Time variable Λ and the accelerating Universe

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We perform a deductive study of accelerating Universe and focus on the importance of variable time-dependent Λ in the Einstein’s field equations under the phenomenological assumption, Λ = αH\textsuperscript{2} for the full physical range of α. The relevance of variable Λ with regard to various key issues like dark matter, dark energy, geometry of the field, age of the Universe, deceleration parameter and barotropic equation of state has been trivially addressed. The deceleration parameter and the barotropic equation of state parameter obey a straight line relationship for a flat Universe described by Friedmann and Raychaudhuri equations. Both the parameters are found identical for α = 1.

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To account for the vast majority of mass in the observable Universe and to explain its accelerated expansion, the physical cosmology of today requires two outstanding concepts: (i) the matter which does not interact with the electromagnetic force - dark matter (ii) and the hypothetical energy that tends to increase the rate of expansion of the Universe - dark energy.

Zwicky \cite{1}, using Virial Theorem, had suggested for a possible existence of dark matter long ago, which was later on supported by the studies of rotation curves \cite{2,3}, gravitational lensing \cite{4,5}, CMB anisotropy \cite{6,7} and bullet clusters \cite{8,9}. The advent of inflationary theory \cite{10,11,12} led to the convincing belief that 96% of matter content of the Universe is hidden mass constituted by 23% dark matter and 73% dark energy \cite{13}. Dark matter plays a central role in early Universe during structure formation and galaxy evolution because of its nature to clump in sub-megaparsec scales. COBE and CMB experiments suggest that baryonic dark matter is not more than a small fraction of the total dark matter present in the Universe \cite{14}. More to it, observational constraints regarding neutrino mass and relic neutrino density \cite{15,16} eliminate the possibility of hot dark matter in favor of cold non-baryonic dark matter \cite{14}.

Observational results for an accelerating Universe \cite{17,18} favored the idea of an accelerating agent that had been referred to as dark energy. In accordance with the data, the already introduced Standard Cold Dark Matter (SCDM) model is found giving way to Λ-CDM model that has an advantage of assuming a nearly scale-invariant primordial perturbations and a Universe with no spatial curvature as predicted by the Inflationary theory \cite{19,20,21}. It is found in good agreement with various observational results \cite{22}. In the Λ-CDM scenario the present acceleration of the Universe cannot be a permanent feature because, structure formation cannot proceed during acceleration. Thus, the Universe must have undergone a decelerating phase prior to the present accelerating phase \cite{23}. Observational evidence \cite{24} also supports this idea. So, the deceleration parameter must have undergone a flip in sign during cosmic evolution.

One of the favorite candidates among the dark energy models is related to dynamic Λ \cite{25,26}. In fact, the concept of dark energy and the physics of accelerating Universe appears to be inherent in the Λ-term of Einstein’s field equations. Herein, we perform a study of accelerating Universe in context of the time-dependent Λ in the field equations and reveal the importance of dynamic Λ while addressing various key issues and some known physical observables like dark matter, dark energy, geometry of the field, deceleration parameter and its sign flip, age of the Universe and equation of state parameter.

For this we consider the Einstein’s field equations,

\[
R^{ij} - \frac{1}{2} R g^{ij} = -8\pi G \left[T^{ij} - \frac{\Lambda}{8\pi G} g^{ij}\right],
\]

which yield Friedmann equation

\[
\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G \rho}{3} + \frac{\Lambda}{3},
\]

for the spherically symmetric FLRW metric. We also get Raychaudhuri equation

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}
\]

in connection to the evolution scenario for expansion of null or time-like geodesic congruences. Here, \(\Lambda = \Lambda(t)\) is the time-dependent function of the erstwhile cosmological constant as introduced by Einstein, \(a = a(t)\) is the scale factor of the Universe and \(k\) is the curvature constant.

The Raychaudhuri equation \cite{23} at once shows that for \(\rho + 3p = 0\), acceleration is initiated by the Λ-term only...
that seems to relate \( \Lambda \) with dark energy. It also shows that for a positive \( \Lambda \), the Universe may accelerate with the condition \( \rho + 3p \leq 0 \) i.e. \( p \) is negative for a positive \( \rho \) with a definite contribution of \( \Lambda \) in the acceleration. The placement of \( \Lambda \) in the field equations itself suggests for it to be a part of total energy momentum \( T^{ij} = T^{ij} - \Lambda g^{ij}/8\pi G \) of the Universe. In fact, for a variable \( \Lambda \), solutions of field equations are possible only, if instead of \( T^{ij} \), the total energy momentum \( \tilde{T}^{ij} \) is conserved \([27]\) and in that case the second term \((-\Lambda g^{ij}/8\pi G)\) of the above equation acts as an additional source term in the field equations. In the observational front, the data set coming from the Supernova Legacy Survey (SNLS) during its first year of observation show that dark energy behaves in the same manner as that of a cosmological constant. Not ignoring the fact that the flat Universe would continue evolving with variable \( \Lambda \) the present day, \( H_0 \), and other related quantities (\( \Lambda_0, p_0, \rho_0 \) and \( q_0 \)) may be treated as dynamic and time variable.

