Abstract. In this short note, we relate the boxicity of graphs (and the dimension of posets) with their generalized coloring parameters. In particular, together with known estimates, our results imply that any graph with no \(K_t\)-minor can be represented as the intersection of \(O(t^2 \log t)\) interval graphs (improving the previous bound of \(O(t^4)\)), and as the intersection of \(\frac{15}{7}t^2\) circular-arc graphs.

1. Introduction

The intersection \(G_1 \cap \cdots \cap G_k\) of \(k\) graphs \(G_1, \ldots, G_k\) defined on the same vertex set \(V\), is the graph \((V, E_1 \cap \cdots \cap E_k)\), where \(E_i\) \((1 \leq i \leq k)\) denotes the edge set of \(G_i\). The boxicity \(\text{box}(G)\) of a graph \(G\), introduced by Roberts [15] in 1969, is defined as the smallest \(k\) such that \(G\) is the intersection of \(k\) interval graphs.

Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two [16] and Thomassen proved in 1986 that planar graphs have boxicity at most three [19]. Outerplanar graphs have no \(K_4\)-minor and planar graphs have no \(K_5\)-minor, so a natural question is how these two results extend to graphs with no \(K_t\)-minor for \(t \geq 6\).

It was proved in [5] that if a graph has acyclic chromatic number at most \(k\), then its boxicity is at most \(k(k-1)\). Using an earlier result of Nešetřil and Ossona de Mendez [13], it implied that graphs with no \(K_t\)-minor have boxicity \(O(t^4 \log t)^2\). Using a more recent result of van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [8], this implies that graphs with no \(K_t\)-minor have boxicity \(O(t^4)\). On the other hand, it was noted in [4] that a result of Adiga, Bhowmick and Chandran [1] (deduced from a result of Erdős, Kierstead and Trotter [3]) implies the existence of graphs with no \(K_t\)-minor and with boxicity \(\Omega(t^{1/2} \log t)\).

In this note, we relate the boxicity of graphs with their generalized coloring numbers (see the next section for a precise definition). Using this connection together with earlier results, we prove the following result.

**Theorem 1.** If \(G\) has no \(K_t\)-minor, then \(\text{box}(G) = O(t^2 \log t)\).

Our technique can be slightly refined (and the bound can be slightly improved) if instead of considering boxicity we consider a variant, in which we seek to represent graphs as the intersection of a minimum number of circular-arc graphs (instead of interval graphs as in the definition of boxicity).

**Theorem 2.** If \(G\) has no \(K_t\)-minor, then \(G\) can be represented as the intersection of \(\frac{15}{7}t^2\) circular-arc graphs.

The dimension of a poset \(\mathcal{P}\), denoted by \(\text{dim}(\mathcal{P})\), is the minimum number of linear orders whose intersection is exactly \(\mathcal{P}\). Adiga, Bhowmick and Chandran [1] discovered a nice connection between the boxicity of graphs and the dimension of posets, which has the following consequence: for any

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poset $\mathcal{P}$ with comparability graph $G_\mathcal{P}$, $\dim(\mathcal{P}) \leq 2\text{box}(G_\mathcal{P})$. In particular, our main result implies the following.

**Theorem 3.** If $\mathcal{P}$ is a poset whose comparability graph $G_\mathcal{P}$ has no $K_t$-minor, then $\dim(\mathcal{P}) = O(t^2 \log t)$.

It should be noted that while Theorem 1 directly implies Theorem 3 (using the result of [1] mentioned above), the converse implication is not as straightforward. Note also that a direct proof of Theorem 3 can be obtained along the same lines as that of Theorem 1 (see [20]).

Our result is based on a connection between the boxicity of graphs and their weak 2-coloring number (defined in the next section). Thus our result can also be seen as a connection between the dimension of posets and the weak 2-coloring number of their comparability graphs. Interestingly, similar connections between the dimension of posets and weak colorings of their cover graphs have recently been discovered. We omit the precise definition of cover graphs here as they will not be mentioned past this introduction. We just mention that the cover graph of a poset $\mathcal{P}$ via transitivity. In particular, the cover graph of $\mathcal{P}$ can be much sparser than the comparability graph of $\mathcal{P}$ (for a chain, the first is a path while the second is a complete graph). However, for posets of height two, the comparability graph and the cover graph coincide.

