Faster and Non-ergodic $O(1/K)$
Stochastic Alternating Direction Method of Multipliers

Cong Fang  
fangcong@pku.edu.cn  
Peking University

Feng Cheng  
fengcheng@pku.edu.cn  
Peking University

Zhouchen Lin  
zlin@pku.edu.cn  
Peking University

Original circulated date: 22th April, 2017.

Abstract

We study stochastic convex optimization subjected to linear equality constraints. Traditional Stochastic Alternating Direction Method of Multipliers Ouyang et al. (2013) and its Nesterov’s acceleration scheme AzadiSra & Sra (2014) can only achieve ergodic $O(1/\sqrt{K})$ convergence rates, where $K$ is the number of iteration. By introducing Variance Reduction (VR) techniques, the convergence rates improve to ergodic $O(1/K)$ Zhong & Kwok (2014); Zheng & Kwok (2016).

In this paper, we propose a new stochastic ADMM which elaborately integrates Nesterov’s extrapolation and VR techniques. We prove that our algorithm can achieve a non-ergodic $O(1/K)$ convergence rate which is optimal for separable linearly constrained non-smooth convex problems, while the convergence rates of VR based ADMM methods are actually tight $O(1/\sqrt{K})$ in non-ergodic sense. To the best of our knowledge, this is the first work that achieves a truly accelerated, stochastic convergence rate for constrained convex problems. The experimental results demonstrate that our algorithm is faster than the existing state-of-the-art stochastic ADMM methods.

1 Introduction

We consider the following general convex finite-sum problems with linear constraints:

$$\min_{x_1, x_2} h_1(x_1) + f_1(x_1) + h_2(x_2) + \frac{1}{n} \sum_{i=1}^{n} f_{2,i}(x_2),$$

$$s.t. \quad A_1 x_1 + A_2 x_2 = b,$$

(1)

where $f_1(x_1)$ and $f_{2,i}(x_2)$ with $i \in \{1, 2, \cdots, n\}$ are convex and have Lipschitz continuous gradients, $h_1(x_1)$ and $h_2(x_2)$ are also convex. We denote that $L_1$ is the Lipschitz constant of $f_1(x_1)$, $L_2$ is the Lipschitz constant of $f_{2,i}(x_2)$ with $i \in \{1, 2, \cdots, n\}$, and $f_2(x) = \frac{1}{n} \sum_{i=1}^{n} f_{2,i}(x)$. We define $F_i(x_i) = h_i(x_i) + f_i(x_i)$ for $i = 1, 2$, $x = (x_1, x_2)$, $F(x_1, x_2) = F_1(x_1) + F_2(x_2)$, and $Ax = \sum_{i=1}^{2} A_i x_i$.

Problem (1) is of great importance in machine learning. The finite-sum function $f_2(x_2)$ is typically a loss over training samples, and the remaining functions control the structure or regularize the model to aid generalization AzadiSra & Sra (2014). The idea of using linear constraints to decouple the loss and regularization terms enables researchers to consider some more sophisticated regularization terms which might be very complicated to solve through proximity operators for Gradient Descent Beck & Teboulle (2009) methods. For example, for multitask learning problems Argyriou et al. (2007); Shen et al. (2015), the regularization term is set as $\mu_1 ||x||_* + \mu_2 ||x||_1$, for most graph-guided fused Lasso and overlapping group Lasso problem Kim et al. (2009); Zheng & Kwok (2016).
Table 1: Convergence rates of ADMM type methods solving Problem (1) (“non-” indicates “non-ergodic”, while “er-” indicates “ergodic”. “Sto.” is short for “Stochastic”, and “Bat.” indicates batch or deterministic algorithms).

| Type | Algorithm          | Convergence Rate            |
|------|--------------------|------------------------------|
| Bat. | ADMM (Davis & Yin 2016) | Tight non-\(O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | LADM-NE (Li & Lin 2016)     | Optimal non-\(O\left(\frac{1}{K}\right)\) |
| Sto. | STOC-ADMM (Ouyang et al. 2013) | er\(-O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | OPG-ADMM (Suzuki 2013)     | er\(-O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | OPT-ADMM (Azadi & Sra 2014) | er\(-O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | SDCA-ADMM (Suzuki 2014)    | unknown                      |
|      | SAG-ADMM (Zhong & Kwok 2014) | Tight non-\(O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | SVRG-ADMM (Zheng & Kwok 2016) | Tight non-\(O\left(\frac{1}{\sqrt{K}}\right)\) |
|      | ACC-SADMM (ours)          | Optimal non-\(O\left(\frac{1}{\sqrt{K}}\right)\) |

the regularization term can be written as \(\mu \|Ax\|_1\), and for many multi-view learning tasks Wang et al. (2016), the regularization terms always involve \(\mu_1\|x\|_2+1+\mu_2\|x\|_*\).

Alternating Direction Method of Multipliers (ADMM) is a very popular optimization method to solve Problem (1), with its advantages in speed, easy implementation, good scalability, shown in lots of literatures (see survey Boyd et al. (2011)). However, though ADMM is effective in practice, the provable convergence rate is not fast. A popular criterion to judge convergence is in ergodic sense. And it is proved in (He & Yuan, 2012; Lin et al., 2015b) that ADMM converges with an \(O\left(\frac{1}{K}\right)\) ergodic rate. Since the non-ergodic results \(O\left(\frac{1}{\sqrt{K}}\right)\), rather than the ergodic one (convex combination of \(x_1, x_2, \ldots, x_K\)) is much faster in practice, researchers gradually turn to analyse the convergence rate in non-ergodic sense. Davis & Yin (2016) prove that the Douglas-Rachford (DR) splitting converges in non-ergodic \(O\left(\frac{1}{\sqrt{K}}\right)\). They also construct a family of functions showing that non-ergodic \(O\left(\frac{1}{\sqrt{K}}\right)\) is tight. Chen et al. (2015) establish \(O\left(\frac{1}{\sqrt{K}}\right)\) for Linearized ADMM. Then Li & Lin (2016) accelerate ADMM through Nesterov’s extrapolation and obtain a non-ergodic \(O\left(\frac{1}{\sqrt{K}}\right)\) convergence rate. They also prove that the lower complexity bound of ADMM type methods for the separable linearly constrained nonsmooth convex problems is exactly \(O\left(\frac{1}{K}\right)\), which demonstrates that their algorithm is optimal. The convergence rates for different ADMM based algorithms are shown in Table 1.

