DIRECT SUMS OF FINITE DIMENSIONAL $\text{SL}_N^\infty$ SPACES

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Abstract. $\text{SL}_n^\infty$ denotes the space of functions whose square function is in $L^\infty$, and the subspaces $\text{SL}_n^\infty$, $n \in \mathbb{N}$, are the finite dimensional building blocks of $\text{SL}_n^\infty$.

We show that the identity operator $\text{Id}_{\text{SL}_n^\infty}$ on $\text{SL}_n^\infty$ well factors through operators $T : \text{SL}_n^\infty \to \text{SL}_n^\infty$ having large diagonal with respect to the standard Haar system. Moreover, we prove that $\text{Id}_{\text{SL}_n^\infty}$ well factors either through any given operator $T : \text{SL}_n^\infty \to \text{SL}_n^\infty$, or through $\text{Id}_{\text{SL}_n^\infty} - T$. Let $X^{(r)}$ denote the direct sum $(\sum_{n \in \mathbb{N}_0} \text{SL}_n^\infty)_r$, where $1 \leq r \leq \infty$. Using Bourgain’s localization method, we obtain from the finite dimensional factorization result that for each $1 \leq r \leq \infty$, the identity operator $\text{Id}_{X^{(r)}}$ on $X^{(r)}$ factors either through any given operator $T : X^{(r)} \to X^{(r)}$, or through $\text{Id}_{X^{(r)}} - T$. Consequently, the spaces $(\sum_{n \in \mathbb{N}_0} \text{SL}_n^\infty)_r$, $1 \leq r \leq \infty$, are all primary.

1. Introduction

Let $\mathcal{D}$ denote the collection of dyadic intervals contained in the unit interval $[0, 1)$; it is given by

$$\mathcal{D} = \{(k-1)2^{-n}, k2^{-n}) : n \in \mathbb{N}_0, 1 \leq k \leq 2^n\}.$$ 

Let $|\cdot|$ denote the Lebesgue measure. For any $N \in \mathbb{N}_0$ we put

$$\mathcal{D}_N = \{I \in \mathcal{D} : |I| = 2^{-N}\} \quad \text{and} \quad \mathcal{D}_N = \bigcup_{n=0}^N \mathcal{D}_n. \quad (1.1)$$

We denote the $L^\infty$-normalized Haar function supported on $I \in \mathcal{D}$ by $h_I$; i.e. if $I_0, I_1 \in \mathcal{D}$ are such that $\inf I_0 < \inf I_1$ and $I_0 \cup I_1 = I$, then

$$h_I = \chi_{I_0} - \chi_{I_1},$$

where $\chi_A$ denotes the characteristic function of $A \subset [0, 1)$. The Rademacher functions $r_n$, $n \in \mathbb{N}_0$ are given by

$$r_n = \sum_{I \in \mathcal{D}_n} h_I, \quad n \in \mathbb{N}_0.$$ 

The non-separable Banach space $\text{SL}_n^\infty$ is given by

$$\text{SL}_n^\infty = \left\{f = \sum_{I \in \mathcal{D}} a_I h_I \in L^2 : \|f\|_{\text{SL}^\infty} < \infty\right\}, \quad (1.2)$$

equipped with the norm

$$\left\| \sum_{I \in \mathcal{D}} a_I h_I \right\|_{\text{SL}_n^\infty} = \left(\sum_{I \in \mathcal{D}} a_I^2 h_I^2\right)^{1/2} \|L^\infty\}. \quad (1.3)$$

(To see that $\text{SL}_n^\infty$ is non-separable, consider any infinite collection of pairwise disjoint dyadic intervals $\{I_j : j \in \mathbb{N}\} \subset \mathcal{D}$, and embed $\ell^\infty$ into $\text{SL}_n^\infty$ by $e_j \mapsto h_{I_j}$.)

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where $e_j$ denotes the $j^{th}$ standard unit vector in $\ell^\infty$. We want to emphasize that throughout this paper, any series in $SL^\infty$ merely represents the vector of coefficients, and it does not indicate any kind of convergence. For variants of the space $SL^\infty$, we refer the reader to [8]. The Hardy space $H^1$ is the completion of $\text{span}\{h_I : I \in \mathcal{D}\}$ under the norm

$$||\sum_{I \in \mathcal{D}} a_I h_I||_{H^1} = \int_0^1 \left( \sum_{I \in \mathcal{D}} a_I^2 h_I^2(x) \right)^{1/2} dx.$$  

We define the duality pairing $(\cdot, \cdot) : SL^\infty \times H^1 \to \mathbb{R}$ by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) dx, \quad f \in SL^\infty, \quad g \in H^1,$$

and note the well-known and obvious inequality (see e.g. [5]):

$$|\langle f, g \rangle| \leq ||f||_{SL^\infty} ||g||_{H^1}, \quad f \in SL^\infty, \quad g \in H^1.$$  

We call a bounded linear map between Banach spaces an operator. By [1,7], the operator $J : SL^\infty \to (H^1)^*$ defined by $g \mapsto (f \mapsto \langle f, g \rangle)$ has norm 1. Now, let $I : H^1 \to (H^1)^{**}$ denote the canonical embedding. Hence, for any given operator $T : SL^\infty \to SL^\infty$, the operator $S : H^1 \to (SL^\infty)^*$, defined by $S := T^* J^* I$ is bounded by $||T||$ and satisfies

$$\langle S g, f \rangle = \langle g, T f \rangle, \quad f \in SL^\infty, \quad g \in H^1,$$

where the duality pairing on the left hand side is the canonical duality pairing between $(SL^\infty)^*$ and $SL^\infty$, and the duality pairing on the right hand side is given by (1.6). For the sake of brevity, we shall write $T^*$ instead of $T^* J^* I$.

Given $n \in \mathbb{N}_0$, we define the following finite dimensional spaces:

$$SL^\infty_n = \text{span}\{h_I : I \in \mathcal{D}^n\} \subset SL^\infty \quad \text{and} \quad H^1_n = \text{span}\{h_I : I \in \mathcal{D}^n\} \subset H^1.$$  

Let $n \in \mathbb{N}_0, \delta > 0$, and let $T : SL^\infty_n \to SL^\infty_n$ denote an operator. We say that $T$ has a $\delta$-large diagonal with respect to the Haar system $(h_I : I \in \mathcal{D}^n)$ if

$$|\langle Th_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}^n.$$  

If unambiguous, we simply say $T$ has large diagonal without explicitly specifying $\delta$ and the system of functions.

## 2. Main Results

Theorem 2.1 asserts that the identity operator on $SL^\infty_n$ factors through any operator $T : SL^\infty_N \to SL^\infty_N$ having large diagonal, where $N$ depends (among other parameters) on $n$ and $||T||$. It is the first step towards a factorization result for direct sums of $SL^\infty_n$ spaces. Theorem 2.1 is a finite dimensional quantitative version of the infinite dimensional factorization result [11, Theorem 2.1].

**Theorem 2.1.** Let $n \in \mathbb{N}_0$, $\Gamma, \eta > 0$ and $\delta > 0$. Then there exists an integer $N = N(n, \Gamma, \eta, \delta)$, such that for any operator $T : SL^\infty_N \to SL^\infty_N$ with $||T|| \leq \Gamma$ and

$$|\langle Th_K, h_K \rangle| \geq \delta |K|, \quad K \in \mathcal{D}^N,$$

there exist operators $R : SL^\infty_n \to SL^\infty_N$ and $S : SL^\infty_N \to SL^\infty_n$ such that the diagram

$$\begin{array}{ccc}
SL^\infty_n & \xrightarrow{\text{id}} & SL^\infty_n \\
R \downarrow & & \downarrow s \\
SL^\infty_N & \xrightarrow{T} & SL^\infty_N
\end{array}$$

(2.1)
is commutative. Moreover, the operators $R$ and $S$ can be chosen in such way that they satisfy $\|R\|\|S\| \leq (1 + \eta)/\delta$.

Our next result is the local factorization Theorem 2.2. The key difference between Theorem 2.2 and Theorem 2.1 is that in Theorem 2.2 we do not require the operator $T$ to have large diagonal. To compensate, we will use additional combinatorics to select a large subset of intervals $\mathcal{L}$, on which either $T$ or $\text{Id} - T$ has large diagonal (see Section 4.2 and 4.3). It is the choice of that $\mathcal{L}$ which determines whether the identity factors through $T$ or $\text{Id} - T$ (see Section 4.3).

**Theorem 2.2.** Let $n \in \mathbb{N}_0$ and $\Gamma, \eta > 0$. Then there exists an integer $N = N(n, \Gamma, \eta)$, such that for any operator $T : SL_n^\infty \to SL_n^\infty$ with $\|T\| \leq \Gamma$, we can find operators $R : SL_n^\infty \to SL_n^\infty$ and $S : SL_n^\infty \to SL_n^\infty$, such that for either $H = T$ or $H = \text{Id} - T$ the diagram

$$
\begin{array}{c}
SL_n^\infty \\
\downarrow R \hspace{1cm} \downarrow S \\
SL_n^\infty \\
\downarrow H \\
SL_n^\infty
\end{array}
$$

is commutative. Moreover, it is possible to choose the operators $R$ and $S$ in such way that they satisfy $\|R\|\|S\| \leq 2 + \eta$.

Recall that a Banach space $X$ is primary if for every bounded projection $Q : X \to X$, either $Q(X)$ or $(\text{Id} - Q)(X)$ is isomorphic to $X$ (see e.g. [13]). In Theorem 2.2 we tie the local results of Theorem 2.2 for the spaces $SL_n^\infty$, $n \in \mathbb{N}_0$, together, to obtain factorization results in $X^{(r)} = (\sum_{n \in \mathbb{N}_0} SL_n^\infty)_r$, $1 \leq r \leq \infty$. Specifically, we obtain that all the spaces $X^{(r)}$, $1 \leq r \leq \infty$, are primary. Moreover, in Section 5.1 we will show that $SL^\infty$ is isomorphic to $X^{(\infty)}$; consequently, $SL^\infty$ is primary, as well.

**Theorem 2.3.** Let $1 \leq r \leq \infty$, put $X^{(r)} = (\sum_{n \in \mathbb{N}_0} SL_n^\infty)_r$ and let $T : X^{(r)} \to X^{(r)}$ denote an operator. Then for any $\eta > 0$, there exist operators $R, S : X^{(r)} \to X^{(r)}$ such that for either $H = T$ or $H = \text{Id} - T$ the diagram

$$
\begin{array}{c}
X^{(r)} \\
\downarrow R \hspace{1cm} \downarrow S \\
X^{(r)} \\
\downarrow H \\
X^{(r)}
\end{array}
$$

is commutative. The operators $R$ and $S$ can be chosen such that $\|R\|\|S\| \leq 1 + \eta$. Consequently, the spaces $(\sum_{n \in \mathbb{N}_0} SL_n^\infty)_r$, $1 \leq r \leq \infty$, as well as $SL^\infty$ are all primary.

We remark that the primarity of $SL^\infty$ has previously been established in [11], by working directly in the non-separable space $SL^\infty$ using infinite dimensional methods instead of Bourgain’s localization method. In contrast, here we will use Bourgain’s localization method to show the primarity of $SL^\infty$.

3. Embeddings, projections and quantitative diagonalization of operators

The main point of this section is to provide the technical main result (see Theorem 3.7) of this paper. Theorem 3.7 permits us to quantitatively almost-diagonalize a given operator $T$ by a block basis of the Haar system. Moreover, it is possible to select the block basis in such way, that if the operator $T$ has large diagonal with
respect to the Haar system, then $T$ has large diagonal with respect to the block basis.

Before we come to the proof of Theorem 3.7, we will discuss several results on embeddings and projections in $SL^\infty$ established in [11], which will play a vital role in the proof of Theorem 3.7. Additionally, we replace the techniques involving qualitative limits of Rademacher functions in [11] with quantitative combinatorics of dyadic intervals (see Lemma 3.6).

3.1. Embeddings and projections.

Here, we briefly discuss the conditions (J1)–(J4) (which go back to Jones [7]) and their consequences. First, we will show that the conditions (J1)–(J4) are stable under reiteration (see Theorem 3.3). Then we will prove that whenever a block basis $(b_I : I \in \mathcal{D}^n)$ of the Haar system $(h_I : I \in \mathcal{D}^N)$ is selected according to (J1)–(J4), $(b_I : I \in \mathcal{D}^n)$ spans a complemented copy of $SL^\infty_n$ (see Theorem 3.4; the constants for the norms of the isomorphism and the projection do not depend on $n$).

Let $J \subseteq \mathcal{D}$ be a collection of dyadic intervals, and let $N$ be a collection of sets. Let $J$ index collections $\mathcal{B}_I \subseteq N$, $I \in J$, and put

$$\mathcal{B} = \bigcup_{I \in J} \mathcal{B}_I \quad \text{and} \quad B_I = \bigcup \mathcal{B}_I, \quad \text{for all } I \in J. \quad (3.1)$$

We say that the (possibly finite) sequence $(\mathcal{B}_I : I \in J)$ satisfies Jones’ compatibility conditions (see [7]) with constant $\kappa_J \geq 1$, if the following conditions (J1)–(J4) are satisfied:

(J1) The collection $N$ consists of measurable sets with finite and positive measure and is nested, i.e. whenever $N_0, N_1 \in N$ with $N_0 \cap N_1 \neq \emptyset$, then $N_0 \subseteq N_1$ or $N_1 \subseteq N_0$. Moreover, for each $I \in J$, the collection $\mathcal{B}_I \subseteq N$ is finite.

(J2) For each $I \in J$, the collection $\mathcal{B}_I$ is non-empty and consists of pairwise disjoint sets. Furthermore, $\mathcal{B}_{I_0} \cap \mathcal{B}_{I_1} = \emptyset$, whenever $I_0, I_1 \in J$ are distinct.

