Strings in PP-Waves and Worldsheet Deconstruction

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Abstract

Based on the observation that \(AdS_5 \times S^5/Z_k\) orbifolds have a maximal supersymmetric PP-wave limit, the description of strings in PP-waves in terms of \(\mathcal{N} = 2\) quiver gauge theories is presented. We consider two different, small and large \(k\), cases and show that the operators in the gauge theory which correspond to stringy excitations are labelled by two integers, the excitation and winding or momentum numbers. For the large \(k\) case, the relation between the space-time and worldsheet deconstructions is discussed. We also comment on the possible duality between these two cases.
1 Introduction

The PP-waves has been shown to be maximal supersymmetric solutions of type IIB supergravity [1] and arise as the Penrose limit [2] of another maximal SUSY supergravity solution, namely $AdS_5 \times S^5$ [3, 4]. The (type IIB Green-Schwarz) string theory in the PP-wave background has been formulated [5] and shown to be exactly solvable [6]. The string $\sigma$-model action written in the light-cone takes a very simple form

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^{2\pi\alpha'p^\perp} d\sigma \left[ \frac{1}{2} \dot{X}_i^2 - \frac{1}{2} X_i'^2 - \frac{1}{2} \mu^2 X_i^2 + i \bar{S}(\Gamma^i \partial_i + \mu I)S \right],$$

where $X_i$'s are eight transverse bosonic string worldsheet fields, $I = \Gamma^{1234}$ and $S$ is a Majorana (space-time) spinor. The parameter $\mu$, which is the mass parameter of the strings, is the field strength of self-dual 5-form of the background PP-wave solution. The maximally SUSY PP-wave background has a $SO(4) \times SO(4)$ rotational symmetry, which is manifest in the action (1).

Since PP-waves are related to the $AdS_5 \times S^5$, Berenstien, Maldacena and Nastase (BMN), have shown how to translate the Penrose limit over the gravity side into the dual $\mathcal{N} = 4$ SYM gauge theory [8]. According to BMN dictionary, taking the Penrose limit corresponds to considering the sector of the gauge theory operators with large R-charge $J$ and large conformal weight $\Delta$, with $\Delta \sim J$ and $\Delta - J$ = positive and finite. In fact BMN explicitly constructed the string creation-annihilation operators through particular gauge theory operators. In this way they conjectured a way to obtain “string bit” formulation of the strings on PP-waves in the gauge theory language (the string bits correspond to certain chiral primary operators in the gauge theory).

Soon after the BMN work it was shown that the other supergravity solutions which are of the form $AdS_5 \times \mathcal{M}^5$, where $\mathcal{M}$ is a smooth Einstein manifold and preserves some SUSY may also lead to the maximal supersymmetric PP-waves in the Penrose limit [3, 10, 11]. Since string theory on these backgrounds is dual to a $\mathcal{N} = 1, D = 4$ super-conformal field theory (SCFT), in the same spirit of the BMN work, there should be particular sub-sectors of the $\mathcal{N} = 1$ gauge theories which shows the enhancement of SUSY and hence R-symmetry.

The PP-wave limit of the $AdS_5 \times S^5/Z_k$ orbifold has also been considered [12, 13, 14]. In [12] we showed that the $AdS_5 \times S^5/Z_k$ orbifold, depending on how we take the Penrose limit, admits two PP-waves. One is half supersymmetric and is the orbifold version of the maximal SUSY PP-wave background. However, the other one is the maximal SUSY PP-wave, as if there were no orbifolding. String theory on $AdS_5 \times S^5/Z_k$ is dual to a $\mathcal{N} = 2, D = 4, SU(N)^k$ quiver gauge theory with bi-fundamental matter fields. In [12, 13] the half SUSY case were considered and the stringy creation-annihilation of the string on the orbifolds of PP-waves were constructed out of the proper dual (“holographic”) gauge theory operators.

\footnote{In the past two months there have been a cascade of papers devoted to this subject [5].}
In this work we would like to consider the maximal SUSY PP-wave limit of the orbifolds and specify the operators in the dual $\mathcal{N} = 2$ gauge theory side which correspond to strings on PP-waves. Therefore, strings on PP-waves would have descriptions in terms of $\mathcal{N} = 4$ [8], $\mathcal{N} = 1$ [9, 10, 11] and also $\mathcal{N} = 2$ super-conformal gauge theories.

