NONLINEAR DIFFUSIVE-DISPERSIVE LIMITS FOR MULTIDIMENSIONAL CONSERVATION LAWS

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Abstract. We consider a class of multidimensional conservation laws with vanishing nonlinear diffusion and dispersion terms. Under a condition on the relative size of the diffusion and dispersion coefficients, we establish that the diffusive-dispersive solutions are uniformly bounded in a space $L^p$ (for arbitrary large, depending on the nonlinearity of the diffusion) and converge to the classical, entropy solution of the corresponding multidimensional, hyperbolic conservation law. Previous results were restricted to one-dimensional equations and specific spaces $L^p$. Our proof is based on DiPerna’s uniqueness theorem in the class of entropy measure-valued solutions.

1. Introduction.

Nonlinear hyperbolic conservation laws arise in the modeling of many problems from continuum mechanics, physics, chemistry, etc. The equations become parabolic when additional dissipation mechanisms are taken into account: diffusion, heat conduction, capillarity in fluids, Hall effect in magnetohydrodynamics, etc. From a mathematical standpoint, hyperbolic equations admit discontinuous solutions while parabolic equations have smooth solutions. Discontinuous solutions, understood in the generalized sense of the distribution theory, are usually non-unique. It is therefore fundamental to understand which solutions are selected by a specific zero diffusion-dispersion limit. In this paper we address this issue for multidimensional, scalar conservation laws, and review previous work on the subject restricted to one-dimensional equations.

Consider the Cauchy problem

$$\partial_t u + \text{div} f(u) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d,$$

where the unknown function $u$ is scalar-valued. Smooth solutions to (1.1) also satisfy an infinite list of additional conservation laws:

$$\partial_t \eta(u) + \text{div} q(u) = 0, \quad \nabla q = \nabla \eta \nabla f,$$

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where $\eta$ is a convex function of $u$. For discontinuous solutions, Kružkov [5] shows that (1.2) should be replaced by the set of inequalities
\[ \partial_t \eta(u) + \text{div} \, q(u) \leq 0, \] which select physically meaningful, discontinuous solutions. The condition (1.3) is called an entropy inequality; it is motivated by the second law of thermodynamics, in the context of gas dynamics. By definition, an entropy solution of problem (1.1) satisfies (1.1) in the sense of distributions, and additionally (1.3) for any entropy pair $(\eta, q)$ with convex $\eta$.

Consider the following approximation of (1.1) obtained by adding a nonlinear diffusion, $b : \mathbb{R}^d \to \mathbb{R}^d$, and a linear dispersion to the right hand side of (1.1a):
\[ \partial_t u + \text{div} \, f(u) = \text{div} \left( \varepsilon b_j(\nabla u) + \delta \partial_{x_j}^2 u \right), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \quad (1.4a) \]
\[ u(x, 0) = u_{0,\varepsilon,\delta}(x), \quad x \in \mathbb{R}^d. \quad (1.4b) \]

Let $u_{\varepsilon,\delta} : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ be smooth solutions defined on an interval $[0, T]$ with a uniform $T$ independent of $\varepsilon, \delta$. In (1.4b), $u_{0,\varepsilon,\delta}$ is an approximation of the initial condition $u_0$ in (1.1b).

Our main purpose is to derive conditions under which, as $\varepsilon > 0$ and $\delta$ tend to zero, the solutions $u_{\varepsilon,\delta}$ converge in a strong topology to the entropy solution of (1.1). When $\varepsilon = 0$, equation (1.4a) is a generalized version of the well-known Korteweg-de Vries (KdV) equation, and the solutions become more and more oscillatory as $\delta \to 0$: the approximate solutions do not converge to zero; see Lax and Levermore [6]. When $\delta = 0$, (1.4a) reduces to a nonlinear parabolic equation, resembling the pseudo-viscosity approximation of von Neumann and Richtmyer [8]; in that regime, the solution converges strongly to the entropy solution. Therefore, to ensure the convergence of the zero diffusion-dispersion approximation (1.4), it is necessary that diffusion dominate dispersion. Indeed the main result of the present paper establishes, under rather broad assumptions (see Section 3, Theorem 3.1–3.3), that the solution of (1.4) tends to the classical entropy solution of (1.1) when $\varepsilon, \delta \to 0$ with $|\delta| < \varepsilon$.

For clarity, the main assumptions made in this paper are collected here. First concerning the flux function we shall assume
\[(H_1) \text{ for some } C_1, C'_1 > 0 \text{ and } m \geq 0, \quad |f'(u)| \leq C_1 + C'_1 |u|^{m-1} \quad \text{for all } u \in \mathbb{R}. \]
For the diffusion term, we fix $r \geq 0$ and assume
\[(H_2) \text{ for some } C_2 > 0, \quad C_2 |\lambda|^{r+1} \leq \lambda \cdot b(\lambda) \leq C_3 |\lambda|^{r+1} \quad \text{for all } \lambda \in \mathbb{R}^d. \]
In the case $0 \leq r < 2$, we will need also
\[(H_3) \quad Db(\lambda) \text{ is a positive definite matrix uniformly in } \lambda \in \mathbb{R}^d. \]
We remark that the diffusion $b_j(\nabla u) = \partial_{x_j} u$ satisfies $(H_3)$.

The case $d = 1$ of one-dimensional equations was treated in the important paper by Schonbek [9], where, in particular, the concept of $L^p$ Young measures is introduced together with an extension of the compensated compactness method for conservation laws. We follow here LeFloch and Natalini [7] who, also for one-dimensional equations, developed another approach based on DiPerna’s uniqueness theorem for entropy measure-valued solutions [2] (see Section 2 for a review). Specifically one uses a generalization of DiPerna’s result to $L^p$ functions, due to Szepessy.
The present paper therefore relies on a method of proof that was successful first in proving convergence of finite difference schemes: Szeppessy ([10] and the references therein by Szeppessy and co-authors) and Coquel and LeFloch [1]. Recent work by Hayes and LeFloch (see [3,4]) treats the transitional case where both terms, the diffusion and the dispersion, are in balance. Convergence results in this regime cannot be obtained by the measure-valued solutions approach.

2. Entropy Measure-Valued Solutions.

We include here, as background for further reference, basic material on Young measures and entropy measure-valued (e.m.-v.) solutions. First of all we will need Schonbek's representation theorem for the Young measures associated with a sequence of uniformly bounded in $L^q$. The corresponding setting in $L^\infty$ was first established by Tartar [11]. In the whole of this subsection, $q \in (1, \infty)$ and $T \leq \infty$ are fixed.

