Some variations of a ‘divergent’ Ramanujan-type $q$-supercongruence

Victor J. W. Guo

School of Mathematics and Statistics, Huaiyin Normal University, Huai’an, People’s Republic of China

ABSTRACT

Using the $q$-Wilf–Zeilberger method and a $q$-analogue of a ‘divergent’ Ramanujan-type supercongruence, we give several $q$-supercongruences modulo the fourth power of a cyclotomic polynomial. One of them is a $q$-analogue of a supercongruence recently proved by Wang: for any prime $p > 3$,

$$
\sum_{k=0}^{p-1} (3k - 1) \left( \frac{1}{2} \right)_k \left( -\frac{1}{2} \right)_k 4^k \equiv p - 2p^3 \pmod{p^4},
$$

where $(a)_k = a(a + 1) \cdots (a + k - 1)$ is the Pochhammer symbol.

1. Introduction

By making use of the Wilf–Zeilberger (abbr. WZ) method [33, 34], Guillera and Zudilin [4] established the following supercongruence: for any odd prime $p$,

$$
\sum_{k=0}^{(p-1)/2} \frac{3k + 1}{16^k} \left( \frac{2k}{k} \right)^3 \equiv p \pmod{p^3}.
$$

We can also sum $k$ in (1) up to $p-1$, since the $p$-adic order of $(\frac{1}{2})_k/k!$ is 1 for $k$ in the range $(p + 1)/2 \leq k \leq p - 1$. In the spirit of [36], the supercongruence (1) corresponds to a divergent Ramanujan-type series for $1/\pi$:

$$
\sum_{k=0}^{\infty} \frac{3k + 1}{16^k} \left( \frac{2k}{k} \right)^3 = -\frac{2i}{\pi}
$$

(see [4, (47)]). Here the summation in (2) must be understood as the analytic continuation of the corresponding hypergeometric series.
Still using the WZ method and the divisibility result: for \( n > 1 \),
\[
2n \binom{2n}{n} \sum_{k=0}^{n-1} (3k + 1) \binom{2k}{k}^3 16^{n-k-1},
\]
which was conjectured by Z.-W. Sun [27] and confirmed by Mao and Zhang [21], B.Y. Sun [26] proved the following result: for \( n > 1 \),
\[
2n \binom{2n}{n} \sum_{k=0}^{n-1} \frac{6k^4}{2k - 1} \binom{2k}{k}^3 16^{n-k-1}.
\]
Motivated by Sun’s work, we found the following supercongruence: for any prime \( p > 3 \),
\[
\sum_{k=0}^{p-1} \frac{6k^4}{16^k(2k - 1)} \binom{2k}{k}^3 \equiv p + 2p^3 \pmod{p^4}.
\]
We shall prove the supercongruence (5) by establishing its \( q \)-analogue. Recall that the \( q \)-shifted factorial is defined by
\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}
\]
for any number \( n \). The \( q \)-integer is defined as \([m] = [m]_q = (1 - q^m)/(1 - q)\) (see [2]), and the \( q \)-binomial coefficients are given by
\[
\left[ \begin{array}{c} M \\ N \end{array} \right] = \begin{cases} \frac{(q; q)_M}{(q; q)_N(q; q)_{M-N}}, & \text{if } 0 \leq N \leq M, \\ 0, & \text{otherwise.} \end{cases}
\]
Moreover, the \( n \)th cyclotomic polynomial \( \Phi_n(q) \) is defined as
\[
\Phi_n(q) = \prod_{1 \leq k \leq n \atop \gcd(k,n)=1} (q - \zeta^k),
\]
where \( \zeta \) is an \( n \)th primitive root of unity. Our \( q \)-analogue of (5) can be stated as follows.

**Theorem 1.1:** Let \( n > 1 \) be an odd integer. Then
\[
\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2}{[2k-1](-q; q)_k^4} \left[ \frac{2k}{k} \right]^3 q^{-(k^2+3k)/2} \equiv [n]q^{-(n+1)/2} + (1 + q)[n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24}[n]^3 q^{-(n+1)/2} \pmod{[n]\Phi_n(q)^3}.
\]
It is easy to see that
\[
\lim_{q \to 1} \left[ \frac{M}{N} \right] = \left( \frac{M}{N} \right), \quad \text{and} \quad \lim_{q \to 1} [m] = m.
\]
Thus, letting \( n = p > 3 \) be a prime and taking \( q \to 1 \) in (6), we obtain the supercongruence (5). Furthermore, we can also deduce from (6) that, for any prime \( p > 3 \) and integer \( r \geq 2 \),
\[
\sum_{k=0}^{p^r-1} \frac{6k^4}{16^k(2k-1)} \binom{2k}{k}^3 \equiv p^r \pmod{p^{r+3}}.
\]
Recently, via the WZ method and the summation package \( \Sigma \) [24], Wang [29] proved the following supercongruence: for any prime \( p > 3 \),
\[
\sum_{k=0}^{p-1} (3k - 1) \frac{(\frac{1}{2})_k (-\frac{1}{2})_k}{k!^3} 4^k \equiv p - 2p^3 \pmod{p^4}, \quad (7)
\]
where \((a)_k = a(a + 1) \cdots (a + k - 1)\) is the Pochhammer symbol. This also extends a conjectural result of the author and Schlosser [12, Conjecture 6.2].

