The stable property of Newton slopes for general Witt towers

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Abstract

We consider the Newton polygon of general Artin-Schrier-Witt tower of curves defined by \( f(x) \in W(F_q[x]) \) with bounded degree at all coordinates, and generalise the result produced by Davis, D.Wan and L.Xiao in 2013. In fact, we proved that the slopes of Newton polygon are unions of arithmetic progressions if the conductor is large enough, and when the conductor increases by 1, the slopes simply scale by \( \frac{1}{p} \).

1 Introduction

Let \( p \) be a rational prime number, \( q = p^a \), where \( a > 0 \) is an integer. Let \( F_q \) be the field of \( q \) elements. For every \( i \geq 0 \), let \( f_i(x) = \sum_{u=0}^{d_i} a_i u x^u \in F_q[x] \) be a polynomial. Assume that \( d_0 > 0 \), and if \( d_i > 0 \), then \( a_i, d_i \neq 0 \). Consider the ring of Witt vectors \( W(F_q[x]) \) over \( F_q[x] \). For any natural number \( j \), we define the map \( \iota_j : F_q[x] \to W(F_q[x]) \) as \( \iota_j(\sum_{u=0}^{d} a_u x^u) = \sum_{u=0}^{d} (0, \ldots, 0, a_u x^u, 0, \ldots) \), where the terms \( a_u x^u \) are sent to the \( (j+1) \)-th coordinates. Now, we define \( f(x) = \sum_{i=0}^{\infty} \iota_i(f_i(x)) \in W(F_q[x]) \). This infinite sum is valid because every coordinate is determined by finite terms.

For any commutative ring \( R \), we can consider the ring of Witt vectors over it, written as \( W(R) \). And, for any positive integer \( m \), we have \( W_m(R) \), the ring of truncated Witt vectors of \( m \) coordinates. There is an obvious truncation map \( \tau_m : W(R) \to W_m(R) \), which sends Witt vector \( (r_1,r_2,\ldots) \) to \( (r_1, r_2, \ldots, r_m) \). Now, let \( R = F_q[x] \). We define \( f^{(m)}(x) = \tau_m(f(x)) \in W_m(F_q[x]) \).

Now, the Artin-Schreier-Witt tower of curves is a sequence of affine curves defined by the following equations

\[
C_m : \frac{y^F_m - y_m}{y_m} = f^{(m)}(x)
\]

for any \( m \), where \( y_m = (y_m^{(1)}, y_m^{(2)}, \ldots, y_m^{(m)}) \) is a Witt vector of \( m \) coordinates, and \( .^F \) means raising each Witt coordinate to its \( p \)-th power. It’s a tower of smooth affine curves \( \ldots \to C_m \to C_{m-1} \to \ldots \to C_0 = A^1_{F_q} \). The inverse limit
$C_\infty$ is defined by the equation $y^F - y = f(x)$, where $y = (y_1, y_2, \ldots)$ is a Witt vector with infinite coordinates. We can build the function fields of $C_m, C_\infty$ over $\mathbb{F}_q$, and the Galois group of curves is defined to be the Galois group of the corresponding function fields. Using the isomorphism

$$W(\mathbb{F}_p) \cong Gal(C_\infty/C_0)$$

$\xi \mapsto \sigma_\xi : y \mapsto y + \xi,$

it’s clear that $Gal(C_\infty/C_0) \cong W(\mathbb{F}_p) \cong \mathbb{Z}_p$. The later isomorphism is defined as $(b_0, b_1, \ldots, b_m, \ldots) \mapsto \hat{b}_0 + \hat{b}_1 p + \ldots + \hat{b}_m p^m + \ldots$, where $\hat{\cdot}$ is the Teichmüller lifting from $\mathbb{F}_p$ to $\mathbb{Z}_p$, $\hat{\cdot}$ also denote the general Teichmüller lifting from $\mathbb{F}_p$ to $\mathbb{Z}$.

Therefore, this tower is a $\mathbb{Z}_p$-tower of curves. Adding the infinite point to each curve, we get a tower of projective curves. The unique ramification point is the infinite point $\infty$.

Next, we review the Weil’s conjectures (certainly, they were already proved).

The zeta function of the affine curve $C_m$ is

$$\zeta(C_m, s) = \exp(2 \pi \sqrt{-1} \sum_{k \geq 1} \frac{s^k}{k}) = \frac{P(C_m, s)}{1 - q^s},$$

where $P(C_m, s)$ is a polynomial of degree $2g(C_m)$, $g(C_m)$ being the genus of $C_m$. $P(C_m, s) = \prod (1 - \alpha_is)$, where the roots $\alpha_i$ are all algebraic, with absolute values at infinite places $q^2$, and the $p'$-adic ($p' \neq p$) absolute values 1. The remaining problem is to obtain the $p$-adic valuations, or equivalently, the $q$-adic Newton polygon of $\zeta(C_m, s)$.

It’s well known that there is a decomposition of L-functions, such that

$$\zeta(C_m, s) = \zeta(C_0, s) \prod_{1 \leq m_\psi \leq m} L(\psi, s) = \frac{\prod_{1 \leq m_\psi \leq m} L(\psi, s)}{1 - q^s},$$

in which $\psi : \mathbb{Z} \to Gal(C_m/C_0) \cong \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}_p^\times$ is an additive character with finite kernel $p^{m_\psi}\mathbb{Z}$, and $L(\psi, s)$ is defined to be

$$\prod_{x \in |A_{\mathbb{F}_q}^1|} \frac{1}{1 - \psi(Frob_x)s^{deg(x)}},$$

where $| \cdot |$ means sets of closed points of schemes, and $Frob_x \in Gal(C_{m_\psi}/C_0)$ is the Frobenius element associated to $x$ such that $Frob_x \alpha \equiv \alpha^{q^{deg(x)}} \pmod{x}$, noting that $x$ can be viewed as an irreducible polynomial over $\mathbb{F}_q$ of one variable, $deg(x)$ is the degree of the polynomial if $x \neq 0$, and $deg(x) = 1$ if $x = 0$. Thus, the problem of determining $q$-adic Newton slope of $\zeta(C_m, s)$ reduces to that of $L(\psi, s)$.
In [3], the authors considered the simplest case \( f(x) = \sum_{i=0}^{d} (a_i x^i, 0, 0, \ldots), \ p \nmid d \), and proved that for \( m \) large enough, the \( q \)-adic Newton slopes of \( L(\psi, s) \) form a union of some arithmetic progressions (Theorem 1.2 of [3]). Now, I will prove a generalization of this result. In fact, we consider the general case \( f(x) = \sum_{i=0}^{\infty} \iota_i(f_i(x)), \) assuming the degrees \( d_i \) have an upper bound.

