Shellability of complexes of directed trees

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Abstract

The question of shellability of complexes of directed trees was asked by R. Stanley. D. Kozlov showed that the existence of a complete source in a directed graph provides a shelling of its complex of directed trees. We will show that this property gives a shelling that is straightforward in some sense. Among the simplicial polytopes, only the crosspolytopes allow such a shelling. Furthermore, we show that the complex of directed trees of a complete double directed graph is a union of suitable spheres. We also investigate shellability of the maximal pure skeleton of a complex of directed trees. Also, we prove that the complexes of directed trees of a directed graph which is essentially a tree is vertex-decomposable. For these complexes we describe the set of generating facets.

1 Introduction

A directed tree with a root $r$ is an acyclic directed graph $T = (V(T), E(T))$ such that for every $x \in V(T)$ there exists a unique path from $r$ to $x$. A directed forest is a family of disjoint directed trees. We say that a vertex $y$ is below vertex $x$ in a directed tree $T$ if there exists a unique path from $x$ to $y$. In this paper we write $\overrightarrow{xy}$ for a directed edge from $x$ to $y$.

Definition 1. Let $D$ be a directed graph. The vertices of the complex of directed trees $\Delta(D)$ are oriented edges of $D$. The faces of $\Delta(D)$ are all directed forests that are subgraphs of $D$.

The investigation of complexes of directed trees was initiated by D. Kozlov in [9]. The complex of directed trees of a graph $G$ is recognized in [5] as a discrete Morse complex of this graph (the authors treat graph as a 1-dimensional complex). Directed forests of $G$ correspond with Morse matchings on $G$. Complexes of directed trees are also studied in [8] and [10].

A $d$-dimensional simplicial complex is pure if every simplex of dimension less than $d$ is a face of some $d$-simplex. For further definitions about simplicial complexes and other topological concepts used in this paper we refer the reader to the textbook [13].

Definition 2. A simplicial complex $\Delta$ is shellable if $\Delta$ is pure and there exists a linear ordering (shelling order) $F_1, F_2, \ldots, F_k$ of maximal faces (facets) of $\Delta$
such that for all \( i < j \leq k \), there exist some \( l < j \) and a vertex \( v \) of \( F_j \), such that

\[
F_i \cap F_j \subseteq F_i \cap F_j = F_j \setminus \{v\}.
\]

(1)

For a fixed shelling order \( F_1, F_2, \ldots, F_k \) of \( \Delta \), the restriction \( R(F_j) \) of the facet \( F_j \) is defined by:

\[
R(F_j) = \{v \text{ is a vertex of } F_j : F_j \setminus \{v\} \subset F_i \text{ for some } 1 \leq i < j\}.
\]

Geometrically, if we build up \( \Delta \) from its facets according to the shelling order, then \( R(F_j) \) is the unique minimal new face added at the \( j \)-th step. The type of the facet \( F_j \) in the given shelling order is the cardinality of \( R(F_j) \), that is, \( \text{type}(F_j) = |R(F_j)| \).

For a \( d \)-dimensional simplicial complex \( \Delta \) we denote the number of \( i \)-dimensional faces of \( \Delta \) by \( f_i \), and call \( f(\Delta) = (f_{d-1}, f_0, f_1, \ldots, f_d) \) the \( f \)-vector. A new invariant, the \( h \)-vector of \( d \)-dimensional complex \( \Delta \) is \( h(\Delta) = (h_0, h_1, \ldots, h_d, h_{d+1}) \) defined by the formula

\[
h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.
\]

If a simplicial complex \( \Delta \) is shellable, then

\[
h_k(\Delta) = |\{F \text{ is a facet of } \Delta : \text{type}(F) = k\}|
\]

is an important combinatorial interpretation of \( h(\Delta) \). This interpretation was of great significance in the proof of the upper-bound theorem and in the characterization of \( f \)-vectors of simplicial polytopes (see chapter 8 in [15]).

If a \( d \)-dimensional simplicial complex \( \Delta \) is shellable, then \( \Delta \) is homotopy equivalent to a wedge of \( h_d \) spheres of dimension \( d \). A set of maximal simplices from a simplicial complex \( \Delta \) is a set of generating simplices if the removal of their interiors makes \( \Delta \) contractible.

For a given shelling order of a complex \( \Delta \) we have that

\[
\{F \in \Delta : F \text{ is a facet and } R(F) = F\}
\]

is a set of generating facets of \( \Delta \). Note that a facet \( F \) is in this set if and only if

\[
\forall v \in F \text{ there exists a facet } F' \text{ before } F \text{ such that } F \cap F' = F \setminus \{v\}.
\]

(2)

The concept of shellability for nonpure complexes is introduced in [4]. In the definition of shellability of nonpure complexes we just drop the requirement of purity from Definition 4.

For a facet \( F \) of a shellable nonpure complex we can define its restriction \( R(F) \) as before. For nonpure complexes the definitions of \( f \)-vector and \( h \)-vector are extended for double indexed arrays. For a nonpure complex \( \Delta \) let

\[
f_{i,j}(\Delta) = |\{A \in \Delta : |A| = j, i = \max\{|T| : A \subseteq T \subseteq \Delta\}\}|
\]

and

\[
h_{i,j}(\Delta) = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}.
\]
The above defined arrays are called the $f$-triangle and the $h$-triangle of $\Delta$. If $\Delta$ is a shellable complex, we have the following combinatorial interpretation of the $h$-triangle: $h_{i,j}(\Delta) = |\{F \text{ a facet of } \Delta : |F| = i, |R(F)| = j\}|$.

