A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component

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Abstract

We study the Cauchy problem for the 3D Navier-Stokes equations, and prove some scalaring-invariant regularity criteria involving only one velocity component.

1 Introduction

We consider the following incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0 \\
\nabla \cdot u &= 0 \\
u = 0 &\quad \text{in } (0, T) \times \mathbb{R}^3, \\
u(0, x) &= u_0(x), \quad \text{in } \mathbb{R}^3.
\end{align*}$$

(1)

Here $u = (u_1, u_2, u_3)$ is the velocity, $p$ is a scalar pressure, $\nu > 0$ is the kinematic viscosity, and $u_0$ with $\nabla \cdot u_0 = 0$ is the initial velocity.

The existence of a global weak solution

$$u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$$

to (1) has long been established by Leray [13], see also Hopf [9]. But the issue of regularity and uniqueness of $u$ remains open. Initialed by Serrin [18, 19] and

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Prodi [17], there has been a lot of literatures devoted to finding sufficient conditions to ensure $u$ to be smooth, see, e.g., [1],[2], [6], [7], [10], [15], [16], [21], [23], [26], [27], [28], [29], and references therein.

Recently, many authors become interested in the regularity criteria involving only one velocity component, or its gradient, even though most of which are not scaling invariant. Let us track the progresses we made during the last decade.

For one component regularity, one is preferred to showing that the condition

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad 2/p + 3/q = \beta, \quad 3/\beta < q \leq \infty, \quad (2)$$

with $\beta = 1$ or

$$\nabla u_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \quad 2/r + 3/s = \gamma, \quad 3/\gamma < q \leq \infty, \quad (3)$$

with $\gamma = 2$, ensures that $u \in C^\infty((0, T] \times \mathbb{R}^3)$.

However, this is quite difficult to prove, and all results are with $\beta < 1$ or $\gamma < 2$, to the authors’ best knowledge. More precisely, $\beta$ (resp. $\gamma$) is first taken to be $1/2$ (resp. $3/2$) in [14] and [22]. Then, using intricate decomposition of the pressure $p$, Kukavica and Ziane [12] was able to show that (2) with $\beta = 5/8$ or (3) with $\gamma = 11/6$ is enough to ensure smoothness of $u$. Later, Cao and Titi [5] extended $\alpha$ to be $3/2 + 2/3q$, by invoking multiplicative Sobolev imbedding inequality:

$$\|f\|_6 \leq C \|\nabla_h f\|_2^{2/3} \|\partial_3 f\|_2^{1/3}, \quad (4)$$

where $\nabla_h = (\partial_1, \partial_2)$ is the horizontal gradient. Finally, Zhou and Pokorný in [24],[25] give another contribution, which states that the condition (2) with $\beta = 3/4 + 1/2q$ or (3) with $\gamma = 23/12$ entails the regularity of $u$, although, there are some restrictions on $s$.

Interestingly enough, regularity in one direction is always scaling invariant, see [3], [11].

The purpose of this paper is to make a further contribution in this direction. Precisely, we have

**Theorem 1.** Let $u_0 \in V$, and $u$ be a weak solution to (1) in $[0, T]$ with initial datum $u_0$. If

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \partial_3 u_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \quad (5)$$

with $1 \leq p, q, r, s \leq \infty, \quad 0 \leq \beta, \gamma < \infty$ satisfying

$$\begin{cases}
2/p + 3/q = \beta, & 2/r + 3/s = \gamma \\
\left(1 - \frac{1}{s}\right)q = \frac{1/r + 3/8}{3/8 - 1/p} = \frac{9/4 - \gamma}{\beta - 3/4} > 1
\end{cases}$$

with $p < \infty$ or $r < \infty$.
then $u$ is smooth in $[0, T] \times \mathbb{R}^3$.

