ASYMPTOTIC ESTIMATES OF THE NORMS OF POSITIVE DEFINITE TÖPLITZ MATRICES AND DETECTION OF QUASI-PERIODIC COMPONENTS OF STATIONARY RANDOM SIGNALS

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Abstract. Asymptotic forms of the Hilbert-Scmidt and Hilbert norms of positive definite Töplitz matrices \( Q_N = (b(j-k))_{j,k=0}^{N-1} \) as \( N \to \infty \) are determined. Here \( b(j) \) are consequent trigonometric moments of a generating non-negative measure \( d\sigma(\theta) \) on \( [-\pi, \pi] \). It is proven that \( \sigma(\theta) \) is continuous if and only if any of those contributions is \( o(N) \). Analogous criteria are given for positive integral operators with difference kernels.

Obtained results are applied to processing of stationary random signals, in particular, neutron signals emitted by boiling water nuclear reactors.

1. Introduction

This paper is motivated by the studies [VGM01] devoted to the problem of detection of hidden unstable components in random neutron signals measured in boiling water nuclear reactors. We assume that a signal from a monitored system forms, during a sufficiently long time interval, a real-valued stationary random process \( \xi(t), t \in \mathbb{Z} \), with discrete time, such that the means

\[
\langle \xi(t) \rangle = 0, \quad \langle \xi^2(t) \rangle = 1.
\]

The correlation function of such a process

\[
b(t) := \langle \xi(t)\xi(0) \rangle = \langle \xi(t+t')\xi(t') \rangle, \quad t, t' \in \mathbb{Z},
\]

is a real-valued sequence, which admits the representation

\[
b(t) = \int_{-\pi}^{\pi} \exp(it\theta) d\sigma(\theta), \tag{1}
\]

where \( \sigma(\theta) \) is a non-decreasing bounded function on \([-\pi, \pi]\) [R67]. By virtue of our assumptions

\[
b(0) = \sigma(\pi) - \sigma(-\pi) = 1,
\]

and for any \( \theta_1, \theta_2 \) such that \( 0 \leq \theta_1 \leq \theta_2 \leq \pi \), we have

\[
\sigma(\theta_2) - \sigma(\theta_1) = \sigma(-\theta_1) - \sigma(-\theta_2). \tag{2}
\]

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In general, the spectral distribution function $\sigma(\theta)$ which determines the process correlation function by (1), can be split into a sum

$\sigma(\theta) = \sigma_c(\theta) + \sigma_d(\theta)$

of a continuous non-decreasing function $\sigma_c(\theta)$ and a non-decreasing step function $\sigma_d(\theta)$, and such a representation is unique up to constant contributions in $\sigma_c(\theta)$ and $\sigma_d(\theta)$ \cite{R67}. Notice that both functions, $\sigma_c(\theta)$ and $\sigma_d(\theta)$, satisfy the condition (2).

Actually, the problem, formulated in \cite{VGM01}, was to find, in a real-time operation mode, whether the spectral distribution function of the random signal $\sigma(\theta)$ contains or does not contain a non-trivial component $\sigma_d(\theta)$. We will call here for brevity the random process (signal) $\xi(t)$ stable if its spectral distribution function $\sigma(\theta)$ is continuous and unstable otherwise. In other words, the process is unstable if and only if

$\sigma_d(\pi) - \sigma_d(-\pi) > 0$.

The main task of the present work is to formulate criteria of stability of the process in terms of its correlation function $b(t)$. As a main tool we use the sequence of positive definiteToeplitz matrices $Q_N = (b(j-k))_{j,k=0}^{N-1}$. This paper is organized in the following way.

In Section 2 we find the principal asymptotic contribution of the Hilbert-Schmidt norm $\|Q_N\|_2$ with $N \to \infty$, and show that the process is stable if and only if this contribution is $o(N)$.

Section 3 contains a similar criterion, but with the Hilbert norm $\|Q_N\|$ instead of $\|Q_N\|_2$. We prove here that if $N \to \infty$, then $\|Q_N\| = m \cdot N + o(N)$, where $m$ is the maximal jump of $\sigma(\theta)$.

In Section 4 both criteria are generalized for continuous time processes or for positive integral operators with difference kernels.

In Section 5 the efficiency of the above stability criteria for signal processing is discussed and illustrated by application to simulated signals and real neutron signals emitted by the Forsmark 1&2 boiling water reactor.

2. ASYMPTOTIC FORM OF THE HILBERT-SCHMIDT NORM OF TRUNCATED CORRELATION MATRICES AND THE STABILITY CRITERION

Let us denote by $\{Q_N\}, N = 1, 2, \ldots$, the sequence of Toeplitz matrices $(b(j-k))_{j,k=0}^{N-1}$ and let $\|Q_N\|_2$ be the Hilbert-Schmidt norm of $Q_N$:

$$\|Q_N\|_2 = \left( \sum_{j,k=0}^{N-1} b^2(j-k) \right)^{\frac{1}{2}} = \left( N b^2(0) + 2 \sum_{k=1}^{N-1} (N-k) b^2(k) \right)^{\frac{1}{2}}.$$

Our assumptions imply that

$$b^2(k) = \left| \int_{-\pi}^{\pi} \exp(ik\theta) d\sigma(\theta) \right|^2 \leq b^2(0) = 1.$$

Therefore $\|Q_N\|_2 \leq N$.