On the other hand, to meet the demands of solving large-scale machine learning problems, stochastic algorithms Bottou (2004) have drawn a lot of interest in recent years. For stochastic ADMM (SADMM), the prior works are from STOC-ADMM Ouyang et al. (2013) and OPG-ADMM Suzuki (2013). Due to the noise of gradient, both of the two algorithms can only achieve an ergodic \(O\left(\frac{1}{\sqrt{K}}\right)\) convergence rate. There are two lines of research to accelerate SADMM. The first is to introduce the Variance Reduction (VR) technique into SADMM. VR methods are widely accepted tricks for finite sum problems which ensure the descent direction to have a bounded variance and so can achieve faster convergence rates. The existing VR based SADMM algorithms include SDCA-ADMM Suzuki (2014), SAG-ADMM Zhong & Kwok (2014) and SVRG-ADMM Zheng & Kwok (2016). SAG-ADMM and SVRG-ADMM can provably achieve ergodic \(O\left(\frac{1}{K}\right)\) rates for Problem (1). However, in non-ergodic sense, their convergence rates are \(O\left(\frac{1}{\sqrt{K}}\right)\) (please see detailed discussions in Section 5.2), while the fastest rate for batch ADMM method is \(O\left(\frac{1}{K}\right)\) Li & Lin (2016). So there still exists a gap between stochastic and batch (deterministic) ADMM. The second line to accelerate SADMM is through the Nesterov’s acceleration Nesterov (1983). This work is from Azadi & Sra (2014), in which the authors propose an ergodic \(O\left(\frac{\mu}{K^2} + \frac{\mu}{K} + \frac{\sigma}{\sqrt{K}}\right)\) stochastic algorithm (OPT-ADMM). The dependence on the smoothness constant of the convergence rate is \(O\left(\frac{1}{K^2}\right)\) and so each term...
in the convergence rate seems to have been improved to optimal. This method is imperfect due to the following two reasons: 1) The worst convergence rate is still the ergodic $O(1/K)$. There is no theoretical improvement in the order $K$. 2) OPT-SADMM is not very effective in practice. The method does not adopt any special technique to tackle the noise of gradient except adding a proximal term $\frac{\|x^{k+1} - x^k\|^2}{\sigma^2}$ to ensure convergence. As the gradients have noise, directly applying the original Nesterov’s extrapolation on the variables often causes the objective function to oscillate and decrease slowly during iteration.

In this paper, we propose Accelerated Stochastic ADMM (ACC-SADMM) for large scale general convex finite-sum problems with linear constraints. By elaborately integrating Nesterov’s extrapolation and VR techniques, ACC-SADMM provably achieves a non-ergodic $O(1/K)$ convergence rate which is optimal for non-smooth problems. So ACC-SADMM fills the theoretical gap between the stochastic and batch (deterministic) ADMM. The original idea to design our ACC-SADMM is by explicitly considering the snapshot vector into the extrapolation terms. This is, to some degree, inspired by [Allen-Zhu 2017] who proposes an $O(1/K^2)$ stochastic gradient algorithm named Katyusha for convex problems. However, there are many distinctions between the two algorithms (please see detailed discussions in Section 5.1). Our method is also very efficient in practice since we have sufficiently considered the noise of gradient into our acceleration scheme. For example, we see detailed discussions in Section 5.1. Our method is also very efficient in practice since we have sufficiently considered the noise of gradient into our acceleration scheme. For example, we adopt extrapolation as $y_s^k = x_s^k + (1 - \theta_{1,s} - \theta_2)(x_s^k - x_s^{k-1})$ in the inner loop, where $\theta_2$ is a constant and $\theta_{1,s}$ decreases after each whole inner loop, instead of directly adopting extrapolation as $y^k = x^k + \frac{\theta^k_k (1-\theta_{1,s}^{k-1})}{\theta_{1,s}^k} (x^k - x^{k-1})$ in the original Nesterov’s scheme as [AzadiSra & Sra 2014] does. So our extrapolation is more “conservative” to tackle the noise of gradients. There are also variants on updating of multiplier and the snapshot vector. We list the contributions of our work as follows:

- We propose ACC-SADMM for large scale convex finite-sum problems with linear constraints which integrates Nesterov’s extrapolation and VR techniques. We prove that our algorithm converges in non-ergodic $O(1/K)$ which is optimal for separable linearly constrained non-smooth convex problems. To our best knowledge, this is the first work that achieves a truly accelerated, stochastic convergence rate for constrained convex problems.

- Our algorithm is fast in practice. We have sufficiently considered the noise of gradient into the extrapolation scheme. The memory cost of our method is also low. The experiments on four benchmark datasets demonstrate the superiority of our algorithm. We also do experiments on the Multitask Learning [Argyriou et al. 2007] problem to demonstrate that our algorithm can be used on very large datasets.

**Notation.** We denote $\|x\|$ as the Euclidean norm, $(x, y) = x^T y$, $\|x\|_2 = \sqrt{x^T x}$, $\|x\|_G = \sqrt{x^T G x}$, and $(x, y)_G = x^T G y$, where $G \succeq 0$. For a matrix $X$, $\|X\|$ is its spectral norm. We use $I$ to denote the identity matrix. Besides, a function $f$ has Lipschitz continuous gradients if $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$, which implies [Nesterov 2013]:

$$f(y) \leq f(x) + \langle \nabla f(x), x - y \rangle + \frac{L}{2}\|x - y\|^2,$$

where $\nabla f(x)$ denotes the gradient of $f$.

## 2 Related Works and Preliminary

### 2.1 Accelerated Stochastic Gradient Algorithms

There are several works in which the authors propose accelerated, stochastic algorithms for unconstrained convex problems. [Nitanda 2014] accelerates SVRG [Johnson & Zhang 2013] through Nesterov’s extrapolation for strongly convex problems. However, their method cannot be extended
to general convex problems. Catalyst\cite{Frostig2015} or APPA\cite{Lin2015a}, reduction also take strategies to obtain faster convergence rate for stochastic convex problems. When the objective function is smooth, these methods can achieve optimal $O(1/K^2)$ convergence rate. Recently, Allen-Zhu\cite{2017} and Hien et al.\cite{2016} propose optimal $O(1/K^2)$ algorithms for general convex problems, named Katyusha and ASMD, respectively. For $\sigma$-strongly convex problems, Katyusha also meets the optimal $O((n + \sqrt{nL/\sigma})\log \frac{1}{\varepsilon})$ rate. However, none of the above algorithms considers the problems with constraints.

2.2 Accelerated Batch ADMM Methods

There are two lines of works which attempt to accelerate Batch ADMM through Nesterov’s acceleration schemes. The first line adopts acceleration only on the smooth term $(f_i(x))$. The works are from Ouyang et al.\cite{2015}; Lu et al.\cite{2016}. The convergence rate that they obtain is ergodic $O(\frac{L^2}{K^2} + \frac{D^2}{K})$. The dependence on the smoothness constant is accelerated to $O(1/K^2)$. So these methods are faster than ADMM but still remain $O(1/K)$ in the ergodic sense. The second line is to adopt acceleration on both $f_i(x)$ and constraints. The resultant algorithm is from Li & Lin\cite{2016}, which is proven to have a non-ergodic $O(1/K)$ rate. Since the original ADMM have a tight $O(1/\sqrt{K})$ convergence rate\cite{Davis2016} in the non-ergodic sense, their method is faster theoretically.

