Non-stationary oscillation of a string on the Winkler foundation subjected to a discrete mass-spring system non-uniformly moving at a sub-critical speed

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Abstract

We consider non-stationary free and forced transverse oscillation of an infinite taut string on the Winkler foundation subjected to a discrete mass-spring system non-uniformly moving at a given sub-critical speed. The speed of the mass-spring system is assumed to be a slowly time-varying function. To describe a non-vanishing free oscillation we use an analytic approach based on the method of stationary phase and the method of multiple scales. The moving oscillator is characterized by a partial frequency, which can be greater or less than the cut-off frequency. Accordingly, a sub-critical uniformly accelerated motion generally has two stages. At the first stage there exists a trapped mode, and, therefore, a part of the wave energy is localized near the moving oscillator and does not propagate away. For this stage we obtain the analytic solution in a simple form describing non-vanishing free oscillation and verify it numerically. For the second stage there is no trapped mode, and all the wave energy propagate away. This stage is investigated numerically, and some unexpected results are obtained. Additionally, we consider the case of the oscillator with a destabilizing spring. The dynamics of the system in the latter case is quite different from the case of commonly used stabilizing spring, since the system loses the stability during an accelerated motion. We also take into consideration the forced oscillation caused by an external load being a superposition of harmonics with time-varying parameters (the amplitude and the frequency).

Keywords: moving load, free and forced oscillation, trapped mode, linear wave localization, the method of stationary phase, the method of multiple scales

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1. Introduction

In the paper we deal with a moving load problem [1, 2]. We consider transverse oscillation of an infinite taut string on the Winkler foundation. The string is equipped with a moving discrete mass-spring oscillator non-uniformly moving at a given sub-critical speed (see Fig. 1 for the schematic of the system).

Most of the results concerning non-uniformly moving loads applied to a string are obtained for the case of an inertialess point loads [3–6]. A famous effect related to a point inertial load was demonstrated by Stokes [7] who considered a uniform motion of a point mass along a finite inertialess string and obtained a paradoxical result concerning discontinuity in the particle trajectory close to the end support (see also [1, 8, 9]). The Stokes paradox was resolved in recent series of studies [10–12] where the dynamic problem in the framework of geometrically non-linear problem statement was investigated following to ideas of [13–17]. Motion of a small point mass along a semi-infinite string without elastic foundation was considered in [18]. Motion of a linear oscillator along a finite string or beam is considered in [19–21] mostly from the numerical point of view. Analytical solutions for infinite, semi-infinite and finite strings without elastic foundation with uniformly moving masses are obtained in [22]. The nonlinear problem concerning a moving mass along a finite suspended cable was considered in [23]. Instability of transversal vibration of a mass moving along a string on the elastic foundation with periodic or stochastic near-periodic stiffness was considered in [24, 25].

In [26] it was suggested a new approach based on the method of stationary phase and the method of multiple scales successively applied to the problem concerning a non-uniform sub-critical motion of a point mass along an infinite string on the Winkler foundation. The method of multiple scales was applied to describe the evolution of the amplitude of a trapped mode of oscillation, due to which a non-stationary free oscillation in such a system does not vanish with time. To the best of our knowledge, such an approach was never used before to investigate the dynamics of systems with moving loads. The extensive bibliography on the phenomenon of trapped modes and localization of linear waves can be found in [27–30]. This method, which is not a pure asymptotic one, has a deep asymptotic motivation (see Sect. 3 for the discussion). It is a general approach,
which allows us to investigate non-stationary free localized oscillation in infinite systems with time-varying parameters in the case if the corresponding system with constant parameters possesses a single trapped mode. The method was applied to several model problems concerning an infinite string on elastic foundation with a discrete inclusion, namely a string with time-varying tension [28], string with time-varying mass [31], string with oscillator of time-varying stiffness [32]. Also, the method was successfully applied to the problem concerning a beam with discrete inclusion of a time-varying stiffness [33], and a resonant solution of a model problem concerning a string [34] was obtained. In all cases we compare our analytic results with numerics and demonstrate an excellent agreement. This allows us to develop some mathematical tricks, which simplify the problem solution.

In this paper we deal with a moving discrete sub-system, namely a moving mass-spring system (a linear oscillator). The moving oscillator is characterized by a partial frequency, which can be greater or less than the cut-off frequency for the infinite waveguide (i.e., the string on elastic foundation). Accordingly, a sub-critical uniformly accelerated motion generally has two stages. At the first stage there exists the trapped mode, and, therefore, a part of the wave energy is localized near the moving oscillator and does not propagate away. For this stage we obtain the analytic solution in a simple form and verify it numerically. For the second stage there is no trapped mode, and all the wave energy propagate away. This stage is investigated numerically, and some unexpected results are obtained (the growth of the force between the load and the string, see Sect 5.2.1 for details). We also provide some ideas how the second stage can be described analytically (though this remains to be a subject of a future work, see Sect. 6). To the best of our knowledge, these two stages of an accelerated motion are not identified and described in the modern literature. Moreover, the corresponding two qualitatively different cases (whether or not there exist trapped modes) in the problem concerning a uniform sub-critical motion are also not discussed (see, e.g., recent study [35]).

We underline that complicated models involving an infinite waveguide and several moving discrete inclusions constructed by spring, mass, and damper elements [35], which are used in engineering to investigate, for example, pantograph-catenary dynamics [36–40], under certain conditions definitely can possess trapped modes in the absence of viscosity in dampers, and, therefore, a non-vanishing free oscillation can be observed there. Note that adding of damper elements with a small viscosity does not change the situation significantly, since in the latter case free localized oscillation vanishes very slowly [41]. A damper element is not taken into account in the current paper, though for the time being we have some ideas for future how to consider it in the framework of our approach. The obtained analytic solution can be used as a reference solution in applications, and brings the understanding that a sub-critical accelerated motion of a discrete system composed
The problem considered in [26] is a limiting case of the current problem. The second stage of an accelerated motion does not exist for a point mass problem investigated in that previous paper. Note that in [26] the solution initially was obtained in extremely complicated form as a product of ten indefinite integrals, though the final result has a very simple structure. In this paper we suggest a mathematical trick, which allows us to significantly simplify the calculations and to obtain the solution of more complicated problems in an easier way. While considering the corresponding limiting case, we have discovered an error in [26], though the behaviour of the erroneous solution is very close to the correct solution (see Sect. 5.2.2 for details). Note that revisitation of [26] has never been a motivation of our current work, since we have been absolutely sure that the old solution is correct being in an excellent agreement with numerical results. In addition to the fact that now we deal with a mechanical system different from the one considered in [26], in this paper we introduce some other extensions to the problem formulation, namely:

- We extend the class of admissible loads applied to the discrete subsystem, for which the analytic solution can be obtained. In [26] the external load is the constant weight of the point mass. Now we introduce a load, which can be represented as the superposition of a “pulse” load, which acts during some time, and a number of harmonics with independent (from the speed) time-varying parameters (the frequency and the amplitude). Thus, the forced oscillation, caused by such a harmonic load, co-existing with a free localized oscillation is under investigation. The obtained analytic solution is verified numerically.

- We consider both the classical case of a stabilizing oscillator spring, and the case when the stiffness of the spring in the oscillator is negative (i.e., we deal with a destabilizing spring). Destabilizing springs can be realized by magnetic forces [42]. They are used, e.g., in multimode control of cable vibration [43, 44] and when constructing metamaterials [45–47]. The dynamics of the system in the latter case is quite different from the case of commonly used stabilizing spring, since the system loses the stability during an accelerated motion.

There are only few known analytic solutions concerning non-stationary oscillation of infinite waveguides caused by non-uniformly moving inertial loads. To the best of our knowledge, our previous paper [26] is the only one paper, where such a solution is obtained in a simple asymptotic form. Almost all results concerning moving loads in infinite dispersive waveguides are obtained by applying the method of integral transforms, which generally cannot yield simple analytic results for systems with time-varying parameters. We have suggested a general alternative approach based on the method of multiple scales. On the other hand, as far as we know, in the modern literature there are no analytic solutions for a system with a non-uniformly moving oscillator, which demonstrates
more complicated and qualitatively different dynamic behaviour. Thus, we suppose that our new results are an important contribution to the understanding of dynamics for systems with moving loads.

The paper is organized as follows. In Sect. 2 we present mathematical formulation for the problem in the fixed (non-moving) co-ordinates (that are used in the paper in the numerical treatment), as well as in the co-moving with the load co-ordinates (that are used in the paper in the analytic treatment). In Sect. 3 we discuss our analytic approach. We briefly discuss what a trapped mode is, and why it is important for our problem. We introduce a class of the admissible loadings applied to the mass-spring system. Thus, in this paper we deal with both free and forced oscillation, and, accordingly, we need to introduce some modifications to our method. The solution is represented in the form that we call the multi-frequency ansatz (Sect. 3.1), which selects a number of harmonics with time-varying frequencies considered to be important. The choice of these frequencies is based on the results got by the method of stationary phase [48, 49] applied to the corresponding system with constant parameters. Every term of the multi-frequency ansatz is represented in the form of an asymptotic series, which we call the single-frequency ansatz (Sect. 3.2). The aim of the analytic work in this paper is to find the principal term of the multi-frequency ansatz. To do this we use a modification of the method of multiple scales [50, 51]. In Sect. 3.3 we present the multi-scale representation for the differential operators. In Sect. 4 we obtain the analytic solution for our problem. In Sect. 4.1 we evaluate free localized oscillation, and this is the most important part of the paper. In Sect. 4.2 we evaluate the modes of forced oscillation. In Sect. 4.3 we calculate the unknown constants in the expression describing the free oscillation. In Sect. 4.4 we obtain the analytic solution for the unknown internal force between the string and the discrete mass. In Sect. 5 we verify the constructed analytic solution numerically. To do this we derive an integral equation for the unknown internal force (Sect. 5.1), and solve it numerically to compare the results (Sect. 5.2) for several qualitatively different cases. The cases of pure free oscillation for a stabilizing and destabilizing discrete oscillator spring are considered in Sects. 5.2.1, 5.2.3, respectively. In Sect. 5.2.2 we compare obtained results with results of previous paper [26] and demonstrate that the old solutions is erroneous, though it has a behaviour, which is very close to the correct solution. In Sect. 5.2.4 we demonstrate numerically that a free oscillation is negligible if the trapped mode does not exist in the system, as generally expected. In Sect. 5.2.5 we verify the analytic solution in the case of co-existing free and forced oscillation. In Sect. 5.3 we very briefly discuss some more qualitatively different cases. In Conclusion (Sect. 6) we discuss the basic results of the paper and its possible generalizations.

The case of a uniform motion of the mass-spring oscillator is not considered to be the subject of this paper. Thus, all necessary auxiliary results are presented in Appendix. The material of
Appendix A–Appendix E mostly involves some known results or their generalizations (see [52, 26, 28, 32, 27, 53, 54]). The final result (Appendix F) concerning the large-time asymptotics of non-stationary free and forced oscillation in the system with constant parameters is the generalization of the results obtained in [41], though we use a bit different technique for the asymptotic evaluation of integrals. Finally, in Appendix G–Appendix H we present the formulae for the fundamental solutions in time domain, which we use to derive the integral equation.

