A^1-CURVES ON LOG SMOOTH VARIETIES

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Abstract. In this paper, we introduce the notion of A^1-connectedness from the logarithmic geometry point of view, and develop a technique for producing very free A^1-curves on quasi-projective varieties. We then classify all A^1-curves on good wonderful compactifications, and study their A^1-connectedness. Finally projective spaces are identified as the distinguished compactifications of affine spaces with ample log tangent bundles.

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1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic char k \geq 0.

1.1. Why A^1-curves? The behavior of rational curves over projective varieties are intensively studied during the past decades. Rationally connectedness introduced and studied by Mori [Mor79a], Campana [Cam92], and Kollár-Miyaoka-Mori [KMM92] plays a central roll in the study of the geometry of higher dimensional varieties. Varieties that are (separably) rationally connected are proved to admit nice arithmetic properties over non-closed fields, such as function fields of curves [GHS03, dJS03, HT06], and large fields [Kol99, KS03].

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Following Iitaka’s philosophy, the great progresses along this line naturally leads to the consideration of non-proper cases. Starting with a smooth quasi-projective variety $U$, it is natural to replace $U$ by a pair $(X, D)$ where $X$ is a proper variety, and $D \subset X$ is a divisor such that $U = X \setminus D$.

It is then natural to ask what are “rational curves” on a pair $(X, D)$? A crucial observation of Keel-McKernan [KM99] suggests that the right notion of log rational curves on log pairs should consists of the following two types of curves:

- A rational curve on a log smooth pair $(X, D)$ is a morphism of pairs $f : (\mathbb{P}^1, \emptyset) \to (X, D)$, i.e., the image of $\mathbb{P}^1$ avoids $D$.
- An $A^1$-curve on a log smooth pair $(X, D)$ is a morphism of pairs $f : (\mathbb{P}^1, \{\infty\}) \to (X, D)$, i.e., the image of $\mathbb{P}^1$ meets $D$ exactly once at $\infty$.

While the first case gets back to the old notion of rationally connectedness, the second one is of different flavor, and is more involved. Luckily, the recent development of stable log maps [GS13, Che14, AC] provides a solid foundation for working with rational curves of the second type, which we call $A^1$-curves. This will require such pair $(X, D)$ to be toroidal or equivalently log smooth. This is the main focus here.

The idea of $A^1$-connectedness already has its application for even the study of type one rational curves — it plays a central roll in the proof of separably rationally connectedness of general Fano complete intersections over all characteristics [CZ]. Many interesting examples of $A^1$-connected varieties with smooth irreducible boundary were provided there [CZ, Corollary 1.10]:

**Theorem 1.1.** Given a log smooth log Fano variety $(X, D)$ with $D$ irreducible, $(X, D)$ is $A^1$-connected if $(X, D)$ is $A^1$-uniruled. Moreover, the latter condition holds when the normal bundle of $D$ corresponds to a nontrivial effective divisor.

One major task of the current paper is to extend the above result to log smooth varieties with general boundaries.

1.2. How to produce $A^1$-curves? Let $X$ be a log smooth variety. An $A^1$-curve is a genus zero stable log map $f : C/S \to X$ with a unique contact marking and irreducible underlying domain curve, see Definition 2.2. Note that the image of $f$ can lie on the boundary of $X$. This allows us to construct $A^1$-curves from degeneration, and is really the advantage of the theory of stable log maps as we will see below.

The notion of $A^1$-connectedness and some of its basic properties will be discussed in Section 2. Intuitively, a log smooth variety $X$ is $A^1$-connected if two general points of $X$ can be connected by an $A^1$-curve on $X$.

While the definition of $A^1$-connectedness is very similar to rationally connectedness, the study of $A^1$-curves is much harder than rational curves in
at least the two following aspects. First, an rational curve on the underly-
ing variety in general cannot be lift to an $\mathbb{A}^1$-curve. Second, the gluing of
two $\mathbb{A}^1$-curves needs not to be an $\mathbb{A}^1$-curve any more. It turns out that the
above two issues enjoys a nice solution in the presence of a center with good
properties. For the first one, we have

**Proposition 1.2.** Let $X$ be a log smooth variety with a proper separably
rationally connected center $Y \subset X$. Then

1. there is a distinguished chart $\beta : P \to \mathcal{M}_Y$; and
2. a rational curve $f : C = \mathbb{P}^1 \to Y$ can be lift to an $\mathbb{A}^1$-curve $f : C/S \to X$ if and only if $c_\beta(C) \in P^\vee$.

The proof of the first statement is given in Lemma 3.11, and the second
statement follows from Corollary 4.9.

The second issue is closely related to the existence of very free $\mathbb{A}^1$-curves
and the $\mathbb{A}^1$-connectedness, which is our major focus. Analyzing the gluing
of $\mathbb{A}^1$-curves leads to the following:

**Theorem 1.3.** Let $X$ be a log smooth variety with a proper separably ra-
tionally connected fully free center $Y$. Then $X$ admits very free $\mathbb{A}^1$-curves
with contact markings supported on $Y$.

We refer to Section 2 and 3 for the notations. The proof of the above
theorem will be given in the Section 4.4.

1.3. $\mathbb{A}^1$-curves on sober spherical varieties. The class of projective ho-
mogeneous spaces with an action of linear algebraic group is a good source
of examples of rationally connected varieties. They are Fano and admit
globally generated tangent bundles.

In our study of $\mathbb{A}^1$-connected varieties, we would like to search their log-
arithmic generalization in the class of (not necessarily projective) homo-
geous spaces. Such a homogeneous space $G/H$ should satisfy the following:

1. it admits a toroidal compactification as a log smooth pair;
2. it is log Fano;
3. the log tangent bundle of the pair is globally generated, i.e., log
homogeneous spaces [Bri07b].

Fortunately this has been classified by Luna-Vust, Brion, and Knop [LV83,
Kno91, Bri12, Bri07b] using the theory of spherical embeddings. Such a
homogeneous space is called sober, and it admits the log smooth wonderful
compactification (the underlying scheme of the compactification could be
singular). Wonderful compactifications are the building blocks of the theory
of spherical varieties, and they include the wonderful compactifications of
semisimple groups and symmetric spaces [DCP83].

Before state our results, we would like to emphasize one important feature
of $\mathbb{A}^1$-connectedness. For many applications of rationally connectedness, one
of the most important thing is to show the existence of very free rational
curves. To exhibit a very free $\mathbb{A}^1$-curve will be still important, but not good
enough in the theory of $\mathbb{A}^1$-connectedness. The first evidence is the fully freeness condition in Theorem 1.3, which will require the existence of free $\mathbb{A}^1$-curves touching the boundary in sufficiently many ways, see Definition 3.15. This will be crucial for Zariski density of integral points in [CZ14b]. Thus, it is important to classify all free $\mathbb{A}^1$-curves.

**Theorem 1.4.** Assume that $\text{char } k = 0$, and $X$ is the log smooth variety associated to a wonderful compactification of an open sober spherical homogeneous space $G/H$ with $G$ simply connected. Further assume $X$ is good, see Definition 5.15. Then we have the following:

1. Any log-admissible effective one-cycle on $X$ with nontrivial contact order (resp., trivial contact order) can be represented by an $\mathbb{A}^1$-curve (resp., a rational curve) on $G/H$. In particular, $\text{NE}(X) = \text{NE}_{\mathbb{A}^1}(X)$.
2. The unique closed orbit is a fully free center of $X$. In particular, $X$ is $\mathbb{A}^1$-connected.
3. The center of $X$ is primitive if and only if $\mathbb{X}^*(H)$ is torsion-free. In particular, when $H$ is connected, the center of $X$ is primitive.

**Remark 1.5.** The condition that $X$ is good as in the above theorem is to guarantee the center $Y$ contains a basis of $\text{NE}(X)$, see Proposition 5.18. It contains many interesting cases, including wonderful compactifications of semisimple groups, and more generally spherical varieties of minimal rank [Bri07a, Res10]. We wish to study $\mathbb{A}^1$-connectedness for general wonderful compactifications by reducing to the case of this paper.

For possibly positive characteristics, we have

**Theorem 1.6.** Let $G$ be a semisimple linear algebraic $k$-group with arbitrary char $k \geq 0$, and $X_G$ be the log smooth variety associated to the wonderful compactification of $G$. Let $\Lambda_{R'}$ be the coroot lattice of $G$ and $C$ be a positive Weyl chamber of the root system of $(G, T)$ with a maximal torus $T$. Then we have the following:

1. Any log-admissible effective one-cycle on $X_G$ can be represented by a free $\mathbb{A}^1$-curve on $G$. In particular, we have

$$\text{NE}(X_G) = \text{NE}_{\mathbb{A}^1}(X_G) = \Lambda_{R'} \cap C^+.$$

2. If further assume that $\text{char } k \nmid |\pi_1(G, T)|$, then the unique closed orbit is a fully free center of $X_G$. In particular, $X_G$ is separably $\mathbb{A}^1$-connected.
3. The unique closed orbit as the center of $X$ is primitive if and only if $G$ is simply connected. In this case, it is separably $\mathbb{A}^1$-connected in arbitrary characteristics.

The above two theorems will be proved in Section 5.5 and 5.6 by verifying the assumptions in a more general set-up in Theorem 5.14. While the study of $\mathbb{A}^1$-connectedness, we also provide some basic properties of wonderful compactifications from the logarithmic point of view in Section 5.2 and 5.3. These will be useful in our subsequent paper [CZ14a].
Remark 1.7. It is an interesting question to study when the conditions in Theorem 5.14 hold over positive characteristics. However, this is beyond the scope of the current paper.

1.4. Log smooth varieties with ample tangent bundles. Affine spaces is another example of homogeneous varieties we shall not neglect here. Recall that projective spaces are identified as smooth proper varieties with ample tangent bundles as conjectured by Hartshorne [Har70], and proved by Mori [Mor79b]. Combining Mori’s idea with the theory of $\mathbb{A}^1$-curves and Keel-McKernan’s work [KM99], we identify projective spaces as a distinguished compactification of affine spaces as follows:

Theorem 1.8. Let $X = (\underline{X}, \Delta_X)$ be a log smooth projective variety such that

1. $-K_X$ is ample on the interior;
2. there exists a point $x \in U$ such that for any $\mathbb{A}^1$ or $\mathbb{P}^1$-curve $f : C \to X$ through $x$, the pull-back $f^*T_X$ is ample.

Then we have:
(a) if there is an $\mathbb{A}^1$-curve through $x$, then $\underline{X} = \mathbb{P}^n$ and $\Delta_X$ is a hyperplane in $\mathbb{P}^n$;
(b) otherwise, there is a $\mathbb{P}^1$-curve through $x$, and $X = \mathbb{P}^n$ with $\Delta_X = \emptyset$.

In particular, this implies that

Corollary 1.9. Let $X$ be a log smooth projective variety. If the log tangent bundle $T_X$ is ample, then $X$ is isomorphic to the log scheme associated to either $(\mathbb{P}^n, \emptyset)$ or $(\mathbb{P}^n, D)$, where $D$ is a hyperplane.

The proof of Theorem 1.8 will be given in the end of Section 6. When the underlying space $\underline{X}$ is smooth, a simple proof of Corollary 1.9 using the result [CP98, Theorem 1.1] can be given. So a major task in the proof is to pass from the singular case to the smooth case. Note that ampleness of $-K_X$ implies the condition ample on the interior, see Definition 6.1. However ampleness on the interior is invariant under log étale birational modification, which is the key to deal with the singular case. We hope the way Theorem 1.8 is stated makes this clear.

1.5. Notations. Throughout this paper, all log structures are assumed to be fine and saturated. We refer to [Kat89] for the basics of logarithmic geometry.

Capital letters like $X, Y$ are reserved for log schemes. The underlying structure of a log scheme $X$ and $Y$ is denoted by $\underline{X}$ and $\underline{Y}$ respectively.

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2. Basic definitions

2.1. Notions of log maps. We first recall some basic terminologies of log maps, and refer to [GS13, Che14, AC, ACMW] for detailed discussions.

