A critical analysis of a recent test of the Lense-Thirring effect with the LAGEOS satellites

L. Iorio,
Viale Unità di Italia 68, 70125
Bari, Italy
tel./fax 0039 080 5443144
e-mail: lorenzo.iorio@libero.it

Abstract

In this paper we quantitatively discuss the impact of the current uncertainties in the even zonal harmonic coefficients $J_\ell$ of the Newtonian part of the terrestrial gravitational potential on the measurement of the general relativistic Lense-Thirring effect using a suitable linear combination of the nodes $\Omega$ of the laser-ranged LAGEOS and LAGEOS II satellites. The 1-sigma systematic error due to the mismodelling in the $J_\ell$ coefficients ranges from $\sim 4\%$ for the EIGEN-GRACE02S gravity field model to $\sim 9\%$ for the GGM02S model. Another important source of systematic error of gravitational origin is represented by the secular variations $\dot{J}_\ell$ of the even zonal harmonics. While the relativistic and $J_\ell$ signals are linear in time, the shift due to $\dot{J}_\ell$ is quadratic. We quantitatively assess their impact on the measurement of the Lense-Thirring effect with numerical simulations obtaining a $10^{-20}\%$ 1-sigma total error over 11 years for EIGEN-GRACE02S. Ciufolini and Pavlis, in a test performed in 2004, claim a total error of 5% at 1-sigma level.

Keywords: Lense-Thirring effect; LAGEOS satellites; GRACE Earth gravity field models; even zonal harmonics and their secular variations

1 Introduction

1.1 The Lense-Thirring effect

The post-Newtonian Lense-Thirring effect (Lense and Thirring 1918, Soffel 1989, Lämmezahl and Neugebauer 2001) is one of the few predictions of the Einsteinian General Relativity Theory (GRT) for which a direct and undisputable test is not yet available.

According to Einstein, the action of the gravitational potential $U$ of a given distribution of mass-energy is described by the metric coefficients
$g_{\mu\nu}, \mu, \nu = 0, 1, 2, 3$ of the space-time metric tensor. They are determined, in principle, by solving the fully non-linear field equations of GRT for such a mass-energy content. These equations can be linearized in the weak-field ($U/c^2 \ll 1$, where $c$ is the speed of light in vacuum) and slow-motion ($v/c \ll 1$) approximation (Mashhoon 2001; Ruggiero and Tartaglia 2002), valid throughout the Solar System, and look like the equations of the linear Maxwellian electromagnetism. Among other things, a noncentral, Lorentz-like force

$$F_{LT} = -2m \left( \frac{v}{c} \right) \times B_g$$  \hspace{1cm} (1)

acts on a moving test particle of mass $m$. It is induced by the post-Newtonian component $B_g$ of the gravitational field in which the particle moves with velocity $v$. $B_g$ is related to the mass currents of the source of the gravitational field and comes from the off-diagonal components $g_{0i}, i = 1, 2, 3$ of the metric tensor. Thanks to such an analogy, the ensemble of the gravitational effects induced by mass displacements is also named gravitomagnetism. Far from a central rotating body of proper angular momentum $L$ the gravitomagnetic field is

$$B_g = \frac{G[3r(r \cdot L) - r^2L]}{cr^5}. \hspace{1cm} (2)$$

One of the consequences of Eq. (1) and Eq. (2) is a gravitational spin–orbit coupling. Indeed, if we consider the orbital motion of a particle in the gravitational field of a central spinning mass, it turns out that the orbital angular momentum $\ell$ of the particle undergoes the Lense–Thirring precession, so that the longitude of the ascending node $\Omega$ and the argument of pericentre $\omega$ of the orbit of the test particle are affected by tiny secular rates $\dot{\Omega}_{LT}, \dot{\omega}_{LT}$ (Lense and Thirring 1918, Cugusi and Proverbio 1978, Soffel 1989, Ashby and Allison 1993, Iorio 2001a)

$$\dot{\Omega}_{LT} = \frac{2GL}{c^2a^3(1 - e^2)^{3/2}}, \hspace{1cm} \dot{\omega}_{LT} = -\frac{6GL\cos i}{c^2a^3(1 - e^2)^{3/2}}. \hspace{1cm} (3)$$

where $a$, $e$ and $i$ are the semimajor axis, the eccentricity and the inclination, respectively, of the orbit and $G$ is the Newtonian gravitational constant. Note that in their original paper Lense and Thirring (1918) used the longitude of the pericentre $\varpi$.

In April 2004 the GP-B mission (Everitt et al. 2001) has been launched. One of its goals is the measurement of another effect induced by the terrestrial angular momentum, i.e. the gravitomagnetic precession of the spins.
(Schiff 1960) of four superconducting gyroscopes carried onboard with an expected accuracy of 1% or better.

However, according to Nordtvedt (2003), the multi-decade analysis of the Moon’s orbit by means of the Lunar Laser Ranging (LLR) technique yields a comprehensive test of the various parts of order $O(c^{-2})$ of the post-Newtonian equation of motion. The existence of the Lense-Thirring signature as predicted by GRT would, then, be indirectly inferred from the high accuracy of the lunar orbital reconstruction. Also the radial motion of the LAGEOS satellite would yield another indirect confirmation of the existence of the Lense-Thirring effect (Nordtvedt 1988).

