Fractional quantum Hall effect in Hofstadter butterflies of Dirac fermions

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Abstract

We report on the influence of a periodic potential on the fractional quantum Hall effect (FQHE) states in monolayer graphene. We have shown that for two values of the magnetic flux per unit cell (one-half and one-third flux quantum) an increase of the periodic potential strength results in a closure of the FQHE gap and appearance of gaps due to the periodic potential. In the case of one-half flux quantum this causes a change of the ground state and consequently the change of the momentum of the system in the ground state. While there is also crossing between low-lying energy levels for one-third flux quantum, the ground state does not change with the increase of the periodic potential strength and is always characterized by the same momentum. Finally, it is shown that for one-half flux quantum the emergent gaps are due entirely to the electron–electron interaction, whereas for the one-third flux quantum per unit cell these are due to both non-interacting electrons (Hofstadter butterfly pattern) and the electron–electron interaction.

Keywords: Hofstadter butterflies, Dirac fermions, fractional quantum Hall effect

(Some figures may appear in colour only in the online journal)
it can be useful for interaction effects in fractal butterflies in other systems as well [1, 24]. Specifically, we choose the external periodic potential to be of the form

$$V(x, y) = V_0[\cos(q_x x) + \cos(q_y y)],$$  \hspace{1cm} (1)

where $V_0$ is the amplitude of the periodic potential and $q_x = q_y = q_0 = 2\pi / a_0$, where $a_0$ is the period of the external potential. Then the many-body Hamiltonian is

$$\mathcal{H} = \sum_i [\mathcal{H}_B + V(x_i, y_i)] + \frac{1}{2} \sum_{i \neq j} V_{ij},$$  \hspace{1cm} (2)

where $\mathcal{H}_B$ is the Hamiltonian of an electron in graphene in a perpendicular magnetic field and the last term is the Coulomb interaction. The electron energy spectrum of graphene has twofold valley and twofold spin degeneracy in the absence of an external magnetic field, the periodic potential and the interaction between the electrons. We disregard the lifting of the valley degeneracy due to the Coulomb interaction and the periodic potential. We consider here the fully spin polarized electron system and focus our attention on the valley $K$. The single-particle Hamiltonian $\mathcal{H}_B$ is then written as [6, 7, 16]

$$\mathcal{H}_B = v_F \left( \begin{array}{ccc} 0 & \pi_+ & 0 \\ \pi_- & 0 & 0 \end{array} \right),$$  \hspace{1cm} (3)

where $\pi_{\pm} = \pi_x \pm i\pi_y$, $\pi_x = p + eA/c$, $p$ is the two-dimensional electron momentum, $A = (0, Bx, 0)$ is the vector potential and $v_F \approx 10^6$ ms$^{-1}$ is the Fermi velocity in graphene [6, 7].

We consider a system of finite number $N_s$ of electrons in a toroidal geometry, i.e. the size of the system is $L_x = M_\alpha a_0$ and $L_y = M_\beta a_0$ ($M_\alpha$ and $M_\beta$ are integers) and apply the periodic boundary conditions (PBC) in order to eliminate the boundary effects. Defining the parameter $\alpha = \phi_0 / \phi$, where $\phi = B\alpha_0^2$ is the magnetic flux through the unit cell of the periodic potential and $\phi_0 = hc/e$ the flux quantum, we have

$$\frac{N_s}{M_\alpha M_\beta} = \frac{1}{\alpha} = \frac{r}{v},$$  \hspace{1cm} (4)

