Covering the large spectrum and generalized Riesz products

James R. Lee∗

Abstract

Chang’s Lemma is a widely employed result in additive combinatorics. It gives bounds on the dimension of the large spectrum of probability distributions on finite abelian groups. Recently, Bloom (2016) presented a powerful variant of Chang’s Lemma that yields the strongest known quantitative version of Roth’s theorem on 3-term arithmetic progressions in dense subsets of the integers. In this note, we show how such theorems can be derived from the approximation of probability measures via entropy maximization.

1 Introduction

Let \( G \) be a finite abelian group. Chang’s Lemma [Cha02] asserts that, for every large subset \( S \subseteq G \), the large Fourier coefficients of the indicator function \( 1_S \) lie in a low-dimensional subspace. This has seen a number of applications in additive combinatorics (in addition to Chang’s original application to Freiman’s theorem).

A theorem of Bloom [Blo16] shows that a large subset of the large spectrum can be contained in an even lower-dimensional subspace. We refer to Section 3 for the formal statements. Bloom employs his theorem as the key tool in obtaining the following quantitative version of Roth’s theorem.

Theorem 1.1. There exists a \( c > 0 \) such that for all sufficiently large \( N \), the following holds: If \( A \subseteq \{1, \ldots, N\} \) contains no non-trivial three-term arithmetic progression, then

\[
|A| \leq c \frac{(\log \log N)^4}{\log N} N.
\]

This improves slightly over Sanders’ [San11] breakthrough result that has \((\log \log N)^4\) replaced by \((\log \log N)^6\).

In this note, we state a general approximation theorem for probability measures on finite spaces equipped with no algebraic structure. From this theorem, Bloom’s result follows easily. While Bloom’s proof uses the additive structure in a seemingly fundamental and intricate way, our argument is elementary and requires only a direct application of the fact that the characters of a finite abelian group are homomorphisms and bounded in \( \ell_\infty \).

The statement and proof are inspired by the “entropy maximization” philosophy: Given a probability measure \( \mu \) and a collection of linear observables \( F \), one can find a “simple” approximator \( \tilde{\mu} \) (with respect to \( F \)) by maximizing the entropy of \( \tilde{\mu} \) over all probability measures having similar behavior on \( F \).

∗University of Washington
Our use of this philosophy is motivated by the work [LRS15] where it is employed in the setting of quantum states and von Neumann entropy. In [IMR14], the authors use a simple entropy argument to prove the special case of Chang’s Lemma when $G = \mathbb{F}_2^n$. The entropy-maximization approach is also related, at least in spirit, to the works [Gow10] and [RTTV08] on “dense model theorems,” and to a long line of works employing an “entropy regularizer” in the setting of convex optimization. For a discussion of these connections, additional applications of our sparse approximation theorem, and further accounts of the use of relative entropy in additive combinatorics, we refer to the forthcoming paper of Wolf [Wol17].

In the next section, we state and prove an approximation theorem in the context of finite probability spaces. In Section 3, we prove the results of Bloom and Chang.

2 An approximation theorem

Let $X$ be a finite set equipped with a probability measure $\mu$. We use $L^2(\mu)$ to denote the Hilbert space of real-valued functions on $X$ equipped with inner product $\langle f, g \rangle = \sum_{x \in X} \mu(x)f(x)g(x)$. For a function $h : X \to \mathbb{R}$, we will use the notation $E_\mu h = \sum_{x \in X} \mu(x)h(x)$. We also denote by $\|h\|_p = (E_\mu |h|^p)^{1/p}$ the $L^p(\mu)$ norm for $p \geq 1$.

Denote the set of densities with respect to $\mu$ by $\Delta_X = \{f : X \to [0, \infty) : \|f\|_1 = 1\}$. For $f \in \Delta_X$, define the relative entropy

$$\text{Ent}_\mu(f) = E_\mu [f \log f].$$

We will also use the notion of the relative entropy between two densities $h, h' \in \Delta_X$:

$$D_\mu(h \| h') = E_\mu \left[ h \log \frac{h}{h'} \right].$$

This definition makes sense whenever supp$(h) \subseteq$ supp$(h')$. Otherwise, we take the value to be $+\infty$. Generalized Riesz products. Suppose that $F \subseteq L^2(\mu)$ is a collection satisfying sup $\phi \in F \|\phi\|_\infty \leq 1$. Define the semi-norm $\|f\|_F = \sup_{\phi \in F} |\langle \phi, f \rangle|$. Say that a function $R \in L^2(\mu)$ is a degree-$d$ Riesz $F$-product if

$$R(x) = \prod_{i=1}^d (1 + \varepsilon_i \phi_i(x))$$

for some $d \geq 1$ and $\phi_1, \ldots, \phi_d \in F$, $\varepsilon_1, \ldots, \varepsilon_d \in \{-1, 0, 1\}$. Observe that every such $R$ is non-negative on $X$.

