Lagrangian and Hamiltonian Mechanical Systems on Para-Quaternionic Kähler Manifolds

Mehmet Tekkoyun *

Department of Mathematics, Pamukkale University,
20070 Denizli, Turkey

January 20, 2010

Abstract

In the present paper, we introduce para-quaternionic Kähler analogue of Lagrangian and Hamiltonian mechanical systems. Finally, the geometrical-physical results related to para-quaternionic Kähler mechanical systems are also given.

Keywords: Para-Quaternionic Kähler Geometry, Lagrangian and Hamiltonian Mechanical Systems.

MSC: 53C15, 70H03, 70H05.

*Corresponding author. E-mail address: tekkoyun@pau.edu.tr; Tel: +902582953616; Fax: +902582963535
1 Introduction

Modern differential geometry plays an important role to explain the dynamics of Lagrangians. So, if \( Q \) is an \( m \)-dimensional configuration manifold and \( L : TQ \to \mathbb{R} \) is a regular Lagrangian function, then it is well-known that there is a unique vector field \( \xi \) on \( TQ \) such that dynamics equations is given by

\[
i_\xi \Phi_L = dE_L
\]

where \( \Phi_L \) indicates the symplectic form. The triple \( (TQ, \Phi_L, \xi) \) is called Lagrangian system on the tangent bundle \( TQ \).

Also, modern differential geometry provides a good framework in which develop the dynamics of Hamiltonians. Therefore, if \( Q \) is an \( m \)-dimensional configuration manifold and \( H : T^*Q \to \mathbb{R} \) is a regular Hamiltonian function, then there is a unique vector field \( X \) on \( T^*Q \) such that dynamic equations are given by

\[
i_X \Phi = dH
\]

where \( \Phi \) indicates the symplectic form. The triple \( (T^*Q, \Phi, X) \) is called Hamiltonian system on the cotangent bundle \( T^*Q \).

Nowadays, there are many studies about Lagrangian and Hamiltonian dynamics, mechanics, formalisms, systems and equations \([1, 2, 3, 4, 5, 6]\) and there in. There are real, complex, paracomplex and other analogues. As we know it is possible to produce different analogous in different spaces.

Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian’s defining relation is most succinctly written as:

\[
i^2 = j^2 = k^2 = ijk = -1
\]
If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [7].

The algebra $B$ of split quaternions is a four-dimensional real vector space with basis $\{1, i, s, t\}$ given by

\[ i^2 = -1, \quad s^2 = 1 = t^2, \quad is = t = -si. \]

This carries a natural indefinite inner product given by $< p, q > = \text{Re} \overline{pq}$, where $p = x + iy + su + tv$ has $\overline{p} = x - iy - su - tv$. We have $\|p\|^2 = x^2 + y^2 - s^2 - t^2$, so a metric of signature $(2, 2)$. This norm is multiplicative, $\|pq\|^2 = \|p\|^2 \|q\|^2$, but the presence of elements of length zero means that $B$ contains zero divisors. The fundamental structures $1, i, s, t$ are not the only split quaternions with square $\pm 1$. Using the multiplication rules for $B$, one can calculate

\[ p^2 = -1 \text{ if and only if } p = iy + su + tv, \quad y^2 - s^2 - t^2 = 1, \]

\[ p^2 = +1 \text{ if and only if } p = iy + su + tv, \quad y^2 - s^2 - t^2 = -1 \text{ or } p = \pm 1. \]

The right $B$-module $B^n \cong \mathbb{R}^{4n}$ inherits the inner product $< \xi, \eta > = \text{Re} \overline{\xi^T \eta}$ of signature $(2n, 2n)$. The automorphism group of $(B^n, \langle \cdot, \cdot \rangle)$ is $Sp(n, B) = \{ A \in M_n(B) : A^T A = 1 \}$ which is a Lie group isomorphic to $Sp(2n, \mathbb{R})$, the symmetries of a symplectic vector space $(\mathbb{R}^{2n}, \omega)$. Especially, $Sp(1, B) \cong SL(2, \mathbb{R})$ is the pseudo-sphere of $B = \mathbb{R}^{2,2}$. The Lie algebra of $Sp(n, B)$ is $sp(n, B) = \{ A \in M_n(B) : A + A^T = 0 \}$, so $sp(1, B) = \text{Im} B$. The group $Sp(n, B) \times Sp(1, B)$ acts on $B^n$ via
\[(A, p)\xi = A\xi \bar{p}.\]  

(3)

For detail see [8].

