We find the general solution to Polchinski’s classical scattering equations for $1+1$ dimensional string theory. This allows efficient computation of scattering amplitudes in the standard Liouville $\times c = 1$ background. Moreover, the solution leads to a mapping from a large class of time-dependent collective field theory backgrounds to corresponding nonlinear sigma models. Finally, we derive recursion relations between tachyon amplitudes. These may be summarized by an infinite set of nonlinear PDE’s for the partition function in an arbitrary time-dependent background.
1. Introduction

Time-independent string backgrounds have been much studied at the classical level. The time-independence of the background allows the use of the methodology of Euclidean 2D conformal field theory. On the other hand, nontrivial time-dependent backgrounds ineluctably lead to problems with negative signature bosons and negative conformal weight vertex operators. The corresponding conformal field theories are expected to be more subtle than their Euclidean cousins.

While there are some isolated examples of time-dependent backgrounds in string theory, a large collection of tractable or solvable backgrounds has only recently become available. As a result of the recent developments in matrix model technology [1] and the parallel developments in Liouville theory [2–6], we now have available for study a tractable but nontrivial example of string theory, namely, string theory in 1 + 1 dimensions. Using the collective field theory obtained from the matrix model [7,8], Polchinski found a large class of nontrivial time-dependent classical backgrounds for 1+1 dimensional string theory [9]. Unfortunately, the physics of these solutions has been partly obscure since the relation of the matrix model coordinate $\tau$ [7,8] to the Liouville coordinate $\phi$ is nontrivial [10]. Thus the mapping of the solutions of [9] to a corresponding nonlinear $\sigma$-model has been unclear, even at the formal level.

In this note we investigate further Polchinski’s scattering equations (equations (2.5) below), and give the general solution of these equations in (3.4). From the classical solution we can extract genus zero $S$-matrix elements. For example, the result (4.4) extends the result of [11] for $1 \rightarrow m$ scattering to $n \rightarrow m$ scattering in a simple kinematic regime. We then propose a (formal) mapping from a given solution of Polchinski’s equations to a nonlinear sigma model in (5.8). Using the solution (3.4) we derive some recursion relations between tachyon amplitudes which are summarized by the nonlinear differential equations (6.6) for the partition function. We conclude with hopes for the future.

2. Classical Solutions of 2D String Theory

Classical 2D string theory can be formulated in terms of a single field theoretic degree of freedom, $\chi(\lambda, t)$ related to the “massless tachyon” degree of freedom. The (collective field theory) Hamiltonian is [7,8,12]

$$H = \int d\lambda \left\{ \frac{g^2}{2} \chi'^2 \pi^2 + \frac{\pi^2}{6g^2} (\chi')^3 + \frac{v(\lambda)}{g^2} \chi' \right\}$$  \hspace{1cm} (2.1)
where \( v(\lambda) \) is the double-scaled matrix model potential. Physically appropriate boundary conditions for the fields have not been carefully investigated. Roughly speaking, we require \( \pi \chi, \chi \) to vanish on a half-axis. Moreover \( \partial_\lambda \chi \geq 0 \). The classical field equations are nonlinear and look formidable, but by defining \( p_\pm \equiv -g^2 \pi \chi \pm \pi \chi' \) the equations separate \[8,9,13,14\]

\[
\partial_t p_\pm = -v'(\lambda) - p_\pm \partial_\lambda p_\pm . \tag{2.2}
\]

In particular, for the potential \( v(\lambda) = -\frac{1}{2} \lambda^2 \) the general solution to these equations was found by Polchinski \[9\]. He also gave a beautiful interpretation in terms of a time-dependent Fermi sea in free fermion phase space with coordinates \((\lambda, p)\), determined by the parametric equations:

\[
\begin{align*}
\lambda &= (1 + a(\sigma)) \cosh(\sigma - t) \quad (2.3a) \\
p &= (1 + a(\sigma)) \sinh(\sigma - t) . \quad (2.3b)
\end{align*}
\]

For \( a(\sigma) \) sufficiently small and sufficiently slowly-varying (see below) \[2.3a\] will have two solutions \( \sigma_\pm(\lambda, t) \), and substitution into \[2.3b\] defines upper and lower branches \( p_\pm(\lambda, t) \) of the sea. Using the change of variables \( \lambda = \cosh \tau, \ \tau > 0 \), we define the asymptotic waveforms \( \psi_\pm(t \pm \tau) \) in the far past and future by the limiting behavior

\[
p_\mp(\lambda, t) \to \mp \lambda \pm \frac{1}{2\lambda} \left( 1 + \psi_\pm(t \pm \tau) \right) + O(1/\lambda^2) \tag{2.4}
\]

where \( \lambda \to +\infty \) holding \( t \pm \tau \) fixed.