2.3 SADMM and Its Variance Reduction Variants

We introduce some preliminaries of SADMM. Most SADMM methods alternately minimize the following variant surrogate of the augmented Lagrangian:

$$L'(x_1, x_2, \lambda, \beta) = h_1(x_1) + \langle \nabla f_1(x_1), x_1 \rangle + \frac{L_1}{2} \|x_1 - x_1^k\|^2_{A_1},$$

$$+ h_2(x_2) + \langle \tilde{\nabla} f_2(x_2), x_2 \rangle + \frac{L_2}{2} \|x_2 - x_2^k\|^2_{A_2} + \frac{\beta}{2} \|A_1 x_1 + A_2 x_2 - b + \lambda\|^2,$$

where $\tilde{\nabla} f_2(x_2)$ is an estimator of $\nabla f_2(x_2)$ from one or a mini-batch of training samples. So the computation cost for each iteration reduces from $O(n)$ to $O(b)$ instead, where $b$ is the mini-batch size. When $f_i(x) = 0$ and $G_i = 0$, with $i = 1, 2$, Problem\cite{1} is solved as exact ADMM. When there is no $h_i(x_i)$, $G_i$ is set as the identity matrix $I$, with $i = 1, 2$, the subproblem in $x_i$ can be solved through matrix inversion. This scheme is advocated in many SADMM methods\cite{Ouyang2013};\cite{Zhang2014}. Another common approach is linearization (also called the inexact Uzawa method)\cite{Lin2011};\cite{Zhang2011}, where $G_i$ is set as $\eta_i I - \frac{\beta}{L_i} A_i^T A_i$ with $\eta_i \geq 1 + \frac{\beta}{L_i} \|A_i^T A_i\|$.

For STOC-ADMM\cite{Ouyang2013}, $\tilde{\nabla} f_2(x_2)$ is simply set as:

$$\tilde{\nabla} f_2(x_2) = \frac{1}{b} \sum_{i_k \in I_k} \nabla f_{2,i_k}(x_2),$$

where $I_k$ is the mini-batch of size $b$ from $\{1, 2, \ldots, n\}$.

VR methods\cite{Suzuki2014};\cite{Zhang2014};\cite{Zheng2016} choose more sophisticated gradient estimator to achieve faster convergence rates. As our method bounds the variance through the technique of SVRG\cite{Johnson2013}, we introduce SVRG-ADMM Zheng & Kwok\cite{2016}, which uses the gradient estimator as:

$$\tilde{\nabla} f_2(x_2) = \frac{1}{b} \sum_{i_k \in I_k} (\nabla f_{2,i_k}(x_2) - \nabla f_{2,i_k}(\tilde{x}_2)) + \nabla f_2(\tilde{x}_2),$$

where $\tilde{x}_2$ is the mini-batch of size $b$ from $\{1, 2, \ldots, n\}$. The dependence on the smoothness constant is accelerated to $O(1/K^2)$. The works

2.3 SADMM and Its Variance Reduction Variants

We introduce some preliminaries of SADMM. Most SADMM methods alternately minimize the following variant surrogate of the augmented Lagrangian:

$$L'(x_1, x_2, \lambda, \beta) = h_1(x_1) + \langle \nabla f_1(x_1), x_1 \rangle + \frac{L_1}{2} \|x_1 - x_1^k\|^2_{A_1},$$

$$+ h_2(x_2) + \langle \tilde{\nabla} f_2(x_2), x_2 \rangle + \frac{L_2}{2} \|x_2 - x_2^k\|^2_{A_2} + \frac{\beta}{2} \|A_1 x_1 + A_2 x_2 - b + \lambda\|^2,$$

where $\tilde{\nabla} f_2(x_2)$ is an estimator of $\nabla f_2(x_2)$ from one or a mini-batch of training samples. So the computation cost for each iteration reduces from $O(n)$ to $O(b)$ instead, where $b$ is the mini-batch size. When $f_i(x) = 0$ and $G_i = 0$, with $i = 1, 2$, Problem\cite{1} is solved as exact ADMM. When there is no $h_i(x_i)$, $G_i$ is set as the identity matrix $I$, with $i = 1, 2$, the subproblem in $x_i$ can be solved through matrix inversion. This scheme is advocated in many SADMM methods\cite{Ouyang2013};\cite{Zhang2014}. Another common approach is linearization (also called the inexact Uzawa method)\cite{Lin2011};\cite{Zhang2011}, where $G_i$ is set as $\eta_i I - \frac{\beta}{L_i} A_i^T A_i$ with $\eta_i \geq 1 + \frac{\beta}{L_i} \|A_i^T A_i\|$.

For STOC-ADMM\cite{Ouyang2013}, $\tilde{\nabla} f_2(x_2)$ is simply set as:

$$\tilde{\nabla} f_2(x_2) = \frac{1}{b} \sum_{i_k \in I_k} \nabla f_{2,i_k}(x_2),$$

where $I_k$ is the mini-batch of size $b$ from $\{1, 2, \ldots, n\}$.

VR methods\cite{Suzuki2014};\cite{Zhang2014};\cite{Zheng2016} choose more sophisticated gradient estimator to achieve faster convergence rates. As our method bounds the variance through the technique of SVRG\cite{Johnson2013}, we introduce SVRG-ADMM Zheng & Kwok\cite{2016}, which uses the gradient estimator as:

$$\tilde{\nabla} f_2(x_2) = \frac{1}{b} \sum_{i_k \in I_k} (\nabla f_{2,i_k}(x_2) - \nabla f_{2,i_k}(\tilde{x}_2)) + \nabla f_2(\tilde{x}_2),$$

where $\tilde{x}_2$ is the mini-batch of size $b$ from $\{1, 2, \ldots, n\}$. The dependence on the smoothness constant is accelerated to $O(1/K^2)$. The works
where $\tilde{x}_2$ is a snapshot vector. An advantage of SVRG [Johnson & Zhang, 2013] based methods is its low storage requirement, independent of the number of training samples. This makes them more practical on very large datasets. In our multitask learning experiments, SAG-ADMM [Zhang & Kwok, 2014] needs 38.2TB for storing the weights, and SDCA-ADMM needs 9.6GB [Suzuki, 2014] for the dual variables, while the memory cost for our method and SVRG-ADMM is no more than 250MB. [Zheng & Kwok, 2016] prove that SVRG-ADMM converges in ergodic $O(1/K)$. Like batch-ADMM, in non-ergodic sense, the convergence rate is tight $O(1/\sqrt{K})$ (see the discussions in Section 5.2).

