ON SUBFIELDS OF THE FUNCTION FIELD OF A GENERAL SURFACE IN $\mathbb{P}^3$

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Abstract. In this paper we study a birational immersion from a general smooth plane curve to a non-rational surface with $p_g = q = 0$ to treat dominant rational maps from a very general surface $X$ of degree $\geq 5$ in $\mathbb{P}^3$ to smooth projective surfaces $Y$. Based on the classification theory of algebraic surfaces, Hodge theory, and the deformation theory, we prove that there is no dominant rational map from $X$ to $Y$ unless $Y$ is rational or $Y$ is birational to $X$.

Riemann-Hurwitz Theorem (Chapter XXI in [3]) said that if $\phi : C \to C'$ is a non-constant morphism from a general curve $C$ of genus $g > 1$ onto a curve $C'$ then either $\phi$ is birational, or else $C'$ is rational. It is interesting to investigate the same statement for higher dimensional varieties of general type under some assumption of generality in a suitable moduli space. Let $X$ be smooth complex projective variety of general type. The dominant rational maps of finite degree $X \rightarrow Y$ to smooth varieties of general type, up to birational equivalence of $Y$ form a finite set. The proof follows from the approach of Maehara [13], combined with the results of Hacon and McKernan [11], of Takayama [14], and of Tsuji [15].

Motivated by this finiteness theorem for dominant rational maps on a variety of general type and by the results obtained in [10] we study dominant rational maps from a very general complex surface $X$ of degree $d \geq 5$ in $\mathbb{P}^3$ to smooth projective surfaces $Y$. The main result of this paper is the following.

Theorem 0.1. Let $X \subset \mathbb{P}^3$ be a very general smooth complex surface of degree $d > 4$. Let $Y$ be a non rational surface. Then there is no dominant rational map $f : X \rightarrow Y$ unless $f$ is a birational map.

We recall that a very general element of $U$ has the property $P$ if $P$ holds in the complement of a union of countably many proper subvarieties of $U$. We get immediately the following completely algebraic version of our theorem.

Theorem 0.2. Let $K$ be the function field of a very general complex surface in $\mathbb{P}^3$ of degree $d > 4$. Let $\mathbb{C} \subseteq K' \subsetneq K$ be a proper subfield of $K$. Then $K'$ is isomorphic either to $\mathbb{C}(x)$, if the transcendental degree of $K'$ is 1, or to $\mathbb{C}(x,y)$ if $K'$ has transcendental degree 2.

If one choose a special surface $X$ in $\mathbb{P}^3$ then it might have a dominant rational map to a surface of general type $Y$. Classical Godeaux surfaces $Y$ (minimal surfaces of general type with $p_g(Y) := h^0(Y,K_Y) = 0$, $q(Y) := h^1(Y,O_Y) = 0$, $K^2 = 1$, and $\pi_1(Y) = \mathbb{Z}_5$) are obtained by the $\mathbb{Z}_5$-quotient of $\mathbb{Z}_5$-invariant quintics [9].

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As far as we know this gives the first examples of fields of transcendence degree 2 of non-ruled surfaces that does not contain any proper non-rational field. This could have applications to the fields theory and to the absolute Galois theory.

To prove our main Theorem, we study a birational immersion from a general smooth plane curve to a non-rational surface with $p_g = q = 0$.

**Theorem 0.3.** Let $S$ be a non-rational surface with $p_g = q = 0$. If $C$ is a general smooth plane curve of degree $d \geq 10$ then there is no birational immersion $\kappa : C \to S$.

Moreover, we obtain

**Theorem 0.4.** (=Theorem 3.4) Let $S$ be an elliptic surface with $p_g = q = 0$ of Kodaira dimension 1. We assume moreover that $\text{Pic}(S)$ is torsion free. Then a general smooth plane curve of degree $d \geq 6$ cannot be birationally immersed in $S$.