(J3) For all $I_0, I_1 \in J$ holds that

$$B_{I_0} \cap B_{I_1} = \emptyset \quad \text{if} \quad I_0 \cap I_1 = \emptyset, \quad \text{and} \quad B_{I_0} \subseteq B_{I_1} \quad \text{if} \quad I_0 \subseteq I_1.$$

(J4) For all $I_0, I \in J$ with $I_0 \subseteq I$ and $N \in \mathcal{B}_I$, we have

$$\left| \frac{N \cap B_{I_0}}{N} \right| \geq \kappa_J^{-1} \frac{|B_{I_0}|}{|B_I|}.$$

In the following Lemma 3.1, we record three facts about collections satisfying (J1)–(J4). It is a straightforward finite dimensional adaptation of [11, Lemma 3.1].

**Lemma 3.1.** Let $n \in \mathbb{N}_0$ and let $(\mathcal{B}_I : I \in \mathcal{D}^n)$ satisfy (J1)–(J4). Then the following statements are true:

(i) $(\mathcal{B}_I : I \in \mathcal{D}^n)$ is a finite sequence of nested measurable sets of finite positive measure.

(ii) Let $I, I_0 \in \mathcal{D}^n$, then

$$B_{I_0} \subseteq B_I \quad \text{if and only if} \quad I_0 \subseteq I.$$

(iii) Let $I_0, I \in \mathcal{D}^n$, with $I_0 \subseteq I$. Then for all $N_0 \in \mathcal{B}_{I_0}$ there exists a set $N \in \mathcal{B}_I$ such that $N_0 \subseteq N$.

**Proof.** Replacing $\mathcal{D}^n$ with $\mathcal{D}$ in the proof of [11, Lemma 3.1] and repeating it, yields the above result. \hfill \qed

**Remark 3.2.** By Lemma 3.1 (ii), we can uniquely identify $I$ with $B_I$. 

The following Theorem 3.3 (which is a finite dimensional version of Theorem 3.2) asserts that Jones’ compatibility conditions \((J1)–(J4)\) are stable under iteration.

**Theorem 3.3.** Let \(n, N \in \mathbb{N}_0\), and let \((A_I : I \in \mathcal{D}^N)\) be a finite sequence of collections of sets that satisfies \((J1)–(J4)\) with constant \(\kappa_J \geq 1\). Put \(M = \bigcup_{I \in \mathcal{D}^N} A_I\) and \(A_I = \bigcup A_I, I \in \mathcal{D}^N\). Let \(N\) denote the collection of nested sets given by
\[
N = \{ A_I : I \in \mathcal{D}^N \}.
\]
For each \(J \in \mathcal{D}^n\) let \(\mathcal{B}_J \subset N\) be such that \((\mathcal{B}_J : J \in \mathcal{D}^n)\) satisfies \((J1)–(J4)\) with constant \(\kappa_J \geq 1\), where we put \(B_J = \bigcup \mathcal{B}_J\). Finally, for all \(J \in \mathcal{D}^n\), we define
\[
C_J = \bigcup_{A_I \in \mathcal{B}_J} A_I \quad \text{and} \quad C_J = \bigcup \mathcal{C}_J,
\]
and we note that \(C_J = B_J\). Then \((C_J : J \in \mathcal{D}^n)\) is a finite sequence of collections of sets in \(M\) satisfying \((J1)–(J4)\) with constant \(\kappa_J^2\).

**Proof.** The proof of Theorem 3.3 follows immediately by replacing \(\mathcal{D}^n\) with \(\mathcal{D}\) in the proof of Theorem 3.2, and repeating the argument.  

Here we establish that if \((\mathcal{B}_I : I \in \mathcal{D}^n)\) with \(\mathcal{B}_I \subset \mathcal{D}^N, I \in \mathcal{D}^n\), satisfies Jones’ compatibility conditions \((J1)–(J4)\), then the block basis \((b_I : I \in \mathcal{D}^n)\) given by
\[
b_I = \sum_{K \in \mathcal{B}_I} h_K, \quad I \in \mathcal{D}^n
\]
spans a complemented copy of \(SL_N^\infty\) (the constants for the norms of the isomorphism and the projection do not depend on \(n\)). Theorem 3.4 is a straightforward finite dimensional adaptation of Theorem 3.3.

**Theorem 3.4.** Let \(n, N \in \mathbb{N}_0\) and \(\mathcal{B}_I \subset \mathcal{D}^N, I \in \mathcal{D}^n\). Assume that the finite sequence of collections of dyadic intervals \((\mathcal{B}_I : I \in \mathcal{D}^n)\) satisfies Jones’ compatibility conditions \((J1)–(J4)\) with constant \(\kappa_J \geq 1\). Let \((b_I : I \in \mathcal{D}^n)\) denote the block basis of the Haar system \((h_I : I \in \mathcal{D}^n)\) given by
\[
b_I = \sum_{K \in \mathcal{B}_I} h_K, \quad I \in \mathcal{D}^n.
\]
Then the operators \(B : SL_N^\infty \to SL_N^\infty\) and \(Q : SL_N^\infty \to SL_N^\infty\) given by
\[
Bf = \sum_{I \in \mathcal{D}^n} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I \quad \text{and} \quad Qg = \sum_{I \in \mathcal{D}^n} \frac{\langle g, h_I \rangle}{\|b_I\|_2^2} h_I,
\]
satisfy the estimates
\[
\|Bf\|_{SL_N^\infty} \leq \|f\|_{SL_N^\infty} \quad \text{and} \quad \|Qg\|_{SL_N^\infty} \leq \kappa_J^{1/2} \|g\|_{SL_N^\infty},
\]
for all \(f \in SL_N^\infty, g \in SL_N^\infty\). Moreover, the diagram
\[
\begin{array}{ccc}
SL_N^\infty & \xrightarrow{Id_{SL_N^\infty}} & SL_N^\infty \\
B \downarrow & & \downarrow Q \\
SL_N^\infty
\end{array}
\]
is commutative. Consequently, the range of \(B\) is complemented by the projection \(BQ\) with \(\|BQ\| \leq \kappa_J^{1/2}\), and \(B\) is an isomorphism onto its range with \(\|B\||B^{-1}| \leq \kappa_J^{1/2}\).

**Proof.** Clearly, the finite dimensional operators \(B\) and \(Q\) in the above theorem are truncated versions of the corresponding infinite dimensional operators in Theorem 3.3. Hence, the result follows by 1-unconditionality of the Haar system in \(SL^\infty\).
Remark 3.5. Let \( n, N \in \mathbb{N} \) and \( B_I \subset \mathcal{D}^N \), \( I \in \mathcal{D}^n \) be such that \( (B_I : I \in \mathcal{D}^n) \) satisfies Jones’ compatibility conditions \((J_1)–(J_4)\) with constant \( \varepsilon \). Recall that in \((3.1)\) we defined \( \varepsilon \in \{\pm 1\} : K \in \mathcal{B} \). Now, given a finite sequence of signs \( \varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in \mathcal{B}) \), we put
\[
b_I^{(\varepsilon)} = \sum_{K \in B_I} \varepsilon_K h_K, \quad I \in \mathcal{D}^n. \tag{3.6}
\]
We call \((b_I^{(\varepsilon)} : I \in \mathcal{D}^n)\) the block basis generated by \((B_I : I \in \mathcal{D}^n)\) and \( \varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in \mathcal{B}) \).

The block basis \((b_I^{(\varepsilon)} : I \in \mathcal{D}^n)\) gives rise to the operators \( B^{(\varepsilon)} : SL_n^\infty \to SL_N^\infty \) and \( Q^{(\varepsilon)} : SL_n^\infty \to SL_N^\infty \):
\[
B^{(\varepsilon)} f = \sum_{I \in \mathcal{D}^n} \frac{\langle f, h_I \rangle}{\|h_I\|_2^2} b_I^{(\varepsilon)} \quad \text{and} \quad Q^{(\varepsilon)} g = \sum_{I \in \mathcal{D}^n} \frac{\langle g, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} h_I, \tag{3.7}
\]
for all \( f \in SL_n^\infty \) and \( g \in SL_N^\infty \). Before we proceed, recall the definitions of the operators \( B, Q \) (see Theorem 3.4). By the 1-unconditionality of the Haar system in \( SL^\infty \) and
\[
Q^{(\varepsilon)} g = Q g^{(\varepsilon)}, \quad \text{for all} \ g \in SL_N^\infty \quad \text{and} \quad g^{(\varepsilon)} = \sum_{K \in \mathcal{B}} \varepsilon_K \frac{\langle g, h_K \rangle}{\|h_K\|_2} h_K,
\]
we obtain
\[
\|B^{(\varepsilon)}\| \leq \|B\| \quad \text{and} \quad \|Q^{(\varepsilon)}\| \leq \|Q\|. \tag{3.8}
\]
Moreover, we have the identity
\[
Q^{(\varepsilon)} B^{(\varepsilon)} = \text{Id}_{SL_N^\infty}. \tag{3.9}
\]
Consequently, the range of \( B^{(\varepsilon)} \) is complemented by the projection \( B^{(\varepsilon)} Q^{(\varepsilon)} \) with \( \|B^{(\varepsilon)} Q^{(\varepsilon)}\| \leq \kappa_{1/2} \), and \( B^{(\varepsilon)} \) is an isomorphism onto its range with \( \|B^{(\varepsilon)}\| \|(B^{(\varepsilon)})^{-1}\| \leq \kappa_{1/2} \).

3.2. A combinatorial lemma.

The following Lemma 3.6 will later be used in the proofs of the local results Theorem 3.7 and Corollary 3.10. We remark that in the infinite dimensional setting (see [14]), we used that the Rademacher functions tend to 0 in a specific way which is related to weak and weak* convergence (see [11] Lemma 4.1, Lemma 4.2). Lemma 3.6 can be viewed as a quantitative substitute for those arguments.

There are versions of Lemma 3.6 involving spaces other than \( H^1 \) and \( SL^\infty \), e.g. bi-parameter \( H^1 \) and bi-parameter BMO (see [12] Lemma 4.1), or mixed norm Hardy spaces (see [10] Lemma 4.1). Their origin is the version for one parameter \( H^1 \) and one parameter BMO (see [14]; see also [15] Lemma 5.2.4).

Lemma 3.6. Let \( i \in \mathbb{N}, \ K_0 \in \mathcal{D}, \) and let \( f_j \in SL^\infty \) and \( g_j \in (SL^\infty)^* \), \( 1 \leq j \leq i \), be such that
\[
\sum_{j=1}^i \|f_j\|_{SL^\infty} \leq 1 \quad \text{and} \quad \sum_{j=1}^i \|g_j\|_{(SL^\infty)^*} \leq |K_0|. \tag{3.10}
\]
The local frequency weight \( \omega : \mathcal{D} \to [0, \infty) \) is given by
\[
\omega(K) = \sum_{j=1}^i |\langle f_j, h_K \rangle| + |\langle h_K, g_j \rangle|, \quad K \in \mathcal{D}. \tag{3.11}
\]
Given \( \tau > 0, \) and \( \nu \in \mathbb{N} \) with \( 2^{-\nu} \leq |K_0|, \) we define the collection of dyadic intervals
\[
\mathcal{G}(K_0) = \{ K \in \mathcal{D} : K \subset K_0, \ |K| \leq 2^{-\nu}, \ \omega(K) \leq \tau |K| \}.
\]
Moreover, we put
\[ \mathcal{G}_k(K_0) = \mathcal{G}(K_0) \cap D_k, \quad k \in \mathbb{N}. \]
Then for each \( \rho > 0 \), there exists an integer \( k \) with
\[ r \leq k \leq \left\lfloor \frac{4}{\rho^2 \tau^2} \right\rfloor + r \tag{3.12} \]
such that
\[ | \bigcup \mathcal{G}_k(K_0) | \geq (1 - \rho) |K_0|. \tag{3.13} \]

See Figure 1 for a depiction of the collections \( \mathcal{G}_k(K_0) \), \( r \leq k \leq \left\lfloor \frac{4}{\rho^2 \tau^2} \right\rfloor + r \).

**Proof.** The proof of Lemma 3.6 is obtained by essentially repeating the argument given for [14, Lemma 4]. Define
\[ \mathcal{H}_k(K_0) = \{ K \in D_k : K \subset K_0, \ K \notin \mathcal{G}_k(K_0) \}, \quad k \in \mathbb{N}_0, \]
and put
\[ A = \left\lfloor \frac{4}{\rho^2 \tau^2} \right\rfloor + r. \]
Assume that the conclusion of Lemma 3.6 is not true, i.e. assume that
\[ | \bigcup \mathcal{H}_k(K_0) | > \rho |K_0|, \quad r \leq k \leq A. \]

On the one hand, summing these above estimates yields
\[ \sum_{k=r}^{A} \left| \bigcup \mathcal{H}_k(K_0) \right| \geq (A - r + 1) \rho |K_0|. \tag{3.14} \]

On the other hand, observe that by definition of \( \mathcal{H}_k(K_0) \) and \( \mathcal{G}(K_0) \), we have
\[ \tau \sum_{k=r}^{A} \left| \bigcup \mathcal{H}_k(K_0) \right| \leq \sum_{j=1}^{i} \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k} |\langle f_j, h_K \rangle| + |\langle h_K, g_j \rangle|. \tag{3.15} \]

Now, we rewrite the right hand side of (3.15) in the following way:
\[ \sum_{j=1}^{i} \left| \left\{ f_j, \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k} \pm h_K \right\} \right| + \left| \left\{ \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k} \pm h_K, g_j \right\} \right|. \tag{3.16} \]
Note the estimates:
\[
\left\| \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k(K_0)} \pm h_K \right\|_{H^1} \leq \sqrt{A - r + 1} |K_0|, \tag{3.17a}
\]
\[
\left\| \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k(K_0)} \pm h_K \right\|_{SL^\infty} \leq \sqrt{A - r + 1}. \tag{3.17b}
\]
Combining (3.15) and (3.16) with (3.17), and using (3.10) yields
\[
\tau \sum_{k=r}^{A} \left| \bigcup \mathcal{H}_k(K_0) \right| \leq \sum_{j=1}^{i} \|f_j\|_{SL^\infty} \left\| \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k(K_0)} \pm h_K \right\|_{H^1}
+ \sum_{j=1}^{i} \|g_j\|_{SL^\infty} \left\| \sum_{k=r}^{A} \sum_{K \in \mathcal{H}_k(K_0)} \pm h_K \right\|_{SL^\infty}
\leq 2\sqrt{A - r + 1} |K_0|
\]
By (3.14) and the above estimate we obtain
\[
A \leq \frac{4}{\rho^2 \tau^2} + r - 1,
\]
which contradicts the definition of \(A\). \(\square\)

3.3. Quantitative diagonalization of operators on \(SL^\infty_N\).
Here, we will show that any given operator \(T\) acting on \(SL^\infty_N\) which has large diagonal with respect to the Haar system, can be almost-diagonalized by a block basis of the Haar system \((b^{(c)}_I : I \in \mathcal{D}^n)\) (see (3.6) for the definition of \(b^{(c)}_I\)) in the space \(SL^\infty_n\), that spans a complemented copy of \(SL^\infty_n\) (the constants for the norms of the isomorphism and the projection do not depend on \(n\); see Theorem 3.7), where the dimensions \(N\) and \(n\) are quantitatively linked. This will be achieved by constructing the block basis \((b^{(c)}_I : I \in \mathcal{D}^n)\) so that Jones’ compatibility conditions (J1)–(J4) are satisfied. In order to keep the diagonal of the operator (with respect to the block basis \((b^{(c)}_I : I \in \mathcal{D}^n)\)) large, we will choose signs appropriately; this technique was introduced by Andrew in [1]. The signs in [1] are selected semi-probabilistically, where in contrast our argument is entirely probabilistic.