**Theorem 2.1** (Sparse approximation theorem). For every $0 < \eta < \frac{1}{e}$ and $f \in \Delta_X$, there is a $g \in \Delta_X$ such that:

1. $\|f - g\|_F \leq \eta$.
2. There is a subset $F' \subseteq F$ with $|F'| \leq 9 \frac{\text{Ent}_\mu(f)}{\eta^2},$ (2.1) and such that $g$ is a non-negative linear combination of degree-$d$ Riesz $F'$-products for

$$d \leq 12 \frac{\text{Ent}_\mu(f)}{\eta} + O \left( \frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}} \right)$$ (2.2)
While Theorem 2.1 yields a result that is closely related to Chang’s Lemma and is sufficient for the case $G = \mathbb{F}_2^n$, it seems that a more delicate property is required to recover the full statement. Say that the family $\mathcal{F}$ is \textit{Laplace pseudorandom} if for every collection $\{\lambda_\phi : \phi \in \mathcal{F}\}$ of real numbers, the following property holds:

$$\log \mathbb{E}_\mu \left[ \exp \left( \sum_{\phi \in \mathcal{F}} \lambda_\phi \phi \right) \right] \leq \frac{1}{2} \sum_{\phi \in \mathcal{F}} \lambda_\phi^2. \quad (2.3)$$

\textbf{Lemma 2.2.} If $\mathcal{F}$ is Laplace pseudorandom then for any $f \in \Delta_X$, it holds that

$$\sum_{\phi \in \mathcal{F}} \langle f, \phi \rangle^2 \leq 2 \text{Ent}_\mu(f).$$

\subsection*{2.1 Duality theory for relative entropy minimization}

Lemma 2.2 and part of Theorem 2.1 can be proved using only elementary properties of duality for optimization of convex functions over polytopes. Establishing the bound (2.1) will require an iterative algorithm described in Section 2.3.

Fix some $f \in \Delta_X$, a finite collection $\mathcal{F}_0 \subseteq L^2(\mu)$, and a parameter $\delta \geq 0$. Consider the optimization:

\begin{align*}
\text{minimize} & \quad \text{Ent}_\mu(g) \quad (2.4) \\
\text{subject to} & \quad g \in \Delta_X \\
& \quad \langle g, \phi \rangle \geq \langle f, \phi \rangle - \delta \quad \forall \phi \in \mathcal{F}_0.
\end{align*}

Note that we are minimizing a strongly convex function over a non-empty, compact polytope (since $f$ itself satisfies all the constraints), and thus (2.4) has a unique optimal solution. The corresponding dual optimization is

\begin{align*}
\text{maximize} & \quad -\log \left( \mathbb{E}_\mu \exp \left( \sum_{\phi \in \mathcal{F}_0} \lambda_\phi \phi \right) \right) + \sum_{\phi \in \mathcal{F}_0} \lambda_\phi \left( \langle f, \phi \rangle - \delta \right) \quad (2.5) \\
\text{subject to} & \quad \lambda_\phi \geq 0 \quad \forall \phi \in \mathcal{F}_0.
\end{align*}

See, for instance, [BV04, §5.2.4].

Let $P^*$ and $D^*$ denote the optimal values of (2.4) and (2.5), respectively. By weak duality, the inequality $P^* \geq D^*$ always holds. Let us use this fact to prove Lemma 2.2.

\textbf{Proof of Lemma 2.2.} Consider the optimizations (2.4) and (2.5) with $\delta = 0$ and

$$\mathcal{F}_0 = \{ \text{sign}(\langle f, \phi \rangle) \phi : \phi \in \mathcal{F} \}$$

so that $\langle f, \phi \rangle \geq 0$ for $\phi \in \mathcal{F}_0$. Then by weak duality:

$$\text{Ent}_\mu(f) \geq P^* \geq D^* \geq -\log \left( \mathbb{E}_\mu \exp \left( \sum_{\phi \in \mathcal{F}_0} \langle f, \phi \rangle \phi \right) \right) + \sum_{\phi \in \mathcal{F}_0} \langle f, \phi \rangle^2,$$

where the last inequality employs the feasible solution $\{\lambda_\phi = \langle f, \phi \rangle : \phi \in \mathcal{F}_0\}$.
Using the assumption that $\mathcal{F}$ is Laplace pseudorandom, this yields
\[
\text{Ent}_\mu(f) \geq \frac{1}{2} \sum_{\varphi \in \mathcal{F}} \langle f, \varphi \rangle^2,
\]
completing the proof. \[\square\]

For $\delta > 0$, the optimization (2.4) is strictly feasible since (as witnessed by $f$), and hence Slater’s theorem implies that strong duality holds and $P^* = D^*$ (see, e.g., [BV04, §5.3.2]). In this case, the KKT conditions hold, i.e., the gradient of the Lagrangian is identically zero at the optimal solution.

