Vacuum driven accelerated expansion*

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Abstract

It has been shown that an improved estimation of quantum vacuum energy can yield not only acceptable but also experimentally sensible results. The very idea consists in a straightforward extraction of gravitationally interacting part of the full quantum vacuum energy by means of gauge transformations. The implementation of the idea has been performed in the formalism of effective action, in the language of Schwinger’s proper time and the Seeley–DeWitt heat kernel expansion, in the background of the Friedmann–Robertson–Walker geometry.

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I. INTRODUCTION

There are three famous problems in modern physics which can, in principle, be treated as independent ones or, just the opposite, (all or any two of them) as mutually related:

- accelerated expansion of the Universe [1] (proven by astrophysicists);
- cosmological constant (very small though non-vanishing) [2, 3];
- quantum vacuum energy (theoretically — very huge, experimentally — very small).

The accelerated expansion of the Universe is by now rather a well-established by astronomical observations fact, in particular, Supernovae Ia data [1]. That mysterious phenomenon still awaits an explanation. There are dozens of candidates for the solution of the problem. One of the possible solutions, and the simplest one, is the introduction of the cosmological constant \( \Lambda \).

Another one (or the same) is quantum vacuum energy. These solutions are, in a sense, traditional, and they seem to be the most natural and simple ones. Their “only drawback” is the fact that, it seems, they do not work well.

The cosmological constant \( \Lambda \) troubles physicists from nearly the very beginning of the existence of general relativity. There are also dozens of candidates for the solution of this problem (they are even catalogued [4]). Unfortunately, explanation of the accelerated expansion by the vanishingly small value of the cosmological constant shifts only the problem rather than solves it. Traditional approach to the issue of the cosmological constant \( \Lambda \) uses quantum vacuum energy as an adequate quantity. But still the mechanism, being very appealing, does not work, it seems, properly. It appears that the traditionally calculated, Casimir-like value of quantum vacuum energy is absolutely too big than accepted and two orders of orders too big than required! Entirely independently of the problem of the accelerated expansion and of the problem of the cosmological constant such a drastically huge value of the vacuum energy density is a big problem in itself.

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This means that one should explain the quantum vacuum energy problem independently whether it could or should be later related to the accelerated expansion and the cosmological constant or not. There is quite a numerous collection of potential explanations of the above issues in literature. It is not our intention to list or review them but it seems to be useful for our further purposes to mention at least some.

One of the early standard ideas was to lower the ultraviolet cutoff scale, using e.g. supersymmetry arguments. It helps a bit, but only a little bit if we want to be in accordance with current experimental facts (roughly, it cuts the order by two [3]).

As another rather a radical solution one should mention the idea assuming that quantum vacuum energy does not, for some unknown reasons, interact with gravity at all. Such an assumption is an arbitrary and an absolutely ad hoc idea which contradicts generally accepted lore. Neither standard general relativity, nor standard quantum field theory presumes such exceptions.

It appears, and the aim of our paper is to show it, that, in principle, it is possible to sensible estimate the value of quantum vacuum energy, and to obtain an acceptable result. Moreover, the result is not only reasonable in itself (the quantum vacuum energy is not huge) but also experimentally. Our approach does not appeal to any more or less clever or exotic assumption. Just the opposite, our idea is supposed to adhere to standard quantum field theory formalism as closely as possible.

II. QUANTUM VACUUM ENERGY

The well-known, standard (but not properly working) approach to estimation of the quantum vacuum energy density $\varrho_vac$ calculates the Casimir-like energy density for the whole Universe. The result of such a calculation for a single boson scalar mode (in mass units) is [2]

$$\varrho_vac = \frac{1}{2} \int_0^\Lambda \frac{4\pi}{(2\pi\hbar)^3} \sqrt{(mc)^2 + k^2} \ k^2 \text{d}k,$$

where $m$ is the mass of the particle. For a large ultraviolet (UV) momentum cutoff $\Lambda$

$$\varrho_vac \approx \frac{1}{(4\pi)^2} \frac{\Lambda^4}{\hbar^2 c}.$$

Setting $\Lambda = \Lambda_p$, where $\Lambda_p$ is the Planck momentum,

$$\Lambda_p = \sqrt{\frac{\hbar c^3}{G}} \approx 6.5 \text{ kg/m/s},$$

and $G$ is the newtonian gravitational constant, we obtain

$$\varrho_vac^p \approx \frac{c^5}{(4\pi)^2 \hbar G^2} \approx 3.4 \times 10^{94} \text{ kg/m}^3,$$

