GENERALIZATION OF THE THEORY OF SEN IN THE
SEMI-STABLE REPRESENTATION CASE

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Abstract. For a semi-stable representation $V$, we will construct a subspace $D_{\pi,\text{Sen}}(V)$ of $C_p \otimes Q_p V$ endowed with a linear derivation $\nabla^{(\pi)}$. The action of $\nabla^{(\pi)}$ on $D_{\pi,\text{Sen}}(V)$ is closely related to the action of the monodromy operator $N$ on $D_{nt}(V)$. Furthermore, in the geometric case, the action of $\nabla^{(\pi)}$ on $D_{\pi,\text{Sen}}(V)$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.

1. Introduction

Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of $K$ and consider its $p$-adic completion $C_p$. By a $p$-adic representation of $G_K = \text{Gal}(\overline{K}/K)$, we mean a finite dimensional vector space $V$ over $Q_p$ endowed with a continuous action of $G_K$. Put $K_\infty = \bigcup_{0 \leq m} K(\zeta_p^m)$ where $\zeta_p$ denote a primitive $p$-th root of unity in $K$ satisfying $(\zeta_p + 1)^p = \zeta_p$. Let $H_K$ denote the kernel of the cyclotomic character $\chi: G_K \to \mathbb{Z}_p^*$ and define $\Gamma_K$ to be $G_K/H_K \simeq \text{Gal}(K_\infty/K)$. Then, for a $p$-adic representation $V$ of $G_K$, Sen constructs a $K_\infty$-vector space $D_{\text{Sen}}(V)$ of dimension $\dim_{Q_p} V$ in $(C_p \otimes Q_p V)^{H_K}$ equipped with the $K_\infty$-linear derivation $\nabla^{(0)}$ which is the $p$-adic Lie algebra of $\Gamma_K$. In the case when $V$ is a Hodge-Tate representation of $G_K$, the set of eigenvalues of $\nabla^{(0)}$ on $D_{\text{Sen}}(V)$ is exactly the same as the set of Hodge-Tate weights of $V$.

Now, we shall state the aim of this article. First, let us fix some notations. Fix a prime $\pi$ of $\mathcal{O}_K$ (the ring of integers of $K$) and for each $1 \leq m$, fix a $p^m$-th root $\pi^{1/p^m}$ of $\pi$ in $\overline{K}$ satisfying $(\pi^{1/p^{m+1}})^p = \pi^{1/p^m}$. Put $K^{\text{BK}}_\infty = \bigcup_{0 \leq m} K(\pi^{1/p^m})$ and $K^{\text{BK}} = \bigcup_{0 \leq m} K(\pi^{1/p^m})$. Here, the letter BK stands for the Breuil-Kisin extension. Let $H$ denote the Galois group $\text{Gal}(\overline{K}/K^{\text{BK}}_\infty)$ and define $\Gamma_{\text{BK}}$ to be $\text{Gal}(K^{\text{BK}}_\infty/K_\infty)$. Then, we have an isomorphism of profinite groups $G_K/H \simeq \Gamma_K \rtimes \Gamma_{\text{BK}}$. In this article, for a semi-stable representation $V$ of $G_K$, we shall construct a $K^{\text{BK}}_\infty$-vector space $D_{\pi,\text{Sen}}(V)$ of dimension $\dim_{Q_p} V$ in
(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H \text{ equipped with the } K_{\text{BK}}^p \text{-linear derivations } \nabla^{(0)} \text{ and } \nabla^{(\pi)}. \text{ Here, } \nabla^{(\pi)} \text{ denotes the } p \text{-adic Lie algebra of } \Gamma_{\text{BK}}. \text{ Then, the action of } \nabla^{(0)} \text{ on } D_{\pi, \text{Sen}}(V) \text{ tells us about Hodge-Tate weights as in the case of } D_{\text{Sen}}(V) \text{ and the action of } \nabla^{(\pi)} \text{ on } D_{\pi, \text{Sen}}(V) \text{ is closely related to the action of the monodromy operator } N \text{ on } D_{\text{st}}(V). \text{ Furthermore, in the case } V = H_{\text{ét}}^m(X \otimes_{\mathcal{O}} \overline{K}, \mathbb{Q}_p) \text{ where } X \text{ denotes a proper smooth scheme over } K, \text{ the action of } \nabla^{(\pi)} \text{ on } D_{\pi, \text{Sen}}(V) \text{ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem.}

