LARGE DEVIATIONS FOR VALUES OF L-FUNCTIONS ATTACHED TO CUSP FORMS IN THE LEVEL ASPECT

MASAHIRO MINE

Abstract. We study the distribution of values of automorphic L-functions in a family of holomorphic cusp forms with prime level. We prove an asymptotic formula for a certain density function closely related to this value-distribution. The formula is applied to estimate large values of L-functions.

1. Introduction

Let q be a prime number. Denote by \( S_2(q) \) the space of holomorphic cusp forms for the congruence subgroup \( \Gamma_0(q) \) of weight 2 with trivial nebentypus. We describe the Fourier series expansion of \( f \in S_2(q) \) at infinity as

\[
f(z) = \sum_{n=1}^{\infty} a_f(n) \sqrt{n} \exp(2\pi i nz)
\]

for \( z \in \mathbb{C} \) with \( \text{Im}(z) > 0 \) so that the automorphic L-function

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}
\]

has the critical strip \( 0 \leq \Re(s) \leq 1 \). The behavior of the values \( L(s, f) \) in the critical strip has received attention by many researchers. Put \( q(s) = q(|s| + 3)^2 \) for \( s \in \mathbb{C} \). By a standard method of L-functions, we have the convexity bound

\[
L(s, f) \ll q(s)^{(1-\sigma)/2+\epsilon}
\]

for \( s = \sigma + it \) with \( 0 \leq \sigma \leq 1 \), where \( \epsilon \) is any positive real number. Then the work of reducing the exponent in (1.1) proceeded, and some results are included in [13, Theorem 5.19]. The Grand Riemann Hypothesis (GRH) is also useful to derive a sharp upper bound of \( \log L(s, f) \) for \( \Re(s) > 1/2 \). Let \( B_2(q) \) be a basis of \( S_2(q) \) consisting of primitive cusp forms. Throughout this paper, \( \log_j \) indicates the \( j \)-fold iterated natural logarithm, that is,

\[
\log_1 = \log \quad \text{and} \quad \log_{j+1} = \log(\log_j)
\]

for \( j \geq 1 \). Then GRH implies

\[
\log L(s, f) \ll \left( \frac{\log q(s)}{2\sigma - 1} \right)^{2-2\sigma} \log_2 q(s)
\]

for \( s = \sigma + it \) with \( 1/2 < \sigma \leq 5/4 \) if \( f \) belongs to \( B_2(q) \); see [13, Theorem 5.19]. It is believed that even (1.2) is not best possible. Moreover, the true order of the

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magnitude of log $L(\sigma, f)$ is expected to be $(\log q)^{1-\sigma+o(1)}$ for $1/2 < \sigma < 1$ by some probabilistic observations. Define the distribution function

$$\Phi_q(\sigma, \tau) = \frac{\# \{ f \in B_2(q) \mid \log L(\sigma, f) > \tau \}}{\# B_2(q)}$$

for $\sigma > 1/2$ and $\tau \in \mathbb{R}$. Then it is reasonable to consider the decay of $\Phi_q(\sigma, \tau)$ with respect to $\tau$ such that $\tau \approx (\log q)^{1-\sigma+o(1)}$ for $1/2 < \sigma < 1$. For technical reasons, the weighted distribution function

$$\tilde{\Phi}_q(\sigma, \tau) = \left( \sum_{f \in B_2(q)} \omega_f \right)^{-1} \sum_{f \in B_2(q)} \omega_f$$

is usually easier to deal with, where $\omega_f = (4\pi \langle f, f \rangle)^{-1}$ is the harmonic weight, and $\langle f, g \rangle$ indicates the Petersson inner product on $\Gamma_0(q) \backslash \mathbb{H}$. Lamzouri [15] proved that there exist positive constants $A(\sigma)$ and $c(\sigma)$ satisfying

$$\tilde{\Phi}_q(\sigma, \tau) = \exp \left( -A(\sigma) \frac{\tau^{1-\sigma}}{1-\sigma} (\log \tau)^{1-\sigma} \left( 1 + O \left( \frac{1}{\sqrt{\log \tau}} + r(\log q, \tau) \right) \right) \right)$$

uniformly in the range $1 \ll \tau \leq c(\sigma)(\log q)^{1-\sigma}(\log_2 q)^{-1}$, where

$$r(y, \tau) = \left( \frac{\tau}{y^{1-\sigma}(\log y)^{-1}} \right)^{(\sigma-\frac{1}{2})/(1-\sigma)}.$$  

One can deduce from [1.3] a similar result for the distribution function $\Phi_q(\sigma, \tau)$. Recall the following facts on cusp forms:

$$\# B_2(q) = \frac{q}{12} + O(1); \quad \sum_{f \in B_2(q)} \omega_f = 1 + O \left( q^{-3/2} \log q \right);$$

$$q^{-1}(\log q)^{-3} \ll \omega_f \ll q^{-1} \log q.$$  

See [24] and [5, (1.16) and (1.17)]. Thus the relation

$$\log \Phi_q(\sigma, \tau) = \log \tilde{\Phi}_q(\sigma, \tau) + O(\log_2 q)$$

follows from the definitions of $\Phi_q(\sigma, \tau)$ and $\tilde{\Phi}_q(\sigma, \tau)$. It implies that (1.3) remains valid for $\Phi_q(\sigma, \tau)$ in a suitable range of $\tau$. The purpose of this paper is to provide a more precise formula of $\Phi_q(\sigma, \tau)$ for $1/2 < \sigma < 1$ in the same range of $\tau$ as in the result of Lamzouri. Furthermore, we consider an analogous issue on $\Phi_q(1, \tau)$. Note that several authors [14, 18, 27] proved asymptotic formulas of $\Phi_q(1, \tau)$ that look quite different from (1.3); see (1.9) below.

1.1. **Statement of results.** Let $\sigma > 1/2$ and $\tau \in \mathbb{R}$. We begin with the limiting distribution function

$$\Phi(\sigma, \tau) = \lim_{q \to \infty} \Phi_q(\sigma, \tau).$$

The author [23] showed the existence of a continuous function $\mathcal{M}_\sigma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $\Phi(\sigma, \tau)$ satisfies the identity

$$\Phi(\sigma, \tau) = \int_{\tau}^{\infty} \mathcal{M}_\sigma(x) \, dx,$$
Nevertheless, we see that Theorem 1.1 yields the formula
\[ \sigma \text{ only on} \]

Furthermore, the polynomial \( A \) where
\( A \) is large enough. Here, the implied constant depends on \( 0 < a < 2 \).

A similar formula was also proved by Liu–Royer–Wu \([18]\) for the L-functions. Then, we define
\[ (1.7) \quad \mathcal{M}_\sigma(\tau + x) = \frac{F_\sigma(\kappa) e^{-\kappa(\tau + x)}}{\sqrt{f_\sigma''(\kappa)}} \left\{ \exp \left( -\frac{x^2}{2 f_\sigma''(\kappa)} \right) + O \left( \kappa^{\frac{1}{2} \sqrt{\log \kappa}} \right) \right\} \]

uniformly for all \( x \in \mathbb{R} \) if \( \tau > 0 \) is large enough, where the implied constant depends only on \( \sigma \).

Remark that the main term of (1.7) dominates the error term only if \( |x| \) is small. Nevertheless, we see that Theorem 1.1 yields the formula
\[ \Phi(\sigma, \tau) = \int_0^\infty \mathcal{M}_\sigma(\tau + x) |dx| \]
\[ = \frac{F_\sigma(\kappa) e^{-\kappa \tau}}{\kappa \sqrt{2 \pi f_\sigma''(\kappa)}} \left\{ 1 + O \left( \kappa^{\frac{1}{2} \sqrt{\log \kappa}} \right) \right\} \]
for \( 1/2 < \sigma \leq 1 \). A similar formula was also proved by Liu–Royer–Wu \([18]\) for the limiting distribution function \( \Phi(1, \tau) := \lim_{\nu \to \infty} \Phi_\nu(1, \tau) \). In Section 3 we study the asymptotic behaviors of \( f_\sigma(\kappa) \) and its derivatives. Then we derive the following corollaries of Theorem 1.1. Throughout this paper, \( I_\nu(u) \) denotes the modified Bessel function of the first kind of order \( \nu \), and we put \( g(u) = \log(I_1(2u))/u \).

**Corollary 1.2.** Let \( 1/2 < \sigma < 1 \) and \( N \in \mathbb{Z}_{\geq 1} \). For \( n = 0, \ldots, N - 1 \), there exist polynomials \( A_{n,\sigma}(x) \) of degree at most \( n \) with \( A_{0,\sigma}(x) \equiv 1 \) such that the formula
\[ \Phi(\sigma, \tau) = \exp \left( -A(\sigma) \tau^{\frac{1}{1-\sigma}} (\log \tau)^{\frac{1}{\sigma-1}} \sum_{n=0}^{N-1} \frac{A_{n,\sigma}(\log_2 \tau)}{(\log \tau)^n} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^N \right) \right) \]
holds if \( \tau > 0 \) is large enough. Here, the implied constant depends on \( \sigma \) and \( N \), and \( A(\sigma) \) is the positive constant determined as
\[ A(\sigma) = (1 - \sigma) \left( \frac{1 - \sigma}{\sigma} \int_0^\infty g(y^{-\sigma}) \, dy \right)^{-\frac{1}{\sigma-1}}. \]

Furthermore, the polynomial \( A_{1,\sigma}(x) \) is obtained as
\[ A_{1,\sigma}(x) = \frac{\sigma}{1 - \sigma} \left\{ x - \log \left( \frac{1 - \sigma}{\sigma} a_0(\sigma) \right) + \frac{1 - \sigma}{\sigma} a_1(\sigma) \right\}, \]
where \( a_0(\sigma) \) and \( a_1(\sigma) \) are the constants represented as
\[ a_0(\sigma) = \int_0^\infty g(y^{-\sigma}) \, dy \quad \text{and} \quad a_1(\sigma) = \int_0^\infty g(y^{-\sigma}) \log y \, dy. \]
To simplify the statement in the case $\sigma = 1$, we put $\tau = 2 \log t + 2\gamma$ by using Euler’s constant $\gamma = 0.577\ldots$. Then we have

$$
\Phi_q(1, \tau) = \frac{\# \{ f \in B_2(q) \mid L(1, f) > (e^\gamma t)^2 \}}{\#B_2(q)}.
$$

For $u > 0$, we define the function $g_*(u)$ as

$$
g_*(u) = \begin{cases} 
g(u) & \text{if } 0 < u \leq 1, \\
g(u) - 2u & \text{if } u > 1. \end{cases}
$$

The following corollary is a variant of [18, Theorem 1.5] which was originally stated for the distribution function $\tilde{\Phi}(1, \tau)$.

**Corollary 1.3.** Let $N \in \mathbb{Z}_{\geq 1}$ and put $\tau = 2 \log t + 2\gamma$. For $n = 0, \ldots, N - 1$, there exist real numbers $a_n$ with $a_0 = 1$ such that the formula

$$
\Phi(1, \tau) = \exp \left( -\frac{e^t - A}{t} \left( \sum_{n=0}^{N-1} a_n \frac{t^n}{n^n} + O \left( \frac{1}{t^N} \right) \right) \right)
$$

holds if $t > 0$ is large enough. Here, the implied constant depends on $N$, and $A$ is the constant determined as

$$
A = 1 + \frac{1}{2} \int_0^{\infty} g_*(y^{-1}) \, dy - \log 2.
$$

Furthermore, the real number $a_1$ is obtained as

$$
a_1 = -\frac{1}{8} a_0^2 + \frac{1}{2} a_1 + \frac{1}{2};
$$

where $a_0$ and $a_1$ are the constants represented as

$$
a_0 = \int_0^{\infty} g_*(y^{-1}) \, dy \quad \text{and} \quad a_1 = \int_0^{\infty} g_*(y^{-1}) \log y \, dy.
$$

Then, we proceed to study the distribution function $\Phi_q(\sigma, \tau)$ for $1/2 < \sigma < 1$. Using a certain asymptotic formula for complex moments of $L(\sigma, f)$ proved in [23], we associate $\Phi_q(\sigma, \tau)$ with $\Phi(\sigma, \tau)$ as follows.

**Theorem 1.4.** Let $B \geq 1$ be a real number.

(i) For $1/2 < \sigma < 1$, there exists a positive constant $c(\sigma, B)$ such that

$$
\Phi_q(\sigma, \tau) = \Phi(\sigma, \tau) \left( 1 + O \left( \frac{1}{(\log q)^B} + \frac{(\tau \log \tau)^{1-\sigma}}{(\log q)^\sigma} \right) \right)
$$

holds uniformly in the range $1 \ll \tau \leq c(\sigma, B)(\log q)^{1-\sigma}(\log_2 q)^{-1}$, where the implied constant depends on $\sigma$ and $B$.

(ii) Put $\tau = 2 \log t + 2\gamma$. Then there exists a positive constant $c(B)$ such that

$$
\Phi_q(1, \tau) = \Phi(1, \tau) \left( 1 + O \left( \frac{1}{(\log q)^B} + \frac{e^t}{(\log q)(\log_2 q \log_3 q)^{-1}} \right) \right)
$$

holds uniformly in the range $1 \ll t \leq \log_2 q - \log_3 q - \log_4 q - c(B)$, where the implied constant depends on $B$.

From Corollary [12] and Theorem [13](i), we deduce the following result on $\Phi_q(\sigma, \tau)$ for $1/2 < \sigma < 1$. It refines [13] in the desired range of $\tau$. 
Corollary 1.5. Let $1/2 < \sigma < 1$ and $N \in \mathbb{Z}_{\geq 1}$. Then there exists a positive constant $c(\sigma)$ such that
\[
\Phi_q(\sigma, \tau) = \exp \left( -A(\sigma) t \frac{1}{\tau} \left( \log \tau \right)^{\frac{1}{2}} \sum_{n=0}^{N-1} A_n,\sigma \left( \log \tau \right)^n + O \left( \frac{\left( \log \tau \right)^N}{\log \tau} \right) \right)
\]
holds uniformly in the range $1 \ll \tau \leq c(\sigma)(\log q)^{1-\sigma}(\log 2q)^{-1}$, where $A(\sigma)$ and $A_n,\sigma(x)$ are as in Corollary 1.3. Here, the implied constant depends on $\sigma$ and $N$.

Remark. (i) For $\sigma > 1/2$ and $\tau \in \mathbb{R}$, we define another distribution function
\[
\Psi_q(\sigma, \tau) = \frac{\# \{ f \in B_2(q) \mid \log L(\sigma, f) < -\tau \}}{\# B_2(q)}.
\]
Then the limiting distribution function $\Psi(\sigma, \tau) := \lim_{q \to \infty} \Psi_q(\sigma, \tau)$ satisfies
\[
\Psi(\sigma, \tau) = \int_{-\infty}^{-\tau} M_\sigma(x) \, dx = \int_{-\infty}^{0} M_\sigma(-\tau + x) \, dx.
\]
For $1/2 < \sigma \leq 1$, one can prove a formula for $M_\sigma(-\tau + x)$ similar to (1.7) by replacing $F_\sigma(\kappa)$ and $f_\sigma(\kappa)$ of (1.6) with $F_\sigma(-\kappa)$ and $f_\sigma(-\kappa)$, respectively. We see that Corollaries 1.2 and 1.3 remain true for $\Psi(\sigma, \tau)$. Furthermore, the method for the proof of Theorem 1.4 is available to compare $\Psi_q(\sigma, \tau)$ with $\Psi(\sigma, \tau)$. As a result, one can prove that $\Psi_q(\sigma, \tau)$ satisfies the same asymptotic formulas as $\Phi_q(\sigma, \tau)$ described in Corollaries 1.5 and 1.6.