**Remark 2.** We make some comments on (6). For the sake of corollaries followed, we write down (6)$_1$. In (6)$_2$, the first equality is due to some Hölder / Young conjugates, see (13)$_{1,2}$ and the derivation of (19), while the second one is some compatibility condition, see (13)$_3$. Finally, (6)$_3$ is to ensure the application of Gronwall inequality in (18).

If we choose $\beta = 1$ or $\gamma = 2$ in Theorem 1, we have the following Serrin-type regularity criterion.

**Corollary 3.** Assume as in Theorem 1. If

$$u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \partial_3 u_3 \in L^r(0, T; L^s(\mathbb{R}^3)),$$

with $1 \leq p, r \leq \infty$, $1 \leq q, s \leq \infty$ satisfying either of the following conditions,

1. \[
\begin{align*}
\frac{2}{p} + \frac{3}{q} = 1, & \quad \frac{2}{r} + \frac{3}{s} < 2 \\
\left(1 - \frac{1}{s}\right)q &= \frac{1/r + 3/8}{3/8 - 1/p} = 9 - 4\left(\frac{2}{r} + \frac{3}{s}\right) \\
p < \infty \text{ or } r < \infty
\end{align*}
\]

2. \[
\begin{align*}
\frac{2}{p} + \frac{3}{q} < 1, & \quad \frac{2}{r} + \frac{3}{s} = 2 \\
\left(1 - \frac{1}{s}\right)q &= \frac{1/r + 3/8}{3/8 - 1/p} = \frac{1}{4(2/p + 3/q) - 3} \\
p < \infty \text{ or } r < \infty
\end{align*}
\]

then $u$ is smooth in $[0, T] \times \mathbb{R}^3$.

Due to Definition 5 in Sect. 2, $u_3 \in L^\infty(0, T; L^2(\mathbb{R}^3))$, we may take $p = \infty$, $q = 2$, $\gamma = 3/4 + 3/2s$ in Theorem 1 to yield

**Corollary 4.** [4] Assume as in Theorem 1. Then the condition

$$\partial_3 u_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{r} + \frac{3}{s} = \frac{3}{4} + \frac{3}{2s}, \quad 1 \leq r < \infty,$$

ensures that $u \in C^\infty([0, T] \times \mathbb{R}^3)$.

The function space $V$, the definition of a weak solution and other often-used notations will be given in Sect. 2.

We shall use method from [4] and [25]. See the details in Sect. 3.
2 Prelimiaries

We gather here some definitions, notations and intricate (in)equalities.

The Lebesgue spaces \( L^q(\mathbb{R}^3) \) is endowed with norm \( \|\cdot\|_q \), with its bold-face counterpart denotes the set of vector-valued functions, and we denote by \( \|\cdot\|_{p,q} \) the norm for anisotropic Lebesgue spaces \( L^p(0,T;L^q(\mathbb{R}^3)) \). The Sobolev spaces \( W^{k,p}(\mathbb{R}^3) \) (resp. \( H^k(\mathbb{R}^3) \)) is equipped with the norm \( \|\cdot\|_{k,p} \) (resp. \( \|\cdot\|_{k,2} \)).

Let \( C_c^\infty(\mathbb{R}^n) \) be the set of smooth vector-valued functions with compact support, we then define

\[ \mathcal{V} = \left\{ v \in C_c^\infty(\mathbb{R}^3); \nabla \cdot v = 0 \right\}, \]

\[ H = \text{the completion of } \mathcal{V} \text{ under the norm } \|\cdot\|_2, \]

\[ V = \text{the completion of } \mathcal{V} \text{ under the norm } \|\cdot\|_{1,2}. \]

With these spaces at hand, we recall the weak formulation of (1), see [20].

