EXISTENCE OF NONNEGATIVE SOLUTIONS TO SINGULAR ELLIPTIC PROBLEMS, A VARIATIONAL APPROACH

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Abstract. We consider the problem
\[-\Delta u = g(.,u) + f(.,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad u \geq 0 \text{ in } \Omega,\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(f : \Omega \times [0, \infty) \to \mathbb{R}\) and \(g : \Omega \times (0, \infty) \to [0, \infty)\) are Carathéodory functions, with \(g(x,.)\) nonnegative, nonincreasing, and singular at the origin. We establish sufficient conditions for the existence of a nonnegative weak solution \(0 \neq u \in H^1_0(\Omega)\) to the stated problem. We also provide conditions that guarantee that the found solution is positive a.e. in \(\Omega\). The problem with a parameter \(\Delta u = g(.,u) + \lambda f(.,u)\) in \(\Omega, \quad u = 0 \text{ on } \partial \Omega, \quad u \geq 0 \text{ in } \Omega\) is also studied. For both problems, the special case when \(g(x,s) := a(x)s^{-\alpha(x)}\), i.e., a singularity with variable exponent, is also considered.

1. Introduction and statement of the problem. Consider the semilinear elliptic problem
\[
\begin{cases}
-\Delta u = g(.,u) + f(.,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{in } \Omega,
\end{cases}
\]
and the related problem with a parameter \(\lambda, \quad \begin{cases}
-\Delta u = g(.,u) + \lambda f(.,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{in } \Omega,
\end{cases}
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\); \(\lambda \in \mathbb{R}\), \(f, g\) are functions defined on \(\Omega \times [0, \infty)\) and \(\Omega \times (0, \infty)\) respectively; and \(g(x,s)\) is allowed to be singular at \(s = 0\).

Singular problems like (1) and (2) appear in applications to chemical catalysts process, non-Newtonian fluids, and in models for the temperature of electrical conductors (see e.g., [9], [5], [15], [17] and the references therein). Problem (1) was treated, when \(g(x,s) = a(x)s^{-\alpha}\) and \(f = 0\), and under various assumptions on \(a\) and \(\alpha\), in [17], [15], [10], [24], [13] and [2]. Problem (2) was studied in [27] with \(g(x,s) = a(x)s^{-\alpha}\) and \(f(.,s) = s^p\), \(\alpha\) and \(p\) in \((0,1)\), \(a\) regular enough. The bifurcation problem \(-\Delta u = g(.,u) + f(.,\lambda u)\) in \(\Omega, \quad u = 0 \text{ on } \partial \Omega, \quad u > 0 \text{ in } \Omega\), was treated in [8] when \(g(.,u) = au^{-\alpha}, \quad a \in C^1(\Omega), \quad a > 0 \text{ in } \Omega\), and \(f \in C^1(\Omega \times [0, \infty))\).

Multi-parameter singular bifurcation problems of the form \(-\Delta u = g(.,u) + \lambda |\nabla u|^p + \mu f(.,u)\) in \(\Omega, \quad u = f \text{ on } \partial \Omega, \quad u > 0 \text{ in } \Omega\), were studied in [20]. In [16] existence and

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nonexistence theorems were stated for Lane–Emden–Fowler equations with convection and singular potential. In [26] existence, nonexistence, and uniqueness theorems were stated for blow-up boundary solutions of logistic equations, and for singular Lane–Emden–Fowler equations with convection term. In [1] existence and uniqueness results were obtained for mild solutions of singular initial value parabolic problems involving the p-Laplacian. The existence of singular subsolutions to singular problems on a punctured ball was studied in [19].

In [7], Cirstea, Ghergu and Rădulescu considered the problem \(-\Delta u = a(x) h(u) + \lambda f(u)\) in \(\Omega\), \(u = 0\) on \(\partial \Omega\), \(u > 0\) in \(\Omega\), when \(\Omega\) is a regular enough domain, \(0 < f \in C^{0,\varepsilon}[0,\infty)\) and \(0 < h \in C^{0,\varepsilon}(0,\infty)\) for some \(\varepsilon \in (0,1)\), \(f\) nondecreasing, \(f(s)/s\) nonincreasing, \(h\) nonincreasing, \(\lim_{s \to 0^+} h(s) = +\infty\). Additionally, they assume that there exist \(\alpha \in (0,1)\), \(\sigma_0 > 0\), and \(c > 0\), such that \(h(s) \leq cs^{-\alpha}\) for \(s \in (0,\sigma_0)\). Under these assumptions they proved existence results for classical solutions in \(C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})\).

Multiplicity results for positive solutions of singular elliptic Dirichlet problems were obtained in [18] and [25]; in both articles the singular term of the considered nonlinearity has the form \(a(x) s^{-\alpha}\), with \(0 \leq a \in L^\infty(\Omega)\), \(a \neq 0\) in \(\Omega\), and \(\alpha\) positive.

Recently, problem (1) has been addressed by Chu, Gao and Gao [6], under the assumptions \(g(x,s) = a(x) s^{-\alpha(x)}\), with \(0 < \alpha \in C(\overline{\Omega})\), satisfying some suitable hypothesis, and \(f(x,s) = 0\) (i.e., singular nonlinearity with a variable exponent).

In [11], existence of solutions to the following free boundary singular bifurcation problems was studied: \(-\Delta u = \chi_{\{u > 0\}} (\alpha + \lambda f(u))\) in \(\Omega\), \(u = 0\) on \(\partial \Omega\), \(u \geq 0\) in \(\Omega\), \(u \neq 0\) in \(\Omega\), where, for \(h : \Omega \times (0,\infty) \to \mathbb{R}\), \(\chi_{\{s > 0\}} h(x,s)\) stands for the function, defined on \(\Omega \times [0,\infty)\), given by \(\chi_{\{s > 0\}} h(x,s) := h(x,s)\) if \(s > 0\), and \(\chi_{\{s > 0\}} h(x,s) := 0\) if \(s = 0\). Here and below, \(u \neq 0\) in \(\Omega\) will mean \(|\{x \in \Omega : u(x) \neq 0\}| > 0\). They assume \(0 < \alpha < 1\), \(\lambda > 0\), and that \(f : \Omega \times [0,\infty) \to [0,\infty)\) is a Carathéodory function \(f = f(x,s)\) that, a.e. \(x \in \Omega\), is nondecreasing and concave in \(s\), and satisfies \(\lim_{s \to \infty} f(x,s)/s = 0\) uniformly on \(x \in \Omega\).

For additional references, and a systematic study of singular problems, we refer the reader to [21], [26], see also [14].

The aim of this work is to prove, under suitable rather general hypothesis on \(g\), and for a wide class of sublinear (at \(\infty\)) functions \(f\), existence results for nonnegative weak solutions (which may be zero on a subset of \(\Omega\) with positive measure) to the following analogous of problems (1) and (2):

\[
\begin{aligned}
-\Delta u &= \chi_{\{u > 0\}} g(u) + f(u) \quad \text{in} \quad \Omega, \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega, \\
\quad u &\geq 0 \quad \text{in} \quad \Omega, \quad u \neq 0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta u_\lambda &= \chi_{\{u_\lambda > 0\}} g(u_\lambda) + \lambda f(u_\lambda) \quad \text{in} \quad \Omega, \\
\quad u_\lambda &= 0 \quad \text{on} \quad \partial \Omega, \\
\quad u_\lambda &\geq 0 \quad \text{in} \quad \Omega, \quad u_\lambda \neq 0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with \(C^{1,1}\) boundary; \(\lambda \in \mathbb{R}\), \(f, g\) are functions defined on \(\Omega \times [0,\infty)\) and \(\Omega \times (0,\infty)\) respectively; with \(s \to g(x,s)\) singular at the origin for \(x\) in a subset of \(\Omega\) with positive measure.

By a weak solution of (3) we mean a solution in the sense of the following definition:
Definition 1.1. Let \( h : \Omega \rightarrow \mathbb{R} \) be a measurable function such that \( h\varphi \in L^1(\Omega) \) for all \( \varphi \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \). We say that \( u : \Omega \rightarrow \mathbb{R} \) is a weak solution to the problem

\[
\begin{cases}
-\Delta u = h \text{ in } \Omega, \\
u = 0 \text{ on } \partial\Omega
\end{cases}
\]

if \( u \in H^1_0(\Omega) \), and \( \int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega h\varphi \) for all \( \varphi \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \). Similarly, we say that

\[
\begin{cases}
-\Delta u \geq h \text{ in } \Omega \text{ (respectively } -\Delta u \leq h \text{ in } \Omega), \\
u = 0 \text{ on } \partial\Omega
\end{cases}
\]

if \( u \in H^1_0(\Omega) \), and \( \int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega h\varphi \) (resp. \( \int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega h\varphi \)) for all nonnegative \( \varphi \) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \).

Let us recall that a function \( h : \Omega \times [0, \infty) \rightarrow \mathbb{R} \) (or \( h : \Omega \times (0, \infty) \rightarrow \mathbb{R} \)) is called a Carathéodory function if \( h \) is measurable for any \( s \in [0, \infty) \), and \( h(\cdot, s) \) is continuous a.e. \( x \in \Omega \).

The following conditions \((g_1)-(g_5)\) and \((f_1)-(f_3)\) will be assumed throughout the paper, except in Theorem 1.3 where a weaker condition \((g'_2)\) will be imposed; and also in Theorem 1.6. Additional conditions on \( f \) will be required in Theorem 1.2, and in Theorems 1.3-1.6.

\((g_1)\) \( g : \Omega \times (0, \infty) \rightarrow [0, \infty) \) is a Carathéodory function, and \( g(\cdot, s) \in L^\infty(\Omega) \) for any \( s > 0 \).

\((g_2)\) \( s \rightarrow g(x, s) \) is non increasing on \([0, \infty)\) a.e. \( x \in \Omega \).

\((g_3)\) There exists a positive constant \( \beta \) such that the following inequality holds a.e. \( x \in \Omega \):

\[
g(x, s) \leq c (g(x, 2s) + 1) \text{ for all } s > 0
\]

\((g_4)\) \( D_M \neq \emptyset \) for any \( M > 0 \), where

\[
D_M := \left\{ u \in H^1_0(\Omega) : 0 \leq u \leq M \text{ and } \int_\Omega |G_M(x, u(x))| \ dx < \infty \right\},
\]

and \( G_M : \Omega \times [0, \infty) \rightarrow R \cup \{-\infty\} \) is defined by \( G_M(x, s) := \int_M^s g(x, \sigma) \ d\sigma \).

\((g_5)\) The set \( \Omega_\infty := \{ x \in \Omega : \lim_{s \to \infty} g(x, s) = \infty \} \) has positive measure.

\((f_1)\) \( f : \Omega \times [0, \infty) \rightarrow \mathbb{R} \) is a Carathéodory function.

\((f_2)\) \( \max_{0 \leq s \leq M} |f(\cdot, s)| \in L^1(\Omega) \) for any \( M > 0 \).

\((f_3)\) \( f(\cdot, 0) \geq 0 \).

As we will see in Remark 2.4, conditions \((g_2)\) and \((g_3)\) imply that, for any \( s_0 > 0 \), there exist positive constants \( A \) and \( \beta \) such that, a.e. in \( \Omega \), we have

\[
g(\cdot, s) \leq As^{-\beta} \text{ for any } s \in (0, s_0]. \tag{5}
\]

Recall that \( \lambda \in \mathbb{R} \) is called a principal eigenvalue for \( -\Delta \) in \( \Omega \), with homogeneous Dirichlet boundary condition and weight function \( b \in L^\infty(\Omega) \), if the problem \( -\Delta \phi = \lambda b\phi \) in \( \Omega \), \( \phi = 0 \) on \( \partial\Omega \), has a solution \( \phi \) (called a principal eigenfunction) such that \( \phi > 0 \) in \( \Omega \). It is a well known fact that if \( b^+ \neq 0 \) then this problem has a unique positive principal eigenvalue, denoted by \( \lambda_1(b) \) (see e.g., Remark 2.8 below and the references therein).

