CATALAN MATROID DECOMPOSITIONS OF CERTAIN POSITROIDS

BRENDAN PAWLOWSKI

Abstract. Given a permutation $w$ in $S_n$, the matroid of a generic $n \times n$ matrix whose non-zero entries in row $i$ lie in columns $w(i)$ through $n+i$ is an example of a positroid. We enumerate the bases of such a positroid as a sum of certain products of Catalan numbers, each term indexed by the 123-avoiding permutations above $w$ in Bruhat order. We also give a similar sum formula for their Tutte polynomials. These are both avatars of a structural result writing such a positroid as a disjoint union of matroids, each isomorphic to a direct sum of Catalan matroids and a matroid with one basis.

1. Introduction

Given a permutation $w \in S_n$, consider a generic $1 \times 2n$ matrix $M_w$ whose nonzero entries in row $i$ are in columns $[w(i), i + n]$. Here $[a, b]$ denotes $\{a, a + 1, \ldots, b\}$ for integers $a$ and $b$; we will also write $[n]$ for $[1, n]$. For example,

$$M_{2143} = \begin{bmatrix}
0 & * & * & * & 0 & 0 \\
* & * & * & * & 0 & 0 \\
0 & 0 & 0 & * & * & * \\
0 & 0 & * & * & * & *
\end{bmatrix}.$$

Let $P_w$ be the set of bases of the matroid associated to $M_w$. That is, $P_w$ is the set of $I \in \binom{[2n]}{n}$ such that the $n \times n$ minor of $M_w$ in columns $I$ is nonzero.

The matroid $P_w$ belongs to (at least) two interesting classes of matroids. First, it is a transversal matroid; see [4] for an introduction. Take a collection $A = \{A_1, \ldots, A_n\}$ of finite sets. A transversal of $A$ is a set $\{x_1, \ldots, x_n\}$ such that $x_i \in A_i$ for each $i$ and all the $x_i$ are distinct. The set of all transversals of $A$ is the set of bases for a matroid. Indeed, if $M_A$ is a generic matrix with $n$ rows whose nonzero entries in row $i$ are in columns $A_i$, then the matroid of $M_A$ is exactly the transversal matroid of $A$. Thus, $M_w$ is the transversal matroid of the set collection $\{[w(i), i + n] : 1 \leq i \leq n\}$.

Second, $P_w$ is a positroid: the matroid of a real matrix whose maximal minors are all nonnegative. Given a rank $k$ positroid $P$ on $[N]$, Knutson, Lam, and Speyer considered the closure of the locus in the complex Grassmannian $\text{Gr}(k, N)$ of $k$-planes having matroid $P$ with respect to a fixed basis [6]. Among other nice properties, these positroid varieties turn out to be exactly the images of Richardson varieties in the complete flag variety under projection $\text{Fl}(N) \to \text{Gr}(k, N)$.

Given any set of intervals $S = \{[a_1, b_1], \ldots, [a_k, b_k]\}$ in $[N]$, taking the rowspans of matrices of the form $M_S$ gives a subset of $\text{Gr}(k, n)$ whose closure is an irreducible variety called a rank variety. Billey and Coskun showed that rank varieties are exactly the images of Richardson varieties in the partial flag variety under projection $\text{Fl}(1, 2, \ldots, k; \mathbb{C}^n) \to \text{Gr}(k, N)$ [2]. Thus, every rank variety is a positroid variety. In particular, $P_w$ is a positroid.

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Having its nonzero entries algebraically independent, say.
Our main results concern the size and structure of $P_w$. An anti-fixed point of $w \in S_n$ is a number $i \in [n]$ such that $w(i) = n - i + 1$. Define a permutation statistic
\[ g(w) = C_{\ell_1 + 1} \cdots C_{\ell_k + 1}, \]
where $\ell_1, \ldots, \ell_k$ are the lengths of the maximal runs of consecutive anti-fixed points in $w$, and $C_j$ is the $j$th Catalan number.

**Theorem 1.1.** For any $w \in S_n$, $P_w$ has size
\[ \sum_{v \geq w \text{ avoids } 123} g(v), \]
where $\geq$ is strong Bruhat order.

Here, a permutation avoids 123 if it has no (not necessarily consecutive) increasing subsequence of length 3. In the special case that $w = w_0 = n(n-1) \cdots 1$, Theorem 1.1 reads $\#P_{w_0} = C_{n+1}$. In fact, $P_{w_0}$ is isomorphic to the rank $n+1$ Catalan matroid $C_{n+1}$ considered by Ardila, whose bases are the Dyck paths of length $2n + 2$, each path viewed as the set of its upsteps [1]. Theorem 1.1 arises from a stronger structural result for $P_w$ (cf. Theorem 3.2 below).

**Theorem 1.2.** There is a partition of $\binom{[2n]}{n}$ into sets $Q_v$ indexed by 123-avoiding permutations $v$ such that for any $w \in S_n$,
\begin{itemize}
  \item $P_w$ is the (disjoint) union $\bigcup_{v \geq w \text{ avoids } 123} Q_v$
  \item If $v$ has runs of consecutive anti-fixed points of lengths $\ell_1, \ldots, \ell_k$, then $Q_v$ is isomorphic to a direct sum of the Catalan matroids $C_{\ell_1+1}, \ldots, C_{\ell_k+1}$ plus a matroid with one basis. In particular, $\#Q_v = g(v)$.
\end{itemize}

In Section 2 we use a bijection of Krattenthaler between 123-avoiding permutations and Dyck paths to prove Theorem 1.1 in the case where $w$ is the identity permutation. This special case will be useful in proving Theorem 1.2 which we do in Section 3. In Section 4 we give a formula for the Tutte polynomial of $P_w$ along the lines of Theorem 1.1. Section 5 concludes with some conjectures about a related family of matroids also indexed by permutations.

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2. Standardizing lattice paths to Dyck paths

Given a positive integer $n$, a Dyck path of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ which only uses steps $(1,1)$ (up-steps) or $(1,-1)$ (down-steps), and which never goes below the line $y = 0$. Let $D_n$ be the set of Dyck paths of length $2n$—it is well-known that $\#D_n$ is the $n$th Catalan number $C_n$, and that this is also the number of 123-avoiding permutations in $S_n$.

If $w \in S_{2n}$ is the identity permutation, then $P_w = \binom{[2n]}{n}$. In this case, Theorem 1.1 reads
\[ \sum_{v \in S_n \text{ avoids } 123} C_{\ell_1 + 1} \cdots C_{\ell_k + 1} = \binom{2n}{n}. \]

where $\ell_1, \ldots, \ell_k$ are the lengths of runs of anti-fixed points of each $v$.

Here is a similar identity for Dyck paths. We can view any $I \in \binom{[2n]}{n}$ as a lattice path from $(0,0)$ to $(2n,0)$ by taking one step for each $i \in 1, 2, \ldots, 2n$, either $(1,1)$ or $(1,-1)$ depending on whether $i \in I$ or $i \notin I$. We say such a lattice path has a peak at step $i$ if step $i$ is an up-step and step $i + 1$ is a down-step. The height of an up-step $i$ (or a peak at $i$) in a Dyck path is the $y$-coordinate of its endpoint; that is, the number of up-steps (weakly) before $i$
minus the number of down-steps before $i$. By the height of a peak $i$ we will mean the height of the corresponding up-step.

**Definition 2.1.** A saw in a lattice path is a maximal consecutive sequence of height 1 peaks. Here “consecutive” means what it should for peaks, that the corresponding step indices are separated by 2. The following identity will be the Dyck path analogue of (1).

**Lemma 2.2.** For any $n$, 
\[ \sum_{D \in \mathcal{D}_n} C_{\ell_1+1} \cdots C_{\ell_k+1} = \binom{2n}{n}, \]
where $2\ell_1, \ldots, 2\ell_k$ are the lengths of the saws of each Dyck path $D$.

This identity is not hard to prove. Suppose $I \in \binom{[2n]}{n}$ is a lattice path. The standardization of $I$ is the Dyck path $\text{st}(I)$ obtained by replacing each maximal segment of $I$ below the $x$-axis with a saw of the same length.

**Example 2.3.** If $n = 9$ and $I = \{1, 5, 6, 7, 9, 10, 14, 16, 17\}$, or
\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \ CD
Corollary 2.7. If \( K(w) \) is represented by a generic \((n)\), \( \ell \) where \( n \) \( \ell \) \( n \). That is, the \( K \) \( w \) \( w(i) \) \( w(j) \). Likewise, the entries weakly below the SW-NE diagonal are the right-to-left maxima: positions \( i \) such that \( j > i \) implies \( w(i) > w(j) \).

Lemma 2.5. Say \( w \in S_n \) avoids 123 and \( j \in [1,n] \). Then \( j \) is a left-to-right minimum if and only if \( w(j) \leq n - j + 1 \), a right-to-left maximum if and only if \( w(j) \geq n - j + 1 \), and an anti-fixed point if and only if it is both.

