A survey on the study of real zeros of flow polynomials

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Abstract
For a bridgeless graph $G$, its flow polynomial is defined to be the function $F(G, q)$, which counts the number of nowhere-zero $\Gamma$-flows on an orientation of $G$ whenever $q$ is a positive integer and $\Gamma$ is an additive Abelian group of order $q$. It was introduced by Tutte in 1950, and the locations of zeros of this polynomial have been studied by many researchers. This paper gives a survey on the results and problems on the study of real zeros of flow polynomials.

KEYWORDS
bridgeless graph, flow polynomial, flow root, nowhere-zero flow, zero-free interval

1 INTRODUCTION

Let $G = (V, E)$ be a finite multigraph with vertex set $V$ and edge set $E$ and let $D$ be an orientation of $G$. For any additive Abelian group $\Gamma$, a $\Gamma$-flow on $D$ is a mapping $\phi: E \to \Gamma$ such that

$$\sum_{e \in A^+(v)} \phi(e) - \sum_{e \in A^-(v)} \phi(e) = 0$$

holds for every vertex $v$ in $G$, where $A^+(v)$ (resp. $A^-(v)$) is the set of arcs in $D$ with tail $v$ (resp. the set of arcs in $D$ with head $v$). A $\Gamma$-flow $\phi$ on $D$ is called a nowhere-zero $\Gamma$-flow on $D$ if $\phi(e) \neq 0$ holds for all $e \in E$.

By applying (1), one can shows that $\sum_{e \in A^+(S)} \phi(e) - \sum_{e \in A^-(S)} \phi(e)$ holds for any $S \subseteq V$ and any $\Gamma$-flow $\phi$ on $D$, where $A^+(S)$ (resp. $A^-(S)$) is the set of non-loop arcs in $D$ with tails in $S$ and heads in $V - S$ (resp. the set of non-loop arcs in $D$ with heads in $S$ and tails in $V - S$). Thus there is no nowhere-zero $\Gamma$-flow on $D$ when $G$ contains a bridge.

For any integer $q \geq 2$, a nowhere-zero $q$-flow on $D$ is defined to be a nowhere-zero $\mathbb{Z}$-flow $\psi$ on $D$ such that $|\psi(e)| \leq q - 1$ holds for all $e \in E$, where $\mathbb{Z}$ is the additive group consisting of
integers. Due to a result of Tutte [1], these two kinds of nowhere-zero flows have the same property of existence.

**Theorem 1.1** ([1]). *For any orientation \( D \) of \( G \) and any positive integer \( q \), there exists a nowhere-zero \( q \)-flow on \( D \) if and only if there exists a nowhere-zero \( \Gamma \)-flow on \( D \), where \( q \) is the order of \( \Gamma \).*

The flow polynomial of a graph was introduced by Tutte [1] in 1950. For any positive integer \( q \), let \( F(G, q) \) be the number of nowhere-zero \( \Gamma \)-flows on \( D \), where \( \Gamma \) is an additive Abelian group of order \( q \). It is not difficult to verify that the definition of \( F(G, q) \) does not depend on the selections of \( D \) and \( \Gamma \) but on \( G \) and \( q \). For any positive integer \( q \), Theorem 1.1 implies that \( F(G, q) > 0 \) if and only if there exists a nowhere-zero \( q \)-flow on \( D \) for any orientation \( D \) of \( G \).

Note that \( F(G, q) \) is not equal to the number of nowhere-zero \( q \)-flows. For example, if \( G \) is the graph with one vertex and one loop, then \( F(G, q) = q - 1 \) while the number of nowhere-zero \( q \)-flows is \( 2(q - 1) \).

The function \( F(G, q) \) can also be determined by the following properties [2]:

\[
F(G, q) = \begin{cases} 
1, & \text{if } E = \emptyset; \\
0, & \text{if } G \text{ has a bridge;} \\
F(G_1, q) \cdots F(G_k, q), & \text{if } G = G_1 \cup \cdots \cup G_k; \\
(q - 1)F(G - e, q), & \text{if } e \text{ is a loop in } G; \\
F(G/e, q) - F(G - e, q), & \text{if } e \text{ is neither a loop nor a bridge,}
\end{cases}
\]  

(2)

where \( G/e \) and \( G - e \) are the graphs obtained from \( G \) by contracting \( e \) and deleting \( e \), respectively, and \( G_1 \cup \cdots \cup G_{k-1} \cup G_k \) is the disjoint union of graphs \( G_1, G_2, ..., G_{k-1} \) and \( G_k \). By applying the properties in (2), \( F(G, q) \) can be expressed in terms of the Tutte polynomial \( T_G(x, y) \) of \( G \):

\[
F(G, q) = \sum_{E \subseteq E} (-1)^{|E| - |E'|} q^{|E'| + c(E') - |V|} = (-1)^{|E| - |V| + c(E')} T_G(0, 1 - q),
\]  

(3)

where \( c(E') \) is the number of components of the spanning subgraph \((V, E')\) of \( G \) and

\[
T_G(x, y) = \sum_{E \subseteq E} (x - 1)^{c(E') - c(E)} (y - 1)^{|E| - |V| + c(E')}. 
\]  

(4)

Both (2) and (3) show that \( F(G, q) \) is a polynomial in \( q \). Thus the variable \( q \) in \( F(G, q) \) can be considered as a real or complex number for the study of its algebraic or other properties.

