FINITE AND INFINITE MALLOWS RANKING MODELS, MAXIMUM LIKELIHOOD ESTIMATOR, AND REGENERATION

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Abstract. In this paper we are concerned with various Mallows ranking models. First we study the statistical properties of the MLE of Mallows’ φ model: \( P_{\theta, \pi_0}(\pi) \propto \exp(-\theta \text{inv}(\pi \circ \pi_0^{-1})) \), where \( \theta \) is the dispersion parameter and \( \pi_0 \) is the central ranking. We prove that (1). the MLE \( \hat{\theta} \) is biased upwards for both Mallows’ φ model and the single parameter IGM model; (2). the MLE \( \hat{\pi}_0 \) converges exponentially for Mallows’ φ model. We also make connections of various Mallows ranking models, encompassing the work of Gnedin and Olshanski \[20\] and Pitman and Tang \[41\]. Motivated by the infinite top-\( t \) ranking model of Meilă and Bao \[36\], we propose an algorithm to select the model size \( t \) automatically. The key idea relies on the regenerative property of such an infinite permutation. Finally, we apply our algorithm to several data sets including the APA election data and the University homepage search data.

Key words: Bias, convergence rate, Infinite Generalized Mallows model, large deviations, Mallows model, maximum likelihood estimator, random permutations, ranked data, regeneration.

1. Introduction and main results

Ranked or permutation data appear in many problems of social choice, user recommendation and information retrieval. Examples include ranking candidates by a large number of voters in political elections (e.g. instant-runoff voting), the preference list of competing items collected from consumers in market research, and the document retrieval problem where one aims to design a meta-search engine according to a ranked list of webpages output by various search algorithms. In the sequel, we use the words ranking and permutation interchangeably. A ranking model is given by a collection of items, and an unknown total ordering of these items.

There is a rich body of work on probabilistic ranking models. The earliest work in this direction is due to Thurstone \[46, 47\] where he ranked items through the order statistics of a Gaussian random vector. Later Bradley and Terry \[4\] introduced an exponential family model by pairwise comparisons, which was extended by Luce and Plackett \[31, 42\] with comparisons of multiple items. See also Hunter \[25\], Cattelan \[6\], Chen and Suh \[7\], Hung and Fithian \[24\], Shah and Wainwright \[44\] and references therein for algorithms and statistical analysis of the Bradley-Terry model and its variants. A more tractable subclass of the Bradley-Terry model was proposed by Mallows \[32\] as follows. For \( n \geq 1 \), let \( S_n \) be the set of permutations of \( [n] := \{1, \ldots, n\} \). The parametric model

\[
P_{\theta, \pi_0, d}(\pi) = \frac{1}{\Psi(\theta, d)} e^{-\theta d(\pi, \pi_0)} \quad \text{for } \pi \in S_n,
\]

\[\text{Date: August 28, 2018.}\]
is referred to as the Mallows model. Here $\theta > 0$ is the dispersion parameter, $\pi_0$ is the central ranking, $d(\cdot, \cdot) : \mathcal{S}_n \times \mathcal{S}_n \to \mathbb{R}_+$ is a discrepancy function which is right invariant:

$$d(\pi, \sigma) = d(\pi \circ \sigma^{-1}, id) \quad \text{for } \pi, \sigma \in \mathcal{S}_n,$$

and $\Psi(\theta, d) := \sum_{\pi \in \mathcal{S}_n} e^{-\theta d(\pi, \pi_0)}$ is the normalizing constant. Note that $\mathbb{P}_{0, \pi_0, d}$ is the uniform distribution on $\mathcal{S}_n$. Mallows primarily considered two special cases of (1.1). That is,

- Mallows’ $\theta$ model, where $d(\pi, \sigma) = \sum_{i=1}^n (\pi(i) - \sigma(i))^2$ is the Spearman’s rho,
- Mallows’ $\phi$ model, where $d(\pi, \sigma)$ is the minimum number of adjacent transpositions to bring $\sigma$ to $\pi$, called the Kendall’s tau.

The general form (1.1) was suggested by Diaconis [14, Chapter 6] along with other discrepancy functions as the Hamming distance, the Cayley distance and so on. Diaconis [14, 15] and Critchlow [11] also pioneered the group representation approach to ranked, discrepancy functions such as the Hamming distance, the Cayley distance and so on. Diaconis [14, 15] and Critchlow [11] also pioneered the group representation approach to ranked, and partially ranked data.

In this paper we are primarily concerned with the statistical properties of the maximum likelihood estimator (MLE) of the Mallows’ $\phi$ model as well as its infinite counterpart. Specializing (1.1) with the Kendall’s tau, the Mallows’ $\phi$ model is expressed as

$$\mathbb{P}_{\theta, \pi_0}(\pi) = \frac{1}{\Psi(\theta)} e^{-\theta \operatorname{inv}(\pi \pi_0^{-1})} \quad \text{for } \pi \in \mathcal{S}_n, \quad (1.2)$$

where $\operatorname{inv}(\pi) := \#\{(i, j) \in [n]^2 : i < j \text{ and } \pi(i) > \pi(j)\}$ is the number of inversions of $\pi$. In Mallows’ paper [32], the model was written as $\mathbb{P}_{\phi, \pi_0}(\pi) = \Psi(-\ln \phi)^{-1} e^{\phi \operatorname{inv}(\pi \pi_0^{-1})}$ with $\phi = e^{-\theta}$. It is easily seen that $\mathbb{P}_{\theta, \pi_0}$ has a unique mode $\pi_0$ if $\theta > 0$. The Mallows’ $\phi$ model (1.2) is of particular interest, since it is an instance of two large classes of ranking models: distance-based ranking models [17] and multistage ranking models [18], both introduced by Fligner and Verducci. In addition, the one parameter model (1.2) has an $n-1$ parameter extension where $\theta$ is replaced with $\tilde{\theta} := (\theta_1, \ldots, \theta_{n-1})$ by factorizing the inversions. This $n-1$ parameter model, called the Generalized Mallows (GM$\tilde{\theta}$,$\pi_0$) model, will be discussed in Section 2. See Critchlow, Fligner and Verducci [12], and Marden [34] for a review of these ranking models.

