A LOCALIZATION THEOREM FOR FINITE W-ALGEBRAS

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ABSTRACT. Following the work of Beilinson-Bernstein [BB] and Kashiwara-Rouquier [KR], we give a geometric interpretation of certain categories of modules over the finite W-algebra. Along the way, we give a new, general Hamiltonian reduction formalism for quantizations and we reprove the Skryabin equivalence.

1. INTRODUCTION

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, and \( U(\mathfrak{g}) \) its enveloping algebra. The Beilinson-Bernstein localization theorem [BB] gives a geometric interpretation of the category of finitely generated modules over \( U(\mathfrak{g}) \) with trivial central character. In particular, this category is equivalent to the category \( \text{Mod}^{\text{coh}}(D(G/B)) \) of coherent \( D \)--modules on the flag variety associated to \( G \). This result can be explained as follows: there is a natural map \( T^*(G/B) \to N \) which is a resolution of singularities. The normality of the variety \( N \) implies that \( \Gamma(N, O_N) = \Gamma(T^*(G/B), O_{T^*(G/B)}) \).

Further, the ring \( U(\mathfrak{g})_0 \) can be thought of as a quantization of the nilpotent cone \( N \), and the sheaf \( D_{G/B} \) can be thought of as a quantization of the variety \( T^*(G/B) \). However, the sheaf \( D_{G/B} \) is not local on \( T^*(G/B) \), only on \( G/B \) itself.

Kashiwara and Rouquier (in [KR]) give a framework for reformulating this theorem using a notion of sheaves of asymptotic differential operators. One can define a sheaf of algebras \( D_h(G/B) \) on the variety \( T^*(G/B) \), which is (in some sense) a quantization. This sheaf is defined over the power series field \( \mathbb{C}((h)) \), and therefore the category of modules over it is not equivalent to a \( \mathbb{C} \)-linear category of modules over \( U(\mathfrak{g}) \). However, this can be corrected by considering the \( \mathbb{C}^* \)-action on \( T^*(G/B) \) given by dilating the fibres. In particular, there is a notion of \( \mathbb{C}^* \)-equivariant \( D_h(G/B) \)-module for which the equivalence \( \text{Mod}^{\text{coh}, \mathbb{C}^*}(D_h(G/B)) \to \text{Mod}^{f.g.}(U(\mathfrak{g})_0) \) holds.

Our goal in this paper is to give a version of this theorem for the finite W-algebras. We give the precise definition of these objects below. For now, we simply note that given a nilpotent element \( e \in N \), there is subvariety \( S_e \subseteq N \) called the transverse slice to the \( G \)-orbit at \( e \). This variety admits a natural \( \mathbb{C}^* \)-action that contracts it to \( e \). Then there is a filtered, noncommutative algebra \( U(\mathfrak{g}, e)_0 \) (the finite W-algebra at \( e \) with trivial central character) such that \( gr(U(\mathfrak{g}, e)_0) \simeq O(S_e) \), where the grading on \( S_e \) is given by the aforementioned \( \mathbb{C}^* \)-action.

There is a resolution of singularities \( \tilde{S}_e \to S_e \) where \( \tilde{S}_e \) is the (set-theoretic) inverse image of \( S_e \) under \( T^*(G/B) \to N \). In addition, there is a \( \mathbb{C}^* \)-action on \( T^*(G/B) \) that preserves \( \tilde{S}_e \) and for which the resolution \( \tilde{S}_e \to S_e \) is equivariant. Then, our main theorem gives a sheaf of algebras on \( \tilde{S}_e \) called \( D_h(0, \chi) \), which is (in a sense) a quantization of \( \tilde{S}_e \) and for which there is the equivalence \( \text{Mod}^{\text{coh}, \mathbb{C}^*}(D_h(0, \chi)) \to \text{Mod}^{f.g.}(U(\mathfrak{g}, e)_0) \). Our proof relies on the fact that the variety \( \tilde{S}_e \) is not only a subvariety of \( T^*(G/B) \) but can also be obtained via the procedure of “Hamiltonian reduction.” We can then obtain the sheaf \( D_h(0, \chi) \) via the procedure of Hamiltonian reduction of the sheaf \( D_h(G/B) \). The proof of the result follows the same lines as the proof of the classical Beilinson-Bernstein theorem.

In the body of the paper, we prove all of the results outlined above, in a slightly more general form. In particular, we work with categories of modules over any anti-dominant central character, not just the trivial one. Further, we give several applications to the theory of W-algebras, including...
reproving the well-known Skryabin equivalence. Beyond this, the results of this paper have been
cited, e.g., in [D], and the paper [DR] provides a generalization to the affine case. Another related
reference is [GI], chapter 6, where different presentation of modules over the W-algebra with a given
central character is given. In that work, the idea is to present coherent sheaves over the quasi-
projective scheme $S_e$ as a Serre quotient of modules over a suitable graded algebra; and then to
present a certain non-commutative deformation of that algebra (called a directed algebra) so that
a suitable Serre quotient of its modules are equivalent to modules over the W-algebra. While there
do not seem to be direct implications in either direction, that work is certainly morally related to
this one.

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improved the paper.

2. W-algebras and Quantum Hamiltonian Reduction

Let $A$ be an associative algebra over $\mathbb{C}$, and let $M$ be a connected affine algebraic group; we set
$\text{Lie}(M) = m$. We suppose that there is an action of $M$ on $A$ that is algebraic (i.e., locally finite), and
respects the algebra structure. We assume given an algebra morphism $\rho : Um \to A$ such that the
adjoint action of $m$ on $A$ (i.e., the action given by $ad(m)(a) = \rho(m)a - \rho(a)m$ for all $m \in m, a \in A$)
is the differential of the $M$ action. Let $I \subseteq Um$ be a two-sided ideal. Then it is easy to see that
$(A/\rho(I))^M$ inherits an algebra structure from $A$, called the quantum Hamiltonian reduction of $A$
with respect to $I$. If there exists a character $\chi$ on $m$ such that $I = \text{ker}(\chi)$ (where we also use the
letter $\chi$ to denote the unique extension of this character to a character of $Um$), then we can describe
the algebra structure on $(A/\rho(I))^M$ via an isomorphism $(A/\rho(I))^M \to \text{End}_A(A/\chi)^{op}$ that takes $u \in (A/\rho(I))^M$ to right multiplication by $u$ in $A/\chi$.

We will now define the finite W-algebra $U(g,e)$ via the quantum Hamiltonian reduction procedure.
For references on everything in this section, see [GG]. We let $e \in g$ be a nonzero nilpotent element.
By the Jacobson-Morozov theorem, there exist $f, h \in g$ such that $\{e, f, h\}$ form an $\mathfrak{sl}_2$-triple, and
we fix such a triple throughout. Given this, the adjoint action makes $g$ into a finite dimensional
$\mathfrak{sl}_2$-module, and we have the corresponding weight decomposition $g = \oplus g(i)$, where $g(i) = \{x \in g | [h, x] = ix\}$. This makes $g$ into a graded Lie algebra. We let $\chi \in g^*$ be the element associated to $e$
under the isomorphism $g \cong g^*$ given by the Killing form. We define a skew-symmetric bilinear form
on $g(-1)$ via

$$<x, y > = \chi([x, y])$$

which is easily seen to be non-degenerate. Thus, $(g(-1), <, >)$ is a symplectic vector space, and we
choose $l \subset g(-1)$ a Lagrangian subspace. We define $m_l = l \oplus \bigoplus_{i \geq -2} g(i)$, a nilpotent Lie algebra
such that $\chi|_{m_l}$ is a character of $m_l$. We let $M_l$ be the unipotent connected algebraic subgroup of $G$
such that $\text{Lie}(M_l) = m_l$. Then $M_l$ acts on $Ug$ via the adjoint action, and we let $I \subseteq Um_l$ be the
kernel of the character $\chi$. So we see that we are in the setup of a quantum Hamiltonian reduction
(where $A = Ug$, and $\rho : Um_l \to Ug$ is the natural inclusion).

**Definition 2.1.** The finite W-algebra associated to $e \in g$, denoted $U(g, e)$, is the quantum Hamiltonian reduction of $Ug$ with respect to $M_l$ and the ideal $I \subset Um_l$.

For example, if $e$ is a regular nilpotent element, then $U(g, e) \cong Z(Ug)$; we always have a canonical
map $Z(Ug) \to U(g, e)$ because $Z(Ug) = U(g)^G \subset U(g)^{M_l}$. In fact, this map is always an isomorphism onto the center of $U(g, e)$ (as explained in [PT2] section 5, footnote 2). In case $e$ is regular,
the map is actually surjective.
We wish to “explain” the finite W-algebra by expressing it as a quantization of the algebra of functions on the Slodowy slice \( S \subset \mathfrak{g}^* \), which is the image under \( \mathfrak{g} = \mathfrak{g}^* \) of the affine subspace \( e + \ker(ad) \).

To make this more precise, we introduce a \( \mathbb{C}^* \) action on \( \mathfrak{g} \) as follows: our chosen \( \mathfrak{sl}_2 \)-triple gives a homomorphism \( \tilde{\gamma} : SL_2(\mathbb{C}) \to G \), and we define \( \gamma(t) = \tilde{\gamma}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \), so that \( Ad(\gamma(t)e) = t^2e \); so we define \( \tilde{\rho}(t) = t^{-2}Ad(\gamma(t)) \), a \( \mathbb{C}^* \)-action on \( \mathfrak{g} \) that stabilizes \( S \) and fixes \( e \) (in fact, the inverse of this action contracts \( S \) to \( e \)). So, this action induces a grading on \( S^*\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] \) and \( \mathbb{C}[S] \) (where we now think of \( S \subseteq \mathfrak{g}^* \) by using the Killing form to identify \( \mathfrak{g} \) and \( \mathfrak{g}^* \), and transport the \( \mathbb{C}^* \)-action accordingly). This grading can now be described explicitly as follows: one writes \( S^*\mathfrak{g} = \bigoplus_{n \geq 0} S^n\mathfrak{g} \), the decomposition using the standard grading, and we let \( S^n\mathfrak{g}(i) = \{ x \in S^n\mathfrak{g} \mid [h, x] = ix \} \), where \( h \in \mathfrak{g} \) is as before, and the bracket denotes the unique extension of the adjoint action of \( h \) on \( \mathfrak{g} \) to a derivation of \( S^*\mathfrak{g} \). The grading defined above is then obtained by setting \( S^n\mathfrak{g}[n] = \text{span}\{ S^i\mathfrak{g}(i) \mid i + 2j = n \} \) for all \( n \in \mathbb{Z} \) (note that negative degrees do in fact occur). Then the grading on \( \mathbb{C}[S] \) is the one inherited from \( S^*\mathfrak{g} \), and it is easy to see that \( \mathbb{C}[S] \) has only positive degrees under this grading.

Now, we define the Kazhdan filtration on \( U\mathfrak{g} \) by first setting \( U_n\mathfrak{g}(i) = \{ x \in U_n\mathfrak{g} \mid [h, x] = ix \} \), where \( U\mathfrak{g} = \bigcup U_n\mathfrak{g} \) is the usual (PBW) filtration, and the bracket is just the bracket in \( U\mathfrak{g} \), and then defining \( F_nU\mathfrak{g} = \text{span}\{ x \in U\mathfrak{g}(i) \mid i + 2j \leq n \} \) for all \( n \in \mathbb{Z} \). Then an easy application of the PBW theorem shows that, considering \( U\mathfrak{g} \) and \( S^*\mathfrak{g} \) with the above filtration and grading, \( \text{Gr}(U\mathfrak{g}) = S^*\mathfrak{g} \).

If we let \( U(\mathfrak{g}, e) \) have the inherited filtration, then we have

**Theorem 2.2.** \( \text{Gr}(U(\mathfrak{g}, e)) = \mathbb{C}[S] \)

This isomorphism also puts a natural Poisson structure on \( \mathbb{C}[S] \), which is described in [GG].

Because of this theorem, the algebra \( U(\mathfrak{g}, e) \) is sometimes referred to as the enveloping algebra of the slice \( S \).

### 2.1. Hamiltonian Reduction for Asymptotic Enveloping Algebras.

In this subsection, we explain a version of the above constructions for so-called asymptotic enveloping algebras- this variant will be used repeatedly below. To set things up, we consider, as above, \( M \), a connected affine algebraic group; we set \( \text{Lie}(M) = \mathfrak{m} \).

**Definition 2.3.** Let \( U_h(\mathfrak{m})(0) \) be the algebra defined as the \( h \)-completion of the algebra \( T^*\mathfrak{m}/I \), where \( I \) is the two-sided ideal in the tensor algebra \( T^*\mathfrak{m} \) over the polynomial ring \( \mathbb{C}[h] \) generated by \( \{ xy - yx - h[x, y] \mid x, y \in \mathfrak{m} \} \). Further, \( U_h(\mathfrak{m}) \) will denote \( U_h(\mathfrak{m})(0)| h^{-1} \). The algebra \( U_h(\mathfrak{m})(0) \) is called the asymptotic enveloping algebra of \( \mathfrak{m} \).

By construction one has

\[
U_h(\mathfrak{m})(0)/h \to S^*\mathfrak{m}
\]

where the object on the right is the symmetric algebra of \( \mathfrak{m} \) over \( \mathbb{C} \). It is not difficult to see that \( U_h(\mathfrak{m})(0) \) is an \( h \)-complete \( \mathbb{C}[[h]] \) algebra, which is is noetherian. In particular, every ideal (either left, right, or two-sided) of \( U_h(\mathfrak{m})(0) \) is finitely generated and \( h \)-complete.

From the adjoint action of the group \( M \) on \( T^*\mathfrak{m} \), we deduce an action (also called the adjoint action) on \( T^*\mathfrak{m}/I \); and therefore an \( M \) action on each quotient \( (T^*\mathfrak{m}/I)/h^n \) for which the quotient maps are equivariant. Passing to the inverse limit, we obtain an \( M \) action on \( U_h(\mathfrak{m})(0) \) that we refer to as the adjoint action; inverting \( h \) also gives an action on \( U_h(\mathfrak{m}) \).

