SCHWARZ LEMMA FOR HARMONIC MAPPINGS IN THE UNIT BALL

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ABSTRACT. We prove the following generalization of Schwarz lemma for harmonic mappings. If \( u \) is a harmonic mapping of the unit ball \( B_n \) onto itself such that \( u(0) = 0 \) and \( \|u\|_p := \left( \int_{S_n} |u(\eta)|^p d\sigma(\eta) \right)^{1/p} < \infty \), \( p \geq 1 \) then \( |u(x)| \leq g_p(|x|) \|u\|_p \) for some smooth sharp function \( g_p \) vanishing in 0. Moreover we provide sharp constant \( C_p \) in the inequality \( \|Du(0)\| \leq C_p \|u\|_p \). Those two results extend some known result from harmonic mapping theory ([1, Chapter VI]).

1. Introduction

On the paper \( \mathbb{R}^m \) is the standard Euclidean space with the norm \( |x| = \sqrt{\sum x_i^2} \). Let \( p \geq 1 \) and assume that \( H^p \) is the Hardy space of the holomorphic mappings on the unit ball \( B_n \subset \mathbb{C}^n \cong \mathbb{R}^{2n} \). In their classical paper [7], Macintyre and Rogosinski proved the following result: Let \( f \) be holomorphic on the unit disk such that \( f(0) = 0 \) and such that \( \|f\|_{H^p} < \infty \) for \( p \geq 1 \), then

\[
|f(z)| \leq \frac{|z|}{(1 - |z|^2)^{1/p}} \|f\|_{H^p}
\]

with extremal functions \( f(w) = \frac{A_w}{(1 - \overline{z}w)^{1/p}} \). This is a generalization of Schwarz lemma (for \( p = \infty \) it coincides with the classical Schwarz lemma).

Then for holomorphic mappings on the unit ball \( B_n \subset \mathbb{C}^n \) we have the following result of Zhu [8, Theorem 4.17]:

\[
|f(z)| \leq \frac{1}{(1 - |z|^2)^{n/p}} \|f\|_p.
\]

Let us sketch the proof of (1.2), which imply (1.1), for \( n = 1 \). By definition

\[
\|f\|_p^p := \|f\|_{H^p}^p = \int_{S} |f(\eta)|^p d\sigma(\eta).
\]

Here as \( d\sigma \) is the normalized rotationally invariant Borel measure on the unit sphere \( S = S_n = \partial B_n \). Choose the holomorphic change \( \eta(w) = \varphi_z(w) \)
in (1.3), where $\varphi_z$ is an automorphism of the unit ball such that $\varphi(0) = z$. Now by using holomorphic mapping

\[(1.4) \quad F_r(w) = f_r(\varphi_z(w)) \left( \frac{1 - |z|^2}{(1 - \langle z, w \rangle)^2} \right)^{n/p}, \]

and by making use of the mean value inequality we obtain (1.2). In order to derive (1.1) with $n = 1$, from (1.2), just alike for the classical proof of Schwarz lemma we make use of holomorphic mapping $g(z) = f(z)/z$, whose $H^p$ norm coincides with the $H^p$ norm of $f$. However an analogous inequality for higher-dimensional case ($n > 1$) cannot be proved in the same way. In this paper we will attack this problem for the class of harmonic mappings, which contain holomorphic mappings.

Assume now that that $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$ and consider the Hardy class $H^p$ of harmonic mappings defined in the unit ball, i.e. of harmonic mappings $f : B^n \to \mathbb{R}^m$ with

\[\|f\|_p := \sup_r \left( \int_{S} |f(r\eta)|^p d\sigma(\eta) \right)^{1/p} < \infty.\]

Here as before $d\sigma$ is the normalized rotationally invariant Borel measure on the unit sphere $S = S^{n-1}$.

It is well known that a harmonic function (and a mapping) $u \in H^p(B)$, $p > 1$, where $B = B^n$ is the unit ball with the boundary $S = S^{n-1}$, has the following integral representation

\[(1.5) \quad u(x) = \mathcal{P}[f](x) = \int_{S^{n-1}} P(x, \zeta) f(\zeta) d\sigma(\zeta),\]

where

\[P(x, \zeta) = \frac{1 - |\zeta|^2}{|x - \zeta|^{n+2}}, \quad \zeta \in S^{n-1}\]

is Poisson kernel and $\sigma$ is the unique normalized rotation invariant Borel measure on $S^{n-1}$ and $| \cdot |$ is the Euclidean norm.

