Chain homogeneous Souslin algebras

Gido Scharfenberger-Fabian*

Ernst-Moritz-Arndt-Universität Greifswald, Institut für Mathematik und Informatik, Walther-Rathenau-Straße 47, 17487 Greifswald, Germany

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Assuming Jensen’s principle ♦+ we construct Souslin algebras all of whose maximal chains are pairwise isomorphic as total orders, thereby answering questions of Koppelberg and Todorčević.

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1 Introduction

Souslin’s problem was published in the first issue of Fundamenta Mathematicae (1920, p. 223):

Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu’un élément) n’empiétant les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?

During the following decades, though still far from being solved, Souslin’s problem was reconsidered by a number of mathematicians who established several equivalent formulations in terms of trees [16, 19], metric spaces [15] or Boolean algebras [18]. When in the 1960’s the independence of ZFC and Souslin’s hypothesis SH, which states that the answer to Souslin’s question is “yes”, was established, general interest was focused on the techniques developed for such independence results. Yet there were also some publications exploring the structural properties of Souslin lines [9] and of Souslin trees [4] which are the counter-examples to SH. Further results on possible representations of subalgebras of Souslin algebras were obtained in [5] and also as a by-product of the famous consistency proof of ZFC with the conjunction of Souslin’s hypothesis SH and the continuum hypothesis CH, cf. [3]. These approaches have been adapted, enhanced and utilized in [21]. This line of research is continued in the present article.

Souslin lines can be analyzed through their partition trees, which are normal Souslin trees, i.e., normal trees of height ω₁ without uncountable antichains. There is also an inverse procedure, which generates a Souslin line from a given Souslin tree. But these operations are highly non-canonical, meaning that one Souslin tree can give rise to up to 2^{ℵ₀} non-isomorphic Souslin lines and (in a restricted sense) vice versa. In this paper we give an example of a Souslin tree to which only one complete and dense linear order type is associated. Moreover, this Souslin tree and its associated Souslin algebra have interesting, strong homogeneity properties. And this is where we meet a second, algebraic motivation for this research.

Homogeneity of a mathematical object generally means that the object locally looks similar, i.e., that sufficiently small parts of the object are all structurally equal to each other. E.g., for a Boolean algebra B homogeneity is defined as the property that all non-trivial relative algebras B ↾ a for a ∈ B⁺ are isomorphic to each other. Of course, many popular structures satisfy stronger homogeneity conditions than ordinary homogeneity, as e.g., (ℝ, <), the linear order of the real numbers has for each pair of countable, dense subsets an auto-homeomorphism that maps one countable, dense subset onto the other.

∗ e-mail: gido.scharfenberger-fabian@uni-greifswald.de

“Is a (linearly) ordered set without jumps nor gaps and such that each disjoint set of its intervals (containing more than one point) is at most countable necessarily an (ordinary) linear continuum?”
The variant of homogeneity studied in the present text is \textit{chain homogeneity for Boolean algebras}, the property of a Boolean algebra $B$ that for any two maximal chains $K_0$ and $K_1$ of $B$ (i.e., subsets of $B$ which are maximal being linearly ordered by the partial order of $B$) there is an isomorphism (of linear orders) between $K_0$ and $K_1$. Note that such an isomorphism of linear orders does not necessarily extend to an automorphism of $B$.

In the case of $\sigma$-complete, atomless Boolean algebras it is easy to see that chain homogeneity implies the c.c.c., so the only possible order types for maximal chains of such algebras are the real unit interval $[0,1]$ or a Souslin line with endpoints. The only known, complete, atomless and chain homogeneous Boolean algebras are the Cohen-algebras and measure algebras, both having only separable maximal chains.

The present article gives the affirmative answer to the question of S. Koppelberg whether there are, under appropriate assumptions, complete and atomless chain homogeneous Boolean algebras with inseparable maximal chains. It is a completely revised version of Chapter 2 of the authors PhD thesis [20].

The organization of the text is quite conventional. In Section 2 we review the basic definitions concerning Souslin lines, trees and algebras and fix the notation. The following three sections introduce the technology we utilize in our Souslin tree constructions, which consists of relations and mappings on the tree and results on the extendibility thereof when a new level is added to the tree. In Section 6 the main results are stated, the proof of which, a somewhat involved Souslin tree construction, is prepared in Section 7 and carried out in Section 8. The paper closes with two sections containing some further remarks and open problems.

\section{Preliminaries}

We collect the basic definitions and results and fix the notation and for later reference. Most proofs are fairly straightforward and therefore omitted, yet references are given. The ambient theory is ZFC, Zermelo-Fraenkel set theory with choice; further hypotheses are explicitly stated.

\subsection{Boolean algebras}

Boolean algebras and the most elementary related notions, such as subalgebras, completeness, atoms will not be defined here, though we will mention some more or less subtle relationships between them. The main reference for looking up undefined notions is the first volume of the \textit{Handbook of Boolean Algebras} [12].

Let $B$ be a Boolean algebra. We call the binary relation on $B$

\[ x \leq_B y :\Longleftrightarrow xy = x \Longleftrightarrow x - y = 0 \]

the \textit{natural} or the \textit{canonical} order of $B$. A subset $X \subset B$, which is totally ordered by $\leq_B$ is a \textit{chain} of $B$. A chain $X$ is a \textit{maximal chain} of $B$ if $X$ is furthermore $\subset$-maximal amongst the chains of $B$. We call a Boolean algebra all of whose maximal chains are pairwise order isomorphic \textit{chain homogeneous}. We let

\[ \text{mc } B = \{ K \subset B \mid K \text{ is a maximal chain of } B \}. \]

An \textit{antichain} of $B$ is a family $X \subset B$ of pairwise disjoint (i.e., $xy = 0$) elements of $B$. We say that $B$ satisfies (or has) the \textit{countable chain condition} (or short c.c.c.) if every antichain of $B$ is at most countable.

\begin{lemma}
If $B$ is $\sigma$-complete and chain homogeneous, then $B$ satisfies the countable chain condition.
\end{lemma}

\begin{proof}
Given an uncountable antichain $X$ of $B$ it is easy to construct well-ordered chains with supremum 1, $K_0 = \{ x_0 <_B x_1 <_B \ldots \}$ of order type $\omega$ and $K_1 = \{ y_0 <_B y_1 <_B \ldots \}$ of order type $\omega_1$. Now for any pair $K, K' \in \text{mc } B$ with $K_0 \subset K$ and $K_1 \subset K'$ there should be an isomorphism $\varphi : K \rightarrow K'$ but then $n \mapsto \min\{ \alpha \mid \varphi(x_n) \leq_B y_n \}$ would give a cofinal countable sequence in $\omega_1$. \hfill \Box
\end{proof}

For any subset $X$ of a complete Boolean algebra $B$ we call

\[ \langle X \rangle^\text{cm} := \bigcap \{ A \mid X \subseteq A \text{ and } A \text{ is a complete subalgebra of } B \} \]

the \textit{subalgebra of $B$ that is completely generated by $X$} and maybe write $\langle X \rangle^\text{cm}_B$ if $B$ is not clear from the context. If the superscript "cm" is omitted, i.e., by $\langle X \rangle_B$, we denote the intersection of all subalgebras containing $X$ as a subset (and not only the complete subalgebras) and call $\langle X \rangle$ the subalgebra of $B$ that is \textit{finitarily generated by $X$}. 

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Note that given an arbitrary Boolean algebra \( A \) which is a subalgebra of a complete Boolean algebra \( B \), the (Dedekind) completion \( \overline{A} \), which is the unique complete Boolean algebra containing \( A \) as a dense subalgebra (cf. [12, Section 4.3]), is not necessarily isomorphic to \( (A)^{\mathbb{Z}} \). A regular subalgebra \( A \) of \( B \) is a subalgebra, such that for all \( M \subset A \) we have \( \sum B M = \sum A M \) if the latter sum exists. A well-known fact is that \( A \) is a regular subalgebra of \( B \) if and only if \( A \) is a dense subset of \( (A)^{\mathbb{Z}} \). In this case \( (A)^{\mathbb{Z}} \) is indeed isomorphic to \( \overline{A} \), the Dedekind completion of \( A \) (cf. [13, Proposition 4]).

A Boolean algebra \( B \) is \( \aleph_0 \)-distributive if for every family \((a_{ij})_{i \in \omega, j \in J}\) with an index set \( J \) of arbitrary size, the following equation holds:

\[
\prod_{i \in \omega} \sum_{j \in J} a_{ij} = \sum \left\{ \prod_{i \in \omega} a_{ij} \mid f \in J \right\}.
\]

We do not need more specific concepts of distributivity.

We frequently consider the regular open algebra \( ROX \) of some topological space \( X \). A subset \( U \) of \( X \) is regular open if the interior of the closure of \( U \) is equal to \( U \) and \( ROX \) is the set of all regular open subsets of \( X \). The regular open algebra of any space \( X \) is a complete Boolean algebra (cf. [12, Theorem 1.37]). Note that because of the different operations of the two algebras, \( ROX \) is not a subalgebra of \( \mathcal{P}(X) \), unless \( X \) is a discrete space. If the topology of \( X \) is not specified, then one of following applies:

- \( X \) is a linear order; then \( X \) carries the order topology,
- \( X = T \) is a tree; then \( T \) carries the partial order topology which is generated by all subsets of the form \( T(t) := \{s \in T \mid t \leq s \} \) for some \( t \in T \) (a definition of our tree notation follows below).

### 2.2 Normal trees

A tree is a well-founded partial order \((T, <_T)\) where the set of predecessors \( \{s \mid s <_T t\} \) is totally (and thus well-) ordered by \( <_T \) for all \( t \in T \). The elements of a tree are called nodes. The height of the node \( t \) in \( T \) is the order type of the set of its predecessors under the ordering of \( T \), i.e., \( \text{ht}_T(t) := \text{ot}(\{s \mid s <_T t\}, <_T) \). For an ordinal \( \alpha \) we let \( T_{\alpha} \) denote the set of nodes of \( T \) with height \( \alpha \). If \( \text{ht}_T(s) > \alpha \) we let \( s|\alpha \) be the unique predecessor of \( s \) in level \( T_{\alpha} \).

The height of a tree \( T \), \( \text{ht} T \), is the minimal ordinal \( \alpha \) such that \( T_{\alpha} \) is empty. An antichain is a set of pairwise incomparable nodes of \( T \), so for \( \alpha < \text{ht} T \), the level \( T_{\alpha} \) is an antichain of \( T \).

If \( c \) is a subset of the height of a tree \( T \), let \( T|c \) denote the tree that consists of all the nodes of \( T \) whose height lies in \( c \) ordered by the inherited tree order \( <_T \):

\[
T|c = \bigcup_{\alpha \in c} T_{\alpha}, \quad \text{ordered by: } s <_{T|c} t : \iff s <_T t.
\]

For a node \( t \in T \) we let \( \text{succ}(t) \) be the set of \( t \)’s immediate successors. By \( T(t) \) we denote the set of all successors of \( t \) in \( T \) (including \( t \)) which forms a tree with root \( t \) under the ordering inherited from \( T \). Nodes that do not have \( <_T \)-successors, are called leaves and \( T \) is called \( \kappa \)-branching, \( \kappa \) a cardinal, if all nodes of \( T \) have exactly \( \kappa \) immediate successors, except for the leaves.

A branch is a subset \( b \) of \( T \) that is linearly ordered by \( <_T \) and closed downwards, i.e., if \( s <_T t \in b \), then \( s \in b \). For \( \alpha < \text{ot}(b, <_T) \) we let \( b|\alpha \) be the unique element of \( T_{\alpha} \cap b \) and so extend the similar notation for nodes and their predecessors to branches and their elements. (For a normal tree this is just natural, since the nodes can be identified with the branches leading to them.) A branch is maximal in \( T \) if it is not properly contained in any other branch of \( T \).