1.2 Aim of the paper

In the case of the Earth, the precessions of Eq. (3) are very tiny: for the laser-ranged LAGEOS satellites they amount to a few tens of milliarcseconds per year (mas yr$^{-1}$ in the following). Extracting such a minute signal from the background of the much larger competing classical effects of gravitational (even zonal harmonics and their secular variations, tides) and non-gravitational (direct solar radiation pressure, Earth albedo, direct Earth infrared radiation, thermal forces like the solar Yarkovsky-Schach effect and the terrestrial Yarkovsky-Rubincam effect) origin is a very challenging task that requires detailed knowledge also from many branches like geodesy, geodynamics and celestial mechanics other than relativistic physics. In turn, the extremely high accuracy required for such relativistic tests in the terrestrial space environment can help in further increasing our knowledge of many aspects traditionally treated by geophysical and space sciences. In this particular case, as we will see, it is especially true for the even zonal harmonic coefficients $J_\ell$ of the terrestrial gravitational potential and their secular variations $\dot{J}_\ell$.

Recently, a test of the Lense-Thirring effect on the orbit of a test particle has been performed by Ciufolini and Pavlis (2004). They analyzed the data of the laser-ranged LAGEOS and LAGEOS II satellites in the gravitational field of the Earth by using an observable proposed in Iorio and Morea (2004). The data analysis spans 11 years; the 2nd generation GRACE-only EIGEN-GRACE02S Earth gravity model (Reigber et al. 2005a) released by the GeoForschungsZentrum (GFZ), Potsdam was adopted. The total accuracy claimed by Ciufolini and Pavlis is 5-10% at 1-3 sigma, respectively, but such estimate is controversial (Iorio 2005a) for various reasons. One of the major critical points is the impact of the systematic error induced by the secular variations $\dot{J}_\ell$ of the even zonal harmonics. Another critical remark
is that only one Earth gravity model was used by Ciufolini and Pavlis in their analysis.

In this paper, we quantitatively investigate these issues. It will be shown that the error budget analysis performed by Ciufolini and Pavlis is optimistic. Our evaluation of the total error in the Lense-Thirring measurement ranges from 19\% to 24\% at 1-sigma level.

2 The LAGEOS-LAGEOS II $J_2$-free combination

The adopted observable is the following combination of the residuals $\delta\Omega_{\text{obs}}$ of the rates of the nodes of LAGEOS and LAGEOS II (Iorio and Morea 2004)

$$\delta\hat{\Omega}_{\text{obs}}^{\text{LAGEOS}} + c_1\delta\hat{\Omega}_{\text{obs}}^{\text{LAGEOS II}} \sim \mu_{\text{LT}} S_{\text{LT}},$$

where $c_1 \sim 0.546$, $S_{\text{LT}} = 48.1$ mas yr$^{-1}$ is the slope of the secular trend according to GRT, and $\mu_{\text{LT}}$ is equal to 1 in Einsteinian theory and 0 in Newtonian mechanics. For the explicit expression of $c_1$ see below Eq. (5).

The combination of Eq. (4) has been built up in order to cancel out the first even zonal harmonic $J_2$, along with its time-varying part which also includes its secular variations $\dot{J}_2$. The impact of the non-gravitational perturbations on the nodes of the LAGEOS satellites is of the order of $\sim 1\%$ of the Lense-Thirring effect (Lucchesi 2001; 2002; 2003; 2004; Lucchesi et al. 2004).

The idea of using only the nodes of the LAGEOS satellites to disentangle the Lense-Thirring effect from $J_2$ was proposed in Ries et al. (2003) in the context of the expected improvements in our knowledge of the Earth’s gravity field from the dedicated GRACE mission. The linear combination approach was adopted for the first time by Ciufolini (1996) for his early, less precise tests with the nodes of the LAGEOS satellites and the perigee of LAGEOS II (Ciufolini et al. 1998).

Let us write the observed residuals of the rate of the node of a satellite $\delta\hat{\Omega}$ in terms of the Lense-Thirring precession and of the classical precession induced by the Earth’s quadrupole mass moment

$$\delta\hat{\Omega}_{\text{obs}} = \dot{\Omega}_{\text{LT}} \mu_{\text{LT}} + \dot{\Omega}_2 J_2 + [\text{higher degree terms}],$$

where $\dot{\Omega}_{\text{LT}}$ is as in Eq. (3) and

$$\dot{\Omega}_2 \equiv \frac{\partial \Omega_{\text{geopotential}}}{\partial J_2} = -\frac{3}{2} n \left(\frac{R}{a}\right)^2 \frac{\cos i}{(1 - e^2)^2},$$