Proof. Suppose \( w(j) \leq n - j + 1 \) but \( j \) is not a left-to-right minimum, so there is \( i < j \) with \( w(i) < w(j) \). Since \( w \) avoids 123, every \( k \) such that \( w(j) < w(k) \) must be in \([j] \setminus \{i\}\). But there are at least \( j \) such values of \( k \), so this is impossible by the pigeonhole principle. Likewise, if \( w(j) \geq n - j + 1 \), then \( j \) is a right-to-left maximum. Every entry of \( w \) is either a left-to-right minimum or a right-to-left maximum (a counterexample would yield a 123 pattern), so the converses hold as well. \( \square \)

So, we can describe \( K(w) \) as follows. Say \( i_1 < \cdots < i_k \) are the left-to-right minima of \( w \). Also define \( w(i_0) = n + 1 = i_{k+1} \). Using \( U \) for an up-step and \( D \) for a down-step, the Dyck path \( K(w) \) is then
\[
U^{w(i_0)-w(i_1)}D^{i_2-i_1}U^{w(i_1)-w(i_2)}D^{i_3-i_2} \cdots U^{w(i_k-1)-w(i_k)}D^{i_{k+1}-i_k}.
\]
From this description it is easy to see that left-to-right minima of \( w \) correspond to peaks of \( K(w) \), and anti-fixed points to peaks of height 1. Thus, applying \( K \) to Lemma 2.2 gives the identity \( \square \).

Lemma 2.6. Suppose \( w \in S_n \) avoids 123. Then \( j \) is a left-to-right minimum of \( w \) if and only if \( K(w) \) has a peak at \( n - w(j) + j \), and this peak has height \( n + 2 - w(j) - j \).

Corollary 2.7. If \( w \) avoids 123 and has runs of anti-fixed points of lengths \( \ell_1, \ldots, \ell_k \), then \( K(w) \) has saws of lengths \( 2\ell_1, \ldots, 2\ell_k \).

Corollary 2.8. For any \( n \),
\[
\sum_{w \in S_n \setminus \{123\}} C_{\ell_1+1} \cdots C_{\ell_k+1} = \binom{2n}{n},
\]
where \( \ell_1, \ldots, \ell_k \) are the lengths of runs of anti-fixed points of each \( v \).

3. The structure of \( \mathcal{P}_w \)

Ardila has shown that the set
\[
\mathcal{C}_n = \{ \text{up-steps of } D : D \in \mathcal{D}_n \}
\]
is the set of bases of a matroid, the Catalan matroid of rank \( n \). He also showed that \( \mathcal{C}_{n+1} \) is represented by a generic \((n+1) \times 2(n+1)\) matrix of the form
\[
A_{n+1} = \begin{bmatrix}
* & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & * & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
* & * & * & * & \cdots & * & * & 0
\end{bmatrix}
\]
That is, the \( n \)-subsets of \([2n]\) where this matrix has invertible \( n \times n \) minors are exactly \( \mathcal{C}_n \).
Write $P_n$ for $P_{w_0}$ where $w_0 \in S_n$. Then $P_n$ is represented by the $n \times 2n$ matrix

$$M_{w_0} = \begin{bmatrix}
0 & 0 & \cdots & 0 & * & * & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & * & * & * & \cdots & \vdots & 0 \\
0 & 0 & \cdots & * & * & * & \cdots & * & 0 \\
* & * & \cdots & * & * & * & \cdots & * & *
\end{bmatrix}$$

Deleting row 1 and columns 1 and $2n+2$ of $A_{n+1}$, then permuting columns, gives the matrix $M_{w_0}$. Hence $P_n$ is isomorphic to $C_{n+1}$. Specifically, say $\alpha : [2, 2n+1] \to [2n]$ is the function sending $2, 3, \ldots, n+1$ to $n+1, n, n+2, n-1, \ldots, 2n, 1$. Then $D \in C_{n+1}$ if and only if $\alpha(D \setminus 1) \in P_n$. Since $C_{n+1}$ is made up of the sets $I \in {\binom{[2n+2]}{n}}$ such that #$(I \cap [2j+1]) \geq j + 1$ for $0 \leq j \leq n$, this isomorphism gives the following description of $P_n$.

**Lemma 3.1.** If $w_0 \in S_n$ is the reverse permutation, $P_n$ is the set of $I \in {\binom{[2n]}{n}}$ such that #$(I \cap [n-j+1, n-j]) \geq j$ for $1 \leq j \leq n$.

We will need to consider shifted versions of $P_n$, for which the following notation will be useful. Given a subset $X = \{x_1 < \cdots < x_k\}$ of $[n]$, write $Z_j X$ for

$$(\{n - x_j + 1, \ldots, n - x_1 + 1, n + x_1 + \ldots, n + x_j\})$$

(note that $Z_j X$ also depends on $n$, but we suppress that in the notation). Now for an interval $K$ in $[n]$, let $f_{K,n}$ be the unique increasing function $[2\#K] \to Z_{\#K} K$. Finally, define $P_{K,n}$ to be $f_{K,n}(P_n)$ where $w_0 \in S_{\#K}$. For example, $P_{[3,4],7}$ is the matroid of a generic matrix

$$\begin{bmatrix}
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Alternatively, we can give a description in the style of Lemma 3.1. $P_{K,n}$ consists of the $\#K$-subsets $I$ of $Z_{\#K} K$ such that

$$\#(I \cap Z_j K) \geq j$$

for each $j$ in $[\#K]$.

Let $L(w)$ be the set of left-to-right minima of $w$ which are not right-to-left maxima, and $R(w)$ the set of right-to-left maxima which are not left-to-right minima. We can now state our main structural result for $P_w$.

**Theorem 3.2.** Say $v, w \in S_n$.

(a) If $v \leq w$ in Bruhat order, then $P_v \subseteq P_w$.

(b) The sets $Q_w = P_w \setminus \bigcup_{v \geq w} P_v$ are pairwise disjoint.

(c) If $w$ contains 123, then $Q_w$ is empty.

(d) If $w$ avoids 123, say $w$ has runs of anti-fixed points $A_1, \ldots, A_k$. Then

$$Q_w = \biguplus_{i=1}^k P_{A_i,n} \oplus \{w(L(w))\} \oplus \{n + R(w)\}.$$

Here, for two families of sets $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$ is the family

$$\{I \cup J : I \in \mathcal{F}, J \in \mathcal{G}\},$$

where $\cup$ is disjoint union. That is, if $\mathcal{F}$ and $\mathcal{G}$ are sets of bases for two matroids, then $\mathcal{F} \oplus \mathcal{G}$ is the set of bases for the direct sum of the two matroids. Also, for a set $A$ and integer $n$, $n + A$ means $\{i + n : i \in A\}$.

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2Mnemonic: $Z_j X$ is an initial segment of $X$ reflected around the center of $[2n]$, but “$C_j$” is taken.
We can describe $Q_w$ directly in terms of matrices. For $w$ avoiding 123, let $N_w$ be a generic matrix whose entries are zero except that

- the entries $(i, w(i))$ for $i \in L(w)$ are nonzero;
- the entries $(i, i + n)$ for $i \in R(w)$ are nonzero; and
- if $w$ has runs of anti-fixed points starting at $a_1, \ldots, a_k$ with lengths $\ell_1, \ldots, \ell_k$, then
  - for each $p$ the submatrix in rows $[a_p, a_p + \ell_p - 1]$ and columns $f_{a_p, f_{p,n}}([1, 2\ell_p])$ is $M_{w_v}$, where $w_0 \in S_{\ell_p}$.

Then $Q_w$ is the matroid of $N_w$.

**Remark 1.** We can also describe $Q_w$ in the manner of Lemma 3.1. Let $A$ be the set of anti-fixed points of $w$, and define

$$G(w) = w(L(w) \cup A) \cup ((R(w) \cup A) + n = w(L(w)) \cup (R(w) + n) \cup Z_{\# A A}.$$ 

This is the groundset of $Q_w$. Now $Q_w$ consists of the $n$-subsets $I$ of $G(w)$ such that for each maximal interval $K \subseteq A$,

$$\#(I \cap Z, K) \geq j$$

for $1 \leq j \leq \# K$. Notice that this implies (but is not equivalent to) $\#(I \cap Z_i, A) \geq j$ for all $j$, or $\#(I \cap Z_i(A \cap [j])) \geq \#(A \cap [j])$.

**Example 3.3.** Take $w = 645312$, which avoids 123. The runs of anti-fixed points occur in positions 1 and 4, and

$L(w) = \{2, 5\}, \quad R(w) = \{6, 3\}.$

Hence

$$Q_w = P_{[1,1],6} \oplus P_{[4,4],6} \oplus \{\{4, 1\}\} \oplus \{\{12, 9\}\}.$$ 

We have $P_{[1,1],6} = \{\{6\}, \{7\}\}$ and $P_{[4,4],6} = \{\{3\}, \{10\}\}$. So, $Q_w$ consists of the four sets

$$13469(12), \quad 13479(12), \quad 1469(10)(12), \quad 1479(10)(12).$$

Because $P_w$ is the disjoint union of $Q_v$ for all 123-avoiding $v \geq w$, and $\# P_{K,n} = C_{\# K + 1}$, we get an immediate enumerative corollary.

**Corollary 3.4.** The size of $P_w$ is

$$\sum_{v \geq w \text{ avoids } 123} C_{\ell_1 + 1} \cdots C_{\ell_k + 1},$$

where $\ell_1, \ldots, \ell_k$ are as in the statement of Theorem 3.2, the lengths of the runs of anti-fixed points in each $v$.

