BIHARMONIC HYPERSURFACES IN A CONFORMALLY FLAT SPACE

LIANG TANG AND YE-LIN OU *

Abstract

Biharmonic hypersurfaces in a generic conformally flat space are studied in this paper. The equation of such hypersurfaces is derived and is used to determine the conformally flat metric $f^{-2}\delta_{ij}$ on the Euclidean space $\mathbb{R}^{m+1}$ so that a minimal hypersurface $M^m \rightarrow (\mathbb{R}^{m+1}, \delta_{ij})$ in a Euclidean space becomes a biharmonic hypersurface $M^m \rightarrow (\mathbb{R}^{m+1}, f^{-2}\delta_{ij})$ in the conformally flat space. Our examples include all biharmonic hypersurfaces found in [Ou1] and [OT] as special cases.

1. Biharmonic maps and submanifolds

All manifolds, maps, and tensor fields studied in this paper are assumed to be smooth.

A map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is called a biharmonic map if it is a critical point of the bienergy functional

$$E^2(\varphi, \Omega) = \frac{1}{2} \int_\Omega |\tau(\varphi)|^2 \, dx$$

for every compact subset $\Omega$ of $M$, where $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ is the tension field of $\varphi$. The biharmonic map equation is the Euler-Lagrange equation of this functional which can be written as (see [Ji1])

$$\tau^2(\varphi) := \text{Trace}_g (\nabla^\varphi \nabla \varphi - \nabla^\varphi_{\nabla \varphi}) \tau(\varphi) - \text{Trace}_g R^N(d\varphi, \tau(\varphi)) d\varphi = 0,$$

where $R^N$ denotes the curvature operator of $(N, h)$ defined by

$$R^N(X, Y) Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

*Research supported by NSF of Guangxi (P. R. China), 2011GXNSFA018127.
A submanifold $M$ of $(N, h)$ is called a **biharmonic submanifold** if its inclusion map $i: (M, i^*h) \rightarrow (N, h)$ is a biharmonic isometric immersion.

From the well-known fact that a harmonic map is a map between Riemannian manifold whose tension field $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ vanishes identically and that an isometric immersion is minimal if and only if it is harmonic we have the following relationships:

\[
\{\text{Harmonic maps}\} \subset \{\text{Biharmonic maps}\},
\]

\[
\{\text{Minimal submanifolds}\} \subset \{\text{Biharmonic submanifolds}\}.
\]

These relationships justify our using the names *proper biharmonic maps* for those biharmonic maps which are not harmonic and *proper biharmonic submanifolds* for those biharmonic submanifolds which are not minimal.

Among the interesting problems in the study of biharmonic submanifolds are the following two conjectures.

**Chen’s conjecture** (see e.g., [Ch1], [CI], [HV], [Di], [CMO1], [Ch2], [NU] and the references therein): any biharmonic submanifold in a Euclidean space is minimal.

**The generalized Chen’s conjecture**: any biharmonic submanifold of $(N, h)$ with $\text{Riem}^N \leq 0$ is minimal (see e.g., [CMO1], [BMO1], [BMO2], [IU], [NU], [Ch2], [OT] and the references therein).

While Chen’s conjecture is still open the generalized Chen’s conjecture has been proved to be false in the authors recent paper [OT]. The main idea in solving the generalized Chen’s conjecture is to construct a proper biharmonic hypersurface in a 5-dimensional conformally flat space with negative sectional curvature. This and the biharmonicity of the product maps (see [Ou2]) are used to construct many examples of proper biharmonic submanifolds in a nonpositively curved manifolds.

A Riemannian manifold $(M^m, g)$ is called a conformally flat space if for any point of $M$ there exists a neighborhood which is conformally diffeomorphic to an open subset of the Euclidean space $\mathbb{R}^m$. More precisely, $\forall p \in M$, there exists a neighborhood $U$, $p \in U \subset M$ and a diffeomorphism $\varphi: (U, g) \rightarrow \varphi(U) \subset (\mathbb{R}^m, h)$, such that $\varphi^*h = e^{2\alpha}g$, where $h$ denotes the standard Euclidean metric on $\mathbb{R}^m$. The following facts are well known:

- any two-dimensional Riemannian manifold is conformally flat;
In the context of biharmonic submanifolds, there has been a growing study on biharmonic submanifolds in space forms (see [Ch1], [Ch2], [CI], [Ji1], [Ji2], [CMO1], [CMO2], [BMO1], [CMO2], [Di], [HV], and the references therein) in recent years. Some interesting examples of biharmonic hypersurfaces in some special conformally flat spaces and their applications in solving the generalized Chen’s conjecture have been obtained in [Ou1] and [OT], and some classifications of biharmonic submanifolds in conformally flat spaces $S^m \times \mathbb{R}$ and $H^m \times \mathbb{R}$ have been given in [OW], [FOR], and [FR].

This paper attempts to study biharmonic hypersurfaces in a generic conformally flat space. We derived the equation for such hypersurfaces which generalizes the equation for the biharmonic hypersurface in a space form. As an application, we use the equation to determine the conformally flat metric $f^{-2}\delta_{ij}$ on the Euclidean space $\mathbb{R}^{m+1}$ so that a minimal hypersurface $M^m \rightarrow (\mathbb{R}^{m+1}, \delta_{ij})$ in a Euclidean space becomes a biharmonic hypersurface $M^m \rightarrow (\mathbb{R}^{m+1}, f^{-2}\delta_{ij})$ in the conformally flat space. Our examples include all biharmonic hypersurfaces found previously in [Ou1] and [OT] as special cases.