in which $R$ and $n$ are the Earth’s mean equatorial radius and the Keplerian mean motion $n = \sqrt{GM/a^3}$, respectively. The coefficients $\dot{\Omega}_\ell$ have been
explicitly worked out in Iorio (2003) up to degree $\ell = 20$; their numerical values for LAGEOS and LAGEOS II, whose orbital parameters are listed in Table 1 can be found in Table 2. If we write Eq. (5) for both LAGEOS and LAGEOS II and solve for $\mu_{LT}$ the so-obtained linear system of two equations in the two unknowns $\mu_{LT}$ and $J_2$ it is possible to obtain Eq. (4) with

$$c_1 \equiv -\frac{\dot{\Omega}_{\text{LAGEOS}}} {\dot{\Omega}_{\text{LAGEOS II}}} = -\frac{\cos i_{\text{LAGEOS}}}{\cos i_{\text{LAGEOS II}}} \left( \frac{1 - e^2_{\text{LAGEOS II}}}{1 - e^2_{\text{LAGEOS}}} \right)^2 \left( \frac{a_{\text{LAGEOS II}}}{a_{\text{LAGEOS}}} \right)^{7/2}.$$  

(7)

Note that $c_1$ is independent of the time span and of the even zonal harmonics: it is only fixed by the orbital parameters $a, e, i$ of LAGEOS and LAGEOS II.

3 The impact of the even zonal harmonics

The combination of Eq. (4) is, by construction, independent of the perturbing effects of degree $\ell = 2$ and order $m = 0$. In the case of $J_2$ it can be checked by using the figures in Table 2 in order to calculate

$$\dot{\Omega}_{\text{LAGEOS}} + c_1 \dot{\Omega}_{\text{LAGEOS II}},$$  

(8)

the result is zero. Note that the same also holds for the time-dependent perturbations like the $\ell = 2, m = 0$ constituent of the 18.6-year tide. Indeed, by calculating the left-hand side of Eq. (8) with the values of the perturbing amplitudes of Table I and Table II in Iorio (2001b) it is possible to explicitly check this fact.

3.1 The error due to the static part of the geopotential

Eq. (4) is affected by all the remaining even zonal harmonics of degree higher than two $J_4, J_6, J_8, \ldots$, along with their secular variations $\dot{J}_4, \dot{J}_6, \ldots$.

The secular rate induced by $J_4, J_6, \ldots$ can be calculated as

$$\sum_{\ell \geq 4} \left( \dot{\Omega}_{\ell}^{\text{LAGEOS}} + c_1 \dot{\Omega}_{\ell}^{\text{LAGEOS II}} \right) J_\ell$$  

(9)

by using Table 2 for $\dot{\Omega}_{\ell}, \ell = 4, 6, \ldots$ and the values for $J_{\ell \geq 4}$ of the chosen Earth gravity field model. Such aliasing shift is quite larger than the Lense-Thirring rate. Indeed, according to, e.g., the EIGEN-CG01C (Reigber et al. 2005b) and EIGEN-CG03C ( Förste et al. 2005) Earth gravity models, which combine data from CHAMP, GRACE and terrestrial measurements, the
nominal rate induced by the static part of the even zonals on the combination of Eq. (4) is $\sim 10^5$ mas yr$^{-1}$. It turns out that for the LAGEOS satellites the precessions due to the zonals with $\ell \geq 16$ can safely be neglected: indeed, their effect on the combination of Eq. (4) amounts to 0.05 mas yr$^{-1}$; the sensitivity is $\sim 1$ mas yr$^{-1}$. Thus, it is necessary to model the action of the low-degree even zonal part of the classical terrestrial gravitational field very accurately. To be more precise, in order to reduce the classical aliasing rates to a level $\sim 1\%$ of the Lense-Thirring effect it would be necessary to know $J_4$ and $J_6$ with an uncertainty of better than $2 \times 10^{-12}$ and $4 \times 10^{-12}$, respectively. The present-day errors in $J_4$ and $J_6$ are, instead, $8.4 \times 10^{-12}$ and $6.6 \times 10^{-12}$, according to the latest model EIGEN-CG03C.

Unfortunately, it turns out that the present-day knowledge of the even zonal harmonics, represented by the latest models based on CHAMP and, especially, GRACE, is not yet good enough to allow for a reliable, model-independent $1\%$ test of the Lense-Thirring effect with the combination of Eq. (4). Indeed, at present GRACE seems to experience some difficulties in getting notable improvements in measuring the even zonal harmonics of low-degree (Wahr et al. 2004), contrary to the medium-high degree even zonals which, instead, do not pose problems to Eq. (4). As summarized in Table 3, the systematic percent bias $\delta \mu_{\text{geopotential}}$ induced by the mismodelled part of the geopotential on the gravitomagnetic effect ranges from $\sim 4\%$ for EIGEN-CG03C and EIGEN-GRACE02S to $\sim 9\%$ for GGM02S (Tapley et al. 2005). These figures are 1-sigma upper bounds obtained by linearly adding the absolute values of the individual mismodelled precessions, according to Eq. (9) with $\sigma_{J_{\ell \geq 4}}$ instead of $J_{\ell \geq 4}$, and then compared to the Lense-Thirring rate $S_{\text{LT}}$. Since the signature of such source of bias is the same as that of the Lense-Thirring effect, it is not possible to fit and remove it from the time series without also affecting the relativistic signal of interest. It is only possible to evaluate as accurately as possible its aliasing impact on the Lense-Thirring trend.

