Focused Bayesian Prediction∗

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Abstract

We propose a new method for conducting Bayesian prediction that delivers accurate predictions without correctly specifying the unknown true data generating process. A prior is defined over a class of plausible predictive models. After observing data, we update the prior to a posterior over these models, via a criterion that captures a user-specified measure of predictive accuracy. Under regularity, this update yields posterior concentration onto the element of the predictive class that maximizes the expectation of the accuracy measure. In a series of simulation experiments and empirical examples we find notable gains in predictive accuracy relative to conventional likelihood-based prediction.

Keywords: Loss-based prediction; Bayesian forecasting, Proper scoring rules; stochastic volatility model; expected shortfall; M4 forecasting competition

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Bayesian prediction quantifies uncertainty about the future value of a random variable using the rules and language of probability. A probability distribution for a future value is produced, conditioned only on past observations; all uncertainty about the parameters of the prediction model, plus any uncertainty about the model itself, having been integrated, or averaged out via these simple rules. Inherent to this natural and coherent approach to prediction, however, is the assumption that the process that has generated the observed data is either equivalent to the particular model on which we condition, or contained in the set of models over which we average. Such a heroic assumption is clearly at odds with reality; in particular in the realm of the social and economic sciences where statistical data arises through complex processes that we can only ever intend to approximate.

In response to this limitation of the conventional paradigm, we propose an alternative approach to Bayesian prediction. A prior is placed over a class of plausible predictive models. The prior is then updated to a posterior over these models, via a criterion function that represents a user-specified measure of predictive accuracy. This criterion replaces the likelihood function in the conventional Bayesian update and, hence, obviates the explicit need for correct model specification. Summarization of the posterior so produced - via its mean, for example - yields a single, representative predictive distribution that is expressly designed to yield accurate forecasts according to the given measure. Alternatively, the full extent of the posterior variation that obtains can be quantified and visualized. Given this deliberate focus on a particular aspect of predictive performance in the building of predictions, we refer to the principle as focused Bayesian prediction, or simply FBP.

To quantify predictive accuracy we use the concept of a scoring rule. (See Gneiting et al., 2007 and Gneiting and Raftery, 2007 for early expositions.) In short, a scoring rule rewards a probabilistic forecast for assigning a high density ordinate (or high probability mass) to the observed value (‘calibration’), subject to some criterion of ‘sharpness’, or some reward for accuracy in a particular part of the predictive support (e.g. the tails; Diks et al., 2011, Opschoor et al., 2017). Under appropriate regularity, we establish that this approach ensures, asymptotically, accurate performance according to the specified measure of predictive accuracy, and without dependence on correct model choice. Extensive numerical results support the theoretical results: focus on predictive accuracy, rather than correct model specification per se, leads to improved predictive performance.

This approach to Bayesian prediction has elements in common with the ‘probably approximately correct’ (PAC)-Bayes approach to prediction adopted in the machine learning literature; see Guedj (2019) for a recent review. The use of Bayesian up-dating per se without reference to a likelihood function also has echoes of the inferential methods proposed by Chernozhukov and
Hong (2003), Bissiri et al. (2016), Giummolè et al. (2017) and Knoblauch et al. (2019), amongst others, in which uncertainty about unknown parameters (in a given model) is updated via a general loss, or score, function. A major challenge in these generalizations of the standard Bayesian paradigm is the calibration of the scale of the loss (or score), which has a direct impact on the resultant variance of posterior of the parameters. Several methods for specifying this scale have been proposed (Bissiri et al., 2016; Giummolè et al., 2017; Holmes and Walker, 2017; Lyddon et al., 2019). Whilst we draw some insights from this literature, we propose new approaches that are informed specifically by the prediction context in which we are working, and which ensure posterior concentration around the predictive that is optimal under the given accuracy measure.¹

The predictive distributions within the plausible class may characterize a single dynamic structure depending on a single vector of unknown parameters. However, they may also be weighted combinations of predictives from distinct models. As such, our approach represents a coherent Bayesian method for estimating weighted combinations of predictives via forecast accuracy criteria, and without the need to assume that the true model lies within the set of constituent predictives. Whilst an established literature on estimating mixtures of predictives exists (see Aastveit et al., 2019 for an extensive review) - including work that invokes Bayesian principles - this is the first work to provide a formal, all-encompassing Bayesian updating scheme for such commonly encountered predictive situations.

After providing theoretical validation of the new method, its efficacy and usefulness is demonstrated through a set of simulation exercises, based on alternative predictive classes for a stochastic volatility model for financial returns. These classes are deliberately chosen to represent, at one end, a very misspecified representation of the (known) true data generating process (DGP) and, at the other end, a less misspecified version. The comparator in all cases is the standard, and misspecified, likelihood-based Bayesian update of the given parametric class. The results are clear: within-sample updating based on a specific measure of predictive accuracy almost always leads to the best out-of-sample performance according to that measure. The degree of superiority depends on the interplay between the model class, including the manner in which the model is misspecified, and the desired measure of accuracy - with animated graphics used to illustrate this point. The differential impact of update choice on posterior variation is also highlighted, via an animated display of posterior distributions for the expected shortfall of both ‘long’ and ‘short’ portfolios in the financial asset.

Two empirical illustrations complete the analysis. In the first, we predict two different series of daily financial returns using predictive classes based on the Gaussian (generalized) autoregressive conditional heteroscedastic ((G)ARCH) class of volatility model, known to be misspecified for

¹Related frequentist work in which forecasts are derived with a view to ‘optimizing’ a specified form of out-of-sample accuracy can be found in Gneiting et al. (2005), Gneiting and Raftery (2007), Elliott and Timmermann (2008), Gneiting (2011b,a) and Patton (2019).
the more complex process driving returns. The series considered are returns on: the U.S. dollar currency index, and the S&P500 stock index. The empirical results mimic those produced by simulation, with predictive accuracy improved by using the focused update - rather than the conventional (likelihood-based) update - in virtually all cases. The increase in predictive accuracy translates into more accurate value at risk (VaR) forecasts: use of an update that focuses on tail accuracy leads to a better match of empirical to nominal VaR coverage (than does the likelihood update) in all cases, and more frequent support of the joint null of correct coverage and independent violations.

In the second empirical example we pit FBP against the best performers in the Makridakis 4 (M4) forecasting competition. We perform the exercise using the 23,000 annual time series from the set of 100,000 series (of varying frequencies) used in the competition. We select as the predictive class, the exponential smoothing model of [Hyndman et al. (2002)](referred to as ETS hereafter): which had ranked highly amongst all twenty-five competitors. Adopting the same preliminary model selection procedure as the authors to specify the components of the ETS class for each of the 23,000 series, we update the chosen class using the mean scaled interval score (MSIS). This measure of predictive accuracy penalizes a prediction interval if the observed value falls outside the interval (appropriately weighted by its nominal coverage), and rewards a narrow interval, and was one of the two measures used to rank methods in the competition. As measured by MSIS, FBP not only almost always outperforms maximum likelihood-based implementation of ETS, but it outperforms all four predictive methods that were previously ranked best in the competition, in a large number of cases.

The rest of the article is organised as follows. In Section 2 we propose our new Bayesian predictive paradigm, and briefly illustrate its ability to produce more accurate predictions using a toy example. In Section 3 asymptotic validation of the method is provided, under the required regularity. More extensive illustration of the new approach via simulation, and visualization of the results, is the content of Section 4 whilst Section 5 illustrates the power of the method in empirical settings. In Section 6 we discuss the implications of our results and future lines of research. The proofs of all theoretical results, and certain computational details, are provided in appendices to the paper.

## 2 Focused Bayesian Prediction

### 2.1 Preliminaries and notation

Consider a stochastic process \( \{y_t : \Omega \to \mathbb{R}, t \in \mathbb{N}\} \) defined on the complete probability space \((\Omega, \mathcal{F}, G)\). Let \( \mathcal{F}_t := \sigma(y_1, \ldots, y_t) \) denote the natural sigma-field, and let \( G \) denote the infinite-dimensional distribution of the sequence \( y_1, y_2, \ldots \).
Throughout, we focus on one-step-ahead predictions and let \( \mathcal{P}^t := \{ P_{\theta}^t : \theta \in \Theta \} \) denote a generic class of one-step-ahead predictive models for \( y_{t+1} \), which are conditioned on time \( t \) information \( \mathcal{F}_t \), and where the generic elements of \( \mathcal{P}^t \) are represented by \( P_{\theta}^t(\cdot) := P(\cdot | \theta, \mathcal{F}_t) \). The parameter \( \theta \) indexes values in the predictive class, with \( \theta \) defined on \((\Theta, \mathcal{T}, \Pi)\), and where \( \Pi \) measures our beliefs about \( \theta \). Our beliefs \( \Pi \) over \( \Theta \) - both prior and posterior - generate corresponding beliefs over the elements in the predictive class \( \mathcal{P}^t \), in the usual manner, and, therefore, throughout the remainder we abuse notation and refer to \( \Pi \) as indexing beliefs over the class \( \mathcal{P}^t \).

Our goal is to construct a sequence of probability measures over \( \mathcal{P}^t \), starting from our prior beliefs \( \Pi \), such that hypotheses in \( \mathcal{P}^t \) that have ‘higher predictive accuracy’, are given higher posterior probability, after observing realizations from \( \{ y_t : t \geq 1 \} \). Given a user-defined measure of accuracy, we demonstrate that such a probability measure can be constructed using a Bayesian updating framework.

Importantly, however, we deviate from the standard approach to the production of Bayesian predictives in that the class \( \mathcal{P}^t \) only represents plausible predictive models for \( y_{t+1} \). At no point in what follows do we make the unrealistic assumption that the true one-step-ahead predictive is contained in \( \mathcal{P}^t \).  

2.2 Bayesian updating based on scoring rules

Using generic notation for the moment, for \( \mathcal{P} \) a convex class of predictive distributions on \((\Omega, \mathcal{F})\), we measure the predictive accuracy of \( P \in \mathcal{P} \) using the scoring rule \( S : \mathcal{P} \times \Omega \to \mathbb{R} \), whereby if the predictive distribution \( P \) is quoted and the value \( y \) eventuates, then the reward, or positively-oriented ‘score’, is \( S(P, y) \). The expected score under the true predictive \( G \) is defined as

\[
S(\cdot, G) := \int_{y \in \Omega} S(\cdot, y) dG(y). \tag{1}
\]

We say that a scoring rule is proper relative to \( \mathcal{P} \) if, for all \( P, G \in \mathcal{P}, S(G, G) \geq S(P, G) \), and is strictly proper, relative to \( \mathcal{P} \), if \( S(G, G) = S(P, G) \iff P = G \). That is, a proper scoring rule is one whereby if the forecaster’s best judgment is indeed the true measure \( G \) there is no incentive to quote anything other than \( P = G \) \cite{Gneiting2007}.

Under the assumption that a given predictive class contains the truth, i.e. that \( G \in \mathcal{P} \), the expectation of any proper score \( S(\cdot, y) \), with respect to the truth \( (G) \), will be maximized at the truth, \( G \). Hence, maximization over \( \mathcal{P} \) of the expected scoring rule \( S(\cdot, G) \), will reveal the true predictive mechanism when it is contained in \( \mathcal{P} \). In practice of course, the expected score \( S(\cdot, G) \) is unattainable, and a sample estimate based on observed data is used to define a sample score-based

\footnote{The treatment of scalar \( y_t \) and one-step-ahead prediction is for the purpose of illustration only, and all the methodology that follows can easily be extended to multivariate \( y_t \) and multi-step-ahead prediction in the usual manner.}
criterion. Maximization of the sample criterion, which implicitly depends on the true predictive process through the observed data, will yield the member of the predictive class that maximizes the relevant sample criterion. However, asymptotically the true predictive distribution will be recovered via any proper score criterion (again, on the assumption that the true predictive lies in the class $\mathcal{P}$).

The very premise of this paper is that, in reality, any choice of predictive class is such that the truth is not contained therein, at which point there is no reason to presume that the expectation of any particular scoring rule will be maximized at the truth or, indeed, maximized by the same predictive distribution that maximizes a different (expected) score. This does not, however, preclude the meaningfulness of a score as a measure of predictive accuracy, or invalidate the goal of seeking accuracy according to this particular measure. Indeed, it renders the distinctiveness of different scoring rules, and what form of forecast accuracy they do and do not reward, even more critical, and provides strong justification for driving predictive decisions by the very score that matters for the problem at hand.

With these insights in mind, we proceed as follows, reverting now to the specific notation that characterizes our problem, as defined in Section 2.1. Given observed data $y_n = (y_1, y_2, \ldots, y_n)'$, our object of interest is $P^n_\theta$, that is, the predictive distribution for $y_{n+1}$, conditional on information known at time $n$, $\mathcal{F}_n$. Given our prior beliefs $\Pi(\cdot)$, over $\mathcal{P}^n$, we update these beliefs using the following coherent posterior measure: for $A \subset \mathcal{P}^n$

$$
\Pi_w(A | y_n) = \frac{\int_A \exp \left[ w_n S_n(P^n_\theta) \right] d\Pi (P^n_\theta)}{\int_{\Theta} \exp \left[ w_n S_n(P^n_\theta) \right] d\Pi (P^n_\theta)},
$$

where

$$
S_n(P^n_\theta) = \sum_{t=0}^{n-1} S(P^n_\theta, y_{t+1}),
$$

and where the scale factor $w_n$ (indexed by $n$), used to define and index the posterior, is to be discussed in detail below. Two more comments regarding notation are useful at this point. First, consistent with our earlier comment, we abuse notation by defining a prior directly over a predictive, $P^n_\theta$, here. In fact, the prior is placed over $\theta$, and the prior over $P^n_\theta$ merely implied. Hence, the incongruous appearance of conditioning data in the prior (through the definition of $P^n_\theta$) is of no concern. It is simply used to define the particular function of $\theta$ ($P^n_\theta$) that is our ultimate object of interest, and that function conditions on (past) observed data, as it is a predictive distribution. Second the criterion function that defines the update in (2) is, of course, comprised

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3The nature of the conditioning set $\mathcal{F}_n$ differs according to the dynamic structure of $P^n_\theta$. For example, in a Markov model of order 1, $\mathcal{F}_n$ comprises the observed $y_n$ only. In contrast, a predictive for a long memory model conditions on all available past observations. The conditioning set, $\mathcal{F}_n$, may also, of course, include observed values of covariates. Hence, we keep the notation for this conditioning set, $\mathcal{F}_n$, which is implicit in the definition of $P^n_\theta$, distinct from that of the observed data, $y_n$, that is used to build the posterior over the elements of $\mathcal{P}^n$. 


of the sequential one-step-ahead predictives, \( P_t^\theta \), for \( t = 0, 1, ..., n - 1 \).