Basically, we will follow the method in [3], with necessary variations. First, we give an upper bound and a lower bound of the \( q \)-adic Newton polygon. Then, an analog of section 3 of [3] gives the result. Our main result is

**Theorem 1** Let \( m_0 = \tilde{m} + \lceil \log_p \left( \frac{1}{p} + \frac{a(d_{i1} - 1)^2}{\delta_{i1}} \right) \rceil, \) where \( \tilde{m} \) is defined such that \( \tilde{m} - 1 \) the first \( i \) such that \( d_i = \max_{j \geq 0} \{d_j\} \), and \( \delta_{i1} = \max_{0 \leq i \leq m - 1} \{p^{\tilde{m} - i - 1} d_i\} \). Let \( 0 < \gamma_1, \ldots, \gamma_{\delta_{i1} p^{m_0} - \tilde{m} - 1} < 1 \) denote the slopes of the \( q \)-adic Newton polygon of \( L(\psi, s) \) for a finite character \( \psi : \mathbb{Z}/p^{m_0} \mathbb{Z} \to \mathbb{C}_p^\times \). Then, for every finite character \( \psi : \mathbb{Z}/p^{m_0} \mathbb{Z} \to \mathbb{C}_p^\times \) such that \( m_\psi \geq m_0 \), the \( q \)-adic Newton polygon of \( L(s, \psi) \) has slopes

\[
\bigcup_{i=0}^{p^{m_0 - m} - 1} \left\{ \frac{\gamma_1 + i}{p^{m_0 - m} - 1}, \frac{\gamma_{\delta_{i1} p^{m_0} - m - 1} + i}{p^{m_0 - m}} \right\} = \{0\}.
\]

In other words, the \( q \)-adic Newton slopes of \( L(\psi, s) \) form a union of \( \delta_{i1} p^{m_0 - \tilde{m}} \) arithmetic progressions, with increment \( p^{m_0 - m_\psi} \).

## 2 Upper bound of \( q \)-adic Newton polygon of \( L \)-functions

In this and the next section, we will assume that \( \psi \) is an additive character, and \( m_\psi = m \) if there is no confusion.

We consider the \( L \)-function defined on the torus. Let

\[
L^*(\psi, s) = \prod_{x \in [G_m(\mathbb{F}_\psi)]} \frac{1}{1 - \psi(Frob_x)s^{deg(x)}} = (1 - \psi(Frob_0)s)L(\psi, s).
\]

Liu Chunlei and Wei Dasheng [1] had proved that for non-degenerated \( f(x) \), \( L^*(\psi, s) \) was a polynomial of \( s \) which had degree

\[
d = \max_{0 \leq i \leq m - 1} \{p^{m - i - 1} d_i\},
\]

and the endpoint of the \( q \)-adic Newton polygon is \( (d, \frac{d - 1}{2}) \). Non-degenerated means

\[
\sum_{\substack{d = p^{m - i - 1} d_i \leq p^{m - i - 1} d_i \neq 0}} d_i d_i p^{m - i - 1} = 0.
\]

Note that \( (1 - \psi(Frob_0)s) \) is a factor, so we know that \( q \)-adic Newton polygon of \( L^*(\psi, s) \) has an initial segment of slope 0. Therefore, we have an upper bound of \( L^*(\psi, s) \): the polygonal line connecting \((0, 0), (1, 0)\) and \((d, \frac{d - 1}{2})\).
3 Exponential sums and $T$-adic L-functions

Now we try to get an upper bound of the Newton polygon, using the method of transforming through $T$-adic L-functions. First, we consider the L-function $L^*(\psi, s)$ above. By [1], we have

$$L^*(\psi, s) = \exp \left( \sum_{k=1}^{\infty} S^*(\psi, k) \frac{s^k}{k} \right),$$

where

$$S^*(\psi, k) = \sum_{x \in \mathbb{F}_{q^k}^*} \psi(\text{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f^m(x))).$$

Note that, in the above expression, $\psi$ is explained as a character $W_m(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{C}_p^\times$. From now on, we will write $\mathbb{Z}_{p^n}$ to denote the integral ring of the unique unramified extension of $\mathbb{Z}_p$ of degree $n$.

Now using the isomorphism given in [1]:

$$\nu_{m, k} : W_m(\mathbb{F}_{q^k}) \rightarrow \mathbb{Z}_p[\mu_{q^k-1}]/(p^m)$$

$$(a_0, \ldots, a_{m-1}) \mapsto \sum_{i=0}^{m-1} a_i^p \cdot p^i \left( \mod p^m \right),$$

where the ♯ means Teichmüller lifting, then, for $x \in \mathbb{F}_{q^k}$,

$$\nu_{m, k}(f^m(x)) \equiv \sum_{i=0}^{m-1} p^i \sum_{u=0}^{d_i} \hat{a}_{iu}^{p^i-1} \cdot x^{up^i-1} \left( \mod p^m \right).$$

Therefore,

$$\nu_{m, k}(\text{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f^m(x))) = \text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(\hat{f}(x)),$$

where

$$\hat{f}(x) = \sum_{i=0}^{m-1} p^i \sum_{u=0}^{d_i} \hat{a}_{iu} x^{u}.$$ 

Write $\sum_{u=0}^{d} b_u x^u = \sum_{u=0}^{d} \hat{b}_u x^u$, for any $\sum_{u=0}^{d} b_u x^u \in \mathbb{F}_q[x]$. Then, we can rewrite that

$$\hat{f}(x) = \sum_{i=0}^{m-1} p^i \hat{f}_i(x).$$

We also define that

$$\hat{f}(x) = \sum_{i=0}^{\infty} p^i \hat{f}_i(x).$$
\( \hat{f}(x) \) is a polynomial because \( \max\{d_i\} \) exist.

It’s routine to check that \( \nu_{m,k}|W_{m}(\mathbb{F}_p) \) is exactly the canonical isomorphism \( W_{m}(\mathbb{F}_p) \cong \mathbb{Z}/p^m\mathbb{Z} \), noting that \( \mathbb{Z}/p^m\mathbb{Z} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p \). Therefore, \( \psi \) can be viewed as a character defined on \( \mathbb{Z}_p \). Note that \( \psi(p^i\alpha) = 1 \) for any \( i \geq m, \alpha \in \mathbb{Z}_p \), so that for any \( m \), we have

\[
\psi(Tr_{W_{m}(\mathbb{F}_p)}/W_{m}(\mathbb{F}_p))(f^{(m)}(x)) = \psi(Tr_{\mathbb{Q}_p/\mathbb{Q}_p}(\hat{f}(\hat{x}))) = \psi(Tr_{\mathbb{Q}_p/\mathbb{Q}_p}(\hat{f}(\hat{x}))).
\]

As a consequence, we have

\[
S^*(\psi, k) = \sum_{x \in \mu_{q^k-1}} \psi(Tr_{\mathbb{Q}_p/\mathbb{Q}_p}(\hat{f}(x))).
\]

For the purpose of generating a lower bound of the \( q \)-adic Newton polygons, we introduce the \( T \)-adic L-function. The basic idea is an analog of [2]. For a positive integer \( k \), we define the \( T \)-adic exponential sum of \( f \) over \( \mathbb{F}_{q^k} \) as the sum

\[
S_f(T, k) = \sum_{x \in \mu_{q^k-1}} (1 + T)^{Tr_{\mathbb{Q}_p/\mathbb{Q}_p}(\hat{f}(x))} = \sum_{x \in \mu_{q^k-1}} \prod_{i=0}^{\infty} (1 + T)^{\nu_{q^k,i}(x)}.
\]

and the \( T \)-adic L-function is

\[
L_f(T, s) = \exp\left( \sum_{k=1}^{\infty} S_f(T, k) \frac{s^k}{k} \right) \in 1 + s\mathbb{Z}_p[[T]][[s]].
\]

The relation between the \( T \)-adic L-functions and the original ones is quite easy to describe. In fact, let \( \pi_{\psi} = \psi(1) - 1 \), we have

\[
L_f(T, s)|_{T=\pi_{\psi}} = L^*(\psi, s).
\]

### 4 Dwork’s trace formula

For the convenience of calculation, we do a variable change. Consider the Artin-Hasse exponential series \( E(X) = \exp(\sum_{i=1}^{\infty} \frac{X^p^i}{p^i}) \). It has some standard properties.

