THE DISTRIBUTION OF SUPERCONDUCTIVITY
NEAR A MAGNETIC BARRIER

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Abstract. We consider the Ginzburg–Landau functional, defined on a two-
dimensional simply connected domain with smooth boundary, in the situation
when the applied magnetic field is piecewise constant with a jump discontinuity
along a smooth curve. In the regime of large Ginzburg–Landau parameter and
strong magnetic field, we study the concentration of minimizing configurations
along this discontinuity.

1. Introduction

1.1. Motivation. The Ginzburg–Landau theory, introduced in [LG50], is a phe-
nomenological macroscopic model describing the response of a superconducting
sample to an external magnetic field. The phenomenological quantities associated
with a superconductor are the order parameter $\psi$ and the magnetic potential $A$,
where $|\psi|^2$ measures the density of the superconducting Cooper pairs and $\text{curl} A$
represents the induced magnetic field in the sample.

In this paper, the superconducting sample is an infinite cylindrical domain subjected
to a magnetic field with direction parallel to the axis of the cylinder. For this specific
geometry, it is enough to consider the horizontal cross section of the sample, $\Omega \subset \mathbb{R}^2$.
The phenomenological configuration $(\psi, A)$ is then defined on the domain $\Omega$.

The study of the Ginzburg–Landau model in the case of a uniform or a smooth non-
uniform applied magnetic field has been the focus of much attention in literature. We
refer to the two monographs [FH10,SS07] for the uniform magnetic field case. Smooth
magnetic fields are the subject of the papers [Att15a,Att15b,HK15,LP99,PK02]. In
this paper, we focus on the case where the applied magnetic field is a step function,
which is not covered in the aforementioned papers. Such magnetic fields are interesting
because they give rise to edge currents on the interface separating the distinct values
of the magnetic field—the magnetic barrier (see [DHS14,HPRS16,RP00]). Our
configuration is illustrated in Figure 2.

In an earlier contribution [AK16], we explored the influence of a step magnetic
field on the distribution of bulk superconductivity, which highlighted the regime
where an edge current might occur near the magnetic barrier. In this contribution,
we will demonstrate the existence of such a current by providing examples where the
superconductivity concentrates at the interface separating the distinct values of the
magnetic field.

1.2. The functional and the mathematical set-up. We assume that the domain
$\Omega$ is open in $\mathbb{R}^2$, bounded and simply connected. The Ginzburg–Landau (GL) free
energy is given by the functional
\[
E_{\kappa,H}(\psi, A) = \int_{\Omega} \left( \left( \nabla - i \kappa H A \right) \psi \right)^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \, dx + \kappa^2 H^2 \int_{\Omega} |\text{curl} A - B_0|^2 \, dx,
\]
with \( \psi \in H^1(\Omega; \mathbb{C}) \) and \( A = (A_1, A_2) \in H^1(\Omega; \mathbb{R}^2) \). Here, \( \kappa > 0 \) is a large GL parameter, the function \( B_0: \Omega \to [-1, 1] \) is the profile of the applied magnetic field and \( H > 0 \) is the intensity of this applied magnetic field.

The parameter \( \kappa \) depends on the temperature and the type of the material. It is a characteristic scale of the sample that measures the size of vortex cores (which is proportional to \( \kappa^{-1} \)). Vortex cores are narrow regions in the sample, which corresponds to \( \kappa \) being a large parameter. That is the reason behind our analysis of the asymptotic regime \( \kappa \to +\infty \), following many early papers addressing this asymptotic regime (see e.g. [SS07]). We work under the following assumptions on the domain \( \Omega \) and the magnetic field \( B_0 \) (illustrated in Figure 1):

Assumption 1.1.

1. \( \Omega_1 \subset \Omega \) and \( \Omega_2 \subset \Omega \) are two disjoint open sets.
2. \( \Omega_1 \) and \( \Omega_2 \) have a finite number of connected components.
3. \( \partial \Omega_1 \) and \( \partial \Omega_2 \) are piecewise smooth with (possibly) a finite number of corners.
4. \( \Gamma = \partial \Omega_1 \cap \partial \Omega_2 \) is the union of a finite number of smooth curves. We will refer to \( \Gamma \) as the magnetic barrier.
5. \( \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^o \) and \( \partial \Omega \) is smooth.
6. \( \Gamma \cap \partial \Omega \) is either empty or finite.
7. If \( \Gamma \cap \partial \Omega \neq \emptyset \), then \( \Gamma \) intersects \( \partial \Omega \) transversely, i.e. \( \nu_{\partial \Omega} \times \nu_{\Gamma} \neq 0 \), where \( \nu_{\partial \Omega} \) and \( \nu_{\Gamma} \) are respectively the unit normal vectors of \( \partial \Omega \) and \( \Gamma \).
8. \( a \in [-1, 1] \setminus \{0\} \) is a given constant.
9. \( B_0 = \mathbb{1}_{\Omega_1} + a \mathbb{1}_{\Omega_2} \).
The ground state of the superconductor describes its behaviour at equilibrium. It is obtained by minimizing the GL functional in (1.1) with respect to \((\psi, A)\). The corresponding energy is called the ground state energy, denoted by \(E_{\text{g, st}}(\kappa, H)\), where

\[
E_{\text{g, st}}(\kappa, H) = \inf \{ E_{\kappa, H}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \}.
\]

The functional in (1.1) enjoys the property of \textit{gauge invariance}. It does not change under the transformation \((\psi, A) \mapsto (e^{i\varphi} \psi, A + \nabla \varphi)\), for any (say smooth) function \(\varphi : \mathbb{R}^2 \to \mathbb{R}\). It follows that the only physically meaningful quantities are the gauge invariant ones, such as \(|\psi|\) and \(\text{curl } A\). The gauge invariance permits us to restrict the GL functional to the space \(H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) where

\[
H^1_{\text{div}}(\Omega) = \{ A \in H^1(\Omega; \mathbb{R}^2) : \text{div } A = 0 \text{ in } \Omega, \ A \cdot \nu_{\partial \Omega} = 0 \text{ on } \partial \Omega \}.
\]

More precisely, the ground state energy can be written as follows (see [FH10 Appendix D])

\[
E_{\text{g, st}}(\kappa, H) = \inf \{ E_{\kappa, H}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega) \}.
\]

This restriction allows us to make profit from some well-known regularity properties of vector fields in \(H^1_{\text{div}}(\Omega)\) (see [AK16 Appendix B]).

Critical points \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) of \(E_{\kappa, H}\) are weak solutions of the following GL equations:

\[
\begin{cases}
(\nabla - i\kappa H A)^2 \psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\
-\nabla \cdot (\text{curl } A - B_0) = \frac{1}{\pi \rho} \text{Im}(\psi(\nabla - i\kappa H A)\psi) & \text{in } \Omega, \\
\nu \cdot (\nabla - i\kappa H A)\psi = 0 & \text{on } \partial \Omega, \\
\text{curl } A = B_0 & \text{on } \partial \Omega.
\end{cases}
\]

Here,

\[
(\nabla - i\kappa H A)^2 \psi = \Delta \psi - i\kappa H (\text{div } A)\psi - 2i\kappa H A \cdot \nabla \psi - \kappa^2 H^2 |A|^2 \psi
\]

and \(\nabla \perp = (\partial_{x_2}, -\partial_{x_1})\) is the Hodge gradient.

1.3. \textbf{Some earlier results for uniform magnetic fields.} The value of the ground state energy \(E_{\text{g, st}}(\kappa, H)\) depends on \(\kappa\) and \(H\) in a non-trivial fashion. The physical explanation of this is that a superconductor undergoes phase transitions as the intensity of the applied magnetic field varies.

To illustrate the dependence on the intensity of the applied magnetic field, we assume that \(H = b \kappa\), for some fixed parameter \(b > 0\). Such magnetic field strengths are considered in many papers, [AH07, LP99, Pan02, SS03].

Assuming that the applied magnetic field is uniform, which corresponds to taking \(B_0 \equiv 1\) in (1.1), the following scenario takes place. If \(b > \Theta_0^{-1}\), where \(\Theta_0 \approx 0.59\) is a universal constant defined in (2.20) below, the only minimizer of the GL functional (up to change of gauge) is the trivial state \((0, \hat{\Psi})\) where \(\text{curl } \hat{\Psi} = 1\) (see [GP99]). This corresponds in Physics to the destruction of superconductivity when the sample is submitted to a large external magnetic field, and occurs when the intensity \(H\) crosses a specific threshold value, the so-called \textit{third critical field}, denoted by \(H_{C_3}\).

Another well-known critical field to be considered is the \textit{second critical field} \(H_{C_2}\), which is much harder to define. When \(H < H_{C_2}\), then the superconductivity is uniformly distributed in the interior of the sample (see [SS03]). This is the bulk superconductivity regime. When \(H_{C_2} < H < H_{C_3}\), then the surface superconductivity regime occurs: the superconductivity disappears from the interior and is localised in a thin layer near the boundary of the sample (see [AH07, HFPS11, Pan02, CR14]). The transition from surface to bulk superconductivity takes place when \(H\) varies around the critical value \(\kappa\), which we informally take as the definition of \(H_{C_2}\) (see [FKII]). One
more critical field left is $H_{c_1}$. It marks the transition from the pure superconducting phase to the phase with vortices. We refer to [SS07] for its definition.

1.4. Expected behaviour of magnetic steps. Let us return back to the case where the magnetic field is a step function as in Assumption 1.1. At some stage, the expected behaviour of the superconductor in question deviates from the one submitted to a uniform magnetic field. Recently, this case was considered in [AK16] and the following was obtained. Suppose that $H = b\kappa$ and $\kappa$ is large. If $b < 1/|a|$ then the bulk superconductivity persists, if $b > 1/|a|$ then superconductivity decays exponentially in the bulk of $\Omega_1$ and $\Omega_2$, and may nucleate in thin layers near $\Gamma \cup \partial \Omega$ (see Assumption 1.1 and Figure 2). The present contribution affirms the presence of superconductivity in the vicinity of $\Gamma$ when $b$ is greater than, but close to the value $1/|a|$, for some negative values of $a$. The precise statements are given in Theorems 1.5 and 1.7 below.

The aforementioned behaviour of the superconductor in presence of magnetic steps is consistent with the existing literature (for instance see [DHS14, HPRS16, HS15, Iwa85, RP00]). Particularly, the case where $a \in [-1, 0)$ is called the trapping magnetic step (see [HPRS16]), where the discontinuous magnetic field may create supercurrents (snake orbits) flowing along the magnetic barrier ($\Gamma$ in our context). On the other hand, no such snake orbits are formed in the case where $a \in (0, 1)$, which is called the non-trapping magnetic step. However, the approach was generally spectral where some properties of relevant linear models were analysed [HPRS16, HS15, Iwa85, RP00], and no estimates for the non-linear GL energy in (1.1) were established.

This contribution together with [AK16] provide such estimates. Particularly in the case where $a \in [-1, 0)$ and $b > 1/|a|$, Theorems 1.5 and 1.7 below establish global and local asymptotic estimates for the ground state energy $E_{g, st}(\kappa, H)$ and the $L^4$-norm of the minimizing order parameter $\psi$. These theorems assert the nucleation of superconductivity near the magnetic barrier $\Gamma$ (and the surface $\partial \Omega$) when $b$ exceeds the threshold value $1/|a|$.

1.5. Main results. Our results are valid under the following additional assumption.

Assumption 1.2. The parameter $H$ depends on $\kappa$ in the following manner

$$H = b\kappa,$$

(1.5)

where $b$ is a fixed parameter satisfying

$$b > \frac{1}{|a|}, \quad a \in [-1, 1) \setminus \{0\}.$$

Remark 1.3. Even though the case $a \in (0, 1)$ is included in Assumption 1.2, it will not be central in our study (the reader may notice this in the majority of our theorems statements). The reason is that, our main concern is to analyse the interesting phenomenon happening when the bulk superconductivity is only restricted to a narrow neighbourhood of the magnetic barrier $\Gamma$, and this only occurs when the values of the two magnetic fields interacting near $\Gamma$ are of opposite signs, that is when $a \in [-1, 0)$, (see Figure 2). This can be seen through the trivial cases in Section 3.2, and is consistent with the aforementioned literature findings (non-trapping magnetic steps). Moreover, the case $b \leq 1/|a|$ is treated previously in [AK16] and corresponds to the bulk regime.

The statements of the main theorems involve two non-decreasing continuous functions:

$$e_a : [|a|^{-1}, +\infty) \to (-\infty, 0] \quad \text{and} \quad E_{\text{surf}} : [1, +\infty) \to (-\infty, 0],$$
respectively defined in (3.5) and (7.1). The energy $E_{\text{surf}}$ has been studied in many papers \cite{AH07,CR14,Pan02,HFPS11,FK11}. We will refer to $E_{\text{surf}}$ as the surface energy. The function $e_a$ is constructed in this paper, and we will refer to it as the barrier energy.

**Remark 1.4.** It is worthy to mention that $e_a(b)$ vanishes if and only if

- $a \in (0,1)$; or
- $a \in [-1,0)$ and $b \geq 1/\beta_a$, where $\beta_a$ is defined in (2.10) below and satisfies $\beta_a \in (0,|a|)$ (see Theorem 2.8).

The surface energy $E_{\text{surf}}(b)$ vanishes if and only if $b \geq \Theta_0^{-1}$, where $\Theta_0$ is the constant in (2.20).

**Theorem 1.5** (Global asymptotics). For all $a \in [-1,1] \setminus \{0\}$ and $b > 1/|a|$, the ground state energy $E_{\text{g, st}}(\kappa,H)$ in (1.3) satisfies, when $H = b\kappa$,

$$E_{\text{g, st}}(\kappa,H) = E_{a}^{L}(b)\kappa + o(\kappa) \quad (\kappa \to +\infty)$$  

where

$$E_{a}^{L}(b) = b^{-\frac{1}{2}}\left( |\Gamma|e_a(b) + |\partial\Omega_1 \cap \partial\Omega|E_{\text{surf}}(b) + |\partial\Omega_2 \cap \partial\Omega| |a|^{-\frac{1}{2}}E_{\text{surf}}(b|a|) \right).$$

Furthermore, every minimizer $(\psi,A) \in H^1(\Omega;\mathbb{C}) \times H^1_{\text{div}}(\Omega)$ of the functional in (1.1) satisfies

$$\int_{\Omega} |\psi|^4 \, dx = -2E_{a}^{L}(b)\kappa^{-1} + o(\kappa^{-1}) \quad (\kappa \to +\infty).$$

**Remark 1.6.** In the asymptotics displayed in Theorem 1.5, the term $|\Gamma|b^{-\frac{1}{2}}e_a(b)$ corresponds to the energy contribution of the magnetic barrier. The rest of the terms indicate the energy contributions of the surface of the sample.

**Discussion of Theorem 1.5**

We will discuss the result in Theorem 1.5 in the interesting case where the magnetic barrier $\Gamma$ intersects the boundary of $\Omega$. Hence we will assume that $\partial\Omega_j \cap \partial\Omega \neq \emptyset$ for $j \in \{1,2\}$. When this condition is violated, the discussion below can be adjusted easily.

Theorem 1.5 leads to the following observation that mainly relies on Remark 1.4 and the order of the values $|a|\Theta_0$, $\Theta_0$, $\beta_a$ and $|a|$.

- For $a = -1$, we have $\beta_a = \Theta_0 < |a|$ (see (2.24)). Consequently, in light of Remark 1.4.
– If \( 1 < b < \Theta_0^{-1} \), then the surface of the sample carries superconductivity and the entire bulk is in a normal state except for the region near the magnetic barrier (see Figure 3). Moreover, the energy contributions of the magnetic barrier and the surface of the sample are of the same order and described by the surface energy, since in this case \( E_a(b) = E_{\text{surf}}(b) \), see (3.75). This behaviour is remarkably distinct from the case of a uniform applied magnetic field.

– If \( b \geq \Theta_0^{-1} \), then all the aforementioned energy contributions vanish, \( E_a(b) = 0 \).

• For \( a \in (-1, 0) \), comparing the values \( \beta_a, \Theta_0 \) and \(|a|\) is more subtle. In (2.16), (2.22) and Theorem 2.8 below, we proved that

\[
\forall a \in (-\Theta_0, 0), \quad |a| \Theta_0 < \beta_a < |a| < \Theta_0.
\]

However, the ordering of \( \beta_a \) and \( \Theta_0 \) is not known yet for \( a \in (-1, -\Theta_0) \).

The inequality \( \beta_a < |a| \) is new and is a slight improvement of the estimates in [HPRS16]. With (1.8) in hand, Theorem 1.5 and Remark 1.4 indicate the following behaviour for \( a \in (-\Theta_0, 0) \) and \( b > |a|^{-1} \):

\[ \text{Figure 4. Superconductivity distribution in the set } \Omega \text{ subjected to a magnetic field } B_0, \text{ in the regime where } a \in (-\Theta_0, 0), H = bs \text{ and respectively } |a|^{-1} < b < \beta_a^{-1} \text{ and } \beta_a^{-1} \leq b < |a|^{-1} \Theta_0^{-1}. \text{ The white regions are in a normal state, while the dark regions carry superconductivity.} \]

– The part of the sample’s surface near \( \partial \Omega \cap \partial \Omega \) does not carry superconductivity.

– If \(|a|^{-1} < b < \beta_a^{-1}\), then surface superconductivity is confined to the part of the surface near \( \partial \Omega \cap \partial \Omega \). At the same time, superconductivity is observed along the magnetic barrier \( \Gamma \) (see Figure 4), its strength is described by the function \( \epsilon_a(b) \). This behaviour is interesting for two reasons. Firstly, it demonstrates the existence of the edge current along the magnetic barrier, which is consistent with physics (see [HPRS16]). Secondly, it marks a distinct behaviour from the one known for uniform applied magnetic fields, in which case the whole surface carries superconductivity evenly (see for instance [HK17], [FP13], [Pan02]).

– If \( \beta_a^{-1} \leq b < |a|^{-1} \Theta_0^{-1} \), then superconductivity only survives along \( \partial \Omega \cap \partial \Omega \) (see Figure 4). Our results then display the strength of the applied magnetic field responsible for the breakdown of the edge current along the barrier.

– If \( b \geq |a|^{-1} \Theta_0^{-1} \), then all energy contributions in Theorem 1.5 disappear.

• For \( a \in (0, 1) \), \( \beta_a = a \) (see (2.17)). When \( b > a^{-1} \), Theorem 1.5 reveals the absence of superconductivity along the magnetic barrier. As for the distribution of superconductivity along the surface of the sample, we distinguish between two regimes:

\[ \text{Figure 4. Superconductivity distribution in the set } \Omega \text{ subjected to a magnetic field } B_0, \text{ in the regime where } a \in (-\Theta_0, 0), H = bs \text{ and respectively } |a|^{-1} < b < \beta_a^{-1} \text{ and } \beta_a^{-1} \leq b < |a|^{-1} \Theta_0^{-1}. \text{ The white regions are in a normal state, while the dark regions carry superconductivity.} \]

For a \( a \in (0, 1) \), \( \beta_a = a \) (see (2.17)). When \( b > a^{-1} \), Theorem 1.5 reveals the absence of superconductivity along the magnetic barrier. As for the distribution of superconductivity along the surface of the sample, we distinguish between two regimes:
Regime 1, $a \in (0, \Theta_0]$. The part of the boundary, $\partial \Omega_1 \cap \partial \Omega$, does not carry superconductivity. It remains to inspect the energy contribution of $\partial \Omega_2 \cap \partial \Omega$. In that respect:
- If $a^{-1} < b < a^{-1} \Theta_0^{-1}$, then superconductivity exists along $\partial \Omega_2 \cap \partial \Omega$.
- If $b \geq a^{-1} \Theta_0^{-1}$, then superconductivity disappears along $\partial \Omega_2 \cap \partial \Omega$.

Regime 2, $a \in (\Theta_0, 1)$. We observe the following:
- If $a^{-1} < b < \Theta_0^{-1}$, then the entire surface of the sample, $\partial \Omega$, is in a superconducting state, though its distribution is not uniform.
- If $\Theta_0^{-1} \leq b < a^{-1} \Theta_0^{-1}$, then only $\partial \Omega_2 \cap \partial \Omega$ carries superconductivity.
- If $b \geq a^{-1} \Theta_0^{-1}$, then all the energy contributions in Theorem 1.5 vanish.

Our next theorem describes the local behaviour of the minimizing order parameter $\psi$. To that end, we define the following distribution in $\mathbb{R}^2$,

$$C^\infty_c(\mathbb{R}^2) \ni \varphi \mapsto T^b(\varphi),$$

where

$$T^b(\varphi) = -2b^{-\frac{1}{2}} \left( \epsilon_n(b) \int_{\Gamma} \varphi ds_\Gamma + E_{\text{surf}}(b) \int_{\partial \Omega_1 \cap \partial \Omega} \varphi ds + |a|^{-\frac{1}{2}} E_{\text{surf}}(b|a|) \int_{\partial \Omega_2 \cap \partial \Omega} \varphi ds \right).$$

Here $ds_\Gamma$ and $ds$ denote the arc-length measures on $\Gamma$ and $\partial \Omega$ respectively.

**Theorem 1.7** (Local asymptotics). For all $a \in [-1, 1] \setminus \{0\}$ and $b > 1/|a|$, if $(\psi, A)_{\kappa, H} \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ is a minimizer of the functional in (1.1) for $H = b k$, then, as $\kappa \to +\infty$,

$$\kappa T^b_{\kappa} \rightharpoonup T \text{ in } D'(\mathbb{R}^2),$$

where $T_{\kappa}$ is the distribution in $\mathbb{R}^2$ defined as follows

$$C^\infty_c(\mathbb{R}^2) \ni \varphi \mapsto T^b_{\kappa}(\varphi) = \int_{\Omega} |\psi|^4 \varphi dx,$$

and the convergence of $T^b_{\kappa}$ to $T$ is understood in the following sense:

$$\forall \varphi \in C^\infty_c(\mathbb{R}^2), \quad \lim_{\kappa \to +\infty} \kappa T^b_{\kappa}(\varphi) = T^b(\varphi).$$

1.6. **Notation.**

- The letter $C$ denotes a positive constant whose value may change from one formula to another. Unless otherwise stated, the constant $C$ depends on the value of $a$ and the domain $\Omega$, and is independent of $\kappa$ and $H$.
- Let $a(\kappa)$ and $b(\kappa)$ be two positive functions, we write:
  - $a(\kappa) \ll b(\kappa)$, if $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to +\infty$.
  - $a(\kappa) \approx b(\kappa)$, if there exist constants $\kappa_0, C_1$ and $C_2$ such that for all $\kappa \geq \kappa_0$, $C_1 a(\kappa) \leq b(\kappa) \leq C_2 a(\kappa)$.
- The quantity $o(1)$ indicates a function of $\kappa$, defined by universal quantities, the domain $\Omega$, given functions, etc and such that $|o(1)| \ll 1$. Any expression $o(1)$ is independent of the minimizer $(\psi, A)$ of (1.1). Similarly, $O(1)$ indicates a function of $\kappa$, bounded by a constant independent of the minimizers of (1.1).
- Let $n \in \mathbb{N}$, $p \in \mathbb{N}$, $N \in \mathbb{N}$, $\alpha \in (0, 1)$, $K \subset \mathbb{R}^N$ be an open set. We use the following Hölder space:

$$C^{n,\alpha}(K) = \left\{ f \in C^n(\overline{K}) \mid \sup_{x \neq y \in K} \frac{|D^n f(x) - D^n f(y)|}{|x - y|^\alpha} < +\infty \right\} .$$

- Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval. We introduce the space:

$$B^n(I) = \{ u \in L^2(I) : x^i D^j u \in L^2(I), \forall i, j \in \mathbb{N} \text{ s.t. } i + j \leq n \} .$$

(1.9)
1.7. Organization of the paper. The rest of the paper is divided into seven sections and one appendix. Section 2 presents some preliminaries, particularly, some a priori estimates, exponential decay results, and a linear 2D operator with a step magnetic field. Theorem 2.8 is an improvement of a result in [HPRS16].

Section 3 introduces a reduced GL energy crucial in the study of superconductivity near the magnetic barrier \( \Gamma \) and we introduce the barrier energy \( e_a(\cdot) \).

In Section 4, we present the Frenet coordinates defined in a tubular neighbourhood of the curve \( \Gamma \). These coordinates are frequently used in the study of surface superconductivity (see [FH10, Appendix F]).

