Local Optimality of Almost Piecewise-Linear Quantizers for Witsenhausen’s Problem

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Abstract

We pose Witsenhausen’s problem as a leader-follower game of incomplete information. The follower makes a noisy observation of the leader’s action (who moves first) and chooses an action minimizing her expected deviation from the leader’s action. Knowing this, leader who observes the realization of the state, chooses an action that minimizes her distance to the state of the world and the ex-ante expected deviation from the follower’s action. We study the perfect Bayesian equilibria of the game and identify a class of “near piecewise-linear equilibria” when leader cares much more about being close to the follower than the state, and the precision of the prior is poor. As a major consequence of this result, we prove the existence of a set of local minima for Witsenhausen’s problem in form of slopey quantizers, which are at most a constant factor away from the optimal cost. We provide supporting theory for numerical investigations in the literature suggesting optimality of almost piecewise-linear strategies.

Index Terms

Decentralized control, optimal stochastic control, incomplete information games, perfect Bayesian equilibrium, asymptotic quantization theory.

I. INTRODUCTION

In his seminal work Witsenhausen (1968), Witsenhausen constructed a simple two-stage Linear-Quadratic-Gaussian (LQG) decentralized control problem where the optimal controller happens to be nonlinear. This example showed for the first time that linear quadratic Gaussian

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This work was supported by ARO MURI W911NF-12-1-0509.
team problems can have nonlinear solutions. Using this counterexample, Başar (1976) produced an example showing that the standard decentralized static output feedback optimal control problem of linear deterministic systems could also admit nonlinear solutions. For nearly half a century, this counterexample has been a subject of intense research across multiple communities (Ho and Chang (1980); Bansal and Başar (1987); Mitter and Sahai (1999); Wu and Verdú (2011); Grover et al. (2015); Gupta et al. (2015)).

The endogenous information structure of Witsenhausen’s counterexample, where the signal observed in the second stage is a noisy version of the control action in the first stage, gives rise to a nonclassical information structure. While the problem looks deceptively simple with quadratic cost, it is actually a very complicated, nonconvex, functional optimization problem. This counterexample has shed light on intricacies of optimal decisions in stochastic team optimization problems with similar information structure. Naturally, this problem has given rise to a large body of literature. For example, Yüksel and Başar (2013) provides a variant of Witsenhausen’s counterexample with discrete primitive random variables and finite support, where no optimal solution exists. Another interesting variant, with the same information structure but different cost function, is the Gaussian test channel (Bansal and Başar (1987); Başar (2008)) where the linear strategies can be shown to be optimal. Interestingly, Rotkowitz (2006) shows that if the objective function is changed to a worst case induced norm, the linear controllers dominate nonlinear policies. While Witsenhausen (1968) proves the existence of an optimal solution using tools from real and functional analysis, other works such as (Wu and Verdú (2011); Gupta et al. (2015)) suggest lifting the problem to an equivalent optimization problem over the space of probability measures and then employing tools from the optimal transport theory Villani (2009).

Although the optimal strategy and optimal cost for Witsenhausen’s counterexample are still unknown, it can be shown that carefully designed nonlinear strategies can largely outperform the linear strategies (see, e.g., the multi-point quantization strategies proposed by Mitter and Sahai (1999)). This result, in particular, implies the fragility of the comparative statics and policies solely derived based on the linear strategies in problems with similar setting. A relevant line of research is to provide error bounds on the proximity to optimality for approximate solutions. Grover and Sahai (2010); Grover et al. (2013) use information theoretic techniques and vector versions of the original problem to provide such bounds. In Saldi et al. (2017), authors provide a general result on when one can approximate a continuous team decision problem with a
finite one through quantized approximations, using which they show that quantized policies are asymptotically optimal for Witsenhausen’s counterexample. There are also several works aiming to approximate the optimal solution. Li et al. (2009); Baglietto et al. (2001); Lee et al. (2001); Mehmetoğlu et al. (2014) employ different heuristic approaches, all confirming what one might intuitively call an almost piecewise-linear form for the optimal controller. However, a complete optimality proof for such strategies has been elusive.

In this paper, following Witsenhausen’s original intuition, we view the problem as a leader-follower coordination game in which the action of the leader is corrupted by an additive noise, before reaching the follower. The leader aims to coordinate with the follower while staying close to the observed state, recognizing that her action is not observed perfectly. As a result, she needs to signal the follower in a manner that can be decoded efficiently.

More than a mere academic counterexample, the above setup could model a scenario where coordination happens across generations and the insights of the leader who is from a different generation is corrupted/lost by the time the message reaches the future generations. If the leader can internalize the fact that her actions will not be observed perfectly, how should she act to make sure coordination occur? When the leader cares far more about coordination with the follower than staying “on the message”, the near piecewise-linear equilibrium strategy of the leader coarsens the observation in well-spaced intervals, rather than merely broadcasting a linearly scaled version of the observed state as the linear strategy would suggest.

To this end, we analyze the perfect Bayesian equilibria of this game and show that strong complementarity\(^1\) between the leader and the follower combined with a prior with poor enough precision can give rise to nonlinear equilibria, and in particular, equilibria in form of what has been deemed in the literature as slopey quantizers Grover et al. (2009). We subsequently show that these equilibria are indeed local minima of the original Witsenhausen’s problem. Using some related results from asymptotic quantization theory (Panter and Dite (1951); Lloyd (1982); Na and Neuhoff (2001)) together with analytical lower bounds on the optimal cost of Witsenhausen’s problem derived in Grover et al. (2013), we further show that these local minima are near-optimal in the sense that their corresponding cost is at most a constant factor away from the optimal one. Our work thus provides an analytical support for the local optimality of slopey quantization

\(^1\)Games of strategic complementarities are those in which the best response of each player is increasing in actions of others Vives (2005).
strategies for Witsenhausen’s counterexample.

The main idea behind the proof is to carefully construct a class of what we informally refer to as near piecewise-linear or slopey quantization strategies for the leader that stays invariant under the best response operator. These strategies can be viewed as small-slope variations of a fixed-rate scalar quantizer minimizing the mean squared quantization error (Panter and Dite (1951); Lloyd (1982); Na and Neuhoff (2001)). Such an optimal quantizer is characterized by optimality conditions on the threshold levels which determine the boundaries of the quantization cells (or segments) and quantization levels: i) quantization levels must be the centroid of the segments, and ii) thresholds in between two adjacent quantization levels must be equidistant from them. For any fixed number of segments, we consider the strategies whose segments are in a vicinity of the optimal MSE quantizer, have a unique fixed point in each segment close to the quantization level, and are almost linear within each segment with a near-zero derivative. For such strategies, leader’s actions remain very close to fixed points of the strategy in each segment. Therefore, well-spaced fixed points (combined with appropriate relative prior of the state in different segments) reveal the leader’s actions to the follower with high probability, making the “signal” easily decodable. As a consequence, we can characterize the best response of the follower to leader’s strategy. Using this characterization, we show that the best response of the leader to follower’s strategy also varies very little, essentially remaining near piecewise-linear over most of the range of the observed signals.

A key challenge in deriving the invariance property for this set of strategies for the leader is to bound and tightly control the displacement in the fixed points and endpoints of the segments of leader’s strategy under the action of the best response operator. One major observation here is that the fixed points of the leader’s best responses are local minimizers of the expected deviation of the leader’s action from the follower. As has been documented in Wu and Verdú (2011), the expected deviation is a non convex functional that is the main reason the problem is hard to solve. This insight allows us to show that the fixed points of the leader’s best response lie in a tight neighborhood of the fixed points of the follower’s strategy. We then show that the fixed points of the follower’s strategy in turn lie in a vicinity of a convex combination of the leader’s fixed points and the expected value of the state of the world within each segment. Combining the two, we can derive an approximate dynamics for the displacement in the fixed points and endpoints of the segments in leader’s strategy under the best response. Using this approximate dynamics,
we then characterize an invariant set of fixed points and interval endpoints for leader’s strategy, which we can then use in order to prove the existence of a near piecewise-linear equilibrium strategy for the leader.

II. Model

We view Witsenhausen’s problem (Witsenhausen (1968)) as a game between a leader $L$ and a follower $F$. Before the agents act, the state of the world $\theta$ is drawn from a normal distribution with zero mean and variance $\sigma^2$. The leader can observe the realization of $\theta$ and acts first. The payoff of the leader is given as follows

$$u_L = -r_L(\theta - a_L)^2 - (1 - r_L)(a_F - a_L)^2,$$

where $a_F$ is the action of the follower and $0 < r_L < 1$. The follower makes a private, noisy observation of the leader’s action, $s = a_L + \delta$ where $\delta \sim N(0, 1)$. The payoff of the follower is given by

$$u_F = -(a_L - a_F)^2.$$

We consider the perfect Bayesian equilibria of the game and show that they reduce to the Bayes Nash equilibria due to the Gaussian noise in the observation.\(^2\) Denote with $a^*_L(\theta)$ and $a^*_F(s)$ the equilibrium strategies, and with $\nu^*(\cdot|s)$ the follower’s belief about leader’s action given $s$. Due to the normal noise in the observation, $\nu^*(\cdot|s)$ is fully determined by $a^*_L(\theta)$ and the prior as there are no off-equilibrium-path information sets. Equilibrium strategies should thus satisfy

$$a^*_F(s) = \mathbb{E}_{\nu^*}[a^*_L|s] = \int_{-\infty}^{\infty} a_L \nu^*(a_L|s) da_L,$$

$$a^*_L(\theta) = \arg\max_{a_L} -r_L(\theta - a_L)^2 - (1 - r_L) \int_{-\infty}^{\infty} (a^*_F(s) - a_L)^2 \phi(s - a_L) ds,$$

where $\phi(\cdot)$ denotes the PDF of the standard normal distribution.

Our model yields the original setup in Witsenhausen (1968) by choosing $\frac{r_L}{1 - r_L} = k^2$. The expected control cost then maps to the (negated) expected payoff of the leader. It is a simple exercise to find the optimal solution to Witsenhausen’s problem in the class of linear strategies

\(^2\)See, e.g., Fudenberg and Tirole (1991) for a definition of perfect Bayesian equilibrium and Bayes Nash equilibrium.
(see Lemma 11 in Witsenhausen (1968)), which is also an equilibrium of the game described above. Witsenhausen (Witsenhausen (1968)) showed that, for sufficiently large $\sigma$, this linear solution is not optimal. In fact, the linear solution can be extremely suboptimal in the sense that the asymptotic ratio of the corresponding cost to the optimal one is infinity (Mitter and Sahai (1999)). Our objective in this paper is to characterize a set of local minima for the problem in Witsenhausen (1968), with a near piecewise-linear strategy for the leader and a cost within a constant factor of the optimal one, given a sufficiently large $\sigma$. To this end, we analyze the equilibria of the game described above in regime $\frac{1}{2} \leq r_L \sigma^2 \leq 1$ and sufficiently large $\sigma$.

III. NONLINEAR EQUILIBRIA

We first prove the existence of a collection of equilibria with a near piecewise-linear strategy for the leader for sufficiently large values of $\sigma$. Our approach is to identify a set of such strategies for the leader which is invariant under the best response operator. We characterize such a set in the next section.

A. An Invariant Set of Near Piecewise-Linear Strategies for the Leader

Given $m \in \mathbb{N}$, consider a partition of the normal distribution $N(0, \sigma^2)$ into $2m + 1$ segments $\bigcup_{k=-m}^{m} B_k^0$, with $B_k^0 = \left[b_k^0, b_{k+1}^0\right]$ for $k \in \mathbb{N}_m$, $B_0^0 = (b_{-1}^0, b_{1}^0)$, and $B_{-k}^0 = (b_{-k-1}^0, b_{-k}^0]$, with $b_{-k}^0 = -b_k^0$ and $b_{m+1}^0 = -b_{m-1}^0 = +\infty$. Denote with $c_k^0$ the centroid of segment $B_k^0$, that is, $c_k^0 = \mathbb{E}_{N(0, \sigma^2)}[\theta | \theta \in B_k^0]$. Clearly, $c_0^0 = 0$ and $c_{-k}^0 = -c_k^0$ for $k \in \mathbb{N}_m$. We now specifically focus on a partition where the interval endpoints $b_k^0$ are equidistant from the centroids adjacent to them, i.e., $b_k^0 = \frac{c_{k+1}^0 + c_{k}^0}{2}$ for $k \in \mathbb{N}_m$. We can show that such a partition exists and is unique. This partition in fact corresponds to the $(2m+1)$-level fixed-rate scalar quantizer that minimizes the mean-square distortion for a source characterized by $\theta \sim N(0, \sigma^2)$ (Panter and Dite (1951); Lloyd (1982); Na and Neuhoff (2001)). The properties of this quantizer as $m \to \infty$ are extensively studied in asymptotic quantization theory, as will be discussed and used in analysing the performance of our proposed local minima in Section IV.

This clearly covers the case $k^2 \sigma^2 = 1$.

From now on and for the sake of brevity, we may avoid reasserting the “sufficiently large $\sigma$” requirement. All the results are derived under this assumption unless specified otherwise.
Fig. 1. Partition of the normal distribution in a \((2m+1)\)-level optimal MSE quantizer, for \(m = 2\).

Roughly speaking, the set of strategies we propose for the leader are a class of \((2m+1)\)-segmented strategies with segments being close to \(B^0_k\) \((-m \leq k \leq m\), with a fixed point in each segment in a certain vicinity of \(c^0_k\) \((-m \leq k \leq m\), and almost linear with a slope close to \(r_L\) over each segment. Before proceeding further, we present some (non-asymptotic) properties of this base configuration, which will facilitate the proof of the invariance property for the proposed set of strategies.

**Lemma 1.** Given \(m \in \mathbb{N}\), consider the partition of the normal distribution \(N(0, \sigma^2)\) in a \((2m+1)\)-level optimal MSE quantizer as described above. Let \(x^0_k = \frac{c^0_k - c^0_{k-1}}{\sigma} \) for \(1 \leq k \leq m\). Then,

i) \( 1 - \left(\frac{x^0_m}{\sigma}\right)^2 \leq \frac{x^0_m}{\sigma^2} \leq 1 \). If \( m \geq 2 \), then \( \frac{3}{4} \leq \frac{x^0_m}{\sigma^2} \leq 1 \).

ii) For \( 1 \leq k < m \),

\[
\frac{\phi\left(\frac{b^0_k}{\sigma}\right)}{\phi\left(\frac{c^0_k}{\sigma}\right)} \leq \left(\frac{x^0_{k+1}}{x^0_k}\right)^2 \leq \frac{\phi\left(\frac{b^0_{k+1}}{\sigma}\right)}{\phi\left(\frac{b^0_{k+1}}{\sigma}\right)}. \tag{1}
\]

As a result, \( 1 \leq \frac{x^0_{k+1}}{x^0_k} \leq e \), for \( 1 \leq k < m \).

iii) For \( 0 \leq j \leq k \leq m \),

\[
\frac{\text{Prob}[\theta | \theta \in B^0_j]}{\text{Prob}[\theta | \theta \in B^0_j]} \leq \frac{1 + e}{2}.
\]
iv) For any \( m \geq 2 \),
\[
\frac{\sqrt{\pi}}{2e^m} \leq \frac{x^0_1}{\sigma} \leq \frac{\sqrt{2\pi e}}{2m},
\]
\[
2\sqrt{\ln m - 3} \leq \frac{c^0_m}{\sigma} \leq 2\sqrt{2\ln m + 2},
\]
\[
\frac{3}{8\sqrt{2\ln m + 2}} \leq \frac{x^0_m}{\sigma} \leq \frac{1}{2\sqrt{\ln m - 3}}.
\]

Proof. See the appendix.

Next, we construct a set of \((2m+1)\)-segmented increasing odd functions, denoted by \( A^n_L(r_L, \sigma) \) satisfying the following properties:

**Property 1.** For every \( a_L(\theta) \in A^n_L(r_L, \sigma) \), there exist \( 2m + 1 \) segments \( B_k = [b_k, b_{k+1}) \), for \( k \in \mathbb{N}_m \), \( B_0 = (-b_1, b_1) \), and \( B_{-k} = (b_{-k-1}, b_{-k}] \), with \( b_{m+1} = -b_{-m-1} = +\infty \) such that:

- \( a_L(\theta) \) is increasing and odd (i.e., \( a_L(-\theta) = -a_L(\theta) \)), and is smooth over each interval.
- \( a_L(\theta) \) has a unique fixed point in each segment. That is, for each interval \( B_k, (-m \leq k \leq m) \), there exists a unique \( c_k \in B_k \) such that \( a_L(c_k) = c_k \), with \( c_0 = 0 \).

We also impose the constraint that interval endpoints \( b_k \) remain close to midpoints of \([c_{k-1}, c_k]\) and that fixed points \( c_k \) remain within certain vicinity of \( c_k^0 \)'s.

**Property 2.** For every \( k \in \mathbb{N}_m \), \( |b_k - \frac{c_{k-1} + c_k}{2}| \leq 0.1r_L \). Moreover, \( |c_k - c_k^0| \leq 2.9 \).

