On Serre’s injectivity question and norm principle

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Abstract. Let $k$ be a field of characteristic not 2. We give a positive answer to Serre’s injectivity question for any smooth connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. We do this by relating Serre’s question to the norm principles proved by Barquero and Merkurjev. We give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the non-trialitarian $D_n$ case and yield a positive answer to Serre’s question for connected reductive $k$-groups whose Dynkin diagrams contain components of (non-trialitarian) type $D_n$ too. We also investigate Serre’s question for quasi-split reductive $k$-groups.

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1. Introduction

Let $k$ be a field. Then the following question of Serre, which is open in general, asks

Question 1.1 (Serre, [13, p. 233]). Let $G$ be any connected linear algebraic group over a field $k$. Let $L_1, L_2, \ldots, L_r$ be finite field extensions of $k$ of degrees $d_1, d_2, \ldots, d_r$ respectively such that $\gcd(d_i) = 1$. Then is the following sequence exact?

$$1 \to H^1(k, G) \to \prod_{i=1}^{r} H^1(L_i, G).$$

The classical result that the index of a central simple algebra divides the degrees of its splitting fields answers Serre’s question affirmatively for the group $\text{PGL}_n$. Springer’s theorem for quadratic forms answers it affirmatively for the (albeit sometimes disconnected) group $\text{O}(q)$ and Bayer–Lenstra’s theorem [2], for the groups of isometries of algebras with involutions. Jodi Black [3] answers Serre’s question positively for absolutely simple simply connected and adjoint $k$-groups of classical type. In this paper, we use and extend Jodi’s result to connected reductive $k$-groups whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$.

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**Theorem 1.2.** Let $k$ be a field of characteristic not 2. Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. Then Serre’s question has a positive answer for $G$.

We also investigate Serre’s question for reductive $k$-groups whose derived subgroups admit quasi-split simply connected covers. More precisely, we give a uniform proof for the following:

**Theorem 1.3.** Let $k$ be a field of characteristic not 2. Let $G$ be a connected quasi-split reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$. Then Serre’s question has a positive answer for $G$.

We relate Serre’s question for $G$ with the norm principles of other closely related groups following a series of reductions previously used by Barquero and Merkurjev to prove the norm principles for reductive groups whose Dynkin diagrams do not contain connected components of type $D_n$, $E_6$ or $E_7$ [1]. We also give a scalar obstruction defined up to spinor norms whose vanishing will imply the norm principle for the (non-trialitarian) $D_n$ case and yield a positive answer to Serre’s question for connected reductive $k$-groups whose Dynkin diagrams contain components of this type also.

In the next section, we begin with some lemmata and preliminary reductions. In Section 3, we introduce intermediate groups $\hat{G}$ and $\tilde{G}$ and relate Serre’s question for $G$ to Serre’s question for $\hat{G}$ and $\tilde{G}$ via the norm principle. In Section 4, we investigate the norm principle for (non-trialitarian) type $D_n$ groups and find the scalar obstruction whose vanishing will imply the norm principle for the (non-trialitarian) $D_n$ case. In the final section, we use the reduction techniques used in Sections 2 and 3 to discuss Serre’s question for connected reductive $k$-groups whose derived subgroups admit quasi-split simply connected covers.

**2. Preliminaries**

We work over the base field $k$ of characteristic not 2. By a $k$-group, we mean a smooth connected linear algebraic group defined over $k$. And mostly, we will restrict ourselves to reductive groups. We say that a $k$-group $\hat{G}$ satisfies $SQ$ if Serre’s question has a positive answer for $\hat{G}$.

**2.1. Reduction to characteristic 0.** Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$, $C_n$ or (non-trialitarian) $D_n$. Without loss of generality we may assume that $k$ is of characteristic 0 [7, p. 47]. We give a sketch of the reduction argument for the sake of completeness.

Suppose that the characteristic of $k$ is $p > 0$. Let $L_1, L_2, \ldots, L_r$ be finite field extensions of $k$ of degrees $d_1, d_2, \ldots, d_r$ respectively such that $\gcd_i(d_i) = 1$ and
let $\xi$ be an element in the kernel of

$$H^1(k, G) \to \prod_{i=1}^r H^1(L_i, G).$$

By a theorem of Gabber, Liu and Lorenzini [5, Thm. 9.2] which was pointed out to us by O. Wittenberg, we note that any torsor under a smooth group scheme $G/k$ which admits a zero-cycle of degree 1 also admits a zero-cycle of degree 1 whose support is étale over $k$. Thus without loss of generality we can assume that the given coprime extensions $L_i/k$ are in fact separable.

By [10, Thms. 1 & 2], there exists a complete discrete valuation ring $R$ with residue field $k$ and fraction field $K$ of characteristic zero. Let $S_i$ denote corresponding étale extensions of $R$ with residue fields $L_i$ and fraction fields $K_i$.