Sections 5 and 6 are devoted for the analysis of the local behaviour of the minimizing order parameter near the magnetic barrier \( \Gamma \), while Section 7 recalls well-known results about the local behaviour the order parameter near the surface \( \partial \Omega \).

Collecting the local estimates established in Sections 6 and 7, we prove in Section 8 our main theorems (Theorems 1.5 and 1.7 above).

Finally, in the appendix, we collect some common spectral results used throughout the paper.

One remarkable aspect of our proofs is that we have not used the a priori elliptic \( L^\infty \)-estimate \( \| (\nabla - i\kappa H A) \psi \|_{L^\infty} \leq C\kappa \). Such estimate is not known to hold in our case of discontinuous magnetic field \( B_0 \). Instead, we used the easy energy estimate \( \| (\nabla - i\kappa H A) \psi \|_{L^2} \leq C\kappa \) and the regularity of the curl-div system (cf. Theorem 2.3). This also spares us the complex derivation of the \( L^\infty \)-estimate (see [FH10, Chapter 11]). We have made an effort to keep the proofs reasonably self-contained.

2. Preliminaries

2.1. A Priori Estimates. We present some celebrated estimates needed in the sequel to control the various errors arising while estimating the energy in (1.1).

We begin by the following well-known estimate of the order parameter:

**Proposition 2.1.** If \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\) is a weak solution to (1.4), then

\[ \| \psi \|_{L^\infty(\Omega)} \leq 1. \]

A detailed proof of Proposition 2.1 can be found in [FH10, Proposition 10.3.1]. The \( L^\infty \)-bound in Proposition 2.1 is crucial in deriving some a priori estimates on the solutions of the Ginzburg-Landau equations (1.4) listed in Theorem 2.3 below.

Recall the magnetic field \( B_0 \) introduced in Assumption 1.1. In the next lemma, we will fix the gauge for the magnetic potential generating \( B_0 \) (see [AK16, Lemma A.1]):

**Lemma 2.2.** Suppose that the conditions in Assumption 1.1 hold. There exists a unique vector field \( F \in H^1_{\text{div}}(\Omega) \) such that

\[ \text{curl } F = B_0. \]

Furthermore, \( F \) is in \( C^\infty(\Omega_i) \) and in \( H^2(\Omega_i) \), \( i = 1, 2 \).

We collect below some useful estimates whose proofs are given in [AK16, Theorem 4.2].

**Theorem 2.3.** Let \( \alpha \in (0, 1) \) be a constant. Suppose that the conditions in Assumption 1.1 hold. There exist two constants \( \kappa_0 > 0 \) and \( C > 0 \) such that, if (1.5) is satisfied and \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) is a solution of (1.4), then

1. \( \| (\nabla - i\kappa H A) \psi \|_{L^2(\Omega)} \leq C\kappa \).
2. \( \| \text{curl}(A - F) \|_{L^2(\Omega)} \leq C/\kappa \).
3. \( A - F \in H^2(\Omega) \) and \( \| A - F \|_{H^2(\Omega)} \leq C/\kappa \).
4. \( A - F \in C^{0,\alpha}(\Omega) \) and \( \| A - F \|_{C^{0,\alpha}(\Omega)} \leq C/\kappa \).
2.2. **Exponential decay of the order parameter.** The following theorem displays a regime for the intensity of the applied magnetic field where the order parameter and the GL energy are exponentially small in the bulk of the domains \( \Omega_1 \) and \( \Omega_2 \).

**Theorem 2.4.** Given \( a \in [-1, 1] \setminus \{0\} \) and \( b > 1/|a| \), there exist constants \( \kappa_0 > 0 \), \( C > 0 \), and \( a_0 > 0 \) such that, if

\[
\kappa \geq \kappa_0, \quad \kappa_0 \kappa^{-1} \leq \ell < 1 \quad \text{and} \quad (\psi, \mathbf{A}) \text{ is a solution of } (1.4) \text{ for } H = b \kappa,
\]

then

\[
\int_{\Omega_j \cap \{\text{dist}(x, \partial \Omega_j) \geq \ell\}} \left( |\psi|^2 + |(\nabla - i \kappa \mathbf{A}) \psi|^2 \right) \, dx \leq Ce^{-a_0 \kappa \ell},
\]

for \( j \in \{1, 2\} \).

The proof of Theorem 2.4 follows from stronger Agmon-type estimates established in [AK16, Theorems 1.5 & 7.3].

2.3. **Families of Sturm–Liouville operators on \( L^2(\mathbb{R}_+) \).** In this section, we will briefly present some spectral properties of the self-adjoint realization on \( L^2(\mathbb{R}_+) \) of the Sturm–Liouville operator:

\[
H[\gamma, \xi] = -\frac{d^2}{dt^2} + (t - \xi)^2,
\]

(2.1)

defined over the domain:

\[
\text{Dom} \left( H[\gamma, \xi] \right) = \{ u \in B^2(\mathbb{R}_+) : u'(0) = \gamma u(0) \},
\]

where \( \xi \) and \( \gamma \) are two real parameters, and the space \( B^2(\mathbb{R}_+) \) was introduced in (1.9).

The quadratic form associated to \( H[\gamma, \xi] \) is

\[
B^1(\mathbb{R}_+) \ni u \mapsto q[\gamma, \xi](u) = \int_0^{+\infty} \left( |u'(t)|^2 + |(t - \xi)u(t)|^2 \right) \, dt + \gamma |u(0)|^2.
\]

Since the embedding of the form domain \( B^1(\mathbb{R}_+) \) in \( L^2(\mathbb{R}_+) \) is compact, the spectrum of \( H[\gamma, \xi] \) is an increasing sequence of eigenvalues tending to \( +\infty \). Denote by \( \mu(\gamma, \xi) \) the first eigenvalue of the operator \( H[\gamma, \xi] \):

\[
\mu(\gamma, \xi) = \inf \text{sp}(H[\gamma, \xi]) = \inf_{u \in B^1(\mathbb{R}_+)} \frac{q[\gamma, \xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}
\]

(2.2)

and let

\[
\Theta(\gamma) = \inf_{\xi \in \mathbb{R}} \mu(\gamma, \xi).
\]

(2.3)

*The Neumann realization.* The particular case where \( \gamma = 0 \) corresponds to the Neumann realization, denoted by \( H^N[\xi] \), with the associated quadratic form \( q^N[\xi] = q[0, \xi] \). The first eigenvalue of \( H^N[\xi] \) is denoted by

\[
\mu^N(\xi) = \inf \text{sp}(H^N[\xi]) = \mu(0, \xi).
\]

(2.4)

*The Dirichlet realization.* Besides the Robin and Neumann realizations, we introduce the Dirichlet realization

\[
H^D[\xi] = -\frac{d^2}{dt^2} + (t - \xi)^2
\]

with domain

\[
\text{Dom} \left( H^D[\xi] \right) = \{ u \in B^2(\mathbb{R}_+) : u(0) = 0 \}.
\]

The associated quadratic form is defined by

\[
B^1(\mathbb{R}_+) \cap H^D_0(\mathbb{R}_+) \ni u \mapsto q^D[\xi](u) = \int_0^{+\infty} \left( |u'(t)|^2 + |(t - \xi)u(t)|^2 \right) \, dt.
\]
We introduce the first eigenvalue of $H^D[\xi]$ as follows

$$
\mu^D(\xi) = \inf \text{sp}(H^D[\xi]) = \inf_{u \in B^1(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+)} \frac{q^D[\xi](u)}{\|u\|_{L^2(\mathbb{R})}^2}.
$$

(2.5)

The perturbation theory [Kat66] ensures that the functions

$$
\xi \mapsto \mu^D(\xi), \quad \xi \mapsto \mu^N(\xi), \quad \text{and} \quad \xi \mapsto \mu(\gamma, \xi)
$$

are analytic.

In addition, recall the following well-known Sturm–Liouville theorems (For instance, see [DH93, RS72]):

**Theorem 2.5.** The function $\xi \mapsto \mu^D(\xi)$ introduced in (2.5) is decreasing,

$$
\lim_{\xi \to -\infty} \mu^D(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \to +\infty} \mu^D(\xi) = 1.
$$

Theorems 2.6 and 2.7 below are proved in [Kac06, Section 2].

**Theorem 2.6.** The following statements hold

1. For all $(\gamma, \xi) \in \mathbb{R}^2$, the first eigenvalue $\mu(\gamma, \xi)$ of $H[\gamma, \xi]$ defined in (2.2) is simple, and there exists a unique eigenfunction $\varphi_{\gamma, \xi}$ satisfying

$$
\begin{align*}
-\varphi''_{\gamma, \xi} + (t - \xi)^2 \varphi_{\gamma, \xi} &= \mu(\gamma, \xi) \varphi_{\gamma, \xi}, \quad \varphi_{\gamma, \xi} > 0 \text{ in } \mathbb{R}_+, \\
\varphi_{\gamma, \xi}(0) &= \gamma \varphi_{\gamma, \xi}(0), \\
\int_{\mathbb{R}_+} |\varphi_{\gamma, \xi}(t)|^2 dt &= 1.
\end{align*}
$$

2. For all $\gamma \in \mathbb{R}$, $\lim_{\xi \to -\infty} \mu(\gamma, \xi) = +\infty$, and $\lim_{\xi \to +\infty} \mu(\gamma, \xi) = 1$

**Theorem 2.7.** Let $\Theta(\cdot)$ be the function defined in (2.3). It holds the following:

1. The function $\Theta(\cdot)$ is continuous and increasing.
2. For all $\gamma \in \mathbb{R}$, $-\gamma^2 \leq \Theta(\gamma) < 1$.
3. For all $\gamma < 0$, $-\gamma^2 \leq \Theta(\gamma) \leq -\gamma^2 + \frac{1}{2\gamma^2}$.
4. For all $\gamma \geq 0$, $\Theta(\gamma) > 0$.
5. There exist $C_0, \gamma_0 > 0$ such that, $\forall \gamma \in [\gamma_0, +\infty)$, $1 - C_0 \gamma \exp(-\gamma^2) \leq \Theta(\gamma)$.
6. For all $\gamma \in \mathbb{R}$, the function $\xi \mapsto \mu(\gamma, \xi)$ admits a unique minimum attained at

$$
\xi(\gamma) = \sqrt{\Theta(\gamma) + \gamma^2}.
$$

Furthermore, this minimum is non-degenerate, $\partial^2_\xi \mu(\gamma, \xi(\gamma)) > 0$.

### 2.4. An operator with a step magnetic field.

Let $a \in [-1, 1) \setminus \{0\}$. We consider the magnetic potential $A_0$ defined by

$$
A_0(x) = (-x_2, 0) \quad (x = (x_1, x_2) \in \mathbb{R}^2)
$$

(2.7)

which satisfies $\text{curl} A_0 = 1$. We define the step function $\sigma$ as follows. For $x = (x_1, x_2) \in \mathbb{R}^2$,

$$
\sigma(x) = 1_{\mathbb{R}_+}(x_2) + a1_{\mathbb{R}_-}(x_2).
$$

(2.8)

We introduce the self-adjoint magnetic Hamiltonian in $L^2(\mathbb{R}^2)$

$$
\mathcal{L}_a = -(\nabla - i\sigma A_0)^2,
$$

(2.9)

where

$$
(\nabla - i\sigma A_0)^2 = (\Delta - 2i\sigma A_0 \cdot \nabla - \sigma^2|A_0|^2) = \partial^2_{x_2} + (\partial_{x_1} + i\sigma x_2)^2.
$$

We denote the ground state energy of the operator $\mathcal{L}_a$ by

$$
\beta_a = \inf \text{sp}(\mathcal{L}_a).
$$

(2.10)
Since the Hamiltonian defined in (2.9) is invariant with respect to translations in the $x_1$-direction, we can reduce it to a family of Shrödinger operators on $L^2(\mathbb{R})$, $h_\alpha[\xi]$, parametrized by $\xi \in \mathbb{R}$ and called fiber operators (see [HPRS16, HS15]). The operator $h_\alpha[\xi]$ is defined by

$$h_\alpha[\xi] = -\frac{d^2}{dt^2} + V_\alpha(\xi, t),$$

with

$$V_\alpha(\xi, t) = \begin{cases} (\xi + at)^2, & t < 0, \\ (\xi + t)^2, & t > 0. \end{cases}$$

(2.11)

The domain of $h_\alpha[\xi]$ is given by:

$$\text{Dom } (h_\alpha[\xi]) = \left\{ u \in B^1(\mathbb{R}) : \left( -\frac{d^2}{dt^2} + V_\alpha(\xi, t) \right) u \in L^2(\mathbb{R}), \ u'(0_+) = u'(0_-) \right\}$$

where $B^1(\mathbb{R})$ is defined in (1.9). The quadratic form associated to $h_\alpha[\xi]$ is

$$q_\alpha[\xi](u) = \int_{\mathbb{R}} \left( |u'(t)|^2 + V_\alpha(\xi, t)|u(t)|^2 \right) dt$$

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$$\text{Dom } (q_\alpha[\xi]) = B^1(\mathbb{R}).$$

(2.13)

The spectra of the operators $\mathcal{L}_\alpha$ and $h_\alpha[\xi]$ are linked together as follows (see [FH10, Sec. 4.3])

$$\text{sp}(\mathcal{L}_\alpha) = \bigcup_{\xi \in \mathbb{R}} \text{sp}(h_\alpha[\xi]).$$

(2.14)

We introduce the first eigenvalue of the fiber operator $h_\alpha[\xi]$,

$$\mu_\alpha(\xi) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{q_\alpha[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}.$$

(2.15)

Consequently, for all $\alpha \in [-1, 1) \setminus \{0\}$, we may express the ground state energy in (2.10) by

$$\beta_\alpha = \inf_{\xi \in \mathbb{R}} \mu_\alpha(\xi).$$

(2.16)

Below, we collect some properties of the eigenvalue $\mu_\alpha(\xi)$.

The case $0 < \alpha < 1$. This case is studied in [HS15, Iwa85]. The eigenvalue $\mu_\alpha(\xi)$ is simple and is a decreasing function of $\xi$. The monotonicity of $\mu_\alpha(\cdot)$ and its asymptotics in Proposition A.3 imply that

$$a < \mu_\alpha(\xi) < 1 \quad (\xi \in \mathbb{R}),$$

and that $\beta_\alpha$ introduced in (2.10) satisfies

$$\beta_\alpha = a.$$
The case \(-1 < a < 0\). See also [HPRS16] for the study of this case. The eigenvalue \(\mu_a(\xi)\) is simple, and there exists \(\zeta_a < 0\) satisfying
\[
|a| \geq \mu_a(\zeta_a) = \min_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2.21}
\]
Moreover, using the min-max principle, one can easily prove that
\[
|a| \Theta_0 < \min_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2.22}
\]
Combining the foregoing discussion in the case \(a \in [-1, 0)\), we get \(\beta_a\) introduced in (2.10) satisfies
\[
|a| \Theta_0 \leq \beta_a \leq |a|, \tag{2.23}
\]
and
\[
\beta_{-1} = \Theta_0. \tag{2.24}
\]
By defining \(\zeta_a = -\sqrt{\Theta_0}\) for \(a = -1\), we get furthermore that, for all \(a \in [-1, 0)\),
\[
\beta_a = \mu_a(\zeta_a) \text{ with } \zeta_a < 0. \tag{2.25}
\]
In the next theorem, we will use a direct approach, different from the one in [HPRS16], to establish the existence of a global minimum \(\zeta_a\) in the case where \(a \in (-1, 0)\) and to prove that \(\beta_a < |a|\). This slightly improves the estimates in [HPRS16] (see Remark 2.9 below). Theorem 2.8 is necessary to validate Assumption (3.7), under which we work in Section 3.

**Theorem 2.8.** For all \(a \in (-1, 0)\), there exists \(\xi < 0\) such that \(\mu_a(\xi)\), the first eigenvalue of the operator \(h_a[\xi]\), satisfies
\[
\mu_a(\xi) < |a|.
\]
Consequently, the function \(\xi \mapsto \mu_a(\xi)\) admits a global minimum satisfying
\[
\min_{\xi \in \mathbb{R}} \mu_a(\xi) < |a|.
\]

**Proof.** The proof here is inspired by [Kac07]. Define the function
\[
u(t) = \begin{cases} \varphi_\gamma(0) \exp(-mt), & t \geq 0, \\ \varphi_\gamma(-\sqrt{|a|} t), & t < 0. \end{cases} \tag{2.26}
\]
where \(\gamma\) and \(m\) are two positive constants to be fixed later, and \(\varphi_\gamma = \varphi_{\gamma, \xi(\gamma)}\) is the normalized eigenfunction defined in Theorem 2.7 and associated to the eigenvalue \(\Theta(\gamma) = \mu(\gamma, \xi(\gamma))\) introduced in (2.3). One can check that \(u \in \text{Dom}(q_a[\xi])\), hence by the min-max principle, for all \(\xi \in \mathbb{R}\),
\[
\mu_a(\xi) \leq \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}. \tag{2.27}
\]
Pick \(\xi \in \mathbb{R}\). We will choose \(\xi\) precisely later. The quadratic form \(q_a[\xi](u)\) defined in (2.12) can be decomposed as follows:
\[
q_a[\xi](u) = q^{(1)}_a[\xi](u) + q^{(2)}_a[\xi](u)
\]
where
\[
q^{(1)}_a[\xi](u) = \int_0^{+\infty} (|u'(t)|^2 + |(t + \xi)u(t)|^2) \, dt,
\]
and
\[
q^{(2)}_a[\xi](u) = \int_{-\infty}^0 (|u'(t)|^2 + |(at + \xi)u(t)|^2) \, dt.
\]
A simple computation gives
\[
q^{(1)}_a[\xi](u) = \left(\frac{m}{2} + \frac{\xi^2}{2m} + \frac{\xi}{2m^2} + \frac{1}{4m^3}\right)|\varphi_\gamma(0)|^2. \tag{2.28}
\]
On the other hand, for $t < 0$, $u(t) = \varphi_\gamma(-\sqrt{|a|}t)$, so we do the change of variable $y = -\sqrt{|a|}t$, which in turn yields

$$g_a^{(2)}[\xi](u) = \sqrt{|a|} \int_0^{+\infty} \left( |\varphi_\gamma'(y)|^2 + \left| y + \frac{\xi}{\sqrt{|a|}} \right| \varphi_\gamma(y) \right)^2 dy.$$  

Now we select $\xi = -\sqrt{|a|} \xi(\gamma)$, where $\xi(\gamma)$ is the value defined in Theorem 2.7. That way we get

$$g_a^{(2)}[\xi](u) = \sqrt{|a|} \left( \Theta(\gamma) - \gamma |\varphi_\gamma(0)|^2 \right). \quad (2.29)$$

The definition of the function $u$ in (2.26) yields

$$\int_{-\infty}^{+\infty} |u(t)|^2 dt = \frac{|\varphi_\gamma(0)|^2}{2m} + \frac{1}{\sqrt{|a|}}. \quad (2.30)$$

Combining the results in (2.28)–(2.30) and using Theorem 2.7, we rewrite (2.27) as follows

$$\mu_a(\xi) \leq \frac{\sqrt{|a|} \Theta(\gamma) + \left( \frac{m}{2} - \sqrt{|a|} \gamma + \frac{\xi^2}{2m} + \frac{\xi}{2m^2} + \frac{1}{4m^3} \right) |\varphi_\gamma(0)|^2}{1/\sqrt{|a|} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$

By Theorem 2.7 for all $\gamma > 0$ we have $0 < \Theta(\gamma) < 1$. Thus

$$\mu_a(\xi) \leq \frac{\sqrt{|a|} \Theta(\gamma) + \left( \frac{m}{2} - \sqrt{|a|} \gamma + \frac{|a|\Theta(\gamma)}{2m} + \frac{|a|^2}{2m^2} - \frac{\sqrt{|a|} \Theta(\gamma)^2}{2m^2} + \frac{1}{4m^3} \right) |\varphi_\gamma(0)|^2}{1/\sqrt{|a|} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$  

To finish the proof of the lemma, we choose $\gamma = \sqrt{1/2|a|(1 - |a|)}$ and $m = \sqrt{|a|} \gamma$ so that the quantity $m/2 - \sqrt{|a|} \gamma + |a|/2m + |a|\gamma^2/2m - \sqrt{|a|} \gamma^2/2m^2 + 1/4m^3$ vanishes. Using the fact that $\Theta(\gamma) < 1$, we obtain

$$\mu_a(\xi) \leq \frac{\sqrt{|a|} \Theta(\gamma)}{1/\sqrt{|a|} + \frac{|\varphi_\gamma(0)|^2}{2m}} < |a|\Theta(\gamma) < |a|. \quad (2.31)$$

Now, the existence of the global minimum becomes a consequence of Theorem A.2 and Proposition A.4 in Appendix A. \hfill \square

**Remark 2.9.** Note that our proof of Theorem 2.8 yields the upper bound

$$\forall \ a \in (-1, 0), \quad \beta_a < |a| \Theta \left( \sqrt{\frac{1}{2|a|(1 - |a|)}} \right),$$

which is stronger than $\beta_a < |a|$.

Collecting (2.17)–(2.22) and the result in Theorem 2.8, we deduce the following facts regarding the bottom of the spectrum of the operator $L_a$ introduced in (2.9).

**Proposition 2.10.** For all $a \in [-1, 1] \setminus \{0\}$, let $L_a$ and $\beta_a$ be as in (2.9) and (2.10) respectively, and $\Theta_0$ be as in (2.20). The following statements hold

1. For all $a \in (0, 1)$, $\beta_a = a$. 
(2) For all \( a \in [-1, 0) \), \( |a| \Theta_0 \leq \beta_a < |a| \), there exist \( \zeta_a < 0 \) and a function \( \phi_a \in L^2(\mathbb{R}) \) such that
\[
\int_{\mathbb{R}} |\phi_a|^2 \, dt = 1, \quad -\phi_a''(t) + V_a(\zeta_a, t)\phi_a(t) = \beta_a\phi_a(t) \quad \text{in} \quad \mathbb{R},
\]
where \( V(\cdot, \cdot) \) is introduced in (2.11).

(3) For all \( a \in [-1, 0) \), the function
\[
\psi_a(x_1, x_2) = e^{i\zeta_a x_1} \phi_a(x_2)
\]
is a bounded eigenfunction of the operator \( \mathcal{L}_a \) and satisfies
\[
\mathcal{L}_a \psi_a = \beta_a \psi_a.
\]

3. Reduced Ginzburg–Landau Energy

3.1. The functional and the main result. Assume that \( a \in [-1, 1) \setminus \{0\} \) is fixed, \( \sigma \) is the step function defined in (2.8) and \( \mathbf{A}_0 \) is the magnetic potential defined in (2.7). For every \( R > 0 \), consider the strip
\[
S_R = (-R/2, R/2) \times (-\infty, +\infty).
\]
We introduce the space
\[
\mathcal{D}_R = \left\{ u \in L^2(S_R) : (\nabla - i\sigma \mathbf{A}_0)u \in L^2(S_R), \ u\left(x_1 = \pm \frac{R}{2}, x_2 \right) = 0 \right\}.
\]
For \( b > 0 \), we define the following Ginzburg–Landau energy on \( \mathcal{D}_R \) by
\[
\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left( b|(|\nabla - i\sigma \mathbf{A}_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) \, dx,
\]
along with the ground state energy
\[
\mathfrak{g}_a(b, R) = \inf_{u \in \mathcal{D}_R} \mathcal{G}_{a,b,R}(u).
\]
Our objective is to prove

**Theorem 3.1.** Assume that \( a \in [-1, 1) \setminus \{0\} \), \( b \geq 1/|a| \), \( R > 0 \), \( \mathfrak{g}_a(b, R) \) is the ground state energy in (3.4), and \( \beta_a \) is defined in (2.10). The following holds:

1. \( \mathfrak{g}_a(b, R) \leq 0 \).
2. If \( a \in (0, 1) \), then \( \mathfrak{g}_a(b, R) = 0 \).
3. If \( a \in [-1, 0) \), then there exists a constant \( \epsilon_a(b) \leq 0 \) such that
\[
\lim_{R \to +\infty} \frac{\mathfrak{g}_a(b, R)}{R} = \epsilon_a(b).
\]
Furthermore, \( \epsilon_a(b) = 0 \) if and only if \( b \geq 1/\beta_a \).
4. For all \( a \in [-1, 0) \), the function \( [1/|a|, +\infty) \ni b \mapsto \epsilon_a(b) \) is monotone non-decreasing and continuous.
5. For all \( a \in [-1, 0) \), there exists a universal positive constant \( C \) such that
\[
\forall R \geq 4, \quad \epsilon_a(b) \leq \frac{\mathfrak{g}_a(b, R)}{R} \leq \epsilon_a(b) + C \frac{b^2}{R^4}.
\]

The proof of Theorem 3.1 will occupy the rest of this section through a sequence of lemmas.
3.2. The trivial case. We start by handling the trivial situation where the ground state energy vanishes:

**Lemma 3.2.** If \( a \in [-1, 1] \setminus \{0\} \) and \( b \geq 1/\beta_a \), then for all \( R > 0 \), \( g_a(b, R) = 0 \).