From the above property, if we define \( \bar{x}_k = x^0_k + 3 \) and \( \underline{x}_k = x^0_k - 3 \) for \( 1 \leq k \leq m \), then \( \bar{x}_k \) and \( \underline{x}_k \) represent upper and lower bounds on the lengths of both half-intervals \([c_{k-1}, b_k]\) and \([b_k, c_{k+1}]\).

Finally, we impose a constraint on the slope of \( a_L(\theta) \) in each interval, keeping the slope very close to \( r_L \), as well as a linear bound on \( a_L(\theta) \) in the tail. More precisely, we impose the following property:

**Property 3.** For every \(-m < k < m\) and \( \theta \in B_k \), \( r \leq \frac{d}{d\theta}a_L(\theta) \leq \bar{r} \), where \( r = r_L(1 - 0.5r^2_L\sigma^2) \) and \( \bar{r} = r_L(1 + 0.5r^2_L\sigma^2) \). For the tail interval \( B_m \), \( r \leq \frac{d}{d\theta}a_L(\theta) \leq \bar{r} \) for \( b_m < \theta < c_m + \sqrt{\sigma}\bar{x}_m \).

For \( \theta > c_m + \sqrt{\sigma}\bar{x}_m \) we have \( a_L(\theta) \leq c_m + 3r_L(\theta - c_m) \).

\(^5\) We state the properties (and in many cases the analysis) only for \( \theta \geq 0 \). The counterpart for \( \theta \leq 0 \) is immediate since the function is odd.
For any $\sigma > 0$, define $M(\sigma) = \{ m \in \mathbb{N} | 2\sqrt{2 \ln \sigma} + 4 < x_1^0 < 4\sqrt{\ln \sigma} \}$. We then claim that the set of strategies $A_m^L(r_L, \sigma)$ for $m \in M(\sigma)$, characterized by Property 1-3, is invariant under the best response operator, for sufficiently large values of $\sigma$ in the regime $\frac{1}{2} \leq r_L \sigma^2 \leq 1$.

B. Best Response Analysis

The first step in verifying the invariance of $A_m^L(r_L, \sigma)$ is to characterize the best response of the follower $a_F(s)$ to the leader’s strategy $a_L(\theta) \in A_m^L(r_L, \sigma)$. We can then use these properties to find the updated best response of the leader to $a_F(s)$, denoted by $\tilde{a}_L(\theta)$ and enforce its inclusion in $A_m^L(r_L, \sigma)$.

The follower’s best response to the strategy of the leader $a_L(\theta)$ is the expected action of the leader given the observation $s = a_L + \delta$, that is $a_F(s) = \mathbb{E}_\delta[a_L|s]$. Following a simple application of Bayes rule we can obtain

$$a_F(s) = \frac{\int_{-\infty}^{\infty} a_L(\theta) \phi(s - a_L(\theta)) \phi(\frac{\theta}{\sigma}) d\theta}{\int_{-\infty}^{\infty} \phi(s - a_L(\theta)) \phi(\frac{\theta}{\sigma}) d\theta}.$$ 

Using this, we can easily show that $a_F(s)$ is analytic and increasing, with $\frac{d}{ds}a_F(s) = \text{Var}[a_L|s]$ (see Witsenhausen (1968) for a proof).

In order to characterize $a_F(s)$, we start by estimating the expected action of the leader and its variance conditioned on the interval to which $\theta$ belongs. Actions of the leader in interval $B_k$ ($k \neq \pm m$) are well-concentrated around $c_k$. In fact $a_L(\theta) \in [c_k - \bar{r}\bar{x}_k, c_k + \bar{r}\bar{x}_{k+1}]$ for $\theta \in B_k$, from which the lemma below follows immediately.

**Lemma 2.** For $0 \leq k < m$, $|\mathbb{E}[a_L(\theta)|s, \theta \in B_k] - c_k| \leq \bar{r}\bar{x}_{k+1}$ and $\text{Var}[a_L(\theta)|s, \theta \in B_k] \leq \bar{r}^2(\frac{\bar{x}_k + \bar{x}_{k+1}}{2})^2$.

**Proof.** See the appendix. \hfill \blacksquare

The analysis is a bit involved in the tail, since for $\theta > c_m$ the leader’s actions are not in a bounded vicinity of $c_m$ anymore. However, we can derive several useful properties for the tail as well.

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6 As we will see in Section IV, this corresponds to local minima with an expected cost within a constant factor of the optimal cost.
Lemma 3. Consider a tail observation by the leader (i.e., $\theta \in B_m$). Then,

$$\mathbb{E}[a_L(\theta)|s, \theta \in B_m] - c_m \leq \bar{r} \bar{x}_{m+1},$$

for $s \leq c_m + \bar{x}_{m+1}$, where $\bar{x}_{m+1} = \sqrt{e} \bar{x}_m$. For $s > c_m + \bar{x}_{m+1}$, we have

$$\mathbb{E}[a_L(\theta)|s, \theta \in B_m] - c_m \leq 3r_L \sigma(s - c_m + 1).$$

Also, $\mathbb{E}[a_L(\theta)|s, \theta \in B_m] - c_m \geq -\bar{r} \bar{x}_m$. As for the variance,

$$\text{Var}[a_L(\theta)|s, \theta \in B_m] \leq \begin{cases} 
\frac{1}{3}, & \text{for } s < c_{m-1} \\
0.75 \bar{r}^2 \left( \frac{\bar{x}_m + \bar{x}_{m+1}}{2} \right)^2, & \text{for } c_{m-1} \leq s \leq c_m + \bar{x}_{m+1} \\
2.5 \bar{r}_L^2 \sigma^2(s - c_m)^2, & \text{for } s > c_m + \bar{x}_{m+1}.
\end{cases}$$

Proof. See the appendix.

Let the signal observed by the follower be between $c_k$ and $c_{k+1}$, i.e., $s = c_k + \delta$ with $0 \leq \delta \leq c_{k+1} - c_k$. Then, we claim that the follower’s posterior on $\theta$ given $s$ has a negligible probability out of the neighboring intervals $B_k \cup B_{k+1}$.

Lemma 4. Let the observed signal by the follower be $s = c_k + \delta$, where $0 \leq \delta \leq c_{k+1} - c_k$, with $k \geq 0$. Then, for any $j \geq 1$,

$$\frac{\text{Prob}[\theta \in B_{k-j}|s]}{\text{Prob}[\theta \in B_k|s]} \leq e^{-\frac{(c_k-c_{k-j})^2}{2} + 3j}.$$ 

Similarly,

$$\frac{\text{Prob}[\theta \in B_{k+j+1}|s]}{\text{Prob}[\theta \in B_{k+1}|s]} \leq e^{-\frac{(c_{k+j+1}-c_{k+1})^2}{2} + 3j}.$$ 

Proof. See the appendix.

Using this lemma and the fact that the fixed points $c_k$ are well-spaced, we can show that the effect of the intervals other than $B_k$ and $B_{k+1}$ on $a_F(s)$ are negligible. In order to characterize the follower’s best response $a_F(s)$, we then need to focus only on the segments adjacent to the observed signal, and in particular figure out the weight of each of these two neighboring intervals in the follower’s posterior on $\theta$. We do this in the following lemma.
Lemma 5. Define

\[ m_{k+1} = \frac{c_k + c_{k+1}}{2} + \frac{1}{\Delta_{k+1}} \ln \left( \frac{\text{Prob}[\theta \in B_k]}{\text{Prob}[\theta \in B_{k+1}]} \right), \]

where \( \Delta_{k+1} = c_{k+1} - c_k \). Also, write the signal observed by the follower as \( s = m_{k+1} + \delta \). Then, for \( 0 \leq k < m - 1 \),

\[ e^{-\Delta_{k+1}(\delta + \bar{x}_{k+1}) - \frac{\delta^2}{2}} \leq \frac{\text{Prob}[\theta \in B_k|s]}{\text{Prob}[\theta \in B_{k+1}|s]} \leq e^{\Delta_{k+1}(\delta - \bar{x}_k) + \frac{\delta^2}{2}}. \]

For the case involving the tail segment \( B_m \),

\[ e^{-\Delta_m(\delta - \bar{x}_m) - \frac{\delta^2}{2}} \leq \frac{\text{Prob}[\theta \in B_{m-1}|s]}{\text{Prob}[\theta \in B_m|s]} \leq 1.16e^{-\Delta_m(\delta - \bar{x}_m) + \frac{\delta^2}{2}}. \]

Proof. See the appendix. ■

It is worth mentioning that \( m_{k+1} \) defined in the above lemma is quite close to the midpoint of \( c_k \) and \( c_{k+1} \). In fact, using Lemma 1 we can show that \( |m_{k+1} - \frac{c_k + c_{k+1}}{2}| < \frac{2}{\Delta_{k+1}} \). We can now characterize the best response of the follower \( a_F(s) \) to the leader’s strategy \( a_L(\theta) \in A_L^m(r_L, \sigma) \) up to the first order.

Lemma 6. Let \( s = m_{k+1} + \delta \), with \( c_k \leq s \leq c_{k+1} \). Then

\[ a_F(s) \geq c_k + \frac{\Delta_{k+1}}{1 + 1.17e^{-\Delta_{k+1}\delta}} - 1.01\bar{x}_{k+2} \]
\[ a_F(s) \leq c_k + \frac{1.17\Delta_{k+1}}{1.17 + e^{-\Delta_{k+1}\delta}} + 1.01\bar{x}_{k+2}. \]

Also,

\[ 0 \leq \frac{d}{ds}a_F(s) \leq 1.17e^{\Delta_{k+1}\delta} \Delta_{k+1}^2 + 1.01\bar{x}^2(\bar{x}_k + \bar{x}_{k+1}) \quad \text{for} \ \delta \leq -0.5 \]
\[ 0 \leq \frac{d}{ds}a_F(s) \leq 1.17e^{-\Delta_{k+1}\delta} \Delta_{k+1}^2 + 1.01\bar{x}^2(\bar{x}_{k+1} + \bar{x}_{k+2}) \quad \text{for} \ \delta > -0.5. \]

Proof. See the appendix. ■

Corollary 1. A useful consequence of Lemma 6 is that

\[ a_F(s) \geq c_{k+1} - 1.17e^{-\Delta_{k+1}\delta}\Delta_{k+1} - 1.01\bar{x}_{k+2} \]
\[ a_F(s) \leq c_k + 1.17e^{\Delta_{k+1}\delta}\Delta_{k+1} + 1.01\bar{x}_{k+2}, \]
and

\[ 0 \leq \frac{d}{ds} a_F(s) \leq 1.17 e^{-\Delta_{k+1}^i |\delta|} \Delta_{k+1}^2 + 1.01 r^2 \bar{x}_{k+2}^2, \tag{2} \]

where \( s = m_{k+1} + \delta \), with \( c_k \leq s \leq c_{k+1} \).

Note that the exponential terms in the above bounds vanish quite fast for large \( \Delta_{k+1} \) and \( |\delta| \). For small \( |\delta| \), another useful upper bound on the derivative of \( a_F(s) \) is

\[ \frac{d}{ds} a_F(s) \leq \frac{1}{4} (\Delta_{k+1} + 2 \bar{r} \bar{x}_{k+2})^2 + 0.01 r^2 \bar{x}_{k+2}^2. \tag{3} \]

**Corollary 2.** Let \( s = m_{k+1} + \delta \), with \( c_k \leq s \leq c_{k+1} \). Then,

\[ c_k - 1.1 \bar{r} \bar{x}_{k+2} \leq a_F(s) \leq c_k + 1.1 \bar{r} \bar{x}_{k+2} \text{ for } \delta < -\frac{2 \sqrt{2 \ln \sigma}}{5}, \]

\[ c_{k+1} - 1.1 \bar{r} \bar{x}_{k+2} \leq a_F(s) \leq c_{k+1} + 1.1 \bar{r} \bar{x}_{k+2} \text{ for } \delta > \frac{2 \sqrt{2 \ln \sigma}}{5}. \]

Roughly speaking, the above corollary says that, if the observed signal by the follower is far enough from the midpoint of \( c_k \) and \( c_{k+1} \), then the optimal action of the follower is well-concentrated around \( c_k \) or \( c_{k+1} \) (whichever that is closer), and changes very slowly according to Lemma 6.\(^7\) However, \( a_F(s) \) may have very high variations for \( s \) close to \( m_{k+1} \) as can be seen from Lemma 6.

The following lemma characterizes \( a_F(s) \) when follower makes a tail observation.

**Lemma 7.** Let \( s = c_m + \delta \), where \( \delta > 0 \). Then,

i) for \( \delta \leq \bar{x}_{m+1} \), \( c_m - 1.01 \bar{r} \bar{x}_{m+1} \leq a_F(s) \leq c_m + \bar{r} \bar{x}_{m+1} \), and \( 0 \leq \frac{d}{ds} a_F(s) \leq 0.8 \bar{r}^2 (\bar{x}_m + \bar{x}_{m+1})^2 \).

ii) for \( \delta > \bar{x}_{m+1} \), \( c_m - 1.01 \bar{r} \bar{x}_m \leq a_F(s) \leq c_m + 3 r_L \sigma (\delta + 1) \), and \( 0 \leq \frac{d}{ds} a_F(s) \leq 3 r_L^2 \sigma^2 \delta^2 \).

**Proof.** See the appendix. \( \blacksquare \)

Lemma 6 and 7 provide the first order characteristics of the best response of the follower to a leader’s strategy \( a_L(\theta) \in A_L^m(r_L, \sigma) \). We are now ready to analyze the leader’s best response \( \tilde{a}_L(\theta) \) to \( a_F(s) \) and see if it stays in \( A_L^m(r_L, \sigma) \). We have \( \tilde{a}_L(\theta) = \arg\max a_L \tilde{u}_L(\theta, a_L) \), where

\[ \tilde{u}_L(\theta, a_L) = -r_L(\theta - a_L)^2 - (1 - r_L) \int_{-\infty}^{\infty} (a_F(s) - a_L)^2 \phi(s - a_L) ds. \tag{4} \]

\(^7\)Note that \( \frac{2 \sqrt{2 \ln \sigma}}{5} < \frac{\delta_1}{\bar{x}_m} < \frac{\Delta_{k+1}}{10} \)
Lemma 8. Consider $\theta \in [c_k, c_{k+1}]$, $0 \leq k < m$. Then, there exists a unique $\tilde{b}_{k+1} \in [c_k, c_{k+1}]$ such that

$$|\tilde{a}_L(\theta) - c_k| < 5\bar{r}\bar{x}_{m+1} \quad \text{for } \theta < \tilde{b}_{k+1},$$

$$|\tilde{a}_L(\theta) - c_{k+1}| < 5\bar{r}\bar{x}_{m+1} \quad \text{for } \theta \geq \tilde{b}_{k+1}.$$ 

Proof. See the appendix. $lacksquare$

The points $\tilde{b}_{k+1}$ determine the segments of the best response strategy $\tilde{a}_L(\theta)$. We can bound the derivative of $\tilde{a}_L(\theta)$ over these segments by incorporating Lemma 6 and Corollary 1 and 2 into the above bound.

Lemma 9. Consider $\theta \in [c_k, c_{k+1}]$, $0 \leq k < m$, with $\theta \neq \tilde{b}_{k+1}$. Then,

$$\frac{d}{d\theta} \tilde{a}_L(\theta) \geq \frac{r_L}{r_L + (1 - r_L)(1 + 0.4r^2\sigma^2)},$$

$$\frac{d}{d\theta} \tilde{a}_L(\theta) \leq \frac{r_L}{r_L + (1 - r_L)(1 - 0.4r^2\sigma^2)}.$$

Proof. See the appendix. $lacksquare$

Using this lemma and the values $r = r_L(1 - 0.5r^2\sigma^2)$ and $\bar{r} = r_L(1 + 0.5r^2\sigma^2)$, we can easily verify that $r \leq \frac{d}{d\theta} \tilde{a}_L(\theta) \leq \bar{r}$. This means that Property 3 is preserved by the best response for $\theta \in [-c_m, c_m]$. We study the tail case later in Lemma 11. Next, we characterize the fixed points of the best response strategy $\tilde{a}_L(\theta)$.

Lemma 10. Define

$$\tilde{J}_L(a_L) = \int_{-\infty}^{\infty} (a_F(s) - a_L)^2 \phi(s - a_L) ds.$$

Then, $\tilde{J}_L(a_L)$ is strongly convex over $[c_k - 5\bar{r}\bar{x}_{m+1}, c_k + 5\bar{r}\bar{x}_{m+1}]$, with $\frac{d}{da_L} \tilde{J}_L(a_L) \geq 2(1 - 0.4r^2\sigma^2)$. Let $\tilde{c}_k$ be the unique solution of

$$\tilde{c}_k = \text{argmin}_{a_L \in [c_k - 5\bar{r}\bar{x}_{m+1}, c_k + 5\bar{r}\bar{x}_{m+1}]} \tilde{J}_L(a_L).$$

Then, $\tilde{a}_L(\tilde{c}_k) = \tilde{c}_k$.

Proof. See the appendix. $lacksquare$

The above lemma implies that Property 1 is also preserved under the best response. Next
Lemma describes the tail properties of \( \tilde{a}_L(\theta) \).