Fligner and Verducci [18] showed that if the central ranking $\pi_0$ is known, the MLE of $\theta$ (or $\tilde{\theta}$) can be easily computed by convex optimization. But it seems to be harder to find the MLE of $\pi_0$, and only a few heuristic algorithms are available. As pointed out by Meilä, Phadnis, Patterson and Bilmes [38], the problem of finding the MLE of $\pi_0$ for the Mallows’ $\phi$ model is the Kemeny’s consensus ranking problem which is known to be NP-hard [3]. They also gave a branch and bound (B&B) search algorithm to estimate simultaneously $\theta$ (or $\tilde{\theta}$) and $\pi_0$. See Cohen, Schapire and Singer [9], Ailon, Charikar and Newman [11], and Mandhani and Meilä [33] for approximation algorithms for consensus ranking problem, and Lebanon and Lafferty [27], Lu and Boutilier [29, 30], and Vitelli, Sørensen, Crispino, Frigessi and Arjas [48] for the approximate Bayesian inference of Mallows’ mixture models for clustering heterogeneous ranked data. Though there have been efforts in developing algorithms to estimate $\theta$ (or $\tilde{\theta}$) and $\pi_0$, not much is known about the statistical properties of the MLE $\hat{\theta}$ and $\hat{\pi}_0$ even for the simplest model (1.2).
Here we provide statistical analysis to the MLE for the model (1.2). One may ask the following questions:

- Are the MLEs $\hat{\theta}$, $\hat{\pi}_0$ consistent?
- Is the MLE $\hat{\theta}$ unbiased?
- How fast do MLEs $\hat{\theta}$, $\hat{\pi}_0$ converge to $\theta$, $\pi_0$?

Mukherjee [39] proved that the MLE $\hat{\theta}$ for the general Mallows model (1.1) is consistent if $\pi_0$ is known. His approach relies on the concept of permutons [21]. The following result shows that the MLE $\hat{\theta}$ is always biased upwards for the Mallows $\phi$ model.

**Theorem 1.1** (Bias of $\hat{\theta}$). Let $P_{\theta, \pi_0}$ be defined by (1.2), and $\hat{\theta}$ be the MLE of $\theta$ with $N$ samples. Then for each $N \geq 1$,

$$E_{\theta, \pi_0} \hat{\theta} > \theta. \quad (1.3)$$

Moreover, the error $\hat{\theta} - \theta$ is of order $1/\sqrt{N}$ with high probability.

The analysis of the MLE $\hat{\pi}_0$ is more subtle since it lives in a discrete space. By general results of Newey and McFadden [40], and Choirat and Seri [8], one can prove the consistency of $\hat{\pi}_0$. Here we take a further step to establish a concentration bound of $\hat{\pi}_0$ at $\pi_0$ from which the consistency is straightforward. This concentration bound also gives a confidence interval for the central ranking in the Mallows’ $\phi$ model.

**Theorem 1.2** (Convergence rate of $\hat{\pi}_0$). Let $P_{\theta, \pi_0}$ be defined by (1.2), and $\hat{\pi}_0$ be the MLE of $\pi_0$ with $N$ samples. Then for $N$ large enough,

$$\frac{1}{1 - e^{-\theta}} \sqrt{\frac{2}{\pi N}} \left( \cosh \frac{\theta}{2} \right)^{-N} \leq P_{\theta, \pi_0}(\hat{\pi}_0 \neq \pi_0) \leq \frac{n(n-1)}{2} \left( \cosh \frac{\theta}{2} \right)^{-N}. \quad (1.4)$$

The proofs of Theorems 1.1 and 1.2 will be given in Section 3. To the best of our knowledge, Theorems 1.1 and 1.2 seem to be new. The same questions can also be asked for the generalized Mallows model, but the analysis is more complicated. We leave these problems for future work.

When the number of items $n$ is large, learning a complete ranking model becomes impracticable. A line of work by Filgner and Verducci [17], Busse, Orbanz and Buhmann [5], Meila and Bao [36], and Meila and Chen [37] focused on the top-orderings for the GM$_{\theta, \pi_0}$ model. Among these work, Meila and Bao [36] proposed a probability model over the top-orderings of infinite permutations, called the Infinite Generalized Mallows (IGM$_{\theta, \pi_0}$) model. More precisely, the IGM$_{\theta, \pi_0}$ model has the form

$$P_{\theta, \pi_0}(\pi) = \frac{1}{\Psi(\tilde{\theta})} \cdot \exp \left( -\sum_{j=1}^{t} \theta_j s_j(\pi \circ \pi_0^{-1}) \right), \quad (1.5)$$

where $s(\pi) := (s_1(\pi), s_2(\pi), \ldots)$ is the inversion table of $\pi$ defined by

$$s_j(\pi) := \pi^{-1}(j) - 1 - \sum_{j' < j} 1_{\{\pi^{-1}(j') < \pi^{-1}(j)\}}, \quad (1.6)$$

and $\Psi(\tilde{\theta}) = \prod_{j=1}^{t} \frac{1}{1 - e^{-\theta_j}}$ is the normalizing constant. The integer ‘$t$’ is referred to as the model size of the IGM model. The model (1.5) is useful to tackle the problem of ranking...
a large number of items (e.g., retrieving information of a target from webpages output by various search engines). If $\theta_1 = \cdots = \theta_t = \theta$, the IGM model is called the single parameter IGM model. Similar to Theorem 1.1, the MLE of $\theta$ is biased upwards for the single parameter IGM model, confirming an observation in [36].

**Theorem 1.3** (Bias of $\hat{\theta}$). Let $P_{\theta, \pi_0}$ be the distribution of the single parameter IGM model, and $\hat{\theta}$ be the MLE of $\theta$ with $N$ samples. Then for each $N \geq 1$,

$$E_{\theta, \pi_0} \hat{\theta} > \theta. \tag{1.7}$$

However, the convergence rate of $\hat{\pi}_0$ for the single parameter IGM model seems to be difficult. We leave the analog of Theorem 1.2 open to interested readers.

As explained in [36], the IGM model (1.5) is the marginal distribution of a random permutation of positive integers. In case of the single parameter IGM, this random infinite permutation, called the infinite $q$-shuffle, was studied in the probability context by Gnedin and Olshanski [20] with parameterization $q = e^{-\theta}$. They provided a nice construction of the infinite $q$-shuffle, which is reminiscent of absorption sampling [26, 43] and the repeated insertion model [16]. The infinite $q$-shuffle was further used by Gladkich and Peled [19] to establish the asymptotic results for the Mallows’ $\phi$ model (1.2), and was extended by Pitman and Tang [41] to the $p$-shifted permutations of positive integers. But the link between Meilà-Bao’s infinite ranking model and infinite $q$-shuffle or $p$-shifted permutations does not seem to have been previously noticed. So we point out this connection which is detailed in Section 2.

The aforementioned top-$t$ ranking models all require choosing ‘$t$’ manually. It was observed in [41] that the random infinite limit of the single parameter IGM model has a remarkable renewal/regenerative property. This suggests a heuristic procedure to select the model size ‘$t$’ automatically based on Meilà-Bao’s search algorithms. We will discuss such an approach in Section 2. See also Lebanon and Mao [28] for a nonparametric ranking model, and Huang and Guestrin [22, 23] and Meek and Meilà [35] for hierarchical ranking models.

**Outline:** The rest of the paper is organized as follows.

- In Section 2 we provide background and make connections of various ranking models. We also propose an algorithm to select the model size ‘$t$’ for the IGM models.
- In Section 3 we study the statistical properties of the MLE $(\hat{\theta}, \hat{\pi}_0)$ for Mallows models. There Theorems 1.1-1.3 are proved.
- In Section 4 we provide experimental results on both synthetic and real-world data.