**Remark 3.3.**

1. Under the assumptions in Lemma 3.2, the function \( u = 0 \in \mathcal{D}_R \) is a minimizer of the functional in (3.3).
2. When \( a \in (0, 1) \), \( \beta_a \) by Proposition 2.10 hence Lemma 3.2 yields that \( g_a(b, R) = 0 \) for all \( b \geq 1/a \) and \( R > 0 \).

**Proof of Lemma 3.3** We have the obvious upper bound \( g_a(b, R) \leq G_{a,b,R}(0) = 0 \). Next we prove the lower bound \( g_a(b, R) \geq 0 \). Pick an arbitrary function \( u \in \mathcal{D}_R \) and extend it by zero on \( \mathbb{R}^2 \). Using the min-max principle, we get

\[
G_{a,b,R}(u) \geq b \beta_a \int_{S_R} |u|^2 \, dx + \int_{S_R} \left( -|u|^2 + \frac{1}{2} |u|^4 \right) \, dx \geq 0 \text{ since } b \geq \frac{1}{\beta_a}.
\]

Minimizing over \( u \in \mathcal{D}_R \), we get \( g_a(b, R) \geq 0 \). \( \square \)

3.3. Existence of minimizers. Now we handle the following case (which is complementary to the one in Lemma 3.2):

\[
-1 \leq a < 0 \quad \text{and} \quad \frac{1}{|a|} \leq b < \frac{1}{\beta_a},
\]

where \( \beta_a \) is the first eigenvalue introduced in (2.10). Under Assumption (3.7), we will demonstrate the existence of a minimizer of the functional in (3.3) along with an estimate of its decay at infinity. This is the content of

**Proposition 3.4.** Assume that (3.7) holds. For all \( R > 0 \), there exists a function \( \varphi_{a,b,R} \in \mathcal{D}_R \) such that

\[
G_{a,b,R}(\varphi_{a,b,R}) = g_a(b, R) \quad \text{and} \quad \|\varphi_{a,b,R}\|_{L^\infty(S_R)} \leq 1.
\]

Here \( G_{a,b,R} \) is the functional introduced in (3.3) and \( g_a(b, R) \) is the ground state energy introduced in (3.4).

Furthermore, there exists a universal constant \( C > 0 \) such that, for all \( R > 0 \), the function \( \varphi_{a,b,R} \) satisfies

\[
\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left( (|\nabla - i \sigma A_0|) \varphi_{a,b,R} + |\varphi_{a,b,R}|^2 \right) \, dx \leq C b R,
\]

\[
\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_{a,b,R}|^4 \, dx \leq C b^2 R,
\]

and

\[
\int_{S_R} \left( b|\nabla - i \sigma A_0| \varphi_{a,b,R} + |\varphi_{a,b,R}|^2 \right) \, dx \leq C b R.
\]

In the proof of Proposition 3.4, we will use the approach in [FKP13, Theorem 3.6] and [Pan02] which can be described in a heuristic manner as follows. The unboundedness of the set \( S_R \) makes the existence of the minimizer \( \varphi_{a,b,R} \) in (3.8) non-trivial. In the following, we consider a reduced Ginzburg–Landau energy \( G_{a,b,R,m} \) defined on the bounded set \( S_{R,m} = (-R/2, R/2) \times (-m, m) \), and we establish some decay estimates of its minimizer \( \varphi_{a,b,R,m} \). Later, using a limiting argument on \( G_{a,b,R,m} \) and \( \varphi_{a,b,R,m} \) for large values of \( m \), we obtain the existence of the minimizer \( \varphi_{a,b,R} \) together with the decay properties in Proposition 3.4.
Since the proof of Proposition 3.4 is lengthy, we opt to divide it into several lemmas. First, for every \( m \in \mathbb{N} \), we introduce the set \( S_{R,m} = (-R/2, R/2) \times (-m, m) \) and the functional

\[
G_{a,b,R,m}(u) = \int_{S_{R,m}} \left( b |(\nabla - i \sigma A_0)u|^2 - |u|^2 + \frac{1}{2} |u|^4 \right) \, dx
\]

(3.12)
defined over the space

\[
D_{R,m} = \left\{ u \in L^2(S_{R,m}) : (\nabla - i \sigma A_0)u \in L^2(S_{R,m}), \quad u\Big(x_1 = \pm \frac{R}{2}, \cdot\Big) = u\Big(\cdot, x_2 = \pm m\Big) = 0 \right\}.
\]

(3.13)

Here \( \sigma \) was defined in (2.8). Now we define the ground state energy

\[
g_a(b, R, m) = \inf_{u \in D_{R,m}} G_{a,b,R,m}(u).
\]

(3.14)

Lemma 3.5. Assume that (3.7) holds. There exists a universal constant \( C > 0 \), and for all \( R > 0 \), \( m \geq 1 \), there exists a function \( \varphi_{a,b,R,m} \in D_{R,m} \) satisfying,

\[
\|\varphi_{a,b,R,m}\|_{L^\infty(S_{R,m})} \leq 1,
\]

(3.15)

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{\ln |x_2|^2} \left( (|\nabla - i \sigma A_0|\varphi_{a,b,R,m}^2 + |\varphi_{a,b,R,m}|^2) \right) \, dx \leq CbR,
\]

(3.16)

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{\ln |x_2|^2} |\varphi_{a,b,R,m}|^4 \, dx \leq Cb^2R,
\]

(3.17)

and

\[
G_{a,b,R,m}(\varphi_{a,b,R,m}) = g_a(b, R, m).
\]

(3.18)

Here \( G_{a,b,R,m} \) is the functional introduced in (3.12) and \( g_a(b, R, m) \) is the ground state energy introduced in (3.14).

Proof. The proof is reminiscent of the one in [Pan02, Theorem 4.1]. The boundedness and the regularity of the domain \( S_{R,m} \) guarantee the existence of a minimizer \( \varphi_m := \varphi_{a,b,R,m} \) of \( G_{a,b,R,m} \) in \( D_{R,m} \), satisfying

\[
-b(\nabla - i \sigma A_0)^2 \varphi_m = (1 - |\varphi_m|^2)\varphi_m \quad \text{in } S_{R,m},
\]

(3.19)

see e.g. [FH10, Chapter 11]. Furthermore, Proposition 10.3.1 in [FH10] ensures that

\[
\|\varphi_m\|_{L^\infty(S_{R,m})} \leq 1.
\]

Next, select \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi(x_2) = 0 \) if \( |x_2| \leq 1 \), and \( \chi(x_2) = |x_2|^2/\ln |x_2| \) if \( |x_2| \geq 4 \). The function \( \chi \) consequently satisfies

\[
0 < |\chi'(x_2)| < \frac{3\sqrt{|x_2|}}{2\ln |x_2|} \quad \text{for all } |x_2| \geq 4.
\]

Multiply (3.19) by \( \chi^2 |\varphi_m| \) and integrate by parts,

\[
\int_{S_{R,m}} \left( b |(\nabla - i \sigma A_0)\chi |\varphi_m|^2 - \chi^2 |\varphi_m|^2 + \chi^2 |\varphi_m|^4 \right) \, dx = b \int_{S_{R,m}} \chi^2 |\varphi_m|^2 \, dx.
\]

(3.20)
Since the function \( x \mapsto \chi(x_2)\varphi_m(x) \) is supported in \( S_{R,m} \cap \{|x_2| \geq 1\} \) where \( \text{curl}(\sigma A_0) = \sigma \), we can apply the spectral inequality in [FH10, Lemma 1.4.1] to get, under the assumption \( 1/|a| \leq b < 1/\beta_a \),

\[
\int_{S_{R,m}} |(\nabla - i\sigma A_0)\chi\varphi_m|^2 \, dx \geq b \int_{S_{R,m}} |\sigma|^2 |\varphi_m|^2 \, dx \\
\geq \int_{S_{R,m}} \chi^2 |\varphi_m|^2 \, dx. \tag{3.21}
\]

It follows from (3.20) and (3.21)

\[
\int_{S_{R,m}} \chi^2(x_2)|\varphi_m|^4 \, dx \leq b \int_{S_{R,m}} \chi^2(x_2)|\varphi_m|^2 \, dx \\
\leq b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \chi^2(x_2)|\varphi_m|^2 \, dx + b \int_{S_{R,m} \cap \{|x_2| < 4\}} \chi^2(x_2)|\varphi_m|^2 \, dx \\
\leq Cb \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^2}{(\ln |x_2|)^2} |\varphi_m|^2 \, dx + CbR. \tag{3.22}
\]

Using the Hölder inequality,

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^2}{(\ln |x_2|)^2} |\varphi_m|^2 \, dx \tag{3.23}
\]

\[
\leq \left( \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{1}{x_2^2(\ln |x_2|)^2} \, dx \right)^{\frac{1}{2}} \left( \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \right)^{\frac{1}{2}} \\
\leq CR^\frac{3}{2} \left( \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \right)^{\frac{1}{2}}. \tag{3.24}
\]

Now, using Cauchy-Schwarz inequality, the properties of \( \chi \) in (3.22) and (3.24), we obtain

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \leq \int_{S_{R,m}} \chi^2(x_2)|\varphi_m|^4 \, dx \\
\leq CR^\frac{3}{2}b \left( \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \right)^{\frac{1}{2}} + CbR \\
\leq Cb^2R + CbR. \tag{3.25}
\]

Consequently, under the assumption \( 1 \leq 1/|a| \leq b < 1/\beta_a \), we get (3.17) Inserting (3.17) into (3.24), we get

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |\varphi_m|^2 \, dx \leq CbR. \tag{3.26}
\]

We still need to establish

\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx \leq CbR. \tag{3.27}
\]

To that end, we select \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(x_2) = 0 \) if \( |x_2| \leq 1 \), and \( \eta(x_2) = \sqrt{|x_2|/\ln |x_2|} \) if \( |x_2| \geq 4 \). Multiplying the equation in (3.19) by \( \eta \varphi_m \) and integrating
over $S_{R,m}$, we get
\[
b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} |(\nabla - i\sigma A_0)\eta(x_2)\varphi_m|^2 \, dx
\]
\[
= \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \left( \eta^2(x_2)|\varphi_m|^2 - \eta^2(x_2)|\varphi_m|^4 + bn^2(x_2)|\varphi_m|^2 \right) \, dx. \tag{3.28}
\]
It is easy to check by a straightforward computation and Cauchy’s inequality that
\[
\eta^2(x_2)|(\nabla - i\sigma A_0)\varphi_m|^2 \leq |(\nabla - i\sigma A_0)\eta(x_2)\varphi_m|^2 + 2|Re \{\varphi_m\eta(x_2), \eta(x_2)(\nabla - i\sigma A_0)\varphi_m\}| - \eta^2(x_2)|\varphi_m|^2,
\]
\[
\leq |(\nabla - i\sigma A_0)\eta(x_2)\varphi_m|^2 + \frac{1}{2}\eta^2(x_2)|(\nabla - i\sigma A_0)\varphi_m|^2 + \eta^2(x_2)|\varphi_m|^2.
\]
Integrating, we get
\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2)|(\nabla - i\sigma A_0)\varphi_m|^2 \, dx \leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} |(\nabla - i\sigma A_0)\eta(x_2)\varphi_m|^2 + \frac{1}{2}\eta^2(x_2)|\varphi_m|^2 \, dx. \tag{3.29}
\]
Combining (3.28) and (3.29), we get
\[
b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2)|(\nabla - i\sigma A_0)\varphi_m|^2 \, dx \leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2)|\varphi_m|^2 \, dx
\]
\[
+ 4b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2)|\varphi_m|^2 \, dx. \tag{3.30}
\]
The definition of $\eta$ yields that, in $S_{R,m} \cap \{|x_2| \geq 4\}$, $\eta^2 = |x_2|/(\ln |x_2|)^2$, and $\eta^2 \leq 4\eta^2$. Hence, (3.26) and (3.30) imply (3.27).

**Corollary 3.6.** There exists a universal constant $C > 0$ such that, if (3.7) holds, the minimizer $\varphi_{a,b,R,m}$ in Lemma 3.5 satisfies, for all $R > 0$, $m \in \mathbb{N}$,
\[
\int_{S_{R,m}} b |(\nabla - i\sigma A_0)\varphi_{a,b,R,m}|^2 + |\varphi_{a,b,R,m}|^2 \, dx \leq CbR. \tag{3.31}
\]

**Proof.** For the sake of brevity, we will write $\varphi_m$ for $\varphi_{a,b,R,m}$. Using (3.26) and the fact that $|x_2|/(\ln |x_2|)^2 \geq 1$, we get
\[
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} |\varphi_m|^2 \, dx \leq CbR.
\]
On the other hand, using $\|\varphi_m\|_{\infty} \leq 1$ and $b > 1$ we get
\[
\int_{S_{R,m} \cap \{|x_2| < 4\}} |\varphi_m|^2 \, dx \leq CbR.
\]
Next, since $\varphi_m$ satisfies
\[
-b(\nabla - i\sigma A_0)^2\varphi_m = (1 - |\varphi_m|^2)\varphi_m \text{ in } S_{R,m},
\]
a simple integration by parts over $S_{R,m}$ yields
\[
\int_{S_{R,m}} b |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx = \int_{S_{R,m}} |\varphi_m|^2 \, dx - \int_{S_{R,m}} |\varphi_m|^4 \, dx
\]
\[
\leq \int_{S_{R,m}} |\varphi_m|^2 \, dx \leq CbR.
\]
Now, we will investigate the regularity of the minimizer $\varphi_{a,b,R,m}$ in Lemma 3.5. We have to be careful at this point since the magnetic field is a step function and therefore has singularities. As a byproduct, we will extract a convergent subsequence of $(\varphi_{a,b,R,m})_{m \geq 1}$.

We will use the following terminology. Let $\Omega \subset \mathbb{R}^2$ be an open set. If $(u_m)_{m \geq 1}$ is a sequence in $H^k(\Omega)$, then by saying that $(u_m)$ is bounded/convergent in $H^k_{{\text{loc}}} (\Omega)$, we mean that it is bounded/convergent in $H^k(K)$, for every $K \subset \Omega$ open and relatively compact. A similar terminology applies for boundedness/convergence in $C^{k,\alpha}_{\text{loc}}(\Omega)$: A sequence $(u_m)_{m \geq 1}$ is bounded/convergent in $C^{k,\alpha}_{\text{loc}}(\Omega)$ if it is bounded/convergent in $C^{k,\alpha}(K)$, for every $K \subset \Omega$ open and relatively compact.

**Lemma 3.7.** Assume that $(3.7)$ holds. Let $R > 0$ and $\alpha \in (0,1)$ be fixed. The sequence $(\varphi_{a,b,R,m})_{m \geq 1}$ defined by Lemma 3.5 is bounded in $H^2_{{\text{loc}}} (S_R)$ and consequently in $C^{1,\alpha}_{\text{loc}}(S_R)$.

**Proof.** For simplicity, we will write $\varphi_m = \varphi_{a,b,R,m}$. The proof is split into three steps. **Step 1.** We first prove the boundedness of $(\varphi_m)$ in $H^2_{{\text{loc}}} (S_R)$. Using (3.19) we may write

$$\Delta \varphi_m = \frac{1}{b}(|\varphi_m|^2 - 1) \varphi_m + 2i\sigma A_0 \cdot \nabla \varphi_m + |\sigma|^2 |A_0|^2 \varphi_m. \quad (3.32)$$

Assume that $K \subset S_R$ is open and relatively compact. Choose an open and bounded set $\tilde{K}$ such that $\overline{K} \subset \tilde{K} \subset S_R$. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $\tilde{K} \subset S_{R,m}$ and by Cauchy’s inequality,

$$\int_{\tilde{K}} |\nabla \varphi_m|^2 dx \leq 2 \int_{\tilde{K}} |(\nabla - i\sigma A_0)\varphi_m|^2 dx + 2 \int_{\tilde{K}} |\sigma|^2 |A_0|^2 |\varphi_m|^2 dx. \quad (3.31)$$

Using $|\varphi_m| \leq 1$, the decay estimate in (3.31) and the boundedness of $\sigma$ and $A_0$ in $\tilde{K}$, we get a constant $C = C(\tilde{K}, R)$ such that

$$\int_{\tilde{K}} |\nabla \varphi_m|^2 dx \leq C,$$

and

$$\int_{\tilde{K}} |\Delta \varphi_m|^2 dx \leq C,$$

in light of (3.32). By the interior elliptic estimates (see for instance [FH10, Section E.4.1]), we get that $\varphi_m \in H^2(K)$ and

$$\|\varphi_m\|_{H^2(K)} \leq C \left( \|\Delta \varphi_m\|_{L^2(\tilde{K})} + \|\varphi_m\|_{L^2(\tilde{K})} \right) \leq \tilde{C}, \quad (3.33)$$

where $\tilde{C}$ is a constant independent from $m$. This proves that $(\varphi_m)_{m \geq 1}$ is bounded in $H^2_{{\text{loc}}} (S_R)$. **Step 2.** Here we will improve the result in Step 1 and prove that $(\varphi_m)_{m \geq 1}$ is bounded in $H^2_{{\text{loc}}} (S_R)$. It is enough to prove that the sequence $(\nabla \varphi_m)_{m \geq 1}$ is bounded in $H^2_{{\text{loc}}} (S_R)$.

Let $\varsigma_m = \partial_{x_2} \varphi_m$. We will prove that $(\Delta \varsigma_m)_{m \geq 1}$ is bounded in $L^2_{{\text{loc}}} (S_R)$. Recall that, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$A_0(x) = (-x_2, 0) \quad \text{and} \quad \sigma(x) = 1_{\mathbb{R}^+}
(x_2) + a 1_{\mathbb{R}^-}
(x_2),$$
hence,
\[
(\sigma A_0)(x) = \left( -x_2 1_{\mathbb{R}_+}(x_2) - a x_2 1_{\mathbb{R}_-}(x_2), 0 \right), \quad (3.34)
\]
\[
(\sigma^2 |A_0|^2)(x) = x_2^2 1_{\mathbb{R}_+}(x_2) + a^2 x_2^2 1_{\mathbb{R}_-}(x_2). \quad (3.35)
\]

Obviously, the functions in (3.34) and (3.35) admit respectively the following weak partial derivatives
\[
\partial_{x_2}(\sigma A_0)(x) = \left( -1_{\mathbb{R}_+}(x_2) - a 1_{\mathbb{R}_-}(x_2), 0 \right) = \left( -\sigma(x), 0 \right), \quad (3.36)
\]
\[
\partial_{x_2}(\sigma^2 |A_0|^2)(x) = 2x_2 1_{\mathbb{R}_+}(x_2) + 2a^2 x_2 1_{\mathbb{R}_-}(x_2) = 2x_2 \sigma^2(x). \quad (3.37)
\]

A straightforward computation using (3.32), (3.36) and (3.37) yields
\[
\Delta m = \partial_{x_2} \Delta \varphi_m
\]
\[
= \frac{1}{b} \varphi_m^2 \partial_{x_2} \varphi_m + \frac{1}{b} |\varphi_m|^2 \partial_{x_2} \varphi_m - 2i \sigma x_2 \partial_{x_2} \partial_{x_1} \varphi_m - 2i \sigma \partial_{x_1} \varphi_m + \sigma^2 x_2^2 \partial_{x_2} \varphi_m + 2a^2 x_2 \varphi_m,
\]
in the sense of weak derivatives. By Step 1, the sequence \((\varphi_m)\) is bounded in \(H^2_{\text{loc}}(S_R)\). Consequently, since \(|\varphi_m| \leq 1\), it is clear that \((\Delta m)_{m \geq 1}\) is bounded in \(L^2_{\text{loc}}(S_R)\). By the interior elliptic estimates, we get that \((m = \partial_{x_2} \varphi_m)_{m \geq 1}\) is bounded in \(H^2_{\text{loc}}(S_R)\).

In a similar fashion, we prove that \((\partial_{x_1} \varphi_m)_{m \geq 1}\) is bounded in \(H^2_{\text{loc}}(S_R)\).

Step 3. Finally, for every relatively compact open set \(K \subset \Omega\), the space \(H^3(K)\) is embedded in \(C^{1,\alpha}(\overline{K})\). Consequently, \((\varphi_m)\) is bounded in \(C^{1,\alpha}_{\text{loc}}(S_R)\).

**Lemma 3.8.** Assume that \(R > 0\) and that (3.7) holds. Let \((\varphi_{a,b,R,m})_{m \geq 1}\) be the sequence defined in Lemma 3.5. There exists a function \(\varphi_{a,b,R} \in H^3_{\text{loc}}(S_R)\) and a subsequence, denoted by \((\varphi_{a,b,R,m})_{m \geq 1'}\), such that
\[
\varphi_{a,b,R,m} \longrightarrow \varphi_{a,b,R} \text{ in } H^2_{\text{loc}}(S_R) \quad \text{and} \quad \varphi_{a,b,R,m} \longrightarrow \varphi_{a,b,R} \text{ in } C^{0,\alpha}_{\text{loc}}(S_R) \quad (\alpha \in (0,1)).
\]

Furthermore, for all \(\alpha \in (0,1)\), \(\varphi_{a,b,R} \in C^{1,\alpha}_{\text{loc}}(S_R)\).

**Proof.** We continue writing \(\varphi_m\) for \(\varphi_{a,b,R,m}\) and \(\varphi\) for \(\varphi_{a,b,R}\). In the sequel, let \(\alpha \in (0,1)\) be fixed.

Let \(K \subset S_R\) be open and relatively compact. By Lemma 3.7 \((\varphi_m)_{m \geq 1}\) is bounded in \(H^3(K)\), hence it has a weakly convergent subsequence by the Banach–Alaoglu theorem. By the compact embedding of \(H^3(K)\) in \(H^2(K)\), and of \(H^2(K)\) in \(C^{0,\alpha}(\overline{K})\), we may extract a subsequence, that we denote by \((\varphi_m)\), such that it is strongly convergent in \(H^2(K)\) and \(C^{0,\alpha}(\overline{K})\). This subsequence and its limit \(\varphi_K\) are independent of \(\alpha\); we will prove that they are actually independent of the relatively compact set \(K\). This will be done by the standard Cantor’s diagonal process that we outline below.

For all \(p \in \mathbb{N}\), set \(K_p = (-R/2, R/2) \times (-p, p)\). Let \(I_0 = \mathbb{N}\). The sequence \((\varphi_m)_{m \in I_0}\) has a subsequence \((\varphi_m)_{m \in I_1}\) such that it is weakly convergent in \(H^3(K_1)\), and strongly convergent in \(H^2(K_1)\) and \(C^{0,\alpha}(\overline{K_1})\). We denote the limit of this sequence by \(\varphi_1\). Note that \(\varphi_1 \in H^3(K_1)\). By iteration, we obtain a collection of functions \((\varphi_p)_{p \in \mathbb{N}}\) and a collection of subsequences, \((\varphi_m)_{m \in I_p}\), such that

- \(I_1 \supset I_2 \supset I_3 \supset \cdots\)
- for every \(p \in \mathbb{N}\), \((\varphi_m)_{m \in I_p}\) is a subsequence of \((\varphi_m)_{m \in I_{p-1}}\).
- for every \(p \in \mathbb{N}\), the subsequence \((\varphi_m)_{m \in I_p}\) converges weakly to \(\varphi_p\) in \(H^3(K_p)\).
- for every \(p \in \mathbb{N}\), the subsequence \((\varphi_m)_{m \in I_p}\) converges strongly to \(\varphi_p\) in \(H^2(K_p)\) and \(C^{0,\alpha}(\overline{K_p})\).
The Sobolev embedding of $H^3(K_p)$ in $C^{1,\alpha}(\overline{K_p})$ yields that $\varphi_p \in C^{1,\alpha}(\overline{K_p})$. It is useful to note that

$$\text{If } p < q, \text{ then } \varphi_p = \varphi_q \text{ in } K_p. \quad (3.38)$$

Indeed, the strong convergence of $(\varphi_m)_{m \in I_q}$ to $\varphi_q$ in $H^2(K_q)$ implies the following pointwise convergence of $(\varphi_m)_{m \in I_q}$ in $K_q$ (along a subsequence)

$$\lim_{m \to +\infty} \varphi_m(x) = \varphi_q(x), \text{ a.e. in } K_q.$$  

But $K_p \subset K_q$, then

$$\lim_{m \to +\infty} \varphi_m(x) = \varphi_q(x), \text{ a.e. in } K_p. \quad (3.39)$$

Similarly, the strong convergence of $(\varphi_m)_{m \in I_p}$ to $\varphi_p$ in $H^2(K_p)$ implies

$$\lim_{m \to +\infty} \varphi_m(x) = \varphi_p(x), \text{ a.e. in } K_p.$$  

Since $I_q \subset I_p$, we get the following pointwise convergence of $(\varphi_m)_{m \in I_q}$ in $K_p$

$$\lim_{m \to +\infty} \varphi_m(x) = \varphi_p(x), \text{ a.e. in } K_p. \quad (3.40)$$

Having in hand the continuity of $\varphi_p$ and $\varphi_q$, (3.38) follows from (3.39) and (3.40).

