Hidden Frustrated Interactions and Quantum Annealing in Trapped Ion Spin-Phonon Chains

Pedro Nevado\(^1\) and Diego Porras\(^1,\)*

\(^1\)Department of Physics and Astronomy, University of Sussex, Falmer, Brighton BN1 9QH, UK

(Dated: June 20, 2014)

We show that a trapped ion chain interacting with an optical spin-dependent force shows strong frustration effects due to the interplay between long-range interactions and the dressing by optical phases. We consider a strong spin-phonon coupling regime and predict a quantum phase diagram with different competing magnetic and structural orders. In a highly frustrated region, the system shows enhanced quantum fluctuations and entanglement characteristic of spin-liquid phases. We propose and describe within a mean-field approach a quantum annealing process to induce a quasi-adiabatic evolution towards the ground state.

PACS numbers: 03.67.Ac, 37.10.Ty, 75.10.Jm, 64.70.Tg

Introduction.- Systems with frustrated interactions play an important role in fields from magnetism \cite{1} to molecular biology \cite{2}. Frustration leads to a complex energy landscape where finding the global energy minimum can be a computationally demanding task. Those systems exhibit fascinating properties, like the strong thermal or quantum fluctuations that characterize spin-liquid phases \cite{1,3}. Some numerical algorithms to seek for ground states of frustrated systems involve the idea of quantum annealing \cite{4}, where quantum fluctuations aid the simulation to explore the energy landscape in an adiabatic evolution towards the classical ground state. Atomic analogical quantum simulators \cite{5} are an ideal test-bed to simulate quantum magnetism \cite{6–8}. Recent experiments have demonstrated the simulation of frustrated Ising models with up to 16 ions \cite{9,10} in linear traps, and the generation of Ising interactions with hundreds of ions in planar Penning traps \cite{11}. The technical progress in the fabrication of two-dimensional arrays of microtraps has an exciting outlook in providing us with synthetic ion crystals of different geometries that can lead to geometrical frustration \cite{8,12}. Zig-zag ion crystal phases have been also proposed to study frustrated quantum magnets \cite{13}.

In this work we depart from schemes simulating effective spin-spin couplings \cite{6} and consider the limit of strong spin-phonon coupling leading to cooperative Jahn-Teller or Rabi-Lattice models \cite{14,15}. We unveil the existence of hidden long-range frustrated interactions leading to the competition between different magnetic and structural orders. Frustration arises by the interplay of two effects: (a) an effective long-range coupling between different ions and (b) the dressing of spin-phonon interactions by means of the optical phases of lasers. Our scheme can be easily implemented with many ions, since it only relies on a single optical force acting on the ion chain. It also allows for higher simulation speeds, since we consider a non-perturbative spin-phonon coupling regime.

The Letter is structured as follows: (i) We present our model, focusing in the case of a trapped ion spin-phonon chain. (ii) We study the classical phase diagram of the model, and show the existence of a set of classical phase transitions separating different structural configurations. (iii) We calculate the quantum phase diagram by means of mean-field theory \cite{16} and the Density Matrix Renormalization Group (D.M.R.G.) method \cite{17}. We identify values of parameters controlling frustration in the ion chain, and identify a highly frustrated regime where quantum fluctuations and entanglement are enhanced. (iv) We propose a quantum annealing scheme that can be used in a trapped ion experiment to search for the ground state, and describe that process within a non-equilibrium mean-field approximation. (v) Finally, we provide a set of experimental parameters for the implementation of our model in state-of-the-art trapped ion setups.

Trapped ion chain with dressed spin-phonon interactions.- We consider a chain of \(N\) trapped ions with two internal levels, \(|0\>\), \(|1\>\), \((j\) is the ion index\). The radial vibrations of the chain are described by a model of hopping phonons (\(\hbar = 1\)) \cite{18},

\[
H_{\text{sph}}\left(\{\omega_j\},\{\delta_{j,l}\}\right) = \sum_j \omega_j a_j^\dagger a_j + \frac{1}{2} \sum_{j,l} \delta_{j,l} a_j^\dagger a_l a_l^\dagger a_j,
\]

where \(a_j\) and \(\omega_j\) are the annihilation operator of phonons and the local trapping frequencies, respectively. Phonon hopping amplitudes are given by \(t_{j,l} = e^{-2/m\omega_{j,l}^\dagger (0)} - e^{\omega_{j,l}^\dagger (0)}\) with \(e\) the electron charge, \(m\) the mass of the ions, \(\omega_j\) the radial trapping frequency, and \(\omega_{j,l}^\dagger\) the equilibrium position of the \(j\) ion. \(H_{\text{sph}}\) can be diagonalized in a basis of collective modes, \(H_{\text{sph}} = (\Omega_j/2) \sum_j \sigma_j^\dagger \sigma_j\). Finally, we consider a couple of lasers inducing a spin-dependent force,

\[
H_{\text{sph}}(\omega) = g \sum_j \sigma_j^\dagger \left( e^{i(\Delta k_j^\dagger (0) - \omega_k) t} a_j + e^{-i(\Delta k_j (0) - \omega_k) t} a_j^\dagger \right),
\]

where \(g\) is the effective spin-phonon coupling, \(\Delta k\) is the projection of the effective Raman wavevector induced by the lasers on the chain axis, and \(\omega_k\) is the frequency of the running-wave created by the lasers \cite{8}. We get rid of the time dependence in the spin-phonon coupling by transforming the total Hamiltonian to a rotating frame, such that \(H_{\text{sph}}(\omega) \rightarrow H_{\text{sph}}^\dagger(0)\), and \(H_{\text{sph}}(\{\omega_j\},\{\delta_{j,l}\}) \rightarrow H_{\text{sph}}(\{\delta_j\},\{\{\delta_{j,l}\}\})\), with effective shifted trapping frequencies \(\delta_j = \omega_j - \omega_k\). Our final