Let us state our results:

Theorem 1.2. Assume that \( f \) and \( g \) satisfy \((f_1)-(f_3)\) and \((g_1)-(g_5)\). Let \( s_0, A, \) and \( \beta \) be positive constants as in \((5)\); and define \( m := \max(2, 1 + \beta) \). Assume also that the following condition holds:

...
Theorem 1.2 remains valid if the hypothesis $\leq f$ by the following:

Let $g$ be defined by

$$
\text{Remark 1. Let us observe that (4) holds for any } \sigma > 0 \text{ if } \sup_{x > 0} \frac{f(x,s)}{s} \leq b \text{ for some } b \in L^\infty(\Omega) \text{ such that } b^+ \neq 0 \text{ and } \lambda_1(b) > m. \text{ In particular, (4) holds if } f \leq 0.
$$

Theorem 1.3. Theorem 1.2 remains valid if the hypothesis (f4) of Theorem 1.2 holds with $b$ satisfying, in addition, $\text{ess inf}_\Omega b > 0$; and the condition (g2) is replaced by the following:

$$
\text{Remark 2. Let us stress that the strength of the singularity has to be limited if there exists a unique solution } u \in H^1_0(\Omega) \cap L^\infty(\Omega) \text{. Additionally, the results in i) hold for any } \lambda \in (0, \infty) \text{ if, in addition, } \lambda \text{ is replaced by } \lambda^* \text{ and } \text{ess inf}_\Omega b > 0 \text{ uniformly on } \Omega.
$$

Theorem 1.4. If (g1)-(g4), (f1)-(f3), and (f4) are satisfied, then

i) For all $\lambda \geq 0$, (4) has a nonnegative weak solution $u_\lambda$ (in the sense of Definition 1.1); moreover $u_\lambda \in L^\infty(\Omega)$.

ii) The results in i) hold for any negative $\lambda$ if, in addition, $\text{ess inf}_\Omega b > 0$, $f(.,0) = 0$ a.e. in $\Omega$, and $\lim_{s \to \infty} \frac{f(x,s)}{s} = 0$ uniformly on $\Omega$.

iii) For $\lambda \geq 0$, the conclusions i) and ii) of Theorem 1.2 hold with $u$ replaced by $u_\lambda$.

Theorem 1.5. Assume (g1)-(g4), (f1)-(f3), and one of the two following conditions:

\begin{itemize}
  \item [(f5)] $\sup_{x > 0} \text{ess sup}_{x \in \Omega} \frac{f(x,s)}{s} < \infty$.
  \item [(f6)] $f \in L^\infty(\Omega \times (0, \sigma))$ for any $\sigma > 0$.
\end{itemize}

Then there exists $\lambda^* > 0$ such that, for any nonnegative $\lambda < \lambda^*$, (4) has a weak solution (in the sense of Definition 1.1) $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$.

In the next theorem we consider the case when the singular part of the nonlinearity has the form $g(x,s) = a(x)s^{-\alpha(x)}$ (i.e., a singularity with variable exponent).

Theorem 1.6. Let $g : \Omega \times (0, \infty) \to [0, \infty)$ be defined by $g(x,s) := a(x)s^{-\alpha(x)}$, where $0 \leq a \in L^\infty(\Omega)$, and $\alpha : \Omega \to \mathbb{R}$ is a nonnegative measurable function such that $\sigma_\lambda := \text{ess sup}_{x \in \Omega} \alpha(x) < 3$ and $|\{x \in \Omega : \alpha(x) > 0\}| > 0$. Then:

i) $g$ satisfies conditions (g1)-(g5), and (g6).

ii) The conclusions of Theorem 1.2 and Theorems 1.3-1.6 hold whenever $f : \Omega \times [0, \infty) \to \mathbb{R}$ satisfies the hypothesis of those theorems.

Remark 2. Let us stress that the strength of the singularity has to be limited if one expects weak solutions in $H^1_0(\Omega)$. Indeed, Lazer and McKenna [24] considered the problem $-\Delta u = au^{-\alpha}$ in $\Omega$, $u = 0$ on $\partial\Omega$, $u > 0$ in $\Omega$, under the assumptions $a \in C^1(\Omega)$, $\text{min}_{\Omega} a > 0$, $\alpha > 0$, and $\Omega$ a bounded regular domain. They proved that there exists a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$; and that $u \in H^1_0(\Omega)$ if, and only if, $\alpha < 3$. A clear-cut simple condition like that is elusive when dealing with a more general singular function $g$; condition (g4) is our core hypothesis in this regard. In examples 1 and 2 of the Section 3, we apply Theorem 1.2 to show that
it is not necessary that (5) holds for some \( \beta \in (0, 3) \) for a weak solution in \( H_0^1(\Omega) \) to exist.

**Remark 3.** Let us recall that, under the assumptions of Theorem 1.2, it is possible that no strictly positive solution exist, whereas a nonnegative solution does exist (see ([22], Example 3.7)).

The existence of nonnegative solutions for restricted versions of problem (3) was investigated in [22] and in [23]. In [22], existence results were obtained for the case \( g(.,u) = au^{-\alpha} \), with \( 0 \leq a \in L^\infty(\Omega) \), \( a \neq 0 \), \( 0 < \alpha < 1 \), and \( f(.,u) = -bu^p \), with \( 0 < p < \frac{\alpha + 2}{\alpha - 2} \), and \( 0 \leq b \in L^r(\Omega) \) for suitable values of \( r \). Those results were extended in [23] (where problem (4) was also considered) to include the case \( 0 < \alpha < 3 \), with \( f \) satisfying conditions similar to those assumed here. Our objective in this manuscript is to obtain similar results for more general singular nonlinearities (including, for example, singular nonlinearities with variable exponent \( g(x,u) = a(x)u^{-\alpha(x)} \)). In order to achieve our objective, in this work we improve the variational approach introduced in [22] and [23]. As in those works, a major obstacle is posed by the fact that, because of the singularity, the energy functional may not be Gateaux differentiable, which prevents the variational method from being applied in its standard form. A further challenge is posed by the fact that the domain of the energy functional is not an open subset of \( H_0^1(\Omega) \). Additionally, expressing \( G_M \) in terms of elementary functions was possible in [22] and in [23], but not here. To overcome these difficulties we consider, for any positive number \( M \), the energy functional \( J_M : D_M \to \mathbb{R} \). We prove that \( J_M \) has a nonnegative minimizer \( u_M \neq 0 \); and that \( \|u_M\|_\infty \leq M \), for some constant \( M \) independent of \( M \). Using these results, and some auxiliary lemmas, we prove Theorem 1.2 in Section 3 by showing that, for \( M \) large enough, \( u_M \) is a weak solution of (3) (in spite of the fact that \( J_M \) may not be Gateaux differentiable at \( u_M \)). Finally, at the end of Section 3, we use Theorem 1.2 to obtain Theorems 1.3 - 1.6.

2. Preliminaries. For any real number \( M > 0 \), let \( J_M : D_M \to \mathbb{R} \) be defined by

\[
J_M(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega G_M(.,u) - \int_\Omega F(.,u),
\]

where \( F(x,s) := \int_0^s f(x,\sigma)\,d\sigma \), and \( G_M \) is defined as in (g4), that is

\[
G_M(x,s) := \int_M^s g(x,\sigma)\,d\sigma.
\]

A straightforward application of Lebesgue’s dominated convergence theorem and the mean value theorem give the following lemma (for a proof see e.g., [23], Lemma 2.3)

**Lemma 2.1.** i) Let \( M > 0 \), and let \( \{u_j\}_{j \in \mathbb{N}} \) be a sequence of measurable functions on \( \Omega \) such that \( 0 \leq u_j \leq M \) for all \( j \in \mathbb{N} \), and \( \lim_{j \to \infty} u_j = u \) a.e. in \( \Omega \) for some \( u : \Omega \to \mathbb{R} \). Then

\[
\lim_{j \to \infty} \int_\Omega F(.,u_j) = \int_\Omega F(.,u).
\]

ii) If \( u,v \) are nonnegative functions in \( L^\infty(\Omega) \), then

\[
\lim_{t \to 0^+} \int_\Omega (F(.,u + tv) - F(.,u)) = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{1}{t} \int_\Omega (F(.,u + tv) - F(.,u)) = \int_\Omega vf(.,u)
\]
If, in addition, \( u - \varepsilon_0 v \geq 0 \) for some \( \varepsilon_0 > 0 \), then
\[
\lim_{t \to 0^-} \frac{1}{t} \int_{\Omega} (F(\cdot, u + tv) - F(\cdot, u)) = 0
\]
and
\[
\lim_{t \to 0^-} \frac{1}{t} \int_{\Omega} (F(\cdot, u + tv) - F(\cdot, u)) = \int_{\Omega} v f(\cdot, u).
\]

**Remark 4.** Let \( M > 0 \), \( s \in [0, M] \) and let \( \{s_j\}_{j \in \mathbb{N}} \) be a sequence in \([0, M]\) such that \( \lim_{j \to \infty} s_j = s \). Since \( g \) is nonnegative, satisfies \((g_1)\), and \( s \to g(x, s) \) is continuous on \((0, \infty)\) a.e. \( x \in \Omega \), we have \( \lim_{j \to \infty} \int_{s_j}^{M} g(\cdot, s) \, ds = \int_{s}^{M} g(\cdot, s) \, ds \), a.e. in \( \Omega \).

**Lemma 2.2.** i) For \( M > 0 \), \( J_M \) is coercive on \( D_M \) with respect to the topology of \( H_0^1(\Omega) \); i.e., \( J_M(\cdot, u) \to \infty \) when \( u \in D_M \), and \( \|\nabla u\|_2 \to \infty \).

ii) \( \inf_{u \in D_M} J_M(\cdot, u) \) is achieved at some \( u \in D_M \).

**Proof.** For \( u \in D_M \) we have \( |\int_{\Omega} F(\cdot, u)| \leq MB_M \), where \( B_M \) is defined by \( B_M := \int_{\Omega} \sup_{0 \leq s \leq M} |f(\cdot, s)| \); and, by \((f_2)\), \( B_M < \infty \). Also, \( \int_{\Omega} G_M(\cdot, u) \leq 0 \). Then \( J_M(\cdot, u) \geq \tfrac{1}{2} \|\nabla u\|_2^2 - MB_M \); therefore i) holds.

To prove ii), let \( \gamma := \inf_{u \in D_M} J_M(\cdot, u) \); note that \( -\infty < \gamma < \infty \). Consider a sequence \( \{u_j\}_{j \in \mathbb{N}} \subset D_M \) such that \( \lim_{j \to \infty} J_M(\cdot, u_j) = \gamma \); it follows from i) that \( \{u_j\}_{j \in \mathbb{N}} \) is bounded in \( H_0^1(\Omega) \). Since the inclusion \( H_0^1(\Omega) \hookrightarrow L^2(\Omega) \) is compact, there exist \( u \in H_0^1(\Omega) \), and a subsequence \( \{u_{j_k}\}_{k \in \mathbb{N}} \), such that \( \{u_{j_k}\}_{k \in \mathbb{N}} \) converges strongly in \( L^2(\Omega) \) and \( \nabla u_{j_k} \) converges weakly to \( \nabla u \) in \( L^2(\Omega, \mathbb{R}^n) \). Taking a further subsequence if necessary, we can assume that \( \{u_{j_k}\}_{k \in \mathbb{N}} \) converges to \( u \) a.e. in \( \Omega \). Thus
\[
\|\nabla u\|_2^2 \leq \limsup_{k \to \infty} \|\nabla u_{j_k}\|_2^2.
\]

Since \( 0 \leq u_{j_k} \leq M \) for all \( k \), we have \( 0 \leq u \leq M \). Also, \( -G_M(\cdot, u_{j_k}) = \int_{u_{j_k}}^{M} g(\cdot, s) \, ds \geq 0 \), and, by Remark 4, \( \lim_{k \to \infty} G_M(\cdot, u_{j_k}) = G_M(\cdot, u) \) a.e. in \( \Omega \), and so, by Fatou’s lemma, \( -\int_{\Omega} G_M(\cdot, u) \leq \limsup_{k \to \infty} \left( -\int_{\Omega} G_M(\cdot, u_{j_k}) \right) \). By Lemma 2.1 i), we also have
\[
\lim_{k \to \infty} \int_{\Omega} F(\cdot, u_{j_k}) = \int_{\Omega} F(\cdot, u).
\]

