On normal subgroups in the fundamental groups of complex surfaces

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Abstract

We show that for each aspherical compact complex surface $X$ whose fundamental group $\pi$ fits into a short exact sequence

$$1 \to K \to \pi \to \pi_1(S) \to 1$$

where $S$ is a compact hyperbolic Riemann surface and the group $K$ is finitely-presentable, there is a complex structure on $S$ and a nonsingular holomorphic fibration $f : X \to S$ which induces the above short exact sequence. In particular, the fundamental groups of compact complex-hyperbolic surfaces cannot fit into the above short exact sequence. As an application we give the first example of a non-coherent uniform lattice in $Isom(\mathbb{H}^2_C)$.

1 Introduction

The goal of this paper is threefold:

(a) We will establish a restriction on the fundamental groups of compact aspherical complex surfaces.

(b) We find the first examples of incoherent uniform lattices in $PU(2,1)$.

(c) We show that the answer to the Question 1 below is negative in the class of uniform lattices in $PU(2,1)$.

Question 1 Is there a Gromov-hyperbolic group $\pi$ which fits into a short exact sequence:

$$1 \to K \to \pi \to Q \to 1$$

where $K$ and $Q$ are closed hyperbolic surface groups?

Suppose that $X$ is an aspherical compact complex surface whose fundamental group $\pi$ fits into a short exact sequence

$$1 \to K \to \pi \to Q = \pi_1(S) \to 1$$

where $S$ is a compact hyperbolic Riemann surface and the group $K$ is finitely-presentable. The main theorem of this paper is

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Theorem 2 Under the above assumptions there a complex structure on $S$ and a nonsingular holomorphic fibration $f : X \to S$ which induces the above short exact sequence.

Remark 3 Actually in Theorem 2 it is enough to assume that $Q$ is a torsion-free group with nonzero $\beta^2_1(Q)$, the 1-st $L_2$-Betti number. On the other hand, in this case we have to assume that $X$ is Kähler. Our proof also works under the assumption that the group $K$ is of the type $FP_2$.

After proving Theorem 2 I have learned that J. Hillman \[10\] proved the same result under stronger assumption that $K$ is the fundamental group of a compact Riemann surface. Our methods seem to be completely different except application of the result of \[10\]. Later it turned out that the same result as Hillman’s was independently proven by D. Kotschick.

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2 Milnor fibration

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a nonconstant holomorphic function, we assume that $0 \in \mathbb{C}^2$ is a critical point of $f$. Let $S_\epsilon = S_\epsilon(0)$ be a sufficiently small metric sphere in $\mathbb{C}^2$ centered at the origin. Let $B_\epsilon(0)$ denote the closed $\epsilon$-ball centered at the origin. Let $K := f^{-1}(0) \cap S_\epsilon$, this is a smooth knot (or link) in the 3-sphere. The Milnor fibration $\phi : S_\epsilon - K \to S^1$ associated with $f$ is defined as $\phi(z,w) = f(z,w)/|f(z,w)|$, see \[16, \S4\].

Below we list some properties of $\phi$ (see \[16, \S4\], \[1\]):

(a) If $\epsilon$ is sufficiently small then $\phi$ determines a smooth fibration of $S_\epsilon - K$ over $S^1$.

(b) Fibers of $\phi$ are connected provided that the germ of $f$ at zero is reduced, otherwise $\phi$ will have disconnected fibers.

(b) The knot (link) $K$ is distinct from a single unknot in $S^3$ unless the germ of $f$ at 0 is isomorphic to $((z,w) \mapsto z^p,(0,0))$.

(c) If $K$ not an unknot, then each component of $\phi^{-1}(t), t \in S^1$, is not simply-connected.

(d) Let $r > 0$ be sufficiently small. Consider $s \in C_r(0)$, a point on the unit circle in $\mathbb{C}$ centered at zero. Let $\mathcal{F}_{\epsilon,s} := f^{-1}(s) \cap B_\epsilon$. The two surfaces $\mathcal{F}_{\epsilon,s}$ and $F_{\epsilon,s} = \phi^{-1}(s/|s|) - f^{-1}(B_\epsilon(0))$ share common boundary. There exists an isotopy of $\mathcal{F}_{\epsilon,r}$ to $F_{\epsilon,r}$ within $B_\epsilon(0)$ which is the identity on the boundary of each surface.