Under the notion of a normal tree we subsume the following four conditions:

(i) there is a single minimal node called the root;
(ii) each node (except for the top level nodes) has at least two immediate successors;
(iii) each node has successors in every higher non-empty level;
(iv) branches of limit height have unique limits (if they are extended in the tree), i.e., if \( s,t \) are nodes of \( T \) of limit height whose sets of predecessors coincide, then \( s = t \).
Whenever we consider a mapping \( \varphi : T \to S \) between trees and call this mapping a tree homomorphism we mean that \( \varphi \) carries \( \leq_T \) to \( \leq_S \) and respects the height function: \( \text{ht}_S(\varphi(s)) = \text{ht}_T(s) \) for all \( s \in T \).

A tree \( T \) is said to be homogeneous, if for all pairs \( s, t \in T \) of the same height there is a tree isomorphism between \( T(s) \) and \( T(t) \), the trees of nodes in \( T \) above \( s \) and \( t \) respectively.

2.3 Souslin lines, trees and algebras

It is common set theoretic knowledge that the existence statements of Souslin algebras, lines and trees are mutually equivalent, yet independent of ZFC.

A Souslin line is a complete, dense linear order that satisfies the countable chain condition but is not separable, i.e., it has no countable dense subset, but all families of pairwise disjoint open intervals are countable.

In this definition the possibility of a Souslin line having a separable non-trivial interval is included. We will, however, never consider a Souslin line with a non-trivial separable interval.

It is a standard result that every Souslin line \( L \) has a dense subset of cardinality \( \aleph_1 \) and therefore the cardinality \( |L| = 2^{\aleph_0} \).

In general, a Souslin tree is a tree \( T \) of height \( \omega_1 \) such that every family of pairwise incomparable nodes and every branch of \( T \) are at most countable. We will only consider normal Souslin trees, where absence of uncountable antichains already implies that the tree has no cofinal branch.

Souslin lines and Souslin trees are tightly related by the following operations (also cf. [23] or [8, 9.14]). If we are given a Souslin tree \( T \) and a total order of its supporting set (or just total orderings of the sets of the form \( \text{succ}(t) \) for all \( t \in T \)) we can equip the set of all maximal branches of \( T \) with the lexicographical ordering. The resulting linear order is dense, c.c.c. and inseparable, i.e., its Dedekind completion is a Souslin line.

Now let \( L \) be a given Souslin line (without separable intervals). Then every normal partition tree of \( L \) is a Souslin tree. A partition tree of a linear order \( L \) is a tree whose nodes are open intervals of \( L \) such that the union of each level is dense in \( L \) and whose tree ordering is given by the inverted inclusion.

Anyway, in general these operations are highly non-canonical; one Souslin tree may be associated to many non-isomorphic Souslin lines and vice versa. To unify considerations one uses the notion of Souslin algebras.

A Souslin algebra is a complete, atomless, c.c.c. and \( \aleph_0 \)-distributive Boolean algebra.2 We will use the following non-standard denotations: A Souslin algebra that has a set of complete generators of size \( \aleph_1 \) will be called small; otherwise it is big. Small Souslin algebras always have \( 2^{\aleph_0} \) elements, while a famous result of Solovay states that the cardinality of (big) Souslin algebras is at most \( 2^{\aleph_1} \) (cf. [8, Theorem 30.20]).

We use letters \( A, B, C \) to denote Souslin algebras. In a Souslin algebra \( B \), a complete subalgebra \( A \) is atomic if and only if \( A \) is completely generated by some countable subset \( X \) of \( B \) (cf. [12, Proposition 14.8]). In the other extreme, since distributivity and the c.c.c. are handed down to complete subalgebras, the atomless, complete subalgebras of a Souslin algebra \( B \) are Souslin algebras as well.

We now review the connections between Souslin algebras and lines and between Souslin algebras and trees, beginning with the lines.

Lemma 2.2 ([12, Section 14, Exercise 5], for a proof cf. [20, 2.1.2-5])

(a) The maximal chains of a Souslin algebra \( B \) are Souslin lines with endpoints and without non-trivial separable intervals.

(b) Let \( L \) be a Souslin line without separable intervals. The regular open algebra of \( L \) is a Souslin algebra. Furthermore, if \( L \) has endpoints, then RO \( L \) has a maximal chain \( K \) which is isomorphic to \( L \) such that \( \langle K \rangle^{cm} = \text{RO} L \).

(c) Let \( B \) be a Souslin algebra and \( K \in \text{mc} B \). Then \( \langle K \rangle \), the subalgebra of \( B \) that is finitarily generated by \( K \), is a regular subalgebra of \( B \). In particular we have \( \langle K \rangle^{cm} \cong \langle K \rangle \cong \text{RO} K \).

It is well known that for a normal Souslin tree \( T \), its regular open algebra RO \( T \) is a Souslin algebra and \( T \) can be densely embedded in RO \( T \). On the other hand, in every Souslin algebra \( B \) that has a family of complete

\[ \text{2 In other contexts than ours, } \kappa \text{-Souslin algebras are defined as complete, atomless, } \kappa \text{-c.c. and } (\langle \kappa \rangle) \text{-distributive Boolean algebras and } \kappa \text{ can be any uncountable cardinal. In this notation the objects of our consideration are called } \aleph_1 \text{-Souslin algebras. But by Lemma (2.1) above, these higher Souslin algebras for } \kappa > \aleph_1 \text{ always have maximal chains of distinct order types and are therefore never chain homogeneous.} \]
generators of cardinality $\aleph_1$ (i.e., is a small Souslin algebra), there is a dense subset $T$ of $\mathbb{B} \setminus \{0\}$ such that $(T, \supseteq_B)$ is a normal Souslin tree (note that $<_T$ is $\supseteq_B$) that is regularly embedded in $\mathbb{B}$ and whose regular open algebra is therefore isomorphic to $\mathbb{B}$ (see e.g. [12, Theorem 14.20]). We will use the following convention established in [3]. Let $T$ be a subset of a Souslin algebra $\mathbb{B}$. Then $T$ is said to be $\textit{souslinise} \mathbb{B}$ or to be a $\textit{souslinisation}$ of $\mathbb{B}$ if $T$ is dense in $\mathbb{B}$ and becomes a Souslin tree under the reverse Boolean order of $\mathbb{B}$. Every level $T_\alpha$ of a souslinisation $T$ of $\mathbb{B}$ is a partition of unity in $\mathbb{B}$ and taking limits in $T$ is simply the evaluation of the corresponding infinite product in $\mathbb{B}$: If $t \in T_\alpha$ is a limit node, let $t_\gamma$ for $\gamma < \alpha$ be the unique $\supseteq_B$-predecessor of $t$ of height $\gamma$. Then $t = \prod\{t_\gamma \mid \gamma < \alpha\}$. The souslinisation is unique up to the elimination of a non-stationary set of levels: Given an isomorphism $\varphi$ between two Souslin algebras with souslinisations $T_1$ and $T_2$ respectively, there is a club $C \subseteq \omega_1$ such that $\varphi$ becomes an isomorphism of trees when restricted to $T_1 \upharpoonright C$. (For a proof of this well-known result, cf. [3, Lemma VIII.9] or [7, Lemma 25.6].)

A souslinisation $T$ of $\mathbb{B}$ provides a natural stratification of $\mathbb{B}$ by countably generated, complete and therefore atomic subalgebras. Fix a souslinisation $T$ of $\mathbb{B}$ and let for $\alpha < \omega_1$

$$\mathbb{B}_\alpha := \langle T_\alpha \rangle^{cm}.$$  

Note that for all $\alpha < \omega_1$ we have $\mathbb{B}_\alpha \cong \mathcal{P}(\omega)$ and $T|(\alpha + 1) \subset \mathbb{B}_\alpha$. Clearly the sequence of the $\mathbb{B}_\alpha$ is increasing. To show $\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}_\alpha$, pick $\alpha \in \mathbb{B} \setminus \{0\}$. There is a maximal pairwise disjoint subset $A$ of $T$ of elements $\leq_B \alpha$, so $\sum A = a$. By the countable chain condition $A$ must be countable and therefore a subset of $\mathbb{B}_\alpha$ for some $\alpha < \omega_1$. We finally note that the sequence of the $\mathbb{B}_\alpha$ is not continuous. For a countable limit ordinal $\alpha$ we have

$$\bigcup_{\gamma < \alpha} \mathbb{B}_\gamma \leq \left( \bigcup_{\gamma < \alpha} \mathbb{B}_\gamma \right)^{cm} = \mathbb{B}_\alpha.$$  

2.4 Jensen’s principle $\diamondsuit^+$

As for most Souslin tree constructions in the literature, we will assume diamond-principles.

Definition 2.3

(a) The sequence $(R_\alpha)_{\alpha < \omega_1}$ of sets $R_\alpha \subseteq \alpha$ for $\alpha < \omega_1$ is a $\diamondsuit$-sequence if for all $X \subseteq \omega_1$ there is some stationary set $s \subseteq \omega_1$ such that, for all $\alpha \in s$ we have $X \cap \alpha = R_\alpha$. The statement “There is a $\diamondsuit$-sequence.” will be denoted by $\diamondsuit$.

(b) The sequence $(S_\alpha)_{\alpha < \omega_1}$ of countable sets $S_\alpha \subseteq \mathcal{P}(\alpha)$ is a $\diamondsuit^+$-sequence if for all $X \subseteq \omega_1$ there is some club set $c \subseteq \omega_1$ s.t., for all $\alpha \in c$ we have $X \cap \alpha, c \cap \alpha \subseteq S_\alpha$. The statement “There is a $\diamondsuit^+$-sequence.” will be denoted by $\diamondsuit^+$.

Since $\diamondsuit^+ \Rightarrow \diamondsuit$ we only need to assume $\diamondsuit^+$ in our statements even though for the sake of convenience we will use both, a $\diamondsuit^+$-sequence and a $\diamondsuit$-sequence, at the same time.

3 Generating chains and wipers

For the representation of a given maximal chain $K$ of a Souslin algebra $\mathbb{B}$ we will use the approximations $K_\alpha = K \cap \mathbb{B}_\alpha$ with respect to some fixed souslinisation $T$ which are chains in $\mathbb{B}_\alpha$, yet not necessarily maximal ones. In the simple case of power set algebras (or their isomorphic copies, such as the $\mathbb{B}_\alpha$) we can give the following characterization for maximal chains.

Proposition 3.1 (Folklore, for a proof cf. [20, 2.1.10]) Let $X$ be a set. Given a $\subset$-chain $K$ of $\mathcal{P}(X)$, define the quasi-ordering $\leq_K$ on $X$ by

$$x \leq_K y :\iff (\forall u \in K)(y \in u \Rightarrow x \in u)$$

and let $\overline{K} := \{ \sum M, \prod M \mid M \subseteq K \}$ be the completion of the linear order $(K, \subset)$ in $\mathcal{P}(X)$. Then the following statements are equivalent:

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In particular, there is a bijective association between the total orders on \( X \) and the maximal chains of \( \mathcal{P}(X) \).

**Definition 3.2** A maximal chain \( K \) of \( \mathcal{B} \) that completely generates \( \mathcal{B} \) will be called a generating chain.

We stress that a chain \( K \) which satisfies \( \langle K \rangle^{\text{cm}} = \mathcal{B} \) will not be called generating unless it is maximal.