**Definition 5.** Let \( u_0 \in H, T > 0 \). A measurable vector-valued function \( u \) defined in \([0,T] \times \mathbb{R}^3\) is said to be a weak solution to (1) if

1. \( u \in L^\infty(0,T;H) \cap L^2(0,T;V); \)
2. (1) holds in the sense of distributions,

\[ (u(t), \phi(t)) + \nu \int_0^t (\nabla u(s), \nabla \phi(s)) \, ds \]
\[ - \int_0^t (u, \partial_t \phi + (u \cdot \nabla) \phi) \, ds = (u_0, \phi(0)), \]

(7)

for all \( \phi \in C_c^\infty([0,T] \times \mathbb{R}^3) \) with \( \nabla \cdot \phi = 0. \)

Here \((\cdot,\cdot)\) is the scalar product in \( L^2(\mathbb{R}^3) \).

Since we are concerned with regularity criteria involving only one velocity component, the following lemma is quite important, see [11].

**Lemma 6.** Assume that \( u \in H^2(\mathbb{R}^3) \) is smooth and divergence free. Then

\[ \sum_{i,j=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta_h u_j \, dx \]

(8)
\[
\sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_i u_j \partial_i u_j \partial_3 u_3 \, dx - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 \, dx \\
+ \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 \, dx \\
= - \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_2^2 u_2 \partial_i u_j \partial_3 u_3 \, dx + \int_{\mathbb{R}^3} \partial_1^2 u_2 \partial_2 u_3 \, dx \\
+ \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2^2 u_2 \partial_3 u_3 \, dx \\
- \int_{\mathbb{R}^3} \partial_2^2 u_2 \partial_2 u_3 \partial_3 u_3 \, dx - \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2^2 u_2 \partial_3 u_3 \, dx,
\]

where \( \Delta_h = \partial_{11}^2 + \partial_{22}^2 \) is the horizontal Laplacian.

It is then immediate that by invoking the divergence free condition (see [4])

**Lemma 7.**

\[
\left| \int_{\mathbb{R}^3} \left((u \cdot \nabla) u\right) \cdot \Delta_h u \, dx \right| \leq \int_{\mathbb{R}^3} |u_3| \cdot |\nabla u| \cdot |\nabla^2 u| \, dx. \tag{9}
\]

We end this section by invoking some interpolation inequalities, see e.g., [22], and a simple revision of an inequality in [4].

**Lemma 8.** Let \( f \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \). Then

\[
f \in L^a(0, T; L^b(\mathbb{R}^3)), \quad 2/a + 3/b \geq 3/2, \quad 2 \leq b \leq 6.
\]

Moreover,

\[
\|f\|_{a,b} \leq C \|f\|_{\infty,2}^{3/b-1/2} \|\nabla f\|_{2,2}^{3/2-3/b}. \tag{10}
\]

**Lemma 9.** For \( f, g, h \in C^\infty_c(\mathbb{R}^3) \), we have

\[
\left| \int_{\mathbb{R}^3} f g h \, dx_1 dx_2 dx_3 \right| \leq C \|f\|_q \|g\|_s \|h\|_1 \|
\]
Proof.

\[
\left| \int_{\mathbb{R}^3} f \, g \, h \, dx_1 \, dx_2 \, dx_3 \right|
\leq \int_{\mathbb{R}^2} \left[ \max_{x_3} |f| \left( \int_{\mathbb{R}} g^2 \, dx_3 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} h^2 \, dx_3 \right)^{\frac{1}{2}} \right] \, dx_1 \, dx_2
\]
(Hölder inequality)

\[
\leq \left[ \int_{\mathbb{R}^2} \left( \max_{x_3} |f| \right)^{\alpha} \, dx_1 \, dx_2 \right] \cdot \left[ \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} g^2 \, dx_3 \right)^{\frac{\alpha}{2}} \, dx_1 \, dx_2 \right]^{\frac{1}{2} \alpha}
\cdot \left( \int_{\mathbb{R}^3} h^2 \, dx_1 \, dx_2 \, dx_3 \right)^{1/2}
\]
(Hölder inequality again)

