A COMBINATORIAL APPROACH TO CLASSICAL REPRESENTATION THEORY

MARTIN RUBEY AND BRUCE W. WESTBURY

Abstract. A problem from invariant theory is, given a representation, to determine the characteristic map of the action of the symmetric group on the invariant subspace of a tensor power. This problem is solved for the defining representations of the symplectic groups, the defining representations of the symmetric groups and the adjoint representations of the general and special linear groups.

Dedicated to Mia and George

Contents

1. Introduction 2
2. Combinatorial species 4
3. Cyclic sieving 8
4. Monoidal categories 10
5. Diagram categories 13
5.1. The permutation category 13
5.2. The Temperley-Lieb category 15
5.3. The general definition 16
5.4. Ideals 18
6. Morita equivalence 19
6.1. Regular algebras 20
6.2. Morita contexts 20
6.3. Idempotents 21
6.4. Inflation 22
7. The Brauer category 25
7.1. Diagram category 25
7.2. Frobenius characters for diagram algebras 25
7.3. Branching rules 27
7.4. Fundamental theorems 29
7.5. Frobenius characters for tensor algebras 35
7.6. Cyclic sieving phenomenon 36
8. The partition category 38
8.1. Diagram category 38
8.2. Frobenius characters for diagram algebras 39
8.3. Branching rules 40

Date: 15 August 2014.
The objective of this paper is to exhibit examples of the cyclic sieving phenomenon involving classical groups and to make this accessible to a reader with a background in combinatorics rather than representation theory. In order to achieve this we have been led to give a contemporary, combinatorial account of the representation theory in [Wey97].

The method used to exhibit the cyclic sieving phenomenon was given in [Wes10]. The method starts from a representation $V$ of a reductive group $G$. Then, for $r \geq 0$, we have the subspace of $\otimes^r V$ invariant under the action of $G$. This subspace has a natural action of the symmetric group, $S_r$. In order to exhibit the cyclic sieving phenomenon we then need to determine the Frobenius character of this representation and to construct a basis invariant under the action of the cyclic group generated by the long cycle. In this paper we apply this method to three families of classical groups; the defining representations of the symplectic groups, the defining permutation representations of the symmetric groups and the adjoint representations of the general linear groups.

It is now well-established that an effective method of understanding the tensor powers of these representations is to use the diagram categories. The primary benefit of the use of diagram categories is that it gives simple conceptual proofs avoiding detailed calculations. A secondary benefit is that the methods work directly with modules which gives stronger results than character calculations. Although in this paper we focus on semisimple algebras this approach lends itself to the general case. For example, the blocks of the centraliser algebras are determined in [CDV11], [CDVM09], [CDVDM08].

The basic problem of decomposing these tensor powers is equivalent to understanding the structure of the endomorphism algebras $\text{End}_G(\otimes^r V)$. In our examples this is a direct sum of matrix algebras. Then the starting point for this method is that there are homomorphisms from the endomorphism algebras in the diagram category onto
these endomorphism algebras. This is known as the first fundamental theorem. The basic problem can then be solved in two steps. The first step is to determine the structure of the endomorphism algebras in the diagram category and to construct the simple modules. This achieved following the method in [HHKP10]. The second step is to determine the kernel. This second step is known as the second fundamental theorem.

The main problem in this paper is a refinement of this basic problem and is solved following the same approach. There are inclusions $K\mathcal{S}_r \to \text{End}_G(\otimes^r V)$, where $K$ is the ground field. The problem now is to consider each simple module of the endomorphism algebra as a $K\mathcal{S}_r$-module and to determine the Frobenius character (a homogeneous symmetric function of degree $r$). The significance of this is that it describes the decomposition of $\otimes^r V$ under the action of $G \times \mathcal{S}_r$. This is equivalent to describing the branching rules for the homomorphism $G \to \text{GL}(V)$ and to describing the decomposition of the Schur functors evaluated at $V$.

In particular the invariant tensors have an action of the symmetric group and we give formulae for their Frobenius characters. For the first two families we describe a basis which is invariant under the long cycle. This then gives examples of the cyclic sieving phenomenon. In the example arising from the symplectic group $\text{Sp}(2n)$, the set is the set of $(n+1)$ noncrossing perfect matchings and the generator of the cyclic group acts by rotation. Our proof of the second fundamental theorem is simpler than existing proofs and avoids the use of algebraic geometry.

The organisation of this paper is as follows. The background needed to make the paper self-contained is presented in the first four sections. The next two sections give an account of the features and methods common to the three examples. The final three sections give the results for each of the three families. These three sections follow the following template:

- The first subsection describes the relevant diagram category.
- The second subsection constructs the simple modules for the endomorphism algebras in the diagram category and determines their Frobenius characters.
- The third subsection gives branching rules for the inclusion of an endomorphism algebra in its successor.
- The fourth subsection gives the evaluation functors and in the first two families describes the kernel and a basis for the image.
- The fifth subsection constructs the simple modules for the endomorphism algebras and relates the parametrisation of simple modules arising from the diagram category with the standard parametrisation.
The sixth subsection discusses the cyclic sieving phenomenon using the second fundamental theorem and the Frobenius characters of invariant tensors.

2. COMBINATORIAL SPECIES

In this section we briefly recall the theory of Joyal’s species, fix notation and give some examples we will use throughout.

Let $\mathcal{B}$ be the category of finite sets with bijections. A combinatorial species (in $d$ variables, also called ‘sorts’) is a functor $F : \mathcal{B}^d \to \mathcal{B}$. By convention, the application of $F$ is written with square brackets: for a $d$-tuple of sets $U_1, \ldots, U_d$

$$F[U_1, \ldots, U_d]$$

is the set of $F$-structures on the given tuple of sets, and, for a $d$-tuple of bijections $\sigma_i : U_i \to V_i$ we write

$$F[\sigma_1, \ldots, \sigma_d] : F[U_1, \ldots, U_d] \to F[V_1, \ldots, V_d]$$

for the so-called transport of structures. For ease of notation let us abbreviate $F[\{1, \ldots, n_1\}, \ldots, \{1, \ldots, n_d\}]$ to $F[n_1, \ldots, n_d]$. Then, $F$ is the same as a family of permutation representations

$$(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_d}) \times F[n_1, \ldots, n_d] \to F[n_1, \ldots, n_d].$$

In particular, if $d = 1$ and $F[k] = \emptyset$ for $k \neq n$, we identify $F$ with the corresponding permutation representation of $\mathfrak{S}_n$.

To every combinatorial species $F$ we associate its cycle index series. This is the symmetric function in $d$ alphabets $x_1, \ldots, x_d$

$$\sum_{n_1, \ldots, n_d} \frac{1}{n_1! \cdots n_d!} \sum_{\sigma_1 \in \mathfrak{S}_{n_1}, \ldots, \sigma_d \in \mathfrak{S}_{n_d}} \text{fix} F[\sigma_1, \ldots, \sigma_d] p_{\lambda(\sigma_1)}(x_1) \cdots p_{\lambda(\sigma_d)}(x_d),$$

where $\text{fix}$ denotes the number of fixed points of a permutation, $\lambda(\sigma)$ is the cycle type of the permutation $\sigma$ and $p$ denotes the power sum symmetric functions.

We will additionally use the following variant of combinatorial species: a tensorial species is a functor $F : \mathcal{B}^d \to \text{Vect}$, where $\text{Vect}$ is the category of finite dimensional vector spaces. Such a functor $F$ is equivalent to a family of linear representations of products of $d$ symmetric groups. Note that we can transform any combinatorial species into a tensorial species by declaring the set $F[U_1, \ldots, U_d]$ to be the basis of a vector space. Conversely, if there is a basis of $F[n_1, \ldots, n_d]$ that is permuted by $F[\sigma_1, \ldots, \sigma_d]$ then we can regard $F$ as a combinatorial species.

For any tensorial species $F$, the associated Frobenius character is the symmetric function $\text{ch} F$, given by

$$\sum_{n_1, \ldots, n_d} \frac{1}{n_1! \cdots n_d!} \sum_{\sigma_1 \in \mathfrak{S}_{n_1}, \ldots, \sigma_d \in \mathfrak{S}_{n_d}} \text{tr} F[\sigma_1, \ldots, \sigma_d] p_{\lambda(\sigma_1)}(x_1) \cdots p_{\lambda(\sigma_d)}(x_d),$$
where \( \text{tr} \) denotes the trace of an endomorphism. Note that the Frobenius character coincides with the cycle index series when \( F \) can be regarded as a combinatorial species.

Since combinatorial and tensorial species are completely analogous concepts we will henceforth use the same notation for either concept, drop the adjectives and simply write of species.

The only point to keep in mind that a tensorial species is determined up to isomorphism by its Frobenius character, while this is not true for combinatorial species. In particular, to show that two combinatorial species are isomorphic it is not sufficient to compare their Frobenius characters, see [BLL98, Section 2.6, Equations (30)–(31)].

For computations with species in several variables it is convenient to introduce a ‘singleton’ species \( X_1, X_2, \ldots, X_d \) for each variable by setting

\[
X_i[U_1, \ldots, U_d] = \begin{cases} 
\{U_i\} & \text{if } |U_i| = 1 \text{ and } U_j = \emptyset \text{ for } j \neq i \\
\emptyset & \text{otherwise} 
\end{cases}
\]

Thus, \( \text{ch} X_i = p_1(x_i) \). In particular, this definition allows us to make the number of variables of a species \( F \) explicit by writing

\[
F(X_1, \ldots, X_d).
\]

A fundamental example for a (univariate) species is \( h_n \), the species of sets with \( n \) elements, which corresponds to the trivial representation of \( \mathfrak{S}_n \). More formally,

\[
h_n[U] = \begin{cases} 
\{U\} & \text{if } |U| = n \\
\emptyset & \text{otherwise} 
\end{cases}
\]

Its cycle index series or Frobenius character, also denoted \( h_n \), is the \( n \)-th complete homogeneous symmetric function.

The interest in the theory of species lies in the fact that it provides combinatorial interpretations for many operations on group actions. For the most important operations on species this is illustrated in great detail in [BLL98], so we will be brief in the following. Since for all operations the definitions of the group actions is always the obvious one we restrict ourselves to the definition of the underlying sets.

Let \( F \) and \( G \) be two species in \( d \) variables. Then \( F + G \) is defined on sets as

\[
(F + G)[U_1, \ldots, U_d] = F[U_1, \ldots, U_d] \amalg G[U_1, \ldots, U_d],
\]

and, for tensorial species,

\[
(F + G)[U_1, \ldots, U_d] = F[U_1, \ldots, U_d] \oplus G[U_1, \ldots, U_d].
\]

Clearly, \( \text{ch}(F + G) = \text{ch} F + \text{ch} G \). Informally, an \( (F + G) \)-structure is either an \( F \)-structure or a \( G \)-structure.
The product $F \cdot G$ of species is defined on sets as
\[(F \cdot G)[U_1, \ldots, U_d] = \prod_{V_i \sqcup W_i = U_i} F[V_1, \ldots, V_d] \times G[W_1, \ldots, W_d].\]

and, for tensorial species,
\[(F \cdot G)[U_1, \ldots, U_d] = \bigoplus_{V_i \sqcup W_i = U_i} F[V_1, \ldots, V_d] \otimes G[W_1, \ldots, W_d].\]

Thus, $\text{ch}(F \cdot G) = (\text{ch} F) \cdot (\text{ch} G)$. Informally, an $(F \cdot G)$-structure on a set $U$ is a pair consisting of an $F$-structure on a set $V$ and a $G$-structure on a set $W$, such that $U$ is the disjoint union of $V$ and $W$.

When $F_m$ and $G_n$ are representations of $S_m$ and $S_n$, their product is precisely the induction product (known also as external, outer or Cauchy product):
\[F_m \cdot G_n = (F_m \times G_n)^{S_m+n}_{S_m \times S_n}.\]

We define the next operation, partitional composition only for univariate species $F$, which is sufficient for our calculations and simpler than the general case. Let $G$ a $d$-variate species. Then $F \circ G$ is the $d$-variate species corresponding to the plethysm of representations:
\[(F \circ G)[U_1, \ldots, U_d] = \prod_{\pi \text{ set partition of } \bigsqcup U_i} F[\pi] \times \prod_{(B_1, \ldots, B_d) \in \pi} G[B_1, \ldots, B_d],\]

where we regard each block of the set partition $\pi$ as a $d$-tuple of sets $(B_1, \ldots, B_d)$ such that $B_i \subseteq U_i$ for $1 \leq i \leq d$. For tensorial species, this is
\[(F \circ G)[U_1, \ldots, U_d] = \bigoplus_{\pi \text{ set partition of } \bigsqcup U_i} F[\pi] \otimes \bigotimes_{(B_1, \ldots, B_d) \in \pi} G[B_1, \ldots, B_d].\]

Alternatively, we will also write $F(G(X_1, \ldots, X_d))$. Note that the notation introduced in equation $\bigsqcup$ is consistent with the definition of partitional composition. For the Frobenius character we have $\text{ch}(F \circ G) = \text{ch} F \circ \text{ch} G$, the plethysm of symmetric functions.

An important special case of the partitional composition arises when $F$ is the species of sets $H = \sum_{n \geq 0} h_n$. An $(H \circ G)$-structure should indeed be thought of a set of $G$-structures. In particular, we will consider the following species:

- the species of perfect matchings $M$, which can be expressed as $H \circ h_2$, since a perfect matching is a set of two-element sets,
- the species of set partitions $P$, which can be expressed as $H \circ H_+$, where $H_+ = \sum_{n \geq 1} h_n$ is the species of non-empty sets, since a set partition is a set of non-empty sets,
- the species of permutations $S$, which can be expressed as $H \circ C_+$, where $C_+$ is the species of cycles, since a permutation is a set of cycles,
• the bivariate species of bijections between a set of sort $X$ and a set of sort $Y$ is $H(X \cdot Y)$.

We will employ two further operations, the diagonal of a bivariate species and the scalar product of two species. These were both introduced in [Red27] and for a textbook exposition see [HP73].

The diagonal of a bivariate species $F$ is the univariate species $\nabla F$ with $\nabla F[U] = F[U^2]$. The Frobenius character of $\nabla F$ is the diagonal of the Frobenius character of $F$, i.e., the linear extension of

$$\nabla \frac{p_\lambda(x) p_\mu(y)}{z^\lambda \cdot z^\mu} = \begin{cases} \frac{p_\lambda(x)}{x^\lambda} & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

To define the scalar product of two species two auxiliary definitions are helpful. Let $F$ be a species which is homogeneous of degree $n$ in its first variable, i.e., $F[U_1, \ldots, U_d]$ is the empty set whenever $|U_1| \neq n$. Then $F|_{X_1=1}$ is the $(d-1)$-variate species defined by sending $[U_2, \ldots, U_d]$ to the set of orbits of $F[[1, \ldots, n], U_2, \ldots, U_d]$ under the action of $S_n$ on $\{1, \ldots, n\}$. As long as the obvious finiteness conditions are met, this definition can be extended by linearity.

The Cartesian product of a $c$-variate species $F$ and a $(c+d)$-variate species $G$ with respect to the first $c$ variables is defined on sets as

$$F \star_{X_1, \ldots, X_c} G[U_1, \ldots, U_{c+d}] = F[U_1, \ldots, U_c] \times G[U_1, \ldots, U_{c+d}],$$

and, for tensorial species

$$F \star_{X_1, \ldots, X_c} G[U_1, \ldots, U_{c+d}] = F[U_1, \ldots, U_c] \otimes G[U_1, \ldots, U_{c+d}].$$

When $d = 0$ we will simply write $F \star G$. In this case, when $F$ and $G$ are representations of $S_n$, their Cartesian product is precisely the Kronecker product of representations (known also as internal product). We remark that several notations for this product are in use: for species it seems that ‘$\times$’ is predominant (e.g., Joyal [Joy81] and Bergeron, Labelle, Leroux [BLL98]), whereas in the context of representation theory, ‘$\star$’ is used more frequently. Curiously, Weyl [Wey97, pg. 20] used ‘$\times$’ for representations.

The scalar product of a $c$-variate species $F$ and a $(c+d)$-variate species $G$ with respect to the first $c$ variables is then defined as the $d$-variate species

$$\langle F, G \rangle_{X_1, \ldots, X_c} = (F \star_{X_1, \ldots, X_c} G)|_{X_1=\ldots=X_c=1}.$$

The representation theoretic interpretation of this operation is as follows: let $F$ be an $\mathfrak{g}_p$-module and let $G$ be an $\mathfrak{g}_p \times \mathfrak{g}_q$-bimodule. Consider $F$ and $G$ as two bivariate species, homogeneous of degree $p$ in the first and homogeneous of degree $q$ in the second variable. Then the species $\langle F, G \rangle_X$ is isomorphic (as a linear representation) to the $\mathfrak{g}_q$-module $F \otimes_k \mathfrak{g}_p G$. 


The following fact concerning the scalar product of species will be useful:

**Lemma 2.1** ([Joy86, Section 2.2, Proposition 5]). For any species $F$, we have
\[
\langle F(X), H(X \cdot Y) \rangle_X = F(Y).
\]
Similarly, we have
\[
\langle F(X_1, X_2), H(X_1 \cdot Y_1 + X_2 \cdot Y_2) \rangle_{X_1, X_2} = F(Y_1, Y_2)
\]
for any bivariate species $F$.

**First proof.** The Schur functions are an orthonormal basis and the Cauchy identity is
\[
H(X \cdot Y) = \sum_{\lambda} s_{\lambda}(X) \cdot s_{\lambda}(Y)
\]
Hence $H(X \cdot Y)$ is a reproducing kernel
\[
\langle F(X), H(X \cdot Y) \rangle_X = \sum_{\lambda} \langle F(X), s_{\lambda}(X) \rangle_X s_{\lambda}(Y) = F(Y)
\]
□

**Second proof.** We only prove the first statement, the second follows by the same reasoning. We first note that $F(X) \ast_X H(X \cdot Y) = F(X \cdot Y)$: for any pair of sets $U_1$ and $U_2$, we find that $H(X \cdot Y)[U_1, U_2]$ is the set of bijections between $U_1$ and $U_2$. Therefore $F(X) \ast_X H(X \cdot Y)[U_1, U_2]$ is the set of pairs whose first component is an $F$-structure on $U_1$ and whose second component is a bijection between $U_1$ and $U_2$. This coincides with the set $F(X \cdot Y)[U_1, U_2]$.

By the definition of the scalar product we now have
\[
\langle F(X), H(X \cdot Y) \rangle_X = (F(X) \ast_X H(X \cdot Y))[X=1] = F(X \cdot Y)[X=1] = F(Y).
\]
□

## 3. Cyclic sieving

The cyclic sieving phenomenon was introduced as a combinatorial theory in [RSW04]. Here we review this theory from the perspective of representation theory.

Let $C$ be a finite cyclic group of order $n$ together with a generator $c \in C$. Put $\omega = \exp(2\pi i/n)$. Then we identify the character ring of $C$ with $\mathbb{Z}[q]/\langle q^n - 1 \rangle$. For $0 \leq k \leq n-1$ the character of the representation $c^k \mapsto \omega^{kp}$ is $q^k \in \mathbb{Z}[q]/\langle q^n - 1 \rangle$. In general, the character of a linear representation $\rho: C \to \text{GL}(U)$, $\chi(\rho) \in \mathbb{Z}[q]/\langle q^n - 1 \rangle$, is characterised by
\[
\chi(\rho)(\omega^p) = \text{tr} \rho(c^p).
\]
for $0 \leq p \leq n - 1$.

