Region crossing change is an unknotting operation

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Abstract

Region crossing change is a local transformation on a knot or link diagram. We show that a region crossing change on a knot diagram is an unknotting operation, and we define the region unknotting numbers for a knot diagram and a knot.

1 Introduction

An unknotting operation is a pattern of local transformation on knot diagrams such that any knot diagram can be transformed into a trivial knot diagram by a finite sequence of this pattern and Reidemeister moves I, I I and III. Unknotting operations play important roles in knot theory, and many unknotting operations are studied. For example, it is known that a delta move and a ♯-move as indicated in Figure 1 are unknotting operations [3], [5], [4]. It is also known that an n-gon move as indicated in Figure 1 is an unknotting operation [1], [6]. In a seminar in Osaka City University, K. Kishimoto proposed a local transformation on a knot or link diagram named region crossing change whose definition is as follows: Let \( D \) be a link diagram on \( S^2 \), and \( |D| \) the four-valent graph obtained from \( D \) by replacing each crossing point with a vertex. We call each component of \( S^2 - |D| \) a
region of $D$. We remark that $D$ with $c$ crossings has $2c$ edges and therefore $c+2$ regions because of Euler characteristic. For example, the diagram $D$ in Figure 2 has five regions $R_1, R_2, \ldots$ and $R_5$. A region crossing change at a region $R$ of $D$ is defined to be the crossing changes at all the crossing points on $\partial R$. For example in Figure 2, we obtain $D'$ (resp. $E'$) from $D$ (resp. $E$) by applying a region crossing change at $R_1$ (resp. $P$). We remark that a $\#$-move and an $n$-gon move on a knot diagram are region crossing changes. Kishimoto raised the following question: Is a region crossing change on a knot diagram an unknotting operation? That is, can we transform any diagram into a diagram of the trivial knot by region crossing changes without Reidemeister moves? We remark that for a link diagram, the answer to the above question is negative. For example, the link diagram in Figure 3 cannot be deformed into a diagram of a trivial link by any region crossing change. We have the following theorem:
**Theorem 1.1.** Let $D$ be a knot diagram, and $c$ a crossing point of $D$. Let $D'$ be the diagram obtained from $D$ by a crossing change at $c$. Then, there exist region crossing changes which deform $D$ into $D'$.

The proof is given in Section 3. Since a crossing change on knot diagram is an unknotting operation, we have the following corollary which answers to Kishimoto’s question:

**Corollary 1.2.** A region crossing change on a knot diagram is an unknotting operation.

Then we can define the region unknotting number $u_R(D)$ of a knot diagram $D$ to be the minimal number of regions of $D$ which are needed to be applied region crossing changes to obtain a diagram of the trivial knot from $D$ without Reidemeister moves. We have the following theorem:

**Theorem 1.3.** Let $D$ be a knot diagram, and $c(D)$ the crossing number of $D$. Then we have

$$u_R(D) \leq \frac{c(D)}{2} + 1.$$  

The proof is given in Section 4. H. A. Miyazawa showed in [1] that for given any knot $K$, there exists an integer $n$ such that a diagram of $K$ can be transformed into a diagram of the trivial knot by one $n$-gon move. In
section 4, we define the region unknotting number of a knot \( K \) which is the minimal number of regions of a minimal diagram \( D \) of \( K \) which are needed to be applied region crossing changes to obtain a diagram of the trivial knot from \( D \) without Reidemeister moves. The rest of this paper is organized as follows: In Section 2, we show properties of a region crossing change which are used in proving Theorem 1.1. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.3 and discuss the region unknotting numbers.

