A matrix trace inequality and its application

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Abstract. In this short paper, we give a complete and affirmative answer to a conjecture on matrix trace inequalities for the sum of positive semidefinite matrices. We also apply the obtained inequality to derive a kind of generalized Golden-Thompson inequality for positive semidefinite matrices.

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1 Introduction

We give some notations. The set of all \( n \times n \) matrices on the complex field \( \mathbb{C} \) is represented by \( M(n, \mathbb{C}) \). The set of all \( n \times n \) Hermitian matrices is also represented by \( M_h(n, \mathbb{C}) \). Moreover the set of all \( n \times n \) nonnegative (positive semidefinite) matrices is also represented by \( M^+(n, \mathbb{C}) \).

Here \( X \in M^+(n, \mathbb{C}) \) means we have \( \langle \phi | X | \phi \rangle \geq 0 \) for any vector \( | \phi \rangle \in \mathbb{C}^n \).

The purpose of this short paper is to give the answer to the following conjecture which was given in the paper [1].

**Conjecture 1.1** ([1]) For \( X, Y \in M^+(n, \mathbb{C}) \) and \( p \in \mathbb{R} \), the following inequalities hold or not?

(i) \( Tr[(I + X + Y + Y^{1/2}XY^{1/2})^p] \leq Tr[(I + X + Y + XY)^p] \) for \( p \geq 1 \).

(ii) \( Tr[(I + X + Y + Y^{1/2}XY^{1/2})^p] \geq Tr[(I + X + Y + XY)^p] \) for \( 0 \leq p \leq 1 \).

We firstly note that the matrix \( I + X + Y + XY = (I + X)(I + Y) \) is generally not positive semidefinite. However, the eigenvalues of the matrix \( (I + X)(I + Y) \) are same to those of the positive semidefinite matrix \( (I + X)^{1/2}(I + Y)(I + X)^{1/2} \). Therefore the expression \( Tr[(I + X + Y + XY)^p] \) always makes sense.

We easily find that the equality for (i) and (ii) in Conjecture 1.1 holds in the case of \( p = 1 \). In addition, the case of \( p = 2 \) was proven by elementary calculations in [1].

Putting \( T = (I + X)^{1/2} \) and \( S = Y^{1/2} \), Conjecture 1.1 can be reformulated by the following problem, because we have \( Tr[(I + X + Y + XY)^p] = Tr[(T^2 + T^2S^2)^p] = Tr[(T^2(I + S^2))^p] = Tr[(T(I + S^2)T)^p] = Tr[(T^2 + TS^2T)^p] \).

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Problem 1.2 For $T, S \in M_+(n, \mathbb{C})$ and $p \in \mathbb{R}$, the following inequalities hold or not?

(i) $\text{Tr}[(T^2 + ST^2S)^p] \leq \text{Tr}[(T^2 + TS^2T)^p]$ for $p \geq 1$.

(ii) $\text{Tr}[(T^2 + ST^2S)^p] \geq \text{Tr}[(T^2 + TS^2T)^p]$ for $0 \leq p \leq 1$.

2 Main results

To solve Problem 1.2, we use the concept of the majorization. See [2] for the details on the majorization. Here for $X \in M_h(n, \mathbb{C})$, $\lambda \downarrow (X) = (\lambda_1 \downarrow (X), \ldots, \lambda_n \downarrow (X))$ represents the eigenvalues of the Hermitian matrix $X$ in decreasing order, $\lambda_1 \downarrow (X) \geq \cdots \geq \lambda_n \downarrow (X)$. In addition $x \prec y$ means that $x = (x_1, \ldots, x_n)$ is majorized by $y = (y_1, \ldots, y_n)$, if we have

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \quad (k = 1, \cdots, n - 1)$$

and

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

We need the following lemma which can be obtained as a consequence of Ky Fan’s maximum principle.