Then
\[
\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} G_M(\cdot, u) - \int_{\Omega} F(\cdot, u)
\leq \lim_{k \to \infty} \frac{1}{2} \|\nabla u_{j_k}\|_2^2 + \lim_{k \to \infty} \left( -\int_{\Omega} G_M(\cdot, u_{j_k}) \right)
+ \lim_{k \to \infty} \left( -\int_{\Omega} F(\cdot, u_{j_k}) \right)
\leq \lim_{k \to \infty} \left( \frac{1}{2} \int_{\Omega} |\nabla u_{j_k}|^2 - \int_{\Omega} G_M(\cdot, u_{j_k}) - \int_{\Omega} F(\cdot, u_{j_k}) \right) = \gamma.
\]

In particular (taking into account that \( \int_{\Omega} |\nabla u|^2 \) and \( \int_{\Omega} F(\cdot, u) \) are finite) \((10)\) gives \( \int_{\Omega} |G_M(\cdot, u)| = \int_{\Omega} G_M(\cdot, u) < \infty \), and so, \( u \in D_M \). Then \( \gamma \leq J_M(\cdot, u) \), and since, by \((10)\), \( J_M(\cdot, u) \leq \gamma \), the lemma follows.$\square$

**Remark 5.** From \((g_3)\) and the fact that \( g(\cdot, s) \in L^\infty(\Omega) \) for any \( s > 0 \), it follows that, for any \( s_0 > 0 \), there exist positive constants \( A, \beta \) such that \((5)\) holds for \( s \leq s_0 \). Indeed, replacing \( c \) by \( \max(1, c) \) if necessary, we can assume that the constant \( c \) in the condition \((g_3)\) satisfies \( c \geq 1 \). Let \( s \in (0, s_0] \), and let \( j \in \mathbb{N} \) satisfy
\(\frac{1}{2}s_0 < s \leq \frac{1}{2}s_0\). Then \(\frac{1}{2}s_0 < 2s \leq s_0\), and so \(g(., 2s) \leq \kappa := \|g(., \frac{1}{2}s_0)\|_{\infty}\). Also, \(\eta \leq \log_2 \left(\frac{s}{s_0}\right)\). Then, a repeated use of \((g_3)\) gives

\[
g(., s) \leq c^j g(., 2s) + \sum_{1 \leq j \leq \beta} c^j \leq c^j (2^j - 1) \leq c^{\log_2 \left(\frac{s}{s_0}\right)} (2^j - 1) \leq c^{\log_2 \left(\frac{s}{s_0}\right)} (\log_2 \left(\frac{s}{s_0}\right) - 1)
\]

which implies that \((5)\) holds for any \(\beta > \log_2 (c)\), and for some positive constant \(A = A(\beta, s_0)\) independent of \(s\).

Incidentally note that any function \(g(x, s) = a(x) s^{-\beta}\), with \(0 \leq a \in L^\infty(\Omega)\) and \(\beta > 0\), satisfies condition \((g_2)\).

**Lemma 2.3.** Let \(M > 0\). Then, for any minimizer \(u\) for \(J_M\) on \(D_M\), and for any nonnegative \(\varphi \in H^1(\Omega) \cap L^\infty(\Omega)\), we have

\[
\int_{\Omega} (\nabla u, \nabla (u \varphi)) \leq \int_{\Omega} \chi_{\{u > 0\}} \left| u \varphi + \int_{\Omega} f(., u) u \varphi \right| (11)
\]

**Proof.** Let \(u\) be a minimizer for \(J_M\) on \(D_M\), let \(\tau \in (-1, 0)\); and let \(\varphi\) be a non-negative function in \(H^1(\Omega) \cap L^\infty(\Omega)\); temporarily assume \(\|\varphi\|_{\infty} \leq 1\) Note that \(u + \tau u \varphi \in D_M\). Indeed, \(0 \leq \frac{1}{2} u \leq u + \tau u \varphi \leq u \leq M\), and \(s \to g(x, s)\) is non-increasing on \((0, \infty)\) almost everywhere \(x \in \Omega\), then, a.e. \(x \in \Omega\), we have

\[
\left| G_M(x, u(x) + \tau u(x) \varphi(x)) \right| = \int_{u(x) + \tau u(x) \varphi(x)}^{\infty} g(x, s) \, ds \leq \int_{\frac{1}{2} u(x)}^{u(x)} g(x, s) \, ds \leq \int_{\frac{1}{2} u(x)}^{u(x)} g(x, s) \, ds + \left| G_M(x, u(x)) \right|.
\]

Condition \((g_3)\), and a change of variables, give

\[
\int_{\frac{1}{2} u(x)}^{u(x)} g(x, s) \, ds \leq c \int_{\frac{1}{2} u(x)}^{u(x)} (g(x, 2s) + 1) \, ds = \frac{c}{2} \int_{u(x)}^{2u(x)} (g(x, \sigma) + 1) \, d\sigma
\]

If \(2u(x) > M\), we have \(\int_{u(x)}^{2u(x)} g(x, \sigma) \, d\sigma \leq \int_{u(x)}^{M} g(x, \sigma) \, d\sigma + \int_{M}^{2u(x)} g(x, \sigma) \, d\sigma \leq |G_M(x, u(x))| + Mg(x, M)\). In the case \(2u(x) \leq M\), we have \(\int_{u(x)}^{2u(x)} g(x, \sigma) \, d\sigma \leq |G_M(x, u(x))| \). Thus, almost everywhere \(x \in \Omega\), we have

\[
\int_{u(x)}^{2u(x)} g(x, \sigma) \, d\sigma \leq \left| G_M(x, u(x)) \right| + Mg(x, M).
\]

Now, from \((12), (13)\) and \((14)\), we get, a.e. in \(\Omega\).

\[
\left| G_M(x, u + \tau u \varphi) \right| \leq \frac{c}{2} u + \left(1 + \frac{c}{2}\right) \left| G_M(x, u) \right| + \frac{c}{2} Mg(x, M)
\]
Since \( u, G_M (\cdot, u) \) and \( g (\cdot, M) \) belong to \( L^1 (\Omega) \), we conclude that \( G_M (\cdot, u + \tau \varphi) \in L^1 (\Omega) \). Thus \( u + \tau \varphi \in D_M \), and so \( J_M (u) \leq J_M (u + \tau \varphi) \), i.e.,

\[
\tau \int_\Omega \langle \nabla u, \nabla (u \varphi) \rangle \geq \int_\Omega (G_M (\cdot, (1 + \tau \varphi) u) - G_M (\cdot, u)) + \int_\Omega (F (\cdot, (1 + \tau \varphi) u) - F (\cdot, u)) - \frac{\tau^2}{2} \int_\Omega u^2 |\nabla \varphi|^2 - \frac{\tau^2}{2} \int_\Omega \varphi^2 |\nabla u|^2 - \tau^2 \int_\Omega u \varphi \langle \nabla u, \nabla \varphi \rangle.
\]  

(16)

Note that \( \frac{d}{ds} G_M (\cdot, s) = g (\cdot, s) \) for \( s > 0 \); since \( s \to g (\cdot, s) \) is nonincreasing, we conclude that \( s \to G_M (\cdot, s) \) is concave on \( (0, \infty) \), therefore

\[
G_M (\cdot, u) - G_M (\cdot, (1 + \tau \varphi) u) \geq -\tau \chi_{\{u > 0\}} g (\cdot, u) \varphi \quad \text{a.e. in } \{u > 0\}.
\]  

(17)

Also \( G_M (x, u (x)) = G_M (x, (1 + \tau \varphi) u (x)) > -\infty \), a.e. \( x \in \{u = 0\} \). Then (17) holds a.e. in \( \Omega \). Also, by Lemma 2.1 ii),

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_\Omega (F (\cdot, u + \tau \varphi u) - F (\cdot, u)) = \int_\Omega \varphi u f (\cdot, u)
\]  

(18)

Dividing (16) by \( \tau \), letting \( \tau \to 0^- \), and using (17) and (18), we get

\[
\int_\Omega \langle \nabla u, \nabla (u \varphi) \rangle \leq \int_\Omega \chi_{\{u > 0\}} \varphi g (\cdot, u) \varphi + \int_\Omega f (\cdot, u) u \varphi.
\]  

(19)

Finally, since both sides in (11) are linear on \( \varphi \), the temporary assumption \( \|\varphi\|_\infty \leq \frac{1}{2} \) can be removed. \( \square \)

**Lemma 2.4.** Let \( M > 0 \), \( m \geq 2 \); and let \( u \) be a minimizer for \( J_M \) on \( D_M \). Then, for any nonnegative \( \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \), we have

\[
\int_\Omega \langle \nabla (u^m), \nabla \varphi \rangle \leq m \int_\Omega \left( \chi_{\{u > 0\}} u^{m-1} g (\cdot, u) + u^{m-1} f (\cdot, u) \right) \varphi.
\]

**Proof.** \( u \in H^1_0 (\Omega) \cap L^\infty (\Omega) \), therefore \( u^m, u^{m-1} \in H^1_0 (\Omega) \), \( \nabla (u^m) = mu^{m-1} \nabla u \), and \( \nabla (u^{m-1}) = (m - 1) u^{m-2} \nabla u \). Also \( u^{m-1} \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \) for any nonnegative \( \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \). Then

\[
\int_\Omega \langle \nabla (u^m), \nabla \varphi \rangle = m \int_\Omega \left( \nabla (u^{m-1}) \varphi \right)
\]  

(20)

\[
= m \int_\Omega \langle \nabla u, \nabla (u^{m-1} \varphi) \rangle - m (m - 1) \int_\Omega u^{m-2} \varphi |\nabla u|^2
\]  

\[
\leq m \int_\Omega \chi_{\{u > 0\}} u^{m-1} g (\cdot, u) + u^{m-1} f (\cdot, u) \varphi,
\]

where the last inequality holds because of Lemma 2.3. \( \square \)

**Remark 6.** Let \( u \in L^1_{\text{loc}} (\Omega) \) be such that \( \nabla u \in L^2 (\Omega) \), and let \( w \in L^\infty (\Omega) \). If \( \int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega w \varphi \) (respectively \( \int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega w \varphi \)) for every nonnegative \( \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \) then, a standard density argument (using the sequence of truncations \( \varphi_j (x) := \min \{ \varphi (x), j \}, j \in \mathbb{N} \) shows that the inequality holds for any nonnegative \( \varphi \in H^1_0 (\Omega) \).
Remark 7. Let us recall some well known properties of principal eigenvalues and principal eigenfunctions (see e.g., [12]). If \( \Omega \) is a \( C^{1,1} \) domain in \( \mathbb{R}^n \), \( b \in L^\infty (\Omega) \), and \( b^+ \neq 0 \), then:

i) There exists a unique positive principal eigenvalue \( \lambda_1 (b) \), and its eigenspace is a one dimensional subspace of \( C^1 (\Omega) \). Moreover, for each positive eigenfunction \( \phi \), there are positive constants \( c_1, c_2 \) such that \( c_1 dQ \leq \phi \leq c_2 dQ \) in \( \Omega \), where \( dQ := \text{dist} (\cdot, \partial \Omega) \). Therefore, for \( \gamma \in \mathbb{R} \), \( \phi^\gamma \) is integrable if, and only if, \( \gamma > -1 \).

ii) If \( 0 < \lambda < \lambda_1 (b) \) and \( h \in L^\infty (\Omega) \), the problem \(-\Delta u = \lambda bu + h \in \Omega, u = 0 \) on \( \partial \Omega \), has a unique solution \( u \in \cap_1 \leq p < \infty W^{2,p} (\Omega) \), and the corresponding solution operator \((-\Delta - \lambda b)^{-1} : L^\infty (\Omega) \to C^0 (\Omega) \) is bounded and strongly positive, i.e., if \( h \in L^\infty (\Omega) \) and \( 0 \leq h \neq 0 \) then \( u \) belongs to the interior of the positive cone of \( C^0 (\Omega) \) where \( C^0 (\Omega) := \{ v \in C^1 (\Omega) : v = 0 \text{ on } \partial \Omega \} \). Moreover, if in addition \( b \geq 0 \) in \( \Omega \), the same property holds for all \( \lambda \in (-\infty, \lambda_1 (b)) \).

iii) If \( b^+ \in L^\infty (\Omega) \) and \( b \leq b^+ \), then \( \lambda_1 (b^+) \leq \lambda_1 (b) \). Also, \( \lambda_1 (kb) = k^{-1} \lambda_1 (b) \) for all \( k \in (0, \infty) \).

Lemma 2.5. If \((g_1)-(g_4)\) and \((f_1)-(f_4)\) hold, then there exists a positive number \( M \) such that, for any \( M > 0 \), and any minimizer \( u \) for \( J_M \) on \( D_M \), \( \| u \|_\infty \leq M \).