2. **Finite and Infinite Mallows Models: ‘$t$’ Selection Algorithm**

In this section we provide background on various Mallows ranking models, encompassing the closely related Gnedin-Olshanski’s $q$-shuffles and Pitman-Tang’s $p$-shifted permutations. We also give an algorithm to select the model size ‘$t$’ for the IGM model (1.5). We follow closely Fligner and Verducci [18], Meilà and Bao [36], and Pitman and Tang [41].
**Finite Mallows models.** Given \( n \) items labelled by \([n]\), a ranking \( \pi \in \mathfrak{S}_n \) is represented by

- the word list \((\pi(1), \pi(2), \ldots, \pi(n))\),
- the ranked list \((\pi^{-1}(1)|\pi^{-1}(2)|\ldots|\pi^{-1}(n))\).

Here \( \pi(i) = j \) means that the item \( i \) has rank \( j \), and conversely \( \pi^{-1}(j) = i \) means that the \( j^{th} \) most preferred is item \( i \). The idea of multistage ranking is to decompose the ranking procedure into independent stages. The most preferred item is selected at the first stage, the best of the remaining at the second stage and so on until the least preferred item is selected. The exactness of the choice at any stage is accessed through a central ranking \( \pi_0 \). For example, if \( \pi_0 = (3|1|2) \), then the ranking \( \pi = (3|2|1) \) gives a correct choice at the first stage, since item 3 is the most preferred in both \( \pi \) and \( \pi_0 \). But at the second stage, among the two remaining items 1 and 2, item 2 is selected by \( \pi \) while the right choice is item 1 according to \( \pi_0 \).

For any ranking \( \pi \in \mathfrak{S}_n \) and \( j \in [n-1] \), let \((s_1(\pi), \ldots, s_{n-1}(\pi))\) be the inversion table of \( \pi \) defined by (1.6). It is easy to see that \( s_j(\pi) \in \{0, \ldots, n-j\} \), and there is a bijection between a ranking \( \pi \) and the inversion table \((s_1(\pi), \ldots, s_{n-1}(\pi))\). The quantity \( s_j(\pi \circ \pi_0^{-1}) \) measures the correctness of the choice at stage \( j \): \( s_j(\pi \circ \pi_0^{-1}) = k \) means that at stage \( j \) the \((k+1)^{th}\) best of the remaining items is selected. In the example with \( \pi_0 = (2,3,1) \) and \( \pi = (3,2,1) \), \( s_1(\pi \circ \pi_0^{-1}) = 0 \) and \( s_1(\pi \circ \pi_0^{-1}) = 1 \).

Fligner and Verducci [17] introduced the multistage ranking models of the form:

\[
\mathbb{P}_{p,\pi_0}(\pi) = \prod_{j=1}^{n-1} p_j \left( s_j(\pi \circ \pi_0^{-1}) \right),
\]  

(2.1)

where \( p_j(\cdot) \) is a probability distribution on \([0, \ldots, n-j]\) at stage \( j \). The choice of \( p_j(k) = (1 - e^{-\theta})e^{-k\theta}/(1 - e^{-(n-j+1)\theta}) \) for \( k \in \{0, \ldots, n-j\} \) in (2.1), and the remarkable identity \( \sum_{j=1}^{n-1} s_j(\pi) = \text{inv}(\pi) \) for any \( \pi \in \mathfrak{S}_n \) yield the Mallows’ \( \phi \) model [12].

This one parameter model has a natural \( n-1 \) parameter extension by simply taking

\[
p_j(k) = \frac{1 - e^{-\theta_j}}{1 - e^{-(n-j+1)\theta_j}} e^{-\theta_j k} \quad \text{for } k \in \{0, \ldots, n-j\}.
\]

The \( n-1 \) parameter model, called the Generalized Mallows \((\text{GM}_{\dot{\theta},\pi_0})\) model, is then defined by

\[
\mathbb{P}_{\dot{\theta},\pi_0}(\pi) = \frac{1}{\Psi(\dot{\theta})} \exp \left( - \sum_{j=1}^{n-1} \theta_j s_j(\pi \circ \pi_0^{-1}) \right) \quad \text{for } \pi \in \mathfrak{S}_n,
\]  

(2.2)

where \( \Psi(\dot{\theta}) = \prod_{j=1}^{n-1} \frac{1 - e^{-(n-j+1)\theta_j}}{1 - e^{-\theta_j}} \) is the normalizing constant. The \( \text{GM}_{\dot{\theta},\pi_0} \) model is also called the Mallows’ \( \phi \)-component model. Note that the model (2.2) can be written in the form

\[
\mathbb{P}_{\dot{\theta},\pi_0}(\pi) = \frac{1}{\Psi(\dot{\theta})} e^{-d_{\dot{\theta}}(\pi,\pi_0)} \quad \text{with } d_{\dot{\theta}}(\pi,\pi_0) := \sum_{j=1}^{n-1} \theta_j s_j(\pi \circ \pi_0^{-1}).
\]

Contrary to the Mallows’ \( \phi \) model, \( d_{\dot{\theta}} \) is not a distance since it does not satisfy the triangle inequality.
Infinite Mallows models. Given a countably infinite items labelled by \( \mathbb{N}_+ := \{1, 2, \ldots\} \), a ranking \( \pi \) over \( \mathbb{N}_+ \) is a bijection from \( \mathbb{N}_+ \) onto itself represented by the word list \( (\pi(1), \pi(2), \ldots) \) or the ranked list \( (\pi^{-1}(1)|\pi^{-1}(2)|\ldots) \). A top-\( t \) ordering of \( \pi \) is the prefix \( (\pi^{-1}(1)|\ldots|\pi^{-1}(t)) \). Motivated by the GM\( \theta,\pi_0 \) model \( (2.2) \), Meilă and Bao \( 36 \) proposed the Infinite Generalized Mallow (IGM\( \theta,\pi_0 \)) model \( (1.5) \), which can also be put in the form

\[
\mathbb{P}_{\theta,\pi_0}(\pi) = \frac{1}{\Psi(\theta)} e^{-d_\theta(\pi,\pi_0)} \quad \text{with} \quad d_\theta(\pi,\pi_0) := \sum_{j=1}^{t} \theta_j s_j(\pi \circ \pi_0^{-1}).
\]

In particular, \( s_j \) is distributed as Geo\( (1 - e^{-\theta_j}) \) on \( \{0, 1, \ldots\} \). As explained in \( 36 \), one can regard \( \pi \) as a top-\( t \) ordering, and \( \pi_0 \) as an ordering over \( \mathbb{N}_+ \). If \( \theta_1 = \cdots = \theta_t \), the model \( (1.5) \) simplifies to

\[
\mathbb{P}_{\theta,\pi_0}(\pi) = \frac{1}{\Psi(\theta)} \exp \left( -\theta \sum_{j=1}^{t} s_j(\pi \circ \pi_0^{-1}) \right), \tag{2.3}
\]

called the single parameter IGM model.

