CONTINUOUS ORBIT EQUIVALENCE UP TO EQUVALENCE RELATIONS

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Abstract. We introduce notions of continuous orbit equivalence and strong (respective, weak) continuous orbit equivalence for automorphism systems of étale equivalence relations, and characterize them in terms of the semi-direct product groupoids, as well as their reduced groupoid C*-algebras with canonical Cartan subalgebras. In particular, we study topological rigidity of expansive automorphism actions on compact (connected) metrizable groups.

1. Introduction

The interplay between orbit equivalence of topological dynamical systems and C*-algebras has been studied by many authors. An early celebrated result in this direction is the work on strong orbit equivalence of minimal homeomorphisms on Cantor sets given by Giordano, Putnam and Skau ([8]). Later, Tomiyama and Boyle-Tomiya studied a generalization of the GPS's result to the case of topologically free homeomorphisms on compact Hausdorff spaces ([3, 28]). In [14], Matsumoto introduced the notion of continuous orbit equivalence of one-sided topological Markov shifts and characterized them in terms of the existence of diagonals preserving *-isomorphisms between the associated Cuntz-Krieger algebras. In [16], Matui and Matsumoto gave a classification result of two-sided irreducible topological Markov shifts in the sense of flow equivalence by means of continuous orbit equivalence of one-sided topological Markov shifts. We can refer to [5, 6] for some generalizations on flow equivalence and study on the relation between topological conjugacy of two-sided shifts of finite type and the associated stabilized Cuntz-Krieger algebras with the canonical Cartan subalgebras and gauge actions. More recently, in [12, 13], Mastumoto introduced notions of asymptotic continuous orbit equivalence, asymptotic conjugacy and asymptotic flip conjugacy in Smale spaces and characterized them in terms of their groupoids and asymptotic Ruelle algebras with their dual actions.

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Our interests lie in group actions. As a topological analogue of the classification results on the probability measure preserving actions in the sense of orbit equivalence, Li introduced the notion of continuous orbit equivalence for continuous group actions and proved that two topologically free systems are continuously orbit equivalent if and only if their associated transformation groupoids are isomorphic (10). By Renault’s result in [22], these conditions are also equivalent to the existence of $C^*$-isomorphism preserving the canonical Cartan subalgebras between the corresponding crossed product algebras. In [7], Li’s rigidity result is generalized to the case of group actions with torsion-free and abelian essential stabilisers.

The local conjugacy relations from expansive group action systems are generalizations of asymptotic equivalence relations of Smale spaces (19, 26). In [9], we characterized continuous orbit equivalence of expansive systems up to local conjugacy relations and showed that two expansive actions are asymptotically continuous orbit equivalent if and only if the associated semi-direct product groupoids of local conjugacy relations are isomorphic.

In this paper we consider continuous orbit equivalence between automorphism systems of étale equivalence relations. Given an étale equivalence relation $\mathcal{R}$ on a compact metrizable space $X$, let $G \acts (X, \mathcal{R})$ be a dynamical system arising from an automorphism action of a countable group $G$ on $\mathcal{R}$ in the sense that each $\alpha_g$ is an automorphism of $\mathcal{R}$ as étale groupoids. Denote by $\mathcal{R} \rtimes_\alpha G$ the associated semi-direct product groupoid. We say that two systems $G \acts (X, \mathcal{R})$ and $H \acts (Y, \mathcal{S})$ conjugate if there exist an isomorphism $\tilde{\varphi} : \mathcal{R} \to \mathcal{S}$ as étale groupoids and a group isomorphism $\theta : G \to H$ such that $\tilde{\varphi}(g\gamma) = \theta(g)\tilde{\varphi}(\gamma)$ for $\gamma \in \mathcal{R}$ and $g \in G$. We call the set $[x]_{G,\mathcal{R}} = \{ y \in X : (gx,y) \in \mathcal{R} \text{ for some } g \in G \}$ the bi-orbit of $x$.

Motivated by the notion of usual orbit equivalence of dynamical systems, we say that $G \acts (X, \mathcal{R})$ and $H \acts (Y, \mathcal{S})$ are orbit equivalent if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi([x]_{G,\mathcal{R}}) = [\varphi(x)]_{H,\mathcal{S}}$ for $x \in X$. We call they are continuously orbit equivalent if there exist a homeomorphism $\varphi : X \to Y$, continuous maps $a : \mathcal{R} \times G \to H$ and $b : \mathcal{S} \times H \to G$ such that both the maps $((x,y),g) \in \mathcal{R} \times G \to (\varphi(x),a((x,y),g)\varphi(g^{-1}y)) \in \mathcal{S}$ and $((x,y),g) \in \mathcal{S} \times H \to (\varphi^{-1}(x),b((x,y),g)\varphi^{-1}(g^{-1}y)) \in \mathcal{R}$ are well-defined and continuous. The followings are main results in this paper.

**Theorem 1.1.** Assume that $G \acts (X, \mathcal{R})$ and $H \acts (Y, \mathcal{S})$ are essentially free. Then the following statements are equivalent.

(i) $G \acts (X, \mathcal{R})$ and $H \acts (Y, \mathcal{S})$ are continuously orbit equivalent;

(ii) $\mathcal{R} \rtimes_\alpha G$ and $\mathcal{S} \rtimes_\beta H$ are isomorphic as étale groupoids;
(iii) there exists a $C^*$-isomorphism $\Phi$ from $C^*_r(\mathcal{R} \rtimes_{\alpha} G)$ onto $C^*_r(\mathcal{S} \rtimes_{\beta} H)$ such that $\Phi(C(X)) = C(Y)$.

Here the notion of essential freeness for $G \blacklozenge_{\alpha} (X, \mathcal{R})$ is a generalization and analogy of topological freeness of dynamical systems. When $\mathcal{R} = \{(x, x) : x \in X\}$ is a trivial étale equivalence relation, or $\mathcal{R}$ is the local conjugacy relation or asymptotical equivalence relation arising from an expansive system $G \blacklozenge_{\alpha} X$ or an irreducible Smale space $(X, \varphi)$, this result is reduced to Theorem 1.2 in [10], Theorem 3.4 in [12] and Theorem 4.2 in [9]. The properties of strong or weak continuous orbit equivalence for automorphism systems are corresponding to two special orbit equivalence with some uniform conditions, and are also analogies of asymptotic flip conjugacy in [12] and (strong) asymptotic conjugacy in [9].

**Theorem 1.2.** Assume that $G \blacklozenge_{\alpha} (X, \mathcal{R})$ and $H \blacklozenge_{\beta} (Y, \mathcal{S})$ are essentially free. Then

(i) $G \blacklozenge_{\alpha} (X, \mathcal{R})$ and $H \blacklozenge_{\beta} (Y, \mathcal{S})$ are weakly continuously orbit equivalent if and only if there is an isomorphism $\Lambda : \mathcal{R} \rtimes_{\alpha} G \to \mathcal{S} \rtimes_{\beta} H$ such that $\Lambda(\mathcal{R}) = \mathcal{S}$ if and only if there is a $C^*$-isomorphism $\Phi : C^*_r(\mathcal{R} \rtimes_{\alpha} G) \to C^*_r(\mathcal{S} \rtimes_{\beta} H)$ such that $\Phi(C(X)) = C(Y)$ and $\Phi(C^*_r(\mathcal{R})) = C^*_r(\mathcal{S})$.

(ii) $G \blacklozenge_{\alpha} (X, \mathcal{R})$ and $H \blacklozenge_{\beta} (Y, \mathcal{S})$ are strongly continuously orbit equivalent if and only if there exist a homeomorphism $\varphi : X \to Y$ and a group isomorphism $\theta : G \to H$ such that $\Lambda : (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G \to (\varphi(x), \theta(g), \varphi(y)) \in \mathcal{S} \rtimes_{\beta} H$ is an isomorphism.

Furthermore, when $\mathcal{R}$ and $\mathcal{S}$ are minimal or $X$ and $Y$ are connected, these two notions are consistent.

The assumption of essential freeness in the above theorems is necessary. Automorphism systems on local conjugacy relations from expansive actions are typical examples. The automorphism systems of local conjugacy relations from a full shift $G \blacklozenge_{\alpha} \mathcal{A}G$ over a finite set $A$ and an irreducible Smale space $(X, \psi)$ are essentially free ([9, 12]). The following result generalizes Matsumoto’s result.

**Theorem 1.3.** Let $\mathcal{R}_{\alpha}$ be the local conjugacy relation from an expansive and transitive action $G \blacklozenge_{\alpha} X$. Assume that $X$ is infinite and has no isolated points and $G$ is an abelian group such that every subgroup generated by $g$ ($g \neq e$) has finite index in $G$. Then $G \blacklozenge_{\alpha} (X, \mathcal{R}_{\alpha})$ is essentially free.

Moreover, if $\mathbb{Z} \blacklozenge_{\alpha} X$ is generated by an expansive homeomorphism $\varphi$ on $X$, then the transitivity condition on $\varphi$ is not necessary.
In [2], S. Bhattacharya proved that topological conjugacy and algebraic conjugacy between two automorphism actions on compact abelian connected metrizable spaces are agreement. We have a rigidity result for automorphism actions on nonabelian groups.

**Proposition 1.4.** Let $G \curvearrowright_{\alpha} (X, R)$ and $H \curvearrowright_{\beta} (Y, S)$ be two systems on local conjugacy relations from topologically free, expansive automorphism actions on compact and connected metrizable groups $X$ and $Y$, respectively. Assume that $\Delta_{\alpha}$ is dense in $X$. Then the following statements are equivalent:

(i) $G \curvearrowright_{\alpha} (X, R)$ and $H \curvearrowright_{\beta} (Y, S)$ are conjugate;

(ii) $G \curvearrowright_{\alpha} (X, R)$ and $H \curvearrowright_{\beta} (Y, S)$ are weakly continuously orbit equivalent;

(iii) $G \curvearrowright_{\alpha} X$ and $H \curvearrowright_{\beta} Y$ are conjugate;

(iv) $G \curvearrowright_{\alpha} X$ and $H \curvearrowright_{\beta} Y$ are algebraically conjugate.

In particular, two hyperbolic toral automorphisms on $\mathbb{R}^n/\mathbb{Z}^n$ are flip conjugate if and only if the $\mathbb{Z}$-actions they generates are continuously orbit equivalent up to the associated local conjugacy relations.

This paper is organized as follows. Section 3 characterizes conjugacy of automorphism systems of étale equivalence relations and the reduced $C^*$-algebra of the associated semi-direct product of equivalence relations. In section 4, we introduce notions of continuous orbit equivalence, strong- and weak- continuous orbit equivalence for automorphism systems, and characterize them in terms of the semi-direct products and the corresponding $C^*$-algebras. In section 5, we discuss essential freeness of automorphism systems on local conjugacy equivalence relations arising from expansive actions, and in section 6, we study topological rigidity of expansive automorphism actions on compact (connected) metrizable groups. As an example, we characterize the structure of the local conjugacy relation from a hyperbolic toral automorphism on $n$-torus.

2. Preliminaries

Unless otherwise specified, all our groups are discrete and countable, their identity elements are denoted by the same symbol $e$, and all topological groupoids are second countable, locally compact and Hausdorff. We refer to [21, 25] for more details on topological groupoids and their $C^*$-algebras, and refer to [17, 29] for $C^*$-dynamical systems.

For a topological groupoid $G$, let $G^{(0)}$ and $G^{(2)}$ be the unit space and the set of composable pairs, respectively. The range and domain maps $r, d$ from
\( \mathcal{G} \) onto \( \mathcal{G}^{(0)} \) are defined by \( r(g) = gg^{-1} \) and \( d(g) = g^{-1}g \), respectively. If \( r \) and \( d \) are local homeomorphisms then \( \mathcal{G} \) is called to be étale. For \( u, v \in \mathcal{G}^{(0)} \), we write \( \mathcal{G}^u = r^{-1}(u) \), \( \mathcal{G}_u = d^{-1}(u) \) and \( \mathcal{G}$$_u^v = \mathcal{G}^u \cap \mathcal{G}_v \). When \( \mathcal{G} \) is étale, these sets are discrete and countable, and \( \mathcal{G}^{(0)} \) is open and closed in \( \mathcal{G} \). Recall that \( \mathcal{G} \) is topologically principle if \( \{ u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{ u \} \} \) is dense in \( \mathcal{G}^{(0)} \).

Each equivalence relation \( \mathcal{R} \subseteq X \times X \) on a topological space \( X \) is a groupoid with multiplication \((x,y)(w,z) = (x,z)\) if \( y = w \) and inverse \((x,y)^{-1} = (y,x)\). If we identify \((x,x)\) with \( x \), then the unit space \( \mathcal{R}^{(0)} \) equals \( X \) and the range (resp. domain) map is defined by \( r(x,y) = x \) (resp. \( d(x,y) = y \)). If there exists a topology on \( \mathcal{R} \) (not necessarily the relative product topology from \( X \times X \)) for which \( \mathcal{R} \) is an étale groupoid, then \( \mathcal{R} \) is called an étale equivalence relation on \( X \). In this case, if every \( \mathcal{R} \)-equivalence class is dense in \( X \) then \( \mathcal{R} \) is minimal.

By a dynamical system, denoted by \( G \curvearrowright X \) (or simply by \( G \curvearrowright X \)), we mean an action \( \alpha \) of a group \( G \) on a second countable, locally compact and Hausdorff space \( X \) by homeomorphisms. The action \( \alpha \) is usually expressed as \( (g,x) \in G \times X \to gx \in X \). The associated transformation groupoid \( X \rtimes G \) is given by the set \( X \times G \) with the product topology, multiplication \((x,g)(y,h) = (x,gh)\) if \( y = g^{-1}x \), and inverse \((x,g)^{-1} = (g^{-1}x, g^{-1})\). Clearly, \( X \rtimes G \) is étale, and if \((x,e)\) is identified with \( x \) then its unit space equals \( X \), range map \( r(x,g) = x \) and domain map \( d(x,g) = g^{-1}x \). A system \( G \curvearrowright X \) is said to be topologically free if for every \( e \neq g \in G \), \( \{ x \in X : gx \neq x \} \) is dense in \( X \). From [10] Corollary 2.3, \( G \curvearrowright X \) is topologically free if and only if \( X \rtimes G \) is topologically principal. Two systems \( G \curvearrowright X \) and \( H \curvearrowright Y \) is conjugate if there are a homeomorphism \( \varphi : X \to Y \) and a group isomorphism \( \theta : G \to H \) such that \( \varphi(gx) = \theta(g)\varphi(x) \) for \( x \in X \) and \( g \in G \).

