Mirror Maps and Instanton Sums for Complete Intersections in Weighted Projective Space

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Abstract

We consider a class of Calabi-Yau compactifications which are constructed as a complete intersection in weighted projective space. For manifolds with one Kähler modulus we construct the mirror manifolds and calculate the instanton sum.

1 Introduction

When considering symmetric $(2, 2)$ super-conformal field theories as internal conformal field theories relevant for string theory, one immediately observes that due to the arbitrariness in the assignment of the relative sign of the two $U(1)$ charges, there is a complete symmetry between two theories for which the $(a, c)$ and the $(c, c)$ rings are interchanged [1][2]. On the geometrical level, when considering conformal field theories which admit an interpretation as compactification on Calabi-Yau manifolds, the fields in the two rings correspond to the Kähler and the complex structure moduli of the Calabi-Yau space, respectively. The observation on the conformal field theory level mentioned above, then lead to the so-called mirror hypothesis. In rough terms, it states that for each Calabi-Yau manifold there should exist one with the two hodge numbers $h_{1,1}$ and $h_{2,1}$ interchanged and that string propagation on these two manifolds should be identical (for further details on mirror symmetry, see [3]). The special case of manifolds with $h_{2,1} = 0$ is discussed in [4].

This mirror hypothesis has to date not been proven yet, but looking at the Hodge numbers of Calabi-Yau spaces constructed so far, one observes a rough symmetry and for some cases...
mirror manifolds have been constructed. Here we add some more examples: manifolds with \( h_{1,1} = 1 \) which are given as complete intersections in weighted projective space.

Aside from being interesting from a mathematical point of view, mirror symmetry can also be put to practical use for the computation of certain physical quantities of the low energy effective field theory such as the Yukawa couplings of charged matter fields or the Kähler metric for the moduli fields. This is so since whereas the Yukawa couplings of three 27's fields, which are related to the (2,1) moduli fields by \( n = 2 \) superconformal symmetry, receive no \( \sigma \)-model corrections (perturbatively and non-perturbatively), the couplings of three 27's, related to the (1,1) moduli, do receive corrections from world sheet instantons and are thus (probably) impossible to compute directly. The former couplings depend on the complex structure moduli, the latter are functions of the Kähler moduli and are given by

\[
\kappa_{ijk} = \int_X b_i^{(1,1)} \wedge b_j^{(1,1)} \wedge b_k^{(1,1)} + \sum_{m,n} m^{-3} e^{-\int_{I_{m,n}} \tau(J)} \int_{I_{m,n}} \tau^*(b_i^{(1,1)}) \int_{I_{m,n}} \tau^*(b_j^{(1,1)}) \int_{I_{m,n}} \tau^*(b_k^{(1,1)}).
\]

Here, \( b_i^{(1,1)} \) are (1,1) forms and \( I_{m,n} \) is the \( m \)-fold cover of a rational curve on \( X \) of degree \( n \), \( \tau \) its inclusion \( I_{m,n} \hookrightarrow X \). The Kähler form \( J = \sum_i h_{1,1} t_i b_i^{(1,1)} \) is expanded in the basis of \( H^3(X, \mathbb{Z}) \) s.t. the moduli \( t_i \) parameterize the complexified Kähler cone. The first term is the sigma model tree level contribution and the second term the contribution from world-sheet instantons.

The mirror hypothesis now amounts to the identification of this coupling with the corresponding coupling on the mirror manifold \( X' \), which is a function of the (2,1) moduli \( t'_i \) of which there is an equal number as there are (1,1) moduli on the original manifold. The relation between \((b_i^{(1,1)}(X), t_i) \leftrightarrow (b_i^{(2,1)}(X'), t'_i)\) is described by the mirror map. The 27^3 couplings on \( X' \) can be computed, at least in principle, from knowledge of the periods of the holomorphic three-form, i.e. the solutions of the Picard-Fuchs equations, and special geometry. We will extend the list of models which have been treated along these lines in refs. to models with \( h(1,1) = 1 \) which are given as complete intersections in weighted projective space.