2. Mathematical formulation

Introduce the following notation: $u(x, t)$ is the displacement of a point of the string at the position $x$ and the time $t$, $M \geq 0$ is the mass in the discrete oscillator, $K$ is the spring stiffness for the discrete oscillator. We assume that $K$ can be positive (stabilizing), negative (destabilizing), or zero:

$$K \preceq 0. \quad (2.1)$$

As we have already discussed in Introduction destabilizing springs are used when constructing metamaterials.

For $t < 0$ the string and the discrete oscillator are at the rest. At the instant $t = 0$ an external non-zero transverse force $p(t)$ applied to the discrete mass appears, and the oscillator starts to move along the string according to the law

$$\ell(t) = \ell(0) + \int_0^t v(T) \, dT, \quad (2.2)$$

where $v(t)$ is the speed of the oscillator (a given function), and $\ell(0)$ is a given initial position for the oscillator. Denote the acceleration of the oscillator as $a(t) = \dot{v}(t)$. We consider a sub-critical regime of the oscillator motion and assume that for all $t$

$$|v(t)| < 1. \quad (2.3)$$

Thus, the motion $\ell(t)$ of the oscillator along the string and the transverse external force $p(t)$ are considered as prescribed quantities. The transverse dynamics of the system is unknown.

Denote the displacement of the point of the string subjected to the moving load as

$$U(t) = u(\ell(t), t). \quad (2.4)$$
The governing equations in the dimensionless form [32] are
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - u = -P(t)\delta(x - \ell(t)), \tag{2.5}
\]
\[
M\frac{d^2 U}{dt^2} + KU = -P(t) + p(t), \tag{2.6}
\]
where \(P(t)\) is the unknown transverse force on the string from the oscillator, \(\delta\) is the Dirac delta-function. Equations (2.5), (2.6) can be rewritten in the co-moving with the discrete oscillator system of co-ordinates \(\xi = x - \ell(\tau), \tau = t:\)
\[
(1 - v^2)u'' + au' + 2vu' - \ddot{u} - u = -P(\tau)\delta(\xi), \tag{2.7}
\]
\[
M\ddot{U} + KU = -P(\tau) + p(\tau). \tag{2.8}
\]
Here and in what follows, we denote by prime and overdot the derivatives with respect to \(\xi\) and \(\tau\), respectively.

According to Eqs. (2.5), (2.6) the following Hugoniot conditions must be satisfied under the moving load:
\[
[u] = 0, \tag{2.9}
\]
\[
[u'] = -\frac{P(\tau)}{1 - v^2}. \tag{2.10}
\]
Here, and in what follows, \([\mu] = \mu(\xi + 0) - \mu(\xi - 0)\) for any arbitrary quantity \(\mu(\xi, \tau)\).

The initial conditions for Eq. (2.7) are zero. They can be formulated in the following form, which is conventional for distributions (or generalized functions) [55]:
\[
u |_{\tau < 0} \equiv 0. \tag{2.11}
\]

3. Method

The main assumption we use to construct the asymptotic solution is that the velocity \(v\) is a slowly varying piecewise monotone function of the slow time \(T:\)
\[
T = \epsilon \tau, \tag{3.1}
\]
\[
v = v(T), \tag{3.2}
\]
and, therefore, acceleration \(a = O(\epsilon)\). Here \(\epsilon\) is a formal small parameter. We look for the asymptotic solution under the following conditions:
• $\epsilon = o(1)$,
• $\tau = O(\epsilon^{-1})$,
• $v(T)$ satisfies restriction (2.3) for all $T$. Note that in what follows, we will formulate more strict restrictions, see Eqs. (E.20), (E.21).

In the paper we restrict ourselves to the external loadings $p$, which can be represented as a superposition of a “pulse” loading, which acts during some time, and a number of harmonics with slowly time-varying frequencies and amplitudes:

$$p(\tau, T) = H(\tau) \left( \frac{\dot{p}(\tau)}{2} + \sum_{i=1}^{N} p^{(\Omega_i)}(T) \exp \left( -i \int_{0}^{T} \Omega_i(T) \, dT \right) \right) + c.c.,$$

where notation c.c. denotes the complex conjugate terms for the whole right-hand side, $H(\cdot)$ is the Heaviside function,

$$\Omega_i \geq 0,$$

$$\Omega_i \neq \Omega_j \quad \forall t \quad \text{if} \quad i \neq j \quad \text{for} \quad i, j = 1, N.$$  \hspace{1cm} (3.4)

We assume the pulse loading $\dot{p}(\tau)$ to be a given real finite integrable function (or a finite generalized function [56, 55]) such that $\dot{p} \equiv 0$ if $\tau < 0$ or $\tau > T$ for certain $T > 0$. The amplitudes $p^{(\Omega_i)}(T)$ ($i = 1, N$) are given smooth complex-valued functions.

3.1. The multi-frequency ansatz

Consider now the unperturbed system ($\epsilon = 0$) with $v = v(0)$, $p^{(\Omega_i)} = p^{(\Omega_i)}(0)$. This case corresponds to the source motion at a constant speed, and the equation of motion (2.7) in the co-moving co-ordinate system is a linear partial differential equations (PDE) with constant coefficients. Usually we expect that due to the wave radiation to infinity, for large times the response of a distributed system subjected to a point excitation transforms into a sum of harmonics with the different frequencies $\Omega_i$:

$$u = \sum_{i=1}^{N} \tilde{u}^{(\Omega_i)} \equiv \sum_{i=1}^{N} \tilde{W}^{(\Omega_i)} \exp(-i\Omega_i \tau) \exp(c.c.) + o(1), \quad \tau \to \infty.$$  \hspace{1cm} (3.6)

The solution in the form of Eq. (3.6) is a pure forced oscillation, and, therefore, constants $\tilde{W}^{(\Omega_i)}$ can be easily found basing on, for example, the stationary formulation for the problem, or by applying the method of stationary phase (see Appendix D). However, generally Eq. (3.6) is not true for the
system under consideration. Under certain conditions (in particular, in the case $M > 0$, $K = 0$ [26]) there is a special frequency $0 < \Omega_0 < \Omega_*$, 

$$\Omega_* \overset{\text{def}}{=} \sqrt{1 - v^2}, \quad (3.7)$$

to which a trapped mode of oscillation corresponds. This mode is localized near the discrete oscillator. The frequency defined in Eq. (3.7) is so-called cut-off (or boundary) frequency, discussed in Appendix Appendix B. According to dispersion relation (B.2) free waves with frequencies upper than the cut-off frequency are sinusoidal propagating waves, whereas free waves with frequencies below than the cut-off frequency are growing inhomogeneous waves, which cannot exist if we require boundedness. The expression for $\Omega_0$, necessary and sufficient conditions for the existence of the trapped mode for the system under consideration are obtained in Appendix Appendix E. If the trapped mode exists, then in the non-resonant case wherein

$$\Omega_i \neq \Omega_0 \quad \text{for} \quad i = 1, N \quad (3.8)$$

instead of Eq. (3.6) we get

$$\mathcal{U} = \sum_{i=0}^{N} \mathcal{U}^{(\Omega_i)} = \sum_{i=0}^{N} \tilde{W}^{(\Omega_i)} \exp(-i\Omega_i \tau) + \text{c.c.} + o(1), \quad \tau \to \infty. \quad (3.9)$$

The additional term $\mathcal{U}^{(\Omega_0)}$ corresponds to “natural” localized oscillation. Constant $\tilde{W}^{(\Omega_0)}$ (the corresponding amplitude) can be found using the method of stationary phase (see Appendix Appendix F). Thus, in the case $\epsilon = 0$, the response of the system under consideration is the superposition of a mode of natural oscillation with frequency $\Omega_0$ and of $N$ modes of forced oscillation, like in the case of a single degree of freedom system. Note that for the problem under consideration Eq. (3.9) in the explicit form is formula (F.8).

Now consider the case of a non-uniform oscillator motion ($\epsilon > 0$). In the non-resonant case (where Eqs. (3.5), (3.8) are fulfilled for all $T \geq 0$) we expect that for large times the displacement under the moving oscillator can be approximately found in the form of the following multi-frequency ansatz:

$$\mathcal{U} \simeq \sum_{i=0}^{N} \mathcal{U}^{(\Omega_i)} = \sum_{i=0}^{N} W^{(\Omega_i)}(T) \exp \left( -i \int_0^T \Omega_i(T) dT \right) + \text{c.c.}, \quad \tau \to \infty. \quad (3.10)$$
where amplitudes $W(\Omega_{i})$ can be represented by the asymptotic expansions

$$W(\Omega_{i})(T) = \sum_{j=0}^{\infty} \epsilon^{j} W_{j}(\Omega_{i})(T).$$  \hspace{1cm} (3.11)

Additionally, we will require that

$$\lim_{T \to +0} W(\Omega_{i})(T) = \hat{W}(\Omega_{i}).$$  \hspace{1cm} (3.12)

The quantity $\Omega_{0}(T)$ in Eq. (3.10) is an immediate value of the trapped mode frequency in the system with fixed $v = v(T)$, where $T$ should be considered as a time-like parameter.

The applicability of ansatz (3.10) as a reasonable approximation for the solution of the problem under consideration is our hypothesis, which should be verified by numerical calculations. This hypothesis is supported by several circumstances, namely:

- For $\epsilon \to 0$ ansatz (3.10) transforms into non-stationary solution (3.9), which can be formally obtained by the method of stationary phase;

- The characteristic time after which the approximation (3.9) becomes practically applicable does not depend on $\epsilon$, and, therefore, is $O(1)$, whereas the characteristic time of change for the velocity $v(T)$ and the external force amplitudes $p_{j}(\Omega_{i})(T)$ ($i = 1, N$) is $O(\epsilon^{-1})$;

- An asymptotic solution for a single degree of freedom system with time-varying coefficients has the form of Eq. (3.10) [50, 51].

However, since the problem under consideration is formulated for a PDE with time-varying coefficients, to prove the applicability of multi-frequency ansatz (3.10) in a formal way is a really hard problem. In this sense, our approach is not a pure asymptotic one, even though accepting (3.10) has a deep asymptotic motivation. Nevertheless, in what follows, we look for the principal terms of the amplitudes $W_{0}(\Omega_{i})(T)$ using a rigorous asymptotic procedure of the method of multiple scales.