In this paper, we shall give a \( q \)-analogue of (7) as follows:

**Theorem 1.2:** Let \( n > 1 \) be an odd integer. Then
\[
\sum_{k=0}^{n-1} [3k - 1] \frac{(q^3)_k (q^{-1})_k^2 (q^2)_k^2}{(q)_k (q^2)_k (q^2)_k} q^{(3k-k^2)/2}
\equiv n[q^{-(n+1)/2} - (1 + q)[n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24}[n]^3 q^{-(n+1)/2}] \pmod{[n]\Phi_n(q)^3}. \quad (8)
\]
Note that \( \lim_{q \to 1} (q^r; q^d)_k/(q^d; q^d)_k = (\frac{r}{d})_k/k! \). As before, letting \( n = p > 3 \) be a prime and taking \( q \to 1 \) in (8), we are led to (7). Moreover, it follows from (8) that, for any prime \( p > 3 \) and integer \( r \geq 2 \),
\[
\sum_{k=0}^{p^r-1} (3k - 1) \frac{(\frac{1}{2})_k (-\frac{1}{2})_k}{k!^3} 4^k \equiv p^r \pmod{p^{r+3}}.
\]
We shall prove Theorems 1.1 and 1.2 by making use of the \( q \)-WZ method [33, 34] and the following \( q \)-supercongruence: for odd \( n \),
\[
\sum_{k=0}^{n-1} [3k + 1] \frac{(q^3)_k q^{(k+1)}}{(q)_k (q^2)_k (q^2)_k}
\equiv q^{(1-n)/2}[n] + \frac{(n^2 - 1)(1 - q)^2}{24}q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}. \quad (9)
\]
This \( q \)-supercongruence was originally conjectured in [5] and recently proved in [7] with the help of the ‘creative microscoping’ method [15] and the Chinese reminder theorem. It
is easy to see that (1) follows from (9) by taking \( n = p \) and \( q \to 1 \). The \( n = p^r \) being an odd prime power and \( q \to 1 \) case of (9) was conjectured by Z.-W. Sun [27]. We point out that some other interesting \( q \)-supercongruences were given in [3, 5–12, 16–20, 22, 23, 25, 28, 30–32, 35].

The paper is organized as follows. We first give two lemmas in the next section. The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4, respectively. Two more similar \( q \)-supercongruences are given in Section 5. Finally, using a result of Ni and Pan [22], we give a \( q \)-analogue of (4) in Section 6.

2. Two lemmas

In this section we give three simple \( q \)-congruences. The first one may be deemed a \( q \)-analogue of Fermat’s little theorem \( 2^{p-1} \equiv 1 \pmod{p} \) for any odd prime \( p \). The third one is a generalization of a recent result of Wang and Ni [30, Lemma 2.2].

**Lemma 2.1:** Let \( n > 1 \) be an odd integer. Then

\[
(-q; q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}.
\]  

**Proof:** It is well known that

\[
\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1} (x - \zeta^k),
\]

where \( \zeta \) is an \( n \)th primitive root of unity. Letting \( x = -1 \) in (11), we get

\[
(-\zeta; \zeta)_{n-1} = 1,
\]

which is equivalent to (10).

**Lemma 2.2:** Let \( n > 1 \) be an odd integer. Then

\[
\frac{(q; q^2)_n}{(1 - q)(q; q)_{n-1}} \equiv [n] \pmod{[n] \Phi_n(q)},
\]  

\[
\frac{(q; q^2)_{n-1}}{(q; q)_{n-1}} \equiv -[n]q \pmod{[n] \Phi_n(q)}.
\]

