0 & 0 & 0 & (B_1; C_1) & 0 \\
0 & 0 & (B_3; C_1) & 0 & 0 & 0 & (B_1; C_1) \\
\end{pmatrix} \]

\[
\begin{cases}
\text{i) of (5.4) implies } & B_1 = B_4 = (1, 0) \\
\text{ii) of (5.4) implies } & B_1 = B_4 = H_2 - H_1 = (1, 0) \\
\text{iii) and iv) imply } & B_2 = 3H_2 - 2H_1 - H_3 = (0, 2) \\
& B_3 = 2B_1 = 2(H_2 - H_1) = (2, 0) 
\end{cases}
\]
Setting the coefficients of $Y^{E_i} = 0$, in (5.3), ($i = 1, \ldots , 4$) we get:

$$
\begin{align*}
& e_4 = -e_1, & e_1 = \frac{(B_2; C_2)}{(B_1; C_1)} = -4 \\
& e_3 = -e_1e_4\frac{(B_1; C_1)}{(B_3; C_1)} = 8, & e_2 = -e_1e_3\frac{(B_3; C_3)}{(B_2; C_2)} = -16/3
\end{align*}
$$

We now use the $4 \times 7$ array $A$ to find the conditions that $S(\Theta)$ has to satisfy, to have a first integral

$$I(\Theta) = e_1Y^{B_1(\Theta)} + \ln \left( 1 + e_2Y^{B_2(\Theta)} + e_3Y^{B_3(\Theta)} + e_4Y^{B_4(\Theta)} \right)$$

where $S(\Theta)$ is given by:

$$S(\Theta) : \ y' = y \left[ \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} Y^{(h_{11}, h_{12})} + \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} Y^{(h_{21}, h_{22})} + \begin{pmatrix} c_{13} \\ c_{23} \end{pmatrix} Y^{(h_{31}, h_{32})} \right]$$

Using $ii)$ of (5.3) we have $B_1 = B_4 = H_2 - H_1$ and looking at the array $A$, we see that $(B_1, B_4; C_j) = 0$ if and only if $j \neq 1$. Thus we have

$$
\begin{align*}
& a) (H_2 - H_1, C_j) = 0 \quad \text{if and only if } j \neq 1. \text{ Similarly,} \\
& b) (B_2; C_1) = 0 \quad \text{implies } (3H_2 - 2H_1 - H_3; C_1) = 0 \\
& c) C_3 = mC_2. \quad \text{For, } (H_2 - H_1; C_2, C_3) = 0 \text{ implies } H_1 = H_2, \\
& \text{if } C_2, C_3 \text{ are linearly independent,} \\
& \text{since, (the order of } S(\Theta) \text{ is two.)}
\end{align*}
$$

Thus $S(\Theta)$ is reduced, by these conditions to a 9 parameter system.

$$I(\Theta) = e_1(\Theta)Y^{B_1(\Theta)} + \ln \left( 1 + \sum_{k=2}^{4} e_k(\Theta)Y^{B_2(\Theta)} \right)$$

and from (5.3), (5.6) we can find $B_k(\Theta), e_k(\Theta)(k = 1, \ldots , 4)$. In the special case when $S(\Theta)$ comes from a second order differential equation, we have the additional conditions:

$$H_1 = (-1,1), \quad C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 \\ c_{22} \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 \\ c_{23} \end{pmatrix}$$

$$S(\Theta) : \ y' = y \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} Y^{(-1,1)} + \begin{pmatrix} 0 \\ c_{22} \end{pmatrix} Y^{(h_{21},1)} + \begin{pmatrix} 0 \\ c_{23} \end{pmatrix} Y^{(3h_{21}+2,h_{32})} \right]$$

writing $S(\Theta)$ as a second order differential equation we get:
\( y'' = c_{22}y^{h_{21}}(y')^2 + c_{32}y^{3h_{21}+2}(y')^{h_{32}+1} \)  

