Delzant’s variation on Scott Complexity

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To Peter Scott, for his 60th birthday

Abstract

We give an exposition of Delzant’s ideas extending the notion of Scott complexity of finitely generated groups to surjective homomorphisms from finitely presented groups.
Introduction

It is an old question of W. Jaco [4] whether every finitely generated indecomposable group has a finitely presented indecomposable cover: given a finitely generated indecomposable group $H$ is there a surjective homomorphism $\phi : G \to H$ from a finitely presented group $G$ such that for any factorization of $\phi = \psi \alpha$, $G \xrightarrow{\alpha} G' \xrightarrow{\psi} H$ with $\alpha$ surjective, we have $G'$ is also indecomposable. Jaco originally raised the question in connection with the coherence of 3-manifold groups. Peter Scott [5] (and independently Peter Shalen) proved the coherence of 3-manifold groups but bypassed the above question. In his proof, Scott used a notion of complexity of finitely generated groups which is sometimes called Scott complexity. Thomas Delzant extended the notion of complexity to surjective homomorphisms of finitely presented groups to finitely generated groups and answered the above question in the affirmative. This leads to a quick proof of the coherence of 3-manifold groups as well as a proof of the acylindrical accessibility theorem of Sela. Delzant knew these arguments for several years and seems to have other applications in mind. Hopefully, he will write up a more complete exposition of his ideas. In view of the interest shown by various people who worked on this problem, we give an exposition of some of his arguments.

1 Complexity

We call a group $G$ indecomposable if $G$ is not a free product and is not isomorphic to $\mathbb{Z}$. Some authors use the term ‘freely indecomposable’. Let $G$ be a finitely generated group and let

$$G = G_1 \ast \cdots \ast G_m \ast G_{m+1} \ast \cdots \ast G_{m+n}$$

be a free product decomposition of $G$ with $G_i$ indecomposable for $i \leq m$ and $G_i$ isomorphic to $\mathbb{Z}$ for $i > m$. The factors $G_i, i \leq m$ are called the indecomposable factors of $G$. These are unique up to isomorphism and there are only finitely many such up to conjugacy in $G$. We call decompositions of $G$ of the above type standard decompositions of $G$. The ordered pair $(m + n, n)$ is called the complexity or Scott complexity of $G$ and is denoted by $c(G)$. There is a partial order on the complexities given by lexicographic order. Scott used Stallings technique of binding ties [7] to prove:
Theorem 1.1 Let $\phi : G \to H$ be a surjective homomorphism of finitely generated groups such that $\phi$ restricted to each indecomposable factor of $G$ is injective. Then $c(G) \geq c(H)$ and $c(G) = c(H)$ if and only if $\phi$ is an isomorphism.

This theorem is proved by first taking a standard decomposition

$$H = H_1 \cdot \ldots \cdot H_m \cdot H_{m+1} \cdot \ldots \cdot H_{m+n}$$

where $C(H) = (m + n, n)$ and then obtaining a decomposition

$$G = G_1 \cdot \ldots \cdot G_m \cdot G_{m+1} \cdot \ldots \cdot G_{m+n}$$

with $\phi(G_i) \subseteq H_i$. This is achieved by the method of binding ties. The hypothesis that $\phi$ is injective on the indecomposable factors of $G$ implies that the $G_j, j \geq m$, are free. Thus $c(G) \geq c(H)$. If $c(G) = c(H)$, clearly $\phi$ is an isomorphism.

The above argument easily extends to the case when $\phi(K)$ is indecomposable for each indecomposable factor of $G$. More generally:

Theorem 1.2 Let $\phi : G \to H$ be a surjective homomorphism of finitely generated groups such that $\phi(K)$ can be conjugated into an indecomposable factor of $H$ for each indecomposable factor $K$ of $G$. Then $c(G) \geq c(H)$. Suppose that $c(G) = c(H) = (m + n, n)$ and

$$H = H_1 \cdot \ldots \cdot H_m \cdot H_{m+1} \cdot \ldots \cdot H_{m+n}$$

is a standard decomposition of $H$. Then there is a standard decomposition

$$G = G_1 \cdot \ldots \cdot G_m \cdot G_{m+1} \cdot \ldots \cdot G_{m+n}$$

such that $\phi(G_i) = H_i$.

