ON THE FORCED EULER AND NAVIER-STOKES EQUATIONS: LINEAR DAMPING AND MODIFIED SCATTERING

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Abstract. We study the asymptotic behavior of the forced linear Euler and nonlinear Navier-Stokes equations close to Couette flow on $T \times I$. As our main result we show that for smooth time-periodic forcing linear inviscid damping persists, i.e. the velocity field (weakly) asymptotically converges. However, stability and scattering to the transport problem fail in $H^s, s > -1$. We further show that this behavior is consistent with the nonlinear Euler equations and that a similar result also holds for the nonlinear Navier-Stokes equations. Hence, these results provide an indication that nonlinear inviscid damping may still hold in Sobolev regularity in the above sense despite the Gevrey regularity instability results of [DM18].

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1. Introduction

The problems of inviscid damping and enhanced dissipation are classical, going back to the works of [Orr07] and Rayleigh [Ray79] around 1900, who studied the linearized Euler and Navier-Stokes equations around Couette flow, $v = (U(y), 0)$. In that case, the linearized inviscid problem in terms of the vorticity is given by

$$\partial_t \omega + y \partial_x \omega = 0$$

and explicitly solvable by the method of characteristics as $\omega(t, x, y) = \omega_0(x - ty, y)$. In particular, one observes that the vorticity converges weakly (but not strongly) in $L^2$ to its $x$ average and as a consequence, the corresponding velocity field converges strongly to a shear flow. This in view of the conservation law structure of Euler’s equations at first very unexpected stabilization mechanism is known as inviscid damping in analogy to Landau damping in plasma physics. However, while the result for Couette flow is classical, until recently little has been known about linearizations around other profiles or the associated nonlinear problem. Following the seminal works of Mouhot and Villani on Landau damping for the Vlasov-Poisson
equation, [MV11], [MV10a], [MV10b], [Lan46], [BMM16] these problems have attracted much renewed interest. For a more extensive discussion of the literature we refer to [BMV16] and briefly mention the following works:

- The problem of nonlinear inviscid damping and enhanced dissipation for Couette flow in an infinite periodic channel has been studied in series of works by Bedrossian, Masmoudi, Vicol, Wang and others [BMV16], [BVW16], [BM15], [BGM15a], [BGM15b], [BMV16], [BM14], [BM13]. Here, in the inviscid setting Gevrey regularity has been shown to be necessary to control the effect of nonlinear resonances, also called *echos*, [DM18].

- Concerning the linearized problem around profiles different from Couette efforts have focused on establishing inviscid damping and enhanced dissipation for more general and degenerate classes of profiles. Here, the works of the author and Coti Zelati, [Zil17b], [Zil17a], [Zil16], [CZZ18] rely on a dispersive approach using multiplier and decompositions. Using a spectral approach Wei, Zhang and Zhao have been able to establish linear inviscid damping for different, more general classes of profiles, [WZZ15], [WZZ18], [WZ18], [WZZ17], [Ste95].

- When considering the setting of a finite periodic channel, $\mathbb{T} \times [0,1]$, or other domains with boundaries one additionally encounters instabilities due to boundary effects, which generally restrict to working with fractional Sobolev spaces $H^s, s < \frac{1}{2}$ or weighted $H^2$ spaces, [Zil16], [Zil17b], [WZZ15], [CZZ18]. This is of particular interest in view of the much higher regularity requirements of present results on the nonlinear problem and known lower bounds on the required regularity [LZ11]. Here, recently Ionescu and Jia [IJ18] have been able to show that for Gevrey regular vorticity compactly supported away from the boundary these blow-up mechanisms can be avoided and nonlinear inviscid damping holds.

- As a related problem damping, mixing and enhanced dissipation for passive scalar problems are an active area of research [ZDE18], [ACM14], [CS17], [CLS17], [Zil18].

Following the results of Bedrossian-Masmoudi [BM15] on inviscid damping in Gevrey regularity, it has been shown by Deng-Masmoudi in [DM18] that Gevrey regularity is necessary for (asymptotic) stability with respect to the linearized dynamics in the setting of an unbounded channel.

However, as also noted there, it remains an open question whether inviscid damping, i.e. convergence in $H^{-1}$, fails otherwise. This question is particularly relevant for the setting of a finite channel, where generally boundary instabilities yield asymptotic blow-up as $t \to \infty$ in $W^{1,\infty}$ and consequently in relatively low Sobolev norms [Zil16].

In this work we make a first modest step in the direction of addressing this question and consider the behavior of the forced equations for shear flows close to Couette flow in a periodic channel:

$$
\begin{align*}
\partial_t v + v \cdot \nabla v &= \nabla p + F + \nu \Delta v, \\
\nabla \cdot v &= 0.
\end{align*}
$$

The velocity field of the fluid is denoted by $v \in \mathbb{R}^2$ and $F \in \mathbb{R}^2$ is a given force field. The pressure $p$ can be interpreted as a Lagrange multiplier ensuring the incompressibility, i.e. $p$ is determined by solving

$$
\Delta p = \nabla \cdot (v \cdot \nabla v) - \nabla \cdot F.
$$

In this article we consider
• the linearized forced Euler equations on $T \times I$ around shear profiles $U(y)$ satisfying linear inviscid damping (c.f. properties (18) to (20) in Section 3),
• the associated consistency problem and
• the nonlinear Navier-Stokes equations on $T \times \mathbb{R}$.

The choice of a bounded interval $I$ for the inviscid setting here allows us to restrict to the case when $U(y)$ is bounded above and below and so that we may restrict to studying unweighted spaces. However, as a severe drawback the comparably low Sobolev regularity available in this setting, makes the consistency problem of Section 3 very challenging. In Section 4, when considering the Navier-Stokes equations, we instead restrict to the case without boundary, $T \times \mathbb{R}$.

We study stability in Sobolev regularity, where we consider smooth deterministic forcing of the following types, where $f = \nabla \times F$ (or the perturbation thereof):

• $f \in H^1_{\text{loc}}(\mathbb{R}^+, H^1(T \times I))$ is of “stationary type” if it is periodic in $t$ with period $T > 0$.
• $f \in H^1_{\text{loc}}(\mathbb{R}^+, H^1(T \times I))$ is “resonant” if $f(t, x + tU(y), y)$ is periodic in $t$ with period $T > 0$.

The types of forcing considered here are intended to mimic the effect of echoes and thus provide insights into what kind of (in)stability results and whether (modified) scattering results can be expected. In particular, we show that a control in just $H^1_{\text{loc}}(\mathbb{R}^+, H^1(T \times I))$ is “too rough” to capture cancellation behavior and that inviscid damping may persists despite instability.

1.1. Main results. As our first main result, we show that for the forced linearized inviscid problem

$$\frac{\partial}{\partial t}\omega + U(y)\partial_x \omega - U''(y)\partial_x \phi = f,$$
$$\Delta \phi = \omega,$$  \hfill (2)

with the above types of forcing both linear inviscid damping and algebraic instability in Sobolev regularity may hold at the same time. Furthermore, this behavior is consistent with the nonlinear equations, where both types of forcing naturally appear.

Our main results for the forced linearized inviscid problem and the associated consistency problem are summarized in the following theorem.

Theorem 1.1. Let $U$ be a flow profile that is bounded above and below on the interval $I \subset \mathbb{R}$ and such that linear inviscid damping holds in the sense that conditions (18) to (20) are satisfied (c.f. Section 3). Suppose further that $f \in H^1_{\text{loc}}(\mathbb{R}^+, H^1(T \times I))$ is periodic in $t$ with period $T > 0$ with vanishing average in $x$, $\int f(t, x, y) dx = 0$. Then we obtain the following results when $f$ is of resonant type:

(1) The evolution for $W(t, x, y) := \omega(t, x + tU(y), y)$ is algebraically unstable in $H^s$ for any $s > -1$.
(2) Linear inviscid damping holds in a weak sense. That is, $v(t)$ is uniformly bounded in $L^2$ and there exists $v_\infty \in L^2$ such that $v(t) \rightharpoonup v_\infty$ as $t \to \infty$.

If instead $f$ is of stationary type and non-degenerate (c.f. Section 3), then

(1) The evolution for $W(t, x, y) := \omega(t, x + tU(y), y)$ is algebraically unstable in $H^s$ for any $s > 0$. 


(2) While \( \omega(t) \) is stable in \( L^2 \), neither \( w(t,x,y) \) nor \( W(t,x,y) \) converge as \( t \to \infty \). Instead there is a sum space decomposition

\[
\omega(t,x,y) = \omega_1(t,x - tU(y),y) + \omega_2(t,x,y),
\]

where both \( \omega_1, \omega_2 \) are stable in \( L^2 \) and converge as \( t \to \infty \).

(3) The evolution of \( \omega \) is asymptotically stable in \( H^s \) for any \(-1 \leq s < 0\). In particular, linear inviscid damping holds.

Let \( \omega(t) \) denote the solution of the forced linearized Euler equations where \( f \) is in the stationary case and assume that \( \omega_0, \partial_t \omega_0, g, \partial_x \omega_0 \in H^{3/2} \cap W^{1,\infty} \). Then the Duhamel integral

\[
\sigma(t,x+tu(y),y) := \int_0^t S(t,\tau)(\nabla \cdot \phi \cdot \nabla \omega)(\tau, x - \tau U(y),y) d\tau
\]
is uniformly bounded in \( H^{-1} \) and weakly asymptotically convergent, but grows unbounded in \( H^s \) for any \( s > -1 \).

**Remark 1.** In view of existing results for the unforced problem, the most important new phenomena and results here are:

- We give an explicit setting of the forced linearized problem, where linear inviscid damping holds, but the equation does not scatter to the transport problem. Considering the nonlinear problem as a forced linear problem and in view of the results of Deng-Masmoudi, [DM18], we consider this as an important first step suggesting that nonlinear inviscid damping may similarly hold in lower regularity in spite of instability.

- Instead of scattering to the transport problem, we observe stability in a sum space consisting of transport-like and stationary-like behavior. However, in the consistency problem it is shown that for the nonlinear problem still further refinements are necessary.

- Stationary streamlines interacting with the forcing and “shear behavior” with respect to frequency in time (cf Section 3) are interesting mathematical effects, which might also be of interest from a physical perspective.

