Dynamical Decoupling Using Slow Pulses: Efficient Suppression of 1/f Noise

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The application of dynamical decoupling pulses to a single qubit interacting with a linear harmonic oscillator bath with 1/f spectral density is studied, and compared to the Ohmic case. Decoupling pulses that are slower than the fastest bath time-scale are shown to drastically reduce the decoherence rate in the 1/f case. Contrary to conclusions drawn from previous studies, this shows that dynamical decoupling pulses do not always have to be ultra-fast. Our results explain a recent experiment in which dephasing due to 1/f charge noise affecting a charge qubit in a small superconducting electrode was successfully suppressed using spin-echo-type gate-voltage pulses.

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The most serious problem in the physical implementation of quantum information processing is that of maintaining quantum coherence. Decoherence due to interaction with the environment can spoil the advantage of quantum algorithms. One of the proposed remedies is the method of “dynamical decoupling”, or “bang-bang” (BB) pulses, in which strong and sufficiently fast pulses are applied to the system. In this manner one can either eliminate or symmetrize the system-bath Hamiltonian so that system and bath are effectively decoupled. The latter assumption is usually stated as:

$$\Delta t \ll 1/\Lambda_{UV},$$

where $\Delta t$ is the pulse interval length and $\Lambda_{UV}$ is the high-frequency cutoff of the bath spectral density $I(\omega)$ [2] [see Eq. (2) below]. It can be shown that the overall action with the environment can spoil the advantage of quantum algorithms [1]. One of the proposed remedies is the method of “dynamical decoupling”, or “bang-bang” (BB) pulses, in which strong and sufficiently fast pulses are applied to the system. In this manner one can either eliminate or symmetrize the system-bath Hamiltonian so that system and bath are effectively decoupled. The latter assumption is usually stated as:

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1/\nu$ and Ohmic baths.

**Decoupling for spin-boson model.**—We consider the linear spin-boson model including periodic decoupling pulses. We first briefly review and somewhat simplify the results derived in \cite{2}. We use $k_B = h = 1$ units. The Hamiltonian is

$$
H = H_S + H_B + H_{SB} + H_P
$$

$$
= \frac{\epsilon}{2} \sigma_z + \sum_k \omega_k b_k^{\dagger} b_k + \sum_k \sigma_z (g_k^{\dagger} b_k + g_k b_k^{\dagger}) + H_P,
$$

where the first (second) term governs the free system (bath) evolution; the third term is the (linear) system-bath interaction in which $b_k$ is the $k$th-mode boson annihilation operator and $g_k$ is a coupling constant; and the last term is the fully controllable Hamiltonian generating the decoupling pulses:

$$
H_P(t) = \sum_{n=1}^{N} V_n(t) e^{i t \sigma_z / 2} \sigma_z e^{- i t \sigma_z / 2},
$$

where the pulse amplitude $V_n(t) = V$ for $t_n \leq t \leq t_n + \tau$ and 0 otherwise, lasting for a duration $\tau \ll \Delta t$, with $t_n = n \Delta t$ being the time at which the $n$th pulse is applied. The properties of the bath are captured by its spectral density

$$
I(\omega) = \sum_k \delta(\omega - \omega_k) |g_k|^2. \tag{2}
$$

The reduced system density matrix is obtained from the total density matrix by tracing over the bath degrees of freedom

$$
\rho_S(t) = \text{Tr}_B [\rho(t)] = \text{Tr}_B [U(t) \rho_S(0) \otimes \rho_B(0) U^{\dagger}(t)],
$$

where we have assumed a factorized initial condition between the system and thermal bath, and $U(t)$ is the time evolution generated by $H$: $U(t) = T \exp \left[ -i \int_0^t ds H(s) \right]$ ($T$ denotes time ordering). We are interested in how decoupling improves the system coherence, defined as $\rho_{01}(t) = \langle 0 | \rho_S(t) | 1 \rangle$. In the interaction picture with respect to $H_S$ and $H_B$ the result in the absence of decoupling pulses (free evolution) is: $\rho_{01}^{\text{f}}(t) = e^{-\Gamma_0(t) \rho_{01}^{\text{f}}(0)}$, where