Now, we are ready to define the limit function $\varphi$ in $S_R = (-R/2,R/2) \times (-\infty, +\infty)$ as follows. Let $x \in S_R$. There exists $p \in \mathbb{N}$ such that $x \in K_p$. We then define $\varphi(x) = \varphi_p(x)$. The function $\varphi$ is well defined by (3.38) and belongs to $H^3_{\text{loc}}(S_R)$, consequently to $C^{1,\alpha}_{\text{loc}}(S_R)$. Next, we will construct a subsequence $(\varphi_m)_{m \in I}$ of $(\varphi_m)_{m \in I_0}$ (with $I \subset I_0$) that converges weakly to the function $\varphi$ in $H^3(K_p)$, for all $p \in \mathbb{N}$. For all $p \geq 1$, the set $I_p \subset \mathbb{N}$ consists of a strictly increasing sequence $\{n_1(p), n_2(p), \ldots\}$: let $n_p$ be the $p^{th}$ element of $I_p$, i.e. $n_p = n_p(p)$. By induction, we can prove that, for all $p, k \in \mathbb{N}$ (with $k \geq 2$), $n_k(p + 1) > n_{k-1}(p + 1) \geq n_{k-1}(p)$. Thus, for all $p \in \mathbb{N}$, $n_{p+1} := n_{p+1}(p + 1) > n_p(p) = n_p$. We define the index set $I = \{n_1, n_2, \ldots\}$ and note that $(\varphi_m)_{m \in I}$ is a subsequence of $(\varphi_m)_{m \geq 1}$, because $n_1 < n_2 < \ldots$. Also, it is a subsequence of $(\varphi_m)_{m \in I_p}$, for every $p \in \mathbb{N}$. Consequently, for all $p \in \mathbb{N}$, the following strong convergence holds

$$\varphi_m \xrightarrow{m \to +\infty} \varphi \text{ in } H^2(K_p) \text{ and } C^{0,\alpha}(\overline{K_p}). \quad (3.41)$$

Finally, if $K \subset S_R$ is an arbitrary open and relatively compact set, then there exists $p \in \mathbb{N}$ such that $K \subset K_p$. Consequently, we inherit from (3.41) that $(\varphi_m)_{m \in I}$ converges to $\varphi$ in $H^2(K)$ and $C^{0,\alpha}(K)$. \qed

**Lemma 3.9.** Assume that $R > 0$ and (3.7) holds. Let $\varphi_{a,b,R}$ be the function defined by Lemma 3.8 The following statements hold:

$$\varphi_{a,b,R} \in \mathcal{D}_R, \quad (3.42)$$

$$|\varphi_{a,b,R}| \leq 1 \quad \text{in } S_R, \quad (3.43)$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left( |(\nabla - i\sigma A_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \quad (3.44)$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_{a,b,R}|^4 dx \leq Cb^2 R, \quad (3.45)$$

$$\int_{S_R} \left( b |(\nabla - i\sigma A_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \quad (3.46)$$

where $C > 0$ is a universal constant and $\mathcal{D}_R$ is the space introduced in (3.2).
Proof. Let \((\varphi_{a,b,R,m})\) be the subsequence in Lemma 3.8. Again, we will use \((\varphi_m)\) and \(\varphi\) for \((\varphi_{a,b,R,m})\) and \(\varphi_{a,b,R}\) respectively.

By Lemma 3.5 the inequality \(|\varphi_m| \leq 1\) holds for all \(m\). The inequality \(|\varphi| \leq 1\) then follows from the uniform convergence of \((\varphi_m)\) stated in Lemma 3.8. By the convergence of \((\varphi_m)\) in \(H^2_{\text{loc}}(S_R)\) and \(C^{0,\alpha}(S_R)\), we get (3.43) from

\[-b(\nabla - i\sigma A_0)\varphi_m = (1 - |\varphi_m|^2)\varphi_m.\]

Now we prove that \(\varphi \in \mathcal{D}_R\). Pick an arbitrary integer \(m_0 \geq 1\). For all \(m \geq m_0\), \(S_{R,m_0} \subset S_{R,m}\). Thus using the decay of \(\varphi_m\) in (3.31) we have

\[
\int_{S_{R,m_0}} |\varphi_m|^2 \, dx \leq \int_{S_{R,m}} |\varphi_m|^2 \, dx \leq CbR.
\]

The uniform convergence of \((\varphi_m)\) to \(\varphi\) gives us

\[
\int_{S_{R,m_0}} |\varphi|^2 \, dx = \lim_{m \to +\infty} \int_{S_{R,m_0}} |\varphi_m|^2 \, dx \leq CbR.
\]

Taking \(m_0 \to +\infty\), we write by the monotone convergence theorem,

\[
\int_{S_R} |\varphi|^2 \, dx \leq CbR.
\]

This proves that \(\varphi \in L^2(S_R)\). Next we will prove that \((\nabla - i\sigma A_0)\varphi \in L^2(S_R)\). In light of the convergence of \((\varphi_m)\) in \(H^1_{\text{loc}}(S_R)\), we can refine the subsequence \((\varphi_m)\) so that

\[(\nabla - i\sigma A_0)\varphi_m \to (\nabla - i\sigma A_0)\varphi \text{ a.e.}\]

Furthermore, by Lemma 3.7 \((\varphi_m)\) is bounded in \(C^1_{\text{loc}}(S_R)\), hence in \(C^1(S_{R,m_0})\), for all \(m_0 \geq 1\). Using the dominated convergence theorem and the estimate in (3.31), we may write, for all \(m_0 \geq 1\),

\[
\int_{S_{R,m_0}} |(\nabla - i\sigma A_0)\varphi|^2 \, dx = \lim_{m \to +\infty} \int_{S_{R,m_0}} |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx \leq CR.
\]

Sending \(m_0 \to +\infty\) and using the monotone convergence theorem, we get

\[
\int_{S_R} |(\nabla - i\sigma A_0)\varphi|^2 \, dx \leq CR.
\]

Thus, we have proved that \(\varphi, (\nabla - i\sigma A_0)\varphi \in L^2(S_R)\). It remains to prove that \(\varphi\) satisfies the boundary condition

\[\varphi \left( x_1 = \pm \frac{R}{2}, x_2 \right) = 0, \quad \text{for all } x_2 \in \mathbb{R}.\]

To see this, let \(x_2 \in \mathbb{R}\). There exists \(m_0\) such that \(x_2 \in (-m_0, m_0)\). By the convergence of \((\varphi_m)\) to \(\varphi\) in \(C^{0,\alpha}(S_{R,m_0})\), we get

\[\varphi \left( x_1 = \pm \frac{R}{2}, x_2 \right) = \lim_{m \to +\infty} \varphi_m \left( x_1 = \pm \frac{R}{2}, x_2 \right) = 0.\]

Finally, we may use similar limiting arguments to pass from the decay estimates of \(\varphi_m\) in (3.16) and (3.17) to the decay estimates of \(\varphi\) in (3.44) and (3.45). \(\square\)

Now, we are ready to establish the existence of a minimizer of the Ginzburg–Landau energy \(G(a,b,R)\) defined in the unbounded set \(S_R\).
Lemma 3.10. Assume that (3.7) holds. For all \( R > 0 \), the function \( \varphi_{a,b,R} \in D_R \) defined in Lemma 3.8 is a minimizer of \( G_{a,b,R} \), that is
\[
G_{a,b,R}(\varphi_{a,b,R}) = g_a(b, R).
\]
Here \( G_{a,b,R} \) is the functional introduced in (3.3) and \( g_a(b, R) \) is the ground state energy defined in (3.4).

Proof. The proof is divided into three steps.

Step 1. (Convergence of the ground state energy). Let \( g_a(b, R, m) \) and \( g_a(b, R) \) be the energies defined in (3.4) and (3.14) respectively. In this step, we will prove that
\[
\lim_{m \to +\infty} g_a(b, R, m) = g_a(b, R).
\]
Let \( u \in D_{R,m} \). We can extend \( u \) by 0 to a function \( \tilde{u} \in D_R \). As an immediate consequence, we get
\[
g_a(b, R, m) \geq g_a(b, R), \quad \text{for all } m \in \mathbb{N}.
\]
Thus,
\[
\liminf_{m \to +\infty} g_a(b, R, m) \geq g_a(b, R).
\]
(3.48)

Next, we will prove that
\[
\limsup_{m \to +\infty} g_a(b, R, m) \leq g_a(b, R).
\]
(3.49)

Consider \( (\varphi_n) \subset D_R \) a minimizing sequence of \( G_{a,b,R} \), that is
\[
g_a(b, R) = \lim_{n \to +\infty} G_{a,b,R}(\varphi_n).
\]
Let \( \vartheta \in C_c(\mathbb{R}) \) be a cut-off function satisfying
\[
0 \leq \vartheta \leq 1 \text{ in } \mathbb{R}, \quad \text{supp } \vartheta \subset (-1, 1), \quad \vartheta = 1 \text{ in } \left[ -\frac{1}{2}, \frac{1}{2} \right].
\]
Consider the re-scaled function \( \vartheta_m(x_2) = \vartheta(x_2/m) \). The function \( \vartheta_m(x_2) \varphi_n(x) \) restricted to \( S_{R,m} \) belongs to \( D_{R,m} \) and consequently
\[
g_a(b, R, m) \leq G_{a,b,R}(\vartheta_m \varphi_n).
\]
(3.50)

By Cauchy’s inequality, for all \( \epsilon \in (0, 1) \)
\[
|\langle \nabla - i\sigma A_0 \rangle \vartheta_m \varphi_n|^2 \leq (1 + \epsilon)|\vartheta_m|\langle \nabla - i\sigma A_0 \rangle \varphi_n|^2 + 2\epsilon^{-1}|\nabla \vartheta_m|^2 |\varphi_n|^2.
\]
Thus, using the definition of the ground state energy \( g_a(b, R, m) \) and the functional \( G_{a,b,R} \) in (3.14) and (3.3) respectively, we obtain
\[
g_a(b, R, m) \leq (1+\epsilon)G_{a,b,R}(\varphi_n) + \frac{2b\epsilon^{-1}}{m^2} \| \vartheta' \|^2_{L_{\infty}(\mathbb{R})} \int_{S_R} |\varphi_n|^2 \, dx + \int_{S_R} (1-\vartheta_m^2 + \epsilon) |\varphi_n|^2 \, dx.
\]
(3.51)

Introducing \( \limsup \) on both sides of (3.51), and using the dominated convergence theorem, we get
\[
\limsup_{m \to +\infty} g_a(b, R, m) \leq (1+\epsilon)G_{a,b,R}(\varphi_n) + \epsilon \int_{S_R} |\varphi_n|^2 \, dx.
\]
Taking the successive limits \( \epsilon \to 0, \) then \( n \to +\infty \), we get (3.49). Combining (3.48) and (3.49), we get (3.47).
Step 2. (The $L^4$-norm of the limit function). Let $(\varphi_m = \varphi_{a,b,R,m})$ be the sequence in Lemma 3.8 which converges to the function $\varphi = \varphi_{a,b,R}$. We would like to verify that the limit function $\varphi$ is a minimizer of the functional $G_{a,b,R}$. To that end, we will prove first that
\[
\lim_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx = \int_{S_R} |\varphi|^4 \, dx.
\] (3.52)

We begin by proving that
\[
\liminf_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx \geq \int_{S_{R,m_0}} |\varphi_m|^4 \, dx.
\] (3.53)

Pick a fixed integer $m_0 \geq 1$. Since $S_{R,m} \supset S_{R,m_0}$ for all $m \geq m_0$, the following inequality holds
\[
\int_{S_{R,m}} |\varphi_m|^4 \, dx \geq \int_{S_{R,m_0}} |\varphi_m|^4 \, dx.
\] (3.54)

In addition, having in hand the uniform convergence of $\varphi_m$ to $\varphi$ on the compact set $S_{R,m_0}$, we get as $m \to \infty$
\[
\int_{S_{R,m_0}} |\varphi_m|^4 \, dx \to \int_{S_{R,m_0}} |\varphi|^4 \, dx.
\] (3.55)

We introduce $\liminf_{m \to +\infty}$ on both sides of (3.54), and we use (3.55) to get
\[
\liminf_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx \geq \int_{S_{R,m_0}} |\varphi|^4 \, dx.
\] This is true for every integer $m_0 \geq 1$. Consequently (3.53) simply follows by applying the monotone convergence theorem.

Next, we prove that
\[
\limsup_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx.
\] (3.56)

Let $C$ be the universal constant in (3.17), $\epsilon > 0$ be fixed, and $R > 0$ be arbitrary. We select an integer $m_0 \geq 1$ such that
\[
\frac{C b^2 R}{m_0} < \epsilon.
\] (3.57)

In light of (3.55), there exists $m_1 \geq m_0$ such that
\[
\forall \ m \geq m_1, \quad \left| \int_{S_{R,m_0}} |\varphi_m|^4 \, dx - \int_{S_{R,m_0}} |\varphi|^4 \, dx \right| \leq \epsilon.
\]

Noticing that
\[
\int_{S_{R,m_0}} |\varphi|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx,
\]
we may write, for all $m \geq m_1$
\[
\int_{S_{R,m_0}} |\varphi_m|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx + \epsilon.
\] (3.58)

On the other hand, for $|x_2| \geq m_0 \geq 1$ we have,
\[
m_0 \leq \frac{|x_2|^3}{(\ln |x_2|)^2}.
\]
Thus, the estimate in (3.17) yields for all $m \geq m_0$,
\[
\int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 \, dx \leq \frac{C \beta^2 R}{m_0} < \epsilon.
\] (3.59)

Combining (3.58) and (3.59), we get for all $m \geq m_1 \geq m_0$
\[
\int_{S_{R,m}} |\varphi_m|^4 \, dx = \int_{S_{R,m_0}} |\varphi_m|^4 \, dx + \int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 \, dx,
\]
\[
\leq \int_{S_R} |\varphi|^4 \, dx + 2\epsilon.
\]

Taking the successive limits $m \to +\infty$ then $\epsilon \to 0_+$, we get (3.56).

Step 3. (The limit function is a minimizer). The convergence in (3.52) is crucial in establishing that $\varphi$ is a minimizer of $G_{a,b,R}$. In light of Eq. (3.19), an integration by parts yields, for all $m \geq 1$,
\[
g_a(b,R,m) = -\frac{1}{2} \int_{S_{R,m}} |\varphi_m|^4 \, dx.
\] (3.60)

By Lemma 3.9, $\varphi \in D_R$ and satisfies (3.43), so after integrating by parts, we get
\[
G_{a,b,R}(\varphi) = -\frac{1}{2} \int_{S_R} |\varphi|^4 \, dx.
\] (3.61)

Comparing (3.60) and (3.61) yields that $G_{a,b,R}(\varphi) = g_a(b,R)$. \hfill \Box

Proof of Proposition 3.4. This proposition is simply a convenient collection in one place of already proved facts in Lemma 3.9 and Lemma 3.10. \hfill \Box

3.4. The limit energy. In this section, we will prove the existence of the limit energy $\epsilon_a(b)$, defined as the limit of $g_a(b,R)/R$ as $R \to +\infty$. After that, we will study, when the parameter $a$ is fixed, some properties of the function $b \mapsto \epsilon_a(b)$.

In the sequel, we assume that $a,b,R$ are constants such that $R \geq 1$ and (3.7) holds.

The next lemma displays some simple, yet very important, translation invariance property of the energy. This property is mainly needed in Theorem 3.1 to establish an upper bound of the limit energy $\epsilon_a(b)$.

Lemma 3.11. Let $n \in \mathbb{N}$. Consider the ground state energy $g_a(b,R)$ defined in (3.4). It holds
\[
g_a(b,nR) \leq n g_a(b,R).
\]

Proof. Let $S_R = (-R/2, R/2) \times \mathbb{R}$ be the strip defined in (3.1). Consider the strip $S_{R,\lambda}$ defined such that
\[
S_{R,\lambda} = S_R + \lambda R, \quad \lambda \in \mathbb{R}.
\]

Let $u \in D_R$, where $D_R$ is the domain defined in (3.2). We define the function $v$ in $S_{R,\lambda}$ as follows
\[
v(x_1, x_2) = u(x_1 - \lambda R, x_2), \quad (x_1, x_2) \in S_{R,\lambda}.
\] (3.62)
An easy computation shows the invariance of the energy under the aforementioned translation, that is
\[
\int_{S_R} b |(\nabla - i \sigma A_0)u|^2 \, dy = \int_{S_{R,\lambda}} b |(\nabla - i \sigma A_0)v|^2 \, dx,
\] (3.63)
and
\[
\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left(b |(\nabla - i \sigma A_0)u|^2 - |u|^2 + \frac{1}{2} |u|^4\right) \, dy = \int_{S_{R,\lambda}} \left(b |(\nabla - i \sigma A_0)v|^2 - |v|^2 + \frac{1}{2} |v|^4\right) \, dx.
\] (3.64)

Now, let \( n \in \mathbb{N} \). Noticing that \( S_{nR} = [-n R/2, n R/2] \times \mathbb{R} = \bigcup_{j \in J} S_{R,j} \), where \( J = \{(1 - n)/2 + k, \ 0 \leq k \leq n - 1\} \), we define a function \( \tilde{u} \) in \( S_{nR} \) as follows
\[
\tilde{u}(x_1, x_2) = u(x_1 - jR, x_2), \quad \text{if} \ (x_1, x_2) \in S_{R,j}.
\]
This definition is consistent, since the sets \( (S_{R,j}) \) are disjoint, their closures cover \( S_{nR} \), and \( u \in D_R \) which yields that \( \tilde{u} \) vanishes on the boundary of every \( S_{R,j} \). Having (3.64), we get consequently
\[
\tilde{u} \in D_{nR} \quad \text{and} \quad \mathcal{G}_{a,b,nR}(\tilde{u}) = n \mathcal{G}_{a,b,R}(u),
\]
This yields that
\[
\mathfrak{g}_a(b,nR) \leq n \mathcal{G}_{a,b,R}(u).
\]

We choose \( u \in D_R \) to be the minimizer \( \varphi_{a,b,R} \) of \( \mathcal{G}_{a,b,R} \), defined in Proposition 3.4 and conclude that
\[
\mathfrak{g}_a(b,nR) \leq n \mathfrak{g}_a(b,R).
\]

Our next result is concerned with the monotonicity of the function \( R \mapsto \mathfrak{g}_a(b,R) \).

**Lemma 3.12.** The function \( R \mapsto \mathfrak{g}_a(b,R) \) defined in (3.4) is monotone non-increasing.

**Proof.** This follows from the domain monotonicity. Indeed, let \( r > 0 \) and \( u \in H_0^1(S_R) \) be a minimizer of \( \mathcal{G}_{a,b,R} \). Consider the function \( \tilde{u} \in H_0^1(S_{R+r}) \) defined as the extension of \( u \) by zero on \( S_{R+r} \setminus S_R \). Obviously
\[
\mathfrak{g}_a(b,R) = \mathcal{G}_{a,b,R}(u) = \mathcal{G}_{a,b,R+r}(\tilde{u}) \geq \mathfrak{g}_a(b,R+r).
\]

The existence of the limit of \( \mathfrak{g}_a(b,R)/R \) as \( R \to +\infty \) will follow from a well known abstract result, see Lemma 3.14 below. To apply this abstract result, we need some bounds on the energy \( \mathfrak{g}_a(b,R) \). These are given in Lemma 3.13 below.

**Lemma 3.13.** Let \( \mathfrak{g}_a(b,R) \) be the ground state energy in (3.4). There exist positive constants \( C_1, C_2, \) and \( C_3 \) dependent solely on \( a \) and \( b \) such that
\[
-C_1 R \leq \frac{\mathfrak{g}_a(b,R)}{1 - b \beta_a} \leq -C_2 R + \frac{C_3}{R}.
\] (3.65)
Proof.

**Upper bound.** Let $\vartheta \in C_c^{\infty}(\mathbb{R})$ be a function satisfying
\[
\text{supp } \vartheta \subset \left(-\frac{1}{2}, \frac{1}{2}\right), \quad 0 \leq \vartheta \leq 1, \quad \vartheta = 1 \text{ in } \left(-\frac{1}{4}, \frac{1}{4}\right),
\]
and let
\[
\theta_R(x) = \vartheta(x/R).
\]
We define the function
\[
v(x) = \theta_R(x_1)\psi_a(x),
\]
where $\psi_a$ is the eigenfunction introduced in (2.33). Recall that $\psi_a(x_1, x_2) = e^{i\xi_a x_1} \phi_a(x_2)$ with $\xi_a$ and $\phi_a$ defined in (2.32), and $\phi_a$ satisfies $\|\phi_a\|_{L^2(\mathbb{R})} = 1$.

The function $\psi_a$ satisfies $-(\nabla - i\sigma A_0)^2 \psi_a = \beta a \psi_a$ in $\mathbb{R}^2$. Multiplying this equation by $v = \theta_R^2 \psi_a$ then integrating on the support of the function $v$, we get
\[
\int_{S_R} |(\nabla - i\sigma A_0)v|^2 \, dx = \beta a \int_{S_R} \theta_R^2(x_1)|\psi_a(x)|^2 \, dx + \int_{S_R} \theta_R^2(x_1)\phi_a^2(x_2) \, dx 
\leq \beta a \int_{S_R} \theta_R^2(x_1)|\psi_a(x)|^2 \, dx + \frac{C}{R}.
\]
Consequently, we get for all $t > 0$
\[
g_a(b,R) \leq G_{a,b,R}(tv)
\leq t^2(b\beta_a - 1) \int_{S_R} \theta_R^2(x_1)\phi_a^2(x_2) \, dx + Cb\frac{t^2}{R} + \frac{t^4}{2} \int_{S_R} \theta_R^2(x_1)\phi_a^4(x_2) \, dx 
\leq t^2(b\beta_a - 1)R + Cb\frac{t^2}{R} + \frac{t^4}{2} \int_{\mathbb{R}} \phi_a^4(x_2) \, dx_2
\]
\[= t^2 R \left( (b\beta_a - 1) + \frac{t^2}{2} \int_{\mathbb{R}} \phi_a^4(x_2) \, dx_2 \right) + Cb\frac{t^2}{R}. \]

We select $t$ such that
\[(b\beta_a - 1) + \frac{t^2}{2} \int_{\mathbb{R}} \phi_a^4(x_2) \, dx_2 = \frac{1}{2}(b\beta_a - 1) < 0 \text{ by (3.7)}. \]
Let $\nu_a = \int_{\mathbb{R}} \phi_a^4(x_2) \, dx_2$. Thus, for $t = \sqrt{(1-b\beta_a)/\nu_a}$ we get
\[
g_a(b,R) \leq \frac{1}{1-b\beta_a} \leq -C_2 R + \frac{C_3}{R},
\]
where $C_2 = (1/2)t^2$ and $C_3 =Cb/\nu_a$ depend only on $a$ and $b$.

