A Meshalkin Theorem for Projective Geometries

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Dedicated to the memory of Lev Meshalkin.

Abstract: Let $\mathcal{M}$ be a family of sequences $(a_1, \ldots, a_p)$ where each $a_k$ is a flat in a projective geometry of rank $n$ (dimension $n-1$) and order $q$, and the sum of ranks, $r(a_1) + \cdots + r(a_p)$, equals the rank of the join $a_1 \lor \cdots \lor a_p$. We prove upper bounds on $|\mathcal{M}|$ and corresponding LYM inequalities assuming that (i) all joins are the whole geometry and for each $k < p$ the set of all $a_k$’s of sequences in $\mathcal{M}$ contains no chain of length $l$, and that (ii) the joins are arbitrary and the chain condition holds for all $k$. These results are $q$-analogs of generalizations of Meshalkin’s and Erdős’s generalizations of Sperner’s theorem and their LYM companions, and they generalize Rota and Harper’s $q$-analog of Erdős’s generalization.

Keywords: Sperner’s theorem, Meshalkin’s theorem, LYM inequality, antichain, $r$-family, $r$-chain-free

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1. INTRODUCING THE PLAYERS

We present a theorem that is at once a \(q\)-analog of a generalization, due to Meshalkin, of Sperner’s famous theorem on antichains of sets and a generalization of Rota and Harper’s \(q\)-analog of both Sperner’s theorem and \(\text{Erdős’s generalization.}\)

Sperner’s theorem \([12]\) concerns a subset \(A\) of \(\mathcal{P}(S)\), the power set of an \(n\)-element set \(S\), that is an antichain: no member of \(A\) contains another. It is part (b) of the following theorem. Part (a), which easily implies (b) (see, e.g., \([1\text{ Section 1.2}]\) was found later by Lubell \([9]\), Yamamoto \([13]\), and Meshalkin \([10]\) (and Bollobás independently proved a generalization \([4]\)); consequently, it and similar inequalities are called LYM inequalities.

**Theorem 1.** Let \(A\) be an antichain of subsets of \(S\). Then:

\[
\text{(a) } \sum_{A \in A} \frac{1}{|A|} \leq 1 \text{ and } \\
\text{(b) } |A| \leq \binom{n}{\lfloor n/2 \rfloor}.
\]

(c) Equality occurs in (a) and (b) if \(A\) consists of all subsets of \(S\) of size \(\lfloor n/2 \rfloor\), or all of size \(\lceil n/2 \rceil\).

The idea of Meshalkin’s insufficiently well known generalization\(^3\) (an idea he attributes to Sevast’yanov) is to consider ordered \(p\)-tuples \(A = (A_1, \ldots, A_p)\) of pairwise disjoint sets whose union is \(S\). We call these weak compositions of \(S\) into \(p\) parts.

**Theorem 2.** Let \(\mathcal{M}\) be a family of weak compositions of \(S\) into \(p\) parts such that each \(\mathcal{M}_k = \{A_k : A \in \mathcal{M}\}\) is an antichain.

\[
\text{(a) } \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|, \ldots, |A_p|}} \leq 1. \\
\text{(b) } |\mathcal{M}| \leq \max_{\alpha_1 + \cdots + \alpha_p = n} \binom{n}{\alpha_1, \ldots, \alpha_p} = \left(\lceil \frac{n}{p} \rceil, \ldots, \lceil \frac{n}{p} \rceil, \lfloor \frac{n}{p} \rfloor, \ldots, \lfloor \frac{n}{p} \rfloor\right).
\]

(c) Equality occurs in (a) and (b) if, for each \(k\), \(\mathcal{M}_k\) consists of all subsets of \(S\) of size \(\lceil \frac{n}{p} \rceil\), or all of size \(\lfloor \frac{n}{p} \rfloor\).

