Analysis and Experimental Verification of Model Solutions of the Pekeris Boundary Problem in the Infrasonic Frequency Range

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Abstract. A comparative assessment of the known solutions of the Pekeris boundary problem is carried out. A detailed analysis of the generalized solution, which is constructed in a non-self-adjoint model formulation, is carried out. One of the features of the generalized solution is the presence of the vortex component of the intensity vector, the level of which increases with decreasing frequency. The vortex component is recorded in the channels of the vector receiver during an experiment with a noisy object.

1. Introduction

The model description of sound fields is based on the solution of boundary problems for the Helmholtz operator. A classical example of such a boundary problem is the Pekeris boundary problem [1]. However, this solution was not the only one, three more solutions were obtained in the class of analytic functions [2-4]. Comparative analysis of solutions in the class of analytic functions, the integral solution in the class of functions that can be represented by the Fourier-Bessel integral, and the generalized solution constructed in the non-self-adjoint model formulation (NMF) was performed in work [5]. A rather detailed analysis of the generalized solution was carried out in works [5-7]. The fundamental differences between these solutions are as follows.

The solution represented by the Fourier-Bessel integral describes the sound field of the standing wave type in the waveguide, and the sound field of the diverging spherical wave type, modulated by the angular spectrum of the source, in the half-space. Only the decision of L.M. Brekhovskikh [2] is correct in the all domain of definition of a waveguide - half-space of the four solutions [1-4] constructed in the class of analytic functions. It defines the sound field, represented by diverging waves, as eigenfunctions of a self-adjoint operator. In this model solution, the waveguide and the half-space are considered as two energetically independent subsystems, and the power flow through the waveguide – half-space interface is identically equal to zero. This circumstance contradicts the fact that the Pekeris waveguide is an open system with energy losses for radiation into the half-space at angles of incidence less than the critical value. The rest of the solutions of this class are correct only when describing the sound field in the waveguide itself, since they all contain leakage waves of the complex spectrum, the amplitude of which grows exponentially in the half-space.

The generalized solution constructed within the framework of the NMF is correct and limited in the all domain of definition. It is discontinuous in pressure and the normal component of the particle
velocity at the transformation horizons, but continuous in impedance, determined through the ratio of integral quantities such as force and volume particle velocity. The alternating normal component of the intensity vector with a half-wavelength period is generated at the transformation horizons due to the discontinuity of the solution. The latter circumstance means that the integral power flow is equal to zero and the generalized solution is correct. The most popular is the classic solution of L.M. Brekhovskikh, built in a self-coupled model formulation. However, the generalized solution constructed within the framework of the NMF is most consistent with the experiment, and this correspondence grows with decreasing frequency.

2. Analysis of the model solutions
Differences in the model description become significant at frequencies lower than the first critical frequency of the model Pekeris waveguide, when leaky waves and the zero mode of the generalized solution become dominant in the total sound field. The zero mode is an inhomogeneous (slow) wave localized at the source horizon. Its speed \( c_0 \) is less than the speed of sound in water \( c_1 \). This velocity corresponds to zero reflection coefficient of the water – liquid seabed interface, which is realized in the region of complex angles of incidence [5].

All poles corresponding to the eigenvalues of the non-self-adjoint operator describing the Pekeris boundary problem appear on the top sheet of the Riemann surface if the cut corresponding to the condition \( \text{Re} k_{32} = 0 \) is used (\( k_{32} \) is the vertical wavenumber in the half-space). Such a cut was first used in [4], and the transition to the NMF is a regularization of the solution [4] in the domain of definition of the waveguide – half-space. The real and complex poles, which correspond to the fours of the eigenvalues of the horizontal wave number (\( \xi, -\xi, \xi^*, -\xi^* \)), appear on the top sheet of the Riemann surface when using this cut. The location of the poles on the plane of the complex variable with a cut corresponding to the condition \( \text{Re} k_{32} = 0 \), and the migration of the poles with a change in the frequency parameter \( k_i h \) are shown by an arrow in Figure 1.

![Figure 1](image_url)

**Figure 1.** The location of the poles on the plane of the complex spectral parameter corresponding to the subsets \( n(1), n(2), n(3) \) of normal waves, \( k_i = \omega/c_i \).