It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics. Therefore, in the present paper, we present equations related to Lagrangian and Hamiltonian mechanical systems on para-quaternionic Kähler manifold.

Throughout this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. \(\mathcal{F}(M), \chi(M)\) and \(\Lambda^1(M)\) denote the set of functions on \(M\), the set of vector fields on \(M\) and the set of 1-forms on \(M\), respectively.

## 2 Para-Quaternionic Kähler Manifolds

Here, we recall hypersymplectic manifolds and para-quaternionic Kähler manifolds given in [8].

Let \(m = 4n\), identify \(\mathbb{R}^{4n}\) with \(\mathbb{B}^n\) and consider \(\hat{G} = \text{Sp}(n, B) \subset \text{GL}(4n, \mathbb{R})\). An \(\text{Sp}(n, B)\)-structure \(\text{Sp}_B(M)\) on \(M\) defines a metric \(g\) of signature \((2n, 2n)\) by \(g(u(v), u(w)) = \langle v, w \rangle\).

The right action of \(i, s\) and \(t\) on \(\mathbb{B}^n\) define endomorphisms \(F, G\) and \(H\) of \(TxM\) satisfying

\[F^2 = -I, \quad G^2 = H^2 = I, \quad FG = H = -GF,\]  

(4)

and the compatibility equations, for \(X, Y \in TxM\)

\[g(FX, FY) = g(X, Y), \quad g(GX, GY) = g(HX, HY),\]  

(5)
where $I$ denotes the identity tensor of type $(1,1)$ in $M$, and $g$ is Riemann metric. Using $(4)$, we obtain three 2-forms $\omega_F, \omega_G$ and $\omega_H$ given by

$$
\omega_F(X,Y) = g(FX,Y), \quad \omega_G(X,Y) = g(GX,Y), \quad \omega_H(X,Y) = g(HX,Y).
$$

The manifold $M$ is said to be hypersymplectic if the 2-forms $\omega_F, \omega_G$ and $\omega_H$ are all closed:

$$
d\omega_F = 0, d\omega_G = 0 \quad \text{and} \quad d\omega_H = 0.
$$

Now we think of the larger structure group $Sp(n, B)Sp(1, B)$ acting on $B^n = R^{4n}$ via $(3)$. Again we have metric of neutral signature $(2n, 2n)$, but now we can not distinguish the endomorphisms $F, G$ and $H$. Instead we have a bundle $G$ of endomorphisms of $TM$ that locally admits a basis $\{F, G, H\}$ satisfying $(4)$ and $(5)$. $\{F, G, H\}$ is called a canonical local basis of the bundle $V$ in any coordinate neighborhood $U$ of $M$. Then $V$ is called a para-quaternionic structure in $M$. The pair $(M, V)$ denotes a para-quaternionic manifold with $V$. A para-quaternionic manifold $M$ is of dimension $m = 4n \quad (n \geq 1)$. A para-quaternionic structure $V$ with such a Riemannian metric $g$ is called a para-quaternionic metric structure. A manifold $M$ with a para-quaternionic metric structure $\{g, V\}$ is called a para-quaternionic metric manifold. The triple $(M, g, V)$ denotes a para-quaternion metric manifold. If $n > 1$, we say that $M$ is para-quaternionic Kähler if its holonomy lies in $Sp(n, B)Sp(1, B)$.

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1,n}$ be a real coordinate system on a neighborhood $U$ of $M$, and let $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}\right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over $R$ of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of $M$, respectively. Taking into
consideration (4), then we can obtain the expressions as follows:

\[
\begin{align*}
F(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_{n+i}},
F(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_i},
F(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{3n+i}},
F(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_{2n+i}}
\end{align*}
\]

\[
\begin{align*}
G(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_{2n+i}},
G(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{3n+i}},
G(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_i},
G(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_{n+i}}
\end{align*}
\]

\[
\begin{align*}
H(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_{3n+i}},
H(\frac{\partial}{\partial x_{n+i}}) = \frac{\partial}{\partial x_{2n+i}},
H(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{n+i}},
H(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_i}
\end{align*}
\]