**Lemma 2.1.** (See [9].) Let $\{u_k\}$ be a bounded sequence in $L^\infty((0, T); L^q(\mathbb{R}^d))$. Then there exists a subsequence still denoted by $\{u_k\}$ and a weakly $\star$ measurable mapping $\nu: \mathbb{R}^d \times (0, T) \to \text{Prob}(\mathbb{R})$ taking its values in the space of non-negative measures with unit total mass (probability measures) such that, for all functions $g \in C(\mathbb{R})$ satisfying

$$g(u) = O(|u|^s) \quad \text{as} \quad |u| \to \infty, \quad \text{for some} \quad s \in [0, q), \quad (2.1)$$

the following limit representation holds

$$\lim_{k \to \infty} \int_{\mathbb{R}^d \times (0, T)} g(u_k(x, t)) \phi(x, t) \, dx \, dt = \int_{\mathbb{R}^d \times (0, T)} \langle \nu(x, t), g \rangle \phi(x, t) \, dx \, dt \quad (2.2)$$

for all $\phi \in L^1(\mathbb{R}^d \times (0, T)) \cap L^\infty(\mathbb{R}^d \times (0, T))$.

Conversely, given $\nu$, there exists a sequence $\{u_k\}$ satisfying the same conditions as above such that (2.2) holds for any $g$ satisfying (2.1).

We use the notation $\langle \nu(x, t), g \rangle := \int_{\mathbb{R}} g(u) \, d\nu(x, t)$, which therefore describes weak $\star$ measurable mapping $\nu: \mathbb{R}^d \times (0, T) \to \text{Prob}(\mathbb{R})$ taking its values in the space of non-negative measures with unit total mass (probability measures) such that, for all functions $g \in C(\mathbb{R})$ satisfying

$$g(u) = O(|u|^s) \quad \text{as} \quad |u| \to \infty, \quad \text{for some} \quad s \in [0, q), \quad (2.1)$$

We use the notation $\langle \nu(x, t), g \rangle := \int_{\mathbb{R}} g(u) \, d\nu(x, t)$, which therefore describes weak $\star$ measurable mapping $\nu: \mathbb{R}^d \times (0, T) \to \text{Prob}(\mathbb{R})$ taking its values in the space of non-negative measures with unit total mass (probability measures) such that, for all continuous $g$ satisfying (2.1). The measure-valued function $\nu(x, t)$ is called a Young measure associated with the sequence $\{u_k\}$. The following result reveals the connection between the structure of $\nu$ and the strong convergence of the subsequence.

**Lemma 2.2.** Suppose that $\nu$ is a Young measure associated with a sequence $\{u_k\}$, bounded in $L^\infty((0, T); L^q(\mathbb{R}^d))$. For $u \in L^\infty((0, T); L^q(\mathbb{R}^d))$, the following statements are equivalent:

(i) $\lim_{k \to \infty} u_k = u$ in $L^s((0, T); L^p_{\text{loc}}(\mathbb{R}^d))$, for all $s < \infty$ and $p \in [1, q]$;

(ii) $\nu(x, t) = \delta_{u(x, t)}$ for a.e. $(x, t) \in \mathbb{R}^d \times (0, T)$.

In (ii) above, the notation $\delta_{u(x, t)}$ is used for the Dirac mass defined by

$$\langle \delta_{u(x, t)}, g \rangle = g(u(x, t)) \quad \text{for all} \quad g \in C(\mathbb{R}) \quad \text{satisfying} \quad (2.1).$$

Following DiPerna [2] and Szeppessy [10], we define the e.m.-v. solutions to the first order Cauchy problem (1.1).
Definition 2.1. Assume that \( f \in C(\mathbb{R})^d \) satisfies the growth condition (2.1) and \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \). A Young measure \( \nu \) associated with a sequence \( \{u_k\} \), which is assumed to be bounded in \( L^\infty((0, T); L^q(\mathbb{R}^d)) \), is called an entropy measure-valued (e.m.-v.) solution to the problem (1.1) if

\[
\partial_t \langle \nu(\cdot), |u - k| \rangle + \text{div} \langle \nu(\cdot), \text{sgn}(u - k)(f(u) - f(k)) \rangle \leq 0 \quad \text{for all } k \in \mathbb{R}, \quad (2.5a)
\]

in the sense of distributions on \( \mathbb{R}^d \times (0, T) \) and

\[
\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu(x,s), |u - u_0(x)| \rangle \, dx \, ds = 0, \quad \text{for all compact set } K \subseteq \mathbb{R}^d.
\]

A function \( u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)) \) is an entropy weak solution to (1.1) in the sense of Kružkov [5] and Volpert [12] if and only if the Dirac measure \( \delta_{u(\cdot)} \) is an e.m.-v. solution. In the case \( q = +\infty \), existence and uniqueness of such solutions was shown in [5]. The following results on e.m.-v. solutions were established in [10]: Proposition 2.3 states that e.m.-v. solutions are actually Kružkov’s solutions. Proposition 2.4 states that the problem has a unique solution in \( L^q \).

Proposition 2.3. Assume that \( f \) satisfies (2.1) and \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \). Suppose that \( \nu \) is an e.m.-v. solution to (1.1). Then there exists a function \( u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)) \) such that

\[
\nu(x,t) = \delta_{u(x,t)} \quad \text{for a.e. } (x,t) \in \mathbb{R}^d \times (0, T).
\]

Proposition 2.4. Assume that \( f \) satisfies (2.1) and \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \). Then there exists a unique entropy solution

\[
u(\cdot) = \delta_{u(\cdot)} \quad \text{for a.e. } \nu(\cdot) \in \mathbb{R}^d \times (0, T).
\]

Proposition 2.4 states that the problem has a unique solution in \( L^q \).

Proposition 2.5. Assume that \( f \) satisfies (2.1) and \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) for \( q > 1 \). Let \( \{u_k\} \) be a sequence bounded in \( L^\infty((0, T); L^q(\mathbb{R}^d)) \) and let \( \nu \) be a Young measure associated with this sequence. If \( \nu \) is an e.m.-v. solution to (1.1), then

\[
\lim_{k \to \infty} u_k = u \quad \text{in } L^s((0, T); L^p_{\text{loc}}(\mathbb{R}^d)) \quad \text{for all } s < \infty \text{ and all } p \in [1, q),
\]

where \( u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)) \) is the unique entropy solution to (1.1).
3. Convergence Results.

Throughout it is assumed \( u_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) and the initial data in (1.4b) are smooth functions with compact support and are uniformly bounded in \( L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) for some \( q > 2 \). While in previous works [9, 7], a single value of \( q \) was treated, we can here handle arbitrary large values of \( q \). For simplicity in the presentation, we will always consider exponents \( q \) of the form

\[
q = 2 + n(r - 1),
\]

where \( n \geq 0 \) is any integer. Therefore, when the diffusion is superlinear, in the sense that \( (H_2) \) holds with \( r > 1 \), then arbitrary large values of \( q \) are obtained. Restricting attention to the diffusion-dominant regime \( \delta = O(\varepsilon) \), we suppose that \( u_0^{\varepsilon,\delta} \) approaches the initial condition \( u_0 \) of (1.1b) in the sense that:

\[
\lim_{\varepsilon \to 0^+} u_0^{\varepsilon,\delta} = u_0 \quad \text{in} \quad L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d),
\]

\[
\|u_0^{\varepsilon,\delta}\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)}.
\]

The following convergence theorems concern a sequence \( u^{\varepsilon,\delta} \) of smooth solutions to problem (1.4), defined on \( \mathbb{R}^d \times [0, T] \) and decaying rapidly at infinity.