an enormously huge value, whereas the experimentally estimated value is of the order of $10^{-26} \text{ kg/m}^3$ ($\approx 0.8 \times 10^{-26} \text{ kg/m}^3$), i.e. 120 orders less! Lowering $\Lambda$ to, say, the supersymmetry scale $\Lambda_{\text{susy}} \sim 1 \text{ TeV/c}$, only slightly improves the situation, namely, $\varrho_vac^{\text{susy}} \approx 1.5 \times 10^{30} \text{ kg/m}^3$, but it does not change the general impression that the whole calculation is principally erroneous. Therefore, as a desperate response to this dramatic situation, an ad hoc idea has emerged that gravitational field is, for some unknown reasons, insensitive to quantum vacuum fluctuations, yielding $\varrho_vac = 0$ exactly.

The both extreme approaches, the ordinary Casimir-like calculation and the insensitiveness idea actually yield an absurd and an incorrect result, which should become obvious for the following reasons. First of all, we claim that the ordinary purely Casimir-like calculation of quantum vacuum energy should not give any measurable contribution to gravitational (or any other) field “by construction”. Actually, in the language of Feynman diagrams, Casimir-like calulus gives rise to contributions coming from closed loops (see, Fig. 1) without any external lines. They do not influence gravitational field because this possibility has not been taken into account — there are no “classical” external lines establishing contact of the internal “matter” loops with the outer gravitational interactions.

Fig.1: A single closed loop representing the Casimir-like contribution of a free matter field.

But one can easily correct the result performing improved calculations. Namely, one should consider contributions coming from closed “matter” loops with classical external gravitational lines attached. Such an approach is not only in full accordance with paradigm of standard quantum field theory, without any additional assumptions, but also, moreover, it could bring us to reasonable results.
III. ESTIMATION

Full contribution coming from a single (non-self-interacting) mode is included in the effective action of the form

\[ S_{\text{eff}} = \pm \hbar \frac{1}{2} \log \det \mathcal{D}, \]  

(5)

where \( \mathcal{D} \) is a non-negative second-order differential operator, and the upper (plus) sign corresponds to bosonic statistics whereas the lower (minus) one corresponds to fermionic statistics, respectively. For simplicity, we work in the euclidean framework throughout. Since (5) is UV divergent we should regularize it. The most convenient and systematic way to control infinities in Eq. (5) is to use Schwinger’s proper-time method. In this approach we can formally rewrite (5) as

\[ S_{\text{eff}} = \pm \hbar \frac{1}{2} \log \det \mathcal{D} = \pm \frac{\hbar}{2} \int_0^\infty \frac{ds}{s} \text{Tr} \ e^{-s \mathcal{D}}, \]  

(6)

and the UV regularized version as

\[ S_{\text{eff}}^\varepsilon = \pm \frac{\hbar}{2} \int_\varepsilon^\infty \frac{ds}{s} \text{Tr} \ e^{-s \mathcal{D}} = \pm \frac{\hbar}{2} \int_\varepsilon^\infty \frac{ds}{s} \mathcal{T}_\mathcal{D}(s), \]  

(7)

where \( \varepsilon \) is an UV cutoff in the units: length to the power two. Next, we have at our disposal the following Seeley–DeWitt heat kernel expansion [5, 6]:

\[ T(s) = \int t(s; x) \sqrt{g} \ d^4x, \]  

(8)

where

\[ t(s; x) = \frac{1}{(4\pi)^2} \sum_{n=0}^\infty a_n(x)s^n. \]  

(9)

The full expansion (7) corresponds to all one-loop Feynman diagrams, those in Fig. 2 and also that in Fig. 1. The purely vacuum diagram in Fig. 1 should certainly excluded as trivial one because, by construction, it does not represent coupling to an external gravitational field. The purely vacuum diagram is contained in the first coefficient of the Seeley–DeWitt expansion, \( a_0(x) \), because only this coefficient survives the vanishing external field limit. For any external gravitational (and not only gravitational) non-vanishing or vanishing field, we have \( a_0(x) = 1 \).

