MOMENTS OF THE RANK OF ELLIPTIC CURVES

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Abstract. Fix an elliptic curve $E/\mathbb{Q}$, and assume the generalized Riemann hypothesis for the $L$-function $L(E_D, s)$ for every quadratic twist $E_D$ of $E$ by $D \in \mathbb{Z}$. We combine Weil’s explicit formula with techniques of Heath-Brown to derive an asymptotic upper bound for the weighted moments of the analytic rank of $E_D$. It follows from this that, for any unbounded increasing function $f$ on $\mathbb{R}$, the analytic rank and (assuming in addition the Birch-Swinnerton-Dyer conjecture) the number of integral points of $E_D$ are less than $f(D)$ for almost all $D$. We also derive an upper bound for the density of low-lying zeros of $L(E_D, s)$ which is compatible with the random matrix models of Katz and Sarnak.

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1. Introduction

Let $E$ be an elliptic curve over $\mathbb{Q}$. The Birch-Swinnerton-Dyer conjecture predicts that

$$ r_{mw}(E) := \text{the rank of the Mordell-Weil group of } E/\mathbb{Q} $$

is equal to the analytic rank

$$ r_{an}(E) := \text{the order at } s = 1 \text{ of the } L\text{-function } L(E, s). $$

This implies in particular the Parity Conjecture:

$$ w(E) = (-1)^{r_{mw}(E)}, $$

where $w(E)$ denotes the sign of the functional equation of $L(E, s)$. Denote by $N_E$ the conductor of $E/\mathbb{Q}$, and by $E_D$ the quadratic twist of $E$ by an integer $D$. If $D$ is square-free

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and is prime to $2N_E$, we have the relation

$$w(E_D) = w(E)\chi_D(-N_E),$$

where $\chi_D$ denotes the quadratic character associated to $Q(\sqrt{D})$. Thus among the square-free integers $D$ prime to $2N_E$, the Parity Conjecture implies that half of the twists $E_D$ have odd Mordell-Weil rank, and the other half, even. Early experimental investigations ([2], [16]) suggest that a positive portion of the quadratic (resp. cubic) twists have rank $\geq 2$. On the other hand, the random matrix models of Katz and Sarnak ([9, §4 and 5], [7, p. 9-10]), which presupposes the Generalized Riemann hypothesis (GRH), predicts that half of the twists should have analytic rank 0, and the other half, analytic rank 1, whence the average analytic rank over all twists should be $1/2$. See [13] for a recent survey on ranks of elliptic curves, and [9] on random matrix theory.

Goldfeld seems to have been the first person to investigate the average rank of elliptic curves in a quadratic twist family. His main tool is Weil’s explicit formula. For the rest of this paper $F$ denotes the triangle function

$$F(x) = \max(0, 1 - |x|).$$

The explicit formula says that the sum over powers of traces of Frobenius of $E_D$, weighted by $F$, is essentially equal to a sum of the Mellin transform of $F$ extended over the non-trivial zeros of $L(E_D, s)$. Under GRH, each term of this latter sum is non-negative. Since $r_{an}(E_D)$ is the order of $L(E_D, s)$ at $s = 1$, to bound the average analytic rank we are led to study the average of the non-Archimedean side of the twisted explicit formula. In this way, Goldfeld [4] shows that under GRH, for $x \gg_{E, \epsilon} 1$ we have

$$\sum_{|D| < x} r_{an}(E_D) \leq (3.25 + \epsilon) \sum_{|D| < x} 1.$$  

He also points out that any improvement of the constant 3.25 to a number strictly less than 2 would imply that a positive portion of the twists would have analytic rank 0, a statement which at present has been proved unconditionally only for special classes of $E$. In his unpublished manuscript, Heath-Brown [6] makes a major breakthrough by improving Goldfeld’s constant, also under GRH, from 3.25 to 1.5, and with $D$ restricted to twists with the same root number. This implies that under GRH, a positive portion of the twists of $E$ have rank 0 and 1, respectively. This improvement is a result of better control over the non-Archimedean side of the twisted explicit formula, so Heath-Brown’s upper bounds are in fact upper bounds for the average of the Archimedean side. By keeping track of the contribution from all the non-trivial zeros and not just $s = 1$, we can apply Heath-Brown’s technique to get an asymptotic formula for all moments of the twisted explicit formula.

For the rest of this paper, the constants involved in any $O$, $o$ and $\ll$ expressions are with respect to the variable $x$ only and depend only on those parameters printed as subscripts next to these symbols. In particular, any unadorned $O$, $o$ and $\ll$ constants are absolute.

**Main Theorem.** Fix a non-negative, thrice continuously differentiable function $W$ compactly supported on $(0, 1)$ or $(-1, 0)$. Fix an elliptic curve $E/Q$, and assume the GRH for every $L(E_D, s)$. For any positive integer $k = o_E(\log \log \log x)$, as $1 + i\tau_D$ runs through the
non-trivial zeros of $L(E_D, s)$ with $\tau_D \neq 0$ we have
\[
\sum_D \left[ r_{an}(E_D) + \sum_{\tau_D \neq 0} \left( \frac{\sin(\tau_D \log x)}{\tau_D \log x} \right)^2 \right] W\left( \frac{D}{x^{k/2} \log^{2k+2} x} \right) \leq \frac{1}{2} \left[ \left( k + \frac{1}{2} + \frac{1}{\sqrt{3}} \right)^k + \left( k + \frac{1}{2} - \frac{1}{\sqrt{3}} \right)^k + o_{E,W}(1) \right] \sum_D W\left( \frac{D}{x^{k/2} \log^{2k+2} x} \right).
\]

Note that the Main Theorem is effective with respect to $k$ so long as $k = o_E(\log \log \log x)$. This allows us to deduce the following result (cf. § 4).

**Corollary 1.** Let $f$ be an unbounded increasing function on $\mathbb{R}$. Fix an elliptic curve $E/\mathbb{Q}$, and assume the GRH for every $L(E_D, s)$. Then the set of integers $D$ for which $r_{an}(E_D) > f(D)$ has density zero.

Conjectures of Lang (and others) giving height bounds for rational and integral points on elliptic curves suggest that ‘most’ elliptic curves have no integral points.\(^1\) Thanks to Corollary 1 and the work of Silverman, we can make this precise for quadratic twist families. Let
\[
E : y^2 = x^3 + Ax + B
\]
be a quasi-minimal model for $E/\mathbb{Q}$ (i.e. $|4A^3 + 27B^2|$ is minimal subject to $A, B \in \mathbb{Z}$). Silverman [15, Theorem A] shows that there exists an absolute constant $\kappa$ such that, if the $j$-invariant of $E/\mathbb{Q}$ is non-integral for $\leq \delta$ primes, then
\[
\left[ \text{the number of } S \text{-integral points on the quasi-minimal model (2)} \right] \leq \kappa^{(1 + r_{mw}(E))(1 + \delta) + \# S}.
\]
Since (2) is quasi-minimal for $E$, up to a bounded power of 2 and 3 the Weierstrass equation
\[
y^2 = x^3 + AD^2 x + BD^3
\]
is quasi-minimal for $E_D$ if $D$ is square-free. Since the $j$-invariant is constant in a quadratic twist family, Silverman’s theorem plus Corollary 1 immediately yields the following conditional result which makes precise for quadratic twist families the heuristic above on integral points (note that $N_{E_D} \ll E^D$).

**Corollary 2.** Fix an elliptic curve $E/\mathbb{Q}$, and assume the GRH and the Birch-Swinnerton-Dyer conjecture for every $L(E_D, s)$. Then for any unbounded increasing function $f$ on $\mathbb{R}$, the set of integers $D$ for which the Weierstrass equation (4) has more than $f(N_{E_D})$ integral points has density zero. \(\square\)

**Question 1.** Brumer [1] shows that the average analytic rank of all elliptic curves over $\mathbb{Q}$, as ordered by their height, is $\leq 2.3$. Is there a higher moment analog of this result? Lang [10] p. 140] conjectures that the number of integral points on a quasi-minimal model of any $E/\mathbb{Q}$ should be bounded solely in terms of $r_{mw}(E_D)$. Silverman [14] p. 251] poses the

\(^1\)I would like to thank Professor Silverman for bringing this to my attention.
more precise conjecture that (3) should hold for all $E$ with no $\delta$-dependence. Silverman’s conjecture plus a higher moment analog of Brumer’s theorem should allow us to extend Corollary 2 uniformly to all elliptic curves over $\mathbb{Q}$.

The two Corollaries above exploit the effectiveness of the Main Theorem with respect to $k$. We now investigate consequences of the Main Theorem for fixed $k$. First, we fix a number $R > 0$ and set $k = [R/e] + 1$ to obtain the following weighted upper bound on the density of large rank twists.