**Lemma 11.** If \( \tilde{b}_m < \theta < \tilde{c}_m + \sigma \bar{x}_{m+1} \), then \( r \leq \frac{d}{d\theta} \tilde{a}_L(\theta) \leq \tilde{r} \). For \( \theta > \tilde{c}_m + \sigma \bar{x}_{m+1} \), we have

\[
\tilde{a}_L(\theta) \leq \tilde{c}_m + 3r_L(\theta - \tilde{c}_m).
\]

**Proof.** See the appendix.

Now, in order to verify that the updated strategy \( \tilde{a}_L(\theta) \) satisfies Property 2, we need to bound the displacements in the fixed points \( \tilde{c}_k \) and endpoints \( \tilde{b}_k \).

**Lemma 12.** For the endpoints of the intervals corresponding to \( \tilde{a}_L(\theta) \), we have

\[
|\tilde{b}_{k+1} - \tilde{c}_k + \tilde{c}_{k+1}| \leq 0.1r_L.
\]

**Proof.** See the appendix.

Bounding the displacement in \( \tilde{c}_k \) can be done in multiple steps: first we need to relate the fixed point of the leader’s best response \( \tilde{a}_L(\theta) \) in interval \( \tilde{B}_k \) to the fixed point of \( a_F(s) \) in \( B_k \) (i.e., \( s_k \)), followed by estimating \( s_k \) in terms of \( c_k \) and \( e_k \) (recall that \( e_k = \mathbb{E}_{N(0,\sigma^2)}[\theta] | \theta \in B_k \), i.e., the expected value of \( \theta \) over \( B_k \)). Finally we bound the displacement in \( e_k \) with the displacement of the interval endpoints using properties of truncated normal distribution.

**Lemma 13.** Let \( s_k \) be the fixed point of \( a_F(s) \) in the interval \([c_k - 5\bar{x}_{m+1}, c_k + 5\bar{x}_{m+1}]\), i.e., \( a_F(s_k) = s_k \). Then,

\[
|\tilde{c}_k - s_k| \leq 0.42r_L^2\left(\frac{\bar{x}_k + \bar{x}_{k+1}}{2}\right)^2 + 0.08r_L^2\bar{x}_1.
\]

**Proof.** See the appendix.

**Lemma 14.** \( s_k \) can be located based on \( c_k \) and \( e_k \) as

\[
|s_k - (1 - r_L)c_k - r_le_k| \leq 1.9r_L^2\bar{x}_{k+1}.
\]

**Proof.** See the appendix.
Using Lemma 12-14, we can reach at
\[
|\tilde{c}_k - (1 - r_L)c_k - r_L\hat{e}_k| \leq 0.42\sigma^2 L \left( \frac{x_k + x_{k+1}}{2} \right)^2 + 2\sigma^2 x_{k+1},
\]
where \( \hat{e}_k = \mathbb{E}_{N(0,\sigma^2)}[\theta|\theta \in \hat{B}_k] \), with \( \hat{B}_k = [\hat{b}_k, \hat{b}_{k+1}] \), \( \hat{b}_k = \frac{c_k - 1 + c_k}{2} \) and \( \hat{b}_{k+1} = \frac{c_k + c_{k+1}}{2} \). We can now use (III-B) and Lemma 12 to verify that Property 2 is also preserved by the best response, completing the proof of the invariance of \( A^{m}_{L}(r_L,\sigma) \) for \( m \in M(\sigma) \) and sufficiently large \( \sigma \) in the regime \( \frac{1}{2} \leq r_L\sigma^2 \leq 1 \). This is carried out in the proof of the following theorem.

**Theorem 1.** Consider the regime \( \frac{1}{2} \leq r_L\sigma^2 \leq 1 \) with \( \sigma > 0 \). Then, the set of \((2m + 1)\)-segmented strategies \( A^{m}_{L}(r_L,\sigma) \) for the leader, characterized by Property 1-3, is invariant under the best response for any \( m \in M(\sigma) = \{m \in \mathbb{N}|2\sqrt{2\ln\sigma} + 4 < x_\sigma^0 < 4\sqrt{\ln\sigma}\} \), for sufficiently large (but finite) \( \sigma \). Moreover, the game described in Section II has an equilibrium for which \( a^{*}_{L}(\theta, r_L, \sigma) \in A^{m}_{L}(r_L, \sigma) \).

**Proof.** See the appendix. ■

### IV. Local Minima and Performance Guarantees

Let
\[
U(a_L, a_F) = -\mathbb{E}_\theta[u_L(\theta, a_L, a_F)] = r_L \int_{-\infty}^{\infty} (\theta - a_L(\theta)) \frac{\phi(\theta)}{\sigma} d\theta
\]
\[
+ (1 - r_L) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (a_F(s) - a_L(\theta))^2 \phi(s - a_L(\theta)) \frac{\phi(\theta)}{\sigma} ds d\theta,
\]
for any two measurable functions \( a_L, a_F : \mathbb{R} \rightarrow \mathbb{R} \). As discussed in Section II, \( U(a_L, a_F) \) defined above maps to the expected cost of the original Witsenhausen’s problem in Witsenhausen (1968). The aim of this section is to study the performance of the equilibrium strategies characterized by Theorem 1 in view of the above cost function.

It is easy to verify that the cost functional \( U(a_L, a_F) \) is Fréchet differentiable (Luenberger (1997)). It follows then that the first variation of \( U \) vanishes at any pair of equilibrium strategies \( (a^{*}_L, a^{*}_F) \) in Theorem 1 (i.e., \( \delta U(a^{*}_L, a^{*}_F) = 0 \)), implying that \( (a^{*}_L, a^{*}_F) \) is a stationary point of the cost functional \( U \). Being an equilibrium also implies that the cost cannot be improved by changing one of the strategies \( a^{*}_L \) or \( a^{*}_F \) while keeping the other fixed, although this does not
rule out the possibility of obtaining a lower cost by simultaneously changing both strategies. With a bit of manipulation, however, we can show that the image of the strategy obtained from an infinitesimal variation in $a^*_L$ also lies within $A^m_L$, using which we can show that $(a^*_L, a^*_F)$ is indeed a local minimum of $U$.

**Lemma 15.** Any pair of equilibrium strategies $(a^*_L, a^*_F)$ characterized by Theorem 1, where $a^*_L \in A^m_L(r_L, \sigma)$ and $a^*_F(s) = \mathbb{E}_\delta[a^*_L|s]^9$, is a local minimum of the cost functional $U$ in (IV).

**Proof.** See the appendix. ■

We now have the first main result of the paper: a near piecewise-linear strategy for the leader (or first controller) that leads to a local minimum of Witsenhausen’s problem. Although the importance of such strategies has been already noticed in the literature (Li et al. (2009); Baglietto et al. (2001); Lee et al. (2001); Mehmetoglu et al. (2014)), no analytical result concerning the optimality of such strategies is reported in the literature. Theorem 1 also presents an important result in the context of two-stage games of incomplete information.

We next aim to evaluate the performance of these local minima with respect to the optimal cost. Looking at the proof of Theorem 1, we can see that $(a^*_L, a^*_F)$ is a minimizer of $U$ over the pair of strategies $(a_L, a_F)$ with $a_L \in A^m_L(r_L, \sigma)$. Therefore, we can use any other pair of strategies with the leader’s strategy being in $A^m_L$ (for which it is easier to evaluate the cost) to find an upper bound for $U(a^*_L, a^*_F)$. For this purpose we use $U(a^*_L, a^*_F) \leq U(a^0_L, a^0_F)$, where $a^0_L$ is the piecewise-linear strategy with segments $B^0_k$ and fixed points $c^0_k$ specified in the base configuration in Section III-A and $\frac{d}{ds}a^0_L(\theta) = r_L$ over each interval, and $a^0_F$ is the optimal $(2m + 1)$-level MSE quantizer (i.e., constant value of $c^0_k$ over segment $B^0_k$). It is easy to see that $(\theta - a^0_L(\theta))^2 = (1-r_L)^2(\theta - a^0_F(\theta))^2$. We can thus write $U(a^0_L, a^0_F) = r_L(1-r_L)^2D^0_L + (1-r_L)D^0_F$, with

$$D^0_L = \int_{-\infty}^{\infty} (\theta - a^0_L(\theta))^2 \frac{\phi(\frac{\theta}{\sigma})}{\sigma} d\theta,$$

$$D^0_F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta)) \frac{\phi(\frac{\theta}{\sigma})}{\sigma} ds d\theta.$$

---

8We thank Anant Sahai for bringing this point into the authors’ attention.

9Recall that $\delta \sim N(0, 1)$ is the noise in the follower’s observation.
$D_F^0$ can be upper-bounded as $D_F^0 \leq 4\sqrt{\frac{2}{e}} \frac{(2-2r_L)^2}{(1-r_L)^2} + r_L^2 D_L^0$ (see the proof of Lemma 16). We can find the exact asymptotic value of $D_F^0$ using results from asymptotic quantization theory (Panter and Dite (1951); Lloyd (1982); Na and Neuhoff (2001)): $D_F^0$ is the mean-square error of an optimal $(2m+1)$-level MSE quantizer for a source $\theta \sim N(0, \sigma)^2$ (see, e.g., Na and Neuhoff (2001)). It is known that for large $m$, $D_L^0 \approx \frac{c_\infty}{(2m+1)^2}$, where $c_\infty$ is the Panter-Dite constant of a normal source given by

\[
c_\infty = \frac{1}{12} \left( \int_{-\infty}^{\infty} \frac{\phi(\frac{\theta}{\sigma})^2}{\sigma^3} d\theta \right)^3 = \frac{\sqrt{3}\pi}{2\sigma^2}.
\]

Another interesting exact asymptotic equality is $(2m+1)^2 x_0^1 \sigma^2 \approx \frac{\sqrt{6}\pi}{2}$ using which we can alternatively write $D_L^0 \approx (x_0^1)^2 \sqrt{\frac{3}{\pi}}$ as $m \to \infty$. We have the following lemma.

**Lemma 16.** For the pair of equilibrium strategies $(a_L^*, a_F^*)$ characterized by Theorem 1, where $a_L^* \in A_{m_L}^*(r_L, \sigma)$ and $a_F^*(s) = \mathbb{E}_a[a_L^*|s]$, we have

\[
\liminf_{m \to \infty} \frac{r_L(1-r_L)(x_1^0)^2}{(2-2r_L)^2} + 4\sqrt{\frac{2}{e}} \frac{(2-2r_L)^2}{(1-r_L)^2} \phi\left(\frac{x_1^0}{\sqrt{2}}\right) \geq 1.
\]

**Proof.** See the appendix. ■

The above asymptotic upper bound on $U(a_L^*, a_F^*)$ is minimized when $x_1^0 \approx 2 \sqrt{2 \ln \sigma} r_L$ for large $\sigma$, yielding a cost $\approx \frac{8r_L \ln \sigma}{\sqrt{3}}$. Recalling that $M(\sigma) = \{m \in \mathbb{N} | 2\sqrt{2 \ln \sigma} + 4 < x_1^0 < 4\sqrt{\ln \sigma}\}$, this implies the existence of a local minimum with near piecewise-linear strategy for the leader with a cost (asymptotically) as low as $\frac{8r_L \ln \sigma}{\sqrt{3}}$. To compare with the optimal solution, we use the lower bounds on the optimal cost of Witsenhausen’s problem derived in Grover et al. (2013). The following lemma is an immediate result of Theorem 4 in Grover et al. (2013).

**Lemma 17.** Denote with $U^*(\sigma)$ the minimum value of the cost functional $U(a_L, a_F)$ given by (IV) in the regime $r_L \sigma^2 = 1$. Then

\[
\limsup_{\sigma \to \infty} \frac{\ln \sigma}{U^*(\sigma)} \leq 1.
\]

**Proof.** See the appendix. ■

\[10\]A relevant open problem is to find the exact asymptotic value of the support region of the optimal quantizer, that is $b_0^m$ in our setting (see Na and Neuhoff (2001) for some related results).

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This lower bound is quite loose (as also pointed out by the authors in Grover et al. (2013)), but still serves our purpose of showing that our proposed local optima are only a constant factor away from the optimal cost as \( \sigma \to \infty \).\(^{11}\) We summarize the main findings of this section in the theorem below.

**Theorem 2.** Any pair of equilibrium strategies \((a^*_L, a^*_F)\) characterized by Theorem 1, where 
\[ a^*_L \in A^m_L(r_L, \sigma) \text{ and } a^*_F(s) = \mathbb{E}[a^*_L | s], \]

is a local minimum of the cost functional \( U \) in (IV). Moreover, the set \( M(\sigma) \) is nonempty for sufficiently large values of \( \sigma \) and,

\[
\liminf_{\sigma \to \infty} \frac{8r_L \ln \sigma}{\min_{m \in M(\sigma)} U(a^*_L, a^*_F)} \geq 1. \quad (7)
\]

In the regime \( r_L \sigma^2 = 1 \), all these local minima are within constant factor of the optimal cost, with at least one being less than 27.8 times away from the optimal value as \( \sigma \to \infty \).

**Proof.** See the appendix. \( \blacksquare \)

**V. Conclusions**

We studied Witsenhausen’s counterexample in a leader-follower game setup where the follower makes noisy observations from the leader’s action and aims to choose her action as close as possible to that of the leader. Leader who moves first and can see the realization of the state of the world chooses her action to minimize her ex-ante distance from the follower’s action as well as the state of the world. We showed the existence of nonlinear perfect Bayesian equilibria in the regime \( \frac{1}{2} \leq r_L \sigma^2 \leq 1 \), where the leader’s strategy is a perturbed slopey variation of an optimal MSE quantizer. We then proved that these equilibria are indeed local minima of the original Witsenhausen’s problem. Incorporating some relevant results from asymptotic quantization theory and lower bounds on the optimal cost of Witsenhausen’s problem from the literature, we showed that the proposed local minima are near-optimal in that they are at most a constant factor away from the optimal one. Our work hence provides a supporting theory for the local optimality of near piecewise-linear strategies for Witsenhausen’s problem.

\(^{11}\) The ratio between the upper and lower bounds in Grover et al. (2013) is almost 100. For the well-known case of \( \sigma = 5 \) and \( k^2 \sigma^2 = 1 \), the lowest known cost \( \approx 0.167 \) is higher than 12.5 times the value obtained from the lower bound.
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APPENDIX

Proof of Lemma 1. To avoid lengthy expressions, we introduce and work with normalized
variables \( \hat{x}_k = \frac{x_k}{\sigma} \), \( \hat{b}_k = \frac{b_k}{\sigma} \), \( \hat{c}_k = \frac{c_k}{\sigma} \), and \( \hat{B}_k = \frac{B_k}{\sigma} \). All the expectations are then taken using \( N(0, 1) \) as the probability measure.

i) Let \( \rho(h) = \frac{\Phi(h)}{1 - \Phi(h)} \) denote the Mill’s ratio. Then, \( 0 \leq 1 - \rho(\rho - h) \leq (\rho - h)^2 \) (see, e.g. Horrace (2015) for a proof). This implies that \( 0 \leq 1 - \hat{c}_m \hat{x}_m \leq \hat{x}_m^2 \). Moreover, \( \hat{x}_m \) is decreasing with \( m \) and it follows from direct calculation that \( \hat{x}_2 = 0.48 < \frac{1}{2} \), hence completing the proof.

ii) Since \( \hat{c}_k \) is the centroid of segment \( \hat{B}_k \), we should have

\[
\int_{b_k}^{\hat{c}_k} (\hat{c}_k - \theta) \phi(\theta) d\theta = \int_{b_k}^{\hat{b}_k+1} (\theta - \hat{c}_k) \phi(\theta) d\theta. \tag{8}
\]

This, together with the fact that \( \int_{b_k}^{\hat{c}_k} (\hat{c}_k - \theta) \phi(\theta) d\theta \leq \phi(\hat{b}_k) \int_{b_k}^{\hat{c}_k} (\hat{c}_k - \theta) d\theta \) and \( \int_{\hat{c}_k}^{\hat{b}_k+1} (\theta - \hat{c}_k) \phi(\theta) d\theta \geq \phi(\hat{b}_k+1) \int_{\hat{c}_k}^{\hat{b}_k+1} (\theta - \hat{c}_k) d\theta \) proves the R.H.S inequality in (1). To derive the L.H.S, let \( p_k^1 = \text{Prob}[\theta | \hat{b}_k \leq \theta \leq \hat{c}_k] \) and \( p_k^2 = \text{Prob}[\theta | \hat{c}_k \leq \theta \leq \hat{b}_{k+1}] \). Noting that \( \phi(\theta) \) (for \( \theta \geq 0 \)) and \( \hat{c}_k - \theta \) are decreasing with \( \theta \), we apply algebraic Chebyshev inequality to (A) to obtain \( p_k^1 \hat{x}_k \leq p_k^2 \hat{x}_{k+1} \). On the other hand,

\[
\frac{p_k^2}{p_k^1} \leq \frac{\hat{x}_{k+1} + \hat{x}_k}{\hat{x}_k} \times \frac{\phi(\hat{c}_k)}{\phi(\hat{b}_k)}. 
\]

Combining the two, we can derive the L.H.S in (1). It then immediately follows from the L.H.S inequality that \( \hat{x}_k \leq \hat{x}_{k+1} \) for \( 1 \leq k \). Applying the result of part (i) to the R.H.S inequality we can easily show that \( \frac{\hat{x}_{k+1}}{\hat{x}_k} \leq e \).