The zeros of the flow polynomial \( F(G, q) \) are called the **flow roots** of \( G \). This paper focuses on giving a review on the study of real flow roots of graphs.

In Section 2, we will introduce some basic results on flow polynomials. Because the flow roots of any plane graph are exactly the non-zero roots of the chromatic polynomial of its dual plane graph (see Theorem 3.1), we will give a short review on the study of real roots of chromatic polynomials in Section 3. The other sections are arranged below:
(i) Section 4: determining maximal zero-free intervals of flow polynomials of the form \((1, a)\) for \(1 < a \leq 2\) and in particular, searching for graphs having no flow roots in the interval \((1, 2)\);

(ii) Section 5: searching for near-cubic graphs, which have no flow roots in the interval \((2, 3)\);

(iii) Section 6: the multiplicity of flow root “2” for near-cubic graphs;

(iv) Section 7: the existence of flow roots larger than 4;

(v) Section 8: the existence of graphs, which have real flow roots only but also contain non-integral real flow roots.

### 2  |  BASIC PROPERTIES OF FLOW POLYNOMIALS

A graph \(G = (V, E)\) is said to be non-separable if either \(|E| \leq |V| = 1\) or \(G\) is connected without loops or cut-vertices, where a vertex \(x\) in \(G\) is called a cut-vertex if \(G - x\) has more components than \(G\) has. We say \(G\) is separable if it is not non-separable.

A block of \(G\) is a maximal subgraph of \(G\) with the property that it is non-separable. Clearly, if \(|E| + |V| \geq 3\), then each loop of \(G\) is considered as a block. A block is said to be trivial if its order is 1 and its size is 0. So a trivial block is an isolated vertex in \(G\). By (2), \(F(G, q) = F(G - V_0, q)\) holds, where \(V_0\) is the set of isolated vertices in \(G\). Thus we may assume that \(G\) has no isolated vertices when we study \(F(G, q)\).

By (2), \(F(G, q) = 0\) whenever \(G\) contains bridges. Also by (2), if \(G\) is separable, then \(F(G, q)\) can be expressed as the multiplication of flow polynomials of its components or blocks.

**Lemma 2.1.** If \(G_1, G_2, ..., G_k\) are the components of \(G\) or the blocks of a connected graph \(G\), then

\[
F(G, q) = \prod_{1 \leq i \leq k} F(G_i, q). \tag{5}
\]

For any \(S \subseteq E\), \(S\) is called an edge-cut of \(G\) if \(S\) is the set of edges with ends in \(V_1\) and in \(V_2\), respectively, where \(\{V_1, V_2\}\) is a partition of \(V\) with \(V_i \neq \emptyset\) for \(i = 1, 2\). Such an edge-cut \(S\) is also written as \(S = (V_1, V_2)\). For any non-separable graph \(G\), \(F(G, q)\) can be further factorized in any one of the following cases, due to Jackson [3] (see [4,6] also):

(i) \(G - e\) is separable for some edge \(e\);

(ii) \(G\) has a proper edge-cut \(S\) with \(2 \leq |S| \leq 3\), where an edge-cut \(S\) is said to be proper if \(G - S\) has no isolated vertices.

For any graph \(G\) and any two vertices \(u\) and \(v\) in \(G\), let \(G + uv\) denote the graph obtained by adding a new edge joining \(u\) and \(v\).

**Lemma 2.2** ([3]). Let \(G\) be a bridgeless connected graph and \(e\) be an edge of \(G\) joining \(u_1\) and \(u_2\). If \(G - e\) is separable and \(H_1\) and \(H_2\) are edge-disjoint subgraphs of \(G - e\) with \(E(H_1) \cup E(H_2) = E(G - e)\), \(V(H_1) \cup V(H_2) = V(G)\), \(V(H_1) \cap V(H_2) = \{v\}\) for some vertex \(v\) of \(G\) and \(u_i \in V(H_i)\) for \(i = 1, 2\), as shown in Figure 1, then
where $G_i = H_i + vu_i$ for $i = 1, 2$.

**Lemma 2.3 ([3]).** Let $S = (V_1, V_2)$ be an edge-cut of a 2-edge connected graph $G$ and let $H_i = G[V_i]$ be the subgraph of $G$ induced by $V_i$ for $i = 1, 2$, as shown in Figure 2 when $|S| = 2$. For $i = 1, 2$, let $G_i$ be obtained from $G$ by contracting $E(H_{3-i})$. If $2 \leq |S| \leq 3$, then

$$F(G, q) = \frac{F(G_1, q)F(G_2, q)}{q - 1}, \quad (7)$$

where $(x)_k$ is the polynomial $x(x-1)\cdots(x-k+1)$.

**Remark.** If $S$ is a non-proper edge-cut with $2 \leq |S| \leq 3$, then $V_i$ contains one vertex only for some $i$, implying that $G_{3-i} \cong G$, where $G_i$ is a graph defined in Lemma 2.3. Thus, only when $S$ is a proper edge-cut of $G$ with $2 \leq |S| \leq 3$, Lemma 2.3 can be applied to express $F(G, q)$ in terms of multiplications of flow polynomials of graphs with smaller orders.

For any edge $e$ of $G$, if one end of $e$ is of degree 2, then the results of (2) imply that $F(G, q) = F(G/e, q)$. Thus, by Lemmas 2.1, 2.2 and 2.3, the study of zeros of flow polynomials can be focused on non-separable and 3-edge connected graphs which do not contain any proper 3-edge-cut and do not contain any edge whose removal results in a separable graph.