\* Corresponding author: D.Porras@sussex.ac.uk
The model has the form of a generalized cooperative Jahn-Teller (or Rabi Lattice) Hamiltonian, with dressed spin-phonon couplings,

\[ H_{\text{MT}}^{\text{JT}} = H_{\text{ph}}(\{\delta_j\}, \{t_{jj}\}) + H_r + H_{\text{ph}}(0). \]  

We focus on homogeneous chains (ions equally spaced by a distance \(d_0\)), a situation that gives an approximate description of a linear Coulomb crystal, or describes linear arrays of ion microtraps. The phonon tunneling is expressed in the homogeneous case like \( t_{j,j} = t_c/|j - l|^3 \), with \( t_c = e^2/ma^2d_0^3 \) [18, 19], and local trapping frequencies are approximately constant \( \omega_j \approx \omega_0 \). Below we study the main properties of model (3), and present later typical values for the involved energy scales in a real trapped ion setup.

Classical limit.- We start by studying the classical phases of \( H_{\text{MT}}^{\text{JT}} \), by setting \( \Omega_\chi = 0 \), and mapping to an effective Ising model by a polaron transformation [6],

\[ H_{\text{MT}}^{\text{JT}}(\Omega_\chi = 0) \rightarrow H_{\text{ph}} + \frac{1}{2} \sum_{j,j'} J_{jj'}^{\text{MT}} \sigma_j^z \sigma_{j'}^z, \]

The Ising interaction is given by

\[ J_{jj'}^{\text{MT}} = -2g^2 \sum_n \xi_n \frac{1}{\delta_n} \Lambda_{j,n} \Lambda_{j',n} \cos(|\Delta k| z_j^{(0)} - z_{j'}^{(0)})), \]

and it is determined by the properties of the vibrational bath in the rotating frame, \( H_{\text{ph}} = \sum_n \delta_n a_n^\dagger a_n \), with shifted normal mode energies, \( \delta_n = \omega_n - \omega_k \). We derive an approximate expression in the long chain limit \( N \gg 1 \) (see Supplemental Material [20]),

\[ J_{jj'}^{\text{MT} = 0} \approx -(1)^{j-l} J_{\text{exp}} e^{-j-l/\xi} + \frac{J_{\text{dip}}}{|j-l|^3}, \]

with \( \xi = \sqrt{\log 2}/2 \sqrt{t_c/\delta_0}, J_{\text{exp}} = 2g^2/(tc \log 2), \) and \( J_{\text{dip}} = g^2 t_c/(\delta_0 + 7\xi(3r/4)^2) \), and \( \delta_0 \) is the lowest normal mode energy. Approximation (6) is very accurate even for moderate \( N \) (see Fig. 1 (a)), and it allows us to describe the approximate behavior of the spin-phonon chain in two extreme cases: (i) Short range limit \( \delta_0 \gg 1, \xi \rightarrow 0 \), \( J_{jj}^{\text{MT}} \approx J_{\text{dip}}/|j-l|^3 \cos(|\Delta k| d_0(j-l)) \). Here the ground state is either antiferromagnetic (AF) or ferromagnetic (F), depending on the sign of the nearest-neighbour couplings: \( \langle \sigma_j^z \rangle_F = \langle \sigma_{j-1}^z \rangle_F \) if \( J_{j,j-1}^{\text{MT}} < 0 \), and \( \langle \sigma_j^z \rangle_{\text{AF}} = -\langle \sigma_{j-1}^z \rangle_{\text{AF}} \) if \( J_{j,j+1}^{\text{MT}} > 0 \). (ii) Long-range limit, \( t_c \gg \delta_0, \xi \gg 1 \). Here we can approximate

\[ J_{jj}^{\text{MT}} = J_{\text{exp}} \sum_{n,m} \chi_n^{\text{MT}} |m|^2 \chi_n^{\text{MT}} \]

with \( \chi_n^{\text{MT}} = (-1)^{j} \cos(\Delta k z_j^{(0)}), \chi_n^{\text{MT}} = (-1)^{j} \sin(\Delta k z_j^{(0)}). \) We arrive thus to an Ising model that resembles the Hopfield model for neural networks [21, 22]. The magnetic order will be governed by the phase imprinted by the lasers, and take two possible values: \( \langle \sigma_j^z \rangle_{\text{Hopf}} = \text{sign}(\chi_n^{\text{MT}}) \), or \( \text{sign}(\chi_n^{\text{MT}}) \), corresponding to the two different basins of attraction of the classical system. The classical phase diagram shows, thus, a competition between different magnetic/structural orders. Note that the AF/F and Hopfield magnetic orders are the same in the case \( \Delta k = 0 \). If \( \Delta k \neq 0 \), \( \langle \sigma_j^z \rangle_{\text{AF/F}} \neq \langle \sigma_j^z \rangle_{\text{Hopf}}, \) and the system will choose one or the other as ground state depending on the interaction range (see Fig. 1 (b)). For intermediate values of \( \xi \), the system will have to choose between competing orders and will be generally in a frustrated phase.

