REFINED ASYMPTOTICS FOR MINIMAL GRAPHS IN THE
HYPERBOLIC SPACE

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ABSTRACT. We study the boundary behaviors of solutions $f$ to the Dirichlet problem
for minimal graphs in the hyperbolic space with singular asymptotic boundaries and
characterize the boundary behaviors of $f$ at the points strictly located in the tangent
cones at the singular points on the boundary. For $n = 2$, we also obtain a refined
estimate of $f$.

1. INTRODUCTION

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Lin [10] studied the Dirichlet problem of the form
\[
\Delta f - \frac{f_i f_j}{1 + |\nabla f|^2} f_{ij} + \frac{n}{f} = 0 \quad \text{in } \Omega,
\]
\[
f > 0 \quad \text{in } \Omega,
\]
\[
f = 0 \quad \text{on } \partial \Omega.
\]

(1.1)

Geometrically, the graph of $f$ is a minimal surface in $\mathbb{H}^{n+1}$ with its asymptotic boundary at infinity given by $\partial \Omega$. For $n = 2$, (1.1) also appears in the study of the Chaplygin gas. See [12] for details.

The existence of a unique solution $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ to (1.1) was shown in [10] with the assumption that $\Omega \subset \mathbb{R}^n$ is a $C^2$-domain and its boundary has nonnegative mean curvature $H_{\partial \Omega}$ with respect to the inward normal direction of $\partial \Omega$. Concerning the higher global regularity, Lin proved if $H_{\partial \Omega} > 0$, then $f \in C^{1/2}(\bar{\Omega})$. In [6], Han and we proved that under the condition $H_{\partial \Omega} \geq 0$, $f \in C^{3/2}(\bar{\Omega})$. Han and we also proved in [6] that (1.1) admits a unique solution $f \in C^{1/2}(\bar{\Omega}) \cap C^\infty(\Omega)$ under the assumption that $\Omega$ is the intersection of finitely many bounded convex $C^2$-domains $\Omega_i$ with $H_{\partial \Omega_i} > 0$.

Concerning asymptotic behaviors of solutions to (1.1), when $\Omega$ is sufficiently smooth, the expansion near the boundary of solution to the Dirichlet problem for minimal graphs in the hyperbolic space is shown in [3]. When $\Omega$ is singular, Han and we [7] studied the asymptotic behaviors of solution $f$ on $\Omega \subset \mathbb{R}^2$ whose boundary are piecewise regular with positive curvatures and derived an estimate of $f$ by comparing it with the corresponding solutions in the intersections of interior tangent balls.

The boundary geometry has great effects on behaviors of solutions to (1.1). When the boundary is regular, asymptotic behaviors are much clearer. For example, if $\Omega$ is a

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bounded $C^2, \alpha$-domain with $H_{\partial \Omega} > 0$, for some $\alpha \in (0, 1)$, then

\begin{equation}
\left| \left( \frac{H_{\partial \Omega}}{2d} \right)^{\frac{1}{2}} f - 1 \right| \leq C d^\alpha,
\end{equation}

where $d$ is the distance function to $\partial \Omega$. Another problem involving positive boundary curvatures is discussed by Jian and Wang [9]. However, difficulties arise when we study asymptotic behaviors of solutions $f$ in domains with singularity. In the general case of singular boundary, it is natural to compare solutions with the corresponding solutions in the tangent cones. This is the approach Han and the first author adopted in the study of the Liouville equation in [4] and the Loewner-Nirenberg problem in [5]. However, for (1.1) in domains with singularity, in light of (1.2), we should abandon this approach, since the boundaries of tangent cones bounded by finitely many hyperplanes have zero mean curvature wherever they are smooth. In a sense, we need to preserve the positivity of the boundary mean curvature. For $n = 2$, in domains whose boundaries are piecewise regular with positive curvatures, Han and we [7] studied the asymptotic behaviors of $f$ to (1.1) and proved that $f$ can be well approximated by the corresponding solutions in the intersections of interior tangent balls.

In this paper, we continue our study of the boundary behaviors of solutions $f$ to (1.1) in general convex domains with singular asymptotic boundaries. We characterize the boundary behaviors of $f$ at the points strictly located in the tangent cones at the singular points on the boundary and prove that $f$ at these points can be well approximated by the corresponding solutions in tangent cones. For $n = 2$, we also obtain a refined estimate of $f$.

Our first main theorem in this paper is the following result.

**Theorem 1.1.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ and, for some $x_0 \in \partial \Omega$ and $R > 0$, $\partial \Omega \cap B_R(x_0)$ consist of $k$ $C^{1,1}$-hypersurfaces $S_i$, $i = 1, \cdots, k$, with the angle between any two of the tangent planes at $x_0$ less than $\pi$. Suppose $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is the solution of (1.1) in $\Omega$ and $f_\nu$ is the corresponding solution in the tangent cone $V$ of $\Omega$ at $x_0$. Then, for any $\delta > 0$ and any $x \in \Omega$ close to $x_0$, with $\text{dist}(x, \partial \Omega) \geq \delta|x - x_0|$, \begin{equation}
|f(x) - f_\nu(x)| \leq C f(x)|x - x_0|,
\end{equation}
where $C$ is some constant depending only on $\delta$ and the geometry of $\partial \Omega$ near $x_0$.