In the maximal SUSY PP-wave limit of the $\text{AdS}_5 \times S^5/Z_k$ orbifolds $k$ remains a free parameter, i.e. the final supergravity background and hence the string theory do not depend on $k$ at all. However, from the field theory side, $k$ is still there. In fact $k$ always appears as $kJ$, where $J$ is the R-charge. In particular one may study small or large $k$ limits, while $kJ$ is fixed and large. The large $k$ limit of the orbifolds have been previously considered in [15, 16] where the interesting effect of “deconstruction” of space-time dimensions was introduced and studied.

This article is organized as follows. In section 2 we briefly review the results of [12] and show how the maximal SUSY PP-wave background arises in the orbifold case. In section 3, we consider the $\mathcal{N} = 2$ quiver gauge theory case and construct the stringy operators corresponding to strings moving in PP-wave background. We consider two different cases of small and large $k$. We also show that in this quiver gauge theory language there is a room for another quantum number. This quantum number can be thought of as momentum (or winding) number for small (or large) $k$ cases. The last section is devoted to discussion and conclusions.

# 2 Review of the PP-wave limit of $\text{AdS}_5 \times S^5/Z_k$

In this section we briefly review the results of [12] where it was shown that $\text{AdS}$ orbifolds admit the maximal SUSY PP-waves as the Penrose limit. To show how it works, consider the supergravity solution corresponding to $\text{AdS}_5 \times S^5/Z_k$ solution

$$l_s^{-2}ds_{10}^2 = R^2 \left( -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + dM_5^2 \right),$$

with

$$dM^2 = d\alpha^2 + \cos^2 \alpha d\theta^2 + \frac{\sin^2 \alpha}{4} (d\gamma^2 + \sin^2 \gamma d\delta^2) + \frac{\sin^2 \alpha}{k^2} [d\chi + \frac{k}{2} (\cos \gamma - 1)d\delta]^2,$$

where $\chi = \hat{y}/l_s$ and $R^4 = 4\pi g_s N k$. $\chi$ ranges from 0 to $2\pi k$ and the $Z_k$ orbifold is defined by the identification $\chi \equiv \chi + 2\pi$. The above metric can be understood as metric for $N$ D3-branes sitting at the orbifold $Z_k$, where D3-branes are along 0123 and $Z_k$ is acting on 4567 directions (as usual) and 89 is the fixed plane (the intersection of 89 plane with $S^5$ is a circle parameterized by angle $\theta$). At the conformal point D3-branes are sitting at the orbifold fixed point ($\alpha = 0$). The IIB self-dual five form is found to be

$$F_{\chi\alpha\theta\gamma\delta} = (\ast F)_{\chi\alpha\theta\gamma\delta} = \frac{R^4}{k} \alpha^2 \cos \alpha \sin^3 \alpha \sin \gamma.$$
The (type IIB) string theory on the above background is dual to a four dimensional \( \mathcal{N} = 2 \) SYM theory with gauge group \( SU(N)^k \) and \( k \) bi-fundamental matter hypermultiplets \( i.e. \) \( (Q_i, \bar{Q}_i), \ i = 1, 2, \ldots, k, \) in \( \mathcal{N} = 1 \) notation. The \( Z_k \) acts as a permutation of the gauge factors. The dimensionless gauge coupling of each \( SU(N) \) part of the gauge group is \( g_{YM}^2 \sim g_s k \). Besides the scalars in bi-fundamental hypermultiplets we also have the complex scalar in the vectormultiplet, \( \varphi_i \).

The maximal SUSY PP-wave background can be obtained by focusing on the particles moving along the \( \chi \) direction. Namely, let us consider the scaling

\[
\begin{align*}
    t &= x^+ + R^{-2}x^- \\
    \frac{1}{k}\chi &= x^+ - R^{-2}x^- \\
    \rho &= \frac{r}{R}, \quad \gamma = \frac{2x}{R}, \quad \alpha = \frac{\pi}{2} - \frac{y}{R}, \quad R \to \infty
\end{align*}
\]

while keeping \( x^+, x^-, r, x \) and \( y \) fixed. In this limit the metric (5) becomes

\[
l_s^{-2}ds^2 = -4dx^+dx^- - \mu^2z^2dx^2 + dz^2,
\]

and

\[
F_{+1234} = F_{+4567} = \mu.
\]

We would like to comment that, although the above solution looks exactly the same as the PP-wave considered by BMN, there is a difference, i.e. the \( x^- \) direction in this case is compact. As we will show this will lead to the possibility of having another (integer) quantum numbers for the string states, namely the light-cone momentum and winding numbers.