iii) A useful property here is that \( g(x) = \frac{1}{x} \int_x^{a+x} \phi(t) dt \) is decreasing in \( x \) for \( a > 0 \). Using this, we can obtain

\[
\begin{align*}
\frac{\text{Prob}[\theta | \theta \in \hat{B}_k]}{\text{Prob}[\theta | \theta \in \hat{B}_j]} & \leq \frac{\hat{x}_k + \hat{x}_{k+1}}{\hat{x}_j + \hat{x}_{j+1}} \times \frac{\phi(\hat{b}_k)}{\phi(\hat{b}_j)} \\
& \leq \frac{\hat{x}_k + \hat{x}_{k+1}}{\hat{x}_j + \hat{x}_{j+1}} \times \frac{\hat{x}_j^2}{\hat{x}_k^2} \\
& \leq \frac{1 + e \hat{x}_j}{2} \times \frac{1 + e}{\hat{x}_k} \leq \frac{1 + e}{2},
\end{align*}
\]

where the last two lines follow from part (ii). We have to modify the proof for \( k = m \). To extend the proof to the case \( k = m \), it suffices to show that \( \text{Prob}[\theta | \theta \in \hat{B}_m] \leq \frac{1 + e}{2} \).
Prob[\theta \in \hat{B}_{m-1}]. We first note that
\[ \text{Prob}[\theta \in \hat{B}_m] = \frac{\phi(\hat{b}_m)}{\hat{c}_m} \leq \frac{4}{3} \phi(\hat{b}_m) \hat{x}_m, \]
where we have used part (i) in the last inequality. The proof now follows from the fact that
\[ \text{Prob}[\theta \in \hat{B}_{m-1}] > (\hat{x}_{m-1} + \hat{x}_m) \phi(\hat{b}_m) \geq (1 + \frac{1}{e}) \hat{x}_m \phi(\hat{b}_m). \]
iv) We start by showing that
\[ e^{\frac{1}{2}(\hat{x}_1 + \ldots + \hat{x}_k)^2} \leq \frac{\hat{x}_{k+1}}{\hat{x}_1} \leq e^{\frac{\hat{e}}{2}(\hat{x}_1 + \ldots + \hat{x}_{k+1})^2}. \] (9)
The LHS easily follows from part (ii), while the RHS requires a more involved analysis as we elaborate below. The idea here is to find an appropriate lower bound for the RHS of (A). Using Jensen’s inequality for the concave function \( e^{-x} \), we can obtain
\[ \int_{\hat{c}_k}^{\hat{b}_{k+1}} (\theta - \hat{c}_k) \phi(\theta) d\theta \geq \frac{\hat{x}_{k+1}^2}{2\sqrt{2\pi}} e^{-\frac{1}{2} \int_{\hat{c}_k}^{\hat{b}_{k+1}} 2\phi(\theta) d\theta}. \]
Combining this with the same upper bound of \( \frac{\hat{x}_k^2}{\hat{x}_1} \phi(\hat{b}_k) \) as in part (ii) for the LHS of (A) and after some simplification we can reach at
\[ \frac{\hat{x}_{k+1}^2}{\hat{x}_k^2} \leq e^{\hat{c}_k \hat{x}_k + \frac{1}{4} \hat{c}_k \hat{x}_{k+1} - \frac{\hat{x}_k^2}{2} + \frac{\hat{x}_{k+1}^2}{4}}. \]
Substituting \( k \) with 1, \ldots, \( k-1 \) and multiplying all these \( k \) inequalities we can prove the RHS inequality in (A).

Incorporating the simple inequality \( \phi(\hat{x}_1 + \ldots + \hat{x}_k) \leq e^{\frac{\hat{e}}{2}(\hat{x}_1 + \ldots + \hat{x}_k)^2} \leq \sqrt{e} \) into (A), we can find that \( \hat{x}_1 \leq \sqrt{2\pi \hat{e} \hat{x}_{k+1} \phi(\hat{x}_1 + \ldots + \hat{x}_{k+1})} \) for \( k = 1, \ldots, m - 1 \). Adding up all these inequalities and \( \hat{x}_1 \leq \sqrt{2\pi \hat{e} \hat{x}_{k+1} \phi(\hat{x}_1)} \) yields
\[ \frac{m \hat{x}_1}{\sqrt{2\pi \hat{e}}} \leq \sum_{k=1}^{m} \hat{x}_k \phi(\hat{x}_1 + \ldots + \hat{x}_k) \leq \int_0^\infty \phi(\theta) d\theta = \frac{1}{2}, \]
proving \( \hat{x}_1 \leq \frac{\sqrt{2\pi \hat{e}}}{2m} \). Based on the RHS of (A) and following a similar approach we can show that
\[ \sum_{k=1}^{m-1} \hat{x}_{k+1} e^{-\frac{\hat{e}}{2}(\hat{x}_1 + \ldots + \hat{x}_k)^2} \leq e(m - 1) \hat{x}_1. \] (10)
On the other hand,
\[
\int_{\hat{x}_1 + \ldots + \hat{x}_m}^{\infty} e^{-\frac{2}{\nu} \theta^2} d\theta \leq \frac{6e^{-\frac{2}{\nu}(\hat{x}_1 + \ldots + \hat{x}_m)^2}}{10(\hat{x}_1 + \ldots + \hat{x}_m)} \\
\leq \frac{6e^{-\frac{2}{\nu}(\hat{x}_1 + \ldots + \hat{x}_m)^2}}{10(\hat{x}_1 + \ldots + \hat{x}_m)} \leq 1.6\hat{x}_1,
\]
where the last inequality follows from (A) and part (i). Putting (A) and (A) together we can show that
\[
e\hat{m} \hat{x}_1 \geq \int_{0}^{\infty} e^{-\frac{2}{\nu} \theta^2} d\theta = \frac{\sqrt{3}}{\pi},
\]
which yields \( \hat{x}_1 \geq \frac{\sqrt{\pi}}{2\hat{m}} \). The bounds on \( \hat{c}_m \) and \( \hat{x}_m \) can then be easily obtained using (A) for \( k = m - 1 \) and that \( \frac{3}{4} \leq \hat{x}_m \hat{c}_m \leq 1 \).

\[\Box\]

Proof of Lemma 2. This is an immediate result of Property 2.

Proof of Lemma 3. We start with the case where \( a_L(c_m + \sigma \bar{x}_{m+1}) \leq s \leq c_m + \bar{x}_{m+1} \). Let \( \theta_c = c_m + \sigma \bar{x}_{m+1} \) and \( \delta_c = s - a_L(\theta_c) \). With some manipulation, we can show that for every \( b_m \leq \theta, \theta' \leq \theta_c \),
\[
\frac{\text{Prob}[\theta'|s]}{\text{Prob}[\theta|s]} = \frac{\phi(s - a_L(\theta') \phi(\frac{\theta'}{\sigma})}{\phi(s - a_L(\theta))} \phi(\frac{\theta}{\sigma}) \phi(\delta_c + r(\theta_c - \theta'))}{\phi(\delta_c + r(\theta_c - \theta))} \geq \frac{\phi(\delta_c + r(\theta_c - \theta'))}{\phi(\delta_c + r(\theta_c - \theta))}.
\]

Integrating with respect to \( \theta' \) and after some simplification, we arrive at
\[
\text{Prob}[\theta|s, b_m \leq \theta \leq \theta_c] \leq \frac{\phi(\delta_c + r(\theta_c - \theta))}{\phi(\delta_c + r(\theta_c - \theta))} \frac{\phi(\theta)}{\Phi(\theta_c) - \Phi(b_m)} \]
\[
\leq \frac{\xi}{\Phi(\theta_c) - \Phi(b_m)},
\]
where \( \phi(\delta_c + r(\theta_c - \theta)) \sim N(\bar{\mu}, \bar{\sigma}^2) \), with \( \bar{\mu} = \frac{\nu^2(\delta_c + r(\theta_c))}{1 + \nu^2 \sigma^2} \) and \( \bar{\sigma}^2 = \frac{\sigma^2}{1 + \nu^2 \sigma^2} \), and
\[
\xi = \frac{\phi(\delta_c + r(\theta_c - b_m))}{\phi(\delta_c + r(\theta_c - b_m))}.
\]

(13)
As for the variance,

\[
\mathbb{E}[a_L(\theta)|s, b_m \leq \theta \leq \theta_c] - c_m \leq \xi \bar{r} \mathbb{E}[\theta - c_m | c_m \leq \theta \leq \theta_c] \\
\leq \xi \bar{r} \mathbb{E}[\theta - c_m | c_m \leq \theta] \\
\leq 1.1 \bar{r} (\mathbb{E}_{N(0,\sigma^2)}[\theta - c_m | c_m \leq \theta] + \frac{4 \bar{r}^2 m \bar{\mu}}{\sigma^2}) \\
\leq 1.1 \bar{r} \bar{x}_m (1 + \frac{\bar{x}_m \bar{\mu}}{\sigma^2}) \\
\leq 1.2 \bar{r} \bar{x}_m < 0.75 \bar{x}_{m+1}.
\]  

(14)

As for the variance,

\[
\text{Var}[a_L(\theta)|s, b_m \leq \theta \leq \theta_c] \leq \xi \bar{r}^2 \text{Var}_\phi[\theta|b_m \leq \theta \leq \theta_c] \leq \xi \bar{r}^2 \text{Var}_{N(\bar{\mu},\sigma^2)}[\theta|b_m \leq \theta].
\]  

(15)

Now, we aim to use the bound \(\text{Var}_{N(\bar{\mu},\sigma^2)}[\theta|b_m \leq \theta] \leq (\mathbb{E}_{N(\bar{\mu},\sigma^2)}[\theta|b_m \leq \theta] - b_m)^2\). We then use \(\mathbb{E}_{N(\bar{\mu},\sigma^2)}[\theta|b_m \leq \theta] \leq \mathbb{E}_{N(0,\sigma^2)}[\theta|b_m \leq \theta] + \frac{4 \bar{r}^2 \bar{\mu}}{\sigma^2}\). This leads to

\[
\text{Var}[a_L(\theta)|s, b_m \leq \theta \leq \theta_c] \leq 1.1 \bar{r}^2 (\bar{x}_m + \frac{4 \bar{r}^2 \bar{\mu}}{\sigma^2})^2 < 1.2 \bar{r}^2 \bar{x}_m^2,
\]  

(16)

for sufficiently large \(\sigma\) and using the easily verifiable fact that \(\bar{\mu} \leq \bar{x}_m + 1\). Similar results to the above can be derived for the case where \(a_L(b_m) \geq b_m \geq a_L(\theta_c)\), using \(\theta_s\) instead of \(\theta_c\), where \(s = a_L(\theta_s)\) with \(b_m \leq \theta_s < \theta_c\). The same for the case \(s < a_L(b_m)\), following a similar argument with \(\bar{\phi}_b \sim N(\bar{\mu}_b, \bar{\nu}^2)\), where \(\bar{\mu}_b = \frac{\bar{r} \sigma^2 \bar{\mu} - \delta_b}{1 + \bar{r} \sigma^2}, \bar{\nu}^2 = \frac{\sigma^2}{1 + \bar{r} \sigma^2}\), and \(\delta_b = a_L(b_m) - s > 0\).

Now we bring into play the tail effect. For every \(\theta \geq \theta_c\), we use

\[
\frac{\text{Prob}[\theta|s]}{\text{Prob}[c_m \leq \theta' \leq \theta_c|s]} = \frac{\phi(s - a_L(\theta))\phi(\frac{\theta}{\sigma})}{\int_{c_m}^{\theta_c} \phi(s - a_L(\theta'))\phi(\frac{\theta'}{\sigma})d\theta'},
\]

using which for \(s = c_m + \delta\) with \(0 \leq \delta \leq \bar{x}_{m+1}\), we get

\[
\frac{\text{Prob}[\theta|s]}{\text{Prob}[c_m \leq \theta' \leq \theta_c|s]} \leq \frac{e^{\frac{\delta^2}{2}}\phi(\frac{\theta}{\sigma})}{\sigma(\Phi(\frac{\theta}{\sigma}) - \Phi(\frac{c_m}{\sigma}))}.
\]  

(17)

Therefore, for \(\theta \geq \theta_c\)

\[
\text{Prob}[\theta|s, \theta \geq c_m] \leq \frac{e^{\frac{\delta^2}{2}}(1 - \Phi(\frac{\theta}{\sigma}))}{\Phi(\frac{\theta}{\sigma}) - \Phi(\frac{c_m}{\sigma})} \times \frac{\phi(\frac{\theta}{\sigma})}{\sigma(1 - \Phi(\frac{c_m}{\sigma}))}.
\]  

(18)
Using the inequality \( h \leq \rho(h) \leq \frac{h^2 + 1}{h} \) for \( h > 0 \), we can show that \( \frac{1 - \Phi(\frac{\theta_c}{\sigma})}{\Phi(\frac{\theta_c}{\sigma}) - \Phi(\frac{cm}{\sigma})} \leq \frac{\phi(\frac{\theta_c}{\sigma})}{\phi(\frac{cm}{\sigma})} \). This, along with (A) and \( 0 \leq \delta \leq \bar{x}_{m+1} \) and \( \theta_c = c_m + \sigma \bar{x}_{m+1} \) yields

\[
\text{Prob}[\theta|s, \theta \in B_m] \leq \text{Prob}[\theta|s, \theta \geq c_m] \leq \frac{e^{-\frac{cm\bar{x}_{m+1}}{\sigma}} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma(1 - \Phi(\frac{\theta_c}{\sigma}))} \leq \frac{e^{-\sigma} \phi\left(\frac{\theta}{\sigma}\right)}{\sigma(1 - \Phi(\frac{\theta_c}{\sigma}))},
\]

(19) for \( \theta \geq \theta_c \). Using this along with (A), we can obtain

\[
E[a_L(\theta)|s, \theta \in B_m] - c_m \leq 0.75\bar{x}_{m+1} + e^{-\sigma}(3r_L(\theta_c - c_m + \sigma(\rho(\frac{\theta_c}{\sigma}) - \frac{\theta_c}{\sigma})))
\leq 0.75\bar{x}_{m+1} + 3r_L\sigma e^{-\sigma}(\bar{x}_{m+1} + \frac{\sigma}{c_m + \sigma \bar{x}_{m+1}}) \leq \bar{r}\bar{x}_{m+1}.
\]

(20)

Also, \( E[a_L(\theta)|s, \theta \in B_m] \geq a_L(b_m) \geq c_m - \bar{r}\bar{x}_m \). To bound the variance, let \( \kappa = E[a_L(\theta)|s, b_m \leq \theta \leq \theta_c] \) (\( \kappa > a_L(b_m) \)). Then,

\[
\text{Var}[a_L(\theta)|s, \theta \in B_m] \leq E[(a_L(\theta) - \kappa)^2|s, \theta \in B_m]
\leq 1.2r^2\bar{x}_m^2 + e^{-\sigma}E_N(0, \sigma^2)[(a_L(\theta) - a_L(b_m))^2|\theta \geq \theta_c]
\leq 1.2r^2\bar{x}_m^2 + e^{-\sigma}E_N(0, \sigma^2)[(3r_L(\theta - c_m) + \bar{r}\bar{x}_m)^2|\theta \geq \theta_c]
\leq 1.2r^2\bar{x}_m^2 + 9r^2e^{-\sigma}((\sigma \bar{x}_{m+1} + 1)^2 + \sigma^2)
\leq 1.3r^2\bar{x}_m^2 \leq 0.75r^2\left(\frac{\bar{x}_m + \bar{x}_{m+1}}{2}\right)^2,
\]

(21)

for sufficiently large \( \sigma \).

For \( s < c_{m-1} \), we use the fact that \( \text{Var}[\theta|s, b_m \leq \theta \leq \theta_c] \leq (\frac{\theta_c - b_m}{2})^2 \) to get

\[
\text{Var}[a_L(\theta)|s, b_m \leq \theta \leq \theta_c] \leq \bar{r}^2(\frac{\bar{r}}{2})^2 < 0.3.
\]

As for the effect of \( \theta > \theta_c \), we can easily see that for \( s < c_m \) (A) becomes

\[
\text{Prob}[\theta|s, \theta \in B_m] \leq \frac{e^{-\sigma}x_{m+1}^2 \phi\left(\frac{\theta}{\sigma}\right)}{\sigma(1 - \Phi(\frac{\theta}{\sigma}))}.
\]

We can use this to bound the variance similar to (A):

\[
\text{Var}[a_L(\theta)|s, \theta \in B_m] \leq \begin{cases} 
0.75\bar{r}^2(\frac{\bar{x}_m + \bar{x}_{m+1}}{2})^2, & c_{m-1} \leq s \leq c_m \\
\frac{1}{3}, & s < c_{m-1}
\end{cases}
\]
For the case where \( s > c_m + \bar{x}_{m+1} \) (i.e., \( \delta > \bar{x}_{m+1} \)), let \( \theta_s = c_m + \sigma \delta \). Then, similar to (A) we can obtain
\[
\text{Prob}[\theta|s, \theta \in B_m] \leq \frac{e^{-\sigma \phi(\frac{\theta}{\sigma})}}{\sigma(1 - \Phi(\frac{\theta}{\sigma}))},
\]
for \( \theta \geq \theta_s \). Using this and similar to (A), we can reach at
\[
\mathbb{E}[a_L(\theta)|s, \theta \in B_m] < c_m + 3r_L \sigma (\delta + 1).
\]
To bound the variance, similar to (A) we can show
\[
\text{Var}[a_L(\theta)|s, \theta \in B_m] < 2.5 r_L^2 \sigma^2 \delta^2,
\]
which completes the proof.