To prove Theorem 3.2, we begin with a characterization of positroids from 3. Suppose $f$ is an affine permutation, that is, a bijection $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(i + n) = f(i) + n$ for some fixed $n$ (the quasiperiod of $f$) and all $i \in \mathbb{Z}$. Notice that an affine permutation is determined completely by the sequence $f(1), f(2), \ldots, f(n)$, and we will write our affine permutations this way. For example, 4721 sends an integer $4k + 1$ to $4k + 4$, $4k + 2$ to $4k + 7$, and so on.

An affine permutation $f$ is bounded if $i \leq f(i) \leq n + i$ for each $i \in \mathbb{Z}$. Postnikov showed, in the slightly different but equivalent language of decorated permutations, that rank $k$ positroids on $[n]$ are indexed by bounded affine permutations $f$ with quasiperiod $n$ and for which exactly $k$ of the values $f(1), \ldots, f(n)$ exceed $n$ 10.

Suppose $f$ is a such a bounded affine permutation, and that exactly $k$ of the values $f(1), \ldots, f(n)$ exceed $n$. The juggling sequence of $f$ is the sequence $(J_1, \ldots, J_n)$ of $k$-subsets of $[n]$ given by

$$J_i = \{f(j) - i + 1 : j < i\} \cap \mathbb{N}.$$ 

Finally, let $\chi$ be the shift permutation $23 \cdots n1 \in S_n$. 
Lemma 3.7. For any \( i \), we must have
\[
\chi^{-i+1}I \geq J_i \text{ for all } i = 1, \ldots, n. \tag{3}
\]

Proof. The same definition gives a partial order on all injective words on \( \mathbb{N} \) of a fixed length, which we will also call Bruhat order. Given
\[
I = \{i_1 < \cdots < i_p \leq n < i_{p+1} < \cdots < i_n\} \in \binom{[n]}{n},
\]
let \( v_I \) be the injective word with \( i_p+1, \ldots, i_n \) in positions \( i_{p+1} - n, \ldots, i_n - n \) (in increasing order), and \( i_p, \ldots, i_1 \) in the remaining positions (in decreasing order). For example, if \( n = 6 \) and \( I = \{1, 2, 5, 7, 10, 11\}, \) then \( v_I = 752(10)(11)1. \)

Lemma 3.6. A set \( I \in \binom{[2n]}{n} \) is in \( \mathcal{P}_w \) if and only if \( B_j(I) \geq w([j]) \) for \( j = 1, \ldots, n. \)

Proof. The test \( \chi^{-i+1}I \geq J_i \) is vacuous for \( i = 1, \ldots, n+1 \) since \( J_i = [n] \). If \( i = n+j+1 \), it reads \( \chi^{-n-j}I \geq [n-j] \cup (w([j]) + n-j) \), which is equivalent to \( B_j(I) \geq w([j]) \).

In fact, we can do better: it turns out that the sets \( B_j(I) \) are nested. Recall the tableau criterion for Bruhat order on the symmetric group: \( v \geq w \) if and only if \( v([j]) \geq w([j]) \) for all \( j \). The same definition gives a partial order on all injective words on \( \mathbb{N} \) of a fixed length, which we will also call Bruhat order. Given
\[
I = \{i_1 < \cdots < i_p \leq n < i_{p+1} < \cdots < i_n\} \in \binom{[2n]}{n},
\]
let \( v_I \) be the injective word with \( i_p+1, \ldots, i_n \) in positions \( i_{p+1} - n, \ldots, i_n - n \) (in increasing order), and \( i_p, \ldots, i_1 \) in the remaining positions (in decreasing order). For example, if \( n = 6 \) and \( I = \{1, 2, 5, 7, 10, 11\}, \) then \( v_I = 752(10)(11)1. \)

Lemma 3.7. For any \( w \in S_n \) and \( I \in \binom{[2n]}{n} \), \( I \in \mathcal{P}_w \) if and only if \( v_I \geq w. \)

Proof. This will follow from the tableau criterion and Lemma 3.6 if we show that \( B_{j-1}(I) \subseteq B_j(I) \) for each \( j \) and that \( B_j(I) \setminus B_{j-1}(I) = \{v_I([j])\} \). For each \( j \geq 0 \), we have
\[
\chi^{-n-j}I = \{i_{p+q(j)} - n < \cdots < i_n - n < i_1 + n < \cdots < i_{p+q(j)-1} + n\} - j,
\]
where \( q(j) \geq 0 \) is such that
\[
i_1 < \cdots < i_{p+q(j)-1} \leq n + j < i_{p+q(j)} < \cdots < i_n.
\]
We must have \( n - (p + q(j)) + 1 \leq 2n - (n + j) \), or equivalently \( p + q(j) - 1 \geq j \). Therefore
\[
B_j(I) = \{i_{p+q(j)-j} < \cdots < i_{p+q(j-1)}\}. \tag{4}
\]
There are two cases now. If \( j = i_{p+r} - n \) for some \( r \geq 1 \), then \( q(j) = q(j-1) + 1 = r + 1 \). One can then see from (4) that \( B_j(I) = B_{j-1}(I) \) and that
\[
B_j(I) \setminus B_{j-1}(I) = \{i_{p+r} - n\} = \{v_I([j])\}.
\]
On the other hand, if \( j \notin \{i_{p+1} - n, \ldots, i_n - n\} \) and \( j \geq 1 \), then \( q(j) = q(j-1) \). Again (4) shows that \( B_j(I) \subseteq B_{j-1}(I) \), and now
\[
B_j(I) \setminus B_{j-1}(I) = \{i_{p+q(j-1)-j}\} = \{i_{p+q(j)-j}\}. \]
Since the sets $B_j(I)$ are nested and get larger by one element with each step, the word formed by the singletons $B_j(I) \setminus B_{j-1}(I)$ must be injective, and its entries are the members of $I$ in some order by (1). We have seen that the entries in positions $\{t_{p+1} - n, \ldots, t_n - n\}$ agree with those for $v_I$. Therefore to show that the remaining entries $t_1, \ldots, t_i$, it suffices to show that they come in decreasing order. This follows from the fact that the function $j \mapsto p + q(j) - j$ is weakly decreasing, since $q(j + 1) - q(j) \in \{0, 1\}$ for each $j$.

Lemma 3.7 says that $P_w$ is the inverse image in $\binom{[2n]}{n}$ of the order ideal above $w$ in the poset of length $n$ injective words under the map $I \mapsto v_I$. The following “dual” perspective will turn out to be useful. Given a fixed $I \in \binom{[2n]}{n}$, let $W_I = \{w \in S_n : I \subseteq \mathcal{P}_w\}$. By Lemma 3.7, $W_I = \{w \in S_n : v_I \geq w\}$. Notice that part (b) of Theorem 3.2 is equivalent to the statement that $W_I$ has a unique maximal element. If $v_I$ and $w_0 = n(n - 1) \cdots 1$ have a greatest lower bound, it will be the unique maximal element of $W_I$. The poset of injective words with Bruhat order is not a lattice, but it turns out that the greatest lower bound exists in this case.

**Lemma 3.8.** Let $v$ be an injective word of length $n$ and $w_0 = n(n - 1) \cdots 1$. For each $j$, define

$$s(j) = \#([v(j)] \cap v([j])) = \{1 \leq i \leq j : v(i) \leq v(j)\}.$$ 

Let $u \in S_n$ be such that $u(j) = v(j)$ if $v(j) \leq n - j + s(j)$, and whose other entries are the other members of $[n]$, in decreasing order. Then $u$ is a greatest lower bound for $v$ and $w_0$ in Bruhat order.

**Proof.** Define

$$E_j = \min(v([j]), w_0([j])) = \{\min(b_1, n - j + 1) < \cdots < \min(b_j, n)\},$$

where $v([j]) = \{b_1 < \cdots < b_j\}$. If the sets $E_j$ are nested, the corresponding injective word will be a greatest lower bound for $v$ and $w_0$, so we must show that $E_{j-1} \subseteq E_j$ and $E_j \setminus E_{j-1} = \{u(j)\}$ for each $j$. The proof will be similar to that of Lemma 3.7.

For each $j$, take $r(j)$ maximal such that $b_{r(j)} \leq n - j + r(j)$, or 0 if there is no such $r$. For a fixed $j$, write $v([j-1]) = \{b_1 < \cdots < b_{r(j-1)}\}$. Then

$$E_{j-1} = \{b_1 < \cdots < b_{r(j-1)} < n - j + r(j - 1) + 2 < \cdots < n\}.$$

Now we consider two cases.

First, suppose $v(j) \leq n - j + s(j)$. Then $r(j) \geq s(j)$, which implies $r(j) = r(j - 1) + 1$, and

$$E_j = \{b_1 < \cdots < b_{r(j-1)} < v(j) < b_s < \cdots < b_{r(j-1)} < n - j + r(j) + 1 < \cdots < n\}$$

$$= \{b_1 < \cdots < b_{r(j-1)} < v(j) < b_s < \cdots < b_{r(j-1)} < n - j + r(j - 1) + 2 < \cdots < n\}$$

$$= E_{j-1} \cup \{v(j)\}.$$ 

Second, suppose $v(j) > n - j + s(j)$. Then $r(j) \leq s(j) - 1$. In this case we have $r(j) \leq r(j - 1)$, and if $r(j) < i \leq r(j - 1)$, then $b_i = n - j + i + 1$. Therefore

$$E_j = \{b_1 < \cdots < b_{r(j)} < n - j + r(j) + 1 < \cdots < n\}$$

$$E_{j-1} = \{b_1 < \cdots < b_{r(j)} < n - j + r(j) + 2 < \cdots < n\},$$

so $E_j = E_{j-1} \cup \{n - j + r(j) + 1\}$.