(ii) Liu–Royer–Wu also showed a result similar to Theorem 1.4 (ii) in a family of cusp forms of weight $k \geq 12$ and level 1 as $k \to \infty$. If we adopt totally the same method for the purpose of comparing $\Phi_q(1, \tau)$ with $\Phi(1, \tau)$, then the admissible range of $t$ is to be obtained as $1 \equiv t \leq T(q)$, where
\[
T(q) = \log_2 q - \frac{5}{2} \log_3 q - \log_4 q - c
\]
with a constant $c > 0$; see [18, Theorem 2]. However, this is narrower than Lamzouri’s range $1 \ll t \leq \log_2 q - \log_3 q - 2\log_4 q$. In this paper, we present a modified method of comparing $\Phi_q(\sigma, \tau)$ with $\Phi(\sigma, \tau)$ for $1/2 < \sigma \leq 1$ so as to fill this gap.
1.2. Related results for other zeta and $L$-functions. The value-distributions of zeta and $L$-functions of degree one are classical topics in analytic number theory. For the Riemann zeta-function $\zeta(s)$, we define the distribution function

$$\Phi_{1,T}(\sigma, \tau) = \frac{1}{T} \text{meas} \{ t \in [0, T] \mid \log |\zeta(\sigma + it)| > \tau \},$$

where $\text{meas}(S)$ is the Lebesgue measure of a set $S \subset \mathbb{R}$. For the Dirichlet $L$-function attached to the quadratic character $\chi_d(n) = (\frac{d}{n})$, we also define

$$\Phi_{2,x}(\sigma, \tau) = \left( \sum_{|d| \leq x} 1 \right)^{-1} \sum_{|d| \leq x} 1,$$

where $\sum^b$ indicates the sum over fundamental discriminants. We explore several results of $\Phi_{1,T}(\sigma, \tau)$ and $\Phi_{2,x}(\sigma, \tau)$ for comparisons with the results described in Section 1.1. First, we note that there exist the limiting distribution functions

$$\Phi_{1}(\sigma, \tau) = \lim_{T \to \infty} \Phi_{1,T}(\sigma, \tau) \quad \text{and} \quad \Phi_{2}(\sigma, \tau) = \lim_{x \to \infty} \Phi_{2,x}(\sigma, \tau)$$

for $\sigma > 1/2$ and $\tau \in \mathbb{R}$, which were essentially proved by Bohr–Jessen $[2,3]$ for $\Phi_{1}(\sigma, \tau)$ and by Chowla–Erdős $[4]$ for $\Phi_{2}(\sigma, \tau)$. The estimates of these distribution functions were improved along with the work of applying methods of probability theory to problems of number theory. In particular, Granville–Soundararajan $[9,10]$ applied the saddle-point method to derive the formulas

$$\Phi_j(1, \tau) = \exp \left( -e^{\tau - A_j} \left( 1 + O \left( \frac{1}{\sqrt{T}} \right) \right) \right),$$

for $j = 1, 2$, where we put $\tau = \log t + \gamma$, and $A_j$ are the constants determined as follows. Define $g_1(u) = \log I_0(u)$ and $g_2(u) = \log \cosh(u)$ for $u > 0$, and put

$$g_{j,*}(u) = \begin{cases} g_j(u) & \text{if } 0 < u \leq 1, \\ g_j(u) - u & \text{if } u > 1 \end{cases}$$

similarly to (1.8). By these functions, the constants $A_j$ are represented as

$$A_j = 1 + \int_{0}^{\infty} g_{j,*}(y^{-1}) \, dy.$$ 

Then, Wu $[26]$ improved formula (1.10) in the form

$$\Phi_2(1, \tau) = \exp \left( -e^{1-A_2} \left( \frac{N-1}{t} \sum_{n=0}^{N-1} \frac{\alpha_n}{t^n} + O \left( \frac{1}{t^{N}} \right) \right) \right),$$

where $\alpha_0, \ldots, \alpha_{N-1}$ are real numbers such that $\alpha_0 = 1$. Wu’s method was also based on the saddle-point method, but the treatment of the saddle-point was slightly different from Granville–Soundararajan’s one. We prove Theorem 1.1 by modifying the method of $[26]$ rather than $[9,10]$. Furthermore, Theorem 1.4 is regarded as an analogue of the formula

$$\Phi_{2,x}(1, \tau) = \Phi_{2}(1, \tau) \left( 1 + O \left( \frac{1}{(\log x)^b} + \frac{e^{t}}{(\log x \log_2 x)(\log_2 x)^2} \right) \right),$$

for $\sigma > 1/2$ and $\tau \in \mathbb{R}$. Wu’s method was also based on the saddle-point method, but the treatment of the saddle-point was slightly different from Granville–Soundararajan’s one. We prove Theorem 1.1 by modifying the method of $[26]$ rather than $[9,10]$. Furthermore, Theorem 1.4 is regarded as an analogue of the formula

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which was shown in [9] uniformly in the range $1 \ll t \leq \log_2 x - 2 \log_3 x + \log_4 q - 20$. Asymptotic behaviors of $\Phi_j(\sigma, \tau)$ for $1/2 < \sigma < 1$ were studied by Lamzouri [15]. Let $A_j(\sigma)$ be the positive constants determined by

$$A_j(\sigma) = (1 - \sigma) \left( \frac{1 - \sigma}{\sigma} \int_0^\infty g_j(y^{-\sigma}) \, dy \right)^{-1}. \tag{2.1}$$

Then he proved the asymptotic formulas

$$\Phi_j(\sigma, \tau) = \exp \left( -A_j(\sigma) \tau^{1-\sigma} (\log \tau)^{\frac{1}{1-\sigma}} \left( 1 + O \left( \frac{1}{\sqrt{\log \tau}} \right) \right) \right)$$

for $j = 1, 2$. Note that a similar result for $j = 1$ was seen in the earlier work of Hattori–Matsumoto [11]. Finally, an analogue of (1.11) for $\Phi_1(\sigma, \tau)$ was achieved by Lamzouri–Lester–Radziwiłł [16, Theorem 1.3]. See also [8] for a refinement.

**Organization of the paper.** This paper consists of five sections.

- **Section 2** is devoted to show some lemmas on certain functions $G_p(z)$ and $G(z)$ defined by the $p$-adic Plancherel measure and the Sato–Tate measure.
- In **Section 3**, we prove asymptotic formulas for the function $f_\sigma(\kappa)$ of (1.6). The main ingredient is to sum up the local components $f_\sigma(p)$ which are approximated by using the functions $G_p(z)$ and $G(z)$ of Section 2.
- **Section 4** is further divided into three subsections. In Section 4.1, we make preparations for the saddle-point method. Then we complete the proof of Theorem 1.1 in Section 4.2. After that, we show corollaries of Theorem 1.1 in Section 4.3.
- In **Section 5**, we present a method of comparing $\Phi_q(\sigma, \tau)$ with $\Phi(\sigma, \tau)$, which is based on the Esseen inequality of probability theory. We finally prove Theorem 1.4 and its corollaries to end this paper.

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## 2. Preliminary Lemmas

Let $p$ be a prime number. Denote by $\mu_p$ the $p$-adic Plancherel measure on the interval $[0, \pi]$ defined as

$$d\mu_p(\theta) = \left( 1 + \frac{1}{p} \right) \left( 1 - \frac{2 \cos 2\theta}{p} + \frac{1}{p^2} \right)^{-1} \frac{2}{\pi} \sin^2 \theta \, d\theta. \tag{2.1}$$

Then $\mu_p$ converges weakly to the Sato–Tate measure $\mu_\infty$ as $p \to \infty$. In this section, we show some preliminary lemmas on the functions

$$G_p(z) = \int_0^\pi \exp(2z \cos \theta) \, d\mu_p(\theta) \quad \text{and} \quad G(z) = \int_0^\pi \exp(2z \cos \theta) \, d\mu_\infty(\theta)$$

defined for $z = u + iv \in \mathbb{C}$. Note that $G(z)$ is an even entire function represented as

$$G(z) = \frac{1}{\pi} \int_0^\pi \exp(2z \cos \theta)(1 - \cos 2\theta) \, d\theta$$

$$= I_0(2z) - I_2(2z) = \frac{I_1(2z)}{z} \tag{2.3}$$

where

$$I_n(z) = \frac{1}{\pi} \int_0^\pi \exp(2z \cos \theta) \cos^n \theta \, d\theta.$$
by definition. Thus, we deduce the following properties of $G(z)$ from the results of
the Bessel functions; see [25].

**Lemma 2.1.**

(i) All zeros of $G(z)$ lie on the imaginary axis. In particular, the
ordinate of the first zero on the upper half-plane is $7.66\ldots$.

(ii) Let $z \in \mathbb{C}$ with $\text{Re}(z) > 0$. Then we have

$$
G(z) = \frac{1}{\sqrt{4\pi}} \frac{\exp(2z)}{z^{3/2}} (1 + O(|z|^{-1}))
$$

as $|z| \to \infty$. Here, the branch of $z^{3/2}$ is chosen so that it is real-valued on
the positive real axis.

(iii) We have $G(z) = 1 + \frac{2}{z^2} + O(|z|^4)$ as $|z| \to 0$.

With regard to the function $G_p(z)$, there seems to be no simple representations by
the Bessel functions such as (2.3). On the other hand, we know that $G_p(z) \to G(z)$
uniformly as $p \to \infty$. More precisely, we prove the following result.

**Lemma 2.2.** Let $z = u + iv$ with $u > 0$ and $|v| \leq u$. Then we have uniformly

$$
G_p(z) = G(z) \left(1 + O\left(p^{-1}\right)\right)
$$

for every prime number $p$.

**Proof.** Since we have

$$
\left(1 + \frac{1}{p}\right) \left(1 - \frac{2\cos 2\theta}{p} + \frac{1}{p^2}\right)^{-1} = 1 + O(p^{-1})
$$

for all $\theta \in [0, \pi]$, the difference between $G_p(z)$ and $G(z)$ is evaluated as

$$
G_p(z) - G(z) \ll p^{-1} \int_0^\pi \exp(2u\cos \theta) \frac{2}{\pi} \sin^2 \theta \, d\theta = p^{-1} G(u).
$$

By Lemma 2.1 (ii), we have $|G(z)| \asymp G(u)$ for $z = u + iv$ with $u > 0$ and $|v| \leq u$. Hence we obtain

$$
\frac{G_p(z) - G(z)}{G(z)} \ll p^{-1},
$$

which yields the desired formula. \hfill \Box

By Lemma 2.1 (i), one can define $g(z) = \log G(z)$ as a holomorphic function
on the right half-plane $\text{Re}(z) > 0$, where the branch of the logarithm is taken so
that $g(u) \in \mathbb{R}$ for $u > 0$. Then we deduce the following estimates of $g(u)$ and its
derivatives from Lemma 2.1 (ii), (iii).

**Lemma 2.3.** We have

$$
g(u) = \begin{cases}
O(u^2) & \text{if } 0 < u \leq 1, \\
2u + O(\log u) & \text{if } u > 1,
\end{cases}
$$

and for all $j \geq 2$,

$$
g^{(j)}(u) = \begin{cases}
O(j!) & \text{if } 0 < u \leq 1, \\
O(2^j j! u^{-j}) & \text{if } u > 1.
\end{cases}
$$
Proof. Let \( z \in \mathbb{C} \) with \( \text{Re}(z) > 1/2 \). Using Lemma 2.1 (ii), we derive the formula

\[
g(z) = 2z - \frac{3}{2} \log z - \frac{1}{2} \log 4\pi + h(z),
\]

where \( h(z) \) is a holomorphic function such that \( h(z) \ll |z|^{-1} \). It yields the desired estimate of \( g(u) \) for \( u > 1 \). We also obtain

\[
g^{(j)}(z) = 2\delta_{j,1} + (j - 1)! \frac{3(-1)^j}{2^j} + h^{(j)}(z)
\]

for all \( j \geq 1 \), where \( \delta_{j,1} \) equals to 1 if \( j = 1 \) and 0 otherwise. For \( u > 1 \), Cauchy’s integral formula yields

\[
h^{(j)}(u) = \frac{j!}{2\pi i} \int \frac{h(z)}{(z-u)^{j+1}} \, dz \ll 2^j j! u^{-j-1},
\]

and therefore the results on the derivatives follow. The results for \( 0 < u \leq 1 \) can be proved similarly. Indeed, we deduce from Lemma 2.1 (iii) that \( g(z) = \frac{1}{2} z^2 + O(|z|^4) \) for \( |z| \leq 1 \). Then the upper bounds of \( g(u) \) and \( g'(u) \) follow immediately. For \( j \geq 2 \), we again use Cauchy’s integral formula to obtain

\[
g^{(j)}(u) = \frac{j!}{2\pi i} \int \frac{g(z)}{(z-u)^{j+1}} \, dz \ll j! u^{-j-1}
\]

by noting that \( g(z) \) is bounded on the disk \( |z| \leq 2 \). \( \square \)

Let \( n \) and \( j \) be non-negative integers. If \( 1/2 < \sigma < 1 \), then the integral

\[
g_{n,j}(\sigma) = \int_0^\infty \frac{g^{(j)}(u)}{u^{\frac{1}{2}+1-j}} (\log u)^n \, du
\]

is finite by Lemma 2.3. If \( \sigma = 1 \), we modify the function \( g(u) \) by \( g_*(u) \) as in (1.8). Then we again deduce from Lemma 2.3 that the integral

\[
g_{n,j} = \int_0^\infty \frac{g_*(j)}{u^{\frac{1}{2}-j}} (\log u)^n \, du
\]

is finite. Moreover, it can be easily check that \( g_{0,j}(\sigma) \ll j! \) and \( g_{0,j} \ll j! \) by the integrations by parts. The constants \( a_0(\sigma), a_1(\sigma), a_0, a_1 \) of Corollaries 1.2 and 1.3 are related to these integrals when \( n = 0, 1 \).

3. Estimates of cumulant-generating functions

Let \( \Theta = (\Theta_p)_p \) be a sequence of independent random variables distributed on the interval \([0, \pi]\) according to the measure \( \mu_p \) of (2.1). Then we define

\[
L(\sigma, \Theta) = \prod_p \left( 1 - 2(\cos \Theta_p)p^{-\sigma} + p^{-2\sigma} \right)^{-1}.
\]

We see that \( L(\sigma, \Theta) \) presents an \( \mathbb{R} \)-valued random variable for \( \sigma > 1/2 \) since the right-hand side of (3.1) converges almost surely. For \( \theta \in [0, \pi] \), we also define

\[
\lambda_{p,\sigma}(\theta) = \sum_{m=1}^\infty \frac{\cos(m\theta)}{m} p^{-m\sigma}.
\]
Then we have \( \log L(\sigma, \Theta) = \sum_p 2\lambda_{p, \sigma}(\Theta_p) \) for \( \sigma > 1/2 \). The random Euler product \( L(\sigma, \Theta) \) is associated with the value-distribution of \( L(\sigma, f) \). Indeed, the limiting distribution function \( \Phi(\sigma, \tau) \) of \( \text{(1.4)} \) is represented as

\[
\Phi(\sigma, \tau) = \mathbb{P}(\log L(\sigma, \Theta) > \tau)
\]

for \( \sigma > 1/2 \) and \( \tau \in \mathbb{R} \), where \( \mathbb{P}(E) \) denotes the probability of an event \( E \). Thus, the \( M \)-function \( M_\sigma \) is a probability density function of the random variable \( \log L(\sigma, \Theta) \).