- Even in the case of particularly simple forcing such a time-independent forcing, the consistency equation exhibits resonant behavior and instability in \( H^s, s > -1 \). However, the evolution is weakly asymptotically stable in \( H^{-1} \).

- A key challenge in the consistency problem is given by the rather low regularity of (forced) solutions and the very different time-dependence of factors in the nonlinearity \( v \cdot \nabla \omega \). In view of the question of persistence of inviscid damping, we hence focus on the analysis in negative Sobolev spaces.

Concerning the full nonlinear problem, we obtain a similar result for the Navier-Stokes equations, where we further restrict the choice of (small) forcing. That is, we choose \( \int f(t,x,y) dx \) to fix the \( x \) average of the vorticity and for simplicity consider the case where \( f - \int f dx \) is stationary.

**Theorem 1.2** (The forced Navier-Stokes problem near Couette flow). Consider the linearized forced Navier-Stokes equations near Couette flow and let \( \omega_0, f_0 \in H^s, s \geq 0 \) with \( \int \omega_0 dx = 0 = \int f_0 dx \). Then in the case of a resonant forcing \( f(t,x,y) = f_0(x + ty,y) \), there exists a decomposition

\[
\omega(t,x,y) = \omega_1(t,x + ty,y) + \omega_2(t,x + ty,y)
\]
such that both \( \omega_1, \omega_2 \) are stable in \( L^2 \) and \( \omega_1 \) exhibits enhanced dissipation, but \( \omega_2 \) generally only exhibits decay at an algebraic rate

\[
\exp(C \nu^3) \| \omega_1(t) \|_{L^2} + \| \omega_2(t) \|_{L^2} \leq C_\nu (\| \omega_0 \|_{L^2} + \| f_0 \|_{L^2}).
\]
If the forcing is stationary, then there exists a stationary solution \( g \in H^1 \) of the linearized problem and any solution with initial data \( \omega_0 \) is damped towards \( g \) at super-exponential rates

\[
\|\omega(t) - g\|_{L^2} \leq \exp\left(-\frac{\nu}{3} t\right)\|\omega_0 - g\|_{L^2}.
\]

For the nonlinear forced Navier-Stokes equations around Couette flow, we consider \( f \in L^2 \) time-independent with \( \|f\|_{L^2} \leq \frac{\nu}{40} \). Then there exists a stationary solution \( g \in H^1 \) of the equation

\[
y \partial_x g + \nu \Delta g + (v_y \cdot \nabla g) = f,
\]
where \((\cdot)_{\neq} \) denotes the \( L^2 \) projection on functions with vanishing \( x \)-average.

Furthermore, solutions \( \omega \) of the forced Navier-Stokes equation with different initial data

\[
\partial_t \omega + y \partial_x \omega + (\nabla \phi \cdot \nabla)_{\neq} - \nu \Delta \omega = f,
\]
with \( f \) chosen to ensure the vanishing \( x \)-average, can be decomposed as

\[
\omega = \omega^* + g,
\]
where \( \omega^* \) decays (at least) as \( \exp(-\frac{\nu}{2} t) \) in \( L^2 \).

1.2. Organization of the Article. The remainder of the article is organized as follows:

- In Section 2 we fix notational conventions and discuss the prototypical case of Couette flow, \( U(y) = y \), where explicit solution formulas are available. In particular, we see that the conditions of the theorem and lemma are necessary in that case.
- In Section 3 we study the problem for more general shear flows exhibiting linear inviscid damping and comment on extensions of these results, resonances and conditions on \( f \).
- In Section 4 we discuss the nonlinear viscous problem. Here, we for simplicity fix \( \int \omega dx \) and consider stationary forcing.

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2. Model Case: Couette Flow

As an instructive model, let us consider the linearization around Couette flow \( (U(y) = 0, F = 0) \) on \( \mathbb{T} \times I \), for which the linearized equation greatly simplifies:

\[
\begin{align*}
\partial_t \omega + y \partial_x \omega &= f, \\
\omega|_{t=0} &= \omega_0.
\end{align*}
\]

(3)

As a further simplification, for the remainder of this section, we will assume that the interval \( I \) is chosen such that \( |y| \geq 1 \) on \( I \). In order to introduce ideas, we here consider two very specific cases of forcing. As an at first sight somewhat artificial case, we consider \( f_0 \in H^1 \) with non-trivial dependence on \( x \) and

\[
f(t, x, y) = f_0(x - ty, y).
\]

(4)

We call this the resonant case, since here the evolution of \( W(t, x, y) := \omega(x + ty, y) \) involves

\[
\int_0^t f_0(x - (\tau - \tau)y, y)d\tau = tf_0(x, y)
\]

involves
and thus mixing and oscillations are canceled due the resonance of the structure of \( f(t,x,y) \) with the underlying shear behavior.

As a second case, we consider \( f \) to be \textit{time-independent}, that is
\[
f(t,x,y) = f_0(x,y),
\]
with \( f_0 \) as above. Both cases should be considered as prototypical and to show that a more fine-grained control than just of the \( L^2 \) norm is necessary to control the long-term behavior of (3).

**Remark 2.** We remark that the stationary and resonant cases discussed in the introduction, Section 1, can be reduced to these cases. For instance, if \( f \) is a given time-periodic forcing, we may apply a Fourier transform in time to decompose it into \( e^{-ikct-ikx} \mathcal{F}_{x,t} f(c,k,y) \). Then \( \omega_k(t,x,y) := e^{-ikct} \mathcal{F}_{x} \omega(t,k,y)e^{ikx} \) solves
\[
\partial_t \omega_k + (y + c_k) \partial_x \omega_k = \mathcal{F}_{x,t} f(c,k,y)e^{ikx}.
\]
Since the equation decouples in \( k \), we may consider it a fixed parameter and interpret this equation as a Galilean transformation \( y \mapsto y + \frac{c_k}{k} \) of the time-independent setting (5).

We in particular remark that in this setting instead of stationary streamlines \( y = 0 \), streamlines of interest are those which move at a speed \( y = -\frac{c_k}{k} \) matching the frequency in time.

In the present special case of Couette flow, we can compute explicit solutions and thus show that in both cases stability and scattering to the transport problem fail in \( H^s, s > 0 \). Moreover, in the first case, stability even fails in \( H^{-1} \). However, (weak) linear inviscid damping, that is (weak) asymptotic stability in \( H^{-1} \) holds. Furthermore, we see that a variant of the resonant case naturally arises in the study of the consistency problem.

**Proposition 2.1.** Let \( \omega_0 \in L^2 \) and \( f_0 \in L^2 \) with \( \int f_0 dx = 0 \). Then the explicit solution of (3) in the resonant case (4) is given by
\[
\omega(t,x,y) = \omega_0(x-ty,y) + tf_0(x-ty,y).
\]
In particular, unless \( f_0 \) has trivial dependence on \( x \), it holds that
- The evolution is algebraically unstable in \( H^s \) for any \( s > -1 \).
- The evolution is stable in \( H^{-1} \) and weakly compact in that space. We interpret this as a weak analogue of linear inviscid damping.

If we instead consider the time-independent case (5) and suppose that additionally
\[
g(x,y) := \frac{1}{y} \partial_x^{-1} f_0(x,y) \in L^2,
\]
then the explicit solution of (3) is given by
\[
\omega(t,x,y) = \omega_0(x-ty,y) - g(x-ty,y) + g(x,y),
\]
and, unless \( g \) has trivial dependence on \( x \), it holds that:
- The evolution is algebraically unstable in \( H^s \) for any \( s > 0 \).
- While stability holds in \( L^2 \), asymptotic stability or scattering fail.
- The evolution is asymptotically stable in \( H^s \) for any \( s < 0 \). In particular, the associated velocity field strongly converges in \( L^2 \) as \( t \to \infty \).
Proof. Using the method of characteristics and Duhamel’s formula, we obtain that the explicit solution of (3) is given by

$$\omega(t, x, y) = \omega_0(x - ty, y) + \int_0^t f(\tau, x - (t - \tau)y) d\tau.$$  

In the resonant case, the integral simplifies to

$$\int_0^t f_0(x - ty, y) d\tau = tf_0(x - ty, y).$$

The algebraic instability in $H^s, s > 0$ follows immediately from this explicit formula. Concerning the behavior in $H^s, s < 0$, we note that

$$tf_0(x - ty, y) = -\frac{d}{dy}g(x - ty, y) + (\partial_y g)(x - ty, y).$$

Using the characterization of $H^{-1}$ as the dual space of $H^1_0$ and integrating by parts, we hence obtain stability in $H^{-1}$ with a bound by $\|g\|_{L^2} + \|\partial_y g\|_{L^2} \leq \|g\|_{H^1}$. Furthermore, $g(x - ty, y)$ and $(\partial_y g)(x - ty, y)$ weakly converge to 0 in $L^2$. Hence, weak compactness in $H^{-1}$ follows and $(\partial_y g)(x - ty, y)$ even converges strongly in $H^{-1}$. Similarly, the algebraic instability of $g(x - ty, y)$ in $H^s, s > 0$ yields instability of $\omega$ in $H^s, s > -1$.

In the time-independent case, the explicit solution is given by

$$\omega(t, x, y) = \omega_0(x - ty, y) - g(x - ty, y) + g(x, y)$$

by the fundamental theorem of calculus. We remark that here the vanishing average of $f_0$ in $x$ is necessary to ensure that the integral

$$\int_0^t f_0(x - (t - \tau)y, y) d\tau = \int_x^{x-ty} \frac{1}{y} f(\xi, y) d\xi$$

is a well-defined periodic function in $x$. We observe that the change of variables $(x, y) \mapsto (x - ty, y)$ is an isometry in $L^2$ and hence $\omega$ is stable in $L^2$ but not asymptotically stable. Furthermore, the evolution is algebraically unstable in $H^s, s > 0$ and asymptotically stable in $H^s, s < 0$. \hfill \Box

We thus see that algebraic instability in $L^2$ and asymptotic stability in $H^{-1}$ can be compatible. One might object that the choice of “resonant” force $f$ is quite artificial. In particular, we note that “non-resonant” time-independent (or time-periodic) force fields result in better stability behavior in Sobolev regularity.