2. Biharmonic hypersurfaces in a conformally flat space

Biharmonic hypersurfaces in a generic Riemannian manifold has been studied in [Ou1] and one of the main results is the following

**Theorem 2.1.** [Ou1] Let $\varphi : M^m \rightarrow N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$. Then $\varphi$ is biharmonic if and only if:

\[
\begin{align*}
\Delta H - H|A|^2 + HRic^N(\xi, \xi) &= 0, \\
2A(\text{grad } H) + \frac{n}{2} \text{grad } H^2 - 2 H (\text{Ric}^N(\xi))^\top &= 0,
\end{align*}
\]

where $\text{Ric}^N : T_qN \rightarrow T_qN$ denotes the Ricci operator of the ambient space defined by $\langle \text{Ric}^N(Z), W \rangle = \text{Ric}^N(Z, W)$ and $A$ is the shape operator of the hypersurface with respect to the unit normal vector $\xi$.

When the ambient space is a conformally flat space we have

**Theorem 2.2.** Let $\varphi : (M^m, g) \rightarrow (C^{m+1}, h = e^{-2\sigma} \bar{h})$ be an isometric immersion of codimension-one with mean curvature vector $\eta = H\xi$, where $(C^{m+1}, h)$
denotes a conformally flat space. Then \( \varphi \) is biharmonic if and only if:

\[
\begin{align*}
\Delta^M H - H|A|^2 + H\{\Delta h \sigma + (m - 1)[\text{Hess}_h(\sigma)(\xi, \xi) - (\xi \sigma)^2 + |\text{grad}_h \sigma|^2]\} = 0,
2A(\text{grad}_h H) + \frac{m}{2}\text{grad}_h H^2 - 2(m - 1)H[\text{grad}_h(\xi \sigma) - (\xi \sigma)\text{grad}_h \sigma + A(\text{grad}_h \sigma)] = 0.
\end{align*}
\]

where \( \sigma \) is the conformal factor of \((C^{m+1}, h)\), \( A \) is the shape operator of the hypersurface with respect to the unit normal vector \( \xi \), \( \Delta_h \) and \( \text{grad}_h \) denote the gradient and the Laplacian of ambient space \( C^{m+1} \) and the hypersurface \( M \) respectively.

Proof. Let \( \nabla, R, \text{Ric}, \text{grad}_h, \Delta_h \) denote the Levi-Civita connection, Riemannian curvature, Ricci curvature, the gradient operator, and the Laplace operator on \((C^{m+1}, h)\) respectively. The same notations with a “\( \bar{} \)” over head will be used for the counterparts on \((C^{m+1}, \bar{h})\). It is well known (see e.g., [Wa]) that the relationship between of the two connections and the Riemannian curvature are given by

\[
\nabla_X Y = \nabla_X Y + (X \sigma)Y + (Y \sigma)X - h(X, Y)\text{grad}_h \sigma,
\]

\[
\overline{R}(W, Z, X, Y) = e^{2\sigma}\{R(W, Z, X, Y) + h(\nabla_X \text{grad}_h \sigma, Z)h(Y, W)
-h(\nabla_Y \text{grad}_h \sigma, Z)h(X, W) + h(X, Z)h(\nabla_Y \text{grad}_h \sigma, W)
-h(Y, Z)h(\nabla_X \text{grad}_h \sigma, W) + [(Y \sigma)(Z \sigma)
-h(Y, Z)|\text{grad}_h \sigma|^2\}h(X, W) - [(X \sigma)(Z \sigma)
-h(X, Z)|\text{grad}_h \sigma|^2\}h(Y, W) + [(X \sigma)h(Y, Z)
-(Y \sigma)h(X, Z)]h(\text{grad}_h \sigma, W)\}.
\]

In local coordinates, we have

\[
e^{-2\sigma}\overline{R}_{ijkl} = R_{ijkl} + h_{il}\sigma_{jk} - h_{ik}\sigma_{jl} + h_{jk}\sigma_{il} - h_{jl}\sigma_{ik} + (h_{il}h_{jk} - h_{ik}h_{jl})|\text{grad}_h \sigma|^2,
\]

where we have used the notation \( \sigma_{jk} = \nabla_k \sigma_j - \sigma_k \sigma_j = \nabla_k \nabla_j \sigma - \sigma_k \sigma_j = \text{Hess}_h(\sigma)(\partial_j, \partial_k) - \sigma_k \sigma_j \) with \( \text{Hess}_h \) denoting the Hessian operator.

The relationship between the Ricci curvatures of the two conformally related metrics is given by

\[
\overline{\text{Ric}}_{jk} = \text{Ric}_{jk} - (m - 1)\sigma_{jk} - h_{jk}[\Delta_h \sigma + (m - 1)|\text{grad}_h \sigma|^2].
\]
It follows that if the metric $\tilde{h}$ is Euclidean (flat), i.e., the metric $h = e^{-2\sigma} \tilde{h}$ is conformally flat, then we obtain the Ricci curvature of a conformally flat space

$$Ric_{jk} = (m - 1)\sigma_{jk} + h_{jk}[\Delta \sigma + (m - 1)|\text{grad}_h \sigma|^2]$$

$$= (m - 1)\text{Hess}_h(\sigma)(\partial_j, \partial_k) - (m - 1)\sigma_k \sigma^j + h_{jk}[\Delta_h \sigma + (m - 1)|\text{grad}_h \sigma|^2].$$