**Lemma 3.3** (cf. [20, 2.2.2]) Given a generating chain \( K \subseteq \mathcal{B} \) the set

\[
C := \{ \alpha < \omega_1 : K \cap \mathbb{B}^\alpha \in \text{mc}(\mathbb{B}^\alpha) \} = \{ \alpha \in \omega_1 : \langle K \cap \mathbb{B}^\alpha \rangle^{\text{cm}} = \mathbb{B}^\alpha \}
\]

is closed and unbounded in \( \omega_1 \).

The proof of the last lemma works by a straightforward catch-up argument using the c.c.c. and the distributive law valid in \( \mathcal{B} \). We now define the means of representation of generating chains of a Souslin algebra as a relation on its souslinisation. Requirement (ii) in this definition is of technical importance when it comes to the construction of isomorphisms between wipers.

**Definition 3.4**

(a) Let \( T \) be a normal tree with countable levels of arbitrary height \( \alpha \leq \omega_1 \) and let \( C \) be a subset of \( \alpha \). A wiper of total orders (or more convenient a wiper) on \( T|C \) is a family \( W = \langle <_{\gamma} \mid \gamma \in C \rangle \) of total orders \( <_{\gamma} \) on \( T_{\gamma} \) such that

\[
W \text{ respects the tree order of } T: \text{ for all } \beta, \gamma \in C, \beta < \gamma \text{ and } s, t \in T_{\beta}, s', t' \in T_{\gamma} \text{ we have }
\]

\[s < T s' \land t < T t' \land s <_{\beta} t \Rightarrow s' <_{\gamma} t',\]

(ii) if \( \beta, \gamma \in C, \beta < \gamma \), and \( s \in T_{\beta} \), then the set \( I_{s, \gamma} = \{ t \in T_{\gamma} \mid s <_{T} t \} \) of successors of \( s \) in level \( \gamma \) is ordered densely without endpoints by \( <_{\gamma} \).

(b) If \( T \) is a souslinisation of a Souslin algebra \( \mathcal{B} \) and \( W \) is a wiper on \( T \), we say that

\[K_{W} := \left\{ \sum M \mid (\exists \alpha \in \omega_1) M \subseteq T_{\alpha} \text{ and } M \text{ is an initial segment of } <_{\alpha} \right\}
\]

is the subset of \( \mathcal{B} \) which is represented by \( W \).

In order to illustrate this last definition, imagine the Souslin tree \( T|C \) printed on the windscreen of your car, the levels on concentric lines around some point at the bottom, where the root of \( T \) is situated and the windscreen wiper is fixed, and if \( s, t \in T|C \) are of height \( \alpha \), then \( s \) stands to the left of \( t \) if and only if \( s <_{\alpha} t \) according to \( <_{\alpha} \in W \). In this picture each position of a windscreen wiper, which has its axis fixed in the root of \( T \), corresponds to a member \( a \) of the generating chain \( K_{W} \) induced by the wiper \( W \). And this member \( a \in K_{W} \) can be calculated as the sum over the “wiped area”, i.e., the sum of all nodes which lie to the left of the windscreen wiper.

**Lemma 3.5** (cf. [20, 2.2.4]) Let \( \mathcal{B} \) be a Souslin algebra with souslinisation \( T \).

(a) If \( W \) is a wiper on \( T \), then \( K_{W} \) is a generating chain of \( \mathcal{B} \).

(b) Let \( K \) be a generating chain of \( \mathcal{B} \). Then there is a club \( C \subseteq \omega_1 \), s.t. there is a wiper \( W \) on \( T|C \), representing \( K \), i.e., with \( K = K_{W} \) as above.

By this correspondence we also see that a small Souslin algebra always has exactly \( 2^{\text{lh}} \) maximal chains.

### 4 Complete subalgebras and tree equivalence relations

In order to represent complete, atomless subalgebras of a Souslin algebra we will use certain equivalence relations defined on a souslinisation. The basic ideas for this method of representation can already be found in work of Jech and Jensen from around 1970. Here we will use the notions developed in [20]3 and [21] which

\[3 \text{ Note that t.e.r.s as defined here are called decent t.e.r.s in [20]. The property of decency of [20, Definition 1.1.3] refers to what is called honesty in the present text.}\]
are more appropriate for our purpose: rendering all Souslin subalgebras isomorphic to the superalgebra. In particular, we will take a topological point of view towards countable trees which simplifies matters significantly. (Cf. [21, Sections 2 and 5] for a more detailed account on this.)

**Definition 4.1** Let $T$ be a normal, $\aleph_0$-branching tree.

(a) We say that an equivalence relation $\equiv$ on $T$ is a tree equivalence relation if $\equiv$ respects levels (i.e., $s \equiv t$ implies $\text{ht } s = \text{ht } t$), is compatible with the tree order of $T$ (i.e., $\text{ht } s > \text{ht } r$ and $s < t \equiv u > r \implies s \equiv r$) and honest (i.e., for each triple $s, s', t \in T$ such that $s < s'$ and $s \equiv t$ we either find a node $t' > t$ such that $s' \equiv t'$ or there is no immediate successor of $t$ equivalent to $s'(|\text{ht}(s) + 1)$, the immediate successor of $s$ below $s'$) and the quotient $T/\equiv$ is a normal and $\aleph_0$-branching tree with the induced order.

(b) A t.e.r. $\equiv$ on $T$ is said to be
- *nice* if for all triples $s, s', t \in T$ with $s \equiv t$ and $s < s'$ there is some $t' > t$ equivalent to $s'$;
- *$\infty$-nice* if it is nice and for a node $s$ and every $\equiv$-class $t/\equiv$ the intersection $T(s) \cap (t/\equiv)$ is either empty or infinite. (Equivalently we could require only $\text{succ}(s) \cap (t/\equiv)$ to be infinite.)

A t.e.r. $\equiv$ on a souslinisation $T$ of $\mathbb{B}$ represents the atomless and complete subalgebra $A_\equiv = \langle \{ t/\equiv \mid t \in T \} \rangle^{cm}$ of $\mathbb{B} = \text{RO } T$. (We denote the $\equiv$-class of $t \in T$ by $t/\equiv$.) A subalgebra of $\mathbb{B}$ is called (indeed, if it is represented by an $\infty$-nice t.e.r. on some souslinisation of $\mathbb{B}$.) Since niceness descends to the restriction to a club set of levels, this is independent of the choice of the souslinisation.\(^4\)

The result [21, Lemma 3] states that for every Souslin algebra $\mathbb{B}$ with a souslinisation $T$ and every complete and atomless subalgebra $A$ of $\mathbb{B}$ there is a t.e.r. $\equiv$ that represents $A$ on $T|C$ for some club $C$ of $\omega_1$. If $A$ is nice or $\infty$-nice, then $\equiv$ can be chosen in a corresponding fashion.

A maximal branch $b$ of a tree $T$ of limit height is called *cofinal* if its order type with respect to $<_T$ coincides with the height of $T$. We let

$$[T] := \{ b \mid b \text{ is a cofinal branch of } T \}.$$

For $s \in T$ set $\hat{s} := \{ b \in [T] \mid s \in b \}$. We consider $[T]$ as a topological space with the base $\{ \hat{s} \mid s \in T \}$. Then for a normal tree $T$ of countable limit height $\alpha$ with only countable levels the space $[T]$ a perfect Polish space, i.e., a separable and completely metrizable space without isolated points.

Every monotone map $\varphi : T \to S$ of trees induces a continuous mapping $\overrightarrow{\varphi} : [T] \to [S]$, $b \mapsto \varphi^* b$. For every club $c \subseteq \text{ht } T$ the inclusion of $T|c$ in $T$ induces a natural homeomorphism between $[T|c]$ and $[T]$ (a cofinal branch of $T$ is uniquely determined by its intersection with $T|c$). If $T$ carries a t.e.r. $\equiv$, then an equivalence relation $\equiv$ on $[T]$ is induced by letting $b \equiv b'$ if and only if for all members $s \in b$ and $t \in b'$ of the same height we have $s \equiv t$. Denoting the induced relation by the same symbol as the t.e.r. is justified by the circumstance that any new limit level on top of $T$ corresponds to a subset of $[T]$.

We now come to state the key property and the associated lemma for the construction of a Souslin algebra with tightly controlled subalgebras.

**Definition 4.2** Let $X$ be a topological space and $\equiv$ an equivalence relation on $X$. We say that a subset $N \subseteq X$ is suitable for $\equiv$ if for every element $x \in X$ the intersection of its $\equiv$-class $x/\equiv$ with $N$ is either empty or dense in $x/\equiv$ (with respect to the subspace topology).

In order to add a new limit level to a countable tree $T$ we will choose a countable, dense subset $Q$ of $[T]$ and assign to every member $x \in Q$ a bounding limit node. A given t.e.r. $\equiv$ will be extended to the new level if and only if the set $Q$ is suitable for $\equiv$ in the sense defined above. Otherwise it would fail to be honest. We will always succeed in extending $\equiv$ if we are allowed to choose the members of $Q$ from a comeagre subset of $[T]$ that is suitable for $\equiv$. The Reduction Lemma guarantees the existence of such a set. The proof uses descriptive set theory and can be found in [21, Lemma 7].

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\(^4\) Nice subalgebras are much easier to handle than general subalgebras. But as chain homogeneous Souslin algebras are necessarily homogeneous (cf. Proposition 9.1) and every homogeneous, small Souslin algebra also has non-nice, atomless and complete subalgebras ([21, Theorems 3 and 4]) we also have to consider this general case in our construction.
Lemma 4.3 (Reduction Lemma) Let $T$ be a countable, normal and $\aleph_0$-branching tree of limit height with a t.e.r. $\equiv$. Let $M \subseteq [T]$ be comeagre. Then there is a comeagre subset $M' \subseteq M$ which is suitable for $\equiv$.

Extending a tree mapping $\varphi$ on $T$ to a new limit requires that the set $Q$ as above be closed under the application of $\varphi$. The following results establish that the Reduction Lemma can also be applied to achieve this.

Lemma 4.4 For a normal, countable tree $T$ of limit height equipped with a t.e.r. $\equiv$ consider the canonical tree epimorphism

$$\pi : T \to [T]/\equiv, \quad \pi(t) = t/\equiv.$$ 

Let the set $Y := \{y \in [T]/\equiv \mid (\exists x \in [T]) y = \pi''x\}$ be equipped with the subspace topology inherited from $[T]/\equiv$. Then the induced map

$$\pi : [T] \to Y, \quad x \mapsto \pi''x$$

is an open and surjective mapping and $Y$ is a comeagre subset of $[T]$.

Proof. Surjectivity of $\pi$ as in the statement of the lemma is trivial. In order to prove that $\pi$ is an open mapping, we show that the images of the basic open sets $t \in T$ are open in $Y$. So let $t$ be a node of $T$. We prove the equation $\pi''t = \bigcup \{\pi(r) \mid r \in \text{succ}(t)\} \cap Y$. Letting $\pi(x) = \pi''x$ with $t \in x \in [T]$ be a member of the left hand side we choose $r$ to be the unique immediate successor of $t$ in $x$ and see that this branch is also a member of the right hand side.

For the converse inclusion let $y \in Y$ be in $\pi''(t)$ for some immediate successor $r$ of $t$. We have to find a branch $x \in [T]$ through $t$ with $\pi(x) = y$. As $y$ is in $Y$ it has a $\pi$-pre-image $x^* \in [T]$. Letting $\gamma := \text{ht}(t)$, take $s$ to be the node of $x^*$ on level $\gamma$ and $s'$ its immediate successor in $x^*$. Then clearly $s \equiv t$ and $s' \equiv r$. Because of $\equiv$’s honesty, there are on every level $\delta > \gamma$ successors of $t$ which stand in relation $\equiv$ to a member of $x^*$. The set of all these nodes forms (together with $t$ and its predecessors) a subtree of $T$ which has a cofinal branch $x$ by König’s Lemma. Clearly we have $\pi(x) = \pi(x^*) = y$, as desired.

