Action and entropy of black holes in spacetimes with cosmological constant

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Abstract

In the Euclidean path integral approach, we calculate the actions and the entropies for the Reissner-Nordström-de Sitter solutions. When the temperatures of black hole and cosmological horizons are equal, the entropy is the sum of one-quarter areas of black hole and cosmological horizons; when the inner and outer black hole horizons coincide, the entropy is only one-quarter area of cosmological horizon; and the entropy vanishes when the two black hole horizons and cosmological horizon coincide. We also calculate the Euler numbers of the corresponding Euclidean manifolds, and discuss the relationship between the entropy of instanton and the Euler number.

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I. INTRODUCTION

No doubt the discovery of Hawking \([1]\) that a black hole emits particles as a blackbody is one of the most important achievements of quantum field theory in curved spacetimes. Due to the Hawking evaporation, classical general relativity, statistical physics, and quantum field theory are connected in the quantum black hole physics. Therefore it is generally believed that the deep investigation of black hole physics would be helpful to set up a satisfactory quantum theory of gravitation.

For a Schwarzschild black hole, it is well known that the hole has a Hawking temperature

\[ T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, \]  

(1.1)

where \(\kappa\) is the surface gravity at the horizon. Shortly after the Hawking’s discovery, Hartle and Hawking \([2]\) rederived the black hole radiance in the path integral formalism. Gibbons and Perry \([3]\) found that the thermal Green function in the black hole background has an intrinsic period \(\beta = 2\pi\kappa^{-1}\), which is just a characteristic feature of finite temperature quantum fields in flat spacetimes.

Before the Hawking’s discovery, Bekenstein \([4]\) already suggested that a black hole should have an entropy proportional to the area of black hole horizon. The work of Hawking made the entropy quantitative:

\[ S_{\text{BH}} = \frac{1}{4} A_{\text{BH}}, \]  

(1.2)

where \(A_{\text{BH}}\) is the area of horizon. In addition, Gibbons and Hawking \([5]\) found that the Hawking evaporation also occurs at the cosmological horizon in the de Sitter space. Furthermore they observed \([6]\) that the cosmological horizon has an associated entropy obeying the area formula:

\[ S_{\text{CH}} = \frac{1}{4} A_{\text{CH}}, \]  

(1.3)

where \(A_{\text{CH}}\) is the area of cosmological horizon.

When the Schwarzschild black hole is embedded into the de Sitter space, one has the Schwarzschild-de Sitter black hole:

\[ ds^2 = -\left(1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{1}{3} \Lambda r^2\right)^{-1} dr^2 + r^2 d\Omega^2, \]  

(1.4)

where \(\Lambda\) is the cosmological constant. As \(9M^2\Lambda < 1\), we have a black hole horizon \(r_B\) and a cosmological horizon \(r_C\) \((r_C > r_B)\). Naturally there exist two different temperatures associated with the black hole horizon and cosmological horizon, respectively. Combining Eqs. (1.2) and (1.3), most people believe that the gravitational entropy of Schwarzschild-de Sitter black hole should be

\[ S = S_{\text{BH}} + S_{\text{CH}} = \frac{1}{4} A_{\text{BH}} + \frac{1}{4} A_{\text{CH}}. \]  

(1.5)

But the derivation of this entropy formula (1.5) is not yet developed. A good method to get the area formula of black hole entropy is the formalism of Gibbons and Hawking \([6]\)
in the Euclidean path integral method of quantum gravity theory. In this formalism the regular gravitational instantons (regular solutions of Euclidean Einstein equations) play a crucial role. However, this formalism cannot apply to the Schwarzschild-de Sitter black holes because one cannot obtain a regular gravitational instanton solutions in this case. By analytically continuing Eq. (1.4) to its Euclidean section, one can employ the same procedure used in the Schwarzschild black hole to remove the conical singularity at the black hole horizon or at the cosmological horizon. But one cannot remove the two singularities simultaneously because the temperatures usually do not equal to each other. Thus the Euclidean manifold is left with a conical singularity. An exception is the Nariai spacetime, which can be regarded as the limiting case of the Schwarzschild-de Sitter black hole as the black hole horizon and cosmological horizon coincide \( (9M^2\Lambda = 1) \). Its Euclidean spacetime represents a regular gravitational instanton with topological structure \( S^2 \times S^2 \), and its metric is

\[
ds^2 = (1 - \Lambda r^2)d\tau^2 + (1 - \Lambda r^2)^{-1}dr^2 + \Lambda^{-1}d\Omega_2^2.
\]  

