Proof Theory for Lax Logic

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For Dick de Jongh, on the occasion of his 80th birthday.

Abstract

In this paper some proof theory for propositional Lax Logic is developed. A cut free terminating sequent calculus is introduced for the logic, and based on that calculus it is shown that the logic has uniform interpolation. Furthermore, a separate, simple proof of interpolation is provided that also uses the sequent calculus. From the literature it is known that Lax Logic has interpolation, but all known proofs use models rather than proof systems.

Keywords: intuitionistic modal logic, Lax Logic, sequent calculus, uniform interpolation

MSC: 03B05, 03B45, 03F03

1 Introduction

Propositional Lax Logic (PLL) is an intuitionistic modal logic that models several phenomena in logic and computer science. The logic has a single unary modal operator $\Box$ and is axiomatized by the principles of intuitionistic propositional logic $\mathbf{IPC}$ plus the following three modal principles:

$$\varphi \rightarrow \Box \varphi \ (\Box R) \quad \Box \Box \varphi \rightarrow \Box \varphi \ (\Box M) \quad \Box \varphi \land \Box \psi \rightarrow \Box (\varphi \land \psi) \ (\Box S).$$

In type theory, the modal operator describes a certain type constructor, and there exists a Curry-Howard like correspondence between Lax Logic and the computational lambda calculus (Benton et al., 1993). In algebraic logic the operator is a so-called nucleus, and in recent work in that area, embeddings of superintuitionistic logics into extensions of PLL have been used to provide new characterizations of subframe superintuitionistic logics (Bezhanishvili et al., 2019), while in (Melzer, 2020) the method of canonical formulas for Lax Logic has been developed.

The origins of the modality can be traced back to Curry (1957) but the logic received its name Lax Logic in the PhD-thesis of Mendler (1993), where the logic was introduced in the setting of hardware verification. In that context Lax Logic is used to model reasoning about statements of the form $\varphi$ holds under some constraint, denoted as $\Box \varphi$. Further investigation led to (Fairtlough and Mendler, 1994, 1997), in which, among other things, a sequent calculus for the logic is provided, a fact that is relevant for this paper.

In this paper we introduce a terminating sequent calculus $\mathbf{G4iLL}$ for PLL which is weakly analytic (in a way explained in Section 2.1) and use it to prove that the logic has uniform interpolation. Results on uniform interpolation for intuitionistic modal logics are rare, in particular for those that are not normal. In (Iemhoff, 2019b) it is shown that the logics $\mathbf{iK}_\Box$ and $\mathbf{iKD}_\Box$ have uniform interpolation, which is generalized in (Jalali and Tabatabai, 2019).
Finally, this work is part of the larger project of Universal Proof Theory, which aims to develop general methods to answer universal questions about proof systems. Questions such as: When does a theory have a certain type of proof system? What is the connection between the logical properties of a theory and its proof systems? How can it be established that a theory has a certain type of proof system? What is the connection between the logical properties of a theory and its proof systems? How can it be established that a theory has a certain type of proof system? What is the connection between the logical properties of a theory and its proof systems?

Besides its use in the proof of uniform interpolation, the terminating sequent calculus that we develop is also interesting in connection with a paper of Howe (2001) in which a decision method for PLL is provided that is based on an annotated sequent calculus for the logic. In that paper it is shown that the calculus is sound and complete for Lax Logic and that backwards proof search in the calculus is terminating. Our sequent calculus G4iLL is also terminating, but consists of standard sequents instead of annotated ones.

Finally, this work is part of the larger project of Universal Proof Theory, which aims to develop general methods to answer universal questions about proof systems. Questions such as: When does a theory have a certain type of proof system? What is the connection between the logical properties of a theory and its proof systems? How can it be established that a theory has uniform interpolation, then the method identifies a class of proof systems that are excluded for the logic. If, on the other hand, a logic does not have uniform interpolation, then the method identifies a class of proof systems that are excluded for the logic. If we develop is also interesting in connection with a paper of Howe (2001) in which a decision method for PLL is provided that is based on an annotated sequent calculus for the logic. In that paper it is shown that the calculus is sound and complete for Lax Logic and that backwards proof search in the calculus is terminating. Our sequent calculus G4iLL is also terminating, but consists of standard sequents instead of annotated ones.

2 Preliminaries

The language $\mathcal{L}$ for propositional lax logic PLL contains a constant $\bot$, propositional variables or atoms $p, q, r, \ldots$, the modal operator $\bigcirc$ and the connectives $\wedge, \vee, \to$, where $\neg \varphi$ is defined as $(\varphi \to \bot)$. $\bot$ is by definition not an atom. A circle formula is a formula of the form $\bigcirc \varphi$.

The set of formulas in $\mathcal{L}$ is denoted by $\mathcal{F}$.

We denote finite multisets of formulas by $\Gamma, \Pi, \Delta, \Sigma$. $\Gamma \cup \Pi$ is the multiset that contains only formulas $\varphi$ that belong to $\Gamma$ or $\Pi$ and the number of occurrences of $\varphi$ in $\Gamma \cup \Pi$ is the sum of the occurrences of $\varphi$ in $\Gamma$ and in $\Pi$. $\Gamma, \Pi$ is short for $\Gamma \cup \Pi$. Multiset $\Gamma, \varphi^n, \Pi$ denotes the union of $\Gamma, \Pi$ and the multiset that consists of $n$ copies of $\varphi$. We let $(\Gamma \Rightarrow \Delta) \subseteq (\Gamma' \Rightarrow \Delta')$ denote that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Furthermore (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_a \Gamma \quad (\Gamma \Rightarrow \Delta)^s \equiv_s \Delta \quad \bigcirc \Gamma \equiv_s \{ \bigcirc \varphi \mid \varphi \in \Gamma \}.$$  

When working with sequents with a superscript, such as $S^s$, then $S'^a$ is short for $(S^s)^a$, and similarly for $S'^s$. We only consider (single-conclusion) sequents, which are expressions $(\Gamma \Rightarrow \Delta)$, where $\Gamma$ and $\Delta$ are finite multisets of formulas and $\Delta$ contains at most one formula. They are interpreted as $I(\Gamma \Rightarrow \Delta) \equiv_{\mathcal{F}} (\bigwedge \Gamma \to \bigvee \Delta)$, where $\bigvee \varnothing$ is interpreted as $\bot$. Expression $(\Gamma \Rightarrow \Delta) : (\Pi \Rightarrow \Sigma)$ denotes $(\Gamma, \Pi \Rightarrow \Sigma, \Delta)$. When sequents are used in the setting of formulas, we often write $S$ for $I(S)$, such as in $\vdash \bigvee_j S_j$, which thus denotes $\vdash \bigvee_j I(S_j)$.

The degree of a formula $\varphi$ is inductively defined as $d(\bot) = 0$, $d(p) = 1$, $d(\bigcirc \varphi) = d(\varphi) + 1$, and $d(\varphi \circ \psi) = d(\varphi) + d(\psi) + 1$ for $\circ \in \{ \wedge, \vee, \to \}$. In the setting of G4iLL we need an order
on sequents based on the weight function \( w(\cdot) \) on formulas from [Dyckhoff, 1992], which is inductively defined as: the weight of an atom and the constant \( \bot \) is 1, \( w(\circ \varphi) = w(\varphi) + 1 \), and \( w(\varphi \circ \psi) = w(\varphi) + w(\psi) + i \), where \( i = 1 \) in case \( \circ \in \{ \lor, \to \} \) and \( i = 2 \) otherwise. We use the following ordering on sequents: \( S_0 \ll S_1 \) if and only if \( S_0 \cup S_0 \ll S_1 \cup S_1 \), where \( \ll \) is the order on multisets determined by weight as in [Dershowitz and Manna, 1979] (where they in fact define \( \gg \)): for multisets \( \Gamma, \Delta \) we have \( \Delta \ll \Gamma \) if \( \Delta \) is the result of replacing one or more formulas in \( \Gamma \) by zero or more formulas of lower weight.

2.1 The calculi G3iLL and G4iLL

One of the standard calculi without structural rules for IPC is G3ip (Figure 2.1), which is the propositional part of the calculus G3i from [Troelstra and Schwichtenberg, 1996]. The calculus G4ip (Figure 2.2) for IPC from [Dyckhoff, 1992] is similar to G3ip except for the left implication rule, which is replaced by rules that make G4ip terminating, a feature that G3ip does not possess (the definition of terminating is given in Section 2.2). The calculus G3iLL is G3ip extended by the two modal rules \( R \) and \( L \) given in Figure 2.3, which are the two modal rules of the calculus GPLL in [Fairtlough and Mendler, 1997]. The calculus G4iLL is G4ip extended by the four rules given in Figure 2.3. G3iLL is an analytic calculus, meaning that it has the subformula property. G4iLL does not have the subformula property, but the words of Dyckhoff (1992) (last paragraph Section 1) about G4ip apply here as well: “Nevertheless, it has it in an obvious weak sense: one can work out what formulae are able to appear in a proof of a particular end-sequent.”

2.2 Properties of calculi

A calculus is terminating with respect to an order \( < \) on sequents if the following hold: the calculus is finite (has finitely many axioms and rules); for all sequents \( S \) there are at most finitely many instances of a rule in the calculus with conclusion \( S \); in every instance of a rule in the calculus the premises come before the conclusion in the order \( < \).

**Fact 1** Calculus G4iLL is terminating in order \( \ll \) and calculus G3iLL is not.

In the section on uniform interpolation we use that G4iLL is a reductive calculus. Here an order on sequents \( < \) is reductive if the following hold: it is well-founded; all proper subsequents of a sequent come before that sequent; whenever all formulas in \( S \) occur boxed in \( S' \), then \( S < S' \); \( (\Gamma, \varphi \Rightarrow \Delta) < (\Gamma, q \Rightarrow \varphi \Rightarrow \Delta) \) for all multisets \( \Gamma, \Delta \), formulas \( \varphi \) and atoms \( q \). Clearly, \( \ll \) is reductive. A calculus is reductive if it is terminating with respect to an order that is reductive.

**Fact 2** Calculus G4iLL is reductive with respect to order \( \ll \).
A typical example of a rule that in general cannot belong to a reductive calculus is the cut rule, as in most common orders on sequents the premisses of that rule do not come before its conclusion.

In (Iemhoff, 2019b) (Section 7.1), the notion of a balanced calculus $G$ is defined as: $G$ is reductive, Cut and Weakening are admissible in it, and $G$ satisfies two conditions that $G_{4iLL}$ trivially satisfies. Therefore, using Fact 2 and Corollary 1 below, we can conclude the following.

