Smarr Mass formulas for BPS multicenter Black Holes

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Mass formulas for multicenter BPS 4D black holes are presented. For example, ADM mass for a two center BPS solution can be related to the intercenter distance \( r \), the angular momentum \( J^2 \), the dyonic charge vectors \( q_i \), and the value of the scalar moduli at infinity \( (\infty) \) by the relation

\[
M_{\text{ADM}}^2 = A \left( 1 + \alpha J^2 \left( 1 + \frac{2M_{\text{ADM}}}{r} + \frac{4}{r^2} \right) \right)
\]

where \( A(Q) \), \( \alpha(q_i) \) are symplectic invariant quantities \( Q \), the total charge vector) depending on the special geometry prepotential defining the theory. The formula predicts the existence of a continuos class, for fixed value of the charges, of BH’s with interdistances \( r \in (0, \infty) \) and \( M_{\text{ADM}} \in (\infty, M_{\infty}) \). Smarr-like expressions incorporating the intercenter distance are obtained from it:

\[
dM \equiv \Omega dJ + \Phi dq_i + F dr,
\]

in addition to an effective angular velocity \( \Omega \) and electromagnetic potentials \( \Phi \), the equation allows to define an effective “force”, \( F \), acting between the centers. This effective force is always negative: at infinity we recover the familiar Newton law \( F \sim 1/r^2 \) while at short distances \( F \sim f_0 + f_1/r^2 \).

Similar results can be easily obtained for more general models and number of centers.

1. Introduction. Extended theories of gravity as Supergravity, and in particular supersymmetric extremal black hole solutions, continues to be central for M-theory, string theory phenomenology, quantum properties of black holes, and the AdS/CFT correspondence. Applications can be found from extensions of the physical particle SM to supersymmetric black hole solutions and strongly coupled systems. Black hole physics, in general relativity or in extended theories as supergravity, is of interest in different backgrounds: from astrophysics to classical general relativity, quantum field theory, particle physics, string and supergravity. Nowadays black holes in Supergravity theories are used to answer to condensed matter questions in particular in strongly correlated fermionic systems including high T superconductors [1-7].