(1.6)

After a coordinate transformation, one can clearly see its topological structure. The metric (1.6) can be rewritten as

\[
ds^2 = \Lambda^{-1}(d\xi^2 + \sin^2\xi d\psi^2) + \Lambda^{-1}d\Omega_2^2,
\]  

(1.7)

where \( 0 \leq \psi \leq 2\pi \), and \( 0 \leq \xi \leq \pi \).

The geometric features of black hole temperature and entropy seem to strongly imply that the black hole thermodynamics is closely related to nontrivial topological structure of spacetime. In recent years, there has been considerable interest in this aspect of black hole physics. Bañados, Teitelboim, and Zanelli [7] showed that in the Euclidean black hole manifold with topology \( R^2 \times S^{d-2} \), the deficit angle of a cusp at any point in \( R^2 \) and the area of the \( S^{d-2} \) are canonical conjugates. The black hole entropy emerges as the Euler class of a small disk centered at the horizon multiplied by the area of the \( S^{d-2} \) there. Hawking, Horowitz, and Ross [8] argued that, due to the very different topologies between the extremal black holes and nonextremal black holes, the area formula of entropy fails for extremal black holes and the entropy of extremal black holes should vanish, despite the non-zero area of horizon. In the Euclidean section of extremal black holes since there does not exist any conical singularity at the black hole horizon, the Euclidean time \( \tau \) can have any period. Teitelboim [9] further confirmed the zero entropy in the Hamiltonian formalism for extremal black holes. He put forward that the vanishing entropy is due to the vanishing Euler characteristic \( \chi \) for extremal black holes. In Ref. [10], Gibbons and Kallosh investigated the relation between the entropy and Euler number for dilaton black holes in some detail. They found that for extremal dilaton black holes an inner boundary should be introduced in addition to the outer boundary to obtain an integer value of the Euler number. Thus the resulting manifolds have (if one identifies the Euclidean time) a topology \( S^1 \times R \times S^2 \) and the Euler number \( \chi = 0 \). For the nonextremal black holes the topology is \( R^2 \times S^2 \) and the Euler number \( \chi = 2 \). More recently, Liberati and Pollifrone [11] have further discussed the relation between the black hole entropy and the Euler number and suggested the following formula

\[
S = \frac{\chi A}{8}.
\]  

(1.8)
They have checked this formula for a wide class of gravitational instantons such as Schwarzschild instanton, dilaton $U(1)$ black holes, de Sitter instanton, Nariai instanton and Kerr black holes.

In this paper, we would like to provide some evidence in favor of the formula (1.3) in the static, spherically symmetric Reissner-Norstrøm (RN)-de Sitter black holes, which are solutions of the Einstein-Maxwell equations with a cosmological constant. When $M = |Q|$, the temperature of black hole horizon is equal to the one of cosmological horizon, we have the so-called lukewarm black hole. Since its analytic continuation to the Euclidean section provides a regular instanton, we can check the formula (1.5) in the standard Euclidean quantum theory of gravitation. In addition since we have two different horizons for this regular instanton in contrast to the Nariai instanton, it is of interest to see how to modify the formula (1.8) in this case.

The plan of this paper is as follows. In Sec. II we first introduce the lukewarm black hole and the regular instanton, and then calculate the action and the entropy of this instanton. In Sec. III we investigate two special cases of RN-de Sitter solutions. One is the case where the inner and outer horizons of black holes coincide. The solution is called a cold black hole. The other is the so-called ultracold solution where the two black hole horizons and the cosmological horizon coincide. In Sec. IV we compute the Euler number for lukewarm black holes, cold black holes and ultracold solutions, respectively, and discuss the relation between our results and Euler numbers. The conclusions and discussions are included in Sec. V.