Inspired by results in [7], we now compare solutions $f$ to (1.1) with those in the intersections of interior tangent balls and prove a refined estimate.

**Theorem 1.2.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^2$ and, for some $x_0 \in \partial \Omega$ and $R > 0$, $\partial \Omega \cap B_R(x_0)$ consist of two $C^{2, \alpha}$-curves $\sigma_1$ and $\sigma_2$ intersecting at $x_0$ with an angle $\mu \pi$, for some constants $\alpha \in (0, 1)$ and $\mu \in (0, 1)$. Assume the curvature $\kappa_i$ of $\sigma_i$ at $x_0$ is positive, $i = 1, 2$. Suppose $f \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ is the solution of (1.1) in $\Omega$ and $f_\nu$
is the corresponding solution in
\[ \Omega_{x_0, \mu, \kappa_1, \kappa_2} = B_{\frac{1}{\kappa_1}} \left( x_0 + \frac{1}{\kappa_1} \nu_1 \right) \cap B_{\frac{1}{\kappa_2}} \left( x_0 + \frac{1}{\kappa_2} \nu_2 \right), \]
where \( \nu_1 \) and \( \nu_2 \) are interior unit normal vector to \( \sigma_1 \) and \( \sigma_2 \) at \( x_0 \), respectively. Then, for any \( \epsilon \in (0, \alpha) \) and \( \delta > 0 \), there exists a constant \( \mu(\epsilon, \alpha) > 0 \), such that, if \( \mu \leq \mu(\epsilon, \alpha) \), then, for any \( x \in \Omega \) close to \( x_0 \), with \( \text{dist}(x, \partial \Omega) \geq \delta |x - x_0| \),
\[ |f(x) - f^*(x)| \leq C f(x) |x - x_0|^{1+\alpha-\epsilon}, \tag{1.4} \]
where \( C \) is a positive constant depending only on \( R, \mu, \alpha, \epsilon, \delta \) and the \( C^{2, \alpha} \)-norms of \( \sigma_1 \) and \( \sigma_2 \) in \( B_R(x_0) \).

The paper is organized as following. In Section 2, we study the boundary behaviors of solutions of (1.1) in bounded convex domains bounded by finitely many \( C^{1,1} \)-hypersurfaces and prove Theorem 1.1. In Section 3, we study \( f \) in domains whose boundaries are piecewise regular with positive curvatures and prove Theorem 1.2.

2. Solutions in Convex Domains Bounded by Hypersurfaces

In this section, we discuss the boundary behaviors of solutions of (1.1) in convex domains bounded by several \( C^{1,1} \) hypersurfaces. We prove, at points strictly located in tangent cones defined at singular points on the boundary, the solutions \( f \) are well approximated by the corresponding solutions in these cones.

First, we discuss (1.1) in infinite cones and prove the existence and uniqueness of solutions of (1.1) in infinite cones. Since this part follows [7] closely, we only sketch the proof.

For some constant \( \mu \in (0, 1) \), define
\[ V_\mu = \{ (r, \theta) \mid r \in (0, \infty), \theta \in (0, \mu \pi) \}. \tag{2.1} \]
This is an infinite cone in \( \mathbb{R}^2 \), expressed in polar coordinates. Then, \( V_\mu := V_\mu \times \mathbb{R}^{n-2} \) is an infinite cone in \( \mathbb{R}^n \). Our goal is to find a solution \( f \) to (1.1) in \( \Omega = V_\mu \), whose form is given by
\[ f = rh(\theta), \tag{2.2} \]
where \( (r, \theta) \) is the polar coordinates in \( \mathbb{R}^2 \). Substituting (2.2) in (1.1), we have
\[ \frac{h'' + h}{r} - \frac{h'^2 (h'' + h)}{r(1 + h^2 + h'^2)} + \frac{n}{rh} = 0. \tag{2.3} \]
In view of (2.3), we set the operator \( \mathcal{L} \) acting on functions \( h = h(\theta), \theta \in (0, \mu \pi) \), by
\[ \mathcal{L}h = h(1 + h^2)(h'' + h) + n(1 + h^2 + h'^2). \tag{2.4} \]
First, we construct supersolutions of \( \mathcal{L} \).
Lemma 2.1. For some constant \( \mu \in (0, 1) \), there exist constants \( A > 0, B \geq 0, \alpha \in [n, +\infty) \) and \( \beta \in (0, 1) \) such that

\[
\mathcal{L} \left( A(\sin \frac{\theta}{\mu})^{1/\alpha} + B(\sin \frac{\theta}{\mu})^{1/\beta} \right) \leq 0 \quad \text{on } (0, \mu \pi).
\]