where $N_s$ is the number of magnetic flux quanta passing through the system and $r$ and $v$ are coprime integers. The filling factor is defined as $v = p/q = N_s / N_s$, where $p$ and $q$ are again coprime integers. For a many-body system only the set of $\{L_{mn} / N_s\}$ of the center-of-mass (CM) translations acts within the same Hilbert space [18, 22]. Here $L_{mn} = mL_x \hat{x} + nL_y \hat{y}$ is a magnetic translation lattice vector and $(L_x, L_y)$ defines the magnetic translation unit cell [21]. Without the PBC, the Hamiltonian (2) has a symmetry of a magnetic translation of the CM by any periodic potential lattice vector. In order to have this symmetry in the thermodynamic limit, the magnetic translation of the CM by the magnetic translation lattice vector should be compatible with the translation by the periodic potential lattice vector [9]. This compatibility results in additional constraints on our system, that $M_\alpha$ and $M_\beta$ are divisible by $v$. These constraints and (4) dictate that $N_s = \kappa_{x,y} M_{x,y}$, where $\kappa_{x,y}$ are integers. Therefore from the set of CM translation $\{L_{mn} / N_s\}$ only those which are also translations by the periodic potential lattice vector will both preserve the Hilbert space and commute with the Hamiltonian (2).

We are seeking the set of appropriate magnetic translations that characterizes the states of the Hamiltonian (2) by its momentum eigenvalues. Based on the considerations above we search for appropriate translations as the CM translations with the translation vector $\mathbf{a}_j = m\beta_1 a_0 \hat{x} + n\beta_2 a_0 \hat{y}$, where $\beta_1$ and $\beta_2$ are integers determined below. In order for these CM translations to be diagonalized simultaneously the following condition must be satisfied $\frac{N_s}{\beta_1 \beta_2} = \pm 1, \pm 2, \ldots$. By choosing for example $\beta_2 = 1$ and demanding the above condition for $\beta_1$, it can be shown that this condition is the same as the one obtained earlier by Kol and Read [9]. Hence $\beta_1 \beta_2$ describes the degeneracy of the system for each value of the CM momentum. We now make the assumption that the application of the normal momentum operator $\mathcal{Q}(\mathbf{Q}) = \sum_i e^{iQ \cdot \mathbf{x}}$ to the many-particle state will increase its momentum by $\mathbf{Q}$ provided that $\mathbf{Q}$ is a magnetic translation reciprocal lattice vector. From the relation

$$T^{CM}(\mathbf{a}_p) \mathcal{Q}(\mathbf{Q}_{st}) = e^{i\mathbf{a}_p \cdot \mathcal{Q}(\mathbf{Q}_{st})} T^{CM}(\mathbf{a}_p),$$  \hspace{1cm} (5)

it follows that the eigenvalues of the CM translation operator will have the form $e^{2\pi i j (M_\alpha / \beta_1) + M_\beta / \beta_2}$, where $s$ and $t$ are integers, which characterize the vector $\mathbf{Q}_{st}$ in a magnetic translation reciprocal lattice. Hence, $s$ and $t$ are defined only modulo $M_\alpha / \beta_1$ and $M_\beta / \beta_2$, respectively, and there are $M_\alpha / \beta_1 \cdot M_\beta / \beta_2$ allowed eigenvalues. It is clear from the discussions above that $s$ and $t$ are related to the CM momentum of the system and also in special cases of the system size, to the relative momentum.

We consider the many-body states $|j_1, j_2, \ldots, j_{N_s}\rangle$ as basis states constructed from the single-particle eigenvectors of the Hamiltonian (3) [6, 7, 16]

$$\Psi_{n, j} = C_n \left( \begin{array}{c} \text{sgn}(n)(-1)^j \varphi_{n\alpha,-1,j} \\ \varphi_{n\beta,j} \end{array} \right),$$  \hspace{1cm} (6)

where $C_n = 1$ for $n = 0$ and $C_n = 1 / \sqrt{2}$ for $n \neq 0$, $\text{sgn}(n) = 1$ for $n > 0$, $\text{sgn}(n) = 0$ for $n = 0$, and $\text{sgn}(n) = -1$ for $n < 0$. Here $\varphi_{n,j}$ is the electron wave function in the $n$-th Landau level (LL) with the parabolic dispersion taking into account the PBC [18, 25].

