Collective modes and ballistic expansion of a Fermi gas in the BCS-BEC crossover

Hui Hu, A. Minguzzi, Xia-Ji Liu, and M. P. Tosi
1 NEST-INFM and Classe di Scienze, Scuola Normale Superiore, I-56126 Pisa, Italy
2 Department of Physics, Tsinghua University, Beijing 100084, China
(Dated: August 25, 2021)

We evaluate the frequencies of collective modes and the anisotropic expansion rate of a harmonically trapped Fermi superfluid at varying coupling strengths across a Feshbach resonance driving a BCS-BEC crossover. The equations of motion for the superfluid are obtained from a microscopic mean-field expression for the compressibility and are solved within a scaling ansatz. Our results confirm non-monotonic behavior in the crossover region and are in quantitative agreement with current measurements of the transverse breathing mode by Kinast et al. [Phys. Rev. Lett. 92, 150402 (2004)] and of the axial breathing mode by Bartenstein et al. [Phys. Rev. Lett. 92, 203201 (2004)].

PACS numbers: 03.75.-b, 03.75.Ss

Current experiments on ultracold Fermi gases are rapidly advancing towards the realization of superfluid states, and Bose-Einstein condensation of dimers has already been achieved. A key tool for the manipulation of atomic gases is the use of a Feshbach resonance to vary the magnitude and sign of the coupling strength. Across the resonance the s-wave scattering length $a$ goes from large positive to large negative values, thus allowing exploration of the crossover from the Bardeen-Cooper-Schrieffer (BCS) state to the Bose-Einstein condensate (BEC) of bound-fermion pairs. As Fermi gases have been demonstrated to be stable also near the resonance, they offer a new opportunity to investigate highly correlated many-body systems.

As in the case of bosonic clouds, the frequencies of collective modes of Fermi gases can be measured to high accuracy and can yield information on the state of the system and on its collisional properties, thus contributing to its characterization in a strong-coupling regime. Another dynamical observable which is readily accessed is the aspect ratio of an expanding cloud released from an anisotropic trap, which depends on its quantum state and on its degree of collisionality. While a normal Fermi gas in the collisionless regime expands spherically, an anisotropic expansion is predicted for a normal gas in the collisional regime as well as for a superfluid gas.

The purpose of this Letter is to present a theory of the collective modes and of the expansion of a trapped superfluid Fermi gas at zero temperature as the coupling strength is varied across a Feshbach resonance. The frequencies of the collective modes are well known in both the BCS and the BEC limit. From non-mean-field perturbative estimates it has been conjectured that the frequency of the transverse breathing mode in a highly elongated trap should exhibit a non-trivial dependence on the scattering length. Two further studies have been based on semi-empirical forms of the equation of state. Here we use a microscopic mean-field description of the BCS-BEC crossover, which at zero temperature is believed to capture the essential physics in all regimes. We calculate the equation of state and the density profiles of the gas under axially symmetric confinement with the help of a local density approximation (LDA), and use them to determine the collective mode frequencies and the expansion rate by means of a simple scaling assumption. Our results do not use an interpolation scheme nor involve adjustable parameters, and show already at mean-field level non-monotonic behaviors across a Feshbach resonance.

The frequencies of radial and axial breathing modes have very recently been measured in elongated clouds of $^6$Li atoms at various values of the coupling strength in the strong-coupling intermediate regime. Quite remarkably, our mean-field approach yields predictions for the frequencies of the breathing modes that are in good quantitative agreement with these experiments.

Equation of state and equilibrium density profile.

Our starting point is to determine the chemical potential $\mu(n)$ as a function of density $n$ for a homogeneous Fermi gas through the BCS-BEC crossover at zero temperature. We use a mean-field theory proposed in the pioneering work of Leggett and of Nozières and Schmitt-Rink. This describes well both the BCS weak-coupling regime and the BEC strong-coupling limit, and is believed to be a reliable approximation in between. It has also been applied to a trapped gas within the LDA. The gas consists of two equally populated spin components that interact by a contact pseudopotential parametrized by the s-wave scattering length $a$. The mean-field theory for the crossover extends the usual BCS gap equation

$$1 = \frac{4\pi\hbar^2a}{m} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2\epsilon_k - \frac{1}{2E_k}} \right]$$

and number equation

$$n = \int \frac{d^3k}{(2\pi)^3} \left( 1 - \frac{\epsilon_k - \mu}{E_k} \right)$$