**Theorem 2.1.** The process $\xi(t)$ is stable if and only if

$$\lim_{N \to \infty} \frac{1}{N} \|Q_N\|_2 = 0.$$
Proof. Introduce

\[ \Phi_N(\theta) = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}N(\theta - \theta')}{N^2 \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma_c(\theta') \]

and

\[ \Psi_N(\theta) = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}N(\theta - \theta')}{N^2 \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma_d(\theta') = \]

\[ \frac{1}{2} \sum_{\alpha} \left( \frac{\sin^2 \frac{1}{2}N(\theta - \theta_\alpha)}{N^2 \sin^2 \frac{1}{2}(\theta - \theta_\alpha)} + \frac{\sin^2 \frac{1}{2}N(\theta + \theta_\alpha)}{N^2 \sin^2 \frac{1}{2}(\theta + \theta_\alpha)} \right) m_\alpha, \]

where \( \{ \pm \theta_\alpha \} \) is the set of jump points of \( \sigma(\theta) \) or, what is the same, of the points of growth of \( \sigma_d(\theta) \),

\[ \frac{1}{2} m_\alpha = \sigma(\theta_\alpha + 0) - \sigma(\theta_\alpha - 0) = \sigma_d(-\theta_\alpha + 0) - \sigma_d(-\theta_\alpha - 0). \]

Due to (1) and (4), we have

\[ \frac{1}{N^2} \| Q_N \|^2 = \frac{1}{N^2} \sum_{j,k=0}^{N-1} b^2(j-k) = \]

\[ (5) \frac{1}{N^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \sum_{j,k=0}^{N-1} \exp(i(j-k)(\theta-\theta')) \right) d\sigma(\theta') d\sigma(\theta) = \]

\[ \int_{-\pi}^{\pi} \Phi_N(\theta) d\sigma_c(\theta) + \int_{-\pi}^{\pi} \Psi_N(\theta) d\sigma_c(\theta) + 2 \int_{-\pi}^{\pi} \Phi_N(\theta) d\sigma_d(\theta) + \int_{-\pi}^{\pi} \Psi_N(\theta) d\sigma_d(\theta). \]

Let

\[ \omega(\delta) = \max_{|\theta-\theta'\leq\delta} |\sigma_c(\theta) - \sigma_c(\theta')|. \]

The continuity of \( \sigma_c(\theta) \) implies \( \omega(\delta) \to 0 \) as \( \delta \to 0 \).

Since

\[ (6) \frac{\sin^2 \frac{1}{2}N(\theta - \theta')}{N^2 \sin^2 \frac{1}{2}(\theta - \theta')} \leq 1 \]

and

\[ |\sin x| \geq \frac{2}{\pi} |x|, \quad |x| \leq \frac{\pi}{2} \pmod{2\pi}, \]

then, for any \( 0 < \delta < \frac{2}{\pi} \), we have

\[ \Phi_N(\theta) = \int_{|\theta-\theta'|<\delta \pmod{2\pi}} \frac{\sin^2 \frac{1}{2}N(\theta - \theta')}{N^2 \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma_c(\theta') + \]

\[ \int_{|\theta-\theta'|>\delta \pmod{2\pi}} \frac{\sin^2 \frac{1}{2}N(\theta - \theta')}{N^2 \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma_c(\theta') \leq \]

\[ (7) 2\omega(\delta) + \frac{\pi^2}{N^2} [\sigma_c(\pi) - \sigma_c(-\pi)] \leq 2\omega(\delta) + \frac{\pi^2}{N^2\delta^2}. \]
Therefore, by an appropriate choice of \( \delta \) and \( N \), the first two integrals on the right-hand part of \( \text{(5)} \) can be made arbitrarily small. Hence, these integrals tend to zero as \( N \to \infty \).

Rewrite now the third integral on the right-hand side of \( \text{(5)} \) in the form

\[
\int_{-\pi}^{\pi} \Psi_N(\theta) \, d\sigma_d(\theta) = \frac{1}{2} \sum_{\alpha} m_\alpha^2 + \frac{1}{2} \sum_{\alpha \neq \alpha'} m_\alpha m_{\alpha'} \left( \frac{\sin^2 \frac{1}{2} N (\theta_\alpha - \theta_{\alpha'})}{\sin^2 \frac{1}{2} (\theta_\alpha - \theta_{\alpha'})} + \frac{\sin^2 \frac{1}{2} N (\theta_\alpha + \theta_{\alpha'})}{\sin^2 \frac{1}{2} (\theta_\alpha + \theta_{\alpha'})} \right),
\]

and assume that \( \sigma_d(\theta) \) has a finite number of jumps, then we can conclude, taking into account \( \text{(6)}, \text{(7)} \) and the inequality

\[
\sum_{\alpha} m_\alpha = \sigma_d(\pi) - \sigma_d(-\pi) \leq \sigma(\pi) - \sigma(-\pi) = 1
\]

that

\[
\lim_{N \to \infty} \frac{1}{N^2} \|Q_N\|_2^2 = \frac{1}{2} \sum_{\alpha} m_\alpha^2.
\]

Thus,

\[
\max_{\alpha} m_\alpha \leq \sqrt{\frac{1}{2} \sum_{\alpha} m_\alpha^2} = \lim_{N \to \infty} \frac{1}{N} \|Q_N\|_2 \leq \sqrt{\max_{\alpha} m_\alpha}.
\]

The generalization of \( \text{(5)} \) and, hence, of \( \text{(4)} \) for the case of \( \sigma_d(\theta) \) having infinitely many jumps can be obtained by continuity using the standard method employed in the next section.

The condition of Theorem 2.1 in special cases can be specified.