Proof. For some \( \alpha > 0 \), set

\[
\varphi(\theta) = \left( \sin \frac{\theta}{\mu} \right)^{1/\alpha}.
\]

By differentiating twice, we have

\[
\varphi' = \frac{\varphi^{-\alpha}}{1 + \alpha \mu} \cos \frac{\theta}{\mu}, \quad \varphi'' = -\frac{1}{\mu^2(1 + \alpha)^2} \varphi - \frac{\alpha}{\mu^2(1 + \alpha)^2} \varphi^{-1 - 2\alpha}.
\]

Then, for some positive constant \( A \),

\[
\mathcal{L}(A\varphi) = A^2 \varphi'(1 + A^2 \varphi^2) \left[ (1 - \frac{1}{\mu^2(1 + \alpha)^2}) \varphi - \frac{\alpha}{\mu^2(1 + \alpha)^2} \varphi^{-2\alpha - 1} \right]
\]

\[
+ n \left[ 1 + A^2 \varphi^2 + \frac{A^2}{\mu^2(1 + \alpha)^2} \varphi^{-2\alpha}(1 - \varphi^{2\alpha}) \right].
\]

We first consider the case \( \mu \leq \frac{1}{1 + n} \). With \( \alpha = n \), we have

\[
\mathcal{L}(A\varphi) = A^2(1 - \frac{1}{(1 + n)^2\mu^2})(1 + n)\varphi^2 + n
\]

\[
+ A^4 \varphi^2 \left[ (1 - \frac{1}{(1 + n)^2\mu^2}) \varphi^2 - \frac{n}{(1 + n)^2\mu^2} \varphi^{-2n} \right].
\]

Hence,

\[
\mathcal{L}(\sqrt{(1 + n)\mu}\varphi) \leq n - n\varphi^{2 - 2n} \leq 0.
\]

Next, we consider the case \( \mu > \frac{1}{1 + n} \). Fix an arbitrary constant \( \alpha \in (n, +\infty) \). Set

\[
\psi = \left( \sin \frac{\theta}{\mu} \right)^{1/\beta},
\]

\[
h = A\varphi + B\psi,
\]

where we take \( \beta = \min\left\{ \frac{1}{2}(\frac{1}{\mu} - 1), \frac{1}{100} \right\} \), \( A \geq 1 \) to be determined, and set \( B = CA \) for a sufficiently large constant \( C \) to be determined. We can compare with the corresponding terms appearing in the proof of Lemma 2.1 in [7]. Then, we proceed similarly as in the proof of Lemma 2.1 in [7] and we just point out a key difference that, for some positive constant \( \tau \), when \( \sin \frac{\theta}{\mu} < \frac{1}{1 + \alpha} \),

\[
\mathcal{L}(A(\sin \frac{\theta}{\mu})^{1/\alpha} + B(\sin \frac{\theta}{\mu})^{1/\beta}) \leq A^2 \frac{n - \alpha}{\mu^2(1 + \alpha)^2} \varphi^{-2\alpha} + n + CA^2 \varphi^{-2\alpha + \tau},
\]

and \( n - \alpha < 0 \). Hence we obtain the desired result. \( \square \)
For any $L > 0$, we define an operator $T_L$ by
\[(2.7) \quad T_L(x_1, \cdots, x_n+1) = \frac{L}{(x_1 - L)^2 + x_2^2 + \cdots + x_{n+1}^2} (L^2 - |x|^2, 2Lx_2, \cdots, 2Lx_{n+1}).\]

Then $T_L$ is an isometric automorphism in $\mathbb{H}^{n+1}$, which maps $(L, 0, \cdots, 0)$ to infinity. Restricted to $\mathbb{R}^n \times \{x_{n+1} = 0\}$, $T_L$ is a conformal transform. We can obtain (2.7) by a combination of some conformal transforms in $\mathbb{R}^{n+1}$. (See [7]). It is obvious that
\[
\frac{2L^2x_{n+1}}{(x_1 - L)^2 + \cdots + x_{n+1}^2} \to 0 \quad \text{as} \quad x_1^2 + \cdots + x_{n+1}^2 \to \infty.
\]

With Lemma 2.1 and $T_L$, we prove the existence and uniqueness of the solution of (1.1) in any cone $V \subseteq \mathbb{R}^n$ by following closely the proof of Theorem 2.3 in [7]. In fact, any cone $V$ is contained in a cone $V'$ bounded by two hyperplanes with an angle less than $\pi$ and the super-solution to (1.1) on $V$ is a upper bound for the solution to (1.1) on $V$ by the maximum principle. From the proof, we also conclude that the solution in $V$ has the form
\[(2.8) \quad f_V(x) = |x|g_V(\theta),\]
with $\theta \in S^{n-1}$.

Next, we turn our attention to (1.1) on domains.