This background, and consequently the strings moving in this background, do not depend on \( k \) and in particular the above limit is true for both small and large \( k \). As we see, the SUSY has been enhanced to the maximal 32 supercharges, and also the \( SU(2) \times U(1) \) R-symmetry has been enhanced to \( SO(4) \cong SU(2) \times SU(2) \). However, from the gauge theory side \( k \) is still meaningful. Therefore in the proper gauge theory operators which describe the stringy operators, for the small \( k \) case, \( k \) can only appear as another quantum number and its corresponding operator should commute with all the string creation operators. We would like to note that in the PP-wave limit (3) we are focusing on the geodesics far from the orbifold point (\( \alpha = 0 \)). From the gauge theory side this means that we are studying the theory far from the conformal point. This in particular implies that in the gauge theory operators corresponding to the string excitations only one of the \( SU(N) \) gauge factors should be manifest.

3 \( \mathcal{N} = 2 \) Description of strings in PP-waves

To construct the string operators in the gauge theory language, let us first very shortly review the BMN arguments [8]. The maximal SUSY type IIB PP-wave background has a \( SO(4) \times SO(4) \) isometry, as well as translations along \( x^+, x^- \) (and
also sixteen other spatial and non-compact isometries \(^{(17)}\)). These isometries from
the gauge theory point of view correspond to the reminiscent of \(SO(4, 2)\) conformal symmetry and \(SO(6) \cong SU(4)\) R-symmetry. Choosing the sub-sector of the
operators of the theory with large R-charge \(J\) under a \(U(1)\) factor of \(SO(6)\), will
effectively reduce the \(SO(6)\) to \(SO(4)\). On the other hand focusing on operators
which are “almost” chiral primary (i.e. operators of large conformal weight, \(\Delta\)
where \(\Delta - J = \text{finite}\) we have also effectively reduced the conformal group to the
\(SO(4)\) subgroup \(^{(18)}\). Although in the PP-wave limit the ’t Hooft coupling is very
large, working with “almost” chiral primary operators has the advantage of many
supersymmetric cancellations so that the effective gauge theory coupling is now

\[
g_{eff}^2 = \frac{g_{YM}^2 N}{J^2} \approx \frac{1}{4\pi} .
\]

(8)

According to BMN the stringy operators can be written in terms of definite
gauge theory operators. Let \(Z\) be the (complex) scalar field which carries one unit
of R-charge. The strings/gauge theory duality can be summarized as follows:

| Gauge Theory                  | String Theory |
|-------------------------------|---------------|
| \(Tr(Z^J)\)                  | vacuum        |
| \(g_{eff}\)                  | \(\frac{1}{\alpha'^{\mu p^+}}\) |
| \(\sum_{l=1}^{J} e^{\frac{2\pi i a_n}{J}} Tr(Z^l \phi \bar{Z} Z^{J-l})\) | \(a_n^\dagger a\) |
| \(\sum_{l=1}^{J} e^{\frac{2\pi i a_n}{J}} Tr(Z^l D a Z Z^{J-l})\) | \(a_n^\dagger (4+a)\) |

Table 1: \(\mathcal{N} = 4\) SYM-String Theory Correspondence

where \(a = 1, 2, 3, 4, \phi \) are the four scalars (other than \(Z, \bar{Z}\)), \(D a\) is the covariant
derivative of the “holographic” gauge theory \(^{(19)}\) and \(p^+\) is the strings light-cone
momentum.

Now let us concentrate on the orbifold model, the case with half supersymmetry.
Starting with a \(\mathcal{N} = 4\) SYM theory, the orbifolding will break \(SU(4)\) R-symmetry to
\(SU(2) \times U(1)\), and the \(Z_k\) is the discrete subgroup of the broken \(SU(2)\) piece. From
the string theory point of view, the \(Z_k\) action is a symmetry of the action \(^{(1)}\) and
one can orbifold four of the \(X_i\) components. Similar to the flat case, this orbifolding
preserves half of SUSY, however, breaks the \(SO(4) \times SO(4)\) symmetry to \(SO(4) \times
SU(2)\). Hence, to construct the corresponding operators from the gauge theory point
of view, one should start with operators which carry the R-charge under the \(U(1)_R\)
part of the \(SU(2) \times U(1)\) R-symmetry. More explicitly the sequence of \(Z\)’s should
be replaced with a proper combination of \( \varphi \)'s, the scalar in the vectormultiplets \( \chi \).

From the supergravity side one can understand this noting that the orbifold of PP-waves can be obtained from the \( AdS_5 \times S^5/Z_k \) solutions and focusing on the geometry seen by particles which are moving very fast near the orbifold point (or conformal point from the corresponding quiver gauge theory side) along the \( S^1 \) (in the \( S^5 \)) which is fixed under orbifold action \[12\].

