SURFACE MOTIONS and FLUID DYNAMICS

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Abstract

A certain class of surface motions, including those of a relativistic membrane minimizing the 3-dimensional volume swept out in Minkowski-space, is shown to be equivalent to 3-dimensional steady-state irrotational inviscid isentropic gas-dynamics. The SU(∞) Nahm equations turn out to correspond to motions where the time \( t \) at which the surface moves through the point \( \mathbf{r} \) is a harmonic function of the three space-coordinates. This solution also implies the linearisation of a non-trivial-looking scalar field theory.

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The dynamics of surfaces is of vital interest to quite a variety of physical problems (see e.g. [1]). Models that have attracted particular attention include the evolution by mean curvature ([2], [3], [4], [5]), the kinetic roughening of growing surfaces and the motion of domain walls [14], developable surfaces and surfaces maintaining constant negative curvature (see e.g. [6], [7]), and in [8], several families of integrable surfaces have been determined, for which the spectral parameter appearing in the zero-curvature condition may be interpreted as a (time-) deformation parameter.

Let me consider a time-dependent 2-dimensional surface $\Sigma_t$ in $\mathbb{R}^3$, whose motion is locally of the form

$$\dot{\hat{r}} = a \cdot \hat{n} ,$$

(1)

where $\hat{r} = \hat{r}(t, \varphi^1, \varphi^2)$ is the position vector, $\hat{n}$ the surface normal, and $a = a(g)$ is assumed to be only a function of the surface area element $\sqrt{g} = |\partial_1 \hat{r} \times \partial_2 \hat{r}|$ ($\partial_1, \partial_2$ and $\cdot$ denoting differentiation with respect to $\varphi^1, \varphi^2$ and $t$, respectively). Two cases of special interest are $a = \sqrt{g}$ and $a = \sqrt{1-g}$, the former corresponding to a reduction of the self-dual Yang-Mills equations in $\mathbb{R}^4$ with the gauge group $\text{SDiff} \Sigma_t$ (see e.g. [9], [10]), while the latter corresponds to the surface sweeping out a minimal 3-dimensional hypersurface in Minkowski-space (this was derived in [11]). In order to show the equivalence of (1) with 3-dimensional steady-state isentropic irrotational gas-dynamics (and explicitly solve the case $a = \sqrt{g}$) let me write (1) in a slightly different way, namely

$$\dot{x} = \gamma\{y, z\} , \quad \dot{y} = \gamma\{z, x\} , \quad \dot{z} = \gamma\{x, y\} ,$$

(2)

where $\gamma = \frac{a(g)}{\sqrt{g}}$ and $\{\ldots\}$ denotes the Poisson-bracket $\frac{\partial}{\partial \varphi^1} \frac{\partial}{\partial \varphi^2} - \frac{\partial}{\partial \varphi^2} \frac{\partial}{\partial \varphi^1}$.

Choosing the unknown functions $x$, $y$ and $z$ as the independent(!) variables,

$$t, \varphi^1, \varphi^2 \rightarrow x^1 = x(t, \varphi^1, \varphi^2) , \quad x^2 = y(t, \varphi^1, \varphi^2) , \quad x^3 = z(t, \varphi^1, \varphi^2)$$

(3)

(possible as long as $\dot{\hat{r}} \neq 0$), and

$$B_1 = \gamma\{y, z\} , \quad B_2 = \gamma\{z, x\} , \quad B_3 = \gamma\{x, y\} ,$$

(4)

expressed in the new coordinates, as the new dependent variables, such that

$$\partial_t \hat{r} = \hat{B} \cdot \hat{\nabla} , \quad \gamma \cdot \{\ldots, \hat{r}\} \hat{\nabla} = \hat{B} \times \hat{\nabla} ,$$

(5)
one finds
\[ \vec{B} \cdot \nabla B_i - B_i \nabla \cdot \vec{B} + \vec{B} \partial_i \vec{B} = B_i \vec{B} \cdot \nabla \frac{\gamma}{\gamma}, \]  
(6)

by taking the time-derivative of (4), using (2) and (5).