**Proof of Lemma 4.** From the properties of the base configuration, it is easy to see that
\[
\frac{\text{Prob}[\theta \in B_{k-j}]}{\text{Prob}[\theta \in B_k]} \leq e^{2j+1} \quad \text{and} \quad \frac{\text{Prob}[\theta \in B_{k+1}]}{\text{Prob}[\theta \in B_{k+j+1}]} \geq \frac{1}{2}.
\]
To prove the lemma for \( k < m \), we write
\[
\frac{\text{Prob}[\theta \in B_{k-j}|s]}{\text{Prob}[\theta \in B_k|s]} \leq \frac{\text{Prob}[\theta \in B_{k-j}] \phi(\delta + (c_k - c_{k-j}) - \bar{r}x_m)}{\text{Prob}[\theta \in B_k] \phi(\delta + \bar{r}x_m)}
\leq \frac{\text{Prob}[\theta \in B_{k-j}]}{\text{Prob}[\theta \in B_k]} e^{-(\frac{(c_k - c_{k-j})}{2})^2 + \bar{r}x_m(c_k - c_{k-j}) - \delta(c_k - c_{k-j}) - 2\bar{r}x_m)}
\leq e^{-(\frac{(c_k - c_{k-j})}{2})^2 + 3j}.
\]
The case \( k = m \) needs separate treatment. Define \( \hat{B}_m = [b_m, c_m + \bar{x}_m] \). Then, it is easy to verify that (A) still holds if we replace \( B_m \) with \( \hat{B}_m \). Therefore, the proof in this case follows from an argument similar to above on noting that \( \text{Prob}[\theta \in \hat{B}_m|s] \leq \text{Prob}[\theta \in B_m|s] \).

**Proof of Lemma 5.** If \( s \geq c_k + \bar{r}x_{k+2} \), then
\[
\frac{\text{Prob}[\theta \in B_k|s]}{\text{Prob}[\theta \in B_{k+1}|s]} \leq \frac{\text{Prob}[\theta \in B_k] \phi(m_{k+1} + \delta - c_k - \bar{r}x_{k+2})}{\text{Prob}[\theta \in B_{k+1}] \phi(m_{k+1} + \delta - c_{k+1} - \bar{r}x_{k+2})}
\leq \frac{\text{Prob}[\theta \in B_k]}{\text{Prob}[\theta \in B_{k+1}]} e^{\Delta_{k+1}(\frac{c_k + c_{k+1}}{2} + \bar{r}x_{k+2} - \delta)}
\leq e^{\Delta_{k+1}(\bar{r}x_{k+2} - \delta)},
\]
where the last inequality follows from the definition of \( m_{k+1} \). However, for the case where \( s < c_k + \bar{x}_{k+2} \), the upper bound on the likelihood \( \text{Prob}[s|\theta \in B_k] \) in the first inequality may be less and hence is replaced by 1, which will thereby lead to

\[
\frac{\text{Prob}[\theta \in B_k|s]}{\text{Prob}[\theta \in B_{k+1}|s]} \leq e^{\Delta_{k+1}(\bar{x}_{k+2} - \delta) + \frac{\bar{x}^2_{k+2}}{2}}.
\]

The other side of the inequality can be proved similarly.

For the case \( k = m - 1 \), the lower bound \( \phi(m_{k+1} + \delta - c_{k+1} - \bar{x}_{k+2}) \) for the likelihood \( \text{Prob}[s|\theta \in B_{k+1}] \) is not valid anymore. To fix this, as in Lemma 4, we use \( \hat{B}_m = [b_m, c_m + \bar{x}_m] \) instead of \( B_m \) to obtain,

\[
\frac{\text{Prob}[\theta \in B_{m-1}|s]}{\text{Prob}[\theta \in B_m|s]} \leq \frac{\text{Prob}[\theta \in B_{m-1}|s]}{\text{Prob}[\theta \in B_m|s]} \leq \frac{\text{Prob}[\theta \in B_m]}{\text{Prob}[\theta \in \hat{B}_m]} e^{\Delta_m(\bar{x}_m - \delta) + \frac{\bar{x}^2_m}{2}}.
\]

On the other hand, we can show that \( \frac{\text{Prob}[\theta \in \hat{B}_m]}{\text{Prob}[\theta \in B_m]} < 1.16 \), which completes the proof. It is easy to see that the inequality in LHS stays as before for \( k = m - 1 \).

**Proof of Lemma 6.** As the first step we bound the effect of intervals other than \( B_k \cup B_{k+1} \). Let \( \eta = \mathbb{E}[a_L(\theta)|s, \theta \in B_k \cup B_{k+1}] \) and \( \eta_j = \mathbb{E}[a_L(\theta)|s, \theta \in B_j] \) for \(-m \leq j \leq m\). Using Lemma 4, we can write

\[
\sum_{j=1}^{k+m} \frac{\text{Prob}[\theta \in B_{k-j}|s]}{\text{Prob}[\theta \in B_k|s]} (\eta - \eta_{k-j}) = \sum_{j=1}^{k+m} (c_k - c_{k-j} + 2\bar{x}_m + 2\bar{x}_m) e^{-\frac{(c_k-c_{k-j})^2}{2} + 3j} 
\]

\[
\leq \sum_{j=1}^{k+m} (2j \bar{x}_1 + 2\bar{x}_m + 2\bar{x}_m) e^{-2j^2 \bar{x}^2_1 + 3j} 
\]

\[
\leq 2e^{-2\bar{x}_1^2 + 3\bar{x}_m} \sum_{j=1}^{k+m} j e^{-2(j^2-1)\bar{x}^2_1 + 3(j-1)} 
\]

\[
\leq 2e^{-2\bar{x}_1^2 + 3\bar{x}_m} \sum_{j=1}^{\infty} e^{-(j-1)^2} 
\]

\[
\leq \frac{2.8e^3}{\sigma^{15}} < 10^{-10}r^2 \bar{x}^2_1, \quad (24)
\]

where we have used the identity \( \sum_{j=0}^{\infty} e^{-j^2} \approx 1.386 \). Similarly, we can bound the effect of
non-neighboring intervals on the variance:

\[
\sum_{j=1}^{k+m} \frac{\text{Prob}[\theta \in B_{k-j}|s]}{\text{Prob}[\theta \in B_k|s]} (\eta - \eta_{k-j})^2 \leq \frac{5.6e^3}{\sigma^{14}}.
\]

On the other hand,

\[
\sum_{j=1}^{k+m} \frac{\text{Prob}[\theta \in B_{k-j}|s]}{\text{Prob}[\theta \in B_k|s]} \text{Var}[a_L(\theta)|s, \theta \in B_{k-j}] \leq \sum_{j=1}^{k+m} \frac{1}{3} e^{- (c_k - c_{k-j})^2/2} + 3j \leq \frac{0.5e^3}{\sigma^{16}}.
\]

Combining the two, we obtain

\[
\sum_{j=-m}^{m} \text{Prob}[\theta \in B_j|s]((\eta - \eta_j)^2 + \text{Var}[a_L(\theta)|s, \theta \in B_j]) \leq 10^{-10} r^2 \bar{x}_1^2. \tag{25}
\]

Therefore, the effect of intervals other than \(B_k\) and \(B_{k+1}\) on \(a_F(s)\) (and its derivative given by \(\text{Var}[a_L|s]\)) is quite negligible. Now, focusing on these two intervals (i.e., \(B_k\) and \(B_{k+1}\)), we have

\[
\mathbb{E}[a_L(\theta)|s, \theta \in B_k \cup B_{k+1}] = p\mathbb{E}[a_L(\theta)|s, \theta \in B_k] + (1-p)\mathbb{E}[a_L(\theta)|s, \theta \in B_{k+1}]
\]

\[
\leq p(c_k + \bar{r}\bar{x}_{k+1}) + (1-p)(c_k + \Delta_{k+1} + \bar{r}\bar{x}_{k+2})
\]

\[
\leq c_k + (1-p)\Delta_{k+1} + \bar{r}\bar{x}_{k+2},
\]

where \(p = \frac{\text{Prob}[\theta \in B_k|s]}{\text{Prob}[\theta \in B_k \cup B_{k+1}|s]}\). The proof for the upper bound on \(a_F(s)\) now follows from Lemma 5. The proof for the lower bound on \(a_F(s)\) is similar. Now, as for the derivative, we first note that \(\frac{d}{ds} a_F(s) = \text{Var}[a_L|s]\). Again, focusing on \(B_k \cup B_{k+1}\), we can write

\[
\text{Var}[a_L|s, \theta \in B_k \cup B_{k+1}] \leq p\text{Var}[a_L|s, \theta \in B_k] + (1-p)\text{Var}[a_L|s, \theta \in B_{k+1}]
\]

\[
+ p(1-p)(\mathbb{E}[a_L|s, \theta \in B_k] - \mathbb{E}[a_L|s, \theta \in B_{k+1}])^2.
\]

The rest easily follows from Lemma 2 and Lemma 5.

\[\blacksquare\]

**Proof of Lemma 7.** Exploiting the term \(e^{-\delta(c_m - c_{m-r} - 2\bar{x}\sigma)}\) in (A) (for \(k = m\)), it is easy to observe that the same upper bounds given by (A) and (A) hold for the effect of intervals other than \(B_m\) on \(a_F(s)\) provided

\[
e^{-\delta(c_m - c_{m-r} - 2\bar{x}\bar{m})} (\mathbb{E}[a_L(\theta)|s, \theta \in B_m] - c_m) < 2\bar{x}_m.
\]

Verifying the above inequality is quite straightforward using Lemma 3, and specially noting that
\[ \mathbb{E}[a_L(\theta)|s, \theta \in B_m] - c_m < 3r_L\sigma(\delta + 1) \text{ for } \delta > \bar{x}_{m+1}. \] The proof of the lemma is now an immediate consequence of Lemma 3.

**Proof of Lemma 8.** We start by showing that
\[ \tilde{J}_L(c_k) = \int_{-\infty}^{\infty} (a_F(s) - c_k)^2 \phi(s - c_k) ds \leq 1.1\bar{r}^2\bar{x}_{k+2}^2. \] (26)

With some manipulation and using \( \rho(h) \geq h \), we can obtain
\[
\begin{align*}
\int_{c_k}^{m_{k+1}} (a_F(s) - c_k)^2 \phi(s - c_k) ds & \leq \int_{0}^{m_{k+1} - c_k} (1.01\bar{r}\bar{x}_{k+2} + 1.17\Delta_{k+1}e^{-\delta\Delta_{k+1}})^2 \phi(m_{k+1} - c_k - \delta)d\delta \\
& \leq \frac{1.01^2\bar{r}^2\bar{x}_{k+2}^2}{2} + \left(\frac{5.48x_1^2}{4x_1 - x_1} + \frac{4.74\bar{r}\bar{x}_{k+2}x_1}{2x_1 - x_1}\right)\phi(x_1) \\
& \leq \frac{1.01^2\bar{r}^2\bar{x}_{k+2}^2}{2} + \frac{0.73x_1 + 1.9\bar{r}\bar{x}_{k+2}}{\sigma^4}.
\end{align*}
\] (27)

Similarly,
\[
\int_{m_{k+1}}^{c_{k+1}} (a_F(s) - c_k)^2 \phi(s - c_k) ds \leq (2\bar{x}_1 + 1.01\bar{r}\bar{x}_{k+2})^2 \frac{\phi(x_1)}{x_1} \leq \frac{5x_1}{\sqrt{2\pi}\sigma^4}.
\]

Using \( a_F(s) \leq s + 1.1\bar{x}_m \), we can show
\[
\begin{align*}
\int_{c_{k+1}}^{\infty} (a_F(s) - c_k)^2 \phi(s - c_k) ds & \leq \frac{\sqrt{2\pi}}{2} \phi(2\bar{x}_1)(1.1\bar{x}_m + 2\bar{x}_1 + 1)^2 \\
& \leq \frac{\sqrt{2\pi}}{2} \phi(4\sqrt{2\ln\sigma})(1.1\bar{x}_m + \bar{x}_1 + 1)^2 \\
& \leq \frac{(1.1\bar{x}_m + \bar{x}_1 + 1)^2}{2\sigma^4} < 10^{-4}\bar{r}^2\bar{x}_{k+2}^2.
\end{align*}
\]

Combining these, we can arrive at
\[
\int_{-\infty}^{\infty} (a_F(s) - c_k)^2 \phi(s - c_k) ds \leq 1.03\bar{r}^2\bar{x}_{k+2}^2 + \frac{5.46x_1 + 3.8\bar{r}\bar{x}_{k+2} + (1.1\bar{x}_m + 2\sqrt{2\ln\sigma} + 1)^2}{\sigma^4}
\leq 1.1\bar{r}^2\bar{x}_{k+2}^2.
\]

For the case of \( k = m \), \( \int_{c_m}^{\infty} (a_F(s) - c_m)^2 \phi(s - c_m) ds \) needs a different treatment. First we note
that
\[
\int_{c_m + \bar{x}_{m+1}}^{\infty} (a_F(s) - c_m)^2 \phi(s - c_m) ds \leq \int_{\bar{x}_{m+1}}^{\infty} 9r_L^2 \sigma^2(\delta + 1)^2 \phi(\delta) d\delta
\]
\[
\leq 9r_L^2 \sigma^2 \phi(\bar{x}_{m+1}) \left(\frac{\bar{x}_{m+1} + 2}{2}\right)^2 < 10^{-4} r^2 \bar{x}_{m+1}^2.
\]
Also,
\[
\int_{c_m}^{c_m + \bar{x}_{m+1}} (a_F(s) - c_m)^2 \phi(s - c_m) ds < \frac{r^2 \bar{x}_{m+1}^2}{2} < 0.5 r^2 \bar{x}_{m+1}^2,
\]
using which it is straightforward to verify that (A) holds for \( k = m \) as well (define \( \bar{x}_{m+2} = \bar{x}_{m+1} \) for consistency).

Let \( \theta = c_k + \epsilon \), with \( 0 \leq \epsilon \leq \frac{\Delta_{k+1}}{2} \). We first show that \( \tilde{a}_L(\theta) \) lies in a \( 5\bar{r} \bar{x}_{m+1} \)-vicinity of either \( c_k \) or \( c_{k+1} \). We begin with the case where \( \tilde{a}_L(\theta) \in [c_k, c_{k+1}] \). Let \( \tilde{a}_L(\theta) = c_k + \epsilon' \), with \( 5\bar{r} \bar{x}_{m+1} \leq \epsilon' \leq \Delta_{k+1} - 5\bar{r} \bar{x}_{m+1} \). We can use Corollary 2 to obtain a lower bound for \( \tilde{J}_L(\tilde{a}_L) \):
\[
\tilde{J}(\tilde{a}_L) \geq (\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2 (1 - \Phi\left(\frac{4\sqrt{2} \ln \sigma}{5}\right)) \geq \frac{5(\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2}{16 \sqrt{\pi} \ln \sigma \sigma_{\frac{16}{\pi}}^2}.
\]
Putting this together with \( \tilde{u}_L(\theta, \tilde{a}_L) \geq \tilde{u}_L(\theta, c_k) \geq -r_L(\frac{\Delta_{k+1}}{2}) - 1.1(1 - r_L) \bar{r}^2 \bar{x}_{k+2}^2 \), it is easy to show that \( \epsilon' - 1.1\bar{r} \bar{x}_{m+1} \leq \frac{\Delta_{k+1}}{\sqrt{\bar{r}} \sigma} \) for sufficiently large \( \sigma \). A second use of Corollary 2 now yields \( \tilde{J}_L(\tilde{a}_L) \geq (\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2 \Phi\left(\frac{\Delta_{k+1}}{4}\right) \geq 0.99(\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2 \) for large \( \sigma \). Therefore,
\[
\tilde{u}_L(\theta, \tilde{a}_L) \leq -r_L(\epsilon' - \epsilon)^2 - 0.99(\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2.
\]
On the other hand, using (A), we get \( \tilde{u}_L(\theta, c_k) \geq -r_L \epsilon^2 - 0.99(1 - r_L)(1.1\bar{r} \bar{x}_{m+1})^2 \). The RHS in (A) is maximized for \( \epsilon^* = \frac{r_L \epsilon + 0.99(1 - r_L) \times 1.1 \bar{r} \bar{x}_{m+1}}{r_L + 0.99(1 - r_L)} \), and having \( \tilde{u}_L(\theta, \tilde{a}_L) \geq \tilde{u}_L(\theta, c_k) \) requires \( \epsilon' \leq 2\epsilon^* \), implying \( \epsilon' < 5\bar{r} \bar{x}_{m+1} \).