However, such resonant forcing naturally appears in the consistency equation. More precisely, we have seen that in the case of time-independent forcing, $\omega$ is of the form

$$\omega(t, x, y) = \omega_1(x - ty, y) + \omega_2(x, y).$$

In the consistency problem we insert this decomposition of $\omega$ into the nonlinearity $\nabla \cdot \phi \cdot \nabla \omega$ and obtain four corresponding products, some of which will be of resonant type.

**Proposition 2.2.** Let $\omega(t, x, y) = \omega_1(x - ty, y) + \omega_2(x, y)$, where $\omega_1, \omega_2, \partial_x \omega_1, \partial_x \omega_2 \in H^{3/2} \cap W^{1, \infty}$. We consider the Duhamel integral

$$\int_0^t (\nabla \cdot \phi \cdot \nabla \omega)(\tau, x + (t - \tau)y, y) d\tau,$$

for the case of a finite periodic channel, $T \times I$.

- If $\omega_1 = 0$, the Duhamel integral (10) is bounded and convergent in $H^s, s < -\frac{1}{2}$.
- If $\omega_1 = 0$, the Duhamel integral is bounded and convergent in $H^s, s < 0$. 

If $\omega_1, \omega_2$ are non-trivial, the Duhamel integral is bounded in $H^{-1}$ and asymptotically converges weakly. However, it grows unbounded in any $H^s$ with $s > -1$.

**Proof.** We consider the problem in Lagrangian variables and thus $W(t,x,y) = \omega(t,x+ty,y)$ can be split as

$$W(t,x,y) = \omega_1(x,y) + \omega_2(x+ty,y).$$

We stress that in the present setting we could restrict to $\omega_1, \omega_2 \in H^s$ for $s$ large in order to allow for easier proofs. However, in the general setting of Section 3, $\omega_1$ and $\omega_2$ will turn out to be time-dependent with suitable control only in $H^s, s < \frac{3}{2}$ and a bound by $\log(t)$ in $W^{1,\infty}$. Our main challenge in the following is thus to handle this lack of higher regularity.

Expressing the Duhamel integral in terms of $W$, we have to estimate

$$\int_0^T \nabla^\perp \Phi(t,x,y) \cdot \nabla W(t,x,y) d\tau,$$

(11) $\Delta_t \Phi := (\partial_{x}^2 + (\partial_y - t\partial_x)^2)\Phi(t,x,y) = W(t,x,y).$

Splitting the stream function as $
\Phi_1(t,x,y) = \Delta_t^{-1} \omega_1(x,y),
\Phi_2(t,x,y) = \Delta_t^{-1} \omega_2(x+ty,y) = (\Delta_t^{-1} \omega_2)(x+ty,y),$

we split (11) into four terms corresponding to different products.

$$\int_0^T \nabla^\perp \Phi(t,x,y) \cdot \nabla W(t,x,y) d\tau
= \int_0^T \nabla^\perp \Delta_t^{-1} \omega_1(x,y) \cdot \nabla \omega_1(x,y) d\tau
+ \int_0^T \nabla^\perp (\Delta_t^{-1} \omega_2)(x+ty,y) \cdot \nabla \omega_2(x+ty,y) d\tau
+ \int_0^T \nabla^\perp \Delta_t^{-1} \omega_1(x,y) \cdot \nabla \omega_2(x+ty,y) d\tau
+ \int_0^T \nabla^\perp (\Delta_t^{-1} \omega_2)(x+ty,y) \cdot \nabla \omega_1(x,y) d\tau
= I + II + III + IV.$$

We remark that if $\omega_1 = 0$, all terms except $II$ vanish. If $\omega_2 = 0$, all terms accept $I$ vanish. If both are non-trivial additionally $III$ and $IV$ are non-trivial. Here, we show $III$ to be bounded in $L^2$, while $IV$ exhibits growth in $H^s$ for any $s > -1$.

Ad I: We consider

$$\int_0^T \nabla^\perp \Delta_t^{-1} \omega_1(x,y) \cdot \nabla \omega_1(x,y) dt,$$

(12) $\nabla^\perp \Delta_t^{-1} \omega_1(x,y)$ asymptotically decays in time, but that our bound

$$\|\nabla^\perp \Delta_t^{-1} \omega_1(x,y)\|_{L^2} \leq C(1 + t)^{-\sigma}\|\omega_1\|_{H^{1+\sigma}}$$

is not sufficient to obtain an integrable decay rate, since we are limited to $1 + \sigma < \frac{3}{2}$.

One way to avoid this obstacle is to consider the asymptotic behavior in negative Sobolev regularity. That is, expressing the nonlinearity as a divergence, we can
We explicitly compute

\[ \| \nabla^\perp \cdot (\Delta_t^{-1} \omega_1(x, y) \cdot \nabla \omega_1(x, y)) \|_{H^1_0} \leq \| \Delta_t^{-1} \omega_1(x, y) \|_{L^2} \| \nabla \omega_1(x, y) \|_{L^\infty} \]

\[ \leq C t^{-3/2} \| \omega_1 \|_{H^{3/2}} \| \nabla \omega_1(x, y) \|_{L^\infty} \]

\[ \| \nabla^\perp (\Delta_t^{-1} \omega_1(x, y) \cdot \nabla \omega_1(x, y)) \|_{L^2} \leq C t^{-1/2} \| \omega_1 \|_{H^{3/2}} \| \nabla \omega_1(x, y) \|_{L^\infty} \]

to obtain that for \( s \in [-1, 0] \)

\[ \| \nabla^\perp \cdot (\Delta_t^{-1} \omega_1(x, y) \cdot \nabla \omega_1(x, y)) \|_{H^s_0} \leq C t^{-1/2 + s} \| \omega_1 \|_{H^{3/2}} \| \nabla \omega_1(x, y) \|_{L^\infty}, \]

and is hence integrable in time for \( s < -\frac{1}{2} \). However, this estimate is not optimal.

As can be seen via characterization in terms of Fourier variables for the setting of an infinite channel or by a basis characterization for a finite channel (c.f. [Zil16]), the linear operator

\[ B_T := \int_0^T \Delta_t^{-1} dt \]

is uniformly bounded in \( L^2 \) and converges as \( T \to \infty \). Hence, we may compute (12) explicitly as

\[ \nabla \omega_1(x, y) B_T \nabla^\perp \omega_1(x, y). \]

Ad II: We explicitly compute II as

\[ \int_0^T \nabla^\perp \Delta_t^{-1} \omega_2(x + ty, y) \cdot \nabla_t \omega_2(x + ty, y) dt \]

\[ = \int_0^T (\nabla^\perp \Delta_t^{-1} \omega_2(x + ty, y) \cdot \nabla \omega_2)(x + ty, y) dt \]

\[ = \frac{1}{y} \partial_x^{-1} (\nabla^\perp \Delta_t^{-1} \omega_2) \cdot \nabla \omega_2)(x + ty, y) \bigg|_{t=0}^T. \]

We in particular note that II consists of one term depending on \( (x + Ty, y) \), which is hence stable in Lagrangian Sobolev spaces but not Eulerian, and another term depending on \( (x, y) \), where the converse holds. Thus, we emphasize that stability should more naturally be considered in sum spaces. This concludes the proof in the case where one of \( \omega_1 \) or \( \omega_2 \) is zero. When considering the general case, one additionally has to estimate the mixed terms III and IV.

Ad III: We express the integral as

\[ \int_0^T \nabla^\perp \Delta_t^{-1} \omega_1(x, y) \cdot \nabla \omega_2(x + ty, y) dx \]

\[ = \int_0^T \nabla^\perp \Delta_t^{-1} \omega_1(x, y) \cdot (\nabla \omega_2)(x + ty, y) \]

Note that, even in the whole space case, the multiplier associated with \( \nabla_t \Delta_t^{-1} \) is not integrable in time, but rather suggests a logarithmic growth. However, we may additionally make use of the oscillation of the second factor and compute:

\[ \nabla^\perp \Delta_t^{-1} \omega_1(x, y) \cdot \frac{1}{y} \partial_x^{-1} (\nabla \omega_2)(x + ty, y) \bigg|_{t=0}^T \]

\[ - \int_0^T \partial_t (\nabla^\perp \Delta_t^{-1}) \omega_1(x, y) \cdot \frac{1}{y} \partial_x^{-1} (\nabla \omega_2)(x + ty, y). \]
We note that \( \partial_t(\nabla^\perp_\tau \Delta_\tau^{-1}) \approx \partial_x \Delta_\tau^{-1} \) exhibits higher decay rates in \( L^2 \) when applied to sufficiently regular functions. In particular,

\[
\|\partial_t(\nabla^\perp_\tau \Delta_\tau^{-1})\omega_1(x,y)\|_{L^2} \leq C(1 + t)^{-3/2 + \delta} \|\omega_1\|_{H^{3/2-\delta}}
\]

and hence the latter integral is absolutely convergent in \( L^2 \).

Ad IV: We split this contribution as

\[
\int_0^T \nabla^\perp_\tau \Delta_\tau^{-1} \omega_2(x + ty, y) \nabla \omega_1(x,y)dt
= \int_0^T \nabla^\perp \phi_2(x + ty, y) \cdot \nabla \omega_1(x,y)dt
= \int_0^T (\nabla^\perp \phi_2)(x + ty, y) \cdot \nabla \omega_1(x,y)dt
+ \int_0^T t\partial_y \phi_2(x + ty, y) \partial_x \omega_1(x,y)dt.
\]

The first integral has an explicit anti-derivative with

\[
\frac{1}{y} \partial_x^{-1}(\nabla^\perp \phi_2)(x + ty, y) \bigg|_{t=0}^T \cdot \nabla \omega_1(x,y).
\]

Similarly, for the second integral we obtain

\[
\frac{T}{y} \phi_2(x + Ty, y) \partial_x \omega_1(x,y)
- \int_0^T \frac{1}{y} \phi_2(x + ty, y) \partial_x \omega_1(x,y)dx.
\]

The last integral is then again of the above form and can be computed as

\[
\frac{1}{y} \partial_x^{-1} \phi_2(x + ty, y) \bigg|_{t=0}^T \partial_x \omega_1(x,y).
\]

We remark that the contributions in (13) and (15) are of the form

\[
a_1(x + ty, y)a_2(x, y) + b_1(x, y)b_2(x, y).
\]

The latter term is time-independent and hence stable, while the first term is of a mixed or product type. In particular, this term is not asymptotically stable in \( L^2 \), neither with respect to Eulerian or Lagrangian coordinates, though each factor is asymptotically stable with respect to one of the coordinates. However, by the Riemann-Lebesgue Lemma the product converges weakly in \( L^2 \) as \( T \to \infty \) and strongly in \( H^s, s < 0 \).

