Generalized fractional Ornstein-Uhlenbeck processes

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Abstract

We introduce an extended version of the fractional Ornstein-Uhlenbeck (FOU) process where the integrand is replaced by the exponential of an independent Lévy process. We call the process the generalized fractional Ornstein-Uhlenbeck (GFOU) process. Alternatively, the process can be constructed from a generalized Ornstein-Uhlenbeck (GOU) process using an independent fractional Brownian motion (FBM) as integrator. We show that the GFOU process is well-defined by checking the existence of the integral included in the process, and investigate its properties. It is proved that the process has a stationary version and exhibits long memory. We also find that the process satisfies a certain stochastic differential equation. Our underlying intention is to introduce long memory into the GOU process which has short memory without losing the possibility of jumps. Note that both FOU and GOU processes have found application in a variety of fields as useful alternatives to the Ornstein-Uhlenbeck (OU) process.

Keywords: Ornstein-Uhlenbeck processes, Lévy process, Stochastic integral, Long memory, Fractional Brownian motion.

1 Introduction

The fractional Brownian motion (FBM) is one of the most popular processes for constructing long-range dependent stochastic processes with continuous path and its fields of applications are very wide. To name just a few, we see FBM models in the fields of telecommunications, signal processes, environmental models and economics. A recent reference is e.g., Doukhan et al. (2003). Statistical methods for FBM have also been studied (see e.g. Beran (1994)).

We review the definition and name properties of FBM.

Definition 1.1 Let $0 < H \leq 1$. A fractional Brownian motion $B_t^H := \{B_t^H\}_{t \in \mathbb{R}}$ is a centered Gaussian process with $B_0^H = 0$ and $\text{Cov}(B_t^H, B_s^H) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H})$, $t, s \in \mathbb{R}$. 

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The stationarity and the convergence or divergence property have been intensively studied. If \( \{\xi_t\}_{t \geq 0} \) and \( \{\eta_t\}_{t \geq 0} \) are independent and \( \int_0^t e^{-\xi_s} d\eta_s \) converges a.s. as \( t \to \infty \) to a finite random variable, \( V := \{V_t\}_{t \geq 0} \) has the stationary version (see e.g. Remark 2.2 of \cite{LindnerMaller05}). The short memory property of \( V \) was also shown in Section 4 of \cite{LindnerMaller05}.

Another extension of the original OU process is the fractional Ornstein-Uhlenbeck process, where FBM is used as integrator. An advantage of using the process is to realize stationary long range dependent processes. Let \( \lambda > 0 \) and an initial random variable \( X_0^H \in L^1 \). A fractional Ornstein-Uhlenbeck (FOU) process is defined as

\[
X_t^H = e^{-\lambda t} \left( X_0^H + \int_0^t e^{\lambda s} dB^H_s \right), \quad t \geq 0.
\]

Here we need a non-semimartingale approach to construct stochastic integrals with FBM. We can find several useful theoretical tools in e.g., \cite{Lin95}, \cite{MikoschNorvaisa00} or \cite{PipirasTaqqu00}. \cite{Cheriditoetal03} has shown that the FOU process is the unique continuous solution of a Langevin equation: \( X_t = X_0^H - \lambda \int_0^t X_s ds + B^H_t, \quad t \geq 0 \) and investigated its dependence properties. The main purpose of this paper is to construct a version of the GOU process which allows for long memory modeling by the use of a FBM.
In order to define a generalized Ornstein-Uhlenbeck process we define a two-sided Lévy process as

\[ \xi_t := \begin{cases} \xi^1_t & \text{if } t \geq 0 \\ -\xi^2_{-t} & \text{if } t < 0, \end{cases} \tag{3} \]

where \( \{\xi^1_t\}_{t \geq 0} \) and \( \{\xi^2_t\}_{t \geq 0} \) are independent copies of \( \{\xi_t\}_{t \geq 0} \). We work throughout with a bivariate complete probability space

\[ (\Omega := \Omega_1 \times \Omega_2, \mathcal{F} := \mathcal{F}_1 \otimes \mathcal{F}_2, P := P_1 \otimes P_2) \tag{4} \]

Let \( \{\xi_t\}_{t \in \mathbb{R}} \) defined on \( (\Omega_1, \mathcal{F}_1, P_1) \) be a Lévy process and a FBM \( \{B^H_t\}_{t \in \mathbb{R}} \) with Hurst index \( H \in (0, 1) \) defined on \( (\Omega_2, \mathcal{F}_2, P_2) \) which is independent of \( \{\xi_t\}_{t \in \mathbb{R}} \). A generalized fractional Ornstein-Uhlenbeck (GFOU) process with initial value \( Y_0 \in L^1(\Omega) \) is defined as

\[ Y_t := e^{-\xi_t} \left( Y_0 + \int_0^t e^{\xi_s} dB^H_s \right), \quad t \geq 0. \tag{5} \]

If the initial variable satisfies

\[ Y_0 = \int_{-\infty}^0 e^{\xi_s} dB^H_s, \tag{6} \]

then, for convenience, we sometimes replace \( Y = \{Y_t\}_{t \geq 0} \) with

\[ \overline{Y}_t := e^{-\xi_t} \int_{-\infty}^t e^{\xi_s} dB^H_s. \tag{7} \]

The process \( Y \) is regarded as an extension of \( V \) given in (1) where the stochastic process of integration \( \{\eta_t\}_{t \geq 0} \) is replaced with \( \{B^H_t\}_{t \geq 0} \) with \( H \in (0, 1) \) and also is regarded as an extended version of the FOU process where the integrand is replaced by the exponential of an independent Lévy process \( \xi_t \). We should remark that \( Y \) has jumps caused by the process \( e^{-\xi_t} \).

The paper is organized as follows. In Section 2.1 we recall the definition of Lévy processes and summarize properties needed. In Sections 2.2.1 and 2.2.2 we review Riemann-Stieltjes integrals for functions with bounded \( p \)-variation and the stochastic integral in the \( L^2(\Omega) \)-sense respectively. In Section 3 we investigate the existence of the integral in the GFOU process in order to justify the definition of the GFOU process. The stationarity condition and the second order behavior of the GFOU process are discussed in Section 4 and we observe the long memory property. Here we also examine stochastic integrals constructed by a single FBM, where \( \xi \) in the process \( Y \) is replaced with \( B^H \) used as the integrator. In Section 5 we obtain a stochastic differential equation, whose solution is given in form of the GFOU process.

We use the following notations throughout. Write \( a.s. \) if equality holds almost surely. We will take the expectations for a bivariate process \( \{(Z^1_t, Z^2_t)\}_{t \in \mathbb{R}} \). If the expectations only for a process \( \{Z^1_t\}_{t \in \mathbb{R}} \) is considered, we write its expectation as \( E_{Z^1} \).

# 2 Preliminaries

## 2.1 Lévy processes

In this subsection we introduce the setup for the Lévy process. Let \( \xi := \{\xi_t\}_{t \geq 0} \) be a Lévy process on \( \mathbb{R} \) with \( (a_\xi, \nu_\xi, \gamma_\xi) \) generating triplet, where \( a_\xi \geq 0 \) and \( \gamma_\xi \in \mathbb{R} \) are constants and a measure \( \nu_\xi \)}
on $\mathbb{R} \setminus \{0\}$ satisfies
\[ \int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^2)\nu_\xi(dx) < \infty. \]
We call $\nu_\xi$ the Lévy measure of $\xi$. Then, the characteristic function of $\xi_t$ at time $t = 1$ is written as
\[ Ee^{iz\xi_1} = \exp \left[ -\frac{a_\xi}{2} z^2 + i\gamma z + \int_{\mathbb{R} \setminus \{0\}} \left( e^{izx} - 1 - izx1_{\{|x|\leq1\}} \right) \nu_\xi(dx) \right], \quad z \in \mathbb{R}. \] (8)
For more on Lévy processes and their properties, we refer to [Sato (1999)]. In later sections we consider several examples related the Lévy measure. Assumptions and results are somewhat different from paper to paper. The first one is a well-known result (e.g. Theorem 21.9 (i) of [Sato (1999)].)