Let $\Omega$ be a bounded convex domain and, for some $x_0 \in \partial \Omega$ and $R > 0$, $\partial \Omega \cap B_R(x_0)$ consist of $k C^{1,1}$-hypersurfaces $S_i$, $i = 1, \cdots, k$, with the angle between any two of the tangent planes at $x_0$ less than $\pi$. Denote by $V_{x_0}$ the tangent cone of $\Omega$ at $x_0$. Then, $V_{x_0}$ is bounded by $P_i$, the tangent plane of $S_i$ at $x_0$, for $i = 1, \cdots, k$. Denote by $\nu_i$ the unit inner normal vector to $P_i$, $i = 1, \cdots, k$.

Assume $x_0 \in \partial \Omega$ is the origin 0. By the convexity, $\Omega$ is a bounded Lipschitz domain and we can assume
\[\Omega \cap B_R(x_0) = \{x \in B_R : x_n > f(x')\},\]
f for some Lipschitz function $f$ on $B'_R \subset \mathbb{R}^{n-1}$, with $f(0) = 0$. Then, there exists a finite circular cone $V_{\theta_0}$ such that $x_0$ is its vertex, the $x_n$-axis its axis, $2\theta_0$ the apex angle, $h$ the height, and
\[V_{\theta_0} \subseteq \overline{\Omega}, \quad -V_{\theta_0} \subseteq \Omega^c.\]
In the following, we denote by $\mu_{x_0, \pi}$ the minimal angle among angles between any two of the tangent planes at $x_0$.

For a positive constant $L$, set
\[(2.9) \quad B_i^L = B_{\frac{L/2}{<\nu_i, e_n>}} \left(x_0 + \frac{L/2}{<\nu_i, e_n>} \nu_i \right).
\]
It is easy to see that
\[(2.10) \quad \bigcap_i \partial B_i^L = \{x_0, q\},\]
where \( q = x_0 + L e_n \). Note \( \langle \nu_i, e_n \rangle \geq \sin \theta_0 \). Hence,

\[
\frac{L}{2} \leq \frac{L/2}{\langle \nu_i, e_n \rangle} \leq \frac{L/2}{\sin \theta_0}.
\]

For some constant \( L \) depending only on \( R \) and the \( C^{1,1} \)-norms of \( S_i \), for \( i = 1, \cdots, k \), we note that each ball \( B^L_i \) is above the corresponding hypersurface \( S_i \), although it is not necessarily in \( \Omega \).

Now we are ready to prove Theorem 1.1

**Proof of Theorem 1.1** Throughout the proof, we always denote by \( C \) some positive constant depending only on \( n, R, \theta_0, \delta, \mu_{x_0}, h \) and the \( C^{1,1} \)-norms of hypersurfaces \( S_i, i = 1, \cdots, k \), near \( x_0 \). Set, for \( L \) sufficiently small,

\[
\tilde{\Omega} = \bigcap_i B^{2L}_i \subset \Omega,
\]

where \( B^{2L}_i \) is defined above.

Then, for \( |x - x_0| \) small with \( \text{dist}(x, \partial \Omega) \geq \delta|x - x_0| \), we have

\[
\text{dist}(x, \partial V) \geq \delta|x - x_0|,
\]

and

\[
\text{dist}(x, \partial \tilde{\Omega}) > \frac{\delta}{2}|x - x_0|.
\]

For convenience, we rotate the coordinates such that \( x_n \)-axis above becomes \( x_1 \)-axis and assume

\[
x_0 = (-L, 0, \cdots, 0), \quad q = (L, 0, \cdots, 0).
\]

Let \( \tilde{f} \) be the solution of (1.1) in \( \tilde{\Omega} \). The maximum principle implies

\[
(2.11) \quad f \geq \tilde{f} \quad \text{in} \quad \tilde{\Omega}.
\]

We note that the tangent cone of \( \Omega \) at \( x_0 \) is also the tangent cone of \( \tilde{\Omega} \) at \( x_0 \). We consider the map \( T_L \) introduced in (2.7). Then, \( T_L|_{\mathbb{R}^n \times \{0\}} \) maps conformally \( \tilde{\Omega} \) to an infinite cone \( \tilde{V} \), which conjugates to \( V \), with

\[
(2.12) \quad \tilde{V} = V + \frac{1}{2} x_0 \tilde{q},
\]

and \( T_L \) maps the minimal graph \( \{(x, \tilde{f}(x))\} \) in \( \mathbb{H}^{n+1} \) to the minimal graph \( \{(y, \tilde{f}_{\tilde{V}}(y))\} \) in \( \mathbb{H}^{n+1} \). By (2.7) and (2.8), we have

\[
JT_L|_{x_0} = \frac{1}{2} I_{(n+1) \times (n+1)},
\]

and, for \( |x - x_0| \) small and \( a \in \{2, \cdots, n\} \),

\[
\left| y_1 - \frac{1}{2} (x_1 + L) \right| \leq C|x - x_0|^2,
\]

\[
\left| y_a - \frac{1}{2} x_a \right| \leq C|x_a||x - x_0|,
\]
and
\[ |\tilde{f}_V(y) - \frac{1}{2}\tilde{f}(x)| \leq C\tilde{f}(x)|x - x_0|.\]