### 3 | REAL ZEROS OF CHROMATIC POLYNOMIALS

The flow polynomial $F(G, q)$ is considered as the dual polynomial of the chromatic polynomial $P(G, q)$, mainly due to their close relation given in Theorem 3.1, where $P(G, q)$ is defined to be the function which counts the number of proper $q$-colorings of $G$ whenever $q$ is a positive integer. Clearly, for a positive integer $q$, $G$ admits a proper $q$-coloring if and only if $P(G, q) > 0$. 

![Figure 1](image1.png)

**FIGURE 1** $G-e$ is separable

![Figure 2](image2.png)

**FIGURE 2** $G$ has a 2-edge-cut
Theorem 3.1 ([7]). \( P(G, q) = qF(G^*, q) \) holds for any connected plane graph \( G \), where \( G^* \) is the dual plane graph of \( G \).

The chromatic polynomial \( P(G, q) \) was introduced by Birkhoff [8] in 1912 in the hope of proving the four-color theorem (i.e., \( P(G, 4) > 0 \) holds for any non-loop planar graph \( G \)). This function \( P(G, q) \) is indeed a polynomial in \( q \), as its definition implies that for any integer \( q \geq 1,

\[
P(G, q) = \sum_{1 \leq k \leq |V(G)|} \alpha_k(G)(q)k,
\]

where \( \alpha_k(G) \) is the number of partitions of \( V(G) \) into exactly \( k \) non-empty independent sets. Thus the variable \( q \) in the function \( P(G, q) \) can be considered as a complex number. If \( P(G, q) = 0 \), then \( q \) is called a chromatic root of \( G \).

By Theorem 3.1, the flow roots and chromatic roots of planar graphs have the same distribution, except that “0” is a chromatic root of every graph but not a flow root of any graph.

Due to Sokal’s result below on complex zeros of chromatic polynomials and Tutte’s result in Theorem 3.1, the complex flow roots of planar graphs are dense everywhere in the whole complex plane with the possible exception of the disc \( |z - 1| < 1 \).

Theorem 3.2 ([9]). The complex zeros of chromatic polynomials of planar graphs are dense everywhere in the whole complex plane with the possible exception of the disc \( |z - 1| < 1 \).

This section focuses on giving a review on the study of real chromatic roots of graphs and, in particular, of planar graphs. For other results or problems on chromatic polynomials, the reader can refer to [10,13].

For general graphs, \((-\infty, 0), (0, 1) \) and \((1, 32/27) \) are the only maximal zero-free intervals for all chromatic polynomials, where an interval is said to be zero-free for a function if it has no zero in this interval.

The first two zero-free intervals for all chromatic polynomials follow directly from parts (i) and (iii) of the following result.

Theorem 3.3 ([29, 30]). Let \( G \) be a non-loop graph of order \( n \) and component number \( c \). Then

(i) \( (-1)^nP(G, q) > 0 \) for all real \( q < 0 \);

(ii) the multiplicity of the root 0 of \( P(G, q) \) is equal to \( c \);

(iii) \( (-1)^{n-c}P(G, q) > 0 \) for all real \( 0 < q < 1 \);

(iv) the multiplicity of the root 1 of \( P(G, q) \) is equal to the number of non-trivial blocks.

The third such zero-free interval \((1, 32/27) \) was due to Jackson [14].

Theorem 3.4 ([14]). Let \( G \) be a non-loop connected graph of order \( n \) and block number \( b \). Then \( (-1)^{n+b-1}P(G, q) > 0 \) holds for all real \( q \) in \((1, 32/27) \).
Thomassen [15] showed that these three intervals are the only zero-free intervals for all chromatic polynomials.

**Theorem 3.5** ([15]). *For any interval* \((a, b)\) *with* \(32/27 \leq a < b\), *there exists a graph that has chromatic roots in the interval* \((a, b)\).

Now we focus on planar graphs. By a combinatorial approach, it is not difficult to prove that every non-loop planar graph admits a proper 5-coloring (ie, \(P(G, 5) > 0\) holds for any non-loop planar graph \(G\)). This result can also be proved by induction with an algebraic approach. Actually, Birkhoff and Lewis [10] proved that \(P(G, q) > 0\) holds for all non-loop planar graphs \(G\) and all real numbers \(q \geq 5\). Thus, \([5, \infty)\) is a zero-free interval for chromatic polynomials of all non-loop planar graphs.

**Theorem 3.6** ([10]). \(P(G, q) > 0\) holds for all non-loop planar graphs \(G\) and all real numbers \(q\) in the interval \([5, \infty)\).

Note that Theorem 3.6 does not hold if \(G\) is not restricted to planar graphs as \(P(G, q) = 0\) whenever \(q\) is an integer with \(0 \leq q < \chi(G)\).

Birkhoff and Lewis [10] also conjectured that \([4, 5)\) is a zero-free interval for chromatic polynomials of all non-loop planar graphs.

**Conjecture 3.1** ([10]). \(P(G, q) > 0\) holds for all non-loop plane graphs \(G\) and all real numbers \(q\) in the interval \([4, 5)\).

Conjecture 3.1 includes the four-color conjecture (ie, the case \(q = 4\)). The four-color conjecture was first proved by Appel and Haken [16] in 1977 and was reproved by Robertson et al [17] in 1997. However, the study of Conjecture 3.1 has no other progress, and it is even unknown if there exists a real number \(\epsilon > 0\) such that \(P(G, q) > 0\) holds for all plane graphs \(G\) and all real \(q \in (5 - \epsilon, 5)\).

