Radiation leading to long-lived current states in a superconducting circuit

Gilberto A Urzuá and Alvaro Restuccia

Abstract. In this work we show that radiation can have interesting effects on a superconducting inductive circuit. Consider that for some time a battery supplies a constant voltage to a ring made of a superconductor wire above its critical temperature. At some instant the battery is disconnected and the ring is immersed in a low temperature bath, its resistance becoming zero. We study the system right after it undergoes the phase transition and enters the superconducting state. The system is modeled as an ideal selfinductance, and the radiation emitted is modeled as magnetic dipole radiation. We analyze qualitatively the differential equation that models the electrical current evolution to obtain some of its main characteristics. We find that, depending upon its initial conditions when it enters the superconducting state, the electrical current can evolve in one out of three very well differentiated sets of states. The first set consists of just one state where the current decays exponentially, in a similar fashion as it would do if the phase transition would have not taken place. The second set is comprised of short-lived states, where the current decays faster than the exponential state, and dies suddenly. The third set is comprised of long-lived states, where the current decays slower than the exponential state and reaches smoothly a constant value different from zero at a finite time. It stays in that state and actually lives in it forever.

Departamento de Física, Universidad de Antofagasta, Casilla 170, Antofagasta, Chile
E-mail: gilberto.urzua@uantof.cl, alvaro.restuccia@uantof.cl

1. Introduction
Consider a physical system consisting of a ring made of a superconducting wire, at the beginning above its critical temperature, that is connected to a battery supplying a constant voltage $V_0$. The battery provides energy to the system and a positive defined electrical current $i > 0$ starts circulating. As the current $i$ grows, some fraction of the provided energy is stored as magnetic energy around the ring, some fraction is dissipated as Joule heat in the wire, and the minor complementary fraction is radiated to infinity. After a while the current asymptotically reaches a steady-state value that maintains a constant magnetic field. In this regimen there is no radiation to infinity [1], and the energy being delivered by the battery is being dissipated as Joule heat.

The battery can be disconnected at any time during this process, ceasing to deliver power. When the battery is disconnected, the current does not drop to zero at once, but decreases smoothly, since the magnetic field around the ring now gives power back to the wire. While the current decays, some fraction of the delivered back energy is dissipated as Joule heat and the complementary fraction is radiated to infinity. Being the resistance finite, the final current is zero and in the end no energy remains stored as magnetic energy around the ring. Usually in this kind of physical system the energy radiated to infinity is ignored, and the system is modeled...
as an RL circuit.

In this paper we are interested in studying what happens to the electrical current after the ring has been cooled through its transition temperature for superconductivity. We assume that once the battery is disconnected, the superconductor ring is immersed in a low temperature bath below its transition temperature, its resistance $R$ dropping essentially to zero in a transition time $\varepsilon$. The fact that now the resistance is zero, implies that energy is not dissipated as Joule heat anymore, and we are compelled to include radiation in our analysis.

Steady electrical currents without attenuation have been observed in superconducting solenoids [2]. The question arises, does radiation plays some role in the birth of such constant current states? We study the system after the phase transition has taken place. The physical system is modeled by a circuit that have an ideal selfinductance, and the radiation emitted is modeled as magnetic dipole radiation. After imposing energy balance, we obtain the differential equation that models the evolution of the current. Analyzing it we find that if, after the system undergoes the phase transition, the current enters the superconducting state satisfying certain conditions, after a finite time it will reach a state of constant current different from zero that stays forever.

2. Background: RL model
In a normal non-superconducting system, usually the energy radiated to infinity is ignored and the system is studied within the RL model: The physical system is modeled by a circuit that have an ideal resistance $R$ and an ideal selfinductance $L$. The model does not incorporate radiation to infinity. Within this model, when the battery is connected the electrical current increases as

$$i = \frac{V_0}{R} [1 - \exp (- (R/L)t)].$$

(1)

When enough time goes by, the current reaches asymptotically the value $i = V_0/R$. If the battery is disconnected, it decays exponentially as

$$i = \frac{V_0}{R} \exp (- (R/L)t).$$

(2)

Being the resistance finite, the final current is zero and in the end no energy remains stored around the ring. Any energy radiated to infinity that might have occurred, in this model is counted as energy dissipated as Joule heat, which becomes overestimated.