First, if seen as a power series in \( \mathbb{Q}_p[[X]] \), \( E(X) \) is a power series with \( p \)-adic integer coefficients, i.e., it’s in \( \mathbb{Z}_p[[X]] \). In fact, \( E(X) \in 1 + X + X^2\mathbb{Z}_p[[X]] \). Next, we describe another property of \( E(X) \). Let \( t = E(X) - 1 \), then it’s routine to prove that \( X \in t + t^2\mathbb{Z}_p[[t]] \).

Now, let \( E(\pi) = 1 + T \), and \( E(\pi_i) = (1 + T)^p^i \), for any \( i \geq 0 \). We have \( \pi_0 = \pi \). By the properties of Artin-Hasse exponential series given above, we have \( T \in \pi + \pi^2\mathbb{Z}_p[[\pi]] \), and \( \pi \in T + T^2\mathbb{Z}_p[[T]] \). Let \( T_i = (1 + T)^p^i - 1 \), then \( \pi_i \in T_i + T_i^2\mathbb{Z}_p[[T_i]] \). But, \( T_i = E(\pi)p^i - 1 \), therefore, \( \pi_i \in p^i\pi + \pi^2\mathbb{Z}_p[[\pi]] \).
For any $h(x) = \sum_{u=0}^{d'} h_u x^u \in \mathbb{F}_q[x]$, and any indeterminate $\eta$, define

$$E_h(x, \eta) = \prod_{u=0}^{d'} E(\eta h_u x^u).$$

Fix a generator $\sigma$ of $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$, and let it act on $\mathbb{Q}_q[[\pi, x]]$, such that $\sigma(\pi) = \pi$, $\sigma(x) = x$. We have the Dwork’s splitting formula (Lemma 4.3 of [2]):

**Lemma 1** If $x \in \mu_{q^k-1}$, $h(x) = \sum_{u=0}^{d'} h_u x^u \in \mathbb{F}_q[x]$, $\eta$ is an indeterminate, then

$$E(\eta) \sum_{j=0}^{ak-1} x^{p^j} = \exp \left( \sum_{i=0}^{\infty} \frac{\eta^{p^i}}{p^i} \sum_{j=0}^{ak-1} x^{p^j} \right).$$

But $\sum_{j=0}^{ak-1} x^{p^j} = \sum_{j=0}^{ak-1} x^{p^{j+i}}$, therefore,

$$E(\eta) \sum_{j=0}^{ak-1} x^{p^j} = \exp \left( \sum_{i=0}^{\infty} \frac{\eta^{p^i}}{p^i} \sum_{j=0}^{ak-1} x^{p^{j+i}} \right) = \prod_{j=0}^{ak-1} E(\eta x^{p^j}).$$

This is the commutative lemma.

Now we prove the theorem. We have

$$E(\eta) \sum_{j=0}^{ak-1} x^{p^j} = \prod_{u=0}^{d'} \sum_{j=0}^{ak-1} x^{p^j} \prod_{j=0}^{ak-1} E(\eta x^{p^j}) = \prod_{u=0}^{d'} \prod_{j=0}^{ak-1} E(\eta h_u x^u) = \prod_{j=0}^{ak-1} E_h(x, \eta).$$

The theorem is proved.

Now, we treat the exponential sum. Define $E_f(x) = \prod_{i=0}^{\infty} E_{f_i}(x, \pi_i)$. First, we need to prove that this infinite product is convergent. Note that $\pi_i \in ((1 + T)^p - 1) + ((1 + T)p - 1)^2 \mathbb{Z}_p[[T]^p - 1]]$. But $(1 + T)p - 1 = \sum_{j=1}^{p^i} (p^i)_j T^j$, so

\[\text{6}\]
in the expansion of $\pi_i$ as a power series of $T$, the coefficient of $T^j$ term is a $\mathbb{Z}_p$-linear combination of the numbers $(p^i_j) \cdots (p^i_{j_1})$, where $j_1 + \ldots + j_i = j$. Suppose $p^{i'}_j \leq j < p^{i'+1} \leq p^i$. Then, by the expression of binomial numbers, $p$-order of $(p^i_j) \cdots (p^i_{j_1})$ is at least $i - i'$. Therefore, the $p$-order of the coefficient of $T^j$ term of $\pi_i$ is also at least $i - i'$. Denote $\max\{d_i\} = D$, and define $a_{iu} = 0$ if $u > d_i$, then we have $E_f(x) = \prod_{u=0}^{\infty} \prod_{i=0}^{\infty} E(\pi_i \alpha_{iu} x^u)$. Expand the internal infinite product, we find that the coefficient of $x^u$ has the form $\sum_{v_{i1} + \ldots + v_{ir} = u} c_{i1,\ldots,i_r} v_{i1}^1 \cdots v_{ir}^{i_r}$, where $c$'s are coefficients in $\mathbb{Z}_q$. By the result involving $p$-order of $\pi_i$ proved above, this sum is convergent. Thus, the definition of $E_f(x)$ by infinite product is valid.

Now, we have

\[
(1 + T)^{Tr_{Q_k/q_p}(f(x))} = \prod_{i=0}^{\infty} (1 + T)^{p^i Tr_{Q_k/q_p}(f_i(x))} = \prod_{i=0}^{\infty} E(\pi_i) Tr_{Q_k/q_p}(f_i(x)) = \prod_{i=0}^{\infty} \prod_{j=0}^{a_{k-1}^i} E_f^j(x^{p^i}).
\]

As a conclusion, we have

**Theorem 2** If $x \in \mu_{q^k-1}$, then

\[
(1 + T)^{Tr_{Q_k/q_p}(f(x))} = \prod_{j=0}^{a_{k-1}^i} E_f^j(x^{p^i}).
\]