Let $(g^*, (\lambda^*_\varphi))$ denote the corresponding optimal primal-dual pair. The gradient condition yields
\[
g^* = \frac{\exp \left( \sum_{\varphi \in \mathcal{F}_0} \lambda^*_\varphi \varphi \right)}{E_\mu \exp \left( \sum_{\varphi \in \mathcal{F}_0} \lambda^*_\varphi \varphi \right)}.
\]

It follows that
\[
\text{Ent}_\mu(f) - D_\mu(f \parallel g^*) = E_\mu \left[ f \log g^* \right] = D^* + \delta \sum_{\varphi \in \mathcal{F}_0} \lambda^*_\varphi,
\]
where the latter equality uses $E_\mu f = 1$.

**Lemma 2.3.** For every $\delta > 0$, the optimal solution $(\lambda^*_\varphi)$ of (2.5) satisfies
\[
\sum_{\varphi \in \mathcal{F}_0} \lambda^*_\varphi \leq \frac{\text{Ent}_\mu(f)}{\delta}.
\]

**Proof.** Note that $D^* \geq 0$ because $\lambda_\varphi \equiv 0$ is a feasible solution. Therefore (2.7) yields
\[
\delta \sum_{\varphi \in \mathcal{F}_0} \lambda^*_\varphi \leq \text{Ent}_\mu(f) - D_\mu(f \parallel g^*) \leq \text{Ent}_\mu(f).
\]

\[\square\]

### 2.2 Truncating the exponential

Let us now move on to the proof of Theorem 2.1.

**Lemma 2.4.** Suppose that $\|\varphi\|_\infty \leq 1$ for $\varphi \in \mathcal{F}_0 \subseteq L^2(\mu)$. Consider non-negative numbers $\{c_\varphi : \varphi \in \mathcal{F}_0\}$ and
\[
h = \exp \left( \sum_{\varphi \in \mathcal{F}_0} c_\varphi (1 + \varphi) \right).
\]

Then for every $0 < \eta < \frac{1}{e^2}$, there is a density $\tilde{h} \in \Delta_X$ that is a non-negative linear combination of degree-$d$ Riesz $\mathcal{F}_0$-products and such that
\[
d \leq 6 \sum_{\varphi \in \mathcal{F}_0} c_\varphi + O \left( \frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}} \right),
\]
and
\[
\left\| \frac{h}{E_\mu h} - \tilde{h} \right\|_1 \leq \eta.
\]
Theorem 2.1 asserts that to obtain a density \( \tilde{\psi} \) without the sparsity constraint (Lemma 2.4) where in the second inequality we have used \( \| \cdot \|_\infty \leq 2c \), where \( c = \sum_{\phi \in F_0} c_\phi \). Denote \( p_m(x) = \sum_{j \leq m} \frac{x^j}{j!} \) and recall from Taylor’s theorem that for \( B \geq 0 \),

\[
\sup_{x \in [0,B]} \frac{|e^x - p_m(x)|}{e^x} \leq \frac{B^{m+1}}{(m+1)!}.
\]

Let us choose \( m \leq 3B + O\left( \frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}} \right) \) so as to make this quantity less than \( \eta/2 \). Thus setting \( B = 2c \) yields

\[
\|e^\psi - p_m(\psi)\|_1 \leq \frac{\eta}{2} \mathbb{E}_\mu[e^\psi].
\]

(2.8)

Now define

\[
\tilde{h} = \frac{p_m(\psi)}{\mathbb{E}_\mu p_m(\psi)},
\]

and note that \( \tilde{h} \) is a non-negative combination of degree-\( m \) Riesz \( F_0 \)-products. Moreover,

\[
\left\| \frac{h}{\mathbb{E}_\mu h} - \tilde{h} \right\|_1 \leq \left\| \frac{h}{\mathbb{E}_\mu h} - p_m(\psi) \right\|_1 + \left\| \frac{p_m(\psi)}{\mathbb{E}_\mu p_m(\psi)} - \tilde{h} \right\|_1 \leq \frac{\eta}{2} + \frac{\mathbb{E}_\mu p_m(\psi)}{\mathbb{E}_\mu h} - 1 \leq \eta.
\]

\( \square \)

We first prove Theorem 2.1 without the sparsity constraint (2.1) since it follows easily from the machinery we already have.