It is a rather trivial problem the existence and construction of extremal BH solutions in a wide number of well known theories. The two-parameters Reissner-Nordstrom (RN) metric, for example, describes black holes of (ADM) mass \( M \) and charge \( Q \) only when the ratio \( Q/M \) is sufficiently small. In the extremal case, the borderline between naked singularities and black hole solutions, the mass and electromagnetic charges \( P,Q \) are related by \( M^2 = P^2 + Q^2 \). This can be considered a (necessary and sufficient) condition on the macroscopic parameters for the existence of a extremal RN BH. Solutions saturating this bound can be considered as the stable final state of Hawking evaporation [23]. For the Majumdar-Papapetrou solution with \( H = 1 + \sum M_i/|x - x_i| \), the conditions are \( M_i > 0 \) with \( M_i^2 = q_i^2 \), \( M_{\text{ADM}} = \sum M_i \). For the three parameter Kerr-Newman extremal case, the mass, charge and angular momentum of the solution are related by the (quartic in the mass) extremal condition \( M^2 = Q^2 + J^2/M^2 \). From this relation we get the limits \( M^2 \sim Q^2 \) for \( J \rightarrow 0 \) and \( M^2 \sim |J| \) for \( J \rightarrow \infty \). In the supergravity dilaton model, a 4d low energy limit of string theory, the presence of the dilaton induces that the transition between black hole and naked singularities occurs at [10] \( M^2 = \frac{1}{2} Q^2 \exp 2\phi_\infty \), where \( \phi_\infty \) is the value of the additional scalar field at spatial infinity. The factor modifying this last formula from the RN relation is related to the existence of a dilaton scalar charge [10]. For the RN metric the extremal value corresponds to the case where gravitational and electromagnetic interactions are balanced. In any supergravity scenario, nearly unavoidably, the scalars contribute with an extra attractive long range force. In the Maxwell-Einstein-dilaton supergravity [8, 9] extremal one center case one obtains a quartic expression for the ADM mass in terms of \( Q, P \), electric and magnetic charges, and \( \phi_\infty \), the value of the dilaton at spatial infinity (the axion is put to zero in addition). In the extremal BPS case the relation reduces to quadratic expression \( M^2 = \frac{1}{2} (p e^{-\phi_\infty} + q e^{\phi_\infty})^2 \). Finally, mass relations are known for one center extremal black hole solutions: For 4d, \( N = 2 \) ungauged SUGRA theories the following constraint between the BH mass, the scalar charges, the scalar metric \( g_{ab} \) evaluated at spatial infinity and the BH potential is well known [11] \( M^2 + G_{ab} \Sigma^a \Sigma^b - V_{bh}(p, q, \phi_\infty) = 0 \). The quantities \( \phi_\infty \), \( \Sigma^a \) are defined by the expansion of the scalar fields at infinity \( \phi^a(r \rightarrow \infty) \sim \phi_{\infty}^a + \Sigma^a/r + \alpha(1/r^2) \). In summary, given a set of parameters \( M, Q, \phi_\infty, J, \Sigma^a \) (or a subset of them) satisfying simple relations as the previous ones, we can build explicitly the solutions from them. By construction such relations connecting ADM masses, charges and possibly other moduli can be considered local neccesary and sufficient conditions for the existence of one center extremal BH solutions.
Multicenter BH solutions are an important ingredient on, for example, counting the right numbers of degree of freedom in Entropy BH computations. However it is not a trivial problem in general the proof of the existence or not of BPS multicenter solutions of given charges, center positions and values of the moduli (scalar fields at spatial infinity). The most general stationary (time independent) 4-dimensional metric compatible with supersymmetry can be written in the IWP form [12-14],

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U}d\mathbf{x}^2. \tag{1}$$

This is in particular the metric of a 4d BPS solution of general $N = 2$ supergravity theories coupled to vectors and scalars. In terms of this metric, some necessary and sufficient, non local conditions for the existence of multicenter extremal solutions are [13]: 1) the fulfillment of certain integrability conditions for the 1-form $\omega$ which imposes restrictions on the allowed center positions; 2) the metric factor $e^{2U}$ is positive at any spacetime point; 3) the scalar field solutions of the theory, $z^a(x)$, must adopt physically consistent values at any spacetime point. In particular the conditions that the charge vectors at any center, $q_i$, define by themselves a single black hole solution (in particular positivity of the associated entropy) are necessary but not sufficient for the existence of a multicenter BPS solution corresponding to that set of vectors.

It would be desirable to have a fully local set of necessary and sufficient BPS existence conditions in terms of “macroscopical” parameters $M_{ADM}, q_i, \phi^a_\infty$, physical parameters appearing, or determining, directly in the field equations and their solutions but this is not known.

Supersymmetric $N = 2$ supergravity solutions, in particular multicenter stationary BPS solutions, can be constructed systematically following well established methods [16-18, 20-22]. The 1-form $\omega$ and the function $e^{-2U}$ are related in these theories to the Kähler potential and connection, $\mathcal{K}, Q$ [14]. We demand asymptotic flatness, $e^{-2U} \to 1$ together with $\omega \to 0$ for $|x| \to \infty$. BPS field equation solutions can be written in terms of real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$. For example $e^{-2U} = (\mathcal{R} | \mathcal{I})$. The $2n_v + 2$ components of $\mathcal{I}$ and $\mathcal{R}$ are $R^3$ harmonic functions. There is an algebraic relation between $\mathcal{R}$ and $\mathcal{I}$, the stabilization equation

$$\mathcal{R} = S\mathcal{I}, \tag{2}$$

where the $S$ matrix is given in terms of the second derivatives of the, assumed quadratic, prepotential defining the theory.