The method of proof combines the classification theory of algebraic surfaces, Hodge theory and the deformation theory. By Hodge theory (as in [10]) one has to consider only dominant map $f : X \to Y$ where $Y$ is simply connected and without 2–holomorphic global forms, that is $p_g(Y) = 0$. From the classification theory one obtains that the Kodaira dimension, $\text{kod}(Y)$, of $Y$, must be $\geq 1$. The case of surfaces of general type, that is $\text{kod}(Y) = 2$, was also considered in [10], where the problem was solved only for $d \leq 11$. The new idea here was to consider the full family of smooth plane curves which are the hyperplane sections of surfaces $X \subset \mathbb{P}^3$. We show that a very general plane curve, up possibly to some small degree cases, cannot be birationally immersed in $Y$. The case of elliptic surfaces, that is when $\text{kod}(Y) = 1$, is similar, but slightly difficult. A careful study of the curves on $Y$ as well as an estimate of the moduli of non-rational elliptic surfaces was necessary.

The method presented here can be used to obtain similar results for the very general point of families of surfaces of general type that contains families of curves of high dimension. This is for instance symmetric product of curves of genus $g \geq 4$ and product of curves (see [5]). In a forthcoming paper we will treat a dominant rational map from a product of curves. It is not clear at the moment how to treat the cases of $\text{kod}(X) = 1$ and $\text{kod}(X) = 0$. For instance the case where $X$ is a very general quartic surface in $\mathbb{P}^3$ needs a different approach.

In this paper we work on the field of complex numbers.

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1. Family of curves on surfaces

We recall a basic result on deformation of curves on a surface (see for instance [1]). Let $C$ be a smooth projective curve of $g(C) \geq 2$ and let $S$ be a smooth projective surface. Let $T_C$ and $T_S$ be the holomorphic tangent bundles.

**Definition 1.1.** A morphism $\kappa : C \to S$ will be called a birational immersion if the induced map $C \to \kappa(C)$ is birational.

The differential $d\kappa : T_C \to \kappa^*T_S$ of a birational immersion $\kappa$ is a sheaf inclusion and hence induces an exact sequence on $C$:

$$0 \to T_C \to \kappa^*T_S \to N \to 0.$$  

The first order deformations of $\kappa$ are classified by the global sections $H^0(C,N)$ of the normal sheaf $N$.

Let $U$ be a Kuranishi family of deformation of maps, assume $\dim U > 0$. Assume that $\kappa : C \to S$ is a general point of $U$. Let $N_{\text{tors}} \subset N$ be the torsion sheaf of $N$ and set $N' = N/N_{\text{tors}}$. One has ([1], Chapter XXI in [3]) that $\dim U \leq h^0(C,N')$. When $\deg \kappa^*(K_S) = m \geq 0$ we have $\deg N' \leq 2g - 2$ where $g = g(C)$. Therefore Clifford theorem gives:

$$h^0(N') \leq g - \frac{1}{2}m.$$  

**Proposition 1.2.** Assume $\dim U > 0$. We have

(a) If $K_S$ is a nef divisor then $\dim U \leq g - \frac{\deg \kappa^*(K_S)}{2}$.
(b) If $1 < \dim U = g - \frac{\deg \kappa^*(K_S)}{2}$, then $N = N'$ and moreover either $\kappa^*(K_S) = \mathcal{O}_C$ or $C$ is hyperelliptic.

We give an immediate application of the above result:

**Proposition 1.3.** Let $S$ be a surface of general type with $p_g = q = 0$. If $C$ is a general smooth plane curve of degree $d \geq 9$ then there is no birational immersion $\kappa : C \to S$.

**Proof.** We can assume that $S$ is minimal. The proof is obtained by contradiction: suppose a general $C$ can be birationally immersed in $S$ with $p_g = q = 0$. Plane curves of degree $d$ depend on

$$\frac{(d+1)(d+2)}{2} - 9$$

dimensional moduli. By ([10], Corollary 2.5.3) surfaces of general type with $p_g = q = 0$ depends on $M \leq 19$ parameters. It follows that on some surface $S$ we can find at least

$$\frac{(d+1)(d+2)}{2} - 28 > 0$$

moduli. Since the dimension of the family is positive and since $C$ is not hyperelliptic we have then by Proposition 1.2

$$\frac{(d+1)(d+2)}{2} - 28 < g - \frac{\deg \kappa^*(K_S)}{2}.$$
Therefore
\[
\frac{(d + 1)(d + 2)}{2} - \frac{(d - 1)(d - 2)}{2} < 28 - \frac{\deg \kappa^*(K_S)}{2}.
\]

Therefore, if \(d \geq 9\) then there is no birational immersion \(\kappa : C \rightarrow S\). □

By the same argument in Proposition 1.3, one can treat also for Enriques surfaces. We note that the dimension of moduli space of Enriques surfaces is 10.

