Scalar Perturbations in a String Inspired Inflationary Scenario

Carlos Eduardo Magalhães Batista*
Instituto Gleb Wathagin
Universidade Estadual de Campinas
Campinas - São Paulo
Brazil
and
Júlio César Fabris†
Departamento de Física
Universidade Federal do Espírito Santo
Vitória - Espírito Santo - CEP29060-900
Brazil

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Abstract

We consider an inflationary model inspired in the low energy limit of string theory. In this model, the scale factor grows exponentially with time. A perturbation study is performed, and we show that there is a mode which displays an exponential growth in the perturbation of the scalar field.

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*e-mail: batista@ifi.unicamp.br
†e-mail: fabris@cce.ufes.br
1 Introduction

One of most outstanding problems today in Cosmology concerns the formation of the structures observed in the Universe\cite{1}. The traditional mechanism for the formations of such structures, the growing of small fluctuations in the density distribution, can not account for the existence of galaxies and clusters of galaxies in the context of the Classical Standard Model. It is now believed that the Inflationary Scenario can lead to some answer to this question\cite{2}. In fact, in order to explain the structure formation, we must have fluctuations in the density distribution of the order of $10^{-5}$ already in the beginning of the radiative era. But the initial statistical fluctuations must be much less than that. So, in the problem of structure formation it is necessary to have a gravitational instability mechanism in the primordial Universe in order to grow the perturbations to the desired value.

So, it seems that to solve the problem of the structure formation, we must have a fast growing of the perturbations during some phase before the radiation dominated period. In this work we investigate such a problem in the context of an inflationary scenario inspired in a low energy effective model comming from string theory. In this case, we have a scalar field $\Psi$ that couples non-minimaly to gravity. We add a potential term. However, in order to have an exponential growing solution for the scale factor of the Universe, we include a cosmological constant. In order to have some generality in our expressions, we introduce this cosmological constant as perfect fluid with an equation of state $p = \alpha \rho$, putting in the end $\alpha = -1$.

So, our Lagrangian is

$$L = \sqrt{-g} e^\Psi \left( R + \nabla^\mu \nabla_\mu - V(\Psi) \right) + L_m. \tag{1}$$

Models with a similar field structure has been considered in the literature\cite{3}. We will show that we can obtain an exponential behaviour for the scale factor. Our interest, however, is the evolution of the perturbed quantities around a classical background, which will be given by the Robertson-Walker metric. Since, we use an equation of state for the false vacuum, the perturbations in $\rho$ are zero\cite{4}. But we can find non-zero perturbations for the scalar field $\Psi$. The fluctuations in this scalar field (which is a dilaton field) will be responsable for the perturbations in matter when the inflationary phase ceases and there is the transition to the radiative phase\cite{5}. We can obtain exponential growing
modes for the perturbations in the scalar field during the inflationary phase. So, this model may contain the desired mechanism for the generation of the seeds of the structure formation.

We organize this paper as follows. In section II, we establish the background model and its solution; in section III, we perform a perturbative analysis and determine analytical solutions for the fluctuations in the physical quantities. We end with the conclusions. We work with the traditional formalism of Lifschitz-Khalatnikov, fixing the synchronous coordinate condition. This has the advantage of simplicity giving the results directly in terms of the physical quantities.

2 The Background Model

We will consider in (1) a potential term of the kind \( V(\Psi) = m^2 \Psi^2 \), where \( m \) is a mass parameter. It comes out to be convenient, in view of the latter perturbation analysis, to perform a field transformation of the kind \( e^\Psi = \phi \). In this case, (1) takes the form,

\[
L = \sqrt{-g} \left( \phi R - \omega \frac{\phi^\mu \phi_{;\mu}}{\phi} - V(\phi) \right) + L_m .
\]

In this expression, we should have \( \omega = -1 \), in order to be in agreement with (1). We work with an arbitrary \( \omega \) in such a way to be in contact with the Brans-Dicke theory, which is the prototype of the scalar-tensorial theories. Moreover, \( V(\phi) = m^2 \phi ln^2 \phi \). Again, \( L_m \) is the lagrangian density associated with matter, which for our purposes will be a perfect fluid matter, with the equation of state \( p = -\rho \). But, we will introduce this hypothesis just in the end.

