Semi-empirical formulation of multiple scattering for the Gaussian beam model of heavy charged particles stopping in tissue-like matter

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Abstract
Dose calculation for radiotherapy with protons and heavier ions deals with a large volume of path integrals involving a scattering power of body tissue. This work provides a simple model for such demanding applications. There is an approximate linearity between RMS end-point displacement and range of incident particles in water, empirically found in measurements and detailed calculations. This fact was translated into a simple linear formula, from which the scattering power that is only inversely proportional to the residual range was derived. The simplicity enabled the analytical formulation for ions stopping in water, which was designed to be equivalent with the extended Highland model and agreed with measurements within 2% or 0.02 cm in RMS displacement. The simplicity will also improve the efficiency of numerical path integrals in the presence of heterogeneity.

1. Introduction
The essence of radiotherapy with protons and heavier ions lies in precise control of incident particles that are designed to stop in tumor volume. The targeting precision will be inevitably deteriorated by multiple scattering in beam modifiers and patient body and such effects must be accurately handled for dose calculations in treatment planning. On the other hand, simplicity and efficiency are also essential in clinical practice and there has always been need for a computational method that balances all these demanding and conflicting requirements.

Fermi and then Eyges (1948) developed a general theory for charged particles that undergo energy loss and multiple scattering in matter. A group of particles is approximated as a Gaussian beam growing in space with statistical variances

$$\theta^2(x) = \theta^2(0) + \int_0^x T(x') \, dx',$$  

(1)
\[
\bar{y}'(x) = \bar{y}'(0) + \bar{y}'(0)x + \int_{0}^{x} (x - x')T(x') \, dx',
\]
(2)

\[
\bar{y}^2(x) = \bar{y}^2(0) + 2\bar{y}'(0)x + \theta^2(0)x^2 + \int_{0}^{x} (x - x')^2T(x') \, dx',
\]
(3)

where \( x \) is the longitudinal position and \( y \) and \( \theta \) are the projected particle position and angle, respectively. The original Fermi–Eyges theory was formulated with the scattering power in the small-angle approximation (Rossi and Greisen 1948),

\[
T = \frac{E^2_{s}}{X_{0}} \left( \frac{z}{p\nu} \right)^{2},
\]
(4)

where \( E_{s} = m_{e}c^{2}\sqrt{2\pi/\alpha} \approx 15.0 \text{ MeV} \) is a constant, \( X_{0} \) is the radiation length of the material, and \( z, p \) and \( \nu \) are the charge, the momentum and the velocity of the particles. The Fermi–Rossi formula (4) totally ignores effects of large-angle single scattering (Hanson et al 1951) and was found to be inaccurate (Wong et al 1990).

Based on formulations by Highland (1975, 1979) and Gottschalk et al (1993), Kanematsu (2008a) proposed a scattering power with correction for the single-scattering effect, although within the Gaussian approximation,

\[
T = 0.970 \left( 1 + \frac{\ln \ell}{20.7} \right) \left( 1 + \frac{\ln \ell}{22.7} \right) \left( \frac{E^2_{s}}{X_{0}} \right) \left( \frac{z}{p\nu} \right)^{2},
\]
(5)

where \( \ell = \int_{0}^{x} dx'/X_{0}(x') \) is the radiative path length. Although it would be difficult to calculate integrals (1)–(3) because of the embedded integral in the \( \ell \) terms, Kanematsu (2008a) further derived an approximate formula for the RMS displacement of incident ions at the end point in homogeneous matter as

\[
\sigma_{y0}(R_{0}) = \frac{E_{s}}{\text{MeV}} \frac{z^{1-\kappa}}{\sqrt{3-\kappa}} \left( \frac{m}{m_{p}} \right)^{\frac{1}{2}} \left( \frac{X_{0}w}{\rho_{S}X_{0}} \right)^{\frac{1}{2}} \left( \frac{R_{0}}{\lambda} \right)^{\frac{1}{2}} \sqrt{\frac{R_{0}^{3}}{\rho_{S}^{3}X_{0}}},
\]
(6)

where \( m/m_{p} \) is the ion mass in units of the proton mass, \( R_{0} \) is the expected in-water range on the incidence, \( \rho_{S} \) is the stopping-power ratio of the matter relative to water, and \( \kappa = 1.08 \) and \( \lambda = 4.67 \times 10^{-4} \text{ cm} \) are constants.

