Joule-Thomson expansion of charged Gauss-Bonnet black holes in AdS space

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Joule-Thomson expansion process is studied for charged Gauss-Bonnet black holes in AdS space. Firstly, in five dimensional space-time, isenthalpic curve in $T - P$ graph is obtained and the cooling-heating region is determined. Secondly, the explicit expression of Joule-Thomson coefficient is obtained from general formulas of enthalpy and temperature. Our methods also suit to van der Waals system as well as other black hole system. Then the inversion curve $\tilde{T}(\tilde{P})$ which separating the cooling region and heating region is obtained and investigated. Thirdly, an interesting dependence of the inversion curves on the charge ($Q$) and the Gauss-Bonnet parameter ($\alpha$) are revealed. In $\tilde{T} - \tilde{P}$ phase, the charge decrease the cooling region, while the Gauss-Bonnet parameter increase the cooling region. Fourthly, applying to our methods, the Joule-Thomson expansion process for $\alpha = 0$ case in four dimension is studied, where the Gauss-Bonnet AdS black hole degenerate into normal AdS black hole. The inversion curves for van der Waals systems consist of two parts, one has positive slope, the other has negative slope. While for black hole systems, the slope of the inversion curves are always positive, which seems to be an universal feature.
I. INTRODUCTION

Thermodynamics of black holes is an interesting and challenging topic since the discovery of black hole’s entropy[1], the four thermodynamic law[2], and the Hawking radiation[3] in 1970s. It has fundamental connections between the classical thermodynamics, general relativity, and quantum mechanics. Specially, due to the development of AdS/CFT duality[4–6], this connection has been deepened and a lot attention has been attracted to the AdS black holes.

In AdS space, there exists Hawking-Page phase transition between stable large black hole and thermal gas[7] which is explained as the confinement/deconfinement phase transition of a gauge field[8]. Considering the AdS black holes are electrically charged, rich phase structures are found by Chamblin et al[9, 10]. It is discovered that the phase transition behavior of charged AdS black hole is reminiscent to the liquid-gas phase transition in a van der Waals system[11]. In the extended phase space where the cosmological constant is identified as pressure[12], further investigation of the $P − V$ critical behavior of a charged AdS black hole support the analogy between the black holes and the van der Waals liquid-gas system. It is found that both the black hole and van der Waals system share not only the same $P − V$ diagram, but also the critical exponents[13]. This analogy has been generalized to different AdS black holes, such as rotating black holes, higher dimensional black holes, Gauss-Bonnet black holes, f(R) black holes, black holes with scalar hair, etc[14–56].

Apart from the phase transition and critical phenomena, the analogy between the black holes and the van der Waals system has been creatively generalized to the well-known Joule-Thomson expansion process[57] recently. In Joule-Thomson expansion of classical thermodynamics, gas at a high pressure passes through a porous plug to a section with a low pressure and during the process enthalpy is unchanged. There is an interesting phenomena during Joule-Thomson expansion process in $T − P$ graph which is divided into two parts, one is cooling region, the other is heating region. There is an inversion curve separating the cooling and heating regions. For charged AdS black holes[57] and Kerr-AdS black holes[58], the isenthalpy expansion process and the inversion curve are investigated. Then the works are generalized to quintessence charged AdS black holes[59], holographic superfluids[60] and charged AdS black holes in f(R) gravity[61]. The results show that the inversion curves $\tilde{T}(\tilde{P})$ for all the above black hole systems have only positive slope. While for the van der Waals
system, the inversion curves have both positive and negative slopes which form a circle in pressure axis.

We are curious about the missing of negative slopes of the inversion curves, and eager to check out that whether the quantum gravity effects will cure this problem or whether this feature for black hole systems is just universal. So we will focus on the Gauss-Bonnet Einstein-Maxwell gravity. Consider the effects of Gauss-Bonnet term is interesting and important, since whatever the quantum gravity may be, there will be higher order corrections to the pure Einstein action, and Gauss-Bonnet is a well combination of terms which have candidate corrections. What’s more, these terms represent part of the $1/N$ correction to the large $N$ limit of the holographically dual $SU(N)$-like gauge field theory\[62\]. As a result, the investigation is interesting in its own right.