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## 2. Preliminaries on basic facts

### 2.1. $p$-adic periods rings and $p$-adic representations. (See [F1] for details.)

Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$. Put $K_0 = \text{Frac}(W(k))$ where $W(k)$ denotes the ring of Witt vectors with coefficients in $k$. Choose an algebraic closure $\overline{K}$ of $K$ and consider its $p$-adic completion $\mathbb{C}_p$. Put

$$
\widehat{E} = \lim_{\xrightarrow{x \rightarrow x_0}} \mathbb{C}_p = \{ (x^{(0)}, x^{(1)}, \ldots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p \}.
$$

For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widehat{E}$, define their sum and product by $(x + y)^{(i)} = \lim_{j \rightarrow \infty} (x^{(i+j)} + y^{(i+j)})^p$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. Let $\epsilon = (\epsilon^{(n)})$ denote an element of $\widehat{E}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widehat{E}$ is a perfect field of characteristic $p > 0$ and is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_E(x) = v_p(x^{(0)})$ where $v_p$ denotes the $p$-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widehat{E}$ is equipped with an action of a Frobenius $\sigma$ and a continuous action of the Galois group $G_K = \text{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_E$. Define $\widehat{E}^+$ to be the ring of integers for this valuation. Put $\widehat{A}^+ = W(\widehat{E}^+)$ and $\widehat{B}^+ = \widehat{A}^+[1/p] = \{ \sum_{k \gg -\infty} p^k x_k \mid x_k \in \widehat{E}^+ \}$ where $\ast$ denotes the Teichmüller lift of $\ast \in \widehat{E}^+$. This ring $\widehat{B}^+$ is equipped with a surjective homomorphism

$$
\theta : \widehat{B}^+ \rightarrow \mathbb{C}_p : \sum p^k x_k \mapsto \sum p^k x_k^{(0)}.
$$
Let \( \tilde{p} \) denote \((p^{(n)}) \in \tilde{E}^+ \) such that \( p^{(0)} = p \). Then, \( \text{Ker}(\theta) \) is the principal ideal generated by \( \omega = [\tilde{p}] - p \). The ring \( B^+_{\text{dR}} \) is defined to be the \( \text{Ker}(\theta) \)-adic completion of \( \tilde{B}^+ \)

\[
B^+_{\text{dR}} = \lim_{\xrightarrow{n\geq 0}} \tilde{B}^+ / (\text{Ker}(\theta))^n.
\]

This is a discrete valuation ring and \( t = \log([\epsilon]) \) which converges in \( B^+_{\text{dR}} \) is a generator of the maximal ideal. Put \( B_{\text{dR}} = B^+_{\text{dR}}[1/t] \). The ring \( B_{\text{dR}} \) becomes a field and is equipped with an action of the Galois group \( G_K \) and a filtration defined by \( \text{Fil}^i B_{\text{dR}} = t^i B^+_{\text{dR}} \ (i \in \mathbb{Z}) \). Then, \((B_{\text{dR}})^G_K\) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a de Rham representation of \( G_K \) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}}(V)).
\]