\[
\leq C \left[ \int_{\mathbb{R}^3} |f|^{\alpha - 1} \left| \partial_3 f \right| \, dx_1 \, dx_2 \, dx_3 \right]^{1/\alpha} \cdot \left[ \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} g^{\frac{2\theta}{\alpha}} \, dx_3 \right)^{\frac{\alpha - 2}{\alpha}} \, dx_3 \right]^{1/2} \cdot \|h\|_2
\]
(Minkowski inequality)

\[
\leq C \|f\|_{q_{\alpha}}^{\frac{a-1}{\alpha}} \left\| \partial_3 f \right\|_{s_{\alpha}}^{1/\alpha} \left\| g \right\|_2^{\frac{\alpha-2}{\alpha}} \left\| \partial_1 g \right\|_2^{1/\alpha} \left\| \partial_2 g \right\|_2^{1/\alpha} \|h\|_2
\]
(Hölder, interpolation inequalities and (4)).

\[\square\]

3 Proof of the main result

In this section, we prove Theorem 1.

**Step 1** Some reductions.

By the classical ”weak = strong” type uniqueness theorem, we need only prove that

\[
\|\nabla u\|_{\infty, 2} < \infty.
\]

(12)

Due to (6), we can take an \(\alpha > 2\) such that

\[
\alpha - 1 = (1 - 1/s)q = \frac{1/r + 3/8}{3/8 - 1/p} = \frac{9/4 - \gamma}{\beta - 3/4}.
\]
i.e.,

\[
\begin{align*}
\alpha - 1 + \frac{1}{p} &= \frac{3(\alpha - 2)}{8}, \\
\frac{\alpha - 1}{q} + \frac{1}{r} &= 1, \\
(\alpha - 1)\beta + \gamma &= \frac{3(\alpha - 2)}{4} + 3.
\end{align*}
\]

**(Step II) \|\nabla_h u\|_2 estimates.**

Taking the inner product of (1)_1 with \(-\Delta_h u\) in \(L^2(\mathbb{R}^3)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_2^2 + \nu \|\nabla\nabla_h u\|_2^2 = \int_{\mathbb{R}^3} \left[ (u \cdot \nabla) u \right] \cdot \Delta_h u dx
\]

\[
\leq C \int_{\mathbb{R}^3} |u_3| \cdot |\nabla u| \cdot |\nabla \nabla_h u| \cdot dx \quad \text{(by (9))}
\]

\[
\leq C \|u_3\|_q^a \|\partial_3 u_3\|_{s}^{1/a} \|\nabla u\|_2^a \|\nabla \nabla_h u\|_2^{a-2} \quad \text{(by (11), (13))}
\]

\[
\leq \frac{\nu}{2} \|\nabla\nabla_h u\|_2^2 + C \|u_3\|_q^{a-1} \|\partial_3 u_3\|_{s}^{2} \|\nabla u\|_2^2.
\]

\text{(14)}

Integrating (14), we deduce

\[
\|\nabla_h u(t)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 ds
\]

\[
\leq \|\nabla_h u_0\|_2^2 + C \int_0^t \|u_3\|_q^{2(\alpha-1)} \|\partial_3 u_3\|_{s}^{2} \|\nabla u(s)\|_2^2 ds,
\]

\text{(15)}

for all \(t \in [0, T]\).

**(Step III) \|\nabla u\|_2 estimates.**

Taking inner product of (1)_1 with \(-\Delta u\) in \(L^2(\mathbb{R}^3)\), we gather, noticing (9) and \(\nabla \cdot u = 0\), that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \nu \|\Delta u\|_2^2 = \int_{\mathbb{R}^3} \left[ (u \cdot \nabla) u \right] \cdot \Delta h u dx + \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \partial_3^2 u dx
\]

\[
\leq C \int_{\mathbb{R}^3} \left[ |u_3| \cdot |\nabla u| \cdot |\nabla \nabla_h u| + |\nabla_h u| \cdot |\partial_3^2 u| \right] dx
\]

\[
\equiv J_1 + J_2.
\]

