TOPICAL REVIEW

A first course on twistors, integrability and gluon scattering amplitudes

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Abstract

These notes accompany an introductory lecture course on the twistor approach to supersymmetric gauge theories aimed at early stage PhD students. It was held by the author at the University of Cambridge during the Michaelmas term in 2009. The lectures assume a working knowledge of differential geometry and quantum field theory. No prior knowledge of twistor theory is required.

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Preface

The course is divided into two main parts: (I) the re-formulation of gauge theory on twistor space and (II) the construction of tree-level gauge theory scattering amplitudes. More specifically, the first few lectures deal with the basics of twistor geometry and its application to free field theories. We then move on and discuss the nonlinear field equations of self-dual Yang–Mills theory. The subsequent lectures deal with supersymmetric self-dual Yang–Mills theories and the extension to the full non-self-dual supersymmetric Yang–Mills theory in the case of maximal $\mathcal{N} = 4$ supersymmetry. Whilst studying the field equations of these theories, we shall also discuss the associated action functionals on twistor space. Having re-interpreted $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on twistor space, we discuss the construction of tree-level scattering amplitudes. We first transform, to twistor space, the so-called maximally-helicity-violating (MHV) amplitudes. Afterwards we discuss the construction of general tree-level amplitudes by means of the Cachazo–Svrček–Witten rules and the Britto–Cachazo–Feng–Witten recursion relations. Some mathematical concepts underlying twistor geometry are summarized in several appendices. The computation of scattering amplitudes beyond tree level is not covered here.
My main motivation for writing these lecture notes was to provide an opportunity for students and researchers in mathematical physics to get a grip of twistor geometry and its application to perturbative gauge theory without having to go through the wealth of text books and research papers but at the same time providing as detailed derivations as possible. Since the present review should be understood as notes accompanying an introductory lecture course rather than as an exhaustive review article of the field, I emphasize that even though I tried to refer to the original literature as accurately as possible, I had to make certain choices for the clarity of presentation. As a result, the list of references is by no means complete. Moreover, to keep the notes rather short in length, I had to omit various interesting topics and recent developments. Therefore, the reader is urged to consult Spires HEP and arXiv.org for the latest advancements and especially the citations of Witten’s paper on twistor string theory, published in Commun. Math. Phys. 252 189 (2004) (arXiv:hep-th/0312171).

Should you find any typos or mistakes in the text, please let me know by sending an email to m.wolf@damtp.cam.ac.uk. For the most recent version of these lecture notes, please also check http://www.damtp.cam.ac.uk/user/wolf.

Literature

Amongst many others (see bibliography at the end of this review), the following lecture notes and books have been used when compiling this review and are recommended as references and for additional reading (chronologically ordered).

**Complex geometry:**
(i) P Griffiths and J Harris 1978 *Principles of Algebraic Geometry* (New York: Wiley)
(ii) R O Wells 1980 *Differential Analysis on Complex Manifolds* (New York: Springer)
(iii) M Nakahara [69] *Geometry, Topology and Physics* (Bristol: Institute of Physics Publishing)
(iv) V Bouchard Lectures on complex geometry, Calabi–Yau manifolds and toric geometry arXiv:hep-th/0702063

**Supermanifolds and supersymmetry:**
(i) Y I Manin [18] *Gauge Field Theory and Complex Geometry* (New York: Springer)
(ii) C Bartocci, U Bruzzo and D Hernandéz-Ruipérez 1991 *The Geometry of Supermanifolds* (Dordrecht: Kluwer)
(iii) J Wess and J Bagger 1992 *Supersymmetry and Supergravity* (Princeton, NJ: Princeton University Press)
(iv) C Sämann [84] *Introduction to Supersymmetry* Lecture Notes (Trinity College, Dublin)

**Twistor geometry:**
(i) R S Ward and R O Wells 1989 *Twistor Geometry and Field Theory* (Cambridge: Cambridge University Press)
(ii) S A Huggett and K P Tod 1994 *An Introduction to Twistor Theory* (Cambridge: Cambridge University Press)
(iii) L. J Mason and N M J Woodhouse 1996 *Integrability, Self-Duality, and Twistor Theory* (Oxford: Clarendon)
(iv) M Dunajski [2] *Solitons, Instantons and Twistors* (Oxford: Oxford University Press)
Tree-level gauge theory scattering amplitudes and twistor theory:

(i) F Cachazo and P Svrˇcek 2005 Lectures on twistor strings and perturbative Yang–Mills theory PoS RTN2005 (2005) 004 (arXiv:hep-th/0504194)

(ii) J A P Bedford 2007 On perturbative field theory and twistor string theory PhD Thesis Queen Mary, University of London (arXiv:0709.3478)

(iii) C Vergu 2008 Twistors, strings and supersymmetric gauge theories PhD Thesis Université Paris IV—Pierre et Marie Curie (arXiv:0809.1807)

Part I: Twistor re-formulation of gauge theory

1. Twistor space

1.1. Motivation

Usually, the equations of motion of physically interesting theories are complicated systems of coupled nonlinear partial differential equations. This thus makes it extremely hard to find explicit solutions. However, among the theories of interest are some which are completely solvable in the sense of allowing for the construction (in principle) of all solutions to the corresponding equations of motion. We shall refer to these systems as integrable systems. It should be noted at this point that there are various distinct notions of integrability in the literature and here we shall use the word ‘integrability’ in the loose sense of ‘complete solvability’ without any concrete assumptions. The prime examples of integrable theories are the self-dual Yang–Mills and gravity theories in four dimensions including their various reductions to lower spacetime dimensions. See e.g. [1, 2] for details.

Twistor theory has turned out to be a very powerful tool in analysing integrable systems. The key ingredient of twistor theory is the substitution of spacetime as a background for physical processes by an auxiliary space called twistor space. The term ‘twistor space’ is used collectively and refers to different spaces being associated with different physical theories under consideration. All these twistor spaces have one thing in common in that they are (partially) complex manifolds, and moreover, solutions to the field equations on spacetime of the theory in question are encoded in terms of differentially unconstrained (partially) complex analytic data on twistor space. This way one may sometimes even classify all solutions to a problem. The goal of the first part of these lecture notes is the twistor re-formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory on four-dimensional flat spacetime.

1.2. Preliminaries

Let us consider $M^4 \cong \mathbb{R}^{p+q}$ for $p + q = 4$, where $\mathbb{R}^{p+q}$ is equipped with a metric $g = (g_{\mu\nu}) = \text{diag}(-\mathbb{I}_p, \mathbb{I}_q)$ of signature $(p, q)$. Here and in the following, $\mu, \nu, \ldots$ run from 0 to 3. In particular, for $(p, q) = (0, 4)$ we shall speak of Euclidean ($\mathbb{E}$) space, for $(p, q) = (1, 3)$ of Minkowski ($\mathbb{M}$) space and for $(p, q) = (2, 2)$ of Kleinian ($\mathbb{K}$) space. The rotation group is then given by $SO(p, q)$. Below we shall only be interested in the connected component of the identity of the rotation group $SO(p, q)$ which is commonly denoted by $SO_0(p, q)$.

If we let $\alpha, \beta, \ldots = 1, 2$ and $\dot{\alpha}, \dot{\beta}, \ldots = \dot{1}, \dot{2}$, then we may represent any real four-vector $x = (x^\mu) \in M^4$ as a $2 \times 2$ matrix $\underline{x} = (x^{\mu\dot{\nu}}) \in \text{Mat}(2, \mathbb{C}) \cong \mathbb{C}^4$ subject to the following
reality conditions:

\[ \begin{align*}
\mathbb{E} : & \quad \bar{x} = -\sigma_2 x \sigma_2^\dagger, \\
\mathbb{M} : & \quad \bar{x} = -x^t, \\
\mathbb{K} : & \quad \bar{x} = x,
\end{align*} \]

where bar denotes complex conjugation, ‘t’ transposition and \( \sigma_i \), for \( i, j, \ldots = 1, 2, 3 \), are the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(1.1)

Recall that they obey

\[ \sigma_i \sigma_j = \delta_{ij} + i \sum_k \varepsilon_{ijk} \sigma_k, \]

(1.3)

where \( \delta_{ij} \) is the Kronecker symbol and \( \varepsilon_{ijk} \) is totally anti-symmetric in its indices with \( \varepsilon_{123} = 1 \).

To be more concrete, the isomorphism \( \sigma : x \mapsto \bar{x} = \sigma(x) \) can be written as

\[ x^{a\beta} = \sigma^{a\beta}_\mu x^\mu \quad \iff \quad x^\mu = \tfrac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{\mu\alpha\beta} x^{\beta\mu}, \]

(1.4a)

where \( \varepsilon_{a\beta} = \varepsilon_{[a\beta]} \) with \( \varepsilon_{12} = -1 \) and \( \varepsilon_{a\alpha} \varepsilon^{\gamma\beta} = \delta^\gamma_\beta \) (and similar relations for \( \varepsilon_{a\beta} \))

\[ \begin{align*}
\mathbb{E} : & \quad (\sigma^{a\beta}_\mu) := (\mathbb{1}, i\sigma_3, -i\sigma_2, -i\sigma_1), \\
\mathbb{M} : & \quad (\sigma^{a\beta}_\mu) := (-i\mathbb{1}, -i\sigma_1, -i\sigma_2, -i\sigma_3), \\
\mathbb{K} : & \quad (\sigma^{a\beta}_\mu) := (\sigma_3, \sigma_1, -i\sigma_2, \mathbb{1}_2).
\end{align*} \]

(1.4b)

The line element \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) on \( M^4 \equiv \mathbb{R}^{p,q} \) is then given by

\[ ds^2 = \det d\bar{x} = \frac{1}{2} \varepsilon^{a\beta} \varepsilon_{a\alpha\beta} dx^a dx^\beta. \]

(1.5)

Rotations (respectively, Lorentz transformations) act on \( x^\mu \) according to \( x^\mu \mapsto x'^\mu = \Lambda^\mu_{\nu} x^\nu \) with \( \Lambda = (\Lambda^\mu_{\nu}) \in SO_0(p, q) \). The induced action on \( \bar{x} \) reads as

\[ \bar{x} \mapsto \bar{x}' = g_{1,2} \bar{x} g_{1,2}^T \quad \text{for} \quad g_{1,2} \in GL(2, \mathbb{C}). \]

(1.6)

The \( g_{1,2} \) are not arbitrary for several reasons. Firstly, any two pairs \( (g_1, g_2) \) and \( (g'_1, g'_2) \) with \( (g'_1, g'_2) = (tg_1, t^{-1} g_2) \) for \( t \in \mathbb{C} \setminus \{0\} \) induce the same transformation on \( \bar{x} \); hence, we may regard the equivalence classes \( \{(g_1, g_2)\} = \{(g'_1, g'_2)\} = \{(tg_1, t^{-1} g_2)\} \). Furthermore, rotations preserve the line element and from \( \det d\bar{x} = \det dx^\mu \) we conclude that \( \det g_1 \det g_2 = 1 \). Altogether, we may take \( g_{1,2} \in SL(2, \mathbb{C}) \) without loss of generality. In addition, the \( g_{1,2} \) have to preserve the reality conditions (1.1). For instance, on \( \mathbb{E} \) we find that \( \bar{g}_{1,2} = -\sigma_1 g_{1,2} \sigma_1^T \). Explicitly, we have

\[ g_{1,2} = \begin{pmatrix} a_{1,2} & b_{1,2} \\ c_{1,2} & d_{1,2} \end{pmatrix} = \begin{pmatrix} a_{1,2} & b_{1,2} \\ -b_{1,2} & a_{1,2} \end{pmatrix}. \]

(1.7)

Since \( \det g_{1,2} = 1 = |a_{1,2}|^2 + |b_{1,2}|^2 \) (which topologically describes a three-sphere) we conclude that \( g_{1,2} \in SU(2) \), i.e. \( g_{1,2}^T = g_{1,2}^{-1} \). In addition, if \( g_{1,2} \in SU(2) \) then also \( \pm g_{1,2} \in SU(2) \) and since \( g_{1,2} \) and \( \pm g_{1,2} \) induce the same transformation on \( \bar{x} \), we have therefore established

\[ \text{SO}(4) \cong (SU(2) \times SU(2)) / \mathbb{Z}_2. \]

(1.8)

Note that for the Kleinian case one may alternatively impose \( \bar{x} = \sigma_1 \sigma_1^Tg_{1,2} \).

We have chosen particle physics literature conventions which are somewhat different from the twistor literature.
One may proceed similarly for \( \mathcal{M} \) and \( \mathcal{K} \) but we leave this as an exercise. Eventually, we arrive at

\[
E : \text{SO}(4) \cong (\text{SU}(2) \times \text{SU}(2))/\mathbb{Z}_2, \quad \text{with } \underline{x} \mapsto g_1 \underline{x} g_2 \quad \text{and} \quad g_{1,2} \in \text{SU}(2),
\]

\[
\mathcal{M} : \text{SO}_0(1,3) \cong \text{SL}(2, \mathbb{C})/\mathbb{Z}_2, \quad \text{with } \underline{x} \mapsto g \underline{x} g^\dagger \quad \text{and} \quad g \in \text{SL}(2, \mathbb{C}),
\]

\[
\mathcal{K} : \text{SO}_0(2,2) \cong (\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))/\mathbb{Z}_2, \quad \text{with } \underline{x} \mapsto g_1 \underline{x} g_2 \quad \text{and} \quad g_{1,2} \in \text{SL}(2, \mathbb{R}).
\]

Note that in general one may write

\[
\text{SO}_0(p,q) \cong \text{Spin}(p,q)/\mathbb{Z}_2,
\]

where \( \text{Spin}(p,q) \) is known as the spin group of \( \mathbb{R}^{p,q} \). In a more mathematical terminology, \( \text{Spin}(p,q) \) is the double cover of \( \text{SO}_0(p,q) \) (for the sum \( p+q \) not necessarily restricted to 4). For \( p = 0,1 \) and \( q > 2 \), the spin group is simply connected and thus coincides with the universal cover. Since the fundamental group (or first homotopy group) of \( \text{Spin}(2,2) \) is non-vanishing, \( \pi_1(\text{Spin}(2,2)) \cong \mathbb{Z} \times \mathbb{Z} \), the spin group \( \text{Spin}(2,2) \) is not simply connected. See, e.g., [3, 4] for more details on the spin groups.

In summary, we may either work with \( x^\mu \) or with \( x^{a\dot{\beta}} \) and making this identification amounts to identifying \( g_{\mu
u} \) with \( \frac{1}{2} \epsilon_{a\beta} \epsilon_{a\dot{\beta}} \). Different signatures are encoded in different reality conditions (1.1) on \( x^{a\dot{\beta}} \). Hence, in the following we shall work with the complexification \( M^4 \otimes \mathbb{C} \cong \mathbb{C}^4 \) and \( \underline{x} = (x^{a\dot{\beta}}) \in \text{Mat}(2, \mathbb{C}) \) and impose the reality conditions whenever appropriate. Therefore, the different cases of (1.9) can be understood as different real forms of the complex version

\[
\text{SO}(4,\mathbb{C}) \cong (\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}))/\mathbb{Z}_2.
\]

For brevity, we denote \( \underline{x} \) by \( x \) and \( M^4 \otimes \mathbb{C} \) by \( M^4 \).

### Exercise 1.1

Prove that the rotation groups on \( \mathcal{M} \) and \( \mathcal{K} \) are given by (1.9).

#### 1.3. Twistor space

In this section, we shall introduce Penrose’s twistor space [5] by starting from complex spacetime \( M^4 \cong \mathbb{C}^4 \) and the identification \( x^\mu \leftrightarrow x^{a\dot{\beta}} \). According to the discussion of the previous section, we view the tangent bundle \( TM^4 \) of \( M^4 \) according to

\[
TM^4 \cong S \otimes \tilde{S},
\]

\[
\partial_{\mu} := \frac{\partial}{\partial x^\mu} \quad \leftrightarrow \quad \partial_{a\dot{\beta}} := \frac{\partial}{\partial x^{a\dot{\beta}}},
\]

where \( S \) and \( \tilde{S} \) are the two complex rank-2 vector bundles called the bundles of dotted and undotted spinors. See appendix A for the definition of a vector bundle. The two copies of \( \text{SL}(2, \mathbb{C}) \) in (1.11) act independently on \( S \) and \( \tilde{S} \). Let us denote undotted spinors by \( \mu^a \) and dotted ones by \( \lambda^a \). On \( S \) and \( \tilde{S} \) we have the symplectic forms \( \epsilon_{a\dot{\beta}} \) and \( \tilde{\epsilon}_{a\dot{\beta}} \) from before which can be used to raise and lower spinor indices:

\[
\mu_a = \epsilon_{a\beta} \mu^\beta \quad \text{and} \quad \lambda_a = \tilde{\epsilon}_{a\dot{\beta}} \lambda^\dot{\beta}.
\]

### Remark 1.1

Let us comment on conformal structures since the identification (1.12) amounts to choosing a (holomorphic) conformal structure. This can be seen as follows. The standard
definition of a conformal structure on a four-dimensional complex manifold $X$ states that a conformal structure is an equivalence class $[g]$, the conformal class, of holomorphic metrics $g$ on $X$, where two given metrics $g$ and $g'$ are called equivalent if $g' = \gamma^2 g$ for some nowhere vanishing holomorphic function $\gamma$. Put differently, a conformal structure is a line subbundle $L$ in $T^*X \otimes T^*X$. Another maybe less familiar definition assumes a factorization of the holomorphic tangent bundle $TX$ of $X$ as a tensor product of two rank-2 holomorphic vector bundles $S$ and $\tilde{S}$, that is, $TX \cong S \otimes \tilde{S}$. This isomorphism in turn gives (canonically) the line subbundle $\Lambda^2 S^* \otimes \Lambda^2 \tilde{S}^*$ in $T^*X \otimes T^*X$ which, in fact, can be identified with $L$. The metric $g$ is then given by the tensor product of the two symplectic forms on $S$ and $\tilde{S}$ (as done above) which are sections of $\Lambda^2 S^*$ and $\Lambda^2 \tilde{S}^*$.

Let us now consider the projectivization of the dual spin bundle $\tilde{S}^*$. Since $\tilde{S}$ is of rank 2, the projectivization $P(\tilde{S}^*) \rightarrow M^4$ is a $\mathbb{C}P^1$-bundle over $M^4$. Hence, $P(\tilde{S}^*)$ is a five-dimensional complex manifold bi-holomorphic to $\mathbb{C}^4 \times \mathbb{C}P^1$. In what follows, we shall denote it by $F_5$ and call it correspondence space. The reason for this name becomes transparent momentarily.

We take $(x^\alpha \dot{\beta}, \lambda^\alpha \dot{\beta})$ as coordinates on $F_5$, where $\lambda^\alpha \dot{\beta}$ are homogeneous coordinates on $\mathbb{C}P^1$.

Remark 1.2. Remember that $\mathbb{C}P^1$ can be covered by two coordinate patches, $U_+ \cup U_-$. If we let $\lambda_\alpha = (\lambda_1, \lambda_2)^t$ be homogeneous coordinates on $\mathbb{C}P^1$ with $\lambda_\alpha \sim t\lambda_\alpha$ for $t \in \mathbb{C} \setminus \{0\}$, $U_\pm$ and the corresponding affine coordinates $\lambda_\pm$ can be defined as follows:

- $U_+$ : $\lambda_1 \neq 0$ and $\lambda_+ := \frac{\lambda_2}{\lambda_1}$
- $U_-$ : $\lambda_2 \neq 0$ and $\lambda_- := \frac{\lambda_1}{\lambda_2}$

On $U_+ \cap U_- \cong \mathbb{C} \setminus \{0\}$ we have $\lambda_+ = \lambda_-^{-1}$.

On $F_5$ we may consider the following vector fields:

$$V_\alpha = \lambda^\beta \partial_\alpha \beta = \lambda^\beta \frac{\partial}{\partial x^\alpha \beta}.$$  \hspace{1cm} (1.14)

They define an integrable rank-2 distribution on $F_5$ (i.e. a rank-2 subbundle in $TF_5$) which is called the twistor distribution. Therefore, we have a foliation of $F_5$ by two-dimensional complex manifolds. The resulting quotient will be twistor space, a three-dimensional complex manifold denoted by $P^3$. We have thus established the following double fibration:

$$
\begin{array}{ccc}
& & F^5 \\
\pi_1 & \nearrow & \pi_2 \\
P^3 & & M^4
\end{array}
$$  \hspace{1cm} (1.15)

The projection $\pi_2$ is the trivial projection and $\pi_1 : (x^{\alpha \beta}, \lambda_\alpha) \mapsto (z^\alpha, \lambda_\alpha) = (x^{\alpha \beta} \lambda_\beta, \lambda_\alpha)$, where $(z^\alpha, \lambda_\alpha)$ are homogeneous coordinates on $P^3$. The relation

$$z^\alpha = x^{\alpha \beta} \lambda_\beta$$  \hspace{1cm} (1.16)

is known as the incidence relation. Note that (1.15) makes clear why $F_5$ is called correspondence space: it is the space that ‘links’ spacetime with twistor space.
Also $P^3$ can be covered by two coordinate patches, which we (again) denote by $U_\pm$ (see also remark 1.2):

\[
U_+ : \quad \lambda_1 \neq 0 \quad \text{and} \quad z_+^\alpha := \frac{z^\alpha}{\lambda_1} \quad \text{and} \quad \lambda_+ := \frac{\lambda_2}{\lambda_1},
\]
\[
U_- : \quad \lambda_2 \neq 0 \quad \text{and} \quad z_-^\alpha := \frac{z^\alpha}{\lambda_2} \quad \text{and} \quad \lambda_- := \frac{\lambda_1}{\lambda_2}.
\]

(1.17)

On $U_+ \cap U_-$ we have $z_+^\alpha = \lambda_+ z_-^\alpha$ and $\lambda_+ = \lambda_-^{-1}$. This shows that twistor space $P^3$ can be identified with the total space of the holomorphic fibration

\[
O(1) \oplus O(1) \to \mathbb{C}P^1,
\]

where $O(1)$ is the dual of the tautological line bundle $O(-1)$ over $\mathbb{C}P^1$:

\[
O(-1) := \{(\lambda_\alpha, \rho_\alpha) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \rho_\alpha \propto \lambda_\alpha \},
\]

i.e. $O(1) = O(-1)^*$. The bundle $O(1)$ is also referred to as the hyperplane bundle. Other line bundles, which we will frequently encounter below, are

\[
O(-m) = O(1)^\otimes m \quad \text{and} \quad O(m) = O(-1)^m \quad \text{for} \; m \in \mathbb{N}.
\]

The incidence relation $z^\alpha = x^\alpha \lambda_\beta \lambda_\beta$ identifies $x \in M^4$ with holomorphic sections of (1.18).

Note that $P^3$ can also be identified with $\mathbb{C}P^3 \setminus \mathbb{C}P^1$, where the deleted projective line is given by $z^\alpha \neq 0$ and $\lambda_\alpha = 0$.

**Exercise 1.2.** Let $\lambda_\alpha$ be homogeneous coordinates on $\mathbb{C}P^1$ and $z$ be the fibre coordinates of $O(m) \to \mathbb{C}P^1$ for $m \in \mathbb{Z}$. Furthermore, let $\{U_\pm\}$ be the canonical cover as in remark 1.2. Show that the transition function of $O(m)$ is given by $\lambda_\alpha^m = \lambda_-^{-m}$. Show further that while $O(1)$ has global holomorphic sections, $O(-1)$ does not.

Having established the double fibration (1.15), we may ask about the geometric correspondence, also known as the Klein correspondence, between spacetime $M^4$ and twistor space $P^3$. In fact, for any point $x \in M^4$, the corresponding manifold $L_x := \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^3$ is a curve which is bi-holomorphic to $\mathbb{C}P^1$. Conversely, any point $p \in P^3$ corresponds to a totally null-plane in $M^4$, which can be seen as follows. For some fixed $p = (z, \lambda) \in P^3$, the incidence relation (1.16) tells us that $x^{\alpha \beta} = x_0^{\alpha \beta} + \mu^{\alpha \beta} \lambda_\beta$ since $\lambda_\alpha \lambda_\beta = \varepsilon^{\alpha \beta} \lambda_\alpha \lambda_\beta = 0$. Here, $x_0$ is a particular solution to (1.16). Hence, this describes a two-plane in $M^4$ which is totally null since any null-vector $x^{\alpha \beta}$ is of the form $x^{\alpha \beta} = \mu^{\alpha \beta} \lambda_\beta$. In addition, (1.16) implies that the removed line $\mathbb{C}P^1$ of $P^3 \equiv \mathbb{C}P^3 \setminus \mathbb{C}P^1$ corresponds to the point ‘infinity’ of spacetime. Thus, $\mathbb{C}P^3$ can be understood as the twistor space of conformally compactified complexified spacetime.

**Remark 1.3.** Recall that a four-vector $x^\mu$ in $M^4$ is said to be null if it has zero norm, i.e. $g_{\mu \nu}x^\mu x^\nu = 0$. This is equivalent to saying that $\det x = 0$. Hence, the two columns/rows of $x$ must be linearly dependent. Thus, $x^{\alpha \beta} = \mu^{\alpha \beta}$.

2. Massless fields and the Penrose transform

The subject of this section is to sketch how twistor space can be used to derive all solutions to zero-rest-mass field equations.
2.1. Integral formulae for massless fields

To begin with, let \( P^3 \) be twistor space (as before) and consider a function \( f \) that is holomorphic on the intersection \( U_+ \cap U_- \subset P^3 \). Furthermore, let us pull back \( f \) to the correspondence space \( F^5 \). The pull-back of \( f(x^\alpha, \lambda_\beta) \) is \( f(x^{\alpha\beta}, \lambda_{\alpha}) \), since the tangent spaces of the leaves of the fibration \( \pi_1 : F^5 \to P^3 \) are spanned by (1.14) and so the pull-backs have to be annihilated by the vector fields (1.14). Then we may consider following the contour integral

\[
\phi(x) = -\frac{1}{2\pi i} \oint_{\bar{C}} \lambda_\alpha \lambda^\alpha f(x^{\alpha\beta}, \lambda_{\alpha}),
\]

where \( \bar{C} \) is a closed curve in \( U_+ \cap U_- \subset \mathbb{C}P^3 \). Since the measure \( d\lambda_\alpha \lambda^\alpha \) is of homogeneity 2, the function \( f \) should be of homogeneity \(-2\) as only then is the integral well defined. Put differently, only if \( f \) is of homogeneity \(-2\), \( \phi \) is a function defined on \( M^4 \).

Furthermore, one readily computes

\[
\Box \phi = 0, \quad \text{with} \quad \Box := \frac{1}{2} \partial_{\alpha\beta} \partial^{\alpha\beta}
\]

by differentiating under the integral. Hence, the function \( \phi \) satisfies the Klein–Gordon equation. Therefore, any \( f \) with the above properties will yield a solution to the Klein–Gordon equation via the contour integral (2.1). This is the essence of twistor theory. Differentially constrained data on spacetime (in the present situation the function \( \phi \)) is encoded in differentially unconstrained complex analytic data on twistor space (in the present situation the function \( f \)).

**Exercise 2.1.** Consider the following function \( f = 1/(z^1 z^5) \) which is holomorphic on \( U_+ \cap U_- \subset P^3 \). Clearly, it is of homogeneity \(-2\). Show that the integral (2.1) gives rise to \( \phi = 1/\det x \). Hence, this \( f \) yields the elementary solution to the Klein–Gordon equation based at the origin \( x = 0 \).

What about the other zero-rest-mass field equations? Can we say something similar about them? Consider a zero-rest-mass field \( \phi_{a_1 \cdots a_3} \) of positive helicity \( h \) (with \( h > 0 \)). Then,

\[
\phi_{a_1 \cdots a_3}(x) = -\frac{1}{2\pi i} \oint_{\bar{C}} \lambda_\alpha \lambda^{a_1} \lambda_{a_1} \cdots \lambda_{a_3} f(x^{\alpha\beta}, \lambda_{\alpha})
\]

solves the equation

\[
\partial^{a_1} \phi_{a_1 \cdots a_3} = 0.
\]

Again, in order to have a well-defined integral, the integrand should have total homogeneity zero, which is equivalent to requiring \( f \) to be of homogeneity \(-2h - 2\). Likewise, we may also consider a zero-rest-mass field \( \phi_{a_1 \cdots a_3} \) of negative helicity \(-h \) (with \( h > 0 \)) for which we take

\[
\phi_{a_1 \cdots a_3}(x) = -\frac{1}{2\pi i} \oint_{\bar{C}} \lambda_\alpha \partial^{a_1} \partial_{\alpha^{a_1}} \cdots \partial_{\alpha^{a_3}} f(x^{\alpha\beta}, \lambda_{\alpha})
\]

such that \( f \) is of homogeneity \( 2h - 2 \). Hence,

\[
\partial^{a_1} \phi_{a_1 \cdots a_3} = 0.
\]

These contour integral formulae provide the advertised Penrose transform [6, 7]. Sometimes, one refers to this transform as the Radon–Penrose transform to emphasize that it is a generalization of the Radon transform⁶.

---

⁵ As before, we shall not make any notational distinction between the coordinate patches covering \( \mathbb{C}P^3 \) and the ones covering twistor space.

⁶ The Radon transform, named after Johann Radon [8], is an integral transform in two dimensions consisting of the integral of a function over straight lines. It plays an important role in computer-assisted tomography. The higher dimensional analogue of the Radon transform is the x-ray transform; see footnote 27.
In summary, any function on twistor space, provided it is of appropriate homogeneity $m \in \mathbb{Z}$, can be used to construct solutions to zero-rest-mass field equations. However, there are a lot of different functions leading to the same solution. For instance, we could simply change $f$ by adding a function which has singularities on one side of the contour but is holomorphic on the other, since the contour integral does not feel such functions. How can we understand what is going on? Furthermore, are the integral formulae invertible? In addition, we made use of particular coverings, so do the results depend on these choices? The tool which helps clarify all these issues is sheaf cohomology\(^7\). For a detailed discussion about sheaf theory, see e.g. \([4, 9]\).

2.2. Čech cohomology groups and Penrose’s theorem—a sketch

Consider some Abelian sheaf $S$ over some manifold $X$, that is, for any open subset $U \subset X$ one has an Abelian group $S(U)$ subject to certain ‘locality conditions’; appendix D collects useful definitions regarding sheaves including some examples. Furthermore, let $\mathcal{U} = \{U_i\}$ be an open cover of $X$. A $q$-cochain of the covering $\mathcal{U}$ with values in $S$ is a collection $f = \{f_{i_0 \ast \cdots \ast i_q}\}$ of sections of the sheaf $S$ over non-empty intersections $U_{i_0} \cap \cdots \cap U_{i_q}$.

The set of all $q$-cochains has an Abelian group structure (with respect to addition) and is denoted by $C^q(\mathcal{U}, S)$. Then we define the coboundary map by

$$\delta_q : C^q(\mathcal{U}, S) \rightarrow C^{q+1}(\mathcal{U}, S),$$

$$(\delta_q f)_{i_0 \ast \cdots \ast i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k r_{i_0 \ast \cdots \ast i_k \ast i_{k+1} \ast \cdots \ast i_{q+1}} f_{i_0 \ast \cdots \ast i_k \ast i_{k+1} \ast \cdots \ast i_{q+1}},$$

where

$$r_{i_0 \ast \cdots \ast i_{q+1}} : S(U_{i_0} \cap \cdots \cap U_{i_k} \cap \cdots \cap U_{i_{q+1}}) \rightarrow S(U_{i_0} \cap \cdots \cap U_{i_{q+1}})$$

is the sheaf restriction morphism and $\hat{i}_k$ means omitting $i_k$. It is clear that $\delta_q$ is a morphism of groups, and one may check that $\delta_q \circ \delta_{q-1} = 0$.

**Exercise 2.2.** Show that $\delta_q \circ \delta_{q-1} \equiv 0$ for $\delta_q$ as defined above.

Furthermore, we see straight away that $\ker \delta_q = S(X)$. Next we define

$$Z^q(\mathcal{U}, S) := \ker \delta_q \quad \text{and} \quad B^q(\mathcal{U}, S) := \text{im} \delta_{q-1}.$$  

(2.8)

We call elements of $Z^q(\mathcal{U}, S)$ $q$-cocycles and elements of $B^q(\mathcal{U}, S)$ $q$-coboundaries, respectively. Cocycles are anti-symmetric in their indices. Both $Z^q(\mathcal{U}, S)$ and $B^q(\mathcal{U}, S)$ are Abelian groups and since the coboundary map is nil-quadratic, $B^q(\mathcal{U}, S)$ is a (normal) subgroup of $Z^q(\mathcal{U}, S)$. The $q$th Čech cohomology group is the quotient

$$H^q(\mathcal{U}, S) := Z^q(\mathcal{U}, S)/B^q(\mathcal{U}, S).$$

(2.9)

In order to get used to these definitions, let us consider a simple example and take the (Abelian) sheaf of holomorphic sections of the line bundle $O(m) \rightarrow \mathbb{C}P^1$. As before we choose the canonical cover $\mathcal{U} = \{U_a\}$ of $\mathbb{C}P^1$. Since there is only a double intersection, all cohomology groups $H^q$ with $q > 1$ vanish automatically. Table 1 then summarizes $H^0$ and $H^1$. Note that when writing $H^q(X, E)$ for some vector bundle $E \rightarrow X$ over some manifold $X$, we actually mean the (Abelian) sheaf $E$ of sections (either smooth or holomorphic depending on the context) of $E$. By a slight abuse of notation, we shall often not make a notational distinction between $E$ and its sheaf of sections $\mathcal{E}$ and simply write $E$ in both cases.

\(^7\) In section 8.3 we present a discussion for Kleinian signature which bypasses sheaf cohomology.
Table 1. Čech cohomology groups for $\mathcal{O}(m) \rightarrow \mathbb{C}P^1$ with respect to the cover $\mathcal{U} = \{ U_k \}$.

| $m$   | $-4$ | $-3$ | $-2$ | $-1$ | 0   | 1   | 2   | $\cdots$ |
|-------|------|------|------|------|-----|-----|-----|----------|
| $H^0(\mathcal{U}, \mathcal{O}(m))$ | $0$  | $0$  | $0$  | $0$  | $\mathbb{C}^1$ | $\mathbb{C}^2$ | $\mathbb{C}^3$ | $\cdots$ |
| $H^1(\mathcal{U}, \mathcal{O}(m))$ | $\mathbb{C}^1$ | $\mathbb{C}^2$ | $\mathbb{C}^1$ | $0$  | $0$  | $0$  | $\cdots$ |

Let us now compute $H^1(\mathcal{U}, \mathcal{O}(\mathcal{O}(m))$ for $m \geq 0$. The rest is left as an exercise. To this end, consider some representative $f = \{ f_{\pm} \}$ defined on $U_+ \cap U_- \subset \mathbb{C}P^1$.\(^8\) Clearly, $\delta_1 f = 0$ as there are no triple intersections. Without loss of generality, $f$ might be taken as

$$f_{+} = \frac{1}{(\lambda_1)^m} \sum_{n=-\infty}^{\infty} c_n \left( \frac{\lambda_2}{\lambda_1} \right)^n. \quad (2.10)$$

This can be re-written according to

$$f_{+} = \frac{1}{(\lambda_1)^m} \sum_{n=-\infty}^{\infty} c_n \left( \frac{\lambda_2}{\lambda_1} \right)^n$$

$$= \frac{1}{(\lambda_1)^m} \left[ \sum_{n=-m}^{-1} + \sum_{n=0}^{\infty} \right] c_n \left( \frac{\lambda_2}{\lambda_1} \right)^n$$

$$= \frac{1}{(\lambda_1)^m} \sum_{n=0}^{m} c_n \left( \frac{\lambda_2}{\lambda_1} \right)^n + \sum_{n=1}^{\infty} c_{-n} \left( \frac{\lambda_1}{\lambda_2} \right)^n$$

$$= r_{-}^{+} f_{-} + r_{+}^{+} f_{+} = f_{-}^{+},$$

where $r_{+}^{\pm}$ are the restriction mappings. Since the $f_{\pm}$ are holomorphic on $U_{\pm}$, we conclude that $f = \{ f_{\pm} \}$ is cohomologous to $f' = \{ f'_{\pm} \}$ with

$$f_{+}' = \sum_{n=1}^{m-1} c_{-n} \left( \frac{\lambda_2}{\lambda_1} \right)^{m-n}. \quad (12.12)$$

There are precisely $m-1$ independent complex parameters, $c_{-1}, \ldots, c_{-m+1}$, which parametrize $f'$. Hence, we have established $H^1(\mathcal{U}, \mathcal{O}(\mathcal{O}(m)) \cong \mathbb{C}^{m-1}$ whenever $m > 1$ and $H^1(\mathcal{U}, \mathcal{O}(\mathcal{O}(m)) = 0$ for $m = 0, 1$.

**Exercise 2.3.** Complete table 1.

Table 1 hints that there is some sort of duality. In fact,

$$H^0(\mathcal{U}, \mathcal{O}(m)) \cong H^1(\mathcal{U}, \mathcal{O}(\mathcal{O}(m) - 2))^*, \quad (2.13)$$

which is a special instance of Serre duality (see also remark 2.4). Here, the star denotes the vector space dual. To understand this relation better, consider $(m \geq 0)$

$$g \in H^0(\mathcal{U}, \mathcal{O}(m)), \quad \text{with} \quad g = g^{a_1 \cdots a_m} \lambda_{a_1} \cdots \lambda_{a_m} \quad (2.14)$$

and $f \in H^1(\mathcal{U}, \mathcal{O}(\mathcal{O}(m) - 2))$. Then, define the pairing

$$(f, g) := \frac{1}{2\pi i} \oint_{\mathcal{C}} d\lambda_{a} \lambda^{a} f(\lambda_a) g(\lambda_a). \quad (2.15)$$

\(^8\) Note that in the preceding sections, we have not made a notational distinction between $f$ and $f_{\pm}$, but strictly speaking we should have.
where the contour is chosen as before. This expression is complex linear and non-degenerate and depends only on the cohomology class of \( f \). Hence, it gives the duality (2.13).

A nice way of writing (2.15) is as \( (f, g) = \int_{\gamma} g \, d\lambda_a \wedge \lambda_{a_1} \wedge \cdots \wedge \lambda_{a_n} f(\lambda_a) \), where

\[
f_{a_1 \cdots a_n} := -\frac{1}{2\pi i} \oint_{\gamma} d\lambda_a \lambda_a^a \lambda_{a_1} \cdots \lambda_{a_n} f(\lambda_a),
\]

such that Penrose’s contour integral formula (2.1) can be recognized as an instance of Serre duality (the coordinate \( x \) being interpreted as some parameter).

**Remark 2.1.** If \( S \) is some Abelian sheaf over some compact complex manifold \( X \) with covering \( \mathcal{U} \) and \( K \), the sheaf of sections of the canonical line bundle \( K := \det T^*X \), then there is the following isomorphism which is referred to as Serre duality (or sometimes to as Kodaira–Serre duality):

\[
H^q(\mathcal{U}, S) \cong H^{d-q}(\mathcal{U}, S^* \otimes K)^*.
\]

Here, \( d = \dim \mathbb{C} X \). See e.g. [9] for more details. In our present case, \( X = \mathbb{C}P^1 \) and so \( d = 1 \) and \( K = \det T^*\mathbb{C}P^1 = T^*\mathbb{C}P^1 \cong \mathcal{O}(-2) \) and furthermore \( S = \mathcal{O}(m) \).

One technical issue remains to be clarified. Apparently all of our above calculations seem to depend on the chosen cover. But is this really the case?

Consider again some manifold \( X \) with cover \( \mathcal{U} \) together with some Abelian sheaf \( S \). If another cover \( \mathcal{V} \) is the refinement of \( \mathcal{U} \), that is, for \( \mathcal{U} = \{U_i\}_{i \in I} \) and \( \mathcal{V} = \{V_j\}_{j \in J} \) there is a map \( \rho : J \to I \) of index sets, such that for any \( j \in J \), \( V_j \subseteq U_{\rho(j)} \), and then there is a natural group homomorphism (induced by the restriction mappings of the sheaf \( S \))

\[
h_{\mathcal{U} \to \mathcal{V}} : H^q(\mathcal{U}, S) \to H^q(\mathcal{V}, S).
\]

We can then define the inductive limit of these cohomology groups with respect to the partially ordered set of all coverings (see also remark 2.5)

\[
H^q(X, S) := \lim_{\mathcal{U}} \text{ind} H^q(\mathcal{U}, S)
\]

which we call the \( q \)th Čech cohomology group of \( X \) with coefficients in \( S \).

**Remark 2.2.** Let us recall the definition of the inductive limit. If we let \( I \) be a partially ordered set (with respect to \( \geq \)) and \( S_i \) a family of modules indexed by \( I \) with homomorphisms \( f^i_j : S_i \to S_j \) with \( i \geq j \) and \( f^i_i = \text{id} \), \( f^i_j \circ f^j_k = f^i_k \) for \( i \geq j \geq k \), then the inductive limit,

\[
\lim_{i \in I} S_i,
\]

is defined by quotienting the disjoint union \( \bigcup_{i \in I} S_i = \bigcup_{i \in I} \langle i, S_i \rangle \) by the following equivalence relation. Two elements \( x_i \) and \( x_j \) of \( \bigcup_{i \in I} S_i \) are said to be equivalent if there exists a \( k \in I \) such that \( f^k_k(x_i) = f^k_k(x_j) \).

By the properties of inductive limits, we have a homomorphism \( H^q(\mathcal{U}, S) \to H^q(X, S) \). Now the question is: When does this becomes an isomorphism? The following theorem tells us when this is going to happen.

**Theorem 2.1** (Leray). Let \( \mathcal{U} = \{U_i\} \) be a covering of \( X \) with the property that for all tuples \( (U_{i_0}, \ldots, U_{i_p}) \) of the cover, \( H^q(U_{i_0} \cap \cdots \cap U_{i_p}, S) = 0 \) for all \( q \geq 1 \). Then

\[
H^q(\mathcal{U}, S) \cong H^q(X, S).
\]

For a proof, see e.g. [9, 10].
Remark 2.3. We have seen that twistor space $L_x$ arises as $T_xM$ if $Y$ is also interested reader to e.g. [4] for details. It therefore lies somewhat far afield from the main thread of development and we refer the reader to the helicity $h$ with $h \geq 0$ of zero-rest-mass field equations on $M^4$, then
$$H^1(P^3, O(\mp 2h - 2)) \cong H^0(M^4, \wedge_{\pm h}).$$

The proof of this theorem requires more work including a weightier mathematical machinery. It therefore lies somewhat far afield from the main thread of development and we refer the interested reader to e.g. [4] for details.

3. Self-dual Yang–Mills theory and the Penrose–Ward transform

So far, we have discussed free field equations. The subject of this section is a generalization of our above discussion to the nonlinear field equations of self-dual Yang–Mills theory on four-dimensional spacetime. Self-dual Yang–Mills theory can be regarded as a subsector of Yang–Mills theory and in fact, the self-dual Yang–Mills equations are the Bogomolnyi

\begin{table}[h]
\centering
\begin{tabular}{ccccccc}
\hline
$m$ & $\cdots$ & $-4$ & $-3$ & $-2$ & $-1$ & $0$ & $1$ & $2$ & $\cdots$
\hline
$H^0(CP^1, O(m))$ & $\cdots$ & $0$ & $0$ & $0$ & $0$ & $C^1$ & $C^2$ & $C^3$ & $\cdots$
$H^1(CP^1, O(m))$ & $\cdots$ & $C^3$ & $C^2$ & $C^1$ & $0$ & $0$ & $0$ & $0$ & $\cdots$
\hline
\end{tabular}
\caption{Čech cohomology groups for $O(m) \to CP^1$.}
\end{table}

Such covers are called Leray or acyclic covers and in fact our two-set cover $\mathcal{U} = \{U_\pm \}$ of $CP^1$ is of this form. Therefore, table 1 translates into table 2.

Remark 2.3. We have seen that twistor space $P^3 \cong CP^3 \setminus CP^1 \cong O(1) \oplus O(1)$, see (1.17) and (1.18). There is yet another interpretation. The Riemann sphere $CP^1$ can be embedded into $CP^3$. The normal bundle $N_{CP^1|CP^3}$ of $CP^1$ inside $CP^3$ is $O(1) \oplus O(1)$ as follows from the short exact sequence:
$$0 \to \varphi_1 \to TCP^1 \xrightarrow{\varphi_2} TCP^3|_{CP^1} \xrightarrow{\varphi_3} N_{CP^1|CP^3} \to 0.$$
equations of Yang–Mills theory. Solutions to the self-dual Yang–Mills equations are always solutions to the Yang–Mills equations, while the converse may not be true.

3.1. Motivation

To begin with, let $M^4$ be $\mathbb{E}$ and $E \rightarrow M^4$ a (complex) vector bundle over $M^4$ with the structure group $G$. For the moment, we shall assume that $G$ is semi-simple and compact. This allows us to normalize the generators $t_a$ of $G$ according to

$$\text{tr} (t_a t_b) = C(r) \delta_{ab}$$

with $C(r) > 0$. Furthermore, let $\nabla : \Omega^p(M^4, E) \rightarrow \Omega^{p+1}(M^4, E)$ be a connection on $E$ with curvature $F = \nabla^2 \in H^2(M^4, \Omega^2(M^4, \text{End} E))$. The Yang–Mills coupling constant is denoted by $g_{YM}$.

The Yang–Mills action functional is defined by

$$S = -\frac{1}{g_{YM}^2} \int_{M^4} \text{tr} (F \wedge \ast F),$$

where $g_{YM}$ is the Yang–Mills coupling constant and `$\ast$' denotes the Hodge star on $M^4$. The corresponding field equations read as

$$\nabla \ast F = 0 \iff \nabla^\alpha F_{\mu \nu} = 0.$$  

Exercise 3.1. Derive (3.3) by varying (3.2).

Solutions to the Yang–Mills equations are critical points of the Yang–Mills action. The critical points may be local maxima of the action, local minima or saddle points. To find the field configurations that truly minimize (3.2), we consider the following inequality:

$$\pm \int_{M^4} \text{tr} [(F \pm \ast F) \wedge (F \pm \ast F)] \geq 0.$$  

A short calculation then shows that

$$- \int_{M^4} \text{tr} (F \wedge \ast F) \geq \pm \int_{M^4} \text{tr} (F \wedge F)$$

and therefore

$$S \geq \pm \frac{1}{g_{YM}^2} \int_{M^4} \text{tr} (F \wedge F) \iff S \geq \frac{8\pi^2}{g_{YM}^2} |Q|,$$

where $Q \in \mathbb{Z}$ is called the topological charge or instanton number,

$$Q = -\frac{1}{8\pi^2} \int_{M^4} \text{tr} (F \wedge F) = -c_2(E).$$

Here, $c_2(E)$ denotes the second Chern class of $E$; see appendix A for the definition.