However, the situation for the maximal PP-wave limit of the \( AdS_5 \times S^5/Z_k \) is completely different. As it is seen from \( \chi \), the particle moving (spinning) very fast along \( \chi \) direction while sitting at \( \rho = 0, \alpha = \frac{\pi}{2} \), feels a smooth metric. Since this metric is exactly what we have in the non-orbifold case (of course except for the compactness of \( x^- \)), one leads to the conclusion that the \( Z_k \) subgroup is replaced by a \( U(1) \) subgroup and in addition, the two other generators of the broken \( SU(2) \) combined with the \( \mathcal{U}(1) \) generates the new \( SU(2) \) so that the R-symmetry is again \( SO(4) \cong SU(2) \times SU(2) \). Furthermore, being far from the conformal point, instead of \( SU(N)^k \) we effectively deal with a single \( SU(N) \) gauge group. This is again what we expect to see from the BMN picture. Noting the scaling \( \chi \) the light cone momenta are

\[
2p^- = i\partial_x^+ = i(\partial_t + k\partial_{\chi}) = \Delta - kJ, \tag{9}
\]
\[
2p^+ = \frac{i\partial_x^-}{R^2} = i\frac{(\partial_t - k\partial_{\chi})}{R^2} = \frac{\Delta + kJ}{R^2}, \tag{10}
\]

where \( J \) is the eigen-value for \(-i\partial_{\chi} \) operator. Observe that what is important and should be large is the effective angular momentum \( J_{\text{eff}} = kJ \). In fact it is this quantity which has to be taken large in the large \( R \) limit namely,

\[
(kJ)^2 \sim R^4 = 4\pi g_s N k \; . \tag{11}
\]

Note that the coupling of the effective \( SU(N) \) gauge theory is of order of \( g_s k \). Hence the effective gauge coupling in our case (both large and small \( k \) cases) is

\[
g_{\text{eff}}^2 = \frac{g_{YM}^2 N}{(kJ)^2} \sim \frac{g_s N k}{R^4} \sim \frac{1}{4\pi} \; . \tag{12}
\]

That is, we can still treat the effective \( SU(N) \) gauge theory perturbatively. Although in the string theory side \( J \) and \( k \) always appear as \( kJ \), from the gauge theory point of view \( J \) and \( k \) still have their usual physical meanings. Here we study two different cases (in the gauge theory side) which should correspond to the same string theory, namely, small \( k \) (and large \( J \)) and large \( k \) (small \( J \)) limits.

### 3.1 Small \( k \) case

As we have discussed, the \( J_{\text{eff}} = kJ \) should be large and scale as \( R^2 \). First we consider the case where \( k \) is of order of one and \( J \sim R^2 \). In order to construct
the chiral operators corresponding to the closed string excitation we first need to identify the vacuum state.

Let us define $Z_i = Q^1_i + iQ^2_i$. It is evident that $Z_i$’s are in the bi-fundamental representation, i.e. under gauge transformations $Z_i \to U_i Z_i U_i^{-1}$. The conformal dimension of $Z_i$ fields is one and their $\chi$ R-charge is $\frac{1}{k}$ (and hence their $J_{eff} = k \cdot \frac{1}{k} = 1$). Next we define a scalar field $Z$,

$$Z = \prod_{i=1}^{k} Z_i$$

which has dimension $k$ and therefore

$$\mathcal{O}_{vac} = \text{Tr}(Z^J)$$

is chiral primary, i.e. $\mathcal{O}_{vac}$ has $\Delta - kJ = 0$. We would like to note that the $\text{Tr}$ in the above expression is made over $N \times N$ matrices. These matrices are in fact in the adjoint representations of only one of the $SU(N)$ factors of our $SU(N)^k$ gauge theory. This $SU(N)$ factor have been chosen to be the one parameterized by $U_1$. It is easy to check that $\mathcal{O}_{vac}$ do not depend on this specific choice of the $SU(N)$ factor. To see that, it is convenient to define the “twist” operator, $\omega$

$$\omega F_i \omega^{-1} = F_{i+1}$$

where $F_i$ is a generic operator of the $i$’th factor of the $SU(N)^k$ gauge group. If the above state corresponds to the string theory vacuum state (or more precisely the state with zero excitation number), we see that it is $k$-fold degenerate.

Similar to the BMN case, the idea is to discretize the string worldsheet into $J$ pieces. However, noting the operator corresponding stringy vacuum state, at each point of the string bit we have a $k$-point ($Z_k$ lattice) internal space. Then to obtain the string oscillatory modes we need to insert proper operators into the $J$-point string of $Z$’s.