We wish to consider the analogue of the Hamiltonian reduction construction in this situation. To do so, we need the correct analogue of the algebra \( A \), equipped with its \( M \)-action. For that we give the
Definition 2.4. Let $A_h$ be an associative $\mathbb{C}[[h]]$-algebra, which is $h$-flat and complete with respect to the $h$-adic topology. Suppose also that $A_h$ is left and right noetherian. Then we say that $A_h$ is $M$-equivariant if there is an algebraic action of $M$, by algebra automorphisms, on each reduction $A_h/h^n$ so that all the quotient maps are $M$-equivariant. In this case there is an induced action of $M$ on $A_h$, and we demand that $m \cdot h = h$ for all $m \in M$.

Suppose we are given a map of algebras $\rho : U_h(m)(0) \to A$, which takes $h$ to $h$ and is continuous with respect to the $h$-adic topology. Such a map is called a comoment map; we say that this map is compatible with the action of $M$ if, upon reduction mod $h^n$ for each $n$, we have that the action $a \to \rho(m)a - \rho(m)\alpha$ is given by $h \cdot d(m)$ (where $d(m)$ denotes the differential of the given action of $M$).

In this set-up, we have the following proposition of Hamiltonian reduction:

Proposition 2.5. Let $J$ be a two-sided ideal of $U_h(m)(0)$. Suppose that $A/A \cdot J$ is $h$-torsion free. Then the complete $\mathbb{C}[[h]]$-module $(A/A \cdot J)^m$ is naturally an algebra so that, if $a, b \in A$ are elements whose images in $A/A \cdot J$ are denoted $\bar{a}$ and $\bar{b}$, then $\bar{a} \cdot \bar{b}$ is the reduction of the product $ab$ in $A$.

Proof. Let $a, b \in A$ whose images $\bar{a}, \bar{b}$ in $A/A \cdot J$ are $\mathbb{m}$-invariant. As $A/A \cdot J$ is $h$-torsion free, the $\mathbb{m}$-invariants are elements invariant under $ad(\rho(m))$ for all $m \in \mathbb{m}$. Thus we can see that, for any $m \in \mathbb{m}$, $\rho(m)a - \rho(m)b = j_m$ and $\rho(m)b - \rho(m)a = k_m$ for some $j_m, k_m \in A \cdot J$. Choose a $\mathbb{C}$-basis for $\mathbb{m}$, which we call $\{x_i\}$. Then any element of $U_h(m)$ can be represented as a formal series

$$\sum_{I, i} c_{I,i} h^i x^I$$

where $c_{I,i} \in \mathbb{C}$, $x^I = x_1^{i_1} \ldots x_n^{i_n}$ for a multi-index $I$, and $i \to \infty$ as $|I| \to \infty$. Repeatedly applying the above formulas for the action of $\rho(x)$, we see that $\rho(x^I)b - \rho(x^I)a = \alpha_{x^I} + k_{x^I}$ where $k_{x^I} \in A \cdot J$ and $\alpha_{x^I}$ is an element of $A \cdot J$ (this is because $J$ is a two-sided ideal of $U_h(m)(0)$, so that $A \cdot J \cdot \rho(m) \subset A \cdot J$ for all $m \in \mathbb{m}$). Therefore, as $\rho(h) = h$ we have

$$\rho\left(\sum_{I, i} c_{I,i} h^i x^I\right) = \sum_{I, i} c_{I,i} h^i (\rho(x^I) + \alpha_{x^I}) = b \sum_{I, i} c_{I,i} h^i + \sum_{I, i} c_{I,i} h^i \alpha_{x^I}$$

and as the latter sum is convergent in the complete ideal $A \cdot J$, we see that $\rho(u)b - \rho(u)a \in A \cdot J$ for any $u \in U_h(m)$.

Now, consider any representatives for the classes $\bar{a}, \bar{b}$; call them $a + j_1, b + k_1$. Then

$$(a + j_1)(b + k_1) = ab + j_1b + ak_1 + j_1k_1$$

and, as $j_1 \in A \cdot J$, we obtain $j_1b \in A \cdot J$ by commuting elements of form $\rho(u)$ past the $b$ via the above claim. Therefore $(a + j_1)(b + k_1)$ is congruent to $ab$ in $A/A \cdot J$ and the result follows. \(\Box\)

Now, in the main case of interest in this paper, $R = A/h$ will be the ring of coordinates of an algebraic variety, on which $M$ acts algebraically. In that case, as $M$ is connected, one has that the $M$-invariants in $R/(\overline{J})$ are the same as the $\mathbb{m}$-invariants. As $A/A \cdot J$ is an infinitesimal deformation of $R/(\overline{J})$, a quick proof by induction shows that $(A/A \cdot J)^M = (A/A \cdot J)^m$. Thus we obtain

Corollary 2.6. With hypotheses as above, the complete $\mathbb{C}[[h]]$-module $(A/A \cdot J)^M$ is naturally an algebra so that, if $a, b \in A$ are elements whose images in $A/A \cdot J$ are denoted $\bar{a}$ and $\bar{b}$, then $\bar{a} \cdot \bar{b}$ is the reduction of the product $ab \in A$.

This type of Hamiltonian reduction will occur throughout the paper.

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1The noetherian condition probably isn’t strictly necessary, but it is always satisfied in this paper and so we assume it for convenience.
3. Differential Operators and Quantization

Let $X$ be a smooth complex algebraic variety. Then the sheaf of differential operators on $X$, $D_X$, is a sheaf of filtered algebras whose associated graded sheaf is isomorphic to $\pi_*(O_{T^*X})$ where $T^*X$ is the cotangent bundle to $X$, and $\pi: T^*X \to X$ is the natural map (see [HTT] for details). So $D_X$ is a quantization of the cotangent bundle of $X$, but it is only local on $X$, not $T^*X$. To correct this, we introduce the following

**Definition 3.1.** (c.f [BK2]) Let $X$ be an affine complex algebraic variety. The we define the algebra of asymptotic differential operators on $X$, $D_h(X)(0)$, to be the $h$-completion of the algebra generated by $O_X$, the global vector fields $\Theta_X$, and the variable $h$, subject to the relations $\xi_h f_1 = f_1 \xi_h$ and $\xi_h \xi = h \xi(f)$. The algebra $D_h(X)(0)$ is a quantization of $T^*X$ in the sense of the definition given below. We may apply localization to this algebra to obtain a sheaf on $T^*X$; for a general algebraic variety we glue this construction to obtain the sheaf of asymptotic differential operators $D_h(X)(0)$. We again emphasize that this is a sheaf on $T^*X$, not on $X$.

The general context for this definition is given by

**Definition 3.2.** (c.f [BK1]) Let $Y$ be a Poisson variety, i.e., an complex algebraic variety equipped with a Poisson bracket on the structure sheaf. A quantization of $Y$, $O_Y$, is a sheaf of associative, flat $C[[h]]$ algebras on $Y$ that is complete with respect to the $h$-adic topology and equipped with an isomorphism $O_Y/hO_Y \to O_Y$. This gives $O_Y$ the structure of a sheaf of Poisson algebras, and we demand that this structure agrees with the given one.

Most of the time (though not always!) in this paper, it will be the case that $Y$ is smooth and the Poisson structure in question comes from a symplectic form on $Y$. In the case of [3.1] the symplectic variety in question is of the form $T^*X$.

As the algebras appearing here are $h$-torsion free, it is sometimes convenient to invert $h$. We define $D_h(X) := D_h(X)(0)[h^{-1}]$ for any algebraic variety; this is a $C((h))$-linear sheaf on $T^*X$. Although not a quantization, this is the sheaf of algebras that we will actually use in this paper, for reasons that will become clear in the next section.

We note at this point that this sheaf is considered (in a somewhat different notation) in the paper [KR]. There, they introduce the formalism of $W$-algebras (no relation to the $W$-algebras in section 1!). To avoid confusion, we will call them $QDO$-algebras, standing for quantized differential operator algebras. We recall now the

**Definition 3.3.** [KR] Let $X$ be a smooth holomorphic symplectic variety of dimension $2n$. A $QDO$-algebra on $X$ is a sheaf of $C((h))$-linear algebras, $D_h$, such that for each $x \in X$, there exists an open neighborhood $U$ of $x$ and a symplectic holomorphic morphism $\phi: U \to T^*\mathbb{C}^n$ such that $D_h|_U \to \phi^* D_h(\mathbb{C}^n)$.

Although this is a convenient definition in the analytic topology, in the Zariski topology it is very poorly behaved. Therefore we will not use this definition in this paper; however, we should note that several of the basic techniques of [KR] do carry over to this situation; in particular, many of their arguments only rely on the fact that a $QDO$ is a quantization in the sense of [3.2] when we quote results from this paper (as we do a few times in the sequel), we only quote results of this type.

Next, we define the categories of modules over quantizations that we will consider. Let us recall from [KS], theorem 1.2.5, that if $O_h$ is a quantization of a smooth variety $Y$, then $O_h$ is a locally noetherian, stalk-wise noetherian sheaf of algebras.

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2We would like to thank the referee for emphasizing this point.
Definition 3.4. Let $O_h$ be a quantization of the smooth Poisson algebraic variety $Y$. An $O_h$-module $M(0)$ is coherent if it is locally finitely generated. Further, any such module is automatically complete in the $h$-adic topology (by [KS], theorem 1.2.5, part 3).

An $O_h[h^{-1}]$ module $M$ is said to be coherent if there exists, globally, a coherent $O_h[h^{-1}]$ module, $M(0)$, such that $M \cong M(0)[h^{-1}]$. An $O_h[h^{-1}]$-module is said to be quasi-coherent if it is a direct limit of coherent modules; i.e., we simply define $\text{Mod}^{\text{coh}}(O_h[h^{-1}])$ as the ind-category of $\text{Mod}^{\text{coh}}(O_h[h^{-1}])$.

As noted above, we may refer the reader to [KS] (chapter 1) for details about modules over quantized algebras in a very general context. In particular, we note that our definition of “coherent” for $D_h(0)$-modules agrees with the one given there. Applying again their theorem 1.2.5, we have

Lemma 3.5. An $O_h$-module $M(0)$ is coherent iff $M(0)$ is $h$-complete and $h^nM(0)/h^{n+1}M(0)$ is a coherent $O_Y$ module for all $n \geq 0$. This is equivalent to the condition of $M(0)$ having bounded $h$-torsion and $M(0)/hM(0)$ being coherent over $O_Y$.

We also note that the categories $\text{Mod}^{\text{coh}}(O_h)$, $\text{Mod}^{\text{coh}}(O_h[h^{-1}])$ and $\text{Mod}^{\text{coh}}(O_h)$ are abelian; the first two by the noetherian property of $O_h$, and the last because it is an ind-category.

To finish this section, we note a key fact about the cohomology of modules over the algebra $O_h$. This is lemma 2.12 in [KR], and its proof goes over mutatis mutandis to the algebraic situation; one may also consult (the proof of) [KS], theorem 1.2.5, part 5.

Lemma 3.6. Let $M(0)$ be a coherent $O_h$ module. Assume that $H^i(X, M(0)/hM(0)) = 0$ for $i > 0$. Then we have

i) The natural map $\Gamma(X, M(0)) \to \Gamma(X, M(0)/hM(0))$ is surjective.

ii) $H^i(X, M(0)) = 0$ for $i > 0$.

4. Equivariance

We suppose now that we have an algebraic group $G$ acting algebraically on our Poisson variety $Y$, in such a way that the group action respects the Poisson bracket. We wish to define equivariant versions of everything introduced in the previous section. We start with a general

Definition 4.1. Let $O_h$ be a quantization of $Y$. Then $O_h$ is said to be $G$-equivariant if each sheaf $O_h/h^nO_h$ (for $n \geq 0$) admits a $G$-equivariant structure (as a sheaf of algebras; i.e., we demand that the action of $G$ respect the multiplication), in such a way that the natural maps $O_h/h^nO_h \to O_h/h^nO_h$ are $G$-morphisms. We demand that $h$ be stable under the action of $G$ in the sense that, on global sections, $\tilde{\rho}_g^{-1}(h) = \chi(g)h$, where $\chi$ is an algebraic character of $G$ (which will usually be the trivial character).

In particular, this definition gives us isomorphisms $O_h \xrightarrow{\rho_g^{-1}} O_h$ (where $\rho_g : Y \to Y$ is the map associated to $g \in G$) for all $g \in G$. This definition extends immediately to the algebra $O_h[h^{-1}]$; we simply extend the action by demanding that $G$ act on $h^{-1}$ by the inverse of the character $\chi$.

To obtain equivariance conditions for coherent modules, we let $M \in \text{Mod}^{\text{coh}}(O_h)$ and let $M(0)$ be a lattice. We further suppose that $O_h$ is a $G$-equivariant sheaf in the above sense. Then we have

Definition 4.2. $M(0)$ is a quasi-$G$-equivariant $O_h$-module if each sheaf $M(0)/h^nM(0)$ (for $n \geq 0$) is $G$-equivariant as a module over $O_h/h^n$; i.e., there is a $G$-action on $M(0)/h^nM(0)$ that is compatible with the $G$-action on $O_h/h^n$, in the sense that the action map $O_h/h^n \otimes M(0)/h^nM(0) \to M(0)/h^nM(0)$ is $G$-equivariant for each $n$; and this action makes $M(0)/hM(0)$ into an equivariant coherent sheaf. We demand that the natural quotient maps are $G$-morphisms. We demand that $h$ be stable under the action of $G$ in the sense that, on global sections, $\rho_g^{-1}(h) = \chi(g)h$ where $\chi$ is an algebraic character of $G$ (which will usually be the trivial character).
We say that $M$ is $G$-equivariant if it admits a $G$-action, and there is a $G$-stable lattice $M(0)$ which is $G$-equivariant in the above sense.

