AFFINE OPEN SUBSETS IN $\mathbb{A}^3$ WITHOUT THE CANCELLATION PROPERTY

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Abstract. We give families of examples of principal open subsets of the affine space $\mathbb{A}^3$ which do not have the cancellation property. We show as a by-product that the cylinders over Koras-Russell threefolds of the first kind have a trivial Makar-Limanov invariant.

Introduction

The generalized Cancellation Problem asks if two algebraic varieties $X$ and $Y$ with isomorphic cylinders $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are isomorphic themselves. Although the answer turns out to be affirmative for a large class of varieties including the case when one of the varieties is the affine plane $\mathbb{A}^2$ [10] [13], counter-examples exist for affine varieties in any dimension $\geq 2$, and the particular case when one of the two varieties is an affine space $\mathbb{A}^n$, $n \geq 3$, still remains a widely open problem.

The first counter-example for complex affine varieties has been constructed by Danielewski [1] in 1989: he exploited the fact that the non isomorphic affine surfaces $S_1 = \{xz = y^2 - 1\}$ and $S_2 = \{x^2z = y^2 - 1\}$ in $\mathbb{A}^3_{\mathbb{C}}$ can be equipped with free actions of the additive group $\mathbb{G}_a$ admitting geometric quotients in the form of non trivial $\mathbb{G}_a$-bundles $\rho_i : S_i \to \mathbb{A}^1$, $i = 1, 2$ over the affine line with a double origin. It then follows that the fiber product $S_1 \times_{\mathbb{A}^1} S_2$ inherits simultaneously the structure of a $\mathbb{G}_a$-bundle over $S_1$ and $S_2$ via the first and the second projection respectively, but since $S_1$ and $S_2$ are both affine, the latter are both trivial, providing isomorphisms $S_1 \times \mathbb{A}^1 \simeq S_1 \times \mathbb{A}^1$ and $S_2 \simeq S_2 \times \mathbb{A}^1$. Since then, Danielewski’s fiber product trick has been the source of many new counter-examples in any dimension [7, 3, 8, 5], some of these being very close to affine spaces either from an algebraic or a topological point of view.

However, a counter-example over the field of real numbers was constructed earlier by Hochster [9] using the algebraic counterpart of the classical fact from differential geometry that the tangent bundle of the real sphere $S^2$ is non trivial but 1-stably trivial. His argument actually applies more generally to the situation when a finitely generated domain $R$ over a field $k$ admits a non trivial projective module $M$ of rank $n - 1 \geq 1$ such that $M \oplus R \simeq R^{\oplus n} \oplus R$. Indeed, these hypotheses immediately imply that the varieties $X = \text{Spec}(k(\text{Sym}(M)))$ and $Y = \text{Spec}(R[x_1, \ldots, x_n])$ are not isomorphic as schemes over $Z = \text{Spec}(R)$ while their cylinders are. Of course, there is no reason in general that $X$ and $Y$ are not isomorphic as $k$-varieties, but this holds for instance when $Z$ does not admit any dominant morphism from an affine space $\mathbb{A}^n_k$ since then any isomorphism between $X$ and $Y$ necessarily descends to an automorphism of $Z$ ([10] [2]). Recently, Jelonek [11] gave revival to Hochster idea by constructing families of examples of non uniruled affine open subsets of affine spaces of any dimension $\geq 8$ with 1-stably trivial but non trivial vector bundles, which fail the cancellation property.

While affine affine open subsets of affine spaces of dimension $\leq 2$ always have the cancellation property (see e.g. loc. cit.), we derive in this note from a variant of Danielewski’s fiber product trick that cancellation already fails for suitably chosen principal open subsets of $\mathbb{A}^3$.

As an application of our construction we also obtain that all cylinders over Koras-Russell threefolds $X_{d,k,l} = \{x^d z = y^l + x - t^k = 0\} \subset \mathbb{A}^3$, $d \geq 2$ and $2 \leq l < k$ relatively prime [12], have a trivial Makar-Limanov invariant [13].