Similarly, the time independent 3-dimensional 1-form $\omega = \omega_i dx^i$ satisfies the equation $d\omega = 2 (\mathcal{I} | \Delta \mathcal{I})$ together with the integrability condition $(\mathcal{I} | \Delta \mathcal{I}) = 0$. As a consequence of this condition, the center interdistances are restricted, $(r_{ab} = |x_a - x_b|$, for any $q_i$

$$\langle \mathcal{I}_\infty | q_i \rangle + \sum_a \frac{\langle q_a | q_i \rangle}{r_{ab}} = 0. \tag{3}$$

The previous equations implies $N = \sum N_b = 0$ where $N_b = \langle \mathcal{I}_\infty | q_b \rangle$ are the “NUT charges”. The solutions for this set of equations give the possible values of the center positions. The asymptotic flatness condition implies $(\mathcal{R}_\infty | \mathcal{I}_\infty) = (S\mathcal{I}_\infty | \mathcal{I}_\infty) = 1$.

In practice, specific solutions are determined by giving a particular, suitable, ansatz for the symplectic vector $\mathcal{I}$. For multicenter BPS solutions we consider [16-18, 20] a symplectic real vector $\mathcal{I}$ of the form

$$\mathcal{I} = \mathcal{I}_\infty + \sum_i \frac{q_i}{x - x_i}. \tag{4}$$

Thus $N = 2$ solutions with $n_c$ centers are specified by $n_c + 1$ symplectic vector $\mathcal{I}_\infty, q_i$ quantities.

The $n_v$ complex moduli $z^a_\infty$ and with them all other macroscopic parameters, $M_{ADM}, \Sigma^a$ and possibly some or all of $r_{ab}$, are complicated, implicit functions of the $2n_v + 2$ real components of $\mathcal{I}_\infty$. As we show in this work, the standard, “non-physical”, $2n_v + 2$ real components of $\mathcal{I}_\infty$ can be exchanged by “contravariant” components which are directly physical quantities and which obeying the conditions of flat asymptoticity and integrability conditions above. Let us note that for a fixed configuration of $n_c$ centers, or charges, we have $n_c$ partial mass parameters $M_i$ and $n_c(n_c - 1)/2$ intercenter parameters $r_{ij}$. In a model with $n_v$ complex scalars, the symplectic space is of dimension $2n_v + 2$...

2. General Mass formulas. Any real symplectic vector $X$ of dim $2n_v + 2$ can be expanded in a basis of charge vectors (or linear combinations of them), and some additional $n_a$ (possibly zero) vectors, $s_a$ [20]. In particular we can write, in terms of eigenvectors of $S^3$, $X = \alpha^k P^+ w_k + \alpha^k P^- w_k \tag{5}$

Associated to the basis we define the “metric” $g$ with components $g_{kk} \equiv \langle P^+_k w_k | P^- w_k \rangle$. The submatrix $g_{ab} \equiv \langle P^+_a s_a | P^- w_b \rangle$ can be chosen diagonal but it is

\[1\] We refer to [18, 20] for further details and notation fixing.

\[2\] The matrix $S(z)$ is nothing else the matrix $M(F)$ appearing in $\mathcal{M}$. The bilinear form $(S q_i | q_j)$ is not the usual defined $I_1$ invariant, but coincides with it at the attractors points (infinity and charge centers). See [20].

\[3\] $w_i = (q_{iuc}, s_a)$, the $s_a$ are chosen such that $(s_a | P^+_i q_j) = 0$ for all the $q_i, s_a$. Projectors onto the $+i$ eigenvalues of $S$ are $P^+_2 = (1 + S)/2$. $P^2_\pm = P^\pm$.


in general indefinite. The “contravariant” components of \( X \) with respect the metric \( g \), are given by
\[
a_j = (X | P_+ w_j) = g_{ij} \alpha^i,
\]
Eq. 6 is complemented by additional consistency equations from the scalar fields. The values of the scalar fields at infinity are related to \( \mathcal{I}_\infty \) by the expression
\[
z_\alpha^\infty = Z_\infty^\alpha / z_\infty^0 \text{ where } (i,j) = (1, n_c).
\]
This is an implicit equation including the quantities \( z_\infty \) at both sides of the equation (as \( P_- = P_- (z_\infty) \)). Equation 11 allows, if needed, to express the “unphysical” quantities \( \lambda_i \) in terms of the scalar fields at infinity, masses, charges and intercenter distances. In the case that the \( \lambda_i \) does not appear, the equations 11 (one complex equation for each of the scalar fields) imposes additional constraints.