$$
\Gamma_0(t) = \int_{\Delta IR} d\omega \coth \left( \frac{\beta \omega}{2} \right) \frac{1 - \cos \omega t_{2N}}{\omega^2} I(\omega) \tag{3}
$$

$$\beta = 1/(k_B T).$$

In the Schrödinger picture there are oscillations at the natural frequency $\epsilon$, i.e., $\rho_{01}(t) = e^{-\epsilon t} \rho_{01}^{\text{f}}(t)$.

Similarly in the presence of the decoupling pulses, at $t_{2N} = 2N \Delta t$, $\rho_{01}(t_{2N}) = e^{-\epsilon t_{2N} \Gamma_{\rho}(N, \Delta t) \rho_{01}^{\text{f}}(0)}$, where we can show from Eqs. (46),(47) of [2] that

$$
\Gamma_{\rho}(N, \Delta t) = \frac{\omega}{2} \coth \left( \frac{\beta \omega}{2} \right) \frac{1 - \cos \omega t_{2N}}{\omega^2} I(\omega) \tan^2 \left( \frac{\omega \Delta t}{2} \right).
$$

The $\tan^2 (\omega \Delta t)$ term (which was not found in [2]) is the suppression factor arising from the decoupling procedure. In effect, the bath spectral density in the presence of decoupling pulses has been transformed from $I(\omega)$ to the singular spectral density $I'(\omega) = I(\omega) \tan^2 (\omega \Delta t / 2)$. Note, however, that the singularity of $\tan^2 \omega / \nu$ at $\omega \Delta t = (2n + 1)\pi$ for an integer $n$ is canceled by the vanishing of $1 - \cos \omega t_{2N}$ at the same points, so $\Gamma_{\rho}$ remains finite. Nevertheless, and as already pointed out in [2], the value $\omega \Delta t = \pi$ is special: In the limit $N \gg 1$ the integrand of Eq. (4) is highly oscillatory for $\omega \Delta t > \pi$, and grows to $16N^2$ at $\omega \Delta t = \pi$. Thus, decoherence suppression is effective when

$$\Lambda_{UV} \Delta t < \pi. \tag{5}
$$

This is an upper bound on $\Delta t$ that is independent of the specific form of $I(\omega)$. Note further that decoupling enhances decoherence from all modes with $(4n + 1)\pi / 2 < \omega \Delta t < (4n + 3)\pi / 2$, since for these values $\tan^2 (\omega \Delta t / 2) > 1$. However, this effect may be quenched if the weight of these modes is sufficiently low; this is indeed what happens in the $1/\nu$ case.

**Results for $1/\nu$ and Ohmic spectral densities.**—Let us now assume that the spectral density has the following form:

$$
I(\omega) = \gamma \omega^\nu, \quad \nu = \pm 1, \tag{6}
$$

with UV cutoff $\Lambda_{UV}$ and IR cutoff $\Lambda_{IR}$. Thus we are comparing $1/\nu$ noise (the case $\nu = -1$) to an Ohmic bath (the case $\nu = 1$, considered in [2]).

