PACKING AND COVERING PROPERTIES OF SEQUENCES OF CONVEX BODIES

GÁBOR FEJES TÓTH

Abstract. This paper surveys different variants of the following problem: Given a convex set $K$ and a sequence $\{C_i\}$ of convex bodies in $E^n$, is it possible to pack the sequence of bodies in $K$ or cover $K$ with the bodies? Algorithmic versions of these problems are on-line packing and on-line covering: The bodies of the sequence are given one at a time and the algorithm is to decide on the placement of the arriving body before the next body is revealed; once placed, the body cannot be moved.

Consider the following problem: Given a convex set $K$ and a sequence $\{C_i\}$ of convex bodies in $E^n$, is it possible to pack the sequence of bodies in $K$ or cover $K$ with the bodies? More specifically, is it possible to find for each $i$ a congruent copy $C_i$ of each $C_i$, so that the bodies $\{C_i\}$ form a packing in, or a covering of, $K$? If so, then we say that the sequence $\{C_i\}$ permits an isometric packing in $K$, or an isometric covering of $K$, respectively. If translates of the bodies $C_i$ are used, then we say that $\{C_i\}$ permits a translative packing in $K$, or a translative covering of $K$.

Problem 10.1 from the Scottish Book (see MAULDIN [1981, p. 74]) reads as follows:

"PROBLEM 10.1: MAZUR, AUERBACH, ULAM, BANACH
Theorem. If $\{K_n\}_{n=1}^\infty$ is a sequence of convex bodies, each of diameter $\leq a$ and the sum of their volumes is $\leq b$, then there exists a cube with the diameter $c = f(a,b)$ such that one can put all the given bodies in it disjointly.
Corollary. One kilogram of potatoes can be put into a finite sack. Determine the function $c = f(a,b)$.”

Because of the corollary, packing problems of this type are sometimes called potato-sack problems.

The Scottish Book does not give a proof of the theorem. A proof was later described by KOŚNISKI [1957], who presented an explicit bound on $f(a,b)$. The main idea is to enclose each $C_i$ in a box, that is in a rectangular parallelepiped, whose volume is greater than that of $C_i$ at most by a constant factor independent from $C_i$, and whose diameter is not too much larger than the diameter of $C_i$, thereby reducing the problem to packing a sequence of boxes in a box. Then it is shown that every sequence of $n$-dimensional boxes of edges at most $D$ and total volume at most $V$ can be packed in a box whose $n-1$ edges are of length $3D$ and the $n$-th one is of length $(V + D^n)/D^{n-1}$. MOON and MOSER [1967] improved the above
bound. They proved that such a family of boxes can be packed in a box of sides $2D, 2D, \ldots, 2D, (2V + D^n)/D^{n-1}$.

Moon and Moser also addressed the problem of covering and proved that a family of $n$-dimensional boxes of edges at most $D$ and total volume $V$ can cover a cube of side $D$ if $V \geq c_n(2D)^n$, where $c_n = \frac{2 \cdot 4 \cdot 8 \cdot \cdots \cdot 2^n}{1 \cdot 3 \cdot 7 \cdot \cdots \cdot (2^n - 1)} < 2.463$. MEIR and MOSER [1968] slightly improved this result. In particular, they showed that any family of $n$-dimensional cubes of total volume $V$ can transitively cover a cube of volume $V/(2^n - 1)$. Concerning packing cubes in a cube they proved that a family of cubes of total volume $V$ can be transitively packed in a cube of volume $2^{n-1}V$ provided that the side-lengths of the cubes do not exceed $2^{n-1}/\sqrt{V}$. The bound $2^{n-1}V$ is best possible, since two cubes of volume $V/2$ cannot be transitively packed in a cube of volume smaller than $2^{n-1}V$.

Let $\mathcal{F}$ be a family of convex bodies. For a given convex body $C$ we define $p_i(C|\mathcal{F})$ as the greatest number such that every sequence of members of $\mathcal{F}$ whose diameter does not exceed the diameter of $C$ and whose total volume is at most $p_i(C|\mathcal{F}) \text{vol}(C)$, permits an isometric packing in $C$. Further, we define $c_i(C|\mathcal{F})$ as the smallest number such that every sequence of members of $\mathcal{F}$ with total volume at least $c_i(C|\mathcal{F}) \text{vol}(C)$ permits an isometric covering of $C$. We define similarly $p_t(C|\mathcal{F})$ and $c_t(C|\mathcal{F})$ where the extreme values of the total volume are taken for translative packings and translative coverings, respectively.