Part (b) is Meshalkin’s theorem \([10]\); the corresponding LYM inequality (a) was subsequently found by Hochberg and Hirsch \([7]\). (In expressions like the multinomial coefficient in (b), since the lower numbers must sum to \(n\), the number of them that equal \(\lceil \frac{n}{p} \rceil\) is the least nonnegative residue of \(n\) modulo \(p + 1\).)

In \([2]\) Wang and we generalized Theorem 2 in a way that simultaneously also generalizes \(\text{Erdős’s theorem on } l\text{-chain-free families: subsets of } \mathcal{P}(S) \text{ that contain no chain of length } l\). (Such families have been called “\(r\)-families” and “\(k\)-families”, where \(r\) or \(k\) is the forbidden length. We believe a more suggestive name is needed.)

**Theorem 3** (\([2\text{ Corollary 4.1}]\)). Let \(\mathcal{M}\) be a family of weak compositions of \(S\) into \(p\) parts such that each \(\mathcal{M}_k\), for \(k < p\), is \(l\)-chain-free. Then:

\[
\text{(a) } \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|, \ldots, |A_p|}} \leq l^{p-1}, \text{ and}
\]

\(^3\text{We do not find it in books on the subject }\([1\text{ 5}]\text{ but only in }\([8]\).}\)
(b) \(|\mathcal{M}|\) is no greater than the sum of the \(l^{p-1}\) largest multinomial coefficients of the form \(\binom{n}{\alpha_1, \ldots, \alpha_p}\).

Erdős’s theorem [3] is essentially the case \(p = 2\), in which \(A_2 = S \setminus A_1\) is redundant. The upper bound is then the sum of the \(l\) largest binomial coefficients \(\binom{n}{j}\), \(0 \leq j \leq n\), and is attained by taking a suitable subclass of \(\mathcal{P}(S)\). In general the bounds in Theorem 3 cannot be attained [2, Section 5].

Rota and Harper began the process of \(q\)-analagizing by finding versions of Sperner’s and Erdős’s theorems for finite projective geometries [11]. We think of a projective geometry be attained [2, Section 5].

A Meshalkin sequence of length \(q\) is a sequence \(\alpha = (\alpha_1, \ldots, \alpha_p)\) of flats whose join is \(\hat{1}\) and whose ranks sum to \(n\). The \(q\)-Gaussian coefficients (usually the “\(q\)” is omitted) are the quantities

\[
\left[ \frac{n}{k} \right] = \frac{n!_q}{k!_q(n-k)!_q} \quad \text{where} \quad n!_q = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1).
\]

They are the \(q\)-analogs of the binomial coefficients. Again, a family of projective flats is \(l\)-chain-free if it contains no chain of length \(l\). Let \(\mathcal{L}_k\) be the set of all flats of rank \(k\) in \(\mathbb{P}^{n-1}(q)\).

**Theorem 4 ([11] p. 200).** Let \(\mathcal{A}\) be an \(l\)-chain-free family of flats in \(\mathbb{P}^{n-1}(q)\).

(a) \(\sum_{a \in \mathcal{A}} \frac{1}{r(a)} \leq l\).

(b) \(|\mathcal{A}|\) is at most the sum of the \(l\) largest Gaussian coefficients \(\left[ \frac{n}{j} \right]\) for \(0 \leq j \leq n\).

(c) There is equality in (a) and (b) when \(\mathcal{A}\) consists of the \(l\) largest classes \(\mathcal{L}_k\), if \(n - l\) is even, or the \(l - 1\) largest classes and one of the two next largest classes, if \(n - l\) is odd.

Our \(q\)-analog theorem concerns the projective analogs of weak compositions of a set. A Meshalkin sequence of length \(p\) in \(\mathbb{P}^{n-1}(q)\) is a sequence \(a = (a_1, \ldots, a_p)\) of flats whose join is \(\hat{1}\) and whose ranks sum to \(n\). The submodular law implies that, if \(a_J := \bigvee_{j \in J} a_j\) for an index subset \(J \subseteq [p] = \{1, 2, \ldots, p\}\), then \(a_I \land a_J = \hat{0}\) for any disjoint \(I, J \subseteq [p]\); so the members of a Meshalkin sequence are highly disjoint.