Define \( B_{\text{HT}} \) to be the associated graded algebra to the filtration \( \text{Fil}^i B_{\text{dR}} \). The quotient \( \text{gr}^i B_{\text{HT}} = \text{Fil}^i B_{\text{dR}}/\text{Fil}^{i+1} B_{\text{dR}} \ (i \in \mathbb{Z}) \) is a one-dimensional \( \mathbb{C}_p \)-vector space spanned by the image of \( t^i \). Thus, we obtain the presentation

\[
B_{\text{HT}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)
\]

where \( \mathbb{C}_p(i) = \mathbb{C}_p \otimes_{\mathbb{Z}_p} (i) \) is the Tate twist. Then, \((B_{\text{HT}})^G_K\) is canonically isomorphic to \( K \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{HT}}(V) = (B_{\text{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a Hodge-Tate representation of \( G_K \) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{HT}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{HT}}(V)).
\]

Let \( \theta : \tilde{A}^+ \to \mathcal{O}_{\mathbb{C}_p} \) be the natural homomorphism where \( \mathcal{O}_{\mathbb{C}_p} \) denotes the ring of integers of \( \mathbb{C}_p \). Define the ring \( A_{\text{cris}} \) to be the \( p \)-adic completion of the PD-envelope of \( \text{Ker}(\theta) \) compatible with the canonical PD-envelope over the ideal generated by \( p \). Put \( B^+_{\text{cris}} = A_{\text{cris}}[1/p] \) and \( B_{\text{cris}} = B^+_{\text{cris}}[1/t] \). These rings are \( K_0 \)-algebras endowed with actions of \( G_K \) and Frobenius \( \varphi \). Furthermore, since these rings are canonically included in \( B_{\text{dR}} \), they are endowed with the filtration induced by that of \( B_{\text{dR}} \). Then, \((B_{\text{cris}})^{G_K}\) is canonically isomorphic to \( K_0 \). Thus, for a \( p \)-adic representation \( V \) of \( G_K \), \( D_{\text{cris}}(V) = (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K} \) is naturally a \( K_0 \)-vector space. We say that a \( p \)-adic representation \( V \) of \( G_K \) is a crystalline representation of \( G_K \) if we have

\[
\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\text{cris}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_{K_0} D_{\text{cris}}(V)).
\]

Fix a prime element \( \pi \) of \( \mathcal{O}_K \) (the ring of integers of \( K \)) and an element \( s = (s^{(n)}) \in \tilde{E}^+ \) such that \( s^{(0)} = \pi \). Then, the series \( \log([s] \pi^{-1}) \) converges to an element \( u_\pi \) in \( B^+_{\text{dR}} \) and the subring \( B_{\text{cris}}[u_\pi] \) of \( B_{\text{dR}} \) depends only on the choice of \( \pi \). We denote this ring by \( B_{\text{st}} \). Since this ring is included in \( B_{\text{dR}} \), it is endowed with the action of \( G_K \) and the filtration induced by those on \( B_{\text{dR}} \). The element \( u_\pi \) is transcendental over \( B_{\text{cris}} \) and we extend the Frobenius \( \varphi \) on \( B_{\text{cris}} \) to \( B_{\text{st}} \),
by putting $\varphi(u_\pi) = pu_\pi$. Furthermore, define the $B_{\text{cris}}$-derivation $B_{\text{st}} \to B_{\text{st}}$ by $N(u_\pi) = -1$. It is easy to verify that we have $N\varphi = p\varphi N$ and that the action of $N$ on $D_{\text{st}}(V)$ is nilpotent. As in the case of $B_{\text{cris}}$, we have $(B_{\text{st}})^{G_K} = K_0$. Thus, for a $p$-adic representation $V$ of $G_K$, $D_{\text{st}}(V) = (B_{\text{st}} \otimes_{Q_p} V)^{G_K}$ is naturally a $K_0$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a semi-stable representation of $G_K$ if we have

$$\dim_{Q_p} V = \dim_{K_0} D_{\text{st}}(V) \quad (\text{we always have } \dim_{Q_p} V \geq \dim_{K_0} D_{\text{st}}(V)).$$

Furthermore, we say that $V$ is a potentially semi-stable representation of $G_K$ if there exists a finite field extension $L/K$ in $\overline{K}$ such that $V$ is a semi-stable representation of $G_L$. Due to the result of Berger [Be1], it is known that $V$ is a potentially semi-stable representation of $G_K$ if and only if $V$ is a de Rham representation of $G_K$.