II. ENTROPY OF LUKEWARM BLACK HOLES

The action of the Einstein-Maxwell theory with a cosmological constant is

$$I = \frac{1}{16\pi} \int_V d^4x \sqrt{-g} (R - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \frac{1}{8\pi} \int_{\partial V} d^4x \sqrt{-h} K, \quad (2.1)$$

where $R$ denotes the scalar curvature, $K$ is the trace of the second fundamental form of surface $\partial V$, $\Lambda$ is the cosmological constant and $F_{\mu\nu}$ is the Maxwell field. In the action (2.1) one has the static, spherically symmetric RN-de Sitter solutions (in this paper we consider the case of $\Lambda > 0$ only, for the case $\Lambda < 0$ results will be trivial):

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2 d\Omega_2^2, \quad (2.2)$$

where

$$N^2(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2, \quad (2.3)$$

$M$ and $Q$ are the mass and the electric charge of the solution, respectively, $d\Omega_2^2$ stands for the line element on a unit 2-sphere. Usually the equation $N^2(r) = 0$ has four roots, three positive real roots and a negative real root. The maximal positive root $r_C$ is the cosmological horizon; the minimal ($r_A$) is the inner (Cauchy) horizon of black hole; and the intermediate ($r_B$) the outer horizon of black hole. Their surface gravities are

$$\kappa_a = \frac{1}{2} |[N^2(r)]'|_{r=r_a}, \quad a = A, B, C \quad (2.4)$$
where a prime denotes the derivative with respect to $r$. Thus the outer horizon of black hole has a Hawking temperature $T_B = \kappa_B/2\pi$, and the cosmological horizon also has a temperature $T_C = \kappa_C/2\pi$ different from $T_B$. Hence this spacetime is unstable quantum mechanically. In its Euclidean section

$$ds^2 = N^2(r)dr^2 + N^{-2}(r)dt^2 + r^2d\Omega_2^2,$$

one has no way to remove simultaneously the two conical singularities at the black hole horizon and cosmological horizon. However, Mellor and Moss [12], and Romans [13] found that when $M^2 = Q^2$ in Eq. (2.3), one has the lukewarm RN solution where

$$T_B = T_C = \frac{\kappa_B}{2\pi}.$$  

In this case the spacetime is in thermal equilibrium with a common temperature $T_C = T_B$, and is stable classically and quantum mechanically. This equality (2.6) provides us a regular instanton in the Euclidean version (2.5), because we can remove simultaneously two conical singularities by requiring the imaginary time $\tau$ has a period $\beta = T_C^{-1} = T_B^{-1}$. The resulting manifold (2.5) has a topology $S^2 \times S^2$, which is the same as that of the Nariai instanton.

Following the standard method of Gibbons and Hawking [6], we now evaluate the Euclidean action and the entropy for this lukewarm instanton solution. The Euclidean action can be obtained by continuing (2.1) to its Euclidean counterpart

$$I_E = -\frac{1}{16\pi} \int_V d^4x \sqrt{g} \left( R - 2\Lambda - F_{\mu\nu}F^{\mu\nu} \right) - \frac{1}{8\pi} \int_{\partial V} d^3x \sqrt{h} \left( K - K_0 \right),$$  

where we have introduced a subtraction term $K_0$, which is the trace of the second fundamental form of the reference background. In the case of lukewarm solution, we do not have to consider the boundary term in the action (2.7), because the topology is $S^2 \times S^2$ which has no boundary. Using the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2 \left( F_{\mu\nu} F^\nu - \frac{1}{4} g_{\mu\nu} F^2 \right),$$

and the scalar curvature

$$R = -g^{-1/2}[g^{1/2}(N^2)']' - 2G_{00}^0,$$

where $G_{00}^0$ is the 0-0 component of the Einstein tensor, we can easily obtain

$$I_E = -\pi r_C^2 - \pi r_B^2 - \beta Q^2 \left( \frac{1}{r_B} - \frac{1}{r_C} \right).$$

