Mean field squared and energy-momentum tensor for
the hyperbolic vacuum in dS spacetime

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Abstract

We evaluate the Hadamard function and the vacuum expectation values (VEVs) of the
field squared and energy-momentum tensor for a massless conformally coupled scalar field in
\((D+1)\)-dimensional de Sitter (dS) spacetime foliated by spatial sections of negative constant
curvature. It is assumed that the field is prepared in the hyperbolic vacuum state. An integral
representation for the difference of the Hadamard functions corresponding to the hyperbolic
and Bunch-Davies vacua is provided that is well adapted for the evaluation of the expectation
values in the coincidence limit. It is shown that the Bunch-Davies state is interpreted as
thermal with respect to the hyperbolic vacuum. An expression for the corresponding density
of states is provided. The relations obtained for the difference in the VEVs for the Bunch-
Davies and hyperbolic vacua are compared with the corresponding relations for the Fulling-
Rindler and Minkowski vacua in flat spacetime. The similarity between those relations is
explained by the conformal connection of dS spacetime with hyperbolic foliation and Rindler
spacetime. As a limiting case, the VEVs for the conformal vacuum in the Milne universe are
discussed.

Keywords: Vacuum polarization, de Sitter spacetime, hyperbolic vacuum

1 Introduction

Canonical quantization of fields is carried out by defining complete set of modes end expanding
the field operator in terms of those modes. The creation and annihilation operators are defined as
the coefficients in the expansion. They allow one to build the Fock space of states starting from
the definition of the vacuum as the state of quantum field nullified by the annihilation operator.
As seen from this construction, the notion of the vacuum depends on the choice of the normal
modes used in the expansion of the field operator (see \cite{1} for a general discussion). Different sets
of modes, in general, lead to different vacuum states. Well-known examples of two inequivalent
vacua in flat spacetime are the Minkowski vacuum and the Fulling-Rindler vacuum. They
 correspond to quantization of fields in terms of the modes for inertial and uniformly accelerated
observers, respectively. Other examples are the Hartle-Hawking, Boulware and Unruh vacua for
Schwarzschild black hole geometry.
Because of high symmetry and cosmological applications, the de Sitter (dS) spacetime is among the most important gravitational backgrounds in quantum field theory. In that geometry there exists a one-parameter family of dS-invariant vacuum states (known as $\alpha$-vacua) [2, 3]. Among these maximally symmetric states the Bunch-Davies (Euclidean) vacuum [4] is distinguished by the property that in the flat spacetime limit it is reduced to the Minkowski vacuum. Its black hole counterpart is the Hartle-Hawking state. The Bunch-Davies (BD) vacuum is naturally realized in the quantization based on the mode functions being the solutions of classical field equations in global and planar (inflationary) coordinates of dS spacetime (for different coordinate systems in dS spacetime, see, for example [5]). Another coordinate system frequently used in considerations of thermal aspects of dS spacetime corresponds to static coordinates. The vacuum state realized by the normal modes in those coordinates is referred to as static vacuum. In the flat spacetime limit it reduces to the Fulling-Rindler vacuum and its black hole counterpart is the Boulware vacuum. The vacuum state in dS spacetime corresponding to the Unruh vacuum in black hole geometry has been recently considered in [6] (Unruh-de Sitter state).

Here we will consider the local characteristics of another vacuum state in dS spacetime referred to as hyperbolic (H-) vacuum [7]. Its flat spacetime counterpart corresponds to the conformal vacuum in the Milne patch of Minkowski spacetime. The H-vacuum is a quantum state that is nullified by the annihilation operator defined in terms of the normal modes being the solutions of the field equation in the coordinate system corresponding to the foliation of dS spacetime by spatial sections having constant negative curvature ($k = -1$ Robertson-Walker coordinatization of dS spacetime). This coordinatization has been employed in investigations of open inflationary models of the early universe (see [8]-[15]) and in discussions of the entanglement entropy in dS spacetime [16]-[20]. The hyperbolic coordinates are well-adapted for studies of long range quantum correlations between two causally disconnected regions separated by a finite region. Recent interest to the open inflation scenario is related to its natural realization in the context of the string landscape through the bubble nucleation. In the thin wall approximation, the influence of the exterior geometry on the dynamics of quantum fluctuations for scalar fields is reduced to the imposition of Robin type boundary condition (see [21, 47] for the dS geometry described in inflationary and static coordinates). The corresponding Casimir densities induced by a spherical boundary in the hyperbolic vacuum of dS spacetime have been recently discussed in [23]. The effect of a delta-functional wall on the entanglement entropy between two causally disconnected regions in dS spacetime with negative curvature spatial foliation is studied in [24].

The paper is organized as follows. In the next section we present the normal modes and Hadamard functions for the H- and BD vacua for a massive scalar field with general curvature coupling parameter. The corresponding expressions are essentially simplified for a conformally coupled massless field and they are given in section 3. The VEVs of the field squared and energy-momentum tensor are investigated in section 4. The main results are summarized in section 5.

2 Hadamard functions in hyperbolic and Bunch-Davies vacua

The $(D + 1)$-dimensional de Sitter spacetime can be visualized as a hyperboloid in $(D + 2)$-dimensional Minkowski spacetime (see, for instance, [5]). The entire hyperboloid is covered by the global coordinates. The corresponding spatial sections are $D$-dimensional spheres. In cosmological applications, in particular, in models of inflation, the most popular coordinate system corresponds to the so-called inflationary or planar coordinates. These coordinates cover a half of the hyperboloid and the spacetime is foliated by flat spatial sections. Introducing spherical spatial coordinates $(r_1, \vartheta, \phi)$, with $\vartheta = (\theta_1, \ldots, \theta_n)$, $n = D - 2$, the line element is
presented in the form
\[ ds^2 = dt_1^2 - e^{2t_1/\alpha} \left( dr_1^2 + r_1^2 d\Omega_{D-1}^2 \right), \]
where \(-\infty < t_1 < +\infty, 0 \leq r_1 < \infty, d\Omega_{D-1}^2\) is the line element on a unit sphere \(S^{D-1}\), and for the angular coordinates one has \(0 \leq \theta_k \leq \pi, k = 1, 2, \ldots, n, 0 \leq \phi \leq 2\pi\). The parameter \(\alpha\) determines the curvature of the spacetime and is related to positive cosmological constant \(\Lambda\) by the formula \(\Lambda = D(D-1)/(2\alpha^2)\).

The foliation of dS spacetime with negative curvature spatial sections is realized by hyperbolic coordinates \((t, r, \vartheta, \phi)\) with \(0 \leq t, r < \infty\). In these coordinates the line element reads
\[ ds^2 = dt^2 - \alpha^2 \sinh^2(t/\alpha) (dr^2 + \sinh^2 r d\Omega_{D-1}^2). \]

The transformation between the inflationary and hyperbolic coordinates is given by
\[ t_1 = \alpha \ln \left[ \cosh(t/\alpha) + \sinh(t/\alpha) \cosh r \right], \]
\[ r_1 = \alpha e^{-t_1/\alpha} \sinh(t/\alpha) \sinh r. \]

The discussion of different regions of the Penrose diagram covered by hyperbolic coordinates can be found in [23, 25]. Passing to a new radial coordinate \(r_{\text{FRW}} = \sinh r\), the line element (2) is written in the form corresponding to open Friedmann-Robertson-Walker cosmological models with the scale factor \(\alpha \sinh(t/\alpha)\). In terms of the hyperbolic coordinates, the geodesic distance \(d(x, x')\) between two spacetime points \(x = (t, r, \vartheta, \phi)\) and \(x' = (t', r', \vartheta', \phi')\) is determined by the dS invariant function
\[ u(x, x') = \cosh(t/\alpha) \cosh(t'/\alpha) - \sinh(t/\alpha) \sinh(t'/\alpha) w, \]
where
\[ w = w(r, r', \theta) = \cosh r \cosh r' - \sinh r \sinh r' \cos \theta, \]
and \(\theta\) is the angle between the directions determined by \((\vartheta, \phi)\) and \((\vartheta', \phi')\). Note that \(w \geq 1\). For \(u(x, x') > 1\) the geodesic distance is given by the relation \(\cosh[d(x, x')/\alpha] = u(x, x')\). Otherwise the hyperbolic function in the left-hand side should be replaced by the cosine function.