Finally, $Y$ is separable, metrizable and the image of the Polish space $[T]$ under the open mapping $\pi$, so $Y$ is Polish by a theorem of Sierpiński (cf. [11, 8.19]) and thus comeagre in $[T]/\equiv$ by Choquet’s Theorem (cf. [11, 8.17(ii)]) being a dense subset of $[T]/\equiv$ that contains a Polish space as a subset. □

Proposition 4.5 Let $X$ and $Y$ be Polish spaces and $h : X \to Y$ a continuous mapping such that $h''X$ is comeagre in $Y$ and the right-hand side restriction of $h$ to its image, i.e., $h : X \to h''X$ is an open mapping.

(a) If $D \subseteq Y$ is meagre, then $h^{-1}D$ is meagre as well.

(b) If $M \subseteq X$ is comeagre, then $h''M$ is comeagre as well.

Proof. The proof of part (a) is straight forward. For part (b) we can assume without loss of generality, that $M$ is a dense $G_\delta$-subset of $X$. Then clearly $h''M$ is an analytic subset of $Y$. So, by a theorem of Lusin and Sierpiński (cf. [8, Theorem 11.18.b] or [11, Theorem 21.6]), $h''M$ has the Baire property, i.e., there is an open subset $U$ of $Y$, such that the symmetric difference of $h''X$ and $U$ is meagre. In particular the sets $h''X \setminus U$ and $U \setminus h''X$ are meagre. If we can show that $U$ is dense in $Y$, then the proof is finished.

So assume to the contrary that there is an open subset $V$ of $Y$ which is disjoint from $U$. Then $V \cup h''X$ is meagre and so is $M \cap h^{-1}V = h^{-1}((V \cup h''M)$ by part (a). But since $M$ is comeagre and $h^{-1}V$ is open, their intersection cannot be meagre — contradiction! □

We combine the Reduction Lemma and the last two propositions and get the tool which is appropriate for our use in the tree construction.

Corollary 4.6 Let $T$ be a countable, normal and $\aleph_0$-branching tree of limit height $\alpha$. Let $H$ be a countable set of triples $h = (c_h, \equiv_h, \varphi_h)$ where $c_h$ is a club subset of $\alpha$, $\equiv_h$ is a t.e.r. on $T|c_h$ (with associated canonical mapping $\pi_h$) and $\varphi_h : (T|c_h)/\equiv_h \to T|c_h$ is an isomorphism. Let furthermore $I$ be a countable set of pairs $i = (c_i, \equiv_i)$ such that $c_i$ is club in $\alpha$ and $\equiv_i$ is an $\infty$-nice t.e.r. on $T|c_i$. If $M$ is a comeagre subset of $[T]$, then there is a comeagre subset $N$ of $M$ such that

(1) for all $h \in H$, $N$ is suitable for $\equiv_h$ and $(\overline{\varphi}_h \circ \pi_h)^{''}N = N$;

(2) $N$ is suitable for all $\equiv_i$ where $i \in I$. 
Proof idea. As images and pre-images of comeagre sets under the mappings from $H$ are all comeagre, repeated intersections deliver $N$. 

Our last technical lemma on t.e.r.s concerns nested $\omega$-nice t.e.r.s and has a straightforward proof. If $\equiv_0$ refines $\equiv_1$ as an equivalence relation on some set $M$, we define the equivalence relation $\equiv_{0/1}$ on $M/\equiv_1$ by

$$(x/\equiv_1) \equiv_{0/1} (y/\equiv_1) :\iff x \equiv_0 y$$

for $x, y \in M$.

Lemma 4.7 Assume we are given two $\omega$-nice t.e.r.'s $\equiv_0$ and $\equiv_1$ on $T$ which is normal, $\aleph_0$-branching and of countable limit height, such that $\equiv_0$ is refined by $\equiv_1$ and the t.e.r. $\equiv_{0/1}$ on the quotient tree $T/\equiv_1$ induced by $\equiv_0$ is $\omega$-nice. Let furthermore $M$ be a comeagre subset of $[T]$ that is suitable for both $\equiv_0$ and $\equiv_1$. Then $M/\equiv_1$ is suitable for $\equiv_{0/1}$.

5 Variations of Kurepa’s Lemma

A striking property of normal trees is formulated in Kurepa’s Isomorphism Lemma for normal trees, cf. [16, p. 102]. The result and its proof are well-known, but since we will use some variations of the argument later on, we also sketch the proof.

Lemma 5.1 (Kurepa’s Isomorphism Lemma) Let $S, T$ be two normal $\kappa$-branching trees, $\kappa \leq \omega$, of the same limit height $\alpha < \omega_1$ with countable levels only. Then $S \cong T$.

Proof. Let the height of $S$ and $T$ be a countable limit ordinal $\alpha$. Choose countable and dense sets $X \subset [S]$ and $Y \subset [T]$ and enumerate them by $(x_i \mid i \in \omega)$ and $(y_i \mid i \in \omega)$ respectively.

We give a back-and-forth-construction of a bijective mapping $f : X \rightarrow Y$ which lifts to a tree isomorphism $\varphi : S \rightarrow T$. Define $f(x_0) = y_0$ and $\varphi(x_0|\gamma) = y_0|\gamma$ for all $\gamma < \alpha$.

As the forth-argument is completely analogous, we only explain how the back-step works. Let $i$ be minimal such that $f^{-1}(y_i)$ has not yet been fixed, and pick the minimal $\gamma$ such that $\varphi^{-1}(y_i|\gamma)$ has not yet been defined.

This $\gamma$ is a successor ordinal, say $\gamma = \delta + 1$, because $S$ and $T$ are assumed to be normal trees.

Now choose an immediate successor $s$ of $\varphi^{-1}(y_i|\delta)$ such that $\varphi(s)$ has not yet been defined. Such a node $s$ exists by the choice of $\gamma$. Finally let $j$ be minimal such that $s \in x_j$. Then $f(x_j) = y_i$ and $s, t \in [T]$. This is consistent with the choices met for $f$ and $\varphi$ so far.

After every step of the construction $\varphi$ is a partial isomorphism between $S$ and $T$. By the choice of $X$ and $Y$, the union of these partial isomorphisms is bijective, so in the end $\varphi : S \rightarrow T$ is an isomorphism.

The argument also shows that for any $\gamma < \alpha$ and every isomorphism $\varphi_0$ between $S\upharpoonright(\gamma + 1)$ and $T\upharpoonright(\gamma + 1)$ there is an extension $\varphi_0 \subset \varphi : S \cong T$.

The denseness requirement in Definition 3.4 of wipers enables us to combine the argument from the proof of Kurepa’s Isomorphism Lemma with that of Cantor’s for the $\aleph_0$-categoricity of the dense linear orders in order to construct isomorphisms between wipers.

Lemma 5.2 Let $T$ and $S$ be two countable, $\aleph_0$-branching, and normal trees of the same height $\alpha < \omega_1$ and let $W_0 = \{\gamma \mid \gamma < \alpha\}$ be a wiper on $T$ and $W_1 = \{\gamma \mid \gamma < \alpha\}$ a wiper on $S$. Furthermore let $\beta < \alpha$ and $\varphi'$ be an isomorphism from $T\upharpoonright(\beta + 1)$ onto $S\upharpoonright(\beta + 1)$ such that for all $\gamma \leq \beta$ and $s, t \in T_\gamma$ we have

$$s \prec_\gamma t \iff \varphi'(s) \prec_\gamma \varphi'(t).$$

Then there is an isomorphism $\varphi$ between $T$ and $S$ extending $\varphi'$, such that for all $\gamma < \alpha$ and $s, t \in T_\gamma$ we have

$$s \prec_\gamma t \iff \varphi(s) \prec_\gamma \varphi(t).$$

Proof. We refer to the proof of Kurepa’s Isomorphism Lemma 5.1 and describe the only manipulation: When it comes to choosing of $\varphi^{-1}(s)$, this choice has to respect the wipers, which is always possible by the denseness requirement.
Our last Kurepa-style lemma states that isomorphisms between quotient trees by $\infty$-nice t.e.r.s can be lifted to isomorphisms of the underlying trees. In order to prove it, we first formulate a regularity property of subalgebras induced by an $\infty$-nice t.e.r.

**Proposition 5.3** Let $T$ be a normal, $\aleph_0$-branching tree of countable limit height carrying an $\infty$-nice t.e.r. $\equiv$. Then for every cofinal branch $b$ of $T$ the set $\{ s \in T \mid (\exists r \in b) \ s \equiv r \}$ is an $\aleph_0$-branching, normal tree.

**Proof.** Unique limits are inherited from $T$, infinite branching follows from the $\infty$-condition, successors in every higher level follow from niceness.

**Proposition 5.4** Let $T$ be a normal, $\aleph_0$-branching tree of limit height $\alpha < \omega_1$ and let $\equiv$ and $\sim$ be $\infty$-nice t.e.r.'s on $T$. Let $\gamma$ be equal to $\alpha$ or else be a successor ordinal below $\alpha$. Let $\varphi'$ be an isomorphism between $(T\upharpoonright\gamma)/\equiv$ and $(T\upharpoonright\gamma)/\sim$. Then there is an automorphism $\varphi$ of $T$ that carries $\equiv$ to $\sim$, s.t. the induced map on $T/\equiv$ is an isomorphism onto $T/\sim$ that extends $\varphi'$.

**Proof.** First use Lemma 5.1 to extend $\varphi'$ to an isomorphism between $T/\equiv$ and $T/\sim$ in the case that $\gamma < \alpha$. So we can assume $\alpha = \gamma$.

We now give a back-and-forth-argument which lifts the isomorphism $\varphi'$ to an automorphism $\varphi$ of $T$. Enumerate $T$ in order type $\omega$ by $s_0, s_1, \ldots$. For the induction step in the forth-direction let $i$ be the minimal index $i$ for which $\varphi(s_i)$ has not yet been defined. Our choice for $\varphi(s_i)$ has to respect the tree order and the equivalence relations. So let $s$ be the predecessor of maximal height whose image under $\varphi$ is already determined by the choices met earlier in the construction, where $s = s_i$ is allowed. In the case that $s <_T s_i$, we want to pick a node $t \in \varphi'(s_i/\equiv)$ above $\varphi(s)$ which exists by the niceness of $\sim$. We furthermore require that $t$ has not yet been assigned as some $\varphi(s_j)$. This choice is possible due to the $\infty$-part in the $\infty$-niceness of $\sim$.

For the back-step we replace in the above argument all $\varphi$ and $\varphi'$ by $\varphi^{-1}$ and $\varphi'^{-1}$.

### 6 Statement of the main results

**Definition 6.1** Let $\mathbb{B}$ be a Souslin algebra. By continuous, $\infty$-nice chain of subalgebras of $\mathbb{B}$ we denote a sequence $(\mathbb{A}_\nu \mid \nu < \beta)$ of $\infty$-nice subalgebras of $\mathbb{B}$ such that

(i) $\mathbb{A}_\mu$ is an $\infty$-nice-subalgebra of $\mathbb{A}_\nu$, for $\mu < \nu < \beta$,

(ii) for limit $\alpha < \beta$ the algebra $\mathbb{A}_\alpha$ is completely generated by the preceeding members $\mathbb{A}_\nu$, $\nu < \alpha$ of the chain, i.e.,

$$\mathbb{A}_\alpha = \left( \bigcup_{\nu < \alpha} \mathbb{A}_\nu \right)^{\text{cm}}.$$  

**Theorem 6.2** Assume $\diamondsuit^+$. Then there is a small, chain homogeneous Souslin algebra $\mathbb{B}$. Furthermore, the construction can be refined to achieve the following additional homogeneity properties.