The update in (2) is coherent in the sense used by Bissiri et al. (2016) in their application of a general loss function in the Bayesian update for the parameters of a given parametric model. That is, the posterior that results from updating the prior using two sets of observations in one step, is the same as that produced by two sequential updates. Proof of this property follows that of Bissiri et al. (2016) and exploits the exponential form of the first term on the right-hand-side of (2), in addition to certain conditions on \( w_n \) to be made explicit below. Indeed, the appearance of \( w_n \) serves to distinguish (2) from what would be an extension (to the predictive setting) of the loss-based inference approach adopted by Chernozhukov and Hong (2003). We elaborate on this point in Section 3.1.4

In the case where \( w_n = 1 \) and \( S(P_t^\theta, y_{t+1}) = \ln p(y_{t+1}|F_t, \theta) \), with \( p(y_{t+1}|F_t, \theta) \) denoting the predictive density (or mass) function associated with the class \( P_t^\theta \), the update in (2) is equivalent to the conventional likelihood-based update of the prior defined over \( \theta \). We refer hereafter to this special case as ‘exact Bayes’, and acknowledge that, given the presumption of misspecification, there is no sense in which exact Bayes remains the ‘gold standard’. This case remains, however, a critical benchmark in the numerical work, in which the degree of misspecification of \( P_t^\theta \) will be seen to influence the relative out-of-sample performance of the conventional Bayesian update.

We can summarize the posterior in (2) by producing a simulation-based estimate of the mean predictive:

\[
E_w[P^n_\theta|y_n] = \int P^n_\theta d\Pi_w(P^n_\theta|y_n).
\]

However, it is equally feasible to construct measures that capture the variability of the posterior, such as quantiles or the posterior variance. Moreover, we can use various graphical techniques to visualize the variation of the predictives themselves and understand the way in which posterior variation over the class \( P_t^\theta \) impacts on predictive accuracy per se.

Before moving on, we quickly demonstrate the usefulness of this new approach to Bayesian prediction, and the predictive gains that it can reap, using a simple toy example.

### 2.3 A toy example: ARCH(1)

We produce predictive distributions for a financial return generated from a latent stochastic volatility model with a skewed marginal distribution, with precise details of this true DGP to be given in Section 4. The predictive class, \( P_t^\theta \), is defined by an ARCH model of order 1 (ARCH(1)) with Gaussian errors, \( y_t = \theta_1 + \sigma_t \epsilon_t, \epsilon_t \sim i.i.d.N(0,1), \sigma_t^2 = \theta_2 + \theta_3 (y_{t-1} - \theta_1)^2 \), with

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4We note that a negatively-oriented score can be viewed as a relevant measure of ‘loss’ in a predictive setting. Moreover, it is also possible to define the loss associated with predictive inaccuracy using functions that are not formally defined as scoring rules (see, for example, Pesaran and Skouras [2004]). However, we give emphasis to scoring rules in this paper, making brief note only of the applicability of our method to more general loss functions in the Discussion.
Table 1: Predictive accuracy of FBP, using the ARCH(1) predictive class. The rows in each panel refer to the update method used. The columns refer to the out-of-sample measure used to compute the average scores. The figures in bold are the largest average scores according to a given out-of-sample measure.

| Updating method | Out-of-sample score |  
|-----------------|----------------------|  
|                 | LS                   | CS<10%  | CS>90%  |
| Exact Bayes     | -1.3605              | -0.4089 | -0.2745 |
| FBP-CS<10%      | -1.4420              | -0.3943 | -0.3833 |
| FBP-CS>90%      | -3.0067              | -1.4157 | -0.2397 |

As noted already, use of the log score (LS), (5), in (2) (with \( w_n = 1 \)) yields the conventional likelihood-based Bayesian update, and we label the results based on this score as exact Bayes as a consequence. The score in (6) is the censored likelihood score (CS) introduced by Diks et al. (2011), and applied by Opschoor et al. (2017) to the prediction of financial returns. This score rewards predictive accuracy over the region of interest \( A \) (with \( A^c \) indicating the complement of this region). Here we report results solely for \( A \) defining the lower and upper tails of the predictive distribution, as determined respectively by the 10% and 90% quantile of the empirical distribution of \( y_t \) computed over an initial sample period. We label the results based on the use of (6) in (2) (also using \( w_n = 1 \)) as FBP-CS<10% and FBP-CS>90%.

Postponing discussion of full design details until Section 4, we record in Table 1 out-of-sample results based on repeated computation of (4) using expanding windows to produce (via Markov chain Monte Carlo) draws from (2). Using a total of 2,000 out-of-sample values, the average LS (computed across the 2,000 mean predictives) and the average CS for the lower and upper 10% tail (denoted by CS<10% and CS>90% respectively) are computed for each of the three different updating methods. The largest average score, according to each out-of-sample evaluation method, is highlighted in bold.

Recalling that we use positively oriented scores, we see that use of the CS rule in the posterior update yields better out-of-sample performance, as measured by that score, in both the upper and lower tails. In absolute terms, the gain of ‘focusing’ is more substantial in the upper tail than the lower tail, and in Section 4 we shall see why this is so. The average LS produced by the exact
Bayes (LS-based) update is also larger than the average LS produced by both FBP-CS\( <10\% \) and FBP-CS\( >90\% \).

In summary, focusing works, and the following theoretical results give some insight into why.

3 Bayesian and Frequentist Agreement

Whilst the elements of \( P^t \) may, in principle, be either parametric or nonparametric conditional distributions, in the remainder we focus on the parametric case to simplify the analysis, leaving rigorous analysis of nonparametric conditionals for later study. However, we remind the reader that this reduction to parametric conditionals covers both the canonical case where the elements of \( P^t \) are indexed by a finite-dimensional parameter, in which case \( \Theta \) is a Euclidean space, as well as the case where the elements in \( P^t \) are (a finite collection of) mixtures of predictives, in which case \( \Theta \) denotes either the weights of the mixture, or the combination of the weights and the unknown parameters of the constituent predictives.

3.1 Choosing \( w_n \)

With reference to the conventional Bayesian approach to inference on the unknown parameters, \( \theta \), which characterize an assumed DGP, the posterior density,

\[
\pi(\theta | y_n) \propto \ell(y_n | \theta) \pi(\theta),
\]

where \( \ell(y_n | \theta) \) denotes the likelihood function, arises via a decomposition of the joint probability distribution for \( \theta \) and the random vector \( y_n \). As such, the representation of \( \pi(\theta | y_n) \) as proportional to the product of a density (or mass) function for \( y_n \), and the prior for \( \theta \), reflects the usual calculus of probability distributions, and provides a natural ‘weighting’ between the likelihood and the prior.

Once one moves away from this conventional framework, and replaces the likelihood with an alternative mechanism through which the data provides information about \( \theta \), this natural weighting is lost. Instead, a subjective choice must be made regarding the relative weight given to prior and data-based information in the production of the posterior, with the scale factor \( w_n \) in (2) denoting this subjective choice of weighting. [Bissiri et al. (2016)] propose several methods for choosing \( w_n \), including annealing methods, hyper-parametrization of \( w_n \), and setting \( w_n \) to ensure the equivalence of the expected ‘loss’ of the prior and data-based components of (2). The authors also suggest choosing \( w_n \) to ensure correct frequentist coverage of posterior credible intervals, plus the use of priors that are conjugate to the weighted data-based criterion.\(^5\)

\(^5\)Further proposals on the choice of \( w_n \) can be found in Holmes and Walker (2017) and Lyddon et al. (2019).
In contrast, our interest is not in inference on $\theta$ per se, but on forecast accuracy. Given this goal, from a theoretical standpoint, our only concern is that, for $\{w_n : n \geq 1\}$ a chosen scaling sequence, the FBP posterior measure concentrates onto the element of $P^n$ that is most accurate in the chosen scoring rule, which is defined by the following value in $\Theta$:

$$\theta_* = \arg \max_{\theta \in \Theta} \lim_{n \to \infty} \mathbb{E}[S_n(P^n_\theta)/n].$$ (8)

As the following result demonstrates, this concentration occurs for any reasonable choice of $w_n$.

**Lemma 1.** Assume Assumptions 2 and 3 in Appendix B are satisfied, and denote the FBP posterior density function by $\pi_w(\theta|y_n)$. If the sequence $\{w_n\}$ satisfies $\lim_n w_n = C$, $0 < C < \infty$, the posterior density $\pi_w(\cdot|y_n)$ converges to $P_\theta := \lim_{n \to \infty} P^n_\theta$, the limit of the predictive defined by $\theta_*$, at rate $1/\sqrt{n}$.

**Remark 1.** The above result demonstrates that, if we restrict our analysis to a class of parametric predictives, FBP asymptotically concentrates, at rate $1/\sqrt{n}$, onto the predictive that is most accurate according to the scoring rule $P^t_\theta \mapsto \lim_{n \to \infty} \mathbb{E}[S(P^n_\theta, \cdot)/n]$.

**Remark 2.** The conditions for the above result are discussed in Appendix B and are similar to the standard regularity conditions for parametric M-estimators, along with some uniform control on the tail of the prior $\Pi$. These conditions are similar to those used elsewhere in the literature, e.g., Chernozhukov and Hong (2003). Interestingly, Lemma 2 is valid for a wide variety of $\{w_n\}$. In Sections 4 and 5, we detail the particular values of $w_n$ that we use to produce our numerical predictions.

### 3.2 Merging

In the previous subsection, we have seen that, for a reasonable choice of $w_n$, the FBP posterior concentrates on the element of $P^n$ that is most accurate for prediction under the chosen scoring rule. In this section, we compare the behavior of predictions obtained from the FBP posterior with those that would be obtained using direct optimization of an expected score criterion to produce a frequentist point estimate of $\theta$, and the associated predictive that conditions on this point estimate.

Define, for $B \in \mathcal{F}_n$, the following predictive measures

$$P^n_w(B) = \int_{\Theta} dP^n_\theta(B) d\Pi_w(P^n_\theta|y_n),$$ (9)

$$P^n_*(B) = \int_{\Theta} dP^n_\theta(B) d\delta_{\theta_*},$$ (10)

where $\delta_{\theta_*}$ denotes the Dirac measure at the point $\theta = \theta_*$, for $\theta_*$ defined in (8). The predictive $P^n_w(\cdot)$ defines a distribution for the random variable $y_{n+1}$, conditional on observed data $y_n$, and
where our uncertainty about the members of the predictive class, $\mathcal{P}^n$, is captured using the posterior $\Pi_w(\cdot|y_n)$, for some choice of tuning sequence $w_n$. In contrast, the predictive $P^*_n(\cdot)$ in (10) denotes the optimal predictive obtained by maximizing the expected score. Clearly, obtaining $P^*_n$ is infeasible in practice. Instead, the following estimated value of $\theta^*$, $\hat{\theta} := \arg\min_{\theta \in \mathcal{P}^n} S_n(P^n_\theta)/n$, is generally used in place of $\theta^*$. Under the same regularity conditions as Lemma 1, we can derive the asymptotic behavior of $\hat{\theta}$.

**Lemma 2.** Under Assumptions 1-3 in Appendix B, if $\hat{\theta}$ is consistent for $\theta^*$, and if $S_n(P^n_{\theta^*}) \geq S_n(P^n_{\hat{\theta}}) + o_p(1/\sqrt{n})$, then $\sqrt{n}(\hat{\theta} - \theta^*) \Rightarrow N(0, W)$, where $W = H^{-1}VH^{-1}$ and

$$V := \lim_{n \to \infty} \text{Var}\left[\frac{\partial}{\partial \theta} S_n(P^n_{\theta^*}) - \mathbb{E}\left[\frac{\partial}{\partial \theta} S_n(P^n_{\theta^*}) \right]\right]; \quad H := \text{plim}_{n \to \infty} \mathbb{E}\left[\frac{\partial}{\partial \theta \partial \theta'} S_n(P^n_{\theta})|_{\theta = \theta^*}\right].$$

Using the estimator $\hat{\theta}$, we can define the following frequentist equivalent to the FBP predictive:

$$P^n_\hat{\theta}(B) = \int_{\Theta} P^n_\theta(B) d\mathcal{N}(\hat{\theta}, W/n), \quad (11)$$

where $\mathcal{N}(\hat{\theta}, W/n)$ denotes the normal distribution function with mean $\hat{\theta}$ and variance-covariance matrix $W/n$. Using Lemmas 1 and 2, we can deduce the following relationship between the frequentist predictive in equation (11) and the FBP predictive in (9).

**Theorem 1.** Under Assumptions 1-3 in Appendix B for $\lim_n w_n = C > 0$, the predictive distributions $P^n_w(\cdot)$ and $P^n_\hat{\theta}(\cdot)$ satisfy:

$$\sup_{B \in \mathcal{B}} |P^n_w(B) - P^n_\hat{\theta}(B)| = o_p(1).$$

**Remark 3.** Theorem 1 states that, for any sequence $\lim_n w_n = C$, the FBP predictive $P^n_w(\cdot)$ and the (feasible) optimal frequentist predictive $P^n_\hat{\theta}(\cdot)$ will agree asymptotically. The above result is colloquially referred to as ‘merging’ (Blackwell and Dubins, 1962). This result states that, in terms of the total variation distance, the predictions obtained by FBP and those obtained by a frequentist making predictions according to an optimal score estimator $\hat{\theta}$ will asymptotically agree.

### 4 Simulation Study: Financial Returns

#### 4.1 Overview of the simulation design

We first illustrate our approach with a simulation exercise that nests the toy example in Section 2.3. Adopting a simulation approach allows us to choose predictive classes that misspecify the (known) DGP to varying degrees, and to thereby measure the relative performance of FBP in different misspecification settings. We use both numerical summaries and animated graphics to
illustrate the predictive accuracy of FBP, using a range of scores to define the update. With a slight abuse of terminology, in what follows we refer to FBP-LS solely as ‘exact Bayes’, reserving the abbreviation FBP for all other instances of the focused method.

We address three questions. First, what sample size is required in practice for the asymptotic results to be on display? That is, how large does $n$ have to be for FBP based on a particular scoring rule to provide the best out-of-sample performance according to that same rule? Second, does the degree of misspecification affect the dominance of FBP over exact Bayes? Third, does misspecification have a differential impact on FBP implemented via different scoring rules?