A canonical local basis \(\{F^*, G^*, H^*\}\) of \(V^*\) of the cotangent space \(T^*(M)\) of manifold \(M\) satisfies the condition as follows:

\[
F^{*2} = -I,\ G^{*2} = H^{*2} = I,\ F^*G^* = H^* = -G^*F^*,
\]

(6)

defining by

\[
\begin{align*}
F^*(dx_i) &= dx_{n+i},\ F^*(dx_{n+i}) = -dx_i,\ F^*(dx_{2n+i}) = dx_{3n+i},\ F^*(dx_{3n+i}) = -dx_{2n+i},
G^*(dx_i) &= dx_{2n+i},\ G^*(dx_{n+i}) = -dx_{3n+i},\ G^*(dx_{2n+i}) = dx_i,\ G^*(dx_{3n+i}) = -dx_{n+i},
H^*(dx_i) &= dx_{3n+i},\ H^*(dx_{n+i}) = dx_{2n+i},\ H^*(dx_{2n+i}) = dx_{n+i},\ H^*(dx_{3n+i}) = dx_i.
\end{align*}
\]

3 Lagrangian Mechanical Systems

Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis \(\{F, G, H\}\) of \(V\) on para-quaternionic Kähler manifold \((M, g, V)\).

Firstly, let \(F\) take a local basis element on the para-quaternionic Kähler manifold \((M, g, V)\), and \(\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}\) be its coordinate functions. Let semispray be the vector field \(X\) determined by

\[
X = X_i \frac{\partial}{\partial x_i} + X_{n+i} \frac{\partial}{\partial x_{n+i}} + X_{2n+i} \frac{\partial}{\partial x_{2n+i}} + X_{3n+i} \frac{\partial}{\partial x_{3n+i}},
\]

(7)
where $X^i = \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time $t$. The vector field defined by

$$V_F = F(X) = X^i \frac{\partial}{\partial x_{n+i}} - X^{n+i} \frac{\partial}{\partial x_i} + X^{2n+i} \frac{\partial}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial}{\partial x_{2n+i}}$$

is named Liouville vector field on the para-quaternionic Kähler manifold $(M, g, V)$. The maps given by $T, P : M \to \mathbb{R}$ such that $T = \frac{1}{2} m_i (\dot{x}_i^2 + \dot{x}_{n+i}^2 + \dot{x}_{2n+i}^2 + \dot{x}_{3n+i}^2), P = m_i g h$ are said to be the kinetic energy and the potential energy of the system, respectively. Here $m_i, g$ and $h$ stand for mass of a mechanical system having $m$ particles, the gravity acceleration and distance to the origin of a mechanical system on the para-quaternionic Kähler manifold $(M, g, V)$, respectively.

Then $L : M \to \mathbb{R}$ is a map that satisfies the conditions: i) $L = T - P$ is a Lagrangian function, ii) the function determined by $E^F_L = V_F(L) - L$, is energy function.

The function $i_F$ induced by $F$ and denoted by

$$i_F \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^r \omega(X_1, ..., F X_i, ..., X_r),$$

is called vertical derivation, where $\omega \in \wedge^r M, X_i \in \chi(M)$. The vertical differentiation $d_F$ is given by

$$d_F = [i_F, d] = i_F d - di_F$$

where $d$ is the usual exterior derivation. For $\Phi^F_L$, the closed para-quaternionic Kähler form is the closed 2-form given by $\Phi^F_L = -dd_F L$ such that

$$d_F = \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{3n+i}} dx_{2n+i} - \frac{\partial}{\partial x_{2n+i}} dx_{3n+i} : \mathcal{F}(M) \to \wedge^1 \mathcal{M}.$$

6
Then we have

\[ \Phi^F_L = -\frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{2n+i} \\
+ \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_n+j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_i \\
- \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i} \\
+ \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{3n+i} \\
- \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} \\
+ \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{3n+i}. \]

Let \( X \) be the second order differential equation (semispray) given by (7). Then we calculate

\[
i_X \Phi^F_L = -X_i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_i + X_i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X_i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_{n+i} \\
- X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + X^{i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{3n+i} \\
- X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_{2n+i} \\
+ X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \\
+ X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+i} \\
- X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \\
+ X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \\
+ X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{n+i} \\
- X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \\
+ X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \\
- X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \]