For a conformally flat space $(C^{m+1}, h)$, we can choose local coordinates $(x^1, x^2, \ldots, x^{m+1})$ with local frame \{\(\partial_\alpha = \frac{\partial}{\partial x^\alpha}\)\}_{\alpha=1,\ldots,m+1}. It follows that \{\(\epsilon_\alpha = e^\sigma \partial_\alpha\)\}_{\alpha=1,\ldots,m+1} form an orthonormal basis on \((C^{m+1}, h = e^{-2\sigma} \tilde{h})\). Let \(\{e_1, \cdots, e_m, \xi\}\) be an orthonormal frame on \((C^{m+1}, h)\) adapted to the hypersurface \(M\) so that \(\{e_i\}_{i=1,\cdots,m}\) tangent and \(\xi\) normal to \(M\). The relationship between the two orthonormal frames is given by

$$\begin{cases} 
  e_i = T_1^\alpha \epsilon_\alpha, & i = 1, 2, \cdots, m. \\
  \xi = T_{m+1}^\alpha \epsilon_\alpha, 
\end{cases}$$

where \((T_\beta^\alpha)\) is an \((m+1) \times (m+1)\) orthogonal matrix.

Using the relation (8) we can compute the Ricci curvatures

$$Ric(\xi, \xi) = e^{2\sigma} T_{m+1}^j T_k^i \text{Ric}_{jk}$$

$$= \Delta_h \sigma + (m - 1)\text{Hess}_h(\sigma)(\xi, \xi) - (\xi \sigma)^2 + |\text{grad}_h \sigma|^2,$$

and

$$[Ric(\xi)]^T = \sum_{i=1}^m e^{2\sigma} T_{m+1}^j T_i^k \text{Ric}_{jk} e_i$$

$$= (m - 1)|\text{grad}_g(\xi \sigma) + \sum_{i=1}^m (Ae_i(\sigma))e_i - (\xi \sigma)|\text{grad}_g \sigma|]$$

$$= (m - 1)|\text{grad}_g(\xi \sigma) - (\xi \sigma)|\text{grad}_g \sigma + \sum_{i,j=1}^m b(e_i, e_j)e_j(\sigma)e_i|]$$

$$= (m - 1)|\text{grad}_g(\xi \sigma) - (\xi \sigma)|\text{grad}_g \sigma + A(\text{grad}_g \sigma)|. $$

Substituting (9) and (10) into the biharmonic hypersurface equation (2) we obtain the theorem. □

As immediate consequences of Theorem 2.2 we have
Corollary 2.3. A constant mean curvature hypersurface in a conformally flat space \( C^{m+1} \) is proper biharmonic if and only if

\[
\begin{aligned}
|A|^2 &= \Delta h \sigma + (m - 1) [\text{Hess}_h(\sigma)(\xi, \xi) - (\xi \sigma)^2 + |\text{grad}_h \sigma|^2], \\
\text{grad}_g \xi(\sigma) - \xi(\sigma) \text{grad}_g \sigma + A(\text{grad}_g \sigma) &= 0.
\end{aligned}
\]

In particular, if \( \xi(\sigma) = 0 \), (11) reduces to

\[
\begin{aligned}
|A|^2 &= \Delta h \sigma + (m - 1) [\text{Hess}_h(\sigma)(\xi, \xi) + |\text{grad}_h \sigma|^2], \\
A(\text{grad}_g \sigma) &= 0.
\end{aligned}
\]

Let \( M \) be a totally umbilical hypersurface in \( C^{m+1} \), i.e., all principal normal curvature at any point \( p \in M \) are equal to the same number \( \lambda(p) \), it follows that

\[
H = \frac{1}{m} \sum_{i=1}^m h(Ae_i, e_i) = \lambda,
\]

\[
A(\text{grad}_g H) = A(\sum_{i=1}^m (e_i \lambda) e_i) = \frac{1}{2} \text{grad}_g H^2,
\]

\[
|A|^2 = m \lambda^2 = m H^2, \quad A(\text{grad}_g \sigma) = \lambda \text{grad}_g \sigma = H \text{grad}_g \sigma.
\]

From this we have

Corollary 2.4. A totally umbilical hypersurface in \( C^{m+1} \) is biharmonic if and only if its mean curvature function \( H \) is a solution of the following PDEs

\[
\begin{aligned}
\Delta^M H - m H^3 + H \{ \Delta \sigma + (m - 1) [\text{Hess}(\sigma)(\xi, \xi) - (\xi \sigma)^2 + |\text{grad}\sigma|^2] \} &= 0, \\
\frac{2 + m}{2} \text{grad}_g H^2 - 2(m - 1) H [\text{grad}_g (\xi \sigma) - \xi \sigma \text{grad}_g \sigma + H \text{grad}_g \sigma] &= 0.
\end{aligned}
\]

if \( H = \xi(\sigma) \), (13) becomes

\[
\begin{aligned}
\Delta^M H - m H^3 + H \{ \Delta \sigma + (m - 1) [\text{Hess}(\sigma)(\xi, \xi) - H^2 + |\text{grad}\sigma|^2] \} &= 0, \\
(m - 4) \text{grad}_g H^2 &= 0.
\end{aligned}
\]

In [Ou1], an interesting foliation of proper biharmonic hypersurfaces was found in a conformally flat space, and later in [OT], counter examples to the generalized Chen’s conjecture were found by using proper biharmonic hypersurfaces in 5-dimensional conformally flat spaces. All those proper biharmonic hypersurfaces were constructed by starting with a hyperplane (a totally geodesic hypersurface) in a Euclidean space and then performing a suitable conformal change of the Euclidean metric into a conformally flat metric which renders the totally geodesic hypersurface a proper biharmonic hypersurface. Now we study the more
general problem: given a minimal hypersurface in a Euclidean space, i.e., a minimal isometric immersion, \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, h) \), under what conditions on \( f \) the hypersurface in the conformally flat space \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, f^{-2}h) \) becomes a proper biharmonic hypersurface?

