CONFORMALLY INVARIANT $\sigma$ MODELS ON $AdS$ SPACES, 
CHERN-SIMONS $p$-BRANES, HADRONIC BAGS AND $W$ GEOMETRY

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314, USA

Revised October, 2000

ABSTRACT

Conformally invariant sigma models in $D = 2n$ dimensions with target non-compact $O(2n,1)$ groups are studied. It is shown that despite the non-compact nature of the $O(2n,1)$ groups, the classical action and Hamiltonian are positive definite. Instanton field configurations are found to correspond geometrically to conformal “stereographic” mappings of $R^{2n}$ into the Euclidean signature $AdS_{2n}$ spaces. Zaikov’s relationship between Self Dual $p$-branes and Chern-Simons $p'$-branes, provided $p = p' + 1$ and the embedding $D = p + 1$-dimensional manifold is Euclidean, is elaborated further. Branes actions can be obtained also from a Moyal deformation quantization of Generalized Yang Mills Theories. Using this procedure, we show how four dimensional $SU(N)$ YM theories contain Chern-Simons membranes and hadronic bags in the large $N$ limit. Since Chern-Simons $p'$-branes have an underlying infinite dimensional algebra containing $W_{1+\infty}$, as shown by Zaikov, we discuss the importance that $W$ geometry should have in the final formulation of $M$ theory.

I. INTRODUCTION

Conformally invariant $O(2n+1)$ $\sigma$ models in $2n$ dimensions were of crucial importance in the construction of conformally invariant Lagrangians, with vanishing world-volume cosmological constant, for bosonic $p$-branes ($p + 1 = 2n$) [1,2]. In particular, Self Dual unit charge instanton solutions were found which correspond to conformal (stereographic) maps from $R^{2n} \to S^{2n}$. These models [1] were the higher-dimensional extension of the $O(5)$ $\sigma$ models of Feisager and Leinaas in 4D [3]. When $p + 1 = odd$, the authors [2] also built Lagrangians for bosonic extendons (branes), however, conformal invariance was lost. These sort of Lagrangians allowed the author to construct a polynomial action for the spinning membrane after a Weyl covariantization process was performed [4]. Another Weyl invariant spinning membrane action was constructed by [5] but it was non-polynomial complicating the quantization program.

Conformal Field Theories have risen to more prominence recently mainly due to Maldacena’s conjecture on the AdS/CFT duality between M/string theory on $AdS_d \times S^{D-d}$ backgrounds and CFT’s living on the projective boundary of the $AdS_d$ spaces [6]. Relevant $\sigma$ models with target spaces on certain supergroups have been used to describe CFT on $AdS$ backgrounds with Ramond-Ramond (RR) Flux [7]. In particular, various exact 2D CFT on $AdS_{2n+1}$ backgrounds have been constructed recently that could be used in building superstring theories propagating on $AdS_{2n+1}$ backgrounds [8]. These $\sigma$ models
were based on the standard $SL(2, R)$ WZNW model; i.e. $\sigma$ models on group manifolds with WZNW terms.

Conventional $\sigma$-models are based on compact groups like $O(2n), SU(n)$,... In general, compact simple groups are used mainly to simplify the quantization program; i.e the quantization is not riddled with the standard problems of ghosts due to the non-positive definite inner products; uniqueness of the WZ functional; solvability of the model; positivity of the Hamiltonian [29,30]. The authors [29] studied $O(N, 1), SU(N, 1)$ non-compact sigma models in two-dimensions and have shown that a dynamical mass generation with asymptotic freedom is possible and that a sensible unitary quantization program is possible by recurring to a selection rule in the Hilbert space of states (only positive definite norms are allowed).

Setting aside these technical subtleties on the compact simple character of the group, and motivated by the recent findings on Anti de Sitter spaces, we will study conformally invariant (noncompact) $O(2n-1, 2)$ $\sigma$-models in $2n$ even dimensions; i.e maps from $R^{2n}$ to $O(2n-1, 2)$ and $O(2n, 1)$. Instanton solutions are found for the latter case, corresponding to conformal, “stereographic “ maps from $R^{2n}$ to $O(2n, 1)$. The $SO(2n, 1)$ group manifold, modulo the action of its maximal compact subgroup $SO(2n)$, $SO(2n, 1)/SO(2n)$, is topologically the Euclidean signature $AdS_{2n}$ whose natural isometry group is $SO(2n, 1)$. To be more specific, we shall concentrate on $O(2n-1, 2)$ and $O(2n, 1)$ $\sigma$-models keeping in mind that the relevant groups are the $SO(2n-1, 2)$ and $SO(2n, 1)$.

The Lorentz signature $AdS_{2n}$ space can be viewed as a hyperboloid embedded in a pseudo-Euclidean $2n + 1$-dim manifold with coordinates $y^A = y^0, y^1, ..., y^{2n}$ and diagonal metric given by $\eta_{AB} = diag (-, +, +, ..., +, -)$ with length squared $y^A y_A$ preserved by the isometry group $SO(2n-1, 2)$. The $AdS_{2n}$ is defined as the geometrical locus:

$$y^A y^B \eta_{AB} = -R^2 = -1.$$  \hspace{1cm} (1)

de Sitter spaces require a change in the sign in the r.h.s.

Signature subtleties are crucial in the construction of instanton solutions. The ordinary Hodge dual star operation is signature dependent. For example, the double Hodge dual star operation acting on a rank $p$ differential form in a 4D space satisfies:

$$**F = s(-1)^{p(4-p)}F. \ s = +1 \ for \ (4, 0); (2, 2). \ s = -1 \ for \ (3, 1); (1, 3)$$  \hspace{1cm} (2)

where we have displayed the explicit signature dependence in the values of $s$. Hence, for a rank two form $F$, in Euclidean 4D space and with signature $(2, 2)$ (Atiyah-Ward spaces) one can find solutions to the (anti) self dual YM equations: $^*F = \pm F$. There are no YM instantons in 4D Minkowski space since in the latter one has imaginary eigenvalues: $^*F = \pm i F$.

The two temporal variables are required in the embedding process of $AdS_{2n}$ into the pseudo-Euclidean space $R^{2n-1, 2}$. We will show, in fact, that it is the noncompact $O(2n, 1)$ $\sigma$-model, instead of the $O(2n-1, 2)$ $\sigma$-model, that has instanton solutions obeying a double self duality condition similar to the one obeyed by the BPST instanton [9]. It was shown [1] that, the Euclidean (compact) $O(2n+1)$ $\sigma$ model instanton solutions are directly related to instanton solutions of $O(2n)$ Generalized YM (GYM) theories in $R^{2n}$ [10] obeying the aforementioned double self duality condition.
In view of the conformally invariant $\sigma$ models/GYM connection in higher dimensions, the next step is to study $p$-branes (with $p + 1 = 2n$) propagating on $AdS_D$ backgrounds. In particular, the critical case when the dimensionality of the target space is saturated, $D = p + 1 = 2n$. Dolan-Tchrakian [1,2] constructed the corresponding conformally invariant Skyrme-like actions, with vanishing world-volume cosmological constant, based on these conformally invariant $\sigma$ models in $2n$-dimensions. Upon the algebraic elimination of the auxiliary world-volume metric, Dolan and Tchrakian have shown that one recovers the Dirac-Nambu-Goto action:

$$ S = T \int d^{2n} \sigma \sqrt{|\det G_{\alpha\beta}|}. \quad G_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu. $$

where $G_{\alpha\beta}$ is the induced world-volume metric resulting from the embedding of the $p + 1 = 2n$ hypervolume into the $D$-dim target spacetime. When the spacetime dimension is saturated: $D = p + 1 = 2n$ the square root of the Dirac-Nambu-Goto action simplifies and one obtains the usual Jacobian for the change of variables from $\sigma$ to $X$. In such case the Nambu-Goto action is topological: there are no physical local transverse degrees of freedom. Such topological actions have been studied by Zaikov [11]. For a review of membranes and other $p-\text{branes}$ with extensive references see [12]. The action is then:

$$ S = T \int d^{2n} \sigma \partial_{\sigma^1} X^{\mu_1} \wedge \ldots \wedge \partial_{\sigma^{2n}} X^{\mu_{2n}}. $$

where $T$ is the $p$-brane (extendon) tension. For a $p$-brane whose world-volume has a natural boundary, an integration (Gauss law) yields:

$$ S = T \int_{\partial V} d^{2n-1} \zeta \ X^{\mu_1} \wedge \partial_\zeta \ X^{\mu_2} \wedge \ldots \wedge \partial_{\zeta^{2n-1}} X^{\mu_{2n}}. $$

one then recovers the action for the Chern-Simons $p'$ brane whose $p'+1$ world-volume variables, $\zeta^a$, $a = 1, 2, \ldots, p'+1$, are integrated over the $2n-1$-dim boundary $\partial V$ of the $2n$-dim domain $V$ associated with the world volume of the open $p$-brane. The value of $p'$ must be such that $p'+1 = p = 2n-1$ [11]. Zaikov concluded that these topological Chern Simons $p'$-branes exit only in target spacetimes of dimensionality $D = p' + 2$.