There exists a smooth $R$-group scheme $\tilde{G}$ with special fiber $\tilde{G}_K$. Now given any torsor $t \in H^1(k, G)$, there exists a torsor $\tilde{t} \in H^1_{\text{ét}}(R, \tilde{G})$ specializing to $t$ which is unique up to isomorphism. This in turn gives a torsor $\tilde{t}_K$ in $H^1(K, \tilde{G}_K)$ by base change, thus defining a map $i_k : H^1(k, G) \to H^1(K, \tilde{G}_K)$ [6, p. 29]. It clearly sends the trivial element to the trivial element. The map $i$ also behaves well with the natural restriction maps, i.e., it fits into the following commutative diagram:

$$
\begin{array}{ccc}
H^1(k, G) & \xrightarrow{i_k} & H^1(K, \tilde{G}_K) \\
\downarrow & & \downarrow \\
\prod H^1(L_i, G) & \xrightarrow{\prod i_{k_i}} & \prod H^1(K_i, \tilde{G}_K).
\end{array}
$$

Let $\tilde{\xi}$ denote the torsor in $H^1_{\text{ét}}(R, \tilde{G})$ corresponding to $\xi$ as above. Therefore $\tilde{\xi}_K := i_k(\tilde{\xi})$ is in the kernel of

$$H^1(K, \tilde{G}_K) \to \prod_{i=1}^r H^1(K_i, \tilde{G}_K).$$

Suppose that $\tilde{G}_K$ satisfies $SQ$. Then $\tilde{\xi}_K$ is trivial. However by [12], the natural map $H^1_{\text{ét}}(R, \tilde{G}) \to H^1(K, \tilde{G}_K)$ is injective and hence $\tilde{\xi}$ is trivial in $H^1_{\text{ét}}(R, \tilde{G})$. This implies that its specialization, $\xi$, is trivial in $H^1(k, G)$.

Thus from here on, we assume that the base field $k$ has characteristic 0.

2.2. Lemmata.

Lemma 2.1. Let $k$-groups $G$ and $H$ satisfy $SQ$. Then $G \times_k H$ also satisfies $SQ$.

Proof. Let $L/k$ be a field extension. Then the map

$$H^1(k, G \times_k H) \to H^1(L, G \times_k H)$$
is precisely the product of the maps
\[ H^1(k, G) \to H^1(L, G) \quad \text{and} \quad H^1(k, H) \to H^1(L, H). \]
This immediately shows that if \( G \) and \( H \) satisfy \( SQ \), so does \( G \times_k H \).

**Lemma 2.2.** Let \( 1 \to Q \to H \to G \to 1 \) be a central extension of a \( k \)-group \( G \) by a quasi-trivial torus \( Q \). Then \( H \) satisfies \( SQ \) if and only if \( G \) satisfies \( SQ \).

**Proof.** Let \( L_i \) be field extensions of \( k \) such that \( \gcd[L_i : k] = 1 \). Since \( Q \) is quasi-trivial, \( H^1(L, Q) = \{ 1 \} \forall L/k \). From the long exact sequence in cohomology, we have the following commutative diagram.

\[
\begin{array}{cccccc}
1 & \to & H^1(k, H) & \to & H^1(k, G) & \to & H^2(k, Q) \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \prod H^1(L_i, H) & \to & \prod H^1(L_i, G) & \to & \prod H^2(L_i, Q) \\
\end{array}
\]

From the above diagram, it is clear that if \( G \) satisfies \( SQ \), so does \( H \).

Conversely, assume that \( H \) satisfies \( SQ \). Let \( a \in H^1(k, G) \) become trivial in \( \prod H^1(L_i, G) \). Then \( \delta_k(a) \) becomes trivial in each \( H^2(L_i, Q) \). Hence the corestriction \( \text{Cor}_{L_i/k} (\delta_k(a)) = \delta_k(a)^{d_i} \) becomes trivial in \( H^2(k, Q) \) where \( d_i = [L_i : k] \). Since \( \gcd(d_i) = 1 \), this implies that \( \delta_k(a) \) is itself trivial in \( H^2(k, Q) \). Therefore \( a \) comes from an element \( b \in H^1(k, H) \) which is trivial in \( \prod H^1(L_i, H) \). (The fact that \( H^1(L_i, Q) = \{ 1 \} \forall L/k \) guarantees that \( b \) is trivial in \( H^1(L_i, H) \).) Since \( H \) satisfies \( SQ \) by assumption, \( b \) is trivial in \( H^1(k, H) \) which implies the triviality of \( a \) in \( H^1(k, G) \).

**Lemma 2.3.** Let \( E \) be a finite separable field extension of \( k \) and let \( H \) be an \( E \)-group satisfying \( SQ \). Then the \( k \)-group \( R_{E/k}(H) \) also satisfies \( SQ \).

**Proof.** Set \( G = R_{E/k}(H) \) and let \( \xi \) be an element in the kernel of \( H^1(k, G) \to \prod H^1(L_i, G) \) where \( \gcd[L_i : k] = 1 \).

Since \( \text{char}(k) = 0 \), \( L_i \otimes_k E \) is an étale \( E \)-algebra and hence isomorphic to \( E_{1,i} \times E_{2,i} \times \cdots \times E_{n_i,i} \) where each \( E_{j,i} \) is a separable field extension of \( E \). Thus \( \sum_{i=1}^{n_i} [E_{j,i} : E] = [L_i : k] \) and therefore \( \gcd(E_{j,i} : E) = 1 \) where \( 1 \leq i \leq r \) and \( 1 \leq j \leq n_i \).

By Eckmann–Faddeev–Shapiro, we have a natural bijection of pointed sets
\[
H^1(k, G) \simeq H^1(E, H),
\]
\[
H^1(L_i, G) \simeq \prod_{j=1}^{n_i} H^1(E_{j,i}, H).
\]

Thus we have that \( \xi \) is in the kernel of \( H^1(E, H) \to \prod_{1 \leq j, i \leq n_i} H^1(E_{j,i}, H) \). Since \( H \) satisfies \( SQ \), we see that \( \xi \) is trivial.
3. Serre’s question and norm principles

3.1. Intermediate groups $\hat{G}$ and $\tilde{G}$. Notations are as in Section 5 of [1].

Let $G$ be our given connected reductive $k$-group whose Dynkin diagram contains
connected components only of type $A_n$, $B_n$, $C_n$ or (non-trialitarian) $D_n$ and let $G'$
denote its derived subgroup. Let $Z(G) = T$ and $Z(G') = \mu$.