2 The manifolds and their mirrors

There exist examples of complete intersections with \( c_1 = 0 \) in ordinary projective complex space: two cubics in \( \mathbb{P}^5 \), a quartic and a quadric in \( \mathbb{P}^5 \), two quadrics and a cubic in \( \mathbb{P}^6 \) and four quadrics in \( \mathbb{P}^7 \). Passing to weighted projective spaces as ambient space, several thousand Calabi-Yau manifolds can be constructed. As in the hypersurface case, the complete intersections inherit, in most of the cases, cyclic quotient singularities from the ambient space and canonical desingularizations are required to obtain a smooth manifold. As the process of the canonical desingularization introduces irreducible exceptional divisors, the dimension of the fourth homology group and hence the number of (1,1)-forms or Kähler moduli for these examples is usually bigger than one. Within the class under consideration we have only the following five families listed in Table 1, which are, as the intersection locus avoids all singularities of the ambient space, one Kähler modulus examples. The Hodge diamond can be calculated by the general formulas of, which automatically take care of the desingularizations, but as we deal here only with smooth intersections, we will use the somewhat simpler adjunction formula

\[\text{[1]}\]

\[\text{The list in II.4.6. of \[14\] is not complete.}\]
to get the Euler characteristic of the \((m - 1) - k\)-dimensional smooth complete intersection
\[ X := X_{(d_1, \ldots, d_k)} \in \mathbb{P}^{m-1}(w) \]
of \(k\) polynomial constraints \(p_1 = \ldots = p_k = 0\) in \(\mathbb{P}^{m-1}(w)\) of degree \(d_1, \ldots, d_k\) as the coefficient of \(J^{m-1}\) in the formal expansion in \(J\) of the quotient
\[
\frac{\prod_{i=1}^{m}(1 + w_i J) \prod_{j=1}^{k} d_j J}{\prod_{j=1}^{k}(1 + d_j J) \prod_{i=1}^{m} w_i}.
\]

(2.1)

| \(N^0\) | \(X = X_{(d_1, d_2)} \subset \mathbb{P}^{5}(w)\) | \(\chi(X)\) | \(h^{1,2}(X)\) | \(h^{1,1}(X)\) |
|---|---|---|---|---|
| 1 | \(X_{(4,4)} \subset \mathbb{P}^{5}(1,1,1,1,1,2)\) | -144 | 73 | 1 |
| 2 | \(X_{(6,6)} \subset \mathbb{P}^{5}(1,2,3,1,2,3)\) | -120 | 61 | 1 |
| 3 | \(X_{(4,3)} \subset \mathbb{P}^{5}(2,1,1,1,1,1)\) | -156 | 79 | 1 |
| 4 | \(X_{(6,2)} \subset \mathbb{P}^{5}(3,1,1,1,1,1)\) | -256 | 129 | 1 |
| 5 | \(X_{(6,4)} \subset \mathbb{P}^{5}(3,2,2,1,1,1)\) | -156 | 79 | 1 |

Table 1: Complete intersections in weighted projective space with one Kähler modulus

To construct the mirror intersections we first have to specify a configuration of the polynomial constraints, which is transversal in \(\mathbb{P}^{m-1}(w)\) for almost all values of the complex deformation parameters. This means that all \(k \times k\) subdeterminants of \(\left( \frac{\partial w_i}{\partial x_j} \right)\) are allowed to vanish for generic values the deformation parameter only at \(x_1 = \ldots = x_m = 0\). As in [14] one can use Bertinis Theorem to formulate a transversality criterium as the requirement that certain monomials occurs in the polynomials. Let \(J = \{j_1, \ldots, j_l\} \in \{1, \ldots, m\}\) be an index set and \(X^{M_j} := x_{j_1}^{m_1} \ldots x_{j_l}^{m_l}\) (where \(m_j \in \mathbb{N}_0\)) a monomial. For \(k = 2\) (see [14] Theorem I.5.7 and [11] for generalisations) transversality requires for all index sets \(J\) that there occur

(a) either a monomial \(X_j^{M_j}\) in \(p_1\) and a monomial \(X_j^{M_j}\) in \(p_2\)

(b) or a monomial \(X_j^{M_j}\) in \(p_1\) and \(|J| - 1\) monomials \(X_j^{M_j} x_{e_i}\) in \(p_2\) with distinct \(e_i\)’s

(c) or (b) with \(p_1\) and \(p_2\) interchanged

(d) or \(|J|\) monomials \(X_j^{M_j} x_{e_i}\) in \(p_1\) with distinct \(e_i\)’s and \(|J|\) monomials \(X_j^{M_j} x_{e_j}\) in \(p_2\) with distinct \(e_j\)’s s.t. \(\{e_1^1, e_2^2\}\) contains at least \(|J| + 1\) distinct elements