**Corollary 3.** Fix an elliptic curve $E/\mathbb{Q}$, and the GRH for every $L(E_D, s)$. Then for any fixed $R > 0$ and $x \gg R 1$, we have

$$\sum_{r_{an}(E_D) \geq R} W\left(\frac{D}{x}\right) \leq \frac{1/2 + o_{E,W,R}(1)}{1.44467^R} \sum_D W\left(\frac{D}{x}\right). \quad \square$$

**Remark 1.** For $k = 1$, the Main Theorem is essentially due to Heath-Brown. More precisely, denote by $\Delta_E(\pm)$ the set of square-free integers $D$ prime to $N_E$ for which $L(E_D, s)$ have root numbers $+1$ and $-1$, respectively. Then Heath-Brown shows that

$$\sum_{D \in \Delta_E(\pm)} r_{an}(E_D) W\left(\frac{D}{x}\right) \leq \left(\frac{3}{2} + o_{E}(1)\right) \sum_D W\left(\frac{D}{x}\right). \quad (5)$$

It then follows that

$$\sum_{D \in \Delta_E(+) \atop r_{an}(E_D) = 0} W\left(\frac{D}{x}\right) \geq \left(\frac{1}{2} + o_{E}(1)\right) \sum_{D \in \Delta_E(+)} W\left(\frac{D}{x}\right), \quad (6)$$

$$\sum_{D \in \Delta_E(-) \atop r_{an}(E_D) = 1} W\left(\frac{D}{x}\right) \geq \left(\frac{3}{4} + o_{E}(1)\right) \sum_{D \in \Delta_E(-)} W\left(\frac{D}{x}\right). \quad (7)$$

The general outline of the proof of the Main Theorem follows that of Heath-Brown; in particular, we make crucial use of his smooth averaging; cf. §5. Our main contribution is in the handling of certain truncated multivariable sums (Proposition 2) and in the arithmetic applications (Corollary 1 to 4). In particular, for $k > 1$ the Main Theorem (and hence Corollary 3) can also be refined to sum over $D \in \Delta_E(\pm)$ only; we can even drop the condition $(D, N_E) = 1$, at the cost of introducing tedious congruence argument on $D$ in the proof of the Main Theorem. Such refinements, however, do not improve the lower bounds (6) and (7), so we will not pursue these issues here.

From the proof of the Main Theorem we see that $x^{k/2} \log^{2k+2} x$ can be replaced by $x^{k/2+\epsilon}$ for any $\epsilon > 0$, provided that we stipulate the $o(1)$-term on the right side be dependent upon
Thus consider the number of non-trivial zeros \( \rho \) of \( L(E_D, s) \) with \(|\text{im}(\rho)| < Y \) is \( \frac{Y \log Y}{2\pi} + O_E(Y + \log |D|) \). This suggests that if the low-lying zeros of \( L(E_D, s) \) are uniformly distributed as \( D \) varies, then removing the factor \( k + \epsilon \) from the \( \tau_D \)-sum in (8) should result in scaling the right side of (8) by a factor of \( (k + \epsilon)^{-k} \). That would mean almost all twists of \( E_D \) would have analytic rank \( \leq 1 + \epsilon \).

**Question 2.** Can we make precise this heuristic argument? Specifically, does random matrix theory provide the proper framework within which to formulate the type of uniform distribution statement required here?

The factor \( k + \epsilon \) in the \( \tau_D \)-sum is due to the fact that the asymptotic formula in (8) sums over \( |D| \ll W x^{k/2 + \epsilon} \). If we can prove a similar formula – even just an upper bound – by summing over \( |D| \ll W x^\alpha \) for some fixed \( \alpha \), uniformly for infinitely many \( k \), then we would be able to prove that almost all \( E_D \) have analytic rank \( \leq 2\alpha + 1 \). The reason we need to take such a long sum is to ensure that the main term dominates the error term (28). Now, our argument leading up to (28) is essentially optimal, except in one step where we estimate a difference of two terms by bounding each term; cf. Remark 4.

**Question 3.** Can we improve this error term (28)?

Corollary 3 gives an upper bound for the weighted average of the multiplicity of the (potential) zero at \( s = 1 \) of \( L(E_D, s) \). This argument can be extended to count non-trivial zeros of bounded height. We begin with some notation. If \( E_D \) is an even twist, then under GRH the non-trivial zeros of \( L(E_D, s) \) come in complex conjugate pairs \( 1 + i\gamma_{E_D,j} \) with \( 0 \leq \gamma_{E_D,1} \leq \gamma_{E_D,2} \leq \cdots \). If \( E_D \) is an odd twist, then \( L(E_D, s) \) has a zero at \( s = 1 \); we label the remaining zeros as \( 1 + i\gamma_{E_D,j} \) with \( 0 \leq \gamma_{E_D,1} \leq \gamma_{E_D,2} \leq \cdots \). Finally, regardless of the parity of \( E_D \), define

\[
\tilde{\gamma}_{E_D,j} = \gamma_{E_D,j} (\log N_{E_D})/2\pi.
\]

Since \( \left( \frac{\sin(\frac{x}{2})/x}{\frac{x}{2}} \right)^2 \) is decreasing for \( 0 < x < 2\pi \), if for some \( |D| \gg E \) 1 we have \( \tilde{\gamma}_{E_D,3k} < 1/2\pi \), then for this \( D \) and for every \( j \leq 3k \),

\[
\left( \frac{\sin((\log |D|)/2)/2}{-\log |D|/2} \right)^2 \left( \frac{\sin(1/2)}{1/2} \right)^2 = 0.9193953884.
\]

Invoke the Main Theorem and we get
Corollary 4. Fix an elliptic curve $E/\mathbb{Q}$, and assume the GRH for every $L(E_D, s)$. For any integer $k > 0$ and $x \gg k^1$, we have
\[
\sum_{\tilde{\gamma}_{E_D, \pm k} < 1/2\pi} W\left(\frac{D}{x}\right) \leq \frac{1 + o_{E,W,k}(1)}{1.402408^k} \sum_D W\left(\frac{D}{x}\right). \quad \square
\]

To put this result into context, recall that random matrix theory \[ 8, \S 6.9, \S 7.5.5 \] furnishes a family of probability measures $v(+, j), v(-, j)$ on $\mathbb{R}$, $j = 1, 2, \ldots$, with respect to which Katz and Sarnak formulate the following conjecture.

Conjecture (Katz-Sarnak). For any integer $j \geq 1$ and any compactly supported complex-value function $h$ on $\mathbb{R}$,
\[
\sum'_{w(E_D) = +1} h(\tilde{\gamma}_{E_D, j}) = \left( \sum'_{w(E_D) = +1} 1 + o_{E,h}(1) \right) \int_{\mathbb{R}} h \cdot dv(+, j),
\]
where $\sum'_D$ signifies that $D$ runs through all square-free integers $D$. Similarly for $v(-, j)$.

As is pointed out in (\[ 9, \text{p. 21}], [7, \text{p. 10}]), this Conjecture implies that almost all even (resp. odd) twists of $E$ have analytic rank 0 (resp. 1). By choosing $h$ to be supported on an arbitrarily small neighborhood of $0 \in \mathbb{R}$, this Conjecture implies that for any fixed $j$ and any $\epsilon > 0$, there exists $\delta_j(\epsilon) > 0$ so that
- $\delta_j(\epsilon) \to 0$ as $\epsilon \to 0$; and
- the set of square-free $D$ for which $\tilde{\gamma}_{E_D, j} < \epsilon$ and $w(E_D) = 1$, has density $< \delta_j(\epsilon)$.

In particular, for any $\epsilon > 0$ the $\delta_j(\epsilon)$ (if they exist) form a non-increasing sequence that converges to 0. With respect to this formalism, Corollary 4 can be viewed as proving the existence of $\delta_j(1/2j)$ under GRH (instead of the full random matrix theory), such that $\delta_j(1/2\pi) \to 0$ as $j \to \infty$. However, our present argument does not allow us to replace $1.402408$ with an arbitrarily large constant by replacing $1/2\pi$ with an arbitrarily small number.

Remark 2. The Main Theorem, and hence the Corollaries, readily extends to cubic and higher order twists; cf. Remark 3. We can also replace $E$ by a newform.

Acknowledgment. I am indebted to Professor Heath-Brown for sending me a copy of his preprint \[ 6]. I would like to thank Professors Hajir, Hoffstein, Rosen and Silverman for many useful discussions.

2. Explicit Formula

Fix a modular elliptic curve $E/\mathbb{Q}$ of conductor $N_E$. Denote by $a_n(E)$ the $n$-th coefficient of $L(E, s)$. For any prime $p \nmid N_E$, denote by $\alpha_p(E)$ and $\overline{\alpha}_p(E)$ the eigenvalues of the Frobenius of $E/F_p$. Define
\[
c_n(E) = \begin{cases} 
\alpha_p(E)^m + \overline{\alpha}_p(E)^m & \text{if } n = p^m > 1 \text{ and } p \nmid N_E; \\
a_p(E)^m & \text{if } n = p^m > 1 \text{ and } p|N_E; \\
0 & \text{otherwise}.
\end{cases}
\]
Note that $c_p(E) = a_p(E)$. For any $\lambda > 0$, define $F_\lambda(x) = F(x/\lambda)$. Denote by $\Phi_\lambda(x)$ the Mellin transform of $F_\lambda$:

$$\Phi_\lambda(u) = \int_{-\infty}^{\infty} F_\lambda(x)e^{(u-1)x} dx.$$ 

Note that if $s = 1 + it$ with $t \in \mathbb{R}$, then

$$\Phi_\lambda(s) = \lambda \left( \frac{\sin(\lambda t/2)}{\lambda t/2} \right)^2.$$ 