**Proof:** It is easy to see that

\[
\frac{(q; q^2)_n}{(1 - q)(q; q)_{n-1}} = \left[\frac{2n}{n}\right] \frac{1}{(-q; q)_n},
\]

and

\[
\left[\frac{2n}{n}\right] \frac{1}{(-q; q)_n} = \frac{(q; q^2)_{(n-1)/2}(q^{n+2}; q^2)_{(n-1)/2}}{(q; q)_{n-1}}
\equiv \frac{(q; q^2)_{(n-1)/2}(q^2; q^2)_{(n-1)/2}}{(q; q)_{n-1}} \equiv 1 \pmod{\Phi_n(q)}
\]

in view of \( q^n \equiv 1 \pmod{\Phi_n(q)} \). Since \( \left[\frac{2n}{n}\right] \) is a polynomial in \( q \) and \( [n] \) is relatively prime to \((-q; q)_n\) for odd \( n \), the \( q \)-congruence (12) immediately follows from (14) and (15).
Observing that
\[
\frac{(q; q^2)_{n-1}}{(q; q)_n} = \frac{1}{[2n - 1]} \frac{(q; q^2)_n}{(1 - q)(q; q)_{n-1}},
\]
[2n − 1] = (1 − q^{2n−1})/(1 − q) ≡ −q^{−1} (mod \ Φ_n(q)),
and [2n − 1] and [n] are relatively prime polynomials in q, we deduce (13) from (12). □

3. Proof of Theorem 1.1

Define two functions \( F(n, k) \) and \( G(n, k) \) as follows:
\[
F(n, k) = [3n + 2k + 1] \frac{(q; q^2)_n(q^{2k+1}; q^2)^2 q^{-(n+1)-(2n+1)k}}{(q; q^3(q; q^2)_n)}, \tag{16}
\]
\[
G(n, k) = - \frac{(1 + q^{n+2k−1})(q; q^2)_n(q^{2k+1}; q^2)^2 q^{−n−1} q^{−(2n−1)−(2k+1)k}}{(1 − q)(q; q^2)_{n−1}q^{−n−1}q^{−(2n−1)−(2k+1)k}}. \tag{17}
\]
It is easy to check that
\[
F(n, k − 1) − F(n, k) = G(n + 1, k) − G(n, k). \tag{18}
\]
That is, the functions \( F(n, k) \) and \( G(n, k) \) form a \( q \)-WZ pair.

We now let \( m > 1 \) be an odd integer. Summing (18) over \( n \) from 0 to \( m − 1 \), we obtain
\[
\sum_{n=0}^{m−1} F(n, k − 1) − \sum_{n=0}^{m−1} F(n, k) = G(m, k) − G(0, k) = G(m, k). \tag{19}
\]
In light of (10) and (12), we have
\[
G(m, 1) = - \frac{(1 + q^{m+1})(q; q^2)_m(q^3(q; q^2))^{m-1}q^{−m/2}−2m+1}{(1 − q)(q; q)_m(q; q^3(q; q^2))^{m-1}q^{−m/2}−2m+1}.
\]
\[
= - \frac{(1 + q^{m+1})(q; q^2)_m(q^3(q; q^2))^{m-1}q^{−m/2}−2m+1}{(1 − q^3)(q; q)_3(q; q^3(q; q^2))^{m-1}q^{−m/2}−2m+1}
\]
\[
≡ - (1 + q)q[m]^3 \pmod{[m]^3Φ_m(q)}, \tag{20}
\]
since \( q^m ≡ 1 \pmod{Φ_m(q)} \). Combining (19) and (20), we conclude that
\[
\sum_{n=0}^{m−1} F(n, 1) \equiv \sum_{n=0}^{m−1} F(n, 0) + (1 + q)q[m]^3 \pmod{[m]^3Φ_m(q)}. \tag{21}
\]
It is easy to see that
\[
\sum_{n=0}^{m-1} F(n, 1) = \sum_{n=0}^{m-1} [3n + 3] \frac{(q; q^2)_n (q^2; q^2)_n q^{-(\frac{n+1}{2})} - (2n+1)}{(q; q^2)_n (q^2; q^2)_n} \\
= \sum_{n=1}^{m} [3n] \frac{(q; q^2)_{n-1} (q^2; q^2)_{n-1} q^{-(\frac{n}{2})} - (2n-1)}{(q; q^2)_{n-1} (q^2; q^2)_{n-1}} \\
= \sum_{n=1}^{m} \frac{[3n][2n][n]^2}{[2n-1]} \frac{2n^3}{n} q^{-(n^2 + 3n)/2+1}. \tag{22}
\]

On the other hand, by (9) we have
\[
\sum_{n=0}^{m-1} F(n, 0) = \sum_{n=0}^{m-1} [3n + 1] \frac{(q; q^2)_n q^{-(\frac{n+1}{2})}}{(q; q^2)_n (q^2; q^2)_n} \\
\equiv q^{(1-m)/2} [m] + \frac{(m^2 - 1)(1-q)^2}{24} q^{(1-m)/2} [m]^3 \pmod{[m] \Phi_m(q)^3}. \tag{23}
\]

Substituting (22) and (23) into (21), and noticing the \(m\)th summand on the right-hand side of (22) is congruent to 0 modulo \([m]^4\), we are led to (6) with \(n \mapsto m, k \mapsto n\) differing only by a factor \(q\).