**Theorem 2.5.** Let \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, h) \) be a minimal hypersurface with the unit normal vector field \( \xi \) in a Euclidean space, then, the hypersurface \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, f^{-2}h) \) in the conformally flat space is a biharmonic hypersurface if and only if

\[
\begin{align*}
\n &\n f \Delta_g(f(\xi f)) - m \nabla_g f(f(\xi f)) - f^2(\xi f)|A|^2_h - 2m(\xi f)^3 \\
&\n + mf(\xi f) \text{Hess}_h(f)(\xi, \xi) = 0, \\
&\n 2fA(\nabla_g(\xi f)) - 2(m - 1)(\xi f)A(\nabla_g f) + (4 - m)(\xi f)\nabla_g(\xi f) = 0,
\end{align*}
\]

where \( h \) denotes the standard Euclidean metric on \( \mathbb{R}^{m+1} \) and \( g \) and \( A \) are the metric and the shape operator of the minimal hypersurface \( M^m \hookrightarrow (\mathbb{R}^{m+1}, h) \).

**Proof.** First of all, with the new notations, the equation \((\mathfrak{M})\) for biharmonic hypersurface in the conformally flat space \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, f^{-2}h) \) becomes

\[
\begin{align*}
\Delta_g^M \bar{H} - \bar{H}|A|^2_h + \bar{H}\{\Delta_h \sigma + (m - 1)[\text{Hess}_h(\sigma)(\xi, \xi) - (\xi \sigma)^2 + |\nabla_h \sigma|^2_h]\} = 0,
\end{align*}
\]

where \( \sigma = \ln f \).

**Claim:** Let \( \phi : M^m \hookrightarrow (N^{m+1}, h) \) be a minimal hypersurface with the unit normal vector field \( \xi \). Then, the mean curvature function \( \bar{H} \) of \( M \) as a hypersurface in \( \phi : M^m \hookrightarrow (N^{m+1}, \bar{h} = f^{-2}h) \) is given by

\[
\bar{H} = \xi f.
\]

**Proof of the Claim:** Let \( \phi : M^m \hookrightarrow (N^{m+1}, h) \) be a hypersurface with mean curvature function \( \bar{H} \) and the unit normal vector field \( \xi \). Let \( \bar{h} = f^{-2}h \) be a conformal change of the metric. Then, a straightforward computation (see e.g., \([\text{Ne}]\)) the mean curvature function \( \bar{H} \) of \( M \) as a hypersurface in \( (N^{m+1}, \bar{h}) \) is given by

\[
\bar{H} = f \bar{H} + f \xi(\ln f),
\]

from which we obtain the Claim.

Let \( \{e_1, \cdots, e_m, \xi\} \) be an orthonormal frame adapted to the minimal hypersurface \( \phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, h) \) with \( \xi \) being the unit normal vector with respect
to the metric $h$. Then, $\{\bar{e}_i = f e_i, \ i = 1, \cdots, m; \bar{\xi} = f \xi\}$ consist an orthonormal frame adapted to the hypersurface $\phi : M^m \hookrightarrow (\mathbb{R}^{m+1}, \bar{h} = f^{-2}h)$ with respect to the metric $\bar{h}$.

Using the relationship (4) between the connections of the two conformally related metrics we have

\begin{align}
\nabla_{\bar{e}_i} \bar{e}_j &= f^2 \nabla_{e_i} e_j - e_j(f)\bar{e}_i + \delta_{ij} \text{grad}_\bar{h} \ln f, \\
\nabla_{\bar{e}_i} \bar{\xi} &= f^2 \nabla_{e_i} \xi - \xi(f)\bar{e}_i. 
\end{align}

A straightforward computation yields

\begin{align}
|A|^2_{\bar{h}} &= \sum_{i=1}^m \bar{h}(\nabla_{\bar{e}_i} \bar{\xi}, \nabla_{\bar{e}_i} \bar{\xi}) \\
&= \sum_{i=1}^m [f^2 h(\nabla_{e_i} \xi, \nabla_{e_i} \xi)] + 2mf\xi(f)\bar{H} + m(\xi f)^2 \\
&= f^2|A|^2_h + m(\xi f)^2.
\end{align}

