Towards the hidden symmetry in Coulomb interacting twisted bilayer graphene: renormalization group approach

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We develop a two stage renormalization group which connects the continuum Hamiltonian for twisted bilayer graphene at length scales shorter than the moire superlattice period to the Hamiltonian for the active narrow bands only which is valid at distances much longer than the moire period. In the first stage, the Coulomb interaction renormalizes the Fermi velocity and the interlayer tunnelings in such a way as to suppress the ratio of the same sublattice to opposite sublattice tunneling, hence approaching the so-called chiral limit. In the second stage, the interlayer tunneling is treated non-perturbatively. Via a progressive numerical elimination of remote bands the relative strength of the one-particle-like dispersion and the interactions within the active narrow band Hamiltonian is determined, thus quantifying the residual correlations and justifying the strong coupling approach in the final step. We also calculate exactly the exciton energy spectrum from the Coulomb interactions projected onto the renormalized narrow bands. The resulting softening of the collective modes marks the propinquity of the enlarged (“hidden”) $U(4) \times U(4)$ symmetry in the magic angle twisted bilayer graphene.

It has been known for some time that the electron-electron Coulomb interactions cause an upward renormalization of the Fermi velocity, $v_F$, upon approaching the charge neutrality point (CNP) of mono-layer graphene.\textsuperscript{11,12} Such momentum dependent steepening of the Dirac cone depends on the graphene’s dielectric environment and is weaker for stronger dielectrics, but even for hexagonal boron nitride (hBN) encapsulated devices the increase can be $\approx 10 \sim 15\%$. Such a small change in $v_F$ would be of limited interest if it weren’t for the recent explosion of research into the magic angle twisted bilayer graphene (TBG)\textsuperscript{13,14}, where the experiments show extremely strong sensitivity of the correlated electron phenomena to the twist angle $\theta$. Even a $\sim 5\%$ change of $\theta$ away from the optimal (magic) value has been reported to produce at least a factor of 2 reduction\textsuperscript{15,16} of the superconducting $T_c$, with even stronger suppression of the correlated insulator states\textsuperscript{17}.

The strong band structure sensitivity is due to the dependence on the dimensionless parameters $w_{0,1}/v_Fk_0$, where $w_0$ and $w_1$ parameterize the interlayer tunneling energy in the AA and AB regions respectively, and where the momentum displacement of the Dirac cones is given by $k_0 = 2k_D \sin \frac{\theta}{2}, \quad k_D = 4\pi/3a_0, \quad a_0 \approx 0.246$nm (in $\hbar = 1$ units\textsuperscript{5}). Therefore, at a fixed magic $\theta$, even a $\sim 10\%$ percent difference in $v_F$ alone would be sufficient to de-tune the system from the optimal flat band condition. As such, if neither of $w_j$ renormalized due to Coulomb interactions, but only $v_F$ did, the magic angle condition would depend on whether the TBG was encapsulated in the hBN, or only from one side, because the different dielectric environments would produce a different strength of Coulomb interactions, former with a dielectric constant of $c_{hBN} \approx 4.4$ and the latter with $c \approx (1 + c_{hBN})/2 = 2.7$. The difference in the $v_F$, and therefore the magic angle, would then be within the sensitivity of the correlated insulating states; no such dependence of the magic angle on the partial or complete encapsulation has been reported.

Here we develop a renormalization group (RG) approach to the Coulomb interactions in the twisted bilayer graphene and show that $w_1$ renormalizes in precisely such a way as to compensate for the growth of $v_F$ making the magic angle largely insensitive to the effective dielectric constant $\epsilon$. Interestingly, we find that $w_0$ does not renormalize due to Coulomb interactions. Therefore, the ratio $w_0/w_1$ shrinks and the system flows closer to the chiral limit described by Tarnopolsky, Kruchkov and Vishwanath\textsuperscript{18,19}. As illustrated in the Fig. 1; the flow from a high energy (with the UV cutoff $E_c$), where the Coulomb interaction and $w_{0,1}$ are perturbative, to a low energy of the narrow bands where neither is, crosses over to a regime where the effects of $w_{0,1}$ become non-perturbative, but the Coulomb interaction is still perturbative. This happens at the energy scale $E^{*}_c \sim O(w_1)$, marking the beginning of the second stage of our RG; the band structure scaling collapse in Fig\textsuperscript{2} shows that the $2^{nd}$ stage seamlessly connects to the $1^{st}$ stage even if $E^{*}_c$ changes. In the $2^{nd}$ stage, we numerically integrate out the two most remote bands, one above and one below the CNP, rotate the remaining states to diagonalize the renormalized kinetic energy and re-express the interaction in terms of the rotated states, iterating the procedure until we reach the narrow bands. If the resulting narrow bands bandwidth (or, more precisely the root-mean-square of the renormalized kinetic energy dispersion) is much smaller than the interaction (or more precisely, the particle-hole charge gap), as we find it is near the magic angle, the final step is treated non-perturbatively in the Coulomb interaction i.e. by solving the interaction-only problem (strong coupling limit) and then treating the renormalized kinetic energy terms as a perturbation.

The condition $w_0 = 0$, and thus the chiral
We begin with the Hamiltonian $H = H_{\text{kin}} + V_{\text{int}}$ where

$$H_{\text{kin}} = \int d^2r \chi^\dagger_\sigma(r) \left( \hat{H}_{\text{BM}} \right) \chi_\sigma(r)$$

$$V_{\text{int}} = \frac{1}{2} \int d^2rd^2r' V(r-r') \chi^\dagger_\sigma(r) \chi^\dagger_\sigma(r') \chi_\sigma(r') \chi_\sigma(r)$$

where $\chi_\sigma = (\psi^\dagger_\sigma, \phi^\dagger_\sigma)$ creates an electron in valley $K$ ($K'$) for its upper (lower) component, and the repeated spin-$\frac{1}{2}$ indices $\sigma$ are summed. The Bistritzer-MacDonald(BM) continuum Hamiltonian\cite{BM13,BM15,BM16} for twist angle $\theta$ is

$$\hat{H}_{\text{BM}} = \begin{pmatrix} V_{\text{F}} \sigma \frac{e}{p} \cdot \mathbf{p} & T(r) \\ T^\dagger(r) & V_{\text{F}} \sigma \frac{e}{p} \cdot \mathbf{p} \end{pmatrix},$$

where the twisted Pauli matrices acting on the sublattice indices are $\sigma_2 = e^{-i\theta z}(\sigma_x, \sigma_y)e^{i\theta z}$.