**Lower bound.** Let $\varphi_{a,b,R}$ be the minimizer in Proposition 3.4. For simplicity, we will write $\varphi = \varphi_{a,b,R}$. It follows from the min-max principle that
\[
g_a(b,R) = G_{a,b,R}(\varphi) \geq (b\beta_a - 1) \int_{S_R} |\varphi|^2 \, dx.
\]
By (3.7), $b\beta_a - 1 < 0$. By (3.11), $\int_{S_R} |\varphi|^2 \, dx \leq CbR$, where $C > 0$ is a universal constant. We choose $C_1 = C/\beta_a$ and get the following inequality
\[g_a(b,R) \geq Cb(b\beta_a - 1)R \geq -C_1(1-b\beta_a)R.
\]
Obviously, $C_1$ depends solely on $a$. \hfill \Box

The next abstract lemma is a key-ingredient in the proof of Theorem 3.1 and more precisely in establishing the existence of the limit energy $\epsilon_a(b)$ introduced in (3.5). Variants of it were used in many papers, see [FK13, FKP13, Pan02, SS03]. Here we use the version from [FK13] Lemma 2.2.
Lemma 3.14. Let $\delta > 0$. Consider a monotone non-increasing function $d : (\delta, +\infty) \to (-\infty, 0]$ such that the function $f : (\delta, +\infty) \ni l \mapsto d(l)/l^2 \in \mathbb{R}$ is bounded. Suppose that there exists a constant $C > 0$ such that the estimate
\[
f(nl) \geq f((1 + \alpha)l) - C \left( \alpha + \frac{1}{\alpha^2l^2} \right)
\]
holds true for all $\alpha \in (0, 1)$, $n \in \mathbb{N}$, and $l \geq \ell_0$. Then $f(l)$ has a limit $A$ as $l \to +\infty$.

Furthermore, for all $l \geq 2\ell_0$, the following estimate holds
\[
f(l) \leq A + \frac{2C}{l^3}.
\]

We will apply Lemma 3.14 on the function $f : R \mapsto g_a(b, R)/R$ in order to define $\epsilon_a(b)$ as $\lim_{R \to +\infty} g_a(b, R)/R$. To that end, we establish that the above choice of $f$ fulfills the conditions in Lemma 3.14. This is the content of Lemma 3.15. There exists a universal constant $C > 0$ such that, for all $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, the ground state energy $g_a(b, R)$ defined in (3.4) satisfies
\[
g_a(b, n^2R)/n^2R \geq g_a(b, (1 + \alpha)^2R)/((1 + \alpha)^2R) - Cb^2\left( \alpha + \frac{1}{\alpha^2R} \right). \tag{3.66}
\]

Proof. Let $n \geq 1$ be a natural number, $\alpha \in (0, 1)$ and consider the family of strips $S_j = \left(-n^2 - 1 - \alpha + (2j - 1)\left(1 + \frac{\alpha}{2}\right), -n^2 - 1 + (2j + 1)\left(1 + \frac{\alpha}{2}\right)\right) \times \mathbb{R}$, $(j \in \mathbb{Z})$

Notice that the width of $S_j$ is $2(1 + \alpha)$, and the width of the overlapping region between two strips, when it exists, is $\alpha$. We consider the partition of unity of $\mathbb{R}^2$:
\[
\sum_j |\chi_j|^2 = 1, \quad 0 \leq \chi_j \leq 1, \quad \sum_j |\nabla \chi_j|^2 \leq \frac{C}{\alpha^2}, \quad \text{supp} \chi_j \subset S_j,
\]
where $C$ is a universal constant. Define
\[
\chi_{R,j}(x) = \chi_j(2x/R)
\]
($\chi_{R,j}$) is then a new partition of unity satisfying
\[
\sum_j |\chi_{R,j}|^2 = 1, \quad 0 \leq \chi_{R,j} \leq 1, \quad \sum_j |\nabla \chi_{R,j}|^2 \leq \frac{C}{\alpha^2R^2}, \quad \text{supp} \chi_{R,j} \subset S_{R,j}, \tag{3.67}
\]
where $S_{R,j} = \{xR/2 : x \in S_j\}$. The family of strips $(S_{R,j})_{j \in \{1, 2, \ldots, n^2\}}$ yields a covering of $S_{n^2R} = (-n^2R/2, n^2R/2) \times \mathbb{R}$ by $n^2$ strips, each of width $(1 + \alpha)R$. Let $\varphi_{a,b,n^2R} \in D_{n^2R}$ be the minimizer in Proposition 3.4. We decompose the energy associated to $\varphi_{a,b,n^2R}$ as follows
\[
g_a(b, n^2R) = G_{a,b,n^2R}(\varphi_{a,b,n^2R})
\]
\[
\geq \sum_{j=1}^{n^2} \left( G_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) - b\|\nabla \chi_{R,j} \varphi_{a,b,n^2R}\|^2_{L^2(S_{n^2R})} \right)
\]
\[
= \sum_{j=1}^{n^2} G_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) - b \int_{S_{n^2R}} \left( \sum_{j=1}^{n^2} |\nabla \chi_{R,j}|^2 \right) |\varphi_{a,b,n^2R}|^2 \, dx
\]
\[
\geq \sum_{j=1}^{n^2} G_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) - Cb^2n^2 \frac{n^2}{\alpha^2R}.
\]

The first inequality above follows from the celebrated IMS localization formula (see [CFKS09, Theorem 3.2]), while the second comes from (3.11) and the properties
of $\chi_{R,j}$ in (3.67). Notice that $\chi_{R,j} \varphi_{a,b,n^2R}$ is supported in an infinite strip of width $(1 + \alpha)R$. By energy translation invariance along the $x_1$-direction (see (3.64)), we have

$$G_{a,b,n^2R}(\chi_{R,j} \varphi_{a,b,n^2R}) \geq g_a(b, (1 + \alpha)R).$$

As a consequence,

$$g_a(b, n^2R) \geq n^2g_a(b, (1 + \alpha)R) - Cb^2\frac{n^2}{\alpha^2R}.$$

For $R \geq 1$, dividing both sides by $n^2R$ and using the monotonicity of $R \mapsto g_a(b, R)$, we get

$$\frac{g_a(b, n^2R)}{n^2R} \geq \frac{g_a(b, (1 + \alpha)R)}{R} - Cb^2\frac{1}{\alpha^2R^2} \geq \frac{g_a(b, (1 + \alpha)^2R)}{(1 + \alpha)^2R} - Cb^2\left(\alpha + \frac{1}{\alpha^2R}\right).$$

\[\square\]

### 3.5. Proof of Theorem 3.1

Here we will verify all the statements appearing in Theorem 3.1. Noticing that $G_{a,b,R}(0) = 0$, we get Item (1). The second item is already proved in Lemma 3.2. Defining $c_a(b) = 0$ for $b \geq 1/\beta_a$, the items (3) and (5) hold trivially since $g_a(b, R) = 0$ in this case. For these two items, we handle now the case where $1/|a| \leq b < 1/\beta_a$.

In the proof of Item (3), we define the two functions $d_{a,b}(l) = g_a(b, l^2)$ and $f_{a,b}(l) = d_{a,b}(l)/l^2$. Using Lemmas 3.12 and 3.13, we see that $d_{a,b}(\cdot)$ is non-positive, monotone non-increasing, and that $f_{a,b}(\cdot)$ is bounded. Reformulating (3.66) by taking $R = l^2$, we get for $\ell \geq 2$

$$f_{a,b}(nl) \geq f_{a,b}((1 + \alpha)l) - Cb^2\left(\alpha + \frac{1}{\alpha^2l^2}\right).$$

Thus, the functions $d_{a,b}(l)$ and $f_{a,b}(l)$ satisfy the assumptions in Lemma 3.14. This assures the existence of a constant $c_a(b) \leq 0$, depending on $a$ and $b$, such that

$$\lim_{l \to +\infty} f_{a,b}(l) = c_a(b),$$

that is, with the choice $l = \sqrt{R}$

$$\lim_{R \to +\infty} \frac{g_a(b, R)}{R} = c_a(b).$$

Moreover, Lemma 3.13 ensures that $c_a(b) < 0$. The upper bound in Item (5) of Theorem 3.1 follows from Lemma 3.14. It remains to establish a lower bound for $g_a(b, R)/R$. Let $n \geq 1$ be an integer. By Lemma 3.11

$$g_a(b, nR) \leq n\ g_a(b, R).$$

Dividing both sides by $nR$ and taking $n \to +\infty$ yields

$$\frac{g_a(b, R)}{R} \geq c_a(b).$$

The monotonicity of the function $c_a(\cdot)$ is straightforward and follows from that of $g_a(\cdot, R)$. Let $\epsilon > 0$, we have $g_a(b + \epsilon, R) \geq g_a(b, R)$. Dividing both sides of this inequality by $R$ then taking $R \to +\infty$ gives us $c_a(b + \epsilon) \geq c_a(b)$. Our final task is to prove the continuity of the function $c_a(\cdot)$. Let $b \in \left[\frac{1}{|a|}, 1/\beta_a\right]$, and $\epsilon > 0$. We will prove that $c_a(\cdot)$ is right continuous at $b$. Since $c_a(\cdot)$ is monotone non-decreasing, $c_a(b + \epsilon) \geq c_a(b)$. Consequently,

$$\lim_{\epsilon \to 0_+} c_a(b + \epsilon) \geq c_a(b).$$
Hence, it is sufficient to prove that $\limsup_{\epsilon \to 0^+} \epsilon_a(b + \epsilon) \leq \epsilon_a(b)$. We may use the following lower bound from (3.6),

$$\epsilon_a(b + \epsilon) \leq \frac{g_a(b + \epsilon, R)}{R},$$

which in turn yields

$$\limsup_{\epsilon \to 0^+} \epsilon_a(b + \epsilon) \leq \limsup_{\epsilon \to 0^+} \frac{g_a(b + \epsilon, R)}{R}. \quad (3.68)$$

Let $u \in D_R$. We have $g_a(b + \epsilon, R) \leq G_{a,b+\epsilon,R}(u)$, where the functional $G_{a,\cdot,R}$ is defined in (3.3). We infer from (3.68) that

$$\limsup_{\epsilon \to 0^+} \epsilon_a(b + \epsilon) \leq \frac{G_{a,b,R}(u)}{R} + \limsup_{\epsilon \to 0^+} \frac{\epsilon}{R} \int |(\nabla - i\sigma A_0)u|^2 \, dx$$

$$= \frac{G_{a,b,R}(u)}{R}. \quad \text{(3.68)}$$

This is true for all $u \in D_R$ and $R \geq 1$. Minimizing over $u \in D_R$ yields, for all $R \geq 1$,

$$\limsup_{\epsilon \to 0^+} \epsilon_a(b + \epsilon) \leq \frac{g_a(b, R)}{R}.$$

Taking $R \to +\infty$, we get the desired inequality.

Let $b \in \left(\frac{1}{|a|}, \frac{1}{\beta a}\right]$ and $\epsilon < 0$. Now we prove the left continuity at $b$. The monotonicity of $\epsilon_a(\cdot)$ yields that $\limsup_{\epsilon \to 0^-} \epsilon_a(b + \epsilon) \leq \epsilon_a(b)$. So, it is sufficient to prove that

$$\liminf_{\epsilon \to 0^-} \epsilon_a(b + \epsilon) \geq \epsilon_a(b).$$

Let $\varphi_{a,b+\epsilon,R}$ be the minimizer of $G_{a,b+\epsilon,R}$ defined in (3.8). Using the upper bound in (3.6) together with (3.11), we get

$$\epsilon_a(b + \epsilon) \geq \frac{g_a(b + \epsilon, R)}{R} - C \frac{(b + \epsilon)^2}{R^\frac{3}{2}}$$

$$\geq \frac{G_{a,b+\epsilon,R}(\varphi_{a,b+\epsilon,R})}{R} - C \frac{(b + \epsilon)^2}{R^\frac{3}{2}}$$

$$\geq \frac{G_{a,b,R}(\varphi_{a,b+\epsilon,R})}{R} + \frac{\epsilon}{R} \int |(\nabla - i\sigma A_0)\varphi_{a,b+\epsilon,R}|^2 \, dx - C \frac{(b + \epsilon)^2}{R^\frac{3}{2}}$$

$$\geq \frac{g_a(b, R)}{R} + C\epsilon - C \frac{(b + \epsilon)^2}{R^\frac{3}{2}}.$$  

Hence,

$$\liminf_{\epsilon \to 0^-} \epsilon_a(b + \epsilon) \geq \frac{g_a(b, R)}{R} - C \frac{b^2}{R^\frac{3}{2}}.$$ 

Taking $R \to +\infty$, we get the desired inequality.

3.6. **An effective one-dimensional energy.** Assume that $a \in [-1, 1) \setminus \{0\}$ and $b > 0$. For all $\xi \in \mathbb{R}$, consider the functional

$$E_{a,b,\xi}^{1D}(f) = \int_{-\infty}^{0} \left( b |f'(t)|^2 + b(at + \xi)^2 |f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) \, dt$$

$$+ \int_{0}^{+\infty} \left( b |f'(t)|^2 + b(t + \xi)^2 |f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) \, dt. \quad (3.69)$$
defined over the space $B^1(\mathbb{R})$, and let
\[
E^{1D}_{a,b}(\xi) = \inf_{f \in B^1(\mathbb{R})} \mathcal{E}^{1D}_{a,b,\xi}(f).
\] (3.70)

We would like to find a relationship between the 2D-energy in (3.4) and the 1D-energy in (3.70) for some specific value of $\xi$. The existing results on the Ginzburg–Landau functional with a uniform magnetic field suggest that we should select $\xi$ so as to minimize the function $\xi \mapsto E^{1D}_{a,b}(\xi)$, see [AH07, CR14, Pan02].

In light of Remark 3.3, we will assume that $a$ and $b$ satisfy
\[
a \in [-1, 0) \quad \text{and} \quad b \geq \frac{1}{|a|}.
\] (3.71)

Under (3.71), the numerical computations indicate that the global minimum $\beta_a$, defined in (2.10), is attained at a non-degenerate unique point, denoted by $c_a$ in (2.21) (see [HPRS16, Section 1.3]). To our knowledge, such a uniqueness result has not been analytically proven yet. In the sequel, we will assume that uniqueness of $c_a$ holds. Under this assumption, Proposition A.4 yields that for each fixed value of $b$ such that $1/|a| < b < 1/\beta_a$, there exist two real numbers $\xi_1(a, b)$ and $\xi_2(a, b)$ satisfying
\[
\xi_1(a, b) < c_a < \xi_2(a, b),
\]
and
\[
(\mu_a)^{-1}\left((\beta_a, b^{-1})\right) = \left(\xi_1(a, b), \xi_2(a, b)\right).
\]

With $\xi_1(a, b)$ and $\xi_2(a, b)$ in hand, we can list some elementary properties of the functional $\mathcal{E}^{1D}_{a,b,\xi}$ in (3.69):

**Theorem 3.16.** Let $a \in [-1, 0)$ and $b \geq 1/|a|$.

1. The functional $\mathcal{E}^{1D}_{a,b,\xi}$ has a non-trivial minimizer in $B^1(\mathbb{R})$ if and only if $1/|a| < b < 1/\beta_a$. Furthermore, one can find a positive minimizing $f_\xi$, dependent on $a$ and $b$, such that any minimizer has the form $cf_\xi$ where $c \in \mathbb{C}$ and $|c| = 1$.

2. (Assuming that $\xi_0$ is unique) For $1/|a| < b < 1/\beta_a$, there exists $\xi_0 \in (\xi_1(a, b), \xi_2(a, b))$, dependent on $a$ and $b$, such that
\[
E^{1D}_{a,b}(\xi_0) = \inf_{\xi \in \mathbb{R}} E^{1D}_{a,b}(\xi).
\]

3. (Feynman–Hellmann)
\[
\int_{-\infty}^{0} (at + \xi_0)|f_\xi(t)|^2 dt + \int_{0}^{+\infty} (t + \xi_0)|f_\xi(t)|^2 dt = 0.
\]

The proof of Theorem 3.16 may be derived exactly as done in [FH10, Section 14.2] devoted to the analysis of the following 1D-functional
\[
\mathcal{E}^{1D}_{b,\xi}(f) = \int_{0}^{+\infty} \left(b|f'(t)|^2 + b(t + \xi)^2|f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4\right) dt,
\]
defined over the space $B^1(\mathbb{R}+)$. We introduce the ground state energy
\[
E^{1D}_{b}(\xi) = \inf_{f \in B^1(\mathbb{R})} \mathcal{E}^{1D}_{b,\xi}(f).
\] (3.72)

The ground state energy in (3.72) plays a crucial role in the study of surface superconductivity under the presence of a uniform magnetic field (see [AH07, FH10, HPFS11, CR14]). Let $E^{\text{unif}}_{a,\xi}(\kappa, H)$ be the ground state energy of the functional in (1.1) for
$B_0 = 1$. Assuming that $H = b\kappa$ and $1 < b < \Theta_0^{-1}$ ($\Theta_0 \in (0,1)$ is a universal (spectral) constant defined in (2.20)), then as $\kappa \to +\infty$,

$$
E_{\text{unif}}(\kappa, H) = |\partial \Omega| b^{\frac{1}{2}} E_{1D}^b + O(1),
$$

(3.73)

where $E_{1D}^b = \inf_{\xi \in \mathbb{R}} E_{1D}^b(\xi)$. That has been conjectured by Pan [Pan02], then proved by Almog-Helffer and Helffer-Fournais-Persson [AH07,HFPS11] under a restrictive assumption on $b$, using a spectral approach. In the whole regime $b \in (1, \Theta_0^{-1})$, the upper bound part in (3.73) easily holds (see [FH10, Section 14.4.2]), while the matching lower bound is more difficult to obtain and has been finally proved by Correggi-Rougerie [CR14]. The proof of Correggi-Rougerie, based on the positivity of a certain cost function, was markedly different from the spectral approach of [AH07,HFPS11].

Going back to our step magnetic field problem and the one dimensional energy in (3.69), it is reasonable to make the following conjecture

**Conjecture 3.17.** Assume that $-1 \leq a < 0$ and $1/|a| < b < 1/\beta_a$, where $\beta_a$ is defined in (2.10). Then, the energy $\varepsilon_a(b)$ introduced in (3.5) satisfies

$$
\varepsilon_a(b) = E_{1D}^{a,b},
$$

where

$$
E_{1D}^{a,b} = \inf_{\xi \in \mathbb{R}} E_{1D}^{a,b}(\xi).
$$

(3.74)

and $E_{1D}^{a,b}(\cdot)$ is defined in (3.70).

By a symmetry argument, Conjecture 3.17 trivially holds in the case $a = -1$, namely

$$
\varepsilon_{-1}(b) = E_{-1,b}^{1D} = E_b^{1D}.
$$

(3.75)

However, there are many points that do not allow us to prove this conjecture in the case where $a \in (-1,0)$. Besides the lack of the uniqueness of the minimum $\zeta_a$, the new potential term

$$(s(t)t + \xi)^2$$

where $s(t) = \begin{cases} 1 & \text{if } t > 0, \\ a & \text{if } t < 0, \end{cases}$

creates computational difficulties preventing the adoption of the proof in [CR14], (in particular, in the positivity proof of the cost function).

4. The Frenet Coordinates

In this section, we assume that the set $\Gamma$ consists of a single smooth curve that may intersect the boundary of $\Omega$ transversely in two points. In the general case, $\Gamma$ consists of a finite number of such curves. By working on each component separately, we reduce to the simple case above.

To study the energy contribution along $\Gamma$, we will use the Frenet coordinates which are valid in a tubular neighbourhood of $\Gamma$. For more details regarding these coordinates, see e.g. [FH10, Appendix F]. We will list the basic properties of these coordinates here.

For $t_0 > 0$, we define the open set

$$
\Gamma(t_0) = \{x \in \Omega : \text{dist}(x, \Gamma) < t_0\}.
$$

(4.1)

We introduce the function $t : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$
t(x) = \begin{cases} \text{dist}(x, \Gamma) & \text{if } x \in \Omega_1, \\ -\text{dist}(x, \Gamma) & \text{if } x \in \Omega_2. \end{cases}
$$

(4.2)
Let \((-|\Gamma|/2, |\Gamma|/2) \ni s \mapsto M(s) \in \Gamma\) be the arc-length parametrization of \(\Gamma\) oriented counter clockwise. The vector
\[
T(s) := M'(s)
\] (4.3)
is the unit tangent vector to \(\Gamma\) at the point \(M(s)\). Let \(\nu(s)\) be the unit inward normal of \(\partial \Omega_1\) at the point \(M(s)\). The orientation of the parametrization \(M\) is displayed as follows
\[
\det(T(s), \nu(s)) = 1.
\]
The curvature \(k_r\) of \(\Gamma\) is defined by
\[
T'(s) = k_r(s)\nu(s).
\]
When \(t_0\) is sufficiently small, the transformation
\[
\Phi : \left( -\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right) \times (-t_0, t_0) \ni (s, t) \mapsto M(s) + t\nu(s) \in \Gamma(t_0)
\] (4.4)
is a diffeomorphism whose Jacobian is
\[
a(s, t) = \det(D\Phi) = 1 - tk_r(s).
\]
The inverse of \(\Phi\), \(\Phi^{-1}\), defines a system of coordinates for the tubular neighbourhood \(\Gamma(t_0)\) of \(\Gamma\),
\[
\Phi^{-1}(x) = (s(x), t(x)).
\]
To each function \(u \in H^1_0(\Gamma(t_0))\), we associate the function \(\tilde{u} \in H^1(\Phi^{-1}(\Gamma(t_0)))\) as follows
\[
\tilde{u}(s, t) = u(\Phi(s, t)).
\] (4.5)
We also associate to any vector field \(A = (A_1, A_2) \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)\), the vector field
\[
\tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in H^1 \left( -\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right) \times (-t_0, t_0), \mathbb{R}^2
\]
where
\[
\tilde{A}_1(s, t) = a(s, t)A(\Phi(s, t)) \cdot T(s) \quad \text{and} \quad \tilde{A}_2(s, t) = A(\Phi(s, t)) \cdot \nu(s).
\] (4.6)
Then we have the following change of variable formulae:
\[
\int_{\Omega_1} |(\nabla - iA)u|^2 \, dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} \left( a^{-2}\left|\partial_a - i\tilde{A}_1\right|\tilde{u}^2 + \left|\partial_t - i\tilde{A}_2\right|\tilde{u}^2 \right) a \, ds \, dt,
\] (4.7)
\[
\int_{\Omega_2} |(\nabla - iA)u|^2 \, dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} \left( a^{-2}\left|\partial_a - i\tilde{A}_1\right|\tilde{u}^2 + \left|\partial_t - i\tilde{A}_2\right|\tilde{u}^2 \right) a \, ds \, dt,
\] (4.8)
and
\[
\int_{\mathbb{R}^2} |u(x)|^2 \, dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} |\tilde{u}|^2 a \, ds \, dt.
\] (4.9)
We define
\[
\bar{B}(s, t) = B(\Phi(s, t)), \quad \text{for all } (s, t) \in \left( -\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right) \times (-t_0, t_0).
\]
Note that
\[
\left( \partial_a \tilde{A}_2(s, t) - \partial_t \tilde{A}_1(s, t) \right) \, ds \wedge dt = B(\Phi(s, t)) \, dx \wedge dy
\]
\[
= (1 - tk_r(s)) \bar{B}(s, t) \, ds \wedge dt
\] (4.10)
which gives us
\[
\text{curl} \tilde{A} = \partial_a \tilde{A}_2 - \partial_t \tilde{A}_1 = (1 - tk_r(s)) \bar{B}(s, t).
\]
In Propositions [1.1] and [1.2] we will construct a special gauge transformation that will allow us to express a given vector field in a canonical manner.

**Proposition 4.1.** For any vector field $A = (A_1, A_2) \in H^1(\Omega, \mathbb{R}^2)$, there exists a $H^2$-function $\omega$ such that the vector field defined by $\bar{A}_{\text{new}} = A - \nabla \omega$ satisfies

$$\left( \bar{A}_{\text{new}} \right)_2 = 0 \quad \text{in } \Gamma \left( \frac{t_0}{2} \right),$$

where $\bar{A}_{\text{new}}$ is the vector field associated to $A_{\text{new}}$ as in (4.6).