Since the closed quaternion Kähler form \( \Phi^F_L \) on \( (M, V) \) is the symplectic structure, it is found

\[ E^F_L = V_F(L) - L = X^i \frac{\partial L}{\partial x^i} - X^{n+i} \frac{\partial L}{\partial x^{n+i}} + X^{2n+i} \frac{\partial L}{\partial x^{2n+i}} - X^{3n+i} \frac{\partial L}{\partial x^{3n+i}} - L \]
and hence

\[ dE^F_L = X_i \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j + X^2_{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{3n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_i \partial x_{2n+i}} dx_j \]

+ \left( X^i \frac{\partial^2 L}{\partial x_{i} \partial x_{n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{i} \partial x_{n+i}} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{i} \partial x_{3n+i}} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{i} \partial x_{2n+i}} dx_{n+j} \right) \]

Using (1), we find the expression as follows:

\[ -X^i \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_i \partial x_{3n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_{2n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{3n+i}} dx_{2n+j} \]

\[ -X^{2n+i} \frac{\partial^2 L}{\partial x_i \partial x_{2n+i}} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_i \partial x_{2n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_i \partial x_{3n+i}} dx_{3n+j} \]

+ \left( X^{i} \frac{\partial^2 L}{\partial x_{i} \partial x_{n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{i} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{i} \partial x_{2n+i}} dx_{3n+j} \right) = 0. \]

If a curve given by \( \alpha : R \to M \) is considered to be an integral curve of \( X \), then we obtain the equation given by

\[ -X^i \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_i \partial x_{3n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_i \partial x_{2n+i}} dx_j \]

\[ + \left( X^i \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_i \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_i \partial x_{2n+i}} dx_{3n+j} \right) = 0, \]
alternatively

\[-X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \]

\[+ X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \]

\[-X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \]

\[+ X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \]}

\[dx_j + \frac{\partial L}{\partial x_j} dx_j \]

\[dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} \]

\[dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} \]

\[dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0. \]

Then we have the equations

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_i} \right) - \frac{\partial L}{\partial x_{n+i}} = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{n+j}} \right) + \frac{\partial L}{\partial x_j} = 0, \] (8)

such that the equations calculated in (8) are named Euler-Lagrange equations constructed on para-quaternionic Kähler manifold \((M, g, V)\) by means of \(\Phi^F_L\) and thus the triple \((M, \Phi^F_L, X)\) is called a mechanical system on para-quaternionic Kähler manifold \((M, g, V)\).

Secondly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of \(\Phi^G_L\) on para-quaternionic Kähler manifold \((M, g, V)\).

Take \(G\). It is another local basis element on the para-quaternionic Kähler manifold \((M, g, V)\). Let \(X\). It is the semispray in (7). In the case, the vector field determined by

\[ V_G = G(X) = X^i \frac{\partial}{\partial x_{2n+i}} - X^{n+i} \frac{\partial}{\partial x_{2n+i}} + X^{2n+i} \frac{\partial}{\partial x_{n+i}} - X^{3n+i} \frac{\partial}{\partial x_{n+i}} \]

is Liouville vector field on the para-quaternionic Kähler manifold \((M, g, V)\). The operator given by \(E^G_L = V_G(L) - L\) is energy function. Then the function \(i_G\) induced by \(G\) and given by

\[ i_G \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., GX_i, ..., X_r) \]

is vertical derivation, where \(\omega \in \Lambda^r M, X_i \in \chi(M)\). The vertical differentiation \(d_G\) is given by

\[ d_G = [i_G, d] = i_G d - d i_G \]

9
where \( d \) is the usual exterior derivation. Since taking into consideration \( G \), the closed para-quaternionic Kähler form is the closed 2-form given by \( \Phi^G_L = -dd_G L \) such that

\[
d_G = \frac{\partial}{\partial x_{2n+i}} dx_i - \frac{\partial}{\partial x_{3n+i}} dx_{n+i} + \frac{\partial}{\partial x_i} dx_{2n+i} - \frac{\partial}{\partial x_{n+i}} d_{3n+i} : F(M) \to \wedge^3 M.
\]