In order to define the in [5], the variation of the Lévy process plays an important role. We give a brief summary based mainly on Section 5.4 of [Dudley and Norvaiša (1998)] and p.408 of [Mikosch and Norvaiša (2000)]. Define the $p$-variation for $0 < p < \infty$ of a process $X := \{X_t\}_{t \in \mathbb{R}}$ on $[t_1,t_2]$ for $t_1 < t_2$ in $\mathbb{R}$ as
\[ v_p(X) := v_p(X,[t_1,t_2]) := \sup_{\Delta} \sum_{i=1}^{n} |X_{s_i} - X_{s_{i-1}}|^p, \] (9)
where $\Delta$ is a partition $t_1 = s_0 < s_1 < \cdots < s_n = t_2$ of $[t_1,t_2]$ and $n \geq 1$. If $v_p(X,[t_1,t_2]) < \infty$ for all $t_1 < t_2$, we say $X$ has bounded $p$-variation, and if $v_1(X,[t_1,t_2]) < \infty$ we say it is of bounded variation. Since every Lévy process is a semimartingale $v_p(\xi) < \infty$ for $p \geq 2$ (see [Lépingle (1976)].)

We will state three useful results which characterize $p$-variation of Lévy process in terms of the Lévy measure. Unfortunately we can not find a result which uniformly characterize the variation in terms of the Lévy measure. Assumptions and results are somewhat different from paper to paper. The first one is a well-known result (e.g. Theorem 21.9 (i) of [Sato (1999)].)

1. Bounded variation
A Lévy process $\{\xi_t\}_{t \geq 0}$ is of bounded variation if and only if
\[ a_\xi = 0 \text{ and } \int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|)\nu_\xi(dx) < \infty. \]

The next result is a combination of Theorems 4.1 and 4.2 of [Blumenthal and Getoor (1961)] and Theorem 2 of [Monroe (1972)].

2.(a) $p$-variation of Lévy processes
Define $\beta = \inf\{\alpha > 0 : \int_{|x| \leq 1} |x|^\alpha\nu_\xi(dx) < \infty\}$. We call $\beta$ the Blumenthal and Getoor index. If the Lévy process $\{\xi_t\}_{t \geq 0}$ has no Gaussian component ($a_\xi = 0$), then
\[ v_p(\xi;[0,1]) < \infty \quad \text{a.s. if } \quad p > \beta \]
\[ v_p(\xi;[0,1]) = \infty \quad \text{a.s. if } \quad p < \beta. \] (10)
The result by Bretagnolle (1972) is a sharpened version of 2.(a) but with zero mean assumption.

2.(b) $p$-variation of Lévy processes

Let $1 < p < 2$ and $\{\xi_t\}_{t \geq 0}$ be a Lévy process without Gaussian component ($a_\xi = 0$). Then $v_p(\xi; [0, 1]) < \infty$ a.s. if and only if

$$\int_{\mathbb{R} \setminus \{0\}} (1 \wedge |x|^p) \nu_\xi(dx) < \infty.$$ 

Otherwise $v_p(\xi; [0, 1]) = \infty$ a.s.

In particular for $\alpha$-stable Lévy processes we have the result by Fristedt and Taylor (1973) which was stated in convenient form in Mikosch and Norvaiša (2000).

3. $p$-variation of $\alpha$-stable Lévy processes

Let $\{\xi^\alpha_t\}_{t \geq 0}$ be $\alpha$-stable Lévy motion. Assume that $\gamma_\alpha = 0$ for $\alpha < 1$ and that the Lévy measure is symmetric for $\alpha = 1$. Then $v_p(\xi^\alpha)$ is finite or infinite with probability 1 according as $p > \alpha$ or $p \leq \alpha$.

For the existence of the infinite interval integral in $\{Y_t\}_{t \geq 0}$ given in (7) we further need the behavior of $\{\xi_t\}_{t \geq 0}$ as $t \to \infty$. Our assumption is that $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$. Doney and Maller (2002) [Theorem 4.4] have obtained an equivalent condition of this in terms of the Lévy measure $\nu_\xi$. Since several papers well explain equivalent conditions, (see p.72 of Erickson and Maller (2005) or p.1704 of Lindner and Maller (2005)) we do not mention it. Actually if $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$ holds, we can assert a stronger result, which is more useful for our aim.

Lemma 2.1 Suppose $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$. Then for almost all $\omega_1 \in \Omega_1$ there exist $\delta > 0$ and $t_0 = t_0(\omega_1) < \infty$ such that

$$\xi_t > \delta t \quad \text{for} \quad t \geq t_0.$$ 

The proof is a combination of Theorem 4.3 and 4.4 in Doney and Maller (2002). Concerning the integral $\int_0^t e^{-\xi_s} \, d\eta_s$ in the GFOU process given by (1), Erickson and Maller (2005) have characterized the convergence of improper integral $\int_0^\infty e^{-\xi_s} \, d\eta_s$ in terms of the Lévy measure of $\{(\xi_t, \eta_t)\}_{t \geq 0}$, in which the condition $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$ was used.

2.2 Integrals with respect to functions with unbounded variation

2.2.1 Riemann-Stieltjes integrals with $p$-variation

We review several useful definitions of integrals of functions which have unbounded variation but bounded $p$-variation. The excellent introduction to this area is given by Dudley and Norvaiša (1998). Let $f$ and $g$ be real functions on $[a, b]$. Define $\kappa = \{u_1, \ldots, u_n\}$ to be an intermediate partition of $\Delta = [a = s_0 < s_1 < \cdots < s_n = b]$ given as in (9), namely, $s_{j-1} \leq u_i \leq s_j$ for $i = 1, \ldots, n$. A Riemann-Stieltjes sum is defined as

$$S_{RS}(f, g, \Delta, \kappa) := \sum_{i=1}^n f(u_i) [g(s_i) - g(s_{i-1})].$$
Then we say that the Riemann Stieltjes integral exists and equals to \( I \), if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[ |S_{RS}(f, g, \Delta, \kappa) - I| < \epsilon \]

for all partition \( \Delta \) with mesh \( \max(s_i - s_{i-1}) < \delta \) and for all intermediate partitions \( \kappa \) of \( \Delta \). The following theorem was proved by Young (1936). (See also Theorem 2.4 of Mikosch and Norvaiša (2000) or Theorem 4.26 of Dudley and Norvaiša (1998).)

**Theorem 2.1** Assume \( f \) has bounded \( p \)-variation and \( g \) has bounded \( q \)-variation on \([a, b]\) for some \( p, q > 0 \) with \( p^{-1} + q^{-1} > 1 \). Then the integral \( \int_a^b f dg \) exists in the Riemann-Stieltjes sense, and the inequality

\[ \left| \int_a^b f dg \right| \leq C_{p,q}(v_p(f))^{1/p}(v_q(g))^{1/q} \]

holds with \( C_{p,q} = \sum_{n \geq 1} n^{-(p^{-1}+q^{-1})} \).