2000 Mathematics Subject Classification. 14R10, 14R20.
Key words and phrases. Cancellation problem; Koras-Russell threefolds.
1. Principal open subsets in $\mathbb{A}^3$ without the cancellation property

For every $d \geq 1$ and $l \geq 2$, we denote by $B_{d,l}$ the surface in $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x,y,z])$ defined by the equation $f_{d,l} = y^l + x - x^d z = 0$ and by $U_{d,l} = \mathbb{A}^3 \setminus B_{d,l}$ its open complement. By construction, $U_{d,l}$ comes equipped with a flat isotrivial fibration $f_{d,l} : U_{d,l} \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$ with closed fibers isomorphic to the surface $S_{d,l} \subset \mathbb{A}^3 = \text{Spec}(\mathbb{C}[X,Y,Z])$ defined by the equation $X^d Z = Y^l + X - 1$. A surface $S_{d,l}$ having no non constant invertible function, an isomorphism $\varphi : U_{d,l} \overset{\simeq}{\rightarrow} U_{d',l'}$ necessarily maps closed fibers of $f_{d,l}$ isomorphically onto that of $f_{d',l'}$. But since $S_{d,l}$ is isomorphic to $S_{d',l'}$, if and only if $(d',l') = (d,l)$ (see e.g. [6] Theorem 3.2 and Proposition 3.6), it follows that the threefolds $U_{d,l}$, $d \geq 1$, $l \geq 2$, are pairwise non isomorphic. In contrast, we have the following result:

**Theorem 1.** For every fixed $l \geq 2$, the fourfolds $U_{d,l} \times \mathbb{A}^1$, $d \geq 1$, are all isomorphic.

**Proof.** We exploit the fact that every $U_{d,l}$ admits a free $G_a$-action defined by the locally nilpotent derivation $x^d \partial_y + y^{l-1} \partial_y$ of its coordinate ring $\mathbb{C}[x,y,z]_{f_{d,l}}$. A free $G_a$-action being locally trivial in the étale topology, it follows that a geometric quotient $\nu_{d,l} : U_{d,l} \rightarrow \mathfrak{S}_{d,l} = U_{d,l}/G_a$ exists in the form of an étale locally trivial $G_a$-bundle over a certain algebraic space $\mathfrak{S}_{d,l}$. Then it is enough to show that for every fixed $l \geq 2$, the algebraic spaces $\mathfrak{S}_{d,l}$ are all isomorphic, say to a fixed algebraic space $\mathfrak{S}_l$. Indeed, if so, then for every $d, d' \geq 1$, the fiber product $U_{d,l} \times \mathfrak{S}_l \times U_{d',l}$ will be simultaneously a $G_a$-bundle over $U_{d,l}$ and $U_{d',l}$ via the first and second projection respectively whereas will be simultaneously isomorphic to the trivial $G_a$-bundles $U_{d,l} \times \mathbb{A}^1$ and $U_{d',l} \times \mathbb{A}^1$ as $U_{d,l}$ and $U_{d',l}$ are both affine.