3 Multicurves

Definition 1 Let $f : X \to S$ be a nonconstant proper holomorphic map from a connected complex surface $X$ to a Riemann surface (i.e. complex curve) $S$. We will say that $f$ is a nonsingular holomorphic fibration if $f$ is a submersion.
Clearly the mapping \( f \) as above is a real-analytic fibration, however in most cases it does not determine a locally trivial holomorphic bundle. If \( f \) is not a submersion we will still think of it as a singular fibration, we shall use the notation \( F_t \) to denote the fiber \( f^{-1}(t) \) of \( f \) over \( t \in S \).

**Definition 2** Let \( f : X \to D^2 \) be a nonconstant proper holomorphic map with connected fibers where \( X \) is a 2-dimensional complex surface and \( D^2 \) is the unit disk in \( \mathbb{C} \). We assume that the origin is the only critical value of \( f \). The singular fiber \( C = f^{-1}(0) \) is called a multicurve if it is a smooth curve of the multiplicity \( > 1 \). In other words, the germ of \( f \) at each point \( c \in C \) is equivalent to the map \( (z, w) \mapsto z^n, n > 0, z, w \in \mathbb{C} \). The number \( n \) is the multiplicity of \( C \).

Let \( t \in D^2 - 0 \). Define the maps
\[
\iota_* : H_2(f^{-1}(t)) \to H_2(X) \cong H_2(C)
\]
\[
\iota_\# : \pi_1(f^{-1}(t)) \to \pi_1(X) \cong \pi_1(C)
\]
induced by the inclusion \( \iota : f^{-1}(t) \hookrightarrow X \).

**Lemma 4** If \( C \) is a multicurve then the map \( \iota_* \) is not surjective. Assume that \( C \) is a non-simply-connected multicurve. Then the map \( \iota_\# \) is not onto.

**Proof.** Consider \( Y = f^{-1}(D) \subset X \) where \( D = \{ z \in \mathbb{C} : |z| \leq |t| \} \) is the closed disk in \( D^2 \) containing \( t \). The inclusion \( Y \hookrightarrow X \) is a homotopy-equivalence so we restrict our discussion to \( Y \). The map \( C \hookrightarrow Y \) is a homotopy-equivalence, thus the fundamental class of \( C \) generates \( H_2(Y) \). The dual generator of \( H_2(Y, \partial Y) \) is represented by 2-disk \( \Delta \subset Y \) which is transversal to the fibers of \( f \) and \( \partial \Delta \subset \partial Y \). Since \( C \) is a multicurve, the algebraic intersection number \( [f^{-1}(t)] \cdot [\Delta] = n > 1 \), where \( n \) is the multiplicity of \( C \). Thus \( [f^{-1}(t)] = n[C] \) which proves the first assertion.

The map \( \iota_\# \) is injective (since \( \iota \) is homotopic to a covering \( f^{-1}(t) \to C \)). Thus \( n = |\pi_1(C) : \iota_\#(\pi_1(f^{-1}(t)))| \), this proves the second assertion. \( \square \)

### 4 Proof of the main theorem

If \( \pi_1(X) \) fits into short exact sequence
\[
1 \to K \to \pi \to Q = \pi_1(S) \to 1
\]
where \( S \) is a hyperbolic Riemann surface then it follows from Kodaira’s classification theorem that \( X \) is a complex-algebraic surface. If \( X \) is assumed to be Kähler, \( Q \) torsion-free and \( \beta^2_1(Q) \neq 0 \), then \( Q \) is the fundamental group of a hyperbolic Riemann surface, moreover if \( \tilde{X} \) is the covering of \( X \) corresponding to \( K \) then there is a discrete faithful conformal action of \( Q \) on \( \mathbb{H}^2 \) and a \( Q \)-equivariant proper holomorphic map
\[
\tilde{f} : \tilde{X} \to \mathbb{H}^2
\]
with connected fibers (see [1]). In particular, the projection \( \pi_1(X) \to Q \) is induced by a holomorphic map \( f : X \to S \), for the complex structure on \( S \) given by \( \mathbb{H}^2/Q \).
The $i$-th $L_2$-Betti number $\beta^{(2)}_i(G)$ of a finitely presentable group $G$ is the dimension of the $i$-th reduced $L_2$-cohomology group $\tilde{H}^i(G)$, we refer the reader to [3, Chapter 8] and [4] for the precise definitions. For our purposes it is enough to know that $\beta^{(2)}_i(Q) > 0$ for each 2-dimensional finitely presentable group $Q$ provided that $\chi(Q) < 0$ (see [3, Chapter 8]). In particular, if $Q$ is the fundamental group of a hyperbolic Riemann surface of finite type then $\beta^{(2)}_1(Q) > 0$. Thus, in any case we have a holomorphic map $f : X \to S$.