$$\varphi_{n,j} = \frac{1}{\sqrt{L_x \pi^{1/2} L_\ell 2^{(\kappa_j + 1)/2}}} \sum_{k=-\infty}^{\infty} e^{i \frac{k}{2L} (x + kL_x)_y} \left( x + kL_x \right) \left( x + kL_x + X_j \right) \left( x + kL_x + X_j + \ell_0 \right),$$

where $X_j = 2\pi j / L_x$, $\ell_0 = \sqrt{\hbar^2 / eB}$ is the magnetic length, and $H_n(x)$ are the Hermite polynomials. The quantum number $j$ characterizes the single-particle momentum in the $y$ direction $k_y = 2\pi j / L_y$, as well as the guiding center position in the $x$ direction. Due to the system boundaries it takes integer values in the range $0 \leq j < N_s$. The eigenvalues of the Hamiltonian (3) corresponding to the eigenvectors (6) are

3 The periodic rectangular geometry was extensively used earlier in the study of the FQHE in various situations. For example, see [25].
\(\epsilon_n = \text{sgn}(n) n \hbar \omega_B \sqrt{|n|},\) where \(\omega_B = \sqrt{2} v_F / \ell_0.\) The many-body state \(|j_1, j_2, \ldots, j_N\rangle\) is characterized by the LL index \(n\) and the spin of the particles. The factorization rule for CM translations
\[
T^{\text{CM}}(a_p) = (-1)^N \beta_1 \beta_{mn}/a T^{\text{CM}}(\beta_1 m a_0 \hat{x}) T^{\text{CM}}(\beta_2 n a_0 \hat{y}),
\]
leads to the relations
\[
T^{\text{CM}}(\beta_1 m a_0 \hat{x}) |j_1, j_2, \ldots, j_N\rangle = e^{i 2\pi \beta_1 m a_0 j_y} |j_1, j_2, \ldots, j_N\rangle,
\]
\[
T^{\text{CM}}(\beta_2 n a_0 \hat{y}) |j_1, j_2, \ldots, j_N\rangle = |j_1 + m \beta_1 k_x, j_2 + m \beta_1 k_x, \ldots, j_N + m \beta_1 k_x\rangle,
\]
where \(t = \sum_{x} j_i \text{mod} (M_x / \beta_2)\) is the total momentum quantum number in the \(y\) direction. Hence, following the procedure outlined in [18], we fix the total momentum \(T\) and construct the set \(T\) of all the \(N_e\) particle states with the momentum \(t\), i.e. \(T = \{ |j_1, j_2, \ldots, j_N\rangle | \sum_i j_i = t \text{ mod } (M_x / \beta_2)\} \). We then divide the set \(T\) into equivalence classes by defining the states \(|j_1, j_2, \ldots, j_N\rangle\) and \(|j_1, j_2, \ldots, j_N\rangle\) equivalent if and only if they are related by the rule
\[
|j_1 + m \beta_1 k_x, j_2 + m \beta_1 k_x, \ldots, j_N + m \beta_1 k_x\rangle
\]
forms the set of the eigenstates of \(T^{\text{CM}}(a_p)\) and is used as a basis for exact diagonalization of the Hamiltonian (2) with fixed quantum numbers \(s\) and \(t\). Hence, the magnetic translation analysis reduces the size of the Hamiltonian matrix roughly by a factor of \(M_x M_y / \beta_1 \beta_2\).

In what follows we consider the system with filling factor \(v = 1/3\). We also consider two cases \(\alpha = 1/2\) and \(\alpha = 1/3\). We then choose the system size based on the condition (4) and the number of electrons. For \(N_e = 4\) the system size is \(M_x = 3\) and \(M_y = 2\) for \(\alpha = 1/2\), and \(M_x = 2\) and \(M_y = 2\) for \(\alpha = 1/3\). For \(N_e = 6\) the system size is \(M_x = 3\) and \(M_y = 2\) for \(\alpha = 1/2\), and \(M_x = 3\) and \(M_y = 2\) for \(\alpha = 1/3\). We evaluate the FQHE gap for two different cases when the \(n = 0\) LL is filled or \(n = 1\) LL is filled and we disregard the interaction between the LLs. The period of the external potential is taken to be \(a_0 = 20\,\text{nm}\) throughout.