### 2.2.2 Integral with respect to FBM in \( L^2 \)-sense

Another definition of the integral is in \( L^2(\Omega_2) \)-sense. Stochastic integrals with respect to FBM is sometimes defined as the \( L^2(\Omega_2) \)-limits of integrals of step functions (see e.g. Lin (1995)). We see this when \( B^H \) is the Brownian motion \( B^{1/2} \). If a function \( f(u) : \mathbb{R} \to \mathbb{R} \) satisfies \( f(u) \in L^2(\mathbb{R}) \), there exist step functions

\[ f_n(u) := \sum_{i=1}^n f_i 1_{\{u_i < u \leq u_{i+1}\}} \], \quad -\infty < u_1 < \ldots < u_{n+1} < \infty, \quad f_i \in \mathbb{R}, \ n \in \mathbb{N}. \]

such that \( f_n \) converges to \( f \) in \( L^2(\mathbb{R}) \). Then the integral \( \int_\mathbb{R} f(u) dB_u^{1/2} \) is the \( L^2(\Omega_2) \)-limit of the integrals of step functions

\[ \int_\mathbb{R} f_n(u) dB_u^{1/2} = \sum_{i=1}^n f_i \left( B_{u_{i+1}}^{1/2} - B_{u_i}^{1/2} \right), \]

since \( L^2(\mathbb{R}) \) and \( L^2(\Omega) \) are isometric and their inner products are equal, namely,

\[ E \left[ \left( \int_\mathbb{R} (f_n(u) - f(u)) dB_u^{1/2} \right)^2 \right] = \int_\mathbb{R} (f_n(u) - f(u))^2 \, du. \]

Pipiras and Taqqu (2000) have investigated a similar characterization for the integral of \( B^H \) when \( H \neq \frac{1}{2} \). We apply this to the existence of the improper integral in the GFOU process \( \{Y_t\} \) afterward. Define the linear space

\[ \mathcal{S}^p(B^H) := \left\{ X : \int_\mathbb{R} f_n(u) dB_u^H \to X \text{ in } L^2(\Omega) \text{ for some sequence } (f_n)_{n \in \mathbb{N}} \text{ (step functions)} \right\}. \]

Pipiras and Taqqu (2000) have analyzed the functional space of the integrand \( f(u) \) in which it can be asymptotically approximated by \( f_n(u) \) and \( \int_\mathbb{R} f(u) dB_u^H \) is well-defined. For \( H \in (0, \frac{1}{2}) \) they succeeded in specified a Hilbert space of functions on the real line which is isometric to \( \mathcal{S}^p(B^H) \). However, for \( H \in (\frac{1}{2}, 1) \) they had difficulty in finding the corresponding isometric space, and
as second best they analyzed inner product spaces in which the integral with $B^H$ ($H \in (\frac{1}{2}, 1)$) is well-defined. We give only one such inner product space and its inner product for $B^H$ with $H \in (\frac{1}{2}, 1)$. Other inner product spaces do not seem to work for our purpose since they require e.g. characteristic function of $f$ or fractional derivative of $f$ which do not exists in our case where $f(u) = e^{\xi u}$. (See Section 7 of Pipiras and Taqqu (2000).) The space is

$$|A|^H = \{ f : \langle |f|, |f| \rangle_{|A|^H} < \infty \}, \quad H \in \left(\frac{1}{2}, 1\right),$$

where

$$\langle f, g \rangle_{|A|^H} = c_H \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u - v|^{2H-2}dudv$$

is the inner product with $c_H := H(2H - 1)$.

## 3 Existence of the integral

In this Section we analyze the existence of the integral in the GFOU process given in (5). The definition of the GFOU process includes an integral with respect to FBM. Since paths of FBM are of infinite variation and since FBM with $H \neq \frac{1}{2}$ is not a semimartingale, the stochastic integral with respect to FBM ($H \neq \frac{1}{2}$) is not an Itô integral. Additionally, the integrand of the GFOU process is random and the infinite interval integral (6) is needed for its stationarity. We apply two approaches of the integral in Section 2.2 in order to cope with these problems.

**Proposition 3.1** Let $B^H := \{B^H_t\}_{t \in \mathbb{R}}$ be a FBM with $H \in (0, 1)$ and $\xi := \{\xi_t\}_{t \in \mathbb{R}}$ be an independent two-sided Lévy process. Assume that $\{\xi_t\}_{t \in \mathbb{R}}$ has bounded $p$-variation for $0 < p < \infty$. Then the integral $\int_0^1 e^{\xi_s - dB^H_s}$, $0 < t < \infty$ exists in the path-wise Riemann-Stieltjes sense if $\frac{1}{p} + H > 1$. Furthermore with $q > \frac{1}{p}$ and $C_{p,q} = \sum_{n \geq 1} n^{-(p^{-1} + q^{-1})}$ we have

$$\left| \int_0^t e^{\xi_s - dB^H_s} \right| \leq C_{p,q} \left( \sup_{s \in [0,t]} e^{\xi_s} \right) (v_p(\xi))^{1/p} (v_q(B^H))^{1/q} \quad P - a.s.$$  

The proof of Proposition 3.1 is obvious from the continuity of the exponential function and we omit it.

**Remark 3.1** If $p = 1$ in Proposition 3.1, $\xi$ has bounded variation and we can define path-wise integrals for all $B^H$ with $H \in (0, 1)$. On the other hand if $H > \frac{1}{2}$, we can define the path-wise integral for any integrands $\xi$ since $v_p(\xi) < \infty$ for $p > 2$.

The reason why we need the path-wise definition besides the $L^2(\Omega)^p$-approach is that we want to define the integral for $H \in (0, \frac{1}{2})$, which we could not obtain in the $L^2(\Omega)$-approach. It also gives several useful tools easily, such as integration by parts or chain rule for analyzing stochastic differential equations in Section 3.

**Example 3.1** Set $\xi := \xi^\alpha$ be an $\alpha$-stable Lévy motion with index $\alpha$ defined in Subsection 2.1. Then under assumption in 3, $p$-variation of $\alpha$-stable Lévy processes and the assumption $\frac{1}{\alpha} + H > 1$, the integral $e^{-\xi^\alpha} \int_0^t e^{\xi^\alpha - dB^H_u}$ exists in the path-wise sense. Hence the GFOU process is well-defined. Both $\xi^\alpha$ and $B^H$ have infinite variation and the former is known to be extremely heavy tailed process.
Recall that we write $Y_t = Y_t(\omega_1, \omega_2)$ to emphasis that it is a function from probability space \([0,1] \times \Omega\) and that we write $\overline{Y}_t$ if initial value $Y_0$ satisfies (6). Define approximating step functions of the integrand $e^{-\xi_t + \xi_u -}$ of $\overline{Y}_t$ as

$$f_{t,n}(u; \omega_1) := \sum_{i=1}^{n} f_i(u; \omega_1) \mathbb{1}_{\{u_{i-1} < u \leq u_i\}},$$

where

$$f_i(u; \omega_1) := e^{-\xi_t(\omega_1) + \xi_{u_{i-1}}(\omega_1)}.$$

Here $u_i$’s are points in $[-N_n, t]$ such that $-N_n = u_0 < u_1 < \cdots < u_n = t$ and as $n \to \infty$ max $(u_i - u_{i-1}) \downarrow 0$ and $N_n \uparrow \infty$. By integrating $f_{t,n}$ with respect to $B_t^H$, we also define approximating sequence as

$$Z_{i,n}(\omega_1, \omega_2) := \int_{-\infty}^{t} f_{t,n}(u; \omega_1) dB_u^H(\omega_2)$$

$$= \sum_{i=1}^{n} f_i(u; \omega_1) (B_{u_i}^H(\omega_2) - B_{u_{i-1}}^H(\omega_2)).$$

In the following theorem we define the integral $\int_{-\infty}^{t} e^{-\xi_t - \xi_u -} dB_u^H$ as the limit in probability of the $Z_{i,n}^n$ as $n \to \infty$. The reason why we need this approach is that with only path-wise definitions we find difficulty to treat improper integrals. For the existence of the improper integral we should consider long time ($t \to \infty$) behavior of both $\{B_t^H\}_{t \in \mathbb{R}}$ and $\{\xi_t\}_{t \in \mathbb{R}}$ path-wisely, which seems to be not an easy task. Additionally, this approach is well-matched with the analysis of the second order behavior.

