ON SUMS OF HOMOGENEOUS LOCALLY NILPOTENT DERIVATIONS

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Abstract. Let \( A \) be a commutative associative integrally closed \( k \)-algebra without zero divisors effectively graded by a lattice. We obtain a criterion of local nilpotency of the sum of two homogeneous locally nilpotent derivations (LNDs) of fiber type on \( A \) in terms of their degrees. The same problem is solved for commutators of two homogeneous LNDs.

INTRODUCTION

Let \( k \) be an algebraically closed field of characteristic zero. We consider an algebraic torus \( T \simeq (k^\times)^n \) acting effectively on a normal affine variety \( X \) and the corresponding grading of the algebra \( A = k[X] \) by the lattice \( M \) of characters of \( T \).

In this paper we study some properties of homogeneous locally nilpotent derivations (LNDs). A derivation \( \partial \) on \( A \) is said to be \textit{locally nilpotent} if for each \( a \in A \) there exists \( n \in \mathbb{Z}_{>0} \) such that \( \partial^n(a) = 0 \). LNDs on \( A \) are in one-to-one correspondence with regular actions of the group \( G_a(k) = (k,+ \rangle \) on \( X \), see \[6\]. It is easy to see that a derivation \( \partial \) on \( A \) is homogeneous if and only if the corresponding \( G_a(k) \)-action is normalized by the torus \( T \).

We use a description of homogeneous LNDs on an \( M \)-graded algebra \( A \). Recall that a homogeneous LND on \( k[X] \) is said to be of \textit{fiber type} if \( \partial(k(X)^T) = 0 \), see Definition\[2\]. In geometric terms, \( \partial \) is of fiber type if and only if generic orbits of the corresponding \( G_a(k) \)-action are contained in the closures of \( T \)-orbits. A complete classification of homogeneous LNDs of fiber type is due to A. Liendo, see \[9\].

A problem that one faces when dealing with locally nilpotent derivations is that the set of all LNDs on an algebra \( A \) admits no obvious algebraic structure. In Epilogue to \[6\] G. Freudenburg poses several natural questions concerning the structure of \( \text{LND}(A) \). Namely, given \( \partial_1, \partial_2 \in \text{LND}(A) \), under what conditions are \( [\partial_1, \partial_2] \) and \( \partial_1 + \partial_2 \) locally nilpotent? These questions are still open in general. Some results were obtained by M. Ferrero, Y. Lequain, and A. Nowicki in \[5\]. Namely, given two commuting locally nilpotent derivations \( d, \delta \in \text{Der}(R) \) and an element \( a \in R \), it is proven that the derivation \( ad + \delta \) is locally nilpotent if and only if \( d(a) = 0 \).

In this note we give a complete answer to the above-mentioned questions for homogeneous locally nilpotent derivations of fiber type. Namely, \( \partial_1 + \partial_2 \) (resp. \([\partial_1, \partial_2]\)) is locally nilpotent if and only if the sum \( \text{deg} \partial_1 + \text{deg} \partial_2 \) of their degrees is not in the weight cone \( \omega_M(A) \), see Theorem\[2\] (resp. Proposition\[1\]).

It should be noted that study of algebraic properties of homogeneous locally nilpotent derivations plays an important role in recent works on automorphisms of algebraic varieties, see, e.g. \[1\], \[2\], \[3\]. One more motivation comes from the question posed by V. Popov, see \[10\] Problem 3.1. He considers two locally nilpotent derivations \( \partial_1 \) and \( \partial_2 \in \text{LND}(k[x_1, \ldots, x_n])\)

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and asks when the minimal closed subgroup of $\text{Aut}_k k[x_1, \ldots, x_n]$ containing the one-dimensional subgroups $\{\exp(t\partial_1) \mid t \in k\}$ and $\{\exp(t\partial_2) \mid t \in k\}$ is finite dimensional. Here it is important to know when $\partial_1 + \partial_2$ and $[\partial_1, \partial_2]$ are locally finite.

In Section 1 we collect some basic definitions and facts about LNDs. In Section 2 we recall generalities on $T$-varieties and corresponding gradings on their coordinate algebras. Section 3 is devoted to background on homogeneous LNDs and to the classification of LNDs of fiber type from [9]. In Sections 4 and 5 we study commutators and sums of homogeneous LNDs of fiber type. Section 6 contains some corollaries of Theorem 2, examples and generalizations.

1. Locally nilpotent derivations

Let $A$ be a commutative associative $k$-algebra without zero divisors. A derivation $\partial : A \to A$ is a linear map satisfying the Leibniz rule:

$$\partial(ab) = a\partial b + b\partial a \quad \text{for all } a, b \in A$$

Denote the set of all derivations on $A$ by $\text{Der}(A)$.

Recall that an algebra is said to be graded by a commutative semigroup $S$ if there is a direct sum decomposition $A = \bigoplus_{s \in S} A_s$ such that $A_s \cdot A_{s'} \subseteq A_{s+s'}$ for all $s, s' \in S$. A derivation $\partial$ is called homogeneous if it sends homogeneous elements to homogeneous ones. We will write $\partial \in A_s$ if $\partial(A_s) \subseteq A_s$. By Leibniz rule, $\partial(ab) = a\partial b + b\partial a \in A_{\partial(s+s')}$, and so

$$\partial(s + s') = s + \partial(s') = s' + \partial(s).$$

Thus, for a homogeneous nonzero derivation $\partial$ there exists $s_0 \in S$ such that $\partial A_s \subseteq A_{s+s_0}$ for all $s \in S$. An element $s_0 \in S$ is called the degree of $\partial$ and is denoted by $\deg \partial$.

**Definition 1.** A derivation $\partial \in \text{Der}(A)$ is called locally nilpotent (LND for short) if for every $f \in A$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\partial^n f = 0$. The set of all locally nilpotent derivations on $A$ is denoted by $\text{LND}(A)$.