**Proposition 1.4.** Let \(S\) be an Enriques surface. If \(C\) is a general smooth plane curve of degree \(d \geq 7\) then there is no birational immersion \(\kappa : C \rightarrow S\).

2. **Elliptic surfaces with** \(p_g = q = 0\) **of Kodaira dimension 1**

In this section, we study a birational immersion from a general smooth plane curve to an elliptic surface with \(p_g = q = 0\) of Kodaira dimension 1. Let \(S\) be a smooth projective elliptic surface \([4]\). An elliptic surface means that there is a smooth projective curve \(B\) and a surjective map \(\pi : S \rightarrow B\) such that the general fiber \(F = \pi^{-1}(b)\) for \(b \in B\) is an elliptic curve.

We will assume moreover that
1. \(p_g(S) = \dim H^0(S, K_S) = 0\).
2. \(q(S) = \dim H^1(S, \mathcal{O}_S) = 0\).
3. \(S\) is non-rational.
4. \(S\) is minimal.

Easy consequence of the above conditions, we have \(\chi(\mathcal{O}_S) = 1\), \(K^2 = 0\) and \(c_2(S) = 12\). Moreover it follows that the fibration is not isotrivial, that yields the \(j\) invariant of the smooth fibers to be non-constant. We keep the above conditions.

Let \(\pi : S \rightarrow \mathbb{P}^1\) be the elliptic fibration. Let \(N\) be the number of multiple fibers, and let \(k_i\) for \(i = 1, \ldots, N\) be their multiplicities.

We first prove the following:

**Lemma 2.1.** Under the previous hypothesis with \(d \geq 10\) we have \(N \leq d + 3\).

Proof. We consider the map \(f = \pi \circ \kappa : C \rightarrow \mathbb{P}^1\). Let \(\alpha\) be the degree of \(f\). Let \(g\) be the genus of \(C\). Then we have from Hurwitz’ formula

\[
2g - 2 = -2\alpha + \left( \sum_{i=1}^{N} \alpha(1 - 1/k_i) \right) + r \geq -2\alpha + N\frac{\alpha}{2} + r
\]

where \(r\) is the number of branch loci which are not multiple fibers. Note that the ramification index corresponding to a multiple fiber is \(\geq \frac{\alpha}{2}\). Clearly \(\alpha \geq d - 1\) the gonality of the smooth plane curve \(C\) (see for instance Chapter I in \([2]\)).

Since general plane curves are obtained by \(\alpha\)-covers of \(\mathbb{P}^1\), we have

\[
N + r - 3 \geq (d + 1)(d + 2)/2 - 9.
\]
And it applies \( r \geq (d + 1)(d + 2)/2 - N - 6 \). Then combining it with the above equation, we conclude
\[
d(d - 3) \geq (d - 1)(N/2 - 2) + (d + 1)(d + 2)/2 - N - 6.
\]
So \( d^2 - 5d + 18 \geq (d - 3)(N - 4) \), and \( N \leq d + 3 \) because \( d \geq 10 \). \( \square \)

We give an estimate on the moduli \( M(S) \) of \( S \). Now we have the following

**Lemma 2.2.** We have \( h^0(\Omega^1_S(nF)) = n - 1 \) for \( n > 0 \).

**Proof.** We remark that we may assume \( F \) to be a general fiber since the base curve is \( \mathbb{P}^1 \).

We have \( h^0(\Omega^1_S) = q = 0 \). We begin by proving that \( h^0(\Omega^1_S(n - 1)F) \leq n - 2 \). We consider the exact sequence:
\[
0 \to \Omega^1_S((n - 1)F) \to \Omega^1_S(nF) \to \Omega^1_S(nF)|_F \to 0.
\]
Since \( \mathcal{O}_F = \mathcal{O}_F(nF) \) we have \( \Omega^1_S(nF)|_F = \Omega^1_S|_F \). We now show that \( h^0(\Omega^1_S|_F) = 1 \), which implies \( h^0(\Omega^1_S(nF)) \leq h^0(\Omega^1_S(n - 1)F) + 1 \).