(a) For every isomorphism $\varphi' : \mathbb{A}_\lambda \cong \mathbb{A}'_\lambda$ of $\infty$-nice subalgebras of $\mathbb{B}$ there is an automorphism $\varphi$ of $\mathbb{B}$ extending $\varphi'$.

(b) There is a continuous, $\infty$-nice chain $(\mathbb{A}_i \mid i < \omega_1)$ of subalgebras of $\mathbb{B}$ such that:

(i) $\bigcup_{i < \omega_1} \mathbb{A}_i = \mathbb{B}$;

(ii) for every continuous, $\infty$-nice chain $(\mathbb{C}_\nu | \nu < \lambda)$ of subalgebras of $\mathbb{B}$ $(\lambda$ a countable limit) there is an isomorphism $\varphi : \mathbb{A}_\lambda \rightarrow \mathbb{C}_\lambda$ such for all $\nu < \lambda$ the restriction of $\varphi$ to $\mathbb{A}_\nu$ is an isomorphism onto $\mathbb{C}_\nu$.

The additional homogeneity properties stated in (a) and (b) of Theorem 6.2 have as primary goal their use in the construction which proves the following theorem. It answers a question of S. Todorčević, who asked on the occasion of a talk the author gave on chain homogeneous Souslin algebras at Toposym X in 2006 in Prague, whether there are also big, chain homogeneous Souslin algebras.

**Theorem 6.3** Assume $\diamondsuit^+$ and let $\mathbb{B}$ be a chain homogeneous Souslin algebra as in Theorem 6.2. Then there is a big, chain homogeneous Souslin algebra $\mathbb{B}^*$ all of whose subalgebras which are small Souslin algebras are isomorphic to $\mathbb{B}$.

As the proof of this latter theorem is rather short, we give it right away.
Proof. (Theorem 6.3 from Theorem 6.2) We realise the big and chain homogeneous Souslin algebra $B^*$ as the union of an increasing chain of small, chain homogeneous Souslin algebras $(B_{α} \ | \ α < ω_2)$. All the small algebras $B_{α}$ on this chain are isomorphic to the initial algebra $B_0 := B$, which we take from Theorem 6.2. The properties of $B^*$ as stated in the theorem then follow from general principles. As all the $B_{α}$ satisfy the c.c.c. and $B^*$ is defined as their direct limit, $B^*$ also satisfies the c.c.c. and is therefore complete. Distributivity and chain homogeneity are also inherited.

As above, denote by $(B_{-i} \ | \ i < ω_1)$ the continuously increasing chain of $ω_1$-nice subalgebras of $B$. For the inductive successor step, if $B_{α}$ is given, choose any isomorphism $φ_α : B_{α} \cong A_0$. Next choose a Souslin algebra $B_{α+1}$ extending $B_{α}$, such that there is an extension $φ_{α+1}$ of $φ_α$ and witnessing $B_{α+1} \cong B$. This extension exists by (a) of Theorem 6.2.

If $λ < ω_2$ is of countable cofinality, we choose a normal sequence $(i_ν \ | \ ν < μ)$ with $\sup_{ν < μ} i_ν = λ$ for some countable limit ordinal $μ$. Inductively construct an increasing chain $(ψ_ν \ | \ ν < μ)$ of isomorphisms $ψ_ν : A_ν \cong B_i ν$. Then choose $B_λ$ as a super-algebra of

$$∪_{ν < μ} B_i = \bigcup_{α < λ} B_{α}$$

isomorphic to $A_μ$ by some extension $ψ_μ$ of $∪_{ν < μ} ψ_ν$.

If $λ$ has uncountable cofinality we simply let $B_λ := ∪_{α < λ} B_{α}$, which in this case is the direct limit. Then we see that $B_λ \cong B_0$ for $λ < ω_2$ by choosing a cofinal sequence $(i_ν \ | \ ν < ω_1)$ and recursively choosing a chain of isomorphisms $ψ_ν : A_ν \cong B_i ν$. This goes through limit stages $ν < ω_1$ by property (b.ii) of Theorem 6.2. Iterating up to $α = ω_2$ finishes the construction of $B^*$.

7 Final preparations

In our Souslin tree construction we want to build additional objects (mappings, tree isomorphisms) on club sets of levels of the tree $T$ to be constructed, that relate given objects (e.g., pairs of wipers or t.e.r.s on $T|C'$) to each other. During a relevant construction step, initial segments of some of the given objects are proposed by a $♡^+$-sequence and in order to extend an additional object we need some pointer to indicate the ordinal stage up to which the recursive construction of the additional object has reached so far. This is the function of $ε$ of the following definition.

**Definition 7.1** ($♡^+$-machinery) Fix a $♡^+$-sequence $(S_α)_{α < ω_1}$.

(a) Let $C_0 := \{α < ω_1 \ | \ ωα = α \neq 0\}$ be the set of positive, countable fixed points of the left-multiplication with $ω$.

(b) Let $α \in C_0$. For $x \in P(α)$ set

$$c(x) = \{γ \in C_0 ∩ α \ | \ x ∩ ω(γ + 1) \setminus ωγ \neq ∅\}.$$  

(This will the set of levels of the tree, to which $x$ refers.)

(c) The set of relevant guesses for stage $α$, $G_α$, is the set of pairs $(x, d) \in S_α × S_α$ such that $c(x)$, $d$ and $c(x) ∩ d$ are club in $α$ and for $γ \in d$ the sets $x ∩ γ$ and $d ∩ γ$ are in $S_γ$.

(d) For $(x, d) \in G_α$ let

$$ε_{x,d} := \{γ \in c(x) ∩ d \ | \ \bigcup (γ ∩ c(x) ∩ d) = γ\}$$

be the Cantor-Bendixson derivative of $c(x) ∩ d$, i.e., the set of its limit points.

(e) Let $ε_{x,d} := ∪ ε_{x,d}$. (Note that $ε_{x,d} = 0$ if $\text{ot}(c(x) ∩ d) = ω$.)

Now if for example $(x, d) \in G_α$ and $x$ codes a pair of wipers on $T|c(x)$, we know that up to stage $ε_{x,d}$ our recursive construction of the additional object—here: an isomorphism between the wipers given by $x$—has been invoked and therefore up to this stage this isomorphism has yet been constructed.
Recall Lemma 5.4, which states that an isomorphism between two quotient trees $T/\equiv_0$ and $T/\equiv_1$ can be lifted to an automorphism of $T$ if the t.e.r.s are $\infty$-nice. In order to get hold of isomorphisms between quotient trees we introduce the notion of an engaging relation. Consider two $\infty$-nice t.e.r.s $\equiv_0$ and $\equiv_1$ on $T|c$, with $c \subset \alpha$ club, and an isomorphism $\varphi : (T|c)/\equiv_0 \to (T|c)/\equiv_1$ between the quotient trees. Then $\varphi$ naturally induces a relation $\Phi$ on $T|c$, that consists of the pairs $s, t \in T|c$ with $\varphi(s/\equiv_0) = t/\equiv_1$. The properties of such a relation $\Phi$ are captured in the following definition.

**Definition 7.2** We say that a relation $\Phi$ on a tree $T$ satisfying points (1–4) below is *engaging*.

1. There is a set $c_\Phi$ such that for all $s, t \in T$ we have $s\Phi t$ only if $\text{ht}(s) = \text{ht}(t) \in c_\Phi$,
2. the left-induced relation $\Phi^0 := \{(s, s') \in T \times T \mid (\exists t) s\Phi t \land s'\Phi t\}$ is an $\infty$-nice t.e.r. on $T|c_\Phi$,
3. the right-induced relation $\Phi^1 := \{(t, t') \in T \times T \mid (\exists s) s\Phi t \land s\Phi t'\}$ also is an $\infty$-nice t.e.r. on $c_\Phi$,
4. $\Phi$ induces an isomorphism $\varphi_\Phi$ between $(T|c_\Phi)/\Phi^0$ and $(T|c_\Phi)/\Phi^1$ via $\varphi_\Phi(s/\Phi^0) = t/\Phi^1$ for any $t$ with $s\Phi t$.

It is clear that the relation $\Phi$ considered above the definition is indeed engaging with $c_\Phi = c$, and $\Phi^0$ is $\equiv_0$ while $\Phi^1$ is $\equiv_1$.

Finally we define a certain sequence of club sets of $\omega_1$ which will act as the family of supporting sets of levels for the t.e.r.s that represent the increasing sequence $(\mathcal{A}_i \mid i < \omega_1)$ of $\infty$-nice subalgebras. Recall the definition of $C_0$, the set of infinite fixed points of the left hand ordinal multiplication with $\omega$ in $\omega_1$:

$$C_0 := \{\alpha < \omega_1 \mid \alpha \neq 0, \omega\alpha = \alpha\}.$$  

Inductively define $C_{i+1}$ for $i < \omega_1$ to be the Cantor-Bendixson-derivative of $C_i$, and for limit ordinals $i$ let $C_i$ be the intersection of the $C_j$ defined so far:

$$C_{i+1} := \{\alpha \in C_i \mid \sup(C_i \cap \alpha) = \alpha\} \quad \text{and} \quad C_i := \bigcap_{j<i} C_j \quad \text{for limit } i.$$

We list some properties of the sequence $(C_i)_{i<\omega_1}$ used in the construction below.

(a) all the $C_i$ are club in $\omega_1$,
(b) the sequence is continuously decreasing and has an empty intersection, hence there is for every $\alpha \in C_0$ a unique $i = i(\alpha)$ with $\alpha \in C_i \setminus C_{i+1}$,
(c) every $\alpha \in C_0$ has a direct predecessor in $C_{i(\alpha)}$, call it $\alpha^-$,
(d) for limit $i < \omega_1$, the minimum of $C_i$ is the supremum of the minima of the $C_j$ for $j < i$.

### 8 Proof of Theorem 6.2

Before we plunge into the details of the construction, we give a brief sketch of the main steps to be carried out at one stage of the recursion. We use the principle $\diamondsuit^+$ during the construction in order to get hold of all kinds of objects we are interested in, which are

- pairs of wipers that represent pairs of generating chains;
- t.e.r.s that represent Souslin subalgebras;
- engaging relations that represent isomorphism between $\infty$-nice subalgebras.

All of these are relations on $T|C$ for some club $C$ of $\omega_1$. In a relevant stage of the construction, say stage $\alpha \in C_0$, we have the tree $T|\alpha$ at hand and carry out five steps:

1. We collect all pairs $(x, d) \in G_\alpha$ that correspond to any of the above items. If necessary, we extend the mappings associated to these objects which were chosen in earlier stages of the construction by virtue of Kurepa’s Lemma and its variations up to below $T_\alpha$. 

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We now begin and construct a tree-order $<_T$ level by level on the set $\omega_1$ such that the resulting tree $T$ will be a normal and $\aleph_0$-branching Souslin tree. The Souslin algebra to be constructed will be the regular open algebra of $T = (\omega_1, <_T)$.