We associate to a derivation $\partial$ a set $\text{Nil}(\partial) = \{f \in A \mid \exists n \in \mathbb{Z}_{\geq 0} : \partial^n f = 0\}$. Thus, $\partial \in \text{LND}(A)$ exactly when $\text{Nil}(\partial) = A$.

Note that if $\partial_1, \partial_2 \in \text{LND}(A)$ and $[\partial_1, \partial_2] = 0$, then $\partial_1 + \partial_2 \in \text{LND}(A)$, i.e. the sum of commuting LNDs is an LND as well.

Recall that locally nilpotent derivations on an affine algebra $A$ are in one-to-one correspondence with regular actions of $\mathbb{G}_a(k)$ on $X = \text{Spec } A$. Indeed, we have a rational representation $\eta : \mathbb{G}_a(k) \to \text{Aut}_k(A)$, where $\eta(t) = \exp(tD)$. In geometric terms this means that $D$ induces a regular $\mathbb{G}_a(k)$-action on $X$. Conversely, let $\rho : \mathbb{G}_a(k) \times X \to X$ be a regular $\mathbb{G}_a(k)$-action. Then $\rho$ induces a locally nilpotent derivation $D = \frac{d}{dt}|_{t=0} \rho^*$, where $\rho^* : A \to A[t]$. For more detail see [6], Section 1.5.

2. $T$-varieties

An algebraic torus $T = \mathbb{T}^n$ of dimension $n$ is the algebraic variety $(k^*)^n$ with the natural structure of algebraic group. A $T$-variety is an algebraic variety endowed with an effective $T$-action.
A character (resp. one-parameter subgroup) of $T$ is a homomorphism of algebraic groups $\chi : T \to k^\times$ (resp. $\lambda : k^\times \to T$). The set of all characters (resp. one-parameter subgroups) form a lattice $M$ (resp. $N$) of rank $n$. For every $m \in M$ we denote by $\chi^m$ the corresponding character of $T$. We also let $M_Q$ and $N_Q$ be the rational vector spaces $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $A$ be an affine algebra, i.e. a commutative associative finitely generated $k$-algebra with unit and without zero divisors. It is well known that effective $T$-actions on an affine variety $X = \text{Spec } A$ are in one-to-one correspondence with effective $M$-gradings on $A$. Thus, for a $T$-variety $X$ we have an effective $M$-grading on $A = k[X]$:

$$A = \bigoplus_{m \in M} \widetilde{A}_m.$$  

Let $K = \text{Frac } A$ be the field of fractions. We consider $K_0 = \left\{ \frac{f}{g} \in K \mid \deg f = \deg g \right\}$. Notice that $K_0$ coincides with the field of $T$-invariant functions $k(X)^T$. Thus, we have a tower of field extensions $k \subseteq k(X)^T \subseteq K$. One may represent $\widetilde{A}_m = A_m\chi^m$, where $A_m \subseteq k(X)^T$. The weight cone $\omega \subseteq M_Q$ of a given $M$-grading is a cone in $M_Q$ spanned by the set $\{m \in M \mid A_m \neq 0\}$. For a cone $\omega \subseteq M_Q$ we denote the set $\omega \cap M$ by $\omega_M$. Finally, we have

$$A = \bigoplus_{m \in \omega_M} A_m\chi^m, \quad A_m \subseteq k(X)^T.$$  

Since $A$ is finitely generated, the cone $\omega$ is polyhedral, and since the $M$-grading is effective, $\omega$ is of full dimension.

Complexity of a $T$-action is the transcendence degree of the field of $T$-invariant rational functions $k(X)^T$ over $k$. In geometric terms, complexity of a $T$-action equals the codimension of the generic $T$-orbit. In particular, for a $T$-variety of complexity zero, $k(X)^T = k$ and $A \subseteq k[M]$, where $k[M]$ stays for the group algebra of the lattice $M$ and is isomorphic to the algebra of Laurent polynomials over $k$. A toric variety is a normal $T$-variety of complexity zero, or, equivalently, $T$ acts with an open orbit. From now on we will consider only normal $T$-varieties.

3. Demazure roots and homogeneous LNDs

Let $M$ and $N$ be the lattices of characters and one-parameter subgroups of a torus $T$. We consider the natural pairing $\langle , \rangle : M \times N \to \mathbb{Z}$ given by

$$\langle \chi, \lambda \rangle = l, \quad \text{if } \chi \circ \lambda(t) = t^l.$$  

This pairing extends in an obvious way to a pairing

$$\langle , \rangle : M_Q \times N_Q \to \mathbb{Q}$$  

between $\mathbb{Q}$-vector spaces. Let $A$ be as before an $M$-graded affine algebra with the weight cone $\omega \subseteq M_Q$. Let $\omega^\vee = \sigma \subseteq N_Q$ be the dual cone. Since $\omega$ is of full dimension, $\sigma$ is a pointed polyhedral cone.

**Lemma 1.** [8, Lemma 1.13] For any homogeneous LND $\partial$ on $A$ the following holds.

1) The derivation $\partial$ extends in a unique way to a homogeneous $k$-derivation on $k(X)^T[M]$. 

2) If $\partial(k(X)^T) = 0$ then the extension of $\partial$ as in 1) restricts to a homogeneous locally nilpotent $k(X)^T$-derivation on $k(X)^T[\omega_M]$.

**Definition 2.** A homogeneous LND $\partial$ on $A$ is said to be of fiber type if $\partial(k(X)^T) = 0$ and of horizontal type otherwise. Derivations of fiber type correspond to $G_a(k)$-actions such that generic $G_a(k)$-orbits are contained in the closures of $T$-orbits.

In [9] A.Liendo gave a complete classification of homogeneous LNDs of fiber type on an arbitrary graded integrally closed affine algebra. We need the following notation to present his results.