Let us formulate the classical Schwarz lemma for harmonic mappings on the unit ball $B^n \subset \mathbb{R}^n$ and assume its image is $\mathbb{R}^m$. Let $f$ be harmonic on the unit ball, and assume that $\|f\|_{\infty} < \infty$ and that $f(0) = 0$, then we have the following sharp inequality

\[|f(x)| \leq U(rN)\|f\|_{\infty}.\]

Here $r = |x|$, $N = (0, \ldots, 0, 1)$ and $U$ is a harmonic function of the unit ball into $[-1, 1]$ defined by

\[(1.6) \quad U(x) = \mathcal{P}[\chi_{S^+} - \chi_{S^-}](x),\]

where $\chi$ is the indicator function and $S^+ = \{ x \in S : x_n \geq 0 \}$, $S^- = \{ x \in S : x_n \leq 0 \}$.

Assume now that $p < \infty$. We are going to find a sharp function $g(r)$ satisfying the condition $g(0) = 0$ in the inequality

\[|f(x)| \leq g_p(r)\|f\|_p, \quad f \in H^p, \quad f(0) = 0.\]
2. The main result

We prove the following theorem

**Theorem 2.1.** Let \( p \geq 1 \), and let \( q \) be its conjugate and define

\[
g_p(r) = \inf_{a \in [0, \infty)} \left( \int_S |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q}.
\]

Then for \( 1 < p < \infty \), \( g_p : [0, 1) \rightarrow [0, \infty) \) is a smooth increasing diffeomorphism with \( g_p(0) = 0 \), and for every \( f \in H^p \) with \( f(0) = 0 \), we have

\[
|f(x)| \leq g_p(|x|) \| f \|_p
\]

and

\[
\| Df(0) \| \leq \left( \frac{\Gamma\left[ \frac{n}{2} \right] \Gamma\left[ \frac{1+q}{2} \right]}{\sqrt{\pi} \Gamma\left[ \frac{n+q}{2} \right]} \right)^{\frac{1}{q}} \| f \|_p.
\]

Both inequalities (2.1) and (2.2) are sharp. For \( p = \infty \), we have \( g_\infty(r) = U(rN) \) which coincides with Schwarz lemma and \( g_\infty \) is an increasing diffeomorphism of \([0, 1]\) onto itself. Here \( Df(0) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is the formal derivative and \( \| Df(0) \| = \sup_{|h|=1} |Df(0)h| \).

**Remark 2.2.** It seems unlikely that we can explicitly express the function \( g_p(r) \) for general \( p \), however we demonstrate some special cases \( p = 1, 2, \infty \) in Section 3 where among the other fact we prove the last part of this theorem. For some sharp pointwise estimates for the first derivative of harmonic mapping we refer to papers [2, 3, 6, 5]. Some optimal estimates of the harmonic function defined in the unit ball has been obtained in [4], but no normalization \( f(0) = 0 \) is imposed, so the obtained inequalities in [4] are not sharp in the context of this paper.

**Proof.** Let \( x \in B_n \) and \( \eta \in S^{n-1} \) with \( r = |x| \) and let

\[
P_x(\eta) = \frac{1 - r^2}{|x - \eta|^n}
\]

and define

\[
P_r(\eta) = \frac{1 - r^2}{(1 + r^2 - 2r\eta_\eta)^{n/2}}.
\]

Then

\[
f(x) = \int_S P_x(\eta)f(\eta)d\sigma(\eta),
\]

where \( S = S^{n-1} \) is the unit sphere. Now since \( f(0) = 0 \), it follows that

\[
\int_S f(\eta)d\sigma(\eta) = 0,
\]

and so

\[
f(x) = \int_S (P_x(\eta) - a)f(\eta)d\sigma(\eta),
\]
for a number $a = a(r)$ not depending on $\eta$.

Hence by using Hölder inequality, and unitary transformations of the unit sphere we have

$$|f(x)| \leq \left( \int_{S} |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q} \|f\|_p.$$  

So we are going to consider the minimum of the following function

$$\Phi_r(a) = \left( \int_{S} |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q}.$$  

Then $\Phi_r(a)$ is convex and it satisfies the conditions $\Phi'_r(0) < 0$ and $\Phi_r(\infty) = \infty$. This implies that there is a unique constant $a^* = a(r) \in (0, \infty)$ such that

$$\Phi_r(a^*) = \min_{a \in \mathbb{R}} \Phi_r(a).$$  

To show that $\Phi_r$ is convex, observe the following simple fact $\Phi'_r(a) = \|P_r - a\|_{L^q}$. So

$$\Phi_r(\lambda a + (1 - \lambda)b) \leq \|\lambda(P_r - a)\|_{L^q} + \|(1 - \lambda)(P_r - b)\|_{L^q} = \lambda \Phi_r(a) + (1 - \lambda) \Phi_r(b).$$  