Let $X$ be a finite set and $c : X \to X$ a bijection of order $n$. Associated to this is $\chi(X) \in \mathbb{Z}[q]/(q^n - 1)$, the character of the linear representation associated to the permutation representation $X$. The characterisation of the character in equation (2) is now written as

$$\chi(X, c)(\omega^k) = \text{fix}(c^k)$$

for $0 \leq k \leq n - 1$.

For example, if $d | n$ then there is a transitive permutation representation of degree $n/d$ and whose stabiliser is the cyclic group of order $d$ generated by $c^{n/d}$. The character of this permutation representation is $(1 - q^n)/(1 - q^d)$.

Put $\chi(X)(q) = \sum_{k=0}^{n-1} a_k q^k$. Then the combinatorial interpretation of the coefficient, $a_k$, is that it is the number of orbits of $C$ whose stabiliser order divides $k$.

A triple $(X, c, P)$ with $P \in \mathbb{Z}[q]$ exhibits the cyclic sieving phenomenon if $P \equiv \chi(X) \mod q^n - 1$.

**Example 3.1.** For each partition $\lambda \vdash r$ put $X_\lambda = S_\lambda \setminus S_r$ where $S_\lambda$ is a Young subgroup. The set $X_\lambda$ has an action of $S_r$ and we take $c_\lambda$ to be the action of long cycle. Then

$$\left( X_\lambda, c_\lambda, \left[ \begin{array}{c} r \\ \lambda \end{array} \right]_q \right)$$

exhibits the cyclic sieving phenomenon where $\left[ \begin{array}{c} r \\ \lambda \end{array} \right]_q$ is the $q$-analogue of the multinomial coefficient.

Our approach to the cyclic sieving phenomenon is based on the fake degree polynomial. For each $r \geq 0$, let $K_r$ be the $\mathbb{Z}$-module of homogeneous symmetric functions of degree $r$. Then, for each $r \geq 0$, $\text{fd}_r : K_r \to \mathbb{Z}[q]$ is a morphism of $\mathbb{Z}$-modules. This is determined by the properties

$$\text{fd}_{r+s}(f \cdot g) = \left[ \begin{array}{c} r + s \\ r \end{array} \right]_q \text{fd}_r(f) \text{fd}_s(g) \quad \text{fd}_r(h_r) = 1$$

where $f \in K_r$, $g \in K_s$ and $h_r$ is the complete homogeneous symmetric function. Alternatively, the fake degree polynomial can be defined in terms of the principal specialisation.

**Theorem 3.2.** For any representation, $\mathcal{S}_r \to \text{GL}(U)$ the restriction to the cyclic subgroup generated by the long cycle has character $\text{fd}_r \left( \text{ch}(U) \right) \mod q^n - 1$.

This can be proved and generalised using Springer’s theory since the symmetric group is a real reflection group and the long cycle is a regular element. Alternatively, there is an elementary proof which deduces the result from Example 3.1. For any partition $\lambda \vdash r$, the characteristic...
map of the linear representation of $X_\lambda$ is $h_\lambda$ and $\text{fd}_r(h_\lambda) = \left[ \lambda \right]_q$. Hence, by Example 3.1, the two linear maps $K_r \to \mathbb{Z}[q]/\langle q^n - 1 \rangle$ agree on a basis and so are equal.

**Corollary 3.3.** Let $U$ be a representation of $\mathfrak{S}_r$ and let $X \subset U$ be a basis which is permuted by the long cycle $c_r$. Take $P = \text{fd}_r(\text{ch}(U))$; then $(X, c, P)$ exhibits the cyclic sieving phenomenon.

The most straightforward application of this theorem is to permutation representations of $\mathfrak{S}_r$. Here the representation and the basis are given so it remains to determine the characteristic map and its fake degree polynomial.

4. **Monoidal categories**

In this section we give a concise account of symmetric monoidal and pivotal categories. There is a profusion of structures on monoidal categories and confusion over the terminology. Here we only discuss the strict versions of these concepts; this is for brevity and simplicity and is justified since all the examples of interest are strict.

Monoidal and symmetric monoidal categories were introduced in [ML63] and a coherence theorem is given in [KL80]. A more recent reference is [Del90, §2]. In this paper we use the diagram calculus for monoidal and symmetric monoidal categories described in [JS91, Sections 1, 2].

**Definition 4.1.** A **strict monoidal category** consists of the data

- A category $C$
- A functor $\otimes: C \times C \to C$
- An object $I \in C$

The functor $\otimes$ is required to be associative. This is the condition that

$$\otimes \circ (\otimes \times \text{id}) = \otimes \circ (\text{id} \times \otimes)$$

as functors $C \times C \times C \to C$. Equivalently, the following diagram commutes:

$$\begin{array}{ccc}
C \times C \times C & \overset{\otimes \times \text{id}}{\longrightarrow} & C \times C \\
\text{id} \times \otimes & \downarrow & \otimes \\
C \times C & \overset{\otimes}{\longrightarrow} & C
\end{array}$$

The object $I$ is a left and right unit. This means that $\text{id}_I \otimes \phi = \phi$ and $\phi \otimes \text{id}_I = \phi$ for all morphisms $\phi$. 

Let $C$ and $C'$ be strict monoidal categories. A functor $F: C \to C'$ is monoidal if $F(I) = I'$ and the following diagram commutes:

$$
\begin{array}{ccc}
C \times C & \xrightarrow{F \times F} & C' \times C' \\
\downarrow & & \downarrow \\
C & \xrightarrow{F} & C'
\end{array}
$$

**Definition 4.2.** A strict symmetric monoidal category is a strict monoidal category $C$ together with natural isomorphisms $\alpha(x, y): x \otimes y \to y \otimes x$ for all objects $x, y$. These are required to satisfy the following conditions:

$$
\alpha(x, y) \circ \alpha(y, x) = \text{id}_{y \otimes x}
$$

for all $x, y$ and

$$
\begin{align*}
\alpha(x, y \otimes z) &= \text{id}_y \otimes \alpha(x, z) \circ \alpha(x, y) \otimes \text{id}_z \\
\alpha(x \otimes y, z) &= \alpha(x, z) \otimes \text{id}_y \circ \text{id}_x \otimes \alpha(y, z)
\end{align*}
$$

for all $x, y, z$.

A symmetric monoidal functor is a functor compatible with this structure.

**Definition 4.3.** Let $x$ and $y$ be objects in a monoidal category. Then $x$ is a dual of $y$ if there are natural isomorphisms

$$
\begin{align*}
\text{Hom}(y \otimes z, w) &\cong \text{Hom}(z, x \otimes w) \\
\text{Hom}(z \otimes x, w) &\cong \text{Hom}(z, w \otimes y)
\end{align*}
$$

**Definition 4.4.** A strict pivotal category is a strict monoidal category with a monoidal antiinvolution $\ast$ such that for all $x$, $x^\ast$ is a dual of $x$.

**Example 4.5.**

- Let $A$ be an algebra. Then the category of $A$-bimodules is a monoidal category.
- Let $A$ be a commutative algebra. Then the category of $A$-modules is a symmetric monoidal category.
- Let $A$ be a commutative algebra. Then the category of finitely generated projective $A$-modules is a symmetric monoidal category and a pivotal category.

Let $x$ be an object in a $K$-linear monoidal category. Then the space of invariant tensors is the $K$-module $\text{Hom}(I, \otimes^n x)$. In a symmetric monoidal category $\otimes^n x$ has an action of $\mathfrak{S}_n$ and so for any $y$, the $K$-module $\text{Hom}(y, \otimes^n x)$ has an action of $\mathfrak{S}_n$. 
If $x$ has a dual $x^*$ then $\text{Hom}(I, \otimes^n x)$ has a natural automorphism $c$ which satisfies $c^n = 1$. This is rotation and is defined as the composite

$$\text{Hom}(I, \otimes^n x) \cong \text{Hom}(x^*, \otimes^{n-1} x)$$

$$\cong \text{Hom}(x^* \otimes I, \otimes^{n-1} x)$$

$$\cong \text{Hom}(I, \otimes^n x)$$

In a category which is both pivotal and symmetric we have two automorphisms of $\text{Hom}(I, x \otimes y)$. The rotation map is defined using the following natural isomorphisms in a pivotal category

$$\text{Hom}(u, x \otimes y) \cong \text{Hom}(x^* \otimes u, y)$$

$$\text{Hom}(u, x \otimes y) \cong \text{Hom}(u \otimes y^*, x)$$

Then rotation of invariant tensors is the composite of the following isomorphisms

$$\text{Hom}(I, x \otimes y) \rightarrow \text{Hom}(y^*, x) \rightarrow \text{Hom}(I, y \otimes x)$$

In a symmetric category we have the action of the symmetry

$$\text{Hom}(I, x \otimes y) \rightarrow \text{Hom}(I, y \otimes x) \quad \phi \mapsto \alpha(x, y)\phi$$

Taking $x = \otimes^{r-1} x$ these are both automorphisms of $\text{Hom}(I, \otimes^r x)$ of order $r$.

**Lemma 4.6.** If $y$ has a non-degenerate symmetric inner product then the two isomorphisms are equal.

*Proof.* For the proof we use diagrammatic notation. In the following diagrams the grey half circle is an invariant tensor and the pink rectangles show the part of the diagram that changes.
The first diagram shows the map given by the symmetry and the final diagram shows the rotation map. The first equation follows since \( y \) has a non-degenerate symmetric inner product. The second equation is the condition that two pairs of tensors commute. The third equation is an application of the condition that in a symmetric monoidal category, given \( \phi: u \rightarrow v \) and an object \( x \) then

\[
\alpha(v, x) \circ (\phi \otimes \text{id}_x) = (\text{id}_x \otimes \phi) \circ \alpha(u, x)
\]

The strategy then for constructing examples of the cyclic sieving phenomenon is the following. Let \( x \) be an object in a symmetric monoidal category with a dual \( x^* \). The combinatorial structure is a basis of \( \text{Hom}(I, \otimes^n x) \) which is preserved by rotation. Then the aim is to apply Theorem 3.2 to obtain an example of the cyclic sieving phenomenon.

5. Diagram categories

In this section we set up a categorical framework that contains the Brauer category and the partition category as special cases. In the following sections we will employ this framework to compute the Frobenius character of tensor powers of the defining representations of the symplectic group and of the symmetric groups when the dimension of the group is large enough. A slight variation allows us to do the same for the adjoint representations of the general linear groups.

Before we give the definitions, we present two fundamental examples.

5.1. The permutation category. The permutation category, \( \mathcal{D}_P \), has as objects the natural numbers \( \mathbb{N} \) and, for two objects \( r \) and \( s \), morphisms

\[
\text{Hom}_{\mathcal{D}_P}(r, s) = \begin{cases} 
K\mathfrak{S}_r & \text{if } r = s \\
\emptyset & \text{if } r \neq s
\end{cases}
\]

The composition of two morphisms is given by group multiplication. The permutation category is a strict symmetric monoidal category, the tensor product is the standard homomorphism \( \mathfrak{S}_r \times \mathfrak{S}_s \rightarrow \mathfrak{S}_{r+s} \). There is also an antiinvolution * of \( \mathfrak{S}_r \) which on the standard generators is given by \( s_i \mapsto s_{n-i} \), but the category is not pivotal, since \( x^* \) is not a dual of \( x \).
The diagram of $\pi \in S_r$ is given by drawing $r$ lines in the rectangle, connecting $r$ marked points on the top edge with $r$ marked points on the bottom edge. For each $i \in [r]$ there is a line with endpoints $i$ and $\pi(i)$. Thus, an alternative way to realise the composition of two morphisms $\pi$ and $\sigma$ is by stacking the diagram of $\pi$ on top of the diagram of $\sigma$ so that the two sets of $r$ endpoints on the middle edge match.

In general, there are many diagrams of a permutation, we consider two diagrams to be equivalent if they represent the same permutation. In particular, we can require without loss of generality that the lines in a diagram are drawn so that all intersections are transversal and any two lines intersect at most once.

The category $D_p$ is generated as a monoidal category by the single element

\begin{equation}
(3)
\end{equation}

This example also illustrates the fundamental theorems. For any vector space $V$ we have a symmetric monoidal functor from the permutation category to vector spaces which on objects is given by $r \mapsto \otimes^r V$. We will refer to this functor as an evaluation functor. On morphisms it is given by the algebra homomorphisms

$ev_n : K\mathfrak{S}_r \rightarrow \text{End}(\otimes^r V)$

These homomorphisms are natural and so can be regarded as homomorphisms

$ev_n : K\mathfrak{S}_r \rightarrow \text{End}_{\text{GL}(V)}(\otimes^r V)$

The first fundamental theorem is that these homomorphisms are surjective.

The second fundamental theorem describes the kernel and the image. For $r \geq 0$, the antisymmetriser $E(r) \in K\mathfrak{S}_r$ is given by

$E(r) = \frac{1}{r!} \sum_{\pi \in \mathfrak{S}_r} \varepsilon(\pi) \pi$

This is the rank one central idempotent corresponding to the sign representation.

Put $n = \text{dim}(V)$. Then $ev_n(E(n+1)) = 0$ and so the evaluation homomorphisms factor through the quotient by the ideal generated by $E(n+1)$ to give

$\overline{ev}_n : K\mathfrak{S}_r / \langle E(n+1) \rangle \rightarrow \text{End}_{\text{GL}(V)}(\otimes^r V)$

The second fundamental theorem is that these are isomorphisms. Furthermore a basis of the image is given by the set

$\{ \overline{ev}_n(\pi) : \pi \in \mathfrak{S}_r, ds(\pi) < (n+1) \}$,
where $ds(\pi)$ is the length of the longest decreasing subsequence of $\pi$. This can be proved using the Yang-Baxter elements

$$R_i(n) = \frac{1}{(n+1)} \left( 1 + ns_i \right)$$

which give a basis of $K\tilde{S}_r$ indexed by permutations.

5.2. **The Temperley-Lieb category.** The objects of the Temperley-Lieb category $D_T$ are again the natural numbers. For $r, s \in \mathbb{N}$, the set of morphisms $\text{Hom}_{D_T}(r, s)$ is the free $K$-module whose basis is the set of noncrossing perfect matchings of $[r] \amalg [s]$. Thus $\text{Hom}_{D_T}(r, s) = \emptyset$ if $r + s$ is odd. If $r + s = 2k$ then $\dim \text{Hom}_{D_T}(r, s)$ is the $k$-th Catalan number.

The diagram of a basis element $x \in \text{Hom}_{D_T}(r, s)$, a noncrossing perfect matching, consists of $(r + s)/2$ nonintersecting arcs in the rectangle. The composition of morphisms is defined on diagrams and then extended bilinearly: let $x \in \text{Hom}_{D_T}(r, s)$ and $y \in \text{Hom}_{D_T}(s, t)$ be two diagrams. Stack the diagram for $x$ on top of the diagram for $y$ so that the two sets of $s$ endpoints on the middle edge match. The resulting diagram possibly has a number of loops, $c(x, y)$. Removing these loops gives the diagram $x \circ y$. The composition of $x$ and $y$ is then defined as

$$x \cdot y = \delta^{c(x, y)} x \circ y.$$

The Temperley-Lieb category is a strict pivotal category. The tensor product $x \otimes y$ of two diagrams is obtained by putting the diagram of $x$ on the left of the diagram of $y$. The dual of an object is the object itself, the required isomorphism between $\text{Hom}_{D_T}(x \otimes z, w)$ and $\text{Hom}_{D_T}(z, x \otimes w)$ is given as follows. Let $n$ be the diagram

$$\begin{array}{c}
\circle \quad x
\end{array}$$

that consists of $x$ nested arcs. Then the map

$$d \mapsto (n \otimes \text{id}_z) \cdot (\text{id}_x \otimes d)$$

is an isomorphism. However, $D_T$ is not symmetric unless $\delta = \pm 2$.

For $\delta = -2$ the symmetric structure is given by the homomorphisms $\tilde{S}_r \to \text{Hom}(r, r)$ given on the standard generators by $\sigma_i \mapsto 1 + u_i$. This gives a representation of $\tilde{S}_r$ on $\text{Hom}(0, r)$ such that the long cycle is rotation. For $r = 2k$, the Frobenius character is the Schur function associated to the partition $(2^k)$. The fake degree of this Schur function is

$$\frac{1}{[k + 1]_q} \left[ \begin{array}{c} 2k \\ k \end{array} \right]_q$$

This is the example of the cyclic sieving phenomenon that is described in [RSW14a].
Moreover, there is an antiinvolution $\ast$, which amounts to rotating the diagram by a half turn.

The category $D_T$ is generated as a monoidal category by the two elements

(5)

\[
\begin{array}{c}
\text{\includegraphics{diagram1.png}} \\
\text{\includegraphics{diagram2.png}}
\end{array}
\]

5.3. The general definition. Let us first define an auxiliary family of categories, the cobordism categories.

**Definition 5.1.** For $r, s \in \mathbb{N}$ let $D(r, s)$ be a finite set, the set of diagrams. For $r, s, t \in \mathbb{N}$, we require two maps:

\[
\circ : D(r, s) \times D(s, t) \to D(r, t) \\
(x, y) \mapsto x \circ y,
\]

the composition of two diagrams, and

\[
c : D(r, s) \times D(s, t) \to \mathbb{N} \\
(x, y) \mapsto c(x, y),
\]

the number of loops that arise when composing two diagrams.

These maps define the cobordism category, $\mathbb{C}$, as follows: the set of objects of $\mathbb{C}$ is $\mathbb{N}$ and the morphisms between $r, s \in \mathbb{N}$ are $\text{Hom}_\mathbb{C}(r, s) = \mathbb{N} \times D(r, s)$. The composition of two morphisms $(a, x) \in \text{Hom}_\mathbb{C}(r, s)$ and $(b, y) \in \text{Hom}_\mathbb{C}(s, t)$ is defined by

\[
(a, x) \cdot (b, y) = (a + b + c(x, y), x \circ y).
\]

To ensure that $\mathbb{C}$ is a category we have to insist that the maps $\circ$ and $c$ are such that the composition is associative, and that for each $p \in \mathbb{N}$ there is a diagram $\text{id}_p \in D(p, p)$ such that $(0, \text{id}_p)$ is the identity morphism.

Furthermore, we require associative maps

\[
\otimes : D(r_1, s_1) \times D(r_2, s_2) \to D(r_1 + r_2, s_1 + s_2) \\
(x, y) \mapsto x \otimes y,
\]

for all $r_1, s_1, r_2, s_2 \in \mathbb{N}$, such that the tensor product on $\mathbb{C}$

\[
(a, x) \otimes (b, y) = (a + b, x \otimes y)
\]

makes the cobordism category into a strict monoidal category.

Finally, we require a monoidal functor $\ast : \mathbb{C} \to \mathbb{C}^{\text{op}}$ which is the identity on objects and satisfies $(x^\ast)^\ast = x$ for all morphisms $x$.