In particular, when the dimensionality of the target spacetime is saturated, $D = p + 1$, one can construct, in addition, self-dual $p$-brane (extendon) solutions obeying the equations of motion and constraints (resulting from $p+1$ reparametrization invariance of the world-volume) that are directly related to these topological Chern-Simons $p'$-branes. This holds provided $p'+1 = p$ and the embedding manifold is Euclidean [12]. Furthermore, when $D = p+1 = 2n$ one has conformal invariance a well [1,2]. It is in this fashion how the relationship between the self-dual $p$-branes and Chern Simons $p' = p-1$ branes emerges. This is roughly the analogy with Witten’s discovery of the one-to-one relationship between 3D nonabelian Chern-Simons theories and 2D rational CFT. [13].

The topological Chern-Simons action (5) and the action (4), obtained by using Gauss law, are among the main ones considered in this work.

In section II we will show that the Euclidean signature $AdS_{2n}$ signature is an instanton solution of the $O(2n, 1)$ $\sigma$-models. In 3.1 we discuss the relation between (open) Self
Dual $p$-branes and Chern-Simons $p'$-branes for $p = p' + 1$. In 3.2 we show that (space-time filling) $p$-branes can be obtained from a Moyal deformation quantization (of the Lie algebraic structure) of Generalized Yang Mills theories [1] (GYM). Adding a topological term makes these theories nontrivial due to boundary dynamics. In particular, we briefly review [35] how Chern-Simons membranes and Hadronic Bags emerge from the large $N$ limit of quenched, reduced $4D$ $SU(N)$ YM (QCD) [36]. In 3.3 W geometry is discussed in connection to these theories. Finally we present our conclusions.

II Euclidean $AdS_{2n}$ as $O(2n, 1)$ $\sigma$-model Instantons

As a prototype we will imagine a Chern-Simons membrane living on the Euclideanized $AdS_{4}$ background, which in turn, will be shown to be the instanton field configuration of the conformally invariant $O(4, 1)$ $\sigma$ model in $R^{4}$ obeying the double self-duality condition to be described below by eq-(8b).

Since the conformally invariant Euclidean $O(5)\sigma$-model in $R^{4}$ has a correspondence with the conformally invariant $O(4)$ YM in $R^{4}$ [1] it is natural to ask whether the analytical continuation from $O(4, 1) \rightarrow O(5)$ will allow to establish the following relationships in four-dimensions among: topological Chern-Simons membrane living in (Euclidean) $AdS_{4}$; instanton configurations of the conformally invariant $O(4, 1)$ $\sigma$-models in $R^{4}$ and $O(4)$ Yang-Mills instantons (obeying a double self-duality condition described below by eq-(8a)) in $R^{4}$.

The importance to establish this web of relationships is because one may generalize this web to higher dimensions with the provision that $D = 2n = 4k$. For example, in $D = 12$ we may have the connections among the Chern-Simons $p = 10$-brane moving on the Euclideanized $AdS_{12}$; instantons of the conformally invariant $O(12, 1)$ $\sigma$-models in $R^{12}$ and $O(12)$ Generalized Yang-Mills (GYM) theories in $R^{12}$. When the group is compact like $O(2n)$, for example, the latter GYM are defined by Lagrangians [1,10] in $R^{D}$ where $D = 2n = 4k$:

$$L = tr \left( F_{\mu \nu}^{a_{1}a_{2}...a_{2k}} \Sigma_{a_{1}a_{2}...a_{2k}} \right)^{2} = (F_{2} \wedge F_{2}... \wedge F_{2})^{2}. \quad 2n = 4k = D \quad (6)$$

with all indices antisymmetrized. $\Sigma_{a_{1}a_{2}...}$ is the totally antisymmetrized product (on the internal indices $\alpha$) of the product of $k$ factors of the $2^{2k-1} \times 2^{2k-1}$ matrices $\Sigma_{a_{1}a_{2}}$ corresponding to the chiral representation of $SO(4k)$.

The exhibiting relationship between the instanton field configurations of the $O(5)\sigma$ model and the $O(4)$ YM system in $R^{4}$ were given by [1]. The order parameter field of the $O(5)\sigma$ model is the $O(5)$ vector $n^{i}(x)$ obeying the constraint $n^{a}(x)n_{a}(x) = 1$. The definition of the gauge fields and field strengths for the YM system in terms of the $n^{i}(x)$ are:

$$A_{\mu}^{ij} = n^{i}(x) \partial_{\mu} n^{j}(x) - n^{j}(x) \partial_{\mu} n^{i}(x). \quad (7a)$$

$$F_{\mu \nu}^{ij}(x) = \partial_{\mu} A_{\nu}^{ij} - A_{\mu}^{ik} A_{\nu}^{lj} \eta_{kl} - \mu \leftrightarrow \nu = \partial_{[\mu} n^{i}[\nu] n^{j]} \partial_{\nu]}. \quad (7b)$$
due to the constraint $n^{a}n_{a} = 1$ the former field strength coincides with the definition given below by eq-(8c). When $n^{i}n_{i} = -1$ one should have $+A_{\mu}^{ik} A_{\nu}^{lj}$ in the definition (7b) instead;
i.e. a change in sign in the coupling constant. These signs subtleties are essential otherwise the field strength will be zero.

An additional constraint allows to reduce $O(5) \rightarrow O(4)$:

$$(\delta^i_j \partial_\mu + A^i_{\mu j}) n^i(x) = 0 \quad (7c)$$

After using eqs-(7a,7b,7c) the authors [1] have shown that $O(4)$ YM instanton solutions obey a double self duality condition satisfied by the BPST instanton [9] :

$$\epsilon_{\alpha_1 \alpha_2 \beta_1 \beta_2} F^{\beta_1 \beta_2}_{\mu_1 \mu_2} = \epsilon_{\mu_1 \mu_2 \nu_1 \nu_2} F^{\nu_1 \nu_2}_{\alpha_1 \alpha_2} \quad (8a)$$

contrasted with the double self-duality condition exhibited by the $O(5)$ $\sigma$ model in $R^4$ [1]:

$$\epsilon_{i_1 i_2 \ldots i_5} n^{i_5}(x) F^{i_5 i_4}_{\mu_1 \mu_2} = \epsilon_{\mu_1 \mu_2 \nu_1 \nu_2} F^{\nu_1 \nu_2}_{i_1 i_2} \quad (8b)$$

that are related to the $\sigma$ model Lagrangian in $D = 4$

$$S = \int d^4x \frac{1}{2(2!)} (F^{ij}_{\mu \nu})^2. F^{ij}_{\mu \nu} = \partial_\mu n^i(x) \partial_\nu n^j(x) - \partial_\mu n^j(x) \partial_\nu n^i(x). \quad (8c)$$

Similar type of actions are generalized for higher rank objects:

$$S = \int d^{2n}x \frac{1}{2(n!)} (F^{i_1 \ldots i_n}_{\mu_1 \ldots \mu_n})^2. F^{i_1 \ldots i_n}_{\mu_1 \ldots \mu_n} = \partial_\mu n^{[i_1}(x) \ldots \partial_\mu n^{i_n]}(x) \quad (8d)$$

The unit charge instanton solutions of eq-(8b) that minimize the action (8c) and correspond to the $O(4)$ YM instanton solutions of eq-(8a) were found to be precisely the ones corresponding to the stereographic projections (conformal mappings) from $R^4 \rightarrow S^4$ [1] :

$$n^a(x) = \frac{2x^a}{1 + x^2}; a = 1, 2, 3, 4. \quad n^5(x) = \frac{x^2 - 1}{x^2 + 1}. \quad \sum_{a=1}^{a=4} (n^a)^2 + (n^5)^2 = 1. \quad (9)$$

Things differ now for the (noncompact) $O(2n, 1)$ $\sigma$-models due to the subtle signature changes. Now we must see whether or not the $O(2n, 1)$ vector $n^i(x)$ with $i = 1, 2, \ldots, 2n+1$ satisfy the double self duality condition

$$\epsilon_{i_1 \ldots i_n i_{n+1} \ldots \ldots i_{2n}} n^{i_{2n+1}} F^{i_{2n+1} i_{n+2} \ldots i_{2n}}_{\mu_{n+1} \ldots \ldots \mu_{2n}} = \epsilon_{\mu_1 \mu_2 \ldots \mu_{n+1} \ldots \mu_{2n}} F^{\mu_1 \mu_2 \ldots \mu_{2n}}_{i_1 i_2 \ldots i_n} \quad (10)$$

The “stereographic” maps from $R^4$ to the Euclidean signature $AdS_4$ defined by:

$$n^a(x) = \frac{2x^a}{1 - x^2}; a = 1, 2, 3, 4. \quad n^5(x) = \frac{1 + x^2}{1 - x^2}. \quad (11a)$$

$$x^2 \equiv (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \quad (11b)$$

obey the condition:
\[ n^i n_i \equiv (n^1)^2 + (n^2)^2 + (n^3)^2 + (n^4)^2 - (n^5)^2 = -1. \] \hfill (11c)

which is just the $O(4, 1)$ invariant norm of the vector \( n^i(x) = (n^1(x), \ldots, n^5(x)) \) and satisfy the double-self duality condition conditions (8b) as we shall show in this section. It is important to emphasize that the double self duality conditions are not derived from the actions (8c, 8d). However, solutions to the double self duality conditions automatically obey the equations of motion, due to the analog of the Bianchi Identities, like it occurs in ordinary YM theories. But the converse is not true.

