ANTICYCLOTOMIC $p$-ADIC $L$-FUNCTIONS AND
THE EXCEPTIONAL ZERO PHENOMENON

SANTIAGO MOLINA

Abstract. Let $A$ be a modular elliptic curve over a totally real field $F$, and let $E/F$ be a totally imaginary quadratic extension. In the event of exceptional zero phenomenon, we prove a formula for the derivative of the multivariable anticyclotomic $p$-adic $L$-function attached to $(A, E)$, in terms of the Hasse-Weil $L$-function and certain $p$-adic periods attached to the respective automorphic forms. Our methods are based on a new construction of the anticyclotomic $p$-adic $L$-function by means of the corresponding automorphic representation.

Contents

1. Introduction 2
1.1. Notation 4
2. Measures and Iwasawa algebras 5
2.1. Distributions and measures 5
2.2. Iwasawa algebras 6
3. Local theory 7
3.1. Local representations 7
3.2. Universal unramified principal series 8
3.3. Local distributions attached to a torus in $GL_2$ 9
3.4. Torus and inner products 10
3.5. Steinberg representations and intertwining operators 11
3.6. Local pairings 12
4. Cohomology of automorphic forms and Shimura curves 16
4.1. Abel-Jacobi map on Shimura curves 17
4.2. Multiplicity one 20
5. $p$-adic global distributions and measures 20
5.1. Definite anticyclotomic distributions 20
5.2. Cohomological interpretation of $\mu_{E,P}^\pi.\Gamma$ 21
5.3. Heegner points and Gross-Zagier-Zhang formula 25
5.4. Indefinite anticyclotomic distributions 26
5.5. Cohomological interpretation of $\mu_{E,P}^\pi.\Gamma$ 30
5.6. $p$-adic measures and $p$-adic $L$-functions 31
6. Automorphic $L$-invariants 32
6.1. Extensions of the Steinberg representation 33
6.2. $L$-invariants 34
6.3. Geometric $L$-invariants 36
6.4. $L$-invariants and Heegner points 38
7. Exceptional zero phenomenon in the split case 40
8. Appendix 1: Local integrals 44
9. Appendix 2: The $(G, K)$-module of Discrete Series 45
References 46
1. Introduction

Let $F$ be a totally real field and let $A$ be an elliptic curve defined over $F$ (although our results apply for general abelian varieties of GL$_2$-type). One of the central research topics in Modern Number Theory is the relation between the arithmetic of $A$ and the analysis of the (Hasse-Weil) $L$-function $L(A, s)$ attached to $A$. The $L$-function $L(A(s), s)$ as and all its twists $L(A, \psi, s)$, where $\psi$ is a finite character of the Galois group $\text{Gal}(F^{ab}/F)$ of the maximal abelian extension $F^{ab}$ of $F$, are $C$-valued functions that satisfy a certain symmetric functional equations relating their values at $s$ and $2 - s$. The well-known Birch and Swinnerton-Dyer conjecture predicts that the rank of the Mordel-Weil group $A(F)$ coincides with the order of vanishing of $L(A, 1)$. Later generalizations by Bloch and Kato predict that the rank of the $\psi$-isotypical component $A(F)[\psi] = A(F^{ab}) \otimes \mathbb{Z}[[\text{Gal}(F^{ab}/F)] \mathbb{C}(\psi)$ agrees with the order of vanishing of $L(A, \psi, 1)$.

If $A$ has either ordinary good or bad multiplicative reduction at all places above $p$, we obtain a better understanding of the arithmetic of $A/F$ if we replace the $C$-analysis of $L(A, s)$ by the a $\mathbb{C}_p$-analysis of its $p$-adic avatar $L_p(A, s)$, the (cyclotomic) $p$-adic $L$-function of $A$. This is a $\mathbb{C}_p$-valued function that interpolates the critical values $L(A, \psi, 1)$, for any finite character $\psi$ of the Galois group $\mathcal{G}_p \simeq \mathbb{Z}_p$ of the cyclotomic $\mathbb{Z}_p$-extension $F^{\text{cycl}}_p$ of $F$ unramified outside $p$ and $\infty$. The function $L_p(A, s)$ is defined as

$$L_p(A, s) = \int_{\mathcal{G}_p} \exp_p(s \ell(\gamma))d\mu_p(\gamma), \quad s \in \mathbb{C}_p,$$

where $\ell : \mathcal{G}_p \to \mathbb{Z}_p$ is a canonical isomorphism and $\mu_p$ is a certain (cyclotomic) $p$-adic measure attached to $A$. By $L_p(A, s)$ interpolates $L(A, \psi, 1)$, we mean that the measure $\mu_p$ satisfies

$$\int_{\mathcal{G}_p} \psi(\gamma)d\mu_p(\gamma) = \epsilon_p(A, \psi)L(A, \psi, 1),$$

for all finite characters $\psi : \mathcal{G}_p \to \mathbb{C}^\times$, where $\epsilon_p(A, \psi)$ is some Euler factor which is non-zero for almost all $\psi$. Observe that the $p$-adic $L$-function is univocally characterized by the $\mathbb{C}_p$-valued measure $\mu_p$. A $p$-adic analog of the Birch and Swinnerton-Dyer conjecture was stated in [11].

Let $E/F$ be a totally imaginary quadratic extension. Some remarkable achievements towards the Birch and Swinnerton-Dyer conjecture have been obtained by means of the rich theory of Heegner points associated with $E/F$. This encourages us to consider $A/E$ as an elliptic curve defined over $E$. Note that in this setting we can consider anticyclotomic abelian extensions of $E$ which are linearly disjoint from $F^{\text{cycl}}_pE$. Indeed, for any prime ideal $\mathcal{P}$ of $F$ dividing $p$, let $E_{\mathcal{P}}^p$ be the maximal abelian extension of $E$ which is unramified outside $\mathcal{P}$ and $\infty$ and so that the complex conjugation $\tau \in \text{Gal}(E/F)$ acts on $\mathcal{G}_{E, \mathcal{P}} := \text{Gal}(E_{\mathcal{P}}^p/E)$ by $-1$. Up to torsion, $\mathcal{G}_{E, \mathcal{P}}$ is isomorphic to $\mathbb{Z}_p^r$, where $r = [\mathcal{F}_{\mathcal{P}} : \mathbb{Q}_p]$. Motivated by the cyclotomic theory, one may ask if there is an analogous construction of $p$-adic $L$-functions attached to such anticyclotomic $\mathbb{Z}_p$-extensions.

The behavior of the local functional equation outside $\mathcal{P}$ provides a dichotomy in our scenario: on the one hand the definite case, where the number of finite places $v$ outside $\mathcal{P}$ with the sign of the local functional equation (i.e. the local root number) $\omega_v(A/E) = -1$ is even; on the other hand the indefinite case, where that number of finite places is odd.

Assume that $A$ has either ordinary good or multiplicative reduction modulo $\mathcal{P}$. Our starting point is the construction of $\mathbb{C}_p$-valued measures of $\mathcal{G}_{E, \mathcal{P}}$ attached to $A$, with good interpolation properties. In the literature, we can find constructions of such measures in some particular cases: for $F = \mathbb{Q}$ and $E/\mathbb{Q}$ not ramified at $p$, this construction is provided by the cyclotomic $L$-functions $L_p(A, s)$.
we have the work of Bertolini-Darmon (see [5] and [6]); for arbitrary \( F \), we have the work of Van Order also under certain restrictions for the ramification of \( E/F \) (see [18] and [17]). In this paper, we provide alternative constructions valid for arbitrary totally imaginary quadratic extensions \( E/F \). Note the novelty of these constructions, even for \( F = \mathbb{Q} \), when \( E/\mathbb{Q} \) ramifies at \( p \). We denote by \( \mu_{E,P}^I \) and \( \mu_{E,P}^{II} \) the corresponding measures in the definite and indefinite situations, respectively. In analogy with the cyclotomic setting, our definite \( p \)-adic measure \( \mu_{E,P}^I \) interpolates the critical values \( L(A/E, \chi, 1) \) for any finite character \( \chi \) of \( \mathcal{G}_{E,P} \) (Theorem 5.9). In the indefinite case, the corresponding \( p \)-adic measure \( \mu_{E,P}^{II} \) interpolates \( p \)-adic logarithms of certain Heegner points whose height is given by the derivative \( L'(A/E, \chi, 1) \) (Theorem 5.10). In both scenarios (and also including the cyclotomic setting), the interpolation property involves an Euler factor \( e_P(A/E, \chi) \) that is not identically zero.

The set of \( \mathbb{C}_p \)-valued measures \( \text{Meas}(\mathcal{G}_{E,P}, \mathbb{C}_p) \) of \( \mathcal{G}_{E,P} \) has a natural structure of \( \mathbb{C}_p \)-algebra called \textit{Iwasawa algebra}. The morphism \( \deg : \text{Meas}(\mathcal{G}_{E,P}, \mathbb{C}_p) \to \mathbb{C}_p \), that maps any measure to the image of the constant map \( 1 \), is indeed a \( \mathbb{C}_p \)-algebra morphism. The ideal \( I = \ker(\deg) \) is called \textit{augmentation ideal} and the maximum power \( I^r \) in which a measure \( \mu \) lies is called \textit{order of vanishing} of \( \mu \). Observe that, in the cyclotomic setting, \( L_p(A, 0) = \deg(\mu_p) \) and the order of vanishing of \( L_p(A, s) \) at \( s = 0 \) coincides with the order of vanishing of \( \mu_p \).

In this paper, we mean by anticyclotomic \( p \)-adic \( L \)-functions the elements \( L^I_p(A, E) \) and \( L^{II}_p(A, E) \) of the Iwasawa algebra of \( \mathcal{G}_{E,P} \) defined by \( \mu_{E,P}^I \) and \( \mu_{E,P}^{II} \), respectively. In the cyclotomic and definite anticyclotomic setting, the interpolation property relates the image \( \deg(L^I_p(A, E)) \) with the critical value of the classical \( L \)-functions at \( s = 1 \) (analogously, in the indefinite anticyclotomic case the classical \( L \)-function is replaced by its derivative). This suggests that the order of vanishing of such \( p \)-adic \( L \)-functions coincides with the order of vanishing of the classical \( L \)-functions at \( s = 1 \) (resp. their derivatives in the indefinite anticyclotomic case). Nevertheless, a surprising phenomenon appears if \( A \) has split multiplicative reduction: the Euler factor \( e_P(A/E, 1) \) vanishes and we observe a zero of the \( p \)-adic \( L \)-function, even when the classical \( L \)-function (or its derivative) is not zero. These extra zeros coming from the vanishing of the Euler factors are called \textit{exceptional zeros}.

A first approach to understand this exceptional zero phenomenon is to compute the derivatives of the corresponding \( p \)-adic \( L \)-functions or, analogously in terms of Iwasawa algebras, their classes in the respective \( \mathbb{C}_p \)-vector spaces \( I^r/I^{r+1} \). Many authors have contributed to this research line:

- In the cyclotomic setting, the order of vanishing of \( L_p(A, s) \) at \( s = 0 \) is at least \( m \), the number of places above \( p \) where \( A \) has split multiplicative reduction [15]. Moreover, a formula that expresses \( \frac{d^m}{ds^m}L_p(A, s) \big|_{s=0} \) as a product of \( L(A, 1) \) with what are known as (geometric) \( \mathcal{L} \)-invariants \( L_P(A) \) (\( P \mid p \) is a prime of split multiplicative reduction) was conjectured by Hida. This formula was established by Greenberg and Stevens in [9] for \( F = \mathbb{Q} \), by Mok in [10] for arbitrary totally real fields and \( m = 1 \), and finally by Spieß in [15] for arbitrary \( m \) and \( F \) under some mild assumptions. In fact, Spieß proves an analogous formula with the \( \mathcal{L} \)-invariants \( L_P(A) \) replaced by certain automorphic \( \mathcal{L} \)-invariants \( L_P(\pi) \), where \( \pi \) is the automorphic cuspidal attached to \( A \).
- In the definite anticyclotomic setting, there have only been results for \( F = \mathbb{Q} \) so far. Assuming that the quadratic extension \( E/\mathbb{Q} \) splits at \( p \), Bertolini and Darmon proved in [5] a similar formula for the image of \( L^I_p(A, E) \) in \( I/I^2 \approx \mathbb{C}_p \) in terms of \( L(A/E, 1) \) and the same \( \mathcal{L} \)-invariant \( L_P(A) \) that appeared in the cyclotomic setting. If \( p \) is inert in \( E \), Bertolini and Darmon proved in
In an analogous formula involving the \( p \)-adic logarithm of a Heegner point (see also [6]). The case \( E/\mathbb{Q} \) ramified at \( p \) is still an open problem even for \( F = \mathbb{Q} \).

- In the indefinite anticyclotomic setting, the only results to date were for the case \( F = \mathbb{Q} \) and \( p \) inert at \( E \). Bertolini and Darmon proved in [3] that the image of \( L_p^{II}(A, E) \) in \( \mathcal{I}/\mathcal{I}^2 \simeq \mathbb{C}_p \) can be expressed in terms of the critical value \( L(A/E, 1) \) (see also [6]). The cases \( E/\mathbb{Q} \) split or ramified at \( p \) have never been considered previously.

The main result of this paper (Theorem 6.5) establishes a formula for the classes of the \( p \)-adic \( L \)-functions \( L_p^{I}(A, E) \) and \( L_p^{II}(A, E) \) in \( \mathcal{I}/\mathcal{I}^2 \), assuming that \( P \) splits in \( E \). We remark that these results are valid for arbitrary totally real fields and, in turn, resolve the indefinite anticyclotomic case for \( F = \mathbb{Q} \).

In order to explain our result, let us introduce \( \pi \), the automorphic representation of \( \mathrm{GL}_2(\mathbb{A}_F) \) associated with \( A \). The (twisted) Hasse-Weil \( L \)-function \( L(A/E, \chi, s) \) coincides with the Rankin-Selberg \( L \)-function \( L(s - 1/2, \pi_E, \chi) \), where \( \pi_E \) is the extension of the representation \( \pi \) to \( \mathrm{GL}_2(\mathbb{A}_E) \). Actually, we will define our anticyclotomic measures by means of a certain Jacquet-Langlands lift \( \pi^{II} \) of \( \pi \). For this reason we denote them by \( \mu_{E, P}^{III} \) and \( \mu_{E, P}^{III} \), respectively \( L_p^I(\pi, E) \) and \( L_p^{II}(\pi, E) \) when considered as elements of the Iwasawa algebra. In [7] we will introduce a new type of \( L \)-invariants, the automorphic \( L \)-invariant vectors \( L_p(\pi) \in \mathcal{I}/\mathcal{I}^2 \). They are defined in terms of the cohomology of \((S_p)\)-arithmetic groups associated with Jacquet-Langlands lifts of \( \pi \) (see [6]). If we denote by \( \nabla L_p^I(\pi, E) \), \( \nabla L_p^{II}(\pi, E) \in \mathcal{I}/\mathcal{I}^2 \) the classes of the corresponding \( p \)-adic \( L \)-functions, we obtain that

\[
\nabla L_p^I(\pi, E) = L_p(\pi) \left( C_E C(\pi_P) \frac{L(1/2, \pi_E, 1)}{L(1, \pi, ad)} \right)^{1/2},
\]

\[
\nabla L_p^{II}(\pi, E) = L_p(\pi) \log_p(P_T),
\]

where \( C_E \) is a non-zero constant, \( C(\pi_P) \) is a non-zero Euler factor, and \( P_T \) is a Heegner point with explicit height depending on \( L(1/2, \pi_E, 1) \).

In the definite setting, this is a generalization of the result by Bertolini and Darmon in [5], for \( F = \mathbb{Q} \). With the spirit of the \( p \)-adic BSD conjecture introduced in [11], Bertolini and Darmon also conjectured in [2, Conjecture 4.6] such a result in the indefinite setting for \( F = \mathbb{Q} \) and \( A(E) \) of rank 1, but replacing the automorphic \( L \)-invariant vector \( L_p(\pi) \) by the corresponding geometric \( L \)-invariant.

Bearing in mind the case \( F = \mathbb{Q} \), we expect the \( L \)-invariant vector \( L_p(\pi) \) to be related to the geometry of \( A/F_P \). Actually, \( L_p(\pi) \) is defined differently in the definite and indefinite settings. In [6,3] we show in the definite setting that \( L_p(\pi) \) is given by the geometric \( L \)-invariants. It is a work in progress to study the relation between indefinite \( L \)-invariant vectors and geometric \( L \)-invariants. Such relation, predicted by Conjecture 6.6 holds for \( F = \mathbb{Q} \) thanks to the work by Greenberg-Stevens [9] and Longo-Rotger-Vigni [10].

The techniques used to prove these results are generalizations of the work by Spieß in [15], where the cyclotomic measure is defined by means of a certain group cohomology cocycle. In our setting, we construct our anticyclotomic measures by means of certain \((S_p)\)-arithmetic cocycles associated with the automorphic representations \( \pi^{II} \) and the torus attached to the extension \( E/F \). In fact, our construction is given in such generality that we recover the Spieß \( p \)-adic measure if we allow \( E/F \) to be the trivial extension \( E = F \times F \). In this direction, our work helps us to understand the link between the cyclotomic and anticyclotomic framework.

Finally, let us remark that the techniques and results described in this paper could also contribute to research on the exceptional zero phenomenon whenever \( P \) does not split in \( E \). A future research line is to compute the classes \( \nabla L_p^I(\pi, E) \) and...
\(\nabla L^H_p(\pi, E)\) in case \(\mathcal{P}\) is inert in \(E\), generalizing the results of Bertolini and Darmon for \(F = \mathbb{Q}\), or \(\mathcal{P}\) ramified in \(E\), which is an open problem even for \(F = \mathbb{Q}\).

1.1. Notation. For any field \(F\), denoted by \(\mathcal{O}_F\) its ring of integers.

We denote the action of \(GL_2(F)\) on the projective line \(\mathbb{P}^1(F)\) by:

\[
GL_2(F) \times \mathbb{P}^1(F) \longrightarrow \mathbb{P}^1(F), \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \longmapsto g \ast \tau := \frac{a\tau + b}{c\tau + d}.
\]

If \(F\) is a local field, we consider the Haar measure \(d^\times x\) of \(F^\times\) introduced in [21].

If \(F\) is a global number field, we denote by \(\mathfrak{a}_F\) the corresponding Haar measure of the localization \(F^\times\). The product of \(d^\times x_v\) defines a Haar measure on \(\mathfrak{a}_F/F^\times\), where \(\mathfrak{a}_F\) is the group of ideles of \(F\).

If \(E/F\) is a quadratic extension of global fields, if \(v\) is a place of \(F\), we denote by \(d^\times t_v\) the corresponding Haar measure on the torus \(E_v^\times/F_v^\times\) given by the quotient measure. The product of \(d^\times t_v\) defines a Haar measure \(d^\times \epsilon\) on \(\hat{E}^\times/E^\times \hat{F}^\times\), where

\[
\hat{F}^\times := \prod_{v \mid \infty} F_v^\times, \quad \hat{E}^\times := \prod_{v \mid \infty} E_v^\times,
\]

are the groups of finite ideles of \(F\) and \(E\), respectively.

Finally, we fix embeddings \(\mathbb{Q} \hookrightarrow \mathbb{C}\) and \(\mathbb{Q} \hookrightarrow \mathbb{C}_p\).

2. Measures and Iwasawa algebras

2.1. Distributions and measures. Let \(\mathcal{X}\) be a totally disconnected locally compact topological space. For a topological Hausdorff ring \(R\), we denote by \(C(\mathcal{X}, R)\) the ring of continuous maps from \(\mathcal{X}\) to \(R\), and by \(C_c(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)\) the subring of compactly supported continuous maps. We denote by \(C(\mathcal{X}, R)_0\) and \(C_c(\mathcal{X}, R)_0\) the corresponding rings if we consider \(R\) with the discrete topology. We denote by \(C_0(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)\) the set of continuous functions \(f\) with \(f(g) \to 0\) as \(g \to \infty\), more precisely, \(f\) can be extended to a continuous function on the one-point compactification of \(\mathcal{X}\) by setting \(f(\infty) = 0\). We have

\[
C_c(\mathcal{X}, R)_0 \subseteq C_c(\mathcal{X}, R) \subseteq C_0(\mathcal{X}, R) \subseteq C(\mathcal{X}, R).
\]

Definition 2.1. Let \(M\) be an \(R\)-module. An \(M\)-valued distribution on \(\mathcal{X}\) is a homomorphism \(\mu : C_c(\mathcal{X}, \mathbb{Z})_0 \to M\). It extends to a \(R\)-linear map

\[
C_c(\mathcal{X}, R)_0 \longrightarrow M, \quad f \longmapsto \int_{\mathcal{X}} f(\gamma)d\mu(\gamma).
\]

We shall denote the \(R\)-module of \(M\)-valued distributions by \(\text{Dist}(\mathcal{X}, M)\).

Assume that \(R = K\) is a \(p\)-adic field, namely, a field of characteristic 0, complete with respect to a nonarchimedean absolute value \(|\cdot| : K \to \mathbb{R}\), whose restriction to \(\mathbb{Q}\) is the usual \(p\)-adic value. We denote the corresponding valuation ring by \(\mathcal{O}_K\).

A norm on a \(K\)-vector space \(V\) is a function \(\|\| : V \to \mathbb{R}\) satisfying:

\[
i) \|av\| = |a|\|v\|, \quad ii) \|v + w\| \leq \max\{\|v\|, \|w\|\}, \quad iii) \|v\| \geq 0, \text{ with equality if, and only if, } v = 0,
\]

Two norms in \(V\) are equivalent if they define the same topology on \(V\). A normed \(K\)-vector space \((V, \|\|)\) is a \((K-)\text{Banach space}\) if \(V\) is complete with respect to \(\|\|\). Recall that any finite dimensional \(K\)-vector space admits a norm, any two norms are equivalent, and it is complete. The \(K\)-vector space \(C_c(\mathcal{X}, K)\) with the supremums norm \(\|f\|_\infty = \sup_{\gamma \in \mathcal{X}} |f(\gamma)|\) is a \(K\)-Banach space.
Definition 2.2. Let \((V, \| \|)\) be a Banach space. An element \(\mu \in \text{Dist}(X, V)\) is a measure if \(\mu\) is continuous with respect to the supremums norm, i.e. there exists \(C \in \mathbb{R}, C > 0\), such that \(\| f_X f d\mu \| \leq C \| f \|_\infty\) for all \(f \in C_c(X, K)_0\). We will denote the space of \(V\)-valued measures on \(X\) by \(\text{Meas}(X, V)\).

An element \(\mu \in \text{Meas}(X, V)\) can be integrated, not only against locally constant functions, but against any \(f \in C_c(X, K)\), since \(C_c(X, K)_0\) is dense in the Banach space \((C_c(X, K), \| \|_\infty)\). We obtain a continuous functional

\[
C_c(X, K) \rightarrow V; \quad f \mapsto \int_X f d\mu.
\]

An \(O_K\)-submodule \(L\) of a \(K\)-vector space \(V\) is a lattice if \(\cup a \in K^* aL = V\) and \(\cap a \in K^* aL = \{0\}\). For a given lattice \(L \subseteq V\) the function \(p_L(v) := \inf_{a \in L} |a|\) is a norm on \(V\). Any other norm \(\| \|\) on \(V\) is equivalent to \(p_L\) if, and only if, \(L\) is open and bounded in \((V, \| \|)\). For any \(L\) open and bounded lattice on a Banach space \((V, \| \|)\), the space \(\text{Meas}(X, V)\) is the image of the canonical inclusion

\[
(2.1) \quad \text{Dist}(X, L) \otimes_{O_K} K \rightarrow \text{Dist}(X, V).
\]

2.2. Iwasawa algebras. Let \(G\) be a commutative pro-\(p\) group and let \(K\) be a \(p\)-adic field with valuation ring \(O_K\). Since \(O_K\) is a lattice in the Banach space \((K, \| \|)\), the \(K\)-vector space \(\text{Meas}(G, K)\) coincides with the tensor product \(\text{Dist}(G, O_K) \otimes_{O_K} K\).

Let us consider the convolution product

\[
* : \text{Dist}(G, O_K) \times \text{Dist}(G, O_K) \rightarrow \text{Dist}(G, O_K) \quad (\mu_1, \mu_2) \mapsto \mu_1 * \mu_2,
\]

where

\[
\int_G f(\gamma) d(\mu_1 * \mu_2)(\gamma) := \int_G \int_G f(\alpha \cdot \beta) d\mu_1(\alpha) d\mu_2(\beta).
\]

One checks that such convolution product endows \(\text{Dist}(G, O_K)\) with structure of \(O_K\)-algebra, where the unit is given by the Dirac measure \(\int_G f d1 := f(1)\) and the structural ring homomorphism corresponds to

\[
s : O_K \rightarrow \text{Dist}(G, O_K), \quad \alpha \mapsto \alpha d1.
\]

We denote by \(O_K[[G]]\) the \(O_K\)-module \(\text{Dist}(G, O_K)\) endowed with such \(O_K\)-algebra structure. The algebra \(O_K[[G]]\) is called Iwasawa algebra of \(G\) with coefficients in \(O_K\). We deduce that \(\text{Meas}(G, K)\) is equipped with the \(K\)-algebra structure \(O_K[[G]] \otimes_{O_K} K\).

Observe that the natural map

\[
d : G \rightarrow (O_K[[G]])^*; \quad \int_G f d\gamma := f(g),
\]

is a group homomorphism and the \(O_K\)-module homomorphism

\[
deg : O_K[[G]] \rightarrow O_K; \quad \mu \mapsto \int_G d\mu,
\]

is indeed an \(O_K\)-algebra homomorphism.

Let us give a different description of the \(O_K\)-algebra \(O_K[[G]]\). We claim that the Iwasawa algebra is the inverse limit of group algebras

\[
O_K[[G]] = \lim_{\rightarrow} O_K[G/H],
\]

where \(H\) runs over open compact subgroups of \(G\). Indeed, given \(\mu \in \text{Dist}(G, O_K)\) we define the element in \(\lim_{\rightarrow} O_K[G/H]\) provided by the expression

\[
\sum_{gH \in G/H} \left(\int_G 1_g H d\mu\right) gH \in O_K[G/H]
\]
where $1_{g\mathcal{H}}$ is the characteristic function of $g\mathcal{H}$. We leave to the reader the verification that this defines an isomorphism of $\mathcal{O}_K$-algebras. It is easy to check that $\mathrm{deg}$ corresponds to the natural degree homomorphism.

Let $\mathcal{I} := \ker(\mathrm{deg})$ be the augmentation ideal. Since $\mathrm{deg}$ is surjective, $\mathcal{I}/\mathcal{I}^2$ has a natural structure of $\mathcal{O}_K[[\mathcal{G}]]/\mathcal{I} \cong \mathcal{O}_K$-module. Moreover, since $\mathcal{G}$ is a pro-$p$ group, it is a $\mathbb{Z}_p$-module and the map

$$\varphi : \mathcal{G} \rightarrow \mathcal{I}/\mathcal{I}^2; \quad g \mapsto dg - d1,$$

is a $\mathbb{Z}_p$-module homomorphism. Indeed,

$$\varphi(g_1 + g_2) = d(g_1 + g_2) - d1 = \varphi(g_1) + \varphi(g_2) + (dg_1 - d1) * (dg_2 - d1).$$

The following result describes $\mathcal{I}/\mathcal{I}^2$ as the tensor product $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K$.

**Proposition 2.3** (Hurewicz Theorem). Assume that $\mathcal{G}$ is a $\mathbb{Z}_p$-module of finite rank, then the map $\varphi$ defines an isomorphism of $\mathcal{O}_K$-modules

$$\varphi : \mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \rightarrow \mathcal{I}/\mathcal{I}^2.$$

**Proof.** Let us consider the dual group $((\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^\vee) := \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K, \mathcal{O}_K)$ and the canonical $\mathcal{O}_K$-module morphism $\iota : \mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \rightarrow ((\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^\vee)^\vee$. We define

$$\psi : \mathcal{I} \rightarrow ((\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^\vee)^\vee; \quad \mu \mapsto \left( \ell \mapsto \int_{\mathcal{G}} \ell d\mu \right),$$

where $\ell \in \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K, \mathcal{O}_K) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{G}, \mathcal{O}_K)$ is seen as a continuous function in $C(\mathcal{G}, \mathcal{O}_K)$. We check that $\psi(\mathcal{I}^2) = 0$; Indeed, if $\mu_1, \mu_2 \in \mathcal{I}$,

$$\int_{\mathcal{G}} \ell d(\mu_1 * \mu_2) = \int_{\mathcal{G}} \int_{\mathcal{G}} \ell(\alpha + \beta) d\mu_1(\alpha) d\mu_2(\beta) = \deg(\mu_2) \int_{\mathcal{G}} \ell d\mu_1 + \deg(\mu_1) \int_{\mathcal{G}} \ell d\mu_2 = 0.$$

Thus we have defined a $\mathcal{O}_K$-module morphism $\psi : \mathcal{I}/\mathcal{I}^2 \rightarrow ((\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_K)^\vee)^\vee$ satisfying $\psi \circ \varphi = \iota$.

Since $\mathcal{G}$ is a $\mathbb{Z}_p$-module of finite rank, $\iota$ is an isomorphism. Thus, in order to prove our result, it is enough to show that $\psi$ is surjective.

Given the description of $\mathcal{O}_K[[\mathcal{G}]]$ as an inverse limit of group algebras, we observe that at each finite level

$$\mathcal{I}_H := \ker(\deg : \mathcal{O}_K[[\mathcal{G}/\mathcal{H}] \rightarrow \mathcal{O}_K) = \left\{ \sum_{g_H \in \mathcal{G}/\mathcal{H}} \alpha_{g_H}(g\mathcal{H} \setminus \mathcal{H}), \quad \alpha_{g\mathcal{H}} \in \mathcal{O}_K \right\}.$$\)

This implies that the corresponding morphism $\varphi_H : \mathcal{G}/\mathcal{H} \otimes_{\mathbb{Z}_p} \mathcal{O}_K \rightarrow \mathcal{I}_H/\mathcal{I}_H^2$ is surjective. We conclude that $\varphi$ is surjective an the result follows. $\square$

**Corollary 2.4.** Assume that $\mathcal{G}$ is a (free) $\mathbb{Z}_p$-module of finite rank and let $\mathcal{G}^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{G}, \mathbb{Z}_p)$. Then the map

$$\psi : \mathcal{I}/\mathcal{I}^2 \rightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{G}^\vee, \mathcal{O}_K), \quad \mu \mapsto \left( \ell \mapsto \int_{\mathcal{G}} \ell d\mu \right),$$

is a $\mathcal{O}_K$-module isomorphism.