**Remark 5.2.** The composition of morphisms of $\mathbb{C}$ in equation (5) is associative if and only if the composition $\circ$ of diagrams in equation (6) is associative and for any three diagrams $x, y, z$ that can be composed we have

\[
c(x, y) + c(x \circ y, z) = c(x, y \circ z) + c(y, z).
\]
Moreover, the morphisms \((0, \text{id}_p) \in \text{Hom}_C(p,p)\) are identities in \(C\) if and only if the diagrams \(\text{id}_p\) are identities for the composition of diagrams and \(c(\text{id}_r, x) = c(x, \text{id}_s) = 0\). In this case, \(c(\sigma, x) = c(x, \tau) = 0\) whenever \((0, \sigma)\) and \((0, \tau)\) are isomorphisms.

We are now ready to define diagram categories.

Definition 5.3. Let \(K\) be a commutative ring. Given a cobordism category and a fixed parameter \(\delta \in K\), the diagram category \(D\) has objects \(N\). For \(r, s \in \mathbb{N}\), the set of morphisms \(\text{Hom}_D(r, s)\) is the free \(K\)-module with basis \(D(r, s)\). The composition in \(D\), which we will refer to as multiplication henceforth, is defined as

\[
x \cdot y = \delta^{c(x,y)} x \circ y,
\]

for basis elements \(x \in D(r, s), y \in D(s, t)\), and extended \(K\)-bilinearly.

Thus, \(D\) is a \(K\)-linear (or pre-additive) category. The tensor product from \(C\) carries over to \(D\) and makes the diagram category into a strict monoidal category.

Definition 5.4. For any \(r \geq 0\), the diagram algebra, \(D_r\) is the \(K\)-algebra \(\text{Hom}_D(r, r)\).

The diagram algebra is a twisted semigroup algebra in the sense of [Wil07].

We require additionally that \(D\) is symmetric. This symmetric structure arises from a monoidal inclusion of the permutation category in \(C\). This then gives, for each \(r \geq 0\), an inclusion of algebras \(K \mathcal{S}_r \to D_r\).

Also, the functor \(*\) yields an antiinvolution on \(D\). We require additionally that for all \(x\), \(x^*\) is a dual of \(x\), so that \(D\) is a strict pivotal category. Moreover in all our examples the nondegenerate inner products are symmetric. This is required for the cyclic sieving phenomenon.

As already hinted at above, in all our examples the sets \(D(r, s)\), the maps \(\circ\) in equation (6), \(c\) in equation (7), as well as the maps \(\otimes\) and \(*\) have combinatorial meaning.

1. We draw a diagram for \(x \in D(r, s)\) in a rectangle with two vertical sides and two horizontal sides, with \(r\) marked points on the top edge and \(s\) marked points on the bottom edge. Depending on the specific example at hand, some of these points will be connected by strands or some other combinatorial structure.

2. To obtain \(x \circ y\) for \(x \in D(r, s)\) and \(y \in D(s, t)\), we put the diagram for \(x\) on top of the diagram for \(y\) with the \(s\) marked points on the bottom edge of \(x\) matching the \(s\) marked points on the top edge of \(y\). The resulting diagram will have a number of loops (or floating connected components). Let \(c(x, y)\) be the number of loops. Then removing these loops gives \(x \circ y\).

3. The diagram for \(x \otimes y\) is obtained by putting the diagram for \(x\) on the left of the diagram for \(y\) identifying the two horizontal edges.
The diagram for $x^*$ is obtained by rotating the diagram for $x$ through a half turn.

**Example 5.5.** The category of partitioned binary relations in [MM13] is a diagram category.

**Example 5.6.** The diagram algebras in [Li14] are endomorphism algebras in a diagram category.

**Example 5.7.** The walled Brauer-Clifford superalgebras in [BGJ14] are endomorphism algebras in a diagram category.

5.4. **Ideals.** We now define a statistic on diagrams, the propagating number, which allows us to break up diagram categories into more manageable pieces:

**Definition 5.8.** For $r, s \geq 0$ and $z \in D(r, s)$, let

$$\text{pr}(z) = \min \{p \in \mathbb{N} : z = x \circ y \text{ for some } x \in D(r, p) \text{ and some } y \in D(p, s)\}$$

be the *propagating number* of $z$. Let

$$D(r, s; p) = \{x \in D(r, s) | \text{pr}(x) = p\}$$

be the set of diagrams with propagating number equal to $p$.

For example, in the permutation category $D_P$ every element of $S_r$ has propagating number $r$. In the Temperley-Lieb category $D_T$ the propagating number of a diagram is the number of through strands, i.e., the number of lines connecting a point on the top edge of the rectangle to a point on the bottom edge.

**Lemma 5.9.** The propagating number has the following properties:

1. $\text{pr}(x) \leq \min(r, s)$ for all $x \in D(r, s)$.
2. $\text{pr}(x \circ y) \leq \min(\text{pr}(x), \text{pr}(y))$ whenever $x \circ y$ is defined.
3. $\text{pr}(x \otimes y) \leq \text{pr}(x) + \text{pr}(y)$ for all $x, y$.
4. $\text{pr}(x^*) = \text{pr}(x)$ for all $x$.

**Proof.** Claim (1) follows from the existence of identities, since then $x = \text{id}_r \circ x = x \circ \text{id}_s$.

Claim (2) follows from the associativity of $\circ$: if $\text{pr}(x) = p$ then there are diagrams $a \in D(r, p)$ and $b \in D(p, s)$ with $x = a \circ b$ and therefore $x \circ y = a \circ (b \circ y)$.

Claim (3) follows from the functoriality of the tensor product: let $x \in D(r, s)$ and $y \in D(m, n)$. Suppose $\text{pr}(x) = p$ and $\text{pr}(y) = q$. Then there exist diagrams $a \in D(r, p)$, $b \in D(p, s)$, $c \in D(m, q)$, $d \in D(q, n)$ with

$$x = a \circ b \quad \text{and} \quad y = c \circ d.$$

By the functoriality of the tensor product, we have

$$x \otimes y = (a \circ b) \otimes (c \circ d) = (a \otimes c) \circ (b \otimes d).$$
Therefore, \( p(r \otimes y) \leq p + q = pr(x) + pr(y) \).

Claim (4) follows from the functoriality of \( \ast \): suppose that \( x = a \circ b \), then \( x^* = b^* \circ a^* \), so their propagating numbers are the same. \( \Box \)

**Definition 5.10.** A **\( K \)-ideal** \( J \) in a \( K \)-linear category \( D \) is a \( K \)-submodule \( \text{Hom}_J(r, s) \subseteq \text{Hom}_D(r, s) \) for all objects \( r, s \) of \( D \), such that for any morphism in the ideal the left and right composition with a morphism in \( D \) is again in the ideal, whenever the composition is defined in \( D \).

**Definition 5.11.** Let \( J(p) \triangleleft D \) be the \( K \)-ideal where \( \text{Hom}_{J(p)}(r, s) \) is generated by the diagrams with propagating number at most \( p \).

Let \( J_r(p) \) be the ideal of the diagram algebra \( D_r \) generated by the diagrams with propagating number at most \( p \).

This is an ideal by Lemma 5.9, Equation (2). Moreover, by Lemma 5.9, Equation (4), \( J(p) \) is closed under \( \ast \), i.e., \( J(p)^\ast \) is an ideal in \( D^{\text{op}} \).

Let \( D(p) = D/J(p-1) \). That is, the quotient map \( D \to D/J(p-1) \) sends the diagrams with propagating number strictly less than \( p \) to zero, and for \( r, s \in \mathbb{N} \) the diagrams in \( D(r, s) \) with propagating number at least \( p \) to a basis of \( \text{Hom}_{D(p)}(r, s) \).

This has a natural extension.

**Definition 5.12.** Let \( D \) be a monoidal category whose monoid of objects is \( \mathbb{N} \). For \( p \geq 0 \), define an ideal \( J(p) \) by

\[
\text{Hom}_{J(p)}(r, s) = \sum_{k \leq p} \text{Hom}_D(r, k) \otimes \text{Hom}_D(k, s) \subseteq \text{Hom}_D(r, s)
\]

The properties in Lemma 5.9 extend to this setting.

**Example 5.13.** Consider the cobordism category of planar trivalent graphs and the free \( K \)-linear category on this cobordism category. The propagating number is defined using cut paths. A cut path is a path across a rectangular diagram such that each point of intersection is transverse. The propagating number is the minimum of the number of intersection points of a cut path.

This category has an evaluation functor to the category of representations of the exceptional Lie group \( G_2 \). This maps the object \([r] \) to \( \otimes^7 V \) where \( V \) is the seven dimensional fundamental representation. Alternatively, \( G_2 \) is the automorphism group of the octonions and \( V \) is the imaginary octonions. This evaluation functor is full; generators for the kernel and the second fundamental theorem are given in [Kup96]. The algebras \( A_p \) are the Temperley-Lieb algebras.

### 6. Morita Equivalence

The original paper on Morita equivalence is [Mor58]. This paper and subsequent treatments in text books, for example [Bas68, Chapter II], assume that algebras have units and that the unit acts on a module as the identity. Here we present an extension of this theory.
This extension was originally given in [Tay82] and is also presented in [Cae98].

6.1. Regular algebras. Let $K$ be a commutative ring with a unit. We will write $\otimes$ instead of $\otimes_K$ for brevity.

Let $A$ be an associative algebra over $K$, possibly without unit. Let $M$ be a left $A$-module with action $\mu_M : A \otimes M \to M$ and $N$ a right $A$-module with action $\mu_N : N \otimes A \to N$. Then the $R$-module homomorphism $N \otimes M \to N \otimes_A M$ is the coequaliser of

$$\mu_N \otimes \operatorname{id}_M, \operatorname{id}_N \otimes \mu_M : N \otimes A \otimes M \to N \otimes M$$

Equivalently, we have the exact sequence

$$N \otimes A \otimes M \xrightarrow{\mu_N \otimes 1 - 1 \otimes \mu_M} N \otimes M \to N \otimes_A M \to 0$$

For any left $A$-module $M$ the action map $\psi_M : A \otimes M \to M$ factors through $A \otimes_A M$ to give the action map $\psi_M : A \otimes_A M \to M$.

Definition 6.1. A left $A$-module $M$ is regular if the multiplication map $\mu : A \otimes_A M \to M$ is an isomorphism. A similar notion applies to right $A$-modules. An algebra $A$ is regular if it is regular as a left or right $A$-module.

Equivalently, we have the exact sequence

$$A \otimes A \otimes A \xrightarrow{\mu \otimes 1 - 1 \otimes \mu} A \otimes A \xrightarrow{\mu} A \to 0$$

In particular, if $A$ has a unit then it is regular. In this case, an $A$-module is regular if and only if the unit of $A$ acts as the identity map, as shown in [Vit96, Proposition 3.2]. It is sufficient to assume that the action map is surjective. The identity is idempotent and the range is the image of the action map.

Further examples of regular algebras are $H$-unital (or homologically unital) algebras defined in [Wod89].

If $A$ is regular then the category of regular $A$-modules is abelian, see [GV98 Corollary 1.3]. However, there are examples in [GFM07] where the category of regular modules is not abelian.

6.2. Morita contexts. Let $A$ and $B$ be two regular $R$-algebras. The data for a Morita context is two regular bimodules, $A^B_P$ and $B^A_Q$ together with two bimodule homomorphisms $f : P \otimes_B Q \to A$ and $g : Q \otimes_A P \to B$. These are required to satisfy the condition that the following two diagrams commute:

$$
\begin{align*}
\begin{array}{ccc}
P \otimes_B Q & \xrightarrow{\operatorname{id}_P \otimes g} & P \otimes_B B \\
\downarrow & & \downarrow \\
A \otimes_A P & \xrightarrow{f \otimes \operatorname{id}_P} & P \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{ccc}
Q \otimes_A P & \xrightarrow{\operatorname{id}_P \otimes f} & Q \otimes_A A \\
\downarrow & & \downarrow \\
B \otimes_B Q & \xrightarrow{g \otimes \operatorname{id}_P} & Q
\end{array}
\end{align*}
$$

This Morita context is strict if $f$ and $g$ are isomorphisms.
**Lemma 6.2.** Let $A$ and $B$ be two regular $R$-algebras and $\ _AP_B$ a regular bimodule. Then we have a functor $P \otimes_B - : M \mapsto P \otimes_B M$ from the category of regular left $B$-modules to regular left $A$-modules.

**Proof.** This is standard except for the statement that the left $A$-module $P \otimes_B M$ is regular. This follows from the commutativity of the square

\[
\begin{array}{ccc}
A \otimes_A (P \otimes_B M) & \longrightarrow & P \otimes_B M \\
\downarrow & & \downarrow \\
(A \otimes_A P) \otimes_B M & \longrightarrow & P \otimes_B M
\end{array}
\]

The top edge is an isomorphism since the other three edges are known to be isomorphisms and the diagram commutes. □

**Theorem 6.3.** Given a strict Morita context then the functors $P \otimes_B -$ and $Q \otimes_A -$ are inverse equivalences between the category of regular $B$-modules and the category of regular $A$-modules.

**6.3. Idempotents.** This section is based on [Gre80] §6.2.

Let $E$ be an algebra, possibly without unit, and let $e \in E$ be an idempotent of $E$. We specialise the situation of the previous subsection by setting $A = eEe$ and $B = EeE$. Then $A$ has unit $e$ and is therefore regular.

Put $P = eE$ and $Q = Ee$. Then $P$ is an $A - B$ bimodule and $Q$ is a $B - A$ bimodule. Let $f : P \otimes_B Q \to A$ and $g : Q \otimes_A P \to B$ be the multiplication maps.

**Proposition 6.4.** This is a Morita context.

Since the unit $e$ of $A$ acts on $P$ and $Q$ as the identity, $P$ is a regular left $A$-module and $Q$ is a regular right $A$-module. It is clear that the diagrams (11) commute.

To show that this is a Morita context, it remains to show that the following surjective homomorphisms of bimodules are isomorphisms:

\[
(12) \quad P \otimes_B B \to P \quad B \otimes_B Q \to Q \quad B \otimes_B B \to B,
\]

that is, we have to show that $P$, $Q$ and $B$ are regular. However, it is more convenient to show first that $f$ is an isomorphism.

**Lemma 6.5.** The map $f : P \otimes_B Q \to A$ is an isomorphism.

**Proof.** This follows from the observation that $f(e \otimes e) = e$ which is the identity of $A$ and the commutativity of the diagrams (11).
Assume $f(\sum_i p_i \otimes q_i) = 0$. 

$$\sum_i p_i \otimes q_i = \sum_i p_i \otimes q_i \cdot f(e \otimes e) = \sum_i p_i \otimes g(q_i \otimes e) \cdot e = \sum_i f(p_i \otimes q_i) \otimes e = 0$$

□

**Lemma 6.6.** $P$ is a regular $B$-module, that is, the map $P \otimes_B B \to P$ is an isomorphism.

*Proof.* This follows from commutativity of the square

$$
\begin{array}{ccc}
P \otimes_B Q \otimes_A P & \longrightarrow & A \otimes_A P \\
\downarrow & & \downarrow \\
P \otimes_B B & \longrightarrow & P
\end{array}
$$

The diagram commutes and the top and right edges are known to be isomorphisms. It follows that the right and bottom edges are isomorphisms. This shows that the following two maps are isomorphisms

$$P \otimes_B Q \otimes_A P \to P \otimes_B B \quad P \otimes_B B \to P$$

□

**Lemma 6.7.** $Q$ is a regular $B$-module, that is, the map $B \otimes_B Q \to Q$ is an isomorphism.

*Proof.* This follows from commutativity of the square

$$
\begin{array}{ccc}
Q \otimes_A P \otimes_B Q & \longrightarrow & Q \otimes_A A \\
\downarrow & & \downarrow \\
B \otimes_B Q & \longrightarrow & Q
\end{array}
$$

The diagram commutes and the top and right edges are known to be isomorphisms. It follows that the right and bottom edges are isomorphisms. This shows that the following two maps are isomorphisms

$$Q \otimes_A P \otimes_B Q \to B \otimes_B Q \quad B \otimes_B Q \to Q$$

□

**Lemma 6.8.** $B$ is a regular algebra, that is, the map $B \otimes_B B \to B$ is an isomorphism.

*Proof.* This follows from commutativity of the square

$$
\begin{array}{ccc}
Q \otimes_A P \otimes_B Q \otimes_A P & \longrightarrow & Q \otimes_A A \otimes_A P \\
\downarrow & & \downarrow \\
B \otimes_B B & \longrightarrow & B
\end{array}
$$
The bottom edge is an isomorphism since the other three edges are known to be isomorphisms and the diagram commutes. □

This completes the proof of Proposition 6.4. This does not show that we have a strict Morita context. The missing property is that \( g : Q \otimes_A P \rightarrow B \) is an isomorphism.

6.4. Inflation. In this section we specialise the setup of the previous section to diagram categories. We specify two assumptions which are will be demonstrated in each of the examples separately.

Let \( E = D_r/J_r(p-1) \) be the algebra of diagrams with propagating number at least \( p \). As before, let \( e_p \in E \) an idempotent and set \( A = A_p = e_pEe_p \) and \( B = B_p = Ee_pE = J_r(p)/J_r(p-1) \).

**Assumption 1.** For \( 0 \leq p \leq r \) the idempotent \( e_p \) is such that

\[
A_P = e_pE = KD(p,r;p),
\]

\[
B_Q = Ee_p = KD(r,p;p),
\]

where \( K \) is the ground field.

**Assumption 2.** For all \( r,s \geq p \geq 0 \), there are subsets \( \bar{D}(r,p;p) \subseteq D(r,p;p) \) and \( \bar{D}(p,s;p) \subseteq D(p,s;p) \) such that the composition map

\[
\circ : D(r,p;p) \times D(p,r;p) \times \bar{D}(p,s;p) \rightarrow D(r,s;p)
\]

is a bijection.

**Lemma 6.9.** Given Assumptions 1 and 2 the Morita context is strict.

**Proof.** We have to show that \( g : Q \otimes_A P \rightarrow B_p \) is an isomorphism.

Since the composition map is a bijection we have \( A_P \cong A_p \otimes KD(p,r;p) \) and \( Q_A \cong K\bar{D}(r,p;p) \otimes A_p \). Hence

\[
Q \otimes_A P \cong (KD(r,p;p) \otimes A_p) \otimes_{A_p} (A_p \otimes K\bar{D}(p,r;p))
\]

\[
\cong K\bar{D}(r,p;p) \otimes A_p \otimes K\bar{D}(p,r;p)
\]

\[
\cong B.
\]

Thus Theorem 6.3 applies. □

It will be useful to introduce the following notation for the functor realising the equivalence.

**Definition 6.10.** The inflation functor \( \text{Inf}^r_p \) is the equivalence from regular \( A_p \)-modules to regular \( J_r(p)/J_r(p-1) \)-modules

\[
V \mapsto V \otimes_{A_p} KD(p,r;p).
\]

**Remark 6.11.** When \( A_p = K\mathcal{S}_p \) the restriction of inflation to \( \mathcal{S}_r \) can be interpreted as an operation on species. In this case we can consider \( V \) as a species of sort \( X \) and \( D(p,r;p) \) as a bivariate species of sorts \( X \) and \( Y \). Then we have

\[
\text{Inf}^r_p V = \langle V(X), D(p,r;p) \rangle_X.
\]
Proposition 6.12. Suppose that, for \(0 \leq p \leq r\), the algebras \(A_p\) are direct sums of matrix algebras. Let \(A_p\) be the set of irreducible representations of \(A_p\). Assume that \(D_r\) is semisimple and that Assumption 2 holds. Then \(D_r\) is a direct sum of matrix algebras and \(A = \{\text{Inf}^r_p V : V \in A_p, 0 \leq p \leq r\}\) is a complete set of inequivalent irreducible representations of \(D_r\).