to all values of the pairing interaction. Here $\epsilon_k =$
inhomogeneous gas. In fact, in the present approximation

\[
\Gamma = (N^{1/6}a/a_{ho})^{-1}
\]

of the dimensionless coupling parameter \(\kappa = (N^{1/6}a/a_{ho})^{-1}\) for a trapped Fermi gas with \(N = 2 \times 10^5\) and \(\Lambda = \omega_z/\omega_{\perp} = 0.05\). The quantity \(\gamma(n)\) and the transverse density profile \(n_0 \equiv n_0(\rho, 0)\) with \(\rho = \sqrt{x^2 + y^2}\) are plotted in the insets for \(\kappa = -1.0, +0.1, +0.5\) and +1.0 (from bottom to top). The coordinate \(\rho\) is in units of \(a_{ho} = \lambda^{-1/6} \sqrt{\hbar/m\omega_{\perp}}\) and \(n_0\) is in units of \(a_{ho}^3\).

\[h^2 k^2/2m, E_k = \sqrt{(\kappa - \mu)^2 + \Delta^2}, \Delta \text{ is the gap parameter, and } n \text{ is the total density for fermions in either spin state (} n_\uparrow = n_\downarrow = n/2\). These equations are to be solved for \(\mu_0\) and \(\Delta\) with a given choice of the dimensionless coupling parameter \(k_Fa_0 = (3\pi^2 n)^{1/3} a_0\). In the weak-coupling limit \((k_Fa_0 \rightarrow 0^-)\) they give back the standard BCS result, and when \(k_Fa_0 \rightarrow 0^+\) they correctly reproduce the binding energy of fermion pairs at leading order \(10\). In the unitarity limit \((a = \pm \infty)\) the model gives \(\mu(n) \propto n^{2/3}\), scaling as predicted by the universality hypothesis \(21\).

We then use the equation of state \(\mu(n)\) to determine the density profile \(n_0(r)\) under the confining potential \(V_{ext}(r) = m [\omega^2_r (x^2 + y^2) + \omega^2_\perp z^2] / 2\) through the implicit LDA equation \(\mu(n_0(r)) + V_{ext}(r) = \mu_0\). Here \(\mu_0\) is the chemical potential of the trapped gas from the normalization condition \(N = \int d^3r n_0(r)\), \(N\) being the total number of fermionic atoms. A useful dimensionless coupling parameter for a trapped gas is \(\kappa = (N^{1/6}a/a_{ho})^{-1} \approx 1.695(k_Fa_0)^{-1}\), where \(k_F\) is the Fermi wave number at the trap center \(11\).

Our results are illustrated in Fig. 1. The left inset reports the function \(\gamma(n) = 1 + n \left[\frac{d^2\mu}{dn^2}\right] / \left\langle d\mu/dn\right\rangle\) in the homogeneous gas for four choices of \(\kappa\), while the right inset shows the LDA density profiles in the transverse direction. In the special case of a power-law dependence of the chemical potential on density \(\mu(n) \propto n^{\gamma}\), \(\gamma(n) = \gamma\). In the general situation \(\gamma(n)\) acts as an effective exponent. The main body of Fig. 1 shows the quantity \(\tilde{\gamma} = (\int d^3r n_0 r^2_\alpha \gamma(n_0)) / (\int d^3r n_0 r^2_\alpha)\) with \(\alpha = x, y,\) or \(z\), to be used below as an averaged effective exponent for inhomogeneous gas. In fact, in the present approxima-

\[\frac{\partial_t n + \nabla \cdot (n \mathbf{v})}{\partial t} = 0,\]  

and the Euler equation

\[m \partial_t \mathbf{v} + \nabla \left[ \mu(n) + V_{ext} + m \mathbf{v}^2/2 \right] = 0.\]  

Here \(n(r, t)\) and \(\mathbf{v}(r, t)\) are the time-dependent density profile and velocity field of the trapped gas, \(\mu(n)\) is the local chemical potential expressed as a function of \(n(r, t)\) in an adiabatic LDA, and \(V_{ext}(r, t) = m \left[ \omega^2_r(t) (x^2 + y^2) + \omega^2_\perp(t) z^2 \right] / 2\) is the axially symmetric trapping potential. The above equations apply as well to a normal Fermi gas in the collisional regime. The difference from a superfluid should emerge from some further knowledge of the collisional behavior of the cloud or from its response under rotation.