From here on, we will regularly identify a dyadic interval \(I \in \mathcal{D}\) with the natural number \(\mathcal{O}(I)\) given by
\[
\mathcal{O}(I) = 2^n - 1 + k, \quad \text{if } I = [(k - 1)2^{-n}, k2^{-n}). \tag{3.18}
\]
Specifically, for \(\mathcal{O}(I) = i\) we identify
\[
\mathcal{B}_I = \mathcal{B}_i \quad \text{and} \quad b^{(c)}_I = b^{(c)}_i.
\]

**Theorem 3.7.** Let \(n \in \mathbb{N}_0\), \(\Gamma, \eta > 0\) and \(\delta \geq 0\). Then there exists an integer \(N = N(n, \Gamma, \eta)\), such that for any operator \(T : SL^\infty_N \to SL^\infty_N\) with \(\|T\| \leq \Gamma\) and
\[
|\langle Th_K, h_K \rangle| \geq \delta |K|, \quad K \in \mathcal{D}^N,
\]
there exists a finite sequence of collections \((\mathcal{B}_I : I \in \mathcal{D}^n)\) and a finite sequence of signs \(\varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in \mathcal{B})\), where \(\mathcal{B} = \bigcup_{I \in \mathcal{D}^n} \mathcal{B}_I\), which generate the block basis of the Haar system \((b^{(c)}_I : I \in \mathcal{D}^n)\) given by
\[
b^{(c)}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \quad I \in \mathcal{D}^n,
\]
such that the following conditions are satisfied:
(i) $\mathcal{B}_I \subset \mathcal{D}^N$, $I \in \mathcal{D}^n$, and $(\mathcal{B}_I : I \in \mathcal{D}^n)$ satisfies Jones’ compatibility conditions $(\text{J1})$–$(\text{J4})$ with constant $\kappa_J = (1 - \eta)^{-1}$.
(ii) $(b^{(c)}_i : I \in \mathcal{D}^n)$ almost-diagonalizes $T$ in such way that $T$ has $\delta$-large diagonal with respect to $(b^{(c)}_i : I \in \mathcal{D}^n)$. To be more precise, we have the estimates
\[
\sum_{j=1}^{i-1} |\langle Tb^{(c)}_j, b^{(c)}_i \rangle| + |\langle b^{(c)}_i, T^*b^{(c)}_j \rangle| \leq \eta 4^{-i} \|b^{(c)}_i\|_2^2, \quad (3.19a)
\]
\[
\langle Tb^{(c)}_i, b^{(c)}_i \rangle \geq \delta \|b^{(c)}_i\|_2^2, \quad (3.19b)
\]
for all $1 \leq i \leq 2^{n+1} - 1$.

**Proof.** The proof is divided into the following steps:

- **Preparation:** setting up the inductive argument;
- **Construction of $\mathcal{B}_n$:** using the combinatorial Lemma 3.6 to select $\mathcal{B}_n$;
- **Choosing the signs $\varepsilon_K \in \pm 1$, $K \in \mathcal{B}_0$: using a probabilistic argument;**
- **$(\mathcal{B}_j : j \leq i_0)$ satisfies Jones’ compatibility conditions:** verifying that $(\text{J1})$–$(\text{J4})$ is satisfied with constant $\kappa_J = (1 - \eta)^{-1}$;
- **$(b^{(c)}_i : i \leq i_0)$ almost-diagonalizes $T$:** showing that $(3.19a)$ is satisfied;
- **Conclusion of the proof:** summarizing the previous steps.

**Preparation.**

Let $n \in \mathbb{N}_0$, $\Gamma, \eta > 0$, $\delta \geq 0$, and define the following constants, which will be used within the proof:

\[
\rho_i = \eta 2^{-i}, \quad \tau_{i+1} = \eta 8^{-i-1} 2^{-m_i}/\Gamma, \quad \tau_i = \frac{\rho_{i+1}}{\tau_{i+1}}, \quad m_1 = 0, \quad m_{i+1} = m_i + 1 + \left\lfloor \frac{4}{\rho_{i+1} \tau_{i+1}} \right\rfloor, \quad N = m_{2^{n+1} - 1}, \quad (3.20)
\]

for all $1 \leq i \leq 2^{n+1} - 1$. Clearly, $N$ depends only on $n, \Gamma$ and $\eta$, i.e. $N = N(n, \Gamma, \eta)$.

Finally, let $T : SL_N^\infty \to SL_N^\infty$ be such that $\|T\| \leq \Gamma$ and
\[
|\langle T h_K, h_K \rangle| \geq \delta |K|, \quad K \in \mathcal{D}^N.
\]

Before we proceed with the proof, observe first, that by 1-unconditionality, we can assume that
\[
\langle T h_K, h_K \rangle \geq \delta |K|, \quad K \in \mathcal{D}^N.
\]

Second, given $K \in \mathcal{D}^N$, we write
\[
T h_K = \alpha_K h_K + r_K, \quad (3.21a)
\]
where
\[
\alpha_K = \frac{\langle T h_K, h_K \rangle}{|K|} \quad \text{and} \quad r_K = \sum_{L \in \mathcal{D}_N^N \setminus \{K\}} \frac{\langle T h_K, h_L \rangle}{|L|} h_L. \quad (3.21b)
\]

Thirdly, note the estimate
\[
\delta \leq \alpha_K \leq \|T\|, \quad K \in \mathcal{D}^N. \quad (3.22)
\]

We will now inductively define the block basis $(b^{(c)}_i : I \in \mathcal{D}^n)$. To begin the induction, we simply put
\[
\mathcal{B}_1 = \mathcal{B}_{[0,1)} = \{[0,1)\} \quad \text{and} \quad b^{(c)}_1 = b^{(c)}_{[0,1)} = h_{[0,1)}.
\]

and note that $\mathcal{B}_1 \subset \mathcal{D}^{m_1}$ by $(3.20)$.

For the inductive step, let $i_0 \geq 2$ and assume that

- we have already chosen finite collections $\mathcal{B}_j$ with $\mathcal{B}_j \subset \mathcal{D}^{m_j \setminus \mathcal{D}^{m_j - 1}}$, $2 \leq j \leq i_0 - 1$;

- note that $\mathcal{B}_j \subset \mathcal{D}^{m_j}$ by $(3.20)$. 

We will now inductively define the block basis $(b^{(c)}_i : I \in \mathcal{D}^n)$ for all $i \geq i_0$.
By Lemma 3.6 and (3.24), we find for each

\[ \omega_{i_0-1}(K) = \sum_{j=1}^{i_0-1} |(f_j, h_K)| + |(h_K, g_j)|, \quad K \in \mathcal{D}. \]  

By Lemma 3.6 and (3.24), we find for each \( K_1 \in \mathcal{B}_l \cap \mathcal{B}_r \) a collection of pairwise disjoint dyadic intervals \( \mathcal{Z}_{i_0}(K_1) \) such that

\[ |\bigcup \mathcal{Z}_{i_0}(K_1)| \geq (1 - \rho_{i_0})|K_1|, \quad \omega_{i_0-1}(K) \leq \tau_{i_0}|K|, \quad K \in \mathcal{Z}_{i_0}(K_1). \]  

Recall that by (3.20) we have \( m_{i_0} = m_{i_0-1} + 1 + \lfloor \frac{1}{\tau_{i_0} \rho_{i_0}} \rfloor \), hence (3.24) yields

\[ \min\{|K| : K \in \mathcal{Z}_{i_0}(K_1)\} \geq |K_1|2^{-\lfloor \frac{1}{\tau_{i_0} \rho_{i_0}} \rfloor} \geq 2^{-m_{i_0}}. \]  

If \( I_0 \) is the left half of \( I_0 \), we put

\[ \mathcal{B}_{i_0} = \mathcal{B}_{I_0} = \bigcup \{ \mathcal{Z}_{i_0}(K_1) : K_1 \in \mathcal{B}_{I_0}^{l} \}, \]
See Figure 2 for a depiction of the collection and if the inclusion for all choices of signs
In either of the cases (3.28a) and (3.28b), we put (with a slight abuse of notation)
Choosing the signs and observe that by (3.31) and (3.22) we obtain
By (3.21), we have that
Now, let then
Taking the average
and if \( I_0 \) is the right half of \( \tilde{I}_0 \), we define
See Figure 2 for a depiction of the collection \( B_{I_0} \). By (3.27b) and (3.28), we have the inclusion
In either of the cases (3.28a) and (3.28b), we put (with a slight abuse of notation)
for all choices of signs \( \varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in B_{I_0}) \).
Continuing with the proof, we obtain from (3.21) that for all choices of signs \( \varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in B_{I_0}) \)
where
Now, put
and observe that by (3.31) and (3.22) we obtain
By (3.21), we have that \( \langle r_K, h_K \rangle = 0 \), hence
Now, let \( E_{\varepsilon} \) denote the average over all possible choices of signs \( \varepsilon = (\varepsilon_K \in \{\pm 1\} : K \in B_{I_0}) \). If \( K_0 \neq K_1 \), then \( E_{\varepsilon} \varepsilon_{K_0} \varepsilon_{K_1} = 0 \); therefore
Taking the average \( E_{\varepsilon} \) in (3.33) we obtain
}\[ E_{\varepsilon} \langle T b_{I_0}^{(e)}, b_{I_0}^{(e)} \rangle \geq \delta E_{\varepsilon} \| b_{I_0}^{(e)} \|^2. \]
Lemma 3.1 (iii) yields

\[ \langle T h^{(\varepsilon)}_0, h^{(\varepsilon)}_0 \rangle \geq \delta \| h^{(\varepsilon)}_0 \|^2. \]  \hspace{1cm} (3.35)

This concludes the constructive part of the inductive step. Next, we will demonstrate that our construction has the properties claimed in the theorem.

\((\mathcal{B}_j : 1 \leq j \leq i_0)\) satisfies Jones’ Compatibility Conditions.

Note that by our induction hypothesis, (3.18) and (3.29), we obtain that \(\mathcal{B}_j \subset \mathcal{D}^N\), \(1 \leq j \leq i_0\). We will now show inductively that the finite sequence of collections of dyadic intervals \((\mathcal{B}_j : 1 \leq j \leq i_0)\) satisfies (J1)–(J4) with constant \(\kappa_J = (1 - \sum_{j=1}^{i_0} \rho_j)^{-1}\). By our choice of \(Z_{i_0}(K_1)\) in (3.27), it should be clear that the properties (J1)–(J3) are satisfied. We will now prove that (J4) is satisfied, as well.