Theorem 2.5 (Low-degree approximation theorem). For every \( 0 < \eta < \frac{1}{c} \) and \( f \in \Delta_X \), there is a \( g \in \Delta_X \) such that:

1. \( \|f - g\|_F \leq \eta \).
2. \( g \) is a non-negative linear combination of degree-\( d \) Riesz \( F \)-products for

\[
d \leq 12 \frac{\text{Ent}_\mu(f)}{\eta} + O \left( \frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}} \right),
\]

(2.9)

Proof. Consider the optimization (2.4) with \( \delta = \eta/2 \) and \( F_0 = \{ \pm \phi : \phi \in F \} \). Let \( (g^*, \{ \lambda^*_\phi \}) \) denote the corresponding optimal primal-dual pair and observe that

\[
g^* = \exp \left( \sum_{\phi \in F_0} \lambda^*_\phi (1 + \phi) \right) \mathbb{E}_\mu \exp \left( \sum_{\phi \in F_0} \lambda^*_\phi (1 + \phi) \right).
\]

Moreover, Lemma 2.3 asserts that \( c = \sum_{\phi \in F_0} \lambda^*_\phi \leq 2 \text{Ent}_\mu(f) \). Thus we can apply Lemma 2.4 to obtain a density \( \tilde{h} \in \Delta_X \) that is a non-negative linear combination of degree-\( d \) Riesz \( F_0 \)-products with

\[
d \leq 12 \frac{\text{Ent}_\mu(f)}{\eta} + O \left( \frac{\log \frac{1}{\eta}}{\log \log \frac{1}{\eta}} \right),
\]

and such that \( \|\tilde{h} - g^*\|_1 \leq \eta/2 \).

Finally, observe that for any \( \phi \in F \), by definition of the optimization (2.4), we have

\[
|\langle \tilde{h} - f, \phi \rangle| \leq |\langle \tilde{h} - g^*, \phi \rangle| + |\langle g^* - f, \phi \rangle| \leq \|\tilde{h} - g^*\|_1 + \frac{\eta}{2} \leq \eta,
\]

where in the second inequality we have used \( \|\phi\|_\infty \leq 1 \). It follows that \( \|\tilde{h} - f\|_F \leq \eta \), completing the proof. \( \square \)
2.3 Mirror descent

We now prove Theorem 2.1 by giving an algorithm that approximately solves the optimization (2.4). The algorithm and analysis are based on the “mirror descent” framework, analyzed using a Bregman divergence (in this case, the relative entropy). See, for instance, the monograph [Bub14]. The sparsity of the solution (captured by (2.1)) is closely related to sparsity properties of the Frank-Wolfe algorithm [FW56].

Assume that \( \eta > 0 \) and \( f \in \Delta_X \) are given as in the theorem. For some value \( T > 0 \), define a family \( \{g_t : t \in [0, T]\} \subseteq \Delta_X \) by

\[
g_t = \frac{\exp \left( \int_0^t \varphi_s \, ds \right)}{\mathbb{E}_\mu \exp \left( \int_0^t \varphi_s \, ds \right)},
\]

(2.10)

where \( s \mapsto \varphi_s \in L^2(\mu) \) is a measurable function to be specified shortly. Observe that \( g_0 = 1 \) is the constant 1 function.

A simple calculation yields: For \( t \in [0, T) \),

\[
\frac{d}{dt} D_\mu(f \parallel g_t) = \langle \varphi_t, g_t - f \rangle.
\]

(2.11)

We define the maps \( s \mapsto \varphi_s \) to be piecewise constant on a finite sequence of intervals. Given the definition on intervals \([0, t_1), [t_1, t_2), \ldots, [t_{i-1}, t_i)\) with \( 0 < t_1 < t_2 < \cdots < t_i \), we define it on an interval \([t_i, t_{i+1})\) as follows.

If there exists a functional \( \varphi \in \mathcal{F} \) such that

\[
\|\langle g_t, \varphi \rangle - \langle f, \varphi \rangle \| > \frac{2\eta}{3},
\]

then we put

\[
\varphi_s = \text{sign} \left( \langle f - g_t, \varphi \rangle \right) \cdot \varphi
\]

(2.12)

for \( s \in [t_i, t_{i+1}) \) where \( t_{i+1} = \inf \{ t \geq t_i : \|\langle g_t, \varphi \rangle - \langle f, \varphi \rangle \| \leq \eta/3\} \). We will see momentarily why such a \( t_{i+1} \) must exist.

If there is no such functional \( \varphi \) at time \( t_i \), then we set \( T = t_i \) and \( i_{\max} = i \). By construction, we have the property that \( \| f - g_T \|_F \leq \frac{2}{3}\eta \).

