The Complexity of Counting Edge Colorings for Simple Graphs

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Abstract
We prove #P-completeness results for counting edge colorings on simple graphs. These strengthen the corresponding results on multigraphs from [4]. We prove that for any $\kappa \geq r \geq 3$ counting $\kappa$-edge colorings on $r$-regular simple graphs is #P-complete. Furthermore, we show that for planar $r$-regular simple graphs where $r \in \{3, 4, 5\}$ counting edge colorings with $\kappa$ colors for any $\kappa \geq r$ is also #P-complete. As there are no planar $r$-regular simple graphs for any $r > 5$, these statements cover all interesting cases in terms of the parameters $(\kappa, r)$.

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1 Introduction

A proper edge $\kappa$-coloring, or simply an edge $\kappa$-coloring, of a graph $G$ is a labeling of its edges using at most $\kappa$ distinct symbols, called colors, such that any two incident edges have different colors. The number of (proper) edge $\kappa$-colorings of $G$ is an important graph parameter, and can be naturally expressed as the Holant value on the graph $G$ where every vertex of $G$ is assigned the ALL-DISTINCT constraint function.

Given a graph $G$, it is an interesting problem to determine how many colors are required to properly edge color $G$. The minimum number needed is called the edge chromatic number, or chromatic index, and is denoted by $\chi'(G)$. An obvious lower bound is $\chi'(G) \geq \Delta(G)$, the maximum degree of the graph. Vizing’s Theorem [13, 5] states that $\Delta(G) + 1$ colors always suffice for simple graphs (i.e., graphs without self-loops or parallel edges). Whether $\Delta(G)$ colors suffice depends on the graph $G$, and is clearly a problem in NP.

Consider the edge coloring problem over 3-regular simple graphs $G$. It is well known that if a 3-regular graph $G$ contains a bridge (i.e., a cut consisting of one edge) then it has no edge 3-coloring. This follows from a parity argument; see [8] and also a simple proof in section 2 of [7]. For bridgeless planar 3-regular simple graphs, Tait [11] proved in 1880 that an edge 3-coloring always exists iff the Four Color Conjecture (now Theorem) holds. Thus, to the existence question of an edge 3-coloring on a planar 3-regular simple graph, the answer is that there is an edge 3-coloring iff the graph is bridgeless. It follows that edge 3-colorability for 3-regular graphs, as a problem in NP, when restricted to planar simple graphs is solvable in P by the trivial algorithm of checking that the graph is bridgeless. However note that the correctness of this algorithm is certainly not trivial: by Tait’s Theorem, this correctness is equivalent to a proof of the Four Color Theorem.

Without the planarity restriction, Holyer [7] proved that determining if a 3-regular simple graph has an edge 3-coloring is NP-complete. Leven and Galil [9] extended this result to all $\kappa \geq 3$: edge $\kappa$-colorability over $\kappa$-regular simple graphs is NP-complete for all $\kappa \geq 3$.

We say a polynomial time reduction $f$ from SAT to a decision problem $\Pi$ is parsimonious if it preserves the number of solutions, i.e., for any instance $\Phi$ of SAT, the number of solutions to the problem $\Pi$ on the instance $f(\Phi)$ is the same as the number of satisfying assignments of $\Phi$. A parsimonious reduction from SAT to $\Pi$ also implies that the counting version of the problem $\Pi$ is $\#P$-complete. Most NP-completeness reductions from SAT to decision problems are, or can be made to be, parsimonious [1, 10].

However, the reductions by Holyer [7] and by Leven-Galil [9] are not parsimonious. In fact we have the following result from [14] (p. 118, attributed to an unpublished work by Edwards and Welsh [6]) that for any $\kappa \geq 4$, no parsimonious reduction exists from SAT to edge $\kappa$-coloring, unless $P = NP$, which we explain below using a reduction due to Blass and Gurevich [2] with a slight modification.

To discuss the (non)-existence of parsimonious reductions from SAT, we define a solution to edge $\kappa$-coloring as a partition of the edge set into $\kappa$ pairwise disjoint matchings (some of which may be empty). For graphs $G$ with maximum degree $\Delta(G) = \kappa$, if $\chi'(G) = \kappa$, then all $\kappa$ matchings must be nonempty. A theorem due to Thomason [12] says that, for $\kappa \geq 4$, among graphs with $\chi'(G) = \kappa$ and ignoring isolated vertices, the only graph that has a unique (proper) edge $\kappa$-coloring is the star $K_{1,\kappa}$ (uniqueness in the sense of counting partitions). Without the assumption $\chi'(G) = \kappa$, and again ignoring isolated vertices, here is a complete list of graphs having a unique edge $\kappa$-coloring (again uniqueness in the sense of counting partitions): Clearly $\chi'(G) \leq \kappa$. Suppose $\chi'(G) < \kappa$. Let $\sigma$ be a proper $\kappa$-coloring using $\chi'(G)$ colors, and let $C_{\alpha}$ be the set of $\alpha$-colored edges. Then every
non-empty \( C_\alpha \) consists of a single edge, for otherwise one can split it by an unused color from \([\kappa]\) and violate the uniqueness. Moreover, for any \( \alpha \neq \beta \), if \( C_\alpha \) and \( C_\beta \) are non-empty then the two edges are incident, for otherwise we can recolor the edge in \( C_\beta \) by \( \alpha \) and violate the uniqueness. It follows that, in addition to \( K_{1,\kappa} \), the graph \( G \) can only be \( C_3 \) (a cycle of length 3) or among \( \{K_{1,j} \mid 1 \leq j < \kappa\} \) or just the empty graph, and indeed these graphs have a unique edge \( \kappa \)-coloring according to the definition by counting partitions. Obviously it is linear time verifiable whether a graph \( G \) is one of \( K_{1,j} \) (1 \( \leq j \leq \kappa \)) or \( C_3 \) or the empty graph, with a finite union of isolated vertices.