Proof. Let \( M > 0 \), and let \( u \) be a minimizer for \( J_M \) on \( D_M \). Let \( s_0 > 0 \), and let \( A, \beta \) be as in Remark 5. Let \( m := \max \{ 2, \beta + 1 \} \), and let \( b \), and \( \sigma_0 \), be as in \((f_4)\). Thus \( m f (., s) \leq mbs \leq \lambda_1 (b) bs \) a.e. in \( \Omega \) for \( s \geq \sigma_0 \). From Lemma 2.4, for \( 0 \leq \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \), we have

\[
\int_\Omega \langle \nabla (u^m) , \nabla \varphi \rangle \leq m \int_\Omega (\chi_{(u>0)} u^{m-1} g (., u) + u^{m-1} f (., u)) \varphi \tag{21}
\]

\[
= m \int_\Omega (\chi_{(u>0)} u^{m-1} g (., u) \chi_{(u<s_0)} + \chi_{(u>0)} u^{m-1} g (., u) \chi_{(u\geq s_0)}) \varphi \\
+ m \int_\Omega (\chi_{(u<s_0)} u^{m-1} f (., u) + \chi_{(u\geq s_0)} u^{m-1} f (., u)) \varphi.
\]

Note that the term on the right of the last equality is bounded by

\[
m \int_\Omega (Au^{m-1-\beta} \chi_{(u<s_0)} + u^{m-1} g (., s_0)) \varphi + \int_\Omega (m \sigma_0^{m-1} Q + mbu^m) \varphi
\]

with \( Q := \text{ess sup}_{\Omega \times (0, \sigma_0)} f \). Therefore, for any nonnegative \( \varphi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \), we have

\[
\int_\Omega \langle \nabla (u^m) , \nabla \varphi \rangle \\
\leq m \int_\Omega (A^{\alpha_0^{m-1-\beta}} + Au^{m-1}s_0^{-\beta}) \varphi + \int_\Omega (m \sigma_0^{m-1} Q + mbu^m) \varphi.
\]

By Remark 6, (22) holds for any nonnegative \( \varphi \in H^1_0 (\Omega) \), i.e.,

\[-\Delta (u^m) \leq m A^{\alpha_0^{m-1-\beta}} + mA^{m-1}s_0^{m-\beta} + m \sigma_0^{m-1} Q + mbu^m \in (H^1_0 (\Omega))' \tag{23}\]

Since \( m < \lambda_1 (b) \), we have that \((-\Delta - m)^{-1} : L^\infty (\Omega) \to H^1_0 (\Omega) \subset L^\infty (\Omega) \) is a bounded positive operator. Let \( C \) denote its norm. Taking into account that \( u \geq 0 \), from (23), we have

\[
\| u \|_\infty^{m} \leq mC \left( A^{\alpha_0^{m-1-\beta}} + A^{s_0^{-\beta}} \| u \|_\infty^{m-1} + \sigma_0^{m-1} Q \right),
\]

and, since all the constants involved in this inequality are independent of \( M \) and \( u \), the assertion of the lemma follows. \( \square \)
Remark 8. i) Let us recall the following form of the Hopf maximum principle (see [4], Lemma 3.2): Suppose $0 \leq h \in L^\infty(\Omega)$; and let $v$ be the solution of $-\Delta v = h$ in $\Omega$, $v = 0$ on $\partial\Omega$. Then $v \geq cd_1 \int_{\Omega} \Psi \, d\Omega$ a.e. in $\Omega$, where $c > 0$ is a constant depending only on $\Omega$.

ii) Moreover, if $0 \leq \Psi \in L^1_{loc}(\Omega)$, and $v \in H^1_0(\Omega)$ satisfies $-\Delta v \geq \Psi$ on $\Omega$ in the distributional sense, then

$$v \geq cd_1 \int_{\Omega} \Psi \, d\Omega \quad \text{a.e. in } \Omega,$$

where $c$ is the constant given in i) (therefore depending only on $\Omega$). Indeed, for $\delta > 0$, let $\Psi_\delta := \min(\frac{1}{\delta} \Psi)$. Then $0 \leq \Psi_\delta \in L^\infty(\Omega)$. Let $v_\delta \in H^1_0(\Omega)$ be the solution of $-\Delta v_\delta = \Psi_\delta$ in $\Omega$, $v_\delta = 0$ on $\partial\Omega$. Then $-\Delta(v - v_\delta) \geq 0$ in $D'(\Omega)$ and so, since $v - v_\delta \in H^1_0(\Omega)$, we have $-\Delta(v - v_\delta) \geq 0$ in $(H^1_0(\Omega))'$. Thus, by the maximum principle, $v \geq v_\delta$ in $\Omega$, and so, by Lemma 3.2 in [4], $v \geq cd_1 \int_{\Omega} \Psi_\delta \, d\Omega$ a.e. in $\Omega$. Taking the limit as $\delta \to 0^+$ we obtain (24).

3. Proof of the main results.

Proof of Theorem 1.2. Let $M$ be as in Lemma 2.5, let $M := M + 1$, and let $u$ be a minimizer for $J_M$ on $D_M$. Let $\psi$ be a nonnegative function in $H^1_0(\Omega) \cap L^\infty(\Omega)$, and let $\varepsilon > 0$. Thus $\frac{\psi}{u + \varepsilon} \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and $\nabla \left( u \frac{\psi}{u + \varepsilon} \right) = \varepsilon \frac{\nabla u}{u + \varepsilon} \psi + \frac{u}{u + \varepsilon} \nabla \psi$. Then Lemma 2.3 (used with $\varphi = \frac{\psi}{u + \varepsilon}$), gives

$$\varepsilon \int_{\Omega} \psi \frac{|\nabla u|^2}{(u + \varepsilon)^2} + \int_{\Omega} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle$$

$$\leq \int_{\Omega} \chi_{\{u > 0\}} u g \left( \frac{\psi}{u + \varepsilon} \right) + \int_{\Omega} f \left( \frac{\psi}{u + \varepsilon} \right).$$

Since $\nabla u = 0$ a.e. in $\{u = 0\}$, (25) can be written as

$$\varepsilon \int_{\{u > 0\}} \psi \frac{|\nabla u|^2}{(u + \varepsilon)^2} + \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle$$

$$- \int_{\{u > 0\}} f \left( \frac{\psi}{u + \varepsilon} \right) \frac{u}{u + \varepsilon} \psi \leq \int_{\{u > 0\}} g \left( \frac{\psi}{u + \varepsilon} \right).$$

Since $\lim_{\varepsilon \to 0^+} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle = \chi_{\{u > 0\}} \langle \nabla u, \nabla \psi \rangle = \langle \nabla u, \nabla \psi \rangle$ a.e. in $\Omega$, and $\left| \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle \right| \leq |\langle \nabla u, \nabla \psi \rangle| \in L^1(\Omega)$, Lebesgue’s dominated convergence theorem gives

$$\lim_{\varepsilon \to 0^+} \int_{\{u > 0\}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle = \int_{\Omega} \langle \nabla u, \nabla \psi \rangle.$$

Since $\lim_{\varepsilon \to 0^+} \frac{u}{u + \varepsilon} g \left( \frac{\psi}{u + \varepsilon} \right) = g \left( \frac{\psi}{u + \varepsilon} \right)$ a.e. in $\{u > 0\}$, and $g \left( \frac{\psi}{u + \varepsilon} \right)$ is nonincreasing in $\varepsilon$, the monotone convergence theorem gives

$$\lim_{\varepsilon \to 0^+} \int_{\{u > 0\}} g \left( \frac{\psi}{u + \varepsilon} \right) \frac{u}{u + \varepsilon} \psi = \int_{\Omega} \chi_{\{u > 0\}} g \left( \frac{\psi}{u + \varepsilon} \right) \psi$$

Also, $\frac{u}{u + \varepsilon} f \left( \frac{\psi}{u + \varepsilon} \right) \leq \sup_{0 \leq s \leq M} |f \left( \frac{\psi}{u + \varepsilon} \right) \psi| \in L^1(\Omega)$ and then, by Lebesgue’s dominated convergence theorem,

$$\lim_{\varepsilon \to 0^+} \int_{\{u > 0\}} f \left( \frac{\psi}{u + \varepsilon} \right) \frac{u}{u + \varepsilon} \psi = \int_{\Omega} \chi_{\{u > 0\}} f \left( \frac{\psi}{u + \varepsilon} \right) \psi \leq \int_{\Omega} f \left( \frac{\psi}{u + \varepsilon} \right) \psi.$$


the last equality because, by (f3), \( f(., 0) \geq 0 \). Then, from (26), (27), (28) and (29), we have

\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} f(., u) \psi \leq \int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} \chi \{ u > 0 \} f(., u) \psi
\]

\[
\leq \lim_{\varepsilon \to 0^+} \left( \int_{\{ u > 0 \}} \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle - \int_{\{ u > 0 \}} f(., u) \frac{u}{u + \varepsilon} \psi \right)
\]

\[
\leq \lim_{\varepsilon \to 0^+} \left( \int_{\{ u > 0 \}} \varepsilon |\nabla u|^2 + \frac{u}{u + \varepsilon} \langle \nabla u, \nabla \psi \rangle - \int_{\{ u > 0 \}} f(., u) \frac{u}{u + \varepsilon} \psi \right)
\]

\[
\leq \lim_{\varepsilon \to 0^+} \int_{\{ u > 0 \}} g(., u) \frac{u}{u + \varepsilon} \psi = \int_{\{ u > 0 \}} \chi \{ u > 0 \} g(., u) \psi.
\]

Thus \( \int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} f(., u) \psi \leq \int_{\Omega} \chi \{ u > 0 \} g(., u) \psi \). To prove the existence assertion of the theorem it only remains to prove that \( \chi \{ u > 0 \} g(., u) \psi \in L^1(\Omega) \), and that

\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} f(., u) \psi \geq \int_{\Omega} \chi \{ u > 0 \} g(., u) \psi,
\]

for any nonnegative \( \psi \in H^1_0(\Omega) \cap L^\infty(\Omega) \).

Assume that \( \psi \) satisfies the temporary condition \( \| \psi \|_\infty \leq \frac{1}{2} \), and let \( t \in (0, 1) \). Note that \( u + t\psi \in \mathcal{M} \). Indeed, by Lemma 2.5 we have \( u \leq \mathcal{M} \), and so \( 0 \leq u + t\psi \leq \mathcal{M} + 1 = M \). Also \( u + t\psi \in H^1_0(\Omega) \). Note that \( |G_M(., u + t\psi)| = \int_{u + t\psi}^M g(., s) ds \leq \int_u^M g(., s) ds = G_M(., u) \in L^1(\Omega) \) (because \( u \in \mathcal{M} \)), and so \( u + t\psi \in \mathcal{M} \). Then \( J_M(., u) \leq J_M(., u + t\psi) \).