**Fact 3** Calculus $G_{4iLL}$ is a balanced calculus with respect to order $\ll$.

A rule is nonflat if its conclusion contains at least one connective or modal operator. A set of rules or a calculus is nonflat if all of its rules are. Clearly, calculi $G_{3iLL}$ and $G_{4iLL}$ are nonflat.

### 2.3 Structural rules

In a calculus *weakening is admissible* if the following two rules are admissible:

\[
\frac{\Gamma \Rightarrow \varphi}{\Gamma, \varphi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \Delta}
\]

In a calculus *contraction is admissible* if the following rule is admissible:

\[
\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}
\]
A calculus has cut-elimination or cut is admissible in it if the following Cut Rule is admissible:

\[
\frac{\Gamma_1 \Rightarrow \varphi, \Gamma_2, \varphi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta} \text{ Cut}
\]

In a calculus the structural rules are admissible if weakening, contraction and cut are admissible in it.

**Lemma 1** Weakening and contraction as well as the following rule are admissible in G3iLL:

\[
\Gamma \Rightarrow \perp
\]

\[
\Gamma \Rightarrow \Delta
\]

**Proof** The proof is standard, analogous to the proof for G3ip in [Troelstra and Schwichtenberg, 1996] and therefore omitted.

\[\dashv\]

## 3 Cut-elimination

In this section we prove that the Cut Rule is admissible in G3iLL. The proof is by induction on the degree and a subinduction on the level of cuts. The degree of an application of the Cut Rule is the degree of the cut formula. The level of a cut in a derivation is the sum of the heights of its two premisses, where the height of a derivation is the length of its longest branch, where branches consisting of one node are considered to have height 1. In the rules of G3iLL, principal formulas are defined as usual for the rules in G3ip [Troelstra and Schwichtenberg, 1996] and for the modal rules $R\Box$ and $L\Box$ as given in Figure 2.3, the principal formulas are $\Box \varphi$ and $\Box \psi$, respectively.

**Theorem 1** (Cut-elimination) The Cut Rule is admissible in G3iLL.

**Proof** Let $\vdash$ denote $\vdash_{\text{G3iLL}}$. Following the corrected version [Troelstra, 1998] of the cut elimination proof for G3ip in [Troelstra and Schwichtenberg, 1996], we successively eliminate cuts from the proof, always considering those cuts that have no cuts above them, the topmost cuts. For this it suffices to show that for cut free proofs $\mathcal{D}_1$ and $\mathcal{D}_2$, the following proof $\mathcal{D}$ of sequent $S = (\Gamma_1, \Gamma_2 \Rightarrow \Delta)$ can be transformed into a cut free proof $\mathcal{D}'$ of the same endsequent:

\[
\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta} \text{ Cut}
\]

This is proved by induction on the degree of the cut formula, with a subinduction to the level of the cut. We use the fact that weakening and contraction are admissible in G3iLL implicitly at various places.

There are three possibilities:

1. at least one of the premises is an axiom;
2. both premises are not axioms and the cut formula is not principal in at least one of the premises;
3. the cut formula is principal in both premises, which are not axioms.

1. As in [Troelstra and Schwichtenberg, 1996], straightforward, by checking all possible cases: If $\mathcal{D}_1$ is axiom $L\perp$ or $\mathcal{D}_2$ is axiom $L\perp$ and $\perp$ is not the cut formula, then we let $\mathcal{D}'$ be the instance $S$ of that axiom. If $\perp$ is the cut formula, then $(\Gamma_1 \Rightarrow \perp)$ has a cut free
derivation. Thus so has \((\Gamma_1, \Gamma_2 \Rightarrow \Delta)\) by Lemma \(\Pi\) where we use the additional fact that in this proof no new cuts are introduced with respect to \(D_1\). Therefore we can let \(D'\) be this proof.

Assume both premises are not instances of \(L \perp\). If \(D_1\) is an instance of \(Ax\), then \(\varphi\) is an atom. If \(D_2\) also is an instance of \(Ax\), then \(S\) is an instance of \(Ax\), so we let \(D'\) be that instance. If \(D_2\) is not an axiom, then \(\varphi\) cannot be principal in its last inference, because it is an atom. We obtain \(D'\) by \textit{cutting at a lower level} in \(D_2\), by which mean the following.

Suppose the last inference of \(D_2\) is \(R \rightarrow\):

\[
\begin{array}{c}
D_1 \\
\Pi_1 \Rightarrow \varphi \\
\Pi_1, \varphi, \psi_1 \Rightarrow \psi_2 \\
\Pi_1, \Gamma_2 \Rightarrow \psi_1 \Rightarrow \psi_2 \\
\Pi_1, \Gamma_2 \Rightarrow \psi_2 \\
\Gamma_1, \Gamma_2 \Rightarrow \psi_1 \Rightarrow \psi_2 \\
\Gamma_1, \Gamma_2 \Rightarrow \psi_2
\end{array}
\]

Consider the proof \(D_3\):

\[
\begin{array}{c}
D_1 \\
\Pi_1 \Rightarrow \varphi \\
\Pi_1, \varphi, \psi_1 \Rightarrow \psi_2 \\
\Pi_1, \Gamma_2, \psi_1 \Rightarrow \psi_2
\end{array}
\]

Since the cut in \(D_3\) is of the same degree but of lower level, there is a cut free proof \(D'_3\) of \((\Gamma_1, \Gamma_2, \psi_1 \Rightarrow \psi_2)\). Hence the following derivation is the desired cut free proof \(D'\):

\[
\begin{array}{c}
D'_3 \\
\Pi_1, \Gamma_2, \psi_1 \Rightarrow \psi_2 \\
\Gamma_1, \Gamma_2 \Rightarrow \psi_1 \Rightarrow \psi_2
\end{array}
\]

The cases where the last inference of \(D_2\) is another rule than \(R \rightarrow\) are treated in a similar way.

The case that \(D_1\) is not an axiom and \(D_2\) is an instance of \(Ax\) remains. Here we also cut at a lower level, the proof is analogous to the case just treated.

2. First, the case that \(\varphi\) is not principal in the last inference of \(D_1\). Thus the last inference in \(D_1\) is one of the nonmodal rules \(R\) of \textsc{G3ill} or \(L\). We treat the latter, where the last part of the proof looks as follows and \(\varphi = \circ \chi\):

\[
\begin{array}{c}
\Pi_1, \psi \Rightarrow \circ \chi \\
\Pi_1, \circ \psi \Rightarrow \circ \chi \\
\Pi_1, \circ \psi, \Gamma_2 \Rightarrow \Delta
\end{array}
\]

where \(\Gamma_1 = \Pi, \circ \psi\). In case \(\Delta\) consists of a circle formula, we can cut at a lower level in \(D_1\):

\[
\begin{array}{c}
\Pi, \psi \Rightarrow \circ \chi \\
\Pi, \circ \psi \Rightarrow \circ \chi \\
\Pi, \psi, \Gamma_2 \Rightarrow \Delta
\end{array}
\]

Thus we obtain a proof of \((\Gamma_1, \Gamma_2 \Rightarrow \Delta)\) with a cut of the same degree as the cut in \(D\) but of lower level, and the induction hypothesis can be applied. In case \(\Delta\) is not a circle formula, first note that \(\circ \chi\) cannot be principal in the last inference of \(D_2\) as that would imply that \(\Delta\) is a circle formula, which by assumption it is not. Thus \(\circ \chi\) is not principal in the last inference of \(D_2\) and we therefore can cut at a lower level in \(D_2\).

In the case that \(\varphi\) is not principal in the last inference of \(D_2\), we only treat the following two cases, where \(\varphi = \circ \chi\):

\[
\begin{array}{c}
\Gamma_1 \Rightarrow \circ \chi \\
\Gamma_2, \circ \chi \Rightarrow \psi \\
\Gamma_1, \Gamma_2 \Rightarrow \circ \psi
\end{array}
\]

\[
\begin{array}{c}
\Pi, \circ \chi, \psi \Rightarrow \Delta \\
\Pi, \circ \chi, \psi' \Rightarrow \Delta
\end{array}
\]

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where on the right side $\Delta$ is a circle formula and $\Gamma_2 = \Pi, \circ \psi'$. Clearly, in both cases one can cut at a lower level in $D_2$ (cf. case 1).

3. The cut formula is principal in both premises, which are not axioms. We distinguish by cases according to the form of the cut formula and only treat the circle formulas. Thus the proof has the following form:

\[
\begin{array}{c}
D'_1 \\
\Gamma_1 \Rightarrow \chi \\
\Gamma_2, \chi \Rightarrow \circ \psi \\
\hline
\Gamma_1, \Gamma_2 \Rightarrow \circ \psi
\end{array}
\quad \begin{array}{c}
D'_2 \\
\Gamma_1 \Rightarrow \circ \chi \\
\Gamma_2, \circ \chi \Rightarrow \circ \psi \\
\hline
\Gamma_1, \Gamma_2 \Rightarrow \circ \psi
\end{array}
\]

This can be replaced by a proof in which the only cut has a lower degree:

\[
\begin{array}{c}
D'_1 \\
\Gamma_1 \Rightarrow \chi \\
\Gamma_2, \chi \Rightarrow \circ \psi \\
\hline
\Gamma_1, \Gamma_2 \Rightarrow \circ \psi
\end{array}
\]

\[\dashv\]

4 Equivalence of G3iLL and G4iLL

In the short note [Iemhoff, 2020], a theorem has been proven that implies that G3iLL and G4iLL are equivalent. To formulate the theorem, we first need some terminology. In that paper, we define a right modal rule as a rule of the form (the $S_i$ are sequents)

\[
\begin{array}{c}
S_1 \ldots S_n \\
\Gamma \Rightarrow \circ \varphi \\
\hline
R
\end{array}
\]

Because under this definition the rule $L \circ$ is a right modal rule, which is an unfortunate clash in terminology, we here use the name succedent modal rule for such rules instead. With such a succedent modal rule we associate the following implication rule

\[
\begin{array}{c}
S_1 \ldots S_n \\
\Gamma, \circ \varphi \Rightarrow \psi \Rightarrow \Delta \\
\hline
R \rightarrow
\end{array}
\]

If $X$ is a set of rules, then $G3iX$ denotes the calculus $G3ip$ extended by the rules in $X$ and $G4iX$ is the calculus $G4ip$ extended by the rules in $X$ plus the rules $R \rightarrow$ for those $R \in X$ that are succedent modal rules. Note that our notation $G3iLL$ and $G4iLL$ is consistent with this definition, that is, $G4iLL$ is the result of adding the two modal rules $R \circ$ and $L \circ$ of $G3iLL$ to $G4ip$ as well as the rules $R \circ \rightarrow$ and $L \circ \rightarrow$, which are added because both $R \circ$ and $L \circ$ have been added.