Quantum phase diagram.- The quantum ground state of \( H_{\text{MT}}^{\text{JT}} \) is determined by the competition between the spin-phonon interaction through the \( \sigma_z^z \)-component, and the transverse field, \( \Omega_\chi \). In a regime of strong spin-phonon couplings (g \( \approx \delta_0 \)), the system is no longer described by an effective Ising model, since the polaron transformation does not commute with the quantum (\( \sigma_z^z \)) term, \( H_S \) [6]. As a first approach, we use a mean-field spin-phonon ansatz [14],

\[ |\Psi_{\text{MF}}\rangle = \bigotimes_{j,n} \langle \theta_j \rangle \exp(\alpha_n a_n^\dagger - \alpha_n^* a_n) |0\rangle_{\text{ph}}, \]

where \( \langle \theta_j \rangle = \cos(\theta_j/2)|+\rangle_j + \sin(\theta_j/2)|-\rangle_j \), is the spin state of ion \( j \), and \( |0\rangle_{\text{ph}} \) is the phonon vacuum. We calculate the values of \{\langle \theta_j, \alpha_n \rangle\} that minimize the energy \( E_{\text{MF}} = \langle \Psi_{\text{MF}} | H | \Psi_{\text{MF}} \rangle \). The mean-field description is exact in the limits \( \Omega_\chi = 0 \), and \( \Omega_\chi \rightarrow \infty \) and is expected to be very accurate for long-range interactions \( \xi \gg 1 \) [16]. We present first for simplicity the case \( N \gg 1 \), thus neglecting boundary effects, and approximate \( t_{j,j} \approx t_c \delta_j \). The latter approximation is well justified for this problem, since it captures the exponential decay of \( t_{j,j} \) in Eq. (6). We find two different situations: (i) \( \Delta k = 0 \) (no frustration). Mean-field theory predicts a critical value of the transverse field, \( \Omega_{\chi,c} = 4g^2/\delta_0 \), that separates a paramagnetic phase \( (\Omega_\chi \gg \Omega_{\chi,c}) \) with \( \langle \sigma_j^z \rangle = \sin \theta_j = 1, \alpha_n = 0 \), from an antiferromagnetic phase \( (\Omega_\chi \ll \Omega_{\chi,c}) \) with \( \langle \sigma_j^z \rangle = \sin \theta_j = \Omega_\chi/\delta_0/4g^2 \), and phonons displaced in the lowest energy mode (\( n = 0 \)), \( \alpha_0 = \cos(\theta) g \sqrt{\delta_0/\delta_0}, \alpha_n = 0 \). Numerical calculations for finite size chains confirm this picture, see Fig. 2 (a). The transition predicted by mean-field theory corresponds to the quantum magnetic equivalent of the zigzag structural phase transition [23, 24]. (ii) \( \Delta k \neq 0 \) (frustrated interactions). The mean-field phase diagram is modified by the appearance of different magnetic orders in the regions of small \( \Omega_{\chi} \) field. At mean-field level, we find that the AF/F order is replaced by the Hopfield order around a typical value.
Quantum phase diagrams of an ion chain with \( N = 20 \) ions. \( \delta_0 = g = 1 \). (a) and (b) are mean-field calculations with \( \Delta k = 0 \) and \( \Delta k = 2\pi/(3d_0) \), respectively. (c) and (d) are D.M.R.G. calculations with \( \Delta k = 0 \) and \( \Delta k = 2\pi/(3d_0) \), respectively. For the D.M.R.G. method we keep \( D = 18 \) states, and we describe the phonon Hilbert space by considering up to \( n = 7 \) phonon states. (e-f). Different possible classical ground states for \( \Delta k = 5\pi/(3d_0) \) (stars) and \( \Delta k = 2\pi/(3d_0) \) (empty circles): (e) \( t_C = t_1 = 0.2 \) (AF), (f,g) \( t_C = t_{2,3} = 0.5, 0.55 \) (highly frustrated regime, around the value \( t_C \approx 0.5 \)), (h) \( t_C = t_4 = 1 \) (Hopfield phase).

For intermediate values of \( t_C \), we find a **highly frustrated regime**, where the magnetic order goes through several intermediate configurations between the AF/F and Hopfield orders (Fig. 2 (e-h)).

To get a complete understanding of the quantum phase diagram, we have carried out a quasi-exact calculation with the D.M.R.G. method. The comparison between the D.M.R.G. and mean-field calculations allows us to assess the importance of quantum correlations. In the \( \Delta k = 0 \) case, we see an overall agreement between the two approaches, see Figs. 2 (a) and 2 (b). However, in the case \( \Delta k \neq 0 \) the quasi-exact D.M.R.G. calculation shows that quantum fluctuations are enhanced in the highly frustrated region \( t_C \approx t_C^c \), as shown by the increase in the mean \( x \)-magnetization, \( m_x = (1/N)\sum_j \langle \sigma_x^j \rangle \). This feature is absent in the mean-field calculation (see Figs. 2 (b) and (d)). The abrupt increase in quantum fluctuations is shown in more detail in Fig. 3 (a), and the D.M.R.G. calculation shows that in the highly frustrated regime, long-range quantum correlations between ions are also enhanced, thus explaining the failure of mean-field theory in this region, see Fig. 3 (b). Thus, frustration in our system leads to both quantum fluctuations and enhanced long-range quantum correlations, something that is characteristic of spin-liquid phases in frustrated magnets [1, 3]. More detailed calculations with larger size systems confirm this picture [20].