From (2), and defining \( \phi W(\phi) = V(\phi) \), we deduce the following field equations:

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8 \pi}{\phi} T_{\mu \nu} + \frac{\omega}{\phi^2} \left( \phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu \nu} \phi_{;\rho} \phi_{;\rho} \right) + \frac{1}{\phi} \left( \phi_{;\mu} \phi_{;\nu} - g_{\mu \nu} \Box \phi \right) - \frac{g_{\mu \nu}}{2} W \phi ; \quad (3)
\]

\[
\Box \phi = \frac{1}{3 + 2\omega} \left( 8\pi T + \phi^2 W' - \phi W \right) ; \quad (4)
\]

\[
T^{\mu \nu} ;_{\mu} = 0 . \quad (5)
\]
In these equations, the prime represent derivation with respect to the field \( \phi \) and \( T^{\mu\nu} \) comes from the variation of \( L_m \) with respect to the metric. We consider the spatially flat Robertson-Walker metric,

\[ ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{6} \]

and the momentum-energy tensor as

\[ T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu} \tag{7} \]

The pressure and energy density are linked by a barotropic equation of state \( p = \alpha \rho \). The differential linking \( a(t), \phi(t), p \) and \( \rho \) are:

\[ 3\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{\phi} \rho + \frac{3}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{3}{2} \frac{\ddot{\phi}}{\phi} + \frac{W}{2} \tag{8} \]

\[ \ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} = \frac{1}{3 + 2\omega} \left(8\pi(\rho + 3p) + \phi^2 W' - \phi W\right) \tag{9} \]

\[ \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{10} \]

Imposing \( p = -\rho, \phi = \phi_0 = cte \), we obtain the following solutions:

\[ a = a_0 e^{H_0 t} \tag{11} \]

\[ 16\pi \rho = m^2 \phi_0 ln(\phi_0) \tag{12} \]

\[ ln(\phi_0) = -1 \pm \sqrt{1 + 12\frac{H^2}{m^2}} \tag{13} \]

These solutions represent an inflationary phase, driven by a cosmological constant (that can be interpreted as a false vacuum state) and a constant scalar field. This scenario is well in spirit of old inflation, with its advantage and disadvantage. We will not discuss here, for example, the mechanism to end with this inflationary phase, which has an abundant literature. Our interest is its consequence for the evolution for small perturbations during this period, which is believed to be limited in time in a more complete theory.

3 Perturbative Analysis

Now, we turn to the problem of the perturbation of physical quantities in this model. Essentially, we have the perturbations in the metric \( \delta g_{\mu\nu} = h_{\mu\nu} \), in
the scalar field $\delta \phi$ and in the matter $\delta \rho$. We employ the Lifschitz-Khalatnikov formalism mainly for two reasons: it is simpler mathematically, and it gives the results directly in terms of the perturbations in the matter and field quantities. Since in this formalism we fix a frame, we must be sure that our results concern physical quantities, and not coordinate artifacts. This can be done by studying the residual coordinate freedom as we will see below.

So, we introduce in the field equations the quantities $g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu}$, $\phi = \phi^0 + \delta \phi$, $\rho = \rho^0 + \delta \rho$, $u^\mu = u^\mu^0 + \delta u^\mu$, i.e., the variables now are the sum of the background solution and of a small perturbation around it. We impose the synchronous coordinate condition $h_{\mu0}^0 = 0$. We will consider that these perturbations are adiabatic, with $p = \alpha \rho$ and $\delta p = \alpha \delta \rho$. The procedure to deduce the perturbed equations are quite standard, and the final results are:

\begin{align}
\dot{\Delta} + (1 + \alpha)(\Theta - \frac{\dot{h}}{2}) &= 0, \quad (14) \\
(1 + \alpha)\ddot{\Theta} + (2 - 3\alpha)(1 + \alpha)\frac{\dot{a}}{a}\Theta + \alpha \frac{\nabla^2 \Delta}{a^2} &= 0, \quad (15) \\
\ddot{\lambda} + \dot{\lambda}(3 \frac{\dot{a}}{a} + 2 \frac{\dot{\phi}}{\phi}) + \frac{\lambda}{3 + 2\omega} \left(\frac{32\pi \rho}{\phi} + W'' \phi^2 - \frac{q^2}{a^2} (3 + 2\omega)\right) + \frac{\dot{h}}{\phi} - \frac{32\pi \delta \rho}{\phi(3 + 2\omega)} &= 0. \quad (16)
\end{align}

In these equations, $h = \frac{h_{\mu\nu}}{a^2}$, $\Delta = \frac{\delta \rho}{\rho}$, $\lambda = \frac{\delta \phi}{\phi}$, $\Theta = \delta u^i_\cdot i$. The term $q^2$ is provenient from a plane wave expansion $\lambda \propto \exp(iqx)$ where $x$ represents the comoving distance and $q$ is the wavenumber of the perturbation. We will work in the conformal time in order to simplify the resolution of the equations, that is $dt = \alpha d\tau$. So, $a = -\frac{1}{H\tau}$.