This paper is organized as follows. In Sec.\[II\] we briefly review the D-dimensional Charged Gauss-Bonnet black holes in AdS space. In Sec.\[III\] firstly we investigate Joule-Thomson expansion for the Gauss-Bonnet parameter $\alpha > 0$ and $D = 5$ dimensional black holes, the isenthalpy curve is studied, two methods are introduced to derive an explicit Joule-Thomson coefficient, the effects of charge and Gauss-Bonnet parameter on the inversion curves are studied. Then we investigate Joule-Thomson expansion for the Gauss-Bonnet parameter $\alpha = 0$ and $D = 4$ dimensional black holes by our new methods. Sec.\[IV\] devotes to conclusion and discussion.

II. A BRIEF REVIEW OF THE D-DIMENSIONAL CHARGED GAUSS-BONNET BLACK HOLES IN ADS SPACE

Consider action in D-dimensional Einstein-Maxwell theory with a cosmological constant $\Lambda$ and a Gauss-Bonnet term:

$$S = \frac{1}{16\pi} \int d^Dx \sqrt{-g}[R - 2\Lambda + \alpha_{GB}(R_{\gamma\delta\mu\nu}R^{\gamma\delta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2) - F^2],$$

(1)

where the Gauss-Bonnet parameter $\alpha_{GB}$ has dimensions of $[\text{length}]^2$ and the cosmological constant $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$. When $\alpha_{GB} = 0$, the action degenerate into Einstein-Maxwell theory in AdS space. When $\alpha_{GB} \neq 0$, we will work in $D \geq 5$, since in $D = 4$ the Gauss-Bonnet term is purely topological. Define $\alpha \equiv (D-3)(D-4)\alpha_{GB}$. 
The action admits a static black hole solution with maximal symmetry as

\[ ds^2 = -Y(r)dt^2 + \frac{dr^2}{Y(r)} + r^2 d\Omega_{D-2}^2, \quad (2) \]

where \( d\Omega_{D-2}^2 \) is the metric on a round \( D - 2 \) sphere with volume \( \omega_{D-2} \), and

\[ Y(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{4\alpha m}{r^{D-1}} - \frac{4\alpha q^2}{r^{2D-4}} - \frac{4\alpha}{l^2}} \right). \quad (3) \]

Notice that since the \( m = q = 0 \) case, defining the vacuum solution, for a given value of \( l \), \( \alpha \) cannot be arbitrary, but must be constrained by \( 0 \leq \frac{4\alpha}{l^2} \geq 1 \) which should be carefully considered in the following investigation.

When \( \alpha = 0 \), using the L’Hospital’s rule, Eq.(3) will become

\[ Y(r) = 1 - \frac{m}{r^{D-3}} + \frac{q^2}{r^{2D-6}} + \frac{r^2}{l^2}, \quad (4) \]

which is the familiar charged AdS black hole. \( m, q \) are related to the ADM mass \( M \), electric charge \( Q \) by

\[ M = \frac{(D - 2)\omega_{D-2}}{16\pi} m, \quad Q = \sqrt{2(D - 2)(D - 3)} \frac{\omega_{D-2}}{8\pi} q. \quad (5) \]

The horizon radius \( r_+ \) of the black hole is the largest root of \( Y(r_+) = 0 \), which gives us an equation for the black hole mass \( M \),

\[ M = \frac{(D - 2)\omega_{D-2}}{16\pi} (\alpha r_+^{D-3} + r_+^{D-5} + \frac{q^2}{r_+^{D-3}} + \frac{16\pi P_{r_+^{D-1}}}{(D - 1)(D - 2)}), \quad (6) \]

where the AdS radius \( l \) is replaced by pressure \( P = \frac{(D - 1)(D - 2)}{16\pi r_+^{D-1}} \), and in this extended phase space the black hole mass is treated as enthalpy instead of internal energy. The temperature is obtained by the first derivative of \( Y(r) \) at the horizon,

\[ T = \frac{Y'(r_+)}{4\pi} = \frac{1}{4\pi r_+(r_+^2 + 2\alpha)} \left( \frac{16\pi P_{r_+^{D-1}}}{D - 2} + (D - 3)r_+^2 + (D - 5)\alpha - (D - 3) \frac{q^2}{r_+^{D-8}} \right). \quad (7) \]

In the following, we will only use Eq.(6) and Eq.(7) to find the isenthalpic curves and determine the inversion temperatures between the cooling and heating regions for the black hole system in \( T - P \) plane.