Then we get

\[
\Phi^G_L = -\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{2n+i} \\
+ \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+i} \partial x_{2n+i}} dx_{n+i} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+i} \partial x_{3n+i}} dx_{n+i} \wedge dx_{2n+i} \\
- \frac{\partial^2 L}{\partial x_{n+i} \partial x_i} dx_{n+i} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+i} \partial x_{2n+i}} dx_{n+i} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{n+i} \partial x_{3n+i}} dx_{n+i} \wedge dx_{2n+i} \\
+ \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_{n+i} \wedge dx_{3n+i} \wedge dx_{2n+i}.
\]

Considering \( [7] \), it holds

\[
i_X \Phi^G_L = -X^n \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} \delta^n_i dx_i + X^n \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} \delta^n_i dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{n+i} \partial x_{3n+i}} \delta^n_i dx_{2n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} \delta^n_i dx_{3n+i} \\
- X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^n \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^n \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \\
+ X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{3n+i} \\
+ X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
+ X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
- X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} + X^n \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+i} \\
Since the closed para-quaternionic Kähler form $\Phi_G^G$ on $M$ is the symplectic structure, it follows

$$E_G^G = V_G(L) - L = X^i \frac{\partial L}{\partial x_2n+i} - X^{n+i} \frac{\partial L}{\partial x_3n+i} + X^{2n+i} \frac{\partial L}{\partial x_i} - X^{3n+i} \frac{\partial L}{\partial x_{n+i}} - L,$$

and thus

$$dE_G^G = X^i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_j + X^{2n+i} \frac{\partial L}{\partial x_j} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_j$$

$$+ X^i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{j} \partial x_{n+i}} dx_{n+j} - X^{3n+i} \frac{\partial L}{\partial x_{n+i} \partial x_{n+i}} dx_{n+j}$$

By means of (1), we find

$$-X^i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_{n+j} - X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{2n+j} + X^i \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_{3n+j}$$

$$-X^{n+i} \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_j + X^{n+i} \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{j} \partial x_{n+i}} dx_{2n+j}$$

$$+ X^{n+i} \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{j} \partial x_{n+i}} dx_{3n+j}$$

$$-X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{j} \partial x_{n+i}} dx_{2n+j}$$

$$+ X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+i} \partial x_{3n+i}} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{j} \partial x_{n+i}} dx_{3n+j}$$

$$+ \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0.$$
If a curve, given by $\alpha : R \to M$, is an integral curve of $X$, then we present

$$-X^i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_n} \, dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \, dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \, dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \, dx_j + X^n i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} \, dx_n + X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \, dx_n + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \, dx_n + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \, dx_n$$

or

$$-[X^i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}] \, dx_j + \frac{\partial L}{\partial x_j} \, dx_j + X^n i \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} \, dx_n + X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \, dx_n + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \, dx_n + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \, dx_n$$

Then the equations are obtained:

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_{2n+i}} = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{n+i}} \right) + \frac{\partial L}{\partial x_{3n+i}} = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{2n+i}} \right) - \frac{\partial L}{\partial x_{2n+i}} = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_{3n+i}} = 0.$$

Hence the equations introduced in (9) are named Euler-Lagrange equations constructed by means of $\Phi^G$ on para-quaternionic Kähler manifold $(M, g, V)$ and hence the triple $(M, \Phi^G, X)$ is said to be a mechanical system on para-quaternionic Kähler manifold $(M, g, V)$.

Thirdly, we present Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi^H$ on para-quaternionic Kähler manifold $(M, g, V)$.

Let $H$ be a local basis element on the para-quaternionic Kähler manifold $(M, g, V)$. Consider $X$ given by (7). So, Liouville vector field on the para-quaternionic Kähler manifold $(M, g, V)$ is the vector field determined by
The function given by $E^H_L = V_H(L) - L$ is energy function. The operator $i_H$ induced by $H$ and given by

$$i_H \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., HX_i, ..., X_r),$$

is named vertical derivation, where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The vertical differentiation $d_H$ is given by

$$d_H = [i_H, d] = i_H d - d i_H,$$

where $d$ is the usual exterior derivation. Taking $H$, the closed para-quaternionic Kähler form is the closed 2-form given by $\Phi^H_L = -dd_H L$ such that

$$d_H = \frac{\partial}{\partial x_{3n+i}} dx_i + \frac{\partial}{\partial x_{2n+i}} dx_{n+i} + \frac{\partial}{\partial x_{n+i}} dx_{2n+i} + \frac{\partial}{\partial x_i} dx_{3n+i} : \mathcal{F}(M) \to \wedge^3 M.$$