Despite the complex involvement of the variable \( R_{0} \) in (6), Kanematsu (2008a) found the \( \sigma_{y0} - R_{0} \) relation for ions in water to be very linear. In fact, Preston and Kohler of Harvard Cyclotron Laboratory knew the linear relation and the derived universal curve \( \sigma_{y}/\sigma_{y0}(R_{0}) = \sqrt{3x_{R}^{2} - 2x_{R} - 2(1 - x_{R})^{2}}\ln(1 - x_{R}) \) with \( x_{R} = \rho_{S}x/R_{0} \) for relative growth of RMS displacement \( \sigma_{y} = (y^2)^{1/2} \) in homogeneous matter in an unpublished work in 1968. Starting with the empirical linear relation, this work is aimed to develop a simple and general multiple-scattering model to improve efficiency of numerical heterogeneity handling and to enable further analytical beam modeling.

2. Materials and methods

2.1. Linear-displacement model

The linear approximation \( \sigma_{y0} \propto R_{0} \) for homogeneous systems greatly simplifies (6) to

\[
\sigma_{y0}(R_{0}) = 0.0224z^{-0.08} \left( \frac{m}{m_{p}} \right)^{-0.46} \left( \frac{X_{0}w}{\rho_{S}X_{0}} \right)^{\frac{1}{2}} \left( \frac{R_{0}}{\rho_{S}R_{0}} \right)^{0.46},
\]
(7)
where $X_{0w} = 36.08$ cm is the radiation length of water, $\sqrt{X_{0w}/(\rho S X_0)}$ is the scattering/stopping ratio of the material relative to water and $R_0/\rho S$ is the geometrical range. Equation (7) was calibrated to (6) for water at $R_0 = X_0$.

2.2. Formulation of new scattering power

Equation $\bar{y}^2 = \sigma^2$ at the end point $x = R_0/\rho S$ associates (3) and (7) as

$$\int_0^{R_0} \frac{R^2}{\rho S^2} dR = 0.02242 z^{-0.16} \left( \frac{m}{m_p} \right)^{-0.92} \frac{X_{0w}}{\rho S X_0} \left( \frac{R_0}{\rho S} \right)^2,$$

(8)

to lead to another scattering power

$$T = f_{mc} \frac{X_{0w}}{X_0} \frac{1}{R}, \quad f_{mc} = (1.00 \times 10^{-3}) z^{-0.16} \left( \frac{m}{m_p} \right)^{-0.92},$$

(9)

where $f_{mc}$ is the particle-type-dependent factor. The scattering power is inherently applicable to any heterogeneous system by numerical integral of (1)–(3).

2.3. Comparison of models and measurements

We examined these Fermi–Rossi (4), extended Highland (5), and linear-displacement (9) models with unpublished measurements by Phillips (Hollmark et al. 2004), those by Preston and Kohler (Kanematsu 2008a), and Molière-Hanson calculations by Deasy (1998). We took growths of RMS displacement $\sigma_y(x)$ with depth $x$ for $R_0 = 29.4$ cm protons, $R_0 = 29.4$ cm helium ions, and $R_0 = 29.7$ cm carbon ions in water and RMS end-point displacements $\sigma_y(R_0)$ for them with a varied incident range $R_0$.

2.4. Formulae for homogeneous systems

For a point mono-directional ion beam with an in-water range $R_0$ incident into homogeneous matter with constant $\rho S$ and $X_0$, equations (1)–(3) are analytically integrated to

$$\bar{y}^2 = f_{mc} \frac{X_{0w}}{\rho S X_0} \ln \frac{R_0}{R},$$

(10)

$$\bar{y} \theta = f_{mc} \frac{X_{0w}}{\rho S X_0} \left( 1 - \frac{R}{R_0} + \frac{R}{R_0} \ln \frac{R}{R_0} \right) \frac{R_0}{\rho S},$$

(11)

$$\bar{y}^2 = f_{mc} \frac{X_{0w}}{\rho S X_0} \left( \frac{1}{2} + \frac{3}{2} \frac{R^2}{R_0^2} - 2 \frac{R}{R_0} - \frac{R^2}{R_0^2} \ln \frac{R}{R_0} \right) \frac{R_0^2}{\rho S^2},$$

(12)

as a function of the residual range $R = R_0 - \rho S x$ at a distance $x$. Equation (12) in fact reduces to the universal curve by Preston and Kohler.