A map \( \Phi : \mathcal{G} \to \mathcal{H} \) between étale groupoids \( \mathcal{G} \) and \( \mathcal{H} \) is a homomorphism if it is continuous and, for all \( (\gamma, \gamma') \in \mathcal{G}^{(2)} \), we have \( (\Phi(\gamma), \Phi(\gamma')) \in \mathcal{H}^{(2)} \) and \( \Phi(\gamma\gamma') = \Phi(\gamma)\Phi(\gamma') \). Moreover, if \( \Phi \) is a homeomorphism such that \( \Phi \) and \( \Phi^{-1} \) are homomorphisms, then it is called an isomorphism. In this case, the restriction, \( \Phi|_{\mathcal{G}^{(0)}} \), of \( \Phi \) to the unit space \( \mathcal{G}^{(0)} \) is a homeomorphism from \( \mathcal{G}^{(0)} \) onto \( \mathcal{H}^{(0)} \). A homomorphism from \( \mathcal{G} \) into a group \( \Gamma \) is also called a cocycle on \( \mathcal{G} \). Two étale equivalence relations \( \mathcal{R} \subseteq X \times X \) and \( \mathcal{S} \subseteq Y \times Y \) are isomorphic if and only if there exists a homeomorphism \( \varphi : X \to Y \) such that \( \varphi \times \varphi : (x,y) \in \mathcal{R} \to (\varphi(x), \varphi(y)) \in \mathcal{S} \) is an isomorphism.

Given an étale groupoid \( \mathcal{G} \), the linear space, \( C_c(\mathcal{G}) \), of continuous complex functions with compact support on \( \mathcal{G} \) is a \(*\)-algebra under the operations: \( f^*(\gamma) = \overline{f(\gamma^{-1})} \) and \( f * g(\gamma) = \sum_{\gamma' \in G_{d(\gamma)}} f(\gamma\gamma'^{-1})g(\gamma') \) for \( f, g \in C_c(\mathcal{G}) \).
The reduced crossed product of the transformation groupoid is isomorphic to the reduced crossed product $\text{Aut}(C_r^*(\mathbb{G}))$ of $\mathbb{G}$.

Given two groups $N, H$ and a homomorphism $\varphi$ from $H$ into the automorphism group $\text{Aut}(N)$ of $N$, the semi-direct product, denoted by $N \rtimes \varphi H$, of $N$ by $H$ is defined as the set $N \times H$ with group law given by the formulas $(n, h)(n_1, h_1) = (n\varphi_h(n_1), hh_1)$ and $(n, h)^{-1} = (\varphi_{h^{-1}}(n^{-1}), h^{-1})$.

3. $C^*$-algebras associated with semi-direct product groupoids

Given an étale equivalence relation $\mathcal{R}$ on a compact metrizable space $X$, we call a dynamical system $G \curvearrowright \mathcal{R}$ an automorphism system if each $\alpha_g$ is an automorphism of $\mathcal{R}$ as étale groupoids. Clearly, this system induces an action, also denoted by $\alpha$, of $G$ on $X$ by homeomorphisms such that $g(x, y) = (gx, gy)$ for $g \in G$ and $(x, y) \in \mathcal{R}$. We use the notation $G \curvearrowright (X, \mathcal{R})$ (or $G \curvearrowright (X, \mathcal{R})$ for short) to denote an automorphism system.

The semi-direct product groupoid, $\mathcal{R} \times_\alpha G$, attached to $G \curvearrowright (X, \mathcal{R})$, is the set $\mathcal{R} \times G$ with inverse $((x, y), g)^{-1} = ((g^{-1}y, g^{-1}x), g^{-1})$, and multiplication $((x, y), g)\text{((u, v), h)} = ((x, gv), gh)$ if $u = g^{-1}y$. The unit space identifies with $X$ by identifying $((x, x), e)$ with $x$. Then $r((x, y), g) = x$ and $d((x, y), g) = g^{-1}y$. Endowed with the relative product topology from $\mathcal{R} \times G$, the groupoid $\mathcal{R} \times_\alpha G$ is étale [21]. The following is another characterization of the semi-direct product groupoid.

**Definition 3.1.** Let

$$\mathcal{R} \times_\alpha G = \{(x, g, y) \mid g \in G, x, y \in X, (x, gy) \in \mathcal{R}\}.$$  

Then, under the following multiplication and inverse,

$$(x, g, y)(y, h, v) = (x, gh, v), \quad \text{and} \quad (x, g, y)^{-1} = (y, g^{-1}, x),$$

$\mathcal{R} \times_\alpha G$ is a groupoid. Define a map $\gamma_0 : \mathcal{R} \times_\alpha G \to \mathcal{R} \times_\alpha G$, by $\gamma_0((x, g, y)) = ((x, gy), g)$, which is a bijection with inverse $\gamma_0^{-1}((x, y), g) = (x, g, g^{-1}y)$. We
transfer the product topology from $R \times_{\alpha} G$ over $R \times_{\alpha} G$. Then $R \times_{\alpha} G$ is an étale groupoid and $\gamma_0$ is an étale groupoid isomorphism.

**Remark 3.2.** If we identify the unit space $(R \times_{\alpha} G)^{(0)}$ with $X$ as topological spaces by identifying $(x, e, x)$ with $x$, then $r(x, g, y) = x$ and $d(x, g, y) = y$.

The equivalence relation $R$ and the transformation groupoid $X \rtimes G$ can be embedded into $R \times_{\alpha} G$ as étale subgroupoids through the identifications $(x, y) \in R \rightarrow (x, e, y) \in R \times_{\alpha} G$ and $(x, g) \in X \rtimes G \rightarrow (x, g, g^{-1}x) \in R \times_{\alpha} G$.

One can check that the map $\rho_\alpha : R \times_{\alpha} G \rightarrow G$, defined by $\rho_\alpha(x, g, y) = g$, is a cocycle.

We call two automorphism systems $G \lhd_{\alpha} (X, R)$ and $H \lhd_{\beta} (Y, S)$ on compact metrizable spaces *conjugate* if there are an isomorphism $\tilde{\varphi} : R \rightarrow S$ and a group isomorphism $\vartheta : G \rightarrow H$ such that $\tilde{\varphi}(g\gamma) = \vartheta(g)\tilde{\varphi}(\gamma)$ for $\gamma \in R$ and $g \in G$. Clearly, this is equivalent to that there is a homeomorphism $\varphi : X \rightarrow Y$ and a group isomorphism $\vartheta : G \rightarrow H$ such that $\varphi \times \varphi : (x, y) \in R \rightarrow (\varphi(x), \varphi(y)) \in S$ is an isomorphism and $\varphi(gx) = \vartheta(g)\varphi(x)$ for $x \in X$ and $g \in G$. In particular, two systems $G \lhd_{\alpha} X$ and $H \lhd_{\beta} Y$ are conjugate.

**Proposition 3.3.** If $G \lhd_{\alpha} (X, R)$ and $H \lhd_{\beta} (Y, S)$ are conjugate, then there is an isomorphism, $\Lambda : R \times_{\alpha} G \rightarrow S \times_{\beta} H$, such that $\Lambda(R) = S$ and $\Lambda(X \rtimes G) = Y \rtimes H$.

Assume that one of the following statements holds:

(i) $X$ and $Y$ are connected.

(ii) $R$ and $S$ are minimal.

Then the above converse holds, i.e., $G \lhd_{\alpha} (X, R)$ and $H \lhd_{\beta} (Y, S)$ are conjugate if and only if there is an isomorphism, $\Lambda : R \times_{\alpha} G \rightarrow S \times_{\beta} H$, such that $\Lambda(R) = S$ and $\Lambda(X \rtimes G) = Y \rtimes H$.

**Proof.** Assume that $G \lhd_{\alpha} (X, R)$ and $H \lhd_{\beta} (Y, S)$ are conjugate by a homeomorphism $\varphi$ from $X$ onto $Y$ and a group isomorphism $\vartheta$ from $G$ onto $H$. Define the map $\Lambda$ from $R \times_{\alpha} G$ into $S \times_{\beta} H$ by $\Lambda(x, g, y) = (\varphi(x), \vartheta(g), \varphi(y))$. Then $\Lambda$ is an isomorphism with inverse $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), \vartheta^{-1}(h), \varphi^{-1}(v))$ and $\Lambda(R) = S$ and $\Lambda(X \rtimes G) = Y \rtimes H$.

On the contrary, let $\Lambda$ be an isomorphism from $R \times_{\alpha} G$ onto $S \times_{\beta} H$ such that $\Lambda(R) = S$ and $\Lambda(X \rtimes G) = Y \rtimes H$. Let $\varphi$ be the restriction of $\Lambda$ to $X$, and let $a = \rho_\beta \Lambda$ and $b = \rho_\alpha \Lambda^{-1}$. Then $\varphi$ is a homeomorphism from $X$ onto $Y$, and $a$ and $b$ are continuous cocycles on $R \times_{\alpha} G$ and $S \times_{\beta} H$, respectively. Moreover, $\Lambda(x, g, y) = (\varphi(x), a(x, g, y), \varphi(y))$, and its inverse $\Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v))$. The fact that $\Lambda(R) = S$ implies
that \(a(x, e, y) = e\) and \(\varphi \times \varphi : (x, y) \in R \to (\varphi(x), \varphi(y)) \in S\) is an isomorphism. The requirement that \(\Lambda(X \rtimes G) = Y \rtimes H\) gives us that

\[
\varphi(x) = a(x, g, g^{-1}x)\varphi(g^{-1}x).
\]

(3.1)

Also since \((x, g, g^{-1}x)(g^{-1}x, e, g^{-1}y)(g^{-1}y, g^{-1}, y) = (x, e, y)\) for \((x, y) \in R\) and \(g \in G\), we have \(a(x, g, g^{-1}x) = a(y, g, g^{-1}y)\). By symmetry, \(b\) has a similar property to \(a\).

Assume that \(X\) and \(Y\) are connected. Since the restriction map \(a|_{X \times G} : X \rtimes G \to H\) is continuous, we have, for every \(g \in G\), the restriction map \(a|_{X \times \{g\}}\) is a constant, and thus \(a(x, g, g^{-1}x) = a(y, g, g^{-1}y)\) for all \(x, y \in X\) and \(g \in G\). Similarly, we have \(b(u, h, h^{-1}u) = b(v, h, h^{-1}v)\) for all \(u, v \in Y\) and \(h \in H\).

Consequently, under the hypothesis of (i) or (ii), there exist two maps \(\theta : G \to H\) and \(\vartheta : H \to G\) such that \(a(x, g, g^{-1}x) = \theta(g)\) and \(b(u, h, h^{-1}u) = \vartheta(h)\) for every \(x \in X\), \(u \in Y\), \(g \in G\) and \(h \in H\). Since \(\Lambda\) is an isomorphism with inverse \(\Lambda^{-1}\), \(\theta\) is a group isomorphism with inverse \(\vartheta\). Moreover, (3.1) implies that \(\varphi(gx) = \theta(g)\varphi(x)\) for \(x \in X\) and \(g \in G\). Hence \(\varrho_{\alpha} (X, R)\) and \(H \varrho_{\beta} (Y, S)\) are conjugate.

\[\square\]

The rest of this section characterizes the reduced groupoid \(C^*\)-algebra of \(R \varrho_{\alpha} G\) by crossed product construction, which is perhaps a well known fact, as we were unable to find an explicit reference, we provide a proof.

For an automorphism system \(G \varrho_{\alpha} (X, R)\), the map \(\alpha_g(f)(x, y) = f(g^{-1}x, g^{-1}y)\) for \(f \in C_c(R)\), \((x, y) \in R\) and \(g \in G\) gives us an \(C^*\)-dynamical system \((C^*_r(R), G, \alpha)\). Let \(C_c(G, C^*_r(R))\) be set of all continuous complex functions from \(G\) to \(C^*_r(R)\) with compact support sets, which is a \(*\)-algebra over \(\mathbb{C}\) under the following multiplicative and convolution:

\[
(\xi * \eta)(g) = \sum_{h \in G} \xi(h)\alpha_h(\eta(h^{-1}g))
\]

\[
\xi^*(g) = \alpha_g(\xi(g^{-1})^*)
\]

for \(\xi, \eta \in C_c(G, C^*_r(R))\) and whose closure, denoted by \(C^*_r(R) \varrho_{\alpha, r} G\), under the reduced crossed norm is the reduced crossed product \(C^*\)-algebra associated to \((C^*_r(R), G, \alpha)\) ([17] [29]).
Theorem 3.4. Let $G \ni \alpha \ (X, R)$ be an automorphism system. Then $C^*_r(R \rtimes \alpha G)$ is isomorphic to $C^*_r(R) \rtimes \alpha, r \ G$.

Proof. To simplify symbol, let $G = R \rtimes \alpha G$. Define

$$\Phi(\xi)(x, g, y) = \xi(g)(x, gy) \quad \text{for} \quad \xi \in C_c(G, C_c(R)) \quad \text{and} \quad (x, g, y) \in G$$

and

$$\Psi(\eta)(g)(x, y) = \eta(x, g, g^{-1}y), \quad \text{for} \quad \eta \in C_c(G) \quad \text{and} \quad g \in G, \ (x, y) \in R.$$ 

One can check that $\Phi : C_c(G, C_c(R)) \to C_c(G)$ and $\Psi : C_c(G) \to C_c(G, C_c(R))$ are $*$-isomorphisms such that $\Phi$ and $\Psi$ are inverse to each other.