3 Our Algorithm

3.1 ACC-SADMM

In this section, we introduce our Algorithm: ACC-SADMM, which is shown in Algorithm 1. For simplicity, we directly linearize both the smooth term $f_i(x_i)$ and the augmented term $\frac{\beta}{2}\|A_1x_1 + A_2x_2 - b + \lambda\|^2$. It is not hard to extend our method to other schemes mentioned in Section 2.3. ACC-SADMM includes two loops. In the inner loop, it updates the primal and dual variables $x_{s,1}^k$, $x_{s,2}^k$ and $\lambda_s^k$. Then in the outer loop, it preserves snapshot vectors $\tilde{x}_{s,1}$, $\tilde{x}_{s,2}$ and $b_s$, and then resets the initial value of the extrapolation term $y_{s+1}^0$. Specifically, in the inner iteration, $x_1$ is updated as:

$$x_{s,1}^{k+1} = \arg\min_{x_1} h_1(x_1) + \langle \nabla f_1(y_{s,1}^k), x_1 \rangle$$

$$+ \langle \frac{\beta}{\theta_1^k} (A_1 y_{s,1}^k + A_2 y_{s,2}^k - b) + \lambda_s^k, A_1^T x_1 \rangle + \left( \frac{L_1}{2} + \frac{\beta\|A_1^T A_1\|}{2\theta_1^k} \right) \|x_1 - y_{s,1}^k\|^2.$$  

And $x_2$ is updated using the latest information of $x_1$, which can be written as:

$$x_{s,2}^{k+1} = \arg\min_{x_2} h_2(x_2) + \langle \nabla f_2(y_{s,1}^k), x_2 \rangle$$

$$+ \langle \frac{\beta}{\theta_1^k} (A_1 x_{s,1}^{k+1} + A_2 y_{s,2}^k - b) + \lambda_s^k, A_2^T x_2 \rangle + \left( \frac{1}{2} + \frac{1}{\theta_2^k} \right) \|x_2 - y_{s,2}^k\|^2,$$

where $\nabla f_2(y_{s,2}^k)$ is obtained by the technique of SVRG [Johnson & Zhang, 2013] with the form:

$$\nabla f_2(y_{s,2}^k) = \frac{1}{b} \sum_{i_k,s \in I_{(s,s)}} (\nabla f_{2,i_k,s}(y_{s,2}^k - \nabla f_{2,i_k,s}(\tilde{x}_{s,2}^k) + \nabla f_2(\tilde{x}_{s,2}^k)).$$

And $y_{s+1}^{k+1}$ is generated as

$$y_{s}^{k+1} = x_{s}^{k+1} + (1 - \theta_{1,s} - \theta_2)(x_{s}^{k+1} - x_{s}^k),$$

when $k \geq 0$. One can find that $1 - \theta_{1,s} - \theta_2 \leq 1 - \theta_{1,s}$. We do extrapolation in a more “conservative” way to tackle the noise of gradient. Then the multiplier is updated through Eq. (9) and Eq. (10). We can find that $\lambda_s^k$ additionally accumulates a compensation term $\frac{\beta\theta_1^{k+1}}{\theta_1^{k+1}} (A_1 x_1 + A_2 x_2 - b_s)$ to ensure $A_1 x_1 + A_2 x_2$ not to go far from $A_1 \tilde{x}_1 + A_2 \tilde{x}_2$ in the course of iteration.

In the outer loop, we set the snapshot vector $\tilde{x}_{s+1}$ as:

$$\tilde{x}_{s+1} = \frac{1}{m} \left( \left[ 1 - \frac{(\tau - 1)\theta_{1,s+1}}{\theta_2} \right] x_{s}^m + \left[ 1 + \frac{(\tau - 1)\theta_{1,s+1}}{(m - 1)\theta_2} \right] \sum_{k=1}^{m-1} x_{s}^k \right).$$
consider the case that there is no linear constraint. Then by the convexity of $y$

where $E$

At the last step, we reset $Zheng & Kwok (2016)$. The way of generating $\tilde{x}$ guarantees a faster convergence rate for the constraints. Then at the last step, we reset $y^0_{s+1}$ as:

$$ y^0_{s+1} = (1 - \theta_2)\tilde{x}_s^m + \theta_2\tilde{x}_{s+1} + \frac{\theta_{1,s+1}}{\theta_{1,s}} \left[ (1 - \theta_{1,s})x^m_s - (1 - \theta_{1,s} - \theta_2)x^m_{s-1} - \theta_2\tilde{x}_s \right]. $$

The whole algorithm is shown Algorithm 1.

**Algorithm 1 ACC-SADMM**

**Input:** epoch length $m > 1$, $\beta$, $\tau = 2$, $c = 2$, $x_0^0 = 0$, $\lambda^0 = 0$, $\tilde{x}^0 = x_0^0$, $\gamma^0 = x_0^0$, $\theta_{1,s} = \frac{1}{c+\tau}$, $\theta_2 = \frac{m-\tau}{\tau(m-1)}$.

for $s = 0$ to $S-1$

for $k = 0$ to $m-1$

\[ \lambda^k_s = \lambda^k_s + \frac{\beta \theta_2}{\theta_{1,s}} \left( A_1x^k_{s,1} + A_2x^k_{s,2} - \tilde{b}_s \right). \] (9)

Update $x^k_{s,1}$ through Eq. (6).

Update $x^k_{s,2}$ through Eq. (7).

Update $y^k_{s+1}$ through Eq. (8).

\[ \tilde{x}_{s+1} = A_1\tilde{x}_{s+1,1} + A_2\tilde{x}_{s+1,2}. \]

Update $y^0_{s+1}$ through Eq. (13).

Output:

$$ \tilde{x}_S = \frac{1}{(m-1)(\theta_{1,s} + \theta_2) + 1} x^m_S + \frac{\theta_{1,s} + \theta_2}{(m-1)(\theta_{1,s} + \theta_2) + 1} \sum_{k=1}^{m-1} x^k_S. $$

\[ \tilde{x}_{s+1} \]

is not the average of $\{x^k_s\}$, different from most SVRG-based methods Johnson & Zhang (2013); Zheng & Kwok (2016). The way of generating $\tilde{x}$ guarantees a faster convergence rate for the constraints. Then at the last step, we reset $y^0_{s+1}$ as:

$$ y^0_{s+1} = (1 - \theta_2)x^m_s + \theta_2\tilde{x}_{s+1} + \frac{\theta_{1,s+1}}{\theta_{1,s}} \left[ (1 - \theta_{1,s})x^m_s - (1 - \theta_{1,s} - \theta_2)x^m_{s-1} - \theta_2\tilde{x}_s \right]. $$

The whole algorithm is shown Algorithm 1.

### 3.2 Intuition

Though Algorithm 1 is a little complex at the first sight, our intuition to design the algorithm is straightforward. To bound the variance, we use the technique of SVRG Johnson & Zhang (2013). Like Johnson & Zhang (2013); Allen-Zhu (2017), the variance of gradient is bounded through

$$ \mathbb{E}_{i_k} \left( \| \nabla f_2(y^k_2) - \nabla f_2(y^k_2) \|^2 \right) \leq \frac{2L_2}{b} \left[ f_2(\tilde{x}_2) - f_2(y^k_2) + (\nabla f_2(y^k_2), y^k_2 - \tilde{x}_2) \right], $$ (13)

where $\mathbb{E}_{i_k}$ indicates that the expectation is taken over the random choice of $i_{k,s}$, under the condition that $y^k_2$, $\tilde{x}_2$ and $x^k_2$ (the randomness in the first $sm + k$ iterations are fixed) are known. We first consider the case that there is no linear constraint. Then by the convexity of $F_1(x_1)$ Beck & Teboulle (2009), we have

$$ F_1(x^{k+1}_1) \leq F_1(u_1) + \frac{L_1}{2} \| x^{k+1}_1 - y^{k+1}_1 \|^2 - L_1 (x^{k+1}_1 - y^{k+1}_1, x^{k+1}_1 - u_1), $$ (14)
Setting $\mathbf{u}_1$ be $\mathbf{x}_1^i$, $\bar{x}_1$ and $\mathbf{x}_1^*$, respectively, then multiplying the three inequalities by $(1 - \theta_1 - \theta_2)$, $\theta_2$, and $\theta_1$, respectively, and adding them, we have