To prove (31) observe that \( J_M(., u) \leq J_M(., u + t\psi) \) can be written as

\[
0 \leq \frac{1}{t} \left( J_M(., u + t\psi) - J_M(., u) \right)
\]

\[
= \int_{\Omega} \langle \nabla u, \nabla \psi \rangle + \frac{1}{t} \int_{\Omega} |\nabla \psi|^2 - \frac{1}{t} \int_{\Omega} \langle G_M(., u + t\psi) - G_M(., u) \rangle
\]

\[
- \frac{1}{t} \int_{\Omega} \langle F(., u + t\psi) - F(., u) \rangle.
\]

Now,

\[
\frac{1}{t} \int_{\Omega} \langle G_M(., u + t\psi) - G_M(., u) \rangle = \int_{\{ \psi > 0 \}} \frac{1}{t} \int_u^{u + t\psi} g(x, s) ds dx.
\]

For \( x \in \{ u > 0 \} \cap \{ \psi > 0 \} \), the mean value theorem gives \( \int_{u(x)}^{u(x)+t\psi(x)} g(x, s) ds = t g(x, u(x) + \sigma_t \psi(x)) \) with \( \sigma_t \) (which depends on \( x, t, u \) and \( \psi \)) such that \( 0 < \sigma_t < t\psi(x) \). Thus

\[
\int_{\{ \psi > 0 \} \cap \{ u > 0 \}} \frac{1}{t} \int_u^{u(x)+t\psi(x)} g(x, s) ds dx = \int_{\{ \psi > 0 \} \cap \{ u > 0 \}} g(x, u(x) + \sigma_t \psi(x)) \psi(x) dx;
\]

also \( g(x, u(x) + \sigma_t \psi(x)) \geq 0 \), and \( \lim_{t \to 0^+} g(x, u(x) + \sigma_t \psi(x)) = g(x, u(x)) \psi(x) \). Therefore, since \( g \) is nonnegative, from (33), (34) and Fatou’s Lemma, we
get
\[
\lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} (G_M (., u + t\psi) - G_M (., u)) \geq \lim_{t \to 0^+} \int_{\{\psi > 0\} \cap \{u > 0\}} \frac{1}{t} \int_{u(x)}^{u(x) + t\psi(x)} g(x, s) \, ds \, dx
\]
\[
\geq \int_{\{u > 0\} \cap \{\psi > 0\}} \lim_{t \to 0^+} g(., u + \sigma_t) \psi
\]
\[
= \int_{\{u > 0\} \cap \{\psi > 0\}} g(., u) \psi = \int_{\Omega} \chi_{\{u > 0\}} g(., u) \psi.
\]
Thus,
\[
\lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} (G_M (., u + t\psi) - G_M (., u)) \geq \int_{\Omega} \chi_{\{u > 0\}} g(., u) \psi. \quad (35)
\]
Also, by Lemma 2.1 ii), we have
\[
\lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} (F (., u + t\psi) - F (., u)) = \int_{\Omega} f(., u) \psi. \quad (36)
\]
and, from (32),
\[
\frac{1}{t} \int_{\Omega} (G_M (., u + t\psi) - G_M (., u)) \leq \int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \frac{1}{t} \int_{\Omega} (F (., u + t\psi) - F (., u)) + \frac{t}{2} \int_{\Omega} |\nabla \psi|^2 \quad (37)
\]
then, taking \(\lim_{t \to 0^+}\) in (38), using (35), (36), and (37), we get
\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} f(., u) \psi \geq \int_{\Omega} \chi_{\{u > 0\}} g(., u) \psi, \quad (39)
\]
which, in particular, gives \(\chi_{\{u > 0\}} g(., u) \psi \in L^1(\Omega)\). Since both sides of (39) are linear on \(\psi\), the temporary assumption \(\|\psi\|_\infty \leq \frac{1}{2}\) can be removed. Then \(u\) is a solution to (3).

To see that \(u > 0\) a.e. in \(\Omega_\infty\), consider a nonnegative function \(\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)\) such that \(\|\psi\|_\infty \leq \frac{1}{2}\), and a number \(t \in (0, 1)\). Thus we have (33), (32) and (37). We claim that \(\{|u = 0\} \cap G_\infty \cap \{|\psi > 0\}\} = 0\). Indeed, if the claim were false, from (33), using Fatou’s lemma, and taking into account that \(g\) is nonincreasing, we would have
\[
\lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} (G_M (., u + t\psi) - G_M (., u)) \geq \int_{\{u = 0\} \cap G_\infty \cap \{\psi > 0\}} \lim_{t \to 0^+} g(x, t\psi(x)) \psi(x) \, dx = \infty
\]
and so, from (32), (37) and (40), we would have
\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle \geq \lim_{t \to 0^+} \int_{\Omega} \frac{1}{t} (G_M (., u + t\psi) - G_M (., u))
\]
Thus, for any $\int_{\Omega} \psi H = 0$ in $\Omega$. From (30), for any nonnegative $\psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} \chi_{(u>0)} f(\cdot,u) \psi \leq \int_{\Omega} \chi_{(u>0)} g(\cdot,u) \psi$$

and, from (39), we have $\int_{\Omega} \langle \nabla u, \nabla \psi \rangle - \int_{\Omega} f(\cdot,u) \psi \geq \int_{\Omega} \chi_{(u>0)} g(\cdot,u) \psi$. Thus $\int_{\Omega} \chi_{(u>0)} f(\cdot,u) \psi \geq \int_{\Omega} f(\cdot,u) \psi$, therefore $\int_{\Omega} f(\cdot,u) \psi \leq 0$. Using (f3) and that $\psi \geq 0$, we get $f(\cdot,0) \psi = 0$ a.e. in $\{u = 0\}$, and this holds whenever $0 \leq \psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$. Then $\{u = 0\} = 0$, i.e., $u > 0$ a.e. in $\Omega$.

**Proof of Theorem 1.3.** Let $s_1$ be as in (g2). Define $\tilde{g} : \Omega \times (0,\infty) \to \mathbb{R}$ by $\tilde{g}(\cdot,s) := g(\cdot,s)$ if $0 < s < s_1$, and $\tilde{g}(\cdot,s) := g(\cdot,s_1)$ otherwise. Define also $\tilde{f} : \Omega \times [0,\infty) \to \mathbb{R}$ by $\tilde{f}(\cdot,s) := f(\cdot,s)$ if $0 < s < s_1$, and $\tilde{f}(\cdot,s) := f(\cdot,s) + g(\cdot,s) - g(\cdot,s_1)$ if $s \geq s_1$. Then (3) can be rewritten as

$$\begin{cases} -\Delta u = \chi_{(u>0)} \tilde{g}(\cdot,u) + \tilde{f}(\cdot,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u \neq 0 & \text{in } \Omega. \end{cases}$$

The theorem will follow from Theorem 1.2 once we show that $\tilde{g}$ and $\tilde{f}$ satisfy the same conditions (g1)-(g3), (f1)-(f3) satisfied by $g$ and $f$. Clearly $\tilde{g}$ satisfies (g1) and (g2). Since $g$ satisfies (g3) and $\tilde{g}(x,s)$ is continuous in $s$, then we have, a.e. in $\Omega$,

$$\tilde{g}(\cdot,s) = g(\cdot,s) \leq c(g(\cdot,2s) + 1) = c(\tilde{g}(\cdot,2s) + 1) \text{ for } 0 < s < \frac{s_1}{2},$$

$$\tilde{g}(\cdot,s) = g(\cdot,s) \leq ||g||_L^\infty(\Omega \times (\frac{s_1}{2},s_1)) \text{ for } \frac{s_1}{2} \leq s < s_1,$$

$$\tilde{g}(\cdot,s) = g(\cdot,s_1) \leq ||g||_L^\infty(\Omega \times (\frac{s_1}{2},s_1)) \text{ for } s \geq s_1.$$

Thus, for any $s > 0$, we have $\tilde{g}(\cdot,s) \leq c'(\tilde{g}(\cdot,2s) + 1)$ a.e. in $\Omega$, where $c' := \max(c, ||g||_L^\infty(\Omega \times (\frac{s_1}{2},s_1)))$; therefore $\tilde{g}$ satisfies (g3).

Let us prove that $\tilde{g}$ satisfies (g4): i.e., that, for $M > 0$, $\tilde{D}_M \neq \emptyset$, where $\tilde{D}_M := \{u \in H^1_0(\Omega) : 0 \leq u \leq M \text{ and } \tilde{G}_M(\cdot,u) \in L^1(\Omega)\}$, with $\tilde{G}_M : \Omega \times [0,\infty) \to \mathbb{R} \cup \{-\infty, \infty\}$ defined by $\tilde{G}_M(x,s) := \int_{\Omega}^s \tilde{g}(x,\sigma) d\sigma$. Let $G_M$ and $D_M$ be as given (for the function $g$) by (g4); therefore $\tilde{D}_M \neq \emptyset$. Let $u \in D_M$. Thus $0 \leq u \leq M$ and $G_M(\cdot,u) \in L^1(\Omega)$ if $M \leq s_1$ we have $\tilde{G}_M(\cdot,u) = G_M(\cdot,u)$ a.e. in $\Omega$, and so $u \in D_M$, which gives $\tilde{D}_M \neq \emptyset$. If $M > s_1$ then

$$\tilde{G}_M(\cdot,u) = G_M(\cdot,u) \text{ a.e. in } \{x \in \Omega : u(x) \leq s_1\} \quad (41)$$

and

$$\tilde{G}_M(\cdot,u) = \int_{s_1}^{s_1} \tilde{g}(\cdot,s) d\sigma + \int_{s_1}^{M} \tilde{g}(\cdot,s) d\sigma + \int_{s_1}^{u} \tilde{g}(\cdot,s) d\sigma \quad (42)$$
\[ = (s_1 - M) \mathcal{G}(s_1) + \int_{s_1}^{M} g(\cdot, \sigma) \, d\sigma + G_M(u) \quad \text{a.e. in} \quad \{ x \in \Omega : u(x) > s_1 \}, \]

therefore \( |\mathcal{G}_M(u)| \leq (s_1 - M) |\mathcal{G}(s_1)| + \int_{s_1}^{M} g(\cdot, \sigma) \, d\sigma + 2 |G_M(u)| \quad \text{a.e. in} \quad \Omega \). Now, \( G_M(u, \sigma) \in L^1(\Omega) \); also, from \((g_1)\) and \((g_2')\), \( g(\cdot, s_1) \in L^\infty(\Omega) \) and \( \int_{s_1}^{M} g(\cdot, \sigma) \, d\sigma \in L^1(\Omega) \). Then \( \mathcal{G}_M(u) \in L^1(\Omega) \) and so \( \bar{D}_M \neq \emptyset \). Since \((g_0)\) holds for \( g \), clearly it holds also for \( \bar{g} \). Also, taking into account that \( g \) satisfies \((g_1)\) and \((g_2')\), it is immediate that \( \bar{f} \) satisfies \((f_1)-(f_3)\).

In order to see that \( \bar{f} \) satisfies \((f_4)\), consider the number \( \sigma_0 \) and the function \( b \) given, for the function \( f \), in \((f_4)\). Let \( s_1 \) be as in \((g_2')\). Note that

\[ \|g\|_{L^\infty(\Omega \times (\sigma_0, \infty))} < \infty \quad \text{for any} \quad \sigma > 0. \]  

Indeed, this is clear when \( \sigma \geq s_1 \), and if \( \sigma < s_1 \) then, by \((g_2')\), \( g(\cdot, s) \leq g(\cdot, \sigma) \) for any \( s \in [\sigma, s_1] \), and, by \((g_1)\), \( g(\cdot, \sigma) \in L^\infty(\Omega) \) and then \( \|g\|_{L^\infty(\Omega \times (\sigma_0, s_1))} < \infty \), and thus since, by \((g_2')\), \( \|g\|_{L^\infty(\Omega \times (s_1, \infty))} < \infty \), we get \((43)\). Now, from \((f_4)\) and \((43)\), we conclude that \( \text{ess sup}_{(\Omega \times (0, \sigma_0))} \int_{\Omega} f(\cdot, s) \leq \infty \) for any \( \sigma > 0 \). Also, \( \lambda_1(1 + \varepsilon) b = \ln \frac{1}{\varepsilon} \lambda_1 b > m \) for \( \varepsilon \) positive and small enough. For such \( \varepsilon \) and for \( \bar{\sigma}_0 \) positive and large enough, we have, a.e. in \( \Omega \),

\[ \sup_{s \geq \bar{\sigma}_0} \int_{\Omega} \frac{f(\cdot, s)}{s} \leq \sup_{s \geq \bar{\sigma}_0} \frac{g(\cdot, s)}{s} + \frac{1}{\bar{\sigma}_0} \|g\|_{L^\infty(\Omega \times (s_1, \infty))} \leq \frac{1}{\varepsilon} b, \]

where we have used that \( \text{ess inf}_0 b > 0 \). Then, for such \( \varepsilon \) and \( \bar{\sigma}_0 \), \( \bar{f} \) satisfies \((f_4)\) with \( \sigma_0 \) replaced by \( \bar{\sigma}_0 \) and \((1 + \varepsilon) b \) respectively.