Given $x \in X$, let $l^2(R_x)$ be the Hilbert space of all square-summable complex-valued functions on the $R$-equivalent class $R_x$ of $x$. We consider two Hilbert spaces $l^2(G, l^2(R_x)) = \{ \varphi : G \to l^2(R_x), \sum_{g \in G} \| \varphi(g) \|^2 < +\infty \}$ and $l^2(G_x) = \{ \psi : G_x \to \mathbb{C}, \sum_{\gamma \in G_x} \| \psi(\gamma) \|^2 < +\infty \}$. Then the map $U_x$, defined by $(U_x \varphi)(y, g, x) = \varphi(g)(g^{-1}y, x)$ for $\varphi \in l^2(G, l^2(R_x))$ and $(y, g, x) \in G_x$, is an unitary operation from $l^2(G, l^2(R_x))$ onto $l^2(G_x)$.

Let $\pi_x$ and $\lambda_x$ be the regular representations of $C_c(G)$ on $l^2(G_x)$ and $C_c(R)$ on $l^2(R_x)$ associated to $x$, respectively. Then we have the direct sums of representations

$$\pi = \bigoplus_{x \in X} \pi_x : C_c(G) \to \bigoplus_{x \in X} B(l^2(G_x)), \quad \lambda = \bigoplus_{x \in X} \lambda_x : C_c(R) \to \bigoplus_{x \in X} B(l^2(R_x)).$$

Then $\pi_x, \lambda_x, \pi$ and $\lambda$ can be extended to their corresponding reduced groupoid $C^*$-algebras and we use the same symbols to denote their extensions. Moreover, $\pi$ and $\lambda$ are faithful representations on $C^*_r(G)$ and $C^*_r(R)$, respectively.

The representation $\lambda$ induces a faithful representation

$$\tilde{\lambda} : \xi \in C_c(G, C^*_r(R)) \to \bigoplus_{x \in X} \tilde{\lambda}_x(\xi) \in \bigoplus_{x \in X} B(l^2(G, l^2(R_x))),$$

where, for each $x \in X$, $\tilde{\lambda}_x$ is the representation of $C_c(G, C^*_r(R))$ on the Hilbert space $l^2(G, l^2(R_x))$, given by $$(\tilde{\lambda}_x(\xi) \phi)(g) = \sum_{h \in G} \lambda_x(\alpha_{g^{-1}}(\xi(h))) \phi(h^{-1}g)$$

for $\xi \in C_c(G, C^*_r(R))$, $\phi \in l^2(G, l^2(R_x))$. Let $\tilde{\lambda}_x(\xi) = U_x \tilde{\lambda}_x(\xi)U_x^*$ for $x \in X$ and $\xi \in C_c(G, C^*_r(R))$. Then

$$\hat{\lambda} : \xi \in C_c(G, C^*_r(R)) \to \bigoplus_{x \in X} \hat{\lambda}_x(\xi) \in \bigoplus_{x \in X} B(l^2(G_x)),$$

is a faithful representation. We can check that $\pi_x \Phi(\xi) = \hat{\lambda}_x(\xi)$ for each $x \in X$, thus $\pi \Phi(\xi) = \hat{\lambda}(\xi)$ for all $\xi \in C_c(G, C_c(R))$. 

In fact, for each \( \varphi \) in \( l^2 (G, l^2 (\mathcal{R}_x)) \), \((y, g, x) \) in \( \mathcal{G}_x \), we have
\[
(\pi_x \hat{\Phi}(\xi) U_x)(\varphi)(y, g, x)
= \sum_{h \in G, (u, h^{-1}g) \in \mathcal{R}} [\Phi(\xi)(y, h, u)][U_x \varphi(u, h^{-1}g, x)]
= \sum_{h \in G, (g^{-1}hu, x) \in \mathcal{R}_x} [\xi(h)(y, hu)][\varphi(h^{-1}g)(v, x)]
= \sum_{h \in G, (v, x) \in \mathcal{R}_x} \xi(h)(y, gv)\varphi(h^{-1}g)(v, x)
\]
and
\[
U_x \left( \hat{\lambda}_x(\xi)(\varphi) \right)(y, g, x) = (\hat{\lambda}_x(\xi)\varphi)(g^{-1}y, x)
= \sum_{h \in G} \sum_{u \in X} \xi(h)(y, gu)\varphi(h^{-1}g)(u, x).
\]
Then, for each \( \xi \in C_c(G, C_c(\mathcal{R})) \), we have
\[
\|\Phi(\xi)\|_{red} = \sup_{x \in X} \|\pi_x(\Phi(\xi))\|_{B(l^2(\mathcal{G}_x))} = \sup_{x \in X} \|\hat{\lambda}_x(\xi)\|_{B(l^2(\mathcal{G}_x))}.
\]
Thus \( \|\Phi(\xi)\|_{red} = \|\xi\|_{red} \) for \( \xi \in C_c(G, C_c(\mathcal{R})) \), \( \Phi \) is an isomorphism.

4. CONTINUOUS ORBIT EQUIVALENCE OF AUTOMORPHISM SYSTEMS

Given an automorphism system \( G \curvearrowright \alpha (X, \mathcal{R}) \) on a compact metrizable space \( X \), for \( x \in X \), we let \( [x]_G := \{gx : g \in G\} \) and \( [x]_\mathcal{R} := \{y \in X : (x, y) \in \mathcal{R}\} \) be the orbits of \( x \) under the action \( \alpha \) and the relation \( \mathcal{R} \), respectively. We call the set \( [x]_G \cap \mathcal{R} = \{y \in X : (gx, y) \in \mathcal{R} \text{ for some } g \in G\} \) the bi-orbit of \( x \). Clearly, \( x]_G \cap \mathcal{R} = \cup_{y \in [x]_G} [y]_\mathcal{R} = \cup_{y \in [x]_\mathcal{R}} [y]_G = d((\mathcal{R} \times \alpha G) x) = r((\mathcal{R} \times \alpha G) x).

Recall that \( G \curvearrowright X \) and \( H \curvearrowright Y \) are orbit equivalent if there exists a homeomorphism \( \varphi : X \to Y \) such that \( \varphi([x]_G) = [\varphi(x)]_H \) for \( x \in X \). They are said to be continuously orbit equivalent if there exist a homeomorphism \( \varphi : X \to Y \) and continuous maps \( a : G \times X \to H \) and \( b : H \times Y \to G \) such that \( \varphi(gx) = a(g, x)\varphi(x) \) for \( x \in X \) and \( g \in G \), and \( \varphi^{-1}(hy) = b(h, y)\varphi^{-1}(y) \) for \( y \in Y \) and \( h \in H \). Motivated by these notions, we introduce the following definitions.

**Definition 4.1.** Two systems \( G \curvearrowright (X, \mathcal{R}) \) and \( H \curvearrowright (Y, \mathcal{S}) \) are orbit equivalent if there exists a homeomorphism \( \varphi : X \to Y \) such that \( \varphi([x]_G \cap \mathcal{R}) = [\varphi(x)]_{H \times \mathcal{S}} \) for \( x \in X \).

In this case, for \( x, y \in X \) and \( g \in G \) with \((gx, y) \in \mathcal{R}, \) there exists \( h \in H \) such that \( (h\varphi(x), \varphi(y)) \in \mathcal{S}. \) Similarly, for \( u, v \in Y \) and \( h \in H \) with
Proof. Let \((hu, v) \in S\), there exists \(g \in G\) such that \((g \varphi^{-1}(u), \varphi^{-1}(v)) \in \mathcal{R}\). Thus, we have the following notion.

**Definition 4.2.** Two systems \(G \acts (X, \mathcal{R})\) and \(H \acts (Y, \mathcal{S})\) are continuously orbit equivalent, we write \(G \acts (X, \mathcal{R}) \sim_{\text{coe}} H \acts (Y, \mathcal{S})\), if there exist a homeomorphism \(\varphi : X \to Y\) and continuous maps \(a : \mathcal{R} \times G \to H\) and \(b : S \times H \to G\) such that the following maps:

\[
((x, y), g) \in \mathcal{R} \times G \to (\varphi(x), a((x, y), g)\varphi^{-1}(y)) \in \mathcal{S}
\]

and

\[
((x, y), g) \in S \times H \to (\varphi^{-1}(x), b((x, y), g)\varphi^{-1}(y)) \in \mathcal{R}
\]

are continuous.

Clearly, continuous orbit equivalence implies orbit equivalence for automorphism systems. Assume a system \(G \acts \alpha X\) is free in the sense that, for \(g \in G\) and \(x \in X\), \(gx = x\) only if \(g = e\). We consider two automorphism systems \(G \acts (X, \mathcal{R}_1)\) and \(G \acts (X, \mathcal{R}_2)\), where \(\mathcal{R}_1 = \{(x, x) : x \in X\}\) is the trivial étale equivalence relation on \(X\) under the relative product topology and \(\mathcal{R}_2 = \{(x, gx) : x \in X, g \in G\}\) is the orbit equivalence relation under \(\alpha\). Noticing that the map \((x, g) \in X \times G \to (x, g^{-1}x) \in \mathcal{R}_2\) is a bijection, we transfer the product topology on \(X \times G\) over \(\mathcal{R}_2\) via this map. Then \(\mathcal{R}_2\) is an étale equivalence relation on \(X\).

**Proposition 4.3.** Assume that \(G \acts X\) is free. Then \(G \acts (X, \mathcal{R}_1)\) and \(G \acts (X, \mathcal{R}_2)\) are continuously orbit equivalent, but not conjugate.

*Proof.* Let \(\varphi\) be the identity map on \(X\), and let \(a((x, x), g) = g\) for \(((x, x), g) \in \mathcal{R}_1 \times G\). For each \((x, y) \in \mathcal{R}_2\), there exists unique an element in \(G\), denoted by \(k(x, y)\), such that \(y = k(x, y)x\). Let \(b((x, y), g) = k(x, y)^{-1}g\) for \(((x, y), g) \in \mathcal{R}_2 \times G\). Then \(\varphi, a\) and \(b\) satisfy the requirements in Definition 4.2, thus \(G \acts (X, \mathcal{R}_1)\) and \(G \acts (X, \mathcal{R}_2)\) are continuously orbit equivalent.

Since \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are never isomorphic, \(G \acts (X, \mathcal{R}_1)\) and \(G \acts (X, \mathcal{R}_2)\) are not conjugate.

\[\square\]

Using the semi-direct product groupoid \(\mathcal{R} \rtimes_\alpha G\) and the canonical homeomorphism \(\gamma_0\), one can check the following lemma.

**Lemma 4.4.** Two systems \(G \acts (X, \mathcal{R}) \sim_{\text{coe}} H \acts (Y, \mathcal{S})\) if and only if there exist a homeomorphism \(\varphi : X \to Y\) and continuous maps \(a : \mathcal{R} \rtimes_\alpha G \to H\) and \(b : S \rtimes_\beta H \to G\) such that the following maps:

\[
\Psi : (x, g, y) \in \mathcal{R} \rtimes_\alpha G \to (\varphi(x), a(x, g, y), \varphi(y)) \in S \rtimes_\beta H
\]

(4.1)
that

\[ \tilde{\Psi} : (u, h, v) \in S \times H \to (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) \in \mathcal{R} \times G \]  \tag{4.2}

are continuous.

Recall that an étale groupoid \( \mathcal{G} \) is topologically principal if \( \{ u \in \mathcal{G}^{(0)} : \mathcal{G}^u_u = \{ u \} \} \) is dense in \( \mathcal{G}^{(0)} \). From \([11, 22]\), \( \mathcal{G} \) is topologically principal if and only if the interior of \( \mathcal{G}' \) is \( \mathcal{G}^{(0)} \), where \( \mathcal{G}' = \cup_{u \in \mathcal{G}^{(0)}} \mathcal{G}^u_u \) is the isotropy bundle of \( \mathcal{G} \). For \( G \curvearrowright (X, \mathcal{R}) \), we have

\[ (\mathcal{R} \times G)' = \{(x, g, x) : x \in X, g \in G, (x, gx) \in \mathcal{R}\} \]

and

\[ (\mathcal{R} \times G)'' = \{(x, gx, g) : x \in X, g \in G, (x, gx) \in \mathcal{R}\}. \]

Moreover, we have that \( \gamma_0((\mathcal{R} \times G)') = (\mathcal{R} \times G)' \). The following definition comes from \([9, 12]\).

**Definition 4.5.** A system \( G \curvearrowright (X, \mathcal{R}) \) is said to be essentially free if for every \( e \neq g \in G \), \( \{ x \in X : (x, gx) \notin \mathcal{R}\} \) is dense in \( X \).

One can easily see that \( G \curvearrowright (X, \mathcal{R}) \) is essentially free, if and only if the interior of \( \{ x \in X : g[x]_\mathcal{R} = [x]_\mathcal{R}\} \) in \( X \) is empty for every \( g \neq e \).

**Lemma 4.6.** A system \( G \curvearrowright (X, \mathcal{R}) \) is essentially free if and only if \( \mathcal{R} \times G \) (or \( \mathcal{R} \times G \)) is topologically principal.

Moreover, one of these conditions implies that both systems \( G \curvearrowright X \) and \( G \curvearrowright \mathcal{R} \) are topologically free.

**Proof.** It follows from the definitions that the topological principality of \( \mathcal{R} \times G \) implies the essential freeness of \( G \curvearrowright (X, \mathcal{R}) \), thus implies the topological freeness of \( G \curvearrowright X \). To see that the essential freeness of \( G \curvearrowright \mathcal{R} \) implies the topological principality of \( \mathcal{R} \times G \), we only need to show that \( ((x, gx), g) \) is not in the interior of \( (\mathcal{R} \times G)' \) in \( (\mathcal{R} \times G) \) for each \( e \neq g \in G \) and \( x \in X \) with \( (x, gx) \in \mathcal{R} \).