We take the conormal sequence of the immersion of \( F \) into \( S \):
\[
0 \to N^*_F \equiv \mathcal{O}_F \to \Omega^1_S|_F \to \Omega^1_F \to 0.
\]
The associated coboundary map \( \partial : H^0(F, \Omega^1_F) \to H^1(F, \mathcal{O}_F) \) is given by the infinitesimal variation of Hodge structure. Since \( F \) is a general fiber the \( j \)-invariant is not constant. We infer that \( \partial \) is not zero. Therefore \( H^0(F, \Omega^1_S|_F) \) is one dimensional.

We now prove that \( h^0(\Omega^1_S(F)) = 0 \) : we consider the log sequence obtain from \( \square \) and the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \Omega^1_S & \to & \Omega^1_S(\log F) & \to & \mathcal{O}_F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S & \to & \Omega^1_S(F) & \to & \Omega^1_S|_F & \to & 0
\end{array}
\]

One has that the coboundary map \( \partial : H^1(F, \mathcal{O}_F) \to H^1(S, \Omega^1_S|_F) \) in the log sequence is not trivial since \( \partial(1) = c_1(F) \). Therefore the coboundary map \( \partial : H^0(\Omega^1_S|_F) \to H^1(\Omega^1_S) \) is also non trivial, this implies \( h^0(\Omega^1_S(F)) = 0 \). And it concludes \( h^0(\Omega^1_S(nF)) \leq n - 1 \) for \( n \geq 1 \).

To get the other inequality we consider the codifferential of \( \pi \). It induces a sheaf injection \((d\pi)^*\mathcal{O}_S(-2F) \equiv \pi^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_S \). Tensoring with \( \mathcal{O}_S(nF) \) we get
\[
0 \to \mathcal{O}_S((n - 2)F) \to \Omega^1_S(nF).
\]
We get \( h^0(\Omega^1_S(nF)) \geq h^0(\mathcal{O}_S(n - 2F)) = n - 1 \) for \( n \geq 1 \). \( \square \)

Similar result is proved for rational elliptic surfaces (Lemma 2 in \[12\]). We recall \[14\]
\[
K_S = -F + \sum_{i=1}^{N}(k_i - 1)F_i = (N - 1)F - \sum_{i}F_i.
\]
Proposition 2.3. Let $S$ be a minimal elliptic surface with $p_g = q = 0$ of Kodaira dimension 1. Let $M$ be a Kuranishi family of the moduli space of $S$. Then $M(S) = \dim M \leq 8 + N$.

Proof. We would like to estimate $h^1(T_S) = \dim H^1(S, T_S)$. We recall that $H^0(S, T_S) = 0$. In fact the $j$ invariant is not constant. This implies that the connected component of the group of automorphisms of $S$ must commute with the fibration. So if it is not trivial then all smooth fibers are isogeneous, again contradicts to the non-triviality of the $j$ invariant. Therefore from Hirzebruch-Riemann-Roch Theorem and Serre duality (see [6]) we have

$$h^1(T_S) = h^2(T_S) + 10 \chi(O_S) - 2K_S^2 = 10 + h^0(\Omega^1_S(K_S)).$$

We get $h^0(\Omega^1_S(K_S)) = h^0(\Omega^1_S((N-1)F-\sum F_i)) \leq h^0(\Omega^1_S((N-1)F)) \leq N - 2$ using Lemma 2.2. We get $h^2(T_S) \leq N - 2$ and then $M(S) \leq 8 + N$. \hfill $\square$

Proposition 2.4. Let $S$ be an elliptic surface with $p_g = q = 0$ of Kodaira dimension 1. Then a general smooth plane curve of degree $d \geq 10$ cannot be birationally immersed in $S$.