2.2. The theory of Sen. Keep the notation and assumption in Introduction. In the article [S3], Sen shows that, for a $p$-adic representation $V$ of $G_K$, the $\hat{K}_\infty(= (C_p)^{H_K})$-vector space $(C_p \otimes_{Q_p} V)^{H_K}$ has dimension $d = \dim_{Q_p} V$ and the union of the finite dimensional $\hat{K}_\infty$-subspaces of $(C_p \otimes_{Q_p} V)^{H_K}$ stable under $\Gamma_K$ is a $K_\infty$-vector space of dimension $d$ stable under $\Gamma_K$ (called $D_{\text{Sen}}(V)$). We have $C_p \otimes_{K_\infty} D_{\text{Sen}}(V) = C_p \otimes_{Q_p} V$ and the natural map $\hat{K}_\infty \otimes_{K_\infty} D_{\text{Sen}}(V) \to (C_p \otimes_{Q_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_K$ is close enough to 1, then the series of operators on $D_{\text{Sen}}(V)$

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

converges to a $K_\infty$-linear derivation $\nabla^{(0)} : D_{\text{Sen}}(V) \to D_{\text{Sen}}(V)$ and does not depend on the choice of $\gamma$. By the following proposition, we can see that the set of eigenvalues of $\nabla^{(0)}$ on $D_{\text{Sen}}(V)$ is exactly the same as the set of Hodge-Tate weights of $V$ if $V$ is a Hodge-Tate representation of $G_K$.

**Proposition 2.1.** If $V$ is a Hodge-Tate representation of $G_K$, there exists a $\Gamma_K$-equivariant isomorphism of $K_\infty$-vector spaces

$$D_{\text{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{Q_p} V} K_\infty(n_j) \quad (n_j \in \mathbb{Z}).$$

**Proof.** Since $V$ is a Hodge-Tate representation of $G_K$, there exists a basis $\{g_j\}_{j=1}^d$ of $C_p \otimes_{Q_p} V$ over $C_p$ such that it gives the Hodge-Tate decomposition

$$C_p \otimes_{Q_p} V \simeq \bigoplus_{j=1}^d C_p(n_j) : g_j \mapsto t^{n_j} \quad (n_j \in \mathbb{Z}).$$

From this presentation, it follows that $\{g_j\}_{j=1}^d$ forms a basis of a $K_\infty$-vector space $X$ which is contained in $(C_p \otimes_{Q_p} V)^{H_K}$ and stable under the action of $\Gamma_K$. Then, since we have $X \hookrightarrow D_{\text{Sen}}(V)$ by definition and both sides have the same dimension
$d$ over $K_{\infty}$, we get the equality $X = D_{\text{Sen}}(V)$. Thus, we obtain the $\Gamma_K$-equivariant isomorphism of $K_{\infty}$-vector spaces $D_{\text{Sen}}(V) \simeq \bigoplus_{j=1}^{d} K_{\infty}(n_j) : g_j \mapsto t^{n_j}$. 

3. Generalization of $D_{\text{Sen}}(V)$

Let us recall notations. Fix a prime $\pi$ of $\mathcal{O}_K$ (the ring of integers of $K$) and for each $1 \leq m$, fix a $p^m$-th root $\pi^{1/p^m}$ of $\pi$ in $\overline{K}$ satisfying $(\pi^{1/p^{m+1}})^p = \pi^{1/p^m}$. Put $K^{BK} = \bigcup_{0 \leq m} K(\pi^{1/p^m})$ and $K^{BK}_{\infty} = \bigcup_{0 \leq m} K_{\infty}(\pi^{1/p^m})$. Let $H$ denote the Galois group $\text{Gal}(\overline{K}/K^{BK}_{\infty})$ and define $\Gamma_{BK}$ to be $\text{Gal}(K^{BK}_{\infty}/K_{\infty})$. Then, we have an isomorphism of profinite groups