First consider the hypothesis \( (H_2) \) with \( r \geq 2 \), that is the case of diffusions with (at least) quadratic growth.

**Theorem 3.1.** Suppose that the flux \( f \) satisfies \( (H_1) \) with \( m < q \) (which is always possible when \( r > 1 \) by taking \( q \) large enough). Suppose that the diffusion \( b \) satisfies \( (H_2) \) with \( r \geq 2 \). If \( \delta = o(\varepsilon^{\frac{p}{r+1}}) \), then the sequence \( u^{\varepsilon,\delta} \) converges in

\[
L^s((0, T); L^p(\mathbb{R}^d)), \quad \text{for all} \quad s < \infty \quad \text{and} \quad p < q,
\]

\[
\text{to a function} \quad u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)),
\]

which is the unique entropy solution to (1.1).

Observe that \( m \) and \( q \) can be arbitrary large in Theorem 3.1. To treat the case \( r < 2 \), we need the additional condition \( (H_3) \) on the diffusion. First for diffusion with linear growth (\( r = 1 \)), we obtain a result in the space \( L^2 \):

**Theorem 3.2.** Suppose that \( f \) satisfies \( (H_1) \) with \( m \leq 1 \), and \( b \) satisfies \( (H_2)-(H_3) \) with \( r = 1 \). If \( \delta = o(\varepsilon^{2}) \), then the sequence \( u^{\varepsilon,\delta} \) converges in

\[
L^s((0, T); L^p(\mathbb{R}^d)), \quad \text{for all} \quad s < \infty \quad \text{and} \quad p < 2,
\]

\[
\text{to a function} \quad u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)),
\]

which is the unique entropy solution to (1.1).

In particular Theorem 3.2 covers the interesting case of a linear diffusion and a linear dispersion with an (at most) linear flux at infinity. More generally, for general \( r \geq 1 \) we establish that:

**Theorem 3.3.** Suppose that \( f \) satisfies \( (H_1) \) with \( m \leq \frac{2r}{r+1} < q \), and \( b \) satisfies \( (H_2)-(H_3) \) for some \( r \geq 1 \). If \( \delta = o(\varepsilon^{\frac{2}{r+1}}) \) then the sequence \( u^{\varepsilon,\delta} \) converges in

\[
L^s((0, T); L^p(\mathbb{R}^d)), \quad \text{for all} \quad s < \infty \quad \text{and} \quad p < q,
\]

\[
\text{to a function} \quad u \in L^\infty((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)),
\]

which is the unique entropy solution to (1.1).

Our method of proof can also be extended to a general diffusion \( b = b(u, \nabla u, D^2 u) \).
4. Convergence Proofs.

The superscripts $\varepsilon$ and $\delta$ are omitted in this section, except when necessary. In the proof, we make frequent use of the following computation. Multiply (1.4a) by $\eta'(u)$ where $\eta : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function and define $q : \mathbb{R} \to \mathbb{R}^d$ by $q'_j = \eta' f'_j$. We have

$$\partial_t \eta(u) = -\eta'(u) \div (u) + \varepsilon \sum_j \partial_{x_j}(\eta'(u) b_j(\nabla u)) - \partial_{x_j} \eta'(u) b_j(\nabla u)$$

$$+ \delta \sum_j \partial_{x_j} \left( \eta'(u) \partial_{x_j}^2 u \right) - \partial_{x_j} \eta'(u) \partial_{x_j}^2 u$$

$$= -\div q(u) + \varepsilon \sum_j \partial_{x_j}(\eta'(u) b_j(\nabla u)) - \varepsilon \eta''(u) \sum_j \partial_{x_j} u b_j(\nabla u)$$

$$+ \frac{\delta}{2} \sum_j 2 \partial_{x_j} \left( \eta'(u) \partial_{x_j}^2 u \right) - \eta''(u) \partial_{x_j}(\partial_{x_j} u)^2,$$

thus

$$\partial_t \eta(u) + \div q(u) = \varepsilon \div(\eta'(u) b(\nabla u)) - \varepsilon \eta''(u) \nabla \cdot b(\nabla u)$$

$$- \delta \sum_j \eta''(u) \partial_{x_j}(\partial_{x_j} u)^2 + \delta \sum_j \partial_{x_j} \left( \eta'(u) \partial_{x_j}^2 u \right), \quad (4.1a)$$

The last two terms in the right hand side of (4.1a) take also the form

$$\frac{\delta}{2} \sum_j \eta''(u) (\partial_{x_j} u)^3 - 3 \partial_{x_j} \left( \eta''(u) (\partial_{x_j} u)^2 \right) + 2 \partial_{x_j}^2 (\eta'(u) \partial_{x_j} u). \quad (4.1b)$$

When $\eta$ is convex, the term containing $\eta''(u)$ has a favorable sign: the diffusion dissipates the entropy $\eta$.

We begin by collecting fundamental energy estimates in several lemma.

**Lemma 4.1.** Let $\alpha \geq 1$ be any real. Any solution of (1.4a) satisfies, for $t \in [0, T]$,

$$\int_{\mathbb{R}^d} \frac{|u(t)|^{\alpha+1}}{\alpha + 1} dx + \alpha \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} \nabla u \cdot b(\nabla u) dx ds$$

$$= \int_{\mathbb{R}^d} \frac{|u_0|^{\alpha+1}}{\alpha + 1} dx - \frac{\alpha}{2} \delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} \sum_j \partial_{x_j}(\partial_{x_j} u)^2 dx ds. \quad (4.2a)$$

For $\alpha \geq 2$, the last term in the above identity can be replaced by

$$\frac{\alpha (\alpha - 1)}{2} \delta \int_0^t \int_{\mathbb{R}^d} \text{sgn}(u) |u|^{\alpha-2} \sum_j (\partial_{x_j} u)^3 dx ds. \quad (4.2b)$$
Proof. Integrate (4.1a) over the whole of $\mathbb{R}^d$ with $\eta(u) = \frac{|u|^{\alpha+1}}{\alpha+1}$:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{|u|^{\alpha+1}}{\alpha+1} \, dx = -\alpha \varepsilon \int_{\mathbb{R}^d} |u|^{\alpha-1} \nabla u \cdot b(\nabla u) \, dx - \frac{\alpha}{2} \int_{\mathbb{R}^d} \sum_j |u|^{\alpha-1} \partial_x j(\partial_x u)^2 \, dx,$$

which yields (4.2a) after integration over $[0,t]$. One may use (4.1b), instead, to obtain (4.2b). □

Choosing $\alpha = 1$ in Lemma 4.1, we deduce immediately a uniform bound for $u$ in $L^\infty((0,T); L^2(\mathbb{R}^d))$ together with a control for both $\nabla u \cdot b(\nabla u)$ in $L^1(\mathbb{R}^d \times (0,T))$ and $\nabla u$ in $L^{r+1}(\mathbb{R}^d \times (0,T))$.