The slow fields at the two valleys $K/K'$ are expanded in this ‘band’ basis fermion annihilation operators $d_{\sigma K/K',n,k}$ with crystal momentum $k$ in first moire Brillouin zone, and the band index $n$ as

$$\chi_\sigma(r) = \left( \psi_\sigma(r), \phi_\sigma(r) \right) = \sum_{n,k} \left( \Psi_{\sigma,k}(r) d_{\sigma,K,n,k} + \Psi_{\sigma,k}(r) d_{\sigma,K',n,k} \right)$$

It will be helpful for us to think about $H_{\text{kin}}$ as a lowest order gradient expansion of a continuum field theory\cite{TM18} with coupling constants that can flow due to $V_{\text{int}}$ under the 1st stage of RG.

As pointed out in Ref.[12] if the small angle rotation in $\sigma_{\theta/2}$ is ignored, then $\hat{H}_{BM}$ enjoys a p-h symmetry for any value of $w_0$ and $w_1$,

$$-i\mu_y \sigma_z \hat{H}_{BM}^{+} \sigma_z i\mu_y = -\hat{H}_{BM},$$

in that if $\Psi_{\sigma,k}(r)$ is an eigenstate of $\hat{H}_{BM}$ at $k$ with eigenvalue $\epsilon_{\sigma,k}$, then $-i\mu_y \sigma_z \Psi_{\sigma,k}(r)$ is an eigenstate at $-k - q_1$ with eigenvalue $-\epsilon_{\sigma,k}$. We will perform our RG assuming this approximate symmetry is present.

Up to an overall shift of the chemical potential, we can rewrite $V_{\text{int}}$ as

$$V_{\text{int}} = \frac{1}{2} \int d^2rd^2r' V(r-r') \delta \rho(r) \delta \rho(r')$$

$$\delta \rho(r) = \chi^\dagger_\sigma(r) \chi_\sigma(r) - \frac{1}{2} (\chi^\dagger_\sigma(r), \chi_\sigma(r)).$$
For a pure Coulomb interaction \( V(r) = e^2/|r| \). The Hamiltonian in Eqs. (1)-(2) is defined at some high energy cut-off \( \pm E_c \) which corresponds to a maximal value of the band index \( n_c \) in our expansion. The parameters \( v_F, w_0, \) and \( w_1 \) should also be thought of as being fixed by a measurement at \( E_c \). The last term in (9) is usually ignored, but for our RG, it will be helpful to express it as

\[
\frac{1}{2} \{ \chi^+_{\sigma}(r), \chi^-_{\sigma}(r) \} = \tilde{\rho}_{E_c}(r) = 2 \sum_{|\mathbf{k}| \leq E_c} \Psi^*_{n,k}(r)\Psi_{n,k}(r).
\]  

(10)

In the 1st stage, we split \( \chi_{\sigma}(r) = \chi^+_{\sigma}(r) + \chi^-_{\sigma}(r) \) and integrate out the fast modes \( \chi^+_{\sigma}(r) \) with kinetic energy \( E'_{c} < |\mathbf{n}, \mathbf{k}| \leq E_c \), such that \( E'_c \gg w_{0,1} \). In this regime, the \( V_{int} \) can be treated perturbatively. Its contribution to the slow mode Hamiltonian is then

\[
V_{int} \to \frac{1}{2} \int d^2 r d^2 r' V(r - r') \delta \rho^< (r) \delta \rho^< (r') + \frac{1}{2} \int d^2 r d^2 r' V(r - r') \chi^<_{\sigma}(r) \delta F(r, r') \chi^<_{\sigma}(r'),
\]

(11)

where \( \delta \rho^< (r) = \chi^<_{\sigma}(r) \chi^<_{\sigma}(r) - \tilde{\rho}_{E_c}(r) \) which follows from the p-h symmetry. The correction to the \( \hat{H}_{BM} \) comes from

\[
\delta F(r, r') = \sum_{E'_c < |\mathbf{n}, \mathbf{k}| \leq E_c} \text{sign}(\epsilon_{nk})(f_{nk}(r, r') - f_{nk}(r, r'))
\]

where \( f_{nk}(r, r') = \Psi_{nk}(r)\Psi^*_{nk}(r') \). We can now write

\[
\sum_{E'_c < |\mathbf{n}, \mathbf{k}| \leq E_c} \text{sign}(\epsilon_{nk}) f_{nk}(r, r') = \int \frac{dz}{C} \frac{d}{2\pi i} (r) \hat{G}(z)|r'|
\]

(13)

where \( \hat{G}(z) = (z - \hat{H}_{BM})^{-1} \), and the contour \( C \) encloses the z-plane real line segment \( (E_c, -E_c) \) in the clockwise, and segment \( (E'_c, E_c) \) in the counterclockwise, sense. As long as \( E'_c \gg w_{0,1} \), the dominant contribution to the contour integral can be found by replacing \( \hat{G}(z) \approx \hat{G}_0(z) + \hat{G}_0(z)\hat{T}\hat{G}_0(z) + O \left( \frac{w_{0,1}}{E_c} \right) \). For small \( E_c - E'_c \), we thus find that in the 1st RG stage,

\[
\frac{dv_F}{d \ln E_c} = -\frac{e^2}{4\epsilon},
\]

(14)

\[
\frac{d w_0}{d \ln E_c} = 0,
\]

(15)

\[
\frac{d w_1}{d \ln E_c} = -w_1 \frac{e^2}{4\epsilon v_F},
\]

(16)

for \( \epsilon^2 \), being the prefactor of a non-analytic term, does not renormalize when high energy modes are eliminated. Integrating the above equations i.e. progressively reducing the cutoff to \( E'_c \) gives

\[
\frac{w_1(E'_c)}{v_F(E'_c)} = \frac{w_1(E_c)}{v_F(E_c)},
\]

(17)

\[
\frac{w_0(E'_c)}{w_1(E'_c)} = \left( 1 + \frac{e^2}{4\epsilon v_F(E_c)} \frac{1}{\ln E_c} \right).
\]

(18)

The Eq. (17) implies that the magic angle condition is largely insensitive to the renormalization. The Eq. (18) shows that even if we start away from the chiral limit at the UV scale \( E_c \), at a lower energy scale \( E'_c \) we approach it. Next, we combine this stage 1 RG with the non-perturbative (in more potential) stage 2 numerical RG at \( 6w_1 \gtrsim E'_c \), but we stress that results are insensitive to the choice of \( E'_c \) as long as \( w_{0,1}/E'_c \) is small so that stage 1 is under control. The scaling collapse of the band structure shown in the Fig. (2) demonstrates this insensitvity for \( w_1/v_F k_0 = 0.5, \epsilon/v_F = 2.2, \) and \( \epsilon = 4.4 \) with several choices of \( n_c \). We also find an increase of the sublattice polarization and steepening of the Wilson loops along the RG evolution, indicating a further approach of the chiral limit during the stage 2.
provides the exact excitation states in the strong coupling limit. The Eq. (19) 

pling limit. The Eq. (19) can be readily solved for a single particle excitation and we show the result in the Ref. [82] Here we focus on the charge neutral excitations (excitons) X = \sum_{mm'k} f_{mm'k}^{\alpha\beta} \phi_{\alpha m}^k \phi_{\beta m'} \langle k-q \rangle_{modg}, 

w_0(E_c) \over w_1(E_c) = 0.83 \quad E_c = 18.2w_1 \quad \epsilon = 4.4

FIG. 3. The strong coupling exciton spectrum after stage 1 and 2 RG, starting the stage 1 with E_c = 18.2w_1 

corresponding to 2eV for \omega_1 = 110meV, \omega_1/(\epsilon_F k_F) = 0.586 (magic angle), and the initial \omega_0/\omega_1 = 0.83. The branch that becomes gapless at \Gamma corresponds to 4 Goldstone modes of 