$$F_1(x_1^{k+1}) \leq (1 - \theta_1 - \theta_2)F_1(x_1^k) + \theta_2 F_1(\bar{x}_1) + \theta_1 F_1(x_1^*) - L_1(x_1^{k+1} - y_1^k, x_1^k + (1 - \theta_1 - \theta_2)x_1^k - \theta_2 \bar{x}_1 - \theta_1 x_1^*) + \frac{L_1}{2}\|x_1^{k+1} - y_1^k\|^2. \tag{15}$$

where $\theta_1$ and $\theta_2$ are undetermined coefficients. Comparing with Eq. (13), we can find that there is one more term $(\nabla f_2(y_2^k), y_2^k - \bar{x}_2)$ that we need to eliminate. To solve this issue, we analyse the points at $\mathbf{w}^k = y_2^k + \theta_3(y_2^k - \bar{x}_2)$ and $\mathbf{z}^{k+1} = x_2^{k+1} + \theta_3(y_2^k - \bar{x}_2)$, where $\theta_3$ is an undetermined coefficient. When $\theta_3 > 0$, $\mathbf{w}^k$ and $\mathbf{z}^{k+1}$ is closer to $\bar{x}_2$ compared with $y_2^k$ and $x_2^{k+1}$. Then by the convexity of $F_2(x_2)$, we can generate a negative $(\nabla f_2(y_2^k), y_2^k - \bar{x}_2)$, which can help to eliminate the variance term. Next we consider the multiplier term. To construct a recursive term of $L(x_1^{k+1}, x_2^{k+1}, \lambda^*) - (1 - \theta_1 - \theta_2)L(x_1^*, x_2^*, \lambda^*) - \theta_2 L(\bar{x}_1, \bar{x}_2, \lambda^*)$, where $L(x_1, x_2, \lambda)$ satisfies Eq. (17), the multiplier should satisfy the following equations:

$$\hat{\lambda}_s^{k+1} - \hat{\lambda}_s^k = \frac{\beta}{\theta_1 s}(A^k x_s^{k+1} - b),$$

and

$$\hat{\lambda}_s^{k+1} = \lambda_s^k + \frac{\beta(1 - \theta_1 s)}{\theta_1 s}(A^k x_s^k - b), \tag{18}$$

where $\hat{\lambda}_s^k$ is undetermined and

$$L(x_1, x_2, \lambda) = F(x_1, x_2) + (\lambda, A_1 x_1 + A_2 x_2 - b), \tag{17}$$

is the Lagrangian function. By introducing a new variable $\hat{\lambda}_s^k$, then setting

$$\hat{\lambda}_s^k = \hat{\lambda}_s^k + \frac{\beta(1 - \theta_1 s)}{\theta_1 s}(A^k x_s^k - b),$$

and with Eq. (9) and Eq. (10), Eq. (16) and Eq. (16) are satisfied. Then Eq. (11) is obtained as we need $\hat{\lambda}_s^0 = \lambda_s^0$ when $s \geq 1$.

### 4 Convergence Analysis

In this section, we give the convergence results of ACC-SADMM. The proof can be found in Supplementary Material. We first analyse each inner iteration. The result is shown in Lemma 1 which connects $x_s^k$ to $x_s^{k+1}$.

**Lemma 1** Assume that $f_i(x_1)$ and $f_2,i(x_2)$ with $i \in \{1, 2, \cdots, n\}$ are convex and have Lipschitz continuous gradients. $L_1$ is the Lipschitz constant of $f_1(x_1)$. $L_2$ is the Lipschitz constant of $f_2,i(x_2)$ with $i \in \{1, 2, \cdots, n\}$. $h_1(x_1)$ and $h_2(x_2)$ is also convex. For Algorithm 1 in any epoch, we have

$$\mathbb{E}_{\mathbf{i}} \left[ L(x_1^{k+1}, x_2^{k+1}, \lambda^*) \right] - \theta_2 L(\bar{x}_1, \bar{x}_2, \lambda^*) - (1 - \theta_2 - \theta_1)L(x_1^*, x_2^*, \lambda^*) \leq \frac{\theta_1}{2\beta} \|\hat{\lambda}_s^k - \lambda^*\|^2 - \mathbb{E}_{\mathbf{i}} \left[ \|\hat{\lambda}_s^{k+1} - \lambda^*\|^2 \right]$$

$$+ \frac{1}{2}\|y_1^k - (1 - \theta_1 - \theta_2)x_1^k - \theta_2 \bar{x}_1 - \theta_1 x_1^*\|_{G_2}^2 - \frac{1}{2}\mathbb{E}_{\mathbf{i}} \left[ \|y_1^k - (1 - \theta_1 - \theta_2)x_1^k - \theta_2 \bar{x}_1 - \theta_1 x_1^*\|_{G_2}^2 \right]$$

$$+ \frac{1}{2}\|y_2^k - (1 - \theta_1 - \theta_2)x_2^k - \theta_2 \bar{x}_2 - \theta_1 x_2^*\|_{G_2}^2 - \frac{1}{2}\mathbb{E}_{\mathbf{i}} \left[ \|y_2^k - (1 - \theta_1 - \theta_2)x_2^k - \theta_2 \bar{x}_2 - \theta_1 x_2^*\|_{G_2}^2 \right],$$

where $L(x_1, x_2, \lambda)$ satisfies Eq. (17) and $\hat{\lambda}$ satisfies Eq. (18), $G_3 = \left( L_1 + \frac{\beta A_1^TA_1}{\theta_1} \right) I - \frac{\beta A_1^TA_1}{\theta_1}$, and

$G_4 = \left( (1 + \frac{1}{\theta_2})L_2 + \frac{\beta A_1^TA_2}{\theta_1} \right) I$. We have ignored subscript $s$ as $s$ is fixed in each epoch.
Then Theorem 1 analyses ACC-SADMM in the whole iteration, which is the key convergence result of the paper.

**Theorem 1** If the conditions in Lemma 1 hold, then we have

\[
\mathbb{E}\left( \frac{\beta m}{\theta_1 S} (\mathbf{A} \hat{x}_S - \mathbf{b}) - \frac{\beta (m-1) \theta_2}{\theta_1, 0} (\mathbf{A} \mathbf{x}_0 - \mathbf{b}) + \lambda_0^* - \mathbf{x}_0^* \right)^2 + \frac{m}{\theta_1 S} (F(\hat{x}_S) - F(\mathbf{x}^*)) + \langle \lambda^*, \mathbf{A} \hat{x}_S - \mathbf{b} \rangle \right) 
\leq C_3 \left( F(\mathbf{x}_0) - F(\mathbf{x}^*) + \langle \lambda^*, \mathbf{A} \mathbf{x}_0 - \mathbf{b} \rangle \right) + \frac{1}{2\beta} \| \hat{\lambda}_0^* + \frac{\beta (1 - \theta_1, 0)}{\theta_1, 0} (\mathbf{A} \mathbf{x}_0 - \mathbf{b}) - \lambda_0^* \|^2 
+ \frac{1}{2} \| \mathbf{x}_{0,1}^0 - \mathbf{x}_1 \|^2 \left( \theta_1, 0 L_1 + \| \mathbf{A}^T \mathbf{A} \| I - \mathbf{A}^T \mathbf{A} \right) + \frac{1}{2} \| \mathbf{x}_{0,2}^0 - \mathbf{x}_2 \|^2 \left( (1 + \frac{1}{\alpha} \sigma_2) \theta_1, 0 L_2 + \| \mathbf{A}^T \mathbf{A} \| I \right) \right)^T \tag{19} \]

where \( C_3 = \frac{1 - \theta_1, 0 + (m-1) \theta_2}{\theta_1, 0} \).