We start the proof with the simple case when $f$ is a holomorphic Morse function, i.e. the germ of $f$ at each critical point is equivalent to $(z,w) \mapsto zw$. The proof in this case is easier and it illustrates the idea of the proof in the general case.

Let $d$ denote the hyperbolic metric on the unit disk in $\mathbb{C}$. We will suppose that the origin $0$ is a regular value of $\tilde{f}$. Direct computations show that the function $\gamma : x \mapsto d(0, \tilde{f}(x))$ is a real Morse function on $\tilde{X}$ away from $\tilde{f}^{-1}(0)$ and the Morse index of $\gamma$ at each critical point in $\tilde{X} - \tilde{f}^{-1}(0)$ is two. It is clear that $r \in \mathbb{R}_+$ is a critical value of $\gamma$ if and only if there is a critical value $z \in \mathbb{H}^2$ of $\tilde{f}$ within the distance $r$ from the origin. Let $F$ denote the generic fiber of $\tilde{f}$. Thus the space $\tilde{X}$ is obtained by attaching 2-handles to $F \times D^2$. Each singular fiber of $\tilde{f}$ is obtained from $F$ by "pinching" a certain collection of disjoint simple loops. Since $\tilde{X}$ is aspherical, each of these loops is homotopically nontrivial and no two such loops are homotopic to each other. (Otherwise $\tilde{X}$ contains a rational curve which then lifts to a homologically nontrivial 2-cycle in the universal cover of $X$.)

We now claim that the group $\pi_1(\tilde{X})$ is finitely generated but not finitely presentable. Our proof follows an argument of Bestvina and Brady [3]. Since $\tilde{X}$ is obtained from $F \times D^2$ by attaching only 2-handles, the fundamental group of $\tilde{X}$ is the quotient of $\pi_1(F)$. Recall that $\pi_1(\tilde{X})$ is finitely presentable, the epimorphism $\pi_1(F) \to \pi_1(\tilde{X})$ determines a finite generating set for $\pi_1(\tilde{X})$ (i.e. the generators of $\pi_1(F)$).

**Lemma 5** Let $G$ be a finitely presentable group and \{y_1,...,y_m\} be a finite generating set for $G$. Then there is a finite number of relators $R_1,...,R_k$ such that $\langle y_1,...,y_m| R_1,...,R_k \rangle$ is a presentation of $G$.

**Proof.** Let $\langle x_1,...,x_s| Q_1,...,Q_n \rangle$ be a finite presentation of $G$. There is a finite sequence of Tietze transformations (see for instance [4, §1.5]) which transform the generating set $X = \{x_1,...,x_s\}$ to $Y = \{y_1,...,y_m\}$, simultaneously they transform system of relators $Q_1,...,Q_n$ for $X$ to a system of relators for $Y$. On each step a finite presentation is transformed to a finite presentation. Hence, in the end we get a finite system of relators $R_1,...,R_k$ for the generating set $X$. \[\square\]