It is instructive to compare the action (2.10) with the one of RN black holes with topology $R^2 \times S^2$ in the grand canonical ensemble [14]. In the latter $\beta M$ replaces the first term in (2.10). So different topological structures would result in very different Euclidean actions. In the action (2.10), the first term is obviously one quarter of the area of cosmological horizon; the second one is one quarter of the area of black hole horizon; and the last one is the difference of electric potential energy between at the cosmological and the black hole
horizons. When \( Q = 0 \) the action (2.10) also includes two special cases in Ref. [6]: For the de Sitter space \((M = 0)\) with \( r_{C} = \sqrt{\Lambda/3} \), the first term gives the same result; for the Nariai instanton, with \( r_{B} = r_{C} = 1/\sqrt{\Lambda} = 3M \), first two terms give the same result.

In order to get the entropy of the instanton, it is helpful to employ the general argument given by Kallosh, Ortín, and Peet [15]. In a thermodynamic system with conserved charges \( C_{i} \) and corresponding chemical potentials \( \mu_{i} \), the starting point to study this system is the grand partition function \( Z \) in a grand canonical ensemble

\[
Z = \text{Tr} e^{-\beta(H - \mu_{i} C_{i})}.
\]  

(2.11)

The thermodynamic potential \( W = E - TS - \mu_{i} C_{i} \) can be obtained from the partition function as

\[
W = -T \ln Z,
\]  

(2.12)

from which the entropy \( S \) is

\[
S = \beta(E - \mu_{i} C_{i}) + \ln Z.
\]  

(2.13)

In the Euclidean quantum theory of gravity, the partition function is [6]

\[
Z = \int D[\phi]D[g] \exp(-I_{E}[g, \phi]),
\]  

(2.14)

where \( \phi \) represents the matter fields and \( I_{E} \) is the Euclidean action. Naturally it is expected that the dominant contribution of the integration in (2.14) comes from the field configurations satisfying the classical Euclidean field equations. Under the zero-loop approximation, Gibbons and Hawking [6] have shown that the partition function is

\[
Z = \exp[-I_{E}(g_{0}, \phi_{0})],
\]  

(2.15)

where \( I_{E}(g_{0}, \phi_{0}) \) denotes the on-shell Euclidean action of the classical instanton. For asymptotically flat or anti-de Sitter nonextremal black holes, the manifold of instantons has only a boundary at the spatial infinity and is regular at the black hole horizon. For extremal black holes, Gibbons and Kallosh [14], and Hawking et al. [8] have argued that the manifold has an inner boundary corresponding to the black hole horizon, in addition to the outer boundary. In our case the manifold is regular both at the black hole horizon and the cosmological horizon. However, for the later use, we label the on-shell action \( I_{E}(g_{0}, \phi_{0}) \) as

\[
I_{E}(g_{0}, \phi_{0}) = I_{E}^{r_{\text{out}}},
\]  

(2.16)

where \( r_{\text{out}} \) and \( r_{\text{in}} \) represent the outer and inner boundaries, respectively. To get the entropy from (2.13), one must calculate the quantity \( E - \mu_{i} C_{i} \) by considering the following amplitude for imaginary time between two surfaces of Euclidean times \( \tau_{1} \) and \( \tau_{2} \) with given boundary conditions [6,15]:

\[
\langle \tau_{1} | \tau_{2} \rangle = e^{-(\tau_{2} - \tau_{1})(E - \mu_{i} C_{i})}.
\]  

(2.17)

From the above equation, one has
\[ \beta(E - \mu_i C_i) = I_E|_{R_{\text{out}}}^{R_{\text{in}}} \] (2.18)

for \( \tau_2 - \tau_1 = \beta \). Here \( R_{\text{in}} \) and \( R_{\text{out}} \) stand for the inner and outer physical boundaries of the spacetime. They are the black hole horizon and cosmological horizon for the lukewarm black hole case, respectively.