Proof. We first observe that the number of moduli of $S$ is

$$M(S) \leq 8 + N \leq 11 + d < \frac{(d+1)(d+2)}{2} - 9 \text{ for } d \geq 10.$$

It follows that $\kappa(C)$ cannot be rigid in $S$.

By the similar argument in the proof of Proposition 1.3 and using Proposition 2.3

$$\frac{(d+1)(d+2)}{2} - 9 - 8 - N < g - \frac{\deg \kappa^*(K_S)}{2}.$$

Therefore by Lemma 2.1 we have

$$\frac{(d+1)(d+2)}{2} - \frac{(d-1)(d-2)}{2} < 17 + d + 3 - \frac{\deg \kappa^*(K_S)}{2}.$$

Therefore, if $d \geq 10$ then there is no birational immersion $\kappa : C \to S$. \hfill $\square$

Combining Propositions 1.3, 1.4, and 2.4, we get Theorem 0.3.

3. Elliptic surfaces without torsions versus plane curves

This section treats a birational immersion from a general smooth plane curve to a special elliptic surface $S$ with $p_g = q = 0$ of Kodaira dimension 1. We will use the same notation and the same assumption in the beginning of Section 2. We will assume moreover that

- $\text{Pic}(S)$ is torsion free.

Let $\equiv$ denote the linear equivalence of divisors which, in our case, is the homological equivalence. Let $F$ be a general fiber of $\pi$ and $\{F_i\}_{i=1,...,N}$ the multiple fibers, i.e. $F_i$ is effective and $k_iF_i \equiv F$ where $k_i \in \mathbb{Z}$, $k_i > 1$.

We remark from ([7], Chapter 2) that

$$\text{Tors}(\text{Pic}(S)) = \ker(\oplus_{i=1}^N \mathbb{Z}/k_i \mathbb{Z} \to \mathbb{Z}/M \mathbb{Z}).$$
where $M = \prod_{i=1}^{N} k_i$, $\psi((a_1, \ldots, a_N)) = \sum a_i M_i \pmod{M}$, $M_i = M/k_i$. It follows $k_i$ and $k_j$ are relatively prime. We may assume $1 < k_1 < k_2 < \cdots < k_N$.

Since $S$ is not rational we have $N \geq 2$. Let $\Lambda \subset \text{Pic}(S)$ be the subgroup generated by the $F_i$; $\Lambda$ is cyclic and let $\lambda$ be the generator such that $F_i = M \lambda$ with $M > 0$. We have

$$ F \equiv M \lambda, \quad F_i = M_i \lambda. $$

Therefore if we set $K_S = \rho \lambda$ then

$$ \rho = ((N - 1)M - \sum_{i=1}^{N} M_i) = M(N - 1 - \sum_{i=1}^{N} \frac{1}{k_i}). $$

Setting $k_0 = 1$ we have $\rho = Ms$ where

$$ s = (N - \sum_{i=0}^{N} \frac{1}{k_i}). $$

One has immediately

**Proposition 3.1.**  
(1) $\rho \geq 3$ with the only one exception $k_1 = 2, k_2 = 3$.

(2) If $N > 2$ then one has $s > 1$ with the only exception $k_1 = 2, k_2 = 3, k_3 = 5$. In this case $K_S = 29 \lambda$.

(3) If $N = 2$ then $K_S = (((k_1 - 1)(k_2 - 1) - 1)\lambda$.

(4) If $N > 2$ then we have $\rho > 6N$.

**Proof.** Only the last part needs a comment. When $N = 3$ with $k_1 = 2, k_2 = 3, k_3 = 5$, we have $\rho = 29 > 18$. Otherwise we have

$$ \rho \geq \prod_{i} k_i, $$

it follows (it is a very rough estimate) $\rho > N! \geq 6N$.

**Remark 3.2.** One has

(1) $K_S = \lambda \iff k_1 = 2$ and $k_2 = 3$.

(2) If $k_1 = 2$ and $k_2 = k$ then $\rho = k - 2$.

(3) If $k_1 = 3$ and $k_2 = 4$ then $\rho = 5$ and $2K_S = 4F_1 + 2F_2$.

Under the extra condition of torsion freeness of $\text{Pic}(S)$, Lemma 2.1 can be improved.