**Proof.** Using (4.6), we get

$$\left( \bar{A}_{\text{new}} \right)_1 = \bar{A}_1 + \partial_s \bar{\omega}, \quad \left( \bar{A}_{\text{new}} \right)_2 = \bar{A}_2 + \partial_t \bar{\omega} \quad (4.11)$$

where $\bar{\omega}(s, t) = \omega(\Phi(s, t))$. Due to the regularity of $t(x)$ in $\Gamma(t_0)$, we may define $\omega$ so that

$$\bar{\omega}(s, t) = -\chi(t) \int_0^t \bar{A}_2(s, \tilde{t}) \, d\tilde{t},$$

where $\chi$ is a cut-off function supported in $[-t_0, t_0]$, satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in $[-t_0/2, t_0/2]$. \hfill \Box

**Proposition 4.2.** Let $a \in [-1, 1) \setminus \{0\}$ and $A = (A_1, A_2)$ be a vector field in $H^1(\Omega, \mathbb{R}^2)$. For any $x_0 \in \Gamma$, there exists a neighbourhood $V_{x_0}$ of $x_0$ and a function $\omega_{x_0} \in H^2(V_{x_0})$ such that the vector field $A_{\text{new}} := A - \nabla \omega_{x_0}$ satisfies

$$\left( \bar{A}_{\text{new}} \right)_2 = 0 \quad \text{in } V_{x_0}, \quad \left( \bar{A}_{\text{new}} \right)_1 = 0 \quad \text{on } \Gamma \cap V_{x_0}.$$

Furthermore, if $\text{curl } A = \mathbb{I}_{\Omega_1} + a \mathbb{I}_{\Omega_2}$, then we have in $V_{x_0}$

$$\left( \bar{A}_{\text{new}} \right)_1 = \begin{cases} -(t - \frac{t^2}{2} k_r(s)), & \text{if } t > 0, \\ -a(t - \frac{t^2}{2} k_r(s)), & \text{if } t < 0. \end{cases} \quad (4.12)$$

**Proof.** After performing a translation, we may suppose that the coordinates of $x_0$ are given by $(s = 0, t = 0)$. Define

$$V_{x_0} = \left\{ x = \Phi(s, t) \in \Gamma \left( \frac{t_0}{2} \right) : -\frac{\ell}{2} < s < \frac{\ell}{2} \right\}.$$

By Proposition [1.1], after performing a gauge transformation, we may assume that $\bar{A}_2 = 0$ in $\Gamma(t_0/2)$. We define the function $\omega_{x_0}$ such that

$$\bar{\omega}_{x_0}(s, t) = \omega_{x_0}(\Phi(s, t)) = -\chi(s) \int_{-\ell}^s \bar{A}_1(\bar{s}, 0) \, d\bar{s}, \quad (4.13)$$

where $\chi$ is a cut-off function supported in $[-\ell, \ell]$, satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in $[-\ell/2, \ell/2]$. Using (4.11), a straightforward computation yields that $\left( \bar{A}_{\text{new}} \right)_1 = 0$ on $\Gamma \cap V_{x_0}$. Now since $\left( \bar{A}_{\text{new}} \right)_2 = 0$ in $V_{x_0}$, then by (4.10)

$$\partial_t \left( \bar{A}_{\text{new}} \right)_1 = -\bar{B}_{\text{new}}(s, t)(1-tk_r(s)) = -\bar{B}(s, t)(1-tk_r(s)) = \begin{cases} -(1 - tk_r(s)), & t > 0, \\ -a(1 - tk_r(s)), & t < 0. \end{cases} \quad (4.14)$$

Integrating (4.14) with respect to $t$ (starting from $t = 0$), we get (4.12). \hfill \Box
5. A Local Energy

In this section, we will introduce a ‘local version’ of the Ginzburg–Landau functional in (1.1). For this local functional, we will be able to write precise estimates of the ground state energy, which in turn will prove useful in estimating the ground state energy of the full functional in (1.1).

We start by introducing various (geometric) notations/assumptions. Select a positive number $t_0$ sufficiently small so that the Frenet coordinates of Section 4 are valid in the tubular neighbourhood $\Gamma(t_0)$ defined in (4.1). Let $0 < c_1 < c_2$ be fixed constants and $\ell$ be a parameter that is allowed to vary in such a manner that

$$c_1\kappa^{-3/4} < \ell < c_2\kappa^{-3/4}. \tag{5.1}$$

We will refer to (5.1) by writing $\ell \approx \kappa^{-3/4}$. We will assume that $\kappa$ is sufficiently large so that $\ell < t_0/2$.

Consider the set

$$\mathcal{V}(\ell) = \left\{ (s,t) \in \Gamma(t_0) : -\ell/2 < s < \ell/2, -\ell < t < \ell \right\}, \tag{5.2}$$

and the magnetic potential $\tilde{F}$ defined in $\mathcal{V}(\ell)$ by

$$\tilde{F}(s,t) = \left( \tilde{F}_1(s,t), 0 \right) = \left( -\sigma \left( t - \frac{t^2}{2}k_r(s) \right), 0 \right), \tag{5.3}$$

where $\sigma = \sigma(s,t)$ is defined for $(s,t) \in \mathbb{R}^2$ by (see (2.8))

$$\sigma(s,t) = 1 - tk_r(s).$$

Consider the domain $D_\ell$:

$$D_\ell = \{ u \in H^1_0(\mathcal{V}(\ell)) \cap L^\infty(\mathcal{V}(\ell)) : \|u\|_\infty \leq 1 \}. \tag{5.4}$$

For $u \in D_\ell$, we define the (local) energy

$$\mathcal{G}(u; \mathcal{V}(\ell)) = \int_{\mathcal{V}(\ell)} \left( a^{-2} |(\partial_s - i\kappa H \tilde{F}_1)u|^2 + |\partial_t u|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right) a \, ds \, dt, \tag{5.5}$$

where $a(s,t) = 1 - tk_r(s)$. Now we introduce the following ground state energy

$$\mathcal{G}_0 = \mathcal{G}_0(\kappa, H, \ell) = \inf_{u \in D_\ell} \mathcal{G}(u; \mathcal{V}(\ell)). \tag{5.6}$$

Using standard variational methods, one can prove the existence of a minimizer $u_0$ of $\mathcal{G}$

$$\mathcal{G}_0 = \mathcal{G}(u_0; \mathcal{V}(\ell)).$$

Our aim is to write matching upper and lower bounds for $\mathcal{G}_0$ as $\kappa \to +\infty$ in the regime

$$H = b\kappa, \quad a \in [-1, 0) \quad \text{and} \quad b \geq \frac{1}{|a|}. \tag{5.7}$$

5.1. Lower bound of $\mathcal{G}_0$.

**Lemma 5.1.** Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on $a$ and $b$ such that, if $\kappa \geq \kappa_0$ and $\ell$ as in (5.1), then

$$\mathcal{G}_0 \geq b^{-\frac{1}{2}}\kappa \epsilon_a(b) - C, \tag{5.8}$$

where $\mathcal{G}_0$ and $\epsilon_a(b)$ are defined in (5.6) and (3.5) respectively.
Proof. Notice that $a(s, t)$ is bounded in the set $V(\ell)$ as follows
\begin{equation}
1 - C\ell \leq a(s, t) \leq 1 + C\ell. \tag{5.9}
\end{equation}
Consequently
\begin{equation}
G(u; V(\ell)) \geq (1 - C\ell)J(u) - C\kappa^2\ell \int_{V(\ell)} |u|^2 \ ds\ dt, \tag{5.10}
\end{equation}
where
\begin{equation}
J(u) = \int_{V(\ell)} \left( |(\partial_s - i\kappa H\bar{F}_1)u|^2 + |\partial_t u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2} |u|^4 \right) \ ds\ dt. \tag{5.11}
\end{equation}
We apply the Cauchy’s inequality to get
\begin{equation*}
| |(\partial_s - i\kappa H\bar{F}_1)u|^2| = \left| |\partial_s + i\sigma\kappa H(t - \frac{\ell^2}{2} k_r(s))u \right|^2 \geq (1 - \kappa^{-\frac{1}{2}})|(\partial_s + i\sigma\kappa H t)u|^2 - \kappa^2 \sigma^2 \kappa^2 H^2 \frac{\ell^4}{4} k_r(s)|u|^2 \geq (1 - \kappa^{-\frac{1}{2}})|(\partial_s + i\sigma\kappa H t)u|^2 - C\kappa^5 \ell^4 H^2 |u|^2 \because |t| \leq \ell.
\end{equation*}
Inserting the previous estimate into (5.11) and using the uniform bound $|u| \leq 1$, we obtain
\begin{equation}
J(u) \geq (1 - \kappa^{-\frac{1}{2}})T(u) - C\mathcal{R}(u), \tag{5.12}
\end{equation}
where
\begin{equation*}
T(u) = \int_{V(\ell)} \left( |(\partial_s + i\sigma\kappa H t)u|^2 + |\partial_t u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2} |u|^4 \right) \ ds\ dt
\end{equation*}
and
\begin{equation*}
\mathcal{R}(u) = \kappa^\frac{3}{2} \ell^2 + \kappa^\frac{5}{2} H^2 \ell^6.
\end{equation*}
We introduce the following parameters
\begin{equation*}
R = \sqrt{\kappa H\ell}, \quad \gamma = \sqrt{\kappa H}s, \quad \tau = \sqrt{\kappa H}t,
\end{equation*}
and define the re-scaled function
\begin{equation*}
\tilde{u}(\gamma, \tau) = \begin{cases} u(s, t) & \text{if } (\gamma, \tau) \in \left(-R, R\right) \times (-R, R), \\ 0 & \text{otherwise}. \end{cases}
\end{equation*}
Recall the parameter $b = H/\kappa$ in (1.5), in the new scale we may write
\begin{equation*}
T(u) = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} \left[ |(\partial_\gamma + i\sigma\tau)\tilde{u}|^2 + |\partial_\tau \tilde{u}|^2 - \frac{\kappa}{H} |\tilde{u}|^2 + \frac{\kappa}{2H} |\tilde{u}|^4 \right] \ d\gamma d\tau
= \frac{1}{b} G_{a, b, R}(\tilde{u}),
\end{equation*}
$G_{a, b, R}$ is the functional introduced in (3.3), and $\tilde{u} \in D_R$ the domain introduced in (3.2) (since $u \in D_\ell$). Invoking Theorem 3.1, we conclude that
\begin{equation}
T(u) \geq \frac{1}{b} R \epsilon_a(b). \tag{5.13}
\end{equation}
We plug the estimates (5.12) and (5.13) in (5.10), then use $\epsilon_a(b) \leq 0$ and the assumptions on $\kappa$ and $\ell$ to finish the proof of Lemma 5.1. \qed
5.2. Upper bound of $\Theta_0$.

**Lemma 5.2.** Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on $a$ and $b$ such that, if $\kappa \geq \kappa_0$ and $\ell$ as in (5.1), then

$$\Theta_0 \leq b^{-\frac{3}{2}} \kappa \ell \varepsilon_a(b) + C \kappa^{\frac{3}{2}},$$

(5.14)

where $\Theta_0$ and $\varepsilon_a(b)$ are defined in (5.6) and (3.5) respectively.

**Proof.** For $R = \ell \sqrt{\kappa H}$, consider $\varphi = \varphi_{a,b,R}$ the minimizer of $G_{a,b,R}$ defined in (3.8). We define the function $u$ in $D_\ell$ as follows

$$u(s,t) = \chi\left(\frac{t}{\ell}\right) \varphi\left(s \sqrt{\kappa H}, t \sqrt{\kappa H}\right),$$

(5.15)

where $\chi$ is a standard smooth cut-off function satisfying

$$0 \leq \chi \leq 1 \text{ in } \mathbb{R}, \quad \chi = 1 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{and} \quad \text{supp} \chi \subset [-1, 1].$$

Next, we define the following function (with the re-scaled variables)

$$v(\gamma, \tau) = u(s,t) \quad (\gamma, \tau) \in \left(\frac{R}{2}, \frac{R}{2}\right) \times (-R, R),$$

with

$$\gamma = \sqrt{\kappa H} s, \quad \tau = \sqrt{\kappa H} t.$$ 

Using the definition of $v$, the decay of $\varphi$ in (3.11), and the bound of $a(s,t)$ in (5.9), we get

$$\Theta(u) \leq (1 + C\ell) J(u) + C\kappa^2 \ell \int_{\gamma(\ell)} |u|^2 \, ds dt,$$

$$\leq (1 + C\ell) K(v) + C\kappa^2 \ell^3,$$

(5.16)

where $J(u)$ was defined in (5.11),

$$K(v) = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left[ \left| \partial_\gamma + i\sigma \left( \tau - \frac{\kappa^2}{2} \tau \right) \chi_R(\tau) \right| |v|^2 + |\partial_\tau v|^2 - \frac{\kappa}{H} |v|^2 + \frac{\kappa}{2H} |v|^4 \right] \, d\gamma d\tau,$$

and $\epsilon = 1/\sqrt{\kappa H}$.

Let $\chi_R(\tau) = \chi(\tau/R) = \chi(t/\ell)$. We will estimate now each term of $K(v)$ apart, using mainly the decay of the minimizer $\varphi$ in (3.11) and the properties of the function $\chi_R$.

We start with

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} |\partial_\tau v|^2 \, d\gamma d\tau = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left| \chi_R(\tau) \partial_\tau \varphi + \varphi \partial_\tau \chi_R(\tau) \right|^2 \, d\gamma d\tau,$$

$$\leq (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left| \chi_R(\tau) \partial_\tau \varphi \right|^2 \, d\gamma d\tau$$

$$+ C\kappa^{\frac{1}{2}} \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left| \varphi \partial_\tau \chi_R(\tau) \right|^2 \, d\gamma d\tau,$$

$$\leq (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{+\infty} |\partial_\tau \varphi|^2 \, d\gamma d\tau + C\kappa^{-\frac{3}{4}} \ell^{-1}.$$
Similarly, we have
\[
\begin{align*}
\int_{-R}^{R} \int_{-R}^{R} & \left( \partial_\gamma + i \sigma \left( \tau - i \frac{\sigma^2}{2} k_\tau \left( \frac{s}{\epsilon} \right) \right) \right) |v|^2 d\gamma d\tau, \\
& \leq (1 + \kappa^{-\frac{1}{4}}) \int_{-R}^{R} \int_{-R}^{R} \left( \partial_\gamma + i \sigma \tau \right) |v|^2 d\gamma d\tau + C \kappa^{\frac{1}{2}} \int_{-R}^{R} \int_{-R}^{R} \sigma^2 \epsilon^2 |v|^2 d\gamma d\tau, \\
& \leq (1 + \kappa^{-\frac{1}{4}}) \int_{-\infty}^{+\infty} \left| \left( \partial_\gamma + i \sigma \tau \right) \varphi \right|^2 d\gamma d\tau + C \kappa^{\frac{1}{2}} \epsilon \because |\tau| \leq R.
\end{align*}
\]

Next, we may select \( R_0 \) sufficiently large so that, for all \( R \geq R_0 \), we have:
\[
|\tau| \geq \frac{R}{2} \implies \frac{|\tau|}{\ln^2 |\tau|} \geq R^\frac{3}{2}.
\]

The decay of \( \varphi \) and (5.17) yield
\[
\int_{-R}^{R} \int_{-R}^{R} |v|^2 d\gamma d\tau = \int_{-R}^{R} \int_{-R}^{R} |\chi_R(\tau) \varphi|^2 d\gamma d\tau
\]
\[
= \int_{-R}^{R} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau + \int_{-R}^{R} \int_{-\infty}^{+\infty} \left( \chi_R(\tau) - 1 \right) |\varphi|^2 d\gamma d\tau
\]
\[
\geq \int_{-R}^{R} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau - \int_{-R}^{R} \int_{|\tau| \geq R/2} |\varphi|^2 d\gamma d\tau
\]
\[
\geq \int_{-R}^{R} \int_{-\infty}^{+\infty} |\varphi|^2 d\gamma d\tau - C \kappa^{\frac{3}{2}} \ell^2.
\]

Finally, we write the obvious inequality
\[
\int_{-R}^{R} \int_{-R}^{R} |v|^4 d\gamma d\tau \leq \int_{-R}^{R} \int_{-\infty}^{+\infty} |\varphi|^4 d\gamma d\tau.
\]

Gathering the foregoing estimates, we get
\[
K(v) \leq \frac{1 + \kappa^{-\frac{1}{4}}}{b} G_{a,b,R}(\varphi) + C \left( \kappa^\frac{3}{2} \ell + \kappa^{-\frac{3}{2}} \ell^{-1} + \kappa^{\frac{13}{4}} \ell^5 + \kappa^\frac{1}{2} \ell^{\frac{3}{2}} \right),
\]
(5.18)

\[
\leq \frac{1 + \kappa^{-\frac{1}{4}}}{b} G_{a,b,R}(\varphi) + C \kappa^\frac{1}{2}.
\]
(5.19)

Invoking Theorem 3.1 we implement (5.19) into (5.16) to get the desired upper bound. \( \square \)

6. Local Estimates

The aim of this section is to study the concentration of the minimizers \((\psi, A)\) of the functional in (1.1) near the set \( \Gamma \) that separates the values of the applied magnetic field (see Assumption 1.1). This will be displayed by local estimates of the Ginzburg–Landau energy and the \( L^4 \)-norm of the Ginzburg–Landau parameter in Theorem 6.1.

We will introduce the necessary notations and assumptions. Starting with the local energy of the configuration \((\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\) in any open set \( D \subset \Omega \) as
follows
\[
\mathcal{E}_0(\psi, A; D) = \int_D \left( |(\nabla - i\kappa H A)|^2 - \kappa^2 |\psi|^2 + \frac{1}{2}\kappa^2 |\psi|^4 \right) \, dx, \\
\mathcal{E}(\psi, A; D) = \mathcal{E}_0(\psi, A; D) + (\kappa H)^2 \int_\Omega |\text{curl} A - B_0|^2 \, dx.
\] (6.1)

Choose \( t_0 > 0 \) sufficiently small so that the Frenet coordinates of Section 4 are valid in the tubular neighbourhood \( \Gamma(t_0) \) defined in (4.1). For all \( x \in \Gamma(t_0) \), define the point \( p(x) \in \Gamma \) as follows
\[
\text{dist}(x, p(x)) = \text{dist}(x, \Gamma).
\]

Let \( \ell \approx \kappa^{-3/4} \) be a parameter in (5.1) (for some fixed choice of the constants \( c_1 \) and \( c_2 \)). Let \( x_0 \in \Gamma \setminus \partial\Omega \) that is allowed to vary in such a manner that
\[
\text{dist}(x_0, \partial\Omega) > 2\ell.
\] (6.2)

Consider the following neighbourhood of \( x_0 \),
\[
\mathcal{N}_{x_0}(\ell) = \{ x \in \Omega : \text{dist}_\Gamma(x_0, p(x)) < \frac{\ell}{2}, -\ell < t(x) < \ell \},
\] (6.3)

where \( t(\cdot) \) is defined in (4.2). For \( \kappa \) sufficiently large (hence \( \ell \) sufficiently small), we get that \( \mathcal{N}_{x_0}(\ell) \) does not intersect the boundary \( \partial\Omega \), thanks to (6.2). As a consequence of the assumption in (6.2), all the estimates that we will write will hold uniformly with respect to the point \( x_0 \).

We assume that \( a \in [-1, 0) \) and \( b > 0 \) are fixed and satisfy
\[
b > \frac{1}{|a|}.
\] (6.4)

When (6.4) holds, we are able to use the exponential decay of the Ginzburg–Landau parameter away from the set \( \Gamma \) and the surface \( \partial\Omega \) (see Theorem 2.4).

**Theorem 6.1.** Let \( a \in [-1, 0) \) and \( b > 1/|a| \). There exists \( \kappa_0 > 0 \) and a function \( \tau : [\kappa_0, +\infty) \to (0, +\infty) \) such that \( \lim_{\kappa \to +\infty} \tau(\kappa) = 0 \) and the following is true. For \( \kappa \geq \kappa_0 \), \( H = b\kappa \) and \( \ell \approx \kappa^{-3/4} \) as in (5.1), for any \( x_0 \in \Gamma \) satisfying (6.2), every minimizer \( (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega) \) of the functional in (1.1) satisfies
\[
\left| \mathcal{E}_0(\psi, A; \mathcal{N}_{x_0}(\ell)) - b^{-\frac{1}{2}} \kappa \tau(\kappa) \right| \leq \kappa \tau(\kappa),
\] (6.5)

and
\[
\left| \frac{1}{\ell} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx + 2b^{-\frac{1}{2}} \kappa^{-1} \epsilon_a(b) \right| \leq \kappa^{-1} \tau(\kappa),
\] (6.6)

where \( \mathcal{N}_{x_0}(\cdot) \) is the set in (6.3), \( \mathcal{E}_0 \) is the local energy in (6.1), and \( \epsilon_a(b) \) is the limiting energy defined in (3.5). Furthermore, the function \( \tau \) is independent of the point \( x_0 \in \Gamma \).

The proof of Theorem 6.1 follows by collecting the results of Proposition 6.3 and Proposition 6.4 below, which are derived along the lines of [HK17, Section 4] in the study of local surface superconductivity.

Part of the proof of Theorem 6.1 is based on the following remark. After performing a translation, we may assume that the Frenet coordinates of \( x_0 \) are \( (s = 0, t = 0) \) (see Section 3). Recall the local Ginzburg–Landau energy \( \mathcal{E}_0 \) introduced in (6.1). Let \( F \) be the vector field introduced in Lemma 2.2. We have the following relation
\[
\mathcal{E}_0(u, F; \mathcal{N}_{x_0}(\ell)) = \mathcal{G}(v; \mathcal{V}(\ell)),
\] (6.7)
where $\mathfrak{S}$ is defined in (5.5), $u \in H^1_0(\mathcal{N}_{x_0}(\ell))$, $\tilde{v}$ is the function associated to $v = e^{-iKx_0}u$ by the transformation $\Phi^{-1}$ (see (4.5)), and $\omega_{x_0}$ is the gauge transformation function defined in Proposition 4.2.

6.1. Lower bound of the local energy. We start by establishing a lower bound for the local energy $\mathcal{E}_0(u, A; N_{x_0}(\ell))$ for an arbitrary function $u \in H^1_0(\mathcal{N}_{x_0}(\ell))$ satisfying $|u| \leq 1$. We will work under the assumptions made in this section, notably, we assume that (6.4) holds, and $\ell \approx \kappa^{-3/4}$ (see (5.1)), and in the regime where $H = bk$.

Proposition 6.2. There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for $\kappa \geq \kappa_0$ and for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If

- $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{dive}(\Omega)$ is a solution of (1.4),
- $u \in H^1_0(\mathcal{N}_{x_0}(\ell))$ satisfies $|u| \leq 1$,

then

$$\mathcal{E}_0(u, A; N_{x_0}(\ell)) \geq b^{-\frac{3}{2}}\kappa \ell \epsilon_a(b) - C,$$

where $N_{x_0}(\cdot)$ is the neighbourhood defined in (6.3), $\mathcal{E}_0$ is the functional defined in (6.1), and $\epsilon_a(b)$ is the limiting energy in (3.5).

Proof. Let $\alpha \in (0, 1)$ and $F$ be the vector field introduced in Lemma 2.2. Define the function $\phi(x_0) = \left(A(x_0) - F(x_0)\right) \cdot x$. As a consequence of the fourth item in Theorem 2.3 we get the following useful approximation of the vector potential $A$

$$|A(x) - \nabla \phi(x_0)(x) - F(x)| \leq \frac{C}{\kappa} \ell^\alpha$$

for $x \in N_{x_0}(\ell)$. (6.8)

We choose $\alpha = 2/3$ in (6.8). Define the function $w = e^{-iK\phi(x_0)u}$. Using (6.8) and Cauchy’s inequality, we may write

$$|(\nabla - iK\mathbf{A})u|^2 \geq (1 - \kappa^{-\frac{3}{2}})(\nabla - iKHF)w|^2 - \kappa^\frac{3}{2} \kappa^2 H^2 |A - \nabla \phi(x_0) - F|^2 |w|^2$$

$$\geq (1 - \kappa^{-\frac{3}{2}})(\nabla - iKHF)w|^2 - \kappa^\frac{3}{2} \kappa^2 |w|^2.$$

By using that $|w| \leq 1$, we get further

$$\mathcal{E}_0(u, A; N_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{3}{2}})\mathcal{E}_0(w, F; N_{x_0}(\ell)) - C \left(\kappa^\frac{3}{2} \ell^2 + \kappa^\frac{3}{2} \ell^\frac{10}{3}\right).$$

Now, define the function $v = e^{-iK\omega_{x_0}w}$, where $\omega_{x_0}$ is introduced in Proposition 4.2. We may use the relation in (6.7) to write

$$\mathcal{E}_0(u, A; N_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{3}{2}})\mathfrak{S}(\tilde{v}; \mathcal{V}(\ell)) - C \left(\kappa^\frac{3}{2} \ell^2 + \kappa^\frac{3}{2} \ell^\frac{10}{3}\right),$$

Finally, we use the lower bound in Lemma 5.1 together with the inequality $\epsilon_a(b) \leq 0$. This finishes the proof of Proposition 6.2. \qed

6.2. Sharp upper bound on $L^4$-norm. We will derive a lower bound of the local energy $\mathcal{E}_0(\psi, A; N_{x_0}(\ell))$ and an upper bound of the $L^4$-norm of $\psi$ valid for any critical point $(\psi, A)$ of the functional in (1.1). Again, we remind the reader that we assume that (6.4) holds, $\ell \approx \kappa^{-3/4}$ (see (5.1)) and $H = bk$.