We then search for a discrete isomorphy group \(G\) which leaves the holomorphic \((3,0)\) form (see [5,11] below) invariant. In the cases where we have succeeded in constructing the mirror, \(G\) is Abelian and can be represented by phase multiplication

\[ x_i \mapsto \exp(2\pi i g_i) x_i \quad i = 1, \ldots, m \]

(2.2)

\(g_i \in \mathbb{Q}\) on the homogeneous \(\mathbb{P}^{m-1}(w)\) coordinates. The singular orbifold \(X/G\) has precisely the same type of Gorenstein singularities which one encounters in the cases of complete intersections.
or hypersurfaces in weighted projective spaces and the canonical desingularization $X' = X/G$ has the “mirror” Hodge diamond with $h_{i,j}(X/G) = h_{i,3-j}(X)$.

Let us give the explicit construction for the two quartics $X_{(4,4)}$ in $\mathbb{P}^5(1,1,2,1,1,2)$. As the one parameter family we choose the following form of the polynomial constraints (comp. [10])

$$p_1 = x_1^4 + x_2^4 + 2x_3^2 - 4\alpha x_4x_5x_6 = 0, \quad p_2 = x_4^4 + x_5^4 + 2x_6^2 - 4\alpha x_1x_2x_3 = 0, \quad (2.3)$$

which is transversal for almost all values of the deformation parameter $\alpha$, except $\alpha = 0$, $\alpha^8 = 1$ or $\alpha = \infty$.

The maximal symmetry group $G$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}$ and we may choose the following generators:

$$g^{(1)} = \frac{1}{2}(0, 1, 1, 0, 0, 0), \quad g^{(2)} = \frac{1}{2}(0, 0, 0, 0, 1, 1), \quad g^{(3)} = \frac{1}{16}(0, 4, 0, 1, 13, 2).$$

The group action is understood as in (2.2). $G$ has four $\mathbb{Z}_2$ subgroups, which leave four curves $C_1^{(-24)}, C_2^{(-24)}, C_4^{(-24)}, C_5^{(-24)}$ of type $X_{(4,4)}(1,1,1,2)$ invariant. They are specified by the subsets of $X$ with $(x_2 = x_3 = 0)$, $(x_1 = x_3 = 0)$, $(x_4 = x_6 = 0)$ and $(x_5 = x_6 = 0)$ respectively and we may calculate their Euler characteristic (indicated as right upper index) by (2.1). The $\mathbb{Z}_2$ acts on the coordinates of the fibers of the normal bundle to the curves by $g = \frac{1}{2}(1,1)$ and the canonical desingularization of the $A_1$-type $\mathbb{C}^2/\mathbb{Z}_2$ singularity in the fibers adds one exceptional divisor per curve. Moreover we have two $\mathbb{Z}_4$ subgroups of $G$ leaving the curves $C_3^{(-8)} (x_1 = x_2 = 0)$ and $C_6^{(-8)} (x_4 = x_5 = 0)$ of type $X_{(4,4)}(1,1,2,2)$ invariant. The corresponding action on the coordinates of the normal fiber is of type $g = \frac{1}{4}(1,3)$ and gives rise to a $A_3$-type singularity, which adds, by canonical desingularization, three exceptional divisors per curve. So we expect 10 irreducible exceptional divisors from desingularization of the fixed curves. Note that the curves $C_1, C_2, C_3$ have triple intersections in eight points, which we group into the following fixed point sets $P^{(2)}_{1,2} (x_1, x_2 = x_4 = x_5 = x_6 = 0)$, $P^{(4)}_3 (x_3 = x_4 = x_5 = x_6 = 0)$, where the multiplicity of the points in $X$ is again indicated as left upper index. It can also be obtained from (2.1) as a point has Euler number one. Obviously to get their multiplicity on the orbifold $X/G$, we have to multiply by a factor $|I|/|G|$ where $|I|$ and $|G|$ are the orders of the isotropy group of the set and of $G$, respectively. Analogous remarks apply to the intersection of $C_4, C_5, C_6$ and $P^{(2)}_{4,5} (x_4 = x_1 = x_2 = x_3 = 0)$, $P^{(4)}_6 (x_6 = x_1 = x_2 = x_3 = 0)$.