With the aim of replicating the stylized features of financial returns data, we generate a logarithmic return, $y_t$, from

$$h_t = \bar{h} + a(h_{t-1} - \bar{h}) + \sigma_h \eta_t$$

$$z_t = e^{0.5 h_t} \varepsilon_t$$

$$y_t = D^{-1}(F_z(z_t)),$$

where $\eta_t \sim i.i.d. N(0, 1)$ and $\varepsilon_t \sim i.i.d. N(0, 1)$ are independent processes, $\{z_t\}_{t=1}^n$ is a latent process with stochastic (logarithmic) variance, $h_t$, and $F_z$ is the implied marginal distribution of $z_t$ (evaluated via simulation). The ‘observed’ return, $y_t$, is then generated as in (14), via the (inverse) distribution function associated with a standardised skewed-normal distribution, $D^{k6}$. This process of inversion imposes on $\{y_t\}_{t=1}^n$ the dynamics of the stochastic volatility model represented by Equations (12) and (13) (via $F_z$) in addition to the negative skewness that is characteristic of the empirical distribution of a financial return.\(^\text{7}\)

We adopt three alternative parametric predictive classes, $\mathcal{P}^r$: i) Gaussian ARCH(1) (reproduced here for convenience and numbered for future reference):

$$y_t = \theta_1 + \sigma_1 \epsilon_t; \epsilon_t \sim i.i.d. N(0, 1); \sigma_1^2 = \theta_2 + \theta_3 (y_{t-1} - \theta_1)^2;$$

ii) Gaussian GARCH(1,1):

$$y_t = \theta_1 + \sigma_1 \epsilon_t; \epsilon_t \sim i.i.d. N(0, 1); \sigma_1^2 = \theta_2 + \theta_3 (y_{t-1} - \theta_1)^2 + \theta_4 \sigma_{t-1}^2;$$

and iii) a mixture of the predictives of two models: an ARCH(1) model with a skewed-normal innovation, and a GARCH(1,1) model with a Gaussian innovation. We represent the elements of class iii) as:

$$p(y_{t+1}|\mathcal{F}_t, \theta_1) = \theta_1 p_1(y_{t+1}|\mathcal{F}_t, \psi_1) + (1 - \theta_1) p_2(y_{t+1}|\mathcal{F}_t, \psi_2).$$

\(^\text{6}\)For more details of the specific skewed-normal specification that we adopt see Azzalini (1985). \(^\text{7}\)See Smith and Manessoonthorn (2018) for discussion of this type of implied copula model.
In (15) and (16) the respective parameter vectors, $\theta = (\theta_1, \theta_2, \theta_3)'$ and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)'$, characterize the specific predictive model, with the GARCH(1,1) model being a more flexible (and, in this sense, ‘less misspecified’) representation of the true DGP than is the ARCH(1) model. In (17) the parameter vectors $\psi_1$ and $\psi_2$ that characterize the constituent predictives in the mixture are taken as known, and the scalar weight parameter $\theta_1$ is the only unknown. The component $p_2(y_{t+1}|F_t, \psi_2)$ is specified according to the model in (16), whilst $p_1(y_{t+1}|F_t, \psi_1)$ represents the predictive associated with the model in (15), but with $\epsilon_t$ distributed as a standardised skewed-normal variable with asymmetry parameter $\gamma$. With the proviso made that the parameters of the constituent models are taken as given, the mixture is arguably the most flexible form of predictive considered here, and defines the least misspecified predictive class in this sense. The prior over each of the three predictive classes is determined by the prior over the relevant parameter (vector) $\theta$, respectively: i) $\pi(\theta) \propto \frac{1}{\theta_2} \times I [\theta_2 > 0, \theta_3 \in [0, 1)]$ (with $I$ the indicator function), ii) $\pi(\theta) \propto \frac{1}{\theta_2} \times I [\theta_2 > 0, \theta_3 \in [0, 1), \theta_4 \in [0, 1)] I(\theta_3+\theta_4 < 1)$, and iii) $\pi(\theta_1) \propto u$ (with $u$ uniform on $(0, 1)$).

We now implement FBP using the two scoring rules in (5) and (6), plus the continuously ranked probability score (CRPS),

$$S_{\text{CRPS}}(P^t_{\theta}, y_{t+1}) = -\int_{-\infty}^{\infty} [P (y_{t+1}|F_t, \theta) - I(y \geq y_{t+1})]^2 dy, \quad (18)$$

where $P (y_{t+1}|F_t, \theta)$ denotes the predictive cumulative distribution function (cdf) associated with $p (y_{t+1}|F_t, \theta)$. Proposed by Gneiting and Raftery (2007), CRPS is sensitive to distance, rewarding the assignment of high predictive mass near to the realized value of $y_{t+1}$. It can be evaluated in closed form for the (conditionally) Gaussian predictive classes i) and ii), using the third equation provided in Gneiting and Raftery (2007, p. 367). For predictive class iii), evaluation is performed numerically using expression (17) in Gneiting and Ranjan (2011, p. 367). In the case of the CS in (6), all components, including the integral $\int_{A^c} p (y_{t+1}|F_t, \theta) dy_{t+1}$, have closed-form representations for predictive classes i) and ii). For the third predictive class, CS is computed as

$$S_{\text{CS}}(P^t_{\theta}, y_{t+1}) = S_{\text{CS}}(P^{t}_{\psi_1}, y_{t+1}) + \log \{\theta_1 + (1-\theta_1) \exp \left[ S_{\text{CS}}(P^{t}_{\psi_2}, y_{t+1}) - S_{\text{CS}}(P^{t}_{\psi_1}, y_{t+1}) \right] \},$$

where $S_{\text{CS}}(P^{t}_{\psi_1}, y_{t+1})$ and $S_{\text{CS}}(P^{t}_{\psi_2}, y_{t+1})$ correspond to the censored scores for the two constituent models, both having closed-form solutions.

As noted in Section 2.3, when either (5) or (6) is used in (2) a scale of $w_n = 1$ is adopted. This is a natural choice, given that use of (5) defines the (misspecified) likelihood function induced by the predictive class, and that use of (6) is comparable to the specification of the likelihood function for a censored random variable (Diks et al., 2011). When (18) is used to define the posterior update however, the interpretation of $\exp \left[ w_n S_n( P_{\theta}^n ) \right]$ as the (unnormalised) probability distribution of a random variable is lost, and $w_n$ must be chosen with reference to some criterion.
for weighting $w_n S_n (P^n_\theta)$ and $\Pi (P^n_\theta)$. We choose to target a value for $w_n$ that ensures a rate of posterior update - when using CRPS - that is similar to that of the update based on LS, by defining

$$w_n = \frac{E_\pi (\theta | y_n) \left[ \sum_{t=0}^{n-1} S_{LS} (P^t_\theta, y_{t+1}) \right]}{E_\pi (\theta | y_n) \left[ \sum_{t=0}^{n-1} S_{CRPS} (P^t_\theta, y_{t+1}) \right]}.$$  \hfill (19)

The subscript $\pi (\theta | y_n)$ indicates that the expectation is with respect to the exact posterior distribution for $\theta$. In practice, $w_n$ is estimated as

$$\hat{w}_n = \frac{\sum_{j=1}^{J} \left[ \sum_{t=0}^{n-1} S_{LS} (P^t_{\theta^{(j)}}, y_{t+1}) \right]}{\sum_{j=1}^{J} \left[ \sum_{t=0}^{n-1} S_{CRPS} (P^t_{\theta^{(j)}}, y_{t+1}) \right]},$$  \hfill (20)

using $J$ draws of $\theta$ from the $\pi (\theta | y_n)$, $\theta^{(j)}$, $j = 1, 2, ..., J$. The link between specifying $w_n$ as in (19) and achieving a rate of posterior update that approximates that of exact Bayes, is detailed in Appendix A.1. All details of the Markov chain Monte Carlo (MCMC) scheme used to perform the posterior sampling are provided in Appendix A.2.

4.2 Simulation results

4.2.1 Summary results based on mean predictives

We generate 2,500 observations of $y_t$ from the DGP in (12)-(14), using parameter values: $a = 0.9$, $\bar{h} = -0.4581$ and $\sigma_h = 0.4173$, while $D$ defines the standardised skewed-normal distribution with shape parameter $\gamma = -5$, which produces an empirically plausible degree of negative skewness. For each predictive class, and for each score update, the exercise begins by using the relevant computational scheme, as described in Appendix A.2, to produce (after thinning) $M = 4,000$ posterior draws of $\theta$, $\theta^{(j)}$, $j = 1, 2, ..., M$, and, hence, $M$ posterior draws of $p (y_{n+1} | F_n, \theta)$, which we denote simply by $p^{(j)}$, as indexed by the $j^{th}$ draw of $\theta$. This first set of posterior draws is produced using the first $n = 500$ values of $y_t$ in the update in (2). For each predictive class, six score updates are employed, corresponding to (5) and (18), plus (6) with the region $A$ defining (approximately) four tails of the predictive distribution: lower 10%, lower 20%, upper 10% and upper 20%.\footnote{As noted in Section 2.3, the set $A$, for any required tail probability, is determined via reference to the \textit{empirical} distribution computed over the initial sample period.} Hence, for each predictive class, draws from six different $\Pi_w (P^n_\theta | y_n)$ are produced.

Referencing the draws, $p^{(j)}$, $j = 1, 2, ..., M$, from any one of the six distinct posteriors, we first estimate the mean predictive in (4) as: $\hat{E}_w [P^n_\theta | y_n] = 1/M \sum_{j=1}^{M} p^{(j)}$, and compute the out-of-sample score of this single predictive, based on the observed value of $y_{n+1}$, for period $n + 1 = 501$. The same six scores used in the in-sample updates are used to produce these out-of-sample scores. The sample is then extended to $n = 501$, and the same exercise is repeated, with the out-of-sample scores computed using the observed value of $y_{n+1}$, for time period $n + 1 = 502$. This exercise is...
repeated 2,000 times, with the final set of out-of-sample scores computed using the observed value of $y_t$ for time period $n + 1 = 2,500$. The average of the 2,000 scores is recorded in Table 2 for each update method, each out-of-sample evaluation method, and each predictive class.

Expanding on the results reported in Section 2.3, Panels A, B and C in the table correspond respectively to the three predictive classes: ARCH(1), GARCH(1,1) and the mixture. The rows in each panel refer to the six distinct update methods, denoted in turn by: exact Bayes ($\equiv$ FBP-LS), FBP-CRPS, FBP-CS$_{<10\%}$, FBP-CS$_{<20\%}$, FBP-CS$_{>80\%}$ and FBP-CS$_{>90\%}$. The columns refer to the out-of-sample measure used to compute the average scores: LS, CRPS, CS$_{<10\%}$, CS$_{<20\%}$, CS$_{>80\%}$ and CS$_{>90\%}$. Numerical validation of the asymptotic results occurs if the largest average scores (bolded) appear in the diagonal positions in the table; that is, if using FBP with a particular focus yields the best out-of-sample performance according to that same measure of predictive accuracy.

The results in Table 2 broadly validate the asymptotic theory. With minor deviations, the expected appearance of bold figures on the main diagonal of each panel is in evidence - most notably in Panels A and B. Hence, with reference to the first question outlined at the beginning of Section 4.1: an initial sample size exceeding $n = 500$, expanded to $n = 2,499$ in the production of 2,000 one-step-ahead predictions, is sufficient for the use of in-sample focusing to reap benefits out-of-sample. When there is a deviation from the strict diagonal pattern, such as in the CS$_{<10\%}$ column of Panel A and in the CS$_{<10\%}$, CS$_{<20\%}$ and CS$_{>80\%}$ columns of Panel C, the difference between the relevant (non-diagonal) bold value and the value on the diagonal is negligible.

With reference to the second question, the out-of-sample dominance of FBP over exact Bayes declines as the predictive class becomes less misspecified. In particular, the results in Panel C - for the mixture predictive class - reveal that the average scores computed using a given out-of-sample measure are very similar for all six update methods. The extent of the misspecification of the true DGP clearly does matter.

With respect to the third question: there are two notable results regarding the differential impact of the degree of misspecification on the different versions of FBP. First, when the degree of misspecification is most severe (as with the ARCH(1) and GARCH(1,1) predictive classes) use of the update that focuses on the upper tail (FBP-CS$_{>80\%}$ or FBP-CS$_{>90\%}$) produces poor

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9We reiterate that in this numerical assessment of predictive performance based on expanding estimation windows, there are two sample sizes that play a role: i) the size of the estimation period on which the posterior (over predictives) is based, and from which the mean (one-step-ahead) predictive and numerical score are extracted; and ii) the size of the out-of-sample period over which the average (one-step-ahead) scores are computed. With expanding estimation windows, an increase in the out-of-sample period goes hand-in-hand with a continued increase in the estimation period, i.e. an increase in $n$.

10For a large enough sample of course, for a predictive class that contains the true DGP, any updating method based on a proper score should (under regularity) recover the true predictive mechanism and, hence, should yield predictive performance out-of-sample (however measured) that matches that of an update based on an alternative proper score. See (Gneiting and Raftery 2007) for an early exposition of this sort of point, in the context of frequentist point estimation using scoring rules.
Panel A: ARCH(1) predictive class

|                          | Out-of-sample score |                       |                       |                       |                       |                       |                       |
|--------------------------|---------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
|                          | Center Focused      | Left Focused          | Right Focused         |                       |                       |                       |                       |
|                          | LS                  | CRPS                  | CS<10%                | CS<20%                | CS>80%                | CS>90%                |                       |
| Updating method          |                     |                       |                       |                       |                       |                       |                       |
| Exact Bayes              | -1.3605             | -0.5299               | -0.4089               | -0.6687               | -0.4716               | -0.2745               |                       |
| FBP-CRPS                 | -1.3663             | -0.5290               | -0.4206               | -0.6774               | -0.4723               | -0.2775               |                       |
| FBP-CS<10%               | -1.4442             | -0.5558               | -0.3943               | -0.6506               | -0.5755               | -0.3833               |                       |
| FBP-CS<20%               | -1.4660             | -0.5652               | -0.3933               | -0.6484               | -0.6062               | -0.4072               |                       |
| FBP-CS>80%               | -2.0422             | -0.5902               | -0.9655               | -1.3470               | -0.4365               | -0.2430               |                       |
| FBP-CS>90%               | -3.0067             | -0.6747               | -1.4157               | -2.0858               | -0.4592               | -0.2397               |                       |

Panel B: GARCH(1,1) predictive class

|                          | Out-of-sample score |                       |                       |                       |                       |                       |                       |
|--------------------------|---------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|                       |
|                          | Center Focused      | Left Focused          | Right Focused         |                       |                       |                       |                       |
|                          | LS                  | CRPS                  | CS<10%                | CS<20%                | CS>80%                | CS>90%                |                       |
| Updating method          |                     |                       |                       |                       |                       |                       |                       |
| Exact Bayes              | -1.3355             | -0.5259               | -0.3941               | -0.6500               | -0.4710               | -0.2747               |                       |
| FBP-CRPS                 | -1.3381             | -0.5258               | -0.3992               | -0.6532               | -0.4734               | -0.2788               |                       |
| FBP-CS<10%               | -1.3801             | -0.5341               | -0.3838               | -0.6387               | -0.5282               | -0.3317               |                       |
| FBP-CS<20%               | -1.4126             | -0.5480               | -0.3840               | -0.6375               | -0.5650               | -0.3710               |                       |
| FBP-CS>80%               | -2.0535             | -0.5918               | -0.9612               | -1.3530               | -0.4318               | -0.2387               |                       |
| FBP-CS>90%               | -3.1207             | -0.6818               | -1.4544               | -2.1502               | -0.4572               | -0.2347               |                       |

Panel C: Mixture predictive class

|                          | Out-of-sample score |                       |                       |                       |                       |                       |                       |
|--------------------------|---------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|                       |
|                          | Center Focused      | Left Focused          | Right Focused         |                       |                       |                       |                       |
|                          | LS                  | CRPS                  | CS<10%                | CS<20%                | CS>80%                | CS>90%                |                       |
| Updating method          |                     |                       |                       |                       |                       |                       |                       |
| Exact Bayes              | -1.2901             | -0.5241               | -0.3898               | -0.6448               | -0.4363               | -0.2447               |                       |
| FBP-CRPS                 | -1.2975             | -0.5234               | -0.3868               | -0.6418               | -0.4476               | -0.2557               |                       |
| FBP-CS<10%               | -1.3048             | -0.5236               | -0.3871               | -0.6422               | -0.4536               | -0.2610               |                       |
| FBP-CS<20%               | -1.3029             | -0.5235               | -0.3871               | -0.6421               | -0.4523               | -0.2599               |                       |
| FBP-CS>80%               | -1.2902             | -0.5250               | -0.3922               | -0.6472               | -0.4325               | -0.2407               |                       |
| FBP-CS>90%               | -1.2902             | -0.5250               | -0.3922               | -0.6472               | -0.4324               | -0.2406               |                       |

Table 2: Predictive accuracy based on the six different mean predictives. Panels A to C report the average out-of-sample scores for the ARCH(1), GARCH(1,1) and mixture predictive class, respectively. The rows in each panel refer to the update method used. The columns refer to the out-of-sample measure used to compute the average scores. The figures in bold are the largest average scores according to a given out-of-sample measure.
out-of-sample accuracy according to the LS and lower tail measures (CS_{<10\%} and CS_{<20\%}). We provide some graphical insight into this specific phenomenon in Section 4.2.2; however, the point is that focusing incorrectly can hurt, in particular when the predictive class is a poor match for the true DGP. Once misspecification of the predictive class is reduced, the performance of both FBP-CS_{>80\%} and FBP-CS_{>90\%} - according to all out-of-sample measures - broadly matches that of the other updating methods, as can be seen in Panel C.