Equality is achieved for configurations that obey

$$F = \pm \ast F \iff F_{\mu \nu} = \pm \frac{1}{2} \varepsilon_{\mu \nu \lambda \rho \sigma} F^{\lambda \rho \sigma}$$

with $\varepsilon_{\mu \nu \lambda \rho \sigma} = \delta_{[\mu \nu \lambda]} \rho \sigma$ and $\varepsilon_{0123} = 1$. These equations are called the self-dual and anti-self-dual Yang–Mills equations. Solutions to these equations with finite charge $Q$ are referred to as
instantons and anti-instantons. The sign of $Q$ has been chosen such that $Q > 0$ for instantons while $Q < 0$ for anti-instantons. Furthermore, by virtue of the Bianchi identity, $V F = 0 \iff \nabla_{[\mu} F_{\nu]1} = 0$, solutions to (3.8) automatically satisfy the second-order Yang–Mills equations (3.3).

Remember from our discussion in section 1.2 that the rotation group $SO(4)$ is given by

$$SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2.$$  

Therefore, the anti-symmetric tensor product of two vector representations $4 \wedge 4$ decomposes under this isomorphism as $4 \wedge 4 \cong 3 \oplus 3$. More concretely, by taking the explicit isomorphism (1.2), we can write

$$F_{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \epsilon_{\alpha\beta} f_{\alpha\beta} + \epsilon_{\alpha\beta} f_{\alpha\beta},$$  

with $f_{\alpha\beta} = f_{\beta\alpha}$ and $f_{\alpha\beta} = f_{\beta\alpha}$. Since each of these symmetric rank-2 tensors has three independent components, we have made the decomposition $4 \wedge 4 \cong 3 \oplus 3$ explicit. Furthermore, if we write $F = F^+ + F^-$ with $F^\pm := \frac{1}{2} (F \pm * F)$, i.e. $F^\pm = \pm * F^\pm$, then

$$F^+ \iff f_{\alpha\beta} \quad \text{and} \quad F^- \iff f_{\alpha\beta}.$$  

Therefore, the self-dual Yang–Mills equations correspond to

$$F = * F \iff F^- = 0 \iff f_{\alpha\beta} = 0$$  

and similarly for the anti-self-dual Yang–Mills equations.

**Exercise 3.2.** Verify (3.10) and (3.11) explicitly. Show further that $F \wedge * F$ corresponds to $f_{\alpha\beta} f_{\gamma\delta} + f_{\alpha\beta} f_{\gamma\delta}$ while $F \wedge F$ to $f_{\alpha\beta} f_{\gamma\delta} - f_{\alpha\beta} f_{\gamma\delta}$.

Most surprisingly, even though they are nonlinear, the (anti-)self-dual Yang–Mills equations are integrable in the sense that one can give, at least in principle, all solutions. We shall establish this by means of twistor geometry shortly, but again we will not be too rigorous in our discussion. Furthermore, $f_{\alpha\beta} = 0$ or $f_{\alpha\beta} = 0$ make perfect sense in the complex setting. For convenience, we shall therefore work in the complex setting from now on and impose reality conditions later on when necessary. Note that contrary to the Euclidean and Kleinian cases, the (anti-)self-dual Yang–Mills equations on Minkowski space only make sense for complex Lie groups $G$. This is so because $\ast^2 = -1$ on two-forms in Minkowski space.

### 3.2. Penrose–Ward transform

The starting point is the double fibration (1.15), which we state again for the reader’s convenience,

$$\pi_1 \quad \overset{F^5}{\longrightarrow} \quad \pi_2$$

$$\quad \overset{\pi_1}{\longrightarrow} \quad P^3 \quad \overset{\pi_2}{\longrightarrow} \quad M^4$$  

(3.13)

Consider now a rank-$r$ holomorphic vector bundle $E \rightarrow P^3$ together with its pull-back $\pi_1^* E \rightarrow F^5$. Their structure groups are thus $GL(r, \mathbb{C})$. We may impose the additional condition of having a trivial determinant line bundle, $\det E$, which reduces $GL(r, \mathbb{C})$ to $SL(r, \mathbb{C})$. Furthermore, we again choose the two-patch covering $U = \{ U_\lambda \}$ of $P^3$. Similarly, $F^5$ may be covered by two coordinate patches which we denote by $\hat{U} = \{ \hat{U}_\lambda \}$. Therefore, $E$ and $\pi_1^* E$ are characterized by the transition functions $f = \{ f_{\alpha\beta} \}$ and $\pi_1^* f = \{ \pi_1^* f_{\alpha\beta} \}$. As before, the pull-back of $f_{\alpha\beta}(x^\mu, \lambda_\alpha)$ is $f_{\alpha\beta}(x^\mu, \lambda_\alpha)$, i.e. it is annihilated by the vector fields
(1.14) and therefore constant along \( π_1 : F^5 → P^3 \). In the following, we shall not make a
notational distinction between \( f \) and \( π^*_1 f \) and simply write \( f \) for both bundles. Letting \( \bar{δ}_F \)
and \( δ_F \) be the anti-holomorphic parts of the exterior derivatives on \( P^3 \) and \( F^5 \), respectively, we
have \( π^*_1 δ_F = δ_F \circ π^*_1 \). Hence, the transition function \( f_{+-} \) is also annihilated by \( \bar{δ}_F \).

We shall also assume that \( E \) is holomorphically trivial when restricted to any projective
line \( L_x = π_1(π^{-1}_1(x)) \hookrightarrow P^3 \) for \( x ∈ M^4 \). This then implies that there exist matrix-valued
functions \( ψ_± \) on \( \hat{U}_± \), which define trivializations of \( π^*_1 E \) over \( \hat{U} \), such that \( f_{+-} \) can be
decomposed as (see also remark 3.7)
\[
f_{+-} = ψ_+^{-1} ψ_-
\]
(3.14)
with \( \bar{δ}_F ψ_± = 0 \), i.e. the \( ψ_± = ψ_±(x, λ_±) \) are holomorphic on \( \hat{U}_± \). Clearly, this splitting is
not unique, since one can always perform the transformation
\[
ψ_± ↦→ g^{-1} ψ_±,
\]
(3.15)
where \( g \) is some globally defined matrix-valued function holomorphic on \( F^5 \). Hence, it is
constant on \( \mathbb{C}P^1 \), i.e. it depends on \( x \) but not on \( λ_± \). We shall see momentarily, what the
transformation (3.15) corresponds to on spacetime \( M^4 \).

Since \( V^± \frac{A}{a} f_a \) are the restrictions of the vector fields \( V_a \) given in (1.14) to the coordinate patches \( \hat{U}_± \), we find
\[
ψ_+ V^*_a ψ_±^{-1} = ψ_- V^*_a ψ_-\]
on \( \hat{U}_± \). Explicitly, \( V^± \frac{A}{a} = \frac{λ_±}{λ_±} A_β \) with \( λ_± := λ_a / λ_1 = (1, λ_+) \) and \( λ_± := λ_a / λ_2 = (λ_-, 1) \). Therefore, by an extension of Liouville’s theorem, expressions (3.16) can be at most linear in \( λ_± \). This can also be understood by noting
\[
ψ_+ V^*_a ψ_±^{-1} = ψ_- V^*_a ψ_-\]
and so it is of homogeneity 1. Thus, we may introduce a Lie algebra-valued one-form \( A \) on
\( F^5 \) which has components only along \( π_1 : F^5 → P^3 \):
\[
V_a, A|_{\hat{U}_±} := A^±_a = ψ_± V^±_a ψ_±^{-1} = \frac{λ_±}{λ_±} A_β, \tag{3.18}
\]
where \( A_β \) is \( λ_± \)-independent. This can be re-written as
\[
(V^±_a + A^±_a) ψ_± = \frac{λ_±}{λ_±} V_β ψ_± = 0, \quad \text{with} \quad V_β := ∂_β + A_β. \tag{3.19}
\]
The compatibility conditions for this linear system read as
\[
[V_a, V_β] + [V_β, V_a] = 0, \tag{3.20}
\]
which is equivalent to saying that the \( f_β \)-part of
\[
[V_a, V_β] = ε_α_β f_α \tag{3.21}
\]
vanishes. However, \( f_α = 0 \) is nothing but the self-dual Yang–Mills equations (3.12) on \( M^4 \).

Note that the transformations of the form (3.15) induce the transformations
\[
A_β ↦→ g A_β g^{-1} + g^{-1} A_α g \tag{3.22}
\]
of \( A_β \) as can be seen directly from (3.19). Hence, these transformations induce gauge
transformations on spacetime and so we may define gauge equivalence classes \([A_β]\), where
two gauge potentials are said to be equivalent if they differ by a transformation of the form
(3.22). On the other hand, transformations of the form
\[
f_{+-} ↦→ h_±^{-1} f_± h_−, \tag{3.23}
\]
in section 5.2, we will formalize these transformations in the framework of non-Abelian sheaf cohomology.
where $h_{\pm}$ are matrix-valued functions holomorphic on $\hat{U}_{\pm}$ with $V_{\alpha}^\pm h_{\pm} = 0$, leave the gauge potential $A_{\alpha \beta}$ invariant. Since $V_{\alpha}^\pm h_{\pm} = 0$, the functions $h_{\pm}$ descend down to twistor space $\mathbb{P}^3$ and are holomorphic on $U_{\pm}$ (remember that any function on twistor space that is pulled back to the correspondence space must be annihilated by the vector fields $V_\alpha$). Two transition functions that differ by a transformation of the form (3.23) are then said to be equivalent, as they define two holomorphic vector bundles which are bi-holomorphic. Therefore, we may conclude that an equivalence class $[f_{\pm -}]$ corresponds to an equivalence class $[A_{\alpha \beta}]$.

Altogether, we have seen that holomorphic vector bundles $E \to \mathbb{P}^3$ over twistor space, which are holomorphically trivial on all projective lines $L_s \hookrightarrow \mathbb{P}^3$, yield solutions of the self-dual Yang–Mills equations on $M^4$. In fact, the converse is also true. Any solution to the self-dual Yang–Mills equations arises in this way. See e.g. [4] for a complete proof. Therefore, we have

**Theorem 3.1 (Ward [13]).** There is a one-to-one correspondence between gauge equivalence classes of solutions to the self-dual Yang–Mills equations on $M^4$ and equivalence classes of holomorphic vector bundles over the twistor space $\mathbb{P}^3$ which are holomorphically trivial on any projective line $L_s = \pi_1(\pi_2^{-1}(x)) \hookrightarrow \mathbb{P}^3$.

Hence, all solutions to the self-dual Yang–Mills equations are encoded in these vector bundles and once more, differentially constrained data on spacetime (the gauge potential $A_{\alpha \beta}$) is encoded in differentially unconstrained complex analytic data (the transition function $f_{\pm -}$) on twistor space. The reader might be worried that our constructions depend on the choice of coverings, but as in the case of the Penrose transform, this is not the case as will become transparent in section 5.2.

As before, one may also write down certain integral formulae for the gauge potential $A_{\alpha \beta}$. In addition, given a solution $A = dx^{\alpha \beta} A_{\alpha \beta}$ to the self-dual Yang–Mills equations, the matrix-valued functions $\psi_{\pm}$ are given by

\[
\psi_{\pm} = P \exp \left( - \int_{\mathbb{C}_{\pm}} A \right),
\]  

(3.24)

where ‘$P$’ denotes the path-ordering symbol and the contour $\mathbb{C}_{\pm}$ is any real curve in the null-plane $\pi_2(\pi_1^{-1}(p)) \hookrightarrow M^4$ for $p \in \mathbb{P}^3$ running from some point $x_0$ to a point $x$ with $x^{\alpha \beta}(s) = x^{\alpha \beta}_0 + s \mu^\alpha \lambda^\beta_\pm$ for $s \in [0, 1]$ and constant $\mu^\alpha$; the choice of contour plays no role, since the curvature is zero when restricted to the null-plane.

**Exercise 3.3.** Show that for a rank-1 holomorphic vector bundle $E \to \mathbb{P}^3$, the Ward theorem coincides with the Penrose transform for a helicity $h = -1$ field. See also appendix D. Thus, the Ward theorem gives a non-Abelian generalization of that case and one therefore often speaks of the Penrose–Ward transform.

Before giving an explicit example of a real instanton solution, let us say a few words about real structures. In section 1.2, we introduced reality conditions on $M^4$ leading to Euclidean, Minkowski and Kleinian spaces. In fact, these conditions are induced from twistor space as we shall now explain. For concreteness, let us restrict our attention to the Euclidean case. The Kleinian case will be discussed in section 8.3. Remember that a Minkowski signature does not allow for real (anti-)instantons.

A real structure on $\mathbb{P}^3$ is an anti-linear involution $\tau : \mathbb{P}^3 \to \mathbb{P}^3$. We may choose it according to

\[
\tau(\xi^\alpha, \lambda_\alpha) := (\bar{\xi}^\beta \mathcal{C}_\beta^\alpha, \bar{\xi}^\beta \bar{\lambda}_\beta).
\]  

(3.25a)
where bar denotes complex conjugation as before and
\[
(C_\alpha^\beta) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (C_\alpha^{\bar{\beta}}) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\] (3.25b)

By virtue of the incidence relation \( z^\alpha = x^\alpha \bar{\lambda}_\beta \), we obtain an induced involution on \( M^4 \)\( ^{10} \)
\[
\tau(x^\alpha \bar{\lambda}_\beta) = -\bar{x}^\beta \bar{C}_\gamma^\alpha \bar{C}_\delta^\beta.
\] (3.26)

The set of fixed points \( \tau(x) = x \) is given by \( x^{11} = \bar{x}^{22} \) and \( x^{12} = -\bar{x}^{12} \). By inspecting (1.1), we see that this corresponds to a Euclidean signature real slice \( \mathcal{E} \) in \( M^4 \). Furthermore, \( \tau \) can be extended to \( E \to \mathbb{P}^3 \) according to \( f_{-} = (f_{-}(\tau(z, \lambda))) \). This will ensure that the Yang–Mills gauge potential on spacetime is real and in particular, we find from (3.19) that \( A_{\mu} = -A_{\mu}^\dagger \). Here, ‘\( \dagger \)’ denotes Hermitian conjugation.

**Remark 3.1.** Let us briefly comment on generic holomorphic vector bundles over \( \mathbb{C}P^1 \): So, let \( E \to \mathbb{C}P^1 \) be a rank-\( r \) holomorphic vector bundle over \( \mathbb{C}P^1 \). The Birkhoff–Grothendieck theorem (see e.g. [9] for details) then tells us that \( E \) always decomposes into a sum of holomorphic line bundles
\[
\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r) \to \mathbb{C}P^1.
\]
Therefore, if \( \mathcal{U} = \{ \mathcal{U}_\lambda \} \) denotes the canonical cover of \( \mathbb{C}P^1 \), the transition function \( f = \{ f_{-} \} \) of \( E \) is always of the form
\[
f_{-} = \psi^{-1}_- \Lambda_- \psi_-,
\] where \( \psi_- \) are holomorphic on \( \mathcal{U}_\lambda \). If \( \text{det} \) \( E \) is trivial then \( \sum k_i = 0 \). If furthermore \( E \) is holomorphically trivial then \( k_i = 0 \) and \( f_{-} = \psi^{-1}_- \psi_- \).

Note that given some matrix-valued function \( f_{-} \) which is holomorphic on \( U \cap U_- \subset \mathbb{C}P^1 \), the problem of trying to split \( f_{-} \) according to \( f_{-} = \psi^{-1}_- \psi_- \) with \( \psi_- \) holomorphic on \( U_- \) is known as the Riemann–Hilbert problem and its solutions define holomorphically trivial vector bundles on \( \mathbb{C}P^1 \). In addition \( f_{-} \) also depends on some parameter (in our above case the parameter is \( \tau \)), then one speaks of a parametric Riemann–Hilbert problem. A solution to the parametric Riemann–Hilbert problem might not exist for all values of the parameter, but if it exists at some point in parameter space, then it exists in an open neighbourhood of that point.

### 3.3. Example: Belavin–Polyakov–Schwarz–Tyupkin instanton

Let us now present an explicit instanton solution on Euclidean space for the gauge group \( SU(2) \). This amounts to considering a rank-2 holomorphic vector bundle \( E \to \mathbb{P}^3 \) holomorphically trivial on any \( L_\lambda \to \mathbb{P}^3 \) with trivial determinant line bundle \( \text{det} \ E \) and to equipping twistor space with the real structure according to our previous discussion.

Then, let \( E \to \mathbb{P}^3 \) and \( \pi^*_4 E \to \mathbb{F}^5 \) be defined by the following transition function \( f = \{ f_{-} \} \) [14]:
\[
f_{-} = \frac{1}{\Lambda^2} \begin{pmatrix} \Lambda^2 - z \frac{z^{12}}{\lambda_1 \lambda_2} & \frac{(z^\dagger)^2}{\lambda_1 \lambda_2} \\ \frac{z^{12}}{\lambda_1 \lambda_2} & \Lambda^2 + z \frac{z^{12}}{\lambda_1 \lambda_2} \end{pmatrix},
\] (3.27)

\(^{10}\) We shall use the same notation \( \tau \) for the anti-holomorphic involutions induced on the different manifolds appearing in (3.13).

\(^{11}\) In fact, the involution \( \tau \) can be extended to any holomorphic function.
where $\Lambda \in \mathbb{R} \setminus \{0\}$. Evidently, $\det f_{++} = 1$ and so $\det E$ is trivial. Furthermore, $f_{--}(z, \lambda) = (f_{--}(\tau(z, \lambda)))^\dagger$, where $\tau$ is the involution (3.2) leading to Euclidean space. The main problem now is to find a solution to the Riemann–Hilbert problem $f_{--} = \psi_+^* \psi_-$. Note that if we succeed, we have automatically shown that $E \to \mathbb{P}^3$ is holomorphically trivial on any projective line $L_x \hookrightarrow \mathbb{P}^3$.

In terms of the coordinates on $U_+$, we have

$$f_{++} = \frac{1}{\Lambda^2} \begin{pmatrix} \Lambda^2 - \frac{z_1^2 z_2^2}{\lambda^+_+} & \frac{(z_1^2)^2}{\lambda^+_+} \\ -\frac{(z_2^2)^2}{\lambda^+_+} & \Lambda^2 + \frac{z_1^2 z_2^2}{\lambda^+_+} \end{pmatrix}. \quad (3.28)$$

As there is no generic algorithm, let us just present a solution [14]:

$$\psi_+ = -\frac{1}{\Lambda} \frac{1}{\sqrt{x^2 + \Lambda^2}} \begin{pmatrix} x^{22} z_1^2  + \Lambda^2 & -x^{22} z_2^2 \\ x^{12} z_2^1  + \Lambda^2 & -x^{12} z_2^1 + \Lambda^2 \end{pmatrix} \quad \text{and} \quad \psi_- = \psi_+ f_{--}, \quad (3.29)$$

where $x^2 := \det x$.

It remains to determine the gauge potential and the curvature. We find

$$A_{11} = \frac{1}{2(x^2 + \Lambda^2)} \begin{pmatrix} x^{22} & 0 \\ 2x^{12} & -x^{22} \end{pmatrix}, \quad A_{12} = \frac{1}{2(x^2 + \Lambda^2)} \begin{pmatrix} x^{12} & -2x^{22} \\ 0 & -x^{12} \end{pmatrix} \quad (3.30)$$

and $A_{22} = 0$. Hence, our choice of gauge $\psi_\pm \mapsto g^{-1} \psi_\pm$ corresponds to gauging away $A_{22}$. Furthermore, the only non-vanishing components of the curvature are

$$f_{11} = \frac{2\Lambda^2}{(x^2 + \Lambda^2)^2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f_{12} = -\frac{\Lambda^2}{(x^2 + \Lambda^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.31)$$

which shows that we have indeed found a solution to the self-dual Yang–Mills equations. Finally, using (3.7), we find that the instanton charge $Q = 1$. We leave all the details as an exercise. The above solution is the famous Belavin–Polyakov–Schwarz–Tyupkin instanton [15]. Note that $\Lambda$ is referred to as the ‘size modulus’ as it determines the size of the instanton. In addition, there are four translational moduli corresponding to shifts of the form $x \mapsto x + c$ for constant $c$. Altogether, there are five moduli characterizing the charge one SU(2) instanton. For details on how to construct general instantons, see e.g. [16, 17].

**Exercise 3.4.** Show that (3.29) implies (3.30) and (3.31) by using the linear system (3.19).

Furthermore, show that $Q = 1$. You might find the following integral useful:

$$\int_E d^4x \frac{x^2}{(x^2 + \Lambda^2)^4} = \frac{\pi^2}{6\Lambda^4},$$

where $x^2 = x_\mu x^\mu$.

### 4. Supertwistor space

Up to now, we have discussed the purely bosonic setup. As our goal is the construction of amplitudes in supersymmetric gauge theories, we need to incorporate fermionic degrees of freedom. To this end, we start by briefly discussing supermanifolds before we move on and introduce supertwistor space and the supersymmetric generalization of the self-dual Yang–Mills equations. For a detailed discussion about supermanifolds, we refer to [18–20].
4.1. A brief introduction to supermanifolds

Let \( R \cong R_0 \oplus R_1 \) be a \( \mathbb{Z}_2 \)-graded ring, that is, \( R_0 R_0 \subset R_0, R_1 R_0 \subset R_1, R_0 R_1 \subset R_1 \) and \( R_1 R_1 \subset R_0 \). We call elements of \( R_0 \) Grassmann even (or bosonic) and elements of \( R_1 \) Grassmann odd (or fermionic). An element of \( R \) is said to be homogeneous if it is either bosonic or fermionic. The degree (or Grassmann parity) of a homogeneous element is defined to be 0 if it is bosonic and 1 if it is fermionic, respectively. We denote the degree of a homogeneous element \( r \in R \) by \( p_r \) (\( p \) for parity).

We define the supercommutator, \([\cdot, \cdot] : R \times R \rightarrow R\), by

\[
[r_1, r_2] := r_1 r_2 - (-)^{p_{r_1} p_{r_2}} r_2 r_1,
\]

for all homogeneous elements \( r_{1,2} \in R \). The \( \mathbb{Z}_2 \)-graded ring \( R \) is called supercommutative if the supercommutator vanishes for all of the ring’s elements. For our purposes, the most important example of such a supercommutative ring is the Grassmann or exterior algebra over \( \mathbb{C}^n \):

\[
R = \Lambda^* \mathbb{C}^n := \bigoplus_p \Lambda^p \mathbb{C}^n, \tag{4.2a}
\]

with the \( \mathbb{Z}_2 \)-grading being

\[
R = \bigoplus_p \Lambda^{2p} \mathbb{C}^n \oplus \bigoplus_p \Lambda^{2p+1} \mathbb{C}^n. \tag{4.2b}
\]

An \( R \)-module \( M \) is a \( \mathbb{Z}_2 \)-graded bi-module which satisfies

\[
rm = (-)^{p_r p_m}mr, \tag{4.3}
\]

for homogeneous \( r \in R, m \in M \), with \( M \cong M_0 \oplus M_1 \). Then there is a natural map\(^{12}\) \( \Pi \), called the parity operator, which is defined by

\[
(\Pi M)_0 := M_1 \quad \text{and} \quad (\Pi M)_1 := M_0. \tag{4.4}
\]

We should stress that \( R \) is an \( R \)-module itself, and as such \( \Pi R \) is also an \( R \)-module. However, \( \Pi R \) is no longer a \( \mathbb{Z}_2 \)-graded ring since \( (\Pi R)_1 (\Pi R)_1 \subset (\Pi R)_1 \), for instance.

A free module of rank \( m \mid n \) over \( R \) is defined by

\[
R^{m\mid n} := R^m \oplus (\Pi R)^n, \tag{4.5}
\]

where \( R^m := R \oplus \cdots \oplus R \). This has a free system of generators, \( m \) of which are bosonic and \( n \) of which are fermionic, respectively. We stress that the decomposition of \( R^{m\mid n} \) into \( R^{m\mid 0} \) and \( R^{0\mid n} \) has, in general, no invariant meaning and does not coincide with the decomposition into bosonic and fermionic parts, \( [R^n_0 \oplus (\Pi R)_1^n] \oplus [R^m_0 \oplus (\Pi R)_1^m] \). Only when \( R_1 = 0 \), are these decompositions the same. An example is \( C^{m\mid n} \), where we consider the complex numbers as a \( \mathbb{Z}_2 \)-graded ring (where \( R = R_0 \) with \( R_0 = \mathbb{C} \) and \( R_1 = 0 \)).

Let \( R \) be a supercommutative ring and \( R^{m\mid n} \) be a freely generated \( R \)-module. Just as in the commutative case, morphisms between free \( R \)-modules can be given by matrices. The standard matrix format is

\[
A = \begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix}, \tag{4.6}
\]

where \( A \) is said to be bosonic (respectively, fermionic) if \( A_1 \) and \( A_4 \) are filled with bosonic (respectively, fermionic) elements of the ring while \( A_2 \) and \( A_3 \) are filled with fermionic

\(^{12}\) More precisely, it is a functor from the category of \( R \)-modules to the category of \( R \)-modules. See appendix C for details.
(respectively, bosonic) elements. Furthermore, \( A_1 \) is a \( p \times m \)-, \( A_2 \) a \( q \times m \)-, \( A_3 \) a \( p \times n \)- and \( A_4 \) a \( q \times n \)-matrix. The set of matrices in standard format with elements in \( R \) is denoted by \( \text{Mat}(m[n], p[q], R) \). It forms a \( \mathbb{Z}_2 \)-graded module which, with the usual matrix multiplication, is naturally isomorphic to \( \text{Hom}(R^{m[n]}, R^{p[q]}) \). We denote the endomorphisms of \( R^{m[n]} \) by \( \text{End}(m[n], R) \) and the automorphisms by \( \text{Aut}(m[n], R) \), respectively. We use further the special symbols \( \text{gl}(m[n], R) \subset \text{End}(m[n], R) \) to denote the bosonic endomorphisms of \( R^{m[n]} \) and \( \text{GL}(m[n], R) \subset \text{Aut}(m[n], R) \) to denote the bosonic automorphisms.

The supertranspose of \( A \in \text{Mat}(m[n], p[q], R) \) is defined according to

\[
A^{st} := \begin{pmatrix}
A_1^t & (-)^{p_s} A_2^t \\
(-)^{p_s} A_3 & A_4^t
\end{pmatrix},
\]

where the superscript ‘\( t \)’ denotes the usual transpose. The supertransposition satisfies \((A + B)^{st} = A^{st} + B^{st} \) and \((AB)^{st} = (-)^{p_s p_t} B^{st} A^{st} \). We shall use the following definition of the supertrace of \( A \in \text{End}(m[n], R) \):

\[
\text{str} A := \text{tr} A_1 - (-)^{p_s} \text{tr} A_4,
\]

where the supertrace of matrices is defined analogously to (4.1), i.e. \([A, B] := AB - (-)^{p_s p_t} BA \) for \( A, B \in \text{End}(m[n], R) \). Then \( \text{str}(A, B) = 0 \) and \( \text{str}A^{st} = \text{str}A \). Finally, let \( A \in \text{GL}(m[n], R) \). The superdeterminant is given by

\[
\text{sdet} A := \text{det}(A_1 - A_2 A_3^{-1} A_4) \text{det} A_4^{-1},
\]

where the right-hand side is well defined for \( A_1 \in \text{GL}(m[0], R_0) \) and \( A_4 \in \text{GL}(n[0], R_0) \). Furthermore, it belongs to \( \text{GL}(1[0], R_0) \). The superdeterminant satisfies the usual rules, \( \text{sdet}(AB) = \text{sdet}A \text{sdet}B \) and \( \text{sdet} A^{st} = \text{sdet} A \) for \( A, B \in \text{GL}(m[n], R) \). Note that sometimes \( \text{sdet} \) is referred to as the Berezinian and also denoted by \( \text{Ber} \).

After this digression, we may now introduce the local model of a supermanifold. Let \( V \) be an open subset in \( \mathbb{C}^n \) and consider \( \mathcal{O}_V(\Lambda^* \mathbb{C}^n) := \mathcal{O}_V \otimes \Lambda^* \mathbb{C}^n \), where \( \mathcal{O}_V \) is the sheaf of holomorphic functions on \( V \subset \mathbb{C}^n \) which is also referred to as the structure sheaf of \( V \). Thus, \( \mathcal{O}_V(\Lambda^* \mathbb{C}^n) \) is a sheaf of supercommutative rings consisting of \( \Lambda^* \mathbb{C}^n \)-valued holomorphic functions on \( V \). Let now \( (x^1, \ldots, x^n) \) be coordinates on \( V \subset \mathbb{C}^n \) and \( (\eta_1, \ldots, \eta_n) \) be a basis of the sections of \( \mathcal{O}_V(\Lambda^* \mathbb{C}^n) \). Then \( (x^1, \ldots, x^n, \eta_1, \ldots, \eta_n) \) are coordinates for the ringed space \( \mathcal{V}^{m[n]} := (V, \mathcal{O}_V(\Lambda^* \mathbb{C}^n)) \). Any function \( f \) can thus be Taylor-expanded as

\[
f(x, \eta) = \sum_I \eta_I f^I(x),
\]

where \( I \) is a multi-index. These are the fundamental functions in supergeometry.

To define a general supermanifold, let \( X \) be some topological space of real dimension \( 2m \), and let \( \mathcal{R}_X \) be a sheaf of supercommutative rings on \( X \). Furthermore, let \( \mathcal{N} \) be the ideal subsheaf in \( \mathcal{R}_X \) of all nil-potent elements in \( \mathcal{R}_X \), and define \( \mathcal{O}_X := \mathcal{R}_X/\mathcal{N} \). Then \( \mathcal{V}^{m[n]} := (X, \mathcal{R}_X) \) is called a complex supermanifold of dimension \( m[n] \) if the following is fulfilled.

(i) \( X^m := (X, \mathcal{O}_X) \) is an \( m \)-dimensional complex manifold which we call the body of \( X^{m[n]} \).

(ii) For each point \( x \in X \) there is a neighbourhood \( U \ni x \) such that there is a local isomorphism \( \mathcal{R}_X|_U \cong \mathcal{O}_X(\Lambda^* (\mathcal{N}/\mathcal{N}^2)|_U) \), where \( \mathcal{N}/\mathcal{N}^2 \) is a rank-\( n \) locally free sheaf of \( \mathcal{O}_X \)-modules on \( X \), i.e. \( \mathcal{N}/\mathcal{N}^2 \) is locally of the form \( \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X \) (\( n \)-times). \( \mathcal{N}/\mathcal{N}^2 \) is called the characteristic sheaf of \( X^{m[n]} \).

Therefore, complex supermanifolds look locally like \( \mathcal{V}^{m[n]} = (V, \mathcal{O}_V(\Lambda^* \mathbb{C}^n)) \). In view of this, we picture \( \mathbb{C}^{m[n]} \) as \( (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m}(\Lambda^* \mathbb{C}^m)) \). We shall refer to \( \mathcal{R}_X \) as the structure sheaf of \( X^{m[n]} \). Instead of \( \mathcal{R}_X \), one often also writes \( \mathcal{R} \) and likewise for \( \mathcal{O}_X \).
of the supermanifold $X^{m|n}$ and to $O_X$ as the structure sheaf of the body $X^n$ of $X^{m|n}$. Later on, we shall use a more common notation and re-denote $\mathcal{R}_X$ by $O_X$ or simply by $O$ if there is no confusion with the structure sheaf of the body $X^n$ of $X^{m|n}$. In addition, we sometimes write $X^{m|n}$ instead of $X^n$. Furthermore, the tangent bundle $TX^{m|n}$ of a complex supermanifold $X^{m|n}$ is an example of a supervector bundle, where the transition functions are sections of the (non-Abelian) sheaf $\mathfrak{GL}(m|n, \mathcal{R}_X)$ (see section 5.2 for more details).

Remark 4.1. Recall that for a ringed space $(X, O_X)$ with the property that for each $x \in X$ there is a neighbourhood $U \ni x$ such that there is a ringed space isomorphism $(U, O_X|_U) \cong (V, O_V)$, where $V \subset \mathbb{C}^n$. Then $X$ can be given the structure of a complex manifold and moreover, any complex manifold can arise in this manner. By the usual abuse of notation, $(X, O_X)$ is often denoted by $X$.

An important example of a supermanifold in the context of twistor geometry is the complex projective superspace $\mathbb{C} P^{m|n}$. It is given by

$$\mathbb{C} P^{m|n} := (\mathbb{C} P^m, O_{\mathbb{C} P^m}(\Lambda^*(O(-1) \otimes \mathbb{C}^n))),$$

(4.11)

where $O(-1)$ is the tautological line bundle over the complex projective space $\mathbb{C} P^m$. It is defined analogously to $\mathbb{C} P^1$ (see (1.19)). The reason for the appearance of $O(-1)$ is as follows. If we let $(z^0, \ldots, z^n, \eta_1, \ldots, \eta_n)$ be homogeneous coordinates on $\mathbb{C} P^m$, a holomorphic function $f$ on $\mathbb{C} P^m$ has the expansion

$$f = \sum \eta_1 \cdots \eta_n f^{i_1 \cdots i_r} (z^0, \ldots, z^n).$$

(4.12)

Surely, for $f$ to be well defined the homogeneity of $f$ must be zero. Hence, $f^{i_1 \cdots i_r} = f^{i_1 \cdots i_r} |$ must be of homogeneity $-r$. This explains the above form of the structure sheaf of the complex projective superspace.

Exercise 4.1. Let $E \rightarrow X$ be a holomorphic vector bundle over a complex manifold $X$. Show that $(X, O_X(\Lambda^* E^*))$ is a supermanifold according to our definition given above.

Supermanifolds of the form as in the above exercise are called globally split. We see that $\mathbb{C} P^{m|n}$ is of the type $E \rightarrow \mathbb{C} P^m$ with $E = O(1) \otimes \mathbb{C}^n$. Due to a theorem of Batchelor [21] (see also e.g. [19]), any smooth supermanifold is globally split. This is due to the existence of a (smooth) partition of unity. The reader should be warned that, in general, complex supermanifolds are not of this type (basically because of the lack of a holomorphic partition of unity).

4.2. Supertwistor space

Now we have all the necessary ingredients to generalize (1.15) to the supersymmetric setting. Supertwisters were first introduced by Ferber [22].

Consider $M^{4|2N} \cong \mathbb{C}^{4|2N}$ together with the identification

$$TM^{4|2N} \cong H \otimes \tilde{S},$$

(4.13)

where the fibres $H_x$ of $H$ over $x \in M^{4|2N}$ are $\mathbb{C}^{2|N}$ and $\tilde{S}$ is again the dotted spin bundle. In this sense, $H$ is of rank 2|N and $H \cong E \oplus \tilde{S}$, where $\tilde{S}$ is the undotted spin bundle and $E$ is the rank-0|N R-symmetry bundle. In analogy to $x^\mu \leftrightarrow x^{\alpha\beta}$, we now have $x^M \leftrightarrow x^{\alpha\beta} = (x^{\alpha\beta}, \eta^{\alpha\beta})$ for $\Lambda = (\alpha, j), B = (\beta, j), \ldots$ and $i, j, \ldots = 1, \ldots, N$. Note that the above factorization

14 Note that they are subject to the identification $(z^0, \ldots, z^n, \eta_1, \ldots, \eta_n) \sim (tz^0, \ldots, tz^n, \eta_1, \ldots, \eta_n)$, where $t \in \mathbb{C} \setminus \{0\}$. 

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They define an integrable rank-$2|\mathcal{N}$ of the tangent bundle can be understood as a generalization of a conformal structure (see remark 1.1) known as the para-conformal structure (see e.g. [23]).

As in the bosonic setting, we may consider the projectivization of $\hat{S}^*\mathcal{N}$ to define the correspondence space $F^{3|2|\mathcal{N}} := \mathbb{P}(\hat{S}^*\mathcal{N}) \cong \mathbb{C}^{4|2|\mathcal{N}} \times \mathbb{C}P^1$. Furthermore, we consider the vector fields

$$V_A = \lambda^a \partial_{\lambda_a} = \lambda^a \frac{\partial}{\partial x^A}. \quad (4.14)$$

They define an integrable rank-$2|\mathcal{N}$ distribution on the correspondence space. The resulting quotient will be the supertwistor $P^{3|\mathcal{N}}$:

$$\begin{array}{ccc}
P^{3|\mathcal{N}} & \xymatrix{\ar@{-->}[r]^\pi_1 & \pi_2 \ar@{-->}[r] & M^{4|2|\mathcal{N}}}
\end{array} \quad (4.15)$$

The projection $\pi_2$ is the trivial projection and $\pi_1 : (x^{Aa}, \lambda_a) \mapsto (z^A, \lambda_a) = (x^{Aa}, \lambda_a)$, where $(z^A, \lambda_a)$ are homogeneous coordinates on $P^{3|\mathcal{N}}$.

As before, we may cover $P^{3|\mathcal{N}}$ by two coordinate patches, which we (again) denote by $U_{\pm}$:

$$\begin{align*}
U_+ : & \lambda_1 \neq 0 \quad \text{and} \quad z^A_+ := \frac{z^A}{\lambda_1} \quad \text{and} \quad \lambda_+ := \frac{\lambda_2}{\lambda_1}, \\
U_- : & \lambda_2 \neq 0 \quad \text{and} \quad z^A_- := \frac{z^A}{\lambda_2} \quad \text{and} \quad \lambda_- := \frac{\lambda_1}{\lambda_2}. \quad (4.16)
\end{align*}$$

On $U_+ \cap U_-$ we have $z^A_+ = \lambda_+ z^A_- \lambda_-$. This shows that $P^{3|\mathcal{N}}$ can be identified with $\mathbb{C}P^{3|\mathcal{N}} \setminus \mathbb{C}P^{1|\mathcal{N}}$. It can also be identified with the total space of the holomorphic fibration

$$\mathcal{O}(1) \otimes \mathbb{C}^{2|\mathcal{N}} \rightarrow \mathbb{C}P^1. \quad (4.17)$$

Another way of writing this is $\mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \mathcal{O}(1) \otimes \mathbb{C}^V \rightarrow \mathbb{C}P^1$, where $\Pi$ is the parity map given in (4.4). In the following, we shall denote the two patches covering the correspondence space $F^{5|2|\mathcal{N}}$ by $\hat{U}_{\pm}$. Note that remark 2.6 also applies to $P^{3|\mathcal{N}}$.

Similarly, we may extend the geometric correspondence: a point $x \in M^{4|2|\mathcal{N}}$ corresponds to a projective line $L_x = \pi_1(\pi_2^{-1}(\lambda_x)) \in P^{3|\mathcal{N}}$, while a point $p = (z, \lambda) \in P^{3|\mathcal{N}}$ corresponds to a $2|\mathcal{N}$-plane in superspace $M^{4|2|\mathcal{N}}$ that is parametrized by $x^{Aa} = x_0^A + \lambda^a \lambda_i^a$, where $x_0^A$ is a particular solution to the supersymmetric incidence relation $z^A = x^{Aa} \lambda_a$.

### 4.3. Superconformal algebra

Before we move on and talk about supersymmetric extensions of self-dual Yang–Mills theory, let us digress a little and collect a few facts about the superconformal algebra. The conformal algebra, $\text{conf}_4$, in four dimensions is a real form of the complex Lie algebra $\mathfrak{sl}(4, \mathbb{C})$. The concrete real form depends on the choice of signature of spacetime. For the Euclidean signature, we have $\mathfrak{so}(1, 5) \cong \mathfrak{su}^*(4)$ while for Minkowski and Kleinian signatures we have $\mathfrak{so}(2, 4) \cong \mathfrak{su}(2, 2)$ and $\mathfrak{so}(3, 3) \cong \mathfrak{sl}(4, \mathbb{R})$, respectively. Likewise, the $\mathcal{N}$-extended conformal algebra—the superconformal algebra, $\text{conf}_{4|\mathcal{N}}$—is a real form of the complex Lie superalgebra $\mathfrak{sl}(4|\mathcal{N}, \mathbb{C})$ for $\mathcal{N} < 4$ and $\mathfrak{psl}(4|4, \mathbb{C})$ for $\mathcal{N} = 4$. For a compendium of Lie superalgebras, see e.g. [24]. In particular, for $\mathcal{N} < 4$ we have $\mathfrak{su}^*(4|\mathcal{N})$, $\mathfrak{su}(2, 2|\mathcal{N})$ and $\mathfrak{sl}(4|\mathcal{N}, \mathbb{R})$ for Euclidean, Minkowski and Kleinian signatures while for $\mathcal{N} = 4$ the superconformal algebras are $\mathfrak{psu}^*(4|4)$, $\mathfrak{psu}(2, 2|4)$ and $\mathfrak{psl}(4|4, \mathbb{R})$. Note that for a Euclidean signature, the number $\mathcal{N}$ of supersymmetries is restricted to be even.
The generators of $\text{conf}_{4|N'}$ are
\begin{equation}
\text{conf}_{4|N'} = \text{span}\{ P_\mu, L_{\mu\nu}, K^\mu, D, R_i, A \mid Q_{i\alpha}, Q_{i\alpha}', S^{\alpha\beta}, S^{\gamma\delta} \}.
\end{equation}
(4.18)
Here, $P_\mu$ represents translations, $L_{\mu\nu}$ (Lorentz) rotations, $K^\mu$ special conformal transformations, $D$ dilations and $R_i$ the $R$-symmetry while $Q_{i\alpha}, Q_{i\alpha}'$ are the Poincaré supercharges and $S^{\alpha\beta}, S^{\gamma\delta}$ their superconformal partners. Furthermore, $A$ is the axial charge which is absent for $N' = 4$. Making use of the identification (1.12), we may also write
\begin{equation}
\text{conf}_{4|N'} = \text{span}\{ P_{\mu\beta}, L_{\mu\alpha\beta}, K^{\alpha\beta}, D, R_i, A \mid Q_{i\alpha}, Q_{i\alpha}', S^{\alpha\beta}, S^{\gamma\delta} \},
\end{equation}
(4.19)
where the $L_{\mu\alpha\beta}, L_{\mu\alpha\beta}$ are symmetric in their indices (see also (3.10)). We may also include a central extension $\hat{z} = \text{span}\{Z\}$ leading to $\text{conf}_{4|N'} \oplus \hat{z}$, i.e. $[\text{conf}_{4|N'}, \hat{z}] = 0$ and $[\hat{z}, \hat{z}] = 0$.

The commutation relations for the centrally extended superconformal algebra $\text{conf}_{4|N'} \oplus \hat{z}$ are
\begin{align*}
\{ Q_{i\alpha}, Q_{i\alpha}' \} &= -\delta_{i\alpha} P_{\rho\beta}, \quad \{ S^{\alpha\beta}, S^{\gamma\delta} \} = -\delta^{\alpha\beta}_{\gamma\delta}, \\
\{ Q_{i\alpha}, S^{\rho\beta} \} &= -i \left[ \delta^\beta_\gamma L_{\alpha\rho} + \frac{1}{2} \delta^\rho_\gamma \delta^\beta_i (D + Z) + 2 \delta^\rho_\gamma R_i + \frac{1}{2} \delta^\rho_\gamma \delta^\beta_i \left( 1 - \frac{4}{N} \right) A \right], \\
\{ Q_{i\alpha}', S^{\rho\beta} \} &= i \left[ \delta^\beta_\gamma L_{\alpha\rho} + \frac{1}{2} \delta^\rho_\gamma \delta^\beta_i (D - Z) - 2 \delta^\rho_\gamma R_i + \frac{1}{2} \delta^\rho_\gamma \delta^\beta_i \left( 1 - \frac{4}{N} \right) A \right], \\
\{ R_i, S^{\rho\beta} \} &= -i \left( \delta^\beta_\gamma S^{\rho\gamma}_{\alpha} - \frac{1}{N} \delta^\beta_\gamma S^{\rho\alpha}_{\beta} \right), \quad [R_i, S^{\alpha\beta}] = i \left( \delta^\beta_\gamma S^{\alpha\gamma}_{\beta} - \frac{1}{N} \delta^\beta_\gamma S^{\alpha\beta}_{\gamma} \right), \\
\{ L_{\alpha\beta}, S^{\gamma\delta} \} &= -i \left( \delta^\delta_\gamma S^{\alpha\gamma}_{\beta} - \frac{1}{N} \delta^\delta_\gamma S^{\alpha\beta}_{\gamma} \right), \quad [L_{\alpha\beta}, S^{\gamma\delta}] = -i \left( \delta^\delta_\gamma S^{\alpha\delta}_{\beta} - \frac{1}{N} \delta^\delta_\gamma S^{\alpha\beta}_{\delta} \right), \\
\{ S^{\alpha\beta}, P_{\rho\beta} \} &= -\delta^\beta_\rho Q_{\alpha'}, \quad [S^{\alpha\beta}, P_{\rho\beta}] = \delta^\beta_\rho Q_{\alpha'}, \\
[D, S^{\alpha\beta}] &= -\frac{i}{2} S^{\alpha\beta}, \quad [D, S^{\gamma\delta}] = -\frac{i}{2} S^{\gamma\delta}, \\
[A, S^{\alpha\beta}] &= \frac{i}{2} S^{\alpha\beta}, \quad [A, S^{\gamma\delta}] = -\frac{i}{2} S^{\gamma\delta}, \\
[R_i, Q_{i\alpha}] &= -\frac{i}{2} \left( \delta^\alpha_\gamma Q_{i\gamma} - \frac{1}{N} \delta^\alpha_\gamma Q_{i\alpha} \right), \quad [R_i, Q^\gamma_{\alpha}] = \frac{i}{2} \left( \delta^\gamma_\alpha Q_{i\gamma} - \frac{1}{N} \delta^\gamma_\alpha Q^\gamma_{i} \right), \\
[L_{\alpha\beta}, Q_{i\gamma}] &= i \left( \delta^\gamma_\beta Q_{i\alpha} - \frac{1}{2} \delta^\gamma_\beta Q_{i\gamma} \right), \quad [L_{\alpha\beta}, Q^\gamma_{\alpha}] = i \left( \delta^\gamma_\alpha Q_{i\gamma} - \frac{1}{2} \delta^\gamma_\alpha Q^\gamma_{i} \right), \\
[Q_{i\alpha}, K^{\rho\beta}] &= \delta^\rho_\alpha S^{\beta}_{\gamma}\delta_{\gamma\delta}, \quad [Q^\rho_{i\alpha}, K^{\rho\beta}] = -\delta^\rho_\alpha S^{\gamma}_{\beta}\delta_{\gamma\delta}, \\
[D, Q_{i\alpha}] &= \frac{i}{2} Q_{i\alpha}, \quad [D, Q^\alpha_{i\alpha}] = \frac{i}{2} Q^\alpha_{i\alpha}, \\
[A, Q_{i\alpha}] &= -\frac{i}{2} Q_{i\alpha}, \quad [A, Q^\alpha_{i\alpha}] = \frac{i}{2} Q^\alpha_{i\alpha}, \\
[R_i, R_i] &= \frac{i}{2} \left( \delta^\alpha_\gamma R_{i\gamma} - \delta^\alpha_\gamma R_{i\alpha} \right), \\
[D, P_{i\alpha}] &= iP_{i\alpha}, \quad [D, K^{\rho\alpha}] = -iK^{\rho\alpha}, \\
[L_{\alpha\beta}, P_{\rho\gamma}] &= i \left( \delta^\rho_\gamma P_{\alpha\gamma} - \frac{1}{2} \delta^\rho_\gamma P_{\rho\gamma} \right), \quad [L_{\alpha\beta}, P^\rho_{\rho\gamma}] = i \left( \delta^\rho_\gamma P_{\alpha\rho} - \frac{1}{2} \delta^\rho_\gamma P_{\rho\rho} \right), \\
[L_{\alpha\beta}, K^{\rho\gamma}] &= -i \left( \delta^\rho_\gamma K^{\beta\gamma} - \frac{1}{2} \delta^\rho_\gamma K^{\rho\gamma} \right), \quad [L_{\alpha\beta}, K^\rho_{\rho\gamma}] = -i \left( \delta^\rho_\gamma K_{\alpha\beta} - \frac{1}{2} \delta^\rho_\gamma K^\beta_{\rho\rho} \right), \\
[L_{\alpha\beta}, L_{\rho\beta}] &= i \left( \delta^\rho_\beta L_{\alpha\beta} - \delta^\rho_\beta L^\rho_{\rho\beta} \right), \quad [L_{\alpha\beta}, L^\rho_{\rho\beta}] = i \left( \delta^\rho_\beta L_{\alpha\beta} - \delta^\rho_\beta L^\rho_{\rho\beta} \right), \\
[P_{i\alpha}, K^{\rho\beta}] &= -i \left( \delta^\rho_\beta L_{\alpha\beta} + \delta^\rho_\beta L_{\rho\beta} + \delta^\rho_\beta L_{\rho\rho} \right). \quad (4.20)
\end{align*}
Note that for $N = 4$, the axial charge $A$ decouples, as mentioned above. Note also that upon choosing a real structure, not all of the above commutation relations are independent of each other. Some of them will be related via conjugation.