**Proposition 4.2.** For any solution of (1.4a) and $t \in [0,T]$, we have

$$\int_{\mathbb{R}^d} u(t)^2 \, dx + 2 \varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla u \cdot b(\nabla u) \, dx \, ds = \int_{\mathbb{R}^d} u_0^2 \, dx \quad (4.3)$$

and, assuming $(H_2)$,

$$\varepsilon \int_0^t \int_{\mathbb{R}^d} |\nabla u|^{r+1} \, dx \, ds \leq C \int_{\mathbb{R}^d} u_0^2 \, dx \quad (4.4)$$

To derive additional a priori estimates, we use another value of $\alpha$, motivated by controlling the dispersive term in (4.2b) with Hölder inequality, as follows:

$$\left| \int_0^t \int_{\mathbb{R}^d} \text{sgn}(u) |u|^{\alpha-2} \sum_j (\partial_x u)^3 \, dx \, ds \right|$$

$$\leq \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-2} |\nabla u|^3 \, dx \, ds \quad (4.5)$$

$$\leq \left[ \int_0^t \int_{\mathbb{R}^d} |u|^{(\alpha-2)p} \, dx \, ds \right]^\frac{1}{p} \left[ \int_0^t \int_{\mathbb{R}^d} |\nabla u|^{3p'} \, dx \, ds \right]^\frac{1}{p'}.$$

To take advantage of (4.4), we can choose $3p' = r + 1$ provided $r \geq 2$. Then $p = \frac{r+1}{r-1}$, so $(\alpha-2)p = (r+1) \frac{\alpha-2}{\alpha+1}$. Therefore it is rather natural to take the exponent $\alpha = r$ for the entropy, where $r$ is given by the diffusion term. Thus we deduce from Lemma 4.1 a natural estimate for $|u(t)|^{r+1}$, involving the combination $\delta \varepsilon^{-\frac{1}{r+1}}$ of $\delta$ and $\varepsilon$.

**Proposition 4.3.** Assume that $(H_2)$ holds with $r \geq 2$ and $u_0 \in L^{r+1}(\mathbb{R}^d)$. For $t \in [0,T]$, we have

$$\int_{\mathbb{R}^d} |u(t)|^{r+1} \, dx + (r+1) \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{r-1} \nabla u \cdot b(\nabla u) \, dx \, ds \leq C_1(u_0) \left( 1 + \delta \varepsilon^{-\frac{1}{r+1}} \max \left\{ 1, \left[ t C_1(u_0) \left( 1 + \delta \varepsilon^{-\frac{1}{r+1}} \right) \right] \frac{r+2}{r+1} \right\} \right)$$

$$:= H_1 \left( \delta \varepsilon^{-\frac{1}{r+1}} \right) \quad (4.6)$$
and

\[ \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{-1} |\nabla u|^{r+1} \, dx \, ds \leq \frac{C}{(r+1)} H_1 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right), \]  

where \( C > 0 \) is some fixed constant and

\[ C_1(u_0) := \max \left\{ \|u_0\|_{L^{r+1}(\mathbb{R}^d)}^{r+1}, (r+1)\left( \frac{r(r-1)}{2} \right) \left( \frac{n\|u_0\|_{L^2(\mathbb{R}^d)}^2}{(r-1)} \right)^{\frac{r}{r+1}} \right\}. \]

In particular Proposition 4.3 shows that, if \( u_0 \in L^2 \cap L^{r+1} \) and \( \delta = O(\varepsilon^{\frac{1}{r+1}}) \), then \( u(t) \in L^{r+1} \) uniformly for all \( t \geq 0 \).

To motivate the forthcoming derivation, let us consider the special case \( r = 2 \). Then (4.6) gives us an \( L^4 \) estimate. Returning to the original inequality (4.5), but now with the new value \( \alpha = 3 \), we can now estimate the dispersive term in (4.2b) directly in view of the estimate (4.7). In this fashion, we deduce an \( L^4 \) estimate. This argument can be continued inductively to reach any space \( L^q \).

Actually Propositions 4.2 and 4.3 are the first two cases of a general result derived now. We define, for \( n \geq 1 \),

\[ H_0 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right) = C_0(u_0) := \|u_0\|_{L^2(\mathbb{R}^d)}^2, \]

\[ H_n \left( \delta \varepsilon^{-\frac{3}{r+1}} \right) := C_n(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \max \left\{ 1, \left[ t \left[ C_n(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \right) \right]^{(r-1)/r}\right) \right\} , \]

and

\[ C_n(u_0) := \max \left\{ \|u_0\|_{L^{n(r-1)+2}(\mathbb{R}^d)}^{n(r-1)+2}, \frac{n(r-1)+2}{[(n-1)(r-1)+2]^{\frac{r}{r+1}}} \frac{n(r-1)+1}{[(n-1)(r-1)+1]^{1+1}} \right\}. \]

Here \( C > 0 \) is some fixed constant. Note that \( H_n \) and \( C_n \) are uniformly bounded in \( \varepsilon, \delta \) provided \( u_0 \in L^2 \cap L^{n(r-1)+2} \) and \( \delta = O(\varepsilon^{\frac{1}{r+1}}) \).

**Proposition 4.4.** Assume that \((H_2)\) holds with \( r \geq 2 \) and \( u_0 \in L^q(\mathbb{R}^d) \). For \( t \in [0, T] \) and \( n \geq 0 \) such that \( n(r-1)+2 \leq q \), we have

\[ \int_{\mathbb{R}^d} |u(t)|^{n(r-1)+2} \, dx \]

\[ + \varepsilon (n(r-1)+2)(n(r-1)+1) \int_0^t \int_{\mathbb{R}^d} |u|^{n(r-1)} \nabla u \cdot b(\nabla u) \, dx \, ds \]

\[ \leq H_n \left( \delta \varepsilon^{-\frac{3}{r+1}} \right), \]

and

\[ \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{n(r-1)} |\nabla u|^{r+1} \, dx \, ds \]

\[ \leq C (n(r-1)+2)^{-1} (n(r-1)+1)^{-1} H_n \left( \delta \varepsilon^{-\frac{3}{r+1}} \right). \]
Proof of Propositions 4.3 and 4.4. Note first that (4.10) is an immediate consequence of (4.9) and the hypothesis (H_2). If n = 0, (4.9) coincides with (4.3) in Proposition 4.2. For n = 1, the estimate is Proposition 4.3.