Another important issue we shall address now is the issue of singularities. The solutions given by Eqs-(11) obey the double self duality conditions and are naturally singular at the points \( x^2 = 1 \). These points have a correspondence with the projective/conformal boundaries at infinity as we will show at the end of this section. Nevertheless, eqs-(8a, 8b) are still satisfied and no singularities occur in the definition of the norm. These singularities are naturally due to the noncompact nature of AdS spaces.

The most famous example is the AdS metric: the AdS conformally flat metric is singular at the points \( x^2 = 1 \) as well. The scalar curvature is well defined and equals the negative cosmological constant \( R = \Lambda \) as it should. However, the Einstein-Hilbert action associated with the AdS metric blows up due to the divergent contributions of the determinant of the metric; i.e the measure of integration diverges at \( x^2 = 1 \):

\[
g_{\mu\nu} = \frac{1}{(1 - x^2)^2} \eta_{\mu\nu} \Rightarrow \int d^{2n} x \sqrt{g} R = \int d^{2n} x \sqrt{g} \Lambda = \Lambda \int d^{2n} x \frac{1}{(1 - x^2)^{2n}}. \] \hfill (12)

Using the standard power counting arguments one can show that the integral diverges due to the pole at \( x^2 = 1 \). The fact that the action diverges is not a reason to abandon the AdS metric solutions to the equations \( R = \Lambda \). By the same token, despite the singularities of the instanton field configurations due to the non-compact nature of \( O(2n, 1) \), one should not disregard them as being irrelevant nor pathological.

A similar reasoning can be applied to the Schwarzschild solution that has a true physical singularity at the origin \( r = 0 \). And also to the Poincare metric associated with the projective disk model in the complex plane. The latter is singular at the boundary of the disc which can be holomorphically be mapped to the real axis of the complex plane.

These singular field configurations have "zero measure " contribution in the Euclidean path integral since they are exponentially supressed \( e^{-S_E} \to 0 \). At the classical level, one can regularize their singular contributions in the classical action. This is achieved by adding compensating surface terms to the bulk actions (8c, 8d) and in this fashion one can cancel out the divergent contributions due to the points \( x^2 = 1 \). These boundary terms are the analogs of the \( \theta \) topological terms in YM theories. Notice that the actions in eqs-(8c,8d) have the YM form: \( F \wedge^* F \). The topological \( \theta \) terms have the form: \( \theta F \wedge F \) where by in our case by the \( * \) operation one means the double star operation with respect to the base manifold indices and target space indices and is given up to factorial numerical factors by:

\[
{\text{\* \[ F_{i_1i_2\ldots i_n}^{\nu_1\nu_2\ldots\nu_n} \equiv \epsilon^{\mu_1\mu_2\ldots\mu_n\nu_1\nu_2\ldots\nu_n} \epsilon_{i_1i_2\ldots i_{2n+1}} \nu^{i_{2n+2}i_{2n+3}\ldots i_{2n}} \]}}
\hfill (13)
\]
The singular field configurations (11) are nothing but the solutions to the double self duality equations, therefore by adding the analog of a theta term (with the suitable sign) of the form: $\theta F \wedge F$, to the YM-type bulk action $F \wedge^* F$, given by eqs-(8c,8d) and evaluated at the singular field configurations (11), one will be able to cancel out those singular contributions to the bulk action due to the behaviour at $x^2 = 1$; i.e to the projective boundaries at infinity.

Another alternative is to simply introduce a natural “cutoff“ in defining/restricting the domain of integration by imposing $x^2 = \lambda^2 < 1$. The analog of this procedure is the selection of the fundamental modular domain of the string one loop (multiloop) path integrals. The Siegel upper half complex plane is noncompact with infinite area. Dividing by the action of the discrete modular group one ends up by having one single copy of the fundamental domain which is still noncompact but has a finite area. An analogous example of this (although not identical) is the action for a point particle. The action naturally diverges if one takes the interval of proper time from 0 to $\infty$. The cut-off simply states to define the action in the compact domain where the proper time is bounded $0, \tau$.

Before we finalize the discussion on the issue of singularities it is very important to emphasize that the solutions to the double self duality conditions are defined modulo conformal transformations of the base space $R^{2n}$. One could re-absorb the singular behaviour of the instanton solutions at $x^2 = 1$ inside these conformal scaling factors. For example, the AdS metric is singular at $x^2 = 1$ because the conformal factor is precisely singular at $x^2 = 1$. The AdS metric is conformally flat and therefore belongs in the same conformal class as the flat metric. Under this procedure the conformally re-scaled field configurations will be well behaved, however the action (8c, 8d) will still diverge due to the singular behaviour of the determinant of the new re-scaled metric. This is similar to what occurs with the Einstein-Hilbert action associated with the AdS metric: the scalar curvature $R = \Lambda$ is finite but the action diverges due to the singularity in the determinant of the metric at $x^2 = 1$.

After this discussion on adding a compensating boundary term to the actions (8c, 8d), etc., if one wishes to regularize the divergent contributions due to the natural singularities at $x^2 = 1$, which correspond to the projective boundaries at infinity, we will proceed to study the signature dependence in the definition of the duality operation. For the particular case when $2n = 4$, the double duality star operation is defined:

$$\ast F \equiv \epsilon^{\mu_1 \mu_2 \nu_3 \nu_4} \epsilon_{i_1 i_2 i_3 i_4 i_5} n^i F_{\mu_1 \mu_2 \nu_3 \nu_4}.$$

(14)

The $\ast \ast$ acting on $F$ is:

$$\ast \ast F = (-1)(-1)^{s_b}(-1)^{s_g}(-1)^f F.$$

(15)

where (i) the first factor of $-1$ stems from the $n^i n_i = -1$ condition. (ii) The second factor stems from the signature of the base space $R^4$. (iii) The third factor stems from the group signature which is $(-1)^1 = -1$ for $O(4,1)$. (iv) The last one, $(-1)^f$, are additional factors resulting from the permutation of the indices in the $\epsilon_{\mu_1 \ldots \mu_2 n}$ and $\epsilon_{i_1 \ldots i_{2n+1}}$ tensors. These are similar to the $(-1)^{r(D-r)}$ factor appearing in the double Hodge operation. Where $r$ represents the rank of $F$. In this case they yield the factor 1.
From (15) one immediately concludes that the signature of the base manifold must be: (+, +, +, +) so that

\[ **F = (-1)(-1)^{\delta b}(1)^{\delta g}F = (-1)(+1)(1)(1)F = F \Rightarrow ^* = \pm 1. \] (16a) 

and one has a well defined double (anti) self duality condition.

\[ ^*F = \pm F. \] (16b) 

For higher rank fields, \( F^{i_1\cdots i_n}_{\mu_1\cdots \mu_n} \) it follows from (15) that we must have in addition that \( n = \text{even} \). Therefore we conclude that the base manifold must be of Euclidean signature type and have for dimension: \( R^{4k} \) and the target group background \( O(4k, 1) \), whose topology is that of the Euclideanized \( AdS_{4k} \). Thus, the maps are signature-preserving and the dimensionality of the base manifold must be a multiple of four, \( D = 4k \).

It still remains to prove that the \( O(4k, 1) \) \( \sigma \)-field \( n^A(x) \) given by eqs-(11) solve the double self duality condition (10); that they saturate the minimum of the action to ensure topological stability, if indeed such minima exit, which is not necessary the case. Also the instanton solutions must yield a finite valued action; i.e the fields must fall off sufficiently fast at infinity so the action does not blow up:....

To satisfy these requirements it is essential to prove firstly that the classical action and classical Hamiltonian are positive definite. A close inspection reveals that one should have an Euclidean signature base manifold which forces one to choose an Euclidean signature \( AdS_{2n} \) space if the condition (16a) is to be satisfied. Therefore, instead of \( O(2n - 1, 2) \) one must have \( O(2n, 1) \). To illustrate the fact that the classical action is positive definite and without loss of generality we study the \( R^{4,0} \rightarrow AdS_{4,0} \) case. (the Euclideanization of \( AdS_4 \)). We must show that:

\[ (F_{\mu \nu}^{ij})^2 > 0. i, j = 1, 2, 3, 4, 5. \mu, \nu = 0, 1, 2, 3. \] (17)

In \( D = 4 \) there are six terms of the form:

\[ g^{00}g^{11}[(F_{01}^{ab})^2 - (F_{01}^{5a})^2] + g^{00}g^{22}[(F_{02}^{ab})^2 - (F_{02}^{5a})^2] + \ldots + \]

\[ g^{11}g^{22}[(F_{12}^{ab})^2 - (F_{12}^{5a})^2] + \ldots \] (18)