To explain the effect of pulses qualitatively, we approximate $\tan^2 x$ by $x^2 (1 - 2x / \pi)^{-1}$, which allows us to obtain an explicit form for $\Gamma_{\rho}$ for $0 < \Lambda_{UV} \Delta t < \pi / 2$. We further expand $\coth x \approx 1 + 2 \exp (-2x) (x > 1)$. Then, the contribution to $\Gamma_{\rho}$ for $1/\nu$ noise at low temperature is the sum of the zero temperature part
\[ \Gamma_p^{(T=0)}(N, \Delta t) = \gamma (\Delta t)^2 \left[ \log \left( \frac{\Lambda_{UV}}{\Lambda_0} \right) - \log \left( \frac{\pi - \Lambda_{UV} \Delta t}{\pi - \Lambda_{IR} \Delta t} \right) - \text{Ci} \left( \Lambda_{UV} t_{2N} \right) + \text{Ci} \left( \Lambda_0 t_{2N} \right) + O(\Delta t) \right], \]  

and the low temperature correction

\[ \Gamma_p^{(T>0)}(N, \Delta t) = \frac{\gamma (\Delta t)^2}{2} \left[ \log \left( 1 + T^2 t_{2N}^2 \right) + \frac{2 \Delta t T}{\pi} \left( 1 - \frac{1}{1 + T^2 t_{2N}^2} \right) + O(T^2) \right], \]

FIG. 1: Temporal behavior of the logarithm of the decoherence factors at \( T = 0 \). The initial coherence \( \rho_0^{(1)}(0) = 1 \). Parameters are: \( \gamma = 0.05, \Lambda_{UV} = 10 \) for Ohmic and \( \gamma = 0.25, \Lambda_{UV} = 80 \) for \( 1/f \), \( \Lambda_{IR} = 1, \Delta t = 0.025 \) for both. Thick solid (dashed) line: \( 1/f \) case with (without) decoupling pulses. Thin solid (dashed) line: Ohmic case with (without) decoupling pulses. Eq. (4) was used for the case without decoupling pulses, while Eq. (5) was used for the case with decoupling pulses at each \( t = t_{2N} \). The dotted line is our analytical result in Eq. (4).

where \( \text{Ci} \) (Si) is the cosine (sine) integral. In Eq. (5), the limits \( \Lambda_{IR} \rightarrow 0 \) and \( \Lambda_{UV} \rightarrow \infty \) are taken. All terms are finite in these limits. The first and second terms in \( \Gamma_p^{(T=0)}(N, \Delta t) \) (independent of \( t_{2N} \)) determine the asymptotic value \( \Gamma_p^{(T=0)}(\infty, \Delta t) \); the remainder is a damped oscillatory part, given by the difference of two cosine integrals, that vanishes at long times. The second logarithmic term diverges as the pulse interval approaches the inverse UV cutoff frequency time scale of the bath leading to decoherence enhancement from the tan\(^2\) term in Eq. (4). These behaviors are reflected in the exact solutions displayed in Fig. 1. The leading order finite temperature correction \( \Gamma_p^{(T>0)}(N, \Delta t) \) can be separated into two terms. The first term characterizes the asymptotic power law decay and the second term gives the initial damping and the asymptotic relaxation to the \( t_{2N} \)-independent constant.

In Fig. 1 the logarithm of the decoherence factors \( \Gamma_0(t) \) (free evolution) and \( \Gamma_p(t) \) (pulsed evolution) for the \( 1/f \) and Ohmic cases are shown. The smaller is \( \Gamma \), the more coherent is the evolution. The apparent oscillations with a frequency given by \( \Lambda_{UV} \) are caused by the use of a sudden cutoff. Given the parameters used in Fig. 1 the standard timescale condition \( \Delta t \ll 1/\Lambda_{UV} \) is not satisfied in the \( 1/f \) case, while it is \( (\Delta t \Lambda_{UV} = 0.25) \) in the Ohmic case. The most striking feature apparent in Fig. 1 is the highly efficient suppression of decoherence in the case of \( 1/f \) noise, in spite of the seemingly unfavorable pulse interval length. In addition, it can be shown that decoherence due to the \( 1/f \) bath is accelerated when the IR cutoff is decreased, and is more sensitive to the IR cutoff than the Ohmic case. This is a direct consequence of the fact that most of the modes in \( 1/f \) spectrum are concentrated around \( \Lambda_{IR} \). For \( 1/f \) baths we therefore expect slow and strong decoherence on a long time scale, that may be efficiently suppressed by relatively slow and strong pulses. A similar conclusion should be applicable to the more general class of baths with \( 1/f^n \) spectral density, since there too most of the bath spectral density is concentrated in the low frequency range.