It is easy to see that the single parameter IGM model \( (2.3) \) is the marginal distribution of a random permutation of positive integers. Formally, this random infinite permutation is distributed as

\[
\mathbb{P}_{\theta,\pi_0}(\pi) = \frac{1}{\Psi(\theta)} \exp \left( -\theta \sum_{j=1}^{\infty} s_j(\pi \circ \pi_0^{-1}) \right), \tag{2.4}
\]

In the terminology of Gnedin and Olshanski \( 20 \), for \( \pi \) defined by \( (2.4) \), \( \pi \circ \pi_0^{-1} \) is the infinite \( e^{-\theta} \)-shuffle. The latter was generalized by Pitman and Tang \( 41 \) to \( p \)-shifted permutations, with \( p = (p_1, p_2, \ldots) \) a discrete distribution on \( \mathbb{N}_+ \). Here we present a further extension of Pitman-Tang’s \( p \)-shifted permutations.

**Definition 2.1.** Let \( P = (p_{ij})_{i,j \in \mathbb{N}_+} \) be a stochastic matrix on \( \mathbb{N}_+ \) with \( p^i = (p_{ij})_{j \in \mathbb{N}_+} \) being the \( i \)th row of \( P \). Assume that

\[
\lim_{n \to \infty} \prod_{i=1}^{n} (1 - p_{i1}) = 0. \tag{2.5}
\]

Call a random permutation \( \Pi \) of \( \mathbb{N}_+ \) a \( P \)-shifted permutation of \( \mathbb{N}_+ \) if \( \Pi \) has the distribution defined by the following construction from the independent sample \( (X_i)_{i \geq 1} \), with \( X_i \) distributed as \( p^i \). Inductively, let

- \( \Pi_1 := X_1 \),
- for \( i \geq 2 \), let \( \Pi_i := \psi(X_i) \) where \( \psi \) is the increasing bijection from \( \mathbb{N}_+ \) to \( \mathbb{N}_+ \setminus \{\pi_1, \pi_2, \cdots, \pi_{i-1}\} \).

For example, if \( X_1 = 2, X_2 = 1, X_3 = 2, X_4 = 3, X_5 = 4, X_6 = 1, \ldots \), then the associated permutation is \( (2, 1, 4, 6, 8, 3, \ldots) \).

The condition \( (2.5) \) guarantees that \( \Pi \) such constructed is almost surely a permutation of \( \mathbb{N}_+ \). Now the aforementioned infinite ranking models are subcases of the \( P \)-shifted permutations.

- If \( p^i = p \) with \( p_1 > 0 \) for all \( i \), then we get Pitman-Tang’s \( p \)-shifted permutation.
• If \( p^i = \text{Geo}(1 - e^{-\theta}) \) on \( \mathbb{N}_+ \) for all \( i \), then we get Gneden-Olshanski’s infinite \( e^{-\theta}-\text{shuffle} \), i.e. \( \Pi \overset{(d)}{=} \pi \circ \pi_0^{-1} \) for \( \pi \) distributed according to (2.4).

• If \( p^i = \text{Geo}(1 - e^{-\theta_i}) \) on \( \mathbb{N}_+ \) for each \( i \), then \( \Pi \overset{(d)}{=} \pi \circ \pi_0^{-1} \) for \( \pi \) an infinite version of the IGM model (1.5).

The following diagram provides a roadmap of the relations between various Mallows ranking models.

Finite ranking models

Mallows’ \( \phi \) (57') \( \subset \) Generalized Mallows (88')

\[ \xrightarrow[45]{} \]

Single parameter IGM (10') \( \subset \) IGM (10') \( \subset \) p-shifted permutations (this paper)

Infinite ranking models

\( \Pi \)

\( \overset{\text{Infinite } q\text{-shuffle (09')}}{\subset} \)

\( \overset{\text{p-shifted permutation (17')}}{\subset} \)

Figure 1. Diagram of Mallows’ type ranking models.

Now we present an algorithm to select the model size ‘\( t \)’ automatically for the top-\( t \) IGM models. The heuristic comes from the renewal/regenerative property of the single parameter IGM model (2.3). We need the following vocabulary.

Let \( \Pi \) be a permutation of \( \mathbb{N}_+ \). Call \( n \in \mathbb{N}_+ \) a splitting time of \( \Pi \) if \( \Pi \) maps \([1, n]\) onto itself, or equivalently, \( \Pi \) maps \([n + 1, \infty)\) onto itself. The set of splitting times of \( \Pi \) is the collection of finite right endpoints of some finite or infinite family of components of \( \Pi \), say \( \{I_j\} \). So \( \Pi \) acts on each of its components \( I_j \) as an indecomposable permutation of \( I_j \), meaning that \( \Pi \) does not act as a permutation on any proper subinterval of \( I_j \). These components \( I_j \) form a partition of \( \mathbb{N}_+ \), which is coarser than the partition by cycles of \( \Pi \). For example, the permutation \( \pi = (1)(2, 4)(3) \in S_4 \) induces the partition by components \([1][2, 3, 4]\).

The idea is to use the single parameter IGM model to preselect ‘\( t \)’, and then train a top-\( t \) ranking model. Pitman and Tang [41] proved that for \( p = (p_1, p_2, \ldots) \) a discrete distribution with \( p_1 > 0 \) and \( \sum_{i \geq 1} i p_i < \infty \), a p-shifted permutation \( \Pi \) is a concatenation of independent and identically distributed (i.i.d.) components. That is, \( \Pi \) is characterized by \( L \) of a distribution on \( \mathbb{N}_+ \), and \( (Q_n)_{n \geq 1} \) a sequence of distributions on indecomposable permutations such that

- the lengths of components \( (L_i)_{i \geq 1} \) are i.i.d. as \( L \),
- given the length of a component, say \( L_i = n_i \), the reduced component defined via conjugation of \( \Pi \) by the shift from the component to \([n_i]\) is distributed as \( Q_{n_i} \).
To illustrate,

\[ L_1 = 4, \quad L_2 = 6, \quad L_3 = 3 \]

Moreover, the probability generating function (PGF) of \( L \) is given by

\[
F(z) = 1 - \frac{1}{1 + \sum_{n=1}^{\infty} u_n z^n} \quad \text{with} \quad u_n := \prod_{i=1}^{n} \sum_{j=1}^{i} p_i.
\] (2.6)

Specializing (2.6) to the single parameter IGM model gives the following result. The proof of is deferred to Section 3.