If $M$ is a quasicoherent $O_h$-module equipped with an action of $G$, we say that it is $G$-equivariant if it is a direct limit of $G$-equivariant coherent modules.

This definition gives us categories $\text{Mod}^{G,:?}(O_h)$ (where $?$ is coherent or quasicoherent), where we demand that the morphisms respect the $G$-structure.

If we go back to our primary example where our symplectic variety is $T^*X$, then the sheaf $D_h(X)$ is $C^*$-equivariant for the $C^*$-action on $T^*X$ given by dilation on the fibers of $\pi : T^*X \to X$. Therefore, we can consider the sheaf of fixed points, denoted $D_h(X)^{C^*}$. This is a sheaf of algebras on $T^*X$, and one verifies $D_h(X)^{C^*} = E_X$, where $E_X$ denotes the sheaf of formal differential operators on $T^*X$. Then the functor $M \to \pi_s(M)^{C^*}$ provides an equivalence of categories between the category of $C^*$-equivariant coherent $D_h(X)$ modules, and that of coherent $D_X$ modules (this fact, or, rather, its analytic analogue, is discussed in [KR, section 2.3.3]).

In fact, $C^*$-equivariant modules will play a major role in this paper. At this point, we note a few facts: we will consider only $C^*$ actions that act on $h$ as some $t^n$ for $n \geq 0$, and we can assume that $n = 1$ (if not, we can simply replace the ground field $\mathbb{C}((h))$ by $\mathbb{C}((h^{1/n}))$, base change everything to this field, and demand that $C^*$ acts on $h^{1/n}$ as $t$).

**Lemma 4.3.** Given such an action on a $D_h$-module $M$, for any $U \subseteq X$ which is affine, open and $C^*$-invariant, $\Gamma(U, M) \neq 0$ implies $\Gamma(U, M)^{C^*} \neq 0$. Further, if $M$ is coherent, $\Gamma(U, M)^{C^*}$ is generated by finitely many sections as a $\Gamma(U, D)^{C^*}$ module.

**Proof.** These facts are proved by using the definition of equivariance given above. To show the existence of an invariant section, we assume WLOG that $M$ is coherent, using that, over an affine open set, a quasicoherent $C^*$-equivariant module is a limit of its equivariant coherent submodules. So, each surjection $\Gamma(U, M(0))/h^{n+1}\Gamma(U, M(0)) \to \Gamma(U, M(0))/h^n\Gamma(U, M(0))$ admits a $C^*$-invariant splitting. Therefore, we can choose $C^*$-homogeneous sections in $\Gamma(U, M(0))$ whose images generate $\Gamma(U, M(0))/h^{n}\Gamma(U, M(0))$. So $\Gamma(U, M) \neq 0$ implies that $h^n \neq 0$ for all $n \in \mathbb{Z}$, for at least one of these sections; and then choosing the correct $n$ gives a $C^*$-invariant section. The fact about finitely generated modules follows from the Nakayama lemma and the fact that $\Gamma(U, M(0))/h\Gamma(U, M(0))$ is finitely generated over $\Gamma(U, D_h(0))/h\Gamma(U, D_h(0))$ for coherent modules by writing out the action of $D_h$ on $M$. 

5. Hamiltonian Reduction for Quantizations

Let $H$ be a connected affine algebraic group, with Lie algebra $\mathfrak{h}$, and suppose that $H$ acts on the algebraic variety $X$. Then the induced action of $H$ on $T^*X$ is Hamiltonian (see [CG] page 44 for details) and so there exists an $H$-equivariant moment map $\mu : T^*X \to \mathfrak{h}^*$. In fact, we can describe explicitly the comorphism on functions as follows: any $y \in \mathfrak{h}$ gives rise to an algebraic vector field on $X$, denoted $\xi_y$, which in turn gives rise to a regular function on $T^*X$, called $f_y$, via the natural pairing of tangent and cotangent vectors. This map extends uniquely to an algebra morphism $O(\mathfrak{h}^*) = S\mathfrak{h} \to O(T^*X)$.

Let $\chi \in \mathfrak{h}^*$ be a character (i.e., suppose that $\chi([\mathfrak{h}, \mathfrak{h}]) = 0$). Then $\mu^{-1}(\chi)$ is an $H$ invariant closed subvariety of $T^*X$. Suppose that $\mu$ has surjective differential at all points in $\mu^{-1}(\chi)$, so that $\mu^{-1}(\chi)$ is a smooth subvariety. Suppose further that there exists a smooth quotient $\mu^{-1}(\chi)/H$ in the sense that there exists a morphism $p : \mu^{-1}(\chi) \to \mu^{-1}(\chi)/H$ making $\mu^{-1}(\chi)$ a principal $H$-bundle (in the Zariski topology) over $\mu^{-1}(\chi)/H$, and we assume that this quotient admits a symplectic form compatible with the reduction. Then this quotient variety is called the Hamiltonian reduction of $T^*X$ with respect to $\chi$.

In this section we’re going to explain how the Hamiltonian reduction procedure for quantizations will allow us to obtain quantizations of various spaces $\mu^{-1}(\chi)/H$. In fact, the procedure we will
discuss sometimes works for quantizations of Poisson varieties as well; so we move now to that level of generality; however, the case discussed above is the key one to keep in mind.

Suppose now that \( Y \) is a smooth Poisson variety, with quantization \( O_h \). Suppose that \( Y \) admits an action of \( H \), and that \( O_h \) is \( H \)-equivariant in the sense of \( \ref{equivariant_modules} \). Suppose further that \( O_h \) admits a quantum comoment map; i.e., there is map of algebras \( \rho : U_h(\mathfrak{h})(0) \to O_h \) satisfying the condition of \( \ref{equivariant_modules} \). In particular there is an \( H \)-equivariant map \( \mu : Y \to \mathfrak{h} \) (which arises by taking \( h \to 0 \) in the quantum comoment map); in the situation discussed above of an action of \( H \) on \( X \) and \( O_h = D_h \) (on \( T^*X \)) then these criteria are fulfilled. The quantum comoment map takes \( x \in \mathfrak{h} \) to the vector field \( \xi_x \), considered as a section of \( D_h \).

Let \( \chi : U_h(\mathfrak{h})(0) \to \mathbb{C}[\hbar] \) be a continuous character. Such a map is determined by its restriction to \( \mathfrak{h} \), so specifying \( \chi \) is the same as specifying a \( \mathbb{C} \)-linear map, which we will abusively denote \( \chi : \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \to \mathbb{C}[\hbar] \). Let \( \chi_0 \) denote the reduction mod \( h \) of \( \chi \); note that even if \( \chi \) is nontrivial then \( \chi_0 = 0 \) is possible. The sheaf of \( O_h \)-modules \( L_\chi := O_h / \sum_{x \in \mathfrak{h}} O_h \cdot (\xi_x - \chi(x)) \) is then supported on \( \mu^{-1}(\chi_0) \). Suppose, as above, that \( \mu^{-1}(\chi_0)/H \) exists, and let \( p : \mu^{-1}(\chi_0) \to \mu^{-1}(\chi_0)/H \) denote the quotient map; which we assume to be an \( H \)-torsor. Then we make the

**Definition 5.1.** The sheaf \( p_*(L_\chi)_H \) (which, under the assumption that \( L_\chi \) is \( h \)-torsion free, is a sheaf of algebras on \( \mu^{-1}(\chi_0)/H \)) by \( \ref{equivariant_modules} \) is called the quantum Hamiltonian reduction of \( O_h \), with respect to \( \chi \).

At this level of generality, it seems that there is not much we can say about about \( p_*(L_\chi)_H \). For instance, it is not obvious\(^3\) whether or not it is actually a quantization of \( \mu^{-1}(\chi_0)/H \). However, in the two main cases of interest in this paper, we shall see that this is indeed the case.

Now let us turn to the relationship between equivariant modules and Hamiltonian reduction.

Let \( M(0) \) be an \( H \)-equivariant coherent \( O_h \)-module, in the sense of the previous section. Let \( \beta : \mathfrak{h} \to \End_{\mathbb{C}[\hbar]}(M(0)) \) denote the derivative of the action map of \( H \) on \( O_h \). We note that, in the case that \( Y = T^*X \) (with \( H \) acting as above) and \( O_h = D_h(X) = M(0) \), then \( h \cdot \beta(x) \) is the map \( [\xi_x, \cdot] \), where \( \xi_x \) is the vector field associated to \( x \) via the action, considered as a section of \( D_h \).

**Definition 5.2.** Let \( \chi : \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \to \mathbb{C}[\hbar] \) be a \( \mathbb{C} \)-linear map. We say that \( M(0) \) has twist \( \chi \) if the map \( h\beta(x) - \rho(x) \) is equal to \( \chi(x) \) on \( M(0) \), for all \( x \in \mathfrak{h} \).

We denote by \( \text{Mod}^{\text{coh}}_H(O_h) \) (resp. \( \text{Mod}^{\text{qcoh}}_H(O_h) \) the category of coherent (resp. quasicoherent) modules over \( O_h \) with twist \( \chi \); this is a full abelian subcategory of the category of all equivariant \( O_h \)-modules. Observe that \( L_\chi \) has twist \( \chi \), by the definition of the comoment map. Furthermore, unpacking the definition yields \( \text{Hom}(L_\chi, M(0)) = M(0)^\hbar \) for any \( h \)-torsion-free \( M(0) \) with twist \( \chi \), and since \( H \) is connected this means \( \text{Hom}(L_\chi, M(0)) = M(0)^H \) as well.

Then we have the following

**Proposition 5.3.** In the situation above, suppose that \( L_\chi \) is \( h \)-torsion free. Suppose further that \( p_*(L_\chi)_H \) is a quantization of the smooth variety \( Z = \mu^{-1}(\chi_0)/H \), and suppose we have

\[ \mathcal{E}xt^i_{O_h}(L_\chi[h^{-1}], L_\chi[h^{-1}]) = 0 \]

for \( i > 0 \). Then

1) We have equivalences of categories \( \text{Mod}^H(p_*(L_\chi)_H[h^{-1}]) \to \text{Mod}^H_\chi(O_h[h^{-1}]) \) (where \( ? \) stands for coherent or quasicoherent) given by

\[ M \to \mathbb{H}^+(M) := L_\chi[h^{-1}] \otimes_{p_*(L_\chi)_H[h^{-1}]} p^{-1}M \]

for \( M \) in \( \text{Mod}^H(p_*(L_\chi)_H) \), and \( S \to \mathbb{H}^+(S) = p_*(\text{Hom}(L_\chi[h^{-1}], S))_H \) for \( S \) in \( \text{Mod}^H_\chi(O_h) \) (where in the second functor we use \( \text{Hom} \) to mean sheaf hom).

\(^3\)To make sense of this we regard \( U_h(\mathfrak{h})(0) \) as a constant sheaf on \( Y \).

\(^4\)We thank the referee for pointing this out.
2) Suppose that there exists a $\mathbb{C}^*$-action on $X$ that preserves $\mu^{-1}(\chi_0)$ and that this drops to an action on $Z$ in such a way that $p_*(L_\chi)^H$ is $\mathbb{C}^*$-equivariant. Suppose that $\mathbb{C}^*$ acts on $H$ in such a way that $H \rtimes \mathbb{C}^*$ acts on $X$. Then we have an equivalence
\[ \text{Mod}_{\mathbb{C}^*}^H(p_*(L_\chi)^H) \cong \text{Mod}_{\mathbb{C}^*}^H(O_h) \]
which is given by the same formulae.

We will prove this momentarily; our argument uses the theory of quantizations of algebraic varieties as developed in [BDMN], and in particular ideas of the proof of the Hamiltonian reduction theorem there; though, due to the group action, this case is technically different and is not implied by the theorem there (or vice versa). Before giving the details, we would like to comment on the relation with the analogous statement proved by Kashiwara and Rouquier, namely [KR], proposition 2.8. Aside from the technical difference of working in the analytic topology as opposed to the Zariski (which is ultimately a minor issue), the main difference between their construction and ours is that we consider a more general notion of twist then they do. To make the two situations consistent let us suppose that we are working over $\mathbb{C}^*$, e.g., the argument of [BDMN], claim 4.23). (of 5.3) We will prove the result for coherent modules; the quasicoherent case follows formally.

Proof. (of 5.3) We will prove the result for coherent modules; the quasicoherent case follows formally. In addition, part 2 follows by the same method as part 1, so we will concentrate on the first case.

Note that the functors in question form an adjoint pair by the usual hom-tensor adjunction.

Let us denote $p_*(L_\chi)^H := O_{Z,h}$, as it is by assumption a quantization of $Z$.

Recall that $L_\chi/hL_\chi = O_{\mu^{-1}(\chi_0)}$. Since $R^ip_*(O_{\mu^{-1}(\chi_0)}) = 0$, we have that $R^ip_*(L_\chi) = 0$ for $i > 0$ as well (by the argument of lemma 2.12 in [KR]). Therefore
\[ p_*(L_\chi)/hp_*(L_\chi) \cong p_*(L_\chi/hL_\chi) \]
is locally projective as a module over $O_Z$ (since $\mu^{-1}(\chi_0)$ is an $H$-torsor over $Z$). From [KS] corollary 1.6.7, we see that $p_*(L_\chi)$ is faithfully flat over $O_{Z,h}$. Therefore the functor $\mathbb{H}^+$ is exact and conservative. It follows also that the map
\[ S \to \mathbb{H} \circ \mathbb{H}^+(S) \]
is an isomorphism for all coherent $S$ over $O_{Z,h}$; indeed, since $Z$ is smooth we may take locally a finite projective resolution of $S$; so the result follows from the claim for $O_{Z,h}$ itself; but this holds by definition.