Corollary 1 directly demonstrates that ACC-SADMM have a non-ergodic \( O(1/K) \) convergence rate.

**Corollary 1** If the conditions in Lemma 2 holds, we have

\[
\mathbb{E}\| F(\hat{x}_S) - F(\mathbf{x}^*) \| \leq O\left( \frac{1}{S} \right), \quad \mathbb{E}\| \mathbf{A} \hat{x}_S - \mathbf{b} \| \leq O\left( \frac{1}{S} \right). \tag{20} \]

We can find that \( \hat{x}_S \) depends on the latest \( m \) information of \( \mathbf{x}_S^k \). So our convergence results is in non-ergodic sense, while the analysis for SVRG-ADMM [Zheng & Kwok 2016] and SAG-ADMM [Zhong & Kwok 2014] is in ergodic sense, since they consider the point \( \hat{x}_S = \frac{1}{mS} \sum_{s=1}^{S} \sum_{k=1}^{m} \mathbf{x}_S^k \), which is the convex combination of \( \mathbf{x}_S^k \) over all the iterations.

Now we directly use the theoretical results of [Li & Lin 2016] to demonstrate that our algorithm is optimal when there exists non-smooth term in the objective function.

**Theorem 2** For the following problem:

\[
\min_{\mathbf{x}_1, \mathbf{x}_2} F_1(\mathbf{x}_1) + F_2(\mathbf{x}_2), \quad \text{s.t.} \quad \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}, \tag{21} \]

let the ADMM type algorithm to solve it be:

- Generate \( \mathbf{x}_2^k \) and \( \mathbf{y}_2^k \) in any way,
- \( \mathbf{x}_1^{k+1} = \text{Prox}_{F_1/\beta}\left( \mathbf{y}_2^k - \frac{\lambda_k}{\beta} \right) \),
- Generate \( \mathbf{x}_1^{k+1} \) and \( \mathbf{y}_1^{k+1} \) in any way,
- \( \mathbf{x}_2^{k+1} = \text{Prox}_{F_2/\beta}\left( \mathbf{y}_1^{k+1} - \frac{\lambda_k}{\beta} \right) \).

Then there exist convex functions \( F_1 \) and \( F_2 \) defined on \( \mathcal{X} = \{ \mathbf{x} \in \mathbb{R}^{6k+5} : \| \mathbf{x} \| \leq B \} \) for the above general ADMM method, satisfying

\[
L \| \hat{x}_2^k - \mathbf{x}_2^k \| + \| F_1(\hat{x}_1^k) - F_1(\mathbf{x}_1^k) + F_1(\hat{x}_2^k) - F_2(\mathbf{x}_2^k) \| \geq \frac{LB}{8(k+1)}, \tag{22} \]

where \( \hat{x}_1^k = \sum_{i=1}^{k} \alpha_i \mathbf{x}_1^i \) and \( \hat{x}_2^k = \sum_{i=1}^{k} \alpha_i \mathbf{x}_2^i \) for any \( \alpha_i \) and \( \alpha_i \) with \( i \) from 1 to \( k \).

Theorem 2 is Theorem 11 in [Li & Lin 2016]. More details can be found in it. Problem (21) is a special case of Problem (1) as we can set each \( F_{2,i}(\mathbf{x}_2) = F(\mathbf{x}_2) \) with \( i = 1, \cdots, n \) or set \( n = 1 \). So there is no better ADMM type algorithm which converges faster than \( O(1/K) \) for Problem (1).
5 Discussions

We discuss some properties of ACC-SADMM and make further comparisons with some related methods.

5.1 Comparison with Katyusha

As we have mentioned in Introduction, some intuition of our algorithm is inspired by Katyusha Allen-Zhu (2017), which obtains an $O(1/K^2)$ algorithm for convex finite-sum problems. However, Katyusha cannot solve the problem with linear constraints. Besides, Katyusha uses the Nesterov’s second scheme to accelerate the algorithm while our method conducts acceleration through Nesterov’s extrapolation (Nesterov’s first scheme). And our proof uses the technique of Tseng (2008), which is different from Allen-Zhu (2017). Our algorithm can be easily extended to unconstrained convex finite-sum and can also obtain a $O(1/K^2)$ rate but belongs to the Nesterov’s first scheme.

5.2 The Importance of Non-ergodic $O(1/K)$

SAG-ADMM Zhong & Kwok (2014) and SVRG-ADMM Zheng & Kwok (2016) accelerate SADMM to ergodic $O(1/K)$. In Theorem 9 of Li & Lin (2016), the authors generate a class of functions showing that the original ADMM has a tight non-ergodic $O(1/\sqrt{K})$ convergence rate. When $n = 1$, SAG-ADMM and SVRG-ADMM are the same as batch ADMM, so their convergence rates are no better than $O(1/\sqrt{K})$. This shows that our algorithm has a faster convergence rate than VR based SADMM methods in non-ergodic sense. One may deem that judging convergence in ergodic or non-ergodic is unimportant in practice. However, in experiments, our algorithm is much faster than VR based SADMM methods. Actually, though VR based SADMM methods have provably faster rates than STOC-ADMM, the improvement in practice is evident only iterates are close to the convergence point, rather than at the early stage. Both Zhong & Kwok (2014) and Zheng & Kwok (2010) show that SAG-ADMM and SVRG-ADMM are sensitive to the initial points. We also find that if the step sizes are set based on the their theoretical guidances, sometimes they are even slower than STOC-ADMM (see Fig. 1) as the early stage lasts longer when the step size is small. Our algorithm is faster than the two algorithms whenever the step sizes are set based on the theoretical guidances or are tuned to achieve the fastest speed (see Fig. 2). This demonstrates that Nesterov’s extrapolation has truly accelerated the speed and the integration of extrapolation and VR techniques is harmonious and complementary.

5.3 The Growth of Penalty Factor $\frac{\beta}{\theta_1}$

One can find that the penalty factor $\frac{\beta}{\theta_1}$ increases linearly with the iteration. This might make our algorithm impractical because after dozens of epoches, the large value of penalty factor might slow down the decrement of function value. However, in experiments, we have not found any bad influence. There may be two reasons. 1. In our algorithm, $\theta_1$ decreases after each epoch ($m$ iterations), which is much slower than LADM-NE Li & Lin (2016). For most stochastic problems, algorithms converge in less than 100 epoches, thus $\theta_1$ is only 200 times of $\theta_0$. The growth of penalty factor works as a continuation technique Zuo & Lin (2011), which may help to decrease the function value. 2. From Theorem 1, our algorithm converges in $O(1/S)$ whenever $\theta_1$ is large. So from the theoretical viewpoint, a large $\theta_1$ cannot slow down our algorithm. We find that OPT-ADMM Azadi-Sra & Sra (2014) also needs to decrease the step size with the iteration. However, its step size decreasing rate is $O(k^2)$ and is faster than ours.