If for describing the system after the disconnection of the battery one improved the model incorporating the radiation to infinity, one would expect that just some fraction of the stored energy in the magnetic field would be dissipated as Joule heat, while the remaining fraction would be radiated to infinity. Again, being the resistance finite, the final current would be zero.

3. Model with selfinductance $L$ plus radiation for the superconducting system
We will set $t = 0$ right after the system leaves the phase transition and enters into the superconducting state, denoting the current by $I(t)$. The current at $t = 0$ is positive, $I|_{t=0} = I_0 > 0$, and we assume that its derivative at $t = 0$ is $dI/dt|_{t=0} = dI/dt|_{0} < 0$. The superconducting system is modeled as an ideal selfinductance $L$ and an ideal resistance $R = 0$. Now we include the radiation emitted by the ring to infinity, modeling it as magnetic dipole radiation. After imposing balance of energy, the power equation gives the second order non-linear differential equation

$$K \left( \frac{d^2I}{dt^2} \right)^2 + LI \frac{dI}{dt} = 0,$$

(3)

for the current $I(t)$, where $K > 0$ is the radiation constant [3]. Being a second order equation, its initial conditions are given by two independents constants: One for the current and another
one for its derivative. For the initial condition \( I|_0 > 0 \) of this problem, Eq. (3) allows as initial conditions \( dI/dt|_0 \leq 0 \).

For \( dI/dt|_0 = 0 \), its second derivative must be \( d^2 I/dt^2|_0 = 0 \), what leads to a constant current equal to its initial value \( I_0 \), which does not dissipate energy by Joule effect, nor by radiation to infinity. It just keep a constant magnetic field around the loop.

For \( dI/dt|_0 < 0 \) its second derivative can be either positive or negative, and to solve Eq. (3) we can proceed as follows:

While \( I(t) \neq 0 \), we can use the logarithmic derivative of the current \( I \) as a variable \( x \),

\[
  x = \frac{1}{I} \frac{dI}{dt},
\]

Taking its time derivative, after rearranging terms we obtain

\[
  \frac{d^2 I}{dt^2} = \left( \frac{dx}{dt} + x^2 \right) I.
\]

Plugging Eqs. (5) and (4) back into Eq. (2), and dividing by \( I^2 \) we get

\[
  K \left( \frac{dx}{dt} + x^2 \right)^2 + Lx = 0.
\]

Being \( K \) and \( L \) positives, Eq. (6) implies that \( x \) must be nonpositive. Equation (6) can be rewritten as

\[
  \frac{dx}{dt} + x^2 = \pm \left( \frac{K}{L} \right)^{-1/2} \sqrt{-x} \quad x \leq 0.
\]

From here on we focus the discussion of Eq. (7) on the plus sign. From Eq. (5) this implies that we will be discussing solutions with \( d^2 I/dt^2 \geq 0 \), i.e. convex solutions for \( I \). Among these ones, those solutions which approach to zero behave more smoothly than the concave ones.

Making the change of variable

\[
  w = \left( \frac{K}{L} \right)^{1/6} \sqrt{-x} \quad w \geq 0,
\]

the Eq. (7) with the plus sign, becomes

\[
  -2 \left( \frac{K}{L} \right)^{1/3} w \frac{dw}{dt} + w^4 = w.
\]

We notice that if \( w = 0 \) at some time \( T \), then \( w = 0 \) for all \( t \geq T \) is a solution of Eq. (9). Since from Eq. (4),

\[
  w = \left( \frac{K}{L} \right)^{1/6} \sqrt{-\frac{1}{I} \frac{dI}{dt}},
\]

it follows that, for \( w = 0 \), \( dI/dt = 0 \), i.e. \( I(t) = I|_T \) for all \( t \geq T \).