Furthermore, for a facet $T$ in (3). $T$, in both cases simplices $\langle i,j \rangle$ such that $x$. $\langle i,j \rangle$ are in the same class if and only if $d_y \leq d_x$. Namely, if $d_y \leq d_x$ and a vertex $v \in V(T)$ let $d_T(v)$ denote the outdegree of $v$, i.e., $d_T(v) = |\{x \in V(T) : \exists \delta \in E(T)\}|$.

In the proof of Theorem 3.1 in [9], the facets of $\Delta(D)$ are ordered by their degree sequences, i.e., trees $T$ and $T'$ are in the same class if and only if $d_T(v) = d_{T'}(v)$ for all $v \in V(D)$. Subsequently, the facets of $\Delta(D)$ are classified by considering the out-degree of the complete source.

Here we consider a directed graph $D$ with a complete source $c$ and detect some nice properties of a shelling described in the above remark. If $|V(D)| = n$ for $i = 0, 1, \ldots, n-1$, we set $\mathcal{F}_i = \{T \text{ a facet in } \Delta(D) : d_T(c) = n - i - 1\}$.

In the same manner as in the proof of Theorem 3.1 in [9] we can verify that the partition $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{n-1}$ fulfills the condition described in Remark 3. Namely, if $d_T(c) > d_{T'}(c)$ and $T \neq T'$, then there exists an edge $xy \in T' \setminus T$ such that $x \neq c$. We define

- $T'' = T' \setminus \{xy\} \cup \{\overline{xy}\}$ if the vertex $c$ is not below $y$ in $T'$; or
- $T'' = T' \setminus \{xy\} \cup \{\overline{xy}\}$ if $c$ is below $y$ in $T'$ and $r$ is the root of $T'$.

In both cases simplices $T, T', T''$ and the vertex $xy$ satisfy condition described in [3].

Furthermore, for a facet $T \in \Delta(D)$ the unique new face for $T$ in the shelling order defined above is $R(T) = \{xy \in T : x \neq c\}$. Therefore, the type of $T$ is $\text{type}(T) = n - 1 - d_T(c)$, and we obtain that $h_i(\Delta(D)) = |\mathcal{F}_i| = |\{T \text{ is a facet of } \Delta(D) : d_T(c) = n - i - 1\}|$.
Corollary 4. Let $G_n$ be the complete directed graph on $n$ vertices. Then, for all $k = 0, 1, \ldots, n-1$ we have

$$h_k(\Delta(G_n)) = \binom{n-1}{k}(n-1)^k.$$  

Remark 5. If a directed graph $D$ has a complete source, then the shelling of $\Delta(D)$ is straightforward in the following sense:

1. We start the shelling with an appropriate facet $F_0$ and let $\mathcal{F}_0 = \{F_0\}$.

2. When we order all of the facets from $\mathcal{F}_{i-1}$, let $\mathcal{F}_i$ denote the set of all facets of $\Delta(D) \setminus (\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{i-1})$ that are neighborly (share a common ridget) to a simplex from $\mathcal{F}_{i-1}$.

3. We continue shelling of $\Delta(D)$ by arranging simplices from $\mathcal{F}_i$ in an arbitrary order.

4. In this shelling order, for any facet $F$ we have that type($F$) = $i$ $\iff$ $F \in \mathcal{F}_i$.

It may be interesting to find more examples of simplicial complexes that allow a shelling with the properties (1)–(4) from the above remark.

Example 6. Let $D_n$ be the directed graph with $V(D_n) = [n]$ and

$$E(D_n) = \{1^i : i \in [n], i \neq 1\} \cup \{2^j : j \in [n], j \neq 2\}.$$  

It is easy to see that $\Delta(D_n)$ is combinatorially equivalent to the boundary of the $(n-1)$-dimensional crosspolytope.

Theorem 7. The only simplicial $d$-dimensional polytope whose boundary admit a shelling as those described in Remark 4 is the crosspolytope.

Proof. Assume that $P$ is a simplicial $d$-polytope with desired shelling. We identify a facet of $P$ with its set of vertices. Let $F_0 = \{v_1, v_2, \ldots, v_d\}$ be the first facet in this shelling.

Let $w_i$ denote the unique new vertex of the facet of $P$ that contains $(d-2)$-dimensional simplex $F_0 \setminus \{v_i\}$. All of the facets of $P$ whose type is 1 belong to $\mathcal{F}_1$ and therefore have the form $F_0 \setminus \{v_i\} \cup \{w_i\}$. We can conclude that the set of the vertices of $P$ is $V(P) = \{v_1, v_2, \ldots, v_d, w_1, w_2, \ldots, w_d\}$.

For any $S \subseteq [d]$ we consider the $(d-1)$-simplex $F_S = \text{conv}(\{v_i : i \notin S\} \cup \{w_j : j \in S\})$.

We do not know that $F_S$ is a facet of $P$, but we use induction on $k$ to show that

$$\mathcal{F}_k = \{F_S : S \subseteq [d], |S| = k\}. \quad (4)$$

Assume that the above statement holds for all $t \leq k-1$. Let $F \notin \mathcal{F}_k$ be a facet (yet not listed) of $P$ that shares a common ridget with a facet $\hat{F}$ from $\mathcal{F}_{k-1}$.

