Spin structures and the divisibility of Euler classes

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Abstract

In this short article we give a geometric meaning of the divisibility of \( KO \)-theoretical Euler classes for given two spin modules. We are motivated by Furuta’s 10/8-inequality for a closed spin 4-manifold. The role of the reducibles is clarified in the monopole equations of Seiberg-Witten theory, as done by Donaldson and Taubes in Yang-Mills theory.

1 Introduction

Since Donaldson’s celebrated work [7], the intersection form of spin 4-manifolds has been one of interests in gauge theory. In the renewal of 4-dimensional topology by Seiberg-Witten theory, P. B. Kronheimer [15] gave a lecture on this problem following the method of Yang-Mills theory. Soon after, M. Furuta [9] extracted an equivariant map from the monopole equation to get the 10/8-inequality through the divisibility of Euler classes in \( K \)-theory [3]. Although Furuta’s approach is more sophisticated than Kronheimer’s one, its geometric meaning seems to have become vague.

In this paper we will build a bridge between the geometry of the moduli spaces and the divisibility of the Euler classes. Our argument seems to be a straightforward extension of Kronheimer’s method. We bypass Furuta’s finite dimensional approximation of the equation, but use the local Kuranishi model of the reducibles. The divisibility of the Euler classes is a consequence of the compactness of the moduli space, together with an equivariant spin structure on it.

We also investigate the divisibility when the first Betti number is positive [10]. In this case we also need to take account of a natural map from the moduli space to the Jacobian torus. In [12] we have already used it to get an invariant from the moduli space. Related topics have been also discussed by H. Sasahira [20].
The main result of this paper is a by-product of the author’s joint work with M. Furuta on improvements of the 10/8-inequality [10]. This type of observation originated from the three proofs of a special case of the inequality in [11].

This paper consists of two parts. In Section 2 we give the main result, which interprets the divisibility in a general setting of cobordism theory. To clarify the argument we divide it into three cases. In Section 3 we set up gauge theory respecting Pin−(2)-symmetry [15], [22] to get the divisibility [9], [10] from our point of view.

The main theorem was announced at University of Minnesota in the summer of 2006. The author thanks to T. J. Li for organizing the seminar, and M. Furuta for his valuable suggestions and announcement of this result. This paper was yielded from a collaborated work with him [10]. Some ideas have been already introduced in it.

2 Main results

Our main result is expressed by a relation between equivariant cobordism theory and $K$-theory. In many variants of both theory, a specific case is applied to Seiberg-Witten theory. However to clarify our argument, we first explain a simpler version.

2.1 Spin$^c$ case

We first recall the definition of equivariant $K$-theory. Let $G$ be a compact Lie group. For a compact Hausdorff $G$-space $B$, we write the equivariant $K$-group as $K_G(B)$. It is the Grothendieck group of isomorphism classes of complex $G$-vector bundles over $B$. For a (possibly) locally compact Hausdorff $G$-space $B$, let $B^+ = B \cup \{+\}$ be the one point compactification of $B$. Then $K_G(B)$ is the kernel of the map $i^* : K_G(B^+) \to K_G(+) = R(G)$ induced from the inclusion $i : \{+\} \to B$. More generally, for a closed $G$-subspace $C$ of $B$, we define $K_G(B, C) = K_G(B \setminus C)$. If we put $K^{-n}_G(B) = K_G(B \times \mathbb{R}^n)$, where $\mathbb{R}^n$ is the $n$-dimensional trivial $G$-module, we have equivariant cohomology theory for locally compact Hausdorff $G$-spaces (c.f. [14]). In the following argument we do not use cohomology theory explicitly, but it helps us to achieve our results.

Let $V$ be a real spin$^c$ $G$-module of dimension $n$. It means that the action on $V$ factors through a given homomorphism $G \to \text{Spin}^c(n) = (\text{Spin}(n) \times U(1))/\{\pm 1\}$.

When $\dim V$ is even, Bott periodicity theorem [1] tells us that the Bott class $\beta(V) \in K_G(V)$ is formed from the irreducible complex Clifford module
for $V$ and $K_G(V)$ is freely generated $\beta(V)$ as an $R(G)$-module. Then the Euler class $e(V) \in R(G)$ is defined to be the restriction of $\beta(V)$ to the zero set.