The contribution in (14), is seen to be in general algebraically unstable in \( H^s, s > -1 \), similarly to the resonant case of Proposition 2.1. However, we may again note that

\[
\frac{T}{y} \phi_2(x + Ty, y) = \frac{1}{y^2} (d/dy - \partial_y) \partial_x^{-1} \phi_2(x + Ty, y),
\]

which can be used to establish uniform boundedness and weak compactness in \( H_0^{-1} \). We remark that, at this point we require that \( \partial_y \partial_x \omega_1(x,y) \in L^\infty \), so that \( \partial_x \omega_1(x,y) \psi \in H_0^1 \), whenever \( \psi \in H_0^1 \). 

\( \square \)
3. The Forced Linear Euler Equations

Following this preparatory example, let us now consider the case of a more general shear flow.

\[
\begin{align*}
\partial_t \phi + U(y)\partial_x \phi - U''(y)\partial_x \phi &= f, \\
\Delta \phi &= \omega,
\end{align*}
\]

We define the solution operator of the unforced linearized Euler equations in Lagrangian coordinates by \( S(t_1, t_2) \).

**Definition 3.1.** Let \( U(y) \in C^3 \) be given and define \( S(t_2, t_1) : L^2 \to L^2 \) to be the solution operator of the unforced scattering problem. That is, consider the case \( f = 0 \) and define \( W(t, x, y) := \omega(t, x + tU(y), y) \). Then \( W \) satisfies equation

\[
\partial_t W - U'' \partial_x (\partial_x^2 + (\partial_y - tU'(y)\partial_x)^2)^{-1}W = 0.
\]

We define the associated solution operator \( S \) by \( S(t_2, t_1)W(t_1) = W(t_2) \).

We stress that establishing the stability of \( S(t_2, t_1) \) in Sobolev spaces \( H^s \) is one of the main results and challenges of works on linear inviscid damping (c.f. [CZZ18], [Zil17a], [Zil17b], [WZZ17], [WZZ18]). In this work we build up on these results to study the forced equations. In particular, we make use of the following properties of \( S(t_1, t_2) \), which are for instance established for a family bilipschitz flows in [Zil16] and for more general flows in [WZZ15]:

\[
\begin{align*}
&(18) \quad \|S(t_2, t_1)\|_{H^s \to H^r} \leq C_s \quad \text{for} \quad 0 \leq t_1 \leq t_2 < \infty, \quad s \in \left( -\frac{3}{2}, \frac{3}{2} \right), \\
&(19) \quad \|\partial_x S(t, 0)\|_{H^s} \leq C_s (1 + |t|)^{-\sigma} \|S(t, 0)\|_{H^s} \quad \text{for} \quad s \in (-3, 3), \quad \sigma \in (0, 2], \\
&(20) \quad S(t_2, t_1)f(x + \tau U(y), y) = (S(t_2 - \tau, t_1 - \tau)f(x, y))(x + \tau U(y), y).
\end{align*}
\]

We remark that (18) in the above cited works is proven for \( 0 \leq s < \frac{3}{2} \) with the upper bound being sharp due to boundary effects. The estimates for the spaces \( H_s^s \), \( s < 0 \) follow by duality. Furthermore, we for simplicity assume for the remainder of this section that \( I \) is chosen in such a way that

\[
(21) \quad U(y) \quad \text{is bounded above and below on} \quad I,
\]

in order to avoid discussion of stability in degenerately weighted spaces (c.f. [CZZ18]). We note that (20) is a consequence of the structure of the equation, since all coefficient functions depend on time by conjugation with the transport dynamics. The other two properties quantify stability of the unforced problem in Sobolev regularity as well as the damping rates of the associated velocity field.

We obtain the following integral formula for the solution to the forced problem by applying Duhamel’s principle.

**Lemma 3.1.** The solution of the forced scattering equation

\[
\begin{align*}
\partial_t W - U'' \partial_x (\partial_x^2 + (\partial_y - tU'(y)\partial_x)^2)^{-1}W &= f(t, x - tU(y), y), \\
W|_{t=0} &= \omega_0
\end{align*}
\]

is given by

\[
\begin{align*}
W(t) &= S(t, 0)\omega_0 + \int_0^t S(t, \tau)f(\tau, x - \tau U(y), y)d\tau \\
&= S(t, 0)\omega_0 + \int_0^t (S(t - \tau, 0)f(\tau, \cdot, \cdot))(x - \tau y, y)d\tau
\end{align*}
\]
We remark that \( f(t, x, y) \) denotes the forcing in Eulerian variables and (22) is stated in Lagrangian variables.

**Proof.** Let \( L_t \) denote the operator
\[
L_t u = -U'' \partial_y (\partial_x^2 + (\partial_y - tU'(y)\partial_x)^2)\frac{1}{u}.
\]
Then it holds that
\[
(\partial_t - L_t)S(t, \tau)u = 0
\]
for any \( t, \tau \in \mathbb{R} \) and \( u \in L^2 \). Hence, it follows by direct computation that
\[
(\partial_t - L_t) \left( S(t, 0)\omega_0 + \int_0^t S(t, \tau)f(\tau, x - \tau U(y), y)d\tau \right)
\]
\[
= 0 + \int_0^t 0d\tau + S(0, 0)f(t, x - tU(y), y) = f(t, x - tU(y), y).
\]
Concerning the interaction with the transport problem, we note that the time-dependence of all coefficient functions and the elliptic operator is given by conjugation with the transport operator, which is also formulated in (20). Hence, this conjugation is equivalent to a time-shift of the solution operator \( S(\cdot, \cdot) \).

We note that the stability and asymptotic behavior of \( S(t, 0)\omega_0 \) has been intensively studied in prior works on the unforced problem. In the following we hence restrict to the case \( \omega_0 = 0 \) and studying the Duhamel integral. Furthermore, we note that the problem decouples with respect to Fourier modes \( k \) in \( x \) and thus consider
\[
\int_0^t S(t, \tau)e^{iktU(y)} \hat{f}(\tau, k, y)e^{ikx}d\tau
\]
(24)
\[
= \int_0^t e^{iktU(y)}S(t - \tau, 0)\hat{f}(\tau, k, y)e^{ikx}d\tau,
\]
where \( k \in \mathbb{Z} \setminus \{0\} \) is arbitrary but fixed.

Following a similar strategy as in Section 2, we are interested in the behavior of (23) for the following model cases of forcing.

**Definition 3.2 (Model cases).** Let \( S(t, \tau) \) denote the solution operator of the unforced problem and let \( f(t, x, y) \) be a given forcing. We then introduce the following model cases:

- **We call the forcing** \( f \) **resonant**, if there exists \( f_0 \) such that \( f(t, x, y) = (S(t, 0)f_0)(x + ty, y) \).
- **We call the forcing** \( f \) **stationary**, if there exists \( f_0 \) such that \( f(t, x, y) = (S(0, -t)f_0)(x, y) \).

Similarly as in Section 2, we observe that at first sight more general settings of \( f(t, x, y) \) or \( f(t, x + tU(y), y) \) being periodic can be reduced to these cases. Following the same Fourier decomposition argument we may restrict our discussion to forces \( e^{ict}f_c(x, y) \) or \( e^{ict}f_0(x + tU(y), y) \) and after applying a Galilean transform we may further reduce to \( c = 0 \). Now note that
\[
f(x, y) = S(t, 0)(S(0, t)f(x, y))
\]
and that \( f_0(t, x, y) := (S(0, t)f(x, y)) \) enjoys the same \( H^s \) regularity as \( f \) by (18) and that \( \partial_t f_0 \) decays in \( L^2 \) with algebraic rates by property (19). Hence, these cases are very slight generalizations of Definition 3.2 with \( f_0 \) depending on time, as discussed in Proposition 3.2. The following proposition considers the case with \( f_0 \) independent of time to allows for a more transparent characterization via Duhamel’s formula.
Proposition 3.1. Let \( \omega_0 \in L^2 \) and suppose that \( f \) is in the resonant case with \( f_0 \in H^1 \). Then the explicit solution of (16) is given by
\[
\omega(t, x, y) = (S(t, 0)\omega_0)(x - ty, y) + t(S(t, 0)f_0)(x - ty, y).
\]
In particular, unless \( f_0 \) has trivial dependence on \( x \), it holds that
- The evolution is algebraically unstable in \( H^s \) for any \( s > -1 \).
- The evolution is stable in \( H^{-1} \) and weakly compact in that space. We interpret this as a weak analogue of linear inviscid damping.

If we instead consider the case where \( f \) is in the stationary setting and \( \int f_0 dx = 0 \), then the explicit solution of (16) is given by
\[
\omega(t, x, y) = (S(t, 0)(\omega_0 - g))(x - ty, y) + S(0, -t)g(x, y),
\]
where \( g \) solves
\[
U(y)\partial_x g(x, y) = f_0(x, y).
\]
Hence, it holds that
- The evolution is algebraically unstable in \( H^s \) for any \( s > 0 \).
- While stability holds in \( L^2 \), asymptotic stability or scattering fail.
- The evolution is asymptotically stable in \( H^s \) for any \( s < 0 \). In particular, the associated velocity field strongly converges in \( L^2 \) as \( t \to \infty \).