Lemma 2.1 (p.35 in [3]) For $A, B \in M_h(n, \mathbb{C})$ and any $k = 1, 2, \cdots, n$, we have

$$\sum_{j=1}^{k} \lambda_j \downarrow (A + B) \leq \sum_{j=1}^{k} \lambda_j \downarrow (A) + \sum_{j=1}^{k} \lambda_j \downarrow (B). \quad (1)$$

Then we have the following theorem.

Theorem 2.2 For $S, T \in M_+(n, \mathbb{C})$, we have

$$\lambda \downarrow (T^2 + ST^2S) \prec \lambda \downarrow (T^2 + TS^2T) \quad (2)$$

Proof: For $S, T \in M_+(n, \mathbb{C})$, we need only to show the following

$$\sum_{j=1}^{k} \lambda_j \downarrow (T^2 + ST^2S) \leq \sum_{j=1}^{k} \lambda_j \downarrow (T^2 + TS^2T) \quad (3)$$

for $k = 1, 2, \cdots, n - 1$, since we have

$$\sum_{j=1}^{n} \lambda_j \downarrow (T^2 + ST^2S) = \sum_{j=1}^{n} \lambda_j \downarrow (T^2 + TS^2T),$$

which is equivalent to $\text{Tr}[T^2 + ST^2S] = \text{Tr}[T^2 + TS^2T]$.

By Lemma 2.1, we have

$$2 \sum_{j=1}^{k} \lambda_j \downarrow (X) \leq \sum_{j=1}^{k} \lambda_j \downarrow (X + Y) + \sum_{j=1}^{k} \lambda_j \downarrow (X - Y). \quad (4)$$
for \( X, Y \in M_n(\mathbb{C}) \) and any \( k = 1, 2, \ldots, n \).

For \( X \in M(n, \mathbb{C}) \), the matrices \( XX^* \) and \( X^*X \) are unitarily similar so that we have \( \lambda_j^X(XX^*) = \lambda_j^{X^*}(X^*X) \). Then we have the following inequality:

\[
2 \sum_{j=1}^{k} \lambda_j^X(T^2 + TS^2T) = \sum_{j=1}^{k} \lambda_j^X(T^2 + ST^2T) + \sum_{j=1}^{k} \lambda_j^X(T^2 + TS^2T)
\]

\[
= \sum_{j=1}^{k} \lambda_j^X((T + iTS)(T - iST)) + \sum_{j=1}^{k} \lambda_j^X((T - iTS)(T + iST))
\]

\[
= \sum_{j=1}^{k} \lambda_j^X((T - iST)(T + iTS)) + \sum_{j=1}^{k} \lambda_j^X((T + iST)(T - iTS))
\]

\[
= \sum_{j=1}^{k} \lambda_j^X(T^2 + ST^2S + i(T^2S - ST^2)) + \sum_{j=1}^{k} \lambda_j^X(T^2 + ST^2S - i(T^2S - ST^2))
\]

\[
\geq 2 \sum_{j=1}^{k} \lambda_j^X(T^2 + ST^2S),
\]

for any \( k = 1, 2, \ldots, n-1 \), by using the inequality (4) for \( X = T^2 + ST^2S \) and \( Y = i(T^2S - ST^2) \). Thus we have the inequality (3) so that the proof is completed.

From Theorem 2.2 we have the following corollary.

**Corollary 2.3** For \( T, S \in M_+(n, \mathbb{C}) \) and \( p \in \mathbb{R} \), the following inequalities hold.

(i) \( \text{Tr}[(T^2 + ST^2S)^p] \leq \text{Tr}[(T^2 + TS^2T)^p] \) for \( p \geq 1 \).

(ii) \( \text{Tr}[(T^2 + ST^2S)^p] \geq \text{Tr}[(T^2 + TS^2T)^p] \) for \( 0 \leq p \leq 1 \).

*Proof:* Since \( f(x) = x^p, (p \geq 1) \) is convex function and \( f(x) = x^p, (0 \leq p \leq 1) \) is concave function, we have the present corollary thanks to Theorem 2.2 and a general property of majorization (See p.40 in [3]).