**Lemma 3.3.** Under the previous hypothesis with $d \geq 5$ we have $N \leq d - 1$.

**Proof.** We consider the map $f = \pi \circ \kappa : C \to \mathbb{P}^1$. Let $\alpha$ be the degree of $f$. Then we have $\alpha = \kappa(C) \cdot F = M(\kappa(C) \cdot \lambda)$. We remark that the point $p \in C$ such that $\kappa(p) \in F_i$ are ramification points of $f$ with multiplicity a multiple of $k_i$. Let $g$ be the genus of $C$. We have from the Hurwitz’ formula

$$ 2g - 2 \geq -2\alpha + \sum_{i=1}^{N} (\alpha - \alpha/k_i) = \alpha((N - 2) - \sum_{i=1}^{N} 1/k_i). $$

Since $d \geq 5$ we may assume $N \geq 4$, and clearly $\alpha \geq M$. Since the $k_i$ are pairwise prime, we get:
2g − 2 ≥ M(N − 2 − \(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\)) − (N − 4)\(\frac{1}{11}\) ≥ (N − 4)(M − 1/11)

This gives
\[d(d − 3) ≥ (N − 4)(M − 1)\]

Now since \(k_1 ≥ 2\) and \(k_i > 2(i − 1)\) we get
\[M > 2^N(N − 1)! ≥ 2^N(N − 1)^{\frac{N−1}{2}}\]

If \(N ≥ d\) then we get
\[d(d − 3) ≥ (d − 4)2^d(d − 1)^{d−1} ≥ (d − 4)2^d(d − 1)^2.\]

We get a contradiction \((d(d − 3) < 2^d)\).

**Theorem 3.4.** Let \(S\) be an elliptic surface with \(p_g = q = 0\) of Kodaira dimension \(1\). We assume moreover that Pic\(S\) is torsion free. Then a general smooth plane curve of degree \(d ≥ 6\) cannot be birationally immersed in \(S\).

**Proof.** We keep the notation of the previous section. Let \(\pi : S → \mathbb{P}^1\) be the elliptic fibration. In particular \(N\) is the number of multiple fibers.

The proof is given by a contradiction. We assume than that \(κ : C → S\) is a birational immersion where \(C\) is a smooth plane curve of degree \(d > 4\).

We will apply Proposition 1.2. We first observe that since \(d ≥ 6\) the number of moduli of \(S\) is

\[M(S) ≤ 8 + N ≤ 7 + d < \frac{(d + 1)(d + 2)}{2} − 9.\]

It follows that \(κ(C)\) cannot be rigid in \(S\). Arguing as in Proposition 1.3 we get again using Proposition 1.2

\[\frac{(d + 1)(d + 2)}{2} − \frac{(d − 1)(d − 2)}{2} < 9 + M(S) − \frac{\deg κ^∗(K_S)}{2}.\]

That is

\[3d < 9 + 8 + N − \frac{\deg κ^∗(K_S)}{2} = 17 + N − \frac{\deg κ^∗(λ)}{2}.\]

We have \(κ^∗(K_S) = ρλ\). Since a positive multiple of \(λ\) is \(F\) that moves linearly, so we have \(\deg κ^∗(K_S) ≥ ρ\). Finally we get

\[3d < 17 + N − \frac{ρ}{2}.\]

When \(N = 2\), \(ρ = (k_1 − 1)(k_2 − 1) − 1\) and \(3d < 19\). So \(d ≤ 6\). When \(N > 2\) then \(ρ/2 > 3N\), and it follows \(3d < 17 − 2N\). So \(d < 4\).