For a ray $\rho$ of a cone $\sigma$ we let $\sigma_\rho$ denote the cone spanned by all the rays of $\sigma$ except $\rho$. From now on we denote by $\rho$ both a ray and its primitive vector. We also let

$$S_\rho = \sigma_\rho^\vee \cap \{ e \in M \mid \langle e, \rho \rangle = -1 \}.$$ 

In other words, $S_\rho$ is the set of lattice vectors $e \in M$ such that $\langle e, \rho \rangle = -1$ and $\langle e, \rho' \rangle \geq 0$ for every other ray $\rho' \subseteq \sigma$.

**Definition 3.** (see [3]) In the above notation the elements of the set $\mathcal{R} = \bigcup_{\rho \subseteq \sigma} S_\rho$ are called Demazure roots of the cone $\sigma$.

Let $e \in S_\rho$ be a Demazure root corresponding to a ray $\rho$ of the cone $\sigma$. Set

$$\Phi_e = \{ f \in K \mid f \cdot A_m \subseteq A_{m+e} \}, \quad \Phi_e^\times = \Phi_e \setminus \{ 0 \}.$$ 

We define a homogeneous derivation $\partial_{\rho,e}$ of degree $e$ on $k(X)^T[M]$ by

$$\partial_{\rho,e}(\chi^m) = \langle m, \rho \rangle \chi^{m+e}.$$ 

Clearly, for $f \in \Phi_e$ the product $f \partial_{\rho,e}$ is a derivation on $k(X)^T[\omega_M]$. Finally, denote $\partial_{\rho,e,f} = f \partial_{\rho,e}|_A$, which is a homogeneous LND on $A$. The classification theorem is as follows.

**Theorem 1.** [9 Theorem 2.4] Every nonzero homogeneous LND $\partial$ of fiber type on $A$ is of the form $\partial = \partial_{\rho,e,f}$ for some ray $\rho \subseteq \sigma$, some Demazure root $e \in S_\rho$ and some function $f \in \Phi_e^\times$.

In particular, for any homogeneous locally nilpotent derivation $\partial$ of fiber type we have $\deg \partial \notin \omega_M$.

**Remark 1.** For a toric variety $X$ the closure of a dense $T$-orbit coincides with $X$, hence all derivations on $A$ are of fiber type. In this particular case Theorem [9] states that all nonzero homogeneous LNDs on $A = k[X]$ are of the form $\partial = \lambda \partial_{\rho,e}$, for some $\lambda \in k^\times$.

**Example 1.** With $N = \mathbb{Z}^n$ we let $\sigma$ be the cone in $N_\mathbb{Q}$ spanned by the basic unit vectors $e_i$. The dual cone $\omega \subseteq M_\mathbb{Q}$ is spanned by the basic unit vectors in $M_\mathbb{Q}$ as well. The corresponding semigroup algebra is a polynomial algebra $A = k[x_1, \ldots, x_n]$, $\deg x_i = e_i$, and the affine toric variety is $X = \mathbb{A}^n$. According to Theorem [9] all nonzero homogeneous LNDs on $A$ are:

$$\partial = \lambda x_1^{i_1} \ldots x_k^{i_k} \ldots x_n^{i_n} \frac{\partial}{\partial x_k}, \quad \lambda \in k^\times, \quad i_1, \ldots, i_n \in \mathbb{Z}_{\geq 0} \quad (k = 1, \ldots, n).$$

Degree of the preceding derivation is $(i_1, \ldots, -1, \ldots, i_n)$, which is a Demazure root of $\sigma$ corresponding to the ray $\rho_k$. 
4. Commutators of two homogeneous LNDs

Suppose \( \partial_1, \partial_2 \in \text{LND}(A) \) are nonzero homogeneous derivations of fiber type. It follows from Theorem 1 that they are given by

\[
\partial_1(\chi^m) = f_1(m, \rho_1)\chi^{m+e_1}, \quad \partial_2(\chi^m) = f_2(m, \rho_2)\chi^{m+e_2},
\]

where \( e_1, e_2 \in \mathbb{R} \) and \( f_1 \in \Phi_{e_1}^\times, f_2 \in \Phi_{e_2}^\times \). According to the definition, we have

\[
[\partial_1, \partial_2](\chi^m) = f_1 f_2 (m + e_2, \rho_1) (m, \rho_2) - (m + e_1, \rho_2) (m, \rho_1) \chi^{m+e_1+e_2} = f_1 f_2 (e_2, \rho_1) (m, \rho_2) - (e_1, \rho_2) (m, \rho_1) \chi^{m+e_1+e_2}.
\]

**Lemma 2.** If \( \partial_1, \partial_2 \in \text{LND}(A) \) correspond to the same ray of the cone \( \sigma \), then \( \partial_1 \) and \( \partial_2 \) commute.

**Proof.** Let \( \rho_1 = \rho_2 = \rho \). Then \( (e_2, \rho) (m, \rho) - (e_1, \rho) (m, \rho) = (1)(m, \rho) - (1)(m, \rho) = 0 \), and hence the commutator equals zero. \( \square \)

**Proposition 1.** Suppose \( \partial_1, \partial_2 \in \text{LND}(A) \) correspond to different rays \( \rho_1, \rho_2 \) of the cone \( \sigma \).

Then the following holds:

1. \( [\partial_1, \partial_2] = 0 \Leftrightarrow \langle e_1, \rho_2 \rangle = 0 \) and \( \langle e_2, \rho_1 \rangle = 0 \),
2. \( [\partial_1, \partial_2] \in \text{LND}(A) \Leftrightarrow e_1 + e_2 \notin \omega_M \), i.e. \( \langle e_1, \rho_2 \rangle = 0 \) or \( \langle e_2, \rho_1 \rangle = 0 \).

**Proof.**
1. Let \( [\partial_1, \partial_2] = 0 \). Then for all \( m \in \omega_M \) we have \( \langle e_2, \rho_1 \rangle \langle m, \rho_2 \rangle = \langle e_1, \rho_2 \rangle \langle m, \rho_1 \rangle \). One can choose \( m \in \omega_M \) such that \( \langle m, \rho_1 \rangle = 0 \) and \( \langle m, \rho_2 \rangle > 0 \). In this case \( \langle e_2, \rho_1 \rangle = 0 \) implies that \( \langle e_2, \rho_1 \rangle = 0 \). Similarly \( \langle e_1, \rho_2 \rangle = 0 \). The inverse implication holds automatically.