**Theorem 3.1** Let $\{B_t^H\}_{t \in \mathbb{R}}$ be a FBM with $H \in (\frac{1}{2}, 1)$ and $\{\xi_t\}_{t \in \mathbb{R}}$ be an independent two-sided Lévy process. If $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$, then for each $t \geq 0$ $Z_t^n$ given in (12) converges in probability to a limit defined by $\overline{Y}_t$ and which does not depend on the sequence $u^n$. If further $E[e^{-2\xi_1}] < 1$, then $Z_t^n$ converges to $\overline{Y}_t$ in $L^2(\Omega)$ and it follows that

$$E[\overline{Y}_t] = E \left[ \int_{-\infty}^{t} e^{-(\xi_t - \xi_u -)} dB_u^H \right] = 0$$

and for $0 < s \leq t$,

$$E[\overline{Y}_s \overline{Y}_t] = \int_{-\infty}^{s} \int_{-\infty}^{t} E \xi [e^{-\xi_t + \xi_u - \xi_t + \xi_u -} | u - v |^{2H-2}] dv du. \quad (12)$$

**Remark 3.2** (a) For $H \in (0, \frac{1}{2})$ we could not validate the existence of the improper integral in $\overline{Y}_t$.

(b) Under the only condition that $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$, we cannot prove the $L^2(\Omega)$ convergence.

(c) The condition $E[e^{-2\xi_1}] < 1$ implies $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$, which is shown in Proposition 4.1 of Lindner and Maller (2003).

**Proof of Theorem 3.1**

We check that for each $\omega_1$ the integrand $e^{-\xi_t(\omega_1) + \xi_u(\omega_1)} \in |\Lambda|^H$ given in (11). Since $\xi_t(\omega_1)$ is constant, we drop it and only show $e^{\xi_u(\omega_1)} \in |\Lambda|^H$. The function $e^{\xi_u(\omega_1)}$ is càglâde and bounded.
on any finite interval. Due to Lemma 2.1 and to the symmetry of two-sided Lévy processes there exists $T(\omega_1) < 0$ such that for all $u \leq T(\omega_1)$, $\xi_u < \delta u$ where $\delta$ is some positive constant. Then for each $\omega_1$,

$$\int_{-\infty}^{t} \int_{-\infty}^{t} e^{\xi_u - (\omega_1) + \xi_v - (\omega_1)} |u - v|^{2H-2} dudv < \infty$$

is obvious. Hence we can utilize $L^2(\Omega_2)$-integral theory in Section 2.2.2. Namely, for each $t \geq 0$ and for each fixed $\omega_1$, $Z^n_t(\omega_1, \cdot)$ converges in $L^2(\Omega_2, P_2)$. Moreover $Z^n_t$ converges in probability on $(\Omega, P)$ for each $t \geq 0$ since sequence $Z^n_t$ satisfies the Cauchy criterion, as seen by

$$
\lim_{n,m \to \infty} P (|Z^n_t - Z^m_t| > \epsilon) = \lim_{n,m \to \infty} \int_{\Omega_1} \int_{\Omega_2} 1_{(|Z^n_t - Z^m_t| > \epsilon)} dP_2 dP_1 = \int_{\Omega_1} \lim_{n,m \to \infty} P_2 (|Z^n_t(\omega_1) - Z^m_t(\omega_1)| > \epsilon) dP_1 = 0.
$$

The limit is called $\overline{Y}_t$ and it is $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable for each $t$.

Now, with $E[e^{-2\xi_1}] < 1$, we prove the $L^2(\Omega)$-convergence. We have $E[e^{-\xi_1}] < 1$ as well, hence

$$\theta_1 := - \log E[e^{-\xi_1}] > 0 \quad \text{and} \quad \theta_2 := - \log E[e^{-2\xi_1}] > 0.$$

Then using the covariance of the FBM, we have

$$E[(Z^n_t)^2] = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\xi_i + \xi_{u_{i-1}} + \xi_{u_{i-1}}}(B^H_{u_i} - B^H_{u_{i-1}})(B^H_{u_j} - B^H_{u_{j-1}}) \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ E \left[ e^{-2\xi_i + \xi_{u_{i-1}} + \xi_{u_{i-1}}} \right] \frac{1}{2} \left( - |u^n_i - u^n_j|^{2H} + |u^n_{i-1} - u^n_{j-1}|^{2H} \right) \right\}$$

$$= c_H \int_{-\infty}^{t} \int_{-\infty}^{t} \Gamma^n_t(u,v) |u - v|^{2H-2} dv dw, \quad (13)$$

where

$$\Gamma^n_t(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ 1_{u^n_{i-1} < u \leq u^n_{i-1}, u^n_{i-1} < v \leq u^n_{j-1}, i \geq j} e^{-\theta_2(t-u^n_{i-1})-\theta_1(u^n_{i-1}-u^n_{j-1})} \right.$$  

$$+ 1_{u^n_{i-1} < u \leq u^n_{i-1}, u^n_{i-1} < v \leq u^n_{j-1}, i < j} e^{-\theta_2(t-u^n_{i-1})-\theta_1(u^n_{i-1}-u^n_{j-1})} \left\},$$

which obviously converges point-wise to

$$1_{\{u \geq v\} e^{-\theta_2(t-u)-\theta_1(u-v)}} + 1_{\{u < v\} e^{-\theta_2(t-v)-\theta_1(v-u)}}.$$
Since $|\Gamma_t(v,w)| < M'$ for some $M' > 0$ uniformly in $n$, $t \leq 0$, (13) is bounded by $c_H M't^H$. Furthermore according to usual Fubini's theorem, it also follows that

\[
E \left[ (\mathbf{Y}_t)^2 \right] = \int_{\Omega_1} \int_{\Omega_2} (\mathbf{Y}_t(\omega_1, \omega_2))^2 dP_2 dP_1
\]

\[
= c_H e_{\xi} \left[ \int_{-\infty}^{\xi_t} \int_{-\infty}^{\xi_u} e^{-2\xi_t+\xi_u-\xi_v} |u-v|^{2H-2} dudv \right]
\]

\[
= 2c_H \int_{-\infty}^{\xi_t} \int_{-\infty}^{\xi_u} e^{-\theta_2(t-u)-\theta_1(v-u)} 1_{\{u \geq v\}} |u-v|^{2H-2} dudv < \infty.
\]

Observe that this integral is finite. Accordingly $E[(Z^n_t)^2] 
= E \left[ (\mathbf{Y}_t)^2 \right]$ as $n \to \infty$. Now we can apply Theorem 4.5.4 of Chung (2001) to $Z^n_t$ and obtain the $L^2(\Omega)$-convergence. In consequence $E[\mathbf{Y}_s \mathbf{Y}_t]$ turns out to be finite and equation (12) follows from Fubini's theorem. Finally $E[Z^n_t] = 0$, $n \in \mathbb{N}$ implies $E[\mathbf{Y}_t] = 0$. Hence the proof is complete.