The essential remark \[9\] is that the incoming and outgoing asymptotic waveforms are related by a diffeomorphism \( x \mapsto \tilde{x} \), determined by the classical solution: \( \tilde{x} = x + \log(1 + \psi_-(x)) \). Thus, the basic equation of scattering, by which we can determine the outgoing waveform \( \psi_- \) in terms of the incoming waveform \( \psi_+ \) is the functional equation

\[
\psi_-(x) = \psi_+(\tilde{x}) = \psi_+(x + \log(1 + \psi_-(x))) . \tag{2.5}
\]

One can study the time-reversed process using the inverse diffeomorphism, i.e., \( \psi_+(y) = \psi_-(\tilde{y}) \), where \( \tilde{y} = y - \log(1 + \psi_+(y)) \). The asymptotic waveform \( \psi_\pm(x) \) completely determines the Fermi sea profile through the relation

\[
a \left( \frac{x + \tilde{x}}{2} \right) = \sqrt{1 + \psi_-(x)} - 1 . \tag{2.6}
\]
Thus, the space of solutions of the equations of motion may be formally identified with the diffeomorphisms of the line.

We end this section with a few comments on the range of validity of (2.5). First note that (2.6) makes sense only for \( \psi_\pm \geq -1 \). Indeed, we can see directly from (2.4) that violating this condition yields solutions in which the Fermi sea extends beyond the quadrant \( \lambda > |p| \); these represent other phases of the theory in which the limiting behavior differs from (2.4). If this constraint is met we find \( a > -1 \), hence from (2.3) \( p_+(x,t) = p_-(x,t) \) has a unique solution (at the classical turning point \( p_\pm = 0 \)) so \( \partial_\lambda \chi = p_+ - p_- \geq 0 \). The second comment is that if \( \psi_- \) satisfies the restriction above, the map \( x \mapsto \tilde{x} \) is a diffeomorphism provided

\[
1 + \frac{\psi_-'}{1 + \psi_-} \geq 0. \tag{2.7}
\]

When this condition is violated, (2.3d) will have more than two solutions for large enough \( t \), representing a “fold” in the Fermi sea (a region where there are four or more branches of the sea at fixed \( \lambda \)). In terms of the original collective field theory, this corresponds to a singularity in the field \( \chi \). The fact that the classical field equations evolve smooth initial data to singular field configurations is reminiscent of singularities occurring in nonlinear field theories, notably, in general relativity. We comment further on this in the conclusions.

3. General Solution to the Classical Scattering Equations

Suppose \( \Psi_\pm \) constitute a solution of the classical scattering equations (2.5), and suppose further that \( \Psi_\pm + \gamma_\pm \) is a nearby solution, where \( \gamma_\pm \) are small. To first order in the variations (2.3) becomes

\[
\gamma_+(\tilde{x})d\tilde{x} = \gamma_-(x)dx
\]

where \( \tilde{x} = x + \log(1 + \Psi_-(x)) \). Taking a Fourier transform of this equation, with

\[
\gamma_{\pm}(x) \equiv \int_{-\infty}^{\infty} \gamma_{\pm}(\xi)e^{i\xi x}d\xi \tag{3.2}
\]

leads to

\[
\gamma_{\pm}(\xi) = \frac{1}{2\pi} \int e^{-i\xi x} \gamma_{\mp}(x)(1 + \Psi_{\mp}(x))^{1+\mp i\xi} dx. \tag{3.3}
\]

This may be regarded as a first-order differential equation in function space. Integrating this equation with the boundary condition \( \psi_+ = 0 \Rightarrow \psi_- = 0 \) we have the general solution of Polchinski’s scattering equations:

\[
2\pi\psi_{\pm}(\xi) = \frac{1}{1+\mp i\xi} \int_{-\infty}^{\infty} e^{-i\xi x} \left[(1 + \psi_{\mp}(x))^{1+\mp i\xi} - 1\right] dx. \tag{3.4}
\]
or, in position space:

\[ \psi_\pm(x) = - \sum_{p \geq 1} \frac{\Gamma(\pm \partial_x + p - 1)}{\Gamma(\pm \partial_x)} \frac{(-\psi_\pm(x))^p}{p!} \] (3.5)