To state the result we need a few more definitions. If \(\mathcal{M}\) is a set of Meshalkin sequences, then for each \(k \in [p]\) we define \(\mathcal{M}_k := \{a_k : (a_1, \ldots, a_p) \in \mathcal{M}\}\). If \(\alpha_1, \ldots, \alpha_p\) are nonnegative integers whose sum is \(n\), we define the \((q-)\text{Gaussian multinomial coefficient}\) to be

\[
\left[ \frac{n}{\alpha} \right] = \left[ \frac{n}{\alpha_1, \ldots, \alpha_p} \right] = \frac{n!_q}{\alpha_1!_q \cdots \alpha_p!_q},
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_p)\). We write

\[
s_2(\alpha) = \sum_{i<j} \alpha_i \alpha_j
\]

for the second elementary symmetric function of \(\alpha\). If \(a\) is a Meshalkin sequence, we write

\[
r(a) = (r(a_1), \ldots, r(a_p))
\]
for the sequence of ranks. We define \( \mathbb{P}^{n-1}(q) \) to be empty if \( n = 0 \), a point if \( n = 1 \), and a line of \( q + 1 \) points if \( n = 2 \).

**Theorem 5.** Let \( n \geq 0 \), \( l \geq 1 \), \( p \geq 2 \), and \( q \geq 2 \). Let \( \mathcal{M} \) be a family of Meshalkin sequences of length \( p \) in \( \mathbb{P}^{n-1}(q) \) such that, for each \( k \in [p-1] \), \( \mathcal{M}_k \) contains no chain of length \( l \). Then

- (a) \( \sum_{a \in \mathcal{M}} \frac{1}{r(a)} q^{x_2(r(a))} \leq l^{p-1} \), and
- (b) \( |\mathcal{M}| \) is at most equal to the sum of the \( l^{p-1} \) largest amongst the quantities \( \left\lfloor \frac{n}{p} \right\rfloor q^{x_2(r(a))} \) for \( \alpha = (\alpha_1, \ldots, \alpha_p) \) with all \( \alpha_k \geq 0 \) and \( \alpha_1 + \cdots + \alpha_p = n \).

The antichain case (where \( l = 1 \)), the analog of Meshalkin and Hochberg and Hirsch’s theorems, is captured in

**Corollary 6.** Let \( \mathcal{M} \) be a family of Meshalkin sequences of length \( p \geq 2 \) in \( \mathbb{P}^{n-1}(q) \) such that each \( \mathcal{M}_k \) for \( k < p \) is an antichain. Then

- (a) \( \sum_{a \in \mathcal{M}} \frac{1}{r(a)} q^{x_2(r(a))} \leq 1 \), and
- (b) \( |\mathcal{M}| \leq \max_{\alpha} \left\lfloor \frac{n}{n/p, \ldots, n/p, n/p, \ldots, n/p} \right\rfloor q^{x_2(r(a))} \)
- (c) Equality holds in (a) and (b) if, for each \( k \), \( \mathcal{M}_k \) consists of all flats of rank \( \left\lfloor \frac{n}{p} \right\rfloor \) or all of rank \( \left\lceil \frac{n}{p} \right\rceil \).

We believe—but without proof—that the largest families \( \mathcal{M} \) described in (c) are the only ones.

Notice that we do not place any condition in either the theorem or its corollary on \( \mathcal{M}_p \).

Our theorem is not exactly a generalization of that of Rota and Harper because a flat in a projective geometry has a variable number of complements, depending on its rank. Still, our result does imply this and a generalization, as we shall demonstrate in Section 4.