For the case \( \tilde{a}_L(\theta) \not\in [c_k, c_{k+1}] \), suppose \( \tilde{a}_L(\theta) < c_k \) (the other case is similar), and let \( \tilde{a}_L(\theta) = c_k - \epsilon' \). Then, it follows from \( \tilde{u}_L(\theta, \tilde{a}_L) \geq \tilde{u}_L(\theta, c_k) \) that \(-r_L \epsilon^2 - 1.1(1 - r_L) \bar{r}^2 \bar{x}_{m+1}^2 \leq -r_L(\epsilon + \epsilon')^2 \), from which it easily follows that \( \epsilon' < 1.1 \). Now an argument similar to the case \( \tilde{a}_L(\theta) \in [c_k, c_{k+1}] \) shows that \( \tilde{J}_L(\tilde{a}_L) \geq 0.99(\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2 \). Combining this with \( |\theta - c_k| < |\theta - \tilde{a}_L| \) and \( \tilde{u}_L(\theta, \tilde{a}_L) \geq \tilde{u}_L(\theta, c_k) \), we get \( 1.1\bar{r}^2 \bar{x}_{m+1}^2 > 0.99(\epsilon' - 1.1\bar{r} \bar{x}_{m+1})^2 \) resulting in \( \epsilon' < 2.5\bar{r} \bar{x}_{m+1} \).

Similar to Lemma 7 in Witsenhausen (1968), we can show that \( \tilde{a}_L(\theta) \) is increasing. The fact that \( \tilde{a}_L(\theta) \) is increasing implies that it cannot swing between the two neighborhoods. Therefore, there exists a unique \( \tilde{b}_{k+1} \) separating the two regimes of \([c_k - 5\bar{r} \bar{x}_{m+1}, c_k + 5\bar{r} \bar{x}_{m+1}] \) and \([c_{k+1} - \ldots\)
Hence, $5\bar{r}\bar{x}_{m+1}, c_{k+1} + 5\bar{r}\bar{x}_{m+1}$, thus completing the proof.

Proof of Lemma 9. Using $\int_{-\infty}^{\infty} (a_F(s) - a_L)^2 \phi(s - a_L) ds = \int_{-\infty}^{\infty} (a_F(s + a_L) - a_L)^2 \phi(s) ds$, we get

$$\frac{d}{da_L} \int_{-\infty}^{\infty} (a_F(s) - a_L)^2 \phi(s - a_L) ds = 2 \int_{-\infty}^{\infty} (a_L - a_F(s + a_L))(1 - \frac{d}{ds}a_F(s + a_L))\phi(s) ds$$

$$= 2 \int_{-\infty}^{\infty} (a_L - a_F(s))(1 - \frac{d}{ds}a_F(s))\phi(s - a_L) ds.$$

Based on this, the first order condition for $\bar{a}_L$ gives

$$r_L(\bar{a}_L - \theta) + (1 - r_L)\int_{-\infty}^{\infty} (\bar{a}_L - a_F(s))(1 - \frac{d}{ds}a_F(s))\phi(s - \bar{a}_L) ds = 0.$$

Getting partial derivative with respect to $\theta$ we get

$$\frac{r_L}{\partial \bar{a}_L(\theta)} = r_L + (1 - r_L)\int_{-\infty}^{\infty} (1 - \frac{d}{ds}a_F(s))\phi(s - \bar{a}_L) ds$$

$$+ (1 - r_L)\int_{-\infty}^{\infty} (\bar{a}_L - a_F(s))(1 - \frac{d}{ds}a_F(s))(s - \bar{a}_L)\phi(s - \bar{a}_L) ds. \quad (29)$$

Similar to (A) and using (1) and (III-B), and a bit of manipulation we can obtain

$$\int_{c_k}^{c_m + \bar{x}_{m+1}} \left( \frac{d}{ds}a_F(s) - 1.01r^2\bar{x}_{k+2}^2 \right)\phi(s - \bar{a}_L) ds < 2 \times 4.68\bar{\xi}_1 + \frac{1}{\sqrt{2\pi}\sigma^{14}\bar{\bar{\xi}}_1} < 0.1r^2\bar{x}_{k+2}^2 \quad (30)$$

for sufficiently large $\bar{x}_{k+2}$ (or equivalently $\sigma$).

For $s > c_m + \bar{x}_{m+1}$, using Lemma 7 and the fact that $\bar{a}_L(\theta) \leq \bar{a}_L(c_m) < c_m + 5rL\sigma$, we get

$$\int_{c_m + \bar{x}_{m+1}}^{\infty} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L) ds \leq 3r^2\sigma^2 \int_{0}^{\infty} (\delta + \bar{x}_{m+1})^2 \phi(\delta + \bar{x}_{m+1} - 5rL\bar{x}_{m+1}) d\delta$$

$$\leq \frac{3\sqrt{2\pi}}{2} r_L(\bar{x}_{m+1} + 1)^2 \phi(\bar{x}_{m+1} - 5rL\bar{x}_{m+1}) < 10^{-4}r^2\bar{x}_{k+2}^2 \quad (31)$$

Hence,

$$\int_{-\infty}^{\infty} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L) ds < 1.3r^2\bar{x}_{k+2}^2.$$

The case $k = m$ is even easier on noting that $\frac{d}{ds}a_F(s) \leq 0.8r^2\bar{x}_{m+1}^2$ over the whole interval $s \in [c_m, c_m + \bar{x}_{m+1]}$. 

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Now, to bound the other term assume \( a_F(\tilde{a}_L) \leq \tilde{a}_L \). Then,

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)(\tilde{a}_L - a_F(s))(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds \leq 6.5\bar{r}x_{k+2}\int_{\tilde{a}_L}^{m_{k+1}} \frac{d}{ds} a_F(s)(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds \\
< 2.65\bar{r}x_{k+2}^3 + \frac{12.14\bar{r}x_{k+2}d_1^2}{\sigma^4} < 0.1\bar{r}x_{k+2}^2.
\]

For the case \( k = m \) and \( a_F(\tilde{a}_L) \leq \tilde{a}_L \),

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)(\tilde{a}_L - a_F(s))(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds \leq (\tilde{a}_L - a_F(\tilde{a}_L))\int_{\tilde{a}_L}^{\infty} \frac{d}{ds} a_F(s)(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds.
\]

We break it into two parts. First, using an approach similar to (A), we can obtain

\[
\int_{cm+\bar{x}_{m+1}}^{\infty} \frac{d}{ds} a_F(s)(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds < 10^{-4}\bar{r}x_{m+1}^2.
\]

And,

\[
\int_{\tilde{a}_L}^{cm+\bar{x}_{m+1}} \frac{d}{ds} a_F(s)(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds \leq \frac{0.8\bar{r}x_{m+1}^2}{\sqrt{2\pi}} < 0.32\bar{r}x_{m+1}^2.
\]

Hence,

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)(\tilde{a}_L - a_F(s))(s - \tilde{a}_L)\phi(s - \tilde{a}_L)ds \leq 6.5\bar{r}x_{m+1}(0.32\bar{r}x_{m+1}^2 + 10^{-4}\bar{r}x_{m+1}^2) < 0.1\bar{r}x_{m+1}^2.
\]

Putting all together, we get

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)(2 + (\tilde{a}_L - a_F(s))(s - \tilde{a}_L))\phi(s - \tilde{a}_L)ds < (2 \times 1.3 + 0.1)\bar{r}x_{k+2}^2 < 0.4\bar{r}x^2(32)
\]

Now for the other side, on noting that \( a_F(s) \leq s + 1.1\bar{x}_{m+1} \) and \( \frac{d}{ds} a_F(s) \leq ((1 + \bar{r})^2 + 1.01\bar{r}^2)x_{m+1}^2 \), we can get

\[
\int_{c_{k+1}}^{cm+\bar{x}_{m+1}} \frac{d}{ds} a_F(s)((a_F(s) - \tilde{a}_L)(s - \tilde{a}_L) - 2)\phi(s - \tilde{a}_L)ds < 10\bar{x}_{m+1}^3\phi(4\sqrt{2}\ln \sigma) < 10^{-4}\bar{r}x_{m+1}^2.
\]

(33)

For \( s \geq cm+\bar{x}_{m+1}, a_F(s) \leq cm + 3r_L\sigma(s - cm + 1) < s \), and \( \frac{d}{ds} a_F(s) \leq 3r_L(s - cm)^2 \). An
On the other hand, hence approach similar to (A) leads to
\[
\int_{c_m + \bar{x}_{m+1}}^{\infty} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 10^{-4}\bar{x}_1^2 d_{m+1}.
\]
(34)

On the other hand,
\[
\int_{m+1 - \frac{2\ln \sigma}{3}}^{c+1} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds \leq 4.4\bar{x}_{k+2}^2 \phi(m_{k+1} - \bar{a}_L - \sqrt{2\ln \sigma})
\]
\[
\leq 40 \times (2\sqrt{2\ln \sigma})^2 \phi\left(\frac{7\sqrt{2\ln \sigma}}{4}\right)
\]
\[
< 0.1\bar{x}^2 \sigma^2.
\]
(35)

for sufficiently large \(\sigma\).

If \(c_k \leq s \leq m_{k+1} - \frac{2\ln \sigma}{3}\), then \(a_F(s) < c_k + 5.8\bar{x}_k + 2\) and thus \((a_F(s) - \bar{a}_L)(s - \bar{a}_L) < 11\bar{x}_k + 5\bar{x}_k + 2 + \frac{2}{3k+1} < 2\). Overall, we arrive at
\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 0.4\bar{x}^2 \sigma^2.
\]
(36)

Similar argument holds for \(k = m\). The proof now follows from (A), (A), and (A).

**Proof of Lemma 10.** We have already calculated the second derivative of \(\tilde{J}_L\) when deriving the partial derivative in (A). The same argument implies \(\frac{d^2}{d\alpha_L} \tilde{J}_L(a_L) \geq 2(1 - 0.4\bar{x}^2 \sigma^2)\). This, in fact implies that \(\tilde{J}_L\) is strongly convex. It’s unique minimizer \(\bar{c}_k\) minimizes both losses in the leader’s payoff, hence it is the fixed point of \(\bar{a}_L(\theta)\), that is \(\bar{a}_L(\bar{c}_k) = \bar{c}_k\).

**Proof of Lemma 11.** The case \(\bar{b}_m < \theta \leq c_m\) follows from Lemma 9, so we only need to consider \(\theta > c_m\). As the first step, we derive some useful lower bounds on \(\tilde{J}_L(a_L)\) for \(a_L = c_m + \epsilon\) with \(\epsilon \geq 0\). In particular, we claim that for \(\epsilon > \bar{x}_{m+1}\) and sufficiently large \(\sigma\), we have
\[
\tilde{J}_L(a_L) \geq 0.99(\epsilon - \bar{x}_{m+1})^2.
\]
(37)

We consider two cases: If \(\epsilon \leq \frac{3}{4}\bar{x}_{m+1}\), then \(a_F(s) - c_m \leq \bar{x}_{m+1}\) for \(s \leq a_L + \frac{1}{4}\bar{x}_{m+1}\), and hence \(\tilde{J}_L(a_L) \geq (\epsilon - \bar{x}_{m+1})^2 \Phi\left(\frac{\bar{x}_{m+1}}{4}\right)\). If \(\epsilon > \frac{3}{4}\bar{x}_{m+1}\), then \(a_L - a_F(a_L + \frac{\bar{x}_{m+1}}{4} - 1) \geq (1 - 4r_L\sigma)\epsilon\), and hence \(\tilde{J}_L(a_L) \geq (1 - 4r_L\sigma)^2 \Phi\left(\frac{\bar{x}_{m+1}}{4} - 1\right)\epsilon^2\). These two observations result in the lower bound in (A) for sufficiently large \(\sigma\).

Consider now \(\theta = c_m + \delta, 0 \leq \delta\). We claim that \(\bar{a}_L(\theta) < c_m + 2.2r_L(\delta + \bar{x}_{m+1})\). Let \(\bar{a}_L = c_m + \epsilon\).
Then,
\[ \bar{u}_L(\theta, \bar{a}_L) \leq -r_L(\epsilon - \delta)^2 - 0.99(1 - r_L)(\epsilon - \bar{r}x_{m+1})^2. \]

Using an approach similar to Lemma 8, we can show that \( \epsilon < 2\epsilon^* \) where
\[ \epsilon^* = \frac{r_L\delta + 0.99(1 - r_L)\bar{r}x_{m+1}}{r_L + 0.99(1 - r_L)}, \]
from which we easily get \( \epsilon < 2.02r_L(\delta + \bar{x}_{m+1}) \). Using this, the proof of the second part of the lemma is quite straightforward.

As for \( \frac{d}{d\theta} \bar{a}_L(\theta) \), the case \( \bar{a}_L(\theta) \leq c_m + 5\bar{r}x_{m+1} \) is covered in Lemma 9. Hence, we study the case \( \bar{a}_L(\theta) > c_m + 5\bar{r}x_{m+1} \). Noting \( \bar{a}_L(\theta) - c_m \leq 2.2r_L(\sigma\bar{x}_{m+1} + 5\bar{r}x_{m+1} + \bar{x}_{m+1}) < 2.4 \) and similar to (A), we can obtain
\[ \int_{c_m + \bar{x}_{m+1}}^{\infty} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L)ds < 10^{-4}\bar{r}^2\bar{x}_{m+1}^2. \]

Also,
\[ \int_{c_m}^{c_m + \bar{x}_{m+1}} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L)ds \leq 0.8\bar{r}^2\bar{x}_{m+1}^2\Phi(\bar{a}_L - c_m). \]

On the other hand, similar to (A), we have
\[ \int_{-\infty}^{c_m} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L)ds < 10^{-4}\bar{r}^2\bar{x}_{m+1}^2 + 1.01\bar{r}^2\bar{x}_{m+1}^2\Phi(c_m - \bar{a}_L). \]

Therefore,
\[ \int_{-\infty}^{\infty} \frac{d}{ds}a_F(s)\phi(s - \bar{a}_L)ds < 0.91\bar{r}^2\bar{x}_{m+1}^2. \]

To bound the other term, using \( \bar{a}_L - a_F(s) < 2.2r_L(\sigma\bar{x}_{m+1} + \bar{x}_{m+1} + 5r_L\sigma) + 1.5\bar{r}x_{m+1} < 2.5 \) for \( s \geq c_m + \bar{x}_{m+1} \) and similar to (A), we can show that
\[ \int_{c_m + \bar{x}_{m+1}}^{\infty} \frac{d}{ds}a_F(s)(\bar{a}_L - a_F(s))(s - \bar{a}_L)\phi(s - \bar{a}_L)ds < 10^{-4}\bar{r}^2\bar{x}_{m+1}^2. \]
For \( s \leq c_m + \bar{x}_{m+1} \), we have \( a_F(s) < \bar{a}_L \). Therefore,

\[
\int_{-\infty}^{c_m+\bar{x}_{m+1}} \frac{d}{ds} a_F(s)(\bar{a}_L - a_F(s))(s - \bar{a}_L)\phi(s - \bar{a}_L)ds \\
\leq \int_{\bar{a}_L}^{c_m+\bar{x}_{m+1}} \frac{d}{ds} a_F(s)(\bar{a}_L - a_F(s))(s - \bar{a}_L)\phi(s - \bar{a}_L)ds \\
\leq \frac{6.5}{\sqrt{2\pi}} \times 0.8r^2 \bar{x}_{m+1}^2.
\]

Putting all together, we get

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)(2 + (\bar{a}_L - a_F(s))(s - \bar{a}_L))\phi(s - \bar{a}_L)ds < 3.9r^2 \bar{x}_{m+1}^2 < 0.4r^2 \sigma^2. \tag{38}
\]

For the other side, similar to (A), we have

\[
\int_{c_m+\bar{x}_{m+1}}^{\infty} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 10^{-4}r^2 \bar{x}_{m+1}^2.
\]

Also, same as (A),

\[
\int_{-\infty}^{c_m-1} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 10^{-4}r^2 \bar{x}_{m+1}^2.
\]

And similar to (A),

\[
\int_{m_m+\sqrt{\ln \sigma}}^{m_m+1} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 0.1r^2 \sigma^2.
\]

Now, for \( s \in [m_m + \sqrt{\ln \sigma}, c_m + \bar{x}_{m+1}] \), it is easy to verify that \( (a_F(s) - \bar{a}_L)(s - \bar{a}_L) \leq 2 \) for \( s \in [\bar{a}_L, c_m + \bar{x}_{m+1}] \). On the other hand, using Lemma 6 and 7, we can verify that \( \frac{d}{ds} a_F(s) \leq 1.1r^2 \bar{x}_{m+1} \) for \( s \in [m_m + \sqrt{\ln \sigma}, \bar{a}_L] \). This yields

\[
\int_{m_m+\sqrt{\ln \sigma}}^{\bar{a}_L} \frac{d}{ds} a_F(s)((a_F(s) - \bar{a}_L)(s - \bar{a}_L) - 2)\phi(s - \bar{a}_L)ds < 1.1r^2 \bar{x}_{m+1}^2.
\]

From the above analysis it follows that

\[
\int_{-\infty}^{\infty} \frac{d}{ds} a_F(s)((a_F(s) - a_L)(s - a_L) - 2)\phi(s - a_L)ds < 0.1r^2 \sigma^2 + 1.2r^2 \bar{x}_{m+1}^2 < 0.4r^2 \sigma^2.
\]

Using this and (A), and following exact same steps as in the proof of Lemma 9, we can show that \( \bar{r} \leq \frac{d}{d\theta} \alpha_L(\theta) \leq \tilde{r} \).