2. Let us prove necessity. Without loss of generality assume \( \langle e_1, \rho_2 \rangle = 0 \), \( \langle e_2, \rho_1 \rangle > 0 \). Then \( \langle e_1 + e_2, \rho_2 \rangle = -1, \langle e_1 + e_2, \rho_1 \rangle > 0 \). Therefore \( e_1 + e_2 \) is a Demazure root corresponding to the ray \( \rho_2 \) of the cone \( \sigma \) and \( [\partial_1, \partial_2] = f \partial_{\rho_2, e_1 + e_2} \), where \( f = f_1 f_2 (e_2, \rho_1) \in \Phi_{e_1 + e_2}^\times \).

To prove necessity assume that the commutator is an LND. Since its degree equals \( e_1 + e_2 \), there exists a ray \( \rho^* \) of the cone \( \sigma \) such that \( \langle e_1 + e_2, \rho^* \rangle = -1 \). As \( e_1, e_2 \in \sigma_{\rho^*}^\times \) for \( \rho \neq \rho_1, \rho_2 \), one obtains \( \rho^* = \rho_1 \) or \( \rho_2 \). This proves the assertion. \( \square \)

**Remark 2.** In general, if \( \partial \) is a homogeneous derivation of a graded algebra \( A = \bigoplus_{m \in \omega_M} \widetilde{A}_m \), and \( \deg \partial \notin \omega_M \), then \( \partial \) is locally nilpotent. Indeed, for any \( m \in \omega_M \) one can find \( k \in \mathbb{Z}_{\geq 1} \) such that \( (m + k \deg \partial) \notin \omega_M \) and, thus, \( \partial^k(\widetilde{A}_m) = 0 \).

5. Sums of two homogeneous LNDs

In this section we establish a necessary and sufficient condition of local nilpotency of the sum of two homogeneous LNDs. Consider two nonzero homogeneous LNDs \( \partial_1, \partial_2 \) of fiber type:

\[
\partial_1(\chi^m) = f_1(m, \rho_1)\chi^{m+e_1}, \quad \partial_2(\chi^m) = f_2(m, \rho_2)\chi^{m+e_2},
\]

where \( e_1, e_2 \in \mathbb{R} \) and \( f_1 \in \Phi_{e_1}^\times, f_2 \in \Phi_{e_2}^\times \). Note that their sum is homogeneous if and only if both \( \partial_1 \) and \( \partial_2 \) are of the same degree, i.e. \( e_1 = e_2 \).

**Theorem 2.** In the above notation let \( \partial_1 \) and \( \partial_2 \) be homogeneous LNDs of fiber type. Then \( \partial_1 + \partial_2 \in \text{LND}(A) \) if and only if \( e_1 + e_2 \notin \omega_M \).
Proof. Let us prove sufficiency. The condition $e_1 + e_2 \notin \omega_M$ means that either both $\partial_1$ and $\partial_2$ correspond to the same ray, or $\langle e_2, \rho_1 \rangle = 0$ (up to permutation of indices). In the first case according to Lemma 2 our derivations commute and hence their sum is an LND.

Let us consider the second case. It suffices to show that for any $m \in \omega_M$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that $(\partial_1 + \partial_2)^n(\chi^m) = 0$. We proceed by induction on the parameter $\langle m, \rho_1 \rangle$.

Let $\langle m, \rho_1 \rangle = 0$. This means that $m \in \rho_1^+ \cap \omega_M$ and $(\partial_1 + \partial_2)(\chi^m) = \partial_2(\chi^m)$. In addition, the equality $\langle e_2, \rho_1 \rangle = 0$ implies that $m + e_2 \in \rho_1^+ \cap \omega_M$. Continuing these arguments and using local nilpotency of $\partial_2$, we obtain the required condition.

Now consider an arbitrary point $m \in \omega_M$. Since $\partial_2 \in \text{LND}(A)$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\partial_2^n(\chi^m) = 0$. Therefore all the images of $m$ under powers of $\partial_2$, namely $m, m + e_2, \ldots, m + (n - 1)e_2$, lie in the hyperplane $H$ perpendicular to $\rho_1$ and given by the equation $\langle x, \rho_1 \rangle = \langle m, \rho_1 \rangle$. Now we apply $(\partial_1 + \partial_2)^n$ to $\chi^m$. One can easily see that applying $\partial_1$ to an arbitrary element of the hyperplane $H$, the image will be in the hyperplane given by the equation $\langle x, \rho_1 \rangle = \langle m, \rho_1 \rangle - 1$. Since each summand (except for $\partial_2^n$) in $(\partial_1 + \partial_2)^n$ contains $\partial_1$ and $\partial_2^n(\chi^m) = 0$ and using the inductive hypothesis, we obtain local nilpotency of the sum $\partial_1 + \partial_2$.

Now let us prove the inverse implication. If $\text{rk}M = 1$, then $\omega_M = \mathbb{Z}$ or $\mathbb{Z}_{\geq 0}$ and the dual cone $\sigma$ is 0 or $\mathbb{Q}_{\geq 0}$ respectively. In the first case $\sigma$ contains no ray, thus there are no derivations of fiber type on $k[X]$. In the second case $\sigma$ consists of one ray and the condition $e_1 + e_2 \notin \omega_M$ follows immediately.