U(4) spin-valley ferromagnet with quadratic dispersion. Another branch, emphasized by the arrow, softens during the RG, eventually also becoming gapless in the chiral limit, with the total of 8 Goldstone modes of U(4) \times U(4) ferromagnet. The red curve is the onsets of the particle-hole continuum.


dure, we can choose \Psi_{n-,k} = -i \mu \sigma \Psi_{n+,k-\alpha}(r). 

Substitution of such field operators [83] gives \rho(r) = \sum_{kk'} \sum_{\sigma \tau \mu \nu} D_{k\sigma}^\dagger \phi_{k\sigma}(r) D_{k'\tau} \phi_{k'\tau}(r), 

where within the narrow band D_{k\sigma} = (d_{k\sigma}^1 d_{k\sigma}^2 d_{k\sigma}^3 d_{k\sigma}^4). 

Suppressing \textbf{k}' and \textbf{r} dependence, \phi = b_0 \phi_1 + b_1 \phi_2 + b_2 \phi_3, 

thus commuting with all 16 generators of spin-valley U(4) symmetry [84], \phi_{\mu} = \phi_{\mu 1} \phi_{\mu 2} \phi_{\mu 3} \phi_{\mu 4}, 

where \mu = 1, 2, 3 and \tau acts on valley, \sigma on band, and s on spin components.

If a state \(|\Omega\rangle\) is annihilated by \delta \rho(r) for all \textbf{r}, then it is a ground state at the strong coupling because \langle V_{int} \rangle

is positive definite [85,86]. Moreover, any state obtained by a global U(4) rotation is also a ground state, and, at the CNP, can be obtained from a fully filled valley polarized state [87,88]. The exact n-body excitations above any one ground state can also be obtained by solving an (n-1)-body problem because \langle V_{int} X | \Omega \rangle = \frac{1}{2} \int d^2r d^2r' V(r-r') [\delta \rho(r), [\delta \rho(r'), X] | \Omega \rangle 

and because the center of mass momentum is conserved. Therefore, solving the operator eigen-equation

\[ EX = \frac{1}{2} \int d^2r d^2r' V(r-r') [\delta \rho(r), [\delta \rho(r'), X] | \Omega \rangle \]

provides the exact excitation states in the strong coupling limit. The Eq. (19) can be readily solved for a single particle excitation and we show the result in the Ref. [82]. Here we focus on the charge neutral excitations (excitons) X = \sum_{mm'k} f_{mm'k}^{\alpha\beta} \phi_{\alpha m}^k \phi_{\beta m'} \langle k-q \rangle_{modg}, with spin/valley labels \alpha, \beta, by finding the eigenfunctions f_{mm'k}^{\alpha\beta} \langle k-q \rangle_{modg}. Due to the spin-valley U(4) invariance of these equations, it is sufficient to solve for one spin and valley projection, the rest can be obtained by the symmetry. The numerically obtained exciton spectrum at the magic angle is shown in the Fig. 3 for the center of mass momentum \textbf{q} along the path shown in the Fig. 1. The quadratically vanishing dispersion of the lowest branch corresponds to the four U(4) ferromagnetic Goldstone bosons [89].

Under RG a second set of four modes softens. This corresponds to approaching the ("hidden") U(4) \times U(4) invariant chiral limit [90] with its 8 Goldstone bosons. Their gap is a measure of the U(4) \times U(4) anisotropy terms and for the parameters in the Fig. 3 this gap is \Delta_{U(4) \times U(4)} \approx 0.2e^2/eL_m \sim 5meV; the gap vanishes at the chiral limit. Note that the modes disperse despite a complete absence of kinetic energy terms due to the non-local structure of the projected density operators [91].

The Hkin breaks the spin-valley U(4) symmetry down to U(2) \times U(2) and causes splitting of the degenerate ground state manifold. We can obtain an upper bound on the resulting anisotropy terms from 2nd order perturbation in (renormalized) kinetic energy (i.e. "superexchange") by replacing the energy of the excited states at \Gamma with the lowest energy exciton that has a non-zero overlap on the kinetic energy operator (E_{min} \approx 2e^2/eL_m for Fig. 3). For a spin independent valley rotation, parameterized by 3 Euler angles, e^{i \tau_1 \alpha \tau_4 \mu} e^{i \omega \tau_2 \tau_1 \nu} e^{i \gamma \tau_3 \tau_1 \lambda}, we find that the energy splitting per unit cell, \Delta_{U(4)} = 0.2e^2/eL_m, is bounded from above by – (sin^2 \omega) \int d^2k d^2k' p_{\alpha m,\beta m'(\pm)} A_{\alpha \beta m m'} (\pm). The lowest energy state for such a rotation is the Kramers inter-valley coherent state [92] |\Omega \rangle \sim \frac{1}{2} \int_\text{\textbf{r}} \phi_{\mu} \phi_{\mu} \phi_{\mu} \phi_{\mu} |\Omega \rangle, at \omega = \frac{\tau}{2}. For the parameters in Fig. 3 we find that \Delta_{U(4)} < 0.7 \times 10^{-3} e^2/eL_m \sim 0.1meV, justifying the strong coupling approach.

The theory presented here can be extended to include the RPA effects and the p-h asymmetry, which will be important for any detailed quantitative comparison with experiments. Nevertheless, the Coulomb RG induced softening of the hidden symmetry collective modes, whose natural energy scale would normally be \sim e^2/eL_m \sim 25meV, suggests that they may not be frozen out even at \sim 50K. Finally, our results offer a significant shift of the perspective in that the chiral limit [93] – previously considered unphysical – gains the status of an attractive mid-IR RG fixed point when E_c/\omega_1 \rightarrow \infty.