Consider the ring $\mathbb{Z}_q[[\pi]][[x]]$. It is a linear space over $\mathbb{Z}_q[[\pi]]$. Choose the monomial basis $\{1, x, x^2, \ldots x^u, \ldots\}$. Define the map $\varphi : \mathbb{Z}_q[[\pi]][[x]] \rightarrow \mathbb{Z}_q[[\pi]][[x]]$ such that $\varphi_p(\sum_{u=0}^{\infty} b_u x^u) = \sum_{u=0}^{\infty} b_{pu} x^u$, where $b_u \in \mathbb{Z}_q[[\pi]]$ for all $u \in \mathbb{N}$. Then, let $\varphi = \sigma^{-1} \circ \varphi_p \circ E_f(x)$, where $E_f(x)$ means multiplying by $E_f(x)$. Write that $E_f(x) = \sum_{u=0}^{\infty} \alpha_u x^u$, where $\alpha_u \in \mathbb{Z}_q[[\pi]]$. Define $\alpha_u = 0$ if $u < 0$. Now, we have

\[
\varphi(x^u) = \sum_{u=0}^{\infty} \sigma^{-1}(\alpha_{pu-v}) x^u,
\]

therefore, we have the matrix $M = (\sigma^{-1}(\alpha_{pu-v}))_{(u,v)}$ of $\varphi$ over the basis $\{1, x, x^2, \ldots\}$. This matrix has only finitely many nonzero element in every row. Therefore, the powers of the matrix $M$ make sense.
Now, we consider the $ak$-th power of $\varphi$. In fact, noting that $\varphi_p \circ G(x^p) = G(x) \circ \varphi_p$ for any $G(x) \in \mathbb{Z}_q[[\pi]][[x]]$, we have
\[
\varphi^r(g) = \varphi^{r-1}(\varphi_p(E_f^{r-1}(x)g^{r-1})) = \varphi^{r-2}(\varphi_p^2(E_f^{r-2}(x^p)E_f^{r-2}(x)g^{r-2})) = \ldots = \varphi_p^{r}(g^{r-1} \prod_{j=0}^{r-1} E_f^{r-j}(x^p)),
\]
for $r \geq 0$, $g \in \mathbb{Z}_q[[\pi]][[x]]$. Let $r = ak$, we have
\[
\varphi^{ak}(g) = \varphi_p^{ak}(g \prod_{j=0}^{ak-1} E_f^j(x^p)),
\]
therefore, we have
\[
\varphi^{ak} = \varphi_p^{ak} \circ \prod_{j=0}^{ak-1} E_f^j(x^p),
\]
where $\prod_{j=0}^{ak-1} E_f^j(x^p)$ means multiplying by $\prod_{j=0}^{ak-1} E_f^j(x^p)$. This map has the matrix $M^{ak}$ under the basis $\{1, x, x^2, \ldots\}$.

We also write that
\[
E_f(x, \pi_i) = \sum_{u=0}^{\infty} \alpha^{(i)}_u x^u,
\]
and
\[
\prod_{j=0}^{ak-1} E_f^j(x^p) = \sum_{u=0}^{\infty} \beta_{u,k} x^u,
\]
where $\alpha^{(i)}_u, \beta_{u,k} \in \mathbb{Z}_q[[\pi]]$. Then, for the $\pi$-order of $\alpha_u, \alpha^{(i)}_u$ and $\beta_{u,k}$, we have the following lemma, which will be used to prove the existence of some traces and determinants below.

**Lemma 2** We repeat that $D = \max_{i \geq 0} \{d_i\}$. We have
\[
\text{ord}_\pi(\alpha^{(i)}_u) \geq \frac{u}{D}, \text{ord}_\pi(\alpha_u) \geq \frac{u}{D},
\]
and
\[
\text{ord}_\pi(\beta_{u,k}) \geq \frac{u}{p^{ak-1}D}.
\]

Proof. Note that $E_f(x, \pi_i) = \prod_{u=0}^{d_i} E(\pi_i \tilde{a}_{i,u} x^u)$. For the expanded form of each factor $E(\pi_i \tilde{a}_{i,u} x^u)$, the $\pi$-order of coefficient of $x^v$ ($v = uu'$) exceeds or equals to $\frac{uu'}{D}$. Therefore, the product also has the same property. By the definition $\prod_{j=0}^{ak-1} E_f^j(x^p) = \sum_{u=0}^{\infty} \beta_{u,k} x^u$, we obtain the inequality for $\beta_{u,k}$ in the lemma.

Now, we are ready to do a deformation for the exponential sum $S_f(T, k)$, making it be trace of a matrix.
Theorem 3 (Dwork’s trace formula) The trace $\text{Tr}(M^{ak})$ of the matrix $M^{ak}$ exists, and

$$S_f(T, k) = \sum_{x \in \mu_{q^{k-1}}} \prod_{j=0}^{ak-1} E_f^{x^j}(x^p) = (q^k - 1)\text{Tr}(M^{ak}).$$

Proof. Write $\prod_{j=0}^{ak-1} E_f^{x^j}(x^p) = \sum_{u=0}^{\infty} \beta_{u, k} x^u$, where $\beta_{u, k} \in \mathbb{Z}_q[[\pi]]$, noting that

$$\varphi^{ak} = \varphi_p^{ak} \circ \prod_{j=0}^{ak-1} E_f^{x^j}(x^p),$$

we have

$$\varphi^{ak}(x^v) = \sum_{u=0}^{\infty} \beta_{q^v u, k} x^u.$$

Therefore, the matrix $M^{ak}$ of $\varphi^{ak}$ on the basis $\{1, x, x^2, \ldots\}$ is $(\beta_{q^v u, k}(u, v))$.

By lemma 2, the trace

$$\text{Tr}(M^{ak}) = \sum_{u=0}^{\infty} \beta_{(q^k - 1)u, k}$$

exists. But, $S_f(T, k) = \sum_{x \in \mu_{q^{k-1}}} \prod_{j=0}^{ak-1} E_f^{x^j}(x^p) = \sum_{x \in \mu_{q^{k-1}}} \sum_{u=0}^{\infty} \beta_{u, k} x^u$, so we have

$$S_f(T, k) = \sum_{u=0}^{\infty} \sum_{x \in \mu_{q^{k-1}}} \beta_{u, k} x^u = \sum_{u=0}^{\infty} \sum_{x \in \mu_{q^{k-1}}} x^u \beta_{u, k}.$$

Using the fact that

$$\sum_{x \in \mu_{q^{k-1}}} x^u = \begin{cases} q^k - 1, & (q^k - 1) \mid u, \\ 0, & (q^k - 1) \nmid u, \end{cases}$$

we get $S_f(T, k) = \sum_{u=0}^{\infty} \beta_{(q^k - 1)u, k}$. Therefore, the theorem is proved.

Let

$$C_f(T, s) = \exp(-\sum_{k=1}^{\infty} \frac{1}{q^k - 1} S_f(T, k) \frac{s^k}{k}) = \prod_{j=0}^{\infty} L(T, q^j s).$$

Then,

$$L_f(T, s) = \frac{C_f(T, s)}{C_f(T, qs)}.$$

We have the following simple description of $C_f(T, s)$. 

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Theorem 4 (Analytic trace formula)

\[ C_f(T,s) = \det(I - sM^a), \]

and the function \( C_f(T,s) \) is \( T \)-adic analytic.