We consider a quantum scalar field \(\varphi(x)\) with curvature coupling parameter \(\xi\) in background of dS spacetime. The dynamics of the field is governed by the equation
\[ (\Box + m^2 + \xi R) \varphi = 0, \]
where \(\Box\) is the d’Alembert operator and for the Ricci scalar one has \(R = D(D+1)/\alpha^2\). The vacuum state depends on the complete set of mode functions used in the expansion of the field operator. The mode functions realizing the H-vacuum are given by the expression (see [20] for the case \(D = 3\) and [23] in general number of spatial dimensions)
\[ \varphi_{\sigma}(x) = c_{\sigma} P_{i z}^{l z} \frac{P_{i z-1/2}^{l z} (\cosh(t/\alpha))}{\sinh((D-1)/2)(t/\alpha)} \frac{P_{i z-1/2}^{l z} (\cosh r)}{\sinh(D/2-1) r} Y(m_p; \vartheta, \phi), \]
with the normalization coefficient determined from
\[ |c_{\sigma}|^2 = \frac{z \Gamma \left( l + \frac{D-1}{2} + iz \right)^2}{2N(m_p) \alpha^{D-1}}. \]
In (7), \(P_{\nu}^l(x)\) is the associated Legendre function of the first kind, \(Y(m_p; \vartheta, \phi)\) are hyperspherical harmonics and we have defined
\[ \nu = \sqrt{D^2/4 - \xi D(D+1) - m^2 \alpha^2}. \]
The modes are specified by the set of quantum numbers $\sigma = (z, m_p)$, where $0 \leq z < \infty$ and for the quantum numbers related to the angular coordinates one has $m_p = (l, m_1, \ldots, m_n)$ with $l = 0, 1, 2, \ldots$. The integers $m_1, m_2, \ldots, m_n$ obey the relations $-m_{n-1} \leq m_n \leq m_{n-1}$ and $0 \leq m_{n-1} \leq m_{n-2} \leq \cdots \leq m_1 \leq l$.

Given the mode functions (12), we can evaluate the Hadamard function for the H-vacuum by using the mode-sum formula

$$G(x, x') = \sum_\sigma \left[ \varphi_\sigma (x) \varphi^*_\sigma (x') + \varphi_\sigma (x') \varphi^*_\sigma (x) \right]. \quad (10)$$

In [23] the following representation has been derived:

$$G(x, x') = \frac{\alpha^{1-D}}{2(2\pi)^{D/2}} \int_0^\infty dz \frac{\Gamma \left( \frac{D-1}{2} + iz \right)^2}{\sinh(t/\alpha) \sinh(t'/\alpha)} P_{i\nu-1/2} \left( \frac{\cosh(t/\alpha)}{\cosh(t'/\alpha)} \right) P_{i\nu-1/2} \left( \frac{\cosh(t'/\alpha)}{\cosh(t/\alpha)} \right) \frac{P_{i\nu-1/2} \left( \frac{\cosh(t'/\alpha)}{\cosh(t/\alpha)} \right)}{(w^2 - 1)^{(D-2)/4}}, \quad (11)$$

where $w$ is defined by (5).

The vacuum state for a scalar field most frequently used in the inflationary coordinates is the BD vacuum. It is realized by the mode functions (for the corresponding mode functions in hyperbolic foliation see [25])

$$\varphi_\sigma(x) = \frac{c_{\lambda \sigma} \eta h_1^{D/2}}{\Gamma(D/2)} H_{\nu_1}^{(1)}(\lambda |\eta|) J_\nu(\lambda r_1) Y(m_p; \vartheta, \phi), \quad (12)$$

where $H_{\nu_1}^{(1)}(y)$ and $J_\nu(y)$ are the Hankel and Bessel functions, respectively, $\eta_1 = -\alpha e^{-t_1/\alpha}$, $-\infty < \eta_1 < 0$, is the inflationary conformal time coordinate. The normalization coefficient is given by

$$|c_{\lambda \sigma}|^2 = \frac{\pi e^{(\nu - \nu^*)/2}}{4N(m_p) \alpha^{D-1}}. \quad (13)$$

The expression for the corresponding Hadamard function reads (for the corresponding Wightman function see [27])

$$G_{BD}(x, x') = \frac{\Gamma(D/2 + \nu) \Gamma(D/2 - \nu)}{(2\pi)^{(D+1)/2} \alpha^{D-1}} \frac{P_{i\nu-1/2}^{(1-D)/2} \left( -u(x, x') \right)}{|u^2(x, x') - 1|^{(D-1)/4}}. \quad (14)$$

In terms of the inflationary coordinates the function $u(x, x')$ is written as

$$u(x, x') = \frac{(\Delta \eta_1)^2 - |\Delta r_1|^2}{2\eta_1 |\eta_1|^2} + 1, \quad (15)$$

where $\Delta \eta = \eta_1' - \eta_1$ and $|\Delta r_1|^2 = r_1^2 + r_1'^2 - 2r_1 r_1' \cos \theta$. Note that passing to the time coordinate $\eta_1$ the line element (11) is written in manifestly conformally flat form as

$$ds^2 = \left( \alpha/\eta_1 \right)^2 \left( d\eta_1^2 - dr_1^2 - r_1^2 d\Omega_3^2 \right). \quad (16)$$

The Hadamard function for the BD vacuum depends on the spacetime points through the geodesic distance and that state is maximally symmetric. As it has been mentioned above, in dS spacetime there is a one-parameter family of maximally symmetric vacuum states called as $\alpha$-vacua.
3 Hadamard function for a conformally coupled massless field

The expressions for the Hadamard functions are simplified for a conformally coupled massless field with $\xi = (D - 1)/(4D)$ and $m = 0$. In this case $\nu = 1/2$ and for the functions in the integrand of (14) we use

$$P_{0}^{\pm i z} (\cosh (t/\alpha)) = \frac{e^{\mp i z \eta/\alpha}}{\Gamma (1 \pm i z)},$$

where $\eta$, $-\infty < \eta \leq 0$, is the conformal time coordinate in the hyperbolic foliation. One has the following relation $\cosh (t/\alpha) = - \coth (\eta/\alpha)$. From (17) we see that time dependence of the normal modes appears in the form $e^{-i z \eta/\alpha}$ and for the energy $E$ related to the conformal time coordinate one gets $E = z/\alpha$. The Hadamard function for the H-vacuum takes the form

$$G (x, x') = \frac{2 [\sinh (\eta/\alpha) \sinh (\eta'/\alpha)]^{(D-1)/2}}{(2\pi)^{D/2+1} \alpha^{D-1} (w^2 - 1)^{(D-2)/2}} \int_{0}^{\infty} dz \sinh (\pi z)$$

$$\times \cos (z \Delta \eta/\alpha) \left| \Gamma \left( \frac{D - 1}{2} + i z \right) \right|^2 P_{i z - 1/2}^{1-D/2} (w),$$

where $\Delta \eta = \eta' - \eta$.