Definition 2.1. Let $X \to B$ be a morphism of log schemes. A stable log map over a log scheme $S$ is a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{} & B
\end{array}
$$

such that $C \to S$ is a family of log curves over $S$ as defined in [Kat96, Ols07] (see also [Che14, Appendix B] for a quick introduction), and the underlying map $f$ is a family of usual stable maps to the underlying family of targets $X/B$.

For the rest of this paper, we will only consider the case when $B$ is a geometric point with the trivial log structure. For simplicity, we may write $f : C/S \to X$ for a family of stable log maps over $S$. When $S$ is a geometric point with the trivial log structure, we write $f : C \to X$ for simplicity.

Consider a geometric fiber $f : C/S \to X$. For each marking $\sigma \in C$, we have a canonical map of monoids

$$
c_\sigma := \overline{f}^\# : f^*M_{X,\sigma} \to \mathbb{N}
$$

as in [AC, Section 3.8]. We call $c_\sigma$ the contact order at $\sigma$. A marking $\sigma$ is called a contact marking if $c_\sigma$ is non-trivial. Otherwise, we call $\sigma$ a usual marking.

Let $X$ be a log scheme, and $\Gamma = (g, n, \beta, \{c_i\})$ the set of data consisting of the genus $g \in \mathbb{N}$, the number of markings $n$, the curve class $\beta \in H_2(X)$, and the set of contact orders $\{c_i\}$. Denote by $\mathcal{M}_\Gamma(X)$ for the stack of stable log maps with discrete data $\Gamma$.

Definition 2.2. An $\mathbb{A}^1$-curve is a genus zero stable log map with exactly one marking with non-trivial contact order, and smooth underlying source curve. An $\mathbb{A}^1$-curve can have arbitrarily many marked points with trivial contact orders.

An $\mathbb{A}^1$-curve $f : C/S \to X$ over a geometric point $S$ is called free (resp. very free) if $f^*T_X$ is semi-positive (resp. ample).

In general, consider a log map: $f : C/S \to X$ with a subset of markings $P = \{\Sigma_i\}$. Then $f$ is called $P$-free if $H^1(f^*T_X(-P)) = 0$, and $f^*T_X(-P)$ is globally generated.
We have the following analogue of [Kol96, II. 3.14]:

**Proposition 2.3.** Let $X$ be a log smooth log scheme with $\dim X = r$, and $f : C \to X$ be a free $\mathbb{A}^1$-curve. Assume the splitting type
\[ f^* T_X = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r), \]
with $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$. Then

1. If $a_2 \geq 1$, then a general deformation of $f$ is an immersion outside the contact marking;
2. If $a_3 \geq 1$, then a general deformation of $f$ is an embedding outside the contact marking.

**Proof.** The proof is identical to [Kol96, II 3.14] by working with the stack of stable log maps and the log deformation theory, see for example [Che14, Section 2.5].

2.2. $\mathbb{A}^1$-connectedness and uniruledness.

**Definition 2.4.** Let $X$ be a proper log smooth scheme. It is called separably $\mathbb{A}^1$-connected (resp. separably $\mathbb{A}^1$-uniruled) if it admits a very free (resp. free) $\mathbb{A}^1$-curve.

**Remark 2.5.** The reader should note that the definition of $\mathbb{A}^1$-connectedness and $\mathbb{A}^1$-uniruled in [KM99] includes the case of complete rational curves in $X^\circ$. However, we find it is more convenient to study the two cases separately, as they require different techniques.

For a log scheme $X$, recall that $X^\circ$ denotes the open subscheme of $X$ with the trivial log structure. The following observation shows that the definition of $\mathbb{A}^1$-connectedness and uniruledness is intrinsic to the open variety $X^\circ$.

**Proposition 2.6.** Let $X$ be a proper log smooth scheme. The following are equivalent:

1. $X$ is separably $\mathbb{A}^1$-uniruled;
2. there exists a family of 1-pointed $\mathbb{A}^1$-curves $f : C/S \to X$ such that the 1-pointed evaluation is dominant and separable.
3. there exists a flat family $\pi : U \to S$ of usual schemes with geometric fibers isomorphic to $\mathbb{A}^1$, and a separable and dominant morphism $f : U \to X^\circ$ such that the geometric fiber $f_s : \mathbb{A}^1 \to X^\circ$ is a proper morphism for each $s \in S$.

When $\text{char } k = 0$, the separable conditions in (2) and (3) can be removed.

If further assume $\text{char } k = 0$ and $k$ is uncountable, then $X$ is $\mathbb{A}^1$-uniruled if and only if there exists an $\mathbb{A}^1$ curve though general points on $X$.

**Proof.** The equivalence of (1) - (3) follows from the standard argument of [Kol96, IV 1.3] for the stack of stable log maps.

Notice that the stack of $\mathbb{A}^1$-curves in $X$ has countably many irreducible components. The last statement follows from the same argument as in [Kol96, IV (1.3.5)].

♠
Proposition 2.7. Let $X$ be a proper log smooth scheme. The following are equivalent:

1. $X$ is separably $\mathbb{A}^1$-connected;
2. there exists a family of two-pointed $\mathbb{A}^1$-curves on $X$ such that the two-pointed evaluation is separable and dominant.
3. there exists a flat family $\pi: U \to S$ of usual schemes with geometric fiber isomorphic to $\mathbb{A}^1$, and a morphism $f: U \to X^\circ$ such that the geometric fiber $f_s: \mathbb{A}^1 \to X^\circ$ is a proper morphism for each $s \in S$, and the two evaluation map

$$f^{(2)}: U \times_S U \to X^\circ \times X^\circ$$

is separable and dominant.

When $\text{char } k = 0$, the separable condition in (2) and (3) can be removed.

If further assume $\text{char } k = 0$ and $k$ is uncountable, then $X$ is $\mathbb{A}^1$-connected if and only if there exists an $\mathbb{A}^1$ curve through general pair of points in $X^\circ$.

Proof. The statements follows from the standard arguments, see for example [Kol96, IV 3.4, 3.5, 3.7]

Definition 2.8. A smooth usual scheme $X^\circ$ is called separably $\mathbb{A}^1$-connected (resp. separably $\mathbb{A}^1$-uniruled) if it satisfies Proposition 2.7(3) (resp. Proposition 2.6(3)).

Corollary 2.9. Let $g^\circ: X^\circ \to Y^\circ$ be a finite étale morphism of two usual smooth schemes of degree prime to $p$. Then any proper $\mathbb{A}^1$-curve in $Y^\circ$ lifts to a proper $\mathbb{A}^1$-curve in $X^\circ$. Furthermore, $X^\circ$ is separably $\mathbb{A}^1$-connected (resp. $\mathbb{A}^1$-uniruled) if and only if $Y^\circ$ is so.

Proof. The first statement follows from the prime-to-$p$ part of the fundamental group of $\mathbb{A}^1$ is trivial [Gro63, XIII.2.12]. The second statement follows from the first one and Definition 2.8.

3. Curves on Deligne-Faltings log schemes

3.1. Effective curve classes in Deligne-Faltings log schemes. We introduce the following type of log structures which will play an important role in our construction.

Definition 3.1. Let $X$ be a log scheme. The log structure $\mathcal{M}_X$ is called a Deligne-Faltings (DF) log structure if there is a fine, saturated, sharp monoid $P$, and a global morphism of sheaves of monoids $P_X \to \overline{\mathcal{M}}_X := \mathcal{M}_X/\mathcal{O}_X^*$ which locally lifts to a chart of $\mathcal{M}_X$. Here $P_X$ is the global constant sheaf of monoid with coefficient in the monoid $P$. We call $P_X \to \overline{\mathcal{M}}_X$ the global chart of the DF log structure $\mathcal{M}_X$. A log scheme $X$ with a global chart is called a DF log scheme.

For convenience, we may write $P$ for the constant sheaf $P_X$ when there is no confusion about the space $X$. 
Example 3.2. When $X$ is a log smooth variety with simple normal crossings boundary, the log structure $\mathcal{M}_X$ is of Deligne-Faltings type, see [Kat89, Complement 1].

Proposition 3.3. Let $X$ be a log scheme with a DF log structure $\mathcal{M}_X$, and a global chart $\beta : P \to \overline{\mathcal{M}}_X$. Then there exists a natural map of monoids

$$\delta \to (q^{-1}(\beta(\delta)))^\vee,$$

where $q : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ is the quotient morphism.

Proof. Note that $(q^{-1}(\beta(\delta)))$, hence its dual is an $O_{\Delta}$-torsor, which corresponds to a unique element in Pic$(X)$. ♠

Since Pic$(X)$ is a group, the monoid morphism $L_\beta$ naturally extends a group morphism, which will be again denoted by $L_\beta$:

$$P^{gp} \to \text{Pic}(X).$$

Definition 3.4. Let $X$ be a generalized Deligne-Faltings log scheme with a fixed global chart $\beta : P \to \overline{\mathcal{M}}_X$. The total $\beta$-contact order of an effective one-cycle $F$ is an element $c_\beta(F) \in (P^{gp})^\vee$ with

$$c_\beta(F)(\delta) = \deg L_\beta(\delta)|_F.$$

This yields the morphism:

$$c_\beta : \text{NE}(X) \to (P^{gp})^\vee.$$

An effective curve $F$ is $\beta$-admissible if $c_\beta(F) \in P^\vee$. Denote by $\text{NE}_\beta(X) \subset \text{NE}(X)$ the semi-group of $\beta$-admissible effective one-cycles on $X$, and by $\text{TC}_\beta(X)$ be set of total $\beta$-contact orders of $\beta$-admissible effective one-cycles, i.e., the image of the following map:

$$c_\beta : \text{NE}_\beta(X) \to P^\vee.$$

For convenience, we assume $0 \in \text{NE}_\beta(X)$. Hence the semi-group $\text{NE}_\beta(X)$ forms a cone.

Remark 3.5. In general, the global chart of a DF log structure needs not to be unique, hence the contact order morphism (3.1.2) is not unique, see for example [AC, Section 3.5]. A canonical treatment of the contact orders has been studied in [ACGM].

Nevertheless, the cone $\text{NE}_\beta(X)$ does not depend on the choice of global charts. To prove this result, we need the following:

Lemma 3.6. Let $X$ be a log scheme with a DF log structure $\mathcal{M}_X$ and a global chart $\beta : P \to \overline{\mathcal{M}}_X$. Assume there is another global chart $\beta' : \mathbb{N} \to \overline{\mathcal{M}}_X$. Then there is a canonical morphism $\psi : \mathbb{N} \to P$ such that $\beta' = \beta \circ \psi$. 
Proof. Denote by \( \delta' \) the generator of \( \mathbb{N} \). Consider the set \( \Lambda := \beta^{-1}(\beta'(\delta')) \). For each \( \delta \in \Lambda \), denote by \( D_\delta \subset X \) the locus where \( \beta(\delta) \) is non-trivial in \( \overline{M}_X \). Note that since \( \beta' \) is a global chart, \( \overline{M}_{X,x} \) is either trivial or isomorphic to \( \mathbb{N} \). Thus we have \( D_\delta \cap D_\delta = \emptyset \) if \( \delta_1 \neq \delta_2 \). Furthermore, along \( D_\delta \), the chart \( \beta_\delta : \mathbb{N} = \langle \delta \rangle \to \overline{M}_X \) must coincide with the chart \( \beta' \). Thus, denote by \( e = \sum_{\delta \in \Lambda} \delta \). Note that this is a finite sum, since \( P \) is fine, saturated and sharp. We then obtain the morphism \( \psi : \mathbb{N} = \langle e \rangle \to P \) as in the statement. ♠

Proposition-Definition 3.7. Let \( X \) be a DF log scheme with a global chart \( P \to \overline{M}_X \). The cone \( \text{NE}_\beta(X) \) does not depend on the choice of the global chart \( \beta \). We may thus write \( \text{NE}(X) \) and omit the chart \( \beta \) in the notation. The effective curve classes in \( \text{NE}(X) \) will be called log admissible.