Proof. The existence of the determinant \( \det(I - sM^a) \) is derived from the fact (lemma 2) that \( \text{ord}_{\pi}(\beta_{u,k}) \geq \frac{u}{p^a - q^a} \). Using Dwork’s trace formula and the well-known formula

\[ \det(I - sA) = \exp(- \sum_{k=1}^{\infty} \text{Tr}(A^k)s^k), \]

we have

\[ C_f(T,s) = \exp(- \sum_{k=1}^{\infty} \frac{1}{q^k - 1} S_f(T,k) \frac{S^k}{k}) \]

\[ = \exp(- \sum_{k=1}^{\infty} \text{Tr}(M^a)^k \frac{S^k}{k}) = \det(I - sM^a). \]

Then, by the fact that \( \text{ord}_{\pi}(\beta_{u,k}) \geq \frac{u}{p^a - q^a} \) again, \( C_f(T,s) \) is \( T \)-adic analytic.

5 Lower bound of \( (p^\theta, T) \)-adic orders

In [2], lower bound of \( T \)-adic Newton polygon is considered. But in our case, when reducing to \( L^*(\psi, s) \), the contribute of \( p \) in the coefficients cannot be ignored. So, when \( d_i > 0 \) for some \( i > 0 \), we consider the \( (p^\theta, T) \)-adic orders for some positive number \( \theta \) instead.

Consider the \( p \)-th power of Artin-Hasse exponential series. If \( E(Y) = E(X)^p \), then \( E(Y) \in 1 + XZ_p[[X]] \), so \( E(Y) \in 1 + X^{p^\theta}Z_p[[X]] + pXZ_p[[X]] \). But if \( E(Y) = 1 + Z \), then \( Z \in YZ_p[[Y]] \). Therefore, we have \( Y \in X^{p^\theta}Z_p[[X]] + pXZ_p[[X]] \).

Using this fact, we get the following lemma.

Lemma 3 Notations \( \pi \) and \( \pi_i \) defined as in section 4. Review that \( \tilde{m} \) is defined to be one plus the first \( i \) such that \( d_i = D \). If \( \tilde{m} \geq 2 \), let \( \theta = \frac{1}{p^\theta - 2(p-1)} \). Define the ideal \( R = (\pi) \) when \( \tilde{m} = 1 \), \( R = (p^\theta, \pi) \) when \( \tilde{m} \geq 2 \). Then,

\[ \text{ord}_R(\pi_i) \geq p^i \]

for any \( 0 \leq i \leq \tilde{m} - 1 \), and,

\[ \text{ord}_R(\pi_i) \geq p^{\tilde{m}-1} \]

for any \( i \geq \tilde{m} - 1 \).
Proof. In the $\tilde{m} = 1$ case, the lemma just says $ord_\pi(\pi) \geq 1$. Now we consider $\tilde{m} \geq 2$ case. The inequality is clearly true for $i = 0$, because $\pi_0 = \pi$. Then we use induction. If we have proved that $ord_R(\pi_i) \geq p^i$ for some $0 \leq i < \tilde{m} - 1$, then $\pi_{i+1} \in \pi_i^r Z_\pi[[\pi]] + p\pi Z_\pi[[\pi]]$. We have $ord_R(\pi_j) \geq p^j$, so $ord_R(\pi_j^p) \geq p^{i+1}$, and $ord_R(p\pi_i) = p^{\tilde{m}-2}(p-1) + ord_R(\pi_i) \geq p^i(p-1) + p^i = p^{i+1}$. Hence $ord_R(\pi_{i+1}) \geq p^{i+1}$. The first part of the lemma is proved. When $i \geq \tilde{m} - 1$, we have $ord_R(\pi_{i+1}) \geq ord_R(\pi_i)$. This implies the second part.

Now, we study the $R$-order of $\alpha_u$.

**Theorem 5** Let $\tilde{m}$ and $R$ be defined as in the above lemma. Let $\delta = \max_{i \geq 0} \{ \frac{d_i}{p^i} \}$, then

$$ord_R(\alpha_u) \geq \frac{u}{\delta}$$

for all nonnegative integer $u$.

Proof. Review that $E_{f_1}(x, \pi_i) = \prod_{u=0}^{d_i} E(\pi_i \xi_{iu} x^u)$, we have

$$ord_{\pi_i}(\alpha^{(i)}_u) \geq \frac{u}{d_i}$$

for any $i \geq 0$. By the lemma above, we have $ord_R(\pi_i) \geq p^i$ for $0 \leq i \leq \tilde{m} - 1$, and $ord_R(\pi_i) \geq p^{\tilde{m}-1}$ for $i \geq \tilde{m} - 1$. Therefore,

$$ord_R(\alpha^{(i)}_u) \geq \frac{u}{d_i} \frac{1}{p^i}$$

for $0 \leq i \leq \tilde{m} - 1$, and

$$ord_R(\alpha^{(i)}_u) \geq \frac{u}{\frac{d_i}{p^i}}$$

for $i \geq \tilde{m} - 1$. But $E_f(x) = \prod_{i=0}^{\infty} E_{f_i}(x, \pi_i)$. Hence,

$$ord_R(\alpha_u) \geq \frac{u}{\delta}$$

Now, we produce a lower bound (called Hodge bound) for the $R$-orders, or equivalently $(p^\theta, T)$-orders of $C_f(T, s)$.

**Theorem 6** Expand $C_f(T, s)$ to a power series of $s$. Then, the $R$-order of the coefficient of $s^n$ is greater than or equal to $\frac{a(p-1)n(n-1)}{25}$. 

Proof. Note that $Q_q$ is a linear space of dimension $a$ over $Q_p$, and there exist an integral basis. Leaving $\pi$ and $s$ fixed, we can viewed $Z_\pi[[\pi]][[s]]$ as a module over $Z_\pi[[\pi]][[s]]$. Consider the norm map of $Z_\pi[[\pi]][[s]]$ to $Z_\pi[[\pi]][[s]]$. For $\kappa \in Z_\pi[[\pi]][[s]],$ we define the norm to be the determinant of the multiplying by $\kappa$ map, viewed as a linear map on $Z_\pi[[\pi]][[s]]$ over $Z_\pi[[\pi]][[s]]$. 