To estimate the term in (4.2b), with \( \alpha = r \), we use (4.5):

\[
\int_{\mathbb{R}^d} |u(t)|^{r+1} \, dx + (r + 1) r \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{r-1} \nabla u \cdot b(\nabla u) \, dx \, ds
\]

\[
\leq \int_{\mathbb{R}^d} |u_0|^{r+1} \, dx + \frac{(r + 1)(r - 1)}{2} \delta \left[ \int_0^t \int_{\mathbb{R}^d} |u|^{r+1} \, dx \, ds \right]^{\frac{r-2}{r+1}} \left[ \int_0^t \int_{\mathbb{R}^d} |\nabla u|^{r+1} \, dx \, ds \right]^{\frac{r}{r+1}}. \tag{4.11}
\]

By (H_2) the second term in the left hand side of (4.11) is positive. Integrate (4.11) over [0, t] and use (4.4):

\[
\|u\|_{L^{r+1}(\mathbb{R}^d \times (0, T))}^{r+1} \leq t \|u_0\|_{L^{r+1}(\mathbb{R}^d)}^{r+1} + \frac{(r + 1)(r - 1)}{2} t \left( C \|u_0\|_{L^2(\mathbb{R}^d)} \right)^{\frac{r+1}{r-1}} \delta \varepsilon \|u\|_{L^{r+1}(\mathbb{R}^d \times (0, T))}^{-2} \|u\|_{L^{r+1}(\mathbb{R}^d \times (0, T))}^{\frac{r-2}{r+1}}.
\]

Observe that, for \( X > 0 \), the inequality

\[
X \leq K \left( 1 + \Delta X^{\frac{1}{\theta+r}} \right).
\]

where \( 0 \leq \theta < r + 1 \) and \( K > 0 \), implies

\[
X \leq \max \left\{ 1, \left[ K (1 + \Delta) \right]^{\frac{1}{\theta+1}} \right\}. \tag{4.12}
\]

Thus we deduce

\[
\|u\|_{L^{r+1}(\mathbb{R}^d \times (0, T))}^{r+1} \leq \max \left\{ 1, \left[ t C_1(u_0) \left( 1 + \delta \varepsilon^{-\frac{1}{r+1}} \right) \right]^{\frac{r+1}{r-1}} \right\}
\]

and, returning to (4.11):

\[
\int_{\mathbb{R}^d} |u(t)|^{r+1} \, dx + (r + 1) r \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{r-1} \nabla u \cdot b(\nabla u) \, dx \, ds
\]

\[
\leq C_1(u_0) \left( 1 + \delta \varepsilon^{-\frac{1}{r+1}} \right) \max \left\{ 1, \left[ t C_1(u_0) \left( 1 + \delta \varepsilon^{-\frac{1}{r+1}} \right) \right]^{\frac{r-2}{r+1}} \right\} := H_1(\delta \varepsilon^{-\frac{1}{r+1}}).
\]

This completes the proof of (4.6).
This argument can be iterated. We return to the dispersive term and make an estimate similar to (4.5), but now having in view to apply (4.10), already established for $n = 1$:

\[
\left| \int_0^t \int_{\mathbb{R}^d} \text{sgn}(u)|u|^{\alpha-2} \sum_j (\partial_x u)^3 \, dx \, ds \right| \\
\leq \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-2} |\nabla u|^3 \, dx \, ds \\
\leq \left[ \int_0^t \int_{\mathbb{R}^d} |u|^{(\alpha-2-\gamma)p} \, dx \, ds \right]^\frac{1}{p} \left[ \int_0^t \int_{\mathbb{R}^d} |u|^{\gamma p'} |\nabla u|^{3p'} \, dx \, ds \right]^\frac{1}{p'},
\]

where we choose $3p' = r+1$ and $\gamma p' = r-1$, so $(\alpha-2-\gamma)p = (\alpha - 2 - 3\frac{r-1}{r-2}) \frac{r+1}{r-2}$.

Then (4.12) gives

\[
\int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \, dx + (\alpha + 1) \alpha x \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} \nabla u \cdot (\nabla u) \, dx \, ds \\
\leq \int_{\mathbb{R}^d} |u_0|^{\alpha+1} \, dx \\
+ \frac{(\alpha + 1)\alpha(\alpha - 1)}{2[(r+1)r]^\frac{1}{r+1}} \left( CH_1 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right) \right)^\frac{1}{r+1} \delta \varepsilon^{-\frac{1}{r+1}} \left[ \int_0^t \int_{\mathbb{R}^d} |u|^{(\alpha-2-\gamma)p} \, dx \, ds \right]^\frac{1}{r+2}.
\]

We choose $\alpha$ so that $\alpha + 1 = (\alpha - 2 - \gamma)p$, i.e., $\alpha = 2r - 1$.

Integrating (4.14) over the interval $[0,t]$, we obtain

\[
\|u\|_{L^2_r(\mathbb{R}^d \times (0,T))}^2 \\
\leq t \|u_0\|_{L^2_r(\mathbb{R}^d)}^2 \\
+ \frac{r(2r-1)(2r-2)}{[(r+1)r]^\frac{1}{r+1}} t \left( CH_1 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right) \right)^\frac{1}{r+1} \delta \varepsilon^{-\frac{1}{r+1}} \left( \|u\|_{L^2_r(\mathbb{R}^d \times (0,T))} \right)^\frac{1}{r+2} \\
\leq t C_2(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \left( \|u\|_{L^2_r(\mathbb{R}^d \times (0,T))} \right)^\frac{1}{r+1} \right),
\]

with $C_2(u_0) := \max \{ \|u_0\|_{L^2_r(\mathbb{R}^d)}^2, \frac{r(2r-1)(2r-2)}{[(r+1)r]^\frac{1}{r+1}} \left( CH_1 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right) \right)^\frac{1}{r+1} \}$.

By (4.12), we obtain again

\[
\|u\|_{L^2_r(\mathbb{R}^d \times (0,T))} \leq \max \left\{ 1, \left[ t C_2(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \right) \right]^\frac{r+1}{r+2} \right\}.
\]

Then (4.14) gives

\[
\int_{\mathbb{R}^d} |u(t)|^{2r} \, dx + 2r(2r-1) \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{2(r-1)} \nabla u \cdot (\nabla u) \, dx \, ds \\
\leq C_2(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \max \left\{ 1, \left[ t C_2(u_0) \left( 1 + \delta \varepsilon^{-\frac{3}{r+1}} \right) \right]^\frac{r+1}{r+2} \right\} \right)
\]

\[
:= H_2 \left( \delta \varepsilon^{-\frac{3}{r+1}} \right).
\]
We are now concerned with the case where the diffusion exponent in \((H_2)\) satisfies \(r < 2\). In this situation, we require the assumption \((H_3)\), for which instance is satisfied by \(b_j(\nabla u) = \partial_{x_j} u\).

