Finding Disjoint Paths on Edge-Colored Graphs: A Multivariate Complexity Analysis

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COCOA 2016
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Outline

Introduction

FPT Vertex Cover

Parameterized Inapproximability

Conclusion
Motivations

- Originates from **Social Network Analysis**.
- Computing the **connectivity between 2 nodes** is an important problem.
  - measurement of information flow,
  - cohesion group and centrality.
Motivations

- Originates from **Social Network Analysis**.
- Computing the **connectivity between 2 nodes** is an important problem.
  - measurement of information flow,
  - cohesion group and centrality.
- Different kind of relationship:
  - **Different colors on the edges**.
  - Integration of different type of information.
    - Different media.
    - Different protocol.
    - ...
MaxCDP

- Monochromatic disjoint paths between 2 nodes.
- Monochromatic: Information spread among relations of the same kind.
- Number: More connected.
- Length: Short paths are considered more significant.
- Vertex disjoint: Security, traffic congestion...

Introduce vertex disjoint and color-disjoint versions.

- How different relations in a network connect 2 vertices.

Co-author
Acting
Playing
T. Pratchett
S. Hawking
MaxCDP

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  - How different relations in a network connects 2 vertices
MaxCDP: Known results

- Not approximable within with $c^{1-\varepsilon}$ [Dondi et al. 13],
  - but $c$-approximable [Wu 12].
- W[1]-hard w.r.t. number of paths [Dondi et al. 13].
  - Even not in XP (NP-C for 2 paths) [Gourves et al. 12].
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▲ $W[1]$-hard w.r.t. number of paths [Dondi et al. 13].
  ▶ Even not in XP (NP-C for 2 paths) [Gourves et al. 12].
▲ When the length of the paths are bounded by $\ell$:
  ▶ Polynomial if $\ell < 4$, NP-C otherwise [Wu 12].
  ▶ FPT w.r.t. number of paths + $\ell$ [Dondi et al. 13].
    ▶ But no polynomial kernel [Golovach Thilikos 11]
Related problems

- 1 color:
  - 1 source-target: Polynomial (flow).
  - \(k\) sources-targets: NP-C but FPT for \(k\). [Robertson Seymour]
Fixed-Parameter Tractability

- Problem in FPT: any instance \((I, k)\) solved in \(f(k) \cdot |I|^c\).

- Examples:
  - Solution of size \(k\) in a \(n\)-vertices graph.
  - \(n\) voters for \(k\) candidates.
  - Requests of size \(k\) in a \(n\)-sized database.
  - ...

- Many ways to parameterize.
  - Solution size.
  - Structure of the input.
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Structural results MaxCDP

- Real-data is **not random** (e.g. small world phenomenon).
- Information on **the structure**.
- Use it in parameterized complexity.
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Parameterized Inapproximability
MaxCDP w.r.t. Vertex Cover number

- Aim: $f(\tau)n^{O(1)}$ exact algorithm.
- $\tau$ computed in FPT time.
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- **Aim:** $f(\tau)n^{O(1)}$ exact algorithm.
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- For all paths $(s, v, t)$ (length 3): remove $v$.
  - Only one path can use $v$ in an optimal solution.
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- In any $s - t$ path (length $> 3$) of the solution:
  - A vertex in IS is adjacent to one in VC or $s$ or $t$. 
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In any $s - t$ path (length $> 3$) of the solution:
- A vertex in IS is adjacent to one in VC or $s$ or $t$.
  - Paths are of length at most $2\tau$.
  - At most $\tau$ different paths.
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  - A vertex in IS is adjacent to one in VC or $s$ or $t$.
    - Paths are of length at most $2\tau$.
    - At most $\tau$ different paths.
    - Known FPT [Bonizzoni et al. 13].
MaxCDDP and Vertex Cover

▶ For MaxCDDP: cannot remove length 3 paths.
MaxCDDP and Vertex Cover

- For MaxCDDP: cannot remove length 3 paths.

- \((s, d, v, e, t) \cup (s, a, b, c, t)\) better than \((s, v, t)\).
Outline

Introduction

FPT Vertex Cover

Parameterized Inapproximability
# Coping with the hardness

|                | Time | Solution Quality |
|----------------|------|------------------|
| **FPT**        | ![Face](image1.png) | ![Face](image2.png) |
| **Poly. Approx.** | ![Face](image3.png) | ![Face](image4.png) |
## Coping with the hardness

| Method       | Time | Solution Quality |
|--------------|------|------------------|
| FPT          | ![Neutral](neutral.png) | ![Happy](happy.png) |
| Poly. Approx. | ![Happy](happy.png) | ![Sad](sad.png) |
| FPT Approx.  | ![Happy](happy.png) | ![Neutral](neutral.png) |
FPT-Approximation

- A (minimization) problem is **fpt-$\rho$-approximable** if for any input $(I, k)$:
  - If $\text{opt}(I) \leq k$, computes a solution of value bounded by $\rho(k) \cdot k$ in time $f(k)|I|^{O(1)}$,
  - Otherwise, output can be arbitrary.
Example: computing treewidth

|       | Time            | Ratio     |
|-------|-----------------|-----------|
| FPT   | $2^{O(k^2)} \cdot n$ | 1         |
| Poly. Approx. | $\text{poly}(n)$     | $O(k\sqrt{\log k})$ |