Noting that $\sigma = \ln f$ we have:

\begin{align}
\nabla_{\bar{e}_i} \bar{e}_i(\sigma) &= f e_i(\ln f) = e_i(f), \\
\nabla_{\bar{e}_i} \bar{e}_i(\sigma) &= f e_i(e_i(f)) = f e_i e_i(f), \\
\nabla_{\bar{e}_i} \bar{\xi}(\sigma) &= \xi(f), \\
\n\text{grad}_\bar{h} \sigma &= \sum_{i=1}^{m+1} \bar{e}_i(\ln f)\bar{e}_i = f \text{grad}_h f, \\
|\text{grad}_h \sigma|^2_{\bar{h}} &= |\text{grad}_h f|^2_h, \\
\n\nabla_{\bar{e}_i} \bar{e}_i(\bar{H}) &= f e_i(f e_i(\bar{H})) = f e_i(f) e_i(\bar{H}) + f^2 e_i e_i(\bar{H}), \\
\n\text{grad}_h \bar{H} &= f \text{grad}_h f + f \text{grad}_h \bar{H}(f). 
\end{align}
A further computation gives

\[
\nabla^M_{\bar{e}_i} \tilde{e}_i(\bar{H}) = \nabla_{\bar{e}_i} \tilde{e}_i(\bar{H}) - b_h(\bar{e}_i, \tilde{e}_i)\xi(\bar{H}),
\]

(23)

\[
\Delta^M_h \bar{H} = f \Delta_g(f \xi f) - f(\xi f)\Delta_g f - m(f(\nabla_g f))(\xi f),
\]

(24)

\[
\Delta_h \sigma = f \Delta_h f - m |\nabla_h f|^2,
\]

(25)

\[
\text{Hess}_h(\sigma)(\xi, \xi) = \tilde{\xi}(\sigma) - \nabla \tilde{\xi}(\sigma)
\]

\[
= f \text{Hess}_h(f)(\xi, \xi) + (\xi \sigma)^2 - |\nabla_h \sigma|^2,
\]

\[
\text{grad}^M_h \bar{H} = \sum_{i=1}^{m} \bar{e}_i(\xi f)\bar{e}_i = f^2 \text{grad}_g(f f),
\]

(26)

\[
A_h(\text{grad}^M_h \bar{H}) = f^2 A(\text{grad}_g(\xi f)) + f^2 \xi(f)\text{grad}_g(\xi f),
\]

(27)

\[
\text{grad}^M_h \sigma = \sum_{i=1}^{m} \bar{e}_i(\ln f)\bar{e}_i = f \sum_{i=1}^{m} e_i(f)e_i = f \text{grad}_g f,
\]

(28)

\[
A_h(\text{grad}^M_h \sigma) = f^2 A(\text{grad}_g f) + f(\xi f)\text{grad}_g f.
\]

Substituting (17), (21), (22), (23), (24), (25), (26), (27), (28) into (16) and simplifying the resulting equation we obtain Equation (15) which completes the proof of the theorem. \(\square\)

As an application of Theorem 2.5 we have the following theorem which give a generalization of Theorem 3.1 in [On1].

**Theorem 2.6.** The hyperplane \(\varphi : \mathbb{R}^m \to (\mathbb{R}^{m+1}, \bar{h} = f^{-2}(x_1, \ldots, x_m, z)(\sum_{i=1}^{m} dx_i^2 + dz^2)\) with \(\varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, c)\) in the conformally flat space is biharmonic if and only if one of the following three cases happens

1) it is minimal i.e., \(f_z = 0;\)

2) \(m = 4\) and \(f\) is solution of the equation

\[
\sum_{i=1}^{4} [f^2 f_{iiz} - 2 f_{fi} f_{iiz} + f f_{fi} f_{iiz} - 4 f_{fi} f_{i}^2] + 4 f_z(f f_{zz} - 2 f_z^2) = 0,
\]

(29)

3) The hyperplane has nonzero constant mean curvature and \(f\) takes the form \(f(x_1, \ldots, x_m, z) = p(x_1, \ldots, x_m) + q(z)\) with \(p\) and \(q\) satisfying the following equation

\[
(p \sum_{i=1}^{m} p_{ii} - m \sum_{i=1}^{m} p_i^2) + m(q_{zz} - 2 q_z^2) + (mp_{zz} + q \sum_{i=1}^{m} p_{ii}) = 0.
\]

(30)

In particular, if \(p\) is constant, then \(f(x_1, \ldots, x_m, z) = (Az + B)^{-1}.\)
Proof. Noting that the hyperplane in the Euclidean space \( \varphi : \mathbb{R}^m \to (\mathbb{R}^{m+1}, h = \sum_{i=1}^{m} dx_i^2 + dz^2) \) with \( \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, c) \) is a totally geodesic hypersurface with unit normal vector field \( \xi = \partial_z \) we have \( \xi f = f_z \), \( A = 0 \), and that the induced metric on the hyperplane is the standard Euclidean metric \( g = \sum_{i=1}^{m} dx_i^2 \). It follows that the second equation of (15) is equivalent to

\[
(m - 4)f_zf_{zi} = 0, \quad i = 1, 2, \ldots, m. \tag{31}
\]

Case I: \( f_z = 0 \). This means the conformally flat metric is actually homothetic to the Euclidean metric and the hyperplane remains to be totally geodesic.

Case II: \( f_{zi} = 0 \), \( i = 1, 2, \ldots, m \). In this case, the mean curvature of the hypersurface is given by \( H = (\xi f)|_{z=d} = \text{constant} \). It follows that \( f \) takes the form \( p(x_1, \ldots, x_m) + q(z) \) and the first equation of (15) becomes

\[
(p \sum_{i=1}^{m} p_{ii} - m \sum_{i=1}^{m} p_i^2) + m(qq_{zz} - 2q_z^2) + (mpq_{zz} + q \sum_{i=1}^{m} p_i) = 0. \tag{32}
\]

In particular, if \( p = \text{constant} \), then \( f \) depends on \( z \) alone and by Theorem 3.1 in [On1] we conclude that \( f(z) = (Az + B)^{-1} \).