5.7

\[
I(\Theta) = e_1 y^{(h_{21}+1)} + \ln \left(1 + e_2 (y')^{(1-h_{32})} + e_3 y^{2(h_{21}+1)} + e_4 y^{(h_{21}+1)}\right)
\]

where

\[
\begin{align*}
e_1 &= \frac{(h_{21}-1)c_{22}}{h_{21}+1}, & e_2 &= \frac{(h_{32}-1)^2c_{32}}{2(h_{21}+1)c_{32}} \\
e_3 &= \frac{(h_{21}-1)c_{32}^2}{2(h_{21}+1)c_{32}}, & e_4 &= \frac{(1-h_{32})c_{22}}{h_{21}+1}
\end{align*}
\]

Setting \( h_{21} = 0, \ h_{32} = -1, \ c_{22} = 2, \ c_{32} = -3 \) in (5.7) and its first integral \( I(\Theta) \), we get (5.1) and its first integral \( I \).

The differential equation (5.7) and its first integral \( I(\Theta) \), is, actually, a special case of the following:

\( y'' = c_{22}y^{h_{21}}(y')^2 + c_{32}y^{\alpha}(y')^\beta \)  

5.8

where \( \alpha = (q-1)h_{21} + q - 2, \ \beta = h_{32} + 1 \).

For any integer \( q \geq 3 \), (5.8) has a first integral:

\[
I = e_1 y^{(h_{21}+1)} + \ln \left(1 + e_2 y^{(1-h_{32})} + \sum_{k=3}^{q} e_k y^{(q-k+1)(h_{31}+1)}\right)
\]

5.9

Setting \( q = 4 \) in (5.8), (5.9), one gets (5.7) and its first integral \( I(\Theta) \).

6  The \( p \times q \) Array (the algebraic case).

In the following we describe the role that the general \( p \times q \) array plays in determining the necessary and sufficient conditions, that the system \( S \) given by (1.3), must satisfy, for the system to have an algebraic first integral given by (1.4).

Let \( I = \sum_{k=1}^{q} e_k Y^{B_k} \) be a first integral of the multinomial system given by (1.3). Making use of the derivative formula \( i \) of (2.1) we get:

\[
I' = \sum_{k=1}^{q} e_k Y^{B_k} \sum_{j=1}^{r} (B_k; C_j) Y^{H_j} \\
= \sum_{k=1}^{q} (\sum_{j=1}^{r} e_k (B_k; C_j) Y^{B_k+H_j}) = 0
\]

23
In the set of \( q_r \) vectors \( B_k + H_j \), leave out all vectors such that \( (B_k; C_j) = 0 \). Let the remaining vectors form a set of \( p \) distinct vectors \( E_1, \ldots, E_p \), then:

\[
I' = \sum_{i=1}^{p} \left[ \sum_{B_k + H_{\alpha_i} = E_i} e_k(B_k; C_{\alpha_i}) \right] Y E_i = 0 \tag{6.1}
\]

\( (B_k; C_{\alpha_i}) \neq 0 \)

Since \( E_i \ (i = 1, \ldots, p) \) are distinct, equation (6.1) yields a system of \( p \) homogeneous linear equations:

\[
\sum_{k=1}^{q} e_k(B_k; C_{\alpha_i}) = 0 \quad (i = 1, \ldots, p) \tag{6.2}
\]

which \( e_1, \ldots, e_q \) must satisfy. The \( p \times q \) array \( A \) is a pictorial representation of (6.1) and is defined as follows:

\[
A = (A_{ik}) \quad (i = 1, \ldots, p; k = 1, \ldots, q), \text{ where:}
\]

\[
A_{ik} = \begin{cases} 
H_{\alpha} & \text{if } B_k + H_{\alpha} = E_i \text{ and } (B_k; C_{\alpha}) \neq 0 \\
\emptyset & \text{if no such } H_{\alpha} \text{ exists}
\end{cases} \tag{6.3}
\]

The symbol \( \emptyset \) stands for the empty spot.