Now suppose for a contradiction that a parsimonious reduction \( f \) exists from \text{SAT} to edge \( \kappa \)-coloring, for some \( \kappa \geq 4 \). Let \( \Phi \) be an instance of \text{SAT} in conjunctive normal form, \( \Phi = C_1 \land C_2 \land \ldots \land C_m \), where each \( C_i \) is a disjunction of literals on the variables \( \{x_1, x_2, \ldots, x_n\} \). Let \( y \) be a new variable. Define another Boolean formula

\[
\Phi' = \neg y \lor C_1 \land \neg y \lor C_2 \land \ldots \land \neg y \lor C_m \land [y \lor x_1] \land [y \lor x_2] \land \ldots \land [y \lor x_n].
\]

Clearly \( \Phi' \) has the following satisfying assignments, \( y = 0, x_1 = x_2 = \ldots = x_n = 1 \), together with \( y = 1 \) and all satisfying assignments \( (x_1, x_2, \ldots, x_n) \) to \( \Phi \). Therefore \( \Phi' \) has exactly one more satisfying assignment than \( \Phi \) does. It follows that if \( \Phi \not\in \text{SAT} \), then \( \Phi' \) has a unique satisfying assignment, which implies that \( f(\Phi') \) has a unique solution for edge \( \kappa \)-coloring; if \( \Phi \in \text{SAT} \), then \( \Phi' \) has more than one satisfying assignments, which implies that \( f(\Phi') \) does not have a unique solution for edge \( \kappa \)-coloring. But uniqueness of solutions to edge \( \kappa \)-coloring is linear time testable by Thomason’s theorem. This would imply \( P = NP \).

**Remark:** We remark that the notion of unique edge \( \kappa \)-coloring in the sense of counting partitions is the standard one; by contrast if we are to count the number of coloring assignments, we would incur a factor \( \kappa! \) for graphs with \( \chi'(G) = \kappa \), and there would be no parsimonious reduction for a trivial reason. In the rest of the paper, we will adopt the notion of counting edge colorings by counting the number of proper edge coloring assignments. For \( \Delta(G) = \kappa \), a fortiori for \( \kappa \)-regular graphs, if \( \chi'(G) = \kappa \), then the number of solutions according to the two definitions differ by exactly a factor \( \kappa! \). Thus our \#P-hardness results in Theorems 1.1 and 1.2 also imply that counting edge \( \kappa \)-coloring is \#P-complete in the sense of counting partitions, by simply restricting to \( \kappa \)-regular graphs.

It is reasonable to suppose that this lack of parsimonious reductions for edge coloring is the reason that it remained an open problem for some decades whether the counting problem for edge coloring is \#P-complete, until it was proved in [4]. Their proof is carried out as part of a classification program on counting assignments, in particular a certain subclass of Holant problems on higher domain sizes. However, as is generally true for Holant problems, the \#P-hardness applies to multigraphs, i.e., parallel edges and loops are allowed. By contrast, a simple graph does not have parallel edges or loops. Our main purpose in this paper is to prove:

**Theorem 1.1.** \#\( \kappa \)-\text{EdgeColoring} is \#P-complete over \( r \)-regular simple graphs for any integers \( \kappa \geq r \geq 3 \).

**Theorem 1.2.** \#\( \kappa \)-\text{EdgeColoring} is \#P-complete over planar \( r \)-regular simple graphs for any integers \( \kappa \geq r \) and \( r \in \{3, 4, 5\} \).

Note that for \( r > 5 \), simple planar \( r \)-regular graphs do not exist, by Euler’s formula. See Theorem 3.1.
Because these are hardness results, the weaker statement also holds without the restriction on regularity, or on planarity, thus for any $\kappa \geq \Delta \geq 3$, $\#\kappa$-EDGECOLORING is $\#P$-complete for simple graphs with maximum degree $\Delta$. Obviously if $\kappa < \Delta$ there are no proper edge $\kappa$-colorings. For $\Delta \leq 2$, the problem can be easily solved in polynomial time, since for maximum degree at most 2, a graph is a disjoint union of cycles and paths. Thus Theorems 1.1 and 1.2 settle the complexity of counting edge colorings for simple graphs for all parameter settings of $(\kappa, r)$ or $(\kappa, \Delta)$.

Our proof of Theorems 1.1 and 1.2 use reductions from the $\#P$-hardness for edge colorings on multigraphs. At the heart of the reductions are some geometrically inspired gadget constructions using the Platonic solids, such as the icosahedron (see Figure 4).

2 Preliminaries

Counting $\kappa$-edge colorings can be naturally expressed as a Holant problem on a domain of size $\kappa$. Let us first recap the framework of Holant problems on the domain $[\kappa] = \{1, 2, \ldots, \kappa\}$.