In [Iemhoff, 2020], the following is shown.

**Theorem 2** (Equivalence)

If $R$ is a set of nonflat rules, the structural rules are admissible in $G3ip + R$, and $G4ip + R$ is terminating and closed under weakening, then $G3ip + R$ and $G4iX + R$ are equivalent (derive exactly the same sequents) and the structural rules are admissible in $G4iX$.

**Corollary 1** $G3iLL$ and $G4iLL$ are equivalent and the structural rules are admissible in both of them. In particular, $G4iLL$ is a terminating cut free sequent calculus for Lax Logic in which the Cut Rule is admissible.
5 Interpolation

That PLL has interpolation follows from the fact that it has uniform interpolation, as shown in the next section. Because the proof of the latter is complicated and the proof of the former is not, we present in this section a simple proof of interpolation for PLL. Also the algorithm to compute interpolants that can be derived from it is much simpler than the one obtained from the proof of uniform interpolation. The method we present here is a straightforward adaptation of the *Maehara method*, a method to prove interpolation based on sequent calculi first presented in [Maehara, 1960], in Japanese, and later in English in [Ono, 1998].

We introduce some terminology well-known from the literature on interpolation. Given multisets $\Gamma, \Delta$ and formula $\varphi$, let $V(\varphi)$ denote all atoms that occur in $\varphi$, let $V(\Gamma) = \bigcup\{V(\varphi) \mid \varphi \in \Gamma\}$, let $V(\Gamma, \Delta)$ be short for $V(\Gamma \cup \Delta)$, and $V(\Gamma, \varphi)$ and $V(\Gamma, \varphi, \psi)$ for $V(\Gamma, \{\varphi\})$ and $V(\Gamma, \varphi, \psi)$, respectively. To be able to refer to the particular partition of the antecedent of a sequent of which a formula $\chi$ is an interpolant, we say that $\chi$ is an interpolant of $\Gamma; \Gamma' \Rightarrow \Delta$ if $\Gamma \Rightarrow \chi$ and $\Gamma', \chi \Rightarrow \Delta$ are derivable in $\text{G3iLL}$ and $V(\chi) \subseteq V(\Gamma) \cap V(\Gamma') \cap \Delta$.

**Theorem 3** $\text{G3iLL}$ has sequent interpolation: if $\Gamma, \Gamma' \Rightarrow \Delta$ is derivable in $\text{G3iLL}$, then there is a formula $\chi$ such that $\Gamma \Rightarrow \chi$ and $\Gamma', \chi \Rightarrow \Delta$ are derivable in $\text{G3iLL}$ and all atoms in $\chi$ occur both in $\Gamma$ and in $\Gamma' \cup \Delta$.

**Proof**

We prove the theorem by constructing an algorithm that given a derivation $\mathcal{D}$ of $\Gamma, \Gamma' \Rightarrow \Delta$ in $\text{G3iLL}$ produces for any partition $\Gamma; \Gamma' \Rightarrow \Delta$ an interpolant. We use induction to the height of $\mathcal{D}$. The case that $\mathcal{D}$ is an instance of an axiom is straightforward, and so are the cases where the last inference of $\mathcal{D}$ is one of the propositional rules. We treat the case that it is an implication rule as an example, followed by the case that it is one of the modal rules.

If the last inference is $R \to$, then $\Delta$ consists of $\varphi \to \psi$. If the partition is $\Gamma; \Gamma' \Rightarrow \varphi \to \psi$, then consider the partition of the premise $\Gamma; \Gamma', \varphi \Rightarrow \psi$. By the induction hypothesis there is an interpolant $\chi$ such that both $\Gamma \Rightarrow \chi$ and $\Gamma', \chi, \varphi \Rightarrow \psi$ are derivable and $V(\chi) \subseteq V(\Gamma) \cap V(\Gamma', \varphi, \psi)$. This implies that $\Gamma', \chi \Rightarrow \varphi \Rightarrow \psi$ is derivable and $V(\chi) \subseteq V(\Gamma', \varphi \Rightarrow \psi)$. Thus proving that $\chi$ is an interpolant for $\Gamma; \Gamma' \Rightarrow \varphi \Rightarrow \psi$.

If the last inference is $L \to$ with principal formula $\varphi \Rightarrow \psi$, then there are two possible partitions of the last sequent: $\Gamma, \varphi \Rightarrow \psi; \Gamma' \Rightarrow \Delta$ and $\Gamma; \varphi \Rightarrow \psi, \Gamma' \Rightarrow \Delta$. In the first case, by the induction hypothesis there are interpolants $\chi_1$ and $\chi_2$ of the partitions of the premises $\Gamma'; \Gamma, \varphi \Rightarrow \psi$ (note the changed order in the partition) and $\Gamma, \psi; \Gamma' \Rightarrow \Delta$. Thus $V(\chi_1) \subseteq V(\Gamma, \varphi \Rightarrow \psi) \cap V(\Gamma'), V(\chi_2) \subseteq V(\Gamma, \psi) \cap V(\Gamma', \Delta)$, and the following are derivable:

$\Gamma' \Rightarrow \chi_1 \quad \Gamma, \varphi \Rightarrow \psi, \chi_1 \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \chi_2 \quad \Gamma', \chi_1 \Rightarrow \Delta$.

This implies that $\Gamma', \chi_1 \Rightarrow \chi_2 \Rightarrow \Delta$ and $\Gamma, \varphi \Rightarrow \psi, \chi_1 \Rightarrow \chi_2$ are derivable. Thus so is $\Gamma, \varphi \Rightarrow \psi \Rightarrow \chi_1 \Rightarrow \chi_2$. Since $V(\chi_1 \Rightarrow \chi_2) \subseteq V(\Gamma, \varphi \Rightarrow \psi) \cap V(\Gamma') \cap \Delta$, this proves that $\chi_1 \Rightarrow \chi_2$ is an interpolant of $\Gamma, \varphi \Rightarrow \psi; \Gamma' \Rightarrow \Delta$.

In the second case, $\Gamma; \varphi \Rightarrow \psi, \Gamma' \Rightarrow \Delta$, the induction hypothesis implies that there are interpolants $\chi_1$ and $\chi_2$ of the partitions of the premises $\Gamma; \varphi \Rightarrow \psi, \Gamma' \Rightarrow \varphi$ and $\Gamma; \psi, \Gamma' \Rightarrow \Delta$. Thus $V(\chi_1) \subseteq V(\Gamma) \cap V(\Gamma', \varphi \Rightarrow \psi) \cap V(\Gamma), V(\chi_2) \subseteq V(\Gamma) \cap V(\Gamma', \psi) \cap \Delta$, and the following are derivable:

$\Gamma \Rightarrow \chi_1 \quad \Gamma', \varphi \Rightarrow \psi, \chi_1 \Rightarrow \varphi \quad \Gamma \Rightarrow \chi_2 \quad \Gamma', \psi, \chi_2 \Rightarrow \Delta$.

It is not hard to see that $\chi = (\chi_1 \land \chi_2)$ is the desired interpolant.

If the last inference is $R \circ$, then for the interpolant of the conclusion one can take the interpolant of the premise.
If the last inference is $L\bigcirc$, then $\Delta$ consists of a circle formula $\bigcirc\psi$ and the principal formula is $\bigcirc \varphi$. For partition $\Gamma; \Gamma', \bigcirc \varphi \Rightarrow \bigcirc \psi$ of the conclusion, let $\chi$ be the interpolant for the corresponding partition of the premise, namely such that the following are derivable:

$$\Gamma \Rightarrow \chi \quad \Gamma', \varphi, \chi \Rightarrow \bigcirc \psi.$$  

Clearly, $\chi$ is an interpolant for the conclusion as well. Similarly, for partition $\Gamma, \bigcirc \varphi; \Gamma' \Rightarrow \bigcirc \psi$ of the conclusion, let $\chi$ be the interpolant for the corresponding partition of the premise, namely such that the following are derivable:

$$\Gamma, \varphi \Rightarrow \chi \quad \Gamma', \chi \Rightarrow \bigcirc \psi.$$  

If $\chi$ is a circle formula, then it is an interpolant for the conclusion. Otherwise we can take $\bigcirc \chi$ instead. 

Corollary 2 Lax Logic has Craig interpolation. In particular, there is an algorithm that given a derivation of $\varphi \Rightarrow \psi$ in $G3iLL$ produces an interpolant for $\varphi \rightarrow \psi$.

6 Uniform interpolation

In [Iemhoff, 2019b] a modular and constructive method was developed to prove uniform interpolation for certain intermediate and intuitionistic modal logics, which was later generalized to other substructural (modal) logics by Jalali and Tabatabai (2018a,b). In this section we prove that Lax Logic, a logic not covered by these previous methods, has uniform interpolation. Our proof is an adaptation of the proof in [Iemhoff, 2019b] that the intuitionistic modal logic $iK\Box$ has uniform interpolation. For background information and motivation we refer the reader to that paper.

6.1 Uniform interpolants for formulas

A logic has uniform interpolation if for any atom $p$ and any set of atoms $P$ not containing $p$, the embedding of $F(P)$ into $F(P \cup \{p\})$ has a right and a left adjoint: For any formula $\varphi$ and any atom $p$ there exist two formulas, usually denoted by $\forall p \varphi$ and $\exists p \varphi$, in the language of the logic, that do not contain $p$ and such that for all $\psi$ not containing $p$:

$$\vdash \psi \rightarrow \varphi \iff \vdash \psi \rightarrow \forall p \varphi \quad \vdash \varphi \rightarrow \psi \iff \vdash \exists p \varphi \rightarrow \psi.$$  

Given a formula $\varphi$, its universal uniform interpolant with respect to $p_1 \ldots p_n$ is $\forall p_1 \ldots p_n \varphi$, which is short for $\forall p_1 (\forall p_2 (\ldots (\forall p_n \varphi) \ldots))$, and its existential uniform interpolant with respect to $p_1 \ldots p_n$ is $\exists p_1 \ldots p_n \varphi$, short for $\exists p_1 (\exists p_2 (\ldots (\exists p_n \varphi) \ldots))$. The requirements above could be replaced by the following four requirements.

$$\vdash \forall p \varphi \rightarrow \varphi \quad \vdash \psi \rightarrow \varphi \Rightarrow \vdash \psi \rightarrow \forall p \varphi.$$  

$$(\forall)$$

$$\vdash \varphi \rightarrow \exists p \varphi \quad \vdash \varphi \rightarrow \psi \Rightarrow \vdash \exists p \varphi \rightarrow \psi.$$  

$$(\exists)$$

In classical logic one only needs one quantifier, as $\exists p$ can be defined as $\neg \forall p \neg$ and vice versa. Although in the intuitionistic setting $\exists p$ can also be defined in terms of $\forall p$, namely as $\exists p \varphi = \forall q (\forall p (\varphi \rightarrow q) \rightarrow q)$ for a $q$ not in $\varphi$, having it as a separate quantifier is convenient in the proof-theoretic approach presented here (we follow [Pitts, 1992], which also uses both quantifiers).