A radiation field at a given $x$ position can be effectively or virtually modeled with a source, ignoring the matter (ICRU-35 1984). The effective extended source is at $x_e = x - \bar{y} \theta(x)/\bar{y}^2(x)$, where $\bar{y}^2$ would be minimum in the vacuum. The virtual point source is at $x_v = x - \bar{y}^2(x)/y \theta(x)$, from which radiating particles would form a field of equivalent divergence. Similarly, the effective scattering point is at $x_s = x - [\bar{y}^2(x)/\bar{y}^2(x)]^{1/2}$, at which a point-like scattering would cause equivalent RMS angle and displacement (Gottschalk et al. 1993).
3. Results

Figure 1(a) shows RMS-displacement growths $\sigma_y(x)$ and figure 1(b) shows RMS end-point displacements $\sigma_{y0}(R_0)$. The present model was virtually identical to the extended Highland model with deviations from measurements or Molière–Hanson calculations within either 2% or 0.02 cm, while the Fermi–Rossi model overestimated the RMS displacements by nearly 10%. Figure 1(c) shows an end-point shape of a pencil beam measured by Preston and Kohler for $R_0 = 11.4$ cm protons in water and curves estimated by Fermi–Rossi and the present formulations with additional $(0.168 \text{ cm})^2$ to $\sigma_y^2$ for incident beam emittance in their experiment. The Gaussian approximation was in fact adequate for the central part.

Figure 2(a) shows analytical variances $\theta^2$, $\bar{y}^2$ and $\bar{y}^2$ growing with depth $x$ for protons in water. Figure 2(b) shows the relative distances from current position $x$ to the effective extended source $x_e$, the virtual point source $x_v$ and the effective scattering point $x_s$. They approach $(x - x_e)/x \to 1/2$, $(x - x_v)/x \to 2/3$ and $(x - x_s)/x \to 1/\sqrt{3}$ at the $x \to 0$ limit. Increase of the scattering power with depth moves these points relatively closer to the current position.

4. Discussion

The linear-displacement model with the Fermi–Eyges theory has brought general formulae for ions, including the universal curve intuitively derived by Preston and Kohler without formulating the scattering power. In the present model, all the kinematic properties are encapsulated in the residual range $R = R_0 - \int_0^x \rho_s \, dx'$ that is always tracked in beam transport. The ion-type dependence in scattering angle and displacement is simply proportional to $\sqrt{m_z}$, which leads to 50% for $^4\text{He}$, 28% for $^{12}\text{C}$ and 24% for $^{16}\text{O}$ with respect to that of protons for a given incident range. These numbers coincide with detailed numerical calculations by Hollmark et al (2004).
The linearity between end-point displacement and range observed for water is the basis of the present model. Its validity for general body-tissue materials is, however, not obvious. We here examine water and two extreme elements hydrogen ($\rho_s X_0 \approx 113$ cm) and calcium ($\rho_s X_0 \approx 14.5$ cm) among major elements of body tissues (ICRU-46 1992), using (6) in the extended Highland model. Figure 3 shows their $R_0-\sigma_y$ relations with geometrical scale correction and indicates that the linearity will generally hold for body-tissue elements. However, the elemental linearity may not truly warrant the validity for systems heterogeneous in atomic compositions. The scattering power (9) only depends on the residual range that is irrelevant to the multiple scattering accumulated in the other upstream materials, whereas the accumulation should influence the single-scattering effect (Kanematsu 2008a). In other words, the present model implicitly assumes heterogeneity in density only.
In the current practice of treatment planning, the patient heterogeneity is normally modeled with variable-density water (Kanematsu et al. 2003), for which the present model is rigorous with further simplified scattering power

\[ T = \frac{f_{mT}}{R} \mu_s \]  

The simplicity will minimize computation of the integrands in path integrals (1)–(3) for demanding dose calculations (Kanematsu et al. 1998, 2006, 2008b).