**Proposition 2.2.** If the process is stable and its spectral distribution function \( \sigma(\theta) \) satisfies the Hölder condition:

\[
|\sigma(\theta_1) - \sigma(\theta_2)| \leq A|\theta_1 - \theta_2|^\nu, \quad 0 < \nu \leq 1, \quad -\pi \leq \theta_1, \theta_2 \leq \pi,
\]

then

\[
\|Q_N\|_2 \sim \frac{1}{N^{\frac{1}{2\pi}}} O \left( N^{\frac{1}{2\pi-\nu}} \right).
\]

**Proof.** Using the estimate \( \text{(7)} \) and the assumptions of the proposition we deduce that

\[
\|Q_N\|_2 \leq \sqrt{2A\delta^\nu N^2 + \frac{\pi^2}{\delta^2}}, \quad 0 < \delta \leq \frac{\pi}{2}.
\]

By the minimization of the latter estimate in \( \delta \) we obtain that

\[
\|Q_N\|_2 \leq \sqrt{3\pi} \frac{1}{2^{\frac{1}{2\nu}}} A \frac{1}{2^{\frac{1}{2\nu}}} N^{\frac{1}{2\pi-\nu}}.
\]
3. Asymptotic Form of the Maximal Eigenvalue of a Truncated Correlation Matrix

Let us denote by $\lambda_m(N)$ the maximal eigenvalue of a positive (i.e., positive definite) matrix $Q_N$, $\lambda_m(N) = \|Q_N\|$. The condition of Theorem 2.1 admits the following weakening.

**Theorem 3.1.** The process $\xi(t)$ is stable if and only if

$$\lim_{N \to \infty} \frac{\lambda_m(N)}{N} = \frac{1}{N} \lim_{N \to \infty} \frac{1}{N} \|Q_N\| = 0.$$ 

**Proof.** The Hilbert norm $\|A\|$ of any square matrix (or any nuclear operator $A$) satisfies the inequality

$$\|A\| \geq \|A\|_1 \left| \|A\|_2 \right|,$$

where $\|A\|_1$ and $\|A\|_2$ are the nuclear and Hilbert-Schmidt norms, respectively. Recall that $\|A\|$ for positive definite $A$ coincides with the maximal eigenvalue $\lambda_m(A)$ of $A$ and for $A \geq 0$ (11) takes the form

$$\lambda_m(A) \geq \frac{\text{Tr} A^2}{\text{Tr} A}.$$

The application of (12) and (8) to $Q_N$ by virtue of the equality $\text{Tr} Q_N = N b(0) = N$ gives

$$\lim_{N \to \infty} \frac{\lambda_m(N)}{N} \geq \frac{1}{2} \sum_{\alpha} m_{\alpha}^2.$$

On the other hand,

$$\lambda_m(N) \leq \sqrt{\text{Tr} Q_N^2}.$$

Hence,

$$\lim_{N \to \infty} \frac{\lambda_m(N)}{N} \leq \frac{1}{2} \sum_{\alpha} m_{\alpha}^2.$$

□

The inequalities (13), (14) can be specified.

**Theorem 3.2.** Given the sequence of maximal eigenvalues (norms) $\{\lambda_m(N)\}$ of positive definite Töplitz matrices $\{Q_N\}$, generated by a non-negative measure $d\sigma(\theta)$ (3), it holds that

$$\lim_{N \to \infty} \frac{\lambda_m(N)}{N} = \max_{\alpha} m_{\alpha}.$$

**Proof.** The Töplitz matrix $Q_N$ generated by the non-decreasing function (3) is the sum of non-negative Töplitz matrices $Q_N^{(c)}$ and $Q_N^{(d)}$, generated by non-decreasing functions $\sigma_c$ and $\sigma_d$, respectively. Let us denote by $\lambda_m^{(c)}(N)$ and $\lambda_m^{(d)}(N)$ the maximal eigenvalues (norms) of the matrices $Q_N^{(c)}$ and $Q_N^{(d)}$, respectively. Since $Q_N \geq Q_N^{(d)}$, then

$$\lambda_m^{(d)}(N) \leq \lambda_m(N) = \|Q_N^{(d)} + Q_N^{(c)}\| \leq \|Q_N^{(d)}\| + \|Q_N^{(c)}\| = \lambda_m^{(d)}(N) + \lambda_m^{(c)}(N).$$
By virtue of Theorem 3.1, 
\[ \lambda^{(c)}_m (N) = o(N) \]. Hence it remains to prove that
\[ \lim_{N \to \infty} \frac{\lambda^{(d)}_m (N)}{N} = \max_{\alpha} m_{\alpha} . \]

To this end, let us consider first the case of \( \sigma_d (\theta) \) having only a finite number \( 2s \) of points of growth. We will not use in this proof the fact that the jump points of \( \sigma_d (\theta) \) are located symmetrically with respect to the point \( \theta = 0 \). The Töplitz matrix \( Q_N^{(d)} = (b_d (j - k))_{j,k=0}^{N-1} , 2s \ll N \), generated by \( \sigma_d \), can be represented in this case in the form
\[
Q_N^{(d)} = \sum_{\alpha=1}^{2s} m_{\alpha} (\cdot, e_{\alpha}) e_{\alpha} ,
\]
where
\[ (\cdot, e_{\alpha}) e_{\alpha} = (\exp (i (j - k) \theta_{\alpha}))_{j,k=0}^{N-1} \]
are \( N \times N \) matrices of unit rank, so that \( Q_N \) transforms a \( N \times 1 \) column vector \( x = (x_j)_{j=0}^{N-1} \) into
\[
Q_N^{(d)} x = \sum_{\alpha=1}^{2s} m_{\alpha} (x, e_{\alpha}) e_{\alpha} ,
\]
where \((\cdot, \cdot)\) is the scalar product in the linear space of \( N \times 1 \) column vectors \( \mathbb{C}_N \) defined in a standard way:
\[
(x, y) = \sum_{j=0}^{N-1} x_j y_j , \quad x = (x_j)_{j=0}^{N-1} , \quad y = (y_j)_{j=0}^{N-1} .
\]