The number \(\tau = \sqrt{5}/2 + 1/2 \approx 1.618033\) is called the golden ratio, which is the positive real root of the equation \(q^2 = q + 1\). The following results due to Tutte [18,19], Thomassen [15] and Perret and Thomassen [20] present some interesting relations between \(\tau\) and chromatic roots of planar graphs.

**Theorem 3.7.**

(i) [18,19] *For any plane triangulation* \(G\) *with* \(n\) *vertices,*

\[
0 < |P(G, \tau + 1)| \leq \tau^{5-n}
\]

and

\[
P(G, \tau + 2) = (\tau + 2)\tau^{3n-10}(P(G, \tau + 1))^2 > 0;
\]
(ii) [15, 20] chromatic roots of planar graphs are dense everywhere in the interval [32/27, 4) with the possible exception of a small interval $(t_1, t_2)$ around the number $\tau + 2 \approx 3.618033$, where $t_1 \approx 3.618032$ and $t_2 \approx 3.618356$.

Note that in Theorem 3.7 (i), $P(G, \tau + 1) \neq 0$. Actually, $\tau + 1 = (3 + \sqrt{5})/2$ cannot be a chromatic root of any graph $G$. Otherwise, $(3 - \sqrt{5})/2 \approx 0.381966$ is also a chromatic root of $G$, contradicting Theorem 3.3 (iii). More generally, for any positive rational numbers $a$ and $b$, if $\sqrt{b}$ is not rational and $a - 1 < \sqrt{b}$, then $a + \sqrt{b}$ is not a chromatic root of any graph.

4 | ZERO-FREE INTERVALS WITHIN (1, 2) FOR FLOW POLYNOMIALS

Let $G = (V, E)$ be a bridgeless connected graph. Applying (2), it is not difficult to show that $(-1)^{|E| - |V| + 1} F(G, q) > 0$ holds for all real $q$ in the interval $(-\infty, 1)$. But $F(G, 1) = 0$ if $E \neq \emptyset$. Thus $(-\infty, 1)$ is a maximal zero-free interval of all flow polynomials. This is part of the results below due to Wakelin [21].

**Theorem 4.1** ([21]). Let $G = (V, E)$ be a bridgeless connected graph with block number $b$. Then

(i) $(-1)^{|E| - |V| + 1} F(G, q) > 0$ holds for all real $q$ in $(-\infty, 1)$;
(ii) $F(G, q)$ has a zero of multiplicity $b$ at $q = 1$;
(iii) $(-1)^{|E| - |V| + b + 1} F(G, q) > 0$ holds for all real $q$ in $(1, 32/27]$.

By Theorem 4.1, $(1, 32/27]$ is the second zero-free interval for all flow polynomials. By Theorems 3.1 and 3.7 (ii), this zero-free interval $(1, 32/27]$ for flow polynomials is also maximal, and there is no other zero-free interval $(a, b)$ for all flow polynomials with $32/27 < a < b \leq 4$, unless $t_1 \leq a < b \leq t_2$, where $t_1, t_2$ are number stated in Theorem 3.7 (ii). But it is unknown if there exists a zero-free interval $(a, b)$ for all flow polynomials with $4 < a < b$. The study of this problem will be reviewed in Section 7.

Now we consider maximal zero-free intervals of flow polynomials of some subsets of graphs. A plane near-triangulation is a non-loop connected plane graph in which at most one face is not bounded by a cycle of order 3. Birkhoff and Lewis [10] proved that any near-triangulation does not have any chromatic root in $(1, 2)$.

**Theorem 4.2** ([10]). If $G$ is a plane near-triangulation, then $G$ has no chromatic roots in $(1, 2)$.

By Theorems 3.1 and 4.2, we have the following equivalent statement.

**Theorem 4.3.** $(1, 2)$ is a zero-free interval for flow polynomials of bridgeless planar graphs which have at most one vertex of degree not equal to 3.

Jackson [3] showed that Theorems 4.3 holds no matter whether $G$ is planar or non-planar, as long as $G$ contains at most one vertex of degree larger than 3.
Theorem 4.4 ([3]). If $G$ is a bridgeless graph with at most one vertex of degree larger than 3, then $G$ has no flow roots in $(1, 2)$.

Theorem 4.4 was further generalized by Dong [22] who showed that the conclusion holds for every bridgeless graph $G$ with $|W(G)| \leq 2$, where $W(G)$ is the set of vertices in $G$ which are of degrees larger than 3. But the conclusion fails for some bridgeless graphs $G$ with $|W(G)| = 3$. Actually, for any integer $k \geq 3$, it fails for some bridgeless graphs $G$ with $|W(G)| = k$. The graph shown in Figure 3 has a flow root around 1.4301..., which is the only real zero of the polynomial $q^3 - 5q^2 + 10q - 7$. This graph has the least size among all graphs, which have flow roots in $(1, 2)$.

![FIGURE 3](image)

The graph with the smallest size and with a flow root in the interval $(1, 2)$.

For any integer $k \geq 0$, let $\Psi_k$ be the set of bridgeless connected graphs with $|W(G)| \leq k$ and let $\xi_k$ be the supremum in $(1, 2)$ such that $(1, \xi_k)$ is a zero-free interval for flow polynomials of graphs in $\Psi_k$. Clearly, $\xi_0, \xi_1, \xi_2, \ldots$ is a non-increasing sequence.