It was proved by Joret, Micek, Ossona de Mendez and Wiechert [9] that if $\mathcal{P}$ is a poset of height at most $h$, and the cover graph of $\mathcal{P}$ has weak $(3h-3)$-coloring number at most $k$, then $\dim(\mathcal{P}) \leq 4k^3$. For posets $\mathcal{P}$ of height $h = 2$, this implies that the dimension is at most $4k^3$, where $k$ is the weak 3-coloring number of the comparability graph of $\mathcal{P}$. This will be significantly improved in Section 2 (see Theorem 1).

The *adjacency poset* of a graph $G = (V, E)$, introduced by Felsner and Trotter [7], is the poset $(W, \leq)$ with $W = V \cup V'$, where $V'$ is a disjoint copy of $V$, and such that $u \leq v$ if and only if $u = v$, or $u \in V$ and $v \in V'$ and $u, v$ correspond to two distinct vertices of $G$ which are adjacent in $G$. It was proved in [5] that for any graph $G$, the dimension of the adjacency poset of $G$ is at most $2\text{box}(G) + \chi(G) + 4$, where $\chi(G)$ is the chromatic number of $G$. Since graphs with no $K_t$ minor have chromatic number $O(t\sqrt{\log t})$ [11] [18], this implies that the dimension of the adjacency poset of any graph with no $K_t$-minor is $O(t^2 \log t)$.

2. Weak coloring

Let $G$ be a graph and let $\Pi(G)$ denote the set of linear orders on $V(G)$. Fix some linear order $\pi \in \Pi(G)$ for the moment. We write $x <_\pi y$ if $x$ is smaller than $y$ in $\pi$, and we write $x \leq_\pi y$ if $x = y$ or $x <_\pi y$. For a set $S$ of vertices, $x \leq_\pi S$ means that for any vertex $y \in S$, $x \leq_\pi y$. When $\pi$ is clear from the context, we omit the subscript $\pi$ and write $<$ and $\leq$ instead of $<_\pi$ and $\leq_\pi$.

For an integer $r \geq 0$, we say that a vertex $u$ is weakly $r$-reachable from $v$ in $G$ if there is a path $P$ of length (number of edges) at most $r$ between $u$ and $v$, such that $u \leq_\pi P$. In particular, $u$ is weakly 2-reachable from $v$ if $u \leq_\pi v$, and either $u = v$, or $u$ and $v$ are adjacent, or $u$ and $v$ have a common neighbor $w$ with $u <_\pi w <_\pi v$ or $u <_\pi v <_\pi w$.

The *weak $r$-coloring number* of a graph $G$, denoted by $\text{wcol}_r(G)$, is the minimum (over all linear orders $\pi \in \Pi(G)$) of the maximum (over all vertices $v$ of $G$) of the number of vertices that are weakly $r$-reachable from $v$ with respect to $\pi$. For more background, on weak coloring numbers the reader is referred to [14].

In this section we will consider the following slight variant of weak coloring: let $\text{wcol}_r^*(G)$ be the minimum $k$ such that for some linear order $\pi \in \Pi(G)$, there exists a coloring of the vertices of $G$
such that for any vertex \( v \in G \), all the vertices distinct from \( v \) that are weakly \( r \)-reachable from \( v \) have a color that is distinct from that of \( v \). Note that the greedy algorithm trivially shows that for any graph \( G \) and integer \( r \geq 0 \), \( \text{wcol}_r^*(G) \leq \text{wcol}_r(G) \).

**Theorem 4.** For any graph \( G \), \( \text{box}(G) \leq 2 \text{wcol}_2^2(G) \).

**Proof.** Let \( G \) be a graph on \( n \) vertices and let \( c := \text{wcol}_2^2(G) \). By definition, there exist a linear order \( \pi \) on \( V(G) \) and a vertex coloring \( \phi \) with colors from the set \( \{1, \ldots, c\} \), such that whenever a vertex \( u \) is weakly 2-reachable from another vertex \( v \) with respect to \( \pi \), then \( \phi(u) \neq \phi(v) \).