**Remark 3.1:** Usually the basic hypergeometric functions satisfying the condition
\[
F(n, k + 1) - F(n, k) = G(n + 1, k) - G(n, k)
\]
are called a \(q\)-WZ pair. It is also reasonable to call the basic hypergeometric functions satisfying (18) a \(q\)-WZ pair (see Zudilin [36]). The \(q\)-WZ pair in the proof of Theorem 1.1 was found by the author [5] in his proof of a weaker form of (9) modulo \([n] \Phi_n(q)^2\). But the corresponding WZ pair (the limiting case \(q \to 1\)) was first given by Guillera and Zudilin [4] in their proof of (1).

**4. Proof of Theorem 1.2**

Let the functions \(F(n, k)\) and \(G(n, k)\) be given by (16) and (17), respectively. Again, let \(m\) be an odd integer greater than 1. In view of (10), (12), (13) and \(q^m \equiv 1 \pmod{\Phi_m(q)}\), we have
\[
G(m, 0) = -\frac{(1 + q^{m-1})(q; q^2)_m (q^2; q^2)_m q^{-(\frac{m}{2})}}{(1 - q)(q; q^2)_{m-1} (-q; q)_{m-1}} \\
\equiv -(1 + q^{-1}) q^2 [m] \pmod{[m]^3 \Phi_m(q)}. \tag{24}
\]

It follows from (19) and (24) that
\[
\sum_{n=0}^{m-1} F(n, -1) \equiv \sum_{n=0}^{m-1} F(n, 0) - (1 + q) q [m]^3 \pmod{[m]^3 \Phi_m(q)}. \tag{25}
\]
By the definition of (16), we have
\[ \sum_{n=0}^{m-1} F(n, -1) = \sum_{n=0}^{m-1} [3n - 1] \frac{(q; q^2)_n(q^{-1}; q^2)_n q^{-(n+1)/2 + (2n+1)}}{(q; q^2)_n(q^2; q^2)_n}. \] (26)

Finally, substituting (23) and (26) into (25), and dividing both sides by \( q \), we arrive at (8) with \( n \mapsto m, k \mapsto n \).

5. More similar \( q \)-supercongruences

From (9) and (19), we can deduce more \( q \)-supercongruences besides (6) and (8). Here we give two such examples.

**Theorem 5.1:** Let \( n > 3 \) be an odd integer. Then, modulo \([n] \Phi_n(q^3)\),
\[ \sum_{k=0}^{n-1} [3k + 5] \frac{(q; q^2)_k(q^5; q^2)_k^2}{(q; q^2)_k^2(q^2; q^2)_k} q^{-(k^2 + 9k)/2} \equiv [n] q^{5-n/2} + (1 + q)q^3 [n]^3 + \frac{(1 + q^3)q^4}{(1 + q + q^2)^2} [n]^3 + \frac{(n^2 - 1)(1 - q^2)^2}{24} [n]^3 q^{(5-n)/2}. \] (27)

**Proof:** Let \( m > 3 \) be an odd integer. Then
\[ \sum_{n=0}^{m-1} F(n, 2) = \sum_{k=0}^{m-1} [3k + 5] \frac{(q; q^2)_k(q^5; q^2)_k^2}{(q; q^2)_k^2(q^2; q^2)_k} q^{-(k^2 + 9k)/2}. \] (28)

By (19), we get
\[ \sum_{n=0}^{m-1} F(n, 2) = \sum_{n=0}^{m-1} F(n, 0) - G(m, 1) - G(m, 2). \] (29)

Moreover, in view of (12), we have
\[ G(m, 2) = -\frac{(1 + q^{m+3})(q; q^2)_m(q^5; q^2)^2_m q^{(-m)/2 - 4m+2}}{(1 - q)(q; q)_m^3(-q; q)_{m-1}} \]
\[ = -\frac{(1 - q^{2m+1})^2(1 + q^{m+3})(q; q^2)_m^3 q^{(-m)/2 - 4m+2}}{(1 - q^3)^2(1 - q)^3(q; q)_m^3(-q; q)_{m-1}} \]
\[ \equiv -\frac{(1 + q^3)q^2}{(1 + q + q^2)^2} [m]^3 \quad (\text{mod } [m] \Phi_m(q^3)). \] (30)