In fact, for otherwise, choose \( e \neq g_0 \in G \) and \( x_0 \in X \) such that \( (x_0, g_0 x_0) \in \mathcal{R} \) and \( ((x_0, g_0 x_0), g_0) \) is an interior point of \( (\mathcal{R} \times G)' \). Then there exists an open neighbourhood \( \tilde{U} \) of \( (x_0, g_0 x_0) \) in \( \mathcal{R} \) such that

\[ ((x_0, g_0 x_0), g_0) \in \tilde{U} \times \{g_0\} \subseteq (\mathcal{R} \times G)'. \]

The last inclusion implies that \( y = g_0 x \) for each \( (x, y) \in \tilde{U} \). Hence \( \{ x \in X : (x, g_0 x) \in \mathcal{R}\} \) contains the non-empty open subset \( r(\tilde{U}) \) of \( X \), which is contrast to the essential freeness of \( G \curvearrowright (X, \mathcal{R}) \).

Assume \( G \curvearrowright (X, \mathcal{R}) \) is essentially free. Given \( e \neq g \in G \) and a non-empty open subset \( U \subseteq \mathcal{R} \), it follows from the openness of \( r(U) \) that there
exists \( x_0 \in r(U) \) with \( (x_0, gx_0) \notin \mathcal{R} \), thus \( x_0 \neq gx_0 \). Choose \( (x_0, y_0) \in U \). Then \( g(x_0, y_0) \neq (x_0, y_0) \), which implies that \( \{(x, y) \in \mathcal{R} : g(x, y) \neq (x, y)\} \) is dense in \( \mathcal{R} \). Hence \( G \curvearrowright \mathcal{R} \) is topologically free.

\[ \square \]

**Remark 4.7.** The topological freeness of neither \( G \curvearrowright X \) nor \( G \curvearrowright \mathcal{R} \) can imply the essential freeness of \( G \curvearrowright (X, \mathcal{R}) \). To see this, if \( G \curvearrowright X \) is free, then both systems \( G \curvearrowright \mathcal{R}_1 \) and \( G \curvearrowright \mathcal{R}_2 \) in Proposition 4.3 are free, and \( G \curvearrowright (X, \mathcal{R}_1) \) is essentially free, but \( G \curvearrowright (X, \mathcal{R}_2) \) is not.

If \( G \curvearrowright_{\alpha} (X, \mathcal{R}) \) and \( H \curvearrowright_{\beta} (Y, \mathcal{S}) \) are essentially free, then the mappings \( a \) and \( b \) in Lemma 4.4 (or in Definition 4.2) are uniquely determined by (4.1) and (4.2). In fact, suppose that \( a' : \mathcal{R} \times_{\alpha} G \rightarrow H \) is another continuous map such that \( \Psi' : (x, g, y) \in \mathcal{R} \times_{\alpha} G \rightarrow (\varphi(x), a'(x, g, y), \varphi(y)) \in \mathcal{S} \times H \) is continuous. Then

\[(x, g, y) \in \mathcal{R} \times_{\alpha} G \rightarrow (a(x, g, y)\varphi(y), a'(x, g, y)\varphi(y)) \in \mathcal{S} \]

is continuous. Hence, from the continuity of \( a, a' \) and \( \rho_{\alpha} \), for \((x, g, y) \in \mathcal{R} \times_{\alpha} G\), there exists an open neighbourhood \( \tilde{U} \) of \((x, g, y)\) such that the map \( d|_{\tilde{U}} : \tilde{U} \rightarrow d(\tilde{U}) \) is a homeomorphism, \( \rho_{\alpha}(\gamma) = g, a(\gamma) = a(x, g, y) \), and \( a'(\gamma) = a'(x, g, y) \) for each \( \gamma \in \tilde{U} \). For each \( z \in \varphi(d(\tilde{U})) \), choose \( \gamma \in \tilde{U} \) such that \( z = \varphi(d(\gamma)) \). The choice of \( \tilde{U} \) implies that we can assume that \( \gamma = (u, g, v) \), thus \( z = \varphi(v) \). Note that \((\varphi(u), a(\gamma)z) \) and \((\varphi(u), a'(\gamma)z) \), thus \((a(\gamma)z, a'(\gamma)z) \) are in \( \mathcal{S} \). Hence \((a(x, g, y)z, a'(x, g, y)z) \in \mathcal{S} \) for each \( z \in \varphi(d(\tilde{U})) \). The essential freeness of \( H \curvearrowright_{\beta} (Y, \mathcal{S}) \) implies \( a(x, g, y) = a'(x, g, y) \). By symmetry, \( b \) is uniquely determined by (4.2).

**Lemma 4.8.** In Definition 4.2, if \( G \curvearrowright_{\alpha} (X, \mathcal{R}) \) and \( H \curvearrowright_{\beta} (Y, \mathcal{S}) \) are essentially free, then the mappings \( a \) and \( b \) are cocycles on \( \mathcal{R} \times_{\alpha} G \) and \( \mathcal{S} \times_{\beta} H \), respectively.

**Proof.** We only need to show that the mappings \( a \) and \( b \) in Lemma 4.4 are cocycles. Let \( \gamma_1 = (x, g, y), \gamma_2 = (y, h, z) \in \mathcal{R} \times_{\alpha} G \) be arbitrary, and write \( \gamma' = \gamma_1\gamma_2 = (x, gh, z) \). From the continuity of \( a \) and \( \rho_{\alpha} \), choose open neighbourhoods \( U, V \) and \( W \) of \( \gamma_1, \gamma_2 \) and \( \gamma' \) in \( \mathcal{R} \times_{\alpha} G \), respectively, such that \( a(\gamma) = a(\gamma_1), \rho_{\alpha}(\gamma) = g \) for each \( \gamma \in U \), \( a(\eta) = a(\gamma_2), \rho_{\alpha}(\eta) = h \) for each \( \eta \in V \), and \( a(\sigma) = a(\gamma'), \rho_{\alpha}(\sigma) = gh \) for each \( \sigma \in W \). Since the multiplication on \( (\mathcal{R} \times_{\alpha} G)^{(2)} \) is continuous at \( (\gamma_1, \gamma_2) \), we can assume that \( \gamma\eta \in W \) when \( \gamma \in U, \eta \in V \) and \( (\gamma, \eta) \in (\mathcal{R} \times_{\alpha} G)^{(2)} \). Also since the range \( r \) and domain \( d \) are local homeomorphisms and \( d(\gamma_1) = r(\gamma_2) = y \), we can assume that the restrictions \( d|_{U} \) and \( r|_{V} \) are homeomorphisms onto their respective ranges and \( d(U) = r(V) \).
For each $\tilde{y} \in \varphi(d(V))$, choose $\eta \in V$ such that $\tilde{y} = \varphi(d(\eta))$. The choice of $V$ permits us to assume that $\eta = (v, h, w)$ and $a(\eta) = a(\gamma_2)$. Hence $\tilde{y} = \varphi(w)$. Since $v \in r(V) = d(U)$, it follows from the choice of $U$ that we have a $\gamma = (u, g, v) \in U$ and $a(\gamma) = a(\gamma_1)$. Hence $\gamma \eta = (u, gh, w) \in W$ and $a(\eta \gamma) = a(\gamma')$. The hypothesis on $\Psi$ in Lemma 4.4 implies that $(\varphi(u), a(\gamma) \varphi(v))$, $(\varphi(v), a(\eta) \varphi(w))$ and $(\varphi(u), a(\gamma) \eta \varphi(w))$ are all in $S$. Thus, $(a(x, g, y) a(y, h, z) \eta y, a(x, gh, z) \eta y)$ is in $S$ for every $\tilde{y} \in \varphi(d(V))$. The essential freeness of $H \lhd_\beta (Y, S)$ implies $a(x, g, y) a(y, h, z) = a(x, gh, z)$, thus $a$ is a cocycle. By a similar way, we can show that $b$ is a cocycle. 

\textbf{Lemma 4.9.} In Definition 4.2, if $G \lhd_\alpha (X, \mathcal{R})$ and $H \lhd_\beta (Y, \mathcal{S})$ are essentially free, then

$$b((\varphi(x), a((x, y), g) \varphi(g^{-1}y)), a((x, y), g)) = g,$$

$$a((\varphi^{-1}(u), b((u, v), h) \varphi^{-1}(1^{-1}v)), b((u, v), h)) = h$$

for every $((x, y), g) \in \mathcal{R} \times G$ and $((u, v), h) \in \mathcal{S} \times H$.

\textbf{Proof.} We only show that the maps $a$ and $b$ in Lemma 4.4 satisfy that

$$b(\varphi(x), a(x, g, y), \varphi(y)) = g, \quad a(\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) = h$$

for every $(x, g, y) \in \mathcal{R} \times_\alpha G$ and $(u, h, v) \in \mathcal{S} \times_\beta H$.

As before, let $\rho_\alpha$ and $\rho_\beta$ be the canonical cocycles on $\mathcal{R} \rtimes_\alpha G$ and $\mathcal{S} \rtimes_\beta H$, respectively. For an arbitrary $(x, g, y) \in \mathcal{R} \rtimes_\alpha G$, we have $(\varphi(x), h, \varphi(y)) \in \mathcal{S} \rtimes_\beta H$, where $h = a(x, g, y)$. From the continuity of $b$ and $\rho_\beta$, there exists an open neighbourhood $U$ of $(\varphi(x), h, \varphi(y))$ in $\mathcal{S} \rtimes_\beta H$ such that $\rho_\beta(\gamma) = h$, $b(\gamma) = b(\varphi(x), h, \varphi(y))$ for every $\gamma \in U$, and $r|_U$, $d|_U$ are homeomorphisms from $U$ onto $r(U)$ and $d(U)$, respectively.

By the continuity of $\rho_\alpha$, $\Psi$ and $a$ at $(x, g, y)$, as well as that of $\varphi$ at $x$ and $y$, there is an open neighbourhood $V$ of $(x, g, y)$ in $\mathcal{R} \rtimes_\alpha G$ such that

(i) $\rho_\alpha(\gamma) = g$, $a(\gamma) = h$ and $\Psi(\gamma) \in U$ for every $\gamma \in V$;

(ii) $r|_V$ and $d|_V$ are homeomorphisms from $V$ onto $r(V)$ and $d(V)$, respectively;

(iii) $\varphi(r(V)) \subseteq r(U)$ and $\varphi(d(V)) \subseteq d(U)$.

For each $v \in d(V)$, let $\gamma \in V$ such that $d(\gamma) = v$. The above condition (i) implies that we can let $\gamma = (u, g, v)$ and have $a(\gamma) = h$, thus $\Psi(\gamma) = (\varphi(u), h, \varphi(v)) \in U$. The map $\tilde{\Psi}$ gives that $(u, b(\varphi(u), h, \varphi(v))v) \in \mathcal{R}$. From the choice of $U$, $b(\varphi(u), h, \varphi(v)) = b(\varphi(x), h, \varphi(y))$. So $(u, b(\varphi(x), h, \varphi(y))v) \in \mathcal{R}$. Also since $(u, gv) \in \mathcal{R}$, we have $(b(\varphi(x), h, \varphi(y))v, gv) \in \mathcal{R}$. The essential freeness of $G \lhd_\alpha (X, \mathcal{R})$ implies that $b(\varphi(x), a(x, g, y), \varphi(y)) = b(\varphi(x), h, \varphi(y)) = g$. 

\[ \square \]
By a similar way, we can show that \( a(\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) = h \) for each \((u, h, v) \in S \rtimes_\beta H\).

\[ \square \]

The following definition comes from [9, Definition 4.1].

**Definition 4.10.** For two étale equivalence relations \( \mathcal{R} \) and \( \mathcal{S} \) on \( X \) and \( Y \), let \( G \curvearrowright X \) and \( H \curvearrowright Y \) be two systems generating two automorphism systems \( G \curvearrowright (X, \mathcal{R}) \) and \( H \curvearrowright (Y, \mathcal{S}) \). We say that \( G \curvearrowright X \) and \( H \curvearrowright Y \) continuously orbit equivalent up to \( \mathcal{R} \) and \( \mathcal{S} \), if there exist a homeomorphism, \( \varphi : X \to Y \), continuous cocycles \( a : X \rtimes G \to H, b : Y \rtimes H \to G, \sigma : \mathcal{R} \to H, \) and \( \tau : \mathcal{S} \to G \) satisfying the following conditions:

(i) \( \sigma(x, y)a(y, g) = a(x, g)\sigma(g^{-1}x, g^{-1}y) \) for \((x, y) \in \mathcal{R} \) and \( g \in G \);

(ii) \( \tau(x, y)b(y, g) = b(x, g)\tau(g^{-1}x, g^{-1}y) \) for \((x, y) \in \mathcal{S} \) and \( g \in H \);

(iii) The map, \( \xi_1 : (x, g) \in X \times G \to (a(x, g)^{-1}\varphi(x), \varphi(g^{-1}x)) \in \mathcal{S} \), is well-defined and continuous; Moreover,

\[
\xi_1(x, g) = \tau(x, y)a(y, g) \quad \text{for} \quad x \in X \quad \text{and} \quad g \in G.
\]

(iv) The map, \( \xi_2 : (x, g) \in Y \times H \to (b(x, g)^{-1}\varphi^{-1}(x), \varphi^{-1}(g^{-1}x)) \in \mathcal{R} \), is well-defined and continuous; Moreover,

\[
\xi_2(x, g) = a(\varphi^{-1}(x), b(x, g))\sigma(x, y) \quad \text{for} \quad x \in Y \quad \text{and} \quad g \in H.
\]

(v) The map, \( \eta_1 : (x, y) \in \mathcal{R} \to (\sigma(x, y)^{-1}\varphi(x), \varphi(y)) \in \mathcal{S} \), is well-defined and continuous; Moreover,

\[
\eta_1(x, y) = b(\varphi(x), \sigma(x, y))\tau(x, y) \quad \text{for} \quad (x, y) \in \mathcal{R}.
\]

(vi) The map, \( \eta_2 : (x, y) \in \mathcal{S} \to (\tau(x, y)^{-1}\varphi^{-1}(x), \varphi^{-1}(y)) \in \mathcal{R} \), is well-defined and continuous; Moreover,

\[
\eta_2(x, y) = a(\varphi^{-1}(x), \tau(x, y))\sigma(\varphi(x), y) \quad \text{for} \quad (x, y) \in \mathcal{S}.
\]

**Proposition 4.11.** Let \( G \curvearrowright_\alpha (X, \mathcal{R}) \) and \( H \curvearrowright_\beta (Y, \mathcal{S}) \) be two automorphism systems. Then \( G \curvearrowright X \) and \( H \curvearrowright Y \) are continuously orbit equivalent up to \( \mathcal{R} \) and \( \mathcal{S} \) if and only if \( \mathcal{R} \rtimes_\alpha G \) and \( \mathcal{S} \rtimes_\beta H \) are isomorphic as étale groupoids.