Let us define the operators

$$\phi^1 = \prod_{i=1}^{k} (\varphi^1_i) Z_i \quad , \quad \phi^2 = \prod_{i=1}^{k} (\varphi^2_i) Z_i \quad ,$$

$$\phi^3 = \prod_{i=1}^{k} \tilde{Q}^1_i \quad , \quad \phi^4 = \prod_{i=1}^{k} \tilde{Q}^2_i ,$$

where $\varphi^1,2$ are the two real scalars of the complex adjoint scalar of the vectormultiplet. We note that each of these scalars $\phi^a$ ($a = 1, 2, 3, 4$) has $\Delta - kJ = k$ and is in the adjoint representation of the first $SU(N)$ gauge factor. Hence, they are the proper operators needed for constructing stringy excitations. It would be more convenient to use $kN \times kN$ notation. In this notation the operators we are dealing
with are of the form
\[ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_k \end{pmatrix}, \quad D_a = \begin{pmatrix} D_{a1} & D_{a2} & \cdots & D_{ak} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ D_{a1} & D_{a2} & \cdots & D_{ak} \end{pmatrix}, \]
all the off-diagonals are zero, and each element is a \( N \times N \) matrix. In the above
\( D_{ai} \) is the covariant derivative corresponding to the \( i \)'th \( SU(N) \) factor. And the
bi-fundamentals in the hypermultiplet as \( kN \times kN \) matrices are
\[ Z = \begin{pmatrix} 0 & Z_1 & 0 & \cdots & 0 \\ 0 & 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & Z_{k-1} \\ Z_k & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & \tilde{Q}_1 & 0 & \cdots & 0 \\ \tilde{Q}_1 & 0 & \tilde{Q}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{Q}_{k-1} \\ \tilde{Q}_k & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \]
Then instead of products over the \( i \) indices one can use determinants over the \( k \times k \) matrices, e.g.
\[ Z = \det \mathcal{Z}. \]
In this notation
\[ \phi^1 = \det(\varphi^1 \mathcal{Z}) , \quad \phi^2 = \det(\varphi^2 \mathcal{Z}) \]
\[ \phi^3 = \det(\tilde{Q}^1) , \quad \phi^4 = \det(\tilde{Q}^2) \]
Also we define
\[ D_a \mathcal{Z} \equiv \det(D_a \mathcal{Z}). \]
These scalars together with the covariant derivative of \( Z \) will give us the basis for
constructing closed string excitation. The corresponding operators with \( \Delta - kJ = k \)
are given by
\[ a_{(q)n}^a \leftrightarrow \mathcal{O}^a_{(q)n} = \sum_{p=1}^{k} \sum_{m=1}^{J} \omega^p \left[ \text{Tr}(Z^m Z^J Z^{J-m}) \right] \omega^{-p} \ e^{2 \pi \imath p} \ e^{2 \pi \imath \omega \cdot \frac{a}{k}} , \]
\[ a_{(q)n}^{4+a} \leftrightarrow \mathcal{O}^{4+a}_{(q)n} = \sum_{p=1}^{k} \sum_{m=1}^{J} \omega^p \left[ \text{Tr}(Z^m D_a Z^{J-m}) \right] \omega^{-p} \ e^{2 \pi \imath p} \ e^{2 \pi \imath \omega \cdot \frac{a}{k}} , \]
with \( a = 1, 2, 3, 4 \). As we see the above operators have two type of indices, \( n \) and \( q \).
The \( q \) indices show the “momentum” quantum number with respect to \( \omega \), i.e.
\[ \omega \mathcal{O}^a_{(q)n} \omega^{-1} = e^{2 \pi \imath \omega \cdot \frac{a}{k}} \mathcal{O}^a_{(q)n}. \]
Note that operators \( \mathcal{O}^a_{(q)n} \) commute with \( \omega \), for any \( n \) and \( q \). Therefore index \( q \) is an internal quantum number.
The above operators are zero due to the cyclicity of the trace. Similar to the BMN case, these operators can be used to construct the excitations of the closed strings in the 8 transverse directions. This can be done by a further $\phi^a$ or $D_a Z$ insertion. For example

$$a_{(q)n}^+ a_{(q)-n}^+ \leftrightarrow O_{(q)n}^{ab} \sum_{p=1}^{k} \sum_{m=1}^{J} \omega^p \left[ \text{Tr}(\phi^a Z^m \phi^b Z^{J-m}) \right] \omega^{-p} e^{\frac{2\pi i m}{kJ}} e^{\frac{2\pi i p}{kJ}}. \quad (23)$$

It is remarkable that

$$\omega O_{(q)n}^{ab} \omega^{-1} = e^{-\frac{2\pi i q}{k}} O_{(q)n}^{ab}. \quad (24)$$

Therefore we can construct states whose $q$ momentum are non-zero, however, the sum of the other quantum number, $n$, for any closed string state should be zero, similar to the BMN case [8]. One can also write down the fermionic operators using the supersymmetry present in the theory.