**III. JOULE-THOMSON EXPANSION**

The Joule-Thomson expansion for black hole system is an isenthalpy process with fixed enthalpy \( H \) which is identified as the black hole mass \( H = M \) in the extended phase space.
Similar to the Joule-Thomson process with fixed particle number for van der Waals gases, we should consider the canonical ensemble with fixed charge $Q$. The Gauss-Bonnet parameter $\alpha$ will also be treated as a constant.

**A. \( \alpha > 0 \) and \( D = 5 \)**

When $\alpha > 0$ and $D = 5$, Eq.(6) and Eq.(7) become

$$
M = \frac{3\pi}{8}(\alpha + r_+^2 + \frac{4Q^2}{3\pi^2 r_+^2} + \frac{4}{3}\pi Pr_+^4),
$$

$$
T = \frac{\pi^2 r_+^4 (3 + 8\pi Pr_+^2) - 4Q^2}{6\pi^3 r_+^3 (2\alpha + r_+^2)}.
$$

(8)

The pressure $P$ can be rewritten as a function of $M$ and $r_+$, then substitute $P(M, r_+)$ into the temperature formula which also become a function of $M$ and $r_+$,

$$
P(M, r_+) = \frac{8\pi Mr_+^2 - 3\pi^2 \alpha r_+^2 - 3\pi^2 r_+^4 - 4Q^2}{4\pi^3 r_+^6},
$$

$$
T(M, r_+) = \frac{16\pi Mr_+^2 - 6\pi^2 \alpha r_+^2 - 3\pi^2 r_+^4 - 12Q^2}{6\pi^3 r_+^3 (2\alpha + r_+^2)}.
$$

(9)

Solving the above equations, one can obtain function $T(M, P)$ which is lengthy and won’t be shown here. However, for particular values of $\alpha = 0.2$, $Q = \sqrt{3}\pi$, the $T(M = 7, P)$ curve is shown in Fig.1. According to the Joule-Thomson coefficient $\mu = \left(\frac{\partial T}{\partial P}\right)_M$, the inversion pressure and temperature between the cooling and heating regions is ($\tilde{P} = 0.1136, \tilde{T} = 0.1641$) which is set by $\mu = 0$. So the most important thing is to find the function expression of $\mu$. By setting $\mu = 0$, one can obtain the inversion points ($\tilde{P}, \tilde{T}$) for different fixed enthalpy $M$.

In Ref.[57], the Joule-Thomson coefficient is given by,

$$
\mu = \left(\frac{\partial T}{\partial P}\right)_M = \frac{1}{C_P} [T(\frac{\partial V}{\partial T})_P - V],
$$

(10)

which is elegant. But in this paper, we will adopt more straightforward methods by using only Eq.(6) and Eq.(7) to derive the Joule-Thomson coefficient $\mu$. From Eq.(8), one can find that temperature is a function of pressure and radius $T(P, r_+)$, radius is a function of pressure and mass $r_+(P, M)$. Then the Joule-Thomson coefficient is given by,

$$
\mu = \left(\frac{\partial T}{\partial P}\right)_M = \left(\frac{\partial T}{\partial P}\right)_{r_+} + \left(\frac{\partial T}{\partial r_+}\right)\frac{\partial r_+}{\partial P}(\frac{\partial r_+}{\partial P})_M = \left(\frac{\partial T}{\partial P}\right)_{r_+} + \left(\frac{\partial T}{\partial r_+}\right)\frac{\partial P}{\partial r_+} = \frac{4r_+^3}{3(2\alpha + r_+^2)} + \frac{r_+^3 (20Q^2 r_+^2 + \pi^2 r_+^6 (8\pi Pr_+^2 - 3) + 6\alpha (4Q^2 + \pi^2 r_+^4 (1 + 8\pi Pr_+^2)))}{3(2\alpha + r_+^2)^2 (12Q^2 + \pi r_+^2 (3\pi (2\alpha + r_+^2) - 16M))}.
$$

(11)
FIG. 1: The isenthalpy process with $M = 7$ in $T - P$ graph at $\alpha = 0.2$, $Q = \sqrt{3}\pi$. The inversion pressure and temperature between the cooling and heating regions is $(\tilde{P} = 0.1136, \tilde{T} = 0.1641)$ where the Joule-Thomson coefficient $\mu = (\frac{\partial T}{\partial P})_M = 0$.