To this end, let us record that by induction hypothesis, we have the estimate

\[ \frac{|K \cap B_{I_0}|}{|K|} \geq \left( 1 - \sum_{j=1}^{i_0} \rho_j \right) \frac{|B_{I_0}|}{|B_I|}, \]  \hspace{1cm} (3.36)

for all \(I_0, I \in \mathcal{D}^n\) with \(\emptyset(I_0) \leq i_0 - 1\), \(I_0 \subset I\) and \(K \in \mathcal{B}_I\). Now, let \(I_0, I \in \mathcal{D}^n\) be such that \(\emptyset(I_0) = i_0\), \(I \supset I_0\) and let \(K \in \mathcal{B}_I\). First, note that (J3), (J2) and Lemma 3.1 (iii) give us

\[ |K \cap B_{I_0}| = |K \cap B_{I_0} \cap B_{I_{0}}| = \sum_{L \in \mathcal{B}_{I_0}} |L \cap B_{I_0}| = \sum_{L \in \mathcal{B}_{I_0}} |L^\ell \cap B_{I_0}| + |L^r \cap B_{I_0}|. \]

Recall that \(L^\ell\) denotes the left half of \(L\), and that \(L^r\) denotes the right half of \(L\). Considering our choice for \(Z_{i_0}(L^\ell)\) and \(Z_{i_0}(L^r)\) in (3.27a), and for \(\mathcal{B}_{I_0}\) in (3.28), we find that

\[ |L^\ell \cap B_{I_0}| \geq (1 - \rho_{i_0}) |L^\ell| \quad \text{and} \quad |L^r \cap B_{I_0}| \geq (1 - \rho_{i_0}) |L^r|, \quad L \in \mathcal{B}_{I_0}. \]

Now observe that by (3.28), \(B_{I_0} \cap L^\ell = \emptyset\), if \(I_0\) is the left half of \(\tilde{I}_0\), and that \(B_{I_0} \cap L^r = \emptyset\), if \(I_0\) is the right half of \(\tilde{I}_0\). Combining everything after (3.36) with Lemma 3.1 (iii) yields

\[ |K \cap B_{I_0}| \geq \frac{1}{2} (1 - \rho_{i_0}) \sum_{L \in \mathcal{B}_{I_0}} |L| = \frac{1}{2} (1 - \rho_{i_0}) |K \cap B_{I_0}|. \]  \hspace{1cm} (3.37)

Similar considerations give us

\[ |B_{I_0}| = \sum_{L \in \mathcal{B}_{I_0}} |L^\ell| = \sum_{L \in \mathcal{B}_{I_0}} |L^r| \geq 2 |B_{I_0}|. \]  \hspace{1cm} (3.38)

Note that by (3.18) \(\emptyset(\tilde{I}_0) \leq \emptyset(I_0) - 1\). Hence, the estimate (3.37), our induction hypothesis (3.36) and (3.38) yield

\[ |K \cap B_{I_0}| \geq \frac{1}{2} (1 - \rho_{i_0}) \left( 1 - \sum_{j=1}^{i_0} \rho_j \right) |K| \frac{|B_{I_0}|}{|B_I|} \]

\[ \geq \left( 1 - \sum_{j=1}^{i_0} \rho_j \right) |K| \frac{|B_{I_0}|}{|B_I|}. \]  \hspace{1cm} (3.39)
Thus, we proved that (3.36) holds true for \( i_0 \) instead of \( i_0 - 1 \).

\[(b_j^{(c)} : 1 \leq j \leq i_0) \text{ ALMOST-DIAGONALIZES } T.\]

By (3.25) and (3.30), we obtain

\[
\sum_{j=1}^{i_0-1} |\langle T^j b_j^{(c)}, b_j^{(c)} \rangle| + |\langle b_j^{(c)}, T^j b_j^{(c)} \rangle| \leq 2^{i_0} 2^{m_{i_0} - 1} \Gamma \sum_{K \in \mathcal{B}_{i_0}} \sum_{j=1}^{i_0-1} |\langle f_j, h_K \rangle| + |\langle h_K, g_j \rangle|.
\]

Recall that by (3.20) \( \tau_{i_0} = \eta 8^{-i_0} 2^{-m_{i_0} - i} / \Gamma \), thus, the above inequality, (3.26), (3.27a), (3.28) and (3.30) yield

\[
\sum_{j=1}^{i_0-1} |\langle T^j b_j^{(c)}, b_j^{(c)} \rangle| + |\langle b_j^{(c)}, T^j b_j^{(c)} \rangle| \leq 2^{i_0} 2^{m_{i_0} - 1} \Gamma \sum_{K \in \mathcal{B}_{i_0}} \tau_{i_0} |K| = \eta 4^{-i_0} ||h_j^{(c)}||^2.
\]

Hence, (3.40) combined with (3.35) shows that (3.19) is true for all \( 1 \leq i \leq i_0 \).

**Conclusion of the proof.**

Thus far, we proved the following:

- we chose finite collections \( \mathcal{B}_j \) with \( \mathcal{B}_j \subset \mathcal{D}^j \setminus \mathcal{D}^{j-1}, 2 \leq j \leq i_0 \) and \( \mathcal{B}_1 \subset \mathcal{D}^1; \)
- the finite sequence of collections \( \{\mathcal{B}_j : 1 \leq j \leq i_0\} \) satisfies (J1)–(J4) with constant \( \kappa_j = (1 - \sum_{j=1}^{i_0} \rho_j)^{-1}; \)
- we chose signs \( \epsilon = (\epsilon_K \in \{\pm 1\} : K \in \bigcup_{j=1}^{i_0} \mathcal{B}_j); \)
- the block basis elements \( b_j^{(c)} \) have been defined by

\[
b_j^{(c)} = \sum_{K \in \mathcal{B}_j} \epsilon_K h_K, \quad 1 \leq j \leq i_0,
\]

and satisfy (3.19) for all \( 1 \leq i \leq i_0 \).

We conclude the proof by stopping the induction process after \( 2^{n+1} - 1 \) steps and considering the definition of the constants in (3.20).

\[\square\]

**Remark 3.8.** We note the following:

(i) Recall that in (3.20) we defined \( \rho_I = \rho_i = \eta 2^{-i}, 1 \leq i \leq 2^{n+1} - 1, \) whenever \( \mathcal{O}(I) = i \). Now, observe that by summing (3.37) over all \( K \in \mathcal{B}_I \), we obtain by (J2) and (J3) that

\[
|B_I| \geq \frac{1}{2} (1 - \rho_I) |B_I|, \quad I \in \mathcal{D}^n \setminus \{[0, 1]\}.
\]

Recall that we chose \( \mathcal{B}_{[0, 1]} = \{[0, 1]\} \), and note that iterating the latter inequality yields

\[
|B_I| \geq |I| \prod_{j \in \mathcal{D}^n} (1 - \rho_j) \geq |I| (1 - \eta), \quad I \in \mathcal{D}^n.
\]

Furthermore, it is a simple observation that we have the estimate \( |B_I| \leq |I|, \quad I \in \mathcal{D}^n \).

(ii) If \( \delta = 0 \), we can choose the signs arbitrarily; in particular, we can choose \( \epsilon_K = 1, K \in \mathcal{D}^N \), and in that case \( b_j^{(c)} = b_j \) (see (3.2)).
3.4. Almost-annihilating finite dimensional subspaces of $SL^n_N$.

When using Bourgain’s localization method, one eventually needs to pass from the local factorization results to factorization results on the direct sum of the finite dimensional spaces (which in our case are $SL^n_N$, $n \in \mathbb{N}_0$). One of the ingredients is showing that in a large enough space, any finite dimensional space can be almost-annihilated by a bounded projection that has a large image, which goes back to [3, Lemma 1]. The following Definition 3.9 is merely an abstract version for sequences of finite dimensional Banach spaces of the corresponding Lemma in [3, Lemma 2].

In [10], the following notion was introduced.

**Definition 3.9.** We say that a non-decreasing sequence of finite dimensional Banach spaces $(X_n)_{n \in \mathbb{N}}$ with $\sup_n \dim X_n = \infty$ has the property that projections almost annihilate finite dimensional subspaces with constant $C_P > 0$, if the following conditions are satisfied:

For all $n, d \in \mathbb{N}$ and $\eta > 0$ there exists an integer $N = N(n, d, \eta)$ such that for any $d$-dimensional subspace $F \subset X_N$ there exists a bounded projection $Q : X_N \to X_N$ and an isomorphism $R : X_N \to Q(X_N)$ such that

(i) $\|Q\| \leq C_P$,
(ii) $\|R\|, \|R^{-1}\| \leq C_P$,
(iii) $\|Rx\| \leq \eta \|x\|$, for all $x \in F$.

In the following Corollary 3.10 we establish that $(SL^n_N)_{n \in \mathbb{N}_0}$ has the property that projections almost annihilate finite dimensional subspaces, which is a crucial ingredient in the proof of Theorem 2.3.

**Corollary 3.10.** Let $n, d \in \mathbb{N}_0$ and $\eta > 0$. Then there exists an integer $N = N(n, d, \eta)$ such that for any $d$-dimensional subspace $F \subset SL^n_N$ there exists a block basis $(b_I : I \in \mathbb{D}^n)$ satisfying the following conditions:

(i) $\mathcal{B}_I \subset \mathbb{D}^N$, for all $I \in \mathbb{D}^n$.
(ii) For every finite sequence of scalars $(a_I : I \in \mathbb{D}^n)$, we have that

$$(1 + \eta)^{-1} \left\| \sum_{I \in \mathbb{D}^n} a_I h_I \right\|_{SL^n_N} \leq \left\| \sum_{I \in \mathbb{D}^n} a_I b_I \right\|_{SL^n_N} \leq (1 + \eta) \left\| \sum_{I \in \mathbb{D}^n} a_I h_I \right\|_{SL^n_N}.$$  \hfill (3.42)

(iii) The orthogonal projection $Q : SL^n_N \to SL^n_N$ given by

$$Qf = \sum_{I \in \mathbb{D}^n} \frac{\langle f, b_I \rangle}{\|b_I\|^2} b_I$$

satisfies the estimates

$$\|Qf\|_{SL^n_N} \leq (1 + \eta) \|f\|_{SL^n_N}, \quad f \in SL^n_N,$$
$$\|Qf\|_{SL^n_N} \leq \eta \|f\|_{SL^n_N}, \quad f \in F.$$  \hfill (3.43)

**Proof.** Corollary 3.10 is obtained by modifying the proof of Theorem 3.7 in the following way: We choose a finite $\eta/2$-net $\{f_1, \ldots, f_{k_0}\}$ of the compact unit ball of the $d$-dimensional subspace $F$, and we use the frequency weight $\omega(K) = \sum_{k=1}^{k_0} |\langle f_k, h_K \rangle|$ in all stages of the proof instead of the ones defined in (3.26).

**Remark 3.11.** Corollary 3.10 implies that for any $\eta > 0$, the sequence of finite dimensional Banach spaces $(SL^n_N)_{n \in \mathbb{N}_0}$ has the property that projections almost annihilate finite dimensional subspaces with constant $1 + \eta$.

4. LOCAL FACTORIZATION OF THE IDENTITY OPERATOR ON $SL^n_N$ THROUGH OPERATORS WITH LARGE DIAGONAL

Here, we will prove the local factorization results Theorem 2.1 and Theorem 2.2. The key ingredients are the finite dimensional quantitative almost-diagonalization Theorem 3.7 and the projection Theorem 3.4.
4.1. Proof of Theorem 2.1

The basic pattern of the following proof was recently employed in [12, 9, 10, 11].

Let \( n \in \mathbb{N}_0, \Gamma, \eta > 0, \delta > 0, \) and let \( \eta_1 = \eta(n, \delta, \eta) \) denote the largest positive constant such that

\[
0 < \eta_1 \leq 1/2, \quad \frac{\eta_1 2^{n+1}}{\delta} \leq 1/2, \quad \frac{1}{1 - \frac{\eta_1 2^{n+1}}{\delta}} \leq 1 + \eta. \tag{4.1}
\]

Thus, by Theorem 3.7, there exists an integer \( N = N(n, \Gamma, \eta_1) = N(n, \Gamma, \eta, \delta) \), such that for any operator \( T : SL_N^\infty \to SL_N^\infty \) with \( \|T\| \leq \Gamma \) and

\[
|\langle T h_I, h_I \rangle| \geq \delta |I|, \quad I \in \mathcal{D}^N,
\]

there exists a finite sequence of collections \( \{ B_I : I \in \mathcal{D}^n \} \) and a finite sequence of signs \( \varepsilon = (\varepsilon_K) \in \{ \pm 1 \} : K \in \mathcal{B} \), where \( \mathcal{B} = \bigcup_{I \in \mathcal{D}^n} \mathcal{B}_I \), which generate the block basis of the Haar system \( (b^{(\varepsilon)}_I : I \in \mathcal{D}^n) \) given by

\[
b^{(\varepsilon)}_I = \sum_{K \in \mathcal{B}_I} \varepsilon_K h_K, \quad I \in \mathcal{D},
\]

such that the following conditions are satisfied:

(i) \( \mathcal{B}_I \subset \mathcal{D}^N \) whenever \( I \in \mathcal{D}^n \), and \( (\mathcal{B}_I : I \in \mathcal{D}^n) \) satisfies Jones’ compatibility conditions \( \{ J_1 \} \text{--} \{ J_4 \} \) with constant \( \kappa_J = (1 - \eta_1)^{-1} \).

(ii) For all \( 1 \leq i \leq 2^{n+1} - 1 \), we have the estimates

\[
\sum_{j=1}^{i-1} |\langle T b^{(\varepsilon)}_I, b^{(\varepsilon)}_j \rangle| + |\langle b^{(\varepsilon)}_I, T^* b^{(\varepsilon)}_j \rangle| \leq \eta_1 4^{-i} \|b^{(\varepsilon)}_I\|^2_2, \tag{4.2a}
\]

\[
(T b^{(\varepsilon)}_I, b^{(\varepsilon)}_I) \geq \delta \|b^{(\varepsilon)}_I\|^2_2. \tag{4.2b}
\]

Recall that \( \mathcal{B}_i = \mathcal{B}_I \) and \( b^{(\varepsilon)}_I = b^{(\varepsilon)}_i \), whenever \( \mathcal{O}(I) = i \).