Therefore there are finitely many elements $\alpha_1,...,\alpha_n$ of $\pi_1(F)$ which normally generate the kernel $Ker(\phi)$ of

$$\phi : \pi_1(F) \to \pi_1(\tilde{X})$$
We shall identify $\alpha_j$ and the corresponding loops on $\mathcal{F}$. Thus there is a closed metric disk $D$ centered at the origin in $\mathbb{H}^2 = \tilde{S}$ such that each $\alpha_j, j = 1, \ldots, n,$ is contractible in $U = \tilde{f}^{-1}(D)$. This implies that each $\alpha \in \text{Ker}(\phi)$ is contractible in $\tilde{f}^{-1}(D)$. We will assume that the boundary of $D$ contains no critical values of $\gamma$. However we have infinitely many critical values of $\tilde{f}$ outside of the disk $D$. Let $z$ be one of them and $D'$ be a closed topological disk in $\mathbb{H}^2$ which contains both $D$ and $z$ and does not contain any critical values of $\tilde{f}$ which are not in $\{z\} \cup D$. Homotopically the Morse surgery corresponding to $z$ amounts to attaching 2-cells along certain loops $\alpha \subset \mathcal{F}$. Thus $\alpha \in \text{Ker}(\phi)$, which implies that $\alpha$ is contractible in $U$. It follows that we get an immersed homotopically nontrivial 2-sphere $\zeta \subset \tilde{f}^{-1}(D')$. The space $\tilde{X}$ is obtained from $\tilde{f}^{-1}(D')$ by attaching only 2-handles, thus the homotopy class $[\zeta]$ is nontrivial in $\pi_2(\tilde{X})$ which contradicts asphericity of $\tilde{X}$. This concludes the proof in the case when $\tilde{f}$ is a complex Morse function.

**Remark 6** J. Kollar had suggested an argument which reduces the general case to the case of holomorphic Morse function provided that no irreducible component of each singular fiber of $\tilde{f}$ has multiplicity $> 1$. Namely, perturb $f : \tilde{X} \to \mathbb{H}^2$ in a $Q$-equivariant manner to a smooth map $g : \tilde{X} \to \mathbb{H}^2$ with connected fibers so that:

(a) The sets of critical values of $g$ and $\tilde{f}$ are equal.

(b) If $s$ is a critical value of $g$ (and $\tilde{f}$) and $C_s \subset \mathbb{H}^2$ is a small circle around $s$ then the 3-manifolds $g^{-1}(C_s), \tilde{f}^{-1}(C_s)$ are homeomorphic.

(c) The mapping $g$ is a holomorphic Morse function near each singular fiber.

Then apply the same arguments as before to the function $g$ to conclude that neither $\tilde{f}$ nor $g$ has critical points.

However, technically it seems (at least to me) easier to apply the direct topological arguments below than to analyze the special case when a singular fiber of $\tilde{f}$ has an irreducible component of multiplicity $> 1$.

We now consider the general case. We will run essentially the same arguments as in the case of holomorphic Morse function. Let $\Sigma = \Sigma(\tilde{f})$ denote the set of critical values of the holomorphic function $\tilde{f}, \tilde{S}' := \tilde{S} - \Sigma$ and $\tilde{X}' := \tilde{f}^{-1}(S')$.

**Lemma 7** (1) The fundamental group of a generic fiber $\mathcal{F}$ of $\tilde{f}$ maps onto $K = \pi_1(\tilde{X})$. (2) No singular fiber of $\tilde{f}$ is a multicurve, i.e. a singular fiber of $\tilde{f}$ cannot be a smooth complex curve.

**Proof.** The restriction $\tilde{f}'$ of $\tilde{f}$ to $\tilde{X}'$ is a (nonsingular) fibration with connected fibers, thus $\pi_1(\mathcal{F})$ is the kernel of the homomorphism

$$\pi_1(\tilde{f}') : \pi_1(\tilde{X}') \to \pi_1(\tilde{S}')$$

In particular, the subgroup $\pi_1(\mathcal{F})$ is normal in $\pi_1(\tilde{X}')$. For each puncture $s_i \in \Sigma$ choose a small loop on $\tilde{S}'$ going once around $s_i$ and choose a homeomorphic lift $\gamma_i$ of this loop to $\tilde{X}'$. Then the group $\pi_1(\tilde{X}')$ is generated by $\pi_1(\mathcal{F})$ and by the loops $\gamma_i, s_i \in \Sigma$. Let $D_{s_i}$ denote a small metric disk on $\mathbb{H}^2$ centered at $s_i \in \Sigma$ (so that $D_{s_i} \cap \Sigma = \{s_i\}$). If for some $s_i$ the fundamental group of $\pi_1(\partial \tilde{f}^{-1}(D_{s_i}))$ does not map
onto $\pi_1(\tilde{f}^{-1}(D_{s_i}))$ then it is true for infinitely many points $s \in \Sigma$ (all the points in the $Q$-orbit of $s_i$), thus the group $K$ cannot be finitely generated. Thus the map