Next we claim that $\text{Ext}_W^i(L_\chi[h^{-1}], \mathbb{H}^+(S)) = 0$ for $i > 0$ and for any coherent $O_{h,Z}[h^{-1}]$-module $S$. For this, we use again that $S$ is locally of finite homological dimension. Therefore we may proceed by induction on the homological dimension; the base case being exactly the assumption (compare, e.g., the argument of [BDMN], claim 4.23).

Now we consider the functor $\mathbb{H}$. We claim that it is conservative on $\text{Mod}_{\mathbb{C}^*}^\text{coh,H}(O_h[h^{-1}])$. It suffices to prove this for a lattice $M(0)$. Note that the condition of having twist $\chi$ means that $\text{Hom}(L_\chi, M(0)) = M(0)^H$. However, $M(0)/h$ is an $H$-equivariant coherent sheaf which is set-theoretically supported on the torsor $\mu^{-1}(\chi_0)$, so $M(0)/h \neq 0$ implies that $M(0)$ has a nontrivial space of $H$-invariants, and that, in fact, $M(0)/hM(0)$ is the pullback of a sheaf on $\tilde{Z}$, an infinitesimal thickening of $Z$. 


Proceeding by induction, consider the sequence
\[ 0 \to M(0)/h \to M(0)/h^n \to M(0)/h^{n-1} \to 0 \]
As there are no higher derived $H$-invariants of $M(0)/h$ by the above, one concludes that $M(0)/h^n$ has a nonzero space of $H$-invariants, which surjects onto the $H$-invariants of $M(0)/h^{n-1}$. Taking the inverse limit, we obtain the claim.

We now finish the proof by showing that the map
\[ H^1 \circ H(M) \to M \]
is an isomorphism. Write $K$ and $C$ for the kernel and cokernel of this map, respectively. By the adjunction between $H$ and $H^1$, we have that
\[ H \circ H^1 \circ H(M) \to H(M) \]
is an isomorphism. Since $H$ is left exact (it is a hom functor), this forces $H(K) = 0$, so $K = 0$ as $H$ is conservative. Thus we have that
\[ 0 \to H^1 \circ H(M) \to M \to C \to 0 \]
is a short exact sequence; note that this implies $H^1 \circ H(M)$ is coherent over $O_h[h^{-1}]$; from this it follows that $H(M)$ is coherent over $O_{Z,h}$. Now, apply $H$ to the exact sequence above, and use the fact that $\text{Ext}_0^{1}(O_\mathcal{L}(L_h[h^{-1}]),(H^1 \circ H(M))) = 0$ as proved above, to see that $H(C) = 0$, so that $C = 0$ and the result follows. $\square$

Now we need to explain that the hypotheses actually hold when we want them to:

**Lemma 5.4.** Suppose that, in the situation of $\mathbf{5.3}$, we have that $Y = T^*X$, $O_h = D_\hbar$, and the Hamiltonian group action of $H$ comes from an action of $H$ on $X$. Suppose also that $\mu^{-1}(\chi_0)/H$ is a smooth symplectic variety. Then the hypotheses of $\mathbf{5.3}$ are satisfied.

**Proof.** Essentially, this is a consequence of the formal Darboux lemma for the quantization $D_\hbar$. Let $x \in \mu^{-1}(\chi_0)$, and denote by $\hat{D}_{\hbar,x}$ the formal completion of $D_\hbar$ at $x$, and by $\hat{O}_x$ the formal completion of the structure sheaf of $T^*X$ at $x$; the formal neighborhood of $x$ in $T^*X$ is then $\text{Spf}(\hat{O}_x)$. Applying the local description of the Hamiltonian reduction in the smooth case (as recalled in $\mathbf{[KR]}$, lemma 2.7), we have an isomorphism of formal symplectic schemes
\[ \text{Spf}(\hat{O}_x) \cong \text{Spf}(\hat{O}_{T^*H,c} \hat{\otimes} \hat{O}_{Z,p(x)}) \]
where by $\hat{O}_{T^*H,c}$ we mean the completion of $O_{T^*H}$ at the identity element, and by $\hat{O}_{Z,p(x)}$ we mean the completion of $Z$ at $p(x)$; this isomorphism is compatible with the $H$-action.

Therefore, by the basic structure theory for $\hat{D}_\hbar$ (c.f. $\mathbf{[BK]}$, lemma 1.5, or, somewhat more elaborately, $\mathbf{[BDMN]}$, proposition 4.9), we have a corresponding isomorphism of quantizations
\[ \hat{D}_\hbar \cong \hat{D}_{\hbar,H,c} \hat{\otimes} \hat{W}_{h,Z} \]
where $\hat{D}_{\hbar,H,c}$ denotes the formal completion of $h$-differential operators on $H$ at the identity of $T^*H$, and $\hat{W}_{h,Z}$ is a formal quantization of $\hat{O}_Z$; it is necessarily isomorphic to a formal Weyl algebra by $\mathbf{[BK]}$, lemma 1.5. By construction we have
\[ \hat{D}_{\hbar,H,c} \cong \hat{W}_{h}(\mathfrak{h} \oplus \mathfrak{h}^*) \]
where on the right hand side we have the $h$-completed formal Weyl algebra on the symplectic vector space $\mathfrak{h} \oplus \mathfrak{h}^*$. Thus in total we have
\[ \hat{D}_\hbar \cong \hat{W}_{h}(\mathfrak{h} \oplus \mathfrak{h}^*) \hat{\otimes} \hat{W}_{h,Z} \]
Under this isomorphism the formal completion of the left ideal \( \sum_{x} D_{h} \cdot (\xi_{x} - \chi(x)) \) is simply given by \( \hat{D}_{h} \cdot \mathfrak{h} \). Therefore, if we formally complete \( L_{\chi} \) at \( x \) we have

\[
\hat{L}_{\chi} = \hat{D}_{h}/\hat{D}_{h} \cdot \mathfrak{h}
\]

which is evidently \( h \)-torsion free. This shows that \( L_{\chi} \) is \( h \)-flat. Furthermore, the invariants

\[
\hat{L}_{\chi}^{H} = \hat{W}_{h,Z}
\]

satisfy \( \hat{L}_{\chi}^{H}/h = \hat{O}_{Z} \); from this it follows that the natural map \( p_{\chi}(L_{\chi}^{H}/h) \to O_{Z} \) is an isomorphism, as it is so after each formal completion; i.e., \( p_{\chi}(L_{\chi}^{H}) \) is a quantization of \( O_{Z} \).

Finally, to obtain the statement on \( \text{Ext} \) vanishing, we note that it too can be checked upon formal completion (compare, e.g., [BDMN], theorem 4.17). Using the isomorphism \( \hat{D}_{h} = \hat{W}_{h}(\mathfrak{h} \otimes \mathfrak{b}^{*}) \otimes \hat{W}_{h,Z} \) we see that the needed vanishing reduced to the triviality of the de Rham cohomology of the affine space \( \mathfrak{b}^{*} \) (use [BDMN], lemma 4.14), whence the result. \[ \square \]

**Remark 5.5.** As this proof is fundamentally local, this result carries over without change to the case where \( O_{h} \) is a quantization of \( T^{*}X \), which is locally (but not necessarily globally) isomorphic to \( D_{h} \). We will encounter such quantizations in the next two sections.

Finally we finish off this section with

**Example 5.6.** Suppose \( H \) is a connected, affine algebraic group, \( B \leq H \) a connected algebraic subgroup, with \( \text{Lie}(B) = \mathfrak{b}^{*} \). Then we have the natural left and right actions of \( B \) on \( H \): which extend to actions on \( T^{*}H \). The moment map (for the right action) in this case can be described as follows: we have an isomorphism \( T^{*}H = H \times \mathfrak{b}^{*} \), and thus a map to \( \mathfrak{b}^{*} \) via \( (h, \xi) \to ad^{*}(h)(\xi) \) (this is the moment map for the action of \( H \) on \( T^{*}H \), denoted \( \mu^{'} \)). So the moment map \( \mu \) for \( B \) is given by the composition \( T^{*}H \to \mathfrak{b}^{*} \to \mathfrak{b}^{*} \). So, we can describe \( \mu^{-1}(\lambda) \) by first noting that \( res^{-1}(\lambda) = \lambda + \mathfrak{b}^{\perp} \) (where \( \mathfrak{b}^{\perp} \) denotes the annihilator of \( \mathfrak{b} \) in \( \mathfrak{b}^{*} \)). The inverse image of this space under \( \mu^{'} \) is the closed subvariety \( \{(h, ad^{*}(h^{-1})(\lambda + \mathfrak{b}^{\perp})) \in H \times \mathfrak{b}^{*}\} \). Now, if \( \lambda = 0 \), then it is immediate that the quotient of this variety by \( B \) is isomorphic to \( H \times_{B} \mathfrak{b}^{\perp} \cong T^{*}(H/B) \). For general \( \lambda \), we obtain a algebraic variety \( T^{*}(H/B)^{\lambda} \) called a twisted cotangent bundle. We note that if \( H = G \), and \( B \) is a Borel subgroup, then \( G/B \) is the full flag variety. In this case, we can consider characters of \( \mathfrak{b} \) which come from \( \mathfrak{h} \) via extension by zero. If such a \( \lambda \) is an integral character, then the variety \( T^{*}(G/B)^{\lambda} \) is isomorphic to \( T^{*}(G/B) \).

6. Localization

We now apply the formalism of the above sections to the finite \( W \) algebras. To do this, we’ll first discuss (a version of) the classical Beilinson-Bernstein localization theorem. We start with the composition \( G/B \) first discuss (a version of) the classical Beilinson-Bernstein localization theorem. We start with the

\[
G/B
\]

and the sheaf of asymptotic differential operators \( D_{h}(G) \). As \( G \) and \( T^{*}G \) are affine, we will need to understand the global sections of this sheaf.

We have \( \Gamma(D_{h}(G)) = O_{G}[[\mathfrak{h}]] \hat{\otimes}_{\mathbb{C}[[\mathfrak{h}]}} U_{h}(\mathfrak{g})(0) \); where the algebra structure on the tensor product is determined by \( (f \otimes x)(g \otimes y) = h f(xg) \otimes y + f g \otimes xy \) for \( f, g \in O_{G} \) and \( x, y \in \mathfrak{g} \), where by \( (xg) \) we mean the action of \( x \) as a left invariant vector field on \( g \). Then \( D_{h}(G) \) admits both a left and a right equivariant structure for \( G \), by the canonical actions of the group on the functions and vector fields.

We shall work with characters \( \lambda \in \mathfrak{b}^{*} \), which satisfy \( \lambda(\mathfrak{n}) = 0 \). We can apply the Hamiltonian reduction procedure as explained above to \( D_{h}(G) \) and \( D_{h}(G)(0) \), where we consider the right action of \( B \) on \( D_{h}(G) \), and we consider the character \( \chi = h\lambda \). In particular, the constructions here are compatible with those of [KR].

From now on, \( X = G/B \). We obtain a sheaf on \( T^{*}(X) \), denoted \( D_{h}(\lambda - \rho) \) and \( D_{h}(\lambda - \rho)(0) \) (where \( \rho \) denotes the sum of the positive roots in \( \mathfrak{g} \)- this notation will become clear later). The latter sheaf can also be written as follows: we can consider the sheaf \( O_{X}[[\mathfrak{h}]] \hat{\otimes}_{\mathbb{C}[[\mathfrak{h}]}} U_{h}(\mathfrak{g})(0) \) on \( T^{*}X \),
and we can take the quotient of this sheaf by the ideal sheaf generated by \( \{ b - h\lambda(b) | b \in b \} \); by the definition of the reduction procedure and the action of \( B \), this is the same sheaf. Under these notations, the sheaf of asymptotic differential operators is \( D_h(-\rho) \). We note that all the sheaves \( D_h(\lambda) \) are \( G \) equivariant with respect to the left \( G \) action on \( T^*X \).

This description allows us to see that there is a “universal” sheaf of algebras mapping to each \( D_h(\lambda) \); in particular, take the quotient of \( O_X[[h]] \hat{\otimes}_{\mathbb{C}[[h]]} U_h(\mathfrak{g})(0) \) by the ideal sheaf generated by the subspace \( N \) of \( U_h(\mathfrak{g})(0) \). Then the resulting sheaf of algebras, called \( D_h(\mathfrak{h})(0) \) can be thought of as a Hamiltonian reduction with respect to the maximal unipotent subgroup \( N \) (it is not a quantization of \( T^*X \), but rather of an \( H \)-bundle over it). The algebra \( D_h(\mathfrak{h}) := D_h(\mathfrak{h})(0)[h^{-1}] \) maps (via the obvious quotient map) to each \( D_h(\lambda) \).

Now, we have a morphism of algebras \( \Phi_{\lambda}(0) : U_h(\mathfrak{g})(0) \rightarrow \Gamma(T^*X, D_h(\lambda)(0)) \) which is defined in the obvious way using the realization of \( D_h(\lambda) \) given above; this gives then a morphism \( \Phi_{\lambda} : U_h(\mathfrak{g}) \rightarrow \Gamma(T^*X, D_h(\lambda)) \).

Now, \( U_h(\mathfrak{g}) = U(\mathfrak{g})/(h) \) as follows: we have a map \( U(\mathfrak{g}) \rightarrow U_h(\mathfrak{g}) \) by sending \( x \in \mathfrak{g} \) to \( h^{-1}x \) in \( U_h(\mathfrak{g}) \); it is easy to see that this map is an isomorphism onto the subalgebra generated by \( h^{-1}\mathfrak{g} \). Then we extend this map to \( U(\mathfrak{g})[[h]] \) by sending \( h \) to \( h \) to achieve the above isomorphism.