In order to prove that Lax Logic has uniform interpolation, the first task is to adapt the above notions to the setting of sequents and sequent calculus $G4iLL$. 

9
6.2 Reductive orders and rank

The set $F_{ex}$ is the smallest set of expressions that contains all formulas in the language $L$, is closed under the connectives and modal operator, and if $S$ is a sequent in $L$ and $p$ an atom, then $\forall pS$ and $\exists pS$ belong to $F_{ex}$. For example, when $S$ is a sequent in $L$ and $\varphi$ a propositional formula, then $(\Gamma, \varphi \Rightarrow \Delta)$ belongs to $F_{ex}$, as does $\circ (\varphi \land \forall pS)$, but $\exists p \exists q S$ does not.

The reductive order $\ll$ on sequents defined in Section 6.2 determines the following order on formulas in $F_{ex}$, denoted $\ll$. For any expression $\varphi$ in $F_{ex}$, let $qf(\varphi)$ denote the multiset consisting of all occurrences of subformulas of the form $QpS$ in $\varphi$, where $Q \in \{\exists, \forall\}$. The order on multisets of the form $qf(\varphi)$ again is in the style of [Dershowitz and Manna, 1979]:

$qf(\varphi) \prec_{qf} qf(\psi)$ if $qf(\varphi)$ is the result of replacing one or more formulas of the form $QpS$ in $qf(\psi)$ by zero or more formulas of the form $Q'qS'$ with $S' \ll S$, where $Q, Q' \in \{\exists, \forall\}$.

This order is well--defined since by definition such $S$ are sequents in $L$ and therefore can be compared via $\ll$. The order on $F_{ex}$ can now be defined as: if $\varphi, \psi \in F$, then $\varphi \prec \psi$ if $(\Rightarrow \varphi) \prec ((\Rightarrow \psi)$; if $\varphi \in F$ and $\psi \not\in F$, then $\varphi \prec \psi$ and not $\psi \prec \varphi$; if $\varphi, \psi \not\in F$, then $\varphi \prec \psi$ if $qf(\varphi) \prec qf(\psi)$. When $\varphi \prec \psi$, we say that $\varphi$ is of lower rank than $\psi$. Clearly, since the order $\ll$ on sequents is well--founded, so is the order $\prec$ on $F_{ex}$.

6.3 Interpolant assignments

For any given calculus, let $G_{ins}$ denote the set of instances of rules in $G$. A sequent $S$ is principal for an instance $R$ of a rule if the conclusion of $R$ is of the form $S' \cdot S$ for some sequent $S'$ and the principal formula of $R$ belongs to $S$. Otherwise $S$ is nonprincipal. For example, suppose $R$ has conclusion $(\Gamma, \varphi \Rightarrow \Delta)$ and $\varphi$ is the principal formula of $R$, then any sequent of the form $(\Gamma', \varphi \Rightarrow \Delta')$, where $(\Gamma' \Rightarrow \Delta') \subseteq (\Gamma \Rightarrow \Delta)$, is principal for $R$.

An interpolant assignment $\iota$ for $G$, assigns, for every atom $p$ and sequent $S$, $\iota pS = \top$ and $\iota \forall p S = \bot$ in case $S$ is empty, and in case $S$ is not empty:

- for every $R \in G_{ins}$ with conclusion $S$, to each of the expressions $\exists \beta S$ and $\forall \beta S$ a formula in $F_{ex}$ that is of lower rank than $\exists \beta S$ (or, equivalently, of lower rank than $\forall \beta S$), which are denoted by $\iota \exists \beta S$ and $\iota \forall \beta S$, respectively, and

- for every $\beta \in G$ such that $S$ is nonprincipal for at least one instance of $\beta$, to each of the expressions $\exists \beta S$ and $\forall \beta S$ a formula in $F_{ex}$ that is of lower rank than $\exists \beta S$, which are denoted by $\iota \exists \beta S$ and $\iota \forall \beta S$, respectively.

Given an interpolant assignment we define the following formulas in $F_{ex}$. Recall that $p$ and $q$ range over atoms.

\[ \forall^+ pS \equiv_{df} \forall \{ \forall \beta S \mid R \in G_{ins}, S \text{ is the conclusion of } R \} \]
\[ \forall^- pS \equiv_{df} \forall \{ \forall \beta S \mid \beta \in G, S \text{ is nonprincipal for some instance of } \beta \} \]
\[ \exists^+ pS \equiv_{df} \exists \{ \exists \beta S \mid R \in G_{ins}, S \text{ is the conclusion of } R \} \]
\[ \exists^- pS \equiv_{df} \exists \{ \exists \beta S \mid \beta \in G, S \text{ is nonprincipal for some instance of } \beta \} \]
\[ \forall^a pS \equiv_{df} \forall \{ q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \top \} \]
\[ \forall^a pS \equiv_{df} \forall \{ q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \top \} \]
\[ \exists^a pS \equiv_{df} \exists \{ q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \bot \} \]
\[ \exists^a pS \equiv_{df} \exists \{ q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \bot \} \]

Observe that there could be more than one instance of a single rule $\beta$ that has $S$ as a conclusion, in which case every instance corresponds to a separate disjunct or conjunct.
of the interpolant assignment. The definition above is well-defined for reductive calculi, because for such calculi all sets over which the big conjunctions and disjunctions range are finite.

We define a rewrite relation $\rightsquigarrow$ on $\mathcal{F}_{\text{ex}}$ that is the smallest relation on $\mathcal{F}_{\text{ex}}$ that preserves the logical operators and satisfies:

$$\forall pS \mapsto \forall^+ pS \lor \forall^- pS \lor \forall^\text{df} pS \quad \exists pS \mapsto \exists^+ pS \land \exists^- pS \land \exists^\text{df} pS.$$ 

After the definition of uniform interpolation for sequents in Section 6.4 below, three examples of the rewrite relation are provided.

As $\text{G4iLL}$ is a reductive calculus, the following lemma follows from Lemma 3 in (Iemhoff, 2019b).

**Lemma 2** If there exists an interpolant assignment for $\text{G4iLL}$, then the relation $\rightsquigarrow$ on $\mathcal{F}_{\text{ex}}$ is confluent and strongly normalizing.

Note that according to this relation, any formula in $\mathcal{L}$ is in normal form, and any expression in $\mathcal{F}_{\text{ex}}$ that is not a formula in $\mathcal{L}$ is not. Let $\overline{\varphi}$ denote the normal form of an expression $\varphi$. For any sequent $S$ in $\mathcal{L}$ we now define its left and right interpolant to be $\overline{\forall pS}$ and $\overline{\exists pS}$, respectively. We omit the overline most of the time, as it will be clear from the context whether the expression in $\mathcal{F}_{\text{ex}}$ is meant or the actual normal form in $\mathcal{L}$, i.e. the interpolants.

### 6.4 Uniform interpolants for sequents

To define uniform interpolants in the setting of sequents, we introduce the notion of a partition, which applies to sequents and to rules. A partition $S$ is an ordered pair $(S', S^r)$ ($i$ for interpolant, $r$ for rest) such that $S = S' \cdot S^r$. It is a $p$-partition if $p$ does not occur in $S^r$. For any sequent $S$ and partition $(S', S^r)$ we use the abbreviation:

$$S' \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset) \equiv_{\text{df}} \left\{ 
\begin{array}{ll}
S' \cdot (\exists pS^i \Rightarrow \forall pS^i) & \text{if } S^s \neq \emptyset \text{ and } S'^s = \emptyset \\
S' \cdot (\exists pS^i \Rightarrow \forall pS^i) & \text{if } S^s = \emptyset \text{ or } S'^s \neq \emptyset.
\end{array}
\right.$$

A $(p\text{-})$partition of an instance $R = (S_1 \ldots S_n/S_0)$ of a rule is a $(p\text{-})$partition of the sequents in the rule. Given such a partition and $* \in \{r, i\}$, let $(R^*, R^i)$ and $R^r$ respectively denote the expressions

$$\begin{pmatrix}
(S_1^*, S_1^i) & \ldots & (S_n^*, S_n^i)
\end{pmatrix}
\begin{pmatrix}
R^r, R^i
\end{pmatrix}
\begin{pmatrix}
S_1^* \ldots S_n^* \\
S_0^* \ldots S_0^i
\end{pmatrix}
R^*$$

In Lemma 2 in (Iemhoff, 2019b), it is shown that given an interpolant assignment an intuitionistic modal logic has uniform interpolation if the following holds:

(∀l) For all $p$: $\vdash S^a, \forall pS \Rightarrow S^*$;

(∃r) For all $p$: $\vdash S^a \Rightarrow \exists pS$;

(∀∃) If $S$ is derivable, for all $p$ and all $p$–partitions $(S^*, S^i)$: $\vdash S^* \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset)$.

Properties (∀l) and (∃r) are the independent (from partitions) interpolant properties, and (∀∃) is the dependent interpolant property. It is also shown that $\forall pS$ and $\exists pS$ are equivalent to $\forall pI(S)$ and $\exists p(\bigwedge S^a)$, respectively. In particular, $\forall p(\Rightarrow \varphi)$ is equivalent to $\forall p\varphi$ and $\exists p(\varphi \Rightarrow )$ to $\exists p\varphi$.

A partition $(S^*, S^i)$ of $S$ satisfies the interpolant properties if, in the case of the independent property, $S$ satisfies them (in which case we also say that $S$ satisfies them), and in case of
the dependent property, it holds for that particular partition. A sequent satisfies a property if every possible partition of the sequent satisfies it.

To show that Lax Logic has uniform interpolation it thus suffices to show that we can define, for all sequents \( S \), formulas \( \forall pS \) and \( \exists pS \) in \( \mathcal{L} \) in such a way that the three interpolant properties hold. Before we do so, three examples of uniform sequent interpolants are given.