Note that $Q_q$ can be obtained by adding a $(p^2 - 1)$-th root of unity to $Q_p$, so there is a normal integral basis of $Q_q$ over $Q_p$, say $\{ \xi_0, ..., \xi_{a-1} \}$. Therefore,
\{\xi_i x^u | 0 \leq i \leq a - 1, u \in \mathbb{Z}^+\} is a basis of the module \(\mathbb{Z}_q[[\pi]][[x]]\) over \(\mathbb{Z}_p[[\pi]]\). Arrange this basis in the lexicographical order \(\{\xi_0, \ldots, \xi_{a-1}, \xi_0 x, \ldots, \xi_{a-1} x, \ldots\}\), define \(N\) to be the matrix of the map \(\varphi\) under this ordered basis. Review that

\[
\varphi(x^v) = \sum_{u=0}^{\infty} \sigma^{-1}(\alpha_{pu-v}) x^u,
\]

we have

\[
\varphi(\xi_j x^v) = \sum_{u=0}^{\infty} \sigma^{-1}(\xi_j \alpha_{pu-v}) x^u.
\]

Let

\[
\sigma^{-1}(\xi_j \alpha_{pu-v}) = \sum_{i=0}^{a-1} \alpha(i,u),(j,v) \xi_i,
\]

where \(\alpha(i,u),(j,v) \in \mathbb{Z}_p[[\pi]]\), we have

\[
\varphi(\xi_j x^v) = \sum_{u=0}^{\infty} \sum_{i=0}^{a-1} \alpha(i,u),(j,v) \xi_i x^u.
\]

Hence, we have \(N = (\alpha(i,u),(j,v))_{0 \leq i,j \leq a-1, u,v \in \mathbb{Z}^+}\). By definition, if \(pu < v\), then \(\alpha(i,u),(j,v) = 0\). Therefore, powers of \(N\) make sense, and the matrix of \(\varphi^r\) is \(N^r\).

Now, we have

\[
\text{Norm}(\det(I - s^a M^a)) = \det(I - s^a N^a) = \prod_{\zeta = 1}^{a} \det(I - \zeta sN),
\]

thus, the \(R\)-orders of \(s\)-coefficients of \(\det(I - s^a M^a)\) coincide with \(\det(I - sN)\).

Note that for any \(\omega \in \mathbb{Z}_q\), \(\omega = \sum_{i=0}^{a-1} \omega_i \xi_i\), if \(\text{ord}_p(\omega) \geq \lambda\), the same will be true for every \(\omega_i\). By the theorem 3 and the equation

\[
\sigma^{-1}(\xi_j \alpha_{pu-v}) = \sum_{i=0}^{a-1} \alpha(i,u),(j,v) \xi_i,
\]

we have

\[
\text{ord}_R(\alpha(i,u),(j,v)) \geq \frac{pu - v}{\delta}.
\]

Hence, the \(R\)-order of coefficient of \(s^a N\) of \(\det(I - sN)\), or \(\det(I - s^a M^a)\), is greater than or equal to \(\frac{a(p-1)n(n-1)}{2^{\delta}}\). But we have \(C_f(T, s) = \det(I - s M^a)\), (theorem 4), so the \(R\)-order of coefficients of \(s^n\) of \(C_f(T, s)\) is greater than or equal to \(\frac{a(p-1)n(n-1)}{2^{\delta}}\). The proof is completed.
6 Periodicity of Newton polygons

In this section, we prove the main theorem. Review that \( \delta = \max_{i \geq 0} \{ d_i p_i \} \). Let \( \bar{m} \) and \( R \) be defined to be the same as the definition in the last section. Choose a character \( \psi_1 : \mathbb{Z} \to \mathbb{Z}/\bar{m}\mathbb{Z} \to \mathbb{C}_p \). Consider \( L^*(\psi_1, s) \). By the facts stated in section 2, the degree of \( L^*(\psi_1, s) \) is
\[ \delta_1 = \max_{0 \leq i \leq \bar{m} - 1} \{ p^{\bar{m} - i - 1} d_i \} = p^{\bar{m} - 1} \delta, \]
and the \( q \)-adic Newton polygon of \( L^*(\psi_1, s) \) has the broken line
\[ \{(0, 0), (1, 0), (\delta_1, \frac{\delta_1 - 1}{2})\} \]
as an upper bound, with the same endpoint \( (\delta_1, \frac{\delta_1 - 1}{2}) \), assuming that \( f(x) \) is non-degenerated at level \( \bar{m} \). We have \( \text{Tr}_{Q_p/Q_p}(x^p) = \text{Tr}_{Q_p/Q_p}(x) \), therefore, we can reduce the degenerated case to the non-degenerated case, with the degree of polynomials not increasing. Hence this upper bound is also valid for degenerated case. This broken line has a segment of length 1, with slope 0, and a segment of length \((\text{horizontal coordinate}) \delta_1 - 1, \text{with slope } \frac{1}{2} \). Now, let
\[ C^*(\psi_1, s) = C_f(T, s)|_{T = \pi_{\psi_1}} = L^*(\psi_1, q^j s). \]

By the upper bound of \( L^*(\psi_1, s) \) above, the \( q \)-adic Newton polygon of \( C^*(\psi_1, s) \) has an upper bound of slopes
\[ 0, \frac{1}{2}, 1, \frac{3}{2}, ..., n - 1, \frac{2n - 1}{2}, ... \]
with multiplicity
\[ 1, \delta_1 - 1, 1, \delta_1 - 1, ..., 1, \delta_1 - 1, ... \]
respectively. Therefore, this upper bound is the broken line
\[ \{(0, 0), (1, 0), (\delta_1, \frac{\delta_1 - 1}{2}), (\delta_1 + 1, \frac{\delta_1 + 1}{2}), ..., \}
\[ (n\delta_1, \frac{n(n\delta_1 - 1)}{2}), (n\delta_1 + 1, \frac{n(n\delta_1 + 1)}{2}), ... \}. \]

Now we consider the \( T \)-adic functions. Let \( C_f(T, s) = 1 + b_{f,1}(T)s + b_{f,2}(T)s^2 + ... + b_{f,n}(T)s^n + ... \), where \( b_{f,n}(T) \in \mathbb{Z}_p[[T]] \) for all \( n \). Theorem \( \Box \) give the inequality
\[ \text{ord}_R(b_{f,n}(T)) \geq \frac{a(p - 1)n(n - 1)}{2\delta}. \]

As \( \text{ord}_T(\pi) = 1 \), \( R \)-order coincides with \( (p^a, T) \)-order exactly. Now, we set \( T = \pi_{\psi_1} = \psi_1(1) - 1 \). It is a standard fact that \( \text{ord}_p(\pi_{\psi_1}) = \frac{1}{p^{\bar{m} - 1}(p - 1)}. \) As a consequence,
\[ \text{ord}_p(b_{f,n}(\pi_{\psi_1})) \geq \frac{a(n - 1)}{2\delta_1}. \]
where \( \lambda \) about the order of coefficients of it. We write

\[
\text{For an integer } k \geq 1, \text{ we find that (2) is also a lower bound of the }
\]

\[
\text{as a lower bound of } q\text{-adic Newton polygon of } C^* (\psi_1, s). \text{ Note that } \frac{n(\delta_1 - 1)}{2} \text{ and } \frac{n(\delta_1 + 1)}{2} \text{, the upper bound and lower bound coincide in segments between horizontal coordinates } n\delta_1 \text{ and } n\delta_1 + 1 \text{ for all } n. \]

Consider an arbitrary character \( \psi : \mathbb{Z}/p^m\mathbb{Z} \rightarrow \mathbb{C}_p^\times \) such that \( m \geq \bar{m} \). By the same method, we find that (2) is also a lower bound of the \( \pi_\psi^{a(p-1)p^{\bar{m}-1}}\text{-adic Newton polygon of } C^* (\psi, s). \)

Now, we are ready to go back to \( T\)-adic functions, and prove some theorems about the order of coefficients of it. We write

\[
b_{f,n}(T) = \sum_{j \geq \lambda_n} b_{f,n}^{(j)} T^j
\]

where \( \lambda_n \) is the smallest number \( j \) such that the coefficient of \( T^j \) in the expansion of \( b_{f,n}(T) \) is nonzero.