Quantum annealing. Our model can be implemented with trapped ions and used to assess the efficiency of quantum annealing in finding a global classical ground state in the presence of frustrated interactions. Here we exploit the mean-field ansatz (8) to predict the quasi-adiabatic evolution of the system. This approach is reliable in the region of the phase diagram where the mean-field and D.M.R.G. results agree, whereas this method is not conclusive in the highly frustrated region. A trapped ion experiment would allow to assess the relevance of quantum correlations in quantum annealing by providing an analogue simulation to be compared with our classical mean-field approach.

\( H_{\text{JT}}^{\text{LT}} \) has a parity symmetry \( (\sigma^z \rightarrow -\sigma^z) \) that is broken by the mean-field description. To study the adiabatic evolution of the system we consider a situation where this symmetry is broken by adding a longitudinal \( z \)-field to the Hamiltonian, \( H_{\text{JT}}^{\text{LT}} \rightarrow H_{\text{JT}}^{\text{LT}} + \Omega z \sum_j \sigma_z^j / 2 \). We consider a slow evolution of the parameters \( \Omega z (t) = \Omega z_0 (0) e^{-t/\tau_v} \), \( \Omega x (t) = \Omega x_0 (0) e^{-t/\tau_v} \), and \( g(t)^2 = g(0)^2 (1 - e^{-t/\tau_v}) \). At \( t = 0 \), there is no spin-phonon coupling, and the ground state can be easily created by initial-izing the spins in a separable state. The system evolves into...
the strongly coupled spin-phonon regime, and at \( t_f \gg \tau_{ev}, \tau'_{ev} \), it reaches a final state that under ideally adiabatic evolution, would correspond to the classical ground state. As we explain below, it is advantageous to choose different evolution times for the transverse field and couplings, \( \tau_{ev} \), that for the longitudinal symmetry breaking field, \( \tau'_{ev} \).

In addition to the mean-field approximation we assume that the evolution is slow enough for phonons to follow adiabatically the spins. To introduce this approximation let us write the free evolution equation of the collective phonon operator in the interaction picture with respect to \( H_{ph} \), \( \hat{a}_n = \hat{a}_n e^{-i \Omega_n t} \). The equation of motion for \( \hat{a}_n \) reads \( d/dt(\hat{a}_n(t)) = -i g \langle t/\tau_{ev} \rangle \sum_j (\sigma_j^z(t)) e^{i \Delta \omega_j} \hat{M}_{j,n} e^{i \delta_n t} \). Since we are starting in the paramagnetic phase where \( \langle \hat{a}_n(0) \rangle = 0 \), we get, after an integration by parts,

\[
\langle \hat{a}_n(t) \rangle = -g \left( \frac{1}{\tau_{ev}} \right) \sum_j (\sigma_j^z(t)) e^{i \Delta \omega_j} \hat{M}_{j,n} e^{i \delta_n t} + \int_0^t \frac{d}{dt} \left[ g \left( \frac{1}{\tau_{ev}} \right) \sum_j (\sigma_j^z(t)) e^{i \Delta \omega_j} \hat{M}_{j,n} e^{i \delta_n t} \right] e^{-i \delta_n t} dt.
\]

(9)

We can drop the second term in the right hand side of Eq. (4) as long as we assume that \( d(t) \langle \sigma_j^z(t) \rangle \ll \delta_n \) and \( \tau_{ev}^{-1}, \tau'_{ev}^{-1} \ll \delta_n \).

The latter requirement is fulfilled for long enough evolution times \( \tau_{ev}, \tau'_{ev} \), but the first condition must be validated in a posteriori for self-consistency (see [20]). Finally we get

\[
\frac{d}{dt}(\sigma_j^z(t)) = -\Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) + 2 \sum_l (1 - e^{-i \pi}) J_{j,l}^z (\sigma_l^z(t)) \langle \sigma_l^z(t) \rangle \bigg( \Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) \bigg)
\]

\[
\frac{d}{dt}(\sigma_j^z(t)) = -\Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) - \Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) \bigg( \Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) \bigg) \bigg( \Omega_z(0) e^{-i \tau'_{ev}} (\sigma_j^z(t)) \bigg).
\]

(10)

Note that in the mean-field approach we recover a description in terms of an effective spin-spin coupling. To optimally hit the classical ground state the typical time scales must fulfill \( \tau'_{ev} \ll \tau_{ev} \), since otherwise the \( \Omega_z \) field modifies the energy landscape of the problem, thus preventing the quantum (transverse field) \( \Omega_z \) to drive the evolution towards the global minimum.