Imposing $\alpha = -1$, we can see that the perturbations in matter are null. The equation for $\lambda$ can be written as

\begin{align}
\lambda'' + \lambda' \left(2 \frac{a'}{a} + 2 \frac{\phi'}{\phi}\right) + \\
+ \frac{\lambda}{3 + 2\omega} \left(\frac{32\pi \rho a^2}{\phi} - 4 \frac{dW}{d\phi} \phi a^2 - \frac{d^2 W}{d^2 \phi} \phi^2 a^2 + q^2 (3 + 2\omega)\right) &= -h\frac{\phi'}{\phi}. \quad (17)
\end{align}
where primes represent now derivatives with respect to conformal time. Considering now the background solutions, this equation takes the form:

\[ \lambda'' - \frac{2}{\tau} \lambda' + \lambda \left( \frac{4}{1 + 2\omega} \left( \frac{9 + A/H^2}{\tau^2} \right) + q^2 \right) = 0 \quad . \tag{18} \]

where \( A \) is:

\[ A = \frac{-(1 + 4\omega) \frac{dW}{d\phi} \phi + 3W}{2(3 + 2\omega)} + \frac{3W}{2} - \frac{d^2W}{d\phi^2} \frac{\phi^2(1 + 2\omega)}{4(3 + 2\omega)} \quad . \tag{19} \]

Writing \( \lambda = \tau^p \gamma \), where \( p = \frac{3}{2} \), is easy to show that our equation becomes:

\[ \gamma'' + \frac{\gamma'}{\tau} + \left[ (-9/4 + 4(9 + A/H^2)/(1 + 2\omega)) \frac{1}{\tau^2} + q^2 \right] \gamma = 0 \quad . \tag{20} \]

The equation (20) has solutions under the form of Bessel’s functions. The final result for \( \lambda \) is,

\[ \lambda = \tau^p \left( J_\nu(q\tau) + J_{-\nu}(q\tau) \right) \quad . \tag{21} \]

where

\[ \nu^2 = \frac{9}{4} + 4 \left( \frac{9 + A/H^2}{1 + 2\omega} \right) \quad . \tag{22} \]

This solution contains just physical modes. In fact, employing the synchronous coordinate condition, we have yet a residual coordinate freedom, given by a coordinate transformation \( x^\mu \rightarrow x^\mu + \chi^\mu \), which preserves the synchronous condition. But, it can be verified that we can not eliminate the above solutions using this coordinate freedom.

In some cases, \( \nu \) can be purely imaginary, namely, for \(-1.5 < \omega < -1.4 \) or \(-0.5 < \omega < 31.4 \). For the cases where \( \omega \) is \(-1.5 \) or \(-0.5 \), the order of the Bessel function goes to infinity; in the two other limits, it goes to zero.

We can have a better insight on the nature of these solutions by investigating their asymptotical behaviour. We consider first the case where the order of the Bessel function is real. In the case of small values of \( \tau \) (which implies large values of the cosmic time \( t \)), we have,

\[ \lambda \propto \tau^{\frac{3}{2} \pm \nu} \propto e^{-(\frac{3}{2} + \nu)Ht} \quad . \tag{23} \]
We can have an exponential growing for the perturbations if \( \frac{3}{2} \pm \nu < 0 \). In the case where \( \frac{H^2}{m^2} \gg 1 \), which represents the condition for having enough inflation, this means \( \omega < -\frac{5}{4} \).

In the case \( \tau \gg 1 \), which means \( t \ll 1 \), we have the asymptotical limit

\[
\lambda \propto \tau (c_1 \cos \tau + c_2 \sin \tau) \propto e^{-Ht} (c_1 \cos(e^{-Ht}) + c_2 \sin(e^{-Ht})) \quad (24)
\]

So, the solutions present an exponential decreasing oscillations in the beginning, and they can evolve to exponential increasing modes, depending on the value of \( \omega \).

For the case, where the order of the Bessel function is purely imaginary, we have an oscillatory behaviour in both asymptotical limits.

4 Conclusions

The Inflationary Scenario we have described here can be obtained by considering a dilaton field plus a false vacuum energy density. This false vacuum is formally represented by a perfect fluid with an equation of state \( p = -\rho \). In one sense, this field structure was inspired in the low energy limit of string theory, for which we have add a potential for the inflaton, and the term representing the false vacuum. We have obtained an exponential growth for the scale factor, following closely the spirit of old inflation.

In general, in the de Sitter phase, the density perturbations is zero; this result is also true in the case of the traditional Brans-Dicke theory. Our model is somehow similar to the Brans-Dicke case, since the gravitational coupling can be a varying function of time. In this case, even if the matter perturbations are zero during the de Sitter phase, inhomogeneities in the scalar field, which is linked with the gravitational coupling, can induce matter inhomogeneities in the end of the inflationary phase.

We have found many different situations for the behaviour of the perturbation of the scalar field. In some cases, in both asymptotical limits, this perturbation has oscillatory behaviour, with constant or decreasing amplitude. On the other hand, we verified that for values of the parameter \( \omega \) lesser than \(-\frac{5}{4}\), when \( \frac{H^2}{m^2} \gg 1 \), we can have an exponential growth for the perturbation of the scalar field.

It has been determined recently some cases where this exponential growth can also occur, but directly for the density contrast of ordinary matter.
In our case, even if this exponential growth does not concern directly ordinary matter, the inhomogeneities in the scalar field can have important consequences for the formation of structures in a latter phase, after the end of the inflationary regime.

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