Equations \([7]\) and \([5]\) yield the barotropic equation of state

\[ \omega_0 = \frac{p_0}{\rho_0} = -\frac{1 - \alpha - 2q_0}{3 - \alpha} = q_0 \bigg|_{\alpha=1} \]  

representing a straight line

\[ q_0 = \frac{3 - \alpha}{2} \omega_0 + \frac{1 - \alpha}{2} \]  

with \( dq_0/d\omega_0 = (3 - \alpha)/2 \). For \( \alpha = 1 \), equation of state parameter equals the deceleration parameter and therefore is expected to obey the same physical conditions. More to it, as a direct consequence of Friedmann and Raychaudhuri equations, we get while adding equations \([7]\) and \([5]\)

\[ p_0 + \rho_0 = \frac{\Lambda_0}{4\pi G \alpha} \left(1 + q_0\right) = \frac{\Lambda_0}{4\pi G \alpha} \left(1 + q_0\right) \bigg|_{\alpha=1}. \]  

This, for the vacuum equation of state, \( p_0 + \rho_0 = 0 \) \([32, 34, 35, 36]\), provides the condition of constant acceleration \( (q_0 = -1) \) \([37]\). In general, the matter-energy density being positive the counterpart negative pressure acts as a repulsive agent and hence the vacuum equation of state has a deep implication in the case of accelerating Universe scenario. Obviously, for a collapsing Universe with positive \( \rho \), we find \( q_0 < -1 \) and for an accelerating Universe we have \( q_0 > -1 \). Its higher limit may be positive depending upon the density. It is evident from equation \([11]\) that the fate of the Universe depends on \( q_0 \).

Bearing in mind the Hubble’s law and the assumption \( \Lambda = \alpha H^2 \), we may directly arrive from equation \([5]\) at \( q_0 = -1 - \Lambda_0/2\alpha H_0^2 \). This demonstrates that for \( \Lambda_0 = 0 \) or for \( \Lambda_0/H_0^2 = \) constant, the Universe has been evolving through a constant acceleration as indicated earlier \([37]\). It is interesting to note here that Einstein initially obtained an expanding Universe with \( \Lambda = 0 \) (and hence to counteract the dynamical effects of gravity, which would cause the matter-filled Universe to collapse, he later on adopted a non-zero \( \Lambda \) to obtain a static model) while de Sitter obtained a similar expanding Universe with constant \( \Lambda \) and devoid of any ordinary matter (which ultimately made Einstein to drop the cosmological constant from his general relativistic field equations). We are also curious about the situation \( \Lambda_0 = 0 \) which, by virtue of the equation \([7]\), makes the energy density to vanish. It seems to correspond to the special relativistic Universe of Milne \([38]\) under the zero-density limit of the expanding FRWL metric with no cosmological constant \([27]\) but with \( \Lambda_0 = \alpha H_0^2 \).

The available observational data for redshift and scale factor have got flexibility, though limited, to distinguish between a time varying and a constant equation.
of state \([33, 40]\). It, therefore, supports a time variable \(\omega_0\). Such a time variable \(\omega_0\) has been used in literature by many authors \([41, 42, 43]\) to predict various physical observables of the Universe. Some useful limits on \(\omega_0\) was suggested by SNIa data, \(-1.67 < \omega_0 < -0.62\) \([44]\) whereas refined values come from combined SNIa data with CMB anisotropy and galaxy clustering statistics which are \(-1.33 < \omega_0 < -0.79\) \([22]\). Moreover, inflation at an early stage scales the parameter \(\omega_0\), which combined with the above data and dark energy constraint \((\omega_0 > -1.0)\) suggests a physical condition, \(-0.46 > \omega_0 > -1.0\) \([43]\). In the light of our previous discussion, \(q_0\) too must obey the same physical conditions as \(\omega_0\) but for \(\alpha = 1\). The experimental limits, \(-0.75 < q_0 < -0.48\) \([45]\) does fall in this range, which supports the view point that a variable \(\omega_0\) provides physical reason for a nonzero value of \(\Lambda_0\) and for a limited time variability for it.

For an accelerating Universe, \(\Lambda_0\) must be positive, so is \(\alpha\). We plot the linear relationship \([10]\) for different values of \(\alpha\) within its physical range in Fig. 1. We choose \(\alpha \) from 0.0 to -1.0 covering the range \(-0.46 > \omega_0 > -1.0\). To reproduce experimental \(q_0\), we require smaller values of \(\alpha\) for smaller values of \(\omega_0\). The hypotenuse and the base of the bold face triangle, respectively represent \(\alpha = 0\) (or \(\Lambda_0 = 0\)) and \(\alpha = 3\), thereby representing the full physical region. We may compare \(\alpha = 0\) plot with other lines of the Figure representing positive values of \(\alpha\) (or respective values of \(\Lambda_0\)) to extract the effect of \(\Lambda_0\) on \(q_0\). The thin solid line shows the case, \(\alpha = 1\).