The two Euclidean actions can be expressed as

\[
I_E|_{R_{\text{out}}}^{R_{\text{in}}} = -\frac{1}{16\pi} \int_V d^4x \sqrt{g} (R + \mathcal{L}_{\text{matter}}) - \frac{1}{8\pi} \int_{R_{\text{in}}}^{R_{\text{out}}} d^3x \sqrt{h} (K - K_0), \] (2.19)

\[
I_E|_{r_{\text{out}}}^{r_{\text{in}}} = -\frac{1}{16\pi} \int_V d^4x \sqrt{g} (R + \mathcal{L}_{\text{matter}}) - \frac{1}{8\pi} \int_{r_{\text{in}}}^{r_{\text{out}}} d^3x \sqrt{h} (K - K_0), \] (2.20)

where \( \mathcal{L}_{\text{matter}} \) is the Lagrangian for matter fields. Substituting (2.19) and (2.20) into (2.13), one has

\[
S = I_E|_{R_{\text{out}}}^{R_{\text{in}}} - I_E|_{r_{\text{out}}}^{r_{\text{in}}}
= -\frac{1}{8\pi} \int_{R_{\text{in}}}^{R_{\text{out}}} d^3x \sqrt{h} (K - K_0) + \frac{1}{8\pi} \int_{r_{\text{in}}}^{r_{\text{out}}} d^3x \sqrt{h} (K - K_0). \] (2.21)

As we have mentioned above, the Euclidean manifold \( S^2 \times S^2 \) has no boundary, consequently the second term in (2.21) can be dropped out. The extrinsic curvature \( K \) in the metric (2.5) for a timelike surface fixed \( r \) (\( r_B < r < r_C \)) is

\[ K = -g^{-1/2}(Ng^{1/2})'. \] (2.22)

Now we choose \( K_0 \) so that the boundary become asymptotically imbeddable as one goes to larger and larger radii in an asymptotically de Sitter space:

\[ K_0 = -r^{-2}(r^2 \sqrt{1 - \Lambda r^2}/3)'. \] (2.23)

Substituting Eqs. (2.22) and (2.23) into (2.21), we obtain

\[
S = \pi r_C^2 + \pi r_B^2
\]

\[ = \frac{A_{\text{CH}}}{4} + \frac{A_{\text{BH}}}{4}. \] (2.24)

Thus we get indeed the entropy formula (1.5) in the lukewarm black hole model. Comparing (2.24) with (2.10), we can clearly see that the Euclidean action is no longer equal to the entropy of instanton. This is very different from that of asymptotically flat black holes, where they are always equal to each other \[\text{[6,15]}\]. In fact, the situation in which they are not equal already appears in the de Sitter space and the Nariai instanton \[\text{[1]}\].

**III. COLD BLACK HOLES AND ULTRACOLD SOLUTIONS**

In Ref. [13] Romans have classified in detail the static, spherically symmetric solutions of the Einstein-Maxwell equations with a cosmological constant. In addition to the lukewarm black hole discussed in the previous section, there exist two kinds of solutions of interest, cold black holes and ultracold solutions. In this section, we will discuss them separately.
A. Cold black holes

When the inner and outer horizons of the RN-de Sitter black hole coincide with each other, it is called a cold black hole, which corresponds to the extremal black hole in the asymptotically flat or anti-de Sitter spacetime. In this case, the temperature of black hole horizon vanishes, but the cosmological horizon has still a nonvanishing Hawking temperature. Since its Euclidean section has only a conical singularity at the cosmological horizon, we can remove it by identifying the imaginary time with the period

$$\beta = T_C^{-1},$$

(3.1)

where $T_C$ is the Hawking temperature of the cosmological horizon. Then the resulting manifold has an inner boundary at the black hole horizon ($r_B$). The Euclidean action becomes

$$I_E = -\pi r_C^2 - \beta Q^2 \left(\frac{1}{r_B} - \frac{1}{r_C}\right),$$

(3.2)

The second term in Eq. (2.10) now is canceled out by the surface term coming from the black hole horizon in Eq. (2.7). In the entropy formula (2.21), the inner boundary $r_\text{in}$ is the same as the boundary $R_\text{in}$, that is, they are both the black hole horizon. $R_\text{out}$ is the cosmological horizon and $r_\text{out}$ is absent. From (2.21) it follows directly that

$$S = \pi r_C^2.$$  

(3.3)

Obviously, the entropy has only the contribution from the cosmological horizon.