$$\pi_1(\tilde{X}') \rightarrow \pi_1(\tilde{X})$$

is onto. Since $\gamma_i$-s belong to the kernel of this map we conclude that the group $\pi_1(F)$ maps onto $\pi_1(\tilde{X})$.

If $F_{s_i} = \tilde{f}^{-1}(s_i), s_i \in \Sigma,$ is a multicurve then

$$\pi_1(\partial \tilde{f}^{-1}(D_{s_i})) \rightarrow \pi_1(\tilde{f}^{-1}(D_{s_i})) = \pi_1(F_{s_i})$$

is not onto (Lemma 4), which contradicts our assumptions. This proves the second assertion of Lemma. □

Now suppose that $f : X \rightarrow S$ is not a nonsingular holomorphic fibration. Thus the map $\tilde{f}$ has at least one fiber which is not a smooth complex curve. (By Lemma 8 each singular fiber has to be of this type.) Our goal is to show that this assumption leads to a contradiction. Let $T \subset \mathbb{H}^2$ be a locally finite embedded tree whose vertex set is $\Sigma$ (this tree of course is not $\pi_1(S)$-invariant). We can assume that edges of $T$ are geodesics in $\mathbb{H}^2$. For each vertex $s \in \Sigma$ of $T$ we choose a small closed metric disk $D_s$ centered at $s$ such that $D_s \cap T$ is equal to the intersection of $D_s$ and open edges of $T$ emanating from $s$. If $T' \subset T$ is a subtree then $N(T')$ will denote the union of $T'$ and disks $D_s$ for those vertices $s$ of $T$ which belong to $T'$. Let $Y(T') := \tilde{f}^{-1}(N(T'))$.

Since $\tilde{f}$ is a smooth fibration away from singular fibers it follows that the inclusion

$$Y(T) \hookrightarrow \tilde{X}$$

is a homotopy-equivalence. Therefore we restrict our attention to the topology of $Y(T)$.

Let $T'$ be a finite subtree of $T$ which is the convex hull of its vertices.

**Lemma 8** The homomorphism

$$\pi_2(Y(T')) \rightarrow \pi_2(Y(T))$$

is injective.

**Proof.** It is enough to prove this assertion for the lifts of $Y(T'), Y(T)$ to the universal cover of $X$. Since $X$ is aspherical, its universal cover $\tilde{X}$ cannot contain compact complex curves, hence the lift of $\tilde{f}^{-1}(t), t \in T - \Sigma$ to $\tilde{X}$ is a noncompact surface. Therefore this lift has trivial $H_2$ and the assertion follows from the Meyer-Vietors sequence. □

Let $s \in \Sigma - T'$ be a vertex of $T$ which is connected to $T'$ by an edge $[ss'], s' \in \Sigma \cap T'$. Note that the inclusions

$$Y(T') \hookrightarrow Y(T' \cup [s's]), \quad Y(s) \hookrightarrow Y([ss'])$$

are homotopy-equivalences. Here and in what follows $[ss']$ denotes the half-open edge connecting $s$ to $s': s \in [ss'), s' \notin [ss')$. 

Lemma 9 Suppose that $\pi_1(Y(T')) \to \pi_1(Y(T))$ is a monomorphism. Then $\pi_2(Y(T' \cup [ss'])) \neq 0$.

Proof. Let $t \in [ss']$ be the midpoint. Then there is a subsurface $\mathcal{F}' \subset \mathcal{F}_t$ such that:

(a) No boundary loop of $\mathcal{F}'$ is nil-homotopic in $\mathcal{F}_t$.
(b) The image of $\pi_1(\mathcal{F}')$ in $\pi_1(Y([ss']))$ is trivial.