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1. D.C. Elias et al., “Dirac cones reshaped by interaction effects in suspended graphene,” Nat. Phys. 7, 701 (2011).
2. G.L. Yu et al., “Interaction phenomena in graphene seen through quantum capacitance,” PNAS 110, 3282 (2013).
3. J. Gonzalez, F. Guinea, and A.H. Vozmediano, “Non-Fermi liquid behavior of electrons in the half-filled honeycomb lattice (A renormalization group approach),” Nucl. Phys. B 424, 595, (1994).
4. O. Vafek, “Anomalous Thermodynamics of Coulomb-Interacting Massless Dirac Fermions in Two Spatial Dimensions” Phys. Rev. Lett. 98, 216401 (2007).
5. D.E. Sheehy and J. Schmalian, “Quantum Critical Scaling in Graphene,” Phys. Rev. Lett. 99, 226803 (2007).
6. G. Borghi, M. Polini, R. Asgari, and A.H. MacDonald, “Fermi velocity enhancement in monolayer and bilayer graphene”, Solid State Comm. 149, 1117 (2009).
7. E. Barnes, E. H. Hwang, R. E. Throckmorton, and S. Das Sarma, “Effective field theory, three-loop perturbative expansion, and their experimental implications in graphene many-body effects” Phys. Rev. B 89, 235431 (2014).
8. R. Bistritzer and A. H. MacDonald, “Moire bands in twisted double-layer graphene,” Proc. Natl. Acad. Sci. U.S.A. 108, 12233 (2011)
9. Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, R. C. Ashoori, and P. Jarillo-Herrero, “Correlated insulator behaviour at half-filling in magic-angle graphene superlattices,” Nature 556, 43 (2018).
10. Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, “Unconventional superconductivity in magic-angle graphene superlattices,” Nature 556, 80 (2018).
11. M. Yankowitz, S. Chen, H. Polshyn, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, “Tuning superconductivity in twisted bilayer graphene,” Science 363, 1050 (2019).
12. A. L. Sharpe, E. J. Fox, A. W. Barnard, J. Finney, K. Watanabe, T. Taniguchi, M. A. Kastner, and D. Goldhaber-Gordon, “Emergent ferromagnetism near three-quarters filling in twisted bilayer graphene,” Science 365, 605 (2019).
13. M. Serlin, C. L. Tschirhart, H. Polshyn, Y. Zhang, J. Zhu, K. Watanabe, T. Taniguchi, L. Balents, and A. F. Young, “Intrinsic quantized anomalous hall effect in a moire heterostructure,” Science science.aa5533 (2019).
14. A. Kerelsky, L. J McGilly, D. M. Kennes, L. Xian, M. Yankowitz, S. Chen, K. Watanabe, T. Taniguchi, J. Hone, C. Dean, et al., “Maximized electron interactions at the magic angle in twisted bilayer graphene,” Nature 572, 95 (2019).
15. X. Lu, P. Stepianov, W. Yang, M. Xie, M. A. Aamir, I. Das, C. Ugell, K. Watanabe, T. Taniguchi, G. Zhang, A. Bachtold, A. H. MacDonald, and D. K. Efetov, “Superconductors, orbital magnets and correlated states in magic-angle bilayer graphene,” Nature 574, 653 (2019).
16. Y. Xie, B. Lian, B. Jack, X. Liu, C.-L. Chiu, K. Watanabe, T. Taniguchi, B. A. Bernevig, and A. Yazdani, “Spectroscopic signatures of many body correlations in magic-angle twisted bilayer graphene,” Nature 572, 101 (2019).
17. S. L. Tomarken, Y. Cao, A. Demir, K. Watanabe, T. Taniguchi, P. Jarillo-Herrero, and R. C. Ashoori, “Electronic compressibility of magic-angle graphene superlattices,” Phys. Rev. Lett. 123, 046601 (2019).
18. Y. Jiang, X. Lai, K. Watanabe, T. Taniguchi, K. Haule, J. Mao, and E. Y. Andrei, “Charge order and broken rotational symmetry in magic-angle twisted bilayer graphene,” Nature 573, 91 (2019).
19. Y. Choi, J. Kenmer, Y. Peng, A. Thomson, H. Arora, R. Polsky, Y. Zhang, H. Ren, J. Alicea, G. Refael, F. von Oppen, K. Watanabe, T. Taniguchi, and S. Nadj-Perge, “Electronic correlations in twisted bilayer graphene near the magic angle,” Nat. Phys. 15, 1174 (2019).
20. P. Stepianov, I. Das, X. Lu, A. Fahninmaya, K. Watanabe, T. Taniguchi, F. H. L. Koppens, J. Lischner, L. Levitov, D. K. Efetov, “Untying the insulating and superconducting orders in magic-angle graphene”, Nature 583, 375 (2020).
21. D. Wong, K. F. Nickolls, M. Oh, B. Lian, Y. Xie, S. Jeon, K. Watanabe, T. Taniguchi, B. A. Bernevig, and A. Yazdani, “Cascade of electronic transitions in magic-angle twisted bilayer graphene,” Nature 582, 198 (2020).
22. U. Zondiner, A. Rozen, D. Rodan-Legrain, Y. Cao, R. Queiroz, T. Taniguchi, K. Watanabe, Y. Oreg, F. von Oppen, A. Stern, E. Berg, P. Jarillo-Herrero, and S. Ilani, “Cascade of Phase Transitions and Dirac Revivals in Magic Angle Graphene,” Nature 582, 203 (2020).
23. Y. Saito, J. Ge, K. Watanabe, T. Taniguchi, and A. F. Young, “Independent superconductors and correlated insulators in twisted bilayer graphene,” Nat. Phys. 16, 926 (2020).
24. Y. Cao, D. Rodan-Legrain, J. M. Park, F. N. Yuan, K. Watanabe, T. Taniguchi, R. M. Fernandes, F. Fu, and P. Jarillo-Herrero, “Nematicity and Competing Orders in Superconducting Magic-Angle Graphene,” arXiv:2004.04114.
25. C. L. Tschirhart, M. Serlin, H. Polshyn, A. Shragai, Z. Xia, J. Zhu, Y. Zhang, K. Watanabe, T. Taniguchi, M. E. Huber, A. F. Young, “Imaging orbital ferromagnetism in a moire Chern insulator”, arXiv:2006.08053.
26. C. Xu and L. Balents, “Topological Superconductivity in Twisted Multilayer Graphene”, Phys. Rev. Lett. 121, 087001 (2018).
27. J. Kang and O. Vafek, “Symmetry, maximally localized Wannier states, and a low-energy model for twisted bilayer graphene narrow bands,” Phys. Rev. X 8, 031088 (2018).
28. M. Koshino, N. F. Q. Yuan, T. Koretsune, M. Ochi, K. Kuroki, and L. Fu, “Maximally localized Wannier or-bitals and the extended hubbard model for twisted bilayer graphene,” Phys. Rev. X 8, 031087 (2018).
29. H. C. Po, L. Zou, A. Vishwanath, and T. Senthil, “Origin of mott insulating behavior and superconductivity in twisted bilayer graphene,” Phys. Rev. X 8, 031089 (2018).
30. C.-C. Liu, L.-D. Zhang, W.-Q. Chen, and F. Yang, “Chiral spin density wave and d + id superconductivity in the magic-angle-twisted bilayer graphene,” Phys. Rev. Lett. 121, 217001 (2018).
31. F. Wu, A. H. MacDonald, and I. Martin, “Theory of phonon-mediated superconductivity in twisted bilayer graphene”, Phys. Rev. Lett. 121, 257001 (2018).
32. Hiroki Isobe, Noah F. Q. Yuan, and Liang Fu, “Unconventional superconductivity and density waves in twisted bilayer graphene,” Phys. Rev. X 8, 041041 (2018).
33. H. Guo, X. Zhu, S. Feng, and R. T. Scalettar, “Pair-
ing symmetry of interacting fermions on a twisted bilayer graphene superlattice”, Phys. Rev. B 97, 235453 (2018).