Our numerical results are summarized in Fig. 4. The mean-field adiabatic evolution of the spin-phonon system is predicted to reach the global ground state far from the highly frustrated regime. For values of the coupling \( t_C \ll t_C \), the evolution hits the F phase, whereas for \( t_C \gg t_C \), the system finds one of the two possible Hopfield order parameters. We find that for values within the highly frustrated regime \( t_C \approx t_C \), the adiabatic mean-field evolution is generally not able to reach the global classical minimum. This is an ideal scenario for experiments to test whether a quantum correlated annealer would yield results which cannot be described by our mean-field ( separable) ansatz.

\[\text{Trapped ions experimental parameters.} \]

Our model \( H_{1k}^{ph} \) can be naturally implemented with state-of-the-art trapped ion setups. Initialization of the system would include cooling the chain close to the vibrational ground state. A typical average distance between ions is \( d_0 = 10 \mu m \). Considering \(^9\text{Be}^+ \) ions and a radial trapping frequency \( \omega_0 = 5 \) \((2\pi)\) MHz, we get a Coulomb coupling \( t_C = 98 \) kHz. The latter is the most important experimental energy scale, limiting the overall speed of an experiment. Axial trapping frequencies for \( N = 20 \) and 50 ions with those parameters are \( \omega_0 = 192 \) and 94 \((2\pi)\) kHz, respectively. To induce the spin-dependent force, Eq. (3), a running wave detuned from the first red sideband transition by \( \delta_0 - \delta_0 \) may be applied. The detuning can be chosen so as to yield a range of values for \( t_C / \delta_0 \) that covers the phase diagram in Fig. 2. Spin-phonon couplings, \( g \), can be in the range of 100 kHz, thus comparable to \( t_C \) [25]. Illuminating the ions with an optical force with effective wavelength \( \lambda \approx 320 \) nm, would require a small misalignment with the direction transverse to the ion chain axis with angle \( \theta \approx 0.6 \) degrees to render the value \( \Delta k = 2\pi /(3d_0) \) used in the examples above. With those values, the evolution time for quantum annealing in Fig. 4 would be of the order of ms. Radial and axial vibrational normal modes cannot overlap to avoid exciting axial modes with the spin-dependent force, a condition that is fulfilled in the range of parameters considered here. We remark that the strongly coupled spin-phonon regime considered here (with \( \delta_0 \approx g \)), allows us to avoid the condition \( g \ll \delta_0 \) [6].

Conclusions and outlook.- We have shown that one of the simplest models that can be experimentally implemented with trapped ions yields a fascinating phenomenology due to the interplay between long-range interactions and frustration induced by optical phases. Our model can be extended to other setups, like for example arrays of superconducting qubits coupled to microwave cavities, where a similar dressing of qubit-phonon couplings has recently been proposed in [26]. Our proposal can be used to experimentally explore the efficiency of quantum annealing in a controllable one-dimensional setup, thus shedding light on applications of this method in related setups [27, 28].

Acknowledgments.- We acknowledge the EU Marie Curie
describing the radial vibrations reads [19]:

\[ H = \sum_j \left( \frac{1}{2} m x_j^2 + \frac{1}{2} m \omega_x^2 x_j^2 + \frac{1}{2} m \omega_z^2 z_j^2 \right) \]

In the harmonic approximation, vibrations in different directions are not coupled. The Hamiltonian direction, respectively. A Taylor expansion of the Coulomb potential around the equilibrium positions up to second order leads

\[ z_j(0) = \frac{e^2}{4\pi\epsilon_0 a_j} \]

where

\[ a_j = \frac{e^2}{8\pi\epsilon_0} \]

[19].

longitudinal components respectively– render a crystal structure which forms a chain for strong radial confinement \( \omega_x \) ions repulsive interaction along with the effective quadratic potential of the trap –with magnitudes \( \omega_x \) and mass \( m \) along the trap axis \( z \). The ions repulsive interaction along with the effective quadratic potential of the trap –with magnitudes \( \omega_x \) and \( \omega_z \) for the transversal and longitudinal components respectively– render a crystal structure which forms a chain for strong radial confinement \( \omega_x \gg \omega_z \) [19].

The position of each ion \( j \), can be described like

\[ \mathbf{r}_j = \delta x_j \mathbf{\hat{x}}_j + (z_j^{(0)} + \delta z_j) \mathbf{\hat{z}}_j, \]

where \( z_j^{(0)} \) are the equilibrium positions along the chain and \( \delta x_j \) and \( \delta z_j \) are the displacement operators in the radial and axial direction, respectively. A Taylor expansion of the Coulomb potential around the equilibrium positions up to second order leads to an harmonic Hamiltonian. In the harmonic approximation, vibrations in different directions are not coupled. The Hamiltonian describing the radial vibrations reads [19]:

\[ H_{ph} = \frac{1}{2m} \sum_j p_{x,j}^2 + \frac{1}{2} m \omega_x^2 \sum_j \delta x_j^2 - \frac{1}{2} \sum_{j \neq l} \frac{e^2}{|z_j^{(0)} - z_l^{(0)}|} (\delta x_j - \delta x_l)^2. \]

We neglect the axial displacements, since they will be weakly coupled to the spin-dependent forces introduced below. Now we second quantize this expression by means of the identifications \( \delta x_j = 1/\sqrt{2m\omega_x}(a_j + a_j^\dagger) \) and \( p_{x,j} = i\sqrt{m\omega_x/2}(a_j^\dagger - a_j) \).
Furthermore, we dismiss the terms that do not conserve the phonon number such as \(a_i^+ a_i^2\) or \((a_i^+)^2\) because they are fast rotating as long as \(\omega_k \gg \Omega_{j,l}\), which in turn is fulfilled because the harmonic potential energy must be greater than the Coulomb energy. Then we are arrive to the final Hamiltonian for the oscillations

\[
H_{\text{ph}} \left( \{ \omega_j \}, \{ t_{j,l} \} \right) = \sum_j \omega_j a_j^+ a_j + \frac{1}{2} \sum_{j,l} t_{j,l} (a_j^+ a_l + \text{H.c.}) ,
\]

with the on-site frequency and Coulomb mediated long range hopping given by

\[
\omega_j = \omega_k - \frac{1}{2} \sum_{p \neq j} \frac{e^2}{m \omega_k |z_j^{(0)} - z_p^{(0)}|^3} , \quad t_{j,l} = \frac{e^2}{m \omega_k |z_j^{(0)} - z_l^{(0)}|^3} .
\]