By using the properties of the associated Legendre function the following relation can be proved:

$$\left| \Gamma \left( \frac{D - 1}{2} + i z \right) \right|^2 \frac{P_{i z - 1/2}^{1-D/2} (w)}{(w^2 - 1)^{D/2-n}} = (-1)^n \left| \Gamma \left( \frac{D - 1}{2} - n + i z \right) \right|^2 \frac{\partial_{w}^{n-1} P_{i z - 1/2}^{1-D/2+n} (w)}{(w^2 - 1)^{D/2-n}},$$

with $n$ being a non-negative integer. This allows to present the Hadamard function in the form

$$G (x, x') = 2 (-1)^n \frac{[\sinh (\eta/\alpha) \sinh (\eta'/\alpha)]^{D-1}}{(2\pi)^{D/2+1} \alpha^{D-1}} \partial_{w}^{n} \int_{0}^{\infty} dz \sinh (\pi z)$$

$$\times \cos (z \Delta \eta/\alpha) \left| \Gamma \left( \frac{D - 1}{2} - n + i z \right) \right|^2 \frac{P_{i z - 1/2}^{1-D/2+n} (w)}{(w^2 - 1)^{D/2-n}},$$

(20)

This expression can be further simplified and we will consider the odd and even values for the spatial dimension $D$ separately.

In terms of the conformal time, by taking into account that $\sinh (t/\alpha) = -1/\sinh (\eta/\alpha)$, the line element (2) reads

$$ds^2 = \sinh^{-2} (\eta/\alpha) \left[ d\eta^2 - \alpha^2 (dr^2 + \sinh^2 r d\Omega_{D-1}^2) \right].$$

(21)

This shows that the geometry we consider is conformally related to a static spacetime foliated by constant negative curvature spatial sections with the conformal factor $\Omega_{st}^2 (\eta) = \sinh^{-2} (\eta/\alpha)$. The expression for the corresponding Hadamard function, denoted here by $G_{st} (x, x')$, is directly obtained from (18) and from the relation

$$G (x, x') = \left[ \Omega_{st} (\eta) \Omega_{st} (\eta') \right]^{1-D/2} G_{st} (x, x').$$

(22)

An alternative expression for $G_{st} (x, x')$ is obtained from (20).
3.1 Odd $D$

For odd values of $D$, we take $n = (D - 3)/2$ and in the right-hand side of (19) we use the relation

$$P_{iz-1/2}^{-1/2} (w) = \sqrt{\frac{2}{\pi}} \frac{\sin (z\zeta)}{z\sinh \zeta},$$

where $\zeta$ is defined in accordance with

$$w = \cosh \zeta.$$

This leads to the expression

$$G (x, x') = \frac{[\sinh (\eta/\alpha) \sinh (\eta'/\alpha)]^D}{(-2\pi)^{D+1} \alpha^{D-1}} \left( \frac{\partial \zeta / \sinh \zeta}{\zeta^2 - (\Delta \eta)^2 / \alpha^2} \right)^{D+3} 2\zeta / \sinh \zeta,$$

for the Hadamard function. The corresponding formula for the static open universe is obtained from (22). In the special case $D = 3$ and $\theta = 0$ it is in agreement with the expression for the Wightman function given in [28]. The corresponding conformally transformed Hadamard function has been used in [7] for the evaluation of the VEV for the energy-momentum tensor in the H-vacuum of dS spacetime.

For the BD vacuum, by using the relation

$$P_0^{-\beta} (-u) = \frac{1}{\Gamma (1 + \beta)} \left( \frac{u + 1}{u - 1} \right)^{\beta/2},$$

from (14) we get

$$G_{BD} (x, x') = \alpha^{1-D} \Gamma ((D - 1)/2) \left( \frac{\partial \zeta / \sinh \zeta}{\zeta^2 - (\Delta \eta)^2 / \alpha^2} \right)^{D+3} 2\zeta / \sinh \zeta,$$

In the case of odd $D$ this expression can be rewritten as

$$G_{BD} (x, x') = \frac{\alpha^{1-D} \Gamma ((D - 1)/2)}{(2\pi)^{D+1} \alpha^{D-1}} \left( \frac{\partial \zeta / \sinh \zeta}{\zeta^2 - (\Delta \eta)^2 / \alpha^2} \right)^{D+3} 2\zeta / \sinh \zeta,$$

For the further transformation of this expression we note that

$$\partial u = - \sinh (\eta/\alpha) \sinh (\eta'/\alpha) \frac{\partial \zeta}{\sinh \zeta},$$

and

$$1 - u(x, x') = \frac{\cosh \zeta - \cosh (\Delta \eta/\alpha)}{\sinh (\eta/\alpha) \sinh (\eta'/\alpha)}.$$

The Hadamard function is expressed as

$$G_{BD} (x, x') = \frac{[\sinh (\eta/\alpha) \sinh (\eta'/\alpha)]^D}{(-2\pi)^{D+1} \alpha^{D-1}} \left( \frac{\partial \zeta / \sinh \zeta}{\zeta^2 - (\Delta \eta)^2 / \alpha^2} \right)^{D+3} 2\zeta / \sinh \zeta,$$

This representation is well adapted for the evaluation of the difference in the VEVs for the H- and BD vacua.

The BD vacuum is conformally related to the Minkowski vacuum in flat spacetime and for a massless conformally coupled scalar field the relation $G_{BD} (x, x') = (\eta_0/\alpha^2)^{(D-1)/2} G_M (x, x')$
is expected between the corresponding Hadamard functions. This can be seen by using the standard expression

\[ G_M(x, x') = \frac{\Gamma ((D - 1)/2)}{2\pi^{(D+1)/2}} (|\Delta r|^{2} - \Delta \eta_I^2)^{\frac{1-D}{2}}, \]  

and passing to the hyperbolic coordinates. That leads to the following representation

\[ G_M(x, x') = \frac{\Gamma ((D - 1)/2)}{(2\pi)^{D+1/2}(\eta_I^2)^{D-1/2}} \left[ \frac{\sinh(\eta/\alpha) \sinh(\eta'/\alpha)}{\cosh \zeta - \cosh (\Delta \eta/\alpha)} \right]^{\frac{D+1}{2}}. \]  

Multiplying this by \((\eta_I^2/\alpha)^{(D-1)/2}\) we get the function (31).