34 F. Guinea and N. R Walet, “Electrostatic effects, band distortions, and superconductivity in twisted graphene bilayers,” Proc. Natl. Acad. Sci. U.S.A. 115, 13174 (2018).
35 A. Thomson, S. Chatterjee, S. Sachdev, and M. S. Scheurer, “Triangular antiferromagnetism on the honeycomb lattice of twisted bilayer graphene”, Phys. Rev. B 98, 075109 (2018).
36 J. F. Dodaro, S. A. Kivelson, Y. Schattner, X. Q. Sun, and C. Wang, “Phases of a phenomenological model of twisted bilayer graphene,” Phys. Rev. B 98, 075154 (2018).
37 L. Zou, H. C. Po, A. Vishwanath, and T. Senthil, “Band structure of twisted bilayer graphene: Emergent symmetries, commensurate approximants, and Wannier obstructions,” Phys. Rev. B 98, 085435 (2018).
38 M. Ochi, M. Koshino, K. Kuroki, “Possible correlated insulating states in magic-angle twisted bilayer graphene under strongly competing interactions”, Phys. Rev. B 98, 081102 (2018).
39 Louk Rademaker and Paula Mellado, “Charge-transfer insolation in twisted bilayer graphene,” Phys. Rev. B 98, 235158 (2018).
40 L. Balents, “General continuum model for twisted bilayer graphene and arbitrary smooth deformations”, SciPost Phys., 7, 48 (2019).
41 J. Ahn, S. Park, and B.-J. Yang, “Failure of nien-successful theorem and fragile topology in two dimensional systems with space-time inversion symmetry: Application to twisted bilayer graphene at magic angle,” Phys. Rev. X 9, 021013 (2019).
42 Z. Song, Z. Wang, W. Shi, G. Li, C. Fang, and B. Andrei Bernevig, “All magic angles in twisted bilayer graphene are topological,” Phys. Rev. Lett. 123, 036401 (2019).
43 K. Hejazi, C. Liu, H. Shapourian, X. Chen, and L. Balents, “Multiple topological transitions in twisted bilayer graphene near the first magic angle," Phys. Rev. B 99, 035111 (2019).
44 J. Liu, J. Liu, and X. Dai, “The pseudo-Landau-level representation of twisted bilayer graphene: band topology and the implications on the correlated insulating phase,” Phys. Rev. B 99, 155145 (2019).
45 J. W. F. Venderbos and R. M. Fernandes, “Correlations and electronic order in a two-orbital honeycomb lattice model for twisted bilayer graphene,” Phys. Rev. B 98, 245103 (2018).
46 J. Gonzalez and T. Stauber, “Kohn-luttinger superconductivity in twisted bilayer graphene,” Phys. Rev. Lett. 122, 026801 (2019).
47 G. Turnopolsky, A. J. Kruchkov, and A. Vishwanath, “Origin of magic angles in twisted bilayer graphene,” Phys. Rev. Lett. 122, 106405 (2019).
48 J. Kang and O. Vafek, “Strong coupling phases of partially filled twisted bilayer graphene narrow bands,” Phys. Rev. Lett. 122, 246401 (2019).
49 K. Seo, V. N. Kotov, and B. Uchoa, “Ferromagnetic Mott state in twisted bilayer graphene at the magic angle,” Phys. Rev. Lett. 122, 246402 (2019).
50 S. Carr, S. Fang, Z. Zhu, and E. Kaxiras, “Exact continuum model for low-energy electronic states of twisted bilayer graphene” Phys. Rev. Research, 1, 013001, (2019).
51 Y.-H. Zhang, D. Mao, Y. Cao, P. Jarillo-Herrero, and T. Senthil, “Nearly flat chern bands in moire superlattices,” Phys. Rev. B 99, 075127 (2019).
52 Q.-K. Tang, L. Yang, D. Wang, F.-C. Zhang, and Q.-H. Wang, “Spin-triplet f-wave pairing in twisted bilayer graphene near $\frac{1}{4}$ filling,” Phys. Rev. B 99, 094521 (2019).
53 J. Y. Lee, E. Khalaf, S. Liu, X. Liu, Z. Hao, P. Kim, and A. Vishwanath, “Theory of correlated insulating behavior and spin-triplet superconductivity in twisted double bilayer graphene,” Nat. Commun. 10, 1 (2019).
54 B. Roy and V. Juričić, “Unconventional superconductivity in nearly flat bands in twisted bilayer graphene,” Phys. Rev. B 99, 121407 (2019).
55 P. Lucignano, D. Alfé, V. Cataudella, D. Ninno, and G. Cantele, “Crucial role of atomic corrugation on the flat bands and energy gaps of twisted bilayer graphene at the magic angle $\theta \sim 1.08\circ$,” Phys. Rev. B 99, 195419 (2019).
56 X.-C. Wu, A. Keselman, C.-M. Jian, K. A. Pawlak, and C. Xu, “Ferromagnetism and spin-valley liquid states in moire correlated insulators,” Phys. Rev. B 100, 024421 (2019).
57 Y.-P. Lin and R. M. Nandkishore, “Chiral twist on the high-Tc phase diagram in moiré heterostructures,” Phys. Rev. B 100, 085136 (2019).
58 Y. H. Zhang, H. C. Po, and T. Senthil, “Landau level degeneracy in twisted bilayer graphene: Role of symmetry breaking” Phys. Rev. B 100, 125104 (2019).
59 S. Liu, E. Khalaf, J. Y. Lee, and A. Vishwanath, “Nematic topological semimetal and insulator in magic angle bilayer graphene at charge neutrality,” [arXiv:1905.07409]
60 Y. Alavirad and J. D. Sau, “Ferromagnetism and its stability from the one-magnon spectrum in twisted bilayer graphene,” [arXiv:1907.13633]
61 J. Liu and X. Dai, “Correlated insulating states and the quantum anomalous Hall phenomena at all integer fillings in twisted bilayer graphene”, [arXiv:1911.03760]
62 F. Wu and S. Das Sarma, “Collective Excitations of Quantum Anomalous Hall Ferromagnets in Twisted Bilayer Graphene”, Phys. Rev. Lett. 124, 046403 (2020).
63 M. Xie, and A. H. MacDonald, “Nature of the Correlated Insulator States in Twisted Bilayer Graphene,” Phys. Rev. Lett. 124, 097601 (2020).
64 N. Bultinck, S. Chatterjee, and M. P. Zaletel, “Mechanism for Anomalous Hall Ferromagnetism in Twisted Bilayer Graphene,” Phys. Rev. Lett. 124, 166601 (2020).
65 C. Repellin, Z. Dong, Y.-H. Zhang, T. Senthil, “Ferromagnetism in narrow bands of moire superlattices,” Phys. Rev. Lett. 124, 187601 (2020).
66 S. Chatterjee, N. Bultinck, and M. P. Zaletel, “Symmetry breaking and skyrmionic transport in twisted bilayer graphene,” Phys. Rev. B 101, 165141 (2020).
67 N. Bultinck, E. Khalaf, S. Liu, S. Chatterjee, A. Vishwanath, and M. P. Zaletel, “Ground State and Hidden Symmetry of Magic Angle Graphene at Even Integer Filling,” Phys. Rev. X 10, 031034 (2020).
68 D. V. Chichinadze, L. Classen, and A. V. Chubukov, “Nematic superconductivity in twisted bilayer graphene,” Phys. Rev. B 101, 224513 (2020).
69 Y. Zhang, K. Jiang, Z. Wang, and F. C. Zhang, “Correlated insulating phases of twisted bilayer graphene at commensurate filling fractions: a Hartree-Fock study,” Phys. Rev. B 102, 035136 (2020).
70 J. Kang and O. Vafek, “Non-Abelian Dirac node braiding and near-degeneracy of correlated phases at odd integer filling in magic angle twisted bilayer graphene”, Phys. Rev. B 102, 035161 (2020).
71 T. Cea and F. Guinea, “Band structure and insulating
states driven by Coulomb interaction in twisted bilayer graphene”, Phys. Rev. B 102, 045107 (2020).