These equations for \( \vec{B}(\vec{x}) \) can be reduced to a single scalar field equation by simply noting that \( \nabla t \) (calculated from (3), using (2)) is actually equal to \( \vec{B}/\vec{B}^2 \). Alternatively, writing \( \vec{B} = \vec{C}/(\vec{C}^2) \) - which is reminiscent of an invariance - transformation of the Lamé system (see e.g. [12]) , and was already used in [10] to linearize (in an equivalent, spinorial, notation) the SU(\( \infty \)) Nahm equations - leaves (6) unchanged, except for a crucial flip of sign in the pure divergence term,

\[ \vec{C} \cdot \nabla C_i - C_i \nabla \cdot \vec{C} - \vec{C} \partial_i \vec{C} = C_i \vec{C} \cdot \nabla \ln \gamma \]  
(7)

Multiplying by \( C_i \) (and summing over \( i = 1, 2, 3 \)) one finds that \( \vec{\nabla} \cdot \vec{C} = -\vec{C} \cdot \vec{\nabla} \ln \gamma \). Hence \( \vec{C} \cdot \nabla C_i = \vec{C} \partial_i \cdot \vec{C} \), which together with

\[ \vec{C} \cdot (\nabla \times \vec{C}) = 0 \]  
(8)

(this being a consequence of the Jacobi-identity \( \{ \{ x, y \}, z \} + \text{cycl.} = 0 \)) implies

\[ \vec{C} = \nabla f \]  
(9)

The only equation to be solved is therefore

\[ \vec{\nabla}(\tilde{\gamma} \vec{\nabla} f) = 0 \]  
(10)

with \( \tilde{\gamma} \) being a definite function of \( (\nabla f)^2 \) (invert \( \gamma(g) \) to obtain \( g(\gamma) \) and solve \( g(\gamma) \cdot \gamma^2 \cdot (\nabla f)^2 = 1 \) for \( \gamma = \tilde{\gamma}((\nabla f)^2) \).

For the case \( \tilde{\gamma} = 1 \), which before has been solved in rather different ways and contexts (e.g., known as the ‘Eden-model’ in the context of growing surfaces, [13]), this means that the time \( t = f(\vec{x}) \), at which the surface passes the point \( \vec{x} \), is a harmonic function of the three space variables \( x, y, \) and \( z \). This observation provides a very intuitive
understanding of the solution(s) of the corresponding equation(s) (1). To illustrate this by an example, let

\[ t(x, y, z) = z^2 - x^2 \, . \]  

(11)

This solution of Laplace’s equation corresponds to two sheets of hyperboloids in the \( x - z \) plane (infinitely extending perpendicular to this plane) which move towards the \( y \)-axis, becoming ‘singular’ at \( t = 0 \), and then moving away from the \( y \)-axis, in directions perpendicular to the incoming ones. In this case every \( \vec{x} \in \mathbb{R}^3 \) is passed exactly once. Generally, any \( t(\vec{x}) \) that is finite for all \( \vec{x} \in \mathbb{R}^3 \) will be a superposition of harmonic, homogeneous polynomials, of course.

Going back to the general case (\( \dot{\gamma} \neq \text{const.} \)), one notes that eq. (10) means that all equations of type (1) can (at least locally) be written as Lagrangian equations of motion for a scalar field \( f(\vec{x}) \) (which is the time at which the surface passes the point \( \vec{x} \)), with the Lagrangian density \( \mathcal{L} \) being \( \frac{1}{2} \) the integral of \( \gamma \) (expressed as a function of \( (\nabla f)^2 \)). So the following interesting correction with fluid dynamics emerges: viewing \( \nabla f = \vec{V} \) as the velocity, and \( \dot{\gamma} \neq \text{const} \) as the mass density \( \rho \) of an irrotational inviscid 3 dimensional gas, (10) is the continuity equation for time independent (‘steady state’) flows, while the Euler equation,

\[ \nabla \dot{f} + \nabla f \cdot \nabla (\nabla f) + \frac{1}{\rho} \nabla (P(\rho)) = 0 \, , \]  

(12)

(for the steady state case, \( \nabla \dot{f} = 0 \)) determines \( \rho = \rho \left( (\nabla f)^2 = w \right) \) in exactly the way needed for a consistent interpretation of \( \dot{\gamma} \) as the mass density, provided the pressure \( P(\rho) \) is chosen such that

\[ \frac{dP(\rho(w))}{dw} = -\frac{1}{2} \rho(w) \, . \]  