FIG. 2: $T$ vs. $P$ and $\tilde{T}$ vs. $\tilde{P}$ for $\alpha = 0.2$, $Q = \sqrt{3}\pi$. The blue ones are isenthalpy curves $T(M, P)$ for $M = 6, 6.5, 7$. The red one is the inversion curve $\tilde{T}(\tilde{P})$. One can find that the inversion points $(\tilde{P}, \tilde{T})$ increase monotonically with enthalpy.

From Eq. (9), one can obtain a more simple expression,

$$\mu = (\frac{\partial T}{\partial P})_M = (\frac{\partial T}{\partial r_+})_M (\frac{\partial r_+}{\partial P})_M = (\frac{\partial T}{\partial r_+})_M (\frac{\partial r_+}{\partial P})_M$$

$$= \frac{r_+^4 (3\pi^2 r_+^6 + 12\pi(\pi\alpha - 4M)r_+^4 + 4\pi\alpha(3\pi\alpha - 8M) + 15Q^2)r_+^2 + 72\alpha Q^2)}{3(2\alpha + r_+^2)(12Q^2 + \pi r_+^2(3\pi(2\alpha + r_+^2) - 16M))}.$$  \hspace{1cm} (12)

All the above methods are equalvalent. Choose one method and set $\mu = 0$, together with
It is of interest to probe the dependence of the inversion points on the charge and the Gauss-Bonnet parameter. Left panel of Fig. 3 shows the relationship between the inversion curves and charge for fixed Gauss-Bonnet parameter $\alpha = 0.1$. The slope of the inversion curves increase with charge, suggesting that the inversion temperature is higher for the Joule-Thomson expansion with a larger charge. Right panel of Fig. 3 shows the relationship between the inversion curves and Gauss-Bonnet parameter for fixed charge $Q = \sqrt{3}\pi$. In contrast to charge, the slope of the inversion curves decrease with Gauss-Bonnet parameter, suggesting that the inversion temperature is lower for the Joule-Thomson expansion with a larger Gauss-Bonnet parameter. So the effect of charge and Gauss-Bonnet parameter are different. In $\tilde{T} - \tilde{P}$ phase, the charge decrease the cooling region, while the Gauss-Bonnet parameter decrease the heating region.
parameter increase the cooling region.

\[ \alpha = 0 \text{ and } D = 4 \]

When \( \alpha = 0 \) and \( D = 4 \), the Gauss-Bonnet AdS black hole degenerate into normal AdS black hole. Its Joule-Thomson expansion process is investigated in Ref.\[57\]. In this section, we will use our methods to reinvestigate the Joule-Thomson expansion and double check these methods.

The start point is still Eq.(6) and Eq.(7) which become
\begin{align*}
M &= \frac{3Q^2 + 3r^2 + 8\pi Pr^4_+}{6r_+}, \\
T &= \frac{-Q^2 + r^2 + 8\pi Pr^4_+}{4\pi r^4_+}, \quad (13)
\end{align*}

or rewritten as
\begin{align*}
P(M, r_+ &= \frac{6Mr_+ - 3Q^2 - 3r^2}{8\pi r^4_+}, \\
T(M, r_+ &= \frac{3Mr_+ - 2Q^2 - r^2_+}{2\pi r^3_+}. \quad (14)
\end{align*}

Then the Joule-Thomson coefficient is obtained as,
\begin{align*}
\mu &= \left(\frac{\partial T}{\partial P}\right)_{r_+} + \left(\frac{\partial T}{\partial r_+}\right)_M \left(\frac{\partial P}{\partial r_+}\right)_M \\
&= \frac{r_+(15Q^2 + r_+(-18M + 5r_+ + 8\pi Pr^4_+))}{3(2Q^2 + r_+(-3M + r_+))} \\
&= \frac{4r_+(-3Q^2 + 2r^2_+ + 8\pi Pr^4_+)}{3(-Q^2 + r^2_+ + 8\pi Pr^4_+)}, \quad (15)
\end{align*}

or
\begin{align*}
\mu &= \left(\frac{\partial T}{\partial r_+}\right)_M \left(\frac{\partial P}{\partial r_+}\right)_M \\
&= \frac{2r_+(6Q^2 + r_+(r_+ - 6M))}{3(2Q^2 + r_+(r_+ - 3M))} \\
&= \frac{4r_+(-3Q^2 + 2r^2_+ + 8\pi Pr^4_+)}{3(-Q^2 + r^2_+ + 8\pi Pr^4_+)}. \quad (16)
\end{align*}