3.2 The error due to the secular variations of the even zonal harmonics

Another source of non-negligible systematic error is represented by the secular changes $\dot{J}_{\ell \geq 4}$ of the even zonal harmonics of low-degree (Cheng et al. 1997; Bianco et al. 1998; Cox and Chao 2002; Dickey et al. 2002; Cox et al. 2003; Cheng and Tapley, 2004). They are, at present, rather poorly known, as can be inferred from, e.g., Table 1 by Cox et al. (2003). Such temporal variations in the even zonals are mainly related to the Earth’s lower mantle
viscosity features (Ivins et al. 1993). This topic has recently received attention mainly due to the observed inversion of the rate of change of the Earth’s quadrupole mass moment coefficient $J_2$ which, since 1998, began increasing (Cox and Chao 2002). It is not yet clear if such an effect is a long-term feature or is short-term in nature; however, $\dot{J}_2$ is not relevant in the present context because Eq. (4) is not sensitive to it. However, the possibility of interannual variations of $J_4$ and $J_6$, depending also on the time span of the particular solution adopted, cannot be ruled out; they would affect, in principle, Eq. (4). In Table 4 we quote the weighted means of the best estimates of Table 1 by Cox et al. (2003) for $\dot{J}_{\ell \geq 4}$. The uncertainties $\sigma_{\dot{J}_{\ell \geq 4}}$ reported in Table 4 are the variances $1/\sigma^2 = \sum_i (1/\sigma_i^2)$ of the distributions of the formal errors $\sigma_i$ of Table 1 by Cox et al. (2003). Note that the so obtained $\sigma_{\dot{J}_{\ell \geq 4}}$ are smaller than such formal errors.

In the following we will perform quantitative evaluations of the impact of $\dot{J}_{\ell \geq 4}$ on Eq. (4). As a first, preliminary approximation, we will use the results of Table 4 which smooth out the possible interannual variations in the even zonals of interest.

The effect of $\dot{J}_\ell$ integrated over the time grows quadratically. A first, a priori, quantitative evaluation of the aliasing impact of $\dot{J}_4, \dot{J}_6$ on the recovery of the Lense-Thirring effect can be performed by calculating the following quantity over a chosen observational time span $T_{\text{obs}}$

$$
\frac{\sum_{\ell=4}^{6} \left( \dot{\Omega}_{\ell, \text{LAGEOS}} + c_1 \dot{\Omega}_{\ell, \text{LAGEOS II}} \right) \left( \frac{J_\ell}{2} \right) T_{\text{obs}}^2}{S_{\text{LT}} T_{\text{obs}}}.
$$

(10)

For $T_{\text{obs}} = 11$ years, it turns out that, according to Table 4, their impact amounts to $\sim 12\%$. Note that since $\dot{J}_2$ does not affect the combination of Eq. (4) and the errors in $\dot{J}_4$ and $\dot{J}_6$ are of the same order of magnitude of the nominal values, the situation does not change if we calculate Eq. (10) with the $\sigma_{\dot{J}_\ell}$ instead of their best estimates.

### 4 Numerical simulations

In this Section we investigate the impact of $\sigma_{\dot{J}_\ell}$ and $\sigma_{\dot{J}_\ell}$ by means of numerical analyses.

#### 4.1 The simulated data

Following Pavlis and Iorio (2002), we simulate with MATLAB a time-series of the combined node residuals of the LAGEOS satellites data (called Input
Model, IM) for the combination of Eq. (4) by including in it

- **LT ≡ S_{LT}t.** Lense-Thirring trend as predicted by GRT according to Eq. (4).

- **ZONDOT ≡ \sum_{\ell=4}^{6} r_{\ell} (\Omega_{\ell}^{\mathrm{LAGEOS}} + c_1 \Omega_{\ell}^{\mathrm{LAGEOS II}}) \left(\frac{\sigma_{\ell}}{2}\right) t^2.** Quadratic term due to the \( \dot{J}_{\ell} \) according to Table 4. The numbers \( r_{\ell} \) are randomly generated from a normal distribution with mean zero and unit standard deviation.

- **ZONALS ≡ pS_{LT} \frac{t}{100}.** Linear trend with a slope of \( x\% \) of the Lense-Thirring according to Table 3. The number \( p \) is randomly generated as \( r_{\ell} \).

- **TIDE ≡ \sum a_c \sigma_{A_c} \cos \left[\left(\frac{2\pi}{P}\right) t + f_c\right] \) Set of various tidal perturbations of known periods \( P \). For the impact of such kind of perturbations on the orbits of the LAGEOS satellites see Iorio (2001b). The numbers \( a_c, f_c \) are randomly generated as \( p \) and \( r_{\ell} \). The tidal constituents included are \( K_1, P_1, S_2, 165.565 \) and the 18.6-year tide

- **NOISE.** We also included the effect of various other sources of non-systematic errors by using a generator of random numbers for all the points of the simulated time series. The resulting noise was a gaussian one and the amplitude could be varied.