It was proved in \([23]\) that the moment-generating function

\[
F_\sigma(s) = \mathbb{E} [\exp(s \log L(\sigma, \Theta))] = \int_{\mathbb{R}} e^{sx} M_\sigma(x) \, dx
\]

is defined for all \( s = u + iv \in \mathbb{C} \) and has the infinite product representation

\[
(3.3) \quad F_\sigma(s) = \prod_p F_{\sigma, p}(s),
\]

where \( F_{\sigma, p}(s) = \mathbb{E} [\exp(2s \lambda_{p, \sigma}(\Theta_p))] \). The goal of this section is to show the following formulas of the cumulant-generating function \( f_\sigma(\kappa) = \log F_\sigma(\kappa) \) and its derivatives.

**Proposition 3.1.** Let \( 1/2 < \sigma < 1 \) and \( N \in \mathbb{Z}_{\geq 1} \). For all \( j \geq 0 \), we obtain

\[
f^{(j)}_\sigma(\kappa) = \frac{\kappa^{j-1}}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{g_{n,j}(\sigma)}{(\log \kappa)^n} + O \left( \frac{2^j j!}{(\log \kappa)^N} \right) \right\}
\]

if \( \kappa > 0 \) is large enough. Here, \( g_{n,j}(\sigma) \) are defined as \( \text{(2.4)} \), and the implied constants depend on \( \sigma \) and \( N \).

**Proposition 3.2.** Let \( N \in \mathbb{Z}_{\geq 1} \). Then we obtain

\[
f_1(\kappa) = 2\kappa(\log_2 \kappa + \gamma) + \frac{\kappa}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{g_{n,0}}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\},
\]

\[
f'_1(\kappa) = 2(\log_2 \kappa + \gamma) + \frac{1}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{g_{n,1}}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\},
\]

and for all \( j \geq 2 \),

\[
f^{(j)}_1(\kappa) = \frac{\kappa^{j-1}}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{g_{n,j}}{(\log \kappa)^n} + O \left( \frac{2^j j!}{(\log \kappa)^N} \right) \right\}
\]

if \( \kappa > 0 \) is large enough. Here, \( g_{n,j} \) are defined as \( \text{(2.5)} \), and the implied constants depend on \( N \).

Let \( 1/2 < \sigma \leq 1 \) and \( N \in \mathbb{Z}_{\geq 1} \). For \( \kappa \geq 6 \), we determine the parameters \( y_1, y_2 \) such that \( 2 \leq y_1 < y_2 \) by the equations

\[
\kappa y_1^{-2\sigma} = \delta \quad \text{and} \quad \kappa y_2^{-\sigma} = (\log \kappa \log_2 \kappa)^{-\sigma N/(2\sigma - 1)},
\]

where \( \delta > 0 \) is a small absolute constant chosen later. Then we show three lemmas on the local factors \( F_{\sigma, p}(s) \) of \( \text{(3.3)} \) toward the proofs of Propositions 3.1 and 3.2.

**Lemma 3.3.** Let \( 1/2 < \sigma \leq 1 \) and \( \kappa \geq 6 \). For any \( p \leq y_1 \), we have

\[
F_{\sigma, p}(s) = \frac{1}{\sqrt{4\pi}} \left( \frac{\exp(2s \lambda_{p, \sigma}(0))}{(s|\lambda''_{p, \sigma}(0)|)^{3/2}} \right)(1 + p^{-1})(1 - p^{-1})^{-2} \left( 1 + O \left( \frac{1}{\sqrt{\kappa}} \right) \right)
\]
uniformly in the disk $|s - \kappa| \leq \kappa/2$, where $\lambda_{p,\sigma}(\theta)$ is defined as (3.2), and the implied constant depends only on the choice of $\delta$.

Proof. Let $\Lambda_{p}(\theta) = (1 + p^{-1})(1 - 2(\cos \theta)p^{-1} + p^{-2})^{-1}$. We represent $F_{\sigma,p}(s)$ as

$$F_{\sigma,p}(s) = \frac{2}{\pi} \int_{0}^{\pi} \exp(2s\lambda_{p,\sigma}(\theta))\Lambda_{p}(2\theta) \sin^{2} \theta \, d\theta.$$ 

Then we estimate the integral by noting that $\lambda_{p,\sigma}(\theta)$ takes the maximum value at $\theta = 0$ in the interval $[0, \pi]$. Recall that $\lambda_{p,\sigma}'(0) = 0$ and $\lambda_{p,\sigma}''(0) < 0$. By the Taylor series expansion, we have

$$\lambda_{p,\sigma}(\theta) = \lambda_{p,\sigma}(0) + \frac{\lambda_{p,\sigma}''(0)}{2} \theta^{2} \mu_{p,\sigma}(\theta),$$

where $\mu_{p,\sigma}(\theta)$ is a function of $\theta$ represented as

$$\mu_{p,\sigma}(\theta) = 1 + \sum_{j=1}^{\infty} \frac{\lambda_{p,\sigma}^{(j+2)}(0)}{(j+2)!} \theta^{j}.$$  

The derivatives of $\lambda_{p,\sigma}(\theta)$ at $\theta = 0$ are evaluated as

$$\lambda_{p,\sigma}''(0) = -\sum_{m=1}^{\infty} mp^{-m\sigma} \approx p^{-\sigma}$$

and $\lambda_{p,\sigma}^{(k)}(0) \ll k!p^{-\sigma}$ for all $k \geq 3$, where the implied constants are absolute. Thus the coefficients of the power series (3.4) are uniformly bounded, and moreover, there exists a small absolute constant $c > 0$ such that $\mu_{p,\sigma}(\theta) = 1 + O(|\theta|)$ holds uniformly for $|\theta| \leq c$.

Then, we consider the integral

$$I_{1} = \int_{0}^{c} \exp \left( s \lambda_{p,\sigma}(\theta) - \lambda_{p,\sigma}(0) \right) \Lambda_{p}(2\theta) \sin^{2} \theta \, d\theta$$

$$= \int_{0}^{c} \exp \left( s \lambda_{p,\sigma}''(0)\theta^{2} \mu_{p,\sigma}(\theta) \right) \Lambda_{p}(2\theta) \sin^{2} \theta \, d\theta.$$ 

We make a change of variables such that $\phi = \theta^{2} \mu_{p,\sigma}(\theta)$. For $|\theta| \leq c$, we have

$$\Lambda_{p}(2\theta) = \Lambda_{p}(0) \left( 1 + O(\phi) \right),$$

$$\sin^{2} \theta = \phi \left( 1 + O(\sqrt{\phi}) \right),$$

$$\frac{d\theta}{d\phi} = \frac{1}{2\sqrt{\phi}} \left( 1 + O(\sqrt{\phi}) \right)$$

with absolute implied constants. Hence we obtain

$$I_{1} = \frac{1}{2} \Lambda_{p}(0) \int_{0}^{c_{1}} \exp \left( s \lambda_{p,\sigma}''(0)\phi \right) \sqrt{\phi} \left( 1 + O(\sqrt{\phi}) \right) \, d\phi,$$

where we put $c_{1} = c^{2} \mu_{p,\sigma}(c)$. To derive the main term, we see that

$$\int_{0}^{\infty} \exp \left( s \lambda_{p,\sigma}''(0)\phi \right) \sqrt{\phi} \, d\phi = \int_{0}^{\infty} \exp \left( -s|\lambda_{p,\sigma}''(0)|\phi \right) \sqrt{\phi} \, d\phi$$

$$= \sqrt{\frac{\pi}{2}} \frac{1}{(s|\lambda_{p,\sigma}''(0)|)^{3/2}}.$$
by noting that $\lambda''_{p,\sigma}(0) < 0$ and $\text{Re}(s) > 0$. Therefore, $I_1$ is calculated as

$$I_1 = \frac{1}{2} \Lambda_p(0) \left( \frac{\sqrt{\pi}}{2} \frac{1}{|s|^{3/2}} - I_{1,1} + I_{1,2} \right),$$

where $I_{1,1}$ and $I_{1,2}$ are the following integrals:

$$I_{1,1} = \int_{c_1}^{\infty} \exp \left( -s|\lambda''_{p,\sigma}(0)||\phi| \right) \sqrt{\phi} \, d\phi;$$

$$I_{1,2} = \int_{0}^{c_1} \exp \left( -s|\lambda''_{p,\sigma}(0)||\phi| \right) O(\phi) \, d\phi.$$

We recall that $\lambda''_{p,\sigma}(0) \asymp p^{-\sigma}$ and $c_1 = \mu_{p,\sigma}(c)e^2 \asymp 1$ for every $p$. Write $s = u + iv$. Then $u \asymp \kappa$ holds in the disk $|s - \kappa| \leq \kappa/2$. Thus, we evaluate these integrals as

$$I_{1,1} \ll \exp \left( -\frac{u|\lambda''_{p,\sigma}(0)||c_1}{2} \right) \frac{1}{|u|^{3/2}} \ll (kp^{-\sigma})^{-2},$$

$$I_{1,2} \ll \int_{0}^{\infty} \exp \left( -u|\lambda''_{p,\sigma}(0)||\phi| \right) \phi \, d\phi \ll (kp^{-\sigma})^{-2}.$$

Note further that $|s| \asymp \kappa$ in the disk $|s - \kappa| \leq \kappa/2$. Hence formula (3.3) yields

$$I_1 = \frac{\sqrt{\pi}}{4} \frac{\Lambda_p(0)}{|s|^{3/2}} \left( 1 + O \left( \frac{(|s|^{-3/2})}{(kp^{-\sigma})^2} \right) \right)$$

$$= \frac{\sqrt{\pi}}{4} \frac{\Lambda_p(0)}{|s|^{3/2}} \left( 1 + O \left( \frac{1}{\sqrt{\kappa}} \right) \right)$$

since $kp^{-\sigma} \gg \sqrt{\kappa}$ for any $p \leq y_1$. The remaining work is to estimate the integral

$$I_2 = \int_{c}^{\pi} \exp \left( 2s(\lambda_{p,\sigma}(\theta) - \lambda_{p,\sigma}(0)) \right) \Lambda_p(2\theta) \sin^2 \theta \, d\theta.$$

The function $\lambda_{p,\sigma}(\theta)$ is decreasing on the interval $[0, \pi]$. Therefore we obtain

$$I_2 \ll \exp \left( 2u(\lambda_{p,\sigma}(c) - \lambda_{p,\sigma}(0)) \right) \int_{0}^{c} |\Lambda_p(2\theta)| \sin^2 \theta \, d\theta$$

$$\ll \exp \left( -u|\lambda''_{p,\sigma}(0)||c^2 \mu_{p,\sigma}(c)| \right).$$

Since $u \asymp \kappa$, $c^2 \mu_{p,\sigma}(c) \asymp 1$, and $\lambda''_{p,\sigma}(0) \asymp p^{-\sigma}$, we have $I_2 \ll (kp^{-\sigma})^{-2}$. It yields

$$I_2 = \frac{\Lambda_p(0)}{(s|\lambda''_{p,\sigma}(0)||)^{3/2}} O \left( \frac{1}{\sqrt{\kappa}} \right)$$

due to $\Lambda_p(0) > 1$ for every $p$. Combining (3.6) and (3.7), we conclude

$$F_{\sigma,p}(s) = 2 \pi \exp(2s\lambda_{p,\sigma}(0))(I_1 + I_2)$$

$$= \frac{1}{\sqrt{4\pi}} \frac{\exp(2s\lambda_{p,\sigma}(0))}{(s|\lambda''_{p,\sigma}(0)||)^{3/2}} \Lambda_p(0) \left( 1 + O \left( \frac{1}{\sqrt{\kappa}} \right) \right)$$

as desired. $\square$

**Lemma 3.4.** Let $1/2 < \sigma \leq 1$ and $\kappa \geq 6$. For any $p > y_1$, we have

$$F_{\sigma,p}(s) = G(sp^{-\sigma}) \left( 1 + O \left( \frac{(kp^{-\sigma})^2 + p^{-1}}{\sqrt{\kappa}} \right) \right)$$

uniformly in the disk $|s - \kappa| \leq \kappa/2$, where $G(z)$ is defined as (2.2).
Proof. Using the formula \( \exp(z) = 1 + O(|z|) \) with \( |z| \leq 1 \), we have
\[
\exp(2s\lambda_{p,\sigma}(\theta)) = \exp \left( 2s(\cos \theta)p^{-\sigma} + O \left( |s|p^{-2\sigma} \right) \right) \\
= \exp \left( 2s(\cos \theta)p^{-\sigma} \right) \left( 1 + O \left( \kappa p^{-2\sigma} \right) \right)
\]
since \( |s|p^{-2\sigma} \ll \kappa p^{-2\sigma} \leq \delta \) is valid in the disk \( |s - \kappa| \leq \kappa/2 \) for any \( p > y_1 \). Hence \( F_{\sigma,p}(s) \) is estimated as
\[
F_{\sigma,p}(s) = \int_0^\pi \exp \left( 2s(\cos \theta)p^{-\sigma} \right) \left( 1 + O \left( \kappa p^{-2\sigma} \right) \right) d\mu_p(\theta) \\
= G_p(sp^{-\sigma}) + O \left( \kappa p^{-2\sigma} G_p(up^{-\sigma}) \right),
\]
where \( G_p(z) \) is defined as (2.2), and we write \( s = u + iv \). By Lemma 2.2 we obtain
\[
F_{\sigma,p}(s) = G(sp^{-\sigma}) \left( 1 + O \left( p^{-1} + \kappa p^{-2\sigma} \frac{G(up^{-\sigma})}{G(sp^{-\sigma})} \right) \right).
\]
Recall that the estimate \( |G(sp^{-\sigma})| \asymp G(up^{-\sigma}) \) holds in the disk \( |s - \kappa| \leq \kappa/2 \) by Lemma 2.2(ii). Thus the result follows. \( \square \)

Lemma 3.5. Let \( 1/2 < \sigma \leq 1 \) and \( \kappa \geq 6 \). For any \( p \geq y_2 \), we have
\[
F_{\sigma,p}(s) = 1 + O \left( \kappa^2 p^{-2\sigma} \right)
\]
uniformly in the disk \( |s - \kappa| \leq \kappa/2 \).

Proof. By the Taylor series expansion, we have
\[
\exp(s\lambda_{p,\sigma}(\theta)) = 1 + 2s(\cos \theta)p^{-\sigma} + O \left( \kappa^2 p^{-2\sigma} \right)
\]
since \( |s|p^{-\sigma} \ll \kappa p^{-\sigma} \leq 1 \) is valid in the disk \( |s - \kappa| \leq \kappa/2 \) for any \( p \geq y_2 \). By simple calculations, we find the equalities
\[
\int_0^\pi d\mu_p(\theta) = 1 \quad \text{and} \quad \int_0^\pi (\cos \theta) d\mu_p(\theta) = 0.
\]
Hence we obtain the conclusion. \( \square \)

Let \( s = u + iv \) with \( |s - \kappa| \leq \kappa/2 \). If \( \kappa > 0 \) is large enough, then we deduce from Lemma 3.3 that \( F_{\sigma,p}(s) \neq 0 \) for \( p \leq y_1 \). Using Lemma 3.4 we also obtain \( F_{\sigma,p}(s) \neq 0 \) for \( p > y_1 \) if \( \delta \) is small enough. Therefore one can define \( f_{\sigma,p}(s) = \log F_{\sigma,p}(s) \) as a holomorphic function on \( |s - \kappa| \leq \kappa/2 \) for any prime number \( p \), where the branch is chosen so that \( f_{\sigma,p}(u) \in \mathbb{R} \) if \( u > 0 \). Then formula (3.3) yields
\[
f_{\sigma}(s) = \sum_p f_{\sigma,p}(s).
\]
Furthermore, we immediately deduce the following results from the above lemmas.

Lemma 3.6. Let \( \kappa > 0 \) be a large real number. Then the following asymptotic formulas hold uniformly in the disk \( |s - \kappa| \leq \kappa/2 \).