The schematic intersection pattern of the fixed set singularities is depicted in Fig. 1.

\[\begin{array}{ccc}
\mathbb{Z}_2 \times \mathbb{Z}_{16} & \mathbb{Z}_2 \times \mathbb{Z}_{16} & \mathbb{Z}_2 \times \mathbb{Z}_8 \\
\mathbb{Z}_2 : C_1^{(-24)} & \mathbb{Z}_2 : C_4^{(-24)} & \mathbb{Z}_2 : C_5^{(-24)} \\
P_1 & P_4 & P_6 \\
P_2 & P_5 & \\
P_3 & & \\
\mathbb{Z}_2 : C_2^{(-24)} & \mathbb{Z}_4 : C_3^{(-8)} & \mathbb{Z}_4 : C_6^{(-8)} \\
\end{array}\]

Fig. 1
The isotropy group $I$ of the points $P_{1,2}$ and $P_{4,5}$ is $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ whose generators in a local coordinate system $(x_1, x_2, x_3)$ with the fixed point as origin may be represented as

$$g^{(1)} = \frac{1}{2}(0, 1, 1), \quad g^{(2)} = \frac{1}{16}(1, 5, 10).$$

That is locally, we have an Abelian $\mathbb{C}^3 / (\mathbb{Z}_2 \times \mathbb{Z}_{16})$ singularity, whose canonical desingularisation can be constructed by toric geometry [15],[16],[17],[11]. The $\mathbb{Z}_2 \times \mathbb{Z}_8$ isotropy group $I$ of the points $P_3, P_6$ is generated by

$$g^{(1)} = \frac{1}{2}(0, 1, 1), \quad g^{(2)} = \frac{1}{8}(1, 2, 5).$$

We show in figure (2 a) and (2 b) the side $\Delta$ (trace) opposite to the apex of the three dimensional simplicial fan which, together with the lattice $\Lambda$, describes the topological data of the local desingularization processes for the two types of fixed points. The points are the intersection of $\Lambda$ with $\Delta$. Their location is given (see eg. [17]) by

$$\mathcal{P} = \left\{ \sum_{i=1}^{3} \tilde{e}_i g_i \left| (g_1, g_2, g_3) \in \mathbb{Q}^3, \begin{pmatrix} e^{2\pi i g_1} & e^{2\pi i g_2} & e^{2\pi i g_3} \end{pmatrix} \in I, \sum_{i=1}^{3} g_i = 1, g_i \geq 0 \right\}, \quad (2.4)$$

where the vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ span the equilateral triangle from its center. According to the general theory [16, 17], points in the inside of the triangle correspond to new exceptional divisors while points on the sides of the triangles correspond to exceptional divisors which are also present over the generic points of the curves, which intersect in the fixed point. Counting the inner points in Fig (2 a,b) we see that each desingularization of the four $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ orbifold points adds 13 new exceptional divisors, while each desingularization of the two $\mathbb{Z}_2 \times \mathbb{Z}_8$ orbifold singularities adds 5 new exceptional divisors. Together with the ones over the curves we have 72 exceptional divisors from the desingularization and adding the pullback of the Kähler form of the ambient space we obtain $h_{1,1}(\hat{X}/G) = 73$.

Fig. 2 a) Trace of the $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ fan.  
Fig. 2 b) Trace of the $\mathbb{Z}_2 \times \mathbb{Z}_8$ fan.
The Euler characteristic $\chi(X/G)$ is most easily calculated by the "orbifold" formula of \[18\], which for the Abelian case simplifies to

$$\chi(X/G) = \frac{1}{|G|} \left( \chi(X) - \sum_I \chi(S_I) \right) + \sum_I \frac{|I|^2}{|G|} \chi(S_I),$$

(2.5)

where the sum is over all subsets $S_I$ of $X$ which are fixed under the isotropy groups $I \in G$. Application to the case at hand yields $\chi(X/G) = \frac{1}{64} (-144 - 2 \cdot 8 - 4 \cdot (-24 - 8) - 2 \cdot (-8 - 8)) + \frac{-16^2 + 8 \cdot 32^2 + 4 \cdot (-32)^2}{64} + \frac{2 \cdot (-16) \cdot x^2}{64} = 144$, hence we have indeed constructed a mirror configuration.