For any $k \geq 2$, let $\Theta_k$ be the set of those graphs in $\Theta$ with exactly $k$ vertices, where $\Theta$ is the set of graphs defined by the following two steps:

(i) $Z_3 \in \Theta$, where $Z_j$ is the graph with two vertices and $j$ parallel edges joining these two vertices; and

(ii) $G(e) \in \Theta$ for every $G \in \Theta$ and every $e \in E(G)$, where $G(e)$ is the graph obtained from $G - e$ by adding a new vertex $w$ and for each end $u_i$ of $e, i = 1, 2$, adding two parallel edges joining $w$ to $u_i$.

It is not difficult to check that $\Theta_k$ has exactly one graph for $k = 2, 3, 4$, and the graph shown in Figure 3 is the only graph in $\Theta_3$.

The value of $\xi_k$ can be determined by graphs in the finite set $\Theta_k$.

Theorem 4.5 ([22]). For any integer $k \geq 2$, $\xi_k$ is the minimum value among the real flow roots in $(1, 2)$ of all graphs in $\Theta_k$.

Applying Theorem 4.5, the values of $\xi_k$’s for $k \leq 5$ are determined.

Corollary 4.1 ([22]). $\xi_k = 2$ for $k = 0, 1, 2, \xi_3 = 1.430\ldots, \xi_4 = 1.361\ldots$ and $\xi_5 = 1.317\ldots$, where the last three numbers are the real zeros of $q^3 - 5q^2 + 10q - 7$, $q^3 - 4q^2 + 8q - 6$ and $q^3 - 6q^2 + 13q - 9$ in $(1, 2)$, respectively.

Corollary 4.1 tells that $(1, 2)$ is a zero-free interval for flow polynomials of all bridgeless graphs $G$ with $|W(G)| \leq 2$. As $\xi_0, \xi_1, \xi_2, \xi_3, \ldots$ is non-increasing, Theorem 4.5 implies that for any $k \geq 3$, there exist graphs $G$ with $|W(G)| = k$ which have flow roots in $(1, 2)$. However, by
the following result, it is not true that every graph \( G \) with \( |W(G)| \geq 3 \) contains flow roots in \((1, 2)\).

For any \( V_0 \subseteq V(G) \), let \( N_G(V_0) \) denote the set \( \bigcup_{v \in V_0} N_G(v) \).

**Theorem 4.6** ([4]). For any bridgeless graph \( G \), if \( G - W(G) \) has a component \( G_0 \) such that \( W(G) \subseteq N_G(V(G_0)) \) holds, then \((1, 2)\) is a zero-free interval of \( F(G, q) \).

Theorem 4.5 and Corollary 4.1 will be applied in the study of Problem 8.1.

## 5 | ZERO-FREE INTERVALS WITHIN \((2, 3)\)

For any vertex-disjoint graphs \( G_1 \) and \( G_2 \), let \( G_1 + G_2 \) denote the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \cup \{xy: x \in V(G_1), y \in V(G_2)\} \).

Woodall [23,24] showed that every plane triangulation has no chromatic roots in the interval \((2, \alpha)\), where \( \alpha = 2.546602... \) is the unique real chromatic root of the graph \( \overline{C}_4 + \overline{K}_2 \) in the interval \((2, 3)\):

\[
P(\overline{C}_4 + \overline{K}_2, q) = q(q - 1)(q - 2)(q^3 - 9q^2 + 29q - 32).
\]

Note that the dual graph of \( \overline{C}_4 + \overline{K}_2 \) is the cube shown in Figure 4A. By Theorem 3.1, \( \alpha_i \) is the flow root of the cube and Woodall’s result can be translated to the following one on flow polynomials.

**Theorem 5.1** ([48, 49]). \((2, \alpha_i)\) is a zero-free interval for flow polynomials of bridgeless cubic planar graphs.

Woodall [24] conjectured that the conclusion of Theorem 5.1 holds for the interval \((2.677814... , 3)\), where \( \cdots 2.677814... \) is the unique real chromatic root of the graph \( \overline{C}_5 + \overline{K}_2 \) in the interval \((2, 3)\):

\[
P(\overline{C}_5 + \overline{K}_2, q) = q(q - 1)(q - 2)(q - 3)(q^3 - 9q^2 + 30q - 35).
\]

**Conjecture 5.1** ([24]). \((2.677814... , 3)\) is a zero-free interval for the flow polynomials of bridgeless cubic planar graphs.

Jackson [6] extended Theorem 5.1 to all cubic graphs.

![Figure 4](image_url) Cube and contracted cube
Theorem 5.2 ([6]). $(2, \alpha_l)$ is a zero-free interval for flow polynomials of all bridgeless cubic graphs.

We call $G$ a near-cubic graph if $|W(G)| \leq 1$. Jackson [3] extended Theorem 4.4 to near-cubic graphs for a smaller interval.

Theorem 5.3 ([3]). $(2, \alpha_2]$ is a zero-free interval for flow polynomials of bridgeless near-cubic graphs, where $\alpha_2 = \frac{2.225}{2}$ is the real zero in $(2, 3)$ of the polynomial $q^4 - 8q^3 + 22q^2 - 28q + 17$.

Note that Theorem 5.3 does not hold if the number $\alpha_2$ is replaced by any larger number, as Jackson [3] has showed that for any $\epsilon > 0$, there exists a near-cubic graph that has a flow root in $(\alpha_2, \alpha_2 + \epsilon)$.