B. Ultracold solutions

When the inner and outer horizons and the cosmological horizon coincide, that is the equation $N^2(r) = 0$ has three same positive roots, this RN-de Sitter solution is called an ultracold one. Then the physical region is $0 \leq r \leq r_C$, where $r_C$ denotes the triple root. The situation is somewhat similar to the de Sitter space. But there are differences:

$$N^2(r)|_{r=r_C} = 0, \quad [N^2(r)]'|_{r=r_C} \neq 0$$

(3.4)

for the de Sitter space;

$$N^2(r)|_{r=r_C} = [N^2(r)]'|_{r=r_C} = [N^2(r)]''|_{r=r_C} = 0, \quad [N^2(r)]'''|_{r=r_C} < 0$$

(3.5)

for the ultracold solution. In addition, another important point is that $r = 0$ is a naked singularity for the ultracold solutions. Similar to the extremal black holes, the Euclidean section of ultracold solution is regular at $r_C$, we can identify the Euclidean time with any period. In this case the outer boundary $R_\text{out}$ is $r_C$. Similar to the extremal black holes in asymptotically flat or anti-de Sitter spacetimes, we must introduce another outer boundary $r_\text{out}$ at $r_C$ in the Euclidean manifold. Furthermore, to remove the naked singularity at $r = 0$, we introduce an inner boundary $R_\text{in} = r_\text{in} = \varepsilon$, where $\varepsilon$ is a small positive quantity. Thus, from Eq. (2.21) we have
\[ S = 0, \quad (3.6) \]

for the ultracold solutions. The Euclidean action, from (2.7) and (2.20), is

\[ I_E = -\beta Q^2 \left( \frac{1}{\varepsilon} - \frac{1}{r_C} \right), \quad (3.7) \]

where \( \beta \) is the period of the Euclidean time \( \tau \). From (3.7) we can see that the action diverges as \( \varepsilon \to 0 \). This reflects the fact that the electric potential energy is divergent for a point-like charge.

### IV. EULER NUMBERS AND ENTROPY OF INSTANTONS

In the previous sections we have obtained that the entropy of lukewarm black holes is one quarter of the sum of the areas of the black hole horizon and the cosmological horizon, the entropy of cold black hole is only one quarter of the area of cosmological horizon, and the entropy of the ultracold solutions vanishes. Evidently the formula (1.8) of Liberati and Pollifrone cannot apply to our results. In order to explain these results it is instructive to investigate the topological properties of manifolds. For the lukewarm black hole, its topological structure is \( S^2 \times S^2 \). So the Euler number of the manifold is \( \chi = 4 \). For our purposes, however, it is helpful to reexamine the Gauss-Bonnet integral.

In the four dimensional Riemannian manifold, the Gauss-Bonnet integral is

\[ S_{\text{volume \ GB}} = \frac{1}{32\pi^2} \int_V \varepsilon_{abcd} R^{ab} \wedge R^{cd}, \quad (4.1) \]

where \( R^{ab} \) is the curvature two-form defined as

\[ R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad (4.2) \]

and \( \omega^a_b \) is the spin connection one-form. For a closed Riemannian manifold without boundary, its Euler number \( \chi \) is given by the Gauss-Bonnet integral (4.1). For a manifold with boundary, the exact Euler number is obtained by adding a term integrated over the boundary to Eq. (4.1), that is

\[ \chi = S_{\text{volume \ GB}} + S_{\text{boundary \ GB}}, \quad (4.3) \]

where

\[ S_{\text{boundary \ GB}} = -\frac{1}{32\pi^2} \int_{\partial V} \varepsilon_{abcd}(2\theta^{ab} \wedge R^{cd} - \frac{4}{3}\theta^{ab} \wedge \theta^e \wedge \theta^{cd}), \quad (4.4) \]

where \( \theta^{ab} \) is the second fundamental form of the boundary \( \partial V \). In the static, spherically symmetric solution (2.5), the volume integral of Gauss-Bonnet action is

\[ S_{\text{volume \ GB}} = \frac{\beta}{2\pi} \int_{R_{\text{in}}}^{R_{\text{out}}} dr \frac{\partial}{\partial r} [(N^2)'(1 - N^2)] \]