Definition (6.3) implies:

\[
\begin{align*}
& i) \ H_{\alpha} \text{ appears in the } k^{th} \text{ column of } A \text{ if and only if } (B_k; C_{\alpha}) \neq 0 \\
& ii) \text{ if } H_{\alpha}, H_{\beta} \text{ appears in the same row in columns } j, k, \text{ respectively, then:} \\
& \quad B_j + H_{\alpha} = B_k + H_{\beta}
\end{align*} \tag{6.4}
\]

We call columns: \( j, k \) linked if \( ii) \) of (6.4) is satisfied and we set

\[
L_{jk} = H_{\alpha} - H_{\beta}
\]
We call $j, k$ of $A$ connected if there exist columns $P_1, \ldots, P_s$ such that $j = P_1, k = P_s$ and columns $P_\alpha, P_{\alpha+1}$ ($\alpha = 1, \ldots, s - 1$) are linked, and we set

$$L_{jk} = \sum_{\alpha=1}^{s-1} L_{P_\alpha P_{\alpha+1}}$$

e.g. In the array:

$$A = \begin{pmatrix} H_u & H_v & \emptyset \\ \emptyset & H_w & H_v \end{pmatrix}$$

columns 1,2 are linked and $L_{12} = H_u - H_v$; columns 2,3 are linked and $L_{23} = H_w - H_v$; columns 1,3 are connected and $L_{13} = L_{12} + L_{23} = H_u + H_w - 2H_v$.

Note, that when the array $A$ is connected, $L_{jk}$ is defined and

$L_{jk} = B_k - B_j$ for all $1 \leq j < k \leq q$

This implies that the difference of any 2 exponent vectors of $I$ is a linear combination of the exponent vectors of the system $S$.

It is not difficult to show that when $A$ is not connected (i.e. when there exist at least two column of $A$ which can not be connected) then $A$ is the array of an integral $I$ of $S$ such that $I = I_1 + I_2$ where $I_1$ and $I_2$ are both first integrals of $S$,[1]. From now on we shall assume that the array $A$ is connected. It follows from the definition of $A$ that interchanging columns $j, k$ of $A$ is equivalent to interchanging the exponents $B_j, B_k$ of $I$ and interchanging rows $i, j$ of $A$ is equivalent to interchanging $E_i, E_j$ of (6.1). Thus we identify all arrays that can be obtained from one another by an interchange of rows and or columns.

Along with $A$ we define the $p \times q$ matrix

$$MA = (a_{ik} \quad (i = 1, \ldots, p; k = 1, \ldots, q)$$

where

$$a_{ik} = (B_k; C_\alpha) \quad \text{if} \quad A_{ik} = H_\alpha$$

$$a_{ik} = 0 \quad \text{if} \quad A_{ik} = \emptyset$$

6.5
There are two kinds of arrays normal and abnormal. An array $A$ is called normal if for every $H_\alpha$ that appears in $A$, there exist at least one column of $A$ which does not contain $H_\alpha$. An array $A$ is called abnormal if there exists at least one $H_\alpha$ which appears in every column of $A$. It is remarkable that when $A$ is normal we can compute $(B_k; C_\alpha)$ without knowing what the $B_k$’s are. For, let $H_\alpha$ fail to appear in column $j$ of $A$ then, for any $k \neq j$ we have

\[
(B_k; C_\alpha) = (B_k - B_j; C_\alpha) + (B_j; C_\alpha) = (L_{jk}; C_\alpha) + 0 = (L_{jk}, C_\alpha)
\]

Thus, $M(A)$, which is the matrix of the system of homogeneous linear equations given by (6.2), can be computed, when $A$ is normal.