On the following transversal one parameter family $p_1 = x_1^6 + 2x_3^3 + 3x_6^2 - 6 \alpha x_1 x_2 x_3 = 0$, $p_2 = x_1^4 + 2x_3^2 + 3x_6^2 - 6 \alpha x_1 x_2 x_3 = 0$ of the second example, we have a group action $g = \frac{1}{36} (6, 0, 0, 1, 14, 21)$, which has two fixed curves of order two, $C^{(-12)}_1$ ($A_1$-type), and two of order three, $C^{(-6)}_2$ ($A_2$-type). $C^{(-12)}_1$ and $C^{(-6)}_2$ intersect in six points namely three of order 12, two of order 18 and one of order 36, which have the following generators for their isotropy groups: \[\frac{1}{12} (1, 2, 9), \frac{1}{18} (1, 14, 3)\) and $\frac{1}{36} (1, 14, 21)$. An analogous pattern occurs for $C^{(-12)}_2$ and $C^{(-6)}_4$.

Applying (2.4) and (2.5) the reader may check as above that the resolved $X/G$ has the mirror Hodge diamond.

3 The Picard-Fuchs equations and their solutions

Since the dimension of the third cohomology of the manifolds $X'$ considered here is four, there must exist a linear relation between the holomorphic three-form $\Omega(\alpha)$ as a function of the single complex structure modulus $\alpha$ and its first four derivatives of the form $\sum_{i=0}^4 f_i(\alpha) \Omega^{(i)}(\alpha) = d\beta$. Integration of this relation over an element of the third homology $H_3(M, \mathbb{Z})$ gives the Picard-Fuchs equation for the periods $\omega(\alpha)$.

As shown in refs \[19\] (see also: \[20\] \[10\] \[21\]) the periods can be written as

$$\omega_i = \int_{\gamma_1} \int_{\gamma_2} \int_{\Gamma_i} \frac{\omega}{p_1 p_2}$$

(3.1)

where

$$\omega = \sum_{i=1}^m (-1)^j x_1 dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_m$$

$\Gamma_i$ is an element of $H_3(X, \mathbb{Z})$ and $\gamma_i$ are small curves around $p_i = 0$ in the $m$-dimensional embedding space.

For the derivation of the Picard-Fuchs equation it is crucial to note that $\frac{\partial}{\partial x_i} \left( \frac{f}{p_1 p_2} \right) \omega$ is exact, which leads to the following partial integration rule:

$$\frac{f \frac{\partial p_1}{\partial x_i}}{p_1^{m-1} p_2^n} = \frac{1}{m-1} \frac{\partial f}{\partial x_i} - \frac{n}{m-1} \frac{f \frac{\partial p_2}{\partial x_i}}{p_1^{m-1} p_2^n}$$

We have used these rules to compute the Picard-Fuchs equations for those models for which we could construct the mirrors.
Let us demonstrate this on the example of the $\mathbb{Z}_2$ torus given by two quadrics $X_{(2,2)}$ in $\mathbb{P}^3(1,1,1,1)$:

$$p_1 = \frac{1}{2}(x_1^2 + x_2^2 - 2\alpha x_3 x_4) = 0, \quad p_2 = \frac{1}{2}(x_3^2 + x_4^2 - 2\alpha x_1 x_2) = 0. \quad (3.2)$$

We use the notation

$$\int \frac{x_1^i \cdots x_4^i}{p_1^m p_2^n} = \binom{i_1 \ i_2 \ i_3 \ i_4 \ m \ m \ n \ n}{n \ n}.$$

We can then integrate by parts with respect to, say, $x_1$, either by writing $x_1 = \partial p_1 / \partial x_1$ or using $x_1 = -\frac{1}{\alpha} \partial p_2 / \partial x_1$. This leads to the following partial integration rules (with similar rules for partial integration w.r.t. $x_{2,3,4}$):

$$(i_1 \ i_2 \ i_3 \ i_4 \ m \ n) = \frac{m-1}{\alpha(n-1)} \left( i_1 - 2 \ i_2 \ i_3 \ i_4 \ m - 1 \ n + 1 \right) + \frac{n}{m-1} \left( i_1 - 1 \ i_2 + 1 \ i_3 \ i_4 \ m - 1 \ n + 1 \right)$$

and

$$= -\frac{11}{\alpha(n-1)} \left( i_1 - 1 \ i_2 - 1 \ i_3 \ i_4 \ m - 1 \ n - 1 \right) + \frac{m}{\alpha(n-1)} \left( i_1 + 1 \ i_2 - 1 \ i_3 \ i_4 \ m + 1 \ n - 1 \right) \quad (b)$$

