Hamilton-Jacobi Method and Effective Actions of D-brane and M-brane in Supergravity

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Abstract
We show that the effective actions of D-brane and M-brane are solutions to the Hamilton-Jacobi (H-J) equations in supergravities. This fact means that these effective actions are on-shell actions in supergravities. These solutions to the H-J equations reproduce the supergravity solutions that represent D-branes in a $B_2$ field, M2 branes and the M2-M5 bound states. The effective actions in these solutions are those of a probe D-brane and a probe M-brane. Our findings can be applied to the study of the gauge/gravity correspondence, especially the holographic renormalization group, and a search for new solutions of supergravity.

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1 Introduction

In this paper, we show that the D-brane effective action (the Born-Infeld action plus the Wess-Zumino action) is a solution to the Hamilton-Jacobi (H-J) equation of type IIA(IIB) supergravity and that the M-brane effective action is a solution to the H-J equation of 11-dimensional (11-d) supergravity. This fact means that the effective actions of D-brane and M-brane are on-shell actions in supergravities. We also show that these solutions to the H-J equations reproduce the supergravity solutions which represent a stack of D-branes in a $B_2$ field, a stack of M2-branes and a stack of the M2-M5 bound states. In fact, we reported the case of D3-brane in our previous publication [1], and in this paper we generalize our previous result to the cases of D$p$-branes and M-branes.

The D-brane effective action on a curved background is obtained in principle by calculating the disk amplitude in superstring on the background. The disk amplitude in the open string picture is translated into the closed string picture as the transition amplitude between the vacuum and the boundary state representing a probe D-brane. This transition amplitude should reduce in the $\alpha' \rightarrow 0$ limit to an on-shell action in type IIA(IIB) supergravity, which is a functional of the values of the fields on a boundary. Therefore, the $\alpha' \rightarrow 0$ limit of the D-brane effective action should be a solution to the H-J equation of type IIA(IIB) supergravity. Considering the gauge invariance, we see that the $\alpha' \rightarrow 0$ limit of the D-brane effective action corresponds to setting the combination of the gauge field plus the NS 2-form field to be zero. Nevertheless, a nontrivial fact we obtain is that the D-brane effective action itself is a solution to the H-J equation. Probably this fact comes from the supersymmetry, since the R-R plays a crucial role in our analysis.

The strategy of our analysis is as follows. We reduce type IIA(IIB) supergravity on $S^{8-p}$, dropping the fermionic degrees of freedom consistently, and obtain a $(p + 2)$-dimensional gravity. Adopting the radial coordinate as time, we develop the canonical formalism based on the ADM decomposition for this $(p + 2)$-dimensional gravity and obtain the H-J equation originating from the Hamiltonian constraint. We solve the H-J equation under the condition that the fields be constant on fixed-time surfaces, and find that the D$p$-brane effective action is a solution to the H-J equation and reproduces the supergravity solution of a stack of D$p$-branes in a $B_2$ field. We note here that the near-horizon limit of this supergravity solution with $p = 3$ is conjectured to be dual to noncommutative Yang Mills (NCYM) [2, 3, 4]
and reduces to $AdS_5 \times S^5$ in the commutative limit, which is dual to $\mathcal{N} = 4$ super Yang Mills \cite{5, 6}. In our formulation, the fixed-time surface whose dimension is $p + 1$ can be interpreted as the worldvolume of a probe D$p$-brane, and the radial time as the position of the probe D$p$-brane. We also reduce 11-d supergravity on $S^7$ and $S^4$, repeat the above steps, and obtain the M2 and M5 brane effective actions as solutions to the H-J equations, respectively. We find that these solutions to the H-J equation reproduce the supergravity solutions of a stack of M2-branes and a stack of the M2-M5 bound states, respectively. Furthermore, by using the $SL(2, R)$ symmetry in type IIB supergravity and the relation of type IIA supergravity with 11-d supergravity, we obtain solutions to the H-J equations that reproduce the supergravity solutions representing $(p, q)$ strings and $(p, q)$ 5-branes in type IIB supergravity and NS 5-branes in type IIA supergravity. Note that the near-horizon limit of supergravity backgrounds reproduced by M2-branes, the M2-M5 bound states and NS 5-branes are conjectured to be dual to three-dimensional $\mathcal{N} = 8$ superconformal field theory, (a noncommutative version of) six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory and Little String Theory, respectively \cite{5, 4, 7}.

As we discuss below, the fact that our solutions to the H-J equations are the on-shell actions around the supergravity backgrounds which conjectured to be dual to various gauge theories motivates us to further investigate the subject in the present paper.

Indeed, our findings can be applied to the study of the gauge/gravity correspondence. A well-understood example of the gauge/gravity correspondence is the AdS/CFT correspondence \cite{5, 6}; in particular, the correspondence between $\mathcal{N} = 4$ super Yang Mills at the conformally invariant point and type IIB supergravity on $AdS \times S^5$. It is relevant to investigate whether this kind of correspondence can be extended to $\mathcal{N} = 4$ super Yang Mills in the Coulomb branch \cite{5, 8} or four-dimensional less supersymmetric ($\mathcal{N} = 0, 1, 2$) gauge theories \cite{2} or higher (lower) dimensional supersymmetric gauge theories \cite{5, 7, 10}. Another relevant problem to be studied from the viewpoint of the gauge/gravity correspondence is a quantum theory of NCYM. Classical aspects of NCYM such as noncommutative instantons \cite{11, 2} are well-understood while little is known about quantum aspects. In particular, renormalizability of NCYM has not been established perturbatively or non-perturbatively. Note that the authors of Ref.\cite{12} verified the renormalizability of two-dimensional bosonic NCYM by performing a numerical simulation of its lattice version. As we describe shortly, solving the H-J
equation and obtaining the on-shell action in supergravity is doubly important for studying the above issues in the gauge/gravity correspondence. In order to study the gauge/gravity correspondence for the less supersymmetric gauge theories, we must generalize our analysis in the present paper; we should reduce supergravities in more complicated ways and search for solutions to the H-J equations of these reduced gravities that reproduce the supergravity solutions conjectured to be dual to the less supersymmetric gauge theories.

One application of the H-J method in supergravity to the study of the gauge/gravity correspondence is to compare the on-shell action of supergravity with the effective action of the dual gauge theory. Suppose that supergravity on a certain background corresponds to a large $\mathcal{N}$ (noncommutative) gauge theory in which one of the Higgs fields has a nontrivial vacuum expectation value (vev) and the $U(N)$ gauge symmetry is spontaneously broken to $U(1) \times SU(N - 1)$. Then, if we interpret the radial time as a position of the probe D-brane, the on-shell action of supergravity around this background should coincide with the effective action of the gauge theory via the identification of the radial time with the vev of the Higgs field. Thus the H-J method is useful for checking this case of the gauge/gravity correspondence. It is actually conjectured in Ref.[13] that the effective action of $\mathcal{N} = 4$ super Yang Mills in the Coulomb branch takes the form of the D3-brane effective action in the ’t Hooft limit. It is important to perform a similar calculation of the effective action in NCYM and to compare the result with the on-shell action obtained in this paper.

The other application is the study of the holographic renormalization group, which is also useful for establishing the gauge/gravity correspondence. The authors of Ref.[14] derived the renormalization group equation in the dual gauge theory from the H-J equation in supergravity. In particular, they found that in their simple examples the lowest term in the derivative expansion of the on-shell action in supergravity plays a role of the counter terms and gives the beta functions and the anomalous dimensions in the dual gauge theory. In this context, the radial time is interpreted as the renormalization scale in the dual gauge theory. Note that the solutions found in this paper are also the lowest term in the derivative expansion. One can check whether supergravity on a certain background corresponds to a large $N$ gauge theory by comparing the beta functions and the anomalous dimensions given by the on-shell action around the background with those in the gauge theory. Also, one can examine the structure of the renormalization of NCYM through the holographic
renormalization group. Although it is not obvious whether the lowest term in the on-shell action is sufficient in more complicated cases we are interested in, our results are at least a first step to the study of the holographic renormalization group in these cases.

Another application of the H-J method is searching for new solutions and classifying the solutions in supergravity. Using the solution to the H-J equation obtained in the present paper, we expect to be able to obtain new solutions in supergravity reduced on higher dimensional spheres under the condition that the fields depend only on the radial coordinate. That is, while the supergravity solution representing $D_p$-branes in a $B_2$ field is obtained by T-dualizing a tilted smeared $D(p-1)$-brane solution [15], we can search supergravity solutions that cannot be obtained by such a T-dualization, as is discussed below. If we find a solution to the H-J equation of supergravity reduced in a different way, we expect to be able to obtain a different kind of new solutions of supergravity. Furthermore, if we find a complete solution to the H-J equation under a certain condition, we can classify the solutions in supergravity under the condition. Hence, our analysis in the present paper should be a first step to the classification of the supergravity solutions.