Let $\rho : \mu \hookrightarrow S$ be an embedding of $\mu$ into a quasi-trivial torus $S$. We denote the
cofibre product $e(G', \rho) = \frac{G' \times S}{\mu}$ by $\hat{G}$. This $k$-group is called an envelope of $G'$.

\[
\begin{array}{ccc}
\mu & \fleq{\delta} & G' \\
\downarrow{\rho} & & \downarrow{\gamma} \\
S & \fleq{\gamma} & \hat{G}
\end{array}
\]

Now the quasi-trivial torus $S = Z(\hat{G})$ and $\hat{G}$ fit into an exact sequence as follows:

\[1 \to S \to \hat{G} \to G'^{\text{ad}} \to 1 \tag{*}\]

where $G'^{\text{ad}}$ corresponds to the adjoint group of $G'$. We now recall the following
result of Jodi Black which addresses Serre’s question for adjoint groups of classical
type.

**Theorem 3.1** (Jodi Black, [3, Thm. 0.2]). Let $k$ be a field of characteristic different
from 2 and let $J$ be an absolutely simple algebraic $k$-group which is not of type $E_8$
and which is either a simply connected or adjoint classical group or a quasi-split
exceptional group. Then Serre’s question has a positive answer for $J$.

Since every adjoint group of classical type is a product of Weil restrictions of
absolutely simple adjoint groups, the above theorem, along with Lemmata 2.1 and 2.3,
implies that $G'^{\text{ad}}$ satisfies $SQ$. Applying Lemma 2.2 to the exact sequence ($*$) above,
we see that $\hat{G}$ satisfies $SQ$. Let us chose such an envelope $\hat{G}$ of $G'$ which satisfies $SQ$.

Define an intermediate abelian group $\tilde{T}$ to be the cofibre product $\frac{T \times S}{\mu}$

\[
\begin{array}{ccc}
\mu & \fleq{} & T \\
\downarrow{\rho} & & \downarrow{\alpha} \\
S & \fleq{\nu} & \tilde{T}
\end{array}
\]

Let the algebraic group $\tilde{G}$ be the cofibre product defined by the following diagram:

\[
\begin{array}{ccc}
G' \times T & \fleq{m} & G \\
\downarrow{d \times \alpha} & & \downarrow{\beta} \\
G' \times \tilde{T} & \fleq{e} & \tilde{G}
\end{array}
\]
Then we have the following commutative diagram with exact rows [1, Prop. 5.1]. Note that each row is a central extension of $\hat{G}$.

\[
\begin{array}{ccc}
1 & \xrightarrow{\mu} & G' \times \hat{T} & \xrightarrow{\epsilon} & \hat{G} & \xrightarrow{id} & 1 \\
\downarrow{\rho} & & \downarrow{\gamma \nu} & & \downarrow{id} & & \\
1 & \xrightarrow{\psi} & S & \xrightarrow{\gamma} & \hat{G} \times \hat{T} & \xrightarrow{\epsilon} & \hat{G} & \xrightarrow{1} & 1
\end{array}
\]

(* *)

Since $\hat{T}$ is abelian, the existence of the co-restriction map shows that $\hat{T}$ satisfies $SQ$. Since $\hat{G}$ satisfies $SQ$, we can apply Lemmata 2.1 and 2.2 to (**) to see that $\hat{G}$ satisfies $SQ$.

### 3.2. Norm principle and weak norm principle

Let $f : G \rightarrow T$ be a map of $k$-groups where $T$ is an abelian $k$-group. Then we have norm maps $N_{L/k} : T(L) \rightarrow T(k)$ for any separable field extension $L/k$.

\[
\begin{array}{ccc}
G(L) & \xrightarrow{f(L)} & T(L) \\
& \downarrow{N_{L/k}} & \\
G(k) & \xrightarrow{f(k)} & T(k)
\end{array}
\]

We say that the **norm principle** holds for $f : G \rightarrow T$ if for all separable field extensions $L/k$,

\[
N_{L/k}(\text{Image } f(L)) \subseteq \text{Image } f(k).
\]

That is, we say that the **norm principle** holds for $f : G \rightarrow T$ if given any separable field extension $L/k$ and any $t \in T(L)$ such that

\[
t \in (\text{Image } f(L) : G(L) \rightarrow T(L)),
\]

then

\[
N_{L/k}(t) \in (\text{Image } f(k) : G(k) \rightarrow T(k)).
\]

Note that the norm principle holds for any algebraic group homomorphism between abelian groups.

We say that the **weak norm principle** holds for $f : G \rightarrow T$ if given any $t \in T(k)$ such that

\[
t \in (\text{Image } f(L) : G(L) \rightarrow T(L)),
\]

then

\[
t^{[L:k]} = N_{L/k}(t) \in (\text{Image } f(k) : G(k) \rightarrow T(k)).
\]

It is clear that if the norm principle holds for $f$, then so does the weak norm principle.
3.3. Relating Serre’s question and norm principle. The deduction of SQ for $G$ from $\hat{G}$ and $\hat{G}$ follows via the (weak) norm principles.