**Proof.** Follows directly from the proof of Proposition 2.3. $\square$

3. Local theory

3.1. **Local representations.** Let $F$ be a nonarchimedean local field with integer ring $\mathcal{O}$, prime ideal $\mathcal{P}$, uniformizer $\varpi$, residual characteristic $q$, and valuation $\nu$ ($\nu(\varpi) = 1$).
3.2. Universal unramified principal series. Let $G = \text{GL}_2(F)$, and let us consider the following subgroups of $G$

$$K = \text{GL}_2(\mathcal{O}), \quad B = \left\{ \begin{pmatrix} u_1 & x \\ u_2 & \end{pmatrix} \in G \right\}, \quad Z = \left\{ \begin{pmatrix} z \\ \end{pmatrix} \in G \right\}.$$ 

Hence $Z$ is the center of $G$, and $K$ is a maximal compact subgroup. For any ideal $\mathfrak{c} \subset \mathcal{O}$ write $K_0(\mathfrak{c}) \subset K$ for the subgroup of triangular matrices modulo $\mathfrak{c}$.

Let $R$ be a topological Hausdorff ring and assume $\alpha \in R^\times$. Let us consider the unramified character $\mu_\alpha : F^\times \to R^\times$, $\mu_\alpha(x) = \alpha^{\nu(x)}$. This provides a character on $B$ and the induced $R$-representation

$$\text{Ind}_B^G(\mu_\alpha^{-1} \otimes \mu_\alpha) = \left\{ \phi : G \to R \text{ continuous} : \phi \left( \begin{pmatrix} t_1 & x \\ t_2 \\ \end{pmatrix} g \right) = \mu_\alpha(t_2/t_1) \phi(g) \right\}.$$ 

If $\alpha = \pm 1$, the representation $\text{Ind}_B^G(\mu_\alpha^{-1} \otimes \mu_\alpha)$ is reducible, since there is an invariant subspace of rank 1 over $R$ generated by $\phi_0(g) = \alpha^{\nu(\det g)}$. We denote by $(\pi_\alpha^R, V_\alpha^R)$ the quotient representation of $\text{Ind}_B^G(\mu_\alpha^{-1} \otimes \mu_\alpha)$ by this rank 1 subrepresentation. In the case $\alpha \neq \pm 1$, we write $(\pi_\alpha^R, V_\alpha^R)$ for the induced representation $\text{Ind}_B^G(\mu_\alpha^{-1} \otimes \mu_\alpha)$.

By Iwasawa decomposition, $G = BK$. Thus, $\phi_0(bk) = \mu_\alpha^{-1} \otimes \mu_\alpha(b), b \in B k \in K$, is an element of $\text{Ind}_B^G(\mu_\alpha^{-1} \otimes \mu_\alpha)_0$ fixed by $K$ and $Z$. Moreover, if $\alpha \neq \pm 1$, the module $(V_\alpha^R)^K$ is the free $R$-module $R\phi_0$. Similarly, if $\alpha = \pm 1$ the class of $\phi_1(bk) = \mu_\alpha^{-1} \otimes \mu_\alpha(b)1_{K_0(\mathfrak{p})(\mathfrak{w})}$ generates $(V_\alpha^R)^{K_0(\mathfrak{w})}$ freely. In both cases, $\text{rank}_R(V_\alpha^R)^U = 1$, where $U = K$ or $K_0(\mathfrak{w})$.

Let us consider

$$\text{Ind}_U^G(1) = \{ \phi \in C(G, R) : \phi(U Z g) = \phi(g), \phi \text{ compactly supported mod } U Z \},$$

and the Hecke algebra $\mathcal{H}_U^R := \text{Ind}_U^G(1_R)^V$. Note that, by Frobenius reciprocity,

$$\mathcal{H}_U^R = \text{Hom}_{U \bar{Z}}(1_R, \text{Ind}_U^G(1_R)) = \text{End}_G(\text{Ind}_U^G(1_R)).$$

Let $\varphi = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$. We consider the element of the Hecke algebra $T \in \mathcal{H}_K^R = (\text{Ind}_U^G(1_R)^V)$ attached to $1_{UZg_\varphi^{-1}U}$, for $U = K$ or $K_0(\mathfrak{w})$.

Again by Frobenius reciprocity, elements $\phi_i \in (V_\alpha^R)^U$ provide $G$-module homomorphisms

$$\varphi_i : \text{Ind}_{ZK}^G 1_R/(T - a) \text{Ind}_{ZK}^G 1_R \to V_\alpha^R,$$

where $a = \alpha + q a^{-1}$ if $i = 0$, and $a = \alpha = \pm 1$ if $i = 1$.

**Lemma 3.1.** [1] Theorem 20] Assume $R$ is a ring endowed with the discrete topology. Then $\varphi_0$ is injective and $\varphi_1$ is an isomorphism. Moreover, if $R$ is a field then $\varphi_0$ is also bijective.

**Remark 3.2.** We emphasize that $\varphi_0$ is not an isomorphism in general. If $R$ is a domain and $L$ its fraction field, we notice that

$$V_\alpha^R \otimes_R L = V_\alpha^L \simeq \text{Ind}_{ZK}^G 1_L/(T - a) \text{Ind}_{ZK}^G 1_L = \text{Ind}_{ZK}^G 1_R/(T - a) \text{Ind}_{ZK}^G 1_R \otimes_R L.$$ 

Hence, we have two distinguished but generally distinct $G$-stable $R$-modules in $V_\alpha^L$, namely $\Lambda = V_\alpha^R$ and $\Lambda' = \text{Ind}_{ZK}^G 1_R/(T - a) \text{Ind}_{ZK}^G 1_R$, satisfying $\Lambda' \subset \Lambda$.

3.3. Local distributions attached to a torus in $\text{GL}_2$. Let $T^2$ be a 2-dimensional torus in $G = \text{GL}_2(F)$. Note that $T^2$ acts on $F^2$ by means of the embedding $T^2 \hookrightarrow G$.

**Definition 3.3.** We say that $T^2$ is *split* if there is an element in $F^2$ that is an eigenvector for the action of all elements of $T^2$. We say that $T^2$ is *non-split* otherwise.
Let $R$ be a topological Haussdorf ring, and let us consider the action of $T := T^2/Z$ on $C(T, R)$, given by $t * f(x) := f(t^{-1}x)$. Assume that $T^2 \not\subset B$, thus $B \cap T^2 = Z$ and the natural map $T \to G/Z \to B/G \simeq \mathbb{P}^1(F)$ is injective. Let $\mu : B \to R^\times$ be any continuous character trivial at $Z$. We can construct the following morphism

$$\tilde{\delta}_T^\mu : C_\circ(T, R) \to \text{Ind}_B^G(\mu); \quad \tilde{\delta}_T^\mu(f)(g) = \begin{cases} \mu(b)f(t^{-1}), & g = bt \in BT^2 \\ 0, & g \notin BT^2, \end{cases}$$

which is well-defined, since $B \cap T = Z$.

Let $\alpha \in R^\times$ be as above, and let us consider the character $\mu = \mu_\alpha^{-1} \otimes \mu : B \to R^\times$. The composition of $\tilde{\delta}_T^\mu$ with the natural projection gives rise to a morphism

$$(3.2) \quad \delta_T : C_\circ(T, R) \to (\pi^R_\alpha, V^R_\alpha).$$

Since both $\text{Ind}_B^G(\mu)$ and $\pi^R_\alpha$ have trivial central character, restriction to $T^2$ provides a natural action of the group $T = T^2/Z$. It is clear that both morphisms $\tilde{\delta}_T^\mu$ and $\delta_T$ are $T$-equivariant.

**Remark 3.4.** Note that we can restrict $\delta_T$ to the subgroup $C_c(T, R)$ of $C_\circ(T, R)$. Moreover, $C_c(T, R) = C(T, R)$ if $T^2$ is non-split.

**Lemma 3.5.** Assume that $R$ is a domain endowed with the discrete topology and $\alpha \neq \pm 1$. Then there exists $\lambda \in R$ such that

$$\lambda(\text{Im}(\delta_T)) \subseteq \text{Ind}_{ZK}^G 1_R/(T - a)\text{Ind}_{ZK}^G 1_R.$$  

**Proof.** If we prove that there exists $H \subseteq T$ an small enough open compact neighborhood of $1$ such that $\text{Im}((\delta_T)) = R[G]\delta_T(1_H)$, where $1_H$ is the characteristic function of $H$, then the result will automatically follow. Indeed, we can choose $\lambda \in R$ to be such that $\lambda\delta_T(1_H) \in \text{Ind}_{ZK}^G 1_R/(T - a)\text{Ind}_{ZK}^G 1_R$.

Notice that the continuous equivariant map

$$\varphi : G \to B \setminus G \to \mathbb{P}^1(F), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g^{-1} \ast \infty = -\frac{d}{c},$$

provides an injection of $T$ in $\mathbb{P}^1$ that sends $1$ to $\infty$. The open subsets $U_n = \{x \in \mathbb{P}^1, \text{ord}(x) < -n\}$ $(n \geq 0)$ form a neighborhood basis for $\infty$ and their preimages give rise to neighborhood basis $\{H_n\}$ of $1 \in T$. We check that

$$g = \begin{pmatrix} \omega \\ 1 \end{pmatrix}$$

satisfies that $g^{-1}U_n = U_{n+1}$. Thus,

$$\varphi(H_{n+1}) = g^{-1}U_n = \varphi(H_ng_\omega), \quad \text{hence} \quad H_{n+1} \subseteq BH_ng_\omega.$$  

Let $n_0 \in \mathbb{N}$ be big enough that, for any $n \geq n_0$, $H_n \subset K \cap g_\omega^{-1}Kg_\omega$. This implies that

$$H_{n+1} \subseteq BH_ng_\omega \cap g_\omega^{-1}Kg_\omega = g_\omega^{-1}(B \cap K)H_ng_\omega.$$  

It is clear that $\delta_T(1_{H_{n+1}})(g) = \pi_\alpha(g_\omega^{-1})\delta_T(1_{H_n})(g) = 0$ if $\varphi(g) \notin U_{n+1}$. Assume that $\varphi(g) \in U_{n+1}$, then $g = bh_{n+1}$ for some $b \in B$ and $h_{n+1} \in H_{n+1}$. The above claim implies that $h_{n+1} = g_\omega^{-1}b'h_ng_\omega$, where $b' \in B \cap K$ and $h_n \in H_n$. We compute

$$\pi_\alpha(g_\omega^{-1})\delta_T(1_{H_{n+1}})(g) = \delta_T(1_{H_n})(bh_{n+1}g_\omega^{-1}) = \delta_T(1_{H_n})(bg_\omega^{-1}b'h_n) = \mu(g_\omega^{-1} \mu(b))$$

$$\delta_T(1_{H_{n+1}})(g) = \delta_T(1_{H_{n+1}})(b'h_{n+1}) = \mu(b)$$

This implies that $\delta_T(1_{H_{n+1}}) = \mu(g_\omega \pi_\alpha(g_\omega^{-1})\delta_T(1_{H_n})) \in R[G]\delta_T(1_{H_n})$. By induction, we deduce that $\delta_T(1_{H_n}) \in R[G]\delta_T(1_{H_{n-1}})$, for any $n > 0$. Since $1_{H_n}$ generate the $T$-module $C_c(T, R)$ and $\delta_T$ is $T$-equivariant, the results follows. \hfill $\square$

---

1Due to L. Gehrmann
3.4. Torus and inner products. For this section, let \( F \) be a local field (either archimedean or nonarchimedean). Let \( D \) be the quaternion division algebra over \( F \), and let \( T^2 \) be a two-dimensional torus as above. We fix an embedding \( T^2 \hookrightarrow D^\times \) whenever it exists. Let \( \chi : T = T^2/Z \to \mathbb{C} \) be any continuous character.

**Proposition 3.6** (Saito-Tunnel \[13, 10\]). Let \( \pi \) be a representation of \( GL_2(F) \) with central character.

- If either \( \pi \) is principal or \( T^2 \) is split, then \( \dim(\text{Hom}_T(\pi \otimes \chi, \mathbb{C})) = 1 \).
- If \( \pi \) is discrete and \( T^2 \) is non-split, then
  \[ \dim(\text{Hom}_T(\pi \otimes \chi, \mathbb{C})) + \dim(\text{Hom}_T(\pi^{JL} \otimes \chi, \mathbb{C})) = 1. \]

where \( \pi^{JL} \) is the Jacquet-Langlands correspondence of \( \pi \) on \( D^\times \).

Assume that \( \pi \) (and thus \( \pi^{JL} \)) is unitarizable, namely, there is an invariant hermitian inner product \( \langle , \rangle \) on \( \pi \) (and \( \pi^{JL} \)). If \( \dim(\text{Hom}_T(\Pi \otimes \chi, \mathbb{C})) \neq 0 \), where \( \Pi = \pi \) or \( \pi^{JL} \), we can consider the following element of \( \text{Hom}_T(\Pi \otimes \chi, \mathbb{C}) \):

\[ \beta_{\Pi, \chi}(u, w) := \int_T \langle \Pi(t)u, w \rangle \chi(t) dt, \quad u, w \in \Pi. \]

**Proposition 3.7** (Waldspurger \[19\]). Given \( \beta_{\pi, \chi} \) or \( \beta_{\pi^{JL}, \chi} \), we have

(i) If \( \dim(\text{Hom}_T(\pi \otimes \chi, \mathbb{C})) \neq 0 \), then \( \beta_{\pi, \chi} \neq 0 \);

(ii) If \( \dim(\text{Hom}_T(\pi \otimes \chi, \mathbb{C})) = 0 \), then \( \beta_{\pi^{JL}, \chi} \neq 0 \);

(iii) If \( \pi \) is spherical, \( \chi \) unramified, \( v \in \pi^K \), \( \langle v, v \rangle = 1 \) and \( dt \) is the Haar measure on \( T/Z \) such that the volume of the maximal open compact subgroup is 1, then

\[ \beta_{\pi, \chi}(v, v) = \frac{\xi(2)L(1/2, \pi, \chi)}{L(1, \eta)L(1, \pi, ad)} \]

where \( \eta \) is the quadratic character attached to \( T \).

Due to this proposition, we can normalize the above pairing as follows:

\[ \alpha_{\pi, \chi} := \frac{L(1, \eta)L(1, \pi, ad)}{\xi(2)L(1/2, \pi, \chi)} \beta_{\pi, \chi}. \]

3.5. Steinberg representations and intertwining operators. In this section, we assume that our representation is Steinberg, namely, \( \alpha = \pm 1 \). We denote by \( \alpha_s := q^s \alpha \) and \( \mu_s := \mu_{\chi}^{-1} \otimes \mu_{\alpha_s} \), for any \( s \in \mathbb{C} \). We observe that the representation \( \text{Ind}_B^G(\mu_1) \) admits an infinite dimensional subrepresentation \( (V_{\alpha_1}^C, \pi_{\alpha_1}^C) \)

\[ V_{\alpha_1}^C = \left\{ \phi \in \text{Ind}_B^G(\mu_1) : \int_K \phi(k) dk = 0 \right\} \subset \text{Ind}_B^G(\mu_1), \]

by Corollary \[8.1\] with \( M = K \), \( g_0 = 1 \) and \( h(g) = \alpha^{\nu(\det(g))} \phi(g) \) (in this case \( BM = BK = G \)).

Let \( T^2 \) be a two-dimensional torus as above, and let \( T = T^2/Z \). Throughout this section, we assume that \( T^2 \not\subset B \) as above, hence \( T^2 \cap B = Z \). By Corollary \[8.4\] (comparison) we have that

\[ V_{\alpha}^C = \left\{ \phi \in \text{Ind}_B^G(\mu_1) : \int_T \alpha^{\nu(\det(t))} \phi(t) dt = 0 \right\} \subset \text{Ind}_B^G(\mu_1), \]

here the expression \( \alpha^{\nu(\det(t))} \) is well-defined for \( t \in T \), since \( \nu(\det(z)) \in 2\mathbb{Z} \), for \( z \in Z \).

Let \( \phi \in \text{Ind}_B^G(\mu_0) \), such that \( \phi = \delta^\mu_T(f) \) for some \( f \in C_\phi(T, \mathbb{C}) \). Then, we can consider its flat section \( (1 \frac{4.5}) \phi_s := \delta^\mu_T(f) \in \text{Ind}_B^G(\mu_s) \). The integral

\[ I(\phi, s, g) := \int_N \phi_s(n \omega g) dn, \quad \text{where} \quad N := \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} : x \in F \simeq F, \]

\[ \omega := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \]
converges absolutely for \( \text{Re}(s) > 1/2 \) and admits analytic continuation to all \( s \in \mathbb{C} \) \cite[Proposition 4.5.6, Proposition 4.5.7]{[112x-238]} By abuse of notation, we also denote its analytic continuation by \( I(\phi, s, g) \).

Recall the map

\[
\varphi : G \longrightarrow B \backslash G \longrightarrow \mathbb{P}^1, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \longmapsto \frac{d}{c},
\]

that provides an injection of \( T \) in \( \mathbb{P}^1 \). We have that \( \varphi(N \omega) = \mathbb{P}^1 \setminus \{ \infty \} = F \). Hence, for any \( t \in T \) but (possibly) one, there is a unique \( n \in N \), such that \( n \omega \in Bt^{-1} \), for any preimage \( \tilde{t} \in T^2 \) of \( t \). We denote this fact by \( n \omega \in Bt^{-1} \).

**Definition 3.8.** We consider the function \( \theta_T(s)(t) := \mu_{s-1}(n \omega t) \), where \( n \omega \in Bt^{-1} \) and \( n \in N \).

**Remark 3.9.** The expression \( \mu_{s-1}(n \omega t) \) is well-defined, since \( n \omega \tilde{t} \in B \), for any preimage \( \tilde{t} \in T^2 \) of \( t \), and \( \mu_{s-1}(n \omega \tilde{t}) \) does not depend on the given preimage.

If \( t \in T \) and \( n \in N \), let \( y \in T \) such that \( n \omega \in B^{-1}y \) (this can be done for all \( n \) but maybe finitely many). Thus,

\[
\begin{align*}
\phi_s(n \omega t) &= \varphi_s(n \omega y \tilde{y}^{-1}t) = \kappa(n \omega y) \mu_{s-1}(n \omega y) f(t^{-1}y) \\
&= \kappa(n \omega y) \theta_T(s)(y)(t * f)(y),
\end{align*}
\]

where \( \kappa : B \rightarrow \mathbb{R} \) is the modular quasi character defined in Appendix \[8\]. Hence, if we define \( h_t \in C(BT, \mathbb{C}) \) by \( h_t(by) := \kappa(b) \theta_T(s)(y^{-1})(t * f)(y^{-1}) \), for \( b \in B \) and \( y \in T \), we have that \( \phi_s(n \omega t) = h_t(n \omega) \). We compute,

\[
\begin{align*}
I(\phi, s, t) &= \int_N \phi_s(n \omega t) dn = \int_N h_t(n \omega) dn = C_T \int_T h_t(y) dy \\
&= C_T \int_T \theta_T(s)(y)(t * f)(y) dy = C_T \int_T \theta_T(s)(y) f(t^{-1}y) dy,
\end{align*}
\]

by Comparison (Corollary \[8.4\]).

We define \( \Lambda(\phi)(g) := I(\phi, 0, g) \). The following result proves that the expression \( \Lambda(\phi) \) provides a well-defined intertwining operator.

**Proposition 3.10.** We have that:

(i) \( I(\phi, s, bg) = \mu_{1-s}(b) I(\phi, s, g) \), thus, \( \Lambda(\phi) \in \text{Ind}_B^G(\mu_1) \).

(ii) The image of the morphism

\[
\Lambda : \text{Ind}_B^G(\mu_0)) \longrightarrow \text{Ind}_B^G(\mu_1), \quad \phi \longmapsto \Lambda(\phi)
\]

lies in fact in \( V_{\alpha_1}^C \).

(iii) Let \( \phi_0 \in \text{Ind}_B^G(\mu_0) \), defined by \( \phi_0(g) = \alpha^{(\text{det}(g))} \), then \( \Lambda(\phi_0) = 0 \).

(iv) The intertwining operator \( \Lambda \) induces an isomorphism

\[
\Lambda : V_{\alpha}^C \xrightarrow{\cong} V_{\alpha_1}^C.
\]

**Proof.** (i) follows from a direct computation. To prove (ii), we deduce from \[3.5\] that, for any torus \( T \) as above,

\[
\begin{align*}
\int_T \alpha^{(\text{det}(t))} I(\phi, s, t) dt &= C_T \int_T \left( \int_T \alpha^{(\text{det}(t))} \theta_T(s)(y) f(t^{-1}y) dy \right) dt \\
&= C_T \left( \int_T \left( \int_T \alpha^{(\text{det}(t))} f(t^{-1}y) dy \right) \theta_T(s)(y) dy \right) dt \\
&= C_T \left( \int_T \alpha^{(\text{det}(t))} f(t) dt \right) \left( \int_T \alpha^{(\text{det}(t))} \theta_T(s)(t) dt \right),
\end{align*}
\]
since \( \alpha^{v_{\det}(t)} = \alpha^{v_{\det(t^{-1})}} \), \( \alpha^{v_{\det}(t\mu)} = \alpha^{v_{\det}(t)} \alpha^{v_{\det}(\mu)} \) and by exchanging the order of the integrals for \( s \) in the region of absolute convergence. We compute \( \int_T \alpha^{v_{\det}(t)} \theta_T(s)(t)dt \) for the split particular case

\[
T_0 := \{ tx \in \GL_2(F), x \in F^\times \}; \quad t_x := \begin{pmatrix} x & 1-x \\ 1 & 1 \end{pmatrix},
\]

to obtain:

\[
\int_{T_0} \alpha^{v_{\det}(t)} \theta_{T_0}(s)(t)dt = \int_{F^\times} \alpha^{v(x)}(\mu_{s-1}) \left( \begin{array}{ll} 1 & x \\ \frac{1}{x} & 1 \end{array} \right) \omega(\begin{array}{ll} x & x \\ 1-x & 1 \end{array}) dx
\]

\[
= \int_{F^\times} \left| \frac{x}{(x-1)^2} \right|^{1-s} d^\times x = \frac{(q^s-1)^2(q^{s-1}+1)}{(1-q^{s-1})(q-1)(q^{2s-1}-1)}.
\]

We conclude

\[
\int_{T_0} \alpha^{v_{\det}(t)} \Lambda(\phi)(t)dt = \int_{T_0} \alpha^{v_{\det}(t)} I(\phi,0,t)dt = 0,
\]

hence \( \Lambda(\phi) \in V^C_{\alpha} \).

Note that, for any torus \( T \), \( \phi_0 = \delta_T^{mu}(f_0) \) where \( f_0(t) = \alpha^{v_{\det}(t)} \). Thus,

\[
I(\phi_0, s, t) = C_T \int_T \theta_T(s)(y)f_0(t^{-1}y)dy = \alpha^{v_{\det}(t)} C_T \int_T \alpha^{v_{\det}(y)} \theta_T(s)(y)dy.
\]

Due to the previous computation with the particular torus \( T_0 \), we conclude \( \Lambda(\phi_0)(t) = I(\phi_0, 0, t) = 0 \) and \( (iii) \) follows.

Finally, part \( (iv) \) follows from the fact that \( V^C_{\alpha} \) is irreducible and \( \Lambda \) is non-zero.

\[\square\]

3.6. **Local pairings.** In the previous sections, we have defined a \( T \)-equivariant morphism \( \delta_T : C_0(T, \mathbb{C}) \rightarrow \langle \pi, V \rangle \), for any representation \( \pi := \pi^C_\alpha \) associated with \( \alpha \in \mathbb{C}^\times \). Moreover, we have defined the pairing \( \beta_{\pi, \chi} : V \times V \rightarrow \mathbb{C} \). The aim of this section is to compute \( \beta_{\pi, \chi}(\delta_T(f_1), \delta_T(f_2)) \), for \( f_1, f_2 \in C_0(T, \mathbb{C}) \).

We assume that either \( \alpha = \pm 1 \) or \( |\alpha|^2 = q \), the cardinal of the residue field of \( F \). We know that the representation is unitarizable, namely, there is an invariant hermitian inner product \( \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C} \). First, let us fix the inner product:

**Definition 3.11.** If either \( \alpha = \pm 1 \) or \( |\alpha|^2 = q \), we denote by \( \langle \cdot, \cdot \rangle \) the hermitian inner product on \( (V, \pi) := (V^C_\alpha, \pi^C_\alpha) \) given by

\[
\langle \vartheta_1, \vartheta_2 \rangle = \begin{cases} \int_K \vartheta_1(k) \overline{\vartheta_2(k)} dk, & |\alpha|^2 = q, \\
\int_K \vartheta_1(k) \Lambda(\vartheta_2)(k) dk, & \alpha = \pm 1,
\end{cases}
\]

for any \( \vartheta_1, \vartheta_2 \in V^C_{\alpha} \). It is \( G \)-invariant, by Corollary 3.3 of the Appendix with \( M/Z_M = K \).

**Remark 3.12.** Note that, in the case \( \alpha = \pm 1 \), the integral \( \int_K \vartheta_1(k) \Lambda(\vartheta_2)(k) dk \) does not depend on the representative of \( \vartheta_1 \) and \( \vartheta_2 \) in \( \text{Ind}^G_B(\mu_{\alpha} \otimes \mu_{\alpha}^{-1}) \), since \( \Lambda(\phi_0) = 0 \) by Proposition 3.10 \( (iii) \) and

\[
\int_K \phi_0(k) \Lambda(\vartheta_2)(k) dk = \int_K \Lambda(\vartheta_2)(k) dk = 0,
\]

by Proposition 3.10 \( (ii) \).

The following result computes the pairing \( \langle \cdot, \cdot \rangle \) in terms of the torus \( T \).

**Proposition 3.13.** Assume that \( T^2 \not\subset B \) is a 2-dimensional torus, \( T = T^2/Z \) and \( \vartheta_1, \vartheta_2 \in V \). Then, there is a constant \( c_T \) satisfying that

\[
c_T^{-1} \langle \vartheta_1, \vartheta_2 \rangle = \begin{cases} \int_T \vartheta_1(t) \overline{\vartheta_2(t)} dt, & |\alpha|^2 = q, \\
\int_T \vartheta_1(t) \Lambda(\vartheta_2)(t) dt, & \alpha = \pm 1,
\end{cases}
\]

where \( dt \) is the Haar measure of \( T \).
Proof. Note that, \( T \cap B = \mathbb{Z} \) because \( T^2 \not\subset B \). Since \( |\alpha|^2 = q \) or 1, we have that
\[
h(g) = \begin{cases} \vartheta_1(g)\overline{\vartheta_2(g)}, & |\alpha|^2 = q, \\ \vartheta_1(g)\overline{\Lambda(\vartheta_2)(g)}, & \alpha = \pm 1, \end{cases}
\]
satisfies \( h(bg) = \kappa(b)h(g) \), where \( \kappa \) is the modular quasicharacter. The result follows now from Corollary 3.4 of the Appendix with \( M_1/\mathbb{Z}M_1 = K, M_2/\mathbb{Z}M_2 = T \).

The following proposition computes the period \( \beta_{\pi, \chi}(\delta_T(f_1), \delta_T(f_2)) \) for \( f_1, f_2 \in C_0(T, \mathbb{C}) \).

**Proposition 3.14.** Assume that \( T^2 \not\subset B, T = T^2/\mathbb{Z} \) and let \( f_1, f_2 \in C_0(T, \mathbb{C}) \). Then, there is a non-zero constant \( c_T \), depending on \( T \), such that
\[
\beta_{\pi, \chi}(\delta_T(f_1), \delta_T(f_2)) = \begin{cases} c_T \left( \int_T f_1(t)\chi(t)dt \right) \left( \int_T \delta_T(f_2)(t)\chi(t)dt \right), & |\alpha|^2 = q, \\ c_T \left( \int_T f_1(t)\chi(t)dt \right) \left( \int_T f_1(\delta_T(f_2))(t)\chi(t)dt \right), & \alpha = \pm 1, \end{cases}
\]

**Proof.** Let \( f_2^* := f_2 \) if \( |\alpha|^2 = q \) and \( f_2^* := \Lambda(\delta_T(f_2)) \) if \( \alpha = \pm 1 \). Using Proposition 3.13 we obtain,
\[
\int_T (\pi(t)\delta_T(f_1), \delta_T(f_2))\chi(t)dt = c_T \int_T \int_T f_1(x^{-1}t^{-1})f_2^*(x^{-1})\chi(t)dxdy
\]
\[
= c_T \int_T \int_T f_1(y)f_2^*(x^{-1})\chi(yx)dydx \quad (y = x^{-1}t^{-1})
\]
\[
= c_T \int_T \int_T f_1(y)f_2^*(x^{-1})\chi(y)(x^{-1})dydx \quad (\chi^{-1} = \chi)
\]
\[
= c_T \left( \int_T f_1(y)\chi(y)dy \right) \left( \int_T f_2^*(y)\chi(y)dy \right), \quad (y = x^{-1})
\]
and the result follows (we are allowed to exchange the order of the integrals, since \( f_i \in C_0(T, \mathbb{C}) \)).