Proposition 6.3. There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{dive}(\Omega)$ is a critical point of the functional in (1.1) for $\kappa \geq \kappa_0$, then

$$\mathcal{E}_0(\psi, A; N_{x_0}(\ell)) \geq b^{-\frac{3}{2}}\kappa \ell \epsilon_a(b) - C \kappa^\frac{3}{10},$$

and

$$\frac{1}{\ell} \int_{N_{x_0}(\ell)} |\psi|^4 \, dx \leq -2b^{-\frac{3}{2}}\kappa^{-1} \epsilon_a(b) + C \kappa^{-\frac{17}{20}}.$$
Here $\mathcal{N}_{\omega}(\cdot)$, $\mathcal{E}_0$, and $\varepsilon_\omega(b)$ are respectively defined in (5.3), (6.1), and (3.5).

**Proof.** In the sequel, $\gamma = \kappa^{-3/16}$ and $\kappa$ is sufficiently large so that $\gamma \in (0, 1)$. We denote by $\hat{\ell} = (1 + \gamma)\ell$.

Consider a smooth function $f$ satisfying

$$f = 1 \text{ in } \mathcal{N}_{\omega}(\hat{\ell}), \quad f = 0 \text{ in } \mathcal{N}_{\omega}(\hat{\ell})^c,$$

$$0 \leq f \leq 1, \quad |\nabla f| \leq C\gamma^{-1}\ell^{-1} \text{ and } |\nabla f| \leq C\gamma^{-2}\ell^{-2} \text{ in } \Omega.$$  \hfill (6.11)

**Proof of (6.9).** We will extract a lower bound of $\mathcal{E}_0 \left( \psi, \mathbf{A}; \mathcal{N}_{\omega}(\hat{\ell}) \right)$. We use the following simple identity (see [KN16, p. 2871])

$$\int_{\mathcal{N}_{\omega}(\hat{\ell})} |(\nabla - i\kappa \mathbf{A})f\psi|^2 \, dx = \int_{\mathcal{N}_{\omega}(\hat{\ell})} |f(\nabla - i\kappa \mathbf{A})\psi|^2 \, dx - \int_{\mathcal{N}_{\omega}(\hat{\ell})} f|\nabla f||\psi|^2 \, dx,$$

obtained using integration by parts. Having in hand (6.12), $|\psi| \leq 1$ and $|\text{supp}(\nabla f)| \leq C\gamma\ell^2$, we can write

$$\int_{\mathcal{N}_{\omega}(\hat{\ell})} |(\nabla - i\kappa \mathbf{A})f\psi|^2 \, dx \leq \int_{\mathcal{N}_{\omega}(\hat{\ell})} |f(\nabla - i\kappa \mathbf{A})\psi|^2 \, dx + \int_{\mathcal{N}_{\omega}(\hat{\ell})} f|\nabla f||\psi|^2 \, dx,$$

$$\leq \int_{\mathcal{N}_{\omega}(\hat{\ell})} |f(\nabla - i\kappa \mathbf{A})\psi|^2 \, dx + C\gamma^{-2}\ell^{-2}\int_{\text{supp}(\nabla f)} |\psi|^2 \, dx,$$

$$\leq \int_{\mathcal{N}_{\omega}(\hat{\ell})} |f(\nabla - i\kappa \mathbf{A})\psi|^2 \, dx + C\gamma^{-1}.$$  \hfill (6.12)

On the other hand, we write

$$\int_{\mathcal{N}_{\omega}(\hat{\ell})} f^2|\psi|^2 \, dx$$

$$= \int_{\mathcal{N}_{\omega}(\hat{\ell})} |\psi|^2 \, dx - \int_{\mathcal{N}_{\omega}(\hat{\ell})} (1 - f^2)|\psi|^2 \, dx$$

$$= \int_{\mathcal{N}_{\omega}(\hat{\ell})} |\psi|^2 \, dx - \int_{\mathcal{N}_{\omega}(\hat{\ell}) \cap \{|t(x)| \leq \gamma\ell\}} (1 - f^2)|\psi|^2 \, dx - \int_{\mathcal{N}_{\omega}(\hat{\ell}) \cap \{|t(x)| > \gamma\ell\}} (1 - f^2)|\psi|^2 \, dx$$

where $t(\cdot)$ is the distance function defined in (4.2). Recall that $\gamma = \kappa^{-3/16}$, then $\gamma\ell \gg \kappa^{-1}$ which, together with (6.4), allow us to use the exponential decay of $|\psi|^2$ in $\mathcal{N}_{\omega}(\hat{\ell}) \cap \{|t(x)| > \gamma\ell\}$ (see Theorem 2.4). Consequently, the integral over $\mathcal{N}_{\omega}(\hat{\ell}) \cap \{|t(x)| > \gamma\ell\}$ in (6.13) is exponentially small when $\kappa \to +\infty$ (having $0 \leq f \leq 1$). In addition, we have

$$|\text{supp}(1 - f^2) \cap \mathcal{N}_{\omega}(\hat{\ell}) \cap \{|t(x)| \leq \gamma\ell\}| = \mathcal{O}(\gamma^2\ell^2).$$

This yields

$$\int_{\mathcal{N}_{\omega}(\hat{\ell})} f^2|\psi|^2 \, dx \geq \int_{\mathcal{N}_{\omega}(\hat{\ell})} |\psi|^2 \, dx - C\gamma^2\ell^2.$$

Therefore,

$$\mathcal{E}_0(f \psi, \mathbf{A}; \mathcal{N}_{\omega}(\hat{\ell})) \leq \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{\omega}(\hat{\ell})) + C\kappa^2\gamma^2\ell^2 + C\gamma^{-1}.$$  \hfill (6.14)
The fact that \( f \psi \in H^1_0(\mathcal{N}_{x_0}(\ell)) \) and \(|f \psi| \leq 1\) allows us to use the lower bound result established in Proposition 6.2 for \( u = f \psi \). This yields together with (6.14)

\[
\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq b^{-\frac{1}{2}} \kappa \ell \varepsilon_a(b) - C \kappa^\frac{3}{4}
\]

(6.15)

This finishes the proof of (6.9), but with \( \hat{\ell} \) appearing instead of \( \ell \). However, this is not harmful, as we could start the argument with \( \ell = (1 + \gamma)^{-1} \ell \) in place of \( \ell \) and then modify \( \hat{\ell} \) accordingly; in this case we would get \( \hat{\ell} = (1 + \gamma) \ell = \ell \) as required.

**Proof of (6.10).** In light of the first equation in (1.4) satisfied by \((\psi, \mathbf{A})\), we get using integration by parts (see [FK11, (6.2)])

\[
\int_{\mathcal{N}_{x_0}(\ell)} \left( |(\nabla - i \kappa \mathbf{A}) f \psi|^2 - |\nabla f| |\psi|^2 \right) \, dx = \kappa^2 \int_{\mathcal{N}_{x_0}(\ell)} \left( |\psi|^2 - |\psi|^4 \right) f^2 \, dx.
\]

Consequently,

\[
\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) = \kappa^2 \int_{\mathcal{N}_{x_0}(\ell)} f^2 \left( -1 + \frac{1}{2} f^2 \right) |\psi|^4 \, dx + \int_{\mathcal{N}_{x_0}(\ell)} |\nabla f|^2 |\psi|^2 \, dx.
\]

(6.16)

Since \( f = 1 \) in \( \mathcal{N}_{x_0}(\ell) \) and \(-1 + 1/2 f^2 \leq -1/2 \) in \( \mathcal{N}_{x_0}(\ell) \), we get

\[
\int_{\mathcal{N}_{x_0}(\ell)} f^2 \left( -1 + \frac{1}{2} f^2 \right) |\psi|^4 \, dx \leq -\frac{1}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx.
\]

We use the previous inequality, (6.16) and the estimate \(|\text{supp } |\nabla f|| \leq C \gamma \ell^2\) to obtain

\[
\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \leq -\frac{\kappa^2}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx + C \gamma^{-1},
\]

\[
\leq -\frac{\kappa^2}{2} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx + C \kappa^{\frac{3}{4}}.
\]

(6.17)

We insert the lower bound in Proposition 6.2 into (6.17) to get the upper bound of the \( L^4 \)-norm in (6.10). \( \Box \)

### 6.3. Sharp lower bound on the \( L^4 \)-norm

Complementary to Proposition 6.3, we will prove Proposition 6.4 below, whose conclusion holds for minimizing configurations only. We continue working under the assumption that (6.4) holds, \( \ell \approx \kappa^{-3/4} \) (see (5.1)) and \( H = b \kappa \).

**Proposition 6.4.** There exist two constants \( \kappa_0 > 1 \) and \( C > 0 \) such that, for all \( x_0 \in \Gamma \) satisfying (6.2), the following is true. If \((\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\text{div}}^1(\Omega)\) is a minimizer of the functional in (1.1) for \( \kappa \geq \kappa_0 \), then

\[
\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \leq b^{-\frac{3}{2}} \kappa \ell \varepsilon_a(b) + C \kappa^{\frac{3}{4}},
\]

(6.18)

and

\[
\frac{1}{\ell} \int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx \geq -2b^{-\frac{3}{2}} \kappa^{-1} \varepsilon_a(b) - C \kappa^{-\frac{17}{4}}.
\]

(6.19)

Here \( \mathcal{N}_{x_0}(\cdot) \), \( \mathcal{E}_0 \), and \( \varepsilon_a(b) \) are respectively defined in (6.3), (6.1), and (3.5).

**Proof.** The proof is divided into five steps.
Step 1. Construction of a test function and decomposition of the energy. The construction of the test function is inspired from that by Sandier and Serfaty in their study of bulk superconductivity in [SS03]. Let \( (\psi, A) \) be a minimizer of the function in (1.1). For \( \gamma = \kappa^{-3/16} \) and \( \ell = (1 + \gamma)\ell \), we define the function

\[
u(x) = 1_{\mathcal{N}_{\ell, 0}(\hat{\ell})}(x)e^{i\kappa H\phi_{\psi}(x)}v_0(x) + \eta(x)\psi(x), \tag{6.20}\]

where \( v_0(x) = e^{i\kappa H\omega_0(x)}u_0 \circ \Phi^{-1}(x) \), \( \phi_{\psi} \) and \( \omega_0 \) are the gauge transformation functions introduced respectively in (6.8) and Proposition 112 \( \Phi \) is the coordinate transformation in (4.4), \( u_0 \) is a minimizer of the functional \( \mathfrak{F}(\cdot, \mathcal{V}(\hat{\ell})) \) defined in (6.5), and \( \eta \) is a smooth function satisfying

\[
\eta = 0 \text{ in } \mathcal{N}_{\ell, 0}(\hat{\ell}), \quad \eta = 1 \text{ in } \mathcal{N}_{\ell, 0}((1 + 2\gamma)\ell),
\]

\[
0 \leq \eta \leq 1, \quad |\nabla \eta| \leq C_\gamma^{-1}\ell^{-1} \text{ and } |\Delta \eta| \leq C_\gamma^{-2}\ell^{-2} \text{ in } \Omega. \tag{6.21}\]

Recalling the energies defined in (1.1) and (6.1), we write

\[
\mathcal{E}_{\kappa, H}(\cdot, A) = \mathcal{E}_0(\cdot, A; \mathcal{N}_{\ell, 0}(\hat{\ell})) + \mathcal{E}(\cdot, A; \mathcal{N}_{\ell, 0}(\hat{\ell})^\circ). \]

We denote by

\[
\mathcal{E}_1(\cdot, A) = \mathcal{E}_0(\cdot, A; \mathcal{N}_{\ell, 0}(\hat{\ell})), \quad \mathcal{E}_2(\cdot, A) = \mathcal{E}_0(\cdot, A; \mathcal{N}_{\ell, 0}(\hat{\ell}) \setminus \mathcal{N}_{\ell, 0}(\hat{\ell})), \quad \mathcal{E}_3(\cdot, A) = \mathcal{E}(\cdot, A; \mathcal{N}_{\ell, 0}(\hat{\ell})^\circ),
\]

where \( \ell = (1 + 2\gamma)\ell \). Hence, we get the following decomposition of the functional in (1.1).

\[
\mathcal{E}_{\kappa, H}(\cdot, A) = \mathcal{E}_1(\cdot, A) + \mathcal{E}_2(\cdot, A) + \mathcal{E}_3(\cdot, A). \tag{6.22}\]

Step 2. Estimating \( \mathcal{E}_1(\cdot, A) \). Using the approximation in (6.8) for \( \alpha = 2/3 \), the Cauchy-Schwarz inequality and the uniform bound |\( v_0 \)| \( \leq 1 \), we may write

\[
\mathcal{E}_1(\cdot, A) \leq (1 + \kappa^{-\frac{1}{2}})\mathcal{E}_0(v_0, F; \mathcal{N}_{\ell, 0}(\hat{\ell})) + \kappa^{-\frac{1}{2}}\kappa^2\int_{\mathcal{N}_{\ell, 0}(\hat{\ell})} |v_0|^2 \, dx
\]

\[
+ \kappa^{\frac{1}{2}}\kappa^2H^2\int_{\mathcal{N}_{\ell, 0}(\hat{\ell})} |A - \nabla \phi_{\psi} - F|^2|v_0|^2 \, dx, \leq (1 + \kappa^{-\frac{1}{2}})\mathcal{E}_0(v_0, F; \mathcal{N}_{\ell, 0}(\hat{\ell})) + C. \tag{6.22}\]

But by (6.7), we have \( \mathcal{E}_0(v_0, F; \mathcal{N}_{\ell, 0}(\hat{\ell})) = \mathfrak{E}(v_0, \mathcal{V}(\hat{\ell})) \). Hence, we insert the upper bound in Lemma 5.2 into (6.22) to get

\[
\mathcal{E}_1(\cdot, A) \leq b^{-\frac{1}{2}}\kappa^{\frac{1}{2}}\kappa^2 + C\kappa^{\frac{1}{2}}. \tag{6.23}\]

Step 3. Estimating \( \mathcal{E}_2(\cdot, A) \). Notice that \( u = \eta \psi \) with \( 0 \leq \eta \leq 1 \) in \( \mathcal{N}_{\ell, 0}(\hat{\ell}) \setminus \mathcal{N}_{\ell, 0}(\hat{\ell}) \). Then, we do a straightforward computation, similar to the one done in the proof of (6.14), replacing \( f \) by \( \eta \) and \( \mathcal{N}_{\ell, 0}(\hat{\ell}) \) by \( \mathcal{N}_{\ell, 0}(\hat{\ell}) \setminus \mathcal{N}_{\ell, 0}(\hat{\ell}) \). This gives the following relation between \( \mathcal{E}_2(u, A) \) and \( \mathcal{E}_2(\psi, A) \)

\[
\mathcal{E}_2(u, A) \leq \mathcal{E}_2(\psi, A) + C\kappa^{\frac{1}{2}}. \tag{6.24}\]

Step 4. Estimating \( \mathcal{E}_1(\psi, A) \). Since \( (\psi, A) \) is a minimizer of the functional \( \mathcal{E}_{\kappa, H} \) defined in (1.1), we write

\[
\mathcal{E}_{\kappa, H}(\psi, A) \leq \mathcal{E}_{\kappa, H}(u, A),
\]

that is

\[
\mathcal{E}_1(\psi, A) + \mathcal{E}_2(\psi, A) + \mathcal{E}_3(\psi, A) \leq \mathcal{E}_1(u, A) + \mathcal{E}_2(u, A) + \mathcal{E}_3(u, A).
\]

Noticing that \( \mathcal{E}_3(u, A) = \mathcal{E}_3(\psi, A) \), we get

\[
\mathcal{E}_1(\psi, A) + \mathcal{E}_2(\psi, A) \leq \mathcal{E}_1(u, A) + \mathcal{E}_2(u, A).
\]
We use the estimate of $\mathcal{E}_2(u, A)$ in \cite{[6.24]} to get
\[ \mathcal{E}_1(\psi, A) \leq \mathcal{E}_1(u, A) + C_{\kappa}^3. \]
We insert the upper bound of $\mathcal{E}_1(u, A)$ in \cite{[6.23]} in the previous inequality to get
\[ \mathcal{E}_1(\psi, A) \leq b^{-\frac{1}{2}} \hat{\kappa} \mathcal{E}_a(b) + C_{\kappa}^3. \] (6.25)
Recalling that $\mathcal{E}_1(\psi, A) = \mathcal{E}_1(\psi, A; \mathcal{N}_{\gamma}(\ell))$, we see that \cite{[6.25]} is nothing but \cite{[6.18]} with $\ell$ appearing instead of $\ell$. Starting the argument with $\ell$ replaced by $\ell = (1 + \gamma)^{-\frac{1}{\ell}}$, we get \cite{[6.25]} for $\ell = (1 + \gamma)\ell = \ell$, as required. Therefore, we finished the proof of \cite{[6.18]}.

**Step 5. Lower bound of the $L^4$-norm of $\psi$.** Consider the function $f$ defined in \cite{[6.11]}. We use the properties of this function, mainly that $f = 1$ in $\mathcal{N}_{\gamma}(\ell)$ and $0 \leq f \leq 1$ in $\Omega$, to obtain
\[
\int_{\mathcal{N}_{\gamma}(\ell)} f^2 \left( -1 + \frac{1}{2} f^2 \right) |\psi|^4 \, dx = -\frac{1}{2} \int_{\mathcal{N}_{\gamma}(\ell)} |\psi|^4 \, dx + \int_{\mathcal{N}_{\gamma}(\ell) \setminus \mathcal{N}_{\gamma}(\ell)} f^2 \left( -1 + \frac{1}{2} f^2 \right) |\psi|^4 \, dx \\
\geq -\frac{1}{2} \int_{\mathcal{N}_{\gamma}(\ell)} |\psi|^4 \, dx - \int_{\mathcal{N}_{\gamma}(\ell) \setminus \mathcal{N}_{\gamma}(\ell)} |\psi|^4 \, dx.
\]
Following an argument similar to the one for \cite{[6.13]}, we divide the set $\mathcal{N}_{\gamma}(\ell) \setminus \mathcal{N}_{\gamma}(\ell)$ into the two sets $(\mathcal{N}_{\gamma}(\ell) \setminus \mathcal{N}_{\gamma}(\ell)) \cap \{|t(x)| \leq \gamma \ell\}$ and $(\mathcal{N}_{\gamma}(\ell) \setminus \mathcal{N}_{\gamma}(\ell)) \cap \{|t(x)| > \gamma \ell\}$, and we use this time the exponential decay of $|\psi|^4$ deduced from Theorem \ref{thm:2.4} to get
\[ \int_{\mathcal{N}_{\gamma}(\ell)} f^2 \left( -1 + \frac{1}{2} f^2 \right) |\psi|^4 \, dx \geq -\frac{1}{2} \int_{\mathcal{N}_{\gamma}(\ell)} |\psi|^4 \, dx - C_{\kappa}^{-\frac{15}{2}}. \] (6.26)
Inserting \cite{[6.26]} into the identity in \cite{[6.16]} gives us
\[ \mathcal{E}_1(f \psi, A) \geq -\frac{\kappa^2}{2} \int_{\mathcal{N}_{\gamma}(\ell)} |\psi|^4 \, dx - C_{\kappa}^{\frac{3}{2}}. \]
The previous inequality together with \cite{[6.14]} and \cite{[6.25]} establish the lower bound of the $L^4$-norm of $\psi$ as $\kappa \to +\infty$. \hfill \Box

6.4. **Proof of Theorem 6.1**. Estimate \cite{[6.5]} in Theorem 6.1 is obtained by gathering results in \cite{[6.9]} and in \cite{[6.18]}, while Estimate \cite{[6.6]} follows from \cite{[6.10]} and in \cite{[6.19]}.

7. Surface Superconductivity

In Section 6, we worked under the assumption
\[ b > 1/|a|, \quad a \in [-1, 0). \]
We investigated the local behaviour of the sample in a tubular neighbourhood of $\Gamma$. In this section, and under the same assumption, we are concerned in the local behaviour of the sample near the boundary of $\Omega$.

The analysis of superconductivity near $\partial \Omega$ in our case of a step magnetic field ($B_0$ satisfying \cite{[1.1]} is essentially the same as that in the uniform field case, since $B_0$ is constant in each of $\Omega_1$ and $\Omega_2$. Thereby, the results presented in this section are well-known in literature since the celebrated work of Saint-James and de Gennes \cite{[SJG63]}. We refer to \cite{[AH07],[CR14],[FG05],[FP01],[FK11],[FKP13],[LP09],[Pan02]} for rigorous results in general 2D and 3D samples subjected to a constant magnetic field, and to \cite{[NSG09]} for recent experimental results. Particularly, local surface estimates were recently established in \cite{[HK17]}, when $B_0 \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. We will adapt these results to our discontinuous magnetic field (see Theorem 7.1 below).
The statement of our main result, Theorem \ref{thm:main}, involves the surface energy $E_{\text{surf}}$ that we introduce in the next section.

7.1. **The surface energy function.** Let $b > 1$ and $R > 0$. Consider the reduced Ginzburg–Landau functional:

$$\mathcal{W}(U_R) \ni \phi \mapsto E_{b,R}(\phi) = \int_{U_R} \left( b |(\nabla - iA_0)\phi|^2 - |\phi|^2 + \frac{1}{2} |\phi|^4 \right) \, d\gamma d\tau,$$

where $(\gamma, \tau) \in \mathbb{R}^2$, $A_0(\gamma, \tau) = (-\tau, 0)$, $U_R = (-R/2, R/2) \times (0, +\infty)$ and

$$\mathcal{W}(U_R) = \{ u \in L^2(U_R) : (\nabla - iA_0)u \in L^2(U_R), u(\pm R, \cdot ) = 0 \}.$$  

Let $d(b, R)$ be the ground state energy defined by

$$d(b, R) = \inf_{\phi \in \mathcal{W}(U_R)} E_{b,R}(\phi).$$

Pan proved in \cite{Pan02} the existence of a non-decreasing continuous function $E_{\text{surf}} : [1, \Theta_0^{-1}) \to (-\infty, 0]$ such that

$$E_{\text{surf}}(b) = \lim_{R \to +\infty} \frac{d(b, R)}{R},$$  \tag{7.1}

where $\Theta_0$ is defined in \cite{AH07}. The surface energy $E_{\text{surf}}$ has also been described by a 1D-problem (see \cite{AH07, FPS11}). Recently, Correggi–Rougerie \cite{CR14} proved that for all $b \in (1, \Theta_0^{-1})$, $E_{\text{surf}} = E_b^{D}$, where $E_b^{D}$ is the energy introduced in \cite{FD05}. One important property of the function $E_{\text{surf}}$ is (see \cite{FH05})

$$E_{\text{surf}}(\Theta_0^{-1}) = 0 \text{ and } E_{\text{surf}}(b) < 0, \text{ for all } b \in [1, \Theta_0^{-1}).$$  \tag{7.2}

This property allows us to extend the function $E_{\text{surf}}$ continuously to $[1, +\infty)$, by setting it to zero on $[\Theta_0^{-1}, +\infty)$. This extension of the surface energy is still denoted by $E_{\text{surf}}$ for simplicity.