The spin-dependent force can be realized by two lasers inducing an ac Stark shift between two effective levels with \(\omega_0 = \omega_k\), with effective spin-phonon coupling \(g\) and \(\eta_k\). In the main text we just drop the subindex \(j\) in the calculations.

Neglecting the spin-phonon coupling to the axial displacements, \(\delta z_j\), has to be justified with care. Even though \(\eta_k \ll \eta_z\), still a residual coupling to the axial modes can spoil an implementation of our proposal, since those are more difficult to cool to the ground state. However, with the range of parameters that we are choosing, the normal axial modes are not resonant with the radial ones, and the spin-dependent force can be slightly off-resonant with respect to the radial normal modes, while far detuned from the axial normal modes. For example, for \(N = 20\), \(\omega_k = 192 (2\pi) \text{ kHz}\), and \(N = 50\) \(\omega_k = 94 (2\pi) \text{ kHz}\), the maximum axial normal mode energy is \(\omega_{z,\text{max}} = 2.29 (2\pi) \text{ MHz}\) and \(\omega_{x,\text{max}} = 2.5 (2\pi) \text{ MHz}\), respectively, off-resonant with the normal modes, centered around \(\omega_k = 5 (2\pi) \text{ MHz}\).
B. DECAY OF THE ISING INTERACTION

For the sake of clarity we consider the case of an homogeneous chain, so the Hamiltonian of the radial phonons in the rotating frame reads,

\[ H_{ph} = \sum_{j,l} (\delta \delta_{j,l} + t_{j,l} a^\dagger_j a_l) = \sum_{j,l} (\delta \delta_{j,l} + \frac{1}{2} t_c |j-l|^3) a^\dagger_j a_l \]  

(20)

assuming normalized distances, laser-shifted on-site phonon energy \( \delta = \omega_k - \omega_a \), and hopping strength \( t_c = e^2/(m \alpha^2_d \alpha_0^3) \). Since \( t_c > 0 \), a previous transformation \( a_j \rightarrow (-1)^j a_j \), and \( t_{j,l} \rightarrow (-1)^{j-l} t_{j,l} \) is convenient to map the problem to the most usual situation with negative hopping. This takes account of the fact that the lowest energy mode is the zig-zag configuration of the chain because of the ions’ repulsive interactions, and leads to the transformation of the effective spin-spin coupling:

\[ f_{j,l}^\delta \rightarrow (-1)^{(j-l)} f_{j,l}^\delta. \]

(21)

In addition, we consider Periodic Boundary Conditions, such that the phonon coupling is diagonalized in the basis of eigenstates \( M_{j,n} = e^{i \frac{2\pi n j}{N}} \), \( n = -N/2, \ldots, N/2 - 1 \), and the dispersion relation of the normal modes is

\[ \delta_n = \delta + t_c \sum_{k=1}^{N-1} \frac{(-1)^k}{k^3} \cos \left( \frac{2\pi n k}{N} \right), \]

(22)

where \( \omega_{a=n} \) is now the lowest energy mode since we work in a staggered basis.

Let us work out the continuum limit of the Ising interaction (we use length units such that \( d_0 = 1 \) in the following)

\[ f_{j,l}^\Delta = \frac{\cos \Delta k |j-l|}{2\pi} \sum_{p=\pi}^{\pi} \frac{-2g^2 \cos p |j-l|}{\delta + t_c \sum_{k=1}^{N-1} \frac{(-1)^p \cos(pk)}{k^3}} \frac{2\pi N^\sim}{\Delta k} \cos \Delta k |j-l| \frac{2\pi}{\Delta k} \int_{-\pi}^{\pi} \frac{-2g^2 \cos x |j-l|}{\delta + t_c \sum_{k=1}^{\infty} \frac{(-1)^p \cos(xk)}{k^3}} dx, \]

(23)

which can be written as

\[ f_{j,l}^\Delta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{-2g^2 e^{ix j-l}}{\delta + \frac{t_c}{2}(\text{Li}_3(-e^{ix}) + \text{Li}_3(-e^{-ix}))} dx \]

(24)

using parity properties, and where we take \( \Delta k = 0 \) since the laser phase does not contribute to the decay of the interaction. The \( \text{Li}_3(z) \) is the polylogarithmic function defined by the series

\[ \text{Li}_3(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^3}, z \in \mathbb{C}, \]