Fix a positive integer $\kappa \geq 1$. A constraint function, or signature, $f$ has an arity $n \geq 0$, and is a mapping from $[\kappa]^n \to \mathbb{C}$. When $n = 0$, we think of $f$ as a scalar. For the reason of Turing computability we assume all signatures take complex algebraic values.

A Holant problem Holant$(\mathcal{F})$ is parameterized by a set of signatures $\mathcal{F}$. An input to the problem Holant$(\mathcal{F})$ is a signature grid $\Omega = (G, \pi)$ consisting of a (multi)graph $G = (V, E)$, where $\pi$ assigns to each vertex $v \in V$ some $f_v \in \mathcal{F}$ of arity $\deg(v)$, and assigns its incident edges to its input variables.

**Definition 2.1.** Given a set of signatures $\mathcal{F}$, we define the counting problem Holant$(\mathcal{F})$ as:

- **Input:** A signature grid $\Omega = (G, \pi)$;
- **Output:** $\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma | E(v))$, where the sum is over all edge assignments $\sigma : E \to [\kappa]$, $E(v)$ denotes the incident edges of $v$ and $\sigma | E(v)$ denotes the restriction of $\sigma$ to $E(v)$.

By default $G$ is a multigraph so it may have self-loops or parallel edges. A simple graph does not have self-loops or parallel edges. We may restrict the Holant problem to signature grids where input graphs $G$ are simple graphs, or simple planar graphs. We use Pl-Holant$(\mathcal{F})$ to denote the problem restricted to planar signature grids.

A function $f_v$ can be represented by listing its values according to the lexicographical order of the input tuples which is a vector in $\mathbb{C}^{\kappa^{\deg(v)}}$. A signature is called symmetric if it is invariant under a permutation of its variables. An example is the EQUALITY signature $=r$ of arity $r$ (on domain $[\kappa]$), which outputs value 1 if all $r$ input values are equal, and 0 otherwise. Here $=0$ is the scalar 1; $=1$ is the univariate function that takes constant value 1. A binary signature, i.e., a signature of arity 2, can be represented by a $\kappa \times \kappa$ signature matrix. E.g., the binary EQUALITY signature $=2$ is represented by the identity matrix $I_{\kappa}$.

Replacing a signature $f \in \mathcal{F}$ by a constant multiple $cf$, where $c \neq 0$, does not change the complexity of Holant$(\mathcal{F})$. It introduces a global nonzero factor to Holant$_\Omega$.

We allow $\mathcal{F}$ to be an infinite set. We say Holant$(\mathcal{F})$ is (P-time) tractable, if Holant$_\Omega$ is computable in polynomial time where the input description of $\Omega$ includes the signatures used in $\Omega$. We say Holant$(\mathcal{F})$ is $\#P$-hard if there exists a finite subset of $\mathcal{F}$ for which the problem is $\#P$-hard. The same definitions apply for Pl-Holant$(\mathcal{F})$ when $\Omega$ is restricted to planar signature grids. We use $\leq_T$ and $\equiv_T$ to denote polynomial time Turing reduction and equivalence, respectively.
We say a signature \( f \) is \textit{realizable} or \textit{constructible} from a signature set \( \mathcal{F} \) if there is a graph fragment, called a gadget, which is a graph with some dangling edges such that each vertex is assigned a signature from \( \mathcal{F} \), and the resulting graph defines \( f \) by the Holant sum with inputs on the dangling edges. If \( f \) is realizable from a set \( \mathcal{F} \), then we can freely add \( f \) into \( \mathcal{F} \) while preserving the complexity. Formally, such a notion is defined by an \( \mathcal{F} \)-gate \([3]\). An \( \mathcal{F} \)-gate is similar to a signature grid \((G, \pi)\) for Holant\((\mathcal{F})\) except that \( G = (V, E, D) \) is a graph with some dangling edges \( D \). The dangling edges define external variables for the \( \mathcal{F} \)-gate. (See Figure 1 for an example.)

We denote the regular edges in \( E \) by 1, 2, \ldots, \( m \) and the dangling edges in \( D \) by \( m + 1, \ldots, m + n \). Then we can define a function \( \Gamma \) for this \( \mathcal{F} \)-gate as

\[
\Gamma(y_1, \ldots, y_n) = \sum_{x_1, \ldots, x_m \in [\kappa]} H(x_1, \ldots, x_m, y_1, \ldots, y_n),
\]

where \((y_1, \ldots, y_n) \in [\kappa]^n\) is an assignment on the dangling edges and \( H(x_1, \ldots, x_m, y_1, \ldots, y_n) \) is the value of the signature grid on an assignment of all edges in \( G \), which is the product of evaluations at all internal vertices. We also call this function \( \Gamma \) the signature of the \( \mathcal{F} \)-gate, or informally, of the gadget.

An \( \mathcal{F} \)-gate is planar if the underlying graph \( G \) is a planar graph, and the dangling edges, ordered counterclockwise corresponding to the order of the input variables, are in the outer face in a planar embedding. A (planar) \( \mathcal{F} \)-gate can be used in a (planar) signature grid as if it is just a single vertex with the particular signature. An \( \mathcal{F} \)-gate is simple if the underlying graph is simple.