**Lemma 2.6.** \( T \leq 3 \frac{\text{Ent}_\mu(f)}{\eta} \).

**Proof.** Simply observe that for \( t \in [0, T) \), the calculation (2.11) combined with the definition of the sequence \( \{t_i\} \) and the choice (2.12) yields

\[
\frac{d}{dt} D_\mu(f \parallel g_t) \leq -\frac{\eta}{3}.
\]

On the other hand, \( D_\mu(f \parallel g_0) = \text{Ent}_\mu(f) \) and \( D_\mu(f \parallel g_t) \geq 0 \) is always true. This yields the claim. \( \square \)

**Lemma 2.7.** It holds that \( i_{\max} \leq 9 \frac{\text{Ent}_\mu(f)}{\eta^2} \).

**Proof.** Fix an interval \([t_{i-1}, t_i)\) with \( i \leq i_{\max} \). Let \( \varphi = \varphi_{t_{i-1}} \). We calculate

\[
\frac{d}{dt} \langle \varphi, g_t \rangle = -\langle \varphi, g_t (\varphi - \langle \varphi, g_t \rangle) \rangle = -\langle \varphi^2, g_t \rangle + \langle \varphi, g_t \rangle^2.
\]

Notice that the latter quantity is at least \(-\|\varphi\|_\infty^2 \|g_t\|_1 \geq -1 \). Therefore \( t_i - t_{i-1} \geq \frac{\eta}{3} \). We conclude that \( i_{\max} \leq 3T/\eta \) and combine this with Lemma 2.6. \( \square \)
Observe now that
\[ g_T = \frac{\exp\left(\int_0^T (1 + \varphi_s) \, ds\right)}{\mathbb{E}_\mu \exp\left(\int_0^T (1 + \varphi_s) \, ds\right)} \quad (2.13) \]

and \( \|f - g_T\|_F \leq 2\eta/3. \)

Note that if we set
\[ \mathcal{F}' = \{ \varphi \in \mathcal{F} : \varphi = \pm \varphi_t \text{ for some } t \in [0, T] \} , \]

then Lemma 2.7 yields \( |\mathcal{F}'| \leq 9 \left(\frac{\text{Ent}_\eta(f)}{\eta}\right). \) The proof of Theorem 2.1 is concluded using Lemma 2.4 in conjunction with Lemma 2.6, just as in the proof of Theorem 2.5.

3 Covering the large spectrum

Let \( G \) be a finite abelian group equipped with the uniform measure \( \mu \), and let \( \hat{G} \) be the dual group. Let 0 denote the identity element in \( G \) and \( \hat{G} \).

For \( \gamma \in \hat{G}, \) let \( u_\gamma : G \to \mathbb{C} \) denote the corresponding character. One can write any \( f : G \to \mathbb{C} \) as \( f = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma) u_\gamma \). We will need the properties that \( u_\gamma u_{\gamma'} = u_{\gamma + \gamma'} \) for all \( \gamma, \gamma' \in \hat{G} \) and \( \max_{x \in G} |u_\gamma(x)| \leq 1. \) One may consult [TV10, Ch. 4] for a treatment of discrete Fourier analysis tailored to applications in additive combinatorics.

For each value \( \delta > 0 \), we define the set
\[ \text{Spec}_\delta(f) = \{ \gamma \in \hat{G} : |\hat{f}(\gamma)| > \delta \} . \]

Say that a subset \( S \subseteq \hat{G} \) is covered by a subset \( \Lambda \subseteq \hat{G} \) if
\[ S \subseteq \left\{ \sum_{\lambda \in \Lambda} \varepsilon_{\lambda} \lambda : \varepsilon_{\lambda} \in \{-1, 0, 1\} \right\} . \]

A subset \( S \subseteq \hat{G} \) is \( d \)-covered if there exists a subset \( \Lambda \subseteq \hat{G} \) with \( |\Lambda| \leq d \) that covers \( S \).

Let us define the family
\[ \mathcal{F} = \{ \text{Re} u_\gamma, \text{Im} u_\gamma : \gamma \in \hat{G} \} \subseteq L^2(\mu) . \]

Note that \( \|\varphi\|_\infty \leq 1 \) for every \( \varphi \in \mathcal{F} \).

**Lemma 3.1.** If \( R \) is a degree-\( d \) Riesz \( \mathcal{F} \)-product, then \( \text{Spec}_0(R) = \{ \gamma \in \hat{G} : \hat{R}(\gamma) \neq 0 \} \) is \( d \)-covered.