The case $\text{rk}M = 2$ follows from results of P. Kotenkova, see [7]. We recall some notation and results from this article. Consider a one-dimensional subtorus $T \subset \mathbb{T}$ given by the equation $e_1 - e_2 = 0$. The torus $T$ also acts on $X$ and every $\mathbb{T}$-homogeneous LND on $k[X]$ is $T$-homogeneous as well. Recall that a $\mathbb{T}$-root of an $M$-graded algebra $A$ is the degree of some homogeneous LND on $A$. Any $\mathbb{T}$-root of $A = k[X]$ can be restricted to some $T$-root and we denote by $\pi$ the restriction map. Denote by $\Gamma_T \subseteq \mathbb{Q}_0^+$ the hyperplane, corresponding to the subtorus $T$. Let $\langle \cdot, m_T \rangle = 0$ be the equation of the hyperplane $\Gamma_T$, where $m_T \in M$. Evidently, the locally nilpotent derivation $\partial_1 + \partial_2$ restricts to a homogeneous LND with respect to the torus $T$. Our case corresponds to Case 3.3 in [7, Proposition 6]. It follows that all $T$-homogeneous LNDs of degree $\pi(e_1)$ are of the following form (see [7, Proposition 5]):

$$\partial(\chi^m) = \chi^{m+e_2}(\alpha\langle \rho_1, m \rangle \chi^{m_T} + \beta\langle \rho_2, m \rangle)(\alpha \chi^{m_T} + \beta \langle \rho_2, m_T \rangle)^{\langle \rho_1, e_2 \rangle},$$

for some $\alpha, \beta \in k$. Now one can easily see that for $\partial = \partial_1 + \partial_2$ the exponent $\langle \rho_1, e_2 \rangle$ has to vanish, so $\langle \rho_1, e_1 + e_2 \rangle = -1$ and $e_1 + e_2 \notin \omega_M$.

Let us consider the general case $\text{rk}M = n$.

Lemma 3. Denote by $\bar{k}$ the algebraic closure of a field $k$. The derivation $\partial$ on a $k$-algebra $A$ extends in a unique way to a derivation $\bar{\partial}$ on $A \otimes_k \bar{k}$. In addition, $\partial$ is locally nilpotent if and only if $\bar{\partial}$ is.

Proof. To prove the first assertion of the lemma we let $x \in \bar{k}$ and assume that $p(t)$ is the minimal polynomial of $x$. Applying $\bar{\partial}$ to the equation $p(x) = 0$, we obtain $\bar{\partial}(x)p'(x) = 0$, hence $\bar{\partial}(x) = 0$. Further, an extension of $\partial$ to $A \otimes_k \bar{k}$ by linearity is unique. Since $k \subset \text{Ker}(\bar{\partial})$, local nilpotency of $\bar{\partial}$ and $\partial$ are equivalent. 

According to the previous lemma we can replace $k(X)^T$ with its algebraic closure $\overline{k(X)^T}$. 


It is required to show that $e_1 + e_2 \notin \omega_M$. If the roots $e_1, e_2$ correspond to the same ray, the condition holds automatically. If $e_1, e_2$ correspond to different rays $\rho_1$ and $\rho_2$, the required condition is equivalent to the following one: at least one of expressions $\langle e_1, \rho_2 \rangle$ or $\langle e_2, \rho_1 \rangle$ vanishes. We carry out the proof by contradiction.

We assume that the vectors $e_1$ and $e_2$ are not collinear. In this case for any $m \in \omega_M$ we denote by $\gamma_m$ a two-dimensional plane passing through $m$ and spanned by vectors $e_1, e_2$. It is obvious that images of $m$ under powers of the derivation $\partial_1 + \partial_2$ are in $\omega_M \cap \gamma_m$. It follows from the formulae defining $\partial_1$ and $\partial_2$ that $\gamma_m \cap \rho_1^\perp \neq \emptyset$ and $\gamma_m \cap \rho_2^\perp \neq \emptyset$. If \( \gamma_m \subset \rho_1^\perp \) then $\langle e_2, \rho_1 \rangle = 0$ and similarly for another permutation of the indices. Otherwise $\dim (\gamma_m \cap \rho_1^\perp) = \dim (\gamma_m \cap \rho_2^\perp) = 1$.

Now we assume that $m = e_1 + e_2$. Denote the lines in which the plane $\gamma_m$ intersects $\rho_1^\perp$ and $\rho_2^\perp$ by $l_1$ and $l_2$ respectively. It follows from local nilpotency of $\partial_1$, $\partial_2$ that there exist $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that

$$m_1 = e_1 + e_2 + k_1 e_1 \in \rho_1^\perp \cap \gamma_m,
\quad m_2 = e_1 + e_2 + k_2 e_2 \in \rho_2^\perp \cap \gamma_m.$$ 

Obviously, $0 \in \rho_1^\perp \cap \rho_2^\perp \cap \gamma_m$. Thus, $l_1$ and $l_2$ are precisely the lines passing through 0 and $m_1$ or $m_2$ respectively. If $l_1$ and $l_2$ are different, we set $\omega_m = \gamma_m \cap \omega$. The two-dimensional cone $\sigma_m$ dual to $\omega_m$ can be considered as embedded in $N_\mathbb{Q}$ and spanned by the rays $\rho_1, \rho_2$. One can easily see that $e_1$ and $e_2$ are Demazure roots of $\sigma_m$. A locally nilpotent derivation on $\mathbf{k}(X)^\mathbb{T}[\omega_M]$ can be restricted to an LND on $\mathbf{k}(X)^\mathbb{T}[\omega_m \cap M]$, hence using the assertion of the theorem in case of dimension two we obtain $e_1 + e_2 \notin \omega_m \cap M$, a contradiction.