Thus, when $c(G) = c(H)$, and $\phi(K)$ are indecomposable for each indecomposable factor $K$ of $G$, the standard decompositions of $H$ can be imitated by standard decompositions of $G$ which respect $\phi$. It is also easy to see that in this case, the standard decompositions of $G$ can be pushed forward.

Remark 1.3 Suppose that $\phi : G \to H$ is a surjective homomorphism of finitely generated groups such that $\phi(K)$ is indecomposable for each indecomposable factor $K$ of $G$ and further assume that $c(G) = c(H) = (m + n, n)$. If

$$G = G_1 \cdot \ldots \cdot G_m \cdot G_{m+1} \cdot \ldots \cdot G_{m+n}$$
is a standard decomposition of $G$, then there is a standard decomposition
\[ H = H_1 * \cdots * H_m * H_{m+1} * \cdots * H_{m+n} \]
such that $\phi(G_i) = H_i$.

To see this start with a standard decomposition
\[ H = H'_1 * \cdots * H'_m * H'_{m+1} * \cdots * H'_{m+n}. \]
Consider $G' = \ast \phi(G_i)$. We have surjective homomorphisms:
\[ G \xrightarrow{\alpha} G' \xrightarrow{\psi} H \]
with $\phi = \psi \alpha$. We also have $c(G') = (m + n, n)$. By the previous theorem, there is a standard decomposition $G' = G'_1 * \cdots * G'_m * G'_{m+1} * \cdots * G'_{m+n}$ with $\alpha(G'_i) = H'_i$. By construction $\alpha$ restricted to the indecomposable factors of $G'$ is injective and thus $\alpha$ is an isomorphism. Thus we can take $H_i = \alpha \psi(G_i)$.

We now give Delzant’s extension of the notion of complexity.

**Definition 1.4** Let $\phi : G \to H$ be a surjective homomorphism of a finitely presented group $G$. Consider factorizations of $\phi = \psi \alpha$, $G \xrightarrow{\alpha} G' \xrightarrow{\psi} H$ with $\alpha$ surjective and $G'$ finitely presented. The complexity of $\alpha$ is by definition the supremum of the complexities $c(G')$ as $G'$ varies over finitely presented groups.

We also mention a folklore result that is used below:

**Proposition 1.5** Let $\phi : G \to H$ be an epimorphism of a finitely presented group $G$ onto a finitely generated group $H$. Suppose that $H$ is a non-trivial free product $H_1 * H_2$. Then there is a factorization $G \xrightarrow{\alpha} G' \xrightarrow{\psi} H$ with $G'$ finitely presented, $\alpha$ surjective, $G' = G_1 * G_2$ and $\psi(G_i) = H_i, i = 1, 2$.

This proposition is easily proved by Stallings’ method of binding ties.

In the above discussion, we considered several times homomorphisms $\phi : G \to H$ which do not have factorization of the form $\phi = \alpha \psi$, where $\psi$ is a surjective homomorphism from $G$ to $G'$ with $G'$ either a non-trivial free product or infinite cyclic. It is natural to call such homomorphisms $\phi$ essential.
2 Main Results

Theorem 2.1 Let $H$ be a finitely generated indecomposable group. Then there is a finitely presented indecomposable group $G$ and a surjective homomorphism $\phi : G \to H$ such that for any factorization: $\phi = \psi \alpha$,

$$G \xrightarrow{\alpha} G' \xrightarrow{\psi} H$$

with $\alpha$ surjective, then $G'$ is also indecomposable.