(i) For any \( p \leq y_1 \), we have
\[
f_{\sigma,p}(s) = 2s\lambda_{p,\sigma}(0) - \frac{3}{2} \log(|s|\lambda''_{p,\sigma}(0)) - \frac{1}{2} \log 4\pi + O \left( \frac{1}{\sqrt{\kappa}} + p^{-1} \right),
\]
where \( \lambda_{p,\sigma}(\theta) \) is defined as (3.2).
(ii) For any \( y_1 < p < y_2 \), we have
\[
f_{\sigma,p}(s) = g(sp^{-\sigma}) + O \left( \kappa p^{-2\sigma} + p^{-1} \right),
\]
where \( g(z) = \log G(z) \) as in Section 2.

(iii) For any \( p \geq y_2 \), we have \( f_{\sigma,p}(s) = O \left( \kappa^2 p^{-2\sigma} \right) \).

Proof of Proposition 3.1. By (3.8), we obtain
\[
(3.9) \quad f_{\sigma}^{(j)}(\kappa) = \left( \sum_{p \leq y_1} + \sum_{y_1 < p < y_2} + \sum_{p \geq y_2} \right) f_{\sigma,p}^{(j)}(\kappa)
\]
for all \( j \geq 0 \). Let \( p \leq y_1 \). We deduce from Lemma 3.6 (i) the formula
\[
f_{\sigma,p}(\kappa) = 2\kappa \lambda_{p,\sigma}(0) + O \left( \log(\kappa p^{-\sigma}) \right)
\]
by recalling \( \lambda_{p,\sigma}'(0) \approx p^{-\sigma} \). Then, we apply Cauchy’s integral formula in a way similar to the proof of Lemma 2.3. We obtain
\[
(3.10) \quad f_{\sigma,p}^{(j)}(\kappa) = 2\kappa \lambda_{p,\sigma}(0) + O \left( \kappa^{-1} \right) \quad \text{and} \quad f_{\sigma,p}^{(j)}(\kappa) \ll 2^j j! \kappa^{-j}
\]
for all \( j \geq 2 \). Therefore, the upper bounds \( f_{\sigma,p}^{(j)}(\kappa) \ll 2^j j! \kappa^{-j+1}p^{-\sigma} \) hold for all \( j \geq 0 \) since \( \kappa p^{-\sigma} \gg \sqrt{\kappa} \) for \( p \leq y_1 \). Recalling that \( y_1 \approx \kappa^{1/2} \), we estimate the first sum of (3.9) as
\[
\sum_{p \leq y_1} f_{\sigma,p}^{(j)}(\kappa) \ll 2^j j! \kappa^{-j+1} \frac{y_1^{1-\sigma}}{\log y_1} \ll 2^j j! \frac{\kappa^{\frac{1}{2} - j}}{\log \kappa}.
\]
Then, we consider the third sum of (3.9). Let \( p \geq y_2 \). We use Lemma 3.6 (ii) to derive the bounds \( f_{\sigma,p}^{(j)}(\kappa) \ll 2^j j! \kappa^{-j+2}p^{-2\sigma} \) for all \( j \geq 0 \). Thus we obtain
\[
\sum_{p \geq y_2} f_{\sigma,p}^{(j)}(\kappa) \ll 2^j j! \kappa^{-j+2} \frac{y_2^{1-2\sigma}}{\log y_2} \ll 2^j j! \frac{\kappa^{\frac{1}{2} - j}}{(\log \kappa)^{N+1}}
\]
from the choice of \( y_2 \). The main term comes from the second sum of (3.9). Using Lemma 3.6 (ii), we have
\[
f_{\sigma,p}^{(j)}(\kappa) = p^{-\sigma} g^{(j)}(kp^{-\sigma}) + O \left( 2^j j! \kappa^{-j-1}p^{-2\sigma} + 2^j j! \kappa^{-j}p^{-1} \right)
\]
for \( y_1 < p < y_2 \). Note that
\[
\sum_{y_1 < p < y_2} \left( 2^j j! \kappa^{1-j}p^{-2\sigma} + 2^j j! \kappa^{-j}p^{-1} \right) \ll 2^j j! \frac{\kappa^{\frac{1}{2} - j}}{\log \kappa} + 2^j j! \kappa^{-j} \log_2 \kappa.
\]
Put \( u(y) = \kappa y^{-\sigma} \). Then we obtain
\[
\sum_{y_1 < p < y_2} f_{\sigma,p}^{(j)}(\kappa) = \kappa^{-j} \sum_{y_1 < p < y_2} u(p)^j g^{(j)}(u(p)) + O \left( 2^j j! \frac{\kappa^{\frac{1}{2} - j}}{\log \kappa} + 2^j j! \kappa^{-j} \log_2 \kappa \right).
\]
From the above, we deduce
\[
(3.10) \quad f_{\sigma}^{(j)}(\kappa) = \kappa^{-j} \sum_{y_1 < p < y_2} u(p)^j g^{(j)}(u(p)) + O \left( 2^j j! \frac{\kappa^{\frac{1}{2} - j}}{(\log \kappa)^{N+1}} \right).
\]
for all \( j \geq 0 \) with implied constants depending on \( \sigma \) and \( N \). We approximate the sum in (3.10) by using the prime number theorem in the form
\[
\pi(y) = \int_{2}^{y} \frac{dy}{\log y} + O \left( y e^{-8\sqrt{\log y}} \right),
\]
where \( \pi(y) \) counts as usual the prime numbers not exceeding \( y \). We obtain
\[
\sum_{y_1 < p < y_2} u(p)^j \sigma^j(u(p)) = \int_{y_1}^{y_2} u(y)^j \sigma^j(u(y)) \frac{dy}{\log y} + E
\]
by the partial summation, where
\[
E \ll u(y_1)^j \sigma^j(u(y_1)) y_1 e^{-8\sqrt{\log y_1}} + u(y_2)^j \sigma^j(u(y_2)) y_2 e^{-8\sqrt{\log y_2}}
\]
\[+ e^{-8\sqrt{\log y_1}} \int_{y_1}^{y_2} u(y)^j \sigma^j(u(y)) dy.\]
We put \( y_1 = \kappa^{1/\sigma} \). By the choices of these parameters, we have \( 0 < u(y_2) < 1 \), \( u(y_3) = 1 \), and \( u(y_1) > 1 \). Hence Lemma 2.2 yields
\[
E \ll 2^j j! u(y_1)^2 y_1 e^{-8\sqrt{\log y_1}} + j! u(y_2)^2 y_2 e^{-8\sqrt{\log y_2}}
\]
\[+ 2^j j! e^{-8\sqrt{\log y_1}} \int_{y_1}^{y_2} u(y)^2 dy + 2^j j! e^{-8\sqrt{\log y_1}} \int_{y_1}^{y_2} u(y)^2 dy
\]
\[\ll 2^j j! \kappa^{1/\sigma} \left( \log \kappa \right)^N.\]
The integral in (3.11) is calculated as follows. Changing the variables, we have
\[
\int_{y_1}^{y_2} u(y)^j \sigma^j(u(y)) \frac{dy}{\log y} = \kappa^j \int_{u_1}^{u_2} \frac{g^{(j)}(u)}{u^{1/2 + 1 - j} \log(u/\kappa)} du,
\]
where \( u_1 = u(y_1) \) and \( u_2 = u(y_2) \). For \( u_2 \leq u \leq u_1 \), the asymptotic formula
\[
\frac{1}{\log(u/\kappa)} = \frac{1}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \left( \frac{\log u}{\log \kappa} \right)^n + O \left( \left( \frac{\log u}{\log \kappa} \right)^N \right) \right\}
\]
is valid, which yields
\[
\int_{y_1}^{y_2} u(y)^j \sigma^j(u(y)) \frac{dy}{\log y} = \kappa^{1/2 - j} \log \kappa \left\{ \sum_{n=0}^{N-1} \frac{1}{(\log \kappa)^n} \int_{u_2}^{u_1} \frac{g^{(j)}(u)}{u^{1/2 + 1 - j}} (\log u)^n du
\]
\[+ O \left( \frac{1}{(\log \kappa)^N} \int_{u_2}^{u_1} \frac{|g^{(j)}(u)|}{u^{1/2 + 1 - j}} |\log u|^N du \right) \right\}.
\]
Let \( g_{n,j}(\sigma) \) denote the constants of (2.4). Then we have
\[
\int_{u_2}^{u_1} \frac{g^{(j)}(u)}{u^{1/2 + 1 - j}} (\log u)^n du = g_{n,j}(\sigma) + E_{n,j}
\]
for all \( n = 0, \ldots, N - 1 \), where \( E_{n,j} \) are evaluated as
\[
E_{n,j} \ll 2^j j! \int_{0}^{u_2} u^{1 - \frac{n}{2}} |\log u|^n du + 2^j j! \int_{u_1}^{\infty} u^{1 - \frac{n}{2}} |\log u|^n du \ll \frac{2^j j!}{(\log \kappa)^N}
\]
As a result, we find that the integral is estimated as

\[ \int_{u_1}^{u_2} \frac{|g^{(j)}(u)|}{u^{1-\frac{1}{\sigma}}} \log u \, du \leq \int_{0}^{\infty} \frac{|g^{(j)}(u)|}{u^{1-\frac{1}{\sigma}}} \log u \, du \ll 2^j j! . \]

As a result, we find that the integral is estimated as

\[ \int_{y_1}^{y_2} u(y)^j g^{(j)}(u(y)) \frac{dy}{\log y} = \frac{\kappa^\frac{1}{\sigma} - j}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{\theta_{n,j}(\sigma)}{(\log \kappa)^n} + O\left( \frac{2^j j!}{(\log \kappa)^N} \right) \right\} . \]

Combined with (3.10) and (3.11), it deduces the asymptotic formula

\[ f_{\sigma}^{(j)}(\kappa) = \kappa^{-j} \int_{y_1}^{y_2} u(y)^j g^{(j)}(u(y)) \frac{dy}{\log y} + O\left( \frac{2^j j!}{(\log \kappa)^{N+1}} \right) \]

which completes the proof of Proposition 3.1.

**Proof of Proposition 3.2** If \( j \geq 2 \), then the desired formula of \( f_{1}^{(j)}(\kappa) \) can be shown by an argument similar to the proof of Proposition 3.1. We hereby present the proof when \( j = 0 \). By Lemma 3.6 (i), we have

\[ f_{1,p}(\kappa) = -2\kappa \log(1 - p^{-1}) + O\left( \log(\kappa p^{-1}) \right) \]

for any \( p \leq y_1 \). Recalling the definition of \( g_*(u) \), we deduce from Lemma 3.6 (ii) the formulas

\[ f_{1,p}(\kappa) = \begin{cases} g_*(\kappa p^{-1}) + 2\kappa p^{-1} + O\left( \kappa p^{-2} + p^{-1} \right) & \text{if } y_1 < p < \kappa, \\ g_*(\kappa p^{-1}) + O\left( \kappa p^{-2} + p^{-1} \right) & \text{if } \kappa \leq p < y_2. \end{cases} \]

Therefore the sum of terms for \( p < y_2 \) is calculated as

\[ \sum_{p < y_2} f_{1,p}(\kappa) = -2\kappa \sum_{p < \kappa} \log(1 - p^{-1}) + \sum_{y_1 < p < y_2} g_*(\kappa p^{-1}) + E, \]

where the error term \( E \) is evaluated as

\[ E = 2\kappa \sum_{y_1 < p < \kappa} \{ \log(1 - p^{-1}) + p^{-1} \} + O\left( \sum_{p < y_1} \log(\kappa p^{-1}) + \sum_{y_1 < p < y_2} (\kappa p^{-2} + p^{-1}) \right). \]

Furthermore, we use the estimate \( \log(1 - x) = -x + O(x^2) \) with \( |x| < 1 \) to obtain

\[ E \ll \sum_{y_1 < p < \kappa} \kappa p^{-2} + \sum_{p < y_1} \log(\kappa p^{-1}) + \sum_{y_1 < p < y_2} (\kappa p^{-2} + p^{-1}) \]

\[ \ll \frac{\kappa}{y_1 \log y_1} + \frac{(\log \kappa)y_1}{\log y_1} + \log(y_2) \ll \sqrt{\kappa}. \]

Applying the asymptotic formula

\[ -\sum_{p < y} \log(1 - p^{-1}) = \log_2 y + \gamma + O\left( e^{-2\sqrt{\log y}} \right), \]
we derive

\[ \sum_{p < y_2} f_{1,p}(\kappa) = 2\kappa(\log_2 \kappa + \gamma) + \sum_{y_1 < p < y_2} g_*(\kappa p^{-1}) + O\left(\kappa e^{-2\sqrt{\log \kappa}}\right). \]

Then we estimate the contribution of terms for \( p \geq y_2 \). By Lemma 3.6 (iii), we obtain

\[ \sum_{p \geq y_2} f_{1,p}(\kappa) \ll \frac{\kappa^2}{y_2 \log y_2} \ll \frac{\kappa}{(\log \kappa)^N+1}, \]

where the implied constant depends on \( N \). Combining (3.12) and (3.13), we obtain

\[ f_1(\kappa) = 2\kappa(\log_2 \kappa + \gamma) + \sum_{y_1 < p < y_2} g_*(\kappa p^{-1}) + O\left(\frac{\kappa}{(\log \kappa)^N+1}\right). \]

The work of estimating the sum in (3.14) remains, but one can show

\[ \sum_{y_1 < p < y_2} g_*(\kappa p^{-1}) = \frac{\kappa}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{g_{0,n}}{(\log \kappa)^n} + O\left(\frac{1}{(\log \kappa)^N}\right) \right\} \]

along the same line as the proof of Proposition 3.1. Hence the desired asymptotic formula of \( f_1(\kappa) \) follows. In addition, the proof for \( f'_1(\kappa) \) is given in a similar way, and we omit the proof. \( \square \)

4. Probability density functions

4.1. Approximation of the saddle-point. As seen in [14, 15, 18], the saddle-point method is useful to show asymptotic formulas such as (1.3) and (1.9). Here, the saddle-point stands for the solution \( \kappa = \kappa(\sigma, \tau) \) to the equation

\[ f'_\sigma(\kappa) = \tau, \]

where \( f_\sigma(\kappa) \) is the cumulant-generating function defined by (1.3). We approximate the saddle-point by extending the method of Liu–Royer–Wu [18]. First, we prove the existence and uniqueness of the solution of (4.1).

**Lemma 4.1.** Let \( \sigma > 1/2 \) and \( \tau > 0 \). Then there exists a unique real number \( \kappa = \kappa(\sigma, \tau) > 0 \) for which (1.1) is satisfied. We have \( \kappa(\sigma, \tau) \to \infty \) as \( \tau \to \infty \).

**Proof.** Since \( f_\sigma(\kappa) = \log F_\sigma(\kappa) \), we calculate its second derivative as

\[ f''_\sigma(\kappa) = \frac{F''_\sigma(\kappa) F_\sigma(\kappa) - F'_\sigma(\kappa)^2}{F_\sigma(\kappa)^2} = \frac{1}{F_\sigma(\kappa)} \int_{\mathbb{R}} (x - f'_\sigma(\kappa))^2 e^{\kappa x}M_\sigma(x) \, dx. \]

This implies \( f''_\sigma(\kappa) > 0 \), and thus \( f'_\sigma(\kappa) \) is strictly increasing for \( \kappa > 0 \). Furthermore,

\[ \mathbb{E} \left[ \log L(\sigma, \Theta) \right] = \sum_p \sum_{m=1}^\infty \frac{1}{m} \mathbb{E} \left[ \cos(m \Theta_p) \right] p^{-m \sigma} = 0 \]

since \( \mathbb{E} \left[ \cos(m \Theta_p) \right] = 0 \) for all \( m \geq 1 \). Hence the value \( f'_\sigma(0) \) is calculated as

\[ f'_\sigma(0) = \frac{F'_\sigma(0)}{F_\sigma(0)} = \mathbb{E} \left[ \log L(\sigma, \Theta) \right] = 0. \]

Therefore we obtain the conclusion. \( \square \)
Then, we apply Propositions 3.1 and 3.2 to derive the following results on the saddle-point \( \kappa = \kappa(\sigma, \tau) \) for \( 1/2 < \sigma \leq 1 \).