As is well-known, a complete solution to the H-J equation that includes as many arbitrary constants as the number of the degrees of freedom is a generator of the canonical transformation that makes the new Hamiltonian vanish. If one finds a complete solution to the H-J equation, one can represent the coordinates as functions of time for an arbitrary initial condition. Namely, the problem can be completely solved. Although our solution to the H-J equation is not a complete solution, it includes several arbitrary constants. We can reduce the original equations of motion in supergravity, which are the second order differential equations, to the first order ones by using the solution, and obtain as many conserved quantities as the number of the arbitrary constants. We may solve the first order equations to find a new solution of supergravity, utilizing these conserved quantities. Alternatively, we can search for a new solution to the H-J equation that generalizes the present solution and includes more arbitrary constants so that we may find a new solution more easily.

The present paper is organized as follows. In section 2, we develop the Hamilton-Jacobi method in general constrained systems. In section 3, we perform reductions of supergravities on higher-dimensional spheres. In section 4, we develop the canonical formalism for the reduced gravities obtained in section 3 to derive the H-J equations. In section 5, we find
that the Dp-brane effective action is a solution to the H-J equation of the reduced gravity and reproduces the supergravity solution of a stack of D3-branes in a $B_2$ field. Section 6 is devoted to the similar calculation in the cases of M2-brane and M5-brane. In section 7, using the SL(2,R) symmetry in type IIB supergravity and the relation between 11-d supergravity and type IIA supergravity, we obtain solutions to the H-J equations that reproduce the supergravity solutions of $(p, q)$ string and $(p, q)$ 5-brane in type IIB supergravity and of NS 5-brane in type IIA supergravity. Section 8 is devoted to discussion and perspective. In appendix A, the equations of motion in supergravities are listed. In appendix B, we write down the ansatz for the fields made in performing a reduction of type IIA(IIB) supergravity on $S^{8-p}$.

2 H-J method in constrained systems

Let us consider a classical system with $n$ degrees of freedom and $k$ first class constraints. Suppose the action of the system is given in the canonical form:

$$I = \int_{t_0}^{t} d\tau (p_i \dot{q}_i - H(p_1, \cdots, p_n, q_1, \cdots, q_n, \tau) - \alpha_a f_a(p_1, \cdots, p_n, q_1, \cdots, q_n, \tau)),$$

where $i$ runs from 1 to $n$, $a$ runs from 1 to $k$, $H$ is the Hamiltonian and the $\alpha_a$ are Lagrange multipliers. By varying the action with respect to $q_i$, $p_i$ and $\alpha_a$, one obtains the following equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + \alpha_a \frac{\partial f_a}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} - \alpha_a \frac{\partial f_a}{\partial q_i},$$

$$f_a(p_1, \cdots, p_n, q_1, \cdots, q_n, \tau) = 0.$$  

(2.2)

Let $q_i = \bar{q}_i(\tau)$ and $p_i = \bar{p}_i(\tau)$ be a solution to the above equations of motion which satisfies the boundary conditions $\bar{q}_i(t) = x_i$. Substituting this classical solution to the action gives the on-shell action, which can be regarded as a function of the boundary positions $x_i$ and the final time $t$.

$$S(x_1, \cdots, x_n, t) = \int_{t_0}^{t} d\tau (\bar{p}_i \dot{\bar{q}}_i - H(\bar{p}_1, \cdots, \bar{p}_n, \bar{q}_1, \cdots, \bar{q}_n, \tau)).$$

(2.3)
The variation of $S$ with respect to $x_i$ and $t$ is given by
\[
\delta S = (\bar{p}_i(t) \dot{\bar{q}}_i(t) - H(\bar{p}(t), \bar{q}(t), t))\delta t + \int_{t_0}^{t} d\tau \left( \delta \bar{p}_i \dot{\bar{q}}_i + \bar{p}_i \delta \dot{\bar{q}}_i - \frac{\partial H(\bar{p}, \bar{q}, \tau)}{\partial \bar{p}_i} \delta \bar{p}_i - \frac{\partial H(\bar{p}, \bar{q}, \tau)}{\partial \bar{q}_i} \delta \bar{q}_i \right). \tag{2.4}
\]

Using the equations of motion (2.2) and integrating the right-hand side of (2.4) partially, one obtains
\[
\delta S = (\bar{p}_i(t) \dot{\bar{q}}_i(t) - H(\bar{p}(t), \bar{q}(t), t))\delta t + \int_{t_0}^{t} d\tau \left( \frac{\partial f_a(\bar{p}, \bar{q}, \tau)}{\partial \bar{p}_i} \delta \bar{p}_i + \frac{\partial f_a(\bar{p}, \bar{q}, \tau)}{\partial \bar{q}_i} \delta \bar{q}_i \right). \tag{2.5}
\]
Here the last line in (2.5) vanishes, since
\[
0 = f_a(\bar{p} + \delta \bar{p}, \bar{q} + \delta \bar{q}, \tau) = f_a(\bar{p}, \bar{q}, \tau).
\]

Noting that $\bar{q}_i(t + \delta t) + \delta \bar{q}_i(t + \delta t) = x_i + \delta x_i$, one obtains $\delta \bar{q}_i(t) = \delta x_i - \dot{\bar{q}}_i(t)\delta t$. Therefore, (2.5) reduces to
\[
\delta S = -H(\bar{p}(t), x, t)\delta t + \bar{p}_i(t)\delta x_i, \tag{2.6}
\]
which is equivalent to the equations
\[
\frac{\partial S}{\partial t} = -H(\bar{p}(t), x, t),
\frac{\partial S}{\partial x_i} = \bar{p}(t)i. \tag{2.7}
\]

Finally, one obtains the equations satisfied by the on-shell action $S$, the H-J equations:
\[
\frac{\partial S}{\partial t} + H \left( \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}, x_1, \ldots, x_n, t \right) = 0,
\]
\[
f_a \left( \frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_n}, x_1, \ldots, x_n, t \right) = 0. \tag{2.8}
\]

Suppose a solution to (2.8) is given. Then, the canonical momenta are represented in terms of $x_i$ by using the second equation in (2.7). It follows that the equations of motion (2.2) reduce to a set of the first order differential equations only for $q_i$. Thus one can simplify the problem of solving the equations of motion.

It is well-known that the H-J equation is in general more powerful as seen in below. First, note that if $S$ is a solution to the above equations, $S + \sigma$ is also a solution to it, where $\sigma$ is
a arbitrary constant. Let the above solution possess \( l \) arbitrary constants \( \beta_1, \cdots, \beta_l \) that are not the trivial additive constant \( \sigma \). Then, the following quantities are conserved quantities:

\[
\gamma_s = \frac{\partial S}{\partial \beta_s} \quad (s = 1, \cdots, l).
\]  

(2.9)

In fact, on one hand,

\[
\frac{d}{dt} \gamma_s = \frac{\partial^2 S}{\partial \beta_s \partial t} + \frac{\partial^2 S}{\partial \beta_s \partial x_i} \dot{x}_i.
\]  

(2.10)

On the other hand, from (2.8) and the arbitrariness of \( \beta_s \), we obtain

\[
0 = \frac{\partial}{\partial \beta_s} \left( \frac{\partial S}{\partial t} + H \right) = \frac{\partial^2 S}{\partial \beta_s \partial t} + \frac{\partial H}{\partial (\partial S/\partial x_i)} \frac{\partial^2 S}{\partial \beta_s \partial x_i}.
\]  

(2.11)

Summing the two equations in (2.11) and using (2.2) and (2.10) gives

\[
\frac{d}{dt} \gamma_s = 0.
\]  

(2.12)

In particular, when \( l = n \), it follows that \( \beta_i \) and \( \gamma_i \) are new canonical momentum and new coordinates which are obtained from the canonical transformations generated by \( S \), respectively, and are constant with respect to the time. One can completely solve the problem by using the second equation in (2.7) and (2.9).

## 3 Reductions of type IIA(IIB) and 11-d supergravities on higher dimensional spheres

In this section, we perform reductions of supergravities on higher dimensional spheres. We will regard the fixed-time surface as the worldvolume of the \( p \)-brane (Dp-brane or M2-brane or M5-brane) later. Hence, we will reduce supergravity to a \((p + 2)\)-dimensional gravity.