If we let $(z^a, \lambda_\alpha) = (z^a, \eta_i, \lambda_\alpha)$ be homogeneous coordinates on $P^{3|N}$, then $\text{conf}_{4|N} \cong \mathbb{Z}$ can be realized in terms of the following vector fields:

$$P_{a\alpha} = \lambda_\alpha \frac{\partial}{\partial z^a}, \quad K^{a\alpha} = z^a \frac{\partial}{\partial \lambda_\alpha}, \quad D = -i \left( z^a \frac{\partial}{\partial z^a} - \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} \right),$$

$$L_\alpha^\beta = -i \left( \frac{z^\beta}{2} \frac{\partial}{\partial z^a} - \frac{1}{2} \delta^\beta_a \frac{\partial}{\partial z^\gamma} \right), \quad L_\alpha^\beta = i \left( \lambda_\beta \delta_\alpha^\beta - \frac{1}{2} \delta^\beta_a \lambda_\gamma \frac{\partial}{\partial \lambda_\gamma} \right),$$

$$R_\gamma^i = - \frac{i}{2} \left( \eta_i \frac{\partial}{\partial \eta_j} - \frac{1}{N} \eta_k \frac{\partial}{\partial \eta_k} \right), \quad A = -i \frac{\eta_i}{2} \frac{\partial}{\partial \eta_i}, \quad (4.21)$$

$$Z = - \frac{i}{2} \left( z^a \frac{\partial}{\partial z^a} + \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} + \eta_i \frac{\partial}{\partial \eta_i} \right), \quad Q^{ia} = \lambda_\alpha \delta_{ia} \frac{\partial}{\partial \lambda_\alpha}, \quad S^{ia} = \delta^{ia} \frac{\partial}{\partial \lambda_\alpha}.$$  

Using $\partial_\alpha z^\beta = \delta_\alpha^\beta$, $\partial_\alpha \lambda_\beta = \delta_\alpha^\beta$, and $\partial_\alpha \eta_i = \delta_\alpha^i$ for $\partial_\alpha := \partial/\partial z^a$, $\partial_\alpha := \partial/\partial \lambda_\alpha$, and $\partial_\alpha := \partial/\partial \eta_i$, one can straightforwardly check that the above commutation relations are satisfied. Furthermore, we emphasize that we work non-projectively. Working projectively, the central charge $Z$ is absent (when acting on holomorphic functions), as is explained in remark 4.9. The non-projective version will turn out to be more useful in our discussion of scattering amplitudes.

**Remark 4.2.** Consider complex projective superspace $\mathbb{C}P^{m|n}$. Then we have the canonical projection $\pi : \mathbb{C}^{m+1|0} \setminus \{0\} \times \mathbb{C}^n \rightarrow \mathbb{C}P^{m|n}$. Let now $(z^a, \eta_i) = (z^0, \ldots, z^m, \eta_1, \ldots, \eta_n)$ be linear coordinates on $\mathbb{C}^{m+1|0} \setminus \{0\}$ (or equivalently, homogeneous coordinates on $\mathbb{C}P^{m|n}$) for $a = 0, \ldots, m$ and $i = 1, \ldots, n$. Then

$$\pi_+ \left( z^0 \frac{\partial}{\partial z^0} + \eta_1 \frac{\partial}{\partial \eta_1} \right) = 0$$

as follows from a direct calculation in affine coordinates which are defined by

$$\mathbb{C}P^{m|n} \ni U_a : z^a \neq 0 \quad \text{and} \quad (z^{a(\alpha)}, \eta_i(\alpha)) := \left( \frac{z^a}{z^0}, \frac{\eta_i}{z^0} \right)$$

for $a = 0, \ldots, m$ and $\alpha \neq a$, i.e. $\mathbb{C}P^{m|n} = \bigcup_a U_a$.

Likewise, we have a realization of $\text{conf}_{4|N} \cong \mathbb{Z}$ in terms of vector fields on the correspondence space $F^{5|2N}$ compatible with the projection $\pi_1 : F^{5|2N} \rightarrow P^{3|2N}$, i.e. the vector fields (4.21) are the push-forward via $\pi_1_+$ of the vector fields on $F^{5|2N}$. In particular, if we take $(x^{a(\alpha)}, \lambda_\alpha) = (x^{a(\alpha)}, \eta_i(\alpha))$ as coordinates on $F^{5|2N}$, where $\lambda_\alpha$ are homogeneous coordinates on $\mathbb{C}P^1$, we have

$$P_{a\alpha} = \frac{\partial}{\partial x^{a(\alpha)}}, \quad K^{a\alpha} = -x^{a(\alpha)} \frac{\partial}{\partial x^{a(\alpha)}} - x^{a(\alpha)} \eta_i^{a(\alpha)} \frac{\partial}{\partial \eta_i^{a(\alpha)}} + x^{a(\alpha)} \lambda_\beta \frac{\partial}{\partial \lambda_\beta},$$

$$D = -i \left( x^{a(\alpha)} \frac{\partial}{\partial x^{a(\alpha)}} + \frac{1}{2} \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} - \frac{1}{2} \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} \right),$$

$$L_\alpha^\beta = -i \left( x^{a(\alpha)} \frac{\partial}{\partial x^{a(\alpha)}} - \frac{1}{2} \delta_\alpha^\beta \frac{\partial}{\partial x^{a(\alpha)}} \right).$$

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\[ L^\alpha_\beta = -i \left( x^{\beta\gamma} \frac{\partial}{\partial x^\alpha} - \frac{1}{2} \delta^\beta_\alpha x^{\gamma\gamma} \frac{\partial}{\partial x^\gamma} \right) - i \left( \eta^\alpha_i \frac{\partial}{\partial \eta^\beta_i} - \frac{1}{2} \delta^\alpha_\beta \eta^\gamma_i \frac{\partial}{\partial \eta^\gamma_i} \right) \] (4.22)

\[ R^j_i = -\frac{i}{2} \left( \eta^\alpha_i \frac{\partial}{\partial \eta^\beta_i} - \frac{1}{N} \eta^\alpha_i \frac{\partial}{\partial \eta^\beta_i} \right), \quad A = -\frac{i}{2} \eta^\alpha_i \frac{\partial}{\partial \eta^\beta_i}, \quad Z = -\frac{i}{2} \lambda_\alpha \frac{\partial}{\partial \lambda_\alpha}. \]

In order to understand these expressions, let us consider a holomorphic function \( f \) on \( F^{5|2N} \) which descends down to \( P^{3|N} \). Recall that such a function is of the form \( f = f(x^{\alpha\beta} \lambda_\alpha, \lambda_\beta) = f(x^{\alpha\beta} \lambda_\alpha, \eta_\alpha^\gamma \lambda_\beta, \bar{\lambda}_\gamma) \) since then \( V_A f = 0 \). Then,

\[ \left. \frac{\partial}{\partial x^{\alpha\beta}} \right|_{\lambda_\alpha} f = \lambda_\alpha \left. \frac{\partial}{\partial z^A} \right|_{\lambda_\alpha} f, \]

\[ \left. \frac{\partial}{\partial \lambda_\alpha} \right|_{\lambda_\alpha} f = \left( \lambda_\alpha \left. \frac{\partial}{\partial z^A} \right|_{\lambda_\alpha} + \left. \frac{\partial}{\partial \lambda_\alpha} \right|_{\lambda_\alpha} \right) f. \] (4.23)

Next let us exemplify the calculation for the generator \( L^\alpha_\beta \). The rest is left as an exercise. Using relations (4.23), we find

\[ \left[ -i \left( x^{\alpha\beta} \frac{\partial}{\partial x^\alpha} - \frac{1}{2} \delta^\alpha_\beta x^{\gamma\gamma} \frac{\partial}{\partial x^\gamma} \right) \right]_{\lambda_\alpha} + i \left( \lambda_\alpha \frac{\partial}{\partial \lambda_\beta} - \frac{1}{2} \delta^\alpha_\beta \lambda_\gamma \frac{\partial}{\partial \lambda_\gamma} \right) \mid_{\lambda_\alpha} f \]

\[ = i \left( \lambda_\alpha \frac{\partial}{\partial \lambda_\beta} - \frac{1}{2} \delta^\alpha_\beta \lambda_\gamma \frac{\partial}{\partial \lambda_\gamma} \right) \mid_{\lambda_\alpha} f. \] (4.24)

Therefore,

\[ \pi_{\lambda_\alpha} \left[ -i \left( x^{\alpha\beta} \frac{\partial}{\partial x^\alpha} - \frac{1}{2} \delta^\alpha_\beta x^{\gamma\gamma} \frac{\partial}{\partial x^\gamma} \right) + i \left( \lambda_\alpha \frac{\partial}{\partial \lambda_\beta} - \frac{1}{2} \delta^\alpha_\beta \lambda_\gamma \frac{\partial}{\partial \lambda_\gamma} \right) \right] \]

\[ = i \left( \lambda_\alpha \frac{\partial}{\partial \lambda_\beta} - \frac{1}{2} \delta^\alpha_\beta \lambda_\gamma \frac{\partial}{\partial \lambda_\gamma} \right), \] (4.25)

what is precisely the relation between the realizations of the \( L^\alpha_\beta \)-generator on \( F^{5|2N} \) and \( P^{3|N} \) as displayed in (4.21) and (4.22).

**Exercise 4.2.** Show that all the generators (4.21) are the push-forward under \( \pi_{\lambda_\alpha} \) of the generators (4.22).

It remains to give the vector field realization of the superconformal algebra on spacetime \( M^{4|2N} \). This is rather trivial, however, since \( \pi_2 : F^{5|2N} \to M^{4|2N} \) is the trivial projection. We find

\[ P^{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}, \quad K^{\alpha\beta} = -x^{\alpha\beta} x^{\rho\gamma} \frac{\partial}{\partial x^{\rho\gamma}} - x^{\alpha\beta} \eta^\rho_i \frac{\partial}{\partial \eta^\rho_i}, \]

\[ D = -i \left( x^{\alpha\beta} \frac{\partial}{\partial x^{\alpha\beta}} + \frac{1}{2} \eta^\rho_i \frac{\partial}{\partial \eta^\rho_i} \right), \]
Furthermore, upon using Bianchi identities, these equations are known as the constraint equations of the theory. We may then parametrize the scalar potentials as

\[ L_{\alpha} = -i \left( \partial_{\alpha \beta} \frac{\partial}{\partial \alpha \beta} \frac{\partial}{\partial \alpha \beta} - \frac{1}{2} \delta_{\alpha \beta} \delta_{\gamma \gamma} \right) \tag{4.26} \]

\[ L_{\dot{a}} = -i \left( \partial_{\dot{a} \dot{b}} \frac{\partial}{\partial \dot{a} \dot{b}} \frac{\partial}{\partial \dot{a} \dot{b}} - \frac{1}{2} \delta_{\dot{a} \dot{b}} \delta_{\dot{c} \dot{c}} \right) - i \left( \dot{\eta}_{\dot{a}} \frac{\partial}{\partial \dot{a}} - \frac{1}{2} \delta_{\dot{a} \dot{b}} \dot{\eta}_{\dot{b}} \frac{\partial}{\partial \dot{b}} \right) \]

\[ R_{\pm} = -\frac{i}{2} \left( \eta_{i} \frac{\partial}{\partial \eta_{j}} - \frac{1}{N} \eta_{i} \frac{\partial}{\partial \eta_{i}} \right), \quad A = -\frac{i}{2} \dot{\eta}_{i} \frac{\partial}{\partial \dot{a}_{i}} \]

\[ Q_{i a} = i \eta_{i} \frac{\partial}{\partial \dot{a}_{i}}, \quad Q'_{i a} = i \frac{\partial}{\partial \eta_{i}} \]

\[ S'^{i} = i x^{a} \frac{\partial}{\partial \dot{a}_{i}}, \quad S'_{i} = -i \eta_{i} x^{a} \frac{\partial}{\partial \dot{a}_{i}} - i \dot{\eta}_{i} \frac{\partial}{\partial \dot{a}_{i}} \]

5. Supersymmetric self-dual Yang–Mills theory and the Penrose–Ward transform

5.1. Penrose–Ward transform

By analogy with self-dual Yang–Mills theory, we may now proceed to construct supersymmetrized versions of this theory within the twistor framework. The construction is very similar to the bosonic setting, so we can be rather brief.

Take a holomorphic vector bundle \( E \to \mathbb{P}^{3|2N} \) and pull it back to \( F^{5|2N} \). Note that although we restrict our discussion to ordinary vector bundles, everything goes through for supervector bundles as well. Then, the transition function is constant along \( \pi_{1} : F^{5|2N} \to \mathbb{P}^{3|2N} \), i.e. \( V_{+}^{a} f_{+} = 0 \) where the \( V_{+}^{a} \) are the restrictions of \( V_{1} \) to the patches \( \hat{U}_{k} \) with \( F^{5|2N} = \hat{U}_{k} \cap \hat{U}_{-} \). Under the assumption that \( E \) is holomorphically trivial on any \( L_{k} = \pi_{1}(\pi_{-1}(x)) \to \mathbb{P}^{3|2N} \), we again split \( f_{+} \) according to \( f_{+} = \psi_{+}^{-1} \psi_{-} \) and hence \( \psi_{+}^{-1} V_{+}^{a} \psi_{+} = \psi_{-}^{-1} V_{-}^{a} \psi_{-} \) on \( \hat{U}_{+} \cap \hat{U}_{-} \). Therefore, we may again introduce a Lie algebra-valued one-form that has components only along \( \pi_{1} : F^{5|2N} \to \mathbb{P}^{3|2N} \):

\[ A_{\pm} = \lambda_{\pm} A_{\pm} = \psi_{+}^{-1} V_{+}^{a} \psi_{+}, \tag{5.1} \]

where \( A_{\pm} \) is \( \lambda_{\pm} \)-independent. Thus, we find

\[ \lambda_{\pm} \nabla_{A_{\pm}} \psi_{\pm} = 0, \quad \nabla_{A_{\pm}} := \partial_{A_{\pm}} + A_{A_{\pm}} \tag{5.2} \]

which together with the compatibility conditions,

\[ [\nabla_{A_{\pm}}, \nabla_{B_{\pm}}] + [\nabla_{A_{\pm}}, \nabla_{B_{\pm}}] = 0. \tag{5.3} \]

These equations are known as the constraint equations of \( \mathcal{N} \)-extended supersymmetric self-dual Yang–Mills theory (see e.g. [25, 26]).

Let us analyse these equations a bit more for \( \mathcal{N} = 4 \). Cases with \( \mathcal{N} < 4 \) can be obtained from the \( \mathcal{N} = 4 \) equations by suitable truncations. We may write the above constraint equations as

\[ [\nabla_{a_{\beta}}, \nabla_{b_{\beta}}] = \epsilon_{a_{\beta}} F_{a_{\beta} b_{\beta}}, \quad \text{with} \quad F_{a_{\beta} b_{\beta}} = (-)^{p_{a} p_{b}} F_{a_{\beta} b_{\beta}}. \tag{5.4} \]

We may then parametrize \( F_{a_{\beta}} \) as

\[ F_{a_{\beta}} = (F_{a_{\beta}}, F_{a_{\beta}}^{i}, F^{ij}) := (f_{a_{\beta}}, \frac{1}{2} \lambda_{a_{\beta}}, -\phi^{ij}). \tag{5.5} \]

Furthermore, upon using Bianchi identities

\[ [\nabla_{a_{\beta}}, \nabla_{b_{\beta}}] \nabla_{C_{\gamma}}] \quad + \quad (-)^{p_{a} (p_{a} + p_{c})} [\nabla_{b_{\beta}}, \nabla_{C_{\gamma}}] \nabla_{a_{\beta}} \]

\[ \quad + \quad (-)^{p_{c} (p_{c} + p_{c})} [\nabla_{C_{\gamma}}, \nabla_{a_{\beta}}] \nabla_{b_{\beta}}] = 0. \tag{5.6} \]
we find two additional fields
\[ \chi_{ia} := -\frac{\sqrt{2}}{2} \nabla_a \phi_{ij} \quad \text{and} \quad G_{a\dot{\beta}} := \frac{1}{\sqrt{2}} \nabla^j \chi_{a\dot{\beta}}, \] (5.7)
where we have introduced the common abbreviation \( \phi_{ij} := \frac{1}{\sqrt{2}} \epsilon_{ijkl} \phi^{kl} \) and parentheses mean normalized symmetrization. Altogether, we have obtained the fields displayed in Table 3. Note that all these fields are superfields, i.e. they live on \( M^{4|8} \cong \mathbb{C}^4 \).

The question is, how can we construct fields and their corresponding equations of motion on \( M^4 \), since that is what we are actually after. The key idea is to impose the so-called transversal gauge condition \([27–29]\)
\[ \eta^a_i A^i_a = 0. \] (5.8)
This reduces supergauge transformations to ordinary ones as follows. Generic infinitesimal supergauge transformations are of the form
\[ \delta A_{a\dot{\beta}} = \nabla_{Aa\dot{\beta}} = \partial_{a\dot{\beta}} + \{A_{a\dot{\beta}}, \epsilon \}, \] where \( \epsilon \) is a bosonic Lie algebra-valued function on \( M^{4|8} \). Residual gauge transformations that preserve (5.8) are then given by
\[ \eta^a_i \delta A^i_a = 0 \quad \implies \quad \eta^a_i \partial_{a\dot{\beta}} \epsilon = 0 \quad \iff \quad \epsilon = \varepsilon(x^{a\dot{\beta}}), \] (5.9)
i.e. we are left with gauge transformations on spacetime \( M^4 \). Then, by defining the recursion operator \( \mathcal{D} := \eta^a_i \nabla^i_a = \eta^a_i \partial_{a\dot{\beta}} \) and by using the Bianchi identities (5.6), after a somewhat lengthy calculation we obtain the following set of recursion relations:
\[ \mathcal{D}A_{a\dot{\beta}} = -\frac{1}{\sqrt{2}} \epsilon_{a\dot{\beta}} \eta^a_i K^i_a, \]
\[ (1 + \mathcal{D})A^i_a = -\epsilon_{a\dot{\beta}} \eta^i_j \phi^{ij}, \]
\[ \mathcal{D} \phi_{ij} = \sqrt{2} \eta^i_j \chi_{j\dot{\alpha}a}, \]
\[ \mathcal{D} \chi_{ia} = \sqrt{2} \eta^i_j \nabla_{a\alpha} \phi^{ij}, \]
\[ \mathcal{D} \chi_{ia} = -\frac{1}{\sqrt{2}} \eta^a_i G_{a\dot{\beta}} + \frac{1}{\sqrt{2}} \epsilon_{a\dot{\beta}} \eta^i_j \phi^{ij}, \]
\[ \mathcal{D} G_{a\dot{\beta}} = \sqrt{2} \eta^i_j \epsilon_{j\alpha a} \chi_{j\dot{\beta}a}, \] (5.10)
where, as before, parentheses mean normalized symmetrization while the brackets denote normalized anti-symmetrization of the enclosed indices. These equations determine all superfields to order \( n + 1 \), provided one knows them to \( n \)th order in the fermionic coordinates.

At this point, it is helpful to present some formulae which simplify this argument a great deal. Consider some generic superfield \( f \). Its explicit \( \eta \)-expansion has the form
\[ f = f + \sum_{k \geq 1} \eta^{j_1}_h \cdots \eta^{j_k}_h f^{j_1 \cdots j_k}. \] (5.11)
Here and in the following, the circle refers to the zeroth-order term in the superfield expansion of some superfield \( f \). Furthermore, we have \( \mathcal{D} f = \eta^{j_1}_h \cdots \eta^{j_k}_h f^{j_1 \cdots j_k} \), where the bracket \( [ \ ] \) is a composite expression of some superfields. For example, we have \( \mathcal{D} A_{a\dot{\beta}} = \frac{1}{\sqrt{2}} \eta^{j_1}_h \{a\dot{\alpha}\} \eta^{j_1}_h \), with \( [a\dot{\alpha}]_{j_1} = -\epsilon_{a\dot{\beta}a} A^a_{\dot{\beta}} \). Now let
\[ \mathcal{D} \{ \}^{j_1 \cdots j_k} = \eta^{j_1}_h \cdots \eta^{j_k}_h \{ \}^{j_1 \cdots j_k}, \] (5.12)
Then we find after a successive application of $\mathcal{D}$

$$f = \tilde{f} + \sum_{k \geq 1} \frac{1}{k!} \eta_0^{h_1} \cdots \eta_k^{h_k} \tilde{1}_{m-h_k}.$$  \hspace{1cm} (5.13)

If the recursion relation of $f$ is of the form $(1 + \mathcal{D}) f = \eta_0^{h_1} \tilde{1}_{m-h_1}$ as it happens to be for $A'_a$, then $\tilde{f} = 0$ and the superfield expansion is of the form

$$f = \sum_{k \geq 1} \frac{k}{(k+1)!} \eta_0^{h_1} \cdots \eta_k^{h_k} \tilde{1}_{m-h_k}.$$  \hspace{1cm} (5.14)

Using these expressions, one obtains the following results for the superfields $A_{aa}$ and $A'_a$:

$$A_{aa} = \tilde{A}_{aa} + \frac{1}{\sqrt{2}} \varepsilon_{aa} \tilde{X}_a \eta_0^b + \cdots,$$

$$A'_a = -\frac{1}{21} \varepsilon_{aa} \tilde{X}_a \eta_0^b - \frac{\sqrt{3}}{3!} \varepsilon_{ijkl} \tilde{X}_k \eta_0^b \eta_0^l \eta_0^j + \frac{3}{4!} \varepsilon_{ijkl} \varepsilon_{aa} (-G_2 \eta_0^m + \varepsilon_{jl} [\phi_{mn}, \phi_{mi}]) \eta_0^b \eta_0^j \eta_0^l \eta_0^m + \cdots.$$  \hspace{1cm} (5.15)

Upon substituting these superfield expansions into the constraint equations (5.3), (5.4), we obtain

$$f_{\tilde{a}\tilde{b}} = 0,$$

$$\varepsilon_{a\tilde{b}} \nabla_{aa} \tilde{X}_a = 0,$$

$$\Box \phi^{\tilde{a}} = \frac{i}{2} \varepsilon_{a\tilde{b}} \left[ \tilde{X}^{\tilde{a}}, \tilde{X}_a \right].$$  \hspace{1cm} (5.16)

These are the equations of motion of $\mathcal{N} = 4$ supersymmetric self-dual Yang–Mills theory. The equations for less supersymmetry are obtained from these by suitable truncations. We have also introduced the abbreviation $\Box := \frac{i}{2} \varepsilon_{a\tilde{b}} \varepsilon_{a\tilde{b}} \nabla_{aa} \nabla_{\tilde{a}\tilde{b}}$. We stress that (5.16) represent the field equations to the lowest order in the superfield expansions. With the help of the recursion operator $\mathcal{D}$, one may verify that they are in one-to-one correspondence with the constraint equations (5.3). For details, see e.g. [27–29].

Altogether, we have a supersymmetric extension of Ward’s theorem 3.3.

**Theorem 5.1.** There is a one-to-one correspondence between gauge equivalence classes of solutions to the $\mathcal{N}$-extended supersymmetric self-dual Yang–Mills equations on spacetime $M^4$ and equivalence classes of holomorphic vector bundles over supertwistor space $P^{3|N}$ which are holomorphically trivial on any projective line $L_x = \pi_1^{-1}(x) \leftrightarrow P^{3|N}$.

**Exercise 5.1.** Verify all equations from (5.10) to (5.16).

Finally, let us emphasize that the field equations (5.16) also follow from an action principle. Indeed, upon varying

$$S = \int d^4 x \text{ tr} \left\{ \tilde{G}^{a\tilde{b}} f_{\tilde{a}\tilde{b}} + \varepsilon^{ia} \nabla_{aa} \tilde{X}_a - \frac{1}{2} \phi_{ij} \Box \phi^{ij} + \frac{1}{2} \phi_{ij} [\tilde{X}_a, \tilde{X}^{\tilde{a}}] \right\},$$  \hspace{1cm} (5.17)
we find (5.16). In writing this, we have implicitly assumed that a reality condition corresponding either to the Euclidean or Kleinian signature has been chosen; see below for more details. This action functional is known as the Siegel action [30].

Remark 5.1. Let us briefly comment on hidden symmetry structures of self-dual Yang–Mills theories. Since Pohlmeyer’s work [31], it has been known that self-dual Yang–Mills theory possess infinitely many hidden non-local symmetries. Such symmetries are accompanied by conserved non-local charges. As was shown in [14, 32–36], these symmetries are affine extensions of internal symmetries with an underlying Kac–Moody structure. See [37] for a review. Subsequently, Popov and Preitschopf [38] found affine extensions of conformal symmetries of Kac–Moody/Virasoro-type. A systematic investigation of symmetries based on twistor and cohomology theory was performed in [39] (see also [40, 41] and the textbook [1]), where all symmetries of the self-dual Yang–Mills equations were derived. In [42, 43] (see [44] for a review), these ideas were extended to \( \mathcal{N} \)-extended self-dual Yang–Mills theory. For some extensions to the full \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory, see [45]. Note that the symmetries of the self-dual Yang–Mills equations are intimately connected with one-loop MHV scattering amplitudes [46–49]. See also part II of these lecture notes.

Exercise 5.2. Verify that the action functional (5.17) is invariant under the following supersymmetry transformations (\( \epsilon^i_\alpha \) is some constant anti-commuting spinor):

\[
\begin{align*}
\delta A_{\alpha} &= -\frac{1}{\sqrt{2}} e_{\alpha \beta} \tilde{\eta} \tilde{\chi}_{\alpha}^i, \\
\delta \tilde{\phi}_{ij} &= \sqrt{2} \tilde{\eta} \tilde{\phi}_{ij} \tilde{\chi}_{\alpha}^i, \\
\delta \tilde{\chi}_a^i &= \sqrt{2} \tilde{\eta} \tilde{\phi}_a \Phi_{ij}^i, \\
\delta \tilde{\chi}_\alpha &= -\frac{1}{\sqrt{2}} e_{\alpha \beta} \tilde{\eta} \tilde{G}_{a \beta} + \frac{1}{\sqrt{2}} e_{\alpha \beta} \eta \tilde{\phi}_{ij}^i [\tilde{\phi}_{ij}, \tilde{\phi}_{\beta}]. \\
\delta \tilde{G}_{a \beta} &= \sqrt{2} \tilde{\eta} \tilde{G}_{a \beta} \tilde{\chi}_{ji}^i \tilde{\phi}_{\beta}.
\end{align*}
\]

5.2. Holomorphic Chern–Simons theory

Let us pause for a moment and summarize what we have achieved so far. In the preceding sections, we have discussed \( \mathcal{N} = 4 \) supersymmetric self-dual Yang–Mills theory by means of holomorphic vector bundles \( E \rightarrow P^{3|4} \) over the supertwistor space \( P^{3|4} \) that are holomorphically trivial on all projective lines \( L_a = \pi_1 P_2^{-1}(x) \rightarrow P^{3|4} \). These bundles are given by holomorphic transition functions \( f = \{ f_a \} \). We have further shown that the field equations of \( \mathcal{N} = 4 \) supersymmetric self-dual Yang–Mills theory arise upon varying a certain action functional on spacetime, the Siegel action. Figure 1 summarizes pictorially our previous discussion.

The question that now arises and which is depicted in figure 1 concerns the formulation of a corresponding action principle on the supertwistor space. Certainly, such an action, if it exists, should correspond to the Siegel action on spacetime. However, in constructing such a twistor space action, we immediately face a difficulty. Our above approach to the twistor re-formulation of field theories, either linear or nonlinear, is intrinsically on-shell: holomorphic functions on twistor space correspond to solutions to field equations on spacetime and vice versa. In particular, holomorphic transition functions of certain holomorphic vector bundles \( E \rightarrow P^{3|4} \) correspond to solutions to the \( \mathcal{N} = 4 \) supersymmetric self-dual Yang–Mills equations. Therefore, we somehow need an ‘off-shell approach’ to holomorphic vector
follows that these sets will be of particular interest. We remark that from the first of these two definitions it are only concerned with ordinary vector bundles (after all we are interested in bundles (which is also known as the Čech approach). Consider a complex (super)manifold such that the on-shell condition is the holomorphicity of these bundles. We may define the subsets of cocycles $Z^n(U, GL(r, O))$ by

$$Z^n(U, GL(r, O)) := \{ f \in C^n(U, GL(r, O)) \mid f_i = f_j \text{ on } U_i \cap U_j \neq \emptyset \},$$

and

$$f_{ij} f_{jk} f_{ki} = 1 \text{ on } U_i \cap U_j \cap U_k \neq \emptyset.$$

These sets will be of particular interest. We remark that from the first of these two definitions it follows that $Z^n(U, GL(r, O))$ coincides with the group of global sections, $H^0(U, GL(r, O))$, of the sheaf $GL(r, O)$. Note that in general the subset $Z^1(U, GL(r, O)) \subset C^1(U, GL(r, O))$ is not a subgroup of the group $C^1(U, GL(r, O))$. For notational reasons, we shall denote elements of $C^0(U, GL(r, O))$ also by $h = [h_i].$

We say that two cocycles $f, f' \in Z^1(U, GL(r, O))$ are equivalent if $f_{ij} = h_i^{-1} f_{ij} h_j$ for some $h \in C^0(U, GL(r, O))$, since one can always absorb the $h = [h_i]$ in a re-definition of the frame fields. Note that this is precisely the transformation we already encountered in (3.23). The set of equivalence classes induced by this equivalence relation is the first Čech cohomology set and denoted by $H^1(U, GL(r, O))$. If the $U_i$ are all Stein (see remark 5.2)—in the case of supermanifolds $X$ we need the body to be covered by Stein manifolds—we have the bijection

$$H^1(U, GL(r, O)) \cong H^1(X, GL(r, O)), \quad (5.19)$$

otherwise one takes the inductive limit (see remark 2.5).\footnote{Basically everything we shall say below will also apply to $GL(r, O)$ and hence to supervector bundles. As we are only concerned with ordinary vector bundles (after all we are interested in $SU(r)$ gauge theory), we will stick to $GL(r, O)$ for concreteness. See e.g. [44] for the following treatment in the context of supervector bundles.}

![Diagram](image)

**Figure 1.** Correspondences between supertwistor space and spacetime.
Remark 5.2. We call an ordinary complex manifold $(X, \mathcal{O})$ Stein if $X$ is holomorphically convex (that is, the holomorphically convex hull of any compact subset of $X$ is again compact in $X$) and for any $x, y \in X$ with $x \neq y$ there is some $f \in \mathcal{O}$ such that $f(x) \neq f(y)$.

To sum up, we see that within the Čech approach, rank-$r$ holomorphic vector bundles over some complex (super)manifold $X$ are parametrized by $H^1(X, \mathcal{O}(r, \mathcal{O}))$. Note that our cover $\{U_{ij}\}$ of the (super)twistor space is Stein and so $H^1(\{U_{ij}\}, \mathcal{O}(r, \mathcal{O})) \cong H^1(\mathcal{O}\ell_X, \mathcal{O}(r, \mathcal{O}))$. This in turn explains that all of our above constructions are independent of the choice of cover.

Another approach to holomorphic vector bundles is the so-called Dolbeault approach. Let $X$ be a complex (super)manifold and consider a rank-$r$ complex vector bundle $E \rightarrow X$. Furthermore, we let $\Omega^p,q(X)$ be the smooth differential $(p, q)$-forms on $X$ and $\overline{\partial} : \Omega^{p,q}(X) \rightarrow \Omega^{p,q+1}(X)$ be the anti-holomorphic exterior derivative. A $(0, 1)$-connection on $E$ is defined by a covariant differential $\nabla^{0,1} : \Omega^{p,q}(X, E) \rightarrow \Omega^{p,q+1}(X, E)$ which satisfies the Leibniz formula. Here, $\Omega^{p,q}(X, E) := \Omega^{p,q}(X) \otimes E$. Locally, it is of the form $\nabla^{0,1} = \overline{\partial} + A^{0,1}$, where $A^{0,1}$ is a differential $(0, 1)$-form with values in $\text{End } E$ which we shall refer to as the connection $(0, 1)$-form. The complex vector bundle $E$ is said to be holomorphic if the $(0, 1)$-connection is flat, that is, if the corresponding curvature vanishes:

$$F^{0,2} = (\nabla^{0,1})^2 = \overline{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0.$$  

(5.20)

In other words, $\nabla^{0,1}$ defines a holomorphic structure on $E$. The group $H^0(X, \mathcal{O}(r, \mathcal{S}))$, where $\mathcal{S}$ is the sheaf of smooth functions on $X$, acts on $A^{0,1}$ by gauge transformations

$$A^{0,1} \mapsto g^{-1} A^{0,1} g + g^{-1} \overline{\partial} g,$$

with $g \in H^0(X, \mathcal{O}(r, \mathcal{S}))$.  

(5.21)

Clearly, such transformations leave (5.20) invariant; hence, they do not change the holomorphic structure on $E$. Therefore, two solutions to $F^{0,2} = 0$ are regarded as equivalent if they differ by such a gauge transformation. We shall then denote the space of equivalence classes by $H^{0,1}_{\psi_0}(X, E)$.

In summary, we have two apparently different approaches. In the Čech approach, holomorphic vector bundles are parametrized by $H^1(X, \mathcal{O}(r, \mathcal{O}))$ while in the Dolbeault approach by $H^{0,1}_{\psi_0}(X, E)$. However, are these approaches really different? In fact, they turn out to be equivalent by virtue of the following theorem.

Theorem 5.2. Let $X$ be a complex (super)manifold with an open Stein covering $\mathcal{U} = \{U_i\}$ and $E \rightarrow X$ be a rank-$r$ complex vector bundle over $X$. Then there is a map $\rho : H^1(X, \mathcal{O}(r, \mathcal{O})) \rightarrow H^1(X, \mathcal{O}(r, \mathcal{S}))$ of cohomology sets, such that $H^{0,1}_{\psi_0}(X, E) \cong \ker \rho$.

This means that given some holomorphic vector bundle $E \rightarrow X$ in the Dolbeault picture, we can always find a holomorphic vector bundle $\tilde{E} \rightarrow X$ in the Čech picture and vice versa, such that $E$ and $\tilde{E}$ are equivalent as complex vector bundles:

$$(E, f = \{f_{ij}\}, \nabla^{0,1}) \sim (\tilde{E}, \tilde{f} = \{\tilde{f}_{ij}\}, \tilde{\nabla}),$$  

(5.22)

with $\tilde{f}_{ij} = \psi_i^{-1} f_{ij} \psi_j$ for some $\psi = \{\psi_i\} \in C^0(\mathcal{U}, \mathcal{O}(r, \mathcal{S}))$. This theorem might be regarded as a non-Abelian generalization of the famous Dolbeault theorem (see remark 5.6).

We shall not prove this theorem here but instead only make the following observation: given $(E, f = \{f_{ij}\}, \nabla^{0,1})$, then any solution $A^{0,1}$ of $F^{0,2} = 0$ is of the form $A^{0,1}|_{U_i} = A_i = \psi_i \tilde{\nabla} \psi_i^{-1}$ for some $\psi = \{\psi_i\} \in C^0(\mathcal{U}, \mathcal{O}(r, \mathcal{S}))$ with

$$A_j = f^{-1}_{ij} \tilde{\partial} f_{ij} + f^{-1}_{ij} A_i f_{ij},$$  

(5.23)

as patching conditions (see also appendix A). Upon substituting $A_i = \psi_i \tilde{\partial} \psi_i^{-1}$ into these equations, we obtain an $\tilde{f}_{ij} = \psi_i^{-1} f_{ij} \psi_j$ with $\tilde{\partial} \tilde{f}_{ij} = 0$. Conversely, starting from
by the vector fields that span the tangent spaces of the leaves of correspondence space. By the definition of the pull-back, the transition function is annihilated of a holomorphic vector bundle on (super)twistor space to those of the pull-back on the correspondence space. In fact, we already encountered a Čech–Dolbeault correspondence before, but we did not allude to it as such. In sections 3.2 and 5.1 we related the transition function to a Lie algebra-valued (differential) one-form with components only along π₁. More precisely, this one-form is the connection one-form of the so-called relative connection along π₁ which by our very construction turned out to be flat. In this picture, the transition function is the Čech representative while the Lie algebra-valued one-form is the Dolbeault representative of a relatively flat bundle on the correspondence space.

The above theorem is good news in that it yields an ‘off-shell’ formulation of holomorphic vector bundles. Given some complex vector bundle \( E \to X \) with a \((0, 1)\)-connection \( \nabla^{0,1} \) that is represented by \( A^{0,1} \), we can turn it into a holomorphic vector bundle provided \( A^{0,1} \) satisfies the ‘on-shell condition’ (5.20). The latter equation is also known as the equation of motion of holomorphic Chern–Simons theory. What about an action? Is it possible to write down an action functional which yields an \( F^{0,2} = 0 \)? The answer is yes, but not always. The action that gives \( F^{0,2} = 0 \) is [50]

\[
S = -\frac{1}{2\pi i} \int_X \Omega \wedge \left\{ A^{0,1} \wedge \bar{\partial} A^{0,1} + \frac{2\pi i}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right\} = -\frac{1}{2\pi i} \int_X \Omega \wedge \omega, \tag{5.24}
\]

where the pre-factor has been chosen for later convenience. A few words are in order. For a moment, let us assume that \( X \) is an ordinary complex manifold. The holomorphic Chern–Simons form

\[
\omega = \text{tr} \left\{ A^{0,1} \wedge \bar{\partial} A^{0,1} + \frac{2\pi i}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1} \right\} \tag{5.25}
\]

is a \((0, 3)\)-form on \( X \). Furthermore, the action (5.24) is invariant under (5.21) provided \( \Omega \) is holomorphic (and \( g \) is homotopic to the identity and \( g \to 1 \) asymptotically). Thus, \( \dim \mathbb{C} X = 3 \) and so \( \Omega \) is a holomorphic \((3, 0)\)-form. In addition, \( \Omega \) should be globally defined, since the Chern–Simons form is. This in turn puts severe restrictions on the geometric properties of \( X \). Complex manifolds that admit such forms are called formal Calabi–Yau manifolds. The name is chosen to distinguish them from ‘honest’ Calabi–Yau manifolds which are (compact) complex manifolds that are Kähler and that admit globally defined holomorphic top-forms. Equivalently, we could say that the canonical bundle \( K := \text{det} T^{1,0} X \) is trivial, since \( \Omega \) is a section of \( K \). Note that sections of \( K \) are also called holomorphic volume forms. Therefore, while the equation \( F^{0,2} = 0 \) makes sense on any complex manifold \( X \), the corresponding action functional is defined only for three-dimensional formal Calabi–Yau manifolds.

Let us take a closer look at the twistor space \( P^3 \). Certainly, it is a three-dimensional complex manifold. But does it admit a globally defined holomorphic volume form \( \Omega_0 \)? In fact it does not, since

\[
\Omega_0 = \frac{i}{4}dz^\alpha \wedge dz_{\bar{\alpha}} \wedge d\lambda_{\rho} \lambda^\beta, \tag{5.26}
\]

which shows that \( K \cong \mathcal{O}(-4) \) which is not a trivial bundle. Therefore, it is not possible to write down an action principle for holomorphic Chern–Simons theory on \( P^3 \). However,

\[\text{[For a beautiful recent exposition on Calabi–Yau manifolds, see e.g. the lecture notes [51].]}\]
somehow we could have expected that as we do not have an action on spacetime either (apart from the Lagrange multiplier-type action $\int d^4x \, G^{\alpha\beta} f_{\alpha\beta}$ with the additional field $G_{\alpha\beta}$; see section 6.1 for more details).

But what about $\mathcal{N} = 4$ supersymmetric self-dual Yang–Mills theory? We know that there is an action principle. The holomorphic volume form $\Omega$ on supertwistor space $P^{3|4}$ is given by

$$\Omega = \Omega_0 \otimes \Omega_1, \quad \text{with} \quad \Omega_1 := \frac{1}{4!} \epsilon^{ijkl} d\eta_i \, d\eta_j \, d\eta_k \, d\eta_l, \quad (5.27)$$

where $\Omega_0$ is as above. To determine whether this is globally defined or not, we have to take into account the definition of the Berezin integral over fermionic coordinates:

$$\int d\eta_i \, \eta_j = \delta_{ij}. \quad (5.28)$$

If one re-scales $\eta_i \mapsto t \eta_i$, with $t \in \mathbb{C} \setminus \{0\}$, then $d\eta_i \mapsto t^{-1} d\eta_i$. Therefore, the $d\eta_i$ transform oppositely to $\eta_i$ and thus they are not differential forms. In fact, they are so-called integrated forms. This shows that (5.27) is indeed globally defined, since $\Omega_0$ is a basis section of $O(-4)$ while $\Omega_1$ is a basis section of $O(4)$ and so $\Omega$ is of homogeneity zero since $O(-4) \otimes O(4) \cong \mathbb{C}$. Hence, the Berezinian line bundle $\text{Ber}$ (which is the supersymmetric generalization of the canonical bundle) of the supertwistor space $P^{3|4}$ is trivial. In this sense, $P^{3|4}$ is a formal Calabi–Yau supermanifold\(^{17}\). In remark 5.4, we show that any three-dimensional complex spin manifold can be associated with a formal Calabi–Yau supermanifold via the LeBrun construction [60]. Note that if we had instead considered $P^{3|N}$, then $\Omega$ would have been of homogeneity $N - 4$. Altogether, we see that the holomorphic Chern–Simons action $A_{43}$ can only be written down for $\mathcal{N} = 4$. In this case, the symmetry group that preserves the holomorphic measure $\Omega$ is $PSL(4|4, \mathbb{C})$. This is the complexification of the $\mathcal{N} = 4$ superconformal group in four dimensions (which is the global symmetry group of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory). See also section 4.3.

We are almost done. We just need to clarify one issue. The connection one-form $A^{0,1}$ is an ‘honest’ differential form, i.e. it is not an integral form like $\Omega_1$. However, in general $A^{0,1}$ also depends on $\tilde{\eta}_i$ and furthermore on the differential $(0, 1)$-forms $d\tilde{\eta}_i$. As there is no way of integrating fermionic differential forms, the integral (5.24) seems not to make sense. However, this is not the case since the supertwistor space $P^{3|4}$ can be regarded as a holomorphic vector bundle over the twistor space $P^3$; see exercise 4.10 and equation (4.11). Hence, there always exists a gauge in which

$$\frac{\partial}{\partial \tilde{\eta}_i} A^{0,1} = 0 \quad \text{and} \quad \frac{\partial}{\partial \tilde{\eta}_i} \tilde{A}^{0,1} = 0, \quad (5.29)$$

since the fibres $P^{3|4} \rightarrow P^3$ do not have sufficient non-trivial topology. Thus, in this gauge $A^{0,1}$ only has a holomorphic dependence on $\eta_i$. We shall refer to this gauge as the Witten gauge [53].

Summarizing our above discussion, we have found an action principle on supertwistor space for holomorphic vector bundles in the case of $\mathcal{N} = 4$ supersymmetry\(^{18}\). Therefore, the missing box in figure 1 can be filled in with the help of (5.24). In the remainder of this section,

\(^{17}\) In fact, $P^{3|4}$ is also an ‘honest’ Calabi–Yau space in the sense of being Kähler and admitting a super Ricci-flat metric [52, 53]. Note that for ordinary manifolds, the Kähler condition together with the existence of a globally defined holomorphic volume form always implies the existence of a Ricci-flat metric. This is the famous Yau theorem. However, an analogue of Yau’s theorem in the context of supermanifolds does not exist. See e.g. [52, 54–59].

\(^{18}\) An analogous action has been found by Sokatchev [61] in the harmonic superspace approach. Recently, Popov gave alternative twistor space action principles in [62]. Interestingly, similar actions also exist for maximal $\mathcal{N} = 8$ self-dual supergravity in the harmonic space approach [63] as well as in the twistor approach [64].
we demonstrate explicitly that (5.24) indeed implies (5.17) on spacetime. Our discussion below is based on [53, 65–67].

For concreteness, let us choose reality conditions that lead to a Euclidean signature real slice in $M^4$. At the end of section 3.2, we saw that they arise from an anti-holomorphic involution on $P^3$. Similarly, we may introduce an anti-holomorphic involution $\tau : P^{3\mid 4} \to P^{3\mid 4}$ leading to Euclidean superspace. In particular,

$$\tau(\varepsilon^A, \lambda_\alpha) := (\bar{\varepsilon}^A, \bar{\lambda}_\alpha) := (\bar{\varepsilon}^B C_B^A, C_\beta^\alpha \lambda_\bar{\beta}), \quad (5.30)$$

where $(C_A^B) = \text{diag}((C_\beta^\alpha), (C_i^j))$ with $C_\beta^\alpha$ and $C_i^j$ given by (3.25b) and

$$(C_i^j) := \left(\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array}\right). \quad (5.31)$$

By virtue of the incidence relation $\varepsilon^A = x^{\alpha a} \lambda_\alpha$, we find

$$\tau(x^{\alpha a}) = -x^{B\bar{\beta}} C_B^A C_\beta^\alpha. \quad (5.32)$$

The set of fixed points $\tau(x) = x$ corresponds to Euclidean superspace $\mathbb{R}^{4\mid 8}$ in $M^{4\mid 8}$.