Notice that the vectors \( \{e_{\alpha}\} \) are linearly independent. Indeed, suppose that there is a set of complex numbers \( \{z_{\alpha}\} \) such that
\[ \sum_{\alpha=1}^{s} z_{\alpha} e_{\alpha} = 0 . \]
Due to \( 18 \), the numbers \( z_{\alpha} \) satisfy the homogeneous system
\[ \sum_{\alpha=1}^{s} \exp (i k \theta_{\alpha}) z_{\alpha} = 0 , \quad k = 0, 1, \ldots, 2s - 1. \]
But the determinant of this system is the Van der Monde determinant, which vanishes if and only if among the numbers \( \{\exp (i k \theta_{\alpha})\} \) there are equals. The latter is impossible by our assumption. Hence all \( z_{\alpha} = 0 \).

Let \( \lambda \) be a non-zero eigenvalue of \( Q_N^{(d)} \) and \( h_{\lambda} \) be a corresponding non-zero eigenvector:
\[
\sum_{\alpha=1}^{s} m_{\alpha} (h_{\lambda}, e_{\alpha}) e_{\alpha} = \lambda h_{\lambda} .
\]
By \( 19 \) \( h_{\lambda} \) admits the representation:
\[ h_{\lambda} = \sum_{\alpha=1}^{s} z_{\alpha} e_{\alpha} , \]
where $z_{\alpha}$ are some complex numbers, not all of which are equal to zero. Put

$$\eta_{\alpha} = \sqrt{m_{\alpha}} ( h_{\lambda}, e_{\alpha}) .$$

By virtue of (19), not all numbers $\eta_{\alpha} = 0$. Taking the scalar products of both sides of (19) with all vectors $\sqrt{m_{\alpha}} e_{\alpha}$, we obtain the following homogeneous system for $\eta_{\alpha}$:

$$\sum_{\alpha' = 1}^{s} \sqrt{m_{\alpha} m_{\alpha'}} (e_{\alpha'}, e_{\alpha}) \eta_{\alpha'} = \lambda \eta_{\alpha} .$$

Thus, the non-zero eigenvalues of $Q_N^{(d)}$ coincide, with account of their multiplicities, with the eigenvalues of the $2s \times 2s$ Hermitian positive definite matrix

$$A_N = (\sqrt{m_{\alpha} m_{\alpha'}} (e_{\alpha'}, e_{\alpha}))^{2s}_{\alpha, \alpha' = 1} .$$

Notice that by definition of the vectors $e_{\alpha}$, we have

$$(e_{\alpha'}, e_{\alpha}) = \frac{\exp (i \pi (\theta_{\alpha'} - \theta_{\alpha})) - 1}{\exp (i (\theta_{\alpha'} - \theta_{\alpha})) - 1}, \alpha' \neq \alpha; (e_{\alpha}, e_{\alpha}) = N .$$

Hence, the matrix $A_N$ is the sum of the diagonal matrix

$$A_{1,N} := (N m_{\alpha} \delta_{\alpha \alpha'})^{2s}_{\alpha, \alpha' = 1}$$

and the Hermitian matrix $A_{2,N}$ with zero diagonal elements and non-diagonal elements $\sqrt{m_{\alpha} m_{\alpha'}} (e_{\alpha'}, e_{\alpha}), \alpha \neq \alpha'$. By (22) the non-diagonal elements of $A_{2,N}$ are uniformly bounded:

$$|\sqrt{m_{\alpha} m_{\alpha'}} (e_{\alpha'}, e_{\alpha})| \leq 2 \left( \max_{\alpha' \neq \alpha} |\theta_{\alpha'} - \theta_{\alpha}|^{-1} \right) \left( \max_{\alpha} m_{\alpha} \right),$$

and, hence,

$$\|A_{2,N}\| \leq (4s - 2) \left( \max_{\alpha' \neq \alpha} |\theta_{\alpha'} - \theta_{\alpha}|^{-1} \right) \left( \max_{\alpha} m_{\alpha} \right).$$

Therefore,

$$\left[ N - (4s - 2) \left( \max_{\alpha' \neq \alpha} |\theta_{\alpha'} - \theta_{\alpha}|^{-1} \right) \left( \max_{\alpha} m_{\alpha} \right) \right] \leq \|A_{1,N}\| - \|A_{2,N}\| \leq N \|A_{1,N}\| + \|A_{2,N}\| \leq \left[ N + (4s - 2) \left( \max_{\alpha' \neq \alpha} |\theta_{\alpha'} - \theta_{\alpha}|^{-1} \right) \left( \max_{\alpha} m_{\alpha} \right) \right].$$

We see that

$$\lambda_{\alpha}^{(d)} (N) = \|A_{N}\| = N \max_{\alpha} m_{\alpha} + O (1).$$

To prove the relation (15) for a non-decreasing step function $\sigma_{d} (\theta)$, $\sigma_{d} (\pi) - \sigma_{d} (-\pi) = 1$, having infinitely many points of jump, we take a small $\varepsilon > 0$ and split $\sigma_{d} (\theta)$ into a sum $\sigma_{1,d} (\theta) + \sigma_{2,d} (\theta)$ of non-decreasing step functions $\sigma_{1,d} (\theta)$ and $\sigma_{2,d} (\theta)$, where, as before, $\sigma_{1,d} (\theta)$ has a finite number of jump points and $\sigma_{2,d} (\theta)$ is such that

$$\int_{-\pi}^{\pi} d\sigma_{2,d} (\theta) < \varepsilon < \max_{\alpha} m_{\alpha} .$$
With respect to this split, we represent the Toeplitz matrix $Q_N^{(d)}$ as the sum $Q_N^{(1,d)} + Q_N^{(2,d)}$ of non-negative Toeplitz matrices generated by $\sigma_{1,d}(\theta)$ and $\sigma_{2,d}(\theta)$, respectively. Notice that by construction

$$\|Q_N^{(2,d)}\| \leq \text{Tr}Q_N^{(2,d)} < N\varepsilon .$$

Besides,

$$\lambda_m^{(1,d)}(N) = \left\|Q_N^{(1,d)}\right\| \leq \lambda_m^{(d)}(N) \leq \lambda_m^{(1,d)}(N) + \left\|Q_N^{(2,d)}\right\| .$$