**Proposition 4.2.** Let \( N \in \mathbb{Z}_{\geq 1} \). Denote by \( \kappa = \kappa(\sigma, \tau) \) the solution of (4.1) with \( 1/2 < \sigma \leq 1 \). For \( n = 0, \ldots, N - 1 \), there exist polynomials \( B_{n, \sigma}(x) \) of degree at most \( n \) with \( B_{0, \sigma}(x) = 1 \) such that the formula

\[
\kappa = B(\sigma)(\tau \log \tau)^{1-\sigma} \left\{ \sum_{n=0}^{N-1} \frac{B_{n, \sigma}(\log_2 \tau)}{(\log \tau)^n} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^N \right) \right\}
\]

holds if \( \tau > 0 \) is large enough. Here, the implied constant depends on \( \sigma \) and \( N \), and \( B(\sigma) \) is the positive constant determined as

\[
\begin{align*}
B(\sigma) &= \left( \frac{1 - \sigma}{\sigma} g_{0,1}(\sigma) \right)^{-\frac{\sigma}{1-\sigma}}. \\
B_{1, \sigma}(x) &= \frac{\sigma}{1 - \sigma} x + \log B(\sigma) - \frac{g_{1,1}(\sigma)}{g_{0,1}(\sigma)},
\end{align*}
\]

where \( g_{0,1}(\sigma) \) and \( g_{1,1}(\sigma) \) are the constants defined by (2.4).

**Proof.** First, we apply Proposition 3.1 with \( N = 1 \). It yields the formula

\[
\tau = g_{0,1}(\sigma) \kappa^{1-\sigma} \left( 1 + O \left( \frac{1}{\log \kappa} \right) \right).
\]

Furthermore, the logarithm is estimated as

\[
\log \tau = \frac{1 - \sigma}{\sigma} (\log \kappa) \left( 1 + O \left( \frac{\log_2 \kappa}{\log \kappa} \right) \right),
\]

and therefore \( \log \tau \asymp \log \kappa \) follows. Put \( \kappa = B(\sigma)(\tau \log \tau)^{1-\sigma} (1 + h_\sigma(\tau)) \). Then we deduce from (4.3) and (4.4) that \( h_\sigma(\tau) \) satisfies

\[
h_\sigma(\tau) \ll \frac{\log_2 \kappa}{\log \kappa} \ll \frac{\log_2 \tau}{\log \tau},
\]

which derives the result when \( N = 1 \). To consider the case \( N = 2 \), we calculate the terms \( \kappa^{1-\sigma} \) and \( \log \kappa \) as

\[
\begin{align*}
\kappa^{1-\sigma} &= B(\sigma) \left( \frac{1-\sigma}{\sigma} \right) (\tau \log \tau) \left\{ 1 + \frac{1 - \sigma}{\sigma} h_\sigma(\tau) + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right) \right\}, \\
\log \kappa &= \frac{\sigma}{1 - \sigma} (\log \tau) \left\{ 1 + \frac{\log_2 \tau + \frac{1-\sigma}{\sigma} \log B(\sigma)}{\log \tau} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right) \right\},
\end{align*}
\]

by using \( \kappa = B(\sigma)(\tau \log \tau)^{1-\sigma} (1 + h_\sigma(\tau)) \) and (4.5). We insert them to the formula

\[
\tau = g_{0,1}(\sigma) \kappa^{1-\sigma} \left\{ 1 + \frac{g_{1,1}(\sigma)}{g_{0,1}(\sigma)} \frac{1}{\log \kappa} + O \left( \frac{1}{(\log \kappa)^2} \right) \right\}
\]
which is a consequence of Proposition 3.1 with $N = 2$. Then the identity
\[
\tau = \tau\left\{ 1 + \frac{1 - \sigma}{\sigma} h_\sigma(\tau) - \frac{\log \tau}{\log \sigma} + \frac{\log \tau}{\log \sigma} + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^2 \right) \right\}
\]
follows. Therefore, we see that $h_\sigma(\tau)$ satisfies
\[
h_\sigma(\tau) = \frac{1}{\log \tau} \left( \frac{\sigma}{1 - \sigma} \log \tau + \log B(\sigma) - \frac{B_{1,1}(\sigma)}{B_{0,1}(\sigma)} \right) + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^2 \right)
\]
\[
= \frac{B_{1,\sigma}(\log \tau)}{\log \tau} + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^2 \right).
\]
It derives the desired result when $N = 2$. For $m \geq 2$, we assume that it is valid further when $N = 1, \ldots, m$. Note that the asymptotic formula
\[
(4.6) \quad \tau = g_{0,1}(\sigma) \kappa_{\frac{1 - \sigma}{\sigma}} \left\{ 1 + \sum_{j=1}^{m} \frac{g_{j,1}(\sigma)}{g_{0,1}(\sigma)} \frac{\log \tau}{(\log \tau) j} \right\} + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^{m+1} \right)
\]
follows from Proposition 3.1 with $N = m + 1$. If we put
\[
\kappa = B(\sigma)(\log \tau)_{\frac{1 - \sigma}{\sigma}} \left\{ 1 + \sum_{n=0}^{m-1} \frac{B_{n,\sigma}(\log \tau)}{(\log \tau) n} + h_{m,\sigma}(\tau) \right\},
\]
then the inductive assumption gives the upper bound
\[
(4.7) \quad h_{m,\sigma}(\tau) \ll \left( \frac{\log \tau}{\log \sigma} \right)^m.
\]
Hence the terms $\kappa_{\frac{1 - \sigma}{\sigma}}$ and $\log \kappa$ are calculated as
\[
\kappa_{\frac{1 - \sigma}{\sigma}} = B(\sigma)(\log \tau)_{\frac{1 - \sigma}{\sigma}} \left\{ 1 + \sum_{n=1}^{m} \frac{C_{n,\sigma}(\log \tau)}{(\log \tau) n} \right\}
\]
\[
+ \frac{1 - \sigma}{\sigma} h_{m,\sigma}(\tau) + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^{m+1} \right),
\]
\[
\log \kappa = \frac{\sigma}{1 - \sigma} (\log \tau)_{\frac{1 - \sigma}{\sigma}} \left\{ 1 + \sum_{n=1}^{m} \frac{C^*_{n,\sigma}(\log \tau)}{(\log \tau) n} \right\} + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^{m+1} \right),
\]
where $C_{n,\sigma}(x)$ and $C^*_{n,\sigma}(x)$ are polynomials of degree at most $n$. Then formula (4.6) deduces the identity
\[
\tau = \tau\left\{ 1 + \sum_{n=1}^{m} \frac{D_{n,\sigma}(\log \tau)}{(\log \tau) n} + \frac{1 - \sigma}{\sigma} h_{m,\sigma}(\tau) + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^{m+1} \right) \right\}
\]
with some polynomials $D_{n,\sigma}(x)$ of degree at most $n$. Therefore we obtain
\[
h_{m,\sigma}(\tau) = -\frac{\sigma}{1 - \sigma} \sum_{n=1}^{m} \frac{D_{n,\sigma}(\log \tau)}{(\log \tau) n} + O\left( \left( \frac{\log \tau}{\log \sigma} \right)^{m+1} \right).
By inductive assumption (4.7), we see that 
\[ D_{1,\sigma}(x) = \cdots = D_{m-1,\sigma}(x) = 0. \] 
Hence the formula 
\[ h_{m,\sigma}(\tau) = \frac{\sigma}{1 - \sigma} \frac{D_{m,\sigma}(\log_2 \tau)}{(\log \tau)^m} + O \left( \left( \frac{\log \tau}{\log \tau} \right)^{m+1} \right) \]
follows, which asserts that the desired result is valid when \( N = m + 1. \) \( \square \)

**Proposition 4.3.** Let \( N \in \mathbb{Z}_{\geq 1} \) and put \( \tau = 2 \log t + 2\gamma. \) Denote by \( \kappa = \kappa(1, \tau) \) the solution of (4.1) with \( \sigma = 1. \) For \( n = 0, \ldots, N - 1, \) there exist real numbers \( b_n \) with \( b_0 = 1 \) such that the formula 
\[ \kappa = \exp \left( t - \frac{1}{2} g_{0,1} \right) \left\{ \sum_{n=0}^{N-1} b_n t^n + O \left( \frac{1}{tN} \right) \right\} \]
holds if \( t > 0 \) is large enough. Here, the implied constant depends on \( N. \) Furthermore, the real number \( b_1 \) is obtained as 
\[ b_1 = -\frac{1}{8} g_{0,1} - \frac{1}{2} g_{1,1}, \]
where \( g_{0,1} \) and \( g_{1,1} \) are the constants defined by (2.5).

**Proof.** By Proposition 3.2 with \( N = 1, \) we have 
\[ 2 \log t = 2 \log_2 \kappa + \frac{g_{0,1}}{\log \kappa} + O \left( \frac{1}{(\log \kappa)^2} \right) \]
since we put \( \tau = 2 \log t + 2\gamma. \) Therefore the asymptotic formula 
\[ t = (\log \kappa) \exp \left( \frac{g_{0,1}}{2 \log \kappa} + O \left( \frac{1}{(\log \kappa)^2} \right) \right) = \log \kappa + \frac{1}{2} g_{0,1} + O \left( \frac{1}{t} \right) \]
follows by noting that \( t \asymp \log \kappa \) holds. If we use Proposition 3.2 with \( N \geq 2, \) then one can prove more generally
\[ t = \log \kappa + \frac{1}{2} g_{0,1} + \sum_{n=1}^{N-1} \beta_n t^n + O \left( \frac{1}{tN} \right) \]
by induction on \( N, \) where \( \beta_n \) are real numbers such that \( \beta_1 = \frac{1}{2} g_{0,1}^2 + \frac{1}{2} g_{1,1}. \) See also the proof of [13, Lemma 8.1] for an analogous argument. From the above, we obtain the asymptotic formula 
\[ \kappa = \exp \left( t - \frac{1}{2} g_{0,1} - \sum_{n=1}^{N-1} \frac{\beta_n}{\log \kappa} + O \left( \frac{1}{tN} \right) \right) \]
\[ = \exp \left( t - \frac{1}{2} g_{0,1} \right) \left\{ \sum_{n=0}^{N-1} b_n t^n + O \left( \frac{1}{tN} \right) \right\} \]
as desired, where \( b_n \) are real numbers such that \( b_0 = 1 \) and \( b_1 = -\beta_1. \) \( \square \)
4.2. **Proof of Theorem 1.1** Let $\sigma > 1/2$ and $\tau > 0$. Denote by $\kappa = \kappa(\sigma, \tau)$ the solution of (1.1). Using the density function $M_\sigma$ of (1.5), we define

$$N_\sigma(x; \tau) = \frac{e^{\kappa(x+\tau)}}{F_\sigma(\kappa)} M_\sigma(x + \tau)$$

for $x \in \mathbb{R}$, where $F_\sigma(\kappa)$ is the moment-generating function of (1.6). To begin with, we show the following lemmas on the function $N_\sigma(x; \tau)$.

**Lemma 4.4.** Let $\sigma > 1/2$ and $\tau > 0$. Then $N_\sigma(w; \tau)$ is a non-negative continuous function satisfying the equalities

$$\int_\mathbb{R} N_\sigma(x; \tau) \, dx = 1 \quad \text{and} \quad \int_\mathbb{R} N_\sigma(x; \tau) x \, dx = 0.$$ 

**Proof.** By the definition of $N_\sigma$, the Fourier transform is represented as

$$\mathcal{N}_\sigma(v; \tau) := \int_\mathbb{R} N_\sigma(x; \tau) e^{ivx} \, dx = e^{-iv\kappa} \frac{F_\sigma(\kappa + iv)}{F_\sigma(\kappa)}.$$ 

Thus we have $\mathcal{N}_\sigma(0; \tau) = 1$, and furthermore,

$$\left. \frac{d}{dv} \mathcal{N}_\sigma(v; \tau) \right|_{v=0} = -i\tau + i f'_\sigma(\kappa) = 0$$

due to (4.11). Hence the result follows since we have the identities

$$\left. \frac{d^k}{dv^k} \mathcal{N}_\sigma(v; \tau) \right|_{v=0} = i^k \int_\mathbb{R} N_\sigma(x; \tau) x^k \, dx$$

for all $k \geq 0$. \hfill \Box

**Lemma 4.5.** Let $1/2 < \sigma \leq 1$ and $\tau > 0$ be a large real number. Then there exist positive constants $c_1(\sigma)$ and $c_2(\sigma)$ such that we have

$$|\mathcal{N}_\sigma(v; \tau)| \leq \exp\left( -c_2(\sigma) v^2 \frac{\kappa^{1/2 - 2}}{\log \kappa} \right)$$

if $|v| \leq c_1(\sigma)\kappa$ is satisfied.

**Proof.** By formula (4.10), it is sufficient to evaluate $|F_\sigma(s)|/F_\sigma(\kappa)$ with $s = \kappa + iv$. Recall that the function $F_\sigma$ satisfies (3.3). Then we obtain

$$\left| \frac{F_\sigma(s)}{F_\sigma(\kappa)} \right| \leq \prod_{Q_1 < p < Q_2} \left| \frac{F_{\sigma,p}(s)}{F_{\sigma,p}(\kappa)} \right|$$

for $Q_1, Q_2 > 0$ since the inequality $|F_{\sigma,p}(s)| \leq F_{\sigma,p}(\kappa)$ holds for every $p$. We deduce from Lemmas 2.1 and 3.3 that

$$F_{\sigma,p}(s) = \frac{1}{\sqrt{4\pi}} \frac{\exp(2sp^{-\sigma})}{(sp^{-\sigma})^{3/2}} (1 + h_{\sigma,p}(s))$$

for any $p > y_1$ in the disk $|s - \kappa| \leq \kappa/2$, where $h_{\sigma,p}(s)$ is a holomorphic function such that $h_{\sigma,p}(s) \ll \kappa^{-1}p^{\sigma} + \kappa p^{-2\sigma} + p^{-1}$. By Cauchy’s integral formula, we have

$$h^{(j)}_{\sigma,p}(\kappa) \ll 2^j j! \kappa^{-j} \left( \kappa^{-1}p^{\sigma} + \kappa p^{-2\sigma} + p^{-1} \right)$$
for all $j \geq 0$. Then, we choose the parameters $Q_1, Q_2 > 0$ as

\[ Q_1 = \left( \frac{\kappa}{\epsilon_1} \right)^{\frac{1}{2}} \quad \text{and} \quad Q_2 = (\epsilon_2 \kappa)^{\frac{1}{2}} \]

with small positive constants $\epsilon_j = \epsilon_j(\sigma)$. For $Q_1 < p < Q_2$, formula (4.12) yields

\[
\log \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} = \frac{3}{2} \Re \log \left( 1 + \frac{i v}{\kappa} \right) + \Re \left( 1 + \frac{h_{\sigma,p}(s) - h_{\sigma,p}(\kappa)}{1 + h_{\sigma,p}(\kappa)} \right) \\
= -\frac{3 v^2}{4 \kappa^2} + \Re \left( h_{\sigma,p}(s) - h_{\sigma,p}(\kappa) \right) + O \left( \frac{v^3}{\kappa^3} + |h_{\sigma,p}(s) - h_{\sigma,p}(\kappa)|^2 \right)
\]

if $|v| \leq c_1 \kappa$ is satisfied with a small positive constant $c_1 = c_1(\sigma)$. Remark that $h'_{\sigma,p}(\kappa)$ is real by definition. By (4.14), we have

\[
\Re (h_{\sigma,p}(s) - h_{\sigma,p}(\kappa)) \leq \sum_{j=2}^{\infty} \frac{|h_{\sigma,p}(\kappa)|}{j!} |v|^j
\]

\[
\leq \frac{v^2}{\kappa^2} (\kappa^{-1} p^\sigma + \kappa p^{-2\sigma} + p^{-1})
\]

and furthermore,

\[
|h_{\sigma,p}(s) - h_{\sigma,p}(\kappa)|^2 \leq \frac{v^2}{\kappa^2} (\kappa^{-1} p^\sigma + \kappa p^{-2\sigma} + p^{-1})
\]

for $Q_1 < p < Q_2$. Therefore we deduce

\[
\log \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} = \left( -\frac{3}{4} + O \left( \frac{|v|}{\kappa} + \kappa^{-1} p^\sigma + \kappa p^{-2\sigma} + p^{-1} \right) \right) \frac{v^2}{\kappa^2} \leq -\frac{1}{2} \frac{v^2}{\kappa^2}
\]

for $Q_1 < p < Q_2$ if $c_1, \epsilon_1, \epsilon_2 > 0$ are small enough, and $\kappa = \kappa(\sigma, \tau) > 0$ is large enough. By the prime number theorem, we obtain the inequality

\[
\sum_{Q_1 < p < Q_2} \log \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} \leq -\frac{1}{4} \frac{v^2}{\kappa^2} \log Q_2
\]

Inserting (4.14), we conclude that

\[
\prod_{Q_1 < p < Q_2} \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} \leq \exp \left( -c_2(\sigma) p^\frac{1}{2} \kappa^{\frac{1}{2} - 2} \frac{1}{\log \kappa} \right)
\]

with some positive constant $c_2(\sigma)$, which completes the proof. \hfill \square

**Lemma 4.6.** Let $1/2 < \sigma \leq 1$ and $\tau > 0$ be a large real number. For any $c > 0$, there exists a positive constant $c_3(\sigma, c)$ such that we have

\[
|\tilde{N}_\sigma(v; \tau)| \leq \exp \left( -c_3(\sigma, c) \frac{|v|^{1/2}}{\log |v|} \right)
\]

if $|v| > ck$ is satisfied.

