Finite groups with two Chermak-Delgado measures

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Abstract

In this note, we study the finite groups whose Chermak-Delgado measure has exactly two values. They determine an interesting class of $p$-groups containing cyclic groups of prime order and extraspecial $p$-groups.

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1 Introduction

Throughout this paper, let $G$ be a finite group and $L(G)$ be the subgroup lattice of $G$. Denote by

\[ m_G(H) = |H||C_G(H)| \]

the Chermak-Delgado measure of a subgroup $H$ of $G$ and let

\[ m^*(G) = \max\{m_G(H) \mid H \leq G\} \quad \text{and} \quad \mathcal{CD}(G) = \{H \leq G \mid m_G(H) = m^*(G)\}. \]

Then the set $\mathcal{CD}(G)$ forms a modular, self-dual sublattice of $L(G)$, which is called the Chermak-Delgado lattice of $G$. It was first introduced by Chermak and Delgado [7], and revisited by Isaacs [9]. In the last years there has been a growing interest in understanding this lattice (see e.g. [3, 4, 5, 6, 8, 10, 11, 12, 13, 15, 18, 20]). We recall several important properties of the Chermak-Delgado measure that will be used in our paper:
if $H \leq G$ then $m_G(H) \leq m_G(C_G(H))$, and if the measures are equal then $C_G(C_G(H)) = H$;

• if $H, K \leq G$ then $m_G(H)m_G(K) \leq m_G(\langle H, K \rangle)m_G(H \cap K)$, and the equality occurs if and only if $\langle H, K \rangle = HK$ and $C_G(H \cap K) = C_G(H)C_G(K)$;

• if $H \in \mathcal{C}D(G)$ then $C_G(H) \in \mathcal{C}D(G)$ and $C_G(C_G(H)) = H$;

• the minimum subgroup $M(G)$ of $\mathcal{C}D(G)$ (called the Chermak-Delgado subgroup of $G$) is characteristic, abelian, and contains $Z(G)$.

We remark that the Chermak-Delgado measure associated to a finite group $G$ can be seen as a function

$$m_G : L(G) \longrightarrow \mathbb{N}^*, H \mapsto m_G(H), \quad \forall H \in L(G).$$

The starting point for our discussion is given by Corollary 3 of [16], which states that there is no finite non-trivial group $G$ such that $\mathcal{C}D(G) = L(G)$. In other words, $m_G$ has at least two distinct values for every finite non-trivial group $G$. This leads to the following natural question:

Which are the finite groups $G$ whose Chermak-Delgado measure $m_G$ has exactly two values?

In what follows, let $\mathcal{C}$ be the class of finite groups satisfying the above property. Its study is the main goal of the current note.

We recall several basic definitions:

- a *generalized quaternion $2$-group* is a group of order $2^n$, $n \geq 3$, defined by the presentation

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} \rangle;$$

- a finite $p$-group $G$ is said to be *extraspecial* if $Z(G) = G' = \Phi(G)$ has order $p$;

- a finite $p$-group $G$ is said to be *outer abelian* if $G$ is non-abelian, but every proper quotient group of $G$ is abelian;
- a finite $p$-group $G$ of order $p^n$ is said to be of maximal class if the nilpotence class of $G$ is $n - 1$.

The following results on $p$-groups will be useful to us. Lemmas 1.1 and 1.2 appear in (4.26) and (4.4) of [14], II, Lemma 1.3 in Corollary 10 of [19], and Lemma 1.4 in Proposition 1.8 of [1].

**Lemma 1.1.** Any group of order $p^4$ contains an abelian subgroup of order $p^3$.

**Lemma 1.2.** A finite $p$-group $G$ has a unique subgroup of order $p^n$ if and only if either it is cyclic or $p = 2$ and $G \cong Q_{2^n}$ for some $n \geq 3$.

**Lemma 1.3.** A finite $p$-group $G$ is outer abelian if and only if $|G'| = p$ and $Z(G)$ is cyclic, and $G$ is one of the following non-isomorphic groups:

a) $M(n, 1) = \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle$, $n \geq 3$;

b) an extraspecial $p$-group;

c) $G = E \ast A$, where $E$ is an extraspecial $p$-group and $A \cong M(n, 1)$, $n \geq 3$;

d) $G = E \ast A$, where $E$ is an extraspecial $p$-group and $A \cong C_{p^t}$, $t \geq 2$

**Lemma 1.4.** A finite $p$-group $G$ is of maximal class if and only if it has a subgroup $A$ of order $p^2$ such that $C_G(A) = A$.

## 2 Main results

Our first result indicates an important property of the groups in $C$.