Moreover, by (4.2) and (4.41) in Remark B.8, we have the estimate

\[
\|b^{(\varepsilon)}_I\|_2^2 = |\mathcal{B}_I| \geq (1 - \eta_1) |I| \geq (1 - \eta_1) 2^{-n}, \quad I \in \mathcal{D}^n. \tag{4.4c}
\]

Since \( (\mathcal{B}_I : I \in \mathcal{D}^n) \) satisfies Jones’ compatibility conditions \( \{ J_1 \} \text{--} \{ J_4 \} \) with \( \kappa_J = (1 - \eta_1)^{-1} \), Remark 3.5 and Theorem 3.4 imply that the operators \( B^{(\varepsilon)} : SL_N^\infty \to SL_N^\infty \) and \( Q^{(\varepsilon)} : SL_N^\infty \to SL_N^\infty \) given by

\[
B^{(\varepsilon)} f = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, h_i \rangle}{\|h_i\|_2^2} b^{(\varepsilon)}_i \quad \text{and} \quad Q^{(\varepsilon)} g = \sum_{i=1}^{2^{n+1}-1} \frac{\langle g, b^{(\varepsilon)}_i \rangle}{\|b^{(\varepsilon)}_i\|_2^2} h_i \tag{4.3}
\]

satisfy the estimates

\[
\|B^{(\varepsilon)} f\|_{SL^\infty} \leq \|f\|_{SL^\infty} \quad \text{and} \quad \|Q^{(\varepsilon)} g\|_{SL^\infty} \leq (1 - \eta_1)^{1/2} \|g\|_{SL^\infty}, \tag{4.4}
\]

\( f \in SL_N^\infty, \ g \in SL_N^\infty \). By (4.3), the operator \( P^{(\varepsilon)} : SL_N^\infty \to SL_N^\infty \) defined as

\[
P^{(\varepsilon)} f = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, b^{(\varepsilon)}_i \rangle}{\|b^{(\varepsilon)}_i\|_2^2} b^{(\varepsilon)}_i, \quad f \in SL_N^\infty. \tag{4.5}
\]

Therefore, \( P^{(\varepsilon)} \) is an orthogonal projection, which, by (4.4), satisfies the estimate

\[
\|P^{(\varepsilon)} f\|_{SL^\infty} \leq (1 - \eta_1)^{-1/2} \|f\|_{SL^\infty}, \quad f \in SL_N^\infty. \tag{4.6}
\]

Let \( Y \) denote the subspace of \( SL_N^\infty \) given by

\[
Y = \{ g = \sum_{i=1}^{2^{n+1}-1} a_i b^{(\varepsilon)}_i : a_i \in \mathbb{R} \},
\]
and note that $Y$ is the image of the projection $P(c)$. By (3.9) and (4.4), $B(c) : SL_\infty^N \to Y$ is an isomorphism; thus we obtain the following commutative diagram:

$$
\begin{array}{cc}
SL_\infty^N & SL_\infty^N \\
\downarrow B(c) & \downarrow B(c)^{-1} \\
Y & Y \\
\downarrow \text{Id}_Y & \downarrow \text{Id}_Y \\
\end{array}
$$

(4.7)

where $\|B(c)\| = 1$ and $\|B(c)^{-1}\| \leq (1 - \eta_1)^{-1/2}$. Now, define $U : SL_\infty^N \to Y$ by

$$
Uf = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, b_i^{(c)} \rangle}{\langle b_i^{(c)}, b_i^{(c)} \rangle} b_i^{(c)},
$$

(4.8)

and note that by (4.2b), the 1-unconditionality of the Haar system in $SL_\infty$ and (4.6)

$$
\|U : SL_\infty^N \to Y\| \leq \frac{1}{\delta(1 - \eta_1)^{1/2}}.
$$

(4.9)

Observe that for all $g = \sum_{i=1}^{2^n+1-1} a_i b_i^{(c)} \in Y$, we have the following identity:

$$
UTg - g = \sum_{i=1}^{2^{n+1}-1} \sum_{j=1}^{2^{n+1}-1} a_i \langle b_i^{(c)}, b_j^{(c)} \rangle b_j^{(c)} + \sum_{i=1}^{2^{n+1}-1} \sum_{j=1}^{2^{n+1}-1} a_i \langle b_i^{(c)}, T_j b_j^{(c)} \rangle b_j^{(c)}.
$$

(4.10)

Using the estimates (4.2) and $|a_j| \leq \|g\|_{SL_\infty}$, together with the above identity (4.10) yields

$$
\|UTg - g\|_{SL_\infty} \leq \frac{\eta_1}{\delta} \left( 1 + \frac{2^n}{1 - \eta_1} \right) \|g\|_{SL_\infty}.
$$

(4.11)

Finally, let $J : Y \to SL_\infty^N$ denote the operator given by $Jy = y$. By our choice of $\eta_1$ in (4.1), the operator $V : SL_\infty^N \to Y$ given by $V = (UTJ)^{-1}U$ is well defined. With these definitions, the following diagram is commutative:

$$
\begin{array}{cc}
Y & Y \\
\downarrow \text{Id}_Y & \downarrow \text{Id}_Y \\
\downarrow UTJ & \downarrow UTJ^{-1} \\
\downarrow V & \downarrow V \\
SL_\infty^N & SL_\infty^N
\end{array}
$$

(4.12)

Merging the diagrams (4.7) and (4.12) concludes the proof.

4.2. Preparation for the proof of Theorem 2.2

Before we come to the proof of Theorem 2.2, we develop some notation and introduce another combinatorial lemma (see Lemma 1.1).

Let $N$ be a nested collection of sets with finite and positive measure. Given $X \subset N$, we define the founding generation $s_0(X)$ of $X$ by

$$
s_0(X) = \{ N \in X : \text{there is no} \ M \in X \text{ with} \ M \supseteq N \}.
$$

(4.13)

We now inductively define the $k^{th}$ generation $s_k(X)$ of $X$. Assuming that $s_0(X), \ldots, s_{k-1}(X)$ have already been defined, we put

$$
s_k(X) = s_0 \left( X \setminus \left( s_0(X) \cup \cdots \cup s_{k-1}(X) \right) \right), \quad k \in \mathbb{N}.
$$

(4.14)

We denote the point-set of $s_k(X)$ by $G_k(X)$, i.e.

$$
G_k(X) = \bigcup s_k(X).
$$

(4.15)
Clearly, \( G_k(\mathcal{X}) \subset G_{k-1}(\mathcal{X}) \), \( k \in \mathbb{N} \). The Carleson constant \( \|\mathcal{X}\| \) of \( \mathcal{X} \) is defined as
\[
\|\mathcal{X}\| = \sup_{N \in \mathcal{X}} \frac{1}{|N|} \sum_{M \in \mathcal{X}} |M|.
\] (4.16)

The dyadic version of Lemma 4.1 (including a proof) can be found in [15, Lemma 3.1.4].

**Lemma 4.1.** Let \( \mathcal{N} \) be a collection of measurable, nested sets with finite and positive measure, \( k \in \mathbb{N} \) and \( \rho > 0 \). If \( \mathcal{X} \subset \mathcal{N} \) is a collection with \( \|\mathcal{X}\| > \frac{k}{\rho} \), there exists a set \( N_0 \in \mathcal{N} \) such that
\[
|G_\ell(\{N \in \mathcal{X} : N \subset N_0\})| > (1 - \rho)|N_0|, \quad 0 \leq \ell \leq k.
\]

**Proof.** Although the proof is completely analogous the dyadic case, we give it here for sake of completeness. Let \( \mathcal{N} \) be a collection of nested sets with finite and positive measure, \( k \in \mathbb{N} \) and \( 0 < \rho < 1 \). Suppose that the lemma fails, i.e. there exists an \( \ell \in \mathbb{N}_0 \) with \( 0 \leq \ell \leq k \) such that
\[
|G_\ell(\{N \in \mathcal{X} : N \subset N_0\})| \leq (1 - \rho)|N_0|, \quad N_0 \in \mathcal{N}.
\]
Certainly, the above inequality implies that
\[
|G_\ell(\{N \in \mathcal{X} : N \subset N_0\})| \leq (1 - \rho)|N_0|, \quad N_0 \in \mathcal{N}.
\]

Now, let \( N_0 \in \mathcal{N} \) be fixed, and put \( \mathcal{X}_0 = \{N \in \mathcal{X} : N \subset N_0\} \). By iterating the above inequality, we obtain
\[
|G_{m k + \ell}(\mathcal{X}_0)| \leq (1 - \rho)^m|N_0|, \quad m \in \mathbb{N}_0, \quad 0 \leq \ell \leq k - 1.
\]
Summing these estimates yields
\[
\sum_{N \in \mathcal{X}_0} |N| = \sum_{\ell=0}^{k-1} \sum_{m \in \mathbb{N}_0} |G_{m k + \ell}(\mathcal{X}_0)| \leq \sum_{\ell=0}^{k-1} \sum_{m \in \mathbb{N}_0} (1 - \rho)^m|N_0| \leq \frac{k}{\rho}|N_0|.
\]
Since \( N_0 \in \mathcal{X} \) was arbitrary, we obtain from the above inequality that \( \|\mathcal{X}\| \leq \frac{k}{\rho} \), which contradicts our hypothesis.

The next Lemma is a monochromatic (and simplified) version of Jones’ argument [7].

**Lemma 4.2.** Let \( n \in \mathbb{N}_0 \) and \( \alpha, \beta \in \mathbb{R} \) be such that
\[
0 < \beta < 1 \quad \text{and} \quad 0 < \alpha < 2^{-n-1} \beta^{n+1}.
\]
Let \( \mathcal{X} \) denote a collection of measurable nested sets with finite and positive measure which satisfies
\[
|G_n(\mathcal{X})| > (1 - \alpha)|G_0(\mathcal{X})|.
\]
Then there exists a subcollection \( \mathcal{Y} \) of \( \mathcal{X} \) such that
\[
|G_n(\mathcal{Y})| > (1 - \alpha k^{n+1})|G_0(\mathcal{X})|,
\] (4.17a)
and
\[
|N \cap G_n(\mathcal{Y})| > (1 - \beta)|N|, \quad N \in \mathcal{Y},
\] (4.17b)

**Proof.** Put \( \mathcal{F}_0 = \mathcal{G}_n(\mathcal{X}), F_0 = \bigcup \mathcal{F}_0 \) and note that by our hypothesis we have the estimate
\[
|F_0| > (1 - \alpha)|G_0(\mathcal{X})|.
\] (4.18)
Let \( j_0 \in \mathbb{N} \) and assume that we already defined \( \mathcal{F}_0, \ldots, \mathcal{F}_{j_0-1} \) and \( F_0, \ldots, F_{j_0-1} \).
To conclude the inductive step, we simply put
\[
\mathcal{F}_{j_0} = \{N \in \mathcal{G}_n(\mathcal{X}) : |N \cap F_0 \cap \ldots \cap F_{j_0-1}| \geq (1 - \beta)|N|\}, \quad F_{j_0} = \bigcup \mathcal{F}_{j_0}.
\] (4.19)
Stopping the induction process after \( n+1 \) steps yields collections of pairwise disjoint sets \( \mathcal{F}_j, 0 \leq j \leq n \), and \( F_j = \bigcup \mathcal{F}_j, 0 \leq j \leq n \).

Let \( 1 \leq j \leq n \) be fixed. We define

\[
\mathcal{E}_j = G_{n-j}(X) \setminus \mathcal{F}_j,
\]

and note that by (4.19) and (4.20) we have \( |N \cap F_0 \cap \cdots \cap F_{j-1}| < (1-\beta)|N| \) for all \( N \in \mathcal{E}_j \). Now, observe that \( |N| = |N \cap F_0 \cap \cdots \cap F_{j-1}| + |N \setminus (F_0 \cap \cdots \cap F_{j-1})| < (1-\beta)|N| + |N \setminus (F_0 \cap \cdots \cap F_{j-1})| \), for all \( N \in \mathcal{E}_j \). Summing the above estimate over all \( N \in \mathcal{E}_j \) yields

\[
|E_j| \leq (1-\beta)|E_j| + |G_0(X) \setminus (F_0 \cap \cdots \cap F_{j-1})|.
\]

Consequently, we obtain

\[
|E_j| \leq \frac{1}{\beta}|G_0(X) \setminus (F_0 \cap \cdots \cap F_{j-1})|.
\]

Combining \( (1-\alpha)|G_0(X)| \leq |G_{n-j}(X)| = |E_j| + |F_j| \) with the latter estimate gives us

\[
|F_j| \geq (1-\alpha)|G_0(X)| - \frac{1}{\beta}|G_0(X) \setminus (F_0 \cap \cdots \cap F_{j-1})|,
\]

\( 1 \leq j \leq n \). (4.21)

We claim that the following inequality holds:

\[
|F_j| \geq (1-\alpha \frac{2j}{\beta^j})|G_0(X)|,
\]

\( 0 \leq j \leq n \). (4.22)

We will prove the inequality (4.22) by induction on \( j \). By (4.18), the inequality (4.22) is true for \( j = 0 \). Now let \( 0 \leq j_0 \leq n-1 \) and assume we have already proved that (4.22) holds for all \( 0 \leq j \leq j_0 \). By (4.21) and our induction hypothesis we obtain

\[
|F_{j_0+1}| \geq (1-\alpha)|G_0(X)| - \frac{1}{\beta}|G_0(X) \setminus (F_0 \cap \cdots \cap F_{j_0})|
\]

\[
\geq (1-\alpha)|G_0(X)| - \frac{1}{\beta} \sum_{j=0}^{j_0} |G_0(X) \setminus F_j|
\]

\[
\geq (1-\alpha)|G_0(X)| - \frac{1}{\beta} \sum_{j=0}^{j_0} \alpha \frac{2j}{\beta^j} |G_0(X)|.
\]

Using \( \alpha \leq \frac{\alpha}{\beta^{j_0+1}} \) to estimate the first term and \( \frac{1}{\beta} \sum_{j=0}^{j_0} \alpha \frac{2j}{\beta^j} \leq \alpha \frac{2^{j_0+1}}{\beta^{j_0+1}} \) for the second term yields (4.22) for \( j_0 + 1 \). Thus, we proved (4.22).

By (4.22), we obtain

\[
|F_0 \cap \cdots \cap F_n| \geq (1-\alpha \sum_{j=0}^{n} \frac{2j}{\beta^j})|G_0(X)| \geq (1-\alpha \frac{2^{n+1}}{\beta^{n+1}})|G_0(X)|.
\]

Finally, we define the subcollection \( y \) of \( X \) by putting

\[
y = \{ N : N \in \mathcal{F}_j, N \cap F_0 \cap \cdots \cap F_n \neq \emptyset, 0 \leq j \leq n \}.
\]

We will now verify that \( y \) satisfies (4.17). First, we will show that

\[
G_n(y) = F_0 \cap \cdots \cap F_n.
\]

To this end, let \( x \in G_n(y) \) and note that in this case, there exist sets \( N_j \in y, 0 \leq j \leq n \), with \( x \in N_0 \subseteq \cdots \subseteq N_n \). Clearly, this is only possible if \( N_j \in \mathcal{F}_j, 0 \leq j \leq n \). Now, since \( N_0 \in y \), we have \( N_0 \cap F_j \neq \emptyset, 0 \leq j \leq n \), which implies \( N_0 \subset F_j, 0 \leq j \leq n \). Consequently, \( x \in N_0 \subset F_0 \cap \cdots \cap F_n \). If on the other hand \( x \in F_0 \cap \cdots \cap F_n \), then we know that there exist \( N_j \in \mathcal{F}_j \) with \( N_j \ni x, 0 \leq j \leq n \).
Hence, \( x \in \mathcal{N}_0 \subseteq \cdots \subseteq \mathcal{N}_n \) and \( \mathcal{N}_j \cap F_0 \cap \cdots \cap F_n \neq \emptyset, 0 \leq j \leq n \). Therefore, we obtain \( \mathcal{N}_j \in \mathcal{Y} \) for all \( 0 \leq j \leq n \), and that \( \mathcal{N}_0 \in \mathcal{G}_n(\mathcal{Y}) \), which shows that \( x \in \mathcal{N}_0 \subseteq \mathcal{G}_n(\mathcal{Y}) \).