We next consider cobordism classes obtained from complex $G$-modules. We first prepare some notations. We take an auxiliary $G$-invariant norm on $V$. Put $D(V) = \{v \in V \mid ||v|| \leq 1\}$ and write its boundary as $S(V)$. Write $E(V) = V \setminus ($interior of $D(V))$.

Let $V_0, V_1$ be two real spin$^c$ $G$-modules. We suppose that

(i) $\text{dim } V_0 \geq \text{dim } V_1 > 0$.

(ii) $\text{dim } V_0 \equiv \text{dim } V_1 \equiv 0 \pmod{2}$.

(iii) the $G$-action on $V_0$ is free except the zero.

Then we can take a $G$-map $\phi : S(V_0) \to V_1$ which is transversal to the zero, so that $\phi^{-1}(0)$ is a closed free $G$-submanifold with a $G$-isomorphism

$$T \phi^{-1}(0) \oplus R \oplus V_1 \cong V_0.$$ 

Thus $\phi^{-1}(0)$ has a unique spin$^c$ $G$-structure such that the above isomorphism is a spin $G$-isomorphism, where $R$ is considered to be the trivial $G$-module $R$.

We let $\Omega_{G, \text{free}}^{\text{spin}^c}$ be the cobordism group of closed spin$^c$ free $G$-manifolds. The cobordism class $[\phi^{-1}(0)] \in \Omega_{G, \text{free}}^{\text{spin}^c}$ is independent of the choice of $\phi$, since $V_1$ is $G$-contractible. So we may write it as $\omega(V_0, V_1)$.

**Theorem 1.** Let $V_0, V_1$ be real spin$^c$ $G$-modules satisfying the condition (i), (ii), (iii) in the above. If the cobordism class $\omega(V_0, V_1) \in \Omega_{G, \text{free}}^{\text{spin}^c}$ is zero, there exists an element $\alpha$ in $R(G)$ such that

$$e(V_1) = \alpha e(V_0).$$

**Lemma 2.** Let $G$ be a compact Lie group, and $V_0$ a real $G$-module. Let $M$ be a compact $G$-manifold with boundary $\partial M$ and $i : \partial M \to S(V_0)$ a $G$-embedding. Then there exist a real $G$-module $V$ and an $G$-embedding $i' : M \to V_0 \oplus V$ such that $i'|\partial M = i$. Moreover if $G$ acts on $M$ freely and $\text{dim } M < \text{dim } V_0$, we can take $i'$ to be $i'(M) \subset E(V_0 \oplus V) \setminus E(0 \oplus V)$.

**Proof.** Using a $G$-color $\partial M \times [0, 1]$ of $\partial M$, we obtain a $G$-embedding $i' : \partial M \times [0, 1] \to V_0 \times R$ whose restriction to $\partial M \times \{0\}$ is $i$. We next use [5, Chapter VI] to get a real $G$-module $V'$ and a $G$-embedding $j : M \to V'$. (One can use the double of $M$ to reduce the case of closed $G$-manifolds.) Let
\( \rho_k : M \to \mathbb{R} \ (k = 1, 2) \) be a smooth \( G \)-invariant cut-off function such that

\[
\begin{cases}
\rho_1(x) = 1 & x \in \partial M \times \left[0, \frac{2}{3}\right], \\
0 < \rho_1(x) < 1 & x \in \partial M \times \left(\frac{2}{3}, \frac{5}{6}\right), \\
\rho_1(x) = 0 & x \notin \partial M \times \left[0, \frac{2}{3}\right],
\end{cases}
\]

\[
\begin{cases}
\rho_2(x) = 0 & x \in \partial M \times \left[0, \frac{1}{6}\right], \\
0 < \rho_2(x) < 1 & x \in \partial M \times \left(\frac{1}{6}, \frac{1}{3}\right), \\
\rho_2(x) = 1 & x \notin \partial M \times \left[0, \frac{1}{6}\right].
\end{cases}
\]

Then \( \iota' = (\rho_1 \iota, \rho_2 j, \rho_1 - 1, \rho_2) : M \to V_0 \oplus V' \oplus \mathbb{R}^2 \) is a desired \( G \)-embedding.