**Remark 5.4.** Let $X$ be an $m$-dimensional complex manifold and $E \to X$ be a rank-$n$ holomorphic vector bundle over $X$. In exercise 4.10 we saw that $X^{m\mid n} = (X, \mathcal{O}_X(\Lambda^* E^*))$ is a complex supermanifold. The Berezinian line bundle is then given by Ber $\equiv \mathcal{K} \otimes \Lambda^n E$, where $\mathcal{K}$ is the canonical bundle of $X$. Holomorphic volume forms on $X^{m\mid n}$ are sections of Ber and the existence of a globally defined holomorphic volume form is equivalent to the triviality of Ber. Next one can show that there is a short exact sequence

$$0 \to \varphi^{-1} \to T^{1,0} X \otimes V \xrightarrow{\varphi^1} \text{Jet}^1 V \xrightarrow{\varphi^3} V \xrightarrow{\varphi^4} 0,$$

where Jet$^1 V$ is the bundle of first-order jets of some other holomorphic vector bundle $V \to X$ (i.e. sections of Jet$^1 V$ are sections of $V$ together with their first-order derivatives). Exactness of this sequence means $\text{im} \varphi_1 = \text{ker} \varphi_{i+1}$. Now let $m = 3$ and furthermore assume that $X$ is a spin manifold. In particular, this means that $\mathcal{K}$ has a square root denoted by $\mathcal{K}^{1/2}$. Then $X$ can be extended to a formal Calabi–Yau supermanifold of dimension $3\mid 4$ by setting $V := \mathcal{K}^{1/2}$ and $E := \text{Jet}^1 \mathcal{K}^{1/2}$, where $\mathcal{K}^{1/2}$ is the dual of $\mathcal{K}^{1/2}$ (note that for line bundles one often denotes the dual by the inverse since the tensor product of a line bundle with its dual is always trivial). To see that Ber is trivial, we use the above sequence, which implies $\text{det} \text{Jet}^1 V \cong \text{det} V \otimes \text{det}(T^{1,0} X \otimes V)$, i.e.

$$\Lambda^4 E \cong \mathcal{K}^{-1/2} \otimes \Lambda^3 (T^{1,0} X \otimes \mathcal{K}^{1/2}) \cong \mathcal{K}^{-1/2} \otimes \Lambda^3 T^{1,0} X \cong \mathcal{K}^{1/2} \otimes \mathcal{K}^{-1/2} \cong \mathcal{K}^{-1}$$

and hence Ber $\cong \mathcal{K} \otimes \Lambda^4 E \cong \mathcal{K} \otimes \mathcal{K}^{-1}$ is trivial. For instance, the supertwistor space $P^{3\mid 4}$ fits into this construction scheme, since $\mathcal{K} \cong \mathcal{O}(-4)$ and Jet$^1 \mathcal{K}^{-1/2} \cong \mathcal{O}(1) \otimes \mathbb{C}^4$. The latter statement follows from the Euler sequence (see also appendix A):

$$0 \to \mathbb{C} \to \mathcal{O}(1) \otimes \mathbb{C}^4 \to TP^3 \to 0.$$

Upon choosing this reality condition, we may invert the incidence relation. In exercise 5.3, you will derive the following result:

$$x^{\alpha a} = \frac{x^{\alpha a} - x^{\alpha a}}{\lambda_\alpha x_\alpha}. \quad (5.33)$$
This shows that as real manifolds, we have the following diffeomorphisms:

$$\mathbb{P}^{314} \cong \mathbb{R}^{4|8} \times \mathbb{C} \mathbb{P}^1 \cong \mathbb{R}^{4|8} \times S^2,$$

(5.34)

and hence we have a non-holomorphic fibration

$$\pi : \mathbb{P}^{314} \to \mathbb{R}^{4|8},$$

$$\left(\tilde{\varepsilon}, \tilde{\lambda}_a\right) \mapsto x^{\tilde{A} a} = e^{\tilde{A} a}_\lambda \tilde{\lambda}_a / \lambda_\tilde{A} \tilde{\lambda}^{\tilde{A}}.$$  

(5.35)

The same holds true for $\mathbb{P}^{314} \cong \mathbb{R}^{4|2} \times \mathbb{C} \mathbb{P}^1 \cong \mathbb{R}^{4|2} \times S^2$. Therefore, in the real setting we may alternatively work with this single fibration instead of the double fibration (4.15) and we shall do so in the following. This will be most convenient for our purposes.

**Exercise 5.3.** Verify (5.33).

In order to write down the $\bar{\partial}$-operator on $\mathbb{P}^{314}$, we need a basis for differential $(0, 1)$-forms and $(0, 1)$-vector fields in the non-holomorphic coordinates $(x^{\tilde{A} a}, \tilde{\lambda}_a)$. Using the conventions

$$\frac{\partial}{\partial \lambda_\beta} \lambda_\beta = \delta^a_\beta, \quad \frac{\partial}{\partial x^{\tilde{A} a}} \lambda_\beta = \delta^\alpha_\beta, \quad \frac{\partial}{\partial x^{\tilde{A} a}} \bar{\lambda}_a = \delta^\beta_\alpha,$$

and

$$\frac{\partial}{\partial \tilde{x}^{\tilde{A} a}} \lambda_\beta = \delta^a_\beta, \quad \frac{\partial}{\partial \tilde{x}^{\tilde{A} a}} \bar{\lambda}_a = \delta^\beta_\alpha,$$

we have

$$\tilde{e}^0 = \frac{\delta^{(a}}{[\lambda \tilde{\lambda}]} \text{ and } \tilde{e}^h = - \frac{\delta_{a)}}{[\lambda \tilde{\lambda}]} \lambda_\tilde{A}_a \tilde{\lambda},$$

(5.37a)

where $[\rho \lambda] := \varepsilon^{\alpha\beta} \rho_\beta \lambda_\alpha = \rho_\alpha \lambda_\beta$, e.g. $[\delta^{a\lambda}] = \delta^A_a \lambda$, together with

$$\tilde{V}_0 = [\lambda \tilde{\lambda}] \lambda_\tilde{A}_a \text{ and } \tilde{V}_A = \lambda^\alpha \lambda_\tilde{A}_a.$$

(5.37b)

In this sense, $\tilde{V}_0 \tilde{e}^0 = 1$, $\tilde{V}_A \tilde{e}^A = \delta^A_A$, $\tilde{V}_0 \tilde{e}^A = 0$ and $\tilde{V}_A \tilde{e}^0 = 0$ and the $\bar{\partial}$-operator is $\bar{\partial} = \tilde{e}^0 \tilde{V}_0 + \tilde{e}^A \tilde{V}_A$. Note that the vector fields (4.14) are $(0, 1)$-vector fields in this real setting. The holomorphic volume form (5.27) takes the form

$$\Omega = \Omega_0 \otimes \Omega_1$$

$$= \frac{1}{4!} \frac{\delta^{(a}}{[\lambda \tilde{\lambda}]} e^0 \wedge e^a \wedge e_\alpha \otimes \Omega_1 = \frac{1}{2!} \delta^{(a}}{[\lambda \tilde{\lambda}]} \wedge e_{a\alpha} \lambda_\beta \lambda_\tilde{A}_a \wedge \delta^\beta_\alpha \wedge \Omega_1,$$

(5.38)

where $\Omega_1$ is given in (5.27). Here, we do not re-write the $d\eta_i$ (or rather the corresponding integral forms) in terms of the $e_i$. The reason for this becomes transparent in our subsequent discussion. Note also that the $e^0$, $e^A$ are obtained from (5.37a) via complex conjugation (or equivalently, via the involution $\tau$).

Working in the Witten gauge (5.29), we have

$$A^{0,1} = \tilde{e}^0 A_0 + \tilde{e}^a A_a,$$

(5.39)

where $A_0$ and $A_a$ have a holomorphic dependence on $\eta_i$. In particular, we may expand $A^{0,1}$ according to

$$A^{0,1} = A^0 + \gamma_i X^i + \frac{1}{2} \gamma_i \gamma_j X^{ij} + \frac{1}{8} \gamma_i \eta_{ij} \xi_{kl} \Omega_{ijkl} + \frac{1}{4} \gamma_i \eta_{ij} \eta_{kl} \xi_{ijkl} \tilde{\Omega}_6,$$

(5.40)

where the coefficients $A^0, X^i, \gamma^i, X$, and $\tilde{\Omega}$ are differential $(0, 1)$-forms of homogeneity $0$, $-1, -2, -3$ and $-4$ that depend only on $x, \lambda$ and $\tilde{\lambda}$. 

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At this stage, it is useful to digress a bit by discussing the linearized equations of motion. At the linearized level, (5.20) reads as $\delta A^{0,1} = 0$ and the gauge transformations (5.21) reduce to $A^{0,1} \mapsto \delta e$. Therefore, the coefficient fields of (5.40) represent Lie algebra-valued elements of the Dolbeault cohomology group $H^{0,1}_p(P^3,\mathcal{O}(-2h-2))$ for $h = -1, \ldots, 1$.

By the Dolbeault theorem, we have $H^{0,1}_p(P^3,\mathcal{O}(-2h-2)) \cong H^1(P^3,\mathcal{O}(-2h-2))$; see remark 5.6 for more details. Therefore, we can apply theorem 2.2 to conclude that on spacetime, these fields correspond to $f_{\alpha \beta}, \chi^i, \phi^i, \chi_{ia}$ and $G_{\alpha \beta}$ which is precisely the field content displayed in table 3. Hence, one superfield $A^{0,1}$ on supertwistor space $P^{3|4}$ encodes the whole particle spectrum of $\mathcal{N} = 4$ self-dual supersymmetric Yang–Mills theory.

As a next step, we need to remove the extra gauge symmetry beyond the spacetime gauge transformations (after all we want to derive the Siegel action). To achieve this, we have to further gauge-fix $A^{0,1}$ (on top of the already imposed Witten gauge). This procedure is in spirit of our discussion in section 5.1, where we have shown how to go from the constraint equations of supersymmetric self-dual Yang–Mills theory to its actual field equations. To this end, we impose a gauge called the spacetime gauge \[65, 67, 68\] on fibrewise co-closed with respect to the Fubini–Study metric on each $L_\alpha$. Residual gauge transformations that preserve (5.41) are then given by those $g \in H^0(P^{3|4},GL(r,s))$ that obey

$$\delta_L g = 0,$$

as one may straightforwardly check by inspecting (5.21). If we let $(\cdot, \cdot)$ be a metric on the space of matrix-valued differential forms on each $L_\alpha$, then $0 = (g, \delta_L g) = (\delta_L g, \delta_L g) \geq 0$. Hence, $\delta_L g = 0$ and so $g$ is holomorphic on each fibre $L_\alpha$ and thus it must be constant (since the $L_\alpha$ are compact). Altogether, $g = g(\lambda^{a \dot{b}})$ and the residual gauge freedom is precisely spacetime gauge transformations.

**Remark 5.5.** Consider $\mathbb{C}P^1$ with the canonical cover $\{U_\pm\}$ as in remark 1.2. In homogeneous coordinates $\lambda$, the Fubini–Study metric is given by

$$ds^2 = \frac{[d\lambda][d\bar{\lambda}]}{|\lambda \bar{\lambda}|^2},$$

while in affine coordinates $\lambda_\pm$ we have

$$ds^2|_{U_\pm} = \frac{d\lambda_\pm d\bar{\lambda}_\pm}{(1 + \lambda_\pm \bar{\lambda}_\pm)^2}, \quad \text{with} \quad ds^2|_{U_+} = ds^2|_{U_-}.$$

Here, $\hat{\lambda}_\alpha$ was defined in (5.30). Note that the Fubini–Study metric is the Kähler metric on $\mathbb{C}P^1$ with the Kähler form

$$K = K(\lambda) = \frac{[d\lambda][d\bar{\lambda}]}{|\lambda \bar{\lambda}|^2}.$$

Since $A^{1,1}$ is holomorphic in $\eta$, the gauge-fixing condition $\delta_\lambda (\delta^0 A_0) = 0$ has to hold for each component,

$$\delta^0 A_0 = \delta^0 \bar{A}_0 + \eta \delta^0 \bar{A}_0 i + \frac{1}{2!} \eta_i \eta_j \delta^0 \phi_{ij} + \frac{1}{3!} \eta_i \eta_j \eta_k \epsilon^{ijkl} \delta^0 \chi_{i,j,k} + \frac{1}{4!} \eta_i \eta_j \eta_k \eta_l \epsilon^{ijkl} \delta^0 G_{ij},$$

(5.43)
\[ \delta L \left( \bar{A}_0 \right) = \bar{\delta} L \left( \bar{e}^0 \bar{\phi}_0 \right) = \bar{\delta} L \left( \bar{e}^0 \bar{\phi}^{ij}_0 \right) = \bar{\delta} L \left( \bar{e}^0 \bar{\Lambda}_0 \right) = \bar{\delta} L \left( \bar{e}^0 \bar{G}_0 \right) = 0. \] (5.44)

**Remark 5.6.** Let \( X \) be a complex manifold and \( E \to X \) be a holomorphic vector bundle. Furthermore, we denote the \( \bar{\partial} \)-harmonic \((p, q)\)-forms on \( X \) with values in \( E \) by \( \text{Harm}^{\bar{\partial}, q}(X, E) \), where \( \bar{\partial} \) is the anti-holomorphic exterior derivative. These are forms \( \omega \in \text{Harm}^{\bar{\partial}, q}(X, E) \) that obey \( \bar{\partial} \omega = 0 = \bar{\partial}^\dagger \omega \). In addition, let \( H_{\bar{\partial}, q}^0(X, E) \) be the \( \bar{\partial} \)-cohomology groups with coefficients in \( E \), i.e., elements of \( H_{\bar{\partial}, q}^0(X, E) \) are \( \bar{\partial} \)-closed \( E \)-valued \((p, q)\)-forms which are not \( \bar{\partial} \)-exact. If \( X \) is compact, the Hodge theorem establishes the following isomorphism:

\[ \text{Harm}^{\bar{\partial}, q}(X, E) \cong H_{\bar{\partial}}^q(X, E). \]

The Dolbeault theorem establishes another isomorphism for \( X \) not necessarily compact

\[ H_{\bar{\partial}}^q(X, E) \cong H^q(X, \Lambda^\partial T^*X \otimes E), \]

where \( H^q(X, \Lambda^\partial T^*X \otimes E) \) is the \( q \)th \( \check{\text{C}} \)ech cohomology group with coefficients in \( \Lambda^\partial T^*X \otimes E \). For details, see e.g. [9, 69].

Note that \( \bar{e}^0 \bar{A}_0 \), \( \bar{e}^0 \bar{\phi}^{ij}_0 \), \( \bar{e}^0 \bar{\Lambda}_0 \), and \( \bar{e}^0 \bar{G}_0 \) take values in \( \mathcal{O}(m) \otimes \text{End} E \) with \( m \) being 0, 1, \ldots, 4. Furthermore, \( \bar{e}^0 \bar{A}_0 \) is also \( \bar{\partial} \)-closed since \( \text{dim}_C L_x = 1 \) and therefore it is harmonic along the fibres. Likewise, all the component fields of \( \bar{e}^0 \bar{A}_0 \) are fibrewise harmonic. Therefore, the Hodge and Dolbeault theorems (see remark 5.6) then tell us that the restriction of these components to the fibres are \( \text{End} E \)-valued elements of \( H^1(\mathbb{C}P^1, \mathcal{O}(-2h - 2)) \) since \( L_x \cong \mathbb{C}P^1 \). From table 2, we conclude that \( H^1(\mathbb{C}P^1, \mathcal{O}(-2h - 2)) = 0 \) for \( h = -1, -\frac{1}{2} \) and hence \( \bar{A}_0 = \bar{\Lambda}_0 = \bar{G}_0 = 0 \). Furthermore, \( H^1(\mathbb{C}P^1, \mathcal{O}(-2h - 2)) \cong \mathbb{C}^{2h+1} \) for \( h \geq 0 \) and therefore, the other components may be taken as (see also [70])

\[ \bar{A} = \bar{e}^0 \bar{A}_0 + \bar{e}^0 \bar{A}_0 = \bar{e}^0 \bar{A}_0, \]

\[ \bar{\Lambda}^{ij} = \bar{e}^0 \bar{\phi}^{ij}_0 + \bar{e}^0 \bar{\Lambda}_0 = \bar{e}^0 \bar{\phi}^{ij}_0, \]

\[ \bar{\phi}^{ij} = \bar{e}^0 \bar{\phi}^{ij}_0 + \bar{e}^0 \bar{\Lambda}_0 = \bar{e}^0 \bar{\phi}^{ij}_0 + \bar{e}^0 \bar{\Lambda}_0 = \bar{e}^0 \bar{\phi}^{ij}_0 + \bar{e}^0 \bar{\Lambda}_0, \]

\[ \bar{G} = \bar{e}^0 \bar{G}_0 + \bar{e}^0 \bar{G}_0 = 2 \bar{e}^0 \bar{\Lambda}_0 \bar{\Lambda}_0 + \bar{e}^0 \bar{\Lambda}_0 \bar{\Lambda}_0, \]

(5.45)

Here, the fields \( \bar{\phi}^{ij}, \bar{\Lambda}_0 \) and \( \bar{G}_{0 \beta} \) depend only on the spacetime coordinates \( \lambda^{\alpha \beta} \) while the remaining fields may still depend on \( \lambda_0 \) and \( \bar{\Lambda}_0 \). Note that we use \( \bar{\phi}^{ij} = \frac{1}{2} \varepsilon^{ijkl} \bar{\phi}_{kl} \), and similarly for its components.

The non-trivial step in writing (5.45) is in setting \( \bar{A}_0 = 0 \) (and \( \bar{\phi}^{ij}_0 = 0 \) by supersymmetry). This implies that \( E \) is holomorphically trivial on any \( L_x \leftrightarrow \mathbb{P}^{3h+1} \), which is one of the assumptions in Ward’s theorem; see theorems 3.3 and 5.1. Note that this is not guaranteed in general, but will follow from the smallness assumption of \( \bar{A}_0 \) in a perturbative context.
In terms of the expansions (5.45), the action (5.24) reads

\[ S = -\frac{1}{2\pi i} \int \frac{\Omega_0 \wedge \Omega_0}{[\lambda \lambda]^2} \text{tr} \left\{ G_{ab} \delta^{\lambda \beta} \delta^{\gamma \delta} \left( -\lambda^\gamma \partial_{\alpha \gamma} \hat{A}^\alpha + \frac{1}{2} [\hat{A}^\alpha, \hat{A}_\alpha] \right) \\
+ \hat{\chi}_{i a} \delta^{\lambda \beta} \lambda^\gamma \lambda^\delta \delta_{\beta \delta} \left( \lambda^\beta \partial_{\alpha \beta} \hat{\chi}^\alpha_{i a} + [\hat{\chi}^\alpha_{i a}, \hat{\chi}^i_{i a}] \right) - \frac{\partial}{\partial \phi_{ij}(\lambda^\beta \partial_{\alpha \beta} \hat{\phi}^{ij} + [\hat{\chi}^\alpha_{i a}, \hat{\chi}^{i a}])} \\
+ \frac{1}{4} \delta_{ij} \lambda^\gamma \lambda^\delta \lambda^\lambda \lambda^\rho \lambda^\sigma \lambda^\tau \hat{\phi}^{ij} \hat{\phi}^{ij} \right\}. \] (5.46)

Note that in fact one should also add a gauge-fixing term and a ghost term. However, the ghost–matter interaction is such that the Fadeev–Popov determinant is independent of $A^{\beta 1}$. See [67] for more details. For the time being, we therefore omit these pieces.

The fields $\hat{\chi}_{i a}$ and $\hat{\chi}^a_i$ appear only linearly, so they are in fact Lagrange multiplier fields. Integrating them out enforces

\[ \hat{V}_0 \hat{A}_\alpha = 0 \text{ and } \hat{V}_0 \hat{\chi}^{\alpha i} = 0 \] (5.47)

and so $\hat{A}_\alpha$ and $\hat{\chi}^{\alpha i}$ must be holomorphic in $\lambda_\alpha$. Hence,

\[ \hat{A}_\alpha = \lambda^\beta \hat{A}_{a \beta} \text{ and } \hat{\chi}^{\alpha i} = \hat{\chi}^i_{\alpha} \] (5.48)

since they are of homogeneity 1 and 0, respectively. Here, $\hat{A}_{a \beta}$ and $\hat{\chi}^i_{\alpha}$ depend only on $\chi^{\alpha \beta}$. Likewise, $\hat{\phi}^{ij}$ can be eliminated as it appears only quadratically with a constant coefficient. We have

\[ \hat{V}_0 \hat{\phi}^{ij} = \lambda^\beta (\partial_{\alpha \beta} \hat{\phi}^{ij} + [\hat{A}_{a \beta}, \hat{\phi}^{ij}]) \] (5.49)

which yields

\[ \hat{\phi}^{ij} = \frac{1}{\lambda^\beta} \lambda^\rho \lambda^\sigma \lambda^\tau \hat{\phi}^{ij}, \text{ with } \hat{V}_0 \hat{\phi}^{ij} := \partial_{\alpha \beta} + \hat{A}_{a \beta}. \] (5.50)

Inserting (5.48) and (5.50) into (5.46), we find

\[ S = -\frac{1}{2\pi i} \int \frac{\Omega_0 \wedge \Omega_0}{[\lambda \lambda]^2} \text{tr} \left\{ \hat{G}_{ab} \hat{f}_{\gamma \delta} \delta^{\lambda \beta} \delta^{\gamma \delta} \left( \lambda^\beta \partial_{\alpha \beta} \hat{\chi}^\alpha_{i a} + [\hat{\chi}^\alpha_{i a}, \hat{\chi}^i_{i a}] \right) \\
+ \frac{1}{4} \delta_{ij} \lambda^\gamma \lambda^\delta \lambda^\lambda \lambda^\rho \lambda^\sigma \lambda^\tau \hat{\phi}^{ij} \hat{\phi}^{ij} \right\}. \] (5.51)

where $\hat{f}_{\gamma \delta}$ is the anti-self-dual part of the curvature of $\hat{A}_{a \beta}$ (see also (3.21)). Using

\[ \frac{\Omega_0 \wedge \Omega_0}{[\lambda \lambda]^2} = d^4x \frac{[d\lambda \lambda][d\lambda \lambda]}{[\lambda \lambda]^2} \] (5.52)

together with the formula derived in exercise 5.4, we can integrate out the fibres to finally arrive at

\[ S = \int d^4x \text{tr} \left\{ \hat{G}_{ab} \hat{f}_{\gamma \delta} \hat{\chi}_{i a} \nabla a \hat{\chi}^i_{\alpha} \hat{\chi}^a_{\alpha} - \frac{1}{2} \delta_{ij} \hat{\phi}^{ij} + \frac{1}{2} \phi_{ij} [\hat{\chi}^{i a} \hat{\chi}^{j a}] \right\}. \] (5.53)

\[ ^{19} \text{Note that if one integrates out generic quadratic pieces in the path integral, then one picks up field-dependent functional determinants. However, in our case the } \phi^{ij} \text{ piece has a constant coefficient and hence, the determinant is constant and thus can be absorbed in the normalization of the path integral measure.} \]
This is nothing but the Siegel action (5.17).

Altogether, the holomorphic Chern–Simons action (5.24) on supertwistor space \( P^{1|4} \) corresponds to the Siegel action (5.17) on spacetime. This concludes figure 1 and our discussion about \( \mathcal{N} = 4 \) supersymmetric self-dual Yang–Mills theory\(^{20}\).

**Exercise 5.4.** Verify (5.52). Furthermore, prove that

\[
-\frac{1}{2\pi i} \int K f_{a_1\cdots a_n} \hat{\lambda}^{a_1} \cdots \hat{\lambda}^{a_n} \frac{\lambda^{\bar{a}_1} \cdots \lambda^{\bar{a}_n}}{[\lambda \bar{\lambda}]^m} = \frac{1}{m+1} f_{a_1\cdots a_n} g^{a_1\cdots a_n},
\]

where \( K \) is the Kähler form given in remark 5.5 and \( g^{a_1\cdots a_n} \) and \( f_{a_1\cdots a_n} \) do not depend on \( \lambda \) and \( \hat{\lambda} \). In fact, this is the Dolbeault version of Serre duality discussed in remark 2.4 (see also remark 5.6):

\[
H^0(\mathbb{C}P^1, \mathcal{O}(m)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(-m-2))^*.
\]

More concretely, \( g = g^{a_1\cdots a_n} \lambda_{a_1} \hat{\lambda}_{a_n} \) represents an element of \( H^0(\mathbb{C}P^1, \mathcal{O}(m)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(m)) \) while

\[
f = (m+1) f_{a_1\cdots a_n} \hat{\lambda}^{a_1} \cdots \hat{\lambda}^{a_n} [d\lambda \bar{\lambda}]^{m-2} \]

is an element of \( H^0(\mathbb{C}P^1, \mathcal{O}(-m-2)) \cong H^0(\mathbb{C}P^1, \mathcal{O}(-m-2)) \). Here, we have used (5.37a). Then the Dolbeault version of the pairing (2.15) is given by

\[
(f, g) = -\frac{1}{2\pi i} \int [d\lambda \bar{\lambda}] \wedge f g = f_{a_1\cdots a_n} g^{a_1\cdots a_n},
\]

which is nothing but the above formula.

6. \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory from supertwistor space

6.1. Motivation

So far, we have shown how \( \mathcal{N} = 4 \) supersymmetric self-dual Yang–Mills theory can be understood in terms of twistor geometry. The field content of this theory is displayed in table 3 and the corresponding action in (5.17). In fact, the full \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory has precisely the same field content but it differs in the interaction terms. The action is given by

\[
S = \int d^4x \text{tr} \left\{ f^{\alpha \beta} \phi_{\alpha \beta} + f^{a \beta} f_{a \beta} + \phi_{ij} \phi^{ij} - 2 \chi^{ia} \nabla_{a i} \phi^{i} - \phi_{ij} \phi^{ij} \right\}
\]

By comparing the Siegel action (5.53) with action (6.1), we realize that the Siegel action provides only parts of the interaction terms. What about the other terms? Can they also be understood in terms of twistors?

Before delving into this, let us first consider purely bosonic self-dual Yang–Mills theory. This theory may be described by the action

\[
S = \int \text{tr}(G \wedge F^-),
\]

\(^{20}\) For twistor constructions including action functionals of other gauge field theories, see [57, 67, 68, 71–77].
since the corresponding field equations are
\[ F^- = 0 \quad \text{and} \quad \nabla \ast G = 0. \tag{6.3} \]
Hence, we obtain the self-dual Yang–Mills equations plus an equation for an anti-self-dual field \(G\) propagating in the self-dual background. Note that \(G\) is nothing but the field \(G_{\alpha\dot{\beta}}\) we already encountered. Let us modify the above action by adding the term \[ S_\varepsilon = -\frac{1}{2\varepsilon} \int \text{tr}(G \wedge G), \quad \text{with} \quad S_{\text{tot}} = S + S_\varepsilon, \tag{6.4} \]
where \(\varepsilon\) is some small parameter. Upon integrating out the field \(G\), we find
\[ S_{\text{tot}} = \frac{1}{2\varepsilon} \int \text{tr}(F^- \wedge F^-) = -\frac{1}{4\varepsilon} \int \text{tr}(F \wedge \ast F) + \frac{1}{4\varepsilon} \int \text{tr}(F \wedge F), \tag{6.5} \]
where we have used \(F^- = \frac{1}{2}(F \ast \ast F)\). Hence, we obtain the Yang–Mills action (3.2) plus a term proportional to the topological charge (3.7), provided we identify \(\varepsilon\) with the Yang–Mills coupling constant \(g_{\text{YM}}^2\). Therefore, small \(\varepsilon\) corresponds to small \(g_{\text{YM}}^2\). As we are about to study perturbation theory, the topological term will not play a role and we may therefore work with (6.4). Consequently, we may re-phrase our question: Can we derive \(\int \text{tr}(G \wedge G)\) or rather its \(N = 4\) supersymmetric extension from twistor space? This will be answered in the next section.

Exercise 6.1. Show that action (6.1) is invariant under the following supersymmetry transformations (\(\varrho_i^\alpha\) and \(\varrho^{\dot{\alpha}}_i\) are constant anti-commuting spinors):
\[
\begin{align*}
\delta A^\alpha_{\dot{\alpha}} &= -\varepsilon_{\alpha\dot{\beta}} \varrho^{\dot{\beta}} \tilde{X}_{\dot{\alpha}} + \varepsilon_{\dot{\alpha}\dot{\beta}} \varrho^\alpha \tilde{X}^\beta, \\
\delta \phi_{ij} &= \varepsilon_{ijk} \varrho^k \tilde{X}^\alpha, \\
\delta \tilde{X}^\alpha &= -2\varrho^{\dot{\alpha}} \tilde{f}_{\dot{\alpha}\beta} + \varepsilon_{\dot{\alpha}\dot{\beta}} \varrho^{\dot{\alpha}} \tilde{f}^\alpha_{\dot{\beta}} \phi_{jk} - 2\varrho^\alpha \varrho^\beta \tilde{f}^\alpha_{\dot{\beta}} \phi_{ij}, \\
\delta \tilde{f}^{\alpha}_{\dot{\beta}} &= 2\varrho^\alpha \varrho^\beta \tilde{f}^{\alpha}_{\dot{\beta}} + 2\varrho^{\dot{\alpha}} \varrho^{\dot{\beta}} \tilde{f}^\alpha_{\dot{\beta}} + \varepsilon_{\alpha\dot{\beta}} \varrho^\alpha \varrho^\beta \tilde{f}^\alpha_{\dot{\beta}} \phi_{ik}.
\end{align*}
\]

6.2. \(N = 4\) supersymmetric Yang–Mills theory from supertwistor space
In parts, the following discussion follows the lines of [66, 67]. To obtain the full \(N = 4\) supersymmetric Yang–Mills action, one considers the following modification:
\[ S_{\text{tot}} = S + S_\varepsilon, \tag{6.6a} \]
where \(S\) is the holomorphic Chern–Simons action (5.24) and \(S_\varepsilon\) is given by [67]
\[ S_\varepsilon = -\varepsilon \int d^{4\overline{8}}x \log \det \nabla^{0,1}_{L_x}. \tag{6.6b} \]
One takes the \((0,1)\)-connection \(\nabla^{0,1}\), restricts it to the fibre \(L_x \hookrightarrow P^{3|4}\), constructs the determinant of this operator and finally integrates the logarithm of this determinant over \(x \in M^{4|8}\) parametrizing \(L_x \cong \mathbb{C}P^1\). Here, we have used the common abbreviation \(d^{4\overline{8}}x := d^4x d^8\eta\). Note that action (6.6b) is invariant under the gauge transformations (5.21). This will be verified in exercise 6.2. Before showing that \(S_\varepsilon\) really gives the missing interactions, let us emphasize that all the constructions presented in the preceding section also apply to the full twistor space action (6.6a) without alteration. We may therefore stick to \(S_\varepsilon\) for the remainder of this section.
You might find it useful to know that the determinant \( \det \nabla^{0,1} \) behaves under infinitesimal gauge transformations \( \delta A^{0,1} = \nabla^{0,1} \epsilon \), with \( \epsilon \) being an infinitesimal gauge parameter, as

\[
\delta \det \nabla^{0,1} \big|_{L_\alpha} = \left( \frac{1}{2\pi} \int_{L_\alpha} \text{tr}(A^{0,1} \wedge \partial \epsilon) \right) \det \nabla^{0,1} \big|_{L_\alpha}.
\]

This formula is, in fact, the so-called chiral anomaly with \( \nabla^{0,1} \) on \( L_\alpha \) being the chiral Dirac operator. From a physical point of view, it can best be understood in a path integral language. If we consider two fermionic fields \( \alpha \) and \( \beta \) of homogeneity \(-1\) on \( L_\alpha \) taking values in \( E \big|_{L_\alpha} \) and \( E^* \big|_{L_\alpha} \), respectively, with the action

\[
S_{\beta \alpha} = \int_{L_\alpha} [d\lambda \lambda'] \wedge \beta \nabla^{0,1} \alpha,
\]

then

\[
\det \nabla^{0,1} \big|_{L_\alpha} = e^{-W[A^{0,1}]} = \int D\alpha D\beta \exp \left( -\int_{L_\alpha} [d\lambda \lambda'] \wedge \beta \nabla^{0,1} \alpha \right),
\]

where \( W[A^{0,1}] \) is the effective action. The above formula of the gauge variation of the determinant arises by studying variations \( A^{0,1} \mapsto A^{0,1} + \delta A^{0,1} = A^{0,1} + \nabla^{0,1} \epsilon \) of the path integral. See e.g. [69, 78] for more details.

In what follows, we shall need Green’s function of the \( \bar{\partial} \)-operator on \( \mathbb{CP}^1 \) and so it will be useful to recapitulate some of its properties first. To this end, note that any differential \((0,1)\)-form on \( \mathbb{CP}^1 \) is automatically \( \bar{\partial} \)-closed since \( \text{dim}_\mathbb{C} \mathbb{CP}^1 = 1 \). Furthermore, the Dolbeault cohomology groups \( H^{0,1}_\bar{\partial}(\mathbb{CP}^1, O(m)) \) vanish for \( m \geq -1 \) since \( H^{0,1}_\bar{\partial}(\mathbb{CP}^1, O(m)) \cong H^1(\mathbb{CP}^1, O(m)) \) and the Čech cohomology groups \( H^1(\mathbb{CP}^1, O(m)) \) are given in table 2. Therefore, any differential \((0,1)\)-form \( \omega \) of homogeneity \( m \geq -1 \) is necessarily \( \bar{\partial} \)-exact, i.e. \( \omega = \bar{\partial} \rho \) for some \( \rho \) of homogeneity \( m \). We shall denote the element \( \rho \) by \( \bar{\partial}^{-1} \omega \), in the following. Again from table 2, we conclude that \( \bar{\partial}^{-1} \omega \) is not unique for \( m \geq 0 \) as one can always add a global holomorphic function \( \omega \) of homogeneity \( m \): \( \bar{\partial}(\bar{\partial}^{-1} \omega + \alpha) = \omega \). For \( m = -1 \), however, there are no global holomorphic functions and so \( \bar{\partial}^{-1} \omega \) is unique. Explicitly, we have

\[
\bar{\partial}^{-1} \omega(\lambda) = -\frac{1}{2\pi i} \int \frac{[d\lambda \lambda']}{[\lambda \lambda']} \wedge \omega(\lambda').
\]

(6.7)

In deriving this result, one makes use of the basis (5.2)

\[
\omega(\lambda) = \bar{\partial}^{\lambda} \alpha_0(\lambda, \hat{\lambda}_\alpha) = \frac{[d\lambda \hat{\lambda}]}{[\lambda \lambda']} \omega_0(\lambda, \hat{\lambda}_\alpha)
\]

together with

\[
\bar{\partial}(\lambda) \frac{1}{[\lambda \lambda']} = 2\pi [d\lambda \hat{\lambda}] \frac{[\xi \hat{\lambda}]}{[\xi \lambda]} \delta^{(2)}([\lambda \lambda'], [\hat{\lambda} \hat{\lambda}']).
\]

(6.9a)

Here, \( \xi^a \) is some constant spinor. The complex delta function is given by

\[
\delta^{(2)}([\lambda \lambda'], [\hat{\lambda} \hat{\lambda}']) := \frac{1}{2} \delta(\text{Re}[\lambda \lambda']) \delta(\text{Im}[\lambda \lambda'])
\]

(6.9b)

and obeys \( \delta^{(2)}(t[\lambda \lambda'], i[\hat{\lambda} \hat{\lambda}']) = (it)^{-1} \delta^{(2)}([\lambda \lambda'], [\hat{\lambda} \hat{\lambda}']) \) for \( t \in \mathbb{C} \setminus \{0\} \). On support of the delta function, \( \lambda_\alpha \propto \lambda_\alpha' \) and therefore the expression (6.9a) is independent of the choice of \( \xi^a \). Note that \( \omega_0(\lambda_\alpha, \hat{\lambda}_\alpha) = t \omega_0(\lambda_\alpha, \hat{\lambda}_\alpha) \) for \( t \in \mathbb{C} \setminus \{0\} \) since \( \omega \) is of homogeneity \(-1\). Note also that one often writes

\[
G(\lambda, \lambda') := -\frac{1}{2\pi i [\lambda \lambda']} \Rightarrow \bar{\partial}^{-1} \omega(\lambda) = \int K(\lambda') G(\lambda, \lambda') \omega_0(\lambda'),
\]

(6.10)
where $G(\lambda, \lambda')$ is referred to as Green’s function of the $\bar{\partial}$-operator or equivalently as the integral kernel of $\bar{\partial}^{-1}$-operator. Furthermore, $K(\lambda)$ is the Kähler form introduced in remark 5.5 and $\omega_0(\lambda)$ is the short-hand notation for $\omega_0(\lambda_0, \hat{x}_0)$.

**Exercise 6.3.** Verify (6.9a). You may find it useful to consider ($\varepsilon \in \mathbb{R}$)

$$
\bar{\partial}(\lambda) \left[ \frac{\bar{\partial}^2}{\lambda\lambda'} \right] + \varepsilon^2
$$

in the limit $\varepsilon \to 0$.

After this digression, let us come back to the action $S_\varepsilon$. To compute its spacetime form, we recall the identity

$$
\log \det \nabla^0|_L = \text{tr} \log \nabla^0|_L,
$$

and consider the following power series expansion:

$$
\text{tr} \log \nabla^0|_L = \text{tr} \log(\bar{\partial} + A^{0,1})|_L
\quad = \text{tr} \left[ \log(\bar{\partial})|_L + \log(1 + \bar{\partial}^{-1}A^{0,1})|_L \right]
\quad = \text{tr} \left[ \log \bar{\partial}|_L + \sum_{r=1}^{\infty} (-1)^r \int L_x \prod_{s=1}^{r} K(\lambda_s)G(\lambda_{s-1}, \lambda_s)A_0(\lambda_s) \right].
$$

with the identification $\lambda_0 \equiv \lambda_\varepsilon$. Here, we have inserted Green’s function (6.10). In addition, we used (5.39), i.e. each $A_0$ is the $\bar{\partial}^0$-coefficient of $A^{0,1} = \bar{\partial}^0A_0 + \bar{\partial}^\alpha A_\alpha$ and restricted to lie on a copy of the fibre over the same point $x \in M^{\text{fib}}$, i.e. $\lambda_x \in L_x \hookrightarrow P^{3|\delta}$ for all $s$. Note that $A_0$ is of homogeneity 2 as follows from the scaling behaviour of $\bar{\partial}^0$.

Adopting the gauge (5.41), we see that the series (6.12) terminates after the fourth term, since in this gauge $A_0$ starts only at second order in the $\eta_\alpha$-expansion (compare (5.40) with (5.45)). In addition, we eventually integrate over $d^4x$, so we only need to keep terms that contain all $\eta_\alpha^2$. In this respect, recall that on $L_x$ we have $\eta_\alpha = \eta_\alpha^0\lambda_\alpha$. Then there are only three types of relevant terms. First, we have the $G_{\alpha\beta}\bar{G}^{\alpha\beta}$-term which is

$$
-\varepsilon \int d^4x \frac{1}{2} \left( \frac{3}{2\pi i} \right)^2 \int_{L_x} \prod_{s=1}^{2} \frac{K(\lambda_s)}{[\lambda_{s-1}\lambda_s]^2} \times \text{tr} \left\{ \frac{1}{4!} \epsilon^{ijkl} \eta_1 \eta_2 \eta_3 \eta_4 \bar{G}_{\alpha\beta} \bar{G}_{\alpha\beta} \right\} \frac{1}{[\lambda_1\lambda_2]^2},
$$

The integration over the fermionic coordinates can be performed using Nair’s lemma

$$
\int d^4\eta (\eta_1)^4(\eta_2)^4 = [\lambda_1, \lambda_2]^4,
$$

with $(\eta_\alpha)^4 := \frac{1}{4!} \epsilon^{ijkl} \eta_{i\alpha} \eta_{j\alpha} \eta_{k\alpha} \eta_{l\alpha}$, while the integration over $L_x$ is performed with the help of the formula derived in exercise 5.4. We find

$$
-\varepsilon \int d^4x \text{tr} \left\{ \bar{G}_{\alpha\beta} \bar{G}^{\alpha\beta} \right\}.
$$

Next let us look at terms of the form $\bar{\phi}^{ij} \bar{\chi}_{i\alpha} \bar{\chi}_{j\beta}$ and terms that are quartic in the scalars $\phi_{ij}$. They come from the expressions

43
\[ -\varepsilon \int \frac{d^4 x}{(2\pi i)^4} \frac{1}{L_{\text{x}}!} \prod_{s=1}^{4} \frac{K(\lambda_s)}{[\lambda_s^{-1} \lambda_s]} \times \text{tr} \left\{ \frac{1}{21!} \epsilon_{123} \eta_1 \phi^{ij} \frac{1}{21!} \epsilon^{ijkl} \eta_2 \eta_3 \eta_4 \chi^{L_i} \chi^{L_j} \right\} (6.16) \]

and

\[ -\varepsilon \int \frac{d^4 x}{(2\pi i)^4} \frac{1}{L_{\text{x}}!} \prod_{s=1}^{4} \frac{K(\lambda_s)}{[\lambda_s^{-1} \lambda_s]} \times \text{tr} \left\{ \frac{1}{21!} \epsilon_{123} \eta_1 \phi^{ij} \frac{1}{21!} \epsilon^{ijkl} \eta_2 \eta_3 \eta_4 \phi^{ij} \right\} (6.17) \]

respectively. In similarity with the $G_a^b G_a^b$-term, we may now integrate out the fermionic coordinates and the fibres. We leave this as an exercise and only state the final result. Collecting all the terms, we find that action (6.6b) reduces to

\[ S_c = -\varepsilon \int d^4 x \text{tr} \left\{ \frac{1}{2} G_a^b G_a^b + \phi^{ij} [\chi_{ij}, \chi^a] - \frac{1}{8} [\phi^{ij}, \phi^{kl}][\phi^{ij}, \phi^{kl}] \right\}. (6.18) \]

Combining this with the Siegel action (5.53) and integrating out $G_a^b$, we arrive at the full $\mathcal{N} = 4$ supersymmetric Yang–Mills action (6.1), modulo the topological term (see also exercise 3.7). Therefore, the twistor action (6.6a) is indeed perturbatively equivalent to the $\mathcal{N} = 4$ supersymmetric Yang–Mills action on spacetime.

**Exercise 6.4.** Complete the calculation by integrating in (6.16), (6.17) over the fermionic coordinates and the fibres to arrive at (6.18). Verify Nair’s lemma (6.14).

This concludes our discussion of the twistor re-formulation of gauge theory. The next part of these lecture notes is devoted to the construction of tree-level gauge theory scattering amplitudes by means of twistors.

**Remark 6.1.** Apart from the approach discussed here, there exist alternative twistor constructions of (full) supersymmetric Yang–Mills theory. See [18, 79–83]. In [76], a twistor action for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory was proposed using the so-called ambidextrous approach that uses both twistors and dual twistors. For an ambidextrous approach to maximal supersymmetric Yang–Mills theory in three dimensions, see [84].

**Remark 6.2.** Since most of the things said above in the supersymmetric setting were sparked off by Witten’s twistor string theory [53], let us make a few comments on that. His work is based on three facts: (I) holomorphic vector bundles over $P^{1|4}$ are related to $\mathcal{N} = 4$ supersymmetric self-dual Yang–Mills theory on $M^3$, (II) the supertwistor space $P^{1|4}$ is a Calabi–Yau supermanifold and (III) the existence of a string theory, the open topological B model, whose effective action is the holomorphic Chern–Simons action [50]. Roughly speaking, topological string theories are simplified versions of string theories obtained by giving different spins to the worldsheet description of ordinary string theories. Topological string theories come in two main versions, the A and B models, which are related by a duality called mirror symmetry. See e.g. [85] for more details. Unlike ordinary string theories, topological string theories are exactly solvable in the sense that the computation of correlation functions of physical observables is reduced to classical questions in geometry.

Witten then interprets the perturbative expansion of the full $\mathcal{N} = 4$ supersymmetric Yang–Mills theory as an instanton expansion of the open topological B model on $P^{1|4}$ (see also...
Table 4. The number of Feynman diagrams relevant for tree-level $n$-gluon scattering.

| $n$ | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| number of diagrams | 4   | 25  | 220 | 2485| 34  | 300 | 559 | 405 |

[86, 87] for alternative formulations). At the end of the day, this corresponds to complementing the holomorphic Chern–Simons action by an additional term, similar to what we have done above. Witten’s string approach leads to $\int \det \nabla^0$. Witten and Berkovits [88] soon realized, however, that $\mathcal{N} = 4$ supersymmetric Yang–Mills theory can be described this way only at tree level in perturbation theory, since at loop level conformal supergravity is inextricably mixed in with the gauge theory. The reason for that is the non-gauge invariance of the string theory formula $\int \det \nabla^0$. Gauge invariance of $\int \det \nabla^0$ may be restored by compensating gauge transformations of fields from the closed string sector [88]. These observations led Boels, Mason and Skinner [67] to suggest the twistor action (6.6a) which is gauge invariant and thus free of conformal supergravity (roughly speaking, the logarithm cancels out the multi-trace contributions responsible for conformal supergravity). However, the derivation of (6.6b) from string theory remains unclear.

Part II: Tree-level gauge theory scattering amplitudes

7. Scattering amplitudes in Yang–Mills theories

7.1. Motivation and preliminaries

In order to compute scattering amplitudes in a quantum field theory, one usually takes the (local) Lagrangian, derives the corresponding set of Feynman rules and constructs the amplitudes order by order in perturbation theory. However, gauge theories—with or without matter—present many technical challenges as the calculational complexity grows rapidly with the number of external states (i.e. the number of particles one is scattering) and the number of loops. For instance, even at tree level where there are no loops to consider, the number of Feynman diagrams describing $n$-particle scattering of gluons in pure Yang–Mills theory grows faster than factorially with $n$ [89, 90], see table 4.

We should stress that these numbers are relevant for the case where one is considering a single-colour structure only (see below). The total number of diagrams after summing over all possible colour structures is much larger.

In contrast to the complexity of the calculation, the final result is often surprisingly simple and elegant. To jump ahead of our story a bit, the prime example is the so-called MHV amplitude describing the scattering of two gluons (say $r$ and $s$) of positive helicity with $n - 2$ gluons of negative helicity. At tree level, the (momentum space) amplitude can be recast as

$$A_{0,n}^\text{MHV}(r^*, s^*) = g_{\text{YM}}^2 (2\pi)^4 \delta^{(4)} \left( \sum_{\alpha = 1}^{n} p^\alpha_{r+} \alpha \right) \frac{|rs|^4}{[12][34] \cdots [n1]}.$$  

(7.1)

Our notation will be explained shortly. This is the famous Parke–Taylor formula, which was first conjectured by Parke and Taylor in [91] and later proved by Berends and Giele in [92]. Equivalently, one could consider scattering of two gluons of negative helicity and of $n - 2$ gluons of positive helicity. This leads to the so-called $\overline{\text{MHV}}$ or ‘googly’ amplitude\(^{21}\), which in a Minkowski signature spacetime is obtained by complex conjugating the MHV amplitude.