Second, note that combining (4.24) with (4.23) yields (4.17a).

Thirdly, let \( \mathcal{N} \in \mathcal{Y} \). Thus, there exists an integer \( j_0 \) with \( 0 \leq j_0 \leq n \) such that \( \mathcal{N} \in \mathcal{Y}_{j_0} \), and \( \mathcal{N} \cap F_0 \cap \cdots \cap F_n \neq \emptyset \). Now, observe that whenever \( j \geq j_0 \) and \( \mathcal{N} \cap F_j \neq \emptyset \), then \( \mathcal{N} \subset F_j \). Hence, \( \mathcal{N} \cap F_0 \cap \cdots \cap F_n = \mathcal{N} \cap F_0 \cap \cdots \cap F_{j_0 - 1} \), and we obtain (4.17b) by (4.19) and (4.24).

\[ 4.3. \text{Proof of Theorem 2.2} \]

The framework for the proof of Theorem 2.2 is similar to that of [11, Theorem 2.2], although here we certainly use quantitative, finite dimensional techniques instead of qualitative, infinite dimensional techniques.

**Proof of Theorem 2.2** Let \( n \in \mathbb{N}_0 \) and \( \Gamma, \eta > 0 \). Let \( \eta_1 = \eta_1(n, \eta) \) denote the largest positive constant satisfying the inequalities

\[ 0 < \eta_1 \leq 1/2, \quad \eta_1 4^{n+3} n \leq 1/2, \quad \frac{1}{1 - \eta_1 4^{n+3} n} \leq 1 + \eta. \]

(4.25)

Define the integers

\[ n_1 = \left\lceil \left( \frac{32n}{\eta_1} \right)^{n+2} \right\rceil + 1, \quad N = N(n_1, \Gamma, \eta_1). \]

(4.26)

where \( N(n_1, \Gamma, \eta_1) \) is the integer in Theorem 3.7 with the parameters \( n_1 \), \( \Gamma \), and \( \eta \), and \( \delta = 0 \). Note that by the above definitions of \( n_1 \) and \( \eta_1 \), we have actually that \( N \) depends only on \( n \), \( \Gamma \) and \( \eta_1 \), i.e. \( N = N(n, \Gamma, \eta) \). Finally, let \( T : SL_N^\infty \rightarrow SL_N^\infty \) with \( \|T\| \leq \Gamma \) be fixed throughout the rest of the proof.

The proof will be divided into the following four steps.

**Step 1** By Theorem 3.7 there exists a finite sequence of collections \( \{B_K : K \in \mathcal{D}^{n_1}\} \) with \( B_K \subseteq \mathcal{D}^{n_1}, K \in \mathcal{D}^{n_1} \), satisfying Jones’ compatibility conditions [11]–[14]. These collections generate the block basis \( \{b_K : K \in \mathcal{D}^{n_1}\} \), which simultaneously almost-diagonalizes the operators \( T \) and \( \text{Id}_{SL_N^\infty} - T \).

**Step 2** One of the two collections

\[ \mathcal{M} = \left\{ B_K : K \in \mathcal{D}^{n_1}, \langle T b_K, b_K \rangle \geq \frac{\|b_K\|^2}{2} \right\}, \]

\[ \mathcal{N} = \left\{ B_K : K \in \mathcal{D}^{n_1}, \langle (\text{Id}_{SL_N^\infty} - T) b_K, b_K \rangle \geq \frac{\|b_K\|^2}{2} \right\}, \]

contains a finite sequence of subcollections \( \{\mathcal{E}_I : I \in \mathcal{D}^n\} \) that satisfies Jones’ compatibility conditions [11]–[14].

**Step 3** Consequently, by the reiteration Theorem 3.3 and Theorem 3.4 the block basis \( \{\tilde{b}_I : I \in \mathcal{D}^n\} \) given by

\[ \tilde{b}_I = \sum_{B_K \in \mathcal{E}_I} b_K, \quad I \in \mathcal{D}^n, \]

spans a complemented copy of \( SL_n^\infty \) (the constants for the norms of the isomorphism and the projection do not depend on \( n \)). Moreover, the operators \( T \) and \( \text{Id}_{SL_N^\infty} - T \) are both almost-diagonalized by \( \{\tilde{b}_I : I \in \mathcal{D}^n\} \), and either \( T \) or \( \text{Id}_{SL_N^\infty} - T \) has large diagonal with respect to \( \{\tilde{b}_I : I \in \mathcal{D}^n\} \) (depending on whether we selected \( \mathcal{E}_I \subseteq \mathcal{M}, I \in \mathcal{D}^n \) or \( \mathcal{E}_I \subseteq \mathcal{N}, I \in \mathcal{D}^n \), in the previous step).
Remark 3.8 (i) and (ii)) with parameters.

By (4.30) and (4.26) we have that

\[ \|R\| \leq 2 + \eta, \]

where \( H \) is either \( T \) or \( \text{Id}_{SL_N^\infty} - T \).

**Step 1.**

By the definition of \( N \) (see (4.26)), the almost-diagonalization Theorem 3.7 (and Remark 3.8 (i) and (ii)) with parameters \( n, \eta \) and \( \delta = 0 \), we obtain a finite block basic sequence \((b_K : K \in \mathcal{D}^n)\) given by

\[ b_K = \sum_{Q \in \mathcal{B}_K} h_Q, \quad K \in \mathcal{D}^n, \quad (4.27) \]

which has the following properties:

(i) \( \mathcal{B}_K \subset \mathcal{D}^N, K \in \mathcal{D}^n \), and the finite sequence of collections \((\mathcal{B}_K : K \in \mathcal{D}^n)\) satisfies Jones’ compatibility conditions (J1)–(J4) with constant \( \kappa_J = (1 - \eta_1)^{-1} \), and

\[ (1 - \eta_1)K \leq |B_K| \leq |K|, \quad K \in \mathcal{D}^n. \quad (4.28) \]

(ii) For any \( 1 \leq i \leq 2^{n_1} - 1 \) we have the estimate

\[ \sum_{j=1}^{i-1} |\langle Tb_j, b_i \rangle| + |\langle b_i, T^*b_j \rangle| \leq \eta_1 \|b_i\|^2. \quad (4.29a) \]

Since \( \langle b_j, b_i \rangle = 0, i \neq j \), we also have

\[ \sum_{j=1}^{i-1} |\langle (\text{Id}_{SL_N^\infty} - T)b_j, b_i \rangle| + |\langle b_i, (\text{Id}_{SL_N^\infty} - T)^*b_j \rangle| \leq \eta_1 \|b_i\|^2. \quad (4.29b) \]

**Step 2.**

We define the collections of measurable, nested sets with finite and positive measure

\[ M = \left\{ B_K : K \in \mathcal{D}^n, \langle Tb_K, b_K \rangle \geq \frac{\|b_K\|^2}{2} \right\}, \]

\[ N = \left\{ B_K : K \in \mathcal{D}^n, \langle (\text{Id}_{SL_N^\infty} - T)b_K, b_K \rangle \geq \frac{\|b_K\|^2}{2} \right\}. \]

Note that by (4.28) we have \( (1 - \eta_1)n \leq [M \cup N] \leq [M] + [N] \) (see (4.16) for a definition of \([\cdot]\)). If \([M] \geq (1 - \eta_1)\frac{n}{2} \), we define \( \mathcal{L} = M \) and \( H = T \); otherwise, we put \( \mathcal{L} = N \) and \( H = \text{Id}_{SL_N^\infty} - T \). We note the estimate

\[ \|\mathcal{L}\| \geq (1 - \eta_1)\frac{n}{2} \quad \text{and} \quad \langle Hb_I, b_I \rangle \geq \frac{\|b_I\|^2}{2}, \quad I \in \mathcal{L}. \quad (4.30) \]

By (4.30) and (4.26) we have that \( \|\mathcal{L}\| \geq (1 - \eta_1)\frac{n}{2} > \frac{n}{(\gamma_1/(32n))^{n+1}}, \) thus Lemma 4.1 (with a parameter setting of \( k = n \) and \( p = (\frac{n}{32n})^{n+1} \)) implies that there exists a set \( B_0 \in \mathcal{L} \) such that

\[ |G_k\{ \{B \in \mathcal{L} : B \subset B_0\}\} > (1 - \left(\frac{\eta_1}{32n}\right)^{n+1})|B_0|, \quad 0 \leq k \leq n. \quad (4.31) \]
Put $\mathcal{L}_0 = \{B \in \mathcal{L} : B \subset B_0\}$. Since $(\frac{n}{2^m})^{n+1} < 2^{-n-1}(\frac{n}{2^m})^{n+1}$, Lemma 4.2 yields a subcollection $\mathcal{L}_1$ of $\mathcal{L}_0$ containing $B_0$ such that
\[ |B \cap G_k(\mathcal{L}_1)| \geq (1 - \frac{\eta}{8^n})|B|, \quad B \in \mathcal{L}_1, \quad 0 \leq k \leq n. \quad (4.32) \]

We will now inductively define a finite sequence of collections $(\mathcal{C}_I : I \in \mathcal{D}^n)$ with $\mathcal{C}_I \subset \mathcal{L}_1$, $I \in \mathcal{D}^n$. To begin, we simply put
\[ \mathcal{C}_1 = \mathcal{C}_{[0,1]} = \{B_0\}. \quad (4.33) \]
Let us assume that we have already defined the collections $\mathcal{C}_1, \ldots, \mathcal{C}_{i-1}$. We will now construct $\mathcal{C}_i$. To this end, let $I_0 \in \mathcal{D}^n$ with $\mathcal{O}(I_0) = i_0$, and define
\[ C^T_{I_0} = \bigcup\{B^T_K : B_K \in \mathcal{C}_{\tilde{T}_0}\} \quad \text{and} \quad C^r_{I_0} = \bigcup\{B^r_K : B_K \in \mathcal{C}_{\tilde{T}_0}\}, \]
where the sets $B^T_K$ and $B^r_K$ are given by
\[ B^T_K = \bigcup\{Q^T : Q \in \mathcal{B}_K\} \quad \text{and} \quad B^r_K = \bigcup\{Q^r : Q \in \mathcal{B}_K\}. \]
If $I_0$ is the left half of $\tilde{T}_0$ and $|I_0| = 2^{-k_0}$, $k_0 \leq n$, we put
\[ \mathcal{C}_{I_0} = \{B \in \mathcal{G}_{i_0}(\mathcal{L}_1) : B \subset C^T_{I_0}\}, \quad (4.34a) \]
and if $I_0$ is the right half of $\tilde{T}_0$ and $|I_0| = 2^{-k_0}$, we define
\[ \mathcal{C}_{I_0} = \{B \in \mathcal{G}_{i_0}(\mathcal{L}_1) : B \subset C^r_{I_0}\}. \quad (4.34b) \]
We stop the induction after the construction of the collections $\mathcal{C}_I$, $I \in \mathcal{D}^n$.

It is easily verified that $(\mathcal{C}_I : I \in \mathcal{D}^n)$ satisfies (J1)–(J3). We will now verify that $(\mathcal{C}_I : I \in \mathcal{D}^n)$ satisfies (J4) with constant $(1 - \eta_i)^{-1}$. First, note that by the inductive construction above (see (4.33) and (4.34)), we have $\mathcal{G}_n(\mathcal{L}_1) = \bigcup_{I \in \mathcal{D}^n} \mathcal{C}_I$, and thus, (4.32) gives us
\[ |B \cap \bigcup_{I \in \mathcal{D}^n} C_I| \geq (1 - \frac{\eta}{8^n})|B|, \quad B \in \bigcup_{I \in \mathcal{D}^n} \mathcal{C}_I. \quad (4.35) \]
Second, let $I \in \mathcal{D}^n \setminus \{[0,1]\}$ and $B \in \mathcal{C}_I$ be fixed (if $I = [0,1]$, there is nothing to show). Choose $J \in \mathcal{D}^n \setminus \{[0,1]\}$ such that $I \cup J = \tilde{I} = \tilde{J}$. Hence, by (4.35) and (J3) we obtain
\[ (1 - \frac{\eta}{8^n})|B| \leq |B \cap C_I| + |B \cap C_J|. \]
Considering (4.34) and that $|B^T| \leq |B|/2$, $|B^r| \leq |B|/2$, the above estimate yields
\[ (1 - \frac{\eta}{8^n})|B| \leq \begin{cases} |B^T \cap C_I| + |B|/2, \quad \text{if } I \text{ is the left half of } \tilde{I}, \\ |B^r \cap C_I| + |B|/2, \quad \text{if } I \text{ is the right half of } \tilde{I}. \end{cases} \]
Since either $B \cap C_I = B^T \cap C_I$ or $B \cap C_I = B^r \cap C_I$, the latter estimate gives us
\[ \frac{1}{2} (1 - \frac{\eta}{4^n})|B| \leq |B \cap C_I| \leq \frac{1}{2} |B|, \quad I \in \mathcal{D}^n \setminus \{[0,1]\}, \quad B \in \mathcal{C}_I. \quad (4.36) \]
The estimate on the right hand side is obvious from the principle of our construction (see (4.33) and (4.34)). Summing (4.36) over all $B \in \mathcal{C}_I$ yields together with (J2) and (J3) that
\[ \frac{1}{2} (1 - \frac{\eta}{4^n})|C_I| \leq |C_I| \leq \frac{1}{2} |C_I|, \quad I \in \mathcal{D}^n, \quad I \neq [0,1]. \]
By iterating the latter inequality we obtain
\[ (1 - \frac{\eta}{4^n})|I| \leq \frac{|C_I|}{|C_{[0,1]}|} \leq |I|, \quad I \in \mathcal{D}^n. \quad (4.37) \]
Let \( I_0, I \in \mathcal{D}^n \) with \( I_0 \subset I \), \(|I_0| \leq |I|\), and \( B \subset \mathcal{C}_I \), then (4.32) and (4.33) imply
\[
|B \cap C_{I_0}| = |B \cap C_{I_0} \cap C_{I_0}| = \sum_{A \in \mathcal{C}_{I_0}} |B \cap A \cap C_{I_0}|. \tag{4.38}
\]