As mentioned in Introduction, Corollary 2.3 implies the following corollary by putting \( T = (I + X)^{1/2} \) and \( S = Y^{1/2} \).

**Corollary 2.4** For \( X, Y \in M_+(n, \mathbb{C}) \) and \( p \in \mathbb{R} \), the following inequalities hold.

(i) \( \text{Tr}[(I + X + Y + Y^{1/2}XY^{1/2})^p] \leq \text{Tr}[(I + X + Y + XY)^p] \) for \( p \geq 1 \).

(ii) \( \text{Tr}[(I + X + Y + Y^{1/2}XY^{1/2})^p] \geq \text{Tr}[(I + X + Y + XY)^p] \) for \( 0 \leq p \leq 1 \).

Thus Conjecture 1.1 was completely solved with an affirmative answer.

### 3 An application

In this section, we give a kind of one-parameter extension of the famous Golden-Thompson inequality [4, 5] for positive semidefinite matrices, applying the obtained result in the previous section. For this purpose, we denote the generalized exponential function by \( \exp_\nu(X) \equiv (I + \nu X)^{1/2} \) for \( \nu \in (0, 1] \) and \( X \in M(n, \mathbb{C}) \) such that \( \text{Tr}[(I + \nu X)^{1/2}] \in \mathbb{R} \). In addition, we use the following inequalities proved in [6].
Lemma 3.1 ([6]) For $X, Y \in M_+(n, \mathbb{C})$, and $\nu \in (0, 1]$, we have

(i) \[ \text{Tr}[\exp_\nu(X + Y)] \leq \text{Tr}[\exp_\nu(X + \nu Y^{1/2}XY^{1/2})]. \tag{5} \]

(ii) \[ \text{Tr}[\exp_\nu(X + Y + \nu XY)] \leq \text{Tr}[\exp_\nu(X) \exp_\nu(Y)]. \tag{6} \]

As mentioned in the below of Conjecture 1.1, the expression of the left hand side in (6) makes also sense, since we have $\text{Tr}[\exp_\nu(X + Y + \nu XY)] = \text{Tr}[\{(I + \nu X)^{1/2}(I + \nu Y)(I + \nu X)^{1/2}\}^{1/\nu}] \geq 0$.

From (i) of Corollary 2.4 and Lemma 3.1 we have the following proposition.

Proposition 3.2 For $X, Y \in M_+(n, \mathbb{C})$ and $\nu \in (0, 1]$, we have

\[ \text{Tr}[\exp_\nu(X + Y)] \leq \text{Tr}[\exp_\nu(X) \exp_\nu(Y)]. \tag{7} \]

Proof: The right hand side of (5) is bounded from the above by applying (i) of Corollary 2.4 and putting $X_1 = \nu X$, $Y_1 = \nu Y$ and $p = \frac{1}{\nu}$:

\[
\begin{align*}
\text{Tr} \left[ \exp_\nu(X + Y + \nu Y^{1/2}XY^{1/2}) \right] &= \text{Tr} \left[ \left\{ (I + \nu(X + Y + \nu Y^{1/2}XY^{1/2}))^{1/\nu} \right\}^p \right] \\
&= \text{Tr} \left[ (I + X_1 + Y_1 + Y_1^{1/2}X_1Y_1^{1/2})^p \right] \\
&\leq \text{Tr} \left[ (I + X_1 + Y_1 + X_1Y_1)^p \right] \\
&= \text{Tr} \left[ \{I + \nu(X + Y + \nu XY)\}^{1/\nu} \right] \\
&= \text{Tr} \left[ \exp_\nu(X + Y + \nu XY) \right],
\end{align*}
\]

which is the left hand side of (6). Thus we have the present proposition thanks to Lemma 3.1.

Note that the inequality (7) can be regarded as a kind of one-parameter extension of the Golden-Thompson inequality for positive semidefinite matrices $X$ and $Y$.

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