The second differential impact of misspecification pertains to the exact (misspecified) Bayesian update, relative to all four tail-focused methods (FBP-CS_{<10\%}, FBP-CS_{<20\%}, FBP-CS_{>80\%} and FBP-CS_{>90\%}). For example, the values of CS_{>90\%} for FBP-CS_{>90\%} (the three bolded figures in the very last column of Table 2) change very little over the three panels, as the degree of misspecification lessens. A similar comment applies to the three values of CS_{<10\%} for FBP-CS_{<10\%}. In contrast, the improvement in performance in the tails (so the values of CS_{>90\%} and CS_{<10\%}) for exact Bayes, as one moves from the most to the least misspecified predictive class, is more marked; which makes sense. The focused methods do not aim to get the model correct; instead, they are deliberately tailored to a particular predictive task (accurate prediction of extreme values in this case). Hence, misspecification of the model per se matters less. The predictive performance of exact Bayes, on the other hand, depends entirely on the match between the model that underpins the method and the truth; there is nothing else that exact Bayes brings to the table; if the model is wrong, prediction (however measured) will be adversely affected by that error.

To gauge the sensitivity of the findings to the size of the out-of-sample evaluation period (and the size of the underlying expanding estimation periods), we plot the average one-step-ahead score as a function of the latter. In the quest for brevity, we perform this task for two out-of-sample measures only: CS_{<10\%} and CS_{>90\%}. In Panel A of Figure 1 the cumulative average of CS_{<10\%} (for 400 to 2,000 out-of-sample periods) is plotted for two forms of in-sample updates only: exact Bayes (the dashed green line) and FBP-CS_{<10\%}, (the blue full line). Each of the three figures (A.1, A.2, A.3) corresponds respectively to results for each of the three predictive classes (ARCH(1), GARCH(1,1) and the mixture). In Panel B (B.1, B.2 and B.3) the corresponding results for the cumulative average of CS_{>90\%} are presented, based on exact Bayes (the dashed green line) and FBP-CS_{>90\%}, (the blue full line). In all figures, the final numerical values plotted correspond to the relevant values reported in Table 2.

Beginning with Figure A.1, we see that a sufficiently large of out-of-sample evaluation period (exceeding approximately 600) is needed for the dominant performance of FBP over exact Bayes to be in evidence visually; with in-sample estimation periods exceeding \( n = 1, 100 \) contributing to these average score results. However, beyond this point, the amount by which the full line exceeds the dashed one stabilizes, reflecting the extent to which FBP-CS_{<10\%} produces more accurate predictions of extreme (lower tail) returns than does exact Bayes, asymptotically. Tallying with the interpretation of the numerical results in Table 2 the extent to which FBP-CS_{<10\%} is superior
to exact Bayes is successively less in Figures A.2 and A.3, with the dashed line ‘moving up’ to match the full line, as the misspecification of the predictive class is reduced. The size of the evaluation period required to produce a visual distinction between the out-of-sample performance of exact Bayes and FBP-CS_{<10\%} is larger, the less misspecified is the class.

In Panel B, the superiority of FBP-CS_{>90\%} over exact Bayes, in terms of accurately predicting extremely large returns is in stark evidence. In this case, the relative performance of the two updating methods is less affected by the move from the ARCH(1) to the GARCH(1,1) predictive class. However, once again, use of the more flexible mixture of predictives to underpin the exact Bayes update brings its performance much closer to that of the focused update, with the accuracy of the latter being reasonably robust to the choice of predictive class.

In the following section we provide some insight into why the focused update in the upper tail reaps more benefit out-of-sample than does the lower-tail update, relative to exact Bayes, and the role that misspecification plays here.

4.2.2 Animation of the mean predictives

In Figures 2 and 3 we display animated plots of the one-step-ahead mean predictives produced using expanding windows of \( n = 500 \) to \( n = 2,499 \), and based solely on the (most misspecified) Gaussian ARCH(1) predictive class. The mean predictives produced by both updating methods (FBP and exact Bayes) are superimposed upon the true predictive, produced using simulation from (12)-(14). Figure 2 presents the results for lower tail focus (FBP-CS_{<10\%} versus exact Bayes) and Figure 3 the results for upper tail focus (FBP-CS_{>90\%} versus exact Bayes). The vertical lines in each plot indicate the return that defines the quantile \( A \) in (6).

Two things are clear from Figure 2: one, the lower tails of both the FBP-CS_{<10\%} and exact Bayes predictives are quite similar; two, both tails are - for some time points - quite good at picking up the shape of the true predictive tail, but with the FBP-CS_{<10\%} predictive tail being a better match most of the time. These plots thus provide some explanation of the summary results in Panel A (column 3) of Table 2 and Panel A.1 of Figure 1, in which FBP-CS_{<10\%} dominates exact Bayes, but with the improvement in forecast accuracy being reasonably small, despite the misspecification of the predictive class.

In Figure 3 however, the animated display is very different. The upper tail of the exact Bayes predictive consistently fails to pick up the shape of the true: the misspecified nature of the Gaussian ARCH(1) model has a marked impact on predictive accuracy in this region of the support of \( y_{n+1} \). In contrast, FBP-CS_{>90\%} has the flexibility to focus only what matters - upper tail predictive accuracy - and, as such, produces predictives with upper tails that are much closer in shape to the true, and which are often visually indistinguishable from the true. These plots thus explain the clear numerical dominance of FBP-CS_{>90\%} over exact Bayes in Panel A (column 6) of Table 2 and Panel B.1 of Figure 1.
We finish by noting that focus on upper tail accuracy does - as highlighted by the relevant figures in the middle columns of Panel A in Table 2 - impact quite severely on the ability of FBP-CS_{>90\%} to pick up the lower tail of the true predictive. This outcome highlights the fact that the *ex-ante* decision as to what form of accuracy to focus on is critical, and most notably so in the very misspecified case.

### 4.2.3 The differential effect of posterior variation

When adopting the conventional Bayesian paradigm for prediction, a single question needs to be addressed: which model (or set of models) is to be used to produce the predictive distribution? Once that model (or set of models) has been chosen, computational methods are used to integrate out the posterior uncertainty associated with that choice, and a single (marginal) predictive distribution thereby produced. Posterior parameter (and model) uncertainty affects the location, shape, and degree of dispersion of the marginal predictive, and any predictive conclusions drawn from it; however, it is not the convention to explicitly quantify the impact of posterior variation on prediction.

Our new proposal introduces an additional choice into the mix: which measure of predictive accuracy is to drive the production of a predictive distribution? Each different form of in-sample update serves as a different ‘window’ through which a choice of predictive model (or mixture of predictive models) - and all posterior uncertainty associated with that choice - impinges on predictive outcomes. For example, one choice of update may yield a posterior distribution over a given predictive class that is very diffuse; another choice may lead to a very concentrated posterior. Hence, finite sample posterior variation itself has import, since it is not unique, even given a particular choice of predictive class.

We illustrate this point using $M = 4,000$ posterior draws from (2) for both FBP-CS_{<10\%} and FBP-CS_{>90\%}, using the Gaussian ARCH(1) predictive class, and for selected values of $n+1$ between 501 and 2,500. From the draws from (2) based on the FBP-CS_{<10\%} update, we produce the corresponding 4,000 values of the expected shortfall for period $n+1$ for a portfolio that is ‘long’ in the asset:

$$
\text{ES}^{n+1}_{0.1}(\theta) = -\int_{y_{n+1} < A_{0.1}} y_{n+1} p(y_{n+1}|F_n, \theta) \, dy_{n+1}.
$$

The integral bound $A_{0.1}$ in (21) denotes the 10\% quantile, and $\text{ES}^{n+1}_{0.1}(\theta)$ denotes the (negative of the) mean of $y_{n+1}$ conditional on the future return falling into the lower 10\% tail of $p(y_{n+1}|F_n, \theta)$; that is, the (negative) expected return in a worst case scenario. For a ‘short’ portfolio, in which extremely large returns are harmful, the relevant function is:

$$
\text{ES}^{n+1}_{0.9}(\theta) = \int_{y_{n+1} > A_{0.9}} y_{n+1} p(y_{n+1}|F_n, \theta) \, dy_{n+1},
$$

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where \( A_{0.9} \) in (22) denotes the 90% quantile, and \( ES_{0.9}^{n+1}(\theta) \) denotes the mean of \( y_{n+1} \) conditional on the future return falling into the upper 10% tail of \( p(y_{n+1}|F_n, \theta) \). In this case we produce 4,000 values of \( ES_{0.9}^{n+1}(\theta) \) using the draws from (2) based on the FBP-CS_{90\%} update. As a comparator in each case, we perform the same exercise but using the exact (likelihood-based) Bayes update. The ‘true’ values of \( ES_{0.1}^{n+1} \) and \( ES_{0.9}^{n+1} \), computed using simulation from (12)-(14), are reproduced on the respective plots as red vertical lines.

From Panel A in Figure 4 we see results that broadly confirm the lack of dominance of the FBP-CS_{<10\%} update over exact Bayes, in terms of accurately reproducing the lower tail of the true predictive. Neither the FBP-based posterior of \( ES_{0.1}^{n+1}(\theta) \) (the full blue curve), nor the exact Bayes posterior (the dashed green curve) is located uniformly closer to the true vertical (red) value than the other, across time. That said, the posterior variation in the exact Bayes posterior is always less than that of the FBP posterior, for the selected time points considered.

In contrast, in Panel B in Figure 4 there is a much more marked tendency for the FBP posterior of \( ES_{0.9}^{n+1}(\theta) \) to be located closer to the true value of \( ES_{0.9}^{n+1} \) than is the exact Bayes posterior, in addition to being much more concentrated. Hence, there are several instances over time in which the FBP posterior is extremely concentrated around - or very near to - the true expected shortfall: providing a further illustration of the benefits reaped by focusing on upper tail accuracy in the update.

## 5 Empirical Illustrations

We now illustrate the potential of the focused approach in empirical settings. In Section 5.1 we produce predictive results for two empirical return series, using the same predictive classes adopted in the simulation experiments above. We document the accuracy of posterior mean predictives in a similar manner to the documentation of the results in Table 2, as well as computing empirical exceedances for predictive VaRs for both long and short portfolios. In Section 5.2 we provide a quite different - and ambitious - empirical illustration, by using FBP to predict the 23,000 annual time series from the 2018 ‘M4’ forecasting competition.

### 5.1 Financial returns

The two series used in the first empirical illustration are: i) 4,000 observations of daily returns on the U.S. dollar currency index (DXY), from 3 Jan 2000 to 3 Nov 2015; and ii) 4,000 observations of daily returns on the S&P500 index, from 3 Jan 1996 to 3 Feb 2012. Both series are supplied by the Securities Industries Research Centre of Asia Pacific (SIRCA) on behalf of Reuters. All returns are continuously compounded. To match the simulation exercise, the last 2,000 observations in each series are used to perform all out-of-sample assessments, with the one-step-ahead mean predictives produced in the same manner as described in Section 4.2.1 (using expanding estimation
samples), apart from the fact that the first estimation sample for both empirical series is of length \( n = 2,000 \) (rather than \( n = 500 \)). We also adopt the same three predictive classes, defined by the Gaussian ARCH(1), Gaussian GARCH(1,1) and mixture models.

As tallying with the typical features exhibited by financial returns, the descriptive statistics reported in Table 3 provide evidence of time-varying volatility (significant serial correlation in squared returns) and marginal non-Gaussianity (significant non-normality in the level of returns) in both series. Hence, we can conclude that the simple Gaussian ARCH(1) and GARCH(1,1) predictive classes are likely to be misspecified, and more so than the more flexible mixture class. As such, we would anticipate accuracy gains - by using FBP rather than exact Bayes - to be most in evidence for the ARCH(1) predictive class, with decreasing relative gains expected as the predictive class becomes less misspecified.

In Table 4, we reproduce results for both the likelihood-based update (exact Bayes) and the FBP update that matches the out-of-sample accuracy measure used. That is, for each of the two empirical series, and for each predictive class, there is a single row of accuracy results labeled ‘FBP’, with the update underlying the FBP figure in any particular column matching the accuracy measure in the column label.

The results confirm our expectations. In Panel A, the figures based on the ARCH(1) predictive class tell a clear story: using an update that focuses on the measure that is assessed out-of-sample reaps accuracy gains. In all cases, and for both series, the exact (misspecified) Bayesian predictives are out-performed by FBP. In Panel B, FBP based on the GARCH(1,1) class continues to dominate exact Bayes uniformly; however the degree of dominance is less marked than in Panel A. The dominance of FBP over exact Bayes is no longer uniform in Panel C, for the case of the (least misspecified) mixture predictive class. Nevertheless, in all four instances, FBP still out-performs exact Bayes in the upper tail.