One then finds

$$\left( \begin{array}{ccc} 2 & 2 & 1 \\ 0 & 0 & 3 \end{array} \right) = -\frac{1}{\alpha} \left( \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & 2 \end{array} \right) + \frac{1}{4\alpha} \left( \begin{array}{ccc} 1 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right) + \frac{1}{2} \left( \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & 2 \end{array} \right).$$

where in the first step we have used rule (b) twice once for $x_1$ and once for $x_2$; to obtain the second line we have substituted $x_1^2 + x_2^2 = 2p_1 + \alpha x_3 x_4$. This leads to $\omega'' = 4 \left( \frac{1}{1} \frac{1}{1} \frac{1}{2} \right) - \frac{1}{\alpha} \omega'$. By repeated use of the partial integration rules, we can express $\left( \begin{array}{ccc} 1 & 1 & 2 \\ 0 & 0 & 2 \end{array} \right)$ in terms of $\omega$ and $\omega'$ and finally arrive at the period equation for the $\mathbb{Z}_2$ torus

$$\alpha(1 - \alpha^4)\omega'' + (1 - 5\alpha^4)\omega' - 4\alpha^3\omega = 0.$$

In the same way we have derived the following differential equations for the periods of the two models under investigation:

$$\alpha^3(1 - \alpha^8)\omega^{(iv)} - 2\alpha^2(1 + 7\alpha^8)\omega''' - \alpha(1 + 55\alpha^8)\omega'' + (9 - 65\alpha^8)\omega' - 16\alpha^7\omega = 0$$

and

$$\alpha^3(1 - \alpha^{12})\omega^{(iv)} - 2\alpha^2(5 + 7\alpha^{12})\omega''' + \alpha(23 - 55\alpha^{12})\omega'' + (49 - 65\alpha^{12})\omega' - 16\alpha^{11}\omega = 0$$

The equations have regular singular points at $\alpha = 0$, $\alpha^{d_1+d_2} = 1$ and $\alpha + \infty$. The solutions of the indicial equations are $\{0_2, 4_2\}$, $\{0, 1_2, 2\}$ and $\{2_4\}$ respectively, for our first model and $\{0_2, 8_2\}$, $\{0, 1_2, 2\}$ and $\{2_4\}$ for our second model (the subscripts denote multiplicities). We note that in contrast to the models considered in $\mathbb{E}$ $\mathbb{E}$ $\mathbb{S}$ there are two logarithmic solutions at $\alpha = 0$.

In terms of the variable $z = \alpha^{-(d_1+d_2)}$ these equations may be rewritten in the form

$$((\Theta^4 - z(\Theta + a_1))(\Theta + a_2)(\Theta + a_3)(\Theta + a_4))\tilde{\omega} = 0$$

where $\Theta = z^{1/4}$ and $\tilde{\omega} = \alpha^2\omega$. The two cases correspond to the parameters $\{a_i\} = \{\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}\}$ and $\{\frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}\}$, respectively. We will drop the tilde in $\tilde{\omega}$ in the following.
4 The Yukawa couplings and the instanton numbers

The simplest way to arrive at the mirror map, the Yukawa couplings and the number of instantons is to follow [6] (see also [10]). This requires merely the knowledge of two solutions of the period equation in the neighbourhood of the singular point $\alpha = \infty (\sim z = 0)$, namely the pure power series solution $\varpi_0(z)$ and the solution with one power of log $z$, $\varpi_1(z)$.

If we normalize $\varpi_0(z)$ as $\varpi_0(z) = 1 + O(z)$, it is given as

$$\varpi_0(z) = \sum_{n=0}^{\infty} \frac{4}{n!} \prod_{i=1}^{k} (a_i)^n z^n \equiv \sum_{n=0}^{\infty} \prod_{i=1}^{k} (d_i n)! (\gamma^{-1} z)^n$$

where $k$ is the number of polynomial constraints (for our threefolds, $d - k = 4$) and $\gamma = 2^{12}$ and $\gamma = 2^{8} 3^{6}$ for the two models, respectively. Notice that this solution may be represented as the multiple contour integral

$$\oint \frac{dx_1 \ldots dx_m}{p_1 p_2},$$

where we expand the integrand for $\alpha \to \infty$. This corresponds to the explicit evaluation of (3.1) with a judicious choice for the cycle $\gamma$.