**Theorem 2.1.** If a finite group $G$ is contained in $C$, then $|Z(G)|$ is a prime.

**Proof.** Assume that $|Z(G)|$ is not a prime.

If $|Z(G)| = 1$, then we have $m_G(1) = m_G(G) = |G|$. Also, $G$ cannot be a $p$-group. It follows that there are at least two distinct primes $p$ and $q$ dividing $|G|$. Let $S_p$ and $S_q$ be a Sylow $p$-subgroup and a Sylow $q$-subgroup of $G$, of orders $p^m$ and $q^n$, respectively. Since $1 \neq Z(S_p) \subseteq C_G(S_p)$, we get $p^{m+1} \mid m_G(S_p)$. Similarly, $q^{n+1} \mid m_G(S_q)$. We infer that $m_G(S_p) \neq m_G(1)$ and $m_G(S_q) \neq m_G(1)$, and so $m_G(S_p) = m_G(S_q)$. Then $p^{m+1} \mid q^n|C_G(S_q)|$, i.e. $p^{m+1} \mid |C_G(S_q)|$, a contradiction.
If there are two distinct primes $p$ and $q$ dividing $|Z(G)|$, then $Z(G)$ contains two subgroups of orders $p$ and $q$, say $H$ and $K$. It results that the following three Chermak-Delgado measures

$$m_G(H) = p|G|, \quad m_G(K) = q|G| \quad \text{and} \quad m_G(Z(G)) = m_G(G) = |Z(G)||G|$$

are distinct, contradicting our hypothesis.

This completes the proof. □

Using Theorem 2.1, we are able to determine the abelian groups in $\mathcal{C}$.

**Corollary 2.2.** The cyclic groups of prime order are the unique abelian groups contained in $\mathcal{C}$.

By Corollary 2.2 we easily infer that $\mathcal{C}$ is not closed under subgroups, homomorphic images, direct products or extensions. Also, from the first part of the proof of Theorem 2.1 we obtain that:

**Corollary 2.3.** All groups contained in $\mathcal{C}$ are $p$-groups.

Since our study can be reduced to $p$-groups and it is completely finished for abelian groups, in what follows we will suppose that $G$ is a non-abelian $p$-group of order $p^n$ ($n \geq 3$) belonging to $\mathcal{C}$. Then:

a) $\text{Im}(m_G) = \{p^n, p^{n+1}\}$, and consequently $m^*(G) = p^{n+1};$

b) $Z(G)$ is the unique minimal normal subgroup of $G$, and consequently $Z(G) \subseteq G' \subseteq \Phi(G);$  

c) $HZ(G) \in \mathcal{CD}(G), \forall H \leq G$ satisfying $Z(G) \not\subseteq H.$

Indeed, for such a subgroup $H$ of $G$ we have $H \cap Z(G) = 1$, and therefore $m_G(H \cap Z(G)) = p^n$. Then the inequality

$$m_G(H)m_G(Z(G)) \leq m_G(HZ(G))m_G(H \cap Z(G))$$

becomes

$$p^{n+1}m_G(H) \leq p^nm_G(HZ(G)),$$

that is

$$p^nm_G(H) \leq m_G(HZ(G)).$$

Clearly, this implies that $m_G(HZ(G)) = p^{n+1}$, i.e. $HZ(G) \in \mathcal{CD}(G).$
There are many examples of finite non-abelian $p$-groups $G$ such that $\mathcal{CD}(G) = \{Z(G), G\}$ (see e.g. Corollary 2.2 and Proposition 2.3 of [5]). Using Corollary 2.2 and the above item c), we are able to prove that the intersection between this class of groups and $\mathcal{C}$ is empty.

**Corollary 2.4.** $\mathcal{C}$ does not contain non-abelian $p$-groups $G$ with $\mathcal{CD}(G) = \{Z(G), G\}$.

**Proof.** Assume that $\mathcal{C}$ contains a non-abelian $p$-group $G$ satisfying $\mathcal{CD}(G) = \{Z(G), G\}$.

If $G$ possesses a minimal subgroup $H \neq Z(G)$, then $HZ(G) \in \mathcal{CD}(G)$ by c). On the other hand, we obviously have $HZ(G) \neq Z(G)$, and since $\mathcal{CD}(G) = \{Z(G), G\}$ we get $HZ(G) = G$. Then $|G| = p^2$, implying that $G$ is abelian, a contradiction.

If $Z(G)$ is the unique subgroup of order $p$ in $G$, then $G$ is a generalized quaternion 2-group by Lemma 1.2, i.e. $p = 2$ and

$$G \cong Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1} \rangle$$ for some $n \geq 3$.

It results that $G$ has a cyclic maximal subgroup $H \cong \langle a \rangle$. So,

$$m_G(H) = 2^{2n-2} \leq 2^{n+1} = m^*(G),$$

which means $n \leq 3$. Since $G$ is non-abelian we get $n = 3$, that is $G \cong Q_8$. Then $\mathcal{CD}(G)$ is a quasi-antichain of width 3, contradicting the hypothesis. $\square$

Next we will focus on giving examples of non-abelian $p$-groups in $\mathcal{C}$.