As $\rho$ runs through the zeros $\rho = \beta + i\gamma$ of $L(E, s)$ with $0 < \beta < 2$, counted with multiplicity, Weil’s explicit formula [12, §II.2] says that

$$\sum \Phi_\lambda(\rho) := \lim_{z \to \infty} \sum_{|\rho| < z} \Phi_\lambda(\rho) = \log N_E - 2 \sum_{p} c_{p^n}(E) \log p F \left( \frac{\log p^n}{\lambda} \right) - 2 \log 2\pi - 2 \int_{0}^{\infty} \left( \frac{F(t/\lambda)}{e^t - 1} - \frac{1}{te^t} \right) dt.$$ 

Note that $|c_{p^n}(E)| \leq 2p^{m/2}$. Since $||F|| \leq 1$, that means

$$\sum_{p, m \geq 3} \frac{c_{p^n}(E) \log p}{p^m} F \left( \frac{\log p^n}{\lambda} \right) \ll \sum_{p, m \geq 3} \frac{\log p}{p^{m/2}} \ll \sum_{n > 1} \frac{\log n}{n^{3/2}} \ll 1.$$ 

For $\lambda \geq 1$, the integral in (10) is bounded from above and below by absolute constants, so the explicit formula now takes the form

$$\sum \Phi_\lambda(\rho) = \log N_E - 2 \sum_{p} \frac{c_p(E) \log p}{p} F \left( \frac{\log p}{\lambda} \right) - 2 \sum_{p} \frac{c_{p^2}(E) \log p}{p} F \left( \frac{\log p^2}{\lambda} \right) + O(1).$$ 

Next, we study how the explicit formula behaves under quadratic twists. If $p \nmid 2N_ED$ (note that $2N_E$ and $D$ need not be coprime and $D$ need not be square-free), then

$$c_p(E_D) = a_p(E) \left( \frac{D}{p} \right), \quad c_{p^2}(E_D) = c_{p^2}(E).$$ 

Since $||F|| \leq 1$,

$$\sum_{p | 2N_ED} \frac{F \left( \frac{\log p}{\lambda} \right) \log p}{p} \left( c_p(E_D) - a_p(E) \left( \frac{D}{p} \right) \right) \ll \sum_{p | 2N_ED} \frac{\log p}{\sqrt{p}},$$

$$\sum_{p | 2N_ED} \frac{F \left( \frac{\log p}{\lambda} \right) \log p}{p} \left( c_{p^2}(E_D) - c_{p^2}(E) \right) \ll \sum_{p | 2N_ED} \frac{\log p}{p}.$$ 

Since $\log p \ll p^{1/4}$, for $|D| \geq 2$ the right side of both expressions above are

$$\ll \sum_{p | 2N_ED} p^{-1/4} \ll \sum_{p < \log(2N_E|D|)} p^{-1/4} \ll E \log^{3/4} |D|.$$
As $\rho_D$ runs through the zeros of $L(E_D, s)$ with $0 < \Re(\rho_D) < 2$, we now have
\[
\sum_{\rho_D} \Phi_\lambda(\rho_D) = \log N_{E_D} - 2 \sum_p \frac{c_p(E) \log p}{p} D(F\left(\frac{\log p}{\lambda}\right)) - 2 \sum_p \frac{c_p(E) \log p}{p^2} F\left(\frac{2\log p}{\lambda}\right) + O(\log^{3/4}|D|).
\]

**Lemma 1.** We have the estimates
\[
\sum_p \frac{c^2_p(E) \log p}{p^2} F\left(\frac{2\log p}{\lambda}\right) = -\lambda/4 + o_E(\lambda),
\]
\[
\sum_p \frac{a_p(E)^2 \log^2 p}{p^2} F\left(\frac{\log p}{\lambda}\right)^2 = \lambda^2/12 + o_E(\lambda^2).
\]

**Proof.** If $p \nmid N_E$, then $c^2_p(E) = a_p(E)^2 - 2p$, so
\[
\sum_{p \nmid 2N_E} \frac{c^2_p(E) \log p}{p^s} = \sum_{p \nmid 2N_E} \frac{a_p(E)^2 \log p}{p^s} - 2 \sum_{p \nmid 2N_E} \frac{\log p}{p^{s-1}}.
\]
Up to the bad primes and a term holomorphic for $\Re(s) > 3/2$, the two sums on the right are $(-1)$ times the logarithmic derivative of, respectively, the Rankin-Selberg $L$-function of the cusp form associated to $E$ with itself, and $\zeta(s-1)$. Each of the convolution $L$-function and $\zeta(s-1)$ has a simple pole at $s = 2$. Tauberian theorem then gives
\[
- \sum_{p < x} \frac{a_p(E)^2 \log p}{p} = \sum_{p < x} \frac{c^2_p(E) \log p}{p} = -x + o_E(1).
\]
The Lemma now follows from partial summation. \(\square\)

Set $\lambda = \log x$ and define
\[
\beta_p = \frac{a_p(E) \log p}{p} F\left(\frac{\log p}{\log x}\right), \quad X_k = x^{k/2} \log^{2k+2} x.
\]
In what follow, we will take $D$ so that $|D| \leq X_k$. From now on, assume\(^2\)
\[
(11) \quad k = o_E(\log \log \log x),
\]
whence $O_E(\log^{3/4} |D|) = o_E(\log x)$. Combine all these and recall that $N_{E_D} \ll N_{E_D}^2$, we now arrive at the final form of the explicit formula for $E_D$:
\[
(12) \quad \sum_{\rho_D} \Phi_{\log x}(\rho_D) \leq \log(D^2) + (\log x)/2 - 2 \sum_p \beta_p \left(\frac{D}{p}\right) + o_E(\log x).
\]
We emphasize again that $D$ need not be coprime to $2N_E$ or square-free.

\(^2\)We choose this $o$-bound for $k$ to simplify the exposition. The optimal choice would be that which renders the $O$-term in Proposition to be $o_E(\log x)$, but such refinements have no material impact on the arithmetic applications of the Main Theorem.
3. Moments of analytic rank

Define

\[ f(x, D) = 2 \log |D| + (\log x)/2, \quad R(x, D) = 2 \sum_p \beta_p \left( \frac{D}{p} \right). \]

Let \( W \) be a thrice continuously differentiable function with compact support on \((1/2, 1)\) or \((-1, -1/2)\). The \( k \)-th moment of the twisted explicit formula, weighted by \( W \), now becomes

\[
\sum_D \left( \sum_{\rho_D} \Phi_{\log x}(\rho_D) \right)^k W \left( \frac{D}{X_k} \right) \leq \sum_D \left( 2 \log |D| + (\log x)/2 + o_E(\log x) \right)^k W \left( \frac{D}{X_k} \right)
\]

\[ + \sum_{r=1}^k \binom{k}{r} (-1)^r \sum_D f(x, D)^{k-r} R(x, D)^r W \left( \frac{D}{X_k} \right) \]

\[ + \sum_{r=1}^k \binom{k}{r} o_{E,k} \left( \sum_i \binom{k-r-i}{r} \log^i x \sum_D f(x, D)^{k-r-i} R(x, D)^r W \left( \frac{D}{X_k} \right) \right) \]

We begin by tackling the first of the three sums on the right.

Lemma 2. For \( l \geq 0 \), we have

\[
\sum_D f(x, D)^l W \left( \frac{D}{X_k} \right) = \left( (k + 1/2) \log x + o_{E,W}(\log x) \right)^l \left[ \sum_D W \left( \frac{D}{X_k} \right) + o(X_k) \right].
\]

Proof. Since \( W(x) = 0 \) if \( |x| \geq 1 \), the sum in the Lemma extends over \( |D| \leq X_k \) only. Thus with \( X' := x^{k/2} \), from (11) we see that

\[
\left( (k + 1/2) \log x + o_{E}(\log x) \right)^l \sum'_{|D| > X'} W \left( \frac{D}{X_k} \right) \geq \sum'_{|D| > X'} f(x, D)^l W \left( \frac{D}{X_k} \right)
\]

\[ \geq \left( (k + 1/2) \log x + o_{E}(\log x) \right)^l \sum'_{|D| > X'} W \left( \frac{D}{X_k} \right). \]

The condition \( |D| > X' \) can be dropped at the cost of introducing a term

\[
\ll \left( (k + 1/2) \log x + o_{E}(\log x) \right)^l \sum'_{|D| \leq X'} W \left( \frac{D}{X_k} \right) \ll \left( (k + 1/2) \log x + o_{E}(\log x) \right)^l x^{k/2},
\]

and the Lemma follows. \( \square \)

The rest of the paper is devoted to prove the following result. The proof of the Main Theorem makes use of the conditional estimate only; we state the unconditional result for comparison.
Proposition 1. For \( r > 0 \), we have the estimate
\[
\sum_{D} f(x, D)^{i} R(x, D)^{r} W\left(\frac{D}{X_{k}}\right)
\begin{cases}
= \left(2 \log X_{k} + \frac{\log x}{2} + o_{E, W}(\log x)\right)^{i} (1 + o(E))^{r/2} \log x \sum_{D} W\left(\frac{D}{X_{k}}\right) + O_{E, W}(4 r^{3} x^{3r}(\log X_{k} + \log x)^{r+1}/T^{2}) & \text{if } r \text{ is even}, \\
= O_{E, W}(4 r^{3} x^{3r}(\log X_{k} + \log x)^{r+1}/T^{2}) & \text{if } r \text{ is odd}.
\end{cases}
\]

If we assume the GRH for every \( L(E_{D}, s) \), then the \( O \)-term can be improved to
\[
O_{E, W}(c_{E} r^{3} x^{3r}(\log X_{k} + \log x)^{r+1}/T^{2})
\]
for some constant \( c_{E} \) depending on \( E \) only.