The proof of this proposition is the same as that of [9, Theorem 4.2] in which the local conjugacy is not necessary. We only provide a brief proof. For details, see [9, Theorem 4.2].

**Proof.** Assume that \( \Lambda : \mathcal{R} \rtimes_\alpha G \to \mathcal{S} \rtimes_\beta H \) is an isomorphism. Let \( \varphi \) be the restriction of \( \Lambda \) to the unit space \((\mathcal{R} \times G)^{(0)}\) and let \( a(x, g) = \rho_\beta \Lambda(x, g, g^{-1}x), \sigma(x, y) = \rho_\beta \Lambda(x, e, y), \) and \( b(u, h) = \rho_\alpha \Lambda^{-1}(u, h, h^{-1}u), \)
\[ \tau(u, v) = \rho_\alpha \Lambda^{-1}(u, e, v). \]

Then \( \varphi, a, b, \sigma \) and \( \tau \) satisfy the requirements in Definition 4.10, thus \( G \curvearrowright X \) and \( H \curvearrowright Y \) are continuously orbit equivalent up to \( \mathcal{R} \) and \( \mathcal{S} \).

Conversely, assume that there are maps \( \varphi, a, b, \sigma \) and \( \tau \) satisfying the requirements in Definition 4.10. Define

\[ \Lambda(x, g, y) = (\varphi(x), a(x, g)\sigma(g^{-1}x, y), \varphi(y)) \quad \text{for} \quad (x, g, y) \in \mathcal{R}_\alpha \times G. \]

Then \( \Lambda \) is an isomorphism from \( \mathcal{R} \times_\alpha G \) onto \( \mathcal{S} \times_\beta H \), whose inverse \( \Lambda^{-1} \) is defined by \( \Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h)\tau(h^{-1}u, v), \varphi^{-1}(v)) \).

**Theorem 4.12.** Assume that \( G \curvearrowright (X, \mathcal{R}) \) and \( H \curvearrowright (Y, \mathcal{S}) \) are essentially free. Then the following statements are equivalent.

1. \( G \curvearrowright (X, \mathcal{R}) \sim_{\cC} H \curvearrowright (Y, \mathcal{S}) \);
2. \( G \curvearrowright X \) and \( H \curvearrowright Y \) are continuously orbit equivalent up to \( \mathcal{R} \) and \( \mathcal{S} \);
3. \( \mathcal{R} \times_\alpha G \) and \( \mathcal{S} \times_\beta H \) are isomorphic as étale groupoids;
4. there exists a \( C^* \)-isomorphism \( \Phi \) from \( C^*_r(\mathcal{R} \times_\alpha G) \) onto \( C^*_r(\mathcal{S} \times_\beta H) \) such that \( \Phi(C(X)) = C(Y) \).

**Proof.** The equivalence of (ii) and (iii) follows from Proposition 4.11. From Lemma 4.6, \( \mathcal{R} \times_\alpha G \) and \( \mathcal{S} \times_\beta H \) are topological principal, thus the equivalence of (iii) and (iv) follows from \([7, 22]\).

Assume (iii) holds, i.e., there is an isomorphism \( \Lambda \) from \( \mathcal{R} \times_\alpha G \) onto \( \mathcal{S} \times_\beta H \). Let \( \varphi \) be the restriction of \( \Lambda \) to the unit space \( X \), and let \( a(x, g, y) = \rho_\beta \Lambda(x, g, y) \) for \( (x, g, y) \in \mathcal{R} \times_\alpha G \), \( b(u, h, v) = \rho_\alpha \Lambda^{-1}(u, h, v) \) for \( (u, h, v) \in \mathcal{S} \times_\beta H \). Then \( \varphi \) is a homeomorphism from \( X \) onto \( Y \), and \( \Lambda(x, g, y) = (\varphi(x), a(x, g, y), \varphi(y)) \) and \( \Lambda^{-1}(u, h, v) = (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) \). So \( \varphi, a \) and \( b \) satisfy the requirements in Lemma 4.4, thus \( G \curvearrowright (X, \mathcal{R}) \sim_{\cC} H \curvearrowright (Y, \mathcal{S}) \), i.e., (i) holds.

Assume (i) holds. From Lemma 4.4, there exist mappings \( \varphi, a \) and \( b \) such that the mappings \( \Psi : (x, g, y) \in \mathcal{R} \times_\alpha G \rightarrow (\varphi(x), a(x, g, y), \varphi(y)) \in \mathcal{S} \times_\beta H \) and \( \tilde{\Psi} : (u, h, v) \in \mathcal{S} \times_\beta H \rightarrow (\varphi^{-1}(u), b(u, h, v), \varphi^{-1}(v)) \in \mathcal{R} \times_\alpha G \) are continuous. From Lemma 4.8 and Lemma 4.9, \( \Psi \) and \( \tilde{\Psi} \) are étale groupoid isomorphisms and inverse to each other, thus (iii) holds.

**Remark 4.13.** For \( G \curvearrowright X \), let \( \mathcal{R}_1 = \{(x, x) : x \in X\} \) be as in Proposition 4.3. Then \( \mathcal{R}_1 \times_\alpha G \) is isomorphic to the transformation groupoid \( X \times G \), and the notions of continuous orbit equivalence for \( G \curvearrowright (X, \mathcal{R}_1) \) and \( G \curvearrowright X \) in the Li’s sense are consistent. Hence Theorem 4.12 is a generalization of Theorem 1.2 in \([10]\).
There are two special cases for orbit equivalence of two systems $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ via a homeomorphism $\varphi : X \to Y$. One is, for each $g \in G$, there is $h \in H$ such that $(h\varphi(x), \varphi(y)) \in \mathcal{S}$ for each $x, y \in X$ with $(gx, y) \in \mathcal{R}$, and by symmetry, for each $h \in H$, there is $g \in G$ such that $(g\varphi^{-1}(x), \varphi^{-1}(y)) \in \mathcal{R}$ for each $x, y \in Y$ with $(hx, y) \in \mathcal{S}$. The other is, for each $g \in G$ and $x \in X$, there is $h \in H$ such that $(h\varphi(x), \varphi(z)) \in \mathcal{S}$ for each $(gx, z) \in \mathcal{R}$, and by symmetry, for each $h \in H$ and $y \in Y$, there is $g \in G$ such that $(g\varphi^{-1}(y), \varphi^{-1}(z)) \in \mathcal{R}$ for each $(hy, z) \in \mathcal{S}$. Inspired by these ideas, we have the following notions, comparing with those of (strong) asymptotic conjugation in [9, Definition 4.4].

**Definition 4.14.** We say $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ **strongly continuously orbit equivalent**, write $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{scoe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$, if they are continuously orbit equivalent and in Definition 4.2 we can take the maps $a(\gamma, g) = a(\gamma', g)$ and $b(\nu, h) = b(\nu', h)$ for all $\gamma, \gamma' \in \mathcal{R}$ and $\nu, \nu' \in \mathcal{S}$.

We say these two systems **weakly continuously orbit equivalent**, write $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{wcoe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$, if they are continuously orbit equivalent and in Definition 4.2 we can take the maps $a(\gamma, g) = a(\gamma', g)$ for $\gamma, \gamma' \in \mathcal{R}$ with $d(\gamma) = d(\gamma')$, and $b(\nu, h) = b(\nu', h)$ for $\nu, \nu' \in \mathcal{S}$ with $d(\nu) = d(\nu')$.

**Remark 4.15.** Clearly, the strong continuous orbit equivalence implies the weak one. If $G \curvearrowright_{\alpha} X$ is free, then $G \curvearrowright_{\alpha} (X, \mathcal{R}_1)$ and $G \curvearrowright_{\alpha} (X, \mathcal{R}_2)$ in Proposition 4.3 are continuously orbit equivalent, but not weakly continuously orbit equivalent, because they do not satisfy the second special case.

The following corollary is an analogy to [9, Proposition 4.5]

**Corollary 4.16.** Assume that $G \curvearrowright_{\alpha} (X, \mathcal{R})$ and $H \curvearrowright_{\beta} (Y, \mathcal{S})$ are essentially free. Then

(i) $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{wcoe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$ if and only if there is an isomorphism $\Lambda : \mathcal{R} \rtimes_{\alpha} G \to \mathcal{S} \rtimes_{\beta} H$ such that $\Lambda(\mathcal{R}) = \mathcal{S}$ if and only if there is a $C^*$-isomorphism $\Phi : C^*_r(\mathcal{R} \rtimes_{\alpha} G) \to C^*_r(\mathcal{S} \rtimes_{\beta} H)$ such that $\Phi(C(X)) = C(Y)$ and $\Phi(C^*_r(\mathcal{R})) = C^*_r(\mathcal{S})$.

(ii) $G \curvearrowright_{\alpha} (X, \mathcal{R}) \sim_{\text{scoe}} H \curvearrowright_{\beta} (Y, \mathcal{S})$ if and only if there exist a homeomorphism $\varphi : X \to Y$ and a group isomorphism $\theta : G \to H$ such that $\Lambda : (x, g, y) \in \mathcal{R} \rtimes_{\alpha} G \to (\varphi(x), \theta(g), \varphi(y)) \in \mathcal{S} \rtimes_{\beta} H$ is an isomorphism.

Furthermore, when $\mathcal{R}$ and $\mathcal{S}$ are minimal or $X$ and $Y$ are connected, these two notions of strong continuous orbit equivalence and weak continuous orbit equivalence are consistent.
Proof. One can check that if the map $a$ in Definition 4.2 is a cocycle on $\mathcal{R} \rtimes_\alpha G$, then $a(\gamma, g) = a(\gamma', g)$ for $\gamma, \gamma' \in \mathcal{R}$ with $d(\gamma) = d(\gamma')$ if and only if $a(\gamma, e) = e$ for all $\gamma \in \mathcal{R}$. By symmetry, $b$ has a similar characterization when it is a cocycle. From Lemma 4.8, Theorem 4.12 and its proof, we can obtain (i) and (ii).

We now show that the weak continuous orbit equivalence of $G \rtimes_\alpha (X, \mathcal{R})$ and $H \rtimes_\beta (Y, \mathcal{S})$ implies the strong one when $\mathcal{R}$ and $\mathcal{S}$ are minimal or $X$ and $Y$ are connected. To see this, by assumption and the first paragraph of this proof, we have a homeomorphism $\varphi$ and two continuous cocycles $a, b$ with $a(x, e, y) = e$ for all $(x, y) \in \mathcal{R}$ and $b(u, e, v) = e$ for $(u, v) \in \mathcal{S}$, satisfying Lemma 4.4.

Assume that $X$ and $Y$ are connected. For each $g \in G$, the map $x \in X \to a(x, g, g^{-1}x) \in H$ is continuous, thus it is a constant. Hence $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$ for all $x, y \in X$. By symmetry, $b$ has a similar property.

Assume that $\mathcal{R}$ and $\mathcal{S}$ are minimal. For $(x, y) \in \mathcal{R}$ and $g \in G$, since $(x, g, g^{-1}x)(g^{-1}x, e, g^{-1}y)(g^{-1}y, g^{-1}y) = (x, e, y)$, we have $a(x, g, g^{-1}x) = a(g^{-1}y, g^{-1}y)^{-1} = a(y, g, g^{-1}y)$. Given arbitrary $x, y \in X$ and $g \in G$, we choose a sequence $\{x_n\}$ in $[x]_\mathcal{R}$ converging to $y$ in $X$. Thus $\{(x_n, g, g^{-1}x_n)\}$ converges to $(y, g, g^{-1}y)$ in $\mathcal{R} \rtimes_\alpha G$, thus the continuity of $a$ implies that $a(x, g, g^{-1}x) = a(y, g, g^{-1}y)$.

Remark that $a(x, g, y) = a(x, g, g^{-1}x)a(g^{-1}x, e, y) = a(x, g, g^{-1}x)$ for $(x, y) \in \mathcal{R} \rtimes_\alpha G$. Consequently, if one of the above two assumptions holds, then $a(x, g, y) = a(u, g, v)$ for $(x, g, y), (u, g, v) \in \mathcal{R} \rtimes_\alpha G$. By a similar way, we can show that $b$ satisfies a similar requirement. Hence $G \rtimes_\alpha (X, \mathcal{R})$ and $H \rtimes_\beta (Y, \mathcal{S})$ are strongly continuously orbit equivalent.

\[\Box\]

5. Local conjugacy relations from expansive systems

A system $G \rtimes_\alpha X$ is called expansive if the action $\alpha$ is expansive, which means for a metric $d$ on $X$ compatible with the topology, there exists a constant $\delta > 0$ such that, for $x, y \in X$, if $d(gx, gy) < \delta$ for all $g \in G$ then $x = y$. For convenience, given a real-valued function $\psi$ on $G$, the notation $\lim_{g \to \infty} \psi(g) = 0$ means that, for any $\epsilon > 0$, there exists a finite subset $F$ of $G$ such that $|\psi(g)| < \epsilon$ for all $g \notin F$.