First, in order to fix the conventions, we write down the actions of type IIA(IIB) and 11-d supergravities. In this paper, we drop the fermionic degrees of freedom consistently. In the following equations, \(|K_q|^2 = \frac{1}{q!} K_{M_1 \cdots M_q} K^{M_1 \cdots M_q} \) for a \( q \)-form \( K_q \), where the appropriate
metric is used for the contractions, and \( C_{p+1} \) is the R-R \((p+1)\)-form. The bosonic part of type IIA supergravity is given by

\[
I_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} \left[ e^{-2\phi} \left( R_G + 4\partial_M \phi \partial^M \phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_2|^2 - \frac{1}{2} |\tilde{F}_4|^2 \right] - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4, \tag{3.1}
\]

where

\[
H_3 = dB_2, \quad F_{p+2} = dC_{p+1} \quad (p = 0, 2),
\]

\[
\tilde{F}_4 = F_4 - C_1 \wedge H_3. \tag{3.2}
\]

The bosonic part of type IIB supergravity is given by

\[
I_{IIB} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-G} \left[ e^{-2\phi} \left( R_G + 4\partial_M \phi \partial^M \phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_1|^2 - \frac{1}{2} |\tilde{F}_3|^2 - \frac{1}{4} |\tilde{F}_5|^2 \right] + \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3, \tag{3.3}
\]

where

\[
H_3 = dB_2, \quad F_{p+2} = dC_{p+1} \quad (p = -1, 1, 3),
\]

\[
\tilde{F}_3 = F_3 + C_0 \wedge H_3,
\]

\[
\tilde{F}_5 = F_5 + C_2 \wedge H_3. \tag{3.4}
\]

One must also impose the self-duality condition

\[
* \tilde{F}_5 = \tilde{F}_5 \tag{3.5}
\]

on the equations of motion derived from the above action. The bosonic part of 11-d supergravity is given by

\[
I_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}X \sqrt{-G} \left( R_G - \frac{1}{2} |F_4^{(M)}|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4^{(M)} \wedge F_4^{(M)}, \tag{3.6}
\]

where

\[
F_4^{(M)} = dA_3,
\]
We list all of the equations of motion and the Bianchi identities in these supergravities in appendix A.

Let us consider a reduction of type IIA (IIB) supergravity on $S^{8-p} (p = 0, \cdots, 7)$, where $p$ takes 0, 2, 4, 6 for type IIA supergravity and 1, 3, 5, 7 for type IIB supergravity. We split the ten-dimensional coordinates $X^M (M = 0, 1, \cdots, 9)$ into two parts, as $X^M = (\xi^\alpha, \theta^i)$ ($\alpha = 0, \cdots, p+1$, $i = 1, \cdots, 8-p$), where the $\xi^\alpha$ are $(p+2)$-dimensional coordinates and the $\theta^i$ parametrize $S^{8-p}$. We make the following ansatz for the ten-dimensional metric, which preserves the $(p+2)$-dimensional general covariance:

$$ds_{10} = G_{MN} dX^M dX^N = h_{\alpha\beta}(\xi) d\xi^\alpha d\xi^\beta + e^{\phi(\xi)} d\Omega_{8-p}.$$ (3.7)

We assume that the dilaton depends only on $\xi^{\alpha_1}$:

$$\phi = \phi(\xi).$$ (3.8)

The ansätzes for the other fields are summarized in appendix A. Here, as an example, we perform the reduction for the $p = 6$ case explicitly. The reductions for the other cases can be performed in the same way as the $p = 6$ case. The ansätzes for the other fields in the $p = 6$ case are

$$H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma + \frac{1}{2!} d_{\alpha}(\xi) \epsilon_{\theta_1 \theta_2} d\xi^\alpha \wedge d\theta_1 \wedge d\theta_2,$$

$$F_2 = \frac{1}{2!} F_{\alpha_1\alpha_2}(\xi) d\xi^{\alpha_1} \wedge d\xi^{\alpha_2} - \frac{1}{2! 8!} e^{\rho/2} \varepsilon^{\alpha_1 \cdots \alpha_8} \tilde{F}_{\alpha_1 \cdots \alpha_8}(\xi) \epsilon_{\theta_1 \theta_2} d\theta_1 \wedge d\theta_2,$$

$$\tilde{F}_4 = \frac{1}{4!} \tilde{F}_{\alpha_1 \cdots \alpha_4}(\xi) d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_4} + \frac{1}{2! 2! 6!} e^{\rho/2} \varepsilon_{\alpha_1 \cdots \alpha_6 \beta_1 \beta_2} \tilde{F}^{\alpha_1 \cdots \alpha_6}(\xi) \epsilon_{\theta_1 \theta_2} d\xi^{\beta_1} \wedge d\xi^{\beta_2} \wedge d\theta_1 \wedge d\theta_2.$$ (3.9)

By substituting (3.7), (3.8) and (3.9) into the equations of motion and the Bianchi identities in type IIA supergravity in appendix A, we obtain the following equations in eight dimensions.

$$R_{\alpha\beta} + \frac{1}{2} \nabla^{(8)}_\alpha \nabla^{(8)}_\beta \phi - \frac{1}{2} \nabla^{(8)}_\alpha \nabla^{(8)}_\beta \rho - \frac{1}{8} \partial_\alpha \rho \partial_\beta \rho - \frac{1}{4} H_{\alpha\beta\gamma\delta} H^\gamma_{\beta\delta} - \frac{1}{2} e^{-\rho} d_{\alpha} d_{\beta}$$

\(^1\)It is sufficient for the purpose of this paper to assume that the fields depend only on $\xi^{p+1} (= r)$. However, we consider more general ansätzes such as (3.7) and (3.8) for further developments.
Applying this procedure to the other case, we obtain $I_{p+2}$. We have thus reduced type IIA(IIB) supergravity on $S^{8-p}$ and obtain the $(p+2)$-dimensional gravities. These reductions are consistent truncations in the sense that every solution of $I_{p+2}$ can be lifted to a solution of type IIA(IIB) supergravity.
Let us write down $I_{p+2}$ explicitly. We first define $\tilde{I}_{p+2}$ by

$$
\tilde{I}_{p+2} = \int d^{p+2}\sqrt{-h} \left[ e^{-2\phi + \frac{8}{3p+8}} \left( R_h + e^{-\frac{2}{3}} R^{(S^8-p)} + 4\partial_\alpha \phi \partial^\alpha \phi + \frac{(8-p)(7-p)}{16} \partial_\alpha \rho \partial^\alpha \rho \right. \right.
$$

$$
- (8-p) \partial_\alpha \phi \partial^\alpha \rho - \frac{1}{2} |H_3|^2 \left. \right) + \frac{1}{2} e^{\frac{8}{3p+8}} \sum_n |\tilde{F}_n|^2 \right], \quad (3.24)
$$

where

$$
\tilde{F}_n = F_n + s C_{n-3} \wedge H_3, \quad s = \begin{cases} -1 & \text{for type IIA (p = 0, 2, 4, 6)} \\ 1 & \text{for type IIB (p = 1, 3, 5, 7)} \end{cases}
$$

Here $n$ takes $2, 4, \ldots, p + 2$ for type IIA supergravity and $1, 3, \ldots, p + 2$ for type IIB supergravity. $\tilde{I}_2$ is obtained by setting $H_3 = 0$ in (3.23). Then,

$$
I_{p+2} = \tilde{I}_{p+2} \quad \text{for} \quad p = 0, 1, \ldots, 5, \quad (3.25)
$$

$$
I_8 = \tilde{I}_8 + \int d^8\sqrt{-h} \left( - \frac{1}{2} e^{-\rho} |d_1|^2 \right) + \int b_0 \left( F_2 \wedge \tilde{F}_6 - \frac{1}{2} F_4 \wedge \tilde{F}_4 \right), \quad (3.26)
$$

$$
I_9 = \tilde{I}_9 + \int d^9\sqrt{-h} \left( - \frac{1}{2} e^{-\frac{2}{3}} |d_2|^2 \right) + \int b_1 \wedge (\tilde{F}_3 \wedge \tilde{F}_5 - F_1 \wedge \tilde{F}_7), \quad (3.27)
$$

where $d_1 = db_0$ and $d_2 = db_1$.