$$G_K/H \simeq \Gamma_K \rtimes \Gamma_{BK}.$$ 

For $\beta \in \Gamma_{BK}$, we have $\beta(\zeta_p^m) = \zeta_p^m$ and define the homomorphism $c : \Gamma_{BK} \to \mathbb{Z}_p$ such that we have $\beta(\pi^{1/p^m}) = \pi^{1/p^m} \zeta_p^{c(\beta)}$. Then, the homomorphism $c$ defines an isomorphism $\Gamma_{BK} \simeq \mathbb{Z}_p$ of profinite groups.

3.1. Construction of $D_{\pi, \text{Sen}}(V)$. Let $V$ be a semi-stable representation of $G_K$. For simplicity, assume that the number of the nilpotent block of the monodromy operator $N$ on $D_{st}(V)$ is 1. In the general case, one can easily construct $D_{\pi, \text{Sen}}(V)$ in the same way. Then, there exists a basis $\{g_j\}_j$ of $D_{st}(V)$ over $K_0$ such that we have

$$g_1 \overset{N}{\to} g_2 \overset{N}{\to} \cdots \overset{N}{\to} g_d \overset{N}{\to} 0.$$ 

By twisting $\{g_j\}_j$ by some powers of $t$ in $B_{st} \otimes_{\mathbb{Q}_p} V$, we obtain a basis $\{f_j\}_j$ of $B^{+}_{\text{cris}}[u_\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B^{+}_{\text{cris}}[u_\pi/t]$. Then, we can write

$$(*) \begin{cases} f_1 = t^{m_1(=0)}(F_1 + (-1)^1 \frac{u_1}{1!} F_2 + (-1)^2 \frac{u_1^2}{2!} F_3 + \cdots + (-1)^{d-1} \frac{u_1^{d-1}}{(d-1)! t^{d-1}} F_d) \\ f_2 = t^{m_2}(F_2 + (-1)^1 \frac{u_1^2}{2!} F_3 + \cdots + (-1)^{d-2} \frac{u_1^{d-2}}{(d-2)! t^{d-2}} F_d) \\ \vdots \\ f_d = t^{md} F_d \end{cases}$$

where $\{F_j\}_j$ denotes a set of elements of $B^{+}_{\text{cris}} \otimes_{\mathbb{Q}_p} V$ and we take $m_j \in \mathbb{Z}$ such that $\{f_j\}_j$ forms a basis of $B^{+}_{\text{cris}}[u_\pi/t] \otimes_{\mathbb{Q}_p} V$ over $B^{+}_{\text{cris}}[u_\pi/t]$.

**Definition 3.1.** With notations as above, let $\{h_j := t^{m_j} F_j\}_j$ denote the image of $\{t^{m_j} F_j\}_j$ by the homomorphism $B^{+}_{\text{cris}} \otimes_{\mathbb{Q}_p} V \to \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. Then, define $D_{\pi, \text{Sen}}(V)$ to be the $K^{BK}_{\infty}$-vector space generated by $\{h_j\}_j$ contained in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$.

**Lemma 3.2.** The elements $\{h_j\}_j$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. In particular, $\{h_j\}_j$ forms a basis of $D_{\pi, \text{Sen}}(V)$ over $K^{BK}_{\infty}$ and its dimension over $K^{BK}_{\infty}$ is equal to $\dim_{\mathbb{Q}_p} V$.

**Proof.** We can show inductively that $\{h_d\}, \{h_{d-1}, h_d\}, \ldots, \{h_1, \ldots, h_d\}$ are linearly independent over $\mathbb{C}_p$ in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$. 