Let us assume for the rest of the section that \( (\pi, V) \) is Steinberg, hence \( \alpha = \pm 1 \). Proposition 3.14 implies that, in order to compute \( \beta_{\pi, \chi}(\delta_T(f_1), \delta_T(f_2)) \), one has to describe the integral \( \int_T \Lambda(\delta_T(f))(t)\chi(t)dt \) in terms of \( f \in C_0(T, \mathbb{C}) \). By Equation 3.15
\[
\int_T I(\delta_T(f), s, t)\chi(t)dt = c_T \int_T \int_T \theta_T(s)(y)f(t^{-1}y)\chi(t)dydt
\]
\[
= c_T \int_T \int_T \theta_T(s)(y)f(x)\chi(x^{-1}y)dtdy \quad (x = t^{-1}y)
\]
\[
= c_T \left( \int_T \theta_T(s)(t)\chi(t)dt \right) \left( \int_T f(t)\chi^{-1}(t)dt \right).
\]
Again, the integral \( I_T(\chi, s) := \int_T \theta_T(s)(t)\chi(t)dt \) converges absolutely for \( \text{Re}(s) > 1/2 \) and admits analytic continuation to all \( s \in \mathbb{C} \). By abuse of notation, denote also by \( I_T(\chi, s) \) its analytic continuation. We conclude
\[
\int_T \Lambda(\delta_T(f))(t)\chi(t)dt = \int_T I(\delta_T(f), 0, t)\chi(t)dt = c_T I_T(\chi, 0) \int_T f(t)\chi^{-1}(t)dt.
\]

**Remark 3.15.** If we assume that \( \chi(t) = \alpha^{(\det(t))} \), for all \( \phi = \delta_T(f) \in V \)
\[
0 = \int_T \alpha^{(\det(t))}\Lambda(\phi)(t)dt = c_T I_T(\chi, 0) \int_T \alpha^{(\det(t))}f(t)dt
\]
by Proposition 3.10 (ii). This implies that \( I_T(\chi, 0) = 0 \) in this case. We call this the Exceptional Zero Phenomenon.
Let $E/F$ be the quadratic extension, such that $T^2 = E^\times$. Then $T^2$ is split if, and only if, $E = F \times F$. If $T^2$ is non-split, we say that $T^2$ is inert if the extension $E/F$ is inert, and we say that $T^2$ ramifies if $E/F$ ramifies. Let $O_E$ be the integer ring of $E$. Let $\eta$ be the quadratic character associated with $T$. Hence $L(s,\eta) = (1-q^s)^{-1}$ if $E/F$ is split, $L(s,\eta) = (1+q^s)^{-1}$ if $E/F$ is inert, or $L(s,\eta) = 1$ if $E/F$ is ramified. Recall the local Riemann zeta function $\zeta(s) = (1-q^s)^{-1}$.

Note that the maximal compact subgroup $K_0$ of $T$ admits a natural composition series $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n \supseteq \cdots$ such that $[K_n : K_{n+1}] = q$, for all $n > 0$. Given a character $\chi$ of $T$, its conductor $n_\chi$ is the integer such that $\chi | K_{n_\chi} = 1$ and $\chi | K_{n_\chi-1} \neq 1$.

**Theorem 3.16.** There is an integer $n_T$, depending on the embedding $T \hookrightarrow \text{GL}_2(F)$, such that the analytic continuation $I_T(\chi, s)$ is given by

$$I_T(\chi, s) = \begin{cases} q^{(2-2s)n_T} L(-1, \eta) \zeta(1-2s) q^{(1-2s)n_\chi} \chi | O_E^\times \neq 1, \\ q^{(2-2s)n_T} L(-1, \eta) \zeta(1-2s) q^{(1-2s)n_\chi} \chi | O_E^\times = 1, \end{cases}$$

where $n_\chi$ is the conductor of $\chi$ and, for an unramified character $\chi$,

$$L(s, \pi, \chi) = \begin{cases} (1 - \alpha \chi(\varpi)q^s)^{-1} (1 - \alpha \chi(\varpi)^{-1}q^s)^{-1} \quad & E/F \text{ splits} \\
(1 - q^{2s})^{-1} \quad & E/F \text{ splits} \\
(1 - \alpha \chi(\varpi q)q^s)^{-1} \quad & E/F \text{ ramifies}, \end{cases}$$

with $\varpi_E$ a uniformizer of $O_E$ in the ramified case.

**Proof.** Let us assume that the embedding $T^2 \hookrightarrow \text{GL}_2(F)$ is given by

$$T^2 \hookrightarrow \text{GL}_2(F); \quad t \mapsto \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$ 

Hence we compute

$$\theta_T(s)(t) = \mu_{s-1} \left( \begin{pmatrix} 1 & \frac{1}{a(t)} \\ \frac{c(t)}{a(t)} & 1 \end{pmatrix} \right) \omega \left( \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \right) = \alpha^{\nu(\det(t))} \left| \frac{\det(t)}{c(t)^2} \right|^{1-s}.$$

**Assume that $T^2$ is split.** Thus $T \simeq F^\times \subset T^2$. Since any split torus is conjugated to the group of diagonal matrices, we compute that $\det(x) = x$ and $c(x) = C(x-1)$, for some $C \in F$. Since $T^2 \not\subset B$, we have $C \neq 0$. Hence we write $n_T = \text{ord}(C) > -\infty$. We compute

$$I_T(\chi, s) = \int_{T^2} \theta_T(s)(t) \chi(t) dt = \int_{F^\times} \chi(x) \alpha^{\nu(x)} \left| \frac{x}{C^2(x-1)^2} \right|^{1-s} d^\times x$$

$$= q^{(2-2s)n_T} \left( \int_{F^\times \setminus O^\times} \alpha^{\nu(x)} \chi(x)|x|^{-s} d^\times x \right. + \int_{O^\times} \chi(x)|x|^{-1}|2^{s-2} d^\times x +$$

$$\left. + \int_{O^\times \setminus O^\times} \alpha^{\nu(x)} \chi(x)|x|^{1-s} d^\times x \right).$$

Two of these integrals can be calculated easily:

$$\int_{F^\times \setminus O^\times} \alpha^{\nu(x)} \chi(x)|x|^{1-s} d^\times x = \sum_{n > 0} q^{(s-1)n} \alpha^n \chi(\varpi) \int_{O^\times} \chi(x) d^\times x$$

$$= q^{s-1} \alpha \chi(\varpi)^{-1} \frac{1}{1-q^{s-1} \alpha \chi(\varpi)^{-1}} \int_{O^\times} \chi(x) d^\times x.$$ 

$$\int_{O^\times \setminus O^\times} \alpha^{\nu(x)} \chi(x)|x|^{1-s} d^\times x = \sum_{n > 0} q^{(s-1)n} \alpha^n \chi(\varpi) \int_{O^\times} \chi(x) d^\times x$$

$$= q^{s-1} \alpha \chi(\varpi) \frac{1}{1-q^{s-1} \alpha \chi(\varpi)} \int_{O^\times} \chi(x) d^\times x.$$
For any \( n \in \mathbb{N} \), write \( \mathcal{O}_n^\times := 1 + \varpi^n \mathcal{O} \). Let \( n_\chi \) be the conductor of \( \chi \), namely, \( \chi \mid \mathcal{O}_n^\times = 1 \), but \( \chi \mid \mathcal{O}_{n_{\chi} - 1}^\times \neq 1 \). By orthogonality \( \int_{\mathcal{O}_n^\times} \chi(x)\,dx = 0 \) if \( n < n_\chi \), and \( \int_{\mathcal{O}_n^\times} \chi(x)\,dx = \text{vol}(\mathcal{O}_n^\times) \) otherwise. Write also \( U_n := \mathcal{O}_n^\times \setminus \mathcal{O}_{n+1}^\times \). Since \( [\mathcal{O}^\times : \mathcal{O}_n^\times] = q^{n-1} (q - 1) \) for \( n > 0 \), choosing the Haar measure with \( \text{vol}(\mathcal{O}_n^\times) = 1 \), we obtain that \( \text{vol}(\mathcal{O}_n^\times) = (q - 1)^{-1} q^{1-n} (n > 0) \), \( \text{vol}(U_n) = q^{-n} (n > 0) \) and \( \text{vol}(U_0) = (q - 1)^{-1} (q - 2) \). Thus,

\[
\int_{\mathcal{O}_n^\times} \chi(x)|x - 1|^{2s-2} dx = \sum_{n \geq 0} q^{(2-2s)n} \int_{U_n} \chi(x)dx = \begin{cases} \frac{q^{(1-2s)(n_\chi - 1)}}{q - 1} + \sum_{n \geq n_\chi} q^{(1-2s)n}, & n_\chi > 0, \\ \frac{q^{-2} + \sum_{n > 0} q^{(1-2s)n}}{q - 1}, & n_\chi = 0 \end{cases},
\]

\[
= \begin{cases} \frac{q^{(1-2s)n_{\chi + 1}} - q^{(1-2s)(n_\chi - 1)}}{(q - 1)^{1-2s}}, & n_\chi > 0, \\ \frac{(1-\alpha_\chi \alpha^{-1})(1-\alpha_\chi \alpha^{-1} q^{-2})}{(1-\alpha_\chi \alpha^{-1})(1-\alpha_\chi \alpha^{-1} q^{-2})}, & n_\chi = 0. \end{cases}
\]

We conclude

\[
I_T(\chi, s) = \begin{cases} \frac{q^{(2-2s)n_T}}{(1-q^{2-2s})d}(q^{1-2s}n_{\chi + 1} - q^{1-2s}(n_\chi - 1)), & n_\chi > 0, \\ \frac{q^{(1-2s)n_{\chi + 1}} - q^{(1-2s)(n_\chi - 1)}}{(q - 1)^{1-2s}}, & n_\chi = 0 \end{cases},
\]

\[
= \begin{cases} \frac{q^{(1-2s)n_{\chi + 1}} - q^{(1-2s)(n_\chi - 1)}}{(q - 1)^{1-2s}}, & n_\chi > 0, \\ \frac{(1-\alpha_\chi \alpha^{-1})(1-\alpha_\chi \alpha^{-1} q^{-2})}{(1-\alpha_\chi \alpha^{-1})(1-\alpha_\chi \alpha^{-1} q^{-2})}, & n_\chi = 0. \end{cases}
\]

We assume that \( T^2 \) is inert. Thus \( T^2 \simeq \varpi^{\delta} \mathcal{O}_E^\times \), where \( \mathcal{O}_E \) is the integer ring of \( E \). We have \( T = T^2 / \mathbb{Z} = \mathcal{O}_E^\times / \mathcal{O}^\times \) and \( \mathcal{O}_E = \mathcal{O} + \alpha \mathcal{O} \), for some \( \alpha \in \mathcal{O}_E^\times \). Let \( n_T := \nu(c(\alpha)) \). If we now consider the subrings \( \mathcal{O}_n := \mathcal{O} + \varpi^n \mathcal{O} \), for every \( n \in \mathbb{N} \), we compute

\[
I_T(\chi, s) = \int_T \theta_T(s)(t)\chi(t)dt = \int_{\mathcal{O}_n^\times / \mathcal{O}^\times} \chi(t)c(t)|t|^{2s-2}dt
\]

\[
= \sum_{n \geq 0} \int_{(\mathcal{O}_n^\times / \mathcal{O}^\times) \setminus (\mathcal{O}_{n+1}^\times / \mathcal{O}^\times)} \chi(t)q^{(n - n_\chi + 1)(2s-2)}dt
\]

Set \( U_n := (\mathcal{O}_n^\times / \mathcal{O}^\times) \setminus (\mathcal{O}_{n+1}^\times / \mathcal{O}^\times) \). If \( \chi \) has conductor \( n_\chi \), namely, it factorizes by \( \mathcal{O}_E^\times / \mathcal{O}_n^\times \), we obtain that

\[
I_T(\chi, s) = \sum_{n_\chi > n \geq 0} q^{(n - n_\chi + 1)(2s-2)} \int_{U_n} \chi(t)dt + \sum_{n < n_\chi} q^{(n - n_\chi + 1)(2s-2)}\text{vol}(U_n)
\]

When the character \( \chi \mid \mathcal{O}_n^\times / \mathcal{O}^\times \) is not trivial \( (n < n_\chi) \), we have that \( \int_{\mathcal{O}_n^\times / \mathcal{O}^\times} \chi(t)dt = 0 \), by orthogonality. This implies that, if \( n < n_\chi - 1 \),

\[
\int_{U_n} \chi(t)dt = \int_{\mathcal{O}_n^\times / \mathcal{O}^\times} \chi(t)dt - \int_{\mathcal{O}_{n+1}^\times / \mathcal{O}^\times} \chi(t)dt = 0.
\]

On the other side, if \( n = n_\chi - 1 \),

\[
\int_{U_{n_\chi - 1}} \chi(t)dt = \int_{\mathcal{O}_{n_\chi - 1}^\times / \mathcal{O}^\times} \chi(t)dt - \int_{\mathcal{O}_{n_\chi}^\times / \mathcal{O}^\times} \chi(t)dt = -\text{vol}(\mathcal{O}_{n_\chi}^\times / \mathcal{O}^\times).
\]

Since \( \mathcal{O}_E^\times \) is compact, we can assume that \( \text{vol}(\mathcal{O}_E^\times / \mathcal{O}^\times) = 1 \). Moreover, since \( [\mathcal{O}_E^\times : \mathcal{O}_n^\times] = (q+1)q^{n-1} \) for \( n > 1 \), we deduce that \( \text{vol}(\mathcal{O}_n^\times / \mathcal{O}^\times) = (q+1)^{-1} q^{1-n} \) whenever \( n > 0 \). Thus \( \text{vol}(U_n) = (q+1)^{-1} q^{1-n}(q - 1) \) if \( n > 0 \), and \( \text{vol}(U_0) = q(q + 1)^{-1} \).

With all these computations we obtain the value of \( I_T(\chi, s) \):

\[
I_T(\chi, s) = \begin{cases} q^{(2-2s)n_T} - q^{(1-2s)(n_\chi - 1)} + (q - 1) \sum_{n \geq n_\chi} q^{(1-2s)n}, & \chi \neq 1 \\ \frac{q^{(2-2s)n_T}}{q + (q - 1) \sum_{n > 0} q^{(1-2s)n}}, & \chi = 1. \end{cases}
\]
We conclude that
\[ I_T(\chi, s) = \begin{cases} 
\frac{q((2s-2)n_T)}{(q+1)(1-q^{(2s-2)n_T})} (q^{(1-2s)n_{\chi}+1} - q^{(1-2s)(n_{\chi}-1)}), & \chi \neq 1 \\
\frac{q((2s-2)n_T)}{(1+q^{-(2s-2)n_T})} (1 - q^{-2s}), & \chi = 1.
\end{cases} \]

Assume that \( T^2 \) ramifies. Thus \( T^2 \simeq \mathbb{Z}/2\mathbb{Z} \), where \( \mathcal{O}_E \) is the integer ring of \( E \) and \( \varpi^2_E \in \varpi \mathcal{O}_E \). This implies \( T = T^2 / \mathbb{Z} \cong \mathbb{Z} / \mathbb{Z} \times \mathcal{O}_E \setminus \mathcal{O}_E \) and \( |\varpi(\mathcal{O}_E)| = q \).

Note that \( \mathcal{O}_E = \mathcal{O} + \varpi \mathcal{O}_E \), and let \( n_T := \text{ord}(c(\varpi_E)) \). Let us consider now the subrings \( \mathcal{O}_n := \mathcal{O} + \varpi^n \mathcal{O}_E \), and let \( U_n := (\mathcal{O}_n / \mathcal{O}) \setminus (\mathcal{O}_{n+1} / \mathcal{O}) \), for \( n \in \mathbb{N} \).

We compute
\[ I_T(\chi, s) = \int_T \theta_T(s) \chi(t) dt = \int_{\mathbb{Z}/\mathbb{Z}} \alpha^{(\det(t))} \chi(t) \left| \frac{\det(t)}{c(t)^2} \right|^{1-s} dt \\
= \int_{\mathbb{Z}/\mathbb{Z}} \mathcal{O}_E / \mathcal{O} \alpha^{q^{-1}} \chi(t) |c(t)|^{2s-2} dt + \int_{\mathcal{O}_E / \mathcal{O}} \chi(t) |c(t)|^{2s-2} dt \\
= q^{(2n_T-1)-(1-s)} \chi(\mathcal{O}_E) \int_{\mathcal{O}_E / \mathcal{O}} \chi(t) dt + \sum_{n \geq 0} \int_{U_n} \chi(t) q^n(t + n)(2-2s) dt \\
Again we denote by \( n_{\chi} \geq 1 \) the conductor of \( \chi \), namely, \( \chi \) factorizes through \( \mathcal{O}_E / \mathcal{O}_{n_{\chi}-1} \) (we reserve \( n_{\chi} = 0 \) for the case \( \chi = 1 \)). By orthogonality, \( \int_{\mathcal{O}_E / \mathcal{O} \times \chi(t) dt} = 0 \) if \( n < n_{\chi} - 1 \). For a well-chosen Haar measure, \( \text{vol}(\mathcal{O}_E / \mathcal{O} \times q^n) = q^{-n} \) and \( \text{vol}(U_n) = q^{-n-1}(q-1) \), for \( n \geq 0 \), since \( [\mathcal{O}_E : \mathcal{O}_E] = q^n \). Some computations analogous to the previous cases yield the formula
\[ I_T(\chi, s) = \begin{cases} 
q^{(2s-2)n_T} \left( \frac{q((1-2s)n_{\chi} - 1)}{1-q^{(1-2s)n_{\chi}}} + \sum_{n \geq n_{\chi}} q((1-2s)n_{\chi} - 1)(q-1) \right), & n_{\chi} > 1 \\
q^{(2s-2)n_T} \alpha^{q^{-1}} \chi(\mathcal{O}_E) + \sum_{n \geq 0} q^{(1-2s)n_{\chi}}(q-1), & n_{\chi} = 1 \\
q^{(2s-2)n_T} \left( \frac{q((1-2s)n_{\chi} + 1)}{1-q^{(1-2s)n_{\chi}}} - q((1-2s)(n_{\chi}-1)) \right), & n_{\chi} > 1 \\
q^{(2s-2)n_T}(1-\alpha^{(\mathcal{O}_E)}q^{-s})(1+\alpha^{(\mathcal{O}_E)}q^{s-1}), & n_{\chi} = 1,
\end{cases} \]

since \( \alpha^{(\mathcal{O}_E)} = 1 \).

\[ \square \]

**Corollary 3.17.** The analytic continuation \( I_T(\chi, s) \) satisfies \( I_T(\chi, 0) = 0 \) if, and only if, \( \chi(t) = \alpha^{(\det(t))} \).

**Proof.** This follows directly from the above result, observing that \( \zeta(1) \neq 0, \zeta(-2)^{-1} \neq 0, L(-1, \pi, \chi) \neq 0 \) and \( L(0, \pi, \chi)^{-1} = 0 \), if, and only if, \( \chi(t) = \alpha^{(\det(t))} \).

\[ \square \]

**Corollary 3.18.** Assume that \( T^2 \not\subseteq B, T = T^2 / \mathbb{Z} \), and let \( f_1, f_2 \in \mathcal{C}(\mathcal{O}_T, \mathbb{C}) \). Then, there exist non-zero constants \( k_T^1, k_T^2 \), depending on \( T \), such that
\[ \alpha_{\pi, \chi}(\delta_T(f_1), \delta_T(f_2)) = \begin{cases} 
k_T \rho(1, \pi, \chi, T) (\int_T f_1(t) \chi(t) dt)(\int_T f_2(t) \chi(t) dt), & |\alpha|^2 = q, \\
k_T \rho(1, \pi, \chi, L(1/2, \pi, \chi)) (\int_T f_1(t) \chi(t) dt)(\int_T f_2(t) \chi(t) dt), & \alpha = \pm 1, \chi |\mathcal{O}_E| = 1, \\
k_T^\prime \rho(1, \pi, \chi, T) (\int_T f_1(t) \chi(t) dt)(\int_T f_2(t) \chi(t) dt), & \alpha = \pm 1, \chi |\mathcal{O}_E| \neq 1,
\end{cases} \]

where \( n_{\chi} \) is the conductor of \( \chi \).

4. COHOMOLOGY OF AUTOMORPHIC FORMS AND SHIMURA CURVES

Let \( F \) be a totally real number field. Let \( G \) be the algebraic group associated with the multiplicative group of a quaternion algebra over \( F \), that can be either totally definite or split at a single archimedean place \( \sigma \). Write \( G(F)^+ \subset G(F) \) for the subgroup of elements of positive norm.
Let $S$ be a finite set of nonarchimedean places, let $F_S = \prod_{v \in S} F_v$ and $F^S = \hat{F} \cap \prod_{v \in S} F_v$. Usually we shall consider $G(F)$ in $G(F^S)$ by means of the natural monomorphism. Given any ring $R$, let $N, M$ be a $R(G(F))$-module and a $R$-module respectively. We define $A_f^S(N, M)$ to be the module of functions $f : G(F^S) / (F^S)^\times \rightarrow \text{Hom}_R(N, M)$ such that there exists an open compact subgroup $U \subseteq G(F^S)$ such that $f(U) = f(\cdot)$. Note that $A_f^S(N, M)$ is equipped with commuting $G(F)$- and $G(F^S)$-actions:

$$ (\gamma \cdot f)(g) = \gamma (f(\gamma^{-1} g)), \quad \gamma \in G(F), $$

$$ (h \cdot f)(g) = f(gh)), \quad h \in G(F^S); $$

where $g \in G(F^S)$, $f \in A_f^S(N, M)$ and we are considering the usual action of $G(F)$ on $\text{Hom}_R(N, M)$. We write $A_f^S(N, M)$ instead of $A_f^S_{(G)}(N, M)$ and $A_f(N, M)$ instead of $A_f^0(N, M)$. Similarly, we define $A_f^S(R, M)$, where $R$ is endowed with trivial $G(F)$-action. Note that, if $M$ and $N$ are $C$-vector space for some field $C$, then $H^q(H, A_f^S(N, M))$ is a smooth $G(F^S)$-representation over $C$, for any subgroup $H \subseteq G(F)$.

**Remark 4.1.** For any $G(F)$-module $M'$ and any $G(F_S)$-representation $M$ over $R$, we have an isomorphism of $(G(F), G(F^S))$-representations:

$$ (4.6) \quad \phi : \text{Hom}_{G(F_S)}(M, A_f(M', N)) \rightarrow A_f^S(M \otimes_R M', N) $$

$$ \varphi \mapsto \phi(\varphi)(m \otimes m') = \varphi(m)(g)(m'), $$

with inverse $\phi^{-1}(f)(m)(gs, g)(m') := f(g)(gsm \otimes m)$, for all $g \in G(F^S)$, $m \in M$, $m' \in M'$ and $gs \in G(F_S)$.

**Lemma 4.2.** Assume that $M$ is an $R$-module and $R \rightarrow R'$ a flat ring homomorphism. Then the canonical map

$$ H^q(G(F), A_f^S(M \otimes_R R')) \rightarrow H^q(G(F), A_f^S(M \otimes_R R')) $$

is an isomorphism for all $q \geq 0$.

**Proof.** This result can be found essentially in [15 Corollary 4.7], let us reproduce its proof in our setting. It is enough to prove that $H^q(G(F), A_f^S(M) \otimes_R R') \simeq H^q(G(F), A_f^S(M \otimes_R R')^U)$, for any compact open subgroup $U$ of $G(F^S)$. Note that, for any $R$-module $N$, $A_f^S(N)^U = \text{Coind}_U^G(F^S) N$. Hence it is enough to prove that the functor $N \mapsto H^q(G(F), \text{Coind}_U^G(F^S) N)$ commutes with direct limits (any module is the direct limit of free modules of finite rank). By the Strong Approximation Theorem, there are only finitely many double cosets $G(F)gU$ in $G(F^S)$. If $\{g_i\}_{i=1}^n \subseteq G(F^S)$ is a system of representatives then

$$ H^q(G(F), \text{Coind}_U^G(F^S) N) = \bigoplus_{i=1}^n H^q(\Gamma_{g_i}, N), \quad \Gamma_{g_i} = G(F) \cap g_i^{-1} U g_i. $$

Since the group $\Gamma_{g_i}$ is $S$-arithmetic, hence of type (VFL), the functor $N \mapsto H^q(\Gamma_{g_i}, N)$ commutes with direct limits (see [14], p.101) and the result follows.

### 4.1. Abel-Jacobi map on Shimura curves

In this section, we assume that $G$ is the multiplicative group of a quaternion algebra that splits at a single place $\sigma | \infty$. Let us consider in this case the $C$-vector space $\mathcal{A}(C)$ of functions $f : G(F_\sigma) / \mathbb{R}^\times \times G(F) / \hat{F}^\times \rightarrow C$ such that:

- There exists an open compact subgroup $U \subseteq G(\hat{F})$ such that $f(\cdot U) = f(\cdot)$.
- $f \mid_{G(F_\sigma)} \in C^\infty(\text{GL}_2(\mathbb{R}), C)$, under a fixed identification $G(F_\sigma) \simeq \text{GL}_2(\mathbb{R})$.
- Fixing $K_\sigma$, a maximal compact subgroup of $G(F_\sigma)$ isomorphic to $O(2)$, we assume that any $f \in \mathcal{A}(C)$ is $K_\sigma$-finite, namely, its right translates by elements of $K_\sigma$ span a finite-dimensional vector space.
• We assume that any $f \in \mathcal{A}(\mathbb{C})$ must be $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the centre of the universal enveloping algebra of $G(F)$. Write $\rho$ for the action of $G(F) \times G(\hat{F})$ given by right translation, $(\mathcal{A}(\mathbb{C}), \rho)$ defines a smooth $G(\hat{F})$-representation and a $(G_{\sigma}, K_{\sigma})$-module, where $G_{\sigma}$ is the Lie algebra of $G(F_{\sigma})$. Moreover, $\mathcal{A}(\mathbb{C})$ is also equipped with the $G(F)$-action:

$$(h \cdot f)(g_{\sigma}, g) = f(h^{-1}g_{\sigma}, h^{-1}g), \quad h \in G(F),$$

where $g \in G(\hat{F})$, $g_{\sigma} \in G(F_{\sigma})$, $f \in \mathcal{A}(\mathbb{C})$. Let us fix an isomorphism $G(F_{\sigma}) \cong \text{GL}_{2}(\mathbb{R})$ that maps $K_{\sigma}$ to $O(2)$. Let $V$ be a $(G_{\sigma}, K_{\sigma})$-module. In analogy with Remark 4.1 we define

$$\mathcal{A}^\sigma(V, \mathbb{C}) := \text{Hom}_{(G_{\sigma}, K_{\sigma})}(V, \mathcal{A}(\mathbb{C})),$$

endowed with the natural $G(F)$- and $G(\hat{F})$-actions. Notice that $\mathcal{A}^\sigma(\mathbb{C}, \mathbb{C}) = \mathcal{A}_f(\mathbb{C})$.

4.1.1. Cohomology of a Shimura curve. For any open compact subgroup $U \subset G(\hat{F})$, let us consider the Shimura curve $X_U$, whose set of non-cuspidal points $Y_U(\mathbb{C})$ is in correspondence with the double coset space:

$$Y_U(\mathbb{C}) = G(F) \backslash \left( (\mathbb{C} \setminus \mathbb{R}) \times G(\hat{F}) / \hat{F} \times U \right) \subseteq X_U(\mathbb{C}).$$

It is well known that the space of holomorphic forms $\Omega^1_{Y_U}$ of $Y_U$ can be identified with $H^0(G(F), \mathcal{A}^\sigma(D, C)^U)$ by means of the morphism

$$H^0(G(F), \mathcal{A}^\sigma(D, C)^U) \rightarrow \Omega^1_{Y_U}; \quad \phi \mapsto \omega_\phi(g_{\infty} i, g_f) := \frac{1}{2\pi i} \phi(f_2)f_2^{-1}(g_{\infty}, g_f)d\tau,$$

where $D$ is the discrete series representation of weight 2, $f_2 \in D$ is a generator (see Appendix 2-34), $g_f \in G(\hat{F})$, $g_{\infty} \in G(F_{\sigma})^+$, and $\tau = g_{\infty} i \in \mathfrak{h}$, where $\mathfrak{h}$ is the Poincaré upper plane. Note that $\phi(f_2)f_2^{-1}$ is a function on $G(F_{\sigma})^+/\text{SO}(2)\mathbb{R}^+ \times G(\hat{F}) = \mathcal{H} \times G(\hat{F})$. It defines a differential form on $Y_U$ because $\phi$ is $G(F)$-invariant. Moreover, $\omega_\phi$ is holomorphic since $L_{f_2} = 0$ (see 9.37), and $\phi$ is a morphism of $(G_{\sigma}, K_{\sigma})$-modules.

We claim that 9.39 and 9.40 provide exact sequences

$$0 \rightarrow \mathcal{A}_f(\mathbb{C}) \xrightarrow{\iota^\pm} \mathcal{A}^\sigma(I^+, \mathbb{C}) \xrightarrow{\text{pr}^+} \mathcal{A}^\sigma(D, \mathbb{C}) \rightarrow 0;$$

$$0 \rightarrow \mathcal{A}_f(\mathbb{C}) \xrightarrow{\iota^\pm} \mathcal{A}^\sigma(I^-, \mathbb{C}) \xrightarrow{\text{pr}^-} \mathcal{A}^\sigma(D, \mathbb{C}) \rightarrow 0.$$

Indeed, since $\text{Hom}_{(G_{\sigma}, K_{\sigma})}(\mathbb{C}, \mathcal{A}(\mathbb{C}))$ is left exact, we only have to check that $\text{pr}^\pm$ is surjective. We note that $I^\pm$ is generated as a $(G_{\sigma}, K_{\sigma})$-module by $f_0$, and $Rf_0 = f_2$, $L_{f_0} = f_{-2}$ by 9.37. Hence, any $\varphi^\pm \in \mathcal{A}^\sigma(I^\pm, \mathbb{C})$ is characterized by $\varphi^\pm(f_0) \in \mathcal{A}(\mathbb{C})$. Given $\phi \in \mathcal{A}^\sigma(D, \mathbb{C})$, to find a pre-image $\varphi^\pm \in (\text{pr}^\pm)^{-1}(\phi)$ is equivalent to find a SO(2)$\mathbb{R}$-invariant $h = \varphi^\pm(f_0) \in \mathcal{A}(\mathbb{C})$, such that $Rh = \phi(f_2)$ and $Lh = \phi(f_{-2})$. Since,

$$Rh = iye^{2i\theta} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) h = 2if_2 \frac{\partial}{\partial \tau} h, \quad Lh = -2if_{-2} \frac{\partial}{\partial \tau} h,$$

we deduce that any pre-image $\varphi^\pm \in (\text{pr}^\pm)^{-1}(\phi)$ is characterized by

$$d\varphi^\pm(f_0) = \frac{1}{2i} \phi(f_2)f_2^{-1}d\tau - \frac{1}{2i} \phi(f_{-2})f_{-2}^{-1}d\bar{\tau}$$

Hence, since any closed differential 1-form in $\mathcal{H}$ is exact, the claim follows.