We perform reductions of 11-d supergravity on $S^7$ and $S^4$ in the same way as those of type IIA(IIB) supergravity. For a $S^7$ reduction, we make the following ansatzes:

$$
d_{s_{11}} = h_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta + e^{\frac{4}{7}} d\Omega_7,
$$

$$
F_4^{(M)} = \frac{1}{4!} F_{\alpha_1 \cdots \alpha_4}(\xi) d\xi^\alpha_1 \wedge \cdots \wedge d\xi^\alpha_4, \quad (3.28)
$$

where $\alpha, \beta$ run form 0 to 3. Then, we obtain a 4-dimensional gravity, which is a consistent truncation of 11-d supergravity,

$$
I_4^{(M)} = \int d^4\sqrt{-h} e^{\frac{2}{7}} \left( R_h + e^{-\frac{2}{3}} R^{(S^7)} + \frac{21}{8} \partial_\alpha \rho \partial^\alpha \rho - \frac{1}{2} |F_4^{(M)}|^2 \right), \quad (3.29)
$$

where $F_4^{(M)} = dA_3$. For a $S^4$ reduction, we make the following ansatzes:

$$
d_{s_{11}} = h_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta + e^{\frac{4}{7}} d\Omega_4,
$$

$$
F_4^{(M)} = F_{\alpha_1 \cdots \alpha_4}(\xi) d\xi^\alpha_1 \wedge \cdots \wedge d\xi^\alpha_4
$$

$$
+ \frac{1}{4!} \frac{1}{7!} e^{\rho} \varepsilon_{\alpha_1 \cdots \alpha_7} \tilde{F}_{\alpha_1 \cdots \alpha_7}(\xi) \varepsilon_{\theta_1 \cdots \theta_4} d\theta_{\alpha_1} \wedge \cdots \wedge d\theta_{\alpha_4}, \quad (3.30)
$$
where \( \alpha, \beta \) run from 0 to 6. Then, we obtain a 7-dimensional gravity, which is also a consistent truncation of 11-d supergravity,

\[
I_7^{(M)} = \int d^4 \xi \sqrt{-h} \, e^\rho \left( R_h + e^{-\frac{\rho}{2}} R^{(S^4)} + \frac{3}{4} \partial_\alpha \rho \, \partial^\alpha \rho - \frac{1}{2} |F_4^{(M)}|^2 - \frac{1}{2} |\tilde{F}_7^{(M)}|^2 \right),
\]

where \( \tilde{F}_7^{(M)} = dA_6 - \frac{1}{2} A_3 \wedge F_4^{(M)} \).

4 Canonical formalism and the H-J equations in the reduced gravities

In this section, we develop the canonical formalism for \( I_{p+2}, I_4^{(M)} \) and \( I_7^{(M)} \) obtained in the previous section. First we rename the \((p+2)-dimensional coordinates:

\[
\xi^\mu = x^\mu \ (\mu = 0, \cdots, p), \quad \xi^{p+1} = r.
\]

Here \( p \) takes 2 and 5 for \( I_4^{(M)} \) and \( I_7^{(M)} \), respectively. Adopting \( r \) as time, we make the ADM decomposition for the \((p+2)-dimensional metric.

\[
d\bar{s}_{p+2}^2 = h_{\alpha\beta} \, d\xi^\alpha \, d\xi^\beta
\]

\[
= (n^2 + g^\mu\nu n_\mu n_\nu) \, dr^2 + 2n_\mu \, dr \, dx^\mu + g_{\mu\nu} \, dx^\mu dx^\nu,
\]

where \( n \) and \( n_\mu \) are the lapse function and the shift function, respectively. Hence force \( \mu, \nu \) run from 0 to \( p \).

In what follows, we consider a boundary surface specified by \( r = \text{const.} \) and impose the Dirichlet condition for the fields on the boundary. Here we need to add the Gibbons-Hawking term [16] to the actions, which is defined on the boundary and ensures that the Dirichlet condition can be imposed consistently. Then, the \((p+2)-dimensional action \( I_{p+2} \) with the Gibbons-Hawking term on the boundary can be expressed in the canonical form as follows. For \( p = 0, 1, \cdots, 5 \),

\[
I_{p+2} = \int dr d^{p+1}x \sqrt{-g} \left( \pi^{\mu\nu} \partial_\tau g_{\mu\nu} + \pi_\phi \partial_\tau \phi + \pi_\rho \partial_\tau \rho + \pi_B^{\mu\nu} \partial_\tau B_{\mu\nu} + \sum_n \pi_{C_{\mu_1\cdots\mu_{n-1}}}^{\mu_1\cdots\mu_{n-1}} \partial_\tau C_{\mu_1\cdots\mu_{n-1}} \right.
\]

\[
- nH - n^\mu H_\mu - B_{\tau\mu} G_\mu^\tau - \sum_n C_{\tau\mu_1\cdots\mu_{n-2}} G_{\mu_1\cdots\mu_{n-2} C_{\mu_1\cdots\mu_{n-1}}}
\]

(4.2)
with

\[ H = -e^{2\phi - \frac{2\pi}{p} \rho} \left( (\pi^{\mu\nu})^2 + \frac{1}{2} \pi_{\mu}^2 + \frac{1}{2} \mu_{\mu} \pi_{\phi} + \frac{4}{8 - p} \pi_{\rho}^2 + \pi_{\phi} \pi_{\rho} \right) \]

\[ + \left( \pi_{\mu B} - \sum_{n \geq 3} \frac{(n-1)(n-2)}{2} C_{\mu_1 \cdots \mu_{n-3} \mu n_{\mu n-3 \mu n}} \right)^2 \right) \]

\[ - e^{\frac{8 - p}{2} \rho} \sum_{n} \frac{(n-1)!}{2} (\pi_{\mu n-1}^{\mu n-1})^2 - \mathcal{L}, \]

\[ H^\mu = -2 \nabla_\nu \pi^{\nu \mu} + \pi_{\phi} \partial_\mu \phi + \pi_{\rho} \partial_\mu \rho + \pi_{\nu B}^\mu \pi_{\nu}^\lambda \pi_{\nu}^\mu \]

\[ + \sum_{n \geq 4} \frac{\pi_{\nu_1 \cdots \nu_{n-1}}^{\nu_1 \cdots \nu_{n-1}}}{C_{n-1}} \left( F_{\nu_1 \cdots \nu_{n-1}}^{\mu} + s \frac{(n-1)!}{(n-4)!} 3! C_{n-1}^{\nu_1 \cdots \nu_{n-1}} H_{\nu_{n-3} \nu_{n-2} \nu_{n-1}} \right), \]

\[ G_B^{\mu B} = -2 \nabla_\nu \pi^{\nu \mu}, \]

\[ G_{C_{n-1}}^{\mu_1 \cdots \mu_{n-2}} = -(n-1) \nabla_\nu \pi_{C_{n-1}}^{\mu_1 \cdots \mu_{n-2}} - s \frac{(n+1)!}{(n-2)!} 3! C_{n+1}^{\mu_1 \cdots \mu_{n-2} \nu_1 \nu_2 \nu_3} H_{\nu_1 \nu_2 \nu_3}, \]

where

\[ \mathcal{L} = e^{-2\phi + \frac{8 - p}{2} \rho} \left( R_g + 4 \nabla_\mu \nabla_\nu \phi - \frac{8 - p}{2} \nabla_\mu \nabla_\nu \rho - 4 \partial_\mu \phi \partial_\nu \phi - \frac{(8 - p)(9 - p)}{16} \partial_\mu \rho \partial_\nu \rho \right) \]

\[ + (8 - p) \partial_\mu \phi \partial_\nu \rho \left( - \frac{1}{2} |H_3|^2 \right) \right) - \frac{1}{2} e^{\frac{8 - p}{2} \rho} \sum_{n} |\tilde{F}_n|^2 + e^{-2\phi + \frac{8 - p}{2} \rho} R(S^{8-p}), \]

\( I_8 \) and \( I_9 \) with the Gibbons-Hawking terms can be rewritten in the form \( 4.2 \) with \( p = 6 \) and \( p = 7 \), respectively, up to the terms including \( b_0 \) and \( b_1 \). The differences will turn out not to be relevant for our purpose, so that we do not write down the precise canonical forms of \( I_8 \) and \( I_9 \) here.