We use an approach [26] based on the modification of the method of multiple scales (Sect. 7.1.6 of [51]) for ordinary differential equations (ODE) with slowly varying coefficients. The corresponding rigorous proofs, which validate such asymptotic approach in the case of a one degree of freedom system, can be found in [50]. However, since we look for the solution of a PDE, we need to continue the multi-frequency ansatz (3.10) to a neighbourhood of the point $\xi = 0$:

$$u(\xi, \tau) = \sum_{i=0}^{N} u^{(\Omega_{i})},$$  \hspace{1cm} (3.13)

$$u^{(\Omega_{i})}(\xi, \tau) \bigg|_{\xi = 0} = U^{(\Omega_{i})}(\tau).$$  \hspace{1cm} (3.14)
3.2. The single-frequency ansatz

We assume that every amplitude \( p^{(\Omega_i)}(T) \) is presented in the form of an asymptotic series:

\[
p(T) = \sum_{j=0}^{\infty} \epsilon^j p_j(T). \tag{3.15}
\]

Here we have dropped the superscript \((\Omega_i)\) near quantities \( p \) and \( p_j \) for the aim of simplicity.

We represent the continuation \( u^{(\Omega_i)} \) of every term \( u^{(\Omega_i)} \) in the right-hand side of (3.10) to \( \xi \leq 0 \) as the following single-frequency ansatz:

\[
u(\xi, \tau) = W(X,T) \exp \phi(\xi, \tau), \tag{3.16}
\]

where

\[
X = \epsilon \xi \tag{3.17}
\]

is the slow spatial co-ordinate; \( \phi(\xi, \tau) \) such that

\[
\phi' = i\omega(X,T), \quad \dot{\phi} = -i\Omega(X,T), \tag{3.18}
\]

\[
\lim_{X \to \pm 0} \Omega = \Omega_i \tag{3.19}
\]

is the phase;

\[
W(X, T) = \sum_{j=0}^{\infty} \epsilon^j W_j(X, T) \tag{3.20}
\]

such that

\[
\lim_{X \to \pm 0} W(X, T) = W(T), \tag{3.21}
\]

\[
\lim_{X \to \pm 0} W_j(X, T) = W_j(T) \tag{3.22}
\]

is the amplitude. The wave-number \( \omega(X, T) \) and the frequency \( \Omega(X, T) \) should satisfy dispersion relation (B.2) and equation

\[
\Omega_X' + \omega_T' = 0 \tag{3.23}
\]

that follows from (3.18) for all \( X \) and \( T \) in a neighbourhood of \( X = 0 \). In this case, the phase
\( \varphi(\xi, \tau) \) can be defined by the formula

\[
\varphi = i \int (\omega \, d\xi - \Omega \, d\tau).
\]  

(3.24)

Additionally, we require that

\[
[W] = 0, \quad [\varphi] = 0.
\]  

(3.25)

**Remark 1.** Representations (3.16)–(3.25) are valid and different for all \( N+1 \) single modes \( u^{(\Omega_i)} \), i.e., we again have dropped in those equations the superscript \( (\Omega_i) \) near quantities \( u, W, W_j, \phi, \Omega, \omega \) for the aim of simplicity. Moreover, the analytic expressions for these quantities are generally different for \( X \leq 0 \) since we additionally require that \( u \) satisfy some boundary conditions at infinity \( (X \to \pm\infty) \). These are vanishing boundary conditions for \( |\Omega_i| < \Omega_* \) (where the wave-numbers are imaginary) and some radiation conditions for \( |\Omega_i| > \Omega_* \) (where the wave-numbers are real). To satisfy these boundary conditions we need to choose for \( X \leq 0 \) different roots (B.6).

The aim of the analytic work in this paper is to find the principal term of multi-frequency ansatz (3.10). We evaluate independently every term of (3.10), which is represented in the form of a single-frequency ansatz (3.16)–(3.25). In what follows, we will get the corresponding solution in the case when initially the trapped mode in the corresponding system with constant \( v = v(0) \) exists, i.e., if Eq. (E.20) or (E.21) is fulfilled for \( T = 0 \). Also, we will demonstrate that the principal term of multi-frequency ansatz (3.10) and the corresponding numerical solution are in excellent agreement unless the trapped mode disappears at a certain \( T \). We also demonstrate that multi-frequency ansatz (3.10) is not practically applicable after this instant. The latter case is beyond the scope of the analytic work in this paper.

Finally, we indicate that the principal zero-order terms for all modes of multi-frequency ansatz (3.10) \( U^{(\Omega_i)} \), \( i = \Gamma, N \), which correspond to a forced oscillation, can be found without consideration of their continuations \( u^{(\Omega_i)} \) (these terms can be found from equations of zero order approximation). Looking for the principal zero-order term only, we really need to introduce the continuation \( W_0^{(\Omega_i)}(X, T) \) only to calculate the evolution of amplitude for the trapped mode \( U_0^{(\Omega_i)}(T) \). In the latter case the equations of the first order approximation are necessary. The corresponding details are given in what follows (see Sect. 4).

### 3.3. The representation for the differential operators

According to the method of multiple scales [51], the slow variables \( X, T \), and the fast phase \( \varphi \) are assumed to be independent variables. In this way, we represent the differential operators with
respect to time and the spatial co-ordinate in the following form:

\[
\dot{\cdot} = -i\Omega \partial_\phi + \epsilon \partial_T, \quad \dot{\cdot}' = i\omega \partial_\phi + \epsilon \partial_X, \\
\dot{\cdot}'' = -\Omega^2 \partial_{\phi\phi} - 2i\Omega \partial_{\phi T} - i\epsilon \partial_\phi + O(\epsilon^2), \\
\dot{\cdot}''' = \omega \Omega^2 \partial_{\phi\phi} - \epsilon \partial_{\phi T} + O(\epsilon^2).
\]  

(3.26)

4. Asymptotic solution

4.1. Contribution from the trapped mode

Accepting of the representation for the contribution from the frequency of the localized oscillation in the form of the single-frequency ansatz (3.16)–(3.25) (wherein \(i = 0\) and the superscript \((\Omega_0)\) is assumed near the corresponding quantities, see Remark 1) implies that

- Frequency equation (E.5) for the trapped mode holds for all \(T\);
- Dispersion relation (B.2) at \(\xi = \pm 0\) holds for all \(T\).

At first, we substitute Eq. (3.16) and representations for differential operators (3.26) into the second Hugoniot condition (2.10), wherein \(P(\tau)\) in the right-hand side is expressed by Eq. (2.6). Since the representation for the solution at \(\xi = 0\) in the form of multi-frequency ansatz (3.9) becomes valid after a certain time, whereas \(\hat{p}\) is a finite or exponentially vanishing function, we take \(p = 0\) here. In this way, taking into account (3.19), one can obtain

\[
[i\omega W + \epsilon W_X] + O(\epsilon^2) = \frac{M(-\Omega_0^2 W(0, T) - 2i\Omega_0 W(0, T)'_T - \epsilon \Omega_0' W(0, T)) + KW(0, T)}{1 - \nu^2}. \quad (4.1)
\]

Now we substitute expansion (3.20) into Eq. (4.1) and equate coefficients of like powers \(\epsilon\). Taking into account frequency equation (E.5) for \(\Omega_0\), one can demonstrate that the equation for the zeroth order approximation is identically satisfied. For the first order approximation one can derive:

\[
[W_0, X]' = \frac{-2iM\Omega_0 W(0, T)'_T - iM\Omega_0' W(0, T)}{1 - \nu^2}. \quad (4.2)
\]

Note that Eq. (4.2) does not involve terms, which depend on \(W_1\), since the common multiplier before all such terms equals zero according to the frequency equation (E.5).

On the other hand, we can define the quantity in the left-hand side of Eq. (4.2) by consideration of Eq. (2.7), wherein the right-hand side is put to zero, in the case \(\xi \equiv 0\), and, in particular at \(\xi \to \pm 0\). To do this, we substitute ansatz (3.16)–(3.25) and representations (3.26) into Eq. (2.7) and equate coefficients of like powers \(\epsilon\). Taking into account dispersion relation (B.2), for the zeroth
order approximation one can again demonstrate that the corresponding equation is identically satisfied. For the first order approximation we obtain:

\[
((1 - v^2)2\omega - 2v\Omega_0) W_0'_{X} + (2v\omega + 2\Omega_0) W_0'_{T} + ((1 - v^2)\omega'_{X} + 2v\omega'_{T} + \Omega_0' + \omega a) W_0 = 0 \quad (4.3)
\]

or

\[
W_0'_{X} = \frac{- (2v\omega + 2\Omega_0) W_0'_{T} + ((1 - v^2)\omega'_{X} + 2v\omega'_{T} + \Omega_0' + \omega a) W_0}{(1 - v^2)2\omega - 2v\Omega_0} \quad (4.4)
\]

at \( \xi = \pm 0 \). Due to Eq. (3.23) one has

\[
\omega'_{X} = \omega'_{T} = -\omega'_{\Omega} \omega'_{T} \quad (4.5)
\]

Using this formula, we can write down:

\[
W_0'_{X} = \frac{- (2v\omega + 2\Omega_0) W_0'_{T} + ((1 - v^2)\omega'_{X} + 2v\omega'_{T} + \Omega_0' + \omega a) W_0}{(1 - v^2)2\omega - 2v\Omega_0} \quad (4.6)
\]

Here \( \omega'_{\Omega} \) and \( \omega'_{T} \) should be calculated in accordance with Eq. (B.6).

Thus, one can obtain

\[
[W_0'_{X}] = -\frac{(\Lambda_0 + \Lambda_2) W_0(0, T) + \Lambda_4 W_0(0, T)'_T}{(1 - v^2)S(\Omega_0)} \quad (4.7)
\]

where

\[
\Lambda_0 \overset{\text{def}}{=} aB(\Omega_0),
\]

\[
\Lambda_1 \overset{\text{def}}{=} 2vB(\Omega_0) + 2\Omega_0,
\]

\[
\Lambda_2 \overset{\text{def}}{=} -(1 - v^2)(B'_{\Omega}(\Omega_0)B'_T(\Omega_0) - S'_{\Omega}(\Omega_0)S'_T(\Omega_0)) + 2vB'_T(\Omega_0) + \Omega'_0
\]

\[
= (1 - v^2)S'_0S'_T + vB'_T + \Omega'_T \quad (4.8)
\]

Now, equating the right-hand sides of Eqs. (4.2) and (4.7) results in the first approximation equation for \( W_0(T) \equiv W_0(0, T) \):

\[
\begin{align*}
\frac{-2M\Omega_0 W_0'_{T} - M\Omega_0' W_0}{(1 - v^2)} &= \frac{(\Lambda_0 + \Lambda_2) W_0 + \Lambda_1 W_0'_{T}}{(1 - v^2)S(\Omega_0)} \quad (4.9)
\end{align*}
\]

The above equation can be transformed to the following one:

\[
\frac{W_0'_{T}}{W_0} = -\frac{\Lambda_0 + \Lambda_2 + MS\Omega_0'_{T}}{\Lambda_1 + 2M\Omega_0 S} \quad (4.10)
\]
Substituting expressions (4.8) for $\Lambda_0, \Lambda_1, \Lambda_2$ into the above equation, we can derive:

$$\frac{W_0'}{W_0} = -\frac{1}{2} \frac{(1 - v^2)S_{\Omega}S_T' + vB_T' + aB + \Omega_0' + M\Omega_0S_T'}{vB + \Omega_0 + M\Omega_0S}.$$  \hspace{1cm} (4.11)