Hence, the empirical results mimic the patterns observed in the simulation setting, and continue to send the clear signal: when model misspecification is marked, FBP is beneficial, and

Table 3: Summary statistics. ‘JB stat’ is the test statistic for the Jarque-Bera test of normality, with a critical value of 5.99. ‘LB stat’ is the test statistic for the Ljung-Box test of serial correlation in the squared returns; the critical value based on a lag length of 3 is 7.82. ‘Skewness’ is the Pearson measure of sample skewness, and ‘Kurtosis’ a sample measure of excess kurtosis. The labels ‘Min’ and ‘Max’ refer to the smallest and largest value, respectively, while ‘Range’ is the difference between these two. The remaining statistics have the obvious interpretations.

|        | Min | Max | Mean | Median | St.Dev | Range | Skewness | Kurtosis | JB stat | LB stat |
|--------|-----|-----|------|--------|--------|-------|----------|----------|---------|---------|
| DXY    | -2.913 | 1.645 | -0.010 | -0.007 | 0.316 | 4.558 | -0.303 | 4.517 | 3461 | 271 |
| S&P500 | -21.070 | 19.510 | 0.035 | 0.164 | 2.590 | 40.581 | -0.260 | 5.762 | 5579 | 2783 |

\[11\] The choice of \( w_n \) for each FBP method remains as described in Section 4.1.
Table 4: Predictive accuracy based on FBP and exact Bayes mean predictives. Panels A to C report the average out-of-sample scores for the ARCH(1), GARCH(1,1) and mixture predictive class, respectively. The rows in each panel refer to the update method used. The columns refer to the out-of-sample measure used to compute the average scores. Each FBP figure is based on an update that matches the given out-of-sample accuracy measure. The bold figures are the largest average scores according to a given out-of-sample measure. The two time series are labelled as DXY and S&P500.
Table 5: Predictive Value at Risk assessment. The figures recorded are the proportion of times that the observed out-of-sample values ‘exceed’ the predictive VaR $\alpha$ indicated by the column heading. Panels A to C, respectively, record results for the data simulated from (12)-(14), the DXY empirical data, and the S&P500 empirical data. The bold value in each column indicates the empirical coverage that is closest to the nominal tail probability, whilst an asterisk indicates rejection of the null hypothesis of independence and correct coverage at the 1% level of significance, using the Christoffersen test.

| Updating method | Panel A: Simulated dataset | Panel B: DXY | Panel C: S&P500 |
|-----------------|---------------------------|--------------|-----------------|
|                 | Out-of-sample exceedances | Out-of-sample exceedances | Out-of-sample exceedances |
| Exact Bayes     | VaR$_{0.1}$ 0.117 0.196 0.217 0.062* 0.073* 0.142* 0.157* 0.085 0.081* 0.152* 0.131* 0.060* | VaR$_{0.1}$ 0.081 0.152 0.131 0.060 | VaR$_{0.1}$ 0.081 0.152 0.131 0.060 |
| FBP-CS$_{<10\%}$ | 0.102 0.198 0.05* 0.002* 0.084 0.237* 0.027* 0.011* 0.096 0.225 0.023* 0.008* | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 |
| FBP-CS$_{<20\%}$ | 0.103 0.203 0.023* 0.000* 0.077* 0.180 0.059* 0.022* 0.083 0.192 0.039* 0.017* | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 |
| FBP-CS$_{>80\%}$ | 0.338* 0.413* 0.197 0.104 0.031* 0.060* 0.204 0.094 0.050* 0.102* 0.164* 0.072* | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 |
| FBP-CS$_{>90\%}$ | 0.471* 0.551* 0.148* 0.085 0.015* 0.037* 0.247* 0.100 0.042* 0.074* 0.182* 0.078* | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 | VaR$_{0.1}$ 0.192 0.039 0.017* 0.083 |

particularly in terms of upper tail accuracy in the case of negatively skewed data. We now assess the practical significance of that benefit by conducting an out-of-sample assessment of the $\alpha \times 100\%$ predictive VaR - or predictive quantile (denoted by VaR$_{\alpha}$) - computed from the relevant mean predictive, and for the most misspecified class only: Gaussian ARCH(1). For comparative purposes, we also perform the exercise for the simulated data analysed in Section 4.2.

For all updating methods and all return series (both empirical and simulated), we first compute the probability of ‘exceedance’, $\hat{\alpha}$, as the proportion of realised out-of-sample values that are less than the predictive VaR$_{\alpha}$ for $\alpha = 0.1$ and $\alpha = 0.2$, as is relevant for a long portfolio. We then compute $1 - \hat{\alpha}$, as the proportion of realised values that are greater than the predictive VaR$_{\alpha}$ for $\alpha = 0.8$ and $\alpha = 0.9$, as pertains to a short portfolio. Table 5 reports the values of $\hat{\alpha}$ (or $1 - \hat{\alpha}$), for the exact Bayes update and four different versions of FBP that focus explicitly on tail accuracy: FBP-CS$_{<10\%}$, FBP-CS$_{<20\%}$, FBP-CS$_{>80\%}$ and FBP-CS$_{>90\%}$. The bold figure in each column indicates the empirical exceedance that is closest to the nominal tail probability. The asterisk then indicates whether the null hypothesis of $\hat{\alpha} = \alpha$ (or $1 - \hat{\alpha} = 1 - \alpha$) and independence in the exceedances is rejected at the 1% significance level by the Christoffersen (1998) test. In this particular illustration there is, of course, no exact match between update and out-of-sample measure; however we would anticipate that the FBP methods that focus on accuracy in the lower tail would yield better predictions of VaR$_{0.1}$ and VaR$_{0.2}$ (and, hence, better associated performance statistics) than would both exact Bayes (with no lower tail focus) and the FBP methods that focus on accuracy in the upper tail; with the corresponding conclusions expected for upper tail focus, and accurate prediction of VaR$_{0.8}$ and VaR$_{0.9}$. Hence the usefulness of reproducing all five sets of results for each scenario.
Looking first at the bold figures in Table 5, a simple conclusion can be drawn: for all three series, an ‘appropriate’ form of FBP (i.e. with an update that rewards accuracy in the relevant tail) produces empirical exceedances that are closer to the nominal values than do both exact Bayes and the ‘inappropriate’ FBP (i.e. with an update that rewards accuracy in the opposite tail). By this measure, focussing correctly reaps VaR accuracy benefits for all three series. For the DXY series (Panel B) we can hone this conclusion further: the updates that focus on accuracy in a particular marginal tail (remembering that the threshold used in the CS score is based on an estimate of a marginal quantile) also yield the best empirical exceedances for the conditional VaR with an equivalent probability. All four bold diagonal exceedances in Panel B are also insignificantly different from the nominal value, and are not associated with rejection of the null of independent violations. Indeed, with the exception of the FBP-CS>80% exceedance for FBP-CS>90%, these four diagonal figures are the only ones associated with a failure to reject the joint null of correct coverage and independent violations.

For the data simulated from (12)-(14) (Panel A), with one exception, all FBP methods that focus on the tail that is relevant for VaR prediction - so the lower tail for VaR0.1 and VaR0.2, and the upper tail for VaR0.8 and VaR0.9 - yield statistics that fail to reject the joint null. Moreover, as is somewhat consistent with the particular dominance of FBP over exact Bayes in the upper tail illustrated in Section 4.2.1, there is a slight tendency for FBP to also be more superior to exact Bayes in terms of prediction of the upper tail VaRα’s. For the S&P500 data, the FBP methods that focus on lower tail accuracy (FBP-CS<10% and FBP-CS<20%) are the only ones that do not formally reject the joint null - when used to predict VaR0.1 and VaR0.2. However, again, in terms of raw exceedances for VaRα’s in a particular tail, the FBP methods that reward accuracy in that same tail outperform everything else.

5.2 M4 forecasting competition

The M4 competition was an exploration of forecast performance organised by the University of Nicosia and the New York University Tandon School of Engineering in 2018. A total of 100,000 time series - of differing frequencies and lengths - were made available to the public. Each forecasting expert (or expert team) was then to submit a vector of h-step ahead forecasts, \( h = 1, 2, ..., H \), for each of the series. The winner of the competition in a particular category was the expert (team) who achieved the best average out-of-sample predictive accuracy according to the measure of accuracy that defined that category, over all horizons and all series.

One category was concerned with predictive interval accuracy, as measured by the mean scaled interval score (MSIS) proposed in Gneiting and Raftery (2007). The MSIS formula used in the

\[ 12 \] Details of all aspects of the competition can be found via the following link: [https://www.m4.unic.ac.cy/wp-content/uploads/2018/03/M4-Competitors-Guide.pdf](https://www.m4.unic.ac.cy/wp-content/uploads/2018/03/M4-Competitors-Guide.pdf) M4 competition details.
competition, defined over the 100 \((1 - \alpha)\)% prediction interval, is given by

\[
\text{MSIS} = \frac{1}{H} \sum_{h=1}^{H} \left( u_{t+h} - l_{t+h} + \frac{2}{\alpha} (l_{t+h} - y_{t+h}) \mathbf{1}\{y_{t+h} < l_{t+h}\} + \frac{2}{\alpha} (y_{t+h} - u_{t+h}) \mathbf{1}\{y_{t+h} > u_{t+h}\}\right),
\]

where \(H\) denotes the longest predictive horizon considered, \(l_{t+h}\) and \(u_{t+h}\) denote the 100 \((\alpha/2)\)% and 100 \((1 - \alpha/2)\)% predictive quantile, respectively, \(y_{t+h}\) is the realised value at time \(t + h\), \(h = 1, 2, ..., H\), and \(m\) denotes the frequency of the data. The overall predictive accuracy according to this score was measured by the mean MSIS over the 100,000 series.

The fourth best performance in this particular forecasting category was achieved by the ‘M4 team’, who produced prediction intervals using a model with an exponential smoothing structure for the level, trend and seasonal components, and a Gaussian distributional assumption, referred to as the ETS model hereafter (Hyndman et al., 2002). All three smoothing components of this model can be either additive or multiplicative in structure, while the trend component can also be specified to be damped; that is, the trend component can be forced to taper off over longer predictive horizons. A variety of ETS specifications were fit to each series using maximum likelihood estimation (MLE), and the best specification then selected via Akaike’s Information Criterion (AIC). The ETS model used to define the predictive interval was then characterized by at most four parameters: \(\theta_1\), the smoothing parameter in the level; \(\theta_2\), the smoothing parameter in the trend; \(\theta_3\), the smoothing parameter in the seasonality; and \(\theta_4\) the damping parameter in the trend; with \(z\) denoting the vector of initial states. The maximum likelihood estimates of the parameters and \(z\), for the selected model, were then ‘plugged into’ the assumed Gaussian predictive.

We now use the ETS model as the predictive class to which FBP is applied, with the MSIS score used to define the update. To keep the exercise computationally feasible we adopt the same (MLE/AIC-based) strategy as the M4 team for choosing the specification of the ETS model for each series. We also perform the exercise only for the 23,000 annual time series, and for \(H = 6\). For each of the \(i\) annual series, \(i = 1, 2, ..., 23,000\), we define: the full sample vector of observations of length \(n_i\) as \(y_{n_i}\); the vector of observations up to time \(t\) as \(y_{i,t}\); the observed value at any particular time point \(t + h\) as \(y_{i,t+h}\); the \(h\)-step-ahead \(i^{th}\) ETS predictive distribution as \(P_{\theta_i}^{t}\) (with \(\theta_i\) denoting the unknown parameters for the selected model for the \(i^{th}\) series); and the 100 \((\alpha/2)\)% and 100 \((1 - \alpha/2)\)% quantiles of the \(i^{th}\) ETS predictive density as \(l_{i,t+h}\) and \(u_{i,t+h}\) respectively. Using this notation (and acknowledging a slight abuse of the notation used earlier for the one-step-ahead predictives and associated scores), we define the (positively-oriented) MSIS-

\[^{13}\text{It is worth noting that for the ETS model there is no variance parameter; instead the variance of the residuals is employed to construct the predictive densities.}\]

\[^{14}\text{We employ the ETS function of the forcast package in R to perform this exercise.}\]
based sample criterion as $S_n(P^n_{\theta_i}) = \sum_{t=0}^{n_i-1} S_{MSIS}(P^t_{\theta_i}, y_{i,t,H})$, where

$$S_{MSIS}(P^t_{\theta_i}, y_{i,t,H}) = -\frac{1}{\min(H,n_i-t)} \sum_{h=1}^{\min(H,n_i-t)} \left( u_{i,t+h} - l_{i,t+h} + \frac{2}{\alpha} (l_{i,t+h} - y_{i,t+h}) I\{y_{i,t+h} < l_{i,t+h}\} \\
+ \frac{2}{\alpha} (y_{i,t+h} - u_{i,t+h}) I\{y_{i,t+h} > u_{i,t+h}\} \right)$$

(23)

and $y_{i,t,H} = (y_{i,t+1}, \ldots, y_{i,t+\min(H,n_i-t)})'$. Because there are 23,000 time series, and associated posteriors of the form of (2), the scale factor for each series $i$, call it $w_{i,n_i}$, needs to be selected via a computationally efficient and automated method. We set each $w_{i,n_i} = n_i d_{\theta_i}/2S_n(P^n_{\theta_i})$, where $\hat{\theta}_i$ denotes the MLE of $\theta_i$. This choice of $w_{i,n_i}$ ensures that the scale and convergence of the FBP posterior variance are anchored to those of a well understood benchmark posterior. More details are provided in Appendix A.3. Posterior draws of the ETS predictives for the $i^{th}$ series are defined by the draws of the underlying $\theta_i$, which are, in turn, produced using the same MCMC scheme described in Appendix A.2 for the GARCH(1,1) model (used in the simulation exercise), with appropriate adjustments made for the bounds on the ETS parameters.$^{15}$

Table 6 documents the accuracy with which five alternative methods forecast the 23,000 annual series, with accuracy measured exclusively by the (positively-oriented) MSIS rule. The methods are the top four performers in the full M4 competition (in which all 100,000 series are forecast, and accuracy is measured by MSIS) - denoted by M4-1$^{st}$, M4-2$^{nd}$, M4-3$^{rd}$ and M4-4$^{th}$ respectively - and the focused Bayesian method with update based on the MSIS rule, denoted simply by FBP. The four first place-getters are, in brief, the hybrid method of exponential smoothing and recurrent neural networks in Smyl (2019) (M4-1$^{st}$), the feature-based forecast combination method by Montero-Manso et al. (2019) (M4-2$^{nd}$), the Card forecasting method in Doornik et al. (2019) (M4-3$^{rd}$), with M4-4$^{th}$, as noted earlier, being the ETS/MLE-based method of the M4 team. We use the forecasts provided by the competitors (available in the M4 package in R) to assess the predictive accuracy of M4-1$^{st}$, M4-2$^{nd}$, M4-3$^{rd}$ and M4-4$^{th}$ for the 23,000 annual series.