In figure 1 the dependence of low-lying energy levels on the amplitude of the periodic potential \(V_0\) is presented for \(N_e = 4\) and \(n = 0\) LL. Both cases of \(\alpha = 1/2\) and \(\alpha = 1/3\) are shown. Here the levels which in the absence of the periodic potential correspond to the ground state and become triply degenerate as \(V_0 \to 0\) are depicted in green, while the level which first crosses those ground states is depicted in red. The same dependence for the case of \(n = 1\) LL is shown in figure 2. For \(n = 0\) and \(n = 1\) LL and for \(V_0 = 0\), the FQHE gap is about 3.36 meV and 4 meV, respectively, for \(\alpha = 1/2\), and 3.69 meV and 4.51 meV, respectively, for \(\alpha = 1/3\). The difference between the gaps for \(\alpha = 1/2\) and \(\alpha = 1/3\) comes from the fact that by fixing \(\alpha\) and \(a_0\) we fix the magnetic field strength \(B\) and hence, these two cases correspond to different values of \(B\). The degeneracy of each level characterized by the CM momentum is \(\beta_1 = \beta_2 = 1\). In spite of that we notice in figures 1(a) and 2(a) that for \(\alpha = 1/2\) the ground state splits into two levels when the periodic potential is present. It should be noted that just as for \(V_0 = 0\), the spectrum as a function of the CM momentum has a full point symmetry of the PBC Bravais lattice. So although these three levels are characterized by different CM momentum, two of those are degenerate due to the PBC rectangular Bravais lattice. This degeneracy is not present for the cases where both \(\beta_1 N_e / M_x\) and \(\beta_2 N_e / M_y\) are integers because, as will be shown below, in these cases the relative momentum is a conserved quantity and the states can be characterized by both CM and relative momentum. When
the relative momentum is a conserved quantity all three ground states correspond to both relative and CM momentum equal to zero and hence cannot be degenerate when the periodic potential is present.

In the single-electron case the inclusion of a periodic potential splits the LL into \( r \) subbands of equal weight [26], if \( \alpha = v/r \). In that case, for \( \alpha = 1/2 \) there is no bandgap between the two subbands. Hence, the appearance of gaps for the ground and excited states for \( \alpha = 1/2 \) is a direct consequence of the Coulomb interaction [8]. The most striking feature in figures 1(a) and 2(a) for \( \alpha = 1/2 \) is the crossing of the excited level with both ground states and change of the ground state at \( V_0 \approx 7 \) meV for \( n = 0 \) LL and at \( V_0 \approx 40 \) meV for \( n = 1 \) LL. The difference in the value of \( V_0 \) where the ground state changes between the \( n = 0 \) and \( n = 1 \) LL is a direct consequence of the robustness of the FQHE state for \( n = 1 \) LL compared to that of \( n = 0 \) LL, which can be clearly seen also by the magnitude of the gaps for both cases above and was found earlier in graphene [11]. These crossings and the change of the ground state result in the crossing of the levels with different CM momentum and also with different relative momentum where the relative momentum is a conserved quantity (see below). In figures 1(b) and 2(b) for \( \alpha = 1/3 \) we observe similar crossing between the levels and closing of the FQHE gap, although there is no ground state change in this case. This is related to the fact that we consider the system with filling factor \( \nu = 1/3 \). For \( \alpha = 1/3 \) this corresponds to the ground state of the system, which will be separated from the excited states by the inclusion of the periodic potential even for non-interacting electrons due to the Hofstadter gap. Even though the inclusion of interaction adds additional gaps to the energy spectra, as can be seen for the excited states in figures 1(b) and 2(b), the Hofstadter gaps are considerably larger and an increase of \( V_0 \) will not result in a change of the ground state. When \( \alpha = 1/2 \) the filling factor \( \nu = 1/3 \) corresponds to the point inside the subband and as was shown earlier [8], although for \( \alpha = 1/2 \) there are no gaps for non-interacting electrons, interaction opens the gaps and the highest gap is observed for \( \nu = 1/2 \), which corresponds to the crossing points of two subbands in the non-interacting case. Hence, this point results in the change of the ground state for \( \alpha = 1/2 \), closure of the FQHE gap and, afterwards, reappearance of the gap due to the periodic potential.