By this lemma, we can easily verify that the following proposition holds.
Proposition 3.3. (c.f. Subsection 2.2) We have $\mathbb{C}_p \otimes_{K^\text{BK}_\infty} D_{\pi\text{-Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $(\mathbb{C}_p)^n \otimes_{K^\text{BK}_\infty} D_{\pi\text{-Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^n$ is an isomorphism.

It follows easily that the $K^\text{BK}_\infty$-vector space $D_{\pi\text{-Sen}}(V)$ is equipped with the $K^\text{BK}_\infty$-linear derivation $\nabla(0) = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ if $\gamma \in \Gamma_K$ is close enough to 1. By the following proposition, we can see that the action of $\nabla(0)$ on $D_{\pi\text{-Sen}}(V)$ tells us about Hodge-Tate weights as in the case of $D_{\text{Sen}}(V)$.

Proposition 3.4. (c.f. Proposition 2.1) For a semi-stable representation $V$ of $G_K$, there exists a $\Gamma_K$-equivariant isomorphism of $K^\text{BK}_\infty$-vector spaces

$$D_{\pi\text{-Sen}}(V) \simeq \bigoplus_{j=1}^d K^\text{BK}_\infty(n_j) : h_j \mapsto t_{n_j} \quad (n_j \in \mathbb{Z}).$$

Furthermore, the set of integers $\{n_j\}_j$ is exactly the same as the set of Hodge-Tate weights of $V$.

Proof. Note that we have $\{\gamma(f_j) = (\chi(\gamma))^n f_j\}_j$ by definition. Then, we can show inductively that we have $\{\gamma(h_d) = (\chi(\gamma))^{n_d} h_d\}, \{\gamma(h_{d-1}) = (\chi(\gamma))^{n_{d-1}} h_{d-1}\}, \ldots$ \{\gamma(h_1) = (\chi(\gamma))^{n_1} h_1\}. The rest is easily verified by Proposition 3.3.

On the other hand, if $\beta \in \Gamma_BK$ is close enough to 1, the series of operators on $D_{\pi\text{-Sen}}(V)$

$$\nabla(\pi) = \frac{\log(\beta)}{c(\beta)} = -\frac{1}{c(\beta)} \sum_{k \geq 1} \frac{(1 - \beta)^k}{k}$$

converges to a $K^\text{BK}_\infty$-linear derivation on $D_{\pi\text{-Sen}}(V)$ does not depend on the choice of $\beta \in \Gamma_BK$. This easily follows from the calculations $\nabla(\pi)(f_j) = 0$ and $\nabla(\pi)(\frac{u_\pi}{t}) = 1$.

Remark 3.5. By using the calculations $\nabla(\pi)(f_j) = 0$ and $\nabla(\pi)(\frac{u_\pi}{t}) = 1$, we obtain $\nabla(\pi)(F_j) = F_{j+1} (j < d)$ and $\nabla(\pi)(F_d) = 0$. Thus, we can rewrite $(\ast)$ as

$$\nabla(\pi)(f_j) = t_{n_j} \left( \sum_{k=0}^{d-j} (-1)^k \frac{u_\pi^k}{k! t^k} (\nabla(\pi))^k(F_j) \right) \quad (1 \leq j \leq d).$$

Compare this formula to the main construction $\{f_\pi^{(j)}\}_j$ in [M1]. In fact, the idea of the construction of $D_{\pi\text{-Sen}}(V)$ is based on the similarity between Corollary 2.1.14 of [Ki] and Main Theorems of [M1] and [M2].