The process $\{\mathbf{Y}_t\}_{t \in \mathbb{R}}$ obtained in Theorem 3.1 is the GFOU process with initial value $\mathbf{Y}_0$. We close this section with the following concluding Remarks.

Remark 3.3 (a) In both Proposition 3.1 and Theorem 3.1 $\xi$ is independent of $B^H$ and we have

\[
e^{-\xi_t} \int_a^b e^{\xi_u-dB_u^H} = \int_a^b e^{\xi_t+\xi_u-dB_u^H}, \quad -\infty \leq a < b < \infty.
\]

This is not allowed in usual theory of stochastic integrals related to semimartingale (Protter (2004)) since $\{\xi_t\}_{t \in \mathbb{R}}$ is not adapted.

(b) From Theorem 3.1 $\lim_{t \to -\infty} \xi_t \xrightarrow{a.s.} +\infty$ implies $|Y_0| = \left| \int_{-\infty}^0 e^{\xi_s} dB_s^H \right| < \infty \ a.s. \ \Omega$. Therefore together with Proposition 3.1 we can also treat $\{\mathbf{Y}_t\}_{t \geq 0} = \{Y_t + Y_0\}_{t \geq 0}$ path-wisely.

(c) We should mention that Erickson and Maller (2005) [p.81] gave another idea for improper integrals of a function of Lévy processes with respect to FBM: $\int_0^\infty g(\xi_t)dB_t^H$. Investigation of their idea is also our next concern.

4 Stationarity and Second order behavior of GFOU processes

Here we investigate the strict stationarity and the second order behavior of the GFOU process $\mathbf{Y} := \{\mathbf{Y}_t\}_{t \geq 0}$. Since we could not validate the existence of $\mathbf{Y}$ with Hurst parameter $H \in (0, \frac{1}{2})$ and since our main concern in this paper is the long memory case, we confine our results to the case $H \in (\frac{1}{2}, 1)$ throughout this section.

The stationarity of $\mathbf{Y}$ is as follows.

Proposition 4.1 If $\lim_{t \to -\infty} \xi_t \xrightarrow{a.s.} +\infty$, then $\mathbf{Y}_t$ exists for all $t$ and the process $\mathbf{Y} := \{\mathbf{Y}_t\}_{t \geq 0}$ is strictly stationary.

Proof of Proposition 4.1

Let $0 \leq t_1 < t_2 < \ldots < t_m < \infty$, $m \in \mathbb{N}$ and $h > 0$. We use the sequence $Z^n_t$ given in (12). Since
both $\{\xi_t\}_{t \in \mathbb{R}}$ and $\{B^H_t\}_{t \in \mathbb{R}}$ have stationary increments, does the pair $\{(\xi_t, B^H_t)\}_{t \in \mathbb{R}}$ as well because of independence. Thus,

$$Z^n_{t_i} = \sum_{j=1}^{n} e^{-\xi_{t} + \xi_{j-1}} (B^H_{u^j_{t_i} - B^H_{u^j_{t_i-1}}}), \quad -N_n = u^1_t < \cdots < u^n_t = t_i$$

$$= \sum_{j=1}^{n} e^{-\xi_{t+h} + \xi_{u^n_{t_i+h}}} (B^H_{u^n_{t_i+h} - B^H_{u^n_{t_i+h-1}}})$$

$$= Z^n_{t_i}, \quad 1 \leq i \leq m,$$

simultaneously in $i$ and we have

$$(Z^n_{t_1}, \ldots, Z^n_{t_m}) \overset{d}{=} (Z^n_{t_1+h}, \ldots, Z^n_{t_m+h}).$$

Then by virtue of Theorem 3.1 as $n \to \infty$, $(Z^n_{t_1}, \ldots, Z^n_{t_m})$ converges in probability to $(Y_{t_1}, \ldots, Y_{t_n})$ and $(Z^n_{t_1+h}, \ldots, Z^n_{t_m+h})$ converges in probability to $(Y_{t_1+h}, \ldots, Y_{t_n+h})$. This yields

$$(Y_{t_1}, \ldots, Y_{t_n}) \overset{d}{=} (Y_{t_1+h}, \ldots, Y_{t_n+h})$$

and the conclusion holds. \hfill \Box

**Remark 4.1** (a) The logic in the proof works even in the case $H \in (0, \frac{1}{2})$. If the assumptions of Proposition 3.1 are satisfied and if the integral $Y_0 = \int_{-\infty}^{0} e^{\xi_s} d\xi_s^H$ with $H \in (0, \frac{1}{2})$ exists, then $Y$ with $H \in (0, \frac{1}{2})$ is defined and is strictly stationary.

(b) In connection with the GOU process $\{V\}_{t \geq 0}$ given in (1). Proposition 4.1 corresponds to Theorem 3.1 of [Carmona et al. (2001)](Carmona2001), where $\xi$ and $\eta$ are independent, and under conditions of a.s. convergence of the integral $\int_{0}^{\infty} e^{\xi_s} d\eta_s$ and $\lim_{t \to \infty} \xi_t \overset{a.s.}{=} +\infty$ the stationary version exists and equals in distribution to $V_{\infty}$.

Next we investigate the second order behavior of $\{\xi_t\}$ and derive the auto-covariance function explicitly. What should be remarked is that while the auto-covariance function of the GOU process $V$ given in (1) decreases exponentially (Theorem 4.2 of [Lindner and Maller (2003)](Lindner2003)), that of the FOU process $\{X^H_t\}_{t \geq 0}$ given in (1) decays like a power function (Theorem 2.4 and Corollary 2.5 in [Cheridito et al. (2003)](Cheridito2003)). Since $Y$ is regarded as a version of GOU processes and FOU processes, this investigation is interesting. We utilize results in Theorem 3.1 and obtain Theorem 4.1 and Corollary 4.1 below. Note that even the existence of $Y$ and the equation (12) are obtained several difficulties still lay in calculating the auto-covariance function. The integrand in (12) is regarded as exponential moment of of 4 dependent random variable, i.e. $E\xi[e^{-\xi_t - \xi_s - \xi_v - \xi_u}]$ and dependencies of these variables are different in the order of $s, u, t$ and $v$. We also require that after $E\xi[e^{-\xi_t - \xi_s - \xi_v - \xi_u}]$ is calculated the double integral in (12) has a suitable representation for our purpose. In the proofs of Theorem 4.1 and Corollary 4.1 we will see how to get over these difficulties. Proofs of Theorem 4.1 and Corollary 4.1 are given in Appendix A since they require a lot of technical and tedious calculations.