Due to the Euclidean nature of the base manifold the \( g^{00}g^{11} > 0 \) so we must show then that the terms inside the brackets in eq-(18) are all positive definite. The \( a, b \) indices run over: 1, 2, 3, 4 and the last one is \( i = 5 \) component \( n^5 \) associated with the non-compact \( O(4, 1) \)-valued \( n^i \) field. Setting \( n^i = (n^a, n^5) \equiv (\vec{\pi}, \sigma) \) the condition \( n^in_i = -1 \) yields:

\[ \sigma^2 = \vec{\pi} \cdot \vec{\pi} + 1 = \pi^2 + 1. \] (19)

which implies for every single component of \( x^\mu; \mu = 0, 1, 2, 3 \) (we are not summing over the \( x^\mu \)):

\[ (\partial_\mu n^5)^2 < \frac{\pi^2}{\sigma^2}(\partial_\mu \vec{\pi}).(\partial_\mu \vec{\pi}); \mu = 0, 1, 2, 3. \] (20)
Then:

\[(F_{01}^{5a})^2 = \frac{1}{\sigma^2} [(\partial_0 \pi^a) (\bar{\pi} \partial_1 \bar{\pi}) - (\partial_1 \pi^a) (\bar{\pi} \partial_0 \bar{\pi})]^2 < \frac{\pi^2}{\sigma^2} (\partial_0 \pi^a \partial_1 \pi^b - \partial_1 \pi^a \partial_0 \pi^b)^2. \quad (21)\]

where we used the relation \((\bar{A}, B)^2 \leq A^2 B^2\). The first term of (18) yields:

\[(F_{01}^{ab})^2 - (F_{01}^{5a})^2 > (1 - \frac{\pi^2}{\sigma^2}) (\partial_0 \pi^a \partial_1 \pi^b - \partial_1 \pi^a \partial_0 \pi^b)^2 > 0. \quad (22)\]

because \((1 - \frac{\pi^2}{\sigma^2}) > 0\) due to the relation (19). The same argument applies for each single component of \((F_{\mu\nu}^{ij})^2\) and hence the classical action and Hamiltonian are positive definite.

This is an essential ingredient for the instanton solution to saturate the minimum of the classical action. From ordinary Yang-Mills in Euclidean space and for compact gauge groups like \(SU(2)\) we have the familiar relation:

\[
\int d^4x \ (F - \ast F)^2 = 2 \int d^4x \ F^2 - 2 \int d^4x \ F\ast F \geq 0 \Rightarrow \int d^4x \ F^2 \geq \int d^4x \ F\ast F
\]

in the self dual case, the last equality holds. The last integral for the \(SU(2)\) group (topologically \(S^3\)) is the winding number \(S^3 \rightarrow S^3\), a topological invariant; hence the self dual instanton obeying \(F = \ast F\) saturates the lower bound of the positive definite action and ensures the topological stability. To finalize one must have that the field strengths fall off sufficiently fast at infinity to ensure that the action doesn’t blow up: \(r^n F_{\mu_1 \ldots \mu_n} \rightarrow 0\) as \(r \rightarrow \infty\).

In the non-compact \(AdS\) case studied here one naturally encounters singularities at \(x^2 = 1\) for the instanton configurations corresponding to the projective boundaries at infinity. The classical action for the \(O(4,1)\) \(\sigma\)-model in \(R^4\) was positive definite and since the instanton solutions saturate the lower bound of the classical action this implies that the action diverges. The source of its divergence is naturally due to the noncompact nature of the \(O(2n,1)\) group. To regularize the action one can add compensating boundary terms and/or restrict the domain of integration to a compact region inside the 2n-dimensional disc \(x^2 = 1\).

Whatever the choices to regularize the action may be this does not change the crux of this work: The instanton solutions found in eqs-(11) are genuine solutions of the double self duality conditions (8b,13) which are an integrable subset of the equations of motion for the conformally invariant \(\sigma\)-models on \(AdS_{2n}\) spaces given by the action of eqs-(8c,8d). The instanton solutions are defined modulo a conformal transformation of the base manifold \(R^{2n}\).

It remains to prove that the solutions (11) obey the double self duality condition (10). This former property is a straightforward extension of the results in [1,3] for the compact case. Given the stereographic projections defined by eq-(11): \(R^{2n} \rightarrow H^{2n}\) one can find a set of independent \(2n + 1\) frame vectors, \(E_p\), representing the pullback of the orthogonal frame vectors of the \(2n + 1\)-dim pseudo-Euclidean manifold onto the Euclidean signature \(AdS_{2n}\) space:
\[ E_1 = \partial_x n^i; \quad E_2 = \partial_x n^i; \quad \ldots E_{2n} = \partial_x n^i. \quad E_{2n+1} = n^i. \quad i = 1, 2, \ldots, 2n + 1 \]  \tag{24}

obeying the orthogonality condition:

\[ g_{\mu\nu} \equiv E^i_{\mu} E^i_{\nu} = \partial_\mu n^i \partial_\nu n^i = \frac{4}{(1-x^2)^2} \eta_{\mu\nu}. \]  \tag{25}

which precisely yields the Euclideanized \( AdS_{2n} \) metric \( g_{\mu\nu} \) defined as the pullback of the embedding metric, satisfying \( R_{\mu\nu} \sim g_{\mu\nu}\Lambda \) and \( R \sim \Lambda \). The latter curvature relations are the hallmark of Anti de Sitter space (negative scalar curvature). The positive definite measure spanned by the collection of \( 2n + 1 \) frame vectors defines the natural volume element of the Euclideanized \( AdS_{2n} \):

\[ \sqrt{\det h} = \det (E_1, E_2, \ldots, E_{2n}, E_{2n+1}). \]  \tag{26}

It was shown by [1, 3] (for the compact case the orthogonality condition is naturally modified to yield the metric of the \( S^{2n} \)) that the orthogonality conditions eqs-(25) are the necessary and sufficient conditions to show that the particular field configurations \( n^i(x) \) given by (11), after extending the four dimensional result to any \( D = 2n \), satisfy the double self-duality conditions eq-(10). The orthogonality relation implies that the angles among any two vectors is preserved and that all vectors after the conformal mapping has been taken have equal length relative to each other (this does not mean that they have the same length as before!).

This is a signal of a conformal mapping. The “stereographic” maps are a particular class of conformal maps. The action of the conformal group in \( R^{2n} \) will furnish the remaining conformal maps [1, 3]. Hence, we see here once again the interrelation between self-duality and conformal invariance. In two dimensions self-duality implies conformal invariance (holomorphicity). In higher dimensions matters are more restricted, they are not equivalent.

Therefore, setting aside the subtleties in the quantization program due to the non-compact nature of \( O(2n, 1) \), we have shown that these \( O(2n, 1) \) \( \sigma \)-model instanton field configurations obtained by means of the “stereographic” (conformal) maps of \( R^{2n} \rightarrow H^{2n} \), correspond precisely to the coordinates, \( y^1, y^2, \ldots, y^4, y^5 \) of the 5-dim pseudo-Euclidean manifold onto which the Euclidean signature \( AdS_4 \) space is embedded. To illustrate this lets focus on Euclidean signature \( AdS_4 \) defined as the geometrical locus:

\[ y^2 \equiv (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 = -\rho^2 = -1. \]  \tag{27}

where we set the \( AdS \) scale \( \rho = 1 \). The \( AdS_4 \) coordinates: \( z^0, z^1, z^2, z^3 \) are related to the \( y^A \) using the “stereographic” projection (see the lectures by Petersen [6]) from the “south” pole of the Euclidean signature \( AdS_4 \) to the equator:

\[ y^\alpha(z) = \rho \frac{2z^\mu}{1 - z^2}; \quad \mu = 0, 1, 2, 3. \quad a = 1, 2, 3, 4. \quad y^5(z) = \rho \frac{1 + z^2}{1 - z^2}. \]  \tag{28}
it is straightforward to verify that $y^2 = -\rho^2 = -1$. The Euclidean signature $AdS$ corresponds to two hyperbolic branches. A north and south branch respectively intersecting the vertical axis at $y^5 = \pm \rho$. The mappings are performed by projecting the rays stemming from the "south pole" $S$, the vertex of the southern hyperbolic branch located at $y^5 = -\rho$, intersecting the equator $R^{2n}$ (passing through the origin dividing the north and south branches) at point $P'$. One extends the rays $SP'$ through the equator upwardly until they intersect the upper hyperbolic branch at point $P$. It is in this fashion how one constructs the "stereographic" mappings (28).

The Euclidean signature $AdS_4$ metric is given:

$$g_{\mu\nu} = \frac{4}{(1-z^2)^2} \eta_{\mu\nu} \quad z^2 = (z^0)^2 + \ldots + (z^3)^2. \quad (29)$$

As mentioned above, the metric blows up at $x^2 = 1$, points that correspond to the projective boundaries at infinity. As we can verify by inspection, there is an exact match with the instanton field configurations $n^i(x)$ (11). The metric (29) also matches (25) after the correspondence: $n^a(x) \leftrightarrow y^\mu(z)$ and $n^5(x) \leftrightarrow y^5(z)$ is made.

Therefore, the main conclusion of this section is: The instanton field configurations of the conformally invariant $O(2n,1)$ $\sigma$-models, obeying the double self-duality conditions (8b,13) correspond geometrically to the conformal "stereographic" maps of $R^{2n}$ into the Euclidean signature $AdS_{2n}$ spaces.