Proof. Let \( F \subset X \) be an effective curve. Assume \( F \) is \( \beta \)-admissible but not \( \beta' \)-admissible for another global chart \( \beta' : P' \to \overline{M}_X \). We may thus assume that \( \deg L_{\beta'}(\delta) \mid F < 0 \). By Lemma 3.6, we known this is impossible if \( P' \cong \mathbb{N} \). We then reduce the general situation to the rank one case as follows.

For any element \( \beta \in P' \), consider the submonoid \( \mathbb{N} \cong \langle \delta \rangle \subset P' \). Denote by \( Q := \beta^{-1}(\beta'(\langle \delta \rangle)) \). Then \( Q \) is a fine, saturated, and sharp monoid. Denote by \( N_\delta \) the preimage of \( \langle \delta \rangle \) under the quotient map \( M_X \to \overline{M}_X \). Thus, we obtain two global charts \( \beta_Q : Q \to \overline{N}_X \) and \( \beta_\delta : \mathbb{N} \to \overline{N}_X \). Now the statement follows by applying the rank one case to the charts \( \beta_Q \) and \( \beta_\delta \). ♠

The major focus of this paper is \( \mathbb{A}^1 \)-curves. For later use, we introduce:

Definition 3.8. Let \( \text{NE}_{\mathbb{A}^1}(X) \subset \text{NE}(X) \) be the set of effective curve classes of \( \mathbb{A}^1 \)-curves. When \( X \) is of DF type with a global chart \( \beta : P \to \overline{M}_X \), denote by \( \text{TC}_{\mathbb{A}^1,\beta}(X) \) the set of \( \beta \)-contact orders of \( \mathbb{A}^1 \)-curves.

3.2. Effective rational curve classes in the center. Let \( X \) be a log scheme. Then by [Ols03, Lemma 3.5] there is a canonical stratification \( \{X_\lambda\}_{\lambda \in \Lambda} \) associated to \( X \) such that

1. \( X_\lambda \to X \) is a connected locally closed subscheme with the pull-back log structure.
2. The sheaf of groups \( \overline{M}_{X,\lambda}^{\text{op}} \) is a locally constant sheaf.
3. \( X = \cup \lambda X_\lambda \) is a disjoint union.

Denote by \( \overline{X}_\lambda \) the closure of \( X_\lambda \) in \( X \). Then \( X_\lambda \) is called a center of \( X \) if \( X_\lambda = \overline{X}_\lambda \).

For later use, we use the letter \( Y \) for a connected center of \( X \), and view \( Y \) as a log scheme with log structure pulled back from \( X \). We observe that when \( X \) is log smooth, its center \( Y \) also has smooth underlying structure, see the construction in [Ols03, Lemma 3.5(ii)].
Definition 3.9. A center $Y$ of a log scheme $X$ is called of Deligne-Faltings (DF) type if the log structure $\mathcal{M}_X|_Y$ is of DF type.

For a DF-type center $Y$, a global chart $\beta : P \to \mathcal{M}_Y$ is called distinguished if the composition $P \to \mathcal{M}_X|_Y \to \mathcal{M}_{X,x}$ is an isomorphism for some $x \in Y$.

Note that the sheaf $\mathcal{M}_Y$ is a locally constant sheaf of monoids. Thus if $Y$ has a distinguished chart $\beta$ as above, then $\beta$ is an isomorphism of sheaves of monoids, and $\mathcal{M}_Y$ is a globally constant sheaf of monoids on $Y$.

The following is an observation which follows directly from toroidal modification:

Lemma 3.10. Let $Y$ be a center of a log smooth variety $X$. Let $\pi : X' \to X$ be any birational log étale morphism, and $Y' \subset X'$ is any center over $Y$. Then

1. the morphism $(\bar{\pi})_*^{gp}|_Y : \pi_*^{gp}\mathcal{M}^{gp}_{Y'} \to \mathcal{M}^{gp}_Y$ is a canonical isomorphism of sheaves of groups.
2. the underlying schemes $Y$ and $Y'$ are smooth in the usual sense;
3. the underlying morphism $\pi|_{Y'} : Y' \to Y$ is finite and étale.

We are particularly interested in the following situation.

Lemma 3.11. Let $X$ be a log smooth variety, and $Y \subset X$ is a center of $X$. Assume $Y$ is proper and separably rationally connected. Then $\mathcal{M}_Y$ is a globally constant sheaf of monoids over $Y$. In particular the center $Y$ is of DF-type with a distinguished global chart.

Proof. Let $X' \to X$ be a birational log étale morphism such that $X'$ has simple normal crossings boundary. Such resolution exists over algebraically closed field of arbitrary characteristics, see [Niz06, ACMW]. Let $Y' \subset X'$ be a center over $Y$. Since $X'$ has simple normal crossings boundary, the sheaf of free monoids $\mathcal{M}_{Y'} \cong \mathbb{N}_k^p$ is a globally constant sheaf of monoids for some positive integer $k$.

By [Deb03, Corollary 3.6], the separably rationally connectedness of $Y'$ implies the underlying morphism $Y' \to Y$ is an isomorphism. Since $\mathcal{M}_Y$ is a constant sheaf of monoids, by Lemma 3.10, the sheaf of groups $\mathcal{M}^{gp}_Y$ is a constant sheaf of monoid. Then the canonical inclusion $\mathcal{M}_Y \subset \mathcal{M}^{gp}_Y$ implies that $\mathcal{M}_Y$ is globally constant sheaf of monoids. This proves the statement.♠

We next observe that total contact order of an effective curve in the center is invariant under log étale resolution.

Lemma 3.12. Let $Y$ be a center of a log smooth variety $X$. Let $\pi : X' \to X$ be any birational log étale morphism, and $Y' \subset X'$ is any center over $Y$. Assume there are two distinguished charts $\beta : P \to \mathcal{M}_Y$ and $\beta' : Q \to \mathcal{M}_{Y'}$. For any effective rational curve $f : C \cong \mathbb{P}^1 \to Y$, let $f' : C' \cong \mathbb{P}^1 \to Y'$ be a lift of $f$. Then we have

$$c_\beta(C) = c_{\beta'}(C')$$
in \((Q^{gp})^\vee = (P^{gp})^\vee\).

**Proof.** Consider the following commutative diagram over \(Y'\):

\[
\begin{array}{c}
\pi^* M_Y \longrightarrow M_{Y'} \\
\downarrow \quad \downarrow \\
\pi^* M_Y' \longrightarrow M_{Y'} \\
\cong \quad \cong \\
P \longrightarrow Q.
\end{array}
\]

We want to determine \(\deg L_{\beta}(\delta)|_{\mathcal{C}}\) for each \(\delta \in Q\). By (3), the bottom arrow induces an isomorphism \(P^{gp} \cong Q^{gp}\). The statement then follows. \(\star\)

Let \(Y\) be a DF-type center of a log smooth variety \(X\) with a chart \(\beta : P \rightarrow \overline{\mathcal{M}}_Y\). Consider the subset \(\mathcal{C}_\beta(Y) \subset P^\vee\) consisting of \(0 \in P^\vee\) and \(\mathcal{C}_\beta(\mathcal{C})\) for any irreducible free rational curve \(\mathcal{C} \rightarrow Y\). Since \(Y\) is a center, it is smooth in the usual sense. Thus any two free rational curves in \(Y\) can glued and smoothed out to another free rational curve in \(Y\). This implies that

**Lemma 3.13.** Under the above assumptions, the subset \(\mathcal{C}_\beta(Y)\) is closed under addition.

**Definition 3.14.** We call \(\mathcal{C}_\beta\) the cone of \(A^1\)-contact orders of free rational curves supported on \(Y\). When \(\beta\) is a distinguished chart, we may use \(\mathcal{C}(Y)\) when there is no confusion about the chart \(\beta\).

The following definition carries an important feature of the case with boundaries:

**Definition 3.15.** Let \(\beta : P \rightarrow \overline{\mathcal{M}}_Y\) be a global chart. A center \(Y \subset X\) is called fully free if \(\mathcal{C}_\beta(Y)\) spans the vector space \(P^\vee \otimes \mathbb{Z}\). A fully free center \(Y\) is called primitive if \(\mathcal{C}_\beta(Y)\) spans the lattice \((P^{gp})^\vee\).

4. **Logarithmic comb construction**

4.1. **Combinatorial obstructions.** We start our comb construction for the case of simple normal crossings boundary. Let \(X\) be a log smooth scheme with simple normal crossings boundary \(D\) given by smooth irreducible components \(D_1, \ldots, D_k\). For any \(\lambda \subset [k] := \{1, \ldots, k\}\), denote by \(D_\lambda = \cap_{i \in \lambda} D_i\). We view \(D, D_i,\) and \(D_\lambda\) as usual schemes with the trivial log structure, and denote by \(D_\lambda^\dagger\) the log scheme over \(D_\lambda\) with log structure \(\mathcal{M}_\lambda := \mathcal{M}_X|_{D_\lambda}\).

**Construction 4.1.** We assume that \(D_\lambda \subset X\) is a center. Write \(n := |\lambda|\). Let \(f : \mathcal{C} \rightarrow D_\lambda\) be a usual stable map such that

1. \(\mathcal{C} = \mathcal{C}_0 \cup \mathbb{P}_1 \cup \cdots \cup \mathbb{P}_m\) is a usual prestable curve with a unique marking \(q_\infty \in \mathcal{C}_0\).
2. \(\mathcal{C}_0\) is a smooth irreducible curve of genus \(g\), and \(\mathbb{P}_i\) is an irreducible rational curve attached to \(\mathcal{C}_0\) at a unique node \(r_i\).
(3) \( c_{ij} := \sum f_0 [P_i] \cap D_j \geq 0 \) for each \( i = 1, \ldots, k \) and \( j \in \lambda \). Write \( \vec{c}_i = (c_i^1, \ldots, c_i^n) \).

(4) \( c_{\infty j} := \sum f_0 [C] \cap D_j \geq 0 \) for all \( j \in \lambda \). Write \( \vec{c}_\infty = (c_\infty^1, \ldots, c_\infty^n) \).

**Remark 4.2.** For the convenience of future use, we slightly generalize our discussion to the case with possibly higher genus handle. However, the major focus of the current paper will be the genus \( g = 0 \) case.

Our goal is trying to understand when \( f \) can be lift a stable log map to \( D_\lambda^1 \), hence to \( X \). The first obstruction to have such lifting is on the level of characteristic monoids, which is known as the admissibility in the rank one case [Che14, Definition 3.3.6].

Denote by \( P := \bigoplus_{j \in \lambda} N_j \). Since \( X \) has simple normal crossings boundary, the log structure \( M_X \) hence \( M_\lambda \) are of Deligne-Faltings type. Furthermore, there is a distinguished chart

\[
\beta : P \to \overline{M}_\lambda
\]

which locally lifts to a chart of \( M_\lambda \) over \( D_\lambda^1 \). We further assume that each copy \( N_i \) corresponds to the log structure from \( D_j \). Write \( c_i := c_i(\mathbb{P}_i) \) for each \( i \), where \( c_i \) is given in (3.1.2). Denote by \( \{\delta_1, \ldots, \delta_n\} \) the standard generator of \( P \). Then we have

\[
c_i(\delta_j) = c_{ij}.
\]

To further proceed, we introduce the following:

**Notation 4.3.**

(1) For each \( \mathbb{P}_i \) we introduce the monoid \( P_i \cong P \) with the set of standard generators \( e_{i1}, \ldots, e_{in} \). Here the \( e_{ij} \) should be viewed as the vertex element of \( P_i \) corresponding to \( D_j \) as in [AC, 4.1.2].

(2) For each node \( r_i \) we introduce the monoid \( N_i = \langle l_i \rangle \cong N \). Here the element \( l_i \) should be viewed as the edge element of the node \( r_i \) as in [AC, 4.1.2].