Using the idea of planar \( \mathcal{F} \)-gates, we can reduce one planar Holant problem to another. Suppose \( g \) is the signature of some planar \( \mathcal{F} \)-gate. Then Pl-Holant\((\mathcal{F} \cup \{g\})\) \( \leq_T \) Pl-Holant\((\mathcal{F})\). The reduction is simple. Given an instance of Pl-Holant\((\mathcal{F} \cup \{g\})\), by replacing every appearance of \( g \) by the \( \mathcal{F} \)-gate, we get an instance of Pl-Holant\((\mathcal{F})\). Since the signature of the \( \mathcal{F} \)-gate is \( g \), the Holant values for these two signature grids are identical.

An arity \( r \) signature on domain size \( \kappa \) is fully specified by \( \kappa^r \) values. A \textit{symmetric} signature of arity \( r \) on domain \([\kappa]\) can be specified by \( \binom{r+\kappa-1}{\kappa-1} \) values, namely the signature value is determined by how many variables \((r_1 \geq 0)\) take the first value in \([\kappa]\), and how many variables \((r_2 \geq 0)\) take the second value in \([\kappa]\), etc., such that \( r_1 + r_2 + \ldots + r_\kappa = r \).

The signature that defines the problem of edge coloring is the \textit{ALL-DISTINCT} function. The signature \text{ALL-DISTINCT}_{r,\kappa}, or \text{AD}_{r,\kappa} for short, has arity \( r \) on domain size \( \kappa \). It outputs value 1 when all inputs are distinct and 0 otherwise. Here \text{AD}_0 is the scalar 1; \text{AD}_1 is the univariate function that takes constant value 1. When \( \mathcal{F} = \{\text{AD}_{r,\kappa} \mid r \geq 0\} \), the problem Holant\((\mathcal{F})\) is called \#\(\kappa\)-\textit{EdgeColoring}.
We prove our #P-completeness results for counting edge colorings over (regular) simple graphs by reducing from the corresponding results for multigraphs (Theorem 1.1 in [4]). We restate it below for easy reference.

**Theorem 2.2.** #\(\kappa\)-EdgeColoring is #P-complete over planar \(r\)-regular multigraphs for any integers \(\kappa \geq r \geq 3\).

In this paper, we show two results, Theorems 1.1 and 1.2, about the counting complexity of edge colorings over (regular) simple graphs, one for the planar case and the other for the general (i.e., not necessarily planar) case. As noted before, the only interesting cases are for \(\kappa \geq r \geq 3\), and in the planar case for \(\kappa \geq r = 3, 4, 5\). We handle both the planar and nonplanar cases together, but we first prove it for the case \(\kappa = r\) in Section 3 and then for the case \(\kappa > r\) in Section 4.

We remark that when the underlying graph has a loop then there is no proper edge colorings so the corresponding Holant value is 0. Hence we may assume that our reduction proof for Theorems 1.1 and 1.2 starts with the problem of edge colorings on loopless multigraphs in Theorem 2.2.

### 3 Case \(\kappa = r\)

We start with the statement of Theorem 2.2 specialized with \(\kappa = r \geq 3\). We want to prove a corresponding statement for simple graphs, i.e., graphs without self-loops or parallel edges. We first recall a well-known fact that for \(r > 5\), there are no planar \(r\)-regular simple graphs. This is a topological fact from Euler’s formula; for completeness we include a simple proof. For any simple plane graph \(G\), i.e., a planar graph with a planar embedding, let \(V, E\) and \(F\) be respectively the number of vertices, edges and faces formed by \(G\). Then Euler’s formula says that

\[ V - E + F = 2. \]

If \(G\) is an \(r\)-regular simple graph, then counting the ends of every edge once we get \(rV = 2E\). Similarly \(rV \geq 3F\), since at every vertex we encounter \(r\) incident faces, but each face is counted at least three times on a simple graph \(G\). Hence \((1 - \frac{2}{r} + \frac{3}{r})V \geq 2\). In particular \(r \leq 5\).

For \(\kappa \in \{3, 4, 5\}\) we prove the following for planar \(\kappa\)-regular simple graphs.

**Theorem 3.1.** #\(\kappa\)-EdgeColoring is #P-complete over planar \(\kappa\)-regular simple graphs, for \(\kappa \in \{3, 4, 5\}\).

**Proof.** We reduce the problem of #\(\kappa\)-EdgeColoring over planar \(\kappa\)-regular multigraphs to planar \(\kappa\)-regular simple graphs in polynomial time, for \(\kappa \in \{3, 4, 5\}\).

Let \(G\) be any planar \(\kappa\)-regular multigraph. As mentioned before, we may also assume \(G\) has no self-loops. We will replace every edge of \(G\) by a suitable gadget with two external edges to obtain a planar \(\kappa\)-regular simple graph \(G'\). The key property of the gadget is as follows: Every proper edge \(\kappa\)-coloring assigns the same color to the two external edges \(e_1\) and \(e_2\), and such a coloring exists. Note that, by permuting the set \([\kappa]\), for any fixed color value for \(e_1\) and \(e_2\) there are the same number \(c\) of extensions to a proper edge \(\kappa\)-coloring of the gadget. With this property, the number of proper edge \(\kappa\)-colorings of \(G'\) is \(c^E\) times that of \(G\), where \(E\) denotes the number of edges of \(G\).