Let $\beta : G \to \hat{G}$ be the embedding of $k$-groups with the cokernel $P$ isomorphic to the torus $\frac{\mathbb{Z}}{\mu}$ where $\hat{G}$ and $G$ are as in Section 3.1. Thus we have the following exact sequence:

$$1 \to G \xrightarrow{\beta} \hat{G} \xrightarrow{\pi} P \to 1.$$ 

**Lemma 3.2.** If the weak norm principle holds for $\pi : \hat{G} \to P$, then $G$ satisfies SQ.

**Proof.** From the long exact sequence of cohomology, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
1 & \to & G(k) & \to & \hat{G}(k) & \xrightarrow{\pi_k} & P(k) & \xrightarrow{\delta_k} & H^1(k, G) & \xrightarrow{\beta_k} & H^1(k, \hat{G}) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \prod G(L_i) & \to & \prod \hat{G}(L_i) & \xrightarrow{\prod \pi_{L_i}} & \prod P(L_i) & \xrightarrow{\prod \delta_{L_i}} & \prod H^1(L_i, G) & \to & \prod H^1(L_i, \hat{G}).
\end{array}
$$

Let $a \in H^1(k, G)$ become trivial in $\prod H^1(L_i, G)$. As $\hat{G}$ satisfies SQ, $\beta_k(a)$ becomes trivial in $H^1(k, \hat{G})$. Hence $a = \delta_k(b)$ for some $b \in P(k)$ and $\delta_{L_i}(b)$ is trivial in $H^1(L_i, G)$. Therefore, there exist $c_i \in \hat{G}(L_i)$ such that $\pi_{L_i}(c_i) = b$.

Showing that $G$ satisfies SQ, i.e. that $a$ is trivial, is equivalent to showing

$$b \in \left(\text{Image } \pi_k : \hat{G}(k) \to P(k)\right).$$

However $b \in \left(\text{Image } \pi_{L_i} : \hat{G}(L_i) \to P(L_i)\right)$. Since the weak norm principle holds for $\pi : \hat{G} \to P$, $b^{d_i} \in \text{Image } \left(\pi_k : \hat{G}(k) \to P(k)\right)$ where $[L_i : k] = d_i$ for each $i$.

As $\gcd_i(d_i) = 1$, this means $b \in \text{Image } \left(\pi_k : \hat{G}(k) \to P(k)\right)$. \hfill $\square$

We recall now the norm principle of Merkurjev and Barquero for reductive groups of classical type.

**Theorem 3.3** (Barquero–Merkurjev, [1]). Let $G$ be a reductive group over a field $k$. Assume that the Dynkin diagram of $G$ does not contain connected components $D_n$, $n \geq 4$, $E_6$ or $E_7$. Let $T$ be any commutative $k$-group. Then the norm principle holds for any group homomorphism $G \to T$.

This shows that the norm principle and hence the weak norm principle holds for the map $\pi : \hat{G} \to P$ for reductive $k$-groups $G$ as in the main theorem (Theorem 1.2). Thus we have concluded the proof for the following:

**Theorem 1.2.** Let $k$ be a field of characteristic not 2. Let $G$ be a connected reductive $k$-group whose Dynkin diagram contains connected components only of type $A_n$, $B_n$ or $C_n$. Then Serre’s question has a positive answer for $G$. 
4. Obstruction to norm principle for (non-trialitarian) $D_n$

4.1. Preliminaries. Let $(A, \sigma)$ be a central simple algebra of degree $2n$ over $k$ and let $\sigma$ be an orthogonal involution. Let $C(A, \sigma)$ denote its Clifford algebra which is a central simple algebra over its center, $Z/k$, the discriminant extension. Let $i$ denote the non-trivial automorphism of $Z/k$ and let $\sigma$ denote the canonical involution of $C(A, \sigma)$.

Recall that, depending on the parity of $n$, $\sigma$ is either an involution of the second kind (when $n$ is odd) or of the first kind (when $n$ is even). Let $\mu : \text{Sim}(C(A, \sigma), \sigma) \to R_{Z/k} \mathbb{G}_m$ denote the multiplier map sending similitude $c$ to $\sigma(c)c$.

Let $\Omega(A, \sigma)$ be the extended Clifford group. Note that this has center $R_{Z/k} \mathbb{G}_m$ and is an envelope of $\text{Spin}(A, \sigma)$ [1, Ex. 4.4]. We recall below the map $\sim : \Omega(A, \sigma)(k) \to Z^*/k^*$ as defined in [9, p. 182].

Given $\omega \in \Omega(A, \sigma)(k)$, let $g \in \text{GO}^+(A, \sigma)(k)$ be some similitude such that $\omega \sim gk^*$ under the natural surjection $\Omega(A, \sigma)(k) \to \text{PGO}^+(A, \sigma)(k)$.

Let $h = \mu(g)^{-1}g^2 \in O^+(A, \sigma)(k)$ and let $\gamma \in \Gamma(A, \sigma)(k)$ be some element in the special Clifford group which maps to $h$ under the vector representation $\chi' : \Gamma(A, \sigma)(k) \to O^+(A, \sigma)(k)$. Then $\omega^2 = \gamma z$ for some $z \in Z^*$ and $\chi(\omega) = zk^*$.

Note that the map $\chi$ has $\Gamma(A, \sigma)(k)$ as kernel. Also if $z \in Z^*$, then $\chi(z) = z^2k^*$.

By following the reductions in [1], it is easy to see that one needs to investigate whether the norm principle holds for the canonical map

$$\Omega(A, \sigma) \to \frac{\Omega(A, \sigma)}{[\Omega(A, \sigma), \Omega(A, \sigma)]}.$$ 

We will need to investigate the norm principle for two different maps depending on the parity of $n$.