**Remark 9.** i) Let \( h : \Omega \to \mathbb{R} \) be a measurable function such that \( h \geq 0 \) in \( \Omega \) and \( h\varphi \in L^1(\Omega) \) for all \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \). Let \( u \in H^1_0(\Omega) \) be a weak solution, in the sense of Definition 1.1, to the problem \( -\Delta u = h \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). Then \( u \) is a weak solution in the standard \( H^1_0 \) sense, i.e., \( u \) satisfies

\[ \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} h \varphi \]  

for all \( \varphi \in H^1_0(\Omega) \) (not only for \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \)). Indeed, let \( \varphi \in H^1_0(\Omega) \). Thus \( \varphi^+ = \varphi^- \) belong to \( H^1_0(\Omega) \). For \( k \in \mathbb{N} \) let \( \varphi_k^- = \text{min}(k, \varphi^-) \). Thus \( \varphi_k^+ \in H^1_0(\Omega) \cap L^\infty(\Omega) \), and \( \{ \varphi_k^+ \}_{k \in \mathbb{N}} \) is a nondecreasing sequence of nonnegative functions that converges to \( \varphi^+ \) in \( H^1_0(\Omega) \). Then \( \int_{\Omega} \langle \nabla u, \nabla (\varphi_k^+) \rangle = \int_{\Omega} h \varphi_k^+ \), and taking the limit as \( k \to \infty \), the monotone convergence theorem gives \( \int_{\Omega} \langle \nabla u, \nabla \varphi^+ \rangle = \int_{\Omega} h \varphi^+ \). Since the left hand side of this inequality is finite, we get, in particular, that \( h \varphi^+ \in L^1(\Omega) \). Similarly, we have \( \int_{\Omega} \langle \nabla u, \nabla \varphi^- \rangle = \int_{\Omega} h \varphi^- \) and \( h \varphi^- \in L^1(\Omega) \). Then \( h \varphi \in L^1(\Omega) \) and \((44)\) holds. ii) From i) it follows that, if \( f \) and \( g \) satisfy the hypothesis of Theorem 1.2, and if, in addition, \( f \geq 0 \), then the solution given by Theorem 1.2 is a weak solution in the standard \( H^1_0 \) sense (i.e., such that \( \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} \varphi \) for all \( \varphi \in H^1_0(\Omega) \)).

As announced in Remark 2, we now present a singular function \( g(x, s) \) which is independent of \( x \), for any \( \alpha < 3 \) satisfies \( \lim_{s \to 0^+} s^\alpha g(s) = \infty \), and such that the problem \(-\Delta u = g(\cdot, u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \) has a positive weak solution in \( H^1_0(\Omega) \).

**Example 1.** Let \( \Omega = B_1(0) \) be the unit open ball in \( \mathbb{R}^n \) centered at the origin, let \( h : (0, \infty) \to \mathbb{R} \) be defined by \( h(s) := \frac{2s^{-3}}{\ln^2(s)} \left( 1 + \frac{2}{\ln(s)} \right) \) for \( s \in (0, e^{-2}) \), and
by $h(s) := 0$ otherwise. Let $g : \Omega \times (0, \infty) \to \mathbb{R}$ be defined by $g(x, s) := h(s)$. Let us show that $g$ satisfies the assumptions $(g_1)$-$(g_5)$. Indeed, $g$ is a nonnegative Carathéodory function, and $\lim_{s \to 0^+} g(x, s) = \infty$ for $x \in \Omega$. Also, \[
abla g = \frac{\partial g}{\partial s} = \frac{2}{s^2 \ln s} \left(3 \ln^2 s + 10 \ln s + 10\right) < 0 \quad \text{for} \quad 0 < s < e^{-2}.
\] Thus $g(x, \cdot)$ is nonincreasing on $(0, \infty)$ for $x \in \Omega$. Then $g$ satisfies $(g_1)$, $(g_2)$ and $(g_5)$. A computation shows that $\lim_{s \to 0^+} \frac{g(x, s)}{g(x, 2s) + 1} = 8$ and that $\lim_{s \to 0^+} \frac{g(x, s)}{g(x, 2s) + 1} = 16 \frac{e^{\delta} \ln^2 (\ln(2 + s^2))}{1 - e^{-2s^2}}$. Since $s \to \frac{g(x, s)}{g(x, 2s) + 1}$ is continuous on $(0, \frac{1}{2} e^{-2})$, it follows that there exists a constant $c_1 > 0$ such that $\frac{g(x, s)}{g(x, 2s) + 1} \leq c_1$ for $0 < s < \frac{1}{2} e^{-2}$. Also, $\frac{g(x, s)}{g(x, 2s) + 1} = \frac{2s^3}{(\ln s)^3} (1 + \frac{2}{1 + s}) \leq h\left(\frac{1}{2} e^{-2}\right)$ for $\frac{1}{2} e^{-2} \leq s < e^{-2}$, and $\frac{g(x, s)}{g(x, 2s) + 1} = 0$ for $s \geq e^{-2}$. Thus $g$ satisfies $(g_3)$ with $c = \max(c_1, h\left(\frac{1}{2} e^{-2}\right))$. Let us show that $g$ also satisfies $(g_4)$. Indeed, for $M > 0$ and $(x, s) \in \Omega \times (0, \infty)$, let $G_M(x, s) = \int_M^s g(x, \sigma) d\sigma$. Then, for $x \in \Omega$, $G_M(x, s) = -\frac{e^{-2}}{\ln(s)} + \frac{M}{\ln^2(M)}$ if $M < e^{-2}$ and $s < e^{-2}$, $G_M(x, s) = -\frac{e^{-2}}{\ln(s)} + \frac{1}{16} e^4$ if $M \geq e^{-2}$ and $s < e^{-2}$, $G_M(x, s) = 0$ if $M \geq e^{-2}$ and $s \geq e^{-2}$. Let $\varphi : [0, 1] \to \mathbb{R}$ be defined by $\varphi(r) := (1 - r)^\frac{1}{2} \left(-\ln (1 - r)\right)^{-\frac{3}{4}}$ if $r \in [0, 1]$, and $\varphi(1) := 0$. Thus $\varphi \in C([0, 1])$, $\varphi \geq 0$ and, for $r \in (0, 1)$, $\varphi'(r) = \frac{1}{6(\ln(1 - r))^2} \frac{3\ln(1 - r) - 4}{\sqrt{1 - r}} < 0$. Then $\varphi$ is nonincreasing on $[0, 1]$. Also, for $0 < \delta < 1$, $\varphi(1 - \delta) = \frac{\delta}{\left(\ln \delta\right)^{\frac{3}{4}}}$. Note that $\ln \delta > 0^+ \frac{\delta^{\frac{3}{4}}}{(-\ln \delta)^{\frac{3}{4}}} = 0$. Then there exists $\delta \in \left(0, e^{-\frac{4}{3}}\right)$ such that $\varphi(1 - \delta) < \min(M, e^{-2})$; fix such a $\delta$, and define $u \in C(\Omega)$ by $u(x) := \varphi(|x|)$ if $1 - \delta < |x| \leq 1$, and $u(x) := \varphi(1 - \delta)$ otherwise. Then $0 \leq u \leq \min(M, e^{-2})$ in $\Omega$. Also, $\nabla u(x) = \frac{\varphi'(|x|) x}{|x|} = \frac{1}{6(\ln(1 - |x|))^2} \frac{3\ln(1 - |x|) - 4}{\sqrt{1 - |x|}} \frac{|x|}{|x|}$ for $1 - \delta < |x| < 1$, and $\nabla u(x) = 0$ if $|x| < 1 - \delta$. Note that $|3\ln (1 - r) - 4| = 4 - 3\ln (1 - r) \leq 2(-3\ln (1 - r))$ for $1 - \delta < r < 1$. Therefore we have
\[
\int_\Omega |\nabla u|^2 = \omega_{n-1} \int_{1-\delta}^1 \frac{1}{1 - r} \frac{3\ln (1 - r) - 4}{\ln(1 - r)} dr \\
\leq \frac{\omega_{n-1}}{6} \int_{1-\delta}^1 (1 - r)^{1 - \frac{3}{4}} (-\ln (1 - r))^{\frac{3}{4}} dr < \infty,
\]
where $\omega_{n-1}$ is the area of the unit sphere $S^{n-1}$. Then $u \in H^1(\Omega) \cap C(\overline{\Omega})$, and $u = 0$ on $\partial \Omega$. Thus $u \in H^1_0(\Omega)$ (see e.g., [3], Theorem 9.17). Also, $0 \leq u \leq e^{-2}$ in $\Omega$, and so $G_M(x, u(x)) = -\frac{(u(x))^2}{\ln^2(u(x))} + C_M$, where $C_M = \frac{M}{\ln^2(M)}$ if $M \geq e^{-2}$, and $C_M = \frac{1}{16} e^4$ if $M \geq e^{-2}$. Then
\[
\int_\Omega |G_M(x, u(x))| dx \leq C_M |\Omega| + \int_\Omega \frac{(u(x))^2}{\ln^2(u(x))} dx \\
= C_M |\Omega| + \omega_{n-1} \frac{(\varphi(1 - \delta))^2}{\ln^4(\varphi(1 - \delta))} \int_{1-\delta}^1 r^{-n-1} dr + \omega_{n-1} \int_{1-\delta}^1 \frac{(\varphi(r))^2}{\ln^4(\varphi(r))} r^{-n-1} dr.
\]
Now,
\[
\int_{1-\delta}^1 \frac{(\varphi(r))^2}{\ln^4(\varphi(r))} r^{-n-1} dr \leq \int_{1-\delta}^1 \frac{(\varphi(r))^2}{\ln^4(\varphi(r))} dr
\]
where in the last equality we have used the change of variables \( t = -\ln (1 - r) \).
Thus \( \int_{\Omega} |G_M (x, u(x))| \, dx < \infty \), therefore \( u \in D_M \). Therefore, by Theorem 1.2, the problem \(-\Delta u = \chi_{\{u > 0\}} g (., u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \) has a weak solution, in the sense of Definition 1.1, \( u \in H^1_0(\Omega) \cap L^\infty (\Omega) \). In particular we have that \( \chi_{\{u > 0\}} g (., u) \Psi \in L^1 (\Omega) \) for any \( \Psi \in H^1_0 (\Omega) \cap L^\infty (\Omega) \), and so \( \chi_{\{u > 0\}} g (., u) \in L^1_{loc} (\Omega) \). Also, as \( u \neq 0 \) in \( \Omega \), we have \( \chi_{\{u > 0\}} g (., u) \neq 0 \). Since \(-\Delta u = \chi_{\{u > 0\}} g (., u) \) in \( D'(\Omega) \), the Hopf maximum principle (as stated in Remark 8 ii)) applies and we get \( u \geq c'd_\Omega \) a.e. in \( \Omega \) for some positive constant \( c' \). Moreover, by Remark 9 ii), \( u \) is a weak solution, in the standard \( H^1_0(\Omega) \) sense, to the problem \(-\Delta u = g (., u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \).

In the following example, for any given \( \beta > 3 \) we construct a function \( g : \Omega \times (0, \infty) \to [0, \infty) \) such that, for any \( s_0 > 0 \), \( \text{ess sup}_{\Omega \times (0, s_0)} s^\beta g (x, s) = \infty \); moreover, for the \( g \) so constructed: (5) does not hold if \( \beta \leq \theta \). We show that the problem \(-\Delta u = g (., u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \), has a positive weak solution in \( H^1_0 (\Omega) \).