If the lines $l_1$ and $l_2$ coincide, then the vectors $e_1 + e_2 + k_1 e_1$ and $e_1 + e_2 + k_2 e_2$ are collinear. This means the following:

$$\frac{k_1 + 1}{k_2 + 1} = 1 \Leftrightarrow (k_1 + 1)(k_2 + 1) = 1 \Leftrightarrow k_1 = k_2 = 0.$$

Thus, $e_1 + e_2 \in \rho_1^\perp \cap \rho_2^\perp$ and $\gamma_m \cap \omega$ is the line spanned by $e_1 + e_2$. Indeed, let $\langle ae_1 + be_2, \rho_1 \rangle \geq 0$. Using that $\langle e_1, \rho_1 \rangle = -1$, $\langle e_2, \rho_1 \rangle = 1$ one obtains $a \geq b$. Similarly we obtain $a \geq b$ and, hence, $a = b$. Consider now an arbitrary element $s \in \omega_M^0 = \text{Int}(\omega_M)$ and the corresponding plane $\gamma_s$. Since $\gamma_s \parallel \gamma_m$, the plane $\gamma_s$ intersects both $\rho_1^\perp$ and $\rho_2^\perp$ in parallel lines $v_1$ and $v_2$ with leading vector $e_1 + e_2$. Our aim now is to construct $f \in \mathbf{k}(X)^\mathbb{T}[\omega_M]$ such that $f \notin \text{Nil}(\partial_1 + \partial_2)$.

Consider a two-dimensional plane containing the lines $v_1$ and $v_2$, and its intersection with the cone $\omega_M$. We obtain a stripe-shaped diagram, see Figure 1.

One can also consider a lattice $S$ passing through $s$ with generating vectors $e_1, e_2$. Since $\langle e_i, \rho_j \rangle = (-1)^{i+j-1}$ ($i, j = 1, 2$), the derivations $\partial_1, \partial_2$ send elements of the lattice $S$ one level up or down respectively. Here levels are of the form \{ $x \in S \mid \langle x, \rho_2 \rangle = \alpha$ \}, $\alpha = 0, \ldots, k$. In particular, 0-level lies on $v_2$ and $k$-level lies on $v_1$. Consider elements $m_1, \ldots, m_r \in S$ as shown on Figure 1. Note that Figure 1 represents the case of even $k$, for odd $k$ arguments are similar. Let us prove that there exist coefficients $a_1, \ldots, a_r \in \mathbf{k}(X)^\mathbb{T}$ such that

$$a_1 \chi^{m_1} + \cdots + a_r \chi^{m_r} \notin \text{Nil}(\partial_1 + \partial_2).$$

Indeed, we apply $(\partial_1 + \partial_2)^2$ to an element with yet undetermined coefficients and write the column vector of coefficients at $\chi^{m_1+e_1+e_2}, \ldots, \chi^{m_r+e_1+e_2}$ after decomposition into graded components:
We try to find such values $a_1,\ldots,a_r$, that these two vectors are proportional. In other words, $(a_1,\ldots,a_r)^T$ has to be an eigenvector of the following three-diagonal matrix $A$:

$$
\begin{pmatrix}
a_1 & a_1 \cdot f_1 f_2 \cdot k \cdot 1 + a_2 \cdot f_2^2 \cdot 2 \cdot 1 \\
a_2 & a_2 \cdot f_1 f_2 \cdot ((k - 2) \cdot 3 + 2 \cdot (k - 1)) + a_3 \cdot f_1^2 \cdot 4 \cdot 3 + a_1 \cdot f_2^2 \cdot k \cdot (k - 1) \\
a_3 & a_3 \cdot f_1 f_2 \cdot ((k - 4) \cdot 5 + 4 \cdot (k - 3)) + a_4 \cdot f_1^2 \cdot 6 \cdot 5 + a_2 \cdot f_2^2 \cdot (k - 2) \cdot (k - 3) \\
\vdots & \vdots \\
a_{r-1} & a_{r-1} \cdot f_1 f_2 \cdot (2 \cdot (k - 1) + (k - 2) \cdot 3) + a_r \cdot f_1^2 \cdot k \cdot (k - 1) + a_{r-2} \cdot f_2^2 \cdot 4 \cdot 3 \\
a_r & a_r \cdot f_1 f_2 \cdot k \cdot 1 + a_{r-1} \cdot f_2^2 \cdot 2 \cdot 1 \\
\end{pmatrix}
$$

Therefore, it suffices to show that the matrix $A$ possesses a nonzero eigenvalue $\lambda$, because otherwise applying even powers of derivation $\partial_1 + \partial_2$ we obtain the following sequence:

$$
\sum_{i=1}^{r} a_i \chi^{m_i} \rightarrow \lambda \left( \sum_{i=1}^{r} a_i \chi^{m_i+e_1+e_2} \right) \rightarrow \lambda^2 \left( \sum_{i=1}^{r} a_i \chi^{m_i+2(e_1+e_2)} \right) \rightarrow \ldots,
$$

whose members do not vanish and thus $\sum_{i=1}^{r} a_i \chi^{m_i} \notin \text{Nil}(\partial_1 + \partial_2)$. We write the characteristic polynomial of $A$:

$$
\chi(\lambda) = \det(\lambda E - A) = \lambda^r - \text{tr}A\lambda^{r-1} + \cdots + (-1)^r \det A
$$

and compute the coefficient at $\lambda^{r-1}$. One can easily see that $\text{tr}A$ is a product of two factors, where the first factor is $f_1 f_2$ and the second one is a sum of positive integers. More precisely,
\[ \text{tr}A = f_1 f_2 \sum_{j=1}^{k} j(k+1-j) = f_1 f_2 \left( (k+1) \frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6} \right) = f_1 f_2 \frac{k(k+1)(k+2)}{6} \neq 0. \]

Hence, \( \text{tr}A \) does not vanish and \( \chi(\lambda) \neq \lambda^t \). Therefore we obtain existence of a nonzero \( \lambda \in k(X)^T \) with \( \chi(\lambda) = 0 \).