From the inductive hypothesis we have $F = F_S$ (for $S \subseteq [n], |S| = k - 1$) and $F = F_S \setminus \{v_i\} \cup \{w_j\}$ for $i, j \notin S$. If $i \neq j$ then the edge $\{v_i, w_j\}$ and the $(k-1)$-simplex $\{w_s : s \in S \cup \{j\}\}$ are two different minimal new faces that $F$ contributes in the shelling of $P$, which is impossible. Therefore, we can conclude that $i = j$, and $F = F_S \setminus \{v_i\} \cup \{w_i\} = F_{S \cup \{i\}}$. 

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We have that any of the facets that belong to $\mathcal{F}_k$ has the form described in [4]. All of the facets from $\mathcal{F}_k$ can be listed in an arbitrary order and any of them has the type $k$. Therefore, we conclude that two facets from $\mathcal{F}_k$ cannot share the same ridget, and we obtain that

$$(d - k + 1)|\mathcal{F}_{k-1}| = k|\mathcal{F}_k|.$$  

The inductive assumption and the above equations complete the proof of [4]. So, we may conclude that $P$ is combinatorially equivalent with $d$-dimensional crosspolytope.

If a directed graph $D$ has a complete source $c$ then the complex $\Delta(D)$ is homotopy equivalent to a wedge of the spheres. In [3], D. Kozlov describes generating facets of $\Delta(D)$ as rooted trees of $D$ having complete source $c$ as a leaf.

Here we study the combinatorics of the spheres in $\Delta(D)$ when $D$ has a complete source. For each tree $T$ that is a generating facet we associate a sphere $S_T \subset \Delta(D)$ that contains $T$ and describe the combinatorial type of $S_T$.

We consider a directed graph $D$ with $n$ vertices. Assume that $c$ is a complete source of $D$. Let $T$ be a rooted spanning tree of $D$ with vertex $c$ as a leaf. If $x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_k \rightarrow c$ is the unique directed path from $x_1$ (the root of $T$) to $c$, let $\sigma_T$ denote the simplex $\{x_1, x_2, \ldots, x_k, c\}$. It is obvious that $\sigma_T \notin \Delta(D)$. Also note that $\partial \sigma_T \subset \Delta(D)$.

Let $A = \{y_1, y_2, \ldots, y_r\} = V(D) \setminus \{x_1, x_2, \ldots, x_k, c\}$, i.e., $A$ contains $r = n - k - 1$ vertices that do not belong to the unique directed path from $x_1$ to $c$ in $T$. For any $y_i \in A$ there exists the unique vertical $z_i$ such that $\overline{z_i y_i} \in E(T)$. Now, we define

$$S_T = \partial \sigma_T \ast \{\overline{z_1 y_1} \ast \overline{z_2 y_2} \ast \cdots \ast \overline{z_r y_r}\}.$$  

It is not complicated to prove that $S_T \subset \Delta(D)$. The sphere $S_T$ is $(n - k - 1)$-folded bipyramid over the boundary of $k$-simplex $\sigma_T$.

**Proposition 8.** If a directed graph $D$ has two complete sources, then $\Delta(D)$ is the union of the spheres defined in [4].

**Proof.** Let us denote two complete sources in $D$ by $c$ and $c'$. If $c$ is a leaf in $T$, then we have $T \in S_T$. If $c$ is not a leaf in a tree $T$, then let $\{x_1, x_2, \ldots, x_k\}$ be the set of all vertices of $D$ such that $\overline{cx_i} \in E(T)$ for all $i = 1, 2, \ldots, k$.

If the vertex $c'$ is not below $c$ in $T$, we define

$$T' = T \setminus \{\overline{cx_1}, \overline{cx_2}, \ldots, \overline{cx_k}\} \cup \{\overrightarrow{cx_1}, \overrightarrow{cx_2}, \ldots, \overrightarrow{cx_k}\}.$$  

In the case when $c'$ is below $c$ (then we have that $c' = x_i$ or $c'$ is below $x_i$) and the root of $T$ is $r$ we define

$$T' = T \setminus \{\overline{cx_1}, \overline{cx_2}, \ldots, \overline{cx_k}\} \cup \{\overrightarrow{x_1}, \ldots, \overrightarrow{x_{i-1}}, \overrightarrow{x_{i+1}}, \ldots, \overrightarrow{x_k}\}.$$  

In both cases the directed tree $T'$ is a generating facet of $\Delta(D)$. Obviously, the facet $T$ is contained in the sphere $S_{T'}$.


We conclude now that $\Delta(G_n)$ is the union of the $(n-k-1)$-folded bipyramids over the boundary of $k$-simplex. A simple calculation and the well-known formulae for the number of forests with $n-1$ vertices and $k$ trees such that $k$ specified nodes belong to distinct trees (Theorem 3.3 in [12]) give us the number of spheres in $\Delta(G_n)$ of the same combinatorial type.

**Corollary 9.** For any $n \geq 1$ the complex $\Delta(G_n)$ is a union of $(n-1)^{n-1}$ spheres of dimension $n-2$. For $0 < k < n$ there are exactly

$$\frac{(n-1)! k(n-1)^{n-k-2}}{(n-k-1)!}$$

of these spheres that are $(n-k-1)$-folded bipyramid over the boundary of $(k-1)$-simplex.

### 3 Shellability of skeleton of $\Delta(D)$

The subcomplex of a complex of directed trees generated by its maximal facets was studied in [1] and [5].