The subsurface $\mathcal{F}'$ appears as follows: let $p \in \mathcal{F}_s$ be a singular point, then $\mathcal{F}'$ is a part of $\mathcal{F}_t$ corresponding to the Milnor fiber in $S_t(p)$, see section 2. If a boundary loop of $\mathcal{F}'$ is nil-homotopic in $\mathcal{F}_t$ then $\hat{X}$ contains a rational complex curve which contradicts the assumption that $\pi_2(X) = 0$.

Therefore, the assumption of Lemma implies that the image of $\pi_1(\mathcal{F}')$ in $\pi_1(Y(T' \cap [ss']))$ is trivial. Consider the total lift $\hat{\mathcal{F}'}$ of $\mathcal{F}'$ to the universal cover $\hat{X}$ of $X$, then $\hat{\mathcal{F}'}$ is contained in the lift $\mathcal{F}_t$ of $\mathcal{F}_t$ to $\hat{X}$. Note that no component of $\hat{\mathcal{F}}_t - \hat{\mathcal{F}'}$ is bounded (otherwise after degeneration of $\hat{\mathcal{F}}_s$ to a singular fiber we will get a compact complex curve in $\hat{X}$ which is impossible). If $\mathcal{F}'$ is not a planar surface then $\hat{\mathcal{F}'}$ contains a non-separating loop, otherwise a component of $\partial \hat{\mathcal{F}'}$ is not nil-homologous in $\hat{\mathcal{F}}_t$. In the both cases we apply Meyer-Vietors arguments to get a homologically nontrivial spherical cycle in $\hat{Y}(T' \cup [ss'])$, thus $\pi_2(Y(T' \cup [ss'])) \neq 0$. □

Since $K$ is assumed to be finitely-presentable, there are finitely many elements $\alpha_i \in \pi_1(\mathcal{F})$ which normally generate the kernel of $\pi_1(\mathcal{F}) \to \pi_1(Y)$. Thus there is a finite subtree $T' \subset T$ such that all the loops $\alpha_i$ are nil-homotopic in $Y(T')$. Since $\pi_1(Y(T'))$ maps onto $\pi_1(Y(T))$ (Lemma 8) it follows that $\pi_1(Y(T')) \to \pi_1(Y(T))$ is an isomorphism. Hence for an edge $[ss']$ of $T$ which has one vertex in $T'$ and the other vertex in $T - T'$ we have:

$$\pi_2(Y(T' \cup [ss'])) \neq 0$$

(according to Lemma 8). Now we apply Lemma 8 to conclude that $\pi_2(Y(T)) \neq 0$. However $\pi_2(Y(T)) \cong \pi_2(\hat{X}) = 0$ since $X$ is aspherical. This contradiction proves Theorem 8. □

5 Complex-hyperbolic surfaces

Let $B \subset \mathbb{C}^2$ be the unit ball. We will give $B$ the Kobayashi metric, this metric can be described as follows. Let $p, q \in B$ be distinct points, there is a unique complex line $L \subset \mathbb{C}^2$ so that $p, q \in B \cap L$. Now identify $B \cap L$ with the hyperbolic plane $\mathbb{H}^2$ where the curvature is normalized to be $-1$. Finally let $d(p, q) := d_{\mathbb{H}^2}(p, q)$. Then the complex-hyperbolic plane $\mathbb{H}^2_C$ is the unit ball $B$ with the Kobayashi distance $d$. It turns out that the Kobayashi distance $d$ is induced by a Riemannian metric $\rho$ on $B$. Below we list some properties of the complex-hyperbolic plane $\mathbb{H}^2_C$, we refer to [8], [2], [4], [20] for detailed discussion.

(a) $\rho$ is Kähler.
(b) The sectional curvature of $\rho$ is pinched between the constants $-1$ and $-1/4$.
(c) The group of biholomorphic automorphisms of $B$ equals the identity component in the isometry group of $\mathbb{H}^2$ which is isomorphic to the $PU(2,1)$ so that $B$ is the symmetric space for the group $PU(2,1)$: $B = PU(2,1)/K$ where $K \cong U(2)$ is a maximal compact subgroup in $PU(2,1)$.