Assuming the GRH-estimate, we then see that
\[
\frac{1}{\log x} \sum_{D} \left(\sum_{\rho_{D}} \Phi_{\log x}(\rho_{D})\right)^{k} W\left(\frac{D}{X_{k}}\right) \leq (k + 1/2 + o_{E}(1))^{k} \sum_{D} W\left(\frac{D}{X_{k}}\right) + \sum_{r = 1}^{k} \binom{k}{r} (1 + o_{E}(1))^{r/2} (k + 1/2 + o_{E}(1))^{k-r} (1/\sqrt{3})^{r} \sum_{D} W\left(\frac{D}{X_{k}}\right)
\]
\[
+ O_{E, W}(k^{3} c_{E}^{k} x^{2k}(\log X_{k} + \log x)^{2k}).
\]
Recall (11) and we see that this \( O \)-term is \( o_{E, W}(X_{k}) \). To write \( \sum_{r \text{ even}} \) is to write \( \frac{1}{2} \sum_{\text{all } r} (1 + (-1)^{r}) \). Expand the rest of the second line above accordingly and recall (11), we get
\[
\sum_{D} \left[r_{an}(E_{D}) + \sum_{\tau_{D} \neq 0} \left(\frac{\sin(\tau_{D}(\log x)/2)}{\tau_{D}(\log x)/2}\right)^{2} \right]^{k} W\left(\frac{D}{X_{k}}\right)
\leq \frac{1}{2} \left[(k + 1/2 + 1/\sqrt{3})^{k} + (k + 1/2 - 1/\sqrt{3})^{k} + o_{E, W}(1)\right] \sum_{D} W\left(\frac{D}{X_{k}}\right),
\]
and the Main Theorem follows.

4. Proof of Corollary 1

Given any subset \( S \subset \mathbb{Z} \), define its lower density to be the lim sup over all numbers \( \sigma \geq 0 \) such that
\[
\#\{s \in S : |s| < x\} > \sigma x \quad \text{for all } x \gg s, \sigma 1
\]
In particular, \( S \) has density zero if and only if it has lower density zero.

Lemma 3. With \( W \) as in the Lemma, there exists a constant \( \lambda_{W} > 0 \) depending on \( W \) only, such that for any subset \( S \subset \mathbb{Z} \) with lower density \( \sigma_{S} \), we have
\[
\sum_{s \in S} W\left(\frac{s}{x}\right) > \lambda_{W} \cdot \sigma_{S} x.
\]
Proof. Without loss of generality, assume that \( W \) is supported on \((0, 1)\). Since \( W \) is continuous and since \( W(0) = W(1) \), there exists an integer \( n > 4/\sigma_s \) such that for some \( 0 < m < n - 1 \) and some \( w_0 > 0 \), we have
\[
W(r) \geq w_0 \text{ for } r \in N_n(m) := \{ t \in \mathbb{R} : m/n \leq t \leq (m+1)/n \}.
\]

For \( 0 \leq i < n \), set
\[
S_n(i, x) := \{ s \in S : i/n \leq s/x \leq (i+1)/n \}.
\]
By the definition of lower density, for \( x \gg \sigma_s \) we have \( \sum \#S_n(i, x) \geq \sigma_s x \). That means \( \#S_n(j, x) \geq \sigma_s x/2n \) for some \( j \neq 0, n - 1 \): otherwise
\[
\sigma_s x \leq \sum \#S_n(i, x) \leq \sum \#S_n(i, x) + \#S_n(0, x) + \#S_n(n - 1, x)
\]
\[
\leq (n - 2)\sigma_s x/2n + 2(x/n + O(1))
\]
\[
= x(\sigma_s/2 - \sigma_s/n + 2/n) + O(1)
\]
\[
< (1 - 1/n)\sigma_s x + O(1),
\]
a contradiction.

Suppose this \( j \geq m \); then for \( s \in S_n(j, x) \),
\[
\frac{m}{n} = \frac{j}{n} \frac{m}{j} \leq \frac{s}{x} \frac{m}{j} \leq \frac{j+1}{n} \frac{m}{j} \leq \frac{m+1}{n},
\]
whence
\[
\sum_{s \in S} W\left(\frac{s}{x \mu}\right) \geq \sum_{s \in S_n(j, x)} W\left(\frac{s}{x \mu}\right) \geq w_0 \frac{\sigma_s x}{n} = w_0 \frac{\sigma_s (x/j) \frac{m}{j}}{n} \geq w_0 \frac{\sigma_s (x/j)}{n^2}. \]

Next, suppose \( j < m \). Then for some \( 0 \leq \mu m \) we have \( \# S_l \geq \sigma_s x/2mn \), where
\[
S_l := \{ s \in S : \frac{1}{n} (j + \frac{l}{m}) \leq s \leq \frac{1}{n} (j + \frac{l+1}{m}) \}.
\]
Multiplication by \( \mu := (m+1)/(j + \frac{l+1}{m}) \) takes the interval \([j + \frac{l}{m}, j + \frac{l+1}{m}]\) inside the interval \([m, m+1]\), whence
\[
\sum_{s \in S} W\left(\frac{s}{x \mu}\right) \geq \sum_{s \in S_l} W\left(\frac{s}{x \mu}\right) \geq w_0 \frac{\sigma_s x}{mn} = w_0 \frac{\sigma_s (x \mu)}{mn \mu} \text{ since } \mu > 1 \text{ and } n > m.
\]

Combine these two cases for \( j \) and we see that
\[
\sum_{s \in S} W\left(\frac{s}{x}\right) \geq \frac{w_0}{n^2} \sigma_s x.
\]
This completes the proof of the Lemma. \( \square \)
Proof of Corollary 1. Without loss of generality we can assume that \( f(x) = o(\log \log \log x) \). Since \( f \) is unbounded and increasing, we can find a sequence \( 0 < x_1 < x_2 < \cdots \) such that
\[
x_n/f(x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.
\]
Define a function \( g \) on \( \mathbb{R} \) by
\[
g(x) = \begin{cases} f(x_1) & x < x_2, \\ f(x_i) & x_{i+1} \leq x < x_{i+2}, i \geq 1. \end{cases}
\]
By (14) we have \( g(x) = o(f(x)) = o(\log \log \log x) \). Finally, set \( k(x) = g(\sqrt{x}) \), so
\[
k(x) = o(f(\sqrt{x})) = o(f(x^{k(x)/2})).
\]
On the other hand,
\[
\sum_D r_{an}(E_D)^{k(x)} W\left(\frac{D}{X_{k(x)}}\right) \gg_{E,W} f(x^{k(x)/2}) \sum_{D \geq x_{k(x)/2}} W\left(\frac{D}{X_{k(x)}}\right) \quad \text{by the Main Theorem}
\]
\[
\sum_{D \geq x_{k(x)/2}} W\left(\frac{D}{X_{k(x)}}\right) \geq f(x^{k(x)/2}) \sum_{r_{an}(E_D) > f(D)} \quad \text{if } f \text{ is increasing}
\]
\[
\gg_{E,W} f(x^{k(x)/2}) \sum_{D \geq x_{k(x)/2}} W\left(\frac{D}{X_{k(x)}}\right) 
\]
In light of (15) this lower density must be zero, and Corollary 1 follows.

5. Poisson summation

In this section we adopt Heath-Brown’s argument to reduce Proposition 1 to a ‘multivariable prime number theorem’ for elliptic curves, to be proved in section 7. We begin with an auxiliary result. Denote by \( \hat{W}_l \) the Fourier transform with respect to \( t \) of
\[
W_l(x, t, X_k) := (\log(t^2 X_k^2) + (\log x)/2)^l W(t).
\]
Note that the integral defining \( \hat{W}_l \) makes sense since \( W(0) = 0 \).