Here we ask about the minimal dimension of the facets of $\Delta(D)$, i.e., we want to determine the maximal $k$ such that the $k$-skeleton of $\Delta(D)$ is pure. Note that for any directed graph $D$ we have that the $k$-skeleton of $\Delta(D)$ is

$$\Delta^{(k)}(D) = \{ F : F \text{ is a rooted forest in } D \text{ with at least } |V(D)| - k - 1 \text{ trees } \}.$$

For a simple graph $G$ let $\overrightarrow{G}$ denote the directed graph obtained by replacing every edge $xy$ of $G$ with two directed edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$. The greatest distance between two vertices of a graph $G$ is the diameter of $G$, denoted by $\text{diam}(G)$. A subset of the vertex set of a graph is independent if no two of its elements are adjacent. The set of neighbors of a vertex $v$ in a graph $G$ is denoted by $N(v)$.

For a graph $G$ we say that $A \subseteq V(G)$ is a strongly independent set if $A$ is independent and $N(u) \cap N(v) = \emptyset$ for all $u, v \in A, u \neq v$. Let $r(G)$ denote the maximal cardinality of a strongly independent subset of $V(G)$.

**Proposition 10.** The $k$-skeleton of $\Delta(\overrightarrow{G})$ is pure if and only if $k \leq |V(G)| - 1 - r(G)$.

**Proof.** Let $F$ be a directed forest of $\overrightarrow{G}$ with roots $x_1, x_2, \ldots, x_t$. If the forest $F$ is a facet of $\Delta(\overrightarrow{G})$, then $\{x_1, x_2, \ldots, x_t\}$ is an independent set in $G$. Further, if $T_i$ denotes the tree of $F$ that contain $x_i$, then $N(x_i) \subseteq V(T_i)$. Therefore we obtain that $\{x_1, x_2, \ldots, x_t\}$ is a strongly independent set.

So, minimal facets of $\Delta(\overrightarrow{G})$ correspond with maximal strongly independent sets of $G$.

**Corollary 11.** For a connected graph $G$, the complex $\Delta(\overrightarrow{G})$ is pure if and only if $\text{diam}(G) \leq 2$.

For a graph $G$ let $m_G$ denote the maximal dimension of skeleton of $\Delta(\overrightarrow{G})$ that is pure. From Proposition 10 we know that $m_G = |V(G)| - r(G) - 1$. Now we examine shellability of $\Delta^{(m_G)}(\overrightarrow{G})$. 
We say that a maximal strongly independent set \( A = \{x_1, x_2, \ldots, x_r\} \) of a graph \( G \) is a complete \( r \)-source if \( V(G) = A \cup N(x_1) \cup N(x_2) \cdots \cup N(x_r) \).

**Theorem 12.** If a graph \( G \) has a complete \( r \)-source, then \( \Delta^{(mG)}(G) \) is shellable.

**Proof.** Let \( A = \{x_1, x_2, \ldots, x_r\} \) be a complete \( r \)-source in \( G \). Assume that the vertex set \( V(G) \) is linearly ordered. For a facet \( F \) of \( \Delta^{(mG)}(G) \) (recall that \( F \) is a directed forest with \( r \) trees) we define \( d_F = (d_F(x_1), d_F(x_2), \ldots, d_F(x_r)) \) and \( S_F = (F_1, F_2, \ldots, F_r) \) where \( F_i = \{v \in N(x_i) : \overrightarrow{x_i} \in E(F)\} \).

Let \( <_L \) denote the lexicographical order on \( \mathbb{N}^r \). We say that \( S_F \preceq S_{F'} \) if and only if \( F_1 = F'_1, \ldots, F_{i-1} = F'_{i-1} \) and \( \min(F_i, F'_i) \in F_i \). Now, we define a partial order on the facets of \( \Delta^{(mG)}(G) \):\[
F < F' \iff \begin{cases} 
d_{F'} <_L d_F, \\
d_F = d_{F'} \text{ and } S_F \preceq S_{F'}.
\end{cases}
\]

The above order induces a partition of the facets of \( \Delta^{(mG)}(G) \). A block in this partition contains all forests of \( \Delta^{(mG)}(G) \) in which the sets of outgoing edges having \( x_i \) as the source are the same for all \( i = 1, 2, \ldots, r \). Note that the relation \( < \) induces a linear order on the blocks. The forest with edges \( \{\overrightarrow{x_i} : x_i \in A, v \in N(x_i)\} \) is the only facet contained in the first block.

Now, we will prove that this partition of the facets of \( \Delta^{(mG)}(G) \) satisfies conditions described in Remark 3. Consider two different forests \( F, F' \in \Delta^{(mG)}(G) \) such that \( F \in F_i, F' \in F_j \) and \( i < j \). Let \( T_1, T_2, \ldots, T_s \) denote the trees of the forest \( F \cap F' \). For \( i = 1, 2, \ldots, s \) let \( r_i \) denote the root of \( T_i \). Note that \( s > r \) and all edges from \( E(F) \setminus E(F') \) have the form \( \overrightarrow{x_i} \). We consider the following three possibilities:

1. There exists an edge \( \overrightarrow{uv} \in E(F') \setminus E(F) \) such that \( u \notin A \). As we have that \( r < s \), we can conclude that there exists \( j \) such that \( r_j \in N(x_i) \) and \( x_i \) is not below \( r_j \) in \( F' \setminus \{\overrightarrow{uv}\} \). Then we set \( F'' = F' \setminus \{\overrightarrow{uv}\} \cup \{\overrightarrow{x_i}\} \).

Now, we assume that \( \overrightarrow{uv} \in E(F') \setminus E(F) \) implies \( u \in A \). Further, let \( i_0 \) denotes the minimal \( i \in [r] \) for which there exists an edge \( \overrightarrow{x_{i_0}z} \in E(F') \setminus E(F) \).