Proof. The semisimplicity of \(D_r\) is equivalent to the condition that for any \(A_p\)-module \(U\) and any \(A_q\)-module \(V\)

\[
\text{Hom}_{D_r}(\text{Inf}^r_p(V), \text{Inf}^r_q(U)) = 0 \quad \text{if } p \neq q.
\]

Thus the elements of \(A\) are all inequivalent.

Then we have a surjective algebra homomorphism

\[
D_r \rightarrow \bigoplus_{p, V \in A_p} \text{End}(\text{Inf}^r_p V)
\]

Therefore, it is sufficient to show that these two algebras have the same dimension.

Using Assumption 2,

\[
\text{Inf}^r_p(V) \cong V \otimes_{A_p} K\tilde{D}(p, r; p)
\]

and so \(\dim \text{Inf}^r_p(V) = \dim V \cdot \dim K\tilde{D}(p, r; p)\).

Using Assumption 2 again, we obtain

\[
\sum_{p=0}^{r} \sum_{V \in A_p} (\dim \text{Inf}^r_p(V))^2 = \sum_{p=0}^{r} \sum_{V \in A_p} (\dim V)^2 (\dim K\tilde{D}(p, r; p))^2
\]

\[
= \sum_{p=0}^{r} \dim A_p (\dim K\tilde{D}(p, r; p))^2
\]

\[
= \sum_{p=0}^{r} \dim K\tilde{D}(r, r; p)
\]

\[
= \dim D_r.
\]

Remark 6.13. Assume that \(A_p\) is a cellular algebra for all \(p \geq 0\) as defined in [GL96] then \(D_r\) is a cellular algebra for all \(r \geq 0\). The proof is inductive with inductive step given by [KX99].

The class of cellularly stratified algebras is introduced in [HHKP10]. The diagram algebras are cellularly stratified since Assumption 2 implies that they satisfy [HHKP10] Definition 2.1.
7. The Brauer category

In this section we discuss the representation theory of the symplectic groups using the combinatorics of perfect matchings and the Brauer category. The Brauer category was introduced for this purpose in [Bra37].

7.1. Diagram category. For $r, s \geq 0$, let $D_{\text{Br}}(r, s)$ be the finite set of perfect matchings on $[r] \sqcup [s]$. In particular, $D_{\text{Br}}(r, s) = \emptyset$ if $r + s$ is odd. An element of $D_{\text{Br}}(r, s)$ is visualised as a set of $(r + s)/2$ strands drawn in a rectangle, with $r$ endpoints on the top edge of the rectangle and $s$ endpoints on the bottom edge.

The Brauer category $C_{\text{Br}}$ is the cobordism category with

$$\text{Hom}_{C_{\text{Br}}}(r, s) = \mathbb{N} \times D_{\text{Br}}(r, s).$$

The topological interpretation of this category is that it is equivalent to the cobordism category whose objects are 0-manifolds and whose morphisms are 1-manifolds.

There are inclusions $C_P \to C_{\text{Br}}$ and $C_T \to C_{\text{Br}}$. Furthermore $C_{\text{Br}}$ is generated as a category by these two subcategories. It follows that $C_{\text{Br}}$ is generated as a monoidal category by the morphisms (3) and (5).

There are three types of strands:

(13)

A strand of the first type is called a through strand. The number of through strands in a diagram $x$ is $\text{pr}(x)$, the propagating number of $x$.

We denote the diagram category corresponding to the cobordism category with $D_{\text{Br}(s)}$.

7.2. Frobenius characters for diagram algebras. In this section we employ the results of Section [6.3] and elementary considerations on diagrams to compute the Frobenius characters of the restriction of modules of the diagram algebras to modules of the symmetric group.

In order to apply the results of Section [6.3] we have to provide a decomposition for the diagrams in $D(r, s; p)$.
Lemma 7.2.1. Any Brauer diagram, \( x \in D(r,s;p) \), has a unique decomposition as

\[
\begin{array}{c}
\text{\( p \)} \\
\hline
\text{\( r-p \)}
\end{array}
\quad
\begin{array}{c}
\text{\( p \)} \\
\hline
\text{\( s-p \)}
\end{array}
\]

The two pale grey rectangles are a \((p,r-p)\)-shuffle and a \((p,s-p)\)-shuffle, the dark grey square is a permutation on \( p \) strands and the two pink half circles are perfect matchings on \( r-p \) and on \( s-p \) points.

Proof. Take the strands of each type. The propagating strands give the permutation and the other two types give the two perfect matchings. Then the two shuffles arrange the boundary points in the prescribed order. \( \square \)

Definition 7.2.2. For \( r \geq p \geq 0 \), let \( \overline{D}(p,r;p) \) be the set of diagrams of the form

\[
\begin{array}{c}
\text{\( p \)} \\
\hline
\text{\( r-p \)}
\end{array}
\quad
\begin{array}{c}
\text{\( p \)} \\
\hline
\text{\( r-p \)}
\end{array}
\]

and define \( \overline{D}(r,p;p) \) in an analogous way.

Corollary 7.2.3. For all \( r, s \geq p \geq 0 \), the composition map

\[ \circ : \overline{D}(r,p;p) \times D(p,p;p) \times \overline{D}(p,s;p) \rightarrow D(r,s;p) \]

is a bijection.

Let \( e_p \in D_r \) be the idempotent \( \delta^{-(r-p)/2} \text{id}_p \otimes u \), where \( \text{id}_p \otimes u \) is the diagram

\[
\begin{array}{c}
\text{\( p \)} \\
\hline
\text{\( (r-p)/2 \)}
\end{array}
\]

(14)

Then, by Lemma 6.9 we have a strict Morita context and inflation is an equivalence from \( K \otimes \mathfrak{S}_p \)-modules to regular \( J_r(p)/J_r(p-1) \)-modules.
We are now ready to prove the main theorem of this section:

**Theorem 7.2.4.** Let $V$ be an $\mathcal{S}_p\mathcal{S}_r$-module. Then $(\text{Inf}^r_p V) \downarrow^{D_r} \mathcal{S}_r$ is isomorphic to $V \cdot M_{r-p}$, where $M_{r-p}$ is the species of perfect matchings of sets of cardinality $r-p$.

**First proof.** Starting with the tensor product definition of induction, we have:

\[
(V \otimes K M_{r-p}) \downarrow^{\mathcal{S}_p \otimes \mathcal{S}_{r-p}} \cong V \otimes K M_{r-p} \otimes_{K \mathcal{S}_p \otimes K \mathcal{S}_{r-p}} K \mathcal{S}_r
\]

\[
\cong V \otimes_{K \mathcal{S}_p} [(K \mathcal{S}_p \otimes K M_{r-p}) \otimes_{K \mathcal{S}_p \otimes K \mathcal{S}_{r-p}} K \mathcal{S}_r]
\]

\[
\cong V \otimes_{K \mathcal{S}_p} K \mathcal{D}(p, r; r)
\]

Let us now provide an alternative proof. This proof also shows that when $V$ is a permutation representation, the isomorphism is an isomorphism of permutation representations.

**Second proof.** Consider the bivariate combinatorial species $H(X \cdot Y) \cdot M(Y)$, where $H$ is the species of sets and $M$ is the species of perfect matchings. Informally, this is the species that matches the set of labels of sort $X$ with a subset of the labels of sort $Y$, and puts a perfect matching on the remaining labels of sort $Y$.

It follows immediately from Corollary 7.2.3 (upon setting $r = p$ and $s = r$) that the restriction of this species to sets of sort $X$ is isomorphic to the $\mathcal{S}_p \times \mathcal{S}_r$-bimodule $K \mathcal{D}(p, r; p)$ regarded as a bivariate species.

As a consequence of the linearity of the scalar product and Lemma 2.1, we have

\[
\langle V, H(X \cdot Y) \cdot M(Y) \rangle_X = V \cdot M.
\]

Restricting to the homogeneous component of degree $r$ we obtain the claim via remark 6.11.

**7.3. Branching rules.** In this section we determine branching rules for the diagram algebras, assuming their semisimplicity. This explains the relevance of oscillating tableaux. This result is also proved in [Wen88].

For $r \geq 0$ we have an inclusion $D_r \to D_{r+1}$. Here we study the decomposition of the irreducible $D_{r+1}$-modules restricted to a $D_r$-module.

**Proposition 7.3.1.** For $r \geq p \geq 0$ and $V$ any $A_p$-module,

\[
\text{Inf}^r_{p+1}(V) \downarrow_{D_{r+1}}^{A_{r+1}} \cong \text{Inf}^r_{p+1}(V \uparrow_{A_r}^{A_{p+1}}) \oplus \text{Inf}^r_{p-1}(V \uparrow_{A_r}^{A_{p-1}})
\]
Proof. Let $V$ be a representation of $A_p = K\mathfrak{S}_p$. Then we consider $V \otimes_{A_p} D(p, r + 1; p)$ restricted to $D_r$. This is given by

$$V \otimes_{A_p} \left( K D(p, r + 1; p) \downarrow_{D_r}^{D_{r+1}} \right)$$

The first observation is that $KD(p, r + 1; p) \downarrow_{D_r}^{D_{r+1}}$ has a decomposition as a $A_p$-$D_r$ bimodule into two summands corresponding to a partition of the diagrams into two disjoint sets. The diagrams are partitioned according to whether the last point on the bottom row is matched with a point on the bottom row or on the top row.

The $A_p$-$D_r$ bimodule generated by the diagrams whose last point on the bottom row is matched with a point on the bottom row is isomorphic to $KD(p + 1, r; p + 1) \uparrow_{\mathfrak{S}_{p+1}}^{\mathfrak{S}_p}$. Informally, the isomorphism can be described as moving the last point from the bottom row to the top row:

Now we have

$$V \otimes_{A_p} (KD(p + 1, r; p + 1) \downarrow_{A_p}^{A_{p+1}}) \cong (V \uparrow_{A_p}^{A_{p+1}}) \otimes_{A_{p+1}} K D(p + 1, r; p + 1)$$

as $D_r$-modules: apply associativity to the tensor product

$$V \otimes_{A_p} A_{p+1} \otimes_{A_{p+1}} K D(p + 1, r; p + 1)$$

and recall that induction is defined as $V \uparrow_{A_p}^{A_{p+1}} = V \otimes_{A_p} A_{p+1}$.

Similarly we have

$$V \otimes_{A_p} (KD(p - 1, r; p - 1) \uparrow_{A_{p-1}}^{A_p}) \cong (V \downarrow_{A_{p-1}}^{A_p}) \otimes_{A_{p-1}} K D(p - 1, r; p - 1)$$

as $A_{p-1}$-modules by applying associativity to the tensor product

$$V \otimes_{A_p} A_p \otimes_{A_{p-1}} K D(p - 1, r; p - 1).$$

□
Corollary 7.3.2. Let $\lambda \vdash p$ and put $U(r, \lambda) = \text{Inf}_p^r(S^\lambda)$, where $S^\lambda$ is the irreducible representation of $A_p$ corresponding to $\lambda$. Then the branching rules are given by

$$U(r + 1, \lambda) \downarrow^{D_{r+1}}_{D_r} \cong \bigoplus_{\mu = \lambda \pm \Box} U(r, \mu).$$

Proof. By Proposition 6.12 \{U(r, \lambda) : \lambda \vdash p, 0 \leq p \leq r\} is a complete set of inequivalent irreducible representations of $D_r$. From Proposition 7.3.1 it follows that

$$\text{Hom}_{D_r} \left( \text{Inf}_p^{r+1}(V) \downarrow^{D_{r+1}}_{D_r}, \text{Inf}_q^r(U) \right) \cong \text{Hom}_{D_r} \left( \text{Inf}_p^{r+1}(V) \uparrow^{A_{p+1}}_{A_p}, \text{Inf}_q^r(U) \right) \cong \begin{cases} \text{Hom}_{A_{p+1}} \left( V \uparrow^{A_{p+1}}_{A_p}, U \right) & \text{if } q = p + 1 \\ \text{Hom}_{A_{p-1}} \left( V \downarrow^{A_{p-1}}_{A_p}, U \right) & \text{if } q = p - 1 \\ 0 & \text{otherwise,} \end{cases}$$

where for the second isomorphism we used that inflation is a fully faithful functor. Now let $U$ and $V$ be irreducible representations of $A_p \cong K \mathfrak{S}_p$. Then the statement follows from the branching rules for the symmetric group.

7.4. Fundamental theorems. The theorems of this section connect the representation theory of the diagram algebras with the representation theory of the tensor powers of the defining representation of the symplectic group by providing an explicit isomorphism of categories.

Let $K$ be a field of characteristic zero and let $V$ be a vector space over $K$ of dimension $2n$, equipped with a non-degenerate skew-symmetric bilinear form $\langle \ , \ \rangle$. The symplectic group $\text{Sp}(V)$ is the group of linear transformations that preserve this form. In fact, it is more convenient to take $V$ to be an odd vector space (as a super vector space) with a non-degenerate symmetric bilinear form. Let $\{b_1, \ldots, b_{2n}\}$ be a basis of $V$ and let $\{\bar{b}_1, \ldots, \bar{b}_{2n}\}$ be the dual basis, i.e., $\langle b_i, b_j \rangle = \delta_{i,j}$.

Let $\mathcal{D}_{\text{Sp}(2n)}$ be the diagram category corresponding to $C_{Br}$ with $\delta = -2n$ and let $\mathcal{T}_{\text{Sp}(2n)}$ be the category of invariant tensors of $\text{Sp}(V)$.

The connection between $\mathcal{D}_{\text{Sp}(2n)}$ and $\mathcal{T}_{\text{Sp}(2n)}$ is established by a functor we now define:

Definition 7.4.1. Let $\text{ev}_{\text{Sp}(2n)}$ be the symmetric and pivotal monoidal functor $\mathcal{D}_{\text{Sp}(2n)} \to \mathcal{T}_{\text{Sp}(2n)}$. Explicitly, $\text{ev}_{\text{Sp}(2n)}$ sends the object $r \in \mathbb{N}$
to $\otimes V$ and is defined on the generators by

\[
ev_{\text{Sp}(2n)}(\begin{array}{c}
\bigcirc
d\end{array}) = u \otimes v \mapsto v \otimes u
\]

\[
ev_{\text{Sp}(2n)}(\begin{array}{c}
\bigcirc
d\end{array}) = 1 \mapsto \sum_i b_i \otimes \bar{b}_i
\]

\[
ev_{\text{Sp}(2n)}(\begin{array}{c}
\bigcirc
d\end{array}) = u \otimes v \mapsto \langle u, v \rangle.
\]

We remark that the first equality is forced by the requirement that $\ev_{\text{Sp}(2n)}$ is a symmetric functor, given that $V$ is an odd vector space. The other two equalities are forced by the requirement that $\ev_{\text{Sp}(2n)}$ is pivotal.

The first fundamental theorem for the symplectic group can now be stated as follows:

**Theorem 7.4.2.** [Bra37, Wey97, Theorem (6.1A)] For all $n > 0$ the functor $\ev_{\text{Sp}(2n)}: D_{\text{Sp}(2n)} \to T_{\text{Sp}(2n)}$ is full.

In the remainder of this section we provide an explicit description of the kernel of $\ev_{\text{Sp}(2n)}$ as an ideal in the diagram category. We will denote this ideal with $\text{Pf}(n)$ because of its intimate connection to the Pfaffian. Moreover we obtain a simple basis for the vector space $\text{Hom}_{D_{\text{Sp}(2n)}}/\text{Pf}(n) (0, r)$. This basis is preserved by rotation, which we will use in Section 7.6 to exhibit a cyclic sieving phenomenon.

The fundamental object involved is an idempotent $E(n+1)$ of the diagram algebra $D_{n+1}$ for which we have $\ev_{\text{Sp}(2n)}(E(n+1)) = 0$. This element can be characterised as follows.

Let $\rho$ be the one dimensional representation of $D_{n+1}$ which on diagrams is given by

\[
\rho(x) = \begin{cases} 
1 & \text{if pr}(x) = n + 1 \\
0 & \text{if pr}(x) < n + 1
\end{cases}
\]

The element $E(n+1)$ of $D_{n+1}$ is determined, up to scalar multiple, by the properties

(15) \[ xE(n+1) = \rho(x)E(n+1) = E(n+1)x \]

It follows that $E(n+1)$ can be scaled so that it is idempotent and these properties now determine $E(n+1)$. It is clear that $E(n+1)$ is a central idempotent and that its rank equals one.

We will give two constructions of $E(n+1)$. The first is as a simple linear combination of diagrams.

**Definition 7.4.3.**

\[
E(n+1) = \frac{1}{(n+1)!} \sum_{x \in D(n+1, n+1)} x
\]
**Definition 7.4.4.** For $1 \leq i \leq n$ define $u_i \in D(n+1, n+1)$ to consist of the pairs $(a, a')$ for $a \notin \{i, i+1\}$ together with the pairs $(i, i+1)$ and $(i', (i+1)')$ and define $s_i \in D(n+1, n+1)$ to consist of the pairs $(a, a')$ for $a \notin \{i, i+1\}$ together with the pairs $(i, (i+1)')$ and $(i', i+1)$:

\[
\begin{align*}
    &u_i = \\
    &\begin{array}{c}
        i - 1 \\
        n - i - 1
    \end{array}
\end{align*}
\]

\[
\begin{align*}
    &s_i = \\
    &\begin{array}{c}
        i - 1 \\
        n - i - 1
    \end{array}
\end{align*}
\]

**Lemma 7.4.5.** The element $E(n+1)$ in Definition 7.4.3 satisfies the properties \[15\].

**Proof.** If $\text{pr}(x) = n+1$ then $x$ acts as a permutation on the set of diagrams $D(n+1, n+1)$, so this case is clear.

The elements $u_i$, $1 \leq i \leq n$ generate the ideal $J_{n+1}(n)$ so it is sufficient to show that $u_i E(n+1) = 0$ and $E(n+1) u_i = 0$ for $1 \leq i \leq n$.

We will now show that $u_i E(n+1) = 0$. The case $E(n+1) u_i = 0$ is similar.

It is clear that $u_i E(n+1)$ is a linear combination of diagrams which contain the pair $(i, i+1)$. Hence it is sufficient to show that the coefficient of each of these diagrams is 0. Let $x$ be a diagram which contains the pair $(i, i+1)$. The set $\{y \in D(n+1, n+1) : u_i y = x\}$ has cardinality $2n$ and there is precisely one diagram $z$ with $u_i z = \delta x$. Therefore the coefficient of $x$ is $\delta + 2n$ and so for $\delta = -2n$ this coefficient is 0.

The set of $2n$ diagrams is constructed as follows. The diagram $x$ contains $n$ pairs other than $(i, i+1)$. For each of these pairs we construct two elements of the set. Take the pair $(u, v)$, and replace the two pairs $(i, i+1)$ and $(u, v)$ by $(i, u)$ and $(i+1, v)$ and by $(i+1, u)$ and $(i, v)$ keeping all the remaining pairs.