Proof. Recall that the explicit solution in Lagrangian variables is given by (23):
\[
W(t) = S(t, 0)\omega_0 + \int_0^t S(t, \tau)f(\tau, x - \tau U(y), y)d\tau.
\]
In the resonant case we hence obtain that
\[
W(t) = S(t, 0)\omega_0 + \int_0^t S(t, \tau)S(\tau, 0)f_0(x, y)d\tau
= S(t, 0)\omega_0 + \int_0^t S(t, 0)f_0(x, y)d\tau
= S(t, 0)\omega_0 + tS(t, 0)f_0(x, y).
\]
Similarly, in the stationary case we obtain that
\[
W(t) = S(t, 0)\omega_0 + \int_0^t S(t, \tau)(S(0, -\tau)f_0)(x - \tau U(y), y)d\tau
= S(t, 0)\omega_0 + \int_0^t S(t, \tau)(S(\tau, 0)f_0(x - \tau U(y), y))d\tau
= S(t, 0)\omega_0 + S(t, 0)\int_0^t f_0(x - \tau U(y), y)d\tau
= S(t, 0)(\omega_0 - g) + S(t, 0)g(x - tU(y), y).
\]
The claimed solution formulas then follow by the change to Eulerian variables and noting that \( S(t, 0)g(x - tU(y), y) = (S(0, -t)g)(x - tU(y), y) \). \( \square \)

3.1. Consistency. When considering the consistency problem, similarly as in Section 2, we encounter terms which we interpret as further forcing. However, unlike in the Couette flow case, the solution operators \( S(t, \tau) \) are generally not compatible with products. Hence, these contributions are similar to the resonant and stationary cases discussed above, but also include explicit time-dependences.
Proposition 3.2. Let $\omega(t), \phi(t)$ denote the solution of the linearized forced Euler equations (16), where $f$ is in the stationary case. Further assume that $\omega_0, g, \partial_t \omega_0, \partial_s g \in H_{\frac{2}{5}} \cap W^{1,\infty}$. Then the Duhamel integral

$$\sigma(t, x + tU(y), y) := \int_0^t S(t, \tau) (\nabla^\perp \phi \cdot \nabla \omega)(\tau, x - \tau U(y), y) d\tau$$

satisfies the following properties:

(1) If either $g = 0$ or $\omega_0 = g$, then $\sigma$ is asymptotically stable in $H^s$ for any $s < -\frac{1}{2}$.

(2) Generally, $\sigma(t, x + tU(y), y)$ is uniformly bounded in $H^{-1}$ and weakly asymptotically convergent, but not in $H^s, s > -1$.

Proof. We recall that $\omega(t, x, y)$ is explicitly given by (26)

$$\omega(t, x, y) = (S(t, 0)(\omega_0 - g))(x - tU(y), y) + S(0, -t)g(x, y),$$

which, in analogy to Proposition 2.2, we rewrite as

$$\omega(t, x, y) = \omega_1(t, x - tU(y), y) + \omega_2(t, x, y)$$

$$\Leftrightarrow W(t, x, y) = \omega_1(t, x, y) + \omega_2(t, x + tU(y), y).$$

We recall that by properties (18) and (19), the time derivatives of $\omega_1(t, x, y)$ and $\omega_2(t, x, y)$ decay in $L^2$ with algebraic rates, depending on the regularity of $\omega_0 - g$ and $g$.

Following a similar strategy as in Section 2, we further split the stream function as

$$\phi(t, x + tU(y), y) = \phi_1(t, x, y) + \phi_2(t, x + tU(y), y),$$

where $\phi_1, \phi_2$ are the solutions of

$$(\partial_x^2 + (\partial_y - tU'(y)\partial_x)^2)\phi_1 = \omega_1,$$

$$\Delta \phi_2 = \omega_2.$$

We note that $\phi_2$ asymptotically decays in $L^2$ by inviscid damping and does so with algebraic rates by (18) and (19). In contrast $\phi_1$ is not expected to decay but rather behave similar to a stationary function.

Using this splitting we decompose $\sigma(t)$ into four terms:

(27)

$$\sigma(t, x + tU(y), y) = \int_0^t S(t, \tau) (\nabla^\perp \phi_1(\tau, x, y) \cdot \nabla \omega_1(\tau, x, y)) d\tau$$

$$+ \int_0^t S(t, \tau) (\nabla^\perp \phi_2(\tau, x + tU(y), y) \cdot \nabla \omega_2(\tau, x + tU(y), y)) d\tau$$

$$+ \int_0^t S(t, \tau) (\nabla^\perp \phi_1(\tau, x, y) \cdot \nabla \omega_2(\tau, x + tU(y), y)) d\tau$$

$$+ \int_0^t S(t, \tau) (\nabla^\perp \phi_2(\tau, x + tU(y), y) \cdot \nabla \omega_1(\tau, x, y)) d\tau$$

$$=: I + II + III + IV.$$
• IV is uniformly bounded in $H^{-1}$ and weakly asymptotically convergent.

**Ad I.** We remark that this case in a sense corresponds to the unforced problem. In [Zil16] it has been discussed for the setting without boundary, where stability in higher Sobolev regularity is available, and in the context of a blow-up result for the setting with boundary. In the present setting, we instead aim to establish stability in negative Sobolev regularity $H^s$, $s < -\frac{1}{2}$, despite the blow-up behavior and when $\omega_1$ is of comparably low regularity.

Since $S(\cdot, \cdot)$ is a bounded linear operator on $H^s$, it suffices to show that

$$
\int_0^t \|\nabla^\perp \phi_1(\tau) \cdot \nabla \omega_1(\tau)\|_{H^s} d\tau
$$

is uniformly bounded in $t$. If higher regularity estimates were available, we could estimate by

$$
\|\nabla^\perp \phi_1(\tau)\|_{L^2} \leq (1 + |\tau|)^{-\sigma} \|\omega_1(\tau)\|_{H^{1+\sigma}},
$$

$$
\|\nabla \omega_1(\tau)\|_{L^\infty} \leq \|\omega_1(\tau)\|_{H^{s}},
$$

with $1 < \sigma \leq 2$ and $\sigma' > \frac{3}{2}$. However, as shown for instance in [Zil16], both estimates fail in the setting with boundary in the sense that $\|\omega_1(\tau)\|_{H^{s'}}$ for $\sigma > \frac{3}{2}$ and $\|\partial_t \omega_1(\tau)\|_{L^\infty}$ generally grow unbounded as $\tau \to \infty$.

Using the null structure of the nonlinearity, we obtain the following estimate by interpolation for any $s \in [-1, 0]$:

$$
\|\nabla^\perp \phi_1(\tau) \cdot \nabla \omega_1(\tau)\|_{L^2} \leq \|\nabla^\perp \phi_1(\tau)\|_{L^2} \|\nabla \omega_1(\tau)\|_{L^\infty},
$$

$$
\|\nabla^\perp \phi_1(\tau) \cdot \nabla \omega_1(\tau)\|_{H^{-1}} = \|\nabla^\perp (\phi_1(\tau) \cdot \nabla \omega_1(\tau))\|_{H^{-1}} \leq \|\phi_1(\tau)\|_{L^2} \|\nabla \omega_1(\tau)\|_{L^\infty}
$$

$\Rightarrow \|\nabla^\perp \phi_1(\tau) \cdot \nabla \omega_1(\tau)\|_{H^{-1}} \leq \|\phi_1(\tau)\|_{H^{1+\sigma}} \|\nabla \omega_1(\tau)\|_{L^\infty}$.

We further recall that, by (19), for $\sigma \in [0, 2]$

$$
\|\phi_1(\tau)\|_{H^{1+\sigma}} \leq C(1 + |\tau|)^{-\sigma} \|\omega_1(\tau)\|_{H^{1+\sigma}}
$$

and further use that by the more detailed analysis of the extremal case $\sigma = \frac{3}{2}$ (c.f. [CZZ18], [Zil16], [WZZ18]) it holds that

$$
\|\nabla \omega_1(\tau)\|_{L^\infty} \leq C \log(2 + \tau).
$$

Hence, choosing $s < -\frac{1}{2}$ and $1 < \sigma \leq 2$ such that $s + 1 + \sigma < \frac{3}{2}$ it follows that (28) is uniformly bounded in $t$.

**Ad II.** We introduce the short notation

$$
f_2(t, x, y) := \nabla^\perp \phi_2(t, x, y) \cdot \nabla \omega_2(t, x, y).
$$

Then $I$ may be written as

$$
\int_0^t S(t, 0) S(0, \tau) f_2(\tau, x - \tau U(y), y) d\tau = S(t, 0) \int_0^t (S(-\tau, 0) f_2)(\tau, x - \tau U(y), y) d\tau.
$$

We note that this term is very similar to forcing in the stationary case except that

$$(S(-\tau, 0) f_2)(\tau, x, y) =: f_2^*(\tau, x, y)$$

explicitly depends on time. Using integration by parts and assuming zero average in $x$, we may compute the time-integral as

$$
S(t, 0) \int_0^t \frac{1}{U(y)} \partial_x^{-1} (f_2^*(t, x - tU(y), y) - f_2^*(0, x, y))
$$

$$
- S(t, 0) \int_0^t \frac{1}{U(y)} \partial_x^{-1} \partial_x f_2^*(\tau, x - \tau U(y), y) d\tau.
$$
We express the integral as

\[ \int_0^t S(t, \tau) \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \cdot (\nabla \omega_2)(\tau, x + \tau U(y), y) d\tau \]

Integrating by parts in the \( x + \tau U(y) \) dependence, we obtain

\[ S(t, \tau) \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \cdot (\nabla \omega_2)(\tau, x + \tau U(y), y) \bigg|_{t=0}^T, \]

which is asymptotically stable in \( H^s, s < 0 \), as well as several error terms

\[
\begin{align*}
\int_0^t & S(t, \tau) \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \cdot \frac{1}{U(y)} \partial_x^{-1}(\nabla \omega_2)(\tau, x + \tau U(y), y) d\tau \\
+ & \int_0^t S(t, \tau)(\nabla_x \Delta_x^{-1})' \omega_1(\tau, x, y) \cdot \frac{1}{U(y)} \partial_x^{-1}(\nabla \omega_2)(\tau, x + \tau U(y), y) d\tau \\
+ & \int_0^t S(t, \tau) \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \cdot \frac{1}{U(y)} \partial_x^{-1}(\nabla \omega_2)(\tau, x + \tau U(y), y) d\tau \\
+ & \int_0^t S(t, \tau) \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \cdot \frac{1}{U(y)} \partial_x^{-1}(\nabla \omega_2)(\tau, x + \tau U(y), y) d\tau.
\end{align*}
\]

We remark that the second and third term can be estimated in \( L^2 \) using that

\[
(\nabla_x \Delta_x^{-1})' \omega_1(\tau, x, y) \in L^2 + \nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \in L^2 \\
\leq C(1 + \tau)^{-3/2+\delta}(\omega_1 \in H^{3/2-\delta} + \| \partial_x \omega_1 \|_{H^{3/2-\delta}}).
\]

Similarly, for the last term, we may use the structure of the nonlinearity to estimate by

\[
\nabla_x \Delta_x^{-1} \omega_1(\tau, x, y) \in L^2 \|
\]

and

\[
\partial_x \Delta_x^{-1} \omega_1(\tau, x, y) \in L^2 \|
\]
since the scalar products involves only pairs of $(\partial_y - tU'(y)\partial_x, \partial_y)$ derivatives. In both (29) and (30), the $L^2$ norm decays with an algebraic rate, while the $L^\infty$ norms are either bounded or grows at most logarithmically with time.