The solution (3.4) is valid for finite (i.e., not infinitesimal) field configurations. The formula breaks down exactly for solutions violating the conditions of the previous section. For example, a waveform \( \psi_-(x) = \beta \cos \omega_0 x \) in the far future corresponds to the incoming waveform:

\[
\begin{align*}
\psi_+(x) &= \text{Re} \left( \sum_{n \geq 0} \psi_+(n)e^{in\omega_0 x} \right) \\
\psi_+(n) &= 2\pi(-1)^n \frac{\Gamma(in\omega_0 + n - 1)}{\Gamma(in\omega_0)n!} \left(\frac{\beta}{2}\right)^n _2F_1\left(\frac{in\omega_0 + n - 1}{2}, \frac{in\omega_0 + n}{2}; n + 1; \beta^2\right).
\end{align*}
\] (3.6)

The hypergeometric function has a branch point at \( \beta = 1 \) reflecting the breakdown of (2.6). Now suppose \( \beta < 1 \) and consider the dependence on \( \omega_0 \). Using stationary phase approximation in (3.4) we find that for

\[ \frac{\beta \omega_0}{\sqrt{1 - \beta^2}} < 1 \] (3.7)

\( \psi_+(n) \) decreases exponentially with \( |n| \) while if condition (3.7) is violated then \( \psi_+(n) \sim O(n^{-3/2}) \) for large \( n \). Thus the Fourier series \( \psi_+(x) \) becomes discontinuous, reflecting the violation of (2.7) (in this periodic solution there are in fact an infinite number of “folds”).

A final interesting pathology occurs for waves formally corresponding to \( \omega = i \). In this case the functional equations (2.5) can be solved directly to give the solution:

\[ \begin{align*}
\pi \chi &= -\frac{\alpha e^t}{2g^2} \theta(\lambda \geq 1 + \frac{1}{2} \alpha e^t) \\
\partial_\lambda \chi &= \frac{1}{\pi} \sqrt{(\lambda - 1 - \frac{1}{2} \alpha e^t)(\lambda + 1 - \frac{1}{2} \alpha e^t)} \theta(\lambda \geq 1 + \frac{1}{2} \alpha e^t).
\end{align*} \] (3.8)

In the infinite past we have the standard static background, but as time increases the Fermi sea is drained, leaving nothing behind. This pathology can be eliminated by the boundary condition \( \partial_\lambda \chi = 0 \Rightarrow \pi \chi = 0 \) forcing zero momentum flow at the edge of the sea. While physically reasonable, it is not clear that this eliminates all pathological solutions.
4. Tree-Level S-Matrix in a Classical Background

4.1. Scattering in the standard background

We now introduce quantum mechanics by considering the \( \gamma \pm \) to be related to incoming and outgoing free quantum fields via:

\[
\gamma_+ \to \sqrt{4\pi g} (\partial_t \pm \partial_\tau) \chi_+ \\
\chi_+ = i \int_{-\infty}^{\infty} \alpha_+(\xi) e^{i\xi(t+\tau)} \frac{d\xi}{\sqrt{4\pi\xi}} \\
\chi_- = i \int_{-\infty}^{\infty} \alpha_-(\xi) e^{i\xi(t-\tau)} \frac{d\xi}{\sqrt{4\pi\xi}}
\]

(4.1)

\[
[\alpha_+(\xi), \alpha_-(\xi')] = -\xi \delta(\xi + \xi').
\]

The general solution (3.4) of the classical scattering equations implicitly summarizes the entire tree-level S-matrix. Following Polchinski, we interpret the expansion of (3.4) around the trivial background as the quantum mechanical operator equation relating in and out fields:

\[
\alpha_\pm(\eta) = \sum_{p \geq 1} (-2g)^{p-1} \frac{\Gamma(1 \mp i\eta)}{\Gamma(2 \mp i\eta - p)} \frac{1}{p!} \int_{-\infty}^{\infty} d^p \xi \delta(\eta - \sum \xi_i) \alpha_\mp(\xi_1) \cdots \alpha_\mp(\xi_p).
\]

(4.2)

The vacuum is defined by \( \alpha_\pm(-\omega)|0\rangle = 0 \). (In this paper \( \omega \) will always stand for positive quantities.) The expression on the rhs of (4.2) is normal-ordered with respect to this vacuum.