Columns 1 to 4 of Panel A in Table 6 report various summaries of the 23,000 MSIS values - mean, median, standard deviation, and multiple quantiles - for each of the four competition methods. Based on the mean results, the ranking of the four methods for the 23,000 annual series is seen to equal their ranking in the full competition (as indicated by the column labels). Column 5 presents the summary statistics for FBP. First, we observe that both the mean and the median of the (positively-oriented) MSIS values for FBP are larger than those for M4-4$^{th}$ (ETS/MLE), which indicates that focusing does enhance predictive performance over and above simple use of the maximizer of the likelihood function of the (inevitably misspecified) ETS model. Second, we

$^{15}$The parameter restrictions for the ETS model are: $0 < \theta_1, \theta_2, \theta_3 < 1$ and $0.8 < \theta_4 < 0.98$.
Panel A: Out-of-sample MSIS summaries

Predictive method

| Summary     | M4-1st | M4-2nd | M4-3rd | M4-4th | FBP  |
|-------------|--------|--------|--------|--------|------|
| Mean        | -23.90 | -27.48 | -30.20 | -34.90 | -34.04 |
| Median      | -16.18 | -16.09 | -18.47 | -15.49 | -14.70 |
| SD          | 48.17  | 65.37  | 65.30  | 76.84  | 75.90 |
| Quant 1%    | -183.17| -297.98| -214.84| -348.52| -340.00|
| Quant 10%   | -32.23 | -28.12 | -52.33 | -66.71 | -65.12 |
| Quant 20%   | -24.14 | -19.52 | -35.07 | -34.50 | -33.14 |
| Quant 30%   | -20.45 | -17.72 | -27.19 | -24.43 | -23.28 |
| Quant 40%   | -18.02 | -16.77 | -22.20 | -19.02 | -17.95 |
| Quant 50%   | -16.18 | -16.09 | -18.47 | -15.49 | -14.70 |
| Quant 60%   | -14.62 | -15.54 | -15.35 | -12.71 | -12.21 |
| Quant 70%   | -13.29 | -15.02 | -12.65 | -10.45 | -10.36 |
| Quant 80%   | -12.01 | -14.41 | -10.43 | -8.95  | -8.96  |
| Quant 90%   | -10.28 | -13.41 | -8.35  | -7.16  | -7.03  |
| Quant 99%   | -5.29  | -7.25  | -4.17  | -3.30  | -3.47  |

Panel B: Grouped predictive performance

Predictive method

| Time series group | M4-1st | M4-2nd | M4-3rd | M4-4th | FBP  |
|-------------------|--------|--------|--------|--------|------|
| Group 1           | 1359   | 1732   | 766    | 304    | 439  |
| Group 2           | 814    | 2136   | 405    | 555    | 690  |
| Group 3           | 989    | 558    | 557    | 1118   | 1378 |
| Group 4           | 522    | 58     | 813    | 1534   | 1673 |
| Group 5           | 338    | 48     | 1006   | 1565   | 1643 |
| Total             | 4022   | 4532   | 3547   | 5076   | 5823 |

Table 6: Predictive accuracy of five competing methods for the 23,000 annual time series from the M4 competition. The labels M4-1st, M4-2nd, M4-3rd and M4-4th denote the first, second, third and fourth best methods (overall) in the MSIS category of M4. The label FBP refers to the focused Bayesian prediction method applied to the ETS predictive class and using the MSIS rule as the update. Panel A records various summaries of the 23,000 (positively-oriented) MSIS values; Panel B reports the number of series for which a particular method performs best in each of the five groups described in the text. The bold font is used to indicate the best performing method according to a given performance measure.
observe that in terms of median MSIS, FBP produces the best results out of all five methods considered. By looking at the quantiles of the distribution of the 23,000 scores for all methods we can glean why this is so. In particular, the values of MSIS at the 1st quantile for M4-4th and FBP are -348.52 and -340.00 respectively, whilst for M4-1st, M4-2nd and M4-3rd the corresponding values are -183.17, -297.98 and -214.84, respectively. In other words both ETS-based methods forecast 1% of the series quite poorly, which impacts on their mean MSIS. In contrast, the ETS-based methods - and particularly FBP - forecast quite accurately those series that produce MSIS values that are in the middle of the distribution, or in the upper tail.

To provide further insight here, we divide the 23,000 series into five different groups of 4,600 series. Group 1 has the 4,600 series with the smallest average (positively-oriented) MSIS, computed across the five methods (i.e. the series with the worst average results according to this measure), while Group 5 comprises the 4,600 series with the largest average MSIS values. The remaining groups span the middle. For each of these groups we count how many series are best predicted by each of the five methods. The results are presented in Panel B of Table 6. First, we observe that for all groups FBP has a larger number of favourable results than does the likelihood-based ETS method (M4-4th). Second, for the last three groups FBP outperforms all four other methods. Last, observe that the method that is best at predicting the highest number of series overall is FBP, with a total of 5,823 series (out of 23,000) for which it is the best performer. Focusing on predictive interval accuracy in the up-date has clearly yielded benefits out-of-sample.

6 Discussion

We have proposed a new paradigm for Bayesian prediction that can deliver accurate predictions in whatever metric is most meaningful for the problem at hand. By replacing the conventional likelihood function in the Bayesian update with an appropriate function of a proper scoring rule, the resultant posterior - by construction - gives high probability mass to predictive distributions that yield high scores. Under regularity, the posterior asymptotically concentrates onto the predictive that maximizes the expected scoring rule and, in this sense, yields the most accurate predictions in the given class. Movement away from the conventional Bayesian approach does involve the choice of a scale factor, which determines the relative weight of the prior and data-based components of the posterior. However, this factor can be chosen via common sense criteria in finite samples.

The choice of predictive class used is not pre-determined, and can be any plausible class of predictives, including combinations of predictives, that captures the features of the data that matter for prediction. In particular, the class can be deliberately selected to be computationally simple; the aim of the approach not being to perfectly capture all aspects of the true data generating process, but to achieve a particular type of predictive accuracy via a suitable Bayesian
update. Whilst the emphasis in the paper has been on classes of parametric predictives, or finite combinations of predictives, the paradigm, in principle, applies to non-parametric predictive settings also. Similarly, the focus on time series data, and ‘forecasting’ future values of random variables indexed by time, has also been a choice on our part, and not something that is intrinsic to the methodology.

The scope and value of our new approach to prediction are demonstrated using illustrations with simulated and empirical data. The benefits reaped by ‘focusing’ on the type of predictive accuracy that matters are stark, with virtually all numerical results showing gains over and above conventional (misspecified) likelihood-based prediction. Whilst a reduction in the degree of misspecification of the predictive class does lead to more homogeneous predictions across methods, for a large enough sample the superiority of FBP is still almost always in evidence, even when the predictive class is a reasonable representation of the true data generating process.

Some obvious avenues of future research remain open. First, the ability to generate accurate predictions via a simplistic representation of the truth means that models with, for example, intractable likelihood functions, can now be treated using exact simulation methods like MCMC, rather than via approximate methods like approximate Bayesian computation (Sisson et al., 2018) or Bayesian synthetic likelihood (Price et al., 2018). Comparison between the results yielded by FBP and those yielded by predictives based on such approximate methods of inference (e.g. Frazier et al., 2019) would be of interest. Second, in cases where the set of unknowns is very large, and an MCMC simulation method is likely to be inefficient, variational methods could be explored. In particular - and in the spirit of the generalized variational inference proposed by Knoblauch et al. (2019) - the choice of variational family, and/or the distance measure to be minimized, could be driven by the specific prediction focus of any problem. Finally, all results in this paper have been based on proper scoring rules only. This is by no means essential, but has simply been a decision made to render the scope of the paper manageable. In practice, any loss function in which predictive accuracy plays a role (e.g. financial loss functions associated with optimal portfolios of predicted returns; asymmetric loss functions associated with predicted energy demand) can drive the Bayesian update, and thereby give high weight to predictive distributions that yield small loss.

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A Computational Details

A.1 Setting $w_n$ for the CRPS rule

Let $\Sigma_{w,n}$ denote the posterior covariance matrix of $\theta$ calculated under the FBP posterior density $\pi_{w}(\theta|y_n)$, and which is associated with an arbitrary choice of $w_n$. As discussed in the main text, the role of the tuning sequence $w_n$ is to control the relative weight given to the sample score criterion and the prior within the update. The choice of $w_n$ is subjective in nature, in particular when the score criterion cannot be interpreted as a likelihood function, as is the case for the CRPS scoring rule. In this case then, we set $w_n$ so that the rate of posterior update of CRPS-based FBP is comparable to that of exact (likelihood-based) Bayes.
To this end, let $\psi(n)$ denote the trace of the covariance matrix of $\theta$ calculated under the exact Bayesian posterior in (7). We then propose finding a sequence $w_n^*$ such that the trace of $\Sigma_{w,n}$, when calculated under this sequence, satisfies $\text{tr}[\Sigma_{w,n}] = \psi(n)$. However, directly computing such a $w_n^*$ would entail solving the computationally intensive optimization problem,

$$w_n^* = \arg\min_{w_n \in W} \left\{ \text{tr}[\Sigma_{w,n}] - \psi(n) \right\}^2.$$

Hence, rather than pursuing $w_n^*$ directly, we seek a computationally efficient approximation $\tilde{w}_n^*$, defined as

$$\tilde{w}_n^* = \frac{\mathbb{E}_{p(\theta|y_n)}[\log p(y_n|\theta)]}{\mathbb{E}_{\pi(\theta|y_n)}[S_n(P_\theta^n)]},$$

(24)

where $\mathbb{E}_{\pi(\theta|y_n)}$ indicates an expectation computed under (7), $\log p(y_n|\theta) = \sum_{t=0}^{n-1} S_{LS}(P_\theta^t, y_{t+1})$ and $S_n(P_\theta^n) = \sum_{t=0}^{n-1} \text{SCRPS}(P_\theta^t, y_{t+1})$. To see how this sub-optimal choice of $w_n$ resembles $w_n^*$, consider the optimally-scaled scoring function $S_n^*(P_\theta^n) = w_n^* S_n(P_\theta^n)$. Substituting $S_n(P_\theta^n) = \frac{S_n(P_\theta^n)}{w_n^*}$ into (24) yields

$$\tilde{w}_n^* = w_n^* \frac{\mathbb{E}_{\pi(\theta|y_n)}[\log p(y_n|\theta)]}{\mathbb{E}_{\pi(\theta|y_n)}[S_n^*(P_\theta^n)]}.$$  

(25)

From (25) we see that as long as the posterior means of the optimally-scaled score $S_n^*(P_\theta^n)$ and $\log p(y_n|\theta)$ are reasonably similar then, $\tilde{w}_n^* \approx w_n^*$, and $\text{tr}[\Sigma_{w,n}^*] \approx \text{tr}[\Sigma_{w,n}] = \psi(n)$ as a consequence. This assumption is not unrealistic, since - by construction - $S_n^*(P_\theta^n)$ and $\log p(y_n|\theta)$ are score functions for which the respective sums of posterior variances are equal. Additionally, this sequence $\tilde{w}_n^*$ converges, as $n \to \infty$, to a constant $C$, and Theorem 1 thus still applies. Of course, in practice, $\tilde{w}_n^*$ itself needs to be estimated via draws from (7). However, as this computation needs to be performed only once, a sufficiently accurate estimate of $\tilde{w}_n^*$ can be produced via a large enough number of posterior draws. To keep the notation streamlined in the main text, in Section 4.1 we simply record the formula in (24) as the setting for $w_n$, and refer to the simulation-based estimate used in the computations as $\tilde{w}_n$.

### A.2 Computational scheme

For the predictive classes i) and ii) all posterior updates are performed numerically, using a Metropolis-Hasting (MH) scheme to select draws of the underlying $\theta$. The MH acceptance ratio, at iteration $j$, is of the form

$$\alpha = \min \left\{ 0, \frac{\exp\left[ \tilde{w}_n S_n(P_\theta^n) \right] \pi(\theta^{(c)})}{\exp\left[ \tilde{w}_n S_n(P_{\theta^{(j-1)}}) \right] \pi(\theta^{(j-1)})} \times \frac{q(\theta^{(j-1)}|\theta^{(j)})}{q(\theta^{(j)}|\theta^{(j-1)})} \right\},$$

(26)

where $q(.)$ denotes the candidate density function, $\theta^{(c)}$ the draw from $q(.)$, $\theta^{(j-1)}$ the previous draw in the chain, and $\tilde{w}_n$ the scale factor defined in (20).

For the Gaussian ARCH(1) predictive class, the candidate density is a truncated normal,
\[ q(\theta^{(c)}|\theta^{(j-1)}) \propto \phi(\theta^{(c)}; \theta^{(j-1)}, \sigma_q^2 I_3) I \left\{ \theta_3^{(c)} \in [0,1), \theta_2^{(c)} > 0 \right\}, \]

where \( \phi \) denotes the normal density function, \( I_3 \) is the three-dimensional identity matrix, and \( \sigma_q \) controls the step size of the proposal, initiated at \( \sigma_q = 0.05 \) and then set adaptively to target acceptance rates between 30% and 70%.

For the GARCH(1,1) predictive class, the four elements of \( \theta \) are not sampled in one block; instead, they are randomly assigned to two pairs at the beginning of each iteration. The pairs are then sampled, one pair conditional on the other, with the two-dimensional candidate density in each case equal to the product of two independent scalar normals, truncated where appropriate to reflect the parameter restrictions for the GARCH(1,1) model: \( \theta_2 > 0, \theta_3 \in [0,1), \theta_3 + \theta_4 < 1, \text{ and } \theta_4 \in [0,1) \). The random assignment into pairs, plus the use of independent normal candidates for each individual parameter, means that the step size - and hence, targeted acceptance rates - for each parameter can be controlled separately. The same form of adaptive scheme as described above is used for each parameter. Because the sampling is conducted via two MH steps, the formula for the acceptance ratio in (26) is modified accordingly.16

Finally, for predictive class iii) the posterior update for the scalar parameter \( \theta_1 \) is approximated numerically over a fine grid for the scalar parameter \( \theta_1 \), on the unit interval. Draws of \( \theta_1 \) from the numerically evaluated posterior then define draws of predictives from the posterior defined over the mixture predictive class.