This allows us to relate the traditional sheaves of twisted differential operators (as defined in [M]) to the sheaves that we have defined. So, let \( U \subseteq X \) be an open subset, and let \( V \subseteq T^*X \) be the inverse image of \( U \) under the natural projection. Then \( D_h(\lambda)(V) = O_v [[h]] \hat{\otimes}_{\mathbb{C}[[h]]} U_h(\mathfrak{g})(0)[h^{-1}] \), while \( D(\lambda)(U) = O_v \otimes_{U(\mathfrak{g})} \mathfrak{g} \). So we get a map \( D(\lambda)(U) \rightarrow D_h(\lambda)(V) \) via the above map \( U(\mathfrak{g}) \rightarrow U_h(\mathfrak{g}) \) and the inclusion \( O_X \rightarrow O_X[[h]] \).

So, if we consider the restriction of \( \Phi_{\lambda} \) to \( U(\mathfrak{g}) \), we obtain a morphism \( \Phi_{\lambda} : U(\mathfrak{g}) \rightarrow \Gamma(T^*X, D(\lambda)) \).

Now, by the results in [M], we have that the kernel of \( \Phi_{\lambda} \) is the ideal \( U(\mathfrak{g})I_{\lambda} \), where \( I_{\lambda} \) is ideal in the center of \( U(\mathfrak{g}) \) corresponding to \( \lambda \) (here we use the fact that \( \text{Spec}(Z(\mathfrak{g})) = \mathfrak{g}/W \); see [M] for details), so that we have \( \Gamma(T^*X, D(\lambda)) = U(\mathfrak{g})/U(\mathfrak{g})I_{\lambda} := U(\mathfrak{g})_{\lambda} \). Therefore, we see that the kernel of \( \Phi_{\lambda} \) contains the ideal \( J_{\lambda} = I_{\lambda}(h) \), and the kernel of \( \Phi_{\lambda}(0) \) contains \( J_{\lambda}(0) := J_{\lambda} \cap U_h(\mathfrak{g})(0) \).

Now we wish to actually calculate the image and kernel of \( \Phi_{\lambda} \). We start by recalling a very general fact:

**Lemma 6.1.** Let \( Z \) be an algebraic variety and let \( O_h \) be a \( \mathbb{C}[[h]] \)-algebra on \( Z \) that is an \( h \)-complete, \( h \)-flat deformation of \( O_Z \). Suppose that \( R^i\Gamma(O_Z) = 0 \) for all \( i > 0 \). Then \( R^i\Gamma(O_h) = 0 \) for all \( i > 0 \) and \( \Gamma(O_h)/h \rightarrow \Gamma(O_Z) \). Further, \( \Gamma(O_h) \) is itself \( h \)-flat and \( h \)-complete.

**Proof.** The vanishing of \( R^i\Gamma(O_h) \) follows from the vanishing of \( R^i\Gamma(O_Z) \) by [KR], lemma 2.12. Then, from the short exact sequence

\[
0 \rightarrow O_h \xrightarrow{h} O_h \rightarrow O_Z \rightarrow 0
\]

and the vanishing of \( R^1\Gamma(O_h) \), we see that \( \Gamma(O_h)/h \rightarrow \Gamma(O_Z) \). Finally, applying [KS], corollary 1.6.2, we see that \( R^i\Gamma(O_h) = R\text{Hom}(O_h, O_h) \) is a cohomologically complete complex of \( \mathbb{C}[[h]] \)-modules. Since it is concentrated in a single homological degree and \( h \)-torsion free (since \( O_h \) is), this implies that it is complete in the \( h \)-adic topology by [KS], lemma 1.5.4.

□

From this we conclude

**Lemma 6.2.** The induced map \( \Phi_{\lambda}(0) : U_h(\mathfrak{g})(0)/J_{\lambda}(0) \rightarrow \Gamma(T^*X, D_h(\lambda)(0)) \) is an isomorphism of algebras.

**Proof.** By definition we have \( U_h(\mathfrak{g})(0)/h \cong S^\bullet(\mathfrak{g}) \), the symmetric algebra of \( \mathfrak{g} \), and we have (c.f. [HTT], proposition 11.3.1) that the image of \( J_{\lambda}(0) \) in \( S^\bullet(\mathfrak{g}) \) is exactly \( I(N) \), the ideal of the nilpotent cone. Therefore

\[
(U_h(\mathfrak{g})(0)/J_{\lambda}(0))/h \cong S^\bullet(\mathfrak{g})/I(N)
\]
bundles. We formulate this twisting by using the Hamiltonian reduction definition of differential

Combining the two functors, we get equivalences of categories

chapter 3, section 5). The key to the argument will be the twisting of Serre’s theorems about ample bundles on projective varieties go through in this case (see [H],

given by

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functor (i.e., \(\Gamma(\mathcal{M})\)) = 0 implies \(\Gamma(M) = 0\) implies \(M = 0\).

The inverse functor is given by the localization of modules: \(M \to D_h(\lambda) \otimes U_{h,\lambda} M\). Here \(U_{h,\lambda}\) represents the constant sheaf on \(T^*X\), and \(M\) as well. \(D_h(\lambda)\) is a module over \(U_{h,\lambda}\) via the identification of global sections.

The proof of this theorem will follow from similar considerations to the classical case. To begin, recall that for each \(\psi \in \mathfrak{b}^*\) which comes from a character of \(\mathfrak{h}^*\), we have the induced line bundle \(O_\psi\) on \(X\). We choose a normalization so that the anti-dominant weights correspond to ample line bundles. By abuse of notation, we shall also denote by \(O_\psi\) the line bundle \(\pi^*O_\psi\) on \(T^*X\). For anti-dominant \(\psi\), these bundles are ample over the base scheme \(N\), as can easily be seen by looking at the morphism to projective space over \(N\) corresponding to \(O_\psi\). Since \(N\) is an affine variety, all of Serre’s theorems about ample bundles on projective varieties go through in this case (see [H], chapter 3, section 5). The key to the argument will be the twisting of \(D_h(\lambda)\) modules by these line bundles. We formulate this twisting by using the Hamiltonian reduction definition of differential operators, following [KR].

In particular, as discussed in the previous section, we have equivalences of categories

\begin{align*}
\text{Mod}^{qc}(D_h(\lambda)) & \sim \text{Mod}^{\mathcal{B},qc}(D_h(T^*G)) \\
\text{Mod}^{coh}(D_h(\lambda)) & \sim \text{Mod}^{\mathcal{B},coh}(D_h(T^*G))
\end{align*}

On \(T^*G\), we have, if \(V\) is any finite dimensional \(\mathcal{B}\)-module, the twist functor

\[\text{Mod}^{\mathcal{B},?}(D_h(T^*G)) \to \text{Mod}^{\mathcal{B},?}(D_h(T^*G))\]

given by \(M \to M \otimes V\) (where \(?\) stands for either coherent or quasi-coherent). In the case where \(V = \mathbb{C}_\psi\), the latter functor is an equivalence of categories

\[\text{Mod}^{\mathcal{B},?}(D_h(T^*G)) \sim \text{Mod}^{\mathcal{B},?}(D_h(T^*G))\].

Combining the two functors, we get equivalences of categories

\[\text{Mod}^\lambda(D_h(\lambda)) \sim \text{Mod}^\lambda(D_h(\lambda + \psi)),\]
and we shall refer to this functor as $F_{\psi}$ (in both the coherent and quasicoherent cases, with the character $\lambda$ being understood). We can describe this functor directly as follows: we denote the quotient morphism by $p: \mu^{-1}(0) \to T^*(G/B)$. Then, we define $V_{\psi} := \mathbb{C}_{\psi} \otimes O_{T^*G}|_{\mu^{-1}(0)}$, with its $B$-equivariant structure defined by the representation $\mathbb{C}_{\psi}$. Then we have

Claim 6.6. The sheaf $p_*(V_{\psi})^B \cong O_{\psi}$.

Proof. To check this, it suffices to show that $p^*(O_{\psi}) \cong V_{\psi}$ (as we are dealing with $B$-equivariant sheaves, and $p$ is a $B$-principal bundle morphism). To check this, it suffices to take the line bundle $O_{\psi}$ on $G/B$, pull back to $G$, and then pull back to $\mu^{-1}(0)$. But the pullback of $O_{\psi}$ to $G$ is the sheaf $\mathbb{C}_{\psi} \otimes O_G$, by the definition of the induced bundle. This proves the claim. \hfill $\Box$

So, given a module $M(0)$ in $Mod_{\chi}^{\text{coh}}(D_h(\lambda)(0))$, our functor $F_{\psi}$ is given by

$$M(0) \to p_*Hom_{D_h(T^*G)}(L_\lambda(0), L_\lambda(0) \otimes_{p^{-1}D_h(\lambda)(0)} p^{-1}M(0) \otimes \mathbb{C}_{\psi})^B$$

Thus, we see that

$$F_{\psi}(M(0))/hF_{\psi}(M(0)) \cong M(0)/hM(0) \otimes O_{T^*G}$$

using the fact that the functor

$$M(0) \to p_*Hom_{D_h(T^*G)}(L_\lambda(0), L_\lambda(0) \otimes_{p^{-1}D_h(\lambda)(0)} p^{-1}M(0))$$

is just the identity (see above).

We can also consider the twist functor in the case of the $B$ module $L(\nu)$, where $L(\nu)$ is the irreducible $G$-module of highest weight $\nu$ (where $\nu$ is supposed to be dominant integral). This gives a functor $G_{\nu} : Mod^B(D_h(h))(0) \to Mod^B(D_h(h))(0)$ (and, of course, a $G_{\nu} : Mod^B(D_h(h))(0) \to Mod^B(D_h(h))(0)$), which, however, does not map a subcategory of the type $Mod^B(D_h(\lambda)(0))$ to another, because the module $L(\nu)$ does not have a $B$-character.

On the other hand, we have, by standard weight theory, a finite $B$-filtration of $L(\nu)$, $\{L(\nu)_i\}$, such that the subquotients $L(\nu)_i/L(\nu)_{i-1}$ are one dimensional $B$-modules. We now let $M(0) \in Mod_{\chi}^{\text{coh}}(D_h(\lambda)(0))$. If we twist $M$ by $L(\nu)$; then the result is

$$M(0) \to p_*Hom_{D_h(T^*G)}(L_\lambda(0), L_\lambda(0) \otimes_{p^{-1}D_h(\lambda)(0)} p^{-1}M(0) \otimes L(\nu))^B := G_{\nu}(M(0))$$

Now, because the module $L(\nu)$ has a $G$-action, the sheaf $p_*(L(\nu) \otimes O_{T^*G}|_{\mu^{-1}(0)})^B$ is actually a trivial vector bundle over $T^*X$. So in this case we conclude that $G_{\nu}(M(0)) \cong M(0) \otimes L(\nu)$; i.e., it is simply a finite direct sum of copies of $M(0)$.

Then, we have a filtration on $G_{\nu}(M(0))$, $\{G_{\nu}(M(0))_i\}$, induced from that on $L(\nu)$; this is a filtration of $D_h(h)(0)$-modules. The important point is the following: the subquotients of this filtration $G_{\nu}(M(0))_i/G_{\nu}(M(0))_{i-1}$ are isomorphic to the sheaf

$$p_*Hom_{D_h(T^*G)}(L_\lambda(0), L_\lambda(0) \otimes_{p^{-1}D_h(\lambda)(0)} p^{-1}M(0) \otimes \mathbb{C}_{\psi})^B := F_{\psi}(M(0))$$

And, of course, the same isomorphism holds after inverting $h$ everywhere.

Now, if we restrict our attention to the form $U(g)$ described above (the one generated by elements of the form $h^{-1}x$ for $x \in g$), then we have that the ideal $I_{\lambda+\nu}$ acts trivially on $F_{\psi}(M)$. If we associate to each $I_\lambda$ the central character $\chi_\lambda$, then we have that for all $\xi \in Z(g)$, the product $\Pi_\lambda(\xi - \chi_{\lambda+\nu}(\xi))$ annihilates $G_{\nu}(M)$. Therefore, we can write $G_{\nu}(M) \cong \bigoplus G_{\nu}(M)|_{\Psi}$ a direct sum of generalized $Z(g)$-eigensheaves.

Repeating the proof of [M] (lemma 1, pg 24) verbatim, we can conclude the following

Lemma 6.7. Let $\lambda$ be an anti-dominant weight, and $\mu$ a dominant integral weight and let $M \in Mod^B(D_h(\lambda))$. Then we have that, $M \cong G_{\mu}F_{\mu}(M)|_{\chi\lambda}$. Therefore, we see that $M$ is a direct summand of $F_{\mu}(M) \otimes L(\mu)$. Further, let $w_0$ denote the longest element of the Weyl group. Then the sheaf $F_{\mu}(M)$ is a direct summand of $G_{-w_0\mu}(M) \cong M \otimes L(-w_0\mu)$. 
Now we can give the

Proof. (of 6.35) We first handle exactness. We note that any \( M \in Mod^{qc}(D_h(\lambda)) \) is a direct limit of coherent \( D_h(\lambda) \)-modules (see section 2) and that cohomology commutes with direct limits on a noetherian space. So WLOG \( M \in Mod^{coh}(D_h(\lambda)) \) with \( \lambda \) anti-dominant, and with \( M(0) \) a lattice. Then \( M(0)/hM(0) \) is a coherent sheaf on \( T^*X \), and by Serre’s theorem, there exists \( \mu >> 0 \) so that \( H^i(\Omega_{T^*X}, M(0)/hM(0) \otimes O_{-\mu}) = 0 \) for all \( i > 0 \). Further, we know that \( M(0)/hM(0) \otimes O_{-\mu} = F_{-\mu}(M(0))/hF_{-\mu}(M(0)) \).

Now, by \( 5.6 \) we have that \( H^i(T^*X, F_{-\mu}(M(0))) = 0 \) for all \( i > 0 \). Therefore we conclude that \( H^i(T^*X, F_{-\mu}(\Sigma)) = 0 \) for all \( i > 0 \) as \( F_{-\mu}(M) = F_{-\mu}(M(0))[h^{-1}] \).