Let $x \in V(G)$. A branch at $x$ is a path $P = x_v_1 v_2...v_m$ such that $d_G(v_i) = 2$ for $1 \leq i \leq m - 1$ and $d_G(v_m) \geq 3$. We say that $v_1, v_2, ..., v_{m-1}$ are inner vertices of this branch at $x$. Let $G^x$ be the graph obtained from $G$ by deleting $x$ and all the inner vertices of every branch at $x$.

An edge-cut $S = (V_1, V_2)$ of a graph $G$ is said to be cyclic if $G[V]$ contains cycles for both $i = 1, 2$. A graph $G$ is said to be cyclically $k$-edge-connected if each cyclic edge-cut of $G$ has at least $k$ edges. A graph is said to be essentially $3$-connected if it is a subdivision of a $3$-connected graph or the graph $Z_k$ for some $k \geq 3$. For essentially $3$-connected near-cubic graphs, Jackson [6] obtained a zero-free interval for their flow polynomials, which is larger than the one in Theorem 5.3.

Theorem 5.4 ([6]). Let $G$ be a near-cubic graph with order $n$ and size $m$ such that $G$ is essentially $3$-connected.

**i)** Suppose that for any $x \in V(G)$ with $d(x) \geq 5$, $G^x$ is cyclically $3$-edge-connected. Then $F(G, q)$ is non-zero with sign $(-1)^{m-n+1}$ for $q \in (2, \alpha_3)$, where $\alpha_3 = 2.43...$ is a flow root of the contracted cube, shown in Figure 4B, whose flow polynomial is $(q - 1)(q - 2)(q^3 - 8q^2 + 23q - 23)$.

**ii)** Suppose that for any $x \in V(G)$ with $d(x) \geq 4$, $G^x$ is cyclically $3$-edge-connected and $B(G^x)$ has at most one component of order $1$, where $B(G^x)$ is the subgraph of $G^x$ induced by the set of vertices of degrees $2$ in $G^x$. Then $F(G, q)$ is non-zero with sign $(-1)^{m-n+1}$ for $q \in (2, \alpha_l)$.

It is mentioned in [6] that $\tau^2 = 2.618...$ is an accumulation point of flow roots of cyclically $4$-edge-connected cubic graphs, as the Cartesian products $C_{2r} \times K_2$ for $r \geq 3$ are cyclically $4$-edge-connected and have flow roots tending to $\tau^2$ from below as $r$ tends to infinity, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Jackson [6] proposed the following conjectures on cyclically $4$-edge-connected cubic graphs.

**Conjecture 5.2** ([6]). For all $\epsilon > 0$, there exist only finitely many cyclically $4$-edge-connected cubic graphs with a flow root in $(2, \tau^2 - \epsilon)$.

**Conjecture 5.3** ([6]). Let $G$ be a cyclically $4$-edge-connected cubic graph. Then $G$ has at most one flow root in $(2, \tau^2)$.
6 | MULTIPLICITY OF THE FLOW ROOT AT 2

For any bridgeless graph $G$ with $\delta(G) \geq 2$, Theorem 4.1 and the third equality in (2) imply that the multiplicity of the flow root at 1 of $G$ is equal to the total number of blocks in $G$. This section focuses on the multiplicity of the flow root at 2.

It is well known that a graph $G$ has a nowhere-zero 2-flow if and only if $G$ is an even graph (ie, every vertex in $G$ is of an even degree). Thus Theorem 1.1 implies that 2 is a flow root of $G$ if and only if $G$ is not an even graph.

Woodall [23] showed that if $G$ is a 3-connected plane triangulation, then the multiplicity of chromatic roots of $G$ at 2 is exactly 1. By Theorem 3.1, Woodall’s result is equivalent to that for any 3-connected cubic planar graph $G$, and the multiplicity of flow roots of $G$ at 2 is exactly 1. This result was extended to near-cubic graphs by Jackson [3].

For any bridgeless graph $G$, let

$$q_2(G, q) = \frac{F(G, q)}{(q - 1)(q - 2)}.$$ 

Jackson [3] showed that a non-separable near-cubic graph $G$ with $|V(G)| \geq 2$ is essentially 3-connected if and only if the multiplicity of the flow root of $G$ at 2 is at most 1.

**Theorem 6.1** ([3]). For any non-separable near-cubic graph $G$ with $|V(G)| \geq 2$, $q_2(G, q)$ is a polynomial in $q$. Furthermore:

(i) if $G$ is not essentially 3-connected, then $q_2(G, 2) = 0$;

(ii) if $G$ is essentially 3-connected, then $q_2(G, 2)$ is non-zero with sign $(-1)^{m-n+1}$, where $n$ and $m$ are the order and size of $G$.

7 | FLOW ROOTS LARGER THAN 4

The Petersen graph has a flow root at $q = 4$, as its flow polynomial is

$$(q - 1)(q - 2)(q - 3)(q - 4)(q^2 - 5q + 10).$$

Thus, (9) and Theorem 1.1 imply that the Petersen graph does not admit a nowhere-zero 4-flow. By (2), $F(G, q) = F(G/e, q)$ holds for any edge $e$ in $G$ that has one end of degree 2. Thus any subdivision of the Petersen graph does not admit a nowhere-zero 4-flow. Tutte [25] guessed that any bridgeless graph without a subdivision of the Petersen graph has a nowhere-zero 4-flow.