(13)

For

\[ \rho = \left( (\nabla f)^2 - 1 \right)^{-1/2} \, , \]  

(14)

which corresponds to \( a = \sqrt{1 - g} \) in eq. (1) (i.e. the relativistic minimal hypersurface case), the equations of motion, (10), read

\[ \nabla^2 f = \frac{1}{2} \frac{\nabla f \cdot \nabla ((\nabla f)^2)}{(\nabla f)^2 - 1} \, , \]  

(15)
corresponding to the Lagrangian

\[ \mathcal{L} = \sqrt{(\nabla f)^2 - 1} \, . \]  

(16)

It is interesting to note that (13) yields the Kármán-Tsien-Chaplygin equation of state,

\[ P(\rho) = -\frac{1}{\rho} \, , \]  

(17)

as long as

\[ \rho = ((\nabla f)^2 + \epsilon)^{-\frac{1}{2}} \, , \]  

(18)

for all \( \epsilon \). This means that the relativistic \((\epsilon = -1)\) and the Euclidean \((\epsilon = +1)\) minimal hypersurface problem, as well as the minimization of \( \int d^3x \, |\nabla f| \) \((\epsilon = 0)\) are all related to Kármán-Tsien irrotational gas-dynamics, - the first corresponding to the supersonic regime \((\nabla f)^2 = (\nabla f)^2 > \frac{dP}{d\rho} \equiv (\text{velocity of sound})^2 \equiv c^2\), the second to the subsonic regime, and the third \((\epsilon = 0)\) corresponding to the case of the Mach-number \( M: = |\nabla f| / c \) being exactly equal to one.

However, as it was demonstrated in [15] that the relativistic minimal hypersurface-problem also corresponds to a time-dependent Kármán-Tsien gas in one space-dimension lower, the rather unique fact emerges that a d-dimensional steady state Kármán-Tsien gas is equivalent to a time-dependent d-1 dimensional one. Of course, this has to do with the hidden relativistic invariance, and is therefore, a posteriori, not surprising (and can easily be checked directly).

Finally, let me also derive (10) by first obtaining a single second-order equation for \( z(t, x, y) \) from (1), and then interchanging the role of \( z \) and \( t \) as dependent, resp. independent, variable: So, consider first the transformation

\[ \varphi^0 = t, \varphi^1, \varphi^2 \rightarrow y^0 = t, \quad y^1 = x(t, \varphi^1, \varphi^2), \quad y^2 = y(t, \varphi^1, \varphi^2) \]  

(19)

for (1), which allows one to derive from (2) a pair of first order differential equations for \( J = \{x, y\} \) and \( z \) (both viewed as functions of the new variables; \( \gamma = \gamma(g) \) with
\[ g = J^2 \cdot (1 + (\vec{\nabla} z)^2), \quad \vec{\nabla} = \left( \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right)^{tr} , \]

\[ \dot{z} = \gamma J (1 + (\vec{\nabla} z)^2), \quad \dot{J} = -J^2 \vec{\nabla} (\gamma \vec{\nabla} z) \]  

(20)

(cp. [11]). One can then verify that the corresponding second order equation for \( z \),

\[ \vec{\nabla} (\tilde{\gamma} \vec{\nabla} z) = \left( \frac{\tilde{\gamma} (1 + (\vec{\nabla} z)^2)}{\dot{z}} \right) , \]  

(21)

follows from the Lagrangian

\[ \tilde{L} = \dot{z} L \left( w = \frac{1 + (\vec{\nabla} z)^2}{\dot{z}^2} \right) \]  

(22)

if \( 2 \frac{\partial \tilde{L}}{\partial w} = \tilde{\gamma} (w) \) (the latter being defined by \( \tilde{\gamma} = \gamma \left( g = \frac{1}{\dot{z}^2} \right) \)). Note that (21) remains nonlinear if \( \tilde{\gamma} = \text{const} \), in contrast with (10), while the transformation

\[ x_1, x_2, x_3 \rightarrow y_0 = f(\vec{x}), \quad y_1 = x_1, \quad y_2 = x_2 \]  