But now, by \( 6.7 \) we have an injection

\[ H^i(T^*X, M) \to H^i(T^*X, F_{-\mu}(M) \otimes L(\mu)) = H^i(T^*X, F_{-\mu}(M)) \otimes L(\mu) = 0 \]

for all \( i > 0 \). So exactness is shown.

We now show that \( \Gamma(T^*X, M) = 0 \) for \( M \in Mod^{qc}(D_h(\lambda)) \). Our assumption is that we have that \( 0 = \Gamma(T^*X, M(0))[h^{-1}] \), which implies that for each global section \( s \), there exists some \( n \geq 1 \) such that \( h^n s = 0 \). Now, we define, for each \( i > 1 \), the subsheaf \( M(i)_0 \), which is the sheaf of local sections of \( M(0) \) which are annihilated by \( h^i \). Then the theorem becomes equivalent to showing \( M(0) = \cup_i M(0)_i \). If not, we consider the quotient sheaf \( N(0) = M(0)/\cup_i M(0)_i \). Then \( N(0) \) is a nontrivial \( D_h(\lambda)(0) \)-module.

We note that by definition, \( M = N(0)[h^{-1}] \). Therefore, \( \Gamma(T^*X, N) = 0 \). Further, the construction of \( N(0) \) implies that the natural map \( N(0) \to N \) is injective (i.e., there are no local sections that are killed by a power of \( h \)). So we see that it suffices to show \( N = 0 \).

Now, there exists some dominant \( \mu \) such that \( \Gamma(T^*X, F_{-\mu}(N(0))/hF_{-\mu}(N(0))) \neq 0 \) by Serre’s theorem’s about ample line bundles (we note that the assumption that \( N(0) \neq 0 \) implies \( N(0)/hN(0) \neq 0 \) by the Nakayama lemma). This implies (by \( 3.4 \) that \( \Gamma(T^*X, F_{-\mu}(N(0))) \neq 0 \). In turn, the module \( F_{-\mu}(N(0)) \) injects to \( F_{-\mu}(N) \) as \( F_{-\mu}(N(0)) \) has no local sections which are killed by a power of \( h \) (this follows from the corresponding fact about \( N(0) \)).

Now, let \( w_0 \) denote the longest element of the Weyl group. Given a dominant weight \( \mu \), we have an injection \( F_{-\mu}(N) \to G_{-w_0\mu}(N) \) (see lemma 5.7). Therefore, the fact that \( 0 \neq \Gamma(T^*X, F_{-\mu}(N)) \) implies

\[ 0 \neq \Gamma(T^*X, G_{-w_0\mu}(N)) = \Gamma(T^*X, N) \otimes L(-w_0\mu) = 0, \]

a contradiction. \( \square \)

We now discuss the \( \mathbb{C}^* \)-equivariance conditions that need to be imposed. The above theorem deals with categories of modules defined over the field \( \mathbb{C}(h) \), whereas the original localization theorem deals with the \( \mathbb{C} \)-linear category of \( U(\mathfrak{g})_\lambda \)-modules. We now show how to recover the original theorem from the one above.

First of all, we have canonical \( \mathbb{C}^* \)-actions on both \( U_{h,\lambda} \) and \( D_h(\lambda) \): for \( U_{h,\lambda} \) we let \( \phi_1(h) = t^{-1}h \) and \( \phi_1(g) = tg \) for all \( g \in \mathfrak{g} \). This is the standard \( \mathbb{C}^* \) action on \( U_{h}(\mathfrak{g}) \) and it induces one on \( U_{h,\lambda} \). For \( D_h(\lambda) \) we start with the sheaf \( D_h \) on \( T^*G \), and consider the action of \( \mathbb{C}^* \) by dilation of the fibers. \( D_h \) is equivariant with respect to this action by setting \( \psi_1(h) = t^{-1}h \), \( \psi_1(\xi) = t\xi \) where \( \xi \) is any global vector field. It is easy to observe that this action preserves the set \( \mu^{-1}(0) \subseteq T^*G \) and commutes with the action of \( B \) on the right. Thus we see that this gives rise to a \( \mathbb{C}^* \)-action on \( T^*X \) with respect to which all the sheaves \( D_h(\lambda) \) are equivariant.

Now we can make some observations about these actions: first, \( U_{h,\lambda}^\mathbb{C}^* \cong U(\mathfrak{g})_\lambda \). This follows from the fact that \( U_{h}(\mathfrak{g})^\mathbb{C}^* \cong U(\mathfrak{g}) \), which is simply the identification of \( U(\mathfrak{g}) \) with the subalgebra of \( U_{h}(\mathfrak{g}) \) generated by \( h^{-1}\mathfrak{g} \) (that these are the \( \mathbb{C}^* \)-fixed elements follows immediately from the description of the \( \mathbb{C}^* \) action given above).
Next, we can observe that, for an open subset $U \subseteq G/B$, if $V = \pi^{-1}(U)$, we have $D_h(\lambda)(V)^{C^*} \cong D(\lambda)(U)$ (by the same reasoning as the above). We can in fact make the stronger statement that we have an equivalence of categories: $\text{Mod}^{f.g.}(\mathbb{C})^\lambda(D_h(\lambda)) \cong \text{Mod}^{f.g.}(\mathbb{C})^\lambda(D(\lambda))$, where the left-hand side denotes the category of $C^*$-equivariant coherent $D_h(\lambda)$ modules. This equivalence is given by taking $C^*$-invariant sections.

Given all this, the statement of the final theorem (the original Beilinson-Bernstein localization) is intuitively clear:

**Theorem 6.8.** For $\lambda$ anti-dominant, we have an equivalences of categories:

$$
\text{Mod}^{f.g.}(U(\mathfrak{g})_\lambda) \cong \text{Mod}^{f.g.}(\mathbb{C})^\lambda(D_h(\lambda)) \cong \text{Mod}^{f.g.}(\mathbb{C})^\lambda(D(\lambda))
$$

**Proof.** This proof follows the mechanics of the previous argument (which we will use along the way). In one direction, we have the functor $M \rightarrow \Gamma(M)^{C^*}$ which takes $C^*$-equivariant coherent $D_h(\lambda)$-modules to $U(\mathfrak{g})_\lambda$ modules. We wish to show that its image lives inside the category of finitely generated $U(\mathfrak{g})_\lambda$ modules. (This argument is more or less standard, but the presence of the $C^*$-action requires some care). To do so, we first to show that $\Gamma^{C^*}$ is an exact and conservative functor. The exactness is clear from the exactness of $\Gamma$ as taking invariants for a $C^*$-action is exact. To show that it is conservative, we again only need to show that taking $C^*$ invariants is conservative; this follows from our discussion of $C^*$-actions in \[3\] (we note that the discussion goes through in this case, as we are taking the invariants functor on the category of $U_{h,\lambda}$-modules). Therefore, we conclude that every $C^*$-equivariant coherent $D_h(\lambda)$-module $M$ is generated by $C^*$-invariant global sections: let $N$ be the sub-$D_h(\lambda)$-module of $M$ generated by the $C^*$-invariant global sections. Then we have the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$, applying our exact functor shows that $\Gamma(M/N)^{C^*} = 0$, so $M/N$ is 0 as required.

To complete the argument about finite generation, we note that our module $M$ is locally finitely generated: for any affine open covering of $T^*X$, $\{U_i\}$, we have that $M|_{U_i}$ is a finitely generated $D_h(\lambda)|_{U_i}$-module. Now, we choose an affine, open, finite $C^*$-invariant cover of $T^*X$ (one can always do this for a normal variety with a $C^*$-action, although in this case it is obvious as we can just take an affine cover of $X$ and pull back to $T^*X$). Then for each $M|_{U_i}$, we have that $(M|_{U_i})^{C^*}$ is finitely generated as a $D_h(\lambda)^{C^*}|_{U_i}$-module by \[3\] By the above, we can choose finitely many $C^*$-invariant global sections that restrict to generators of $(M|_{U_i})^{C^*}$. By the finiteness of the cover, we have found finitely many global sections which generate the $D_h(\lambda)^{C^*}$-module $M^{C^*}$. Therefore, these elements generate the $U(\mathfrak{g})$-module $\Gamma(M)^{C^*}$.

Now, the functor in the opposite direction is given by $V \rightarrow D_h(\lambda) \otimes_{U(\mathfrak{g})_\lambda} V$. This is clearly a (quasi)coherent, $C^*$-equivariant $D_h(\lambda)$-module (with the $C^*$-action given via the one on $D_h(\lambda)$). Now the proof that these two functors are inverse is totally standard. \qed

Our goal in the rest of this section is to explain how localization works when one replaces the usual $C^*$-action with the action that one needs to study the finite $W$-algebras. We note that the above proof doesn’t depend on the particular $C^*$-action, but that both the algebra of invariants and the sheaf of invariant operators do.

Fix a nilpotent element $e \in \mathfrak{g}$. As recalled above in \[2\] there is a natural homomorphism $\gamma : \mathbb{C}^* \rightarrow G$, which leads to a homomorphism $\rho : \mathbb{C}^* \rightarrow \mathbb{C}^* \times G$, defined as

$$\rho(t) = (t^{-2}, \gamma(t))$$

Now, for any $\lambda$ as above, the sheaf $D_h(\lambda)$ is equivariant over $\mathbb{C}^* \times G$, where the $\mathbb{C}^*$ action is the one described above, and the $G$ action is the natural one on differential operators. Therefore, we may restrict this action to $\mathbb{C}^*$ via $\rho$ to obtain our new $\mathbb{C}^*$ action on $D_h(\lambda)$. 
Via the adjoint action of $G$ on $\mathfrak{g}$, and the natural $\mathbb{C}^*$ action on $U_h(\mathfrak{g})$ via the grading, we obtain an action of $\mathbb{C}^* \times G$ on $U_h(\mathfrak{g})$. Again restricting via $\rho$ we obtain our required $\mathbb{C}^*$-action on $U_h(\mathfrak{g})$.

It is worth describing this action explicitly: we have the decomposition $\mathfrak{g} = \oplus \mathfrak{g}(i)$ which was the weight decomposition for our chosen $\mathfrak{sl}_2$-triple. Then for $g \in \mathfrak{g}(i)$, we put $\sigma_\rho(g) = t^{i+2}g$, and we let $\sigma_t(h) = t^2h$, and extend this to all of $U_h(\mathfrak{g})$ in the natural way. This corresponds to the Kazhdan filtration on $U(\mathfrak{g})$.

Because $h$ has degree 2, we work from now on with the extended ring $U_h(\mathfrak{g}) \otimes_{\mathbb{C}(h)} \mathbb{C}(h^{1/2})$, and we similarly extend the sheaf $D_h(\lambda)$.

**Lemma 6.9.** This action preserves the ideal $J_\lambda \subset U_h(\mathfrak{g})$.

**Proof.** To show this, we describe a generating set for $J_\lambda$ as follows: the Killing form is a perfect pairing between $\mathfrak{g}(i)$ and $\mathfrak{g}(-i)$. We choose bases in these spaces which are dual to each other; this then gives a basis of $\mathfrak{g}$, for a basis element $X_i$ we let $\hat{X}_i$ denote its dual element. Let $\phi$ be any finite dimensional representation of $\mathfrak{g}$. According to [Kn] (prop 5.32, proof of theorem 5.44), a generating set for the ideal $I_\lambda \subseteq U(\mathfrak{g})$ is given by elements of the form

$$\sum_{i_1, \ldots, i_n} Tr(\lambda X_{i_1} \cdots \lambda X_{i_n}) (\lambda(\hat{X}_{i_1}) \cdots \lambda(\hat{X}_{i_n})).$$

Therefore, we conclude that a generating set for $J_\lambda$ is given by

$$\sum_{i_1, \ldots, i_n} h^{-n}Tr(\lambda X_{i_1} \cdots \lambda X_{i_n}) (\lambda(\hat{X}_{i_1}) \cdots \lambda(\hat{X}_{i_n})).$$

Now, the only way that $\sigma_t(h)$ acts non-trivially in $U_h(\mathfrak{g})$ is traceless. Now, since $\hat{X}_i \in \mathfrak{g}(j_i)$, we let $\sum_{i=1}^n j_i = 0$: this follows from the fact that the representation $\phi$ inherits a grading from the same $\mathfrak{sl}_2$-action; and any matrix that shifts the grading non-trivially is traceless. Now, since $\hat{X}_i$ lives in degree $-j_i$, it must be that the element $\hat{X}_i \cdots \hat{X}_n$ also has degree 0 with respect to this $\mathfrak{sl}_2$ action. By the definition of the $\mathbb{C}^*$-action we are working with, we see that $h^{-j_i+2} \hat{X}_i$ is $\mathbb{C}^*$-invariant, and so it follows that the generating set considered above is in fact $\mathbb{C}^*$-invariant; and so, therefore, is the ideal $J_\lambda$. □

We consider now the ring of invariants with respect to this action. Clearly, this ring consists of series, infinite in positive powers of $h$, whose terms are products of elements of the form $h^{-(i+2)/2}g$ with $g \in \mathfrak{g}(i)$. Therefore, this ring is not isomorphic to the enveloping algebra $U(\mathfrak{g})$ in particular, it will include infinite series whose terms come from $\oplus_{i \leq -3} \mathfrak{g}(i)$ (which, we note, is a subalgebra of $\mathfrak{m}_i$), and in fact, it is clear that this algebra is the completion of $U(\mathfrak{g})$ with respect to the nilpotent Lie subalgebra $\oplus_{i \leq -3} \mathfrak{g}(i)$ (one can consult [C2] section 5 for details on this notion of completion; however, we will not use this). Therefore, it follows from our computation of the global sections of $D_h(\lambda)$ above that $\Gamma(T^*X, D_h(\lambda)^{\mathbb{C}^*}) \cong U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$. To quantify this, we consider the copy of $U(\mathfrak{g}) \subseteq U_h(\mathfrak{g})^{\mathbb{C}^*}$ (just the algebra generated by $h^{-(i+2)/2}g$ for $g \in \mathfrak{g}(i)$); and we note that $J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$ is generated by the elements given in the proof of [LM] which are simply generators for the ideal $I_\lambda \subseteq U(\mathfrak{g})$. So the ideal $J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$ is the ideal in $U_h(\mathfrak{g})^{\mathbb{C}^*}$ generated by $I_\lambda$.