The difference between exact sequences (4.7) and (4.8) is the action of complex conjugation. We know that complex conjugation acts on $Y_U(\mathbb{C})$ by sending $(\tau, g_f)$ to $(\overline{\tau}, g_f)$, for any $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $g_f \in G(\hat{F})$. Given $\gamma \in G(F) \setminus G(F)^+$ and $g_{\infty} i = \tau \in \mathfrak{h}$,

$$\omega_\phi(\tau, g_f) := \omega_\phi(\gamma \tau, \gamma g_f) = \frac{1}{2\pi i} \phi(f_2)f_2^{-1}(g_{\infty} \omega, g_f)d\tau = \frac{1}{2\pi i} \phi(\omega f_2)f_{-2}^{-1}(g_{\infty}, g_f)d\bar{\tau}.$$
As it is shown in Appendix 2 \( \phi(\omega f_2) = \pm \phi(f_{-2}) \) depending whether we have chosen exact sequence (4.7) or (4.8). We deduce that
\[
(4.9) \quad d\varphi^\pm(f_0) = \pi(\omega_\varphi \mp \bar{\omega}_\varphi).
\]

The exact sequences (4.7) and (4.8) provide morphisms in cohomology
\[
\partial^\pm : H^0(G(F), A^\varphi(D, \mathbb{C})^U) \rightarrow H^1(G(F), A_f(C)^U).
\]
We can identify \( H^1(G(F)^+, A_f(C)^U) \) with the singular cohomology of \( X_U \) with coefficients in \( \mathbb{C} \). Moreover, once we interpret \( H^0(G(F), A^\varphi(D, \mathbb{C})^U) \) as the holomorphic differentials of \( Y_U \), the compositions
\[
(4.10) \quad H^0(G(F), A^\varphi(D, \mathbb{C})^U) \xrightarrow{\partial^\pm} H^1(G(F), A_f(C)^U) \xrightarrow{\text{res}} H^1(G(F)^+, A_f(C)^U),
\]
correspond to the morphisms
\[
C^\pm : \Omega^1_{X_U} \rightarrow H^1(X_U, \mathbb{C}); \quad \omega_\varphi \mapsto \left( c \mapsto \int_c (\omega_\varphi \mp \bar{\omega}_\varphi) \right),
\]
by (4.9).

**Remark 4.3.** Note that \( H^1(X_U, \mathbb{Q}) = H^1(X_U, \mathbb{Q})^+ \oplus H^1(X_U, \mathbb{Q})^- \), where \( H^1(X_U, \mathbb{Q})^\pm \) is the subspace where complex conjugation acts by \( \pm 1 \), respectively. Then it is clear that both \( H^1(X_U, \mathbb{C})^\pm \simeq H^1(G(F), A_f(C)^U) \) and \( C^\pm : \Omega^1_{X_U} \rightarrow H^1(X_U, \mathbb{C})^\pm \).

Let \( H \subseteq (\mathbb{C} \setminus \mathbb{R}) \) be a \( G(F) \)-subset. Write \( \Delta_H = \mathbb{Z}[H] \), equipped with the natural degree morphism \( \text{deg} : \Delta_H \rightarrow \mathbb{Z} \). If \( \Delta_H^0 \) is the kernel of \( \text{deg} \), then both \( \Delta_H \) and \( \Delta_H^0 \) are \( G(F) \)-modules. Moreover, we have the exact sequence
\[
(4.11) \quad 0 \rightarrow A_f^\varphi(M) \xrightarrow{\text{deg}^*} A_f^\varphi(\Delta_H, M) \rightarrow A_f^\varphi(\Delta_H^0, M) \rightarrow 0.
\]

**Lemma 4.4.** The well defined \( G(F) \)-equivariant morphism
\[
ev^\pm : A^\varphi(I^\pm, \mathbb{C}) \rightarrow A_f(\Delta_H, \mathbb{C}); \quad \ev^\pm(\phi)(\tau) = \frac{1}{\pi} \phi(f_0)(\tau, g),
\]
makes the following diagram commutative
\[
\begin{array}{cccccc}
0 & \rightarrow & A_f(\mathbb{C}) & \xrightarrow{\iota^\pm} & A^\varphi(I^\pm, \mathbb{C}) & \xrightarrow{\text{pr}^\pm} & A^\varphi(D, \mathbb{C}) & \rightarrow & 0 \\
\downarrow{\text{id}} & & \downarrow{\ev^\pm} & & \downarrow{\text{deg}^*} & & \downarrow{\text{deg}^*} & & \downarrow{\ev_0^\pm} & \\
0 & \rightarrow & A_f(\mathbb{C}) & \xrightarrow{\text{deg}^*} & A_f(\Delta_H, \mathbb{C}) & \rightarrow & A_f(\Delta_H^0, \mathbb{C}) & \rightarrow & 0
\end{array}
\]

**Proof.** We compute that
\[
\ev^\pm(\iota^\pm(f))(g)(\tau) = \frac{1}{\pi} \iota^\pm(f)(f_0)(\tau, g) = \frac{1}{\pi} \left( \int_0^\pi d\theta \right) f(g) = f(g) = \text{deg}^* f(\phi(g)(\tau).
\]
Hence, \( \text{deg}^* = \ev^\pm \circ \iota^\pm \). The existence of \( \ev_0^\pm \) follows from a diagram chasing. \( \square \)

The corresponding morphism in cohomology
\[
\ev_0^\pm : H^0(G(F), A^\varphi(D, \mathbb{C})^U) \rightarrow H^0(G(F), A_f(\Delta_H^0, \mathbb{C})^U),
\]
has also a geometric interpretation. Indeed, for any \( m = \sum n_i \tau_i \in \Delta_H^0 \) and \( g \in G(\hat{F}) \), we have the divisor \( (m, gU) = \sum n_i (\tau_i, gU) \in \text{Div}^0(Y_U) \). By (4.9), we have that
\[
(4.13) \quad \ev_0^\pm(\phi)(g)(m) = \int_{(m, gU)} (\omega_\varphi \mp \bar{\omega}_\varphi).
\]
Hence, \( \ev_0^- \) and \( \ev_0^+ \) define the real and imaginary part, respectively, of the image of \( (m, gU) \in \text{Div}^0(Y_U) \) under the Abel-Jacobi map.
4.2. Multiplicity one. Let \( \Pi \) be an automorphic weight 2 representation of \( G(\mathbb{A}_F) \) with trivial central character, let \( \Pi^S \) be its restriction to \( G(F^S) \), and let \( L \) be its field of definition. Thus, there is a smooth irreducible \( L \)-representation \( V \) of \( G(F^S) \), such that \( \Pi^S \cong V \otimes_L \mathbb{C} \). For a field \( C \) containing \( L \) and a smooth semi-simple \( C \)-representation \( W \) of \( G(F^S) \), we write \( W_\Pi \) for \( \text{Hom}_{G(F^S)}(\Pi^S, W) : = \text{Hom}_{G(F^S)}(V \otimes_L C, W) \).

**Definition 4.5.** A \( (F^S) \)-representation \( W \) over \( C \) is of automorphic type if \( W \) is smooth and semi-simple and the only irreducible subrepresentations of \( W \) are either the one-dimensional representations or the \( (F^S) \)-representations over \( C \), attached to automorphic weight 2 representations of \( G(\mathbb{A}_F) \) with a trivial central character.

Let \( W \) be a \( (F^S) \)-representation over \( C \) of automorphic type. By strong multiplicity one, \( W_\Pi \) is independent of the set \( S \) in the following sense: if \( S' \supset S \) and \( U_{S'} = \prod_{v \in S' \setminus S} U_v \) is the open subgroup of \( G(F_{S'}) \) such that \( \dim(\Pi^S_{S'}) = 1 \), then we have

\[
\text{Hom}_{G(F^S)}(\Pi^S_{S'}, W_{U_{S'}}) = \text{Hom}_{G(F^S)}(\Pi^S, W).
\]

**Proposition 4.6.** Let \( G \) be a quaternion algebra over \( F \) that is either definite or splitting at a single archimedean place. Then, for any field \( C \) containing the field of definition of \( \Pi \), the \( (F) \)-representation \( H^k(G(F), \mathcal{A}_f(C)) \) is of automorphic type for all \( k \in \mathbb{Z} \). Moreover,

\[
\begin{align*}
H^0(G(F), \mathcal{A}_f(C))_\Pi &\cong C \quad \text{and} \quad H^k(G(F), \mathcal{A}_f(C))_\Pi = 0 \quad (k \neq 0); & \text{if } G \text{ definite} \\
H^1(G(F), \mathcal{A}_f(C))_\Pi &\cong C \quad \text{and} \quad H^k(G(F), \mathcal{A}_f(C))_\Pi = 0 \quad (k \neq 1); & \text{otherwise.}
\end{align*}
\]

**Proof.** Let \( U \subset G(\hat{F}) \) be an open compact subgroup.

In the definite case, the double coset space \( X_U = G(F) \setminus G(\hat{F})/U \hat{F}^\times \) is a finite set of points. The group \( H^k(G(F), \mathcal{A}_f(C))^U \) can be identified with the singular cohomology of \( X_U \) with coefficients in \( C \). Thus, we deduce that \( H^k(G(F), \mathcal{A}_f(C)) = 0 \) for \( k > 0 \). Moreover, \( H^0(G(F), \mathcal{A}_f(C)) \) is in correspondence with the set of modular forms \( \phi : G(F) \setminus G(\mathbb{A}_F)/U G(F_{\infty}) \hat{F}^\times \rightarrow C \), for some open compact subgroup \( U \subset G(\hat{F}) \). This proves that \( H^k(G(F), \mathcal{A}_f(C)) \) is of automorphic type and, by multiplicity one, \( H^0(G(F), \mathcal{A}_f(C))_\Pi = C \).

In case \( G \) splits at a single archimedean place \( \sigma \), the groups \( H^k(G(F)^+, \mathcal{A}_f(C))^U \) can be identified with the singular cohomology of the Shimura curve \( X_U \) with coefficients in \( C \). Thus, \( H^k(G(F)^+, \mathcal{A}_f(C)) = 0 \), for \( k > 2 \), \( H^0(G(F)^+, \mathcal{A}_f(C)) \) and \( H^2(G(F)^+, \mathcal{A}_f(C)) \) contain only one-dimensional irreducible subrepresentations and, by Eichler-Shimura, \( H^1(G(F)^+, \mathcal{A}_f(C))_\Pi = C \otimes C \). Moreover, the groups \( H^k(G(F)^+, \mathcal{A}_f(C)) \) come equipped with an action of \( G(F)/G(F)^+ \cong \mathbb{Z}/2\mathbb{Z} \) given by conjugation. We can identify \( H^k(G(F), \mathcal{A}_f(C)) \) as the elements of \( H^k(G(F)^+, \mathcal{A}_f(C)) \) fixed by \( G(F)/G(F)^+ \). This implies that \( H^k(G(F), \mathcal{A}_f(C)) \) is of automorphic type and \( H^1(G(F), \mathcal{A}_f(C))_\Pi \cong C \).

5. \( p \)-adic global distributions and measures

As in the previous section, \( F \) will be a totally real number field. Fix a prime \( \mathcal{P} \) with uniformizer \( \varpi \) and residue characteristic \( q \). We will denote by \( \mathcal{O}_{\mathcal{P}} \) the integer ring of \( F_{\mathcal{P}} \).

5.1. Definite anticyclotomic distributions. Throughout this section, \( E/F \) will be either a totally imaginary quadratic extension or the trivial extension \( E = F \times F \). We will denote by \( T \) the torus \( E^\times/F^\times \), likewise, \( \tilde{T} := \hat{E}^\times/\hat{F}^\times \), \( \mathbb{A}_T := \mathbb{A}_{E/F} \), \( T_v = E_v^\times/F_v^\times \), and \( T' = \prod_{v \neq w} E_w^\times/F_w^\times \), where \( \prod' \) denotes the restricted product with respect to \( \mathcal{O}_{E,w} / \mathcal{O}_{F,w} \). Let \( \mathcal{P} \) be a prime ideal of \( F \) above \( p \), and denote by \( \Sigma_\infty \) the set of infinite places of \( F \). Fixing an open subgroup \( \Gamma \subseteq T^\mathcal{P} \), we shall
consider the $p$-adic abelian extension of $E$ associated with $\Gamma$, namely, the maximal abelian $p$-adic Galois group $G_{E,p}^\Gamma$ such that the Artin map $\rho_A : k_T/T \to G_{E,p}^\Gamma$ factors through $\Gamma$.

**Remark 5.1.** Note that here we are considering either cyclotomic extensions of $F$, whenever $E = F \times F$ and therefore $G_{E,p}^\Gamma$ is a quotient of $A \times A$, or anticyclotomic extensions of $E$, whenever $E/F$ is totally imaginary and $G_{E,p}^\Gamma$ is a quotient of $A \times A$.

Let $G$ be the algebraic group associated with the multiplicative group of a quaternion algebra $A$ over $F$ that splits at $P$. Let $(\Pi, V_{\Pi})$ be an automorphic representation of $G(\mathbb{A}_F)$ with a trivial central character, and let us identify $V_{\Pi}$ with a subrepresentation of $H^0(G(F), A(\mathbb{C}))$. Let also assume that $(\Pi_P, V_{\Pi_P}) \simeq (\pi_{\alpha}^{'}, V_{\alpha}^{'})$, where $\alpha = \pm 1$ or $|\alpha| = q$. By the Tensor Product Theorem, we have a $G(F_P)$-equivariant morphism

$$(5.14) \quad V_{\alpha}^C \simeq V_{\Pi_P} \rightarrow V_{\Pi_P}^{U_P}.$$  

**Remark 5.2.** Note that the above morphism is not unique in general. Nevertheless, since $\dim_{C}(\bigotimes_{v \mid \infty} V_{\Pi_v})^{U} = 1$, this morphism is unique up to constant if $V_{\Pi}$ is trivial for all $v \mid \infty$. This will be the case when the quaternion algebra $A$ is totally definite and the automorphic representation $\Pi$ is of parallel weight 2.

Assume that there is a fixed embedding $i : E \hookrightarrow A$. Hence, we can consider $E^\times$, $E_0^\times$, $A_0^\times$, and $A_1^\times$ inside $G(F_0)$, $G(F_1)$, and $G(A_k)$, respectively, by means of $i$. Let us also assume that $(E_0^\times \cap U_P)/(E_1^\times \cap U_P) = \Gamma$. We choose $B_P \subset G(F_P)$, some conjugate of the group of upper triangular matrices, in such a way that $E^\times \not\subset B_P$. Let $R \subset \mathbb{C}$ be any ring (endowed with the discrete topology) such that $\alpha \in R^\times$. The map $\alpha \mapsto \Gamma$ provides a $T_P$-equivariant $R$-module morphism

$$(5.15) \quad \delta_{T_P} : C_c(T_P, R) \rightarrow \text{Ind}_{B_P}^{G(F_P)}(\mu_{1}^{-1} \otimes \mu_{\alpha}) \rightarrow V_{\alpha}^R,$$

and it induces the $T_P$-equivariant morphism

$$\delta : C_c(T_P, R) \rightarrow V_{\alpha}^R \subseteq V_{\Pi_P} \xrightarrow{\delta_{T_P}} V_{\Pi_P}^{U_P}.$$  

Finally, we define the distribution $\mu_{\Pi}^H_{E,p}$ on $G_{E,p}^\Gamma$ as follows: for $g \in C(G_{E,p}^\Gamma, \mathbb{C}) = C_c(G_{E,p}^\Gamma, \mathbb{C})_0$,

$$\int_{G_{E,p}^\Gamma} g(\gamma) d\mu_{\Pi}^H_{E,p}(\gamma) := [H_P : H] \int_{k_T/T} g(\rho_A(x))\delta(1_H)(x)d^\times x,$$

where $\rho_A : k_T/T \to G_{E,p}^\Gamma$ is the Artin map, $H \subset T_P$ is an open subgroup small enough that $g \circ \rho_A$ is $H$-invariant, $1_H \in C_c(T_P, \mathbb{Z}) \subset C_c(T_P, \mathbb{C})$ is the characteristic function of $H$, and $H_P$ is the maximal open subgroup of $T_P$. Note that the above definition does not depend on the choice of $H$. Indeed, for any other $H' \subseteq H$, we
have $1_H = \sum_{h \in H/H'} 1_{hH'} = \sum_{h \in H/H'} h \ast 1_{H'}$, hence (if $g \circ \rho_A$ is $H$-invariant)

$$\int_{G_{E,p}^H} g(\gamma) d\mu_{E,p}^H(\gamma) = [H_P : H] \int_{k_T/T} g(\rho_A(x)) \delta \left( \sum_{h \in H/H'} h \ast 1_{H'} \right) (x) d^x x$$

$$= [H_P : H] \sum_{h \in H/H'} \int_{k_T/T} g(\rho_A(x)) \delta(1_{H'}) (xh) d^x x$$

$$= [H_P : H] \sum_{h \in H/H'} \int_{k_T/T} g(\rho_A(xh^{-1})) \delta(1_{H'}) (x) d^x x$$

$$= [H_P : H] [H : H'] \int_{k_T/T} g(\rho_A(x)) \delta(1_{H'}) (x) d^x x,$$

where the second equality is obtained from the $T_P$-equivariance of $\delta$.

In the cyclotomic setting ($E = F \times F$ and $T = F^\times$), the maximal abelian extension of $F$ unramified outside $\infty$ and $p$ is isomorphic to $\mathbb{Z}_p$. The following lemma describes the free part of the Galois group $G_{E,p}^T$, if $E/F$ is a totally imaginary quadratic extension, and establishes a big difference between the cyclotomic and the anticyclotomic setting:

**Lemma 5.3.** Assume that $E/F$ is a totally imaginary quadratic extension and let $G_{E,p}^T$ be the torsion subgroup of $G_{E,p}^G$. Then $G_{E,p}^T = G_{E,p}^G \times G_{E,p}$, where $G_{E,p} = \mathbb{Z}_p[F_0 \otimes \mathbb{Q}_q]$.

**Remark 5.4.** Observe that $G_{E,p}$ does not depend on $\Gamma$.

**Proof.** First note that, in this case, the Artin map $\rho_A : k_T/TT \rightarrow G_{E,p}^G$ factors through $\hat{T}/TT$. Since $\Gamma \subseteq T^P$ is open, $\hat{T}/\Gamma = T_P \times T^P / \Gamma$ where $T^P / \Gamma$ is discrete. Moreover,

$$\text{rank}_\mathbb{Z}(O_E^T) = \frac{1}{2} [E : \mathbb{Q}] - 1 = [F : \mathbb{Q}] - 1 = \text{rank}_\mathbb{Z}(O_E^T) .$$

Hence $T$ is a discrete subgroup of $\hat{T}$. We deduce that the $\mathbb{Z}_p$-rank of $G_{E,p}^T$ coincides with the $\mathbb{Z}_p$-rank of $O_E^T / O_E^T$, which is clearly $[F_P : \mathbb{Q_P}]$. \hfill $\square$

From now on, we shall consider the distributions $\mu_{E,p}^H$ restricted to functions supported on $G_{E,p}$.

5.1.1. Waldspurger formula and interpolation properties. Let $E$ be a quadratic extension of $F$, as before. Again, we denote by $T$ the torus $E^\times / F^\times$. Let us consider $(\pi, V_\pi)$, an irreducible cuspidal automorphic representation in $L^2(GL_2(F) \backslash GL_2(k_F))$ with trivial central character and parallel weight $(2, 2, \cdots, 2)$, and let $\chi$ be a finite character of $\mathbb{A}_F / T$. Write $L(s, \pi_E, \chi)$ for the Rankin-Selberg $L$-series associated with $\pi, \chi$ and $E$. We also consider the finite sets of places of $F$

$$\Sigma_\pi^\chi := \{ \nu : \dim(\text{Hom}_{T^\chi}(\pi_\nu \otimes \chi_\nu, \mathbb{C})) = 0 \},$$

$$\Sigma_\pi^\infty := \{ \nu \mid \infty : \dim(\text{Hom}_{T^\infty}(\pi_\nu \otimes \chi_\nu, \mathbb{C})) = 0 \} .$$

Let $A$ be a quaternion algebra with ramification set $\Sigma_A$ and let $G$ be the algebraic group associated with the multiplicative group of $A$. For any place $\nu$ of $F$, denote by $(\pi^{\nu}_{\nu}, V^{\nu}_{\nu})$ the Jacquet-Langlands lift of the local representation $\pi_\nu$ on $G(F_\nu)$ (if $\nu \notin \Sigma_A$, then $\pi^{\nu}_{\nu} = \pi_\nu$). We also consider the global Jacquet-Langlands lift $(\pi_{\nu}^{\nu}, V_{\nu}^{\nu})$ of $\pi$ on $G(k_F)$. Invoking again the Tensor Product Theorem, $(\pi_{\nu}^{\nu}, V_{\nu}^{\nu}) \simeq (\otimes_{\nu}^\prime \pi_{\nu}^{\nu}, \otimes_{\nu}^\prime V_{\nu}^{\nu})$. We define the following pairing on $\pi_{\nu}^{\nu}$

$$\beta^{\nu}_{\nu}^{\prime \nu}, \chi_\nu := \frac{\xi(2)}{L(1, \eta)L(1, \pi, ad)} \prod_{\nu} \alpha^{\nu}_{\nu}^{\nu}, \chi_\nu ,$$

where $\xi(s)$ is the Riemann zeta function.
where \( \eta : \mathbb{A}_F^\times / F^\times N_E/F (\mathbb{A}_E^\times) \to \{ \pm 1 \} \) is the quadratic character associated with the extension \( E/F \) (\( \eta \) is trivial if \( E = F \times F \)) and \( \alpha_{\pi, \chi} \) are the local pairings defined in \( \text{[3,4]} \). The pairing \( \beta_{\pi, \chi, \chi} \) is well-defined by Proposition \( \text{[3,7]} \). The following result is due to Waldspurger \( \text{[19]} \).

**Theorem 5.5** (Waldspurger). Assume that \( \phi \in V_{\pi, \chi} \) corresponds to \( \otimes_v f_v \in \bigotimes_v' V_{\pi, \chi} \) under the isomorphism \( \pi^{\text{JL}} \simeq \bigotimes_v' \pi_v^{\text{JL}} \), then

\[
\left\| \int_{k_{\tau}/T} \chi(\epsilon) \phi(\epsilon) d^\times \epsilon \right\|^2 = \frac{1}{2} L(1/2, \pi_E, \chi)L(1, \eta)\beta_{\pi, \chi, \chi}(\otimes_v f_v, \otimes_v f_v).
\]

Moreover, the above expression is 0 unless \( \Sigma_A = \Sigma_E^\chi \).

By Saito-Tunnel (Proposition \( \text{[3,6]} \)), if \( T^2 \) does not split at some place \( \sigma \mid \infty \), then \( \Sigma^\chi_{\infty} \subseteq \Sigma^\chi \) for every finite character \( \chi \). Since the \( \mathbb{C} \)-vector space in \( C(\mathcal{G}_E, \mathbb{C}) \) is generated by the set of finite characters with trivial component outside \( \mathcal{P} \), we deduce the following direct consequence of the Waldspurger formula:

**Corollary 5.6.** Let \( E/F \) be a totally imaginary quadratic extension. Let \( G \) be the algebraic group associated with the multiplicative group of a quaternion algebra \( A \). Then \( \Sigma^\chi_{\infty} \setminus \{ \mathcal{P} \} = \Sigma_A \) is a necessary condition for the Jacquet-Langlands lift \( \pi^{\text{JL}} \) and the distribution \( \tilde{\mu}_{\pi, \chi}^{\text{JL}} \) to be non-zero. In particular, \( A \) is totally definite and \#(\( \hat{\Sigma}^\chi_n \setminus \{ \mathcal{P} \} \)) + [\mathbb{F} : \mathbb{Q}] \) is even.

**Remark 5.7.** The above result implies that whenever \#(\( \hat{\Sigma}^\chi_n \setminus \{ \mathcal{P} \} \)) + [\mathbb{F} : \mathbb{Q}] \) is odd we will not be able to construct non-zero anticyclotomic \( p \)-adic distributions of \( \mathcal{G}_E, \mathcal{P} \) using the previous procedure. In \( \text{[3,4]} \) we will explain a different procedure to deal with this situation.

From now on, we assume that \( E/F \) is a totally imaginary quadratic extension, thus, we deal with the anticyclotomic situation. For a complete description using similar techniques to the cyclotomic case (interpolation property and exceptional zero phenomenon) see \( \text{[15]} \).

**Definition 5.8.** Let \( (\pi, V_\tau) \) be an irreducible cuspidal automorphic representation in \( L^2(\text{GL}_2(F) \setminus \text{GL}_2(\mathbb{A}_F)) \) with trivial central character and parallel weight \( (2, 2, \cdots, 2) \), and assume that \( (\pi_\mathcal{P}, V_{\pi_\mathcal{P}}) \simeq (\pi^\mathcal{P}_E, V^\mathcal{P}_E) \), where \( \alpha = \pm 1 \) or \( |\alpha| = q \).

Under the assumption that \#(\( \hat{\Sigma}^\chi_n \setminus \{ \mathcal{P} \} \)) + [\mathbb{F} : \mathbb{Q}] \) even, we define \( \mu_{\pi, \chi} \) to be the distribution \( \tilde{\mu}_{\pi_\mathcal{P}} \) of \( \mathcal{G}_E, \mathcal{P} \), where \( \pi^{\text{JL}} \) is the corresponding Jacquet-Langlands automorphic representation attached to the totally definite quaternion algebra with ramification set \( (\hat{\Sigma}^\chi_n \setminus \{ \mathcal{P} \} \) \( \cup \Sigma_\infty \).

**Theorem 5.9** (Interpolation Property). There is a non-zero constant \( C_E \) depending on \( E/F \) such that, for any continuous character \( \chi : \mathcal{G}_E, \mathcal{P} \to \mathbb{C}^\times \),

\[
\left\| \int_{\mathcal{G}_E, \mathcal{P}} \chi(\gamma) d\mu_{\pi, \chi}(\gamma) \right\|^2 = C_E C(\pi_\mathcal{P}, \chi_\mathcal{P}) \frac{L(1/2, \pi_E, \chi)}{L(1, \pi, \text{ad})},
\]

where

\[
C(\pi_\mathcal{P}, \chi_\mathcal{P}) = \begin{cases} 
\frac{L(1, \pi_\mathcal{P}, \text{ad})}{L(1/2, \pi_\mathcal{P}, \chi_\mathcal{P})}, & |\alpha|^2 = q; \\
\frac{L(-1, \pi_\mathcal{P}, \chi_\mathcal{P})L(1, \pi_\mathcal{P}, \text{ad})}{L(1, \pi_\mathcal{P}, \text{ad})L(1/2, \pi_\mathcal{P}, \chi_\mathcal{P})}, & \alpha = \pm 1, \chi_\mathcal{P} |\mathcal{O}_E^\chi = 1, \\
q^{n_\chi} \frac{L(1, \pi_\mathcal{P}, \chi_\mathcal{P})}{L(1/2, \pi_\mathcal{P}, \chi_\mathcal{P})}, & \alpha = \pm 1, \chi_\mathcal{P} |\mathcal{O}_E^\chi \neq 1,
\end{cases}
\]

and \( n_\chi \) is the conductor of \( \chi_\mathcal{P} \).

**Proof.** Let \( \pi^{\text{JL}} \) be the Jacquet-Langlands automorphic representation attached to the totally definite quaternion algebra with ramification set \( \hat{\Sigma}^\chi_n \setminus \{ \mathcal{P} \} \). Let \( G \) be the multiplicative group of such quaternion algebra. In order to define \( \mu_{\pi, \chi} = \tilde{\mu}_{\pi_\mathcal{P}}^{\text{JL}} \).
we have to choose a $G(F_p)$-equivariant morphism $V_\alpha^C \to V_{\pi,\iota}^{U_p}$, which is unique up to constant by Remark 5.2. Assume that $\delta(1_H)$ corresponds to $\bigotimes_v f_v \in \bigotimes_v V_{\pi,\iota}^{U_p}$ under the isomorphism of the Tensor Product Theorem. Thus, $f_v \in V_{\pi,\iota}^{U_p}$ is a generator of $V_{\pi,\iota}^{U_p}$ as $\mathbb{C}[G(F_v)]$-module for all $v \nmid \infty P$, and $f_P = \delta_{T_P}(1_H)$. We compute that

$$\left| \int_{G_{E,P}} \chi(\gamma) d_\mu_{G_{E,P}}(\gamma) \right|^2 = [H_P : H]^2 \int_{kT/T} \chi(\rho_A(x)) \delta(1_H)(x) d^\times x \right|^2 = C_\phi[H_P : H]^2 \frac{L(1/2, \pi_{E}, \chi)}{L(1, \pi, ad)} \alpha_{\pi_{E}, \chi, \phi}(\delta_{T_P}(1_H), \delta_{T_P}(1_H)),$$

where $C_\phi = \frac{1}{2} \xi(2) \prod_{v \nmid P} \alpha_{\pi_j \chi, \chi_v}(f_v, f_v)$. Since $\chi$ is a character of $G_{E,P}$, the components $\chi_v$ at $v \nmid P$ are trivial. By Proposition 3.6 the choice of the quaternion algebra ensures that $\alpha_{\pi_j \chi, \chi_v}(f_v, f_v) \neq 0$ for $v \nmid P$, thus $C_\phi \neq 0$. By Corollary 3.15

$$\alpha_{\pi_{E}, \chi, \phi}(\delta_{T_P}(1_H), \delta_{T_P}(1_H)) = k_{T_P} C(\pi_{E}, \chi \phi) \text{vol}(H_P)^2,$$

for some non-zero constant $k_{T_P}$ depending on $T_P$. Finally, the result follows from the fact that $\text{vol}(H)[H_P : H] = \text{vol}(H_P)$ is a fixed constant. \hfill \Box

5.2. Cohomological interpretation of $\mu_{G_{E,P}}^{T,\iota}$. We showed in the proof of Lemma 5.3 that $T$ is a discrete subgroup in $\tilde{T}$. The Artin map provides an isomorphism $G_{E,P} \cong \tilde{T}/\Gamma_T$, for some open compact subgroup $\Gamma \subset T_{\mathbb{P}}$. Hence, for any Hausdorff topological ring $R$, we have an isomorphism

$$\partial : C(G_{E,P}, R) \xrightarrow{\approx} H_0(T, C_e(\tilde{T}/\Gamma, R)).$$

Moreover, the natural map

$$C_e(T_P, R) \otimes_R C_c(T_{\mathbb{P}}/\Gamma, R) \xrightarrow{\cong} C_c(\tilde{T}/\Gamma, R); \quad (f_P \otimes f_{\mathbb{P}}) \longmapsto f_P \cdot f_{\mathbb{P}},$$

is an isomorphism and, since $T_{\mathbb{P}}/\Gamma$ is discrete, any $f_{\mathbb{P}} \in C_c(T_{\mathbb{P}}/\Gamma, R)$ can be written as a finite combination of characteristic functions of $T_{\mathbb{P}}/\Gamma$.