Let $0$ be the root of $T$ and in every successor step fix $\aleph_0$ distinct direct successors for each maximal node in such a way that for every $\alpha \in \omega_1 \setminus \{0\}$ the level $T_\alpha$ consists of the next $\omega$ many ordinals not yet used in the construction. So we have $T_0 = \omega_1 \setminus \{0\}$, $T_0 = \omega n \setminus \omega(n-1)$ for all natural numbers $n \geq 2$ and finally $T|\alpha = T_\alpha$ for all infinite, countable ordinals $\alpha$.

Now let $\alpha < \omega_1$ be a limit ordinal. By the inductive assumption we have so far constructed a normal tree order $(T | \alpha, <_T)$ on the supporting set $\omega_\alpha$.

If $\alpha < \omega_\alpha$ we simply choose a countable dense subset $Q_\alpha$ of $T|\alpha$ and embed $Q_\alpha$ onto $\omega(\alpha + 1) \setminus \omega_\alpha$, i.e., we choose a bijection between $Q_\alpha$ and $\omega(\alpha + 1) \setminus \omega_\alpha$ and extend $<_T$ on $\omega(\alpha + 1)$ in the obvious way.

If $\alpha = \omega_\alpha$, we want to choose a countable dense subset $Q_\alpha$ of $T|\alpha$, too, but this time our set also has to seal certain maximal antichains of $T|\alpha$ and extend enough tree isomorphisms. Recall the definitions from the last section concerning the use of the $\Diamond^+$-sequence $(G_\alpha, c(x), \varepsilon_{x,d}, \text{etc.)}$.

From now on, if $(x,d) \in G_\alpha$ is fixed, we write $c$ for $c(x)$. Denote by $E_\alpha$ the subset of $S_\alpha \times S_\alpha$ that consists of all pairs $(x,d) \in G_\alpha$ such that $g'_d x$ and $g''_d x$ are wipers $W_0 = \langle \times \gamma; \gamma \in c(x) \rangle$ and $W_1 = \langle \times \gamma; \gamma \in c(x) \rangle$ on $T|c(x)$ respectively. So $(x,d) \in E_\alpha$ if $x$ codes a pair of wipers on $T|c$ and is guessed correctly by the $\Diamond^+$-sequence along with a club $d$ on the members of $d$ itself.

Now we take care of the homogeneity property stated in part (a) of Theorem 6.2. We define

$E'_\alpha := \{(x,d) \in G_\alpha \mid x \text{ codes an engaging relation } \Phi_\alpha \text{ on } T|c(x)\}.$

Every $(x,d) \in E'_\alpha$ induces a pair of $\aleph_1$-nice t.e.r.s $\equiv_0^\alpha$ and $\equiv_1^\alpha$ and an isomorphism $\varphi'_\alpha$ between $(T|c)/\equiv_0^\alpha$ and $(T|c)/\equiv_1^\alpha$.

The guesses of the $\Diamond^+$-sequence for t.e.r.s are collected in the set $F_\alpha$:

$F_\alpha := \{(x,d) \in G_\alpha \mid x \text{ codes a t.e.r. on } c(x)\}.$

Here we code e.g. with respect to the map $g_0$. For $(x,d) \in F_\alpha$ let $\equiv_x$ denote the t.e.r. which is coded by $x$. It induces the $\aleph_0$-branching, normal tree $(T|c)/\equiv_x$.

Keep in mind that $\alpha \cap d$ is unbounded in $\alpha$ if $(x,d) \in E_\alpha$ or $(x,d) \in F_\alpha$ by the definition of the relevant stages set $G_\alpha$.

The inductive hypothesis is that $T|\alpha$ is a countable and $\aleph_0$-branching tree of height $\alpha$ and

(1) for every pair $(x,d) \in E_\alpha$ there is a $c$-chain $\langle \varphi_{x,d}; \gamma \in c(x) \rangle$ and each $\varphi_{x,d}$ is a tree automorphism of $T|c \cap d \cap \gamma + 1$ that was fixed in induction step $\gamma \in c(x)$ and which carries $W_0 x, d \cap c \cap d \cap \gamma + 1 = \langle \times \delta; \delta \in c \cap d \cap \gamma + 1 \rangle$ to $W_1 x, d \cap c \cap d \cap \gamma + 1 = \langle \times \delta; \delta \in c \cap d \cap \gamma + 1 \rangle$. In short: $\varphi_{x,d}$ is an isomorphism of wipers between $W_0 x, d \cap c \cap d \cap \gamma + 1$ and $W_1 x, d \cap c \cap d \cap \gamma + 1$ for all $\gamma \in c(x)$; for every pair $(x,d) \in E_\alpha$ there is an automorphism of $T|c \cap d \cap \varepsilon_{x,d}$ which extends $\varphi'_\alpha$ as defined above.
(2) For every pair \((x, d) \in F\alpha\) there is a \(\subseteq\)-chain \((\psi_{x \cap \gamma, d \cap \gamma} : \gamma \in \varepsilon_{x, d})\) of tree isomorphisms

\[
\psi_{x \cap \gamma, d \cap \gamma} : T|(c \cap d \cap \gamma + 1)/\varepsilon_{x, d} \rightarrow T|(c \cap d \cap \gamma + 1)
\]

each of them fixed in induction step \(\gamma \in \varepsilon_{x, d}\).

(3) For every \(i < \omega_1\) with \(\alpha \cap C_i \neq \emptyset\) there is an \(\infty\)-nice t.e.r. \(\equiv_i\) on \(T|(\alpha \cap C_i)\) such that

- for all two such \(j < i\) the relation \(\equiv_{j/i}\) is an \(\infty\)-nice t.e.r. on \((T|\alpha \cap C_i)/\equiv\) and
- for each such limit \(i\) the t.e.r. \(\equiv_i\) is given by \(s \equiv_i t \iff (\forall j < i) s \equiv_j t\), for all \(s, t \in T\), with \(\gamma \in C_i\).

Recall that \(c = c(x)\) and \(\varepsilon_{x, d}\) is the set of limit points of \(c \cap d\). For those \((x, d)\) in \(E\alpha, E\alpha\) or in \(F\alpha\) but with \(\varepsilon_{x, d} < \alpha\) we need to choose extensions for the maps granted by the inductive hypothesis as follows.

Fix \((x, d) \in E\alpha\). If \(\varepsilon_{x, d} = \alpha\) let

\[
\varphi_{x, d} : \bigcup_{\gamma \in \varepsilon_{x, d}} \varphi_{x \cap \gamma, d \cap \gamma}.
\]

Otherwise extend \(\varphi_{x \cap \varepsilon_{x, d}, d \cap \varepsilon_{x, d}}\) by Proposition 5.2 to some isomorphism \(\varphi_{x, d}\) between \(W^0_{x, d}|c \cap d\) and \(W^1_{x, d}|c \cap d\).

For \((x, d) \in E\alpha\) use Lemma 5.4 to lift \(\varphi_{x, d}\) to an automorphism \(\varphi_{x, d}\) of \(T|c(x) \cap d\) carrying \(\Phi^0\) to \(\Phi^1\) which extends \(\varphi_{x, d}\).

Let \((x, d) \in F\alpha\) and set \(\psi_{x, d} := \bigcup_{\gamma} \psi_{x \cap \gamma, d \cap \gamma}\) if \(\varepsilon_{x, d} = \alpha\). Otherwise extend the union of the chain by Kurepa’s Lemma 5.1 to some isomorphism

\[
\psi_{x, d} : T|(c \cap d)/\equiv_x \rightarrow T|c \cap d.
\]

### 8.2 Arranging the scene for the \(\infty\)-nice chain

We now describe how to embed the increasing \(\omega_1\)-sequence of \(\infty\)-nice subalgebras \(\mathbb{A}_i\) in \(\mathbb{B}\). Recall the definition of the decreasing sequence \((C_i \mid i \in \omega_1)\) of club subsets of \(\omega_1\) as well as the derived definitions of \(i(\alpha)\) and \(\alpha^-\) from the final part of Section 7. The \(\infty\)-nice t.e.r. \(\equiv_i\) representing \(\mathbb{A}_i\) will be defined on \(T|C_i \cup \{0\}\) in the course of the construction of \(T\). The main requirements to meet are (as stated in clause (3) of the inductive hypothesis):

- \(\equiv_i\) is \(\infty\)-nice for all \(i < \omega_1\),
- for \(i > j\) the restriction of \(\equiv_j\) to \(T|C_i\) is refined by \(\equiv_i\) in a way such that the induced t.e.r. \(\equiv_{j/i}\) on the normal Souslin tree \((T|C_i)/\equiv_i\) is \(\infty\)-nice,
- for limit \(i < \omega_1\) we want to have \(\mathbb{A}_i = \bigcup_{j<i} \mathbb{A}_j\)\(^{\text{cm}}\), so \(\equiv_i\) shall be the conjunction of the \(\equiv_{j}\) for \(j < i\) in this case:

\[
s \equiv_i t \iff (\forall j < i) s \equiv_j t.
\]

For any \(i\), on level \(T_0 = \{\text{root}\}\) the relation \(\equiv_i\) is of course trivial. On level \(T_{\min C_i}\) we define \(\equiv_i\) to be the identity, i.e., \(s \equiv_i t\) if and only if \(s = t\) for \(s, t \in T_{\min C_i}\). This is a minor violation of the \(\infty\)-niceness requirement we posed on \(\equiv_i\). But this is easily remedied by deleting \(\min C_i\) from the club set \(C_i\). On the other hand, by this convention we directly see that in the end \(\bigcup \mathbb{A}_i\) will be a dense subset of \(\mathbb{B}\), because \(\{\min C_i \mid i < \omega_1\}\) is unbounded in \(\omega_1\).

In level \(\alpha \in C_0\) we have that for all \(j < i := i(\alpha)\) the set \(C_j \cap \alpha\) is club in \(\alpha\) and the t.e.r. \(\equiv_j\) on \(T|(C_j \cap \alpha)\) has by normality of the quotient tree a unique t.e.r.-extension to \(T_\alpha\). So, to satisfy the niceness condition for the t.e.r.s \(\equiv_j\) with \(j < i(\alpha)\), level \(T_\alpha\) has to be chosen suitably with the aid of the Reduction Lemma 4.3.

Let now \(\alpha \in C_0\) be such that \(i = i(\alpha)\) is a limit ordinal. For \(j < i\) define \(\equiv_j\) on \([T|\alpha]\) as above and let for \(x, y \in [T|\alpha]\)

\[
x \equiv_i y \iff (\forall j < i) x \equiv_j y.
\]

This is the coarsest possibility to extend \(\equiv_i\) to \(T_\alpha\) and the only one by our requirement that \(\bigcup_{j<i} \mathbb{A}_j\) completely generate \(\mathbb{A}_i\).
We now show that the relation $\equiv_i$ on $[T|\alpha]$ is induced by a (single) $\infty$-nice t.e.r. $\simeq_i$. Therefore the Reduction Lemma 4.3 can be applied to $\equiv_i$. The t.e.r. $\simeq_i$ will be defined as a diagonal along the $\equiv_j$ for $j < i$.

Fix $\delta_0$ in $C_0$ between $\alpha^-$ and $\alpha$, so $j_0 := i(\delta_0) < i$. Then define recursively $j_\nu := i(\delta_\nu)$ and

$$
\delta_{\nu+1} := \min(C_{j_\nu+1} \setminus \delta_{\nu}) = \min(C_{j_\nu+1} \setminus \delta_0) \quad \text{and} \\
\delta_\mu := \sup\{\delta_\nu \mid \nu < \mu\} \quad \text{for limit } \mu \text{ as long as } i(\delta_\nu) < i \text{ for all } \nu < \mu.
$$

The recursion stops at the limit ordinal $\mu$ with $i(\delta_\mu) = i$. Then the $i(\nu)$ are the ordinals from $j_0$ up to $i$, and the final $\delta_\mu$ is just $\alpha$.