With this in hand, we can repeat verbatim the proof of [LM] and obtain

**Theorem 6.10.** For $\lambda$ anti-dominant, we have equivalences of categories

$$Mod^{f-g}(U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}) \cong Mod^{coh, \mathbb{C}^*}(D_h(\lambda))$$

$$Mod(U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}) \cong Mod^{p-c, \mathbb{C}^*}(D_h(\lambda))$$

On the face of it, this theorem is not very useful, because of our lack of knowledge of the category appearing on the left. However, this category becomes quite tractable after one additional modification: we have the adjoint action of the group $M_t$ on the algebra $U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$, and we can consider the category $Mod^{M_t,f-g}(U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*})$ of $\chi$-twisted $M_t$-equivariant finitely generated modules. It is easy to see that this is just the category of modules $V$ such that for all $m \in \mathfrak{m}_t$, $m-\chi(m)$ acts locally nilpotently on $V$. Now, by definition, $\chi|_{\oplus_{i \leq -3} \mathfrak{g}(i)} = 0$. Therefore, for a module in $Mod^{M_t,f-g}(U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*})$, all of the infinite series in the ring $U_h(\mathfrak{g})^{\mathbb{C}^*}/J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$ simply...
act via finitely many terms. Therefore, combining this with our above description of $J_\chi \cap U_h(\mathfrak{g})^{C^*}$, we see that we have a canonical equivalence of categories
\[ \text{Mod}_{M_t,f.g.}(U_h(\mathfrak{g})^{C^*})/J_\chi \cap U_h(\mathfrak{g})^{C^*} \rightarrow \text{Mod}_{\chi,f.g.}(U(\mathfrak{g})) \]

So, a localization theorem for this category would need to consider $M_t$-equivariant $D_h(\lambda)$-modules. It is clear, by restriction of the $G$-action, that there is an $M_t$-equivariant structure on the algebra $D_h(\lambda)$. Unfortunately, the above $C^*$-action and the $M_t$ action do not commute. However, we can express the structure we want by looking at the adjoint action of the one parameter group $C^*$ on the group $M_t$, via the morphism $\gamma(t)$. This allows us to form the semi-direct product $M_t \rtimes C^*$. Then, adding an $M_t$-equivariance condition to the category on the right of the above theorem is the same as looking $D_h(\lambda)$-modules that are equivariant with respect to $M_t \rtimes C^*$. If we consider $\chi$ as a map $\chi : m \rightarrow \mathbb{C}[[h]]$, then we can consider the category of $M_t \rtimes C^*$-equivariant modules over $D_h(\lambda)$ so that the $M_t$ action has twist $\chi$, in the sense of $\chi$.

For any module in $M \in \text{Mod}_{M_t \times C^*, coh}(D_h(\lambda))$, its $C^*$-invariant global sections are a module over $U_h(\mathfrak{g})^{C^*}$. The condition that $M$ be $M_t$-equivariant with twist $\chi$ ensures that $\Gamma(M)^{C^*}$ admits an $M_t$-action which integrates the $\chi$-twisted action of $m \in U_h(\mathfrak{g})^{C^*}$, where the embedding is chosen as above; i.e., taking $m \in m_\chi$ to $h^{-(1/2)}m$. So, combining the above observations with the $D_h(\lambda)$-affineness of $T^*X$ gives:

**Theorem 6.11.** For $\lambda$ anti-dominant, we have equivalences of categories
\[ \text{Mod}_{\chi,f.g.}(U(\mathfrak{g})_{\lambda}) \rightarrow \text{Mod}_{\chi,f.g.}(D_h(\lambda)) \]
\[ \text{Mod}_{\chi}(U(\mathfrak{g})_{\lambda}) \rightarrow \text{Mod}_{\chi}(D_h(\lambda)) \]

7. Localization for $W$-Algebras and the Skryabin Equivalence

In this section we weave together the major threads of the paper and prove the main results. We begin by applying the Hamiltonian reduction formalism of $\chi$ to the case of semisimple Lie algebras and flag varieties.

We let $O_{h^*}$ denote the sheafification of $U_h(\mathfrak{g})(0)$ over the variety $\mathfrak{g}^*$. This is a quantization of a smooth Poisson variety, which admits an action of $G$. In particular, after fixing a nilpotent element $e \in \mathfrak{g}$, we obtain the action of $M_t$ on $O_{h^*}$; as detailed in the previous section, we have also an action of $M_t \rtimes C^*$. The natural map $U_h(m_\chi)(0) \rightarrow U_h(\mathfrak{g})(0)$ upon sheafification gives a comoment map $U_h(m_\chi)(0) \rightarrow O_{h^*}$. Consider the character on $U_h(m_\chi)(0)$ determined by $\chi : m_\chi \rightarrow \mathbb{C}$, which we will also denote it by $\chi$. Therefore we can consider $M_t \rtimes C^*$-equivariant modules over $O_{h^*}$, whose $M_t$-action has twist $\chi$. We have

**Lemma 7.1.** The functor $\Gamma^{C^*}$ induces an equivalence of categories
\[ \text{Mod}_{\chi}(O_{h^*}[h^{-1}]) \rightarrow \text{Mod}_{\chi,f.g.}(U(\mathfrak{g})) \]

The analogous result holds upon replacing $\text{coh}$ with $\text{gcoh}$ on the left and finitely generated modules with all modules on the right.

**Proof.** As $\mathfrak{g}^*$ is an affine variety, the global section functor is an equivalence from $\text{Mod}_{\text{coh}}(O_{h^*})$ to $\text{Mod}_{f.g.}(U_h(\mathfrak{g})(0))$, and the same holds upon inverting $h$. Let $M(0)$ be an $M_t \rtimes C^*$-equivariant module over $U_h(\mathfrak{g})(0)$. Then, as explained above, $M(0)[h^{-1}]^{C^*}$ is a module over a certain completion of $U(\mathfrak{g})$, along an ideal generated by a Lie-subalgebra of $m_\chi$. We thus have the action of $m_\chi$ on $M(0)[h^{-1}]^{C^*}$, and the condition that $M(0)$ has twist $\chi$ ensures that the $\chi$-twisted action of $M_t$ on $M(0)[h^{-1}]^{C^*}$ integrates to an action of $M_t$.

As $M_t$ is a unipotent group, this implies that $m - \chi(m)$ acts locally nilpotently on $M(0)[h^{-1}]^{C^*}$, which implies that this module is actually a $U(\mathfrak{g})$-module, and so we see $\Gamma^{C^*}$ takes values in the correct category.
To obtain the inverse functor, we note that any module in $\text{Mod}_{\chi}^{M_l}((U(\g))$ is necessarily locally finite with respect to the action of $m - \chi(m)$ and so the $U(\g)$-action extends uniquely to the structure of a module over $U_h(\g)^{C^*}$, and therefore it is the $C^*$-invariants of a unique coherent $C^*$-equivariant module over $U_h(\g)$, which necessarily has an action of $M_l$ with twist $\chi$. Localizing this module over $g^*$ yields the required inverse functor. \hfill $\Box$

Now we apply the machinery of [3]

**Proposition 7.2.** The assumptions of [5.3] are satisfied for the action of $M_l$ on $O_{h,g^*}$ with respect to the character $\chi$. Therefore we have a quantization $O_{h,S}$ of $S$ given by Hamiltonian reduction and an equivalence of categories

$$\text{Mod}_{\chi}^{M_l \times C^*,coh}(O_{h,g^*}[h^{-1}]) \cong \text{Mod}_{\chi}^{C^*,coh}(O_{h,S}[h^{-1}])$$

Furthermore, there is an equivalence of categories

$$\text{Mod}_{\chi}^{C^*,coh}(O_{h,S}[h^{-1}]) \cong \text{Mod}_{\chi}^{f,g}(U(\g,e))$$

The analogous statements hold when $coh$ is replaced by $qcoh$.

We will prove this shortly. Note, however, that, when combined with the previous result, it gives the Skryabin equivalence

**Corollary 7.3.** There is an equivalence of categories

$$\text{Mod}_{\chi}^{M_l}(U(\g)) \cong \text{Mod}(U(\g,e))$$

given by $V \mapsto V^{M_l}$. It induces an equivalence on the finitely generated modules on each side.

In fact, it is worth noting here that the group $M_l$ is unnecessary in the category on the left: because it is a connected unipotent group, any module which is locally nilpotent for $m_l$ (with respect to character $\chi$) will carry a unique action of $M_l$. However, it does make the statement of the functor cleaner.

**Proof.** (of [7.2]) The proof of the first statement is rather similar in spirit to the proof of [5.3] Although there is no formal Darboux theorem for Poisson varieties, we do have that the Poisson variety $g^*$ is an affine space, and the Poisson structure comes from a linear Poisson bracket. The moment map $g^* \to m^*_l$ is simply the dual to the inclusion, and the space $\mu^{-1}(\chi)$ is isomorphic to $S \times M_l$ as an $M_l$-variety.

Let $x \in \mu^{-1}(\chi)$. By the above discussion the formal completion of $O_{h,g^*}$ at the point $x$, denoted $\hat{O}_{h,g^*}$, is isomorphic to the completion of $U_h(\g)(0)$ at the ideal generated by $h$ and $\g$. We may therefore choose coordinates, denoted $\{x_i, y_i, z_i\}$ that are a basis of $\g$ such that the $x_i$ span $m_l$, and the $z_i$ correspond to the dual of $m_l$ under the Killing form. Elements of $\hat{O}_{h,g^*}$ are then uniquely represented as series

$$\sum_{i,I,J,K} a_{i,I,J,K} h^i x^I y^J z^K$$

where $a_{i,I,J,K} \in \mathbb{C}$ (here $I, J, K$ represent multi-indices). The image of the comoment map in this formal completion is then simply the set of series for which each term has a nonzero contribution of $x^I$. Thus the quotient by this ideal is $h$-torsion free, and the set of $M_l$-invariants in this quotient is $h$-torsion free and quantizes a formal neighborhood of a point in $S$. It follows that the first two conditions of [5.3] are satisfied. For the final condition (the Ext vanishing), we have that the $x_i$ and the $z_i$ are dual bases with respect to the Poisson bracket. This allows one to again reduce the question to the vanishing of de Rham cohomology for affine space (exactly as in [GG], proposition 5.1, or the very similar argument in [BDMN], lemma 4.14) and the result follows.
Finally let us give the equivalence

\[ \text{Mod}^{C^*:\text{coh}}(O_{h,S}[h^{-1}]) \rightarrow \text{Mod}^{f,g}(U(\mathfrak{g},e)) \]

As \( S \) is affine we have

\[ \text{Mod}^{C^*:\text{coh}}(O_{h,S}[h^{-1}]) \rightarrow \text{Mod}^{C^*:f,g}(\Gamma(O_{h,S}[h^{-1}])) \rightarrow \text{Mod}^{f,g}(\Gamma(O_{h,S}[h^{-1}])) \]

and, as the \( C^* \)-action on \( S \) contracts \( S \) to the point \( \{e\} \), we have that associated filtration on \( \Gamma(O_{h,S}[h^{-1}])C^* \) is concentrated in positive degrees, and the associated graded of this filtration is therefore isomorphic to \( \Gamma(O_{h,S})/h\Gamma(O_{h,S}) \). Applying (6.1) to \( O_{h,S} \) we have that

\[ \Gamma(O_{h,S})/h\Gamma(O_{h,S}) \rightarrow \Gamma(O_{h,S}/h) = \Gamma(O_S) \]

On the other hand, from the definition of \( U(\mathfrak{g},e) \) as a Hamiltonian reduction, there clearly exists a map \( U(\mathfrak{g},e) \rightarrow \Gamma(O_{h,S}[h^{-1}]) \) deduced from the map \( U(\mathfrak{g}) \rightarrow \Gamma(O_{h,g}[h^{-1}]) \), which takes an element \( g \in \mathfrak{g}(i) \) to \( h^{-(i+2)}/2g \) (as discussed below (6.9)). This map has image contained in the \( C^* \)-fixed locus, and thus the map \( U(\mathfrak{g},e) \rightarrow \Gamma(O_{h,S}) \) actually lands in \( \Gamma(O_{h,S}[h^{-1}])C^* \). Taking associated graded with respect to the induced filtrations on each side, we obtain a map

\[ \text{gr}(U(\mathfrak{g},e)) \rightarrow \Gamma(O_S) \]

which, from the very construction of \( U(\mathfrak{g},e) \), must be the identity map. Therefore the map \( U(\mathfrak{g},e) \rightarrow \Gamma(O_{h,S}[h^{-1}])C^* \) is an isomorphism, and the result follows. \( \square \)

Now that this has been done, we aim describe our localization theorem for the finite \( W \)-algebras, and give several proofs of it. We first need to recall some of the relevant geometry. Let \( e \in N \). Letting \( S \) denote the Slodowy slice as above, we have the singular algebraic variety \( S \cap N := S_e \), where \( N \) is the nilpotent cone in \( \mathfrak{g} \). Then, if \( \mu : T^*X \rightarrow N \) is the Springer resolution, we have that \( S_e := \mu^{-1}(S_e) \rightarrow S_e \) is also a resolution of singularities. In particular, \( e \) is a regular value for the moment \( \mu \), a crucial fact for our considerations (c.f. [G1], chapter 2, for a detailed proof).