\[ = \frac{\beta}{2\pi} \{[(N^2)'(1 - N^2)]_{r=R_{\text{in}}} - [(N^2)'(1 - N^2)]_{r=R_{\text{out}}}, \quad (4.5) \]
where $\beta$ is the period of the Euclidean time $\tau$, and the boundary term (4.4) is

$$S_{\text{boundary}}^{\text{GB}} = \frac{\beta}{2\pi} \left[ ((N^2)'(1 - N^2))|_{r_{\text{in}}}^{r_{\text{out}}} \right],$$

(4.6)

(i) For the lukewarm black holes, the topology of manifold is $S^2 \times S^2$, that is, it is a compact manifold without boundary. The Euclidean time has an intrinsic period

$$\beta = T_B^{-1} = T_C^{-1} = 4\pi[(N^2)'|_{r=r_B}]^{-1} = 4\pi[(N^2)'|_{r=r_C}]^{-1},$$

(4.7)

Thus the Euler number is only given by Eq. (4.5). Substituting (4.7) into (4.5) yields

$$\chi = 2 - (-2) = 2 + 2 = 4.$$  

(4.8)

Clearly, we can understand that the first 2 in (4.8) comes from the black hole horizon and the second from the cosmological horizon.

(ii) For the cold black holes, the Euclidean time has no intrinsic period at the black hole horizon because there is no conical singularity there, so the period $\beta$ can be arbitrary. But there is a conical singularity at the cosmological horizon, in order to remove the singularity, we must identify the $\tau$ with an intrinsic period

$$\beta = 4\pi[(N^2)'|_{r=r_C}]^{-1}.$$  

(4.9)

In this case, the resulting manifold has the topology $R^2 \times S^2$, we must consider the boundary term (4.6). The outer boundary $r_{\text{out}}$ is absent. However, we must introduce an inner boundary at $r_{\text{in}} = r_B + \epsilon$, as in the extremal black holes in the asymptotically flat space [10]. Combining (4.5) and (4.6), we have

$$\chi = 0 - (-2) = 2,$$

(4.10)

from which we can see clearly that the cosmological horizon has the contribution to the Euler number only.

(iii) For ultracold solutions, the Euclidean time has no intrinsic period. Thus $\beta$ can be an arbitrary finite value, but

$$N^2(r) = [N^2(r)]' = 0,$$

(4.11)

at $r = r_C$. In this case, the Euclidean manifold has the topological structure $S^1 \times R^1 \times S^2$, we must introduce not only the outer boundary $R_{\text{out}} = r_{\text{out}} = r_C$ but also the inner boundary $R_{\text{in}} = r_{\text{in}} = \epsilon$, because $N^2(r)$ and $[N^2(r)]'$ are both divergent at $r = 0$. From (4.5) and (4.6), we obtain

$$\chi = 0 + 0 = 0.$$  

(4.12)

What is the relation between the black hole entropy and the Euler number? The relation (1.8) of Liberati and Pollifrone does not apply to our case. In order to relate the entropy
of instantons to the Euler numbers, an interesting suggestion is to divide the Euler number into two parts:

$$\chi = \chi_1 + \chi_2.$$  \hspace{1cm} (4.13)

For example, we could think that $\chi_1$ is the contribution of the black hole horizon and $\chi_2$ comes from the cosmological horizon. From the calculations made above, this division seems reasonable. Thus we have the following relation

$$S = \frac{\chi_1}{8} A_{BH} + \frac{\chi_2}{8} A_{CH}.$$ \hspace{1cm} (4.14)

This formula contains some known special cases. For asymptotically flat or anti-de Sitter nonextremal black holes, the cosmological horizon is absent. Thus we have $\chi_1 = 2$ and $\chi_2 = 0$. The outer boundary contributes a zero result, the black hole entropy is $S = \chi_1 A_{BH}/8$. For the extremal black holes, $\chi_1$ and $\chi_2$ both vanish: $\chi_1 = \chi_2 = 0$, therefore $S = 0$. For the de Sitter space, the black hole horizon is absent, we have $\chi_1 = 0$ and $\chi_2 = 2$, and the entropy of de Sitter space is $S = \chi_2 A_{CH}/8$; For the Nariai instanton which can be regarded as the limiting case of the Schwarzschild-de Sitter black holes, $\chi_1 = \chi_2 = 2$, and the area of black hole horizon equals to the one of cosmological horizon. The entropy of the Nariai instanton obeys the relation (4.14). For lukewarm black holes, cold black holes and ultracold solutions, their entropy satisfies manifestly the formula (4.14).