In the case the number of centers is large enough, one can select a subset of charge vectors to form a basis where to expand \( \mathcal{I}_\infty \). This procedure can be repeated for different subsets generating different equations similar to Eq. 11 which should be simultaneously satisfied.

### 3. Two center mass relations.
For the sake of concreteness, let us consider now a particular, non-trivial case: a model with one scalar \( (n = 2) \) and two centers (with charge vectors \( q_{1,2} \)). The dimensionality of the symplectic space is \( 2n = 4 \), and the degrees of freedom \( dof = 2 \). The hermitian matrix \( 2(\langle q_j | q_i \rangle) \) is of signature \((1, 1)\), therefore with negative determinant: \( \det(2(\langle q_j | q_i \rangle)) = \det(S) - J^2 < 0 \). A, S are the real, imaginary parts of \( q_{ij} \), in particular \( S_{ij} = \langle S_{ji} | q_i \rangle \). We have introduced the (signed) module, J, of the angular momentum.

The \( \omega \)-form compatibility equations or absence of NUT charge condition for this case read:
\[
-N_2 = N_1 = \frac{\langle q_1 | q_2 \rangle}{r} \equiv \frac{J}{r^2}.
\]
Eq. 13 becomes for this two center case
\[
M^2 = M_0^2 \left( 1 + \frac{J^2}{-\det(S)} \left( 1 + \frac{2M}{r^2} + \frac{A^2}{r^2} \right) \right).
\]
The solution to this quadratic equation for \( M \) has a real and positive solution, assuming \( A > 0 \), only if \( \det(S) < 0 \). In this case, there exists a unique, solution for any intercenter distance \( r \) and \( J \). The quotient \( J^2 / (-\det(S)) \) is always positive that implies that \( M^2 \geq 0 \) if \( M_0^2 > 0 \). The quantities \( \det(S), A, M_0^2 \) depends on the charges and the.

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\(^4\) \( X^{ij} \equiv \text{Re} \left( \langle g^{ij} \rangle Y^{ij} \right) \text{Im} \left( \langle g^{ij} \rangle \right) \text{ for } i, j = 1, n_c.
\(^5\) The relation \( \overline{J} \) is quadratic in \( M \). The reality and positivity of \( M \) imposes an upper bound on the contribution of the extra vectors \( s_i \lambda_i \lambda_i^\alpha \leq 1 + (b^2 - 4ac)/4a \). Eq. 8 is also quadratic on the angular momentum components \( J_{ik} \equiv \langle q_i | q_k \rangle \) or its module \( J \). It can be written as \( 1 = aM^2 + bJ^2 + cJ^2 + \Delta^2 \). Similarly, reality of \( J \) imposes conditions on the coefficients of the quadratic expression.

\(^6\) We define \( J \) as
\[
J = \langle q_1 | q_2 \rangle \frac{x_1 - x_2}{|x_1 - x_2|}.
\]
moduli at infinity, in addition $M_0^2$ depends on the relative masses of the centers. In the limit of large intercenter distance, $r \to \infty$, the equation \cite{14} becomes

$$M_\infty^2 = M_0^2 \left(1 + \frac{J^2}{|\det(S)|}\right)$$  \hspace{1cm} (15)$$

where $M_\infty$, if real, is the total mass that would have such configuration of charges with an infinity intercenter distance. At large distances $M \sim M_\infty + J^2/(|\det(S)|r)$ while at short distances the ADM mass behaviour is of the type $M \sim A/r + Br$.