The difference of the Hadamard functions, that determines the difference in the local VEVs, is presented as

\[ G(x, x') - G_{BD}(x, x') = \left[ \frac{\sinh(\eta/\alpha) \sinh(\eta'/\alpha)}{\cosh \zeta - \cosh (\Delta \eta/\alpha)} \right]^\frac{D+1}{2} \times \left( \frac{\partial_\zeta}{\sinh \zeta} \right)^\frac{D-3}{2} \alpha^{D-1} \]  

Note that in this formula the part

\[ G_{st}(x, x') = \left[ \frac{\sinh(\eta/\alpha) \sinh(\eta'/\alpha)}{\cosh \zeta - \cosh (\Delta \eta/\alpha)} \right]^\frac{D+1}{2} \times \left( \frac{\partial_\zeta}{\sinh \zeta} \right)^\frac{D-3}{2} \frac{2\zeta}{\zeta^2 - (\Delta \eta)^2/\alpha^2} \]  

with \(x = (\eta, r, \vartheta, \phi)\), is the Hadamard function for a scalar field in static hyperbolic universe with the line element given by the expression in the square brackets of (21). By using the result

\[ \sum_{n=-\infty}^{\infty} \frac{a}{n^2 + a^2} = \pi \coth(\pi a), \]  

the following relation is proved:

\[ \sum_{n=-\infty}^{\infty} G_{st}(x_n, x') = \frac{\alpha^{1-D}}{(2\pi)^{D+1/2}} \left( \frac{\partial_\zeta}{\sinh \zeta} \right)^\frac{D-3}{2} \frac{y \sinh(y\zeta)}{\cosh(y\zeta) - \cosh(y\Delta \eta/\alpha)}, \]  

where \(x_n = (\eta + in\beta, r, \vartheta, \phi)\) and \(y = 2\pi\alpha/\beta\). Taking \(\beta = 2\pi\alpha\), we see that the state corresponding to the part with the second term in the square brackets of (34) is interpreted as a thermal state with temperature \(1/(2\pi\alpha)\). This result for \(D = 3\) has been discussed in [7].

### 3.2 Even \(D\)

For even values of \(D\), taking \(n = D/2 - 1\), the expression of the Hadamard function for the H-vacuum reads

\[ G(x, x') = -\frac{\alpha^{1-D}}{(2\pi)^{D/2}} \left[ \frac{\sinh(\eta/\alpha) \sinh(\eta'/\alpha)}{\cosh \zeta - \cosh (\Delta \eta/\alpha)} \right]^\frac{D-1}{2} \times \partial_w^{D/2-1} \int_0^\infty dz \tanh(\pi z) \cos(z\Delta \eta/\alpha) P_{\frac{D-1}{2}}(w). \]  

\[ (38) \]
For the BD vacuum one has the expression (27). By taking into account the relation
\[ \frac{1}{y^{D/2-1}} = (-1)^{D/2-1} \pi^{1/2} \frac{1}{\Gamma((D-1)/2)} \frac{1}{\sqrt{y}} \]  
and (30), it can be rewritten in the form
\[ G_{\text{BD}}(x, x') = -\frac{[\text{sinh}(\eta/\alpha) \text{sinh}(\eta'/\alpha)]^{D-1}}{\sqrt{2}} \frac{1}{\alpha^{D-1}} \frac{1}{\sqrt{w - \cosh(\Delta\eta/\alpha)}}. \]  
Next we use the relation [29]
\[ \int_0^\infty dz \cos(z\Delta\eta/\alpha) P_{iz-1/2}(w) = \frac{1}{\sqrt{2} \cosh(\Delta\eta/\alpha)^{1/2}}, \]  
to present the Hadamard function in the form
\[ G_{\text{BD}}(x, x') = -\frac{\alpha^{1-D}}{(-2\pi)^{D/2}} [\text{sinh}(\eta/\alpha) \text{sinh}(\eta'/\alpha)]^{D-1} \times \partial_{D/2-1}^{D/2-1} \int_0^\infty dz \cos(z\Delta\eta/\alpha) P_{iz-1/2}(w). \]  
For the difference of the Hadamard functions we get
\[ G(x, x') - G_{\text{BD}}(x, x') = \frac{2\alpha^{1-D}}{(-2\pi)^{D/2}} [\text{sinh}(\eta/\alpha) \text{sinh}(\eta'/\alpha)]^{D-1} \times \int_0^\infty dz \frac{\cos(z\Delta\eta/\alpha)}{e^{2\pi z} + 1} \frac{P_{iz-1/2}(w)}{(w^2 - 1)^{1/2}}, \]  
where we have introduced the associated Legendre function of the first kind in accordance with \( \partial_{D/2-1}^{D/2-1} P_{iz-1/2}(w) = (w^2 - 1)^{(2-D)/4} P_{iz-1/2}(w) \).

In the discussion above we have considered the difference in the Hadamard functions for the H- and BD vacua. Similar expressions are obtained for the differences of other two-point functions. For example, the difference in the Wightman functions for odd and even values of spatial dimension is given by the right-hand sides of (34) and (43) with an additional coefficient $1/2$. The corresponding formulas can be used for the investigation of the response of particle detectors.

4 Vacuum expectation values

In this section we evaluate the VEVs of the field squared and of the energy-momentum tensor for the H-vacuum by using the formulas (34) and (43).

4.1 Mean field squared

We start the investigation for local VEVs from the mean field squared. Having the difference between the Hadamard functions, it can be evaluated by making use of the formula
\[ \langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{\text{BD}} + \frac{1}{2} \lim_{x' \rightarrow x} [G(x, x') - G_{\text{BD}}(x, x')]. \]
The important thing to be mentioned here is that the last term is finite and the renormalization of the divergences is reduced to the one for $\langle \phi^2 \rangle_{\text{BD}}$. The latter procedure is widely discussed in the literature. Here we note that, because of the maximal symmetry of the BD vacuum, the VEV $\langle \phi^2 \rangle_{\text{BD}}$ does not depend on the spacetime coordinates.

We start from the case of even $D$. By taking into account that for $w \to 1^{+}$ one has

$$P^\mu_{iz-1/2}(w) \approx \frac{\Gamma(iz + 1/2 + \mu)}{\Gamma(\mu + 1)\Gamma(iz + 1/2 - \mu)} \left(\frac{w - 1}{2}\right)^{\mu/2},$$

we get

$$\lim_{w \to 1^{+}} \frac{P^{D/2-1}_{iz-1/2}(w)}{(w^2 - 1)^{D/4}} = \frac{2^{1-D/2}\Gamma(iz + (D-1)/2)}{\Gamma(D/2)\Gamma(iz - (D-3)/2)}.$$  

By using the relations for the gamma function, the VEV is presented as

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_{\text{BD}} - \frac{[\alpha \sinh(t/\alpha)]^{1-D}}{2^{D/2-1/2}D!} \int_{0}^{\infty} dz \ e^{-\pi z} |\Gamma((D-1)/2 + iz)|^2. \quad (47)$$

An equivalent expression is given by

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_{\text{BD}} - \frac{2[\alpha \sinh(t/\alpha)]^{1-D}}{(4\pi)^{D/2}D!} \int_{0}^{\infty} dz \ \frac{z^{D-2}}{e^{2\pi z} + 1} \prod_{l=0}^{D/2-2} \left(\frac{l+1/2}{z}\right)^2, \quad (48)$$

where the integral is expressed in terms of the Riemann zeta function. Note that the difference $\langle \phi^2 \rangle - \langle \phi^2 \rangle_{\text{BD}}$ is negative.