**Proposition 4.5.** Suppose that \((H_1)\)–\((H_3)\) hold with \(m\) and \(r\) such that \(m \leq \frac{2r}{r+1}\) and \(r \geq 1\). For \(t \in [0, T]\), we have

\[
\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + \varepsilon \int_0^T \int_{\mathbb{R}^d} |D^2 u|^2 \, dxdt \leq C,
\]

\[
\int_{\mathbb{R}^d} |u(t)|^2 + \varepsilon \int_0^T \int_{\mathbb{R}^d} |\nabla u|^{r+1} \, dxdt \leq C \left( 1 + \delta \frac{m^2}{r+1} \right).
\]

**Proof.** We differentiate \((1.4a)\) with respect to the space variable \(x\) :

\[
\partial_t \nabla u + \div (f'(u) \cdot \nabla u) = \varepsilon \nabla \sum_j \partial_{x_j} (b_j(\nabla u)) + \delta \sum_j \partial_{x_j}^2 (\nabla u)
\]

and, then, we multiply it by \(\nabla u\) and integrate over \(\mathbb{R}^d\). After further integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx - \int_{\mathbb{R}^d} \Delta u f'(u) \cdot \nabla u \, dx
\]

\[
= -\varepsilon \int_{\mathbb{R}^d} \sum_k \nabla \partial_{x_k} u \cdot \nabla u \partial_{x_k} u \, dx - \delta \sum_j \int_{\mathbb{R}^d} \partial_{x_j} \left( \sum_k \partial_{x_k}^2 (\nabla u) \right) \, dx.
\]

Thus, integrating on \([0, t]\) using \((H_1)\) yields

\[
\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + 2 \varepsilon \int_{\mathbb{R}^d} \sum_k \nabla \partial_{x_k} u \cdot \nabla u \partial_{x_k} u \, dx
\]

\[
\leq \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + 2C_1 \int_0^t \int_{\mathbb{R}^d} |D^2 u|^2 |u|^{m-1} |\nabla u| \, dxdt
\]

\[
\leq \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + C \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{2m-2} |\nabla u|^2 \, dxdt + C_4 \varepsilon \int_0^t \int_{\mathbb{R}^d} |D^2 u|^2 \, dxdt,
\]

and so, using \((H_3)\),

\[
\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + C_5 \varepsilon \int_0^t \int_{\mathbb{R}^d} |D^2 u|^2 \, dxdt
\]

\[
\leq \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + \frac{C}{\varepsilon} \int_0^t \int_{\mathbb{R}^d} |u|^{2m-2} |\nabla u|^2 \, dxdt.
\]

By Hölder inequality and for \(m \leq \frac{r+1}{r-1}\) :

\[
\int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + C_5 \varepsilon \int_0^t \int_{\mathbb{R}^d} |D^2 u|^2 \, dxdt
\]

\[
\leq \int_{\mathbb{R}^d} |\nabla u_0|^2 \, dx + C \varepsilon^{-1} \left[ \int_0^t \int_{\mathbb{R}^d} |\nabla u|^{r+1} \, dxdt \right]^{\frac{2}{r+1}} \left[ \int_0^t \int_{\mathbb{R}^d} |u|^2 \, dxdt \right]^{\frac{r-1}{r+1}},
\]
and now (4.15) follows from (4.3)-(4.4).

To establish (4.16)-(4.17) we use (4.2a) for \( \alpha \geq 1 \):

\[
\int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \, dx + C \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |\nabla u|^{r+1} \, dx \, dt \\
\leq \int_{\mathbb{R}^d} |u_0|^{\alpha+1} \, dx + C' \delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |\nabla u| |D^2 u| \, dx \, dt.
\]

We evaluate the last term using (H2):

\[
\delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |\nabla u| |D^2 u| \, dx \, dt \\
\leq \delta \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} \left( \frac{C_2 \varepsilon}{(r+1) \delta} |\nabla u|^{r+1} + \frac{r}{r+1} \left( \frac{\delta}{C_2 \varepsilon} \right)^{\frac{1}{r+1}} |D^2 u|^{\frac{r+1}{r}} \right) \, dx \, dt \\
\leq \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |\nabla u|^{r+1} \, dx \, dt + C'' \delta \varepsilon^{\frac{r+1}{r+1}} \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |D^2 u|^{\frac{r+1}{r}} \, dx \, dt.
\]

So, we have

\[
\int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \, dx + C \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |\nabla u|^{r+1} \, dx \, dt \\
\leq \int_{\mathbb{R}^d} |u_0|^{\alpha+1} \, dx + C' \delta \varepsilon^{\frac{r+1}{r+1}} \int_0^t \int_{\mathbb{R}^d} |u|^{\alpha-1} |D^2 u|^{\frac{r+1}{r}} \, dx \, dt.
\]

Taking \( \alpha = 1 + \frac{r+1}{r} \), we deduce

\[
\int_{\mathbb{R}^d} |u(t)|^{2r+1} \, dx + C \varepsilon \int_0^t \int_{\mathbb{R}^d} |u|^{r+1} |\nabla u|^{r+1} \, dx \, dt \\
\leq \int_{\mathbb{R}^d} |u_0|^{2r+1} \, dx + C' \delta \varepsilon^{\frac{r+1}{r+1}} \left[ \int_0^t \int_{\mathbb{R}^d} |u|^2 \, dx \, dt \right]^{\frac{r+1}{2}} \left[ \int_0^t \int_{\mathbb{R}^d} |D^2 u|^2 \, dx \, dt \right]^{\frac{r+1}{2r}}.
\]

The conclusion follows now easily. □

**Proof of Theorems 3.1.** We first derive (2.5a), based on the conservation law (4.1b) with an arbitrary convex function, \( \eta \), where we assume \( \eta', \eta'', \eta''' \) bounded functions on \( \mathbb{R} \). We claim that there exists a bounded measure \( \mu \leq 0 \) such that

\[
\partial_t \eta(u) + \text{div} \, q(u) \longrightarrow \mu \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \times (0, T)).
\]

From (4.1b), we obtain

\[
\partial_t \eta(u) + \text{div} \, q(u) = \varepsilon \text{div}(\eta'(u) \, b(\nabla u)) - \varepsilon \eta''(u) \, \nabla u \cdot b(\nabla u) \\
+ \frac{\delta}{2} \sum_j \eta'''(u) \left( \partial_j u \right)^3 - 3 \partial_j \left( \eta''(u) \left( \partial_j u \right)^2 \right) + 2 \partial_{j}^2 \left( \eta'(u) \, \partial_j u \right)
\]

\[:= \mu_1 + \mu_2 + \mu_3,
\]
with obvious notation. For each positive \( \theta \in C_0^\infty(\mathbb{R}^d \times (0, T)) \) we evaluate \( \langle \mu_i, \theta \rangle \) for \( i = 1, 2, 3 \). To treat \( \mu_1 \), we use Hölder inequality with the exponent \( \frac{r+1}{r} \). In view of \((H_2)\) and \((4.4)\) of Proposition 4.2 and assumption \((3.1)\), we get

\[
|\langle \mu_1, \theta \rangle| \leq \varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_j |\eta'(u) b_j(\nabla u) \partial_x \theta| \, dx dt
\]

so

\[
|\langle \mu_1, \theta \rangle| \leq C \varepsilon \int_0^T \int_{\mathbb{R}^d} |\nabla \theta| |b(\nabla u)| \, dx dt
\]