The map $\mu_*$ for $n$ odd. Let $U \subset \mathbb{G}_m \times R_{Z/k} \mathbb{G}_m$ be the algebraic subgroup defined by

$$U(k) = \{(f, z) \in k^* \times Z^* | f^4 = N_{Z/k}(z)\}.$$ 

Recall the map $\mu_* : \Omega(A, \sigma) \to U$ defined in [9, p. 188] which sends

$$\omega \sim (\mu(\omega), ai(a)^{-1} \mu(\omega)^2),$$

where $\omega \in \Omega(A, \sigma)(k)$ and $\chi(\omega) = a k^*$. This induces the following exact sequence [9, p. 190]

$$1 \to \text{Spin}(A, \sigma) \to \Omega(A, \sigma) \xrightarrow{\mu_*} U \to 1.$$ 

Since the semisimple part of $\Omega(A, \sigma)$ is $\text{Spin}(A, \sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map $\mu_*$. 


The map $\mu$ for $n$ even. Recall the following exact sequence induced by restricting $\underline{\mu}$ to $\Omega(A, \sigma)$ [9, p. 187]

$$1 \to \text{Spin}(A, \sigma) \to \Omega(A, \sigma) \xrightarrow{\mu} R_{Z/k}\mathbb{G}_m \to 1.$$ 

Since the semisimple part of $\Omega(A, \sigma)$ is Spin$(A, \sigma)$, the above exact sequence shows that it suffices to check the norm principle for the map $\underline{\mu}$.

4.2. An obstruction to being in the image of $\mu_*$ for $n$ odd. Given $(f, z) \in U(k)$, we would like to formulate an obstruction which prevents $(f, z)$ from being in the image $\mu_*(\Omega(A, \sigma)(k))$. Note that for $z \in Z^*$, $\mu_*(z) = (N_{Z/k}(z), z^4)$ and hence the algebraic subgroup $U_0 \subseteq U$ defined by

$$U_0(k) = \{(N_{Z/k}(z), z^4)|z \in Z^*\}$$

has its $k$-points in the image $\mu_*(\Omega(A, \sigma)(k))$.

Let $\mu_{4[Z]}$ denote the kernel of the norm map $R_{K/k}\mu_n N \to \mu_n$ where $K/k$ is a quadratic extension. Note that $\mu_{4[Z]}$ is the center of Spin$(A, \sigma)$ as $n$ is odd. Also recall that [9, Prop. 30.13, p. 418]

$$H^1(k, \mu_{4[Z]}) \cong \frac{U(k)}{U_0(k)}.$$

Thus, we can construct the map $S : \text{PGO}^+(A, \sigma)(k) \to H^1(k, \mu_{4[Z]})$ induced by the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
1 & \longrightarrow & Z^* & \longrightarrow & \Omega(A, \sigma)(k) & \longrightarrow & \text{PGO}^+(A, \sigma)(k) & \longrightarrow & 1 \\
& & \downarrow{\mu_*} & & \downarrow{\chi'} & & \downarrow{S} & & \\
1 & \longrightarrow & U_0(k) & \longrightarrow & U(k) & \longrightarrow & H^1(k, \mu_{4[Z]}) & \longrightarrow & 1
\end{array}
$$

The map $S$ also turns out to be the connecting map from $\text{PGO}^+(A, \sigma)(k) \to H^1(k, \mu_{4[Z]})$ [9, Prop. 13.37, p. 190] in the long exact sequence of cohomology corresponding to the exact sequence

$$1 \to \mu_{4[Z]} \to \text{Spin}(A, \sigma) \to \text{PGO}^+(A, \sigma) \to 1.$$ 

Since the maps $\mu_* : Z^* \to U_0(k)$ and $\chi' : \Omega(A, \sigma)(k) \to \text{PGO}^+(A, \sigma)(k)$ are surjective, an element $(f, z) \in U(k)$ is in the image $\mu_*(\Omega(A, \sigma)(k))$ if and only if its image $[f, z] \in H^1(k, \mu_{4[Z]})$ is in the image $S(\text{PGO}^+(A, \sigma)(k))$. 


Therefore we look for an obstruction preventing \([f, z]\) from being in the image \(S(PGO^+(A, \sigma)(k)).\) Recall the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
1 & \to & \mu_2 & \to & Spin(A, \sigma) & \xrightarrow{\chi} & O^+(A, \sigma) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_4[Z] & \to & Spin(A, \sigma) & \xrightarrow{\chi'} & PGO^+(A, \sigma) & \to & 1 \\
\end{array}
\]

The long exact sequence of cohomology induces the following commutative diagram (Figure 1) with exact columns [9, Prop. 13.36, p. 189], where

\[
\begin{array}{ccccccc}
O^+(A, \sigma)(k) & \xrightarrow{Sn} & k^* & \xrightarrow{\mathcal{S}} & \mu & \xrightarrow{\mu_4[Z]} & PGO^+(A, \sigma)(k) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
O^+(A, \sigma)(k) & \xrightarrow{\mu} & k^* & \xrightarrow{\mathcal{S}} & H^1(k, \mu_4[Z]) \\
\end{array}
\]

\[\mu: PGO^+(A, \sigma)(k) \to \frac{k^*}{k^{*2}}\) is induced by the multiplier map \(\mu: GO^+(A, \sigma) \to \mathbb{G}_m\)

\[i: \frac{k^*}{k^{*2}} \to H^1(k, \mu_4[Z]) = \frac{U(k)}{U_0(k)}\) is the map sending \(f k^{*2} \sim [f, f^2]\)

\[j: \frac{U(k)}{U_0(k)} \to H^1(k, \mu_4[Z]) \to \frac{k^*}{k^{*2}}\) is the map sending \([f, z] \sim N(z_0)k^{*2},\)

where \(z_0 \in Z^*\) is such that \(z_0i(z_0)^{-1} = f^{-2}z.\)

**Definition 4.1.** We call an element \((f, z) \in U(k)\) to be special if there exists a \([g] \in PGO^+(A, \sigma)(k)\) such that \(j([f, z]) = \mu([g]).\)
Let \((f, z) \in U(k)\) be a special element and let \([g] \in \text{PGO}^+(A, \sigma)(k)\) be such that \(j((f, z)) = \mu([g])\). From the discussion above, it is clear that \((f, z)\) is in the image \(\mu_*(\Omega(A, \sigma)(k))\) if and only if \([f, z]\) is in the image \(S(\text{PGO}^+(A, \sigma)(k))\).