We also assume the additional condition. Then \( j(M) \cap \{0\} = \emptyset \), and we may assume \( j(M) \subset E(V') \) and \( i'(M) \subset E(V_0 \oplus V' \oplus \mathbb{R}^2) \). Moreover we can take \( i'(M) \) to be transversal to \( V' \oplus \mathbb{R}^2 \) in \( V_0 \oplus V' \oplus \mathbb{R}^2 \). The dimension condition implies that \( i'(M) \cap (0 \oplus V' \oplus \mathbb{R}^2) = \emptyset \).

(Proof of Theorem \( \mathbf{1} \)) Pick a \( G \)-map \( \varphi : S(V_0) \to V_1 \) as above, so that \( [\varphi^{-1}(0)] = \omega(V_0, V_1) \). By assumption we have a compact spin\(^c\) free \( G \)-manifold \( M \) with \( \partial M = \varphi^{-1}(0) \) as spin\(^c\) \( G \)-manifolds. Let \( i : \partial M \to V_0 \) be the inclusion map. We use Lemma \( \mathbf{2} \) to find a \( G \)-module \( V \) and a \( G \)-embedding \( i' : M \to E(V_0 \oplus V) \setminus E(0 \oplus V) \) such that \( i' \partial M = i \). By adding some \( G \)-module, we may suppose that \( V \) is spin\(^c\) \( G \)-module and \( \dim V \equiv 0 \) (mod 2). For instance, \( V \oplus V \cong V \otimes \mathbb{C} \) will do, if we use the canonical Spin\(^c\) structure on \( V \otimes \mathbb{C} \) (c.f. \([2]\)).

Then the \( G \)-normal bundle \( N \) to \( M \) naturally inherits a spin\(^c\) \( G \)-structure, whose restriction to \( \partial M \) is the one induced by \( \varphi \). It implies that \( \varphi^*_V \beta(V_1 \oplus V) = j^* \beta(N) \), where \( j : S(V_0 \oplus V) \to V_0 \oplus V \) is the inclusion and \( \varphi_V : S(V_0 \oplus V) \to V_1 \oplus V \) is the join of \( \varphi \) and \( 1_V \). We thus get an element

\[
\gamma \in K_G(E(V_0 \oplus V) \cup_{\varphi_V} (V_1 \oplus V), E(0 \oplus V) \cup_{1_{S(0\oplus V)}} E(0 \oplus V)). \quad (2.1)
\]
However, \( \varphi_V \) is obviously \( G \)-homotopic to the map
\[
0_V : S(V_0 \oplus V) \to D(V_1 \oplus V),
\]
\[(u, v) \mapsto (0, v).
\]
So we may replace \( \varphi_V \) in (2.1) by \( 0_V \). The resulting space \( E(V_0 \oplus V) \cup_{0_V} (V_1 \oplus V) \) is \( G \)-homeomorphic to \( (V_0 \oplus V) \cup_{1_{D(0\oplus V)}} (V_1 \oplus V) \), since one can shrink the sphere \( S(V_0) \times v \) into the point \( (0, v) \) for each \( v \in V \). More explicitly it is a \( G \)-map from \( E(V_0 \oplus V) \) to \( V_0 \oplus V \) which takes the form of \((u, v) \mapsto (r(u, v)u, v)\), where
\[
r(u, v) = \begin{cases} 
\frac{||u|| - \sqrt{1 - ||v||^2}}{||u||}, & ||v|| \leq 1, \\
1, & ||v|| \geq 1.
\end{cases}
\]

So we may also suppose the class \( \gamma \) is in
\[
K_G((V_0 \oplus V) \cup_{1_{D(0\oplus V)}} (V_1 \oplus V), E(0 \oplus V) \cup_{1_{S(0\oplus V)}} E(0 \oplus V)).
\]

Under the natural identification \( K_G(V_1 \oplus V, E(0 \oplus V)) \cong K_G(V_1 \oplus V) \), the restriction of \( \gamma \) to \( K_G(V_1 \oplus V) \) is \( \beta(V_1 \oplus V) \), while its restriction to \( K_G(V_0 \oplus V, E(0 \oplus V)) \cong K_G(V_0 \oplus V) \) is \( \alpha \beta(V_0 \oplus V) \) for some \( \alpha \in K(V) \). In \( K_G(D(0 \oplus V), S(0 \oplus V)) \cong K_G(V) \) we have \( \alpha e(V_0)\beta(V) = e(V_1)\beta(V) \), since the both sides are the same restriction of \( \gamma \).