\footnote{We follow Tseng (2008) to name the extrapolation scheme as Nesterov’s first scheme and the three-step scheme Nesterov (1988) as the Nesterov’s second scheme.}
6 Experiments

We conduct experiments to show the effectiveness of our method\(^2\). We compare our method with the following the-state-of-the-art SADMM algorithms: (1) STOC-ADMM [Ouyang et al., 2013], (2) SVRG-ADMM [Zheng & Kwok, 2016], (3) OPT-SADMM [AzadiSra & Sra, 2014], (4) SAG-ADMM [Zheng & Kwok, 2014]. We ignore the comparison with SDCA-ADMM [Suzuki, 2014] since there is no analysis for it on general convex problems and it is also not faster than SVRG-ADMM [Zheng & Kwok, 2016]. Experiments are performed on Intel(R) CPU i7-4770 @ 3.40GHz machine with 16 GB memory.

6.1 Lasso Problems

We perform experiments to solve two typical Lasso problems. The first is the original Lasso problem:

\[
\min_{x} \mu \|x\|_1 + \frac{1}{n} \sum_{i=1}^{n} \| h_{i} - x^T a_{i} \|^2,
\]

where \( h_{i} \) and \( a_{i} \) are the tag and the data vector, respectively. The second is Graph-Guided Fused Lasso model:

\[
\min_{x} \mu \|Ax\|_1 + \frac{1}{n} \sum_{i=1}^{n} l_{i}(x),
\]

\(^2\)We will put our code online once our paper is accepted.
where $l_i(x)$ is the logistic loss on sample $i$, and $A = [G; I]$ is a matrix encoding the feature sparsity pattern. $G$ is the sparsity pattern of the graph obtained by sparse inverse covariance estimation [Friedman et al. (2008)]. The experiments are performed on four benchmark data sets: a9a, covertype, mnist and dna.

The details of the dataset and the mini-batch size that we use in all SADMM are shown in Table 2. And like Zhong & Kwok (2014) and Zheng & Kwok (2016), we fix $\mu$ to be $10^{-5}$ and report the performance based on $(x_i, Ax_i)$ to satisfy the constraints of ADMM. Results are averaged over five repetitions. And we set $m = \frac{2\beta}{\gamma}$ for all the algorithms. To solve the problems by ADMM methods, we introduce an variable $y = x$ or $y = Ax$. The update for $x$ can be written as: $x^{k+1} = x^k - \gamma(\nabla f_2(x_{2k}) + \beta A^T (Ax - y) + A^T \lambda)$, where $\gamma$ is the step size, which depends on the penalty factor $\beta$ and the Lipschitz constant $L_2$. For the original Lasso (Eq. (23)), $L_2$ has a closed-form solution, namely, we set $L_2 = \max_i ||a_i||^2 = 1$. So in this task, the step sizes are set through theoretical guidances for each algorithm. For Graph-Guided Fused Lasso (Eq. (24)), we regard $L_2$ as a tunable parameter and tune $L_2$ to obtain the best step size for each algorithm, which is similar to Zheng & Kwok (2016) and Zhong & Kwok (2014). Except ACC-SADMM, we use the continuation technique [Zuo & Lin (2011)] to accelerate algorithms. We set $\beta_0 = \min(10, \rho^5 \beta_0)$. Since SAG-ADMM and SVRG-ADMM are sensitive to initial points, like Zheng & Kwok (2016), they are initialized by running STOC-ADMM for $\frac{3n}{\alpha}$ iterations. SAG-ADMM is performed on the first three datasets due to its large memory requirement. More details about parameter setting can be found in Supplementary Materials.

The experimental results of original Lasso and Graph-Guided Fused Lasso are shown in Fig. 1 and Fig. 2 respectively. From the results, we can find that SVRG-ADMM performs much better than STOC-ADMM when the step size is large while the improvement is not large when the step size is small.

---

Figure 2: Experimental results of solving the Graph-Guided Fused Lasso problem (Eq. (24)) (Top: objective gap; Bottom: testing loss). The step size is tuned to be the best for each algorithm. The computation time has included the cost of calculating full gradients for SVRG based methods. SVRG-ADMM and SAG-ADMM are initialized by running STOC-ADMM for $\frac{3n}{\alpha}$ iterations. “-ERG” represents the ergodic results for the corresponding algorithms.

---

3a9a, covertype and dna are from [http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/](http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/), and mnist is from [http://yann.lecun.com/exdb/mnist/](http://yann.lecun.com/exdb/mnist/)

4We normalize the Frobenius norm of each feature to 1.
size is small as it has the cost of calculating the full gradient. SAG-ADMM encounters a similar situation as $x$ is not updated on the latest information. OPT-ADMM performs well on the small step size. However, when the step size is large, the noise of the gradients limits the effectiveness of the extrapolation. Our algorithm is faster than other SADMM on both problems. More experimental results where we set a larger fixed step size and the memory costs of all algorithms are shown in Supplementary Materials.

### 6.2 Multitask Learning

We perform experiments on multitask learning [Argyriou et al. (2007)]. A similar experiment is also conducted by Zheng & Kwok (2016). The experiment is performed on a 1000-class ImageNet dataset [Russakovsky et al. (2015)] (see Table 2). The features are generated from the last fully connected layer of the convolutional VGG-16 net [Simonyan & Zisserman (2014)]. More detailed descriptions on the problem are shown in Supplementary Materials.

Fig. 3 shows the objective gap and test error against iteration. Our method is also faster than other SADMM.

### 7 Conclusion

We propose ACC-SADMM for the general convex finite-sum problems. ACC-SADMM integrates Nesterov’s extrapolation and VR techniques and achieves a non-ergodic $O(1/K)$ convergence rate. We do experiments to demonstrate that our algorithm is faster than other SADMM.

Table 2: Details of datasets. (Dim., Cla, and mini., are short for dimensionality, class, and minibatch, respectively. Las. is short for the Lasso problem. Mul. is short for Multitask Learning.)

| Pro. | Dataset | # training | # testing | Dim. $\times$ Cla. | # mini. |
|------|---------|------------|-----------|-------------------|---------|
| Las. | a9a     | 72,876     | 72,875    | $74 \times 2$    | 100     |
|      | covertype | 290,506   | 290,506   | $54 \times 2$    |         |
|      | mnist   | 60,000     | 10,000    | $784 \times 10$  |         |
|      | dna     | 2,400,000  | 600,000   | $800 \times 2$   | 500     |
| Mul. | ImageNet | 1,281,167 | 50,000    | $4,096 \times 1,000$ | 2,000   |
References

Allen-Zhu, Zeyuan. Katyusha: The first truly accelerated stochastic gradient method. In *Annual Symposium on the Theory of Computing*, 2017.