However it is not clear from this construction that $\ev_{\text{Sp}(2n)} \left( E(n+1) \right) = 0$. We now give an alternative construction which makes this clear.

**Definition 7.4.6.** For $k \in \mathbb{Z}$ and $1 \leq i \leq n$ define $R_i(k) \in D_{n+1}$ by

\[
R_i(k) = \frac{1}{k+1} \left( 1 + ks_i - \frac{2k}{\delta + 2k - 2} u_i \right)
\]

**Proposition 7.4.7.** These elements satisfy

\[
R_i(h) R_{i+1}(h+k) R_i(k) = R_{i+1}(h) R_i(h+k) R_{i+1}(h)
\]

\[
R_i(h) R_j(k) = R_j(k) R_i(h) \quad \text{for } |i - j| > 1
\]

**Proof.** The second relation is clear. The first relation is known as the Yang-Baxter equation and is checked by a direct calculation. This calculation can be carried out using generators and relations or by using a faithful representation. □
Next we construct an element of $D_{n+1}$ for each reduced word in the standard generators of $\mathfrak{S}_{n+1}$. Let $s_i s_j \ldots s_i$ be a reduced word. Then the associated element of $D_{n+1}$ is of the form

$$R_{i_1}(k_1) R_{i_2}(k_2) \ldots R_{i_l}(k_l)$$

The integers $k_1, \ldots, k_l$ are determined as follows. First draw the string diagram of the reduced word, where each letter of the reduced word corresponds to a crossing of two adjacent strings. Since the word is reduced, any two strings can cross at most once. We number the crossings by their corresponding position in the reduced word, and we label each string with the number of its starting point on the top of the diagram. For the $p$-th crossing let the numbers of the two strings be $a_p$ and $b_p$. Then we have $a_p < b_p$ for a reduced diagram and we put $k_p = b_p - a_p$.

**Example 7.4.8.** The reduced word $s_3 s_5 s_4 s_5 s_3 s_6 s_5$ gives the element

$$R_3(1) R_5(1) R_4(3) R_5(2) R_6(2) R_5(4) R_5(2)$$

The following is Matsumoto’s theorem, [Mat64].

**Proposition 7.4.9.** Two reduced words represent the same permutation if and only if they are related by a finite sequence of moves of the form $s_i s_{i+1} s_i \rightarrow s_{i+1} s_i s_{i+1}$, $s_i s_{i+1} s_i \rightarrow s_i s_{i+1} s_i$ and $s_i s_j \rightarrow s_j s_i$ for $|i - j| > 1$.

**Lemma 7.4.10.** If two reduced words represent the same permutation then the associated elements of $D_{n+1}$ are the same.

**Proof.** By Proposition 7.4.9 it is sufficient to show that the element is not changed when a reduced word is changed by one of these moves. This follows from Proposition 7.4.7. □

**Definition 7.4.11.** Let $E^\delta(n + 1)$ be the element of $D_{n+1}$ associated to the longest length permutation.

**Lemma 7.4.12.** $E^\delta(n + 1)$ satisfies the properties

$$xE^\delta(n + 1) = \rho(x) E^\delta(n + 1) = E^\delta(n + 1)x.$$  

In particular, $E^\delta(n + 1)$ is idempotent and for $\delta = -2n$ we have $E^\delta(n + 1) = E(n + 1)$.

**Proof.** Choose a reduced word for the longest length permutation that begins with $s_j$. Then $E^\delta(n + 1)$ regarded as a word in the $R_i(k)$ begins with $R_j(1)$, because the strings $i$ and $i+1$ cross first. Since $s_j R_j(1) = R_j(1)$ and $u_i R_j(1) = 0$ we have $xE^\delta(n + 1) = \rho(x) E^\delta(n + 1)$. The second equation is proved similarly, noting that a reduced word for the longest length permutation that ends with $s_j$ leads to a word in the $R_i(k)$ that ends with $R_j(1)$. □
Proposition 7.4.13.

\[ \text{ev}_{\text{Sp}(2n)}(E(n+1)) = 0 \]

Proof. Because \( \text{ev}_{\text{Sp}(2n)} \) is a functor, \( \text{ev}_{\text{Sp}(2n)}(E(n+1)) \) is an idempotent in \( T_{\text{Sp}(2n)} \), too. Moreover, \( \text{ev}_{\text{Sp}(2n)} \) is pivotal and therefore preserves the trace of morphisms. Recall that the (diagrammatic) trace \( \text{tr}_n \) of a diagram \( \alpha \) in \( D_n \) is defined as

\[ \eta_{2n} \cdot (\alpha \otimes \text{id}_n) \cdot \eta_{2n}^* \]

where \( \eta_{2n} \in D(0, 2n) \) is the diagram that consists of \( n \) nested arcs. The following properties are easily verified.

\[ \text{tr}_{n+1}(\alpha \otimes \text{id}_1) = \delta \text{tr}_n \alpha \]

\[ \text{tr}_{n+1} \alpha s_n \beta = \text{tr}_n \alpha \beta \]

\[ \text{tr}_{n+1} \alpha u_n \beta = \text{tr}_n \alpha \beta \]

for \( \alpha, \beta \) diagrams on \( n \) strings.

The rank of an idempotent is equal to its trace so it is sufficient to show that \( \text{tr}_{n+1} E^\delta(n+1) = 0 \) for \( \delta = -2n \).

A particular choice of the reduced word for the longest length permutation yields

\[ E^\delta(n+1) = R_1(1) \ldots R_n(n) E^\delta(n). \]

We now compute the trace by expanding \( R_n(n) = \frac{1}{n+1} (1 + ns_n + nu_n) \):

\[ \text{tr}_{n+1} E^\delta(n+1) = \frac{1}{n+1} (\delta + n + n) \text{tr}_{n+1} R_1(1) \ldots R_{n-1}(n-1) E^\delta(n). \]

Substituting \(-2n\) for \( \delta \) gives \( \text{tr}_{n+1} E(n+1) = 0. \) \( \square \)

We will now describe the kernel of \( \text{ev}_{\text{Sp}(2n)} \) in terms of diagrammatic Pfaffians.

Definition 7.4.14 ([Gav99, Definition 3.4 (b)]). Let \( S \) be a \( 2(n+1) \)-element subset of \( r \sqcup [s] \) and let \( f \) be a perfect matching of \( r \sqcup [s] \setminus S \). Then the diagrammatic Pfaffian of order \( 2(n+1) \) corresponding to \( f \) is the element of the diagram algebra \( D_r(-2n) \)

\[ \text{Pf}(f) = \sum_{s \text{ a perfect matching of } S} (s \cup f). \]

Note that for \( r, s = n+1 \) and \( f = \emptyset \) we have, by Definition 7.4.3,

\[ E(n+1) = \frac{1}{(n+1)!} \text{Pf}(\emptyset). \]

Definition 7.4.15. For \( r, s \geq 0 \), the subspace \( \text{Pf}^{(n)}(r, s) \) in the ideal \( \text{Pf}^{(n)} \) of \( D_{\text{Sp}(2n)} \) is spanned by the set

\[ \text{Pf}^{(n)}(r, s) = \{ \text{Pf}(f) : f \text{ a perfect matching of a subset of } [r] \sqcup [s] \}. \]

Example 7.4.16. For \( n = 1 \) we have 2-noncrossing diagrams which means noncrossing or planar. The basic Pfaffian, \( \text{Pf}(\emptyset) \) is
For $\delta = -2$, the quotient of the Brauer category by this relation is the Temperley-Lieb category.

The ideal $\text{Pf}^{(n)}$ is also characterised as the pivotal symmetric ideal generated by $\text{Pf}(\emptyset)$. It then follows from Proposition 7.4.13 that we have a pivotal symmetric quotient functor

$$\mathcal{D}_{\text{Sp}(2n)} \to \mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}}$$

**Definition 7.4.17.** Let $\overline{ev}_{\text{Sp}(2n)} : \mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}} \to \mathcal{T}_{\text{Sp}(2n)}$ be the functor that factors $ev_{\text{Sp}(2n)}$ through the quotient.

We can now state the main theorem of this section, also known as the second fundamental theorem for the symplectic group:

**Theorem 7.4.18.** The functor $\overline{ev}_{\text{Sp}(2n)} : \mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}} \to \mathcal{T}_{\text{Sp}(2n)}$ is an isomorphism of categories.

**Proof.** Since $\overline{ev}_{\text{Sp}(2n)}$ is obviously bijective on objects and full by the first fundamental theorem, it is sufficient to show

$$\dim \text{Hom}_{\mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}}}(r, s) \leq \dim \text{Hom}_{\mathcal{T}_{\text{Sp}(2n)}}(r, s).$$

This is achieved by combining Lemma 7.4.19, Lemma 7.4.21, Theorem 7.4.23 and Lemma 7.4.24 below. □

We first restrict our attention to the invariant tensors:

**Lemma 7.4.19.** We have isomorphisms of vector spaces

$$\text{Hom}_{\mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}}}(r, s) \cong \text{Hom}_{\mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}}}(0, r + s)$$

$$\text{Hom}_{\mathcal{T}_{\text{Sp}(2n)}}(r, s) \cong \text{Hom}_{\mathcal{T}_{\text{Sp}(2n)}}(0, r + s).$$

Let us recall an indexing set for the basis of $\text{Hom}_{\mathcal{T}_{\text{Sp}(2n)}}(0, r)$:

**Definition 7.4.20.** An $n$-symplectic oscillating tableau of length $r$ (and final shape $\emptyset$) is a sequence of partitions

$$\emptyset = \mu^0, \mu^1, \ldots, \mu^r = \emptyset$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell and every partition $\mu^i$ has at most $n$ non-zero parts.

**Lemma 7.4.21.** $\text{Hom}_{\mathcal{T}_{\text{Sp}(2n)}}(0, \otimes^r V)$ has a basis indexed by $n$-symplectic oscillating tableaux.

**Proof.** This follows immediately from the branching rule for tensoring the defining representation with an irreducible representation of $\text{Sp}(2n)$.

Next we exhibit a set of diagrams that span $\text{Hom}_{\mathcal{D}_{\text{Sp}(2n)/\text{Pf}^{(n)}}}(0, r)$. In fact, this is the key observation.
Definition 7.4.22. Let $d$ be a diagram in $D(0,r)$. Then an $n$-crossing in $d$ is a set of $n$ distinct strands such that every pair of strands crosses, i.e., $d$ contains strands $(a_1,b_1), \ldots, (a_n,b_n)$ with $a_1 < a_2 < \cdots < a_n < b_1 < b_2 < \cdots < b_n$. The diagram is $(n+1)$-noncrossing if it contains no $(n+1)$-crossing.

Theorem 7.4.23. The set of $(n+1)$-noncrossing diagrams form a basis of $\text{Hom}_{D_{\text{Sp}(2n)}}(0,r)$. 

Proof. We only need to show that the set spans. For each $f$ we write $\text{Pf}(f)$ as a rewrite rule. The term that is singled out is the perfect matching of $S$ in which every pair of strands crosses. The diagrams which cannot be simplified using these rewrite rules are the $(n+1)$-noncrossing diagrams. The procedure terminates because the number of pairs of strands which cross decreases. □

We can now use a bijection due to Sundaram [Sun86, Lemma 8.3] to finish the proof of our main theorem.

Lemma 7.4.24. For all $n$ and $r$ there is a bijection between the set of $n$-symplectic oscillating tableaux of length $r$ and the set of $(n+1)$-noncrossing diagrams in $D(0,r)$. 

Corollary 7.4.25. The diagram algebra $D_r = \text{Hom}_{D_{\text{Sp}(2n)}}(r,r)$ is semisimple for $n \geq r$.

Proof. When $n \geq r$ the ideal $\text{Pf}(n)(r,r)$ is empty. Therefore $\text{ev}_{\text{Sp}(2n)}$ is an isomorphism of categories. Since $\text{Hom}_{T_{\text{Sp}(2n)}}(r,r)$ is semisimple, so is $\text{Hom}_{D_{\text{Sp}(2n)}}(r,r)$. □

7.5. Frobenius characters for tensor algebras.

Theorem 7.5.1. Let $\mu$ be a partition of length at most $n$ and let $W(\mu)$ be the irreducible representation of $\text{Sp}(2n)$ with highest weight $(\mu_1 - \mu_2, \ldots, \mu_{n-1} - \mu_n, \mu_n)$. Then for $n \geq r$ there is a natural isomorphism

$$\text{Hom}_{\text{Sp}(V)}(W(\mu), \otimes^r V) \cong \text{Inf}_{\mu}^r(S^\mu)$$

of $D_r$-modules.

Proof. The idea is to compare the classical branching rule for $V \otimes W(\mu)$, see Littlewood [Lit58]

$$W(\mu) \otimes V \cong \bigoplus_{\lambda=\mu\pm\square} W(\lambda).$$

with Corollary 7.3.2.

Let $U(r,\mu) = \text{Hom}_{\text{Sp}(V)}(W(\mu), \otimes^r V)$, i.e.,

$$\otimes^r V \cong \bigoplus_{\mu} W(\mu) \otimes U(r,\mu),$$

(17)
let \( \tilde{U}(r, \mu) = \text{Inf}_p(S^\mu) \) and let \( \tilde{W}(r, \mu) = \text{Hom}_{D_r}(\tilde{U}(r, \mu), \otimes^r V) \), i.e.,
\[
\otimes^r V \cong \bigoplus_{\mu} \tilde{W}(r, \mu) \otimes \tilde{U}(r, \mu).
\]

We then find, using the classical branching rule and Corollary 7.3.2 respectively,
\[
\begin{align*}
\text{Hom}_{D_r}(U(r, \mu), \otimes^{r+1} V) & \cong \bigoplus_{\lambda \vdash \mu+\square \atop \lambda \vdash \mu} W(\lambda) \quad \text{and} \\
\text{Hom}_{D_r}(\tilde{U}(r, \mu), \otimes^{r+1} V) & \cong \bigoplus_{\lambda \vdash \mu+\square \atop \lambda \vdash \mu} \tilde{W}(r+1, \lambda).
\end{align*}
\]

We now use induction on \( r \). For \( r \leq 2 \) and \( |\mu| \leq r \) it can be checked directly that \( U(r, \mu) \cong \tilde{U}(r, \mu) \) and \( W(\mu) \cong \tilde{W}(r, \mu) \).

Suppose now that \( r \geq 2 \) and \( U(r, \mu) \cong \tilde{U}(r, \mu) \) and \( W(\mu) \cong \tilde{W}(r, \mu) \) for \( |\mu| \leq r \). Thus, for all partitions \( \mu \) of \( r \) we have
\[
\bigoplus_{\lambda \vdash \mu+\square \atop \lambda \vdash \mu} W(\lambda) \cong \bigoplus_{\lambda \vdash \mu+\square \atop \lambda \vdash \mu} \tilde{W}(r+1, \lambda).
\]

Note that the set of partitions covered by a partition \( \lambda \) determines \( \lambda \) when \(|\lambda| > 2\). Furthermore, note that for \(|\lambda| = r+1\), the modules \( W(\lambda) \) and \( \tilde{W}(r+1, \lambda) \) occur precisely in those equations corresponding to partitions \( \mu \) that are covered by \( \lambda \). Therefore, \( W(\lambda) \) and \( \tilde{W}(r+1, \lambda) \) must be isomorphic.

This in turn implies via Equations (17) and (18) that \( U(r+1, \lambda) \) and \( \tilde{U}(r+1, \lambda) \) are isomorphic, too. \( \square \)

7.6. Cyclic sieving phenomenon. We now use the results obtained so far to exhibit an instance of the cyclic sieving phenomenon. Let the \( X \) be the set of \((n+1)\)-noncrossing perfect matchings of \( \{1, \ldots, 2r\} \), and let \( \rho : X \to X \) be the map that rotates a matching, i.e., \( \rho \) acts on \( \{1, \ldots, 2r\} \) as \( \rho(i) = i \pmod{2r} + 1 \).

By Theorem 7.4.23 \( X \) is a basis for \( \text{Hom}_{\text{Sp}(2n)/\text{Pr}(n)}(0, 2r) \). Since the functor \( \text{ev}_{\text{Sp}(2n)} \) is pivotal and symmetric, we can apply Lemma 4.6 and Theorem 3.2.

For \( n \geq r \) the set \( X \) coincides with \( D(0, 2r) \), i.e., the set of all perfect matchings of \( \{1, \ldots, 2r\} \). In this case the cyclic sieving phenomenon follows directly from Corollary 3.3.

**Corollary 7.6.1.** Let \( X \) be the set of perfect matchings on \( \{1, \ldots, 2r\} \) and let \( \rho \) be the rotation map acting on \( X \). Let
\[
P(q) = \text{fd}(h_r \circ h_2) = \sum_{\lambda \vdash \mu \vdash \square \atop \lambda \vdash \mu \vdash \square} \text{fd} \ s_{\lambda}.
\]

Then the triple \((X, \rho, P(q))\) exhibits the cyclic sieving phenomenon.
Proof. The only observation to make is that the species of perfect matchings on \{1, \ldots, 2r\} is the composition of the species of sets of cardinality \(r\) with the species of sets of cardinality 2. □

For \(n < r\), we cannot realise \((n+1)\)-noncrossing perfect matchings of \{1, \ldots, 2r\} as a combinatorial species. In particular, the module \(\text{Hom}_{\mathbb{D}_{\text{Sp}(2n)/\text{Pf}(n)}}(0, 2r)\) restricted to \(\mathfrak{S}_2\), is not a permutation representation. However, combinatorial descriptions of the Frobenius character of \(\text{Hom}_{\text{Sp}(V)}(W(\mu), \otimes^r V)\) were obtained in \cite{Sun86} (using combinatorics) and in \cite{Tok87} (using representation theory). For the special case of the invariant tensors, a geometric proof can be found in Procesi \cite[Equation 11.5.1.6]{Pro07}.

**Lemma 7.6.2.**

\[
\text{ch} \text{Hom}_{\mathbb{D}_{\text{Sp}(2n)}}(0, 2r) = \sum_{\lambda \vdash 2r \text{ columns of even length } \ell(\lambda) \leq 2n} s_{\lambda^t}
\]

**Remark 7.6.3.** The partitions indexing the Schur functions appearing in the Frobenius character are all transposed, since we defined \(V\) to be an odd vector space.

**Theorem 7.6.4.** Let \(X = X_{r,n}\) be the set of \((n+1)\)-noncrossing perfect matchings on \{1, \ldots, 2r\} and let \(\rho\) be the rotation map acting on \(X\). Let

\[
P(q) = \sum_{\lambda \vdash 2r \text{ columns of even length } \ell(\lambda) \leq 2n} \text{fd } s_{\lambda^t}
\]

Then the triple \((X, \rho, P(q))\) exhibits the cyclic sieving phenomenon.

**Remark 7.6.5.** Let \(T\) be an oscillating tableau. The descent set of \(T\) is defined in \cite{RSW14b} and is denoted by \(\text{Des}(T)\). The major index of \(T\) is

\[
\text{maj}(T) = \sum_{i \in \text{Des}(T)} i
\]

Then the main result of \cite{RSW14b} is that

\[
P_{r,n}(q) = \sum_T q^{\text{maj}(T)}
\]

where the sum is over \(n\)-symplectic oscillating tableaux of length \(2r\) and weight 0.