2. If \( E(F') \setminus E(F) \) also contains an edge \( \overrightarrow{x_j}\) such that \( i_0 < j \), then from \( d_F(x_{i_0}) \geq d_F(x_j) \) we conclude that there exists \( x_{i_0}z \in E(F) \setminus E(F') \). The vertex \( z \) is the root in \( F' \) and \( x_{i_0} \) is not below \( z \) in \( F' \). Otherwise we have an edge \( \overrightarrow{x_{i_0}z} \) in \( E(F') \setminus E(F) \), such that \( x \notin A \) (or \( z \notin A \)). In this case we set \( F'' = F' \setminus \{\overrightarrow{x_j}\} \cup \{\overrightarrow{x_{i_0}z}\} \).

3. If \( E(F') \setminus E(F) = \{x_{i_0}v_1, x_{i_0}v_2, \ldots, x_{i_0}v_m\} \), we have that there exists the edge \( \overrightarrow{x_{i_0}u} \in E(F) \setminus E(F') \) such that \( u \) is smaller than any of \( v_i \) in the linear order defined on \( V(G) \). Again \( u \) is the root in \( F' \), and we set \( F'' = F' \setminus \{x_{i_0}v_1\} \cup \{x_{i_0}u\} \).

In any of the cases considered above, it is clear that the forests \( F, F', F'' \) satisfy $ \Delta $.

Now, we investigate shellability of \( \Delta^{(mC_n)}(G_n) \), where \( C_n \) denotes a cycle with \( n \) vertices.
Theorem 13. A complex $\Delta^{(m_{C_n})}(\overline{C}_n)$ is shellable if and only if $n = 3k$ or $n = 3k + 1$.

Proof. Note that $r(C_n) = \lceil \frac{n}{3} \rceil$ and therefore we have that

$$m_{C_n} = \begin{cases} 2k - 1, & \text{if } n = 3k; \\ 2k, & \text{if } n = 3k + 1; \\ 2k + 1, & \text{if } n = 3k + 2. \end{cases}$$

Let $C_n$ denote the simplicial complex with $n$ vertices indexed by $\mathbb{Z}_n$ and $F \subseteq \mathbb{Z}_n$ is a face if and only if it does not contain $\{i, i + 1\}$ for $i \in \mathbb{Z}_n$. It is obvious that $\Delta(\overline{C}_n) = C_{2n} \setminus \{1, 3, \ldots, 2n - 1\}$. So, we conclude that $\Delta(2k+1)\overline{C}_n$ is shellable.

If $n = 3k$, then $\{1, 4, 7, \ldots, 3k - 2\}$ is a complete $r$-source for $C_n$ and from Theorem 12 we know that $\Delta^{(2k+1)}(\overline{C}_{3k})$ is shellable.

If $n = 3k + 1$, we will prove that the lexicographical order of the facets of $C_{6k+2}$ defined by $A < B$ if and only if $\min(A \cup B) \in A$ is a shelling order.

For $A = \{a_0, a_1, \ldots, a_{2k}\} < B = \{b_0, b_1, \ldots, b_{2k}\}$ let $a_i = \min(A \setminus B) \in A$ and $b_j = \min B \setminus A$. We consider $C = (B \setminus \{b_j\}) \cup \{a_i\}$. Note that $C$ is not contained in $C_{6k+2}$ if and only if $a_0 = 1, b_0 = 2, b_{2k} = 6k + 2$. In that case, because $1 \in A$ we have that $6k + 2 \notin A$.

If $b_{2k} = 6k + 2$ and $b_{2k+1} < 6k$, then we define $C = B \setminus \{6k + 1\}$.

If $b_0 = 2, b_{2k+1} = 6k + 2$, then there exists $s \in \{1, 2, \ldots, 2k - 1\}$ such that $b_s - b_{s-1} > 3$. Then, we let $C = B \setminus \{6k + 2\} \cup \{b_{s-1} + 2\}$. It is easy to check that the condition described in (1) is satisfied in any of the above cases. So, we can conclude that $\Delta^{(2k+1)}(\overline{C}_{3k+1})$ is shellable.

For $n = 3k + 2$ we consider complex $\Delta^{(2k+1)}(\overline{C}_{3k+2}) = C_{6k+4}$. We know that $C_{6k+4}$ is homotopy equivalent with a $2k$-dimensional sphere (see Proposition 5.1 in [2]). From the proof of this proposition we can identify this sphere with the boundary of $(2k+1)$-dimensional crosspolytope $\{1, 2\} \ast \{4, 5\} \ast \cdots \ast \{6k+1, 6k+2\}$.

Obviously, this sphere is contained in $C_{6k+4}$.

However, $C_{6k+4}$ also contains $(2k+1)$-dimensional spheres (boundaries of $(2k + 2)$-simplex $\{1, 3, 5, \ldots, 4k + 5\}$ in $C_{6k+4}$).

Therefore, we obtain that this complex is homotopy equivalent to a wedge of spheres of different dimensions. So, we conclude that $\Delta^{(2k+1)}(\overline{C}_{3k+2})$ is not shellable.

$\Box$

4 Trees

For a simple graph $G = (V, E)$ the independency complex $I(G)$ is the simplicial complex with vertex set $V$ and with faces the independent sets of $G$. The independence complex has been previously studied in [7], [11].