We now introduce the variable $x = z/\gamma$ and normalize $\varpi_1$ such that

$$t(x) \equiv \frac{\varpi_1(x)}{\varpi_0(x)} \sim \log x \quad \text{for} \quad x \to 0$$

This relation describes the mirror map. More explicitly, if we make the ansatz $\varpi_1(x) = \sum_{n=0}^{\infty} d_n x^n + c \varpi_0(x) \log x$ we find $c = 1$, and $d_0 = 0$, and for the $d_n, n > 0$ a recursion relation. The fully instanton corrected Yukawa coupling on the manifolds $X'$ is then

$$\kappa_{\text{ttt}} = -\kappa_0^{\text{ttt}} W^3 (\gamma x(q) - 1)$$

where $W = \varpi_0 \Theta \varpi_1 - \varpi_1 \Theta \varpi_0$, $x(q)$ is the inverse function of $q(x) \equiv \exp(t(x))$ and $\kappa_0^{\text{ttt}}$, the infinite radius limit of the Yukawa coupling, is the intersection number $\kappa_0^{\text{ttt}} = \int_M J^3 = \prod_{i=1}^{k} d_i / \prod_{j=1}^{d} w_j \ (\{4; 1\} in our examples).

As conjectured in [5] and proven in [22], $\kappa_{\text{ttt}}$ can be expanded as

$$\kappa_{\text{ttt}} = \kappa_0^{\text{ttt}} + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d}$$

where $n_d$ denotes the number of rational curves of degree $d$. (The denominator arises from summing over all multiple covers.)

With the given information it is now straightforward to compute the $n_d$. We have listed the first few in Table 2.

5 Discussion

There are three models left in Table 1. For those we have not been able to construct the mirror by finding a symmetry group $G$ such that the mirror manifold $X'$ is given as $X' = \hat{X}/G$.

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2We would like to thank P. Candelas for discussions on this point and for providing us with a preliminary version of [4].
get information about the mirrors of these models we may, inspired by the success of ref. [10], extrapolate our knowledge of the parameters $a_i$ of the period equation to these models. This leads to the following choices for the three models in turn: $\{a_i\} = \{1, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}\}, \{\frac{1}{6}, \frac{2}{3}, \frac{5}{6}, \frac{1}{2}\}, \{\frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}\}$. The corresponding values for $\gamma$ are $\gamma = \{3^32^6; 3^32^8; 3^32^{10}\}$. With $\kappa_{ttt}^0 = \{6; 4; 2\}$ we do indeed find integer $n_d$’s. The first few are listed in Table 2. We thus conjecture that with these choices for the parameters we do correctly describe the periods and thus the Yukawa couplings of the mirror manifolds, even though we do not know an explicit description for them.

| $N^0 = 1$ | $N^0 = 2$ | $N^0 = 3$ | $N^0 = 4$ | $N^0 = 5$ |
|---|---|---|---|---|
| $n_0$ | 4 | 1 | 6 | 4 |
| $n_1$ | 3712 | 67104 | 15552 | 4992 |
| $n_2$ | 982464 | 847288224 | 223560 | 2388768 |
| $n_3$ | 683478144 | 28583248229280 | 64754568 | 2732060032 |
| $n_4$ | 699999511744 | 1431885139218998016 | 27482893704 | 4599616564224 |
| $n_0$ | 6 | 4 | 4 | 2 |
| $n_1$ | 1944 | 4992 | 15552 | 27904176 |
| $n_2$ | 223560 | 2388768 | 223560 | 13388454688 |
| $n_3$ | 64754568 | 2732060032 | 2732060032 | 950676829466832 |

Table 2: The numbers of rational curves of low degree

In closing, we want to remark that the method of [23] can in principle be used to determine also the number of elliptic curves on the manifolds. Unfortunately, the results of [23] can, however, not be immediately applied here. For the cases they consider, the manifolds, and thus the index $F_1$, is regular at $\alpha = 0$, which was used as an input. In contrast to this, our manifolds cease to be transversal at $\alpha = 0$.

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Note that in all cases considered here and in [23] the $a_i$ are determined by $\{a_i\} = \{\frac{\alpha^2}{\beta^2}; 0 < \frac{\alpha}{\beta} < 1; \frac{\alpha}{\beta} \neq \frac{\beta}{\alpha}; \forall p = 1, \ldots, k, q = 1, \ldots, m, \alpha, \beta \in \mathbb{N}\}$
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