(23)

(resp. \( y_0, y_1, y_2 \rightarrow x_1 = y_1, \quad x_2 = y_2, \quad x_3 = z(y_0, y_1, y_2) \)) provides a one to one correspondence between

\[ S = \int d^3y \, \dot{z} \cdot \tilde{L} \left( \frac{(\vec{\nabla} z)^2 + 1}{\dot{z}^2} \right) \]  

(24)

and

\[ S = \int d^3x \tilde{L} ((\vec{\nabla} f)^2) , \]  

(25)

hence confirming \( f(\vec{x}) \) as the time at which the surface passes \( \vec{x} = (x_1, \ x_2, \ x_3) \).

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Addendum:

Contemplating the possibility that special choices of $\gamma$ may correspond to integrable 3-manifolds that are connected to the SU($\infty$) Nahm case by generalized Bäcklund-transformations, it might be useful to note various ‘zero-curvature or Lax-type’ representations for the case of constant $\gamma$, and axially symmetric surfaces. Firstly, (if $\gamma = 1$) (21) can be written in Lax-form (with spectral parameter, $\lambda$),

$$\dot{L} = [L, M],$$

by defining

$$L(\lambda) = \frac{\dot{z}}{1 + \partial z \partial \bar{z}}((\partial z + i\lambda)\bar{\partial} - (\bar{\partial} z + i\lambda)\partial),$$

$$M(\lambda) = \frac{\dot{z}}{1 + \partial z \partial \bar{z}}(i\lambda\partial - (\partial z)\bar{\partial}) ,$$

where $\partial := \partial_x - i\partial_y$, $\bar{\partial} = \partial_x + i\partial_y$. One may also just take

$$L = \frac{\dot{z}}{1 + \partial z \partial z} \bar{\partial},$$

$$M = \frac{-\dot{z}}{1 + \partial z \partial z} (\partial z)\bar{\partial} .$$

Actually, any equation of motion of the form

$$\frac{d}{dt} F(x, y; z, \partial z, \bar{\partial} z, \dot{z}, \cdots) = \nabla^2 z$$

is representable in Lax-form by letting

$$L = \frac{1}{F} \bar{\partial} , \quad M = \frac{-1}{F} (\partial z)\bar{\partial} .$$

To obtain conserved quantities for the SU($\infty$)-Nahm equations (with compact $\Sigma_t$) it is easiest to represent the original equations, (2), in Lax-form on the Poisson-Lie algebra of functions (on $\Sigma_t$), rather than vectorfields,

$$\mathcal{L} = \frac{1}{\lambda} (x + iy) - 2iz + \lambda(x - iy)$$

$$\mathcal{M} = iz - \lambda(x - iy)$$

$$\dot{\mathcal{L}} = \{\mathcal{L}, \mathcal{M}\}$$
which implies that

\[ Q_{lm} := \left( \frac{\partial^m}{\partial \lambda^m} \int_{\Sigma_t} d\varphi^1 d\varphi^2 \, L(\lambda)^f \right) |_{\lambda=0} \]  

(32)

is time-independent; the conserved densities are just the harmonic homogenous polynomials (note that \((\nabla L)^2 = 0\)). (10)\(_{\gamma=1}\) on the other hand, is equivalent to

\[ [(L_1 + iL_2) + \lambda(L - iL_3), \ (L + iL_3) - \lambda(L_1 - iL_2)] = 0 \ , \]  

(33)

\[ \hat{L} := \frac{\nabla^2 f}{(\nabla f)^2} \times \hat{\nabla} \ , \quad L := \frac{\nabla f}{(\nabla f)^2} \cdot \hat{\nabla} \]  

(34)

due to the identity

\[ [L, L_i] + \frac{1}{2} \epsilon_{ijk} [L_j, \ L_k] = -\frac{\nabla^2 f}{(\nabla f)^2} L_i \ . \]  

(35)

This leads to the possibility of representing axially symmetric surface motions by

\[ [\gamma^{-1}(L_1 + iL_2), \ L + iL_3] = 0 \ , \]  

(36)

as \([\gamma, \ L_3] = 0\) if \(f = f(\sqrt{x^2 + y^2}, \ z)\).
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