Incoming (outgoing) states are created by the action of \( \alpha_+(\omega) \) \( \alpha_-(\omega) \). We compute the tree level approximation to the connected S-matrix element

\[
S_c(\sum \omega_i \to \sum \omega_i') = \langle 0| \prod_i \alpha_-(\omega_i') \prod_j \alpha_+(\omega_j)|0\rangle_c
\]

(4.3)

by inserting (4.2) and extracting the connected contribution at lowest nonvanishing order in \( g \). The connected amplitude for \( n \to m \) scattering is of order \( g^{m+n-2} \), and calculations of tree-level scattering amplitudes reduce to combinatorics. For example, consider \( S(\sum_1^n \omega_i \to \sum_1^m \omega_i') \) in the kinematic regime \( \forall k, \omega_n > \omega'_k > \sum_{j=1}^{n-1} \omega_j \). Using (4.2) one finds the simple expression

\[
S = -i (-2g)^{m+n-2} \prod_{j=1}^n \omega_j \prod_{k=1}^m \omega_k' \frac{\Gamma(-i\omega_n)}{\Gamma(1 - m - i\omega_n)} \frac{\Gamma(1 - m - i\Omega)}{\Gamma(3 - n - m - i\Omega)}
\]

(4.4)

where \( \Omega = \sum_{j=1}^n \omega_j \). Setting \( n = 1 \), we recover the amplitudes computed in [11] using a continuum formulation.

---

1 Actually, the normal ordering in (4.2) is a convenient choice; to lowest order in \( g \) the operator ordering is irrelevant.
4.2. Scattering in nontrivial backgrounds

The formalism described above also allows us to compute tree level quantum scattering amplitudes in nontrivial backgrounds. There are two physical effects. The classical background \( \psi_{\pm} \) can emit and absorb quanta, and the quanta themselves can interact. The description of these two processes must be self-consistent. We will assume the existence of a classical detector which can exist in the background and measure the presence of individual quanta \([13]\).

The effects of the classical background are taken into account by writing the total in- and out-fields as a classical piece plus a quantum piece: \( \hat{\psi}_{\pm} = \psi_{\pm} + \partial_\tau \chi_{\pm} \). Thus the total field \( \hat{\psi}_+ \) is related to the quantum field by a unitary transformation

\[
S[\psi_+] \equiv e^{-\frac{1}{2g} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \alpha_+ (\xi) \psi_+ (-\xi)}
\]

such that \( S^\dagger [\psi_+] \gamma_+ (\xi) S[\psi_+] = \gamma_+ (\xi) + \psi_+ (\xi) \). An analogous transformation \( S[\psi_-] \) shifts \( \gamma_- \). The effect of this unitary transformation on \( \hat{\psi}_\mp \) is quite nontrivial and leads to an equation generalizing (4.2). A classically evolving detector measures the \( S \)-matrix given by

\[
S(\sum \omega_i \rightarrow \sum \omega'_i; \psi) = \langle 0 | S^\dagger [\psi_-] \prod \alpha_- (-\omega_i) \prod \alpha_+ (\omega_i) S[\psi_+] | 0 \rangle
\]

which may be computed using the above formulae.

It is also of interest to consider particle creation in the state which is described as an initial classical state with no “extra” quanta \( |\psi_+ \rangle \equiv S[\psi_+] | 0 \rangle \). For example we find

\[
\langle \psi_+ | \alpha_- (\omega) \alpha_- (-\omega) | \psi_+ \rangle = \\
\int_0^\infty d\eta \eta^\omega \left[ \sum_{p=2}^\infty \frac{\Gamma(1 + i\omega)}{\Gamma(2 + i\omega - p)(p - 1)!} \int d^{p-1} \xi \delta (\omega + \eta - \sum_{i=1}^{p-1} \xi_i) \prod_1^{p-1} \psi_+ (\xi_i) \right]^2
\]

to lowest order in \( g \) and all orders in the background.