**Proof.** Similarly to (4.11), we have the inequality

\[
\frac{|F_\sigma(s)|}{F_\sigma(\kappa)} \leq \prod_{p > Q_1} \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)}
\]
for $Q_3 > 0$. If the condition $|s|p^{-\sigma} < \delta$ is valid with a small positive constant $\delta = \delta(\sigma, c)$, then $F_{\sigma,p}(s)$ is calculated as

$$F_{\sigma,p}(s) = 1 + 2sp^{-\sigma}E[\cos \Theta_p] + s^2p^{-2\sigma}E[(\cos \Theta_p)^2] + O(|s|^3p^{-3\sigma})$$

$$= 1 + \frac{1}{4}s^2p^{-2\sigma} + O(|s|^2p^{-2\sigma-1} + |s|^3p^{-3\sigma})$$

by using the equalities

$$E[\cos \Theta_p] = 0 \quad \text{and} \quad E[(\cos \Theta_p)^2] = \frac{1}{4}\left(1 + \frac{1}{p}\right).$$

It yields the asymptotic formula

$$\log F_{\sigma,p}(s) = \frac{1}{4}s^2p^{-2\sigma} + O(|s|^2p^{-2\sigma-1} + |s|^3p^{-3\sigma})$$

if $Q_3$ is large and $\delta$ is small. Then, we choose the parameter $Q_3 > 0$ as

$$(4.15) \quad Q_3 = \left(\frac{4}{\delta^2}v\right)^{1/\sigma}$$

so that the condition $|s|p^{-\sigma} < \delta$ is satisfied for $p > Q_3$. Since $\text{Re}(s^2) = \kappa^2 - v^2$ with $s = \kappa + iv$, we obtain

$$\log \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} = \left(-1 + O\left(p^{-1} + vp^{-\sigma}\right)\right)v^2p^{-2\sigma} \leq -\frac{1}{8}v^2p^{-2\sigma}$$

for $p > Q_3$ if $\delta > 0$ is sufficiently small, and $\kappa = \kappa(\sigma, \tau) > 0$ is sufficiently large. By the prime number theorem, we have

$$\sum_{p > Q_3} \log \frac{|F_{\sigma,p}(s)|}{F_{\sigma,p}(\kappa)} \leq -\frac{v^2}{8} \sum_{p > Q_3} p^{-2\sigma} \leq -\frac{v^2}{16(2\sigma - 1) \log Q_3} Q_3^{1-2\sigma}$$

$$(\text{4.15})$$

Inserting $$(\text{4.15})$$ to this, we derive the inequality

$$\prod_{p > Q_3} \left|\frac{F_{\sigma,p}(s)}{F_{\sigma,p}(\kappa)}\right| \leq \exp\left(-c_3(\sigma, c)\frac{|v|^{1/\sigma}}{\log |v|}\right)$$

with some positive constant $c_3(\sigma, c)$ if $\kappa > 0$ is large. Hence the result follows. \hfill \Box

**Proof of Theorem 1.1.** Let $1/2 < \sigma \leq 1$. By the definition of the function $\mathcal{N}_\sigma$, the desired result follows if we have

$$(4.16) \quad \mathcal{N}_\sigma(x; \tau) = \frac{1}{\sqrt{f''_\sigma(\kappa)}} \left\{ \exp\left(-\frac{x^2}{2f''_\sigma(\kappa)}\right) + O\left(\kappa^{-\frac{1}{2}}\sqrt{\log \kappa}\right) \right\}.$$  

To show this, we use the inverse formula

$$(4.17) \quad \mathcal{N}_\sigma(x; \tau) = \int_{\mathbb{R}} \tilde{N}_\sigma(v; \tau)e^{-ixv}dv$$

which is justified by Lemmas 4.3 and 4.5. Note that $\tilde{N}_\sigma(v; \tau)$ is represented as

$$\tilde{N}_\sigma(v; \tau) = \exp\left(-\frac{f''_\sigma(\kappa)}{2}v^2\right)W(iv)$$
by (4.10), where $W$ is the following entire function:
\[
W(z) = \exp \left( -\tau z - \frac{f''(\kappa)}{2} z^2 \right) \frac{F_\sigma(z + \kappa)}{F_\sigma(\kappa)}
\]

Put $\lambda = \kappa^{1 - \frac{i}{\sigma}}$. Then we have
\[
\int_{-\lambda}^{\lambda} \hat{N}_\sigma(v; \tau) e^{-ixv} |dv| = \int_{-\lambda}^{\lambda} \exp \left( -\frac{f''(\kappa)}{2} v^2 \right) e^{-ixv} |dv| \\
+ \int_{-\lambda}^{\lambda} \exp \left( -\frac{f''(\kappa)}{2} v^2 \right) (W(iv) - 1) e^{-ixv} |dv|
= I_{1,1} + I_{1,2},
\]
say. For any $a, b > 0$ with $ab^2 > 1$, we have
\[
\int_{-\infty}^{\infty} \exp(-av^2) e^{-ixv} |dv| = \frac{1}{\sqrt{2a}} \exp \left( -\frac{x^2}{4a} \right), \\
\int_{b}^{\infty} \exp(-av^2) |dv| \ll \frac{1}{\sqrt{2a}} \exp(-ab^2).
\]
Therefore the first integral $I_{1,1}$ is estimated as
\[
I_{1,1} = \frac{1}{\sqrt{f''(\kappa)}} \left\{ \exp \left( -\frac{\sigma}{2 f''(\kappa)} \right) + O \left( \exp \left( -\frac{f''(\kappa)}{2} \lambda^2 \right) \right) \right\}
\]
since $f''(\kappa) \lambda^2 \leq \kappa^{\frac{3}{2}} (\log \kappa)^{-1} \to \infty$ as $\kappa \to \infty$ by Propositions 3.1 and 3.2. The second integral $I_{1,2}$ is evaluated as follows. We see that $W(z)$ is represented as
\[
W(z) = \exp \left( f_\sigma(z + \kappa) - f_\sigma(\kappa) - f'_\sigma(\kappa) z - \frac{f''(\kappa)}{2} z^2 \right) \\
= \exp \left( \sum_{j=3}^{\infty} \frac{f^{(j)}(\kappa)}{j!} z^j \right)
\]
at least near the origin. Thus we have $W(z) = 1 + \sum_{j \geq 3} w_j z^j / j!$, where
\[
w_j = \sum_{k=1}^{\lfloor j/3 \rfloor} \frac{1}{k!} \sum_{j_1 + \cdots + j_k = j, j_k \geq 3} \binom{j}{j_1, \ldots, j_k} f^{(j_1)}(\kappa) \cdots f^{(j_k)}(\kappa).
\]
Additionally, we have the inequality
\[
1 + \sum_{j \geq 3} \frac{|w_j|}{j!} |z|^j \\
\leq 1 + \sum_{j \geq 3} \frac{1}{j!} \left\{ \sum_{k=1}^{\lfloor j/3 \rfloor} \frac{1}{k!} \sum_{j_1 + \cdots + j_k = j, j_k \geq 3} \binom{j}{j_1, \ldots, j_k} |f^{(j_1)}(\kappa)| \cdots |f^{(j_k)}(\kappa)| \right\} |z|^j \\
= \exp \left( \sum_{j=3}^{\infty} \frac{|f^{(j)}(\kappa)|}{j!} |z|^j \right)
\]
for any \( z \in \mathbb{C} \). Then, we deduce from Propositions 3.1 and 3.2 the upper bounds

\[
f^{(j)}(\sigma) \leq C_j \frac{\kappa^{\frac{1}{2}-j}}{\log \kappa}
\]

for all \( j \geq 3 \) by recalling that \( g_{0,j}(\sigma) \leq j! \) and \( g_{0,j} \leq j! \). By this, we obtain

\[
\sum_{j=3}^{\infty} \frac{|f^{(j)}(\sigma)|}{j!} |v|^j \leq \frac{C_3}{\log \kappa} \sum_{j=3}^{\infty} \left( \frac{|v|}{\kappa} \right)^j \leq \frac{C_3}{\log \kappa} |v|^3
\]

for \( |v| \leq \lambda \) with the implied constant depending only on \( \sigma \). Remark that \( \kappa^{\frac{1}{2}-3} |v|^3 \) is bounded for \( |v| \leq \lambda \). Hence we obtain from (4.19) that

\[
\sum_{j=3}^{\infty} \frac{|w_j|}{j!} |v|^j \leq \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{C_3}{\log \kappa} |v|^3 \right)^n \leq \frac{C_3}{\log \kappa} |v|^3,
\]

where \( C = C(\sigma) \) is a positive constant. As a result, we evaluate \( I_{1,2} \) as

\[
|I_{1,2}| \leq \int_{-\lambda}^{\lambda} \exp \left( -\frac{f''(\sigma)}{2} v^2 \right) \left( \sum_{j=3}^{\infty} \frac{|w_j|}{j!} |v|^j \right) |dv| \leq \frac{C_3}{\log \kappa} \int_{0}^{\infty} \exp \left( -\frac{f''(\sigma)}{2} v^2 \right) v^3 dv \leq \frac{C_3}{\log \kappa} \frac{1}{f''(\sigma)(\kappa)^2}.
\]

We further note that Propositions 3.1 and 3.2 provide the bound

\[
\frac{1}{\log \kappa} \frac{1}{f''(\sigma)(\kappa)} \leq \kappa^{-\frac{1}{2\pi}} \sqrt{\log \kappa}.
\]

Therefore, the integral \( I_{1,2} \) is evaluated as

\[
I_{1,2} \leq \frac{1}{\sqrt{f''(\sigma)}} \kappa^{-\frac{1}{2\pi}} \sqrt{\log \kappa}.
\]

Combining (4.18) and (4.20), we derive

\[
\int_{-\lambda}^{\lambda} \tilde{N}(v; \tau) e^{-ixv} |dv| = \frac{1}{\sqrt{f''(\sigma)}} \left\{ \exp \left( -\frac{x^2}{2f''(\sigma)} \right) + O \left( \kappa^{-\frac{1}{2\pi}} \sqrt{\log \kappa} \right) \right\}.
\]

Thus, the remaining work is to bound the integral

\[
I_2 = \int_{|v| > \lambda} \tilde{N}(v; \tau) e^{-ixv} |dv|.
\]

Let \( c_1(\sigma), c_2(\sigma), \) and \( c_3(\sigma, c) \) denote the positive constants of Lemmas 4.5 and 4.6 and put \( c_3(\sigma) = c_3(\sigma, c_1(\sigma)) \). By these lemmas, we have

\[
I_2 \leq \int_{\lambda}^{c_1(\sigma)\kappa} \exp \left( -c_2(\sigma) v^2 \frac{\kappa^{\frac{3}{2}}}{\log \kappa} \right) dv + \int_{c_1(\sigma)\kappa}^{\infty} \exp \left( -c_3(\sigma) \frac{v^2}{\log v} \right) dv \leq \exp \left( -c_2(\sigma) \frac{\kappa^{\frac{3}{2}}}{\log \kappa} \right) + \exp \left( -c_3(\sigma) \frac{\kappa^{\frac{3}{2}}}{\log \kappa} \right),
\]
where \( c_2'(\sigma) \) and \( c_3'(\sigma) \) are positive constants. Finally, we again use Propositions 3.1 and 3.2 to deduce

\[
I_2 \ll \frac{1}{\sqrt{f''_\sigma(\kappa)}} \kappa^{-\frac{1}{2}} \sqrt{\log \kappa}.
\]

Hence we obtain the result by formula (4.17).

\[
\square
\]

4.3. Corollaries. As stated in Section 1.1, one can deduce from Theorem 1.1 several results on the distribution function \( \Phi(\sigma, \tau) \) for \( 1/2 < \sigma \leq 1 \).

**Corollary 4.7.** With the same assumption as in Theorem 1.1, we obtain

\[
\Phi(\sigma, \tau) = \frac{F_\sigma(\kappa)e^{-\kappa \tau}}{\kappa \sqrt{2\pi f''_\sigma(\kappa)}} \left\{ 1 + O \left( \kappa^{-\frac{1}{2}} \sqrt{\log \kappa} \right) \right\}
\]

if \( \tau > 0 \) is large enough, where the implied constant depends on \( \sigma \).

**Proof.** By Theorem 1.1, we calculate \( \Phi(\sigma, \tau) \) as

\[
\Phi(\sigma, \tau) = \int_0^\infty M_\sigma(\tau + x) |dx|
\]

\[
= \frac{F_\sigma(\kappa)e^{-\kappa \tau}}{\sqrt{f''_\sigma(\kappa)}} \left\{ \int_0^\infty \exp \left( -\kappa x - \frac{x^2}{2f''_\sigma(\kappa)} \right) |dx| + O \left( \kappa^{-\frac{1}{2}} \sqrt{\log \kappa} \right) \int_0^\infty e^{-\kappa x} |dx| \right\}
\]

\[
= \frac{F_\sigma(\kappa)e^{-\kappa \tau}}{\kappa \sqrt{2\pi f''_\sigma(\kappa)}} \left\{ \int_0^\infty \exp \left( -x - \frac{x^2}{2\kappa^2 f''_\sigma(\kappa)} \right) dx + O \left( \kappa^{-\frac{1}{2}} \sqrt{\log \kappa} \right) \right\}.
\]

Furthermore, we have the asymptotic formula

\[
\int_0^\infty \exp \left( -x - \frac{x^2}{2\kappa^2 f''_\sigma(\kappa)} \right) dx = 1 + O \left( \frac{1}{\kappa^2 f''_\sigma(\kappa)} \right)
\]

\[
= 1 + O \left( \kappa^{-\frac{1}{2}} \log \kappa \right)
\]

by using Propositions 3.1 and 3.2. Hence we obtain the conclusion. \( \square \)

We further deduce from Corollary 4.7 the formula

\[
\log \Phi(\sigma, \tau) = f_\sigma(\kappa) - \kappa \tau + O \left( \log \kappa + \log f''_\sigma(\kappa) \right)
\]

\[
= f_\sigma(\kappa) - \kappa f''_\sigma(\kappa) + O(\log \kappa)
\]

for \( 1/2 < \sigma \leq 1 \) by recalling \( \tau = f''_\sigma(\kappa) \) and \( f''_\sigma(\kappa) \geq \kappa^{\frac{1-2\sigma}{2}} (\log \kappa)^{-1} \). By this, we prove Corollaries 1.2 and 1.3 as below.