As we expect, these operators have the same structure as in the $\mathcal{N} = 4$ case. However, one should note that $\Delta - kJ$ for these operators is $k$ (instead of one). This can be understood as follows:

In the Penrose limit which leads to the maximal SUSY case we are considering geometry seen by particle far from the orbifold fixed point ($\alpha = \pi/2$). In other words from that particle point of view it seems that we are working in the covering space of the orbifold and hence to compare our results with that of BMN we should rescale $p^+$ by a factor of $k$. This in particular implies that the dimension of the operators in the subsector of the theory we are focusing on, is always a multiple of $k$. More intuitively, in the BMN picture the length $R$ have been divided into $J$ pieces and since $J \sim R^2$ the momentum along this direction should be measured in $\frac{1}{R^2} \sim \frac{J}{R^2}$ units. However, in our case $kJ \sim R^2$, while still we have divided the length $R$ into $J$ pieces. Therefore the momentum units are now $\frac{kR}{J}$. To summarize, taking a fast moving particle in the $\chi$ direction breaks the Lorentz group $SO(2,4)$ to $SO(4)$ and effectively reduces the gauge group $SU(N)^k$ to $SU(N)$ while enhances the R-symmetry from $U(1)_R \times SU(2)_R$ to a global $SO(4)$ symmetry.

### 3.2 Large $k$ case

In this part we focus on the large $k$ case. Let us first consider $J = 1$ case while $k$ is scaling like $R^2$. Here the closed string worldsheet is (de)constructed from $k$-points which are $k$ equidistant points around a circle of radius $R$ (the distance between the string bits is going to zero as $\frac{1}{R}$ in the large $R$, PP-wave, limit). The operator (14) (with $J = 1$) corresponds to the vacuum of string theory. Then the stringy creation operators can be found by insertion of $\varphi_i$'s (scalars in the vectormultiplet), $\tilde{Q}_i$'s (other bi-fundamental scalars), and the covariant derivative in the sequence of $Z_i$'s in (14). More explicitly:

$$O_{q}^{1,2} = \sum_{p=1}^{k} e^{\frac{2\pi i q}{kJ}} STr(Z_1 \cdots Z_{p-1}\varphi_p Z_p \cdots Z_k)$$
\[ O_{q}^{3,4} = \sum_{p=1}^{k} e^{\frac{2\pi i p q}{k}} STr(Z_1 \cdots Z_{p-1}Z_p\tilde{Q}_pZ_pZ_{p+1} \cdots Z_k) \]

\[ O_{q}^{4+a} = \sum_{p=1}^{k} e^{\frac{2\pi i p q}{k}} STr(Z_1 \cdots Z_{p-1}D_aZ_p \cdots Z_k), \tag{25} \]

where \(a = 1, \cdots 4\). In the above \(STr\) has been defined as

\[ STr(A_1A_2 \cdots A_{p-1}B_p \cdots A_k) = \frac{1}{k} \sum_{p=1}^{k} Tr(A_1A_2 \cdots A_{p-1}B_p \cdots A_k) \]

\[ = \frac{1}{k} \sum_{p=1}^{k} \omega^p Tr(B_1A_1A_2 \cdots A_k)\omega^{-p}, \tag{26} \]

where \(\omega\) is the twist operator. The \(STr\) defined as above, guarantees the necessary "cyclicity" condition of the trace, crucial for constructing the stringy operators. Therefore the operators (25) are identically zero. Using \(STr\) is further justified noting that \(STr\) is actually the trace over the diagonal \(SU(N)\) factor and the fact that the maximal supersymmetric PP-wave limit corresponds to quiver gauge theory far from its conformal point. Identifying (14) with the string theory vacuum state gives a non-zero expectation value to the vacuum operator. This in particular implies that the original \(SU(N)^k\) gauge theory is higgsed down to the diagonal \(SU(N)\) and hence the traces should be made over that subsector only. One can proceed to construct closed string operators by insertion of another \(\varphi_i\) or \(\tilde{Q}_i\) in the \(Z_i\) chain.