Now we turn to the case of odd $D$. In the evaluation of the coincidence limit for (34) first we put $\theta = 0$ and $\Delta \eta = 0$. In this case one gets $\zeta = \Delta r = r' - r$. For the mean field squared we find

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_{\text{BD}} + \frac{(-2\pi)^{-D+1}}{2[\alpha \sinh(t/\alpha)]^{D-1}} \lim_{w \to 1^{+}} \frac{\partial^{D-3}}{\partial w^{D-3}} g(w), \quad (49)$$

with the function

$$g(w) = \frac{2}{\arccosh(w) \sqrt{w^2 - 1}} - \frac{1}{w - 1}. \quad (50)$$

We define the constants $a_l$, $l = 0, 1, 2, \ldots$ in accordance with

$$g(w) = \frac{1}{6} \sum_{l=0}^{\infty} a_l (w - 1)^l. \quad (51)$$

For the first five coefficients one has

$$a_0 = -1, \ a_1 = \frac{11}{30}, \ a_2 = -\frac{191}{1260}, \ a_3 = \frac{2497}{37800}, \ a_4 = -\frac{14797}{498960}. \quad (52)$$

The mean field squared is expressed as

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_{\text{BD}} - \frac{(2\pi)^{-D+1} b_D}{12[\alpha \sinh(t/\alpha)]^{D-1}}. \quad (53)$$

where

$$b_D = (-1)^{D-1} \Gamma((D-1)/2) a_{(D-3)/2}. \quad (54)$$
Note that the constants $b_D$ can be directly expressed as

$$b_D = 6 (-1)^{D-1} \lim_{w \to 1} \partial_{w^{D-3}} g(w).$$

(55)

For the first five coefficients we get

$$b_3 = 1, \quad b_5 = \frac{11}{30}, \quad b_7 = \frac{191}{630}, \quad b_9 = \frac{2497}{6300}, \quad b_{11} = \frac{14797}{20790}.$$  

(56)

Similar to the previous case, the difference $\langle \varphi^2 \rangle - \langle \varphi^2 \rangle_{BD}$ is negative. It can be checked that, similar to (48), the result (53) is written in the integral form as

$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{BD} - \frac{2 [\alpha \sinh(t/\alpha)]^{1-D}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty dz \frac{z^{D-2}}{e^{2\pi z} - 1} \prod_{l=0}^{D-3/2} \left[ \left( \frac{l}{z} \right)^2 + 1 \right].$$

(57)

Combining the formulas given above, for the mean field squared we write

$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{BD} - \frac{B_D}{[\alpha \sinh(t/\alpha)]^{D-1}},$$

(58)

where the coefficient is given by the expression

$$B_D = \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dz \frac{z^{D-2} A_D(z)}{e^{2\pi z} + (-1)^D}.$$  

(59)

Here we have introduced the function

$$A_D(z) = \prod_{l=0}^{l_m} \left[ \left( \frac{l+1/2 - \{D/2\}}{z} \right)^2 + 1 \right],$$

(60)

where $l_m = D/2 - 2 + \{D/2\}$ and $\{D/2\}$ stands for the fractional part of $D/2$.

The relation (58) between the VEVs in the H- and BD vacua is similar to the corresponding relations between the Fulling-Rindler and Minkowski vacua in flat spacetime. In the Rindler coordinates $(\tau_R, \chi, x_R)$, with $x_R = (x^2_R, \ldots, x^D_R)$, the Minkowskian line element is written as

$$ds_M^2 = \chi^2 d\tau^2_R - dx^2 - dx_R^2,$$

(61)

and the mean field squared for a scalar massless field in the Fulling-Rindler and Minkowski vacua are connected by the relation [31]

$$\langle \varphi^2 \rangle_{FR} = \langle \varphi^2 \rangle_M - B_D \chi^{1-D},$$

(62)

with the same coefficient $B_D$ from (59). The worldline with fixed spatial coordinates $(\chi, x_R)$ describes an observer with constant proper acceleration $1/\chi$.

### 4.2 VEV of the energy-momentum tensor

Another important local characteristic of the vacuum state is the VEV of the energy-momentum tensor. The difference in the VEVs for the H- and BD vacua, $\Delta \langle T_{ik} \rangle = \langle T_{ik} \rangle - \langle T_{ik} \rangle_{BD}$, can be evaluated by making use of the formula

$$\Delta \langle T_{ik} \rangle = \frac{1}{2} \lim_{x' \to x} \partial_{x'} \partial_k \left[ G(x, x') - G_{BD}(x, x') \right] + \left[ \xi - \frac{1}{4} g_{ik} \nabla_p \nabla^p - \xi \nabla_i \nabla_k - \xi R_{ik} \right] (\langle \varphi^2 \rangle - \langle \varphi^2 \rangle_{BD}),$$

(63)
where \( \xi = (D - 1)/(4D) \) for a conformally coupled field, \( \nabla_i \) is the covariant derivative operator and \( R_{ik} = Dg_{ik}/\alpha^2 \) is the Ricci tensor for the dS spacetime. The expression in the right-hand side is finite and a renormalization is not required. It does not contain ambiguities that are present in various schemes of renormalization for separate parts \( \langle T_{ik} \rangle_\text{BD} \) and \( \langle T_{ik} \rangle_\text{BD} \) (for discussion of renormalization ambiguities in the VEV of the energy-momentum tensor see \[32, 33, 34\] and references therein). From the symmetry of the problem we expect that the VEV \( \langle T_i^k \rangle \) is a function of the time coordinate alone and the vacuum stresses are isotropic \( \Delta \langle T^0_1 \rangle = \Delta \langle T^2_2 \rangle = \cdots = \Delta \langle T^D_D \rangle \). The continuity equation \( \nabla_k \langle T^k_i \rangle = 0 \) leads to the relation

\[
\Delta \langle T^1_1 \rangle = \frac{\partial_{t/\alpha} [\sinh^{D-1}(t/\alpha)]}{\alpha \sinh(t/\alpha) \cosh(t/\alpha)},
\]

between the energy density and the stresses. We consider the case of a conformally coupled massless field and the tensor \( \langle T_i^k \rangle_\text{BD} \) is traceless \( \Delta \langle T^1_1 \rangle = 0 \). This leads to the relation \( \Delta \langle T^0_0 \rangle = -D \Delta \langle T^1_1 \rangle \). Combining this with (64) we conclude that the tensor \( \langle T^k_i \rangle \) has the structure

\[
\langle T^k_i \rangle_\text{BD} + C_D \frac{\text{diag}(1, -1/D, \ldots, -1/D)}{\alpha \sinh(t/\alpha)^{D+1}},
\]

where the VEV for the BD vacuum has the form \( \langle T^k_i \rangle_\text{BD} = C_D^{(BD)} \delta^k_i \alpha^{-D-1}/(D+1) \) with a numerical constant \( C_D^{(BD)} \). Note that the latter is completely determined by the trace anomaly: \( C_D^{(BD)} = \alpha^{D+1} \langle T^1_1 \rangle_\text{BD} \). In odd dimensional spacetimes the trace anomaly is absent and \( C_D^{(BD)} = 0 \). For \( D = 3 \) one has \( C_3^{(BD)} = 1/(240\pi^2) \) and the trace anomaly in odd dimensional dS spacetimes with \( D > 3 \) has been investigated in \[35\]. In particular, \( C_5^{(BD)} = -5/(4032\pi^3) \) and \( C_7^{(BD)} = 23/(34560\pi^4) \). The sign of the coefficient \( C_D^{(BD)} \) is determined by \( (D-3)/2 \). The conformal anomaly in spaces with hyperbolic spatial sections was considered in \[36\] (for a recent discussion of renormalization of the energy-momentum tensor in a general spacetime of arbitrary dimension see \[34\]). As seen from (65), for the evaluation of the difference in the VEVs for the BD and H-vacua it is sufficient to evaluate one of the components in (63).