2 Properties of a region crossing change

In this section, we show properties of a region crossing change. Let \( D \) be a link diagram, and \( R \) a region of \( D \). We denote by \( D(R) \) the diagram obtained from \( D \) by a region crossing change at \( R \). For two regions \( R_1 \) and \( R_2 \) of \( D \), we have \((D(R_1))(R_2) = (D(R_2))(R_1) \) and \((D(R))(R) = D \) because crossing changes do not depend on the order, and twice crossing changes at a crossing point are canceled. Then, for regions \( R_1, R_2, \ldots, R_n \) of \( D \) and the set of regions \( P = R_1 \cup R_2 \cup \cdots \cup R_n \), we denote by \( D(P) \) the diagram obtained from \( D \) by region crossing changes at \( R_1, R_2, \ldots, R_n \). A link diagram \( D \) on \( S^2 \) is reduced if \( D \) has no crossings as shown in Figure 4, where each square means a diagram of a tangle. We call such a crossing a reducible crossing, and the set of a reducible crossing and a square a reducible part. We have

![Figure 4:](image)

the following lemma:

**Lemma 2.1.** Let \( D \) be a link diagram, and \( R_1, R_2 \) regions of \( D \). Let \( c \) be a crossing point of \( D \). If \( c \) is not a reducible crossing and satisfies \( c \in \partial R_1 \cap \partial R_2 \), then the region crossing changes at \( R_1 \) and \( R_2 \) do not change the over/under information of \( c \).
From Lemma 2.1, we have the following proposition:

**Proposition 2.2.** For a link diagram \( D \), there exists a crossing \( c \) of \( D \) such that the regions \( R_1, R_2, R_3 \) and \( R_4 \) around \( c \) as shown in Figure 5 satisfy \( R_1 = R_3 \) or \( R_2 = R_4 \) if and only if \( D \) is a non-reduced link diagram.

Checkerboard coloring for a link diagram \( D \) is a coloring of all the regions of \( D \) with two colors black and white such that two regions which are adjacent by an edge of \( |D| \) have always disjoint colors. From Lemma 2.1 and Proposition 2.2 we have the following corollary:

**Corollary 2.3.** Let \( D \) be a reduced link diagram with a checkerboard coloring, and \( D' \) the diagram obtained from \( D \) by region crossing changes at all the regions colored black (or white). Then, \( D = D' \).

We remark that Corollary 2.3 does not hold for a non-reduced link diagram (see, for example, the diagram \( E \) in Figure 2). From Corollary 2.3 we have the following corollary:

**Corollary 2.4.** Let \( D \) be a reduced link diagram, and \( B \) the set of all the regions of \( D \) colored black with a checkerboard coloring. Let \( R \) be a subset of \( B \) consisting of regions of \( D \). Then, \( D(R) = D(B - R) \).

From Lemma 2.1 and that \( (D(R))(R) = D \), we have the following corollary:

**Corollary 2.5.** Let \( D \) be a link diagram, and \( c \) a crossing point of \( D \). Let \( R_1, R_2, R_3 \) and \( R_4 \) be regions of \( D \) around \( c \) as shown in Figure 3. If \( R_1 \neq R_3 \) and \( R_2 \neq R_4 \), then the region crossing changes at \( R_1 \cup R_2 \cup R_3 \cup R_4 \) do not change the over/under information of \( c \).
3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $D$ be a knot diagram, and $c$ a crossing point of $D$. Let $D'$ be the diagram which is obtained from $D$ by a crossing change at $c$. Then, we can find a set of regions $P$ such that we obtain $D'$ from $D$ by region crossing changes at $P$ by the following procedure:

**Step 1.** We splice $D$ at $c$ by giving $D$ an orientation (see Figure 6). Then, we obtain a diagram $D_s$ of a two-component link.

**Step 2.** We apply a checkerboard coloring for one component $D^1$ of $D_s$ by ignoring another component $D^2$. Then, if there exists a reducible part $S$ of $D_s$ which does not include the spliced arcs, and whose outer part is colored black and opposite region is colored white, we recolor all the regions of $S$ with black as illustrated in Figure 7. If there exists a reducible part $T$ of $D_s$ which does not include the spliced arcs, and whose outer part and opposite region are colored black, we recolor $T$ with the coloring such that $T$ is colored with a checkerboard coloring and the outer part of $T$ is colored black when
we splice at the reducible crossing of $T$. If there exist two reducible parts $T_1$ and $T_2$ such that they do not have the spliced arcs and $T_1 \subset T_2$, we apply the above recoloring for $T_2$ before $T_1$.