The algebraic spaces $\mathfrak{S}_{d,l}$ can be described explicitly as follows: one checks that the isotrivial fibration $f_{d,l} : U_{d,l} \rightarrow \mathbb{A}^1$, $u \mapsto t = u^l$, with isomorphism $\Phi_{d,l} : S_{d,l} \times \mathbb{A}^1 \overset{\sim}{\rightarrow} U_{d,l} \times \mathbb{A}^1$ given by $(X,Y,Z,u) \mapsto (u^l X, uY, u^{l-1}Z, u)$ and the $\mu_l$-invariant morphism $\pi_{d,l} = \text{pr}_1 \circ \Phi_{d,l} : S_{d,l} \times \mathbb{A}^1 \rightarrow U_{d,l}$ descends to an isomorphism $(S_{d,l} \times \mathbb{A}^1)/\mu_l \simeq U_{d,l}$. The $G_a$-action on $U_{d,l}$ lifts via the proper étale morphism $\pi_{d,l}$ to the free $G_a$-action on $S_{d,l} \times \mathbb{A}^1$ commuting with that of $\mu_l$ defined by the locally nilpotent derivation $u^{ld-1}(X^d \partial_Y + Y^{l-1} \partial_Z)$ of its coordinate ring $\mathbb{C}[X,Y,Z]/(X^d Z - Y^l - X + 1) \{u \pm 1\}$. The principal divisor $\{X = 0\}$ of $S_{d,l} \times \mathbb{A}^1$ is $G_a$-invariant and it decomposes into the disjoint union of irreducible divisors $D_{\eta} = \{X = Y - \eta = 0\}_{\eta \in \mu_l} \simeq \text{Spec}(\mathbb{C}[Z]/\{u \pm 1\})$ on which $\mu_l$ acts by $D_{\eta} \ni (Z,u) \mapsto (Z, \pm u) \in \mathfrak{S}_l$. Now a similar argument as in [7] Lemma 1.2 implies that for every $\eta \in \mu_l$, the $G_a$-invariant morphism $\text{pr}_X \times \text{id} : S_{d,l} \times \mathbb{A}^1 \rightarrow \mathbb{A}^3 \times \mathbb{A}^1$ restricts on $(S_{d,l} \times \mathbb{A}^1) \setminus \bigcup_{\eta \in \mu_l} \{D_{\eta}\}$ to a trivial $G_a$-bundle over $\mathbb{A}^3 \times \mathbb{A}^1$. Letting $C(l)$ be the scheme over $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[X])$ obtained by gluing $l$ copies $C_{\eta}$, $\eta \in \mu_l$, of $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[X])$ outside their respective origins, it follows that $\text{pr}_X \times \text{id}$ factors through a $\mu_l$-equivariant $G_a$-bundle $\rho_{d,l} \times \text{id} : S_{d,l} \times \mathbb{A}^1 \rightarrow C(l) \times \mathbb{A}^1$, where $\mu_l$ acts freely on $(C(l) \times \mathbb{A}^1)$ by $C_{\eta} \times \mathbb{A}^1 \ni (x,u) \mapsto (x, \pm u) \in C_{\eta} \times \mathbb{A}^1$

A quotient $(C(l) \times \mathbb{A}^1)/\mu_l$ exist in the category of algebraic spaces in the form of a principal $\mu_l$-bundle $\sigma_l : C(l) \times \mathbb{A}^1 \rightarrow \mathfrak{S}_l$, and the above description implies that $\rho_{d,l} \times \text{id}$ descends to an étale locally trivial $G_a$-bundle $\nu_{d,l} : U_{d,l} \rightarrow \mathfrak{S}_l$ for which the diagram

$$
\begin{array}{ccc}
S_{d,l} \times \mathbb{A}^1 & \overset{\pi_{d,l}}{\rightarrow} & U_{d,l} \simeq (S_{d,l} \times \mathbb{A}^1)/\mu_l \\
\rho_{d,l} \times \text{id} & \downarrow & \\
C(l) \times \mathbb{A}^1 & \overset{\sigma_l}{\rightarrow} & \mathfrak{S}_l,
\end{array}
$$

is cartesian. By virtue of the universal property of categorical quotients one has necessarily $\mathfrak{S}_{d,l} \simeq \mathfrak{S}_l$ for every $d \geq 1$. In particular, the isomorphism type of $\mathfrak{S}_{d,l}$ depends only on $l$, which completes the proof. □