Relation to space-time (de)construction

An interesting feature of our model in the large \(k\) and worldsheet deconstruction, is its close relation with space-time deconstructing dimension scenario [15]. In fact the model we are considering here is exactly the one studied in [14] which exhibits a new dimension for large \(k\) limit. In [16] it has been shown that the quiver model we have been studying here in its Higgs branch flows to a five dimensional \(SU(N)\) gauge theory. In our point of view it can be understood through T-duality where our solution will be a configuration of NS5 and wrapped D4-branes. The supergravity solution of this model, as well as its uplift to M-theory, have been considered in [20]. The theory has a Higgs phase where the gauge group \(SU(N)^k\) breaks to its diagonal \(SU(N)\) factor and hence is effectively the theory on the worldvolume of D4-branes wrapped on a circle with the radius proportional to \(k\).

To establish the relation between worldsheet and space-time deconstructions we note the usual KK expansion of a generic field of five dimensional theory reduced on a circle, i.e.

\[ A(\vec{X}, y) = \sum_{p=1}^{\infty} A_p(\vec{X})e^{ipy}, \]

where \(\vec{X}\) are parametrizing the four dimensional part and \(y\) which is ranging from 0 to \(2\pi\), the compact fifth direction. Now if we replace the circle with a discretized
version consisting of \( k \) points, \( p \) in the KK expansion ranges from 1 to \( k \) and \( y \) can be replaced with \( \frac{2\pi q}{k} \), \( q \) ranging from 1 to \( k \). In this discretized form the usual KK expansion is essentially the same as the expansion we have used to construct the stringy operators (cf. Eq.\((23))\). This could lead to a close relation between string theory on the PP-wave and this five dimensional theory. Definitely this consideration deserves more study and we are going to push this direction in our future work \( [21] \).

\( J > 1 \) case

Here we will show that \( J > 1 \) opens the possibility for strings to have “winding” modes. To visualize this, we recall the usual string windings. Let \( \sigma \) be the closed string worldsheet parameter which ranges from 0 to \( 2\pi \). Then, the winding number \( w \) is the number of times a string repeats itself along a compact direction in the \( 2\pi R \) units (\( R \) is the radius of the compact direction), i.e. \( X(2\pi) - X(0) = 2\pi R w \), where \( X \) is the string worldsheet field. Usually the winding number, unlike the momentum modes, is not associated with a (quantum) operator and basically quantization of winding number is a classical effect. However, one can think of winding to be eigenvalue of an “identity” operator \( \Omega \), where \( \Omega X(0)\Omega^{-1} = X(2\pi) \). Essentially \( \Omega \) is translation operator on the worldsheet by \( 2\pi \) and \( \Delta \Omega X = 2\pi R w \). Now let us go back to the gauge theory case. As we have shown for \( J = 1 \) case we (de)construct worldsheet from \( Z_i \) \( 1 \leq i \leq k \). In other words index \( i \) plays the role of worldsheet coordinate \( \sigma \). In particular twist operator \( \omega \) generates infinitesimal translations along the worldsheet. However, to see the role of \( J \) it is simpler to extend the range of \( i \) index from \( k \) to \( kJ \) and then in the end we re-identify \( Z_i \) with \( Z_{i+k} \). To avoid possible confusions we will denote this extended index by \( r \) \( 1 \leq r \leq kJ \). Then any \( r \) can be written as \( r = mk + q \) where \( 0 \leq m \leq J - 1 \) and \( 1 \leq q \leq k \). Similar to the previous case we use the operator \( \omega \) to move in the \( q \) part of the index \( r \), moreover we introduce the new operator \( \Omega \) which changes \( r \) in \( k \) units:

\[
\begin{align*}
\Omega Z_r \Omega^{-1} &= Z_{r+k}, \quad \Omega^J \equiv 1, \\
\omega Z_r \omega^{-1} &= Z_{r+1}, \quad \omega^k \equiv 1, \quad Z_r \equiv Z_{r+kJ}.
\end{align*}
\]

(27)

Equipped with the above operators, \( \omega, \Omega \) and the extended index \( r \), we are ready to write the proper operators we are looking for. The proper vacuum state is still given by \((14)\). This vacuum is \( J \)-fold degenerate (it is invariant under \( \Omega \)). The operators corresponding to excited strings can be defined as

\[
\mathcal{O}_\alpha^{(n)q} = \sum_{p=1}^{k} \sum_{m=0}^{J-1} STr(Z^m Z_1^{\alpha} Z^{J-m-1}) e^{2\pi ipq/k} e^{2\pi i mn J}, \quad \alpha = 1, 2, \cdots, 8,
\]

(28)

where \( Z \) is defined by \((13)\) and

\[
Z^m \equiv \prod_{s=0}^{m-1} \Omega^s Z \Omega^{-s},
\]

(29)
In the Eq.(30) $X^a_1$ can be chosen among $D_{a1}, \varphi^1_1, \varphi^2_1, \tilde{Q}^1_1$ and $\tilde{Q}^2_1$. \(STr\) is defined by (28). We should emphasize that after making all the manipulations again we set $Z_{r+k}^{(r)} \equiv Z_{r}$ (or equivalently, $\Omega$ is set to identity).