Thus \(S([g])[f, z]^{-1}\) is in kernel \(j = \text{Image } i\) and hence there exists some \(\alpha \in k^*\) such that
\[
[f, z] = S([g])[\alpha, \alpha^2] \in \frac{U(k)}{U_0(k)}.
\]

Note that if \(g\) is changed by an element in \(O^+(A, \sigma)(k)\), then \(\alpha\) changes by a spinor norm by Figure 1 above. Thus given a special element, we have produced a scalar \(\alpha \in k^*\) which is well defined up to spinor norms.

This happens if and only if there exists \(w \in \Omega(A, \sigma)(k)\) such that
\[
\alpha = \underline{\mu}(w),
\]
\[
\alpha^2 = \underline{x}(w)i(\underline{x}(w))^{-1}\underline{\mu}(w)^2.
\]

This implies \(\underline{x}(w) \in k^*\) and hence \(w \in \Gamma(A, \sigma)(k)\). Thus \(\alpha\) is a spinor norm, being the similarity of an element in the special Clifford group. Also note if \(\alpha\) is a spinor norm, then \(\alpha = \underline{\mu}(\gamma)\) for some \(\gamma \in \Gamma(A, \sigma)(k)\) and \(\mu_*(\gamma) = \left(\underline{\mu}(\gamma), \underline{\mu}(\gamma)^2\right)\).

Thus a special element \((f, z)\) is in the image of \(\mu_*\) if and only if the produced scalar \(\alpha\) is a spinor norm. We call the class of \(\alpha\) in \(k^*/\text{Sim}(A, \sigma)\) to be the scalar obstruction preventing the special element \((f, z) \in U(k)\) from being in the image \(\mu_*(\Omega(A, \sigma)(k))\).

### 4.3. An obstruction to being in the image of \(\mu\) for \(n\) even

Given \(z \in Z^*\), we would like to formulate an obstruction which prevents \(z\) from being in the image \(\mu(\Omega(A, \sigma)(k))\). Note that for \(z \in Z^*, \underline{\mu}(z) = z^2\) and hence the subgroup \(Z^{*2}\) is in the image \(\underline{\mu}(\Omega(A, \sigma)(k))\).

Like in the case of odd \(n\), we can construct the map \(S: \text{PGO}^+(A, \sigma)(k) \to \frac{Z^*}{Z^{*2}}\) induced by the following commutative diagram with exact rows [9, Def. 13.32, p. 187]:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & Z^* & \longrightarrow & \Omega(A, \sigma)(k) & \longrightarrow & \text{PGO}^+(A, \sigma)(k) & \longrightarrow & 1 \\
\mu & & \downarrow & & \mu & & \downarrow & & \underline{s} & & \\
1 & \longrightarrow & Z^{*2} & \longrightarrow & Z^* & \longrightarrow & \frac{Z^*}{Z^{*2}} & \longrightarrow & 1 \\
\end{array}
\]

Again by the surjectivity of the maps, \(\underline{\mu} : Z^* \to Z^{*2}\) and \(\chi' : \Omega(A, \sigma)(k) \to \text{PGO}^+(A, \sigma)(k)\), an element \(z \in Z^*\) is in the image \(\underline{\mu}(\Omega(A, \sigma)(k))\) if and only
if its image $[z] \in \frac{Z^*}{k^{\times}}$ is in the image $S(PGO^+(A, \sigma) (k))$. Therefore we look for an obstruction preventing $[z]$ from being in the image $S(PGO^+(A, \sigma) (k))$. And as before, we arrive at the following commutative diagram (Figure 2) with exact rows and columns [9, Prop. 13.33, p. 188], where

\[
\begin{array}{ccc}
O^+(A, \sigma) (k) & \xrightarrow{S_n} & k^* \\
\downarrow{\pi} & & \downarrow{j} \\
PGO^+(A, \sigma) (k) & \xrightarrow{S} & \frac{Z^*}{k^{\times}} \\
\downarrow{\mu} & & \downarrow{j} \\
\frac{k^*}{k^{\times}} & = & \frac{k^*}{k^{\times}} \\
\end{array}
\]

Figure 2. Spinor norms and $S$ for $n$ even

$\mu : PGO^+(A, \sigma)(k) \to \frac{k^*}{k^{\times}}$ is induced by the multiplier map $\mu : GO^+(A, \sigma) \to \mathbb{G}_m$

$i : \frac{k^*}{k^{\times}} \to \frac{Z^*}{k^{\times}}$ is the inclusion map

$j : \frac{Z^*}{k^{\times}} \to \frac{k^*}{k^{\times}}$ is induced by the norm map from $Z^* \to k^*$.

**Definition 4.2.** We call an element $z \in Z^*$ to be special if there exists a $[g] \in PGO^+(A, \sigma) (k)$ such that $j([z]) = \mu([g])$.