Such a $\phi : G \to H$ is called a finitely presented indecomposable cover of $H$. In the terminology introduced at the end of the previous section, Theorem 2.1 reads:

Theorem 2.2 If $H$ is a finitely generated indecomposable group, then $H$ admits essential epimorphisms from finitely presented groups.

In view of Proposition 1.5, Theorem 2.2 easily follows from:

Theorem 2.3 Let $\phi_i : G_i \to H$, $G_i \xrightarrow{\alpha_i} G_{i+1} \xrightarrow{\phi_i} H$ be such that $\phi_i = \phi_{i+1} \alpha_i$, and all maps are surjective homomorphisms with $G_i$ finitely presented. Suppose further that $G$ is the direct limit of the $G_i$, that is, if $\phi_1(g) = 1$, then there is an $n$ such that $\alpha_n \alpha_{n-1} \cdots \alpha_1(g) = 1$. Then there is an integer $K$ such that $G_i$ is indecomposable for $i \geq K$.

We now present the proof of the second theorem.

Proof. Clearly $c(\phi_i) \geq c(\phi_{i+1})$. By going to a subsequence if necessary, we may assume that $c(\phi_i)$ are all equal to $(m+n, n)$. If $(m+n, n) = (1, 0)$, there is nothing to prove. Otherwise, we will arrive at a contradiction. Consider standard decompositions of $G_i$.

$$G_i = G_1^i * \cdots * G_m^i * G_{m+1}^i * \cdots * G_{m+n}^i.$$ 

We want to arrange these so that $\alpha_i(G_j^i) \subseteq G_{j+1}^i$. We claim that $\alpha_i(G_j^i)$ is indecomposable. Firstly $\alpha_i(G_j^i)$ cannot be isomorphic to $\mathbb{Z}$ since $c(G_j^i) = c(g_{i+1})$. If $\alpha_i(G_j^i)$ is a free product, then $\alpha_i$ factors through a finitely presented group with at least $(n+m+1)$ factors by Proposition 1.5. Hence $\alpha_i(G_j^i)$ are indecomposable for all $i$ and $j \leq m$. Hence, by Remark 1.3, we can arrange the standard decompositions of $G_i$ so that $\alpha_i(G_j^i) = G_{j+1}^i$. To complete the proof of Theorem 2.2, we observe that since $G$ is a direct limit of $G_i$, $G$ is
a free product of the direct limits $G^j$ of $G^i_j$. Since $G$ is indecomposable, all but one of $G^j$ must be trivial. But none of $G^j$ can be trivial since all $G^i_j$ are finitely presented and triviality of $G^j$ implies that the complexity of some $G_i$ for large $i$ is smaller than $(m + n, n)$. Clearly $(m + n, n) \neq (0, 1)$. This completes the proof of Theorem 2.2.

3 Applications

We sketch quick proofs of two applications. The first is the Scott-Shalen theorem [5]:

**Theorem 3.1** If the fundamental group of a 3-manifold is finitely generated, then it is finitely presented.

The argument goes as follows. Let $H = \pi_1(M)$ be the fundamental group of a 3-manifold $M$ and suppose that $H$ is finitely generated. We may assume that $H$ is indecomposable. Let $\phi : G \to H$ be a finitely presented indecomposable cover of $H$. We represent $G$ as the fundamental group of a finite simplicial complex $K$ and construct a piecewise linear map $f$ which induces $\phi$. Let $N_1$ be a regular neighbourhood of $f(K)$ and $G_1$ be the image of $\pi_1(K)$ in $\pi_1(N_1)$. By Theorem 1.1, $G_1$ is indecomposable. If the boundary $\partial N_1$ of $N_1$ is not incompressible in $M$, then there is a Dehn disc $D_1$ contained in either $N_1$ or in $C_1$, the closure of the complement of $N_1$ in $M$ such that $D_1$ intersects $\partial N_1$ in exactly $\partial D_1$. If $D_1$ is contained in $C_1$, we add thickened $D_1$ to $N_1$ to obtain $N_2$ and call $G_2$, the image of $G$ in $\pi_1(N_2)$. If $D_1$ is in $N_1$, we split $N_1$ along $D_1$. Then one of the pieces (one may be empty if $D_1$ is non-separating), say $N_2$ contains $G_1$ up to conjugacy. Call this $G_2$. So, we can homotope $f$ so that the image of $G$ is $G_2$ under the induced map in the fundamental groups. After a finite number of steps, we find $N_k$ which is incompressible in $M$. Hence $\pi_1(N_k)$ maps injectively to $\pi_1(M)$. Since $\pi_1(N_k)$ contains $G_k$ which maps onto $H = \pi_1(M)$, we see that $\pi_1(N_k)$ is isomorphic to $\pi_1(M)$. Since $N_k$ is compact, we see that $\pi(M)$ is finitely presented.