Theorem 7.6.4 can be generalised. Informally, \(X = X(r, n, k)\) is the set of \(k\)-regular graphs (with loops prohibited but multiple edges allowed) on \(r\) vertices which are also \((n+1)\)-noncrossing. More precisely, consider \{1, \ldots, kr\} as a cyclically ordered set and partition this set into \(r\) blocks of size \(k\) by \(i \mapsto [(i - 1)/k]\). Then the set \(X\) is the set of \((n+1)\)-noncrossing perfect matchings with the additional properties
that there is no pair contained in a block and if two pairs cross then the four elements are in four distinct blocks. Let $\rho$ be rotation by $k$ points i.e. $\rho(i) = i + k - 1 \mod kr + 1$. The Frobenius character is given by an application of [Wes14, Theorem 1] and is

$$\sum_{\lambda \vdash kr \text{ columns of even length} \ell(\lambda) \leq 2n} \langle h_r(X \cdot h_k(Y)), s_\lambda(Y) \rangle_Y$$

Define $P = P(r, n, k)$ to be the fake degree of this symmetric function. Then $(X, \rho, P)$ exhibits the cyclic sieving phenomenon.

The case $k = 1$ is Theorem 7.6.4 and the case $k = 2$ is related to the invariant tensors of the adjoint representation in [Han85]. The case $n = 1$ is implicit in [PK97] and the case $n = 2$ is related to the $C_2$ webs in [Kup96].

8. The partition category

In this section we discuss the representation theory of the symmetric groups using the combinatorics of set partitions and the partition category. The partition category was introduced in [Mar94] and also in [Jon94] where the relationship with the representation theory of the symmetric groups was observed.

8.1. Diagram category. For $r, s \geq 0$, let $D_\rho(r, s)$ be the finite set of set partitions of $[r] \sqcup [s]$ and let $D_\Sigma(r, s)$ be the subset of set partitions with no singletons.

Let $x$ be a set partition of $[r] \sqcup [s]$ and let $y$ be a set partition of $[s] \sqcup [t]$. Consider these as equivalence relations. These generate an equivalence relation on $[r] \sqcup [s] \sqcup [t]$. For each block $\beta$ such that $\beta \cap ([r] \sqcup [t]) \neq \emptyset$, $\beta \cap ([r] \sqcup [t])$ is a block of $x \circ y$. Define $c(x, y)$ to be the number of blocks that do not contain any elements of $[r] \sqcup [t]$.

The partition category $C_\rho$ is the cobordism category with

$$\text{Hom}_{C_\rho}(r, s) = \mathbb{N} \times D_\rho(r, s).$$

There is an inclusion $C_{\Sigma r} \to C_\rho$. The category $C_\rho$ is generated, as a monoidal category, by the morphisms

A through block is a block $\beta$ such that $\beta \cap [r] \neq \emptyset$ and $\beta \cap [s] \neq \emptyset$. The propagating number of a diagram is the number of through blocks.

We denote the diagram category corresponding to the cobordism category with $D_\rho(\delta)$. Although we do not make use of this in this paper we remark that the object [1] in the partition category is a commutative Frobenius algebra.
8.2. Frobenius characters for diagram algebras.

Lemma 8.2.1. Any partition diagram, \( x \in D(r, s; p) \), has a unique decomposition as

\[
\begin{array}{c}
\begin{array}{c}
\quad r \\
\uparrow \\
\quad a \\
\downarrow \\
\quad p \\
\downarrow \\
\quad b \\
\downarrow \\
\quad s \\
\end{array}
\end{array}
\]

The two pale grey rectangles are a \((a, r - a)\)-shuffle and a \((b, s - b)\)-shuffle, the dark grey square is a permutation on \(p\) strands, the two pink half circles are set partitions on \(r - a\) and on \(s - b\) points and the two pink rectangles are set partitions on \(a\) and on \(b\) points into \(p\) blocks.

Proof. The upper shuffle sorts the points on the top edge according to whether they belong to a through block or not. The \(a\) points which belong to a through block form a set partition into \(p\) blocks. These blocks are joined via a permutation to the \(p\) blocks of the set partition formed by the \(b\) points on the bottom edge which belong to a through block.

\[\square\]

Definition 8.2.2. For \(r \geq p \geq 0\) let \(\bar{D}(p, r; p)\) be the set of diagrams of the form

\[
\begin{array}{c}
\begin{array}{c}
\quad p \\
\uparrow \\
\quad a \\
\downarrow \\
\quad r \\
\end{array}
\end{array}
\]

and define \(\bar{D}(r, p; p)\) in an analogous way.

Corollary 8.2.3. For all \(r, s \geq p \geq 0\), the composition map

\[\circ : \bar{D}(r, p; p) \times D(p, p; p) \times \bar{D}(p, s; p) \rightarrow D(r, s; p)\]

is a bijection.
Let \( e_p \in D_r \) be the idempotent \( \delta^{-(r-p)/2} \text{id}_p \otimes u \), where \( \text{id}_p \otimes u \) is the diagram
\[
\begin{array}{c|c|c}
\text{p} & & \text{r-p} \\
\hline
\end{array}
\]
(19)

Then, by Lemma 6.9 we have a strict Morita context and inflation is an equivalence from \( K \mathfrak{S}_p \)-modules to regular \( J_r(p) / J_r(p-1) \)-modules.

We can now prove the main theorem of this section:

**Theorem 8.2.4.** Let \( V \) be an \( \mathfrak{S}_p \)-module. Then \( (\text{Inf}_r^p V)^{D_r} \) is isomorphic to the restriction of \((V \circ H_+) \cdot P\) to the homogeneous component of degree \( r \), where \( H_+ \) is the species of non-empty sets and \( P \) is the species of set partitions.

**Proof.** Consider the bivariate combinatorial species
\[
H \circ (X \cdot H_+(Y)) \cdot P(Y),
\]
where \( P = H \circ H_+ \) is the species of set partitions. It follows immediately from Corollary 8.2.3 that the restriction of this species to sets of sort \( X \) of cardinality \( p \) and sets of sort \( Y \) of cardinality \( r \) is isomorphic to the \( \mathfrak{S}_r \times \mathfrak{S}_p \)-bimodule \( KD(r, p; p) \), regarded as a bivariate species.

It remains to remark that we have
\[
\left\langle V, H(X \cdot H_+(Y)) \cdot P(Y) \right\rangle_X = (V \circ H_+) \cdot P
\]
as a consequence of the linearity of the scalar product and Lemma 2.1, where we substitute \( H_+(Y) \) for \( Y \).

**8.3. Branching rules.** In this section we determine branching rules for the diagram algebras, assuming their semisimplicity. This explains the relevance of vacillating tableaux.

In order to describe the branching rules we introduce intermediate algebras, \( D'_r \subset D_r \subset D'_r+1 \) and describe the branching rules for each inclusion separately. The algebra \( D'_{r+1} \subset D_{r+1} \) is the subalgebra with basis diagrams in which the last points on the top and bottom row are in the same block. This gives algebras \( A'_p \subset A_p \subset A'_{p+1} \) where the inclusion \( A_p \subset A'_{p+1} \) is the identity map.

The inflation functor from \( A'_p \)-modules to \( D'_r \)-modules is constructed using the bimodule which as a subspace of \( KD(p, r; p) \) has basis the set of diagrams in which the last points on the top and bottom row are in the same block. Denote this set by \( D'(p, r; p) \). This inflation functor is denoted by \( \text{Inf}'_p \).
Proposition 8.3.1. For \( r \geq p \geq 0 \) and \( V \) any \( A_p \)-module, there is a short exact sequence

\[
0 \to \operatorname{Inf}^r_p(V \downarrow_{A_p^r}^{A_p}) \to \operatorname{Inf}^r_p(V) \downarrow_{D_r}^{D_r'} \to \operatorname{Inf}^r_{p+1}(V \uparrow_{A_p^{p+1}}^{A_p}) \to 0.
\]

Proof. Consider the \( A_p \)-\( D'_r \) bimodule \( KD(p, r; p) \). The set of diagrams in \( D(p, r; p) \) whose last point in the bottom row is in a through block generates a submodule isomorphic to \( KD'(p, r; p) \uparrow_{A_p'}^{A_p} \).

The quotient of \( KD(p, r; p) \) by this module is isomorphic to \( KD'(p+1, r; p+1) \): the isomorphism is given on diagrams by adding a point to the top row and connecting it to the block containing the last point of the bottom row. If the last point of the bottom row is in a through block the image under the isomorphism is zero because the propagating number of the resulting diagram is then strictly less than \( p + 1 \).

By the definition of inflation we have

\[
V \otimes_{A_p} KD'(p, r; p) \uparrow_{A_p}^{A_p} \cong V \otimes_{A_p} A_p \otimes_{A_p'} KD'(p, r; p) \cong \operatorname{Inf}^r_p(V \downarrow_{A_p'}^{A_p})
\]

and

\[
V \otimes_{A_p} KD'(p+1, r; p+1) \cong \operatorname{Inf}^r_{p+1}(V \uparrow_{A_p^{p+1}}^{A_p}).
\]

Corollary 8.3.2. Assume that \( D'_r \) is semisimple. Then, for \( r \geq p \geq 0 \) and \( V \) any \( A_p \)-module,

\[
\operatorname{Inf}^r_p(V) \downarrow_{D'_r} \cong \operatorname{Inf}^r_{p+1}(V \uparrow_{A_p^{p+1}}^{A_p}) \oplus \operatorname{Inf}^r_p(V \downarrow_{A_p'}^{A_p})
\]

Proposition 8.3.3. For \( r \geq p \geq 0 \) and \( V \) any \( A_{p+1} \)-module,

\[
\operatorname{Inf}^r_{p+1}(V) \downarrow_{D_r} \cong \operatorname{Inf}^r_{p+1}(V \uparrow_{A_{p+1}^{p+1}}^{A_{p+1}}) \oplus \operatorname{Inf}^r_p(V \downarrow_{A_p'}^{A_p})
\]

Proof. Consider the \( A_{p+1} \)-\( D_r \) bimodule \( KD'(p+1, r+1; p+1) \downarrow_{A_p^{p+1}}^{A_{p+1}^{p+1}} \). The set of diagrams in \( D'(p+1, r+1; p+1) \) whose last points in the top and in the bottom row are a block generate a submodule isomorphic to \( KD(p, r; p) \) where the isomorphism is given by removing the last points in the top and bottom row.

The remaining diagrams generate a submodule isomorphic to \( KD(p+1, r; p+1) \downarrow_{A_{p+1}^{p+1}}^{A_{p+1}^{p+1}} \) where the isomorphism is given by removing the last point in the bottom row. Note that the block containing this point by construction contains another in the bottom row, so the resulting diagram has indeed propagating number \( p + 1 \).

Finally we have

\[
V \otimes_{A_{p+1}} KD(p, r; p) \cong \operatorname{Inf}^r_p(V \downarrow_{A_p}^{A_{p+1}})
\]

and

\[
V \otimes_{A_{p+1}} KD(p+1, r; p+1) \cong \operatorname{Inf}^r_{p+1}(V \uparrow_{A_{p+1}^{p+1}}^{A_{p+1}^{p+1}}).
\]

\( \square \)
Corollary 8.3.4. Let $\lambda \vdash p$ and put

$$U(r, \lambda) = \text{Inf}_p^r(S(\lambda)) \text{ and } U'(r, \lambda) = '\text{Inf}_p^{r+1}(S(\lambda)).$$

Then the branching rules are given by

$$U(r, \lambda) \downarrow^D_{\lambda=\mu+\Box} \bigoplus_{\mu=\lambda+\Box or \mu=\lambda} U'(r, \mu) \text{ and }$$

$$U'(r+1, \lambda) \downarrow^D_{\lambda=\mu+\Box} \bigoplus_{\mu=\lambda+\Box or \mu=\lambda} U(r, \mu).$$

Proof. The proof is completely analogous to the proof of Proposition 7.3.2 and therefore omitted.

8.4. Fundamental theorems. Let $K$ be a field of characteristic zero and let $V$ be the defining permutation representation of the symmetric group $S_n$.

Let $D_{S_n}$ be the diagram category corresponding to $C_{P}$ with $\delta = n$ and let $T_{S_n}$ be the category of invariant tensors. There is a functor from $D_{S_n}$ to $T_{S_n}$ that allows us to study the representations of $\otimes^r V$ via diagrams and vice versa. Let $\{v_1, \ldots, v_n\}$ be the basis of $V$ as a permutation representation.

Definition 8.4.1 ([Jon94], [HR05, Equation 3.2], [Wey97, Chapter II A §3]). Let $\text{ev}_{S_n} : D_{S_n} \to T_{S_n}$ be the symmetric and pivotal monoidal functor $D_{S_n} \to T_{S_n}$.

Explicitly, for a diagram $x$ let

$$\text{ev}_{S_n}(x)_{i_{r+1}, \ldots, i_{r+s}} = \begin{cases} 1 \text{ if } i_k = i_\ell \text{ whenever } k \text{ and } \ell \text{ are in the same block of } x \\ 0 \text{ otherwise.} \end{cases}$$

Then $\text{ev}_{S_n}$ sends the object $r \in \mathbb{N}$ to $V^\otimes r$ and

$$\text{ev}_{S_n}(x) = \left(v_{i_1} \otimes \cdots \otimes v_{i_r} \mapsto \sum_{1 \leq i_{r+1}, \ldots, i_{r+s} \leq n} (x)_{i_{r+1}, \ldots, i_{r+s}}^{i_1, \ldots, i_r} v_{i_{r+1}} \otimes \cdots \otimes v_{i_{r+s}}\right).$$

Since $\text{ev}_{S_n}$ is a monoidal functor it is determined by its values on the generators. These are:

- $\text{ev}_{S(n)}(\begin{array}{c} \Box \\ \end{array}) = u \otimes v \mapsto v \otimes u$
- $\text{ev}_{S(n)}(\begin{array}{c} \Box \\ \end{array}) = v_i \mapsto 1$
- $\text{ev}_{S(n)}(\begin{array}{c} \Box \\ \end{array}) = v_i \mapsto v_i \otimes v_i,$
- $\text{ev}_{S(n)}(\begin{array}{c} \Box \\ \end{array}) = v_i \otimes v_j \mapsto \delta_{i,j} v_i,$
- $\text{ev}_{S(n)}(\begin{array}{c} \Box \\ \end{array}) = 1 \mapsto \sum_j v_j.$
The first fundamental theorem for the symmetric group can be stated as follows:

**Theorem 8.4.2** ([HR05, Theorem 3.6]). *For all* \( n > 0 \) *the functor* \( \text{ev}_{\mathfrak{S}_n} : D_{\mathfrak{S}_n} \to T_{\mathfrak{S}_n} \) *is full.*

A second fundamental theorem is also available. For any \( d \in D(r, r) \) define an element \( x_d \in D_r \) via

\[
d = \sum_{d' \leq d} x_{d'},
\]

where \( d' \leq d \) means that every block of the partition \( d \) is contained in a block of the partition \( d' \).

**Definition 8.4.3.** For \( r, s \geq 0 \), the subspace \( P^{(n)}(r, s) \) in the ideal \( P^{(n)} \) of \( D_{\mathfrak{S}(n)} \) is spanned by the set

\[
P^{(n)}(r, s) = \{ x_d : d \text{ has more than } n \text{ blocks} \}.
\]

**Definition 8.4.4.** Let \( \overline{\text{ev}}_{\mathfrak{S}(n)} : D_{\mathfrak{S}(n)}/P^{(n)} \to T_{\mathfrak{S}(n)} \) be the functor that factors \( \text{ev}_{\mathfrak{S}(n)} \) through the quotient.

**Theorem 8.4.5** ([HR05, Theorem 3.6]). The functor \( \overline{\text{ev}}_{\mathfrak{S}(n)} : D_{\mathfrak{S}(n)}/P^{(n)} \to T_{\mathfrak{S}(n)} \) is an isomorphism of categories.

**Theorem 8.4.6.** The set of partitions with less than \( n + 1 \) blocks form a basis of \( \text{Hom}_{D_{\mathfrak{S}(n)}}/P^{(n)}(0, r) \).

**Corollary 8.4.7.** The diagram algebra \( D_r = \text{Hom}_{D_{\mathfrak{S}_n}}(r, r) \) is semisimple for \( n \geq 2r \).

We note that it should be possible to derive a second fundamental theorem also using the methods that succeeded in the case of the Brauer category. To this end we have the following conjecture.

For \( r \geq 0 \), let \( \rho \) be the one dimensional representation of \( D_r \) which on diagrams is given by

\[
\rho(x) = \begin{cases} 
  x & \text{if } \text{pr}(x) = r \\
  0 & \text{if } \text{pr}(x) < r
\end{cases}
\]

Note that \( \text{pr}(x) = r \) if and only if \( x \) is a permutation.

The element \( E(r) \) is determined, up to scale by the properties

\[
x E(r) = \rho(x) E(r) = E(r)x
\]

The scale is determined by the property that \( E(r) \) is idempotent. Then \( E(r) \) is a rank one central idempotent. This central idempotent can be constructed using [MW99]. It is also, up to scale, the element \( x_d \) associated to the element \( d \in D(r, r) \) consisting of \( 2r \) singletons.

Similarly, the restriction of \( \rho \) to \( D'_r \) is a one dimensional representation and we have a corresponding rank one central idempotent, \( E'(r) \).
Definition 8.4.8. The generators \( h_i, s_i \) and \( p_i \) are:

\[
\begin{align*}
   h_i &= i - (n - i - 1) - (n - i - 1) \\
   s_i &= i - 1 - n - i - 1 \\
   p_i &= i - 1 - n - i
\end{align*}
\]

Conjecture 8.4.9. The idempotents \( E(r) \in D_r \) and \( E'(r) \in D'_r \) are constructed recursively by \( E'(1) = 1 \)

\[
E(r) = \frac{1}{r} E'(r) \left[ 1 + (r - 1) s_{r-1} - \frac{1}{\delta - 2r + 2} p_r \right] E'(r)
\]

\[
E'(r + 1) = E(r) \left[ 1 - r \left( \frac{\delta - 2r + 2}{\delta - 2r + 1} \right) h_r \right] E(r)
\]

Example 8.4.10.

\[
\begin{align*}
   E(1) &= (1 - \frac{1}{\delta} p_1) \\
   E'(2) &= E(1) \left( 1 - \frac{\delta}{\delta - 1} h_1 \right) E(1) \\
   E(2) &= \frac{1}{2} E'(2) \left( 1 + s_1 - \frac{1}{\delta - 2} p_2 \right) E'(2) \\
   E'(3) &= E(2) \left( 1 - 2 \frac{\delta - 2}{\delta - 3} h_2 \right) E(2) \\
   E(3) &= \frac{1}{3} E'(3) \left( 1 + 2 s_2 - \frac{1}{\delta - 4} p_3 \right) E'(3)
\end{align*}
\]

Conjecture 8.4.11. The ideal \( P^{(n)} \) of \( D_{S(n)} \) is the pivotal symmetric ideal generated by \( E(n + 1) \).

8.5. Frobenius characters for tensor algebras.

Theorem 8.5.1. Let \( \mu \) be a partition of \( n \) and let \( S^\mu \) be the irreducible representation of \( S_n \) corresponding to \( \mu \). Then for \( n \geq 2r \) there is a natural isomorphism

\[
\text{Hom}_{S_n}(S^\mu \otimes V) \cong \text{Inf}_p^r(S^{\tilde{\mu}})
\]

of \( D_r \)-modules, where \( \tilde{\mu} \) is the partition \( \mu \) with the first part removed.