In order to prove that $\Phi'_r(0) < 0$, by calculation we find out that

$$\Phi'_r(0) = -\int_{S} |P_r|^{q-1} d\sigma(\eta) \left( \int_{S} |P_r|^q d\sigma(\eta) \right)^{1/q-1} < 0.$$  

Furthermore

$$\Phi'_r(a) = -\left( \int_{S} |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q-1} F(r, a),$$  

where

$$F(r, a) = \int_{S} (P_r(\eta) - a)(P_r(\eta) - a)^{q-2}d\sigma(\eta).$$  

So

$$F_a(r, a) = (q - 1) \int_{S} |P_r(\eta) - a|^{q-2}d\sigma(\eta) > 0,$$

and this implies in particular that $F(r, a)$ as a function of $a$ is strictly increasing, so $\Phi_r$ has only one stationary point which is its minimum which we denote by $a^* = a(r)$.

Since

$$F(r, a(r)) = 0,$$

the implicit function theorem implies that there is a smooth function $a^*$ depending on $r$ such that

$$\frac{\partial a(r)}{\partial r} = -\frac{\frac{\partial F}{\partial r}}{\frac{\partial F}{\partial a}}.$$
Thus the function $g_p(r) = \Phi_r(a(r))$ is smooth function of $r$ as a composition of smooth functions. It satisfies the condition $g_p(0) = 0$ and we have

$$|f(x)| \leq g_p(|x|)\|f\|_{L^p}.$$  

Moreover

$$g_p'(0) = \left( \int_S |n\eta_n|^q d\sigma(\eta) \right)^{1/q} = n \left( \frac{\Gamma \left[ \frac{n}{2} \right] \Gamma \left[ \frac{1+q}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{n+q}{2} \right]} \right)^{\frac{1}{q}}$$

So

$$\|Df(0)\| \leq n \left( \frac{\Gamma \left[ \frac{n}{2} \right] \Gamma \left[ \frac{1+q}{2} \right]}{\sqrt{\pi} \Gamma \left[ \frac{n+q}{2} \right]} \right)^{\frac{1}{q}}$$

Since $g_p(r) = \Phi(r, a(r))$ we have

$$\partial_r g_p(r) = \partial_1 \Phi(r, a(r)) + \partial_2 \Phi(r, a(r)) \partial_r a(r) = \partial_1 \Phi(r, a(r)).$$

Since

$$\Phi(r, a) = \left( \int_S |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q},$$

we have that

$$\partial_1 \Phi(r, a) = \int_S \partial_r P_r(\eta)(P_r(\eta) - a)|P_r(\eta) - a|^{q-2} d\sigma(\eta) \left( \int_S |P_r(\eta) - a|^q d\sigma(\eta) \right)^{1/q - 1}.$$  

Since

$$P_r(\eta) = \frac{1 - r^2}{(1 + r^2 - 2r\eta_n)^{n/2}},$$

it follows that

$$P_r(\eta) - a = \frac{1 - r^2}{(1 + r^2 - 2r\eta_n)^{n/2}} - a = n\eta_n r + (1 - a) + O(r^2),$$

and hence

$$\partial_1 \Phi(r, 1) = O(r) + \left( \int_S |n\eta_n|^q d\sigma(\eta) \right)^{1/q}.$$  

Next we have $\lim_{r \to 0} \partial_1 \Phi(r, a(r)) = \partial_1 \Phi(0, 1)$, and so

$$\partial_r g_p(0) = \left( \int_S |n\eta_n|^q d\sigma(\eta) \right)^{1/q}.$$  

In order to show that the inequality is sharp, fix $x$ and without loosing the generality assume that $x = RN$, with $R = |x|$ and let

$$f_R(\eta) = |P_R(\eta) - a(R)|^{q/p} \text{sign}(P_R(\eta) - a(R))$$

and let $u_R(y) = P[f_R](y)$. From \eqref{eq:25}

$$F(R, a) = \int_S (P_R(\eta) - a(R))|P_R(\eta) - a(R)|^{q-2} dt = 0.$$
Hence we obtain that $u_R(0) = 0$. Now, the Hölder inequality $\text{(2.3)}$ is an equality for $u_R$ in $x$. This implies the sharpness of inequality. In order to prove that for $p < \infty$, $\lim_{r \to 1} g_p(r) = \infty$, let $f \in \mathcal{H}_p \setminus \mathcal{H}_\infty$ and assume without losing of generality that $f(0) = 0$. Then

$$\sup_r g_p(r) \geq \sup_x |f(x)| \|f\|_p = \infty.$$ 

Finally prove that $g_p$ is strictly increasing. Let $r < s$ and choose $\|f\|_p = 1$ such that $g_p(r) = |f(x_0)| = \max_{|x| \leq r} |f(x)|$. Clearly $f$ is not a constant function. Then by maximum principle $|f(x_0)| < \max_{|x|=s} |f(x)| \leq g_p(s)$. So $g_p$ is a strictly increasing function.