In application of Bragg peaks to radiotherapy, the end-point displacement is the most important, for which the Gaussian approximation was valid in the proton experiment by Preston and Kohler. Incidentally, the single-scattering effect generally reduces with depth in a thick target (Kanematsu 2008a). Although ions suffer nuclear interactions with resultant fragments that generally scatter at large angles (Matsufuji et al. 2005), their contributions are relatively less significant at the Bragg peaks.

5. Conclusions

A novel multiple-scattering model has been formulated based on the fact that the RMS end-point displacement is proportional to the incident range in water. The model was designed to be equivalent with the extended Highland model for stopping ions in water and agreed with measurements within 2% or 0.02 cm in RMS displacement.

The resultant scattering-power formula that is only inversely proportional to the residual range is much simpler than former formulations and can be used in the framework of the Fermi–Eyges theory for Gaussian-beam transport in tissue-like matter. The simplicity enables analytical beam modeling for homogeneous systems and improves efficiency of numerical path integrals for heterogeneous systems. The present model is ideal for demanding dose calculations in treatment planning of heavy-charged-particle radiotherapy.

References

Deasy J O 1998 A proton dose calculation algorithm for conformal therapy simulations based on Molière’s theory of lateral deflections Med. Phys. 25 476–83
Eynes L 1948 Multiple scattering with energy loss Phys. Rev. 74 1534–5
Gottschalk B, Koehler A M, Schneider R J, Sisterson J M and Wagner M S 1993 Multiple Coulomb scattering of 160 MeV protons Nucl. Instrum. Methods B 74 467–90
Hanson A O, Lanzl L H, Lyman E M and Scott M B 1951 Measurement of multiple scattering of 15.7-MeV electrons Phys. Rev. 84 634–7
Highland V L 1975 Some practical remarks on multiple scattering Nucl. Instrum. Methods 129 497–9
Highland V L 1979 Some practical remarks on multiple scattering Nucl. Instrum. Methods 161 171 (erratum)
Holmmark M, Uhrdin J, Dž B, Gudowska I and Brahme A 2004 Influence of multiple scattering and energy loss straggling on the absorbed dose distributions of therapeutic light ion beams: I. Analytical pencil beam model Phys. Med. Biol. 49 3247–65
ICRU-35 1984 Radiation dosimetry: electron beams with energies between 1 and 50 MeV ICRU Report 35 (Bethesda, MD: ICRU)
ICRU-46 1992 Photon, electron, proton and neutron interaction data for body tissues ICRU Report 46 (Bethesda, MD: ICRU)
Kanematsu N 2008a Alternative scattering power for Gaussian beam model of heavy charged particles Nucl. Instrum. Methods B 266 5056–62
Kanematsu N, Akagi T, Futami Y, Higashi A, Kanai T, Matsufuji N, Tomura H and Yamashita H 1998 A proton dose calculation code for treatment planning based on the pencil beam algorithm Japan. J. Med. Phys. 18 88–103
Kanematsu N, Akagi T, Takatani Y, Yonai S, Sakamoto H and Yamashita H 2006 Extended collimator model for pencil-beam dose calculation in proton radiotherapy Phys. Med. Biol. 51 4807–17
Kanematsu N, Matsufuji N, Kuhno R, Minohara S and Kanai T 2003 A CT calibration method based on the polybinary tissue model for radiotherapy treatment planning Phys. Med. Biol. 48 1053–64
Kanematsu N, Yonai S and Ishizaki A 2008b The grid-dose-spreading algorithm for dose distribution calculation in heavy charged particle radiotherapy Med. Phys. 35 602–8
Matsufuji N et al 2005 Spatial fragment distribution from a therapeutic pencil-like carbon beam in water Phys. Med. Biol. 50 3393–403
Rossi B and Greisen K 1941 Cosmic ray theory Rev. Mod. Phys. 13 240–309
Wong M, Schimmerling W, Phillips M H, Ledewigt B A, Landis D A, Walton J T and Curtis S B 1990 Med. Phys. 17 163–71