### A.3 Setting \( w_{n_i} \) for the MSIS rule

For the M4 competition example in Section 5.2 we require the setting of \( w_{n_i} \) for \( S_{n_i}(P_{\theta_i}^{n_i}) \) defined in terms of the MSIS scoring rule in (23), for all \( i = 1, 2, ..., 23,000 \). In principle, the same approach could be adopted as described in Section 4.1 for the case of the CRPS scoring rule. However, that would entail 23,000 preliminary runs of MCMC to estimate each \( \hat{w}_{n_i} \). Hence, we propose a more computationally efficient way of setting each \( w_{n_i} \) for this example. The basic idea is to control for the scale of \( S_{n_i}(P_{\theta_i}^{n_i}) \) directly by imposing \( w_{n_i} S_{n_i}(P_{\theta_i}^{n_i}) = \gamma(n_i) \), where \( \gamma(n_i) \) is a user-defined deterministic function that serves as a benchmark scale for the term entering the exponential function in the expression for the FBP posterior, and \( \theta_i^\dagger \) is a representative parameter value, taken as the maximum likelihood estimate of \( \theta_i \) in the empirical example. Via a series of arguments that amount to adopting an approximating Gaussian model, we set \( \gamma(n_i) = \frac{1}{2} n_id_{\theta_i} \), where \( d_{\theta_i} \) is the dimension of \( \theta_i \). With this choice of \( \gamma(n_i) \), the scaling constant \( w_{n_i} \) is thus set to \( w_{n_i} = n_id_{\theta_i}/2S_{n_i}(P_{\theta_i}^{n_i}) \). This choice of \( w_{n_i} \) converges asymptotically to a constant \( C \) and, as such, Theorem 1 still applies.

16See Roberts and Rosenthal (2009) for details on adaptive MCMC, and Smith (2015) for an illustration of its use with random allocation.
B Assumptions and Proofs of Main Results

Consider a stochastic process \{y_t : \Omega \to \mathbb{R}, t \in \mathbb{N}\} defined on the complete probability space \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{F}_t := \sigma(y_1, \ldots, y_t)\) denote the natural sigma-field. The model predictive class is given by \(\mathcal{P}^t := \{P^t_\theta : \theta \in \Theta\}\), where the parameter space is a (possibly non-compact) subset of the Euclidean space \(\Theta \subseteq \mathbb{R}^d\), and for any \(A \subseteq \mathcal{F}\), \(P^t_\theta(A) := P(A | \mathcal{F}_t, \theta)\).

Define the \(\delta\)-neighborhood of \(\Theta\), around the point \(\theta_*\), as \(\mathcal{N}_\delta(\theta_*) := \{\theta : \|\theta - \theta_*\| \leq \delta\}\), where \(\|\cdot\|\) denotes the Euclidean norm. We say two sequences \(y_\theta\) and \(y_{\theta_*}\) are close if \(\sup_{n} \|y_\theta - y_{\theta_*}\| \leq \delta\).

Recall the empirical scoring rule calculated from \(\mathcal{P}^t\):

\[
S_n(\theta) := \sum_{t=0}^{n-1} S(P^t_\theta, y_{t+1}),
\]

and recall the limit optimizer,

\[
\theta_* := \arg \max_{P^t_\theta \in \mathcal{P}} \left[ \liminf_{n \to \infty} \sum_{t=0}^{n-1} S(P^t_\theta, y_{t+1})/v_n \right].
\]

The value \(\theta_*\) corresponds to the predictive \(P^t_\theta \in \mathcal{P}^t\) such that, in the limit, we minimize the expected (i.e., asymptotic) loss, in terms of \(S(\cdot, y)\), between the assumed predictive class, \(\mathcal{P}\), and the true data generating process that underlies the observed values of \(y_{t+1}\).

We consider the following regularity conditions on the function \(S_n\) and the prior \(\pi(\cdot)\).

**Assumption 1.** The prior density \(\pi(\theta)\) is continuous on \(\|\theta - \theta_*\| \leq \delta\), for some \(\delta > 0\), and positive for all \(\theta \in \Theta\). In addition, for any \(t > 0\), and any \(\kappa, 0 < \kappa < \infty\), there exists some \(C > 0\), and \(p > \kappa\) such that

\[
\Pi(\|\theta - \theta_*\| > t) \leq C t^{-p}.
\]

**Assumption 2.** There exists a sequence \(v_n\) diverging to \(\infty\) such that, for any \(\delta > 0\) there exists some \(\epsilon > 0\),

\[
\lim_{n \to \infty} \Pr \left\{ \sup_{\theta \in \mathcal{N}_\delta(\theta_*)} \frac{1}{v_n^2} \left[ S_n(\theta) - S_n(\theta_*) \right] \leq -\epsilon \right\} = 1.
\]

**Assumption 3.** For some \(\delta > 0\), the following are satisfied uniformly for \(\theta \in \mathcal{N}_\delta(\theta_*):\) There exists a sequence \(v_n\) diverging to \(\infty\), a vector function \(\Delta_n(\theta)\), matrices \(H_n\) and \(V_n\), such that

1. \(S_n(\theta) - S_n(\theta_*) = v_n (\theta - \theta_*)' \Delta_n(\theta_*) / v_n - \frac{1}{2} v_n (\theta - \theta_*)' H_n v_n (\theta - \theta_*) + R_n(\theta)\)
2. \(V_n^{-1/2} \Delta_n(\theta_*) / v_n \Rightarrow N(0, I)\) under \(P\).
3. For \(n\) large enough, \(H_n\) is positive-definite and, for some matrix \(H\), \(H_n \to_p H\).
4. For any \(\epsilon > 0\), there exists \(M \to \infty\) and \(\delta = o(1)\) such that

\[
\limsup_{n \to \infty} \Pr \left[ \frac{\sup_{\|\theta - \theta_*\| \leq M} |R_n(\theta)|}{\sup_{\|\theta - \theta_*\| \leq \delta} \frac{v_n^2}{v_n} \|\theta - \theta_*\|^2} > \epsilon \right] < \epsilon \text{ and } \limsup_{n \to \infty} \Pr \left[ \frac{\sup_{\|\theta - \theta_*\| \leq M} |R_n(\theta)|}{\sup_{\|\theta - \theta_*\| \leq \delta} \frac{v_n^2}{v_n} \|\theta - \theta_*\|^2} > \epsilon \right] = 0
\]
Remark 4. The use of \( v_n \) in the assumptions allows us to capture cases where the scoring rules may converge at rates other than the canonical \( \sqrt{n} \). Such instances include situations where the data are trended or otherwise non-stationary.

B.1 Posterior concentration

Consider the FBP posterior pdf

\[
\pi_w(\theta | y_n) = \frac{\exp \left[ w_n S_n(\theta) \right] \pi(\theta)}{\int_{\Theta} \exp \left[ w_n S_n(\theta) \right] \pi(\theta) d\theta}.
\]  

(27)

Under Assumptions 1-3, we prove a generalization of the result in Lemma 2, which demonstrates that the scaled posterior \( \tilde{\pi}_w \left( \eta | y_n \right) := \pi_w(\theta | y_n) / v_n \), where \( \pi_w(\theta | y_n) \) is defined in (27), converges to a Gaussian version in the total variation of moments norm. The result of Lemma 1 then follows by taking \( v_n = \sqrt{n} \) and considering the standard definition of \( S_n(\theta) \).

Proposition 1. For

\[
\eta := v_n (\theta - \theta^*) - H_n^{-1} \Delta_n(\theta^*) / v_n,
\]

and

\[
\phi (\eta | H_n) := Q_n \exp \left[ -\frac{C}{2} \eta' H_n^{-1} \eta \right]
\]

with \( Q_n := \sqrt{C \det[H_n] (2\pi)^{-d_{\theta}}} \), if Assumptions 1-3 are satisfied and \( \lim w_n = C, 0 < C < \infty \), then

\[
\int \left[ 1 + \| \eta \|^\kappa \right] | \tilde{\pi}_w(\eta | y_n) - \phi (\eta | H_n) | d\eta = o_p(1).
\]

Proof. Define

\[
L_n := \theta^* + H_n^{-1} \Delta_n(\theta^*) / v_n^2
\]

and note that by a change of variables \( \theta \mapsto \eta \), we obtain the posterior

\[
\tilde{\pi}[\eta | y_n] := \frac{\pi (\eta / v_n + L_n) \exp \left[ w_n S_n(u / v_n + L_n) \right]}{\int \pi (u / v_n + L_n) \exp \left[ w_n S_n(u / v_n + L_n) \right] du}.
\]

Define

\[
\gamma(\eta) := S_n(\eta / v_n + L_n) - S_n(\theta^*) - \frac{1}{2 v_n^2} \Delta_n(\theta^*)' H_n^{-1} \Delta_n(\theta^*)
\]

and note that

\[
\tilde{\pi}[\eta | y_n] := \frac{\pi (\eta / v_n + L_n) \exp (w_n \gamma(\eta))}{C_n},
\]

for

\[
C_n = \int \pi (u / v_n + L_n) \exp (w_n \gamma(\eta)) du.
\]

Note that we can write

\[
\int \| \eta \|^\kappa | \tilde{\pi}(\eta | y_n) - \phi (\eta | H_n) | d\eta = \frac{1}{C_n} \int \| \eta \|^\kappa e^{w_n \gamma(\eta)} \pi (\eta / v_n + L_n) - \phi (\eta | H_n) C_n d\eta
\]

\[
= J_n / C_n.
\]
where
\[
J_n = \int \|\eta\|^\kappa \left|e^{\nu_n \gamma(\eta) \pi}(\eta/v_n + L_n) - \phi(\eta|H_n) C_n\right| \, d\eta. \tag{28}
\]

Now, bound \(J_n\) as follows
\[
J_n \leq \int \|\eta\|^\kappa \left\{\left|e^{\nu_n \gamma(\eta) \pi}(\eta/v_n + L_n) - \pi(\theta_*) \phi(\eta|H_n) Q_n^{-1}\right| + \left|\pi(\theta_*) Q_n^{-1} - C_n\right| \phi(\eta|H_n)\right\} \, d\eta,
\]
\[
\leq J_{1n} + J_{2n},
\]
where
\[
J_{1n} := \int \|\eta\|^\kappa \left|e^{\nu_n \gamma(\eta) \pi}(\eta/v_n + L_n) - \pi(\theta_*) \phi(\eta|H_n) Q_n^{-1}\right| \, d\eta \tag{29}
\]
\[
J_{2n} := \left|\pi(\theta_*) Q_n^{-1} - C_n\right| \int \|\eta\|^\kappa \phi(\eta|H_n) \, d\eta \tag{30}
\]

The result follows if we can prove that \(J_{1n} = o_p(1)\) since, taking \(\kappa = 0\), \(J_{1n} = o_p(1)\) implies that
\[
\left|C_n - \pi(\theta_*) Q_n^{-1}\right| = \left|\int e^{\nu_n \gamma(\eta) \pi}(\eta/v_n + L_n) \, d\eta - \pi(\theta_*) Q_n^{-1} \int \phi(\eta|H_n) \, d\eta\right|,
\]
\[
= o_p(1).
\]

which implies that \(J_{2n} = o_p(1)\) and therefore \(J_n = o_p(1)\).

Using Assumption 3.1, the fact that \((\theta - \theta_*) = (\eta - \eta^*_n)/v_n\) with \(\eta^*_n := -H_n^{-1} \Delta_n/v_n\), and the definitions of \(L_n\) and \(\gamma(\eta)\), we re-express \(\gamma(\eta)\) as
\[
\gamma(\eta) = S_n(\theta) - S_n(\theta_*) - \frac{1}{2v_n^2}\Delta_n(\theta_*)' H_n^{-1} \Delta_n(\theta_*)
\]
\[
= \frac{(\eta - \eta^*_n)'}{v_n} \Delta_n(\theta_*) - \frac{v_n^2}{2} \frac{(\eta - \eta^*_n)'}{v_n} H_n \frac{(\eta - \eta^*_n)}{v_n} - \frac{1}{2v_n^2}\Delta_n(\theta_*)' H_n^{-1} \Delta_n(\theta_*) + R_n(\eta/v_n + L_n)
\]
\[
= -\frac{1}{2} \eta' H_n \eta + R_n(\eta/v_n + L_n).
\]

To demonstrate that \(J_{1n} = o_p(1)\), we split the integral for \(J_{1n}\) into three regions: for \(0 < M < \infty\), and some \(\delta > 0\),
1. \(\|\eta\| \leq M\)
2. \(M < \|\eta\| \leq \delta v_n\)
3. \(\|\eta\| \geq \delta v_n\)
Area 1: Over $\|\eta\| \leq M$,

\[
\sup_{\|\eta\| \leq M} |\pi (\eta/v_n + L_n) - \pi(\theta_*)| = o_p(1),
\]
\[
\sup_{\|\eta\| \leq M} |R_n(\eta/v_n + L_n)| = o_p(1).
\]

The first result follows from continuity of $\pi(\cdot)$ in Assumption 1, the convergence of $\Delta_n(\theta_*)$ in Assumption 3.2, and the definition of $L_n$. The second result follows from the second part of Assumption 3.4. The dominated convergence theorem then allows us to deduce that $J_{1n} = o_p(1)$ over $\|\eta\| \leq M$.

Area 2: Over $M \leq \|\eta\| \leq \delta v_n$, the second term in the integral can be made arbitrarily small by taking $M$ large enough and $\delta = o(1)$. It therefore suffices to show that, for $M$ large enough and $\delta$ small enough,

\[
J_{1n} := \int_{M \leq \|\eta\| \leq \delta v_n} \|\eta\|^\kappa \exp \left[ w_n \gamma(\eta) \right] \pi(\eta/v_n + L_n) \, d\eta = o_p(1).
\]

From the definition of $\gamma(\eta)$, it follows that

\[
\exp \left[ w_n \gamma(\eta) \right] \leq \exp \left[ -\frac{w_n}{2} \eta' H_n \eta + w_n |R_n(\eta/v_n + L_n)| \right].
\]

By the first part of Assumption 3.4, on the set $\{\eta : M \leq \|\eta\| \leq \delta v_n\}$,

\[
|R_n(\eta/v_n + L_n)| = o_p(\|v_n^{-1}(\eta - \eta_n^*)\|^2) = o_p \left( \|\eta + \frac{1}{v_n} H^{-1} \Delta_n^2 \| \right).
\]

Since $\|\eta_n^*\|^2 = O_p(1)$, we conclude that, for some $C > 0$, on the set $A_{M,\delta} = \{\eta : M \leq \|\eta\| \leq \delta v_n\}$,

\[
\exp \left[ w_n \gamma(\eta) \right] \leq C \exp \left[ -\frac{w_n}{2} \eta' H_n \eta + o_p(\|\eta\|^2) \right].
\]

We then have that, for some $M \to \infty$, $J_{1n} \leq K_{1n} + K_{2n} + K_{3n}$, where

\[
K_{1n} := \int_{A_{M,\delta}} \|\eta\|^\kappa \exp \left( -\frac{w_n}{2} \eta' H_n \eta \right) \sup_{\|\eta\| \leq M} \left[ \exp \left[ o_p(\|\eta\|^2) \right] \pi(\eta/v_n + L_n) - \pi(\theta_*) \right] \, d\eta,
\]
\[
K_{2n} := \int_{A_{M,\delta}} \|\eta\|^\kappa \exp \left[ -\frac{w_n}{2} \eta' H_n \eta + o_p(\|\eta\|^2) \right] \pi(\eta/v_n + L_n) \, d\eta,
\]
\[
K_{3n} := \pi(\theta_*) \int_{A_{M,\delta}} \|\eta\|^\kappa \exp \left[ -\frac{w_n}{2} \eta' H_n \eta + o_p(\|\eta\|^2) \right] \, d\eta
\]

For any fixed $M$, $K_{1n} = o_p(1)$, hence, for some sequence $M \to \infty$, by the dominated convergence theorem, it follows that $K_{1n} = o_p(1)$. For the second and third terms, we note the following:

(i) For any $0 < \kappa < \infty$, on the set $\{\eta : \|\eta\| \geq M\}$, there exists some $M'$ large enough such that for all $M > M'$:

\[
\|\eta\|^\kappa \exp (-\eta' H_n \eta) = O(1/M).
\]
(ii) From the definition of $L_n$, $L_n \to_p \theta_*$ and by continuity of $\pi(\cdot)$, on the set $\{\eta : M \leq \|\eta\| \leq \delta v_n\}$, for $\delta = o(1)$, we conclude that

$$\pi(L_n + \eta/v_n) = \pi(\theta_*) + o_p(1).$$

Applying (i) yields $K_{3n} = o_p(1)$. Applying (i) and (ii) together yields $K_{2n} = o_p(1)$.