Substituting (20), (23), (28) and (30) into (29), we arrive at (27) with \( n \mapsto m, k \mapsto n \) differing only by a factor \( q^{-2} \).
Theorem 5.2: Let \( n > 3 \) be an odd integer. Then, modulo \( [n]\Phi_n(q)^3 \),
\[
\sum_{k=0}^{n-1} [3k - 3] \frac{(q; q^2)_k(q^{-3}; q^2)_k^2}{(q; q)_k^2(q^2; q^2)_k} q^{(7k-k^2)/2} \equiv [n]q^{-(n+3)/2} - \frac{1 + q}{q} [n] - \frac{1 + q^3}{(1 + q + q^2)^2} [n]^3 + \frac{(n^2 - 1)(1 - q)^2}{24} [n]^3 q^{-(n+3)/2}.
\]

Proof: The proof is similar to that of Theorem 5.1. This time we need to use
\[
\sum_{n=0}^{m-1} F(n, -2) = \sum_{n=0}^{m-1} F(n, 0) + G(m, 0) + G(m, -1),
\]
and
\[
G(m, -1) = -\frac{(1 + q^{m-3})(q; q^2)_m(q^{-1}; q^2)_m q^{-(m^2)/2} + 2m-1}{(1 - q)(q; q)_{m-1}(-q; q)_{m-1}} \equiv -\frac{(1 + q^3)q^2}{(1 + q + q^2)^2} [m] \quad (\text{mod } [m]\Phi_m(q)^3).
\]

6. A \( q \)-an analogue of (4)

A \( q \)-analogue of (3) was conjectured by the author [5, Conjecture 1.7], and was recently confirmed by the author and Wang [14]. In this section we shall give a \( q \)-analogue of (4). To this end, we need some lemmas. The first one is a special case of [22, Lemma 3.2].

Lemma 6.1: Let \( s \) and \( t \) be non-negative integers with \( 0 \leq t \leq d-1 \). Then
\[
\frac{(q; q^2)_{sd+t}}{(q^2, q^2)_{sd+t}} \equiv \frac{1}{4^s} \binom{2s}{s} \left( \frac{q}{q^2} \right)^t \quad (\text{mod } \Phi_d(q)).
\]

The second one is the following \( q \)-congruence.

Lemma 6.2: Let \( d \) be a positive odd integer. Let \( s \) and \( t \) be non-negative integers. Then
\[
(-q; q)_{sd+t} \equiv 2^s (-q; q)_t \quad (\text{mod } \Phi_d(q)).
\]

Proof: By Lemma 2.1, we have \( (-q; q)_d \equiv 2 \quad (\text{mod } \Phi_d(q)) \) [13]. The proof then follows from the equality \( (-q; q)_{sd+t} = (-q^{d+1}; q)_t \prod_{j=0}^{s-1} (-q^{jd+1}; q)_d \) and the \( q \)-congruence \( q^d \equiv 1 \quad (\text{mod } \Phi_d(q)) \).
To state the third lemma, we have to introduce some notation. For any positive integer \( n \), let

\[
S(n) = \left\{ d \geq 3 : d \text{ is odd and } \left\lfloor \frac{n - \frac{d+1}{2}}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},
\]

where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). It is clear that, for any integer \( d > 2n - 1 \), we have \((d + 1)/2 > n\), and so \( d \notin S(n) \). That is, the cardinality of \( S(n) \) is finite. Let

\[
A_n(q) = \prod_{d \in S(n)} \Phi_d(q), \quad C_n(q) = \prod_{d \mid n, d > 1, \text{\( d \) is odd}} \Phi_d(q),
\]

It is easily seen that, if \( d \mid n \), then \( d \notin S(n) \). Hence, the polynomials \( A_n(q) \) and \( C_n(q) \) are relatively prime. We need the following result, which is a special case of [22, Theorem 2.1].