One can rewrite this equation as follows:

$$\frac{W_0'}{W_0} = -\frac{1}{2} \frac{(1 - v^2)S_{\Omega}S_T' + vB_T' + aB + \Omega_0' + M\Omega_0S_T' - M\Omega_0S_T'}{vB + \Omega_0 + M\Omega_0S}.$$  \hspace{1cm} (4.12)

or, equivalently:

$$\frac{W_0'}{W_0} = -\frac{1}{2} \frac{(1 - v^2)S_{\Omega}S_T' + (vB + \Omega_0 + M\Omega_0S)' - M\Omega_0S_T'}{vB + \Omega_0 + M\Omega_0S}.$$  \hspace{1cm} (4.13)

Thus, we obtain:

$$\frac{W_0'}{W_0} = -\frac{1}{2} \frac{(1 - v^2)S_{\Omega}S_T' - M\Omega_0S_T'}{vB + \Omega_0 + M\Omega_0S}.$$  \hspace{1cm} (4.14)

Taking into account Eqs. (B.7),(B.5), one can demonstrate that

$$\frac{(1 - v^2)S_{\Omega}S_T' - M\Omega_0S_T'}{vB + \Omega_0 + M\Omega_0S} = -\frac{1}{S}.$$  \hspace{1cm} (4.15)

Finally, Eq. (4.14) can be transformed to the following one:

$$\frac{W_0'}{W_0} = \frac{1}{2} \frac{S_T'}{S} - \frac{1}{2} \frac{(vB + \Omega_0 + M\Omega_0S)'_{T}}{vB + \Omega_0 + M\Omega_0S}.$$  \hspace{1cm} (4.16)

Thus, we represent the right-hand side of our equation in the form of a total differential of the logarithm of a certain function. The solution of Eq. (4.16) is as follows:

$$W_0 = C\sqrt{\frac{S}{vB + \Omega_0 + M\Omega_0S}},$$  \hspace{1cm} (4.17)

where $C$ is an arbitrary complex constant. Taking into account Eqs. (B.7),(B.5), one obtains:

$$W_0 = C\frac{(1 - v^2 - \Omega_0^2)^{1/4}}{\Omega_0^{1/2} (1 + M\sqrt{1 - v^2 - \Omega_0^2})^{1/2}} = C\sqrt{\frac{M\Omega_0^2 - K}{\Omega_0(M^2\Omega_0^2 - KM + 2)}}.$$  \hspace{1cm} (4.18)

To obtain the second equality in the last formula we have taken into account frequency equa-
tion (E.5). In the particular case \( M = 0 \) Eq. (4.18) yields:

\[
W_0 = C \left( \frac{1 - v^2 - \Omega_0^2}{\Omega_0^{1/2}} \right)^{1/4} = C \sqrt{-\frac{K}{2\Omega_0}}
\]  

(4.19)

or

\[
W_0 = \frac{\tilde{C}}{\sqrt{\Omega_0}},
\]

(4.20)

where \( \tilde{C} = C \sqrt{-K/2} \) is a constant.

**Remark 2.** Note that according to frequency equation (E.5) \( K \) should be negative if the trapped mode exists in the case \( M = 0 \).

**Remark 3.** Formula (4.20), which describes the evolution for the amplitude of the trapped mode in the case \( M = 0, K < 0 \), coincides with the corresponding formula for a linear oscillator with spring of slowly time-varying stiffness [51] (the Liouville–Green approximation). The particular case \( M = 0, K < 0 \) of the problem under consideration is the only one known for us system [33, 28, 32, 31, 26] with trapped mode for which the corresponding formula has this simple classical form.

In the particular case \( K = 0 \) Eq. (4.18) yields:

\[
W_0 = C \frac{(1 - v^2 - \Omega_0^2)^{1/4}}{\Omega_0^{1/2}} \left( 1 + M \sqrt{1 - v^2 - \Omega_0^2} \right)^{1/2} = C \sqrt{\frac{M\Omega_0}{M^2\Omega_0^2 + 2}}.
\]  

(4.21)

### 4.2. Contribution from the modes of forced oscillation

Again, we represent every mode of forced oscillation in the form of the single-frequency ansatz (3.16)–(3.25) (wherein \( i = 1, N \) and superscript \((\Omega_i)\) is assumed near the corresponding quantities, see Remark 1). In this way, one gets analogously to Eq. (4.1)

\[
[i\omega W] + O(\epsilon) = \frac{(-M\Omega_0^2 + K)W(0, T) - p}{1 - v^2}.
\]  

(4.22)

Now we substitute expansion (3.20) into Eq. (4.22) and equate coefficients of like powers \( \epsilon \). For the zeroth order approximation one obtains a non-trivial equation:

\[
[i\omega] W_0 = \frac{(-M\Omega_0^2 + K)W_0 - p_0(T)}{1 - v^2}.
\]  

(4.23)

Resolving the last equation yields

\[
W_0 = p_0(T) G(0, \Omega_i),
\]

(4.24)

where \( G(0, \Omega_i) \) is the Green function defined by (D.4) or (D.5).
In the particular case, where \( \Omega_i = 0 \) and \( p_0 = \text{const} \) is the weight, this yields

\[
W_0 = \frac{p_0}{2\sqrt{1 - v^2} + K}.
\] (4.25)

4.3. Calculation of the unknown constants

Taking into account the contributions from the frequency of localized oscillation \( \Omega_0 \), from the frequencies of forced oscillation \( \Omega_i \), and the corresponding complex conjugate terms, one obtains

\[
U = \sum_{i=0}^{N} U^{(\Omega_i)} + \text{c.c.} + O(\epsilon) = U_{\text{forced}} + U_{\text{free}} + O(\epsilon),
\] (4.26)

\[
U_{\text{forced}} \overset{\text{def}}{=} \sum_{i=1}^{N} U^{(\Omega_i)} + \text{c.c.} = 2 \sum_{i=1}^{N} \left| p^{(\Omega_i)}(T) \mathcal{G}(0, \Omega_i(T)) \right| \nonumber
\times \cos \left( \int_0^T \Omega_i(T) \, dT - \arg \left( p^{(\Omega_i)}(T) \mathcal{G}(0, \Omega_i(T)) \right) \right),
\] (4.27)

\[
U_{\text{free}} \overset{\text{def}}{=} U^{(\Omega_0)} + \text{c.c.} = C_0 \frac{(1 - v^2(T) - \Omega_0^2(T))^{1/4} \sin \left( \int_0^T \Omega_0(T) \, dT - D_0 \right)}{\Omega_0^{1/2}(T) \left( 1 + M \sqrt{1 - v^2(T) - \Omega_0^2(T)} \right)^{1/2}}.
\] (4.28)

To find the unknown real constants \( C_0 \) and \( D_0 \) related to the complex constant \( C \) (introduced in (4.17)) as

\[
C_0 = 2|C|, \quad D_0 = \arg C,
\] (4.29)

one should match the right-hand sides of Eq. (4.26) taken at \( T = 0 \) and Eq. (F.8). In this way the right-hand side of Eq. (4.27) transforms into the first term in the right-hand side of Eq. (F.8). Equating the second terms yields

\[
C_0 = \frac{(1 - v^2(0) - \Omega_0^2(0))^{1/4} |\mathcal{F}\{p(\tau, 0)\}(\Omega_0(0))|}{\Omega_0^{1/2}(0) \left( 1 + M \sqrt{1 - v^2(0) - \Omega_0^2(0)} \right)^{1/2}},
\] (4.30)

\[
D_0 = \arg \mathcal{F}\{p(\tau, 0)\}(\Omega_0(0)).
\] (4.31)

Here and in what follows \( \mathcal{F} \) is a symbol of the Fourier transform with respect to time \( \tau \), see the details in Appendix F.
4.4. Analytic expression for the unknown internal force

Using Eqs. (2.8), (4.26)–(4.28) one gets:

\[ P = p + P_{\text{forced}} + P_{\text{free}} + O(\epsilon), \]  
(4.32)

\[ P_{\text{forced}} \overset{\text{def}}{=} 2 \sum_{i=1}^{N} \left| p^{(i)}(t) \mathcal{G}(0, \Omega_i(T)) \right| (M\Omega_i^2(T) - K) \times \cos \left( \int_0^T \Omega_i(T) \, dT - \arg \left( p^{(i)}(T) \mathcal{G}(0, \Omega_i(T)) \right) \right), \]  
(4.33)

\[ P_{\text{free}} \overset{\text{def}}{=} C_0 (M\Omega_0^2(T) - K) \frac{(1 - v^2(T) - \Omega_0^2(T))^{1/4} \sin (\int_0^T \Omega_0(T) \, dT - D_0)}{\Omega_0^{1/2}(T) \left( 1 + M \sqrt{1 - v^2(T) - \Omega_0^2(T)} \right)^{1/2}}. \]  
(4.34)

5. Numerics

5.1. Integral equation for the unknown internal force

The solution satisfying Eq. (2.5), and initial conditions (2.11) can be written as the convolution over \( x \) and \( t \) of the fundamental solution of the Klein-Gordon PDE (G.1) with the right-hand side of Eq. (2.5). At the point under the moving inclusion \( x = \ell(t) \) the solution is as follows [5, 6]:

\[ \mathcal{U}(t) = P(t) \delta(x - \ell(t)) \ast \Phi(x, t) \bigg|_{x=\ell(t)} = \frac{H(t)}{2} \int_0^t H \left( 1 - \frac{|\ell(t) - \ell(\tau)|}{t - \tau} \right) P(\tau) J_0 \left( \sqrt{(t - \tau)^2 - (\ell(t) - \ell(\tau))^2} \right) \, d\tau, \]  
(5.1)

where \( J_0(\cdot) \) is the Bessel function of the first kind of zero order, and the asterisk is the symbol for the convolution operator. In the subcritical case (2.3) the Heaviside function in the integrand in the right-hand side of (5.1) is equal to one.

At the same time at the point under the inclusion in the case \( M > 0 \) the same solution can be found as the convolution over \( t \) of the fundamental solution (H.1) of ODE describing the linear oscillator with the right-hand side of Eq. (2.6):

\[ \mathcal{U}(t) = (-P(t) + p(t)) \ast \Psi(t) = H(t) \int_0^t (p(\tau) - P(\tau)) \Psi(t - \tau) \, d\tau. \]  
(5.2)

Equating the right-hand sides of Eqs. (5.1) and (5.2) one gets a Volterra integral equation of the first kind for the unknown internal force \( P(t) \), which is valid for \( t > 0 \). Differentiating this equation
with respect to $t$ yields the following Volterra integral equation of the second kind:

$$
P(t) = \int_0^t P(\tau) \left( \frac{t - \tau - (x(t) - \ell(\tau))\ell'(t)}{\sqrt{(t - \tau)^2 - (\ell(t) - \ell(\tau))^2}} J_1 \left( \sqrt{(t - \tau)^2 - (\ell(t) - \ell(\tau))^2} \right) - 2\Psi'(t - \tau) \right) d\tau + 2 \int_0^t p(\tau)\Psi'(t - \tau) d\tau, \quad (5.3)$$

where $J_1(\cdot)$ is the Bessel function of the first kind of the first order. The last formula is written in the simplified form, which takes into account that restriction (2.3) is satisfied.