Now let \( e_1 \) and \( e_2 \) be collinear, i.e. \( e_1 = te_2, \ t \in \mathbb{Q}^* \). Combining the conditions
\[ \langle e_1, \rho_2 \rangle = \langle te_2, \rho_2 \rangle = -t \Rightarrow t \in \mathbb{Z}_{<0} \]
and
\[ \langle e_2, \rho_1 \rangle = \left\langle \frac{e_1}{t}, \rho_1 \right\rangle = -\frac{1}{t} \Rightarrow \frac{1}{t} \in \mathbb{Z}_{<0}, \]
we see that \( t = -1 \) and \( e_1 = -e_2 \). The construction from the previous case allows us to obtain an element of the semigroup algebra \( k(X)^T[\omega_M] \) that does not belong to \( \text{Nil}(\partial_1 + \partial_2) \). The proof is completed. \( \square \)

6. Concluding remarks

The next corollary follows directly from Lemma 2 Propostion 1 and Theorem 2

Corollary 1. Let \( \partial_1 \) and \( \partial_2 \) be two nonzero homogeneous LNDs on \( A \) of fiber type. Then \( \partial_1 + \partial_2 \) is a locally nilpotent derivation if and only if \( [\partial_1, \partial_2] \) is.

Example 2. Let us illustrate the obtained results for \( A = k[x_1, \ldots, x_n] \) in the settings of Example 1 Using for simplicity multiindices \( (x^l = x_1^{i_1} \ldots x_n^{i_n}) \), we set
\[ \partial_1 = \lambda_1 x^l \frac{\partial}{\partial x_i}, \quad \partial_2 = \lambda_2 x^j \frac{\partial}{\partial x_j}, \]
where \( x^l \) and \( x^j \) do not contain factors \( x_i \) and \( x_j \) respectively. Then the following conditions hold:

1) \( [\partial_1, \partial_2] = 0 \iff x^l \) does not depend on \( x_j \) and \( x^j \) does not depend on \( x_i \);
2) \( [\partial_1, \partial_2] \in \text{LND}(A) \iff \partial_1 + \partial_2 \in \text{LND}(A) \iff x^l \) does not depend on \( x_j \) or \( x^j \) does not depend on \( x_i \).

Note that in the settings of Example 2 Theorem 2 can be easily obtained from the following proposition.

Proposition 2. [5 Principle 5] Let \( \partial \in \text{LND}(A) \) and \( f_1, \ldots, f_m \in A \ (m \geq 1) \). Suppose there exists a permutation \( \sigma \in S_m \) such that \( \partial f_i \in f_{\sigma(i)}A \) for each \( i \). Then in each orbit of \( \sigma \) there is an index \( i \) with \( \partial f_i = 0 \).

Indeed, suppose \( \partial = \partial_1 + \partial_2 \in \text{LND}(A) \). We take \( f_1 = x_i, \ f_2 = x_j \). Then \( \partial f_1 = \lambda_1 x^l \neq 0 \) and \( \partial f_2 = \lambda_2 x^j \neq 0 \). Using Proposition 2 we obtain the required condition. The converse is immediate.

Corollary 2. Let \( \partial_1 \) and \( \partial_2 \) be two nonzero homogeneous LNDs on \( A \) of fiber type. Suppose \( e_1 + e_2 \notin \omega_M \). Then the Lie algebra \( L \) generated by \( \partial_1 \) and \( \partial_2 \) over \( k \) is finite dimensional and consists of locally nilpotent derivations.
Proof. We can assume without loss of generality that \( \langle e_1, \rho_2 \rangle = 0 \) and \( \langle e_2, \rho_1 \rangle = n \geq 0 \). Then the algebra \( L \) is linearly generated by the derivations \( \partial_1, \partial_2 \) and \( \partial_2^{(1)}, \ldots, \partial_2^{(n)} \), where \( \partial_2^{(i)} = \text{ad}(\partial_1)^i \partial_2 \). So the dimension of \( L \) equals to \( n + 2 \). Note also that \( \partial_2^{(i)}, i = 1, \ldots, n, \) are locally nilpotent derivations of degree \( e_2 + ie_1 \) respectively, which are Demazure roots corresponding to the ray \( \rho_2 \). Let us assume that a derivation \( \partial \) in \( L \) is a linear combination of \( \partial_2, \partial_2^{(1)}, \ldots, \partial_2^{(n)} \). Then \( \partial \) is locally nilpotent because all these LNDs commute. If \( \partial = \partial_1 + \lambda_0 \partial_2 + \lambda_1 \partial_2^{(1)} + \cdots + \lambda_n \partial_2^{(n)} \), the proof is similar to the proof of sufficiency in Theorem 2 \( \square \)

At the moment the author does not know whether the condition \( \deg \partial_1 + \deg \partial_2 \not\in \omega_M \) implies that \( \partial_1 + \partial_2 \in \text{LND}(A) \) for locally nilpotent derivations \( \partial_1, \partial_2 \) of horizontal type. Let us give a particular result in this direction.

Proposition 3. Consider an effectively graded affine algebra

\[
A = \bigoplus_{m \in \omega_M} \tilde{A}_m.
\]

Let \( \partial_1 \) and \( \partial_2 \) be two locally nilpotent derivations on \( A \) of degrees \( e_1, e_2 \in M \) respectively. If \( \{te_1 + (1 - t)e_2, \ t \in \mathbb{Q}\} \cap \omega_M = \emptyset \), then \( \partial_1 + \partial_2 \in \text{LND}(A) \).

Proof. Let us consider the grading of \( A \) given by the quotient group \( M' = M/(e_1 - e_2) \), and denote by \( \pi : M \rightarrow M' \) the factorization map. Derivations \( \partial_1 \) and \( \partial_2 \) considered as derivations on \( M' \)-graded algebra are homogeneous of the same degree \( \pi(e_1) = \pi(e_2) \). Hence, the sum \( \partial_1 + \partial_2 \) is \( M' \)-homogeneous of degree \( \pi(e_1) \) as well.

The condition \( \{te_1 + (1 - t)e_2, \ t \in \mathbb{Q}\} \cap \omega_M = \emptyset \) means that the line passing through the lattice points \( e_1, e_2 \in M \) does not intersect the cone \( \omega_M \). This implies that \( \pi(e_1) \not\in \pi(\omega_M) \).