In both cases we see that $E_{j-1} \subseteq E_j$, so the permutation $u$ defined by $E_j \setminus E_{j-1} = \{u(j)\}$ is a greatest upper bound for $w_0$ and $v$. Moreover, if $v(j) \leq n - j + s(j)$, then $u(j) = v(j)$. If $j$ is, on the other hand, such that $v(j) > n - j + s(j)$, then $u(j) = n - j + r(j) + 1$; since $r(j) - j$ is a weakly decreasing function, we see that $u$ is weakly decreasing on such positions $j$, as claimed. \qed
Corollary 3.9. For any $I \in \binom{[2n]}{n}$, the set $W_I$ has a unique maximal element $u_I$.

Proof. The unique maximal element of $W_I$ is the greatest lower bound of $v_I$ and $w_0$ given by Lemma 3.8

Remark 2. Calculating $u_I$ is even simpler than Lemma 3.8, might lead one to believe, because the entries of $v_I$ in $[n]$ form a decreasing sequence, so in the case that $v_I(j) \leq n - j + s(j)$, we actually have $s(j) = 1$. Hence $u_I$ is the permutation in $S_n$ such that $u_I(j) = v_I(j)$ when $v_I(j) \leq n - j + 1$, and whose other entries form a decreasing subsequence. For example, say $I = \{1, 3, 4, 6, 9, 10, 15, 16, 17\}$. Then $v_I = (10)9643(15)(16)(17)1$ and $u_I = 986437521$. In fact, the next lemma shows that $u_I$ is completely determined by even less information.

Lemma 3.10. For any $I \in \binom{[2n]}{n}$, we have $u_I(j) < n - j + 1$ if and only if $v_I(j) < n - j + 1$ (and in this case $u_I(j) = v_I(j)$). Moreover, the permutation $u_I$ is uniquely determined by the pairs $(j, v_I(j))$ for which $v_I(j) < n - j + 1$.

Proof. The description of $u_I$ in Remark 2 shows that $u_I$ is the union of two decreasing subsequences, and so it avoids the pattern 123. Any 123-avoiding permutation $z \in S_n$ is completely determined by the pairs $(j, z(j))$ for which $z(j) < n - j + 1$, because the other entries will be right-to-left maxima and come in decreasing order. Thus it suffices to prove the first claim.

By Remark 2 if $v_I(j) < n - j + 1$ then $u_I(j) = v_I(j)$. Suppose that $u_I(j) < n - j + 1$ but that $u_I(j) \neq v_I(j)$. This implies $v_I(j) > n - j + 1$. By the pigeonhole principle, there must be $k > j$ such that $u_I(k) > n - k + 1$, and the minimal such $k$ must satisfy $u_I(k) > u_I(j)$. For such a $k$ we have $v_I(k) > n - k + 1$. But then the construction of $u_I$ implies that $u_I(j)$ and $u_I(k)$ are part of the same decreasing subsequence. This is a contradiction, since $j < k$ and $u_I(j) < u_I(k)$.

We now restate and prove Theorem 3.2.

Theorem (Theorem 3.2). Say $v, w \in S_n$. Then

(a) If $v \leq w$ in Bruhat order, then $P_w \subseteq P_v$.
(b) The sets $Q_w = P_w \setminus \bigcup_{v \succ w} P_v$ are pairwise disjoint.
(c) If $w$ contains 123, then $Q_w$ is empty.
(d) If $w$ avoids 123, say $w$ has runs of anti-fixed points $A_1, \ldots, A_k$. Then

$$Q_w = \bigoplus_{i=1}^{k} P_{A_i \cup n} \oplus \{w(L(w))\} \oplus \{n + R(w)\}.$$  \hfill (5)

Proof. (a) This follows immediately from Lemma 3.7.

(b) We have $I \in Q_w$ if and only if $w$ is a maximal element of $W_I = \{w \in S_n : I \in P_w\}$. Since Corollary 3.9 says that $W_I$ has a unique maximal element, the sets $Q_w$ must be pairwise disjoint.

(c) Suppose $I \in Q_w$. As in (b), this is equivalent to $u_I = w$. The description of $u_I$ in Remark 2 shows that $u_I$ is the union of two decreasing subsequences, and so it avoids 123. (One can also prove this on the level of matrices, by using a 123 pattern in $w$ and a transversal of $M_w$ in columns $I$ to find some transposition $t$ such that $w \leq wt$ along with a transversal of $M_{wt}$ in columns $I$.)

(d) Let $\hat{Q}_w$ be the set on the right-hand side of (5), and suppose $I \in \hat{Q}_w$. Let us see that $w = u_I$, which implies $I \in Q_w$. By part (c), $u_I$ avoids 123. As mentioned in the proof of Lemma 3.10 a 123-avoiding permutation $z$ is completely determined by the pairs $(j, z(j))$ where $z(j) < n - j + 1$. Hence, it is enough to show that if $w(j) < n - j + 1$
or \( u_I(j) < n - j + 1 \), then \( w(j) = u_I(j) \). By Lemma 3.10, this is equivalent to: if \( w(j) < n - j + 1 \) or \( v_I(j) < n - j + 1 \), then \( w(j) = v_I(j) \).

Observe that \( v_I(j) = k < n - j + 1 \) if and only if \( n + j \notin I \) and \( k \) is the \((j - \#(I \cap [n + 1, n + j]))^{th}\) largest element in \( I \cap [n] \). Equivalently, \( n + j \notin I \) and \( \#(I \cap [k, n + j]) = j \). Thus, we want either of \( w(j) < n - j + 1 \) or \( v_I(j) < n - j + 1 \) to imply \( n + j \notin I \) and \( \#(I \cap [w(j), n + j]) = j \). The first condition is easy: if \( n + j \in I \), then \((1) j \) must be a right-to-left maximum of \( w \), so \( w(j) \geq n - j + 1 \), and \((2) \) the construction of \( v_I \) implies \( v_I(j) > n \).

Let \( A \) be the set of anti-fixed points of \( w \). Recall that \( I \) is the disjoint union of \( w(L(w)) \), \( R(w) + n \), and \( I \cap Z_{\#A} A \). We treat these three cases separately. First, since \( j \in L(w) \), we have
\[
\#(w(L(w)) \cap [w(j), n + j]) = \#(w(L(w)) \cap [w(j), n]) = \#(L(w) \cap [j]).
\]
Next,
\[
\#((R(w) + n) \cap [w(j), n + j]) = \#(R(w) \cap [j]).
\]
Finally, since \( j \) is a left-to-right minimum, \([w(j), n + j] \supseteq [n - j + 1, n + j] \), and the description of \( \tilde{Q}_w \) given in Remark 1 then says that
\[
\#(I \cap Z_{\#A} A \cap [w(j), n + j]) \geq \#(A \cap [j]).
\]
Putting these three pieces of \( I \cap [w(j), n + j] \) together,
\[
\#(I \cap [w(j), n + j]) \geq \#(L(w) \cap [j]) + \#(R(w) \cap [j]) + \#(I \cap Z_{\#A} A \cap [w(j), n + j])
\]
\[
\geq (j - \#(A \cap [j])) + \#(A \cap [j]) = j.
\]

For the reverse inequality, we use the easy direction of Hall’s marriage theorem. Let \( D_i \) be the set of *’s in column \( i \) of the matrix \( N_w \) representing \( \tilde{Q}_w \). That is,
\[
D_i = \begin{cases} 
\{w^{-1}(i)\} & \text{if } i \in w(L(w)) \\
\{i - n\} & \text{if } i \in R(w) + n \\
[a_\ell - \ell + k, a_\ell] & \text{if } i \text{ or } 2n - i + 1 \text{ is } k^{th} \text{ in a run of anti-fixed points } a_1, \ldots, a_\ell.
\end{cases}
\]
Since \( N_w \) has a transversal in columns \( I \), we must have
\[
\#(I \cap [w(j), n + j]) \leq \# \bigcup_{i \in I \cap [w(j), n + j]} D_i.
\]
Notice that if \( w(i) < i' \leq n \) or \( n > i' > i + n \), the contents of \( D_{i'} \) are bounded above by \( i \). Therefore \( \bigcup_{i \in I \cap [w(j), n + j]} D_i \subseteq [j] \), and we get the desired inequality.

We have now shown that \( \tilde{Q}_w \subseteq Q_w \) for all 123-avoiding \( w \). By parts (a) and (b), the non-empty \( Q_w \) partition \( P_{12 \ldots n} = \binom{[2n]}{n} \). Thus to get \( Q_w = Q_w \), it is enough to show that
\[
\sum_{w \in N_w \text{ avoids } 123} \#Q_w = \binom{2n}{n},
\]
which we have done in Corollary 2.8.

\[ \square \]

We conclude with a few results on symmetries of positroids which will be useful in the next section.

**Theorem 3.11.** Let \( \bar{x} = 2n + 1 - x \). Then \( \overline{P_w} = P_{w_0 w^{-1} w_0} \) for any permutation \( w \).

**Proof.** This follows immediately from the matrix identity \( w_0 w^{-1} (2n + 1 - M_w) = M_{w_0 w^{-1} w_0} \).

\[ \square \]
Corollary 3.12. For any \( I \in \binom{[2n]}{n} \), \( \tau_I = w_0 u_I^{-1} w_0 \).