(3) For each \( i \in \{1, \ldots, m\} \), we introduce a morphism of monoids

\[
\phi_i := id \oplus c_i : P_0 \to P_i \oplus N_i
\]

given by

\[
\phi_i(\delta) = \delta + c_i(\delta) \cdot l_i
\]

for \( \delta \in P_0 \cong P_i \). This should be viewed as the edge equation as in [AC, 4.1.2].

Note that the collection of monoids \( \{P_0, P_1 \oplus N_1, \ldots, P_m \oplus N_m\} \) with the set of morphisms \( \{\phi_i\} \) forms a finite system in the category of monoids, denoted by \( \Phi \). Denote by

\[
\overline{M} := \lim \limits_{\to} \Phi
\]

the direct limit of \( \Phi \) in the category of fine saturated sharp monoids. Such limit exists as the finite colimit exists in the category of finitely generated
monoids [Ogu06]. One checks that \( \overline{M} \) is same as the minimal monoid as introduced in [AC, 4.1.2]. By the formulation, we have the natural maps

\[ \chi_i : P_i \to \overline{M} \]

and

\[ \Theta_i : N_i \to \overline{M}. \]

**Lemma 4.4.** Both \( \chi_i^{-1}(0) \) and \( \Theta_i^{-1}(0) \) are the trivial monoid for any \( i \).

**Proof.** Consider the set of morphisms

\[ g_0 = id_P : P_0 \to P \]

and

\[ g_i := id \oplus 0 : P_i \oplus N_i \to P \]

where \( g_i \) is the projection to its first factor for \( i \neq 0 \). Thus the set \( \{ g_i \}_{i=0}^m \) induces a morphism from the system \( \Phi \) to \( P \). Since \( P \) is a fine and saturated sharp monoids, we obtain a morphism

\[ g : \overline{M} \to P. \]

By the choice of \( g_i \), for any \( \delta \in P_i \) we have \( g_i(\delta) = 0 \) if and only if \( \delta = 0 \) in \( P_i \). This implies that \( \chi_i^{-1}(0) \) for any \( i \).

For each \( i \neq 0 \), we introduce another set of morphisms

\[ h_0 := c_i : P_0 \to \mathbb{N}, \quad h_i := 0 \oplus id : P_i \oplus N_i \to \mathbb{N} \]

and

\[ h_j := c_j \oplus 0 : P_j \oplus N_j \to \mathbb{N}, \quad \text{for } j \neq i. \]

Again the set of morphisms \( \{ h_i \}_{i=0}^m \) induces a morphism from the system \( \Phi \) to \( \mathbb{N} \), hence a morphism \( \overline{M} \to \mathbb{N} \). Furthermore, we check that the composition \( N_i \to \overline{M} \to \mathbb{N} \) is the identity. This proves that \( \Theta_i^{-1}(0) \) is trivial. \( \qed \)

**Remark 4.5.** If \( f \) can be lifted to a stable log map \( f : C/S \to X \), then the base monoid \( \overline{M}_S \) will automatically satisfy the conditions in Lemma 4.4. Such property of \( \overline{M} \) is called admissible as in [Che14]. This lemma shows that the obstruction on the level of the monoid vanishes.

For later use, we will identify the elements \( l_i \) and \( e_{ij} \) with their images \( \chi_i(e_{ij}) \) and \( \Theta_i(l_i) \) in \( \overline{M} \) when there is no confusion about the monoid.

### 4.2. Lift to stable log maps.

We adopt the notations as in the previous section.

Let \( C^\sharp \to S^\sharp \) be the log curve with the standard log structure associated to the underlying pre-stable curve \( (C, q_\infty) \). We refer to [Ols07] and [Kat96] for more details of the standard log curves. We may fix a chart

\[ \beta^\sharp : P^\sharp := \oplus_{i=1}^m N_i \to \mathcal{M}_{St}. \]

Using the canonical map \( \oplus_i \Theta_i : P^\sharp \to \overline{M} \) and the chart \( \beta^\sharp \), we form a new log structure over \( S^\sharp := S^\sharp \):

\[ \mathcal{M} := \overline{M} \oplus_{P^\sharp} \mathcal{M}_{St}. \]
Note that the inclusion
\[ \beta : \overline{M} \to M \]
defines a chart of \( M \). Denote by \( S = (\mathcal{S}, \mathcal{M}) \). The morphism \( M_{S^\sharp} \to M \) defines a morphism of log schemes
\[ S \to S^\sharp. \]
Denote by
\[ C := (C, \mathcal{M}_C) = C^\sharp \times_{S^\sharp} S \]
where the fiber product is taken in the category of log schemes. We thus obtain the log curve \( \pi : C \to S \) over the underlying prestable curve \((C, q_\infty)\). We now need to construct a log map \( f : C/S \to X \). This amounts to construct the map of log structures:
\[ f^\# : f^* \mathcal{M}_X \to \mathcal{M}_C. \]
Recall we have a standard basis \( \{\delta_j\} \) of \( P \) such that \( \delta_j \) corresponds to the divisor \( D_j \) when there is no danger of confusion. Denote by \( \sigma \) the local coordinate near \( q_\infty \), and \( \log \sigma \) the corresponding image in \( \mathcal{M}_C \). For each node \( r_i \), denote by \( x_i \) and \( y_i \) the two coordinates of \( r_i \) on \( C_0 \) and \( \mathbb{P}_i \) respectively. Let \( \log x_i \) and \( \log y_i \) be the corresponding images in \( \mathcal{M}_C \). Choosing the coordinates properly, we may assume that
\[ \log x_i + \log y_i = l_i \]
where \( l_i \) is identified with its image \( \beta(l_i) \) in \( \mathcal{M}_C \). On the level of characteristic monoids, we have

**Lemma 4.6.** There exists a unique morphism of sheaves of monoids
\[ \tilde{f}^\# : \tilde{f}^* \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C \]
determined by
\[
(1) \quad \tilde{f}^\#(\delta_j) = e_{0j} + c_{\infty j} \cdot \log \sigma \text{ at the marking } q_\infty; \\
(2) \quad \tilde{f}^\#(\delta_j) = e_{ij} + c_{ij} \cdot \log y_i \text{ at the node } r_i.
\]
Here we identify \( \log \sigma \) and \( \log y_i \) with their corresponding images in \( \overline{\mathcal{M}}_C \).

**Proof.** It suffices to check the compatibility over the non-marked, smooth locus of \( C \). One may then check that the compatibility is precisely the minimality of the monoid \( \overline{\mathcal{M}} \) given by (4.1.3).

Denote by \( \psi_X : \tilde{f}^* \mathcal{M}_X \to \tilde{f}^* \overline{\mathcal{M}}_X \) and \( \psi_C : \mathcal{M}_C \to \overline{\mathcal{M}}_C \) the quotient morphisms. The inverse images \( T_j := \psi^{-1}(\delta_j) \) and \( \mathcal{T}_j := \psi_C^{-1}(\tilde{f}^\#(\delta_j)) \) forms a \( \mathcal{O}_C^\ast \)-torsor. We observe that

**Lemma 4.7.** To define a log map \( f : C/S \to X \) with \( \tilde{f}^\# \) described as in Lemma 4.6 is equivalent to have a set of isomorphisms of torsors
\[ T_j \to \mathcal{T}_j, \quad \text{for each } j \in \lambda. \]
Proposition 4.8. There exists a stable log map \( f : C/S \to X \) over the underlying map as in Construction 4.1 if and only if for each \( j \in \lambda \), there is an isomorphism of line bundles

\[
N_{D_j}|_{\mathcal{C}_0} \cong \mathcal{O}_{\mathcal{C}_0}(c_{\infty j} \cdot q_{\infty} - \sum_{i=1}^{m} c_{ij} \cdot r_i), \quad \text{over } \mathcal{C}_0.
\]

Proof. We notice that the restriction \((T_j)|_{\mathcal{C}_0}\) is the torsor associated to \(N_{D_j}|_{\mathcal{C}_0}\), and \(T_j|_{\mathcal{C}_0}\) is the torsor associated to \(\mathcal{O}_{\mathcal{C}_0}(-c_{\infty j} \cdot q_{\infty} + \sum_{i=1}^{m} c_{ij} \cdot r_i)\). Furthermore, the restriction \((T_j)|_{\mathbb{P}_i}\) is the torsor associated to \(\mathcal{O}_{\mathbb{P}_i}(-c_{ij})\), and \(T_j|_{\mathbb{P}_i}\) is the torsor associated to \(\mathcal{O}_{\mathbb{P}_i}(-c_{ij} r_i)\). Since the curve \(\mathcal{C}\) is a comb with \(\mathbb{P}_i \cong \mathbb{P}^1\), the statement follows from Lemma 4.7.

Corollary 4.9. Situations as in Construction 4.1, if \(g = 0\) then the lifting \(f\) over \(f\) always exists.

Proof. In the genus zero case, the existence of isomorphisms (4.2.2) follows from the degree consideration.

In general, we have

Proposition 4.10. Let \(X\) be a log smooth variety and \(Y \subset X\) a connected center. Consider a usual genus zero stable map \(f : \mathcal{C} \cong \mathbb{P}^1 \to Y\). Assume that there is a log étale birational modification \(\overline{X'} \to X\) with a connected center \(Y' \subset X'\) over \(Y\) such that

1. \(Y'\) is of DF-type with a distinguished global chart \(\beta : P \to \mathcal{M}_Y\), see Definition 3.9; and
2. \(V_{\beta}(f'_*|_{\mathcal{C}}) \in P'\) for some lift \(f'_* : \mathcal{C} \to Y'\) of \(f\).

Then there is an \(\mathbb{A}^1\)-curve \(f : C/S \to X\) over \(f\).

Proof. By Corollary 4.9, we could lift \(f'_*\) to an \(\mathbb{A}^1\)-curve \(f' : C/S \to Y'\). Composing \(f'\) with \(Y' \to Y \to X\) yields the \(\mathbb{A}^1\)-curve as needed.

4.3. Deformation of Combs. Consider the target \(X\) with simple normal crossings boundary as in Section 4.1. For any \(\lambda \subset [k] := \{1, \cdots, k\}\), recall that \(D_{\lambda} = \cap_{i \in \lambda} D_i\). Let \(X^\dagger_{\lambda}\) be the log scheme associated to the pair \((D_{\lambda}, \cup_{i \in \tilde{\emptyset}(\lambda)}(D_{\lambda} \cap D_i))\), where \(\tilde{\emptyset}(\lambda)\) is the collection of index \(i\) such that \(D_i \cap D_{\lambda}\) is a smooth divisor in \(D_{\lambda}\). Let \(X'\) be the log scheme associated to the pair \((X, D = \cup_{i \notin \lambda} D_i)\). We learned the following lemma from a discussion with Professor Yi Hu.

Lemma 4.11. Notations as above, there is a natural exact sequence

\[
0 \to \Omega_{X^\dagger_{\lambda}} \to \Omega_X|_{D_{\lambda}} \to \mathcal{O}_{D_{\lambda}}^{\oplus|\lambda|} \to 0.
\]

Proof. The strict log map \(D_{\lambda} \to X'\)
induces an exact sequence

\[(4.3.1) \quad 0 \to N_{D_\lambda/X}' \to \Omega_{X'}|_{D_\lambda} \to \Omega_{X|_{D_\lambda}} \to 0\]

On the other hand, there is an exact sequence

\[(4.3.2) \quad 0 \to \Omega_{X'} \to \Omega_X \to \sum_{i \in \lambda} \mathcal{O}_{D_i} \to 0\]

Tensoring with \(\mathcal{O}_{D_\lambda}\), we obtain the exact sequence:

\[(4.3.3) \quad 0 \to \sum_{i \in \lambda} \text{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \to \Omega_{X'}|_{D_\lambda} \to \Omega_X|_{D_\lambda} \to \sum_{i \in \lambda} \mathcal{O}_{D_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_\lambda} \to 0\]

For the term on the right, we have

\[\sum_{i} \mathcal{O}_{D_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_\lambda} = \sum_{i \in \lambda} \mathcal{O}_{D_\lambda}.\]

To calculate the \(\text{Tor}^1_{\mathcal{O}_X}(-, -)\), we take the resolution

\[0 \to \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0\]

Tensoring with \(\mathcal{O}_{D_\lambda}\) we have the exact sequence:

\[(4.3.4) \quad 0 \to \text{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \to \mathcal{O}_{D_\lambda}(-D_i) \to \mathcal{O}_{D_i} \to \mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda} \to 0\]

Note that if \(i \notin \lambda\), then we have

\[\text{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) = 0\]

since they intersect transversally.