For \(\kappa = 3\), consider the gadget \(H_3\) in Figure 2. Replacing every edge of \(G\) by \(H_3\) we get a planar 3-regular simple graph \(G'\). To prove the key property for \(H_3\), suppose \(e_1\) is assigned a color \textit{Red}, then its two adjacent edges \(e_3\) and \(e_4\) must be assigned the other two colors \textit{Blue} and \textit{Green}.
in some order. Then the middle vertical edge \( e_5 \) is assigned Red, and its remaining two adjacent edges \( e_6 \) and \( e_7 \) must be assigned Blue and Green in a unique way (depending on the colors of \( e_3 \) and \( e_4 \)). It follows that \( e_2 \) is also assigned Red, and we found exactly \( c = 2 \) proper edge 3-colorings.

For \( \kappa = 4 \), consider the gadget \( H_4 \) in Figure 3. Replacing every edge of \( G \) by \( H_4 \) we get a simple planar 4-regular graph \( G' \). Suppose the middle horizontal edge \( e \) is assigned Red. Then \( e_3 \) and \( e_4 \) are assigned two other colors, say Blue and Green, in some order. Let the 4th color be Yellow. Then \( e_5 \) and \( e_6 \) must be assigned Yellow and Red in some order. In particular exactly one of \( e_5 \) or \( e_6 \) is assigned Yellow. It can be easily shown that the color (Blue or Green) of the edge \( (e_3 \) or \( e_4) \) that is on the same internal face as this edge colored Yellow \( (e_5 \) or \( e_6)) \) must be the color of both the external edges \( e_1 \) and \( e_2 \), and once \( e_5 \) or \( e_6 \) is chosen to be colored Yellow there is a unique extension. This implies that the gadget \( H_4 \) satisfies the key property with the constant \( c = 12 \): If the two external edges \( e_1 \) and \( e_2 \) are to be Blue, say, then \( e \) must be not Blue, and one of \( e_3 \) and \( e_4 \) must be Blue, and the rest are all forced, with the unique choice of \( e_5 \) or \( e_6 \) that shares a face with the Blue edge in \( \{e_3, e_4\} \) colored Yellow. Thus there are \( 3 \times 2 \times 2 = 12 \) choices.

We remark that the gadgets \( H_3 \) and \( H_4 \) are obtained from breaking one edge in a tetrahedron and an octahedron, respectively.

For \( \kappa = 5 \), we use a similar construction, but we will argue slightly differently. Let \( H \) be any plane 5-regular simple graph that can be edge 5-colored. Such graphs exist; for concreteness we can take the icosahedron which is shown in Figure 4 and a specific 5-coloring can be seen in Figure 5. Because \( H \) is 5-regular, \( 5V = 2E \), so \( V \) is even. Now remove one edge of \( H \) on the external face, say between vertices \( u \) and \( v \). This defines a graph \( H' \). Define \( H_5 \) to be the graph obtained from \( H' \) by adding two external edges \( e_1 \) and \( e_2 \) from \( u \) and \( v \) (in the case of the icosahedron see Figures 6
We claim that $H_5$ satisfies the key property. To prove that we only need to show that any edge 5-coloring of $H_5$ must use the same color on the two external edges. For a contradiction, suppose this is not so. Then for some 5-coloring of $H_5$, there are two distinct colors Red and Green, say, such that the two external edges are colored Red and Green, respectively. Consider the set $S$ of edges in $H_5$ colored Red or Green. Clearly the two external vertices are incident to edges in $S$. Since all other vertices of $H_5$ have degree 5, all vertices of $H_5$ are also incident to edges in $S$. $S$ must be a vertex disjoint union of (zero or more) alternating Red-Green cycles, and exactly one alternating path starting and ending with the two external edges. This alternating path starts and ends with distinct colors, and therefore it has an even number of edges and thus an odd number of vertices. The alternating cycles have an even number of vertices. This is a contradiction to $V \equiv 0 \pmod{2}$. (The exact value $c > 0$ for $H_5$ is a constant and is unimportant to the existence of this reduction.)

The idea of the proof can also be adapted to prove the following theorem without the planarity.
resuirement.

**Theorem 3.2.** $\#\kappa$-EdgeColoring is $\#P$-complete over $\kappa$-regular simple graphs, for all $\kappa \geq 3$.

**Proof.** Let $H$ be a $\kappa$-regular simple graph that admits a proper edge $\kappa$-coloring. For now assume such a graph exists. If we consider any color, the edges with that color form a perfect matching of $H$, because the number of colors and the regularity parameter are both $\kappa$. Hence $H$ has an even number of vertices.

Take any two adjacent vertices $u$ and $v$. Now remove one edge $(u, v)$ of $H$; this defines a graph $H'$. Define $H^*$ to be the graph obtained from $H'$ by adding two external edges $e_1$ and $e_2$ incident to $u$ and $v$, respectively. $H^*$ will be the gadget replacing every edge in a $\kappa$-regular multigraph $G$ to produce a $\kappa$-regular simple graph $G'$. By the same proof above we see that in any edge $\kappa$-coloring of $H'$, the $\kappa - 1$ incident edges at $u$ receive the same set of $\kappa - 1$ colors as the $\kappa - 1$ incident edges at $v$. Thus $H^*$ satisfies the key property stated in the proof of Theorem 3.1. This completes the proof, provided we exhibit one such graph $H$.