We aim to show that \( G \) is the intersection of \( 2c \) interval graphs \( I_1, \ldots, I_{2c} \). We associate to each color \( i \in [c] \) the two interval graphs \( I_i \) and \( I_i + c \). So let us fix color \( i \) for the moment. We explicitly define the intervals representing the vertices of \( V(G) \) in \( I_i \) and \( I_i + c \), respectively. Let us consider the vertices \( v_1, \ldots, v_\ell \) that received color \( i \) by \( \phi \). By relabelling the vertices if needed, we may assume that \( v_1 < \cdots < v_\ell \) holds in \( \pi \).

We start with \( I_i \). Here, we map \( v_j (1 \leq j \leq \ell) \) to the point \( \{j\} \); and for every vertex \( u \) that is not colored with \( i \), we consider two cases: if \( u \) has no neighbor colored \( i \) we map \( u \) to the point \( \{n\} \), and otherwise we consider the minimal \( k \) \( (1 \leq k \leq \ell) \) such that \( u \) is adjacent to \( v_k \), and then we map \( u \) to the interval \( [k, n] \). Notice that \( I_i \) is a supergraph of \( G \).

We now proceed with \( I_i + c \). Here, we reverse the order of the vertices with color \( i \), that is, we map \( v_j (1 \leq j \leq \ell) \) to the point \( \{\ell - j + 1\} \); and for every vertex \( u \) not colored with \( i \), we again map \( u \) to the point \( \{n\} \) if \( u \) has no neighbor colored \( i \), and otherwise we consider the maximal \( k' \) \( (1 \leq k' \leq \ell) \) such that \( u \) is adjacent to \( v_{k'} \), and then we map \( u \) to the interval \( [\ell - k', 1, n] \). Notice that \( I_i + c \) is also a supergraph of \( G \). In Figure 1 the two interval graphs \( I_i \) and \( I_i + c \) are illustrated by their induced box representation in dimensions \( i \) and \( i + c \).

![Figure 1](image_url)

**Figure 1.** Illustration of \( I_i \) and \( I_i + c \) as the corresponding box representation. Vertices with color \( i \) are mapped to the red points. Projections onto the two axis yield the intervals representing the vertices.

Next, we show that \( G \) is the intersection of \( I_1, \ldots, I_{2c} \). Since all involved interval graphs are supergraphs of \( G \), we only need to show that for each pair of non-adjacent vertices \( u, v \in V(G) \) there is an interval graph \( I_j (1 \leq j \leq 2c) \) in which the two vertices are mapped to disjoint intervals.

So suppose that \( u \) and \( v \) are non-adjacent in \( G \). We may assume without loss of generality that \( u < v \) in \( \pi \). If both \( u \) and \( v \) have the same color \( i \), then their intervals are distinct points...
in \( I_i \) (and also in \( I_{i+k} \)) and thus disjoint. So suppose that \( u \) and \( v \) have distinct colors \( i \) and \( j \), respectively. We assume for a contradiction that the intervals of \( u \) and \( v \) intersect in every interval graph \( I_1, \ldots, I_{2c} \). This holds in particular in \( I_i \) and \( I_{i+k} \) (where \( u \) is mapped to a point and \( v \) to an interval containing point \( \{n\} \)); and from this we deduce that there are distinct vertices \( x \) and \( y \) with color \( i \) such that \( v \) is adjacent to both of them and \( x < u < y \) in \( \pi \). However, together with our assumption \( u < v \) in \( \pi \) this implies that \( x \) is weakly 2-reachable from \( y \) with respect to \( \pi \), as is witnessed by the path \( x, v, y \). This is a contradiction to the properties of the coloring \( \phi \).

We conclude that \( G \) is indeed the intersection of \( I_1, \ldots, I_{2c} \), and thus \( \text{box}(G) \leq 2c \). \( \square \)

It is proven implicitly in [13] that if every minor of a graph \( G \) has average degree at most \( d \), then the star-chromatic number \( \chi_s(G) \) is \( O(d^2) \). A closer look at the proof contained in the paper reveals that this bound also holds for \( wcol_2^*(G) \). (See Theorem 2.1 in this paper; the authors do not state this observation, but it directly follows from the constructed conflict graph that contains edges between any two vertices which are weakly 2-reachable from each other.) Since graphs with no \( K_t \)-minor have average degree \( O(t^{1/2} \log t) \), it follows that these graphs have \( wcol_2^*(G) = O(t^2 \log t) \). We thus obtain Theorem 1 as a direct consequence of Theorem 4.