They are consistent with each other. Setting \( \mu = 0 \), the only positive and real root of \( r_+ \) is
\begin{equation}
r_+ = \frac{\sqrt{\sqrt{1 + 24\pi Q^2\tilde{P}} - 1}}{2\sqrt{2\pi \tilde{P}}}. \quad (17)
\end{equation}
Substituting this root into the temperature formula above, finally the analytical inversion curve $\tilde{T}(\tilde{P})$ is obtained

$$\tilde{T} = \sqrt{\frac{\tilde{P}}{2\pi} \left( \frac{1 + 16\pi Q^2 \tilde{P} - \sqrt{1 + 24\pi Q^2 \tilde{P} - 1}}{(1 + 24\pi Q^2 \tilde{P} - 1)^{3/2}} \right)}, \quad (18)$$

which is exactly same with Eq.(44) in Ref.[57]. Same with $\alpha > 0, D = 5$ case, the slopes of the inversion curves $\tilde{T}(\tilde{P})$ are also always positive.

IV. CONCLUSION AND DISCUSSION

In this paper, we studied the Joule-Thomson expansion for charged Gauss-Bonnet black holes in AdS space. In the extended phase, the cosmological is identified as pressure and the black hole mass as enthalpy. Thus we considered the expansion process with constant mass. Firstly, the Joule-Thomson expansion process (the isenthalpy curve) is depicted in Fig.1. The inversion point $(\tilde{P}, \tilde{T})$ separating the cooling region and heating region locates at $\mu = 0$ where the temperature is highest during the Joule-Thomson expansion process. Left part with positive slope is the cooling region, while right part with negative slope is the heating region.

Secondly, the explicit expression of Joule-Thomson coefficient is obtained from general formulas of enthalpy and temperature. Our methods also suit to van der Waals gas as well as other black hole system. The isenthalpy curves for different enthalpy and the inversion curve for specific Gauss-Bonnet parameter $\alpha = 0.2$ and charge $Q = \sqrt{3\pi}$ are depicted in Fig.2. The inversion curve divides the $T - P$ graph into two branches. The branch above the inversion curve is the cooling region, and the branch below the inversion curve is the heating region. The slope of the inversion curve is always positive.

Thirdly, the dependence of the inversion curves on the charge and the Gauss-Bonnet parameter are investigated. An interesting result is depicted in Fig.3 where the slope of the inversion curves increase with charge, but decrease with Gauss-Bonnet parameter. As a result, the effect of charge and Gauss-Bonnet parameter are different. In $\tilde{T} - \tilde{P}$ phase, the charge decrease the cooling region, while the Gauss-Bonnet parameter increase the cooling region.

Finally, we checked that the Gauss-Bonnet AdS black hole degenerate into normal AdS black hole when $\alpha = 0$. By applying to our methods, the Joule-Thomson expansion process
is reinvestigated in four dimensional space-time. An analytical expression of the inversion curves is obtained which is exactly same with that in Ref. [57]. The slope of the inversion curves are also always positive.

In the end, we would like to point out that the inversion curves of Joule-Thomson expansion for van der Waals gas system consist of two parts which forms a circle in pressure axis. One part is a lower one with positive slope, the other is an upper one with negative slope. For the details, one can refer to Ref. [57], or any textbook on thermodynamics and statistical physics. But as far as we know, the inversion curves of Joule-Thomson expansion for black hole system, such as charged AdS black holes [57], Kerr-AdS black holes [58], quintessence charged AdS black holes [59], charged AdS black holes in f(R) gravity [61], as well as our charged AdS black holes in Gauss-Bonnet gravity, are only with positive slope. This seems to be an universal feature for black hole system. The under physics behind the missing of the negative slope or the difference of Joule-Thomson expansion for black hole system and van der Waals system deserve to be disclosed in the future research.

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