Lemma 4. There exists a constant \( \gamma_W > 0 \) depending on \( W \) only, so that for \( l > 0, m \neq 0 \) and \( X_k > 2 \), as \( t \to \infty \),
\[
(i) \quad \|W\|, |\hat{W}_l| \text{ and } |\hat{W}_l| \text{ all satisfy } \gamma_W l^3 (\log X_k + \log x)^l \min(1, |t|^{-3});
\]
\[
(ii) \quad \int_2^x \frac{\partial^3}{\partial t^3} \left( \hat{W}_l \left( x, \frac{X_k m}{t}, X_k \right) \right) dt < \gamma_W l^3 (\log X_k + \log x)^l (T|m|)^{-1/2} \min\left(1, \left(\frac{x^r}{X_k m}\right)^{3/2}\right).
\]

Proof. For the rest of this proof, \( \gamma_l \) denotes a constant depending on \( W \) only. Since \( W(t) = 0 \) is zero around an open neighborhood of 0 and since \( W \) has compact support,
\[
\frac{\partial^3}{\partial t^3} W_l(x, t, X_k) < \gamma_l l^3 (\log X_k + \log x)^l.
\]
Apply integration by parts three times and recall that $W$ has compact support, we get

$$
\dot{W}(x, t, X_k) < \frac{1}{|t|^3} \int_{-\infty}^{\infty} \frac{\partial^3}{\partial y^3} W_i(x, y, X_k) dy < \gamma_3 t^3 (\log X_k + \log x)^t \min(1, |t|^{-3}).
$$

The same argument yields the same estimate for $\frac{\partial}{\partial t} \dot{W}_i(x, t, X_k)$ (with different constant). Consequently,

$$
\begin{align*}
\frac{\partial}{\partial t} \left[ \dot{W}_i(x, X_k) \frac{1}{|t|^{3/2}} \right] &= \left( \frac{\partial}{\partial t} \dot{W}_i \right) + \dot{W}_i \left( \frac{X_k}{t} \right) t^{-3/2} + \frac{\partial}{\partial t} \dot{W}_i \left( \frac{X_k}{t} \right) t^{-3/2} \\
&= \left\{ \begin{array}{ll}
\gamma_4 t^3 (\log X_k + \log x)^t \left[ (\frac{X_k}{t})^{-3} \frac{X_k}{t^{3/2}} + (\frac{X_k}{t})^{-3} t^{-3/2} \right] & \text{if } |X_k m/t| \geq 1, \\
\gamma_5 t^3 (\log X_k + \log x)^t \left[ \frac{X_k m}{t^{3/2}} + t^{-3/2} \right] & \text{if } |X_k m/t| < 1
\end{array} \right.
\end{align*}
$$

So if $|X_k m| \geq x^r$, the integral in the Lemma becomes

$$
< \gamma_7 t^3 (\log X_k + \log x)^t \int_{2}^{x^r} t^{-3/2} \frac{t^2}{|X_k m|^2} dt < \gamma_8 t^3 (\log X_k + \log x)^t \frac{x^{3r/2}}{|X_k m|^2}.
$$

On the other hand, if $|X_k m| \leq x^r$, then splitting the integral as $\int_{2}^{x^r} + \int_{x^r}^{X_k m}$ gives

$$
< \gamma_9 t^3 (\log X_k + \log x)^t \left( (X_k m)^{-1/2} + \int_{X_k m}^{x^r} t^{-3/2} dt \right) < \gamma_{10} t^3 (\log X_k + \log x)^t (X_k m)^{-1/2}.
$$

Take $\gamma_W$ to be the maximum of the $\gamma_i$ and the Lemma follows.

Recall the definition of $R(x, D)^r$ and we get

$$
\sum_D f(x, D)^i R(x, D)^r W \left( \frac{D}{X_k} \right) = 2^r \sum_D f(x, D)^i W \left( \frac{D}{X_k} \right) \sum_{p_1 \cdots p_r > 2} \beta_{p_1} \cdots \beta_{p_r} \left( \frac{D}{p_1} \right) \cdots \left( \frac{D}{p_r} \right).
\tag{16}
$$

Note that the primes $p_1, \ldots, p_r$ in the inner-sum above need not be distinct. In particular, the product of the quadratic symbols is a non-trivial character precisely when $p_1 \cdots p_r$ is not a square. We proceed accordingly.

**Contribution to (16) from those $(p_1, \ldots, p_r)$ whose product is a square**

Then every prime in the $r$-tuple appears with even multiplicity, which means (i) $r$ is even, and (ii) the product of quadratic characters in (16) is 1 if every $p_i \nmid D$, and is zero otherwise.
Thus the contribution in question is

\[ 2^r \sum_{p_1, \ldots, p_{r/2}} (\beta_{p_1} \cdots \beta_{p_{r/2}})^2 \sum_{D \neq 0(p_i)} f(x, D)^i W\left(\frac{D}{X_k}\right) \]

(17)

\[ = 2^r \sum_{p_1, \ldots, p_{r/2}} (\beta_{p_1} \cdots \beta_{p_{r/2}})^2 \sum_{\delta \mid \pi'} \mu(\delta) \sum_d f(x, d)^i W\left(\frac{d\delta}{X_k}\right), \]

where \( \pi' = p_1 \cdots p_{r/2} \) and \( \mu = \text{Möbius function} \). The terms in (17) with \( \delta > 1 \) sum to

\[ 2^r \left( \sum_p \beta_p^2 \right)^{r/2} \sum_d \left( 2 \log |d| + (\log x)/2 \right)^i W\left(\frac{d}{X_k}\right). \]

(18)

By Lemma 1 and Lemma 2 this is

\[ \leq \left( 2 \log X_k + (\log x)/2 + o(\log x) \right)^i \left( 1/3 + o_{E,r}(1) \right)^{r/2} \log^r x \sum_d W\left(\frac{d}{X_k}\right). \]

On the other hand, the terms in (17) with \( \delta = 1 \) sum to

\[ \ll 2^r \sum_{p_1, \ldots, p_{r/2}} (\beta_{p_1} \cdots \beta_{p_{r/2}})^2 \sum_{\delta \mid \pi'} \sum_d f(x, d)^i W\left(\frac{d\delta}{X_k}\right) \]

\[ \ll 2^r \left( 2 \log X_k + \frac{\log x}{2} + o(1) \right)^i \sum_{p_1, \ldots, p_{r/2}} (\beta_{p_1} \cdots \beta_{p_{r/2}})^2 \sum_{\delta \mid \pi'} \sum_{|d| \leq X_k/\delta} 1 + \sum_{|d| > X_k/\delta} \left( \frac{X_k}{d\delta} \right)^3 \]

(19) \[ \ll 2^r \left( 2 \log X_k + \frac{\log x}{2} + o(1) \right)^i \sum_{p_1, \ldots, p_{r/2}} (\beta_{p_1} \cdots \beta_{p_{r/2}})^2 \sum_{\delta \mid \pi'} X_k/\delta, \]

where in the second line we use Lemma 3(a). The number of \( \delta \mid \pi' = p_1 \cdots p_{r/2} \) is \( \leq 2^{r/2} \), so (13) is

\[ \ll X_k 2^{3r/2} \left( 2 \log X_k + (\log x)/2 + o_{E,i}(1) \right)^i \sum_p \frac{\beta_p^2}{p} \left( \sum_q \beta_q^2 \right)^{r/2-1} \]

\[ \ll X_k 2^{r/2} \left( 1/3 + o_E(1) \right)^{r/2-1} \left( 2 \log X_k + (\log x)/2 + o_{E,i}(1) \right)^i \log^{r-2} x. \]

Keeping in mind that \( \sum_D W(D/X_k) \ll W X_k \), we see that if \( r \) is even, then the terms in (16) coming from those \((p_1, \ldots, p_r)\) whose product is a square, is

\[ \left( 2 \log X_k + \frac{\log x}{2} + o_{E,i}(\log x) \right)^i \left( 1/3 + o_E(1) \right)^{r/2} \left( \log^r x + O_w(2^{r/2} \log^{r-2} x) \right) \sum_d W\left(\frac{d}{X_k}\right). \]

**Contribution to (16) from those \((p_1, \ldots, p_r)\) whose product is not a square**
Set
\[
\begin{align*}
\pi &= p_1 \cdots p_r, \\
\pi_0 &= \text{the largest perfect square divisor of } \pi \text{ such that } (\pi, \pi/\pi_0) = 1, \\
\pi_1 &= \text{the square-free part of } \pi_0, \\
\pi_2 &= \text{the square-free part of } \pi_1.
\end{align*}
\]

Then the contribution in question is equal to
\[
2^r \beta_{p_1} \cdots \beta_{p_r} \sum_{j(\pi_1\pi_2)} \left( \frac{j}{\pi_2} \right) \sum_{m=-\infty}^\infty f(x, j + m\pi_1\pi_2) i^W \left( \frac{j + m\pi_1\pi_2}{X_k} \right).
\]

Set \( e(z) = \exp(2\pi iz) \). Apply Poisson summation and we get
\[
2^r \sum_{p_1, \ldots, p_r \atop \pi_2 > 1} \beta_{p_1} \cdots \beta_{p_r} \sum_{j(\pi_1\pi_2)} \left( \frac{j}{\pi_2} \right) \sum_{m=-\infty}^\infty \hat{W}_i \left( x, \frac{X_k m}{\pi_1\pi_2}, X_k \right) \frac{X_k}{\pi_1\pi_2} e \left( \frac{mj}{\pi_1\pi_2} \right)
\]
\[
= 2^r X_k \sum_{p_1, \ldots, p_r \atop \pi_2 > 1} \beta_{p_1} \cdots \beta_{p_r} \sum_{m=-\infty}^\infty \hat{W}_i \left( x, \frac{X_k m}{\pi_1\pi_2}, X_k \right) \sum_{j(\pi_1\pi_2)} \left( \frac{j}{\pi_2} \right) e \left( \frac{mj}{\pi_1\pi_2} \right).
\]

Since \( \pi_2 > 1 \), if \( \pi_1\pi_2 \mid m \) then the \( j \)-sum in (21) is zero. So suppose \( \pi_1\pi_2 \nmid m \); in particular, \( m \neq 0 \). For \( l = 1, 2 \), set
\[
\delta_l = (\pi_l, m), \quad \pi_l = \delta_l \pi_l', \quad m = \delta_l m_l.
\]