(25)

that converges in the circle \( |z| < 1 \). However, we are interested in its analytic continuation to any \( z \in \mathbb{C} \), as we intend to approximate the former integral by continuing it to the complex plane and applying the Residue Theorem, which assures that

\[ f_{j,l}^\Delta = \frac{1}{2\pi} \left( 2\pi i \sum_{i=1}^{n} \text{Res} \left( \frac{-2g^2 e^{ix j-l}}{\delta + \frac{t_c}{2}(\text{Li}_3(-e^{ix}) + \text{Li}_3(-e^{-ix}))}, z_i \right) - \int_{-\pi}^{\pi} \frac{-2g^2 e^{ix j-l}}{\delta + \frac{t_c}{2}(\text{Li}_3(-e^{ix}) + \text{Li}_3(-e^{-ix}))} dx \right), \]

(26)

where \( z_i \), \( i = 1, \ldots, n \), are the isolated singularities of the function inside a closed contour containing them. From inspection of the absolute value over the complex plane of the integrand, we conclude that a convenient choice of contour may be a segment on the real axis from \(-\pi\) to \(\pi\), along with a curve on the upper part of the complex plane such as (i) it encloses the singularity lying on the imaginary axis (point \( z_+ \) in Figure 5) and, (ii) it perpendicularly (the steepest descent direction) joins to the ends \(-\pi\) and \(\pi\) on the real axis.

The first contribution to the integral is the first term in the right hand side of Eq. (26), and it is determined by the residue in the pole, which can be approximated by taking the Taylor series of the denominator for \( z \ll 1 \),

\[ \delta + \frac{t_c}{2} \left( -\frac{3}{2} \zeta(3) \right) \log(2) z_{\pm}^2 = 0 \iff z_{\pm} = \pm i \sqrt{\frac{\delta - 3/4 \zeta(3) t_c}{1/2 \log(2) t_c}}, \]

(27)

so the first term in (26) gives an exponential decay \( \approx -2g^2 \xi/(t_c \log(2)) e^{-|j-l|/\xi} \) with length scale \( \xi = |z_+|^{-1} = \sqrt{\log(2/\xi \sqrt{t_c/\delta_0}), \text{ and } \delta_0 = \delta - 3/4 \zeta(3) t_c} \) is the lowest energy of the phonons’ normal modes spectrum.
The second term in the r.h.s. of Eq. (26) has to be evaluated with a choice for the contour $\gamma$ to close the contour integration in the complex plane. In the limit $|j - l| \gg 1$ we can linearize the integrand around $-\pi$ and $\pi$, $f(z) \approx f(\pm \pi + it)$, $t \in [0, +\infty)$, and let the contour $\gamma$ consist of two straight lines from both points in the real axis to $+i\infty$. This yields a contribution of the form $1/|j - l|^3$, which together with the residue contribution and undoing the transformation (21) yields our final result,

$$J_{J,I}^{\Delta k = 0} \sim -(-1)^{j-I} J_{\exp} e^{-|j-I|/\xi} + J_{\text{dip}}, \quad J_{\exp} = 2g^2 \xi^2 \frac{t_c}{t_c \log 2}, \quad J_{\text{dip}} = \frac{g^2 t_c}{(\delta_0 + \frac{7\xi}{3} t_c)^2}. \quad (28)$$

The exponential and dipolar contributions meet for a particular distance $d$ that can be upper-bounded as $d \leq 6\xi \log 3\xi + \xi \log J_{\exp}/J_{\text{dip}}$.

### C. QUANTUM ANNEALING IN THE MEAN-FIELD APPROXIMATION

In order to carry out the quantum annealing process, we have to add suitable annealing schedules [27] to drive the evolution of our Hamiltonian

$$H_{\text{ph}} = \sum_n \bar{a}_n \tilde{a}_n^\dagger a_n, \quad H_s = \frac{\Omega_s}{2} \sum_j \sigma_j^z + \frac{\Omega_z}{2} \sum_j \sigma_j^x, \quad H_{\text{ph}} = g \left( \frac{1}{\tau_{\text{cv}}} \right) \sum_{j,n} \bar{a}_n (M_{j,n} a_n e^{-i\Delta k j} + \text{H.c.}), \quad (29)$$

where we have added an extra field $\Omega_s (t/\tau_{\text{cv}})$. This term breaks the $\mathbb{Z}_2$ symmetry, what is necessary to avoid a static spurious solution stemming from the mean field approximation we are going to impose on the Hamiltonian. Also, the time dependence stands for the generic adiabatic drivings of the coupling and fields acting on the system to make it evolve from the paramagnetic regime to the symmetry-broken region of the phase map. We stress that the symmetry breaking term has a different typical rate change $\tau_{\text{cv}}$. It turns out that such subtlety is essential to optimally evolve to the global ground state, since having $\tau_{\text{cv}} > \tau_{\text{cv}}'$ allows the system to explore the manifold of quasi-degenerate states for a longer time during the quasi-adiabatic evolution.