5. Vertex Operator Calculations

We would like to describe the scattering in time-dependent backgrounds in terms of vertex operator correlators in time-dependent \( \sigma \)-model backgrounds. First, we relate the
above $S$-matrix amplitudes to the vertex operator correlators of the Euclidean $c = 1 \times$ Liouville system with action:

$$S = \int d^2 z \sqrt{\hat{g}} \left[ \frac{1}{8\pi} (\nabla \phi)^2 + \frac{\mu}{8\pi \gamma^2} e^{\gamma \phi} + \frac{Q}{8\pi} \phi R(\hat{g}) + \frac{1}{8\pi} (\hat{\nabla} X)^2 \right]$$  \hspace{1cm} (5.1)

Vertex operator correlators for the theory (5.1) have been calculated in [11]. The gravitationally dressed vertex operators are

$$V_q = \int e^{i q X / \sqrt{2}} e^{\sqrt{2}(1 - \frac{1}{2} |q|) \phi}$$

$$\bar{V}_q = \int e^{i q X / \sqrt{2}} e^{\sqrt{2}(1 + \frac{1}{2} |q|) \phi}$$  \hspace{1cm} (5.2)

where $q \in \mathbb{R}$. The operators $\bar{V}_q$ are known to be subtle. It is argued in [3] that they either do not exist as quantum operators or that they decouple from ordinary amplitudes. The correlators of the $V_q$'s may be analytically continued to obtain the $S$-matrix elements of section (4.1) above. Specifically, the amplitudes

$$\mathcal{R}(q_i; q'_i) \equiv \langle \prod_{i=1}^{n} V_{q_i} \prod_{i=1}^{m} V_{q'_i} \rangle$$  \hspace{1cm} (5.3)

where $q_i < 0$, $q'_i > 0$, and $\sum q_i + \sum q'_i = 0$ are related to the connected Minkowskian $S$-matrix elements by the analytic continuation [1,16]

$$\mathcal{R}(q_i = i \omega_i; q'_i = -i \omega'_i) = \prod_{i=1}^{n} \frac{\Gamma(i \omega_i)}{\Gamma(-i \omega_i)} \prod_{i=1}^{m} \frac{\Gamma(i \omega'_i)}{\Gamma(-i \omega'_i)} S_c(\sum_{i=1}^{n} \omega_i \rightarrow \sum_{i=1}^{m} \omega'_i) .$$  \hspace{1cm} (5.4)

Since the sign of $q$ is crucial to the Minkowskian interpretation we must distinguish four different Minkowskian vertex operators:

$$T_{-\omega}^+ \equiv \frac{\Gamma(-i \omega)}{\Gamma(i \omega)} \int_{\Sigma} e^{2 \varphi e^{i \omega (t+\varphi)}}$$

$$T_{-\omega}^- \equiv \frac{\Gamma(-i \omega)}{\Gamma(i \omega)} \int_{\Sigma} e^{2 \varphi e^{-i \omega (t-\varphi)}}$$

$$T_{+\omega}^+ \equiv \frac{\Gamma(i \omega)}{\Gamma(-i \omega)} \int_{\Sigma} e^{2 \varphi e^{-i \omega (t+\varphi)}}$$

$$T_{+\omega}^- \equiv \frac{\Gamma(i \omega)}{\Gamma(-i \omega)} \int_{\Sigma} e^{2 \varphi e^{i \omega (t-\varphi)}}$$  \hspace{1cm} (5.5)

\footnote{We follow the notation of [3].}
where $\varphi = \phi/\sqrt{2}$. We see that $T_+^-\omega, T_-^+\omega$ are the continuation of the Seiberg branch $V_q$, while $T_+^\omega, T_-^-\omega$ are the continuation of the other branch $\overline{V}_q$. Thus, in Minkowski space we see that for scattering on the left half-line the Seiberg branch is simply the branch for which the incoming (outgoing) particles are right- (left-) moving. Assuming that the Minkowskian amplitudes are correlation functions of analytically continued vertex operators, we identify the vertex operators creating incoming and outgoing tachyons in Minkowski space as

$$V_{\text{in}}(\omega) = T_-\omega$$
$$V_{\text{out}}(\omega) = T_+\omega.$$  \hspace{1cm} (5.6)