In most of the investigated cases $\mathcal{F}$ consists of homothetic or similar copies of a convex body. For a given convex body $C$ let $C_h$ and $C_s$ denote the family of (positive and negative) homothetic copies of $C$ and the family of similar copies of $C$, respectively. Further, let $C_h^+$ and $C_h^-$ be the families of positive homothetic copies and negative homothetic copies of $C$, respectively.

0.1. Packing and covering cubes and boxes. Let $I^n(s)$ denote an $n$-dimensional cube of edge length $s$. If the size is irrelevant, we simply write $I^n$. Then the theorems of MEIR and MOSER [1968] mentioned above state that

\[ p_i(I^n|I_h^n) = 2(1/2)^n \]

and

\[ c_i(I^n|I_h^n) = 2^n - 1. \]

The latter equality was independently proved also by A. BEZDEK and K. BEZDEK [1984].

Better bounds have been obtained under the assumption that the sets used for packing and covering are uniformly bounded and the container is large. Let $\mathcal{B}^n$ denote the family of $n$-dimensional boxes of edge length at most $1$. GROEMER [1982] proved that for $s \geq 3$

\[ p_i(I^n(s)|\mathcal{B}^n) \geq (s - 1)^n - \frac{s - 1}{s - 2}((s - 1)^{n-2} - 1) \]

and

\[ c_i(I^n(s)|\mathcal{B}^n) \leq (s + 1)^n - 1. \]
1. Results for General Convex Bodies

Macbeath [1951] proved that to every \(n\)-dimensional convex body \(C\) there exist two boxes, \(Q_1\) and \(Q_2\), with

\[n^n V(Q_1) \geq V(C) \geq \frac{1}{n!} V(C),\]

such that \(Q_1 \subset C \subset Q_2\). Let \(\mathcal{F}^n\) denote the family of \(n\)-dimensional convex bodies whose diameters are at most 1. The combination of Macbeath’s theorem with the above mentioned theorems of Groemer yields

\[p_i(I^n(s)|\mathcal{F}^n) \geq \frac{1}{n!}((s - 1)^n - \frac{s - 1}{s - 2}((s - 1)^n - 2))\]

and

\[c_i(I^n(s)|\mathcal{F}^n) \leq n^x((s + 1)^n - 1).\]

Macbeath’s theorem was also proved by Hadwiger [1955a]. The part about the box containing \(C\) was proved also by Kosiński [1957] and the part about the box contained in \(C\) by Chakerian [1975]. Lassak [1993] improved the bound for the volume of the box \(Q_1\) by a factor of \(2\). In particular, every \(n\)-convex disk \(C\) of area \(a\) in the plane is contained in a rectangle of area \(2a\). This was previously proved also by Radziszewski [1952] and Süss [1955].

Since two copies of \((\frac{1}{2} + \varepsilon)C\) cannot be packed transitively in \(C\) we have \(p_i(C|\mathcal{C}^n_h) \leq \frac{1}{2}\). Soifer [1999a] and Novotný [2001] conjectured that \(p_i(C|\mathcal{C}^n_h) = \frac{1}{2}\) for every convex disk \(C\). We are far from being able to prove or disprove this conjecture. The best known lower bound for general convex disks is \(p_i(C|\mathcal{C}^n_h) \geq \frac{1}{2}\) due to Januszewski [2007a]. Concerning packing positive and negative homothetic copies in a convex disk Januszewski [2008a] established the inequality \(p_i(C|\mathcal{C}^n_h) \geq \frac{7}{39}\).

The stronger conjecture of Soifer [1999a] that we have even \(p_i(C|\mathcal{C}^n_h) = \frac{1}{2}\) for every convex disk \(C\) was refuted by Novotný [2001] by showing that for the rectangle \(R\) with sides \(3^{1/4}\) and \(2^{1/4}\), \(p_i(R|R_h) = \sqrt{3/8}\).