By (2.8), when \( \dist(x, \partial \Omega) \geq \delta|x - x_0| \),
\[ |\nabla \tilde{f}_V| \leq C(\tilde{V}, \delta) \]
and
\[ \tilde{f}_V(y) \geq \tilde{f}_V\left(\frac{1}{2}(x_1 + L), \frac{1}{2}x_2, \ldots, \frac{1}{2}x_n\right) - C(\tilde{V}, \delta)|x - x_0|^2 \]
\[ \geq \frac{1}{2}|x - x_0|g_V(\theta)(1 - C|x - x_0|), \]
where we used the fact that \( g_V(\theta) \geq c \), for some positive constant \( c \) depending on \( \tilde{V} \) and \( \delta \), when \( \dist(x, \partial \Omega) \geq \delta|x - x_0| \) and \( x \) is close to \( x_0 \), by noting \( g_V(\theta) > 0 \). Therefore, combining (2.11) and the fact \( g_V = g_V \) by (2.12), we have
\[ (2.13) \quad f(x) \geq |x - x_0|g_V(\theta)(1 - C|x - x_0|). \]
Also, by the maximum principle, we have, for any \( x \in \Omega \),
\[ (2.14) \quad f(x) \leq f(x) = |x - x_0|g_V(\theta). \]
This finishes the proof. \( \square \)

3. Refined expansion

In [7], we studied asymptotic behaviors of \( f \) in the hyperbolic space with singular asymptotic boundaries under the assumption that the boundaries are piecewise regular with positive curvatures and approximated such solutions by the corresponding solutions in the intersections of interior tangent balls up to an order \( |x|^\beta \), with \( \beta \in (0, \frac{1}{2}) \). On the other hand, Theorem 1.1 demonstrates that, at points strictly located in tangent cones defined at the singular points on the boundary, the solutions \( f \) are well approximated by the corresponding solutions in these cones up to the order \( |x| \). In light of this, we expect that the corresponding solutions in the interior tangent balls should provide a refined estimate over the estimate in [7].

To this end, we need a localization lemma which provides more information on the local properties of asymptotic expansions near singular boundary points up to certain orders. Compare with Lemma 3.1 in [7].

**Lemma 3.1.** Let \( \Omega \) and \( \Omega_* \) be two convex domains in \( \mathbb{R}^2 \) such that, for some \( x_0 \in \partial \Omega \) and some \( R_0 \in (0, 1) \),
\[ \Omega \cap B_{R_0}(x_0) = \Omega_* \cap B_{R_0}(x_0), \]
and that \( \partial \Omega \cap B_{R_0}(x_0) \) consists of two \( C^{1,1} \)-curves \( \sigma_1, \sigma_2 \) intersecting at \( x_0 \), with the angle between the tangent lines of \( \sigma_1 \) and \( \sigma_2 \) given by \( \mu \pi \), for some \( \mu \in (0, 1) \). Suppose that \( f \) and \( f_* \) are solutions of (1.1) for \( \Omega \) and \( \Omega_* \), respectively. Then, for any \( \beta > 0 \) and
\( \delta > 0 \), there exists a constant \( \mu(\beta) \) such that, for any \( \mu \in (0, \mu(\beta)] \) and any \( x \in \Omega \) close to \( x_0 \), with \( \text{dist}(x, \partial \Omega) \geq \delta|x - x_0| \),

\[ |f(x) - f_*(x)| \leq C f(x) \left( \frac{|x - x_0|}{R_0} \right)^{2+\beta}, \]

where \( C \) is a positive constant depending only on \( \mu \), \( \delta \) and the \( C^{1,1} \)-norms of \( \sigma_1 \) and \( \sigma_2 \) in \( B_{R_0}(x_0) \).

**Proof.** We note that the equation in (1.1) is invariant under the scaling \( f \mapsto f(R \cdot)/R \). Without loss of generality, we assume \( x_0 = 0 \) and \( R_0 = 1 \) and prove, for any \( x \in \Omega \) close to \( x_0 \), with \( \text{dist}(x, \partial \Omega) \geq \delta|x| \),

\[ |f(x) - f_*(x)| \leq C f(x)|x|^{2+\beta}. \]

For any \( x \in \Omega \cap B_{r_0} \) with \( \text{dist}(x, \partial \Omega) \geq \delta|x| \), we have

\[ C_{\mu}|x| \geq f(x) \geq c|x|, \quad C_{\mu}|x| \geq f_*(x) \geq c|x|, \]

where \( r_0 \) and \( c \) are small positive constants obtained by Theorem 2.1 and \( C_{\mu} \) is a positive constant obtained by the maximum principle and (2.8). Hence, for any \( r_1 \in (0, r_0) \), (3.2) holds for any \( x \in \Omega \cap (B_{r_0} \setminus B_{r_1}) \), with \( \text{dist}(x, \partial \Omega) \geq \delta|x| \) by taking \( C \) in (3.2) large.