These conformally invariant $O(2n,1)$ sigma model actions in $R^{2n}$ are just the sigma model generalizations of the MacDowell-Mansouri action for ordinary gravity based on gauging the conformal group $SO(4,2)$ in $D = 4$ Minkowski space and the Anti de Sitter group $SO(3,2)$. The ordinary Lorentz spin connection and the tetrad in four dimensions are just pieces of the $SO(3,2)$ gauge connection. Upon setting the torsion to zero one recovers the 4D Einstein-Hilbert action with a cosmological constant and the Gauss-Bonnet topological term. The conformally invariant sigma models actions [1] are the ones required in building conformally invariant bosonic $p = 2n$-brane actions in flat or curved target backgrounds. It would be interesting to see what sort of actions can be derived following the analog of a MacDowell-Mansouri procedure to obtain Einstein gravity from a gauge theory.

III.

In this section we shall discuss in detail the connection between Self-Dual $p$-branes, Chern-Simons $p'$-Branes (for $p = p' + 1$) and $W$ Geometry. We will also outline the straightforward steps to derive (spacetime-filling) $p$-branes from Moyal Deformation Quantization of Generalized Yang Mills Theories [1, 34]. In particular, we shall show how Chern-Simons membranes and Hadronic bags emerge from Quenched Large $N$ QCD. We leave all the details for the references [34, 35].

3.1 Self Dual $p$-branes

As pointed in the introduction, Zaikov noticed that self-dual $p$-branes, when $p + 1 = D = 2n$, are related to Chern-Simons $p'$-branes ($p' + 1 = p$) provided the embedding man-
ifold is Euclidean [11]. The Euclidean $AdS_{2n}$ space can be seen, not only as a $O(2n, 1)$ $\sigma$-model instanton, but also as a self-dual $p$-brane whose world volume is the $AdS_{2n}$ space ($p + 1 = 2n = D$ dimensional). To see this we must first write down Dolan-Tchrakian action for a bosonic $p$-brane with $p + 1 = 2n$ embedded in a flat/curved target space of dimensionality $D = p + 1 = 2n$. Dolan-Tchrakian action is valid for flat or curved target spacetime metrics [2].

\[
S = T \int d^{2n} \sigma \left[ E^{p_1} \wedge E^{p_2} \wedge ... \wedge E^{p_n} \right] \wedge * \left[ (E^{q_1} \wedge E^{q_2} \wedge ... \wedge E^{q_n}) \right] \eta_{p_1q_1} \eta_{p_2q_2} ... ... \tag{30}
\]

the action is YM like $F \wedge * F$ and was derived based on the conformally invariant $O(2n+1)$ $\sigma$-model actions. The star operation in (30) is the standard Hodge duality one defined w.r.t the $p + 1$-dim world-volume metric of the $p$-brane: $h_{ab}$. The world-volume one-forms, $E^p; p = 1, 2, ... 2n$ are obtained as the pullbacks of the $D = 2n = p + 1$ one forms, $e^A; A = 1, 2, ... 2n$ associated with the $D = 2n$ dimensional flat/curved target space onto which we embed the world-volume of the $p$-brane. A self-dual $p$-brane (self dual w.r.t the Hodge star operation) obeys:

\[
* (E^{q_1} \wedge E^{q_2} \wedge ... \wedge E^{q_n}) \wedge (E^{q_1} \wedge E^{q_2} \wedge ... \wedge E^{q_n}) = [E^{p_1} \wedge E^{p_2} \wedge ... \wedge E^{p_n}]. \tag{31}
\]

a self-dual $p$-brane will automatically satisfy the equations of motion. This is the $p$-brane generalization of the relation $d*F = *j = 0$ in the absence of sources in the YM equations. When $*F = F$ the latter equations become the Bianchi identities $dF = 0$. The action (30) for a self-dual $p$-brane becomes then the integral over the ordinary Jacobian ($p + 1 = 2n$ volume form) associated with the change of variables $\sigma \rightarrow X$ and hence is topological. If the world volume has a natural boundary an integration by parts (Gauss law) will yield the Chern-Simons $p' = p - 1$-brane. In this way we can see once more how the self-dual $p$-brane is related to the Chern-Simons $p' = p - 1$-brane (at least on shell).

This is the first step. One needs to show now that the self-duality condition (31) is directly related to the self-duality condition formulated by Zaikov when $p + 1 = 2n$ and, furthermore, that it is also related to the double-self duality conditions (10) associated with the conformally invariant $O(2n, 1)$ $\sigma$-models in $R^{2n}$. The self-duality condition formulated by Zaikov when the embedding manifold is Euclidean and when $p + 1 = 4$, for example, is [11]:

\[
\mathcal{F}_{ab}^{JK} = \partial_{a} X^{J} \partial_{b} X^{K} - \partial_{a} X^{J} \partial_{b} X^{K} = \frac{1}{4} \epsilon_{abcd} \epsilon^{JKLM} \mathcal{F}_{cd}^{LM}. \tag{32}
\]

where $X^I(\sigma^a)$ are the embedding coordinates of the $p$-brane into the target space, $X^I; I = 1, 2, ... 2n$. Eq-(32) is the self-duality condition for case when $p = 3$ ; $p + 1 = 4$ obtained from Dirac-Nambu-Goto actions. The Dolan-Tchrakian action (30) is equivalent at the classical level to the Dirac-Nambu-Goto actions upon elimination of the auxiliary world-volume metric, $h_{ab}$. Whether or not this equivalence occurs Quantum Mechanically is another issue. Hence, classically, Zaikov’s self-duality conditions (32) are equivalent to the self-duality conditions (31) of the self-dual extendon furnished by Dolan-Tchrakian actions (30).
Now we must proceed with the connection between the latter self-duality for extendons (p-branes) and the double self duality conditions associated with the $O(2n, 1)$ $\sigma$-models. The self-duality condition of Zaikov has exactly the same form as the double self-duality condition for the $O(4)$ YM instanton (8a), obtained from the compact $O(2n + 1)$ $\sigma$ model, after using the constraint (7c), $D_{\mu}(A)n^i = 0$ ; $i = 1, 2, ..., 2n, 2n + 1$ (for the particular case $n = 2$), and the relations (7a,7b) which allow to reduce $O(5)$ to $O(4)$ and show that the $O(4)$ YM field is a composite field expressed in terms of the $n^i$. We need now to establish the relations/correspondences between the p-brane coordinates $X_I(\sigma^a)$ of (32) and the $O(4)$ YM composite fields of eqs-(7a,7b) appearing in eq-(8a). The ensuing relation is:

$$A_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta} \leftrightarrow X^J(\sigma^a) . \quad F_{\mu\nu}^{\alpha\beta}\Sigma_{\alpha\beta} \leftrightarrow F_{ab}^{JK} = \partial_{\sigma^a}X^J\partial_{\sigma^b}X^K - \partial_{\sigma^b}X^J\partial_{\sigma^a}X^K . \quad (33)$$

where $\Sigma_{\alpha\beta}$ are the $2 \times 2$ matrices (Pauli) corresponding to the chiral representation of $SO(4) \sim SU(2) \times SU(2)$. The $\mu, \nu$,.. indices in the l.h.s of (33) run over the four dimensional base manifold $R^4$ where the $O(4)$ YM fields live. The $I, J, K$... in the r.h.s of (33) run over the four-dimensional embedding target space. The $\sigma^a$ indices run over the four-dimensional world volume of the $p = 3$-brane.

The last equation is reminiscent of the chiral model approaches to Self Dual Gravity based on Self Dual Yang Mills [31] theories. A Moyal deformation quantization of the Nahm equations associated with a $SU(2)$ YM [32] theory yields the classical $N \to \infty$ limit of the $SU(N)$ YM Nahm equations directly, without ever using the $\infty \times \infty$ matrices of the large $N$ matrix models. By simply taking the classical $\hbar = 0$ limit of the Moyal brackets, the ordinary Poisson bracket algebra associated with area-preserving diffs algebra $SU(\infty)$ is automatically recovered. This supports furthermore the fact that the underlying geometry may be noncommutative [18,32] and that p-branes are essentially gauge theories of area/volume...preserving diffs [14,20,21].

The way this is attained is the following. A Moyal quantization takes the operator $\hat{A}_\mu(x^\mu)$ into $A_\mu(x^\mu; q, p)$ and commutators into Moyal brackets. A dimensional reduction to one temporal dimension brings $A_\mu(t, q, p)$, which precisely corresponds to the membrane coordinates $X_\mu(t, \sigma^1, \sigma^2)$ after identifying $\sigma^a$ with $q, p$. The $\hbar = 0$ limit turns the Moyal bracket into a Poisson one. It is in this fashion how the large $N$ $SU(N)$ matrix model bears a relation to the physics of membranes. The Moyal quantization explains this in a straightforward fashion without having to use $\infty \times \infty$ matrices!

Therefore, the self-duality conditions of the Dolan-Tchrakian p-brane actions in Euclidean embedding manifolds (for the case $p+1 = 4$) (31) have a one-to-one correspondence with Zaikov’s self-duality conditions (32) (for Euclidean embedding manifolds) and with the double self-duality conditions of the $O(4)$ YM instanton (8a), if, and only if, the gauge fields/target space coordinates correspondence (33) is used.