The set of the $\delta_\nu$ joined with $\{0\}$ is the club set of $\alpha$ on which we now define the t.e.r. $\simeq_i$. For $s, t \in T_{\delta_\nu}$ define

$$
s \simeq t \iff (\forall \beta < \delta_0)(i(\beta) \leq i \Rightarrow s|\beta \equiv_{i(\beta)} t|\beta) \quad \text{and} \quad s \equiv_{j_\nu} t
$$

To see that $\simeq$ is an $\infty$-nice t.e.r., fix $s \simeq t$ on level $\delta_\xi$ and consider $s' > s$, where $s' \in T_{\delta_\nu}$. In the successor case, letting $\nu = \nu^- + 1$, i.e., $j_\nu = j_{\nu^-} + 1$, this follows from the definition of $\equiv_{j_{\nu^-}}$, which has only classes that are dense subsets of the $\equiv_{j_{\nu^-}}$-classes. To argue for niceness in the limit case we refer to the inductive hypothesis, which implies that $T_{\delta_\nu}$ has been chosen in a way that guarantees that the $\equiv_{j_{\nu^-}}$-classes lie densely in the $\equiv_{j_{\nu}}$-classes for $j < j_{\nu}$. The $\infty$-part of $\infty$-niceness is trivially satisfied on limit stages when satisfied everywhere below. Since the $j_\nu$ are cofinal in $i$, the $\infty$-nice t.e.r. $\simeq_i$ induces $\equiv_i$ on $[T|\alpha]$.

However, for successor $i(\alpha)$ the definition of $\equiv_{i(\alpha)}$ on $T_\alpha$ involves the choice of the new level $T_\alpha$. So we continue by giving the rules for the choice of the new level $T_\alpha$.

### 8.3 Suitably reducing the focus

Now Lemma 4.4 comes into play. For $(x, d) \in F_\alpha$ let

$$
\pi_{x, d} : T|c \cap d \longrightarrow (T|c \cap d)/\equiv_x, \quad s \longmapsto s/\equiv_x
$$

be the canonical mapping associated to $\equiv_x$ and define

$$
\rho_{x, d} := \psi_{x, d} \circ \pi_{x, d} : T|c \cap d \longrightarrow T|c \cap d.
$$

Since $\psi_{x, d}$ is an isomorphism, the induced continuous map

$$
\overline{\pi}_{x, d} : [T|\alpha] \longrightarrow [T|\alpha], \quad b \longmapsto \{s \mid (\exists t \in b \cap (T|c \cap d)) s \not\leq_T \rho_{x, d}(t)\}
$$

has a comeagre image in $[T|\alpha]$ and is an open mapping when the range is restricted to $\overline{\pi}_{x, d}''[T|\alpha]$.

The $\bigwedge$-sequence $(R_\nu)_{\nu < \omega_1}$ proposes candidates for maximal antichains in the usual way. If $R_\alpha$ is a maximal antichain of $T|\alpha$, then we have to ensure that each member of $T_\alpha$ is a $<_T$-successor of some element of $R_\alpha$. That means $Q_\alpha$ has to be a subset of

$$
M_\alpha := \{x \in [T|\alpha] : (\exists \gamma < \alpha) x|\gamma \in R_\alpha\}
$$

which is itself an open dense subset of $[T|\alpha]$, because $R_\alpha$ is a maximal antichain. If $R_\alpha$ is not a maximal antichain in $T|\alpha$, we simply set $M_\alpha = [T|\alpha]$. We apply the Corollary 4.6 on reduction to the sets $M := M_\alpha$ and

$$
H := \{\overline{\pi}_{x, d} \mid (x, d) \in E_\alpha \cup E'_\alpha\} \cup \{\overline{\pi}_{x, d} \mid (x, d) \in F_\alpha\}
$$

(where $\overline{\pi}_{x, d}$ is the autohomeomorphism on $[T|\alpha]$ induced by $\varphi_{x, d}$ for $(x, d) \in E_\alpha \cup E'_\alpha$) and

$$
I := \begin{cases} 
\{\equiv_j \mid j < i(\alpha)\}, & \text{if } i(\alpha) \text{ is a successor ordinal} \\
\{\equiv_j \mid j \leq i(\alpha)\}, & \text{if } i(\alpha) \text{ is a limit ordinal.}
\end{cases}
$$
Since the $E_0 \cup E'_0$-part of $H$ consists of homeomorphisms and the $F_\alpha$-part is subject to Lemma 4.4, the hypotheses of Corollary 4.6 are satisfied. The result is a comeagre subset $N_\alpha$ of $M_\alpha$ such that

1. for all $(x, d) \in E_0 \cup E'_0$ we have $\varphi_{x,d}'' N_\alpha = N_\alpha$;
2. for each $(x, d) \in F_\alpha$, $N_\alpha$ is suitable for $\equiv_x$ and $\varphi_{x,d}'' N_\alpha = N_\alpha$;
3. $N_\alpha$ is suitable for all $\equiv_\alpha$.

The inductive claim is that there is a choice for $Q_\alpha$, i.e., a countable and dense subset of $M_\alpha$ that shares properties (1–3) of $N_\alpha$ stated above. For then, $T | \alpha$ can be extended by a level $T_\alpha$ of nodes corresponding to the branches in $Q_\alpha$, in a way that guarantees that the relevant t.e.r.s and tree homomorphisms extend to the new level.

The construction of $Q_\alpha$ is now easy: We first pick a right-inverse $\sigma_{x,d} : N_\alpha \rightarrow N_\alpha$ of $\varphi_{x,d}$ for every pair $(x, d) \in F_\alpha$ and then choose a countable and dense subset $Z_\alpha$ of $N_\alpha$. Then extend $Z_\alpha$ in countably many steps in order to close it under the application of the mappings

\[ \varphi_{x,d} \text{ and } \varphi_{x,d}^{-1} \text{ for } (x, d) \in E_0 \cup E'_0 \]

and

\[ \varphi_{x,d} \text{ and } \sigma_{x,d} \text{ for } (x, d) \in F_\alpha \]

and render the set suitable for all t.e.r.s under consideration. Finally, to define $T_\alpha$, choose a bijection $j : Q_\alpha \rightarrow \omega(\alpha + 1) \setminus \omega \alpha$ and extend $<_T$ in the obvious way.

### 8.4 Extending $\equiv_i$ for successor $i$

Next we define $\equiv_i$ on $T_\alpha$ in the case that $i = j + 1$. (Recall that for limit $i = i(\alpha)$ the relation $\equiv_i$ on $T_\alpha$ is ruled its behavior below.) Letting $\alpha^\ast$ be the maximum of $\alpha \cap C_i$, $t \in T_\alpha$ and $r := t | \alpha$, we easily see that we have an infinite set $\check{t} = t \cap \alpha^\ast$ (here we view $T_\alpha$ as a dense subset of $[T | \alpha]$ which we partition into $\aleph_0$ infinite sets $P(t \equiv_j, r, n)$, each of them dense in $\check{t} \cap t \equiv_j$. Note that the definition of $P$ has to be independent of the choice of $t$. Finally we define $\equiv_i$ on $T_\alpha$ by letting $s \equiv_i t$ if and only if

\[ s | \alpha^- \equiv_i t | \alpha^- \text{ and ex. } n \in \omega \text{ s.t. } s \in P(s \equiv_j, s | \alpha^-, n) \text{ and } t \in P(t \equiv_j, t | \alpha^-, n). \]

Then by construction, $\equiv_i$ is $\alpha$-nice as well as $\equiv_{j+1}$.

### 8.5 Check of isomorphisms

To finish the induction step, we check the various homogeneity properties claimed in the theorem. We start and show that our choice of $T_\alpha$ admits extensions of the tree isomorphisms $\psi_{x,d}$ for all $(x, d) \in F_\alpha$. Fix $(x, d) \in F_\alpha$ and set $c' = (c(x) \cap d) \cup \{ \alpha \}$. First of all extend $\equiv_x$ to $T_\alpha$ by letting $s \equiv_x t$ if and only if $s | \gamma \equiv_x t | \gamma$ for all $\gamma \in c(x)$. (This is the only t.e.r. extending $\equiv_x$ on $T_\alpha$ because of the normality requirement in the definition of t.e.r.)

We can identify $T_\alpha$ with $Q_\alpha$ via the bijection $j$. This in mind, we show that the unique extension $j$ of $\psi_{x,d}$ to $(T | c')/\equiv_x$ is an isomorphism onto $T | c'$: for $s \in T_\alpha$ define $\psi'(s/\equiv_x)$ as the unique $t \in T_\alpha$ with $\psi_{x,d}(s | \gamma/\equiv_x) = t | \gamma$ for all $\gamma \in c' \setminus \{ \alpha \}$ and let $\psi := \psi_{x,d} \cup \psi'$. We check the soundness of this definition. For $s \in T_\alpha$ there is

$\psi(s/\equiv_x) = j \circ \varphi_{x,d} \circ j^{-1}(s)$. The normality of $T/\equiv_x$ guarantees that the definition of $\psi(s/\equiv_x)$ is independent of the choice of the representative $s$. So it is clear that $\psi$ is indeed a monotone tree mapping.

Now, by the definition of $\equiv_x$ on $T_\alpha$, if $s \neq t$ for $s, t \in T_\alpha$, then there is some $\gamma \in c' \setminus \{ \alpha \}$ with $s | \gamma \neq_t t | \gamma$ and so $\psi(s/\equiv_x) \neq \psi(t/\equiv_x)$. On the other hand, for every $s \in T_\alpha$ we have

$\psi(j \circ \sigma_{x,d} \circ j^{-1}(s)/\equiv_x) = s$.

So $\psi$ is bijective also on the top level of $(T | c')/\equiv_x$ and therefore a tree isomorphism.

For reference in later induction steps we denote this tree isomorphism $\psi$ by $\psi_{x,d}$. This completes the inductive construction.
8.6 Chain homogeneity

It remains to show that the above construction yields a Souslin tree \( T \) whose regular open algebra \( B \) is chain homogeneous. We omit the standard argument proving that \( T \) is Souslin.

If we are given two generating chains \( K, K' \) of \( B \) let \( X \subset \omega_1 \) be a code with respect to the map \( g \) (fixed at the very beginning of the construction) for a pair of wipers on the club \( C \subset \{ \alpha : \omega \alpha = \alpha \} \) representing \( K \) and \( K' \).

Let \( D \) be a club set in \( \omega_1 \) associated to \( X \) by \( \Diamond^* \). Then for each \( \alpha \in E = \{ \gamma \in C \cap D \mid \gamma = \bigcup(\gamma \cap C \cap D) \} \) the construction of the tree gives us a tree automorphism \( \varphi_{X \cap \alpha, D \cap \alpha} \) of \( T | (C \cap D \cap \alpha) \), and the union of that increasing chain,

\[
\varphi = \bigcup_{\alpha \in E} \varphi_{X \cap \alpha, D \cap \alpha},
\]

extends to an automorphism \( \varphi \) of \( B \) that carries \( K \) to \( K' \).

Now let \( A \) be a complete and atomless subalgebra of \( B \) represented by the decent t.e.r. \( \equiv \) on \( T | C \). Let \( D \) be a club such that the \( \Diamond^* \)-sequence guesses \( D \cap \alpha \) and \( X \cap \alpha \) for a code \( X \) for \( \equiv \) for all \( \alpha \in D \). Then the construction yields a tree isomorphism \( \psi : (T | C \cap D) / \equiv \rightarrow T | C \cap D \) and therefore a Boolean isomorphism between \( B \) and \( A \).