Let \( g \) be the solution to (1.1) in \( \Omega \cap B_1 \). By the maximum principle, we have

\[ f \geq g, \quad f_* \geq g \quad \text{in} \quad \Omega \cap B_1. \]

Write \( r = |x| \). We claim, there exists a small \( r_{1\mu} \) such that

\[ g \geq f - Ar^{3+\beta}, \quad g \geq f_* - Ar^{3+\beta} \quad \text{in} \quad \Omega \cap B_{r_{1\mu}}. \]

By combining (3.3), (3.4), and (3.5), we have, for any \( x \in \Omega \cap B_{r_{1\mu}} \), with \( \text{dist}(x, \partial \Omega) \geq \delta|x| \),

\[ f_*(x) \geq g(x) \geq f(x)(1 - C|x|^{2+\beta}), \quad f(x) \geq g(x) \geq f_*(x)(1 - C|x|^{2+\beta}). \]

Hence, we obtain (3.2) for any \( x \in \Omega \cap B_{r_{1\mu}} \), with \( \text{dist}(x, \partial \Omega) \geq \delta|x| \).

We now prove the first inequality in (3.5). First, we consider the boundary condition. Proceeding as in [7], we have

\[ g \geq \left(1 - \frac{1}{r_{1\mu}^3} + 2r^\alpha \right) f \quad \text{in} \quad \Omega \cap B_{r_{1\mu}}, \]

and

\begin{align*}
  f & \leq C_{\mu}r \quad \text{in} \quad \Omega, \\
  f & = 0 \quad \text{on} \quad \partial \Omega, \\
  C_{\mu} & \leq \sqrt{3\mu} \quad \text{for} \quad \mu \leq \frac{1}{3},
\end{align*}

where \( r_{1\mu} \) is the small positive constant defined in Lemma 3.1 in [7] and the subscript \( \mu \) indicates its dependence on \( \mu \). In the following, we assume \( \mu \) is small. Then by (3.7),

\[ C_{\mu} \leq 1. \]
Next, we require, for some small $r_1\mu < r_0\mu$,

$$f\left(1 - \left(\frac{1}{r_0^\alpha} + 2\right)r^\alpha\right) \geq f - Ar^{3+\beta}$$
on $\Omega \cap \partial B_{r_1\mu}$.

To this end, take

$$A = C\mu\left(\frac{1}{r_0^\alpha} + 2\right)r_1^{\alpha-2-\beta}.$$ 

Combining with (3.6), the boundary condition is satisfied.

Next, set

$$h = f - Ar^{3+\beta},$$

and

$$Q(h) = \left(\delta_{ij} - \frac{h_i h_j}{1 + |\nabla h|^2}\right)h_{ij} + \frac{n}{h}.$$ 

We will prove $Q(h) \geq 0$ in $\Omega \cap B_{r_1\mu}$, for the general dimension $n$. Take $r_1\mu$ sufficiently small, with $r_1\mu \ll r_0\mu$. We have, for $r \leq r_1\mu$,

$$|\nabla Ar^{3+\beta}| \leq A(3 + \beta)r^{2+\beta} \leq (3 + \beta)(\frac{1}{r_0^\alpha} + 2)r_1^{\alpha}(\frac{r}{r_1\mu})^{2+\beta} \ll 1.$$ 

We claim that

$$\left(\delta_{ij} - \frac{h_i h_j}{1 + |\nabla h|^2}\right)f_{ij} \geq \left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2}\right)f_{ij}(1 + C|\nabla Ar^{3+\beta}|),$$

where $C$ is a positive constant independent of $\mu$. In fact, we have $C = 2.1 + 2(n - 1)$ from the proof of (3.11). Assuming (3.11), we have, by (3.7),

$$Q(h) \geq (\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2})f_{ij}(1 + C|\nabla Ar^{3+\beta}|)$$

$$+ (\delta_{ij} - \frac{h_i h_j}{1 + |\nabla h|^2})\partial_{ij}(-Ar^{3+\beta}) + \frac{n}{f - Ar^{3+\beta}}$$

$$\geq \frac{n}{f - Ar^{3+\beta}} - \frac{n}{f}(1 + C|\nabla Ar^{3+\beta}|) - n(3 + \beta)(2 + \beta)A^{1+\beta}$$

$$\geq A^{1+\beta}\left(\frac{n}{C^2\mu}(1 - C\mu(3 + \beta)C) - n(3 + \beta)(2 + \beta)\right).$$

By (3.7), we choose $\mu$ small so that $C\mu$ is small. Therefore, $Q(h) \geq 0$.