**Remark 2.** The algebraic spaces $\mathfrak{S}_l = (C(l) \times \mathbb{A}^1)/\mu_l$, $l \geq 2$, considered in the proof above cannot be schemes: indeed, otherwise the image in $\mathfrak{S}_l$ of the point $(0,1) \in \mathbb{C}^l \times \mathbb{A}^1 \subset C(l) \times \mathbb{A}^1$ would have a Zariski open affine neighborhood $V$. But then the inverse image of $V$ by the finite étale cover $\sigma_l : C(l) \times \mathbb{A}^1 \rightarrow \mathfrak{S}_l$ would be a $\mu_l$-invariant affine open neighborhood of $(0,1)$ in $C(l) \times \mathbb{A}^1$, which
is absurd since $(0,1)$ does not even have a separated $μ_1$-invariant open neighborhood in $C(ι) × A^1_i$. This implies in turn that the free $G_a$-action on $U_{d,l}$ defined by the locally nilpotent derivation $x^d_3∂_y + y^{l-1}_3∂_z$ is not locally trivial in the Zariski topology. In contrast, the latter property holds for its lift to $S_{d,l} × A^1_i$ via the étale Galois cover $π_{d,l} : S_{d,l} × A^1_i → U_{d,l}$. Remark 3. In Danielewski’s construction for the surfaces $S_i = \{ x^i z = y^2 - 1 \} ⊂ A^3$, $i = 1, 2$, the geometric quotients $S_i / G_a ≃ A^1$, $i = 1, 2$, were obtained from the categorical quotients $S_i / G_a = Spec(C[x,y,z]^{G_a}_μ)$ taken in the category of affine schemes by replacing the origin by two copies of itself, one for each orbit in the zero fiber of the quotient morphism $q = pr_2 : S_i → A^1$. For the threefolds $U_{d,l}$, the difference between the quotients $U_{d,l} / G_a = Spec(C[x,y,z]^{G_a}_μ)$ in the category of (affine) schemes and the geometric quotients $S_i = U_{d,l} / G_a$ is very similar: indeed, we may identify $U_{d,l}$ with the closed subvariety of $A^3 × A^1_i = Spec(C[x,y,z][t^{±1}])$ defined by the equation $x^d_3 z = y^2 + x - t$ in such a way that $f_{d,l} : U_{d,l} → A^1_i$ coincides with the projection $pr_{d,l} |_{U_{d,l}}$. Then, the kernel of the locally nilpotent derivation $x^d_3∂_y + y^{l-1}_3∂_z$ of the coordinate ring of $U_{d,l}$ coincides with the subalgebra $C[x,y,z][t^{±1}]$ and so, the $G_a$-invariant morphism $q = pr_{d,l} : U_{d,l} → A^1_i × L = Spec(C[x,z][t^{±1}])$ is a categorical quotient in the category of affine schemes. One checks easily that $q$ restricts to a trivial $G_a$-bundle over the principal open subset $\{ x ≠ 0 \}$ of $A^3 × A^1_i$, whereas the inverse image of the punctured line $\{ x = 0 \} ≃ L$ is isomorphic to $A^1_i × L = Spec(C[y,z][t^{±1}]/(y^2 - t)z)$ where $G_a$ acts by translations on the second factor. So we may interpret the geometric quotient $S_i = U_{d,l} / G_a$ as being obtained from $U_{d,l} / G_a = A^3 × L$ by replacing the punctured line $\{ x = 0 \} ≃ L$ not by $l$ disjoint copies of itself but, instead, by the total space $L$ of the nontrivial étale Galois cover $pr_{d,l} : L → L$. The Koras-Russell threefolds $X_{d,k,l}$ are smooth complex affine varieties defined by equations of the form $x^d_3 z = y^l + x - t^k$, where $d ≥ 2$ and $2 ≤ l < k$ are relatively prime. While all diffomorphic to the euclidean space $R^6$, none of these threefold is algebraically isomorphic to the affine $A^3$. Indeed, it was established by Kaliman and Makar-Limanov [13, 12] that they have fewer algebraic $G_a$-actions than the affine space $A^3$: the subring $ML(X_{d,k,l})$ of their coordinate ring consisting of regular functions invariant under all algebraic $G_a$-actions on $X_{d,k,l}$ is equal to the polynomial ring $C[x]$, while $ML(A^3)$ is trivial, consisting of constants only. However, it was observed by the author in [3] that the Makar-Limanov invariant $ML$ fails to distinguish the cylinder over the so-called Russell cubic $X_{2,2,3}$ from the affine space $A^3$. This phenomenon holds more generally for cylinders over all Koras-Russell threefolds $X_{d,k,l}$: Corollary 4. All the cylinders $X_{d,k,l} × A^1$ have a trivial Makar-Limanov invariant. Proof. We consider $X_{d,k,l} × A^1$ as the subvariety of $Spec(C[x,y,z,t][v])$ defined by the equation $f_{d,l} = t^k = 0$. While $ML(X_{d,k,l} × A^1) ⊂ ML(X_{d,k,l}) = C[x]$, it is enough to construct a locally nilpotent derivation of $C[x,y,z,t][v]/(f_{d,l} - t^k)$ which does not have $x$ in its kernel. One checks easily that $ML(U_{d,l} × A^1) = C[f_{d,l}^{±1}]$ is the intersection of the kernels of the locally nilpotent derivations $x∂_y + y^{l-1}_3∂_z$ and $y^{l-1}_3∂_x + (z-1)∂_y$ of $C[x,y,z][f_{d,l}]$. Theorem 3 above implies in particular that $ML(U_{d,l} × A^1) ≃ ML(U_{d,l} × A^1) = C[f_{d,l}^{±1}]$ and so, there exists a locally nilpotent derivation $δ_{d,l}$ of $Γ(U_{d,l} × A^1, C_{U_{d,l} × A^1}) = C[x,y,z][f_{d,l}]$ which does not have $x$ in its kernel. Up to multiplying it by a suitable power of $f_{d,l} ∈ Ker(δ_{d,l})$, we may further assume that $δ_{d,l}$ is the extension to $C[x,y,z][f_{d,l}]$ of a locally nilpotent derivation of $C[x,y,z][v]$ which has $f_{d,l}$ but not $x$ in its kernel. This implies in particular that $B_{d,l} × A^1 = Spec(C[x,y,z]/(f_{d,l}))$ is invariant under the corresponding $G_a$-action on $A^3 = Spec(C[x,y,z][v])$. The projection $p = pr_{x,y,z,v} : X_{d,k,l} × A^1 → A^3$ being a finite Galois cover with branch locus $B_{d,l} × A^1$, it follows that the $G_a$-action on $A^3$ lifts to a one on $X_{d,k,l} × A^1$ for which $p : X_{d,k,l} × A^1 → A^3$ is $G_a$-equivariant. By construction, the corresponding locally nilpotent derivation of $C[x,y,z][v]/(f_{d,l} - t^k)$ does not have $x$ in its kernel. In the proof above, we used the following classical fact that we include here because of a lack of an appropriate reference. 