In the special case $M = 0$ Eq. (2.6) is not an ODE any more. In the latter case, the Volterra integral equation for unknown $P(t)$ can be obtained by substituting of Eq. (5.1) into Eq. (2.6). Provided that (2.3) is satisfied, this yields

$$
P(t) = -\frac{K}{2} \int_0^t P(\tau) J_0 \left( \sqrt{(t - \tau)^2 - (\ell(t) - \ell(\tau))^2} \right) d\tau + p(t). \quad (5.4)$$

The numerical solution of integral equation (5.3) or (5.4) can be compared with the analytic expression for the internal force $P$ (4.32).

The methodology to solve the obtained integral equation numerically is completely analogous to the one used in our previous paper [33], where it is discussed in detail. To obtain the numerical solution for the displacement $U(t)$, we use formula (5.1) and compute the convolution of the numerically obtained internal force $P(t)$ with the fundamental solution of the Klein-Gordon equation.

5.2. Comparison between analytic and numerical results

5.2.1. Pure free oscillation in the case $M > 0, K > 0$

Considering a pure free localized oscillation we take

$$
\hat{p}(\tau, T) = \tau_0^{-1} (H(\tau) - H(\tau - \tau_0)), \quad p^{(\Omega_0)} \equiv 0, \quad \text{(5.5)}
$$

where $\tau_0$ is a small positive constant. Quantity $\hat{p}(\tau)$ weakly converges to $\delta(\tau)$ as $\tau_0 \to +0$. In the limiting case we have

$$
\mathcal{F}\{p(\tau, 0)\}(\Omega_0(0)) = 1. \quad \text{(5.6)}
$$

The last equation should be used to define the unknown constants (4.30), (4.31).

As generally, we assume that initially in the system with $v = v(0)$ condition (E.20) is satisfied and the trapped mode exists, i.e., we can use the constructed analytic solution. In Fig. 2 we compare the results obtained for a monotonically increasing $v(T)$. We present results for the internal force
$P(\tau)$ at sub-plot (a) and displacement $U(\tau)$ at sub-plot (b). The yellow span in each sub-plot corresponds to the time interval, where the solution is still sub-critical, but restriction (E.20) is not satisfied. Thus, our asymptotic solution is defined only for the first stage of the motion (for the time values to the left of the yellow span). The left boundary of the span corresponds to the instant when the immediate value of the localized mode frequency approaches the cut-off frequency (3.7) (this corresponds to the disappearing of the trapped mode). The right boundary of the span corresponds to the instant of overcoming the critical speed ($\nu = 1$). The asymptotic solution approaches the numerical one very quickly. One can observe that variation of the amplitude of the localized oscillation is more pronounced in the case of the internal force $P(\tau)$ than for the displacement $U(\tau)$. The divergence between the asymptotic and numerical solutions begins again considerable in a left neighbourhood of the yellow span. At the left boundary of the yellow span the amplitudes of both analytic solutions $P(\tau)$ and $U(\tau)$ become zero. It is interesting that the amplitude of the internal force $P$ observable for the numerical solution begins again to grow within the span, whereas the corresponding amplitude of the displacement $U$ decreases monotonically within the span. For the time being, the nature of the characteristic frequency which corresponds to the oscillation within the span is not absolutely clear for us, but according to our hypothesis the oscillation after disappearing of the trapped mode can be described as a resonant solution describing overcoming the cut-off frequency.

5.2.2. Pure free oscillation in the case $M > 0$, $K = 0$: comparison with the results of the previous paper [26]

The particular case under consideration was considered in the previous paper [26], where the approach based on the method of multiple scales was suggested at the first time. The final formula describing the evolution of the amplitude of the localized oscillation was

$$W^0_{old} = C \sqrt{\frac{1 - \nu^2}{\Omega_0 (M^2 \Omega_0^2 + 2)}},$$

see [26], Eq. (5.15) in that paper. This formula is in the contradiction with the result, which we have got in this paper, see Eq. (4.21). Thus, one of these two formulae is definitely erroneous. The problem is that the analytic calculations in [26] are extremely complicated comparing to the current paper, where the technique based on representing of the right-hand side of the first approximation equation in the form of the total differential is suggested and applied. Note that previously in [26] the corresponding formula was obtained as a product of ten indefinite integrals. We were sure that old asymptotics is correct since it describes the numerical results quite well. Therefore, analysing the contradiction, we want to check which asymptotics corresponds to numerical results
Figure 2: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34) with the corresponding numerical solution for the accelerating oscillator moving at speed $v = T$: $\epsilon = 0.01$, $M = 5$, $K = 3$. (a) The internal force, (b) the displacement. The yellow span in each sub-plot corresponds to the second stage of the motion, when the solution is still sub-critical, but restriction (E.20) is not satisfied.
Figure 3: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34), the old asymptotic solution obtained in [26], and the corresponding numerical solution for the accelerating oscillator moving at the speed $v = T$: $\epsilon = 0.01$, $M = 2$, $K = 0$. (a) The internal force, (b) the displacement. The yellow vertical line in each sub-plot corresponds to the instant of overcoming the critical speed $v = 1$. 22
better. First, we compare the old asymptotics, the new one and numerics in the case considered in the plots presented in [26], namely $M = 2$, see Fig. 3. This figure does not allow one to make the decision, since both asymptotic formulae seem to work well, at least for times, when trapped mode frequency $\Omega_0$ is far enough from the cut-off frequency $\Omega_*$. To analyse the difference between the results, we have introduced the ratio $\kappa$ of the normalized amplitudes\footnote{We have introduced and used the normalized amplitudes investigating other problem considered in [32].} $W_0/(W_0|_{v=0})$ and $W_0^{\text{old}}/(W_0^{\text{old}}|_{v=0})$:

$$\kappa = \frac{W_0}{W_0^{\text{old}} \left( \frac{W_0^{\text{old}} |_{v=0}}{W_0 |_{v=0}} \right)^{-1}}. \quad (5.8)$$

The plots of the coefficient $\kappa(M)$ for various values of $v$ are presented in Fig. 4. One can see that the more mass $M$, the more ratio $\kappa$. In Fig. 5 we compare the old asymptotics, the new one and numerics in the case $M = 100$. One can see that the new asymptotics is definitely better describes numerics, thus, we make a decision that the error is in the old calculations.

Finally, we carefully analyse the calculations in [26] and discovered the error, which emerges when calculating quantity $\Phi_3$, see Eq. (5.10) in that paper. In [26], to calculate $\Phi_3$, the frequency
Figure 5: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34), the old asymptotic solution obtained in [26], and the corresponding numerical solution for the accelerating oscillator moving at the speed $v = T$: $\epsilon = 0.01$, $M = 100$, $K = 0$. (a) The internal force, (b) the displacement. The yellow vertical line in each sub-plot corresponds to the instant of overcoming the critical speed $v = 1$. 
equation for the trapped mode differentiated with respect to \( \Omega_0 \) was used for the aim of equations simplification. Since the frequency equation is defined only for the trapped mode frequency (and it is not valid in the neighbourhood of this frequency), this operation is senseless. We also carefully analysed our subsequent studies [32, 31, 33], where the approach based on the method of multiple scales was applied to a number of problems, and now we are sure that the same error was never repeated.

5.2.3. Pure free oscillation in the case \( K < 0 \)

Again, we assume that initially in the system with \( v = v(0) \) condition (E.21) is satisfied and the trapped mode exists, i.e., we can use the constructed analytic solution. In the case \( K < 0, \; M > 0 \) the kernel and the free term of integral equation (5.3) grow exponentially as \( t \to \infty \). This leads to an oscillatory numerical instability, which is observed after certain value of time even in the case of a uniform motion (or even for non-moving mass-spring system), see Fig. 6. Since in the case of a uniform motion, Eq. (F.8) is an “exact” asymptotics got by the method of stationary phase, we guess that we observe a numerical instability and not a physical one.

![Figure 6: Numeric instability observed when solving integral equation (5.3) in the case \( K < 0, \; M > 0: \; v = 0, \; M = 2, \; K = -1 \).](image)

Figure 6: Numeric instability observed when solving integral equation (5.3) in the case \( K < 0, \; M > 0: \; v = 0, \; M = 2, \; K = -1 \).

Apparently, we can get rid of this kind of instability using floating-point arithmetics with higher precision. Increasing of \( M \) leads to the more moderate growth of the kernel and the free term.
On this way we can get the numerical results for bigger $\tau$. Note that in the special case $M = 0$ the integral equation has the form of Eq. (5.4), which does not involve any exponentially growing functions; therefore, we do not observe any instability.

In Fig. 7 we compare the results for the internal force $P(\tau)$ (a) and the displacement $U(\tau)$ (b) obtained for big enough mass $M = 12$. The yellow span in each sub-plot corresponds to the time interval, where the solution is still sub-critical, but restriction (E.21) is not satisfied. Thus, our asymptotic solution is defined only for the time values to the left of the yellow span. The left boundary of the span corresponds to the instant when immediate value of the trapped mode frequency approaches zero (this corresponds to the string buckling). The right boundary of the span corresponds to the instant of overcoming the critical speed ($v = 1$).

Again, the asymptotic solution approaches the numerical one very quickly. The numerical and asymptotic solutions diverge just at the left boundary of the span. The asymptotics approaches infinity at the left boundary, whereas the numerical solution demonstrates a growth within the span. At the right boundary of the span the displacement becomes to be prescribed by the perturbations, radiated in the past, during the sub-critical stage of the motion [6]. After that instant the displacement $U(\tau)$ decreases.

5.2.4. Pure free oscillation in the case when the trapped mode does not exist initially

Assume now that initially in the system with $v = v(0)$ condition (E.20) is not satisfied, and the trapped mode does not exist, i.e., the constructed analytic solution is not valid. Thus, the first stage of the motion does not exist. Note that we have zero second term describing the free oscillation in the corresponding system with constant parameters (see Eq. (F.8)); therefore, we expect that the free oscillation in the system with time-varying $v(T)$ is negligible. We will discuss numerics for the practically more important case $M > 0$, $K > 0$. In Fig. 8 we present the numerical results obtained for a monotonically increasing $v(T)$. One can see that free oscillation quickly vanishes as expected due to analysis performed for the system with constant parameters. The amplitude of oscillation is negligible comparing with the case, where the trapped mode exists, and slightly increases before overcoming the critical speed $v = 1$.