So let \( K \) be a maximal chain of \( B \), \( A = (K)^{cm} \) and \( \psi : B \cong A \). Now \( K \) is isomorphic to the generating chain \( \psi^{-1}K \) and is thus of that unique order type.

8.7 Verification of the properties of the \( \infty \)-nice chain

Let \( i \) be a countable limit ordinal. Then, by our definition of \( \equiv \) (in the limit case), for every node \( s \in T | C_i \), its \( \equiv \)-class is just the intersection over the family \( (s/\equiv_j)_{j < i} \). This shows that \( A_i = \bigcup_{j < i} A_j \)^{cm}, i.e., continuity of the chain.

For the proof of (b.ii) let a continuous, \( \infty \)-nice chain \( (C_\nu \mid \nu \leq \lambda) \), \( \lambda \) a countable limit ordinal, be given. By chain homogeneity of \( B \) we can choose an isomorphism \( \varphi_0 : A_0 \rightarrow C_0 \), and then inductively extend a given isomorphism \( \varphi_\nu : A_\nu \rightarrow C_\nu \) to \( \varphi_{\nu+1} : A_{\nu+1} \cong C_{\nu+1} \rightarrow C_{\nu+1} \) and extend \( \psi ': \varphi_{\nu+1}^{-1} \circ \varphi_\nu \) by virtue of condition (a) to an automorphism \( \psi \) of \( A_{\nu+1} \). Then set \( \varphi_{\nu+1} := \varphi_{\nu+1}^0 \circ \psi \).

At a limit stage \( \mu \), the argument given above shows that

\[
\left( \bigcup_{\nu < \mu} C_\nu \right)^{cm} = C_\mu = \left( \bigcup_{\nu < \mu} \varphi_\nu'' A_\nu \right)^{cm}.
\]

So there is a unique isomorphism \( \varphi_\mu \) extending \( \bigcup_{\nu < \mu} \varphi_\nu \) with domain \( A_\mu = \left( \bigcup_{\nu < \mu} A_\nu \right)^{cm} \) and range \( C_\mu = \left( \bigcup_{\nu < \mu} C_\nu \right)^{cm} \).

This finishes the proof of Theorem 6.2.

\[ \square \]

9 Features and remarks

9.1 Features of chain homogeneous Souslin algebras

It is not hard to see that by omission of the additional steps (for properties (a) and (b) of Theorem 6.2) in the construction we arrive at a small Souslin algebra which is plainly chain homogeneous. The following two propositions state properties shared by all chain homogeneous Souslin algebras.

Proposition 9.1 A small and chain homogeneous Souslin algebra \( B \) is homogeneous in the following strong sense. For every pair \( A_0, A_1 \) of Souslin subalgebras and \( x \in A_0 \) and \( y \in A_1 \) where \( 0 < B x, y < B 1 \), there are \( 2^{|A_0|} \) distinct isomorphisms \( \varphi : A_0 \rightarrow A_1 \) with \( \varphi(x) = y \).

Proof. We first argue that \( B \) is weakly homogeneous. \( B \) is assumed to be small. So for every pair \( a, b \) of non-zero elements of \( B \) there are generating chains \( K, K' \) of \( B \) with \( a \in K \) and \( b \in K' \). Each isomorphism \( \varphi \) between \( K \) and \( K' \) satisfies \( \varphi(a) \cdot b \neq 0 \) and extends to an automorphism of \( B \).
By a theorem proved independently by Koppelberg and by Solovay (cf. [22, Theorem 18.4.1]), every complete and weakly homogeneous Boolean algebra is a power of a homogeneous factor, which in our case is isomorphic to $\mathbb{B}$, because of the c.c.c. satisfied by $\mathbb{B}$.

As there are $2^{\aleph_1}$ distinct generating chains, chain homogeneity implies that $\mathbb{B}$ has $2^{\aleph_1}$ automorphisms. The rest is then routine.

**Proposition 9.2**

(a) If $\mathcal{A}$ is an atomless, complete subalgebra of the Souslin algebra $\mathbb{B}$, and $\mathcal{A}$ and $\mathbb{B}$ are isomorphic, then no atomless, complete subalgebra of $\mathbb{B}$ can be independent over $\mathcal{A}$.

(b) A chain homogeneous Souslin algebra has no independent pair of atomless and complete subalgebras.

**Proof.** The proof of (a) is straight forward applying the fact that a Souslin algebra cannot have an independent pair of mutually isomorphic Souslin subalgebras. For the proof of (b) note that here we have not assumed $\mathbb{B}$ to be small. In the case of a small Souslin algebra, an application of part (a) suffices to prove (b). But also a big Souslin algebra $\mathbb{B}$ has only maximal chains, that completely generate subalgebras which are small Souslin algebras. So assume that $\mathcal{A}_0$, $\mathcal{A}_1$ form an independent pair of complete and atomless subalgebras of $\mathbb{B}$. Then there are maximal chains $K_0 \subseteq \mathcal{A}_0$ and $K_1 \subseteq \mathcal{A}_1$, which are isomorphic to each other. The isomorphism between the chains extends to an isomorphism between the two subalgebras that are completely generated by the chains:

$$C_0 := \langle K_0 \rangle^{cm} \subseteq \mathcal{A}_0 \quad \text{and} \quad C_1 := \langle K_1 \rangle^{cm}.$$ 

But then again, we have an isomorphic pair of subalgebras $C_0, C_1$ of $\mathbb{B}$ that cannot be independent unless $\mathbb{B}$ fails to satisfy the countable chain condition.

**9.2 The number of non-isomorphic maximal chains**

Classical questions about Souslin trees often circulated around the number isomorphism types or the number of automorphisms. In this respect, it is worth noting that a Souslin algebra which is not chain homogeneous has at least $2^{\aleph_0}$ non-isomorphic maximal chains and that the regular open algebras of most Souslin trees considered in set theoretic literature, such as strongly homogeneous or free Souslin trees have $2^{\aleph_1}$ non-isomorphic maximal (even generating) chains.

**9.3 On the hypothesis**

Concerning the hypothesis $\langle \diamondsuit \rangle^{+}$ met for our constructions, we remark the following. In [2, Section 6] and [1, Section 4] a model of $\text{ZFC} + \neg \text{SH}$ is constructed, in which there is no homogeneous Souslin tree. Yet a (chain) homogeneous Souslin algebra always has a homogeneous souslinisation. So we have found a ZFC-model where Souslin’s hypothesis fails, but which has no chain homogeneous Souslin algebras.

It is open whether we could make do with strictly less than $\diamondsuit^{+}$, e.g., if the assumption of $\diamondsuit$ is sufficient to guarantee the existence of a chain homogeneous Souslin algebra, cf. the open problems section below.

However, we can show that the existence of chain homogeneous Souslin algebras with properties as strong as stated in Theorem 6.2 cannot be obtained from $\diamondsuit$ alone.

**Proposition 9.3** Let $\mathbb{B}^*$ be the big and chain homogeneous algebra of Theorem 6.3. Forcing with $\mathbb{B}^*$ turns any ground model Souslin tree $T$, which is regularly embeddable into $\mathbb{B}^*$, into a Kurepa tree, i.e., $T$ has at least $\aleph_2$ cofinal branches in the generic extension.

**Proof.** Let $G \subseteq \mathbb{B}_{\omega_2}$ be a $V$-generic filter, and let $\mathcal{A} := \langle T \rangle^{cm}$. For $\alpha < \omega_2$ fix in $V$ an isomorphism $\varphi_\alpha : \mathbb{B}_\alpha \cong \mathcal{A}$ and in $V[G]$ define $G_\alpha := \mathbb{B}_\alpha \cap G$.

Staying in $V[G]$, for $\alpha < \omega_2$, the set

$$b_\alpha := \{ t \in T \mid (\exists g \in G_\alpha) \varphi_\alpha(g) \leq_A t \}$$

is an $\omega_1$-branch of $T$, and we have $V[G_\alpha] = V[b_\alpha]$ for all $\alpha < \omega_2$. Finally, if $\gamma < \beta < \omega_2$, then $b_\beta \in V[b_\beta] \setminus V[b_\gamma]$, because the map $\varphi_\gamma$ is in $V$. In particular we have $b_\beta \neq b_\gamma$. 

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Since the Iteration Theorem 6.3 only assumes the strong properties of the Souslin algebra of Theorem 6.2 we have the following result on the hypotheses used.

**Corollary 9.4** Assume that there is a Mahlo cardinal. Then there is a model of $\mathsf{ZFC} + \diamond$ in which there is no Souslin algebra with the properties stated in Theorem 6.2.

**Proof.** In [10], Jensen considered the generic Kurepa hypothesis $\mathsf{GKH} := “There is a c.c.c. forcing that forces the Kurepa hypothesis $\mathsf{KH}$ in the generic extension”. He shows that if $\kappa$ is a Mahlo cardinal, then $\mathsf{GKH}$ is false in the Levy-style generic extension collapsing $\kappa$ to become $\aleph_2$. This partial order always forces $\diamond$ (cf. [14, Exercises VIII.J.5/6]). Finally, $\mathbb{B}_{\omega_2}$ is c.c.c. and forces, by the last Proposition, $\mathsf{KH}$. $\Box$

As far as we know, the proofs of Theorems 6.2 and 6.3 give the first construction of a big Souslin algebra assuming the principle $\diamond^+$ only. Jensen’s constructions use $\diamond$ and $\square$, cf. [3] or [10]. In [6, Section 5] Jech gives a forcing that adjoins a so-called Souslin mess. This is a partial order of partial functions generalising the notion of a Souslin tree. The regular open algebra of a large enough Souslin mess is a big Souslin algebra. Laver has constructed a Souslin mess, only using $\diamond$ and Silver’s principle $W$. ($W$ is a strengthening of Kurepa’s Hypothesis $\mathsf{KH}$; cf. [7, (24.16)].)

### 10 Open problems

#### 10.1 Weaker hypotheses?

*Assuming $\diamond$, is there a chain homogeneous Souslin algebra?*

This is only a sample question, as $\diamond$ could be replaced by any of its variants known to be strictly weaker than $\diamond^+$.

In [3, Section V.3], a Souslin tree with at least $\aleph_2$ automorphisms (a property in common with chain homogeneous Souslin algebras) is constructed under $\diamond^+$, and on p. 51 the authors remark: *It is doubtful whether this is provable from $\diamond$. Though it is not hard to see that the regular open algebra of this tree is homogeneous for generating chains, we do not know whether it is fully chain homogeneous. As argued above, it is impossible to carry out a construction with a result as strong as the one given here under $\diamond$ alone.*

#### 10.2 On uniqueness for Souslin lines

*Is there a model of $\mathsf{ZFC}$ with exactly one Souslin line (up to isomorphism, without separable intervals)?*

Such a property could be paraphrased as “Souslin’s hypothesis minus one” as there would be essentially exactly one counter-example to $\mathsf{SH}$. In such a model the associated Boolean algebra would have to be chain homogeneous. Anyway, also if we refocus the question on Souslin trees or Souslin algebras, then the respective algebra must admit isomorphisms to all of its Souslin subalgebras, a feature in common with the chain homogeneous Souslin algebras.

In attempts to construct such a model by variations of the well-known iterated forcing techniques of Solovay-Tennenbaum (variation for a strongly homogeneous Souslin tree by Larson, cf. [17, Section 4]) or that of Jensen (variation for a free Souslin tree by Abraham, cf. [1, Section 4]), which leave one a priori specified Souslin tree intact, the main obstacle is to ensure that enough of the homogeneity properties of the preserved Souslin algebra remain valid in the final model.

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