Now we have only to consider the case \(d = 6\) and \(N = 2\). Otherwise \(ρ ≥ 3\) by Proposition 3.1. This is possible only when \(N = 2\), \(k_1 = 2\) and \(k_2 = 3\). Therefore \(ρ = 1\), \(κ^∗(K_S) = O_C(p)\) for \(p ∈ C\). Set \(κ(C) = D\) we have \(D · K_S = 1\) and \(D · F = 6\). The composition \(π \circ κ : C → \mathbb{P}^1\) has degree \(6\). Since the only rational map of degree \(6\) are obtained by projecting from a point. So \(O_C(6p) = κ^∗O_S(F)\) is the \(a_6^p\) on \(C\). But this concludes that \(p\) a flex point of maximal order \(6\) on \(C\) which contradicts to with fact that \(C\) is a general plane curve of degree \(6\). (Alternatively intersecting with the multiple fibre there are a three-tangent and a bitangent to \(2\) flexes).
Remark 3.5. By using the same computation one can prove that the very general curve of genus $\geq 6$ (in the sense of moduli) cannot be immersed in any elliptic surface with $p_g = q = 0$ of Kodaira dimension 1 and $\text{Pic}(S)$ torsion free. In fact if $N > 4$ we still have the inequality:

$$2g - 2 > (N - 4)(M - 1) \geq 2^N(N - 4)(N - 1)!$$

that gives $N \leq \max(4, \sqrt{g})$ then the inequality $3g - 3 \geq g - 1 + N - 2 + 10$ gives $2g - 2 < \max(12, 8 + \sqrt{g})$. This gives $g \leq 6$. The case $g = 6$ can then be analyzed easily.

4. Proof of Main Theorem

Now we are ready to prove our main Theorem (Theorem 0.1 in Introduction). The proof will be obtained by a contradiction. Assume that a dominant rational map $f : X \dasharrow Y$ exists.

From [10] we may assume:

1. $p_g(Y) = q(Y) = 0$ and $Y$ is simply connected.
2. $\text{Pic}(X) = \mathbb{Z}[L]$, where $L$ is the hyperplane section of $X$.

From the classification theory of algebraic surfaces, we have two cases for $Y$: either $Y$ is of general type, or $Y$ is an elliptic surface with Kodaira dimension 1.

Lemma 4.1. Let $C$ be a general hyperplane section of $X$. If $f : X \dasharrow Y$ is dominant then $f_C : C \to Y$ is birational onto its image.

Proof. Since $C$ is a general hyperplane of $X$, we may assume that a general point of $Y$ belongs $f_C(C)$. We recall that the Jacobian of a general hyperplane section $C$ is simple. Then if $f_C : C \to f_C(C)$ is not birational then the normalization of $f_C(C)$ is rational. Since $Y$ is not ruled it follows that $f$ is not dominant. Therefore we get a contradiction. □

Proof. of Theorem 0.1 Assume by contradiction that for a general surface of degree $d$ we have a dominant rational map to $Y$. We get a birational immersion from a general plane curve into a surface $Y$ with $p_g = q = 0$. The $Y$ is either of general type or an elliptic surface with Kodaira dimension 1. In the first case we have $d \leq 9$ by Proposition 1.3. But from [10] we have that this is impossible. If $Y$ is an elliptic surface then we get $d < 6$ by Theorem 3.4. In fact, simply connected implies $N = 2$ [8] (N is as usual the number of the multiple fibers), and we also get $d \leq 7$ by the similar proof in Proposition 1.3.

So $d = 5$ is the only remained case. We recall from [10] that after a resolution of the singularities of $f$

$$Z \xrightarrow{g} Y \xrightarrow{f} X$$

Let $E$ be the exceptional divisor. We may assume:

1. $g^*K_Y = rL - W$ where $r \geq 0$ and $W$ is an effective divisor supported on $E$
2. $K_Z = g^*K_Y + R$, $R = sL + E + W$, $r + s = d - 4 = 1$, and $s \geq 0$. 


Since \( g^*(F) \) moves on \( Y \), so \( r > 0 \). And since \( g^*(K_Y) = \rho g^*(\lambda) \) we get \( \rho \leq r \leq d - 4 = 1 \). Therefore \( \rho = 1 \). We have then \( N = 2, k_1 = 2, \) and \( k_2 = 3 \). When \( d = 5 \) we have \( r = \rho = 1, \) and \( s = 0 \). The branch divisor is contained in the image \( g(E) \) of the exceptional divisor. This is a rigid divisor. Let \( M(X) \) be the number of moduli of \( X \) and \( M(Y) \) the number of moduli of \( Y \), this implies \( 10 = M(Y) \geq M(X) = 40 \). Then we get a contradiction.

\[ \square \]

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