We will consider the component \( \Delta \langle T_{11} \rangle \). First of all it can be seen that

\[
\nabla_\rho \nabla^\rho \left( \langle \varphi^2 \rangle - \langle \varphi^2 \rangle_\text{BD} \right) = -\frac{D-1}{\alpha^2} \left( \langle \varphi^2 \rangle - \langle \varphi^2 \rangle_\text{BD} \right),
\]

\[
\nabla_1 \nabla_1 \left( \langle \varphi^2 \rangle - \langle \varphi^2 \rangle_\text{BD} \right) = (D-1) \cosh^2(t/\alpha) \left( \langle \varphi^2 \rangle - \langle \varphi^2 \rangle_\text{BD} \right).
\]

In the evaluation of the part in (63) containing the coincidence limit we will consider the cases of even and odd \( D \) separately. For even values of \( D \), the difference of the Hadamard functions is expressed as \( \text{Hadamard functions} \). As before, putting \( \theta = 0 \) and \( \Delta \eta = 0 \), the term with the coincidence limit is presented as

\[
\lim_{x' \to x} \nabla_1 \nabla_1 \left[ G(x, x') \right] = -\frac{2(-2\pi)^{-D/2}}{\alpha \sinh(t/\alpha)^{D+1}} \int_0^\infty dz \frac{1}{e^{2\pi z} + 1} \times \lim_{x' \to x} \partial^2_{\Delta r} \left( \frac{P_{iz-1/2}^{D/2-1}(w)}{(w^2 - 1)^{(D-2)/4}} \right).
\]

By taking into account \( \partial^2_{\Delta r} = (\sqrt{w^2 - 1}^2 \partial w)^2 \) and by using the recurrence relations for the associated Legendre function, it can be shown that

\[
\left( \sqrt{w^2 - 1} \frac{\partial}{\partial w} \right)^2 \left( \frac{P^{D/2-1}_{iz-1/2}(w)}{(w^2 - 1)^{(D-2)/4}} \right) = \frac{P^{D/2+1}_{iz-1/2}(w)}{(w^2 - 1)^{(D-2)/4}} + w \frac{P^{D/2}_{iz-1/2}(w)}{(w^2 - 1)^{(D-2)/4}}.
\]
From (45) it follows that the contribution of the first term in the right-hand side of (68) vanishes in the limit \( w \to 0 \) and one obtains
\[
\lim_{x' \to x} \partial_1 \partial_1 \left[ G(x, x') - G_{BD}(x, x') \right] = -\frac{[\alpha \sinh(t/\alpha)]^{1-D}}{2^D \pi^{D/2} \Gamma(D/2 + 1)} \times \int_0^\infty dz \ e^{-\pi z} |\Gamma((D + 1)/2 + i z)|^2. \tag{69}
\]
With this result, from (63) we find the expression for \( \Delta \langle T^1_1 \rangle \). For the coefficient in (65) one gets
\[
C_D = -\frac{\pi^{-D/2-1}}{2^D \Gamma(D/2)} \int_0^\infty dz \ e^{-\pi z} z^2 |\Gamma((D - 1)/2 + i z)|^2
= -\frac{2^{1-D} \pi^{-D/2}}{\Gamma(D/2)} \int_0^\infty dz \ \frac{z^D A_D(z)}{e^{2\pi z} - 1}, \tag{70}
\]
with \( A_D \) defined in (60). In the special case \( D = 4 \) we obtain
\[
C_4 = -\frac{3}{2^9 \pi^7} \left[ \pi^2 \zeta(3) + 15 \zeta(5) \right] , \tag{71}
\]
with \( \zeta(x) \) being the Riemann zeta function.

For odd values of \( D \) we can put \( \theta = 0 \) and \( \Delta \eta = 0 \) in (54). The limit is reduced to
\[
\lim_{x' \to x} \partial_1 \partial_1 \left[ G_0(x, x') - G_{0(BD)}(x, x') \right] = -\frac{(-2\pi)^{-D+1}}{\Gamma(D-1) \alpha \sinh(t/\alpha)} \lim_{x' \to x} \partial_2 \partial_{\omega^2} \omega^2 \ g(w). \tag{72}
\]
Combining this result with (66) and by using the expansion (51) we can show that
\[
C_D = \frac{(D - 1)^2 b_D - 4Db_{D+2}}{48(2\pi)^{D/2+1}}. \tag{73}
\]
In particular, one has
\[
C_3 = -\frac{1}{480 \pi^2}, \quad C_5 = -\frac{31}{60480 \pi^3}. \tag{74}
\]
The result for \( D = 3 \) coincides with that found in [7]. As seen, in both cases of odd and even \( D \) the energy density in the H-vacuum is smaller than the one for the BD vacuum.

Similar to the case of the field squared, it can be shown that for odd \( D \) the coefficient \( C_D \) is presented in the integral form
\[
C_D = -\frac{2^{1-D} \pi^{-D/2}}{\Gamma(D/2)} \int_0^\infty dz \ \frac{z^D A_D(z)}{e^{2\pi z} - 1}, \tag{75}
\]
where \( A_D(z) \) is given by (60). Combining the results for even and odd \( D \), the coefficient in the expression (65) for the VEV of the energy-momentum tensor is written as
\[
C_D = \frac{2^{1-D} \pi^{-D/2}}{\Gamma(D/2)} \int_0^\infty dz \ \frac{z^D A_D(z)}{e^{2\pi z} + (-1)^D}. \tag{76}
\]
4.3 Density of states and asymptotics

Introducing the energy $E = \frac{z}{\alpha}$, the energy density for the H-vacuum is written in the form

$$\langle T_0^0 \rangle_{\text{BD}} = \langle T_0^0 \rangle_{\text{BD}} - \frac{\sinh^{-D-1}(t/\alpha)}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty dE \frac{E^D A_D(\alpha E)}{e^{2\pi\alpha E} + (-1)^D}. \quad (77)$$

This shows the thermal nature of the BD vacuum with respect to the H-vacuum with the temperature $T = 1/(2\pi\alpha)$. Denoting by $\rho(E)dE$ the number of states in the energy range $(E, E + dE)$, from (77) we read the density of states

$$\rho(E) = \frac{2E^{D-1}A_D(\alpha E)}{(4\pi)^{D/2}\Gamma(D/2)}. \quad (78)$$

The same expression is obtained when one considers the thermal properties of the Minkowski vacuum with respect to the Fulling-Rindler vacuum in flat spacetime. Note that the density of states $\rho_M(E)$ for zero spin massless particles in Minkowski spacetime is obtained by integrating the number of states $d^DP/(2\pi)^D$ over the angles determining the direction of the momentum $p$ and is related to (78) by the formula $\rho(E) = \rho_M(E)A_D(\alpha E)$. It is of interest to note that in even number of spatial dimensions the average number of particles is given by Fermi-Dirac distribution. Similar features for scalar Rindler particles in Minkowski vacuum and in the response of particle detectors have been already discussed in the literature [37, 38, 39, 40].