Then we find

$$
\Phi^H_L = -\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_i - \frac{\partial^2 L}{\partial x_{2n+i} \partial x_{2n+i}} dx_j \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{n+i} \partial x_{n+i}} dx_j \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{3n+i}.$$

Using (7), we calculate
\[ i X \Phi^H = -X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta^j_i dx_{3n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{n+i} \]

\[ + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta^j_i dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+i} \]

\[ + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \]

\[ - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{3n+i} \]

\[ + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \]

\[ + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{3n+i} \]

\[ - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta^j_i dx_{2n+i} \]

Since the closed para-quaternionic Kähler form \( \Phi^H \) on \( M \) is the symplectic structure, it is obtained

\[ E^H_L = V_H(L) - L = X^i \frac{\partial L}{\partial x_{3n+i}} + X^{n+i} \frac{\partial L}{\partial x_{2n+i}} + X^{2n+i} \frac{\partial L}{\partial x_{n+i}} + X^{3n+i} \frac{\partial L}{\partial x_i} - L. \]

Thus we get

\[ dE^H_L = X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \]

\[ + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{n+j} \]

\[ + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} \]

\[ + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{3n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{3n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+j} \]

\[ - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j}. \]
Using (1), we calculate the expression as follows:

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

If a curve, shown by \( \alpha : R \rightarrow M \), is an integral curve of \( X \), then it follows

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

or alternatively

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

\[-X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^n + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j
\]

Then we obtained the equations

\[ \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{n+i}} \right) = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{2n+i}} \right) = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{n+i}} \right) = 0, \quad \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_{3n+i}} \right) = 0. \]
Thus the equations introduced by (10) infer Euler-Lagrange equations constructed by means of \( \Phi^H_L \) on para-quaternionic Kähler manifold \((M, g, V)\) and then the triple \((M, \Phi^H_L, X)\) is named a mechanical system on para-quaternionic Kähler manifold \((M, g, V)\).

## 4 Hamiltonian Mechanical Systems

Here, we present Hamiltonian equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on para-quaternionic Kähler manifold \((M, g, V^*)\).

Firstly, let \((M, g, V^*)\) be a para-quaternionic Kähler manifold. Suppose that an element of para-quaternionic structure \(V^*\), a Liouville form and a 1-form on para-quaternionic Kähler manifold \((M, g, V^*)\) are shown by \(F^*, \lambda_{F^*}\) and \(\omega_{F^*}\), respectively.

Consider \(\omega_{F^*} = \frac{1}{2}(x_idx_i + x_{n+i}dx_{n+i} + x_{2n+i}dx_{2n+i} + x_{3n+i}dx_{3n+i})\). Then we have \(\lambda_{F^*} = F^*(\omega_{F^*}) = \frac{1}{2}(x_idx_{n+i} - x_{n+i}dx_i + x_{2n+i}dx_{3n+i} - x_{3n+i}dx_{2n+i})\). It is concluded that if \(\Phi_{F^*}\) is a closed para-quaternionic Kähler form on para-quaternionic Kähler manifold \((M, g, V^*)\), then \(\Phi_{F^*}\) is also a symplectic structure on para-quaternionic Kähler manifold \((M, g, V^*)\).

Take \(X\). It is Hamiltonian vector field associated with Hamiltonian energy \(H\) and determined by (7).

Then

\[
\Phi_{F^*} = -d\lambda_{F^*} = dx_{n+i} \wedge dx_i + dx_{3n+i} \wedge dx_{2n+i},
\]

and

\[
i_X \Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i}dx_i - X^idx_{n+i} + X^{3n+i}dx_{2n+i} - X^{2n+i}dx_{3n+i}. \tag{11}
\]

Furthermore, the differential of Hamiltonian energy is obtained by

\[
dH = \frac{\partial H}{\partial x_i}dx_i + \frac{\partial H}{\partial x_{n+i}}dx_{n+i} + \frac{\partial H}{\partial x_{2n+i}}dx_{2n+i} + \frac{\partial H}{\partial x_{3n+i}}dx_{3n+i}. \tag{12}
\]
With respect to (2), if equaled (11) and (12), the Hamiltonian vector field is found as follows:

\[
X = -\frac{\partial H}{\partial x_{n+i}} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_{n+i}} - \frac{\partial H}{\partial x_{3n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial H}{\partial x_{2n+i}} \frac{\partial}{\partial x_{3n+i}}. \tag{13}
\]