Note that \( 4.2 \) takes the form of \( 2.1 \). Indeed, \( n, n_\mu \) and \( C_{\mu_1 \cdots \mu_{n-2}} \) play the roles of \( \alpha_a \) in \( 2.1 \). They give the constraints, \( H = 0 \), \( H^\mu = 0 \) and \( G_{C_{n-1}} = 0 \), which are called the Hamiltonian constraint, the momentum constraint and the Gauss law constraint, respectively, and correspond to \( f_a = 0 \) in \( 2.2 \). Note that \( 4.2 \) has no analogue of \( H \) in \( 2.1 \). It follows from the arguments in section 2 that the H-J equations of this system are given by

\[ \frac{\partial S_{p+1}}{\partial \tilde{F}} = 0, \]

\[ H = 0, \]

\[ H^\mu = 0, \]

\[ G_{C_{n-1}} = 0 \]

(4.5)
with
\[
\pi^{\mu\nu}(x) = \frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta g_{\mu\nu}(x)}, \quad \pi_\phi(x) = \frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta \phi(x)}, \quad \pi_\rho(x) = \frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta \rho(x)},
\]
\[
\pi^B(x) = \frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta B_{\mu\nu}(x)}, \quad \pi_{C_{\mu_1\cdots\mu_{n-1}}}(x) = \frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta C_{\mu_1\cdots\mu_{n-1}}(x)}.
\]  
(4.6)

where \( \tilde{r} \) is the boundary value of \( r \), and \( g_{\mu\nu}(x), \phi(x), \cdots, C_{\mu_1\cdots\mu_{n-1}}(x) \) are the boundary values of the corresponding fields. The first equation in (4.5) indicates that \( S_{p+1} \) does not depend on the boundary ‘time’ explicitly. The last three equations in (4.5) give functional differential equations for \( S_{p+1} \). The third and fourth ones imply that \( S_{p+1} \) must be invariant under the diffeomorphism in \( p+1 \) dimensions and the \( U(1) \) gauge transformations (See appendix C in Ref.[1]). The second one is a nontrivial equation that can determine the form of \( S_{p+1} \). Hereafter, we call this equation the Hamilton-Jacobi equation.

The canonical form of \( I_4^{(M)} \) with the Gibbons-Hawking term is
\[
I_4^{(M)} = \int d^4 \xi \sqrt{-g} \left( \pi^{\mu\nu} \partial_\tau g_{\mu\nu} + \pi_\rho \partial_\tau \rho + \pi^{\mu\nu\lambda} \partial_\tau A_{\mu\nu\lambda} - nH - n_\mu H^\mu - A_{\mu\nu} G^{\mu\nu}_{A_3} \right)
\]  
(4.7)

with
\[
H = e^{- \tilde{\tau} \rho} \left( (\pi^{\mu\nu})^2 - \frac{1}{9} (\pi^\mu)_\mu \right)^2 \frac{8}{63} \pi^\rho \pi_\rho + \frac{4}{9} \pi^\mu \pi_\rho - 3 (\pi^{\mu\nu\lambda})^2 \right) - \mathcal{L},
\]
\[
H^\mu = -2 \nabla_\mu \pi^{\mu\nu} + \pi_\rho \partial^\mu \rho + \pi^{\mu\nu\lambda} \partial_\tau F^{(M)\mu}_{\nu\lambda\rho},
\]
\[
G^{\mu\nu}_{A_3} = -3 \nabla_\lambda \pi^{\lambda\mu\nu}_{A_3},
\]  
(4.8)

where
\[
\mathcal{L} = e^{\tilde{\tau} \rho} \left( R - \frac{7}{2} \nabla_\mu \nabla^\mu \rho - \frac{7}{2} \partial_\mu \rho \partial^\mu \rho - \frac{1}{2} |F_4^{(M)}|^2 \right) + e^{\tilde{\tau} \rho} R^{(S^7)}.
\]  
(4.9)

The canonical form of \( I_7^{(M)} \) with the Gibbons-Hawking term is
\[
I_7^{(M)} = \int d^7 \xi \sqrt{-g} \left( \pi^{\mu\nu} \partial_\tau g_{\mu\nu} + \pi_\rho \partial_\tau \rho + \pi^{\mu\nu\lambda} \partial_\tau A_{\mu\nu\lambda} + \pi^{\mu_1\cdots\mu_6}_{A_6} \partial_\tau A_{\mu_1\cdots\mu_6}
\]
\[
- nH - n_\mu H^\mu - A_{\mu\nu} G^{\mu\nu}_{A_3} - A_{\mu_1\cdots\mu_5} G^{\mu_1\cdots\mu_5}_{A_6} \right)
\]  
(4.10)

with
\[
H = -e^{- \rho} \left( (\pi^{\mu\nu})^2 - \frac{1}{9} (\pi^\mu)_\mu \right)^2 + \frac{5}{9} \pi^\rho \pi_\rho - \frac{4}{9} \pi^\mu \pi_\rho + 3 (\pi^{\mu\nu\lambda})^2 + 10 \pi^{\mu_1\mu_2\nu_1\nu_2 \mu_3}_{A_6} A_{\nu_1\nu_2\nu_3} \right)^2
\]
\[ H^\mu = -2\nabla_\nu \pi^{\nu\mu} + \pi_\rho \partial^\mu \rho + \pi^{\mu\lambda\rho} F^{(M)}_{\mu\nu_1...\nu_6} \left( \frac{15}{2} A^{\mu}_{\nu_1\nu_2} F^{(M)}_{\nu_3...\nu_6} \right) \pi^{\nu_1...\nu_6}, \]

\[ G^\mu_3 = -3\nabla_\lambda \pi^{\mu\lambda\nu} \]

\[ G^\mu_{A_6} = -6\nabla_\nu \pi^{\mu\nu_{15}}, \]

where

\[ \mathcal{L} = e^\rho \left( R - 2\nabla_\mu \nabla^\mu \rho - \frac{5}{4} \partial_\mu \rho \partial^\mu \rho - \frac{1}{2} |F^{(M)}|^2 - \frac{1}{2} |\tilde{F}^{(M)}|^2 \right) + e^{\frac{2}{8-p}} R^{(8^p)} \]

The H-J equations for \( I_4^{(M)} \) and \( I_7^{(M)} \) are derived in the same way as the one for \( I_{p+2} \).

## 5 Dp-brane effective action as a solution to the H-J equation

### 5.1 Dp-brane effective action as a solution

In this subsection, we find a solution to the H-J equation obtained in the previous section. We assume that the fields are constant on the fixed-time surface. Let \( S_{p+1}^{(0)} \) be a solution to the H-J equation under this assumption. We can drop \( b_0 \) and \( b_1 \) consistently in the H-J equations for \( S_{7}^{(0)} \) and \( S_{8}^{(0)} \). That is, after this simplification, the H-J equations for \( S_{7}^{(0)} \) and \( S_{8}^{(0)} \) coincide with the ones derived from (4.2) with \( p = 6 \) and \( p = 7 \), respectively. We see from (4.3), (4.4), (4.5) and (4.6) that \( S_{p+1}^{(0)} \) satisfies the equation

\[ e^{\frac{2\rho-8-p}{4p}} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta g_{\mu\nu}} \right)^2 + \frac{1}{2} \frac{\delta S_{p+1}^{(0)}}{\delta \rho} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta \phi} \right)^2 \]

\[ + \frac{4}{8-p} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta \rho} \right)^2 + \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta \phi} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta \rho} \]

\[ + \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta B_{\mu\nu}} - \sum_n \frac{(n-1)(n-2)}{2} C_{\mu_1...\mu_{n-3}} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta C_{\mu_1...\mu_{n-3\mu}}} \right) \]

\[ + e^{\frac{2\rho-8-p}{4p}} \frac{1}{2} \frac{\delta S_{p+1}^{(0)}}{\delta \phi} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta \rho} \]

\[ = 0. \] (5.1)
In what follows, we show that the form

\[ S_{p+1}^{(0)} = S_{p+1}^{c} + S_{p+1}^{BI} + S_{p+1}^{WZ} + \sigma_{p+1} \]  

(5.2)

is a solution to (5.1), with

\[ S_{p+1}^{c} = \alpha_{p+1} \int d^{p+1}x \sqrt{-g} e^{-2\phi + \frac{7-8}{4} \rho}, \]

\[ S_{p+1}^{BI} = \beta_{p+1} \int d^{p+1}x e^{-\phi} \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})}, \]

\[ S_{p+1}^{WZ} = \gamma_{p+1} \int \sum_{n} C_{n} - 1 \wedge e^{F} \]

\[ = \gamma_{p+1} \int \left( C_{p+1} + C_{p-1} \wedge F + \frac{1}{2} C_{p-3} \wedge F \wedge F + \cdots \right), \]  

(5.3)

where \( F_{\mu\nu} = B_{\mu\nu} + F_{\mu\nu}^{(p+1)} \), \( F_{\mu\nu}^{(p+1)} \) is an arbitrary constant anti-symmetric tensor, and \( \sigma_{p+1} \) is an arbitrary constant. As we discussed in section 6 in Ref.\[1\], \( S_{p+1}^{BI} + S_{p+1}^{WZ} \) is the effective action of a probe Dp-brane while \( S_{p+1}^{c} \) should be interpreted as the vacuum to vacuum amplitude and does not contribute to the effective action of the probe Dp-brane. Furthermore, \( F_{\mu\nu}^{(p+1)} \) is interpreted as the \( U(1) \) gauge field strength in the world-volume.