### 2. Proof of Theorem \( \mathbf{5} \)

The proof of Theorem \( \mathbf{5} \) is adapted from the short proof of Theorem \( \mathbf{3} \) in [3]. It is complicated by the multiplicity of complements of a flat, so we require the powerful lemma of Harper, Klain, and Rota (\[8, Lemma 3.1.3\], improving on \[11, Lemma on p. 199\]; for a short proof see \[2, Lemmas 3.1 and 5.2\]) and a count of the number of complements.

**Lemma 7.** Suppose given real numbers \( m_1 \geq m_2 \geq \cdots \geq m_N \geq 0 \), other real numbers \( q_1, \ldots, q_N \in [0, 1] \), and an integer \( P \) with \( 1 \leq P \leq N \). If \( \sum_{k=1}^{N} q_k \leq P \), then

(1) \[ q_1m_1 + \cdots + q_Nm_N \leq m_1 + \cdots + m_P . \]

Let \( m_{P'+1} \) and \( m_{P''} \) be the first and last \( m_k \)'s equal to \( m_P \). Assuming \( m_P > 0 \), there is equality in (1) if and only if

\[ q_k = 1 \text{ for } m_k > m_P, \quad q_k = 0 \text{ for } m_k < m_P, \quad \text{and} \quad q_{P'+1} + \cdots + q_{P''} = P - P' . \]
Lemma 8. A flat of rank $k$ in $\mathbb{P}^{n-1}(q)$ has $q^{k(n-k)}$ complements.

Proof. The number of ways to extend a fixed ordered basis $(P_1, \ldots, P_k)$ of the flat to an ordered basis $(P_1, \ldots, P_n)$ of $\mathbb{P}^{n-1}(q)$ is

$$\frac{q^n - q^k}{q - 1} \frac{q^n - q^{k+1}}{q - 1} \cdots \frac{q^n - q^{n-1}}{q - 1}.$$ 

Then $P_{k+1} \vee \cdots \vee P_n$ is a complement and is generated by the last $n - k$ points in

$$\frac{q^{n-k} - 1}{q - 1} \frac{q^{n-k} - q}{q - 1} \cdots \frac{q^{n-k} - q^{n-k-1}}{q - 1}$$

of the extended ordered bases. Dividing the former by the latter, there are

$$q^{\left(\binom{n}{2} - \binom{k}{2}\right) - \binom{n-k}{2}} = q^{k(n-k)}$$

complements. □

Proof of (a). We proceed by induction on $p$. For a flat $f$, define

$$\mathcal{M}(f) := \{(a_2, \ldots, a_p) : (f, a_2, \ldots, a_p) \in \mathcal{M}\}$$

and also, letting $c$ be another flat, define

$$\mathcal{M}^c(f) := \{(a_2, \ldots, a_p) \in \mathcal{M}(f) : a_2 \vee \cdots a_p = c\}.$$

For $a \in \mathcal{M}$, we write $r_1 = r(a_1)$. Finally, $C(a_1)$ is the set of complements of $a_1$. If $p > 2$, then

$$\sum_{a_1 \in \mathcal{M}} \sum_{a \in \mathcal{M}} q^{s_2(r(a))} = \sum_{a_1 \in \mathcal{M}_1} \sum_{a' \in \mathcal{M}(a_1)} \sum_{c \in C(a_1)} q^{s_2(r(a'))}$$

$$\leq \sum_{a_1 \in \mathcal{M}_1} \sum_{a' \in \mathcal{M}(a_1)} \sum_{c \in C(a_1)} q^{s_2(r(a'))}$$

by induction, because $\mathcal{M}^c(a_1)$ is a Meshalkin family in $c \cong \mathbb{P}^{r(c)-1} = \mathbb{P}^{n-r_1-1}$ and each $\mathcal{M}_k^c(a')$ for $k < p - 1$, being a subset of $\mathcal{M}_{k+1}$, is $l$-chain-free,

$$\leq \sum_{a_1 \in \mathcal{M}_1} \sum_{a' \in \mathcal{M}(a_1)} q^{r_1(n-r_1)l^{p-2}}$$

by Lemma □

$$\leq l \cdot l^{p-2}$$

by the theorem of Rota and Harper.

