Bethe Ansatz for Lattice Analogues of $N = 2$ Superconformal Theories

Z. Maassarani

Physics Department
University of Southern California
Los Angeles, CA 90089-0484

Abstract

The critical Boltzmann weights for lattice analogues of the $N = 2$ superconformal coset models $G_1 \times SO(d_H(G/H))$ were given in [1]. In this paper Bethe Ansatz methods are employed to calculate the spectrum of the transfer matrix obtained from these Boltzmann weights. From this the central charge and conformal weights are obtained by calculating finite-size corrections to the free energy per site. The results agree with those obtained from the superconformal model.

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1 Introduction

The lattice analogues of $N=2$ superconformal models were constructed in \cite{1}. Specifically, Coulomb gas techniques were used to obtain the critical Boltzmann weights corresponding to the models

$$G_1 \times SO_1 \left( \dim_R(G/H) \right) \over H_{g-h+1}$$

(1)

where $G$ is simply laced and of level one, $G/H$ is a hermitian symmetric space and $g$ and $h$ are the dual Coxeter numbers of $G$ and $H$. The central charge is $c = \frac{3d}{g+1}$, where $d$ is the complex dimension of $G/H$. The Boltzmann weights were explicitly constructed for the grassmannian model with $G = SU(m+n)$ and $H = SU(m) \times SU(n) \times U(1)$. While the arguments of \cite{1} were fairly compelling, it is necessary to perform some fundamental checks on the results. More precisely, the lattice model, at the critical temperature, should exhibit conformal invariance at large distances. It is therefore the purpose of this paper to determine the central charge and conformal weights from the Bethe Ansatz solution to the lattice model. The results agree with the coset model predictions.

The starting point of \cite{1} is to consider the lattice analogues of the $G_{k,1} = \frac{G_k \times G_1}{O_{k+1}}$ models. These conformal models only appear at the critical point of the lattice model \cite{2,3,4}. In the vertex description of the lattice model one associates to each edge of the $45^{\circ}$ lattice a copy of the fundamental representation, $V$, of $G$. The Hilbert space of a vertical slice of $2L$ edges is then $V^\otimes 2L$. The continuum limit can be argued to correspond to a model consisting of $r$ free bosons, where $r$ is the rank of $G$. The transfer matrix $\tau(u,q)$, which depends on a spectral parameter $u$ and a deformation parameter $q$, gives the ‘time’ evolution of this slice from left to right. It is given by

$$\tau(u,q) = \left[ \prod_{p=1}^{L} X_{2p-1}(u,q) \right] \left[ \prod_{p=1}^{L-1} X_{2p}(u,q) \right]$$

(2)

where the matrix $X_p(u,q) = \frac{1}{2\pi} \tilde{R}(u,q)$ acts on the tensor product of the $p^{th}$ and $(p+1)^{th}$ copies of $V$. The matrix $\tilde{R}$ is the ‘$R$-matrix’ of $U_q(G)$. For $G = SU(n)$, $V$ is $n$-dimensional and one has \cite{5}:

$$\tilde{R}(u,q) = (xq - x^{-1}q^{-1}) \frac{n}{k=1} E_{kk} \otimes E_{kk} + (x - x^{-1}) \frac{n}{k,l=1} E_{kl} \otimes E_{lk}$$

$$+ (q - q^{-1}) \left[ x \frac{n}{k>l} + x^{-1} \frac{n}{k<l} \right] E_{kk} \otimes E_{ll}, \quad 1 \leq k, l \leq n,$$

(3)

where $x = e^{iu}$ and $(E_{kl})_{i,j} = \delta_{k;i} \delta_{l;j}$. The matrix $\tilde{R}$ satisfies the Yang-Baxter equation

$$(\tilde{R}(u,q) \otimes I)(I \otimes \tilde{R}(u+v,q))((\tilde{R}(v,q) \otimes I) = (I \otimes \tilde{R}(v,q))((\tilde{R}(u+v,q) \otimes I)(I \otimes \tilde{R}(u,q)).$$

(4)

The nearest-neighbor spin-chain Hamiltonian is given by

$$\mathcal{H} = \tau^{-1}(u) \frac{\partial \tau(u)}{\partial u} \bigg|_{u=0}.$$  

(5)

It contains a boundary term

$$\mathcal{H}_{bdry} = -\frac{2i}{g} \rho_G (h^{(1)} - h^{(2L)})$$

(6)

where $\rho_G$ is the Weyl vector of $G$ and $h^{(j)}$ is the Cartan subalgebra generator acting on the $j^{th}$ edge of the lattice. This boundary term ensures the commutation of $\mathcal{H}$ with the generators of $U_q(G)$.
It also shifts the ground state energies and is thus the discrete counterpart of introducing a boundary charge proportional to \( \rho_G \) into a gaussian model. One now takes \( q \) such that

\[
q^{k+g+1} = -1. \tag{7}
\]

One can then perform the quantum group truncation by using the modified trace

\[
\text{Tr}[\mathcal{O}] = \text{tr}[\mathcal{O} \mu \otimes ... \otimes \mu], \tag{8}
\]

where

\[
\mu = q^{2\rho_G,h}, \tag{9}
\]

to compute the partition function and the correlation functions. Because the transfer matrix commutes with the generators of \( U_q(G) \), it preserves this truncation.

In order to obtain the lattice analogue of the \( N = 2 \) superconformal model \([1]\) one first sets \( k = 0 \) in \( G_{k,1} \) and eq. \((6)\). One then replaces the \( R \)-matrix by a ‘conjugated’ one \([3]\), namely

\[
\tilde{R}'(x,q) \equiv [I \otimes x - \frac{2i}{\sqrt{g}}(\rho_G - \rho_H)h] \tilde{R}(x,q) [x^\frac{2i}{\sqrt{g}}(\rho_G - \rho_H)h] \otimes I]. \tag{10}
\]

The matrix \( \tilde{R}' \) satisfies the Yang-Baxter equation and it commutes with \( U_q(H') \) where \( H' \) is the semi-simple factor of \( H \), i.e. \( H = H' \times U(1) \). One should note that the foregoing transformation is not a similarity transformation. One then only performs the quantum group truncation with respect to \( U_q(H') \). This can be implemented by employing an \( H \)-modified trace to compute correlators; that is, \( \mu \) in \((9)\) is replaced by

\[
\mu' = q^{2\rho_H,h}. \tag{11}
\]

The conjugation of the \( R \)-matrix amounts to adding boundary terms to the transfer matrix. This is most easily seen on the Hamiltonian; \( \mathcal{H}_{bdry} \) of eq. \((3)\) is replaced by

\[
\mathcal{H}'_{bdry} = -\frac{2i}{g} \rho_H (h^{(1)} - h^{(2L)}), \tag{12}
\]

that is \( \rho_G \) has been replaced by \( \rho_H \). As we shall see, this produces the shift from \( c = 0 \) for \( G_{0,1} \) to \( c = \frac{3d}{g+1} \) for the coset model of \( \[1\] \).

### 2 Central charge and conformal weights from the Bethe Ansatz

The transfer matrix written in eq. \((2)\) corresponds to free boundary conditions. This choice of boundary conditions is essential to ensure the commutation of \( \tau \) with the generators of the quantum group \( U_q(G) \) \([1]\). However two such transfer matrices with two different spectral parameters do not commute. As the commutation of the transfer matrices is an essential ingredient of the method of the algebraic Bethe Ansatz one would like to consider periodic boundary conditions. It was shown in \([1]\) how the quantum group structure can be exhibited for the \( SU(2) \) spin-chain with twisted periodic boundary conditions after properly choosing the twist parameters and the magnetization of the states acted upon. I shall assume that this type of investigation generalizes to other groups. The use of a 90° lattice with a row-to-row transfer matrix is another ingredient of the Bethe Ansatz. The transfer matrices differ from those corresponding to the 45° lattices. I shall consider here the continuum limit of the chains with twisted periodic boundary conditions with a twist matrix \( \mu' \).
One more remark is in order before proceeding to the calculation. The ‘conjugation’ operation appearing in eq. (10) can be interpreted as a ‘gauge transformation’ [8, 9]. The transfer matrix in eq. (2) is not invariant under gauge transformations as was seen above. The transfer matrix constructed in [9] and the induced Hamiltonian are for free boundary conditions and certain surface terms. They are invariant under gauge transformations. Hence the foregoing transfer matrix is not equivalent to that of ref. [9]. For both types of transfer matrices the corresponding Bethe Ansatz equations have not been written for $SU(n)$ with $n > 2$. This precludes for the moment a direct approach.

I shall write the Bethe Ansatz equations for $G = SU(n)$ but the method works for other simply laced groups and the final results are still valid with the appropriate modifications. The row-to-row transfer matrix with twisted periodic boundary conditions is constructed out of a diagonal twist matrix $M$ and operator matrices $L_{ai}$ where

$$L_{ai} = (xq - x^{-1}q^{-1})^{-1}P_{ai} \hat{R}_{ai} \tag{13}$$

and $P$ is the permutation operator on $V \otimes V$. Recall from equation (3) that $\hat{R}$ and hence $P \hat{R}$ act on $V \otimes V$. The index $a$ correspond to a single copy of $V$, the ‘auxiliary’ space, and $i$ to one copy of $V$ at each site, a ‘quantum’ space, of the $L$-site chain. Therefore the operator $L_{ai}$ acts non-trivially at the site $i$. The transfer matrix is given by

$$\tau(u) = tr_a(M_a T(u)) \tag{14}$$

where

$$T(u) = L_{a1} L_{a2} \ldots L_{aL} \tag{15}$$

is the ‘monodromy’ matrix. The trace is taken on the auxiliary space. Untwisted periodic boundary conditions correspond to a matrix $M$ proportional to the identity on $V$. From the Yang-Baxter equation (4) one can derive

$$\hat{R}(u - v, q) T(u) \otimes T(v) = T(v) \otimes T(u) \hat{R}(u - v, q) \tag{16}$$

where $\hat{R}$ and the tensor product correspond to two copies of an auxiliary space (see ref. [11] for a review). A multiplication with respect to quantum indices, at each site, is implicit. For any diagonal $M$ one has $[M \otimes M, \hat{R}] = 0$. One can then easily show that two transfer matrices with different spectral parameters commute. One can then construct a common set of eigenvectors for these transfer matrices using the nested algebraic Bethe Ansatz method [10]. The Ansatz for the eigenvector consists of a linear combination of vectors obtained by applying certain ‘creation’ operators obtained from the monodromy matrix $T$ taken at yet undetermined spectral parameters $u_i$ on an ‘initial’ eigenvector of $\tau$. The algebraic relations given in (16) are used to obtain certain conditions on the parameters $u_i$ and on the coefficients of the linear combination; a similar Ansatz is then made $r - 2$ times (the nesting). The complete set of conditions on the $u_i$’s constitute the Bethe Ansatz equations. I have modified the calculations accordingly to take into account the twist matrix $M$. In what follows $M$ is equal to $\mu'$ and is diagonal with elements $m_1, \ldots, m_n$. Taking the logarithm of the nested Bethe Ansatz equations for twisted periodic boundary conditions gives:

$$\sum_{j=1}^{k+1} \phi(\lambda^{(k)}_i - \lambda^{(k+1)}_j, \gamma) - \sum_{j=1}^{k} \phi(\lambda^{(k)}_i - \lambda^{(k)}_j, \gamma) + \sum_{j=1}^{k-1} \phi(\lambda^{(k)}_i - \lambda^{(k-1)}_j, \gamma) = i \log\left(\frac{m_k}{m_{k+1}}\right) + 2\pi I_i^{(k)}, \quad 1 \leq k \leq n - 1, \tag{17}$$
where
\[ 0 = p_n \leq p_{n-1} \leq \ldots \leq p_0 = L, \quad \lambda_j^{(k)} = -i(u_j^{(k)} - k^2/2), \quad \lambda_i^{(0)} = 0, \quad q = e^{i\gamma} \]
and \( \phi(z, \alpha) \) is defined by
\[ \phi(z, \alpha) = i \log \left( \frac{\sinh(z + i\alpha)}{\sinh(z - i\alpha)} \right) \quad (18) \]
with \( \phi(0, \alpha) = \pi \). One can show that the integers \( j_i^{(k)} \) are bounded; one then solves for the roots of equations (17) for each set of integers within the bounds. For \( 0 < \gamma < \frac{\pi}{2} \) one has a critical regime whose continuum limit is gapless and corresponds to a conformal field theory. The eigenvalues are given by:
\[ \Lambda_M(u) = \prod_{i=1}^{p_0} \frac{1}{a(u - u_i^{(0)})} \sum_{j=1}^{n} m_j \prod_{l=1}^{p_{j-1}} a(u - u_l^{(j-1)}) \prod_{m=1}^{p_j} a(u_m^{(j)} - u) \quad (19) \]
where \( a(u) = \frac{\sin(\gamma - u)}{\sin(u)} \). The equations (17) encode the root structure of the \( A_n \) Dynkin diagram (see ref. [11] for instance). One has similar equations for the other Lie algebras; they involve the simple roots and the highest weight of the representation considered [13]. For large \( L \), fixed ratios \( p_i/L \) (\( i = 1, \ldots, n - 1 \)) and \( 0 < u < \gamma/2 \), one can show that the leading term in \( \Lambda_M \) is given by the first term in (19) for which \( j = 1 \); the remaining terms give exponentially small corrections which do not affect the finite-size expansion in \( 1/L \).

The central charge and conformal weights can then be extracted following a procedure similar to that of refs. [12] [13]. The calculations were modified to take into account the twist matrix contribution. One takes large values of \( L \) and then calculates finite-size corrections in \( 1/L \) for the free energy per site \( f_L(u) = -\frac{1}{L} \log \Lambda_M(u) \). As \( L \) becomes large the set of solutions of the Bethe Ansatz equations take a specific distribution in the complex plane. The exact distribution is not known. However a conjecture supported by numerical computations is usually made for the form of the roots of eqs. (17); it is called the ‘string hypothesis’. One then postulates the existence of some densities for the root distribution and replaces sums by integrals in the continuum limit. These densities satisfy a set of equations obtained from eqs. (17). One then calculates the corrections up to \( 1/L^2 \) for \( f_L - f_{\infty} \). The form of the leading terms of the \( \frac{1}{L} \)-expansion were predicted in [14] by conformally mapping the complex plane onto a strip of width \( L \). The central charge and conformal weights appear in this expansion. Upon comparing the two expansions I obtain:
\[ c = r - \frac{12}{\pi(\pi - \gamma)} A^{-1} t, \quad (20) \]
\[ \Delta = \frac{1}{2(1-\gamma/\pi)} (\tilde{h}^+ - \frac{\gamma}{2\pi} \tilde{S} + \frac{1}{4} \tilde{t}) A^{-1} (\tilde{h}^+ - \frac{\gamma}{2\pi} \tilde{S} + \frac{1}{4} \tilde{t}) + \frac{c_4}{4}, \quad (21) \]
\[ \overline{\Delta} = \frac{1}{2(1-\gamma/\pi)} (\tilde{h}^- - \frac{\gamma}{2\pi} \tilde{S} - \frac{1}{4} \tilde{t}) A^{-1} (\tilde{h}^- - \frac{\gamma}{2\pi} \tilde{S} - \frac{1}{4} \tilde{t}) + \frac{c_4}{4}, \quad (22) \]
where \( t_j = \frac{1}{2L} \log\left( \frac{m_j}{m_{j+1}} \right) \), \( A \) is the Cartan matrix of \( G \) and \( r \) its rank, \( \tilde{h}^\pm \) are \( r \)-dimensional vectors of integers labeling excitations (holes and complex strings) and \( \frac{1}{2} \tilde{S}_j = \frac{p_{j+1} - p_j}{2} - p_j \) is the ‘spin’ at the Ansatz-level \( j \). The vectors \( \tilde{h}^\pm \) and \( \tilde{S} \) are not independent, namely one must have \( \tilde{h}^+ + \tilde{h}^- = \tilde{S} \). The number of sites \( L \) is taken to be a multiple of \( m + n \). This is a natural requirement if one considers the ground state of the statistical model, for which \( \tilde{S} = \tilde{h}^\pm = 0 \). With \( p_0 = L \) and \( p_L = 0 \) this implies \( p_i = L - \frac{1}{g} i \) which should be integers. Therefore \( L \) should be a multiple of \( m + n \). Intuitively, the ground state should be made up of a linear combination of vectors with
equal number of $SU(m + n)$ ‘spins’, or vectors in the canonical basis of $\mathbb{R}^{m+n}$, in each directions; for $SU(2)$ an equal number of spins up and spins down. The Bethe Ansatz equations for various twisted periodic boundary conditions for $SU(2)$ were analyzed in [12]. There, an even number of sites was considered implying integer spin.

One sees that the central charge has been decreased from its untwisted value of $r$. These formulae are valid for any simply laced Lie group.

Consider now the lattice grassmannian models where $G = SU(m + n)$ and $H = SU(m) \times SU(n) \times U(1)$, with $m + n \geq 3$. The twist matrix is equal to

$$M = q^{2\rho_H} = \text{diag} (q^{m-1}, q^{m-3}, ..., q^{-m-1}, q^{-n-1}, q^{-n+1}).$$

With the choices made for the roots of $G$ and $H$ one has $\rho_G - \rho_H = \frac{m+n}{2} \lambda_m$ where $\lambda_m$ is the $m$th fundamental weight of $G$. Define $x_i \equiv \frac{1}{\gamma} \log m_i$, then $t_i = \frac{c}{2} (x_i - x_{i+1})$ and one can rewrite the central charge as

$$c = m + n - 1 - \frac{3\gamma^2}{\pi(\pi - \gamma)(m + n)} \sum_{1 \leq i, j \leq m+n} (x_i - x_j)^2$$

$$= m + n - 1 - \frac{3\gamma^2}{\pi(\pi - \gamma)} \sum_{1 \leq i \leq m+n} x_i^2$$

where the last equality follows from $\sum_{i=1}^{m+n} x_i = 0$. Since $\gamma = \frac{\pi}{g+1} = \frac{\pi}{m+n+1}$, $x_i = m + 1 - 2i$ for $i = 1$ to $m$ and $x_i = 2m + n + 1 - 2i$ for $i = m + 1$ to $m + n$ one obtains $c = \frac{3\gamma}{m+n+1}$. Therefore one recovers the central charge for the corresponding coset model.

### 3 The continuum model

The conformal weights for the $N = 2$ superconformal coset model of equation (1) can be read off the modified $A$-type modular invariant gaussian partition function [11] :

$$Z = \frac{1}{2 |W(H)| |Z(G)| |\eta(\tau)|^{2r}} \sum_{w \in \mathcal{W}(G)} \sum_{v \in \mathcal{M}(G)^*} \sum_{v_1, v_2} \sum_{\Gamma} \sum_{\xi=0,1}^{\eta=0,1} \epsilon(w) q^{2(v_L + \eta s)^2} e^{-4\pi i s.(\xi_L v_L - \xi_R v_R)} e^{-2\pi i \xi s.(v_L - v_R)} ,$$

where

$$\Gamma = \sqrt{g(g + 1)M(G)}$$

$$v_L = v + u + v_1 , \; v_R = w(v) + u + v_2$$

and

$$s = \frac{1}{\sqrt{g(g + 1)}} (\rho_G - \rho_H) .$$

The lattices appearing in the sums are the root $(M(G))$ and weight $(M(G)^*)$ lattices of $G$. The foregoing partition function represents $\text{Tr}[q^{L_0 - c/24} \tau^{L_0 - c/24} e^{-2\pi i (\xi_L J_0 - \xi_R J_0)}]$ taken over the entire
There does not seem to be a general way of identifying the integer vectors form, it must be remembered that the various component parts are constrained integer vectors. The Bethe Ansatz method and finite-size techniques were used to study the continuum limit of a lattice that should exhibit at criticality in the continuum limit an $N=2$ superconformal behaviour. The central charge obtained from the Bethe Ansatz is identical with that of the coset model to which the lattice model is expected to flow in the continuum limit. The conformal weights obtained from the Bethe Ansatz weights is not worrying. One has no reason to expect the weights obtained from the Bethe Ansatz to be organized exactly as those of the coset model. Indeed the Bethe Ansatz was done for twisted periodic boundary conditions instead of free boundary conditions and the analysis of the excitations does not seem naturally related to the coset model labels.

4 Conclusion

The Bethe Ansatz method and finite-size techniques were used to study the continuum limit of a lattice that should exhibit at criticality in the continuum limit an $N=2$ superconformal behaviour. The central charge obtained from the Bethe Ansatz is identical with that of the coset model to which the lattice model is expected to flow in the continuum limit. The conformal weights obtained for the specific set of excitations considered, correspond to a subset of the full set of weights of the $N=2$ superconformal theory (31). This confirms the identification of the finite lattices at the critical temperature with the $N=2$ superconformal models to which they are expected to flow in the continuum limit. Twisted periodic boundary conditions were used assuming that an analysis

\[ \Delta = \frac{1}{2} (v_L + \eta s)^2 + \frac{c - r}{24}, \quad \Delta = \frac{1}{2} (v_R + \eta s)^2 + \frac{c - r}{24}. \]  

One can write $v_L$ and $v_R$ as follows:

\[ v_L = \sum_{i=1,n} \left( \frac{1}{\beta} v_i \lambda_i + \beta u_i \lambda_i + \frac{g}{\beta} r_{i}^{(1) \alpha_i} \right), \quad v_R = \sum_{i=1,n} \left( \frac{1}{\beta} v_i w(\lambda_i) + \beta u_i \lambda_i + \frac{g}{\beta} r_{i}^{(2) \alpha_i} \right), \]

where $\vec{v}$, $\vec{u}$, $\vec{r}_1$ and $\vec{r}_2$ belong to $\mathbb{Z}^r$, the $\alpha_i$'s are the simple roots of $G$ ($\alpha_i^2 = 2$) and the $\lambda_i$'s are the fundamental weights of $G$. While equations (21), (22) and equations (29) are of an identical form, it must be remembered that the various component parts are constrained integer vectors. There does not seem to be a general way of identifying the integer vectors $\vec{v}$, $\vec{u}$, $\vec{r}_1$ and $\vec{r}_2$ with the vectors $\vec{S}$, $\vec{h}^\pm$ and $\vec{f}$ in the Bethe Ansatz. One can consider particular scalar fields ($\Delta = \overline{\Delta}$) for which $\vec{S} = 0 = \vec{h}^+ + \vec{h}^-$. The vectors $\vec{h}^\pm$ can be negative because they label holes and string excitations. Using $\lambda_i \alpha_j = \delta_{i,j}$, $\lambda_i = A_{ij}^{-1} \alpha_j$ and the symmetry of the Cartan matrix for simply laced groups it is easy to see that the weights (21) and (22) can be put in the form (29) provided that:

\[ v_i = h_i^+ + \frac{x_i - x_{i+1} - (m + n) \delta_{i,m} \eta}{2} - k_i g, \quad u_i = \frac{-x_i + x_{i+1} + (m + n) \delta_{i,m} \eta}{2} + k_i (g + 1), \]

\[ r_{i}^{(1)} = r_{i}^{(2)} = 0, \quad k_i \in \mathbb{Z}, \]

where $\eta = 0$ (the NS sector) if $m + n$ is even and $\eta = 1$ (the Ramond sector) if $m + n$ is odd. Indeed, for even $m + n$ the twist integers $(x_i - x_{i+1})$ are even as can be seen from the matrix $M$ in eq. (23). For odd $m + n$ only $x_m - x_{m+1}$ is odd.

The Bethe Ansatz conformal weights (21) and (22) were found to be identical in form to those of the grassmannian coset model. Specific subsets of the set of weights of the coset model can be obtained from the Bethe Ansatz conformal weights. This is another confirmation that one has a lattice analogue of the coset model. The observation that there does not seem to be a general way of identifying the components of the coset model conformal weights with the parameters of the Bethe Ansatz weights is not worrying. One has no reason to expect the weights obtained from the Bethe Ansatz to be organized exactly as those of the coset model. Indeed the Bethe Ansatz was done for twisted periodic boundary conditions instead of free boundary conditions and the analysis of the excitations does not seem naturally related to the coset model labels.
similar to that of ref. [7] for $SU(2)$ generalizes to $SU(n)$ with $n > 2$. Therefore one expects an overlap between the set of states of the twisted periodic transfer matrix and those of the transfer matrix $[4]$ constructed from $R^\prime$. Furthermore Bethe Ansatz equations have not yet been derived for free boundary conditions and surface terms for $SU(n)$ with $n > 2$. Such an Ansatz would provide the full set of conformal weights corresponding to the coset model. It would then provide another valuable verification that one has a lattice analogue of an $N = 2$ superconformal coset model. A further and delicate step in this identification consists of studying the operator content, including the operator multiplicities, of the statistical model.

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