**Proof of Lemma 12.** We start by finding an upper bound for \( \tilde{J}_L(\tilde{c}_k) \). We use \( \tilde{J}_L(\tilde{c}_k) \leq \tilde{J}_L(s_k) \),
where we recall that \( a_F(s_k) = s_k \). Noting that the upper bound on the derivative of \( a_F(s) \) in (1) is increasing for \( s_k \leq s \leq m_{k+1} \), we can write

\[
\int_{s_k}^{m_{k+1}} (a_F(s) - a_F(s_k))^2 \phi(s - s_k) ds \leq \frac{1.012^2 \bar{x}^4_{k+2}}{2} + 2.4 \bar{x}^2_{k+2} \Delta^2_{k+1} \frac{(m_{k+1} - s_k)^2 \phi(m_{k+1} - s_k)}{\Delta_{k+1} - m_{k+1} + s_k} + 1.37 \Delta^4_{k+1} \frac{(m_{k+1} - s_k)^2 \phi(m_{k+1} - s_k)}{2 \Delta_{k+1} - m_{k+1} + s_k} \leq \frac{1.012^2 \bar{x}^4_{k+2}}{2} + (9.6 \bar{x}^2_{k+2} x_1^3 + 7.31 x_1^5) \phi(x_1) \leq \frac{1.012^2 \bar{x}^4_{k+2}}{2} + \frac{(9.6 \bar{x}^2_{k+2} x_1^3 + 7.31 x_1^5) e^{-\left(x_1 - 1\right)}}{\sqrt{2 \pi} \sigma^4}.
\]

Similarly, using (III-B) and Lemma 7 we can show that

\[
\int_{m_{k+1}}^{\infty} (a_F(s) - s_k)^2 \phi(s - s_k) ds < \frac{2 x_1^5 e^{-\left(x_1 - 1\right)}}{\sqrt{2 \pi} \sigma^4}.
\]

These two yield \( \tilde{J}_L(s_k) < 0.2 \bar{r}^2_L \) for sufficiently large \( \sigma \). It is easy to verify that the same hold when \( k = m \). The analysis in this case is even simpler on noting that \( \frac{d}{ds} a_F(s) \leq 0.8 \bar{r}^2 \bar{x}_{m+1}^2 \) over the whole interval \( s \in [c_m, c_m + \bar{x}_{m+1}] \).

Applying the Envelope’s theorem to (III-B), we get

\[
\frac{d}{d \theta} \bar{u}_L(\theta, \bar{a}_L(\theta)) = 2 r_L(\theta - \bar{a}_L(\theta)).
\]

Integrating this, along the inequality below

\[
r \theta + (1 - r) \bar{c}_k \leq \bar{a}_L(\theta) \leq \bar{r} \theta + (1 - \bar{r}) \bar{c}_k,
\]

we get

\[
r_L(1 - \bar{r})(\theta - \bar{c}_k)^2 \leq \bar{u}_L(\theta, \cdot) - \bar{u}_L(\bar{c}_k, \cdot) \leq r_L(1 - \bar{r})(\theta - \bar{c}_k)^2,
\]

where we use \( \bar{u}_L(\theta, \cdot) \) as a short-note for \( \bar{u}_L(\theta, \bar{a}_L(\theta)) \). Now, we note that at the endpoint \( \theta = \bar{b}_{k+1} \), the above should also hold for \( \bar{c}_{k+1} \). Using this and noting \( 0 < \bar{u}_L(\bar{c}_k, \cdot) < 0.2 r^2_L \) and \( 0 < \bar{u}_L(\bar{c}_{k+1}, \cdot) < 0.2 r^2_L \), we can arrive at

\[
(1 - r_L)|(\bar{b}_{k+1} - \bar{c}_{k+1})^2 - (\bar{b}_{k+1} - \bar{c}_k)^2| < 0.2 r_L + \frac{r^2_L}{2} (\bar{c}_{k+1} - \bar{c}_k)^2,
\]
using which the proof of the lemma is trivial. ■

Proof of Lemma 13. Using Corollary 2, it is straightforward to show that $|s_k - c_k| < 1.1\bar{r}\bar{x}_{m+1}$.

Evaluating the derivative of $\tilde{J}_L(a_L)$ at $a_L = s_k$, we get

$$\frac{d}{da_L} \tilde{J}_L(s_k) = 2\int_{-\infty}^{\infty} (a_F(s) - a_F(s))(1 - \frac{d}{ds} a_F(s))\phi(s - s_k)ds,$$

where we have also used $s_k = a_F(s_k)$. We consider the case $s_k \leq \bar{c}_k$ yielding $\frac{d}{da_L} \tilde{J}_L(s_k) \leq 0$ (the other case is quite similar). Noting that for $s < s_k$, $\frac{d}{ds} a_F(s) > 1$ requires $s < m_k + 0.5$, we can obtain

$$-\frac{d}{da_L} \tilde{J}_L(s_k) \leq 2 \int_{s_k}^{\infty} (a_F(s) - a_F(s_k))\phi(s - s_k)ds + 2 \int_{-\infty}^{m_k + 0.5} (a_F(s) - a_F(s))(1 - \frac{d}{ds} a_F(s))\phi(s - s_k)ds.$$

With a bit of manipulation, we can obtain

$$\int_{s_k}^{\infty} (a_F(s) - a_F(s_k))\phi(s - s_k)ds \leq \frac{1.01\bar{r}^2(\bar{x}_k + \bar{x}_{k+1})^2}{4\sqrt{2\pi}} + \frac{4.68x_1^2e^{-(x_1-1)}\sqrt{2\pi}}{2\pi\sigma^4}.$$

As for the second term, similarly

$$\int_{-\infty}^{m_k + 0.5} (a_F(s) - a_F(s))(1 - \frac{d}{ds} a_F(s))\phi(s - s_k)ds < (1 + \bar{r})^2\bar{x}_{m+1}^2\phi(2\bar{x}_1 - 1.1\bar{r}\bar{x}_{m+1})$$

$$+ (1 + \bar{r})^2\bar{x}_1^2\phi(\bar{x}_1 - 0.5 - 1.1\bar{r}\bar{x}_{m+1}).$$

By putting all together it is easy to verify that for sufficiently large $\sigma$

$$0 \leq -\frac{d}{da_L} \tilde{J}_L(s_k) \leq \frac{1.01\bar{r}^2(\bar{x}_k + \bar{x}_{k+1})^2}{2\sqrt{2\pi}} + 0.1\bar{r}^2\bar{x}_1.$$

On the other hand, as we showed before $\frac{d^2}{da_L^2} \tilde{J}_L(s_k) \geq 2(1 - 0.4\bar{r}^2\sigma^2)$, meaning that $s_k$ is at most

$$\frac{1.01\bar{r}^2(\bar{x}_k + \bar{x}_{k+1})^2}{4\sqrt{2\pi}(1 - 0.4\bar{r}^2\sigma^2)} + \frac{0.1\bar{r}^2\bar{x}_1}{2(1 - 0.4\bar{r}^2\sigma^2)} < 0.42\bar{x}_1^2\frac{(x_k + x_{k+1})^2}{2} + 0.08r_1^2d_1$$

away from the minimizer at which the first derivative is zero. This completes the proof. ■

Proof of Lemma 14. We start by finding the fixed point of $E[a_L(\theta)|s, \theta \in B_k]$, that is $E[a_L(\theta)|\hat{s}_k, \theta \in B_k] = \hat{s}_k$. $\hat{s}_k$ is the solution of the following equation

$$\int_{b_k}^{b_{k+1}} (a_L(\theta) - \hat{s}_k)\phi(a_L(\theta) - \hat{s}_k)\phi(\theta\sigma)d\theta = 0.$$
Noting that \( \hat{s}_k \) is close to \( c_k \) and in particular \(|a_L(\theta) - \hat{s}_k| < 1 \) for \( \theta \in B_k \), and that \( x\phi(x) \) is increasing for \(|x| < 1 \), together with the fact that \( a_L(\theta) \leq a_L(b_k) + \bar{r}(\theta - b_k) \), we obtain
\[
\int_{b_k}^{b_{k+1}} (\bar{r}(\theta - b_k) - (\hat{s}_k - a_L(b_k))) \phi(\bar{r}(\theta - b_k) - (\hat{s}_k - a_L(b_k))) \phi(\frac{\theta}{\sigma}) d\theta > 0.
\]
Therefore by finding the solution of
\[
\int_{b_k}^{b_{k+1}} \bar{r}(\theta - y) \phi(\bar{r}(\theta - y)) \phi(\frac{\theta}{\sigma}) d\theta = 0, \tag{39}
\]
we can upper-bound the fixed point \( \hat{s}_k \) as \( \hat{s}_k \leq a_L(b_k) + \bar{r}(y - b_k) \). Simplifying (A) yields
\[
y = E_{\bar{\psi}_y}[\theta | \theta \in B_k], \tag{40}
\]
where \( \bar{\psi}_y \sim N(\frac{\bar{r}^2 \sigma^2}{1 + \bar{r}^2 \sigma^2} y, \frac{\sigma^2}{1 + \bar{r}^2 \sigma^2}) \). A quick bound for \( y \) can be obtained from \( E_{\bar{\psi}_y}[\theta | \theta \in B_k] \leq e_k + \frac{\bar{r}^2 \sigma^2}{1 + \bar{r}^2 \sigma^2} y \), which yields \( y - e_k \leq \bar{r}^2 \sigma^2 e_k < 0.1 \bar{x}_k \). An alternative representation of (A) is
\[
\frac{y}{\sqrt{1 + \bar{r}^2 \sigma^2}} = E_{N(0, \sigma^2)}[\theta | b_k - \Delta b_k(y) \leq \theta \leq b_{k+1} - \Delta b_k(y) + \frac{\bar{r}^2 \sigma^2(b_{k+1} - b_k)}{1 + \sqrt{1 + \bar{r}^2 \sigma^2}}],
\]
where \( \Delta b_k(y) = \frac{\bar{r}^2 \sigma^2}{1 + \bar{r}^2 \sigma^2 + \sqrt{1 + \bar{r}^2 \sigma^2}} y + \frac{\bar{r}^2 \sigma^2}{1 + \sqrt{1 + \bar{r}^2 \sigma^2}} (y - b_k) > 0 \). A useful property of normal distribution is that \( \frac{\partial}{\partial a} \frac{\partial}{\partial b} E_{N(0, \sigma^2)}[\theta | a \leq \theta \leq b] = 1 - \text{Var}_{N(0, \sigma^2)}[\theta | a \leq \theta \leq b] \geq 1 - \frac{(b-a)^2}{(1+\sqrt{3})^2 \sigma^2} \).

Also, \( \frac{\partial}{\partial a} E_{N(0, \sigma^2)}[\theta | a \leq \theta \leq b] \leq \frac{1}{2} \). Applying these to the above equation we can obtain
\[
0 \leq \frac{y}{\sqrt{1 + \bar{r}^2 \sigma^2}} + \Delta b_k(y) - e_k \leq \frac{\bar{r}^2 \sigma^2(b_{k+1} - b_k)}{2(1 + \sqrt{1 + \bar{r}^2 \sigma^2})} + \frac{(b_{k+1} - b_k)^2}{(1 + \sqrt{3})^2 \sigma^2} \Delta b_k(y),
\]
where \( e_k = E_{N(0, \sigma^2)}[\theta | \theta \in B_k] \). Simplifying this, along with \( y(b_{k+1} - b_k) < 2.2 \sigma^2 \) (which follows from \( \epsilon_m^0(\epsilon_m^0 - b_m^0) \leq 1 \)), we can arrive at
\[
|y - e_k| \leq 0.4 \bar{r}^2 \sigma^2 (b_{k+1} - b_k).
\]
Recalling \( \hat{s}_k \leq a_L(b_k) + \bar{r}(y - b_k) \), and that \( a_L(b_k) \leq c_k \) and \( c_k - r(c_k - b_k) \leq c_k - r_L(c_k - b_k) + (r_L - r) \bar{x}_k \), we can reach at
\[
\hat{s}_k \leq (1 - r_L)c_k + r_L e_k + r_L^2 \bar{x}_k + 0.4 \bar{r}^3 \sigma^2 (\bar{x}_k + \bar{x}_{k+1}). \tag{41}
\]
Following a similar argument to lower-bound \( \hat{s}_k \), we can show that \( |\hat{s}_k - (1 - r_L)c_k - r_L e_k| \leq r_L^2 \bar{x}_k + 0.4 \bar{r}^3 \sigma^2 (\bar{x}_k + \bar{x}_{k+1}) \).

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To find the fixed point of $a_F(s)$ in $B_k$ (that is $a_F(s_k) = s_k$), we first note that $s_k$ lies within the interval $[c_k - 2\bar{\bar{x}}_{m+1}, c_k + 2\bar{\bar{x}}_{m+1}]$. Moreover, for $s \in [c_k - 2\bar{\bar{x}}_{m+1}, c_k + 2\bar{\bar{x}}_{m+1}]$, using Lemma 6 we can obtain $\frac{d}{ds}a_F(s) < 0.01$. This along with $|a_F(s_k) - s_k| < 1.5\bar{\bar{x}}_{m+1}$ implies that for $s \in [c_k - 2\bar{\bar{x}}_{m+1}, c_k + 2\bar{\bar{x}}_{m+1}]$, we have $a_F(s) \in [c_k - 2\bar{\bar{x}}_{m+1}, c_k + 2\bar{\bar{x}}_{m+1}]$. Therefore,

$$ |s_k - s_k| < \frac{|s_k - a_F(s_k)|}{1 - 0.01} = \frac{|\mathbb{E}[a_L(\theta)|s_k, \theta \in B_k] - \mathbb{E}[a_L(\theta)|\hat{s}_k]|}{1 - 0.01}. $$

Assume $s_k \geq c_k$ (the other case is similar). We have already shown as part of Lemma 6 that while observing $s \in [c_k, c_k+1]$, the effect of intervals other than $B_k \cup B_{k+1}$ on $a_F(s)$ is negligible (as given by (A)). Similarly, and by using Lemma 5, we can show that

$$ \text{Prob}\{\theta \in B_{k+1} | \hat{s}_k\} \left[ |\mathbb{E}[a_L(\theta)|s_k, \theta \in B_k] - \mathbb{E}[a_L(\theta)|\hat{s}_k] \right] \leq 1.17e^{-\Delta_{k+1}}(\Delta_{k+1} + 2\bar{\bar{x}}_{m+1}), $$

where $\delta = m_{k+1} - \hat{s}_k$. Combining this and (A), we can arrive at

$$ |\mathbb{E}[a_L(\theta)|s_k, \theta \in B_k] - \mathbb{E}[a_L(\theta)|\hat{s}_k]| < 10^{-4}r^2_L\bar{x}_1. $$

After all, we have

$$ |s_k - (1 - r_L)c_k - r_L e_k| < 1.02r^2_L\bar{x}_k + 0.41\bar{\bar{x}}^2(\bar{x}_k + \bar{x}_{k+1}) < 1.9r^2_L\bar{x}_{k+1}. \quad (42) $$

The proof has to be modified for the tail case $k = m$, since in the tail $|a_L(\theta) - \hat{s}_m| < 1$ does not hold for all $\theta \in B_m$. To handle this, we define $\mathbb{E}[a_L(\theta)|\hat{s}_m, \theta \in \hat{B}_m] = \hat{s}_m$, where $\hat{B}_m = [b_m, b_m + \sigma^2 - \bar{x}]$. It is easy to verify that $\bar{r}(\sigma^2 - \bar{x}) < 1$ and hence $|a_L(\theta) - \hat{s}_m| < 1$ holds in this interval. We also need to find an alternative to the variance inequality $\text{Var}_{N(0, \sigma^2)}[\theta|\theta \in B_k] \leq \frac{(b_{k+1} - b_k)^2}{(1 + \sqrt{3})^2 \sigma^2}$. Here we use $\text{Var}_{N(0, \sigma^2)}[\theta|\theta \in B_m] \leq (e_m - b_m)^2$. Also, we can show $\Delta b_m(y) < \bar{r}e_m$, from which and a bit manipulation we can reach at $\text{Var}_{N(0, \sigma^2)}[\theta|\theta \geq b_m - \Delta b_m(y)] \leq \frac{(e_m - b_m)^2}{(1 - \bar{r})^2}$. We can then show that the results derived above also holds for $\hat{B}_m$. More precisely,

$$ |\hat{s}_m - (1 - r_L)c_m - r_L\mathbb{E}_{N(0, \sigma^2)}[\theta|\theta \in \hat{B}_m]| < r^2_L\bar{x}_m + 0.4\bar{\bar{x}}^2(\bar{x}_m + \bar{x}_{m+1}). $$
Let $\theta_b = b_m + \sigma^2 - \sigma$. We can easily observe that

$$e_m - \mathbb{E}_{N(0,\sigma^2)}[\theta|\theta \in \hat{B}_m] \leq \text{Prob}[\theta \geq \theta_b|\theta \in B_m](\mathbb{E}_{N(0,\sigma^2)}[\theta|\theta \geq \theta_b] - \mathbb{E}_{N(0,\sigma^2)}[\theta|\theta \in \hat{B}_m])$$

$$\leq \frac{\phi(\frac{\theta_b}{\sigma})}{\phi(\frac{b_m}{\sigma})}(\theta_b + \frac{\sigma^2}{\theta_b} - b_m) < 10^{-4}r_L^3 \sigma.$$ 

We also need to bound $|\mathbb{E}[a_L(\theta)\hat{s}_m, \theta \in \hat{B}_m] - \mathbb{E}[a_L(\theta)\hat{s}_m]|$. We start with $|\mathbb{E}[a_L(\theta)\hat{s}_m, \theta \in \hat{B}_m] - \mathbb{E}[a_L(\theta)\hat{s}_m, \theta \in B_m]|$. We first note that

$$a_L(\theta_b) \geq c_m + \epsilon(\sigma^2 - 2\sigma) > c_m + 3\bar{r}\sigma,$$

which along $|\hat{s}_m - c_m| < \bar{r}\sigma$ implies that $|a_L(\theta_b) - \hat{s}_m| > 2\bar{r}\sigma$. Consequently, $|\hat{s}_m - a_L(\theta')| < |a_L(\theta_b) - \hat{s}_m|$ for all $\theta' \in \hat{B}_m$. Similar to (A), we can hence derive

$$\text{Prob}[\theta|\hat{s}_m, \theta \in B_m] \leq \frac{e^{-\frac{(\sigma-1)^2}{2} - \phi(\frac{\theta}{\sigma})}}{\sigma(1 - \Phi(\frac{\theta_b}{\sigma}))},$$

for $\theta \geq \theta_b$. Using this along with $\mathbb{E}[a_L(\theta)|\hat{s}_m, \theta \in \hat{B}_m] \geq c_m - \bar{r}\sigma$, we can then obtain

$$\mathbb{E}[a_L(\theta)|\hat{s}_m, \theta \in B_m] - \mathbb{E}[a_L(\theta)|\hat{s}_m, \theta \in \hat{B}_m] < 10^{-4}r_L^3 \sigma.$$ 

Therefore, the same steps as in the non-tail case can be followed in this case as well, resulting in (A) to also hold for $k = m$. 