72 Y. D. Liao, J. Kang, C. N. Breio, X. Y. Xu, H.-Q. Wu, B. M. Andersen, R. M. Fernandes, and Z. Y. Meng, “Correlation-induced insulating topological phases at charge neutrality in twisted bilayer graphene”, [arXiv:2004.12536]

73 D. V. Chichinadze, L. Classen, and A. V. Chubukov, “Orbital antiferromagnetism, nematicity, and density wave orders in twisted bilayer graphene”, [arXiv:2007.00871]

74 R. M. Fernandes and J. W. F. Venderbos, “Nematicity with a twist: rotational symmetry breaking in a moiré superlattice,” Science Advances 6, eaba8834 (2020).

75 L. Balents, C. R. Dean, D. K. Efetov, and A. F. Young, “Superconductivity and strong correlations in moiré flat bands”, Nat. Phys. 16, 725 (2020).

76 E. Brillaux, D. Carpentier, A. A. Fedorenko, and L. Savary, “Nematic insulator at charge neutrality in twisted bilayer graphene,” [arXiv:2008.05401]

77 T. Soejima, D. E. Parker, N. Bultinck, J. Hauschild, and M. P. Zaletel, “Efficient simulation of moiré materials using the density matrix renormalization group”, [arXiv:2009.02354]

78 B. M. Hunt et al., “Direct measurement of discrete valley and orbital quantum numbers in bilayer graphene”, Nat. Commun. 8, 948 (2017); see Table I in the SI.

79 Y. Ren et al., “WKB estimate of bilayer graphene’s magic twist angles”, [arXiv:2006.13292]

80 S. Becker et al., “Mathematics of magic angles in a model of twisted bilayer graphene” [arXiv:2008.08489]

81 N.N.T. Nam and M. Koshino, “Lattice relaxation and energy band modulation in twisted bilayer graphene” Phys. Rev. B, 96, 075311, (2017).

82 See Supplemental Material for details of the derivation of the RG equations, RG evolution of the sublattice polarization and Wilson loop eigenvalues, as well as the single particle excitation spectrum in the strong coupling.

83 I.F. Herbut, “Quantum Critical Points with the Coulomb Interaction and the Dynamical Exponent: When and Why z=1” Phys. Rev. Lett. 87, 137004 (2001); although there are non-logarithmic corrections to the dielectric function, whose full consideration is beyond the scope of this work, they do not change the main conclusions.

84 H. Watanabe, “Counting Rules of Nambu-Goldstone Modes” Ann. Rev. of Cond. Mat. Phys., 11, 169 (2020).
Supplemental Material for “Towards the hidden symmetry in Coulomb interacting twisted bilayer graphene: renormalization group approach”

Appendix A: Details of the 1st stage RG derivation for Coulomb interacting Bistritzer-MacDonald model

For the contour $C$ enclosing the $z$-plane real line segment $(-E_c, -E'_c)$ in the clockwise, and segment $(E'_c, E_c)$ in the counterclockwise, sense

$$\oint_C \frac{dz}{2\pi i} (\mathbf{r}|\hat{G}(z)|\mathbf{r}') = \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}_c} e^{-ik\cdot\mathbf{r}'} \oint_C \frac{dz}{2\pi i} (k|\hat{G}(z)|k').$$  \hspace{1cm} (A1)

For $E'_c \gg w_{0,1}$, we can expand the Green’s function to first non-trivial order in $\hat{T}$ and find

$$\oint_C \frac{dz}{2\pi i} (k|\hat{G}(z)|k') = \oint_C \frac{dz}{2\pi i} (k|\hat{G}_0(z)|k') + \oint_C \frac{dz}{2\pi i} (k|\hat{G}_0(z)\hat{T}\hat{G}_0(z)|k').$$  \hspace{1cm} (A2)

Without loss of generality, we can focus on one valley only, the contribution from the second one can be determined by time reversal symmetry. The particle-hole symmetric BM Hamiltonian, acting on Bloch functions, is

$$\hat{H} = \begin{pmatrix} v_F \sigma \cdot \mathbf{p} & T(\mathbf{r})e^{i\mathbf{q}_1\cdot\cdot} \\ e^{-i\mathbf{q}_1\cdot\cdot} T(\mathbf{r})^\dagger (\mathbf{p} + \mathbf{q}_1) \end{pmatrix} = \hat{H}_0 + \hat{T},$$  \hspace{1cm} (A3)

and the interlayer tunneling term is

$$T(\mathbf{r}) = \sum_{j=1}^{3} T_j e^{-i\mathbf{q}_j\cdot\cdot}; T_{j+1} = w_{0,1} + w_1 \begin{pmatrix} 0 & e^{i\pi j} \\ e^{-i\pi j} & 0 \end{pmatrix}. \hspace{1cm} (A4)$$