For \( w \neq 0 \), Eq. (9) can be divided by \( w \) to get

\[
  2 \left( \frac{K}{L} \right)^{1/3} \frac{dw}{dt} = w^3 - 1, \quad w > 0.
\]

In the following we study the system of Eqs. (10)-(11), that is equivalent to the original power equation given in Eq. (3).
4. Exponential solution
From Eq. (11), we see that the static solution for \( w \) is the constant \( w = 1 \). Squaring Eq. (10) for \( w = 1 \) one obtains
\[
\frac{1}{I} \frac{dI}{dt} = -\left( \frac{K}{L} \right)^{-1/3},
\]
whose solution is the exponential function
\[
I(t) = I_0 \exp\left( -\frac{K}{L} \right)^{-1/3} t),
\]
where \( I_0 \equiv I\big|_{t=0} > 0 \).

5. Sudden death solution
We now study the case for \( w_0 > 1 \), where \( w_0 \equiv w\big|_{t=0} \). Evaluating Eq. (10) on \( t = 0 \) and squaring it, we obtain the initial condition
\[
\left( \frac{1}{I} \frac{dI}{dt} \right) \bigg|_{0} < -\left( \frac{K}{L} \right)^{-1/3}.
\]
After separating variables in Eq. (11) and integrating, we obtain
\[
\int_{w_0}^{w_1} \frac{dw}{w^3 - 1} = \frac{1}{2} \left( \frac{K}{L} \right)^{-1/3} t,
\]
where \( w_t \equiv w\big|_{t} \). When \( t \) increases, the right side increases and therefore the integral increases. Since in the neighborhood of \( w_0 > 1 \) the argument of the integral is positive, it must hold that \( w_t > w_0 \). For this integral, as \( w_t \to \infty \), the area under the curve remains finite, i.e.
\[
\int_{w_0}^{\infty} \frac{dw}{w^3 - 1} < \infty,
\]
and therefore, when \( w_t \to \infty, \ t \to T_1 \) finite. We can analyze the implications of this for the current \( I \) as \( t \to T_1 \). From Eq. (5) and Eq. (7) with the plus sign we have that
\[
\frac{1}{I} \frac{d^2I}{dt^2} = \left( \frac{K}{L} \right)^{-1/2} \sqrt{-x} > 0.
\]
Since \( I > 0 \), then \( d^2I/dt^2 > 0 \). The fact that \( d^2I/dt^2 > 0 \) implies that the initial negative value \( dI/dt\big|_0 \) is a lower bound of the nonpositive quantity \( dI/dt \), for all \( t \). Moreover from Eq. (10),
\[
w_t = \left( \frac{K}{L} \right)^{1/6} \sqrt{-\frac{1}{I} \frac{dI}{dt}}
\]
Since \( dI/dt \) is bounded below, \( -dI/dt \) is bounded above. Therefore if \( w_t \to \infty \) then necessarily \( I \to 0 \). Therefore
\[
as t \to T_1, \quad I(t) \to 0.
\]
The previous analysis is valid up to \( t \to T_1 \). As the solution is continuous when \( I \to 0 \), then
\[
for \quad t = T_1, \quad I(T_1) = 0.
\]
Now we need to analyze the solution for \( t > T_1 \). From the power equation (3)

\[
K \left( \frac{d^2 I}{dt^2} \right)^2 + \frac{1}{2} L \frac{d(I^2)}{dt} = 0,
\]

integrating we get

\[
\frac{1}{2} LI^2 \bigg|_t^T + \int_{T_1}^t K \left( \frac{d^2 I}{dt^2} \right)^2 = \frac{1}{2} LI^2 \bigg|_{T_1}.
\]

If at the instant \( T_1 \) the current is zero \( I^2 \bigg|_{T_1} = 0 \Rightarrow I = 0 \) para todo \( t \geq T_1 \): The solution \( I(t) \) decreases to \( I = 0 \) in a finite time \( T_1 \) and it stays \( I = 0 \) for all \( t > T_1 \). We have found similar behavior in capacitive system when radiation is considered [4].