Armed with this identification we can now write (4.6) as a $\sigma$-model correlator. We first note that $S[\psi_{\pm}]$ may be normal ordered to give

$$S[\psi_{\pm}] = e^{-\frac{1}{4g^2} \int_0^\infty \frac{d\xi}{\xi} |\psi_{\pm}(\xi)|^2} e^{-\frac{i}{2g} \int_0^\infty \frac{d\xi}{\xi} \alpha_\pm(\xi)\psi_{\pm}(-\xi)} e^{-\frac{i}{2g} \int_{-\infty}^0 \frac{d\xi}{\xi} \alpha_\pm(\xi)\psi_{\pm}(\xi)}.$$  \hspace{1cm} (5.7)

Inserting this converts (4.6) to a sum of correlation functions of the form (5.4), and we may thus write

$$\langle \psi_- | \prod \alpha_-(\omega_i) \prod \alpha_+(\omega'_i) | \psi_+ \rangle = e^{-\frac{1}{4g^2} \int_0^\infty \frac{d\xi}{\xi} (|\psi_+(\xi)|^2 - |\psi_-(\xi)|^2)}$$

$$\langle \prod V_{\text{out}}(\omega_i) \prod V_{\text{in}}(\omega'_i) e^{-\frac{i}{2g} \int_0^\infty \frac{d\xi}{\xi} (\psi_-(\omega)V_{\text{out}}(\omega) + \psi_+(\omega)V_{\text{in}}(\omega))} \rangle_{\sigma\text{-model}}$$

realizing the time-dependent tachyon background explicitly as a modification of the $\sigma$-model action.

We close with some comments on the identification (5.6) of the vertex operators. The first remark is that, if we assume that the analytically continued $\overline{V}_q$ decouple as in Euclidean space, there is an ambiguity in the identification because mixing with these will not affect amplitudes. This might resolve a puzzle regarding the relation of the vertex operators to the asymptotics of the the Wheeler-de-Witt wavefunction \[17,10\]. A related point is that if one attempts to construct a $\sigma$-model formulation of amplitudes like (4.7) one is forced to include the “wrong” branch operators and abandon the hypothesis of decoupling.

\[3\] We thank N. Seiberg for lively correspondence on this and related questions.
6. Relations for tachyon amplitudes

6.1. Recursion relations

As a final application of (4.2), we derive some “Ward identities” for insertion of a tachyon operator in Euclidean vertex operator correlators. In this section we scale \(-2g\) to one for simplicity.

The identities are most simply written in terms of 
\[ T_q = \frac{\Gamma(|q|)}{\Gamma(1-|q|)} V_q. \]

Consider first the insertion of a “special tachyon,” with \(q \in \mathbb{Z}_+\). If we continue \(\omega \to in\) with \(n \in \mathbb{Z}_+\) then the series (4.2) truncates after \(n+1\) terms. These terms have a “universal” effect in correlation functions. Specifically, an insertion of \(T_n\) is given by

\[
\langle T_n \prod_{i=1}^{m} T_{q_i} \rangle = \sum_{n=1}^{\min(m,n+1)} \frac{\Gamma(n)}{\Gamma(2+n-k)} \]  

The notation is the following: Let \(S = \{q_1 \ldots q_m\}\), and let \(S^-\) denote the subset of \(S\) of negative momenta. Denote \(m_- = |S^-|\). The sum on \(T\) is over subsets of \(S^-\) of order \(l\). The subsequent sum is over distinct disjoint decompositions \(S_1 \Pi \ldots \Pi S_{k-1} = S \setminus T\). \(q(T)\) denotes the sum of momenta in the set \(T\). The momenta \(q_i\) are taken to be generic so that the step functions are unambiguous. This entails no loss of generality since the amplitudes are continuous (but not differentiable) across kinematic boundaries.

The first two examples of (6.1) are:

\[ \langle T_1 \prod_{i=1}^{n} T_{q_i} \rangle = \sum_{q_i < -2} \langle T_{q_i} \prod_{j \neq i} T_{q_j} \rangle = \sum_{q_i < -1} \langle T_{q_i} \prod_{j \neq i} T_{q_j} \rangle \]  

\[ \langle T_2 \prod_{i=1}^{m} T_{q_i} \rangle = \sum_{q_i < -2} \langle T_{q_i} \prod_{j \neq i} T_{q_j} \rangle + \sum_{q_i + q_j < -2} \langle T_{q_i} T_{q_j} \rangle \]

\[ + \sum_{q_i + q_j < 0} \theta(q(S_1))q(S_1)\theta(q(S_2))q(S_2)\langle T_{q(S_1)} \prod_{S_1} T_{q_j} \rangle \langle T_{q(S_2)} \prod_{S_2} T_{q_j} \rangle \]  

\[ \]  