### 2.2 Spin case

The setting is similar to the previous one. We denote by \( KO_G(B) \) the equivariant \( KO \)-group, that is, the Grothendieck group of isomorphism classes of real \( G \)-vector bundles over \( B \). By putting \( KO_G^m(B) = KO_G(B \times \mathbb{R}^m) \) for the trivial \( G \)-module \( \mathbb{R}^m \), we have cohomology theory \( KO_G^*(B) \).

Let \( V \) be a real spin \( G \)-module of dimension \( n \). It means that the action on \( V \) factors through a given homomorphism \( G \to \text{Spin}(n) \). When \( \dim V \equiv 0 \pmod{8} \), \( KO_G(V) \) is freely generated by the Bott class \( \beta(V) \). So we may write \( KO_G^m(B) = KO_G(B \times \mathbb{R}^m) \), if \( n + m \equiv 0 \pmod{8} \).

In [10] we extend Bott periodicity as follows: Put \( \beta(V) = \beta(V \oplus \mathbb{R}^m) \in KO_G^m(V) \) for \( n + m \equiv 0 \pmod{8} \). Bott periodicity theorem indicates that the total cohomology ring \( KO^*(V) \) is freely generated by \( \beta(V) \) as a \( KO_G^*(pt) \)-module. The Euler class \( e(V) \in KO_G^0(pt) \) is defined to be its restriction to \( \{0\} \oplus \mathbb{R}^m \).

For two real spin \( G \)-modules \( V_0, V_1 \) satisfying the condition (i), (iii) in the above, we have a cobordism class \( \omega(V_0, V_1) \) in the cobordism group \( \Omega^\text{spin}_{G, \text{free}} \) of closed spin free \( G \)-manifolds. Our proof of the following theorem is nothing but reputation.
Theorem 3. Let $V_0, V_1$ be real spin $G$-modules satisfying the condition (i), (iii) in the above. If the cobordism class $\omega(V_0, V_1) \in \Omega_{G, \text{free}}^{\text{spin}}$ is zero, there exists an element $\alpha$ in $KO_G^d(\text{pt})$ ($d = \dim V_1 - \dim V_0$) such that

$$e(V_1) = \alpha e(V_0).$$

Note that the condition (ii) is not necessary in our extension of Euler classes.

2.3 Bundle cases

We extend Theorem 3 to pairs of spin $G$-vector bundles. Let $B$ be a compact Hausdorff $G$-space. By a spin $G$-vector bundle $V$ over $B$, we mean a spin structure on $V$, together with a lift of the $G$-action on $V$ to it. When $\text{rank } V \equiv 0 \pmod{8}$, $KO_G(V)$ is freely generated by the Bott class $\beta(V)$ as a $KO_G(B)$-module. For arbitrary $n = \text{rank } V$, put $\beta(V) = \beta(V \oplus \mathbb{R}^m) \in KO_G(V)$ for $n + m \equiv 0 \pmod{8}$. The Euler class $e(V) \in KO_G^0(B)$ is the restriction of $\beta(V)$ to $B \times \mathbb{R}^m$. Here we identify the zero section with the base space $B$.

From now we assume that $B$ is a spin $G$-manifold. Let $V_0, V_1$ be spin $G$-bundles over $B$. We suppose that the $G$ action is free on $V_0 \setminus B$, and so we can take a fiber-preserving $G$-map $\varphi : S(V_0) \to V_1$ which is transversal to the zero section $B$. Then $\varphi^{-1}(B)$ is a closed free $G$-submanifold in $S(V_0)$.