\text{(16)}
We estimate $J_1$ as in (14),

$$J_1 \leq \frac{\nu}{2} \|\Delta u\|_2^2 + C \|u_3\|_{\frac{2(\alpha-1)}{2-\alpha}} \|\partial_3 u_3\|_{\frac{2}{2-\alpha}} \|\nabla u\|_2^2.$$  

Meanwhile, using Hölder, interpolation inequalities and (4), $J_2$ is dominated as

$$J_2 \leq C \|\nabla h u\|_2 \|\nabla u\|_2^2 \leq C \|\nabla h u\|_2 \|\nabla u\|_2^{1/2} \|\Delta u\|_2 \|\Delta u\|_2^{1/2}.$$  

Replacing these two last displaced inequalities into (16) yields

$$\frac{d}{dt} \|\nabla u\|_2^2 + \nu \|\Delta u\|_2^2 \leq C \|u_3\|_{\frac{2(\alpha-1)}{2-\alpha}} \|\partial_3 u_3\|_{\frac{2}{2-\alpha}} \|\nabla u\|_2^2 + C \|\nabla h u\|_2 \|\nabla u\|_2^{1/2} \|\nabla \nabla u\|_2 \|\Delta u\|_2^{1/2}. \quad (17)$$

Integrating the above inequality, and invoking Hölder inequality, we have

$$\|\nabla u(t)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 \, ds \leq \|\nabla u_0\|_2^2 + C \int_0^t \|u_3\|_{\frac{2(\alpha-1)}{2-\alpha}} \|\partial_3 u_3\|_{\frac{2}{2-\alpha}} \|\nabla u\|_2^2 \, ds + C \sup_{0 \leq s \leq t} \|\nabla h u(s)\|_2 \cdot \left( \int_0^t \|\nabla u(s)\|_2^2 \, ds \right)^{1/4} \cdot \left( \int_0^t \|\Delta u(s)\|_2^2 \, ds \right)^{1/4}.$$  

Thanks to (15), we get

$$\|\nabla u(t)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 \, ds \leq \|\nabla u_0\|_2^2 + C \int_0^t \|u_3\|_{\frac{2(\alpha-1)}{2-\alpha}} \|\partial_3 u_3\|_{\frac{2}{2-\alpha}} \|\nabla u\|_2^2 \, ds \quad + C \left[ \|\nabla h u_0\|_2^2 + C \int_0^t \|u_3\|_{\frac{2(\alpha-1)}{2-\alpha}} \|\partial_3 u_3\|_{\frac{2}{2-\alpha}} \|\nabla u(s)\|_2^2 \, ds \right] \cdot \left( \int_0^t \|\Delta u(s)\|_2^2 \, ds \right)^{1/4}.$$  

By Young and Hölder inequalities, and the fact $u \in L^2(0, T; V)$, we find

$$\|\nabla u(t)\|_2^2 + \frac{\nu}{2} \int_0^t \|\Delta u(s)\|_2^2 \, ds.$$
\[
\leq C \|\nabla u_0\|_2^2 + C \int_0^t \|u_3\|_{q-\frac{2}{r-2}}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_s^{\frac{2}{r-2}} \|\nabla u(s)\|_2^2 \, ds \\
+ C \int_0^t \|u_3\|_{q-\frac{2}{r-2}}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_s^{\frac{2}{r-2}} \|\nabla u(s)\|_2^2 \, ds. \tag{18}
\]

Thanks to \((13)_1\) and Young inequality, we obtain
\[
\|\nabla u(t)\|_2^2 + \frac{\nu}{2} \int_0^t \|\Delta u(s)\|_2^2 \, ds \\
\leq C \|\nabla u_0\|_2^2 + C \int_0^t \left( \|u_3\|_q^p + \|\partial_3 u_3\|_r^s + 1 \right) \|\nabla u(s)\|_2^2 \, ds. \tag{19}
\]

Therefore, by Gronwall inequality and \((5)\), we have \((12)\) as desired. The proof is completed.

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