We adopt the following scaling form of the time-dependent density profile,

\[n(r, t) = \left[ \prod_{\alpha} b_{\alpha}(t) \right]^{-1} n_0(x/b_\alpha(t), y/b_\alpha(t), z/b_\alpha(t))\]  

where \(n_0(r)\) is again the equilibrium density profile and the dependence of the profile \(n(r, t)\) on time is entirely contained in the scaling parameters \(b_{\alpha}(t)\) \(22\). This ansatz correctly reflects the expansion and compression along each axis and therefore is appropriate for the description of the breathing modes and of the ballistic expansion of the cloud. The corresponding form of the velocity field is fixed by the equation of continuity as \(v_{\alpha}(r, t) = b_\alpha(t) r_\alpha / b_\alpha(t)\).

The scaling ansatz in Eq. \(5\) is an exact solution of the equations of motion if the equation of state is a power law \(\mu(n) \propto n^\gamma\). In this case the scaling parameters obey the coupled differential equations

\[b_\alpha + \omega^2_\alpha b_\alpha - \left( \omega^2_\alpha / b_\alpha \right) \left( \prod_{\beta} b_\beta \right)^{-\gamma} = 0.\]  

However, with a more general form of the equation of state the scaling solution is not satisfied at every position \(r\). A useful approximation is to take a spatial average with the weight function \(r^2_\alpha n_0\) \(23\). After some straightforward algebra we obtain

\[
\bar{b}_\alpha + \omega^2_\alpha b_\alpha - \left( \omega^2_\alpha / b_\alpha \right) \left( \prod_{\beta} b_\beta \right)^{-\gamma} = 0.\]  

\[
\frac{\int d^3r n_0 r^2_\alpha \left[ (d\mu/dn)_{n=n_0} \prod_{\beta} b_{\beta}\right]^{-1} / (d\mu/dn)_{n=n_0}}{\int d^3r n_0 r^2_\alpha} = 0.\]
FIG. 2: (color online) The frequency of the axial and transverse breathing modes in an elongated trapped Fermi gas as functions of $\kappa = (N^{1/6}a/a_{ho})^{-1}$. The solid circles and empty triangles with error bars are the experimental results given by Kinast et al. \[17\] and by Bartenstein et al. \[18\], respectively. Here we plot only the measured frequencies with low decay rates in the resonance region, for instance, for the axial mode we use the data of the inset of figure 1 in Ref. \[18\]. The resonance position $B_0$ has been set at 837 G. The error bar in the experimental data in the horizontal ($\kappa$) axis is due to the uncertainty of $B_0$ and we have taken $\Delta B_0 = \pm 15$ G.

which reduces to Eq. (6) if $\mu(n) \propto n^\gamma$.

Breathing modes — We evaluate the transverse and axial breathing modes by taking periodic variations of the trapping frequencies $\omega_{\perp, z}(t) = \omega_{\perp, z}(1 + \epsilon \cos(\omega t))$ with $\epsilon \ll 1$. Linearization of Eq. (6) yields the linearized form of Eq. (6) with $\gamma$ replaced by $\bar{\gamma}$. One immediately finds three mode frequencies, one of which lies at $\omega = \sqrt{2}\omega_{\perp}$ independently of the coupling strength and belongs to a surface mode with projected angular momentum $m = \pm 2$. The other two mode frequencies are given by

$$\omega_{\pm}^2/\omega_0^2 = 1 + \bar{\gamma} + (2 + \bar{\gamma}) \lambda^2/2$$

$$\pm \sqrt{[1 + \bar{\gamma} + (2 + \bar{\gamma}) \lambda^2/2]^2 - 2(2 + 3\bar{\gamma}) \lambda^4}$$

where $\lambda = \omega_0/\omega_{\perp}$ and the $\pm$ signs refer to the transverse and axial mode, respectively.