**Example 2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^{1,1} \) boundary, and let \( \{U_j\}_{j \in \mathbb{N}, j \neq 0} \) be a sequence of subsets of \( \Omega \) such that \( \Omega = \bigcup_{j=0}^{\infty} U_j \) with \( U_i \cap U_j = \emptyset \) if \( i \neq j \), \( U_j \neq \emptyset \) for all \( j \), \( U_0 \) is a bounded domain with \( C^{1,1} \) boundary, and \( U_j \) is measurable for all \( j \). Let \( \beta_0 \in (0, 3) \), \( \theta > 3 \), \( \gamma > \theta \), and let \( \{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1) \) be a sequence such that \( \lim_{j \to \infty} \varepsilon_j = 0 \). Let \( g : \Omega \times (0, \infty) \to (0, \infty) \) be defined by \( g(x, s) := s^{\beta_0} \) if \( x \in U_0 \), \( s > 0 \); and for \( j \geq 1 \), \( g(x, s) := s^\gamma \) if \( x \in U_j \), \( s \geq \varepsilon_j \); and \( g(x, s) := \varepsilon_j^\gamma \) if \( x \in U_j \), \( 0 < s < \varepsilon_j \). Clearly \( g \) satisfies \( (g_1), (g_2) \), and it is easy to check that \((g_3)\) holds with constant \( c = 2^\gamma \). \((g_3)\) also holds with \( \Omega_\infty = U_0 \). Moreover, if in addition, the following inequalities hold

\[
\sum_{j=1}^{\infty} \frac{1}{\gamma - 1} \varepsilon_j^{1-\gamma} |U_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \varepsilon_j^{\gamma} |U_j| < \infty,
\]

then \( D_M \neq \emptyset \) for all \( M > 0 \). Indeed, let \( \varphi \) be the positive principal eigenfunction for \(-\Delta\) in \( U_0 \) with homogeneous Dirichlet boundary condition, with weight function 1, and normalized such that \( \|\varphi\|_\infty = M^{1+\beta_0} \). Note that \( \nabla \left( \varphi^{\frac{2}{1+\beta_0}} \right) = \left( \frac{2}{1+\beta_0} \right)^2 \varphi^{\frac{2(1-\beta_0)}{1+\beta_0}} \left| \nabla \varphi \right|^2 \), and that \( \frac{2(1-\beta_0)}{1+\beta_0} > -1 \). Since, for some positive constant \( k \), \( \varphi \geq kd_{U_0} \) in \( U_0 \), it follows that \( \Phi_0 := \varphi^{\frac{2}{1+\beta_0}} \in H^1_0 (U_0) \). Also, \( \int_{U_0} \Phi_0^{1-\beta_0} = \int_{U_0} \varphi^{\frac{2(1-\beta_0)}{1+\beta_0}} < \infty \), and \( 0 \leq \Phi_0 \leq M \). Let \( \Phi : \Omega \to \mathbb{R} \) be the extension by zero of \( \Phi_0 \). Then \( \Phi \in H^1_0 (\Omega) \), \( 0 \leq \Phi \leq M \) and

\[
\int_{\Omega} |G_M (., \Phi)| = \int_{U_0} \left| \int_M s^{-\beta_0} ds \right| \, dx.
\]
\[ + \sum_{j=1}^{\infty} \int_{U_j} \left( \int_{B_j} s^{-\gamma} \, ds \right) \, dx + \sum_{j=1}^{\infty} \int_{U_j} \left( \int_{B_j} \varepsilon_j^{-\gamma} \, ds \right) \, dx \\
= \int_{U_0} \left| \frac{1}{1-\beta_0} \left( \Phi_0^{1-\beta_0} - M^{1-\beta_0} \right) \right| \\
+ \sum_{j=1}^{\infty} \left| \frac{1}{1-\gamma} \left( \varepsilon_j^{1-\gamma} - M^{1-\gamma} \right) \right| |U_j| + M \sum_{j=1}^{\infty} \varepsilon_j^{-\gamma} |U_j|. \]

Since \( \lim_{j \to \infty} \varepsilon_j = 0 \), we have \( \varepsilon_j < M \) for \( j \) large enough, and so, for any such \( j \), we have \( \left| \varepsilon_j^{1-\gamma} - M^{1-\gamma} \right| = \varepsilon_j^{1-\gamma} - M^{1-\gamma} \leq \varepsilon_j^{1-\gamma} \). Then, taking into account (45), we get \( \int_{\Omega} |G_M (., \Phi)| < \infty \), i.e., \( \Phi \in D_M \), and so \( (g_4) \) holds.

Thus Theorem 1.2 (applied with \( f = 0 \)) says that the problem

\[
\begin{cases}
-\Delta u = \chi_{\{u>0\}} g (., u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega, \\
u \geq 0 \text{ in } \Omega, \ n \neq 0
\end{cases}
\]

has a weak solution (in the sense of Definition 1.1) \( u \in H^1_0 (\Omega) \cap L^\infty (\Omega) \). Reasoning as we did at the end of the Example 1, we conclude that \( u \) is a positive weak solution, in the standard \( H^1_0 (\Omega) \) sense, to the problem \(-\Delta u = g (., u) \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \), and that \( u \geq c' \delta_\Omega \text{ a.e. in } \Omega \) for some positive constant \( c' \).

Notice that there are no positive constants \( C, \tau_0 \), such that \( g (x, s) \leq Cs^{-\theta} \) in \( \Omega \times (0, \tau_0) \). Indeed, if such constants existed, taking \( x \in U_j \) and \( s = \varepsilon_j \) we would get \( \varepsilon_j^{1-\gamma} \leq C\varepsilon_j^{-\theta} \) for all \( j \in \mathbb{N} \), and so \( \varepsilon_j^{1-\gamma} \leq C \); and taking \( \lim_{j \to \infty} \), we would obtain \( \theta - \gamma > 0 \), which is a contradiction.

An example of a domain \( \Omega \subset \mathbb{R}^n \), sequences \( \{ U_j \}_{j \in \mathbb{N} \cup \{ 0 \} } \), \( \{ \varepsilon_j \}_{j \in \mathbb{N} } \), \( \beta_0 \), \( \gamma \), that satisfy all the above conditions (in particular those in (45)) can be given as follows: Let \( \Omega = B_2 (2, 0) \) (where \( B_r (x) \) denotes the open ball with radius \( r \) centered at \( x \)). Let \( U_0 = B_1 (1, 0) \) and, for \( j \geq 1 \), let \( U_j = B_{r_j} (r_j, 0) - B_{r_j-1} (r_j-1, 0) \), with \( r_j = 2 - 2^{-j} \), let \( \beta_0 \in (0, 3) \), let \( \gamma > \max \{ 3, 2n \} \), and, for \( j \in \mathbb{N} \), let \( \varepsilon_j = \left( \frac{1}{1-\gamma} 2^j \alpha (n) \left( r_j^n - r_{j-1}^- \right) \right)^{1/\gamma} \) where \( \alpha (n) \) stands for the volume of the unit ball in \( \mathbb{R}^n \). Then \( \Omega = \cup_{j \in \mathbb{N} \cup \{ 0 \} } U_j \), and the only facts to be verified are those in (45), and that \( \lim_{j \to \infty} \varepsilon_j = 0 \). Note that \( 0 < \varepsilon_j \leq \left( \frac{1}{1-\gamma} 2^j \alpha (n) \left( r_j^n - r_{j-1}^- \right) n^{2n-1} \right)^{1/\gamma} \), the last inequality holds because of the mean value theorem. Therefore \( \lim_{j \to \infty} \varepsilon_j = 0 \).

Also, \( \sum_{j=1}^{\infty} \varepsilon_j^{1-\gamma} |U_j| = \sum_{j=1}^{\infty} \frac{1}{1-\gamma} \left( r_j^n - r_{j-1}^- \right) \alpha (n) (r_j^n - r_{j-1}^-)^{1-\gamma} \leq \sum_{j=1}^{\infty} \left( \frac{1}{1-\gamma} \right)^{1-\gamma} 2^{-\frac{j\gamma}{1-\gamma}} (\alpha (n) (r_j^n - r_{j-1}^-))^{1-\gamma} < \infty \), therefore the first inequality in (45) holds. Finally, since \( r_j^n - r_{j-1}^- \geq (r_j - r_{j-1})^n \) and \( \gamma > 2n \), we have, for some positive constant \( c \),

\[
\sum_{j=1}^{\infty} \varepsilon_j^{1-\gamma} |U_j| = \sum_{j=1}^{\infty} \left( \frac{1}{1-\gamma} \right)^{1-\gamma} 2^{-\frac{j\gamma}{1-\gamma}} (\alpha (n) (r_j^n - r_{j-1}^-))^{1-\gamma} \leq \sum_{j=1}^{\infty} \left( \frac{1}{1-\gamma} \right)^{1-\gamma} 2^{-\frac{j\gamma}{1-\gamma}} (\alpha (n) (r_j^n - r_{j-1}^-))^{1-\gamma} = c \sum_{j=1}^{\infty} 2^{-\frac{j\gamma}{1-\gamma}} 2^{jn} < \infty
\]

and then the second inequality of (45) also holds.
Proof of Theorem 1.4. For $\lambda \geq 0$, $\lambda f$ satisfies the same assumptions fulfilled by $f$, and so i) follows from Theorem 1.2. If, in addition, $f(.,0) = 0$ and $\lim_{s \to \infty} \frac{f(x,s)}{s} = 0$ uniformly on $\Omega$, and if the function $b$ provided by $(f_4)$ satisfies $\text{ess inf}_\Omega b > 0$, then $-f$ satisfies $(f_1)$-$(f_4)$; therefore, for $\lambda < 0$, writing $\lambda f = -\lambda (-f)$, ii) follows from i). Finally, iii) follows from Theorem 1.2.

In order to emphasize the dependence on $f$, we will sometimes write $J_{M,f}$ for the functional $J_M$.

Proof of Theorem 1.5. Assume that $(f_5)$ holds. Let $\beta$ be as in Remark 5, $m := \max \{2,1+\beta\}$, and let $k > \max \left\{0, \sup_{s>0} \text{ess sup}_{x \in \Omega} \frac{f(x,s)}{s}\right\}$. Thus $\frac{\lambda f(x,s)}{s} \leq \lambda k$ a.e. in $\Omega$, for all $s > 0$. For $0 < \lambda < \frac{\lambda k}{m}$ we have $\lambda_1 (\lambda k 1) = \frac{\lambda k}{\lambda} > m$, where $\lambda_1$ and $\lambda_1 (\lambda k 1)$ denote the first eigenvalues for $-\Delta$ on $\Omega$, with homogeneous Dirichlet condition, and weight functions $1$ and $\lambda k 1$ respectively. Thus Theorem 1.2 gives, for such a $\lambda$, a weak solution to problem (4). Note also that, for $\lambda = 0$, (4) reduces to $-\Delta u = \chi_{\{u>0\}} g(.,u)$ in $\Omega$, $u \geq 0$ in $\Omega$, $u = 0$ on $\partial \Omega$; and, again by Theorem 1.2, this problem has a positive weak solution $u \in H_0^1 (\Omega) \cap L^\infty (\Omega)$. Therefore the lemma holds with $\lambda^* := \frac{\lambda_1 (\Omega)}{\lambda k}$.

Assume now that $(f_6)$ holds. Let $V := \{k \in (0,\infty) : f(.,k) \in L^\infty (\Omega)\}$. Since $f \in L^\infty (\Omega \times (0,\sigma))$ for all $\sigma > 0$, we have that $\mathbb{R} \setminus V$ has zero Lebesgue measure. For $k \in V$ to be chosen latter, let $f_k : \Omega \times (0,\infty)$ be defined by $f_k(.,s) := f(.,s)$ if $0 \leq s \leq k$, and by $f_k(.,s) := f(.,s)$ otherwise, and let $\lambda > 0$. Clearly $\lambda f_k$ satisfies the conditions $(f_1)$-$(f_3)$. Since $f(.,k) \in L^\infty (\Omega)$, we have $\lim_{s \to \infty} \frac{\lambda f_k(.,s)}{s} = 0$ uniformly on $\Omega$, and so $\lambda f_k$ satisfies also $(f_4)$. Let $u_\lambda \in H_0^1 (\Omega) \cap L^\infty (\Omega)$ be the solution to the problem:

$$
\begin{cases}
-\Delta u_\lambda = \chi_{\{u_\lambda>0\}} a u_\lambda^{-\alpha} + \lambda f_k(x,u_\lambda) \quad \text{in} \ \Omega, \\
\int_{\Omega} u_\lambda = 0 \quad \text{on} \ \partial \Omega, \\
u_\lambda \geq 0 \quad \text{in} \ \Omega, \\
\end{cases}
$$

provided by Theorem 1.2. Thus $u_\lambda \neq 0$ and, by the proof of Theorem 1.2, $u_\lambda$ is a minimizer for $J_{M,\lambda f_k}$ on $D_M$ for some $M$ positive and large enough. Let $s_0 > 0$, let $\beta$, and $A$ be as given in Remark 5, let $m := \max \{2,1+\beta\}$, and let $\eta \in (0,1)$. Take $\Lambda \in (0,\infty)$, and define $s_1 := \max \left\{\frac{\eta_1 m}{\eta_1 m}, k\right\}$. Thus, for $s > s_1$ and $0 \leq \lambda < \Lambda$, $\frac{\lambda |f_k(.,s)|}{s} \leq \frac{\Lambda |f(.,k)|}{s_1} \leq \frac{\eta_1 m}{\eta_1 m} \leq \frac{\lambda_1 m}{m}$ a.e. in $\Omega$.