Let $n = \kappa!$. Since $\kappa \geq 3$, we have $n/2 \geq \kappa$. We take the vertex set of $H$ to be the additive group $\mathbb{Z}_n$. We define $\kappa$ edge disjoint perfect matchings, and the edge set of $H$ as their union. For $1 \leq \ell \leq \lfloor \kappa/2 \rfloor$, let $E_\ell = \{(i, i + \ell) \mid 0 \leq i < \ell\}$, and $E'_\ell = \{(i + \ell, i + 2\ell) \mid 0 \leq i < \ell\}$. Let $M_\ell$ and $M'_\ell$ be respectively the orbits of $E_\ell$ and $E'_\ell$ by the subgroup $(2\ell)\mathbb{Z}_n \simeq \mathbb{Z}_{n/(2\ell)}$. Thus

\[
M_\ell = \bigcup_{i=0}^{\ell-1}\{(i, i + \ell), (i + 2\ell, i + 3\ell), \ldots, (i + n - 2\ell, i + n - \ell)\},
\]

\[
M'_\ell = \bigcup_{i=0}^{\ell-1}\{(i + \ell, i + 2\ell), (i + 3\ell, i + 4\ell), \ldots, (i + n - \ell, i + n)\}.
\]

Note that $2\ell \leq \kappa$ and so $2\ell \mid n$. If $\kappa$ is odd, then add one more perfect matching $M = \{(j, j + n/2) \mid 0 \leq j < n/2\}$. This defines $\kappa$ edge disjoint perfect matchings. \hfill \Box
4 Case $\kappa > r$

In this section, we prove similar results for simple graphs where $\kappa > r$. We use $J_\kappa$ to denote the all-1 $\kappa$ by $\kappa$ matrix. We call a gadget $r$-regular if its internal vertices all have degree $r$.

**Theorem 4.1.** $\#\kappa$-EdgeColoring is $\#P$-complete over planar $r$-regular simple graphs for all $\kappa > r$, and $r \in \{3, 4, 5\}$.

**Proof.** We reduce the problem $\#\kappa$-EdgeColoring over planar $r$-regular multigraphs, which is $\#P$-complete by Theorem 2.2, to planar $r$-regular simple graphs in polynomial time. Then the statement of the theorem follows.

For now let us suppose we can construct a binary planar $r$-regular simple gadget $f$ consisting of only AD$_{r,\kappa}$ signatures. We say a function is domain invariant if it is invariant under any permutation of the input domain elements. Clearly AD$_{r,\kappa}$ is domain invariant, and so $f$ is also domain invariant. Hence its $\kappa$ by $\kappa$ signature matrix has the form $A = (a - b)I_\kappa + bJ_\kappa$ for some nonnegative integers $a$ and $b$, i.e., it has values $a$ and $b$ on and off the main diagonal, correspondingly. In particular, $A$ is symmetric. For any $n \geq 1$, consider a binary gadget $f_n$ obtained by connecting $f$ sequentially $n$ times, with $f_1 = f$ (see Figure 8). Note that the order in which the dangling edges of $f$ are connected to each other makes no difference in this construction because $A$ is symmetric. Obviously, each gadget $f_n$ for $n \geq 1$ is still a planar $r$-regular simple gadget, with a domain invariant signature matrix $A^n$. Assume further that we can satisfy $a \neq b$. 

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**Figure 7:** A proper edge 5-coloring of the icosahedron gadget where $e_1, e_2$ have the same color.

**Figure 8:** Recursive construction to interpolate the binary EQUALITY signature $=2$. All vertices are assigned the gadget $f$. The dangling edges of $f$ can be connected in any order.
Let \( \Omega = \Omega(G) \) be an input instance to Pl-Holant(AD_{r,κ}), where \( G = (V, E) \) is a planar \( r \)-regular multigraph. Here every vertex in \( V \) is assigned a symmetric signature \( AD_{r,κ} \). Let \( F \subseteq E \) be the subset of the edges of \( G \) each of which is parallel to at least one other edge. For any \( n \geq 1 \), we construct \( \Omega_n = \Omega(G_n) \) to be the instance of Pl-Holant(AD_{r,κ}) obtained by replacing every edge in \( F \) by the gadget \( f_n \). Clearly, the underlying graph \( G_n \) of \( \Omega_n \) is simple planar \( r \)-regular and has size polynomial in \( n \) and the size of \( \Omega \). Let \( m = |F| \), which is also the number of the gadgets \( f_n \) used in \( \Omega_n \). Let \( \Omega' = \Omega(G') \) be a signature grid obtained from \( \Omega \) by placing the binary EQUALITY signature \( =_2 \) in the middle of every edge in \( F \). Clearly, Holant_{\Omega'} = Holant_\Omega. We show how to compute this value from the values Holant_{\Omega_n}, where \( n \geq 1 \) in polynomial time.

The matrix \( A = (a-b)I_\kappa + bI_\kappa \) has eigenvalues \( \lambda_1 = a-b+\kappa b \) and \( \lambda_i = a-b \) for \( 2 \leq i \leq \kappa \). Since it is real symmetric, it can be orthogonally diagonalized over \( \mathbb{R} \), i.e., there exist a real orthogonal matrix \( S \) and a real diagonal matrix \( D = (\lambda_i)_{i=1}^\kappa \) such that \( A = S^TDS \). As \( a \neq b \) we have \( \lambda_i = a-b \neq 0 \) for \( 2 \leq i \leq \kappa \). Also \( \lambda_1 = a+(\kappa-1)b > 0 \). It follows that the matrices \( A \) and \( D \) are nondegenerate. Clearly \( A^n = S^T D^n S \), for all \( n \geq 1 \), and is also symmetric.