Proof. Theorem 3.11 is equivalent to \( W_I = w_0 W_I^{-1} w_0 \). Since \( w \mapsto w_0 w^{-1} w_0 \) is an automorphism of Bruhat order,

\[
u_I = \max W_I = w_0 (\max W_I)^{-1} w_0 = w_0 u_I^{-1} w_0.\]

\( \square \)

If \( M \) is a matroid with groundset \( E \), then \( \{ E \setminus I : I \text{ a basis of } M \} \) is also the set of bases for a matroid, the dual of \( M \), written \( M^* \).

Theorem 3.13. For any permutation \( w \), \( \mathcal{P}_w^* \) is isomorphic to \( \mathcal{P}_{w^{-1}} \).

Proof. Set \( w_n^* = (n+1) \cdots (2n)1 \cdots n \in S_n \). Let us see that \( w_n^* \mathcal{P}_w^* = \mathcal{P}_{w^{-1}} \). Since inversion is an automorphism of Bruhat order and \( \mathcal{P}_w = \bigcup_{v \geq w} \mathcal{Q}_v \) by Theorem 3.2, it is enough to show that \( w_n^* \mathcal{Q}_v^* = \mathcal{Q}_{w^{-1}} \). Let \( A \) be the set of anti-fixed points of \( v \), and \( A_1, \ldots, A_k \) the maximal intervals in \( A \). Since \([n]\) is the disjoint union \( L(v) \cup R(v) \cup A \), we have

\[
\mathcal{Q}_v^* = \{ vR(v) \} \oplus \{ L(v) + n \} \oplus \bigoplus_{i=1}^k \mathcal{P}_{A_i,n}^*.
\]

Also, \( L(v) = v^{-1}L(v^{-1}) \) and \( R(v) = v^{-1} R(v^{-1}) \), so

\[
w_n^* \mathcal{Q}_v^* = \{ v^{-1}L(v^{-1}) \} \oplus \{ R(v^{-1}) + n \} \oplus \bigoplus_{i=1}^k w_n^* \mathcal{P}_{A_i,n}^*.
\]

The anti-fixed points of \( v^{-1} \) are \( n+1-A \), so all we need to do is show \( w_n^* \mathcal{P}_{K,n} = \mathcal{P}_{n+1-K,n} \) for any interval \( K \subseteq [n] \). When pushed through the isomorphism of \( \mathcal{P}_{K,n} \) with \( C_{#K+1} \) given at the beginning of this section, this identity becomes \( w_0 C_{#K+1}^* = C_{#K+1} \), where \( w_0 \in S_{#K+1} \). But the latter identity is certainly true: complementing and reversing is an automorphism of the set of Dyck paths. \( \square \)

4. The Tutte polynomial of \( \mathcal{P}_w \)

Theorem 3.2 writes \( \mathcal{P}_w \) as the disjoint union of matroids \( \mathcal{Q}_v \) over 123-avoiding permutations \( v \) above \( w \) in Bruhat order, with each \( \mathcal{Q}_v \) isomorphic to a direct sum of Catalan matroids and a matroid with one basis. In this section we give an analogous formula for the Tutte polynomial of \( \mathcal{P}_w \), writing it as a sum over 123-avoiding permutations \( v \) above \( w \) of certain modifications of the Tutte polynomials of \( \mathcal{Q}_v \). First we recall one definition of the Tutte polynomial.

Definition 4.1. Given a matroid \( M \) with groundset \( S \), the rank of a subset \( I \subseteq S \) is the maximal size of an intersection of \( I \) with a basis of \( M \). Write \( \text{rank}_M(I) \) for this number. The Tutte polynomial of \( M \) is then the bivariate generating function

\[
T_M(x, y) = \sum_{I \subseteq S} (x - 1)^{\text{rank}(M) - \text{rank}_M(I)} (y - 1)^{|I| - \text{rank}_M(I)}.
\]

Here \( \text{rank}(M) \) is the size of any basis of \( M \).

Let \( T_n(x, y) \) be the Tutte polynomial of the matroid \( \mathcal{P}_n \). If \( M \) is the matroid on \( \{2n+1, 2n+2\} \) with bases \( \{\{2n+1\}\} \), then \( M \oplus \mathcal{P}_n \) is isomorphic to \( C_{n+1} \). The Tutte polynomial of \( M \) is \( xy \), and Tutte polynomials are multiplicative on direct sums, so \( T_n(x, y) \) is the Tutte polynomial of \( C_{n+1} \) divided by \( xy \).
Given a Dyck path $D$, let $a(D)$ be the height of the first peak and $b(D)$ the number of times $D$ touches the $x$-axis, not counting the first. In [1], Ardila shows that
\[
\sum_{D \in D_n} x^{a(D)} y^{b(D)}.
\]
is the Tutte polynomial of $C_n$. Hence
\[T_n(x, y) = \sum_{D \in D_{n+1}} x^{a(D)-1} y^{b(D)-1}.
\]

It is more natural to give $T_n(x, y)$ as a sum over $P_n$ using the bijection to $C_{n+1}$ given at the beginning of Section 3. Define a total order $\prec$ on $[2n]$ by
\[n + 1 < n < n + 2 < n - 1 < \cdots < 2n < 1.
\]
For $I \in P_n$, define $c(I)$ as the length of the longest $\prec$-initial segment of $[2n]$, and $d(I)$ as the number of integers $j \in [2n]$ such that $\#(I \cap [n+1, n+j]) = \#(I \cap [n-j+1, n-1])$. Then Ardila’s formula (6) translates to
\[T_n(x, y) = \sum_{I \in P_n} x^{c(I)} y^{d(I)}.
\]

Given an interval $K \subseteq [n]$, define a modified version of $T_n$ as follows:
\[T_{K,n}(x, y) = \begin{cases} T_{\#K}(x, y) & \text{if } K = [n] \\ T_{\#K}(x, 1) & \text{if } 1 \in K \text{ and } n \notin K \\ T_{\#K}(1, y) & \text{if } 1 \notin K \text{ and } n \in K \\ T_{\#K}(1, 1) & \text{if } 1, n \notin K \end{cases}
\]
Notice that $T_{\#K+1}(1, 1) = C_{\#K+1}$, the number of bases in $P_{K,n}$. Also, given a 123-avoiding $w \in S_n$ with runs of anti-fixed points $A_1, \ldots, A_k$, define
\[U_w(x, y) = \prod_{i=1}^k T_{A_i,n}(x, y).
\]

**Theorem 4.2.** For any permutation $w \in S_n$, the Tutte polynomial of $P_w$ is
\[U_{w_0}(x, y) + (1 - (x-1)(y-1)) \sum_{w \leq w_0 \text{ w avoids } 123} U_w(x, y).
\]

We start with a characterization of ranks in $P_w$. Recall that Lemma 3.7 associates to each $n$-subset $I$ of $[2n]$ an permutation $u_I \in S_n$ in such a way that $I \in P_w$ if and only if $u_I \succeq w$. We will follow a similar strategy here, and construct, for any nonnegative integer $r$ and any $I \subseteq [2n]$, a permutation $u_I^r$ such that $I$ has rank at least $r$ in $P_w$ if and only if $u_I^r \succeq w$.

Say $I \subseteq [2n]$ has size at least $r$. Define $J_r(I)$ to be the $\lessdot$-lexicographically smallest $n$-set such that $\#(J_r(I) \cap I) \geq r$. Explicitly, if
\[I = \{i_1 \prec i_2 \prec \cdots \} \quad \text{and} \quad [2n] \setminus \{i_1, \ldots, i_r\} = \{j_1 \prec j_2 \prec \cdots \},
\]
then $J_r(I) = \{i_1, \ldots, i_r, j_1, \ldots, j_{n-r}\}$. Now define $u_I^r = u_{J_r(I)}$.

**Theorem 4.3.** The set $I$ has rank at least $r$ in $P_w$ if and only if $J_r(I) \in P_w$, or equivalently, $u_I^r \succeq w$.

**Remark 3.** What is really important here is the partial order $n + 1, n \prec n + 2, n - 1 \prec \cdots \prec 2n, 1$: one can show that although $J_r(I)$ depends on the choice of linear extension of this partial order to a total order, $u_I^r$ does not (indeed, this is a consequence of Theorem 4.3).
We postpone the proof of Theorem 4.4 since it is somewhat involved, and move on to its consequences for ranks in $P_w$. Let $P^r_w = \{I \subseteq [2n] : I \text{ has rank at least } r \text{ in } P_w\}$, and $Q^r_w = P^r_w \setminus \bigcup_{r'>w} P^r_w$. Theorem 4.4 shows that $Q^r_w$ is the set of $I$ such that $u^*_I = w$. Equivalently, if we think of $J_r$ as a function $2^{[2n]} \to \binom{[2n]}{r}$, then $Q^r_w = \bigcup_{K \in Q_w} J^{-1}_r(K)$, and we can give a reasonable description of $J^{-1}_r(K)$ for a fixed $K$.

**Lemma 4.4.** Let $K$ be an $n$-subset of $[2n]$, and $0 \leq r \leq n$. Write $K = E \cup F$ where $E$ is the maximal initial segment of $[2n]$ in $K$ (in the order $\prec$). Then $J^{-1}_r(K)$ is the collection of sets of the form $E' \cup F \cup G$, where $E' \in \binom{E}{\# E - n + r}$ and $G \subseteq [2n]$ satisfies $\min(G) > \max(F)$.