**Proposition 2.2.** Let \( \Pi \) be a random permutation of \( \mathbb{N}_+ \) distributed as \( \mathbb{P}_{\theta, \text{id}} \) defined by (2.4). Let \( L \) be the common distribution of lengths of components of \( \Pi \). Then

\[
\mathbb{E}L = \frac{1}{(e^{-\theta}; e^{-\theta})_{\infty}},
\] (2.7)

\[
\text{Var} L = \frac{1}{(e^{-\theta}; e^{-\theta})_{\infty}} \left( 2 \sum_{k \geq 1} \frac{e^{-k\theta}}{(e^{-\theta}; e^{-\theta})_k (1 - e^{-k\theta})} + 1 - \frac{1}{(e^{-\theta}; e^{-\theta})_\infty} \right),
\] (2.8)

where \((a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k)\) is the Q-Pochhammer function.

![Figure 2](image-url)  
**Figure 2.** Plot of \( \theta \to 1/(e^{-\theta}; e^{-\theta})_{\infty} \) for \( \theta \in [0, 5] \).

Pitman-Tang’s theory indicates that for the single parameter model (2.4), the first component of \( \pi \circ \pi_0^{-1} \) has length \( L \) whose expectation is \( 1/(e^{-\theta}; e^{-\theta})_{\infty} \). Given the dispersion parameter \( \theta \), a complete permutation is expected to occur at some place close to \( 1/(e^{-\theta}; e^{-\theta})_{\infty} \). The latter can be regarded as the effective length of the random infinite permutation, which suggests a candidate for ‘t’ in top-\( t \) ranking models. Since \( \theta \) is unknown, we would like to find the ‘t’ closest to the effective length. More precisely, given \( t \) in a suitable range \( T \), we fit the single parameter IGM model (2.3) to get the MLE \( \hat{\theta}(t) \) by the algorithms in Meilă and Bao [36]. Those algorithms also work for partially ranked data. Then we search for

\[
t := \min \left\{ \arg\min_{t \in T} \left| t - \frac{1}{(e^{-\hat{\theta}(t)}; e^{-\hat{\theta}(t)})_{\infty}} \right|, \frac{N}{4} \right\},
\] (2.9)
where the cutoff $N/4$ is used to avoid overfitting. Here we can take $T = [t_{\text{min}}, t_{\text{max}}]$, with $t_{\text{min}}$ (resp. $t_{\text{max}}$) the minimum (resp. maximum) length of permutations in the data. Practically, one needs to search $t'$ with small values to narrow down the choices for the effective length. With $t'$ selected according to (2.9), we can then fit the IGM model (1.5).

**Algorithm 1** ‘$t$’ selection algorithm

```plaintext
procedure t_SEL($T$)
    Err ← $\infty$, $t_{\text{SEL}}$ ← 0  # Initialization
    for $t$ in $T$ do
        $\theta$ ← MB($t$)  # Run Meilă-Bao’s algorithm
        if $|t - 1/(e^{-\theta}; e^{-\theta})_\infty| < Err$ then
            Err ← $|t - 1/(e^{-\theta}; e^{-\theta})_\infty|$
            $t_{\text{SEL}}$ ← $t$
        end if
    end for
    return min($t_{\text{SEL}}, N/4$)  # The selected ‘$t$’ is $t_{\text{SEL}}$
end procedure
```

3. Statistical properties of the MLE

In this section we prove Theorems 1.1-1.3 and Proposition 2.2.

3.1. Bias of the MLE $\hat{\theta}$. We start by showing that the MLE $\hat{\theta}$ of the dispersion parameter $\theta$ is biased upwards for the Mallows’ $\phi$ model (1.2), and for the single parameter IGM model (2.3).

**Proof of Theorem 1.4** We first consider the case where the central ranking $\pi_0$ is known. Assume w.l.o.g. that $\pi_0 = \text{id}$ by suitably relabelling the items. Then the model (1.2) simplifies to

$$P_{\theta, \text{id}}(\pi) = \exp(-\theta \text{inv}(\pi) - \ln \Psi(\theta)), \tag{3.1}$$

where $\Psi(\theta) := \sum_{\pi \in \mathcal{S}_n}\exp(-\theta \text{inv}(\pi))$ is the normalizing constant. Given $N$ samples $(\pi_i)_{1 \leq i \leq N}$, the MLE $\hat{\theta}$ is the solution to the following equation:

$$-\frac{\Psi'(\theta)}{\Psi(\theta)} = \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i). \tag{3.2}$$

Now by writing $\Psi(\theta) = f(e^{-\theta})$ with $f(q) := \sum_{\pi \in \mathcal{S}_n}q^{\text{inv}(\pi)}$, we have

$$-\frac{\Psi'(\theta)}{\Psi(\theta)} = g(e^{-\theta}), \text{ with } g(q) := \frac{qf'(q)}{f(q)}.$$

As a consequence, $\hat{\theta} = -\log g^{-1}(\frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i))$. The function $f(q)$ is known as the $q$-factorial [45, Proposition 1.10.12]:

$$f(q) = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).$$
Thus,
\[
g(q) = q \sum_{k=1}^{n-1} \frac{1 - (k + 1)q^k + kq^{k+1}}{(1-q)(1-q^{k+1})}. \tag{3.3}
\]
As observed by Mallows [32], and Gnedin and Olshanski [20, Proposition 3.2], for \( \pi \) distributed according to (3.1), the number of inversions has the same distribution as a sum of independent truncated geometric random variables. That is,
\[
\text{inv}(\pi) \sim G_{e^{-\theta},1} + \cdots + G_{e^{-\theta},n} \quad \text{for} \quad \pi \sim P_{\theta,id},
\]
where \( P(q,k) = q^i(1-q)/(1-q^k) \) for \( i \in \{0,1,\ldots,k-1\} \). Thus, \( \mathbb{E}\text{inv}(\pi) = g(e^{-\theta}) \).

By elementary analysis, \( \theta \mapsto g(e^{-\theta}) \) is strictly convex and decreasing. So its inverse function \( q \mapsto -\log g^{-1}(q) \) is strictly convex. By Jensen’s inequality,
\[
-\mathbb{E}\log g^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i) \right) > -\log g^{-1} \left( \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i) \right) \right),
\]
which implies that \( \mathbb{E}\hat{\theta} > \theta \).