Applying the estimate (23) to $Q_N^{(1,d)}$ and taking into account the inequality (24) for $N \to \infty$ yields

$$N \cdot \max_{\alpha} m_{\alpha} + O(1) = \lambda_m^{(1,d)}(N) \leq \lambda_m^{(d)}(N) \leq N \cdot \left( \max_{\alpha} m_{\alpha} + \varepsilon \right) + O(1) .$$

Finally,

$$\max_{\alpha} m_{\alpha} \leq \lim_{N \to \infty} \frac{\lambda_m^{(d)}(N)}{N} \leq \lim_{N \to \infty} \frac{\lambda_m^{(d)}(N)}{N} \leq \max_{\alpha} m_{\alpha} + \varepsilon ,$$

where $\varepsilon > 0$ can be taken arbitrarily small. \hfill \Box

**Remark 3.3.** For the Hilbert norm $\|C\|$ of any $N \times N$ matrix $C = (c_{jk})_{j,k=0}^{N-1}$ the following estimate holds:

$$\|C\| \leq \max \left\{ \max_{j} \sum_{p=0}^{N-1} |c_{jp}| , \max_{k} \sum_{p=0}^{N-1} |c_{pk}| \right\} .$$

If $C = (c_{j-k})_{j,k=0}^{N-1}$ is a Toeplitz matrix, then by (26)

$$\|C\| \leq |c_0| + \sum_{p=1}^{N-1} (|c_p| + |c_{-p}|) .$$

Thus for the Toeplitz matrices $Q_N$, which are under consideration here, Theorem 3.2 by virtue of (26) gives

$$\frac{1}{2} \max_{\alpha} m_{\alpha} = \lim_{N \to \infty} \frac{1}{2N} \|Q_N\| \leq \lim_{N \to \infty} \frac{1}{2N} \sum_{p=0}^{N-1} |b(p)| .$$

Therefore the condition

$$\lim_{N \to \infty} \frac{1}{N} \sum_{p=0}^{N-1} |b(p)| = 0$$

is sufficient for the process stability.

### 4. Extension to Continuous Time Processes

Real signals are, certainly, continuous time processes, $\xi(t)$. The correlation function $b(t)$ of a process having a finite second moment $\langle \xi^2(t) \rangle$ is a Hermitian positive function. As such, it admits the representation

$$b(t) = \int_{-\infty}^{\infty} \exp(i\lambda t) d\theta(\lambda) ,$$
where \( \vartheta(\lambda) \) is a bounded non-decreasing function on the real axis. Like for the discrete time processes, \( \vartheta(\lambda) \) can be represented, in general, as the sum

\[
\vartheta(\lambda) = \vartheta_c(\lambda) + \vartheta_d(\lambda)
\]

of a non-decreasing continuous function \( \vartheta_c(\lambda) \) and a non-decreasing step function \( \vartheta_d(\lambda) \), and we call the process stable if \( \vartheta(\lambda) \) is continuous and unstable otherwise.

To investigate the instability characteristics of a continuous time process, we consider instead of the Töplitz matrices \( Q_N \), the set of non-negative integral operators

\[
(B_T f)(t) = \int_0^T b(t-s) f(s) \, ds, \quad 0 < T < \infty,
\]

in the Hilbert spaces \( L_2(0,T) \). Since \( b(t) \) is a continuous function, all these operators are nuclear and their nuclear and Hilbert-Schmidt norms \( \|B_T\|_1 \) and \( \|B_T\|_2 \) are given by the expressions

\[
\|B_T\|_1 = T b(0) = \int_{-\infty}^{\infty} d\vartheta(\lambda),
\]

\[
\|B_T\|_2 = \sqrt{2T \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2 \, dt} =
\]

\[
\sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4}{(\lambda - \lambda')^2} \sin^2 \left(\frac{\lambda - \lambda'}{2}\right) d\vartheta(\lambda') \, d\vartheta(\lambda)}. \quad (32)
\]

Let us denote, as before, by \( \{m_\alpha\} \) the set of jumps of \( \vartheta(\lambda) \). Using (31), (32) and the arguments similar to those employed in the proofs of Theorems 2.1 and 3.2, we obtain the following criteria of stability of a continuous time process \( \xi(t) \).

**Theorem 4.1.** A stationary continuous time process \( \xi(t) \) is stable if and only if its correlation function \( b(t) \) possesses the property:

\[
\lim_{T \to \infty} \frac{2}{T} \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2 \, dt = 0.
\]

Otherwise,

\[
\lim_{T \to \infty} \frac{2}{T} \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2 \, dt = \sum_\alpha m_\alpha^2.
\]

**Theorem 4.2.** A stationary continuous time process \( \xi(t) \) is stable if and only if the Hilbert norms \( \|B_T\| \) of integral operators (30), where \( b(t) \) is the correlation function of the process, are such that

\[
\lim_{T \to \infty} \frac{1}{T} \|B_T\| = 0.
\]

Otherwise,

\[
\lim_{T \to \infty} \frac{1}{T} \|B_T\| = \max_\alpha m_\alpha. \quad (33)
\]
Remark 4.3. For the norm of the integral operator $B_T$ the following estimate:

$$\|B_T\| \leq 2 \int_0^T |b(t)| \, dt$$

holds. As it stems from (33) the relation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |b(t)| \, dt = 0 ,$$

guarantees the stability of the process $\xi(t)$.