**Proof.** Say $I = E' \cup F \cup G$ where $E'$, $F$, $G$ are as in the statement of the lemma, and write $I = \{i_1 \prec i_2 \prec \cdots \}$. Since $\# E' + \# F = r$, we have $\{i_1, \ldots, i_r\} = E' \cup F$. Thus $[2n] \setminus \{i_1, \ldots, i_r\}$ contains $E \setminus E'$, which has size $n - r$. Since $E$ is an initial segment, the smallest $n - r$ elements of $[2n] \setminus \{i_1, \ldots, i_r\}$ are exactly $E \setminus E'$, so $J_r(I) = (E' \cup F) \cup (E \setminus E') = K$.

Conversely, suppose $J_r(I) = K$, with $[2n] \setminus I = \{j_1 \prec j_2 \prec \cdots \}$ as in the definition of $J_r$. Let $E'$ consist of the $\prec$-first $\# E - n + r$ elements of $I$ (noting that $\# E - n + r \leq r \leq \# I$). Since $E$ is an initial segment of size $\# E' + n - r$, we must have $E' \cup \{j_1 \prec j_2 \prec \cdots \} \cup \{i_r+1 \prec \cdots \} = E$. But this forces $F = \{i_{\# E' + 1} \prec \cdots \prec i_r\} \subseteq I$, and then defining $C = \{i_{r+1} \prec \cdots \prec i_n\}$ gives the desired decomposition $I = E' \cup F \cup G$.

Finally, we will need a description of $U_w$ in the style of (7). As above, let $c(K)$ be the length of the largest $\prec$-initial segment of $[2n]$ contained in $K$, and let $\bar{c}(K)$ be the length of the largest $\prec$-final segment in $[2n]$ \setminus $K$.

**Lemma 4.5.** For any 123-avoiding $w \neq w_0$, $U_w(x, y) = \sum_{K \in Q_w} c(K) x^{\bar{c}(K)} y^{|K|}$.

**Proof.** Suppose $w$ avoids 123, and has runs of anti-fixed points $A_1, \ldots, A_k$. Then any $K \in Q_w$ is a disjoint union $L_1 \cup \cdots \cup L_k \cup w(L(w)) \cup (R(w) + n)$, where $L_i \in P_{A_i, n}$.

Suppose $K \in Q_w$ contains as a maximal $\prec$-initial segment $E = \{n + 1, \ldots, n + \alpha, n, n - 1, \ldots, n - \beta + 1\}$ for some $\alpha, \beta$. By definition of $Q_w$, this means $w$ has right-to-left maxima in positions $1, \ldots, \alpha$. But this is only possible if $w$ has anti-fixed points in those positions. Likewise, $w$ has left-to-right minima with values $n, n - 1, \ldots, n - \beta + 1$, hence anti-fixed points in positions $1, \ldots, \beta$. This shows that $E \subseteq L_1$ if $w(1) = n$, and that $E = \emptyset$ if $w(1) \neq n$. Hence $c(K) = c(L_1)$ if $w(1) = n$, and $c(K) = 0$ otherwise. An analogous argument shows that $\bar{c}(K) = \bar{c}(L_r)$ if $w(n) = 1$, and $\bar{c}(K) = 0$ otherwise. Now we see that:

- If $w(1) \neq n$ and $w(n) \neq 1$, then $\sum_{K \in Q_w} x^{c(K)} y^{\bar{c}(K)} = \# Q_w = \prod_{i=1}^k T_{\# A_i}(1, 1) = U_w(x, y)$.
- If $w(1) = n$ and $w(n) \neq 1$, then using (7), $\sum_{K \in Q_w} x^{c(K)} y^{\bar{c}(K)} = \sum_{L \in P_{\neq A_1}} x^{c(L)} \prod_{i=2}^k T_{\# A_i}(1, 1)$
  $= T_{\# A_1}(x, 1) \prod_{i=2}^k T_{\# A_i}(1, 1) = U_w(x, y)$. 


• If \( w(1) \neq n \) and \( w(n) = 1 \), then

\[
\sum_{K \in \mathcal{Q}_w} x^{e(K)} y^{\tau(K)} = \sum_{L \in \mathcal{P}_{w A_1}} y^{\tau(L)} \prod_{i=1}^{k-1} T_{\# A_i}(1, 1)
\]

\[
= \sum_{L \in \mathcal{P}_{w A_1}} y^{\tau(L)} \prod_{i=1}^{k-1} T_{\# A_i}(1, 1)
\]

\[
= T_{\# A_1}(y, 1) \prod_{i=1}^{k-1} T_{\# A_i}(1, 1)
\]

To get the second equality, we use the fact from Theorem 3.13 that \( I \mapsto w_n^*(\{2n\} \setminus I) \) is an automorphism of \( \mathcal{P}_n \), and that it exchanges the statistics \( e \) and \( \bar{c} \). Taking the dual of a matroid corresponds to switching the variables in the Tutte polynomial, so since \( \mathcal{P}_n \) is self-dual, \( T_n(x, y) = T_n(y, x) \). Thus

\[
\sum_{K \in \mathcal{Q}_w} x^{e(K)} y^{\tau(K)} = T_{\# A_1}(1, y) \prod_{i=1}^{k-1} T_{\# A_i}(1, 1) = U_w(x, y).
\]

• If \( w(1) = n \) and \( w(n) = 1 \), then \( k > 1 \) since \( w \neq w_0 \), and

\[
\sum_{K \in \mathcal{Q}_w} x^{e(K)} y^{\tau(K)} = \sum_{L \in \mathcal{P}_{w A_1}} x^{e(L)} \sum_{L \in \mathcal{P}_{w A_1}} y^{\tau(L)} \prod_{i=2}^{k-1} T_{\# A_i}(1, 1)
\]

\[
= T_{\# A_1}(x, 1) T_{\# A_1}(1, y) \prod_{i=2}^{k-1} T_{\# A_i}(1, 1) = U_w(x, y).
\]

\[\square\]

Let \( T_w(x, y) \) be the Tutte polynomial of \( \mathcal{P}_w \). The Möbius function of Bruhat order on \( S_n \) is \( \mu(w, v) = (-1)^{\ell(v) - \ell(w)} \), so by Möbius inversion, for any particular \( w \in S_n \), Theorem 4.2 is equivalent to

\[
\sum_{v \geq w} (-1)^{\ell(v) - \ell(w)} T_v(x, y) = \begin{cases} U_{w_0}(x, y) = T_n(x, y) & \text{if } w = w_0 \\ (1 - (x - 1)(y - 1))U_w & \text{if } w \neq w_0 \text{ avoids 123} \\ 0 & \text{if } w \text{ contains 123} \end{cases}
\]

**Proof of Theorem 4.2** Write \( \text{rank}_v(I) \) for the rank of \( I \) in \( \mathcal{P}_v \). By Theorem 4.3, \( \text{rank}_v(I) = r \) if and only if \( v \leq u_r^I \) and \( v \not\leq u_r^{I+1} \). Thus,

\[
\sum_{v \geq w} (-1)^{\ell(v) - \ell(w)} T_v(x, y) = \sum_{v \geq w} (-1)^{\ell(v) - \ell(w)} \sum_{I \subseteq [2n]} (x - 1)^{n - \text{rank}_v(I)} (y - 1)^{#I - \text{rank}_v(I)}
\]

\[
= \sum_{I \subseteq [2n]} \sum_{r = 0}^{n} (x - 1)^{n - r} (y - 1)^{#I - r} \sum_{v \in [w, u_r^I] \setminus [w, u_r^{I+1}]} (-1)^{\ell(v) - \ell(w)}.
\]

The term \( (x - 1)^{n - r} (y - 1)^{#I - r} \) will occur frequently, so we will simply write \( f \) for it in the rest of the proof.
Any Bruhat interval with more than one element has the same number of elements with even length as with odd length \[3\], so
\[
\sum_{v \in [w, u^r[w, u^{r+1}]} (-1)^{\ell(v) - \ell(w)} = \begin{cases} 
0 & \text{if } w \neq u^r \text{ and } w \neq u^{r+1} \\
1 & \text{if } w = u^r > u^{r+1} \\
-1 & \text{if } u^r > u^{r+1} = w
\end{cases}
\]

Observe that \(w = u^r > u^{r+1}\) if and only if \(I \in \mathcal{Q}_w \setminus \mathcal{Q}_w^{r+1}\), and \(u^r > u^{r+1} = w\) if and only if \(I \in \mathcal{Q}_w^{r+1} \setminus \mathcal{Q}_w\). Therefore

\[
\sum_{v \geq w} (-1)^{\ell(v) - \ell(w)}T_v(x, y) = \sum_{r=0}^{n} \left[ \sum_{I \in \mathcal{Q}_w \setminus \mathcal{Q}_w^{r+1}} f - \sum_{I \in \mathcal{Q}_w^{r+1} \setminus \mathcal{Q}_w} f \right]
\]

\[
= \sum_{r=0}^{n} \left[ \sum_{I \in \mathcal{Q}_w} f - \sum_{I \in \mathcal{Q}_w^{r+1}} f \right]
\]

\[
= \sum_{r=0}^{n} \sum_{I \in \mathcal{Q}_w} f - (x-1)(y-1) \sum_{r=1}^{n+1} \sum_{I \in \mathcal{Q}_w} f.
\]