Case III: \( m = 4 \). In this case, one can easily (use the fact that both \( g \) and \( h \) are Euclidean metrics) that the first equation of (15) is equivalent to (29). Summarizing the above results we obtain the theorem. \( \Box \)

Example 1. For any \( A, B, C, D > 0 \), let \( M^4 = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 \mid x_i > 0, i = 1, 2, 3, 4 \} \) and \( N^5 = M \times [0, \infty) \). Then, the isometric immersion \( \varphi : M^4 \to (N^5, h = (Ax_i + B)(Cz + D)^2(\sum_{i=1}^{m} dx_i^2 + dz^2)) \) with \( \varphi(x_1, \ldots, x_4) = (x_1, \ldots, x_4, K) \) and \( K > 0 \) is a proper biharmonic hyper surface in the conformally flat space.

In fact, this can be obtained by looking for a solution of (29) of the form \( f = p(x_i)q(z) \). In this case, one can easily check that

\[
f_z = p'(x_i)q(z), \quad f_z = p(x_i)q'(z), \quad f_{ii} = p''(x_i)q(z),
\]

\[
f_{iz} = p'(x_i)q'(z), \quad f_{zz} = p(x_i)q''(z), \quad f_{iiz} = p''(x_i)q'(z).
\]

Substituting these into the equation (29) we have

\[
p(x_i)q'(z)q^2(z)[p(x_i)p''(x_i) - 3(p'(x_i))^2] + 2p^3(x_i)q'(z)[q(z)q''(z) - 2(q'(z))^2] = 0,
\]

By looking for the solutions satisfying \( p(x_i)p''(x_i) - 3(p'(x_i))^2 = 0 \) and \( q(z)q''(z) - 2(q'(z))^2 = 0 \) we obtain special solutions \( p(x_i) = \frac{1}{\sqrt{Ax_i + B}} \) and \( q(z) = \frac{1}{Cz + D} \) with positive constants \( A, B, C, D \). From this we obtain the example.
Recall that the main idea used in constructing a counter example to the generalized Chen’s conjecture (see [OT]) is to find a conformal change of the Euclidean metric on \( \mathbb{R}^{m+1} \) so that certain hyperplane becomes a proper biharmonic hypersurface and at the same time the conformally flat metric has nonpositive sectional curvature. In [On1], the author starts with a plane perpendicular to the last coordinate axis and searches for a conformally flat metric whose conformal factor depends only on the last coordinate. It turns out that for this type of the metric the hyperplane does become proper biharmonic but the metric cannot be non-positively curved. Later in [OT], we succeed in finding many counter examples by using a hyperplane in a more general position and searching for the same type of the conformally flat metrics. Our next theorem shows that even we starts with a hyperplane that perpendicular to the last coordinate axis we can find conformally flat metrics depending on more variables which then give negative sectional curvature and turn the hyperplane into a proper biharmonic hypersurface.

Before stating and proving our next theorem we prove the following lemma which has its own interest.

**Lemma 2.7.** Let \((C^m, h = f^{-2}(x_1, \cdots, x_m) \sum_{i=1}^m dx_i^2)\) be a conformally flat space and let \(\{e_i = f \frac{\partial}{\partial x_i}, i = 1, 2, \cdots, m\}\) denote an orthonormal frame on \(C^m\). Let \(P\) be a plan section at a point and suppose that \(P\) is spanned by an orthonormal basis \(X, Y\). Then, the sectional curvature \(K(P)\) of \(C^m\) is given by

\[
K(P) = \sum_{i,j=1}^m (a_i a_j + b_i b_j) f f_{ij} - \sum_{i=1}^m f_i^2. 
\]  
(33)

where \(a_i = h(X, e_i), \ b_i = h(Y, e_i), \ f_i = \frac{\partial f}{\partial x_i}, \ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}\).

**Proof.** One can easily check (see, e.g., [Wa] or [OT]) that the sectional curvature \(K(P)\) of \(C^m\) is given by

\[
K(P) = h(\nabla_X \text{grad}\sigma, X) + h(\nabla_Y \text{grad}\sigma, Y) + |\text{grad}\sigma|^2 - (X\sigma)^2 - (Y\sigma)^2 
\]  
(34)

\[
\quad + XX(\sigma) - (\nabla_X X)(\sigma) - (X\sigma)^2 
\]  

\[
\quad + YY(\sigma) - (\nabla_Y Y)(\sigma) - (Y\sigma)^2 + |\text{grad}\sigma|^2, 
\]