Assume that a curve

\[\alpha : I \subset \mathbb{R} \to M\]

be an integral curve of the Hamiltonian vector field \(X\), i.e.,

\[X(\alpha(t)) = \dot{\alpha}, \quad t \in I. \tag{14}\]

In the local coordinates, it is obtained that

\[\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i})\]

and

\[\dot{\alpha}(t) = \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt} \frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt} \frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt} \frac{\partial}{\partial x_{3n+i}}. \tag{15}\]

Taking (14), if we equal (13) and (15), it holds

\[\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dx_{2n+i}}{dt} = -\frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_{2n+i}}. \tag{16}\]

Hence, the equations introduced in (16) are named \textit{Hamiltonian equations} with respect to component \(F^*\) of para-quaternionic structure \(V^*\) on para-quaternionic Kähler manifold \((M, g, V^*)\), and then the triple \((M, \Phi_{F^*}, X)\) is said to be a \textit{Hamiltonian mechanical system} on para-quaternionic Kähler manifold \((M, g, V^*)\).

Secondly, let \((M, g, V^*)\) be a para-quaternionic Kähler manifold. Assume that a component of para-quaternion structure \(V^*\), a Liouville form and a 1-form on para-quaternionic Kähler manifold \((M, g, V^*)\) are denoted by \(G^*\), \(\lambda_{G^*}\) and \(\omega_{G^*}\), respectively.
Take $\omega_{G^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} - x_{2n+i} dx_{2n+i} - x_{3n+i} dx_{3n+i})$. Then we calculate $\lambda_{G^*} = G^*(\omega_{G^*}) = \frac{1}{2}(x_i dx_{2n+i} - x_{n+i} dx_{3n+i} - x_{2n+i} dx_i + x_{3n+i} dx_{n+i})$. It is well-known if $\Phi_{G^*}$ is a closed para-quaternionic Kähler form on para-quaternionic Kähler manifold $(M, g, V^*)$, then $\Phi_{G^*}$ is also a symplectic structure on para-quaternionic Kähler manifold $(M, g, V^*)$.

Let $X$ a Hamiltonian vector field related to Hamiltonian energy $H$ and given by (7).

Taking into consideration $\Phi_{G^*} = -d\lambda_{G^*} = dx_{2n+i} \wedge dx_i \wedge dx_{3n+i}$, then we calculate

$$i_X \Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i} dx_i - X^i dx_{2n+i} + X^{n+i} dx_{3n+i} - X^{3n+i} dx_{n+i}.$$  \hspace{1cm} (17)

According to (2), if we equal (12) and (17), it yields

$$X = - \frac{\partial H}{\partial x_{2n+i}} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_{3n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial H}{\partial x_{i}} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial H}{\partial x_{n+i}} \frac{\partial}{\partial x_{3n+i}}.$$  \hspace{1cm} (18)

Taking (14), (15) and (18) are equal, we find equations

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{2n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}.$$  \hspace{1cm} (19)

Finally, the equations found in (19) are called Hamiltonian equations with respect to component $G^*$ of para-quaternionic structure $V^*$ on para-quaternionic Kähler manifold $(M, g, V^*)$, and then the triple $(M, \Phi_{G^*}, X)$ is named a Hamiltonian mechanical system on para-quaternionic Kähler manifold $(M, g, V^*)$.

Thirdly, let $(M, g, V^*)$ be a para-quaternionic Kähler manifold. By $H^*$, $\lambda_{H^*}$ and $\omega_{H^*}$, we give a element of para-quaternion structure $V^*$, a Liouville form and a 1-form on para-quaternionic Kähler manifold $(M, g, V^*)$, respectively.
Let $\omega_{H^*} = \frac{1}{2}(x_idx_i + x_{n+i}dx_{n+i} - x_{2n+i}dx_{2n+i} - x_{3n+i}dx_{3n+i})$. Then we find $\lambda_{H^*} = H^*(\omega_{H^*}) = \frac{1}{2}(x_idx_{3n+i} + x_{n+i}dx_{2n+i} - x_{2n+i}dx_{n+i} - x_{3n+i}dx_i)$. We know that if $\Phi_{H^*}$ is a closed para-quaternionic Kähler form on para-quaternionic Kähler manifold $(M, g, V^*)$, then $\Phi_{H^*}$ is also a symplectic structure on para-quaternionic Kähler manifold $(M, g, V^*)$.