Let $z \in Z^*$ be a special element and let $[g] \in PGO^+(A, \sigma) (k)$ be such that $j([z]) = \mu([g])$. As before a special element $z \in Z^*$ is in the image $\mu(\Omega (A, \sigma) (k))$ if and only if $[z]$ is in the image $S(\Omega (A, \sigma) (k))$.

Thus $S([g])[z]^{-1}$ is in kernel $j = \text{Image } i$ and hence there exists some $\alpha \in k^*$ such that

$[z] = S([g])[\alpha] \in \frac{Z^*}{k^{\times}}$.

Note that if $g$ is changed by an element in $O^+(A, \sigma) (k)$, then $\alpha$ changes by a spinor norm by Figure 2 above. Thus given a special element, we have produced a scalar $\alpha \in k^*$ which is well defined up to spinor norms.

$[z] \in S(\Omega (A, \sigma) (k)) \iff \alpha \in S(\Omega (A, \sigma) (k))$

Since $\alpha \in k^*$ also, this is equivalent to $\alpha$ being a spinor norm [9, Prop. 13.25, p. 184].

We call the class of $\alpha$ in $\frac{k^*}{\text{Sgn}(A, \sigma)}$ to be the scalar obstruction preventing the special element $z \in Z^*$ from being in the image $\mu(\Omega (A, \sigma) (k))$. 
4.4. Scharlau’s norm principle for $\mu : \text{GO}^+(A, \sigma) \to \mathbb{G}_m$. Let $\mu : \text{GO}^+(A, \sigma) \to \mathbb{G}_m$ denote the multiplier map and let $L/k$ be a separable field extension of finite degree. Let $g_1 \in \text{GO}^+(A, \sigma)(L)$ be such that $\mu(g_1) = f_1 \in L^*$. Let $f$ denote $N_{L/k}(f_1)$. We would like to show that $f$ is in the image $\mu(\text{GO}^+(A, \sigma)(k))$.

Note that by a generalization of Scharlau’s norm principle ([9, Prop. 12.21]; [3, Lemma 4.3]) there exists a $\tilde{g} \in \text{GO}(A, \sigma)(k)$ such that $f = \mu(\tilde{g})$. However we would like to find a proper similitude $g \in \text{GO}^+(A, \sigma)(k)$ such that $\mu(g) = f$.

We investigate the cases when the algebra $A$ is non-split and split separately.

**Case I: $A$ is non-split.** Note that $g_1 \in \text{GO}^+(A, \sigma)(L)$. If $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \text{GO}^+(A, \sigma)(k)$. By a generalization of Dieudonné’s theorem [9, Thm. 13.38, p. 190], we see that the quaternion algebras

\[ B_1 = (Z, f_1) = 0 \in \text{Br}(L), \]
\[ B_2 = (Z, f) = A \in \text{Br}(k). \]

Since $A$ is non-split, $B_2 \neq 0 \in \text{Br}(k)$. However co-restriction of $B_1$ from $L$ to $k$ gives a contradiction, because

\[ 0 = \text{Cor} B_1 = (Z, N_{L/k}(f_1)) = B_2 \in \text{Br}(k). \]

Hence $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$.

**Case II: $A$ is split.** Since $A$ is split, $A = \text{End} V$ where $(V, q)$ is a quadratic space and $\sigma$ is the adjoint involution for the quadratic form $q$. Again, if $\tilde{g} \in \text{GO}^+(A, \sigma)(k)$, we are done. Hence assume $\tilde{g} \notin \text{GO}^+(A, \sigma)(k)$. That is

\[ \det(\tilde{g}) = -f^{2n/2} = -(f^n). \]

Since $A$ is of even degree ($2n$) and split, there exists an isometry\(^1\) $h$ of determinant $-1$. Set $g = \tilde{g}h$. Then $\det(g) = f^n$ where $\mu(g) = f$. Thus we have found a suitable $g \in \text{GO}^+(A, \sigma)(k)$ which concludes the proof of the following:

**Theorem 4.3.** The norm principle holds for the map $\mu : \text{GO}^+(A, \sigma) \to \mathbb{G}_m$.

4.5. Spinor obstruction to norm principle for non-trialitarian $D_n$. Let $L/k$ be a separable field extension of finite degree. And let $w_1 \in \Omega(A, \sigma)(L)$ be such that for

\[ n \text{ odd : } \mu_+(w_1) = \theta \text{ which is equal to } (f_1, z_1) \in U(L), \]
\[ n \text{ even : } \mu_-(w_1) = \theta \text{ which is equal to } z_1 \in \left(R_{Z/k}\mathbb{G}_m \right)(L). \]

\(^1\text{Since } V \text{ is of even dimension } 2n, \text{ } h \text{ can be chosen to be a hyperplane reflection for instance} \)
We would like to investigate whether $N_{L/k}(\theta)$ is in the image of $\mu_* (\Omega (A, \sigma) (k))$ (resp. $\mu (\Omega (A, \sigma) (k))$) when $n$ is odd (resp. even) in order to check if the norm principle holds for the map $\mu_* : \Omega (A, \sigma) \to U$ (resp. $\mu : \Omega (A, \sigma) \to R_{Z/k} \mathbb{G}_m$).

Let $[g_1] \in \text{PGO}^+ (A, \sigma) (L)$ be the image of $w_1$ under the canonical map $\chi': \Omega (A, \sigma) (L) \to \text{PGO}^+ (A, \sigma) (L)$. Clearly $\theta$ is special and let $g_1 \in \text{GO}^+ (A, \sigma) (L)$ be such that $\mu ([g_1]) = j ([\theta])$.