**Proof of Theorem 1.** The first step is to use (III-B) to verify that Property 2 is also preserved by the best response, completing the proof of the invariance of $A_L^m$. From (III-B) we get

$$|\tilde{c}_k - c_k^0| \leq (1 - r_L)|c_k - c_k^0| + r_L|\hat{e}_k - c_k^0| + 0.42r_L^3(\frac{\bar{x}_k + \bar{x}_{k+1}}{2})^2 + 2r_L^2\bar{x}_{k+1}.$$ 

Thus to prove that Property 2 is preserved under the best response, it suffices to show that

$$|\hat{e}_k - c_k^0| + 0.42r_L(\frac{\bar{x}_k + \bar{x}_{k+1}}{2})^2 + 2r_L\bar{x}_{k+1} \leq 2.9. \quad (43)$$

To bound $|\hat{e}_k - c_k^0|$, we first note that $\frac{\partial}{\partial b_k}\hat{e}_k + \frac{\partial}{\partial b_{k+1}}\hat{e}_k = 1 - \frac{\text{Var}_{N(0,\sigma^2)}[\theta|\theta \in \hat{B}_k]}{\sigma^2}$. Using the inequality $\text{Var}[\theta|\theta \in \hat{B}_k] > (\hat{e}_k - \mathbb{E}[\theta|\hat{b}_k \leq \theta \leq \hat{e}_k])(\mathbb{E}[\theta|\hat{e}_k \leq \theta \leq \hat{b}_{k+1}] - \hat{e}_k)$ and Lemma 1 we can show
yielding $|\hat{e}_k - e^0_k| < 2.9(1 - \frac{\bar{x}_k \bar{x}_{k+1}}{2(1+\sqrt{e})})$. Therefore to have (A), it suffices to have

$$\frac{0.42(\bar{x}_k + \bar{x}_{k+1})^2}{4\bar{x}_k \bar{x}_{k+1}} + \frac{2\bar{x}_{k+1}}{\bar{x}_k \bar{x}_{k+1}} \leq \frac{2.9}{2(1+\sqrt{e})},$$

which is satisfied for sufficiently large values of $\sigma$ on noting that $\frac{0.42(e+1)^2}{4e} \leq \frac{2.9}{2(1+\sqrt{e})}$. This completes the proof of the invariance of $A^m_L$. To show the existence of an equilibrium with the leader’s strategy in $A^m_L$, consider the pair of strategies $(a^*_L(\theta), a^*_F(s))$ maximizing the ex-ante expected payoff of the leader, where $a_L(\theta) \in A^m_L$ and $a_F(s) \in C^\omega$, that is,

$$(a^*_L(\theta), a^*_F(s)) \in \arg \max_{a_L \in A^m_L, a_F \in C^\omega} E[\theta][u_L(\theta, a_L(\theta)) | a_F(\cdot)]. \quad (44)$$

Then, using the invariance property of $A^m_L$, we can show that $(a^*_L(\theta), a^*_F(s))$ is indeed an equilibrium of the game. To see this, we first note that $a^*_F(s)$ almost surely matches the best response of the follower to the leader’s strategy $a^*_L(\theta)$. Otherwise, replacing it with the best response will improve the ex-ante expected payoff of the leader. Noting that both $a^*_F(s)$ and the best response to $a^*_L(\theta)$ are analytic, almost surely equivalence implies exact equivalence. Similarly, $a^*_L(\theta)$ has to coincide with the best response to $a^*_F(s)$ (which again lies in $A^m_L$ due to the invariance of $A^m_L$) almost surely. If not, the best response of the leader to $a^*_F(s)$ will improve the ex-ante expected payoff of the leader. If two strategies in $A^m_L$ match almost surely, then they have to be exactly identical. To see this, first note that for two such strategies the segments have to be identical. Within each segment, both strategies are analytic and hence have to be identical if they coincide almost surely. This shows that the pair of strategies $(a^*_L(\theta), a^*_F(s))$ given by (A) are best responses to each other, and hence correspond to an equilibrium of the game.

**Proof of Lemma 15.** We need to prove that there does not exist an infinitesimal variation of $(a^*_L, a^*_F)$, namely $(a^0_L, a^0_F)$, for which $U(a^0_L, a^0_F) < U(a^*_L, a^*_F)$. Noting that $U(a^0_L, E[a_L^0 | s]) \leq U(a^*_L, a^*_F)$, we only need to consider the strategies in which the follower’s action is the expected action of the leader given the observation $s$ (i.e., $E[a_L^0 | s]$). The idea is to show that for a sufficiently small $\delta_L > 0$ and any strategy $a^0_F$ with $\|a^0_L - a^*_L\|_{\infty} < \delta_L$, the best response image obtained from $a^0_L \rightarrow a^0_F \rightarrow \tilde{a}_L^0$ lies in $A^m_L$. The proof then follows from the fact that
\[ U(\hat{a}_L^\delta, \mathbb{E}[\hat{a}_L^\delta|s]) \leq U(a_L^\delta, \mathbb{E}[a_L^\delta|s]), \]
and that \((a_L^\delta, a_F^\delta)\) is the minimizer of \(U\) over all pair of strategies \((a_L, a_F)\) with \(a_L \in A_L^n\) (see the proof of Theorem 1).

To prove the inclusion of \(\hat{a}_L^\delta\) in \(A_L^n\), it suffices to show that all the properties for the follower’s best response to a strategy in \(A_L^n\) given specifically by Lemma 6-7 and Lemma 14 also hold for \(a_F^\delta\), noting that these are all we need to deduce Property 1-3 for the leader’s best response (which define the set \(A_L^n\)). What is left is then to show that the properties for \(a_F^*\) given by Lemma 6-7 and Lemma 14 also hold for \(a_F^\delta(s) = \mathbb{E}[a_L^\delta|s]\) for sufficiently small \(\delta\). The proof easily follows from a couple of simple observations. First, it is straightforward to verify that all the bounds given for \(a_F^*\) in the aforementioned lemmas are indeed strict. Therefore, by recasting the corresponding inequalities as continuous functions of \(\delta_L\) we can ensure that all of them will still hold for sufficiently small \(\delta_L\). We elaborate on this in more details in what follows.

We start by verifying that Lemma 2-3 also hold for \(a_L^\delta\) for small enough \(\delta_L\). In Lemma 2,

\[
a_L^\delta(\theta) - c_k^* \leq \delta_L + \bar{r}(b_{k+1}^* - c_k^*) \leq \delta_L + 0.1\bar{r}_L + \bar{r}\frac{c_{k+1}^* - c_k^*}{2} \leq \delta_L + 0.1\bar{r}_L + \bar{r}(x_{k+1}^0 + 2.9) \leq \bar{r}x_{k+1},
\]

for small enough \(\delta_L\) where we recall that \(\bar{x}_{k+1} = x_{k+1}^0 + 3\). Similarly we can show that \(a_L^\delta(\theta) - c_k^* \geq -\bar{r}\bar{x}_k\), hence Lemma 2 also holds for \(a_L^\delta\). Next, we study the effect of \(\delta_L\) in Lemma 3. As for (A), using

\[
\frac{\phi(s - a_L^\delta(\theta))}{\phi(s - a_L^\delta(\theta))} \geq \frac{\phi(s - a_L^\delta(\theta))}{\phi(s - a_L^\delta(\theta))} e^{-\delta_L^2 - 2\delta_L x_{m+1}},
\]

the RHS of the inequality will be multiplied by \(e^{-\delta_L^2 - 2\delta_L x_{m+1}}\). As a result, the value of \(\xi\) in (A) will be multiplied by \(e^{\delta_L^2 + 2\delta_L x_{m+1}}\). (A) will then become

\[
\mathbb{E}[a_L^\delta(\theta)|s, b_m^* \leq \theta \leq \theta_c^*] - c_m^* \leq \delta_L + 1.2e^{\delta_L^2 + 2\delta_L x_{m+1}} \bar{r}\bar{x}_m < 0.75\bar{r}\bar{x}_m+1,
\]

for sufficiently small \(\delta_L\). For the bound on variance in (A)-(A), let \(\mu^* = \mathbb{E}[a_L^*(\theta)|s, b_m^* \leq \theta \leq \theta_c^*]\). Then,

\[
\text{Var}[a_L^\delta(\theta)|s, b_m^* \leq \theta \leq \theta_c^*] \leq \text{Var}[a_L^*(\theta)|s, b_m^* \leq \theta \leq \theta_c^*] + \delta_L^2 + 2\delta_L \sqrt{\text{Var}[a_L^*(\theta)|s, b_m^* \leq \theta \leq \theta_c^*]}
\]
Hence, \((A)\) becomes

\[ \text{Var}[\rho^0_L(\theta)|s, b^0_m \leq \theta \leq \theta^*_c] \leq 1.1e^{\frac{\delta^2_L + 2\delta_L \bar{x}_m + r^2}{\sigma^2}} + \delta^2_L + 2.2\delta_L \bar{x}_m < 1.2\bar{x}_m, \]

for sufficiently small \(\delta_L\). As for the modification required in the tail effect, \(e^{\frac{\delta^2_L}{\sigma^2}}\) in \((A)\) has to be replaced with \(e^{\frac{(\delta + \delta_L)^2}{\sigma^2}}\), using which we can verify that \((A)\) still holds for small enough \(\delta_L\). The rest of the changes are similar.

Lemma 4-5 are based on Lemma 2-3, and Lemma 6-7 are derived using Lemma 2-5, hence also hold for \(\rho^0_F\). Finally, in Lemma 14 which is about the fixed points of the follower’s strategy, noting \(\rho^0_L(b^*_k) \leq \rho^0_L(b^*_k) + \delta_L\), we need to add \(\delta_L\) to the RHS of \((A)\). Using this, we can easily verify that this lemma also holds for \(\rho^0_F\). Therefore, all the properties required for the follower’s strategy to deduce Property 1-3 for the leader’s best response are satisfied for \(\rho^0_F\) for sufficiently small \(\delta_L\), indicating that \(\rho^0_F\) lies in \(A^m_L\). This completes the proof. \(\blacksquare\)

**Proof of Lemma 16.** Using

\[ \lim_{m \to \infty} \frac{(x^0_m)^2}{D^0_L} = 1, \]

it suffices to show that \(D^0_F \leq 4\sqrt{2}\left(\frac{2-r_L}{1-r_L}\right)^2 \phi\left(\frac{x^0_0}{\sqrt{2}}\right) + r^2_L D^0_L\). Consider an interval \(B^0_k\) and some \(\theta \in B^0_k\). For any \(j > k\) (similarly for \(j < k\), we have

\[ \frac{(c^0_j - a^0_L(\theta))^2}{(b^0_j - a^0_L(\theta))^2} \leq \frac{(2x^0_j - r_L x^0_j)^2}{(x^0_j - r_L x^0_j)^2} = \frac{(2-r_L)^2}{(1-r_L)^2}. \]

Using this, we can obtain

\[ \int_{s \notin B^0_k} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta))ds \leq \frac{(2-r_L)^2}{(1-r_L)^2} \int_{s \notin B^0_k} (s - a^0_L(\theta))^2 \phi(s - a^0_L(\theta))ds. \]

Combining this with the inequality \(\int_a^\infty x^2 \phi(x)dx \leq 2 \max(xe^{-\frac{x^2}{2}})\phi\left(\frac{a}{\sqrt{2}}\right)\), we arrive at

\[ \int_{s \notin B^0_k} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta))ds \leq 4\sqrt{\frac{2}{e}} \phi\left(\frac{x^0_0}{\sqrt{2}}\right). \]

On the other hand,

\[ \int_{s \in B^0_k} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta))ds \leq (c^0_k - a^0_L(\theta))^2 = r^2_F(a^0_F(\theta) - \theta)^2, \]
implying that
\[
\int_{-\infty}^{\infty} \int_{s \in B_0^k} (a^0_F(s) - a^0_L(\theta))^2 \phi(s - a^0_L(\theta)) \frac{\phi(\theta)}{\sigma} ds d\theta \leq 
\]
\[
r_L^2 \int_{-\infty}^{\infty} (\theta - a^0_F(\theta))^2 \frac{\phi(\theta)}{\sigma} d\theta = r_L^2 D_0^k,
\]
which completes the proof. \(\blacksquare\)

**Proof of Lemma 17.** First we note that the minimum value of the cost functional \(U(a_L, a_F)\) with \(r_L \sigma^2 = 1\) is asymptotically the same as the optimal cost of Witsenhausen’s problem for \(k^2 \sigma^2 = 1\). Using the inequality given by (17) in the proof of Theorem 4 in Grover et al. (2013), we can obtain
\[
U^*(\sigma) > \min_{P^* > 0.5} \{k^2 P^* + \frac{1}{15} e^{-12 P^*}\},
\]
noting that in the scalar version of Witsenhausen’s problem we have \(m = 1\). Minimizing the RHS above we can find \(U^*(\sigma) > \frac{\ln \sigma}{6 \sigma^2} + \frac{1 - \ln 1.25}{12 \sigma^2}\), which completes the proof. \(\blacksquare\)

**Proof of Theorem 2.** The first part of the theorem follows directly from Lemma 15. Using
\[
M(\sigma) = \{m \in \mathbb{N} | 2\sqrt{2 \ln \sigma} + 4 < x_1^0 < 4 \sqrt{\ln \sigma}\}
\]
and that \(m \frac{x_1^0}{\sigma} \approx \sqrt{\frac{6\pi}{4}}\) for large \(m\), we get
\[
M(\sigma) \approx \{m \in \mathbb{N} | \sqrt{\frac{6\pi \sigma}{16 \sqrt{\ln \sigma}}} < m < \frac{\sqrt{6\pi \sigma}}{8 \sqrt{2 \ln \sigma} + 16}\},
\]
which is clearly nonempty for sufficiently large \(\sigma\). Denote with \(x_1^*\) the minimizer of the asymptotic upper bound on \(U(a_L^*, a_F^*)\) given by Lemma 16. Then, it is easy to verify that \(\lim_{\sigma \to \infty} \frac{x_1^*}{2 \sqrt{2 \ln \sigma}} = 1\), which clearly intersects \(M(\sigma)\) for large \(\sigma\). The corresponding asymptotic cost is \(\approx \frac{8 r_L \ln \sigma}{\sqrt{3}}\), hence proving (2). We can use Lemma 16 and Lemma 17 to show that equilibrium strategies corresponding to \(m \in M(\sigma)\) are within a constant factor of the optimal cost. Specially, the equilibrium corresponding to \(\arg \min_{m \in M(\sigma)} U(a_L^*, a_F^*)\) is at most
\[
\frac{8 r_L \ln \sigma}{\sqrt{3} \frac{\ln \sigma}{6 \sigma^2}} = 16 \sqrt{3} < 27.8
\]
away from the optimal cost as \(\sigma \to \infty\). \(\blacksquare\)