**Theorem 7** For an integer \( k \equiv 0, 1 \pmod{\delta_1} \), the \( R\)-order of \( b_{f,k}(T) \) is exactly

\[
\Lambda_k = \frac{a(p-1)k(k-1)}{2\delta_1}, \quad \Lambda_k \text{ is an integer, and } b_{f,k}^{(\Lambda_k)} \in \mathbb{Z}_p.
\]

Proof. We know the \( q\)-adic Newton polygon for \( C^* (\psi_1, s) \) passes through

\[
(n\delta_1, \frac{n(n\delta_1 - 1)}{2}), (n\delta_1 + 1, \frac{n(n\delta_1 + 1)}{2})
\]

for any \( n \). By the lower bound (2), the slope of the preceding segment must be less than or equal to \( n - \frac{1}{\delta_1} \), and the slope of the following segment must be greater than or equal to \( n + \frac{1}{\delta_1} \). Thus, \( (n\delta_1, \frac{n(n\delta_1 - 1)}{2}) \) and \( (n\delta_1 + 1, \frac{n(n\delta_1 + 1)}{2}) \) must be vertices of the \( q\)-adic Newton polygon. Therefore, \( ord_q(b_{f,k}^{(\Lambda_k)}) = \frac{k(k-1)}{2\delta_1} \) for all \( k \equiv 0, 1 \pmod{\delta_1} \). But \( ord_q(p^\theta) = \frac{\theta}{a(p-1)p^{\bar{m}-1}} \) for any case, so we have

\[
ord_q(b_{f,k}^{(\Lambda_k)}) = \frac{k(k-1)}{2\delta_1} \geq \frac{1}{a(p-1)p^{\bar{m}-1}} ord_R(b_{f,k}(T)),
\]

therefore,

\[
ord_R(b_{f,k}(T)) \leq \frac{a(p-1)p^{\bar{m}-1}k(k-1)}{2\delta_1} = \frac{a(p-1)k(k-1)}{2\delta_1}.
\]

By theorem 6 the equality holds. If the equality is attached by a monomial \( b_{f,k}^{(j)} T^j \) such that \( ord_p(b_{f,k}^{(j)}) > 0 \), then \( ord_q(b_{f,k}^{(j)} \pi_\psi^k) > \frac{k(k-1)}{2\delta_1} \). So, there must exist a monomial attaching the equality only by \( T \). The theorem follows.
Theorem 8 Let \( \psi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}_p^\times \) be a character, where \( m \geq \tilde{m} \). Then, the \( \pi_{\psi}^{a(p-1)p^{\tilde{m}-1}} \)-adic Newton polygon of \( C^*(\psi, s) \) contains the line segments connecting \( (n\delta_1, \frac{n(n\delta_1-1)}{2}), (n\delta_1 + 1, \frac{n(n\delta_1+1)}{2}) \) for all \( n \). Therefore, \( \Pi \) is also an upper bound for \( \pi_{\psi}^{a(p-1)p^{\tilde{m}-1}} \)-adic Newton polygon of \( C^*(\psi, s) \).

Proof. Consider \( k \equiv 0, 1 \pmod{\delta_1} \). By theorem 7 we have \( \text{ord}_R(b_{f,k}^{(j)}T^j) \geq \frac{a}{2(p-1)k(k-1)} \) for any \( j \geq \lambda_k \), and equality holds for some \( j \), including \( j = \Lambda_k = \frac{a(p-1)k(k-1)}{2}\delta_1 \). For \( \tilde{m} = 1 \), we have \( \lambda_k = \Lambda_k \), therefore,

\[
\text{ord}_{\pi_{\psi}^{a(p-1)p^{\tilde{m}-1}}}(b_{f,k}(\pi_{\psi})) = \frac{\Lambda_k}{a(p-1)p^{\tilde{m}-1}} = \frac{k(k-1)}{2\delta_1}.
\]

Now we consider the \( \tilde{m} \geq 2 \) case. If there is some \( j \neq \Lambda_k \) such that the equality holds, then \( \text{ord}_q(b_{f,k}^{(j)}T^j) > 0 \). Note that

\[
\text{ord}_R(b_{f,k}^{(j)}T^j) = \frac{a}{\theta} \text{ord}_q(b_{f,k}^{(j)}) + j \geq \Lambda_k,
\]
and

\[
\text{ord}_q(b_{f,k}^{(j)}\pi_{\psi}^j) = \text{ord}_q(b_{f,k}^{(j)}) + \text{ord}_q(\pi_{\psi}^j),
\]

we deduce that

\[
\text{ord}_q(b_{f,k}^{(j)}\pi_{\psi}^j) = \frac{\theta}{a} \text{ord}_R(b_{f,k}^{(j)}T^j) - \left( \frac{\theta}{a} - \text{ord}_q(\pi_{\psi}) \right)j.
\]

Now \( \frac{\theta}{a} = \frac{1}{ap^{m-2}(p-1)} > \text{ord}_q(\pi_{\psi}) = \frac{1}{ap^{m-1}(p-1)} \). For \( j > \Lambda_k \), we have

\[
\text{ord}_q(b_{f,k}^{(j)}\pi_{\psi}^j) = \text{ord}_q(b_{f,k}^{(j)}) + \text{ord}_q(\pi_{\psi})j > \text{ord}_q(\pi_{\psi})\Lambda_k.
\]

For \( \lambda_k \leq j \leq \Lambda_k \), we have

\[
\text{ord}_q(b_{f,k}^{(j)}\pi_{\psi}^j) \geq \frac{\theta}{a} \Lambda_k - \left( \frac{\theta}{a} - \text{ord}_q(\pi_{\psi}) \right)j \geq \Lambda_k \text{ord}_q(\pi_{\psi}),
\]
and all equalities hold only when \( j = \Lambda_k \). Therefore, we have

\[
\text{ord}_{\pi_{\psi}^{a(p-1)p^{\tilde{m}-1}}}(b_{f,k}(\pi_{\psi})) = \frac{1}{a(p-1)p^{\tilde{m}-1}}\Lambda_k = \frac{k(k-1)}{2\delta_1}.
\]

In any case, \( \Pi \) is an upper bound of \( \pi_{\psi}^{a(p-1)p^{\tilde{m}-1}} \)-adic Newton polygon of \( C^*(\psi, s) \), which coincides with the lower bound \( \Pi_2 \) periodically on the line segments connecting \( (n\delta_1, \frac{n(n\delta_1-1)}{2}), (n\delta_1 + 1, \frac{n(n\delta_1+1)}{2}) \) for all \( n \). The theorem follows.