Finally, if the Euclidean signature $AdS_{2n}$ space is to be seen as a self-dual p-brane; i.e. as the Euclideanized world-volume of an open self-dual p-brane such $p + 1 = 2n$, in addition to an $O(2n, 1)$ $\sigma$-model instanton in $R^{2n}$, we must analytically continue $O(2n, 1)$ to $O(2n + 1)$ and, afterwards, follow the same procedure as above, relating the $O(5)$ $\sigma$-model to $O(4)$ YM and its Generalized Yang-Mills extensions in $D = 2n = 4k$ dimensions [10]. For example, one could first reduce $O(4, 1)$ to $O(3, 1)$ using equations like (7a,7b,7c).
and perform afterwards the analytical continuation $O(3,1)$ to $O(4)$. It is important to have a compact group like $O(4)$ otherwise we will run into the same problems of non-positive definite actions, Hamiltonians...

To sum up, the arguments ranging from eqs-(31-33), are the necessary ones required that would allow us to view the Euclidean $AdS_{2n}$ space as the Euclideanized world volume of an open self-dual $p$-brane ($p + 1 = 2n$) in addition to being an $O(2n,1)$ $\sigma$-model instanton. In section 3.2 we will present further evidence which suggests that there is a formal equivalence between the two definitions of duality; i.e after using the gauge fields/coordinates correspondence (33) one may be able to show that the Euclidean $AdS_{2n}$ space is the Euclideanized world volume of an open self-dual $p$-brane ($p + 1 = 2n$) in addition to being an $O(2n,1)$ $\sigma$-model instanton.

After having presented the discussion of the self-dual $p$-branes /Chern-Simons $p' = p-1$-branes relationship in target spacetimes of dimension $D = p+1 = p'+2 = 2n = 4k$, we will choose a particular higher dimensional example. The topological Chern-Simons action for the $p = 10$-brane is defined over the $d = 11$-boundary domain of $R^{12}$. Boundary to-boundary maps are a special class, among the infinite family of embeddings of the Chern-Simons $p = 10$-brane into $AdS_{12}$, which conformally map the $d = 11$-dim boundary domain into the projective boundary of the $AdS_{12}$ space: which is topologically $S^1 \times S^{10}$.

The Euclidean signature $AdS_{12}$ has, instead, for projective boundary the $S^{11}$. The signature matching condition (16a) imposes the signature of the base manifold to be Euclidean. If one views $R^{12}$ as the interior of a twelve-dimensional ball, $B_{12}$ of very large (\(\infty\)) radius, the $d = 11$ boundary domain is effectively $S^{11}$. Therefore, for the very special case associated with boundary to-boundary maps, the Chern-Simons $p = 10$-brane can be characterized by the winding number of the mappings of $S^{11} \rightarrow S^{11}$. In this special case, the Chern-Simons $p = 10$-brane lives effectively on the projective boundary of Euclidean $AdS_{12}$. We emphasize that this is just a particular case, since in general, the topological Chern-Simons $p = 10$-brane lives on the whole bulk of $AdS_{12}$. Since the Chern-Simons $p = 10$-brane theory is topological, it has no local bulk degrees of freedom, only global, that are naturally confined to the boundary. In this sense the theory is holographic [27,28].

Furthermore, since $AdS_{12}$ can be seen as a hyperboloid embedded in a $d = 13$-dim pseudo-Euclidean manifold we have $d = 11,12,13$ for characteristic dimensions; hence this construction may be relevant to understand the intricate relations of $M,F,S$ Theory [25,26] and to shed some light into the geometrical, holographic and topological underpinnings behind the AdS/CFT duality conjecture.

A Chern-Simons $p = 11$-brane living on a Euclidean signature $AdS_{13}$ embedded in a $d = 14$-dim pseudo Euclidean space has for world volume a 12-dim manifold. In this case the relevant dimensions would be $11,12,13,14$. The main difference is that the odd dimensional $AdS_{13}$ will not corespond to an instanton configuration associated with an $O(13,1)$ $\sigma$-model because in odd dimensions conformally invariant sigma models of the Dolan-Tchrakian type cannot be constructed. Higher dimensional topological actions have been proposed by Chapline to describe unique theories of gravity and matter [24].

### 3.2 Branes from Moyal Deformation Quantization of GYM Theories
Chern Simons Membranes and Hadronic Bags from Quenched large N QCD

SU(N) reduced, quenched gauge theories have been shown to be related to string theories in the large N limit. We will review very briefly the steps that show how a 4D YM reduced, quenched, theory supplemented by a topological \( \theta \) term can be related, through a Weyl Wigner Groenowold Moyal (WWGM) quantization procedure, to an open 3 – brane model [35]. The bulk is the interior of a hadronic bag and the boundary is the Chern-Simons world volume of a membrane. The boundary dynamics is not trivial despite the fact that there are no transverse bulk dynamics associated with the interior of the bag. For further details we refer to [35].

The reduced and quenched YM action in \( D = 4 \) is [36]:

\[
S = -\frac{1}{4} \left( \frac{2\pi}{a} \right)^4 \frac{N}{g_{YM}^2} Tr \ F_{\mu\nu} F^{\mu\nu}. \quad F_{\mu\nu} = [i D_\mu, i D_\nu].
\]

(34)

Notice that the reduced quenched action is define at a “ point “ \( x_o \). For simplicity we are omitting the matrix \( SU(N) \) indices.

The \( \theta \) term is:

\[
S = -\frac{\theta N g_{YM}^2}{16\pi^2} \left( \frac{2\pi}{a} \right)^4 \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} Tr \ F_{\mu\nu} F_{\rho\sigma}.
\]

(35)

The WWGM quantization establishes a one-to-one correspondence between a linear operator \( D_\mu \) acting on the Hilbert space \( \mathcal{H} \) of square integrable functions on \( R^D \) and a smooth function \( A_\mu(x,y) \) which is the inverse Fourier transform of \( A_\mu(q,p) \). The latter is obtained by evaluating the Trace of the operator \( D_\mu \) by means of summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space \( \mathcal{H} \). Under the WWGM correspondence the matrix product turns into the Moyal \(*\) product and the commutator turns into the Moyal bracket:

\[
\frac{1}{\hbar} [A_\mu, A_\nu] \rightarrow \{A_\mu, A_\nu\}.
\]

(36)

The WWGM deformation of the actions is:

\[
S = -\frac{1}{16} \left( \frac{2\pi}{a} \right)^4 \frac{N}{g_{YM}^2} \frac{1}{2} \int d^2D \sigma \ F_{\mu\nu} * F^{\mu\nu}.
\]

(37)

The corresponding WWGM deformation of the \( \theta \) term is:

\[
S_\theta = -\frac{\theta N g_{YM}^2}{64\pi^2} \left( \frac{2\pi}{a} \right)^4 \frac{1}{2} \int d^2D \sigma \ \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} * F_{\rho\sigma}.
\]

(38)

where the Moyal mappings of the initial field strenghts are explicit \( \sigma \) dependent (phase space) quantities:

\[
F_{\mu\nu}(\sigma) \quad \text{where } \sigma \text{ runs over the } 2D\text{-dimensional phase space spanned by } q^i, p^i; \quad i = 1, 2, 3...D.
\]

By making the gauge fields/coordinates correspondence:

\[
A_\mu(\sigma) \rightarrow \left( \frac{2\pi}{N} \right)^{1/4} X_\mu(\sigma). \quad F_{\mu\nu}(\sigma) \rightarrow \left( \frac{2\pi}{N} \right)^{1/2} \{X_\mu(\sigma), X_\nu(\sigma)\}
\]

(39a)
performing the correspondence between the Trace and phase space integration:

\[
\frac{(2\pi)^4}{N^3} Tr \to \int d^{2D}\sigma.
\]

(39b)

and by taking the deformation parameter "\(\hbar\)" of the WWM quantization to be \(2\pi/N = \hbar = 0\) then the "classical limit" is nothing but the equivalent of the large \(N\) limit. Hence, the Moyal brackets turn into Poisson brackets, the factors of \(N\) decouple and the first action (37) is just equivalent to the Dolan-Tchrakian action evaluated in the conformal gauge. The second action (38) is the Chern-Simons Zaikov action for a membrane whose world volume lives on 3-dim the boundary of the hadronic bag. Upon identifying the inverse lattice spacing of the large \(N\) quenched, reduced QCD with the QCD scale of \((2\pi/a) = \Lambda_{QCD} = 200\ Mev\) we obtained the value of the dynamically generated bag pressure \(\mu_o\) given by:

\[
\mu^4_o = \frac{1}{4\pi} (200\ Mev)^4 \frac{1}{(g^2_{YM}/4\pi)}.
\]

(40a)

taking for the conventionally assumed value of \(g^2_{YM}/4\pi \sim 0.18\) we get an actual value for the hadronic bag constant \(\mu_o\) close to the value of \(\Lambda_{QCD} = 200\ Mev\) which falls within the range of the phenomenologically accepted values of \(\mu_o \sim 110\ Mev\), if we take into account the uncertainty on the value of the \(\Lambda_{QCD}\) which lies in the range of: \(120\ Mev < \Lambda_{QCD} < 350\ Mev\). One should notice also the interesting fact. If one sets the QCD scale \(a\) to be:

\[
a^4 = NL^4_{Planck} \ln \mu^4_o \sim \frac{1}{a^4 g^2_{YM}} \Rightarrow \mu^4_o \sim \frac{1}{L^4_{Planck} (Ng^2_{YM})}.
\]

(40b)

one makes contact with Maldacena’s celebrated result which relates the size of the Anti de Sitter space throat \(a^4\) with the 't Hooft coupling \(Ng^2_{YM}\) and the Planck scale \(L^4_{Planck}\) (Gravity). The large \(N\) limit, or strong coupling limit, is implemented by the double scaling relation \(L^4_{Planck} N = a^4\) finite as \(N \to \infty\) and \(L^4_{Planck} \to 0\). Hence, relations (40a, 40b) very naturally reflect Maldacena’s result which support his AdS/CFT duality conjecture obtained from a completely different approach that did not require compactifications. True, we need to work further to show why \(a^4 = NL^4_{Planck}\), how gravity emerges. Since we obtained brane actions from YM (GYM) theories via the Moyal deformation quantization and branes do contain gravity it is not surprising to arrive at similar results.