We can write the \((i,j)\) entry \((A^n)_{ij}\) by a formal expansion, for every \( n \geq 1 \) and some real \( \alpha_{ij\ell} \)'s are dependent on \( S \), but independent of \( n \) and \( \lambda_i \), where \( 1 \leq i, j, \ell \leq \kappa \). By the formal expansion of the symmetric matrix \( A^\tau \) above, we have \( \alpha_{ij\ell} = \alpha_{ij\ell} \).

In the evaluation of Holant_{\Omega_n}, we stratify the (edge) assignments in \( \Omega_n \) based on the colors assigned to the dangling edges of the binary gadgets \( f_n \) in \( \Omega_n \) as follows. Denote by \( \tau = (t_{ij})_{1 \leq i \leq j \leq \kappa} \) a nonnegative integer tuple with entries indexed by ordered pairs of numbers and that satisfy \( \sum_{1 \leq i < j \leq \kappa} t_{ij} = m \). Let \( c_\tau \) be the sum over all assignments of the products of all signatures in Holant_{\Omega_n}, except the contributions by the gadgets \( f_n \) such that the endpoints of precisely \( t_{ij} \) constituent gadgets \( f_n \) receive the assignments \((i,j)\) (in either order of the end points) for every \( 1 \leq i \leq j \leq \kappa \). Let \( T \) denote the set of all such possible tuples \( \tau \), where \( |T| = \binom{m+\kappa(\kappa+1)/2-1}{\kappa(\kappa+1)/2-1} \).

Then

\[
\text{Holant}_{\Omega_n} = \sum_{\tau \in T} c_\tau \prod_{1 \leq i < j \leq \kappa} ((A^n)_{ij})^{t_{ij}} = \sum_{\tau \in T} c_\tau \prod_{1 \leq i < j \leq \kappa} \left( \sum_{\ell=1}^\kappa \alpha_{ij\ell} \lambda_\ell^n \right)^{t_{ij}}.
\]

Expanding out the last sum and rearranging the terms we get

\[
\text{Holant}_{\Omega_n} = \sum_{i_1+\ldots+i_\kappa = m} b_{i_1,\ldots,i_\kappa} \left( \prod_{\ell=1}^\kappa \lambda_\ell^{i_\ell} \right)^n, \tag{4.1}
\]

for some numbers \( b_{i_1,\ldots,i_\kappa} \) independent of \( \lambda_\ell \)'s.

This can be viewed as a linear system with the unknowns \( b_{i_1,\ldots,i_\kappa} \) with the rows indexed by \( n \).

The number of unknowns is \( \binom{m+\kappa-1}{\kappa-1} \) which is polynomial in \( n \) and the size of the input instance \( \Omega \), since \( \kappa \) is a constant. The values \( \prod_{\ell=1}^\kappa \lambda_\ell^{i_\ell} \) can all be computed in polynomial time.

On the other hand, it is clear that

\[
\text{Holant}_{\Omega'} = \sum_{i_1+\ldots+i_\kappa = m} b_{i_1,\ldots,i_\kappa}.
\]

We show next how to compute the value Holant_{\Omega'} from the values Holant_{\Omega_n} where \( n \geq 1 \) in polynomial time. The coefficient matrix of the linear system (4.1) is Vandermonde. However, when \( m \geq 1 \), it is not of full rank because the coefficients \( \prod_{\ell=1}^\kappa \lambda_\ell^{i_\ell} \) are not pairwise distinct, and
therefore it has repeating columns. Nevertheless, when there are two repeating columns we replace
the corresponding unknowns \( b_{i_1,\ldots,i_n} \) and \( b_{i'_1,\ldots,i'_m} \) with their sum as a new variable; we repeat this
replacement procedure until there are no repeating columns. Since all \( \lambda_\ell \neq 0 \), for \( 1 \leq \ell \leq \kappa \),
after the replacement, we have a Vandermonde system of full rank. Therefore we can solve this
modified linear system in polynomial time and find the desired value \( \text{Holant}(\Omega) = \text{Holant}(\Omega') = 
\sum_{i_1+\ldots+i_n=m \atop i_1,\ldots,i_n \geq 0} b_{i_1,\ldots,i_n} \).

**Remark:** The above proof did not use the fact in this specific case \( \lambda_2 = \ldots = \lambda_\kappa \). The general
Vandermonde argument does not need this property.

We are left to prove that we can construct a binary planar \( r \)-regular simple gadget \( f \) with
distinct diagonal and off diagonal values \( a \neq b \) in its signature matrix \( A \). From the previous section
we know that for every \( r \in \{3,4,5\} \) there exists a binary connected planar \( r \)-regular simple gadget.
For example, one can take \( f \) to be the gadget \( H_r \) described in Theorem 3.1, i.e., the ones shown
in Figures 2, 3 and 6, correspondingly. The connectedness of each \( H_3, H_4 \) and \( H_5 \) is self-evident.
In the proof of Theorem 3.1, we also showed that in each respective case \( r \in \{3,4,5\} \), \( H_r \) can be
\( r \)-colored so that the two dangling edges are colored with the same color. Fix such a coloring. Since
now we are given \( \kappa > r \), we can modify this coloring to change the color of one of the dangling
edges to one of the remaining \( \kappa - r \geq 1 \) colors different from the \( r \) colors used. This still produces
a proper \( \kappa \)-edge coloring but in which the dangling edges are now colored differently. Therefore
\( b \neq 0 \). If \( a \neq b \), then we are done.