**Theorem 2.5.** All extraspecial $p$-groups are contained in $\mathcal{C}$.

**Proof.** Let $G$ be an extraspecial $p$-group. It is well-known that $\mathcal{CD}(G)$ consists of all subgroups $H$ of $G$ containing $Z(G)$ (see e.g. Example 2.8 of [8] or Theorem 4.3.4 of [17]). Consequently, all these subgroups have the same Chermak-Delgado measure. On the other hand, by Lemma 2.6 of [2] any subgroup $H$ of $G$ with $Z(G) \nsubseteq H$ satisfies $m_G(H) = |G|$. Thus the function $m_G$ has exactly two values, as desired. $\square$

Using GAP, we are also able to give an example of a non-extraspecial non-abelian $p$-group in $\mathcal{C}$, namely SmallGroup(32,8):

$$G = \langle a, b, c \mid a^4 = 1, b^4 = a^2, c^2 = bab^{-1} = a^{-1}, ac = ca, cbc^{-1} = a^{-1}b^3 \rangle.$$
Note that the nilpotence class of $G$ is 3. Also, $CD(G)$ is described in Lemma 4.5.16 and Corollaries 4.5.20 and 4.5.21 of [17].

We observe that all non-abelian groups of order $p^3$ belong to $C$ because they are extraspecial. The same thing cannot be said about non-abelian groups of order $p^4$: by Lemma 1.1 such a group $G$ has an abelian subgroup $A$ of order $p^3$, and so $m^*(G) \geq m_G(A) = p^6 > p^5$, implying that $G$ is not contained in $C$. This argument can be extended in the following way.

**Proposition 2.6.** If a non-abelian group of order $p^n$ contains an abelian subgroup of order $\geq p^{\left\lceil \frac{m+n}{2} \right\rceil}$, then it does not belong to $C$.

Since any group of order 64 contains an abelian subgroup of order 16, by Proposition 2.6 we infer that:

**Corollary 2.7.** $C$ does not contain non-abelian groups of order 64.

Another application of Proposition 2.6 is the following:

**Theorem 2.8.** Let $G$ be a finite $p$-group of nilpotence class 2 contained in $C$. Then $G$ is extraspecial.

**Proof.** Since the nilpotence class of $G$ is 2, we have that $G/Z(G)$ is abelian and so $G' \subseteq Z(G)$. By Theorem 1 we get $G' = Z(G)$, which implies that $G$ is an outer abelian $p$-group. Then $G$ belongs to one of the four classes of groups in Lemma 1.3.

We observe that $M(n, 1)$ has a cyclic subgroup of order $p^n$, namely $\langle a \rangle$, and $n \geq \left\lceil \frac{m+n}{2} \right\rceil$ for $n \geq 3$. Thus it cannot be contained in $C$ by Proposition 2.6. Also, it is easy to see that a central product $E \ast A$, where $E$ is an extraspecial $p$-group of order $p^{2m+1}$ and $A \cong M(n, 1), n \geq 3$, always has an abelian subgroup of order $p^{m+n}$. Since $m + n \geq \left\lceil \frac{2m+n+4}{2} \right\rceil$ for $n \geq 3$, by Proposition 2.6 we infer that $E \ast A$ does not belong to $C$. Similarly, a central product $E \ast A$, where $E$ is an extraspecial $p$-group of order $p^{2m+1}$ and $A \cong C_{p^t}$ with $t \geq 2$, always has an abelian subgroup of order $p^{m+t}$. If $t \geq 3$ then $m + t \geq \left\lceil \frac{2m+t+4}{2} \right\rceil$, implying that $E \ast A$ is not contained in $C$. If $t = 2$, it suffices to observe that the center of $E \ast A$ is of order $p^2$, and consequently $E \ast A$ is not contained in $C$ by Theorem 1. These shows that the unique possibility is that $G$ be an extraspecial $p$-group, as desired. \hfill \Box

Our last result shows that the non-abelian groups of order $p^3$ are in fact the unique $p$-groups of maximal class in $C$. 
Theorem 2.9. Let $G$ be a finite $p$-group of maximal class contained in $C$. Then $G$ is non-abelian of order $p^3$.

Proof. Obviously, $G$ is non-abelian. Let $|G| = p^n$. By Lemma 1.4 we know that $G$ possesses a subgroup $A$ of order $p^2$ such that $C_G(A) = A$. It follows that $m_G(A) = p^4$, and therefore we have either $n = 3$ or $n = 4$. Since the case $n = 4$ is impossible, we get $n = 3$, as desired.

Inspired by the above examples, we end this note by indicating the following open problem.

Open problem. Which are the pairs $(p, n)$, where $p$ is a prime and $n$ is a positive integer, such that $C$ contains groups of order $p^n$?

Note that all pairs $(p, n)$ with $n$ odd satisfy this property by Corollary 2.2 and Theorem 2.5.

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