Since \( \tilde{I}_0 \subset I \), we know that \( A \subset B \) whenever \( A \in \mathcal{C}_{I_0}, B \in \mathcal{C}_I \) and \( B \cap A \neq \emptyset \). Hence, (4.38), (4.34) and (4.32) yield
\[
|B \cap C_{I_0}| = \sum_{A \subset B} |A \cap C_{I_0}| \geq \frac{1}{2} (1 - \eta) |B \cap C_{I_0}|. \tag{4.39}
\]

Iterating (4.38) and (4.39) while using (4.36) in each of those iterations, we obtain
\[
|B \cap C_{I_0}| \geq (1 - \frac{\eta}{4}) |I_0| |B \cap C_{[0,1]}|, \quad B \in \mathcal{C}_I.
\]

Combining the latter estimate with (4.37) and noting that \( B \subset C_{[0,1]} \) yields
\[
|B \cap C_{I_0}| \geq (1 - \frac{\eta}{2}) \frac{|I_0|}{|C_I|} |B|, \quad B \subset \mathcal{C}_I, \tag{4.40}
\]
whenever \( I_0, I \in \mathcal{D}^n \) with \( I_0 \subset I \). By (4.40), the finite sequence \( \{\mathcal{C}_I : I \in \mathcal{D}^n\} \) satisfies Jones’ compatibility conditions (J1)–(J4) with constant \( \kappa_J = (1 - \frac{\eta}{2})^{-1} \). Furthermore, in Step 1 we showed that \( \mathcal{B}_K : K \in \mathcal{D}^{n_1} \) satisfies Jones’ compatibility conditions (J1)–(J4) with constant \( \kappa_J = (1 - \eta_1)^{-1} \). Consequently, if we put
\[
\tilde{b}_I = \bigcup_{B_K \in \mathcal{B}_I} B_K \quad \text{and} \quad \tilde{B}_I = \bigcup_{\mathcal{B}_I} \tilde{b}_I, \quad I \in \mathcal{D}^n, \tag{4.41}
\]
the reiteration Theorem 3.3 implies that \( \tilde{B}_I : I \in \mathcal{D}^n \), satisfies (J1)–(J4) with constant \( \kappa_J = (1 - \eta_1)^{-2} \).

**Step 3.**

Define the block basis \( \tilde{b}_I : I \in \mathcal{D}^n \) of \( (b_K : K \in \mathcal{D}^{n_1}) \) by putting
\[
\tilde{b}_I = \sum_{B_K \in \mathcal{B}_I} b_K = \sum_{B_K \in \mathcal{B}_I} \sum_{Q \in \mathcal{B}_K} \sum_{Q \in \mathcal{E}_I} h_Q = \sum_{Q \in \mathcal{E}_I} h_Q, \quad I \in \mathcal{D}^n. \tag{4.42}
\]

Now, let \( I \in \mathcal{D}^n \) be fixed. By (4.32), (4.34) and (4.33), we obtain the diagonal estimate
\[
\langle H \tilde{b}_I, \tilde{b}_I \rangle \geq \frac{1}{2} \|\tilde{b}_I\|^2 - \sum_{B_K, B_L \in \mathcal{E}_I, \mathcal{O}(K) \cap \mathcal{O}(L)} |\langle H b_K, b_L \rangle| + |\langle b_L, H^* b_K \rangle| \\
\geq \frac{1}{2} \|\tilde{b}_I\|^2 - \eta_1 \sum_{B_L \in \mathcal{E}_I} \|b_L\|^2 - (1 - \eta_1) \|\tilde{b}_I\|^2.
\]

We summarize what we proved so far:
\[
\langle H \tilde{b}_I, \tilde{b}_I \rangle \geq \left( \frac{1}{2} - \eta_1 \right) \|\tilde{b}_I\|^2, \quad I \in \mathcal{D}^n. \tag{4.43}
\]

For the off diagonal estimate, we define
\[
\mathcal{E} = \bigcup \{\mathcal{E}_J : J \in \mathcal{D}^n, \mathcal{O}(J) \varsubsetneq \mathcal{O}(I)\},
\]
and note that by (4.32) we have the estimate
\[
\sum_{J \in \mathcal{D}^n \atop \mathcal{O}(J) \varsubsetneq \mathcal{O}(I)} |\langle H b_I, b_I \rangle| \leq \sum_{B_L \in \mathcal{E}_I} \sum_{B_K \in \mathcal{E}_I, \mathcal{O}(K) \varsubsetneq \mathcal{O}(L)} |\langle H b_K, b_L \rangle| + \sum_{B_L \in \mathcal{E}_I} \sum_{B_K \in \mathcal{E}_I, \mathcal{O}(K) \varsubsetneq \mathcal{O}(L)} |\langle b_K, H^* b_L \rangle|. \tag{4.44}
\]
Estimating the first sum on the right hand side of (4.44) by (4.29) yields
\[ \sum_{B_L \in \mathcal{E}_I} \sum_{B_K \in \mathcal{E}_{O(K) < O(L)}} |\langle Hb_K, b_L \rangle| \leq \eta_1 \sum_{B_L \in \mathcal{E}_I} \|b_L\|_2 = \eta_1 \|\tilde{b}_I\|_2^2. \]  
(4.45)

The latter equality follows from (J2). By (4.29), we obtain the following estimate for the second term on the right hand side of (4.44):
\[ \sum_{B_K \in \mathcal{E}} \sum_{O(K) > O(L)} \sum_{B_L \in \mathcal{E}_{O(K) > O(L)}} |\langle b_K, H^*b_L \rangle| \leq \eta_1 \sum_{B_K \in \mathcal{E}} \|b_K\|_2^2. \]  
(4.46)

(J2) and (4.28) gives us
\[ \sum_{B_K \in \mathcal{E}} \|b_K\|_2^2 \leq \sum_{B_K \in \mathcal{E}} |K| \leq \sum_{J \in D^n} \sum_{J \subseteq \mathcal{D^2}} |C_J| \leq n\|C_{[0, 1]}\|. \]  
(4.47)

Combining (4.46) with (4.47) yields
\[ \sum_{B_L \in \mathcal{E}_I} \sum_{B_K \in \mathcal{E}_{O(K) > O(L)}} |\langle b_K, H^*b_L \rangle| \leq \eta_1 n\|C_{[0, 1]}\|. \]  
(4.48)

Collecting the estimates (4.48), (4.45) and (4.44) we obtain
\[ \sum_{J \in D^n} \sum_{J \subseteq \mathcal{D^2}} |\langle H\tilde{b}_J, \tilde{b}_I \rangle| \leq 2\eta_1 n\|C_{[0, 1]}\|. \]  
(4.49)

Repeating the above argument with the roles of H and H* reversed yields
\[ \sum_{J \in D^n} \sum_{J \subseteq \mathcal{D^2}} |\langle \tilde{b}_I, H^*\tilde{b}_J \rangle| \leq 2\eta_1 n\|C_{[0, 1]}\|. \]  
(4.50)

Adding (4.49) and (4.50) yields our desired off-diagonal estimate
\[ \sum_{J \in D^n} \sum_{J \subseteq \mathcal{D^2}} |\langle H\tilde{b}_J, \tilde{b}_I \rangle| + |\langle \tilde{b}_I, H^*\tilde{b}_J \rangle| \leq 4\eta_1 n\|C_{[0, 1]}\|, \quad I \in \mathcal{D^n}. \]  
(4.51)

**Step 4.**
As usual, we identify \( \tilde{b}_i = \tilde{b}_I \), whenever \( \mathcal{O}(I) = i \). Thus, (4.51) and (4.43) read as follows:
\[ \sum_{j=1}^{i-1} |\langle H\tilde{b}_j, \tilde{b}_i \rangle| + |\langle \tilde{b}_i, H^*\tilde{b}_j \rangle| \leq 4\eta_1 n\|C_{[0, 1]}\|, \quad 1 \leq i \leq 2^{n+1} - 1, \]  
(4.52a)
\[ \langle H\tilde{b}_i, \tilde{b}_i \rangle \geq \left( \frac{1}{2} - \eta_1 \right)\|\tilde{b}_i\|_2^2, \quad 1 \leq i \leq 2^{n+1} - 1. \]  
(4.52b)

Moreover, by (J2) and (4.37), we have that
\[ \|\tilde{b}_i\|_2^2 \leq \|\tilde{b}_i\|_2^2 \geq \left( 1 - \frac{\eta_1}{4} \right)\|\tilde{b}_i\|_2^2 \geq \left( 1 - \frac{\eta_1}{4} \right)2^{-n}\|C_{[0, 1]}\|, \]  
(4.52c)

for all \( 1 \leq i \leq 2^{n+1} - 1 \) and \( I \in \mathcal{D^n} \) with \( \mathcal{O}(I) = 2^i \). Comparing (4.52) with (4.1), the only relevant difference is the presence of the additional factor \( \|C_{[0, 1]}\| \) on the right hand sides of (4.52a) and (4.52c). We will now repeat the proof of Theorem 2.1 with \( \tilde{b}_i \) in place of \( b_i^{(c)} \). By (4.41) in Step 2 (\( \tilde{B}_I : I \in \mathcal{D^n} \)) satisfies (J1)–(J4).
with constant $\kappa_J = (1 - \eta_1)^{-2}$. Thus, by (4.42) and Theorem 4.4, the operators $\tilde{B} : SL_n^\infty \to SL_N^\infty$ and $\tilde{Q} : SL_N^\infty \to SL_n^\infty$ given by

$$\tilde{B}f = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, \tilde{b}_i \rangle}{\|b_i\|^2} \tilde{b}_i \quad \text{and} \quad \tilde{Q}g = \sum_{i=1}^{2^{n+1}-1} \frac{\langle g, \tilde{b}_i \rangle}{\|b_i\|^2} b_i \quad (4.53)$$

satisfy the estimates

$$\|\tilde{B}f\|_{SL^\infty} \leq 1 \|f\|_{SL^\infty} \quad \text{and} \quad \|\tilde{Q}g\|_{SL^\infty} \leq (1 - \eta_1)^{-1} \|g\|_{SL^\infty}, \quad (4.54)$$

for all $f \in SL_n^\infty$, $g \in SL_N^\infty$.

Moreover, the diagram

$$SL_n^\infty \xrightarrow{\tilde{B}} SL_N^\infty \xrightarrow{\tilde{Q}} SL_n^\infty \quad (4.55)$$

is commutative. By (4.53), the bounded projection $\tilde{P} : SL_N^\infty \to SL_N^\infty$ given by $\tilde{P} = \tilde{B}\tilde{Q}$ has the form

$$\tilde{P}f = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, \tilde{b}_i \rangle}{\|b_i\|^2} b_i, \quad f \in SL_N^\infty, \quad (4.56)$$

and is therefore an orthogonal projection. By (4.54), $\tilde{P}$ satisfies the estimate

$$\|\tilde{P}f\|_{SL^\infty} \leq (1 - \eta_1)^{-1} \|f\|_{SL^\infty}, \quad f \in SL_N^\infty. \quad (4.57)$$

Now, we define the subspace $\tilde{Y}$ of $SL_N^\infty$ by

$$\tilde{Y} = \left\{ g = \sum_{i=1}^{2^{n+1}-1} a_i \tilde{b}_i : a_i \in \mathbb{R} \right\},$$

which is the image of the projection $\tilde{P}$. By (4.54) and (4.55), we obtain the following commutative diagram:

$$SL_n^\infty \xrightarrow{\tilde{B}} SL_N^\infty \xrightarrow{\tilde{Q}|\tilde{Y}|} \tilde{Y} \quad \text{and} \quad \|\tilde{B}\|\|\tilde{Q}|\tilde{Y}| \leq (1 - \eta_1)^{-1}. \quad (4.58)$$

Now, define $\tilde{U} : SL_N^\infty \to \tilde{Y}$ by

$$\tilde{U}f = \sum_{i=1}^{2^{n+1}-1} \frac{\langle f, \tilde{b}_i \rangle}{\langle Hb_i, b_i \rangle} \tilde{b}_i, \quad f \in SL_N^\infty. \quad (4.59)$$

and note that by (4.52b), the 1-unconditionality of the Haar system in $SL^\infty$ and (4.57), the operator $\tilde{U}$ has the upper bound

$$\|\tilde{U} : SL_N^\infty \to \tilde{Y}\| \leq \frac{2}{1 - 3\eta_1}. \quad (4.60)$$

Observe that for all $g = \sum_{i=1}^{2^{n+1}-1} a_i \tilde{b}_i \in \tilde{Y}$, the following identity is true:

$$\tilde{U}Hg - g = \sum_{i=1}^{2^{n+1}-1} \sum_{j=1}^{2^{n+1}-1} a_i \frac{\langle H\tilde{b}_j, \tilde{b}_i \rangle}{\langle Hb_i, b_i \rangle} \tilde{b}_i + \sum_{i=1}^{2^{n+1}-1} \sum_{j=1}^{2^{n+1}-1} a_i \frac{\langle \tilde{b}_i, H^*b_j \rangle}{\langle Hb_i, b_i \rangle} \tilde{b}_j. \quad (4.61)$$
Thus, the following diagram is commutative:

\[ \eta \]

Finally, let \( J : \tilde{Y} \to SL_n^\infty \) denote the operator given by \( Jy = y \). By our choice of \( \eta_1 \) in (4.25), the operator \( \tilde{V} : SL_n^\infty \to \tilde{Y} \) given by \( \tilde{V} = (UHJ)^{-1}U \) is well defined. Thus, the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\mathrm{Id}_Y} & \tilde{Y} \\
\downarrow{UHJ} & & \downarrow{UHJ} \\
\tilde{Y} & \xrightarrow{(UHJ)^{-1}} & \tilde{Y} \\
\downarrow{J} & & \downarrow{J} \\
SL_n^\infty & \xrightarrow{H} & SL_n^\infty
\end{array}
\]

The estimate for \( \tilde{V} \) follows from the choice we made for \( \eta_1 \) in (4.25). Merging the diagrams (4.58) and (4.63) concludes the proof. \( \square \)

5. Direct sums of \( SL_n^\infty \) spaces are primary

First, we establish that \( SL_n^\infty \) is isomorphic to the \( \ell^\infty \) direct sum of its finite dimensional building blocks \( SL_n^r \). Second, we combine the finite dimensional factorization results Theorem 2.2 for \( SL_n^\infty \), \( n \in \mathbb{N}_0 \) to obtain the factorization result Theorem 2.3 for \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_r \), \( 1 \leq r \leq \infty \). Consequently, we obtain that \( SL_n^\infty \) is primary.