**Conjecture 7.1** ([26]). Any bridgeless graph without a subdivision of the Petersen graph has a nowhere-zero 4-flow.

By Theorem 3.1, this conjecture, called *Tutte’s four-flow conjecture*, is obviously stronger than the four-color theorem, as each plane graph does not have a subdivision of the Petersen graph. For more details on Tutte’s four-flow conjecture, the reader can refer to the surveys by Jaeger [27] and Younger [28].
Tutte [26] also conjectured that any bridgeless graph admits a nowhere-zero 5-flow, known as Tutte’s five-flow conjecture.

**Conjecture 7.2** ([26]). Any bridgeless graph admits a nowhere-zero 5-flow.

Conjecture 7.2 is still unproven and so far the best-known result on this conjecture is due to Seymour [29], who showed that every bridgeless graph has a nowhere-zero 6-flow. For other progress regarding this conjecture, the reader may refer to [30].

Seymour’s result in [29] does not imply that $F(G, q) > 0$ holds for all bridgeless graphs $G$ and all real numbers $q \geq 6$, although this inequality does hold for all positive integers $q \geq 6$. This situation is similar to the case that the four-color theorem (ie, $P(G, 4) > 0$ holds for all non-loop plane graphs $G$) does not imply Conjecture 3.1.

The study whether the inequality $F(G, q) > 0$ holds for all bridgeless graphs $G$ and real numbers $q > c$, where $c$ is a constant, was initiated by Welsh [35], who proposed the following conjecture in 1970s.

**Conjecture 7.3** ([35]). For any bridgeless graph $G$, $F(G, q) > 0$ holds for all real numbers $q \in (4, \infty)$.

Clearly Conjecture 3.1 is weaker than Conjecture 7.3, as it is equivalent to the special case of Conjecture 7.3 when the graphs $G$ are restricted to planar graphs and the interval for $q$ is restricted to $(4, 5)$. Counter-examples to Conjecture 7.3 have been found while Conjecture 3.1 has neither any counter-example nor any result confirming it, even for a small interval $(a, b)$ within $(4, 5)$.

The first counter-example to Conjecture 7.3 was due to Haggard et al [36], who showed that the generalized Petersen graph $G_{16,6}$ has real flow roots at around 4.0252205 and 4.2331455, where the generalized Petersen graph $G_{n,k}$ for $n \geq 3$ and $1 \leq k \leq [(n - 1)/2]$ is the graph with vertex set $\{u_i, v_i: 1 \leq i \leq n\}$ and edge set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+k}: 1 \leq i \leq n\}$. $u_{r+1}$ is considered as $u_1$ and $v_s$ for $s > n$ is considered as $v_1$ and $t$ is the integer with $1 \leq t \leq n$ such that $s - t$ is a multiple of $n$. These graphs were introduced by Coxeter [37] and named by Watkins [38]. Clearly, $G_{5,2}$ is the Petersen graph.

There may be other counter-examples in the family of generalized Petersen graphs to Conjecture 7.3. But Haggard et al [36] believed that any counter-example to Conjecture 7.3 does not have flow roots greater than or equal to 5. They proposed the following conjecture by changing the interval to $[5, \infty)$.

**Conjecture 7.4** ([36]). For any bridgeless graph $G$, $F(G, q) > 0$ holds for all real numbers $q \in [5, \infty)$.

A few years ago, the above-mentioned conjecture was disproved by Jacobsen and Salas [39], who found counter-examples by studying the generalized Petersen graphs $G_{6n,6}$ and $G_{7n,7}$ for $n \geq 2$. 
Theorem 7.1 ([39]).

(i) The value \( q = 5 \) is an isolated accumulation point of real flow roots of graphs from the set \( \{ G_{6n,6}, G_{7n,7}; n \geq 3 \} \).

(ii) The value \( \hat{q} \approx 5.235261 \) (where \( \approx \) means “within 10\(^{-6}\)”) is an accumulation point of real zeros of \( F(G_{7n,7}, q) \).

Jacobsen and Salas [39] further modified Conjecture 7.4 by changing the interval to \([6, \infty)\).

Conjecture 7.5 ([39]). For any bridgeless graph \( G \), \( F(G, q) > 0 \) holds for all real numbers \( q \in [6, \infty) \).

As pointed out by Jacobsen and Salas [39], Conjecture 7.5 might be false and it might even be the case that there does not exist any finite upper bound for the real flow roots of general graphs. Now we propose the following conjecture that is much weaker than Conjecture 7.5.

Conjecture 7.6. There exists a constant \( c \) such that for any bridgeless graph \( G \), \( F(G, q) > 0 \) holds for all real numbers \( q \geq c \).

Jackson [5] showed that for any bridgeless graph \( G \) of order \( n \), all real flow roots of \( G \) are smaller than \( n \frac{2}{\log 2} \).

Theorem 7.2 ([5]). For any bridgeless graph \( G \) of order \( n \), \( F(G, q) > 0 \) holds for all real \( q \geq 2\log_2 n \).

Theorem 7.2 is actually a special case of a more general result on the characteristic polynomial \( C(M, q) \) of a matroid \( M = (E, r) \), where

\[
C(M, q) = \sum_{A \subseteq E} (-1)^{|A|} q^{r(M) - r(A)}. \tag{10}
\]

For any graph \( G \), if \( M_G \) and \( M_G^* \) are the cycle matroid and the cocycle matroid of \( G \), respectively, then \( C(M_G, q) = q^{-P(G, q)} \), where \( c \) is the number of components of \( G \), and \( C(M_G^*, q) = F(G, q) \). Thus \( C(G, q) \) is an extension of both \( P(G, q) \) and \( F(G, q) \).