Summarizing these facts, we see that \( \partial \) is a homogeneous derivation on \( M' \)-graded algebra \( A \) and its degree \( \pi(e_1) \) does not lie in the weight cone \( \pi(\omega_M) \). By Remark 2 this is sufficient for the local nilpotency of \( \partial \). \( \square \)

Unlike the case of derivations of fiber type another implication of Theorem 2 does not hold for derivations of horizontal type.

Example 3. Consider a \( T \)-variety \( \mathbb{A}^2 \) with the following action of a one-dimensional torus \( T \): \( t \cdot (x, y) = (tx, ty) \). The algebra of regular functions \( A = k[x, y] \) is graded by the weight cone \( \omega_M = \mathbb{Z}_{\geq 0} \) and \( \deg x = \deg y = 1 \). Moreover, \( k(X)^T = k \left( \frac{x}{y} \right) \).

Consider a homogeneous locally nilpotent derivation \( \partial = y \frac{\partial}{\partial x} \) on \( A \). Note that \( \partial \left( \frac{x}{y} \right) = 1 \), hence \( \partial \) is of horizontal type. Let us show that two copies of the derivation \( \partial \) give a counterexample to Theorem 2. Condition for commutators given in Proposition 1 for derivations of fiber type also does not hold. Indeed, note that \( \deg \partial = 0 \in \omega_M \). Taking \( \partial_1 = \partial_2 = \partial \), we obtain \( \deg \partial_1 + \deg \partial_2 = 2 \in \omega_M \), though \( \partial_1 + \partial_2 = 2\partial \in \text{LND}(A) \) and \( [\partial_1, \partial_2] = [\partial, \partial] = 0 \in \text{LND}(A) \).

Finally, we would like to discuss torus actions and homogeneous LNDs coming from actions of reductive algebraic groups, cf. [1].
Let $G$ be a reductive algebraic group and $\mathfrak{g} = \text{Lie } G$. We consider a maximal torus $\mathbb{T} \subset G$ and $\mathfrak{h} = \text{Lie } \mathbb{T}$. The reductive Lie algebra $\mathfrak{g}$ admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\Delta \subset \mathfrak{h}^*$ is the system of roots and $\mathfrak{g}_\alpha = \langle e_\alpha \rangle$.

Starting with a regular action of $G$ on an affine variety $X$ we obtain the $\mathbb{T}$-action on $X$ and the corresponding grading of $A = k[X]$ by the lattice $M$ of characters of the torus $\mathbb{T}$. For every $\alpha \in \Delta$, the nilpotent element $e_\alpha \in \mathfrak{g}$ defines a homogeneous locally nilpotent derivation $\partial_\alpha$ on $k[X]$ of degree $\alpha$.

**Lemma 4.** Let $\alpha, \beta \in \Delta$. Then $e_\alpha + e_\beta$ is nilpotent if and only if $\alpha + \beta \neq 0$.

**Proof.** Assume $\beta = -\alpha$. In this case $e_\alpha$, $e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}]$ form an $\mathfrak{sl}_2$-triple and, thus, $e_\alpha + e_{-\alpha}$ is semisimple.

For $\alpha \neq -\beta$ we can consider a hyperplane in $\mathfrak{h}^*$ such that $\alpha$ and $\beta$ are contained in the same open half-space. This hyperplane determines a one-dimensional torus $T$, and the roots $\alpha$ and $\beta$ are positive with respect to $T$. Looking at the weights occuring in the decomposition of the element $\text{ad}(e_\alpha + e_\beta)^N(x)$ for some homogeneous $x \in \mathfrak{g}$ we obtain that $e_\alpha + e_\beta$ is nilpotent. □

Using this lemma and the result of Theorem 2 we obtain the following

**Corollary 3.** Assume that $\alpha + \beta \neq 0$. If $\partial_\alpha$ and $\partial_\beta$ are of fiber type and $\partial_\alpha + \partial_\beta \in \text{LND}(A)$, then $\alpha + \beta$ does not belong to the weight cone $\omega_M$ corresponding to the action of the maximal torus $\mathbb{T}$ in $G$ on $A$.

**Example 4.** Consider a natural $GL_n(k)$-action on the affine space $A^n$. The affine space $A^n$ gets a structure of a toric variety under the action of the maximal torus

$$\mathbb{T}^n = \{\text{diag}(t_1, \ldots, t_n)\} \subseteq GL_n(k).$$

The weights of $x_1, \ldots, x_n$ are $\varepsilon_1, \ldots, \varepsilon_n$ respectively. A root $\alpha$ determines an LND $\partial_\alpha$:

$$\alpha = \varepsilon_i - \varepsilon_j, i \neq j \quad \leadsto \quad \partial_\alpha = x_i \frac{\partial}{\partial x_j}.$$ 

It is easy to see that

$$\partial_\alpha + \partial_\beta = x_i \frac{\partial}{\partial x_j} + x_k \frac{\partial}{\partial x_l} \in \text{LND}(A) \quad \iff \quad (i, j) \neq (l, k) \quad \iff \quad \alpha \neq -\beta.$$

Moreover, $\alpha + \beta = \varepsilon_i - \varepsilon_j + \varepsilon_k - \varepsilon_l \notin \omega_M = \mathbb{Z}_{\geq 0}^n$ for $(i, j) \neq (l, k)$. This illustrates Corollary 3.

**Example 5.** Consider a natural $SL_n(k)$-action on the affine space $A^n$. The maximal torus

$$\mathbb{T}^{n-1} = \{\text{diag}(t_1, \ldots, t_n) \mid \prod_i t_i = 1\} \subseteq SL_n(k)$$

acts on $A^n$ with complexity one. The weights of $x_1, \ldots, x_n$ are $\varepsilon_1, \ldots, \varepsilon_{n-1}$ and $-\varepsilon_1 - \cdots - \varepsilon_{n-1}$ respectively, thus, $\omega_M = M$. Therefore, there exist no locally nilpotent derivations of fiber type. Taking $\partial_\alpha + \partial_\beta$ with $\alpha + \beta \neq 0$ we obtain one more counterexample to Theorem 2 in case of derivations of horizontal type.
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