BIHARMONIC HYPERSURFACES IN A CONFORMALLY FLAT SPACE
where $\sigma = \ln f$. By hypothesis and a straightforward computation we have
\[
\sum_{i=1}^{m} a_i^2 = 1, \quad \sum_{i=1}^{m} b_i^2 = 1,
\]
\[
e_i(\sigma) = f_i,
\]
\[
\nabla e_i e_j = \delta_{ij} \sum_{k=1}^{m} f_k e_k - f_j e_i,
\]
(35)
\[
|\text{grad}\sigma|^2 = \sum_{i=1}^{m} f_i^2.
\]
A further computation gives
\[
X(\sigma) = \sum_{i=1}^{m} a_i e_i(\sigma) = \sum_{i=1}^{m} a_i f_i,
\]
\[
XX(\sigma) = \sum_{i,j=1}^{m} a_i e_i(a_j) f_j + \sum_{i,j=1}^{m} a_i a_j f f_{ij},
\]
\[
\nabla_X X = \sum_{i,j=1}^{m} a_i e_i(a_j) e_j + \sum_{i,j=1}^{m} a_i a_j \nabla e_i e_j,
\]
(36)
\[
(\nabla_X X)(\sigma) = \sum_{i,j=1}^{m} a_i e_i(a_j) f_j + \sum_{i,j=1}^{m} a_i a_j (\nabla e_i e_j)(\sigma)
\]
\[
= \sum_{i,j=1}^{m} a_i e_i(a_j) f_j + \sum_{i,j=1}^{m} f_i^2 - \sum_{i,j=1}^{m} a_i a_j f_i f_j,
\]
\[
XX(\sigma) - (\nabla_X X)(\sigma) - (X\sigma)^2 = \sum_{i,j=1}^{m} a_i a_j f f_{ij} - \sum_{i=1}^{m} f_i^2.
\]

Similarly, we have
\[
YY(\sigma) - (\nabla_Y Y)(\sigma) - (Y\sigma)^2 = \sum_{i,j=1}^{m} b_i b_j f f_{ij} - \sum_{i=1}^{m} f_i^2.
\]

Substituting (35), (36), (37) into (34) we have
\[
K(P) = \sum_{i,j=1}^{m} (a_i a_j + b_i b_j) f f_{ij} - \sum_{i=1}^{m} f_i^2,
\]
which gives the Lemma.

Now we are ready to prove the following theorem which provides many counter examples to the generalized Chen’s conjecture.
Theorem 2.8. For constants $A, B, C, K$, with $A^2 + B^2 \neq 0$ we use $\Sigma$ to denote the hyperplane in $\mathbb{R}^5 = \{(x_1, \cdots, x_4, z)\}$ defined by $A \sum_{i=1}^{4} x_i + Bz + C = 0$. Let $M^4 = \mathbb{R}^4 \setminus \Sigma$ and $C^5 = \mathbb{R}^5 \setminus \Sigma$ be the conformally flat space with the metric $h = (A \sum_{i=1}^{4} x_i + Bz + C)^{-2t}(\sum_{i=1}^{m} dx_i^2 + dz^2)$. Then,

1) The isometric immersion $\varphi : M^4 \rightarrow (C^5, h = (A \sum_{i=1}^{4} x_i + Bz + C)^{-2t}(\sum_{i=1}^{m} dx_i^2 + dz^2))$ with $\varphi(x_1, \cdots, x_4) = (x_1, \cdots, x_4, K)$ is a proper biharmonic hypersurface for $t = -1$ or $t = \frac{2A^2}{4A^2 + B^2}$.

2) For $A \neq 0$ and any $t$ with $0 < t < 1$, the conformally flat space $(C^5, h = (A \sum_{i=1}^{4} x_i + Bz + C)^{-2t}(\sum_{i=1}^{m} dx_i^2 + dz^2))$ has negative sectional curvature.

Proof. By Theorem 2.6 we look for the conformal factor $f$ of the form $f(x_1, \cdots, x_4, z) = (A \sum_{i=1}^{4} x_i + Bz + C)^t$. A simple calculation one checks that for $i, j = 1, 2, 3, 4, i \neq j$,

\begin{align}
  f_i &= A t (A \sum_{i=1}^{4} x_i + Bz + C)^{t-1}, \\
  f_z &= B t (A \sum_{i=1}^{4} x_i + Bz + C)^{t-1}, \\
  f_{ii} &= A^2 t (t-1)(A \sum_{i=1}^{4} x_i + Bz + C)^{t-2}, \\
  f_{ij} &= A^2 (t-1)(A \sum_{i=1}^{4} x_i + Bz + C)^{t-2}, \\
  f_{iz} &= A B t (t-1)(A \sum_{i=1}^{4} x_i + Bz + C)^{t-2}, \\
  f_{zz} &= B^2 t (t-1)(A \sum_{i=1}^{4} x_i + Bz + C)^{t-2}, \\
  f_{iiz} &= A^2 B t (t-1)(t-2)(A \sum_{i=1}^{4} x_i + Bz + C)^{t-3}.
\end{align}

Substituting these into the equation (29) we have $(4A^2 + B^2)t^2 + (2A^2 + B^2)t - 2A^2 = 0$, which has solutions $t = -1$ or $t = \frac{2A^2}{4A^2 + B^2}$. This give the first statement of the theorem.
For the second statement, we substitute \( f_i, f_{ij} \) in (38) and (39) into the (33) to have
\[
K(P) = \sum_{i,j=1}^{5} (a_i a_j + b_i b_j) f f_{ij} - \sum_{i=1}^{5} f_i^2
\]
\[
= A^2 \{ [(\sum_{i=1}^{5} a_i)^2 + (\sum_{i=1}^{5} b_i)^2] t (t - 1) - 5 t^2 \} (A \sum_{j=1}^{4} x_j + B z + C)^{2t-2},
\]
which is strictly negative since \([ (\sum_{i=1}^{5} a_i)^2 + (\sum_{i=1}^{5} b_i)^2] t (t - 1) - 5 t^2 < 0\) for \( 0 < t < 1 \), and \( A^2 (A \sum_{j=1}^{4} x_j + B z + C)^{2t-2} > 0 \) for \( A \sum_{j=1}^{4} x_j + B z + C \neq 0 \).
This completes the proof of the theorem.

\[\square\]

To conclude the paper, we would like to point out that by using Theorem 2.5 to a totally geodesic hypersurface one can very easily prove the following theorem which was proved in [OT] using a different method that involves a lengthy computation.