**Example 1** Suppose the calculus only contains the rule \( \mathcal{R} \) for conjunction on the left:

\[
\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \land \psi \Rightarrow \Delta}
\]

We provide the uniform interpolants for \( S = (p \land q, r, s \Rightarrow t) \). We have

\[
\forall pS \rightarrow \forall pS \lor \forall pS \lor \forall apS \quad \exists pS \rightarrow \exists pS \land \exists pS \land \exists apS,
\]

where \( R = S_1/S \) stands for the instance of \( \mathcal{R} \) with \( p \land q \) as the principal formula. Thus \( S_1 = (p, q, r, s \Rightarrow t) \). The standard interpolant assignment introduced below satisfies \( \forall pS = \forall pS_1, \exists pS = \exists pS_1, \forall apS = \top \), and \( \exists apS = \top \). This implies that

\[
\forall pS \rightarrow \forall pS_1 \lor \top \quad \exists pS \rightarrow \top \land r \land s.
\]

Since \( S_1 \) cannot be the conclusion of any rule, we have \( \forall pS_1 = \forall pS \lor \forall apS_1 = \top \lor t \) and \( \exists pS_1 = \exists pS \land \exists apS_1 = \top \land q \land r \land s \). Therefore

\[
\forall pS \rightarrow t \lor \top \land t \quad \exists pS \rightarrow \top \land q \land r \land s \land \top \land r \land s.
\]

Indeed, with \( \forall pS = t \) and \( \exists pS = q \land r \land s \), the three interpolant properties are satisfied for \( S \).

**Example 2** For the same calculus as in the previous example, the uniform interpolants of the sequent \( S = (p_0 \land q_0, p_1 \land q_1 \Rightarrow r) \) are computed as follows, where we abbreviate \( p_i \land q_i \) with \( \psi_i \). For \( i = 0, 1 \), let \( R_i \) stand for the instance of \( \mathcal{R} \) with \( \psi_i \) as the principal formula, and define sequents \( S_0 = (p_0, q_0, \psi_1 \Rightarrow r) \) and \( S_1 = (\psi_0, p_1, q_1 \Rightarrow r) \), which are the premises of these instances. By the above definition,

\[
\forall pS \rightarrow \forall R_0S \lor \forall R_1S \lor \forall pS \lor \forall apS \quad \exists pS \rightarrow \exists R_0S \land \exists R_1S \land \exists pS \land \exists apS.
\]

The disjuncts and conjuncts can be computed as in the previous example.

**Example 3** Suppose the calculus consists only of the rule \( \mathcal{R} \) for disjunction on the right:

\[
\frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \lor \varphi_1} \quad (i = 0, 1)
\]

We provide the uniform interpolants for \( S = (r \Rightarrow p \lor q) \). Let \( R_p, R_q \) stand for the instances of \( \mathcal{R} \) with \( p \) and \( q \) as the conclusion and premises \( S_p = (r \Rightarrow p) \) and \( S_q = (r \Rightarrow q) \), respectively. By the above definition,

\[
\forall pS \rightarrow \forall R_pS \lor \forall R_qS \lor \forall pS \lor \forall apS \quad \exists pS \rightarrow \exists R_pS \land \exists R_qS \land \exists pS \land \exists apS.
\]

The standard interpolant assignment introduced below satisfies \( \forall apS = \forall pS_1 \) for any instance \( R = S_1/S \) of \( \mathcal{R} \), as well as \( \forall R_pS = \bot \) and \( \exists R_qS = \top \) for any \( S \). This, together with the fact that \( \forall apS = \bot \), implies

\[
\forall pS \rightarrow \forall R_pS \lor \forall R_qS \land \bot \land \bot \quad \exists pS \rightarrow \exists R_pS \land \exists R_qS \land \bot.
\]
Since $S_p$ and $S_q$ cannot be the conclusion of any rule, we have

$$
∀pS_p = ∇∀pS_p \lor ∀^pS_p = ∅ \lor ⊥ \quad ∀pS_q = ∇∀pS_q \lor ∀^pS_q = ∅ \lor q
$$

$$∃pS_p = ∇∃pS_p \land ∃^pS_p = T \land r = ∇∃pS_q \land ∃^pS_q = ∃pS_q.
$$

Therefore $∀pS → ⊥ \lor ⊥ \lor ⊥ \lor q \land ⊥$ and $∃pS → T \land r \land T \land r \land T \land r$. Indeed, with $∀pS = q$ and $∃pS = r$, the three interpolant properties are satisfied for $S$.

### 6.5 The inductive properties

For a modular development of uniform interpolation, we introduce the following six properties of rules, where $∅ \vdash ϕ$ should be read as $\vdash ϕ$. Given an instance $R = (S_1 \ldots S_n/S_0)$ of a rule $\mathcal{R}$, we define

$$
\mathcal{I}_R = \{ i \cdot (∀pS_i \Rightarrow ), (S^a \Rightarrow ∃pS_j) \mid 1 ≤ j ≤ n \} \cup
\{ (S^a \Rightarrow ∃p(S^a \Rightarrow )) \mid S \subseteq S_0 \lor S \subseteq S_j \text{ for some } 1 ≤ j ≤ n \}
$$

$$
\mathcal{D}_R = ∪_{j=1}^n \{ (S^j \cdot (∃pS_j \Rightarrow ∀pS^j \mid ∅)) \mid (S^j, S_j) \text{ a } p\text{-partition of } S_j \}.
$$

For $\mathcal{D}_R$, note that it contains the sequent $S^j \cdot (∃pS^j \Rightarrow ∀pS^j \mid ∅)$ for any possible $p$-partition $(S^j, S_j)$ of a premiss $S_j$ of $R$. And that for $S$ with empty succedent, $S^* \cdot (∃pS^a \Rightarrow )$ derives $S^* \cdot (∃pS^a \Rightarrow ∀pS^a)$.

The sets $\mathcal{I}_R$ and $\mathcal{D}_R$ contain the sequents to which, in a proof of the interpolant properties that uses induction along $\prec$, the induction hypothesis could be applied. In such a proof, the assumption that the interpolant properties hold for all sequents below $S$ implies that the sequents in $\mathcal{I}_R$ and $\mathcal{D}_R$ are derivable. Note that in the case that $R$ is an instance of an axiom, the latter set is empty, but the former is not, as it contains all sequents of the form $S^a \Rightarrow ∃p(S^a \Rightarrow )$ for $S$ such that $S \subseteq S_0$ or $S \subseteq S_j$.

(IPP)$^\mathcal{R}$ $\mathcal{I}_R \vdash S \cdot (∀pS \Rightarrow )$ for every instance $R$ of $\mathcal{R}$ with conclusion $S$.

(IPN)$^\mathcal{R}$ If $S$ is nonprincipal for some instance of $\mathcal{R}$, then the assumption that all sequents below $S$ satisfy the interpolant properties implies $\vdash S \cdot (∀pS \Rightarrow )$.

(IPP)$^\mathcal{R}$ $\mathcal{I}_R \vdash (S^a \Rightarrow ∃pS)$ for every instance $R$ of $\mathcal{R}$ with conclusion $S$.

(IPN)$^\mathcal{R}$ If $S$ is nonprincipal for some instance of $\mathcal{R}$, then the assumption that all sequents below $S$ satisfy the interpolant properties implies $\vdash (S^a \Rightarrow ∃pS)$.

(DPP)$^\mathcal{R}$ For every sequent $S$ that has a derivation of which the last inference is an instance $R$ of $\mathcal{R}$, and for every $p$-partition $(S^*, S^j)$ such that sequent $S^j$ is principal for $R$: $\mathcal{D}_R \vdash S^* \cdot (∃pS^i \Rightarrow ∀pS^j \mid ∅)$.

(DPN)$^\mathcal{R}$ For every sequent $S$ that has a derivation of which the last inference is an instance $R$ of $\mathcal{R}$, and for every $p$-partition $(S^*, S^j)$ such that sequent $S^j$ is nonprincipal for $R$: if all sequents that are below $S$ satisfy the interpolant properties, then $\vdash S^* \cdot (∃pS^i \Rightarrow ∀pS^j \mid ∅)$.

These six properties are called the inductive properties in this paper. “IP” stands for independent property, “DP” for dependent property, “P” and “N” for principal and nonprincipal, respectively.

For an intuitionistic modal logic $\mathcal{L}$ with calculus $\mathcal{G}$, an interpolant assignment is sound for a rule $\mathcal{R}$ in $\mathcal{G}$, if the six inductive properties hold for $\mathcal{R}$, where “$\vdash$” equals “$\vdash^\mathcal{L}$”. It is sound for a calculus if it is sound for all the rules of the calculus.
Remark 1 The following observation will be used to prove (DPP)_R and (DPN)_R. Consider a sequent \( S \) with partition \( (S^r, S^i) \), which has a derivation of which the last inference is an instance \( R = (S_1 \ldots S_n)/S \) of \( \mathcal{R} \). To prove (DPP)_R, thus in case \( S^i \) is principal for \( R \), in order to prove
\[
\mathcal{D}_R^p \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset)
\]
it suffices to show that
\[
\mathcal{D}_R^p \vdash S^r \cdot (\exists \neg pS^i \Rightarrow \forall \neg pS^i \mid \emptyset)
\]
for some partition \((R^r, R^i)\) of \( R \) with conclusion \((S^r, S^i)\) such that \( R^i \) is an instance of \( \mathcal{R} \). The reason being, that for such an \( R^i \), \( \exists \neg pS^i \) is a conjunct of \( \exists pS^i \) and \( \forall \neg pS^i \) a disjunct of \( \forall pS^i \). Likewise, to prove (DPN)_R, thus in case \( S^i \) is nonprincipal for \( R \), to prove
\[
\Gamma \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset)
\]
it suffices to prove that
\[
\Gamma \vdash S^r \cdot (\exists \neg pS^i \Rightarrow \forall \neg pS^i \mid \emptyset).
\]

Corollary 32 and Theorem 31 in (Iemhoff, 2019b) state the following.

Theorem 4 Any intuitionistic modal logic \( \mathcal{L} \) with a balanced calculus that consists of \( G4iK_{\Box} \), focused rules, and modal focused rules, has uniform interpolation.

Theorem 5 Any intuitionistic modal logic \( \mathcal{L} \) with a balanced calculus that contains \( G4iK_{\Box} \) and has an interpolant assignment that is sound with respect to all rules that neither are focused, nor modal focused, nor belong to \( G4iK_{\Box} \), has uniform interpolation.

Given the fact that all rules of \( G3ip \) are focused, as shown in (Iemhoff, 2019b), to prove that \( PLL \) has uniform interpolation it suffices to show that the four extra rules of \( G4iLL \) are sound with respect to an interpolant assignment that is an extension of the one from (Iemhoff, 2019b), which is called the standard interpolant assignment. This is shown in the next section.

6.6 Interpolant assignment for PLL

Here the interpolant assignment for the four modal rules of \( G4iLL \) is introduced. Its soundness is proved in the next section.