Now consider the case where the central ranking \( \pi_0 \) is unknown. The MLE \( \hat{\theta} \) is given by
\[
\hat{\theta} = -\log g^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i \circ \hat{\pi}_0^{-1}) \right), \tag{3.4}
\]
where \( g \) is defined as in (3.3), and \( \hat{\pi}_0 \) is the MLE of \( \pi_0 \). By the definition of \( \hat{\pi}_0 \),
\[
\sum_{i=1}^{N} \text{inv}(\pi_i \circ \hat{\pi}_0^{-1}) \leq \sum_{i=1}^{N} \text{inv}(\pi_i \circ \pi_0^{-1}).
\]
Since \( q \mapsto -\log g^{-1}(q) \) is strictly convex and decreasing, we get
\[
\mathbb{E}\hat{\theta} = -\mathbb{E}\log g^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i \circ \hat{\pi}_0^{-1}) \right)
\geq -\mathbb{E}\log g^{-1} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i \circ \pi_0^{-1}) \right)
> -\log g^{-1} \left( \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \text{inv}(\pi_i \circ \pi_0^{-1}) \right) \right) = \theta,
\]
where the last equality follows from the fact that \( \pi_i \circ \pi_0^{-1} \) is distributed according to (3.1) for each \( i \). Finally, the error \( \hat{\theta} - \theta \) is of order \( 1/\sqrt{N} \) with high probability is due to the central limit theorem. \( \square \)

Proof of Theorem 1.3 The argument is similar to that of the Mallows model. If \( \pi_0 \) is known, set \( \pi_0 = id \) by relabelling the items. So the distribution of top-\( t \) orderings of the IGM model is
\[
P_{\theta,id}(\pi) = \exp \left( -\theta \sum_{j=1}^{t} s_j(\pi) - \frac{t}{1-e^{-\theta}} \right), \tag{3.5}
\]
and $E_{s_j}(\pi) = 1/(1 + e^\theta)$ for each $j$. Given $N$ samples $(\pi_i)_{1 \leq i \leq N}$ with the corresponding $(s_j^i(\pi); 1 \leq j \leq t)_{1 \leq i \leq N}$, a simple computation leads to

$$\hat{\theta} = \log \left( 1 + \frac{Nt}{\sum_{i=1}^{N} \sum_{j=1}^{t} s_j^i(\pi)} \right).$$

By strict convexity of $\theta \mapsto \log(1 + 1/\theta)$, we get $E\hat{\theta} > \theta$. Now if $\pi_0$ is unknown, we have

$$\hat{\theta} = \log \left( 1 + \frac{Nt}{\sum_{i=1}^{N} \sum_{j=1}^{t} s_j^i(\pi \circ \pi_0^{-1})} \right),$$

where $\hat{\pi}_0$ is the MLE of $\pi_0$. By the definition of $\hat{\pi}_0$,

$$\sum_{i=1}^{N} \sum_{j=1}^{t} s_j^i(\pi \circ \pi_0^{-1}) \leq \sum_{i=1}^{N} \sum_{j=1}^{t} s_j^i(\pi \circ \pi_0^{-1}).$$

Again strict convexity and monotonicity of $\theta \mapsto \log(1 + 1/\theta)$ yield the desired result as in the Mallows’ $\phi$ model. $\square$

### 3.2. Convergence rate of the MLE $\hat{\pi}_0$

Next we study the convergence rate of the MLE $\hat{\pi}_0$ of the central ranking $\pi_0$ for the Mallows’ $\phi$ model [1,2]. Given $N$ samples $(\pi_i)_{1 \leq i \leq N}$, the MLE $\hat{\pi}_0$ is given by

$$\hat{\pi}_0 = \arg\min_{\pi \in \mathfrak{S}_n} \sum_{i=1}^{N} \text{inv}(\pi \circ \pi^{-1}).$$

**Proof of Theorem 1.2**. Assume w.l.o.g. that the true central ranking $\pi_0 = id$. We aim to find the bounds of $P_{\theta,id}(\hat{\pi}_0 \neq id)$ given the dispersion parameter $\theta$.

For $\pi, \pi' \in \mathfrak{S}_n$, we say that $\pi$ is more likely than $\pi'$, denoted $\pi \succeq \pi'$, if $\sum_{i=1}^{N} \text{inv}(\pi \circ \pi^{-1}) \leq \sum_{i=1}^{N} \text{inv}(\pi \circ \pi^{-1}')$. For $1 \leq j < k \leq n$, let $(j, k)$ be the transposition of $j$ and $k$. By the union bound, we get

$$P_{\theta,id}((1, 2) \succeq id) \leq P_{\theta,id}(\hat{\pi}_0 \neq id) \leq \sum_{1 \leq j < k \leq N} P_{\theta,id}((j, k) \succeq id). \quad (3.8)$$

Now it suffices to analyze the two-sided bounds of (3.8).

**Lower bound**: For any permutation $\pi \in \mathfrak{S}_n$, if $\pi(1) > \pi(2)$, then $\text{inv}(\pi \circ (1, 2)) = \text{inv}(\pi) - 1$, and if $\pi(1) < \pi(2)$, then $\text{inv}(\pi \circ (1, 2)) = \text{inv}(\pi) + 1$. Thus,

$$\sum_{i=1}^{N} \text{inv}(\pi_i \circ (1, 2)) = \sum_{i=1}^{N} \text{inv}(\pi_i) + \# \{i : \pi_i(1) < \pi_i(2)\} - \# \{i : \pi_i(1) > \pi_i(2)\}.$$ 

Consequently,

$$P_{\theta,id}((1, 2) \succeq id) = P_{\theta,id} \left( \# \{i : \pi_i(1) < \pi_i(2)\} \leq \# \{i : \pi_i(1) > \pi_i(2)\} \right)$$

$$= P_{\theta,id} \left( \# \{i : \pi_i(1) > \pi_i(2)\} \geq \frac{N}{2} \right). \quad (3.9)$$
Note that $P_{\theta, id}(\pi_i(1) > \pi_i(2)) = e^{-\theta}P_{\theta, id}(\pi_i(1) < \pi_i(2))$ which implies that
\[
P_{\theta, id}(\pi_i(1) > \pi_i(2)) = \frac{1}{1 + e^\theta}.
\] (3.10)

Combining (3.9) and (3.10) yields
\[
P_{\theta, id}(1, 2) \succeq id = P\left(\text{Bin}\left(N, \frac{1}{1 + e^\theta}\right) \geq \frac{N}{2}\right) \sim \frac{1}{1 - e^{-\theta}} \sqrt{\frac{2}{\pi N}} \left(\cosh \frac{\theta}{2}\right)^{-N},
\] (3.11)

where Bin($N, p$) is the binomial random variable with parameters $N$ and $p$, and the last estimate follows from the large deviation asymptotics \cite[Theorem 2]{2} that for $p < a$,
\[
P(\text{Bin}(N, p) > aN) \sim \frac{(1 - p)\sqrt{a}}{(a - p)\sqrt{2\pi(1 - a)N}} e^{-NH(a, p)},
\]

where $H(a, p) := a \log \left(\frac{a}{p}\right) + (1 - a) \log \left(\frac{1 - a}{1 - p}\right)$ is the relative entropy, or the Kullback-Leibler distance between Bin($N, p$) and Bin($N, a$).