In a nutshell

\[
\text{IM} = \text{LT} + \text{ZONDOT} + \text{ZONALS} + \text{TIDE} + \text{NOISE.} \tag{11}
\]

We include in our model the possibility of varying the length of the time series \( T_{\text{obs}} \), the temporal step \( \Delta t \) which simulates the orbital arc length, the amplitude of the noise and of the mismodelling in the perturbations and the initial phases of the sinusoidal terms in order to simulate different initial conditions and uncertainties in the dynamical force models of the orbital processors. The so-built IM represents the basis of our subsequent analyses.

In order to check the reliability of the adopted procedure we first try to obtain some of the quantitative features of the real signal as released by Ciufolini and Pavlis (2004). For example, they fitted the ‘raw’ curve of Fig. 2 (a) of their paper with a straight line only obtaining a Root Mean Square (RMS) of the post-fit residuals of 15 mas. To this aim we perform 5000 runs for \( T_{\text{obs}} = 11 \) years and \( x = 4 \) (EIGEN-GRACE02S) by fitting the simulated signals with a straight line (LF) \( a_0 + St \). We calculate the RMS
of the residuals about the straight line fit. The averaged RMS amounts to 15.6 mas. In Fig. 1 we plot, among other things, the simulated time series and the fitted straight line for one such run representing a generic set of initial conditions and mismodelling in the force models. It turns out that the RMS for such IM is 14.8 mas. This shows that our strategy represents a good starting point.

The recovery of the slope $S$ with LF is reliable because over 5000 runs the averaged error amounts to

$$\left\langle \frac{\sigma_{S_{LF}}}{S_{LF}} \right\rangle_{\text{5000 runs}} = 0.8\%. \quad (12)$$

### 4.2 Linear and quadratic fits

In regard to the determined slope of the fitted data, a departure of $\sim 7\%$ from the predicted Lense-Thirring effect, which is present in IM, occurs for $x = 4$ (EIGEN-GRACE02S), i.e.

$$\left\langle \frac{S_{LF} - S_{LT}}{S_{LT}} \right\rangle_{\text{5000 runs}} \sim 7\%. \quad (13)$$

The same holds for $x = 3.8$ (EIGEN-CG03C). For $x = 6.3$ (EIGEN-CG01C) and $x = 8.7$ (GGM02S) the departures amount to $\sim 8\%$ and $\sim 10\%$, respectively. Instead, Ciufolini and Pavlis (2004) claim that the slope of their linear fit amounts to 47.4 mas yr$^{-1}$, i.e. a $\sim 1\%$ departure from the predictions of GRT.

In the same set of 5000 runs, we also fit the basic IM, i.e. without any rescaling, with a quadratic polynomial (QF) $a_0 + St + Qt^2$, finding

$$\left\langle \frac{S_{QF} - S_{LT}}{S_{LT}} \right\rangle_{\text{5000 runs}} \sim 14\%. \quad (14)$$

In order to assess the impact of the secular variations of the even zonal harmonics, we make the following preliminary remark. From a simple visual inspection of the plot of the simulated IM in Fig. 1 which refers to a IM built with $1-\sigma_{\dot{J}_t}$, it is not possible to infer anything about the presence or not of a quadratic term attributable to the secular variations of the even zonal harmonics. The parabolic signal is, indeed, very smooth and tends to affect the linear signal, especially as far as the early data are concerned. To be more quantitative, we assess the systematic error due to the rates of the even zonals by calculating for every run the difference between the
slopes obtained with the previously described LF and QF. Note that we do not include the sinusoidal perturbations in the fits because they are a minor problem. For example, the $\ell = 2, m = 0$ 18.6-year tide does not affect Eq. (4) and over 11 years all the most relevant perturbations describe many full cycles; the longest period is that of the solar $K_1$ tide perturbation on the node of LAGEOS, i.e. 2.85 years. Note that the approach of comparing the slopes of the fits performed with and without the features of which we want to assess the effect is the same adopted by Pavlis and Iorio (2002) and Ciufolini and Pavlis (2004) for the time-dependent harmonic perturbations. However, the problem of the secular rates of the even zonals was not treated in Ciufolini and Pavlis (2004), which only dealt with the time-dependent sinusoidal perturbations finding a 2% maximum error.

We get

$$\left\langle \frac{|S_{QF} - S_{LF}|}{S_{LT}} \right\rangle_{T_{obs}=11 \text{ years}} \sim 13\%.$$ (15)

Such results depend on the size of the parabolic signal introduced in the simulated IM. Indeed, if we conservatively triple the $\sigma_{\dot{J}_\ell}$ of Table 4 we get a discrepancy of 15%. The complete results of our simulations are shown in Tables 5, 6 and 7 for $1\sigma_{\dot{J}_\ell}$, $3\sigma_{\dot{J}_\ell}$ and $5\sigma_{\dot{J}_\ell}$, respectively. In Fig. 1 we plot the linear components of LF and QF for one of such runs.