Proof. The proof is very similar to the proof of Theorem 7.5.1. Recall that \( V \cong S^{(n)} \oplus S^{(n-1,1)} \) and the classical branching rule is

\[
S^\mu \otimes V \cong \bigoplus_{\mu = \nu + \square} S^\lambda.
\]

Let \( U(r, \mu) = \text{Hom}_{S_n}(S^\mu \otimes V) \), i.e.,

\[
\otimes^r V \cong \bigoplus_{\mu} S^\mu \otimes U(r, \mu), \tag{20}
\]
let $\tilde{U}(r, \mu) = \text{Inf}_p (S^\mu)$ (where $\tilde{\mu} \vdash p$ and let $\tilde{W}(r, \mu) = \text{Hom}_{D_r}(\tilde{U}(r, \mu), \otimes^r V)$, i.e.,
\begin{equation}
\otimes^r V \cong \bigoplus_{\mu} \tilde{W}(r, \mu) \otimes \tilde{U}(r, \mu).
\end{equation}

We then find, using the classical branching rule and Corollary 8.3.4 respectively,
\[
\text{Hom}_{D_r}(U(r, \mu), \otimes^{r+1} V \downarrow^{D_{r+1}}) \cong \bigoplus_{\lambda = \mu + \square} S^\lambda \quad \text{and}
\]
\[
\text{Hom}_{D_r}(\tilde{U}(r, \mu), \otimes^{r+1} V \downarrow^{D_{r+1}}) \cong \bigoplus_{\lambda = \mu + \square} \tilde{W}(r+1, \lambda).
\]
The proof now proceeds exactly as the proof of Theorem 7.5.1: for $r \leq 2$ we have to check directly that $U(r, \mu) \cong \tilde{U}(r, \mu)$ and $S^\mu \cong \tilde{W}(r, \mu)$, whereas for $r > 2$ we use induction. To this end, note that for a given $\lambda$ the multiset of partitions obtained by first removing a box from $\lambda$ and then adding a box uniquely determines $\lambda$, see for example [Sta88, Corollary 4.2].

8.6. Cyclic sieving phenomenon. As in Section 7.6 we will now exhibit an instance of the cyclic sieving phenomenon. Let $X$ be the set of set partitions $\{1, \ldots, r\}$ into at most $n$ blocks, and let $\rho : X \to X$ be the rotation map, i.e., $\rho$ acts on $\{1, \ldots, r\}$ as $\rho(i) = i \pmod{r} + 1$.

In contrast to the situation with $(n+1)$-noncrossing perfect matchings, the set partitions into at most $n$ blocks can be understood as a combinatorial species.

Let $H_n = \sum_{k=0}^n h_k$ be the species of sets of cardinality at most $n$ and let $(H_n \circ H_+)_r$ be the homogeneous part of degree $r$.

**Theorem 8.6.1.** Let $X = X_{r,n}$ be the set of set partitions on $\{1, \ldots, r\}$ into at most $n$ blocks and let $\rho$ be the rotation map acting on $X$. Let
\[
P(q) = \text{fd ch}(H_n \circ H_+)_r,
\]
Then the triple $(X, \rho, P(q))$ exhibits the cyclic sieving phenomenon.

For completeness, we remark that as in the case of the symplectic group, the Frobenius characters of the isotypical components are known:

**Theorem 8.6.2** ([ST94, Theorem 5.1] and [Sta99, Exercise 7.74]). For any $\mu \vdash n$,
\[
\sum_{r \geq 0} \chi_{\text{Hom}_{\mathfrak{S}_n}(S^\mu, \otimes^r V)} = s_\mu \circ H.
\]

As in Section 7.6, we can generalise Theorem 8.6.1. Let $X = X(r, n, k)$ be the set of multiset partitions of $\{1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r\}$ where each label occurs $k$ times. Let $\rho$ be the rotation action, i.e., $\rho$ acts on
the labels as $\rho(i) = i \pmod{r} + 1$. The Frobenius character is given by an application of \cite[Theorem 1]{Wes14} and is

$$\langle h_r(X \cdot h_k(Y)), (H_n \circ H_\oplus)_{kr}(Y) \rangle_Y.$$ 

Let $P = P(r, n, k)$ be the fake degree of this symmetric function. Then $(X, \rho, P)$ exhibits the cyclic sieving phenomenon.

9. The directed Brauer category

In this section we discuss the invariant theory of the adjoint representation of the general linear groups using the combinatorics of directed perfect matchings, or bijections, and the directed Brauer category; alternatively, the combinatorics of walled Brauer diagrams and the walled Brauer category. This invariant theory was developed in \cite{Sta84}, \cite{Han85}, \cite{Ste87}, \cite{BCH+94}. The use of diagrams in this context emerged from the skein relation approach to the HOMFLYPT knot polynomial.

9.1. Diagram category. For $r, s \geq 0$, let $D_{\mathcal{Br}}(r, s)$ be the set of directed perfect matchings of $[r] \Pi [s]$, i.e., the set of partitions of $[r] \Pi [s]$ into ordered pairs. In terms of diagrams an element of $D_{\mathcal{Br}}(r, s)$ is a Brauer diagram in $D_{\mathcal{Br}}(r, s)$ together with an orientation of each strand.

The directed cobordism category $C_{\mathcal{Br}}$ has as objects the set of words in the alphabet $\{+, -\}$, i.e., $\mathbb{N} \times \mathbb{N}$. The set of morphisms is $\mathbb{N} \times \mathbb{N} \times D_{\mathcal{Br}}(r, s)$, since there are two oriented loops. For an element $x \in \mathbb{N} \times \mathbb{N} \times D_{\mathcal{Br}}(r, s)$ the domain of $x$ is the word $u_1, \ldots, u_r$ with

$$u_i = \begin{cases} + & \text{if } i \text{ is the initial point of a pair} \\ - & \text{if } i \text{ is the final point of a pair} \end{cases}$$

and the codomain of $x$ is the word $v_1, \ldots, v_s$ with

$$v_i = \begin{cases} + & \text{if } i \text{ is the final point of a pair} \\ - & \text{if } i \text{ is the initial point of a pair.} \end{cases}$$

Composition is defined in the same way as for the Brauer category.

The topological interpretation of this category is that it is equivalent to the cobordism category whose objects are oriented 0-manifolds and whose morphisms are oriented 1-manifolds.

We denote with $D_{\mathcal{Br}(\delta)}$ the corresponding diagram category. Here both oriented loops give a factor of $\delta$.

Then $D_{\mathcal{Br}(\delta)}$ is generated as a monoidal category by the diagrams obtained from (3) and (5) by putting orientations on the strands.

There are now six types of strand since each type of strand in (13) can be directed in two possible ways. In particular there are two types of through strand. The two types of through strand determine two propagating numbers each of which satisfies the conditions in Lemma 5.9.
We will use boldface letters to denote pairs of numbers. This means that objects will typically be denoted by \( r = (r_1, r_2) \) and propagating numbers by \( p = (p_1, p_2) \).

Any object of the directed Brauer category is isomorphic to \( +^n -^m \) for a unique pair \( (n, m) \) with \( n, m \in \mathbb{N} \). The full subcategory on these objects is called the walled Brauer category. By construction, the inclusion of the walled Brauer category in the directed Brauer category is fully faithful and essentially surjective and hence is an equivalence.

For any \( p \), the algebra \( A_p \) is \( K\mathfrak{S}_p = K(\mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2}) \). Also for any \( r \) we have an inclusion \( K\mathfrak{S}_r \subseteq D_r \).

The directed Brauer category has a decomposition as a direct sum of subcategories. The components are indexed by \( \mathbb{Z} \). A sign sequence with \( r \) plus signs and \( s \) minus signs is an object of block \( k \) where \( k = r - s \). The zero component is a symmetric monoidal subcategory.

The intersection of the zero component with the walled Brauer category has objects \((r, r)\). Another subcategory is the full subcategory with objects \((+, -)^r\) for \( r \geq 0 \). These two inclusions are both fully faithful and essentially surjective and so are equivalences. In fact both subcategories are isomorphic. This category is denoted \( D_A \). This a diagram category in the sense of section 5. An alternative description of this as a diagram category is given in [Thu04].

Since \( D_A \) is a symmetric monoidal category we have inclusions \( \mathfrak{S}_r \to D_r \). These inclusions are given by the composition of the diagonal map \( \mathfrak{S}_r \to \mathfrak{S}_r \times \mathfrak{S}_r \) with the inclusions above \( \mathfrak{S}_r \times \mathfrak{S}_r \to D_r \).

Although we do not make use of this in this paper we remark that the object \([1]\) in the category \( D_A \) is a symmetric Frobenius algebra.

9.2. **Frobenius characters for diagram algebras.** In this section we employ the results of Section 6.4, adapted in the obvious way, to compute the Frobenius characters of the restriction of modules of the diagram algebras \( D_r \) to \( K(\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2})\-modules.

Let us first provide a decomposition for the diagrams in \( D(r, s; p) \).
**Lemma 9.2.1.** Any walled Brauer diagram in $D(r, s; p)$ has a unique decomposition as

![Diagram](image)

The four pale grey rectangles are shuffles and the four dark grey squares are permutations.

**Definition 9.2.2.** For $r_1 \geq p_1 \geq 0$ and $r_2 \geq p_2 \geq 0$ let $\bar{D}(p; r; p)$ be the set of walled Brauer diagrams of the form

![Diagram](image)

and define $\tilde{D}(r, p; p)$ in an analogous way.

**Corollary 9.2.3.** For $r_1, s_1 \geq p_1 \geq 0$ and $r_2, s_2 \geq p_2 \geq 0$ the composition map

$$\circ: \bar{D}(r, p; p) \times D(p, p; p) \times \tilde{D}(p, s; p) \to D(r, s; p)$$

is a bijection.

Let us now proceed in obvious analogy to Section [6.4]. Let $E = D_r/(J_r(p_1 - 1, p_2) \oplus J_r(p_1, p_2 - 1))$. Note that for any diagram in $D_r$ with propagating numbers $p$ we have $r_1 - p_1 = r_2 - p_2$. Therefore, the element of $D_r$ corresponding to the diagram

![Diagram](image)

(22)
multiplied with $\delta^{-(r_1-p_1)}$ is an idempotent, $e_p$. Then, analogous to Lemma 6.9, we have a strict Morita context and inflation is an equivalence from $K\mathcal{S}_{p_1} \times \mathcal{S}_{p_2}$-modules to regular $J_r(p)/(J_r(p_1 - 1, p_2) \oplus J_r(p_1, p_2 - 1))$-modules.

The following are the main theorems of this section.

Theorem 9.2.4 is implicit in the results of [Han85] and the corresponding character formula is [Hal96, Corollary 7.24].

**Theorem 9.2.4.** Let $V = V_1 \otimes V_2$ be an $\mathcal{S}_{p_1} \times \mathcal{S}_{p_2}$-module. Then the $\mathcal{S}_{r_1} \times \mathcal{S}_{r_2}$-module $(\text{Inf}^r_p V) \downarrow_{\mathcal{S}_{r_1} \times \mathcal{S}_{r_2}}^{D_r}$ is isomorphic to

$$\bigoplus_{\lambda \vdash k} V_1 \cdot S^\lambda \otimes V_2 \cdot S^\lambda,$$

where we denote $r_1 - p_1 = r_2 - p_2$ by $k$.

**Remark 9.2.5.** Using the notation for species, formula (23) can be rewritten as

$$\sum_{\lambda \vdash k} V_1(Y_1) \cdot S^\lambda(Y_1) \cdot V_2(Y_2) \cdot S^\lambda(Y_2).$$

**Proof.** Assume $K$ is a field of characteristic zero. There is a natural isomorphism of $K\mathcal{S}_{p_1} \cdot K\mathcal{S}_{r_1}$ bimodules

$$D(r, p; p) \cong K\mathcal{S}_p \otimes_{K\mathcal{S}_r} K\mathcal{S}_r$$

This is then isomorphic to

$$\bigoplus_{\lambda \vdash k} K\mathcal{S}_{p_1} \cdot S^\lambda \otimes K\mathcal{S}_{p_2} \cdot S^\lambda.$$

The version using combinatorial species looks quite different:

**Theorem 9.2.6.** For an arbitrary $\mathcal{S}_{p_1} \times \mathcal{S}_{p_2}$-module $V(Y_1, Y_2)$, the $\mathcal{S}_{r_1} \times \mathcal{S}_{r_2}$-module $(\text{Inf}^r_p V) \downarrow_{\mathcal{S}_{r_1} \times \mathcal{S}_{r_2}}^{D_r}$ is isomorphic, as a species, to the homogeneous part of degree $r_1$ in sort $Y_1$ and $r_2$ in sort $Y_2$ in the bivariate species

$$V(Y_1, Y_2) \cdot H(Y_1 \cdot Y_2).$$

When $V(Y_1, Y_2)$ is a permutation representation, the isomorphism is an isomorphism of permutation representations.

**Proof.** Consider the combinatorial species in four variables

$$H(X_1 \cdot Y_1 + X_2 \cdot Y_2 + Y_1 \cdot Y_2).$$

It follows immediately from Corollary 9.2.3 that the restriction of this species to sets of sort $X_i$ of cardinality $p_i$ and sets of sort $Y_i$ of cardinality $r_i$ is isomorphic to the $\mathcal{S}_{p_1} \times \mathcal{S}_{p_2} \times \mathcal{S}_{r_1} \times \mathcal{S}_{r_2}$-module $KD(p, r; p)$, regarded as a species.
As a consequence of the linearity of the scalar product and Lemma 2.1 we have

$$\langle V(X_1, X_2), H(X_1 \cdot Y_1 + X_2 \cdot Y_2 + Y_1 \cdot Y_2) \rangle_{X_1, X_2} = V(Y_1, Y_2) \cdot H(Y_1 \cdot Y_2),$$

which implies the claim by remark 6.11.

Theorems 9.2.4 and 9.2.6 are essentially the same result. The character identity that arises from comparing the results is the Cauchy identity.

Let us now specialise to the diagram algebra $D_r$ in the category $D_A$. This is the diagram algebra $D_{r,r}$ in the category $D_{\mathcal{B}(\delta)}$ with the diagonal action of the symmetric group $\mathcal{S}_r$. We first record a direct consequence of Theorem 9.2.4.

**Theorem 9.2.7.** Let $V = V_1 \otimes V_2$ be an $\mathcal{S}_p \times \mathcal{S}_p$-module. Then the $\mathcal{S}_r$-module $(\text{Inf}_p^r V) \downarrow_{\mathcal{S}_r} D_r$ is isomorphic to

$$\bigoplus_{\lambda \vdash k} V_1 \cdot S^\lambda \ast V_2 \cdot S^\lambda.$$

**Remark 9.2.8.** Using the notation for species, formula (24) can be rewritten as

$$\sum_{\lambda \vdash k} V_1 \cdot S^\lambda \ast V_2 \cdot S^\lambda.$$

Taking the diagonal of the result of Theorem 9.2.6 requires significantly more work. The following theorem is derived by means of symmetric functions by Stanley [Sta84, Theorem 6.2], see also [Ste87, Proposition 4.8, Proposition 7.8 and Corollary 8.5].

Recall that $L_+$ is the species of non-empty linear orders and $S = H \circ C_+$ is the species of permutations.

**Theorem 9.2.9.** Let $V(Y_1, Y_2)$ be an arbitrary $\mathcal{S}_p \times \mathcal{S}_p$-module. Then the $\mathcal{S}_r$-module $(\text{Inf}_p^r V) \downarrow_{\mathcal{S}_r} D_r$ is isomorphic to

$$S \cdot (\nabla V(Y_1, Y_2))(L_+).$$

**Proof.** Given Theorem 9.2.6 it remains to show that

$$\nabla (V(Y_1, Y_2) \cdot H(Y_1 \cdot Y_2)) = S \cdot (\nabla V(Y_1, Y_2))(L_+),$$

where $L_+$ is the species of non-empty linear orders and $S$ is the species of permutations.

Let $U$ be a finite set and consider an element $w$ of

$$V(Y_1, Y_2) \cdot H(Y_1 \cdot Y_2)[U, U].$$

By the definition of ‘·’, this is an element of

$$V[U_1, U_3] \times H(Y_1 \cdot Y_2)[U_2, U_4],$$

for some sets $U_1$–$U_4$ with $U = U_1 \cup U_2 = U_3 \cup U_4$ and $U_1 \cap U_2 = U_3 \cap U_4 = \emptyset$. We can identify $w$ with an element of $S[V_1] \times (\nabla V(Y_1, Y_2))(L_+)[V_2]$ for two disjoint sets $V_1$ and $V_2$ with union $U$ as follows:
Let \( w' \) be the projection of \( w \) to \( H(Y_1 \cdot Y_2)[U_2, U_3] \). Thus, \( w' \) can be interpreted as a bijection between \( U_2 \) and \( U_3 \). Let \( V_1 \) be the maximal subset of \( U_2 \cap U_3 \) which is permuted by \( w' \). The restriction of \( w' \) to \( H(Y_1 \cdot Y_2)[V_1, V_1] \) can then be identified with an element of \( S[V_1] \).

Let \( V_2 = U \setminus V_1 \) and consider the set \( (\nabla V(Y_1, Y_2))(L_+)[V_2] \). By the definition of the composition of species this set equals

\[
\nabla V(Y_1, Y_2)[\pi] \times L_+[\pi_1] \times \cdots \times L_+[\pi_k]
\]

for some set partition \( \pi = (\pi_1, \ldots, \pi_k) \) of \( V_2 \). We regard the elements of \( U_1 \cap U_3 \) as singletons of \( \pi \). The remaining linearly ordered blocks of the set partition are obtained by choosing an element \( a \) in \( U_2 \cap U_3 \), and then repeatedly applying \( w' \) until we obtain an element in \( U_1 \cap U_4 \):

\[
a, w'(a), w'(w'(a)), \ldots
\]

Thus, \( U_1 \) consists precisely of the final elements of the blocks (including singletons), while \( U_3 \) is the collection of initial elements of the blocks (including singletons). Therefore, there is a natural bijection between \( \nabla V(Y_1, Y_2)[\pi] \) and \( V(Y_1, Y_2)[U_1, U_3] \), and a natural bijection between \( S[V_1] \times L_+[\pi_1] \times \cdots \times L_+[\pi_k] \) and bijections \( w' \).

Theorems 9.2.7 and 9.2.9 are essentially the same result. Hence comparing the answers gives the character identity

\[
\sum_{\lambda} \left( s_{\alpha} s_{\lambda} \right) * \left( s_{\beta} s_{\lambda} \right) = \left( \prod_{k \geq 1} (1 - p_k)^{-1} \right) \cdot \left( s_{\alpha} * s_{\beta} \circ \frac{s_1}{1 - s_1} \right),
\]

where we have used

\[
H \circ C_+ = \sum_{\lambda} p_{\lambda} = \left( \prod_{k \geq 1} (1 - p_k)^{-1} \right)
\]

9.3. Branching rules. For \( r \geq 0 \) we have an inclusion \( D_{(r_1, r_2)} \rightarrow D_{(r_1+1, r_2)} \). Here we study the decomposition of the irreducible \( D_{(r_1, r_2+1)} \)-modules restricted to a \( D_{(r_1, r_2+1)} \)-module.