\(^{21}\) The term ‘googly’ is borrowed from cricket and refers to a ball thrown with the opposite of the natural spin.
Note that our conventions are somewhat opposite from the scattering theory literature, where $n$-gluon amplitudes with two positive helicity and $n-2$ gluons of negative helicity are actually called MHV amplitudes. The MHV amplitudes are of phenomenological importance. For instance, the $n = 4$ and $n = 5$ MHV amplitudes dominate the two-jet and three-jet production in hadron colliders at very high energies. The tree-level $n$-gluon amplitudes with all gluons of the same helicity or all but one gluon of the same helicity are even simpler: they vanish. This follows, for instance, from supersymmetric Ward identities (see also [90, 98] for reviews).

It is natural to wonder why the final form of these amplitudes (and others not mentioned here) is so simple despite that their derivation is extremely complicated. One of the main reasons for this is that the Feynman prescription involves (gauge-dependent) off-shell states. The question then arises: Are there alternative methods which do not suffer these issues? The remainder of these lecture notes is devoted to precisely this question in the context of supersymmetric Yang–Mills theory. But before delving into this, let us give some justification for why it is actually very useful to consider the construction of ‘physical’ quantities like scattering amplitudes in such a ‘non-physical’ theory.

It is a crucial observation that $\mathcal{N} = 4$ amplitudes are identical to or at least part of the physical amplitudes. For instance, gluon scattering amplitudes at tree level are the same in both pure Yang–Mills theory and $\mathcal{N} = 4$ supersymmetric Yang–Mills theory as follows from inspecting the interaction terms in action (6.1). Hence, pure Yang–Mills gluon scattering amplitudes at tree level have a ‘hidden’ $\mathcal{N} = 4$ supersymmetry,

$$A_{0,n}^{\text{YM}} = A_{0,n}^{\mathcal{N} = 4}.$$

and it does not matter which one of the two theories we use to compute them. The same can of course be said about any supersymmetric gauge theory with adjoint matter fields when one is concerned with scattering of external gluons at tree level. If there is no confusion, we will occasionally omit either one or even both of the subscripts appearing on the symbol $A$ which denotes a scattering amplitude.

Likewise, we can find a supersymmetric decomposition for gluon scattering at one-loop. It is

$$A_{1,n}^{\text{YM}} = 4A_{1,n}^{\mathcal{N} = 4} + 2A_{1,n}^{\text{chiral}} + 2A_{1,n}^{\text{scalar}}.
$$

In words this says that the $n$-gluon amplitude in pure Yang–Mills theory at one-loop can be decomposed into three terms. First, a term where the whole $\mathcal{N} = 4$ multiplet propagates in the loop. This is represented by $A_{1,n}^{\mathcal{N} = 4}$. Second, there is the term $A_{1,n}^{\text{chiral}}$ where an $\mathcal{N} = 1$ chiral multiplet propagates in the loop. Lastly, there is the term $A_{1,n}^{\text{scalar}}$ where a scalar propagates in the loop. The reason for this becomes most transparent when one considers the multiplicities of the various particle multiplets in question. Recall that the $\mathcal{N} = 4$ multiplet is $h_{m}^{\mathcal{N} = 4} = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$; the subscript $m$ denotes the multiplicity of the respective helicity-$h$ field. The $\mathcal{N} = 1$ chiral multiplet is $h_{m}^{\text{chiral}} = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$ while the scalar multiplet is just $h_{m}^{\text{scalar}} = (-1, 0, \frac{1}{2}, 0, 1)$. The pure Yang–Mills multiplet contains of course just a vector field, i.e. $h_{m}^{\text{YM}} = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$. Altogether,

$$(-1, -\frac{1}{2}, 0, \frac{1}{2}, 1) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1) - 4(-1, 0, \frac{1}{2}, 0, 1) + 2(-1, 0, \frac{1}{2}, 0, 1).
$$

This is true for $n \geq 4$. For complex momenta or for signatures other than the Minkowski signature, the three-gluon amplitude with all but one gluon of the same helicity is not necessarily zero but just a special case of the MHV or MHV amplitude [53]. See also below.
The left-hand side of (7.3) is extremely complicated to evaluate. However, the three pieces on the right-hand side are easier to deal with. The first two pieces are contributions coming from supersymmetric field theories and these extra supersymmetries greatly help to reduce the complexity of the calculation there. Much of the difficulty is thus pushed into the last term which is the most complex of the three, but still far easier than the left-hand side, mainly because a scalar instead of a gluon is propagating in the loop and thus one does not have to deal with polarizations.

The upshot is then that supersymmetric field theories are not only simpler toy models which one can use to try to understand the gauge theories of the Standard Model of Particle Physics, but relevant theories themselves which contribute parts and sometimes the entirety of the answer to calculations in physically relevant theories.

The next two sections will set up some notation and conventions used in later discussions.

7.2. Colour ordering

One important simplification in computing amplitudes comes from the concept of colour ordering. For concreteness, let us consider SU(N) gauge theory. Let $t_a$ be the generators of SU(N) and $f_{abc}$ be the structure constants; $a, b, \ldots = 1, \ldots, N^2 - 1$. Then $[t_a, t_b] = f_{abc} t_c$. We shall assume that the $t_a$ are anti-Hermitian, i.e. $t_a^\dagger = -t_a$. In a given matrix representation, we will write $(t_a)^m_n$ for $m, n, \ldots = 1, \ldots, d(r)$, where $d(r)$ is the dimension of the representation. For instance, $d(r) = N$ for the fundamental representation while $d(r) = d(G) = N^2 - 1$ for the adjoint representation. Furthermore, $g_{ab} = \text{tr} (t_a t_b) = -\text{tr} (t_b t_a) = C(r) \delta_{ab}$, with $C(r) = \frac{1}{2}$ for the fundamental representation and $C(r) = N$ for the adjoint representation. Using $g_{ab}$, we may re-write the structure constants $f_{abc}$ as

$$f_{abc} = -\text{tr} ([t_a, t_b] t_c).$$

(7.5)

In re-writing $f_{abc}$ this way, all the colour factors appearing in the Feynman rules can be replaced by strings of the $t_a$ and their traces, e.g.

$$\sum g^{bc} \text{tr} (\cdots t_a t_b \cdots) \text{tr} (\cdots t_c \cdots) \text{tr} (\cdots t_d \cdots)$$

(7.6)

if we only have external gluons (or adjoint matter as in $N = 4$ supersymmetric Yang–Mills theory). Here, $g^{ab}$ is the inverse of $g_{ab}$, i.e. $g_{ac} g^{cb} = \delta^b_a$. Likewise, if external matter in a different representation is present, we have sums of the type

$$\sum g^{bc} g^{de} (t_a \cdots t_b)^m_n (t_c \cdots t_d)^l k \text{tr} (t_e \cdots t_f)^k_l \cdots$$

(7.7)

but this case will not be of further interest in the following, as we will solely be dealing with scattering amplitudes in pure Yang–Mills theory or in its $N = 4$ supersymmetric extension.

In order to simplify the number of traces, let us recall the following identity in the fundamental representation of su(N) ($d(r) = N$):

$$g^{ab} (t_a)^m_n (t_b)^l_k = \delta^a_m \delta^b_n - \frac{1}{N} \delta^a_m \delta^b_k.$$  

(7.8)

This is nothing but the completeness relation for the generators $t_a$ in the fundamental representation.

Exercise 7.1. Check (7.8).

As an immediate consequence of (7.8), we have

$$g^{ab} \text{tr} (X t_a) \text{tr} (t_b Y) = \text{tr} (XY) - \frac{1}{N} \text{tr} X \text{tr} Y.$$  

(7.9)
The term in (7.8), (7.9) proportional to $1/N$ corresponds to the subtraction of the trace part in $u(N)$ in which $su(N)$ is embedded. As such, terms involving it disappear at tree level after one sums over all the permutations present. Hence, if one considers gluon scattering at tree level, the amplitude contains only single-trace terms,

$$A_{0,n} = \frac{g^{n-2}}{g_{YM}} \sum_{\sigma \in S_n/Z_n} \text{tr} (t_{\sigma(1)} \cdots t_{\sigma(n)}) A_{0,n}(\sigma(1), \ldots, \sigma(n)),$$

where $S_n$ is the permutation group of degree $n$ and $Z_n$ is the group of cyclic permutations of order $n$. Note that $A_{0,n}$ is assumed to contain the momentum conserving delta function (we shall take all momenta as incoming). Furthermore, we have re-scaled the gauge potential $A_\mu$ according to $A_\mu \rightarrow g_{YM}A_\mu$ to bring the Yang–Mills action (3.2) into a form suitable for perturbation theory. Therefore, the three-gluon vertex is of order $g_{YM}$ while the four-gluon vertex is of order $g_{YM}^2$ meaning that tree-level $n$-gluon amplitudes scale like $g_{YM}^{n-2}$. The amplitude $A_{0,n}$ is called the colour-stripped or partial amplitude and it is this object which we will be interested in our subsequent discussion.

Let us note in passing that there is a similar colour decomposition at one-loop. In pure Yang–Mills theory at one-loop, we have the following expression for the $n$-gluon amplitude [99]:

$$A_{1,n} = \frac{g_{YM}}{N} \sum_{\sigma \in S_n/Z_n} \text{tr} (t_{\sigma(1)} \cdots t_{\sigma(n)}) A_{1,n}^{(1)} (\sigma(1), \ldots, \sigma(n)) + \frac{1}{N} \sum_{c=1}^{[n/2]} \sum_{\sigma \in S_n/(Z_c \times Z_{n-c})} \text{tr} (t_{\sigma(1)} \cdots t_{\sigma(c)}) \text{tr} (t_{\sigma(c+1)} \cdots t_{\sigma(n)}) A_{1,n}^{(c)} (\sigma(1), \ldots, \sigma(n)).$$

(7.11)

Hence, if one is only interested in the so-called large $N$-limit which is also referred to as the planar limit (see [100]), only the single-trace term survives. In fact, it is a general feature that in the planar limit the amplitude to $\ell$-loop order is given by a single-trace expression. Of course, for finite $N$ this is not the case and the colour structure of the amplitudes will look much more complicated. Note that due to a remarkable result of Bern, Dixon, Dunbar and Kosower [101], the double-trace expressions $A_{1,n}^{(c)}$ in (7.11) are obtained as sums over permutations of the single-trace term $A_{1,n}^{(1)}$. This also applies to gauge theories with external particles and those running in the loop both in the adjoint representation in general and to $N = 4$ supersymmetric Yang–Mills theory in particular.

### 7.3. Spinor-helicity formalism re-visited

In the conventional description as given in most standard textbooks, scattering amplitudes are considered as a function of the external momenta of the particles (in fact the Mandelstam invariants) together with the spin information such as polarization vectors if one considers photons or gluons (leaving aside colour degrees of freedom). Furthermore, see e.g. Weinberg’s book [102] for the details. However, as we shall see momentarily, there is a more convenient way to encode both the momentum and spin information. In the following, we continue to follow the philosophy of working in a complex setting for most of the time. Reality conditions are imposed (explicitly or implicitly) whenever needed.

Consider a null-momentum $p_{a\dot{a}}$. From remark 1.3 in section 1.3 we know that $p_{a\dot{a}}$ can be written as $p_{a\dot{a}} = \tilde{k}_a k_\dot{a}$ for two co-spinors $\tilde{k}_a$ and $k_\dot{a}$. Clearly, this decomposition is not unique since one can always perform the following transformation:

$$(k_a, \tilde{k}_a) \mapsto (t^{-1} \tilde{k}_a, tk_\dot{a}) \quad \text{for} \quad t \in \mathbb{C} \setminus \{0\}.$$
This is in fact the action of the little group of $\text{SO}(4,\mathbb{C})$ on the co-spinors. Therefore, one associates a helicity of $-1/2$ with $\tilde{k}_a$ and a helicity of $1/2$ with $k_\alpha$, respectively, so that $p_{\alpha\beta}$ has helicity zero. In addition, we say that a quantity carries (a generalized) helicity $h$ if and only if $\tilde{\epsilon}^{-\beta}_\alpha p_{\alpha\beta} = h$. In particular, $\tilde{k}_a$ and $k_\alpha$ are labelled by helicities $-1/2$ and $1/2$, respectively.

Let us then consider a collection of null-momenta $p_{raa}$ labelled by $r, s, \ldots \in \{1, \ldots, n\}$. Each of these can then be decomposed as $p_{raa} = \tilde{k}_{raa}k_{raa}$. Next we define the spinor brackets $\langle \tilde{k}_r \tilde{k}_s \rangle := \epsilon_{a\beta}k^a_r k^\beta_s$ and $\{k_r k_s \} := [r s] := \epsilon_{\alpha\beta\gamma} k_{raa} k_{\gamma\beta\beta}$. (7.14) Recall that the second of these products was already introduced in (5.2). The inner product of a null-momentum $p_{raa}$ and $p_{raa}$, is then

$$\langle p_r \cdot p_s \rangle = 2\epsilon_{a\beta}k^a_r k^\beta_s p_{raa}p_{\beta\beta} = -2\langle r s \rangle [r s].$$

Once we are given a null-momentum and its co-spinor decomposition, we have enough information to describe the different helicity-$h$ wavefunctions (plane-wave solutions) modulo gauge equivalence [103]. Since we are interested in gauge theories, we restrict our attention to $|h| \leq 1$. Let us start with helicity $h = 0$. The wavefunction of a scalar field is just

$$\phi(x) = e^{i\tilde{\epsilon}^a_{\alpha}\tilde{k}_a},$$

i.e. $a_{aa}a^{aa}\phi = 0$, for some fixed null-momentum $p_{a\beta} = \tilde{k}_a k_\beta$.

Likewise, the helicity $h = \pm \frac{1}{2}$ plane waves are

$$\chi_a(x) = k_a e^{i\alpha_{\alpha}\tilde{k}_a} \quad \text{and} \quad \chi_a(x) = \tilde{k}_a e^{i\alpha_{\alpha}\tilde{k}_a},$$

i.e. $a_{aa}x^a = 0 = a_{aa}x^a$.

For a massless particle of helicity $h = \pm 1$, the usual method is to specify the polarization co-vector $\epsilon_{a\beta}$ in addition to its momentum $p_{a\beta}$ together with the constraint $p_{a\beta}\epsilon^{a\beta} = 0$. This constraint is equivalent to the Lorentz gauge condition and deals with fixing the gauge invariance inherent in gauge theories. It is clear that if we add any multiple of $p_{a\beta}$ to $\epsilon_{a\beta}$ then this condition is still satisfied and we have the gauge invariance

$$\epsilon_{a\beta} \rightarrow \epsilon_{a\beta}' = \epsilon_{a\beta} + \rho p_{a\beta} \quad \text{for} \quad \rho \in \mathbb{C}. \tag{7.18}$$

Since we have the decomposition of the momentum co-vector into two co-spinors, we can take the $h = \pm 1$ polarization co-vectors to be [53, 104]

$$\epsilon^+_{a\beta} = \frac{\tilde{\mu}_a \tilde{k}_\beta}{(\tilde{\mu} k)} \quad \text{and} \quad \epsilon^-_{a\beta} = \frac{\tilde{k}_a \mu_\beta}{[k \mu]}, \tag{7.19}$$

where $\tilde{\mu}_a$ and $\mu_\beta$ are arbitrary co-spinors. Note that the $\epsilon^\pm_{a\beta}$ do not depend on the choice of these co-spinors—modulo transformations of the form (7.18). This can be seen as follows [53]. Consider $\epsilon^+_{a\beta}$. Any change of $\tilde{\mu}_a$ is of the form

$$\tilde{\mu}_a \rightarrow \tilde{\mu}_a + a\tilde{\mu}_a + b\tilde{k}_a \quad \text{for} \quad a, b \in \mathbb{C} \quad \text{and} \quad a \neq -1. \tag{7.20}$$

since the space of possible $\tilde{\mu}_a$ is two dimensional and $\tilde{\mu}_a$ and $\tilde{k}_a$ are linearly independent by assumption (if they were linearly dependent then $(\tilde{\mu} k) = 0$ and the above expression would not make sense). However, under such a change, $\epsilon^+_{a\beta}$ behaves as

$$\epsilon^+_{a\beta} \rightarrow \epsilon^+_{a\beta} + \frac{b}{1 + a} \tilde{k}_a \tilde{k}_\beta = \epsilon^+_{a\beta} + \rho p_{a\beta}, \quad \text{with} \quad \rho = \frac{b}{1 + a}. \tag{7.21}$$

23 Note that since the metric is $\epsilon_{a\beta}p_{a\beta} = \frac{1}{2} \epsilon_{a\beta}e_{a\beta}$, the inverse metric is $g^{a\beta a\beta} = 2\epsilon^{a\beta}e_{a\beta}$. 

\[ \text{J. Phys. A: Math. Theor. 43 (2010) 393001} \]
which is precisely \((7.18)\). Likewise, a similar argument goes through for \(\epsilon_{a\bar{\beta}}^-\). Furthermore, under the scaling \((7.12)\) we find that \(\epsilon_{a\bar{\beta}}^\pm \mapsto \ell^2 \epsilon_{a\bar{\beta}}^\pm\) and so we may conclude that the \(\epsilon_{a\bar{\beta}}^\pm\) indeed carry helicity \(\pm 1\). Therefore, the helicity \(h = \pm 1\) plane waves are

\[
A_{a\bar{\beta}}^\pm(x) = \epsilon_{a\bar{\beta}}^\pm e^{i\ell^2 k_\alpha k_\beta}.
\]

\((7.22)\)

**Exercise 7.2.** Consider the curvature \(F_{a\bar{\alpha}\bar{\beta}} = \partial_{a\bar{\alpha}} A_{\bar{\beta}} - \partial_{a\bar{\beta}} A_{\bar{\alpha}} = \epsilon_{a\bar{\alpha}} f_{\bar{\alpha}\bar{\beta}} + \epsilon_{a\bar{\beta}} f_{\bar{\alpha}\bar{\beta}}\). Show that for the choice \(A_{a\bar{\alpha}}^+\) as given in \((7.22)\), we have \(F_{a\bar{\alpha}\bar{\beta}}^+ = \imath \epsilon_{a\bar{\alpha}} k_\alpha k_\beta e^{i\ell^2 k_\alpha k_\beta}\) while for \(A_{a\bar{\alpha}}^-\) we have \(F_{a\bar{\alpha}\bar{\beta}}^- = \imath \epsilon_{a\bar{\beta}} k_\alpha k_\beta e^{i\ell^2 k_\alpha k_\beta}\).

In summary, the different wavefunctions scale as \(r^h\) under \((7.12)\). Therefore, instead of considering the scattering amplitude \(A\) as a function of the momenta and spins, we may equivalently regard it as a function of the two co-spinors specifying the momenta and the helicity. Hence, we may write

\[
A = A([k_{ra}, k_{\bar{r}a}, h_r]) \quad \text{for} \quad r = 1, \ldots, n.
\]

\((7.23)\)

In what follows, we shall also make use of the notation \(A = A(1^{h_1}, \ldots, n^{h_n})\) where the co-spinor dependence is understood. When formulated in this way, the amplitude obeys an auxiliary condition for each \(r\) (i.e. no summation over \(r\)) \([53]\):

\[
\left(-\dot{k}_{ra} \frac{\partial}{\partial k_{ra}} + k_{ra} \frac{\partial}{\partial k_{\bar{r}a}}\right) A = 2h_r A.
\]

\((7.24)\)

This is easily checked by using \((7.13)\).

**Remark 7.1.** Above we have considered wavefunctions in position space, but we may equivalently view them in momentum space and in fact, in sections 8.2 and 8.3, this will be the more convenient point of view. We follow \([105, 106]\). First, the Klein–Gordon equation in momentum space is \(p^2 \phi(p) = 0\), where \(p^2 = 4 \det p_{a\bar{\beta}}\), see \((7.15)\) and footnote 23. Therefore, any solution is of the form \(\phi(p) = \delta(p^2) \phi_0(\vec{k}, k)\) for some function \(\phi_0(\vec{k}, k)\) defined on the null-cone in momentum space, where \(p_{a\bar{\beta}} = \vec{k}_a k_{\bar{\beta}}\). Second, we can discuss the other helicity fields. The momentum space versions of the field equations \((2.4), (2.6)\) are given by \(p^{a\bar{\alpha}} \phi_{a\bar{\alpha} - a\bar{\alpha}}(p) = 0\) and \(p^{r\bar{a}} \phi_{r\bar{a} - a\bar{\alpha}}(p) = 0\). Away from the null-cone \(p^2 = 0\), the matrix \(p^{a\bar{\beta}}\) is invertible. Therefore, the general solutions are given by \(\phi_{a\bar{\alpha} - a\bar{\alpha}}(p) = \delta(p^2) k_{a1} \ldots k_{a2n} \phi_{-2n}(\vec{k}, k)\) and \(\phi_{r\bar{a} - a\bar{\alpha}}(p) = \delta(p^2) k_{\bar{r}a1} \ldots k_{\bar{r}a2m} \phi_{-2m}(\vec{k}, k)\), where again the \(\phi_{-2n}(\vec{k}, k)\) are some functions defined on the null-cone (with \(h > 0\)). Note that the \(\phi_{-2h}(\vec{k}, k)\) for the \(h \geq 0\) scale under \((7.12)\) like \(\phi_{-2h}(i^{-1} \vec{k}, tk) = i^{2h} \phi_{-2h}(\vec{k}, k)\) in order to compensate the scaling of the co-spinor pre-factors. We shall refer to \(\phi_{-2h}(\vec{k}, k)\) as the on-shell momentum space wavefunctions of helicity \(\mp h\).

**Exercise 7.3.** Choose the Minkowski signature. Show that the Fourier transform of the solution \((7.16)\) can be written as \(\tilde{\phi}(\vec{p}) = \delta(\vec{p}^2) \Theta(p_0) 2|\vec{p}| \delta^{(3)}(\vec{p} - \vec{k})\), where \(p_0\) is the ‘time component’ of \(p_{a\bar{\beta}}\) and \(\vec{p}\) and \(\vec{k}\) are the ‘spatial components’ of \(p_{a\bar{\beta}}\) and \(k_{a\bar{\beta}} := \vec{k}_a k_{\bar{\beta}}\), respectively. Here, \(\Theta(x)\) is the Heaviside step function with \(\Theta(x) = 1\) for \(x > 0\) and zero otherwise. Note that on the support of the delta function we have \(p_{a\bar{\beta}} = \vec{k}_a k_{\bar{\beta}}\).
8. MHV amplitudes and twistor theory

8.1. Tree-level MHV amplitudes

In section 7.1, we already encountered some colour-stripped tree-level $n$-gluon scattering amplitudes (7.10), and with the above discussion the notation should be clear. For the reader’s convenience, let us re-state them (in the complex setting):

\[ A_{0,n}(1^\pm, \ldots, n^\pm) = 0, \quad \text{for } n \geq 4 \quad A_{0,n}(1^\pm, 2^\mp, \ldots, n^\mp) = 0, \quad (8.1a) \]

\[ A_{0,n}(\ldots, (r - 1)^-, r^+, (r + 1)^-, \ldots, (s - 1)^-, s^+, (s + 1)^-, \ldots) \]

\[ = g_{\text{YM}}^n (2\pi)^4 \delta^{(4)} \left( \sum_{\nu=1}^n k_{\nu} \epsilon_{\nu} k_{\nu} \right) \frac{\langle r s \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad (8.1c) \]

\[ A_{0,n}(\ldots, (r - 1)^+, r^-, (r + 1)^+, \ldots, (s - 1)^-, s^+, (s + 1)^-, \ldots) \]

\[ = g_{\text{YM}}^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{\nu=1}^n k_{\nu} \epsilon_{\nu} k_{\nu} \right) \frac{\langle r s \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}. \quad (8.1d) \]

As already mentioned, for a Minkowski signature spacetime and for $n = 3$, the amplitudes (8.1c) and (8.1d) actually vanish.

The amplitude (8.1c) is referred to as the MHV amplitude or MHV amplitude in short. We shall also write $A_{0,4}^{\text{MHV}}(r^+, s^+)$ and $A_{0,4}^{\text{MHV}}(r^-, s^-)$ for a Minkowski signature, they are the complex conjugates of each other. In addition, $n$-gluon scattering amplitudes with three positive helicity gluons and $n - 3$ negative helicity gluons are called next-to-MHV amplitudes or NMHV amplitudes in short. Analogously, one considers $N^0 \text{MHV}$ and $N^0 \text{MHV}$ amplitudes. Hence, an $N^0 \text{MHV}$ has a ‘total helicity’ of $-n + 2(k + 2)$.

It is a straightforward exercise to show that the amplitudes (8.1) indeed obey condition (7.24).

Exercise 8.1. Consider four-gluon scattering. Show that we indeed have $A_{0,4}^{\text{MHV}}(1^+, 2^+) = A_{0,4}^{\text{MHV}}(3^-, 4^-)$ or more generally $A_{0,4}^{\text{MHV}}(r^+, s^+) = A_{0,4}^{\text{MHV}}(r^-, s^-)$ for $\{r, s\} \neq \{r', s'\}$. In order to demonstrate this, you will need to use overall momentum conservation.

As a next step, it is instructive to verify the invariance of the MHV amplitudes under the action of the conformal group. Recall from section 4.3 that the conformal group is generated by translations $P_{ra}$, Lorentz rotations $L_a^\beta$, dilatations $D$ and special conformal transformations $K^aa$. In terms of the co-spinors $\tilde{k}_{ra}$ and $k_{ra}$ for $r = 1, \ldots, n$, these generators may be represented by [53]

\[ P_{ra} = -i \sum_r \tilde{k}_{ra} k_{ra}, \quad K^aa = -i \sum_r \frac{\partial^2}{\partial k_{ra} \partial \tilde{k}_{ra}}, \]

\[ D = \frac{i}{2} \sum_r \left( \tilde{k}_{ra} \frac{\partial}{\partial k_{ra}} + k_{ra} \frac{\partial}{\partial \tilde{k}_{ra}} + 2 \right), \quad (8.2) \]

\[ L_a^\beta = i \sum_r \left( k_{ra} \frac{\partial}{\partial k_{\beta r}} - \frac{1}{2} \delta_a^\beta \tilde{k}_{r\gamma} \frac{\partial}{\partial \tilde{k}_{r\gamma}} \right), \quad L_a^\beta = i \sum_r \left( \tilde{k}_{ra} \frac{\partial}{\partial \tilde{k}_{\beta r}} - \frac{1}{2} \delta_a^\beta k_{r\gamma} \frac{\partial}{\partial k_{r\gamma}} \right). \]
If not indicated otherwise, summations over $r, s, \ldots$ always run from 1 to $n$. Upon using (7.13), one may straightforwardly check that these generators obey the bosonic part of (4.20). Next we observe that the amplitude is manifestly invariant under (Lorentz) rotations. It is however, invariant under translations because of the momentum-conserving delta function. What therefore remains is for us to verify dilatations and special conformal transformations. Let us write

$$D = \sum_r D_r = \frac{1}{2} \sum_r (\hat{D}_r + 2),$$

where $\hat{D}_r$ is given by

$$\hat{D}_r := \frac{\partial}{\partial k_{ra}} + k_{ra} \frac{\partial}{\partial k_{ra}}.$$

Then

$$D \left( \delta^{(4)} \left( \sum_{s'} \tilde{k}_{ra} k_{s'\beta} \right) \frac{[rs]^4}{[12][23] \cdots [n1]} \right)$$

$$= \frac{i}{2} \sum \left[ \hat{D}_{r'} \left( \delta^{(4)} \left( \sum_{s'} \tilde{k}_{r'a} k_{s'\beta} \right) [rs]^4 \right) \frac{1}{[12][23] \cdots [n1]} \right]$$

$$+ \delta^{(4)} \left( \sum_{s'} \tilde{k}_{r'a} k_{s'\beta} \right) [rs]^4 (\hat{D}_{r'} + 2) \frac{1}{[12][23] \cdots [n1]} \right].$$

(8.4a)

However,

$$\sum \hat{D}_{r'} \left( \delta^{(4)} \left( \sum_{s'} \tilde{k}_{r'a} k_{s'\beta} \right) [rs]^4 \right) = -8 + 8 = 0,$$

(8.4b)

since under the rescaling $\delta^{(4)} (t \sum \tilde{k}_{ra} k_{r\beta}) = t^{-4} \delta^{(4)} (\sum \tilde{k}_{ra} k_{r\beta})$ for $t \in \mathbb{C} \setminus \{0\}$. Therefore, combining (8.4a) with (8.4b) we find that $D A^{\text{MHV}} = 0$ as claimed.

**Exercise 8.2.** Show that $K^{aa}_0 A^{\text{MHV}} = 0$.

Representation (8.2) of the generators of the conformal group is rather unusual in the sense of being a mix of first-order and second-order differential operators and of multiplication operators. The representation given in (4.21) on twistor space is more natural and indeed, following Witten [53], we can bring (8.2) into the form of (4.21) by making the (formal) substitutions

$$\tilde{k}_{ra} \mapsto i \frac{\partial}{\partial z_r^a} \quad \text{and} \quad \frac{\partial}{\partial k_{ra}} \mapsto i z_r^a$$

(8.5)

and by re-labelling $k_{ra}$ by $\lambda_{ra}$ for all $r = 1, \ldots, n$. This can also be understood by noting that after this substitution, the transformation (7.12) will become the usual rescaling of projective space, i.e. $(z_r^a, \lambda_{ra}) \sim (t z_r^a, t \lambda_{ra})$ for $t \in \mathbb{C} \setminus \{0\}$. Substitution (8.5) will be referred to as Witten’s half Fourier transform to twistor space. The reason for this name becomes transparent in section 8.3. For the sake of brevity, we shall also refer to it as the Witten transform$^{24}$.

Before giving a more precise meaning to (8.5), we can already make two observations about the Witten transform of scattering amplitudes. First, in making the substitution (8.5), we have chosen to transform $\tilde{k}_{ra}$ instead of $\tilde{k}_{ra}$. Naturally, the symmetry between $\tilde{k}_{ra}$ and $k_{ra}$

$^{24}$ Note that some people also use the terminology twistor transform. However, the name twistor transform is reserved for the transformation that acts between twistor space and dual twistor space.
is lost and therefore parity symmetry is obscured. Henceforth, scattering amplitudes with, say, \( m \) positive helicity gluons and \( n \) negative helicity gluons will be treated completely differently from those with \( m \) and \( n \) interchanged. Second, let \( \mathcal{A} \) be the Witten transform of an \( n \)-particle scattering amplitude \( A \). According to our above discussion, the quantity \( \mathcal{A} \) will live on the \( n \)-particle twistor space \( \mathcal{T}^\otimes n \), which is defined as

\[
\mathcal{T}^\otimes n := P^1_1 \times \cdots \times P^4_n,
\]

together with the canonical projections \( pr_n : \mathcal{T}^\otimes n \to P^4_n \). Each \( P^4_n \) is equipped with homogeneous coordinates \((\zeta^a, \lambda_{\alpha a})\). The auxiliary condition (7.24) then translates into

\[
\left( \frac{\partial}{\partial \zeta^a} + \lambda_{\alpha a} \frac{\partial}{\partial \lambda_{\alpha a}} \right) \mathcal{A} = (2h - 2)\mathcal{A}.
\]

In other terms this says that \( \mathcal{A} \) should be regarded as a section of

\[
O(2h_1 - 2, \ldots, 2h_n - 2) := pr^*_1 O(2h_1 - 2) \otimes \cdots \otimes pr^*_n O(2h_n - 2).
\]

Thus, like solutions to zero-rest-mass field equations, scattering amplitudes can also be interpreted as holomorphic functions of a certain homogeneity on twistor space (or more precisely on the multi-particle twistor space (8.6)). One may naturally wonder whether one can then also give scattering amplitudes a sheaf cohomological interpretation as done for solutions to the zero-rest-mass field equations (see section 2). This would in turn yield a precise definition of the Witten transform. Unfortunately, this issue has not yet been completely settled and we will therefore not concern ourselves with it any further in these notes. When choosing a Kleinian signature spacetime, on the other hand, one can bypass sheaf cohomology and give the Witten transform a precise meaning and this will be the subject of section 8.3. Firstly, however, we would like to extend the above ideas to the \( \mathcal{N} = 4 \) supersymmetric theory.

8.2. Tree-level MHV superamplitudes

In \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory one has different kind of particles, the gluons \( f_{\alpha \dot{\beta}} \), \( f_{\dot{\alpha} \beta} \), the gluinos \( \chi_{\alpha} \), \( \chi_{\dot{\alpha}} \) and the scalars \( \phi^{i j} = \frac{i}{2} \epsilon^{i j k l} \phi_{k l} \). According to remark 7.1, the corresponding momentum space wavefunctions may be taken as follows:

\[
\begin{aligned}
\tilde{f}_{\alpha \dot{\beta}}(p) &= \delta(p^2) k_{\alpha} \tilde{k}_{\dot{\beta}} \tilde{f}^+(\tilde{k}, k), \\
\tilde{\phi}_{ij}(p) &= \delta(p^2) \phi_{ij}(\tilde{k}, k), \\
\tilde{\chi}_{\alpha}(p) &= \delta(p^2) k_{\alpha} \tilde{\chi}_{\dot{\alpha}}^-(\tilde{k}, k), \\
\tilde{f}_{\dot{\alpha} \beta}(p) &= \delta(p^2) k_{\dot{\alpha}} \tilde{k}_\beta \tilde{f}^-(\tilde{k}, k),
\end{aligned}
\]

(8.9)

where \( \tilde{f}^+, \tilde{\chi}^{+\dot{\alpha}}, \tilde{\phi}_{ij}, \tilde{\chi}_{\dot{\alpha}}^- \) and \( \tilde{f}^- \) are the on-shell momentum space wavefunctions of helicity 1, \( \frac{1}{2}, 0, -\frac{1}{2} \) and \(-1\), respectively. These particles can then scatter off each other in many different ways eventually leading to a large variety of amplitudes. Despite this fact, supersymmetry relates many of the amplitudes. It would therefore be desirable to have a formulation of scattering amplitudes in which \( \mathcal{N} = 4 \) supersymmetry is manifest. Fortunately, Nair [107] proposed a particular type of an \( \mathcal{N} = 4 \) on-shell superspace to 20 years ago, used later by Witten [53], that leads to a manifestly supersymmetric formulation of the scattering amplitudes and in addition forms the basis of the latter’s twistor re-formulation.

The key idea of Nair’s is to introduce additional spinless fermionic variables \( \psi^i \) for \( i = 1, \ldots, 4 \), and to combine the different wavefunctions (8.9) into one single superwavefunction \( \Phi = \Phi(\tilde{k}_\alpha, \psi^i, k_\alpha) \):

\[
\Phi := \tilde{f}^- + \psi^i \tilde{\chi}^- + \frac{1}{2!} \psi^i \psi^j \tilde{\phi}_{ij} + \frac{1}{3!} \psi^i \psi^j \psi^k \epsilon_{ijkl} \tilde{\chi}^+ + \frac{1}{4!} \psi^i \psi^j \psi^k \psi^l \epsilon_{ijkl} \tilde{f}^+.
\]

(8.10)

Note that this superspace has a close relationship with the light-cone superspace of [108, 109].
Moreover, in view of our twistor space application, it is useful to combine

\[ (\tilde{k}_a, \psi^i, k_a) \mapsto (t^{-1}\tilde{k}_a, t^{-1}\psi^i, tk_a) \quad \text{for} \quad t \in \mathbb{C} \setminus \{0\}. \tag{8.11} \]

Consequently, the super wavefunction scales as \( \tilde{\Phi}(t^{-1}\tilde{k}_a, t^{-1}\psi^i, tk_a) = t^{-2}\tilde{\Phi}(\tilde{k}_a, \psi^i, k_a) \), i.e. it carries helicity \(-1\). Note that this may be re-expressed as a differential constraint

\[ \left( -\frac{\partial}{\partial k_a} - \psi^i \frac{\partial}{\partial \psi^i} + k_a \frac{\partial}{\partial k_a} + 2 \right) \Phi = 0. \tag{8.12} \]

Due to the scaling property of \( \tilde{\Phi} \), we shall also write \( \tilde{\Phi}_{-2} \), in the following. At this point it is worthwhile recalling the twistor space expression (5.40) and comparing it with expression (8.10). Their resemblance is not coincidental as will be demonstrated in the next section. Moreover, in view of our twistor space application, it is useful to combine \( \tilde{k}_a \) and \( \psi^i \) into a super co-spinor \( \tilde{k}_A = (\tilde{k}_a, \psi^i) \).

In this superspace formulation, \( n \)-particle scattering amplitudes—also referred to as superamplitudes or generating functions—are regarded as functions of \( k_{rA} \) and \( k_{ra} \) and we will denote them by \( \mathcal{F} \):

\[ \mathcal{F} = \mathcal{F}(\Phi_1, \ldots, \Phi_n) = \mathcal{F}(\tilde{k}_{rA}, k_{ra}) \quad \text{for} \quad r = 1, \ldots, n, \tag{8.13} \]

where \( \Phi_r := \tilde{\Phi}(\tilde{k}_{rA}, k_{ra}) \). Likewise, colour-stripped superamplitudes will be denoted by \( F \).

Thus, a term in a superamplitude that is of the 26 This definition is consistent with the definition of the Berezin integral over fermionic coordinates: \( \int d\psi \delta(\psi) = \int d\psi \psi = 1 \) for more than one fermionic coordinate, one has \( \int d\psi \delta^{(n)}(\psi) = 1 \) with \( d\psi = d\psi_1 \cdots d\psi_n \) and \( \delta^{(n)}(\psi) = \psi_n \cdots \psi_1 \).
Obviously, this expression obeys (8.15). Moreover, in order to recover (8.1c) from (8.17), we simply need to pick the term that is of fourth order in both \(\psi^i_j\) and \(\psi^i_s\), corresponding to the two positive helicity gluons; see (8.10). This is most easily accomplished by noting the identity

\[
\delta^{(0|8)} \left( \sum_{r=1}^{n} \psi^i_j k_{ra} \right) = \frac{1}{16} \prod_{i=1}^{4} \sum_{r,s=1}^{n} \left[ r s \right] \psi^i_j \psi^s_j
\]

(8.18)

and by using (8.14). Then, we find

\[
A_{0,n}^{\text{MHV}} (r^+, s^+) = D_r D_s F_{0,0}^{\text{MHV}}.
\]

(8.19)

Note that since \(\hat{\Phi}\) carries helicity \(-1\), the superamplitude carries a total helicity of \(-n\) and since \(D_r\) carries helicity 2 (remember that each \(\psi^i_j\) carries helicity \(-1/2\)), we recover the correct helicity of \(-n+4\) for an MHV amplitude.

Analogously, we may construct other MHV amplitudes. For instance, the \(n\)-particle amplitude involving one positive helicity gluon, say particle \(r\), two gluinos of opposite helicity, say particles \(s\) and \(t\) and \(n-3\) negative helicity gluons is obtained by

\[
A_{0,n}^{\text{MHV}} (r^+, s^-, t^+; r^+, s^-, t^+) = D_r D_s D_t F_{0,0}^{\text{MHV}}
\]

\[
= \delta^{n-6} (2\pi)^{3/2} \delta^{(4)} \left( \sum_{r=1}^{n} k_{ra} k_{rb} \right) \frac{[r s][r t]^3}{[12][23]\cdots [n1]}.
\]

(8.20)

In general, all MHV amplitudes are of the type \([r_1 s_1][r_2 s_2][r_3 s_3][r_4 s_4]/([12][23]\cdots [n1])\) and are obtained as

\[
A_{0,n}^{\text{MHV}} = D^{(8)} F_{0,0}^{\text{MHV}},
\]

(8.21)

where \(D^{(8)}\) is an eighth-order differential operator made from the operators (8.14).

**Exercise 8.3.** Verify (8.19) and (8.20).

In the preceding section, we have shown that the MHV amplitudes are invariant under the action of the conformal group. Likewise, the MHV superamplitudes are invariant under the action of the \(\mathcal{N} = 4\) superconformal group. Recall from section 4.3 that the superconformal group for \(\mathcal{N} = 4\) is generated by translations \(P_{aa}\), Lorentz rotations \(L_a^\beta\) and \(L_a^\beta\), dilatations \(D\) and special conformal transformations \(K^{aa}\) together with the \(R\)-symmetry generators \(R_r^j\) and the Poincaré supercharges \(Q_{ia}, Q_a^i\) and their superconformal partners \(S_{ia}^j, S^j_i\). In terms of \((\bar{k}_r A, k_a) = (\bar{k}_r A, \psi^i_j, k_a)\) for \(r = 1, \ldots, n\), these generators may be represented by (8.2) together with [53]:

\[
Q_{ia} = i \sum_r \bar{k}_r a \frac{\partial}{\partial \psi^i_r}, \quad Q_a^i = \sum_r \bar{k}_r a \psi^i_r,
\]

\[
S_{ia}^j = -i \sum_r \psi^i_r \frac{\partial}{\partial k_{ra}}, \quad S_i^j = - \sum_r \frac{\partial^2}{\partial \psi^i_r \partial k_{ra}},
\]

(8.22a)

\[
R_r^j = \frac{i}{2} \sum_r \left( \psi^i_r \frac{\partial}{\partial \psi^j_r} - \frac{1}{4} \delta^i_1 \psi^j_r \frac{\partial}{\partial \psi^1_r} \right).
\]

Furthermore, the central extension is given by

\[
Z = \frac{i}{2} \sum_r \left( \bar{k}_r a \frac{\partial}{\partial k_{ra}} + \psi^i_r \frac{\partial}{\partial \psi^i_r} - k_{ra} \frac{\partial}{\partial k_{ra}} - 2 \right).
\]

(8.22b)

Following the analysis presented in section 8.1 in the bosonic setting, one may demonstrate the invariance of (8.17) under superconformal transformations. We leave this as an exercise.
Exercise 8.4. Verify that the MHV superamplitude (8.17) is invariant under the action of \( N = 4 \) superconformal group.

In the supersymmetric setting we can again see that the representation of the generators of the superconformal group is rather unusual. However, we can bring (8.2) into the canonical form (4.21) by making the (formal) substitutions [53]

\[
\bar{k}_{rA} \mapsto \frac{i}{\partial \bar{z}_r^A} \quad \text{and} \quad \frac{\partial}{\partial \bar{k}_{rA}} \mapsto i(-)^{p_A} \bar{z}_r^A
\]

and by re-labelling \( \bar{k}_{rA} \) by \( \bar{\lambda}_{rA} \) for all \( r = 1, \ldots, n \). After this substitution, the transformation (8.11) becomes the usual rescaling of projective superspace, i.e. \( (z_r^A, \bar{\lambda}_{rA}) \sim (t z_r^A, t \lambda_{rA}) \) for \( t \in \mathbb{C} \setminus \{0\} \). As before, we shall refer to (8.23) as the Witten transform. Moreover, if we let \( \mathcal{W}[F] \) be the Witten transform of an \( n \)-particle superamplitude \( F \), then \( \mathcal{W}[F] \) will live on the \( n \)-particle supertwistor space \( \mathcal{X}^{3|4n+4} \), which is defined as

\[
\mathcal{X}^{3|4n+4} := P_1^{3|4} \times \cdots \times P_n^{3|4},
\] (8.24)

together with the canonical projections \( \pi_r \) : \( \mathcal{X}^{3|4n+4} \rightarrow P_r^{3|4} \). Each \( P_r^{3|4} \) is equipped with homogeneous coordinates \((z_r^A, \lambda_{rA})\). The auxiliary condition (8.15) then translates into

\[
\left( z_r^A \frac{\partial}{\partial z_r^A} + \lambda_{rA} \frac{\partial}{\partial \lambda_{rA}} \right) \mathcal{W}[F] = 0.
\]

(8.25)

In words this says that \( \mathcal{W}[F] \) should be regarded as a holomorphic function of homogeneity zero on \( \mathcal{X}^{3|4n} \). This is again analogous to what we encountered in section 5.2, where one superfield (5.40) of homogeneity zero on supertwistor space encodes all the wavefunctions.

Remark 8.1. In verifying the invariance of the MHV (super)amplitudes under the action of the (super)conformal group here and in section 8.1, one should actually be a bit more careful. In the complex setting, where there is no relation between the co-spinors \( \bar{k} \) and \( k \) or for Kleinian signature spacetimes where \( k \) and \( \bar{k} \) are real and independent of each other (see also the subsequent section), the above procedure of verifying (super)conformal invariance works as discussed. However, for a Minkowski signature spacetime there is a subtlety. The Minkowski reality conditions (1.1) imply that \( \bar{k} \) and \( k \) are complex conjugates of each other. For a null co-vector \( p_{\alpha\beta} = \bar{k}_\alpha k_\beta \), the reality condition \( p_{\alpha\beta} = -p_{\beta\alpha} \) follows from \( \bar{k}_\alpha = i k_\alpha \). This yields

\[
\frac{\partial}{\partial \bar{k}_\alpha} \frac{1}{|k k'|} = -2\pi \bar{k}_\alpha \frac{\bar{\xi} \hat{k}'}{|ar{\xi} \hat{k}'|} \delta^{(2)}([k k'], \langle \hat{k} \rangle')
\]

for some arbitrary \( \hat{\xi} \); see also (6.9a). Note that on the support of the delta function, \( k \propto k' \) and \( \bar{k} \propto \bar{k}' \) and so this expression is independent of \( \hat{\xi} \). Therefore, even though the functions multiplying the overall delta functions in the MHV (super)amplitudes are independent of \( \hat{\xi} \), one produces delta functions whenever two co-spinors become collinear, that is, whenever \([r s] = 0\) for some \( r \) and \( s \). This is known as the holomorphic anomaly [110] and the problem carries over to generic (super)amplitudes. One can resolve this issue by modifying the generators of the (super)conformal group (8.2), (8.2), such that (super)conformal invariance holds also for Minkowski signature spacetime. See [111–113]. At tree level, the holomorphic anomaly does not really matter as long as one sits at generic points in momentum space. At loop level, however, the anomaly becomes important as one has to integrate over internal momenta.
8.3. Witten’s half Fourier transform

The subject of this section is to give the Witten transform \( (8.5), (8.23) \) a more precise meaning. We shall closely follow the treatment of Mason and Skinner [106]. As we have already indicated, we will focus on reality conditions that lead to a Kleinian signature spacetime.

To jump ahead of our story a bit, the two main reasons for doing this are as follows. First, for Kleinian signature (super)twistor space becomes a subset of real projective (super)space. Second, solutions to zero-rest-mass field equations are represented in terms of straightforward functions rather than representatives of sheaf cohomology groups. This therefore simplifies the discussion. At this point it should be noted that as far as perturbation theory is concerned, the choice of signature is largely irrelevant as the scattering amplitudes are holomorphic functions of the Mandelstam variables.