Since \((\pi_1, \pi_2) = 1\), by the Chinese remainder theorem the \( j \)-sum in (21) is
\[
\left[ \sum_{j_1(\pi_1)} e \left( \frac{m_j}{\pi_1} \right) \right] \left[ \sum_{j_2(\pi_2)} e \left( \frac{m j_2}{\pi_2} \right) \right] = \left[ \sum_{l_1(\pi_1)} e \left( \frac{n_1 l_1}{\pi_1} \right) \sum_{j_1(\pi_1) \atop j_1 \equiv l_1(\pi_1')} \left[ \sum_{l_2(\pi_2) \atop j_2(\pi_2) \atop j_2 \equiv l_2(\pi_2')} e \left( \frac{l_2}{\pi_2} \right) e \left( \frac{n_2 l_2}{\pi_2} \right) \sum_{j_2(\pi_2)} \left( \frac{j_2}{\pi_2} \right) \right] \right].
\]

Note that the \( j_2 \)-sum in (22) is zero unless \( \delta_2 = 1 \), and the \( j_1 \)-sum is \( \delta_1 \). Moreover, \( \pi_1 \), and hence \( \pi_1' \), is square-free, so (22) is
\[
= (-1)^{\#(p|\pi_1')} \delta_1 \sum_{j(\pi_2)} \left( \frac{j}{\pi_2} \right) e \left( \frac{n \delta_1 j}{\pi_2} \right)
\]
\[
= (-1)^{\#(p|\pi_1')} \delta_1 \frac{\sqrt{\pi_2}}{1 + \sqrt{-1}} \left( \frac{n \delta_1}{\pi_2} \right) \left( 1 - \sqrt{-1} \frac{1}{\pi_2} \right),
\]

by the standard quadratic Gauss sum calculation. Recall that \( m = n \delta_1 \neq 0 \) and we see that (21) is
\[
\ll 2^r X_k \sum_{p_1, \ldots, p_r \atop \pi_2 > 1} \beta_{p_1} \cdots \beta_{p_r} \sum_{\delta_1|\pi_1 \atop \pi_1|n \neq 0} \sum_{p_l|n} \hat{W}_i \left( x, \frac{T \delta_1 n}{\pi_1\pi_2}, X_k \right) \delta_1 \left( \frac{\pm n \delta_1}{\pi_2} \right)
\]
\[
\ll 2^r X_k \sum_{|n| \neq 0 \atop \pi_2 > 1} \sum_{p_1, \ldots, p_r \atop (p_l, n) = 1} \beta_{p_1} \cdots \beta_{p_r} \sum_{\delta_1|\pi_1} \left( \frac{\pm n \delta_1}{\pi_2} \right) \left( \frac{1}{\pi_1|\sqrt{\pi_2}} \right) \hat{W}_i \left( x, \frac{T n}{\pi_1\pi_2}, X_k \right).
\]
We now estimate (23) in two ways, first unconditionally and then invoke the GRH.

**Unconditional Estimate**

For the unconditional estimate we will take the test function $F$ to be $F_3$, in which case $||F_3|| \leq 1$, whence $|\beta_p| \leq (2 \log p)/\sqrt{p}$. There are $\leq 2^r$ terms in the $\delta$-sum in $Q(p_1, \ldots, p_r, n)$. Since $F_3$ vanishes outside $(-1, 1)$, we have $\beta_p = 0$ if $p > x$. Use Lemma 4(a) to bound $\hat{W}_i$ and we see that

$$\ll_W \quad 2^r X_k \sum_{|n| \neq 0, p_1, \ldots, p_r < x} 2^r \frac{\log p_1 \cdots \log p_r (p_1 \cdots p_r)^3}{p_1 \cdots p_r} \frac{\log X_k + \log x}{T^3|n|^3} x^3 |n|^3 i^3 (\log X_k + \log x)^i$$

$$\ll_W \quad 4^r r^3 (\log X_k + \log x)^{r+i} x^{3r}/X_k^2.$$

**GRH Estimate**

First, rewrite (23) as

$$(24) \quad 2^r X_k \sum_{|n| \neq 0} \sum_{u \geq 2} \hat{W}_i(x, T u^{-1}, X_k) \frac{1}{u} \sum_{p_1 \cdots p_r, \pi_2 > 1} \beta_{p_1} \cdots \beta_{p_r} \sum_{\delta_1 | \pi_1} \left( \frac{\pm n \delta_1}{\pi_2} \right) \frac{1}{\sqrt{\pi_1}}$$

Note that if $p_1 \cdots p_r \geq x^r$, then $Q(p_1, \ldots, p_r, n) = 0$ for any $n$. In particular, the $u$-sum in (24) is a finite sum. To evaluate this $u$-sum we proceed by partial summation. That calls for the following estimate, to be proved in sections 6 and 7.

**Proposition 2.** Assume the GRH for every $L(E_D, s)$. Then there exists a constant $\tilde{c}_E$ depending on $E$ only so that, for any integers $m, r > 0$, as $p_1, \ldots, p_r$ run through all prime numbers,

$$\sum_{p_1 \cdots p_r \leq U} Q(p_1, \ldots, p_r, n) \ll \tilde{c}_E \left[ \log N_E + 3 \log |n| + 3 \log(U + 2) \right]^r \log^{2r+1} x.$$
Assuming this, the \( u \)-sum in \((24)\) is
\[
(26) \quad \sum_{p_1 \cdots p_t \leq x^r \atop (p_i, n) = 1 \atop \pi_2 > 1} Q(p_1, \ldots, p_t, n) \left\{ W_i \left( x, \frac{X_k n}{x^r}, X_k \right) \right\} \frac{1}{\sqrt{x^r}}
\]
\[
(27) \quad - \int_0^{x^r} \left\{ \sum_{p_1 \cdots p_t \leq t \atop (p_i, n) = 1 \atop \pi_2 > 1} Q(p_1, \ldots, p_t, n) \left\{ \frac{d}{dt} \left( W_i \left( x, \frac{X_k n}{t}, X_k \right) \right) \right\} \right\} dt
\]
\[
\ll_W r^3 \left( 3 \tilde{c}_E \right)^r (\log X_k + \log x)^r \left[ \log |n| + \log (U + 2) \right] \log^{2r+1} x \times \left( \log X_k + \log x \right)^i
\]
\[
\ll_{E,W} r^3 (3 \tilde{c}_E)^r (\log X_k + \log x)^i \left[ \log |n| + \log (U + 2) \right] \frac{\log^{2r+1} x}{\sqrt{|X_k n|}} \min \left( 1, \left| \frac{x^r}{X_k n} \right| \right).
\]
Recall that \( U \leq x^r \). Consequently, \((24)\) becomes
\[
(28) \quad \ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^i (\log |n| + \log x)^r \sum_{|n| \neq 0} \frac{\sqrt{X_k}}{\sqrt{|m|}} \min \left( 1, \left| \frac{x}{X_k n} \right| \right).
\]
Thus the contribution to the \( n \)-sum from those \( |n| \geq x^r / X_k \) is
\[
\ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^i (\log |n| + \log x)^r \sqrt{X_k} \sum_{|n| \geq x^r / X_k} \frac{1}{\sqrt{\log |n|}} \left( \frac{x}{X_k n} \right)^{3/2}
\]
\[
\ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^r \sqrt{X_k} \left( \frac{x}{X_k} \right)^{3/2} \sum_{|n| \geq x^r / X_k} \frac{\log^r |n|}{n^2}
\]
\[
\ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^r x^r / 2.
\]
On the other hand, the contribution from those \( |n| < x^r / X_k \) is
\[
\ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^r \sum_{0 < |n| < x / X_k} \frac{\log^r |n|}{\sqrt{|n|}}
\]
\[
\ll_{E,W} r^{r+3} (3 \tilde{c}_E)^r (\log X_k + \log x)^r x^r / 2.
\]
This completes the proof of Proposition \([1]\) \( \square \)

**Remark 3.** The argument in this section readily extends to higher order twist families. The main difference, say for the cubic twist family \( \mathcal{E}_m : x^3 + y^3 = m \), is that the argument now proceeds according to whether \( p_1 \cdots p_t \) is a perfect cube or not. The rest of the argument, including Proposition \([2]\) extends with no change. As a result, the Main Theorem extends to the cubic twist family \( \mathcal{E}_m \) with the factor 1/2 replaced by 1/3.

**Remark 4.** While Proposition \([2]\) gives an essentially optimal bound for the size of the \( Q \)-sum, we have no control over the sign of this \( Q \)-sum as \( u \) varies. Because of that, to estimate \((26)\) and \((27)\) using Proposition \([2]\) we are forced to put absolute value signs everywhere. This is
essentially the only place in the proof of the Main Theorem where we might lose information (the $\ll$ in (23) does not have any material impact on the rest of the proof).

6. A COMPLEX PRIME NUMBER THEOREM

The results in this section are elliptic curves analog of classical estimates; we provide the details for lack of a good reference. As is customary, given a complex number $s$ we denote by $\sigma$ and $t$ its real and imaginary part, respectively.