Figure 1 corresponds to the representation of the vertical wavenumber in the half-space in the form \( k_{32} = a_2 + i\alpha_2, \ a_2 \geq 0 \). The bold line in Figure 1 corresponds to the poles located on the upper (physical) sheet of the Riemann surface (\( a_2 \geq 0 \)). Individual poles are marked with symbols \( n(1), \ n(2), \ n(3) \).
for diverging waves and symbols \( \bar{n}(2) \), \( \bar{n}(3) \) for backward waves, also included in the total solution. The subset \( n(1) \) includes normal waves that continue into the half-space as an inhomogeneous wave with a decreasing amplitude. For such normal waves, the half-space input impedance is reactive, inertial. The subset \( n(2) \) includes normal waves that continue into the half-space as an inhomogeneous wave with an increasing amplitude. For such normal waves, the half-space input impedance is reactive, elastic. The subset \( n(3) \) includes normal waves of the complex spectrum (leakage waves), which continue into the half-space as an inhomogeneous wave, the amplitude of which decreases along the horizontal coordinate and increases along the vertical coordinate. For such normal waves, the input impedance of the half-space is active-reactive, elastic with a positive definite real part.

The main differences in the structure of the sound field for the generalized solution and the classical solution are shown in Figure 2 for a frequency greater than the first critical frequency, and in Figure 3 for a frequency less than the first critical frequency. The Figures 2, 3 show the sound pressure fields (with the excluded cylindrical divergence) in coordinates normalized to the waveguide depth \( h \).

A characteristic feature of the generalized solution is the presence of a zero mode localized at the source horizon. Another feature is the continuous transition of leaky normal waves of the family \( n(3) \) of the complex spectrum with wavenumbers \( (\xi, -\xi^*) \), which form a standing wave in the waveguide, into normal waves of the families \( n(1), n(2) \), captured by the waveguide. This transition occurs at the frequencies of the longitudinal resonance, which are multiple roots of the dispersion equation.

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**Figure 2.** Generalized solution (a) and classical solution (b); \( k/h = 20 \), \( z_0 = 0.5 \).
Figure 3. Generalized solution (a) and classical solution (b); $k_h = 2.7$, $z_{01} = 0.5$.

The appearance of backward waves with a positive definite group velocity in the total solution is a condition for the realization of longitudinal resonance. Such a mechanism is well known in the theory of solid waveguides with a free surface [8], in which quasi-transverse waves are generated at transverse resonance frequencies, quasi-longitudinal waves are generated at longitudinal resonance frequencies, and the boundary value problems for solid waveguides are always described within the framework of the NMF. The unity of the formation mechanism of wave motion in liquid and solid waveguides indicates that the phenomenon of longitudinal resonance is a fundamental phenomenon. A model description of longitudinal resonances in a wedge-shaped waveguide with a small opening angle is given in work [6], where experimental confirmation of this phenomenon is described.

3. **Experimental verification of model solutions**

Receiving systems based on combined receivers (CR) are widely used in the study of sound fields generated by natural or simulated sources. An example is the work [10-15], in which extensive experimental studies of the main characteristics of sound fields with the use of combined receivers were carried out. In this work, the emphasis of experimental studies is placed on the comparison of experimental data and various model descriptions of the sound field in the Pekeris waveguide, which are listed above.

An experimental study of the sound field generated by a noisy object in a shallow sea was carried out using a receiving system equipped with combined receivers. The sea depth at the site of the experiment was 105 m, the combined receiver was located at a horizon of 50 m. The noisy object moved along a straight line relative to the receiving system at a constant speed. The traverse point corresponds to the moment of time 08 hours 05 minutes. The pass-through characteristics in the channels of the combined receiver are shown in Figures 4, 5. The top line refers to the sound pressure channel. The pressure level of the summed sound field ($S + N$) is shown in red, the noise level $N$ is shown in blue. The lower lines display the level of the intensity vector components $I_x$, $I_y$, $I_z$, and positive definite logarithms are plotted towards positive values along the ordinate axis, if the component is positive, and towards negative values, if the component is negative.
Figure 4. Pass-through characteristics in the channels of the CR, frequency band 4-8 Hz.

Figure 5. Pass-through characteristics in the channels of the CR, frequency band 50-51 Hz.

It is clearly seen that at frequencies of the infrasonic range of 4-8 Hz in all vector channels, the vortex (alternating) component of the intensity vector prevails, which is generated in structures such as
standing waves. This effect can only be described within the framework of the NMF, and is well confirmed experimentally. With increasing frequency, the level of the vortex component of the intensity vector decreases and the level of the horizontal (potential) component increases. The presence of horizontal power flows in the waveguide at frequencies lower than the first critical frequency is confirmed by experiment, and also requires a transition to the description of the sound field in the framework of the NMF.

4. Conclusion
The comparison of the model description of sound fields in the framework of the generalized solution constructed in the non-self-adjoint model formulation and the classical solution has been completed. The fundamental differences between these solutions, which become significant in the infrasonic frequency range, are noted. The experimental data obtained using the CR, which confirm the preference of the model solution constructed in the framework of the NMF, are presented.

5. References
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