The case \(m_- = m\) is exceptional ((6.1) vanishes while the correlator does not) but the amplitude is known from [4.2]. This ungainly feature will be remedied below.
Note that in (6.3) there is a change in tachyon number by one in the second line, and the product of two correlators in the third line. The pattern continues for higher \( n \): there are terms with \(|T| = l = k - 1\) removing \( l \) incoming tachyons, which are linear in the correlators, and terms with a product of \( k - l \) correlators.

A little reflection shows that it is easy to generalize the identity (6.1) to the case of an insertion of \( T_p \) with \( p \in \mathbb{I} \mathbb{R} \) merely by allowing the sum on \( k \) to run to \( m \). The resulting tachyon recursion relations give an efficient way to calculate amplitudes since applying (6.1) to the tachyon of largest absolute momentum reduces the number of tachyon insertions by at least two. For example, the reader may quickly calculate the five point function in an arbitrary kinematic configuration.

6.2. Differential equations

The identities (6.1) are elegantly expressed in terms of the generating functional for connected amplitudes

\[
\mathcal{F}[t, \bar{t}] = \langle 0 | e^{\int_0^\infty (t(p) T_p + \bar{t}(p) T_{-p}) dp} | 0 \rangle_c.
\]  

(6.4)

Introducing the nonlinear differential operator\( \mathcal{D}_n \)

\[
\mathcal{D}_n \mathcal{F} \equiv \sum_{a=1}^{n} \sum_{b=0}^{n+1-a} \frac{(n-1)!}{a!b!(n+1+a-b)!} \int_0^\infty \prod_{1}^{a} dp'_i \prod_{1}^{b} dp_j \delta(\sum_{1}^{a} p'_i - \sum_{1}^{b} p_j - n) \prod_{1}^{a} \bar{t}(p'_i) \prod_{1}^{b} (p_j \frac{\delta \mathcal{F}}{\delta \bar{t}(p_j)}) .
\]

(6.5)

and its conjugate \( \mathcal{D}_n \) we may summarize (6.1) as the set of differential equations:

\[
\frac{\delta \mathcal{F}}{\delta \bar{t}(n)} = \mathcal{D}_n \mathcal{F},
\]

\[
\frac{\delta \mathcal{F}}{\delta t(n)} = \mathcal{D}_n \mathcal{F} .
\]

(6.6)

By (Euclidean) time reversal invariance \( \mathcal{F}(t, \bar{t}) = \mathcal{F}(\bar{t}, t) \), so only half the equations in (6.6) are independent.

\(^5\) For \( b = 0 \) (corresponding to the exceptional case mentioned below (6.1)) the last product is taken to be one.
The equations \((6.6)\), being equivalent to recursion relations for tachyon amplitudes, uniquely determine \(F\), at least for small \(t, \bar{t}\). Although there are \(\aleph_0\) equations in \(\aleph_1\) variables the correlation functions of \(T_p\) are polynomials in \(p\), and hence completely specified by their values for \(p \in \mathbb{Z}\). Thus one may restrict attention to \(t(p), \bar{t}(p)\) with support on \(\mathbb{Z}_+\). Alternatively, one can write a larger set of differential equations by generalizing the operators to \(D_p\) for \(p \in \mathbb{R}\). The essential difference is that the sum on \(a, b\) becomes infinite; in practice, in the calculation of a given amplitude the series terminates.

The consistency conditions obtained by equating mixed partial derivatives in \((6.6)\) are complicated. We may write

\[
\mathcal{D}_n = \int_0^\infty dp\bar{t}(p+n)p \frac{\delta \bar{t}(p)}{\delta t(p)} + \cdots
\]

Thus the “first” term \((a = b = 1)\) in \((6.5)\) defines a set of operators satisfying half a Virasoro algebra, but the higher terms introduce essential complications. It would be interesting to find a concise algebraic description of the algebra of operators generated by the equations \((6.6)\).

6.3. Relation to other work

Ward identities related to special states and special tachyons have been the subject of much study over the past year \([10, 19–29]\). We will now discuss the relation of \((6.6)\) to some of these works.