We find \(q_0\) negative within the bold face triangular region demarcated by physical conditions \(0.0 < \alpha < 3.0\) and \(-0.46 > \omega_0 > -1.0\). The values \(\alpha > 9/4\) (or \(\Omega_\Lambda_0 > 3/4\)) lies below the experimental \(q_0\). Towards the smaller values of \(\alpha\) \((\alpha < 0.7)\) and higher values of \(\omega_0\) \((-0.2 < \omega_0 < 0.0)\), \(q_0\) is observed to flip the sign in the accelerating epoch. Before the present cosmic acceleration, which had started only recently (a few Gyr earlier), the Universe was expanding with deceleration. So, at the turnover stage (from deceleration to acceleration), the deceleration parameter must have changed its sign. It is worth noting here that \(q_0\) and \(\alpha\) (hence \(\Lambda_0\)) determine the fate of the Universe. One may arrive at the same conclusion through recognizing the fact that density is directly proportional to \(\Lambda_0\), one of the predictions of the inflation theory is a flat Universe with a large value of \(\Lambda\) in the early stages of the Universe. One of the predictions of the inflation theory is a flat Universe with a large value of \(\Lambda\) in the early stages of the Universe. Thus, one may argue that it is the cosmological parameter that determined the geometry of the Universe and made it flat during inflation.

We now focus on the implications of \(\Lambda_0\) on the hidden mass. We may, from equation \([7]\), deduce that \(\Lambda_0 \propto H_0^2\) is equivalent to \(\Lambda_0 \propto \rho_0\) as have been suggested for other type of phenomenological \(\Lambda\)-model \([27]\). It may also be shown that the ratio \(\rho_G/\rho_0\) gives a measure for the hidden mass of the Universe. On the basis of the phenomenological model \(\Lambda \simeq H^2\) \([31]\), the present density has been obtained as \(\rho_0 = 3.3 \times 10^{-30} \text{gcm}^{-3}\), which is close to the lower limit of the value obtained by Guth \([32]\): \(4.5 - 18 \times \ldots\).
10^{-30} \text{gmcm}^{-3}. Therefore, knowing the measured galactic mass density, $\rho_G = 4.5 \times 10^{-30} \text{gmcm}^{-3}$, one finds $\rho_G \sim 0.025\rho_0 - 0.15\rho_0$ suggesting thereby that the galactic mass density is about 2.5% - 15% of the total mass density of the present Universe. Hence, according to the present model there is hidden mass ranging from 85% - 97.5%, which seemingly approves the value of Ref. [1]. Variation of $\rho_G/\rho_0$ is shown in the lower panel of Fig. 2 with respect to $\Lambda_0$ and $\alpha$. Higher values of $\Lambda_0$ and lower values of $\alpha$ demonstrate large missing mass.

On the basis of the most recent observations Weinberg [37] reports the age of the Universe as $\approx 12.4 - 14.7$ Gyr. Dynamic $\Lambda$ model has also been used to estimate the age [25, 48], however it seems to suffer either from low-age like 5.4 Gyr [27] that is less than the estimated globular cluster ages, $12.5 \pm 1.2$ Gyr [47] or it is as high as $27.4 \pm 5.6$ Gyr [48]. With the variations of $\alpha$ and $\Lambda_0$ we plot our calculations in the upper panel of Fig. 2. We notice that all the values of $\alpha$ within its physical range reproduce observed age of the Universe but with a suitable $\Lambda_0$. One requires, smaller values of $\Lambda_0$, for the smaller values of $\alpha$ in order to be close with observation. This sheds light on the important time variability of $H_0$ and its correlation to $H_0$, $\Lambda_0 \propto H_0$.

In the present paper, it has been possible to arrive at various interesting physical ideas of modern cosmology through simple considerations of time variability of the observables of flat Universe, specially $\Lambda_0$. We have considered the phenomenological assumption, $\Lambda_0 = \alpha H_0^2$, for the full physical range of $\alpha$ and hence the relevance of time variable $\Lambda$ with regard to various key issues like dark matter, dark energy, geometry of the field, age of the Universe, deceleration parameter and barotropic equation of state has been trivially addressed. Interestingly, the deceleration parameter $q_0$ and the barotropic equation of state parameter $\omega_0$ have been found to obey a straight line relationship (equation (10)) with slope $dq_0/d\omega_0 = (3 - \alpha)/2$ for a flat Universe described by Friedmann and Raychaudhuri equations. For $\alpha = 1$, both the parameters, $\omega_0$ and $q_0$, are equal and hence are expected to obey the same physical conditions. The assumption, $\Lambda_0 = \alpha H_0^2$, seems to represent the Milne Universe. It has been shown that within the physical limits of accelerating Universe, $q_0$ may flip its sign towards lower end values of $\alpha$ and higher end values of $\omega_0$.

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