For \( R = S_1/S \) being an instance of \( \mathcal{R} = R \circ \) or \( \mathcal{R} = L \circ \), the interpolant assignment is defined as follows:
\[
\exists \neg pS \equiv_{df} \exists pS_1 \quad \forall \neg pS \equiv_{df} \forall pS_1
\]
\[
\exists \neg pS \equiv_{df} \top \quad \forall \neg pS \equiv_{df} \bot
\]

For \( R \) being an instance
\[
\frac{S_1 \quad S_2}{S} = \frac{\Gamma \Rightarrow \varphi \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \Box \varphi \Rightarrow \psi \Rightarrow \Delta}
\]
of \( \mathcal{R} = R \circ \neg \), the interpolant assignment is defined as follows:
\[
\exists \neg pS \equiv_{df} \exists pS_1 \land (\forall pS_1 \rightarrow \exists pS_2) \quad \forall \neg pS \equiv_{df} \forall pS_1 \land \forall pS_2
\]
\[
\exists \neg pS \equiv_{df} \top \quad \forall \neg pS \equiv_{df} \bot \quad \text{if } S^* = \emptyset
\]
\[
\exists \neg pS \equiv_{df} \exists p(S^a \Rightarrow) \quad \forall \neg pS \equiv_{df} \bot \quad \text{if } S^* \neq \emptyset
\]

For \( R \) being an instance
\[
\frac{S_1 \quad S_2}{S} = \frac{\Gamma, \chi \Rightarrow \Box \varphi \quad \Gamma, \Box \chi, \psi \Rightarrow \Delta}{\Gamma, \Box \chi, \Box \varphi \Rightarrow \psi \Rightarrow \Delta}
\]
of \( R = L \lor^\cdot \), the interpolant assignment is defined as follows:

\[
\exists^R S \equiv_{df} \exists p S_1 \land (\forall \forall p S_1 \rightarrow \exists p S_2) \quad \forall^R S \equiv_{df} \forall \forall p S_1 \land \forall p S_2.
\]

For \( \exists^R S \) and \( \forall^R S \), first define for \( \gamma = \circ \alpha \rightarrow \beta \):

\[
S^{\gamma 0}_{df} \equiv \{ S^\alpha \setminus \{ \gamma \} \} \rightarrow \circ \alpha \quad S^{\gamma 1}_{df} \equiv \{ S^\alpha \setminus \{ \gamma \}, \beta \rightarrow S^\gamma \}
\]

Then let

\[
\exists^R S \equiv_{df} \begin{cases} S^{\gamma} & \text{if } S^{\gamma} = \emptyset \\ \exists p (S^\alpha \rightarrow ) \land S^{\gamma} & \text{if } S^{\gamma} \neq \emptyset \end{cases}
\]

\[
\forall^R S \equiv_{df} \forall \gamma = \circ \alpha \rightarrow \beta \in S^{\gamma} \exists \forall p S^{\gamma 0} \land \forall p S^{\gamma 1}
\]

### 6.7 Soundness of the interpolant assignment

**Lemma 3** For \( R = R \circ \) all independent inductive properties and (DPP)\(_R\) hold.

**Proof** That (IPN)_R^\circ and (IPN)_R^\gamma hold is clear. For (IPP)_R^\circ and (IPP)_R^\gamma, let \( R \) be an instance

\[
\frac{S \gamma_1}{S} = \frac{S_1}{S} = \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \circ \varphi}
\]

of \( R = R \circ, \exists^R S \) and \( \forall^R S \) are defined as \( \exists p \exists S_1 \) and \( \forall \forall p S_1 \), respectively. Since \( (\Gamma \Rightarrow \exists p S_1) \) and \( (\Gamma, \forall p S_1 \Rightarrow \varphi) \) belong to \( \mathcal{F}_R^\circ \), an application of \( R \) shows that \( \mathcal{F}_R^\circ \vdash \Gamma \Rightarrow \exists^R S \) holds, and an application of \( R \) followed by an application of \( L \circ \) shows that \( \mathcal{F}_R^\circ \vdash \Gamma, \forall^R S \Rightarrow \circ \varphi \) holds.

For (DPP)_R, in a partition of \( S \) in which \( S^i \) is principal, \( S^i \) has to be of the form \( S^i = (\Gamma^i \Rightarrow \circ \varphi) \), where \( \Gamma^i \subseteq \Gamma \). We have to show that \( \mathcal{D}_R^\circ \vdash S^i \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \emptyset) \), which in this case means

\[
\mathcal{D}_R^\circ \vdash \Gamma^i, \exists p S^i \Rightarrow \forall p S^i.
\]

Let \( S^i_1 = (\Gamma^i \Rightarrow \varphi) \). Thus \( \Gamma^i \Rightarrow \exists \forall p S^i_1 \Rightarrow \exists \forall p S^i_1 \Rightarrow \forall \forall p S^i_1 \Rightarrow \forall \exists p S^i_1 \Rightarrow \forall \forall p S^i_1 \Rightarrow \forall \forall p S^i_1 \Rightarrow \forall p S^i \Rightarrow \forall p S^i \).

### Lemma 4** For \( R = L \circ \) all independent inductive properties and (DPP)_R hold.

**Proof** That (IPN)_R^\circ and (IPN)_R^\gamma hold is clear. The proofs of (IPP)_R^\circ and (IPP)_R^\gamma are similar to the one for \( R \circ \), but we include them nevertheless. For \( R \) being an instance

\[
\frac{S \gamma_1}{S} = \frac{\Gamma, \psi \Rightarrow \circ \varphi}{\Gamma, \circ \psi \Rightarrow \circ \varphi}
\]

of \( R = L \circ, \exists^R S \) and \( \forall^R S \) are defined as \( \exists p \exists S_1 \) and \( \forall \forall p S_1 \), respectively. Thus \( (\Gamma, \psi \Rightarrow \exists p S_1) \) and \( (\Gamma, \psi, \forall p S_1 \Rightarrow \circ \varphi) \) belong to \( \mathcal{F}_R^\circ \). An application of \( R \circ \) followed by an application of \( R \) to the former shows that \( \mathcal{F}_R^\circ \vdash \Gamma, \circ \psi \Rightarrow \exists^R S \) holds. Two applications of \( R \) to the latter show that \( \mathcal{F}_R^\circ \vdash \Gamma, \forall p S^i \Rightarrow \circ \varphi \) holds.

For (DPP)_R, we distinguish the case that \( S^{\circ \alpha} \) is empty and that it is not. Note that \( S^{\circ \alpha} \) has to contain \( \circ \psi \) because \( S^i \) is supposed to be principal for \( L \circ \). In the first case, \( S^i = (\Gamma^i, \circ \psi \Rightarrow ) \) for some \( \Gamma^i \subseteq \Gamma \). We have to show that \( \mathcal{D}_R^\circ \vdash S^i \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \emptyset) \), which in this case means

\[
\mathcal{D}_R^\circ \vdash \Gamma^i, \exists p S^i \Rightarrow \circ \varphi.
\]
Consider the partition of $S_1$ such that $S_1^i = (\Gamma^i, \psi \Rightarrow \varphi)$. Thus $\Gamma^r, \exists p S_1^i \Rightarrow \varphi$ belongs to $\mathcal{D}_R^p$ and an application of $L \circ$ gives $\mathcal{D}_R^p \vdash \Gamma^r, \exists p S_1^i \Rightarrow \varphi$. Since $S_1^i/S_1^i$ is an instance of $R$, $\emptyset \exists p S_1^i$ is a conjunct of $\exists p S_i^i$. This implies (2).

In the second case, $S_i = (\Gamma^i, \varphi \Rightarrow \varphi)$ for some $\Gamma^i \subseteq \Gamma$. We have to show that $\mathcal{D}_R^p \vdash S_i \cdot (\exists p S_i^i \Rightarrow \psi \Rightarrow \Delta)$, which in this case means

$$\mathcal{D}_R^p \vdash \Gamma^r, \exists p S_i \Rightarrow \psi \Rightarrow \Delta. \tag{3}$$

Consider the partition of $S_1$ such that $S_1^i = (\Gamma^i, \psi \Rightarrow \varphi)$. Thus $\Gamma^r, \exists p S_1^i \Rightarrow \varphi$ belongs to $\mathcal{D}_R^p$. An application of $R \circ$ followed by $L \circ$ proves $\mathcal{D}_R^p \vdash \Gamma^r, \exists p S_1^i \Rightarrow \varphi$. Since $S_i^i/S_i^i$ is an instance of $R$, $\emptyset \exists p S_i^i$ and $\forall p S_i^i$ are a conjunct and a disjunct of $\exists p S_i^i$ and $\forall p S_i^i$, respectively. This implies (4).

**Lemma 5** For $R = R \circ \rightarrow$ all independent inductive properties and $(DPP)_R$ hold.

**Proof** That $(IPN)_R$ holds is clear. For $(IPP)_R^3$ and $(IPP)_R^\forall$, let $R$ be an instance

$$\frac{S_1, S_2}{S} = \frac{\Gamma \Rightarrow \varphi}{\Gamma, \varphi \Rightarrow \Delta}$$

of $R = R \circ \rightarrow$. Since $\exists p S$ and $\forall p S$ are defined as $\exists p S_1 \land (\forall p S_1 \rightarrow \exists p S_2)$ and $\forall p S_1 \land \forall p S_2$, respectively, then we have to show that

$$\frac{\exists p S}{\exists p S} \vdash \Gamma, \varphi \Rightarrow \exists p S_1 \land (\forall p S_1 \rightarrow \exists p S_2) \quad \frac{\forall p S}{\exists p S} \vdash \Gamma, \varphi \Rightarrow \forall p S_1 \land \forall p S_2, \varphi \Rightarrow \psi \Rightarrow \Delta.$$  

This follows from applications of $R$ to sequents $(\Gamma \Rightarrow \exists p S_1)$, $(\Gamma, \varphi \Rightarrow \exists p S_2)$, $(\Gamma, \forall p S_1 \Rightarrow \varphi)$ and $(\Gamma, \varphi \Rightarrow \forall p S_2 \Rightarrow \Delta)$, that all belong to $\mathcal{D}_R^p$.

For $(IPN)_R^3$ only the case that $\Delta$ is not empty is nontrivial. Thus suppose that $S$ is not principal for $R$ and that all sequents lower than $S$ satisfy the interpolant properties. Let $S_0 = (S^0 \Rightarrow \cdot )$. Hence $\exists p S = \exists p S_0$. Because $S_0 \ll S$, $S_0$ satisfies the interpolant properties, and thus $S_0^i \Rightarrow \exists p S_0$ is derivable. Since $S_0 = S_0^i$, it follows that $\vdash S_0^i \Rightarrow \exists p S_0$.