We shall realize this resolution as a Hamiltonian reduction of the left action of the group \( M_l \) on the space \( T^*X \). We think of \( \mu : T^*X \rightarrow \mathfrak{g}^* \) (using the original definition of \( \mu \) as a moment map), and we note that the moment map for \( M_l, \mu' \), is given by the composition \( T^*X \rightarrow \mathfrak{g}^* \rightarrow \mathfrak{m}_l^+ \), where the second map is the restriction of functions. We consider \( \chi \in \mathfrak{m}_l^+ \). Then, using the alternate description of \( T^*X \) as an incidence variety, we have \( \mu^{-1}(\chi) = \{(x,b) \in \mathfrak{g} \times X | x \in b, x \in (\mathfrak{m}_l^+ + \chi) \cap N\} \), where \( \mathfrak{m}_l^+ \) denotes the annihilator of \( \mathfrak{m}_l \) in \( \mathfrak{g} \) under the Killing form (so this corresponds to those functionals in \( \mathfrak{g}^* \) which die on \( \mathfrak{m}_l \), the kernel of the restriction map \( \mathfrak{g}^* \rightarrow \mathfrak{m}_l^+ \)). But now, according to [GG], we have an isomorphism \( M_l \times S \rightarrow \mathfrak{m}_l^+ + \chi \) which is simply the adjoint action \( (m,s) \rightarrow ad(m)(s) \). Therefore, under the same map, we have an isomorphism \( M_l \times (S_e) \rightarrow (\mathfrak{m}_l^+ + \chi) \cap N \).

Now, the action of \( M_l \) on \( T^*X \) (thinking of \( T^*X \) as an incidence variety), is given as follows:

\[ m(x,b) = (\text{ad}(m)(x), m\mathfrak{m}_l^{-1}) \]

Further, we write any element of \( (\mu')^{-1}(\chi) \) uniquely as \( (\text{ad}(m)(y), b) \) (with \( y \in S_e \)), and we have a map \( (\mu')^{-1}(\chi) \rightarrow \hat{S}_e, (\text{ad}(m)(y), b) \rightarrow (y, m^{-1}bm) \). We see immediately that this map is in fact a principal \( M_l \)-bundle, with \( M_l \times \hat{S}_e \rightarrow (\mu')^{-1}(\chi) \) via \( (m, (y,b)) \rightarrow (\text{ad}(m)(y), m\mathfrak{m}_l^{-1}) \). Therefore we have identified \( \hat{S}_e \) as a Hamiltonian reduction, and, therefore, a symplectic variety. We note that by the results in [SI] (see also [G1]), the moment map \( \hat{S}_e \rightarrow S_e \) is a resolution of singularities, and the base variety \( S_e \) is normal.

The next step is to consider the Hamiltonian reduction of differential operators. We have the action of \( M_l \) on \( T^*X \), and we consider modules with twist \( \chi \) as above. As the variety \( S_e \) is smooth symplectic and \( \mu^{-1}(\chi) \) is a principal \( M_l \)-bundle over it, we see that the conditions of (6.3) (and therefore (6.3)) are satisfied. We can thus apply Hamiltonian reduction to \( D_h(\lambda) \) we obtain a quantization of \( \hat{S} \) which we call \( D_h(\lambda, \chi) \).
Theorem 7.4. For \( \lambda \) anti-dominant, we have equivalences of categories
\[
\text{Mod}^{\mathbb{C}^*,\text{coh}}(D_h(\lambda, \chi)) \rightarrow \text{Mod}^f(U(\mathfrak{g}, e)_\lambda)
\]
\[
\text{Mod}^{\mathbb{C}^*,\text{qcoh}}(D_h(\lambda, \chi)) \rightarrow \text{Mod}(U(\mathfrak{g}, e)_\lambda)
\]
given by \( \Gamma^{\mathbb{C}^*} \).

Proof. By the results above, the category \( \text{Mod}^f(U(\mathfrak{g}, e)_\lambda) \) is equivalent to
\( \text{Mod}^{\mathbb{M}_l,f}(U(\mathfrak{g}, e)_\lambda) \), which, by applying the localization theorem 6.11, is equivalent to \( \text{Mod}^{\mathbb{M}_l \times \mathbb{C}^*,\text{coh}}(D_h(\lambda)) \).

Applying Hamiltonian reduction (i.e., 5.3), we obtain that this category is equivalent to \( \text{Mod}^{\mathbb{C}^*,\text{coh}}(D_h(\lambda, \chi)) \).

The first result follows, and the second then follows by taking ind categories on both sides. \( \square \)

In the final part of this chapter, we’re going to give another proof of this theorem, which has a more geometric flavor. In particular, we shall show that the analogues of several key results in the Beilinson-Bernstein theory also hold in the \( W \)-algebra context, leading to a direct proof of the result.

We start by considering global sections. We define the algebra \( U_h(\mathfrak{g}, e)_\lambda(0) := \Gamma(O_{h,S}) \), where \( O_{h,S} \) is the quantization of \( S \) defined above via Hamiltonian reduction. Let \( U_h(\mathfrak{g}, e) = U_h(\mathfrak{g}, e)(0)[h^{-1}] \).

In the course of proving 7.2 we showed that \( U_h(\mathfrak{g}, e)(0)/h \rightarrow \Gamma(O_S) \) and that \( U_h(\mathfrak{g}, e)^{\mathbb{C}^*} = U(\mathfrak{g}, e) \).

Further, for any character \( \lambda \) of \( \mathfrak{b} \) as considered in the previous chapter, we had the ideals \( J_\lambda \subseteq U_h(\mathfrak{g}) \), which had the property that \( J_\lambda(0)/hJ_\lambda(0) = I(N) \) (\( N \) as usual is the nilpotent cone). So we can consider the image of \( J_\lambda \) in \( U_h(\mathfrak{g}, e) \), called \( B_\lambda \), and we see that \( B_\lambda(0)/hB_\lambda(0) = I(S_\psi) \) (this is implied by the fact that \( M_l \times (S \cap N) \rightarrow (\mathfrak{m}^l + \chi) \cap N \)).

Now, we have a map \( \Psi_\lambda : U_h(\mathfrak{g}, e) \rightarrow \Gamma(S_\psi, D_h(\lambda, \chi)) \), which results from the map \( \Phi_\lambda \) as both sides are defined by Hamiltonian reduction. We are now in a situation completely parallel to that of 6.2 i.e., the technical criterion of 6.1 applies on \( \tilde{S}_\psi \), and so we can conclude

Lemma 7.5. \( \Psi_\lambda \) is surjective, and \( \ker(\Psi_\lambda) = B_\lambda \).

We now give the necessary modifications of the proof of 6.8 so that we may obtain another proof of 7.4. Recall that we have equivalences of categories \( \text{Mod}^f(D_h(\lambda)) \rightarrow \text{Mod}^f(D_h(\lambda + \psi)) \) (where ? stands for coherent or quasicoherent). These equivalences were obtained by first lifting an \( M(0) \in \text{Mod}^f(D_h(\lambda)) \) to an element of \( \text{Mod}^{\mathbb{N}^l,f}(D_h(T^*G)) \), then twisting upstairs, and then pushing back down. Since the \( B \)-action we’re considering on \( T^*G \) is on the right, if we consider categories of the form \( \text{Mod}^{\mathbb{M}_l}(D_h(\lambda)) \), then as the \( M_l \) action is on the left, this process gives us equivalences
\[
\text{Mod}^{\mathbb{M}_l}(D_h(\lambda)) \rightarrow \text{Mod}^{\mathbb{M}_l}(D_h(\lambda + \psi))
\]
Furthermore, we also have equivalences
\[
\text{Mod}^{\mathbb{M}_l}(D_h(\lambda)) \rightarrow \text{Mod}^f(D_h(\lambda, \chi))
\]
Combining these, we get equivalences
\[
\text{Mod}^f(D_h(\lambda, \chi)) \rightarrow \text{Mod}^f(D_h(\lambda + \psi, \chi))
\]
As before, we call the resulting functor \( F_\psi \), and we can give a description of how it acts: if we let \( M(0) \in \text{Mod}^f(D_h(\lambda, \chi)(0)) \), then it follows from the definitions that
\[
F_\psi(M(0))/hF_\psi(M(0)) \cong M(0)/hM(0) \otimes p_*(O_{\psi}|_{\mu^{-1}(\chi)})^{M_l}
\]
But we have that the space \( \tilde{S}_\psi \) is a subscheme of \( \hat{N} \) as well as a Hamiltonian reduction. So, if we consider \( O_{\psi}|_{\tilde{S}_\psi} \), then we have that \( p^*(O_{\psi}|_{\tilde{S}_\psi}) \cong O_{\psi}|_{\mu^{-1}(\chi)} \), since \( O_{\psi} \) is an \( M_l \)-equivariant bundle and \( \mu^{-1}(\chi) \cong M_l \times \tilde{S}_\psi \) (as explained above). So now it follows that \( p_*(O_{\psi}|_{\mu^{-1}(\chi)})^{M_l} \cong O_{\psi}|_{\tilde{S}_\psi} \).

The next step is to define the analogue of the functors \( G_\mu \). This is done in the natural way: for any \( M(0) \in \text{Mod}^f(D_h(\lambda, \chi)) \), we can consider the pullback to a module \( N(0) \in \text{Mod}^{\mathbb{M}_l}(D_h(\lambda)(0)) \); we
then apply the functor $G_\nu$ to obtain $G_\nu(M(0)) \in \text{Mod}^{\omega, M_l}(D_h(\mathfrak{g})(0))$. In order to take the reduction of such a module, we first note that the reduction functor of $\text{mod}^{\omega, M_l}$ (which we only defined for an object of a particular $\text{Mod}(D_h(\lambda)(0))$) can actually be defined as $S \to p_*(\hat{S})$ where $\hat{S}$ is the subsheaf consisting of local sections $m$ of $S$ such that $\xi_x \cdot m = \chi(x)m$ for all $x \in \mathfrak{m}$. So, this functor actually makes sense for any object in $\text{Mod}(D_h(\mathfrak{g}))$. We again call the resulting functor $G_\nu$. We note that $G_\nu(M(0))$ is now just a sheaf of abelian groups. However, as the functor $G_\nu$ (for sheaves on $T^*X$) simply amounted to taking a finite direct sum of copies of the input sheaf, we conclude that the same is true of the new $G_\nu$. So, we obtain:

$$G_\nu(M(0)) \cong M(0) \otimes L(\nu)|_{\hat{S}_e}$$

Finally, since this reduction procedure is (at the very least) an additive functor on sheaves of abelian groups, we can conclude from 6.7.

**Lemma 7.6.** Let $\lambda$ be an anti-dominant weight, and $\mu$ a dominant integral weight, and let $M \in \text{Mod}^{\text{coh}}(D_h(\lambda, \chi))$. Then $G_\mu F_{-\mu}(M)$ has $M$ as a direct summand. Further, let $w_0$ denote the longest element of the Weyl group. Then the sheaf $F_{-\mu}(M)$ is a direct summand of $G_{-w_0\mu}(M) \cong M \otimes L(-w_0\mu)$.

Now, given an anti-dominant weight $\psi$, $O_{\psi}$ is an ample line bundle on $N$ (with respect to the base scheme $N$). Therefore, its restriction to $\hat{S}_e$ is ample with respect to $S_e$. So we see that we have all the ingredients that gave us the proof of 6.8 (i.e., the proof that we gave followed formally from the above lemmas and general facts about quantized sheaves of algebras). Thus, we can conclude:

**Theorem 7.7.** Let $\lambda$ be an anti-dominant weight. Then

$$\Gamma : \text{Mod}^{\text{proj}}(D_h(\lambda, \chi)) \to \text{Mod}^{\text{proj}}(U_h(\mathfrak{g}, e)/B_\chi)$$

is an equivalence of categories. Further, $\Gamma$ takes coherent $D_h(\lambda, \chi)$ modules to finitely generated $U_h(\mathfrak{g}, e)$ modules, and we have that

$$\Gamma : \text{Mod}^{\text{coh}}(D_h(\lambda, \chi)) \to \text{Mod}^{\text{coh}}(U_h(\mathfrak{g}, e)/B_\lambda)$$

is an equivalence of categories as well.

Of course, this theorem is not really what we want. To put things in their final form, we need to consider a $\mathbb{C}^*$-action on the category of modules. Fortunately, we have that the Hamiltonian reduction procedure respects the Gan-Ginzburg $\mathbb{C}^*$-action on $D_h(\lambda)$: the ideal $I_\chi$ is clearly $\mathbb{C}^*$-invariant, and the process of taking $M_l$-invariants respects the $\mathbb{C}^*$-action because of the commutation relations between $M_l$ and $\mathbb{C}^*$. Therefore, $D_h(\lambda, \chi)$ is $\mathbb{C}^*$-equivariant with respect to the $\mathbb{C}^*$ action on $\hat{S}_e$.

This will allow us to identify the $\mathbb{C}^*$-invariant global sections of $D_h(\lambda, \chi)$ as follows: we have seen above that $U_h(\mathfrak{g}, e)$ carries a natural $\mathbb{C}^*$ action with respect to which

$$(U_h(\mathfrak{g}, e))^{\mathbb{C}^*} = U(\mathfrak{g}, e)$$

We also concluded above that $J_\lambda \cap U_h(\mathfrak{g})^{\mathbb{C}^*}$ was the ideal generated by the classical ideal $I_\lambda$. So it follows that $B_\lambda \cap (U_h(\mathfrak{g}, e))^{\mathbb{C}^*}$ is the image of this ideal in $U(\mathfrak{g}, e)$. But we have an identification of the center of $U(\mathfrak{g}, e)$ with the center of $U(\mathfrak{g})$ (the natural map $Z(\mathfrak{g}) \to Z(\mathfrak{g}, e)$ is an isomorphism, see 272 section 5, footnote 2). So in fact we can conclude that $\Gamma(\hat{S}_e, D_h(\lambda, \chi))^{\mathbb{C}^*} = U(\mathfrak{g}, e)/I_\lambda := U(\mathfrak{g}, e)_\lambda$. Thus, after taking into account the $\mathbb{C}^*$-actions, the previous theorem immediately implies another proof of 7.4.
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