**Fig. 2:** Closed matter loops influencing classical gravitational field via attached external lines.

Other \( a_n(x) \)'s (for \( n > 0 \)) contain various powers and derivatives of curvature with dimensionality governed by \( n \). In particular, \( a_1(x) = \frac{1}{12} R \) for an ordinary massless scalar mode, where \( R \) is the scalar curvature, and it finitely renormalizes or induces [7] (dependently on the point of view) the classical Hilbert–Einstein action. For this mode, the philosophy of the induced gravity yields, by virtue of (7)–(9),

\[ S_{\text{ind}} = -\frac{\hbar}{2} \frac{1}{\varepsilon^2 (4\pi)^2} \int \frac{1}{6} R \sqrt{g} \ d^4x \]  

\[ = -\frac{\hbar}{12} \frac{1}{L_p^2 (4\pi)^2} \int R \sqrt{g} \ d^4x \]  

\[ = -\frac{1}{12\pi} \frac{\hbar c^3}{G} \int R \sqrt{g} \ d^4x, \]  

(10)

where

\[ L_p^2 = \frac{\hbar G}{c^3}, \]

is the Planck length squared. Therefore, the induced coupling constant for a single mode is 12 times less than the standard classical value! The next term, \( a_2(x) \) (and also further terms), yields quantum corrections to the classical theory and thus is uninteresting for us.

As is well-known, (effective) cosmological constant or dark energy can be induced by the zeroth term, \( a_0 \), and therefore we will concentrate on that term henceforth. The zeroth term yields Casimir-like contribution of the form

\[ S_{\text{Cas}} = \mp \frac{\hbar}{2} \frac{1}{2\varepsilon^2 (4\pi)^2} \int \sqrt{g} \ d^4x \]  

\[ = \mp \frac{\hbar}{4} \frac{1}{L_p^2 (4\pi)^2} \int \sqrt{g} \ d^4x \]  

\[ = \mp \frac{1}{4} \frac{\hbar c^3}{G} \int \sqrt{g} \ d^4x. \]  

(11)
In the flat space limit it corresponds to the value 4 times less than that calculated earlier in (1). This difference is coming from different regularization procedures, and also we should remember that Hamiltonian and Lagrangian are different objects. According to our strategy we have to extract from Eq. (11) only the part corresponding to gravitational field.

For technical simplicity but without any experimental consequences, we can take the metric of the spatially flat Friedmann–Robertson–Walker (FRW) form with the scale factor $a(t)$. For calculational purposes, let us assume that the coordinate time $t = 0$ corresponds to the present moment, and

$$a(0) = 1. \quad (12)$$

Now, power-series expanding around $t = 0$, we have

$$a(t) = 1 + H_0 t - \frac{1}{2} g_0 H_0^2 t^2 + O(t^3), \quad (13)$$

where $H_0$ is the present Hubble expansion rate,

$$H_0 = \dot{a}(0),$$

and $g_0$ is the present deceleration parameter,

$$g_0 = - H_0^{-2} \ddot{a}(0).$$

Hence

$$\sqrt{g} = \left[ a^2(t) \right]^{\frac{1}{2}} = \left[ 1 + 2 H_0 t + (1 - q_0) H_0^2 t^2 + O(t^3) \right]^{\frac{1}{2}}. \quad (14)$$

It is of vital importance for our further considerations to show that the second term in (14), linear in $t$, can be thrown out by virtue of gauge symmetry. Physically, such a potential possibility corresponds to the obvious fact that not any perturbation of flat metric represents genuine gravitational field but only those which are gauge nontrivial. The reasoning goes as follows. Infinitesimal gauge transformations around flat metric are given by

$$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (15)$$

where $\xi_\mu = (\xi \equiv \xi_0, \xi_i)$ are gauge parameters. Explicitly, the first equation is

$$\delta g_{00} = 2 \dot{\xi} = 0, \quad (16)$$

because $g_{00} = 1$ should be constant. A general solution of Eq. (16) is then $\xi = \xi(x)$. The second equation is of the form

$$\delta g_{0i} \equiv \delta g_{i0} = \dot{\xi}_i + \partial_i \xi = 0, \quad (17)$$

because $g_{0i} = 0$ should also be left intact. Hence

$$\dot{\xi}_i = - \partial_i \xi(x) \quad \text{and} \quad \xi_i = - t \partial_i \xi(x) + \eta_i(x). \quad (18)$$