Theorem 9 For a finite character \( \psi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{C}_p^\times \) with \( m_{\psi} \geq \tilde{m} + \log_p(\frac{1}{p} + \frac{a(d_1-1)^2}{s1}) \), the \( \pi_{\psi}^{a(p-1)p^{\tilde{m}-1}} \)-adic Newton polygon of \( C^*(\psi, s) \) is independent of the character \( \psi \).
Proof. We have proved that the $\pi_a^{(p^{-1})p^m-1}$-adic Newton polygon of $C^*(\psi, s)$ has an upper bound (1) and a lower bound (2). These two broken lines coincides periodically, at segments connecting points with horizontal coordinates $n\delta_1$ and $n\delta_1 + 1$. It’s routine to calculate that the difference between these two lines is a period function, having the maximal value not exceeding $\frac{(\delta_1 - 1)^2}{8\delta_1}$. For $n \neq 0, 1 \mod \delta_1$, let $\lambda_n'$ be the first integer $j$ such that $b_j^{(j)} \in \mathbb{Z}_p^\times$. If such $j$ does not exist, define $\lambda_n' = \infty$. By definition, the $\pi_a^{(p^{-1})p^m-1}$-adic Newton polygon of $C^*(\psi, s)$ is the convex hull of points $(n, ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}(\pi_\psi)))$ for all $n \geq 0$. Assume $\bar{m} \geq 2$. By theorem [4] we have

$$ord_R(b_{j,n}^{(j)}T_j) = p_{\bar{m}}^{-2}(p-1)ord_R(b_{j,n}^{(j)}) + j \geq \frac{a(p-1)n(n-1)}{2\delta_1},$$

dividing by $a(p-1)p^m$, the inequality becomes

$$\frac{1}{ap}ord_R(b_{j,n}^{(j)}) + \frac{j}{a(p-1)p^{m-1}} \geq \frac{n(n-1)}{2\delta_1}. \tag{3}$$

Note that the inequality $n_{\psi} \geq \bar{m} + \log_p(\frac{1}{p} + \frac{a(\delta_1 - 1)^2}{8\delta_1})$ is equivalent to $\frac{1}{a}(p^{m_{\psi} - \bar{m} - \frac{1}{p}}) \geq \frac{(\delta_1 - 1)^2}{8\delta_1}$. Now,

$$ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}^{(j)}\pi_\psi) = \frac{ord_p(b_{j,n}^{(j)})}{ord_p(\pi_\psi)a(p-1)p^m} + \frac{j}{a(p-1)p^{m-1}}$$

$$= \frac{p^{m_{\psi} - \bar{m}}}{a} + \frac{j}{a(p-1)p^{m-1}}$$

$$\geq \frac{n(n-1)}{2\delta_1} + \frac{1}{a}(p^{m_{\psi} - \bar{m} - \frac{1}{p}})ord_p(b_{j,n}^{(j)}).$$

Therefore, if $b_{j,n}^{(j)} \notin \mathbb{Z}_p^\times$, then we have

$$ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}^{(j)}\pi_\psi) \geq \frac{n(n-1)}{2\delta_1} + \frac{(\delta_1 - 1)^2}{8\delta_1},$$

greater than or equal to the upper bound (1).

If $\bar{m} = 1$, then we trash the first term of the left side of (3), and repeat the calculation. We also get the same result.

Thus, if the point $(n, \frac{\lambda_n'}{a(p-1)p^{m-1}})$ is lower than the upper bound line (1), then

$$ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}(\pi_\psi)) = \frac{\lambda_n'}{a(p-1)p^{m-1}}.$$

Conversely, if $(n, ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}(\pi_\psi)))$ is lower than (1), then we also have the same equality. Hence the set of points

$$\{(n, ord_{\pi_a^{(p^{-1})p^m-1}}(b_{f,n}(\pi_\psi)))|n \geq 0, and the point is below to (1)\}$$
is fixed, because $\lambda'_n$ is clearly independent of $\psi$. The theorem follows from the convexity.

Now, we prove the main theorem. Let $m_0 = \tilde{m} + \lceil \log_{p}(\frac{1}{p} + \frac{\delta_1 - 1}{801}) \rceil$, and $\psi_0$ is a finite character $\psi_0 : \mathbb{Z}/p^{m_0}\mathbb{Z} \hookrightarrow \mathbb{C}_p^\times$. Let $0 < \gamma_1, \ldots, \gamma_{\delta_1 p^{m_0 - \tilde{m} - 1}} < 1$ denote the slopes of the $q$-adic Newton polygon of the L-function $L(\psi_0, s)$. Then $0, \gamma_1, \ldots, \gamma_{\delta_1 p^{m_0 - \tilde{m} - 1}}$ are the slopes of the $q$-adic Newton polygon of $L^*(\psi_0, s)$, and hence

$$\bigcup_{i \geq 0} \{i, \gamma_1 + i, \ldots, \gamma_{\delta_1 p^{m_0 - \tilde{m} - 1}} + i\}$$

are the slopes of the $q$-adic Newton polygon of $C^*(\psi_0, s)$. Since $\text{ord}_q(\pi_{\psi_0}) = \frac{1}{a(p-1)p^{m_0-\tilde{m}}}$, the $\pi^{a(p-1)p^{m_0-\tilde{m}}}$-adic Newton polygon of $C^*(\psi_0, s)$ is rescaled to

$$\bigcup_{i \geq 0} \{p^{m_0 - \tilde{m}} i, p^{m_0 - \tilde{m}} (\gamma_1 + i), \ldots, p^{m_0 - \tilde{m}} (\gamma_{\delta_1 p^{m_0 - \tilde{m} - 1}} + i)\}. \quad (4)$$

By theorem 9, for any finite character $\psi : \mathbb{Z}/p^{m_\psi}\mathbb{Z} \hookrightarrow \mathbb{C}_p^\times$ such that $m_\psi \geq m_0$, (4) is also the slopes of $\pi^{a(p-1)p^{m_\psi - 1}}$-adic Newton polygon of $C^*(\psi, s)$. By rescaling again, the $q$-adic Newton slopes of $C^*(\psi, s)$ is

$$\bigcup_{i \geq 0} \{\frac{i}{p^{m_\psi - m_0}}, \frac{\gamma_1 + i}{p^{m_\psi - m_0}}, \ldots, \frac{\gamma_{\delta_1 p^{m_\psi - m_0 - 1}} + i}{p^{m_\psi - m_0}}\}.$$

Use the relation

$$L(\psi, s) = \frac{1}{1 - \psi(Frob_0) s} \frac{C^*(\psi, s)}{C^*(\psi, qs)},$$

we find that the $q$-adic Newton polygon of $L(\psi, s)$ has slopes

$$\bigcup_{i=0}^{p^{m_\psi - m_0} - 1} \{\frac{i}{p^{m_\psi - m_0}}, \frac{\gamma_1 + i}{p^{m_\psi - m_0}}, \ldots, \frac{\gamma_{\delta_1 p^{m_\psi - m_0 - 1}} + i}{p^{m_\psi - m_0}}\} - \{0\}.$$

The theorem is proved.

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