**Lemma 5.** Assume the GRH for $L(E,s)$. Then for $\sigma \geq 1 + 1/\log x$ and $|t| \geq 2$, we have the estimate

$$
L'(E,s)/L(E,s) \ll (\log N_E + \log(|s|+2)) \log x.
$$

**Proof.** We have the basic relation

$$
\frac{L'(E,s)}{L(E,s)} = \log \sqrt{\frac{N_E}{2\pi}} + \frac{\Gamma'(s)}{\Gamma(s)} - B_E - \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right),
$$

where $B_E$ is a constant depending only on $E$, and $\rho$ runs through the non-trivial zeros of $L(E,s)$. Since $\overline{L(E,s)} = L(E,\overline{s})$, complex conjugation takes the zeros of $L(E,s)$ to themselves; from (29) we see that $B_E$ is real, and that as in [3, p. 83],

$$
B_E = -\sum_{\rho} \text{Re} \left( \frac{1}{\rho} \right).
$$

The $\Gamma$-term in (29) is $\ll \log |t|$ if $|t| \geq 2$ and $1 \leq \sigma \leq 3$. It follows that

$$
\text{Re} \left( \frac{L'(E,s)}{L(E,s)} \right) \ll \left( \log N_E + \log(|t|+2) \right) - \sum_{\rho} \text{Re} \left( \frac{1}{s-\rho} \right).
$$

Since $L'(E,s)/L(E,s)$ is bounded on the line $\text{Re}(s) = 2$, for such $s$ we get

$$
\sum_{\rho} \text{Re} \left( \frac{1}{s-\rho} \right) \ll \log N_E + \log(|t|+2).
$$

Write $\rho = \beta + i\gamma$. Then for $s = 2 + it$,

$$
\text{Re} \left( \frac{1}{s-\rho} \right) = \frac{2 - \beta}{(2 - \beta)^2 + (t - \gamma)^2} \geq \frac{1/2}{1 + (t - \gamma)^2},
$$

whence

$$
\sum_{\rho} \frac{1}{1 + (t - \gamma)^2} \ll \log N_E + \log(|t|+2).
$$

Standard argument then shows that

$$
\frac{L'(E,s)}{L(E,s)} = \sum_{\rho} \frac{1}{s-\rho} + O(\log N_E + \log(|t|+2)),
$$

where the sum runs over those $\rho$ for which $|T-\gamma| < 1$. By (30) there are $\ll \log N_E + \log(|t|+2)$ such $\rho$, and under GRH, $|s-\rho| \geq 1/\log x$ if $\sigma \geq 1 + 1/\log x$. The Lemma then follows. □
Lemma 6. Assume the GRH for \(L(E,s)\). For \(j \geq 0, x \gg_{E} 1\) and \(1 + 1/\log x \leq \sigma \leq 2\), we have the estimate
\[
\frac{1}{\log^3 x} \sum_{p \leq x} \frac{a_p(E) \log^{1+j} p}{p^s} \ll (\log N_E + \log(|s| + 2)) \log^2 x.
\]

Recall the definition of \(F\) and we get immediately

Corollary 5. Assume the GRH for \(L(E,s)\). Then there exists a constant \(\epsilon_E > 0\) such that, for \(1 + 1/\log x \leq \sigma \leq 2\), we have the estimate
\[
\sum_{p \leq x} \frac{a_p(E) \log p}{p^s} F\left(\frac{\log p}{\log x}\right) \leq \epsilon_E (\log N_E + \log(|s| + 2)) \log^2 x. \quad \square
\]

Proof of Lemma 6. By partial summation it suffices to take \(j = 0\). To handle that case we mimic the proof of the prime number theorem under the Riemann hypothesis. Recall the definition of \(c_n(E)\) in section 2. Set \(c = 1/2 + 1/\log x\). Apply the Perron formula \([3, (2) on p. 104]\) and we get, for \(\sigma > 1\),
\[
\left| \int_{c-\sqrt x}^{c+\sqrt x} \frac{L'(E,\sigma + it + \xi) x^\xi}{L(E,\sigma + it + \xi) \xi} d\xi - \sum_{n \leq x} c_n(E) \log n \right|
\ll \sum_{n=1 \atop n \neq x}^{\infty} \frac{\Lambda(n)}{n^{\sigma+1/2}} \left(\frac{x}{n}\right)^c \min\left(1, \frac{1}{\sqrt x|\log \frac{x}{n}|}\right) + \frac{c \Lambda(n)}{\sqrt x n^{\sigma-1/2}},
\]
(31)
where \(\Lambda\) denotes the usual von Mangoldt function, and the last term on the right side of (31) is present only if \(x\) is a prime power.

If \(n \geq \frac{5}{4} x\) or if \(n \leq \frac{3}{4} x\) then \(|\log \frac{x}{n}|\) has a positive lower bound. Thus the contribution of such \(n\) to the right side of (31) is (recall that \(\sigma > 1\))
\[
\ll \sum_{n=1 \atop n \neq x}^{\infty} \frac{\Lambda(n)}{n^{1+1/\log x}} \ll \frac{\zeta'(1 + 1/\log x)}{\zeta(1 + 1/\log x)} \ll \log x.
\]
The argument in \([3, p. 107]\) shows that the contribution from those \(n\) such that \(\frac{3}{4} x < n < \frac{5}{4} x\), \(x \neq\) prime power, is
\[
\ll \frac{\log x}{\sqrt x} \min\left(1, \frac{x}{\sqrt x(n)}\right) + \log^2 x.
\]
Putting everything together and we get
\[
\left| \int_{c-\sqrt x}^{c+\sqrt x} \frac{L'(E,\sigma + it + \xi) x^\xi}{L(E,\sigma + it + \xi) \xi} d\xi - \sum_{n \leq x} c_n(E) \log n \right| \ll \log^2 x + \frac{\log x}{\sqrt x} \min\left(1, \frac{x}{\sqrt x(n)}\right).
\]
Our next step is to estimate the integral. Since \(\sigma \geq 1\), under the GRH the integrand has no pole inside the rectangle with vertices
\[
c + \sigma + it \pm iT, 1 + \frac{1}{\log x} + it \pm i\sqrt x.
\]
Thus it remains to estimate the integral along the other three edges of this rectangle.
The integral along the top edge is (recall $1 < \sigma \leq 2$)
\[
\int_{c}^{\infty} \frac{-L'(E, \sigma + it + \xi + i\sqrt{x})}{L(E, \sigma + it + \xi + i\sqrt{x})} \frac{x^{\xi+i\sqrt{x}}}{\xi+i\sqrt{x}} d\xi
\]
\[
\leq \frac{\sqrt{x}}{\sqrt{x}} \int_{c}^{\infty} \left[ \log N_E + \log(|\xi + \sigma + it + i\sqrt{x}| + 2) \right] \log x d\xi \quad \text{by Lemma 5}
\]
\[
\leq \frac{\sqrt{x} \log x}{\sqrt{x}} \left[ \log N_E + \log(|t| + 2) \right].
\]

The same bound holds for the integral along the bottom edge. As for the vertical edge,
\[
\int_{-\sqrt{x}}^{\sqrt{x}} \frac{L'(E, 1 + \frac{1}{\log x} + it + i\tau)}{L(E, 1 + \frac{1}{\log x} + it + i\tau)} \frac{x^{1/\log x}}{\frac{1}{\log x} + i\tau} d\tau
\]
\[
\leq \left( \log N_E + \log(1 + \frac{1}{\log x} + |t| + \sqrt{x} + 2) \right) \log x \left( \int_{0}^{2} \log x d\tau + \int_{2}^{\sqrt{x}} \frac{d\tau}{\tau} \right)
\]
\[
\leq \left( \log N_E + \log(|t| + \sqrt{x} + 2) \right) \left( \log^2 x + \log x \log \sqrt{x} \right).
\]

Putting everything together, we get, for $\sigma \geq 1 + 1/\log x$,
\[
\sum_{n<x} \frac{c_n(E) \log n}{n^{\sigma+it}} \ll \log^2 x + \frac{\log x}{\sqrt{x}} + \left( \log N_E + \log(|t| + \sqrt{x} + 2) \right) \log^2 x.
\]

Since $\sigma > 1$, the contribution to the sum on the left side from non-prime $n$ is $\ll \sum_{m<\sqrt{x}} \frac{\log m}{m^{3/2}} \ll 1$, so we are done.

### 7. Proof of Proposition 2

When $r = 1$, Brumer [11 (2.13)] deduces Proposition 2 from the explicit formula in conjunction with an estimate of a weighted sum of zeros of $L(E_D, s)$. Another (essentially equivalent) way is to apply the Perron formula as in the proof of the prime number theorem to the logarithmic derivative of $L(E, s)$. The explicit formula approach does not seem to generalize to $r > 1$, but the approach via the Perron formula does, with the key analytic estimate provided by Corollary 5. We prove Proposition 2 in several steps.