Here $\mathbf{r}$ and $\mathbf{p}$ should be understood to be (first quantized) operators. Therefore,

$$\hat{G}_0(z) = \begin{pmatrix} \hat{g}_0(z, \mathbf{p}) & 0 \\ 0 & \hat{g}_0(z, \mathbf{p} + \mathbf{q}_1) \end{pmatrix},$$ \hspace{1cm} (A5)

where the intra-layer Green’s function is

$$\hat{g}_0(z, \mathbf{p}) = (\omega - v_F \sigma \cdot \mathbf{p})^{-1} = \frac{1}{2} \sum_{s = \pm 1} \frac{1 + s\sigma \cdot \mathbf{p}}{z - sv_F p}$$  \hspace{1cm} (A6)

and

$$\hat{G}_0(z)\hat{T}\hat{G}_0(z) = \begin{pmatrix} 0 & \hat{g}_0(z, \mathbf{p})T(\mathbf{r})e^{i\mathbf{q}_1\cdot\cdot}\hat{g}_0(z, \mathbf{p} + \mathbf{q}_1) \\ \hat{g}_0(z, \mathbf{p} + \mathbf{q}_1)e^{-i\mathbf{q}_1\cdot\cdot} T(\mathbf{r})^\dagger \hat{g}_0(z, \mathbf{p}) \end{pmatrix}. \hspace{1cm} (A7)$$

Now,

$$\langle k|\hat{g}_0(z, \mathbf{p})T(\mathbf{r})e^{i\mathbf{q}_1\cdot\cdot}\hat{g}_0(z, \mathbf{p} + \mathbf{q}_1)|k'\rangle = \frac{1}{4} \sum_{ss' = \pm} \frac{1 + s\sigma \cdot \mathbf{k}}{z - sv_F k} \langle k|T(\mathbf{r})e^{i\mathbf{q}_1\cdot\cdot}|k'\rangle \frac{1 + s'\sigma \cdot \mathbf{k}' + \mathbf{q}_1}{z - sv_F |k' + \mathbf{q}_1|}$$

$$= \frac{1}{4} \sum_{j=1}^{3} \delta_{\mathbf{k}', \mathbf{k} + \mathbf{q}_j - \mathbf{q}_1} \sum_{ss' = \pm} \frac{1 + s\sigma \cdot \mathbf{k}}{z - sv_F k} T_j \frac{1 + s'\sigma \cdot \mathbf{k}' + \mathbf{q}_1}{z - sv_F |k + \mathbf{q}_j|}, \hspace{1cm} (A8)$$

where we used

$$\langle k|T(\mathbf{r})e^{i\mathbf{q}_1\cdot\cdot}|k'\rangle = \sum_{j=1}^{3} T_j (2\pi)^2 \delta (\mathbf{k}' - (\mathbf{k} + \mathbf{q}_j - \mathbf{q}_1)) \equiv \sum_{j=1}^{3} T_j \delta_{\mathbf{k}', \mathbf{k} + \mathbf{q}_j - \mathbf{q}_1}. \hspace{1cm} (A9)$$

Because it is novel, let us focus on the second term on the RHS in $\langle A2 \rangle$. Although formally the contribution to the contour integral comes from $s = s'$ and $s = -s'$, only the latter contributes to the RG flow. To see this, note that for $s = s'$ there is no contribution whatsoever if $v_F k$ and $v_F |k + \mathbf{q}_1|$ both lie inside, or both outside, the interval $(E'_c, E_c)$. There is a contribution only if one of them is outside of the interval. But, in that case, we can imagine extending...
the interval until both poles are included. This shows, that the contribution from adjacent shells cancels if \( s = s' \). Therefore, consider only \( s = -s' \). Then, because \( v_F|q_j| \ll E_c' \), we have

\[
\int \frac{dz}{2\pi i} \frac{1 + \sigma \cdot \frac{k}{k} T_j \left( \frac{1 - \sigma \cdot \frac{k + q_j}{k + q_j}}{z - v_F k} + \frac{1 - \sigma \cdot \frac{k}{k} T_j \left( \frac{1 + \sigma \cdot \frac{k + q_j}{k + q_j}}{z + v_F k} \right) \right)} {v_F k (T_j - \sigma \cdot \frac{k}{k} T_j \sigma \cdot \frac{k}{k})} \approx \Theta (E_c - v_F k) \Theta (v_F k - E_c'),
\]

where \( \Theta(x) \) is the Heaviside step function. The component of \( T_j \) proportional to \( w_0 \) is an identity matrix, which of course commutes through \( k \cdot \sigma \). And because \( k \cdot \sigma k \cdot \sigma = k^2 \), there is no contribution to the renormalization of \( w_0 \). So,

\[
\frac{1}{2} \int d^2r d^2r' V(r - r') \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} e^{i k \cdot r - i k' \cdot r'} \psi^\dagger_{\sigma}(r) \left( \int \frac{dz}{2\pi i} (k) \hat{G}_0(z) \hat{T} \hat{G}_0(z) |k' \right) \psi_{\sigma}(r') =
\]

\[
\sum_{j=1}^3 \frac{d^2q}{(2\pi)^2} \left( \psi^\dagger_{\sigma, q} \right) \left( \begin{array}{c} 0 \\ 0 \\ \varUpsilon_j (q, E_c, E'_c) \end{array} \right) \psi_{\sigma, q} - q - q_i + \psi^\dagger_{\sigma, q + q_j - q_i} \left( \begin{array}{c} 0 \\ 0 \\ \varUpsilon_j (q, E_c, E'_c) \end{array} \right) \psi_{\sigma, q}.
\]

where

\[
\varUpsilon_j (q, E_c, E'_c) = \int \frac{d^2k}{(2\pi)^2} \frac{\Theta (E_c - v_F k) \Theta (v_F k - E'_c)}{4v_F k} (T_j - \sigma \cdot \frac{k}{k} T_j \sigma \cdot \frac{k}{k}).
\]