[Bodlaender 96] [Feige et al. 05]
**Example: computing treewidth**

|                          | Time         | Ratio       | Reference                  |
|--------------------------|--------------|-------------|----------------------------|
| **FPT**                  | $2^{O(k^2)} \cdot n$ | $1$         | [Bodlaender 96]            |
|                          | ![Sad Face]  | ![Happy Face] |                            |
| **Poly. Approx.**        | $\text{poly}(n)$ | $O(k \sqrt{\log k})$ | [Feige et al. 05]          |
|                          | ![Happy Face] | ![Sad Face]  |                            |
| **FPT Approx.**          | $2^{O(k)} \cdot n$ | $5$         | [Bodlaender et al. 13]     |
|                          | ![Happy Face] | ![Happy Face] |                            |
Threshold Set

\[
\begin{array}{ccc}
U & S & w(S_i) \\
1 & S_1 & 2 \\
2 & S_2 & 1 \\
3 & S_3 & 2 \\
4 & & \\
\end{array}
\]
Threshold Set

A maximum solution: \( T = \{1, 2\} \subseteq U \)
Threshold Set

A maximum solution: $T = \{1, 2\} \subseteq U$

- **Independent Set** when $U = V$, $S = E$, weights all 1.
**Threshold Set**

A maximum solution: \( T = \{1, 2\} \subseteq U \)

- **INDEPENDENT SET** when \( U = V, S = E \), weights all 1.
- No fpt cost \( \rho \)-approximation, for any \( \rho \) function (unless FPT=W[1]) [Marx 2013].
Reduction from Threshold Set

$U$

\[1\]
\[2\]
\[3\]
\[4\]

\[s\]

\[u_1\]
\[u_2\]
\[u_3\]
\[u_4\]

A maximum solution:

$T = \{1, 2\} \subseteq U$

MaxCDP (and MaxCDDP due to the $s_i$) are not $\text{fpt-}\rho$-approximable, for any function $\rho$ (unless $\text{FPT}=\text{W}[1]$).
Reduction from Threshold Set

\[ U \quad S \quad w(S_i) \]

1 \quad \(S_1\) \quad 2

2

3

4

\(u_2\)

\(u_3\)

\(u_4\)

\(u_1\)

\(S_1\)

\(S_1\)

\(S_1\)

\(S_2\)

\(\text{Reduction with one-to-one correspondence between solutions.}\)

\(\text{MaxCDP (and MaxCDDP du to the } s_i \text{) are not fpt-} \rho \text{-approximable, for any function } \rho \text{ unless FPT=W[1].}\)
Reduction from Threshold Set

|   |   |   |
|---|---|---|
| 1 | S<sub>1</sub> | 2 |
| 2 | S<sub>2</sub> | 1 |
| 3 | S<sub>3</sub> | 2 |
| 4 | S<sup>2</sup><sub>1</sub> |   |
|   | S<sup>2</sup><sub>2</sub> |   |
|   | S<sub>3</sub> |   |
|   | S<sup>1</sup><sub>1</sub> |   |
|   | S<sup>1</sup><sub>2</sub> |   |
|   | S<sup>1</sup><sub>3</sub> |   |
|   | t |   |

Reduction with one-to-one correspondence between solutions.
MaxCDP (and MaxCDDP due to the $s_i$) are not fpt-\(\rho\)-approximable, for any function \(\rho\) (unless FPT=W[1]).
Reduction from Threshold Set

\[ U \quad S \quad w(S_i) \]

1. \( S_1 \quad 2 \)
2. \( S_2 \quad 1 \)
3. \( S_3 \quad 2 \)

Reduction with one-to-one correspondence between solutions.

MaxCDP (and MaxCDDP due to the \( s_i \)) are not \( \text{fpt-} \rho \)-approximable, for any function \( \rho \) unless \( \text{FPT=W[1]} \).
Reduction from Threshold Set

\[
\begin{align*}
U & \quad S & w(S_i) \\
1 & \quad S_1 & 2 \\
2 & \quad S_2 & 1 \\
3 & \quad S_3 & 2 \\
4 & & \\
\end{align*}
\]

Reduction with one-to-one correspondence between solutions.

\[
\text{MaxCDP (and MaxCDDP due to the} \quad s_i \quad \text{are not FPT-}\rho\text{-approximable, for any function} \quad \rho \quad \text{unless FPT=\text{W}[1].}
\]
Reduction from Threshold Set

U   S   \( w(S_i) \)

\begin{align*}
1 & \quad S_1 & 2 \\
2 & \quad S_2 & 1 \\
3 & \quad S_3 & 2 \\
4 & \\
\end{align*}

A maximum solution: \( T = \{1, 2\} \subseteq U \)
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▶ Reduction with one-to-one correspondence between solutions.

▶ MaxCDP (and MaxCDDP due to the \( s_i \)) are not
fpt-\( \rho \)-approximable, for any function \( \rho \) (unless FPT=W[1]).
Open questions

- Complexity on **special class** of graphs? (planar + 2 colors ?)
- Parameterized complexity w.r.t. **feedback vertex set**? (XP ? FPT ?)
- Fine grained complexity lower bounds?
谢谢