![Figure 7](image)

**Step 3.** We apply the coloring for $D$ which corresponds to the coloring of $D_s$. We call the set of all the regions of $D$ colored black $P$.

![Figure 8](image)

An example of the above procedure is shown in Figure 8. By Corollary 2.3 and Corollary 2.5, a crossing point $p$ of $D$ which corresponds to a self-crossing point of $D^1$ or $D^2$ is not changed the over/under information by region crossing changes at $P$. By Lemma 2.1 a crossing point $p$ of $D$ which corresponds...
to a crossing point between $D^1$ and $D^2$ is not changed the over/under information by the region crossing change at $P$. Hence the region crossing changes at $P$ changes the over/under information of only $c$. □

We give an example on Theorem 1.1 and Corollary 1.2:

Example 3.1. For the knot diagram $D$ in Figure 9, we can change the crossing at $p$ (resp. $q$) by region crossing changes at the set of regions $P_1$ (resp. $P_2$). Since $(D(R))(R) = D$ for each region $R$, we can change the crossings at $p$ and $q$ by region crossing changes at $P$.

![Figure 9:](image)

4 Region unknotting number

In this section, we discuss the region unknotting numbers of a knot diagram and a knot. We prove Theorem 1.3.

Proof of Theorem 1.3. For a knot diagram $D$ with a checkerboard coloring, we denote by $b$ (resp. $w$) the number of regions colored black (resp. white). We have

$$u_R(D) \leq \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{w}{2} \right\rfloor \\
\leq \frac{b + w}{2}$$
because of, for example, Corollary 2.4, where \( \lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\} \).
Remark that for a non-reduced diagram, the inequality also holds. Since \( b + w \) means the number of regions of \( D \), we have
\[
u_R(D) \leq \frac{c(D) + 2}{2}.
\]

\[\square\]

Remark. 4.1. From the proof of Theorem 1.3, it can also be said that the region unknotting number of a knot diagram \( D \) is less than or equal to half the number of regions of \( D \).

We show an example of region unknotting numbers of knot diagrams.

**Example 4.2.** In Figure 11 we list all the knot diagrams based on Rolfsen's knot table [7] with the crossing number eight or less and their region unknotting numbers. We denote by \( D_n \) the diagram of \( m_n \) in Rolfsen's knot table (for example, we denote by \( D_3 \) the diagram of \( 3_1 \)).

Let \( K \) be a knot. The **region unknotting number** \( u_R(K) \) of \( K \) is the minimal \( u_R(D) \) for all diagrams \( D \) of \( K \) with the minimal crossing numbers. Let \( c(K) \) be the crossing number of \( K \). From Theorem 1.3, we have the following corollary:

**Corollary 4.3.** For a knot \( K \), we have
\[
u_R(K) \leq \frac{c(K)}{2} + 1.
\]

In the following example, we show the region unknotting numbers of all the prime knots with crossing number nine or less:

**Example 4.4.** The knots \( 7_1, 8_2, 8_7, 8_9, 8_{18}, 9_3, 9_6, 9_{35}, 9_{40} \) have region unknotting number one or two. The knot \( 9_1 \) have the unknotting number one, two or three. The other prime knots with the crossing number nine or less have the region unknotting number one.
For twist knots, we have the following proposition:

**Proposition 4.5.** A twist knot $K$ has $u_R(K) = 1$.

*Proof.* From a diagram of $K$ as shown in Figure 10, we can obtain a diagram of the trivial knot by a region crossing change at the region $P$ or $Q$. \hfill \qed

![Figure 10:](image)

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Figure 11: