Morse Groups in Symmetric Spaces Corresponding to the
Symmetric Group

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December 1, 1997

1 Introduction

1.1 Nilcones in Symmetric Spaces

Let $\theta : g \to g$ be an involution of a complex semisimple Lie algebra, $t \subset g$ the
fixed points of $\theta$, and $V = g/t$ the corresponding symmetric space. The adjoint
form $K$ of $t$ naturally acts on $V$. The orbits and invariants of this representation
were studied by Kostant and Rallis in [KR]. Let $X = K \backslash V$ be the invariant theory
quotient, and $f : V \to X$ be the quotient map. The space $X$ is isomorphic to $C^r$.

When $g = g' \oplus g'$ and $\theta$ acts by interchanging the factors, $K \mid V$ is just the adjoint
representation of $G'$. A detailed study of the singularities of $f$ in this case leads to
the Springer representations of the Weyl group $W'$ (see [BM], [M], [Sp], [Sl]).

More precisely, if $P$ is the nearby cycles sheaf of $f$ along the nilcone $N = f^{-1}(0)$,
then one obtains representations of $W'$ on the stalks of $P$.

In [Gr] we studied the singularities of $f$ for an arbitrary symmetric space (in
fact, for a more general class of objects). The main result of [Gr] is a description of
the nearby cycles sheaf $P$ in terms of the Fourier transform.

In this paper, we study the microlocal geometry of $f$ in three particular examples:
$V^I = sl_n$, $V^{II} = sl_n/so_n$, and $V^{III} = sl_{2n}/sp_{2n}$. Of all the classical symmetric spaces,
these are the ones that have the symmetric group $\Sigma_n$ as their small Weyl group.
The questions we discuss are equally interesting for other symmetric spaces, but for
the time being, they remain open outside of the three examples above. Our main
results (Theorems 1.2 and 1.3) describe the Morse groups of $P$ with two monodromy
structures (see Section 1.2). Experts in singularity theory may also find technical
Lemma 2.2 to be of some independent interest.

In the case $V = sl_n$, we recover the computation by Evens and Mirković [EM]
of the local Euler obstructions for the nilcone in $sl_n$. We also borrow from [EM] the
idea of exploiting torus symmetry (see our proof of Lemma 4.1).

In his Ph.D. thesis [Groj], Grojnowski studied a class of equivariant perverse
sheaves on symmetric spaces which is related to the nearby cycles sheaf $P$ we dis-
cuss. He also proposed the question of studying the characteristic varieties of these sheaves.

Discussions with Sam Evens, Ivan Mirković, and Ian Grojnowski have been of great value to me. I also wish to thank Robert MacPherson for his guidance and support.

1.2 Morse Groups of the Nearby Cycles

We now introduce the geometric setup for studying the singularities of an algebraic map microlocally (see [Lê] for an early appearance of these ideas, and [Gi], [GM], [KS] for a systematic treatment of Morse groups). A more technical discussion will be given in Section 2.

Let \( f : V = \mathbb{C}^d \to X = \mathbb{C}^n \) be a dominant algebraic map, such that 0 is in the image of \( f \), and is a critical value. Let \( X^{\text{reg}} \subset X \) be the set of regular values of \( f \) (we do not count a point \( \lambda \in X \) as a regular value if \( f^{-1}(\lambda) = \emptyset \)). Assume that \( f \) is a map without blowing up along the fiber \( E = f^{-1}(0) \) (this is a kind of a 'well-behavedness' condition; see [Hi] and Section 2.1 below for a precise definition). Fix a sufficiently fine stratification of \( E \).

Associated to the pair \((f, S)\), there is a non-negative integer \( m \) defined as follows. Fix a point \( p \in S \). Take any smooth function \( g : V \to \mathbb{R} \), such that \( p \) is a critical point of the restriction \( g|_S \). Assume \( g \) is generic among all such functions (more precisely, we need the 2-jet of \( g \) at \( p \) to be generic). Fix a small \( \lambda \in X^{\text{reg}} \) and let \( F_\lambda = f^{-1}(\lambda) \). Then \( m \) is the number of critical points of \( g|_{F_\lambda} \) near \( p \). Note that if \( S \) is open in \( E \), and consists of regular points of \( f \), then \( m = 1 \).

In the language of sheaf theory, the number \( m \) is the multiplicity of the conormal bundle \( \Lambda_S = T^*_S X \subset T^* X \) in the characteristic cycle \( SS(P) \) of the nearby cycles sheaf \( P \) of \( f \). It is an important invariant of \( f \).

The multiplicity \( m \) is, in fact, the dimension of a vector space \( M_\xi(P) \) which depends on \( p \) and \( g \) only through the differential \( \xi = d_p g \in \Lambda_S \). The vector space \( M_\xi(P) \) is called the Morse group of \( P \) relative to \( \xi \). It is defined as follows.

Fix a normal slice \( N \) to \( S \) through \( p \), and small numbers \( 0 < \delta \ll \epsilon \ll 1 \). Let \( B_\epsilon \) be the \( \epsilon \)-ball around \( p \), and \( c \) be the complex codimension of \( S \) in \( E \). Then

\[
M_\xi(P) = H^c(N \cap F_\lambda \cap B_\epsilon, \{ x | g(x) = g(p) + \delta \}).
\]

Here, we need to assume that \( |\lambda| \ll \delta \), and that \( \xi \) lies in a certain Zariski open subset \( \Lambda^0_S \subset \Lambda_S \), called the set of generic conormals to \( S \).

By construction, there are two commuting monodromy actions on \( M_\xi(P) \). First, there is an action of the fundamental group \( \pi_1(\Lambda^0_S) \), coming from the dependence of the Morse group on \( \xi \). This is called the microlocal monodromy action. Second, there is an action of \( \pi_1(X^{\text{reg}} \cap B_0) \), where \( B_0 \) is a small ball around the origin in \( X \). This comes from the choice of \( \lambda \); we call it the monodromy in the family \( f \). The
Morse groups $M_\xi(P)$, along with these two kinds of monodromy, give a great deal of information about the (perverse) sheaf $P$. In many specific problems, they suffice to completely determine the structure of $P$ as an object in the abelian category of perverse sheaves on $E$. This, in turn, can be used to analyze other questions about the singularities of $f$.

### 1.3 Statement of Results

Returning to the situation where $V$ is one of the symmetric spaces $V^{I,II,III}$, we now identify the geometric ingredients of Section 1.2.

The fundamental group $\pi_1(X_{\text{reg}} \cap B_0)$ is the classical braid group $B_n$ on $n$ strands. The zero fiber $E = N$ of the quotient map $f$ is naturally stratified by $K$-orbits. These orbits are parametrized by the partitions of $n$ (in the case $V = V^{II}$, there is, sometimes, an additional sign parameter). The partition corresponding to an orbit $O \subset N$ is given by the Jordan normal form of a point in $O$ (see Lemma 3.1).

Fix a partition $\bar{n}$: $n = n_1 + \ldots + n_k$. Let $\bar{O} \subset N$ be an orbit corresponding to $\bar{n}$. Fix a point $A \in O$, and let $\Lambda_A^0 = O \cap T^*_A V$, the set of generic covectors at $A$.

We can not identify the set $\Lambda_A^0$ explicitly. Instead, we will work with a certain open subset $\tilde{\Lambda}_A \subset \Lambda_A^0$.

Order the numbers $n_i$ so that:

$$n_1 = \ldots = n_{m_1} < n_{m_1+1} = \ldots = n_{m_1+m_2} < \ldots < n_{m_1+\ldots+m_{l-1}+1} = \ldots = n_{m_1+\ldots+m_l},$$

with $m_1 + \ldots + m_l = k$. Let $B_{\bar{n}}$ be the group of braids on $k$ strands, colored in $l$ colors, with $m_j$ strands of $j$-th color ($j = 1, \ldots, l$). The following lemma will be proved in Section 3.

**Lemma 1.1** There exists a Zariski open subset $\tilde{\Lambda}_A \subset \Lambda_A^0$, and a natural homomorphism $\rho : \pi_1(\tilde{\Lambda}_A) \to B_{\bar{n}}$, such that $\rho$ is an isomorphism when $V = V^I$ or $V^{III}$, and a surjection when $V = V^{II}$.

Fix a basepoint $\xi \in \tilde{\Lambda}_A$. Theorem 1.2 describes the Morse group $M_\xi(P)$ as a $B_{\bar{n}}$-module. Let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard generators of $B_{\bar{n}}$. We write $\Sigma_n = B_n/(\sigma_1^2 - 1)$ for the symmetric group on $n$ letters, and $\mathcal{H}_{-1}(\Sigma_n) = \mathbb{C}[B_n]/(\sigma_1 - 1)^2$ for the Hecke algebra of $\Sigma_n$, specialized at $q = -1$. Note that $\mathcal{H}_{-1}(\Sigma_n)$ has a well defined trivial representation of dimension one, in which all the $\sigma_i$ act by identity. We denote this representation by $1$.

**Theorem 1.2 (i)** When $V = V^I$ or $V^{III}$, we have:

$$M_\xi(P) \cong \text{Ind}^{\Sigma_n}_{\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}} 1,$$
as $B_n$-modules. Here, $B_n$ acts on the right-hand side through the natural homomorphism $B_n \to \Sigma_n$.

(ii) When $V = V^{II}$, we have:

$$M_\xi(P) \cong \text{Ind}_{\mathcal{H}_-1(\Sigma_n)}^{\mathcal{H}_-1(\Sigma_{n_1}) \times \cdots \times \mathcal{H}_-1(\Sigma_{n_k})} 1,$$

as $B_n$-modules. Here, $B_n$ acts on the right-hand side through the natural (semi-group) homomorphism $B_n \to \mathcal{H}_-1(\Sigma_n)$.

In the case $V = V^I$, Theorem 1.2 is equivalent to Theorem 0.2 of [EM]. Theorem 1.3 describes the action of $\pi_1(\tilde{\Lambda}_A)$ on $M_\xi(P)$, Let $\psi : B_n \to \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ be the natural map. By part (i) of Theorem 1.2, in the case $V = V^{I}$ or $V^{III}$, there is a natural action $\phi$ of the product $\Sigma_{m_1} \times \cdots \times \Sigma_{m_l}$ on $M_\xi(P)$, commuting with the action of $\Sigma_n$. Let

$$\chi : \Sigma_{m_1} \times \cdots \times \Sigma_{m_l} \to \{1, -1\}$$

be the character taking a simple transposition in $\Sigma_{m_j}$ to $(-1)^{n_{m_1} + \cdots + m_{j}}$.

**Theorem 1.3 (i)** In the case $V = V^{I}$ or $V^{III}$, the microlocal monodromy action of $\pi_1(\tilde{\Lambda}_A)$ on $M_\xi(P)$ is given as $(\phi \otimes \chi) \circ \psi \circ \rho$.

For the case $V = V^{II}$, note that there is a natural map $\zeta : B_k \to B_n$, obtained by collecting the $n$ strands into $k$ ‘ropes,’ consisting of $n_1, \ldots, n_k$ strands. More precisely, if $\kappa_1, \ldots, \kappa_{k-1}$ are the standard generators of $B_k$, then

$$\zeta(\kappa_i) = \sigma_{m_1+\cdots+m_i} \sigma_{m_1+\cdots+m_{i-1}+1} \cdots \sigma_{m_1+\cdots+m_{i-1}+1}$$

$$\sigma_{m_1+\cdots+m_{i+1}} \sigma_{m_1+\cdots+m_{i}} \cdots \sigma_{m_1+\cdots+m_{i-2}} \sigma_{m_1+\cdots+m_{i-1}}.$$

The right action of $B_n$ on itself descends to an action $\eta$ of the image $\zeta(B_n)$ on $M_\xi(P)$ (cf. part (ii) of Theorem 1.2). Define a homomorphism $\phi : B_n \to B_n$ by $\phi : \sigma_i \mapsto \sigma_i^{-1}$. Note that $\phi$ preserves the image $\zeta(B_n)$.

**Theorem 1.3 (ii)** In the case $V = V^{II}$, the microlocal monodromy action of $\pi_1(\tilde{\Lambda}_A)$ on $M_\xi(P)$ is given as $\eta \circ \phi \circ \zeta \circ \rho$.

Theorems 1.2 and 1.3 will be proved in Section 5.

## 2 Geometric Preliminaries

In this section, we recall the basic definitions pertaining to Morse groups and nearby cycles, and prove a technical result (Lemma 2.2) about the curvature of a general fiber near a point singularity.
2.1 Nearby Cycles

Let $V, X$ be smooth, connected algebraic varieties over $\mathbb{C}$ with $\dim V = d$ and $\dim X = r$, and let $f : V \to X$ be a dominant map. Write $X^{\text{reg}} \subset X$ for the set of regular values of $f$ (we do not count a point $\lambda \in X$ as a regular value if $f^{-1}(\lambda) = \emptyset$).

Let $V^0 \subset V$ be the preimage $f^{-1}(X^{\text{reg}})$; note that it is a manifold. Fix a point $x \in f(V) \setminus X^{\text{reg}}$, and let $E = f^{-1}(x)$. Assume that $f$ is a map without blowing up along $E$, i.e., that there exists a stratification $\mathcal{E}$ of $E$, such that for any stratum $S \subset E$, Thom’s $A_f$ condition holds for the pair $(S, V^0)$. Recall that the $A_f$ condition says that for any sequence of points $v_i \subset V^0$, converging to a limit $e \in S$, if there exists a limit

$$\Delta = \lim_{i \to \infty} T_{v_i} f^{-1}(f(v_i)),$$

then $\Delta \supset T_e S$. This implies, in particular, that $\dim E = d - r$ (see [Hi] for a detailed discussion of the $A_f$ condition).

In this setting, we have a well defined nearby cycles sheaf $P = P_f$ of the map $f$ along $E$. It is defined as follows. Let $U$ be a small neighborhood of 0 in $\mathbb{C}$. Choose an algebraic arc $\gamma : U \to X$, such that $\gamma(0) = x$, and $\gamma(\tau) \in X^{\text{reg}}$, for $\tau \neq 0$. We may form the pull-back family $f_\gamma : V_\gamma \to U$, where $V_\gamma = V \times_X U$ and $f_\gamma$ is the projection onto the second factor. Set $P_\gamma = \psi_{f_\gamma} \mathcal{C}_{V_\gamma} [d - r]$, the nearby cycles of the functions $f_\gamma$ with constant coefficients (see [KS] for a discussion of the nearby cycles functor $\psi_g$ for a complex analytic function $g$).

**Proposition-Definition 2.1** [Gr, Proposition 2.4]

(i) The sheaves $P_{f_\gamma}$ for different $\gamma$ are all isomorphic. We may therefore omit the subscript $\gamma$, and call the sheaf $P_f = P_{f_x}$ the nearby cycles of $f$. It is a perverse sheaf on $E$, constructible with respect to $\mathcal{E}$.

(ii) The local fundamental group $\pi_1(X^{\text{reg}} \cap B_x)$, where $B_x \subset X$ is a small ball around $x$, acts on $P_f$ by monodromy. We denote this action by $\mu : \pi_1(X^{\text{reg}} \cap B_x) \to \text{Aut}(P_f)$.

2.2 Morse Groups

For a stratum $S \in \mathcal{E}$, let $\Lambda_S = T^*_SV \subset T^*V$ be the conormal bundle to $S$. The conormal variety $\Lambda_{\mathcal{E}} \subset T^*V$ to the stratification $\mathcal{E}$ is defined by

$$\Lambda_{\mathcal{E}} = \bigcup_{S \in \mathcal{E}} \Lambda_S.$$

Let $\Lambda^0_{\mathcal{E}}$ be the smooth part of $\Lambda_{\mathcal{E}}$. Note that $\Lambda_{\mathcal{E}}^0 \subset T^*V$ is $C^*$-conic. For $S \in \mathcal{E}$ we write $\Lambda_{\mathcal{E}}^0_S = \Lambda_S \cap \Lambda^0_{\mathcal{E}}$; this is called the set of generic conormals to $S$.

Any perverse sheaf $R$ on $E$, constructible with respect to $\mathcal{E}$, gives rise to a local system $M(R)$ on $\Lambda^0_{\mathcal{E}}$, called the Morse local system of $R$. The definition is as follows. Fix a stratum $S \in \mathcal{E}$ and a point $p \in S$. Let $\Lambda^0_p = T^*_pV \cap \Lambda^0_S$, and choose a covector
\(\xi \in \Lambda^0_p\). Let \(g : V \to \mathbb{R}\) be any smooth function with \(g(p) = 0\), and \(d_p g = \xi\). Fix a normal slice \(N \subset V\) to \(S\) through \(p\), and let \(j : N \cap E \to E\) be the inclusion. Choose positive numbers \(0 < \delta \ll \epsilon \ll 1\). Let \(B_{p, \epsilon} \subset V\) be the \(\epsilon\)-ball around \(p\) (in some fixed Hermitian metric), and \(c\) be the complex codimension of \(S\) in \(E\). The stalk \(M_{\xi}(R)\) is defined by:

\[
M_{\xi}(R) = \mathbb{H}^{c - r + d}(E \cap N \cap B_{p, \epsilon}, \{g \geq \delta\}; \ j^* R),
\]

where the right-hand side is a relative hypercohomology group with coefficients in \(j^* R\).

Lemma 2.1 below identifies the Morse groups of the nearby cycles sheaf \(P = P_f\).

Fix an algebraic arc \(\gamma : U \to X\), and let \(P = P_f\gamma\) (this fixes the ‘up-to-isomorphism’ ambiguity in the definition of \(P\)). For \(\tau \in U \setminus \{0\}\), let \(F_{p, \tau} = f^{-1}_\gamma(\tau) \cap N \cap B_{p, \epsilon}\).

The space \(F_{p, \tau}\) is a manifold with boundary. It is called a Milnor fiber of \(f_\gamma\) at \(p\) (see [Mi]).

**Lemma 2.1 (i)** The Morse group \(M_{\xi}(P)\) may be identified as follows. In addition to the choices made above, pick a number \(0 < \tau \ll \delta\). Then:

\[
M_{\xi}(P) = H^c(F_{p, \tau}, \{g \geq \delta\}; \ \mathbb{C}),
\]

where the right-hand side is an ordinary relative cohomology group.

**(ii)** Assume \(g|_{B_{p, \epsilon}}\) is the real part of a complex algebraic function \(\tilde{g} : B_{p, \epsilon} \to \mathbb{C}\). Then the set \(C\) of critical points of \(\tilde{g}|_{F_{p, \tau}}\) is finite.

**(iii)** Write \(C = \{C_i\}\), and let \(m_i\) be the multiplicity of the critical point \(C_i\) (see [Mi] for a definition). Then we have: \(\dim M_{\xi}(P) = \sum_i m_i\).

**Proof:** Part (i) is an immediate consequence of the definitions. For part (ii), note that Thom’s \(A_f\) condition and the fact that \(\xi\) is in \(\Lambda^0_\xi\) imply that the set \(C\) does not come near the boundary \(\partial F_{p, \tau}\). But \(C\) is an intersection of an affine variety with a closed ball. Therefore, we must have \(\dim C = 0\). Part (iii) is an application of Morse theory.

\[\square\]

### 2.3 Curvature of the Nearby Fiber

Assume now \(V \cong \mathbb{C}^d\) is a Hermitian affine space. Given a covector \(\xi \in \Lambda^0_p\), we will denote by the same letter the corresponding affine functional \(\xi : V \to \mathbb{C}\), with \(\xi(p) = 0\). We then have the following result about the curvature of the Milnor fibers \(F_{p, \tau}\).

**Lemma 2.2** There is a Zariski open, dense subset \(\Lambda^1_p \subset \Lambda^0_p\), such that for any compact \(\Delta \subset \Lambda^1_p\) and any \(\kappa > 0\), there exists a number \(\tau_0 > 0\), such that for any \(\tau \in \mathbb{C}^*\) with \(|\tau| < \tau_0\) and any \(\xi \in \Delta\), every critical point \(C\) of \(\xi|_{F_{p, \tau}}\) has the following
strong non-degeneracy property: all eigenvalues of the Hessian of $\xi|_{F_{p,\tau}}$ at $C$ are greater than $\kappa$.

**Proof:** Consider the cotangent bundle $\pi : T^*N \to N$. Fix a number $0 < \tau_1 \ll \epsilon$. For each $\tau \in \mathbb{C}^*$ with $|\tau| < \tau_1$, let $\Omega_{\tau} \subset T^*N$ be the conormal bundle to $F_{p,\tau} \setminus \partial F_{p,\tau}$. Each $\Omega_{\tau}$ is a manifold, obtained by intersecting an algebraic variety with the cylinder $\pi^{-1}(N \cap B_{p,\epsilon}^0)$, where $B_{p,\epsilon}^0$ is the interior of $B_{p,\epsilon}$. Let $\Omega_0 \subset \pi^{-1}(B_{p}^0)$ be the limit of the family $\{\Omega_{\tau}\}$ as $\tau \to 0$. We obtain a 1-parameter family $q : \Omega \to \{||\tau|| < \tau_1\}$, with $q^{-1}(\tau) = \Omega_{\tau}$. The set $\Omega$ is an intersection of an irreducible algebraic variety with an open region in $T^*N$. Note that $\Omega^0 = \Omega \setminus \Omega_0$ is smooth.

From Lemma 2.3, we see that unless $M_\xi(P) = 0$ for $\xi \in \Lambda^0_p$, the fiber $\Omega_0$ contains $T^*_pN$ as an irreducible component. When $M_\xi(P) = 0$, the restriction $\xi|_{F_{p,\tau}}$ has no critical points, and we have nothing to prove. Assume now $M_\xi(P) \neq 0$. By a result of Hironaka [Hi, p. 248, Corollary 1] there is an algebraic stratification of $\Omega_0$ such that for any stratum $\Sigma \in \Omega$, Thom’s $A_q$ condition holds for the pair $(\Sigma, \Omega^0)$. Such a stratification will contain a stratum $\Sigma_0$ which is an open subset of $T^*_pN$. We set:

$$\Lambda^1_p = \{\xi \in \Lambda^0_p | \xi|_{T_pN} \in \Sigma_0\}.$$  

The lemma follows from chasing the meaning of the $A_q$ condition.

\[\square\]

### 3 Some Linear Algebra

In this section, we discuss some geometric preliminaries about the map $f : V \to X$ ($V = V^{I,II,III}$). The main result here is Proposition 3.3 which gives a normal form for a general conormal vector to a $K$-orbit in the nilcone.

We regard $V^I = \mathfrak{sl}_n$ as the space of all trace zero endomorphisms of a vector space $U^I \cong \mathbb{C}^n$. We regard $V^{II} = \mathfrak{sl}_n/\mathfrak{so}_n$ as the space of all self-adjoint trace zero endomorphisms of a vector space $U^{II} \cong \mathbb{C}^n$, endowed with a non-degenerate quadratic form $\nu$. Lastly, we think of $V^{III} = \mathfrak{sl}_{2n}/\mathfrak{sp}_{2n}$ in the following way. Let $(U^{III}, \omega)$ be a complex symplectic $2n$-space. Then $V^{III}$ is the space of all trace zero endomorphisms $A$ of $U^{III}$, satisfying $\omega(Au_1, u_2) = \omega(u_1, Au_2)$, for all $u_1, u_2 \in U^{III}$. We will omit the superscripts $^{I,II,III}$, whenever a statement applies to all three, or when a particular case is specified.

The map $f$ is described as follows. Let $V = V^I$ or $V^{II}$, and $A : U \to U$ be an endomorphism in $V$. Then the components of the image $f(A) \in X \cong \mathbb{C}^{n^2}$ are given by the elementary symmetric functions in the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $A$. Let now $V = V^{III}$. Then any endomorphism $A \in V$ has a spectrum with even multiplicities. Write this spectrum as $\{\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n\}$. Then the
components of the image \( f(A) \) are given by the elementary symmetric functions in \( \{ \lambda_1, \lambda_2, \ldots, \lambda_n \} \).

We now describe the \( K \)-orbits in the nilcone \( \mathcal{N} = f^{-1}(0) \subset V \). To each \( K \)-orbit \( \mathcal{O} \subset \mathcal{N} \), we associate a partition \( \bar{n}(\mathcal{O}) \) of \( n \) as follows. If \( V = V^I \) or \( V^III \), then \( \bar{n}(\mathcal{O}) \) is the partition \( n = n_1 + \ldots + n_k \), where \( n_1, \ldots, n_k \) are the sizes of the blocks in the Jordan normal form of an element \( A \in \mathcal{O} \). When \( V = V^{III} \), the Jordan form of any \( A \in \mathcal{O} \) must have an even number of blocks of each size. Write \( n_1, n_1, n_2, \ldots, n_k, n_k \) for the sizes of these blocks. Then \( \bar{n}(\mathcal{O}) \) is the partition \( n = n_1 + \ldots + n_k \). Given a partition \( \bar{n} : n = n_1 + \ldots + n_k \), we will write \( m_1, \ldots, m_l \) for the multiplicities in the collection \( \{ n_1, \ldots, n_k \} \), as in Section 1.3. We will call an orbit \( \mathcal{O} \subset \mathcal{N} \) (and any point \( A \in \mathcal{O} \) regular, if \( \bar{n}(\mathcal{O}) \) is the partition \( n = n \). Equivalently, \( A \in \mathcal{N} \) is regular whenever \( \text{rank} (d_A f) = n - 1 \).

**Lemma 3.1** (i) When \( V = V^I \) or \( V^III \), the correspondence \( \mathcal{O} \mapsto \bar{n}(\mathcal{O}) \) sets up a bijection between the \( K \)-orbits in \( \mathcal{N} \) and the partitions of \( n \).

(ii) When \( V = V^{II} \), the correspondence \( \mathcal{O} \mapsto \bar{n}(\mathcal{O}) \) gives a surjection from the set of \( K \)-orbits to the set of partitions. If all of the multiplicities \( m_1, \ldots, m_l \) for a partition \( \bar{n} \) are even, then there are two \( K \)-orbits with \( \bar{n}(\mathcal{O}) = \bar{n} \). Otherwise, there is only one.

We now discuss the conormal variety \( \Lambda \) to the orbit stratification of \( \mathcal{N} \). Use the non-degenerate bilinear form \( \text{tr}(AB) \) on \( V \) to identify \( V \) and \( V^* \), and to regard \( \Lambda \) as a subset of \( V \times V \) \((\cong V \times V^* \cong T^*V)\). The following is a simple exercise.

**Lemma 3.2** Let \( A \in \mathcal{N} \), and \( B \in V \). Then \( (A, B) \in \Lambda \) if and only if \( AB = BA \).

Fix a partition \( \bar{n} : n = n_1 + \ldots + n_k \), and an orbit \( \mathcal{O} \subset \mathcal{N} \) with \( \bar{n}(\mathcal{O}) = \bar{n} \). Let \( \Lambda_\mathcal{O} \subset \Lambda \) be the conormal bundle to \( \mathcal{O} \), and \( \Lambda^0_\mathcal{O} \subset \Lambda_\mathcal{O} \) be the set of generic conormals to \( \mathcal{O} \). Define \( \tilde{\Lambda}_\mathcal{O} \) to be the set of all \( (A, B) \in \Lambda_\mathcal{O} \), such that \( B \) has \( k \) distinct eigenvalues.

**Proposition 3.3** (i) The set \( \tilde{\Lambda}_\mathcal{O} \) is a Zariski open, dense subset of \( \Lambda^0_\mathcal{O} \).

(ii) Fix a point \( \xi = (A, B) \in \tilde{\Lambda}_\mathcal{O} \). Let \( U = U_1 \oplus \ldots \oplus U_k \) be the generalized eigenspace decomposition for \( B \). This decomposition is orthogonal with respect to \( \nu \) when \( V = V^{II} \), and to \( \omega \) when \( V = V^{III} \). After a suitable reordering, we have: \( \dim U_i = n_i \) (cases \( V = V^I \) or \( V^{II} \)) or \( \dim U_i = 2n_i \) (case \( V = V^{III} \)). The endomorphism \( A \) preserves each \( U_i \), and the restriction \( A_{|U_i} \) is regular.

(iii) When \( V = V^{III} \), there is an \( A \)-invariant decomposition \( U_i = U_i^+ \oplus U_i^- \), with \( U_i^\pm \) Lagrangian in \( U_i \).
(iv) For each \( i = 1, \ldots, k \), there is a polynomial \( P_i \) of degree \( n_i \), such that \( B|_{U_i} = P_i(A|_{U_i}) \).

**Proof:** This is an exercise in linear algebra using Lemma 3.2.

**Remark 3.4** The reason the results of this paper do not directly generalize to other classical symmetric spaces is the failure of the analog of part (ii) of Proposition 3.3.

**Proof of Lemma 1.1:** The set \( \tilde{\Lambda}_A \) is defined as \( \tilde{\Lambda}_O \cap T_A^* V \). Let \( \Pi_\eta \) be the space of all unordered \( k \)-tuples \( \tilde{u} \) of distinct points in \( \mathbb{C} \), colored in \( l \) colors, with \( m_j \) points of \( j \)-th color. Then \( B_\eta = \pi_1(\Pi_\eta) \). By Proposition 3.3, we have a natural map \( r : \tilde{\Lambda}_A \to \Pi_\eta \) which sends any pair \((A, B) \in \tilde{\Lambda}_A\) to the spectrum of \( B \), with each eigenvalue colored according to its multiplicity. We define \( \rho = r_* : \pi_1(\tilde{\Lambda}_A) \to B_\eta \).

It is easy to see that \( r \) is a fiber bundle with connected fiber. Furthermore, one can check that the fiber is simply connected when \( V = V^I \) or \( V^I \). The lemma follows.

**4 Identifying the Critical Points**

The main idea of the proof of Theorems 1.2 and 1.3 is to pick a generic covector \( \xi = (A, B) \in \tilde{\Lambda}_O \), then compute the Morse group \( M_\xi(P) \) using a normal slice \( N \) to \( O \) at \( A \), which is of a very special form depending on \( B \).

We first describe the construction of \( N \) in the case \( V = V^I \). Fix a covector \( \xi = (A, B) \in \tilde{\Lambda}_O \). Let \( U = \bigoplus_{i=1}^{k} U_i \) be the generalized eigenspace decomposition for \( B \). For \( 1 \leq i, j \leq k \), let \( V_{i,j} = Hom(U_i, U_j) \), so that:

\[
V = \bigoplus_{1 \leq i,j \leq k} V_{i,j}.
\]

Let \( T \subset V \) be the parallel translate of the tangent space \( T_A O \) through the origin. One can check that \( T \) splits as:

\[
T = \bigoplus_{1 \leq i,j \leq k} T_{i,j},
\]

where \( T_{i,j} = T \cap V_{i,j} \). Pick a complement \( \tilde{N}_{i,j} \) to \( T_{i,j} \) in \( V_{i,j} \), and let

\[
\tilde{N} = \bigoplus_{1 \leq i,j \leq k} \tilde{N}_{i,j}.
\]
Then $N$ is the parallel translate of $\bar{N}$, passing through $A$. It is easy to check that $N$ is a normal slice to $\mathcal{O}$. We will need to consider a subset $N_{bd} \subset N$ ("bd" stands for block-diagonal), defined as the parallel translate of $\bar{N}_{bd} = \bigoplus_{i=1}^{k} \bar{N}_{i,i}$.

In the case $V = V^{II}$, we construct $N = N^{II}$ using the inclusion $j : V^{II} \to V^{I}$, which comes from identifying $U^{I}$ and $U^{II}$. The image $j(V^{II})$ is the anti-fixed points of the involution $\theta : A \mapsto -A^*$ on $V^{I}$. We have $\xi = (A, B) \in 1 \mathcal{O} \subset V^{II} \times V^{II}$. Apply the construction described above (in the case $V = V^{I}$) for the pair $(j(A), j(B))$, to obtain a normal slice $N^{I} \subset V^{I}$. Require, in addition, that $N^{I}$ should be $\theta$-invariant. Then set $N^{II} = j^{-1}(N^{I})$, and $N^{II}_{bd} = j^{-1}(N_{bd}^{I})$.

In the case $V = V^{III}$, we proceed as follows. Given a covector $\xi = (A, B) \in 1 \mathcal{O}$, let $U_{i}, U_{i}^{+}, U_{i}^{-} (i = 1, \ldots, k)$ be as in Proposition 3.3. For $1 \leq i < j \leq k$, let:

$$
V_{i,j}^{++} = (\text{Hom}(U_{i}^{+}, U_{j}^{+}) \oplus \text{Hom}(U_{j}^{-}, U_{i}^{-})) \cap V, \\
V_{i,j}^{-+} = (\text{Hom}(U_{i}^{+}, U_{j}^{-}) \oplus \text{Hom}(U_{j}^{+}, U_{i}^{-})) \cap V, \\
V_{i,j}^{+-} = (\text{Hom}(U_{i}^{-}, U_{j}^{+}) \oplus \text{Hom}(U_{j}^{-}, U_{i}^{+})) \cap V, \\
V_{i,j}^{--} = (\text{Hom}(U_{i}^{-}, U_{j}^{-}) \oplus \text{Hom}(U_{j}^{+}, U_{i}^{+})) \cap V.
$$

For $1 \leq i \leq k$, let:

$$
V_{i}^{+} = (\text{Hom}(U_{i}^{+}, U_{i}^{+}) \oplus \text{Hom}(U_{i}^{-}, U_{i}^{-})) \cap V, \\
V_{i}^{++} = \text{Hom}(U_{i}^{+}, U_{i}^{-}) \cap V, \\
V_{i}^{--} = \text{Hom}(U_{i}^{-}, U_{i}^{+}) \cap V.
$$

Let $T \subset V$ be the parallel translate of $T_{A} \mathcal{O}$ through zero. For all the possible values of the subscript and the superscript, choose a linear complement $N_{i}^{*}$ to $T \cap V_{i}^{*}$ in $V_{i}^{*}$. Let $\tilde{N}$ be the direct sum of all the $N_{i}^{*}$, and let $\bar{N}_{bd} = \bigoplus_{i=1}^{k} \bar{N}_{i}^{+}$. We define $N (\bar{N}_{bd})$ to be the parallel translate of $\tilde{N} (\bar{N}_{bd})$ through $A$. This completes the construction of the normal slice $N$.

Choose a regular value $\lambda \in X^{reg}$, corresponding to an $n$-tuple $\{\lambda_{1}, \ldots, \lambda_{n}\}$ of distinct eigenvalues ($\sum \lambda_{i} = 0$). For $\tau \in C$, let $\tau \cdot \lambda \in X^{reg}$ be given by the $n$-tuple $\{\tau \cdot \lambda_{1}, \ldots, \tau \cdot \lambda_{n}\}$.

Consider a curve $\gamma : C \to X$, defined by $\gamma : \tau \mapsto \tau \cdot \lambda$, and form the pull-back family $f_{\tau} : V_{\gamma} \to C$. Fix a small ball $B_{A, \epsilon} \subset V$ around $A$, and let $F_{A,\tau} = f_{\tau}^{-1}(\tau) \cap N \cap B_{A,\epsilon}$, for $0 < |\tau| \ll \epsilon$. Think of $\xi$ as a linear function on $V$, with $\xi(A) = 0$.

**Lemma 4.1** The set of critical points of $\xi|_{F_{A,\tau}}$ is equal $F_{A,\tau} \cap \bar{N}_{bd}$.

**Proof:** By the construction of $N$, any point of the intersection $F_{A,\tau} \cap \bar{N}_{bd}$ is critical for $\xi|_{F_{A,\tau}}$. To prove the opposite inclusion, consider first the case $V = V^{I}$. Let $H \cong (\mathbb{C}^{*})^{k-1}$ be the torus consisting of all $h \in SL_{n}$ which act by a scalar on each $U_{i}$. The torus $H$ acts on $V$ as a subgroup of $K = SL_{n}$, i.e., by conjugation. By
construction, the normal slice $N$ is preserved by this $H$-action. So are the covector $\xi$ and the fiber $f^{-1}_\gamma(\tau)$.

We claim that any critical point $C$ of $\xi|_{F_{A,\tau}}$ must be fixed by $H$. To prove this, assume $C \in F_{A,\tau}$ is a critical point with $\dim H \cdot C \geq 1$. Then the intersection $(H \cdot C) \cap F_{A,\tau}$ consists entirely of critical points of $\xi|_{F_{A,\tau}}$, which contradicts part (ii) of Lemma 2.1.

It remains to note that the fixed points of $H$ in $F_{A,\tau}$ are precisely the intersection $F_{A,\tau} \cap N_{bd}$. This completes the proof for $V = V^I$.

For $V = V^{II}$, we use the inclusion $j : V^{II} \to V^I$. Assume $C \in F_{A,\tau}^{II}$ is a critical point of $\xi^{II}$. Then $j(C) \in F_{j(A),\tau}^{II}$ is a critical point for the covector $\xi^I : V^I \to \mathbb{C}$, given by $j(B)$ (this is because the normal slice $N^I$ is invariant with respect to the involution $\theta$). Applying the lemma in the case $V = V^I$, we find that $j(C)$ is in $N_{bd}^I$. For the case $V = V^{III}$, let $H \sim C^k$ be the set of all $h \subset Sp_{2n}$ which act by a scalar $\kappa^\pm$ on each $U^\pm_i$ (note that we must have $\kappa^+_i \cdot \kappa^-_i = 1$). The rest is exactly as in the case $V = V^I$. 

\[
\square
\]

The idea of using torus symmetry to study microlocal geometry in this setting is due to Evens and Mirković [EM].

The intersection $F_{A,\tau} \cap N_{bd}$ is easy to describe combinatorially. Denote by $B$ the set of all maps $\beta : \{\lambda_1, \ldots, \lambda_n\} \to \{1, \ldots, k\}$ with $\# \beta^{-1}(i) = n_i$. We will sometimes use a shorthand: $\beta(i) = \beta(\lambda_i)$, for $i \in \{1, \ldots, n\}$.

**Lemma 4.2** For each $\beta \in B$, there is a unique endomorphism $C_\beta \in F_{A,\tau} \cap N_{bd}$, such that $C_\beta$ preserves each vector space $U_i$, and the spectrum of $C_\beta|U_i$ is equal to $\tau \cdot \beta^{-1}(i)$ (with multiplicities doubled when $V = V^{II}$). The assignment $\beta \mapsto C_\beta$ gives a bijection between $B$ and $F_{A,\tau} \cap N_{bd}$.

**Proof:** This follows from the construction of $N_{bd}$. 

\[
\square
\]

**Lemma 4.3** (i) Each of the critical points $C_\beta$ is Morse, i.e., the Hessian of $\xi|_{F_{A,\tau}}$ at $C_\beta$ is non-degenerate.

(ii) We have $\dim M_\xi(P) = \# B = \frac{n!}{n_1! \cdots n_k!}$.

**Proof:** This is a general position argument. However, we need to be careful, because the pair $(\xi, N)$ is far from generic in our construction.

Fix an affine normal slice $N_0$ to $\mathcal{O}$ at $A$. Apply Lemma 2.3 to the family $f_\gamma$ and the normal slice $N_0$ to obtain an open set $\Lambda^1_A \subset \Lambda^0_A$. 

11
Claim: Assume $\xi \in \Lambda_A^1$. Then, for $0 < |\tau| \ll \epsilon$, all the critical points of $\xi|_{F_{A,\tau}}$ are Morse.

To prove the claim, we use the $K$-action. Let $\mathfrak{t}_A \subset \mathfrak{k} = \text{Lie}(K)$ be the stabilizer of $A$. Choose a complement $\mathfrak{t}_A^1$ to $\mathfrak{t}_A$ in $\mathfrak{k}$. Assuming the number $\epsilon > 0$ is sufficiently small, there exists a closed neighborhood $U \subset N_0$ of $A$, and a unique diffeomorphism $\psi : U \to N \cap B_c$, such that $y = \psi(x)$ if and only if $y = \exp(t) x$, for some $t \in \mathfrak{t}_A^1$.

Note that $f \circ \psi = f$, so that $F_{A,\tau} = \psi(f_{\gamma}(\tau)) \cap U$.

Choose a compact neighborhood $\Delta \subset \Lambda^1_A$ of $\xi$. Note that the tangent spaces $T_A N_0$ and $T_A N$ are naturally identified with the quotient $T_A V/T_A \mathcal{O}$, and therefore with each other. Under this identification, the differential $d_A \psi$ is the identity. It follows that we can choose a $\tau_1 > 0$, such that for any $\tau \in \mathbb{C}^*$ with $|\tau| < \tau_1$, and any critical point $C$ of $\xi|_{F_{A,\tau}}$, we have: $d_{\psi^{-1}(C)}^* \psi(\xi) \in \Delta$ (we use the identifications $T^*_C N \cong T^*_{\psi^{-1}(C)} N_0 \cong (T_A V/T_A \mathcal{O})^*$). Next, choose a number $\kappa > 0$ which is large compared to the second derivatives of $\psi$. With these choices, we may use Lemma 2.2 to select a $\tau_0 > 0$. The claim for $|\tau| < \min\{\tau_0, \tau_1\}$ then follows from Lemma 2.2.

The claim, together with Lemmas 2.1, 4.2, and 4.3, implies that $\dim M_\xi(P) = \#B$, if $\xi \in \Lambda_A^1$. However, $\dim M_\xi(P)$ is independent of $\xi$, for $\xi \in \Lambda_A^0$. Applying Lemma 2.1 in the other direction, we conclude that for any $\xi \in \Lambda_A$, all the critical points $C_\beta$ are Morse.

5 Picard-Lefschetz Theory

Given Lemmas 4.1, 4.2, and 4.3, the proofs of Theorems 1.2 and 1.3 are obtained as applications of Picard-Lefschetz theory.

Continuing with the situation of Section 4, assume that the numbers $\{\lambda_1, \ldots, \lambda_n\}$ satisfy $\lambda_1 < \ldots < \lambda_n$. Assume also that the covector $\xi = (A, B)$ is chosen so that the endomorphism $B$ is semisimple, with eigenvalues $u_1 < \ldots < u_k$, and that $B|_{U_i} = u_i$.

Let $\Pi_n$ denote the set of all unordered $n$-tuples of distinct points in $C$. We think of $B_n$ as the fundamental group $\pi_1(\Pi_n, \lambda)$, where $\lambda = \{\lambda_i\}$. Then $\Sigma_n$ is the permutations of the $\{\lambda_i\}$, and it acts on the index set $\mathcal{B}$ by $w : \beta \mapsto w\beta = \beta \circ w^{-1}$. Let $\beta_0 \in \mathcal{B}$ be the unique map with $\beta_0(\lambda_i) \leq \beta_0(\lambda_j)$, for all $1 \leq i < j \leq n$. Every $\beta \in \mathcal{B}$ is of the form $w\beta_0$, for some $w \in \Sigma_n$.

Fix small numbers $0 < \tau \ll \delta \ll \epsilon \ll 1$, define $F_{A,\tau} = f^{-1}_\gamma(\tau) \cap N \cap B_{A,\epsilon}$, and recall (cf. Lemma 2.1) that

$$M_\xi(P) = H^c(F_{A,\tau}, \{\xi \geq \delta\}; \mathbb{C}),$$

where $c$ is the complex codimension of $\mathcal{O}$ in $\mathcal{N}$.

We now recall a standard Picard-Lefschetz construction of classes in $M_\xi(P)$ (see [AGV] for a detailed discussion of Picard-Lefschetz theory). Let $C = C_\beta$ be
one of the critical points of $\xi|_{F_{A,\tau}}$ described in Lemma 1.2. Note that $\xi(C) = \tau \cdot \sum_{i=1}^{n} \sigma_i \cdot u_{\beta(i)}$. Fix a smooth path $\alpha : [0, 1] \to \mathbb{C}$ such that:

(i) $\alpha(0) = \xi(C)$, and $\alpha(1) = \delta$;
(ii) $|\alpha(t)| \leq \delta$, for any $t \in [0, 1]$;
(iii) $\alpha(t) \neq \xi(C_{\beta})$, for any $t > 0$, $\beta' \in \mathcal{B}$;
(iv) $\alpha(t_1) \neq \alpha(t_2)$, for $t_1 \neq t_2$;
(v) $\alpha'(t) \neq 0$, for $t \in [0, 1]$.

Let $\mathcal{H} : T_C F_{A,\tau} \to \mathbb{C}$ be the Hessian of $\xi|_{F_{A,\tau}}$ at $C$, and let $T_C[\alpha] \subset T_C F_{A,\tau}$ be the positive eigenspace of the (non-degenerate) real quadratic form

$$\text{Re}(\mathcal{H}/\alpha'(0)) : T_C F_{A,\tau} \to \mathbb{R}.$$ 

Note that $\dim_{\mathbb{R}} T_C[\alpha] = \dim_C F_{A,\tau} = c$. Fix an orientation $\mathcal{O}$ of $T_C[\alpha]$. The triple $(C, \alpha, \mathcal{O})$ defines a homology class

$$[C, \alpha, \mathcal{O}] \in H_c(F_{A,\tau}, \{\xi \geq \delta\}; \mathbb{C}) = M_{\xi}(P)^*.$$

Namely, the class $[C, \alpha, \mathcal{O}]$ is represented by an embedded $c$-disc

$$\kappa : (D^c, \partial D^c) \to (F_{A,\tau}, \{\xi \geq \delta\}),$$

such that the image of $\kappa$ projects onto the image of $\alpha$ and is tangent to $T_C[\alpha]$ at $C$. The sign of $[C, \alpha, \mathcal{O}]$ is given by the orientation $\mathcal{O}$. It is a standard fact that $[C, \alpha, \mathcal{O}] \neq 0$.

Note that a choice of $\sqrt{-1}$ gives an isomorphism $M_{\xi}(P) \cong M_{-\xi}(P)$. On the other hand, the intersection pairing on $F_{A,\tau}$ induces a perfect duality between $M_{\xi}(P)$ and $M_{-\xi}(P)$. Therefore, the Morse group $M_{\xi}(P)$ is canonically its own dual, and we may regard the Picard-Lefschetz class $[C, \alpha, \mathcal{O}]$ as an element of $M_{\xi}(P)$.

Before we begin the proofs of Theorems 1.2 and 1.3, we need to recall a result of [Gr].

**Theorem 5.1** [Gr, Theorems 3.1, 6.4]

(i) In the case $V = V^I$ or $V^{III}$, the monodromy action $\mu : B_n \to \text{Aut}(P)$ factors through $\Sigma_n$.

(ii) In the case $V = V^{II}$, the monodromy action $\mu : B_n \to \text{Aut}(P)$ factors through $\mathcal{H}_{-1}(\Sigma_n)$.

In the case $V = V^I$, Theorem 5.1 goes back to the work of Slodowy in [Sl], and appears in [M] in its present form.

**Proof of Theorem 1.2:** Begin with part (i) of the theorem. By part (i) of Theorem 5.1, the monodromy action $\mu_\ast : B_n \to \text{Aut}(M_{\xi}(P))$ factors through an action $\bar{\mu} : \Sigma_n \to \text{Aut}(M_{\xi}(P))$. Write $C_0 = C_{\beta_0}$. Let $\alpha_0 : [0, 1] \to \mathbb{C}$ be the straight line path connecting $\xi(C_0)$ to $\delta$. Pick an orientation $\mathcal{O}_0$ of $T_{C_0}[\alpha_0]$. Let $\epsilon_0 = [C_0, \alpha_0, \mathcal{O}_0] \in$
\( M_\xi(P) \). Consider the product \( \Sigma_{n_1} \times \ldots \times \Sigma_{n_k} \) as a subgroup of \( \Sigma_n \) in the obvious way.

**Claim 1:** The image of \( e_0 \) under the action \( \tilde{\mu} \) spans \( M_\xi(P) \) as a vector space.

**Claim 2:** For \( w \in \Sigma_{n_1} \times \ldots \times \Sigma_{n_k} \subset \Sigma_n \), we have: \( \tilde{\mu}(w) e_0 = e_0 \).

Part (i) of the theorem follows immediately from the two claims and the computation of \( \dim M_\xi(P) \) in Lemma 1.3. The proofs of both claims are standard applications of Picard-Lefschetz theory. Begin with Claim 1. For each \( M \), pick an element \( w_\beta \in \Sigma_n \) with \( w_\beta \beta_0 = \beta \). Then \( \tilde{\mu}(w_\beta) e_0 \) is of the form \([C_\beta, \alpha_\beta, O_\beta]\), where \( \alpha_\beta \) is some path connecting \( \xi(C_\beta) \) to \( \delta \), and \( O_\beta \) is an orientation of \( T_{C_\beta} [\alpha_\beta] \). The set \( \{ \tilde{\mu}(w_\beta) e_0 \}_{\beta \in \mathcal{B}} \) contains one Picard-Lefschetz class for each critical point of \( \xi |_{F_{A,\tau}} \). Therefore, it is a basis of \( M_\xi(P)^* \).

To prove Claim 2, let \( s \) be a simple reflection in \( \Sigma_{n_1} \times \ldots \times \Sigma_{n_k} \). Note that \( s \beta_0 = \beta_0 \). By tracing what happens to the critical values \( \xi(C_\beta) \) as we permute the two eigenvalues corresponding to \( s \), it is not hard to see that the path \( \alpha_0 \) does not change, and we have: \( \tilde{\mu}(s) e_0 = \pm e_0 \). A further argument with the Hessian \( H_0 : T_{C_0} F_{A,\tau} \to \mathbb{C} \) of \( \xi |_{F_{A,\tau}} \) at \( C_0 \) shows that, in fact, \( \tilde{\mu}(s) e_0 = e_0 \).

The analysis of \( H_0 \) is based on the fact that the decomposition of \( T_{C_0} F_{A,\tau} \) by the intersections with the subspaces \( \tilde{N}_{i,j} \subset T_{C_0} V \cong V \) used in the construction of \( N \) (see Section 4) has the following properties:

(i) \( T_{C_0} F_{A,\tau} \cap \tilde{N}_{i,i} = 0 \), and

(ii) \( T_{C_0} F_{A,\tau} \cap \tilde{N}_{i,j} \) is orthogonal to \( T_{C_0} F_{A,\tau} \cap \tilde{N}_{l,m} \) with respect to \( H_0 \), unless \( i = m \) and \( j = l \).

We omit the details of the analysis of \( H_0 \).

Part (ii) of the theorem is proved similarly.

**Proof of Theorem 1.3:** This is similar to the proof of Theorem 1.2. Continuing with the notation of that proof, we give an outline of the argument.

**Step 1.** Denote the microlocal monodromy action of \( \pi_1(\tilde{A}_A) \) on \( M_\xi(P) \) by \( h \). Since \( h \) commutes with the monodromy action \( \mu_* \), and \( e_0 \) generates \( M_\xi(P) \) under \( \mu_* \), it is enough to verify the claim of the theorem about \( h(v) e_0 \), for any \( v \in \pi_1(\tilde{A}_A) \).

**Step 2.** We show that \( h \) factors through the homomorphism \( \rho \) of Lemma 1.1. In the case \( V = V^I \) or \( V^{III} \), this is clear since \( \rho \) is an isomorphism. Consider now the case \( V = V^II \). Choose an element \( v \in \text{Ker} \rho \). Recall that the homomorphism \( \rho : \pi_1(\tilde{A}_A) \to B_0 \) is defined as the push-forward \( r_* \) by the map \( r : \tilde{A}_A \to \Pi_0 \) which takes a covector \((A, B)\) to the spectrum of \( B \). Therefore, \( v \) may be represented by a loop lying inside a fiber of \( r \). It follows that \( h(v) \) fixes any Picard-Lefschetz class up to sign. A verification with the Hessian \( H_0 \) shows that, in fact, \( h(v) e_0 = e_0 \). We may therefore write \( h = \tilde{h} \circ \rho \).
Step 3. We now specify a set of generators for the group $B_{\tilde{n}}$. Let $\tilde{u} = r(\xi) = \{u_1, \ldots, u_k\} \in \Pi_{\tilde{n}}$. We identify $B_{\tilde{n}}$ with the fundamental group $\pi_1(\Pi_{\tilde{n}}, \tilde{u})$. Note that $B_{\tilde{n}}$ is naturally a subgroup of $B_k$. Let $\kappa_1, \ldots, \kappa_{k-1}$ be the standard generators for $B_k$. For each $1 \leq i < j \leq l$, define an element $\varsigma_{i,j} \in B_{\tilde{n}}$ by

$$\varsigma_{i,i+1} = \kappa_{m_1+\ldots+m_i}^2,$$

and

$$\varsigma_{i,j} = \kappa_{m_1+\ldots+m_{j-1}}^{-1} \kappa_{m_1+\ldots+m_{j-1}} \kappa_{m_1+\ldots+m_i}^{-1} \kappa_{m_1+\ldots+m_i}^2 \kappa_{m_1+\ldots+m_{i+1}} \kappa_{m_1+\ldots+m_{i+2}} \ldots \kappa_{m_1+\ldots+m_{j-1}},$$

when $j - i > 1$. Note that $\varsigma_{i,j}$ is just a braid moving the point $u_{m_1+\ldots+m_{j-1}+1}$ once around $u_{m_1+\ldots+m_i}$. It is not hard to check that the $\{\varsigma_{i,j}\}$, together with the $\{\kappa_i | \kappa_i \in B_{\tilde{n}}\}$ give a set of generators for $B_{\tilde{n}}$.

Step 4. Let $\upsilon$ be one of the generators of $B_{\tilde{n}}$ constructed in Step 3. By tracing what happens to the path $\alpha_0$ used in the definition of $e_0$, as we vary $\tilde{u} \in \Pi_{\tilde{n}}$ along a path representing $\upsilon$, we can check that

$$\tilde{h}(\upsilon) e_0 = \pm \mu_* \circ o \circ \zeta(\upsilon^{-1}) e_0.$$  

Step 5. A verification with the Hessian $H_0$ shows that the sign in the above equation is a plus, except when $V = V^I$ or $V^{III}$, and $\upsilon \in \{\kappa_i | \kappa_i \in B_{\tilde{n}}\}$.

Step 6. Recall that under the identification of $M_{\xi}(P)$ with a quotient of $\mathbb{C}[B_n]$ in the proof of Theorem 1.2, the element $e_0$ is the image of $1 \in \mathbb{C}[B_n]$. Therefore, Steps 4 and 5 verify the claim of the theorem about the action of $\tilde{h}(\upsilon)$ on $e_0$.

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16