In the highly elongated traps of current experimental interest ( $\lambda \ll 1$) the frequencies of the breathing modes reduce to $\omega_+ = \sqrt{2 + 2\bar{\gamma}}$ and $\omega_0 = \sqrt{\omega_{\perp}}$, yielding $\omega_+ = \sqrt{10/3}\omega_{\perp} \approx 1.83\omega_{\perp}$ in a dilute Fermi gas ($\bar{\gamma} = 2/3$) and $\omega_+ = 2\omega_{\perp}$ in a dilute BEC ($\bar{\gamma} = 1$). The mode frequencies in the crossover region are presented in Fig. 2 as solid lines displaying a non-trivial dependence on the coupling parameter. In particular, a dip appears near the unitarity limit on the BCS side, as a result of the pairing that enhances the compressibility of the gas. On the BEC side the frequencies monotonically increase towards the BEC value, this behavior being related in our approximate scheme to the increase of the effective exponent $\bar{\gamma}$ with the coupling constant $\kappa$ for $\kappa > 0$. In Fig. 2 we have also compared our predictions for the axial mode with experimental data by Bartenstein et al. \[18\] and those for the transverse mode with measurements by Kinast et al. \[17\] and by Bartenstein et al. \[18\]. Two sets of the experimental data, i.e., the axial mode in Ref. \[18\] and the transverse mode in Ref. \[17\], are in quantitative agreement with our mean-field results.

However, the remaining set of data for the transverse mode measured by Bartenstein et al. \[18\] contradicts our prediction. The measured frequencies have an abrupt change around the resonance, accompanied by large damping rates. This is not expected by the theory. As noted by Grimm, this might be due to the accidental coincidence between the collective energy and the low-lying quasiparticle energy. As a result, the superfluidity of the Fermi gas could be destroyed by the excitations of quasiparticles. Nevertheless, the tendency of monotonic increase of the predicted mode frequency in the BEC limit, as a natural outcome of the mean-field theory, agrees qualitatively with the experimental observations. This is to be contrasted with the findings by Stringari \[11\] which predict a decrease in the transverse mode frequency towards the BEC limit.

In our calculations the collective mode frequencies in the trap depend mainly on the dependence of the chemical potential on density through an effective power-law exponent. We have not taken account of non-mean-field corrections to the compressibility, which were discussed in Ref. \[11\]. These can be estimated in a perturbative way for $\kappa \to \pm \infty$, and can be shown to be very small near the BEC limit from the Lee-Huang-Yang correction.
The expansion of an ideal gas is also shown in the inset by a thin solid line. The other parameters are \( N = 2 \times 10^5 \) and \( \lambda = 0.05 \).

The calculated collective modes as functions of the anisotropy parameter \( \lambda \) are shown in Fig. 3 for four values of the coupling strength. The sensitivity of the transverse mode to the coupling is preserved for all values of \( \lambda \), while the axial mode is less sensitive to the interactions. For a spherical trap with \( \lambda = 1 \) the breathing modes are known as the monopole and the quadrupole mode and we have \( \omega_+ / \omega_\perp = \sqrt{2 + \gamma} \) and \( \omega_+ / \omega_\perp = \sqrt{4 + 6\gamma} / (2 + \gamma) \).

Expansion of the Fermi gas. — Finally, we turn to study the expansion of the Fermi gas after suddenly switching off the confining potential, by setting \( \omega_\perp(t > 0) = 0 \). We follow the time evolution of the aspect ratio \( R_\perp(t)/R_\perp(0) = \lambda b_\perp(t)/b_\perp(0) \) by numerically solving Eq. 17 with the initial configurations \( b_\parallel(0) = 1 \) and \( b_\perp(0) = 0 \).

The results are shown in Fig. 4. The aspect ratio of the cloud at a given expansion time shows non-monotonic behavior as a function of the coupling strength \( \kappa \), similarly to what is observed in the collective modes frequencies. This reflects again the emergence of important interaction effects in the hydrodynamic behavior.

Conclusions. — In summary, we have presented a microscopic theory of the dynamics of a superfluid Fermi gas through the whole BEC-BCS crossover. Our analysis is based on the equation of state of the gas in a mean-field approach. For a gas under anisotropic confinement we have found that both the frequencies of the breathing modes and the aspect ratio of the expanding cloud show non-monotonic behavior as functions of the coupling constant. We have also made quantitative contact with current experiments.

This work has been partially supported by INFM under the PRA-Photonmatter program. X.-J. L. was supported by NSF-China under Grant No. 10205022 and by the National Fundamental Research Program under Grant No. 001CB309308. We thank Prof. J. Thomas for useful discussions on the experimental data. A.M. thanks Prof. G. V. Shlyapnikov for useful discussions and the LPTMS Orsay, where this work was completed, for their hospitality.

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