(d) Let $\Gamma$ be a torsion-free uniform lattice in $PU(2,1)$. The quotient $B/\Gamma$ is a compact Kähler surface which is actually a smooth complex algebraic surface. The quotient $B/\Gamma$ is called a complex-hyperbolic surface.

(e) For each compact complex-hyperbolic surface we have the following identity between the Chern classes: $c_1^2 = 3c_2$, i.e. $\chi = 3\tau$ where $\chi$ is the Euler characteristic and $\tau$ is the signature.

(f) If $X$ is a smooth compact complex algebraic surface for which the equality $c_1^2 = 3c_2$ holds, then the universal cover of $X$ is biholomorphic to either $\mathbb{H}^2_\mathbb{C}$, or $\mathbb{C}^2$, or the complex-projective plane $\mathbb{P}^2_\mathbb{C}$.

The key fact about complex-hyperbolic surfaces which will be used in this paper is the following recent theorem of K. Liu [12]:

**Theorem 10** Let $X$ be a compact complex-hyperbolic surface. Then $X$ does not admit nonsingular holomorphic fibrations over complex curves.

### 6 Incoherent example

Recall that a group $\Gamma$ is called coherent if every finitely-generated subgroup $\Gamma' \subset \Gamma$ is also finitely presentable. Examples of coherent groups include free groups, surface groups, 3-manifold groups (see [19]) and certain groups of cohomological dimension 2 (see [7], [15]). The simplest example of noncoherent group is $F_2 \times F_2$, where $F_2$ is the free group on two generators. (The finitely generated infinitely presentable subgroup in $F_2 \times F_2$ is the kernel of the homomorphism $\phi : F_2 \times F_2 \to \mathbb{Z}$ where $\phi$ maps each free generator of each $F_2$ to the generator of $\mathbb{Z}$.) Thus there is a uniform lattice in the Lie group $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ which is not coherent. The first example of noncoherent discrete geometrically finite subgroup of $Isom(\mathbb{H}^4)$ was constructed in [11], [18]. Later on this example was generalized in [4] to a uniform lattice in $Isom(\mathbb{H}^4)$.

As an application of the main result of this paper we show that certain uniform lattices in $PU(2,1) = Isom(\mathbb{H}^2_\mathbb{C})$ are not coherent (these are the first known examples of incoherent discrete subgroups of $PU(2,1)$). The groups which we consider were known before (see [13], [2], [5]) however their incoherence was unknown.

**Lemma 11** Suppose that $X$ is a compact complex-hyperbolic surface whose fundamental group $\pi$ fits into a short exact sequence

$$1 \to K \to \pi \to Q = \pi_1(S) \to 1$$

where $S$ is a compact hyperbolic Riemann surface and the group $K$ is finitely generated. Then $K$ is not finitely presentable.
Proof. Suppose that $K$ is finitely presentable. The surface $X$ is aspherical since its universal cover is the complex ball. Then by Theorem 2 the projection $\pi \to Q$ is induced by a nonsingular holomorphic fibration of the surface $X$. On the other hand, complex-hyperbolic surfaces do not admit such fibrations by [12]. □

Now we describe an example the fundamental group of a complex-hyperbolic surface satisfying the conditions of Lemma 11 following [13]. Define automorphisms $\phi, \psi$ of the free group on three generators $A_1, A_2, A_3$ by

$$\phi(A_1) = A_1 A_2 A_1^{-1}, \phi(A_2) = A_1 A_3 A_1^{-1}, \phi(A_3) = A_1$$
$$\psi(A_1) = (A_1 A_2) A_3 (A_1 A_2)^{-1}, \psi(A_2) = A_1 A_2 A_1^{-1}, \psi(A_3) = A_1$$