**Proof of Corollary 1.2.** Let \( 1/2 < \sigma < 1 \). Then it follows from Proposition 3.1 that

\[
\log \Phi(\sigma, \tau) = -\frac{\kappa^{\frac{1}{2}}}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{c_n(\sigma)}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\}
\]

\[
= -c_0(\sigma) \kappa^{\frac{1}{2}} \left\{ 1 + \sum_{n=1}^{N-1} \frac{c_n(\sigma)}{c_0(\sigma)} \frac{1}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\}
\]
holds for any $N \in \mathbb{Z}_{\geq 1}$, where we put $c_n(\sigma) = g_{n,1}(\sigma) - g_{n,0}(\sigma)$. Here, we interpret $\sum_{n=1}^{N-1} = 0$ if $N = 1$. Moreover, we have

$$\kappa^\frac{1}{2} = B(\sigma)^\frac{1}{2}(\tau \log \tau)^\frac{1}{2\sigma} \left\{ 1 + \sum_{n=1}^{N-1} \frac{D_{n,\sigma}(\log_2 \tau)}{(\log \tau)^n} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^N \right) \right\},$$

$$\log \kappa = \frac{\sigma}{1 - \sigma} (\log \tau) \left\{ 1 + \sum_{n=1}^{N-1} \frac{D^*_n(\log_2 \tau)}{(\log \tau)^n} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^N \right) \right\}$$

by Proposition 4.2, where $B(\sigma)$ is determined as (4.2), and $D_{n,\sigma}(x)$ and $D^*_n(x)$ are polynomials of degree at most $n$. One can calculate the polynomials when $n = 1$ as

$$D_{1,\sigma}(x) = \frac{1}{\sigma} B_{1,\sigma}(x)$$

$$= \frac{1}{1 - \sigma} \left( \frac{1 - \sigma}{\sigma} g_{0,1}(\sigma) \right) - \frac{1}{\sigma} g_{1,1}(\sigma),$$

$$D^*_{1,\sigma}(x) = x + \frac{1 - \sigma}{\sigma} \log B(\sigma)$$

$$= x - \log \left( \frac{1 - \sigma}{\sigma} g_{0,1}(\sigma) \right)$$

with $B_{1,\sigma}(x)$ as in Proposition 4.2. Then, (4.21) derives

$$\log \Phi(\sigma, \tau) = -A(\sigma) \tau^{\frac{1}{\sigma} (\log \tau)} + \sum_{n=0}^{N-1} A_n(\log_2 \tau) \left( \frac{\log_2 \tau}{\log \tau} \right)^n + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^N \right),$$

where $A(\sigma)$ is given by $A(\sigma) = c_0(\sigma) B(\sigma)^\frac{1}{2}(1 - \sigma)/\sigma$, and $A_n(\sigma)$ are polynomials of degree at most $n$ such that $A_{0,\sigma}(x) = 1$. Note that $g_{0,0}(\sigma)$ and $g_{0,1}(\sigma)$ satisfy the relations

$$g_{0,0}(\sigma) = \frac{1}{\sigma} g_{0,1}(\sigma) \quad \text{and} \quad g_{0,1}(\sigma) = \int_0^\infty g(y^{-\sigma}) \, dy$$

by definition. Hence $A(\sigma)$ is represented as

$$A(\sigma) = c_0(\sigma) B(\sigma)^\frac{1}{2}(1 - \sigma)/\sigma = (1 - \sigma) \left( \frac{1 - \sigma}{\sigma} g_{0,1}(\sigma) \right)^{-\frac{1}{\sigma}}.$$

Finally, we calculate the polynomial $A_{1,\sigma}(x)$. By formula (4.21), we have

$$\log \Phi(\sigma, \tau) = -c_0(\sigma) B(\sigma)^\frac{1}{2}(\tau \log \tau)^\frac{1}{\sigma} \left\{ 1 + \frac{D_{1,\sigma}(\log_2 \tau)}{\log \tau} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right) \right\}$$

$$\times \frac{1 - \sigma}{\sigma} (\log \tau)^{-1} \left\{ 1 - \frac{D^*_{1,\sigma}(\log_2 \tau)}{\log \tau} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right) \right\}$$

$$\times \left\{ 1 + \frac{c_1(\sigma)}{c_0(\sigma)} \frac{1 - \sigma}{\sigma} (\log \tau)^{-1} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right) \right\}$$

$$= A(\sigma) \tau^{\frac{1}{\sigma} (\log \tau)} + O \left( \left( \frac{\log_2 \tau}{\log \tau} \right)^2 \right),$$
where
\[
A_{1,\sigma}(x) = D_{1,\sigma}(x) - D_{1,\sigma}^*(x) + \frac{c_1(\sigma) 1 - \sigma}{c_0(\sigma)} \sigma
\]
\[
= D_{1,\sigma}(x) - D_{1,\sigma}^*(x) + \frac{1}{\sigma} g_{1,1}(\sigma) - g_{1,0}(\sigma).
\]

Using (4.22) and (4.23), we have
\[
A_{1,\sigma}(x) = \frac{\sigma}{1 - \sigma} x - \frac{\sigma}{1 - \sigma} \log \left( \frac{1 - \sigma}{\sigma} g_{0,1}(\sigma) \right) - \frac{1}{\sigma} g_{1,0}(\sigma).
\]
Thus we obtain the desired representation of \( A_{1,\sigma}(x) \) by noting that
\[
g_{1,0}(\sigma) = -\sigma \int_0^\infty g(y^{-\sigma}) \log y \, dy.
\]
\[\square\]

**Proof of Corollary 1.2.** In this case, we apply Proposition 3.2 to deduce
\[
\log \Phi(1, \tau) = -\frac{\kappa}{\log \kappa} \left\{ \sum_{n=0}^{N-1} \frac{c_n}{(\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\}
\]
\[
= -\frac{2\kappa}{\log \kappa} \left\{ 1 + \sum_{n=1}^{N-1} \frac{c_n}{2 (\log \kappa)^n} + O \left( \frac{1}{(\log \kappa)^N} \right) \right\},
\]
where \( c_n = g_{n,1} - g_{n,0} \). Here, we remark that \( c_0 = 2 \) since we have
\[
g_{0,1} = \lim_{\epsilon \to 0} \left[ \frac{g_*(u)}{u} \right]_{0}^{1-\epsilon} + \left[ \frac{g_*(u)}{u} \right]_{1+\epsilon}^{\infty} + \int_0^\infty \frac{g_*(u)}{u^2} \, du = 2 + g_{0,0}
\]
by integrating by parts. Furthermore, we deduce from Proposition 4.3 and (4.8) the asymptotic formulas
\[
\kappa = \exp \left( t - \frac{1}{2} g_{0,1} \right) \left\{ 1 + \frac{b_1}{t} + \sum_{n=2}^{N-1} \frac{b_n}{t^n} + O \left( \frac{1}{t^N} \right) \right\},
\]
\[
\log \kappa = t \left\{ 1 - \frac{g_{0,1}}{2t} - \sum_{n=2}^{N-1} \frac{\beta_{n-1}}{t^n} + O \left( \frac{1}{t^N} \right) \right\},
\]
where \( b_n \) and \( \beta_n \) are real numbers such that \( b_1 = -\beta_1 = -\frac{1}{2} g_{0,1} - \frac{1}{2} g_{1,1} \). Using these formulas, we obtain
\[
\log \Phi(1, \tau) = -\frac{e^{t-A}}{t} \left\{ \sum_{n=0}^{N-1} \frac{a_n}{t^n} + O \left( \frac{1}{t^N} \right) \right\}
\]
by (4.24), where \( A \) is given by
\[
A = \frac{1}{2} g_{0,1} - \log 2 = 1 + \frac{1}{2} g_{0,0} - \log 2,
\]
and \( a_n \) are real numbers such that \( a_0 = 1 \) and
\[
a_1 = b_1 + \frac{1}{2} g_{0,1} + \frac{1}{2} c_1 = -\frac{1}{8} g_{0,1} - \frac{1}{2} g_{1,0} + \frac{1}{2}.
\]
Finally, we see that $g_{0,1}$ and $g_{1,0}$ are represented as

$$g_{0,1} = \int_{0}^{\infty} g_*(y^{-1}) \, dy \quad \text{and} \quad g_{1,0} = - \int_{0}^{\infty} g_*(y^{-1}) \log y \, dy.$$  

Hence we complete the proof. \qed

**Remark 4.8.** The constant $A$ of this paper is consistent with the constant $A_k$ of Lamzouri [14, Theorem 0.2] when $k = 1$, while he represented it in a slightly different form. Especially, we have

$$A_1 = 1 + \int_{0}^{\infty} \frac{h_*(u)}{u^2} \, du$$

according to the representation by Lamzouri, where $h_*(u)$ is defined as

$$h_*(u) = \begin{cases} 
    h(u) & \text{if } 0 < u < 1, \\
    h(u) - u & \text{if } u \geq 1 
\end{cases}$$

by using the cumulant-generating function

$$h(u) = \log \left( \frac{2}{\pi} \int_{0}^{\pi} \exp(u \cos \theta) \sin^2 \theta \, d\theta \right).$$

Then we have $h(2u) = g(u)$, and therefore, we see that

$$A_1 = 1 + \frac{1}{2} \int_{0}^{\infty} \frac{h_*(2u)}{u^2} \, du$$

$$= 1 + \frac{1}{2} \int_{0}^{\infty} \frac{g_*(u)}{u^2} \, du + \frac{1}{2} \int_{1/2}^{1} \frac{h_*(2u) - g_*(u)}{u^2} \, du$$

$$= 1 + \frac{1}{2} \int_{0}^{\infty} \frac{g_*(u)}{u^2} \, du - \log 2$$

which equals to $A$ of this paper. We further remark that $A$ should be consistent with the constant $\gamma_0$ of Liu–Royer–Wu [18, Corollary 1.5]. However, it appears that they miscalculated the value of $\gamma_0$ by forgetting the term $- \log 2$.

5. Comparisons of distribution functions

In this section, we prove Theorem [14] and its corollaries. For this, we apply the following asymptotic formula on the complex moments of $L(\sigma, f)$ which was obtained in the previous paper of the author [23].

**Proposition 5.1.** Let $1/2 < \sigma \leq 1$ and $B \geq 1$. Then there exist positive constants $a = a(\sigma, B)$, $b = b(\sigma, B)$ and a subset $\mathcal{E}_q = \mathcal{E}_q(\sigma, B)$ of $B_2(q)$ such that

$$\frac{1}{\#B_2(q)} \sum_{f \in B_2(q) \setminus \mathcal{E}_q} L(\sigma, f)^s = F_\sigma(s) + O \left( \frac{F_\sigma(\kappa)}{\log q} B^2 + 2 \right)$$

holds uniformly for $s = \kappa + iv \in \mathbb{C}$ with $|s| \leq aR_\sigma(q)$, where we define

$$R_\sigma(q) = \begin{cases} 
    (\log q)^\sigma & \text{if } 1/2 < \sigma < 1, \\
    (\log q)(\log q \log_2 q \log_3 q)^{-1} & \text{if } \sigma = 1. 
\end{cases}$$
Furthermore, we have

\begin{equation}
\#\mathcal{E}_q \ll q \exp \left(-b \frac{\log q}{\log_2 q}\right).
\end{equation}

The implied constants in (5.1) and (5.2) depend on \(\sigma\) and \(B\).

Note that similar results were obtained by Cogdell–Michel [3] and Lamzouri [15] for the averages weighted by \(\omega_f = (4\pi(f, f))^{-1}\). The results were applied to study the weighted distribution functions \(\Phi_q(\sigma, \tau)\) and \(\Phi(\sigma, \tau)\) in [14, 15]. However, the method of this paper is different from theirs in terms of our using the following Esseen inequality.

**Lemma 5.2** (Esseen inequality [19]). Let \(P\) and \(Q\) be two probability measures on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) with distribution functions \(\Phi(\xi) = P((-\infty, \xi])\) and \(\Psi(\xi) = Q((-\infty, \xi])\), respectively. Assume that \(\Psi\) is differentiable, and that \(K = \sup_{\xi \in \mathbb{R}} |\Psi'(\xi)|\) is finite. Then we have

\begin{equation}
\sup_{\xi \in \mathbb{R}} |\Phi(\xi) - \Psi(\xi)| \leq \frac{2}{\pi} \int_0^R \frac{\phi(v) - \psi(v)}{v} dv + \frac{24K}{\pi} R^{-1}
\end{equation}

for every \(R > 0\), where \(\phi\) and \(\psi\) are the characteristic functions defined as

\[\phi(v) = \int_{\mathbb{R}} e^{ivx} dP(x)\quad \text{and} \quad \psi(v) = \int_{\mathbb{R}} e^{ivx} dQ(x).\]

Let \(1/2 < \sigma \leq 1\) and \(B \geq 1\). Define the functions \(U\) and \(V\) on \(\mathbb{R}\) as

\[U(\xi) = \frac{\# \{ f \in B_{2}(q) \setminus \mathcal{E}_q \mid \log L(\sigma, f) \leq \xi \}}{\#B_{2}(q) \setminus \mathcal{E}_q},\]

\[V(\xi) = \int_{-\infty}^\xi M_{\sigma}(x) \, dx,\]

where \(E_q = E_q(\sigma, B)\) is the subset of \(B_{2}(q)\) as in Proposition 5.1. Then we apply Lemma 5.2 with the probability measures defined as

\[P(A) = \frac{\int_{\mathbb{R}} 1_{A+\tau}(\xi) e^{\xi} dU(\xi)}{\int_{\mathbb{R}} e^{\xi} dU(\xi)}\quad \text{and} \quad Q(A) = \frac{\int_{\mathbb{R}} 1_{A+\tau}(\xi) e^{\xi} dV(\xi)}{\int_{\mathbb{R}} e^{\xi} dV(\xi)}\]

for \(A \in \mathcal{B}(\mathbb{R})\), where \(\kappa = \kappa(\sigma, \tau)\) is the solution to equation (4.1). Here, we denote by \(A + \tau\) the set \(\{a + \tau \mid a \in A\}\), and \(1_S\) is the indicator function of a set \(S \subset \mathbb{R}\). We further put \(\Phi(\xi) = P((-\infty, \xi])\) and \(\Psi(\xi) = Q((-\infty, \xi])\) as above.

**Lemma 5.3.** With the notation above, we have

\[\sup_{\xi \in \mathbb{R}} |\Phi(\xi) - \Psi(\xi)| \ll \frac{1}{(\log q)^{B+1}} + \frac{1}{R_\sigma(q) \sqrt{T_\sigma(q)}},\]

if the condition \(\kappa \leq aR_\sigma(q)/2\) is satisfied with the positive constant \(a = a(\sigma, B)\) of Proposition 5.1. Here, the implied constant depends on \(\sigma\) and \(B\).