A triple $(U, V, \gamma)$, consisting of open subsets $U, V$ of $X$ and a homeomorphism $\gamma : U \to V$, is called a local conjugacy, if $\lim_{g \to \infty} \sup_{z \in U} d(gz, g\gamma(z)) = 0$. Two points $x$ and $y$ in $X$ are said to be locally conjugate, if there exists a
local conjugacy \((U, V, \gamma)\) such that \(x \in U, y \in V\) and \(\gamma(x) = y\). Let

\[ R_\alpha = \{(x, y) \in X \times X : x \text{ and } y \text{ are locally conjugate}\} \]

be the local conjugacy relation on \(X\). From [21] (also see [26]), \(R_\alpha\) is an étale equivalence relation on \(X\) under the topology whose base consists of the sets of the form

\[ \{(x, \gamma(x)) : x \in U\}, \]

where \((U, V, \gamma)\) is a local conjugacy. Moreover, \(G \acts X\) induces an automorphism system \(G \acts \alpha R_\alpha\): \(g(x, y) = (gx, gy)\) for \(g \in G\) and \((x, y) \in R_\alpha\). Thus we have an automorphism system \(G \acts (X, R_\alpha)\).

**Remark 5.1.** If two expansive systems \(G \acts X\) and \(H \acts Y\) are conjugate by a homeomorphism \(\varphi\) from \(X\) onto \(Y\) and a group isomorphism \(\rho\) from \(G\) onto \(H\), then \((\varphi(U), \varphi(V), \varphi\gamma\varphi^{-1}|_{\varphi(U)})\) is a local conjugacy for each local conjugacy \((U, V, \gamma)\), thus \(\varphi \times \varphi : (x, y) \in R_\alpha \rightarrow (\varphi(x), \varphi(y)) \in R_\beta\) is an isomorphism. Hence \(G \acts \alpha (X, R_\alpha)\) and \(H \acts \beta (Y, R_\beta)\) are conjugate, thus two notions of conjugacy for \(G \acts \alpha (X, R_\alpha)\) and \(G \acts \alpha X\) are consistent.

From [9] and [12], the automorphism systems of local conjugacy relations associated to a full shift \(G \acts A^G\) over a finite set \(A\) and an irreducible Smale space \((X, \psi)\) are essentially free. The following result generalizes Matsumoto’s result in the Smale space case to the \(\mathbb{Z}\)-expansive system case.

**Theorem 5.2.** Let \(R_\varphi\) be the local conjugacy relation from an expansive system \(\mathbb{Z} \acts \alpha X\) generated by a homeomorphism \(\varphi\) on \(X\). Assume that \(X\) is infinite and has no isolated points. Then \(\mathbb{Z} \acts \alpha (X, R_\varphi)\) is essentially free.

**Proof.** For an arbitrary integer \(p \geq 1\), we first claim that the set

\[ X_p = \{x \in X : \lim_{n \to \infty} \varphi^{pn}(x) \text{ and } \lim_{n \to \infty} \varphi^{-pn}(x) \text{ exist}\} \]

is countable.

In fact, when \(p = 1\), it follows from [11] Theorem 2.2.22 that \(X_1\) is countable. Moreover, the expansiveness of \(\varphi^p\) implies that \(X_p\) is also countable for every \(p\). For completion, we provide a proof for the claim. Since \(\varphi\) is expansive, it follows that the \(p\)-periodic point set \(F_p(\varphi) = \{x \in X : \varphi^p(x) = x\}\) is finite, say \(F_p(\varphi) = \{y_1, y_2, \ldots, y_k\}\). For each \(x \in X_p\), let \(\lim_{n \to \infty} \varphi^{pn}(x) = y\) and \(\lim_{n \to \infty} \varphi^{-pn}(x) = z\). One can see that \(y, z \in F_p(\varphi)\). For \(1 \leq i, j \leq k\), set \(X_p(i, j) = \{x \in X : \lim_{n \to \infty} \varphi^{pn}(x) = y_i, \lim_{n \to \infty} \varphi^{-pn}(x) = y_j\}\). Given \(x \in X_p(i, j)\), we have \(\lim_{n \to \infty} \varphi^{pn+r}(x) = \varphi^r(y_i)\) and \(\lim_{n \to \infty} \varphi^{-pn-r}(x) = \varphi^{-r}(y_j)\) for each \(0 \leq r \leq p - 1\). Hence there exists an integer \(N \geq 2\) such that
\[ d(\varphi^n(x), \varphi^n(y)) < \frac{c}{2} \] for all \( n \geq N \) and \( d(\varphi^{-n}(x), \varphi^{-n}(y)) < \frac{c}{2} \) for all \( n \leq -N \), where \( c \) is an expansive constant for \( \varphi \). Set
\[
X_{p,N}(i,j) = \left\{ x \in X : \begin{array}{l}
\alpha d(\varphi^n(x), \varphi^n(y)) < \frac{c}{2} \text{ for all } n \geq N \\
\alpha d(\varphi^{-n}(x), \varphi^{-n}(y)) < \frac{c}{2} \text{ for all } n \leq -N
\end{array} \right\}.
\]
Thus \( X_p(i,j) = \bigcup_{N=2}^\infty X_{p,N}(i,j) \) and \( X = \bigcup_{1 \leq i,j \leq k} X_p(i,j) \). To finish the claim, we show, for each \( N \geq 2 \) and \( 1 \leq i,j \leq k \), the set \( X_{p,N}(i,j) \) is finite.

For otherwise, \( X_{p,N}(i,j) \) is infinite for some \( i,j,N \). Choose \( \delta < \frac{c}{2} \) such that if \( d(y,z) \leq \delta \) for \( y,z \in X \) then \( d(\varphi^l(y), \varphi^l(z)) < \frac{c}{2} \) for each integer \( l \) with \( |l| \leq N - 1 \). Since \( X_{p,N}(i,j) \) is infinite, there are two different \( y, z \) in \( X_{p,N}(i,j) \) such that \( d(y, z) < \delta \). Thus \( d(\varphi^l(y), \varphi^l(z)) < c \) for every integer \( l \), which implies that \( y = z \) by expansiveness of \( \varphi \) and is a contradiction. We have established the claim.

For each \( p \geq 1 \), we next claim that if \( x \in X \) with \( (x, \varphi^p(x)) \in \mathcal{R}_\varphi \), then \( x \in X_p \).

We use the method in [20], Lemma 5.3 to complete the claim. Assume that \( z \) is a limit point of \( \{ \varphi^{pn}(x) \mid n \geq 1 \} \). Choose a subsequence \( \{ m_n \} \) of positive integers such that \( n \to \infty d(z, \varphi^{pn_0}(x)) = 0 \). Thus \( \lim_{n \to \infty} d(\varphi^p(z), \varphi^{pn_0}\varphi^p(x)) = 0 \). Since \( (x, \varphi^p(x)) \in \mathcal{R}_\varphi \), we have \( \lim_{n \to \infty} d(\varphi^n(x), \varphi^n(\varphi^p(x))) = 0 \). Consequently, \( \varphi^p(z) = z \), which implies that each limit point of \( \{ \varphi^{pn}(x) \mid n \geq 1 \} \) is in \( F_p(\varphi) = \{ y_1, y_2, \ldots, y_k \} \). Choose an open neighbourhood \( U_i \) of \( y_i \) such that \( U_i \cap \overline{U_j} = \emptyset \) and \( \varphi^p(U_i) \cap \overline{U_j} = \emptyset \) for \( i \neq j \), where \( \overline{U_i} \) is the closure of \( U_i \). The limit point property of \( \{ \varphi^{pn}(x) \mid n \geq 1 \} \) shows that there exists \( N \geq 1 \) such that \( \varphi^{pn}(x) \in U_{i_0}^k \) for \( n \geq N \). If \( \varphi^{pn}(x) \in U_{i_0} \) for some \( i_0 \), then, by the choice of \( U_i \), \( \varphi^{pn}(x) \in U_{i_0} \) for all \( n \geq N \). Hence the sequence \( \{ \varphi^{pn}(x) \mid n \geq 1 \} \) has a unique limit point \( z_{i_0} \), thus it converges.

By a similar argument, one can obtain that \( \{ \varphi^{-pn}(x) \mid n \geq 1 \} \) converges. Thus \( x \in X_p \) and the claim is established. So for each nonzero integer \( p \), we have \( \{ x \in X : (x, \varphi^p(x)) \in \mathcal{R}_\varphi \} = \{ x \in X : (x, \varphi^{|p|}(x)) \in \mathcal{R}_\varphi \} \subseteq X_{|p|} \), thus \( \{ x \in X : (x, \varphi^p(x)) \in \mathcal{R}_\varphi \} \) is countable. Since \( X \) is infinite and has no isolated points, it follows that \( \{ x \in X : (x, \varphi^p(x)) \notin \mathcal{R}_\varphi \} \) is dense in \( X \) for each nonzero integer \( p \). Consequently, \( \mathbb{Z} \curvearrowright (X, \mathcal{R}_\varphi) \) is essentially free.

Recall that the action \( G \curvearrowright X \) is \textit{(topologically) transitive} if for all nonempty open set \( U, V \subseteq X \), there exists \( s \in G \) such that \( sU \cap V \neq \emptyset \).

In this case, choose a countable basis \( \{ U_n : n = 1, 2, \ldots \} \) for the topology on \( X \). The transitivity of \( \alpha \) implies that each open subset \( W_n = \bigcup_{g \in G} gU_n \) is dense in \( X \). It follows from the Baire category theorem that \( \bigcap_{n=1}^\infty W_n \) is dense in \( X \). Thus the set of points in \( X \) with dense orbit is dense.
Proposition 5.3. Let $\mathcal{R}_\alpha$ be the local conjugacy relation from an expansive and transitive action $G \curvearrowright \alpha X$. Assume that $X$ is infinite and has no isolated points and $G$ is an abelian group such that every subgroup generated by $g$ ($g \neq e$) has finite index in $G$. Then $G \curvearrowright \alpha (X, \mathcal{R}_\alpha)$ is essentially free.

Proof. Given $g \in G$, $g \neq e$, let $H^g$ be the subgroup generated by $g$ in $G$. Then $H^g$ has finite index, thus $H^g \curvearrowright \alpha |_{H^g} X$ is expansive, where $\alpha |_{H^g}$ is the restriction of $\alpha$ to $H^g$. So the set $F_{H^g}(\alpha) := \{ x \in X : hx = x, h \in H^g \}$ is finite. From hypothesis, $X$ is uncountable.

If we take an enumeration $s_1, s_2, \cdots$ of the elements of $G$, then $gs_1, gs_2, \cdots$ is also an enumeration of the elements of $G$. Let $x \in X$ with $(x, gx) \in \mathcal{R}_\alpha$. Assume that $z$ is a limit point of the sequence $\{ gs_n x : n = 1, 2, \cdots \}$. By a similar argument to the above theorem, one can see that $gz = z$, thus $z \in F_{H^g}(\alpha)$.

Assume that there exists a $g \in G$, $g \neq e$ such that the interior of $\{ x \in X, (x, gx) \in \mathcal{R}_\alpha \}$ is non-empty. Then the transitivity of $G \curvearrowright \alpha X$ implies that there exists a point $x \in X$ with $(x, gx) \in \mathcal{R}_\alpha$ and having dense orbit, i.e., $\{ gs_n x : n = 1, 2, \cdots \}$ is dense in $X$. From the second paragraph, every limit point of $\{ gs_n x : n = 1, 2, \cdots \}$ is contained in the finite set $F_{H^g}(\alpha)$. Thus the closure of $\{ gs_n x : n = 1, 2, \cdots \}$ in $X$ is countable, which contradicts the fact that $X$ is uncountable. Consequently, $G \curvearrowright \alpha (X, \mathcal{R}_\alpha)$ is essentially free. $\square$

6. Expansive automorphism actions on compact groups

Let $X$ be a compact metrizable group with an invariant compatible metric $d$, i.e., $d(xy, xz) = d(yx, zx) = d(y, z)$ for $x, y, z \in X$. Assume that $G \curvearrowright \alpha X$ is an expansive automorphism system in the sense that it is expansive and each $\alpha g$ is a continuous automorphism on $X$. Let

$$\Delta_\alpha = \{ x \in X : \lim_{g \to \infty} d(\alpha g(x), \alpha g(e)) = 0 \}$$

be the associated homoclinic group, which is an $\alpha$-invariant countable subgroup of $X$ in the sense that $\alpha g(a) \in \Delta_\alpha$ for every $a \in \Delta_\alpha$ and $g \in G$ ([26]). Denote by $\sigma$ the left-multiplication action of $\Delta_\alpha$ on $X$:

$$\sigma_u(x) = ux, \text{ for } u \in \Delta_\alpha \text{ and } x \in X,$$

and by $X \rtimes_\sigma \Delta_\alpha$ the associated transformation groupoid. Let $G \curvearrowright \alpha (X, \mathcal{R})$ be the automorphism system associated to the local conjugacy equivalence relation as in Section 5. The following facts are referred to [26] Lemma 3.7].

Lemma 6.1. Let $G \curvearrowright \alpha X$ be an expansive automorphism system. Then
(1) two elements \( x \) and \( y \) in \( X \) are locally conjugate, if and only if they are homoclinic, i.e., \( \lim_{g \to \infty} d(gx, gy) = 0 \), if and only if \( xy^{-1} \in \Delta_{\alpha} \), if and only if \( x^{-1}y \in \Delta_{\alpha} \), if and only if \( x^{-1} \) and \( y^{-1} \) are locally conjugate.

(2) The map \( \Lambda : (x, y) \in \mathcal{R} \to (x, xy^{-1}) \in X \rtimes_{\sigma} \Delta_{\alpha} \) is an étale groupoid isomorphism.