**Theorem 2.9.** [OT] Let \( a_i, i = 1, 2, \ldots, m \) and \( c \) be constants. Then, the isometric immersion \( \varphi : \mathbb{R}^m \to (\mathbb{R}^{m+1}, h = f^{-2}(z)(\sum_{i=1}^{m} dx_i^2 + dz^2)) \) with \( \varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \sum_{i=1}^{m} a_i x_i + c) \) is biharmonic if and only if one of the following three cases happens

1. \( f' = 0 \), in this case \( \varphi \) is minimal (actually, totally geodesic), or
2. \( m = 4 \) and \( f \) is a solution of the equation

\[
(40) \quad \sum_{i=1}^{4} a_i^2 f^2 f''' + (4 - \sum_{i=1}^{4} a_i^2) f f' f'' - 4(2 + \sum_{i=1}^{4} a_i^2)(f')^3 = 0,
\]
or
3. \( a_i = 0, i = 1, \ldots, m \) and \( f(z) = \frac{1}{Az + B} \), where \( A \) and \( B \) are constants.
In this case each hyperplane is a proper biharmonic hypersurface. This recovers a result (Theorem 3.1) obtained earlier in [Ou1].

**References**

[BMO1] A. Balms, S. Montaldo and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. 168 (2008), 201–220.
[BMO2] A. Balms, S. Montaldo, C. Oniciuc, *Biharmonic submanifolds in space forms*, Symposium Valenceiennes (2008), 25–32.
[CMO1] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of \( S^3 \)*, Internat. J. Math. 12 (2001), no. 8, 867–876.
[CMO2] R. Caddeo, S. Montaldo and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. 130 (2002), 109–123.
[Ch1] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. 17 (1991), no. 2, 169–188.

[Ch2] B. Y. Chen, *Pseudo-Reimannian Geometry, δ-Invariants and Applications*, World Scientific Publishing Co. Pte Ltd, 2011.

[CI] B. Y. Chen and S. Ishikawa, *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math. 52 (1998), no. 1, 167–185.

[Di] I. Dimitrić, *Submanifolds of $E^n$ with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica 20 (1992), no. 1, 53–65.

[FOR] D. Fetcu, C. Oniciuc, H. Rosenberg, *Biharmonic submanifolds with parallel mean curvature vector in $S^n \times \mathbb{R}$*, preprint, arXiv:1109.6138, 2011.

[FR] D. Fetcu and H. Rosenberg, *Surfaces with parallel mean curvature in $S^3 \times \mathbb{R}$ and $H^3 \times \mathbb{R}$*, preprint, arXiv:1103.6254, 2011.

[Ha] L. Habermann, *Riemannian metrics of constant mass and moduli spaces of conformal structures*. Lecture Notes in Mathematics, 1743. Springer-Verlag, Berlin, 2000.

[HV] T. Hasanis and T. Vlachos, *Hypersurfaces in $E^4$ with harmonic mean curvature vector field*, Math. Nachr. 172 (1995), 145–169.

[IIU] T. Ichiyama, J. Inoguchi and H. Urakawa, *Classifications and Isolation Phenomena of Bi-Harmonic Maps and Bi-Yang-Mills Fields*, arXiv:0912.4806, Preprint, 2009.

[Ji1] G. Y. Jiang, 2-Harmonic maps and their first and second variational formulas, Chin. Ann. Math. Ser. A 7(1986) 389-402.

[Ji2] G. Y. Jiang, *Some non-existence theorems of 2-harmonic isometric immersions into Euclidean spaces*, Chin. Ann. Math. Ser. 8A (1987) 376-383.

[NU] N. Nakauchi and H. Urakawa, *Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature*, Annals of Global Analysis and Geometry, 40(2), 2011, 125-131.

[Ne] B. Nelli, *Hypersurfaces de courbure constante dans l’espace hyperbolique*, Thése de Doctorat, Paris VII, 1995.

[Ou1] Y.-L. Ou, *Biharmonic hypersurfaces in Riemannian manifold*, Pacific J. of Math. 248(1), 2010, 217-232.

[Ou2] Y.-L. Ou, *Some constructions of biharmonic maps and Chen’s conjecture on biharmonic hypersurfaces*, J. Geom. Phys., 62, (2012), 751-762.

[OT] Y.-L. Ou and L. Tang, *On the generalized Chen’s conjecture on biharmonic submanifolds*, to appear in Mich. Math. J., 2012.

[OW] Y.-L. Ou and Z. -P. Wang, *Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries*, J. Geom. Phys., 61 (2011) 1845-1853.

[Sa] T. Sasahara, *Stability of biharmonic Legendrian submanifolds in Sasakian space forms*, Canad. Math. Bull. 51 (2008), no. 3, 448–459.

[Wa] G. Walschap, *Metric structures in differential geometry*. Graduate Texts in Mathematics, 224. Springer-Verlag, New York, 2004.

Department of Mathematics & Physics, Hunan University of Technology, Zhuzhou Hunan, P. R. China
DEPARTMENT OF MATHEMATICS
GUANGXI UNIVERSITY FOR NATIONALITIES
NANNING 530006,
P. R. CHINA