If \(i \in \lambda\), we have

\[\mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda} = \mathcal{O}_{D_\lambda}\]

Thus the arrow

\[\mathcal{O}_{D_\lambda} \to \mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda}\]

in (4.3.4) is an isomorphism. This yields

\[\text{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{E_i}) = \mathcal{O}_{D_\lambda}(-D_i) = N_{D_i/X}'|_{D_\lambda}\]

hence

\[\sum_{i} \text{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \cong N_{D_\lambda/X}'|_{D_\lambda}.\]

We deduce

\[(4.3.5) \quad 0 \to N_{D_\lambda/X}'|_{D_\lambda} \to \Omega_{X'}|_{D_\lambda} \to \Omega_X|_{D_\lambda} \to \sum_{i \in \lambda} \mathcal{O}_{D_i} \to 0\]
Putting (4.3.5) and (4.3.1) together, we have the following commutative diagram of solid arrows with the exact row and column:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \rightarrow & N_{D_{\lambda}/X}^Y & \rightarrow & \rightarrow & \Omega_{X|D_{\lambda}} & \rightarrow & \Omega_{X|D_{\lambda}} & \rightarrow & \sum_{i \in \lambda} O_{D_{\lambda}} & \rightarrow & 0 \\
& & & \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & & \approx & & & \rightarrow & & \rightarrow & & \rightarrow & & \\
0 & \rightarrow & \rightarrow & N_{D_{\lambda}/X}^Y & \rightarrow & \rightarrow & \Omega_{X|D_{\lambda}} & \rightarrow & \Omega_{X|D_{\lambda}} & \rightarrow & \sum_{i \in \lambda} O_{D_{\lambda}} & \rightarrow & 0 \\
\end{array}
\]

The above diagram solid arrows induces the dashed arrow making the whole diagram commutative. This proves the statement.

We next study the comb-smoothing technique under the logarithmic setting. Consider the following situation:

**Notation 4.12.** Let \( f : C/S \rightarrow X \) be a genus \( g \) stable log map over the geometric point \( S \) with contact markings \( q_{\infty 1}, \cdots, q_{\infty k} \) such that

1. \( C = \bigcup_{i=0}^{n} C_i \) with smooth irreducible component \( C_i \cong \mathbb{P}^1 \) for \( i \neq 0 \), and a smooth genus \( g \) component \( C_0 \).
2. For each \( i \neq 0 \), we have a unique node \( r_i \in C \) joining \( C_i \) and \( C_0 \). All nodes of \( C \) are of this form. Let \( p_i \) and \( q_i \) denote the special point on \( C_i \) and \( C_0 \) corresponding to \( r_i \).
3. \( f(C_0 \setminus \{q_{\infty 1}, \cdots, q_{\infty k}\}) \subset X_{\lambda} \).
4. \( \bar{c}_i := (c_{ij})_{j \in \lambda} \) where \( c_{ij} = f_*[C_i] \cap D_j \).

Note that the third condition implies that there is a natural log map

\[
f_0 : C'_0 \rightarrow X'_{\lambda}
\]

over the underlying map \( f|_{C_0} \).

We further assume that \( \bar{c}_i \in N_k \) is a non-trivial integral vector, and \( f \) is a local immersion away from the nodes and the contact markings.

Consider the following sequences:

\[
df : f^* \Omega_X \rightarrow \Omega_{C/S}.
\]

and

\[
df_0 : f_0^* \Omega_{X'_{\lambda}} \rightarrow \Omega_{C'_0}.
\]
By [Ols05], the above sequences governs the deformation of $f$ and $f_i$ respectively. The assumption on $\vec{c}_i$ and the local immersion of $f$ implies that $df$ and $df_i$ are surjective with locally free kernel $N_f^\vee$ and $N_{f_i}^\vee$ respectively. We now consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & N_{f_0}^\vee & \to & N_{f_0}^\vee|_{C_0} & \to & V & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & f_0^*\Omega_{X_1} & \to & f_0^*\Omega_X|_{C_0} & \to & \mathcal{O}_{\square}|_{X_0} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega_{C_0} & \to & \Omega_{C}|_{C_0} & \to & \sum_i k_{q_i} & \to & 0 \\
\end{array}
\]

where each rows and columns are exact. A local calculation at $q_i$ shows that

**Lemma 4.13.** The restriction $\phi|_{q_i}$ is given by the $(1 \times |\lambda|)$-matrix $(-\vec{c}_i)$. In particular, the vector bundle $V^\vee$ is the elementary transform of the trivial vector bundle along the logarithmic direction $\vec{c}_i$'s.

4.4. **Proof of Theorem 1.3.** Let $X' \to X$ be a log étale projective birational modification. Since $X$ admits a DF-type fully free center, by Lemma 3.12 $X'$ also admits a fully free center. Replacing $X$ by a log étale resolution, we may assume that $X$ has the normal crossings boundary.

We then pick a very free rational curve $f_0 : C_0 \to Y$. We may further assume that $f_0$ is an immersion by [Kol96, II. 1.8]. This can be achieved by for example replacing $X$ by $X \times \mathbb{P}^2$. Note that if we could construct the desired very free $A^1$-curve in $X \times \mathbb{P}^2$, then taking the composition with the projection $X \times \mathbb{P}^2 \to X$ will yield the desired very free $A^1$-curve in $X$.

Since the center $Y$ is fully free, we may glue teeth $f_i : P_i \to Y$ for $i = 1, \cdots, m$ to the handle $C_0$ such that $f_i$ is a free rational curve in $Y$ with non-negative and sufficiently general $\vec{c}_i$ as in Construction 4.1(3). Furthermore, using the product trick as above, we may assume that each $f_i$ is an immersion. This yields a underlying stable map $f : C \to Y$ as in Construction 4.1. We may pick an arbitrary smooth point $q_{\infty} \in C_0$. By Corollary 4.9, we may lift $f_0$ to a stable log map $f : C/S \to X$. By Lemma 4.13 and (4.3.7), since the choice of $\vec{c}_i$ is sufficiently general, and $m$ is sufficiently large, we may assume that $N_f|_{C_0}$ is ample. Note that $f^*T_X$ is semi-positive. Thus $f$ is unobstructed. A general deformation of $f$ produces a very free $A^1$-curve.
This finishes the proof of Theorem 1.3.

5. WONDERFUL COMPACTIFICATIONS

5.1. Basics on spherical varieties. Let $G$ be a linearly reductive $k$-group. Let $T$ be a maximal torus of $G$ and $B$ be a Borel subgroup containing $T$. For any subgroup $H \subset G$, denote by $H^u$ the unipotent radical of $H$. We write $\mathcal{X}^*(H)$ and $\mathcal{X}_*(H)$ for the character and cocharacter groups of $H$ respectively.

**Definition 5.1.** Let $\mathcal{X}$ be a $G$-variety. The variety $\mathcal{X}$ is a (resp. separably) spherical variety if it is normal, and contains a dense open (resp. separable) $B$-orbit. A subgroup $H \subset G$ is (resp. separably) spherical if $G/H$ is so.

For the rest of this section, $G/H$ will denote a spherical homogeneous space. We fix a point $o \in G/H$ whose $B$-orbit is dense. We use $\mathcal{X}$ to denote a spherical variety which contains $G/H$ as the open $G$-orbit.

For reader’s convenience, here we collect a list of standard terminologies and results that are known to experts. Those might be found in standard literatures, see for example [Kno91].

1. Denote by $k(\mathcal{X})^0(B)$ the set of $B$-eigenfunctions ($B$-semi-invariant functions) on $\mathcal{X}$ given by
   \[ \{ f \in k(\mathcal{X}) \setminus \{0\} \mid f(o) = 1, bf = \chi(b)f, \forall b \in B, \text{ where } \chi \in \mathcal{X}^*(B) \}. \]

2. Let $\Lambda := \Lambda(\mathcal{X})$ be the set of weights of $k(\mathcal{X})^0(B)$. It is a finitely generated free abelian group. Its rank is called the rank of $G/H$. Indeed we have
   \[ k(G/H)^0(B) \cong \Lambda. \]

   We use $f_\lambda$ to denote the $B$-eigenfunction determined by $\lambda \in \Lambda$.

3. Define the valuation space $\mathcal{N}(\mathcal{X}) := \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ with the integral structure $\mathcal{N}(\mathcal{X})^\mathbb{Z} = \Lambda^\vee$.

4. Let $\Delta(\mathcal{X})$ be the set of $B$-stable but not $G$-stable prime divisors on $\mathcal{X}$, called the set of colors. Let $D_G(\mathcal{X})$ be the set of $G$-invariant valuations on $\mathcal{X}$.

5. Any discrete valuation $\nu : k(\mathcal{X})^* \to \mathbb{Q}$ gives an element $\rho_\nu$ in $\mathcal{N}(\mathcal{X})$ by restriction to $k(\mathcal{X})^0(B)$.

6. The valuation map $D_G(\mathcal{X}) \to \mathcal{N}(\mathcal{X})$ is injective. Denote by $\mathcal{V}(\mathcal{X})$ the $\mathbb{Q}$-cone generated by the image of $D_G(\mathcal{X})$ in $\mathcal{N}(\mathcal{X})$. This is a convex cone, called the valuation cone.

7. In fact, those data $\Lambda(\mathcal{X}), \mathcal{N}(\mathcal{X}), \Delta(\mathcal{X}), \mathcal{V}(\mathcal{X})$ defined above only depend on the dense $G$-orbit $G/H$. We simply omit $\mathcal{X}$ and use $\Lambda, N, \Delta, \text{ and } \mathcal{V}$.

8. The valuations given by $\Delta$ induce a map $\rho|_\Delta : \mathbb{Z}^\Delta \to N_{\mathbb{Z}}$.

9. A spherical variety $\mathcal{X}$ is called toroidal if no $D \in \Delta$ contains a $G$-orbit in its closure. It is called simple if it has a unique closed $G$-orbit. We say that $\mathcal{X}$ is a compactification of $G/H$ if it is proper.
A spherical subgroup $H \subset G$ is called sober if $N_G(H)/H$ is finite.

When $H$ is sober and separably spherical, the valuation cone $\mathcal{V}$ associated to $G/H$ is a strictly convex full dimensional cone in $N$, see [Kno91, Corollary 5.3, Theorem 6.1]. In this case, any $G$-equivariant toroidal compactification of $G/H$ is uniquely determined by a fan $\Sigma$ supported on $\mathcal{V}$. The fan $\Sigma$ and $\mathcal{V}$ are called the colored fan and colored cone respectively.

For a sober and separably spherical subgroup $H$, the compactification $X$ of $G/H$ associated to the colored fan $(\mathcal{V}, \emptyset)$ is called the wonderful compactification of $G/H$. In this case $X$ is both toroidal and simple with a unique closed orbit $Y \subset X$. We may assume that $G$ is semisimple [Bri03, Lemma 1].

5.2. Log structures on toroidal embeddings.

The goal of this section is to define a natural $G$-equivariant log structure on toroidal spherical varieties and to find a good criterion for a spherical variety to be log homogeneous in the following sense.

Definition 5.2. Let $H$ be an algebraic group. A log scheme $Y = (\overline{\mathcal{Y}}, \mathcal{M}_Y)$ is a log $H$-scheme if both the underlying scheme $\overline{\mathcal{Y}}$ and the sheaf of monoids $\mathcal{M}$ admit an $H$-action such that the structure morphism $\mathcal{M}_Y \to \mathcal{O}_{\overline{\mathcal{Y}}}$ is $H$-equivariant.