\[^1\]These are called Koras-Russell threefolds of the first kind in [15].
Lemma 5. Let $X$ be a variety defined over a field of characteristic zero and equipped with a non trivial $\mathbb{G}_a$-action, let $Z$ be a normal variety and let $p : Z \to X$ be a finite surjective morphism. Suppose that there exists a $\mathbb{G}_a$-invariant affine open subvariety $U$ of $X$ over which $p$ restricts to an étale morphism. Then there exists a unique $\mathbb{G}_a$-action on $Z$ for which $p : Z \to X$ is a $\mathbb{G}_a$-equivariant morphism.

Proof. The induced $\mathbb{G}_a$-action on the invariant affine open subvariety $U$ of $X$ is determined by a locally nilpotent derivation $\partial$ of $\Gamma(U, \mathcal{O}_U)$. Since $p : p^{-1}(U) \to U$ is étale and proper, $p^{-1}(U)$ is an affine open subvariety of $Z$ and $\partial$ lifts in a unique way to a derivation of $\Gamma(p^{-1}(U), \mathcal{O}_{p^{-1}(U)})$ which is again locally nilpotent by virtue of [17]. By construction, the latter defines a $\mathbb{G}_a$-action on $p^{-1}(U)$ for which the restriction of $p$ to $p^{-1}(U)$ is equivariant. Now the assertion follows from [16, Lemma 6.1] which guarantees that the $\mathbb{G}_a$-action on $p^{-1}(U)$ can be uniquely extended to a one on $Z$ with the desired property. □

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