Now we prove (3.11). Note that

$$\left(\delta_{ij} - \frac{h_i h_j}{1 + |\nabla h|^2}\right)f_{ij} \quad \text{and} \quad \left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2}\right)f_{ij}$$

are invariant under constant orthogonal transforms. Hence, in a neighborhood of any point $p \in \Omega \cap B_{r_1\mu}$, by a rotation, we can assume $\nabla h(p) = h_1(p)$ and proceed to calculate
at $p$ in such coordinates. Set $i, j \in \{2, \cdots, n\}$ and
\[ a_{ij}(f) = \delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2}, \]
(3.13)
\[ a_{ij}(h) = \delta_{ij} - \frac{h_i h_j}{1 + |\nabla h|^2}. \]
Then,
\[ a_{11}(h) = \frac{1}{1 + h_1^2}, \quad a_{ii}(h) = 1, \quad a_{ij}(h) = 0 \text{ for } i \neq j, \]
(3.14)
\[ a_{11}(f) = 1 + \sum_i \frac{f_i^2}{1 + |\nabla f|^2} = \frac{1}{1 + |\nabla f|^2} (1 + \sum_i (\partial_i Ar^{3+\beta})^2), \]
\[ a_{ii}(f) = 1 - \sum_i \frac{f_i^2}{1 + |\nabla f|^2} = \frac{1}{1 + |\nabla f|^2} (1 - \sum_i (\partial_i Ar^{3+\beta})^2), \]
\[ |a_{ij}(f)| = \left| -\frac{f_i f_j}{1 + |\nabla f|^2} \right| \leq \frac{|\nabla f| |\nabla Ar^{3+\beta}|}{1 + |\nabla f|^2} \quad \text{for } i \neq j, \quad i, j \in \{1, \cdots, n\}, \]
where we used the fact that $h_i = 0$ implies
(3.15)
\[ f_i = \partial_i Ar^{3+\beta}. \]
Note $a_{ii}(h), a_{ii}(f)$ are nonnegative by definition. Hence, by (3.15) again and (3.10),
\[ 1 + h_1^2 = 1 + |f_1 - \partial_1 (Ar^{3+\beta})|^2 \]
\[ = 1 + |\nabla f|^2 + |\partial_1 (Ar^{3+\beta})|^2 - 2 f_1 \partial_1 (Ar^{3+\beta}) - \sum_i |\partial_i (Ar^{3+\beta})|^2 \]
\[ \geq (1 + |\nabla f|^2) \cdot \left( 1 - \frac{f_1^2 |\nabla Ar^{3+\beta}| + |\nabla (Ar^{3+\beta})| + \sum_i |\partial_i (Ar^{3+\beta})|^2}{1 + |\nabla f|^2} \right) \]
\[ \geq (1 + |\nabla f|^2) \cdot (1 - |\nabla Ar^{3+\beta}| - |\nabla Ar^{3+\beta}|^2) \]
\[ \geq \frac{1 + |\nabla f|^2}{1 + 2|\nabla Ar^{3+\beta}|}, \]
and hence
\[ a_{11}(h) = \frac{1}{1 + h_1^2} \leq \frac{1}{1 + |\nabla f|^2} (1 + 2|\nabla Ar^{3+\beta}|) \]
(3.16)
\[ = a_{11}(f) \frac{1 + 2|\nabla Ar^{3+\beta}|}{1 + \sum_i (\partial_i Ar^{3+\beta})^2} \]
\[ \leq a_{11}(f) (1 + 2|\nabla Ar^{3+\beta}|), \]
and
\[ a_{ii}(h) \frac{1}{1 + 2|\nabla Ar^{3+\beta}|} \leq a_{ii}(h) (1 - |\nabla Ar^{3+\beta}|^2) \leq a_{ii}(f). \]
Next, we consider $a_{ij}(f)f_{ij}$ for $i \neq j$. Note that $i \neq j$ implies $i \neq 1$ or $j \neq 1$. Without loss of generality, we may assume $j \neq 1$. By (3.14) and the concavity of $f$ from [6], we have

$$|a_{ij}(f)f_{ij}| \leq \frac{\nabla f \cdot \nabla A_{r_3+\beta}}{1 + |\nabla f|^2} \sqrt{|f_{ij}|^2 |f_{ii}|}$$

(3.18)

Comparing the coefficients of $|f_{ii}|$ in the last inequality with $a_{ii}(f)$, we have

$$\sum_{i,j=1}^n a_{ij}(f)f_{ij} = \sum_{i=1}^n a_{ii}(f)f_{ii} + \sum_{i \neq j} a_{ij}(f)f_{ij}$$

$$\leq \sum_{i=1}^n a_{ii}(f)f_{ii} - \frac{1}{2} \sum_{i \neq j} |\nabla A_{r_3+\beta}|(a_{ii}(f)f_{ii} + a_{jj}(f)f_{jj}) \frac{1}{1 - |\nabla A_{r_3+\beta}|^2}$$

$$\leq \sum_{i=1}^n a_{ii}(f)f_{ii}(1 - 1.1(n - 1)|\nabla A_{r_3+\beta}|).$$

Combining the concavity of $f$, (3.10), (3.16), (3.17), and (3.19), we have at $p$,

$$\sum_{i,j=1}^n a_{ij}(h)f_{ij} = \sum_{i=1}^n a_{ii}(h)f_{ii}$$

$$\geq \sum_{i,j=1}^n a_{ij}(f)f_{ij} \frac{1 + 2|\nabla A_{r_3+\beta}|}{1 - 1.1(n - 1)|\nabla A_{r_3+\beta}|}$$

$$\geq \sum_{i,j=1}^n a_{ij}(f)f_{ij}(1 + (2.1 + 1.2(n - 1))|\nabla A_{r_3+\beta}|).$$

Therefore, we complete the proof of (3.11), with $C = 2.1 + 1.2(n - 1)$. □

**Remark 3.2.** In the above proof, we can fix a sufficiently small constant $\varepsilon_0$ independent of $\mu$, and then take $r_{1\mu} = r_{0\mu} \varepsilon_0$. Hence, $r_{1\mu}$ depends on $\mu$ continuously as $r_{0\mu}$ does, which is drawn from [7].