Hence, it only remains to consider the $\hat{S}$ contribution. We recall that by the structure of the equation and assuming that $U'' \in L^\infty$, estimating $\hat{S}$ reduces to estimating

$$\| \partial_t \Delta^{-1}_\tau \left( \nabla_\tau^+ \Delta^{-1}_\tau \omega_1(\tau, x, y) \cdot \frac{1}{U(y)} \partial_x^{-1} \nabla_\tau \omega_2(\tau, x + \tau U(y), y) \right) \|_{L^2} \leq \| \partial_t \Delta^{-1}_\tau \omega_1(\tau, x, y) \|_{L^2} \| \nabla_\tau \omega_2(\tau, x + \tau U(y), y) \|_{L^\infty} \leq t^{-3/2+\delta} \| \partial_x \omega_1 \|_{H^{3/2-\delta}} \log(t).$$

Here, we again used the structure of the nonlinearity as $\nabla_\tau^+ \cdot (a \nabla_x b)$ in order to absorb $\nabla_\tau^+$ into $\Delta^{-1}_\tau$.

**Ad IV:** In view of the leading term in the case of Proposition 2.2, we claim that $\| IV \|_{H^{-1}}$ is uniformly bounded in time. Here, compared to Section 2 additional challenges are given by the logarithmic growth bound on $\| \nabla \omega_1 \|_{L^\infty}$ and the time-dependence via $S(\cdot, \cdot)$.

For this purpose, we rephrase the integral as

$$\int_0^t S(t, \tau)(\nabla_\tau^+ (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y)) \cdot \nabla \omega_1(\tau, x, y) d\tau$$

$$= S(t, \tau) \left( \nabla_\tau^+ \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y)) \cdot \nabla \omega_1(\tau, x, y) \right)_{\tau=0}^t$$

$$- \int_0^t \frac{d}{d\tau} S(t, \tau)(\nabla_\tau^+ \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y)) \cdot \nabla \omega_1(\tau, x, y) d\tau$$

$$= - \int_0^t S(t, \tau)(\nabla_\tau^+ \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y)) \cdot \nabla \omega_1(\tau, x, y) d\tau$$

We note that the first term, again using the structure of the nonlinearity, can be estimated in $H^{-1}$ by

$$\| \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y) \nabla \omega_1(\tau, x, y) \|_{L^2}$$

We observe that control by

$$\| \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x + \tau U(y), y) \|_{L^\infty} \| \omega_1(\tau, x, y) \|_{H^1}$$

yields a uniform bound in $H^{-1}$. Concerning asymptotic stability of this term, we note that $S(\cdot, \cdot)$ is (strongly) convergent in $H^{-1}$ and it thus suffices to consider

$$\left( \nabla_\tau^+ \frac{1}{U(y)} \partial_x^{-1} (\Delta^{-1}_\tau \omega_2)(\tau, x, y)) \cdot \nabla \omega_1(\tau, x + \tau U(y), y) \right)_{\tau=0}^t$$

where we switched to Eulerian coordinates. Using the null structure of the nonlinearity and that the first factor is bounded in $W^{1,\infty}$ by the Sobolev embedding, we obtain weak convergence via the duality formulation of $H^{-1}$ and the oscillation of $\omega_1(\tau, x + \tau U(y), y)$. 


We remark that the operator norm of $\Delta^{-1}$ we need to estimate
\[
\int_0^t \|\Delta^{-1}_x \left( \nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y) \right) \cdot \nabla \omega_1(\tau, x, y) \|_{H^{-1}} d\tau.
\]
We remark that the operator norm of $\Delta^{-1}_x : L^2 \rightarrow H^{-1}$ is bounded by $C(1 + \tau)^{-1}$, which yields a bound $\|IV\|_{H^{-1}} \leq C \log(2 + \tau)$. In order to improve this estimate we first consider the case when $\omega_2$ is restricted to Fourier modes
\[
\Omega_1(t) := \{(k, \eta) : |\eta| \geq ct\}
\]
with respect to $x, z = U(y)$ with a small constant $c > 0$ to be fixed later. Then it holds that
\[
\|\Delta^{-1}P_{\Omega_1(t)}\omega_2\|_{L^2} \leq C t^{-2}\|P_{\Omega_1(t)}\omega_2\|_{L^2}.
\]
In this case, we control
\[
\|\Delta^{-1}_x \left( \nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y) \right) \cdot \nabla \omega_1(\tau, x, y) \|_{L^2} \leq \|\Delta^{-1}_x P_{\Omega_1} \omega_2\|_{L^2}\|\omega_1(\tau, x, y)\|_{L^\infty} \leq Ct^{-2}.
\]
Similarly, if we restrict $\omega_1$ to $\Omega_1$, we may estimate
\[
\|\Delta^{-1}_x \left( \nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y) \right) \cdot \nabla \tau P_{\Omega_1} \omega_1(\tau, x, y) \|_{L^2} \leq \|\nabla_x \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y)\|_{L^\infty}\|P_{\Omega_1} \omega_1\|_{L^2} \leq Ct^{-3/2+\delta}\|\omega_1\|_{H^{3/2-\delta}}.
\]
Hence, it remains to estimate the case when both $\omega_1(\tau, x, y)$ and $\omega_2(\tau, x, y)$ are projected onto the complement of $\Omega_1(\tau)$. Here, we note that, due to the shear in $x$ and the assumed vanishing $x$ average of $\omega_2$,
\[
\partial_x^{-1}(\Delta^{-1}_x P_{\Omega_1} \omega_2)(\tau, x + \tau U(y), y)
\]
is highly oscillatory and close to resonant, i.e. $|\eta - kt| \leq ct$. However, since $\omega_2$ also vanishes in $x$, this implies that the product is localized at frequencies $(k + l, \eta + \xi)$ such that
\[
(|\eta + \xi - (k + l) t|) \geq |l| t - 2ct \geq (1 - 2c)t.
\]
Hence, the product is non-resonant in this case, and we may estimate
\[
\|\Delta^{-1}_x \left( \nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x P_{\Omega_1} \omega_2)(\tau, x + \tau U(y), y) \right) \cdot \nabla \tau P_{\Omega_1} \omega_1(\tau, x, y) \|_{H^{-1}} \leq t^{-2}\|\nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x P_{\Omega_1} \omega_2)(\tau, x + \tau U(y), y) \cdot \nabla \tau P_{\Omega_1} \omega_1(\tau, x, y)\|_{H^{-1}} \leq C t^{-3/2+\delta}\|\omega_1\|_{H^{3/2-\delta}}.
\]
Finally, let us consider the last term. We note that an estimate by
\[
\|\nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y) \cdot \nabla \tau \omega_1(\tau, x, y)\|_{H^{-1}} \leq \|\nabla^\perp \frac{1}{U(y)} \partial_x^{-1}(\Delta^{-1}_x \omega_2)(\tau, x + \tau U(y), y)\|_{L^\infty}\|\nabla \tau \omega_1(\tau, x, y)\|_{L^2} \leq C t^{-1},
\]
is again just barely insufficient to establish boundedness of the $\tau$ integral and that the estimate of the last $L^2$ norm can not be expected to be better than $t^{-1}$ even if $\omega_1$ were arbitrarily regular. However, we note that compared to $IV$ the structure
of this term is better in that $\dot{\omega}_1$ exhibits additional decay. We hence, repeat our integration by parts argument to obtain

$$S(t,\tau)(\nabla_\tau^\perp \frac{1}{U(y)^2} \partial_x^{-2}(\Delta^{-1}\omega_2)\tau, x + \tau U(y), y) \cdot \nabla\dot{\omega}_1(\tau, x, y))d\tau|_{\tau = 0}$$

$$+\int_0^t S(t,\tau)(\nabla_\tau^\perp \frac{1}{U(y)^2} \partial_x^{-2}(\Delta^{-1}\omega_2)\tau, x + \tau U(y), y) \cdot \nabla\dot{\omega}_1(\tau, x, y))d\tau$$

$$-\int_0^t S(t,\tau)(\nabla_\tau^\perp \frac{1}{U(y)^2} \partial_x^{-2}(\Delta^{-1}\omega_2)\tau, x + \tau U(y), y) \cdot \nabla\dot{\omega}_1(\tau, x, y))d\tau$$

The first three terms can then be estimated as above, while for the last term, we may control by

$$\|\nabla_\tau^\perp \frac{1}{U(y)^2} \partial_x^{-2}(\Delta^{-1}\omega_2)\tau, x + \tau U(y), y\|_{L^2} \leq \|\dot{\omega}_1(\tau, x, y)\|_{L^2}$$

$$\leq C_T^{-3/2+\delta} (\|\omega_1\|_{H^{3/2-\delta}} + \|\partial_x\omega_1\|_{H^{3/2-\delta}}).$$

In summary, we thus obtain that also this final term is integrable and hence $\|IV\|_{H^{-1}} \leq C$, as claimed. \[\square\]

4. The Forced Navier-Stokes Problem

We are interested in the long-time asymptotic behavior of the forced Navier-Stokes equations on the infinite periodic channel $\mathbb{T} \times \mathbb{R}$ near Couette flow

$$\partial_t \omega + y \partial_y \omega - \nu \Delta \omega = f - v \cdot \nabla \omega,$$

$$v = \nabla_\perp \omega,$$

$$\Delta \phi = \omega,$$

where $\omega$ denotes the nonlinear perturbation to Couette flow and $v$ the associated perturbation of the velocity field. Here, we will mostly focus on forcing analogous to the stationary and resonant cases of Section 2.