The same procedure has been repeated by rescaling the data of IM in order to shift the zero point of the time series in the middle of the data span. As a result, for $T_{obs} = 11$ years the slopes of LF remain unchanged, showing the same $7 - 10\%$ discrepancy with respect to the GTR value as before according to the different Earth gravity models used, while the slopes of QF do change in such a way that they are now identical to those of LF. In this case the magnitude of $\sigma_{\dot{J}_\ell}$ adopted in the IM does not influence the results. In Fig. 2 we plot the rescaled IM, the Lense-Thirring trend predicted by GTR and the linear components of LF and QF for one run. The results for different time spans and Earth’s gravity models are shown in Table 8.

5 Discussions and conclusion

In this paper we addressed the problem of a quantitative evaluation of the impact of the current lingering uncertainty in the static and varying parts $J_\ell$ and $\dot{J}_\ell$ of the even zonal harmonics of the terrestrial gravitational field on the measurement of the general relativistic Lense-Thirring effect with the combination of Eq. (4) involving the nodes of LAGEOS and LAGEOS II.
To this aim, we realistically simulated the time series of the LAGEOS satellites data and we performed 5000 runs randomly varying the noise level, the initial conditions and the mismodelling of the included dynamical effects within the range of the currently known uncertainties. In each of such runs we fitted the basic simulated signal with a straight line and with a quadratic polynomial. The same procedure was also repeated by manipulating the simulated data rescaling them so that the zero point falls in the middle of the data set.

5.1 The total error budget

We propose the following error budgets for a data span of 11 years (all the obtained figures are 1-sigma bounds)

- From a-priori analytical calculation (EIGEN-GRACE02S)
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(4 + 12 + 2)^2 + 2^2} = 18\%. \]  
  \[(16)\]

  The same also holds for EIGEN-CG03C. For EIGEN-CG01C we have
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(6.3 + 12 + 2)^2 + 4^2} = 20\%. \]  
  \[(17)\]

  GGM02S yields
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(8.7 + 12 + 2)^2 + 2^2} = 23\%. \]  
  \[(18)\]

  We have included in our evaluations also the errors due to the tides (2%) and the non-gravitational perturbations (2%).

- From numerical simulations and fits of the basic simulated data (EIGEN-GRACE02S)
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(7 + 13 + 2)^2 + 2^2} = 22\%. \]  
  \[(19)\]

  The same also holds for EIGEN-CG03C. For EIGEN-CG01C we have
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(8 + 13 + 2)^2 + 2^2} = 23\%. \]  
  \[(20)\]

  GGM02S yields
  \[ \delta \mu_{LT}^{total} \equiv \sqrt{(grav)^2 + (nongrav)^2} \sim \sqrt{(10 + 13 + 2)^2 + 2^2} = 25\%. \]  
  \[(21)\]
From numerical simulations and fits of the rescaled simulated data (EIGEN-GRACE02S)
\[
\delta \mu_{LT}^{\text{total}} \equiv \sqrt{(\text{grav})^2 + (\text{nongrav})^2} \sim \sqrt{(7 + 2)^2 + 2^2} = 9\%.
\] (22)

The same also holds for EIGEN-CG03C. For EIGEN-CG01C we have
\[
\delta \mu_{LT}^{\text{total}} \equiv \sqrt{(\text{grav})^2 + (\text{nongrav})^2} \sim \sqrt{(8 + 2)^2 + 2^2} = 10\%.
\] (23)

GGM02S yields
\[
\delta \mu_{LT}^{\text{total}} \equiv \sqrt{(\text{grav})^2 + (\text{nongrav})^2} \sim \sqrt{(10 + 2)^2 + 2^2} = 12\%.
\] (24)

In regard to the evaluation of the error of gravitational origin, the secular variations of the even zonal harmonics were not solved for in the latest Earth gravity field models based on CHAMP and GRACE: they were held fixed to default values obtained from multi-year data analysis of the data of the constellation of laser-ranged geodetic satellites. Thus, the recovered values for \( J_\ell \) are not independent of \( \dot{J}_\ell \). Thus, caution advises to linearly sum the biases of the static part of the geopotential and of their secular variations in assessing the total systematic error of gravitational origin in the Lense-Thirring test.

Instead, Ciufolini and Pavlis (2004) at the end of the Section Total uncertainty, pag. 960, and in the Supplementary Information .doc file add in quadrature the doubled error due to the static part of the geopotential (i.e. \( 2 \times 4\% \) value obtained from the sum of the individual error terms), their optimistic evaluation of the error due to the time dependent part of the Earth gravity field (2\%, not doubled) and the non-gravitational error (2\%, not doubled) getting \( \sqrt{8^2 + 4^2 + 4^2} = 10\% \). On the other hand, in the Supplementary Information .doc file it seems that they triple the 3\% error due to the static part of the even zonal harmonics obtained optimistically with a root-sum-square calculation and add it in quadrature to the other (not tripled) errors getting \( \sqrt{9^2 + 2^2 + 2^2} \leq 10\% \).