**Area 3:** Over $\|\eta\| \geq \delta v_n$. For $\delta v_n$ large

$$Q^{-1}_{\|\eta\| \geq \delta v_n} \int \|\eta\|^{\kappa} \phi(\eta|H_n)\pi(\theta_*) d\eta = o(1).$$

Now, focus on

$$\bar{J}_1 := \int_{\|\eta\| \geq \delta v_n} \|\eta\|^{\kappa} e^{\delta v_n \gamma(\eta)} \pi(\eta/v_n + L_n) d\eta.$$ The change of variables $\theta = \eta/v_n + L_n$, then yields

$$\bar{J}_1 = v_n^{d_\theta + \kappa} \int_{\|\theta - L_n\| \geq \delta} \|\theta - L_n\|^{\kappa} e^{\delta v_n \gamma(\theta)} v_n^{d_\theta + \kappa} \left[ S_n(\theta) - S_n(\theta_*) - \frac{1}{2v_n} \Delta_n(\theta_*) H_n^{-1} \Delta_n(\theta_*) \right] \pi(\theta) d\theta.$$

Use the fact that $L_n = \theta_* + o_p(1)$, and bound $\bar{J}_1$ by

$$C e^{-\frac{\delta v_n}{2v_n^2} H_n^{-1} \Delta_n(\theta_*)} \int_{\|\theta - \theta_*\| \geq \delta} \|\theta - \theta_*\|^{\kappa} e^{\delta v_n \gamma(\theta)} \pi(\theta) d\theta,$$

From Assumption 3.2, $\frac{1}{2v_n} \Delta_n(\theta_*) H_n^{-1} \Delta_n(\theta_*) = O_p(1)$ so that $e^{-\frac{\delta v_n}{2v_n^2} H_n^{-1} \Delta_n(\theta_*)} = O_p(1)$. By Assumption 2, for any $\delta > 0$, there exists some $\epsilon > 0$ such that

$$\lim_{n \to \infty} Pr \left\{ \sup_{\|\theta - \theta_*\| \geq \delta} [S_n(\theta) - S_n(\theta_*)] \leq -v_n^2 \epsilon \right\} = 1.$$ Applying the above conclusion to $\bar{J}_1$ then yields, for $w_n$ a positive convergence sequence, for $n$ large enough,

$$\bar{J}_1 \lesssim e^{-w_n v_n^2} v_n^{d_\theta + \kappa} \int_{\|\theta - \theta_*\| \geq \delta} \|\theta - \theta_*\|^{\kappa} \pi(\theta) d\theta.$$

Apply the above bound, Assumption [ ] and the dominated convergence theorem to deduce

$$\bar{J}_1 \lesssim e^{-w_n v_n^2} v_n^{d_\theta + \kappa} \int_{\|\theta - \theta_*\| \geq \delta} \|\theta - \theta_*\|^{\kappa} \pi(\theta) d\theta \lesssim e^{-w_n v_n^2} v_n^{d_\theta + \kappa} \int_{\|\theta - \theta_*\| \geq \delta} \|\theta - \theta_*\|^{\kappa} \pi(\theta) d\theta + o_p(1) \leq e^{-w_n v_n^2} v_n^{d_\theta + \kappa} \int_{\|\theta - \theta_*\| \geq \delta} \|\theta - \theta_*\|^{\kappa} \pi(\theta) d\theta + o_p(1) \lesssim e^{-w_n v_n^2} v_n^{d_\theta + \kappa} + o_p(1).$$
B.2 Bayesian and frequentist agreement

We now prove the result of Lemma 2.

Proof of Lemma 2. We first prove that the sequence \( \|\hat{\theta} - \theta_*\| = O_p(v_n^{-1}) \). By assumption,

\[
o_p(v_n^{-1}) \leq S_n(\hat{\theta}) - S_n(\theta_*).
\]

Applying 3.1 to the above we obtain

\[
o_p(v_n^{-1}) \leq v_n \left(\hat{\theta} - \theta_*\right)^\prime \Delta_n(\theta_*) / v_n - \frac{1}{2} v_n \left(\hat{\theta} - \theta_*\right)^\prime H_n v_n \left(\hat{\theta} - \theta_*\right) + R_n(\hat{\theta}). \tag{31}
\]

From the consistency of \( \hat{\theta} \), for any \( \epsilon > 0 \), the following holds with probability at least \( 1 - \epsilon \): there exists some sequence \( \delta_{\epsilon,n} \to 0 \), such that \( \|\hat{\theta} - \theta_*\| < \delta_{\epsilon,n} \); by Assumption 3.2, there exists some \( K_{\epsilon,n} \) such that \( \|\Delta_n(\theta_*)/v_n\| < K_{\epsilon,n} \); for \( \delta_{\epsilon,n} \) as before and for \( H \) as in the statement of the theorem, \( \|H_n - H\| < \delta_{\epsilon,n} \). Define \( h_n := v_n(\theta - \theta_*) \). There then exists a sequence \( K_{\epsilon,n}^* \) such that if we apply the above inequalities to equation (31) we obtain

\[
o_p(v_n^{-1}) \leq -\|H^{1/2}h_n\|^2/2 + K_{\epsilon,n}^* (\|h_n\| + o_p(\|h_n\|^2)),
\]

where the last term follows by applying the second part of Assumption 3.2 and the consistency of \( \theta \). We can rearrange this equation to obtain

\[
o_p(v_n^{-1}) + \|H^{1/2}h_n\|^2/2 - K_{\epsilon,n}^* (\|h_n\| + o_p(\|h_n\|^2)) \leq 0.
\]

The above implies that, for some \( K_{\epsilon,n}^{**} = O_p(1) \), \( \|h_n\| < K_{\epsilon,n}^{**} \), which yields the result.

The result now follows along lines similar to Theorem 5.23 in van der Vaart (1998). For \( h_n := v_n(\theta - \theta_*) \), apply Assumption 3.3 to obtain

\[
S(\theta_* + h_n/v_n) - S_n(\theta_*) = h_n^\prime \Delta_n(\theta_*) / v_n + h_n^\prime [-H_n] h_n / 2 + o_p(1),
\]

\[
S(\theta_* + H^{-1}\Delta_n/v_n^2) - S_n(\theta_*) = -\frac{1}{2} v_n^2 \Delta_n^\prime [-H_n^{-1}] \Delta_n(\theta_*) + o_p(1),
\]

where the last term in the first equation follows from the \( v_n \)-consistency of \( \hat{\theta} \) and in the second instance by Assumption 3.2. By the definition of \( \theta \) the LHS of the first equation is greater than, up to an \( o_p(1) \) term, the LHS of the second equation. Therefore, subtracting the second equation from the first, and completing the square we have

\[
(h_n + H^{-1}\Delta_n/v_n)^\prime [-H_n] (h_n + H^{-1}\Delta_n/v_n) + o_p(1) \geq 0.
\]

Because \( [-H_n] \) converges in probability to a negative definite matrix, it must follow that \( \|h_n - H^{-1}\Delta_n/v_n\| = o_p(1) \). The result then follows from the asymptotic normality in Assumption 3.2.

Proof of Theorem 7. Let \( \rho_H \) denote the Hellinger metric: for absolutely continuous probability
measures $P$ and $G$,
\[
\rho_H\{P, G\} = \left\{ \frac{1}{2} \int \left[ \sqrt{dP} - \sqrt{dG} \right]^2 d\mu \right\}^{1/2}, \quad 0 \leq \rho_H\{P, G\} \leq 1,
\]
for $\mu$ the Lebesgue measure, and define $\rho_{TV}$ to be the total variation metric,
\[
\rho_{TV}\{P, G\} = \sup_{B \in F} |P(B) - G(B)|, \quad 0 \leq \rho_{TV}\{P, G\} \leq 2.
\]
Recall that, according to the definition of merging in Blackwell and Dubins (1962), two predictive measures $P$ and $G$ are said to merge if
\[
\rho_{TV}\{P, G\} = o_p(1).
\]
Recall the definitions of $P^n_w, P^n_\star$ in equations (9) and (10) in the main text. Likewise, consider the following frequentist version of the FBP predictive: Let $\Pi[\cdot|y_n] := N(\hat{\theta}, H^{-1}VH^{-1}/v^2_n)$ denote a normal measure with mean $\hat{\theta}$ and variance $H^{-1}VH^{-1}/v^2_n$ and consider the frequentist equivalent of the FBP predictive:
\[
P^n_\star = \int_\Theta P^n_\star d\Pi[\theta|y_n]. \tag{32}
\]
The result follows similar arguments to those given in the proof of Theorem 1 in Frazier et al. (2019). Fix $\epsilon > 0$ and define the set $V_\epsilon := \{\theta \in \Theta : \rho_H^2\{P^n_\star, P^n_\theta\} > \epsilon/4\}$. By convexity of $\rho_H^2\{P^n_\star, \cdot\}$, and Jensen’s inequality,
\[
\rho_H^2\{P^n_\star, P^n_w\} \leq \int_\Theta \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi_w[\theta|y_n]
\]
\[
= \int_{V_\epsilon} \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi_w[\theta|y_n] + \int_{V_\epsilon^c} \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi_w[\theta|y_n]
\]
\[
= \Pi_w[V_\epsilon|y_n] + \frac{\epsilon}{4} \Pi_w[V_\epsilon^c|y_n].
\]
By definition, $\theta_\star \notin V_\epsilon$ and therefore, by the posterior concentration of $\Pi_w[\cdot|y_n]$ in Lemma 2, $\Pi_w[V_\epsilon|y_n] = o_p(1)$. Hence, we can conclude:
\[
\rho_H^2\{P^n_\star, P^n_w\} \leq o_p(1) + \frac{\epsilon}{4}. \tag{33}
\]
Likewise, for $\Pi[\cdot|y_n]$ defined in equation (32), a similar argument yields
\[
\rho_H^2\{P^n_\star, P^n_\theta\} \leq \int_\Theta \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi[\theta|y_n]
\]
\[
= \int_{V_\epsilon} \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi[\theta|y_n] + \int_{V_\epsilon^c} \rho_H^2\{P^n_\star, P^n_\theta\} d\Pi[\theta|y_n]
\]
\[
= \Pi[V_\epsilon|y_n] + \frac{\epsilon}{4} \Pi[V_\epsilon^c|y_n]
\]
\[
= o_p(1) + \frac{\epsilon}{4}, \tag{34}
\]
where the concentration in (34) follows as a result of Lemma 2 and the definition of $\Pi[\cdot|y_n]$. Now, note that

$$\frac{1}{2} \left[ \rho^2_H\{P^n_w, P^n_\theta\} + \rho^2_H\{P^n_\theta, P^n_\theta\} \right] \geq \frac{1}{4} \left[ \rho_H\{P^n_w, P^n_\theta\} + \rho_H\{P^n_\theta, P^n_\theta\} \right]^2$$

$$\geq \frac{1}{4} \left[ \rho_H\{P^n_w, P^n_\theta\} \right]^2,$$

where the first line follows from the Cauchy-Schwartz inequality and the second line from the triangle inequality. Applying equations (33) and (34), we then obtain

$$\rho^2_H\{P^n_w, P^n_\theta\} \leq \epsilon + o_p(1).$$

Recall that, for probability distributions $P, G$,

$$0 \leq \rho^2_{TV}\{P, G\} \leq 4 \cdot \rho^2_H\{P, G\}.$$

Applying this relationship between $\rho^2_H$ and $\rho^2_{TV}$, yields the stated result.
Figure 1: Out-of-sample performance of exact Bayes and CS-based FBP: relative predictive accuracy in the lower tail (Panel A) and the upper tail (Panel B). The three plots in Panel A depict the cumulative average (over an expanding evaluation period) of $\text{CS}_{<10\%}$ for the exact Bayes (dashed green line) and $\text{FBP-CS}_{<10\%}$ (full blue line) updates, using the ARCH(1), GARCH (1,1) and mixture predictive classes respectively. The three plots in Panel B display the cumulative average of $\text{CS}_{>90\%}$ for the exact Bayes (dashed green line) and $\text{FBP-CS}_{>90\%}$ (full blue line) updates, using the three different predictive classes.
Figure 2: Animation over time of mean predictives based on the exact Bayes (dashed green curve) and FBP-CS$_{<10\%}$ (full blue curve) updates. The red curve is the true predictive, produced using simulation from [12]–[14]. The predictive class used is Gaussian ARCH(1). The black vertical line denotes the threshold used in the FBP-CS$_{<10\%}$ update, in the production of the mean predictive for time period $n+1$.

Figure 3: Animation over time of mean predictives based on the exact Bayes (dashed green curve) and FBP-CS$_{>90\%}$ (full blue curve) updates. The red curve is the true predictive, produced using simulation from [12]–[14]. The predictive class used is Gaussian ARCH(1). The black vertical line denotes the threshold used in the FBP-CS$_{>90\%}$ update, in the production of the mean predictive for time period $n+1$. 

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Figure 4: Animation over time of the posterior densities for: ES\textsuperscript{\textit{n+1}}\textsubscript{0,1}(\theta) (Panel A), and ES\textsuperscript{\textit{n+1}}\textsubscript{0,9}(\theta) (Panel B). In Panel A, the posterior densities are based on the exact Bayes (dashed green curve) and FBP-CS\textsubscript{<10\%} (full blue curve) updates. In Panel B, the posterior densities are based on the exact Bayes (dashed green curve) and FBP-CS\textsubscript{>90\%} (full blue curve) updates. The red vertical line in each panel is the true expected shortfall (ES\textsuperscript{\textit{n+1}}\textsubscript{0,1} in Panel A and ES\textsuperscript{\textit{n+1}}\textsubscript{0,9} in Panel B), produced using simulation from (12)-(14).