Suppose \( a = b \), so we can write \( A = bJ_\kappa \). \( J_\kappa \) can be written as the column vector \( (1,1,\ldots,1)^T \)
times the row vector \( (1,1,\ldots,1) \). Clearly, in terms of evaluating signature value, replacing an
internal edge within the gadget \( f \) with another copy of the gadget \( f \) itself is equivalent, up to a
nonzero scalar \( b \), to cutting this edge in half and assigning the unary signature \( (1,1,\ldots,1)^{\ni} \) to the
two new degree one vertices. Now choose a path \( P \) in the gadget \( f = H_r \) starting in one dangling
edge and ending in the other. Since \( f \) is connected, such a path exists. Suppose it has \( s \) internal
nodes, where \( s \geq 1 \). Let \( g \) be the binary gadget obtained by replacing every edge in \( f \) that is \( \text{not} \) in
\( P \) by a new copy of \( f \) (by the symmetry of \( f \), the order of the dangling edges of \( f \) in the replacement
bears no difference). It is easy to see that \( g \) is a binary connected planar \( r \)-regular simple gadget.
Note that connecting \( (1,1,\ldots,1)^{\ni} \) to \( AD_{n,\kappa} \) gives us the signature \( (\kappa - n + 1) \ AD_{n-1,\kappa} \) where
\( 1 \leq n \leq \kappa \) so it is the signature \( AD_{n-1,\kappa} \), up to the nonzero scalar \( \kappa - n + 1 \). Thus \( g \), as a constraint
function, is equivalent to simply a path \( P \) with \( AD_{n-1,\kappa} \equiv (\neq 2) \) assigned to the intermediate vertices.
Hence, (up to an easily computable nonzero constant) its signature matrix is \( (J_\kappa - I_\kappa)^s \). It is easy
to verify that \( (J_\kappa - I_\kappa)^s \) has the form \( (-1)^s I_\kappa + c_{s,\kappa} J_\kappa \) for some \( c_{s,\kappa} \), and \( (-1)^s \) is the
difference of the diagonal value and the off diagonal value, in particular nonzero. Hence all the necessary
conditions on \( g \) are satisfied and we can use \( g \) instead of \( f \).

Finally, we deal with the nonplanar case for \( \kappa > r \geq 3 \).

**Theorem 4.2.** \( \#\kappa\text{-EdgeColoring} \) is \#P-complete over \( r \)-regular simple graphs for \( \kappa > r \geq 3 \).

**Proof.** As in the proof of Theorem 4.1, we reduce the problem \( \#\kappa\text{-EdgeColoring} \) over planar
\( r \)-regular multigraphs, which is \#P-complete by Theorem 2.2, to (non-necessarily planar) \( r \)-regular
simple graphs in polynomial time. Then the statement of the theorem follows.

Following the steps of the proof of Theorem 4.1, we see that it suffices to produce a binary
connected \( r \)-regular simple gadget \( f \) that is \( r \)-colorable with the additional condition that the
dangling edges are colored with the same color. The only difference is that we do not require \( f \) to be a planar gadget (and for \( r > 5 \) this would be impossible).

Let \( U = \{u_1, u_2, \ldots, u_r\}, V = \{v_1, v_2, \ldots, v_r\} \) and \( \{u, v\} \) be pairwise disjoint sets of vertices. We take the vertex set of \( f \) to be \( U \cup V \cup \{u, v\} \) where \( u, v \) will be the external endpoints of the two dangling edges. Next, we define the edge set of \( f \) to be \( A \cup B \cup C \cup D \) where

\[
A = \{(u_r, u_i) \mid 1 \leq i \leq r-1\}, \\
B = \{(v_r, v_i) \mid 1 \leq i \leq r-1\}, \\
C = \{(u_i, v_j) \mid 1 \leq i, j \leq r-1\}, \\
D = \{(u, u_r), (v_r, v)\}.
\]

Here \((u, u_r)\) and \((v_r, v)\) are the dangling edges of \( f \); there is a complete bipartite graph \( K_{r-1,r-1} \) between \( \{u_1, u_2, \ldots, u_{r-1}\} \) and \( \{v_1, v_2, \ldots, v_{r-1}\} \), while \( u_r \) is connected to all of the former, and \( v_r \) is connected to all of the latter. It is easy to see that \( f \) is indeed a binary connected \( r \)-regular simple gadget.

We will exhibit a proper \( r \)-edge coloring of \( f \). For this, we first label the \( r \) colors by the elements of the additive group \( \mathbb{Z}_r = \{0, 1, \ldots, r-1\} \). Now define the edge coloring of \( f \) as follows: each \((u_i, v_j) \in B\) is colored \( i + j \mod r \) for \( 1 \leq i, j \leq r-1 \), each \((u_r, u_i) \in A\) and each \((v_j, v_r) \in C\) is colored \( j \), for \( 1 \leq j \leq r-1 \), and \((u, u_r), (v_r, v) \in D\) are colored 0. It is easy to see that this coloring satisfies all the necessary requirements.

\[ \square \]

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