4.1. The Linearized Problem. As a simple introductory model let us consider the linearized equation

$$\partial_t \omega + y \partial_y \omega - \nu \Delta \omega = f,$$

$$v = \nabla_\perp \omega,$$

$$\Delta \phi = \omega.$$
In particular, we note for \( k \neq 0 \),
\[
\nu \int_0^t k^2 + (\eta - k\tau)^2 d\tau \geq \frac{1}{3} k^2 \tau^3
\]
and thus the solution of the unforced problem decays with rate \( \exp(-C\nu t^3) \), which is much faster than the at first expected exponential decay rate of the heat equation. One says that this problem exhibits enhanced dissipation.

Proof. We note that \( W(t, x, y) := \omega(t, x + ty, y) \) satisfies the equation

\[
\partial_t W - \nu(\partial_x^2 + (\partial_y - t\partial_z)^2)W = f(t, x + ty, y) \\
\sim \partial_t \tilde{W}(t, k, \eta) + \nu(k^2 + (\eta - kt)^2)\tilde{W}(t, k, \eta) = \tilde{f}(t, k, \eta + kt).
\]

The claimed solution formula hence follows by an application of Duhamel’s principle to this family of inhomogeneous ordinary differential equations. \(\square\)

As in Section 2, we consider as prototypical cases \( f(t, x, y) = f_0(x, y) \) being stationary and \( f(t, x, y) = f_0(x - ty, y) \) being resonant.

Corollary 4.1. Let \( \omega_0, f_0 \in L^2(\mathbb{T} \times \mathbb{R}) \) with \( \int \omega_0 dx = 0 = \int f_0 dx \) and let \( f(t, x, y) = f_0(x - ty, y) \), then the solution of (39) with initial datum \( \omega_0 \) is given by

\[
\tilde{\omega}(t, k, \eta + kt) = \exp\left(-\nu \int_0^t k^2 + (\eta - k\tau)^2 d\tau\right) \tilde{\omega}_0(k, \eta) \\
+ \tilde{f}_0(k, \eta) \int_0^t \exp\left(-\nu \int_\tau^t k^2 + (\eta - ks)^2 ds\right) d\tau \\
=: \exp\left(-\nu \int_0^t k^2 + (\eta - k\tau)^2 d\tau\right) \tilde{\omega}_0(k, \eta) \\
+ b(t, k, \eta + kt) \tilde{f}_0(k, \eta).
\]

Here, \( b(t, k, \eta) \geq 0 \) is monotonically increasing in \( t \), bounded and \( b_\infty(t) := \lim_{t \to \infty} b(t, k, \eta) \) only decays with an algebraic rate \( (k^2 + \eta^2)^{-1} \). In particular, it follows that

- The evolution (of \( \omega(t, x, y) \) or \( \omega(t, x + ty, y) \)) is stable in \( H^s, s \leq 0 \) and the solution asymptotically converges to zero. However, unless \( f_0 \) is trivial, the decay rate is only algebraic.
- The associated velocity field \( v(t) = \nabla^\perp \Delta^{-1} \omega(t) \) is stable in \( L^2 \) and converges to zero in \( L^2 \) as \( t \to \infty \).

If we instead consider the case where \( f(t, x, y) = f_0(x, y) \) is stationary with \( \int f_0 dx = 0 \), then there exists a stationary solution \( g \) of (39) and any solution \( \omega \) with initial data \( \omega_0 \in L^2 \), \( \int \omega_0 dx = 0 \) satisfies

\[
\|\omega(t) - g\|_{L^2} \leq \exp(-\nu t^3)\|\omega_0 - g\|_{L^2}.
\]

Proof. Concerning the resonant case, we note that \( \tilde{f}(\tau, k, \eta + k\tau) = \tilde{f}_0(k, \eta) \) does not depend on time and hence obtain the claimed solution formula as a corollary.
of Lemma 4.1. We further note that using several changes of variables
\[
\begin{aligned}
b(t, k, \eta) &= \int_0^t \exp \left(-\nu \int_0^{t-s} k^2 + (\eta + k(t-s))^2 \, ds \right) \, d\tau \\
&= \int_0^t \exp \left(-\nu \int_0^{t-\tau} k^2 + (\eta + k\sigma)^2 \, d\sigma \right) \, d\tau \\
&= \int_0^t \exp \left(-\nu \int_0^{\xi} k^2 + (\eta + k\sigma)^2 \, d\sigma \right) \, d\xi \\
&= \int_0^t \exp \left(-\nu k^2 \xi - \nu \left(\frac{\eta + k\xi^3 - \eta^3}{3k} \right) \right) \, d\xi.
\end{aligned}
\]

In particular, we note that \( b(t, k, \eta) \) is non-negative and monotonically increasing in \( t \) and uniformly bounded by \( \| \exp(-\nu k^2 \xi) \|_{L^2} = \frac{1}{\nu^2} \) and thus there exists \( b_\infty(k, \eta) = \lim_{t \to \infty} b(t, k, \eta) \). We claim that for \( |\eta| \geq 1 \), it holds that
\[
(38) \quad \frac{c}{k^2 + \eta^2} \leq b_\infty(k, \eta) \leq \frac{\nu^{-1}}{k^2 + \eta^2}.
\]

In particular, it follows that for large \( t \), \( \| b(t, k, \eta) f_0(k, \eta - kt) \|_{L^2} \approx \| f_0(x - ty, y) \|_{H^{-\frac{1}{2}}} \) generally only decays with algebraic rates depending on the regularity of \( f_0 \). We remark that we have seen in Section 2, that for \( \nu \downarrow 0 \), \( b(t, k, \eta) = t \) is unbounded in \( t \) but independent of \( \nu, k, \eta \).

It remains to establish (38). Here, we note that for \( \xi < \frac{|\eta|}{10|k|} \) it holds that
\[
\frac{(\eta + k\xi)^3 - \eta^3}{3k\xi} \approx \eta^2 \xi,
\]
since the difference quotient approximates the derivative. Hence, on that interval
\[
\int_0^{\xi \frac{|\eta|}{10|k|}} \exp \left(-\nu k^2 \xi - \nu \left(\frac{(\eta + k\xi)^3 - \eta^3}{3k} \right) \right) \, d\xi \approx -\frac{1}{k^2 + \eta^2} \nu^{-1} \exp(-\nu(k^2 + \eta^2)\xi) \bigg|_{\xi = 0}.
\]

We remark that for \( k, \eta \) small compared to \( \nu^{-1} \) the \( \nu^{-1} \exp(\cdot) \) difference can be estimated by the difference quotient in 0 and hence can be bounded below by a uniform constant. If instead \( (k^2 + \eta^2) \frac{|\eta|}{10|k|} \) is of size \( \nu^{-1} \) or larger, the difference of the exponentials is bounded below and hence the integral on this interval is comparable to \( \frac{\nu^{-1}}{k^2 + \eta^2} \). On the complement, \( \xi \geq \frac{|\eta|}{10|k|} \), we simply estimate
\[
0 \leq \int_{\xi \geq \xi \frac{|\eta|}{10|k|}} \exp \left(-\nu k^2 \xi - \nu \left(\frac{(\eta + k\xi)^3 - \eta^3}{3k} \right) \right) \, d\xi \leq \int_{\xi \geq \xi \frac{|\eta|}{10|k|}} \exp \left(-\nu k^2 \xi \right) \, d\xi \leq \frac{\nu^{-1}}{k^2} \exp(-\nu \frac{\eta}{10}).
\]

This concludes our proof for the resonant case.

In the stationary case, we claim that
\[
Lg := y \partial_y g + \nu \Delta g = f_0,
\]
possesses a unique weak solution \( g \in H^1(\mathbb{T} \times \mathbb{R}) \) and that \( \| g \|_{H^1} \leq 2\nu^{-1} \| f_0 \|_{L^2} \). Taking \( g \) as a particular solution, the claim then follows immediately by the enhanced dissipation acting on the particular (and thus unforced) solution with initial data
\[ \omega_0 = g. \] It remains to prove the existence and uniqueness of solutions. As \( y\partial_x \) is an unbounded operator on \( H^1(T \times \mathbb{R}) \) we argue via a family of auxiliary problems. For each \( R > 0 \) we define \( g_R \in H^1_0(T \times (-R, R)) \) with \( \int g_R dx = 0 \) to be the unique solution of
\[
y\partial_x g + \nu \Delta g = f_0 \text{ in } T \times (-R, R), \\
g|_{y=-R,R} = 0.
\]
We note that since \( g \) is bounded on \( T \times (-R, R) \),
\[
B_R(h_1, h_2) = \nu \langle \nabla h_1, \nabla h_2 \rangle_{L^2} + \langle h_1, y\partial_x h_2 \rangle_{L^2}
\]
is a bounded bilinear form on \( H^1_0(T \times (-R, R)) \) and that
\[
B_R(h, h) = \nu \| \nabla h \|_{L^2}^2 + \int \frac{1}{2} \partial_x |h|^2 = \nu \| \nabla h \|_{L^2}^2 \geq \frac{\nu}{2} \| h \|_{H^1}^2,
\]
is coercive, where we used that \( \int g_R dx = 0 \). Hence, \( g_R \) exists by the Lax-Milgram theorem and by testing the equation we further obtain that
\[
\| g_R \|_{H^1} \leq \nu^{-1} \| f_0 \|_{L^2}.
\]
The bounded sequence \( (g_R) \) hence posses a weak limit \( g \in H^1(T \times \mathbb{R}) \) along some subsequence \( R_j \to \infty \). We claim that \( g \) is a weak solution. Indeed, let \( \phi \in C_c^\infty(T \times \mathbb{R}) \) be a test function whose support is contained in \( T \times [-L, L] \). Then for any \( R > L \) it holds that
\[
\nu \int \nabla g \cdot \nabla \phi + \int y\partial_x \phi g_R = \int \phi f,
\]
which implies
\[
\nu \int \nabla g \cdot \nabla \phi + \int y\partial_x \phi g = \int \phi f
\]
by letting \( R = R_j \to \infty \). Concerning the uniqueness, taking differences of two solutions \( g_1, g_2 \) it suffices to show that the problem with \( f_0 \) only has a trivial solution. Suppose not and let \( g \in H^1(T \times \mathbb{R}) \) be non-trivial such that
\[
y\partial_x g + \nu \Delta g = 0.
\]
Let \( \chi(y) \) be a standard smooth, symmetrically decreasing cut-off function and \( \chi_R(y) = \chi\left( \frac{R}{y} \right) \). Then \( \chi_R(y) g \in H^1 \) is compactly supported test function and
\[
\int gy\partial_x \chi_R(y) g + \nu \int \nabla g \nabla (\chi_R(y) g) = 0
\]
\[
\Leftrightarrow 0 + \int \chi_R(y) |\nabla g|^2 + \frac{1}{R} \int \chi' \left( \frac{y}{R} \right) |g|^2 = 0.
\]
Here, the first integral vanished as a total derivative in \( x \), the second integral converges to \( \| \nabla g \|_{L^2}^2 \) by monotone or dominated convergence and the last integral is bounded by \( \frac{c}{R} \| g \|_{L^2}^2 \) and hence tends to zero. Thus \( \| \nabla g \|_{L^2} = 0 \) and hence \( g = 0 \), which concludes the proof. \( \square \)