These considerations and our results show that the 1-sigma 5\% total error claimed by Ciufolini and Pavlis (2004), who used only one Earth’s gravity model, is optimistic. Moreover, the present-day uncertainty in the knowledge of the static part of the geopotential does not yet allow for a fully reliable, model-independent determination of the gravitomagnetic effect with the combination of Eq. (11). It should also be considered that the evaluations presented here might turn out to be optimistic because they are
based on the smoothed values of Table 4. The impact of possible interannual variations of $J_4$ and $J_6$, depending also on the particular observational time span considered, has not been considered here and should deserve further investigations.

An inspection of Tables 5-7 shows that the secular variations of the even zonal harmonics may represent a limiting factor also for longer and shorter time spans. Unfortunately, the expected improvements in the knowledge of the static part of the geopotential from the forthcoming GRACE solutions will not ameliorate the situation with respect to the problem of the $J_\ell$.

5.2 How to improve the accuracy and the reliability of the Lense-Thirring tests

Among possible alternatives the most promising one is the launch of another satellite of LAGEOS-type, like LARES or OPTIS (Lämmerzahl et al. 2004). As shown in Iorio (2005b), the improvements in our knowledge of the gravitational field from the GRACE mission would allow to make the requirements on the orbital geometry of such a new laser target much less stringent than in the originally proposed LARES mission. In particular, by combining the data of the new spacecraft with those of LAGEOS and LAGEOS II, it would be possible to reduce its semimajor axis from 12270 km to 7500-8000 km: this would allow to greatly reduce the costs of the mission. Also the inclination could experience large departures from the value of LARES ($i = 70$ deg) without affecting the outcome of the experiment. The accuracy in measuring the Lense-Thirring effect with a three nodes combination including also LAGEOS and LAGEOS II is $\sim 1\%$ at 1-sigma level.
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Table 1: Orbital parameters of the existing LAGEOS and LAGEOS II satellites and their Lense-Thirring node precessions in mas yr$^{-1}$.

| Satellite    | $a$ (km) | $e$   | $i$ (deg) | $\dot{\Omega}_{LT}$ (mas yr$^{-1}$) |
|--------------|---------|-------|-----------|-------------------------------------|
| LAGEOS       | 12270   | 0.0045| 110       | 31                                  |
| LAGEOS II    | 12163   | 0.0135| 52.64     | 31.5                                |
Table 2: Coefficients $\hat{\Omega}_\ell$ of the node classical even zonal precessions of LAGEOS and LAGEOS II up to degree $\ell = 20$ in mas yr$^{-1}$.

| $\ell$ | LAGEOS                          | LAGEOS II                        |
|-------|---------------------------------|----------------------------------|
| 2     | $4.191586788514 \times 10^{11}$ | $-7.669274920758 \times 10^{11}$ |
| 4     | $1.544030247472 \times 10^{11}$ | $-5.58637864293 \times 10^{10}$  |
| 6     | $3.25092246054 \times 10^{10}$  | $4.99185703735 \times 10^{10}$   |
| 8     | $2.1343038821 \times 10^{9}$    | $1.0707933989 \times 10^{10}$    |
| 10    | $-1.4885315218 \times 10^{9}$   | $-2.2176133068 \times 10^{9}$    |
| 12    | $-7.703165634 \times 10^{8}$   | $-1.1555006405 \times 10^{9}$    |
| 14    | $-2.097322521 \times 10^{8}$   | $2.5803602 \times 10^{6}$       |
| 16    | $-3.04891722 \times 10^{7}$    | $8.81906969 \times 10^{7}$      |
| 18    | $2.7037212 \times 10^{6}$     | $1.25437446 \times 10^{7}$      |
| 20    | $3.3458376 \times 10^{6}$     | $-4.8988704 \times 10^{6}$      |

Table 3: Uncertainty of the systematic error $\delta \mu_{\text{geopot}}$ due to the static part of the even zonal harmonics on the Lense-Thirring effect for the combination of Eq. (4) according to the variance matrices of the Earth gravity models EIGEN-CG03C, EIGEN-GRACE02S, EIGEN-CG01C and GGM02S. The linear sum of the absolute values of the individual terms, according to Eq. (9) with $\sigma_{J_{\ell}}$ instead of $J_{\ell}$, have been used in order to give realistic 1-sigma upper bounds.

|                  | EIGEN-CG03C | EIGEN-GRACE02S | EIGEN-CG01C | GGM02S |
|------------------|-------------|----------------|-------------|--------|
| $\delta \mu_{\text{geopot}}$ | 3.8%        | 4.1%           | 6.3%        | 8.7%   |
Table 4: The reported values for $\dot{J}_2$, $\dot{J}_4$ and $\dot{J}_6$ are the weighted means of the best estimates of Table 1 by Cox et al. (2003) in units of $10^{-11}$ yr$^{-1}$. The uncertainties $\sigma_{\dot{J}_\ell}$ are the variances of the distributions of the formal errors of Table 1 by Cox et al. (2003) in the same units.

| $\ell$   | $\dot{J}_\ell$ | $\sigma_{\dot{J}_\ell}$ |
|---------|-----------------|--------------------------|
| $\ell = 2$ | -2.113         | 0.0810                   |
| $\ell = 4$ | -0.6992        | 0.2029                   |
| $\ell = 6$ | -0.3594        | 0.1765                   |