In the formulation of \([14, 18]\) one extracts the correlators of tachyons as the coefficients of nonanalytic powers \(\ell^{|p|}\) in the small \(\ell\) expansion of macroscopic loop amplitudes. The macroscopic loop operator carrying momentum \(p\) has an expansion as a sum of local operators \([10]\)

\[
W_{in}(\ell, p) = -\mathcal{T}_p \frac{\pi |p|}{\sin \pi |p|} \mu^{-|p|/2} I_{|p|}(2\sqrt{\mu} \ell) - \sum_{r=1}^\infty \hat{B}_{r,p} \frac{2(-1)^r r}{r^2 - p^2} \mu^{-r/2} I_r(2\sqrt{\mu} \ell) .
\]

(6.8)

The operators \(\hat{B}_{r,p}\) are redundant for \(p \notin \mathbb{Z}\), and obey Ward identities organized by \(W_{1+\infty}\) \([10]\). Examining \((6.8)\), we see that the smoothness of macroscopic loop amplitudes at \(p \in \mathbb{Z}\) implies the identification \(\mathcal{T}_{\pm n} = \hat{B}_{n, \pm n}\) in correlators. The identities we find are thus related to the boundary operator Ward identities of \([10]\). For example, taking the

\[\text{undoubtedly related to } W_\infty\]
\( \ell \to 0 \) limit of the boundary operator \((B_1)\) Ward identity of [10] we obtain (6.2). In general, (6.1) is not identical to the Ward identities of [10]. A detailed examination shows that the added terms in (6.1) are absent in the cases which were explicitly proved in [10] by manipulation of macroscopic loop amplitudes. Evidently, in the general case there are complications in the behavior of the measure for the fermion path integral under \(W_\infty\) transformations.

Other works have made direct use of the continuum Liouville formulation. In [24] the method of bulk amplitudes is applied to obtain linear relations on correlation functions at finite string coupling. As noted in [24] it is difficult to control the contact terms which correct these relations so we cannot compare.

More recent studies, e.g., [26,28,29], have examined the theory at \(-2g = \frac{1}{\mu} = \infty\) so it is not clear we should compare with results at \(-2g = \frac{1}{\mu} = 1\). Nevertheless, we will try. In (6.1) the term \(l = k - 1\) corresponds to a process where \(l\) (incoming) tachyons in the set \(T\) have been eliminated from the correlation function. This is strongly reminiscent of the fact that \(W_\infty\) symmetry generators do not preserve tachyon number.\(^7\) The Ward identities of [28,29] constitute a set of beautiful quadratic relations on amplitudes with insertions of all BRST invariant dimension zero operators. In addition to the dressed tachyons of ghost number two there are other BRST invariant operators [30–32]. It is possible that upon elimination of amplitudes involving the non-tachyon operators one will be left with a set of identities for the tachyon correlators equivalent to (6.6). A second possibility, suggested by the product of more than two correlators in (6.1), is that there are singularities in the measure on moduli space\(^9\) on high codimension boundaries, and that the simple quadratic relations of [28,29] must be modified. A third possibility, of course, is that our relations are distinct from previous results.

7. Conclusions

We began this paper, and our investigations, with the motivation of understanding better the time-dependent backgrounds of string theory. We have gained a much better understanding of the classical \(S\)-matrix of the theory in two dimensions, found an efficient

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7 Reference [25] argued that this identity is analogous to the puncture operator equation at \(c < 1\).
8 This nonlinearity has been noted in [16,24,26,28].
9 or of the differential form \(\Theta\) of [28].
formalism for calculation of individual amplitudes, and obtained some interesting differential equations for the partition function $F$ on the space of coupling constants $t(p)$. Solving these equations would lead to a solution of the sigma model (5.8) through conformal perturbation theory. Thus it is interesting to find an effective procedure for solving these equations or equivalently the recursion relations (6.1). Indeed, the (partly experimental) results of [33] suggest that some relatively simple and interesting solutions of (6.6) await discovery.

It is also noteworthy that the collective field equations are incomplete since they evolve a large class of smooth initial data to singular field configurations. This might have some bearing on the issue of classical singularities in string theory. In particular, the existence of the free fermion formalism which provides a smooth description of such “pathological” Fermi seas is an indication that the corresponding singularities in the Das-Jevicki formalism are not true singularities of the classical string theory, just of the method of describing the solution.

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