Definition 5.3. A log smooth log $H$-variety $X$ is called log homogeneous if the morphism of sheaves $\mathfrak{h} \otimes \mathcal{O}_X \to T_X$ is surjective, where $\mathfrak{h}$ is the Lie algebra of $H$ and $T_X$ is the log tangent bundle.

The main result of this subsection is the following:

Proposition 5.4. Let $X$ is a toroidal compactification of a separably spherical homogeneous space $G/H$. Assume all closed $G$-orbits of $X$ are separable. Then

1. the pair $(X, X \smallsetminus G/H)$ is toroidal, which yields a log smooth log $G$-variety $X$;
2. the log smooth log $G$-variety $X$ is log homogeneous.

We first recall the following results on the local structure of spherical varieties. For any spherical $G$-variety $X$ and any $G$-orbit $Y \subset X$, we define the open set:

$$X_{Y,B} = X \setminus \cup D,$$

where the union is taken over all $B$-stable prime divisors that do not contain $Y$.

Theorem 5.5 (Local structure theorem). Let $X$ be a spherical $G$-variety and $Y \subset X$ a $G$-orbit containing $x$ such that $Bx$ is open in $Y$. Define the parabolic subgroup $Q$ to be the stabilizer of $X_{Y,B}$ and choose a Levi subgroup $L$ of $Q$. Then:
(1) $X_{Y,B}$ is affine $B$-stable;
(2) There exists an affine $T$-stable closed subvariety $M$ of $X_{Y,B}$ such that the action morphism $\mu : Q^u \times M \to X_{Y,B}$ is finite surjective and $\mu^{-1}(x) = \{(e,x)e\}$, where $e$ is the identity;
(3) If we further assume that $Y$ is a separable $G$-orbit, the action morphism $\mu$ is an isomorphism;
(4) If $Y$ is $G$-separable and $X$ is toroidal, the variety $M$ is an affine toric embedding of a torus $A$ given by the quotient $L/L_0$, where $L_0 \subset L$ is a subgroup containing the derived group of $L$. Furthermore, we have $\Lambda \cong X^*(A)$ via the restriction of $B$-eigenfunctions on $M$. The toric variety $M$ is determined by the colored cone associated to the orbit $Y$;
(5) With the same assumptions in (4), every $G$-orbit of $X_{Y,B}$ is of the form $Q^u Y M'$, where $M'$ is an $A$-orbit. In particular, there is a bijection between $G$-orbits in $X$ and $A$-orbits in $M$.

Proof. Statement (1) and (2) are proved in [Kno93, Theorem 1.2].

For (3), since we can $G$-equivariantly embed $X$ into $\mathbb{P}(V)$, where $V$ is a simple $G$-module, it suffices to prove the case when $X = \mathbb{P}(V)$ (not necessarily spherical) and $X_{Y,B}$ is the complementary in $\mathbb{P}(V)$ of the unique $B$-stable hyperplane. This is proved in [Hur14, Proposition 1.3].

The statement (5) and the first assertion of (4) are proved in [Hur14, Theorem 1.4] (requiring $M$ to be smooth is not necessary).

By (3) and (5), it is easy to see that there is a bijection between $B$-eigenfunctions on $X$ and $A$-eigenfunctions on $M$ via restriction, which preserve the order given by inclusion of orbits. Then the second assertion of (4) follows.

Lemma 5.6. A spherical variety $X$ is covered by $G$-translations of $X_{Y,B}$ for any $G$-orbit $Y$. Furthermore, when $X$ is proper, one may choose $Y$ to be any closed $G$-orbits.

Proof. The first statement follows from [Kno91, Section 2.1]. For each $Y$, the simple embedding $G X_{Y,B}$ of $G/H$ gives a strictly colored cone $(C,F)$ [Kno91, Theorem 3.3]. When $X$ is proper, it belongs to a strictly colored cone of maximal dimension given by $G X_{Y',B}$ for a closed $G$-orbit $Y'$ with $G X_{Y,B} \subset G X_{Y',B}$. The second statement then follows.

Proof of Proposition 5.4. The first statement follows from Theorem 5.5 and Lemma 5.6.

Now consider the second statement. When $X$ is smooth, this is proved in [Hur14, Thm. 1.8]. By further subdividing the colored fan of $X$, we may take a $G$-equivariant log étale birational morphism $f : Z \to X$ such that $Z$ is smooth. Since the log structure is Zariski and $f$ is $G$-equivariant, any closed orbit of $Z$ maps isomorphically onto a closed orbit on $X$. Thus the $G$-action on the closed orbits of $Z$ is separable. Since $f^* T_X = T_Z$ and toric
singularities are rational, the projection formula implies that $f_* T_Z = T_X$.
This proves the statement of the proposition.

5.3. **Global charts of wonderful compactifications.** We now turn to
the situation of wonderful compactifications. Let $G$ be a semi-simple group,
and $H \subset G$ be a sober and separable spherical subgroup. Denote by $X$
the wonderful compactification of $G/H$ as in Section 5.1(10),(11), and (12).
Since $X$ is simple, denote by $Y \subset X$ its unique closed orbit. We consider
the following situation.

We assume that $Y$ is $G$-separable. Then we obtain a log smooth variety
$X = (X, M)$ with the $G$-action by Proposition 5.4. Denote by $X_1, \cdots, X_s$
the irreducible boundary components of $X$, and $\Delta = \{D_1, \cdots, D_s\}$
the set of colors. Let $X_\circ := X \setminus \Delta$ be the dense $B$-stable open subset,
and $X_\circ$ the corresponding log variety. The stabilizer group of $X_\circ$ is a parabolic subgroup
$Q$. Denote by $P := \mathcal{V}^u \cap \Lambda$.

**Lemma 5.7.** Assume $Y$ is $G$-separable. With the same notation as in
Theorem 5.5, for each $B$-eigenfunction $f_\lambda$ such that $\lambda \in P$, we have the
following:

1. $f$ is regular on $X_\circ$.
2. $p_2^*(f|_M) = f$, where $p_2 : X_\circ \cong Q^u \times M \to M$.

In particular, we obtain a chart

$P \to M_\circ := M|_{X_\circ}, \quad \lambda \mapsto f_\lambda.$

**Proof.** (1) is proved in [Kno91, Theorem 2.5]. For (2), it suffices to show
that each $B$-eigenfunction is constant on $Q^u \times \{m\}$ for every $m \in M$.
This follows from the fact that $Q^u$ is a unipotent subgroup of $B$ which only has
the trivial character.

**Proposition 5.8.** Assume $Y$ is $G$-separable. There exists a global chart
$\beta : P \to \overline{M}$ induced by (5.3.1), whose restriction to the unique center
$Y \subset X$ induces the distinguished chart as in Definition (3.9).

**Proof.** We first notice that the sheaf of monoids $\overline{M}|_Y$ is a globally constant
sheaf of monoid in $P$. Thus, it suffices to construct a global chart $P \to \overline{M} := M/O^\ast$.

By Lemma 5.6, $X$ is covered by $\{gX_\circ\}_{g \in G}$. For each open subset $gX_\circ$,
we construct the morphism $P \to \overline{M}|_{gX_\circ}$ sending $\lambda$ to $gf_\lambda$ similar to (5.3.1).
To check this local morphism gives a global morphism from $P$ to $\overline{M}$, it
suffices to show that $gf_\lambda$ and $f_\lambda$ differ by an invertible function on the
common intersection. Consider the rational function $gf_\lambda/f_\lambda$ on $X_\circ \cap gX_\circ$.
By Lemma 5.7, it has nontrivial zeros or poles only at the boundary divisors
$X_1 \cap X_\circ \cap gX_\circ$'s. Since $X_\circ$ is normal, and each prime boundary divisor
corresponds to a $G$-invariant valuation, $gf_\lambda/f_\lambda$ is invertible on $X_\circ \cap gX_\circ$. ♠

The above global chart $\beta$ is natural in the following sense:
Proposition 5.9. Assume Y is G-separable, and consider the morphism as in (3.1.1)

\[ L : \Lambda \cong P^{op} \to \text{Pic} X \]

induced by the global chart \( \beta \). Recall that in this situation, the group \( \text{Pic} X \) is freely generated by the colors by [Bri03, Theorem 1]. Then we have

(1) \( L(\lambda) = \mathcal{O}_X(\sum_{i=1}^t \nu_{X_i}(f_\lambda)X_i) \);
(2) the map \(-L^\vee\) is the valuation morphism on colors \( \rho|_\Delta : \mathbb{Z}^\Delta \to \mathbb{N} \) as in Section 5.1(8).

Proof. The first statement follows from the proof of Proposition 5.8. Since each eigenfunction \( f_\lambda \) has only zeros or poles along \( X_i \)'s or \( D_i \)'s, we have:

\[ \text{div} f_\lambda = \sum_{i=1}^t \nu_{X_i}(f_\lambda)X_i + \sum_{j=1}^s \nu_{D_j}(f_\lambda)D_j. \]

Thus we have the second statement. ♠

5.4. \( \mathbb{A}^1 \)-curves on wonderful compactifications. We next introduce two technique assumptions, which greatly simplifies the structure of the set of \( \mathbb{A}^1 \)-curve classes on \( X \). Later we will verify those two assumptions for a large class of interesting cases.

Hypothesis 5.10. There exists \( B \)-invariant irreducible rational curves \( B_1, \ldots, B_s \) on \( Y \) such that \( B_i \cap D_j = \delta_{ij} \) and \( \text{NE}(X) = \mathbb{N}(B_1, \ldots, B_s) \).

Lemma 5.11. Assume \( Y \) is G-separable, and Hypothesis 5.10 hold. For \( i = 1, \ldots, s \), we have

(1) \( \deg L(\lambda)|_{B_i} = -\nu_{D_i}(f_\lambda) \), and hence \( c(B_i) = -\nu_{D_i} = -\rho|_\Delta(D_i) \) with \( \epsilon \) introduced in Definition 3.4;
(2) \( \text{NE}(X) = \{ F = \sum_{i=1}^s a_i B_i \mid a_i \in \mathbb{Z}_{\geq 0} - \sum_{i=1}^s a_i \rho|_\Delta(D_i) \in V \} \);
(3) \( \text{TC}(X) = -\rho|_\Delta(\mathbb{N}^s) \cap V \).

Proof. This follows from Proposition 5.9. ♠

We further consider the following:

Hypothesis 5.12. The opposite of the valuation cone \(-V\) is contained in the \( \mathbb{Q} \)-cone generated by \( \rho(D_i) \), for \( i = 1, \ldots, s \).

Lemma 5.13. Assume Hypothesis 5.12 holds. Then the morphism \( L : \Lambda \to \text{Pic}(X) \) is injective. Furthermore, the following are equivalent:

(1) \( -\rho|_\Delta : \mathbb{Z}^\Delta \to \Lambda^\vee \) is surjective;
(2) The cokernel of \( L : \Lambda \to \text{Pic}(X) \cong \mathbb{Z}^\Delta \) is torsion-free.

Proof. Since \( V \) is a strictly convex cone in \( \mathbb{N} \), see Section 5.1(11), the injectivity follows from Proposition 5.9, and the observation that the cokernel of \( -\rho|_\Delta \) is finite. The rest of the statement is a direct consequence of the injectivity. ♠
We summarize the discussion as follows:

**Theorem 5.14.** Let $X$ be the log smooth variety associated to the wonderful compactification of a sober separably spherical homogeneous space $G/H$ with the unique $G$-separable closed orbit, see Proposition 5.4. Assume Hypothesis 5.10 and 5.12 hold. Then we have:

1. Any log-admissible effective one-cycle on $X$ can be represented by a free $\mathbb{A}^1$-curve on $G/H$; In particular, this implies $\text{NE}_{\mathbb{A}^1}(X) = \text{NE}(X)$, see Definition 3.8.
2. If $\text{char } k \nmid [\Lambda^\vee : \rho|_{\Delta}(\mathbb{Z}^\Delta)]$, then the unique $G$-closed orbit is a fully free center of $X$. In particular, $X$ is separably $\mathbb{A}^1$-connected.
3. Furthermore, the center of $X$ is primitive if and only if the cokernel of $L : \Lambda \to \text{Pic}(X) \cong \mathbb{Z}^\Delta$ is torsion-free.