**Proof.** First, we check the assumption on \(\Psi\) of Lemma 5.2. Note that the identity

\[Q(A) = \frac{1}{F_\sigma(\kappa)} \int_{A+\tau} e^{\kappa x} M_{\sigma}(x) \, dx = \int_{A} N_{\sigma}(x; \tau) \, dx\]
holds for the function $N_\sigma$ of (4.9). Thus $\Psi(\xi) = \int_{-\infty}^{\xi} N_\sigma(x; \tau) \, dx$ is differentiable, and we have

$$K = \sup_{\xi \in \mathbb{R}} |\Psi'(|\xi|) = \sup_{\xi \in \mathbb{R}} \frac{N_\sigma(\xi; \tau)}{\sqrt{2\pi}} \ll \frac{1}{\sqrt{f_\sigma^2(\kappa)}} < \infty$$

by asymptotic formula (4.16). Hence Lemma 5.2 is available for the probability measures $P$ and $Q$. Next, we have the formula

$$\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dU(\xi) = \frac{1}{\# B_2(q) \setminus \mathcal{E}_q} \sum_{f \in B_2(q) \setminus \mathcal{E}_q} L(\sigma, f)^{\nu + iv}$$

$$= \frac{1}{\# B_2(q)} \sum_{f \in B_2(q) \setminus \mathcal{E}_q} L(\sigma, f)^{\nu + iv} + O\left(\exp\left(-\frac{b}{\log q} \log q \right) F_\sigma(\kappa)\right)$$

by applying (5.2). Furthermore, the equality

$$\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dV(\xi) = F(\nu + iv)$$

is valid by definition. With the above preparations, we determine the parameter $R > 0$ as $R = aR_\sigma(q)/2$ so that $|\kappa + iv| \leq aR_\sigma(q)$ is satisfied for $0 < v < R$. Then Proposition 5.1 yields

$$\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dU(\xi) = \int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dV(\xi) + O\left(\frac{F_\sigma(\kappa)}{(\log q)^{B + 2}}\right)$$

for $0 < v < R$. In addition, we see similarly that

$$\int_{\mathbb{R}} e^{\nu \xi} \, dU(\xi) \simeq \int_{\mathbb{R}} e^{\nu \xi} \, dV(\xi) = F_\sigma(\kappa)$$

holds. Using these formulas, we evaluate the integral of the right-hand side of (5.3) as follows. The characteristic functions $\phi$ and $\psi$ are calculated as

$$\phi(v) = \frac{\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dU(\xi)}{\int_{\mathbb{R}} e^{\nu \xi} \, dU(\xi)} e^{-iv\tau} \quad \text{and} \quad \psi(v) = \frac{\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, dV(\xi)}{\int_{\mathbb{R}} e^{\nu \xi} \, dV(\xi)} e^{-iv\tau}.$$

Hence we obtain

$$|\phi(v) - \psi(v)| \leq \frac{\left|\int_{\mathbb{R}} e^{(\nu + iv)\xi} \, d(U - V)(\xi)\right|}{\int_{\mathbb{R}} e^{\nu \xi} \, dV(\xi)} \leq \frac{\left|\int_{\mathbb{R}} e^{\nu \xi} \, d(U - V)(\xi)\right|}{\int_{\mathbb{R}} e^{\nu \xi} \, dV(\xi)} \ll (\log q)^{-B - 2}$$

for $0 < v < R$ by using (5.20) and (5.6). Put $r = \exp(-L \log q / \log_2 q)$ with a constant $L = L(\sigma, B) > 0$ chosen later. Then we have

$$\int_{r}^{R} \frac{|\phi(v) - \psi(v)|}{v} \, du \ll \left(\log \frac{R}{r}\right) (\log q)^{-B - 2} \ll (\log q)^{-B - 1}.$$
For $0 < v \leq r$, we estimate $\phi(v)$ and $\psi(v)$ by using the formula $e^{i \theta} = 1 + O(|\theta|)$ with arbitrary $\theta \in \mathbb{R}$. We have
\[
\int_{\mathbb{R}} e^{(\kappa + iv) \xi} dU(\xi) = \int_{\mathbb{R}} e^{\kappa \xi} dU(\xi) + O \left( v \int_{\mathbb{R}} |\xi| e^{\kappa \xi} dU(\xi) \right).
\]
By the Cauchy–Schwarz inequality and (5.6), we obtain
\[
\int_{\mathbb{R}} |\xi| e^{\kappa \xi} dU(\xi) \ll \sqrt{M_q} F_\sigma(2\kappa),
\]
where we put
\[
M_q = \int_{\mathbb{R}} |\xi|^2 dU(\xi) = \frac{1}{\#B_2(q) \setminus \mathcal{E}_q} \sum_{f \in B_2(q) \setminus \mathcal{E}_q} |\log L(\sigma, f)|^2.
\]
Thus, $\phi$ is estimated as
\[
\phi(v) = e^{-iv\tau} \left( 1 + O \left( v \sqrt{M_q} \frac{\sqrt{F_\sigma(2\kappa)}}{F_\sigma(\kappa)} \right) \right)
\]
by recalling (5.7). In a similar way, we obtain the formula
\[
\psi(v) = e^{-iv\tau} \left( 1 + O \left( v \sqrt{M} \frac{\sqrt{F_\sigma(2\kappa)}}{F_\sigma(\kappa)} \right) \right),
\]
where $M$ is the constant represented as
\[
M = \int_{\mathbb{R}} |\xi|^2 dV(\xi) = \int_{\mathbb{R}} |x|^2 M_\sigma(x) |dx|.
\]
One can deduce from Proposition 5.1 the estimate $M_q - M \ll F_\sigma(\kappa)(\log q)^{-B}$ by differentiating both sides of (5.1) in $s$. Additionally, we use Propositions 3.1 and 3.2 to derive the bounds
\[
\sqrt{F_\sigma(2\kappa)} = \exp \left( \frac{1}{2} f_\sigma(2\kappa) \right) \leq \begin{cases} 
\exp \left( \frac{L_1}{\log \kappa} \right) & \text{if } 1/2 < \sigma < 1, \\
\exp (L_1 \log_2 \kappa) & \text{if } \sigma = 1,
\end{cases}
\]
where $L_1 = L_1(\sigma)$ is a positive constant. Since $\kappa \leq aR_\sigma(q)/2$, there exists a positive constant $L_2 = L_2(\sigma, B)$ such that
\[
\sqrt{F_\sigma(2\kappa)} \leq \exp \left( L_2 \log q \frac{\log_2 q}{\log_2 q} \right)
\]
in both cases $1/2 < \sigma < 1$ and $\sigma = 1$. Hence, we evaluate the difference between $\phi$ and $\psi$ as
\[
|\phi(v) - \psi(v)| \ll v \left| \sqrt{M_q} - \sqrt{M} \right| \frac{\sqrt{F_\sigma(2\kappa)}}{F_\sigma(\kappa)} \ll v \exp \left( L_2 \log q \frac{\log q}{\log_2 q} \right)
\]
by (5.9) and (5.10). Then, we choose the constant $L > 0$ in the definition of $r$ as $L = 2L_2$. We have
\[
\int_0^r \frac{|\phi(v) - \psi(v)|}{v} du \ll \exp \left( -L_2 \log q \frac{\log q}{\log_2 q} \right).
\]
Combining (5.4), (5.8), and (5.11), we deduce from Lemma 5.2 the desired result.
Proof of Theorem 1.4. For $1/2 < \sigma \leq 1$, we define
\[
\Phi^*_q(\sigma, \tau) = \frac{\# \{ f \in B_2(q) \setminus E_q \mid \log L(\sigma, f) > \tau \}}{\# B_2(q) \setminus E_q},
\]
where $E_q = E_q(\sigma, B)$ is the subset of $B_2(q)$ as in Proposition 5.1. Then the identities
\[
\Phi^*_q(\sigma, \tau) = e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} dU(\xi) \int_0^\infty e^{-\kappa \xi} d\Phi(\xi),
\]
\[
\Phi(\sigma, \tau) = e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} dV(\xi) \int_0^\infty e^{-\kappa \xi} d\Phi(\xi)
\]
hold by the definitions of the probability measures $P$ and $Q$. Therefore we have
\[
(5.12) \quad |\Phi^*_q(\sigma, \tau) - \Phi(\sigma, \tau)| \leq e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} d\Phi(\xi) \left| \int_{\mathbb{R}} e^{\kappa \xi} d(U - V)(\xi) \right|
\]
\[
+ e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} dU(\xi) \left| \int_0^\infty e^{-\kappa \xi} d(\Phi - \Psi)(\xi) \right|.
\]
Suppose that the condition $\kappa \leq aR_\sigma(q)/2$ is satisfied. Since $\Psi(\xi)$ is represented as $\Psi(\xi) = \int_{-\infty}^{\xi} N_\sigma(x; \tau) \, dx$, the upper bound
\[
\int_0^\infty e^{-\kappa \xi} d\Psi(\xi) = \int_0^\infty e^{-\kappa \xi} N_\sigma(x; \tau) \, dx \ll \frac{1}{\kappa \sqrt{f''_\sigma(\kappa)}}
\]
follows from asymptotic formula (4.16). Using (5.5), we further deduce
\[
\int_{\mathbb{R}} e^{\kappa \xi} d(U - V)(\xi) \ll \frac{F_\sigma(\kappa)}{(\log q)^B}.
\]
Thus the first term of (5.12) is estimated as
\[
(5.13) \quad e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} d\Phi(\xi) \left| \int_{\mathbb{R}} e^{\kappa \xi} d(U - V)(\xi) \right| \ll \frac{\Phi(\sigma, \tau)}{(\log q)^B}
\]
by Corollary 4.7. Next, we estimate the second term of (5.12) by applying (5.6) and Lemma 5.3. We have
\[
\int_0^\infty e^{-\kappa \xi} d(\Phi - \Psi)(\xi) \ll \sup_{\xi \in \mathbb{R}} |\Phi(\xi) - \Psi(\xi)|
\]
\[
\ll \frac{1}{(\log q)^{B+1}} + \frac{1}{R_\sigma(q) \sqrt{f''_\sigma(\kappa)}}
\]
by the integration by parts. Hence we derive
\[
e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} dU(\xi) \left| \int_0^\infty e^{-\kappa \xi} d(\Phi - \Psi)(\xi) \right| \ll \Phi(\sigma, \tau) \left( \frac{\kappa \sqrt{f''_\sigma(\kappa)}}{(\log q)^{B+1}} + \frac{\kappa}{R_\sigma(q)} \right)
\]
by using Corollary 4.7 again. Furthermore, in both cases $1/2 < \sigma < 1$ and $\sigma = 1$, the estimate $\kappa \sqrt{f''_\sigma(\kappa)} \ll \log q$ follows from Propositions 3.1 and 3.2 since we suppose that $\kappa$ satisfies $\kappa \leq aR_\sigma(q)/2$. It yields
\[
(5.14) \quad e^{-\tau \kappa} \int_{\mathbb{R}} e^{\kappa \xi} dU(\xi) \left| \int_0^\infty e^{-\kappa \xi} d(\Phi - \Psi)(\xi) \right| \ll \Phi(\sigma, \tau) \left( \frac{1}{(\log q)^B} + \frac{\kappa}{R_\sigma(q)} \right).
\]
By (5.13) and (5.14), we obtain
\[ \Phi_q^*(\sigma, \tau) - \Phi(\sigma, \tau) \ll \Phi(\sigma, \tau) \left( \frac{1}{(\log q)^B} + \frac{\kappa}{R_{\sigma}(q)} \right). \]

The difference between \( \Phi_q(\sigma, \tau) \) and \( \Phi_q^*(\sigma, \tau) \) can be evaluated as
\[ \Phi_q(\sigma, \tau) - \Phi_q^*(\sigma, \tau) \ll \exp\left(-b\frac{\log q}{\log_2 q}\right) \]
by using (5.2). Hence we arrive at the formula
\[ \Phi_q(\sigma, \tau) = \Phi(\sigma, \tau) \left( 1 + O\left( \frac{1}{(\log q)^B} + \frac{\kappa}{R_{\sigma}(q)} \right) \right) + O\left( \exp\left(-b\frac{\log q}{\log_2 q}\right) \right) \]
for \( \frac{1}{2} < \sigma \leq 1 \). In the case \( \frac{1}{2} < \sigma < 1 \), we recall that \( \kappa \) satisfies \( \kappa \asymp (\tau \log \tau)^{\frac{1}{1-\sigma}} \)
by Proposition 4.2. Thus one can take a small positive constant \( c(\sigma, B) \) so that the condition \( \kappa \leq aR_{\sigma}(q)/2 \) holds for \( 1 \ll \tau \leq c(\sigma, B)(\log q)^{1-\sigma}(\log_2 q)^{-1} \). Furthermore, Corollary 1.2 yields
\[ \Phi(\sigma, \tau) \gg \exp\left(-b\frac{\log q}{2\log_2 q}\right) \]
in the range \( 1 \ll \tau \leq c(\sigma, B)(\log q)^{1-\sigma}(\log_2 q)^{-1} \) if \( c(\sigma, B) \) is sufficiently small. Thus, the desired conclusion
\[ \Phi_q(\sigma, \tau) = \Phi(\sigma, \tau) \left( 1 + O\left( \frac{1}{(\log q)^B} + \frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{(\log q)^{\sigma}} \right) \right) \]
follows from (5.15). If \( \sigma = 1 \), then we have \( \kappa \asymp e^t \) by Proposition 4.3, where we put \( \tau = 2\log t + 2\gamma \). Hence there exists a large positive constant \( c(B) \) such that \( \kappa \leq aR_1(q)/2 \) holds for \( 1 \ll t \leq \log_2 q - \log_3 q - \log_4 q - c(B) \). In this case, we have the lower bound
\[ \Phi(1, \tau) \gg \exp\left(-b\frac{\log q}{2\log_2 q}\right) \]
by Corollary 1.3 in the range \( 1 \ll t \leq \log_2 q - \log_3 q - \log_4 q - c(B) \) with \( c(B) \) large enough. Therefore (5.15) yields
\[ \Phi_q(1, \tau) = \Phi(1, \tau) \left( 1 + O\left( \frac{1}{(\log q)^B} + \frac{e^t}{(\log q)(\log_2 q \log_3 q)^{-1}} \right) \right) \]
as desired.

Proof of Corollaries 1.5 and 1.6 If we put \( \Phi_q(\sigma, \tau) = \Phi(\sigma, \tau)(1 + E_q(\sigma, \tau)) \), then Theorem 1.4 (i) deduces the bound
\[ E_q(\sigma, \tau) \ll \frac{1}{\log q} + \frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{(\log q)^{\sigma}}. \]
for $1/2 < \sigma < 1$ in the range $1 \ll \tau \leq c(\sigma, 1)(\log q)^{1-\sigma}(\log_2 q)^{-1}$. We further apply Corollary 1.2 to obtain
\[
\log \Phi_q(\sigma, \tau) = \log \Phi(\sigma, \tau) + O \left( |E_q(\sigma, \tau)| \right)
\]
\[
= -A(\sigma) \tau^{1-\sigma} \left( \log \tau \right)^{-1} \sum_{n=0}^{N-1} A_{n, \sigma} \frac{\log_2 \tau}{\log \tau} + O \left( \left( \log \tau \log \tau \right)^N \frac{\log \tau}{\log \tau} \right)
\]

It completes the proof of Corollary 1.5 since we obtain
\[
\tau^{1-\sigma} \left( \log \tau \right)^{-1} |E_q(\sigma, \tau)| \ll \left( \frac{\log \tau}{\log \tau} \right)^N
\]
in the range $1 \ll \tau \leq c(\sigma, 1)(\log q)^{1-\sigma}(\log_2 q)^{-1}$ by (5.16). Corollary 1.6 can be proved similarly. \hfill \square

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Faculty of Science and Technology, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo 102-8554, Japan

Email address: m-mine@sophia.ac.jp