The initial case, $p = 2$, is similar except that the innermost sum in the second step equals 1. □
Lemma 9. Let $\alpha = (\alpha_1, \ldots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$. The number of all Meshalkin sequences $a$ in $\mathbb{P}^{n-1}$ with $r(a) = \alpha$ is $\left[ \frac{n}{\alpha} \right] q^{s_2(\alpha)}$.

Proof. If $p = 1$, then $a = \vec{1}$ so the conclusion is obvious. If $p > 1$, we get a Meshalkin sequence of length $p$ in $\mathbb{P}^{n-1}$ with rank sequence $r(a) = \alpha$ by choosing $a_1$ to have rank $\alpha_1$, then a complement $c$ of $a_1$, and finally a Meshalkin sequence $a'$ of length $p-1$ in $c \cong \mathbb{P}^{r(c)-1} = \mathbb{P}^{n-\alpha_1-1}$ whose rank sequence is $\alpha' = (\alpha_2, \ldots, \alpha_p)$. The first choice can be made in $\left[ \frac{n-\alpha_1}{\alpha'} \right]$ ways, the second in $q^{s_1(n-\alpha_1)}$ ways, and the third, by induction, in $\left[ \frac{n-\alpha_1}{\alpha'} \right] q^{s_2(\alpha')}$ ways. Multiply.

$\blacksquare$

Proof of (b). Let $N(\alpha)$ be the number of $a \in \mathcal{M}$ for which $r(a) = \alpha$. In Lemma 7, take

$$q_\alpha = \frac{N(\alpha)}{\left[ \frac{n}{\alpha} \right] q^{s_2(\alpha)}} \quad \text{and} \quad m_\alpha = \left[ \frac{n}{\alpha} \right] q^{s_2(\alpha)},$$

and number all possible $\alpha$ so that $m_{\alpha_1} \geq m_{\alpha_2} \geq \cdots$.

Lemma 9 shows that all $q_\alpha \leq 1$ so Lemma 7 does apply. The conclusion is that

$$|\mathcal{M}| = \sum_{i=1}^{N} q_\alpha m_\alpha \leq \left[ \frac{n}{\alpha_1} \right] q^{s_2(\alpha_1)} + \cdots + \left[ \frac{n}{\alpha_P} \right] q^{s_2(\alpha_P)},$$

where $N = \binom{n+p-1}{p-1}$, the number of sequences $\alpha$, and $P = \min(p, N)$. $\blacksquare$

3. Strangeness of the LYM Inequality

There is something odd about the LYM inequality in Theorem 5(a). A normal LYM inequality would be expected to have denominator $\left[ \frac{n}{r(a)} \right]$ without the extra factor $q^{s_2(r(a))}$. Such an LYM inequality does exist; it is a corollary of Theorem 5(a); but it is not strong enough to give the upper bound on $|\mathcal{M}|$. We prove this weaker inequality here.

Proposition 10. Assume the hypotheses of Theorem 5(a) that is: $n \geq 0$, $l \geq 1$, $p \geq 2$, and $q \geq 2$; and $\mathcal{M}$ is a family of Meshalkin sequences of length $p$ in $\mathbb{P}^{n-1}(q)$ such that, for each $k \in [p-1]$, $\mathcal{M}_k$ contains no chain of length $l$. Then $\sum_{a \in \mathcal{M}} \left[ \frac{n}{r(a)} \right]^{-1}$ is bounded above by the sum of the $l^{-1}$ largest expressions $q^{s_2(\alpha)}$ for $\alpha = (\alpha_1, \ldots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$.