Since the representation $(\pi_{J,\iota}, V_{\pi,\iota})$ satisfies $(\pi_{J,\iota}^{U_P}, V_{\pi,\iota}^{U_P}) \cong (\pi_{J,\iota}^{U_{\mathbb{P}}}, V_{\pi,\iota}^{U_{\mathbb{P}}})$, equation (5.14) provides an element $\phi \in \text{Hom}_{G(F_P)}(V_{\alpha}^C, A_f(\mathbb{C})^{U_P}) = A_D^P(V_{\alpha}^C, \mathbb{C})^{U_P}$, which is unique up to constant by Remark 5.2. Since $\pi_{J,\iota}$ is an automorphic representation $\phi \in H^0(G(F), A_D^P(V_{\alpha}^C, \mathbb{C})^{U_P}).$

Lemma 5.10. We have a natural isomorphism

$$H^k \left( G(F), A_D^P \left( \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R/\mathbb{I}, R} \right) \right) \cong H^k(G(F), A_f^P \left( \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R/\mathbb{I}, R} \right) \otimes_{R'} R),$$

for any ideal $\mathbb{I} \subseteq \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R}$, any $k \in \mathbb{N}$ and any flat ring homomorphism $R' \to R$.

Proof. Let $\phi \in H^k \left( G(F), A_D^P \left( \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R/\mathbb{I}, R} \right) \right)$. By Remark 4.11

$$A_f^P \left( \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R/\mathbb{I}, R} \right) \cong \text{Hom}_{G(F_P)} \left( \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_{R/\mathbb{I}, A_f(R)} \right).$$

Moreover, $\phi(1_{F_{\wp}^\circ U_p}) \in H^k(G(F), A_f(R)) \otimes_{R'} R$ by Lemma 4.2 hence the result follows. \hfill \Box

Since $V_{\alpha}^C = \text{Ind}^{G(F_P)}_{F_{\wp}^\circ U_p} 1_C/(T - a)$ ($C = \mathbb{C}$ or $\mathbb{Q}$) by Lemma 3.11 the above result implies that we can assume that $\phi \in H^0(G(F), A_D^P(V_{\alpha}^C, \mathbb{Q})^{U_P})$. The $T_P$-equivariant $\mathbb{Q}$-module morphism

$$\delta_{T_P} : C_e(T_P, \mathbb{Q}) \to V_{\alpha}^\mathbb{Q},$$

then the result follows.
provides a $T$-equivariant morphism
\begin{equation}
\kappa : A_f^p(V_\alpha^\omega, C_p)^{U_p} \to C_e(T / \Gamma, C_{p_0}^\omega); \quad \langle \kappa(\phi), f_p \otimes f_p \rangle := \sum_{x \in T / \Gamma} f_p(x) \phi(x)(\delta_p(f_p)).
\end{equation}

We deduce that
\[ \int_{g \in \mathbb{G}, \tau} g(\gamma) d\mu_{E, \tau}(\gamma) = \kappa(\text{res}\phi) \cap \partial g, \]
where res : $H^0(G(F), A_f^p(V_\alpha^\omega, \mathbb{Q})) \to H^0(T, A_f^p(V_\alpha^\omega, \mathbb{Q}))$ is the restriction map.

5.3. **Heegner points and Gross-Zagier-Zhang formula.** As in previous sections, let $G$ be the multiplicative group of a quaternion algebra $A / F$ that splits at a single archimedean place $\sigma$. Let $E / F$ be a totally imaginary quadratic extension admitting an embedding $E \hookrightarrow A$.

Assume that the automorphic representation $\pi$ admits a Jacquet-Langlands lift $\pi^{JL}$ to $G$. Since $\pi$ has parallel weight 2, dim $H^0(G(F), A^\sigma(D, \mathbb{C}))_{\pi^{JL}} = 1$. Hence, for some open compact subgroup $U \subset G(\mathbb{F})$, there exists $\phi \in H^0(G(F), A^\sigma(D, \mathbb{C}))^{U}$ that generates $\pi^{JL}|_{G(\mathbb{F})}$. Moreover, $\phi$ is unique up to constant.

We have explained how $\phi$ defines an holomorphic differential form $\omega_\phi \in \Omega^1_{\mathcal{Y}_U / C}$, and since $\pi$ is cuspidal, in fact $\omega_\phi \in \Omega^1_{\mathcal{X}_U / C}$. Let $\Omega^1_{\mathcal{X}_U}$ be the cotangent space of $X_U / F$, and let $\Omega^1_{\mathcal{X}}$ be the $\mathbb{Q}$-vector space
\[ \Omega^1_{\mathcal{X}} = \lim_{U \to \mathbb{F}} \Omega^1_{\mathcal{X}_U} \subset \lim_{U \to \mathbb{F}} \Omega^1_{\mathcal{Y}_U / C} \simeq H^0(G(F), A^\sigma(D, \mathbb{C})). \]

This is a $\mathbb{Q}$-representation of $G(\mathbb{F})$ that decomposes as $\Omega^1_{\mathcal{X}} = \bigoplus \rho$, where $\rho$ are irreducible $\mathbb{Q}$-representations. Let $L$ be the field of definition of $\pi^{JL}$, then there exists an irreducible $L$-representation $\rho_\pi \subset \Omega^1_{\mathcal{X}}$ such that
\[ \rho_\pi \otimes L \subset \pi^{JL}|_{G(\mathbb{F})}, \quad \rho_\pi \otimes \mathbb{Q} \subset \bigoplus_{\tau \in \text{Gal}(L / \mathbb{Q})} \tau \pi^{JL}|_{G(\mathbb{F})}, \]
where $\tau \pi^{JL} = \rho_\pi \otimes \tau(L) \subset H^0(G(F), A^\sigma(D, \mathbb{C}))$. After scaling conveniently, we can assume that $\omega_\phi \in \rho_\pi$. The above equation allows to embed $\rho_\pi$ in different $\mathbb{C}$-representations $\tau \pi^{JL}|_{G(F)}$. For all $\tau \in \text{Gal}(L / \mathbb{Q})$, we write $\tau \omega_\phi$ for the image of $\omega_\phi$. The $\mathbb{Z}$-module
\[ \Lambda_\pi := \left\{ \sum_{\tau \in \text{Gal}(L / \mathbb{Q})} \tau \omega_\phi, \quad c \in H^1(X_U, \mathbb{Z}) \right\} \subset L \otimes \mathbb{Q} \simeq \mathbb{C}^{[L : \mathbb{Q}]}, \]
defines a lattice. Moreover, the complex torus $(L \otimes \mathbb{Q} \subset \Lambda_\pi)$ defines an abelian variety of $GL_2$-type $A_\pi$ defined over $F$ such that $\text{End}_{\mathbb{Q}}(A_\pi) = L$. The Abel-Jacobi map
\begin{equation}
AJ : \text{Div}^0(X_U) \to A_\pi(\mathbb{C}); \quad m \mapsto \left( \int_m^\tau \omega_\phi \right)_{\tau \in \text{Gal}(L / \mathbb{Q})} \mod \Lambda_\pi,
\end{equation}
provides, in fact, a morphism $\text{Jac}(X_U) \to A_\pi$ defined over $F$. Since we are interested in points up to torsion, write $A_\mathbb{Q}^0(M) = A_\pi(M) \otimes_{\text{End}(A)} L$, for any field $M$. Hence $A_\mathbb{Q}^0(\mathbb{C}) \simeq (L \otimes \mathbb{Q} / \mathbb{Z})$, where $A_\mathbb{Z}^0 = A_\mathbb{Q} \otimes \mathbb{Z} / \mathbb{Q}$.

We write $\Delta = \mathbb{Z}[\mathbb{C} \setminus \mathbb{R}]$ and $\Delta^0 = \ker(\deg : \Delta \to \mathbb{Z})$. Let us consider the morphism $\lambda_\pi : A_f(\Delta^0, \mathbb{C}) \to A_f(\Delta^0, A_\pi^0(\mathbb{C}))$ given by
\[ \lambda_\pi \psi(m, g) = 1 \otimes \psi(m, g) \mod \Lambda_\pi \subset (L \otimes \mathbb{Q} \subset A_\mathbb{Q}^0(\mathbb{C})). \]
Recall the morphisms $e_{\mathbb{Q}}^+ : A^\sigma(D, \mathbb{C}) \to A_f(\Delta^0, \mathbb{C})$ of Lemma 4.3. Thus, the composition of $e_0 := e_{\mathbb{Q}}^+ + e_{\mathbb{Q}}^-$ and $\lambda_\pi$ provides a morphism of $G(\mathbb{F})$-representations over $L$:
\[ \varphi : \rho_\pi \to H^0(G(F), A_f(\Delta^0, A_\pi^0(\mathbb{C}))), \quad \text{hence} \quad \varphi \in H^0(G(F), A_f(\Delta^0, A_\pi^0(\mathbb{C}))^{\pi^{JL}}). \]
Let us consider the $L$-module $A^0_\pi(\mathbb{C})$, and the exact sequence

$$0 \rightarrow A_f(A^0_\pi(\mathbb{C})) \rightarrow A_f(\Delta, A^0_\pi(\mathbb{C})) \rightarrow A_f(\Delta^0, A^0_\pi(\mathbb{C})) \rightarrow 0.$$ 

This provides an exact sequence in cohomology

$$H^0(G(F), A_f(A^0_\pi(\mathbb{C}))) \rightarrow H^0(G(F), A_f(\Delta, A^0_\pi(\mathbb{C}))) \rightarrow H^0(G(F), A_f(\Delta^0, A^0_\pi(\mathbb{C}))).$$

**Lemma 5.11.** We have that $H^0(G(F), A_f(A^0_\pi(\mathbb{C}))) = 0$. Moreover, $\partial \varphi = 0$.

**Proof.** Note that $A^0_\pi(\mathbb{C})$ is a free $L$-module of infinite rank, hence the first claim follows from the fact that $H^0(G(F), A_f(L)) = 0$ (Proposition 4.6).

On the other side, we know that $\rho_\pi$ is generated by $\omega_\phi \in \rho^U_\pi$. Thus, if we prove that $\partial(\varphi(\omega_\phi)) = 0$ the assertion will follow.

By Remark 4.13 we have that

$$\partial(\varphi(\omega_\phi)) = \lambda_\pi(\partial^+ \phi + \partial^- \phi) = \lambda_\pi \left( C^+ \omega_\phi + C^- \omega_\phi \right),$$

under the identifications given in 4.11.1

For all $c \in H_1(X_U, \mathbb{Z})$,

$$C^+ \omega_\phi(c) + C^- \omega_\phi(c) = 2 \left( \int \omega_\phi \right) \tau \in \Lambda_\pi.$$ 

Hence, the result follows. \qed

The above Lemma implies that there exists a unique $\bar{\varphi} \in H^0(G(F), A_f(\Delta, A^0_\pi(\mathbb{C})))$ extending $\varphi$. Let us describe $\bar{\varphi}(\omega_\phi)$ geometrically in two different ways: On the one hand,

$$\bar{\varphi}(\omega_\phi) : (\mathbb{C} \setminus \mathbb{R}) \times \hat{G} \rightarrow X_U(\mathbb{C}) \rightarrow A^0_\pi[\mathbb{C}]$$

can be described as the extension of the composition of the $F$-morphism $\iota : X_U \rightarrow \text{Jac}(X_U)$, given by a suitable multiple of the Hodge class (resp., of the cusp at infinity if $A = M_2(\mathbb{Q})$), with the modular parametrization $\text{Jac}(X_U) \rightarrow A_\pi$. Indeed, on the degree zero divisors,

$$\bar{\varphi}(\omega_\phi)(z_1 - z_2, g) = \lambda_\pi \circ (ev^+_0 + ev^+_1) \omega_\phi)(z_1 - z_2, g) = \left( \int_{(z_1,gU)} \tau \omega_\phi \right) \mod \Lambda_\pi,$$

by 4.13. On the other hand, let $\mathcal{H}_U = (\text{Ind}_{U, F \times}^U 1_2)^U$ be the Hecke algebra of compactly supported and $U$-bi-invariant functions. By Frobenius reciprocity, $\mathcal{H}_U$ acts on the 1-dimensional space $\rho^U_\pi$, providing a morphism of algebras

$$\lambda : \mathcal{H}_U \rightarrow L, \quad \lambda(T) \omega_\phi = T * \omega_\phi, \text{ for all } T \in \mathcal{H}_U.$$ 

We write

$$\mathcal{H}^0_U = \left\{ T \in \mathcal{H}_U : \int_{N^{-1}(F^g a_N(U))} \mathcal{T}(g) dg = 0, \text{ for all } a \in \hat{F}^g \right\},$$

where $N : G(\hat{F}) \rightarrow \hat{F}^g$ is the norm map. Notice that both $\mathcal{H}_U$ and $\mathcal{H}^0_U$ act on $\text{Div}(X_U)$.

**Lemma 5.12.** For any $T \in \mathcal{H}^0_U$ and any $(z, gU) \in \text{Div}(X_U)$,

$$\mathcal{T} * (z, gU) = \text{vol}(U)^{-1} \int_{G(\hat{F})} \mathcal{T}(h)(z, ghU) dh \in \text{Div}^0(X_U).$$
Shimura’s reciprocity law asserts that the image of $\Phi$ where $\rho \not\equiv \chi$ such a remark.

Given by $\text{Ind}_{E}^{G}f(z, ghU)$, where $\gamma_{gh} \in G(F)$ satisfies $\gamma_{gh}g_{\text{aN}(g)}U = ghU$. By definition of $\mathcal{H}_{U}^0$, the divisor

$$\text{vol}(U)^{-1} \int_{\text{N}^{-1}(a)} T(h)(\gamma_{gh}z)dh \in \mathbb{Z}[\mathbb{C} \setminus \mathbb{R}]$$

has degree zero, hence $T * (z, gU) \in \text{Div}^{0} (X_U)$.

Let $T \in \mathcal{H}_{U}^0$ such that $\lambda(T) \neq 0$. Then,

$$\bar{\varphi}(\omega_{\phi})(z, g) = \lambda(T)^{-1} AJ(T * (z, gU)),$$

where $AJ$ is the Abel-Jacobi map of (5.22). Moreover, the above description does not depend on $T$.

**Remark 5.13.** Such a $T \in \mathcal{H}_{U}^0$ with $\lambda(T) \neq 0$ exists. Indeed, let $Q$ be a prime of $F$ such that $U_Q \simeq \text{GL}_{2}(O_{F_Q})$, where $O_{F_Q}$ is the integer ring of $F_Q$, and the class of $Q$ is trivial in $F^{\times} / N(U) F^{\times}$. Write

$$T = T_{Q} - (q + 1)1 U \in \mathcal{H}_{U}, \quad T_{Q} = \text{characteristic function of } U \left( \begin{array}{c} \omega_{q} \\ 1 \end{array} \right) U,$$

where $\omega_{q} \in O_{F_Q}$ is an uniformizer and $q = \#(O_{F_Q} / \omega_{q})$. It is easy to check that $T \in \mathcal{H}_{U}^0$. Moreover, $\lambda(T) = a_{Q} - q - 1$, where $|a_{Q}| \leq q^{1/2}$, hence $\lambda(T) \neq 0$.

Let $t_{\tau_{E}} \in \mathbb{h} \subset \mathbb{C} \setminus \mathbb{R}$ be the unique point in the Poincaré hyperplane such that $t_{\tau_{E}} = \tau_{E}$, for all $t \in E^{\times}$. Such $\tau_{E}$ defines a $G(F)$-equivariant monomorphism $\text{Ind}_{E}^{G} \zeta_{E} \hookrightarrow \Delta$. Restricting $\bar{\varphi}$ we obtain,

$$\Phi_{T} : \rho_{\pi} \longrightarrow H^{0}(G(F), \text{C} \hbox{oi}n_{E}^{G}f(A_{f}(A_{\pi}(\mathbb{C}))) = H^{0}(E^{\times}, A_{f}(A_{\pi}(\mathbb{C}))).$$

Shimura’s reciprocity law asserts that the image of $\Phi_{T}(\omega_{\phi})$ lies in $A_{\pi}^{0}(E^{ab})$, where $E^{ab}$ is the maximal abelian extension of $E$, and the Galois action of $\text{Gal}(E^{ab} / E)$ is given by

$$\rho_{\lambda}(t) \Phi_{T}(\omega_{\phi})(g) = \Phi_{T}(\omega_{\phi})(tg), \quad t \in \hat{E}^{\times} / E^{\times}, \quad g \in G(\hat{F}),$$

where $\rho_{\lambda} : \hat{E}^{\times} / E^{\times} \to \text{Gal}(E^{ab} / E)$ is the Artin map. Thus,

$$\Phi_{T} \in H^{0}(E^{\times}, A_{f}(A_{\pi}(E^{ab})))_{\pi^{\perp \Lambda}}.$$

In analogy with (5.1.1) we consider the pairing on $\pi^{\perp \Lambda} |_{G(\hat{F})} \simeq \bigotimes_{v \mid \infty} V_{\pi^{v\perp \Lambda}}$

$$\hat{\beta}_{\pi^{\perp \Lambda}, \chi} = \frac{\xi(2)}{L(1, \eta)L(1, \pi, ad)} \prod_{v \mid \infty} \alpha_{\pi^{v\perp \Lambda}, \chi},$$

and let us also consider the Neron-Tate pairing on $A_{\pi}$:

$$\langle \,, \rangle : A_{\pi}(\mathbb{Q}) \times A_{\pi}(\mathbb{Q}) \to \mathbb{R}.$$
Theorem 5.14 (Gross-Zagier-Zhang). For any finite character \( \chi \) and any \( \otimes_v f_v \in \bigotimes_{v \mid \infty} V_{\psi^L} \), we have

\[
\left| \int_{T/T} \chi(\epsilon) \Phi_T(\otimes_v f_v)(\epsilon) d^\infty \epsilon \right|^2 = L'(1/2, \pi_E, \chi) L(1, \eta) \beta_{\pi^L, \chi}(\otimes_v f_v, \otimes_v f_v).
\]

Moreover, the above expression is 0 whether \( \hat{\Sigma}_A \neq \hat{\Sigma}_\chi \).

Proof. The expression is 0 whether \( \hat{\Sigma}_A \neq \hat{\Sigma}_\chi \) by Proposition 3.7.

In order to prove the above formula, we need to introduce some notations from [20]. While \( V \) runs over the set of open compact subgroups of \( G(\hat{F}) \), the curves \( X_V \) form a natural projective system. Let \( X \) be its projective limit, and let \( \text{Alb}(X) \) be the projective limit of the Albanese varieties \( \text{Alb}(X_V) \). The \( F \)-morphisms \( \iota : X_V \to \text{Jac}(X_V) \simeq \text{Alb}(X_V) \), provide an \( F \)-morphism \( \iota : X \to \text{Alb}(X) \). The element \( \tau_E \in \mathbb{C} \setminus \mathbb{R} \) defines a point \( P_E := \iota(\tau_E, 1) \in \text{Alb}(X)(E^{ab}) \). We define formally

\[
P_X := \int_{E^s \times E^s} \chi(\epsilon)(\rho(\epsilon) P_E) d^\infty \epsilon \in (\text{Alb}(X)(E^{ab}) \otimes \mathbb{C})^X.
\]

Let us consider \( C_c(G(\hat{F})/\hat{F}^\times, \mathbb{C}) \) the set of locally constant and compactly supported functions. Any \( f \in C_c(G(\hat{F})/\hat{F}^\times, \mathbb{C}) \), provides a map

\[
T(f) : \text{Alb}(X) \otimes \mathbb{C} \to \text{Jac}(X_U) \otimes \mathbb{C}, \quad \iota(\tau, g) \mapsto \sum_{h \in G(\hat{F})/U \hat{F}^\times} f(h^{-1}) \iota(\tau, gh).
\]

Let us consider the modular parametrization \( AJ : \text{Jac}(X_U) \otimes \mathbb{C} \to \mathbb{C} \otimes \mathscr{O}_L A_\pi \).

By [20], the Neron-Tate pairing

\[
NT_{\pi^L, \chi} : \mathcal{H}(\mathbb{C})^U \times \mathcal{H}(\mathbb{C})^U \to \mathbb{C}; \quad NT_{\pi^L, \chi}(f_1, f_2) := \langle AJ(T(f_1) P_\chi), AJ(T(f_2) P_\chi) \rangle,
\]

factors through the natural morphism

\[
pr : \mathcal{H}(\mathbb{C})^U \to \pi^L |_{G(\hat{F})} \simeq \bigotimes_{v \mid \infty} \pi^L_v; \quad f \mapsto \int_{G(\hat{F})} f(g)(g * \phi) dg.
\]

Hence,

\[
NT_{\pi^L, \chi} \in \text{Hom}_{E^s \times E^s} \left( \bigotimes_{v \mid \infty} \pi^L_v \otimes \chi, \mathbb{C} \right) \otimes \text{Hom}_{E^s \times E^s} \left( \bigotimes_{v \mid \infty} \pi^L_v \otimes \chi^{-1}, \mathbb{C} \right).
\]

The Gross-Zhagier-Zhang formula asserts that

\[
NT_{\pi^L, \chi} = L'(1/2, \pi_E, \chi) L(1, \eta) \beta_{\pi^L, \chi}
\]

(see [20]). The result follows if we prove that

\[
AJ(T(f) P_\chi) = \int_{E^s \times E^s} \chi(\epsilon) \Phi_T(pr(f))(\epsilon) d^\infty \epsilon,
\]

for any \( f \in C_c(G(\hat{F})/\hat{F}^\times, \mathbb{C})^U \), or equivalently \( \Phi_T(pr(f))(\epsilon) = AJ(T(f)(\rho(\epsilon) P_E)) \).

In fact, we can show that \( \Phi_T(pr(f))(g) = AJ(T(f) (\iota(\tau_E, g))) \), for all \( g \in G(\hat{F}) \). Indeed, \( \Phi_T \) is a morphism of \( G(\hat{F}) \)-representations, \( \pi^L |_{G(\hat{F})} \) is generated by \( pr(1_U) \), and

\[
AJ(T(1_U)(\iota(\tau_E, g))) = AJ(\iota(\tau_E, g)) = \Phi_T(\omega_\phi)(g) = \Phi_T(pr(1_U))(g),
\]

for all \( g \in G(\hat{F}) \).
5.4. Indefinite anticyclotomic distributions. As above, let \((\pi, V_\pi)\) be an irreducible cuspidal automorphic representation in \(L^2(\GL_2(F)\backslash \GL_2(\A_F))\) with trivial central character and parallel weight \((2, 2, \ldots, 2)\), and assume that \((\pi_p, V_\pi_p) \simeq (\pi_\alpha, V_\alpha^\C)\), where \(\alpha = \pm 1\) or \(|\alpha| = q\). Let \(E/F\) be a totally imaginary quadratic extension.

In contrast with Definition 5.3, we assume that \(\#(\Sigma^1_\alpha \setminus \{P\}) + [F : \Q]\) is odd. Let \(A\) be the quaternion algebra with ramification set \((\Sigma^1_\alpha \setminus \{P\}) \cup (\Sigma_\infty \setminus \{\sigma\})\), for a fixed archimedean place \(\sigma\). Let \(\pi^{JL}\) be the corresponding Jacquet-Langlands lift to \(G\), the algebraic group associated with the multiplicative group of \(A\).

The previous conditions imply that \(A\) admits an embedding \(i : E \hookrightarrow A\), that we fix once for all. Let us denote by \(T\) the torus associated with \(E^\times /F^\times\). As above, let \(U = U_P \times U_P \subset G(\hat{F})\) be an open compact subgroup such that \(\dim_\C \left( \bigotimes_{v|\infty} \pi_v^{JL} \right)_U = 1\) and \(U_P = \GL_2(O_P)\) or \(K_0(\varpi)\). We have constructed a morphism of \(G(\hat{F})\)-representations

\[
\Phi_T : \bigotimes_{v|\infty} \pi_v^{JL} \simeq \pi^{JL} |_{G(\hat{F})} \rightarrow H^0(T, A_J(A^0_\pi(E^{ab}) \otimes_L \C)).
\]

Let \(R \subset \C\) be any ring (endowed with the discrete topology) such that \(\alpha \in R^\times\). In analogy with the definite situation, we consider the \(T_P\)-equivariant morphism

\[
\hat{\delta} : C_c(T_P, R) \xrightarrow{\delta_{T_P}} V^R_\alpha \subseteq V^{\pi^{JL}} \xrightarrow{U_P^\vee} \left( \bigotimes_{v|\infty} \pi_v^{JL} \right). \]

Write \(\Gamma = (E_P^\times \cup U^P) / (F_P^\times \cup U^P) \subset T^P\). Similarly as above, we decompose \(G^\Gamma_{E, P} = G_{E, T}^P \times G_{E, P}\), where \(G_{E, P}\) is the torsion subgroup. By Lemma 5.3, \(G_{E, P} \simeq \pi'_P/Q_\infty\) does not depend on \(\Gamma\). We will consider the anticyclotomic distributions of functions supported on \(G_{E, P}\). We know that the Artin map \(\rho_A : T/T \rightarrow G_{E, P}^\Gamma\) factors through \(\Gamma\). For \(g \in C(G_{E, P}, \C)\), we define

\[
\int_{G_{E, P}} g(\gamma)d\mu^{11}_{E, P}(\gamma) := [H_P : H]\int_{T/T} g(\rho_A(x))\Phi_T(\hat{\delta}(1_H))(x)d^\times x \in A^0_\pi(E^{ab}) \otimes_L \C,
\]

where \(H \subset T_P\) is an open compact subgroup small enough that \(g \circ \rho_A\) is \(H\)-invariant, \(1_H \in C_c(T_P, \Z) \subset C_c(T_P, \C)\) is the characteristic function of \(H\), and \(H_P\) is the maximal open compact subgroup of \(T_P\). Again, this definition does not depend on \(H\). Indeed, for any \(H' \subseteq H\), we have \(1_H = \sum_{h \in H/H'} 1_h 1_{H'} = \sum_{h \in H/H'} h \ast 1_{H'}\), hence (if \(g \circ \rho_A\) is \(H\)-invariant)

\[
\int_{G_{E, P}} g(\gamma)d\mu^{11}_{E, P}(\gamma) = [H_P : H]\int_{T/T} g(\rho_A(x))\Phi_T \circ \hat{\delta} \left( \sum_{h \in H/H'} h \ast 1_{H'} \right) (x)d^\times x = [H_P : H]\sum_{h \in H/H'} \int_{T/T} g(\rho_A(x))\Phi_T (\hat{\delta}(1_{H'}))(xh)d^\times x = [H_P : H]\sum_{h \in H/H'} \int_{T/T} g(\rho_A(x)^{-1})\Phi_T (\hat{\delta}(1_{H'}))(x)d^\times x = [H_P : H][H : H']\int_{T/T} g(\rho_A(x))\Phi_T (\hat{\delta}(1_{H'}))(x)d^\times x,
\]

where the second equality is obtained from the \(T_P\)-equivariance of \(\Phi_T \circ \hat{\delta}\).

Remark 5.15. By Theorem 6.14, a necessary condition for the Jacquet-Langlands lift \(\pi^{JL}\) and the above expression to be non-zero is precisely that the ramification set of \(A\) is \((\Sigma^1_\alpha \setminus \{P\}) \cup (\Sigma_\infty \setminus \{\sigma\})\).
Theorem 5.16 (Interpolation Property). There is a non-zero constant $C_E$ depending on $E/F$ such that, for any continuous character $\chi : G_{E,F} \to \mathbb{C}^\times$,
\[
\left| \int_{G_{E,F}} \chi(\gamma) d\mu_{E,F}^{\pi,II}(\gamma) \right|^2 = C_E C(\pi_F, \chi_F) \frac{L(1/2, \pi_E, \chi)}{L(1, \pi, \chi)},
\]
where
\[
C(\pi_F, \chi_F) = \begin{cases} 
\frac{L(1, \pi_F, \chi_F)}{L(1/2, \pi_F, \chi_F)}, & |\alpha|^2 = q, \\
\frac{L(1, \pi_F, \chi_F)}{L(1/2, \pi_F, \chi_F)}, & |\alpha|^2 = 1, \\
q^n & |\alpha|^2 = 1, \quad \alpha \neq 1,
\end{cases}
\]
and $n_{\chi}$ is the conductor of $\chi_F$.