**Lemma 6.3:** Let \( v_0(q), v_1(q), \ldots \) be a sequence of rational functions in \( q \), such that, for any odd integer \( d > 1 \),

(i) \( v_k(q) \) is \( \Phi_d(q) \)-integral for each \( k \geq 0 \), i.e., the denominator of \( v_k(q) \) is relatively prime to \( \Phi_d(q) \);
(ii) for any non-negative integers \( s \) and \( t \) with \( t \leq d - 1 \),

\[
v_{s,d+t}(q) \equiv \mu_s(q) v_t(q) \pmod{\Phi_d(q)},
\]

where \( \mu_s(q) \) is a \( \Phi_d(q) \)-integral rational function only dependent on \( s \);
(iii)

\[
\sum_{k=0}^{d-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} v_k(q) \equiv 0 \pmod{\Phi_d(q)}.
\]

Then, for all positive integers \( n \),

\[
\sum_{k=0}^{n-1} \frac{(q; q^2)_k}{(q^2; q^2)_k} v_k(q) \equiv 0 \pmod{A_n(q)C_n(q)}. \tag{32}
\]

For Sun’s divisibility result (4), we find that the following stronger version holds:

\[
4n \binom{2n}{n} \left| \sum_{k=0}^{n-1} \frac{6k^4}{2k-1} \binom{2k}{k}^3 16^{n-k-1} \right. . \tag{33}
\]

Here we shall prove the following \( q \)-analogue of (33).

**Theorem 6.1:** Let \( n > 1 \) be an integer. Then, modulo \( (1 + q^{n-1})^3 [n]_{\binom{2n-1}{n-1}} \),

\[
\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2}{[2k-1]} \left[ \frac{2k^3}{k} \right] (-q^{k+1}; q)_{n-k-1}^4 q^{-(k^2 + 3k)/2} \equiv 0. \tag{34}
\]
Note that $\frac{1}{[2k-1]} \left[ \begin{array}{c} 2k \\ k \end{array} \right]$ is always a polynomial in $q$ with integer coefficients.

**Proof:** It is clear that, for $0 \leq k \leq n-2$, we have

$$(-q^{k+1}; q)_{n-k-1} \equiv 0 \pmod{1 + q^{n-1}}.$$ 

Moreover, for $k = n-1$, there holds $\left[ \begin{array}{c} 2k \\ k \end{array} \right] \equiv 0 \pmod{1 + q^{n-1}}$. This means that the left-hand side of (34) is divisible by $(1 + q^{n-1})^4$.

We now need to prove that the left-hand side of (34) is divisible by $[n]\left[ \begin{array}{c} 2n-1 \\ n-1 \end{array} \right]$. It is well known that the $q$-binomial coefficient $\left[ \begin{array}{c} n \\ k \end{array} \right]$ can be factorized into (see [1, Lemma 1]):

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \prod_{d \in D_{n,k}} \Phi_d(q),$$

where

$$D_{n,k} := \left\{ d \geq 2 : \left\lfloor \frac{k}{d} \right\rfloor + \left\lfloor \frac{n-k}{d} \right\rfloor \leq \left\lfloor \frac{n}{d} \right\rfloor \right\}.$$ 

It is not hard to see that $1 < d \in D_{2n-1,n-1}$ is odd if and only if $d \in S(n)$, and so

$$[n]\left[ \begin{array}{c} 2n-1 \\ n-1 \end{array} \right] = A_n(q)C_n(q) \prod_{d|n \atop d \geq 2 \text{ even}} \Phi_d(q) \cdot \prod_{d \in S(n) \atop d \text{ is even}} \Phi_d(q).$$

For $k \geq 0$, let

$$v_k(q) = \frac{[3k][2k][k]^2(q; q^2)^2_k(-q; q^2)_k^2}{[2k-1](q^2; q^2)_k^2} q^{-(k^2+3k)/2}.$$ 

In view of Lemmas 6.1 and 6.2, for any positive odd integer $d$ and non-negative $s$ and $t$ with $t \leq d-1$, we have

$$v_{sd+t}(q) = \frac{[3(sd+t)][2(sd+t)][sd+t]^2(q; q^2)_{sd+t}^2(-q; q^2)_{sd+t}^2}{[2(sd+t)-1](q^2; q^2)_{sd+t}^2} q^{-((sd+t)^2+3(sd+t))/2}$$

$$\equiv \frac{1}{4^s} \binom{2s}{s}^2 v_t(q) \pmod{\Phi_d(q)}.$$ 

Moreover, by (6), we get

$$\sum_{k=0}^{d-1} \frac{[3k][2k][k]^2(q; q^2)_k^3(-q; q^2)_k^2}{[2k-1](q^2; q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{\Phi_d(q)}.$$ 

Therefore, applying Lemma 6.3, we conclude that

$$\sum_{k=0}^{n-1} \frac{[3k][2k][k]^2(q; q^2)_k^3(-q; q^2)_k^2}{[2k-1](q^2; q^2)_k^3} q^{-(k^2+3k)/2} \equiv 0 \pmod{A_n(q)C_n(q)}.$$ 

(35)