The last part of the proof, i.e. the case $p = \infty$, follows from the previous proof and the next section. $\square$

As a corollary of our main result we obtain

**Corollary 2.3.** If $f \in \mathcal{H}^2$, then $\|Df(0)\| \leq \sqrt{n} \sqrt{\|f\|_2^2 - |f(0)|^2}$.

*Proof.* Let $g(x) = f(x) - f(0)$, then $Df(0) = Dg(0)$, on the other hand

$$\|g\|_2^2 = \langle f - f(0), f - f(0) \rangle = \|f\|^2 + |f(0)|^2 - 2 \langle f, f(0) \rangle = \|f\|_2^2 - |f(0)|^2.$$ 

On the other hand

$$|Dg(0)| \leq \lim_{r \to 0} g'_2(r) \|g\|_2 = \sqrt{n} \|g\|_2.$$ 

The result follows. $\square$

3. Special cases

3.1. The case $p = \infty$. In this case we deal with the extremal problem

$$\inf_a \int_S |P_r(\eta) - a| d\sigma(\eta).$$ 

Let $a_0 = \frac{1-r^2}{(1+r^2)^{n/2}}$. Then

$$\int_S |P_r(\eta) - a_0| d\sigma(\eta) = \int_{S^+} P_r(\eta) d\sigma(\eta) - \int_{S^-} P_r(\eta) d\sigma(\eta)$$

$$= \int_{S^+} (P_r(\eta) - a_0) d\sigma(\eta) - \int_{S^-} (P_r(\eta) - a_0) d\sigma(\eta)$$

$$\leq \inf_a \int_S |P_r(\eta) - a_0| d\sigma(\eta).$$

So $a^* = \frac{1-r^2}{(1+r^2)^{n/2}}$.

This implies that

$$\|f(z)\| \leq U(rN) \|f\|_\infty,$$

which is known as the classical Schwarz lemma for harmonic mappings.
3.2. **The case** $p = 2$. In this case we deal with the extremal problem

$$g_p(r) = \left( \inf_a \int_S |P_r(\eta) - a|^2 d\sigma(\eta) \right)^{1/2}.$$

We have (see [1, p. 140])

$$\int_S |P_r(\eta) - a|^2 d\sigma(\eta) = \int_S P_r^2(\eta) d\sigma(\eta) + a^2 \int_S d\sigma(\eta) - 2a \int_S P_r(\eta) d\sigma(\eta)$$

$$= \frac{1 - |x|^4}{(1 - 2|x|^2 + |x|^4)^{n/2}} + a^2 - 2a.$$

So $a^* = 1$ and

$$\left( \inf_a \int_S |P_r(\eta) - a|^2 d\sigma(\eta) \right)^{1/2} = \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)^{n-1}}} - 1.$$

This implies that

$$|f(x)| \leq \left( \sqrt{\frac{1 + r^2}{(1 - r^2)^{n-1}}} - 1 \right) \|f\|_2$$

which coincides with analogous statement in [1, p. 140].

3.3. **The case** $p = 1$. In this case we deal with the extremal problem

$$g_p(r) = \inf_a \sup_\eta |P_r(\eta) - a|.$$

Since

$$\max_\eta P_r(\eta) = \frac{1 - r^2}{(1 - r)^n}$$

and

$$\min_\eta P_r(\eta) = \frac{1 - r^2}{(1 + r)^n},$$

we easily conclude that

$$g_p(r) = \frac{1}{2} \left( \frac{1 - r^2}{(1 - r)^n} - \frac{1 - r^2}{(1 + r)^n} \right).$$

(In this case $a^* = \frac{1}{2} \left( \frac{1 - r^2}{(1 - r)^n} + \frac{1 - r^2}{(1 + r)^n} \right)$.) So

$$|f(x)| \leq \frac{1}{2} \left( \frac{1 - r^2}{(1 - r)^n} - \frac{1 - r^2}{(1 + r)^n} \right) \|f\|_1.$$
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