L. Fejes Tóth conjectured (see Brass, Moser and Pach [2005, p. 131]) that \(c_i(C|\mathcal{C}^n_h) \leq 3\) for every convex disk \(C\). The first upper bound of \(c_i(C|\mathcal{C}^n_h) \leq 12\) for general convex disks \(C\) is due to A. Bezdek and K. Bezdek [1984]. Bálint Bálintová, Branická, Grešák, Hrínko, Novotný and Stacho [1993] lowered this bound to \(9(9 - \sqrt{17})/4\). Further improvements were achieved by Januszewski [1998a, 2001, 2003a] bringing down the upper bound to \(6.5\). Januszewski [1998a] considered the \(n\)-dimensional case as well, and showed that \(c_i(C|\mathcal{C}^n_h) \leq (n + 1)^n - 1\) for every convex body \(C\) in \(E^n\). Naszódi [2010] improved this bound to \(6^n\) for general convex bodies and to \(3^n\) if \(C\) is centrally symmetric. A further improvement was achieved recently by Livshyts and Tikhomirov [2020b]: \(c_i(C|\mathcal{C}^n_h) \leq 2^n n \log n(1 + o(n))\) if \(C\) is centrally symmetric, and \(c_i(C|\mathcal{C}^n_h) \leq \frac{1}{\sqrt{\pi n}} 4^n n \log n(1 + o(n))\) otherwise.

Solnán (see Brass, Moser and Pach [2005, Sect. 3.2, Conjecture 2]) formulated the following conjecture: If \(C\) is a convex body in \(E^n\) and \(C \subset \cup \lambda_i C_i\) for some positive coefficients \(\lambda_i < 1\) then \(\sum \lambda_i = n\). Solnán and Vásárhelyi [1993] proved the conjecture for \(n = 2\) and showed that equality characterizes parallelograms. The special case of the conjecture when \(C\) is a triangle or a parallelogram was also proved by Dumitrescu and Jiang [2008]. Solnán and Vásárhelyi settled the conjecture also for the case when the number of copies covering \(C\) is at most \(n + 1\). Naszódi [2010] proved the following asymptotic version of the conjecture:
For any $\nu < 1$ there is an $n_0$ such that if $n > n_0$ then for every $n$-dimensional convex body $C$, if some homothetic copies of $C$ of ratios $0 < \lambda_1, \lambda_1, \ldots, \lambda_m < 1$ cover $C$ then $\sum_{i=1}^{m} \lambda_i \geq \nu n$. GLAZYRIN [2019] proved the conjecture for balls.

AMBRUS [2022] proved that if an $n$-dimensional simplex is covered by its negative homothetic copies then the sum of the absolute values of the coefficients is at least $n$. Suggested by this result we raise the question: Is it true that if $C$ is a convex body in $E^n$ and $C \subseteq \cup \lambda_i C_i$ for some coefficients $-1 < \lambda_i < 0$ then $\sum |\lambda_i| \geq n$?

Since, in general, it is more difficult to cover a convex body by its negative than by its positive homothetic copies, we risk the conjecture that the answer is affirmative. Maybe, the same conclusion holds if we only suppose that the absolute value of the coefficients is less than 1.

2. On-line packing and covering

The problem of packing a container with a sequence of convex bodies has an algorithmic version: The bodies of the sequence are given one at a time, as on a conveyor belt, and the algorithm is to decide on the placement of the arriving body before the next body is revealed; once placed, the body cannot be moved. We call this an on-line packing problem. On-line covering problems are defined similarly.

Research in this direction was initiated by LASSAK and ZHANG [1991] for packing and by W. KUPERBERG [1994] for covering. Analogously to the quantities $p_t(C|\mathcal{F})$, $c_t(C|\mathcal{F})$, $p_t(C|\mathcal{F})$ and $c_t(C|\mathcal{F})$ we define $p^*_t(C|\mathcal{F})$, $c^*_t(C|\mathcal{F})$, $p^*_t(C|\mathcal{F})$ and $c^*_t(C|\mathcal{F})$ where the extreme values are taken for on-line arrangements only.