Noting that

\[ \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta B_{\mu\nu}} - \sum_{n} \frac{(n-1)(n-2)}{2} C_{\mu_{1} \cdots \mu_{n-2}} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{(0)}}{\delta g_{\mu_{1} \cdots \mu_{n-3}\mu_{n}}} = \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{BI}}{\delta B_{\mu\nu}} \]

and

\[ \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{WZ}}{\delta g_{\mu\nu}} = 0, \]

one can see that the left-hand side of (5.1) can be decomposed into the four parts

\[ \text{L.H.S. of (5.1)} = e^{2\phi - \frac{8-8\rho}{4}} \times (1) + (2) + (3) + e^{\frac{8-8\rho}{4}} \times (4) + e^{-2\phi + \frac{8-8\rho}{4}} R(S^{8-p}), \]  

(5.4)

with

\[ (1) = \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta g_{\mu\nu}} \right)^{2} + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta g_{\mu\nu}} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta \phi} + \frac{1}{2} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta \phi} \right)^{2} \]

\[ + \frac{4}{8 - p} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta \rho} \right)^{2} + \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta \phi} \frac{1}{\sqrt{-g}} \frac{\delta S_{p+1}^{c}}{\delta \rho}, \]
Then, we have

\[ (2) = 2g_{\mu\nu}g_{\nu\rho} \frac{1}{\sqrt{-g}} \delta S^c_{p+1} \frac{1}{\sqrt{-g}} \delta g_{\mu\rho} + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \delta g_{\mu\rho} \frac{1}{\sqrt{-g}} \delta g_{\rho\nu} + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \delta g_{\mu\rho} \frac{1}{\sqrt{-g}} \delta g_{\rho\nu} + \frac{1}{2} g_{\mu\nu} \frac{1}{\sqrt{-g}} \delta g_{\mu\rho} \frac{1}{\sqrt{-g}} \delta g_{\rho\nu} \]

In order to calculate (2) and (3), we introduce the \((p+1)\times(p+1)\) matrices \(G\) and \(B\):

\[
(G)_{\mu\nu} = g_{\mu\nu}, \quad (B)_{\mu\nu} = F_{\mu\nu}.
\]

Then, we have

\[
\begin{align*}
\frac{1}{\sqrt{-g}} \delta S^c_{p+1} &= \frac{1}{2} \alpha_{p+1} e^{-2\phi + \frac{7}{2} \rho} \left( \frac{1}{G} \right)^\mu_\nu, \\
\frac{1}{\sqrt{-g}} \delta S^B_{p+1} &= \frac{7}{4} \alpha_{p+1} e^{-2\phi + \frac{7}{2} \rho}, \\
\frac{1}{\sqrt{-g}} \delta g_{\mu\nu} &\quad = \frac{1}{2} \beta_{p+1} e^{-\phi} \sqrt{\text{det}(G + B)} \left( \frac{1}{G} G^1 B \frac{1}{G - B} \right)^\mu_\nu, \\
\frac{1}{\sqrt{-g}} \delta g_{\mu\rho} &\quad = \beta_{p+1} e^{-\phi} \frac{\sqrt{\text{det}(G + B)}}{\text{det} G} \left( \frac{1}{G + B} B \frac{1}{G - B} \right)^\mu_\nu, \\
\frac{1}{\sqrt{-g}} \delta B_{\mu\nu} &\quad = -\beta_{p+1} e^{-\phi} \frac{\sqrt{\text{det}(G + B)}}{\text{det} G}.
\end{align*}
\]

Using this notation, we can express each term in (2) and (3) in terms of the trace of the \((p+1)\times(p+1)\) matrix and calculate (2) and (3) as follows:

\[
(2) = \alpha_{p+1} \beta_{p+1} e^{-3\phi + \frac{7}{2} \rho} \sqrt{\text{det}(G + B)} \left( \frac{1}{2} \text{tr} \left( \frac{1}{G} G \frac{1}{G + B} G \frac{1}{G - B} G \right) - \frac{p+1}{4} \right)
\]
From (5.1), (5.4), (5.6) and (5.8), we conclude that $S_{p+1}^{(0)}$ satisfies the H-J equation (5.1) if

$$\alpha_{p+1}^2 = \frac{4(8-p)}{7-p} R_{(S^{(p-1)})} = 4(8-p)^2$$

and

$$\beta_{p+1}^2 = \gamma_{p+1}^2.$$  

### 5.2 Dp-brane in a $B_2$ field

In this subsection, we see that $S_{p+1}^{(0)}$ obtained in the previous subsection reproduces the supergravity solution representing a stack of Dp-branes in a constant $B_2$ field. First, we examine the cases in which $2 \leq p \leq 6$. For simplicity, let us consider the supergravity solutions with only $B_{p-1}$ non-vanishing. These supergravity solutions were constructed in Ref.[15], and they are also solutions of $I_{p+2}$ taking the following forms:

$$ds_{p+2}^2 = f^{-\frac{1}{2}}(\eta_{\mu\hat{\nu}}dx^\mu dx^{\hat{\nu}} + h\delta_{ab}dx^a dx^b) + f^{\frac{3}{2}}dr^2,$$

$$e^{2\phi} = g_{st}^2 f^{-\frac{p-2}{2}} h, \quad e^{\hat{\phi}} = r^{\frac{3}{2}} f^{\frac{1}{2}}, \quad B_{p-1} = \tan f^{-1} h,$$

$$C_{01\cdots p-2} = (-1)^{p+1} g_{st}^{-1} \sin \theta f^{-1}, \quad C_{01\cdots p} = (-1)^{p+1} g_{st}^{-1} \cos \theta f^{-1} h,$$

where

$$\hat{\mu}, \hat{\nu} = 0, 1, \cdots, p-2, \quad a, b = p - 1, p,$$

$$f = 1 + \frac{Q}{r^{7-p}}, \quad h^{-1} = \sin^2 \theta f^{-1} + \cos^2 \theta.$$  

(5.10)

Note that these supergravity solutions preserve 16 supersymmetries, and reduce to the ordinary Dp-brane solutions when $\theta = 0$. 
By varying $I_{p+2}$ with respect to the canonical momenta, we obtain the relations between the canonical momenta and the $r$-derivatives of the fields. Using these relations, we calculate the values of the canonical momenta on the boundary specified by $r = \bar{r}$:

$$\pi_{00} = \pi_{11} = \cdots = \pi_{p-2p-2} = \frac{f^{p+2}}{g_{st}^2} \left( (8 - p)\bar{r}^{7-p} \bar{f} + \frac{1}{2} \bar{r}^{8-p} \partial_r \bar{f} \right),$$

$$\pi_{p-1p+1} = \pi^{I_{p+2}}_{pp} = \frac{f^{p+2}}{g_{st}^2} \left( (8 - p)\bar{r}^{7-p} \bar{f} + \frac{1}{2} \cos^2 \theta \bar{r}^{8-p} \partial_r \bar{f} \right),$$

$$\pi_\phi = \frac{f^{p+3}}{g_{st}^2} \left( 4(p - 8)\bar{r}^{7-p} \bar{f} - \bar{r}^{8-p} \partial_r \bar{f} \right),$$

$$\pi_\rho = \frac{(7 - p)(8 - p)}{2g_{st}^2} \bar{r}^{7-p} \bar{f}^{\frac{2}{p+1}}, \quad \pi^{I_{p+2}}_{Bp-1p} = 0,$$

$$\pi_{C_{p-1p+1}...p+1} = \frac{(-1)^{p+1}}{(p-1)!} \bar{r}^{8-p} \bar{f}^{\frac{p+1}{4}} \partial_r \bar{f},$$

$$\pi_{C_{p+1p+1}...p+1} = \frac{(-1)^{p+1}}{(p+1)!} \bar{r}^{8-p} \bar{h}^{\frac{p+1}{4}} \partial_r \bar{f},$$