5. Application to processing of random signals

5.1. Detection of quasi-periodic components in a random signal. The jumps of the spectral distribution function $\vartheta(\lambda)$ of a stationary process is, in general, a sign of appearance of undamped oscillation components in a signal and the points of discontinuity of $\vartheta(\lambda)$ are either the frequencies of such components themselves or directly related to them. Notice that, due to physical reasons, the measurement of $\xi(t)$ is possible only at discrete moments of time with a step $\Delta$. If $t$ in (29) is an integer multiple of $\Delta$, then it is evident that

$$b(t) = \int_{-\Omega}^{\Omega} \exp(it\theta) \, d\sigma(\theta), \quad \Omega = \frac{\pi}{\Delta},$$

$$\sigma(\theta) = \sum_{n=\pm\infty} \left[ \vartheta(\theta + 2n\Omega) - \vartheta(2n\Omega - \Omega) \right], \quad -\Omega \leq \theta < \Omega .$$

The function $\sigma(\theta)$ is bounded and non-decreasing in the interval $[-\Omega, \Omega]$. If $\vartheta(\lambda)$ loses its continuity at the points $\pm\lambda_1, \pm\lambda_2, \ldots$, then $\sigma(\theta)$ has a non-void set of discontinuity points

$$\left\{ \pm \theta_j' \right\} = \left\{ \mathbb{E} \left( \pm \frac{\lambda_j}{2\Omega} + \frac{1}{2} \right) - \frac{1}{2} \right\} 2\Omega \subset [-\Omega, \Omega] ,$$

where $\mathbb{E}(x)$ is the fractional part of the number $x$. (In general, $\pm \theta_j'$ coincides with every $\pm \theta_j'$ such that $\lambda_j - \lambda_j$ is a multiple of $2\Omega$.) Therefore, in general, the jump of $\sigma(\theta)$ at a point $\theta_j'$ is the sum of the jumps of $\vartheta(\lambda)$ at all co-images of $\theta_j'$ under the mapping (35). Taking $\Delta$ as the time measurement unit, we return to the representation (1) of $b(t)$ for integer $t$. Thus, the spectral distribution function $\sigma(\theta)$ inherits all discontinuities of $\vartheta(\lambda)$ from the interval $[-\Omega, \Omega]$ and also may get new ones at the points (calculated according to (35)) related to the discontinuity points of $\vartheta(\lambda)$ outside this interval. We see that the spectral distribution function for the discrete time process obtained in such a way from a continuous time process, has a non-trivial component $\sigma_d$ if and only if the corresponding spectral distribution function of the initial discrete time process has non-zero jumps on some set of points. In other words, the values of a random continuous time process measured at discrete moments of time, form a stable discrete time process if and only if the initial process is stable.
The correlation function of the discrete time process delivers not only the described gauge of instability of the process, but also the following tool for the detection of the points \( \{ \pm \theta_{\alpha} \} \), which are the discontinuity points of \( \sigma ( \theta ) \). Put

\[
\Theta_N ( \theta ) = b ( 0 ) + 2 \sum_{k=0}^{N-1} \left( 1 - \frac{k}{N} \right) b ( k ) \cos k \theta = \\
= \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2} N ( \theta - \theta')}{N^2 \sin^2 \frac{1}{2} ( \theta - \theta')} d\sigma ( \theta') .
\]  

(36)

It is not difficult to see that

\[
\lim_{N \to \infty} \frac{1}{N} \Theta_N ( \theta ) = \sigma ( \theta + 0 ) - \sigma ( \theta - 0 ) .
\]

Further, take a sufficiently large \( N \gg 1 \) and split the interval \([-\pi, \pi]\) into equal segments of longitude \( \delta \) such that \( N \delta \ll 1 \). Let \( \sigma ( \theta ) \) have a jump \( m_{\alpha} \) within the interval \( (l\delta, (l+1)\delta) \). Since

\[
\frac{5}{6} |x| \leq |x| \left( 1 - \frac{x^2}{6} \right) \leq |\sin x| \leq |x| ,
\]

then, for \( |\theta - l\delta| \sim \delta \), we have

\[
\Theta_N ( \theta ) \geq \int_{l\delta}^{(l+1)\delta} \frac{\sin^2 \frac{1}{2} N ( \theta - \theta')}{N^2 \sin^2 \frac{1}{2} ( \theta - \theta')} d\sigma ( \theta') \geq \frac{5}{6} \int_{l\delta}^{(l+1)\delta} d\sigma ( \theta') \geq \frac{5}{6} N m_{\alpha} .
\]

On the other hand, if the continuous part \( \sigma_c ( \theta ) \) of \( \sigma ( \theta ) \) satisfies the condition of Proposition 2.2, then one can use the arguments similar to those employed in the proof of this proposition to show that

\[
\Theta_N ( \theta ) = O \left( N^{\frac{3}{4}} \right)
\]

at the points remote from the jumps of \( \sigma ( \theta ) \). Hence, the values of \( \Theta_N (l\delta) \), where \( l \) is an integer which satisfies the condition \(-\pi \leq l\delta \leq \pi\), at the distances \( \sim \delta \) from the jump points of \( \sigma ( \theta ) \), must be visible as larger than those at the distances \( \sim l_0 \delta \), where \( Nl_0 \delta \sim 1 \).