5.2.5. Free and forced oscillation

Now we take

$$\dot{p}(\tau) = 0, \quad N = 1, \quad p^{([2])}(T) \in \mathbb{R}, \quad (5.9)$$

and, therefore, the external force $p(T, \tau)$ defined by Eq. (3.3) is as follows:

$$p(T, \tau) = 2H(\tau) p^{([2])}(T) \cos \left( \int_0^T \Omega_1(T) \, dT \right). \quad (5.10)$$
Figure 7: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34) and the corresponding numerical solution for the accelerating oscillator moving at the speed $v = T$: $\epsilon = 0.01$, $M = 12$, $K = -1$. (a) The internal force, (b) the displacement. The yellow span in each sub-plot corresponds to the second stage of the motion, when the solution is still sub-critical, but restriction (E.21) is not satisfied.
Figure 8: Numerical solution for the accelerating oscillator moving at the speed $v = T$: $\epsilon = 0.01$, $M = 1$, $K = 3$. (a) The internal force, (b) the displacement. The yellow span in each sub-plot corresponds to the time interval, when the solution is still sub-critical, but restriction (E.20) is not satisfied.
We put also
\[ p^{(\Omega_1)}(T) = \alpha_0 + \alpha_1 T, \quad \Omega_1(T) = \gamma_0 + \gamma_1 T. \] (5.11)

One has (see (F.3))
\[ F\{p(\tau, 0)\}(\Omega_0(0)) = p^{(\Omega_1)}(0) \frac{2i\Omega_0(0)}{\Omega_0^2(0) - \Omega_1^2(0)}. \] (5.12)

The last equation should be used to define the unknown constants (4.30), (4.31).

We will compare the asymptotic and numerical results for the practically more important case \( M > 0, \ K > 0, \) and assume that initially in the system with \( v = v(0) \) condition (E.20) is satisfied and the trapped mode exists. Since we deal with a non-resonant excitation only (for which (3.8) is fulfilled for all \( T \)), there are two qualitatively different cases, namely the low-frequency case \( \Omega_1 < \Omega_0 \) (the forced oscillation is localized near the mass-spring system) and the high frequency case \( \Omega_1 > \Omega_0 \) (the forced oscillation forms propagating waves). In Figs. 9 & 10 we compare the asymptotic and numerical results obtained in the low frequency case, where we take
\[ \alpha_0 = 0.5, \quad \alpha_1 = 0.5, \quad \gamma_0 = 0, \quad \gamma_1 = 0.4, \] (5.13)
and in the high frequency case, where
\[ \alpha_0 = 0.5, \quad \alpha_1 = 0.5, \quad \gamma_0 = 1.2, \quad \gamma_1 = 0.4, \] (5.14)
respectively. In Fig. 11 we present the plot for the trapped mode frequency \( \Omega_0 \), the cut-off frequency \( \Omega_* \), and the external excitation frequencies \( \Omega_1 \) versus time \( \tau = T/\epsilon \) for the problems illustrated by Figs. 9 & 10. The yellow span in all these figures corresponds to the time interval, where the solution is still sub-critical, but restriction (E.20) is not satisfied. Thus, our asymptotic solution for the free oscillation is defined only for the time values to the left of the yellow span. One can see that the asymptotic solution is in a very good agreement with the numerical one for such values of time. The cyan vertical line corresponds to the instant when the excitation frequency becomes equal the cut-off frequency (in the low-frequency case). According to Eq. (4.32) the asymptotic solution for the internal force \( P(\tau) \) is the superposition of the free oscillation, the forced oscillation, and the external force \( p(\tau) \). At the same time, the displacement \( U(\tau) \) is the superposition of the free oscillation and the forced oscillation. These individual components are also shown in the plots. One can observe an excellent agreement between the analytic and numerical results.

5.3. Final remarks

Remark 4. One can see in Fig. 11 that restriction (3.8), which guaranties a non-resonant character of the solution under consideration, is fulfilled for all admissible \( \tau \). If we break this requirement,
Figure 9: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34) represented as the superposition of the free oscillation, the forced one, and the external force (for the plot (a)) with the corresponding numerical solution for the accelerating oscillator moving at the speed \( v = T \): \( \epsilon = 0.01, M = 5, K = 3 \) in the case of the low-frequency excitation. (a) The internal force, (b) the displacement. The yellow span in each sub-plot corresponds to the second stage of the motion, when the solution is still sub-critical, but restriction (E.20) is not satisfied. The cyan vertical line corresponds to the instant when the excitation frequency becomes equal to the cut-off frequency.
Figure 10: Comparing the asymptotic solution in the form of Eqs. (4.26)–(4.34) represented as the superposition of the free oscillation, the forced one, and the external force (for the plot (a)) with the corresponding numerical solution for the accelerating oscillator moving at the speed \( v = T \); \( \epsilon = 0.01 \), \( M = 5 \), \( K = 3 \) in the case of the high-frequency excitation. (a) The internal force, (b) the displacement. The yellow span in each sub-plot corresponds to the second stage of the motion, when the solution is still sub-critical, but restriction (E.20) is not satisfied.
Figure 11: The trapped mode frequency $\Omega_0$, the cut-off frequency $\Omega_\ast$, and the external excitation frequencies $\Omega_1$ versus time $\tau = T/\epsilon$ for the problems illustrated by Figs. 9 & 10 (the low-frequency and the high frequency cases, respectively). The yellow span corresponds to the second stage of the motion, when the solution is still sub-critical, but restriction (E.20) is not satisfied. The cyan vertical line corresponds to the instant when the excitation frequency becomes equal to the cut-off frequency (in the low-frequency case).
e.g., by taking of $\Omega_1(0)$ in the narrow interval between the trapped mode frequency and the cut-off frequency, then the constructed asymptotic solution becomes practically inapplicable.

**Remark 5.** The particular case, when $\Omega_1 = 0$ and the weight $p = p_0(0) = \text{const}$, is very similar to the case of the pure free oscillation considered in Sect. 5.2.1–5.2.3. According to Eqs. (4.25), (4.26), (4.32) the internal force and the displacement are the superpositions of a free oscillation and a constant quantity. The unknown constants can be found by formulae (4.30), (4.31) and (5.12) wherein $\Omega_1 = 0$. Due to Eqs (4.31), (5.6), (5.12) the shift of the phase between a free oscillation in the case under consideration and a free oscillation considered in Sect. 5.2.1–5.2.3 is $\pi/2$.

**Remark 6.** One can consider more complicated problems, where the external force is a superposition of several harmonics. The possible practically important example corresponds to the case, when both the weight and an external sinusoidal excitation are applied to the mass-spring system. The corresponding asymptotic solution will be in a good agreement with the numerical one.

**Remark 7.** For the aim of simplicity, we have considered in Sect. 5.2.1–5.2.5 the test problems, wherein the quantities $v(T), \Omega_1(T), p^{(\Omega_1)}(T)$, independently vary in a uniform way. One can consider a more complicated problem, where these parameters vary in a non-uniform, and even in a non-monotonous, way, keeping their values in the corresponding admissible intervals. For enough small $\epsilon$, the corresponding asymptotic solutions will be in a good agreement with numerical ones (an example of non-monotonically changing parameters for another problem investigated by the same method can be found in [28], see Fig. 3 there).

**Remark 8.** For the aim of simplicity, we have considered in Sect. 5.2.1–5.2.3 the test problems, wherein the pulse loading is the function (5.5) approximating $\delta(\tau)$. One can consider a more complicated problem, where this function even is not a finite, but an exponentially vanishing at infinity function. For enough small $\epsilon$, the corresponding asymptotic solution will be in a good agreement with numerical one (an example of exponentially vanishing pulse loading for another problem investigated by the same method can be found in [28], see Fig. 4 there).

### 6. Conclusion

In the paper we have considered forced and localized free oscillation of a string on the Winkler foundation subjected to a discrete mass-spring system non-uniformly moving at a sub-critical speed. We have used the analytic approach first time suggested in [26]. Now we have suggested some mathematical trick, namely, transforming of the right-hand side of the first approximation equation to the form of a total differential of the logarithm of a certain function, see Eq. (4.16). This allows us to significantly simplify calculations and obtain the analytic solution of a quite complicated problem. We note that our analytic approach is not a pure asymptotic one, since we
guess which harmonics we need to take into account in the multi-frequency ansatz (see Sect. 3.1). Nevertheless, such an approach has a deep asymptotic motivation. We successively apply two quite different asymptotic methods: the method of stationary phase to the system with constant parameters and the method of multiple scales to the system with time-varying parameters, and then use a simple matching procedure to find unknown constants. It is very difficult and perhaps impossible to prove that we obtain asymptotically correct results in this way, thus, we need to verify the obtained results numerically. This has been done in Sect. 5.

The specific feature of the solution describing oscillation of the string subjected to a uniformly accelerated oscillator is that there exist two qualitatively different stages of the motion. Indeed, assuming that the trapped mode initially exists, one can observe that it disappears at a certain instant of time before overcoming the critical speed \( v = 1 \). In the case \( K > 0 \), at this instant the trapped mode frequency \( \Omega_0(T) \) approaches the cut-off frequency \( \Omega_*(T) \), which is the boundary frequency for the continuous spectrum of natural frequencies. In the case \( K < 0 \), at this instant the trapped mode frequency approaches zero, and we observe the dynamic instability and the buckling.

The main analytic result of the paper is Eqs. (4.26)–(4.34), which are valid in the absence of resonance of any type (see Eqs. (3.8), (3.5)). Herewith, the mass-spring system speed \( v \), the amplitudes \( p^{(\Omega_i)} \) and the frequencies \( \Omega_i \) for the harmonics of the external force are assumed to be independent slowly time-varying functions. These formulae describe the first stage of the sub-critical motion, where the non-vanishing free localized oscillation is observed. We have demonstrated that the analytic solution is in a very good agreement with the numerical one in the corresponding admissible interval of the speed \( v \). The free oscillation at the second stage is described only numerically. For the system with positive \( K \), it has been shown that the amplitude of the internal force \( P \) can begin again to grow after the instant when the trapped mode disappears (but before overcoming the critical speed), see Fig. 2. This fact is a bit unexpected. At the same time, the corresponding amplitude of the displacement \( U \) decreases monotonically. For the system with negative \( K \), numerical post-buckling solution grows until the overcoming the critical speed \( v = 1 \).

After that the displacement \( U \) becomes to be prescribed by the perturbations, radiated in the past \([6]\), during the sub-critical stage of the motion, and the displacement begins to decrease, see Fig. 7. While considering the limiting case \( K = 0 \), we have discovered an error in previous paper \([26]\) (see Sect. 5.2.2 for the corresponding discussion). Note that the erroneous solution obtained in \([26]\) has a behaviour, which is very close to the correct solution. The dynamical behaviour of the system considered in \([26]\) is much simpler, in particular the second stage of a uniformly accelerated motion does not exist.