Let $\pi : V_0 \to B$ be the projection. We put a spin $G$-structure on $TV_0$ from an isomorphism $\pi^*(V_0 \oplus TB) \cong TV_0$ and on $\varphi^{-1}(0)$ from an isomorphism

$$T\varphi^{-1}(0) \oplus \mathbb{R} \oplus \pi^*V_1|_{\varphi^{-1}(0)} \cong TV_0|_{\varphi^{-1}(0)}.$$

Let $\Omega_{G, \text{free}}^{\text{spin}}(B)$ be the cobordism group of $G$-maps from a closed spin free $G$-manifold to $B$. We may write the cobordism class of $\pi|_{\varphi^{-1}(0)}$ as $\omega(V_0, V_1) \in \Omega_{G, \text{free}}^{\text{spin}}(B)$, since it does not depend on the choice of $\varphi$.

Theorem 4. Let $B$ be a closed spin $G$-manifold. Let $V_0, V_1$ be spin $G$-bundles over $B$, satisfying the condition (i), (iii) on each fiber. If $\omega(V_0, V_1)$ is zero in $\Omega_{G, \text{free}}^{\text{spin}}(B)$, there exists an element $\alpha \in KO_G^d(B)$ ($d = \text{rank } V_1 - \text{rank } V_0$) such that

$$e(V_1) = \alpha e(V_0).$$

Proof. We take a fiber-preserving $G$-map $\varphi : S(V_0) \to V_1$ as above, so that $[\varphi^{-1}(0)] = \omega(V_0, V_1)$. We can take a $G$-bundle $V$ such that $V_0 \oplus V \cong V'$ for some real $G$-module $V'$ [11]. Since the $G$-action on $\varphi^{-1}(0)$ is free, we may use $G$-isotopy for the composite $\varphi^{-1}(0) \to S(V_0) \to V_0 \oplus V \to V'$ to be a $G$-embedding by adding some $G$-module to $V$. By assumption we have a compact spin$^c$ free $G$-manifold $M$ and a $G$-map $\pi_M : M \to B$ with $\partial M = \varphi^{-1}(0)$ as
spin\(^c\) \(G\)-manifolds and \(\pi_M|\partial M = \pi|\varphi^{-1}(0)\). It follows from Lemma \(2\) that there exists a \(G\)-module \(V''\) and a \(G\)-embedding \(i: M \to V' \oplus V''\) such that \(i'|\partial M = i\). Then the product map \(\pi_M \times i: M \to B \times (V' \oplus V'')\) is a \(G\)-embedding, which commutes with projection to \(B\). On the other hand, we may assume that \(V \oplus V''\) is a spin \(G\)-bundle. If one continues to take account of projection to \(B\), the rest of the proof is very similar to Theorem \(3\).

3 Applications

Let \(X\) be a connected closed oriented spin 4-manifold. Take a Riemannian metric on \(X\). Then the spinor bundle for \(X\) has a quaternionic structure. We apply our result for the moduli space of monopoles, which is known to be compact. We denote by \(W\) the framed moduli space, on which \(\text{Pin}^{-}(2) = \langle U(1), j \rangle \subset H\) acts as the scalar multiplications on spinors of \(X\), and acts on forms of \(X\) via the projection \(\text{Pin}^{-}(2) \to \{\pm 1\}\). We may suppose that \(W\) is smooth except the reducibles, since the \(\text{Pin}^{-}(2)\)-action is free (c.f. \([16]\)).

If \(b_1(X) = 0\), there is only one gauge equivalent classes of reducibles, which is represented as the pair of the trivial connection and the zero section. The Kuranishi model around it is a \(\text{Pin}^{-}(2)\)-equivariant map

\[
\Phi: H^{k+p} \to H^p \oplus \hat{R}^l \tag{3.1}
\]

where \(k = -\sigma(X)/16\), \(l = b_2^+(X)\) and \(\hat{R}\) is the non-trivial one dimensional real irreducible \(\{\pm 1\}\)-module. We assume \(l > 0\).