Recall that $p_t(I^n|\mathcal{I}^n_k) = 2(\frac{1}{2})^n$. Improving on an earlier result by LASSAK [1997], JANUSZEWSKI and LASSAK [1997] proved that for $n \geq 5$ an equally efficient on-line algorithm exists:

$$p^*_t(I^n|\mathcal{I}^n_k) = 2(1/2)^n.$$  

For $n = 3$ and $n = 4$ they proved the somewhat weaker result $p^*_t(I^n|\mathcal{I}^n_k) \geq \frac{3}{2}(\frac{1}{2})^n$. The 4-dimensional case was settled recently by JANUSZEWSKI and ZIELONKA [2018a]: $p^*_t(I^4|\mathcal{I}^n_k) = 1/8$. The question whether $p^*_t(I^n|\mathcal{I}^n_k) = 2(\frac{1}{2})^n$ for $n = 2$ and $n = 3$ remains open.

Concerning packing boxes in a cube, LASSAK [1997c] proved that $p^*_t(I^n|\mathcal{B}_{nh}) \geq (1 - \sqrt{3}/2)^{n-1}$, which was improved by JANUSZEWSKI and ZIELONKA [2020] to $p^*_t(I^n|\mathcal{B}_{nh}) \geq (3-2\sqrt{2})^{-n}$. It is an open question whether $p^*_t(I^n|\mathcal{B}_{nh}) = p_t(I^n|\mathcal{B}_{nh}) = 2(\frac{1}{2})^n$ for $n \geq 2$.

The algorithm by W. KUPERBERG [1994] yields $c^*_t(I^n|\mathcal{I}^n_k) \leq 4^n$. Better algorithms by JANUSZEWSKI and LASSAK [1994] and by LASSAK [1995] provide the bound $c^*_t(I^n|\mathcal{I}^n_k) \leq 2^n (1+o(n))$. The breakthrough was achieved by JANUSZEWSKI, LASSAK, ROTE and WOEINGER [1996a], who constructed an on-line algorithm showing that $c^*_t(I^n|\mathcal{I}^n_k) \leq 2^n + 3 - \frac{2^{n+2} - 8}{2^{n+2} - 2}$. This bound comes astoundingly close to the best value for off-line coverings. LASSAK [2002] further improved this bound to

$$c^*_t(I^n|\mathcal{I}^n_k) \leq 2^n + \frac{5}{3} (1 + 2^{-n}).$$

For the 3-dimensional case this yields $c^*_t(I^3|\mathcal{I}^n_k) \leq 8 + 15/8 = 9.875$.

The main tool used in the two articles cited above is the $q$-adic on-line algorithm for covering the interval $[0,1]$ with a sequence of segments $S_i$ of length $\delta_i$ ($i = 1, 2, \ldots$), where $q \geq 2$ is an integer, $\delta_i \in \{q^{-1}, q^{-2}, \ldots\}$, and each $S_i$ must be placed on one of the intervals of the form $[k\delta_i, (k+1)\delta_i] \subseteq [0,1]$, for some integer $k$. This
approach was earlier suggested by W. Kuperberg [1994], explicitly for \( q = 2 \) and implicitly for all \( q \geq 2 \). The suggestion was put in the form of a question asking for the existence of a winning algorithm in a 2-adic interval covering game between two players. The solution of the problem appeared in Januszewski, Lassak, Rote and Woeginger [1996b].

3. Special convex disks

A considerable amount of research has been devoted to packing and covering problems involving sequences of special convex disks, such as squares, rectangles, or triangles.

For the special case of a square \( S \) the theorem of Meir and Moser [1968] mentioned above states that \( p_t(S|S_h) = \frac{1}{2} \) and \( c_t(S|S_h) = 3 \). Januszewski [2002a] proved that

\[
c_t(S|S_h) = 2
\]

and in his papers [2007b, 2008b] he proved that \( c_t(S|S_h) = 3 \).