The dynamical evolution is given by the Heisenberg equations of motion $d/dt \langle A \rangle = i\langle [H_{J^2}, A] \rangle$,

$$\begin{align*}
\frac{d}{dt} (\bar{a}_n(t)) &= -i \omega_n (\bar{a}_n(t)) - ig \left( \frac{1}{\tau_{\text{cv}}} \right) \sum_j (\sigma_j^x(t)) M_{j,n}^* e^{i\Delta k j}, \\
\frac{d}{dt} (\tilde{a}_n(t)) &= i \omega_n (\tilde{a}_n(t)) + ig \left( \frac{1}{\tau_{\text{cv}}} \right) \sum_j (\sigma_j^x(t)) M_{j,n} e^{-i\Delta k j}, \\
\frac{d}{dt} (\sigma_j^z(t)) &= -\Omega_z \left( \frac{1}{\tau_{\text{cv}}} \right) (\sigma_j^z(t)) - 2g \left( \frac{1}{\tau_{\text{cv}}} \right) \sum_n (\sigma_j^z(t)) (M_{j,n} (\bar{a}_n(t)) e^{-i\Delta k j} + \text{c.c.}), \\
\frac{d}{dt} (\sigma_j^x(t)) &= \Omega_s \left( \frac{1}{\tau_{\text{cv}}} \right) (\sigma_j^x(t)) - \Omega_s \left( \frac{1}{\tau_{\text{cv}}} \right) (\sigma_j^x(t)) + 2g \left( \frac{1}{\tau_{\text{cv}}} \right) \sum_n (\sigma_j^x(t)) (M_{j,n} (\bar{a}_n(t)) e^{-i\Delta k j} + \text{c.c.}), \\
\frac{d}{dt} (\sigma_j^y(t)) &= \Omega_s \left( \frac{1}{\tau_{\text{cv}}} \right) (\sigma_j^y(t)).
\end{align*} \quad (30)$$
The adiabatic evolution is driven by the exponential factors with time scale $\tau_{ev}$. Now we note that the two first equations can be rewritten if we express the boson operators in the interaction picture. For example, the first equation would read

$$\frac{d}{dt} \langle \tilde{a}_n(t) \rangle = -ig \left( \frac{1}{\tau_{ev}} \right) \sum_j \langle \sigma_j^z(t) \rangle e^{i\Delta k j M_{j,n}^*} e^{i\bar{\omega} n t}, \quad \tilde{a}_n := \bar{a}_n e^{i\bar{\omega} n t}. \tag{31}$$

Here we would carry out an integration by parts as showed in the Letter. However, as stated there, in order to claim

$$\langle \tilde{a}_n(t) \rangle - \langle \tilde{a}_n(0) \rangle = -g \left( \frac{1}{\tau_{ev}} \right) \sum_j \langle \sigma_j^z(t) \rangle e^{i\Delta k j M_{j,n}^*} e^{i\bar{\omega} n t} \bigg|_0^t + \int_0^t dt' \frac{d}{dt'} \left[ g \left( \frac{1}{\tau_{ev}} \right) \sum_j \langle \sigma_j^z(t') \rangle e^{i\Delta k j M_{j,n}^*} \right]_{t'=t} e^{i\bar{\omega} n t'}$$

$$\approx -g \left( \frac{1}{\tau_{ev}} \right) \sum_j \langle \sigma_j^z(t) \rangle e^{i\Delta k j M_{j,n}^*} e^{i\bar{\omega} n t} \tag{32}$$

the condition $d/dt \langle \sigma_j^z(t) \rangle \ll \bar{\omega}_n$ must hold on top of the adiabatic condition $\tau_{ev}^{-1}, \tau_{ev}'^{-1} \ll \bar{\omega}_n$. Fortunately, numerical computations show this is indeed the case (see Figure 6), so the hypothesis that the dynamics of the phonons follow adiabatically the dynamics of the spins turns out to be correct, i.e., the variational constraints are fulfilled at every instant $t$ during the evolution.

### D. LONG-RANGE QUANTUM CORRELATIONS IN THE HIGHLY FRUSTRATED REGIME

The highly frustrated region of the quantum phase diagram of our model show enhanced quantum fluctuations and long-range correlations characteristics of spin-liquid phases. Here we show some values of the spin-spin correlation functions for a chain of $N = 50$ ions that provide additional numerical evidence of this properties. In Fig. 7, our results show an abrupt change in the range of correlations when entering the highly frustrated regime, as well as different magnetic configurations in the AF, frustrated and Hopfield phases.

![Figure 6. Evolution of the expectation values depicting the typical oscillations caused by the asymmetry in the typical times of evolution between the transverse field $\Omega_x$ and the symmetry breaking term $\Omega_z$. We estimate $d/dt \langle \sigma_j^z(t) \rangle$ as the product of the amplitude of the oscillation divided by its period, so that $d/dt \langle \sigma_j^z(t) \rangle \sim 10^{-3} \ll \min \bar{\omega}_n = \bar{\delta}_0 = 1$ in this case. For the calculations in the figure $t = 2, g = 1, \Delta k = 0, \Omega_x = \Omega_z = 5, \tau_{ev} = 8 \tau_{ev}' = 10^{-4}$.](image-url)
Figure 7. Results for the ground state of a chain with \( N = 50 \) ions calculated with a D.M.R.G. algorithm (number of states kept \( D = 18 \), maximum number of phonons \( n = 9 \)), with \( g = 1, \beta_0 = 1, \Omega_x = 0.01 \), and \( \Delta k = 5\pi/(3d_0) \). (a) Correlations functions \( C_{xx}^{\sigma^x}\sigma^x = \langle \sigma^x_j \sigma^x_l \rangle - \langle \sigma^x_j \rangle \langle \sigma^x_l \rangle \). Note the abrupt change in correlation range for the value \( t_C = 0.49 \). (b-e) Different configurations for: (b) \( t_C = 0.48 \) - AF phase, (d) \( t_C = 0.49 \) - highly frustrated, (c) \( t_C = 0.59 \) - highly frustrated, and (e) \( t_C = 1.5 \) - Hopfield phase.