As it has been mentioned above, the relation between the VEVs in the BD and H-vacuum states is similar to that for the Minkowski and Fulling-Rindler vacua in flat spacetime. This is related to the conformal connection between the dS and Rindler spacetimes. To show that (see also [1] for the case $D = 3$) let us consider the coordinate transformation $(\tau_R, \chi, x_R) \rightarrow (\eta, r, \theta, \phi)$ with

$$\tau_R = \frac{\eta}{\alpha}, \quad \chi = \frac{\alpha}{\cosh r - \sinh r \cos \theta_1}, \quad x_R^l = \chi w^l \sinh r, \quad l = 2, \ldots, D. \quad (79)$$

In these relations

$$w^2 = \sin \theta_1 \cos \theta_2, \ldots, w^{D-2} = \cos \theta_{D-2} \prod_{i=1}^{D-3} \sin \theta_i,$$

$$w^{D-1} = \cos \phi \prod_{i=1}^{D-2} \sin \theta_i, \quad w^D = \sin \phi \prod_{i=1}^{D-2} \sin \theta_i. \quad (80)$$

The Rindler line element (61) is transformed to

$$ds_R^2 = \chi^2 (d\eta^2/\alpha^2 - dr^2 - \sinh^2 r d\Omega_{D-1}^2). \quad (81)$$

Comparing with (21), the conformal relation

$$ds^2 = \Omega_R^2 ds_R^2, \quad \Omega_R^2 = \frac{(\cosh r - \sinh r \cos \theta_1)^2}{\sinh^2 (\eta/\alpha)}, \quad (82)$$

is seen between the dS and the Rindler spacetimes. The conformal counterpart of the H-vacuum in dS spacetime is the Fulling-Rindler vacuum in flat spacetime and this explains the above mentioned similarity of the corresponding relations between the VEVs. The conformal relation between the Rindler and dS spacetimes has been used in [41] for the investigation of the vacuum average value of the energy-momentum tensor induced by a curved brane in dS spacetime.
Now let us consider the flat spacetime limit of the results given above. In that limit \( \alpha \to \infty \) and the line element (2) is reduced to the one for the Milne universe

\[
s_{\text{Milne}}^2 = dt^2 - t^2 (dr^2 + \sinh^2 r d\Omega_{D-1}^2).
\]  

(83)

The flat spacetime counterpart of the H-vacuum is the conformal vacuum in the Milne universe (for the investigation of the corresponding VEVs induced by a spherical boundary see [42]). The flat spacetime limit for the BD vacuum corresponds to the Minkowski vacuum. Assuming that the VEVs for the latter are renormalized to zero, for the VEVs in the conformal vacuum of the Milne universe from (58) we get

\[
\langle \phi^2 \rangle_{\text{Milne}} = -\frac{B_D}{t^{D-1}},
\]

\[
\langle T^k_{\text{Milne}} \rangle = \frac{C_D}{t^{D+1}} \text{diag} (1, -1/D, \cdots, -1/D),
\]

(84)

with the coefficients (59) and (76). These expressions give the leading terms in the asymptotic expansions for the VEVs of the field squared and energy-momentum tensor for a massive scalar field with general curvature coupling parameter at early stages of the cosmological expansion \( t \to 0 \). As seen, at early stages the VEVs are large and the backreaction of quantum effects is essential.

At late stages of the expansion, \( t \gg \alpha \), the difference in the VEVs for the H- and BD vacua is suppressed by the exponential factors \( e^{-D-1} t/\alpha \) and \( e^{-(D+1)} t/\alpha \) for the field squared and energy-momentum tensor, respectively. This is an example of the general result from [43], stating that under the condition \( m^2 + \xi R > 0 \) the BD vacuum is the future attractor for cosmological solutions driven by a scalar field with mass \( m \) and curvature coupling parameter \( \xi \) (for a more recent discussion of the attractor properties of the BD state in interacting field theories see [44]).

We have considered the VEV of the energy-momentum tensor in a fixed background. Among the interesting topics of quantum field theory in curved spacetime is the investigation of backreaction of quantum effects on the classical geometry (see, for example, [1, 45]). That is done on the base of semiclassical Einstein equations with the expectation value of the energy-momentum tensor of quantum fields as an additional gravitational source. In particular, motivated by stability issues of dS spacetime, the backreaction of quantum fields on the properties of the BD state has been widely considered in the literature by using variety of methods (see, e.g., [6, 46] and references cited there). The part in (65) corresponding to the BD vacuum state renormalizes the bare cosmological constant. From the point of view of backreaction, interesting effects come from the contribution in (65) corresponding to the difference in the properties of the BD and H-vacua. Its role becomes essential at early stages of the expansion and the backreaction effects should be taken into account. The energy density for that part is negative and it violates the energy conditions in general relativity. The corresponding equation of state is of the radiation type \( p = \varepsilon/D \), with the pressure \( p \) and the energy density \( \varepsilon \), though with the negative energy density. We expect that in a more general background, described by the line element (2) with a general scale factor \( a(t) \) instead of \( \alpha \sinh (t/\alpha) \), the difference of the vacuum energy-momentum tensors for a conformally coupled massless field in two homogeneous vacuum states will have a structure similar to the last term in (65) with the same replacement \( \alpha \sinh (t/\alpha) \to a(t) \). The corresponding cosmological dynamics can be investigated based on the Friedmann equation for open cosmological models in the presence of a positive cosmological constant. We plan to return to this point in future work.

The results obtained above can be applied to dS bubbles with different vacua in the interior and exterior regions. For those geometries the last term in the right-hand side of (65) is present
in the region with the H-vacuum and is absent in the region with the BD vacuum. Note that in those models additional contributions to the VEVs come from the boundary, separating two phases. These are the Casimir type contribution to the vacuum characteristics (see [47] for a general discussion).

5 Conclusion

We have investigated the mean field squared and the VEV of the energy-momentum tensor for a scalar field prepared in the H-vacuum of dS spacetime. Bearing in mind possible applications in field-theoretical models with extra spatial dimensions, a general number of spacetime dimension is considered. The properties of the vacuum state are described by two-point functions and as the first step the Hadamard function is discussed. The corresponding expression is decomposed into two contributions. The first one presents the Hadamard function for the BD vacuum and the second one describes the difference in the correlations of the vacuum fluctuations in those two vacua. With that representation, the renormalization of the VEVs for the H-vacuum is reduced to the renormalization for the BD vacuum state. The latter is well investigated in the literature. The expression for the Hadamard function is essentially simplified in the special case of a conformally coupled massless scalar field. The corresponding expressions for odd and even values of the spatial dimension are given by (25) and (38), respectively. The Hadamard function for the BD vacuum is expressed as (31) and (40).

Given the Hadamard functions, the differences in the mean field squared and the VEV of the energy-momentum tensor for the H- and BD vacua are obtained by making use of the formulas (44) and (63). The divergences in the coincidence limit are the same for those states and the renormalization is not required for the differences. The corresponding expressions for general case of spatial dimension are given by (58) and (65). The coefficients in those formulas are expressed as (59) and (76). These expressions prove the thermal nature of the BD state with respect to the H-vacuum. The corresponding density of states is expressed as (78) with the factor $A_D(\alpha E)$ defined by (60). The latter is interpreted in terms of the ratio of the densities of states for a conformally coupled massless field in dS spacetime and for massless zero spin particles in Minkowski spacetime. It is of interest to note that in odd number of spacetime dimensions the thermal distribution is of Fermi-Dirac type. The essential difference of the dynamics of quantum fields in odd and even dimensional dS spacetimes is also seen in the analysis of the particle production process. As it has been discussed in [48], there is no particle production in odd dimensions and particles are created in even number of dimensions. At late stages of the dS expansion the difference between the VEVs in the H- and BD vacua is exponentially suppressed and this is in agreement with the result on the BD vacuum as the future attractor for a general class of cosmological solutions. At early stages the contributions corresponding to the BD vacuum are subdominant and the behavior of the VEVs is essentially different. At those stages the vacuum energy-momentum tensor is large and the backreaction of quantum effects on the spacetime geometry should be taken into account.

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References

[1] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, UK, 1982).