Proof. Again we apply Lemma 7 this time with $q_\alpha = N(\alpha)/\left[ \frac{n}{\alpha} \right] q^{s_2(\alpha)}$ and $M_\alpha = q^{s_2(\alpha)}$. $\blacksquare$

4. A “Partial” Corollary

We deduce Theorem 4(a) from the case $p = 2$ of Theorem 5(a). Our purpose is not to give a new proof of Theorem 4 but to show that we have a generalization of it.

The key to the proof is that $\mathcal{M}_2$ in our theorem is not required to be $l$-chain-free. Therefore if we have an $l$-chain-free set $\mathcal{A}$ of flats in $\mathbb{P}^{n-1}$, we can define

$$\mathcal{M} = \{(a, c) : a \in \mathcal{A} \text{ and } c \in \mathcal{C}(a)\} ;$$

and $\mathcal{M}$ will satisfy the requirements of Theorem 5. The LYM sum in Theorem 5(a) then equals the LYM sum in Theorem 4(a), and we are done.
The same argument gives a general corollary. A partial Meshalkin sequence of length \( p \) is a sequence \( a = (a_1, \ldots, a_p) \) of flats in \( \mathbb{P}^{n-1}(q) \) such that \( r(a_1 \lor \cdots \lor a_p) = r(a_1) + \cdots + r(a_p) \). We simply do not require the join \( \hat{a} = a_1 \lor \cdots \lor a_p \) to be \( \hat{1} \). The generalized Rota–Harper theorem is:

**Corollary 11.** Let \( p \geq 1, l \geq 1, q \geq 2, \) and \( n \geq 0 \). Let \( \mathcal{M} \) be a family of partial Meshalkin sequences of length \( p \) in \( \mathbb{P}^{n-1}(q) \) such that, for each \( k \in [p] \), \( \mathcal{M}_k \) contains no chain of length \( l \). Then

\[
(a) \quad \sum_{a \in \mathcal{M}} \frac{1}{\binom{n}{r(\hat{a})} \binom{r(\hat{a})}{r(a)}} q^{s_2(r(a))} \leq l^p \text{ and }
(b) \quad |\mathcal{M}| \text{ is at most equal to the sum of the } l^p \text{ largest amongst the quantities } \binom{n}{\alpha} q^{s_2(\alpha)} \text{ for } \alpha = (\alpha_1, \ldots, \alpha_{p+1}) \text{ with all } \alpha_k \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_{p+1} = n.
\]

As a special case we generalize the \( q \)-analog of Sperner’s theorem. (The \( q \)-analog is the case \( p = 1 \)).

**Corollary 12.** Let \( \mathcal{M} \) be a family of partial Meshalkin sequences of length \( p \geq 1 \) in \( \mathbb{P}^{n-1} \) such that each \( \mathcal{M}_k \) is an antichain. Then:

\[
(a) \quad \sum_{a \in \mathcal{M}} \frac{1}{\binom{n}{r(\hat{a})} \binom{r(\hat{a})}{r(a)}} q^{s_2(r(a))} \leq 1.
(b) \quad |\mathcal{M}| \leq \binom{n}{\alpha} q^{s_2(\alpha)} \text{, in which } \alpha = (\left\lfloor \frac{n}{p+1} \right\rfloor, \left\lfloor \frac{n}{p+1} \right\rfloor, \left\lfloor \frac{n}{p+1} \right\rfloor, \ldots, \left\lfloor \frac{n}{p+1} \right\rfloor) \text{ where the number of terms equal to } \left\lfloor \frac{n}{p+1} \right\rfloor \text{ is the least nonnegative residue of } n \text{ modulo } p + 1.
(c) \quad Equality holds in (a) and (b) if, for each } k, \mathcal{M}_k \text{ consists of all flats of rank } \left\lfloor \frac{n}{p+1} \right\rfloor \text{ or all flats of rank } \left\lceil \frac{n}{p+1} \right\rceil.
\]

We conjecture that the largest families \( \mathcal{M} \) described in (c) are unique.

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