Proof. The proof is completely analogous to the proof of Theorem 5.9 if the Waldspurger formula (Theorem 5.5) is replaced by the Gross-Zagier-Zhang formula (Theorem 5.14).

\[\square\]

5.5. Cohomological interpretation of $\mu_{E,F}^{\pi,II}$. Recall that
\[
\rho_{\pi} \otimes L \mathbb{C} \simeq \pi^{\text{ab}} |_{G(F)} \simeq \bigotimes_{v \mid \infty} \pi_v^{\text{ab}}.
\]
Notice that the composition
\[
\text{Ind}_{U_p F^p}^{G(F_p)} 1_{\tilde{Q}}/\langle T - a \rangle \text{Ind}_{U_p F^p}^{G(F_p)} 1_{\tilde{Q}} \rightarrow \left( \bigotimes_{v \mid \infty} \pi_v^{\text{ab}} \right)^{U_p} \rightarrow \left( \pi^{\text{ab}} |_{G(F)} \right)^{U_p}
\]
sends $f$ to $\sum g \in G(F_p)/U_p F_{p} \pi^{\text{ab}} f(g)g \ast \omega_\phi$. Since $\Phi_T(\omega_\phi) \in H^0(T, \mathcal{A}_f(A_0^0(E^{ab})))$, we deduce that the image of
\[
V_{\alpha}^{\tilde{Q}} \rightarrow \left( \pi^{\text{ab}} |_{G(F)} \right)^{U_p} \Phi_{\alpha} \rightarrow H^0(T, \mathcal{A}_f(A_0^0(E^{ab}) \otimes \mathbb{Q})^{U_p})
\]
lies in $H^0(T, \mathcal{A}_f(A_0^0(E^{ab}) \otimes L \mathbb{Q})^{U_p})$. Given the fixed embeddings $\mathcal{O}_L \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{C}$, let
\[
\log_{\rho_{\pi}} = \log_{\omega_\phi} : A_0^0(E^{ab}) \otimes L \mathbb{Q} \rightarrow \mathbb{C},
\]
be the formal group logarithm attached to the differential $\omega_\phi \in \Omega^1_{X_L}$. Composing with such formal logarithm, we obtain a morphism
\[
V_{\alpha}^{\tilde{Q}} \rightarrow H^0(T, \mathcal{A}_f(A_0^0(E^{ab}) \otimes L \mathbb{Q})^{U_p}) \rightarrow H^0(T, \mathcal{A}_f(\mathcal{O}_L^{\alpha})^{U_p}).
\]
From now on, we denote by $\mu_{E,F}^{\pi,II} \in \text{Dist}(G_{E,F}, \mathcal{O}_L)$ (again denoted the same way by abuse of notation) the $p$-adic distribution
\[
\left( g(\gamma) d\mu_{E,F}^{\pi,II}(\gamma) \right) := [H_{\rho} : H] \int_{\tilde{G} / T} g(\rho_{\lambda}(x)) \log_{\rho_{\pi}} \left( \Phi_T(\delta(1_H)(x)) \right) dx.
\]
Thus $\mu_{E,F}^{\pi,II} \in \text{Dist}(G_{E,F}, \mathcal{O}_L)$ can be interpreted as the logarithm of the corresponding $(A_0^0(E^{ab}) \otimes L \mathbb{C})$-valued distribution.

Analogously to (5.22), equation (5.25) provides an element
\[
\log \phi \in H^0(T, \mathcal{A}_f(V_{\alpha}^{\tilde{Q}}, \mathcal{O}_L)^{U_p}).
\]
If we recall the $T_p$-equivariant morphism (5.21),
\[
\kappa : \mathcal{A}_f(V_{\alpha}^{\tilde{Q}}, \mathcal{O}_L)^{U_p} \rightarrow C_c(T / G, \mathcal{O}_L),
\]
we deduce that
\[
\int_{\tilde{G} / T} g(\gamma) d\mu_{E,F}^{\pi,II}(\gamma) = \kappa(\log \phi) \cap \partial g.
\]
5.6. $p$-adic measures and $p$-adic $L$-functions. Let $E/F$ be a totally imaginary quadratic extension. Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_2$ with parallel weight 2 and a trivial central character. Let $\Sigma_\pi$ be the finite set of places in previous sections.

In the definite case $(\#(\Sigma_\pi \setminus \{P\}) + [F : \mathbb{Q}]$ even), we have constructed the distribution $\mu_{E,P}^{\pi,I}$ defined by

$$g \in C(G_{E,P}, \mathbb{C}_p)_0 \mapsto \kappa(\text{res} \phi) \cap \partial(g),$$

where $\phi \in H^0(G(F), \mathbb{A}_F^P(V_{\alpha}^\mathbb{Q}, \overline{\mathbb{Q}})^{U_P})$ is some generator of the Jacquet-Langlands lift to a totally definite quaternion algebra.

In the indefinite case $(\#(\Sigma_\pi \setminus \{P\}) + [F : \mathbb{Q}]$ odd), we have constructed the distribution $\mu_{E,P}^{\pi,II}$ defined by

$$g \in C(G_{E,P}, \mathbb{C}_p)_0 \mapsto \kappa(\text{log} \phi) \cap \partial(g),$$

where $\log \phi \in H^0(T, \mathbb{A}_F^P(V_{\alpha}^\mathbb{Q}, \mathbb{C}_p)^{U_P})$ has been obtained from the $p$-adic formal logarithm of Heegner points associated with $E$ on the Shimura curve attached to a quaternion algebra that splits at a single place.

**Definition 5.17.** We say that $\pi$ is $\mathcal{P}$-ordinary if $\pi_{\mathcal{P}}$ is of the form $V_{\alpha}^\mathbb{C}$ where $\alpha \in O_{\mathbb{C}_p} \cap \overline{\mathbb{Q}}$.

**Proposition 5.18.** If $\pi$ is $\mathcal{P}$-ordinary then both $\mu_{E,P}^{\pi,I}$ and $\mu_{E,P}^{\pi,II}$ are $p$-adic measures.

**Proof.** Let $\mathcal{O} := \overline{\mathbb{Q}} \cap O_{\mathbb{C}_p}$, $V^{\mathcal{O}} := \text{Ind}_{U_P F_{\mathcal{P}}}^{G(F)} 1_{\mathcal{O}}/(\mathcal{T} - a) \text{Ind}_{U_P F_{\mathcal{P}}}^{G(F)} 1_{\mathcal{O}} \subset V_{\alpha}^\mathbb{C}$ and write

$$V := \{v \in \mathbb{A}_F^P(V_{\alpha}^\mathbb{Q}, \mathbb{C}_p)^{U_P} : \psi(T^P)(v) \subseteq O_{\mathbb{C}_p}, \text{ for all } v \in V^{\mathcal{O}}\}.$$

We claim that, if $\psi$ is in the image of the natural monomorphism

$$H^0(T, V) \otimes O_{\mathbb{C}_p} \rightarrow H^0(T, \mathbb{A}_F^P(V_{\alpha}^\mathbb{Q}, \mathbb{C}_p)^{U_P}),$$

then the distribution $g \mapsto \kappa(\psi) \cap \partial(g)$ is a $p$-adic measure. Indeed, by (5.19), the restriction of $\partial$ provides an isomorphism

$$\partial : C(G_{E,P}^\mathcal{O}, O_{\mathbb{C}_p})_0 \iso \rightarrow H_0(T, C_c(\hat{T}/\Gamma, O_{\mathbb{C}_p})_0).$$

Moreover, by (5.20), $C_c(\hat{T}/\Gamma, O_{\mathbb{C}_p})_0 \simeq C_c(T_P, O_{\mathbb{C}_p})_0 \otimes_{O_{\mathbb{C}_p}} C_c(T_P^P/\Gamma, O_{\mathbb{C}_p})_0$. Since $\alpha \in \mathcal{O}^\mathcal{O}$ (ordinary), by Lemma 5.5, there exists a fixed $\lambda \in \mathcal{O}$ such that the restriction of the morphism $\kappa$ of (5.21) factors through

$$\kappa : V \rightarrow \lambda^{-1} C_c(\hat{T}/\Gamma, O_{\mathbb{C}_p})_0,$$

where $\lambda^{-1} C_c(\hat{T}/\Gamma, O_{\mathbb{C}_p})_0 := \text{Hom}_{O_{\mathbb{C}_p}}(C_c(\hat{T}/\Gamma, O_{\mathbb{C}_p})_0, \lambda^{-1} O_{\mathbb{C}_p})$. We conclude that, for any $\psi \in H^0(T, V)$, the map $g \mapsto \kappa(\psi) \cap \partial(g)$ is a distribution in $\text{Dist}(G_{E,P}, \lambda^{-1} O_{\mathbb{C}_p})$. The claim follows from (2.7).

In the definite case, $\psi = \text{res} \phi$, where $\phi \in H^0(G(F), \mathbb{A}_F^P(V_{\alpha}^\mathbb{Q}, \overline{\mathbb{Q}})^{U_P})$. Let $v_0 = [1_{U_P F_{\mathcal{P}}}^\mathcal{O}] \in V^{\mathcal{O}}$. Note that $\phi(v_0)$ is an automorphic form on a totally definite quaternion algebra, hence its image is a finite set of values. This implies that, after scaling, we can assume that $\phi(v_0)$ has values in $\mathcal{O}$. Since $V^{\mathcal{O}} \iso \mathcal{O}[G(F)]v_0$, we deduce that $\phi(f)$ has values in $\mathcal{O}$, for any $f \in V^{\mathcal{O}}$. This implies that $\text{res} \phi \in H^0(T, V)$. Hence $\mu_{E,P}^{\pi,II} \in \text{Meas}(G_{E,P}, C_p)$.

In the indefinite case, $\psi = \text{log} \phi$. By (5.22),

$$(\text{log} \phi)(g)(g_P v_0) = \text{log}_{g_P}(\Phi_T(\omega_\phi(g_P, g)) = \lambda(\tau_P)^{-1}(\text{log}_{g_P}(A\tau_T \ast (\tau_E, (g_P, g) U))),$$

where $\lambda$ is an Artin conductor.
where \( g \in G(F^\mathcal{P}) \), \( g_P \in G(F_P) \) \(((g_P, g) \in G(\hat{F}))\), \( \mathcal{T} \) is some auxiliary Hecke operator in \( H^I_L \), and \( AJ \) is the Abel-Jacobi map \[\text{(5.22)}\]. By Shimura's reciprocity law,
\[
\rho_{\lambda}(T) \Phi_T(\omega_\phi)(G(F_P) \times T^P) = \Phi_T(\omega_\phi)(G(F_P) \times T^P) \Gamma = \Phi_T(\omega_\phi)(G(F_P) \times T^P).
\]
Thus, the Galois action on \( \lambda(T) \Phi_T(\omega_\phi)(G(F_P) \times T^P) \subseteq A_\pi(E^{ab}) \) factors through \( G^P_{E,P} \). This implies that
\[
\lambda(T) \Phi_T(\omega_\phi)(G(F_P) \times T^P) \subseteq A_\pi(E^I), \quad \text{where} \quad \text{Gal}(E^I/E) = G^I_{E,P}.
\]
Let \( E^P_0 \subseteq C_p \) be the local field extension generated by \( E^I \), let \( \mathcal{P}^I \) above \( \mathcal{P} \) be its maximal ideal and let \( k \) be its residue field. Since \( G^P_{E_P} = G^P_{E,P} \times G^P_{E,p} \), where \( G^P_{E,p} \) is finite and \( G_{E,P} \) is the Galois group of an extension totally ramified at \( \mathcal{P} \), \( k \) is finite. Note that the formal logarithm (after normalization) is given by a formal series in \( \mathcal{O}[|t|] \). We have the exact sequence
\[
0 \longrightarrow \mathcal{G}_{A_*}(\mathcal{P}^I) \longrightarrow A_\pi(E^I_P) \xrightarrow{red} A_\pi(k) \longrightarrow 0,
\]
where \( \mathcal{G}_{A_*} \) is the formal group of \( A_* \) and \( A_\pi(k) \) is finite. This implies that the set \( \operatorname{log}_p(A_\pi(E^P_{0})) \) has bounded denominators. We conclude that there exists \( \lambda' \in \mathbb{C}_p \) such that \( \operatorname{log}_p(\lambda(T) \Phi_T(\omega_\phi)(G(F_P) \times T^P)) \subseteq (\lambda')^{-1}\mathcal{O}_{C_p} \). Again, since \( V^\mathcal{O} = \mathcal{O}(G(F_P)) \),
\[
\lambda(T) \Phi_T(\omega_\phi)(G(F_P) \times T^P) \subseteq (\lambda')^{-1}1 \mathcal{O}_{C_p}.
\]
Thus, \( \operatorname{log}_p \in H^0(T, V) \otimes \mathcal{O}_{C_p} \subseteq \mathbb{C}_p \), and \( \mu^{II}_{E,P} \in \operatorname{Meas}(\mathcal{G}_{E,P}, C_p) \).

Hence, in the \( \mathcal{P} \)-ordinary case, \( \mu^{I}_{E,P}, \mu^{II}_{E,P} \in \operatorname{Meas}(\mathcal{G}_{E,P}, \mathbb{C}_p) \) define elements \( L^I_P(\pi, E) \) and \( L^{II}_P(\pi, E) \) in the Iwasawa algebra \( \mathcal{O}_{C_p}[\mathcal{G}_{E,P}] \otimes \mathcal{O}_{C_p} \mathbb{C}_p \), called (anticyclo
totic) \( p \)-adic \( L \)-functions. Interpolation properties (Theorem \[\text{5.9}\] and Theorem \[\text{5.10}\]) and the vanishing of the local factor \( L(0, \pi_\mathcal{P}, 1)^{-1} \) when \( \pi_\mathcal{P} \) is the Steinberg representation with \( \alpha = 1 \), show the appearance of what are known as \text{Exceptional Zeros}. In terms of Iwasawa algebras, this means that both \( L^I_P(\pi, E) \) and \( L^{II}_P(\pi, E) \) lie in the augmentation ideal \( I := \ker(\operatorname{deg}) \), where
\[
\operatorname{deg} : \mathcal{O}_{C_p}[\mathcal{G}_{E,P}] \otimes \mathcal{O}_{C_p} \mathbb{C}_p \simeq \mathcal{Meas}(\mathcal{G}_{E,P}, \mathbb{C}_p) \longrightarrow \mathbb{C}_p, \quad \mu \longmapsto \int_{\mathcal{G}_{E,P}} du.
\]

\textbf{Corollary 5.19 (Exceptional Zero).} \textit{Assume that} \( \pi_\mathcal{P} \) \textit{is Steinberg with} \( \alpha = \alpha(\pi_\mathcal{P}) = 1 \). \textit{Then the} \( p \)-adic \( L \)-\textit{functions satisfy}
\[
L^I_P(\pi, E) \in I = \ker(\operatorname{deg}), \quad i = I, II,
\]
in both definite and indefinite cases.

\section{Automorphic \( \mathcal{L} \)-invariants}

Let \( G \) be the algebraic group associated with the multiplicative group of a quaternion algebra over \( F \) that can be either definite or split at a single archimedean place. In addition, assume that \( G(F_P) = GL_2(F_P) \), where \( \mathcal{P} \) is the prime ideal of \( F \) dividing \( p \). Let \( (\Pi, V_{II}) \) be an automorphic representation of \( G(\mathcal{A}_F) \) of weight 2 with a trivial central character, and assume that \( (\Pi_{\mathcal{P}}, V_{II_{\mathcal{P}}}) \) is the Steinberg representation \( (\pi^C_{\alpha}, V_{II}^C) \) with \( \alpha = 1 \). Let \( U \subseteq G(\hat{F}) \) be an open compact subgroup such that \( \dim_{\mathbb{C}}(\Pi_{\mathcal{P}} |_{G(\hat{F})})^{U} = 1 \). In this section, we will assume that \( T^2 \) is a torus in \( G(F) \) associated with a totally imaginary extension that splits at \( \mathcal{P} \). Write \( T = T^2/Z \) as usual.
6.1. Extensions of the Steinberg representation. The main references in this section are [15, §2.7] and [8]. Let $G := G(F_p)$ and $B$ is its Borel subgroup. Recall that the Steinberg representation $V^C := V^C_1$ is defined by means of the exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow \text{Ind}^G_1 \rightarrow V^C \rightarrow 0,$$

where $\phi_0(g) = 1$ for all $g \in G(F_p)$. For any topological ring $R$, we defined in [8, proposition 6.2] its $R$-valued analogue $(\pi^R, V^R) := (\pi^R_1, V^R_1)$.

Since $T_p$ splits, there are two eigenvectors $v_1, v_2 \in F_p^2$ and two eigenvalue morphisms

$$\lambda_1, \lambda_2 : T_p^2 \rightarrow F_p^\times,$$

such that $tv_t = \lambda(t)v_t$ and $\text{det}(t) = \lambda_1(t)\lambda_2(t)$, for all $t \in T_p^2$. We fix the isomorphism $\psi : T_p \cong F_p^\times$ provided by $t \mapsto \lambda_1(t)/\lambda_2(t)$. Notice that we have the map

$$(6.26) \quad \varphi : G \rightarrow \mathbb{P}^1(F_p); \quad g = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \mapsto g^{-1} \ast \infty = -\frac{d}{c},$$

that identifies $B \backslash G$ with $\mathbb{P}^1(F_p)$. If $T_p \not\subset B$, the restriction of $\varphi$ provides an injection

$$\varphi |_{T_p} : T_p \hookrightarrow \mathbb{P}^1(F_p), \quad \text{such that} \quad \mathbb{P}^1(F_p) \setminus \varphi(T_p) = \{x_1, x_2\},$$

where $x_1$ and $x_2 \in \mathbb{P}^1(F_p)$ correspond to the spaces generated by $v_1$ and $v_2$, respectively. Since $\varphi(1) = \infty \in \varphi(T_p)$, the points $x_i \neq \infty$ can be seen as values in $F_p$. If $B_{x_i} := \varphi^{-1}(x_i) \subset G$, we have the maps

$$\Lambda_i : G \backslash B_{x_i} \rightarrow F_p^\times; \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \mapsto d + x_i c.$$

It is easy to check that $\Lambda_i(gt) = \Lambda_i(g)\lambda_i(t)$, for all $t \in T_p^2$. Let $H \subset T_p$ be the maximal open subgroup. Thus $T_p/H \cong \varpi^\mathbb{Z}$ for some $\varpi \in T_p$. Assume that $x_1 = \lim_{n \rightarrow \infty} x_n$ in $\mathbb{P}^1(F_p)$ and let $U = x_1 \cup \bigcup_{n \in \mathbb{N}} \varpi^n H$.

Let $\ell : F_p^\times \rightarrow R$ be any continuous group homomorphism. Then, we can define the cocycle $c_\ell \in H^1(G, V^R)$ associated with the extension of $R[G]$-modules

$$0 \rightarrow V^R \xrightarrow{\phi \mapsto (\phi, 0)} \mathcal{E}(\ell) \xrightarrow{\pi \phi \mapsto \pi} R \rightarrow 0,$$

where

$$\mathcal{E}(\ell) := \left\{ (\phi, y) \in C(G(F), R) \times R : \phi \left(\begin{array}{c} t_1 \\ t_2 \end{array}\right) = \phi(g) + \ell(t_2)y \right\} / R\phi_0,$$

with $G$-action $g \ast (\phi, y) = (g \ast \phi, y)$ and $g \ast \phi(h) = \phi(hg)$. Note that the above sequence is exact, since $\pi$ is surjective, indeed, we can define $(\phi_1, 1) \in \mathcal{E}(\ell)$, where

$$\phi_1(g) = \left\{\begin{array}{ll} \ell(\Lambda_2(g)) & \text{if } \varphi(g) \in U, \\ \ell(\Lambda_1(g)) & \text{if } \varphi(g) \not\in U. \end{array}\right.$$  

**Remark 6.1.** The identification $B \backslash G \cong \mathbb{P}^1(F_p)$ provides an $R$-module isomorphism $\text{Ind}^G_1 \cong C(\mathbb{P}^1(F_p), R)$. Thus, we can consider $V^R$ as a quotient of $C(\mathbb{P}^1(F_p), R)$.

**Proposition 6.2.** Write $\tilde{\ell} := \ell \circ \psi : T_p \rightarrow R$. The restriction $\text{res}(c_\ell) \in H^1(T_p, V^R)$ coincides with the cocycle

$$\text{res}(c_\ell) = z_i; \quad z_i(t) := (1 - t)\tilde{\ell}1_U \in C(\mathbb{P}^1(F_p), R).$$

**Proof.** We compute $\text{res}(c_\ell)(x)$, for $x \in \mathbb{P}^1(F_p)$,

$$\text{res}(c_\ell)(t)(x) = t \ast \phi_1(g_x) - \phi_1(g_x) = \phi_1(g_x t) - \phi_1(g_x),$$
for any choice \( g_x \in \varphi^{-1}(x) \). Since \( \phi_1(g) = \ell(A_1(g)) + \ell\left(\frac{\lambda(t)}{\lambda_1(t)}\right)1_U(\varphi(g)) \), we have
\[
\text{res}(\phi_1)(t)(x) = \ell(\lambda_1(g_x) - \lambda(A_1(g_x)) + (1 - t)\ell1_U(x) = \\
\ell(\lambda_1(t)) + (1 - t)\ell1_U(x) = \ell(\lambda_1(t))1_{\varphi}(x) + (1 - t)\ell1_U(x),
\]
since \( \ell\left(\frac{\lambda(t)}{\lambda_1(t)}\right) \) coincides with \( \ell(t) \) whenever \( x = \varphi(t) \) with \( t \in T \). The result follows because the constant function \( \ell(\lambda_1(t))1_{\varphi}(x) \) corresponds to \( \ell(\lambda_1(t))\phi_0 \) under the identification of Remark 6.1.

6.2. \( \mathcal{L} \)-invariants. As in [472] let \( C \) be a field endowed with the discrete topology and containing the field of definition of \( \Pi \). Remark 4.1 implies that
\[
H^k(G(F), \mathcal{A}_f^P(V^C, C))\Pi \simeq H^k(G(F), \mathcal{A}_f(C))\Pi.
\]
Let \( c_{\text{ord}} \in H^1(G(F), V^Z) \) be the restriction of the class associated with the continuous morphism \( \text{ord} : F^\times \rightarrow \mathbb{Z} \) (here the ring \( \mathbb{Z} \) is considered with the discrete topology).

**Proposition 6.3.** The cup product by \( c_{\text{ord}} \) provides an isomorphism of 1-dimensional \( C \)-vector spaces
\[
H^k(G(F), \mathcal{A}_f^P(V^C, C))\Pi \xrightarrow{\cup c_{\text{ord}}} H^{k+1}(G(F), \mathcal{A}_f^P(C))\Pi,
\]
where \( k = 0 \), if \( G \) is definite, and \( k = 1 \), if \( G \) splits at a single archimedean place.

**Proof.** Let us describe the extension \( \mathcal{C}(\ell) \) in the particular case \( \ell = \text{ord} : F^\times \rightarrow \mathbb{Z} \subset C \). By Iwasawa decomposition \( G = BK \), where \( K \) is the maximal compact open subgroup \( GL_2(O_{F_p}) \), hence there exists an element \( (\phi_K, 1) \in \mathcal{C}(\text{ord}) \) defined by
\[
\phi_K\left(\begin{array}{c} t_1 \\
\quad x \\
\quad t_2 \end{array}\right) = \text{ord}(t_2),
\]
since \( t_2 \in O^\times_p \) if \( \left(\begin{array}{c} t_1 \\
\quad x \\
\quad t_2 \end{array}\right) \in B \cap K \). This element is clearly \( K \)-invariant, thus, by Frobenius reciprocity,
\[
(\phi_K, 1) \in \mathcal{C}(\text{ord})^{KZ} = \operatorname{Hom}_{KZ}(1_C, \mathcal{C}(\text{ord})) = \operatorname{Hom}_G(\text{Ind}^G_{KZ, 1_C}, \mathcal{C}(\text{ord})).
\]
Let \( T \in \text{End}_G(\text{Ind}^G_{KZ, 1_C}) = (\text{Ind}^G_{KZ, 1_C})^K \) be the Hecke operator corresponding to \( 1_{K_{qg}K_{qZ}} \). It is easy to compute that \( T(\phi_K, 1) = (q + 1)(\phi_K, 1) \), where \( q = O_F/P \), thus \( (\phi_K, 1) \) provides an homomorphism of \( C[G]\)-modules
\[
\varphi_K : \text{Ind}^G_{KZ, 1_C}/(T - a)\text{Ind}^G_{KZ, 1_C} \rightarrow \mathcal{C}(\text{ord}), \quad a = q + 1.
\]
On the other side, the identity in \( \operatorname{Hom}_{KZ}(C, C) = \operatorname{Hom}_G(\text{Ind}^G_{KZ, 1_C}, C) \) is compatible with \( T - a \) \((a = q + 1)\) and provides a surjective homomorphism
\[
\deg : \text{Ind}^G_{KZ, 1_C}/(T - a)\text{Ind}^G_{KZ, 1_C} \rightarrow C
\]
which makes the following diagram commutative
\[
\begin{array}{ccc}
0 & \xrightarrow{i} & V^C & \xrightarrow{\pi} & C & \xrightarrow{0} \\
\varphi_K & & & & \text{id} & \downarrow \deg \\
\text{Ind}^G_{KZ, 1_C} & \xrightarrow{(T - a)\text{Ind}^G_{KZ, 1_C}} & \text{deg}
\end{array}
\]
Write \( K_0 \ := K_0(\mathbb{C}) \), and let \( U \in \text{End}_G(\text{Ind}^G_{K_{0,Z}, 1_C}) \simeq (\text{Ind}^G_{K_{0,Z}, 1_C})^{K_0} \) be the Hecke operator corresponding to \( 1_{K_{0g}K_{0Z}} \). The two natural forgetful morphism \( v_1, v_2 : \text{Ind}^G_{K_{0,Z}, 1_C} \rightarrow \text{Ind}^G_{K_{Z}, 1_C} \) provides a surjective intertwining operator \( v_2 - v_1 : \text{Ind}^G_{K_{0,Z}, 1_C} \rightarrow \ker(\deg) \). It is easy to check that \( av_2 \circ (U - 1) = TV_1 \circ (U - 1) \), hence the above morphism gives rise to an isomorphism
\[
V^C \simeq \text{Ind}^G_{K_{0,Z}, 1_C}/(U - 1)\text{Ind}^G_{K_{0,Z}, 1_C} \simeq \ker(\deg).
\]
This implies that the map $\varphi_K$ is an isomorphism of $\mathbb{C}[G]$-modules.
Notice that (6.28) provides an exact sequence

$$0 \to \mathcal{A}_f^P(C) \to \mathcal{A}_f^P(\mathcal{O}(\text{ord}), C) \to \mathcal{A}_f^P(V^C, C) \to 0,$$

and the corresponding long exact sequence in cohomology

$$\ldots \to \mathcal{H}^k(G(F), \mathcal{A}_f^P(\mathcal{O}(\text{ord}), C)) \to \mathcal{H}^k(G(F), \mathcal{A}_f^P(V^C, C))$$

$$\cup_{\text{ord}} \to \mathcal{H}^{k+1}(G(F), \mathcal{A}_f^P(C)) \to \mathcal{H}^{k+1}(G(F), \mathcal{A}_f^P(\mathcal{O}(\text{ord}), C)) \to \ldots$$

Moreover, by the above description of $\mathcal{O}(\text{ord})$, we have

$$0 \to \mathcal{A}_f^P(\mathcal{O}(\text{ord}), C) \to \mathcal{A}_f^P(\text{Ind}_{K^Z}^G 1_C, C) \xrightarrow{T_p - a} \mathcal{A}_f^P(\text{Ind}_{K^Z}^G 1_C, C) \to 0.$$

The isomorphism (4.6) identifies

$$\mathcal{A}_f^P(\text{Ind}_{K^Z}^G 1_C, C) \cong \text{Hom}_G(\text{Ind}_{K^Z}^G 1_C, \mathcal{A}_f(C)) \cong \mathcal{A}_f(C)^K$$

Since $\Pi_f^G = 0$ and $H^*(G(F), \mathcal{A}_f(C))$ is of automorphic type, we have by strong multiplicity one

$$H^*(G(F), \mathcal{A}_f^P(\text{Ind}_{K^Z}^G 1_C, C)) = \text{Hom}_{G(F)}(\Pi^G, H^*(G(F), \mathcal{A}_f(C)^K))$$

$$= \text{Hom}_{G(F)}(\Pi, H^*(G(F), \mathcal{A}_f(C)^K)) = 0.$$

Hence $H^*(G(F), \mathcal{A}_f^P(\mathcal{O}(\text{ord}), C))_\Pi = 0$ and the result follows. □

**Remark 6.4.** Let $R_0$ denote the topological ring $R$ endowed with the discrete topology. Since $B \backslash G \simeq \mathbb{P}^1(F_p)$ (here $B$ is the Borel subgroup), we can identify $V^H$ with $C_c(F_p, R_0)$ and $V^R$ with $C_c(F_p, R)$, both inside $C(\mathbb{P}^1(F_p), R)$.

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$ be the valuation ring of $\mathbb{C}_p$. The above remark implies that there is a canonical pairing

$$\langle \cdot, \cdot \rangle : \mathcal{A}_f^P(V^{C_0}, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \times V^{Z_p} \to \mathcal{A}_f^P(\mathbb{C}_p),$$

since $\text{Hom}_G(V^{C_0}, \mathcal{O})$ is identified with a space of bounded distributions. Moreover, if we equip $V^{C_0}$ with the action of $G(F)$ provided by the inclusion $G(F) \hookrightarrow G(F_p)$, the above pairing is $G(F)$-equivariant.

Recall that, by Lemma 5.1 and Lemma 5.10, we can identify

$$H^k(G(F), \mathcal{A}_f^P(V^{\mathcal{O}_0}, \mathcal{O})) \simeq H^k(G(F), \mathcal{A}_f^P(V^{C_0}, \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p).$$

Given a continuous homomorphism $\ell : F_p^\times \to \mathbb{Z}_p$, the cup product by $c_\ell \in H^1(G(F), V^{Z_p})$ on the $\Pi$-isotypic component induces a morphism

$$\langle \cdot \cup c_\ell \rangle_\Pi : H^k(G(F), \mathcal{A}_f^P(V^{\mathcal{O}_0}, \mathcal{O}))_\Pi \to H^{k+1}(G(F), \mathcal{A}_f^P(\mathbb{C}_p))_\Pi,$$

where $k = 0$ if $G$ is definite, and $k = 1$ if $G$ splits at a single archimedean place. By Proposition 6.3, both $H^k(G(F), \mathcal{A}_f^P(V^{\mathcal{O}_0}, \mathcal{O}))_\Pi$ and $H^{k+1}(G(F), \mathcal{A}_f^P(\mathbb{C}_p))_\Pi$ are one-dimensional $\mathbb{C}_p$-vector spaces with a fixed isomorphism $\langle \cdot \cup c_{\text{ord}} \rangle_\Pi$.