**Proof.** Write \( R = \prod_{i=1}^d (1 + \varepsilon_i \varphi_i) \) for \( \{\varphi_i\} \subseteq \mathcal{F} \) and \( \{\varepsilon_i\} \subseteq \{-1, 0, 1\} \). For each \( i \), let \( \gamma_i \in \hat{G} \) be such that \( \varphi_i = \text{Re} u_{\gamma_i} \) or \( \varphi_i = \text{Im} u_{\gamma_i} \). Since we can write \( \text{Re} u_\gamma = \frac{1}{2}(u_\gamma + u_{-\gamma}) \) and \( \text{Im} u_\gamma = \frac{1}{2i}(u_\gamma - u_{-\gamma}) \), upon expanding the product defining \( R \), we see that every \( \gamma \in \hat{G} \) with \( \hat{R}(\gamma) \neq 0 \) is a sum of at most \( d \) elements from the multiset \( \Gamma_0 := \{ \gamma_1, \ldots, \gamma_d, -\gamma_1, \ldots, -\gamma_d \} \subseteq \hat{G} \). (We are using the convention here that the empty sum is equal to the identity of \( \hat{G} \) in order to handle \( \hat{R}(0) \neq 0 \).) But we can replace \( \Gamma_0 \) by an actual set \( \Gamma \subseteq \hat{G} \) as follows: For each \( i = 1, \ldots, d \), if \( \gamma_i \) occurs \( t \) times in \( \Gamma_0 \), we replace the \( t \) occurrences of \( \pm\gamma_i \) by the elements \( \pm\gamma_i, \pm 2\gamma_i, \ldots, \pm t\gamma_i \). \( \square \)
3.1 Bloom’s theorem

Recall that $\Delta_G = \{ f : G \to [0, \infty) : \mathbb{E}_\mu f = 1 \}$.

**Theorem 3.2** (Bloom). For every $f \in \Delta_G$ and $0 < \delta < \frac{1}{2}$, there exists a subset $S \subseteq \text{Spec}_\delta(f)$ such that $|S| \geq \frac{\delta}{2} |\text{Spec}_0(f)|$ and $S$ is $d$-covered for

$$d \leq 24 \sqrt{2} \frac{\text{Ent}_\mu(f)}{\delta} + O\left( \frac{\log \frac{1}{\delta}}{\log \log \frac{1}{\delta}} \right).$$

**Proof.** Setting $\eta = \delta/(2 \sqrt{2})$ and applying Theorem 2.1, there exists a $g \in \Delta_G$ such that

$$g = \sum_{i=1}^N c_i R_i$$

with $N \geq 1$, $c_1, \ldots, c_N > 0$, and where $R_1, \ldots, R_N$ are degree-$d$ Riesz $\mathcal{F}$-products for $d$ as in (2.2) and furthermore $\|f - g\|_F \leq \eta$.

Observe that since $g \in \Delta_G$, we have $\sum_{i=1}^N c_i \mathbb{E}_\mu R_i = \mathbb{E}_\mu g = 1$. Thus we can define a random variable $Z \in \{1, 2, \ldots, N\}$ so that

$$\mathbb{P}[Z = i] = c_i \mathbb{E}_\mu R_i.$$

Since $\|f - g\|_F \leq \eta$, we deduce that if $\gamma \in \text{Spec}_{\sqrt{2}\eta}(f)$, then $\gamma \in \text{Spec}_{\sqrt{2}\eta}(g)$. For such $\gamma$, we have

$$\mathbb{E}_z \left[ \left| \langle \gamma, \frac{R_i}{\mathbb{E}_\mu R_i} \rangle \right| \right] = \sum_{i=1}^N c_i \mathbb{E}_\mu \left| \langle \gamma, \frac{R_i}{\mathbb{E}_\mu R_i} \rangle \right| \geq |\langle \gamma, g \rangle| \geq \sqrt{2}\eta = \frac{\delta}{2}.$$

Because $\left| \langle \gamma, \frac{R_i}{\mathbb{E}_\mu R_i} \rangle \right| \leq 1$, we conclude that

$$\mathbb{P}_z \left( \hat{R}_z(\gamma) \neq 0 \right) = \mathbb{P}_z \left( |\langle \gamma, R_z \rangle| > 0 \right) \geq \frac{\delta}{2}.$$

By linearity, $\mathbb{E}_z |\text{Spec}_\delta(R_z)| \geq \frac{\delta}{2} |\text{Spec}_0(f)|$. Moreover, by Lemma 3.1, every set $\text{Spec}_\delta(R_i)$ is $d$-covered. Thus there exists at least one such set that completes the proof of the theorem. \hfill $\Box$