Finally, let us discuss how the results of the paper can be generalized. In our opinion, to describe oscillation at the second stage in the case of the positive spring stiffness \( K \) it would be useful to
construct a resonant solution describing overcoming the cut-off frequency in spirit of [34], where a model resonant solution for a system possessing a trapped mode is obtained. Nevertheless, for the problem under consideration, the generalization of the results of [34] does not seem to be straightforward. The solutions describing the passage through resonance, where \( \Omega_i \simeq \Omega_0 \), apparently, also can be obtained in such a way.

The method used in the paper without essential changes can be applied to another systems involving a string on the Winkler foundation and moving discrete mass-spring systems, e.g., ones considered in [35, 52], under condition that a unique trapped mode initially exists.

**Declaration of competing interest**

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**Appendix A. The system with constant parameters: basic assumptions**

In Appendix A–Appendix F we present some auxiliary results related to the governing equations in the form of Eqs. (2.7), (2.8) in the case of constant parameters. In such a way we introduce several quantities and relations, which are necessary in order to apply our asymptotic and numerical approaches. The loading is assumed to be as follows:

\[
v_T' \equiv 0 \quad \Longrightarrow \quad v(T) \equiv v(T_0); \tag{A.1}
\]

\[
\Omega_i' \equiv 0, \quad i = \overline{1,N} \quad \Longrightarrow \quad \Omega_i(T) = \Omega_i(T_0); \tag{A.2}
\]

\[
p'_{(\Omega_i)} \equiv 0, \quad i = \overline{1,N} \quad \Longrightarrow \quad p(\tau, T) \equiv p(\tau, T_0) \tag{A.3}
\]

for certain \( T_0 = \text{const} \). The oscillator speed is assumed to be sub-critical, i.e., Eq. (2.3) is fulfilled.
Appendix B. The dispersion relation for the Klein-Gordon equation in the moving co-ordinates

Consider properties of the linear differential operator in the left-hand side of Eq. (2.7) in the case (A.1). Assuming that \( P(\tau) = 0 \) and

\[
    u(\xi, \tau) = We^{-i(\Omega \tau + \omega \xi)},
\]

we get the dispersion relation for the operator in the left-hand side of Eq. (2.7) in the following form:

\[
    \omega^2 - 2B(\Omega) \omega + A^2(\Omega) = 0.
\]

Here \( \omega \) is the wave-number,

\[
    A^2(\Omega) \equiv \frac{1 - \Omega^2}{\Omega_*^2},
\]

\[
    B(\Omega) \equiv \frac{v\Omega}{\Omega_*},
\]

where \( \Omega_* \) is the cut-off frequency defined by Eq. (3.7). Put

\[
    S(\Omega) \equiv \sqrt{A^2(\Omega) - B^2(\Omega)} = \frac{\sqrt{\Omega_*^2 - \Omega^2}}{\Omega_*^2},
\]

where the principal branch of the square root is chosen. Thus, from the dispersion relation (B.2) one obtains the expression for the wavenumbers \( \omega \):

\[
    \omega = B(\Omega) \pm iS(\Omega).
\]

One can see that according to dispersion relation (B.2) free waves with frequencies upper than the cut-off frequency are sinusoidal propagating waves, whereas free waves with frequencies below than the cut-off frequency are growing inhomogeneous waves, which cannot exist if we require boundedness.

Note that quantity \( B \) defined by Eq. (B.4) satisfies equality:

\[
    vB + \Omega = \frac{\Omega}{1 - v^2}.
\]
Appendix C. The Green function in the frequency domain for the system without oscillator

Consider now Eq. (2.7) and assume that

\[ u(\xi, \tau) = W(\xi) e^{-i\Omega \tau}, \]  
\[ P(\tau) = e^{-i\Omega \tau}. \] (C.1) (C.2)

Let substitute expressions (C.1) into Eq. (2.7). This yields

\[ (1 - v^2) (W'' - 2iB(\Omega)W' - A^2(\Omega)W) = -\delta(\xi), \] (C.3)

where \( A^2(\Omega) \) and \( B(\Omega) \) are defined by (B.3) and (B.4), respectively. The steady-state solution \( W = G \) of the obtained equation is the Green function in the frequency domain for the system without oscillator. This Green function expresses the displacement of the string subjected to a moving inertialess oscillating load. One can show that the Green function has the following form:

\[ G(\xi, \Omega) = \exp\left(\frac{iB\xi - S|\xi|}{2\Omega^2 S}\right), \quad |\Omega| < \Omega_*; \] (C.4)
\[ G(\xi, \Omega) = -\exp\left(\frac{iB\xi - S\text{sign}(\Omega)|\xi|}{2\Omega^2 \text{sign}(\Omega) S}\right), \quad |\Omega| > \Omega_. \] (C.5)

Expression (C.4) satisfies vanishing boundary conditions at infinity, whereas Eq. (C.5) satisfies the Sommerfeld radiation conditions.

Note that the cut-off frequency \( \Omega_* \) is a resonant frequency for the system without oscillator: the amplitude of forced oscillation becomes infinity as \( \Omega \to \Omega_* \) due to the quantity \( S \) defined by Eq. (B.5) in the denominator of the right-hand side of Eq. (C.4). The corresponding non-stationary solution grows as \( t \to \infty \) [57–59].

Appendix D. The Green function in the frequency domain for the system with oscillator

Consider now Eqs. (2.7), (2.8), wherein

\[ p(\tau) = e^{-i\Omega \tau}, \] (D.1)

assuming that Eq. (C.1) and

\[ U(\tau) = We^{-i\Omega \tau} \] (D.2)
are fulfilled.

Let substitute expressions (C.1), (D.2) into Eqs. (2.7), (2.8). This yields

\[(1 - v^2) (W'' - 2iB(\Omega)W' - A^2(\Omega)W) = -((M\Omega^2 - K)W + 1) \delta(\xi), \quad (D.3)\]

where \(A^2(\Omega)\) and \(B(\Omega)\) are defined by (B.3) and (B.4), respectively. The corresponding steady-state solution \(W = \mathcal{G}\) is the Green function in the frequency domain for the system with the oscillator. One can show that the Green function has the following form:

\[\mathcal{G}(\xi, \Omega) = \begin{cases} \exp \left(\frac{i B \xi - S|\xi|}{2\Omega^2 S + K - M\Omega^2}\right), & |\Omega| < \Omega_*; \\ -\exp \left(\frac{i B \xi - S \text{sign}(\Omega)|\xi|}{2\Omega^2 \text{sign}(\Omega) S + M\Omega^2 - K}\right), & |\Omega| > \Omega_*. \end{cases} \quad (D.4)\]

Expression (D.4) satisfies vanishing boundary conditions at infinity, whereas Eq. (D.5) satisfies the Sommerfeld radiation conditions. Note that the trapped mode frequency \(\Omega_0\) introduced in Sect. Appendix E is a root of the denominator in the right-hand side of Eq. (D.4) according to frequency equation (E.5), and, therefore, this is a resonant frequency. Thus, the cut-off frequency generally is not a resonant frequency for the system with oscillator.

**Appendix E. Spectral problem for a trapped mode**

Put \(p = 0\) and consider the steady-state problem concerning the natural oscillations of the system described by Eqs. (2.7), (2.8), assuming that Eqs. (C.1), (D.2) are fulfilled. We consider only sub-critical speeds, i.e., Eq. (2.3) is fulfilled.

A distributed system with discrete inclusions can possess a mixed spectrum of natural frequencies [27, 29, 30]. In our case, clearly, there exists a continuous spectrum of frequencies, which lies higher than the cut-off (or boundary) frequency: \(|\Omega| \geq \Omega_*\). Here \(\Omega_*\) is given by Eq. (3.7). The modes corresponding to the frequencies from the continuous spectrum are harmonic waves. Trapped modes are modes with finite energy, therefore, we require

\[\int_{-\infty}^{+\infty} W^2 \, d\xi < \infty, \quad \int_{-\infty}^{+\infty} W'^2 \, d\xi < \infty. \quad (E.1)\]

In the framework of the problem under consideration, they correspond to the frequencies from the discrete part of the spectrum, which lies lower than the cut-off frequency: 0 < \(|\Omega| < \Omega_*\). We want to demonstrate that for the problem under consideration under certain conditions the one and only one trapped mode with corresponding frequency \(\Omega_0 > 0\) can exist.
Let substitute expressions (C.1), (D.2) into Eqs. (2.7), (2.8). This yields

\[(1 - v^2) (W'' - 2iB(\Omega)W' - A^2(\Omega)W) = -(M\Omega^2 - K)W \delta(\xi),\]

(E.2)

where \(A^2(\Omega)\) and \(B(\Omega)\) are defined by (B.3) and (B.4), respectively. The solution of the above equation can be written as follows:

\[W(\xi) = (M\Omega^2 - K)WG(\xi, \Omega),\]

(E.3)

where \(G(\xi, \Omega)\) is the Green function (C.4). This expression can be transformed to the following equivalent form:

\[W(\xi) = \frac{(M\Omega^2 - K)W}{2\sqrt{1 - v^2 - \Omega^2}} \exp \left( \frac{i\Omega v\xi - \sqrt{1 - v^2 - \Omega^2}\xi}{1 - v^2} \right).\]

(E.4)

Calculating Eq. (E.4) at \(\xi = 0\) yields the frequency equation for the trapped mode frequency \(\Omega_0\)

\[2\sqrt{\Omega_0^2 - \Omega^2} = M\Omega_0^2 - K,\]

(E.5)

where

\[\Omega_0^2 > 0.\]

(E.6)

Here Eq. (3.7) is taken into account.

At first, consider the case \(M > 0\). Equation (E.5) with condition (E.6) can be equivalently rewritten as the system of the following bi-quadratic equation

\[M^2\Omega_0^4 - 2(KM - 2)\Omega_0^2 + K^2 - 4\Omega_*^2 = 0\]

(E.7)

together with inequalities (E.6) and

\[\frac{K}{M} < \Omega_0^2,\]

(E.8)

It follows from (E.5), (E.6), and (E.8) that

\[0 < \Omega_0^2 < \Omega_*^2\]

(E.9)
and

\[ 0 \leq K < M \Omega_0^2 \quad \text{if} \quad K \geq 0. \quad (E.10) \]

**Remark 9.** Note that from inequality (E.10) it follows that \( \Omega_0^2 > K/M \), where \( K/M \) is the squared partial frequency for the oscillator.

The discriminant for bi-quadratic equation (E.7) is

\[ D = \frac{16(\Omega_0^2 M^2 - KM + 1)}{M^4}. \quad (E.11) \]

The discriminant is positive if and only if

\[ K < \frac{1}{M} + M \Omega_0^2. \quad (E.12) \]

This inequality is clearly true for \( K < 0 \), and it is true due to the second inequality in (E.10) for \( K > 0 \). Thus, the frequency equation (E.5) is equivalent to the system of the following equation for the squared frequency:

\[ \Omega_0^2 = \frac{KM - 2 \pm 2 \sqrt{\Omega_0^2 M^2 - KM + 1}}{M^2}. \quad (E.13) \]

and inequalities (E.6), (E.8).