For \( \mu_2 \), we use \((H_2)\) and the convexity of \( \eta \):

\[
\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathbb{R}^d} \sum_j \theta \eta''(u) \nabla u \cdot b(\nabla u) \, dx dt \leq 0.
\]

For \( \mu_3 \), we use again Hölder inequality, as follows

\[
|\langle \mu_3, \theta \rangle| \leq \frac{\delta}{2} \int_0^T \int_{\mathbb{R}^d} \sum_j \left| \theta \eta'''(u) (\partial_x u)^3 + 3 \eta''(u) (\partial_x u)^2 \partial_x \theta + 2 \eta'(u) \partial_x u \partial_x^2 \theta \right| \, dx dt
\]

so

\[
|\langle \mu_3, \theta \rangle| \leq C \delta \|\theta\|_{L^{r+1}(\mathbb{R}^d \times (0, T))} \left[ \int_{\text{supp } \theta} |\nabla u|^{r+1} \, dx dt \right]^\frac{1}{r+1}
\]

\[
+ C \delta \|\nabla \theta\|_{L^{r+1}(\mathbb{R}^d \times (0, T))} \left[ \int_{\text{supp } \theta} |\nabla u|^{r+1} \, dx dt \right]^\frac{1}{r+1}
\]

\[
+ C \delta \left[ \int_0^T \int_{\mathbb{R}^d} \left( \sum_j |\partial_x^2 \theta| \right)^{\frac{r+1}{r+1}} \, dx dt \right]^\frac{1}{r+1} \left[ \int_{\text{supp } \theta} |\nabla u|^{r+1} \, dx dt \right]^{\frac{1}{r+1}}.
\]

Therefore, we conclude that

\[
|\langle \mu_3, \theta \rangle| \leq C \delta \left( \varepsilon^{-\frac{1}{r+1}} + \varepsilon^{-\frac{2}{r+1}} + \varepsilon^{-\frac{3}{r+1}} \right) \leq C \delta \varepsilon^{-\frac{1}{r+1}}.
\]
Finally the condition $\delta = o(\varepsilon^{\frac{1}{d}})$ is sufficient to imply the desired conclusion.

Using a standard regularization of $\text{sgn}(u)$ and $|u - k|$ (for $k \in \mathbb{R}$), which fulfill the growth condition (2.1), we apply the limit representation (2.2) and conclude that $\nu$ satisfies (2.5a).

To show (2.5b) we follow DiPerna [2] and Szepessy [10]'s arguments. We have to check that, for each compact $K$ of $\mathbb{R}^d$,

$$
\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K |u(x, s), |u - u_0(x)|\, dx ds
$$

$$
= \lim_{t \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{t} \int_0^t \int_K |u(x, s) - u_0(x)|\, dx ds = 0.
$$

By Jensen's inequality, where $m(K)$ stands for Lebesgue measure of $K$, we have

$$
\frac{1}{t} \int_0^t \int_K |u^\varepsilon(x, s) - u_0(x)|\, dx ds
$$

$$
\leq m(K)^{1/2} \left( \frac{1}{t} \int_0^t \int_K (u^\varepsilon(x, s) - u_0(x))^2\, dx ds \right)^{1/2}.
$$

We will establish that

$$
\lim_{\varepsilon \to 0^+} \lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K (u^\varepsilon(x, s) - u_0(x))^2\, dx ds = 0.
$$

Let $K_i \subset K_{i+1}$ ($i = 0, 1, ...$) be an increasing sequence of compact sets such that $K_0 = K$ and $\bigcup_{i \geq 0} K_i = \mathbb{R}^d$. We use the identity $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$:

$$
\frac{1}{t} \int_0^t \int_{K_i} (u^\varepsilon(x, s) - u_0)^2\, dx ds
$$

$$
\leq \frac{1}{t} \int_0^t \left( \int_{K_i} |u^\varepsilon(x, s)|^2\, dx - \int_{K_i} u_0^2\, dx - 2 \int_{K_i} u_0 (u^\varepsilon(x, s) - u_0)\, dx \right)\, ds
$$

$$
\leq \int_{\mathbb{R}^d \setminus K_i} u_0^2\, dx + \frac{2}{t} \int_0^t \int_{K_i} u_0 (u^\varepsilon(x, s) - u_0)\, dx\, ds
$$

for all $i = 0, 1, ...$, where we used (4.3)-(3.1).

Since

$$
\lim_{i \to \infty} \int_{\mathbb{R}^d \setminus K_i} u_0^2\, dx = 0,
$$

we only consider the last term above. Take $\{\theta_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$
\lim_{n \to \infty} \theta_n = u_0 \quad \text{in } L^2(\mathbb{R}^d),
$$

Cauchy-Schwarz inequality gives

$$
\left| \int_{K_i} u_0 (u^\varepsilon(x, s) - u_0)\, dx \right|
$$

$$
\leq \int_{K_i} |u_0 - \theta_n| |u^\varepsilon(x, s) - u_0|\, dx
$$

$$
+ \left| \int_{K_i} \theta_n (u_0^\varepsilon - u_0) + \int_{K_i} \theta_n (u^\varepsilon(x, s) - u_0^\varepsilon)\, dx \right|
$$

$$
\leq \|u_0 - \theta_n\|_{L^2(\mathbb{R}^d)} (\|u^\varepsilon(x, s)\|_{L^2(\mathbb{R}^d)} + \|u_0\|_{L^2(\mathbb{R}^d)})
$$