Given a graph \( H \), a subdivision of \( H \) is a graph obtained from \( H \) by subdividing some of the edges of \( H \) (i.e. replacing them by paths). The subdivision is said to be an \( \leq 1 \)-subdivision if each edge is subdivided at most once (i.e. replaced by a path on at most 2 edges). It was observed by Sebastian Siebertz (personal communication), that the proof of [13] relating the star-chromatic number \( \chi_s(G) \) (and \( wcol_2^*(G) \)) with the density of minors of \( G \) indeed shows the following slightly stronger statement: if for any graph \( H \) such that an \( \leq 1 \)-subdivision of \( H \) appears as a subgraph of \( G \), \( H \) has average degree at most \( d \), then \( wcol_2^*(G) = O(d^2) \). A classic result [2,10] states that graphs with no subdivision of \( K_t \) have average degree \( O(t^2) \). An immediate consequence is the following.

**Theorem 5. If** \( G \) **has no subdivision of** \( K_t \), **then** \( \text{box}(G) = O(t^4) \).

### 3. Strong coloring and circular-arc graphs

The purpose of this section is to prove that if we consider a slightly larger class of graphs (circular-arc graphs instead of interval graphs), we can gain a multiplicative factor of \( \log t \) in Theorem 1.

A circular interval is an interval of the unit circle, and a circular-arc graph is the intersection graph of a family of circular intervals. Equivalently, we can define a circular interval of \( \mathbb{R} \) as being either an interval of \( \mathbb{R} \), or the (closed) complement of an interval of \( \mathbb{R} \). Note that this defines the same intersection graphs, and we will use whatever formulation is the most convenient, depending on the situation.

The toroidal boxicity of a graph \( G \), denoted by \( \text{box}^s(G) \), is the minimum integer \( k \) such that \( G \) can be represented as the intersection of \( k \) circular-arc graphs. Since every interval graph is a circular-arc graph, \( \text{box}^s(G) \leq \text{box}(G) \) for any graph \( G \).

Let \( G \) be a graph. Given a linear order \( \pi \in \Pi(G) \) and an integer \( r \geq 0 \), we say that a vertex \( u \) is strongly \( r \)-reachable from \( v \) if there is a path \( P \) of length at most \( r \) between \( u \) and \( v \), such that \( u \leq_{\pi} P \) and all the internal vertices of \( P \) are larger than \( v \) in \( \pi \). In particular, \( u \) is strongly 2-reaching from \( v \) if \( u \leq_{\pi} v \), and either \( u = v \), or \( u \) and \( v \) are adjacent, or \( u \) and \( v \) have a common neighbor \( w \) with \( u \leq_{\pi} v \leq_{\pi} w \).
The strong \( r \)-coloring number of a graph \( G \), denoted by \( \text{col}_r(G) \), is the minimum (over all linear orders \( \pi \in \Pi(G) \)) of the maximum (over all vertices \( v \) of \( G \)) of the number of vertices that are strongly \( r \)-reachable from \( v \).

**Theorem 6.** For any graph \( G \), \( \text{box}^c(G) \leq 3\text{col}_2(G) \).

*Proof.* The proof proceeds similarly as the proof of Theorem 4. Let \( n \) be the number of vertices in \( G \). We consider a total order \( \pi \in \Pi(G) \) on the vertices of \( G \) such that for any \( v \), at most \( c = \text{col}_2(G) \) vertices are strongly \( 2 \)-reachable from \( v \). Again, any notion of order between the vertices of \( G \) in this proof will implicitly refer to \( \pi \). As before, we start by greedily coloring \( G \), with at most \( c \) colors, such that for any \( v \) and any vertex \( u \neq v \) that is strongly \( 2 \)-reachable from \( v \), the colors of \( u \) and \( v \) are distinct. For each color class \( 1 \leq i \leq c \), we consider the two interval graphs \( I_i \) and \( I_{i+c} \) of the proof of Theorem 4 and a circular arc graph \( I_{i+2c} \) defined as follows. Let \( v_1 < \ldots < v_{2t} \) be the vertices colored \( i \) in \( G \). Again, each vertex \( v_j \) (\( 1 \leq j \leq t \)) is mapped to the point \( \{j\} \). Each vertex \( v \) not colored \( i \) is mapped (1) to the point \( \{n\} \) if \( v \) has no neighbor colored \( i \), (2) to the interval \( [j,n] \) if \( v_j \) is the unique neighbor of \( v \) colored \( i \), and otherwise (3) to the complement of the open interval \( (j,k) \), where \( v_j \) and \( v_k \) are the smallest and second smallest neighbors of \( v \) colored \( i \) (with respect to \( \pi \)).