5.1. Isomorphisms and non-isomorphisms of direct sums of \( SL_n^\infty \).

We show that taking direct sums of \( SL_n^\infty \) with different parameters produces isomorphically different spaces, and that \( SL_n^\infty \) is isomorphic to the \( \ell^\infty \) direct sum of \( SL_n^\infty \). We give two proofs for the latter fact: one using the Hahn-Banach theorem, and another using a compactness argument. Both rely on Pełczyński’s decomposition method [17]; we refer the reader to [21].

**Lemma 5.1.** The spaces \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_r \), \( 1 \leq r \leq \infty \), are all mutually non-isomorphic. The spaces \( SL_n^\infty \) and \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_\infty \) are isomorphic.

**Proof.** By a version of Pitt’s theorem for direct sums of finite dimensional Banach spaces (see e.g. [III] Theorem 5.6) for more details), the spaces \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_r \), \( 1 \leq r \leq \infty \), are mutually non-isomorphic.

Using Pełczyński’s decomposition method, we will now show that \( SL_n^\infty \) is isomorphic to \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_\infty \). First, we will show that \( SL_n^\infty \) contains a complemented copy of \( (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_\infty \). To this end, consider the collections \( A_n \subseteq \mathbb{D} \) given by

\[ A_n = \{ I \in \mathbb{D} : I \subseteq [1 - 2^{-n}, 1 - 2^{-n-1}] \} \]

and define \( E : (\sum_{n\in\mathbb{N}_0} SL_n^\infty)_\infty \to SL_n^\infty \) by

\[
(f_n)_{n=0}^\infty \mapsto \sum_{n=0}^\infty \chi_{[1-2^{-n},1-2^{-n-1}]}f_n \circ \varphi_n,
\]

where \( \varphi_n \) is the affine linear transformation that maps \([1 - 2^{-n}, 1 - 2^{-n-1}]\) into \([0,1]\), and therefore \( \varphi_n(A_n) = \mathbb{D} \). Note that

\[
\|E((f_n)_{n=0}^\infty)\|_{SL_n^\infty} = \sup_{n \in \mathbb{N}_0} \|\chi_{[1-2^{-n},1-2^{-n-1}]}f_n \circ \varphi_n\|_{SL_n^\infty} = \sup_{n \in \mathbb{N}_0} \|f_n\|_{SL_n^\infty} = \|(f_n)_{n=0}^\infty\|_{(\sum_{n\in\mathbb{N}_0} SL_n^\infty)_\infty},
\]
Pełczyński’s decomposition method (see e.g. [21]) yields that the proof was taken from [15, Theorem 2.2.3] and adapted to fit our purpose. Above proof could be replaced by a standard argument, which goes back to [21, 5.2 Remark].

Let \( L : \ell^\infty \to \mathbb{R} \) be a norm 1 functional such that \( L(1, 1, \ldots) = 1 \). Now, we define the operator \( Q : (\sum_{n \in \mathbb{N}_0} SL_n^\infty) \to SL^\infty \) by

\[
\left( \sum_{I \in \mathcal{D}} a_{n,I} h_I \right)_{n=0}^{\infty} \mapsto \sum_{I \in \mathcal{D}} L((a_{n,I})_{n=0}^{\infty}) h_I,
\]

and claim that \( Q \) has norm 1. Let \( f_n = \sum_{I \in \mathcal{D}} a_{n,I} h_I, \, n \in \mathbb{N}_0 \) be such that \((f_n)_{n=0}^{\infty}\) is in the unit ball of \((\sum_{n \in \mathbb{N}_0} SL_n^\infty) \), i.e. \( \sup_{n \in \mathbb{N}_0} \|f_n\|_{SL^\infty} \leq 1 \). Assume that \( \|Q((f_n)_{n=0}^{\infty})\| > 1 \). Thus, there exists a \( I_0 \in \mathcal{D} \) such that

\[
\sum_{I \supset I_0} \left( L((a_{n,I})_{n=0}^{\infty}) \right)^2 > 1.
\]

But then again, we have the estimate

\[
\sum_{I \supset I_0} \left( L((a_{n,I})_{n=0}^{\infty}) \right)^2 = L\left( \sum_{I \supset I_0} L((a_{k,I})_{k=0}^{\infty}) (a_{n,I})_{n=0}^{\infty} \right)
\leq \sup_{n \in \mathbb{N}_0} \left( \sum_{I \supset I_0} L((a_{k,I})_{k=0}^{\infty}) a_{n,I} \right) \left\| \sum_{I \supset I_0} a_{k,I} n_{I_0} \right\|
\leq \sup_{n,k \in \mathbb{N}_0} \left( \sum_{I \supset I_0} a_{k,I}^2 \right)^{1/2} \left( \sum_{I \supset I_0} a_{n,I}^2 \right)^{1/2} \leq 1,
\]

which contradicts (5.1). Hence, \( \|Q\| \) has norm 1. By definition of \( G \) and \( Q \), the following diagram is commutative:

\[
\begin{array}{ccc}
SL^\infty & \xrightarrow{\text{Id}_{SL^\infty}} & SL^\infty \\
\downarrow G & & \downarrow Q \\
(\sum_{n \in \mathbb{N}_0} SL_n^\infty) \to & & \|G\|, \|Q\| = 1.
\end{array}
\]

Consequently, \( SL^\infty \) is isomorphic to a complemented subspace of \((\sum_{n \in \mathbb{N}_0} SL_n^\infty) \), as claimed.

Finally, since \((\sum_{n \in \mathbb{N}_0} SL_n^\infty) \) is isomorphic to \((\sum_{m \in \mathbb{N}_0} (\sum_{n \in \mathbb{N}_0} SL_n^\infty)) \), Pelczyński’s decomposition method (see e.g. [21]) yields that \( SL^\infty \) is isomorphic to \((\sum_{n \in \mathbb{N}_0} SL_n^\infty) \).

Remark 5.2. We want to point out that the construction of the operator \( Q \) in the above proof could be replaced by a standard argument, which goes back to [21] Theorem III.1.E.18], [20], [6] and [18]. We will now present another alternative. The following proof was taken from [15] Theorem 2.2.3 and adapted to fit our purpose.

Let \( C \) denote the closed unit ball of \( SL^\infty \), and let \( \mathcal{T} \) denote the smallest topology on \( C \) such that every functional of the form \( \langle \cdot, h_I \rangle : C \to \mathbb{R}, I \in \mathcal{D} \) is continuous. We will now prove that \((C, \mathcal{T})\) is a compact topological space. We endow \([-1, +1]^\mathcal{D} \) with
the product topology $\mathcal{P}$, which, by Tychonov’s theorem, is a compact topological space, and define the map $\Phi : (C, \mathcal{T}) \to ([-1, +1]^{\mathbb{D}}, \mathcal{P})$ by putting

$$f \mapsto \left( \frac{f(h_I)}{|I|} \right)_{I \in \mathbb{D}}.$$  

By \cite{1.7}, $\Phi$ is well defined, and one can easily check that $\Phi$ is a topological embedding. We will now verify that $\Phi(C)$ is closed. To this end, let $(f_n)$ denote a net in $C$, such that $\Phi(f_n)$ converges to some $(a_I)_{I \in \mathbb{D}} \in [-1, +1]^{\mathbb{D}}$. We define $f = \sum_{I \in \mathbb{D}} a_I h_I$, and assume that $\|f\|_{SL^\infty} > 1$. Then there exists an $I_0 \in \mathbb{D}$ such that

$$\sum_{I \supseteq I_0} a_I^2 > 1$$

We now write $(a_{\alpha,I})_{I \in \mathbb{D}} = \Phi(f_n)$ for each $\alpha$, and note that by our hypothesis $a_{\alpha,I} \to a_I$, for each $I \in \mathbb{D}$. Consequently, there exists an $\alpha_0$ such that

$$\|f_{\alpha_0}\|_{SL^\infty}^2 \geq \sum_{I \supseteq I_0} a_{\alpha_0,I}^2 > 1,$$

which contradicts $f_{\alpha_0} \in C$. Thus, we know that $\Phi(C)$ is a closed subset of the compact topological space $([-1, +1]^{\mathbb{D}}, \mathcal{P})$, and is therefore itself compact. Since $\Phi$ is a topological embedding, $(C, \mathcal{T})$ is a compact topological space.

As in the proof of Lemma 5.1, let $G : SL^\infty \to (\sum_{n \in \mathbb{N}_0} SL_n^\infty)_{\infty}$ denote the isometric embedding given by

$$f \mapsto \left( \sum_{I \in \mathbb{D}^n} \frac{f(h_I)}{|I|} h_I \right)_{n=0}^\infty.$$ 

Let $B$ denote the closed unit ball of $(\sum_{n \in \mathbb{N}_0} SL_n^\infty)_{\infty}$, and for each $m \in \mathbb{N}_0$ define $R_m : B \to C$ by

$$(f_n^\infty)_{n=0} \mapsto f_m.$$ 

In other words, $R_m \in C^B$, $m \in \mathbb{N}_0$, and since $C^B$ is compact in the product topology by Tychonov’s theorem, there exists a subnet $(R_{\alpha_0})$ of $(R_m)_{m=1}^{\infty}$ converging to, say, $R$. This means that for each $(f_n^\infty)_{n=0} \in B$, we have that $R_{\alpha_0}((f_n^\infty)_{n=0}) \to R((f_n^\infty)_{n=0})$ in the topology $\mathcal{T}$ of $C$. Now let $f \in C$, then

$$\langle R(Gf), h_I \rangle = \lim_{\alpha} \langle R_{\alpha}(Gf), h_I \rangle = \langle f, h_I \rangle, \quad I \in \mathbb{D}.$$ 

i.e. $RG = 1_{dC}$. Certainly, we can extend $R$ to an operator on $(\sum_{n \in \mathbb{N}_0} SL_n^\infty)_{\infty}$. We claim that $R$ is bounded by 1. Assume to the contrary, that there exists a $(f_n)_{n=0}^{\infty} \in B$ with $\|R((f_n)_{n=0}^{\infty})\|_{SL^\infty} > 1$. Then there is an interval $I_0 \in \mathbb{D}$ such that

$$\sum_{I \supseteq I_0} \left( \frac{\langle R((f_n)_{n=0}^{\infty}), h_I \rangle}{|I|} \right)^2 > 1.$$ 

Since $R_{\alpha_0}((f_n)_{n=0}^{\infty}) \to R((f_n)_{n=0}^{\infty})$ in $(C, \mathcal{T})$, there exists an $\alpha_0$ such that

$$1 < \sum_{I \supseteq I_0} \left( \frac{\langle R_{\alpha_0}((f_n)_{n=0}^{\infty}), h_I \rangle}{|I|} \right)^2 = \|R_{\alpha_0}((f_n)_{n=0}^{\infty})\|_{SL^\infty}^2 = \|f_{\alpha_0}\|_{SL^\infty}^2,$$

which contradicts $(f_n)_{n=0}^{\infty} \in B$. Thus, $R$ has norm 1.

The above $R$ would be a suitable replacement for $Q$ in the proof of Lemma 5.1.
5.2. Proof of Theorem 2.3

When using Bourgain’s localization method, see e.g. [3, 2, 15, 19, 16, 12], we eventually have to pass from the local result to the global result; in our case from Theorem 2.2 to Theorem 2.3 below, which includes diagonalizing operators on the sum of the finite dimensional spaces. For us, these sums are \( \left( \sum_{n \in \mathbb{N}} SL_{\infty}^n \right)_r \), \( 1 \leq r \leq \infty \).

Due to a gliding hump argument, this diagonalization process does not require any additional hypothesis if \( r < \infty \); if \( r = \infty \) however, we require that the sequence of spaces \( (SL_{\infty}^n)_{n \in \mathbb{N}} \) has the property that projections almost annihilate finite dimensional subspaces (which it has), see Definition 3.9, Corollary 3.10 and Remark 3.11.

For more details, we refer the reader to [10, Section 5].

**Proof of Theorem 2.3.** By Theorem 2.2, the first part of the hypothesis of [10, Proposition 5.4] is satisfied. Moreover by Remark 3.11 for each \( \eta > 0 \), the sequence of finite dimensional Banach spaces \( (SL_{\infty}^n)_{n \in \mathbb{N}} \) has the property that projections almost annihilate finite dimensional subspaces with constant \( 1 + \eta \) (which is only needed in the case \( r = \infty \)). Thus, applying [10, Proposition 5.4] yields the diagram (2.3).

It is easily verified that for fixed \( 1 \leq r \leq \infty \), the Banach space \( X^{(r)} \) is isomorphic to \( \left( \sum_{m \in \mathbb{N}} X^{(r_m)} \right)_r \), hence, by Pelczynski’s decomposition method (see e.g. [21]) and diagram (2.3), we obtain that \( X^{(r)} \) is primary.

Finally, by Lemma 5.1 \( SL_{\infty} \) is isomorphic to the primary Banach space \( X^{(\infty)} \), and is thereby itself primary. \( \square \)

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