We may as well assume \(w \neq w_0\), in which case \(\mathcal{Q}_w^0 = \emptyset\). Also, \(\mathcal{Q}_w^{n+1} = \emptyset\) for any \(w\), so

\[
\sum_{v \geq w} (-1)^{\ell(v) - \ell(w)}T_v(x, y) = [1 - (x-1)(y-1)] \sum_{r=0}^{n} \sum_{I \in \mathcal{Q}_w} f
\]

\[
= [1 - (x-1)(y-1)] \sum_{r=0}^{n} \sum_{K \in \mathcal{Q}_w} \sum_{I \in \mathcal{J}_r^{-1}(K)} f.
\]

As in Lemma \[4.5\] let \(c(K)\) denote the length of the largest initial segment of \([2n] \setminus K\) in the order \(\prec\), and \(\bar{c}(K)\) the length of the largest final segment in \([2n] \setminus K\). By Lemma \[4.4\] a member of \(\mathcal{J}_r^{-1}(I)\) with size \(j + r\) corresponds to a choice of (1) a \((c(K) - n + r)\)-subset of a set of size \(c(K)\), and (2) a \(j\)-subset of the maximal \(\prec\)-final segment of \([2n] \setminus K\). Hence

\[
\sum_{I \in \mathcal{J}_r^{-1}(K)} f = \sum_{I \in \mathcal{J}_r^{-1}(K)} (x-1)^{n-r}(y-1)^{\#I-r}
\]

\[
= \binom{c(K)}{c(K) - n + r} (x-1)^{n-r} \sum_{j=0}^{\bar{c}(K)} \binom{\bar{c}(K)}{j} (y-1)^j
\]

\[
= \binom{c(K)}{c(K) - n + r} (x-1)^{n-r} y^{\bar{c}(K)}.
\]

Continuing on,

\[
\sum_{v \geq w} (-1)^{\ell(v) - \ell(w)}T_v(x, y) = [1 - (x-1)(y-1)] \sum_{K \in \mathcal{Q}_w} \sum_{r=0}^{n} \binom{c(K)}{c(K) - n + r} (x-1)^{n-r} y^{\bar{c}(K)}
\]

\[
= [1 - (x-1)(y-1)] \sum_{K \in \mathcal{Q}_w} \sum_{r=0}^{n} \binom{c(K)}{c(K) - n + r} (x-1)^{n-r} y^{\bar{c}(K)}
\]

This is equal to \([1 - (x-1)(y-1)]U_w(x, y)\) by Lemma \[4.5\]
To prove Theorem \ref{thm:bruhat} we will need some lemmas giving a Bruhat relation between \( u_I \) and \( u_J \) for two sets \( I \) and \( J \). Write \( I \sqsubseteq J \) if \( I \) is the \( \prec \)-lexicographically minimal \#\( I \)-subset of \( I \cup J \). Equivalently, \( I \sqsubseteq J \) if and only if \( J \cap I = J \cap [\max \prec(I)] \). This is a partial order on finite subsets of \( \mathbb{N} \) of a fixed size.

**Lemma 4.6.** Suppose \( I, J \in \binom{[2n]}{n} \) are such that either

(a) \( I \sqsubseteq J \), or
(b) \( J = I \setminus i \cup j \), where \( i \) is contained in a \( \prec \)-initial segment in \( I \) and \( i \leq j \).

Then \( u_I \geq u_J \).

**Proof.** For the case where \( I \sqsubseteq J \), we may assume that \( J = I \setminus i \cup j \) where \( i \in I \) and \( j > \max \prec(I) \), since this is the covering relation for \( \subseteq \). Recall the injective word \( v_I \), with the property that \( u_J \) is the greatest lower bound of \( v_I \) and \( w_0 \), and whose entries are \( I \cap [n+1,2n] \) in increasing order together with \( I \cap [n] \) in decreasing order.

Suppose for the moment that \( j \leq n \). In passing from \( v_I \) to \( v_J \), we remove one entry \( i \) and insert a new entry \( j \) into the decreasing subsequence formed by \( I \cap [n] \) in the unique way that keeps the subsequence decreasing, and then shift part of the subsequence either right or left to fill the gap left by \( i \). If \( i \leq n \), then \( j > i \) implies \( j < i \). Thus, \( j \) enters right of the gap left by \( i \), so we shift leftward. This means that \( v_J \) is entrywise less than or equal to \( v_I \), which implies the weaker statement that \( v_J \leq v_I \) in Bruhat order. Therefore \( u_I \leq u_J \).

Next suppose that \( j \leq n \) still, but now \( i > n \). We consider cases (a) and (b) separately. In case (b), where \( i \) is contained in a \( \prec \)-initial segment in \( I \), \( v_I \) begins \( (n+1)(n+2) \cdots (n+b) \cdots \), with \( i \) being one of those first \( b \) entries. Thus, every entry of the decreasing sequence is right of \( i \), and in particular \( j \) does enter to the right of it when we pass to \( v_J \). In case (a), we have \( j > \max \prec(I) \), which implies \( j \leq \min(I) \) (in the usual order), so \( j \) will be the last entry in the decreasing sequence in \( v_J \). In particular, \( j \) enters right of the gap where \( i \) was. In both cases we end up with \( v_J \) entrywise less than or equal to \( v_I \) as before, as in the last paragraph.

Finally, assume that \( j > n \). We will apply the map \( x \mapsto 2n+1-x \) and use Corollary \ref{cor:bruhat}. The arguments above only depend on \( \prec \) being a linear extension of the partial order

\[
\begin{align*}
n + 1, n &< n + 2, n - 1 < \cdots < 2n, 1 \\
n &< n + 1 < n - 1 < \cdots < 2 < 1 \quad \text{for } n \geq 2.
\end{align*}
\]

and so they still go through if we replace \( \prec \) with the total order \( \preceq \) defined by

\[
\begin{align*}
n &\preceq n + 1 \preceq n - 1 \preceq n + 2 \preceq \cdots \preceq 1 \preceq 2n.
\end{align*}
\]

The hypotheses of the lemma still hold for \( \overline{I} \), \( \overline{j} \), and \( \overline{I} \cup \overline{j} \) using the order \( \preceq \).

As \( \overline{J} \leq n \), the previous arguments show that \( w_{\overline{I}} \geq w_{\overline{J}} \), or

\[
w_0u_{\overline{I}}^{-1}w_0 \geq w_0u_{\overline{J}}^{-1}w_0
\]

by Corollary \ref{cor:bruhat}. Since \( w \mapsto w_0w^{-1}w_0 \) is an automorphism of Bruhat order, this is equivalent to \( u_I \geq u_J \).

\[\square\]

**Lemma 4.7.** Say \( I, I' \in \binom{[2n]}{r} \), where \( r \leq n \). If \( I \sqsubseteq I' \), then \( J_r(I) \sqsubseteq J_r(I') \).

**Proof.** As in the proof of Lemma \ref{lem:bruhat}, we can assume that \( I' = I \setminus i \cup j \), where \( i \in I \) and \( j > \max \prec(I) \). Write \( I = \{i_1 < \cdots < i_r\} \) and \( [2n] \setminus I = \{j_1 < j_2 < \cdots\} \), so \( J_r(I) = \{i_1, \ldots, i_r, j_1, \ldots, j_{n-r}\} \). There are several cases.

- If \( j \leq j_{n-r} \), then \( J_r(I') = J_r(I) \).
- If \( i \leq j_{n-r} < j \), then \( J_r(I') = J_r(I) \setminus j_{n-r} \cup j \). Here \( j > \max \prec(I) \) and \( j \geq j_{n-r} \), so \( j > \max \prec(I) \).
- If \( j < i \leq j \), then \( J_r(I') = J_r(I) \setminus i \cup j \). Once again, \( j > \max \prec(I) \) and \( j \geq j_{n-r} \), so \( j > \max \prec(I) \).
- If \( j < i \leq j \), then \( J_r(I') = J_r(I) \setminus i \cup j \).
Proof of Theorem 4.3

Define \( r < n \) its unique maximum by Theorem 3.2. Suppose \( W \) be done: the unique maximum of \( u \). It is clear from Theorem 3.2 that implies \( u \) for some \( I \). If \( x \) therefore, is that if \( J \). By induction, each \( W \) is the matroid of \( \tilde{w} (I) \). In particular, if \( x \) \( n \). Thus if we knew that each \( W \), so Lemma 4.6 shows that \( u \), so Lemma 4.6 implies \( u \), \( u \). Equivalently, \( u \), \( u \), \( u \), \( u \). This way we can assume \( #I = r \).

Now we induct (downward) on \( r \), assuming \( #I = r \). If \( r = n \), then \( W = W \) has \( u = u \) as its unique maximum by Theorem 3.2. Suppose \( r < n \). Then \( I \in \mathcal{P}^r \) if and only if \( I \cup x \in \mathcal{P}^{r+1} \) for some \( x \notin I \), or equivalently,

\[
W^r_I = \bigcup_{x \notin I} W^r_{I \cup x}
\]

By induction, each \( W^r_{I \cup x} \) has \( u^{r+1} \) as its unique maximal element. What we want to show, therefore, is that if \( x \notin I \), then \( u \geq u^{r+1} \), with equality holding for some \( x \).