The second application is Sela’s acylindrical accessibility theorem (see [6] and [8]). Delzant has given an elementary proof of a more general result in the finitely presented case [2]. A simplicial action of a group $H$ on a simplicial tree $T$ is said to be $k$-acylindrical, if the stabilizers of segments of length $k$ are trivial.
Theorem 3.2 (Sela) Let $H$ be a finitely generated indecomposable group and $k$ a positive integer. Then there is a number $n(k,H)$ such that for any $k$-acylindrical minimal action of $H$ on a simplicial tree $T$, the number of vertices of $T/H$ is bounded by $n(k,H)$.

We recall Delzant’s generalization in the finitely presented case.

Definition 3.3 Let $C$ be a family of subgroups of group $G$ which is closed under conjugation and taking subgroups. We say that a $G$-tree $T$ is $(k,C)$-acylindrical if the stabilizers of segments of length $k$ are in $C$.

Delzant shows:

Theorem 3.4 (Delzant [1]) Suppose $G$ is a finitely presented group and $C$ is a family of subgroups of $G$ which is closed under conjugation and taking subgroups. Moreover, suppose that $G$ does not split over an element of $C$. If $T$ is a minimal $(k,C)$-acylindrical tree, then there is a number $n(k,G)$ such that the number of vertices of $T/G$ is bounded by $n(k,G)$.

The number $n(k,G)$ is defined in terms of the triangular presentations of $G$. In Weidmann’s proof [8], the bound is defined in terms of minimal number generators of $G$. Delzant’s argument gives a more general result:

Theorem 3.5 Suppose that $G$ is a finitely presented group and $C$ a family of subgroups of $G$ closed under conjugation and taking subgroups. For any positive integer $k$, there is a positive integer $n(k,G)$ such that the following holds. For any $(k,C)$-acylindrical, minimal $G$-tree $T$, $G$ has a graph of groups decomposition with edge groups in $C$ (the decomposition may be trivial) such that for any of the vertex groups $G_v$ then for a minimal $G_v$-subtree $T_v$ of $T$, $T_v/G_v$ has at most $n(k,G)$ vertices. In particular, if $G$ does not split over an element of $C$, then $T/G$ has less than $n(k,G)$ vertices.

The proof seems to be inspired by Dunwoody’s ideas from [3]. To deduce Sela’s acylindrical accessibility theorem from the above theorem, let $\phi : G \to H$ be a finitely presented indecomposable cover of $H$ and take $C$ to be the family of subgroups of the kernel of $\phi$. Let $\Gamma$ be the graph of groups given by Delzant’s theorem [1]. Since every element of $C$ fixes $T$, not all the vertex groups of $\Gamma$ can be in $C$. Choose a vertex group $G_v$ with $\phi(G_v) \neq 1$. We claim that $\phi(G_v) = H$, for otherwise $\phi$ factors through a free product. This completes the proof of Sela’s theorem [3].
Delzant’s results and arguments seem to give a procedure for deducing results for finitely generated indecomposable groups from similar results in the finitely presented case. Delzant has used similar ideas in [2] to study conjugacy classes of homomorphic images of a finitely presented group in hyperbolic group.

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