3.2 Chang’s theorem

**Theorem 3.3** (Chang). For every $f \in \Delta_G$ and $\delta > 0$, the set $\text{Spec}_\delta(f)$ is $d$-covered for

$$d \leq 4 \frac{\text{Ent}_\mu(f)}{\delta^2}.$$

Note that Theorem 2.1 implies there is a density $g \in \Delta_G$ such that $\text{Spec}_\delta(f) \subseteq \text{Spec}_\delta(g)$ and from (2.1), one can write $g(x) = \psi(u_{\gamma_1}(x), \ldots, u_{\gamma_k}(x))$ for some function $\psi$ and $\gamma_1, \ldots, \gamma_k \in \hat{G}$ with $k \leq O(\text{Ent}_\mu(f)/\delta^2)$. In the special case $G = \mathbb{F}_2^n$, this implies that

$$\text{Spec}_\delta(g) \subseteq \text{span}_{\mathbb{F}_2}(\gamma_1, \ldots, \gamma_k) = \left\{ \sum_{i=1}^k \epsilon_i \gamma_i : \epsilon_i \in \{-1, 0, 1\} \right\},$$

yielding Theorem 3.3 for $G = \mathbb{F}_2^n$. For general finite abelian $G$, this no longer holds, and one obtains instead the following statement.
Lemma 3.4. For every \( f \in \Delta_G \) and \( 0 < \delta < \frac{1}{e} \), there is a set \( \Lambda \subseteq \hat{G} \) with

\[
|\Lambda| \leq 18 \frac{\text{Ent}_\mu(f)}{\delta^2}
\]

and such that every element \( \gamma \in \text{Spec}_\delta(f) \) can be written

\[
\gamma = \sum_{i=1}^d \varepsilon_i \gamma_i, \quad (\gamma_1, \ldots, \gamma_d) \in \Lambda^d, \varepsilon_1, \ldots, \varepsilon_d \in \{-1, 0, 1\}.
\]

with

\[
d \leq 12 \sqrt{2} \frac{\text{Ent}_\mu(f)}{\delta} + O\left(\frac{\log \frac{1}{\delta}}{\log \log \frac{1}{\delta}}\right).
\]

This should be compared to [Shk06, Thm. 4] which achieves a worse bound on \( |\Lambda| \) but the significantly better bound \( d \leq O(\text{Ent}_\mu(f)) \).

**Proof.** As in the proof of Theorem 3.2, set \( \eta = \delta/\sqrt{2} \) and apply Theorem 2.1 to obtain a density \( g = \sum_{i=1}^N c_i R_i \) where each \( R_i \) is a degree-\( d \) Riesz \( F \)-product with \( d \) as in (2.2). Now \( \gamma \in \text{Spec}_\delta(f) \) implies \( \gamma \in \text{Spec}_0(g) \), which means that \( \gamma \in \text{Spec}_0(R_i) \) for some \( i = 1, \ldots, N \).

To conclude, observe that every element of \( \text{Spec}_0(R_i) \) can be written as \( \sum_{i=1}^d \varepsilon_i \gamma_i \) for some tuple \( (\gamma_1, \ldots, \gamma_d) \in \Lambda^d \) (recall the proof of Lemma 3.1). \( \square \)

In order to prove Theorem 3.3 for general \( G \), we recall the following definition. Say that a subset \( \Lambda \subseteq \hat{G} \) is **disassociated** if

\[
\sum_{\gamma \in \Lambda} \varepsilon_{\gamma} \gamma = 0 \quad \text{and} \quad \{\varepsilon_{\gamma}\} \subseteq \{-1, 0, 1\} \implies \varepsilon_{\gamma} = 0 \quad \forall \gamma \in \Lambda.
\]

If \( \Lambda \subseteq \text{Spec}_\delta(f) \) is a **maximal** disassociated subset, then \( \text{Spec}_\delta(f) \) is covered by \( \Lambda \). Thus the following lemma finishes the proof of Theorem 3.3. The argument is based on a a proof of Rudin’s inequality credited to I. Z. Ruzsa in [Gre04].

**Lemma 3.5.** If \( \Lambda \subseteq \text{Spec}_\delta(f) \) is disassociated, then

\[
|\Lambda| \leq 4 \frac{\text{Ent}_\mu(f)}{\delta^2}.
\]

**Proof.** Let \( F_1 = \{\text{Re} u_{\gamma} : \gamma \in \Lambda\}, F_2 = \{\text{Im} u_{\gamma} : \gamma \in \Lambda\} \).

**Claim 3.6.** The families \( F_1 \) and \( F_2 \) are Laplace pseudorandom.

Given Claim 3.6, we have

\[
|\Lambda| \delta^2 \leq \sum_{\varphi \in F_1 \cup F_2} \langle f, \varphi \rangle^2 \leq 4 \text{Ent}_\mu(f),
\]

where the first inequality follows from \( \Lambda \subseteq \text{Spec}_\delta(f) \) and the second is Lemma 2.2.