Noting the operator (28) following remarks are in order:

i) As previous cases, the \(STr\) is over the $N \times N$ matrices of the diagonal $SU(N)$ gauge factor. So, effectively the subsector of the operators we are dealing with has a single $SU(N)$ gauge symmetry.

ii) $\Omega \mathcal{O}^a_{(n)q} \Omega^{-1} = e^{-\frac{2\pi in}{J}} \mathcal{O}^a_{(n)q}$, $n = 0, 1, \cdots, J - 1$. This in particular implies that $n$ is another quantum number related to strings center of mass degrees of freedom and can be identified with the winding number.

iii) Because of the cyclicity of the trace, operator (28) is zero. The higher stringy excitations can be found by more insertions of $Z^a_1$ (operators defined in Eq.(30)).

iv) The same extension for the range of indices from $k$ to $kJ$ can also be used for the large $J$, small $k$ case. However, then the proper decomposition of index $r$ is $r = qJ + m$, $(0 \leq q < k, \ 0 \leq m < J)$.

4 Discussions

In this paper we have studied the Penrose limit of $AdS_5 \times S^5/Z_k$ which leads to the maximally supersymmetric PP-wave background in type IIB string theory. Since the PP-wave we have found from orbifold model has maximal SUSY, we would expect to find a subsector of $\mathcal{N} = 2$ $SU(N)^k$ quiver theory which exhibits the supersymmetry enhancement. In fact we have shown that there is such a subsector which is parameterized by conformal weight $\Delta$ and R-charge $kJ$ such that both $\Delta$ and $kJ$ are parameterically large while their difference $\Delta - kJ$ is finite.

The important point in this case is that the quantity which has to be taken large in the large $R$ limit is $kJ$ and not the angular momentum of the particle $J$. In fact, as far as the oscillator modes of the string theory (and also the background supergravity) are concerned $J$ and $k$ always appear as $kJ$. Therefore for large $kJ$ limit we have to different choices: we can take either $k$ large and finite $J$ or large $J$ and finite $k$.

For both cases we have studied the gauge invariant chiral operators in the quiver model and identified them with the string excitations of type IIB string theory on the maximally supersymmetric PP-wave. This subsector of the quiver model has the same structure as the one considered for $\mathcal{N} = 4$ indicating that there is a supersymmetry enhancement and furthermore the $SU(N)^k$ gauge symmetry is higgsed down to the diagonal $SU(N)$. 

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We note, however, that although the PP-wave solution looks very much like the one considered in [8], there are some differences which have to be taken into account and these differences make the quiver model have richer structure. The first difference we would like to mention is: the direction which is combined with $t$ is compact and hence the maximally supersymmetric PP-wave we find in this case has compact $x^-$ direction. Moreover, since in the gauge theory side we have two quantum numbers $k$ and $J$, the states in the string theory should have an other extra quantum number in comparison to the ordinary maximally PP-wave background studied by BMN. Therefore the operators we introduced here are labelled by two integers, the excitation and winding or momentum numbers. We have argued that in the large $J$, finite $k$ case there is an operator which commutes with all the stringy oscillators and its eigen-values (which range from 0 to $k - 1$) can be identified as momentum modes. For the large $k$, finite $J$ case, however, we have shown that the other (quantum) number, which ranges from 0 to $J - 1$, corresponds to the winding modes.

We have also discussed the relation between the deconstruction of space-time dimensions through the “theory space” [15, 16] and our worldsheet deconstruction. It would be quite interesting to make this arguments more explicit and also extend the same argument to the M-theory, similar to that of Ref. [16].

The other interesting open question we address here is the possible duality between the large $J$ and large $k$ cases. Besides the similarities in the form of the operators in these two cases (Eqs.(22) and (28)), there is the usual T-duality which exchanges winding and momentum modes. We also expect to see such a duality here, under which the $k$ and $J$ parameters are replaced while their product is kept fixed and scale like $R^2$. We hope to come back to this question in future publications.

**Note added:** While we were preparing our paper for submission we received the paper [22] where the same question has been studied.

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