**Definition 6.5.** Given a continuous group homomorphism $\ell : F_p^\times \to \mathbb{Z}_p$, the automorphic $L$-invariant associated with $\ell$ and $\Pi$ is the unique $\mathcal{L}_P(\Pi, \ell) \in \mathbb{C}_p$ such that

$$\langle \cdot \cup c_\ell \rangle_\Pi = \mathcal{L}_P(\Pi, \ell) \langle \cdot \cup c_{\text{ord}} \rangle_\Pi.$$
Conjecture 6.6. Let \( G_1 \) and \( G_2 \) be the multiplicative groups of two quaternion algebras over \( F \) that are either definite or split at a single archimedean place, but both split at \( P \). If \( \Pi_1 \) and \( \Pi_2 \) are two automorphic representations of \( G_1 \) and \( G_2 \), respectively, related by the Jacquet-Langlands correspondence, then the \( L \)-invariants coincide, namely,

\[
\mathcal{L}_P(\Pi_1, \ell) = \mathcal{L}_P(\Pi_2, \ell),
\]

for any continuous morphism \( \ell : F_P^* \to \mathbb{Z}_p \).

6.3. Geometric \( L \)-invariants. Let \( A_\Pi \) be the abelian variety of \( \text{GL}_2 \)-type associated with \( \Pi \), and let \( B_\Pi \) be its dual abelian variety. Let \( L \) be the field of definition of \( \Pi \). Since \( \Pi \) is Steinberg with \( \alpha = 1 \) at \( P \), the abelian varieties \( A_\Pi \) and \( B_\Pi \) have purely multiplicative reduction at \( P \), hence they admit the following analytic description: There is a pairing (determined up to canonical isomorphism)

\[
X \times Y \xrightarrow{j} F_P^*,
\]

where \( X \) and \( Y \) are free abelian groups of rank \( d = [L : \mathbb{Q}] \), and \( j \) is a bi-multiplicative mapping such that the composition \( \text{ord}_P \circ j \) tensored with \( \mathbb{Q} \) gives a perfect duality of \( \mathbb{Q} \)-vector spaces

\[
(6.30) \quad X \otimes \mathbb{Q} \times Y \otimes \mathbb{Q} \xrightarrow{\text{ord}_P \circ j} \mathbb{Q},
\]

moreover, such that there is a pair of exact sequences of \( \text{Gal}(\overline{\mathbb{Q}}_p/F_P) \)-modules (\( X \) and \( Y \) endowed with trivial Galois action)

\[
(6.31) \quad 0 \to X \to \text{Hom}(Y, \overline{\mathbb{Q}}_p^\times) \xrightarrow{\iota_A} A_\Pi(\overline{\mathbb{Q}}_p) \to 0
\]

\[
(6.32) \quad 0 \to Y \to \text{Hom}(X, \overline{\mathbb{Q}}_p^\times) \xrightarrow{\iota_B} B_\Pi(\overline{\mathbb{Q}}_p) \to 0
\]

where the morphisms on the left are induced by \( j \).

Let \( \mathcal{O}_L \subset L \) be the endomorphism ring of \( A_\Pi \) (and \( B_\Pi \)). Thus, \( X \) and \( Y \) are \( \mathcal{O}_L \)-modules and \( j(\alpha x, y) = j(x, \alpha y) \) for all \( x \in X \), \( y \in Y \) and \( \alpha \in \mathcal{O}_L \). The non-degenerate pairing \((6.30)\) provides an isomorphism of 1-dimensional \( L \)-vector spaces

\[
\alpha : X \otimes \mathbb{Q} \to \text{Hom}(Y \otimes \mathbb{Q}, \mathbb{Q})
\]

Given a continuous morphism \( \ell : F_P^* \to \mathbb{Z}_p \), we consider the corresponding bilinear pairing

\[
X \otimes \mathbb{Q}_p \times Y \otimes \mathbb{Q}_p \xrightarrow{\ell \circ j} \mathbb{Q}_p,
\]

and the corresponding homomorphism of \( L \otimes \mathbb{Q}_p \) modules (free of rank 1)

\[
\beta_\ell : X \otimes \mathbb{Q}_p \to \text{Hom}_{\mathbb{Q}_p}(Y \otimes \mathbb{Q}_p, \mathbb{Q}_p).
\]

Hence there exist \( \mathcal{L}_P(A_\Pi, \ell)' \in L \otimes \mathbb{Q}_p \) such that

\[
\beta_\ell = \mathcal{L}_P(A_\Pi, \ell)' \alpha_p,
\]

where \( \alpha_p = \alpha \otimes 1 : Y \otimes \mathbb{Q}_p \to \text{Hom}_{\mathbb{Q}_p}(X \otimes \mathbb{Q}_p, \mathbb{Q}_p) \). We define \( \mathcal{L}_P(A_\Pi, \ell) \in \mathbb{C}_p \) to be the image of \( \mathcal{L}_P(A_\Pi, \ell)' \) under the homomorphism \( L \otimes \mathbb{Q}_p \to \mathbb{C}_p \) given by the fixed embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \).

Assume we are in the definite case, hence \( \Pi \) is generated by an automorphic form \( \phi \in H^0(G(F), A_f(\mathcal{O}_L)^U) \), where \( G \) is the multiplicative group of a totally definite quaternion algebra. Since \( \mathcal{O}_L \simeq \mathbb{Z}^d \), \( \phi \) can be seen as \( \phi = (\phi_i)_{i=1, \ldots, d} \), where \( \phi_i \in H^0(G(F), A_f(\mathbb{Z})^U) \). Since \( \Pi_\mathbb{F}_p \) is Steinberg with \( \alpha = 1 \), each \( \phi_i \) define an element \( f_i \in H^0(G(F), A_f^p(V^\mathbb{Z}, \mathbb{Z})^U) \).

Fix \( g \in G(F_P) \). Since \( V^R \) is the quotient of the space \( C(\mathbb{P}^1(F_P), R) \) modulo the subspace generated by 1, we can interpret \( f_i(g) \in \text{Hom}(V^\mathbb{Z}, \mathbb{Z}) \) as a distribution of \( \mathbb{P}^1(F_P) \) with integral values and such that \( \int_{\mathbb{P}^1(F_P)} df_i(g) = 0 \). Hence, it makes sense to consider the corresponding multiplicative integral. Let \( \mathcal{H}_p(\overline{\mathbb{Q}}_p) := \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus \mathbb{P}^1(F_P) \) be the \( p \)-adic Poincaré hyperplane, write \( \Delta_P = \mathbb{Z}[\mathcal{H}_P] \).
equipped with the natural degree morphism \( \text{deg} : \Delta_p \to \mathbb{Z} \), and let \( \Delta_p^0 \) be the kernel of \( \text{deg} \). We can identify \( Y \) with the \( \mathbb{Z} \)-module generated by the \( \{ f_i \}_{i=1, \ldots, d} \).

Similarly as in [4.1.1], we define \( \text{ev}_p(\phi) \in H^0(G(F), \mathcal{A}_f^p(\Delta^0_p, \text{Hom}(Y, \mathbb{Q}_p^\times))^{U_p}) \) by

\[
\text{ev}_p(\phi)(g)(z_1 - z_2)(f_i) := \int_{\mathbb{P}^1(F_p)} \left( \frac{z_2 - t}{z_1 - t} \right) \, df_i(g)(t), \quad g \in G(F_p), \, z_1, z_2 \in \mathcal{H}_p.
\]

Note that, in this situation, we also have the exact sequence given by the degree map \( \text{deg} \):

(6.33)

\[
0 \longrightarrow \mathcal{A}_f^p(\text{Hom}(Y, \mathbb{Q}_p^\times))^{\text{deg}} \rightarrow \mathcal{A}_f^p(\Delta_p, \text{Hom}(Y, \mathbb{Q}_p^\times)) \rightarrow \mathcal{A}_f^p(\Delta^0_p, \text{Hom}(Y, \mathbb{Q}_p^\times)) \rightarrow 0.
\]

We consider the image of \( \text{ev}_p(\phi) \) under the connection morphism

\[
H^0(G(F), \mathcal{A}_f^p(\Delta^0_p, \text{Hom}(Y, \mathbb{Q}_p^\times))^{U_p}) \rightarrow H^1(G(F), \mathcal{A}_f^p(\text{Hom}(Y, \mathbb{Q}_p^\times))^{U_p}).
\]

Using the \( p \)-adic uniformization of the Shimura curve \( X_U \), one can show that, in fact, \( \partial(\text{ev}_p(\phi)) \in H^1(G(F), \mathcal{A}_f^p(X)^{U_p}) \).

**Proposition 6.7.** Let \( \ell : F_p^\times \to \mathbb{Z}_p \) be a continuous homomorphism. We have that

\[
f_i \cup c_\ell = \beta_\ell(\partial(\text{ev}_p(\phi)))(f_i) \in H^1(G(F), \mathcal{A}_f^p(C_p)^{U_p}),
\]

for any \( i = 1, \ldots, d \).

**Proof.** Let \( \tilde{\ell} : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times \) be any extension of \( \ell \), namely a continuous additive morphism such that \( \tilde{\ell} \mid _{F_p^\times} = \ell \) (one can always find such an extension composing \( \ell \) with the norm map on each finite extension of \( F_p \) and dividing by the corresponding degree). For any \( z \in \mathcal{H}_p \), we have the class of \( (c_\ell(z), 1) \in \mathcal{E}(\ell) \), where

\[
c_\ell(z) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tilde{\ell}(cz + d),
\]

It is easy to check that

\[
c_\ell(z)(gh) = c_\ell(gz)(h) + c_\ell(z)(g), \quad g, h \in G(F_p).
\]

Since the function \( h \mapsto c_\ell(z)(g) \) is obviously constant, we deduce that \( g(c_\ell(z), 1) = (c_\ell(gz), 1) \), for all \( g \in G(F_p) \).

Let \( \tilde{f}_i \in \mathcal{A}_f^p(\mathcal{E}(\ell), \mathbb{Q}_p) \) be any pre-image of \( f_i \in H^0(G(F), \mathcal{A}_f^p(V^Z, \mathbb{Z})) \). Hence, for all \( \gamma \in G(F) \) and all \( g \in G(F_p) \),

\[
(f_i \cap c_\ell)(\gamma)(g) = \gamma \tilde{f}_i(g)(c_\ell(z), 1) - \tilde{f}_i(g)(c_\ell(z), 1) = \tilde{f}_i(g)(c_\ell(z) - c_\ell(z)) + b_i(\gamma)(g) = \beta_\ell(\partial(\text{ev}_p(\phi)))(f_i)(\gamma)(g),
\]

where \( b_i \) corresponds to the \( 1 \)-coboundary \( b_i(\gamma) = (1 - \gamma)\tilde{f}_i(c_\ell(z), 1) \). Hence the result follows.

This proposition implies that the automorphic \( L \)-invariant \( L_p(\Pi, \ell) \) coincides with the geometric \( L \)-invariant \( L_p(A_{\Pi}, \ell) \) in the definite setting. Such a claim in the indefinite setting is equivalent to Conjecture 6.6.
6.4. *L*-invariants and Heegner points. Throughout this section, assume that $G$ is the multiplicative group attached to a quaternion algebra $A$ that splits at a single archimedean place $\sigma$. Let $\phi \in H^0(G(F), A^r(D, \mathbb{C}))$ be a generator of $\Pi \mid G(F)$, corresponding to a differential form $\omega_\phi \in \Omega^1_{X_0}$ of the cotangent space of $X_0/F$. Recall that the abelian variety of GL$_2$-type $A_\Pi$ associated with $\Pi$ is provided by the complex torus

$$A_\Pi \simeq (\mathbb{C} \otimes \mathbb{Q})/\Lambda_\Pi, \quad \Lambda_\Pi = \left\{ \left( \int_c \tau \omega_\phi \right)_{\tau \in \text{Gal}(L/Q)}, \ c \in H_1(X_U, \mathbb{Z}) \right\}.$$ 

Let $T$ be the torus associated with $E^\times/F^\times$, where $E/F$ is an imaginary quadratic extension admitting an embedding $E \hookrightarrow A$. Let $\Delta_T = \mathbb{Z}[G(F)\tau_E] \simeq \text{Ind}^G(\mathbb{F}/1)_{\mathbb{Z}}$ be the set of divisors supported on $G(F)\tau_E$, and $\Delta_T^0 = \ker(\text{deg} : \Delta_T \to \mathbb{Z})$ the set of degree zero divisors.

Let

$$H_1(X_U, \mathbb{Z})^\pm := \{ c \in H_1(X_U, \mathbb{Z}) ; \ c = \pm \bar{c} \},$$

where $\bar{c}$ is the complex conjugated path. Then, we have that

$$I_\Pi^\pm := \left\{ \frac{C^\pm(\omega_\phi)}{\left( \int_c \tau \omega_\phi \right)_{\tau \in \text{Gal}(L/Q)}, \ c \in H_1^\pm(X_U, \mathbb{Z}) \right\}$$

is a $O_L$-module locally free of rank 1, where $\text{End}(A) = O_L \subset L$ is the order generated by the Hecke eigenvalues of $\Pi$. It is clear that $I_\Pi^+ + I_\Pi^- \subset A_\Pi$ has finite index. Moreover, we have the following surjective morphisms with finite cokernel,

$$(\mathbb{R} \otimes \mathbb{Q})/I_\Pi^+ \longrightarrow A_\Pi(\mathbb{R}), \quad (\mathbb{R}i \otimes \mathbb{Q})/I_\Pi^- \longrightarrow A_\Pi(\mathbb{R}i).$$

Since $I_\Pi^+ \otimes \mathbb{Q} = L \Omega^2_{\mathbb{Q}}$, for some $\Omega^2_{\mathbb{Q}} \in \mathbb{C} \otimes \mathbb{Q}$, the exact sequences

$$0 \longrightarrow L \longrightarrow \mathbb{R} \otimes \mathbb{Q} \longrightarrow A_\Pi^0(\mathbb{R}) \longrightarrow 0, \quad 0 \longrightarrow \Delta_T^0 \longrightarrow \text{Ind}^G(\mathbb{F}/1)_{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

provide the commutative diagram:

\begin{equation}
H^0(G(F), A_f(A_\Pi^0(\mathbb{R}))) \longrightarrow H^1(G(F), A_f(L)) \quad \text{res}
\end{equation}

Moreover, we have an analogous commutative diagram for $A_\Pi(\mathbb{R}i)$. The evaluation $ev_\Pi^+(4, 12)$ of $\tau \omega_\phi, \tau \in \text{Gal}(L/Q)$, gives rise to an element $\varphi \in H^0(G(F), A_f(\Delta_T^+, \mathbb{R} \otimes \mathbb{Q} L))$ such that $d_1(\varphi) \neq 0$ (see Lemma 4.2) and $p_1 \circ d_1(\varphi) = 0$ (see Lemma 5.11). Hence, we obtain $\tilde{\varphi} \in H^0(T, A_f(A_\Pi^0(\mathbb{R})))$, which is unique because $H^0(G(F), A_f(A_\Pi^0(\mathbb{R}))) = 0$. We claim that $d_2(\tilde{\varphi}) \neq 0$. Indeed, if $d_2(\tilde{\varphi}) = 0$ then there should exists some pre-image of $\varphi$ in $H^0(T, A_f(A_\Pi^0(\mathbb{R})))$, and this contradicts the fact that $d_1(\varphi) \neq 0$.

A similar diagram choosing argument shows that $r \circ d_2(\tilde{\varphi}) = 0$. Thus, there exists $\phi_1^+ \in H^1(G(F), A_f(L))$ such that $\text{res}(\phi_1^+) = d_2(\tilde{\varphi})$.

The analogous commutative diagram for $A_\Pi(\mathbb{R}i)$ produces an element $\tilde{\varphi}^- \in H^0(T, A_f(A_\Pi^0(\mathbb{R}))),$ such that $\text{res}(\phi^-_1) = d_2^- \left( \tilde{\varphi}^- \right)$, for some $\phi^-_1 \in H^1(G(F), A_f(L))$.
is analogous to $d_2$.

By equation (5.23) and Shimura’s reciprocity law, $\tilde{\varphi} + \tilde{\varphi}^-$ defines, in fact, an element of $\Phi_T \in H^0(T, \mathcal{A}_T(A_{\Pi}(\bar{\mathbb{Q}})) \otimes_{\mathcal{O}_{\Pi}} L)$. Rescaling the generator $\phi \in \Pi |\phi(f)|$ if necessary, we can always assume that $\Phi_T(\phi) = E^0(T, \mathcal{A}_T(A_{\Pi}(\bar{\mathbb{Q}}))).$ Since $\Pi_T$ is Steinberg with $\alpha = 1$, we have that $\phi \in G(F_T) \phi \cong \text{Ind}_{K_0(F_T)}^F \mathcal{O}_F \cong (\mathbb{Q}/L) / (U - 1) \cong V \cup \mathbb{Q}$. Hence $\Phi_T(\phi)$ defines an element $\Phi_T \mid \mathbb{V}_{\mathcal{O}_L} \in H^0(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}(\mathbb{C}_p))).$ By means of such identifications, we consider

$$\Phi_T \in H^0(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}(\mathbb{C}_p)) \otimes L)\Pi.$$ 

Notice that, multiplication by $p^n$ provides the following exact sequence of $T$-modules

$$0 \rightarrow \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}[p^n](\bar{\mathbb{Q}})) \rightarrow \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}(\mathbb{C}_p)) \rightarrow \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}(\mathbb{C}_p)) \rightarrow 0.$$ 

Since $A_{\Pi}/p^nA_{\Pi} \cong A_{\Pi}[p^n](\bar{\mathbb{Q}})$, we obtain the connection morphism

$$d^n : H^0(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}(\mathbb{C}_p))) \rightarrow H^1(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}/p^nA_{\Pi})), $$

for all $n \in \mathbb{N}$. It is easy to check that

$$d^n(\Phi_T \mid \mathbb{V}_{\mathcal{O}_L}) = \Omega_{\Pi}^n d_2(\tilde{\varphi} \mid \mathbb{V}_{\mathcal{O}_L}) + \Omega_{\Pi}^n d_2(\tilde{\varphi}^- \mid \mathbb{V}_{\mathcal{O}_L}) \mod p^n.$$ 

Recall that the abelian variety $A_{\Pi}(\mathbb{C}_p) \cong \text{Hom}(\mathbb{Y}, \mathbb{C}_p^\times)/X$, for some locally free $\mathcal{O}_L$-modules of rank one $X$ and $Y$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$ be the integer ring of $\mathbb{C}_p$. Since $\alpha : X \otimes \mathbb{Q} \rightarrow \text{Hom}(\mathbb{Y} \otimes \mathbb{Q}, \mathbb{Q})$ is an isomorphism, we can identify the image of $A_{\Pi}(\mathbb{C}_p)$ in $A_{\Pi}(\mathbb{C}_p) = \text{Hom}(\mathbb{Y}, \mathbb{C}_p)$ with the image of the composition

$$\text{Hom}(\mathbb{Y}, \mathcal{O}) \xrightarrow{\exp_p} \text{Hom}(\mathbb{Y}, \mathbb{C}_p^\times)/X \cong A_{\Pi}(\mathbb{C}_p) \rightarrow A_{\Pi}(\mathbb{C}_p),$$

where $\exp_p : \mathcal{O} \rightarrow \mathbb{C}_p^\times$ is the $p$-adic exponential morphism. Hence, we can assume that $\Phi_T \mid \mathbb{V}_{\mathcal{O}_L} \in H^0(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, \text{Hom}(\mathbb{Y}, \mathcal{O}))).$

Given a continuous homomorphism $\ell : F_{\mathcal{P}}^E \rightarrow \mathbb{Z}_p$, we consider $c_{\ell} \in H^1(G(F_{\mathcal{P}}), V^{\mathcal{P}_{\ell}})$ as above. Since $\text{Hom}(\mathbb{V}_{\mathcal{O}_L}, \text{Hom}(\mathbb{Y}, \mathcal{O}))$ can be interpreted as a space of bounded distributions, the cup product by $c_{\ell}$ provides following commutative diagram

$$H^0(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, \text{Hom}(\mathbb{Y}, \mathcal{O}))) \xrightarrow{d^n} H^1(T, \mathcal{A}_T^p(\mathbb{V}_{\mathcal{O}_L}, A_{\Pi}/p^nA_{\Pi})) \xrightarrow{\cup c_{\ell}} H^1(T, \mathcal{A}_T^p(\text{Hom}(\mathbb{Y}, \mathcal{O}))) \xrightarrow{d^n} H^2(T, \mathcal{A}_T^p(A_{\Pi}/p^nA_{\Pi}))$$

Let $\mathcal{L}_T(\Pi, \ell)' \in L \otimes \mathbb{Q}_p$ be the element whose components are $\mathcal{L}_T(\Pi^\tau, \ell, \tau) \in \text{Gal}(L/\mathbb{Q})$. We compute

$$d^n_1(\Phi_T \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} = d^n(\Phi_T \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} = \Omega_{\Pi}^n d_2(\tilde{\varphi} \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} + \Omega_{\Pi}^n d_2(\tilde{\varphi}^- \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} \mod p^n = \Omega_{\Pi}^n \text{res}(\tilde{\varphi}^+ \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} + \Omega_{\Pi}^n \text{res}(\tilde{\varphi}^- \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} \mod p^n \equiv \mathcal{L}_T(\Pi, \ell)' \left( \Omega_{\Pi}^n \text{res}(\tilde{\varphi}^+ \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} + \Omega_{\Pi}^n \text{res}(\tilde{\varphi}^- \mid \mathbb{V}_{\mathcal{O}_L}) \cup c_{\ell} \right) \mod p^n = \mathcal{L}_T(\Pi, \ell)' d^n_1(\Phi_T \mid \mathbb{V}_{\mathcal{O}_L}).$$

By (6.26), $\ker(d^n) = p^n H^1(T, \mathcal{A}_T^p(A_{\Pi}(\mathbb{C}_p)))$. Thus the image of $\Phi_T \mid \mathbb{V}_{\mathcal{O}_L} \cup c_{\ell} - \mathcal{L}_T(\Pi, \ell)'(\Phi_T \mid \mathbb{V}_{\mathcal{O}_L} \cup c_{\ell})$ in $H^1(T, \mathcal{A}_T^p(\text{Hom}(\mathbb{Y}, \mathcal{O}/p^n\mathcal{O})))$ is 0 for all $n$, we conclude

$$\Phi_T \cup c_{\ell} = \mathcal{L}_T(\Pi, \ell)'(\Phi_T \cup c_{\ell}).$$

When we apply $\log_p$ the formal group logarithm attached to the differential $\omega_\phi$, we recover the component corresponding to the fixed embedding $\bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p$. Hence

$$\log_p(\Phi_T \cup c_{\ell}) = \mathcal{L}_T(\Pi, \ell')(\log_p(\Phi_T \cup c_{\ell})).$$
7. Exceptional zero phenomenon in the split case

The aim of the rest of the paper is to compute the class of \( L^1_{\pi}(\pi, E) \) and \( L^1_{\pi}(\pi, E) \) in \( T/T^2 \) in the presence of the Exceptional Zero phenomenon and in case \( T_P \) splits. We will invoke Corollary 2.4 and so such class will be characterized by the integrals

\[
\int_{G_E,P} \ell d\mu_{E,P} = \kappa(\psi_i) \cap \partial(\ell), \quad i = I, II,
\]

for all \( \ell \in \text{Hom}_{Z_p}(G_{E,P}, \mathbb{Z}_p) = G_{E,P}^\vee \).

Let \( H \) be the maximal open compact subgroup of \( T_P \simeq F_p^\times \). We have the exact sequence

\[
0 \rightarrow H \rightarrow T_P \xrightarrow{\text{ord}} \mathbb{Z} \rightarrow 0.
\]

Then we can consider the real manifold \( M = \mathbb{R} \times T^P/\Gamma \) with the following natural action of \( T \):

\[
T \times M \rightarrow M; \quad (t, (x, t^P)) \mapsto (x + \text{ord}(\ell_P(t)), t^P(t)t^P).
\]

Since we can identify \( H_0(M, \mathbb{Z}) \) with \( C_c(T^P/\Gamma, \mathbb{Z}) \), we can consider the fundamental class \( \vartheta \) of the oriented compact manifold \( M/T \) as an element of \( H_1(T, C_c(T^P/\Gamma, \mathbb{Z})) \) by means of the identifications

\[
\vartheta \in H_1(M/T, \mathbb{Z}) = H_1(T, H_0(M, \mathbb{Z})) = H_1(T, C_c(T^P/\Gamma, \mathbb{Z})).
\]

Let \( \ell : G_{E,P} \rightarrow \mathbb{Z}_p \) be a continuous group homomorphism (\( \ell \in C(G_{E,P}, \mathbb{Z}_p) \) not necessarily locally constant). It corresponds to a continuous homomorphism \( \hat{\ell} : \hat{T} \rightarrow \mathbb{Z}_p \) that factors through \( \Gamma T \). Let \( \ell_P : T_P \rightarrow \mathbb{Z}_p \) and \( \ell^P : T^P \rightarrow \mathbb{Z}_p \) be its corresponding restriction to \( T_P \) and \( T^P \). By topological reasons, \( \ell^P = 0 \). Let \( \mathcal{F} \) be a fundamental compact domain for \( \hat{T}/\Gamma \) under the action of \( T \). Hence \( \partial \ell = [\hat{1}_\mathcal{F}] \), where \([\hat{1}_\mathcal{F}] \in H_0(T, C_c(\hat{T}/\Gamma, \mathbb{Z}_p))\) is the image of \( \hat{1}_\mathcal{F} \) and \( 1_P \in C_c(\hat{T}/\Gamma, \mathbb{Z}) \) is the characteristic function of \( \mathcal{F} \). As above, we consider the natural injections \( \ell_P : T \hookrightarrow T_P \) and \( \ell^P : T \hookrightarrow T^P \).

Let us consider again the open compact subset \( U = P_1 \cup \bigcup_{n \in \mathbb{N}} \omega^n H \) of \( \mathbb{P}^1 \). Since \( \ell^P = 0 \), the morphism \( \ell \) depends only on \( \ell_P \). Hence it makes sense to consider the cocycle \( z_{\ell_P} \in H^1(T, C_c(T_P, \mathbb{Z}_p)) \) defined by

\[
z_{\ell_P}(t) := (1 - \ell_P(t))(\ell_P 1_U).
\]

**Remark 7.1.** Note that \( z_{\ell_P}(t) \in C_c(T_P, \mathbb{Z}_p) \subset C_c(T_P, \mathbb{Z}_p) \). Indeed,

\[
z_{\ell_P}(t)(x) = \ell_P(x)1_U(x) - \ell_P(t^P(t)^{-1}x)1_U(t^P(t)^{-1}x)
= \ell_P(x)1_U(x) - (\ell_P(x) - \ell_P(t^P(t)))1_U(t^P(t)^{-1}x)
= \pm \ell_P 1_U(x) + \ell_P(t^P(t))1_U(t^P(t)^{-1}x),
\]

where \( U_t = U \setminus \ell_P(t^P(t)^{-1})1_U \subset T_P \) is clearly open and compact. Since \( \hat{\ell}(T) = \ell_P(t^P(t)) = 0 \), we deduce that \( z_{\ell_P}(t) = \pm \ell_P 1_U(x) \in C_c(T_P, \mathbb{Z}_p) \).

**Proposition 7.2.** We have that

\[
\partial(\ell) = \vartheta \cap z_{\ell_P} \in H_0(T, C_c(\hat{T}/\Gamma, \mathbb{Z}_p)).
\]

**Proof.** Let \( \mathcal{F} \subset \hat{T}/\Gamma \) be an open compact fundamental domain for the action of \( T \) such that \( HF = \mathcal{F} \). By definition, \( \partial(\ell) \) is the image of the compactly supported function \( \hat{1}_{\mathcal{F}} \) in \( H_0(T, C_c(\hat{T}/\Gamma, \mathbb{Z}_p)) \).

Let us consider a finite index subgroup of the form \( \mathcal{T} = \ell^2 \times T^P \subset T \), such that \( \ell_P(T^P) \subset H \) and \( \ell^P(t) \in \Gamma \). Let \( G \subset T_P/H \) be the subgroup generated by \( t \) and \( T^P \) a fundamental domain for \( T_P/H \) under the action of \( \ell^P \). Hence, as above, \( U_G \times T^P \) is a fundamental domain for \( \hat{T}/\Gamma \) under the action of \( T \). Note that, in this situation, \( C_c(T^P/\Gamma, \mathbb{Z}) \simeq C(F^P, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Z}[T^P] \simeq C(F^P, \mathbb{Z}) \otimes_{\mathbb{Z}[T]} \mathbb{Z}[T] \). Thus,
by Shapiro’s Lemma, $H_1(T, C_c(T^p/\Gamma, \mathbb{Z})) = H_1(t^\mathbb{Z}, C(\mathcal{F}_p, \mathbb{Z}))$. We describe $\vartheta$ as the co-restriction of the class in $H_1(t^\mathbb{Z}, C(\mathcal{F}_p, \mathbb{Q}))$ defined by the cocycle

$$f : t^\mathbb{Z} \to C(\mathcal{F}_p, \mathbb{Q}); \quad f(t^n) = \begin{cases} 1/|T| & n = 1, \\ 0, & n \neq 1. \end{cases}$$

Therefore, we compute

$$\vartheta \cap z_{1_p} = \frac{1}{|T : T|} z_{1_p}(t) \otimes 1_{\mathcal{F}_p} = \frac{1}{|T : T|} \ell_p t \varpi_1 \otimes 1_{\mathcal{F}_p} = \frac{1}{|T : T|} \hat{\ell}_1 \varpi_1 \otimes 1_{\mathcal{F}_p},$$

where the second equality has been obtained from Remark 7.1. Since $1_{\mathcal{F}_p} = \sum_{g \in T/T} g 1_{\mathcal{F}_p}$ and $\hat{\ell}$ is trivial on $T$, we deduce that the class of $\hat{\ell}_1 \varpi_1 \otimes 1_{\mathcal{F}_p} = \sum_{g \in T/T} g(\hat{\ell}_1 f)$ coincides with the class of $[T : T] \hat{\ell}_1 f$ in $H_0(T, C_c(T/\Gamma, \mathbb{Z}_p))$, hence the result follows.

Let us consider the subset $T_1 = \{ t \in T : \iota_T(t) \in H \} \subset T$ and let $\mathcal{F}_1$ be a fundamental domain for $T^p/\Gamma$ under the action of $T_1$. We can consider the subgroup

$$\mathcal{X} := U \times \mathcal{F}_1 \subset T.$$

As in previous results, let us identify $V^\mathbb{Z}$ and $C_c(\mathbb{P}_1, \mathbb{Z})$. Hence $\operatorname{ord} : T \to \mathbb{Z}$ defines a cocycle $z_{\operatorname{ord}} \in H^1(T, \mathcal{F}_1, (V^\mathbb{Z}))$, where $z_{\operatorname{ord}}(t) = (1 - \iota_T(t))(\operatorname{ord} 1_U)$.