When \(l \equiv 0 \pmod{4}\), we can put a spin \(\text{Pin}^{-}(2)\)-structure on \(\hat{R}^l\). Recall that the class \([T(W \setminus \text{reducible})]\) in \(KO_{\text{Pin}^{-}(2)}\)-theory is the index bundle of the linearization of the monopole equation, which extends over any compact set of the ambient space (c.f. \([18]\)). Since the ambient space is \(\text{Pin}^{-}(2)\)-contractible to the reducible, at which the index is \([H^k] - [\hat{R}^l] \in KO_{\text{Pin}^{-}(2)}(pt)\), there is a \(\text{Pin}^{-}(2)\)-isomorphism

\[
T(W \setminus \text{reducible}) \oplus \hat{R}^l \oplus V \cong H^k \oplus V
\]

for some \(\text{Pin}^{-}(2)\)-module \(V\). We may suppose that \(V\) is a spin \(G\)-module. Then it implies that \(W \setminus \text{reducible}\) has a spin \(\text{Pin}^{-}(2)\)-structure whose restriction to a neighborhood of the reducible is that defined by (3.1). From Theorem \(3\) we have the following divisibility of Euler classes

\[
e(H^p \oplus \hat{R}^l) = \alpha e(H^{k+p}) \tag{3.2}
\]

for some \(\alpha \in KO_{\text{Pin}^{-}(2)}^{-4k}(pt)\).
When \( l \equiv 0 \pmod{2} \), we can put a spin \( \Gamma \)-structure on \( \hat{R}^l \), and we have a divisibility \( (3.2) \) in \( KO^l_\Gamma(pt) \), where \( \Gamma \) is the 2-fold covering of \( \text{Pin}^-(2) \) obtained from the pull-back diagram:

\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \text{Pin}^-(1) \\
\downarrow & & \downarrow \\
\text{Pin}^-(2) & \longrightarrow & O(1),
\end{array}
\]

and \( \text{Pin}^-(1) \) is the pinor group \( \text{Pin}(1) \) in [2].

**Remark 5.** What we obtained here is a cobordism class of \( G \)-manifolds with \( G \)-equivariant stable parallelization, which is stronger than spin structure. Any corresponding cohomology theory is available to get divisibility, if one knows Euler class.

In general, the reducibles consist of pairs of flat \( U(1) \)-connection on the trivial bundle and the zero section of the half spinor bundle. We may identify it with the Jacobian torus

\[
J_X = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}).
\]

We can associate each flat \( U(1) \)-connection \( a \) to the Dirac operator \( D_a \) on the spinor bundle. We regard \( J_X \) as a Real space. Let \( Ksp(J_X) \) be the \( Ksp \)-group, that is, the \( K \)-group of symplectic bundles, equivalently, complex vector bundles with anti-complex linear action \( j \) with \( j^2 = -1 \). Then the index bundle \( \text{Ind} \mathbb{D} \) over \( J_X \) constructed from the family \( \mathbb{D} = \{ D_a \} \) is an element in \( Ksp(J_X) \). We may write it as \( \text{Ind} \mathbb{D} = [\text{Ker} \mathbb{D}^+] - [\text{Ker} \mathbb{D}^-] \). Then the Kuranishi model around \( J_X \) is a fiber preserving \( \text{Pin}^-(2) \)-equivariant map

\[
\Phi : \text{Ker} \mathbb{D}^+ \to H^+(X; \mathbb{R}) \oplus \text{Coker} \mathbb{D}^-.
\]

We showed in [10] that one can put a spin \( \text{Pin}^-(2) \)-structure on symplectic vector bundles over \( J_X \). Since the ambient space is \( \text{Pin}^-(2) \)-contractible to \( J_X \), Theorem \( 3 \) tells us the following divisibility of Euler classes

\[
e(\text{Ker} \mathbb{D}^+ \oplus H^+(X; \mathbb{R})) = \alpha e(\text{Ker} \mathbb{D}^+)
\]

for some \( \alpha \) in \( KO_{\text{Pin}^-}^{l-4k}(J_X) \), or \( KO_{\Gamma}^{l-4k}(J_X) \).

**Remark 6.** The above divisibility may depend on the representation of the index bundle as \( \text{Ind} \mathbb{D} = [\text{Ker} \mathbb{D}^+] - [\text{Ker} \mathbb{D}^-] \in Ksp(J_X) \). Thus \( \alpha \) is determined in the localization in \( KO_{\text{Pin}^-}^{l-4k}(J_X) \), or \( KO_{\Gamma}^{l-4k}(J_X) \). Our calculation in [10] shows that the equation \( (3.3) \) determines \( \alpha \) in the above localization.
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