4.2. The Nonlinear Problem. In the following we discuss the nonlinear forced Navier-Stokes equations
\[
\partial_1 \omega + y\partial_2 \omega - \nu \Delta \omega = f, \\
\nu = \nabla^\perp \phi, \\
\Delta \phi = \omega.
\] (39)

We remark that in the linearized problem the evolution of the \( x \)-average (and hence the underlying shear) and its \( L^2 \)-orthogonal complement decoupled and that there we could hence without loss of generality restrict to study the case with vanishing
We then note that\[ \parallel g \parallel_{\nu L^2} \text{ denotes the } L^2 \text{ projection on non-zero frequencies with respect to } x \text{ and } f = f_\neq. \]

Given any sufficiently small given stationary forcing \( f(t,x,y) = f_0(x,y), \) these equations then admit a stationary solution.

**Proposition 4.1.** Suppose that \( \| f \|_{H^1} \leq C_1 \nu^2 \) with \( C_1 \leq \frac{1}{30} \), then there exists \( g \in H^2 \) a weak solution of
\[
\begin{aligned}
y \partial_x g + \nu \Delta g + (v[g] \cdot \nabla g)_{\neq 0} &= f, \\
\| g \|_{H^1} &\leq 2C_1 \nu \quad \text{and} \quad \| g \|_{H^2} \leq 2C_1.
\end{aligned}
\]

**Proof.** We argue by fixed point iteration. Let \( L = y \partial_x + \nu \Delta \) be the linear operator on the space
\[ X := \{ u \in H^1(\mathbb{T} \times \mathbb{R}) : \langle u \rangle_x = 0 \}. \]

We then note that \( \nu \Delta \) is a symmetric operator on this space with respect to the \( L^2 \) inner product, possesses a spectral gap and that \( y \partial_x \) is anti-symmetric. We recall that well-posedness of \( L^{-1} \) and its mapping properties have been studied in the proof of Corollary 4.1.

Rephrasing our equation as
\[ Lg = f - (v[g] \cdot \nabla g)_{\neq}, \]
we intend to show that there exists a unique fixed point \( g \) of the mapping
\[ \Psi : g \mapsto L^{-1}(f - (v[g] \nabla g)_{\neq}). \]

In particular, we show that under the conditions of the proposition, there exists \( c = c(C_1) > 0 \) such that \( \Psi : B_{cv}(0) \rightarrow B_{cv}(0) \) and that \( \Psi \) is a contraction. The existence of \( g \) then follows by the Banach fixed point theorem.

Let thus \( c > 0 \) to be fixed later and \( g \in X \) with \( \| g \|_{H^1} \leq cv \). Then \( v[g] \in L^\infty \) with \( \| v[g] \|_{L^\infty} \leq 10cv \) by the Sobolev embedding and \( \| \nabla g \|_{L^2} \leq cv \), thus
\[ \| f - (v[g] \nabla g)_{\neq} \|_{L^2} \leq C_1 \nu^2 + 10c^2 \nu^2. \]

Let now \( u = \Psi(g) \) and assume for the moment that \( u \) is decaying sufficiently quickly such that \( \int u_0 \partial_x u = 0 \). Formally testing with \( -u \) it follows that
\[ \nu \| \nabla u \|_{L^2}^2 = -\int u(f - (v[g] \nabla g)_{\neq}) \leq \| u \|_{L^2} \| f - (v[g] \nabla g)_{\neq} \|_{L^2} \]
\[ \Rightarrow \| \nabla u \|_{L^2} \leq \frac{1}{\nu} \| f - (v[g] \nabla g)_{\neq} \|_{L^2} \leq C_1 \nu + 10c^2 \nu, \]

where we used that \( \| u \|_{L^2} \leq \| \partial_x u \|_{L^2} \leq \| \nabla u \|_{L^2} \) due to the vanishing \( x \) average. Hence, \( \Psi \) maps \( B_{cv}(0) \) into itself provided \( C_1 + 10c^2 \leq c \), which is satisfied by \( c = 2C_1 \leq \frac{3}{30} \). We remark that, as in Corollary 4.1, \( y \partial_x \) is an unbounded operator on \( H^1 \) and that hence, we may only formally test with \(-u\). In order to make the preceding argument rigorous, we may thus again consider a standard cut-off function \( \Psi_R(y) = \Psi_1(\frac{y}{R}) \) on \( \mathbb{R} \) and test with \(-\Psi_R(y)u\) and subsequently take the limit.
Concerning the contraction property, let $g_1, g_2 \in X$ with $\|g_1\|_{H^1}, \|g_2\|_{H^1} \leq cv$. Then by the previous estimate and the quadratic structure of the nonlinearity:

$$\|\Psi(g_1) - \Psi(g_2)\|_{H^1}^2 \leq \frac{1}{\nu^2} \|\nu|\nabla g_1\|^2 - \|\nu|\nabla g_2\|^2 \
\leq \frac{C}{\nu^2} \|g_1 + g_2\|_{H^1}^2, \|g_1 - g_2\|_{H^1}^2 \leq C(2c)^2 \|g_1 - g_2\|_{H^1}^2,$$

where $C \leq 10$ is the constant of the Sobolev embedding. By our choice of $c$, this is hence a contraction. Thus, a solution $g \in X$ with $\|g\|_{H^1} \leq cv, c = 2C_1$ exists by the Banach fixed point theorem. In order to establish improved regularity, we formally test the fixed point equation with $\nabla g$ to obtain that

$$\nu\|\Delta g\|_{L^2}^2 + \int \Delta g \partial_t \phi \leq \|\Delta g\|_{L^2} \|f - v \cdot \nabla g\|_{L^2}.\]

Furthermore, integrating by parts we may estimate

$$\int \Delta g \partial_t \phi = - \int \phi \partial_t \|\nabla g\|^2 / 2 + \int \partial_t g \partial_t \phi = 0 + \int \partial_t \phi \leq \|\nabla g\|_{L^1}^2,$$

which is controlled by the previous estimate.

We have thus constructed one particular stationary solution of the nonlinear, viscous, forced problem. We in particular stress that due to the forcing the stationary state $g$ is a stable asymptotic state and small perturbation are damped to $g$.

**Proposition 4.2.** Let $f(t, x, y) = f_0(x, y)$ satisfy the assumptions of Proposition 4.1 and let $g$ be the associated stationary solution with $\|\nabla g\|_{L^\infty} \leq cv$. Suppose that $\omega$ is a global solution of (36) with $\langle \omega(t) \rangle = 0$. Then $\omega^*$ satisfies the equation

$$\partial_t \omega^* + y \partial_x \omega^* - \nu \Delta \omega^* + (v^* \cdot \nabla \omega^* - \nu|\nabla \omega^* - v^* \cdot \nabla g)_{\mathcal{P}} = 0,$$

where $\mathcal{P}$ denotes the $L^2$ projection onto functions with vanishing $x$-average. Furthermore, $\omega^*$ is nonlinearly stable in $L^2$ and decays at an (at least) exponential decay rate $\exp(-C\nu t)$.

**Proof.** We note that the equation immediately follows by subtracting (41) from (36). The $L^2$ stability then follows from a formal energy argument. That is, testing with $\omega^*$ we obtain that

$$\frac{d}{dt}\|\omega^*\|_{L^2}^2 + 20 + \nu\|\nabla \omega^*\|_{L^2}^2 + 0 = 0 - \langle \omega^*, v^* \cdot \nabla g \rangle = 0.$$

Note that here we use the assumption that $\omega$ and $\omega^*$ is sufficiently regular such that $v^* \cdot \nabla \omega^*$ is well-defined and that we may integrate by parts.

Using that $\|\nabla g\|_{L^\infty} \leq cv$ for a small constant $c$, then we can estimate

$$\langle \omega^*, v^* \cdot \nabla g \rangle = \langle \omega^*, \nabla \phi^* \cdot \nabla \delta g \rangle = - \langle \nabla \omega^*, \phi^* \nabla \delta g \rangle \leq cv \|\nabla \omega^*\|_{L^2} \|\phi^*\|_{L^2}.$$

Since $\omega$ and hence $\omega^*$ and $\phi^*$ possess a vanishing average in $x$ and hence can be controlled using the Poincaré inequality. Thus, it follows that

$$\frac{d}{dt}\|\omega^*\|_{L^2}^2 + \nu\|\nabla \omega^*\|_{L^2}^2 \leq - \nu\|\nabla \omega^*\|_{L^2}^2 \leq - \nu\|\omega^*\|_{L^2}^2,$$

which implies (at least) exponential decay with rate $\exp(-C\nu t)$. \qed
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