**Proposition 1.** Provided that \( M > 0 \), \( K \geq 0 \) the root \( \Omega_0^2(+) \) satisfies inequalities (E.6), (E.8) if and only if \( K \) satisfies inequality (E.10), and the root \( \Omega_0^2(-) \) does not satisfy inequality (E.8). Thus, the trapped mode exists and unique if (E.10) is true, and does not exist otherwise.

**Proof.** If (E.10) is not true, then (E.8) is not true and frequency equation (E.5) does not have any positive roots \( \Omega_0^2 \).

On the other hand, let us prove that if (E.10) is true, then Eqs. (E.6), (E.8) wherein \( \Omega_0^2 = \Omega_0^2(+) \) is true. Since (E.6) follows from (E.8), we need to check (E.8) only. We substitute \( \Omega_0^2(+) \) given by (E.13) into Eq. (E.8) and get

\[
\frac{K}{M} < \frac{KM - 2 + 2 \sqrt{\Omega_0^2 M^2 - KM + 1}}{M^2}
\]

\[ \iff \quad KM < KM - 2 + 2 \sqrt{\Omega_0^2 M^2 - KM + 1}
\]

\[ \iff \quad 1 < \sqrt{\Omega_0^2 M^2 - KM + 1}. \quad (E.14) \]

Since discriminant (E.11) is positive \( (D > 0) \) provided that (E.10) is true, the last inequality is
equivalent to Eq. (E.10).

For $\Omega_{0(-)}^2$ one gets

$$KM < KM - 2 - 2\sqrt{\Omega^2_* M^2 - KM + 1},$$

(E.15)

and this inequality is obviously not true.

\[\square\]

**Proposition 2.** Provided that $M > 0$, $K < 0$ the root $\Omega_{0(+)}^2$ satisfies inequalities (E.6), (E.8) if and only if $K$ satisfies inequality

$$- 2|\Omega_*| < K < 0,$$

(E.16)

and the root $\Omega_{0(-)}^2$ does not satisfy inequality (E.8). Thus, the trapped mode exists and unique if (E.16) is true, and does not exist otherwise.

**Proof.** Since (E.8) follows from (E.6) we need to check (E.6) only. Let us prove that Eq. (E.6) (wherein $\Omega_{0}^2 = \Omega_{0(+)}^2$) is true if and only if (E.16) is true:

$$0 < KM - 2\sqrt{\Omega_*^2 M^2 - KM + 1}$$

$$\iff (2 - KM)^2 < 4(\Omega_*^2 M^2 - KM + 1).$$

$$\iff K^2 < 4\Omega_*^2.$$  

(E.17)

Since $K < 0$, the last inequality is equivalent to (E.16).

The root $\Omega_{0(-)}^2$ clearly does not satisfy Eq. (E.6). \[\square\]

Consider the special case $M = 0$. The frequency equation (E.5) can be equivalently rewritten as follows

$$\Omega_{0}^2 = \Omega_*^2 - \frac{K^2}{4},$$

(E.18)

$$K < 0.$$  

(E.19)

**Proposition 3.** Provided that $M = 0$, $K < 0$ the root (E.18) satisfies inequality (E.6) if and only if $K$ satisfies inequality (E.16). Thus, the trapped mode exists and unique if (E.16) is true, and does not exist otherwise.

**Proof.** Inequality $\Omega_*^2 - \frac{K^2}{4} > 0$ is clearly equivalent to Eq. (E.16). \[\square\]

Finally, the conditions for the existence of the trapped mode are given by inequalities (E.10) and (E.16) for $K \geq 0$ and $K < 0$, respectively. These conditions can be written in the following
equivalent forms in the terms of the speed $v$:

$$v^2 < 1 - \frac{K}{M} \quad \text{if} \quad M > 0 \text{ and } K \geq 0; \quad (E.20)$$

$$v^2 < 1 - \frac{K^2}{4} \quad \text{if} \quad M \geq 0 \text{ and } K < 0. \quad (E.21)$$

Consider the limiting case $v^2 = 1 - K/M - \lambda$, where $\lambda > 0$ is a formal small parameter. One can see that due to Eq. (3.7) the squared cut-off frequency $\Omega_0^2 = K/M + \lambda$. The asymptotics for the squared trapped mode frequency is

$$\Omega_0^2 = \frac{K}{M} + \lambda - \frac{1}{4} \lambda^2 M^2 + o(\lambda^2). \quad (E.22)$$

Thus, a zone between the trapped mode frequency and the cut-off frequency is quite narrow (see Fig. 11).

Note that for the special case $K = 0$ the critical value of the moving load speed is $v = 1$ (the speed of sound). One can see that for $K \neq 0$ the critical value of $v$ is less than the speed of sound. The special case $K = 0$ was considered by S.N. Gavrilov and D.A. Indeitsev in [26]. The special case $v = 0, M \neq 0$ was considered in [32, 53]. The special case $v = 0, M = 0$ was considered in [28].

**Appendix F. Non-stationary free and forced oscillation**

Consider the case when $T_0 = 0$, $p(\tau, T) \equiv p(\tau, 0) \neq 0$. Applying the Fourier transform with respect to time $\tau$, we get

$$\mathcal{F}\{u\}(\xi, \Omega) = \mathcal{F}\{p(\tau, 0)\} \mathcal{G}(\xi, \Omega), \quad (F.1)$$

where symbol $\mathcal{F}\{\cdot\}$ denotes the Fourier transform of the corresponding quantity. Applying the inverse transform yields

$$U = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}\{p(\tau, 0)\} e^{-i\Omega T} d\Omega, \quad (F.2)$$

where

$$\mathcal{F}\{p(\tau, 0)\} = \mathcal{F}\{\hat{p}\} + \sum_{i=1}^{N} \left( p^{(\Omega_i)}(0) \frac{i}{\Omega - \Omega_i + i0} - \bar{p}^{(\Omega_i)}(0) \frac{i}{-\Omega - \Omega_i - i0} \right) \quad (F.3)$$

due to (3.3) [60]. Here and in what follows accent ($\hat{\cdot}$) denotes the complex conjugation.

---

2We understand the Fourier transform here as the Fourier transform for generalized functions (or distributions) [56, 55].
Now we want to apply the method of stationary phase to integral (F.2). We are interested to obtain the non-vanishing as $\tau \to \infty$ terms. These terms are defined by contributions from infinitesimal neighbourhoods of poles just below the real axis [48]. Note that $\mathcal{F}\{\hat{p}\}$ does not have any poles since $\hat{p}$ is a finite function. There are poles of $\mathcal{F}\{p(\tau, 0)\}$ at $\Omega = \pm \Omega_i - i 0$ and, due to (E.5), there are poles of the denominator of the right-hand side of Eq. (F.2) at $\Omega = \pm \Omega_0$. The latter couple of poles must be addressed in a special way to find their positions with respect to the real axis. To do this we use the limiting absorption principle in the same way as it was done in [6]. We have to add the dissipative viscous term into governing equation (2.5), repeat all the calculations and find the roots of the denominator for right-hand side of Eq. (F.2) modified in such a way. Then we consider a limiting case of zero friction to define the positions of the poles with respect to the real axis. One can show that the poles are shifted into the lower half-plane of the complex plane: $\Omega = \pm \Omega_0 - i 0$. We apply the residue theorem, Jordan’s lemma, and the method of stationary phase to asymptotic evaluation of the integral in the right-hand side of (F.2). This results in [48, 6]

$$U = -i \sum_{i=0}^{N} \sum_{\Omega_i = \pm \Omega_i} \text{Res} \left( \frac{\mathcal{F}\{p(\tau, 0)\}}{2\sqrt{1 - v^2 - \Omega_i^2 + K - M\Omega_i^2}} \Omega \right) e^{-i\Omega \tau} + o(1), \quad \tau \to \infty. \quad \text{(F.4)}$$

One has

$$\text{Res} \left( \frac{1}{2\sqrt{1 - v^2 - \Omega_i^2 + K - M\Omega_i^2}} , \pm \Omega_i \right) = \mp \frac{\sqrt{1 - v^2 - \Omega_0^2}}{2\Omega_0 \left( 1 + M \sqrt{1 - v^2 - \Omega_0^2} \right)}, \quad \text{(F.5)}$$

$$\text{Res}(\mathcal{F}\{p(\tau, 0)\}, \Omega_i) = i p^{(\Omega_i)}(0), \quad \text{(F.6)}$$

$$\text{Res}(\mathcal{F}\{p(\tau, 0)\}, -\Omega_i) = i\bar{p}^{(\Omega_i)}(0). \quad \text{(F.7)}$$

Finally, we obtain the following asymptotic formula for the displacements:

$$U = 2 \sum_{i=1}^{N} |p^{(\Omega_i)}(0) G(0, \Omega_i)| \cos \left( \Omega_i \tau - \arg (p^{(\Omega_i)}(0) G(0, \Omega_i)) \right)$$

$$+ \frac{\sqrt{1 - v^2 - \Omega_0^2} |\mathcal{F}\{p(\tau, 0)\}(\Omega_0)|}{\Omega_0 \left( 1 + M \sqrt{1 - v^2 - \Omega_0^2} \right)} \sin \left( \Omega_0 \tau - \arg (\mathcal{F}\{p(\tau, 0)\}(\Omega_0)) \right) + o(1), \quad \tau \to \infty. \quad \text{(F.8)}$$

Hence, for the large times, the non-stationary response of the system under consideration is the superposition of the undamped modes of the forced oscillation with frequencies $\Omega_i, i = 1, N$ (the first term in (F.8)) and the undamped mode with the trapped mode frequency $\Omega_0$ (the second term in (F.8)).
Appendix G. The fundamental solution for the Klein-Gordon PDE

Consider Eq. (2.5) in the case $P(t) = \delta(t)$, $\ell(t) = 0$ with initial conditions (2.11). The corresponding solution $u = \Phi$ is the Green function in time domain for the Klein-Gordon equation (the fundamental solution). It has the form (see, e.g., [61]):

$$\Phi(x, t) = \frac{1}{2} H(t - |x|) J_0(\sqrt{t^2 - x^2}).$$ (G.1)

Here $J_0(\cdot)$ is the Bessel function of the first kind of zero order.

Appendix H. The fundamental solution for the ODE describing a linear oscillator

Consider a free linear oscillator described by Eq. (2.6) wherein $P(t) = 0$, $M > 0$, which is subjected to the force $p(t) = \delta(t)$. The initial conditions are formulated in the following form: $U|_{t<0} = 0$. The corresponding solution $U(t) = \Psi(t)$ is the fundamental solution for the ODE describing a linear oscillator [55]:

$$\Psi(t) = \begin{cases} 
\frac{H(t) \sin \left( \frac{K}{\sqrt{M}} t \right)}{\sqrt{KM}}, & K > 0, \\
\frac{H(t) \sinh \left( \frac{-K}{\sqrt{M}} t \right)}{\sqrt{-KM}}, & K < 0, \\
\frac{H(t) t}{M}, & K = 0.
\end{cases} \quad \text{(H.1)}$$

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