Shellability and vertex-decomposability of independency complexes is discussed in [6] and [14]. A complex $\Delta$ is vertex decomposable if it is a simplex or (recursively) $\Delta$ has a shedding vertex $v$ such that $\Delta \setminus \{v\}$ and $\text{link}_\Delta v$ are vertex decomposable. It is well-known that any vertex decomposable complex is shellable too.

A chordless cycle of length $n$ in a graph $G$ is a cycle $v_1, v_2, \ldots, v_n, v_1$ in $G$ with no chord, i.e. with no edges except $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. 

\[ 
\]
We will use the following Theorem.

**Theorem 14** (Theorem 1, [13]). If $G$ is a graph with no chordless cycles of length other than 3 or 5, then $I(G)$ is vertex decomposable (hence shellable and sequentially Cohen-Macaulay.)

We follow Kozlov [9] and say that a digraph $D$ is essentially a tree if it becomes an undirected tree when one replaces all directed edges (or pairs of directed edges going in opposite directions) by an edge.

**Theorem 15.** Let $D = (V(D), E(D))$ be essentially a tree. Then $\Delta(D)$ is vertex decomposable and hence shellable.

**Proof.** For a given tree $D$ we define a simple graph $G$ in the following way. For $v \in V(D)$ let $d^-(v) = |\{x \in V(D) : x \rightarrow v \in E(D)\}|$ denote the in-degree of $v$ in $D$. We replace every $v \in V(D)$ with a complete graph $K_{d^-(v)}$ whose vertices correspond with directed edges having $v$ as sink. Further, if both of directed edges $\overrightarrow{xy}, \overrightarrow{yx}$ are contained in $E(D)$, then the corresponding vertices of $K_{d^-(v)}$ and $K_{d^-(w)}$ are adjacent in $G$. Formally, we define $V(G) = E(D)$, and edges with the same sink $\overrightarrow{x\ell}, \overrightarrow{yx}$ are adjacent in $G$. Also, if $\overrightarrow{ab}, \overrightarrow{ba} \in E(D)$ they are adjacent as vertices of $G$.

Note that $A \subset V(G)$ is an independent set in $G$ if and only if $A$ is the set of edges of a directed forest in $D$. Therefore we have that $\Delta(D) = I(G)$. Moreover, the construction of $G$ and the assumption that $D$ is essentially a tree guaranteed that $G$ does not contain a chordless cycle of length other than 3. Now, the statement of our theorem follows from Theorem 14.

We describe a way to find an explicit shelling of $\Delta(D)$. Let $D$ be a directed graph and let $v \in V(D)$ be a leaf in $D$. In other words there exists the unique vertex $x \in V(D)$ such that $\overrightarrow{vx}$ or $\overrightarrow{xv}$ or both of them are in $E(D)$ and there are no other edges where $v$ is a source or a sink.

Let $D' = D \setminus \{v\}$ and let $\{y_1, y_2, \ldots, y_k\} = \{y \in V(D') : \overrightarrow{vy} \in E(D')\}$. Furthermore, let $D_0 = D' \setminus \{\overrightarrow{y_1x}, \overrightarrow{y_2x}, \ldots, \overrightarrow{y_kx}\}$ and assume that $\overrightarrow{yix} \in E(D)$ for $i = 1, 2, \ldots, s$. Now, for $p = 1, 2, \ldots, k$ we set $D_p = D_0 \setminus \{\overrightarrow{y_ix}\}$. Note that $D_p = D_0$ for $p > s$.

We know that the complexes $\Delta(D')$, $\Delta(D_0)$ and $\Delta(D_p)$ are shellable. Assume that:

(i) $F_1, F_2, \ldots, F_t$ is a shelling of $\Delta(D')$;

(ii) $H_1, H_2, \ldots, H_s$ is a shelling of $\Delta(D_0)$;

(iii) $G_1^p, G_2^p, \ldots, G_t^p$ is a shelling of $\Delta(D_p)$ (for $p = 1, 2, \ldots, k$).

We use the above notation in the next proposition.

**Proposition 16.** We consider three possible cases.

(a) If $\overrightarrow{xy} \in E(D)$ and $\overrightarrow{yx} \notin E(D)$, then $F_1 \cup \{\overrightarrow{xy}\}, F_2 \cup \{\overrightarrow{xy}\}, \ldots, F_t \cup \{\overrightarrow{xy}\}$ is a shelling of $\Delta(D)$. Also, we have that $h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D'))$.
(b) If $\overline{w} \notin E(D)$ and $\overline{v} \in E(D)$, then

$$H_1 \cup \{\overline{v}\}, \ldots, H_s \cup \{\overline{v}\}, G^1_1 \cup \{\overline{y}_1\}, \ldots, G^1_t \cup \{\overline{y}_1\}, G^2_k \cup \{\overline{y}_2\}$$

is a shelling of $\Delta(D)$. Furthermore, we have that

$$h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D')) + \sum_{p=1}^{k} h_{i-1,j-1}(\Delta(D_p)).$$

(c) If $\overline{v}, \overline{w} \in E(D)$, then

$$F_1 \cup \{\overline{v}\}, F_2 \cup \{\overline{v}\}, \ldots, F_t \cup \{\overline{v}\}, H_1 \cup \{\overline{v}\}, H_2 \cup \{\overline{v}\}, \ldots, H_s \cup \{\overline{v}\}$$

is a shelling of $D$. In that case we have that

$$h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D')) + h_{i-1,j-1}(\Delta(D_0)).$$

Proof.