To extend these results to higher \(p\)-branes than \(p = 3\) one needs to perform the Moyal deformation of Generalized Yang Mills Theories (GYM) as shown by \([1, 34, 35]\). The GYM are based on \(SO(4k)\) in \(R^D\) where \(D = 4k\):

\[
L = \frac{1}{g^2} tr (F^\alpha_{\mu_1\mu_2...\mu_{2k}} \Sigma_{\alpha_1...\alpha_{2k}})^2.
\]

(41)

where \(g\) is a dimensionless coupling constant and:

\[
\Sigma_{\alpha_1...\alpha_{2k}} = \Sigma_{\alpha_1\alpha_2} \Sigma_{\alpha_3\alpha_4}...
\]

(42)
is an anti-symmetrized product of $k$ factors of the $2^{2k-1} \times 2^{2k-1}$ matrices $\Sigma_{\alpha\beta}$ corresponding to the chiral representation of $SO(4k)$. For $k=1$ one has the usual $SO(4) \sim SO(3) \otimes SO(3)$ whose double cover is $SU(2) \otimes SU(2)$. For further details we refer to [1,34].

The analogous procedure of quenching and reduction is attained by looking only at the zero mode sector of the theory. One can write down the action for those field strength configurations that are everywhere constant. A local gauge transformation will rotate them to other field strength configurations that are space-time dependent. Since the invariant action requires performing a group trace to attain gauge invariance, one may simply focus, from the start, on those field strength configurations that are everywhere constant. Performing a WWGM deformation of the Lie algebraic structure in each single one of the subfactors $\Sigma_{\alpha\beta}$ appearing in the definition of the $SO(4k)$-valued field strengths and integrating out the spacetime dependence one arrives [34]:

$$\frac{\Omega_{4k}}{g^2} \int d^{4k} \sigma \left[ \{ A_{\mu_1}, A_{\mu_2} \} \{ A_{\mu_3}, A_{\mu_4} \} \ldots \{ A_{\mu_{2k-1}}, A_{\mu_{2k}} \} \right]^2. \tag{43}$$

where $\Omega_{4k}$ is the spacetime volume resulting from the spacetime integration of constant field configurations (zero modes). Taking the $\hbar = 0$ classical limit and establishing the gauge fields/coordinates $A_{\mu} \rightarrow (X_{\mu}/l_p)$ correspondence (units require dividing by the Planck scale $l_p$) one arrives at the following $p$-brane action and Tension:

$$\frac{\Omega_{4k}}{g^2 l_p^8} \int d^{4k} \sigma \left[ \{ X_{\mu_1}, X_{\mu_2} \} \{ X_{\mu_3}, X_{\mu_4} \} \ldots \{ X_{\mu_{2k-1}}, X_{\mu_{2k}} \} \right]^2. \quad T_p = \frac{\Omega_{4k}}{g^2 l_p^8}. \tag{44}$$

Notice that the fields in the reduced-quenched $SU(N)$ QCD action do not have the canonical dimensions. They are rescaled [35]. The action (44) is equivalent to the DT action [34] in the conformal gauge. As before, this bulk action does not have transverse degrees of freedom: one simply can choose the orthonormal gauge to see that this action reduces to a pure bulk volume term. equivalent to the DT action after choosing the conformal gauge. However, the analog of a the topological theta terms are not trivial. The have true boundary dynamics and these correspond naturally to the Chern-Simons $p'$-branes discussed earlier. Following the same steps as we did for the theta terms in the reduced, quenched large $N$ QCD action one can verify that the Lagrangian density for the $\theta$ terms:

$$L_\theta \sim \theta g^2 \epsilon^{\mu_1 \ldots \mu_n \nu_1 \ldots \nu_n} tr \left( F^{a_1 a_2 \ldots a_{2k}}_{\mu_1 \mu_2 \ldots \mu_{2k}} \Sigma_{\alpha_1 \alpha_2 \ldots \alpha_{2k}} (F^{\beta_1 \beta_2 \ldots \beta_{2k}}_{\nu_1 \nu_2 \ldots \nu_{2k}} \Sigma_{\beta_1 \beta_2 \ldots \beta_{2k}} \right). \tag{45}$$

will yield the Zaikov Chern-Simons $p'$-brane actions after performing the WWGM deformation of the Lie algebraic structure. Upon taking the $\hbar = 0$ limit; identifying the traces with integrals over the phase space variables which later are identified with the world volume coordinates, and establishing the gauge field/coordinates correspondence one ends up with:

$$S_{CS} \sim \frac{\Omega_{4k} \theta g^2}{l_p^8} \int d^{4k} \sigma \ dX_{\mu_1} \wedge dX_{\mu_2} \wedge \ldots \wedge dX_{\mu_{4k}}. \tag{46}$$
after an integration by parts that will yield the Zaikov Chern-Simons $p'$-brane action integrated over the $4k-1$-dim boundary so that the value of $p'$ must obey $p' + 1 = 4k - 1$.

Having shown that certain brane actions and the associated Chern-Simons branes, can be obtained from a Moyal deformation of the Lie algebraic bracket structure of the YM (GYM) theories and that Chern-Simons membranes and Hadronic Bags emerge from a reduced, quenched large $N$ QCD in four dimensions, for example, allows to confirm once more what have established in this work:

- Topological Chern-Simons $p$-Branes and Conformally Invariant $\sigma$ models on $AdS$ spaces are related and connected with $SO(4k)$ Generalized YM theories in $R^{4k}$ (where $k = 2n$).
- We have closed the web among three related theories: (i) Chern-Simons $p$-branes on $AdS_{2n}$ spaces with $p + 2 = 2n$; (ii) $O(2n, 1)$ Conformally invariant $\sigma$ models in $R^{2n}$ with target Euclideanized $AdS_{2n}$ backgrounds and (iii) obtained branes from a Moyal deformation of $SO(4k)$ GYM theories in $R^{2n} = R^{4k}$.
- Also, the relationship between self duality and conformal invariance has been established: The Euclidean signature $AdS_{2n}$ space was shown to correspond precisely to instanton field configurations of the noncompact $O(2n, 1)$ $\sigma$-models in $R^{2n}$, obeying the double self duality condition (10), which was the analog of the BPST instanton [9].

### 3.3 $W$ Geometry

To finalize we shall mention our belief in the importance that $W$ symmetry and its higher dimensional extensions should have in understanding $M$ theory. $W$ geometry was viewed as the geometry associated with the Moyal-Fedosov Deformation program associated with the symplectic geometry of the cotangent bundles of 2D Riemannian surfaces; the role of 4D Self Dual Gravity was also emphasized in [18]. Geometric induced actions for $W_\infty$ gravity based on the coadjoint orbit method associated with $SL(\infty, R)$ WZNW models were constructed by Nissimov, Pacheva and Vaysburd [23]. $W_\infty$ gravity has a hidden $SL(\infty, R)$ Kac-Moody symmetry. Likewise, the $SL(\infty)$ Toda model obtained from a rotational Killing symmetry reduction of 4D Self Dual Gravity (an effective 3D theory) has $W_\infty$ symmetry. Once again we can see the intricate relationship between self duality and conformal field theory in higher dimensions.

To this end we concentrate now on what perhaps is the most significant and salient feature of Chern-Simons $p$-branes: the fact that they admit an infinite number of secondary constraints which form an infinite dimensional closed algebra with respect to the Poisson bracket. [11] Such algebra contains the classical $w_{1+\infty}$ as a subalgebra. The latter algebra corresponds to the area-preserving diffeomorphisms of a cylinder.; the $w_\infty$ algebra corresponds to the area-preserving diffs of a plane; $su(\infty)$ for a sphere.....[14]. These $w_\infty$ algebras are the higher conformal spin $s = 2, 3, 4, ... \infty$ algebraic extensions of the 2D Virasoro algebra.

Higher Spin Algebras (superalgebras) in dimensions greater than two have been furnished by Vasiliev and in [15] were used to describe higher spin gauge interactions of massive particles in $AdS_3$ spaces. These higher spin algebras have been instrumental lately in [16] to construct the $N = 8$ Higher Spin Supergravity in $AdS_4$ which is conjectured to be the field theory limit of M theory on $AdS_4 \times S^7$. 