**Theorem 4.1** Let $\{B^H_t\}_{t \in \mathbb{R}}$ be a FBM with $H \in (\frac{1}{2}, 1)$ and $\{\xi_t\}_{t \in \mathbb{R}}$ be an independent two-sided Lévy process. Suppose that $E[e^{-2\xi_t}] < 1$. Then the stationary version $Y := \{Y_t\}_{t \geq 0}$ exists and for
any $s > 0, t \geq 0$, we have
\[
\text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) = \text{Cov}(\overline{Y}_0, \overline{Y}_s)
\]
\[
= c_H \left( \frac{2e^{-\theta_1 s}}{\theta_2 \theta_1^{2H-1}} \Gamma(2H - 1) - \frac{e^{-\theta_1 s}}{2\theta_1^{2H}} \Gamma(2H - 1) + \frac{e^{-\theta_1 s}}{\theta_2 \theta_1^{2H-1}} F_1(2H - 1, 1; \theta_1 s) + \frac{e^{2\theta_1 s}}{2\theta_2^{2H}} \Gamma(2H - 1, \theta_1 s) \right)
\]
\[
= c_H \left( \frac{2e^{-\theta_1 s}}{\theta_2 \theta_1^{2H-1}} \Gamma(2H - 1) - \frac{e^{-\theta_1 s}}{2\theta_1^{2H}} \Gamma(2H - 1) + \frac{s^{2H-1}}{2\theta_1 (2H - 1)} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} (2H - k) \left( (\theta_1 s)^{-(n+1)} - (-\theta_1 s)^{-(n+1)} \gamma(n + 1, \theta_1 s) \right) \right),
\]
where $\theta_1 := -\log(E[e^{-\xi}]) > 0$ and $\theta_2 := -\log(E[e^{-2\xi}]) > 0$. Here $\gamma(\cdot, \cdot)$ and $\Gamma(\cdot, \cdot)$ are incomplete gamma functions in 8.350 of Gradshteyn and Ryzhik (2000) and $F_1(\cdot, \cdot, \cdot)$ is the confluent hyper-geometric function in 9.210 of Gradshteyn and Ryzhik (2000).

Note that since $\gamma(n + 1, s\theta_1) \to \Gamma(n + 1) = n!$ as $s \to \infty$, we have
\[
\text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) = H \sum_{n=1}^{\infty} \prod_{k=1}^{2n-1} (2H - k) \theta_1^{-2n} s^{2H-2n} + O(e^{-\theta_1 s}) = O(s^{2H-2}).
\]

This conclude that $\overline{Y}$ with $H \in (\frac{1}{2}, 1)$ is a long memory process. While we obtained $\text{Cov}(\overline{Y}_t, \overline{Y}_{t+s})$ using special functions in Theorem 1.1 it is mainly for numerical purpose since for such functions useful softwares are available.

Next we investigate long time dependence of $\{Y_t\}_{t \geq 0}$ with the initial value $Y_0 := X \in L^2(\Omega)$ where $X$ is independent of $\{\xi_t\}_{t \geq 0}$ and $\{B^H_t\}_{t \geq 0}$.

**Corollary 4.1** Let $Y := \{Y_t\}_{t \geq 0}, H \in (\frac{1}{2}, 1)$ be a GFOU process with the initial value $X \in L^2(\Omega)$, where $X$ is independent of $\xi := \{\xi_t\}_{t \geq 0}$ and $B^H := \{B^H_t\}_{t \geq 0}$. Then for fixed $t \geq 0$ as $s \to \infty$.
\[
\text{Cov}(Y_t, Y_{t+s}) = H \sum_{n=1}^{\infty} \prod_{k=1}^{2n-1} (2H - k) \theta_1^{-2n} \left( s^{2H-2n} - e^{-\theta_1 t (t + s)} \right)^{2H-2n} + O(s^{2H-2N-2}).
\]

We see what happens to the second order behavior of the process $Y$ if $\xi$ in $Y$ is replaced with $B^H$ which is the same process as the variable of integration. Although we expect long memory this does not hold. Note that we need only the probability space $(\Omega_2, \mathcal{F}_2, P_2)$ here. For $H \in (\frac{1}{2}, 1)$ and the initial random variable $X \in L^2(\Omega_2)$ independent of $\{B^H_t\}_{t \in \mathbb{R}}$, define
\[
W_t = e^{-B^H_t}(X + \int_0^t e^{B^H_u} dB^H_u).
\]
To analyze $\{W_t\}_{t \geq 0}$ we use the path-wise integral theory (see Subsection 2.3.1). Let $f$ be continuous differentiable and $F(x) = F(0) + \int_0^x f(y)dy$. Then with $H \in (\frac{1}{2}, 1)$ it follows that
\[
F(B^H_t) - F(B^H_0) = \int_0^t f(B^H_u)dB^H_u \ a.s.
\]
By setting $f = e^t$ in above we obtain

$$W_t = 1 + e^{-BH}(X - 1).$$

**Proposition 4.2** Let $H \in (\frac{1}{2}, 1)$ and $t, s \geq 0$. Define $M_1 := (E[X] - 1)^2$ and $M_2 := E[(X - 1)^2]$. The process $\{W_t\}_{t \geq 0}$ has the following auto-covariance and correlation functions.

$$\text{Cov}(W_t, W_{t+s}) = e^{\frac{1}{2}(2H + (t+s)^2H)} \left\{ M_2 e^{\frac{1}{2}(2H + (t+s)^2H - s^2H)} - M_1 \right\}$$

$$= O \left( e^{\frac{1}{2}((t+s)^2H + s^2H - 1)} \right) \text{ as } s \to \infty.$$

$$\text{Corr}(W_t, W_{t+s}) = \frac{M_2 e^{\frac{1}{2}(2H - s^2H)} - M_1 e^{-\frac{1}{2}(t+s)^2H}}{\sqrt{M_2 e^{tH} - M_1 e^{-(t+s)^2H}}}$$

$$= O(e^{-\frac{1}{2}s^H}) \text{ as } s \to \infty.$$

We also consider the drift added process

$$\hat{W}_t = e^{-(B_t^H + at)} \left( X + \int_0^t e^{B_u^H + au} d(B_u^H + au) \right),$$

where $a > 0$ and $X \in L^2(\Omega_2)$ is independent of $\{B_t^H\}_{t \geq 0}$. Even if a drift is added, usual path-wise integral works and

$$\hat{W}_t = 1 + (X - 1)e^{-(B_t^H + at)}$$

holds. Then the auto-covariance and correlation functions of $\{\hat{W}_t\}_{t \geq 0}$ are calculated is a similar manner and become

$$\text{Cov}(\hat{W}_t, \hat{W}_{t+s}) = e^{-a(2t+s)} \text{Cov}(W_t, W_{t+s}),$$

$$\text{Corr}(\hat{W}_t, \hat{W}_{t+s}) = \text{Corr}(W_t, W_{t+s}).$$

Thus our conclusion here is that even if a drift is added it does not have long memory.

## 5 Stochastic differential equation related with GFOU processes

We analyze a stochastic differential equation of which a solution is given by the GFOU process. Let $U := \{U_t\}_{t \geq 0}$ be a Lévy process with generating triplet $(a_U, \nu_U, \gamma_U)$. Assume that the Lévy measure $\nu_U$ has no mass on $(-\infty, -1]$. The Doléans-Dade exponential of $U_t$ is written as $E(U_t) = e^{-\xi_t}$ where

$$\xi_t = -U_t + \frac{a_t}{2}t - \sum_{0 < s \leq t} \left( \log(1 + \Delta U_s) - \Delta U_s \right).$$

See Section 2.2 of [Erickson and Maller (2005)](https://example.com). Here $\xi_t$ is the Lévy processes.
Proposition 5.1  Under the assumption in Proposition 3.1, GFOU \( \{Y_t\}_{t \geq 0} \) with the initial value \( Y_0 \in L^1(\Omega) \) satisfies the stochastic differential equation:
\[
dY_t = Y_t \, dU_t + dB_t^H,
\]
where \( \mathcal{E}(U_t) = e^{-\xi_t} \).