Let $A$ be an array of the exponent vectors of a system $S$, given by (1.3). The following is a set of necessary and sufficient conditions that $A$ must satisfy for $A$ to be an integral array (i.e. there exists a first integral $I$ of $S$ such that $A$ satisfies (6.3)).
a) Each row of $A$ must contain at least two distinct $H_\alpha$'s of $S$ and no $H_\alpha$ may appear more than once in any row or column of $A$.

b) The $L_{jk} \ (1 \leq j < k \leq q)$ are well defined and do not equal $\bar{0}$ ($\bar{0}$ = the zero vector).

c) If $H_\alpha$ fails to appear in columns $j, k$ of $A$, then $(L_{jk}; C_\alpha = 0$

d) If $H_\alpha$ appears in column $j$ but fails to appear in column $k$, then $(L_{jk}; C_\alpha) \neq 0$.

e) Rank of any $q - 1$ columns of $M(A)$ equals rank of $M(A) = q - 1$.

f) $B_1$ satisfies the linear system:

$$(B_1; C_\alpha) = 0 \quad \text{if } H_\alpha \text{ fails to appear in column } 1.$$  

$$(B_1; C_\alpha) = (L_{j1}; C_\alpha) \quad \text{if } H_\alpha \text{ appears in column } 1, \text{ but fails to appear in column } j \text{ (the existence of such a } "j" \text{ is guaranteed by the fact that } A \text{ is normal).}$$

Condition $a)$ is necessary, since $I' = 0$ implies that the coefficients of $Y^{E_i} \ (i = 1, \ldots, p)$ must vanish, thus if only one $H_\alpha$ appears in row $i$ then the coefficient of $Y^{E_i}$ is $= 0$ contrary to assumption. Also,

$H_\alpha = A_{ij} = A_{ik}$ implies $B_j = B_k$ and

$H_\alpha = A_{ik} = A_{jk}$ implies $E_i = E_j$

c) is a strong restriction on $S$ e.g. if

$$A = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$$

then $L_{12} = H_1 - H_2 = H_3 - H_4$ yields a linear relation among the exponents. Also, $L_{jk} = 0$ implies $B_j = B_k$. 

27
Conditions c), d) are implied by \( L_{jk} = B_k - B_j \), thus if \( H_{\alpha} \) does not appear in either column, we have \( (L_{jk}; C_{\alpha}) = (B_k; C_{\alpha}) - (B_j; C_{\alpha}) = 0 + 0 = 0 \). Similarly if \( H_{\alpha} \) appears in column \( k \) but not in column \( j \), \( (L_{jk}; C_{\alpha}) = (B_k; C_{\alpha}) - (B_j; C_{\alpha}) = (B_k, C_{\alpha}) - 0 \neq 0 \).

Condition c) is the requirement that the system of homogeneous linear equations in \( e_k \) \( (k = 1, \ldots, q) \), given by

\[
\sum_{k=1}^{q} a_{ik} e_k = 0 \quad i = (1, \ldots, p),
\]

has a solution such that \( \prod_{k=1}^{q} e_k \neq 0 \).

Condition f) follows from (6.3) and (6.6) by setting \( k = 1 \).

We now show that conditions a), \ldots, f) are also sufficient for the construction of a first integral \( I = \sum_{k=1}^{q} e_k Y^{B_k} \) of \( S \) given by (1.3).

Let \( A = (A_{ik}), \ (M(A)) = (a_{ik}) \ (i = 1, \ldots, p ; k = 1, \ldots, q) \), where \( a_{ik} = 0 \) if \( A_{ik} = 0 \), \( a_{ik} = (L_{jk}; C_{\alpha}) \) if \( A_{ik} = H_{\alpha} \) and \( j \) is such that \( H_{\alpha} \) does not appear in column \( j \) of \( A \). Let \( A, \ M(A) \) satisfy conditions a), \ldots, f) then we constant a first integral \( I \) as follows:

i) Let \( B_1 \) be a solution of the linear system as in f).

ii) Set \( B_k = B_1 + L_{1k} \)

iii) Set \( e_1, \ldots, e_q \) to be a solution of the homogeneous linear system:

\[ \sum_{k=1}^{q} a_{ik} e_k = 0, \] (a solution is guaranteed by e).

iv) Set \( I = \sum_{k=1}^{q} e_k Y^{B_k} \).