Let \( S' \) be the family of squares with diagonals parallel to the sides of \( S \). Januszewski [2010b, 2002b] proved that \( p_t(S|S') = \frac{4}{9} \) and

\[
c_t(S|S') = 2.5.
\]

Concerning packing rectangles into rectangles Yuan, Xu and Ding [2004] proved the following: If \( R_a \) denotes the family of rectangles of side lengths not greater than \( a \), and \( R_{ab} \) denotes a rectangle with sides \( a \) and \( b \geq a \), then

\[
p_i(R_{ab}|R_a) = \frac{ab}{2}.
\]

The case \( a = b \) was proved earlier by Januszewski [2000].

For on-line packing squares into a square Januszewski and Lassak [1997] established the bound \( p^*_t(S|S_h) \geq \frac{5}{16} \). The lower bound on \( p^*_t(S|S_h) \) was subsequently improved to \( \frac{1}{2} \) by Han, Iwama and Zhang [2008], to \( \frac{11}{27} \) by Fekete and Hoffmann [2017] and, finally, to \( \frac{2}{3} \) by Brubach [2013]. Concerning on-line packing rectangles in a square, Lassak [1997c] proved the bound \( p^*_t(S|R) \geq \frac{5}{32} \), which was improved by Januszewski and Zielonka [2018b] to \( p^*_t(S|R) \geq 0.28378 \).

For on-line covering a square with squares Januszewski and Lassak [1995a] proved \( c^*_t(S|S_h) \leq \frac{7}{2} \sqrt{39} + \frac{13}{8} \approx 5.265 \). This was improved to \( c^*_t(S|S_h) \leq 4 \) by Januszewski [2009a]. Concerning on-line covering a square by rectangles Januszewski [1996] proved \( c^*_t(S|R) \leq \frac{17}{9} \).

Richardson [1995] proved that \( p_i(T|T_s) \geq \frac{1}{2} \) for every triangle \( T \). In fact, his algorithm used only positive and negative homothetic copies of \( T \) and he conjectured that the packing is possible by using positive homothetic copies only. This conjecture is equivalent to the statement that \( p_t(T|T_h^+) = \frac{1}{2} \). In 2002c Januszewski established the bound \( p_t(T|T_h^+) \geq \frac{10}{69} \) and later in 2009b he refined the method and confirmed the Richardson’s conjecture. On the other hand, Januszewski disproved Soifer’s conjecture [1999b] that \( p_t(T|T_s) = \frac{1}{2} \) for every triangle \( T \). In 2003b he showed that \( p_i(T|T_s) = \frac{1}{2} \) if and only if \( T \) is equilateral. Moreover, in 2003c he proved that if \( T \) is an isosceles right triangle then
0.511 ≤ \frac{1}{5}(5 - 1\sqrt{3}) ≤ p_t(T|\mathcal{T}_h) ≤ \frac{7}{5} - 2\sqrt{2} ≤ 0.6715. For translative packing of positive and negative homothetic copies of a triangle JANUSZEWSKI [2006] proved
\[ p_t(T|\mathcal{T}_h) = \frac{2}{9}. \]

Concerning covering a triangle \(T\) with homothetic copies VÁSÁRHELYI [1984] proved that
\[ c_l(T|\mathcal{T}_h^-) = 4 \]
and FÜREDI [2003] proved that
\[ c_l(T|\mathcal{T}_h^+) = 2. \]

JANUSZEWSKI [1998b] strengthened Vásárhelyi’s result by showing that
\[ c_l(T|\mathcal{T}_h) = 4. \]

For a right isosceles triangle \(T\) FÜREDI [2007] established the equality
\[ c_t(T|\mathcal{T}_h) = \frac{1 + \sqrt{2}}{2}. \]

Denote by \(T_\varphi\) the triangle obtained from \(T\) by a rotation through the angle \(\varphi\). VÁSÁRHELYI [1993] proved that
\[ c_l(T_{30^\circ}|T_{30^\circ}h) = 4 \]
and \(c_l(T_\varphi|T_\varphi h) < 4\) for every triangle that is not regular.