**Upper bound:** To bound the r.h.s. of (3.8), we need the following comparison result.

**Proposition 3.1.** For $j, k \in \{1, \ldots, n\}$ and $j < k$, we have
\[
P_{\theta, id}(j, k) \succeq id \leq \left(\cosh \frac{\theta}{2}\right)^{-N}.
\] (3.12)

By Proposition 3.1, we get an upper bound of the r.h.s. of (3.8):
\[
\sum_{1 \leq j < k \leq N} P_{\theta, id}(j, k) \succeq id \leq \frac{n(n - 1)}{2} \left(\cosh \frac{\theta}{2}\right)^{-N}.
\]

\[\square\]

In the rest of this subsection, we prove Proposition 3.1.

**Proof of Proposition 3.1.** A similar argument as before shows that
\[
P((j, k) \succeq id) = P((1, 2) \succeq id) \quad \text{for } k - j = 1.
\]

Now let $\ell := k - j \in \{1, \ldots, n - 1\}$. It is not hard to see that for any permutation $\pi \in \mathfrak{S}_n$, the number of inversions $\text{inv}(\pi \circ (j, k))$ can take $2\ell$ values. That is,
\[
\text{inv}(\pi \circ (j, k)) \in \{\text{inv}(\pi) \pm m : m = 1, 3, \ldots, 2\ell - 1\}.
\]

For $m \in \{-2\ell + 1, -2\ell + 3, \ldots, 2\ell - 3, 2\ell - 1\}$, let
\[
p_m := P_{\theta, id}\left(\text{inv}(\pi \circ (j, k)) = \text{inv}(\pi) + m\right).
\]
Observe that
\[ P_{\theta, id}((j, k) \succeq id) = P_{\theta, id} \left( \sum_{i=1}^{N} \left( \text{inv}(\pi_i \circ (j, k)) - \text{inv}(\pi_i) \right) \leq 0 \right) \]
\[ = P \left( \sum_{i=1}^{N} Z_i \leq 0 \right) , \]  
(3.13)
where \( Z_i \) are independent and identically distributed multinomial random variables such that \( Z_i = m \) with probability \( p_m \) for \( m \in \{-2\ell + 1, -2\ell + 3, \ldots, 2\ell - 3, 2\ell - 1\} \). By Cramer’s theorem [13, Section 2.2.1],
\[ P \left( \sum_{i=1}^{N} Z_i \leq 0 \right) \leq \exp \left( -N \sup_{\lambda} \{- \log F(\lambda)\} \right) , \]
where
\[ F(\lambda) := \sum_{m=-2\ell+1, \text{m odd}}^{2\ell-1} p_m e^{\lambda m} , \]
is the moment generating function of \( Z_i \). Note that for \( m > 0 \), we have \( p_{-m} = e^{\theta m} p_m \). Therefore,
\[ \sum_{m=1, m \text{ odd}}^{2\ell-1} p_m (1 + e^{\theta m}) = 1 . \]
(3.15)
Moreover,
\[ \sum_{m=1, m \text{ odd}}^{2\ell-1} p_m (1 + e^{\theta m}) = \sum_{m=1, m \text{ odd}}^{2\ell-1} p_m e^{\frac{\theta m}{2}} (e^{\frac{\theta m}{2}} + e^{-\frac{\theta m}{2}}) \]
\[ \geq \sum_{m=1, m \text{ odd}}^{2\ell-1} p_m e^{\frac{\theta m}{2}} (e^{\frac{\theta}{2}} + e^{-\frac{\theta}{2}}) \]
(3.16)
Combining (3.15) and (3.16) yields \( \sum_{m=1, m \text{ odd}}^{2\ell-1} p_m e^{\frac{\theta m}{2}} \leq \frac{1}{2} (\cosh \frac{\theta}{2})^{-1} \). As a consequence,
\[ \sup_{\lambda} \{- \log F(\lambda)\} \geq - \log F \left( \frac{\theta}{2} \right) \]
\[ = - \log \sum_{m=1, m \text{ odd}}^{2\ell-1} 2 p_m e^{\frac{\theta m}{2}} \]
\[ \geq - \log \left( \cosh \frac{\theta}{2} \right)^{-1} . \]
(3.17)
By (3.13), (3.14) and (3.17), we get the estimate (3.12).

3.3. Effective length \( L \). To conclude this section, we compute the expectation and the variance of the effective length for the single parameter IGM model.
Proof of Proposition 2.2. To simplify the notation, let $q := e^{-\theta}$. Note that $p_i = (1 - q)q^{i-1}$ for $i \geq 1$, so

$$u_n = \prod_{i=1}^{n} (1 - q^i) := (q;q)_n.$$ 

Let

$$f_q(z) := 1 + \sum_{n \geq 1} (q;q)_nz^n \quad \text{and} \quad g_q(z) := \frac{1}{f_q(z)},$$

By (2.6), we have

$$\mathbb{E}L = -g'_q(1) \quad \text{and} \quad \mathbb{E}L^2 = -g''_q(1) - g'_q(1). \quad (3.18)$$

It follows from the $q$-binomial theorem that

$$f_q(x) = (q;q)_{\infty} \sum_{n \geq 0} x^n (q^{n+1};q)_{\infty}$$

$$= (q;q)_{\infty} \sum_{n \geq 0} x^n \sum_{k \geq 0} q^{(n+1)k} (q;q)_k$$

$$= (q;q)_{\infty} \sum_{k \geq 0} q^k (q;q)_k \frac{1}{1 - xq^k},$$

for all $|q| < 1$ and $|x| < 1$. Thus for $x$ sufficiently small,

$$\frac{f_q(1 + x)}{(q;q)_{\infty}} = -\frac{1}{x} + \sum_{k \geq 1} q^k (q;q)_k \frac{1}{1 - q^k - xq^k}$$

$$= -\frac{1}{x} + \sum_{r \geq 0} x^r \sum_{k \geq 1} \frac{1}{(q;q)_k} \left( \frac{q^k}{1 - q^k} \right)^{r+1},$$

which implies that

$$\frac{x f_q(1 + x)}{(q;q)_{\infty}} = -1 + \sum_{r \geq 1} x^r \sum_{k \geq 1} \frac{1}{(q;q)_k} \left( \frac{q^k}{1 - q^k} \right)^r := -1 + \sum_{r \geq 1} A_r(q)x^r.$$

Therefore,

$$g_q(1 + z) = -\frac{1}{(q;q)_{\infty}} \left( z + z^2 \sum_{k \geq 1} \frac{1}{(q;q)_k} \frac{q^k}{1 - q^k} + \ldots \right). \quad (3.19)$$

Combining (3.18) with (3.19) yields (2.7) and (2.8). \qed

4. Experimental results

In this section we provide experimental results on synthetic data and two real-world data: APA election data (large $N$, small $t_{\text{max}}$), and University's homepage search (small $N$, large $t_{\text{max}}$).