**Smarr like relations.** Let us use the mass differential $dM$ to define different quantities à la Smarr

$$dM \equiv \Omega dJ + \Phi idq_i + Fdr,$$

in addition to an effective angular velocity $\Omega$ and electromagnetic potentials $\Phi_i$, the equation allows to define an effective “force”, $F$, acting between the centers. This force is always negative due to the sign of $\det(S)$ enforced by Eq. (14). At infinity $dM/dr \sim -J^2M_0^2/(|\det(S)|r^2) \to 0$ while at short distances $dM/dr \sim f_0 + f_1/r^2$. The positivity of $1/M_0^2 = (S^{-1})_{ij}m_im_j$ implies restrictions on the allowed values of the relative parameters $m_i$. The allowed $m_i$ are in an interval $m_{i,min} \leq \sqrt{|\det(S)|}/A$. The relative masses that minimizes $M_\infty^2$ for fixed $J$ and $S$ are given in this case exactly by the expression

$$m_{i,min} = \frac{\langle S_0_i | Q \rangle}{\langle SQ | Q \rangle},$$  \hspace{1cm} (16)$$

and the value of the total mass at minimum is given by

$$(M_\infty^2)_{min} = A.$$  \hspace{1cm} (17)$$

For this mass configuration we finally get ($\alpha = 1/|\det(S)|$)

$$M^2 = A\left(1 + \alpha J^2\left(1 + \frac{2M}{r} + \frac{A}{r^2}\right)\right).$$  \hspace{1cm} (18)$$

Then $(M_\infty^2)_{min} = A\det(2i\mu S^{-1})$. At large distances $M \sim M_{\infty,min} + J^2A/(|\det(S)|r)$ and $dM/dr \sim -J^2A/(|\det(S)|r^2) \to 0$.

As in the general case, these mass relations have to be complemented by the implicit equations for the moduli at infinity.

In conclusion we have derived new mass formulas for multicolor BPS 4D black holes. They are obtained in the context of $N = 2, d = 4$ supergravity coupled to any number of vectors and with any number of BH centers.

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Extremal mass relations (14) or (18) are valid for two center BHs in any N2 SUGRA. In Fig. (1) we present some explicit results for a toy model with a complex scalar field \( n = 2 \) theory with prepotential \( F = -iX_0X_1 \) (see further details in [18, 20]). For this quadratic prepotential the matrix \( S \) is scalar independent. The only scalar of the theory is \( \chi + i e^{-\phi} \equiv -iz \). The Kahler potential and scalar metric are \( K = -\log \Re (z) \), \( G_{zz} = (2\Re (z))^{-2} \). Let us first take a configuration with the charge sympletic vectors \( q_1 = (1, 8, 0, -1)q_0 \) and \( q_2 = (1, 8, -4, 1)q_0 \). In this case \( A > 0 \), \( -\det(S) > 0 \). The minimal ADM mass configuration corresponds to relative masses \( m_i = M_i/M_{ADM} = (9/16, 7/16) \). The limiting mass is \( M_{2\infty}^2 = \langle SQ \mid Q \rangle \left( 1 - J^2/\det(S) \right) = 8\sqrt{26/17} q_0 \). The initial free parameters for this fixed configuration of charges are \( M_1, M_2, r, z_\infty \) which have to satisfy three real equations (the mass relation, (14) and a complex scalar equation of the type (11)). If in addition we choose a minimal ADM mass configuration it remains only one free parameter (see figure). Let us take a second exemplary configuration with \( q_1 = (1, 8, 0, -1)q_0 \) and \( q_2 = (1, 8, -8, 0)q_0 \). In this case \( A = 80q_0^2 \), \(-\det(S) = 320q_0^4\), \( J^2 = 0 \). Now the intercenter distance is unrestricted while the scalar at infinity is fixed \((z_\infty = 8)\).