The assertion of Theorem 3.2 can be used for the detection of symptoms of emerging instabilities of a random process, which can be considered as stationary for long time intervals. The method consists in the construction of the correlation function of the process from a piece of its time series from the beginning of observation to a rather far off moment of time \( \Upsilon \) in the future. Set, as usually,

\[
b ( k ) = \frac{1}{\Upsilon - k} \sum_{p=0}^{\Upsilon - k} \xi ( p + k ) \xi ( p ) - m^2 , \\
m = \frac{1}{\Upsilon} \sum_{p=0}^{\Upsilon} \xi ( p ) ,
\]

and compute, for a sufficiently large \( N \ll \Upsilon \), the numbers

\[
\frac{1}{N^2} \| Q_N \|_2^2 = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \left( 1 - \frac{k}{N} \right) b^2 ( k ) b^2 ( 0 ) + \frac{1}{N} \sum_{p=0}^{N-1} \left| \frac{b ( p )}{b ( 0 )} \right| ,
\]

(38)
or the numbers
\[ \frac{1}{T} \int_0^T \left( 1 - \frac{t}{T} \right) |b(t)|^2 \, dt, \quad \frac{1}{T} \int_0^T |b(t)| \, dt \]
for a continuous time process. An explicit tendency of any of these numbers to be bounded, for increasing \( N \), from below by certain positive numbers, is a serious evidence of the process instability.

The following example demonstrates that the manifestation of such a tendency begins the sooner in \( N \) the larger is the contribution of the oscillating components generated by \( d\sigma_d(\theta) \) into \( b(0) \).

Let the correlation function of a stationary random process be given by the expressions:

(39) \[ b(0) = 1; \quad b(k) = b(-k) = \sum_{\alpha=1}^s m_\alpha \cos k\theta_\alpha, \quad k = 1, 2, \ldots \]

\[ 1 \leq s < \infty, \quad 0 < \sum_{\alpha=1}^s m_\alpha < 1, \quad 0 < \theta_1, \ldots, \theta_s < \pi \, . \]

The spectral distribution function \( \sigma(\theta) \) of such a process is the sum of the spectral distribution function \( \sigma_c(\theta) \) of the "white noise",

\[ d\sigma_c(\theta) = \frac{p}{2\pi} d\theta, \quad p = 1 - \sum_{\alpha=1}^s m_\alpha \, , \]

and the step function \( \sigma_d(\theta) \), the jump points of which are \( \{ \pm \theta_\alpha \} \), and

\[ \sigma_d(-\theta_\alpha + 0) - \sigma_d(-\theta_\alpha - 0) = \sigma_d(\theta_\alpha + 0) - \sigma_d(\theta_\alpha - 0) = \frac{1}{2} m_\alpha \, . \]

To calculate the right-hand side of (5) in this special case, notice that

\[ \frac{1}{2\pi} \pi \int_{-\pi}^\pi \frac{\sin^2 \frac{1}{2} N (\theta - \theta')}{\sin^2 \frac{1}{2} (\theta - \theta')} \, d\theta' \equiv N \, . \]

Therefore we see that now \( \Phi_N(\theta) = pN^{-1} \) and thus

(40) \[ \frac{1}{N^2} \|Q_N\|_2^2 = \frac{1}{2} \sum_{\alpha=1}^s m_\alpha^2 + \frac{p(2-p)}{N} + \]

\[ \frac{1}{2N^2} \sum_{\alpha' \neq \alpha} m_\alpha m_{\alpha'} \left( \frac{\sin^2 \frac{1}{2} N (\theta_\alpha - \theta_{\alpha'})}{\sin^2 \frac{1}{2} (\theta_\alpha - \theta_{\alpha'})} + \frac{\sin^2 \frac{1}{2} N (\theta_\alpha + \theta_{\alpha'})}{\sin^2 \frac{1}{2} (\theta_\alpha + \theta_{\alpha'})} \right) \, . \]

Hence, the first term on the right hand side of (40) becomes dominant for

\[ N \gtrsim 2p(2-p) \left( \sum_{\alpha=1}^s m_\alpha^2 \right)^{-1} \, . \]
5.2. **Numerical results.** Let us apply the latter results to the investigation of real signals obtained from the Forsmark 1\&2 boiling water reactor (BWR) [VGM01], and also to simulated random signals.

In Fig.1 and Fig.2 we display the sequences of the values

\[
\frac{1}{N^2} \|Q_N\|^2 = \frac{1}{N^2} \sum_{j,k=0}^{N-1} b^2 (j-k)
\]

for the Töplitz matrices constructed for two different shots of real signals obtained in the Forsmark BWR, named in [VGM01] as \(c4_{lprm3}\) and \(c4_{lprm22}\). The sampling interval of these signals equal 0.08 s, and both of them consist of 4209 points. In [VGM01] several methods were used to obtain the corresponding decay ratio (DR) value. The decay ratio is a parameter related to the signal stability: the signal is more unstable if its DR is higher, see Appendix for details. The mean values of the DR given by these methods are 0.90 for \(c4_{lprm3}\) and 0.51 for \(c4_{lprm22}\).

We observe the difference between a more unstable shot, \(c4_{lprm3}\) and a more stable one, \(c4_{lprm22}\). As expected, for higher DR, the sequence tends to zero slower.