[2] B. Allen, Vacuum states in de Sitter space. Phys. Rev. D 32, 3136 (1985).

[3] E. Mottola, Particle creation in de Sitter space. Phys. Rev. D 31, 754 (1985).

[4] T. S. Bunch and P. C. W. Davies, Quantum field theory in de Sitter space: renormalization by point-splitting. Proc. Roy. Soc. Lond. A 360, 117 (1978).

[5] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein’s General Relativity* (Cambridge University Press, Cambridge, England, 2009).

[6] L. Aalsma, M. Parikh and J. P. van der Schaar, Back(reaction) to the future in the Unruh-de Sitter state. J. High Energy Phys. 11 (2019) 136.

[7] J. D. Pfautsch, A new vacuum state in de Sitter space. Phys. Lett. B 117, 283 (1982).

[8] J. R. Gott, Creation of open universes from de Sitter space. Nature 295, 304 (1982).

[9] M. Bucher, A. S. Goldhaber, and N. Turok, An open universe from inflation. Phys. Rev. D 52, 3314 (1995).

[10] K. Yamamoto, M. Sasaki, and T. Tanaka, Large angle CMB anisotropy in an open universe in the one bubble inflationary scenario. Astrophys. J. 455, 412 (1995).

[11] D. H. Lyth and A. Woszczyna, Large scale perturbations in the open universe, Phys. Rev. D 52, 3338 (1995).

[12] J. García-Bellido, J. Garriga, and X. Montes, Quasiopen inflation. Phys. Rev. D 57, 4669 (1998).

[13] D. Yamauchi, A. Linde, A. Naruko, M. Sasaki, and T. Tanaka, Open inflation in the landscape. Phys. Rev. D 84, 043513 (2011).

[14] K. Sugimura, D. Yamauchi, and M. Sasaki, Multi-field open inflation model and multi-field dynamics in tunneling. JCAP 01 (2012) 027.

[15] H. Matsui, F. Takahashi, and W. Yin, QCD axion window and false vacuum Higgs inflation. J. High Energy Phys. 05 (2020) 154.

[16] J. Maldacena and G. L. Pimentel, Entanglement entropy in de Sitter space. J. High Energy Phys. 02 (2013) 038.

[17] S. Kanno, J. Murugan, J. P. Shock, and J. Soda, Entanglement entropy of α-vacua in de Sitter space. J. High Energy Phys. 07 (2014) 072.

[18] N. Iizuka, T. Noumi, and N. Ogawa, Entanglement entropy of de Sitter space α-vacua. Nucl. Phys. B 910, 23 (2016).

[19] S. Kanno, M. Sasaki, and T. Tanaka, Vacuum state of the Dirac field in de Sitter space and entanglement entropy. J. High Energy Phys. 03 (2017) 068.
[20] S. Bhattacharya, S. Chakrabortty, and S. Goyal, Emergent α-like fermionic vacuum structure and entanglement in the hyperbolic de Sitter spacetime. Eur. Phys. J. C 79, 799 (2019).

[21] K. A. Milton and A. A. Saharian, Casimir densities for a spherical boundary in de Sitter spacetime. Phys. Rev. D 85, 064005 (2012).

[22] S. Bellucci, A. A. Saharian, and A. H. Yeranyan, Casimir densities from coexisting vacua. Phys. Rev. D 89, 105006 (2014).

[23] A. A. Saharian and T. A. Petrosyan, Casimir densities induced by a sphere in the hyperbolic vacuum of de Sitter spacetime. Phys. Rev. D 104, 065017 (2021).

[24] A. Albrecht, S. Kanno, and M. Sasaki, Quantum entanglement in de Sitter space with a wall and the decoherence of bubble universes. Phys. Rev. D 97, 083520 (2018).

[25] M. Sasaki, T. Tanaka, and K. Yamamoto, Euclidean vacuum mode functions for a scalar field on open de Sitter space. Phys. Rev. D 51, 2979 (1995).

[26] F. V. Dimitrakopoulos, L. Kabir, B. Mosk, M. Parikh, and J. P. van der Schaar, Vacua and correlators in hyperbolic de Sitter space. J. High Energy Phys. 06 (2015) 095.

[27] P. Candelas and D. J. Raine, General-relativistic quantum field theory: An exactly soluble model. Phys. Rev. D 12, 965 (1975).

[28] T. S. Bunch, Stress tensor of massless conformal quantum fields in hyperbolic universes. Phys. Rev. D 18, 1844 (1978).

[29] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series (Gordon and Breach, New York, 1986), Vol. II.

[30] F.W. Olver et al, NIST Handbook of Mathematical Functions (Cambridge University Press, USA, 2010).

[31] A. A. Saharian, Polarization of the Fulling-Rindler vacuum by a uniformly accelerated mirror. Class. Quantum Grav. 19, 5039 (2002).

[32] V. Moretti, Comments on the stress-energy tensor operator in curved spacetime. Commun. Math. Phys. 232, 189 (2003).

[33] S. Hollands and R. M. Wald, Conservation of the stress tensor in perturbative interacting quantum field theory in curved spacetime. Rev. Math. Phys. 17, 227 (2005).

[34] Y. Décanini and A. Folacci, Hadamard renormalization of the stress-energy tensor for a quantized scalar field in a general spacetime of arbitrary dimension. Phys. Rev. D 78, 044025 (2008).

[35] E. J. Copeland and D. J. Toms, The conformal anomaly in higher dimensions. Class. Quantum Grav. 3, 431 (1986).

[36] A. A. Bytsenko, E. Elizalde, and S. D. Odintsov, The conformal anomaly in N-dimensional spaces having a hyperbolic spatial section. J. Math. Phys. 36, 5084 (1995).

[37] S. Tagaki, On the response of a Rindler-particle detector. II. Prog. Theor. Phys. 74, 142 (1985).
[38] H. Ooguri, Spectrum of Hawking radiation and the Huygens principle. Phys. Rev. D 33, 3573 (1986).

[39] S. Tagaki, Vacuum noise and stress induced by uniform acceleration. Prog. Theor. Phys. Suppl. 88, 1 (1986).

[40] D. Jennings, On the response of a particle detector in anti-de Sitter spacetime. Class. Quantum Grav. 27, 205005 (2010).

[41] A. A. Saharian and M. R. Setare, Casimir energy-momentum tensor for a brane in de Sitter spacetime. Phys. Lett. B 584, 306 (2004).

[42] A. A. Saharian and T. A. Petrosyan, The Casimir densities for a sphere in the Milne universe. Symmetry 12, 619 (2020).

[43] P. R. Anderson, W. Eaker, S. Habib, C. Molina-París, and E. Mottola, Attractor states and quantum instabilities in de Sitter space. Int. J. Theor. Phys. 40, 2217 (2001).

[44] A. Buchel and A. Karapetyan, De Sitter vacua of strongly interacting QFT. J. High Energy Phys. 03 (2017) 114.

[45] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, Effective Action in Quantum Gravity (IOP Publishing, Bristol, UK, 1992).

[46] H. Matsui, Instability of de Sitter spacetime induced by quantum conformal anomaly. JCAP 01 (2019) 003.

[47] S. Bellucci, A. A. Saharian, and A. H. Yeranyan, Casimir densities from coexisting vacua. Phys. Rev. D 89, 105006 (2014).

[48] R. Bousso, A. Maloney, and A. Strominger, Conformal vacua and entropy in de Sitter space. Phys. Rev. D 65, 104039 (2002).