Let $X$ a Hamiltonian vector field connected with Hamiltonian energy $H$ and given by (7). Calculating

$$\Phi_{H^*} = -d\lambda_{H^*} = dx_{3n+i} \wedge dx_i + dx_{2n+i} \wedge dx_{n+i}, \quad (20)$$

we have

$$i_X\Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i}dx_i - X^i dx_{3n+i} + X^{2n+i}dx_{n+i} - X^{n+i}dx_{2n+i}. \quad (21)$$

With respect to (2), we equal (12) and (21), we find the Hamiltonian vector field given by

$$X = -\frac{\partial H}{\partial x_{3n+i}} \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial H}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial H}{\partial x_i} \frac{\partial}{\partial x_{3n+i}}. \quad (22)$$

Considering (14), (15) and (22) are equaled, it yields

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{n+i}}{dt} = -\frac{\partial H}{\partial x_{2n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_{n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_i}. \quad (23)$$

In the end, the equations introduced in (23) are named Hamiltonian equations with respect to element $H^*$ of para-quaternion structure $V^*$ on para-quaternionic Kähler manifold $(M, g, V^*)$, and then the triple $(M, \Phi_{H^*}, X)$ is called a Hamiltonian mechanical system on para-quaternionic Kähler manifold $(M, g, V^*)$.

5 Conclusion

From above, Lagrangian mechanical systems have intrinsically been described taking into account a canonical local basis $\{F, G, H\}$ of $V$ on para-quaternionic Kähler manifold $(M, g, V)$. 

19
The paths of semispray $X$ on the para-quaternionic Kähler manifold are the solutions Euler-Lagrange equations raised in (8), (9) and (11), and introduced by a canonical local basis $\{F, G, H\}$ of vector bundle $V$ on para-quaternionic Kähler manifold $(M, g, V)$.

Also, Hamiltonian mechanical systems have intrinsically been described with taking into account the basis $\{F^*, G^*, H^*\}$ of para-quaternionic structure $V^*$ on para-quaternionic Kähler manifold $(M, g, V^*)$. The paths of Hamilton vector field $X$ on the para-quaternionic Kähler manifold are the solutions Hamiltonian equations raised in (16), (19) and (23), and obtained by a canonical local basis $\{F^*, G^*, H^*\}$ of vector bundle $V^*$ on para-quaternionic Kähler manifold $(M, g, V^*)$.

Lagrangian and Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. One can be proved that the obtained equations are very important to explain the rotational spatial mechanical-physical problems. Therefore, the found equations are only considered to be a first step to realize how para-quaternionic geometry has been used in solving problems in different physical area.

For further research, the Lagrangian and Hamiltonian mechanical equations derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.

References

[1] M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, vol.152, Elsevier, Amsterdam, 1989.
[2] M. De Leon, P.R. Rodrigues, Second-Order Differential Equations and Non-Conservative Lagrangian Mechanics, J. Phys. A: Math. Gen. 20(1987)5393-5396.

[3] M. Zambine, Hamiltonian Perspective on Generalized Complex Structure, arXiv: 0502137.

[4] M. Tekkoyun, On Para-Euler-Lagrange and Para-Hamiltonian Equations, Phys. Lett. A, Vol. 340, Issues 1-4, 2005, pp. 7-11.

[5] M. Tekkoyun, Lagrangian Mechanics on Quaternion Kähler Manifolds, arXiv: 0902.4079.

[6] M. Tekkoyun, Hamiltonian Mechanics on Quaternion Kähler Manifolds, arXiv: 0902.3727.

[7] D. Stahlke, Quaternions in Classical Mechanics, Phys 621. http://www.stahlke.org/dan/phys-papers/quaternion-paper.pdf

[8] A. S. Dancer - H. R. Jørgensen - A. F. Swann, Metric Geometries over the split Quaternions, arXiv: 0412.215.