So let us turn to the proof of Claim 3.6. We prove it for \( F_1 \) as the proof for \( F_2 \) is essentially identical. We require the following two basic facts: For any \( t \in \mathbb{R} \) and \( x \in [-1, 1] \),

\[
e^{tx} \leq \frac{e^t + e^{-t}}{2} + xe^t - e^{-t} = \cosh(t) + x \sinh(t),
\]

(3.1)
\[
\cosh(t) = \sum_{k \geq 0} \frac{t^{2k}}{(2k)!} \leq \sum_{k \geq 0} \frac{t^{2k}}{2^k k!} = e^{t^2/2}.
\] (3.2)

The first uses the fact that \( x \mapsto e^{tx} \) is convex.

Now write
\[
\mathbb{E}_\mu \left[ \exp \left( \sum_{\varphi \in \mathcal{F}_1} \lambda_\varphi \varphi \right) \right] \leq \mathbb{E}_\mu \prod_{\varphi \in \mathcal{F}_1} \left( \cosh(\lambda_\varphi) + \varphi \sinh(\lambda_\varphi) \right) \] (3.3)

Recalling that every \( \varphi \in \mathcal{F}_1 \) is of the form \( \varphi = \text{Re} u_\gamma = \frac{1}{2}(u_\gamma + u_{-\gamma}) \) for some \( \gamma \in \Lambda \), we see that the right-hand side of (3.3) breaks into a linear combination of characters \( u_\alpha \) such that
\[
\alpha = \sum_{\gamma \in \Lambda} \varepsilon_\gamma \gamma, \quad \varepsilon_\gamma \in \{-1, 0, 1\}.
\]

But \( \mathbb{E}_\mu[u_\alpha] = 0 \) unless \( \alpha = 0 \). By disassociativity of \( \Lambda \), this can only happen if \( \varepsilon_\gamma = 0 \) for all \( \gamma \in \Lambda \). In particular, we conclude that
\[
\mathbb{E}_\mu \left[ \exp \left( \sum_{\varphi \in \mathcal{F}_1} \lambda_\varphi \varphi \right) \right] \leq \prod_{\varphi \in \mathcal{F}_1} \cosh(\lambda_\varphi) \leq \exp \left( \frac{1}{2} \sum_{\varphi \in \mathcal{F}_1} \lambda_\varphi^2 \right),
\]
implying that \( \mathcal{F}_1 \) is Laplace pseudorandom and completing the argument. \( \square \)

**Acknowledgements**

We thank Thomas Bloom, Prasad Raghavendra, and Julia Wolf for enlightening discussions, and Thomas Vidick for detailed comments on an initial draft of this manuscript. We are also grateful to Julia Wolf for pointing out the reference [Shk06].

**References**

[Blo16] T. F. Bloom. A quantitative improvement for Roth’s theorem on arithmetic progressions. *J. Lond. Math. Soc. (2)*, 93(3):643–663, 2016.

[Bub14] S. Bubeck. Theory of convex optimization for machine learning. *arXiv:1405.4980*, 2014.

[BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, 2004.

[Cha02] Mei-Chu Chang. A polynomial bound in Freiman’s theorem. *Duke Math. J.*, 113(3):399–419, 2002.

[FW56] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. *Naval Res. Logist. Quart.*, 3:95–110, 1956.

[Gow10] W. T. Gowers. Decompositions, approximate structure, transference, and the Hahn-Banach theorem. *Bull. Lond. Math. Soc.*, 42(4):573–606, 2010.
[Gre04] Ben Green. Spectral structure of sets of integers. In *Fourier analysis and convexity*, Appl. Numer. Harmon. Anal., pages 83–96. Birkhäuser Boston, Boston, MA, 2004.

[IMR14] Russell Impagliazzo, Cristopher Moore, and Alexander Russell. An entropic proof of Chang’s inequality. *SIAM J. Discrete Math.*, 28(1):173–176, 2014.

[LRS15] James R. Lee, Prasad Raghavendra, and David Steurer. Lower bounds on the size of semidefinite programming relaxations. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, STOC 2015, Portland, OR, USA, June 14-17, 2015, pages 567–576, 2015.

[RTTV08] Omer Reingold, Luca Trevisan, Madhur Tulsiani, and Salil P. Vadhan. Dense subsets of pseudorandom sets. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 76–85, 2008.

[San11] Tom Sanders. On Roth’s theorem on progressions. *Ann. of Math. (2)*, 174(1):619–636, 2011.

[Shk06] ID Shkredov. On sets of large exponential sums. In *Doklady Mathematics*, volume 74, pages 860–864. Springer, 2006.

[TV10] Terence Tao and Van H. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[Wol17] Julia Wolf. Some applications of relative entropy in additive combinatorics. 2017. In preparation.