**Step 1.** Define
\[
L_x(E, s) = \sum_{p<x} \frac{a_p(E) \log p}{p^s} F\left( \frac{\log p}{\log x} \right).
\]
This is a finite sum, and hence it is holomorphic for all $s$. Apply the Perron formula as in the proof of Lemma 3 we get
\[
\left| \int_{\frac{1}{\log x} - \sqrt{-x}}^{\frac{1}{\log x} + \sqrt{-x}} L_x(E, s + 1) \frac{U^s}{s} ds - \sum_{p_1 \cdots p_r \leq U} \beta_{p_1}(E) \cdots \beta_{p_r}(E) \right| \ll \log^2 x.
\]
As for the integral, Corollary 5 shows that it is
\[
\leq \epsilon_E^r (\log N_E + \log (U + 2))^r \log^{2r} x \left( U^{1/\log x} \left[ \int_0^2 \frac{dt}{\log x + it} \right] + \int_2^\sqrt{x} \frac{dt}{t} \right).
\]

Recall that \( U \leq x^r \) and we get
\[
(32) \quad \sum_{p_1 \cdots p_r \leq U} \beta_{p_1}(E) \cdots \beta_{p_r}(E) \ll (\epsilon_E^r \epsilon^{r}) (\log N_E + \log (U + 2))^r \log^{2r+1} x.
\]

**Step II.** Fix an integer \( m \neq 0 \). With \( \pi_2 \) defined as in (20), we claim that
\[
(33) \quad \sum_{p_1 \cdots p_r \leq U} \beta_{p_1}(E) \cdots \beta_{p_r}(E) \ll (2\epsilon_E^r \epsilon^{r}) (\log N_E + 2 \log |m| + \log (U + 2))^r \log^{2r+1} x.
\]

To say that \( \pi_2 = 1 \) means that \( r \) is even and \( \pi = (p_1 \cdots p_{r/2})^2 \), so
\[
\sum_{p_1 \cdots p_r \leq U} \beta_{p_1}(E) \cdots \beta_{p_r}(E) \leq \left( \sum_{p \leq x} \beta_p^2 \right)^{r/2} \text{ since } \beta_p = 0 \text{ if } p \geq x
\]
\[
\leq (\log x)^{r/2} = (4 \log^2 x)^{r/2},
\]
which is satisfactory. Thus it remains to study (33) without the additional condition \( \pi_2 > 1 \).

If \( p \nmid 2N_E m \) then \( a_p(E)(\frac{m}{p}) = a_p(Em) \), so the left side of (33) without the \( \pi_2 \) condition is
\[
= \sum_{p_1 \cdots p_r \leq U} \beta_{p_1}(Em) \cdots \beta_{p_r}(Em) + O \left[ \sum_{j=1}^r \sum_{p_1, \ldots, p_j \mid 2N_E m} \log p_1 \cdots \log p_j \sum_{q_1, \ldots, q_{r-j} \leq U/p_1 \cdots p_j} \beta_{q_1}(E) \cdots \beta_{q_{r-j}}(E) \left( \frac{m}{q_1 \cdots q_{r-j}} \right) \right].
\]

We estimate the first sum above using (32), and we estimate each of the inner \( q \)-sum in the \( O \)-term by induction. All together, this yields
\[
\ll (\epsilon_E^r \epsilon) (\log N_{Em} + \log (U + 2))^r \log^{2r+1} x
\]
\[
+ \sum_{j=1}^r 2^j (\epsilon_E^r \epsilon)^{r-j} \left[ \log N_E + 2 \log |m| + \log (U + 2) \right]^{r-j} \log^{2(r-j)+1} x \left( \sum_{p \mid 2N_E m} \log p \right)^j.
\]

Back in section 2 we saw that the \( p \)-sum is \( \ll \log^{3/4} (2N_E m) \). Also, \( N_{Em} \ll N_E m^2 \), and Step II follows.
Step III. Fix an integer $n \neq 0, 1$. We claim that

$$
\sum_{\substack{p_1, \ldots, p_r \leq U \\ (n, p_j) = 1 \\ \pi_2 > 1}} \beta_{p_1} \cdots \beta_{p_r} \left( \frac{m}{p_1 \cdots p_r} \right) \ll (2\epsilon_E)^r \left( \log N_E + 2 \log |m| + \log |n| + \log(U + 2) \right)^r \log^{2r+1} x.
$$

By (33),

$$
\sum_{\substack{p \leq U \\ (n, p) = 1 \atop \rho \not \mid n \atop \pi_2 > 1}} \beta_p \left( \frac{m}{p} \right) \ll (2\epsilon_E) \left( \log N_E + 2 \log |m| + \log(U + 2) \right) \log^3 x + \sum_{p \not \mid n} \frac{\log p}{\sqrt{p}}
$$

$$
\ll (2\epsilon_E) \left( \log N_E + 2 \log |m| + \log(U + 2) \right) \log^3 x + \log |n|.
$$

This gives the case $r = 1$. In general,

$$
\sum_{\substack{p_1, \ldots, p_r \leq U \\ (n, p_j) = 1 \atop \pi_2 > 1}} \beta_{p_1} \cdots \beta_{p_r} \left( \frac{m}{p_1 \cdots p_r} \right) = \sum_{\substack{p_1, \ldots, p_r \leq U \\ \rho \not \mid n \atop \pi_2 > 1}} \beta_{p_1} \cdots \beta_{p_r} \left( \frac{m}{p_1 \cdots p_r} \right)
$$

\[+ O \left( \sum_{j=1}^r \left( \sum_{p_1, \ldots, p_j \leq U \atop \rho \mid n} \frac{\log p_1 \cdots \log p_j}{\sqrt{p_1 \cdots p_j}} \right) \sum_{\substack{q_1, \ldots, q_{r-j} \leq U/p_1 \cdots p_j \atop (q_i, n) = 1 \atop \pi_2 > 1}} \beta_{q_1} \cdots \beta_{q_{r-j}} \left( \frac{m}{q_1 \cdots q_{r-j}} \right) \right).\]

Step III now follows from (33) plus induction on $r$.

Step IV. Finally we come to prove Proposition 2. We proceed by induction on $r$, the case $r = 1$ being automatic.

By (34), the sum of terms with $\pi_1 = 1$ is

$$
\ll (2\epsilon_E)^r \left( \log N_E + 2 \log |m| + \log |n| + \log(U + 2) \right)^r \log^{2r+1} x.
$$

It remains to account for terms with $\pi_1 > 1$. That happens precisely when $\pi$ is exactly divisible by an even prime power. Then the contribution from these terms is therefore equal to ($\lfloor z \rfloor := \text{the largest integer } \leq z$)

$$
\sum_{\lambda = 1}^{\lfloor r/2 \rfloor} \sum_{\substack{p \leq U \atop \rho \not \mid n \atop \pi_2 > 1}} \beta_{p}^{2\lambda} \sum_{q_1, \ldots, q_{r-2\lambda} \leq U/p^{2\lambda} \atop (q_j, n) = 1 \atop \pi_2 > 1} \beta_{q_1} \cdots \beta_{q_{r-2\lambda}} \sum_{\delta_1 | \pi_1 \atop \delta_2 | \pi_2} \left( \pm n \delta_1 \right) \frac{1}{\sqrt{\pi'_1} \sqrt{\pi'_2}}.
$$
where \( \pi_1 \) and \( \pi_2 \) above are defined with respect to the \( r \)-tuple \((q_1, . . . , q_r)\) and \((r-2\lambda)\)-tuple \((q_1, . . . , q_r-2\lambda)\). If we denote by \( \pi_1(q) \) and \( \pi_2(q) \) the corresponding quantities in (20) with respect to the \((r-2\lambda)\)-tuple \((q_1, . . . , q_r-2\lambda)\), then \( \pi_2 = \pi_2(q) \) and \( \pi_1 = \pi_1(q)p \), so (36) is equal to

\[
\sum_{\lambda=1}^{\lfloor r/2 \rfloor} \sum_{p \leq U} \sum_{p \mid n} \sum_{q_1 . . . q_r-2\lambda \leq U/p^{2\lambda}} \beta_{q_1} . . . \beta_{q_r-2\lambda} \sum_{\delta_1 | \pi_1(q)} \left( \sum_{\delta_1 | \pi_1(q)} \frac{1}{\sqrt{\pi_1 p}} \right) + \sum_{\delta_1 | \pi_1(q)} \frac{1}{\sqrt{\pi_1 p}}
\]

(37) \( \ll \sum_{\lambda=1}^{\lfloor r/2 \rfloor} \sum_{p \leq x} \left( \frac{4 \log^2 p}{p} \right)^{\lambda} \left( \frac{1}{\sqrt{p}} \sum_{q_1 . . . q_r-2\lambda \leq U/p^{2\lambda}} \beta_{q_1} . . . \beta_{q_r-2\lambda} \sum_{\delta_1 | \pi_1(q)} \left( \frac{1}{\sqrt{\pi_1 p}} \right) \right)
\)

Note that

\( n \delta_1 p \leq n \pi_1 \leq nU. \)

By induction, each of the two inner \( q \)-sums is

\( \ll (2 \epsilon_E e)^{r-2\lambda} \left( \log N_E + 2 \log(\pi_1) + \log(\pi_2) + \log(U + 2) \right)^{r-2\lambda} \log^{2(r-2\lambda)+1} x. \)

Also,

\[
\sum_{\lambda=1}^{\lfloor r/2 \rfloor} \sum_{p \leq x} \left( \frac{4 \log^2 p}{p} \right)^{\lambda} \ll \log^2 x + \sum_{\lambda=2}^{\lfloor r/2 \rfloor} \sum_{p} \left( \frac{4 \log^2 p}{p} \right)^{\lambda} \ll \log^2 x.
\]

Thus (37) is

\( \ll (2 \epsilon_E e)^{r-2} \left( \log N_E + 2 \log(\pi_1) + \log(\pi_2) + \log(U + 2) \right)^{r-2} \log^{2r-1} x. \)

Combine this with (33) and recall that \( U \leq x^r \), we are done.

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