(5.12)

where

$$\bar{f} = 1 + \frac{\bar{Q}}{\bar{r}^{7-p}}, \quad \bar{h}^{-1} = \sin^2 \theta \bar{f}^{-1} + \cos^2 \theta.$$

On the other hand, $S^{(0)}_{p+1}$ gives the following canonical momenta:

$$\pi_{\mu \nu} = g_{\mu \lambda}g_{\nu \rho} \frac{1}{\sqrt{-g}} \delta g_{\lambda \rho} \delta S^{(0)}_{p+1}$$

$$= \frac{1}{2} \alpha_{p+1} e^{-2\phi + \frac{2}{p+1} \rho} g_{\mu \nu} + \frac{1}{2} \beta_{p+1} e^{-\phi} \sqrt{\frac{\det(G + B)}{\det G}} \left( G \frac{1}{G + B} G \frac{1}{G - B} G \right)_{\mu \nu},$$

$$\pi_\phi = \frac{1}{\sqrt{-g}} \delta S^{(0)}_{p+1} \phi = -2\alpha_{p+1} e^{-2\phi + \frac{2}{p+1} \rho} - \beta_{p+1} e^{-\phi} \sqrt{\frac{\det(G + B)}{\det G}},$$

$$\pi_\rho = \frac{1}{\sqrt{-g}} \delta S^{(0)}_{p+1} \rho = \frac{7 - p}{4} \alpha_{p+1} e^{-2\phi + \frac{2}{p+1} \rho},$$

$$\pi_{B \mu \nu} = g_{\mu \lambda}g_{\nu \rho} \frac{1}{\sqrt{-g}} \delta B_{\lambda \rho} \delta S^{(0)}_{p+1}$$

$$= \frac{1}{2} \beta_{p+1} e^{-\phi} \sqrt{\frac{\det(G + B)}{\det G}} \left( G \frac{1}{G + B} B \frac{1}{G - B} G \right)_{\mu \nu}$$

$$+ \gamma_{p+1} e_{\mu \nu \lambda_1 ... \lambda_{p-1}} \frac{1}{(p+1-2k)!2^k(k-1)!} C^{\lambda_1 \lambda_2 ... \lambda_{p+1-2k}} \times \mathcal{F}^{\lambda_{p+1-2k+1} \lambda_{p+1-2k+2} ... \lambda_{p-2} \lambda_{p-1}},$$

19
\[
\pi_{C_{n-1,\mu_1\mu_2...\mu_{n-1}}} = g_{\mu_1\lambda_1}g_{\mu_2\lambda_2} \cdots g_{\mu_{n-1}\lambda_{n-1}} \frac{1}{\sqrt{-g \delta C_{\lambda_1\lambda_2...\lambda_{n-1}}}} \delta S_{p+1}^{(0)} \]
\[
= \gamma_{p+1} \frac{1}{(n-1)!2^{\frac{p-n+2}{2}}(\frac{p-n+2}{2}!)^2} \varepsilon_{\mu_1\mu_2...\mu_{p+1}} F_{\mu_1\mu_2...\mu_{p+1}} F_{\mu_1\mu_2...\mu_{p+1}}. \tag{5.13}
\]

We substitute the values of the fields in (5.10) into the right-hand sides of (5.13), setting

\[F_{\mu\nu}^{(p+1)} = 0.\]

It can be verified that the right-hand sides of (5.13) reproduce the right-hand sides of (5.12) if

\[\alpha_{p+1} = 16 - 2p \quad \text{and} \quad \beta_{p+1} = (-1)^p \gamma_{p+1} = \frac{(p - 7) \tilde{Q} \cos \theta}{g_{st}}, \tag{5.14}\]

These conditions are consistent with (5.9).

Next, we compare the value of the on-shell action with that of \(S_{p+1}^{(0)}\) directly. Substituting the values of the fields with \(r = \bar{r}\) in (5.10) into (5.3), we obtain

\[S_{p+1}^c = \frac{\alpha_{p+1} V_{p+1} r_0^{7-p}}{g_{st}^2}, \]
\[S_{p+1}^{BI} = \frac{\beta_{p+1} V_{p+1} f_{0-1}}{g_{st} \cos \theta}, \]
\[S_{p+1}^{WZ} = (-1)^{p+1} \gamma_{p+1} V_{p+1} f_{0-1} \frac{1}{g_{st} \cos \theta}, \tag{5.15}\]

where \(V_{p+1} = \int d^{p+1}x\). Here, we set

\[\sigma_{p+1} = 0.\]

Then, it follows from (5.2), (5.14) and (5.15) that

\[S_{p+1}^{(0)} = S_{p+1}^c + S_{p+1}^{BI} + S_{p+1}^{WZ} = \frac{(16 - 2p)V_{p+1} r_0^{7-p}}{g_{st}^2}. \tag{5.16}\]

We calculate the values of the on-shell actions for (5.10) by substituting (5.12) with \(\bar{r}\) replaced by \(r\) into (4.2). Noting that the constraints in (4.2) are satisfied on shell, we reproduce the value of \(S_{p+1}^{(0)}\) for (5.10) as follows:

\[I_{on-shell}^{(p+2)} = \int_0^r dr d^{p+1}x \sqrt{-g(\pi^{\mu\nu} \partial_r g_{\mu\nu} + \pi_{\phi} \partial_r \phi + \pi_{\rho} \partial_r \rho + \pi^{\mu\nu} \partial_r B_{\mu\nu}}
\]
\[+ \sum_n \pi_{C_{n-1}} \partial_r C_{\mu_1...\mu_{n-1}}) \]
\[= \frac{2(7-p)(8-p) V_{p+1} r_0^{7-p}}{g_{st}^2} \int_0^r dr r^{6-p}
\]
\[= \frac{2(8-p) V_{p+1} r^{7-p}}{g_{st}^2}. \tag{5.17}\]
Thus, we have shown that \( S_{p+1}^{(0)} \) with \( F_{\mu\nu}^{(p+1)} = 0 \) and \( \sigma_{p+1} = 0 \) reproduces the supergravity solution (5.10) when \( \alpha_{p+1} \) and \( \beta_{p+1} \) take the values in (5.14). We also verified that \( S_1^{(0)} \) and \( S_2^{(0)} \) reproduce the supergravity solutions representing a stack of D0-branes and of D1-branes, respectively. Note that \( S_4^{(0)} \) also reproduces the near-horizon limit of (5.10) with \( p = 3 \) \([1]\), which is conjectured to be dual to NCYM \([2,3]\).

6 Effective actions of M2-brane and M5-brane as solutions to the H-J equations

6.1 M2-brane case

We assume again that the fields are constant on the fixed-time surface. Let us denote a solution to the H-J equation under this assumption by \( S_{M2}^{(0)} \). It follows from (4.7), (4.8) and (4.9) that \( S_{M2}^{(0)} \) satisfies the equation

\[
\left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta g_{\mu\nu}} \right)^2 - \frac{1}{9} \left( g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta g_{\mu\nu}} \right)^2 + \frac{8}{63} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta \rho} \right)^2 - \frac{4}{g} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta g_{\mu\nu}} \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta \rho} + 3 \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M2}^{(0)}}{\delta A_{\mu\nu\lambda}} \right)^2 + e^{3\rho} R^{(S^7)} = 0. \tag{6.1}
\]

One can easily verify that the form

\[
S_{M2}^{(0)} = S_{M2}^c + S_{M2}^{NG} + S_{M2}^{WZ} + \sigma_{M2} \tag{6.2}
\]

with

\[
S_{M2}^c = \alpha_{M2} \int d^3x \sqrt{-g} e^{\frac{2}{3}\rho},
S_{M2}^{NG} = \beta_{M2} \int d^3x \sqrt{-g},
S_{M2}^{WZ} = \gamma_{M2} \int A_3,
\tag{6.3}
\]

where \( \sigma_{M2} \) is an arbitrary constant, satisfies the H-J equation if

\[
\alpha_{M2}^2 = \frac{14}{3} R^{(S^7)} = 196 \quad \text{and} \quad \beta_{M2}^2 = \gamma_{M2}^2. \tag{6.4}
\]
Note that $S_{M2}^{NG} + S_{M2}^{WZ}$ is interpreted as a probe M2-brane effective action as in the case of the Dp-brane.