$$
+ \|\theta_n\|_{L^2(\mathbb{R}^d)} \|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} + \left| \int_0^s \int_{K_i} \theta_n \partial_s u^\varepsilon\, dx dt \right|.
$$
In view of (4.3) and (3.1)
\[ \| u_0 \|_{L^2(\mathbb{R}^d)} + \| u_0 \|_{L^2(\mathbb{R}^d)} \]
\[ \leq 2 \| u_0 \|_{L^2(\mathbb{R}^d)} \| u_0 \|_{L^2(\mathbb{R}^d)}, \]
which tends to zero when \( n \to \infty \), and since \( \lim_{\varepsilon \to 0^+} \| u_0 \|_{L^2(\mathbb{R}^d)} = 0 \) by (3.1), it remains only to see that
\[ \lim_{t \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{t} \int_0^t \int_0^s \theta_n \partial_s u_{\varepsilon, \delta} \, dx \, dt = 0. \]
We have, by (1.4a),
\[ \left| \int_0^s \int_{K_i} \theta_n \partial_s u \, dx \, dt \right| = \left| \int_0^s \int_{K_i} \theta_n (-\text{div} f(u) + \varepsilon \text{div} b - \delta \sum_j \partial_{x_j}^3 u) \, dx \, dt \right| 
\[ = \left| \int_0^s \int_{K_i} (\nabla \theta_n \cdot f(u) - \varepsilon \nabla \theta_n \cdot b + \delta \sum_j \partial_{x_j}^3 \theta_n) u \, dx \, dt \right| 
\[ := \mu_1 + \mu_2 + \mu_3. \]
To deal with \( \mu_1 \), we use Hölder inequality and (H1)
\[ \int_0^s \int_{K_i} \| \nabla \theta_n \| \| f(u) \| \, dx \, dt \]
\[ \leq C \int_0^s \int_{K_i} \| \nabla \theta_n \| \, dx \, dt + C \left[ \int_0^s \int_{K_i} \| \nabla \theta_n \|^{\frac{n}{2}} \, dx \, dt \right]^{\frac{n}{n-m}} \left[ \int_0^s \int_{K_i} \| u \|^{q} \, dx \, dt \right]^{\frac{m}{n}} 
\[ \leq C s \| \nabla \theta_n \|_{L^1(\mathbb{R}^d)} + C s^{\frac{m}{n-m}} \| \nabla \theta_n \|_{L^\frac{m}{m}(\mathbb{R}^d)} \| u \|_{L^m(\mathbb{R}^d \times (0,T))}. \]
For \( \mu_2 \), using (H2) and once more Hölder inequality with (4.4)-(3.1), we get
\[ \varepsilon \int_0^s \int_{K_i} |\nabla \theta_n| \| b \| \, dx \, dt \]
\[ \leq C_3 \varepsilon \int_0^s \int_{K_i} |\nabla \theta_n| \| \nabla u \|^{r} \, dx \, dt \]
\[ \leq C_3 \varepsilon \left[ \int_0^s \int_{K_i} |\nabla \theta_n|^{r+1} \, dx \, dt \right]^{\frac{1}{r+1}} \left[ \int_0^s \int_{K_i} \| u \|^{r+1} \, dx \, dt \right]^{\frac{1}{r+1}} 
\[ \leq C \varepsilon^{1-\frac{1}{r+1}} s^{\frac{1}{r+1}} \| \nabla \theta_n \|_{L^{r+1}(\mathbb{R}^d)}. \]
Finally, for \( \mu_3 \), we use Cauchy-Schwarz inequality with (4.3)-(3.1):
\[ \delta \int_0^s \int_{K_i} u \sum_j \partial_{x_j}^3 \theta_n \, dx \, dt \]
\[ \leq \delta \left[ \int_0^s \int_{K_i} |u|^2 \, dx \, dt \right]^{\frac{1}{2}} \left[ \int_0^s \int_{K_i} \left| \sum_j \partial_{x_j}^3 \theta_n \right|^2 \, dx \, dt \right]^{\frac{1}{2}} 
\[ \leq \delta s \| \nabla^3 \theta_n \|_{L^2(\mathbb{R}^d)} \| u_0 \|_{L^2(\mathbb{R}^d)}. \]
and so
\[
\theta_n \partial_\epsilon u_{\xi, \epsilon} \frac{dx}{dt}\Big|_{\theta_n} = X
\]
\[
\leq \lim_{\epsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_\mathbb{R} \theta_n \partial_\epsilon u_{\xi, \epsilon} \frac{dx}{dt} \bigg|_{\theta_n} \, ds
\]
\[
\leq C \left( \frac{C}{2} t^2 \|\nabla \theta_n\|_{L^1(\mathbb{R}^d)}^m \right)
\]
\[
+ C \left( \frac{q}{q - m} + 1 \right)^{-1} \left( \frac{\varepsilon}{\rho + 1} \|\nabla \theta_n\|_{L^\infty(\mathbb{R}^d)} \right)
\]
\[
+ C \left( \frac{1}{r + 1} + 1 \right)^{-1} T \left( \frac{\varepsilon}{\rho + 1} \|\nabla \theta_n\|_{L^\infty(\mathbb{R}^d)} \right)
\]
and so
\[
X \leq C_n \left( t + \int_0^t \lim_{\epsilon \rightarrow 0^+} \left| \mu_{\epsilon, \delta} \right|_{L^\infty(\mathbb{R}^d \times (0, T))} \right)
\]
where we have used (4.9) in Proposition 4.4. The desired conclusion when $t \rightarrow 0^+$ follows.

**Proof of Theorems 3.2 and 3.3.** In the previous proof, to establish (2.5a) we started with the identity (4.1) and the condition $\delta = o(\varepsilon^{1/\epsilon})$ as required, in particular to control the term in (4.1b). We now keep the form (4.1a) instead (4.1b). We only need discuss $\mu_3$. It has now the form:
\[
-\delta \sum_j \sum \partial_x_j (\partial_\epsilon u)^2 + \delta \sum_j \partial_x_j (\partial_\epsilon u)^2 u.
\]
The first term is bounded as follows
\[
\left| \int_0^T \int_\mathbb{R} \sum \partial_x_j (\partial_\epsilon u)^2 u \right| \leq C \delta \int_0^T \int_\mathbb{R} \left| \nabla u \right| \frac{D^2 u}{dt} \, dx \, dt
\]
\[
\leq C \delta \int_0^T \int_\mathbb{R} \mu \frac{D^2 u}{dt} \, dx \, dt + \frac{1}{\mu} \left| \left( \nabla u \right) \right|^2 \, dx \, dt
\]
\[
\leq C \delta \left( \mu \varepsilon^{-\frac{\rho + 1}{\rho}} + \frac{1}{\mu} \varepsilon^{-\frac{\rho}{\rho - 1}} \right)
\]
using (4.15) and (4.4), and we take $\mu = \varepsilon$ and $\delta = o(\varepsilon^{1/\epsilon})$.

The second term in $\mu_3$ behaves better:
\[
\delta \int_0^T \int_\mathbb{R} \partial_x_j (\partial_\epsilon u)^2 u \, dx \, dt =: Y
\]
\[
\leq \delta \int_0^T \int_\mathbb{R} \sum \partial_x_j \left( \partial_\epsilon u \right)^2 \, dx \, dt + \delta \int_0^T \int_\mathbb{R} \sum \partial_x_j \left( \partial_\epsilon u \right)^2 u \, dx \, dt
\]
\[
\leq C \delta \int_0^T \int_\mathbb{R} \left| \nabla u \right| \frac{D^2 u}{dt} \, dx \, dt + C \delta \int_0^T \int_\mathbb{R} \left| \nabla u \right|^2 \frac{D^\theta}{dt} \, dx \, dt
\]
thus

\[ Y \leq C \delta \left[ \int_0^T \int_{\mathbb{R}^d} |\nabla u|^{r+1} \, dx \, dt \right]^{\frac{1}{r+1}} + C \delta \left[ \int_0^T \int_{\mathbb{R}^d} |\nabla u|^r \, dx \, dt \right]^{\frac{1}{r}} \]

\[ \leq C \delta \varepsilon^{-\frac{r+1}{r}}. \]

This completes the proof of Theorems 3.2 and 3.3. □

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