Proof of Proposition 5.1
Since the condition of Theorem 2.1 is satisfied, the integral \( \int_0^t B_s^H \, d\xi_s \) also exists in the Riemann-Stieltjes path-wise sense. We use the integration by parts formula to \( Y_t \) and obtain
\[
Y_t = e^{-\xi_t} \left( Y_0 - \int_0^t B_s^H \, d\xi_s \right) + B_t^H.
\]

If we put \( Q_t := Y_t - B_t^H \), the equation above becomes
\[
Q_t = e^{-\xi_t} \left( Q_0 - \int_0^t B_s^H \, d\xi_s \right),
\]
where \( Q_0 = Y_0 \). Since \( \{e^{\xi_s}\}_{s \geq 0} \) is a semimartingale and \( \{B_s^H\}_{s \geq 0} \) is continuous and adapted, the process \( \{Q_s\}_{s \geq 0} \) is also semimartingale. We set \( R_t := e^{-\xi_t} \) and \( S_t := Q_0 - \int_0^t B_s^H \, d\xi_s \). Then the integration by parts formula for semimartingales (e.g. Corollary 2, II of Protter (2004)) yields
\[
Q_t - Q_0 = R_t S_t - R_0 S_0
= \int_{0+}^t R_s \, dS_s + \int_{0+}^t S_s \, dR_s + [R, S]_t - R_0 S_0
= - \int_{0+}^t e^{-\xi_s} B_s \, d\xi_s - \int_{0+}^t Q_s e^{-\xi_s} d\xi_s - \int_{0+}^t B_s^H d[e^{-\xi}, e^\xi]_s
= \int_{0+}^t Q_s \, dU_s - \int_{0+}^t B_s^H \left( e^{-\xi_s} \, d\xi_s - d[e^{-\xi}, e^\xi]_s \right).
\]

Observe the relation between \( e^{-\xi_t} \) and \( U_t \);
\[
1 = e^{\xi_s} e^{-\xi_s}
= \int_{0+}^t e^{\xi_s} \, d\xi_s + \int_{0+}^t e^{-\xi_s} \, d\xi_s + [e^\xi, e^{-\xi}]_t
= \int_{0+}^t dU_s + \int_{0+}^t e^{-\xi_s} \, d\xi_s + [e^\xi, e^{-\xi}]_t.
\]

Using this we obtain
\[
Q_t - Q_0 = \int_{0+}^t (Q_s + B_s^H) \, dU_s,
\]
which is equivalent to
\[
Y_t - Y_0 = \int_{0+}^t Y_s \, dU_s + B_t^H.
\]
The proof is now complete. \( \square \)
Remark 5.1 The Lévy measure of $\{\xi_t\}_{t \geq 0}$ is obtained from that of $\{U_t\}_{t \geq 0}$:
\[
\nu_\xi((x, \infty)) = \nu_U((\infty, e^{-x} - 1)) \quad \text{and} \quad \nu_\xi(\mathbb{R} \setminus \{0\}) = \nu_U((e^x - 1, \infty)).
\]
See again Section 2.2 of [Erickson and Maller (2005)]. Hence if $\nu_T$ is concretely given, using criterion of $p$-variation in Section 2.1, we can check the condition of Proposition 3.1.

The following technical Lemma is not difficult but useful for the existence of $\{Y_t\}_{t \geq 0}$ which is directly constructed from the stochastic differential equation (14). The proof is only a calculation and we omit it.

Lemma 5.1 Assume that $\{U_t\}_{t \geq 0}$ is a Lévy process and that $\{\xi_t\}_{t \geq 0}$ satisfies $\mathcal{E}(U_t) = e^{-\xi_t}$. Then for $0 < \delta < 2$, convergence and divergence of
\[
\int_{|x| < 1} |x|^{\delta} \nu_\xi(dx) \quad \text{and} \quad \int_{|x| < 1} |x|^{\delta} \nu_U(dx)
\]
are equivalent.

Example 5.1 As an example we consider the stochastic differential equation (14), where $U_t$ is given by an $\alpha$-stable Lévy motion $\xi_t^\alpha$ (see Section 2.1). From remark above the Lévy measure $\nu_\xi$ is given by
\[
\nu_\xi(dx) = \begin{cases} 
  c_2 (1 - e^{-x})^{1-\alpha} e^{-x} dx & \text{on } (0, \infty) \\
  c_1 (e^{-x} - 1)^{1-\alpha} e^{-x} dx & \text{on } (-\infty, 0).
\end{cases}
\]
Observe that
\[
\nu_\xi(dx) \sim |x|^{-1-\alpha} dx \quad \text{as } |x| \downarrow 0
\]
and hence variation property of $\xi_t$ is the same as that of $U_t = \xi_t^\alpha$. As a result $v_p(\xi)$ is finite if $p > \alpha$.

A Proofs of Section 4

Proof of Proposition 4.1

From the stationary version $\overline{Y}$ is definable. By virtue of the stationarity of $\overline{Y}$ and Fubini’s theorem, we have
\[
\begin{align*}
\text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) & = \text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) \\
& = c_H \int_{-\infty}^0 \int_{-\infty}^0 E_\xi[e^{-\xi_s - \xi_u - \xi_v}] |u-v|^{2H-2} du dv \\
& \quad + c_H \int_{-\infty}^0 \int_0^s E_\xi[e^{-\xi_s - \xi_u - \xi_v}] |u-v|^{2H-2} du dv \\
& := I + II.
\end{align*}
\]
First we consider the integral $I$. The independent increments property of $\{\xi_t\}_{t \in \mathbb{R}}$ gives
\[
\begin{align*}
E[e^{-\xi_s - \xi_u - \xi_v}] & = E[e^{-\xi_s}e^{-\xi_u}e^{-\xi_v}] \\
& = E[1_{\{u \geq v\}} e^{-\xi_u}e^{-\xi_v} - 2(\xi_0 - \xi_u - (\xi_u - \xi_v))] + E[1_{\{u < v\}} e^{-\xi_u}e^{-\xi_v} - 2(\xi_0 - \xi_u - (\xi_v - \xi_u))] \\
& = 1_{\{u \geq v\}} E[e^{-\xi_u}e^{-\xi_v}] - E[e^{2(\xi_s - \xi_u - \xi_v)}] E[e^{-\xi_u - \xi_v}] + E[e^{2(\xi_s - \xi_u - \xi_v)}] E[e^{-\xi_u - \xi_v}] \\
& + 1_{\{u < v\}} E[e^{-\xi_v}e^{-\xi_u}] - E[e^{2(\xi_s - \xi_u - \xi_v)}] E[e^{-\xi_u - \xi_v}] + E[e^{2(\xi_s - \xi_u - \xi_v)}] E[e^{-\xi_u - \xi_v}]
\end{align*}
\]
The integrand $E[e^{-(\xi_s - \xi_u - \xi_v - \xi_{-v})}]$ is symmetric with respect to $u$ and $v$, and hence
\[
I = 2c_He^{-\theta_1s} \int_{-\infty}^{0} \int_{-\infty}^{0} 1_{\{u \geq v\}} e^{\theta_1u+\theta_1(v-u)}|u-v|^{2H-2}dudv.
\]

Then further calculation shows
\[
I = 2c_He^{-\theta_1s} \int_{-\infty}^{0} \int_{-\infty}^{0} 1_{\{u \geq v\}} e^{\theta_2u+\theta_1(v-u)}|u-v|^{2H-2}dudv
\]
(By change of variables; $x = u - v$)
\[
= 2c_He^{-\theta_1s} \int_{-\infty}^{0} e^{\theta_2u}du \int_{0}^{\infty} e^{-\theta_1x}x^{2H-2}dx
\]
\[
= 2c_He^{-\theta_1s} \frac{\theta_2\theta_1^{2H-1}}{\Gamma(2H-1)}\Gamma(2H-1).
\]