Now we are ready to prove the refined expansions.

**Proof of Theorem 1.2.** Fix any point $x \in \Omega$ close to $x_0$, with $\text{dist}(x, \partial \Omega) \geq \delta |x - x_0|$. Set

$$\Omega_{x_0,\mu,\kappa_1,\kappa_2} = B_{\frac{1}{\kappa_1}} \left( x_0 + \frac{1}{\kappa_1} \nu_1 \right) \cap B_{\frac{1}{\kappa_2}} \left( x_0 + \frac{1}{\kappa_2} \nu_2 \right),$$
and
\[
\tilde{\Omega} = \{ x' | x + (1 + |x - x_0|^{1 + \alpha - \epsilon})^{-1} (x' - x) \in \Omega_{x_0, \mu, \kappa_1, \kappa_2} \},
\]
\[
\hat{\Omega} = \{ x' | x + (1 - |x - x_0|^{1 + \alpha - \epsilon})^{-1} (x' - x) \in \Omega_{x_0, \mu, \kappa_1, \kappa_2} \}.
\]

Let \( \tilde{f}, \hat{f}, f_* \) be the solutions of (1.1) for \( \Omega = \tilde{\Omega}, \hat{\Omega}, \Omega_{x_0, \mu, \kappa_1, \kappa_2} \), respectively. Then,
\[
\tilde{f}(x) = (1 + |x - x_0|^{1 + \alpha - \epsilon}) f_*(x),
\]
and
\[
\hat{f}(x) = (1 - |x - x_0|^{1 + \alpha - \epsilon}) f_*(x),
\]

Write \( \tilde{p} = x + (1 - |x - x_0|^{1 + \alpha - \epsilon})^{-1} (x_0 - x) \). For \( |x - x_0| \) small, it is straightforward to verify
\[
\tilde{\Omega}' \equiv \Omega \bigcap_{C_0} B_{C_0|x - x_0|^{2 + \alpha - \epsilon}}(x_0) \subset \tilde{\Omega},
\]
and
\[
\hat{\Omega}' \equiv \hat{\Omega} \bigcap_{C_0} B_{C_0|x - x_0|^{2 + \alpha - \epsilon}}(\hat{p}) \subset \Omega,
\]
where \( C_0 \) is some constant depending only on \( R, \mu, \alpha, \epsilon, \delta, h \) and the \( C^{2, \alpha} \)-norms of \( \sigma_1 \) and \( \sigma_2 \) in \( B_R(x_0) \).

Let \( f', \hat{f}' \) be the solutions of (1.1) on \( \tilde{\Omega}' \) and \( \hat{\Omega}' \), respectively. We choose \( 2 + \beta = \frac{(2 + \alpha)(1 + \alpha - \epsilon)}{\epsilon} \) in Lemma 3.1. Then,
\[
|f'(x) - f(x)| \leq \frac{|x - x_0|^{(2 + \alpha)(1 + \alpha - \epsilon)}}{C_0|x - x_0|^{2 + \alpha - \epsilon}} \leq \frac{C f(x)|x - x_0|^{1 + \alpha - \epsilon}},
\]
and
\[
|\hat{f}'(x) - \hat{f}(x)| \leq \frac{|x - \hat{p}|^{(2 + \alpha)(1 + \alpha - \epsilon)}}{C_0|x - x_0|^{2 + \alpha - \epsilon}} \leq \frac{C \hat{f}(x)|x - x_0|^{1 + \alpha - \epsilon}},
\]
where we took \( R_0 = C_0|x - x_0|^{2 + \alpha - \epsilon} \) in (3.1). By the maximum principle, we have
\[
f'(x) \leq \tilde{f}(x), \quad \hat{f}'(x) \leq f(x).
\]

Hence,
\[
\begin{align*}
\tilde{f}(x)(1 - C|x - x_0|^{1 + \alpha - \epsilon}) & \leq f(x), \\
f(x)(1 - C|x - x_0|^{1 + \alpha - \epsilon}) & \leq \hat{f}(x).
\end{align*}
\]
Therefore,
\[
(3.20) \quad f_*(x)(1 - C|x - x_0|^{1 + \alpha - \epsilon}) \leq f(x) \leq f_*(x)(1 + C|x - x_0|^{1 + \alpha - \epsilon}).
\]
This completes the proof. □
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