For our pure dephasing case at finite temperature, there is the thermal time scale \( t_\beta \equiv T^{-1} \) at which thermal fluctuations start affecting the system’s coherence. In particular, for \( T \gg \Lambda_{UV} \), decoherence is governed by the thermal fluctuations. In Fig. 1 a finite temperature result is shown. The decoupling pulses enhance the decoherence for the Ohmic bath even at low temperatures, since for the parameters chosen the condition (1) is not satisfied. On the other hand, decoherence suppression in the \( 1/f \) case is highly effective. At high temperature, it has been argued on the basis of the Ohmic case, that decoupling pulses faster than the thermal frequency \( T \) are required to suppress decoherence. Once again, the nature of the bath can qualitatively modify this conclusion. Thus decoupling by relatively slow pulses that obey the condition \( \Lambda_{UV} \Delta t \sim 1 \), can still be effective for decoherence suppression at elevated temperatures. However, as the temperature increases, the effective spectrum shifts toward low frequencies, and at the same time, the influence of the environment increases. Overall, BB becomes ineffective irrespective of the type of bath. This explains the breakdown of decoherence suppression at \( T = 1000 \) in Fig. 1. Note from the figure that the suppression of decoherence for the \( 1/f \) bath is more effective than for the Ohmic bath throughout the whole temperature regime.
For the Ohmic bath, as the interval approaches the threshold value from below, there is a crossover from decoherence suppression to decoherence enhancement, as shown in Fig. 3. For the $1/f$ bath, suppression is still effective for longer pulse intervals as long as $\Delta t\Lambda_{UV} < \pi$ is satisfied.

It is of interest to compare our results with the gate voltage pulse experiment performed in [16] in a Cooper-pair box. The corresponding parameter values in Eq. (4) are: $\gamma = 2E_J^2\alpha^2/\hbar^2 c^2$, with the Josephson charging energy $E_C = 122 \mu$eV and the constant $\alpha = (1.3 \times 10^{-9})^2$ determined by the noise measurement. To achieve 90% coherence with $\Lambda_{UV}$ determined by the noise measurement. To achieve 90% coherence with $\Lambda_{IR} = 100$ [Hz] and $\Lambda_{UV} = 10$ [GHz] at $k_B T = 5 \mu$eV, the pulse interval $\Delta t \sim 0.25$ [ns] is required from our analysis based on Eq. (4) with $N = 1$. Although the pulse sequence of [16] differs from ours (theirs is the $\pi/2 - \pi - \pi/2$ spin-echo sequence), they play essentially the same role. Our $\Delta t$ value roughly agrees with their value, $\Delta t \sim 0.5$ [ns], deduced from Fig. 2 in [16]. This agreement nicely illustrates the experimental feasibility of BB in the case of $1/f$ noise. The effectiveness of spin-echo type pulses in relation to superconducting qubits was also recently discussed in [17]. The spin-boson model is appropriate for the study of $1/f$ noise due to a large number of weakly coupled background charges [18].

Conclusions.— We have shown that the speed requirement of the decoupling method should be stated relative to the type of bath spectral density, and not just in terms of its upper cutoff (baths with bimodal distributions provide another example of this [3]). Most significantly, our exact results have demonstrated that BB can be expected to be highly effective in suppressing decoherence due to the ubiquitous $1/f$ noise, without having to satisfy the stringent time constraints that may render the method overly difficult to implement in other instances. We expect this to have significant implications, e.g., for suppression of noise due to charge fluctuations in electrodes providing control voltages in quantum computation. Such a result has already been obtained experimentally in a Cooper-pair box experiment [16], and is predicted to apply to trapped-ion quantum computation as well [11].

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