By Theorem 4.3, there exists a $g \in \text{GO}^+ (A, \sigma) (k)$ such that

$$\mu ([g]) = N_{L/k} (j [\theta]) = j ([N_{L/k} \theta]) .$$

Hence $N_{L/k} (\theta)$ is special.

By Subsection 4.2 (resp. 4.3), $N_{L/k} (\theta)$ is in the image of $\mu_*$ (resp $\mu$) if and only if the scalar obstruction $\alpha \in \frac{k^*}{\text{Sn}(A, \sigma)}$ defined for $N_{L/k} (\theta)$ vanishes. Thus we have a spinor norm obstruction given below.

**Theorem 4.4** (Spinor norm obstruction). Let $L/k$ be a finite separable extension of fields. Let $f$ denote the map $\mu_*$ (resp $\mu$) in the case when $n$ is odd (resp. even). Given $\theta \in f (\Omega (A, \sigma) (L))$, there exists scalar obstruction $\alpha \in k^*$ such that

$$N_{L/k} (\theta) \in f (\Omega (A, \sigma) (k)) \iff \alpha = 1 \in \frac{k^*}{\text{Sn}(A, \sigma)} .$$

Thus the norm principle for the canonical map

$$\Omega (A, \sigma) \to \frac{\Omega (A, \sigma)}{[\Omega (A, \sigma), \Omega (A, \sigma)]}$$

and hence for non-trialitarian $D_n$ holds if and only if the scalar obstructions are spinor norms.

**5. Quasi-split groups**

Let $G$ be a connected reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$ and let $G'$ denote its derived subgroup. Let $G^{sc}$ denote the simply connected cover of $G'$. Then one has the exact sequence $1 \to C \to G^{sc} \to G' \to 1$, where $C$ is a finite $k$-group of multiplicative type, central in $G^{sc}$. Assuming that $G^{sc}$ is quasi-split, we would like to show that $G$ satisfies $SQ$ by following the reduction techniques used in Sections 2 and 3.

**Lemma 5.1.** Let $G$ be a connected reductive $k$-group. If $G^{sc}$ is quasi-split, then there exists an extension $1 \to Q \to H \xrightarrow{\phi} G \to 1$, where $Q$ is a quasi-trivial $k$-torus, central in reductive $k$-group $H$ with $H'$ simply connected and quasi-split.

---

2The map $f$ commutes with $N_{L/k}$ in both cases.
Proof. Recall that there is a central extension (called a z-extension) of $G$ by a quasi-trivial torus $Q$ such that $H'$ is semisimple and simply connected ([11, Prop. 3.1] and [4, Lemma 1.1.4]).

$$1 \to Q \to H \to G \to 1.$$ 

The restriction $\psi|_{H'} : H' \to G$ yields the fact that $H'$ is the simply connected cover of $G'$ and hence is quasi-split. \hfill \Box

Lemmas 2.2 and 5.1 imply that we can restrict ourselves to connected reductive $k$-groups $G$ such that $G'$ is simply connected and quasi-split.

**Lemma 5.2.** Let $H$ be any reductive $k$-group such that its derived subgroup $H'$ is semisimple simply connected and quasi-split. Let $T$ denote the $k$-torus $H/H'$. Then the natural exact sequence $1 \to H' \to H \overset{\phi}{\to} T \to 1$ induces surjective maps $\phi(L) : H(L) \to T(L)$ for all field extensions $L/k$. In particular, the norm principle holds for $\phi : H \to T$.

Proof. There exists a quasi-trivial maximal torus $Q_1$ of $H'$ defined over $k$ [8, Lem. 6.7]. Let $Q_1 \subset Q_2$, where $Q_2$ is a maximal torus of $H$ defined over $k$. The proof of [8, Lem. 6.6] shows that $\phi|_{Q_2} : Q_2 \to T$ is surjective and that $Q_2 \cap H'$ is a maximal torus of $H'$. Since $Q_2 \cap H' \subseteq Q_1$, we get the following extension of $k$-tori

$$1 \to Q_1 \to Q_2 \to T \to 1$$

Since $Q_1$ is quasitriivial, $H^1(L, Q_1) = 0$ for any field extension $L/k$ which gives the surjectivity of $\phi(L) : Q_2(L) \to T(L)$ and hence of $\phi(L) : H(L) \to T(L)$. \hfill \Box

Let $\hat{G}$ be an envelope of $G'$ defined using an embedding of $\mu = Z(G')$ into a quasi-trivial torus $S$. Note that $G'$ is assumed to be simply connected and quasi-split and is also the derived subgroup of $\hat{G}$ by construction.

Thus, we get an exact sequence $1 \to G' \to \hat{G} \to \hat{G}/G' \to 1$ to which we can apply Lemma 5.2 to conclude that the norm principle holds for the canonical map $\hat{G} \to [\hat{G}, \hat{G}]$.

Constructing the intermediate group $\hat{G}$ as in Section 3.1, we see that the norm principle also holds for the natural map $\hat{G} \to \hat{G}/G$ [1, Prop. 5.1]. Then using Theorem 3.1 [3], Lemma 3.2, and a remark from Gopal Prasad that $G^{sc}$ is quasi-split if and only if $G$ is quasi-split, we can conclude that Theorem 1.3 (restated below) holds.
Theorem 1.3. Let $k$ be a field of characteristic not 2. Let $G$ be a connected quasi-split reductive $k$-group whose Dynkin diagram does not contain connected components of type $E_8$. Then Serre’s question has a positive answer for $G$.

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