Table 5: Results of our tests for the evaluation of the 1-sigma level impact of $\dot{J}_\ell$ on the measurement of the Lense-Thirring effect with the combination of Eq. (4). No rescaling has been applied to the simulated time series. $\Delta_{\text{LF-QF}}$ is the averaged difference between the slopes of the fitted linear trends in the linear fit-only (LF) and the quadratic fit (QF). $\Delta_{\text{LF-LT}}$ is the averaged difference between the slopes of the fitted linear trend in the linear fit-only (LF) and the Lense-Thirring trend as predicted by GRT. $\Delta_{\text{QF-LT}}$ is the averaged difference between the slopes of the linear part of the quadratic fit (QF) and the Lense-Thirring trend as predicted by GRT. A time step of $\Delta t = 15$ days and EIGEN-GRACE02S have been used.

| $T_{\text{obs}}$ (yr) | $\Delta_{\text{LF-QF}}$ (%) | $\Delta_{\text{LF-LT}}$ (%) | $\Delta_{\text{QF-LT}}$ (%) |
|----------------------|-----------------------------|-----------------------------|-----------------------------|
| 3                    | 22                          | 11                          | 24                          |
| 6                    | 10                          | 8                           | 14                          |
| 11                   | 13                          | 7                           | 14                          |
| 20                   | 12                          | 7                           | 11                          |
| 30                   | 9                           | 9                           | 4                           |

Table 6: Results of our tests for the evaluation of the 3-sigma level impact of $\dot{J}_\ell$ on the measurement of the Lense-Thirring effect with the not-rescaled simulated time series. A time step of $\Delta t = 15$ days and EIGEN-GRACE02S have been used.

| $T_{\text{obs}}$ (yr) | $\Delta_{\text{LF-QF}}$ (%) | $\Delta_{\text{LF-LT}}$ (%) | $\Delta_{\text{QF-LT}}$ (%) |
|----------------------|-----------------------------|-----------------------------|-----------------------------|
| 3                    | 22                          | 10                          | 25                          |
| 6                    | 11                          | 9                           | 14                          |
| 11                   | 16                          | 10                          | 14                          |
| 20                   | 18                          | 14                          | 11                          |
| 30                   | 20                          | 20                          | 4                           |
Table 7: Results of our tests for the evaluation of the 5-sigma level impact of $\dot{J}_\ell$ on the measurement of the Lense-Thirring effect with the not-rescaled simulated time series. A time step of $\Delta t = 15$ days and EIGEN-GRACE02S have been used.

| $T_{\text{obs}}$ (yr) | $\Delta_{\text{LF-QF}}$ (%) | $\Delta_{\text{LF-LT}}$ (%) | $\Delta_{\text{QF-LT}}$ (%) |
|------------------------|-------------------------------|-------------------------------|-------------------------------|
| 3                      | 22                            | 11                            | 24                            |
| 6                      | 12                            | 11                            | 14                            |
| 11                     | 18                            | 14                            | 14                            |
| 20                     | 25                            | 23                            | 11                            |
| 30                     | 33                            | 33                            | 5                             |

Table 8: Results of our tests with the rescaled simulated time series. $\Delta_{\text{LF-QF}}$ is now zero and $\Delta_{\text{LF-LT}}$ and $\Delta_{\text{QF-LT}}$ are equal and denoted as $\Delta$ (in percent). There is no dependence on the size of $\sigma_{\dot{J}_\ell}$. A time step of $\Delta t = 15$ days and different Earth’s gravity models have been used.

| $T_{\text{obs}}$ (yr) | $\Delta$ (EIGEN-GRACE02S) | $\Delta$ (EIGEN-CG01C) | $\Delta$ (GGM02S) |
|------------------------|----------------------------|-------------------------|-------------------|
| 3                      | 10                         | 12                      | 13                |
| 6                      | 9                          | 10                      | 11                |
| 11                     | 7                          | 8                       | 10                |
| 20                     | 4                          | 5                       | 7                 |
| 30                     | 3                          | 5                       | 7                 |
Figure 1: Basic, not-rescaled simulated time series (·), predicted Lense-Thirring trend (− −), trend of the straight line-only fit (− −) and trend of the quadratic fit (· − ·) for $T_{obs} = 11$ years, $\Delta t = 15$ days and EIGEN-GRACE02S. The effect of $J_\ell$ is present at 1-sigma level, according to Table 4 in the simulated time series. The difference between the slopes of the two fitted linear trends amount to 13% of the Lense-Thirring effect. A 7% discrepancy between the straight-line only fit and the predicted LT slope occurs. See Table 5.
Figure 2: Rescaled simulated time series (•), predicted Lense-Thirring trend (−), trend of the straight line-only fit (−−−) and trend of the quadratic fit (− − −) for $T_{obs} = 11$ years, $\Delta t = 15$ days and EIGEN-GRACE02S. The $\dot{J}_\ell$ effect is present at 1-sigma level, according to Table 4 in the simulated time series. The difference between the slopes of the two fitted linear trends is now negligible. A 7% discrepancy with the predicted LT slope occurs. See Table 8.