18
Crucial in the construction of the Vasiliev higher spin algebras is the Moyal star products and the fact that these algebras required an Anti de Sitter space. For the relevance of Moyal Brackets in M theory we refer to Fairlie [19]. It has been speculated that the \( W_\infty \)-symmetry of \( W_\infty \) strings after a Higgs-like spontaneous symmetry breakdown yields the infinite massive tower of string states. In particular, anomaly free non-critical (super) \( W_\infty \) strings required \( (D = 11) \) \( D = 27 \) dimensions [22] which are precisely the alleged critical dimensions of the (super) membrane.

Moyal star products are non-local due to the infinite number of derivatives. This nonlocality in conjunction with the fact that Anti de Sitter spaces are required may be relevant in understanding more properties of singleton, doubleton...field theories which are very important in the \( AdS/CFT \) duality conjecture. The massless excitations of the \( CFT \) living on the projective boundary of Anti de Sitter space, associated with the propagation of the supermembrane on \( AdS_d \times S^{D-d} \), are composites of singleton, doubletons...fields [17]. The \( O(5) \) sigma models actions (8c) and the corresponding \( O(4)YM \) fields in eqs-(7a,7b) are based on composite fields ; i.e made out of the \( n^i(x) \). New actions for all \( p \)-branes where the \( \text{analogs of } S \) and \( T \) duality symmetries were built in, already from the start, were given in [20] based on the composite antisymmetric tensor field theories of the volume-preserving diffs group of Guendelman, Nissimov, Pacheva and the local field theory reformulation of extended objects given by Aurilia, Spallucci and Smailagic [21]. This supports the idea that compositeness may be a crucial ingredient in the formulation of \( M \) theory.

4. Concluding Remarks

Based on the relation established in this work among Conformally Invariant \( \sigma \) models on \( AdS \) spaces, Chern-Simons \( p \)-branes, \( W \) geometry , including the Moyal deformation quantization of Generalized Yang Mills Theories (GYM) ( plus \( \theta \) terms ) to yield brane actions ( plus Chern-Simons branes ) that have as an example the Chern-Simons membrane and Hadronic Bag solutions to the quenched large \( N \) limit of \( SU(N)D = 4 \) \( QCD \), we believe that \( M \) theory should have some of the features described below :

- Higher dimensional topological and holographic origins. A particular example of this theory is the topological Chern-Simons \( p = 10\)-brane living on the bulk of \( AdS_{12} \), without local degrees of freedom , with only global degrees of freedom confined to the \( d = 11 \) dimensional boundary. Since \( AdS_{12} \) can be embedded in a \( d = 13 \) dimensional pseudo Euclidean space, \( d = 13 \) is a relevant dimension [24].
- A \( W \) geometric framework extending the role of ordinary 2D CFT, as explained above, to higher dimensions.
- A compositeness structure, like the conformally invariant sigma models in \( R^{2n} \) with target noncompact \( O(2n, 1) \) group manifolds and where the Euclidean signature \( AdS_{2n} \) space is an instanton solution, obeying a double self duality condition, and a self-dual \( p \)- extendon, with \( p + 1 = 2n \) ( assuming that the there is a one-to-one correspondence between the two).
- The Supersymmetric case has not been discussed here but it must be included.
- We close this work with an interesting thought. The \( n \rightarrow \infty \) limit of the \( O(2n, 1) \) \( \sigma \)-models is connected with the \( D = 2n \rightarrow \infty \) limit of the \( AdS_{2n} \) space. Interestingly enough,
Zaikov has pointed out that in the $D = \infty$ limit these Chern-Simons $p$-branes acquire true local dynamics!

**Acknowledgements**

I wish to thank George Chapline for many fruitful discussions at the early stages of this work; to M. Peskin for his assistance at SLAC, Stanford; to E. Abdalla, M. Gomes, M. Sampaio, E. Valadar for their help at the Institute of Physics, Sao Paulo and the UFMG, Belo Horizonte. Special thanks goes to M. L Soares de Castro and M. Fernandes de Castro for their warm hospitality in Belo Horizonte, Brazil where this work was completed. Finally, many thanks to M. Pavsic for his warm invitation to the Jozef Stefan Institute in Ljubljna, Slovenia.

**References**

1. B. Dolan, D.H Tchrakian : Phys. Letts B 198 (4) (1987) 447. D.H Tchrakian : Jour. Math. Phys 21 (1980) 166.
2. B. Dolan, Tchrakian : Phys. Letts B 202 (2) (1988) 211.
3. B. Feisager, J.M Leinass : Phys. Letts B 94 (1980) 192.
4. C. Castro : "Remarks on Spinning Membrane Actions " [hep-th/0007031].
5. Lindstrom, Rocek : Phys. Letts B 218 (1988) 207.
6. J. Maldacena : Adv. Theor. Math. Phys 2 (1998) 231. [hep-th/9711200].
7. N. Berkovits, C. Vafa, E. Witten : "Conformal Field Theory of AdS Background with Ramond-Ramond Flux" [hep-th/9902095].
8. J. de Boer, S.L Shatashvili : "Two-dimensional Conformal Field Theory on $AdS_{2n+1}$ Backgrounds " [hep-th/9905032].
9. A. Belavin, A.M Polyakov, A.S Schwarz and Y. S. Tyupkin : Phys. Letts B 59 (1975) 85.
10. D.H Tchrakian : Jour. Math. Phys. 21 (1980) 166
11 R.P Zaikov : Phys. Letts B 266 (1991) 303. Phys. Letts B 263 (1991) 209.
12. Y. Ne’eman, E. Eiznberg : "Membranes and Other Extendons ($p$-branes) "World Scientific Lecture Notes in Physics vol. 39 1995.
13. E. Witten : Comm. Math. Phys. 121 (1989) 351. Nucl. Phys. B 322 (1989) 629.
14. P. Bouwknegt, K. Schouetens : "$W$-symmetry in Conformal Field Theory "Phys. Reports 223 (1993) 183-276.
15. M. Vasiliev, S. Prokushkin : "3D Higher-Spin Gauge Theories with Matter". [hep-th/9812242, hep-th/9806236].
16. E Sezgin, P. Sundell : "Higher Spin $N = 8$ Supergravity in $AdS_4$". [hep-th/9805125, hep-th/9903020].
17. M. Duff : "Anti de Sitter Spaces, Branes, Singletons, Superconformal Field Theories and All That" [hep-th/9808100].
S. Ferrara, A. Zaffaroni : "Bulk Gauge Fields in AdS Supergravity and Supersingletons" [hep-th/9807090].
M. Flato, C. Fronsdal : Letts. Math. Phys. 44 (1998) 249.
M. Gunaydin, D. Minic, M. Zagermann : "4D Doubleton Conformal Field Theories, CPT and IIB String on \( AdS_5 \times S^5 \)." [9806042].
18. C. Castro : Jour. Geometry and Physics. 33 (2000) 173.
19. D. Fairlie : "Moyal Brackets in M theory" [hep-th/9707190]. Mod. Phys. Letts A 13 (1998) 263.
20. C. Castro : Int. Jour. Mod. Phys. A 13 (6) (1998) 1263.
21. E.I Guendelman, E. Nissimov, S. Pacheva : "Volume-Preserving Diffs versus Local gauge Symmetry" [hep-th/9505128].
H. Aratyn, E. Nissimov, S. Pacheva : Phys. Lett B 255 (1991) 359.
A. Aurilia, A. Smailagic, E. Spallucci : Phys. Rev. D 47 (1993) 2536.
22. C. Castro : Jour. Chaos, Solitons and Fractals 7 (5) (1996) 711.
23. E Nissimov, S. Pacheva, I. Vaysburd : "\( W_\infty \) Gravity, a Geometric Approach" [hep-th/9207048].
24. G. Chapline : Jour. Chaos, Solitons and Fractals 10 (2-3) (1999) 311. Mod. Phys. Letts A 7 (1992) 1959. Mod. Phys. Letts A 5 (1990) 2165.
25. C. Vafa : "Evidence for F Theory" [hep-th/9602022].
26. I. Bars : "Two times in Physics." [hep-th/9809034].
27. P. Horava : "M Theory as a Holographic Field Theory" [hep-th/9712130].
28. L. Smolin : "Chern-Simons theory in 11 dimensions as a non-perturbative phase of M theory".
29. M. Gomes, Y. K Ha : Physics Letts 145 B (1984) 235.
Phys. Rev. Lett 58 (23) (1987) 2390.
30. P. Tran-Ngoc-Bich : Private Communication.
31. H. Garcia-Compean, J. Plebanski, M. Przanowski : "Geometry associated with SDYM and chiral approaches to Self Dual Gravity" [hep-th/9702046].
32. A. Connes, M. Douglas, A. Schwarz : "Noncommutative Geometry and Matrix Theory : Compactification on Tori" [hep-th/9711162].
33. MacDowell Mansouri :
34. C. Castro : "Branes from Moyal Deformation Quantization of GYM Theories" [hep-th/9908115].
35- S. Ansoldi, C. Castro, E. Spallucci : "Chern Simons Hadronic Bag from Quenched Large N QCD" submitted to Physics Letters B.
36- D. Gross, Y. Kitawa : Nuc. Phys B 206 (1982) 440.
Y. Makeenko : "Large N Gauge Theories" [hep-th/0001047].