For $(DPP)_R$, in a partition of $S$ in which $S_i$ is principal, $S_i$ has to be of the form $S_i = (\Gamma^i, \varphi \Rightarrow \psi \Rightarrow \Delta^i)$, where $\Gamma^i \subseteq \Gamma$ and $\Delta^i \subseteq \Delta$. We treat the case that $\Delta^i$ is empty, the other case being analogous. We have to show that

$$\frac{\exists p S_i \Rightarrow \forall p S^i}{\exists p S_i} \vdash \Gamma^r, \exists p S_i \Rightarrow \forall p S^i \Rightarrow \psi \Rightarrow \Delta_i. \tag{4}$$

Let the partitions of the premisses be given by $S_1^i = (\Gamma^i, \varphi)$ and $S_2^i = (\Gamma^i, \psi \Rightarrow \Delta^i)$. Thus $\Gamma^r, \exists p S_1^i \Rightarrow \forall p S^i$ and $S_2^i \cdot (\exists p S_1^i \Rightarrow \forall p S^i)$ belong to $\mathcal{D}_R^p$. As $(S_1^i, S_2^i/S_i^i)$ is an instance of $R$, $\exists p S^i$ contains a conjunct $\exists p S_1^i \land (\forall p S_1^i \rightarrow \exists p S_2^i)$. Therefore $\mathcal{D}_R^p$ derives $(\Gamma^r, \exists p S^i \Rightarrow \forall p S_1^i)$ and $(\Gamma^r, \exists p S_i^i \Rightarrow \forall p S_2^i)$. Since $\forall p S_i^i$ contains a disjunct $\forall p S_1^i \land \forall p S_2^i$, (4) follows.

**Lemma 6** For $R = L \circ \rightarrow$ all independent inductive properties and $(DPP)_R$ hold.

**Proof** For $(IPP)_R^3$ and $(IPP)_R^\forall$, let $R$ be an instance

$$\frac{S_1, S_2}{S} = \frac{\Gamma, \chi \Rightarrow \varphi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \psi \Rightarrow \Delta}$$

of $R = L \circ \rightarrow$. Given the definition of $\exists p S$ and $\forall p S$ in this case, we have to show that

$$\frac{\exists p S}{\exists p S} \vdash \Gamma, \chi \Rightarrow \varphi \Rightarrow \exists p S_1 \land (\forall p S_1 \rightarrow \exists p S_2) \quad \frac{\forall p S}{\exists p S} \vdash \Gamma, \chi \Rightarrow \forall p S_1 \land \forall p S_2, \varphi \Rightarrow \psi \Rightarrow \Delta.$$  

This follows, via applications of $R$, $R \circ$ and $L \circ$, from the fact that $(\Gamma, \chi \Rightarrow \exists p S_1)$ and $(\Gamma, \chi \Rightarrow \forall p S_2)$ as well as $(\Gamma, \chi \Rightarrow \exists p S_1 \Rightarrow \varphi)$ and $(\Gamma, \chi \Rightarrow \forall p S_2 \Rightarrow \Delta)$ belong to $\mathcal{D}_R^p$.

For $(IPN)_R^3$ we have to show three things, under the assumption that all sequents lower than $S$ satisfy the interpolant properties:
(1) \( \vdash S^a \Rightarrow \exists p(S^a \Rightarrow ) \) in case \( S^a \neq \emptyset \);

(2) \( \vdash S^a \Rightarrow \circ \exists p(S^a \setminus \{\alpha\}, \alpha \Rightarrow ) \) for all \( \alpha \in S^a \);

(3) \( \vdash S^a \Rightarrow \exists pS^0 \land (\circ \forall pS^0 \rightarrow \exists pS^1) \) for all \( \gamma = \circ \alpha \rightarrow \beta \) in \( S^a \).

(1) follows immediately from the assumption that all sequents lower than \( S \) satisfy the interpolant properties and the fact that \( (S^a \Rightarrow ) \ll S \), which holds if \( S^a \neq \emptyset \).

(2) Note that \( \{S^a \setminus \{\alpha\}, \alpha \Rightarrow \exists p(S^a \setminus \{\alpha\}, \alpha \Rightarrow )\} \) is derivable because all sequents lower than \( S \) are assumed to satisfy the interpolant properties. An application of \( R \circ \) followed by an application of \( L \circ \) gives the desired result.

(3) Assume that \( S = (\Pi, \gamma \Rightarrow \Sigma) \), where \( \gamma = \circ \alpha \rightarrow \beta \). We use the following abbreviations:

\[
S_\alpha \equiv_u S^0 = (S^a \setminus \{\gamma\} \Rightarrow \circ \alpha) = (\Pi \Rightarrow \circ \alpha)
\]

\[
S_\beta \equiv_u S^1 = (S^a \setminus \{\gamma\}, \beta \Rightarrow S^1) = (\Pi, \beta \Rightarrow \Sigma).
\]

Thus we have to show that \( (S^a \Rightarrow \exists pS_\alpha) \) and \( (S^a, \circ \forall pS_\alpha \Rightarrow \exists pS_\beta) \) are derivable. The former follows immediately from the fact that \( S_\alpha \) and \( S_\beta \) satisfy the interpolant properties, since they are below \( S \). By the same fact, \( (S^0_\alpha \Rightarrow \exists pS_\beta) \) is derivable. As \( (S^0_\beta \Rightarrow \exists pS_\beta) \) is equal to \( (\Pi, \beta \Rightarrow \exists pS_\beta) \), sequent \( (\Pi, \beta, \circ \forall pS_\alpha \Rightarrow \exists pS_\beta) \) is derivable too. An application of \( L \circ \gamma \) to this sequent and \( (S^0_\alpha, \forall pS_\alpha \Rightarrow S^1_\alpha) = (\Pi, \forall pS_\alpha \Rightarrow \circ \alpha) \), which by the same fact is derivable, shows that \( \vdash \Pi, \gamma, \forall pS_\alpha \Rightarrow \exists pS_\beta \), which is what we had to show.

For property \((\text{IPN})_\mathcal{R}^v\) we use the same notation as in (3) of the previous case and have to show that

\[
\vdash S^a, \circ \forall pS_\alpha, \forall pS_\beta \Rightarrow \exists pS^a.
\]

By the assumption that all sequents lower than \( S \) satisfy the interpolant properties, both \( (S^a_\alpha, \forall pS_\alpha \Rightarrow S^a_\alpha) = (\Pi, \forall pS_\alpha \Rightarrow \circ \alpha) \) and \( (S^a_\beta, \forall pS_\beta \Rightarrow S^a_\beta) = (\Pi, \beta, \forall pS_\beta \Rightarrow \Sigma) \) are derivable. Clearly, applications of weakening, \( L \circ \) and \( L \circ \gamma \) show that \( \vdash \Pi, \circ \alpha \rightarrow \beta, \circ \forall pS_\alpha, \forall pS_\beta \Rightarrow \Sigma \), which is what we had to show.

For \((\text{DPP})_\mathcal{R}^v\), in a partition of \( S \) in which \( S^i \) is principal, \( S^i \) has to be of the form \( S^i = (\Gamma^i, \circ \chi, \circ \varphi \rightarrow \psi \Rightarrow \Delta^i) \), where \( \Gamma^i \subseteq \Gamma \) and \( \Delta^i \subseteq \Delta \). We have to show that

\[
\mathcal{D}_R^p \vdash S^i \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset) \quad (5)
\]

We treat the case that \( \Delta^i = \emptyset \), the other case is analogous. Let the partitions of the premisses be given by \( S_1^i = (\Gamma^i, \chi \Rightarrow \varphi) \) and \( S_2^i = (\Gamma^i, \circ \chi, \psi \Rightarrow \Delta^i) \). Thus \( (\Gamma^i, \exists pS_1^i \Rightarrow \forall pS_1^i) \) and \( S_2^i \cdot (\exists pS_2^i \Rightarrow \forall pS_2^i) \) belong to \( \mathcal{D}_R^p \). Hence \( \mathcal{D}_R^p \) derives \( (\Gamma^i, \circ \exists pS_1^i \Rightarrow \forall pS_1^i) \). As \( (S_1^1 S_2^2 / S^i) \) is an instance of \( \mathcal{R} \), \( \exists pS^i \) contains a conjunct \( \circ \forall pS_1^i \land (\circ \exists pS_1^i \Rightarrow \exists pS_2^i) \). Therefore \( \mathcal{D}_R^p \) derives \( (\Gamma^i, \exists pS^i \Rightarrow \forall pS_1^i) \) and \( (\Gamma^i, \exists pS^i \Rightarrow \forall pS_2^i) \). Since \( \forall pS^i \) contains a disjunct \( \circ \forall pS_1^i \lor \forall pS_2^i \), \ref{5} follows.

**Lemma 7** For all modal and implication-modal rules \( \mathcal{R} \), the interpolant assignment satisfies \((\text{DPN})_\mathcal{R}^v\).

**Proof** \((\text{DPN})_\mathcal{R}^v\) is proved with induction to the depth of the derivation \( \mathcal{D} \) of \( S \) for all rules at the same time. Thus we have to show for any partition \( (S^i, S^i) \) of \( S \) such that \( S^i \) is nonprincipal for the last inference \( R \) of \( \mathcal{D} \): if all sequents below \( S \) satisfy the interpolant properties, then

\[
\vdash S^i \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \emptyset). \quad (6)
\]

Note that for the rules in \( \text{G4ip} \) \((\text{DPN})^v\_\mathcal{R}^v\) has already been established in [Lemhauer 2019a](#). If \( \mathcal{D} \) has depth one, \( S \) is an instance of an axiom. If the axiom is \( \Pi \_\perp \), then because \( S^i \) is not an instance of it, \( \perp \in S^\alpha \). Hence \ref{6}. If the axiom is \( Ax \), say \( S = (\Gamma, q \Rightarrow q, \Delta) \), then if \( q \in S^\alpha \cap S^* \), we have \ref{6}. Therefore assume otherwise: \( q \notin S^\alpha \cap S^* \). Because \( S^i \) is
not an instance of $Ax$, $q \notin S'^{ia} \cap S'^{is}$. Thus $q \in S'^{ia} \cap S'^{is}$ or $q \in S'^{ia} \cap S'^{is}$. Since $q$ occurs in $S'$, $q \neq p$. Therefore $\vdash q \Rightarrow S'^{is}$ and $q \in S'^{ia}$, and thus $\vdash \exists q S'^{is}$, or $\vdash S'^{ia} \Rightarrow q$ and \( q \in S'^{is} \), and thus $\vdash q \Rightarrow \forall q S'^{is}$. In both cases (6) follows, which completes the basis case of the induction.