Because \( v_F q \ll E_c \), we can expand in powers of \( v_F q / E_c \). Moreover, because

\[
\sigma \cdot k \sigma_1 \sigma_2 \cdot k = \pm (k_x^2 - k_y^2) \sigma_{1,2} + 2k_x k_y \sigma_{2,1},
\]

the term \( \sigma \cdot k T_j \sigma \cdot k \) will not contribute to the leading term in which \( q \) is set to 0 due to the angular integration. For Coulomb interaction \( V_k = 2\pi e^2 / (\epsilon k) \) we find

\[
\varUpsilon_j (q, E_c, E'_c) = \frac{e^2}{4\epsilon v_F} T_j w_0 = \frac{E_c}{E_c} \ln \frac{E_c}{E_c} + \ldots
\]

where \( \ldots \) are higher order terms in \( v_F q / E_c \). Therefore, we find the RG equations for the interlayer couplings

\[
\frac{d w_0}{d \ln E_c} = 0,
\]

\[
\frac{d w_1}{d \ln E_c} = - \frac{e^2}{4\epsilon v_F} w_1.
\]

Clearly, as long as \( v_F q_j \ll E_c \), the expansion is in powers of \( w_{0,1}/E_c \) and higher order terms in the expansion of BM Green’s function, i.e. terms beyond \( G_0 T G_0 \) will be suppressed by powers of \( w_{0,1}/E_c \) and higher order gradients. The above term is the dominant correction to the BM interlayer tunneling as is consistent with the notion of the continuum model being a field theory expanded in powers of gradients.}

The contribution from the \( G_0(\omega) \) term is standard and leads to

\[
\frac{d v_F}{d \ln E_c} = - \frac{e^2}{4\epsilon}.
\]
Because \( \frac{dE}{d \ln E_c} = 0 \), the above equations are readily integrated. If we stop the renormalization at the scale \( E^*_c \ll E_c \) we find

\[
\begin{align*}
  w_1(E^*_c) &= w_1(E_c) \left(1 + \frac{e^2}{4\epsilon v_F(E_c)} \ln \frac{E_c}{E^*_c}\right) \\
  v_F(E^*_c) &= v_F(E_c) + \frac{e^2}{4\epsilon} \ln \frac{E_c}{E^*_c}
\end{align*}
\]

\[ (A19) \]

\[
\begin{align*}
  \Rightarrow \frac{w_1(E^*_c)}{v_F(E^*_c)} &= \frac{w_1(E_c)}{v_F(E_c)} \\
  \frac{w_0(E^*_c)}{w_1(E^*_c)} &= \frac{w_0(E_c)}{w_1(E_c)} \left(1 + \frac{e^2}{4\epsilon v_F(E_c)} \ln \frac{E_c}{E^*_c}\right).
\end{align*}
\]

\[ (A20) \]

\[ (A21) \]

\[ (A22) \]

The above shows that even when we start away from the chiral limit \( E^*_c \) at the UV scale \( E_c \), at a lower energy scale \( E^*_c \) we approach it. In practice, we find that, after we combine this stage 1 RG with the non-perturbative stage 2 numerical RG, our results are insensitive to the choice of \( E^*_c \) as long as \( w_1/E^*_c \) remains small.

**Appendix B: Sublattice polarization and Wilson loop evolution under RG; and the single particle dispersion in the strong coupling limit.**

![Graph](image)

FIG. S1. The positive eigenvalue of the the sublattice polarization operator \( \sigma_3 \) projected onto the two narrow bands at different \( k \)-points in BZ for un-renormalized BM model with \( w_1/v_F k_\theta = 0.586 \) (magic angle) and \( w_0/w_1 = 0.83 \) (a), and \( w_0/w_1 = 0.83 \) after 1st and 2nd stage RG. The increase marks the approach of the chiral limit which is perfectly sublattice polarized i.e. the eigenvalue is 1 for each \( k \).

As explained in the e.g. Ref. [3] the eigenstates of the projected operator

\[
\hat{O} = \hat{P} e^{-\frac{i}{\hbar} \frac{1}{2}, g_1 \cdot r} \hat{P},
\]

are hybrid Wannier states. Here \( \hat{P} \) is the projection operator onto the narrow bands, \( g_1 \) is the primitive vector of the reciprocal lattice shown in the Fig 1b of the main text, and \( N_1 \) is the number of unit cells along the direction of \( L_1 \) in the entire lattice with periodic boundary conditions. We thus have

\[
\hat{O}|w_\alpha(n, k g_2)\rangle = e^{-2\pi i \frac{1}{2}, (n+(\pm)_k/L_\alpha)|w_\alpha(n, k g_2)\rangle}.
\]

\[ (B2) \]

The hybrid WSs \( |w_\alpha(n, k g_2)\rangle \) are labeled by their momentum \( k \) along \( g_2 \) which is conserved by \( \hat{O} \) and the index \( n \) of the unit cell along \( L_1 \); \( \alpha = \pm 1 \) labels their winding number. Unlike the familiar lowest Landau level wavefunctions in the Landau gauge, the shapes of our hybrid WSs for the narrow bands depend on the momentum index \( k \).

The \( \langle x_\pm \rangle \) physically represents the average of the position operator within each 1D unit cell whose dependence on the conserved momentum \( k \) is shown in the Fig. [3] for various stages of renormalization. The two curves display the winding numbers of \( \pm 1 \) as the momentum \( k \) increases from 0 to \( g_2 \) i.e. the average position of one set of states slides to the right and the other set of states to the left under the increase of the wavenumber \( k \), similar to Landau.
gauge Landau level states in opposite magnetic field. The monotonic steepening of these curves under RG marks the approach of the chiral limit (see also Fig 3 of Ref. 3).

![Graph](image)

**FIG. S2.** The RG evolution of the Wilson loop eigenvalues for the two narrow bands in valley K for the unrenormalized narrow bands with $w_0/w_1 = 0.83$ (blue), renormalized after stage 1 (red), after stage 1 and 9th step of stage 2 RG (purple), after stage 1 and 14th step of stage 2 RG (green) and after both stage 1 and stage 2 (black). The steepening of the Wilson loop marks the approach of the chiral limit. The parameters for the RG are the same as the Fig 3 in the main text.

![Graph](image)

**FIG. S3.** (a) The single particle dispersion in the strong coupling limit for $w_0/w_1 = 0.83$ and $w_1/\nu_F k_\theta = 0.586$ after the 1st and 2nd stage RG. The BZ cut is shown in the main text Fig 1b. (b) The chemical potential $\mu_0$ as the function of the filling factor $\nu$ assuming the single particle excitations are non-interacting.

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1 L. Balents, “General continuum model for twisted bilayer graphene and arbitrary smooth deformations”, SciPost Phys., 7, 48 (2019).
2 G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, “Origin of magic angles in twisted bilayer graphene,” Phys. Rev. Lett. 122, 106405 (2019).
3 J. Kang and O. Vafek, “Non-Abelian Dirac node braiding and near-degeneracy of correlated phases at odd integer filling in magic angle twisted bilayer graphene”, Phys. Rev. B 102, 035161 (2020).