Ron Livne [13] constructed a uniform lattice $\Gamma_{d,N}$ in $PU(2,1)$ with the presentation

$$\langle x, y, A_1, A_2, A_3 | x A_i x^{-1} = \phi(A_i), y A_i y^{-1} = \psi(A_i) \ (1 \leq i \leq 3), x^3 = y^2 = A_1 A_2 A_3, (A_1 A_2 A_3)^{3d} = A_1^2 = A_2^2 = A_3^2 = (yx^{-1})^N = 1 \rangle$$

where $(N,d) \in \{(7,7), (8,4), (9,3), (12,2)\}$. Note that the subgroup $K_d$ generated by $A_1, A_2, A_3$ in $\Gamma_{d,N}$ is normal and finitely generated, the quotient $\Gamma_{d,N}/K_d$ is the hyperbolic triangle group

$$\Delta_N := \langle x, y | x^3 = y^2 = (yx^{-1})^N = 1 \rangle$$

since $N \geq 7$. Now fix a pair $(N,d)$ from the above list and let $\Gamma := \Gamma_{d,N}, \Delta := \Delta_N, K := K_d$. Let $\Delta' < \Delta$ be a torsion-free subgroup of finite index and $\Gamma' < \Gamma$ be the pull-back of $\Delta'$ to $\Gamma$. Then $\Gamma'$ fits into short exact sequence

$$1 \to K \to \Gamma' \to \Delta' \to 1$$

The group $\Gamma'$ still has torsion, so let $\pi$ be a torsion-free subgroup of $\Gamma'$, $K' := \pi \cap K, Q := \pi/K'$. Clearly $K'$ is finitely generated and $Q$ is the fundamental group of a compact hyperbolic Riemann surface. The group $\pi$ acts freely discretely cocompactly on $\mathbb{H}_C^2$, and hence is the fundamental group of the compact complex-hyperbolic surface $X = \mathbb{H}_C^2/\pi$. By Lemma [11] the group $K'$ is not finitely presentable.

Remark 12 Bill Goldman had told me long ago about Livne’s example as a candidate for non-coherence, however until recently I did not know how to prove that the group $K$ is not finitely presentable.

Note that the group $K$ is not geometrically finite and its limit set is the whole sphere at infinity of $\mathbb{H}_C^2$ (since $K$ is normal in $\Gamma$).

Question 13 Let $\Gamma \subset PU(2,1)$ be a finitely generated discrete subgroup whose limit set is not the whole sphere at infinity of $\mathbb{H}_C^2$. Is $\Gamma$ finitely-presentable? Is $\Gamma$ geometrically finite?
Remark 14 There are several reasons why it is difficult to construct finitely generated geometrically infinite subgroups of $PU(2,1)$. One of them is the following result due to M. Ramachandran:

Let $\Gamma$ be a discrete subgroup of $PU(2,1)$ which does not contain parabolic elements and which acts cocompactly on a component $\Omega_0$ of the domain of discontinuity $\Omega(\Gamma) \subset \partial_\infty \mathbb{H}^2_\mathbb{C}$. Then $\Gamma$ is geometrically finite and $\Omega_0 = \Omega(\Gamma)$.

(Instead of assuming that $\Gamma$ contains no parabolic elements it is enough to assume that each maximal parabolic subgroup of $\Gamma$ is isomorphic to a lattice in the 3-dimensional Heisenberg group.)

Question 15 Is there a compact real-hyperbolic 4-manifold $X$ whose fundamental group fits into a short exact sequence:

$$1 \to K \to \pi_1(X) \to Q \to 1$$

where $K$ is finitely presentable or even a surface group and $Q$ is a hyperbolic surface group?

More generally:

Question 16 Is there a Gromov-hyperbolic group $\pi$ which fits into a short exact sequence:

$$1 \to K \to \pi \to Q \to 1$$

where $K$ and $Q$ are closed hyperbolic surface groups?

Note that Lee Mosher \cite{mosher} constructed similar example when $K$ is a closed hyperbolic surface group and $Q$ is a free nonabelian group.

Question 17 Let $\Gamma_g$ be the mapping class group of a compact surface of genus $g$. Is there $g$ and a finitely generated non-free subgroup $Q$ of $\Gamma_g$ which consists only of the identity and pseudo-Anosov elements?

Mosher’s example comes from a “Schottky-type” subgroup $Q$ in $\Gamma_g$ where $K$ is the fundamental group of a genus $g$ surface.

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