3.2. Some properties of differential operators. We shall describe the actions of derivations $\nabla(0)$ and $\nabla(\pi)$ on $D_{\pi\text{-Sen}}(V)$. First, by a standard argument, we can show that, if $x \in D_{\pi\text{-Sen}}(V)$, we have

$$\nabla(0)(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla(\pi)(x) = \lim_{\beta \to 1} \frac{\beta(x) - x}{c(\beta)}.$$
Proposition 3.7. The action of the first, note that we have the relation $\gamma \beta = \beta^{\chi(\gamma)} \gamma$. Then, since we have

$$\lim_{h \to 0} \frac{a^{h+1} - a}{(h + 1) - 1} = a \log(a),$$

we obtain

$$[\nabla^{(0)}, \nabla^{(\pi)}](*) = \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1} \frac{\beta - 1}{c(\beta)}(*),$$

$$= \lim_{\beta \to 1} \frac{\gamma\beta - \gamma - \beta + 1}{(\chi(\gamma) - 1)c(\beta)}(*)$$

$$= \frac{\beta^{\chi(\gamma)} - \beta}{(\chi(\gamma) - 1)c(\beta)}(*)$$

$$= \frac{\beta\log(\beta)}{c(\beta)}(*)$$

$$= \nabla^{(\pi)}(*).$$

Proposition 3.7. The action of the $K_{\infty}^{BK}$-linear derivation $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is nilpotent.

Proof. From the equality $\nabla^{(0)}\nabla^{(\pi)} - \nabla^{(\pi)}\nabla^{(0)} = \nabla^{(\pi)}$, we get $\nabla^{(0)}(\nabla^{(\pi)})^r - (\nabla^{(\pi)})^r\nabla^{(0)} = r(\nabla^{(\pi)})^r$ and $\text{tr}(r(\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. Since the characteristic of $K_{\infty}^{BK}$ is 0, we obtain $\text{tr}((\nabla^{(\pi)})^r) = 0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the $K_{\infty}^{BK}$-linear derivation $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is nilpotent.

Proposition 3.8. For an element $x \in D_{\pi-Sen}(V)$ such that $\nabla^{(0)}(x) = nx$ ($n \in \mathbb{Z}$), we have $\nabla^{(0)}(\nabla^{(\pi)}(x)) = (n + 1)\nabla^{(\pi)}(x)$, that is, the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weight by 1.

Proof. This follows easily from the relation $[\nabla^{(0)}, \nabla^{(\pi)}] = \nabla^{(\pi)}$.

There are many choices of a $K_{\infty}^{BK}$-subspace of dimension $\dim_{\mathbb{Q}_p} V$ in $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$ equipped with derivations $\nabla^{(0)}$ and $\nabla^{(\pi)}$. The aim of this article is, however, to construct a differential module in $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ which is closely related to the module $D_{\text{St}}(V)$. Thus, the following proposition says that the choice $D_{\pi-Sen}(V)$ may be a reasonable one.

Proposition 3.9. For a crystalline representation $V$ of $G_K$, the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is trivial.
Proof. In the case when $V$ is a crystalline representation of $G_K$, we can take $\{ f_j \}$ as a basis of $D_{\pi-Sen}(V)$ over $K^\text{BK}_\infty$. We can see that the action of $\Gamma_{\text{BK}}$ on this basis is trivial and thus the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is trivial. \hfill \Box

Conversely, there is a semi-stable representation $V$ of $G_K$ such that the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is non-trivial. The next example is the prototype of such a semi-stable representation.

Example 3.10. Let $V$ be a $p$-adic representation of $G_K$ attached to the Tate curve $K'/\pi$. We can take a basis $\{e, f\}$ of $V$ over $\mathbb{Q}_p$ such that the action of $g \in G_K$ is given by

$\begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}$.

It is easy to see that $\{h_1 = 1 \otimes f, h_2 = 1 \otimes e\} (\subset \mathbb{C}_p \otimes_{\mathbb{Q}_p} V)$ forms a basis of $D_{\pi-Sen}(V)$ over $K^\text{BK}_\infty$. As indicated by Proposition 3.4, we have

$\nabla^{(0)}(h_1) = 0$ and $\nabla^{(0)}(h_2) = h_2$.

that is, the Hodge-Tate weights of $V$ are $\{0, 1\}$. Furthermore, the action of $\nabla^{(\pi)}$ on this basis is given by

$h_1 \overset{\nabla^{(\pi)}}{\longrightarrow} h_2 \overset{\nabla^{(\pi)}}{\longrightarrow} 0$.