**V. DISCUSSION**

In this paper we have calculated the action and entropy of lukewarm black holes, cold black holes and ultracold solutions in the Einstein-Maxwell theory with a cosmological constant. Our calculations have been performed in the formalism of the Euclidean quantum theory of gravitation under the zero-loop approximation. We have found that in the lukewarm black holes, the action and entropy are no longer equal to each other. The entropy of lukewarm black holes is the sum of entropies of black hole horizon and cosmological horizon, which provides an evidence in favor of the entropy formula (1.3) of de Sitter black holes. For the cold black holes, the gravitational entropy is contributed by only the cosmological horizon, and is one quarter of the area of the cosmological horizon. For the ultracold solutions, although the spacetime is somewhat similar to the de Sitter space, the entropy vanishes identically, as in the case of extremal black holes in the asymptotically flat or anti-de Sitter spacetimes. Further, we have investigated the topological properties of manifolds corresponding to the respective solutions and calculated their Euler numbers. In order to relate the entropy with the Euler number, we present an interesting relation (4.14), in which we divide the Euler number of manifolds into two parts: One comes from the black hole horizon and the other from the cosmological horizon. Of course, it should be stressed here that some results related to the cold black hole and ultracold solution strongly rely on the some arguments about extremal black holes proposed in Refs. [8–10]. Here it also should be noticed that in the semiclassical canonical quantum theory of gravity Broz and Kiefer [17] obtained that the extremal RN black holes have zero entropy.

Over the past two years, however, the statistical explanation of black hole entropy in the string theory has shown that the extremal black hole entropy still obeys the area formula [18].
How to understand the two seemingly contradictory conclusions obtained in the Euclidean path integral method and in the string theory, respectively? It may be helpful to consider the conditions under which the two different results are deduced. In the string theory, some extremal black holes correspond to the BPS saturated states, in which $M$ and $Q$ are already quantized quantities. That is, that the entropy of extremal black holes still obeys the area formula in the string theory is obtained by taking the extremality condition $M = |Q|$ after quantization. In the Euclidean path integral method, the conclusion of extremal black holes having zero entropy is derived under first fixing the topology of Euclidean manifold corresponding to the extremal black holes and then quantizing the theory [8–10]. Just as argued recently by Ghosh and Mitra [19], if one does not first fix the topology and quantizes the theory under the certain boundary conditions in the path integral formalism, and then takes the extremality condition for the extremal black holes, the resulting entropy of extremal black holes satisfies the area formula. The entropy in fact comes from the topology of non-extremal black holes. Ref. [19] clearly demonstrates how the topology of non-extremal black holes enters into the partition function of extremal black holes. Note that the result of zero entropy for extremal black holes in the semiclassical canonical quantum gravity is also derived under first fixing the topological structure (extremality condition) of extremal black holes before the quantization [17]. The results in string theory show that the corrections of strings may affect drastically the geometry near the horizon of black holes. Therefore, the two conclusions are not in contradiction with each other in the sense that the entropy vanishes for the fixed topology, extremal black holes. Thus, the entropy of extremal black holes seems to become relevant to one’s understanding of the extremal black holes. Obviously, further understanding for extremal black holes is needed.

As for the general black holes in de Sitter space, an important problem is to develop a satisfactory method to derive the gravitational entropy. As for this point, the off-shell approach seems a promising direction. Although the relation (1.8) or (4.14) can explain some known results, to what extent does the relation remain valid? These issues are currently under investigation. Finally, we would like to point out that some of our conclusions are also valid for lukewarm black holes in the Kerr-de Sitter solutions [20].

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