**Proof.** Hypothesis 5.12 implies that both $\text{NE}(X)$ and $\text{TC}(X)$ are nonempty. Consider the first statement. For any element $F = \sum_{i=1}^s a_i B_i \in \text{NE}(X)$, there exists a rational curve $f : \mathbb{P}^1 \to Y$ whose curve class is $F$ since $Y$ is a homogeneous space. By Proposition 1.2, there exists a log map $f$ lifting $f$ as an $\mathbb{A}^1$-curve on $X$. Since by Proposition 5.4, $f$ is unobstructed in the moduli space of $\mathbb{A}^1$-curves, $f$ can be deformed to an $\mathbb{A}^1$-curve on $X$ which intersects the locus with the trivial log structure. This implies (1). The second statement follows from the above argument and Theorem 1.3. Finally (3) is a direct consequence of Lemma 5.13 and Definition 3.15.

5.5. **The characteristic zero case.** In this section, we adopt the notations as in Section 5.3, and assume $\text{char } k = 0$. The goal is to give a proof of Theorem 1.4 by verifying the assumptions in Theorem 5.14. We may assume $G$ is simply connected for what follows.

We first introduce a sufficient condition for Hypothesis 5.10. Let

$$\lambda : \mathbb{G}_m \to T$$

be a regular cocharacter. For each color $D \in \Delta$, it is proved in [Bri03, Theorem 2(1)] that there is a point $x_D \in X$ fixed under the $\lambda$-action such that $D$ is the closure of

$$\{x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x = x_D\}.$$

**Definition 5.15.** Notations as above, we say $X$ is *good* if there exists a regular cocharacter $\lambda$ such that $x_D \in Y$ for every color $D$.

To verify Hypothesis 5.10, we recall the idea of Luna on spherical closure of $H$.

**Definition 5.16.** [Lun01, Section 6.1] Let $K$ be a spherical subgroup of $G$. The automorphism group $N_G(K)/K$ naturally acts on $\Delta$. The *spherical closure* $\overline{K}$ of $K$ is the kernel of the action of $N_G(K)$ on $\Delta$. We say that a subgroup $L$ is *spherically closed* if $\overline{T} = L$.

Recall the following facts from [Tim11, Remark 30.1]:

...
Lemma 5.17. Let $\overline{\mathcal{P}}$ be the spherical closure of $H$, and $\mathbf{X}'$ be the log variety associated to the wonderful compactification of $G/\overline{\mathcal{P}}$. There is a $G$-equivariant morphism of log varieties $\pi: \mathbf{X} \to \mathbf{X}'$ with the following properties:

1. $\mathbf{X}'$ is smooth, see [Kno96, Corollary 7.6];
2. Let $\Lambda'$ and $\mathcal{V}'$ be the weight lattice of $\mathbf{X}'$ and the valuation cone of $\mathbf{X}'$ respectively. We have that $\mathcal{V} = \mathcal{V}'$ and $\Lambda' \subset \Lambda$. In particular, $\pi$ is a finite log étale morphism (or equivalently Kummer étale), and $\pi|_Y$ is an isomorphism onto the closed $G$-orbit $Y' \subset \mathbf{X}'$;
3. the set of colors $\Delta' = \{D'_1, \cdots, D'_s\}$ on $\mathbf{X}'$ is identified with $\Delta$ by pullback, i.e., $\pi^{-1}(D'_i) = D_i$.

Proposition 5.18. Assume $\text{char } k = 0$. Then Hypothesis 5.10 holds when $\mathbf{X}$ is good.

Proof. Consider the $G$-equivariant morphism $\pi: \mathbf{X} \to \mathbf{X}'$ as in Lemma 5.17. By [Bri03, Theorem 2], there exists $B$-invariant curves $B_1, \cdots, B_s$ on $Y$ and $B'_1, \cdots, B'_s$ on $Y'$ such that:

1. $\pi$ maps $B_i$ isomorphically onto $B'_i$;
2. $D'_i \cap B'_j = \delta_{ij}$.

By the projection formula, we have $D_i \cap B_j = \delta_{ij}$. By [Bri03, Theorem 2(3)], the effective divisors $D_1, \cdots, D_s$ forms a $\mathbb{N}$-linear basis of the cone of nef divisors. Hence dually $B_1, \cdots, B_s$ forms a $\mathbb{N}$-linear basis of the cone of effective curve classes of $\mathbf{X}$.

Proposition 5.19. Assume $\text{char } k = 0$. Then Hypothesis 5.12 holds.

Proof. Consider the $G$-equivariant morphism $\pi: \mathbf{X} \to \mathbf{X}'$ as in Lemma 5.17. Notice that $\mathbf{X}$ and $\mathbf{X}'$ share the same valuation cone and the same set of colors via pullback. It suffices to check statement on $\mathbf{X}'$, which is given by [Bri07c, Lemma 2.1.2].

Proposition 5.20. When $\text{char } k = 0$, we have the following short exact sequence:

$$0 \to \Lambda \to \text{Pic}(\mathbf{X}) \to \mathfrak{x}^*(H) \to 0.$$

Proof. The case when $\mathbf{X}$ is smooth is proved in [Bri07c, Proposition 2.2.1]. The proof for the general case is similar. Let $\psi: G \to G/H$ be the quotient map. The following short exact sequence in [Bri07c, (2.2.5)] still holds in our case:

$$0 \to \mathfrak{x}^*(B)^{B \cap H} \to \mathfrak{x}^*(B) \times_{\mathfrak{x}^*(B \cap H)} \mathfrak{x}^*(H) \to \mathfrak{x}^*(H) \to 0.$$
$f_D(o) = 1$, see [Lun01, Lemma 6.2.2]. Furthermore, the following diagram commutes because $\nu_D(f) = \nu_{\psi^{-1}(D)}(\pi^*f)$.

\[
\begin{array}{ccc}
\mathcal{X}^*(B)^{B\cap H} & \longrightarrow & \mathcal{X}^*(B) \times_{\mathcal{X}^*(B\cap H)} \mathcal{X}^*(H) \\
\Lambda & \longrightarrow & \text{Pic}(X)
\end{array}
\]

The proposition then follows. ♠

**Proof of Theorem 1.4.** The statements follow from combining Theorem 5.14 and Proposition 5.18, 5.19 and 5.20. Note that $H$ is connected implies $\mathcal{X}^*(H)$ is torsion-free.♠

### 5.6. Semisimple groups

We next turn to wonderful compactifications of semisimple groups. Let $G$ be a semisimple linear algebraic $k$-group of rank $r$, and $B$ a Borel subgroup containing a maximal torus $T$. Denote by $E := \mathcal{X}^*(T) \otimes \mathbb{Q}$ the $\mathbb{Q}$-vector space with the Euclidean product $(\cdot, \cdot)$ given by the Killing form. Let $(E, \Phi)$ be the root system associated to $(G, T)$ with a set of positive simple roots $\Delta := \{\alpha_1, \cdots, \alpha_r\} \subset \Phi$ associated to $B$. Let $\Lambda_R$ be the root lattice generated by $\Delta$. Let $\Lambda_{R'}$ be the coroot lattice in $E$ generated by the coroots

\[
\alpha_j^\vee = \frac{2}{(\alpha_j, \alpha_j)} \alpha_j, \quad j = 1, \cdots, r.
\]

Let $\mathcal{C}_+$ be the positive Weyl chamber spanned by the positive linear combination of fundamental weights. Denote by $\mathcal{C}_- := -\mathcal{C}_+$ the negative Weyl chamber.

The group $G$ is a separable spherical homogeneous space under $G \times G$ by $(g, h).k = gkh^{-1}$, because it contains a $B \times B$-dense orbit by Bruhat decomposition. Denote $\overline{X}_G$ by the wonderful compactification of $G$ and denote $X_G$ by the associated log variety.

**Proposition 5.21.** We have the following properties:

1. $\Lambda \cong \mathcal{X}^*(T)$ and $N \cong \mathcal{X}_+(T) \otimes \mathbb{Q}$;
2. $\mathcal{V}$ is the negative Weyl chamber;
3. the set of colors $\Delta$ maps bijectively to the set of simple coroots under $\rho|_{\Lambda}$ and the image of $\rho|_{\Delta}$ is the coroot lattice in $N$.

**Proof.** In characteristic zero, the first two statements were proved by Brion [Bri12, Section 3.1, 4.1]. Since Brion’s argument only uses basic properties of spherical embeddings [Kno91] and the local structure theorem 5.5. Therefore we have (1) and (2) in arbitrary characteristics.

For (3), it suffices to show the case when $G$ is simply connected. In this case, we know that each defining equation $f_i$ of $D_i$ is a $B \times B$-eigenfunction with the weight $(\chi_i, -\chi_i)$, where $\chi_i$ is a fundamental weight [BK05, Proposition 6.1.11]. Since $\Lambda$ is generated by the fundamental weights and $\nu_{D_i}(f_j) = \delta_{ij}$, we conclude that $\rho(D_i)$ is a simple coroot.♠
Proof of Theorem 1.6. We check all hypotheses in Theorem 5.14 are satisfied. Clearly the center $Y \cong G/B \subset X_G$ is $G$-separable. Hypothesis 5.12 follows from Proposition 5.21. Hypothesis 5.10 can be verified by an argument similar to the proof of Proposition 5.18. We may consider the $G$-equivariant morphism $X_G \to X_G^a$ where the latter is smooth [BK05, Theorem 6.1.8]. By Bruhat decomposition [Bor91, Theorem 22.6], the pullback of a color on $X_G^a$ is a (reduced) color on $X_G$. Applying the projection formula as in the proof of Proposition 5.18, now the theorem follows from Lemma 5.11, Proposition 5.21, and Theorem 5.14.

6. Log smooth varieties with positive log tangents

In this section, we fix a connected log smooth (possibly with singular underlying structure) variety $X$. Let $r : Y \to X$ be a birational log étale morphism of connected log smooth varieties with $Y$ smooth. In particular, $Y$ has normal crossings boundary. Denote by $\Delta_X$ and $\Delta_Y$ the boundary of $X$ and $Y$ respectively. Write $U = X \setminus \Delta_X = Y \setminus \Delta_Y$.

Definition 6.1. A line bundle $L$ over $X$ is called ample on the interior if $L \cap C > 0$ for every irreducible proper curve $C$ on $X$ such that $U \cap C \neq \emptyset$.

Lemma 6.2. If $-K_X$ is ample on the interior, then $-K_X'$ is also ample on the interior for any log étale birational morphism $\phi : X' \to X$.

Proof. This is follows from the projection formula and $\phi^*K_X = K_{X'}$. ♠

Lemma 6.3. Assume that $-K_X$ is ample on the interior. For any point $x \in U$, there is an irreducible rational curve $Z \to X$ such that $x \in Z$, and $-K_X \cap Z \leq \dim X + 1$, where the equality holds only if $Z \to X$ factors through $U$.

Proof. We first choose an arbitrary usual stable map $f : C \to X$ through $x$ with $C$ smooth and irreducible, hence a stable log map $f : C \to X$. Taking proper transform, we obtain a stable log map $f' : C \to Y$ with $f = f' \circ r$. Since $r$ is log étale, we obtain:

$$(-K_Y) \cap C = (-K_Y) \cap C + \Delta_Y \cap C \geq -K_Y \cap C = -K_X \cap C > 0.$$  

By Mori’s Bend-and-Break, there is a rational curve $Z \to Y$ through $x$ with $(-K_Y) \cap Z \leq \dim Y + 1$. Composing with $r$, we obtain a rational curve $Z \to X$ through $x$. Then we calculate

$$(-K_X) \cap Z = (-K_Y) \cap Z = (-K_Y) \cap Z - \Delta_Y \cap Z \leq \dim X + 1.$$  

Here the equality holds only if $\Delta_Y \cap Z = 0$. This concludes the proof. ♠

A genus zero stable log map $f : C/S \to X$ is called a $\mathbb{P}^1$-curve if it has an irreducible source curve with no contact markings. The next result is due to Keel-McKernan [KM99]. Here we slightly strengthen the result to fit our needs.
Proposition 6.4. Assume that $-K_X$ is ample on the interior. Then for any closed point $x \in U$ there is a genus zero stable log map $f : C \to X$ through $x$, which is minimal with respect to a fixed polarization $H$ among all rational curves in $X$, and satisfying one of the following properties:

1. $f$ is a $\mathbb{P}^1$-curve with $\deg f^*(-K_X) \leq n + 1$;
2. $f$ is an $\mathbb{A}^1$-curve with $\deg f^*(-K_X) \leq n$.