This means that the action of $\nabla^{(\pi)}$ on $D_{\pi-Sen}(V)$ is nilpotent (Proposition 3.7) and that the action of $\nabla^{(\pi)}$ increases the Hodge-Tate weights by 1 (Proposition 3.8). Thus, we can know more than Hodge-Tate weights by using the $K^\text{BK}_\infty$-vector space $D_{\pi-Sen}(V)$ equipped with $\nabla^{(\pi)}$.

4. Geometric aspect of $D_{\pi-Sen}(V)$

Let $X$ be a proper smooth scheme over $K$. Then, it is known that the $p$-adic étale cohomology $V^m = H^m_{\text{ét}}(X \otimes_K K, \mathbb{Q}_p)$ is a de Rham representation of $G_K$. Furthermore, due to the result of Berger, we can see that $V^m$ is a potentially semi-stable representation of $G_K$. Let $L/K$ be a finite field extension of $K$ in $\overline{K}$ such that $V^m$ is a semi-stable representation of $G_L$ and let $V^m_L$ denote the restriction of $V^m$ to $G_L$.

In this section, we shall study the geometric aspect of $D_{\pi-Sen}(V^m_L)$ and see that the action of $\nabla^{(\pi)}$ describes an analogy of the infinitesimal variations of Hodge structures and satisfies formulae similar to the Griffiths transversality and the local monodromy theorem. First, by Proposition 3.4, we obtain the $\Gamma_K$-equivariant isomorphism of $L^\text{BK}_\infty$-vector spaces $D_{\pi-Sen}(V^m_L) \simeq \bigoplus_{j=1}^{\dim_{\mathbb{Q}_p}V} L^\text{BK}_\infty(\eta_j)$.

With this presentation, define the subspace $D^{e,t}_{\pi-Sen}(V^m_L)$ of $D_{\pi-Sen}(V^m_L)$ to be

$D^{e,t}_{\pi-Sen}(V^m_L) = \{ x \in D_{\pi-Sen}(V^m_L) | \nabla^{(0)}(x) = tx \}$ ($t \in \mathbb{Z}$). It follows easily that we obtain the decomposition

$D_{\pi-Sen}(V^m_L) = \bigoplus_{s+t=m} D^{e,t}_{\pi-Sen}(V^m_L)$. 

The next proposition claims that the action of $\nabla^{(\pi)}$ on $D_{\pi\mbox{-Sen}}(V_m^L)$ satisfies a formula similar to Griffiths transversality.

**Proposition 4.1.** (Transversality) With notations as above, we have

$$\nabla^{(\pi)}(D_{\pi\mbox{-Sen}}^{s,t}(V_m^L)) \subset D_{\pi\mbox{-Sen}}^{s-1,t+1}(V_m^L).$$

*Proof.* This follows easily from Proposition 3.8. \qed

By the same argument, we can see that an analogy of the local monodromy theorem holds for the $L^\infty_{BK}$-vector space $D_{\pi\mbox{-Sen}}(V_m^L)$ equipped with $\nabla^{(\pi)}$.

**Proposition 4.2.** (Local monodromy theorem) With notations as above, the $L^\infty_{BK}$-linear operator $\nabla^{(\pi)}$ satisfies

$$(\nabla^{(\pi)})^{m+1} | D_{\pi\mbox{-Sen}}(V_m^L) = 0.$$ 

Furthermore, if we put $h_{s,t} = \dim_{L^\infty_{BK}} D_{\pi\mbox{-Sen}}^{s,t}(V_m^L)$ and define $h_m = \sup \{b-a \mid \forall i \in [a,b], h_{i,m-i} \neq 0 \}$, we have

$$(\nabla^{(\pi)})^{h_m+1} | D_{\pi\mbox{-Sen}}(V_m^L) = 0.$$ 

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