In sections 3.2 and 5.2, we saw that Euclidean reality conditions arise from a certain anti-holomorphic involution on the (super)twistor space. We can discuss Kleinian reality conditions in an analogous manner. To this end, we introduce the following anti-holomorphic involution \( \tau : P^{3|4} \to P^{3|4} \) on the supertwistor space:

\[
\tau(z^A, \lambda_\alpha) := (\bar{z}^A, \check{\lambda}_\alpha) := (\bar{z}^A, \check{\lambda}_\alpha).
\]

By virtue of the incidence relation \( z^A = x^{A\alpha} \lambda_\alpha \), we find

\[
\tau(x^{A\alpha}) = \bar{x}^{A\alpha}
\]

such that the set of fixed points \( \tau(x) = x \), given by \( x = \bar{x} \) and denoted by \( M^{2|8}_t \subset M^{4|8}_t \), corresponds to Kleinian superspace \( \mathbb{R}^{2,2|8} \) in \( M^{4|8}_t \). Furthermore, it is important to emphasize that unlike the Euclidean involution \((5.30)\), the involution \((8.26)\) has fixed points on \( P^{3|4} \) since \( \tau^2 = 1 \) on \( P^{3|4} \). In particular, the set of fixed points \( \tau(z, \lambda) = (\bar{z}, \bar{\lambda}) \), given by \( (\bar{z}, \bar{\lambda}) = (\check{z}, \check{\lambda}) \)
and denoted by \( P^{3|4} \subset P^{3|4} \), is an open subset of real projective superspace \( \mathbb{R} P^{3|4} \) that is diffeomorphic to \( \mathbb{R} P^{3|4} \setminus \mathbb{R} P^{1|4} \) and fibred over \( S^1 \cong \mathbb{R} P^1 \subset \mathbb{C} P^1 \). Hence, the double fibration \((4.15)\) becomes

\[
\begin{array}{c}
\pi_1 \\
\downarrow \\
F^{5|8}_t \subseteq \mathbb{R}^{2,2|8} \times S^1 \cong \mathbb{R}^{2,2|8} \times \mathbb{R} P^1 \\
\downarrow \\
\pi_2 \\
\downarrow \\
M^{4|8}_t
\end{array}
\]

where the correspondence space is \( F^{5|8}_t \cong \mathbb{R}^{2,2|8} \times S^1 \cong \mathbb{R}^{2,2|8} \times \mathbb{R} P^1 \). The projection \( \pi_2 \) is again the trivial projection and \( \pi_1 : (x^{A\alpha}, \lambda_\alpha) \mapsto (z^A, \lambda_\alpha) = (x^{A\alpha} \lambda_\alpha, \lambda_\alpha) \). Furthermore, the geometric correspondence in this real setting is as follows. A point \( x \in M^{2|8}_t \cong \mathbb{R}^{2,2|8} \) corresponds to a real projective line \( \mathbb{R} P^1 \cong L_x = \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^{3|4} \), while a point \( P = (\bar{z}, \bar{\lambda}) \in P^{3|4} \) corresponds to a real 2|4-plane \( P^{4|4}_t \) in \( M^{4|8}_t \) that is parametrized by \( x^{A\alpha} = \chi^{A\alpha}_t + \mu^A \lambda_\alpha \), where \( \chi^{A\alpha}_t \) is a particular solution to \( z^A = x^{A\alpha} \lambda_\alpha \). Note that one has a similar double fibration and geometric correspondence in the non-supersymmetric setting (one simply drops the fermionic coordinates). For a discussion on the subtleties of the Penrose–Ward transform in the Kleinian setting, see e.g. [65].

Now we are in a position to give the Witten transform \((8.5), (8.23)\) a precise meaning. Let us start with \((8.5)\) and consider Penrose’s integral formula

\[
\phi(x) = -\frac{1}{2\pi i} \int [d\lambda \lambda] f_{-2}(x^\alpha \lambda_\beta, \lambda_\alpha),
\]

which was introduced in section 2.1. Note that in this real setting, one also refers to this transformation as the x-ray transform\(^27\). Recall that \( \phi \) obeys the Klein–Gordon equation

\(^27\) The x-ray transform derives its name from x-ray tomography because the x-ray transform of some given function \( f \) represents the scattering data of a tomographic scan through an inhomogeneous medium whose density is represented by \( f \). It is closely related to the Radon transform; see footnote 6.
\( \Box \phi = 0 \). From remark 7.1, we also know that the Fourier transform \( \tilde{\phi}(p) \) of \( \phi(x) \) is of the form \( \tilde{\phi}(p) = 2\pi i \delta(p^2) \phi_0(k, k) \), where \( p_{a\beta} = \tilde{k}_a k_\beta \). The pre-factor of \( 2\pi i \) has been chosen for later convenience. Then we may write

\[
2\pi i \delta(p^2) \phi_0(k, k) = \int d^4x \, e^{ip \cdot x} \phi(x) = -\frac{1}{2\pi i} \int d^4x \, e^{ip \cdot x} \oint [d\lambda \lambda] \, f_{-2}(x^{a\beta} \lambda_\beta, \lambda_a). \tag{8.30}
\]

Next let us choose some constant \( \mu_a \) with \( \mu_a = \tilde{\mu}_a \) and decompose \( x^{a\beta} \) as

\[
x^{a\beta} = \frac{\varepsilon^{a\beta} \mu^\beta - w^\beta \lambda^\beta}{[\lambda \mu]^2}. \tag{8.31}
\]

A short calculation shows that \( \varepsilon^{a\beta} = \lambda^{a\beta} \lambda^\beta \) and \( w^a = \lambda^{a\beta} \mu_\beta \). Furthermore, \( \tilde{x} = x \) is satisfied. Note that the choice of \( \mu_a \) does not really matter as will become clear momentarily. The measure \( d^4x \) is then given by

\[
d^4x = \frac{d^2z \, d^2w}{[\lambda \mu]^4}. \tag{8.32}
\]

Making use of the short-hand notation \( \langle z | p | \mu \rangle = \varepsilon^{a\beta} p_{a\beta} \mu^\beta \) etc, we can integrate out \( w^a \) to obtain

\[
2\pi i \delta(p^2) \phi_0(k, k) = -\frac{1}{2\pi i} \int d^4x \, [d\lambda \lambda] \, e^{ip \cdot x} \, f(x^{a\beta} \lambda_\beta, \lambda_a) = \frac{1}{2\pi i} \int d^2z \, [d\lambda \lambda] \exp \left( \frac{i \langle z | p | \mu \rangle - \langle w | p | \lambda \rangle}{[\lambda \mu]} \right) f_{-2}(z, \lambda).
\]

The delta function may be converted into

\[
\delta^{(2)} \left( \frac{p|\lambda}{[\lambda \mu]} \right) = \frac{[k \mu][k \lambda]}{[\lambda \mu]} \delta(p^2) \delta([k \lambda]), \tag{8.34}
\]

where \( p_{a\beta} = \tilde{k}_a k_\beta \) on the support of \( \delta(p^2) \). Next we may perform the integral over \( \lambda_a \):

\[
2\pi i \delta(p^2) \phi_0(k, k) = 2\pi i \delta(p^2) \int d^2z \, [d\lambda \lambda] \, \left( \frac{[k \mu]}{[\lambda \mu]} \right) \delta([k \lambda]) \exp \left( \frac{i \langle z | k | \mu \rangle}{[\lambda \mu]} \right) f_{-2}(z, \lambda) = 2\pi i \delta(p^2) \int d^2z \, e^{i(k \cdot z)} f_{-2}(z, k). \tag{8.35}
\]

Altogether, we arrive at

\[
\phi_0(k, k) = \int d^2z \, e^{i(k \cdot z)} f_{-2}(z, \lambda), \tag{8.36}
\]

\[
f_{-2}(z, \lambda) = \frac{1}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} e^{-i(k \cdot z)} \phi_0(k, k).
\]

Here, we have re-labelled \( k \) by \( \lambda \) as before.

The integral formulae for other helicity fields introduced in section 2.1 may be treated similarly. Recall that they are given by \((h > 0)\)

\[
\phi_{a_1 \cdots a_{2n}}(x) = -\frac{1}{2\pi i} \int [d\lambda \lambda] \lambda_{a_1} \cdots \lambda_{a_{2n}} f_{-2h-2}(x^{a\beta} \lambda_\beta, \lambda_a),
\]

\[
\phi_{a_1 \cdots a_{2n}}(x) = -\frac{1}{2\pi i} \int [d\lambda \lambda] \frac{\partial}{\partial z^{a_1}} \cdots \frac{\partial}{\partial z^{a_{2n}}} f_{-2h-2}(x^{a\beta} \lambda_\beta, \lambda_a). \tag{8.37}
\]

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According to remark 7.1, the corresponding Fourier transforms may be taken as
\[ \hat{\phi}_{a_1 \ldots a_2}(p) = 2\pi i \delta (p^2) k_{a_1} \cdots k_{a_2} \hat{\phi}_{a_2 b}(\tilde{k}, k), \]
\[ \hat{\phi}_{a_1 \ldots a_2}(p) = 2\pi (-i)^{2h-1} \delta (p^2) k_{a_1} \cdots k_{a_2} \hat{\phi}_{a_2 b}(\tilde{k}, k), \]  
where the additional factor of $(-i)^{2h}$ has been inserted for convenience. Repeating the above steps, one can show that
\[ \hat{\phi}_{a_2 b}(\tilde{k}, k) = \int d^2 z e^{i (z \cdot \tilde{k})} f_{a_2 b - 2}(z, \lambda), \]
\[ f_{a_2 b - 2}(z, \lambda) = \int \frac{d^2 \tilde{k}}{(2\pi)^2} e^{-i (\tilde{k} \cdot \lambda)} \hat{\phi}_{a_2 b}(\tilde{k}, k). \]  

In summary, we may rephrase theorem 2.2 as follows [106]:

**Theorem 8.1.** Consider a Kleinian signature spacetime $M^4_{\pm} \cong \mathbb{R}^3 \setminus \mathbb{R}^1$. Let $\hat{\phi}_{a_2 b}(\tilde{k}, k)$ be a (sufficiently well-behaved) on-shell momentum space wavefunction of helicity $\mp h (h \geq 0)$ with $p_{a\delta} = \tilde{k}_a k_\delta$. Furthermore, let $(\zeta^a, \lambda)$ be homogenous coordinates on $P^3_z$. Then one can uniquely associate with $\hat{\phi}_{a_2 b}(\tilde{k}, k)$ a function $f_{a_2 b - 2}(z, \lambda)$ of homogeneity $\mp 2h - 2$ on $P^3_z$ according to
\[ f_{a_2 b - 2}(z, \lambda) = \int \frac{d^2 \tilde{k}}{(2\pi)^2} e^{-i (\tilde{k} \cdot \lambda)} \hat{\phi}_{a_2 b}(\tilde{k}, k), \]
\[ \hat{\phi}_{a_2 b}(\tilde{k}, k) = \int d^2 z e^{i (z \cdot \tilde{k})} f_{a_2 b - 2}(z, \lambda). \]  

For the sake of brevity, we shall write
\[ f_{2h - 2}(z, \lambda) = \int \frac{d^2 \tilde{k}}{(2\pi)^2} e^{-i (\tilde{k} \cdot \lambda)} \hat{\phi}_{2b}(\tilde{k}, k), \]
\[ \hat{\phi}_{2b}(\tilde{k}, k) = \int d^2 z e^{i (z \cdot \tilde{k})} f_{2h - 2}(z, \lambda). \]  

and allow $h$ to run over $\frac{1}{2}\mathbb{Z}$.

Recall that by virtue of (7.24), the scattering amplitude $A = A(\{k_{\rho a}, k_{\rho \delta}, h_r\})$ scales as $t^{-2h}$ under $(k_{\rho a}, k_{\rho \delta}) \mapsto (t^{-1} k_{\rho a}, k_{\rho \delta})$. Also recall that the Witten transform (8.5) of $A$, which we denoted by $\mathcal{W}[A]$, is of homogeneity $2h - 2$ in $(\zeta^a, \lambda)$ on the multi-particle twistor space $\mathcal{P}^3_z$; see (8.7). Therefore, $\mathcal{W}[A]$ is obtained by performing the transformation (8.40) on each $k_{\rho a}$, that is,
\[ \mathcal{W}[A](\{\zeta^a, \lambda, h_r\}) = \int \left( \prod_{r=1}^n \frac{d^2 k_r}{(2\pi)^2} \right) e^{-i \sum_{r=1}^n (z^r \cdot k_r)} A(\{k_{\rho a}, k_{\rho \delta}, h_r\}). \]  

This explains the name ‘half Fourier transform’ since we are essentially Fourier-transforming only $k_r$ and not the ‘whole’ of $p_r$. The inverse Witten transform is just
\[ A(\{k_{\rho a}, k_{\rho \delta}, h_r\}) = \int \left( \prod_{r=1}^n d^2 z_r \right) e^{i \sum_{r=1}^n (z^r \cdot k_r)} \mathcal{W}[A](\{\zeta^a, \lambda, h_r\}). \]  

Giving a precise meaning to (8.23) is now a rather small step. As shown in the preceding section, the superamplitude $F = F(\{k_{\rho A}, k_{\rho a}\})$ scales as $t^{-2}$ under $(k_{\rho A}, k_{\rho a}) \mapsto (t^{-1} k_{\rho A}, t k_{\rho a})$. Furthermore, the measure $d^{2h} k_r := d^2 k_r d^2 \psi_r$ scales as $t^2$ because of the definition of the Berezin integral over fermionic coordinates. If we define
\[ \langle z, k_r \rangle := z^a \tilde{k}_{\rho a} = z^a \psi^r \tilde{k}_{\rho r} + \eta_{\rho r} \psi^r, \]  

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\[
\mathcal{F}[(z^A, \lambda_{ra})] = \int \left( \prod_{r=1}^{n} \frac{d^{d}k_{r}}{(2\pi)^{d}} \right) e^{-i \sum_{r=1}^{n} (\xi_{A}^{r} - \chi^{A}k_{rA})} \mathcal{F}(\{k_{rA}, k_{ra}\})
\]

(8.43)

is of homogeneity zero on the multi-particle supertwistor space \(\mathbb{P}_{\mathbb{R}|\mathbb{R}A}^{4n}\). This is essentially the implication of (8.25).

Finally, we would like to address the relationship between the twistor space expression (5.40) and the on-shell super wavefunction (8.10). In section 5.2 we worked in the Dolbeault picture where everything is encoded in a differential (0, 1)-form (5.40). Upon imposing the linearized field equations \(\delta A^{0,1} = 0\), one can use the Penrose transform to get the whole \(N = 4\) multiplet on four-dimensional spacetime. Now let \(f_0 = \tilde{f}_0(z, \lambda)\) be the \(\check{C}\)ech representative corresponding to \(A^{0,1}\), that is, \(f_0\) is a Lie algebra-valued element of \(H^1(P^3, \mathcal{O})\). Like \(A^{0,1}\), \(f_0\) can also be expanded in powers of \(\eta_i\):

\[
f_0 = \Lambda + \eta_1 x^1 + \frac{1}{2} \eta_1 \eta_1 \Phi_{11} + \frac{1}{3!} \eta_1 \eta_1 \eta_1 \Phi_{11} e_{ijkl} x^i + \frac{1}{4!} \eta_1 \eta_1 \eta_1 \eta_1 \Phi_{ijkl} G,
\]

(8.44)

where the component fields are Lie algebra-valued elements of \(H^1(P^3, \mathcal{O}(\mathcal{O}_{\mathbb{C}} (-2h - 2)^i})\) for \(h = -1, \ldots, 1\). Upon imposing Kleinian reality conditions, the map between \(f_0\) and \(\tilde{\Phi}_{-2}\) is simply the \(N = 4\) supersymmetric version of theorem 8.1 [106]:

\[
f_0(z, \lambda) = \int \frac{d^{d}k}{(2\pi)^{2}} e^{-i k_{\alpha} \tilde{\Phi}_{-2}(\tilde{k}, k)},
\]

\[
\tilde{\Phi}_{-2}(\tilde{k}, k) = \int d^{d}z e^{i k_{\alpha} z} f_0(z, \lambda).
\]

(8.45)

8.4. MHV superamplitudes on supertwistor space

Having given a precise meaning to the Witten transform, we are now in a position to explore the properties of the MHV amplitude (8.1c) and its \(N = 4\) supersymmetric extension (8.17) when re-interpreted in terms of twistors. Since (8.1c) is just a special case of (8.17), we shall focus on the superamplitude only.

In order to transform (8.17) to \(\mathbb{P}_{\mathbb{R}|\mathbb{R}A}^{3n|4n}\), let us re-write the overall delta function as

\[
\delta^{(4|8)} \left( \sum_{r=1}^{n} \hat{k}_{rA} k_{ra} \right) = \int \frac{d^{4|8} x}{(2\pi)^{4}} \exp \left( \sum_{n=1}^{1} x^{Aa} \hat{k}_{rA} k_{ra} \right),
\]

(8.46)

where \(x\) is some dummy variable to be interpreted. Therefore, upon substituting the superamplitude (8.17) into (8.41a), we find

\[
\mathcal{F} \left[ f_{0,n}^{\text{MHV}} \right] = g_{\text{YM}}^{n-2} \int d^{4|8} x \int \left( \prod_{r=1}^{n} \frac{d^{d}k_{r}}{(2\pi)^{d}} \right) e^{-i \sum_{r=1}^{n} (\xi_{A}^{r} - \chi^{A}k_{rA})k_{rA}} \frac{1}{[12][23] \cdots [n]}
\]

(8.47)

Let us interpret this result. The MHV superamplitude is supported on solutions to the equations

\[
z_{r}^{A} = x^{Aa} \lambda_{ra} \quad \text{for} \quad r = 1, \ldots, n
\]

(8.48)

on \(\mathbb{P}_{\mathbb{R}|\mathbb{R}A}^{3n|4n}\), where we have re-labelled \(k_{ra}\) by \(\lambda_{ra}\). One may rephrase this by saying that the MHV superamplitude ‘localizes’ on solutions to these equations. We have encountered relation (8.48) on numerous occasions. For each \(x^{Aa}\), the incidence relation \(z^{A} = x^{Aa} \lambda_{a}\) defines a
Figure 2. The MHV (super)amplitudes localize on projective lines in (super)twistor space. Here, a five-gluon MHV amplitude is depicted as an example.

real projective line inside supertwistor space. Then (8.47) says that the MHV superamplitude vanishes unless equations (8.48) are satisfied, that is, unless there is some real projective line inside supertwistor space which is determined by $x^{Aa}$ via the incidence relation and which contains all $n$ points $(z^A_r, \lambda^r_\alpha)$. Put differently, the supertwistors $Z^I_r := (z^A_r, \lambda^r_\alpha)$ are collinear (see figure 2). The $x$-integral in (8.47) is then to be understood as the integral over the moduli space of this projective line. It is important to stress that the $x^{Aa}$ should not be confused with the particle coordinates in superspacetime even though they have the same mass dimension as the coordinates in superspacetime.

Finally, let us perform the moduli space integral (8.47) to obtain the explicit expression of the Witten transform of the MHV superamplitude. To this end, we use the formula

$$\delta^{(m|n)}(f(x)) = \sum_{\{x^r\mid f(x^r) = 0\}} \frac{\delta^{(m|n)}(x - x^r)}{|\text{sdet} f'(x^r)|},$$

where ‘sdet’ is the superdeterminant (4.9) and re-write two of the delta functions in (8.47) according to

$$\delta^{(2|4)}(z^A_r - x^{Aa} k_{ra}) \delta^{(2|4)}(z^A_s - x^{Aa} k_{sa}) = [rs]^2 \delta^{(4|8)}(x^{Aa} - \frac{z^A_r \lambda^a_{r\alpha} - z^A_s \lambda^a_{s\alpha}}{[rs]}).$$

If we choose $r = 1$ and $s = 2$, for instance, we end up with

$$\mathcal{C}[F^{\text{MHV}}_{0,n}] = \delta^{n-2}_{\text{YM}} \frac{[12][23][n]}{[12][23][n]} \delta^{(2|4)}(z^A_1 [2r] + z^A_s [r1] + z^A_3 [12]).$$

Of particular interest will be the $n = 3$ case, which we record here:

$$\mathcal{C}[F^{\text{MHV}}_{0,3}] = \delta^{\text{YM}} \frac{[12][23][31]}{[12][23][31]} \delta^{(2|4)}(z^A_1 [23] + z^A_3 [31] + z^A_1 [12]).$$

Summarizing, we see that MHV superamplitudes localize on projective lines in supertwistor space. These are special instances of algebraic curves. In section 9.5, we shall see that general superamplitudes also localize on algebraic curves in supertwistor space. These localization properties of scattering amplitudes are among the key results of [53]. This concludes our discussion about MHV amplitudes.

Remark 8.2. Here, we have worked in a split signature spacetime. However, one may complexify and show that in a complex setting, the MHV (super)amplitudes also localize
on degree-1 algebraic curves in (super)twistor space [53]. Put differently, the MHV (super)amplitudes are supported on complex projective lines \( \mathbb{C}P^1 \) in complex (super)twistor space.

9. MHV formalism

In the preceding sections, we have extensively discussed (colour-stripped) tree-level MHV amplitudes in both pure Yang–Mills theory and \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory. In the remainder of these lecture notes we would like to extend this discussion beyond the MHV case and construct general tree-level scattering amplitudes.

One of the striking developments that came out of Witten’s twistor string theory is the MHV formalism developed by Cachazo, Svrˇcek and Witten [114]. Recall that in the usual Feynman diagram expansion, the interaction vertices are obtained from the (local) Lagrangian description of the theory. In particular, interactions are supported on points in spacetime. In the first part of these lecture notes we have seen that points in spacetime correspond to projective lines in twistor space via the incidence relation (1.16). Furthermore, as explained in section 8.4, MHV amplitudes localize on projective lines in twistor space. So one can think of these amplitudes as representing, in some sense, a generalization of local interaction vertices. To take this analogy further, one can try to build more complicated amplitudes from MHV amplitudes by gluing together the latter in an appropriate way. As we shall see momentarily, the gluing is done with the help of simple scalar propagators. In order for this to work, however, one has to continue the MHV amplitudes off-shell. The resulting diagrammatic expansion of scattering amplitudes is referred to as the MHV diagram expansion.

We begin by studying the MHV formalism in the pure Yang–Mills setting and then generalize it to the \( \mathcal{N} = 4 \) case. Before deriving this formalism from twistor space, however, we first set up the basics and discuss some examples.

9.1. Cachazo–Svrˇcek–Witten rules

As indicated, we must continue the MHV amplitudes off-shell in order to use them as vertices for general amplitudes. To this end, consider a generic momentum co-vector \( p \) with \( p^2 \neq 0 \). On general grounds, it can always be decomposed into the sum of two null-momenta. We shall adopt the following decomposition [115, 116]:

\[
p = q + t\ell,
\]

where \( q^2 = 0 \), and \( \ell \) is fixed but arbitrary with \( \ell^2 = 0 \). Here, \( t \) is some complex number which is determined as a function of \( p \) according to

\[
t = \frac{p^2}{2(p \cdot \ell)}.
\]

Since \( q \) and \( \ell \) are null, we may use co-spinors to represent them as \( q_\alpha = \tilde{k}_\alpha k_\beta \) and \( \ell_\alpha = \tilde{\xi}_\alpha \xi_\beta \).

It then follows that

\[
\tilde{k}_\alpha = \frac{p_{\alpha\beta}\xi^\beta}{[k\xi]} \quad \text{and} \quad k_\alpha = \frac{\tilde{\xi}_\beta p_{\beta\alpha}}{(k\xi)}
\]

or in our above short-hand notation,

\[
\tilde{k} = \frac{p[\xi]}{[k\xi]} \quad \text{and} \quad k = \frac{\tilde{\xi}p}{(k\xi)}.
\]
These equations define our off-shell continuation (note that $p^2 \neq 0$ in the above formulae). The co-spinors $k$ corresponding to the legs in the MHV amplitudes that are going to be taken off-shell are represented by (9.4). Note that the denominators in (9.4) turn out to be irrelevant for our applications since the expressions we are dealing with are homogeneous in the $k$ that are continued off-shell (see also below). Thus, we may discard the denominators and simply write

\[ \tilde{k} = p|\xi\] and \[k = \langle \tilde{\xi}|p. \] (9.5)

This is our off-shell formulation of MHV amplitudes. When continued off-shell, we shall often refer to MHV amplitudes as MHV vertices. We complete the definition of the MHV or Cachazo–Svrček–Witten rules, by taking $1/p^2$ for the propagator connecting the MHV vertices.

Cachazo–Svrček–Witten rules. The rules for computing colour-stripped tree-level scattering amplitudes by gluing together MHV amplitudes are as follows.

(i) Use the MHV amplitudes as vertices.
(ii) For each leg of a vertex that joins with a propagator carrying momentum $p$, define its corresponding co-spinor by $k = \langle \tilde{\xi}|p$, where $\tilde{\xi}$ is some fixed spinor.
(iii) Glue the vertices together using scalar propagators $1/p^2$, where $p$ is the momentum flowing between the vertices. The two ends of any propagator must have opposite helicity labels, that is, plus at one end and minus at the other.

The reader may wonder whether the amplitudes constructed this way actually depend on the choice reference spinor $\tilde{\xi}$. In fact, it can be shown that this is not the case [114] and the resulting overall amplitudes are indeed (Lorentz) covariant. This will also become transparent in section 9.3, where we deduce the above rules from the twistor action (6.6a): $\tilde{\xi}$ will arise as an ingredient of a certain gauge condition. BRST invariance in turn guarantees that the overall amplitudes are independent of $\tilde{\xi}$.

Note that a direct proof of these rules was given by Risager in [117]. See figure 3.
9.2. Examples

The Cachazo–Svrˇcek–Witten rules for joining together the MHV amplitudes are probably best illustrated with an example.

A tree-level diagram with \( n_V \) MHV vertices has \( 2n_V \) positive-helicity legs, \( n_V - 1 \) of which are connected by propagators. According to rule (iii), each propagator must subsume precisely one positive-helicity leg and therefore, we are left with \( 2n_V - (n_V - 1) = n_V + 1 \) external positive helicities. Put differently, if we wish to compute a scattering amplitude with \( q \) positive-helicity gluons, we will need \( n_V = q - 1 \) MHV vertices. Clearly, the MHV case is \( q = 2 \) and thus \( n_V = 1 \). In addition, this construction makes the vanishing of the amplitudes (8.1a) and (8.1b) (both for the upper sign choice) manifest. In the former case, \( n_V = -1 \) while in the latter case \( n_V = 0 \).

The first non-trivial examples are obtained for \( q = 3 \) which are the NMHV amplitudes. We shall denote them by \( A^{NMHV}_{0,n} (r^*, s^*, t^*) \) or simply by \( A^{NMHV}_{0,n} \) if there is no confusion.

Thus, we have schematically

\[
A^{NMHV}_{0,n} = \sum_{\{\Gamma_L, \Gamma_R\} | n_L + n_R = n + 2} A^{MHV}_{0,n_L} (\Gamma_L) \frac{1}{p_{L,R}^2} A^{MHV}_{0,n_R} (\Gamma_R)
\]

where the sum is taken over all possible MHV vertices \( \Gamma_{L,R} \) contributing to the amplitude such that \( n_L + n_R = n + 2 \) and \( n_{L,R} \geq 3 \). One may check that there are a total of \( n(n-3)/2 \) distinct diagrams contributing to this sum (see e.g. [118, 119]).

The simplest example of an NMHV amplitude is the four-gluon amplitude \( A^{NMHV}_{0,4} (r^*, s^*, t^*) = A_{0,4} (1^-, 2^*, 3^*, 4^*) \). This amplitude is of type (8.1b) for the lower sign choice and hence, it should vanish. Let us demonstrate this by using the above rules. The relevant diagrams contributing to this amplitude are depicted in figure 4. For each diagram we should write down the MHV amplitudes corresponding to each vertex and join them together with the relevant propagator, remembering to use the above off-shell continuation to deal with the co-spinors associated with the internal lines. Let us start with diagram (a). We have

\[
k = (\tilde{\xi} | p)
\]

and therefore

\[
(a) = \frac{[2k]^4}{[k1][12][2k]} \frac{[34]^4}{p^2 [4k][k3][34]} = \frac{[2k]^3}{[k1][12]} \frac{1}{p^2 [4k][k3]},
\]

where

\[
p = -p_1 - p_2 = p_3 + p_4
\]

due to momentum conservation. Note that \( p_r^2 = 0 \) and therefore, \( p_{ra\tilde{b}} = \tilde{k}_r a_k \delta_{a\tilde{b}} \) for \( r = 1, \ldots, 4 \). Next we define \( e_r := (\tilde{\xi} | k_r) \). Hence,

\[
k = -e_1 k_1 - e_2 k_2 = e_3 k_3 + e_4 k_4.
\]
which follows from (9.8) upon contracting with $\hat{\xi}$. This last equation can in turn be used to compute the quantities $[kr]$ which can then be substituted into (9.7). Noting that $p^2 = -2p_1 \cdot p_2$, we eventually arrive at

$$(a) = -\frac{c_1^3}{4c_2c_3c_4} \langle 12 \rangle \langle 13 \rangle (9.10)$$

Going through the same procedure for the second diagram in figure 4 gives

$$(b) = \frac{c_1^3}{4c_2c_3c_4} \langle 23 \rangle (9.11)$$

and therefore,

$$(a) + (b) = -\frac{c_1^3}{4c_2c_3c_4} \left( \langle 34 \rangle \langle 12 \rangle - \langle 23 \rangle \langle 14 \rangle \right). (9.12)$$

Momentum conservation is $\sum r_pr_r = \sum \hat{k}_r k_r = 0$ and upon contracting this equation with $k_3$, one realizes that $\langle 34 \rangle \langle 14 \rangle = \langle 23 \rangle \langle 12 \rangle$. Altogether, $(a) + (b) = 0$ as expected. For all the NMHV amplitudes, see e.g. [115, 118–120].

**Exercise 9.1.** Derive the five-gluon NMHV amplitude

$$A_{0,5}^{NMHV}(2^+, 4^+, 5^+) = A_{0,5}(1^-, 2^+, 3^-, 4^+, 5^+)$$

using the above rules. You should find $5(5 - 3)/2 = 5$ diagrams that contribute to the amplitude. Note that this amplitude is actually an MHV amplitude and according to (8.1d), you should find $\langle 13 \rangle^4 / (\langle 12 \rangle \langle 23 \rangle \cdots \langle 51 \rangle)$.

The MHV diagram expansion has been implemented both for amplitudes with more external gluons and amplitudes with more positive helicities. In both cases the complexity grows, but the number of diagrams one has to consider follows a power growth for large $n$ (e.g. $n^2$ for the NMHV diagrams [114]) which is a marked improvement on the factorial growth of the number of Feynman diagrams needed to compute the same process; see table 4.

**9.3. MHV diagrams from twistor space**

Let us now examine the Cachazo–Svrček–Witten rules in the context of the twistor action (6.6a). The canonical way of deriving a perturbative or diagrammatic expansion of some theory that is described by some action functional is to fix the gauge symmetries first and then to derive the corresponding propagators and vertices. As we shall see momentarily, there
exists a gauge for the twistor space action (6.6a) that leads directly to the Cachazo–Svrček–Witten rules, that is, to the perturbative expansion of Yang–Mills theory in terms of MHV diagrams. For concreteness, we again assume that Euclidean reality conditions have been imposed. The subsequent discussion follows [66, 67] closely (see also [121, 122]). For a treatment of perturbative Chern–Simons theory, see e.g. [123, 124].

As we wish to describe pure Yang–Mills theory here, we assume that only $A$ and $G$ are present in the superfield expansion (5.40). Thus,

$$A^{0,1} = A + \frac{1}{2} \eta_i \eta_j \eta_k \eta_l \epsilon^{ijkl} G = A + (\eta)^4 G,$$

(9.13)

where we have used the notation of (6.14). As before, we shall use the circles to denote the component fields. Upon substituting the above expansion into the holomorphic Chern–Simons part of action (6.6a), we arrive at

$$S = \int \Omega_0 \wedge \text{tr}[G \wedge (\bar{\partial}A + A \wedge A)]$$

(9.14)

once the fermionic coordinates are integrated over. Note that we have re-scaled the measure $\Omega_0$ in order to get rid of certain numerical pre-factors which are not essential for the discussion that follows.

Furthermore, using

$$A = \bar{e}^0 A_0 + \bar{e}^a A_a \quad \text{and} \quad G = \bar{e}^0 G_0 + \bar{e}^a G_a,$$

(9.15)

we may perform an analysis similar to (6.12) to find the contribution coming from (6.6b):

$$\text{tr} \log \nabla^{0,1}|_{L_x} = \text{tr} \log(\bar{\partial} + A + (\eta)^4 G)|_{L_x}$$

$$= \text{tr} \left[ \log(\bar{\partial} + A)|_{L_x} + \log[1 + (\eta)^4(\bar{\partial} + A)^{-1} G]|_{L_x} \right]$$

$$= \text{tr} \left\{ \log(\bar{\partial} + A)|_{L_x} + \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \right\}$$

$$\left( \int_{L_x} \prod_{j=1}^{r} K(\lambda_j) (\eta_j)^4 G_A(\lambda_{j-1}, \lambda_j) \bar{G}_A(\lambda_j) \right).$$

(9.16)

Here, $G_A(\lambda, \lambda')$ is Green’s function of the ($\bar{\partial} + A$)-operator on the projective line $L_x$:

$$(\bar{\partial} + A)^{-1} \omega(\lambda)|_{L_x} = \int_{L_x} K(\lambda') G_A(\lambda, \lambda') \omega(\lambda') \quad \text{for} \quad \omega = \bar{e}^0 \omega_0 + \bar{e}^a \omega_a,$$

(9.17)

and $K(\lambda)$ is the Kähler form on $L_x$ which was introduced in remark 5.5. Now we may employ Nair’s lemma (6.14) and integrate over $d^4\eta$ in (9.16) to obtain the following form of action (6.6b):

$$S_r = -\frac{1}{2} \int d^4x \int_{L_x} K(\lambda_1) K(\lambda_2) [12]^4 \text{tr}[G_A(\lambda_2, \lambda_1) \bar{G}_A(\lambda_1) G_A(\lambda_1, \lambda_2) \bar{G}_A(\lambda_2)].$$

(9.18)

Altogether, the twistor space action (6.6a) reduces to the Mason twistor action [66]:

$$S_{\text{tot}} = \int \Omega_0 \wedge \text{tr} \left[ G \wedge (\bar{\partial}A + A \wedge A) \right]$$

$$= \frac{1}{2} \int d^4x \int_{L_x} K(\lambda_1) K(\lambda_2) [12]^4 \text{tr}[G_A(\lambda_2, \lambda_1) \bar{G}_A(\lambda_1) G_A(\lambda_1, \lambda_2) \bar{G}_A(\lambda_2)],$$

(9.19)

of pure Yang–Mills theory. Finally, we need to find an explicit expression for Green’s function $G_A(\lambda, \lambda')$. Obviously, for $A_0 = 0$, $G_A(\lambda, \lambda')$ coincides with Green’s function $G(\lambda, \lambda')$ for the $\bar{\partial}|_{L_x}$-operator as given in (6.10). For $A_0 \neq 0$, we have the expansion

$$(\bar{\partial} + A)^{-1}|_{L_x} = \bar{\partial}^{-1}|_{L_x} - \bar{\partial}^{-1} \bar{A} \bar{\partial}^{-1}|_{L_x} + \bar{\partial}^{-1} \bar{A} \bar{\partial}^{-1} \bar{A} \bar{\partial}^{-1}|_{L_x} + \cdots.$$
With the help of (6.10) and (9.17), this can be re-written according to

\[ G_A(\lambda, \lambda') = G(\lambda, \lambda') + \sum_{r=1}^{\infty} (-1)^r \int_{L_r} \left( \prod_{s=1}^{r} K(\lambda_s, \lambda_{s-1}) A_0(\lambda_s) \right) G(\lambda_r, \lambda'), \]

(9.21)

with \( \lambda_0 \equiv \lambda \). This expression may now be substituted into the Mason action (9.19) to obtain

\[ S_{\text{tot}} = \int \Omega_0 \wedge \text{tr} \left[ G \wedge (\partial A + A \wedge A) \right] \]

\[ - \frac{1}{2} \epsilon \int d^4x \sum_{r=2}^{\infty} (-1)^r g_{\text{YM}}^2 \int_{L_r} \left( \prod_{s=1}^{r} K(\lambda_s, \lambda_{s-1}) G(\lambda_r, \lambda'), \overline{G}(\lambda_s, \lambda_{s-1}) \right) \]

\[ \times \sum_{r=2}^{\infty} [1/r^4 \text{tr} \left\{ G_0(\lambda_1) A_0(\lambda_2) \cdots A_0(\lambda_{r-1}) \overline{G}_0(\lambda_1) \overline{A}_0(\lambda_{r+1}) \cdots \overline{A}_0(\lambda_r) \right\}]. \]

(9.22)

Note that the infinite sum strongly resembles a sum of MHV vertices (recall that \( G \) carries helicity +1 while \( A \) carries helicity −1). Also note that (9.22) is invariant under the following gauge transformations:

\[ \delta A = \partial \alpha + [A, \alpha] \quad \text{and} \quad \delta \overline{G} = \partial \beta + [g, \beta] + [G, \alpha], \]

(9.23)

where \( \alpha \) and \( \beta \) are smooth functions of homogeneities 0 and −4, respectively. Note that these transformations are an immediate consequence of (5.21).

Next we would like to restore the dependence on the Yang–Mills coupling constant \( g_{\text{YM}} \) in \( S_{\text{tot}} \) that is compatible with the perturbation theory we have discussed so far. Remember from our discussion in section 6.1 that in this perturbative context, the parameter \( \epsilon \) is identified with \( g_{\text{YM}}^2 \). Rescaling \( A \) as \( A \mapsto g_{\text{YM}} A \) and \( G \) as \( G \mapsto g_{\text{YM}}^{-1} G \), respectively, we find

\[ S_{\text{tot}} = \int \Omega_0 \wedge \text{tr} \left[ G \wedge (\partial A + A \wedge A) \right] \]

\[ - \frac{1}{2} \epsilon \int d^4x \sum_{r=2}^{\infty} (-1)^r g_{\text{YM}}^2 \int_{L_r} \left( \prod_{s=1}^{r} K(\lambda_s, \lambda_{s-1}) G(\lambda_r, \lambda'), \overline{G}(\lambda_s, \lambda_{s-1}) \right) \]

\[ \times \sum_{r=2}^{\infty} [1/r^4 \text{tr} \left\{ G_0(\lambda_1) A_0(\lambda_2) \cdots A_0(\lambda_{r-1}) \overline{G}_0(\lambda_1) \overline{A}_0(\lambda_{r+1}) \cdots \overline{A}_0(\lambda_r) \right\}]. \]

(9.24)

Let us now decompose this expression as

\[ S_{\text{tot}} = S_{\text{kin}} + S_{\text{int}}, \]

(9.25a)

where

\[ S_{\text{kin}} := \int \Omega_0 \wedge \text{tr} \left[ G \wedge \overline{G} \right] \]

\[ S_{\text{int}} := g_{\text{YM}} \int \Omega_0 \wedge \text{tr} \left[ G \wedge A \wedge A \right] \]

\[ - \frac{1}{2} \epsilon \int d^4x \sum_{r=2}^{\infty} (-1)^r g_{\text{YM}}^2 \int_{L_r} \left( \prod_{s=1}^{r} K(\lambda_s, \lambda_{s-1}) G(\lambda_r, \lambda'), \overline{G}(\lambda_s, \lambda_{s-1}) \right) \]

\[ \times \sum_{r=2}^{\infty} [1/r^4 \text{tr} \left\{ G_0(\lambda_1) A_0(\lambda_2) \cdots A_0(\lambda_{r-1}) \overline{G}_0(\lambda_1) \overline{A}_0(\lambda_{r+1}) \cdots \overline{A}_0(\lambda_r) \right\}]. \]

(9.25b)

Note that the first term in the infinite sum of \( S_{\text{int}} \) contains a term that is quadratic in \( G \) and that involves no \( A \) fields. We are always free to treat this term either as a vertex or as part of the kinetic energy, but as our re-writing already suggests, we shall do the former.
After some algebraic manipulations, the interaction part $S_{\text{int}}$ can be further re-written as

$$S_{\text{int}} = g_{\text{YM}} \int \Omega_0 \wedge \tr \left[ G \lands A \lands A \right] - \int d^4x \sum_{r=2}^{\infty} \frac{(-1)^r}{r!} g_{\text{YM}}^{r-2} \sum_{\sigma \in \mathbb{S}_r / \mathbb{Z}_r} \int L_i \left( \prod_{i=1}^{r} K(\lambda_{\sigma(i)}) G(\lambda_{\sigma(i-1)}, \lambda_{\sigma(i)}) \right) \times \sum_{s, t} |\sigma(s)\sigma(t)|^4 \tr \left[ \partial_0(\lambda_{\sigma(1)}) \cdots \partial_0(\lambda_{\sigma(r-1)}) G_0(\lambda_{\sigma(r)}) \right] \cdots \partial_0(\lambda_{\sigma(r+1)}) \cdots \partial_0(\lambda_{\sigma(r)}) \right],$$

(9.26)

Here, $S_r$ is the permutation group of degree $r$ and $Z_r$ is the group of cyclic permutations of order $r$.

In order to proceed further, we need to fix the gauge symmetries; see (9.23). In sections 5.2 and 6.2 we partially fixed the gauge symmetries by imposing the spacetime gauge (5.41). This partial gauge-fixing turned out to be suitable for deriving the usual spacetime action from the twistor action. However, we may certainly impose other gauges and the one that will eventually yield the Cachazo–Svrček–Witten rules is the following:

$$\tilde{\xi}_a^\alpha V_\alpha A = 0 = \tilde{\xi}_a^\alpha \tilde{V}_\alpha G$$

(9.27)

for some fixed but arbitrary spinors $\tilde{\xi}_1$ and $\tilde{\xi}_2$. This is an axial gauge, so the corresponding ghost terms will decouple. Note that this gauge choice differs somewhat from the one of [66, 67]28 since here we are using two a priori different spinors $\tilde{\xi}_1$ and $\tilde{\xi}_2$. Any solution to these gauge constraints is of the form

$$\Delta_\alpha = \tilde{\xi}_\alpha A \quad \text{and} \quad G_{\alpha} = \tilde{\xi}_\alpha G$$

(9.28)

for some smooth functions $A$ and $G$ of homogeneities 1 and $-3$ (remember that $\Delta_\alpha$ and $G_{\alpha}$ are of these homogeneities).

Having chosen gauge-fixing conditions, we are now in a position to derive the propagator or Green’s function of the $\partial$-operator on twistor space $P^3$. Upon using (9.15), $S_{\text{kin}}$ is explicitly given by

$$S_{\text{kin}} = \int \frac{\Omega_0 \lands \hat{\Omega}_0}{[\lambda \bar{\lambda}]^4} \tr(G^\alpha (V_0 A_\alpha - \tilde{V}_\alpha A)) + G_0 \tilde{V}_\alpha A^\alpha).$$

(9.29)

On solutions (9.28) to the gauge constraints (9.27), this becomes

$$S_{\text{kin, gf}} = \int d^4x \frac{K(\lambda)}{(2\pi)^4} K(\lambda) \tr \left[ (G_0, G) \left( \frac{0}{-\tilde{\xi}_1^\alpha \tilde{V}_\alpha} \right) \right]$$

$$= \int d^4p \frac{K(\lambda)}{(2\pi)^4} K(\lambda) \tr \left[ (\tilde{G}_0, \tilde{G}) (p) \left( \frac{0}{-i\tilde{\xi}_1^\alpha p|\lambda|} \right) \bar{\delta}(\lambda) \right] (-p),$$

(9.30)

where in the first step we have used (5.52) to re-write the measure in (9.29) while in the second step we have performed a Fourier transform on the $x$-coordinates.