JANUSZEWSKI [2009c] proved that for an equilateral triangle \(T\) and a square \(S\) having a side parallel to a side of \(T\) we have
\[ p_t(T|S_h) = 2\sqrt{3} - 3 \]
and
\[ c_t(T|S_h) = 2\sqrt{3}. \]

The theorem concerning covering was extended by LU and SU [2018] to covering an isosceles triangle \(T(h)\) with base length 1 and with height \(h\) by homothetic copies of a square \(S\) one side of which is parallel to the base of \(T(h)\). They showed that
\[ c_t(T(h)|S_h) = \begin{cases} \frac{2}{h} & \text{if } \frac{\sqrt{2}}{2} \leq h \leq 1, \\ 4h & \text{if } \frac{\sqrt{2}}{2} \leq h \leq 1, \\ \frac{h}{2} & \text{if } 1 \leq h < \sqrt{2}, \\ 2 & \text{if } \sqrt{2} \leq h. \end{cases} \]

For a right triangle \(T_0\) with legs 1 and \(\sqrt{2}\) and a square \(S\) with sides parallel to the legs of \(T\) FU, LIAN and ZHANG [2019] proved the inequality
\[ p_t(T_0|S_h) \geq \frac{16 - 6\sqrt{2}}{23}. \]

These authors also investigate the problem of covering a tetrahedron \(T_r\) with three mutually perpendicular edges of lengths 1, 1, and \(\sqrt{2}\) by homothetic copies of a cube \(C\) with sides parallel to the edges of \(T_r\). They prove that \(c_l(T_r|C_h) \leq 6\sqrt{2} + 1\).

Concerning packing circles in a circle FEKETE, KELDENICH and SCHIEFER [2019] proved that
\[ p_t(B^2|B^2_h) = \frac{1}{2}. \]

For the on-line case JANUSZEWSKI [2011] established the bound
\[ p_t^*(B^2|B^2_h) > 0.197. \]

For the corresponding covering problem DUMITRESCU and JIANG [2010] proved that \(c_t(B^2|B^2_h) \leq 2.97\) thereby confirming the conjecture of L. Fejes Tóth
for the first convex disk that is not a polygon. They also considered the on-line version of the problem and proved that \( c^*_t(B^2_2|B^2_2) \leq 9.763 \). Januszewski \( 2011 \) improved the latter bound to \( c^*_t(B^2_2|B^2_2) < 6.488 \).

Fekete, Morr and Scheffer \( 2019 \) investigated the problem of packing sequences of circles in a square or triangle. They proved that
\[
    p_t(S|B^2_h) = \frac{\pi}{3 + 2\sqrt{2}}.
\]
Further, if \( T \) is a non-acute triangle with an incircle of area \( a \) then
\[
    p_t(T|B^2_h) = a.
\]
For on-line packing circles in a square Fekete, von Höveling and Scheffer \( 2019 \) proved the inequality
\[
    p^*_t(S|B^2_h) \leq 0.350389.\]
For packing squares in a circle Fekete, Gurunathan, Juneja, Keldenich, Kleist, and Scheffer \( 2019 \) established the equality
\[
    p_t(B^2_2|S_h) = \frac{8}{5\pi}.
\]
The problem of covering the square by a sequence of circular disks was solved by Fekete, Gupta, Keldenich, Scheffer and Shah \( 2020 \). They proved that
\[
    c_t(S|B^2_h) = \frac{195\pi}{256}.
\]
More generally, they gave an algorithm for covering the rectangle \( R(1, \lambda) \) with sides 1 and \( \lambda \geq 1 \), and showed that there is a threshold value \( \lambda_0 = \sqrt{7}/2 - 1/4 = 1.035797 \ldots \), such that for \( \lambda < \lambda_0 \)
\[
    c_t(R(1, \lambda)|B^2_h) = 3\pi \left( \frac{\lambda^2}{16} + \frac{5}{32} + \frac{9}{256}\lambda^2 \right),
\]
and for \( \lambda \geq \lambda_0 \)
\[
    c_t(R(1, \lambda)|B^2_h) = \frac{(\lambda^2 + 2)\pi}{4}.
\]