To determine the points of discontinuity of \(\sigma (\theta)\) we have also calculated the function

\[
\frac{1}{N}\Theta_N (\theta) = \frac{b(0)}{N} + \frac{2}{N} \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) b(k) \cos k\theta
\]
for different values of $N$. The corresponding results are provided at Fig. 3 and Fig. 4, they demonstrate different behavior of $\frac{1}{N} \Theta_N (\theta)$ with $N$ growing, and for different degrees of stability measured by $DR$. In both cases the number of segments into which the interval $[-\pi, \pi]$ was split, was equal to 3000. It can be observed that for $c4_{lprm22}$, a more stable signal, the function $\frac{1}{N} \Theta_N (\theta)$ tends to zero with increasing $N$ much more rapidly.

The peaks of the signal correspond to the instability frequencies. The main peak for $c4_{lprm3}$ is obtained for $\theta = 0.24 \text{ rad}$, which corresponds to the frequency of $f = 0.48 \text{ Hz}$. On the other hand, for $c4_{lprm22}$ the main peak is located at $\theta = 0.27 \text{ rad}$ ($f = 0.53 \text{ Hz}$). We can compare these results with those obtained in [VGM01]. The range of main instability frequencies obtained there is $[0.480, 0.495] \text{ Hz}$ for $c4_{lprm3}$ and $[0.450, 0.557] \text{ Hz}$ for $c4_{lprm22}$. The method allows to locate secondary instability frequencies situated quite close. In Fig. 3 there is a secondary peak located at $\theta = 0.27 \text{ rad}$, it coincides with the main peak of the shot $c4_{lprm22}$. Notice also that the present approach to the detection of instabilities is model-free.

In addition, we have also simulated signals with different degrees of stability using the model suggested in [SVM88], i.e., from the following continuous Langevin model:

$$\ddot{\xi} (t) + c \dot{\xi} (t) + U (\xi) = F (t),$$
with $F(t)$ being a Gaussian colored external force such that

$$
\dot{F}(t) + \tau^{-1}F(t) = \tau^{-1}W(t),
$$

and

$$
U(\xi) = a_1\xi + a_2\xi^2 + a_3\xi^3.
$$

Here $c$ is a damping constant, $a_j$, $j = 1, 2, 3$, are some model constant parameters, and $\tau$ is the correlation time, while $W(t)$ is a white Gaussian noise with the correlation function

$$
\langle W(t)W(t') \rangle = D\delta(t-t').
$$

The parameters of the model, $c$, $a_1$, $a_2$, $a_3$, $D$ and $\tau$, are directly related to the stability of the signal: $a_1$ coincides with $w^2$, $w = 2\pi f$ being the fundamental frequency, and the parameters $c$ and $a_1$ determine $DR$, see Appendix.

In order to compare the results obtained for the simulated signals with those obtained previously for the real ones, we constructed the simulated signals with the values of the parameters chosen to produce the values of $f$ and $DR$ similar to those of the signals $c_{4 \text{prm}3}$ and $c_{4 \text{prm}22}$, and with the same sampling interval, 0.08 s, and the same number of points, 4209. In Fig.5 and Fig.6 the sequences of the values of the Töplitz matrices for the models with $c = 0.689$ ($DR = 0.5$) and $c = 0.105$ ($DR = 0.9$) are gathered (the values of other parameters are: $a_1 = 9.87$, which corresponds to $f = 0.5$ Hz, $a_2 = a_3 = 0$, $D = 500$, $\tau = 0.6$).

As expected, the behavior of these signals is close to that obtained for the real signals, the sequences tend slower to zero for higher $DR$. 

Figure 3. Function $\Theta_N(\theta)$, for the signal $c_{4 \text{prm}3}$, $N = 100$, $N = 300$, $N = 500$. 

$\begin{align*}
\Theta_N(\theta) &= \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_j),
\end{align*}$

where $\theta_j$ are the angles corresponding to the points of the signal $c_{4 \text{prm}3}$.
For the same simulated signals, the discontinuity points are shown in Fig. 7 and Fig. 8. As before, the number of segments in which the interval $[-\pi, \pi]$ was split was equal to 3000.

Again, for lower $DR$ the function $\frac{1}{N} \Theta_N(\theta)$ tends to zero with $N$ much more rapidly. The main peak is now obtained at $\theta = 0.24$ ($f = 0.48$ Hz) for the model with $DR = 0.5$, and at $\theta = 0.25$ rad ($f = 0.50$ Hz) for the model with $DR = 0.9$.

5.3. Appendix. The notion of decay ratio ($DR$), a basic parameter in the analysis of reactor stability, is deduced from the oscillatory model (42), but with (44)

$$U(\xi) = w^2 \xi.$$  

The $DR$ is a measure of the system damping defined as the ratio between two consecutive maxima of the signal, for the model (42) it is a constant parameter,

$$DR = \exp \left\{ -\frac{2\pi c}{\sqrt{4\omega^2 - c^2}} \right\}.$$  

Neutronic signals are very noisy and, in general, their behavior cannot be fitted to that of a continuous second-order system, hence the $DR$ in reality is not a constant, and its value depends on the model used to evaluate it. Nevertheless, it gives a hint on the system stability.
Figure 5. Sequences of $\frac{1}{N^2} \| Q_N \|^2_2$ for the model signals, linear scale of $N$.

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Figure 6. Sequences of $\frac{1}{N^2} \|Q_N\|^2_2$ for the model signals, logarithmical scale of $N$. 
Figure 7. Function $\frac{1}{N} \Theta_N(\theta)$, for the model with $DR = 0.5$, $N = 100, N = 200, N = 400$. 
Figure 8. Function $\frac{1}{N} \Theta_N (\theta)$, for the model with $DR = 0.9$, $N = 100, N = 200, N = 400$. 