Since $u_\lambda$ is a minimizer for $J_{M,\lambda f_k}$ on $D_M$, from Lemma 2.4 we have, for $0 \leq \lambda < \Lambda$, and in the weak sense of Definition 1.1 (i.e., with test functions in $H_0^1 (\Omega) \cap L^\infty (\Omega)$),

$$
-\Delta (u_\lambda^m) \leq m u_\lambda^{m-1} g(.,u_\lambda) + m u_\lambda^{m-1} \lambda f_k(.,u_\lambda)
= m u_\lambda^{m-1} g(.,u_\lambda) \chi_{\{u_\lambda < s_0\}} + m u_\lambda^{m-1} g(.,u_\lambda) \chi_{\{u_\lambda \geq s_0\}}
+ m u_\lambda^{m-1} \lambda f_k(.,u_\lambda) \chi_{\{u_\lambda < s_1\}} + m u_\lambda^{m-1} \lambda f_k(.,u_\lambda) \chi_{\{u_\lambda \geq s_1\}}
\leq m A s_0^{m-1-\beta} + m A u_\lambda^{m-1} s_0^{-\beta} + m A D s_1^{m-1} + m A D s_1^{m-1} + \eta \lambda_1 u_\lambda^m
$$

where $D := \|f\|_{L^\infty (\Omega \times (0,s_1))} < \infty$. Since $m A s_0^{m-1-\beta} + m A u_\lambda^{m-1} s_0^{-\beta} + m A D s_1^{m-1} + \eta \lambda_1 u_\lambda^m \in L^\infty (\Omega)$, Remark 6 says that the inequality

$$
-\Delta (u_\lambda^m) \leq m A s_0^{m-1-\beta} + m A u_\lambda^{m-1} s_0^{-\beta} + m A D s_1^{m-1} + \eta \lambda_1 u_\lambda^m
$$

(47)
holds in the usual $H^1_0(\Omega)$ weak sense (i.e., for any test function in $H^1_0(\Omega)$). Now, $\eta\lambda_1 < \lambda_1$, and so $(-\Delta - \eta\lambda_1)^{-1} : L^\infty(\Omega) \to L^\infty(\Omega)$ is a well defined, bounded, and positive operator; let $C := \|(-\Delta - \eta\lambda_1)^{-1}\|_{L^\infty(\Omega),L^\infty(\Omega)}$. Then, since $u_\lambda$ is nonnegative, from (47) we get $\|u_\lambda\|_\infty \leq C \left( mAs_0^{-\beta} \|u_\lambda\|_\infty^{-1} + mAs_0^{-1-\beta} + m\lambda D_1 \right)$ and so, either $\|u_\lambda\|_\infty \leq 3CmAs_0^{-\beta} \|u_\lambda\|_\infty^{-1}$ or $\|u_\lambda\|_\infty \leq 3CmAs_0^{-1-\beta}$ or $\|u_\lambda\|_\infty \leq 3Cm\lambda D_1^{-1}.

Now we choose $k \in V$ such that $k > \max \left\{ 3CmAs_0^{-\beta}, \left( 3CmAs_0^{-1-\beta} \right)^{\frac{3}{2}} \right\}$. If either $\|u_\lambda\|_\infty \leq 3CmAs_0^{-\beta} \|u_\lambda\|_\infty^{-1}$ or $\|u_\lambda\|_\infty \leq 3CmAs_0^{-1-\beta}$, then $\|u_\lambda\|_\infty \leq k$; therefore $f_k(\cdot, u_\lambda) = f(\cdot, u_\lambda)$, and so $u_\lambda$ is a solution to (4). If $\|u_\lambda\|_\infty \leq 3Cm\lambda D_1^{-1}$, then $\|u_\lambda\|_\infty \leq \lambda^{\frac{c}{m}} \left( 3CmD_1^{-1} \right)^{\frac{3}{2}}$. Define $\lambda^* := \min \left\{ \lambda, \frac{c}{3CmD_1^{-1}} \right\}$. If $\lambda \in [0, \lambda^*)$ then $u_\lambda \leq k$ and so $f_k(\cdot, u_\lambda) = f(\cdot, u_\lambda)$, which implies that $u_\lambda$ solves (4). Finally, the assertion $u_\lambda \neq 0$ follows from Theorem 1.2 applied with $f$ replaced by $\lambda f$.

**Remark 10.** If $(g_1)-(g_5)$ and $(f_1)-(f_3)$ hold and, in addition, $f \leq 0$, then (4) has a weak solution (in the sense of Definition 1.1) for all $\lambda \geq 0$. This follows from Theorem 1.2 applied with $\lambda f$ instead of $f$.

**Proof of Theorem 1.6.** Clearly it is enough to prove i). Conditions $(g_1)$, $(g_2)$, $(g_3)$, and $(g_5)$ are readily verified. In order to see that $(g_3)$ is satisfied, note that, for $x \in \Omega$ and $s > 0$, since $g(x,s) = a(x) s^{-\alpha(x)} = 2^\alpha g(x,2s)$, $(g_3)$ is satisfied with constant $c := 2^\alpha$. Let us now prove that the remaining condition $(g_4)$ holds. Consider, for $M > 0$, the positive principal eigenfunction $\phi$ for $-\Delta$ in $\Omega$, with homogenous Dirichlet boundary condition and weight function $a$, normalized by $\|\phi\|_\infty = M^\frac{1-\alpha}{2\alpha}$. Now consider $u := \phi^\frac{1-\alpha}{2\alpha} \in L^2(\Omega)$. Let us see that $\nabla u \in L^2(\Omega)$: as $\nabla u = \frac{1}{2\alpha} \phi^\frac{1-\alpha}{\alpha} \nabla \phi$, and $\nabla \phi \in L^\infty(\Omega)$, it is enough to note that $\phi^\frac{1-\alpha}{\alpha} \in L^2(\Omega)$, which is true, since $0 \leq \alpha < 3$ implies $\frac{1-\alpha}{\alpha} > -\frac{3}{2}$. Moreover, $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial \Omega$. Thus $u \in H_0^1(\Omega)$, and clearly $u(x) > 0$ for $x \in \Omega$, and $u \leq M$ in $\Omega$. We claim that $u \in D_M$. In order to see this, define $\Omega' := \{ x \in \Omega : \alpha(x) \neq 1 \}$ and $\Omega_1 := \{ x \in \Omega : \alpha(x) = 1 \}$.

We have,

$$G_M(x,u(x)) = \frac{a(x)}{1-\alpha(x)} \left( u(x)^{1-\alpha(x)} - M^{1-\alpha(x)} \right) \quad (48)$$

$$= a(x) M^{1-\alpha(x)} \frac{1}{1-\alpha(x)} \left( \frac{u(x)}{M} \right)^{1-\alpha(x)} - 1 \right) \text{ for } x \in \Omega' ,$$

$$G_M(x,u(x)) = a(x) \ln \left( \frac{u(x)}{M} \right) \text{ for } x \in \Omega_1 .$$

For a fixed $t \in [0,1]$ and $\tau \in \mathbb{R} \setminus \{0\}$, the mean value theorem gives $\frac{1}{\tau} (t^\tau - 1) = t^\theta \ln t$ for some $\theta$ belonging to the open interval with endpoints 0 and $\tau$. In particular, $\theta > 0$ and so, since $t \in [0,1]$, there exists a positive constant $C$ such that $\left| \frac{1}{\tau} (t^\tau - 1) \right| \leq C \ln t$ for any $t \in (0,1]$ and $\tau \in \mathbb{R} \setminus \{0\}$. From this fact, and from (48), it follows that there exists a positive constant $C'$ such that $|G_M(x,u(x))| \leq C' \ln \left| u(x) - \ln M \right|$ for any $x \in \Omega$. Notice that $u^{-\delta} \in L^1(\Omega)$ for $0 < \delta < \frac{1-\alpha}{2\alpha}$, and that (because $u \leq M$ in $\Omega$) $u^{\delta} \ln |u|$ is bounded on $\Omega$. Thus
\[ u \in L^1(\Omega) \text{ and so } G_M(\cdot, u) \in L^1(\Omega). \] Thus \( u \in D_M \) and then \( g \) satisfies also (g4).

**REFERENCES**

[1] B. Bougherara and J. Giacomoni, Existence of mild solutions for a singular parabolic equation and stabilization, *Adv. Nonlinear Anal.*, 4 (2015), 123–134.

[2] B. Bougherara, J. Giacomoni and J. Hernández, Existence and regularity of weak solutions for singular elliptic problems, *2014 Madrid Conference on Applied Mathematics in honor of Alfonso Casal*, Electron. J. Diff. Equ., 22 (2015), 19–30.

[3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 1st edition, Springer-Verlag, New York, 2011.

[4] H. Brezis and X Cabré, Some simple nonlinear pde’s without solutions, *Bollettino dell’Unione Matematica Italiana*, 1 (1998), 223–262.

[5] A. Callegari and A. Nachman, A nonlinear singular boundary-value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.*, 38 (1980), 275–281.

[6] Y. Chu, Y. Gao and W. Gao, Existence of solutions to a class of semilinear elliptic problem with nonlinear singular terms and variable exponent, *Journal of Function Spaces*, 2016, Art. ID 9794739, 11 pp.

[7] F. Cirstea, M. Ghergu and V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type, *Math. Pures Appl.*, 84 (2005), 493–508.

[8] M. M. Coclite and G. Palmieri, On a singular nonlinear Dirichlet problem, *Comm. Part. Differ. Equat.*, 14 (1989), 1315–1327.

[9] D. S. Cohen and H. B. Keller, Some positive problems suggested by nonlinear heat generators, *J. Math. Mech.*, 16 (1967), 1361–1376.

[10] M. G. Crandall, P. H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Part. Differ. Equations*, 2 (1977), 193–222.

[11] J. Dávila and M. Montenegro, Positive versus free boundary solutions to a singular elliptic equation, *J. Anal. Math.*, 90 (2003), 303–335.

[12] D. G. De Figueiredo, Positive solutions of semilinear elliptic equations, in Lect. Notes Math. 957, *Differential Equations* (eds. D. G. de Figueiredo and C. S. H. A. Nig), Springer-Verlag, New York, (1982), 34–87.

[13] M. A. del Pino, A global estimate for the gradient in a singular elliptic boundary value problem, *Proc. R. Soc. Edinburgh Sect. A*, 122 (1992), 341–352.

[14] J. I. Díaz and J. Hernández, Positive and free boundary solutions to singular nonlinear elliptic problems with absorption; An overview and open problems, *Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems* (2012). *Electron. J. Diff. Equ.*, 21 (2014), 31–44.

[15] J. Díaz, M. Morel and L. Oswald, An elliptic equation with singular nonlinearity, *Comm. Part. Differ. Eq.*, 12 (1987), 1333–1344.

[16] L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane-Emden-Fowler equations with convection and singular potential*, J. Math. Pures Appl., 87 (2007), 563–581.

[17] W. Fulks and J. S. Maybee, A singular nonlinear equation, *Osaka Math. J.*, 12 (1960), 1–19.

[18] L. Gasinski and N. S. Papageorgiou, *Nonlinear elliptic equations with singular terms and combined nonlinearities*, *Ann. Henri Poincaré*, 13 (2012), 481–512.

[19] M. Ghergu, V. Liskevich and Z. Sobol, Singular solutions for second-order non-divergence type elliptic inequalities in punctured balls, *J. Anal. Math.*, 123 (2014), 251–279.

[20] M. Ghergu and V. D. Rădulescu, Multi-parameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term, *Proc. Royal Soc. Edinburgh*, Sect. A, 135 (2005), 61–84.

[21] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, 1st edition, Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, No 37, 2008.

[22] T. Godoy and A. Guerin, Nonnegative solutions of a singular elliptic problem, *Electron. J. Diff. Equ.*, 2016 (2016), 1–16.

[23] T. Godoy and A. Guerin, Existence of nonnegative solutions for some singular elliptic problems, *Journal of Nonlinear Functional Analysis*, 2017, Article ID 11, 1–23.
[24] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary value problem, Proc. Amer. Math. Soc., 111 (1991), 721–730.

[25] N. S. Papageorgiou and G. Smyrlis, Nonlinear elliptic equations with singular reaction, Osaka J. Math., 53 (2016), 489–514.

[26] V. D. Rădulescu, Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities, in Handbook of Differential Equations: Stationary Partial Differential Equations, Vol. 4 (ed. M. Chipot), North-Holland Elsevier Science, Amsterdam, (2007), 483–591.

[27] J. Shi and M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. R. Soc. Edinburgh, Sect A, 128 (1998), 1389–1401.

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