\textbf{Proof.} We only give a proof for (2). One can see that \( \Lambda \) is an algebraic isomorphism from \( \mathcal{R} \) onto \( X \rtimes_{\sigma} \Delta_{\alpha} \) with inverse map \( \Lambda^{-1} \), defined by \( \Lambda^{-1}(x, u) = (x, u^{-1}x) \) for \( (x, u) \in X \rtimes_{\sigma} \Delta_{\alpha} \). Given \( (x, y) \in \mathcal{R} \), for \( S \subseteq \Delta_{\alpha} \) and an open subset \( U \subseteq X \) with \( x \in U \) and \( xy^{-1} \in S \), we define \( \gamma(z) = yx^{-1}z \) for \( z \in U \). Then \( (U, \gamma(U), \gamma) \) is a local conjugacy from \( x \) to \( y \), and \( \Lambda(\{(z, \gamma(z)) : z \in U\}) \subseteq U \times S \), thus \( \Lambda \) is continuous at \( (x, y) \). By a similar way, we show that \( \Lambda^{-1} \) is continuous, thus \( \Lambda \) is a homeomorphism. \( \square \)

\textbf{Definition 6.2.} Let \( \Gamma = \Delta_{\alpha} \rtimes G \) be the semi-direct product of \( \Delta_{\alpha} \) by \( G \). Define the action \( \tilde{\alpha} \) of \( \Gamma \) on \( X \) as follows. For \( (a, g) \in \Gamma \) and \( x \in X \),

\[ \tilde{\alpha}_{(a,g)}(x) = a \alpha_{g}(x). \]

One can check that \( \Gamma \rtimes_{\tilde{\alpha}} X \) is an expansive affine system. Remark that \( \Delta_{\alpha} \) and \( G \) can be contained in \( \Gamma \) as subgroups by identifying \( a \in \Delta_{\alpha} \) with \( (a, e) \in \Gamma \), and \( g \in G \) with \( (e, g) \in \Gamma \), thus the restrictions of \( \tilde{\alpha} \) to \( \Delta_{\alpha} \) and \( G \) are the same as \( \sigma \) and \( \alpha \), respectively. Hence the transformation groupoid \( X \rtimes_{\tilde{\alpha}} \Gamma \) contains \( X \rtimes_{\sigma} \Delta_{\alpha} \) and \( X \rtimes_{\alpha} G \) as open subgroupoids.

\textbf{Proposition 6.3.} The map \( \Lambda : (x, g, y) \mapsto (x, (x \alpha_{g}(y^{-1}), g)) \) is an isomorphism of \( \mathcal{R} \rtimes_{\alpha} G \) onto \( X \rtimes_{\tilde{\alpha}} \Gamma \) as étale groupoids. Moreover, \( \Lambda(\mathcal{R}) = X \rtimes_{\sigma} \Delta_{\alpha} \), and \( \Lambda(X \rtimes_{\alpha} G) = X \rtimes_{\tilde{\alpha}} G \).

\textbf{Proof.} From Lemma 6.1, \( \Lambda \) is well-defined and injective. For each \( (x, (a, g)) \) in \( X \rtimes_{\tilde{\alpha}} \Gamma \), we have \( (x, g, \alpha_{g^{-1}}(a^{-1}x)) \in \mathcal{R} \rtimes_{\alpha} G \) and \( \Lambda(x, g, \alpha_{g^{-1}}(a^{-1}x)) = (x, (a, g)) \), thus \( \Lambda \) is bijective. For \( (x, g, y), (u, h, z) \) in \( \mathcal{R} \rtimes_{\alpha} G \), we have \( (x, g, y) \) and \( (u, h, z) \) are composable in \( \mathcal{R} \rtimes_{\alpha} G \), if and only if \( u = y \), if and only if \( \Lambda(x, g, y) \) and \( \Lambda(y, h, z) \) are composable in \( X \rtimes_{\tilde{\alpha}} \Gamma \). Moreover,

\[ \Lambda(x, g, y)\Lambda(y, h, z) = (x, (x \alpha_{g}(y^{-1}), g))(y, (y \alpha_{h}(z^{-1}), h)) = (x, (x \alpha_{gh}(z^{-1}), gh)) = \Lambda((x, g, y)(y, h, z)). \]

The continuity of \( \Lambda \) can be implied by Lemma 6.1 and the canonical homeomorphism \( \gamma_{0} \) from \( \mathcal{R} \rtimes_{\alpha} G \) onto \( \mathcal{R} \times G \). Hence \( \Lambda \) is an étale groupoid isomorphism. \( \square \)
Proposition 6.4. (i) The system $G \curvearrowright_{\alpha} X$ is topologically free, if and only if $\Gamma \curvearrowright_{\tilde{\alpha}} X$ is topologically free, if and only if $R \curvearrowleft_{\alpha} G$ is topologically principal, if and only if $G \curvearrowleft_{\alpha} (X, R)$ is essentially free.

(ii) If $G$ is torsion-free and $\Delta_{\alpha}$ is dense in $X$, then $G \curvearrowleft_{\alpha} X$ is topologically free.

Proof. (i) It follows from [10, Corollary 2.3], Lemma 4.6 and Proposition 6.3 that we only need to show that the topological freeness for $\alpha$ and $\tilde{\alpha}$ is consistent. Since $G$ can be embedded into $\Gamma$ as a subgroup and the restriction of $\tilde{\alpha}$ to $G$ is the same as the action $\alpha$, the topological freeness of $\tilde{\alpha}$ implies that of $\alpha$.

To see the contrary, it is sufficient to show that, for arbitrary $(e, e) \neq (a, g) \in \Gamma$ and non-empty open subset $U$ of $X$, there exists $x \in U$ such that $a \alpha_{g}(x) \neq x$.

In fact, since the restriction of $\tilde{\alpha}$ to $\Delta_{\alpha}$ is free, we can assume that $g \neq e$ and $a \neq e$. Clearly, we can also assume that there exists $y \in U$ such that $a \alpha_{g}(y) = y$. The topologically freeness of $\alpha$ implies there is $z \in y^{-1}U$ such that $\alpha_{g}(z) \neq z$. Let $z = y^{-1}x$ for $x \in U$. Then $a \alpha_{g}(x) \neq x$.

(ii) Given $g \in G$, assume there exists an open subset $U$ of $X$ such that $\alpha_{g}(z) = z$ for every $z \in U$. We can let $e \notin U$. Since $\Delta_{\alpha}$ is dense in $X$, there is $x_{0} \in U \cap \Delta_{\alpha}$, thus $\lim_{h \to \infty} d(\alpha_{h}(x_{0}), e) = 0$. If $g \neq e$, then, from the torsion-freeness of $G$, the set $\{g^{n} : n \in \mathbb{Z}\}$ is infinite, we have $\lim_{n \to \infty} d(\alpha_{g^{n}}(x_{0}), e) = 0$, which contradicts the fact $x_{0} \neq e$ and $\alpha_{g^{n}}(x_{0}) = x_{0}$ for all $n \in \mathbb{Z}$. Consequently, $g = e$, thus $\alpha$ is topologically free.

Recall that two automorphism systems $G \curvearrowright_{\alpha} X$ and $H \curvearrowright_{\beta} Y$ on compact metrizable groups are said to be algebraically conjugate if there exist a continuous isomorphism $\varphi : X \to Y$ and an isomorphism $\rho : G \to H$ such that $\varphi(\alpha_{g}(x)) = \beta_{\rho(g)}(\varphi(x))$ for $g \in G$ and $x \in X$. Form [2], when $X$ and $Y$ are abelian, two notions of algebraical conjugacy and conjugacy for automorphism systems are consistent. In the following we have a similar result for automorphism actions on nonabelian groups.

Proposition 6.5. Let $G \curvearrowright_{\alpha} (X, R)$ and $H \curvearrowright_{\beta} (Y, S)$ be two automorphism systems on local conjugacy relations from topologically free, expansive automorphism actions on compact and connected metrizable groups $X$ and $Y$, respectively. Then the following statements are equivalent:

(i) $G \curvearrowright_{\alpha} (X, R)$ and $H \curvearrowright_{\beta} (Y, S)$ are conjugate;
(ii) $G \curvearrowright_{\alpha} (X, R) \sim_{\text{wco}} H \curvearrowright_{\beta} (Y, S)$;
(iii) $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are continuously orbit equivalent;
(iv) $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are conjugate.

Moreover, if $\Delta_\alpha$ is dense in $X$, then the above conditions are equivalent to the following statement.
(v) $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are algebraically conjugate.

Proof. Since $X$ and $Y$ are connected, the continuous orbit equivalence and conjugacy of $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are consistent. To complete the proof, we only need to prove that (ii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (v) when $\Delta_\alpha$ is dense in $X$. From Corollary 4.16 and Proposition 6.3, there is an étale groupoid isomorphism $\Lambda : X \rtimes \alpha (\Delta_\alpha \rtimes G) \rightarrow Y \rtimes \beta (\Delta_\beta \rtimes H)$ such that $\Lambda(X \rtimes \sigma \Delta_\alpha) = Y \rtimes \sigma' \Delta_\beta$, where $\sigma$ and $\sigma'$ are the left-multiplication actions, and $\tilde{\alpha}$ and $\tilde{\beta}$ are as in Definition 6.2. Since $X$ and $Y$ are connected, there are a homeomorphism $\varphi : X \rightarrow Y$ and a group isomorphism $\theta : \Delta_\alpha \rtimes G \rightarrow \Delta_\beta \rtimes H$ such that

$$\varphi(a\alpha_g(x)) = \tilde{\beta}_{\theta(a,g)}(\varphi(x)) \quad \text{for every} \ (a, g) \in \Delta_\alpha \rtimes G \ 	ext{and} \ x \in X,$$

and $\theta(\Delta_\alpha) = \Delta_\beta$, where $\Delta_\alpha$ and $\Delta_\beta$ are subgroups of the semi-direct groups as before. Define two maps $\xi : G \rightarrow \Delta_\beta$ and $\rho : G \rightarrow H$ by $\theta(e, g) = (\xi(g), \rho(g))$ for $g \in G$. One can check that $\rho$ is a group isomorphism by considering the inverse isomorphism $\theta^{-1}$.

Letting $a = e$, the identity of $X$, in (6.1), we have $\varphi(\alpha_g(x)) = \xi(g)\beta_{\rho(g)}(\varphi(x))$ for every $g \in G$ and $x \in X$. In particular, $\varphi(e) = \xi(g)\beta_{\rho(g)}(\varphi(e))$. Thus

$$\varphi(\alpha_g(x)) = \varphi(e)\beta_{\rho(g)}(\varphi(e)^{-1}\varphi(x)) \quad \text{for} \ x \in X \ 	ext{and} \ g \in G.$$ Define $\tilde{\varphi}(x) = \varphi(e)^{-1}\varphi(x)$ for $x \in X$. Then $\tilde{\varphi} : X \rightarrow Y$ is a homeomorphism and

$$\tilde{\varphi}(\alpha_g(x)) = \beta_{\rho(g)}(\tilde{\varphi}(x)) \quad \text{for} \ x \in X \ 	ext{and} \ g \in G.$$ Consequently, $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are conjugate.

Assume that $\Delta_\alpha$ is dense in $X$. Remark that $\theta(a, e) \in \Delta_\beta$, thus $\tilde{\beta}_{\theta(a,e)}(y) = \theta(a, e)y$ for $a \in \Delta_\alpha$ and $y \in Y$. Letting $g = e$, the identity of $G$, and letting $x = e$, the identity of $X$, in (6.1), one can see that $\varphi(a) = \theta(a, e)\varphi(e)$ for $a \in \Delta_\alpha$. Thus, by putting $g = e$ in (6.1), we have $\varphi(ax) = \theta(a, e)\varphi(x) = (\varphi(a)\varphi(e)^{-1})\varphi(x)$, which implies that $\tilde{\varphi}(ax) = \tilde{\varphi}(a)\tilde{\varphi}(x)$ for every $a \in \Delta_\alpha$ and $x \in X$. From the density of $\Delta_\alpha$ in $X$, the map $\tilde{\varphi} : X \rightarrow Y$ is a continuous isomorphism. So $G \curvearrowright \alpha X$ and $H \curvearrowright \beta Y$ are algebraically conjugate.

Proposition 6.6. Let $G \curvearrowright \alpha (X, \mathcal{R})$ be an automorphism system on local conjugacy relation from a topologically free, expansive automorphism action. Then the following statements are equivalent.
(i) $C_r^*(\mathcal{R})$ is simple;
(ii) $C_r^*(\mathcal{R})$ has a unique tracial state;
(iii) $\Delta_\alpha$ is dense;
(iv) $C_r^*(\mathcal{R} \rtimes_\alpha G)$ is simple;
(v) $C_r^*(\mathcal{R} \rtimes_\alpha G)$ has a unique tracial state.

**Proof.** For the equivalence of (i), (ii) and (iii), we refer to [26, Corollary 3.9]. From Proposition 6.3, $C_r^*(\mathcal{R} \rtimes_\alpha G)$ is isomorphic to $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$, thus they have the same simplicity and the uniqueness of tracial states. From Proposition 6.4 and [10], $X \rtimes_{\bar{\alpha}} \Gamma$ is topologically principal, thus there is a one-to-one correspondence between the family of ideals of $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$ and that of $\bar{\alpha}$-invariant open subsets of $X$ ([21]).

Assume (iii) holds. Since each non-empty $\bar{\alpha}$-invariant open subset $U$ in $X$ is invariant by the left-multiplicative by elements in $\Delta_\alpha$, we have $U = X$. Hence $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$ is simple, thus (iv) holds. On the contrary, if (iv) holds, then $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$ is simple, which leads to the fact that the $\bar{\alpha}$-invariant open $X \setminus \overline{\Delta_\alpha}$ of $X$ is empty, where $\overline{\Delta_\alpha}$ is the closure of $\Delta_\alpha$ in $X$. Thus $\overline{\Delta_\alpha} = X$, i.e., (iii) holds.