Multiplying the left-hand side of (35) by $(-q; q^4)_{n-1}$, and noticing the relation

$$\frac{(q; q^2)_k}{(q^2; q^2)_k} (-q; q^2)_{n-1} = \left[ \begin{array}{c} 2k \\ k \end{array} \right] (-q^{k+1}; q)_{n-k-1},$$

we see that (34) is true modulo $A_n(q)C_n(q)$. 
It remains to show that (34) is also true modulo
\[
\prod_{d \mid n \atop d \geq 2 \text{ is even}} \Phi_d(q) \cdot \prod_{d \in D_{2n-1,n-1} \atop d \text{ is even}} \Phi_d(q).
\]

Firstly, let \( d \mid n \) be an even integer. Then
\[
1 + q^{d/2} = \frac{1 - q^d}{1 - q^{d/2}} \equiv 0 \pmod{\Phi_d(q)}.
\]

It follows that, for \( 0 \leq k < n - d/2 \), the \( q \)-shifted factorial \((-q^{k+1}; q)_{n-k-1}\) incorporates the factor \( 1 + q^{n-d/2} \) and is therefore congruent to 0 modulo \( \Phi_d(q) \). On the other hand, for \( n - d/2 \leq k \leq n-1 \), we have \( d \in D_{2k,k} \), i.e.
\[
\left[ \begin{array}{c} 2k \\ k \end{array} \right] \equiv 0 \pmod{\Phi_d(q)}. \tag{36}
\]

Thus, for \( 0 \leq k \leq n - 1 \), we always have
\[
\left[ \begin{array}{c} 2k \\ k \end{array} \right] (-q^{k+1}; q)_{n-k-1} \equiv 0 \pmod{\Phi_d(q)}. \tag{37}
\]

This means that (34) is true modulo \( \prod_{d \mid n \atop d \geq 2 \text{ is even}} \Phi_d(q) \). Secondly, let \( d \in D_{2n-1,n-1} \) be even. Write \( n = ud + v \) with \( 0 \leq v \leq d - 1 \). Then \( v > d/2 \), and so the polynomial \((-q^{k+1}; q)_{n-k-1}\) contains the factor \( 1 + q^{ud+d/2} \) (which is divisible by \( \Phi_d(q) \)) for \( 0 \leq k < ud + d/2 \). Moreover, for \( ud + d/2 \leq k \leq n - 1 \), we have \( d \in D_{2k,k} \), i.e. (36) holds. Therefore, for \( 0 \leq k \leq n - 1 \), the \( q \)-congruence (37) always holds. This proves that (34) is true modulo \( \prod_{d \in D_{2n-1,n-1} \atop d \text{ is even}} \Phi_d(q) \).

Since \([n]\left[\begin{array}{c} 2n-1 \\ n-1 \end{array}\right] = (1 + q^{n-1})[2n - 1]\left[\begin{array}{c} 2n-3 \\ n-2 \end{array}\right]\), and \((1 + q^{n-1})\) is relatively prime to \([2n - 1]\left[\begin{array}{c} 2n-3 \\ n-2 \end{array}\right]\), the least common multiple of \((1 + q^{n-1})^4\) and \([n]\left[\begin{array}{c} 2n-1 \\ n-1 \end{array}\right]\) is just \((1 + q^{n-1})^3[n]\left[\begin{array}{c} 2n-1 \\ n-1 \end{array}\right]\). This proves the theorem.

Acknowledgments

The author thanks the anonymous referees for their careful readings of the paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was partially supported by the National Natural Science Foundation of China (grant 11771175).

ORCID

Victor J. W. Guo  http://orcid.org/0000-0002-4153-715X
References

[1] W.Y.C. Chen and Q.-H. Hou, *Factors of the Gaussian coefficients*, Discrete Math. 306 (2006), pp. 1446–1449.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, 2004.

[3] C.-Y. Gu and V.J.W. Guo, *Two q-congruences from Carlitz’s formula*, Period. Math. Hungar. 82 (2021), pp. 82–86.

[4] J. Guillera and W. Zudilin, “Divergent” Ramanujan-type supercongruences, Proc. Am. Math. Soc. 140 (2012), pp. 765–777.

[5] V.J.W. Guo, *q-Analogues of two “divergent” Ramanujan-type supercongruences*, Ramanujan J. 52 (2020), pp. 605–624.

[6] V.J.W. Guo, *Proof of some q-supercongruences modulo the fourth power of a cyclotomic polynomial*, Results Math. 75 (2020), Art. 77.

[7] V.J.W. Guo, *q-Supercongruences modulo the fourth power of a cyclotomic polynomial via creative microscoping*, Adv. Appl. Math. 120 (2020), Art. 102078.