We now prove that \( G \) is precisely the intersection of the graphs \( I_i \) for \( 1 \leq i \leq 3k \), which will show that \( G \) is the intersection of at most \( 3k = 3\text{col}_2(G) \) circular-arc graphs. We already proved in the previous section that for each \( 1 \leq i \leq 2k \), the graphs \( I_i \) are supergraphs of \( G \), and it is also clearly the case for the graphs \( I_i \) with \( 2k+1 \leq i \leq 3k \) (in the graph \( I_{i+2c} \), any vertex \( v \) not colored \( i \) is adjacent to all the vertices not colored \( i \) and to all the vertices colored \( i \) that are not strictly between its smallest and second smallest neighbor colored \( i \) ). So, it is sufficient to prove that each non-edge \( uv \) of \( G \) is also a non-edge in a graph \( I_i \) for some \( 1 \leq i \leq 3k \). Consider two non-adjacent vertices \( u < v \) in \( G \). We can assume that \( u \) and \( v \) have distinct colors \( i \) and \( j \), respectively (otherwise \( uv \) is a non-edge in the two graphs \( I_i \) and \( I_{i+c} \) corresponding to their common color class). If \( v \) has at most one neighbor colored \( i \), then note that the set of neighbors of \( v \) colored \( i \) is the same in \( G \) and \( I_i \cap I_{i+c} \), so we can assume that \( v \) has at least two neighbors colored \( i \). As in the proof of Theorem 4 (using the definition of \( I_i \) and \( I_{i+c} \)) we can assume that \( v \) has two neighbors \( x \) and \( y \) colored \( i \), such that \( x < u < y \). Take \( x \) and \( y \) minimal (with respect to \( \pi \)) with this property.

By the definition of the strong \( 2 \)-coloring number, we can assume that at most one neighbor of \( v \) colored \( i \) precedes \( v \) (in \( \pi \)), (since otherwise the smaller neighbor would be strongly \( 2 \)-reachable from the larger neighbor, via \( v \), which would contradict the fact that the two neighbors have the same color). Hence, it follows that \( x \) and \( y \) are respectively the smallest and second smallest neighbors of \( v \). But since \( x < u < y \), it follows that \( u \) and \( v \) are non-adjacent in \( I_{i+3c} \), as desired. \( \square \)

The following result was recently proved by van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [8].

**Theorem 7.** If \( G \) has no \( K_t \)-minor, then \( \text{col}_2(G) \leq \frac{2}{3}(t-1)(t-2) \).

Together with Theorem 3, this immediately implies Theorem 2.

The log \( t \) factor between Theorems 1 and 2 raises some interesting question about the parameter \( \text{box}^c \). It is known that every \( n \)-vertex graph has boxicity at most \( n/2 \), and equality holds only for the complete graph \( K_n \) (\( n \) even) minus a perfect matching. However this graph is a circular-arc graph (see Figure 2) and thus has toroidal boxicity equal to 1.

**Question 8.** What is the maximum toroidal boxicity of a graph on \( n \) vertices?
Figure 2. A circular-arc graph representation of the complete graph $K_{10}$ minus a perfect matching.

It was observed by Andrey Kupavskii (personal communication) that a simple counting argument shows that there are $2^\Theta(bn \log n)$ $n$-vertex graphs of (toroidal) boxicity at most $b$, and thus almost all $n$-vertex graphs have toroidal boxicity $\Omega(n/\log n)$.

It is known that every graph of maximum degree $\Delta$ has boxicity $O(\Delta \log^{1+o(1)} \Delta)$ [17], while there are graphs of maximum degree $\Delta$ with boxicity $\Omega(\Delta \log \Delta)$ [1].

Question 9. What is the maximum toroidal boxicity of a graph of maximum degree $\Delta$?

Since there are $2^\Theta(\Delta n \log n)$ $n$-vertex graphs of maximum degree $\Delta$, it follows that almost all graphs of maximum degree $\Delta$ have (toroidal) boxicity $\Omega(\Delta)$.

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