Proof. By Lemma 6.3, we choose an irreducible rational curve in $X$ through $x$. We then have a genus zero log map $f : C \to Y$. We may assume that $\deg f^*H$ is minimal. By the usual Bend-and-Break lemma, we may further assume that $(-K_Y) \cap C \leq n + 1$. The size at a contact marking is the length of $f^*\Delta_Y$. We may assume that $f$ achieves the maximal size at a contact marking among all such genus zero stable log maps through $x$.

Assume that $f$ is not an $\mathbb{A}^1$ or $\mathbb{P}^1$-curve. Let $k \geq 2$ be the number of contact markings of $f$. We also mark one point $t$ mapped to $x$. Let $\mathcal{M}$ be the moduli stack of stable log maps with discrete data giving by $f$. Note that

$$(-K_Y) \cap C = (-K_X) \cap C > 0.$$  

We then calculate

$$\dim_{[f]} \mathcal{M}(t) \geq (-K_Y) \cap C + k + 1 - 3 > k - 2 \geq 0,$$

where $\mathcal{M}(t) \subset \mathcal{M}$ denotes the closed substack fixing the marking $t$. Thus the image of $f$ moves fixing $t \mapsto x$.

Since $f$ is minimal with respect to a polarization $H$, we obtain a ruled surface $\pi : S \to B$ with $B$ a smooth proper curve, a morphism $h : S \to Y$ obtained by deforming $f$, and at least three sections of $\pi$ with one $\Sigma_1$ given by the marking $t$, and at least two other contact markings, write $\Sigma_2$ and $\Sigma_3$ with one given by the contact marking of maximal size. Since $\Sigma_1$ is contractible, it has negative self-intersection. Since $S$ is a ruled surface, $\Sigma_2$ and $\Sigma_3$ have positive self-intersections. But the choice of maximal size implies that $\Sigma_2 \cap \Sigma_3 = 0$. This is a contradiction, since the effective cone of $S$ is two dimensional.

Finally the statement on the degree of $f^*(-K_X)$ can be deduced from the following calculation:

$$\deg f^*(-K_X) = \deg f^*(-K_Y) = (-K_Y) \cap C - \Delta_Y \cap C \leq n + 1 - \Delta_Y \cap C.$$

Proof of Theorem 1.8. We first observe the following:

Lemma 6.5. Notations and assumptions as in Theorem 1.8. Let $f : C \to Y$ be an $\mathbb{A}^1$ or $\mathbb{P}^1$-curve through $x$ as in Proposition 6.4. Then then $f$ is an immersion and

$$f^*T_Y = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1).$$
Proof. The natural inclusion $T_Y \to T_Y$ induces the following exact sequence:

$$0 \to f^*T_Y \to f^*T_Y \to \mathcal{T} \to 0$$

where $\mathcal{T}$ is a torsion sheaf supported on the contact marking. Since $f^*T_Y$ is ample, the pull-back $f^*T_Y$ is ample as well. By Proposition 6.4, we have $\deg f^*T_Y \leq n + 1$. Since $f^*T_Y$ is ample, the splitting type of $f^*T_Y$ then follows. The immersion follows from [Kol96, Chapter IV 2.11].

Take the stable log map $f : C \to Y$ as in Lemma 6.5. If $f$ is a very free minimal $\mathbb{P}^1$-curve through $x$, then any deformation of $f$ can not break or completely fallen into the boundary, as it is minimal, and through a point in $U$. Thus any deformation of $f$ does not meet $\Delta_Y$. Using Lemma 6.5 and the identical argument as in [Kol96, Chapter V 3.7], we deduce that $Y \cong \mathbb{P}^n$. Then $\Delta_Y$ hence $\Delta_X$ is empty as any effective divisor in $\mathbb{P}^n$ is ample. This implies that there is an étale birational map $\mathbb{P}^n \to X$, from which we deduce that $X = \mathbb{P}^n$.

Assume $f : C \to Y$ is an $A^1$-curve, and is minimal as in Lemma 6.5. The amleness of $f^*T_Y \cong (r \circ f)^*T_X$ implies that $\deg f^*(-K_Y) \geq n$, hence $\deg f^*(-K_Y) = n$ by Proposition 6.4. We then calculate

$$\Delta_Y \cap [C] = (-K_Y) \cap C - (-K_Y) \cap C = n + 1 - n = 1.$$

This implies that $f$ intersects $\Delta_Y$ transversally at the unique contact marking. We also observe that any deformation of $f$ can not break or completely lie on the boundary. Thus, deforming $f$ is the same as deforming the underlying stable maps, i.e. they are parameterized by the same moduli space. The same argument as in [Kol96, Chapter V 3.7] implies $Y \cong \mathbb{P}^n$. The degree consideration implies that $\Delta_Y$ is a hyperplane in $\mathbb{P}^n$. Thus $r$ has to be an isomorphism.

References

[AC] Dan Abramovich and Qile Chen, Stable logarithmic maps to Deligne-Faltings pairs II, The Asian Journal of Mathematics, accepted, arXiv:1102.4531.

[ACGM] D. Abramovich, Q. Chen, D. Gillam, and S. Marcus, The evaluation space of logarithmic stable maps, arXiv:1012.5416v1, preprint.

[ACMW] Dan Abrmovich, Qile Chen, Steffen Marcus, and Jonathan Wise, Boundedness of the space of stable log maps, arXiv:1408.0869.

[BK05] Michel Brion and Shrawan Kumar, Frobenius splitting methods in geometry and representation theory, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005. MR 2107324 (2005k:14104)

[Bor91] Armand Borel, Linear algebraic groups, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012 (92d:20001)

[Bri03] Michel Brion, The cone of effective one-cycles of certain $G$-varieties, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser, Basel, 2003, pp. 180–198. MR 2017584 (2004m:14008)

[Bri07a] ____, Construction of equivariant vector bundles, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 83–111. MR 2348903 (2008k:14101)
A1-CURVES ON LOG SMOOTH VARIETIES

[Bri07b] ______, Log homogeneous varieties, Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007, pp. 1–39. MR 2500349 (2010m:14063)

[Bri07c] ______, The total coordinate ring of a wonderful variety, J. Algebra 313 (2007), no. 1, 61–99. MR 2326138 (2008d:14067)

[Bri12] ______, Spherical varieties, Highlights in Lie algebraic methods, Progr. Math., vol. 295, Birkhäuser/Springer, New York, 2012, pp. 3–24. MR 2866845 (2012j:14072)

[Cam92] F. Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 539–545. MR 1191735 (93k:14050)

[Che14] Qile Chen, Stable logarithmic maps to deligne-faltings pairs I, Annals of Math 180 (2014), 455–521, arXiv:1008.3090.

[CP98] Frédéric Campana and Thomas Peternell, Rational curves and ampleness properties of the tangent bundle of algebraic varieties, Manuscripta Math. 97 (1998), no. 1, 59–74. MR 1642626 (99g:14051)

[CZ] Qile Chen and Yi Zhu, Very free curves on fano complete intersections, Algebraic Geometry, accepted, arXiv:1311.7189.

[CZ14a] Qile Chen and Yi Zhu, On the irreducibility of the space of genus zero stable log maps to wonderful compactifications, to appear.

[CZ14b] ______, On the Zariski density of k1-connected varieties over function field of curves, to appear.

[DCP83] C. De Concini and C. Procesi, Complete symmetric varieties, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 1–44. MR 718125 (85e:14070)

[Deb03] Olivier Debarre, Variétés rationnellement connexes (d’après T. Graber, J. Harris, J. Starr et A. J. de Jong), Astérisque (2003), no. 290, Exp. No. 905, ix, 243–266, Séminaire Bourbaki. Vol. 2001/2002. MR 2074059 (2005g:14096)

[dJS03] A. J. de Jong and J. Starr, Every rationally connected variety over the function field of a curve has a rational point, Amer. J. Math. 125 (2003), no. 3, 567–580. MR 1981034 (2004h:14018)

[GHS03] Tom Graber, Joe Harris, and Jason Starr, Families of rationally connected varieties, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[Gro63] Alexander Grothendieck, Revêtements étales et groupe fondamental. Fasc. I: Exposés I à 5, Séminaire de Géométrie Algébrique, vol. 1960/61, Institut des Hautes Études Scientifiques, Paris, 1963. MR 0217087 (36 #179a)

[GS13] Mark Gross and Bernd Siebert, Logarithmic Gromov-Witten invariants, J. Amer. Math. Soc. 26 (2013), no. 2, 451–510. MR 3011419

[Har70] Robin Hartshorne, Ample subvarieties of algebraic varieties, Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-New York, 1970, Notes written in collaboration with C. Musili. MR 0282977 (44 #211)

[HT06] Brendan Hassett and Yuri Tschinkel, Weak approximation over function fields, Invent. Math. 163 (2006), no. 1, 171–190. MR 2208420 (2007b:14109)

[Hur14] Mathieu Huruguen, Log homogeneous compactifications of some classical groups, arXiv:1403.4337 (2014).

[Kat89] Kazuya Kato, Logarithmic structures of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR MR1463703 (99b:14020)

[Kat96] Fumiharu Kato, Log smooth deformation theory, Tohoku Math. J. (2) 48 (1996), no. 3, 317–354. MR 1404507 (99a:14012)

[KM99] Seán Keel and James McKernan, Rational curves on quasi-projective surfaces, Mem. Amer. Math. Soc. 140 (1999), no. 669, viii+153. MR 1610249 (99m:14068)
János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), no. 3, 429–448. MR 1158625 (93i:14014)

Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 225–249. MR 1131314 (92m:14065)

János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)

János Kollár, *Rationally connected varieties over local fields*, Ann. of Math. (2) 150 (1999), no. 1, 357–367. MR 1715330 (2000h:14019)

János Kollár and Endre Szabó, *Rationally connected varieties over finite fields*, Duke Math. J. 120 (2003), no. 2, 251–267. MR 2019976 (2005h:14090)

D. Luna, *Variétés sphériques de type A*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 161–226. MR 1896179 (2003f:14056)

D. Luna and Th. Vust, *Plongements d’espaces homogènes*, Comment. Math. Helv. 58 (1983), no. 2, 186–245. MR 705534 (85a:14035)

Shigefumi Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) 110 (1979), no. 3, 593–606. MR 554387 (81j:14010)

Shigefumi Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) 110 (1979), no. 3, 593–606. MR 554387 (81j:14010)

Wiesława Nizioł, *Toric singularities: log-blow-ups and global resolutions*, J. Algebraic Geom. 15 (2006), no. 1, 1–29. MR 2177194 (2006i:14015)

Arthur Ogus, *Lectures on logarithmic algebraic geometry*, TeXed notes (2006).

Martin C. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 5, 747–791. MR 2032986 (2004k:14018)

Martin C. Olsson, *The logarithmic cotangent complex*, Math. Ann. 333 (2005), no. 4, 859–931. MR 2195148 (2006j:14017)

Martin C. Olsson, *Logarithmic twisted curves*, Compos. Math. 143 (2007), no. 2, 476–494. MR 2309994 (2008d:14021)

N. Ressayre, *Spherical homogeneous spaces of minimal rank*, Adv. Math. 224 (2010), no. 5, 1784–1800. MR 2646110 (2011h:14071)

Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. 138, Springer, Heidelberg, 2011, Invariant Theory and Algebraic Transformation Groups, 8. MR 2797018 (2012e:14100)

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