**Proposition 7.3.** The class of the characteristic function $1_X$ satisfies

$$[1_X] = \vartheta \cap z_{\operatorname{ord}} \in H_0(T, \mathcal{F}_c(T/\Gamma, \mathbb{C}_p)).$$

**Proof.** Since $\iota_T(T)$ is dense in $T$, we have the exact sequence

$$0 \to T_1 \to T \xrightarrow{\operatorname{ord}} \mathbb{Z} \to 0,$$

where, by abuse of notation, we also denote by $\operatorname{ord}$ the composition $\iota_T \circ \operatorname{ord}$. Note that $z_{\operatorname{ord}}$ is the image through the inflation map of an element $\tilde{z}_{\operatorname{ord}} \in H_1(T/T_1, (V^\mathbb{Z}))$.

On the one hand, let us consider $1_{\mathcal{F}_1}$ the characteristic function of $\mathcal{F}_1$. By strong approximation, given $t \in T$ there exists $t_1 \in T_1$ such that $\iota_T(t^{-1} t_1) \in \Gamma$. This implies that the image $[1_{\mathcal{F}_1}] \in H_0(T_1, C_c(T^p/\Gamma, \mathbb{Z}))$ lies in fact in $[1_{\mathcal{F}_1}] \in H^0(T/T_1, H_0(T_1, C_c(T^p/\Gamma, \mathbb{Z})))$.

On the other hand, let us consider $\vartheta_1 \in H_1(T/T_1, \mathbb{Z}) \simeq H_1(\mathbb{R}/T, \mathbb{Z})$ given by the fundamental class of $\mathbb{R}/T$ ($t \in T$ acts on $\mathbb{R}$ by $tx = x + \operatorname{ord}(\iota_T(t))$ as above).

Then it is clear that $[1_{\mathcal{F}_1}] \otimes \vartheta_1$ is mapped to $\vartheta$ by means of the composition

$$H^0(T/T_1, H_0(T_1, C_c(T^p/\Gamma, \mathbb{Z}))) \otimes H_1(T/T_1, \mathbb{Z}) \xrightarrow{\tilde{z}_{\operatorname{ord}} \cap \vartheta_1} H_1(T/T_1, H_0(T_1, C_c(T^p/\Gamma, \mathbb{Z}))) \xrightarrow{\cong} H_1(T, C_c(T^p/\Gamma, \mathbb{Z})).$$

Thus, the result follows if we show that $\tilde{z}_{\operatorname{ord}} \cap \vartheta_1 = [1_U] \in H_0(T/T_1, (V^\mathbb{Z})_T)$.

Indeed, if $t \in T/T_1$ is a generator then

$$\tilde{z}_{\operatorname{ord}} \cap \vartheta_1 = \tilde{z}_{\operatorname{ord}}(t) = (1 - \iota_T(t))(\operatorname{ord} 1_U) = \operatorname{ord} 1_U - (\iota_T(t)\operatorname{ord})(\iota_T(t) 1_U)$$

$$= \operatorname{ord}(1_U - \iota_T(t)) 1_U + \operatorname{ord}(\iota_T(t)) \iota_T 1_U = \operatorname{ord} \cdot 1_H + \iota_T(t) 1_U = \iota_T(t) 1_U. \quad \Box$$

For any continuous $\mathbb{Z}_p$-module homomorphism $\ell : \mathcal{G}_{E, \mathbb{P}} \to \mathbb{Z}_p$, let $\hat{\ell} : \hat{T} \to \mathbb{Z}_p$ be the composition $\ell \circ \hat{\rho}_A$, where $\hat{\rho}_A : \hat{T} \to \mathcal{G}_{E, \mathbb{P}}$ is the Artin map, and let $\tilde{\ell}_p : T \to \mathbb{Z}_p$ be its restriction to $T_p$ (recall that $\ell^p = 0$). We will identify $T_p$ with $\mathbb{F}_p^*$ by means of the isomorphism $\psi$ of $\mathcal{L}$. Recall the automorphic $L$-invariant $\mathcal{L}_p(x^{1_p}, \ell_p)$ introduced in Definition 6.3.
Definition 7.4. By Corollary [2.4] the morphism in \( \text{Hom}_{\mathcal{G}_E}([G_E, \pi], \mathbb{C}_p) \) that maps \( \ell \in \mathcal{G}_{E, p} \) to \( \mathcal{L}_P(\pi, \ell) \in \mathbb{C}_p \) defines a class \( \mathcal{L}_P(\pi) \in \mathcal{I}/\mathcal{T}^2 \), where \( \mathcal{I} \) is the augmentation map of \( \mathcal{O}_{\mathcal{C}_p}([G_E, \pi]) \otimes \mathcal{O}_{\mathcal{C}_p} \mathbb{C}_p \). Such class \( \mathcal{L}_P(\pi) \) is called automorphic \( \mathcal{L} \)-invariant vector attached to \( \pi \).

We have shown that \( \mathcal{L}_P(\pi, E) \in \mathcal{I} \) whenever \( \pi_P \) Steinberg with \( \alpha = 1 \). The main result of this section computes the class of \( \nabla \mathcal{L}_P(\pi, E) \) in \( \mathcal{I}/\mathcal{T}^2 \) in terms of the automorphic \( \mathcal{L} \)-invariant vector.

Theorem 7.5. Let \( C_E \) be the non-zero constant of Theorem 5.9 or Theorem 6.10, depending on whether we are in the definite or indefinite case. If \( T_P \) splits and \( \alpha = 1 \) then \( \mathcal{L}_P(\pi, E) \in \mathcal{I} \), with \( i = I, II \), and its class \( \nabla \mathcal{L}_P(\pi, E) \) in \( \mathcal{I}/\mathcal{T}^2 \) is given by the formula

\[
\nabla \mathcal{L}_P(\pi, E) = \mathcal{L}_P(\pi) \left( C_E C(\pi_P) \frac{L(1/2, \pi_E, 1)}{L(1, \pi, ad)} \right)^{1/2},
\]

\[
\nabla \mathcal{L}_P^I(\pi, E) = \mathcal{L}_P(\pi) \log(P_T),
\]

where \( C(\pi_P) = \frac{-L(1, \pi, ad)(\pi_E(-1)^2)}{L(1, \pi, 1)} \neq 0 \) and \( P_T \in A_\phi(\bar{Q}) \otimes \mathcal{O}_L \bar{Q} \) is a Heegner point with Neron-Tate canonical height

\[
\langle P_T, P_T \rangle = |P_T|^2 = C_E C(\pi_P) L'(1/2, \pi_E, 1) L(1, \pi, ad).
\]

Proof. By Corollary [2.4] in order to obtain the class \( \nabla \mathcal{L}_P(\pi, E) \in \mathcal{I}/\mathcal{T}^2 \) we have to compute \( \frac{\partial \mathcal{L}_P(\pi, E)}{\partial \ell} := \int_{\mathcal{G}_{E, p}} \ell d\mu_{E, p} \), for all \( \ell \in \mathcal{G}_{E, p} \). By Proposition 7.2

\[
\frac{\partial \mathcal{L}_P(\pi, E)}{\partial \ell} = \kappa(\psi_1) \cap \partial = \kappa(\psi_1) \cap (\partial \cap z_\ell) = (\kappa(\psi_1) \cup z_\ell) \cap \partial.
\]

In the definite case (following the notation of [1]), \( \psi_I = \text{res}(\phi) \) is the image of

\[
\phi \in \text{Hom}_{G(F_p)}(V_\Phi, H^0(G(F), A_f(\bar{Q})))_{\gamma, 1L} \simeq H^0(G(F), A_{\Phi}(V_\Phi, \bar{Q}))_{\gamma, 1L}.
\]

through the restriction morphism. This implies that, by Proposition 6.2

\[
\kappa(\psi_I) \cup z_\ell = \frac{1}{2} \kappa_P(\text{res}(\phi \cup c_\ell)) = \frac{1}{2} \mathcal{L}_P(\pi_{1L}, \ell_P) \kappa_P(\text{res}(\phi \cup c_{ord}))
\]

\[
= \mathcal{L}_P(\pi_{1L}, \ell_P)(\kappa(\psi_I) \cup z_{\text{ord}}),
\]

where \( \kappa_P : A_{\Phi}(\mathbb{C}_p) \rightarrow C_c(T^P/\mathcal{I}, \mathbb{C}_p) \) is given by \( \langle \kappa_P(\phi), f \rangle = \sum_{t \in T_P/\mathcal{I}} f(t) \phi(t) \).

In the indefinite case, \( \psi_{II} = \log(\phi) = \log_{\omega_{\Phi}}(\Phi_T) \), where \( \Phi_T \in H^0(T, A_{\Phi}(V_\Phi, A_{\Phi}(\bar{Q})))_{\gamma, 1L} \) and \( \log_{\omega_{\Phi}} \) is the formal logarithm attached to \( \omega_{\Phi} \). By [6.30], we have that

\[
\kappa(\psi_{II}) \cup z_\ell = \frac{1}{2} \kappa_P(\log_{\omega_{\Phi}}(\Phi_T \cup c_\ell)) = \frac{1}{2} \mathcal{L}_P(\pi_{1L}, \ell_P) \kappa_P(\log_{\omega_{\Phi}}(\Phi_T \cup c_{\text{ord}}))
\]

\[
= \mathcal{L}_P(\pi_{1L}, \ell_P)(\kappa(\psi_{II}) \cup z_{\text{ord}}).
\]

In any of our settings (definite or indefinite) \( \kappa_I \cup z_\ell = \mathcal{L}_P(\pi_{1L}, \ell_P)(\kappa_I \cup z_{\text{ord}}) \), thus

\[
\frac{\partial \mathcal{L}_P(\pi, E)}{\partial \ell} = \mathcal{L}_P(\pi_{1L}, \ell_P)(\kappa(\psi_I) \cup z_{\text{ord}}) \cap \partial = \mathcal{L}_P(\pi_{1L}, \ell_P) \int \mathcal{G}_{E, p} [1_X](\gamma) d\mu(\gamma),
\]

by Proposition 7.2.

We compute that, since \( 1_X = 1_{U \times F_1} \) is \( H \)-invariant,

\[
\int \mathcal{G}_{E, p} [1_X](\gamma) d\mu(\gamma) = \int_{\mathcal{I}/\mathcal{T}} [1_X](x) \delta(1_H)(x) dx,
\]

\[
\int \mathcal{G}_{E, p} [1_X](\gamma) d\mu_{11}(\gamma) = \int_{\mathcal{I}/\mathcal{T}} [1_X](x) \delta_{\Phi}(1_H)(x) dx.
\]
Recall that $T_p/H = \mathbb{Q}^n$, $U = \bigcup_{n \in \mathbb{N}} \mathbb{n} H$ and $H \times \mathcal{F}_1$ contains a finite number of fundamental domains of $\hat{T}$ under the action of $T$. This implies that

$$\int_{\hat{G}_E, p}[1_X](\gamma) d\mu_I(\gamma) = \int_{T/T} \sum_{n \in \mathbb{N}} [1_{H \times \mathcal{F}_1}] (\mathbb{n}^{-n} x) \delta(1_H)(x) d^\times x = \int_{H \times \mathcal{F}_1} \pi^J \mathbb{n} \delta(1_H)(x) d^\times x = \int_{T/T} \delta(1_U)(x) d^\times x.$$

Similarly,

$$\int_{\hat{G}_E, p}[1_X](\gamma) d\mu_I(\gamma) = \int_{T/T} \log_p(\delta_p(1_U)(x)) d^\times x.$$

Using the Waldspurger and Gross-Zagier formulas, we compute

$$\left( \int_{T/T} \delta(1_U)(x) d^\times x \right)^2 = C_\phi \frac{L(1/2, \pi_E, 1)}{L(1, \pi, ad)} \alpha_{p, 1} \delta_{T_p}(1_U), \delta_{T_p}(1_U)),$$

$$\left| \int_{T/T} \delta_p(1_U)(x) d^\times x \right|^2 = C_\phi \frac{L(1/2, \pi_E, 1)}{L(1, \pi, ad)} \alpha_{p, 1} \delta_{T_p}(1_U), \delta_{T_p}(1_U)).$$

Thus, we need to compute the pairing $\beta_{p, 1}(\delta_{T_p}(1_U), \delta_{T_p}(1_U))$. By Proposition 3.13,

$$\beta_{p, 1}(\delta_{T_p}(1_U), \delta_{T_p}(1_U)) = \alpha \int_{T_p} \int_{T_p} 1_U(t^{-1} x) \Lambda(\delta_{T_p}(1_U))(x) d^\times x d^\times t.$$

Hence by (3.5), $\beta_{p, 1}(\delta_{T_p}(1_U), \delta_{T_p}(1_U)) = \alpha \int_{T_p} F(0)(t) d^\times t$, where $F(s)(t)$ is the analytic continuation of the expression

$$F(s)(t) = \int_{T_p} 1_U(t^{-1} x) \int_{T_p} \theta_{T_p}(s)(y) 1_U(x^{-1} y) d^\times x d^\times y.$$

As we showed in the proof of Theorem 3.10, fixing an isomorphism $T_p \simeq F_p^\times$ (Note that $U$ is mapped to $\mathcal{O}_p \setminus 0$), there is a non-zero constant $C$, such that

$$\theta_{T_p}(s)(y) = \frac{y}{C^2(1-y)^2} \left| 1-s \right|^1.$$

We deduce that

$$\begin{align*}
F(s)(t) &= \int_{F_p^\times} 1_{\mathcal{O}_p}(t^{-1} x) \int_{F_p^\times} \left| \frac{y}{C^2(1-y)^2} \right|^{1-s} 1_{\mathcal{O}_p}(x^{-1} y) d^\times x d^\times y \\
&= q^{(2-2s)n_T} \int_{\mathcal{O}_p} \int_{x \mathcal{O}_p} \left| \frac{y}{(1-y)^2} \right|^{1-s} d^\times x d^\times y \\
&= q^{(2-2s)n_T} \sum_{m_x = \text{ord}(t)} \sum_{m_y = m_x} q^{(s-1)m_y} \int_{\mathcal{O}_p^{\times}} \frac{1}{(1-y)^2 d^\times y},
\end{align*}$$

where $n_T = \text{ord}(C)$ and $q = \#(\mathcal{O}_p/\mathbb{n})$. Moreover,

$$\int_{\mathcal{O}_p^{\times}} \frac{1}{(1-y)^{2-2s} d^\times y} = \begin{cases} 
1, & m_y > 0; \\
q^{(2-2s)m_y}, & m_y < 0; \\
q^{2-2s+q^{1-2s}}, & m_y = 0.
\end{cases}$$

By means of a tedious but straightforward computation that will be left to the reader, we obtain that

$$F(0)(t) = \begin{cases} 
q^{2n_T} q^{-\text{ord}(t)} (1-q^{-1})^{-2}, & \text{ord}(t) > 0; \\
q^{2n_T} q^{\text{ord}(t)} (1-q)^{-2}, & \text{ord}(t) \leq 0.
\end{cases}$$
Lemma 8.1. [7, Proposition 2.1.5] Let \( H \) and the result follows.

This implies that
\[
\alpha_{\pi,1}(\delta_{T_p}(\mathbf{1}_U), \delta_{T_p}(\mathbf{1}_U)) = \frac{L(1, \eta P)L(1, \pi P, ad)}{\zeta_p(2)L(1/2, \pi P, 1)} c_T C_T \int_{T_p} F(0)(t) dt
\]
\[
= \frac{c_T C_T q^{2\nu} L(1, \eta P)L(1, \pi P, ad)}{\zeta_p(2)L(1/2, \pi P, 1)} (\sum_{n>0} q^{-n} + \sum_{n\leq 0} q^{n-2})
\]
\[
= \frac{c_T C_T q^{2\nu} L(1, \eta P)L(1, \pi P, ad)}{\zeta_p(2)L(1/2, \pi P, 1)} q^{-1}(1 + q^{-1})
\]
\[
(1 - q^{-1})^3,
\]
and the result follows. \( \square \)

8. Appendix 1: Local integrals

Lemma 8.1. [7] Proposition 2.1.5] Let \( H \) be a locally compact group and \( N \) a compact subgroup of \( H \). Then there is a positive regular Borel measure on the quotient space \( H/N \) that is invariant under the action of \( H \) by left translation. This measure is unique up to a constant multiple.

Let \( F \) be a nonarchimedean local field with absolute value \( |\cdot| : F \to \mathbb{R} \) and integer ring \( \mathcal{O} \). Let \( G = \text{GL}_2(F) \), \( B \) its Borel subgroup, \( Z \) its center and \( K = \text{GL}_2(\mathcal{O}) \) a maximal compact subgroup. Let us consider the modular quasicharacter

\[
\kappa : B \to \mathbb{R} \quad \kappa \left( \begin{array}{cc} t_1 & x \\ t_2 & \end{array} \right) = |t_1/t_2|.
\]

It is a group homomorphism that satisfies \( d_R(b) := \kappa(b)d_L(b) \), where \( d_R \) and \( d_L \) are right and left Haar measures on \( B \).

Lemma 8.2. Let \( M \) be a closed subgroup of \( G \), such that the product \( BM \) is open in \( G \) and its complement has zero measure. Let \( Z_M \subset Z \cap M \) be any subgroup, such that the quotient \( (B \cap M)/Z_M \) is compact. Let \( h \in C_c(G, \mathbb{C}) \), such that \( h(bg) = \kappa(b)h(g) \), for all \( b \in B \) and \( g \in G \). Then, there exists \( \phi \in C_c(G, \mathbb{C}) \), such that \( \phi(g) = \int_B \phi(bg)d_L b \), for all \( g \in G \). Moreover, for any such \( \phi \),

\[
\int_{M/Z_M} h(m) d_R M = C \int_B \phi(g) dg,
\]

where \( C = C_M/Z_M \in \mathbb{R} \) is a certain constant and \( d_R \) is the right Haar measure on \( M/Z_M \).

Proof. We choose a right \((B \cap K)\)-invariant compactly supported function \( \phi_0 \in C_c(B, \mathbb{C}) \), such that \( \int_B \phi_0(b)d_L b = 1 \). Since \( h \) is also left \((B \cap K)\)-invariant, both \( \phi_0 \) and \( h \) provide a function \( \phi \in C_c(G, \mathbb{C}) \) defined by \( \phi(bk) = \phi_0(b)h(k) \), for all \( b \in B \) and \( k \in K \). We compute \((g = b'k', b' \in B, k' \in K)\)

\[
\int_B \phi(bg)d_L b = h(k') \int_B \phi_0(bb') \kappa^{-1}(b)d_R(b) = h(k') \kappa(b') = h(g).
\]

Let \( H := (B \times M)/Z_M \), and \( N := (B \cap M)/Z_M \). We consider \( N \) embedded diagonally in \( H \). Observe that there is a homeomorphism \( H/N \to BM \) given by \((b,m)N \mapsto bm^{-1}\). This induces a linear isomorphism \( C_c(BM, \mathbb{C}) \cong C_c(H/N, \mathbb{C}) \). The linear functional on \( C_c(BM, \mathbb{C}) \) corresponding to the Haar measure of \( G \) gives rise to a linear functional on \( C_c(H/N, \mathbb{C}) \), which is invariant under the left action of \( H \). Another left \( H \)-invariant linear functional on \( C_c(H/N, \mathbb{C}) \) is given by \( \int_{M/Z_M} f((b,m^{-1})N)d_L bd_R m \). By Lemma 8.1, both linear functionals must coincide (up to constant). Hence

\[
C \int_G \phi(g) dg = C \int_{BM} \phi(g) dg = \int_{M/Z_M} \phi(bm)d_L bd_R m = \int_{M/Z_M} h(m) d_R m,
\]
and the result follows. \( \square \)
Corollary 8.3 (Invariance). Let \( h \in C_c(G, \mathbb{C}) \) and \( M \) be as above. Then, for any \( g \in G \),
\[
\int_{M/Z_M} h(mg) d_R m = \int_{M/Z_M} h(m) d_R m.
\]
Proof. Write \( h' \in C_c(G, \mathbb{C}) \) for the translation \( h'(g') = h(g'g) \), for all \( g' \in G \). By the above lemma, there exists \( \phi \in C_c(G, \mathbb{C}) \), such that \( h(g') = \int_B \phi(bg') d_L b \). Thus,
\[
h'(g') = h(g'g) = \int_B \phi(bg'g) d_L b = \int_B \phi'(bg') d_L b,
\]
where \( \phi'(g') := \phi(g'g) \). Applying the second part of the lemma,
\[
\int_{M/Z_M} h(mg) d_R m = \int_{M/Z_M} h'(m) d_R m = \int_G \phi'(g') d g' = \int_G \phi(g') d g'
\]
and the result follows.

Corollary 8.4 (Comparison). Let \( M/Z_M = M_1/Z_{M_1} \) or \( M_2/Z_{M_2} \) and \( h \in C_c(G, \mathbb{C}) \) be as above. Then there is a constant \( C = C(M_1/Z_{M_1}, M_2/Z_{M_2}) \), such that
\[
\int_{M_1/Z_{M_1}} h(m_1) d_R m_1 = C \int_{M_2/Z_{M_2}} h(m_2) d_R m_2.
\]
Proof. By Lemma 8.2 there is \( \phi \in C_c(G, \mathbb{C}) \), which satisfies \( h(g) = \int_B \phi(bg) d_L b \), for all \( b \in B \) and \( g \in G \). This implies that
\[
\int_{M_1/Z_{M_1}} h(m_1) d_R m_1 = \int_G \phi(g) d g = \int_{M_2/Z_{M_2}} h(m_2) d_R m_2,
\]
and the result follows.

9. Appendix 2: The \((G, K)\)-module of Discrete Series

Let \( \mathcal{G} \) be the Lie algebra of \( GL_2(\mathbb{R}) \), let us consider the maximal compact subgroup \( K = O(2) \), and let \( H := SO(2) \subset K \). Denote by \( B \) the Borel subgroup
\[
B = \left\{ u \left( \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ y^{-1/2} & \end{array} \right) \in GL_2(\mathbb{R})^+ \right\}
\]
For our purposes, we will be interested in the \((G, K)\)-module \( D(2) \) of discrete series of weight 2. It can be described as a subrepresentation of an induced representation from a character of the Borel subgroup \( B \). Indeed, since any \( g \in GL_2(\mathbb{R})^+ \) can be written uniquely as \( g = u \cdot \tau(x, y) \cdot \kappa(\theta) \), where
\[
u \in \mathbb{R}^+, \quad \tau(x, y) = \left( \begin{array}{cc} y^{1/2} & xy^{-1/2} \\ y^{-1/2} & \end{array} \right) \in B, \quad \kappa(\theta) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \in SO(2),
\]
we write
\[
\bar{I} = \{ f \in C^\infty(GL_2(\mathbb{R})^+, \mathbb{C}) \text{, such that } f(u \cdot \tau(x, y) \cdot \kappa(\theta)) = yf(\kappa(\theta)) \}.
\]
Then \( I \) is the \((G, H)\)-module of admissible vectors in \( \bar{I} \), namely, the set of \( f \in \bar{I} \) such that the Fourier series of \( f(\kappa(\theta)) \) is finite. Thus,
\[
I = \bigoplus_{k \in \mathbb{Z}} C f_{2k}; \quad f_{2k}(u \cdot \tau(x, y) \cdot \kappa(\theta)) = ye^{2k\theta}.
\]
The \((G, H)\)-module structure of \( I \) can be described as follows: Let \( L, R \in \mathcal{G} \) be the Maass differential operators defined in [7, §2.2]
\[
L = e^{-2i\theta} \left( -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right), \quad R = e^{2i\theta} \left( iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right).
\]
Then, the \((\mathcal{G}, H)\)-module \(I\) is characterized by the relations:

\[
\begin{align*}
(9.37) & \quad Rf_{2k} = (1 + k)f_{2k+2}; & Lf_{2k} = (1 - k)f_{2k-2}; \\
(9.38) & \quad \kappa(t)f_{2k} = e^{2k\pi it}f_{2k}; & uf_{2k} = f_{2k},
\end{align*}
\]

for any \(\kappa(t) \in \text{SO}(2)\) and \(u \in \mathbb{R}^+ \subset \text{GL}_2(\mathbb{R})^+\). Since \(f_{2k} \in \mathbb{C}R^k f_0\), if \(k > 0\), and \(f_{2k} \in \mathbb{C}L^{-k} f_0\), if \(k < 0\), we have that \(I\) is generated by \(f_0\).

To provide structure of \((\mathcal{G}, K)\)-module, we have to define the action of \(\omega = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \in O(2) \setminus \text{SO}(2)\). Therefore, we have to define \(\omega \in \text{End}(I)\), such that

\[(i) \quad \omega f_{2k} \in \mathbb{C}f_{-2k}; \quad (ii) \quad \omega^2 = 1; \quad (iii) \quad \omega R = L\omega.\]

If we write \(\omega f_{2k} = \lambda(k)f_{-2k}\), condition (ii) implies that \(\lambda(k)\lambda(-k) = 1\). Moreover, condition (iii) implies that \(\lambda(k)\lambda(1 - k) = 1\), if \(k \neq 1\). Assuming that \(\lambda(0) = 1\), we obtain two possible \((\mathcal{G}, K)\)-module structures for \(I\): Letting \(\lambda(k) = 1\) for all \(k \in \mathbb{Z}\), or letting \(\lambda(k) = -1\) for all \(k \neq 0\). We denote by \(I^+\) and \(I^-\) the corresponding \((\mathcal{G}, K)\)-module structures.

Note that we have a well defined morphism of \((\mathcal{G}, H)\)-modules

\[I_2 \rightarrow \mathbb{C}, \quad f \mapsto \int_{0}^{\pi} f(\kappa(\theta))d\theta.\]

Moreover, since we have chosen \(\lambda(0) = 1\), it defines a morphism of \((\mathcal{G}, K)\)-modules \(I^+ \rightarrow \mathbb{C}\). The kernels of both morphisms are isomorphic as \((\mathcal{G}, K)\)-modules, its isomorphism class is called \textit{discrete series representation} \(D\). It is an irreducible \((\mathcal{G}, K)\)-module generated by \(f_2\). Moreover, we have constructed two different extensions of \(D\):

\[
\begin{align*}
(9.39) & \quad 0 \rightarrow D \rightarrow I^+ \rightarrow \mathbb{C} \rightarrow 0; \\
(9.40) & \quad 0 \rightarrow D \rightarrow I^- \rightarrow \mathbb{C} \rightarrow 0.
\end{align*}
\]

\section*{References}

[1] L. Barthel and R. Livné. Modular representations of \(\text{GL}_2\) of a local field: the ordinary, unramified case. \textit{J. Number Theory}, 55(1):1–27, 1995.

[2] M. Bertolini and H. Darmon. Heegner points on mumford-tate curves. \textit{Invent. Math.}, 126:413–456, 1996.

[3] M. Bertolini and H. Darmon. A rigid analytic Gross-Zagier formula and arithmetic applications. \textit{Ann. of Math. (2)}, 146(1):111–147, 1997. With an appendix by Bas Edixhoven.

[4] M. Bertolini and H. Darmon. Heegner points, \(p\)-adic \(L\)-functions, and the Cerednik-Drinfeld uniformization. \textit{Invent. Math.}, 131(3):453–491, 1998.

[5] M. Bertolini and H. Darmon. \(p\)-adic \(L\)-functions and the \(p\)-adic uniformization of Shimura curves. \textit{Duke Math. J.}, 98(2):305–334, 1999.

[6] M. Bertolini and H. Darmon. The \(p\)-adic \(L\)-functions of modular elliptic curves, 2000.

[7] D. Bump. \textit{Automorphic forms and representations}, volume 55 of \textit{Cambridge Studies in Advanced Mathematics}. Cambridge University Press, Cambridge, 1997.

[8] L. Gertham. Extensions of the steinberg representation. \textit{Short Note}.

[9] R. Greenberg and G. Stevens. \(p\)-adic \(L\)-functions and \(p\)-adic periods of modular forms. \textit{Invent. Math.}, 112(2):407–447, 1993.

[10] M. Longo, V. Rotger, and S. Vigni. On rigid analytic uniformizations of Jacobians of Shimura curves. \textit{American J. Math.}, 134(5):1197–1246, 2012.

[11] B. Mazur, J. Tate, and J. Teitelbaum. On \(p\)-adic analogues of the conjectures of Birch and Swinnerton-Dyer. \textit{Invent. Math.}, 84(1):1–48, 1986.

[12] C. P. Mok. The exceptional zero conjecture for Hilbert modular forms. \textit{Compositio Mathematica}, 145:1–55, 1 2009.

[13] H. Saito. On tannell’s formula for characters of \(gl(2)\). \textit{Compositio Mathematica}, 85(1):99–108, 1993.

[14] J.-P. Serre. Cohomologie des groupes discrets. In \textit{Séminaire Bourbaki, 23ème année (1970/1971)}, Exp. No. 399, pages 337–350. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971.

[15] M. Spieß. On special zeros of \(p\)-adic \(L\)-functions of Hilbert modular forms. \textit{Invent. Math.}, 196(1):69–138, 2014.
ANTICYCLOTOMIC $p$-ADIC $L$-FUNCTIONS AND THE EXCEPTIONAL ZERO PHENOMENON

[16] J. Tunnell. Local $\epsilon$-factors and characters of $gl(2)$. *American Journal of Mathematics*, 105(6):pp. 1277–1307, 1983.

[17] J. Van Order. $p$-adic interpolation of automorphic periods for $gl_2$. *Preprint*.

[18] J. Van Order. On the quaternionic $p$-adic $L$-functions associated to Hilbert modular eigenforms. *Int. J. Number Theory*, 8(4):1005–1039, 2012.

[19] J.-L. Waldspurger. Sur les valeurs de certaines fonctions $L$ automorphes en leur centre de symétrie. *Compositio Math.*, 54(2):173–242, 1985.

[20] X. Yuan, S.-W. Zhang, and W. Zhang. *The Gross-Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

[21] S.-W. Zhang. Heights of Heegner points on Shimura curves. *Ann. of Math. (2)*, 153(1):27–147, 2001.

CENTRE DE RECERCA MATEMÀTICA

E-mail address: smolina@crm.cat