Argyriou, Andreas, Evgeniou, Theodoros, and Pontil, Massimiliano. Multi-task feature learning. *Proc. Conf. Advances in Neural Information Processing Systems*, 2007.

Azadi Sra, Samaneh and Sra, Suvrit. Towards an optimal stochastic alternating direction method of multipliers. In *Proc. Int’l. Conf. on Machine Learning*, 2014.

Beck, Amir and Teboulle, Marc. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.

Bottou, Léon. Stochastic learning. In *Advanced lectures on machine learning*, pp. 146–168. 2004.

Boyd, Stephen, Parikh, Neal, Chu, Eric, Peleato, Borja, and Eckstein, Jonathan. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends® in Machine Learning*, 3(1):1–122, 2011.

Chen, Caihua, Chan, Raymond H, Ma, Shiqian, and Yang, Junfeng. Inertial proximal ADMM for linearly constrained separable convex optimization. *SIAM Journal on Imaging Sciences*, 8(4):2239–2267, 2015.

Davis, Damek and Yin, Wotao. Convergence rate analysis of several splitting schemes. In *Splitting Methods in Communication, Imaging, Science, and Engineering*, pp. 115–163. 2016.

Defazio, Aaron, Bach, Francis, and Lacoste-Julien, Simon. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Proc. Conf. Advances in Neural Information Processing Systems*, 2014.

Friedman, Jerome, Hastie, Trevor, and Tibshirani, Robert. Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441, 2008.

Frostig, Roy, Ge, Rong, Kakade, Sham, and Sidford, Aaron. Un-regularizing: approximate proximal point and faster stochastic algorithms for empirical risk minimization. In *Proc. Int’l. Conf. on Machine Learning*, 2015.

He, Bingsheng and Yuan, Xiaoming. On the $O(1/n)$ convergence rate of the Douglas–Rachford alternating direction method. *SIAM Journal on Numerical Analysis*, 50(2):700–709, 2012.

Hien, Le Thi Khanh, Lu, Canyi, Xu, Huan, and Feng, Jiashi. Accelerated stochastic mirror descent algorithms for composite non-strongly convex optimization. *arXiv preprint arXiv:1605.06892*, 2016.

Johnson, Rie and Zhang, Tong. Accelerating stochastic gradient descent using predictive variance reduction. In *Proc. Conf. Advances in Neural Information Processing Systems*, 2013.

Kim, Seyoung, Sohn, Kyung-Ah, and Xing, Eric P. A multivariate regression approach to association analysis of a quantitative trait network. *Bioinformatics*, 25(12):i204–i212, 2009.

Li, Huan and Lin, Zhouchen. Optimal nonergodic $O(1/k)$ convergence rate: When linearized ADM meets nesterov’s extrapolation. *arXiv preprint arXiv:1608.06366*, 2016.

Lin, Hongzhou, Mairal, Julien, and Harchaoui, Zaid. A universal catalyst for first-order optimization. In *Proc. Conf. Advances in Neural Information Processing Systems*, 2015a.

Lin, Zhouchen, Liu, Risheng, and Su, Zhixun. Linearized alternating direction method with adaptive penalty for low-rank representation. In *Proc. Conf. Advances in Neural Information Processing Systems*, 2011.

Lin, Zhouchen, Liu, Risheng, and Li, Huan. Linearized alternating direction method with parallel splitting and adaptive penalty for separable convex programs in machine learning. *Machine Learning*, 99(2):287–325, 2015b.
Lu, Canyi, Li, Huan, Lin, Zhouchen, and Yan, Shuicheng. Fast proximal linearized alternating direction method of multiplier with parallel splitting. In Proc. AAAI Conf. on Artificial Intelligence, 2016.

Nesterov, Yurii. A method for unconstrained convex minimization problem with the rate of convergence $O(1/k^2)$. In Doklady an SSSR, volume 269, pp. 543–547, 1983.

Nesterov, Yurii. On an approach to the construction of optimal methods of minimization of smooth convex functions. Ekonomika i Mateaticheskie Metody, 24(3):509–517, 1988.

Nesterov, Yurii. Introductory lectures on convex optimization: A basic course, volume 87. 2013.

Nitanda, Atsushi. Stochastic proximal gradient descent with acceleration techniques. In Proc. Conf. Advances in Neural Information Processing Systems, 2014.

Ouyang, Hua, He, Niao, Tran, Long, and Gray, Alexander G. Stochastic alternating direction method of multipliers. Proc. Int'l. Conf. on Machine Learning, 2013.

Ouyang, Yuyuan, Chen, Yunmei, Lan, Guanghui, and Pasiliao Jr, Eduardo. An accelerated linearized alternating direction method of multipliers. SIAM Journal on Imaging Sciences, 8(1):644–681, 2015.

Russakovsky, Olga, Deng, Jia, Su, Hao, Krause, Jonathan, Satheesh, Sanjeev, Ma, Sean, Huang, Zhiheng, Karpathy, Andrej, Khosla, Aditya, Bernstein, Michael, et al. Imagenet large scale visual recognition challenge. Int'l. Journal of Computer Vision, 115(3):211–252, 2015.

Schmidt, Mark, Le Roux, Nicolas, and Bach, Francis. Minimizing finite sums with the stochastic average gradient. Mathematical Programming, pp. 1–30, 2013.

Shen, Li, Sun, Gang, Lin, Zhouchen, Huang, Qingming, and Wu, Enhua. Adaptive sharing for image classification. In Proc. Int'l. Joint Conf. on Artificial Intelligence, 2015.

Simonyan, Karen and Zisserman, Andrew. Very deep convolutional networks for large-scale image recognition. arXiv preprint arXiv:1409.1556, 2014.

Suzuki, Taiji. Dual averaging and proximal gradient descent for online alternating direction multiplier method. In Proc. Int'l. Conf. on Machine Learning, 2013.

Suzuki, Taiji. Stochastic dual coordinate ascent with alternating direction method of multipliers. In Proc. Int'l. Conf. on Machine Learning, 2014.

Tseng, Paul. On accelerated proximal gradient methods for convex-concave optimization. In Technical report, 2008.

Wang, Kaiye, He, Ran, Wang, Liang, Wang, Wei, and Tan, Tieniu. Joint feature selection and subspace learning for cross-modal retrieval. IEEE Trans. on Pattern Analysis and Machine Intelligence, 38(10):1–1, 2016.

Zhang, Xiaoyun, Burger, Martin, and Osher, Stanley. A unified primal-dual algorithm framework based on bregman iteration. Journal of Scientific Computing, 46:20–46, 2011.

Zheng, Shuai and Kwok, James T. Fast-and-light stochastic admm. In Proc. Int'l. Joint Conf. on Artificial Intelligence, 2016.

Zhong, Wenliang and Kwok, James Tin-Yau. Fast stochastic alternating direction method of multipliers. In Proc. Int'l. Conf. on Machine Learning, 2014.

Zuo, Wangmeng and Lin, Zhouchen. A generalized accelerated proximal gradient approach for total variation-based image restoration. IEEE Trans. on Image Processing, 20(10):2748, 2011.