We first show that \( (B_k; C_{\alpha}) = 0 \) if and only if \( H_{\alpha} \) does not appear in the \( k^{th} \) column of \( A \). By i) this holds for \( k = 1 \), let \( k > 1 \). Let \( H_{\alpha} \) not appear in the \( k^{th} \) column of \( A \) and let \( H_{\alpha} \) also fail to appear in the first column of \( A \), then

\[
(B_k; C_{\alpha}) = (B_1; C_{\alpha}) + (L_{1k}; C_{\alpha}) = 0 + 0 = 0
\]

\( (B_1; C_{\alpha}) = 0 \) follows from the definition of \( B_1 \) given by i), \( (L_{1k}; C_{\alpha}) = 0 \) follows from condition c) that \( A \) satisfies. Let \( H_{\alpha} \) fail to appear in column \( k \) but appear in column 1, then \( (B_1; C_{\alpha}) = (L_k; C_{\alpha}) \) by i), so that

\[
(B_k; C_{\alpha}) = (B_1; C_{\alpha}) + (L_{1k}; C_{\alpha}) = (L_{k1}; C_{\alpha}) + (L_{1k}; C_{\alpha}) = 0.
\]

Now, let \( H_{\alpha} \) appear in column \( k \) but fail to appear in column 1, then

\[
(B_k; C_{\alpha}) = (B_1; C_{\alpha}) + (L_{1k}; C_{\alpha}) = 0 + (L_{1k}; C_{\alpha}) \neq 0,
\]
by condition $d$) that $A$ satisfies.

Finally, if $H_\alpha$ appears in column $k$ and column 1, but fails to appear in some column $j$ (such a $j$ exists because $A$ is normal) then

$$(B_k; C_\alpha) = (B_1; C_\alpha) + (L_{1k}; C_\alpha) = (L_{j1}; C_\alpha) = (L_{jk}; C_\alpha) \neq 0,$$

by condition $d$).

This proves our assertion for all $1 \leq k \leq q$. Let $H_\alpha = A_{ij}$, $H_\beta = A_{ik}$, we show that

$$B_j + H_\alpha = B_k + H_\beta = E_i \quad (i = 1, \ldots, p)$$

For,

$$B_k - B_j = B_1 + L_{1k} - (B_1 + L_{1j}) \quad (by \ ii)$$

$$= L_{1k} + L_{j1} = L_{jk}$$

and $H_\alpha - H_\beta = L_{jk}$ (by definition of $L_{jk}$ when columns $j, k$ are linked) Thus

$$H_\alpha - H_\beta = B_k - B_j \quad and$$

$$H_\alpha + B_j = H_\beta + B_k = E_i$$

We can, now, show that

$$I = \sum_{k=1}^{q} e_k Y^{B_k}$$

is a first integral of $S$. For,

$$I' = \sum_{k=1}^{q} e_k (Y^{B_k}) = \sum_{k=1}^{q} e_k \sum_{j=1}^{r} (B_k, C_j) Y^{B_k + H_j}$$

But $(B_k; C_j) = 0$ for any $H_j$ which does not appear in column $k$, therefor the sum of the $(B_k, C_j)$'s is restricted to summing along the columns of $A$. If instead we sum along the rows of $A$, we get:

$$I' = \sum_{i=1}^{k} (\sum_{k=1}^{q} e_k (B_k, C_j)) Y^{E_i}$$

For each $H_j$ in $A$, let $\alpha_j$ be the column of $A$ in which $H_j$ fails to appear, then:
\[ (B_k; C_j) = (B_k - B_{\alpha j}; C_j) + (B_{\alpha j}; C_j) \]
\[ = (L_{\alpha j,k}; C_j) + 0 = (L_{\alpha j,k}; C_j) \text{ as in (6.4)} \]
\[ = a_{ik} \text{ by (6.4)} \]

Thus

\[ I' = \sum_{i=1}^{k} (\sum_{k=1}^{q} c_k a_{ik}) Y^E_i = 0 \]

by the choice of \( e_k \) (\( k = 1, \ldots, q \)) in iii).