(a) This is obvious, because $\Delta(D)$ is a cone over $\Delta(D')$ with apex $\overline{v}$. Therefore, we have $\mathcal{R}_D(F_i \cup \{\overline{v}\}) = \mathcal{R}_D(F_i)$ and $\Delta(D)$ is contractible.

(b) If a facet $F$ of $\Delta(D)$ contains $\overline{v}$, then $F$ does not contain any of the edges $\{\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_s\}$. So, in that case we have that $F = H \cup \{\overline{v}\}$, for a facet $H$ of $\Delta(D_0)$. If a facet $F'$ of $\Delta(D)$ does not contain $\overline{v}$, then $F'$ must contain exactly one of the edges $\{\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_s\}$. Therefore, we have that $F' = G \cup \{\overline{y}_p\}$ for a facet $G$ of $D_p$.

The supposed shelling of $\Delta(D_0)$ provides that for $i \leq j$ and facets $H_i, H_j$ of $\Delta(D_0)$ there exists $k \leq j$ and $\overline{w} \in H_j$ such that

$$(H_i \cup \{\overline{v}\}) \cap (H_j \cup \{\overline{v}\}) \subseteq (H_k \cup \{\overline{v}\}) \cap (H_j \cup \{\overline{v}\}) = H_j \cup \{\overline{v}\} \setminus \{\overline{w}\}.$$

Note that for any $p$ such that $1 \leq p \leq s$ and for any facet $G^p_k$ of $\Delta(D_p)$ there exists a facet $H_k$ of $\Delta(D_0)$ such that $G^p_k \subseteq H_k$. Therefore, for any facet $H_i$ of $\Delta(D_0)$ we have

$$(H_i \cup \{\overline{v}\}) \cap (G^p_k \cup \{\overline{y}_p\}) \subseteq (H_k \cup \{\overline{v}\}) \cap (G^p_k \cup \{\overline{y}_p\}) = G^p_j.$$

Also, for $q \leq p$ and a facet $G^q_j$ of $\Delta(D_0)$ we have

$$(G^q_j \cup \{\overline{y}_q\}) \cap (G^p_k \cup \{\overline{y}_p\}) \subseteq (H_k \cup \{\overline{v}\}) \cap (G^p_k \cup \{\overline{y}_p\}) = G^p_j.$$

So, we obtain that the order defined in (b) is a shelling order for $\Delta(D)$. In this order we have that the restriction of the facets of $\Delta(D)$ is

$$\mathcal{R}_D(H_i \cup \{\overline{v}\}) = \mathcal{R}_{D_0}(H_i)$$

and

$$\mathcal{R}_D(G^p_k \cup \{\overline{y}_p\}) = \mathcal{R}_{D_p}(G^p_k) \cup \{\overline{y}_p\}.$$

(c) In this case a facet of $\Delta(D)$ has the form

$$\{\overline{v}\} \cup F,$$

for a facet $F$ of $\Delta(D')$ or $\{\overline{v}\} \cup H$, for a facet $H$ of $\Delta(D_0)$.

Again, for a facet $H_j$ of $\Delta(D_0)$ there exists a facet $F_i$ of $\Delta(D')$ such that $H_j \subseteq F_i$. In the similar manner as in (b) we can prove that the considered order is a shelling order. Further, the restriction in this order is

$$\mathcal{R}_D(F_i \cup \{\overline{v}\}) = \mathcal{R}_{D'}(F_i)$$

and

$$\mathcal{R}_D(H_i \cup \{\overline{v}\}) = \mathcal{R}_{D_0}(H_i) \cup \{\overline{v}\}. $$
Remark 17. Now, we can identify generating facets of $\Delta(D)$. If $\vec{x} \notin E(D)$ and $\vec{v} \notin E(D)$, then $\Delta(D)$ is contractible. If $\vec{x} \notin E(D)$ and $\vec{v} \in E(D)$, let $G_p$ denote a set of generating faces of $\Delta(D_p)$ for $p = 1, 2, \ldots, s$. Then, a generating set of facets of $\Delta(D)$ is
\[
\bigcup_{p=1}^{s} \{ G \cup \{ \vec{y}_p \} : G \in G_p \}.
\]
If $\vec{x}, \vec{v} \in E(D)$, then a set of generating facets of $\Delta(D)$ is
\[
\{ H \cup \{ \vec{v} \} : H \text{ is a generating facet of } \Delta(D_0) \}.
\]

A directed acyclic graph is a directed graph without directed cycles. By successive applications of Proposition 16 and Remark 17 we obtain the following result of A. Engström.

Theorem 18 (Theorem 2.10, [8]). If $D$ is a directed acyclic graph, then $\Delta(G)$ is homotopy equivalent to a wedge of $\prod_{v \in V(G)} (d^-(v) - 1)$ spheres of dimension $|V(G)| - |R| - 1$, where $R$ is the set of vertices without edges directed to them.

Now, we investigate homotopy type of $\Delta(D)$ when $D$ is a double directed tree.

Definition 19. A tree $T$ with $2n$ vertices (n leaves and n non-leaves) such that every non-leaf is adjacent to exactly one leaf we call basic tree. Also, we say that a tree with exactly two vertices is a basic tree. We say that the edge connecting a non-leaf and a leaf is peripheral.

We can produce a basic tree if we start with an arbitrary tree $T'$ and add a leaf to each node of $T'$. We use description of generating facets from Remark 17 to obtain the following proposition.

Proposition 20. Let $D$ be a directed tree with $2n$ vertices obtained from a basic tree $T$ by replacing every edge of $T$ by a pair of directed edges going in opposite directions. Then we have that $\Delta(D) \cong S^{n-1}$.