Next it is useful to introduce certain differential $(0, 1)$-form-valued weighted delta functions of spinor products [67, 114]

$$\delta_{(m)}(\lambda_1 \lambda_2) := \left( \frac{[\lambda_1 \xi_1]}{[\lambda_2 \xi_1]} \right)^{m+1} \bar{\delta}(\lambda_2) \frac{1}{[\lambda_1 \lambda_2]}.$$  

(9.31)
where ξ is some constant spinor. Using (6.9a), this is explicitly
\[
\bar{\delta}_{(m)}(|\lambda_1\lambda_2]) = \frac{[d\lambda_2\hat{\lambda}_2]}{[\lambda_2\hat{\lambda}_2]^2} (-2\pi)[\lambda_2\hat{\lambda}_2]^2 \left[ \frac{[\xi_\lambda_1]}{[\xi_\lambda_2]} \right]^{m+1} \left[ \frac{[\xi_\hat{\lambda}_1]}{[\xi_\lambda_2]} \right] \delta^{(2)}([\lambda_1\lambda_2], [\hat{\lambda}_1\hat{\lambda}_2])
\]
\[
= \bar{\delta}^0(\lambda_2) \bar{\delta}^{(m)}_{(m)}(|\lambda_1\lambda_2]).
\] (9.32)

This expression is independent of ξ since on support of the delta function we have λ_1 ∝ λ_2. Note that under a rescaling λ_1,2 → t_1,2λ_1,2 for t_1,2 ∈ C \ {0}, we have
\[
\bar{\delta}_{(m)}(|\lambda_1\lambda_2]) \mapsto t_1^{m+2} t_2^{-m-2} \bar{\delta}_{(m)}(|\lambda_1\lambda_2]) \quad \text{and} \quad \bar{\delta}^{(m)}_{(m)}(|\lambda_1\lambda_2]) \mapsto t_1^{m} t_2^{-m} \bar{\delta}^{(m)}_{(m)}(|\lambda_1\lambda_2]).
\] (9.33)

Using (9.31), S_{kin, gf} given by (9.30) becomes
\[
S_{kin, gf} = \int \frac{d^4p}{(2\pi)^4} K(\lambda_1) K(\lambda_2) \times \text{tr} \left( \begin{pmatrix} \bar{G}_i \bar{G} \end{pmatrix} (p, \lambda_1) \begin{pmatrix} 0 & \mathcal{K}_1(\lambda_1, \lambda_2) & \mathcal{K}_1(\lambda_1, \lambda_2) \\ \mathcal{K}_2(\lambda_1, \lambda_2) & \mathcal{K}_3(\lambda_1, \lambda_2) & \bar{A}_0 \bar{A} \end{pmatrix} (-p, \lambda_2) \right).
\] (9.34a)

where
\[
\mathcal{K}(\lambda_1, \lambda_2) := \begin{pmatrix} 0 & \mathcal{K}_1(\lambda_1, \lambda_2) \\ \mathcal{K}_2(\lambda_1, \lambda_2) & \mathcal{K}_3(\lambda_1, \lambda_2) \end{pmatrix}
\]
\[
:= -\frac{1}{2\pi} \begin{pmatrix} 0 & \mathcal{K}_3(\lambda_1, \lambda_2) \xi_1 \xi_2 p | \lambda_2 \rangle \\ \mathcal{K}_1(\lambda_1, \lambda_2) \xi_1 \xi_2 p | \lambda_2 \rangle & \mathcal{K}_2(\lambda_1, \lambda_2) \xi_1 \xi_2 p | \lambda_2 \rangle \end{pmatrix}.
\] (9.34b)

Green’s function or propagator
\[
\bar{G}(\lambda_1, \lambda_2) := \begin{pmatrix} \bar{G}_1(\lambda_1, \lambda_2) & \bar{G}_2(\lambda_1, \lambda_2) \\ \bar{G}_3(\lambda_1, \lambda_2) & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} \bar{A}_0(p, \lambda_1) \bar{B}_0(-p, \lambda_2) & \bar{A}_0(p, \lambda_1) \bar{B}(-p, \lambda_2) \\ \langle A(p, \lambda_1) \bar{B}_0(-p, \lambda_2) & 0 \end{pmatrix}
\] (9.35)

is determined by
\[
\int K(\lambda_3) \mathcal{K}(\lambda_1, \lambda_3) \bar{G}(\lambda_3, \lambda_2) = \mathcal{I}(\lambda_1, \lambda_2),
\] (9.36)

where \(\mathcal{I}(\lambda_1, \lambda_2)\) is the identity
\[
\mathcal{I}(\lambda_1, \lambda_2) := -\frac{1}{2\pi} \begin{pmatrix} \mathcal{K}_3(\lambda_1, \lambda_2) & 0 \\ 0 & \mathcal{K}_2(\lambda_1, \lambda_2) \end{pmatrix}.
\] (9.37)

A short calculation shows that
\[
\bar{G}_1(\lambda_1, \lambda_2) = \frac{1}{2\pi} \langle \xi_1 | p | \lambda_1 \rangle \langle \bar{V}_0(\lambda_1) \frac{1}{\langle \xi_1 | p | \lambda_1 \rangle} \mathcal{K}_3(\lambda_1, \lambda_2),
\]
\[
\bar{G}_2(\lambda_1, \lambda_2) = \frac{1}{2\pi i} \langle \xi_2 | p | \lambda_2 \rangle \mathcal{K}_3(\lambda_1, \lambda_2),
\]
\[
\bar{G}_3(\lambda_1, \lambda_2) = -\frac{1}{2\pi i} \langle \xi_1 | p | \lambda_1 \rangle \mathcal{K}_3(\lambda_1, \lambda_2).
\] (9.38)
The propagator $G_t$ may be simplified further to give

$$G_t(\lambda_1, \lambda_2) = \frac{1}{2\pi} \langle \xi_1 | p | \lambda_1 \rangle \delta^0_{(2)}(\xi_1) \delta^0_{(2)}(\xi_1 | p | \lambda_2)$$

$$= -\frac{4}{2\pi} \frac{1}{p^2} \delta^0_{(-2)}(\xi_1 | p | \lambda_1 \lambda_2) \delta^0_{(2)}(\xi_1 | p | \lambda_2), \quad (9.39)$$

where in the first step we have inserted definition (9.32) while in the second step we have used the fact that $\lambda_1 \propto \langle \xi_1 | p | \lambda_1 \rangle$ and the identity $p_\alpha p^{\alpha} = \frac{1}{2} p^2 \delta_{\alpha}^{\beta}$. Altogether, we arrive at

$$G(\lambda_1, \lambda_2) = \frac{1}{2\pi} \left( \frac{4}{p^2} \delta^0_{(-2)}(\xi_1 | p | \lambda_1 \lambda_2) \delta^0_{(2)}(\xi_1) \right) \left( \frac{1}{i(p_\alpha p^{\alpha})} \delta^0_{(-3)}(\xi_1 | p | \lambda_2) \right) \right). \quad (9.40)$$

A particularly simple form of the propagator is achieved for the choice $\xi \rightarrow \hat{\xi}$ and $\bar{\xi} \rightarrow \xi$. We shall refer to this gauge as the Cachazo–Svrček–Witten gauge, and the reason for its name will become apparent momentarily. Upon multiplying (9.35), (9.40) by the appropriate differential forms $\delta^0$ and $\bar{\delta}^0$ and by $\xi^0$, our final result for the propagator is [67]

$$\langle \hat{A}(p, \lambda_1) \wedge \hat{B}(-p, \lambda_2) \rangle = -\frac{1}{2\pi} \left( \frac{4}{p^2} \delta^0_{(-2)}(\xi_1 | p | \lambda_1 \lambda_2) \delta^0_{(2)}(\xi_1) \right) \left( \frac{1}{i(p_\alpha p^{\alpha})} \delta^0_{(-3)}(\xi_1 | p | \lambda_2) \right). \quad (9.41)$$

Having constructed the propagator, we shall now take a closer look at the vertices appearing in (9.26). One pleasing feature of the Cachazo–Svrček–Witten gauge is that the interaction vertex $G \wedge A \wedge A$ does not contribute. Therefore, the only remaining interactions come from the infinite sum in (9.26). We shall now show that these are the MHV vertices, i.e. the on-shell version of (9.26) does indeed give rise to the MHV amplitudes.

Recall from theorem 2.2 that solutions to the zero-rest-mass field equations of helicity $h = \frac{1}{2}$ are given by elements of the Čech cohomology group $H^1(P^3, \mathcal{O}(-2h - 2))$. By remark 5.6 we know that $H^1(P^3, \mathcal{O}(-2h - 2)) \cong H^0_0(P^3, \mathcal{O}(-2h - 2))$. Therefore, solutions to free field equations may equivalently be described in terms of differential (0, 1)-forms $\omega_{-2h - 2}$ of homogeneity $-2h - 2$. In particular, the plane-wave solutions of momentum $p = k \hat{k}$ which were constructed in section 7.3 arise from

$$\omega_{-2h - 2}(z, \lambda) = \delta_{(2b)}(k \lambda \xi) \exp \left( i(z \xi \kappa_{(1)}) \right) \quad (9.42)$$

by suitably integrating over $\int_L [\delta \lambda \lambda]$ for more details, see exercise 9.2. As before, the parameter $\xi$ plays no role since on support of the delta function $\lambda \propto k$.

**Exercise 9.2.** Show that wavefunction (7.16) arises from $\int_L [\delta \lambda \lambda] \wedge \omega_{-2}$. Show further that the $h = \pm \frac{1}{2}$ wavefunctions (7.17) are given by $\int_L [\delta \lambda \lambda] \wedge \lambda_{\omega_{-2}}$ and $\int_L [\delta \lambda \lambda] \wedge \lambda_{\omega_{-2}}$ where $\partial_\alpha := \partial / \partial \xi^{(\alpha)}$.

Note that the twistor space wavefunctions (9.42) obey the Cachazo–Svrček–Witten gauge condition, since there are no components along $\bar{\xi}$.

Therefore, we may take

$$\hat{A}(x, \lambda_{(s)}) = t_{(s)} \delta_{(-2)}(k_{(s)} \lambda_{(s)}) \exp \left( i(z_{(s)} \xi_{(s)}) \right),$$

$$\hat{G}(x, \lambda_{(s)}) = t_{(s)} \delta_{(2)}(k_{(s)} \lambda_{(s)}) \exp \left( i(z_{(s)} \xi_{(s)}) \right). \quad (9.43)$$
where $t_{\sigma(i)}$ is an arbitrary element of the Lie algebra of the gauge group. Inserting these wavefunctions and Green’s function (6.10) into (9.26), one may trivially perform the $\lambda$-integrals by replacing $\lambda_{\sigma(s)}$ by $k_{\sigma(s)}$. The final $x$-integral then gives an overall delta function of momentum conservation. The result of these rather straightforward manipulations is

$$S_{\text{int},\text{gf}} = \sum_{r=0}^{\infty} (i g_{\text{YM}})^{r-2} (2\pi)^{4-d} \left( \sum_{i=1}^{r} k_i k_i \right) \times \frac{1}{r!} \sum_{\sigma \in S/r \setminus Z_r} \sum_{s<t}$$

\[
\left[ k_{\sigma(s)} k_{\sigma(t)} \right] \left[ k_{\sigma(r)} k_{\sigma(1)} \right] \left[ k_{\sigma(1)} k_{\sigma(2)} \right] \cdots \left[ k_{\sigma(r-1)} k_{\sigma(r)} \right] \text{tr}\{t_{\sigma(1)} \cdots t_{\sigma(r)}\}. \tag{9.44}
\]

This, however, is a sum over all the MHV amplitudes (8.1c) including the appropriate colour-trace factors (see also (7.10)).

Let us now summarize these results. We have shown\(^{29}\) that the twistor space action (6.6a) indeed gives the MHV diagram expansion once the Cachazo–Svrček–Witten gauge (9.27) with $\tilde{\xi}_{1,2} \rightarrow \tilde{\xi}$ has been imposed. First, for this choice of gauge, the interaction vertices come solely from the second part of (9.26) and as we have just seen, these vertices coincide with the MHV amplitudes when put on-shell. This is the first of the Cachazo–Svrček–Witten rules. Second, the delta functions in the propagator (9.41) lead to the prescription for the insertion of $k = \langle \tilde{\xi} | p \rangle$ as the co-spinor corresponding to the off-shell momentum $p$. This is the second Cachazo–Svrček–Witten rule. Lastly, the third rule also follows from (9.41) due to the explicit appearance of $1/p^2$. Thus, we have obtained the Cachazo–Svrček–Witten rules directly from twistor space. As already indicated, since the spinor $\tilde{\xi}$ arises here as part of the gauge condition, BRST invariance ensures that the overall amplitudes will not depend on it.

Finally, we would like to emphasize that the above procedure makes manifest the equivalence between the traditional Feynman diagram expansion and the MHV diagram expansion for Yang–Mills theory. The spacetime Yang–Mills action including the resulting Feynman diagram expansion on the one side and the action or generating functional yielding the MHV diagram expansion on the other side are merely consequences of different gauge choices for the twistor space action. This is possible since the gauge symmetry of the twistor space action, being defined on a real six-dimensional manifold, is much larger than that of the Yang–Mills action on four-dimensional spacetime. Note that the described equivalence can only be valid at tree level in perturbation theory, since in pure Yang–Mills theory at loop level there are diagrams that cannot be constructed from MHV vertices and propagators alone. See e.g. [126] for a full discussion. Therefore, even though the spacetime gauge (5.41) yields the fully fledged Yang–Mills action, the above derivation of the MHV diagrams appears to work only at tree level (and so do the Cachazo–Svrček–Witten rules stated in the preceding section)\(^{30}\). Presumably, the issue with loops is due to the following two problems. The first problem, as argued in [67], lies in the change of the path integral measure. When going from the spacetime gauge to the Cachazo–Svrček–Witten gauge one must perform a complex gauge transformation while the path integral is only invariant under real gauge transformations. The second problem has to do with certain regularization issues as discussed in [132]. As we are only concerned with tree-level scattering amplitudes in these lecture notes, we shall not worry about these issues any further.

**Remark 9.1.** Let us point out that there is an alternative (but equivalent [132]) approach that maps the ordinary Yang–Mills action to a functional where the vertices are explicitly the MHV vertices. This approach has been developed by Mansfield in [133] (see also [126, 134–136]).

\(^{29}\) See also [125] for a more detailed exposition of this material.

\(^{30}\) See [127–131] for an extension of the Cachazo–Svrček–Witten rules to one-loop level.
For a recent review, see [137]. This approach works directly in four-dimensional spacetime and involves formulating Yang–Mills theory in light-cone gauge and performing a non-local field transformation. In similarity with the twistor space approach above, the spinor $\tilde{\xi}$ arises in fixing the light-cone gauge. Finally, let us mention in passing that the above procedure has also been extended to gravity [122] (see also [138–142]) using the gravity twistor space action of [64] (see also [143]).

9.4. Superamplitudes in the MHV formalism

So far, we have discussed the MHV formalism in pure Yang–Mills theory only. Let us now briefly explain how the above discussion extends to the $\mathcal{N}=4$ supersymmetric theory.

In section 8.2, we have seen that all the different scattering amplitudes occurring in $\mathcal{N}=4$ supersymmetric Yang–Mills theory are encoded in terms of certain superamplitudes which we also called generating functions. In particular, all the different (colour-stripped) MHV amplitudes follow from Nair’s formula (8.17) via (8.21). The question then arises of: How do we generalize (8.17) to $N$ MHV scattering amplitudes, that is, how can we encode all the $N$ MHV amplitudes in terms of $N$ MHV superamplitudes with a relationship similar to (8.21)?

To answer this question, let us first recall that the total helicity of an $n$-particle $N$ MHV scattering amplitude is $-n + 2k + 4$ while an $n$-particle superamplitude has always a total helicity of $-n$. Note that $k = 0, \ldots, n - 4$ since $A^N_{n\text{MHV}} = A^\text{MHV}_n$ and $A^N_{n-4\text{MHV}} = A^\text{MHV}_n$. Furthermore, as we have explained in section 8.2, the differential operators (8.14) will pick from a superamplitude the appropriate particles one wishes to scatter (i.e. the external states). This therefore suggests that an $n$-particle $N$ MHV superamplitude $\mathcal{F}_n^{N\text{MHV}}$ is a homogeneous polynomial of degree $2(2k + 4) = 4k + 8$ in the fermionic coordinates $\psi_i^r$. Each $\psi_i^r$ carries a helicity of $-1/2$ and therefore, a differential operator $D^{(4k+8)}$ of order $4k + 8$ that is made of the operators (8.14) will have a total helicity of $2k + 4$. Consequently, $D^{(4k+8)}\mathcal{F}_n^{N\text{MHV}}$ will carry a helicity of $-n + 2k + 4$, as desired.

Besides this, the superamplitudes are supersymmetric by construction and so they are annihilated by the Poincaré supercharges $Q_i\alpha$ and $Q_i\dot{\alpha}$ and by the generator of spacetime translations $P_{\alpha\dot{\beta}}$. Since $Q_i\alpha$ and $P_{\alpha\dot{\beta}}$ are represented by multiplication operators (8.2), (8.2) when acting on superamplitudes, the latter must be proportional to the overall delta function (8.16) of (super)momentum conservation. Collecting what we have said so far, we may write [105]

$$\mathcal{F}_n = \mathcal{F}^{\text{MHV}}_n + \mathcal{F}^{N\text{MHV}}_n + \cdots + \mathcal{F}^{N-4\text{MHV}}_n$$

(9.45a)

for a generic superamplitude, together with

$$\mathcal{F}^{N\text{MHV}}_n = \frac{\alpha_s}{2\pi} \left(2\pi\right)^4 \delta^{(4|8)} \left(\sum_{\tau=1}^n \tilde{\xi}_\tau k_{\tau\beta} \right) \mathcal{P}^{(4k)}_n,$$

(9.45b)

where $\mathcal{P}^{(4k)}_n$ is a polynomial of degree $4k$ in the fermionic coordinates $\psi_i^r$. Hence,

$$A^{N\text{MHV}}_n = D^{(4k+8)} \mathcal{F}_n|_{\psi=0} = D^{(4k+8)} \mathcal{F}^{N\text{MHV}}_n.$$

(9.46)

We should emphasize that we are suppressing $R$-symmetry indices here; see e.g. (8.20). As before, we will now restrict our attention to the colour-stripped amplitudes which will be
denoted by $F_{n}^{\text{NHV}}$:

$$F_{n}^{\text{NHV}} = g_{YM}^{n-2} (2\pi)^{4} \delta^{(4)}(\sum_{r=1}^{n} k_{r} \cdot k_{r}) P_{n}^{(4k)}. \quad (9.47)$$

According to the Cachazo–Svrček–Witten rules, $N^{4}$ MHV amplitudes in pure Yang–Mills theory are obtained by gluing together $k + 1$ MHV vertices. While there is basically only one type of vertex in pure Yang–Mills theory, in the $\mathcal{N} = 4$ supersymmetric extension we have plenty of different MHV vertices corresponding to the different MHV amplitudes (8.21). Nevertheless, the Cachazo–Svrček–Witten rules can be extended to this case rather straightforwardly as we simply need to glue together the different MHV vertices that follow from (8.21) when continued off-shell appropriately. The types of vertices we need to glue together in order to obtain a particular $N^{4}$MHV amplitude will depend on the $R$-symmetry index structure of the external states for this amplitude. A generic $N^{4}$MHV amplitude $A_{n,n}^{N^{4}\text{MHV}}$ is given by

$$A_{n,n}^{N^{4}\text{MHV}} = \sum_{\{\Gamma_{1},\ldots,\Gamma_{k+1}\}|n_{1}+\ldots+n_{k+1}=n+2k} \frac{A_{0,n_{1}}^{\text{MHV}}(\Gamma_{1}) \cdots A_{0,n_{k+1}}^{\text{MHV}}(\Gamma_{k+1})}{p_{\Gamma_{1}}^{2} \cdots p_{\Gamma_{k+1}}^{2}}, \quad (9.48)$$

where the different MHV vertices correspond to the different MHV amplitudes (8.21). Here, $p_{\Gamma_{r}}$ represents an internal line flowing into the vertex $\Gamma_{r}$ and is given by the sum over the remaining momenta entering $\Gamma_{r}$ (see also below). As before, we are suppressing $R$-symmetry indices. For instance, we have

$$A_{0,n}^{N^{4}\text{MHV}} = \sum_{\{\Gamma_{1},\Gamma_{2}\}|n_{1}+n_{2}=n+2} \frac{A_{0,n_{1}}^{\text{MHV}}(\Gamma_{1})}{p_{\Gamma_{1}}^{2}} A_{0,n_{2}}^{\text{MHV}}(\Gamma_{2}) \quad (9.49a)$$

$$= \sum_{\Gamma_{1} \Gamma_{2}} \Gamma_{1} \Gamma_{2} \frac{1}{p_{\Gamma_{1}}^{2}}$$

$$A_{0,n}^{N^{4}\text{MHV}} = \sum_{\{\Gamma_{1},\Gamma_{2},\Gamma_{3}\}|n_{1}+n_{2}+n_{3}=n+4} \frac{A_{0,n_{1}}^{\text{MHV}}(\Gamma_{1})A_{0,n_{2}}^{\text{MHV}}(\Gamma_{2})A_{0,n_{3}}^{\text{MHV}}(\Gamma_{3})}{p_{\Gamma_{1}}^{2}p_{\Gamma_{1}}^{2}p_{\Gamma_{3}}^{2}} \quad (9.49b)$$

$$= \sum_{\Gamma_{1} \Gamma_{2} \Gamma_{3}} \Gamma_{1} \Gamma_{2} \Gamma_{3} \frac{1}{p_{\Gamma_{1}}^{2}} \frac{1}{p_{\Gamma_{2}}^{2}}$$
for the NMHV and N²MHV amplitudes, while for the N³MHV amplitudes we have two different graph topologies:

\[
A_{0,n}^{N^3\text{MHV}} = \sum_{\{\Gamma_1, \ldots, \Gamma_4\}|n_1 + \ldots + n_4 = n+6} \frac{A_{0,n_1}^{\text{MHV}}(\Gamma_1) \cdots A_{0,n_4}^{\text{MHV}}(\Gamma_4)}{p_{\Gamma_1}^2 p_{\Gamma_2}^2 p_{\Gamma_3}^2 p_{\Gamma_4}^2}
\]

\[
= \sum \Gamma_1 p_{\Gamma_1}^2 \Gamma_2 p_{\Gamma_2}^2 \Gamma_3 p_{\Gamma_3}^2 + \sum \Gamma_1 p_{\Gamma_1}^2 \Gamma_4 p_{\Gamma_4}^2 \Gamma_3 p_{\Gamma_3}^2
\]

(9.49c)

Similarly, one may draw the diagrams contributing to the N⁴MHV amplitude for larger and larger \( k \) but, of course, the complexity in the graph topology will grow. In this notation, figure 4 becomes what is illustrated above in figure 5.

The above procedure of gluing MHV vertices together is not manifestly \( \mathcal{N} = 4 \) supersymmetric. Instead we would like to directly glue MHV supervertices\(^{31}\) together as this would give a manifestly supersymmetrical formulation of the N⁴MHV amplitudes. In order to do this, we need one additional ingredient since each MHV supervertex is of eighth order in the fermionic coordinates such that \( k + 1 \) of such vertices are of order \( 8k + 8 \). However, above we have argued that an N⁴MHV superamplitude is of order \( 4k + 8 \) in the fermionic

\(^{31}\) That is, the vertices that follow from the MHV superamplitudes after an appropriate off-shell continuation.
coordinates $\psi^I$. Elvang, Freedman and Kiermaier [119] propose the introduction of additional fermionic coordinates $\tilde{\psi}^I$, for each internal line $p_I$, and the definition of a fourth-order differential operator

$$D^{(4)}_I := \frac{1}{4!} \varepsilon^{ijkl} \frac{\partial^4}{\partial \psi_I^j \partial \psi_I^k \partial \psi_I^l \partial \psi_I^m}.$$  

(9.50)

All the $N^4$ MHV amplitudes $A_{0,n}^{N\text{MHV}}$ in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory are then given by the following formulae [119]:

$$A_{0,n}^{N\text{MHV}} = D^{(4k+8)}F_{0,n}^{N\text{MHV}},$$  

(9.51a)

with

$$F_{0,n}^{N\text{MHV}} = \sum_{[(\Gamma_1, \ldots, \Gamma_{k+1}) |_{\Gamma_1 \oplus \cdots \oplus \Gamma_k = n+2k]} } \left( \prod_{r=1}^{k} D^{(4)}_{\Gamma_r} \right) \left( \frac{F_{\text{YM}}^{(\Gamma_1)} \cdots F_{\text{YM}}^{(\Gamma_{k+1})}}{p_{\Gamma_1}^{2} \cdots p_{\Gamma_{k+1}}^{2}} \right).$$  

(9.51b)

The justification of these formulae is tied to the action of the differential operator $D^{(4k+8)}$ which is made from (8.14) and which selects the external states of the amplitude. The $R$-symmetry index structure of the external states uniquely determines the $R$-symmetry indices of the internal states (i.e. the internal lines). The operator $D^{(4)}_I$ then uniquely splits into two factors that correspond to the required split of the $R$-symmetry index structure at each end of the internal line.

The expression (9.51b) can be simplified further. We shall just state the result here and refer the interested reader to [119] for the explicit derivation. We have

$$F_{0,n}^{N\text{MHV}} \equiv g_{\text{YM}}^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{r=1}^{n} \tilde{k}_r \cdot \tilde{k}_r \right) p_{0,n}^{(4k)},$$  

(9.52a)

with

$$p_{0,n}^{(4k)} = \sum_{[(\Gamma_1, \ldots, \Gamma_{k+1}) |_{\Gamma_1 \oplus \cdots \oplus \Gamma_k = n+2k]} } \frac{\prod_{r=1}^{k} \frac{1}{p_{\Gamma_r}} \sum_{\text{cyc} \in \Gamma_{\Gamma_1} \cdots \text{cyc} \in \Gamma_{k+1}} \tilde{\xi}_{[p_I]} \xi_{[\tilde{\xi}_{[\xi_{\tilde{\xi}_{[\xi]}]}]}]}^4}{\text{cyc}(\Gamma_1) \cdots \text{cyc}(\Gamma_{k+1})}. $$  

(9.52b)

Here, we have used the notation (6.14) for $\psi^I$ and ‘cyc’ denotes the usual denominator of the spinor brackets appearing in the MHV amplitudes, e.g. cyc$(1, \ldots, n) := [12] \cdots [n1]$. In addition, the external momenta are $p_I = \tilde{k}_r \cdot \tilde{k}_r$ for $r = 1, \ldots, n$. Note that the result for the tree-level NMHV superamplitude was first obtained by Georgiou, Glover and Khoze in [120]. In [144], a refined version of (9.52b) was given in which the number of diagrams one needs to sum over is reduced substantially.

Next one should show that this expression is independent of the choice of the spinor $\tilde{\xi}$ that defines the off-shell continuation. This was done explicitly in [119]. Note, however, that analogously to our discussion presented in section 9.3, $\tilde{\xi}$ can also be shown to arise from a gauge-fixing condition of the twistor space action (6.6a). This time, of course, one should include the full supermultiplet (5.40). Therefore, BRST invariance will guarantee the $\tilde{\xi}$-independence of the overall scattering amplitudes.

**Remark 9.2.** Note that there is an alternative method [145] for constructing the polynomials $\mathcal{P}_n^{(4k)}$ occurring in (9.4). Drummond and Henn [145] express the $\mathcal{P}_n^{(4k)}$ in terms of invariants of the so-called dual superconformal symmetry group [105, 146–160] (see also [161–165] and [166–168] for recent reviews). The dual superconformal symmetry is a recently discovered hidden (dynamical) symmetry planar scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory appear to exhibit.
9.5. Localization properties

In this section, we would like to comment on the localization properties of generic scattering amplitudes in twistor space. In section 8.4, we saw that MHV superamplitudes localize on genus-0 degree-1 curves $\Sigma$ in supertwistor space: $\Sigma \cong \mathbb{C}P^1$ in the complex setting or $\Sigma \cong \mathbb{R}P^1$ in the Kleinian setting. According to the MHV formalism, generic tree-level superamplitudes are obtained by gluing the MHV superamplitudes together (after a suitable off-shell continuation). This is depicted in figure 6 for the case of NMHV amplitudes.

This therefore suggests that tree-level $N^k$MHV superamplitudes localize on maximally disconnected rational curves $\Sigma$ of genus zero, that is, each $\Sigma$ is the union of $k + 1$ (projective) lines in supertwistor space. Therefore, the total degree $d$ of each $\Sigma$ is

$$d = k + 1.$$  \hfill (9.53)

This picture is the so-called maximally disconnected description of scattering amplitudes. As was observed in [169–171], scattering amplitudes may also be described by completely connected rational curves. In particular, tree-level NMHV superamplitudes may also be described by connected genus-0 degree-2 rational curves as is depicted in figure 7. This is referred to as the connected description of scattering amplitudes. It was argued, however, by
Gukov, Motl and Neitzke in [172] that the two descriptions—that is, the disconnected and connected prescriptions—are in fact equivalent.

In general, the conjecture put forward by Witten [53] is that \( \ell \)-loop \( N^k \) MHV superamplitudes localize on rational curves in supertwistor space which are of genus \( g \leq \ell \) and degree

\[
d = k + 1 + g.
\]

(9.54)

Following the earlier work of [106, 173–175], Korchemsky and Sokatchev [176] (see also [177–179]) performed the Witten transform (8.43) of all tree-level scattering amplitudes explicitly. In fact, they transformed the Drummond and Henn expressions [145] of the quantities \( \mathcal{P}^{(4)}_a \) appearing in (9.4) for the \( N^k \) MHV superamplitudes to supertwistor space. They found that the \( N^k \) MHV superamplitudes are supported on \( 2k + 1 \) intersecting lines in supertwistor space in contradistinction to what we have said above (9.53). This puzzle was resolved very recently by Bullimore, Mason and Skinner in [180] (see also [181]) by proving that the Drummond and Henn expressions for the tree-level amplitudes actually contain loop-level information. They showed that the \( 2k + 1 \) intersecting lines form a curve of genus \( g = k \) and therefore, \( d = k + 1 + k = k + 1 + g \). Their result generalizes the fact [155, 178] that the tree-level NMHV superamplitudes can be written in terms of loop-level information. Altogether, the result of Korchemsky and Sokatchev [176] is therefore in support of Witten’s conjecture (9.54).

10. Britto–Cachazo–Feng–Witten recursion relations

Central to this final section will be the discussion of a powerful method for constructing tree-level amplitudes first introduced by Britto, Cachazo and Feng in [191] (stemming from observations made in [192]) and later proven by Britto, Cachazo, Feng and Witten in [193]. We shall start with pure Yang–Mills theory and closely follow the treatment of [193] before moving on to the \( \mathcal{N} = 4 \) supersymmetric extension developed by [142, 154, 194].

10.1. Recursion relations in pure Yang–Mills theory

The basic philosophy of the method developed by Britto, Cachazo, Feng and Witten is to construct tree-level scattering amplitudes in terms of lower-valence on-shell amplitudes and a scalar propagator. Therefore, one also speaks of recursion relations since one constructs higher-point amplitudes from lower-point ones. Recursion relations have been known for some time in field theory since Berends and Giele [92] proposed them in terms of off-shell currents. However, the recursion relations we are about to discuss are somewhat more powerful as they directly apply to on-shell scattering amplitudes and in addition are particularly apt when the amplitudes are written in the spinor-helicity formalism (see our above discussion).

To derive a recursion relation for scattering amplitudes, let \( A_{0,0}(t) \) be a complex one-parameter family of \( n \)-particle colour-stripped scattering amplitudes at tree level with \( t \in \mathbb{C} \) such that \( A_{0,0}(t = 0) \) is the amplitude we are interested in. As before, we shall work in the complex setting, i.e. all momenta are taken to be complex. One can then consider the contour integral [195]

\[
c_\infty := \frac{1}{2\pi i} \oint_{\gamma_\infty} dt \frac{A_{0,0}(t)}{t}.
\]

(10.1)

32 More concretely, they can be written in terms of the so-called leading singularities of the one-loop amplitudes; see also [182–190] for a discussion on the generalized unitarity and leading singularity methods.
where the integration is taken counterclockwise around a circle $C_{\infty}$ at infinity in the complex $t$-plane; see figure 8. Using the residue theorem, we may write this contour integral as

$$c_{\infty} = A_{0,n}(t = 0) + \sum_{k \geq 1} \frac{1}{t_k} \text{Res}_{t_k = 0} A_{0,n}(t),$$

where we have performed the contour integral around $C_0$ explicitly; see figure 8. Equivalently, we have

$$A_{0,n}(t = 0) = c_{\infty} - \sum_{k \geq 1} \frac{1}{t_k} \text{Res}_{t_k = 0} A_{0,n}(t).$$

So far, we have not given the definition of the one-parameter family $A_{0,n}(t)$. There are some obvious requirements for $A_{0,n}(t)$. The main point is to define the family $A_{0,n}(t)$ such that poles in $t$ correspond to multi-particle poles in the scattering amplitude $A_{0,n}(t = 0)$. If this is done, the corresponding residues can be computed from factorization properties of the scattering amplitudes; see e.g. [98, 102].

In order to accomplish this, Britto, Cachazo, Feng and Witten [191, 193] defined $A_{0,n}(t)$ by shifting the momenta of two external particles (gluons) in the original scattering amplitude. Obviously, for this to make sense, one has to ensure that even with these shifts, overall momentum conservation is preserved and that all particles remain on-shell. Thus, let us shift the momenta of the particles $r$ and $s$ according to

$$p_r(t) := p_r + t\ell$$
$$p_s(t) := p_s - t\ell.$$  

Clearly, momentum conservation is maintained by performing these shifts. To preserve the on-shell conditions, $p_r^2(t) = 0 = p_s^2(t)$, we choose

$$\ell_{\alpha\beta} = \bar{k}_{\alpha\bar{\beta}}k_{\beta\bar{\alpha}},$$

where $\bar{k}, k$, and $\bar{k}, k$, are the co-spinors for the momenta $p_r$ and $p_s$, respectively. This then corresponds to shifting the co-spinors

$$\bar{k}_{\alpha\bar{\alpha}}(t) := \bar{k}_{\alpha\bar{\alpha}} + tk_{\alpha\bar{\alpha}}$$
$$k_{\beta\beta}(t) := k_{\beta\beta} - tk_{\beta\beta}.$$  

33 Alternatively, one may take $\ell_{\alpha\beta} = \bar{k}_{\alpha\bar{\beta}}k_{\beta\bar{\alpha}}$. 

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with \( k_{\mu a} \) and \( \tilde{k}_{\mu b} \) unshifted. As a result, we can define
\[
A_{0,n}(t) := A_{0,n}(p_1, \ldots, p_{r-1}, p_r(t), p_{r+1}, \ldots, p_{r+1}, \ldots, p_n).
\] (10.7)

The right-hand side is a physical on-shell amplitude for all \( t \in \mathbb{C} \) (since all the momenta are null).

The family \( A_{0,n}(t) \) is a rational function of \( t \). According to our above discussion, the original tree-level amplitude is a rational function of the spinor brackets and thus, performing the shifts (10.6) clearly renders \( A_{0,n}(t) \) a rational function in \( t \). In addition, \( A_{0,n}(t) \) can only have simple poles as a function of \( t \), since singularities can only come from the poles of a propagator in a Feynman diagram. In tree-level Yang–Mills theory, the momentum in a propagator is always a sum of momenta of adjacent external particles, say \( p = \sum_{r' \in R} p_{r'} \), where \( R \) denotes the set of these external particles. A propagator with this momentum is 1

Let now \( s \in R \) but \( r \notin R \). Then, \( p(t) = p - t\ell \) and \( p^2(t) = p^2 - 2t(p \cdot \ell) \) vanishes at
\[
t = t_p := \frac{p^2}{2(p \cdot \ell)}.
\] (10.8)

These are the only poles of \( A_{0,n}(t) \). Note that there might be poles in \( t \) arising from the shifts (10.6) on the denominators of the polarization vectors (7.19), but these may always be removed by an appropriate choice of \( \tilde{\mu}, \mu \) in (7.19) and hence they are merely gauge artefacts. Thus, \( A_{0,n}(t) \) only has simple poles in \( t \), as claimed. Below we shall also denote \( p(t) \) by \( \tilde{p} \). Since \( \tilde{p} \) is null, we may introduce the co-spinors \( \tilde{k}_p, k_{\tilde{p}} \) and write \( \tilde{p} = k_p k_{\tilde{p}} \).

Now we would like to evaluate the sum in (10.3). To get a pole at \( p^2(t) = 0 \), a tree-level diagram must contain a propagator that divides it into a ‘left sub-amplitude’ that contains all external particles not in \( R \) and a ‘right sub-amplitude’ that contains all external particles that are in \( R \). The internal line connecting the left and the right parts has momentum \( p(t) \) and we need to sum over the helicity \( h = \pm 1 \) at, say, the left of this line. Therefore,
\[
\text{Res}_{t = t_p} A_{0,n}(t) = \sum_h A_h^b(t_p) \text{Res}_{t = t_p} \left\{ \frac{1}{p^2 - 2t(p \cdot \ell)} \right\} A_R^{-h}(t_p)
\]
\[
= -t_p \sum_h A_h^b(t_p) \frac{1}{p^2} A_R^{-h}(t_p)
\] (10.9)
and (10.3) becomes
\[
A_{0,n}(t = 0) = c_\infty + \sum_p \sum_h A_h^b(t_p) \frac{1}{p^2} A_R^{-h}(t_p).
\] (10.10)

Furthermore, in [193] it was shown that \( A_{0,n}(t) \to 0 \) for \( |t| \to \infty \) for pure Yang–Mills theory; see also [196] for a more detailed treatment. This results in the remarkable property that \( c_\infty = 0 \). Altogether, we obtain the following recursion relation [191, 193]:
\[
A_{0,n} = \sum_p \sum_h A_h^b(t_p) \frac{1}{p^2} A_R^{-h}(t_p),
\] (10.11a)
which is depicted in figure 9. Here,
\[
A_h^b(t_p) = A_h^b(\ldots, p_{r-1}, \hat{p}_r, p_{r+1}, \ldots),
\]
\[
A_R^{-h}(t_p) = A_R^{-h}(\hat{p}_r, \ldots, p_{r-1}, \hat{p}_r, p_{r+1}, \ldots),
\] (10.11b)
where \( \hat{p}_r := p_r(t_p) \) and \( \tilde{p}_r := p_r(t_p) \), respectively. Note that ultimately (10.1) tells us that every scattering amplitude reduces to a sum of products of MHV and \( \overline{\text{MHV}} \) amplitudes. This is exemplified in the exercise below.
Exercise 10.1. Consider the MHV amplitude

\[ A_{\text{MHV}}^{\alpha,0}(1^+, r^+) = g_{\text{YM}}^2 (2\pi)^4 \delta^{(4)} \left( \sum_r k_r \tilde{k}_r \right) \frac{[1r]^4}{[12] \cdots [n1]} \]

Assume that this formula (and the one for the \( \overline{\text{MHV}} \) amplitude) is valid up to \( n \) gluons. Use the above recursion relation to prove its validity for \( n+1 \) gluons by appropriately shifting two of the co-spinors. Note that you will need to use both MHV and \( \overline{\text{MHV}} \) amplitudes as left and right sub-amplitudes entering the recursion relation.

10.2. Recursion relations in maximally supersymmetric Yang–Mills theory

Finally, we would like to extend the above recursion relation to the \( \mathcal{N} = 4 \) supersymmetric setting. In section 9.4, we saw that all the different scattering amplitudes in \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory are best understood when formulated in terms of superamplitudes or generating functions \( F \). In order to formulate a supersymmetric version of the recursion relation for the tree-level superamplitudes \( F_{0,n} \), we follow [142, 154, 194] and consider the shifts

\[ \tilde{k}_r A(t) := k_r A(t) + t \tilde{k}_r, \quad k_{\alpha}(t) := k_{\alpha} - tk_{\alpha}, \quad (10.12) \]

with \( k_{\alpha} \) and \( \tilde{k}_{\alpha} \) unshifted. Indeed, this is a direct supersymmetric extension of (10.6).

The supersymmetric recursion relations then follow from arguments similar to those which led to (10.1). We therefore just record the result here [154]:

\[ F_{0,n} = \sum_p \int d^4 \psi_p F_L(t_p) \frac{1}{p^2} F_R(t_p). \quad (10.13) \]

Here, the \( \psi_j \) are the fermionic variables associated with the internal on-shell line with momentum \( \hat{p} \). Note that the sum over helicities gets replaced by the integral over the fermionic coordinates. The superamplitudes \( F_L \) and \( F_R \) carry total helicities of \(-n_L - 1\) and \(-n_R - 1\) where \( n_L \) and \( n_R \) are the number of external states on the left and right sub-amplitudes; see also section 8.1. Since the measure \( d^4 \psi_p \) carries a helicity of +2 (remember that each \( \psi_j \) carries a helicity of \(-1/2\)), the total helicity carried by the right-hand side of (10.13) is \(-n_L - n_R \) and since \( F_{0,n} \) has helicity \(-n \) we have \( n = n_L + n_R \) as it should be.

In order to derive the recursion relations (10.13), one needs to verify that the one-parameter family \( F_{0,n}(t) \) of superamplitudes induced by (10.12) vanishes as \( |t| \to \infty \). This was explicitly shown in [154] (see also [142, 197] for an earlier account). In fact, as proved by Arkani-Hamed, Cachazo and Kaplan in [194], any \( \mathcal{N} = 4 \) superamplitude behaves as

\[ F_{0,n}(t) \to t^{-1} \quad \text{for} \quad |t| \to \infty. \quad (10.14) \]

34 See also Arkani-Hamed’s talk at ‘Wonders of Gauge Theory and Supergravity’, Paris 2008.
35 As also shown in [194], all superamplitudes in \( \mathcal{N} = 8 \) supergravity behave as \( t^{-2} \) implying similar recursion relations for maximal supergravity. See [142, 198–200] for an earlier account of gravity recursion relations.
In the pure Yang–Mills setting, we have seen that the recursion relations imply that all amplitudes are given in terms of MHV and \( \text{MHV} \) amplitudes. Likewise, (10.13) implies that every superamplitude is given in terms of MHV and \( \text{MHV} \) superamplitudes. So far, we have only given the \( \text{MHV} \) superamplitudes (8.17) explicitly. Therefore, in order to use (10.13) we also need the \( \text{MHV} \) superamplitudes. Working—for a moment—in the Minkowski signature, the \( \text{MHV} \) superamplitudes are obtained from (8.17) via complex conjugation; see also remark 8.1. Thus, we may write

\[
F_{0,n}^{\text{MHV}} = g_{\text{YM}}^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{r=1}^n \bar{k}_r \cdot k_r \right) \delta^{(0)} \left( \sum_{r=1}^n \bar{\psi}_r \cdot \bar{k}_r \right) \frac{1}{(12)(23)\cdots(n1)},
\]

where we have used \( \bar{k}_r = i\bar{k}_r \). This, however, is not yet quite the form of the \( \text{MHV} \) superamplitudes that is suitable for the recursion relations as the left and right sub-amplitudes entering (10.13) are holomorphic in the fermionic coordinates. To get an expression which depends holomorphically on the fermionic coordinates, we simply perform a Fourier transform on all the \( \bar{\psi}_r \) in (10.15) (see [105, 118, 155, 171] for general amplitudes), that is,

\[
F_{0,n}^{\text{MHV}} = \int \left( \prod_{r=1}^n \tilde{d}^4 \bar{\psi}_r \right) e^{-i\sum_{r=1}^n \bar{\psi}_r \cdot \bar{k}_r} \tilde{F}_{0,n}^{\text{MHV}}.
\]

Using identity (8.18), we straightforwardly find (see also the exercise below) [118]

\[
F_{0,3}^{\text{MHV}} = g_{\text{YM}}^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{r=1}^n \bar{k}_r \cdot k_r \right) \left( \frac{4}{\prod_{i=1}^n |r_i|} \frac{\delta^2}{\partial \bar{\psi}_r \partial \psi_{r'}^\dagger} \psi_{r'} \right) \frac{1}{(12)(23)(31)}.
\]

We may now return to the complex setting by analytically continuing this expression, i.e. all the \( \bar{k}_r, k_r, \) and \( \psi_r^\dagger \) are to be regarded as complex. We have thus found a formula for the \( \text{MHV} \) superamplitudes which depends holomorphically on the fermionic coordinates and which can therefore be used in the superasymmetric recursion relations (10.13). A special case of (10.17) is the three-particle \( \text{MHV} \) superamplitude [154]

\[
F_{0,3}^{\text{MHV}} = g_{\text{YM}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3) \delta^{(0)} \left( \psi_1(23) + \psi_2(31) + \psi_3(12) \right).
\]

**Exercise 10.2.** Verify (10.17) and (10.18).

We have now provided all the necessary ingredients to construct general superamplitudes from the MHV and \( \text{MHV} \) superamplitudes. In fact, the recursion relation generates all the tree-level amplitudes from the three-particle amplitudes \( F_{0,3}^{\text{MHV}} \) and \( F_{0,3}^{\text{MHV}} \). The interested reader might wish to consult the cited references (see e.g. [154, 194]) for explicit examples. Note that the three-particle amplitudes just follow from helicity information and Lorentz invariance and without actually referring to a Lagrangian [201].

**Remark 10.1.** The Britto–Cachazo–Feng–Witten recursion relations have also been investigated from the point of view of twistor theory. The first twistor formulation of these recursion relations was given in terms of twistor diagrams by Hodges in [173, 174, 202]. The Hodges construction is an ambidextrous approach as it uses both twistors and dual twistors. This approach has been re-considered by Arkani-Hamed, Cachazo, Cheung and Kaplan in [175]. Mason and Skinner [106] discuss the recursion relations in terms of twistors only. In particular, if we let \( \gamma/F_{0,\delta}(Z_1^\dagger, \ldots, Z_n^\dagger) \) be the Witten transform (8.43) of any tree-level \( n \)-particle superamplitude \( F_{0,\delta} \) (in Kleinian signature), where \( Z_i^\dagger := (z_i^\Lambda, \lambda z_i) \) is the supertwistor.
corresponding to particle \( r \), then the shift (10.12) with \( r = 1 \) and \( s = n \) corresponds to the simple shift \( \mathcal{H} [\mathcal{F}_{0,n} (Z_1', \ldots, Z_n') \rightarrow \mathcal{H} [\mathcal{F}_{0,n} (Z_1', \ldots, Z_n' - tZ_1') \). See [106] for more details.

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**Appendices.** The main purpose of the subsequent appendices is to give an overview of some of the mathematical concepts underlying these lecture notes that the reader might not be so familiar with. For obvious reasons, it is impossible to give full-length explanations here. For a detailed account of the following material, we refer to the literature cited at the beginning of these notes.

**Appendix A. Vector bundles**

Let \( X \) be a manifold (either smooth or complex). A complex rank-\( r \) vector bundle over \( X \) is defined to be a manifold \( E \) together with a mapping \( \pi : E \rightarrow X \) such that the following is fulfilled:

(V1) \( \pi \) is surjective,

(V2) for all \( x \in X \), \( E_x := \pi^{-1}(x) \) is a complex vector space of dimension \( r \), i.e. there exists an isomorphism \( h_x : E_x \rightarrow \mathbb{C}^r \)

(V3) and for all \( x \in X \), there exists a neighbourhood \( U \ni x \) and a diffeomorphism \( h_U : U \times \mathbb{C}^r \rightarrow \pi^{-1}(U) \) such that \( \pi \circ h_U : U \times \mathbb{C}^r \rightarrow U \) with \( (x, v) \mapsto x \) for any \( v \in \mathbb{C}^r \).

The manifold \( E \) is called total space and \( E_x \) is the fibre over \( x \in X \). Because of (V3), \( E \) is said to be locally trivial and \( h_U \) is called a trivialization of \( E \) over \( U \). A trivial vector bundle \( E \) is globally of this form, i.e. \( E \cong X \times \mathbb{C}^r \). A map \( s : X \rightarrow E \) satisfying \( \pi \circ s = \text{id}_X \) is called a section of \( E \). Note that we could replace \( \mathbb{C}^r \) in the above definition by \( \mathbb{R}^r \) in which case one speaks of real vector bundles of real rank \( r \).

A useful description of vector bundles is in terms of transition functions. Let \( E \rightarrow X \) be a complex vector bundle and \( \{ U_i \} \) be a covering of \( X \) with trivializations \( h_i : U_i \times \mathbb{C}^r \rightarrow \pi^{-1}(U_i) \) of \( E \) over \( U_i \). If \( U_i \cap U_j \neq \emptyset \) then the functions \( f_{ij} := h_i^{-1} \circ h_j : U_i \cap U_j \times \mathbb{C}^r \rightarrow U_i \cap U_j \times \mathbb{C}^r \) are called transition functions with respect to the covering. By definition, they are diffeomorphisms. Furthermore, they define maps \( \tilde{f}_{ij} : U_i \cap U_j \rightarrow \text{GL}(r, \mathbb{C}) \) via \( f_{ij}(x, v) = (x, \tilde{f}_{ij} v) \). In the following, we shall not make a notational distinction between \( f_{ij} \) and \( \tilde{f}_{ij} \) and simply write \( f_{ij} \). By construction, the transition functions obey

\[
\begin{align*}
    f_{ii} &= 1 \text{ on } U_i, & f_{ij} &= f_{ji}^{-1} \text{ on } U_i \cap U_j \neq \emptyset, \\
    f_{ij} \circ f_{jk} &= f_{ik} \text{ on } U_i \cap U_j \cap U_k \neq \emptyset.
\end{align*}
\]
The last of these equations is called the cocycle condition. Conversely, given a collection of functions \( f_{ij} \) which obey (A.2), one can construct a vector bundle such that the \( f_{ij} \) are its transition functions. See e.g. [4] for a proof. So far, we have talked about complex vector bundles. Likewise, one has holomorphic vector bundles \( E \to X \), where \( X \) is a complex manifold and the transition functions of \( E \) are assumed to be bi-holomorphic.