For the implication (v) $\Rightarrow$ (iii), assume that $C_r^*(\mathcal{R} \rtimes_\alpha G)$, thus $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$, has a unique tracial state. If $\Delta_\alpha$ is not dense in $X$, then the Haar measure $\nu$ on $\overline{\Delta_\alpha}$ extends a Borel probability measure $\hat{\nu}$ on $X$ different from the Haar measure $\mu_0$ on $X$. Since $\mu_0$ is invariant under the actions $\sigma$, $\alpha$ and $\bar{\alpha}$, for a Borel subset $E$ of $X$ and $(a, g) \in \Gamma$, we have $\hat{\nu}(\bar{\alpha}(a,g)(E)) = \nu(\bar{\alpha}(a,g)(E \cap \overline{\Delta_\alpha})) = \hat{\nu}(E)$, thus $\hat{\nu}$ is $\bar{\alpha}$-invariant. The probability measures $\hat{\nu}$ and $\mu_0$ produce two different tracial states on $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$, which is a contradiction.

For the implication (iii) $\Rightarrow$ (v), assume that $\Delta_\alpha$ is dense in $X$. Then the Haar measure $\mu_0$ on $X$ is the unique $\bar{\alpha}$-invariant Borel probability measure on $X$. From [27, Proposition 3.2.4], $C(X) \rtimes_{r,\bar{\alpha}} \Gamma$, and thus $C_r^*(\mathcal{R} \rtimes_\alpha G)$ has a unique tracial state. \hfill \Box

**Example 6.7** (Hyperbolic toral automorphisms). For $n \geq 2$, we consider an expansive $\mathbb{Z}$-action on the $n$-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ generated by a single hyperbolic toral automorphism $\alpha$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the usual quotient map. Recall that $\mathbb{R}^n/\mathbb{Z}^n$ is a compact and connected additive group under the following metric compatible with the quotient topology:

$$d(\pi(x), \pi(y)) = \inf_{z \in \mathbb{Z}^n} \|x - y - z\|, \text{ for } x, y \in \mathbb{R}^n,$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^n$. The elements in $\mathbb{R}^n$ are denoted by column vectors or row vectors.
Let $A$ be the hyperbolic matrix in $GL(n,\mathbb{Z})$ with $\det(A) = \pm 1$ and having no eigenvalues of modules 1, such that

$$\alpha(\pi(x)) = \pi(Ax) \text{ for } x \in \mathbb{R}^n.$$ Then $\mathbb{R}^n = E^s \oplus E^u$, where $E^s = \{ x \in \mathbb{R}^n : \lim_{k \to +\infty} A^k x = 0 \}$ and $E^u = \{ w \in \mathbb{R}^n : \lim_{k \to +\infty} A^{-k} w = 0 \}$ are two invariant subspaces of the linear map on $\mathbb{R}^n$ determined by $A$. Remark that $E^s \cap \mathbb{Z}^n = \{ 0 \}$, $E^u \cap \mathbb{Z}^n = \{ 0 \}$, and both subgroups $\pi(E^s)$ and $\pi(E^u)$, as well as the homoclinic group $\Delta_\alpha = \pi(E^s) \cap \pi(E^u)$ induced by $\alpha$, are dense in $\mathbb{R}^n/\mathbb{Z}^n$. Moreover, the system $\mathbb{Z} \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ generated by $\alpha$ is topologically free (II).

Each $m \in \mathbb{Z}^n$ has the unique decomposition $m = m_\mathbb{Z} - m_\mathbb{R} \in E^s \oplus E^u$. Then the map $\theta : \mathbb{Z}^n \to \Delta_\alpha$ by $\theta(m) = \pi(m_\mathbb{Z}) (= \pi(m_\mathbb{R}))$ is a group isomorphism. As before, we let $\sigma$ be the translation action of $\Delta_\alpha$ on $\mathbb{R}^n/\mathbb{Z}^n$:

$$\sigma_u(x) = u + x \text{ for } u \in \Delta_\alpha, x \in \mathbb{R}^n/\mathbb{Z}^n.$$ Let $\tau$ be the action of $\mathbb{Z}^n$ on $\mathbb{R}^n/\mathbb{Z}^n$ by homeomorphisms:

$$\tau_n(x) = \theta(n) + x \text{ for } n \in \mathbb{Z}^n, x \in \mathbb{R}^n/\mathbb{Z}^n.$$ Then $\mathbb{Z}^n \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ and $\Delta_\alpha \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ are conjugate.

Denote by $\mathbb{Z}^n \rtimes \mathbb{Z}$ the semi-direct product of $\mathbb{Z}^n$ by the automorphism given by $A : m \in \mathbb{Z}^n \to Am \in \mathbb{Z}^n$. Let $\gamma$ be the action of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on $\mathbb{R}^n/\mathbb{Z}^n$:

$$\gamma(m,k)(x) = \theta(m) + \alpha^k(x) \text{ for } (m,k) \in \mathbb{Z}^n \rtimes \mathbb{Z} \text{ and } x \in \mathbb{R}^n/\mathbb{Z}^n.$$ So $\mathbb{Z}^n \rtimes \mathbb{Z} \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ and $\Delta_\alpha \rtimes \tilde{\alpha} \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ are conjugate, where $\tilde{\alpha}$ is given by Definition 6.2.

We consider the multiplicative coordinate system on the $n$-dimensional torus by $\mathbb{T}^n := \{ (z_1, z_2, \cdots, z_n) : z_i \in \mathbb{C}, |z_i| = 1, \text{ for } 1 \leq i \leq n \}$. The correspondence

$$\varphi : [(x_1, x_2, \cdots, x_n)] \in \mathbb{R}^n/\mathbb{Z}^n \to (e^{2\pi i x_1}, e^{2\pi i x_2}, \cdots, e^{2\pi i x_n}) \in \mathbb{T}^n$$

is an isomorphism between two representations, where $[(x_1, x_2, \cdots, x_n)] = \pi(x_1, x_2, \cdots, x_n)$ for $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Using this coordinate system, we can rewrite the above toral automorphism $\alpha$ and the actions $\tau, \gamma$ as follows. Let $A = (a_{ij})$ and $A^{-1} = (b_{ij})$. Define the automorphism $\beta$ of $\mathbb{T}^n$ by

$$\beta(z_1, z_2, \cdots, z_n) = (z_1^{a_{11}} z_2^{a_{12}} \cdots z_n^{a_{1n}}, z_1^{a_{21}} z_2^{a_{22}} \cdots z_n^{a_{2n}}, \cdots, z_1^{a_{n1}} z_2^{a_{n2}} \cdots z_n^{a_{nn}})$$

for $(z_1, z_2, \cdots, z_n) \in \mathbb{T}^n$, the rotation action $\rho$ of $\mathbb{Z}^n$ on $\mathbb{T}^n$ by

$$\rho_m(v) = \varphi(\theta(m))v \text{ for } m \in \mathbb{Z}^n \text{ and } v \in \mathbb{T}^n,$$

and the action $\tilde{\gamma}$ of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on $\mathbb{T}^n$ by

$$\tilde{\gamma}(m,k)(v) = \varphi(\theta(m))\beta^k(v) \text{ for } (m,k) \in \mathbb{Z}^n \rtimes \mathbb{Z} \text{ and } v \in \mathbb{T}^n.$$
Then $\mathbb{Z}^n \rtimes_\gamma \mathbb{R}^n/\mathbb{Z}^n$ and $\mathbb{Z}^n \rtimes_\rho \mathbb{T}^n$ are conjugate, and $\mathbb{Z}^n \rtimes \mathbb{Z} \rtimes_{\tilde{\gamma}} \mathbb{T}^n$ and $\mathbb{Z}^n \rtimes \mathbb{Z} \rtimes_{\tilde{\gamma}} \mathbb{R}^n/\mathbb{Z}^n$ are conjugate.

From Lemma 6.1, Proposition 6.3 and the above, the local conjugacy relation $\mathcal{R}$ given by $\alpha$ and the associated semi-direct product $\mathcal{R} \rtimes_\alpha \mathbb{Z}$ are isomorphic to the transformation groupoids $\mathbb{T}^n \rtimes_\rho \mathbb{Z}^n$ and $\mathbb{T}^n \rtimes_{\tilde{\gamma}} (\mathbb{Z}^n \rtimes \mathbb{Z})$, respectively.

We still denote by $\rho$ the automorphism action of $\mathbb{Z}^n$ on $C(\mathbb{T}^n)$ induced by the system $\mathbb{Z}^n \rtimes_\rho \mathbb{T}^n$:

$$\rho_m(f)(v) = f(\theta(\phi(m))^{-1}v)$$

for $m \in \mathbb{Z}^n$, $f \in C(\mathbb{T}^n)$ and $v \in \mathbb{T}^n$. Let $e_k$, $1 \leq k \leq n$, be the canonical basis of $\mathbb{Z}^n$ and $\theta(e_k) = [(\theta_{k1}, \cdots, \theta_{kn})] \in \Delta_\alpha$, where $\theta_{kj} \in [0, 1]$. Let $U_j$, $1 \leq j \leq n$, be the unitaries in $C(\mathbb{T}^n)$ defined by $U_j(z_1, \cdots, z_n) = z_j$ for $(z_1, \cdots, z_n) \in \mathbb{T}^n$, and let $V_k$, $1 \leq k \leq n$, be the unitaries implementing the $C^*$-automorphism $\rho e_k$ on $C(\mathbb{T}^n)$. One can check that

$$U_j U_k = U_k U_j, \quad V_j V_k = V_k V_j, \quad U_j V_k = e^{2\pi i \theta_{kj}} V_k U_j$$

(6.2)

for $1 \leq j, k \leq n$. From Proposition 6.4 and 6.6, $C^*_r(\mathcal{R})$, thus $C(\mathbb{T}^n) \rtimes_\rho \mathbb{Z}^n$ are simple and have unique tracial states. Hence $C(\mathbb{T}^n) \rtimes_\rho \mathbb{Z}^n$ is generated by $U_j$, $V_j$, $1 \leq j \leq n$, thus is the $2n$-dimensional noncommutative torus $A^\Theta_\alpha$ for a $2n \times 2n$ real skew-symmetric matrix $\Theta = (\bar{\theta}_{kl})$ defined by $\bar{\theta}_{kl} = 0$ for $1 \leq k, l \leq n$ or $n + 1 \leq k, l \leq 2n$, $\bar{\theta}_{kl} = \theta_{k(l-n)}$ for $1 \leq k \leq n$ and $n + 1 \leq l \leq 2n$, and $\bar{\theta}_{kl} = -\theta_{(k-l)}$ for $n + 1 \leq k \leq 2n$ and $1 \leq l \leq n$ ([23]). From [18], $C^*_r(\mathcal{R})$ is an AT-algebra with real rank zero and the range of the unique tracial state acting on $K_0(C^*_r(\mathcal{R}))$ is an isomorphism invariant.

Similarly, we also denote by $\tilde{\gamma}$ the automorphism action of $\mathbb{Z}^n \rtimes \mathbb{Z}$ on $C(\mathbb{T}^n)$ induced by the system $\mathbb{Z}^n \rtimes \mathbb{Z} \rtimes_{\tilde{\gamma}} \mathbb{T}^n$:

$$\tilde{\gamma}(m,k)(f)(v) = f((\phi(A^{-k}m))^{-1} \cdot \beta^{-k}(v))$$

for $(m,k) \in \mathbb{Z}^n \rtimes \mathbb{Z}$, $f \in C(\mathbb{T}^n)$ and $v \in \mathbb{T}^n$. Let $U_j$, $j = 1, 2, \cdots, n$, be the generating set of $C(\mathbb{T}^n)$ as above, and let $V'_j$, $j = 1, 2, \cdots, n$ and $W$ be the unitaries implementing the the automorphisms $\tilde{\gamma}(e_j,0)$ and $\tilde{\gamma}(0,1)$ associated to the generating set $(e_j,0)$ for $1 \leq j \leq n$ and $(0,1)$ of $\mathbb{Z}^n \rtimes \mathbb{Z}$. Then we have

$$U_j U_k = U_k U_j, \quad V'_j V'_k = V'_k V'_j, \quad U_j V'_k = e^{2\pi i \theta_{kj}} V'_k U_j$$

$$W U_j W^* = \prod_{l=1}^{n} U_l^{b_{lj}}, \quad W V'_j W^* = \prod_{l=1}^{n} V'_l^{a_{lj}}$$

(6.3)

for $1 \leq j, k \leq n$. Since $C(\mathbb{T}^n) \rtimes_{\tilde{\gamma}} (\mathbb{Z}^n \rtimes \mathbb{Z})$ is simple from Proposition 6.4 and 6.6, it is generated by the unitaries $U_j, V'_j$, $1 \leq j \leq n$, and $W$ satisfying the above relations.
The following results generalize [15, Theorem 2.9 and Proposition 6.1] and for the statement (1), we can refer to [26, Theorem 3.33 and 3.36].

**Proposition 6.8.** Let $\alpha$ be a hyperbolic toral automorphism on $\mathbb{R}^n/\mathbb{Z}^n$ defined by a hyperbolic matrix $A$. Let $\mathcal{R}$ be the local conjugacy relation induced by $\alpha$. Then

1. $C^*_\mathcal{R}(\mathcal{R})$ is generated by the unitaries $U_j, V_j$, $1 \leq j \leq n$, satisfying the relations (6.2), thus is isomorphic to a simple $2n$-dimensional noncommutative torus and is an AT-algebra with real rank zero.

2. $C^*_\mathcal{R}(\mathcal{R} \rtimes_\alpha \mathbb{Z})$ is generated by unitaries $U_j, V'_j$, $1 \leq j \leq n$, and $W$ satisfying the relations (6.3).

Moreover, two hyperbolic toral automorphisms on $\mathbb{R}^n/\mathbb{Z}^n$ are flip conjugate if and only if the $\mathbb{Z}$-actions they generates are continuously orbit equivalent up to the associated local conjugacy relations.

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