[8] V.J.W. Guo, *Proof of a generalization of the (C.2) supercongruence of Van Hamme*, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 115 (2021), Art. 45.

[9] V.J.W. Guo, A further q-analogue of Van Hamme’s (H.2) supercongruence for primes $p \equiv 3 \pmod{4}$, Int. J. Number Theory (in press). Available at https://doi.org/10.1142/S1793042121500329.

[10] V.J.W. Guo and M.J. Schlosser, *Proof of a basic hypergeometric supercongruence modulo the fifth power of a cyclotomic polynomial*, J. Difference Equ. Appl. 25 (2019), pp. 921–929.

[11] V.J.W. Guo and M.J. Schlosser, *A new family of q-supercongruences modulo the fourth power of a cyclotomic polynomial*, Results Math. 75 (2020), Art. 155.

[12] V.J.W. Guo and M.J. Schlosser, *Some q-supercongruences from transformation formulas for basic hypergeometric series*, Constr. Approx. 53 (2021), pp. 155–200.

[13] V.J.W. Guo and S.-D. Wang, *Some congruences involving fourth powers of central q-binomial coefficients*, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020), pp. 1127–1138.

[14] V.J.W. Guo and S.-D. Wang, *Factors of certain sums involving central q-binomial coefficients*, preprint (2021), arXiv:2102.01861.

[15] V.J.W. Guo and W. Zudilin, *A q-microscope for supercongruences*, Adv. Math. 346 (2019), pp. 329–358.

[16] V.J.W. Guo and W. Zudilin, *Dwork-type supercongruences through a creative q-microscope*, J. Combin. Theory, Ser. A 178 (2021), Art. 105362.

[17] L. Li, *Some q-supercongruences for truncated forms of squares of basic hypergeometric series*, J. Difference Equ. Appl. 27 (2021), pp. 16–25.

[18] L. Li and S.-D. Wang, *Proof of a q-supercongruence conjectured by Guo and Schlosser*, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 114 (2020), Art. 190.

[19] J.-C. Liu, *On a congruence involving q-Catalan numbers*, C. R. Math. Acad. Sci. Paris 358 (2020), pp. 211–215.

[20] J.-C. Liu and F. Petrov, *Congruences on sums of q-binomial coefficients*, Adv. Appl. Math. 116 (2020), Art. 102003.

[21] G.-S. Mao and T. Zhang, *Proof of Sun’s conjectures on super congruences and the divisibility of certain binomial sums*, Ramanujan J. 50 (2019), pp. 1–11.

[22] H.-X. Ni and H. Pan, *Divisibility of some binomial sums*, Acta Arith. 194 (2020), pp. 367–381.

[23] H.-X. Ni and H. Pan, *Some symmetric q-congruences modulo the square of a cyclotomic polynomial*, J. Math. Anal. Appl. 481 (2020), Art. 123372.

[24] C. Schneider, *Symbolic summation assists combinatorics*, Sémin. Lothar. Comb. 56 (2007), p. B56b.

[25] A. Straub, *Supercongruences for polynomial analogs of the Apéry numbers*, Proc. Am. Math. Soc. 147 (2019), pp. 1023–1036.

[26] B.Y. Sun, *Note on a new divisibility property of binomial sums*, Util. Math. 107 (2018), pp. 131–135.
[27] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. 54 (2011), pp. 2509–2535.

[28] R. Tauraso, *Some q-analogs of congruences for central binomial sums*, Colloq. Math. 133 (2013), pp. 133–143.

[29] C. Wang, *Symbolic summation methods and hypergeometric supercongruences*, J. Math. Anal. Appl. 488 (2020), Art. 124068.

[30] C. Wang and H.-X. Ni, Some q-congruences arising from certain identities, to appear in Period. Math. Hungar.

[31] X. Wang and M. Yue, *A q-analogue of the (A.2) supercongruence of Van Hamme for any prime p ≡ 3 (mod 4)*, Int. J. Number Theory 16 (2020), pp. 1325–1335.

[32] X. Wang and M. Yue, *Some q-supercongruences from Watson’s 8φ7 transformation formula*, Results Math. 75 (2020), Art. 71.

[33] H.S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities*, Invent. Math. 108 (1992), pp. 575–633.

[34] H.S. Wilf and D. Zeilberger, *Rational function certification of multisum/integral/“q” identities*, Bull. Am. Math. Soc. (N.S.) 27 (1992), pp. 148–153.

[35] W. Zudilin, *Congruences for q-binomial coefficients*, Ann. Combin. 23 (2019), pp. 1123–1135.

[36] W. Zudilin, *Ramanujan-type supercongruences*, J. Number Theory 129 (2009), pp. 1848–1857.