Let \( E \) and \( E' \) be two vector bundles over \( X \). A morphism (bundle map) \( \phi : E \to E' \) is a mapping such that \( \phi \) restricted to the fibre \( E_x \) is a linear mapping to the fibre \( E'_x \) for all \( x \in X \). We call \( \phi \) a monomorphism if it is one-to-one on the fibres, an epimorphism if it is surjective on the fibres and an isomorphism if it is one-to-one and surjective on the fibres. The vector bundle \( E \) is a subbundle of \( E' \) if \( E \) is a submanifold of \( E' \) and \( E_x \) is a linear subspace of \( E'_x \) for all \( x \in X \).

An important concept we use throughout these notes is that of a pull-back. Let \( E \to X \) be a complex vector bundle and \( g : Y \to X \) be a smooth mapping; then, the pull-back bundle \( g^*E \) is a complex vector bundle over \( Y \) of the same rank as \( E \) such that

\[
g^*E \to E
\]

\[
Y \to X
\]

is commutative. Put differently, the fibre of \( g^*E \) over \( y \in Y \) is just a copy of the fibre of \( E \) over \( g(y) \in X \). Furthermore, if \([U_i]\) is a covering of \( X \) and \( f_{ij} \) are the transition functions of \( E \) then \( \{ g^{-1}(U_i) \} \) defines a covering of \( Y \) such that \( g^*E \) is locally trivial. The transition functions \( g^*f_{ij} \) of the pull-back bundle \( g^*E \) are then given by \( g^*f_{ij} = f_{ij} \circ g \).

Moreover, given two complex vector bundles \( E \) and \( E' \) over \( X \), we can form new vector bundles. For instance, we can form the dual vector bundles, the direct sum of \( E \) and \( E' \), the tensor product of \( E \) and \( E' \), the symmetric tensor product of \( E \) and \( E' \), which we respectively denote by

\[
E^*, \quad E \oplus E', \quad E \otimes E', \quad E \otimes E', \quad \text{and} \quad E \otimes E',
\]

or if \( E \) is a subbundle of \( E' \), one can form the quotient bundle \( E'/E \). We can also form the exterior product of a vector bundle

\[
\Lambda^k E \quad \text{for} \quad k = 0, \ldots, \text{rank} \ E.
\]

In each case, one defines the derived bundles fibrewise by the linear algebra operation indicated. For example, the fibres of \( E \oplus E' \) are defined by \( (E \oplus E')_x := E_x \oplus E'_x \) for all \( x \in X \). If \( r = \text{rank} \ E \) then \( \Lambda^r E \) is given a special symbol \( \text{det} \ E \) and called the determinant line bundle since the transition functions of \( \Lambda^r E \) are given by the determinants of the transition functions of \( E \). In the case where \( E \) is the cotangent bundle \( T^*X \) for some manifold \( X \), \( \text{det} \ T^*X \) is called the canonical bundle and denoted by \( K \) or \( K_X \), respectively.

Another important concept is that of exact sequences. Let \( E_1, E_2 \) and \( E_3 \) be three complex vector bundles over \( X \). Then the sequence

\[
E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \quad \text{(A.6)}
\]

is exact at \( E_2 \) if \( \ker \psi = \text{im} \phi \). A short exact sequence is a sequence of the form

\[
0 \to E_1 \to E_2 \to E_3 \to 0 \quad \text{(A.7)}
\]

which is exact at \( E_1, E_2 \) and \( E_3 \). We say that the sequence splits if \( E_2 \cong E_1 \oplus E_3 \). Hence, one can understand \( E_2 \) in (A.7) as a deformation of the direct sum \( E_1 \oplus E_3 \).
Let us give an example. Let \( X = \mathbb{C}P^m \). Then
\[
0 \rightarrow \mathbb{C} \xrightarrow{1} \mathcal{O}(1) \otimes \mathbb{C}^{m+1} \xrightarrow{\psi} T\mathbb{C}P^m \rightarrow 0
\] (A.8)
is a short exact sequence which is called the Euler sequence (see also remark 5.4). This can be understood as follows. Let \( \pi : \mathbb{C}^{m+1} \setminus \{0\} \rightarrow \mathbb{C}P^m \) be the canonical projection and let \( z^a \), for \( a = 0, \ldots, m \), be linear coordinates on \( \mathbb{C}^{m+1} \) (or equivalently, homogeneous coordinates on \( \mathbb{C}P^m \)). In remark 4.9 we saw that \( \pi_*(z^m \partial_{a}) = 0 \), where \( \partial_{a} := \partial/\partial z^a \). If we let \( s = (s^0, \ldots, s^m) \) be a section of \( \mathcal{O}(1) \otimes \mathbb{C}^{m+1} \), then the mapping \( \psi \) in (A.8) is given by \( \psi(s) = \pi_*(s^a(z) \partial_{a}) \). Clearly, \( \psi \) is surjective and moreover, its kernel is the trivial line bundle spanned by the section \( s_0 = (z^0, \ldots, z^m) \), i.e. \( \psi(s_0) = 0 \). Thus, (A.8) is indeed a short exact sequence as claimed.

The next concept we need is that of connections and curvature. Let \( E \rightarrow X \) be a rank-\( r \) complex vector bundle and let \( \Omega^p(X, E) \) be the differential \( p \)-forms on \( X \) with values in \( E \). A connection \( \nabla \) on \( E \) is a differential operator
\[
\nabla : \Omega^p(X, E) \rightarrow \Omega^{p+1}(X, E)
\] (A.9)
which obeys the Leibniz rule. For \( f \) some function on \( X \) and \( s \) a section of \( T^rX \otimes E \), we have
\[
\nabla(fs) = df \wedge s + f \nabla s.
\] (A.10)

Suppose now that \( e = \{e_1, \ldots, e_r\} \) is a local frame field of \( E \) over \( U \subset X \), i.e. the \( e_a \) are sections of \( E \) over \( U \) for \( a = 1, \ldots, r \), and for all \( s \in U \), \( \{e_1(s), \ldots, e_r(s)\} \) is a basis for \( E_s \). Then we may introduce the connection one-form (or also referred to as the gauge potential) \( A \) according to
\[
\nabla e_a = A_a^b e_b,
\] (A.11)
i.e. \( A \) is a differential one-form with values in the endomorphism bundle \( \text{End} \, E \cong E^* \otimes E \) of \( E \). Thus, \( \nabla \) is locally of the form \( \nabla = d + A \). Let \( \{U_i\} \) be a covering of \( X \) which trivializes \( E \) and let \( e_i := e|_{U_i} = \{e_1|_{U_i}, \ldots, e_r|_{U_i}\} \) be the frame field with respect to \( \{U_i\} \). On \( U_i \cap U_j \neq \emptyset \) we have \( e_i = f_{ij}e_j \), where the \( f_{ij} \) are the transition functions of \( E \). Upon substituting this into (A.11), we find
\[
A_j = f_{ij}^{-1}d f_{ij} + f_{ij}^{-1}A_i f_{ij}, \quad \text{with} \quad A_j := A|_{U_i}.
\] (A.12)

Hence, patching together the \( A_i \) according to this formula, we obtain a globally defined connection one-form \( A \). The curvature (also referred to as the field strength) of \( \nabla \) is defined as
\[
F := \nabla^2
\] (A.13)
and is a section of \( \Lambda^2 T^*X \otimes \text{End} \, E \), i.e. under a change of trivialization, \( F \) behaves as
\[
F_j = f_{ij}^{-1}F_i f_{ij}, \quad \text{with} \quad F_i := F|_{U_i}.
\] (A.14)
Furthermore, the curvature is locally of the form
\[
F = dA + A \wedge A.
\] (A.15)

Note that \( \nabla F = 0 \), which is the so-called Bianchi identity.

**Appendix B. Characteristic classes**

In this section, we shall define certain characteristic classes. However, we shall avoid the general definition in terms of invariant polynomials and cohomology classes and instead focus on the example of Chern classes and characters.
Let $E \to X$ be a complex vector bundle and $F$ be the curvature two-form of a connection $\nabla = d + A$ on $E$. The total Chern class $c(E)$ of $E$ is defined by
\[ c(E) := \det \left( 1 + \frac{i}{2\pi} F \right). \tag{B.1} \]
Since $F$ is a differential two-form, $c(E)$ is a sum of forms of even degrees. The Chern classes $c_k(E)$ are defined by the expansion
\[ c(E) = 1 + c_1(E) + c_2(E) + \cdots. \tag{B.2} \]
Note that one should be more precise here, since the $c_k(E)$ written here are actually the Chern forms. They are even differential forms which are closed. The Chern classes are defined as the cohomology classes of the Chern forms and so the Chern forms are representatives of the Chern classes. However, we shall loosely refer to the $c_k(E)$ as Chern classes, i.e. $c_k(E) \in H^{2k}(X, \mathbb{Z})$ (see also section 2.2 for the definition of the cohomology groups). Note also that the above definitions do not depend on the choice of connection. If $E = TX$ then one usually writes $c_k(X)$.

If $2k > \dim_X X$ then $c_k(E) = 0$. Furthermore, if $k > \text{rank } E$ then $c_k(E) = 0$, as well. Hence, if $E$ is a line bundle then $c(E) = 1 + c_1(E)$ and if in addition $c_1(E) = 0$ then $E$ is a trivial bundle. A few explicit Chern classes are
\[ c_0(E) = 1, \]
\[ c_1(E) = \frac{i}{2\pi} \text{tr } F, \]
\[ c_2(E) = \frac{1}{2} \left( \frac{i}{2\pi} \right)^2 [\text{tr } F \wedge \text{tr } F - \text{tr} (F \wedge F)], \tag{B.3} \]
\[ \vdots \]
\[ c_r(E) = \left( \frac{i}{2\pi} \right)^r \det F, \]
where $r = \text{rank } E$.

If we consider the direct sum $E_1 \oplus E_2$ then $c(E_1 \oplus E_2) = c(E_1) \wedge c(E_2)$ as follows from the properties of the determinant. Furthermore, this property is deformation independent, i.e. it is also valid for the short exact sequence (A.7). This is known as the splitting principle. An immediate consequence is then that $c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2)$. This is an important fact that we used in section 5.2 when talking about Calabi–Yau spaces. Finally, if $g^* E$ denotes the pull-back bundle of $E$ via some map $g$ then $c(g^* E) = g^* c(E)$. This is called naturality or functoriality (see below).

A related quantity is the total Chern character. It is defined by
\[ \text{ch}(E) := \text{tr} \exp \left( \frac{i}{2\pi} F \right) = \text{ch}_0(E) + \text{ch}_1(E) + \text{ch}_2(E) + \cdots, \tag{B.4} \]
where the $k$th Chern character is
\[ \text{ch}_k(E) = \frac{1}{k!} \text{tr} \left( \frac{i}{2\pi} F \right)^k. \tag{B.5} \]
The Chern characters can be expressed in terms of the Chern classes according to
\[ \text{ch}_0(E) = \text{rank } E, \]
\[ \text{ch}_1(E) = c_1(E), \]
\[ \text{ch}_2(E) = \frac{1}{2} [c_1(E) \wedge c_1(E) - 2c_2(E)], \tag{B.6} \]
\[ \vdots \]
The total Chern character has the following properties. Let $E_1$ and $E_2$ be two complex vector bundles. Then
\[ \text{ch}(E_1 \oplus E_2) = \text{ch}(E_1) + \text{ch}(E_2) \]
and
\[ \text{ch}(E_1 \otimes E_2) = \text{ch}(E_1) \land \text{ch}(E_2) \]
If $g^* E$ denotes the pull-back bundle of $E$ via $g$ then $\text{ch}(g^* E) = g^* \text{ch}(E)$ which is also called naturality or functoriality (see below).

Likewise, we may introduce Chern classes and characters for supervector bundles over supermanifolds. Given a rank-$r|s$ complex vector bundle $E$ over a complex supermanifold $X$, we define the $k$th Chern class of $E$ to be
\[ c_k(E) := \frac{1}{k!} \left[ \frac{d^k}{dt^k} \right]_{t=0} \text{sdet} \left( 1 + t \frac{i}{2\pi} F \right) \quad \text{for} \quad k \leq r + s, \]
where $F$ is again the curvature two-form of a connection $\nabla$ on $E$ and ‘sdet’ is the superdeterminant defined in (4.9). The first few Chern classes are given by
\[ c_0(E) = 1, \]
\[ c_1(E) = \frac{i}{2\pi} \text{str} F, \]
\[ c_2(E) = \frac{1}{2} \left( \frac{i}{2\pi} \right)^2 \left[ \text{str} F \land \text{str} F - \text{str}(F \land F) \right], \]
\[ \vdots \]
Here, ‘str’ is the supertrace defined in (4.8). The total Chern class is then
\[ c(E) := \sum_{k=0}^{r+s} c_k(E). \]
In a similar fashion, we may also introduce the $k$th Chern character according to
\[ \text{ch}_k(E) := \frac{1}{k!} \left[ \frac{d^k}{dt^k} \right]_{t=0} \exp \left( t \frac{i}{2\pi} F \right) \quad \text{for} \quad k \leq r + s. \]

More details can be found, e.g., in the book by Bartocci et al [19].

**Appendix C. Categories**

A category $\mathcal{C}$ consists of the following data:
(C1) a collection Ob$(\mathcal{C})$ of objects,
(C2) sets Mor$(X,Y)$ of morphisms for each pair $X,Y \in \text{Ob}(\mathcal{C})$, including a distinguished identity morphism $\text{id}_X \in \text{Mor}(X,X)$ for each $X$,
(C3) a composition of morphisms function $\circ : \text{Mor}(X,Y) \times \text{Mor}(Y,Z) \rightarrow \text{Mor}(X,Z)$ for each triple $X,Y,Z \in \text{Ob}(\mathcal{C})$ satisfying $\text{id} \circ f = f = \text{id} \circ f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

There are many examples of categories, as follows.

(i) The category of vector spaces over $\mathbb{R}$ or $\mathbb{C}$ consists of all vector spaces over $\mathbb{R}$ or $\mathbb{C}$ (=objects). The morphisms are linear maps.
(ii) The category of topological spaces consists of all topological spaces (=objects). The morphisms are continuous maps.
(iii) The category of $C^k$-manifolds consists of all $C^k$-manifolds (=objects). The morphisms are $C^k$-maps. Note that $k$ can also be infinity in which case one speaks of smooth manifolds.
(iv) The category of complex manifolds consists of all complex manifolds (=objects). The morphisms are holomorphic maps.
(v) The categories of complex/holomorphic vector bundles consist of all complex/holomorphic vector bundles (=objects). The morphisms are the smooth/holomorphic bundle maps.
(vi) The category of Lie algebras over $\mathbb{R}$ or $\mathbb{C}$ consists of all Lie algebras over $\mathbb{R}$ or $\mathbb{C}$ (=objects) and the morphisms are those linear maps respecting the Lie bracket.
The category of Lie groups consists of all Lie groups (=objects). The morphisms are the Lie morphisms, which are smooth group morphisms.

Besides the notion of categories, we need the so-called functors which relate different categories. A functor $F$ from a category $\mathcal{C}$ to another category $\mathcal{D}$ takes each object $X$ in $\text{Ob}(\mathcal{C})$ and assigns an object $F(X)$ in $\text{Ob}(\mathcal{D})$ to it. Similarly, it takes each morphism $f$ in $\text{Mor}(X, Y)$ and assigns a morphism $F(f)$ in $\text{Mor}(F(X), F(Y))$ to it such that $F(\text{id}) = \text{id}$ and $F(f \circ g) = F(f) \circ F(g)$. Pictorially, we have

$$
\begin{array}{c}
X \\
\downarrow^f \\
F(X)
\end{array}
\begin{array}{c}
Y \\
\downarrow^F \\
F(Y)
\end{array}
$$

What we have just defined is a covariant functor. A contravariant functor differs from the covariant functor by taking $f$ in $\text{Mor}(X, Y)$ and assigning the morphism $F(f)$ in $\text{Mor}(F(Y), F(X))$ to it with $F(\text{id}) = \text{id}$ and $F(f \circ g) = F(g) \circ F(f)$.

The standard example is the so-called dual vector space functor. This functor takes a vector space $V$ over $\mathbb{R}$ or $\mathbb{C}$ and assigns a dual vector space $F(V) = V^*$ to it and to each linear map $f : V \to W$ it assigns the dual map $F(f) : W^* \to V^*$, with $\omega \mapsto \omega \circ f$ and $\omega \in W^*$, in the reverse direction. Hence, it is a contravariant functor.

Another functor we have already encountered is the parity map $\Pi$ defined in (4.4), which is a functor from the category of $R$-modules to the category of $R$-modules for $R$ being a supercommutative ring. To understand how it acts, let us take a closer look to morphisms between $R$-modules. An additive mapping of $R$-modules, $f : M \to N$, is called an even morphism if it preserves the grading and is $R$-linear. We denote the group of such morphisms by $\text{Hom}_0(M, N)$. On the other hand, we call an additive mapping of $R$-modules odd if it reverses the grading, $pf(m) = pm + 1$, and is $R$-linear, that is, $f(rm) = (-)^p rf(m)$ and $f(m r) = f(m) r$. The group of such morphisms is denoted by $\text{Hom}_1(M, N)$. Then we set

$$
\text{Hom}(M, N) := \text{Hom}_0(M, N) \oplus \text{Hom}_1(M, N)
$$

and it can be given an $R$-module structure. Then $\Pi$ is defined in (4.4), where we implicitly assumed that (i) addition in $\Pi M$ is the same as in $M$, (ii) right multiplication by $R$ is the same as in $M$ and (iii) left multiplication differs by a sign, i.e. $r \Pi m = (-)^p \Pi (rm)$ for $r \in R$, $m \in M$ and $\Pi(m) \in \Pi M$. Corresponding to the morphism $f : M \to N$, we let $f^\Pi : \Pi M \to \Pi N$ be the morphism which agrees with $f$ as a mapping of sets. Moreover, corresponding to the morphism $f : M \to N$, we can find morphisms

$$
\begin{align*}
\Pi f & : M \to \Pi N \quad \text{with} \quad (\Pi f)(m) := \Pi(f(m)), \\
f \Pi & : \Pi M \to N \quad \text{with} \quad (f \Pi)(\Pi m) := f(m),
\end{align*}
$$

and hence $f^\Pi = \Pi f \Pi$. Therefore, $\Pi$ is a covariant functor.

Appendix D. Sheaves

Let $X$ be a topological space. A pre-sheaf $\mathcal{S}$ of Abelian groups on $X$ consists of the following data:

(P1) for all open subsets $U \subset X$, an Abelian group $\mathcal{S}(U)$ and
(P2) for all inclusions $V \subset U$ of open subsets of $X$, a morphism of Abelian groups $r^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$
subject to the conditions:

(P3) \( r(\emptyset) = 0 \),

(P4) \( r_U^U = \text{id}_U : S(U) \to S(U) \) and

(P5) \( W \subset V \subset U \), then \( r_V^W \circ r_V^U = r_U^W \).

Put differently, a pre-sheaf is just a contravariant functor from the category \( \mathfrak{Top}(X) \) (objects = open sets of \( X \), morphisms = inclusion maps) to the category \( \mathfrak{Ab} \) of Abelian groups. In fact, we can replace \( \mathfrak{Ab} \) by any other fixed category \( C \).

Let \( S \) be a pre-sheaf on \( X \), then \( S(U) \) are the sections of \( S \) over \( U \). The \( r_U^V \) are called restriction maps.

A pre-sheaf \( S \) on a topological space is a sheaf if it satisfies

(S1) \( U \) open, \( \{ V_a \} \) open covering of \( U \), \( s \in S(U) \) such that \( r_{U \cap V_a}(s) = 0 \) for all \( a \), then \( s = 0 \),

(S2) \( U \) open, \( \{ V_a \} \) open covering of \( U \), \( s_a \in S(V_a) \), with \( r_{U \cap V_a}(s_a) = r_{U \cap V_b}(s_b) \), then there exists an \( s \in S(U) \) such that \( r_{U \cap V_a}(s) = s_a \) for all \( a \).

Point (S1) says that any section is determined by its local behaviour while (S2) means that local sections can be pieced together to give global sections.

Examples are as follows.

(i) Let \( X \) be a real manifold and \( U \subset X \). Then \( S(U) := \{ \text{smooth functions on } U \} \), \( \Omega^p(U) := \{ \text{smooth } p\text{-forms on } U \} \), etc, are pre-sheaves, where the restriction mappings are the usual restriction mappings. They are also sheaves.

(ii) Let \( X \) be a complex manifold and \( U \subset X \). Then, \( \mathcal{O}(U) := \{ \text{holomorphic functions on } U \} \), \( \Omega^{p,q}(U) := \{ \text{smooth } (p,q)\text{-forms on } U \} \), etc, are pre-sheaves, where the restriction mappings are the usual restriction mappings. They are also sheaves.

(iii) Let \( E \to X \) be a complex vector bundle over some manifold \( X \) and \( U \subset X \). Then \( S(E)(U) := \{ \text{smooth sections of } E \text{ on } U \} \) is a pre-sheaf, where the restriction mappings are the usual restriction mappings. It is also a sheaf.

(iv) Let \( E \to X \) be a holomorphic vector bundle over some complex manifold \( X \) and \( U \subset X \). Then, \( \mathcal{O}(E)(U) := \{ \text{holomorphic sections of } E \text{ on } U \} \) is a pre-sheaf, where the restriction mappings are the usual restriction mappings. It is also a sheaf.

(v) Let \( R \) be a ring, \( X \) a topological space and \( U \subset X \). Then \( R(U) := \{ \text{locally constant continuous functions on } U \} \). This determines a pre-sheaf that is a sheaf. We call this the constant sheaf \( R \) on \( X \), e.g. \( R = \mathbb{R}, \mathbb{C}, \ldots \).

(vi) Consider \( \mathcal{B}(U) := \{ \text{bounded holomorphic functions on } U \subset \mathbb{C} \} \). Then \( U \to \mathcal{B}(U) \) is a pre-sheaf but not a sheaf.

The remainder of this appendix collects some basic notions regarding (pre-)sheaves. Firstly, we need the notion of morphisms between (pre-)sheaves. A morphism of (pre-)sheaves \( \phi : S \to S' \) consists of a morphism of Abelian groups \( \phi_U : S(U) \to S'(U) \) for all open subsets \( U \) such that whenever \( V \subset U \) is an inclusion, the diagram

\[
\begin{array}{ccc}
S(U) & \xrightarrow{\phi_U} & S'(U) \\
\downarrow r_V^U & & \downarrow r_V^U' \\
S(V) & \xrightarrow{\phi_V} & S'(V)
\end{array}
\]

is commutative. An isomorphism is a morphism that has a two-sided inverse. A typical example is the de Rham complex on a real manifold, where the sheaf morphism is the usual exterior derivative.
Let $S$ be a pre-sheaf and

$$S_x := \bigcup_{U \ni x} S(U).$$

Then we say that two elements of $\tilde S_x$, $f \in S(U)$, $U \ni x$ and $g \in S(V)$, $V \ni x$, are equivalent if there exists an open set $W \subset (U \cap V)$, with $x \in W$ such that

$$r_{W}^{U}(f) = r_{W}^{V}(g).$$

We define the stalk $S_x$ to be the set of equivalence classes induced by this equivalence relation. Of course, $S_x$ inherits the algebraic structure of the pre-sheaf $S$, i.e. we can add elements in $S_x$ by adding representatives of equivalence classes. We shall let

$$\pi := r_{x}^{U} : S(U) \rightarrow S_x$$

be the natural restriction mapping to stalks.

Let $\phi : S \rightarrow S'$ be a morphism of sheaves on a topological space $X$. Then $\phi$ is an isomorphism if and only if the induced map on the stalks $\phi_x : S_x \rightarrow S'_x$ is an isomorphism for all $x \in X$. Note that this is not true for pre-sheaves.

Second, let us say a few words about exact sequences. We say that a sequence of morphisms of sheaves

$$S_1 \xrightarrow{\phi} S_2 \xrightarrow{\psi} S_3$$

on a topological space $X$ is exact at $S_2$ if the induced sequence

$$S_{1,x} \xrightarrow{\phi_x} S_{2,x} \xrightarrow{\psi_x} S_{3,x}$$

is exact, i.e. $\ker \psi_x = \operatorname{im} \phi_x$ for all $x \in X$. A short exact sequence of sheaves is a sequence of the form

$$0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow 0$$

which is exact at $S_1$, $S_2$ and $S_3$.

Before moving on, let us pause and give an example. Let $O^\ast$ be the sheaf of non-vanishing holomorphic functions on a complex manifold $X$. Then

$$0 \rightarrow \mathbb{Z} \rightarrow O \xrightarrow{\exp} O^\ast \rightarrow 0$$

is a short exact sequence, where $\exp(f) := e^{2\pi i f}$ for $f \in O(U)$ with $U \subset X$. This sequence is called the exponential sheaf sequence.

In section 2.2, we introduced the notion of sheaf cohomology, so let us state a basic fact about the sheaf cohomology for short exact sequences. Let $X$ be a topological space together with a short exact sequence of the form (D.7). Then (D.7) always induces a long exact sequence of Čech cohomology groups according to

$$0 \rightarrow H^0(X, S_1) \rightarrow H^0(X, S_2) \rightarrow H^0(X, S_3) \rightarrow 0$$

For a proof, see e.g. [4, 9].

Let us come back to the exponential sheaf sequence. We have

$$\cdots \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, O) \xrightarrow{\exp} H^1(X, O^\ast) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \rightarrow \cdots$$

By virtue of our discussion in section 5.2, $H^1(X, O^\ast)$ parametrizes the holomorphic line bundles $E \rightarrow X$. The image $c_1(E) \in H^2(X, \mathbb{Z})$ of a line bundle $E \in H^1(X, O^\ast)$ is the
first Chern class. If \( X = \mathbb{C}P^m \) then \( H^1(\mathbb{C}P^m, \mathcal{O}) = 0 = H^2(\mathbb{C}P^m, \mathcal{O}) \) and the above long exact cohomology sequence yields \( H^1(\mathbb{C}P^m, \mathcal{O}^*) \cong H^2(\mathbb{C}P^m, \mathbb{Z}) \cong \mathbb{Z} \), i.e. we have a classification of all holomorphic line bundles on complex projective space \( \mathbb{C}P^m \). Another related example we already encountered is given for the choice \( X = P^3 \) and \( E \to P^3 \) a holomorphic line bundle over \( P^3 \). Then \( H^1(P^3, \mathcal{O}) = 0 \) since \( P^3 \) is simply connected and if in addition \( c_1(E) = 0 \), i.e. \( L_x \) is holomorphically trivial on any \( L_x \to P^3 \), then we conclude from the above sequence that

\[
H^1(P^3, \mathcal{O}) \cong \{ E \in H^1(P^3, \mathcal{O}^*) \mid c_1(E) = 0 \}.
\] (D.11)

This is basically the content of exercise 3.8.

References

[1] Mason L J and Woodhouse N M J 1996 Integrability, Self-Duality, and Twistor Theory (Oxford: Clarendon)
[2] Dunajski M 2009 Solitons, Instantons and Twistors (Oxford: Oxford University Press)
[3] Lawson H B Jr and Michelsohn M L Spin Geometry (Princeton, NJ: Princeton University Press)
[4] Ward R S and Wells R O 1990 Twistor Geometry and Field Theory (Cambridge: Cambridge University Press)
[5] Penrose R 1967 Twistor algebra J. Math. Phys. 8 345
[6] Penrose R 1969 Solutions of the zero-rest-mass equations J. Math. Phys. 10 38
[7] Penrose R 1977 The twistor program Rep. Math. Phys. 12 65
[8] Radon J 1917 Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. 69 262
[9] Griffiths P and Harris J 1978 Principles of Algebraic Geometry (New York: Wiley)
[10] Hitchin N J 1986 Topological Methods in Algebraic Geometry (New York: Springer)
[11] Burns D 1979 Some background and examples in deformation theory Complex Manifold Techniques in Theoretical Physics ed D Lerner and P Sommers (San Fransisco, CA: Pitman)
[12] Kodaira K 1986 Complex Manifolds and Deformation of Complex Structures (New York: Springer)
[13] Ward R S 1977 The structure of supermanifolds Trans. Am. Math. Soc. 253 329
[14] Bailey T N and Eastwood M G 1991 Complex paraconformal manifolds—their differential geometry and twistor theory Forum Math.
[15] Prasad M K 1980 Instantons and monopoles in Yang–Mills gauge field theories Physica D 1 167
[16] Tong D 2005 TASI lectures on solitons: instantons, monopoles, vortices and kinks Tasi Lecture (Boulder, CO) (arXiv:hep-th/0509216)
[17] Manin Y I 1988 Gauge Field Theory and Complex Geometry (New York: Springer)
[18] Bartocci C, Bruzzo U and Hernández-Ruipérez D 1991 The Geometry of Supermanifolds (Dordrecht: Kluwer)
[19] DeWit B S 1992 Supermanifolds (Cambridge: Cambridge University Press)
[20] Frappat L, Sorba P and Sciarrino A 1996 Dictionary on Lie superalgebras arXiv:hep-th/9607161
[21] Semikhatov A M 1982 Supersymmetric instanton JETP Lett. 35 560
[22] Pohlmeyer K 1980 On the Lagrangian theory of anti-(self)-dual fields in four-dimensional Euclidean space Commun. Math. Phys. 72 37
[23] Chang L-L, Ge M-l and Wu Y-s 1982 The Kac–Moody algebra in the self-dual Yang–Mills equation Phys. Rev. D 25 1086
[33] Chau L-L and Yong-shi W 1982 More about hidden symmetry algebra for the self-dual Yang–Mills system Phys. Rev. D 26 3581
[34] Chau L-L, Ge M-L, Sinha A and Wu Y-S 1983 Hidden symmetry algebra for the self-dual Yang–Mills equation Phys. Lett. B 121 391
[35] Ueno K and Nakamura Y 1982 Transformation theory for anti-(self)-dual equations and the Riemann–Hilbert problem Phys. Lett. B 117 208
[36] Dolan L 1982 A new symmetry group of real self-dual Yang–Mills Phys. Lett. B 113 3581
[37] Dolan L 1984 Kac–Moody algebras and exact solvability in hadronic physics Phys. Rep. 109 1
[38] Popov A D and Preitschopf C R 1996 Conformal symmetries of the self-dual Yang–Mills equations Phys. Lett. B 374 71 (arXiv:hep-th/9512130)
[39] Popov A D 1999 Self-dual Yang–Mills: symmetries and moduli space Rev. Math. Phys. 11 1091 (arXiv:hep-th/9803183)
[40] Popov A D 1999 Holomorphic Chern–Simons–Witten theory: from 2D to 4D conformal field theories Phys. Rep. 109 1
[41] Ivanova T A 1998 On current algebra of symmetries of the self-dual Yang–Mills equations J. Math. Phys. 39 79 (arXiv:hep-th/9702144)
[42] Wolf M 2005 On hidden symmetries of a super gauge theory and twistor string theory J. High Energy Phys. JHEP02(2005)018 (arXiv:hep-th/0412163)
[43] Wolf M 2005 Twistor theory and integrability of self-dual SYM theory Proc. Int. Workshop on Supersymmetries and Quantum Symmetries vol 1 p 448 (arXiv:hep-th/0511230)
[44] Wolf M 2006 On supertwistor geometry and integrability in super gauge theory PhD Thesis Leibniz Universität Hannover (arXiv:hep-th/0611103)
[45] Popov A D and Wolf M 2007 Hidden symmetries and integrable hierarchy of the $N=4$ supersymmetric Yang–Mills equations Commun. Math. Phys. 275 685 (arXiv:hep-th/0608225)
[46] Bardeen W A 1996 Self-dual Yang–Mills theory, integrability and multiparton amplitudes Proc. Theor. Phys. Suppl. 123 1
[47] Cangemi D 1997 Self-dual Yang–Mills theory and one-loop maximally-helicity-violating multi-gluon amplitudes Nucl. Phys. B 484 521 (arXiv:hep-th/9605208)
[48] Cangemi D 1997 Self-duality and maximally-helicity-violating QCD amplitudes Int. J. Mod. Phys. A 12 1215 (arXiv:hep-th/9610021)
[49] Rosly A A and Selivanov K G 1997 On amplitudes in self-dual sector of Yang–Mills theory Phys. Lett. B 399 135 (arXiv:hep-th/9611101)
[50] Witten E 1995 Chern–Simons gauge theory as a string theory Prog. Math. 133 637 (arXiv:hep-th/9207094)
[51] Bouchard V 2007 Lectures on complex geometry, Calabi–Yau manifolds and toric geometry arXiv:hep-th/0702063
[52] Sethi S 1994 Supermanifolds, rigid manifolds and mirror symmetry Nucl. Phys. B 430 31 (arXiv:hep-th/9404186)
[53] Witten E 2004 Perturbative gauge theory as a string theory in twistor space Commun. Math. Phys. 252 189 (arXiv:hep-th/0312171)
[54] Roček M and Wadhwia N 2005 On Calabi–Yau supermanifolds Adv. Theor. Math. Phys. 9 315 (arXiv:hep-th/0408188)
[55] Zhou C-G 2005 On Ricci-flat supermanifolds J. High Energy Phys. JHEP02(2005)004 (arXiv:hep-th/0410047)
[56] Roček M and Wadhwia N 2004 On Calabi–Yau supermanifolds: II arXiv:hep-th/0410081
[57] Sámann C 2005 The topological B model on fattened complex manifolds and subsectors of $N=4$ self-dual Yang–Mills theory J. High Energy Phys. JHEP01(2005)042 (arXiv:hep-th/0410292)
[58] Lindström U, Roček M and von Unge R 2006 Ricci-flat supertwistor spaces J. High Energy Phys. JHEP01(2006)163 (arXiv:hep-th/0509211)
[59] Ricci R 2007 Super Calabi–Yau’s and special Lagrangians J. High Energy Phys. JHEP03(2007)048 (arXiv:hep-th/0511284)
[60] LeBrun C R 2004 Geometry of twistor spaces Simons Workshop Lecture (Stony Brook)
[61] Sokatchev E 1996 An action for $N=4$ supersymmetric self-dual Yang–Mills theory Phys. Rev. D 53 2062 (arXiv:hep-th/9509099)
[62] Popov A D 2010 Hermitian Yang–Mills equations and pseudo holomorphic bundles on nearly Kähler and nearly Calabi–Yau twistor 6-manifolds Nucl. Phys. B 828 594 (arXiv:0907.0106)
[63] Karnas S and Ketov S V 1998 An action of $N=8$ self-dual supergravity in ultra-hyperbolic harmonic superspace Nucl. Phys. B 526 597 (arXiv:hep-th/0712151)
[64] Mason L J and Wolf M 2009 Twistor actions for self-dual supergravities Commun. Math. Phys. 288 97 (arXiv:0706.1941)

[65] Popov A D and Sämann C 2005 On supertwistors, the Penrose–Ward transform and $\mathcal{N} = 4$ super Yang–Mills theory Adv. Theor. Math. Phys. 9 931 (arXiv:hep-th/0405123)

[66] Mason L J 2005 Twistor actions for non-self-dual fields: a derivation of twistor string theory J. High Energy Phys. JHEP10(2005)009 (arXiv:hep-th/0507269)

[67] Boels R, Mason L and Skinner D 2007 Supersymmetric gauge theories in twistor space J. High Energy Phys. JHEP02(2007)014 (arXiv:hep-th/0604040)

[68] Boels R, Mason L and Skinner D 2007 From twistor actions to MHV diagrams Phys. Lett. B 648 90 (arXiv:hep-th/0702035)

[69] Nakahara M 2002 Geometry, Topology and Physics (Bristol: Institute of Physics Publishing)

[70] Woodhouse N M J 1985 Real methods in twistor theory Class. Quantum Grav. 2 257

[71] Popov A D and Wolf M 2004 Topological B model on weighted projective spaces and self-dual models in four dimensions J. High Energy Phys. JHEP09(2004)007 (arXiv:hep-th/0406224)

[72] Giombi S, Kalazizzi M, Ricci R, Robles-Llada D, Trancanelli D and Zoubos K 2005 Orbifolding the twistor string Nucl. Phys. B 719 234 (arXiv:hep-th/0411171)

[73] Chiou D-W, Ganor O J, Hong Y P, Kim B S and Mitra I 2005 Massless and massive three-dimensional super Yang–Mills theory and mini-twistor string theory Phys. Rev. D 71 125016 (arXiv:hep-th/0502076)

[74] Popov A D, Sämann C and Wolf M 2005 The topological B model on a mini-supertwistor space and supersymmetric Bogomolny monopole equations J. High Energy Phys. JHEP10(2005)058 (arXiv:hep-th/0505161)

[75] Lechtenfeld O and Sämann C 2006 Matrix models and D branes in twistor string theory J. High Energy Phys. JHEP03(2006)002 (arXiv:hep-th/0511130)

[76] Mason L J and Skinner D 2006 An ambitwistor Yang–Mills Lagrangian Phys. Lett. B 636 60 (arXiv:hep-th/0510262)

[77] Bedford J, Papageorgakis C and Zoubos K 2007 Twistor strings with flavour J. High Energy Phys. JHEP11(2007)088 (arXiv:0708.1248)

[78] Bertlmann R A 1996 Anomalies in Quantum Field Theory (Oxford: Clarendon)

[79] Witten E 1978 An interpretation of classical Yang–Mills theory Phys. Lett. B 77 394

[80] Isenberg J, Yasskin P B and Green P S 1978 Non-self-dual gauge fields Phys. Lett. B 78 462

[81] Eastwood M G 1987 Supersymmetry, twistors, and the Yang–Mills equations Class. Quantum Grav. 4 1823

[82] Howe P S and Hartwell G G 1995 A superspace survey Class. Quantum Grav. 12 1823

[83] Sämann C 2009 On the mini-supersymambitwistor space and $\mathcal{N} = 8$ super Yang–Mills theory Adv. Math. Phys. 2009 784215 (arXiv:hep-th/0508137)

[84] Hori K, Katz S, Klemm A, Rahal P, Thomas R, Vafa C, Vakil R and Zaslow E 2003 Mirror symmetry (Providence, RI: American Mathematical Society)

[85] Berkovits N 2004 An alternative string theory in twistor space for $\mathcal{N} = 4$ super Yang–Mills theory Phys. Rev. Lett. 93 041601 (arXiv:hep-th/0402045)

[86] Mason L and Skinner D 2008 Heterotic twistor string theory Nucl. Phys. B 795 105 (arXiv:0708.2276)

[87] Berkovits N and Witten E 2004 Conformal supergravity in twistor string theory J. High Energy Phys. JHEP08(2004)009 (arXiv:hep-th/0406051)

[88] Kleiss R and Kuijf H 1999 Multi-gluon cross-sections and five-jet production at hadron colliders Nucl. Phys. B 507 615

[89] Kunszt Z 1986 Combined use of the CALKUL method and recursive calculations for processes with three-jet production at hadron colliders J. High Energy Phys. JHEP02(1986)014 (arXiv:hep-th/0702035)

[90] Mangano M L and Parke S J 1991 Multi-parton amplitudes in gauge theories Phys. Rep. 200 301 (arXiv:hep-th/0509223)

[91] Parke S J and Taylor T R 1986 An amplitude for $n$-gluon scattering Phys. Rev. Lett. 56 2459

[92] Bern Z, Dixon L J, Dunbar D C and Kosower D A 2009 One-loop self-dual and $\mathcal{N} = 4$ super Yang–Mills JHEP02(2009)014 (arXiv:hep-th/0604040)
Bianchi M, Brandhuber A, Spence B J and Travaglini G 2005 Non-supersymmetric loop amplitudes and MHV vertices Nucl. Phys. B 712 59 (arXiv:hep-th/0412108)

Glover E W N, Mastrolia P and Williams C 2008 One-loop phi-MHV amplitudes using the unitarity bootstrap: the general helicity case J. High Energy Phys. JHEP08(2008)017 (arXiv:0804.4149)

Boels R and Schwinn C 2008 Deriving CSW rules for massive scalar legs and pure Yang–Mills loops J. High Energy Phys. JHEP07(2008)007 (arXiv:0805.1197)

Mansfield P 2006 The Lagrangian origin of MHV rules J. High Energy Phys. JHEP03(2006)037 (arXiv:hep-th/0511264)

Gorsky A and Rosly A 2006 From Yang–Mills Lagrangian to MHV diagrams J. High Energy Phys. JHEP01(2006)101 (arXiv:hep-th/0510111)

Ettle J H and Morris T R 2006 Structure of the MHV rules Lagrangian J. High Energy Phys. JHEP08(2006)003 (arXiv:hep-th/0605121)

Feng H and Huang Y-t 2009 MHV Lagrangian for $N = 4$ super Yang–Mills J. High Energy Phys. JHEP04(2009)047 (arXiv:hep-th/0611164)

Ettle J H 2008 MHV Lagrangians for Yang–Mills and QCD PhD Thesis University of Southampton (arXiv:0808.1975)

Giombi S, Ricci R, Robles-Llana D and Trancanelli D 2004 A note on twistor gravity amplitudes J. High Energy Phys. JHEP07(2004)059 (arXiv:hep-th/0405086)

Nair V P 2005 A note on MHV amplitudes for gravitons Phys. Rev. D 71 121701 (arXiv:hep-th/0501143)

Bjerrum-Bohr N E J, Dunbar D C, Ita H, Perkins W B and Risager K 2006 MHV vertices for gravity amplitudes J. High Energy Phys. JHEP01(2006)009 (arXiv:hep-th/0509016)

Nasti A and Travaglini G 2007 One-loop $N = 8$ supergravity amplitudes from MHV diagrams Class. Quantum Grav. 24 6071 (arXiv:0706.0976)

Bianchi M, Elvang H and Freedman D Z 2008 Generating tree amplitudes in $N = 4$ SYM and $N = 8$ SG J. High Energy Phys. JHEP09(2008)063 (arXiv:0805.0757)

Wolf M 2007 Self-dual supergravity and twistor theory Class. Quantum Grav. 24 6287 (arXiv:0705.1422)

Kiermaier M and Naculich S G 2009 A super MHV vertex expansion for $N = 4$ SYM theory J. High Energy Phys. JHEP05(2009)072 (arXiv:0903.0377)

Drummond J M and Henn J M 2009 All tree-level amplitudes in $N = 4$ SYM J. High Energy Phys. JHEP04(2009)018 (arXiv:0808.2475)

Drummond J M, Henn J, Smirnov V A and Sokatchev E 2007 Magic identities for conformal four-point integrals J. High Energy Phys. JHEP01(2007)064 (arXiv:hep-th/0607160)

Alday L F and Maldacena J M 2007 Gluon scattering amplitudes at strong coupling J. High Energy Phys. JHEP06(2007)064 (arXiv:0705.0303)

Ricci R, Tseytlin A A and Wolf M 2007 On T-duality and integrability for strings on AdS backgrounds J. High Energy Phys. JHEP12(2007)082 (arXiv:0711.0707)

Drummond JM, Korchemsky GP and Sokatchev E 2008 Conformal properties of four-gluon planar amplitudes and Wilson loops Nucl. Phys. B 795 385 (arXiv:0707.0243)

Drummond L F and Maldacena J 2007 Comments on gluon scattering amplitudes via AdS/CFT J. High Energy Phys. JHEP11(2007)068 (arXiv:0710.1060)

Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2010 Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes Nucl. Phys. B 826 337 (arXiv:0712.1223)

Berkovits N and Maldacena J 2008 Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection J. High Energy Phys. JHEP09(2008)062 (arXiv:0807.3196)

Beisert N, Ricci R, Tseytlin A A and Wolf M 2008 Dual superconformal symmetry from $AdS_4 \times S^3$ superstring integrability Phys. Rev. D 78 126004 (arXiv:0807.3228)

Brandhuber A, Heslop P and Travaglini G 2008 A note on dual superconformal symmetry of the $N = 4$ super Yang–Mills S-matrix Phys. Rev. D 78 125005 (arXiv:0807.4097)

Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2008 Generalized unitarity for $N = 4$ superamplitudes arXiv:0808.0491

Drummond J M, Henn J M and Plefka J 2009 Yangian symmetry of scattering amplitudes in $N = 4$ super Yang–Mills theory J. High Energy Phys. JHEP05(2009)046 (arXiv:0902.2987)

Beisert N 2009 T-Duality, dual conformal symmetry and integrability for strings on $AdS_4 \times S^3$ Fortschr. Phys. 57 329 (arXiv:0903.0609)

Alday L F, Henn J M, Plefka J and Schuster T 2009 Scattering into the fifth dimension of $N = 4$ super Yang–Mills J. High Energy Phys. JHEP01(2010)077 (arXiv:0908.0684)

Henn J M, Naculich S G, Schnitzer H J and Spradlin M 2010 Higgs-regularized three-loop four-gluon amplitude in $N = 4$ SYM: exponentiation and Regge limits J. High Energy Phys. JHEP04(2010)038 (arXiv:1001.1358)
[191] Britto R, Cachazo F and Feng B 2005 New recursion relations for tree amplitudes of gluons Nucl. Phys. B 715 499 (arXiv:hep-th/0412308)

[192] Roiban R, Spradlin M and Volovich A 2005 Dissolving $\mathcal{N} = 4$ loop amplitudes into QCD tree amplitudes Phys. Rev. Lett. 94 102002 (arXiv:hep-th/0412265)

[193] Britto R, Cachazo F, Feng B and Witten E 2005 Direct proof of tree-level recursion relation in Yang–Mills theory Phys. Rev. Lett. 94 181602 (arXiv:hep-th/0501052)

[194] Arkani-Hamed N, Cachazo F and Kaplan J 2008 What is the simplest quantum field theory? arXiv:0808.1446

[195] Bern Z, Dixon L J and Kosower D A 2005 On-shell recurrence relations for one-loop QCD amplitudes Phys. Rev. D 71 105013 (arXiv:hep-th/0501240)

[196] Arkani-Hamed N and Kaplan J 2008 On tree amplitudes in gauge theory and gravity J. High Energy Phys. JHEP04/2008076 (arXiv:0801.2385)

[197] Luo M-x and Wen C-k 2005 Recursion relations for tree amplitudes in super gauge theories J. High Energy Phys. JHEP03(2005)004 (arXiv:hep-th/0501121)

[198] Bedford J, Brandhuber A, Spence B J and Travaglini G 2005 A recursion relation for gravity amplitudes Nucl. Phys. B 721 98 (arXiv:hep-th/0502146)

[199] Cachazo F and Svrček P 2005 Tree level recursion relations in general relativity arXiv:hep-th/0502160

[200] Benincasa P, Boucher-Veronneau C and Cachazo F 2007 Taming tree amplitudes in general relativity J. High Energy Phys. JHEP11(2007)057 (arXiv:hep-th/0702032)

[201] Benincasa P and Cachazo F 2008 Consistency conditions on the S-matrix of massless particles arXiv:0705.4305

[202] Hodges A P 2006 Scattering amplitudes for eight gauge fields arXiv:hep-th/0603101