As in the definition of \( J_r(I) \), write \( I = \{i_1 \prec \cdots \prec i_r\} \) and \( [2n] \setminus I = \{j_1 \prec j_2 \prec \cdots\} \), so that \( J_r(I) = \{i_1, \ldots, i_r, j_1, \ldots, j_{n-r}\} \). Then

\[
J_{r+1}(I \cup x) = \begin{cases} J_r(I) & \text{if } x \preceq j_{n-r} \\ J_r(I) \setminus j_{n-r} \cup x & \text{if } x \succeq j_{n-r} \end{cases}
\]

In particular, if \( x \) is \( \prec \)-minimal in \( [2n] \setminus I \), then \( x \preceq j_{n-r} \), so \( u^r_I = u^{r+1}_{I \cup x} \).

We can now assume that \( x \succeq j_{n-r} \). By definition, \( j_{n-r} \) is part of a \( \preceq \)-initial segment in \( J_r(I) \), so Lemma 4.6 shows that

\[
u_I^r \geq u_{J_{r+1}(I \cup x)} = u^{r+1}_{I \cup x}.
\]

5. Transversal matroids associated to permutation diagrams

In this section we give some conjectures to the effect that results like Theorem 3.2 and Theorem 4.2 hold for another family of rank \( n \) matroids on \([2n]\) indexed by \( S_n \).

Definition 5.1. The diagram of \( w \in S_n \) is

\[
D(w) = \{(i, w(j)) \in [n] \times [n] : i < j, w(i) > w(j)\}.
\]

Given \( w \in S_n \), let \( \tilde{M}_w \) be a generic \( n \times 2n \) matrix \([A \mid I_n]\), where \( I_n \) is an \( n \times n \) identity matrix, and \( A \) is \( n \times n \) with \( A_{ij} = 0 \) whenever \((i, j) \in D(w)\). The diagram matroid \( DM_w \) of \( w \) is the matroid of \( \tilde{M}_w \).
Example 5.2. Say $w = 31524$. Then

$$
\begin{array}{ccccccc}
\circ & \circ & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
$$

where we use matrix coordinates, and $\circ$ for lattice points in $D(w)$, $\cdot$ for those not in $D(w)$. The diagram matroid of $w$ is then the matroid of a generic matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \ast & \ast & \ast \\
0 & 1 & 0 & 0 & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 1 & 0 & 0 & \ast & \ast & 0 & \ast & \ast \\
0 & 0 & 0 & 1 & 0 & \ast & \ast & \ast & \ast & \ast \\
0 & 0 & 0 & 0 & 1 & \ast & \ast & \ast & \ast & \ast \\
\end{bmatrix}
$$

Conjecture 5.3. Theorem 1.1 holds for $DM_w$. That is, for any $w \in S_n$, the number of bases of $DM_w$ is

$$
\sum_{v \geq w \text{ avoids } 123} C_{\ell_1+1} \cdots C_{\ell_k+1},
$$

where $\ell_1, \ldots, \ell_k$ are the lengths of the runs of anti-fixed points of $v$.

However, Theorem 3.2 no longer holds, as it can happen that $w \leq v$ but $DM_w \not\subseteq DM_v$. One can still hope to prove Conjecture 5.3 by Möbius inversion, but a less trivial sign-reversing involution is required.

Of course, if $DM_w$ and $P_w$ were isomorphic, then Conjecture 5.3 would be true. However, this need not be the case.

Conjecture 5.4. The matroids $DM_w$ and $P_w$ are isomorphic if and only if $w$ avoids the pattern $21354$.

This conjecture has been verified through $S_7$. Despite this, their Tutte polynomials seem to agree, also verified through $S_7$.

Conjecture 5.5. For any $w \in S_n$, the Tutte polynomial of $DM_w$ is equal to the Tutte polynomial of $P_w$.

There is a combinatorial procedure called shifting that relates $DM_w$ and $P_w$ (and which has geometric connections making it useful in studying positroid varieties and other subvarieties of Grassmannians [5, 9]). Given integers $i$ and $j$, and a set $I$, let

$$
\Pi_{i \rightarrow j} I = \begin{cases} 
I \setminus i \cup j & \text{if } i \in I \text{ and } j \notin I \\
I & \text{else}
\end{cases}
$$

If $X$ is a collection of sets, and $I \in X$, then we define

$$
\Pi_{i \rightarrow j, X} I = \begin{cases} 
\Pi_{i \rightarrow j} I & \text{if } \Pi_{i \rightarrow j} I \neq I \text{ and } \Pi_{i \rightarrow j} I \notin X \\
I & \text{else}
\end{cases}
$$

Finally, define $\Pi_{i \rightarrow j} X$ to be $\{\Pi_{i \rightarrow j, X} I : I \in X\}$. 
Let $B(A)$ denote the matroid of a matrix $A$. We can also apply shifting to matrices. Let $\Pi_{i \rightarrow j} A$ be the matrix of the same size as $A$ such that

$$A_{pq} = \begin{cases} A_{pi} & \text{if } q = j \text{ and } A_{pj} = 0 \\ 0 & \text{if } q = i \text{ and } A_{pj} = 0 \\ A_{pq} & \text{else} \end{cases}$$

We have $\# \Pi_{i \rightarrow j} X = \# X$, but it need not be the case that $B(\Pi_{i \rightarrow j} A) = \Pi_{i \rightarrow j} B(A)$. For example, if $A$ is a $2 \times 2$ identity matrix, then $\Pi_{2 \rightarrow 1} B(A) = \{12\}$, while $B(\Pi_{2 \rightarrow 1} A)$ is empty. In general, we only get a containment.

**Lemma 5.6.** If the entries of $A$ are algebraically independent, then $B(\Pi_{i \rightarrow j} A) \subseteq \Pi_{i \rightarrow j} B(A)$.

**Proof.** Suppose $I \in B(\Pi_{i \rightarrow j} A)$, where $I = \{b_1 < \cdots < b_n\}$. Then there is a transversal of $A$ in columns $I$, i.e. a bijection $\pi : I \rightarrow [n]$ such that $(\Pi_{i \rightarrow j} A)_{\pi(b_n)} \neq 0$ for each $p$. We consider various cases.

- If $i, j \notin I$, then $I \in B(A)$ and $\Pi_{i \rightarrow j} I = I$, so $I \in \Pi_{i \rightarrow j} B(A)$.
- If $i \in I$, $j \notin I$, then again $I \in B(A)$, because $\Pi_{i \rightarrow j} A$ restricted to columns $I$ is $A$ restricted to columns $I$ with some nonzero entries made zero. Since $(\Pi_{i \rightarrow j} A)_{\pi(i)}$ is nonzero, $A_{\pi(i)}$ must be nonzero. Therefore by the bijection $\pi' : I \setminus i \cup j \rightarrow [n]$ which agrees with $\pi$ on $I \setminus i$ and having $\pi'(j) = \pi(i)$ is a transversal of $A$. This shows that $B(A)$ also contains $\Pi_{i \rightarrow j} I$. But then $I \in \Pi_{i \rightarrow j} B(A)$.
- Suppose $i \notin I$, $j \in I$. If $A_{\pi(i)} \neq 0$, then modifying $\pi$ appropriately as in the last case will give a transversal of $A$ in columns $I \setminus j \cup i$. Then $I = \Pi_{i \rightarrow j} (I \setminus j \cup i) \in \Pi_{i \rightarrow j} B(A)$.
- If $A_{\pi(i), j} \neq 0$, then the $A_{\pi(j)j} \neq 0$, and so $I \in B(A)$. Then $I = \Pi_{i \rightarrow j} I \in B(A)$.
- Suppose $i, j \in I$. Since the $(\Pi_{i \rightarrow j} A)_{\pi(i)}$ is nonzero, the same entry of $A$ is also nonzero. Therefore if $A_{\pi(j)j} \neq 0$, then $\pi$ is still a transversal of $A$ in columns $I$.

Now say $A_{\pi(i)j} = 0$. Then, since the $(\Pi_{i \rightarrow j} A)_{\pi(i)}$ is nonzero, so is $A_{\pi(i)j}$. Also, since $(\Pi_{i \rightarrow j} A)_{\pi(i)}$ is nonzero, so is $A_{\pi(i)}$. Therefore the bijection $\pi' : I \rightarrow [n]$ agreeing with $\pi$ on $I \setminus \{i, j\}$, and having $\pi'(j) = \pi(i)$, $\pi'(j) = \pi(i)$, is a transversal of $A$ in columns $I$.

Either way we see that $I \in B(A)$, and so $I \in \Pi_{i \rightarrow j} I \in \Pi_{i \rightarrow j} B(A)$.

The matrices $M_w$ and $	ilde{M}_w$ defining $P_w$ and $DM_w$ turn out to be related by a sequence of shifts. Let $\Pi_{w}$ be the composition

$$\Pi_{2n \rightarrow w(n)} \cdots \Pi_{n+2 \rightarrow w(2)} \Pi_{n+1 \rightarrow w(1)}.$$

**Lemma 5.7** ([9], Theorem 5.5). For any permutation $w$, $\Pi_{w} M_w = \tilde{M}_w$.

Thus, Lemma 5.6 shows that $DM_w \subseteq \Pi_{w} P_w$. Since shifting preserves the size of a collection of sets, we see that Conjecture 5.3 is equivalent to:

**Conjecture 5.8.** For any permutation $w$, $\Pi_{w} P_w = DM_w$.

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