Next we consider II. A similar conclusion as above gives
\[
E[e^{-(\xi_s - \xi_v - (\xi_0 - \xi_u))}] = E[e^{-(\xi_s - \xi_u - (\xi_0 - \xi_v))}]
\]
\[
= e^{-\theta_1(s-v)+\theta_1(u+v)}
\]
\[
= e^{-\theta_1s+\theta_1(u+v)}
\]
and we have
\[
II = c_He^{-\theta_1s} \int_{-\infty}^{0} \int_{-\infty}^{0} (u-v)^{2H-2}dudv
\]
(By change of variables; $x = v - u$)
\[
= c_He^{-\theta_1s} \int_{-\infty}^{0} \int_{-\infty}^{0} 1_{\{-x < u < s-x\}} e^{2\theta_1u+\theta_1x}x^{2H-2}dx
\]
\[
= c_He^{-\theta_1s} \left\{ \int_{0}^{s} \left( \frac{1}{2\theta_1} - \frac{1}{2\theta_1} e^{-2\theta_1x} \right) e^{\theta_1x}x^{2H-2}dx 
+ \int_{s}^{\infty} \left( \frac{1}{2\theta_1} e^{2\theta_1(s-x)} - \frac{1}{2\theta_1} e^{-2\theta_1x} \right) e^{\theta_1x}x^{2H-2}dx \right\}
\]
\[
= c_He^{-\theta_1s} \left\{ \int_{0}^{s} \frac{1}{2\theta_1} e^{\theta_1x}x^{2H-2}dx - \frac{1}{2\theta_1} \int_{0}^{\infty} e^{-\theta_1x}x^{2H-2}dx + \frac{e^{2\theta_1s}}{2\theta_1} \int_{s}^{\infty} e^{-\theta_1x}x^{2H-2}dx \right\}
\]
\[
= c_He^{-\theta_1s} \left\{ \frac{s^{2H+1}}{2\theta_1(2H - 1)} \text{F}_1(2H - 1, 2H; \theta_1s) - \frac{1}{2\theta_1^{2H}} \Gamma(2H - 1) + \frac{e^{2\theta_1s}}{2\theta_1^{2H}} \Gamma(2H - 1, \theta_1s) \right\}
\]
\[
= 16
\]
Combining I and II, we obtain the first assertion.

The series representation of the incomplete Gamma function, i.e.

\[ \Gamma(\alpha, x) = x^{\alpha-1} e^{-x} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{x^n} (\alpha - 1) \cdots (\alpha - n) \right) \]
gives

\[ \Gamma(2H - 1, \theta_1 s) = \frac{s^{2H-1}}{2\theta_1 (H - 1)} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} (2H - k)(\theta_1 s)^{-(n+1)} \]  \hspace{1cm} (20)

We apply the binomial expansion

\[(1 - u/s)^{2H-1} = \sum_{n=0}^{\infty} \frac{(2H - 2)(2H - 3) \cdots (2H - n - 1)}{n!} \left(-\frac{u}{s}\right)^n, \quad (0 < u < s)\]
to the representation

\[ \frac{e^{-\theta_1 s} s^{2H-1}}{2\theta_1 (H - 1)} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} (2H - k)(\theta_1 s)^{-(n+1)} \]

Then we exchange the infinite sum and integral by usual Fubini’s theorem and obtain

\[ \frac{e^{-\theta_1 s} s^{2H-1}}{2\theta_1 (H - 1)} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} (2H - k)(\theta_1 s)^{-(n+1)} \]

Then we exchange the infinite sum and integral by usual Fubini’s theorem and obtain

\[ \frac{e^{-\theta_1 s} s^{2H-1}}{2\theta_1 (H - 1)} \sum_{n=0}^{\infty} \prod_{k=1}^{n+1} (2H - k)(\theta_1 s)^{-(n+1)} \]

Thus substituting these expansions 20 and 21 in the previous representation of covariance we obtain the result.

\[ \square \]

**Proof of Corollary 4.1**

Since \( X \) is independent it follows that

\[ E[Y_t]E[Y_{t+s}] = (E[X])^2 e^{-\theta_1 (2t+s)}. \]  \hspace{1cm} (22)

We divide \( E[Y_t, Y_{t+s}] \) into piece as follows.

\[ E[Y_t Y_{t+s}] = E \left\{ X e^{-\xi_t} + \int_0^t e^{-(\xi_t - \xi_u)} dB_u \right\} \left\{ X e^{-\xi_{t+s}} + \int_0^{t+s} e^{-(\xi_{t+s} - \xi_v)} dB_v \right\} \]

\[ = E[X^2 e^{-(\xi_t + \xi_{t+s})}] + E \left[ X e^{-\xi_{t+s}} + \int_0^{t+s} e^{-(\xi_{t+s} - \xi_v)} dB_v \right] \]

\[ + E \left[ X e^{-\xi_{t+s}} \int_0^t e^{-(\xi_t - \xi_u)} dB_u \right] + E \left[ \int_0^t \int_0^{t+s} e^{-(\xi_t - \xi_{t+s} + \xi_v - \xi_u)} dB_u dB_v \right]. \]
The first term is calculated as
\[
E[X^2e^{-(\xi_t+\xi_{t+s})}] = E[X^2]E[e^{-(\xi_t+\xi_{t+s}+2\xi_t)}]
= E[X^2]E[e^{-\xi_t}]E[e^{-2\xi_t}]
= E[X^2]e^{-\theta_1 s-\theta_2 t}.
\] (23)

From Theorem 4.1, the second and the fourth terms are 0. We only need the last term, which is calculated as
\[
\text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) - E\left[\int_{-\infty}^{0} e^{-(\xi_t-\xi_{u-})} dB_H^u \int_{t}^{t+s} e^{-(\xi_{t+s}-\xi_{u-})} dB_H^v\right]
\]
\[
- E\left[\int_{-\infty}^{0} e^{-(\xi_t-\xi_{u-})} dB_H^u \int_{t}^{t+s} e^{-(\xi_{t+s}-\xi_{u-})} dB_H^v\right]
\]
\[
- E\left[\int_{0}^{t} e^{-(\xi_t-\xi_{u-})} dB_H^u \int_{-\infty}^{0} e^{-(\xi_{t+s}-\xi_{u-})} dB_H^v\right]
\]
\[
= \text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) - \{e^{-\theta_1 t}\text{Cov}(\overline{Y}_0, \overline{Y}_{t+s}) - e^{-\theta_1 (t+s)}\text{Cov}(\overline{Y}_0, \overline{Y}_t)\}
\]
\[
- e^{-\theta_1 s}E[e^{-\xi_t} \overline{Y}_0 \overline{Y}_t] - e^{-\theta_1 s}E[e^{-\xi_t} \overline{Y}_0 \overline{Y}_0 t]
\] (24)

where \(Y_t^0\) is \(Y_t\) with initial value 0.

By adding up (22), (23) and (24), we have
\[
\text{Cov}(Y_t, Y_{t+s}) = \text{Cov}(\overline{Y}_t, \overline{Y}_{t+s}) - e^{-\theta_1 t}\text{Cov}(\overline{Y}_0, \overline{Y}_{t+s})
\]
\[
+ e^{-\theta_1 s}\left(E[X^2]e^{-\theta_1 t} - (E[X])^2 e^{-2\theta_1 t} + e^{-\theta_1 t}\text{Cov}(\overline{Y}_0, \overline{Y}_t)\right)
\]
\[
- E[e^{-\xi_t} \overline{Y}_0 \overline{Y}_t] - E[e^{-\xi_t} \overline{Y}_0 \overline{Y}_0 t]
\]

Hence via Theorem 4.1 the conclusion follows. \(\square\)

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