7.2. **Local surface superconductivity.** Let $t_0 > 0$ and $j \in \{1, 2\}$. We define the following set:

$$\Omega_j(t_0) = \{ x \in \Omega_j : \text{dist} (x, \partial\Omega_j \cap \partial\Omega) < t_0 \}. $$  \tag{7.3}

Assume that $t_0$ is sufficiently small, then for any $x \in \Omega_j(t_0)$, there exists a unique point $p(x) \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\text{dist} (x, \partial\Omega_j \cap \partial\Omega) = \text{dist} (x, p(x)).$$

Let $\ell \approx \kappa^{-3/4}$ be the parameter in \cite{FD05}. Choose $x_0 \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\text{dist}(x_0, \Gamma) > 2\ell.$$  \tag{7.4}

We introduce the following small neighbourhood of $x_0$:

$$N_{x_0}^\ell = \{ x \in \Omega_j : \text{dist}_{\partial\Omega} (x_0, p(x)) < \ell \} \subset \text{dist}_{\partial\Omega} (x, p(x)) < \ell \}.$$  \tag{7.5}

The assumption on $x_0$ in (7.4) guarantees that $N_{x_0}^\ell$ and $\Gamma$ do not intersect, for sufficiently large $\kappa$ (hence sufficiently small $\ell$). Consequently, the estimates in this section (and particularly in Theorem \ref{thm:main} below) hold uniformly with respect to the point $x_0$.

Recall the magnetic field $B_0$ defined in \cite{FD05} ($B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$).

**Theorem 7.1.** Let $a \in [-1, 0)$ and $b > 1/|a|$. There exists $\kappa_0 > 0$ and a function $\bar{r} : [\kappa_0, +\infty) \to (0, +\infty)$ such that $\lim_{\kappa \to +\infty} \bar{r}(\kappa) = 0$ and the following is true. For
\[ \kappa \geq \kappa_0, \ H = b\kappa, \ell \text{ as in (5.1)}, \ j \in \{1, 2\}, \ x_0 \in \partial \Omega_j \cap \partial \Omega \text{ satisfying (7.4)}, \ \text{and every minimizer } (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega) \text{ of the functional in (1.1), we have} \]

\[ \left| \mathcal{E}_0(\psi, A; N_{x_0}^j(\ell)) - b^{-\frac{3}{2}}|B_0(x_0)|^{-\frac{1}{2}} \kappa \ell \mathcal{E}_{\text{surf}}(b|B_0(x_0)|) \right| \leq \kappa \ell \bar{r}(\kappa), \]

and

\[ \left| \frac{1}{\ell} \int_{N_{x_0}^j(\ell)} |\psi|^4 \, dx + 2b^{-\frac{3}{2}}|B_0(x_0)|^{-\frac{1}{2}} \kappa^{-1} \mathcal{E}_{\text{surf}}(b|B_0(x_0)|) \right| \leq \kappa^{-1} \bar{r}(\kappa), \]

where \( N_{x_0}^j(\cdot) \) is defined in (7.5), and \( \mathcal{E}_0 \) is the local energy defined in (6.1). Furthermore, the function \( \bar{r} \) is independent of the point \( x_0 \).

The estimates in Theorem 7.1 are already established in [HK17] when \( B_0 \in C^{0,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \). They still hold in our case because \( B_0 \) is constant in \( \Omega_1 \) and \( \Omega_2 \) (see Assumption 1.1) by repeating the proof given in [HK17].

8. Proof of Main Results

8.1. Proof of Theorem 1.5. We will work under the assumptions of Theorem 1.5 restricted to the non-trivial case

\[ a \in [-1, 0) \ \text{and} \ b > 1/|a|, \]

and we will gather the results of the two previous sections to establish, as \( \kappa \) tends to \( +\infty \), asymptotic estimates of the global ground state energy \( E_{\text{g.st}}(\kappa, H) \) in (1.3) and of the \( L^4 \)-norm of the order parameter \( \psi \), where \( (\psi, A) \) is a minimizer of this energy.

![Figure 5](image)

**Figure 5.** Schematic representation of the sets \( \Gamma^*(\ell), \Omega_1^*(\ell), \Omega_2^*(\ell), \Omega_{\text{bulk}} \) and \( T \), where \( \ell \approx \kappa^{-3/4} \). In the regime where \( a \in [-1, 0) \), \( H = b\kappa \) and \( 1/|a| < b \), the blue region is in the normal state, while the other regions may carry superconductivity.

8.1.1. Energy lower bound. Let \( (\psi, A) \) be a minimizer of (1.1). The definition of \( E_{\text{g.st}}(\kappa, H) \) obviously implies that

\[ E_{\text{g.st}}(\kappa, H) = \mathcal{E}_0(\psi, A; \Omega) + \kappa^2 H^2 \int_{\Omega} |\text{curl} \ A - B_0|^2 \, dx \geq \mathcal{E}_0(\psi, A; \Omega), \]

where \( \mathcal{E}_0 \) is defined in (6.1) and \( B_0 \) is defined in (1.1). Hence, it suffices to find a relevant lower bound of \( \mathcal{E}_0(\psi, A; \Omega) \). To that end, we will decompose the sample \( \Omega \) into the sets \( \Gamma^*(\ell), \Omega_1^*(\ell), \Omega_2^*(\ell), \Omega_{\text{bulk}} \) and \( T \) introduced later in this section (see Figure 5) and will establish a lower bound of the energy \( \mathcal{E}_0(\psi, A; \cdot) \) in each of the
We assume \( \ell \) to be the parameter in (5.1) which satisfies \( \ell \approx \kappa^{-3/4} \).

**Lower bound in a neighbourhood of the magnetic barrier.** We start by introducing the set \( \Gamma^*(\ell) \) which covers almost all of the set \( \Gamma \). Recall the assumption that \( \Gamma \) consists of a finite collection of simple smooth curves that may intersect \( \partial \Omega \) transversely. For the simplicity of the exposition, we will focus on the particular case of a single curve intersecting \( \partial \Omega \) at two points. The construction below may be adjusted to cover the general case by considering every single component of \( \Gamma \) separately. We may select two constants \( \ell_0 \in (0, 1) \) and \( c > 2 \), and for all \( \ell \in (0, \ell_0) \), a collection of pairwise distinct points \( \{x_i\}_{i=1}^N \subset \Gamma \) such that,

\[
(x_i)_{i=1}^N \subset \{ u \in \Gamma : \text{dist}(u, \partial \Omega) > 2\ell \}, \tag{8.1}
\]

\[
\forall i \in \{1, ..., N - 1\}, \quad \text{dist}_\Gamma(x_i, x_{i+1}) = \ell, \tag{8.2}
\]

\[
|\Gamma|\ell^{-1} - c \leq N \leq |\Gamma|\ell^{-1}, \tag{8.3}
\]

and

\[
\{ x \in \Omega : \text{dist}(x, \Gamma) < \ell, \text{dist}(x, \partial \Omega) > c\ell \} \subset \Gamma^*(\ell) := \left( \bigcup_{i=1}^N N_{x_i}(\ell) \right)^\circ, \tag{8.4}
\]

where \( N_{x_i}(\ell) \) is the set introduced in (6.3). Note that the family \( \{N_{x_i}(\ell)\}_{1 \leq i \leq N} \) consists of pairwise disjoint sets. Consequently,

\[
\mathcal{E}_0(\psi, A; \Omega^*(\ell)) = \sum_{i=1}^N \mathcal{E}_0(\psi, A; N_{x_i}(\ell)).
\]

A uniform lower bound for the local energies \( \left( \mathcal{E}_0(\psi, A; N_{x_i}(\ell)) \right)_{1 \leq i \leq N} \) is already established in Theorem 6.1. Using the results of the aforementioned theorem, we write

\[
\mathcal{E}_0(\psi, A; \Gamma^*(\ell)) \geq N \left( b^{-\frac{1}{2}} \kappa \mathcal{E}_a(b) - \kappa \mathcal{E}(\kappa) \right),
\]

\[
\geq |\Gamma|b^{-\frac{1}{2}} \kappa \mathcal{E}_a(b) - C\kappa \mathcal{E}(\kappa). \tag{8.5}
\]

The last inequality follows from (8.3) and the fact that \( \mathcal{E}_a(b) \leq 0 \).

**Lower bound in a neighbourhood of the boundary.** Now, we define the two sets \( \Omega^1_1(\ell) \) and \( \Omega^2_2(\ell) \) which cover almost all of the set \( \partial \Omega \). In a similar fashion of the definition of \( \Gamma^*(\ell) \), we fix \( \ell_0 \in (0, 1) \) and \( c > 2 \) and we select collections of points \( \{y_j\}_{j=1}^{N_1} \subset \partial \Omega_1 \) and \( \{z_k\}_{k=1}^{N_2} \subset \partial \Omega_2 \) such that,

\[
(y_j)_{j=1}^{N_1} \subset \{ u \in \partial \Omega_1 : \text{dist}(u, \Gamma) > 2\ell \}, \quad (z_k)_{k=1}^{N_2} \subset \{ u \in \partial \Omega_2 : \text{dist}(u, \Gamma) > 2\ell \}, \tag{8.6}
\]

\[
\forall j \in \{1, ..., N_1 - 1\}, \quad \text{dist}_{\partial \Omega_1}(y_j, y_{j+1}) = \ell, \quad \forall k \in \{1, ..., N_2 - 1\}, \quad \text{dist}_{\partial \Omega_2}(z_k, z_{k+1}) = \ell, \tag{8.7}
\]

\[
|\partial \Omega_1|\ell^{-1} - c \leq N_1 \leq |\partial \Omega_1|\ell^{-1}, \quad |\partial \Omega_2|\ell^{-1} - c \leq N_2 \leq |\partial \Omega_2|\ell^{-1}, \tag{8.8}
\]

and

\[
\{ x \in \Omega : \text{dist}(x, \partial \Omega_1) < \ell, \text{dist}(x, \Gamma) > c\ell \} \subset \Omega_1^1(\ell) := \left( \bigcup_{j=1}^{N_1} N_{y_j}(\ell) \right)^\circ, \tag{8.9}
\]

\[
\{ x \in \Omega : \text{dist}(x, \partial \Omega_2) < \ell, \text{dist}(x, \Gamma) > c\ell \} \subset \Omega_2^2(\ell) := \left( \bigcup_{k=1}^{N_2} N_{z_k}(\ell) \right)^\circ. \tag{8.10}
\]
where $\mathcal{N}_{y_1}(\ell)$ and $\mathcal{N}_{z_1}(\ell)$ were defined in (7.3). Hence following similar steps as in (8.5), we use the uniform lower bound in Theorem 7.1 together with the estimates in (8.8) to get
\[
E_0(\psi, A; \Omega_1^1(\ell)) \geq |\partial \Omega_1 \cap \partial \Omega| b^{-\frac{1}{2}} E_{\text{surf}}(b) - C\kappa e\xi(\kappa),
\]
and
\[
E_0(\psi, A; \Omega_2^1(\ell)) \geq |\partial \Omega_2 \cap \partial \Omega| b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} E_{\text{surf}}(b|a|) - C\kappa e\xi(\kappa).
\]
**Lower bound in the bulk.** Next, we introduce the set $\Omega_{\text{bulk}}$ representing the bulk of the sample. Let
\[
\Omega_{\text{bulk}} = \{ x \in \Omega : \text{dist}(x, \partial \Omega_1 \cup \partial \Omega_2) > \ell \}.
\]
Under our assumptions on $b$ in (6.4) and $\ell$ in (5.1), the exponential decay in Theorem 2.4 allows us to neglect the energy contribution in the bulk, and to particularly write
\[
|E_0(\psi, A; \Omega_{\text{bulk}})| \leq C\xi(\kappa),
\]
where $\xi$ is a real-valued function satisfying $\lim_{\kappa \to +\infty} \xi(\kappa) = 0$.

**Lower bound in a neighbourhood of the $T$-zone.** We finally introduce the remaining set in the decomposition of $\Omega$, the neighbourhood $T$ of $\Gamma \cap \partial \Omega$
\[
T = \Omega \setminus \left( \bigcup_{j=1}^{2} \Omega_j^* + \Gamma(\ell) \cup \Omega_{\text{bulk}} \right).
\]
The definition of the sets $\Gamma^*(\ell)$, $\Omega_1^*(\ell)$, $\Omega_2^*(\ell)$ and $\Omega_{\text{bulk}}$ in (8.4), (8.9), (8.10) and (8.13) ensures that $|T| = O(\ell^2)$ as $\ell \to 0$. This small size of $T$ together with $|\psi| \leq 1$ and the first item in Theorem 2.3 imply the following
\[
|E_0(\psi, A; T)| \leq C\kappa \ell^2,
\]
\[
\leq C \kappa^{\frac{1}{2}}.
\]
We used the assumption $\ell \approx \kappa^{-3/4}$ in the last inequality.

Now it is time to gather pieces, and to derive a lower bound for the global GL energy $E_0(\psi, A; \Omega)$. So, we use the estimates in (8.5), (8.11), (8.12), (8.14) and (8.15) to conclude
\[
E_0(\psi, A; \Omega) = E_0(\psi, A; \Gamma^*(\ell)) + E_0(\psi, A; \Omega_1^*(\ell)) + E_0(\psi, A; \Omega_2^*(\ell)) + E_0(\psi, A; A; T)
\geq |\Gamma| b^{-\frac{1}{2}} \kappa e\xi(b) + |\partial \Omega_1 \cap \partial \Omega| b^{-\frac{1}{2}} \kappa E_{\text{surf}}(b)
\geq |\partial \Omega_2 \cap \partial \Omega| b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} \kappa E_{\text{surf}}(b|a|) - C\kappa R(\kappa).
\]
where $R(\kappa) = r(\kappa) + \tilde{r}(\kappa) + \kappa^{-1/2} = o(1)$, when $\kappa \to +\infty$.

### 8.1.2. Energy upper bound.

The upper bound of the ground state energy $E_{\text{st}}(\kappa, H)$ can be derived by the help of a suitable trial configuration.

In this section, **we are still considering the parameter $\ell$ as in (5.1)**. Let $F$ be the magnetic potential introduced in Lemma 2.2. We define the function $u_\ell \in H^1(\Omega; \mathbb{C}) \cap H_0^1(\Gamma^*(\ell))$
\[
w_\ell(x) = \sum_{i=1}^N \mathbf{1}_{\mathcal{N}_{y_i}(\ell)}(x) v_i(x),
\]
where $\Gamma^*(\ell)$ is the set defined in (8.4), $\mathcal{N}_{y_i}(\ell)$ is the set in (6.3), $v_i(x) = e^{i\kappa H \omega_{z_i}(x)} u_i \circ \Phi^{-1}(x)$, $\omega_{z_i}$ is the gauge transformation function defined in Proposition 1.2, $\Phi$ is the coordinate transformation in (4.4). $u_i$ is defined by $u_i(s, t) = u_0(s - s_i, t)$ where
\((s_i, t_i) = \Phi^{-1}(x_i)\), and \(u_0\) is the minimizer of the functional \(G(\cdot, V(L))\) defined in (5.5). From the definition of \(v_i\), we derive the following identity (see (6.7))

\[
E_0(v_i, F; N_{x_i}(L)) = G(u_0, V(L)),
\]

where \(E_0\) is the energy in (6.1). We use this identity, the results in Lemma 5.2, (8.3) and \((L \approx \kappa^{-3/4})\) to derive the following upper bound

\[
E_0(w_T, F; \Omega) = \sum_{i=1}^{N} E_0(v_i, F; N_{x_i}(L)) \leq |\Gamma|b^{-\frac{1}{2}} \kappa \epsilon(a) + C \kappa^{\frac{3}{2}}. \tag{8.17}
\]

Similarly, for \(j \in \{1, 2\}\), using the results of Theorem 7.1, one may define a function \(w_j \in H^1(\Omega; C) \cap H^1_{\text{loc}}(\Omega_j^*(L))\) satisfying

\[
E_0(w_1, F; \Omega_j^*(L)) \leq |\partial \Omega_1 \cap \partial \Omega|b^{-\frac{1}{2}} \kappa E_{\text{surf}}(b) + C \kappa \epsilon(\kappa),
\]

\[
E_0(w_2, F; \Omega_2^*(L)) \leq |\partial \Omega_2 \cap \partial \Omega|b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} \kappa E_{\text{surf}}(b|a|) + C \kappa \epsilon(\kappa), \tag{8.18}
\]

where \(\Omega_j^*(L)\) is defined in (8.9) and (8.10), and \(r_j\) is a real-valued function tending to zero when \(\kappa \to +\infty\). Now, we define the trial function

\[
w(x) = \mathbb{1}_{\Gamma^*(L)}(x)w_T(x) + \mathbb{1}_{\Omega_1^*(L)}(x)w_1(x) + \mathbb{1}_{\Omega_2^*(L)}(x)w_2(x),
\]

Gathering results in (8.17) and (8.18), we get

\[
E(w, F; \Omega) = E_0(w, F; \Omega) = E_0(w_T, F; \Gamma^*(L)) + E_0(w_1, F; \Omega_1^*(L)) + E_0(w_2, F; \Omega_2^*(L)) \leq |\Gamma|b^{-\frac{1}{2}} \kappa \epsilon(a) + |\partial \Omega_1 \cap \partial \Omega|b^{-\frac{1}{2}} \kappa E_{\text{surf}}(b) + |\partial \Omega_2 \cap \partial \Omega|b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} \kappa E_{\text{surf}}(b|a|) + C \kappa \tilde{R}(\kappa),
\]

where \(\tilde{R}(\kappa) = r_1(\kappa) + r_2(\kappa) + \kappa^{-1/12} = o(1)\), when \(\kappa \to +\infty\). We finally derive the upper bound in (1.16) from the fact that \(E_{\text{st}}(\kappa, H) \leq E(w, F; \Omega)\).

8.1.3. \(L^4\)-norm asymptotics. In light of (1.4), any minimizing configuration \((\psi, A)\) of the functional in (1.4) satisfies

\[
(\nabla - i \kappa HA)^2 \psi = \kappa^2 (|\psi|^2 - 1) \psi \text{ in } \Omega. \tag{8.19}
\]

We multiply both sides of (8.19) by \(\overline{\psi}\) and integrate by parts, using the boundary condition in (1.4). We get

\[
E_0(\psi, A; \Omega) = -\frac{1}{2} \kappa^2 \int_{\Omega} |\psi|^4 \, dx. \tag{8.20}
\]

Having

\[
E_0(\psi, A; \Omega) \leq E_{\text{st}}(\kappa, H),
\]

the lower bound in (1.7) follows from (8.20) and (1.6). The upper bound is derived from (8.20) and (8.16).

8.2. Proof of Theorem 1.7. The proof of Theorem 1.7 follows from the local estimates stated in Theorems 6.1 and 7.1 together with the decay result in Theorem 2.4.
APPENDIX A. SOME SPECTRAL PROPERTIES OF FIBER OPERATORS

Let \( a \in [-1, 1) \setminus \{0\} \) and \( \xi \in \mathbb{R} \). Recall the operator \( h_a[\xi] \) introduced in (2.11) and its associated quadratic form \( q_a[\xi] \) defined in (2.12). The embedding of the domain of \( q_a[\xi] \) is compact in \( L^2(\mathbb{R}) \), hence the spectrum of \( h_a[\xi] \) is an increasing sequence of eigenvalues converging to \(+\infty\). The first eigenvalue \( \mu_a(\xi) \) of this operator was defined in (2.15) by the min-max principle.

The result in the following proposition may be derived similarly as done in [FH10, Section 3.2.1]:

**Proposition A.1.** The first eigenvalue \( \mu_a(\xi) \) of \( h_a[\xi] \) is simple. Furthermore, there exists a positive eigenfunction \( f_{a,\xi} \) normalized with respect to the norm of \( \| \cdot \|_{L^2(\mathbb{R})} \).

**Notation.** For the positive eigenfunction \( f_{a,\xi} \) defined in the previous proposition, we associate the de Gennes parameter:

\[
\gamma_a(\xi) = \left( \frac{f'_{a,\xi}}{f_{a,\xi}} \right)(0+) = \left( \frac{f'_{a,\xi}}{f_{a,\xi}} \right)(0-)
\]

The next theorem calls some regularity properties:

**Theorem A.2.** The functions \( \xi \mapsto \mu_a(\xi), \xi \mapsto f_{a,\xi}, \) and \( \xi \mapsto \gamma_a(\xi) \) are in \( C^\infty \).

**Proof.** Note that the domain of the operator \( h_a[\xi] \) can be expressed, independently of \( \xi \), by

\[
\text{Dom}(h_a[\xi]) = \left\{ u \in B^1(\mathbb{R}) : \left( -\frac{d^2}{dt^2} + s^2 u^2 \right) u \in L^2(\mathbb{R}), \; u'(0+) = u'(0-) \right\},
\]

where \( s(t) = 1_{\mathbb{R}^+}(t) + a1_{\mathbb{R}^-}(t) \). Also, given a function \( u \in \text{Dom}(h_a[\xi]) \), the function

\[
\mathbb{R} \ni \xi \mapsto h_a[\xi](u)
\]

is holomorphic. Hence, \( \left( h_a[\xi] \right)_\xi \) is a self-adjoint holomorphic family of type (A) with compact resolvent defined for \( \xi \in \mathbb{R} \). The regularity results follow then from the perturbation theory of Kato (see [FH10, Theorem C.2.2]). \( \square \)

Next, we establish some bounds of \( \mu_a(\xi) \) involving the first Neumann and Dirichlet eigenvalues \( \mu^N(\cdot) \) and \( \mu^D(\cdot) \) respectively defined in (2.4) and (2.5). These bounds are used to set down important limits in Proposition A.4.

**Lemma A.3.** Let \( a \in [-1, 1) \setminus \{0\} \). It holds

* If \( a \in (0, 1) \), then

\[
\min \left( \mu^N(-\xi), \ a \mu^N \left( \frac{\xi}{\sqrt{a}} \right) \right) \leq \mu_a(\xi) \leq \min \left( \mu^D(-\xi), \ a \mu^D \left( \frac{\xi}{\sqrt{a}} \right) \right).
\]

* If \( a \in [-1, 0) \), then

\[
\min \left( \mu^N(-\xi), \ |a| \mu^N \left( \frac{\xi}{\sqrt{|a|}} \right) \right) \leq \mu_a(\xi) \leq \min \left( \mu^D(-\xi), \ |a| \mu^D \left( \frac{\xi}{\sqrt{|a|}} \right) \right).
\]

**Proof.** The proof is similar to the one done in [Kac06, Lemma 3.2.2]. \( \square \)

**Proposition A.4.** Let \( a \in [-1, 1) \setminus \{0\} \). We have

* For \( a \in (0, 1) \),

\[
\lim_{\xi \to -\infty} \mu_a(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to +\infty} \mu_a(\xi) = a.
\]
• For \( a \in [-1, 0) \),
\[
\lim_{\xi \to -\infty} \mu_a(\xi) = |a| \quad \text{and} \quad \lim_{\xi \to +\infty} \mu_a(\xi) = +\infty.
\]

**Proof.** It is sufficient to apply Lemma 2.3 and Theorems 2.5 and 2.6. □

We conclude this section by determining the derivative of \( \mu_a(\xi) \) with respect to \( \xi \). Note that we are interested in establishing the result only in the case where \( a \in [-1, 0) \).

**Proposition A.5.** For any \( a \in [-1, 0) \) and \( \xi \in \mathbb{R} \) we have
\[
\partial_{\xi} \mu_a(\xi) = \left( 1 - \frac{1}{a} \right) \left( \gamma_a^2(\xi) + \mu_a(\xi) - \xi^2 \right) |f_a(\xi)|^2 \quad \text{(A.2)}
\]
where \( f_{a,\xi} \) is the eigenfunction in Proposition A.1 and \( \gamma_a(\xi) \) as in \( (A.1) \).

**Proof.** (Feynman-Hellmann). For simplicity, we write \( \mu, f \) and \( h \) respectively for \( \mu_a(\xi), f_{a,\xi} \), and \( h_a[\xi] \). Differentiating with respect to \( \xi \) and integrating by parts in
\[
(h - \mu)f = 0 \quad \text{(A.3)}
\]
we get
\[
\left\langle (\partial_{\xi} h - \partial_{\xi} \mu)f, f \right\rangle + \left\langle (h - \mu)\partial_{\xi} f, f \right\rangle = 0.
\]
Hence using
\[
\left\langle (h - \mu)\partial_{\xi} f, f \right\rangle = \left\langle \partial_{\xi} f, (h - \mu)f \right\rangle = 0,
\]
and recalling that \( f \) is normalized, we obtain
\[
\partial_{\xi} \mu = \left\langle \partial_{\xi} h, f \right\rangle = 2 \int_{-\infty}^{0} (\xi + at) f^2 dt + 2 \int_{0}^{+\infty} (\xi + t) f^2 dt \quad \text{(A.4)}
\]
Integrating by parts the right hand side of \( (A.4) \), and using \( (A.3) \) together with the Neumann condition \( (f'(0_-) = f'(0_+)) \) establish the result. □

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