Suppose the derivation of $S$ has depth bigger than one and ends with an application of $\mathcal{R}$. We show that $(DPN)$ holds for $S$. If $\mathcal{R}$ is not a modal or implication-modal rule, then we already know that $(DPN)_{\mathcal{R}}$ holds for all $S$. Therefore suppose $\mathcal{R}$ is such a rule. We consider the four rules separately.

If $\mathcal{R}$ is $\mathcal{R} \circ \mathcal{R}$, $S = (\Gamma \Rightarrow \varnothing \varphi)$ and the premise of the last inference is $S_1 = (\Gamma \Rightarrow \varnothing \varphi)$. Since we consider $(DPN)_{\mathcal{R}}$, $S' = (\Gamma' \Rightarrow \varnothing)$. We have to show that $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\exists \varphi S' \Rightarrow \varnothing \varphi$. Consider the partition of $S_1$ with $S'_1 = S_1$. We have $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\exists \varphi S'_1 \Rightarrow \varnothing$ by the induction hypothesis, which implies what we had to show via the consecutive application of $\mathcal{R} \circ \mathcal{R}$ and $\mathcal{L} \circ \mathcal{R}$.

If $\mathcal{R}$ is $\mathcal{L} \circ \mathcal{R}$, $S = (\Gamma, \varnothing \psi \Rightarrow \varnothing \varphi)$ and the premise of the last inference is $S_1 = (\Gamma, \psi \Rightarrow \varnothing \varphi)$. Consider a partition of $S$ such that $S'$ is not principal for $\mathcal{L} \circ \mathcal{R}$. Thus $S'_1 = (\Gamma' \Rightarrow \varnothing \varphi)$ or $S'_2 = (\Gamma' \Rightarrow \varnothing \psi)$ for some $\Gamma' \subseteq \Gamma$. We have to show that $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\varnothing \varphi$, $\exists \varphi S' \Rightarrow \forall \varphi S'$. In the first case, and $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\varnothing \varphi$, $\exists \varphi S' \Rightarrow \varnothing \varphi$. In both cases we can partition $S'_2$ such that $S'_1 = S_1$. If $S'_2 = (\Gamma' \Rightarrow \varnothing \varphi)$, we have $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\varnothing \psi$, $\exists \varphi S'_2 \Rightarrow \varnothing \varphi$ by the induction hypothesis. An application of $\mathcal{L} \circ \mathcal{R}$ gives $\mathcal{D}^p_\mathcal{R} \vdash \Gamma'$, $\varnothing \psi$, $\exists \varphi S'_2 \Rightarrow \varnothing \varphi$, and since $S'_2 = S_1$, this is what we had to show. The case that $S'_2 = (\Gamma' \Rightarrow \varnothing \varphi)$ is analogous.

If $\mathcal{R}$ is $\mathcal{L} \circ \mathcal{R}^{-1}$, the last inference is of the following form:

$$\frac{S_1 \quad \frac{S_2}{\Gamma \Rightarrow \varnothing \varphi \quad \Gamma, \varnothing \psi \Rightarrow \Delta}}{\Gamma, \varnothing \varphi \Rightarrow \psi \Rightarrow \Delta}$$

As $\mathcal{R}$ is not applicable to $S'$, $S' = (\Gamma \Rightarrow \Delta')$ and $S'' = (\Gamma', \varnothing \psi \Rightarrow \Delta')$. We treat the case that $\Delta' = \varnothing$, the other case is analogous. Thus to show (6), we have to show that

$$\vdash \Gamma', \varnothing \varphi \Rightarrow \psi, \exists \varphi S' \Rightarrow \forall \varphi S'.$$

(7)

Let $S'_1 = (\Gamma' \Rightarrow \varnothing \varphi)$ and $S'_2 = S'$. By the induction hypothesis,

$$\vdash \Gamma', \exists \varphi S'_1 \Rightarrow \varnothing \varphi \quad \vdash \Gamma', \psi, \exists \varphi S'_2 \Rightarrow \forall \varphi S'_2.$$ 

Therefore $\vdash \Gamma', \varnothing \varphi \Rightarrow \psi, \exists \varphi S'_1, \exists \varphi S'_2 \Rightarrow \forall \varphi S'$. Since $S'_2 = S'_1$, for (7) it suffices to show that $\exists \varphi S''$ derives $\exists \varphi S'_1$. If $S'' = \varnothing$, then $S' = S'_1$, and if $S'' \neq \varnothing$, then $\exists \varphi S'_1$ is a conjunct of $\exists \varphi S''$, which is a conjunct of $\exists \varphi S'$.

If $\mathcal{R}$ is $\mathcal{L} \circ \mathcal{R}^{-1}$, the last inference is of the following form:

$$\frac{S_1 \quad \frac{S_2}{\Gamma, \chi \Rightarrow \varnothing \varphi \quad \Gamma, \varnothing \chi, \psi \Rightarrow \Delta}}{\Gamma, \varnothing \chi, \varnothing \varphi \Rightarrow \psi \Rightarrow \Delta}$$

We distinguish three cases.

(1) If $S' = (\Gamma \Rightarrow \Delta')$, then let $S'_1 = (\Gamma \Rightarrow \varnothing \varphi)$ and $S'_2 = S'$. We treat the case that $\Delta' \neq \varnothing$, the other case is analogous. Thus to show (6), we have to show that

$$\vdash \Gamma', \varnothing \varphi \Rightarrow \psi, \exists \varphi S' \Rightarrow \Delta'.$$

(8)

By the induction hypothesis,

$$\vdash \Gamma', \varnothing \varphi \Rightarrow \psi, \exists \varphi S'_1 \Rightarrow \Delta'. $$

Therefore $\vdash \Gamma', \varnothing \varphi \Rightarrow \psi, \exists \varphi S'_1, \exists \varphi S'_2 \Rightarrow \Delta'$. If $\Delta' = \varnothing$, then $S' = S'_1 = S'_2$, which implies (6). Therefore assume that $\Delta' = S''$ is not empty. Thus $\exists \varphi S'_1$ is a conjunct of $\exists \varphi S'$, which is a conjunct of $\exists \varphi S''$. Together with $S'_2 = S'$ this implies (6).
(2) If $S^i = (\Gamma^i \land \chi \Rightarrow \Delta^i)$, then let $S^i_1 = (\Gamma^i, \chi \Rightarrow )$ and $S^i_2 = S^i$. We treat the case that $\Delta^r = \emptyset$, the other case is analogous. Thus to show (9), we have to show that
\[ \vdash \Gamma^r, \, \circ \varphi \rightarrow \psi, \exists p S^i \Rightarrow \forall p S^i. \] (9)
By the induction hypothesis, $\vdash \Gamma^r, \exists p S^i_1 \Rightarrow \circ \varphi$ and $\vdash \Gamma^r, \psi, \exists p S^i_2 \Rightarrow \forall p S^i_2$ are derivable. An application of $L \circ \rightarrow$ gives
\[ \vdash \Gamma^r, \, \circ \varphi \rightarrow \psi, \circ \exists p S^i_1, \exists p S^i_2 \Rightarrow \forall p S^i_2. \]
Note that because $S^i_1 = (S^i \setminus \{ \circ \chi \}, \chi \Rightarrow )$, $\circ \exists p S^i_1$ is a conjunct of $\exists p S^i$, which is a conjunct of $\exists p S^i$. Since also $\exists p S^i_1 \Rightarrow \exists \exists p S^i_1$ is derivable and $S^i_2 = S^i$, we obtain (9).

(3) If $S^i = (\Gamma^i, \circ \varphi \rightarrow \psi \Rightarrow \Delta^i)$, then let $S^i_1 = (\Gamma^i \Rightarrow \circ \varphi)$ and $S^i_2 = (\Gamma^i, \psi \Rightarrow \Delta^i)$. We treat the case that $\Delta^r = \emptyset$, the other case is analogous. Thus to show (10), we have to show that
\[ \vdash \Gamma^r, \, \circ \chi, \exists p S^i \Rightarrow \forall p S^i. \] (10)
By the induction hypothesis, both $(\Gamma^r, \chi, \exists p S^i_1 \Rightarrow \forall p S^i_1)$ and $S_3 = (\Gamma^r, \circ \chi, \exists p S^i_2 \Rightarrow \forall p S^i_2)$ are derivable. Thus so are $S_4 = (\Gamma^r, \chi, \exists p S^i_1 \Rightarrow \circ \forall p S^i_1)$ and $(\Gamma^r, \circ \chi, \exists p S^i_2 \Rightarrow \circ \forall p S^i_2)$ by applications of $R \circ$ and $L \circ$. An application of $L \circ \rightarrow$ to $S_4$ and $S_3$ shows the derivability of $(\Gamma^r, \circ \chi, \exists p S^i_1, \circ \forall p S^i_1 \Rightarrow \exists p S^i_2 \Rightarrow \forall p S^i_2)$. Therefore
\[ \vdash \Gamma^r, \, \circ \chi, \exists p S^i_1, \circ \forall p S^i_1 \Rightarrow \exists p S^i_2 \Rightarrow \circ \forall p S^i_1 \land \forall p S^i_2. \]
Let $\gamma = \circ \varphi \rightarrow \psi$. Since $S^i_1 = (S^i)_{\circ 0}$ and $S^i_2 = (S^i)_{\circ 1}$, $\exists p S^i_1$ and $\circ \forall p S^i_1 \rightarrow \exists p S^i_2$ are conjuncts of $\exists p S^i$, which is a conjunct of $\exists p S^i$. And since $\circ \forall p S^i_1 \land \forall p S^i_2$ is a disjunct of $\exists p S^i$ and thus of $\exists p S^i$, we can conclude (10).

6.8 Uniform interpolation for Lax Logic

Theorem 6 PLL has uniform interpolation.

Proof As the results in Section 6.7 show, the interpolant assignment for GLL given in Section 6.6 is sound. Theorem 5 then implies the result.

7 Conclusion

In this paper we have developed proof theory for propositional Lax Logic PLL. In particular, we have provided a cut free terminating sequent calculus G4iLL for the logic, which is used to prove that Lax Logic has uniform interpolation. The latter was established by a proof-theoretic method that defines uniform interpolants in terms of the rules of the calculus.

There are many open questions related to uniform interpolation in intuitionistic modal logics. We hope that the methods in this paper and the closely related (Iemhoff, 2019; Jalali and Tabatabai, 2018) will be useful in the study of other logics in this class, and perhaps also in the investigation of some of the more general questions in Universal Proof Theory.

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