Synthetic data. In this experiment, we generate 50 sets of $N = 1000$ rankings from the IGM model (1.5) with $\theta = (1, 0.9, 0.8, 0.7, 0.6, 0.5$) and $\pi_0 = \text{id}$. We fix the length
of all observed rankings to be \( t = 6 \). Table 1 displays the percentage of the model sizes selected by Algorithm 1.

| Model size \( t \) | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------------|---|---|---|---|---|---|
| Percentage (%)    | 0 | 65 | 35 | 0 | 0 | 0 |

**TABLE 1.** Percentage of model sizes selected for 50 simulated data sets.

Now we apply the IGM model with the preselected model size \( t \). The estimate \( \hat{\theta}_1 \) lies in the range \([0.94, 1.08]\) with mean 1.0 and standard deviation 0.03, and \( \hat{\theta}_2 \) in the range \([0.84, 0.95]\) with mean 0.9 and standard deviation 0.02. Figure 3 gives a synopsis of \((\hat{\theta}_1, \hat{\theta}_2)\) for the 50 data sets. Moreover, the estimated central rankings restricted to the top-6 ranks are always \((1\,|\,2\,|\,3\,|\,4\,|\,5\,|\,6)\).

**Figure 3.** Estimates of \((\theta_1, \theta_2)\) for 50 simulated data sets.

**APA data set.** We consider the problem of ranking with a small number of items and large sample size. The data consists of \( N = 15449 \) rankings over \( t_{\text{max}} = 5 \) candidates during the American Psychological Association’s presidential election in 1980. Among these rankings, there are only 5738 complete rankings and the number of distinct rankings is \( n = 207 \). Table 2 gives an extract of the data. See Coombs, Cohen and Chamberlin [10] and Diaconis [15] for further background.

| Ranking | # votes | Ranking | # votes | Ranking | # votes |
|---------|---------|---------|---------|---------|---------|
| 1       | 1022    | 31200   | 32      | 12345   | 30      |
| 100     | 1198    | 21003   | 17      | 13452   | 37      |
| 21      | 143     | 21030   | 31      | 23145   | 72      |
| 2001    | 70      | 1023    | 55      | 34152   | 87      |
| 32001   | 41      | 1230    | 9       | 31542   | 30      |

**TABLE 2.** An extract of the APA data.

Table 3 displays the estimate of \( \theta \) for the single parameter IGM with different model sizes. Applying Algorithm 1 yields the selection of \( t = 2 \). Then by fitting the IGM model with \( t = 2 \), we get \( \hat{\theta}_1 = 0.46, \hat{\theta}_2 = 0.54 \) and \( \hat{\pi}_0 = (3\,|\,1\,|\,5\,|\,4\,|\,2) \). By the second
order analysis, Diaconis [15] argued that there is a strong effect of choosing candidates θ{1,3} and θ{4,5}, with candidate θ{2} in the middle. Our result suggests that the pair of candidates θ{1,3} be in a more favorable position.

| Model size t | 1   | 2   | 3   | 4   | 5   |
|--------------|-----|-----|-----|-----|-----|
| Estimated θ  | 0.47| 0.50| 0.54| 0.62| 0.72|

TABLE 3. Estimates of θ for the single parameter IGM model with t ∈ [1, 5].

University’s homepage search. We consider the problem of learning a domain-specific search engine with the data collected by Cohen, Schapire and Singer [9]. The data consists of 157 universities, the queries, and 21 search engines, the experts. The experts search for the university’s name appearing as a phrase, see Table 4 for a summary.

| NAME | NAME title:“homepage” | “NAME” url:index.html |
|------|------------------------|-----------------------|
| “NAME” | NAME welcome | “NAME” url:home.html |
| “NAME” PLACE | NAME url:index.html | “NAME” PLACE title:“home page” |
| title:NAME | NAME url:home.html | “NAME” PLACE title:“homepage” |
| title:“NAME” | “NAME” title:“homepage” | “NAME” PLACE welcome |
| title:“NAME” PLACE | “NAME” title:“home page” | “NAME” PLACE url:index.html |
| NAME title:“home page” | “NAME” welcome | “NAME” PLACE url:home.html |

TABLE 4. 21 university search engines.

Each expert search engine outputs a ranked list of up to $t_{max} = 30$ URLs when queried with the university’s name. The target output is the university’s homepage. There are 10 universities without data, and some search engines return empty list. So there are 147 ranking problems with sample size $N ≤ 21$, and each sample has length ranging from 1 to 30. Figure 4 below provides a summary of the number of distinct URLs $n$, and the number of outputs $T$ given by $N$ search engines for the 147 universities.

![Figure 4](image-url)

**Figure 4.** Left: Histogram of the number of distinct outputs $n$. Right: Histogram of the total number of outputs $T$.

For each query, we apply the IGM model (1.5) with Algorithm 1 to calculate the rank of the university’s homepage under the estimated central ranking $\hat{π}_0$. This rank measures
the correctness of the model. If the target homepage is not among the URLs returned by the search engines, we put it to the end of the list. The central ranking is estimated by the SORTR heuristic, see Meilă and Bao [36, Figure 2]. Table 5 below is an extract of the estimated rank of the target homepages.

| University’s name                      | Estimated rank |
|----------------------------------------|----------------|
| Oregon Health Sciences University      | 2              |
| Ouachita Baptist University            | 1              |
| Our Lady of the Lake University        | 6              |
| Pacific Baptist University             | 2              |
| Pacific Christian College              | 8              |
| Pacific Lutheran University            | 1              |
| Worcester State College                | 1              |
| Yeshiva University                     | 19             |
| York College of Pennsylvania           | 6              |
| Young Harris College                   | 1              |

TABLE 5. A list of 10 universities and their estimated ranks.

Our training model is slightly different from that in Meilă and Bao [36, Section 6.3], since they used the parametrization \( \vec{\theta} = (\theta_1, \ldots, \theta_{t-1}, \theta_t, \ldots, \theta_t) \) for the top-\( t \) IGM model. Figure 5 provides the estimates of \( \theta \) for the single parameter IGM model with different model sizes. By Algorithm 1, 68% select \( t = 5 \), 20% select \( t = 1 \), and 11% select \( t = 2 \) over all 147 queries. We also computed the rank of the homepage for each query and each model size \( t \). Table 6 below summarizes the mean and the median of the target rank for these models.

![Figure 5. Estimates of \( \theta \) for the single parameter IGM model with \( t \in \{1, 2, 3, 6, 10, 30\} \).](image)

| Model size \( t \) | 1 | 2 | 3 | 6 | 10 | 30 | Algo 1 |
|--------------------|---|---|---|---|----|----|--------|
| Mean rank          | 3.7| 4.2| 4.9| 8.0| 13.2| 18.9| 6.8    |
| Median rank        | 2 | 2 | 2 | 3 | 5 | 6 | 3      |

TABLE 6. Mean and median rank of the target homepage under the IGM models.
Note that Meilă and Bao [36, Table 2] got a mean rank around 15 and a median rank around 10. Compared to their results, our experiments give better target rank for small model sizes. This is reasonable because for each query there are only $N \leq 21$ samples and large model sizes may cause overfitting. Small median ranks in Table 6 also supports the validity of the IGM model.

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