Oxley [40] showed that if every cocircuit of \( M \) has a size at most \( d \), then \( C(M, q) > 0 \) holds for all real numbers \( q \geq d \). Jackson [5] noticed that the idea in Oxley’s proof can be applied to get a more general result.

A simple minor of \( M \) is a minor that contains no loops or circuits of length two.

Theorem 7.3 ([5]). Let \( M \) be a matroid. If every simple minor of \( M \) has a cocircuit of size at most \( d \), then \( C(M, q) > 0 \) for all real numbers \( q \geq d \).

As \( F(G, q) = C(M_G^*, q) \), Theorem 7.3 implies that for any bridgeless graph \( G \), if every 3-edge-connected minor of \( G \) has a circuit of length at most \( d \), then \( F(G, q) > 0 \) holds for all real numbers \( q \geq d \). By Balbuena and García-Vázquez’s result in [41], every 3-connected graph \( G \) of order \( n \) contains a circuit of length at most \( 2\log_2 n \). Thus Theorem 7.2 follows from Theorem 7.3.
A graph \( G = (V, E) \) is said to be a chordal graph if for any \( S \subseteq V \) with \( |S| \geq 4 \), the subgraph of \( G \) induced by \( S \) is not isomorphic to any cycle. Dirac [42] showed that \( G \) is chordal if and only if there is an ordering \( u_1, u_2, ..., u_n \) of its vertices such that for all \( i = 1, 2, ..., n \), the subgraph of \( G \) induced by \( \{u_i\} \cup (N_G(u_i) \cap \{u_1, u_2, ..., u_i\}) \) is a clique. By this result, for any chordal graph \( G \), there exist positive integers \( k_i \)'s such that

\[
P(G, q) = \prod_{i=0}^{k} (q - i)^{k_i},
\]

where \( k = \chi(G) - 1 \) (see [12,13]). Thus all chromatic roots of a chordal graph are non-negative integers. Some non-chordal graphs also have this property (see [11,13,43,45]).

By Theorem 3.1 and (11), if \( G \) is a plane graph and its dual \( G^* \) is chordal, then all flow roots of \( G \) are positive integers. Kung and Royle [46] showed that the converse statement also holds.

**Theorem 8.1** ([46]). If \( G \) is a bridgeless graph, then its flow roots are integral if and only if \( G \) is the dual of a chordal and plane graph.

Clearly, the key point in Theorem 8.1 is its necessity, that is, if \( G \) has integral flow roots only, then \( G \) is the dual of a chordal and plane graph. Does this conclusion hold when the condition “\( G \) has integral flow roots only” is replaced by “\( G \) has real flow roots only”? So far it is unknown if there exists a bridgeless graph having real flow roots only which also contains non-integral real flow roots. In [47], Dong studies the following problem.

**Problem 8.1.** Is it true that any bridgeless graph containing real flow roots only has integral flow roots only?

So far Problem 8.1 is even open for planar graphs. Let \( GR \) (resp. \( GI \)) be the set of bridgeless graphs which have real (resp. integral) flow roots only. Clearly, \( GI \subseteq GR \). Problem 8.1 asks if \( GR = GI \) holds. Dong [47] obtained the following results on the study of Problem 8.1.

**Theorem 8.2** ([47]). Assume that \( G = (V, E) \) is a graph in \( GR \). If some flow roots of \( G \) are not in the set \( \{1, 2, 3\} \), then \( |V| + 17 \leq |E| < (32|V| - 49)/5 \) and \( G \) has at least 9 flow roots in the interval \((1, 2)\).

Theorems 8.1 and 8.2 imply some equivalent statements on graphs in \( GR \).

**Corollary 8.1** ([47]). For any graph \( G \in GR \), the following statements are equivalent:

(i) \( G \) is the dual of some plane chordal graph;
(ii) each flow root of \( G \) is in the set \( \{1, 2, 3\} \);
(iii) \( G \) has no flow roots in the interval \((1, 2)\).

By Corollary 8.1, \( GI \) is actually the set of bridgeless graphs \( G \) whose flow roots are in the set \( \{1, 2, 3\} \).
By Lemmas 2.1 and 2.3, to study Problem 8.1, it suffices to consider 3-edge connected non-separable graphs in \(GR\) which do not contain any proper 3-edge-cut. Dong [47] also showed that if \(G\) is such a graph in \(GR - GT\), then \(|W(G)| \geq 3\) and \(G\) contains at least \(f(|W(G)|)\) flow roots in \((1, 2)\), where \(f(k)\) has values 9, 11, 14, respectively, for \(k = 3, 4, 5\), and \(f(k) = \left\lfloor \frac{27k}{11} - \frac{27}{22} \right\rfloor\) for all integers \(k \geq 6\).

So far there is no research conducted on counting the number of flow roots of a graph in the interval \((1, 2)\), except some work of determining those graphs which have no real flow roots in \((1, 2)\), as mentioned in Section 4. We end this paper with the following problem, which is related to Problem 8.1.

**Problem 8.2.** Is there a bridgeless graph \(H\) with at least \(f(k)\) flow roots in \((1, 2)\), where \(k = |W(H)| \geq 3\) and the number of flow roots of \(H\) counts their multiplicities?

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