Proof. Assume that $v_1, v_2, \ldots, v_n$ are leaves of $T$. We label the rest of the vertices of $T$ with $u_1, u_2, \ldots, u_n$ so that $v_i u_i \in E(T)$ for all $i = 1, 2, \ldots, n$. By applying Remark 17 successively we obtain that the set of peripheral edges $\{ \vec{v}_i \vec{u}_i : i = 1, 2, \ldots, n \}$ is the unique generating facet of $\Delta(D)$.

We denote the unique generating facet for a basic tree $T$ by $G_T$, that is, $G_T = \{ \vec{v}_1 \vec{u}_1, \vec{v}_2 \vec{u}_2, \ldots, \vec{v}_n \vec{u}_n \}$.

Let $D$ be a double directed tree obtained from a tree $T$. We describe a bijection between generating simplices of $\Delta(D)$ and decompositions of $T$ into basic trees.

Let $v_1, v_2, \ldots, v_n$ be a fixed linear order of $V(D)$ and choose the first leaf $v \in V(D)$ in this order. Assume that $N(v) = \{ x \}$ and $N(x) = \{ y_1, y_2, \ldots, y_k \}$. From (c) of Remark 17 we know that all generating facets of $\Delta(D_0)$ have to contain the edge $\vec{x}$. Next, we are looking for generating facets of complex $\Delta(D_0)$ where $D_0 = (D \setminus \{ v \}) \setminus \{ \vec{y}_1 \vec{y}, \ldots, \vec{y_k} \}$. From (b) of Proposition 16 we
have that a generating facet of $\Delta(D_0)$ must contain edges $\overrightarrow{z_1y_1}, \overrightarrow{z_2y_2}, \ldots, \overrightarrow{z_ky_k}$ where $z_i \in N(y_i)$ and $z_i \neq x$. If $\deg_T(y_i) = 2$ for all $i = 1, 2, \ldots, k$, we consider a subtree of $T$ spanned by $\{v, x, y_1, z_1, \ldots, y_k, z_k\}$. In the case when $N_T(y_i) = \{x, z_i, u_1, \ldots, u_r\}$, a generating facet of $\Delta(D)$ that contains $\{\overrightarrow{vx}, \overrightarrow{z_1y_1}, \ldots, \overrightarrow{z_ky_k}\}$ also contains edges $\overrightarrow{w_ju_j}$ for $j = 1, 2, \ldots, r$.

By repeating this procedure we obtain a subtree $B_1$ of $T$ such that

1. $B_1$ is a basic tree and $v \in V(B_1)$,
2. for any $x \in V(B_1)$ that is not a leaf in $B_1$ we have that $d_{B_1}(x) = d_T(x)$,
3. $|V(B_1)| \geq 2$ whenever $|V(T)| \geq 2$.

Note that there can be more possibilities for a basic tree $B_1$, see Figure 1. If we can not find a subtree $B_1$ that satisfies the above conditions, then we obtain that $\Delta(D)$ is contractible. After we choose a basic tree $B_1$ that satisfies (1)–(3) we proceed in the same way with $T' = T \setminus \{x \in V(B_1) : d_{B_1}(x) = d_T(x)\}$. Note that $T'$ is a forest or a tree.

Let $v'$ be the first leaf of $T'$ and let $T_1$ be the maximal tree of $T'$ that contains $v'$. Now, we are looking for $B_2$, a subtree of $T_1$ that satisfies (1)–(3). If we can decompose $T$ into $B_1, B_2, \ldots, B_m$ we say that $(B_1, B_2, \ldots, B_m)$ is an ordered decomposition of $T$ into $m$ basic trees.

An ordered decomposition $(B_1, B_2, \ldots, B_m)$ of $T$ that satisfies (1)–(3) produces a generating facet $G_{B_1} \cup G_{B_2} \cup \cdots \cup G_{B_m}$ of $\Delta(D)$.

Figure 1: A tree and its decompositions into basic trees. Oriented edges represent generating facets of $\Delta(D)$. Note that $\Delta(D) \simeq S_5 \vee S_6$.

**Theorem 21.** Let $D$ be a double directed tree with $n$ vertices. Let $\mu_m$ denote the number of ordered decompositions of $D$ into $m$ basic trees. Then we have that

$$\Delta(D) \simeq \bigvee_m \left( \bigvee \mu_m S_{\frac{3m - 3}{2}} \right).$$

**Proof.** We described above a bijection between generating sets of $\Delta(D)$ and ordered decompositions of $D$ that satisfy (1)–(3). Consider such an ordered partition $(B_1, B_2, \ldots, B_m)$ with $m$ basic trees. If a basic tree $B_i$ contains $2s_i$ vertices (and $2s_i - 1$ edges) it contains $s_i$ edges of a generating set of $\Delta(D)$. 

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Then we have $2s_1 - 1 + 2s_2 - 1 + \cdots + 2s_m - 1 = n - 1$, and this decomposition corresponds with

$$s_1 + s_2 + \cdots + s_m - 1 = \frac{n + m - 1}{2} - 1$$

dimensional generating facet of $\Delta(D)$.

**Corollary 22.** Among all double directed trees $D$ with $n$ vertices the biggest dimension of nontrivial homology is $\lceil \frac{n-2}{2} \rceil$. Smallest nontrivial homology for all trees with $n$ vertices appears in the dimension $n - \lfloor \frac{n}{3} \rfloor - 2$.

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