6. Long-lived solution
Finally we study the case with \( w_0 < 1 \). Let us consider again the integral

\[
\int_{w_0}^{w_1} \frac{dw}{w^3 - 1} = \frac{1}{2} \left( \frac{K}{L} \right)^{-1/3} \ln \frac{w}{w_0}.
\]

When \( t \) increases, the right side increases and therefore the integral increases. Since in the neighborhood of \( w_0 < 1 \) the argument of the integral is negative, it must hold that \( w_t < w_0 \).

For this integral, as \( w_t \to 0 \), the area under the curve remains finite, i.e.

\[
\int_{w_0}^0 \frac{dw}{w^3 - 1} < \infty,
\]

and, therefore as \( w_t \to 0 \), \( t \to T_2 \) finite. From Eq. (18) if \( w_t \to 0 \) then \( x \to 0 \). From the definition

\[
x = \frac{1}{I} \frac{dI}{dt},
\]

as \( I \) is bounded above by its initial value \( I_0 \), if \( x \to 0 \) necessarily \( dI/dt \to 0 \). Therefore, there exists a value \( I_f \) such that

\[
as t \to T_2, \quad I \to I_f,
\]

and \( dI/dt \to 0 \). By continuity

\[
I(T_2) = I_f.
\]

The current \( I_f \) is different from zero because for \( t > 0 \) the solution with the initial condition \( I(0) = I_0 \) and \( w_0 < 1 \) never crosses the curve \( I(t) = I_0 \exp(-K/L)^{-1/3}t \), that corresponds to \( w = 1 \). In fact, during the evolution \( w < 1 \) for all \( t > 0 \) and initially the slope of \( I(t) \) is lower than the slope of the exponential.

Let us analyze what does occur for \( t \geq T_2 \). We have to find the solution of the power equation (3)

\[
K \left( \frac{d^2 I}{dt^2} \right)^2 + LI \frac{dI}{dt} = 0,
\]

with the initial conditions at \( t = T_2 \): \( I(T_2) = I_f \) and \( dI/dt \bigg|_{T_2} = 0 \). It is straightforward to notice that the solution is

\[
I(t) = I_f \neq 0 \quad \text{for } t \geq T_2.
\]

We notice that at \( t = T_2 \), \( w = 0 \). We call this the long-lived solution.
7. Conclusions
We have studied an inductive physical system after it has undergone a phase transition to a superconducting state, its resistance being zero. We have modeled the system as a circuit with an ideal selfinductance $L$. We have included the radiation emitted to infinity modeling it as magnetic dipole radiation. After imposing energy balance, we have obtained the differential equation that models the system after the phase transition. Having analyzed it qualitatively we have found that, among the convex electrical current states available, the current can live in one of three classes of well differentiated states, depending upon its initial conditions when it enters the superconducting state after the phase transition. The relevant parameter is the value of its logarithmic derivative. If the electrical current enters the superconducting state decaying with a negative logarithmic derivative equal to $-\left(\frac{K}{L}\right)^{-1/3}$, then it will keep decaying exponentially. If it enters decaying with a logarithmic derivative more negative than $-\left(\frac{K}{L}\right)^{-1/3}$ it will keep decaying faster than the exponential, and will have a short life and a sudden death at a finite time. If the electrical current enter the superconducting state decaying with a logarithmic derivative less negative than $-\left(\frac{K}{L}\right)^{-1/3}$ it will keep decaying slower than the exponential, but after a finite time it will reach smoothly a longed-lived state of constant current different from zero. Actually it would live in that state forever, in agreement with the experimental results shown by File and Mills [2].

References
[1] Landau L D and Lifshitz E M 1987 Classical Theory of Fields (Butterworth-Heinemann, Oxford)
[2] File J and Mills R G 1963 Phys. Rev. Lett. 10(93)
[3] Boykin T B, Hite D and Singh N 2002 Am. J. Phys. 70(4) 415–420
[4] Urzúa G A, Jiménez O, Maass F and Restuccia A Journal of Phys. Conference Series 720