For purely spatial indices we have

$$\delta g_{ij} = \partial_i \xi_j + \partial_j \xi_i = - 2 t \partial_i \xi_j(x) + \partial_i \eta_j(x) + \partial_j \eta_i(x). \quad (19)$$

Now

$$\delta g_{ij} = \begin{cases} 0, & \text{for } i \neq j, \\ f(t), & \text{for } i = j. \end{cases} \quad (20)$$

From Eq. (19) it immediately follows that the most general function $f(t)$ which can be gauged away is linear in $t$. As a final solution of our problem we could assume the particular one

$$\xi(x) = \sum_{i,j=1}^3 \xi_{ij} x^i x^j, \quad \eta(x) = 0, \quad (21)$$

where the constant matrix $\xi_{ij}$, in view of Eq. (20), should be scalar, i.e. $\xi_{ij} = \frac{1}{2} H_0 \delta_{ij}$ after convenient normalization.

Coming back to Eq. (14), we have for small $t$

$$\sqrt{g} = 1 + \frac{3}{2} (1 - q_0) H_0^2 t^2 + O(t^3). \quad (22)$$

Since the integrand in Eq. (11) is now (only) $t$-dependent we can still divide by the spatial volume but dividing by the time coordinate is nothing but an averaging procedure with respect to $t$. As our analysis is perturbative in the time $t$, the longer the time the smaller the reliability of the analysis. The shortest possible time, in the realm of quantum field theory, is $t = T_P$ (the Planck time). Therefore, averaging around present moment $(t = 0)$ means

$$\lim_{T \to T_P} \frac{1}{T} \int_0^T \text{d}t \, (\cdot). \quad (23)$$

Thus, the estimated density is according to Eq. (11) of the order

$$\rho = \mp \frac{1}{4} \frac{e^5 (4 \pi)^2 \hbar G^2}{T \to T_P} \lim_{T \to T_P} \frac{1}{T} \int_0^T \text{d}t \, (\sqrt{g} - 1) \approx \mp \frac{1}{4} \frac{e^5 (4 \pi)^2 \hbar G^2}{2} (1 - q_0) H_0^2 T_P^2, \quad (24)$$

where the subtracation in the upper term corresponds to throwing out of gravitationally non-interacting part of the effective action. Equivalently, in Eq. (5) one should consider $\text{det} \mathcal{D}/\text{det} \mathcal{D}_0$ instead of $\text{det} \mathcal{D}$, where
$D_0$ is a flat space version of $D$. Therefore, the subtraction is not, as it could seem, an ad hoc procedure. Since $T_p^2 = \hbar G/c^5$, and $H_0^2 = \frac{8}{\pi}G\varrho_{\text{crit}}$, where $\varrho_{\text{crit}}$ is the critical density of the Universe, we finally obtain

$$\varrho \approx \mp \frac{1}{48\pi}(1 - q_0)\varrho_{\text{crit}}. \quad (25)$$

Eq. (25) predicts a highly reasonable result. Inserting $q_0 = -0.7$, which is phenomenologically rather a sensible assumption [8], yields the following numerical result

$$\varrho \approx \mp 0.01 \varrho_{\text{crit}}. \quad (26)$$

The experimental value is $\varrho_{\text{exp}} \approx 0.76 \varrho_{\text{crit}}$, therefore $0.01 \varrho_{\text{crit}}$ per single mode is a very good estimation in our opinion. Taking into account the remark directly following Eq. (11), $\varrho$ could be just as well $0.04 \varrho_{\text{crit}}$. We should necessary remember that our analysis is only an estimation.

IV. CONCLUSIONS

Using coordinate gauge freedom we have managed to extract from the full quantum vacuum term induced in an external classical gravitational background by a fluctuating mode of a matter field the fraction corresponding to interaction with gravitational field. An explicit calculus has been performed in the framework of the spatially flat FRW geometry. The value of the contribution coming from a single mode, which appears to be of the order of one hundredth of the critical value, seems to be rather intriguing. Thus, the old primary expectation that quantum vacuum fluctuations could be of physical interest, as e.g. in the Casimir effect, is not unjustified.

We would like to stress that we do not claim that we have found a solution of the problem of the accelerated expansion. We are aware that there are a lot of more or less sensible competing proposals in that area. All of them, including ours, have some drawbacks. Our approach should be just interpreted in this context as a voice in the discussion, indicating a direction for a possible further study.

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