In figure 3 the dependence of low-lying energy levels on \( V_0 \) is shown for \( N_e = 6 \) and \( n = 0 \) LL. Both cases of \( \alpha = 1/2 \) and \( \alpha = 1/3 \) are presented. Similar dependence for \( n = 1 \) LL is shown in figure 4. The FQHE gap for \( n = 0 \) and \( n = 1 \) LL and \( V_0 = 0 \) is \( \sim 4.02 \) meV and \( \sim 4.85 \) meV, respectively, for \( \alpha = 1/2 \), \( \sim 4.54 \) meV and \( \sim 5.52 \) meV for \( \alpha = 1/3 \), respectively. Similar to the case of \( N_e = 4 \) and \( \alpha = 1/2 \), we observe the change of the ground state at \( V_0 \approx 16.5 \) meV for \( n = 0 \) LL and \( V_0 \approx 91 \) meV for \( n = 1 \) LL (not shown in figure 4(a)). For \( \alpha = 1/3 \) we again observe
considerably with the number of electrons, the value of \( \beta \), and hence with the Hamiltonian (2), unless the difference between the ground state, but do not observe any ground state change. The only difference between the ground state and the excited state, but do not observe any ground state change. The only difference between these two \( \alpha \) is a result of the origin of the gaps for the energy levels. For \( \alpha = 1/2 \) the emergent gaps are due to the electron–electron interaction only, whereas for \( \alpha = 1/3 \) these are both due to the non-interacting Hofstadter butterfly pattern and the electron–electron interaction.

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Figure 4. Same as in figure 2 but for \( N_e = 6 \).

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a crossing between the highest ground state and the excited state, but do not observe any ground state change. The only difference between \( N_e = 6 \) and \( N_e = 4 \) is the observation of complete lifting of the degeneracy of ground state with the inclusion of the periodic potential, and is related to the fact that for \( N_e = 6 \) the relative momentum is a conserved quantity for all cases considered. It should be noted that although the value of \( V_0 \) at which the ground state changes for \( \alpha = 1/2 \) varies considerably with the number of electrons, the value of \( V_0 \) at which the closure of the FQHE gap appears is almost the same for both systems.

Just as for \( V_0 = 0 \), we can define the relative magnetic translations \( T^{R}(N,a_p) = T^{R}(N,a_p)T^{CM}(a_p) \), which generally do not commute with the Coulomb interaction term \( V_{ij} \) and hence with the Hamiltonian (2), unless \( \beta_1 N_e/M_e \) and \( \beta_2 N_e/M_e \) are integers and, in that case, the vector \( N,a_p \) is a magnetic translation lattice vector. These conditions are satisfied for all cases considered here, except for \( N_e = 4 \) and \( \alpha = 1/2 \). When these conditions are satisfied the absolute values of the relative momentum and the CM momentum eigenvalues are equal and the state can be characterized both by the CM and the relative momentum eigenstates. As is well known [18, 22], without the periodic potential the triply degenerate ground state is characterized by zero relative momentum. Hence, for the cases when the above conditions are satisfied and the relative momentum is a conserved quantity, we can state that the three ground states (depicted in green in all figures) correspond to both the relative and the CM momentum equal to zero for all \( V_0 \) and the crossing observed in the figures for the ground states result in the change of the value of relative momentum eigenstate of the ground state.

In conclusion, we have performed magnetic translation analysis to study the effect of a periodic potential on the FQHE in graphene for filling factor \( \nu = 1/3 \). For \( \alpha = 1/2 \) and \( \alpha = 1/3 \), increasing the periodic potential strength \( V_0 \) results in a closure of the FQHE gap and the appearance of gaps due to the periodic potential. We also find that for \( \alpha = 1/2 \) this results in a change of the ground state and consequently in the change of the ground state momentum. For \( \alpha = 1/3 \), despite the observation of the crossing between the low-lying energy levels, the ground state does not change with an increase of \( V_0 \) and is always characterized by zero momentum. The difference between these two \( \alpha \) is a result of the origin of the gaps for the energy levels. For \( \alpha = 1/2 \) the emergent gaps are due to the electron–electron interaction only, whereas for \( \alpha = 1/3 \) these are both due to the non-interacting Hofstadter butterfly pattern and the electron–electron interaction.
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