**Proposition 9.3.1** ([Hal96 Theorem 3.16]). For \( r \geq p \geq 0 \) and \( V \) any \( A_p \)-module,

\[
\text{Inf}_{p}^{D_{(r_1, r_2+1)}}(V) \downarrow_{D_{(r_1, r_2)}}^{D_{(r_1, r_2+1)}} \cong \text{Inf}_{(p_1, p_2)}^{A_{(p_1+1, p_2)}}(V) \oplus \text{Inf}_{(p_1, p_2-1)}^{A_{(p_1, p_2-1)}}(V). \]

**Proof.** Let \( V \) be a representation of \( A_p = K(\mathfrak{S}_{p_1} \times \mathfrak{S}_{p_2}) \). We decompose the \( A_p \)-\( D_{(r_1, r_2+1)} \) bimodule \( KD(p, (r_1, r_2+1); p) \) restricted to \( D_r \) into a direct sum.

The set of diagrams in \( D(p, (r_1, r_2+1); p) \) whose last point in the bottom row to the right of the wall is matched with a point in the bottom row (necessarily to the left of the wall) generates a submodule...
isolomorphic to $KD((p_1 + 1, p_2), r; (p_1 + 1, p_2)) \downarrow A_{(p_1, p_2)}$. Informally, the isomorphism is given by moving the last point from the bottom row just to the left of the wall in the top row.

The remaining diagrams in $D(p, (r_1, r_2 + 1); p)$, whose last point in the bottom row is matched with a point in the top row generate a submodule isomorphic to $KD((p_1, p_2 - 1), r; (p_1, p_2 - 1)) \uparrow A_{(p_1, p_2 - 1)}$.

Tensoring with $V$ and applying the definition of induction we obtain

$$V \otimes_A K D((p_1 + 1, p_2), r; (p_1 + 1, p_2)) \downarrow A_{(p_1 + 1, p_2)}$$

$$\cong V \uparrow A_{(p_1 + 1, p_2)} \otimes A_{(p_1 + 1, p_2)} K D((p_1 + 1, p_2), r; (p_1 + 1, p_2))$$

and

$$V \otimes_A K D((p_1, p_2 - 1), r; (p_1, p_2 - 1)) \uparrow A_{(p_1, p_2 - 1)}$$

$$\cong V \uparrow A_{(p_1, p_2 - 1)} \otimes A_{(p_1, p_2 - 1)} K D((p_1, p_2 - 1), r; (p_1, p_2 - 1)).$$

\[ \square \]

**Corollary 9.3.2.** For $r \geq p \geq 0$ and $\lambda \vdash p$ put $U(\lambda, r) = \text{Inf}_p (S^\lambda \otimes S^\lambda)$. Then the branching rules are given by

$$U((r_1, r_2 + 1), \lambda) \downarrow_{r}^{(r_1, r_2 + 1)} \cong \bigoplus_{\mu = (\lambda_1 + \square, \lambda_2) \text{ or } \lambda = (\mu_1, \mu_2 + \square)} U(r, \mu).$$

**Proof.** The proof is completely analogous to the proof of Proposition 7.3.2 and therefore omitted. \[ \square \]

### 9.4. Fundamental theorems.

Let $T_{M(n)}$ be the category of mixed tensors, i.e., the monoidal category generated by the vector spaces $V$ and $V^*$, where $V$ is the defining representation of $\text{GL}(n)$ and $V^*$ the dual representation.

Let $D_{M(n)}$ be the diagram category $D_{\text{Br}(\delta)}$ with $\delta = n$.

Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ and $\{v^1, \ldots, v^n\}$ the dual basis.

**Definition 9.4.1.** For each $n > 0$, there is a functor $ev_{M(n)} : D_{M(n)} \to T_{M(n)}$ uniquely determined by the properties that $ev_{M(n)}(+) = V$ and $ev_{M(n)}(-) = V^*$ and that the functor is pivotal symmetric monoidal.

This functor is also determined by the condition that it is monoidal and its values on the monoidal generators. The monoidal generators are the diagrams obtained from [5] and [3] by taking all orientations. The values on the four diagrams obtained by orienting the diagrams in [5] are $\phi \times v \mapsto \langle \phi, v \rangle$ and $v \otimes \phi \mapsto \langle v, \phi \rangle$ for $v \in V$ and $\phi \in V^*$ together with $1 \mapsto \sum_i v_i \otimes \phi^i$ and $1 \mapsto \sum_i \phi^i \otimes v_i$ where $\{v_i\}$ is a basis of $V$ and $\{\phi^i\}$ is the dual basis of $V^*$.

This functor restricts to the walled Brauer category. This restriction is given on objects by $(r, s) \mapsto (\otimes^r V) \otimes (\otimes^s V^*)$. 
The first fundamental theorem for the mixed tensors can be stated as follows: see [Eld04], Theorem 9.4.2.

**Theorem 9.4.2.** [Wey97, Theorem (2.6A)] For $n > 0$, the evaluation functor $ev_{M(n)}: D_{M(n)} \to T_{M(n)}$ is full.

**Theorem 9.4.3** ([Han85], [BCH+94, Theorem 5.8]). Let $ev = ev_{M(n)}$ be the functor described above. Then the map $ev^{(s_1,s_2)}_{(r_1,r_2)}$ is injective for $2n \geq r_1 + r_2 + s_1 + s_2$.

The rank one central idempotents are constructed in [Wer14].

9.5. **Frobenius characters for tensor algebras.** Recall that the irreducible rational representations of $GL(n)$ are indexed by staircases of length $n$, i.e., sequences $\mu$ of $n$ integers satisfying $\mu_1 \geq \cdots \geq \mu_n$.

There is a staircase $[\alpha, \beta]_n$ for each pair of partitions $[\alpha, \beta]$ with $n \geq |\alpha| + |\beta|$. This is given by

$$[\alpha, \beta]_n = \left( \sum_j \alpha_j e_j \right) - \left( \sum_j \beta_j e_{n-j+1} \right)$$

**Theorem 9.5.1.** Let $\mu$ be a staircase of length $n$ and let $W(\mu)$ be the irreducible rational representation of $GL(n)$ with highest weight $\mu$. Then for $n \geq r_1 + r_2$ there is a natural isomorphism

$$\Hom_{GL(V)}(W(\mu) \otimes \otimes^{r_1} V \otimes \otimes^{r_2} V^*) \cong \text{Inf}_p^r(S^\mu)$$

of $D_r$-modules.

**Proof.** The classical branching rules are

$$W(\mu) \otimes V \cong \bigoplus_{\mu = (\lambda_1, \lambda_2)} \bigoplus_{\mu \geq (\lambda_1, \lambda_2)} W(\lambda)$$

$$W(\mu) \otimes V^* \cong \bigoplus_{\mu = (\lambda_1, \lambda_2)} \bigoplus_{\mu \geq (\lambda_1, \lambda_2)} W(\lambda),$$

the remainder of the proof is as in the other sections. \hfill $\square$

9.6. **Cyclic sieving phenomenon.** In contrast to the results in Section 7.6 and Section 8.6 we can only exhibit an instance of the cyclic sieving phenomenon for the case when $n \geq 2r$.

Let $X = X(r)$ be the set of permutations of $\{1, \ldots, r\}$ and let $\rho$ be the long cycle, i.e., $\rho$ acts on $\{1, \ldots, r\}$ as $\rho(i) = i \pmod{r} + 1$. Then conjugation by $\rho$ generates an action of the cyclic group of order $r$ on $X$.

Take a disc with $2r$ boundary points labelled clockwise by $1, 1', 2, 2', \ldots, r, r'$. Let $X$ be the set of directed perfect matchings directed from an unprimed label to a primed label; equivalently, $X$ is a bijection from the set of unprimed boundary points to the set of primed boundary points.
Let $\rho$ be the rotation map which moves each boundary point clockwise two places.

Then the map which sends a permutation $\pi$ to the directed matching

$$\{(1, \pi(1)'), (2, \pi(2)'), \ldots, (r, \pi(r)')\}$$

is a bijection compatible with the two cyclic actions.

These both have an action of $\mathfrak{S}_r$ and the bijection is compatible with these actions. The Frobenius character of this permutation representation is

$$\sum_{\lambda \vdash r} p_\lambda = \sum_{\lambda \vdash r} s_\lambda \ast s_\lambda$$

where $p_\lambda$ is a power sum function.

**Theorem 9.6.1.** Let $X$ be the set of permutations of $\{1, \ldots, r\}$ and let $\rho : X \to X$ be the rotation map, i.e., $\rho$ acts on $\{1, \ldots, r\}$ as $\rho(i) = i \pmod r + 1$. Let

$$P(q) = \text{fd ch} \ S_r = \text{fd} \sum_{\lambda \vdash r} p_\lambda,$$

where $S_r$ is the species of permutations of sets of cardinality $r$ and $p_\lambda$ are the power sum symmetric functions. Then the triple $(X, \rho, P(q))$ exhibits the cyclic sieving phenomenon.

When $n < 2r$, we can still compute the Frobenius character of $\text{Hom}_{TGL(n)}(0, r)$. Note that it is clear that $\text{Hom}_{T_{M(n)}}((0, 0), (r, r))$ is zero if $r_1 \neq r_2$.

**Theorem 9.6.2.** As a representation of $\mathfrak{S}_r \times \mathfrak{S}_r$,

$$\text{Hom}_{T_{M(n)}}((0, 0), (r, r)) \cong \bigoplus_{\lambda \vdash r, \ell(\lambda) \leq n} S^{\lambda} \otimes S^{\lambda}$$

**Proof.** This uses the results in §5.1.

Consider $\text{Hom}_{T_{M(n)}}((0, 0), (r, r))$ as a $K\mathfrak{S}_r$-$K\mathfrak{S}_r$ bimodule. This bimodule is isomorphic to the quotient of the regular representation of $K\mathfrak{S}_r$.

This quotient representation is isomorphic to the bimodule

$$\bigoplus_{\lambda \vdash r, \ell(\lambda) \leq n} S^{\lambda} \otimes S^{\lambda}$$

□

**Theorem 9.6.3.** There is an isomorphism of $\mathfrak{S}_r$-modules

$$U(r, [\emptyset, \emptyset]) \cong \sum_{\lambda \vdash r, \ell(\lambda) \leq n} S^{\lambda} \ast S^{\lambda}$$
There is a cyclic sieving phenomenon associated to each tensor power of the adjoint representation in [Wes10]. In order to have a combinatorial interpretation of these cyclic sieving phenomena we require a basis of the invariant tensors which is invariant under rotation. For $n = 2$ such a basis can be constructed using Temperley-Lieb diagrams, see [FK97]. For $n = 3$ such a basis is constructed in [Kup96]. A basis invariant under rotation is constructed for all $n$ in [Lus93] but this basis and the rotation map are not explicit.

References

[Bas68] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[BCH+94] Georgia Benkart, Manish Chakrabarti, Thomas Halverson, Robert Leduc, Chanyoung Lee, and Jeffrey Stroomer. Tensor product representations of general linear groups and their connections with Brauer algebras. *J. Algebra*, 166(3):529–567, 1994.

[BGJ+14] Georgia Benkart, Nicolas Guay, Ji Hye Jung, Seok-Jin Kang, and Stewart Wilcox. Quantum walled brauer-clifford superalgebras, 2014.

[BLL98] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[Bra37] Richard Brauer. On algebras which are connected with the semisimple continuous groups. *Ann. of Math. (2)*, 38(4):857–872, 1937.

[Cae98] Stefaan Caenepeel. *Brauer groups, Hopf algebras and Galois theory*, volume 4 of *K-Monographs in Mathematics*. Kluwer Academic Publishers, Dordrecht, 1998.

[CDV11] Anton Cox and Maud De Visscher. Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra. *J. Algebra*, 340:151–181, 2011.

[CDVDM08] Anton Cox, Maud De Visscher, Stephen Doty, and Paul Martin. On the blocks of the walled Brauer algebra. *J. Algebra*, 320(1):169–212, 2008.

[CDVM09] Anton Cox, Maud De Visscher, and Paul Martin. The blocks of the Brauer algebra in characteristic zero. *Represent. Theory*, 13:272–308, 2009.

[Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[Eld04] Alberto Elduque. A modified Brauer algebra as centralizer algebra of the unitary group. *Trans. Amer. Math. Soc.*, 356(10):3963–3983 (electronic), 2004.

[FK97] Igor B. Frenkel and Mikhail G. Khovanov. Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$. *Duke Math. J.*, 87(3):409–480, 1997.

[Gav99] Fabio Gavarini. A Brauer algebra-theoretic proof of Littlewood’s restriction rules. *J. Algebra*, 212(1):240–271, 1999.

[GFM07] Juan González-Férez and Leandro Marín. The category of firm modules need not be abelian. *J. Algebra*, 318(1):377–392, 2007.

[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.
[Gre80] James A. Green. Polynomial representations of $GL_n$, volume 830 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1980.

[GV98] F. Grandjean and E. M. Vitale. Morita equivalence for regular algebras. Cahiers Topologie Géom. Différentielle Catég., 39(2):137–153, 1998.

[Hal96] Tom Halverson. Characters of the centralizer algebras of mixed tensor representations of $GL(r, \mathbb{C})$ and the quantum group $U_q(gl(r, \mathbb{C}))$. Pacific J. Math., 174(2):359–410, 1996.

[Han85] P. Hanlon. On the decomposition of the tensor algebra of the classical Lie algebras. Adv. in Math., 56(3):238–282, 1985.

[Hartmann et al.] Robert Hartmann, Anne Henke, Steffen Koenig, and Rowena Paget. Cohomological stratification of diagram algebras. Math. Ann., 347(4):765–804, 2010.

[HR05] Tom Halverson and Arun Ram. Partition algebras. European J. Combin., 26(6):869–921, 2005.

[Jon94] V. F. R. Jones. The Potts model and the symmetric group. In Subfactors (Kyuzeso, 1993), pages 259–267. World Sci. Publ., River Edge, NJ, 1994.

[Joy81] André Joyal. Une théorie combinatoire des séries formelles. Adv. in Math., 42(1):1–82, 1981.

[Joy86] André Joyal. Foncteurs analytiques et espèces de structures. In Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985), volume 1234 of Lecture Notes in Math., pages 126–159. Springer, Berlin, 1986.

[JS91] André Joyal and Ross Street. The geometry of tensor calculus. I. Adv. Math., 88(1):55–112, 1991.

[KL80] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. J. Pure Appl. Algebra, 19:193–213, 1980.

[Kup96] Greg Kuperberg. Spiders for rank 2 Lie algebras. Comm. Math. Phys., 180(1):109–151, 1996.

[KX99] Steffen König and Changchang Xi. Cellular algebras: inflations and Morita equivalences. J. London Math. Soc. (2), 60(3):700–722, 1999.

[Li14] Yanbo Li. Cellularity of a new class of diagram algebras. Comm. Algebra, 42(10):4172–4181, 2014.

[Lit58] D. E. Littlewood. Products and plethysms of characters with orthogonal, symplectic and symmetric groups. Canad. J. Math., 10:17–32, 1958.

[Lus93] George Lusztig. Introduction to quantum groups, volume 110 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1993.

[Mar94] Paul Martin. Temperley-Lieb algebras for nonplanar statistical mechanics—the partition algebra construction. J. Knot Theory Ramifications, 3(1):51–82, 1994.

[Mat64] Hideya Matsumoto. Générateurs et relations des groupes de Weyl généralisés. C. R. Acad. Sci. Paris, 258:3419–3422, 1964.

[ML63] Saunders Mac Lane. Natural associativity and commutativity. Rice Univ. Studies, 49(4):28–46, 1963.

[MM13] Paul Martin and Volodymyr Mazorchuk. Partitioned binary relations. Math. Scand., 113(1):30–52, 2013.
[Mor58] Kiiti Morita. Duality for modules and its applications to the theory of rings with minimum condition. *Sci. Rep. Tokyo Kyokai Daigaku Sect. A*, 6:83–142, 1958.

[MW99] P. Martin and D. Woodcock. On central idempotents in the partition algebra. *J. Algebra*, 217(1):156–169, 1999.

[Pro07] Claudio Procesi. *Lie groups*. Universitext. Springer, New York, 2007. An approach through invariants and representations.

[Red27] J. Howard Redfield. The Theory of Group-Reduced Distributions. *Amer. J. Math.*, 49(3):433–455, 1927.

[RSW04] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1):17–50, 2004.

[RSW14a] Victor Reiner, Dennis Stanton, and Dennis White. What is . . . cyclic sieving? *Notices Amer. Math. Soc.*, 61(2):169–171, 2014.

[RSW14b] Martin Rubey, Bruce E. Sagan, and Bruce W. Westbury. Descent sets for symplectic groups. *J. Algebraic Combin.*, 40(1):187–208, 2014.

[ST94] Thomas Scharf and Jean-Yves Thibon. A Hopf-algebra approach to inner plethysm. *Adv. Math.*, 104(1):30–58, 1994.

[Sta84] Richard P. Stanley. The stable behavior of some characters of SL(n, C). *Linear and Multilinear Algebra*, 16(1-4):3–27, 1984.

[Sta88] Richard P. Stanley. Differential posets. *J. Amer. Math. Soc.*, 1(4):919–961, 1988.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ste87] John R. Stembridge. Rational tableaux and the tensor algebra of gl_n. *J. Combin. Theory Ser. A*, 46(1):79–120, 1987.

[Sun86] Sheila Sundaram. *ON THE COMBINATORICS OF REPRESENTATIONS OF THE SYMPLECTIC GROUP*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[Tay82] Joseph L. Taylor. A bigger Brauer group. *Pacific J. Math.*, 103(1):163–203, 1982.

[Thu04] Dylan Thurston. From dominoes to hexagons. 2004.

[Tok87] Takeshi Tokuyama. Highest weight vectors associated with some branchings. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 541–545. Amer. Math. Soc., Providence, RI, 1987.

[Vit96] Enrico M. Vitale. The Brauer and Brauer-Taylor groups of a symmetric monoidal category. *Cahiers Topologie Géom. Différentielle Catég.*, 37(2):91–122, 1996.

[Wen88] Hans Wenzl. On the structure of Brauer’s centralizer algebras. *Ann. of Math. (2)*, 128(1):173–193, 1988.

[Wer14] Mathias Werth. Symmetrizer for the Quantized Walled Brauer Algebras. *Comm. Algebra*, 42(11):4839–4853, 2014.

[Wes10] Bruce W. Westbury. Invariant tensors and the cyclic sieving phenomenon, 2010.

[Wes14] Bruce W. Westbury. On enumeration in classical invariant theory, 2014.

[Wey97] Hermann Weyl. *The classical groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Their invariants and representations, Fifteenth printing, Princeton Paperbacks.
Stewart Wilcox. Cellularity of diagram algebras as twisted semigroup algebras. *J. Algebra*, 309(1):10–31, 2007.

Mariusz Wodzicki. Excision in cyclic homology and in rational algebraic $K$-theory. *Ann. of Math. (2)*, 129(3):591–639, 1989.

Fakultät für Mathematik und Geoinformation, TU Wien, Austria

E-mail address: martin.rubey@math.uni-hannover.de

Department of Mathematics, University of Warwick, Coventry, CV4 7AL

E-mail address: Bruce.Westbury@warwick.ac.uk