4. Packing in and covering of the whole space

The investigation of covering the whole space by sequences of convex bodies was initiated by Chakerian \( 1975 \). Clearly, for a sequence \( \{C_i\} \) of convex bodies to permit a covering of space it is necessary that the sum of the volumes \( V(C_i) \), as well as the sum of the widths \( w(C_i) \) be divergent. These conditions are not sufficient. Chakerian and Groemer \( 1974 \) gave necessary and sufficient conditions for a sequence of convex disks to permit a covering of the plane. A sequence \( \{C_i\} \) of convex disks permits a covering of the plane if and only if either the total area of the subsequence with diameter at most 1 is infinite or the total width of the subsequence with diameter greater than 1 is infinite. The sequence \( \{C_i\} \) of convex bodies is **bounded** if the set of the diameters of the bodies is bounded. In particular, it follows that a bounded sequence of convex disks permits a covering of the plane if and only if the total area of the disks is infinite. The analogous statement for \( E^n \), \( n \geq 3 \) was proved by Groemer \( 1976 \). Chakerian and Groemer \( 1978 \) gave necessary and sufficient conditions for a sequence of convex bodies to permit a covering of \( E^n \). Groemer \( 1979 \) proved that for a sequence of convex bodies to permit a covering of \( E^n \) it is necessary and sufficient that the sequence permits a transitive covering of almost all points of \( E^n \). Groemer \( 1980, 1983a \) investigated
coverings of space by finite sequences of unbounded convex sets and in [1983b] he gave an upper bound for the total volume of a sequence of convex bodies permitting a covering of the \( n \)-dimensional sphere.

As consequences of the results of Groemer [1982] mentioned in Section 1.1, we note the following. If \( \{C_i\} \) is a bounded sequence of \( n \)-dimensional convex bodies such that \( \sum V(C_i) = \infty \), then it permits an isometric covering of the \( n \)-dimensional space with density \( \frac{1}{n} \) and an isometric packing with density \( \frac{1}{n^2} \). Moreover, if all the sets \( C_i \) are boxes, then \( \{C_i\} \) permits a translative covering of space and a translative packing in space with density \( \frac{1}{n} \) and \( \frac{1}{n^2} \), respectively. It is an open problem whether for \( n > 2 \) every bounded sequence \( \{C_i\} \) of \( n \)-dimensional convex bodies of infinite total volume permits a translative covering of \( \mathbb{E}^n \).

Groemer [1988] investigated the question under which conditions a sequence of convex bodies \( \{C_i\} \) in \( \mathbb{E}^n \) permits an isometric packing or covering of density 1. He showed that the conditions \( \sum V(C_i) \) and \( \lim_{i \to \infty} d(C_i) = 0 \) on the volume and diameter of the sets \( C_i \) are sufficient for constructing such packings and coverings.

### 5. Covering with slabs

Concerning the problem of covering space with a sequence of slabs, Groemer [1981a, 1981b] proved that every sequence of slabs of widths \( w_i \) in \( \mathbb{E}^n \) for which \( \sum w_i^{(n+1)/2} = \infty \) permits a translative covering. Makai and Pach [1983] conjectured that a sequence of slabs in \( \mathbb{E}^n \) permits a translative covering if and only if the sum of their widths is infinite. They verified the conjecture for the case of the plane. In higher dimensions the conjecture is still unresolved. Kupavskii and Pach [2017] proved that if \( w_1 \geq w_2 \geq \ldots \) is an infinite sequence of positive numbers such that

\[
\limsup_{i \to \infty} \frac{w_1 + w_2 + \ldots + w_i}{\ln(1/w_i)} > 0,
\]

then every sequence of slabs of width \( w_i \) \( (i = 1, 2, \ldots) \) permits a translative covering of \( \mathbb{E}^n \). With this result they got rather close to the proof of the conjecture: For example, it implies that a sequence of slabs of width \( w_i = 1/i \) \( (i = 1, 2, \ldots) \) permits a translative covering, while this is false for the sequence of widths \( w_i = 1/i^{1+\varepsilon} \) for any \( \varepsilon > 0 \).

A detailed account on the topic of packing and covering properties of sequences of convex bodies is found in the survey by Groemer [1985]. For surveys on on-line algorithms see Lassak [1997a] and Csirik and Woeginger [1998].

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ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, REALTANODA U. 13-15., H-1053, BUDAPEST, HUNGARY

Email address: gfejes@renyi.hu