The supergravity solution representing a stack of M2-brane is also a solution of $I^{(M)}$, which is given by

$$ds_4^2 = f^{-\frac{2}{7}} \eta_{\mu\nu} dx^\mu dx^\nu + f^\frac{1}{7} dr^2, \quad f = 1 + \frac{\tilde{Q}}{r^6},$$

$$e^{\frac{2}{7}} = r^2 f^\frac{2}{7}, \quad A_{012} = f^{-1},$$

where $\mu, \nu$ run from 0 to 3. We verified that $S_{M2}^{(0)}$ with $\sigma_{M2} = 0$ reproduces this solution when

$$\alpha_{M2} = 14, \quad \beta_{M2} = -\gamma_{M2} = -6 \tilde{Q},$$

which is consistent with (6.4).

### 6.2 M5-brane case

We adopt again the assumption that the fields are constant on the fixed-time surface. Let $S_{M5}^{(0)}$ be a solution to the H-J equation under this assumption. It follows from (4.10), (4.11) and (4.12) that $S_{M5}^{(0)}$ satisfies the equation

$$\left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta g_{\mu\nu}} \right)^2 - \frac{1}{9} \left( g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta g_{\mu\nu}} \right)^2 + \frac{5}{9} \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta \rho} \right)^2$$

$$- \frac{4}{9} g_{\mu\nu} \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta g_{\mu\nu}} \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta \rho} + 3 \left( \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta A_{\mu\nu\lambda}} + 10 A_{\rho_1\rho_2\rho_3} \frac{1}{\sqrt{-g}} \frac{\delta S_{M5}^{(0)}}{\delta A_{\mu\nu\lambda\rho_1\rho_2\rho_3}} \right)^2$$

$$+ \frac{6!}{2} \left( \frac{\delta S_{M5}^{(0)}}{\delta A_{\mu_1\cdots\mu_6}} \right)^2 + e^{\frac{3}{2} \rho} R^{(S^4)} = 0.$$

Let us consider the following form:

$$S_{M5}^{(0)} = S_{M5}^c + S_{M5}^{BI} + S_{M5}^{WZ} + \sigma_{M5},$$

with

$$S_{M5}^c = \alpha_{M5} \int d^6 x \sqrt{-g} e^{\frac{2}{7}} \rho,$$

$$S_{M5}^{BI} = \beta_{M5} \int d^6 x \sqrt{-g} \sqrt{1 + \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + \frac{1}{288} (F_{\mu\nu\rho} F^{\mu\nu\rho})^2 - \frac{1}{96} F_{\mu\nu\lambda} F^{\nu\lambda\rho} F^{\rho\sigma\tau} F_{\sigma\tau\mu}},$$

$$S_{M5}^{WZ} = \gamma_{M5} \int \left( A_6 + \frac{1}{2} A_3 \wedge F_3 \right),$$

$$S_{M5} = S_{M5}^c + S_{M5}^{BI} + S_{M5}^{WZ} + \sigma_{M5},$$

(6.9)
where $F_{\mu\nu\lambda} = A_{\mu\nu\lambda} + F^{(M5)}_{\mu\nu\lambda}$, $F^{(M5)}_{\mu\nu\lambda}$ is an arbitrary constant completely anti-symmetric tensor, and $\sigma_{M5}$ is an arbitrary constant. We verified that (6.8) satisfies (6.7), up to the constraint
\[-\frac{1}{3!} \varepsilon_{\mu_1\cdots\mu_6} F_{\mu_4\mu_5\mu_6} \]
\[= 12 \frac{\delta}{\delta A_{\mu_1\mu_2\mu_3}} \sqrt{1 + \frac{1}{12} F_{\mu_\rho\nu} F^{\mu_\rho\nu} + \frac{1}{288} (F_{\mu_\rho\nu} F^{\mu_\rho\nu})^2 - \frac{1}{96} F_{\mu_\rho\nu\lambda} F^{\nu\lambda\rho} F^{\rho\sigma\tau} F_{\sigma\tau\mu}},\]
(6.10)
if
\[\alpha^2_{M5} = \frac{16}{3} R^{(S^4)} = 64 \quad \text{and} \quad \beta_{M5} = -\gamma_{M5}.\]
(6.11)

The equations of motion satisfied by M5-brane are determined by the space-time supersymmetry and the kappa symmetry \[18, 19, 20\]. These equations of motion are equivalent to the equations derived from the M5-brane effective action and the non-linear self-duality condition \[21, 22\]. This M5-brane effective action and the non-linear self-duality condition reduce to $S_{M5}^{BI} + S_{M5}^{WZ}$ and (6.10) in our case in which the worldvolume of the M5-brane is the time-fixed surface and the static gauge is taken.

The supergravity solution of the M2-M5 bound state is given in Ref.\[17\], and it is also a solution of $I^{(M)}_7$ which takes the following form:
\[ds^2_6 = f^{-\frac{1}{3}} k^{\frac{1}{3}} \eta_{\hat{\mu}\hat{\nu}} d\hat{x}^{\hat{\mu}} d\hat{x}^{\hat{\nu}} + f^{\frac{1}{4}} k^{-\frac{1}{2}} \delta_{ab} dx^a dx^b + f^{\frac{1}{2}} k^{\frac{1}{2}} dr^2,\]
\[e^{\hat{f}} = r^2 f^{\frac{1}{2}} k^{\frac{1}{3}}, \quad A_{012} = \sin \theta f^{-1}, \quad A_{345} = \tan \theta k^{-1},\]
\[\tilde{F}^{(M)}_{012345r} = 3 \cos \theta \tilde{Q} r^{-4} f^{-1} k^{-1},\]
(6.12)
where
\[\hat{\mu}, \hat{\nu} = 0, 1, 2, \quad a, b = 3, 4, 5,\]
\[f = 1 + \frac{\tilde{Q}}{r^3}, \quad k = \sin^2 \theta + \cos^2 \theta f.\]
(6.13)

We verified that (6.8) with $F^{(M5)}_{\mu\nu\lambda} = 0$ and $\sigma_{M5} = 0$ reproduces this solution when $\alpha_{M5} = 8$ and $\beta_{M5} = -\gamma_{M5} = -3\tilde{Q} \cos \theta$, which is consistent with (6.11).

\[\text{2For a review, see Refs.\[23, 24\]. Also, for an alternative formulation of the M5-brane effective action, see Refs.\[25\].}\]


7 (p, q) string and (p, q) 5-brane in type IIB supergravity and NS 5-brane in type IIA supergravity

7.1 (p, q) string and (p, q) 5-brane

Let us recall the $SL(2, R)$ symmetry in type IIB supergravity. It is convenient for this purpose to work in the Einstein frame and redefine the R-R 4-form. The Einstein metric is given by

$$G_{MN}^{(E)} = e^{-\frac{2}{3}\phi} G_{MN}, \quad (7.1)$$

and the new R-R 4-form is given by

$$C_{4}^{new} = C_4 + \frac{1}{2} C_2 \wedge B_2. \quad (7.2)$$

The type IIB supergravity action (3.3) is rewritten in terms of the Einstein metric and the new R-R 4-form:

$$I_{IIB} = \frac{1}{2\kappa_1^2} \int d^{10}X \sqrt{-G^{(E)}} \left( R_{G^{(E)}} - \frac{1}{2} \partial_M \tau \partial_M \bar{\tau} - \frac{1}{2} \mathcal{M}_{ij} F_3^i \cdot F_3^j - \frac{1}{4} |\bar{F}_5|^2 \right) + \frac{\varepsilon_{ij}}{8\kappa_1^2} \int C_{4}^{new} \wedge F_3^i \wedge F_3^j, \quad (7.3)$$

where

$$\tau = C_0 + i e^{-\phi},$$
$$\mathcal{M}_{ij} = \frac{1}{\text{Im } \tau} \left( \begin{array}{cc} |\tau|^2 & \text{Re } \tau \\ \text{Re } \tau & 1 \end{array} \right),$$
$$C_2^i = \left( B_2 \atop C_2 \right), \quad F_3^i = dC_2^i,$$
$$\bar{F}_5 = dC_{4}^{new} - \frac{1}{2} \varepsilon_{ij} C_2^i \wedge F_3^j. \quad (7.4)$$

One can easily check that the action (7.3) and the self-duality condition (3.5) is invariant under the following $SL(2, R)$ transformation.

$$\tau' = \frac{a\tau + b}{ct + d} \quad a, b, c, d; \text{ real}, \quad ad - bc = 1,$$
$$C_2'^i = \Lambda^i_\bar{j} C_2^j, \quad \Lambda^i_\bar{j} = \left( \begin{array}{cc} d & -c \\ -b & a \end{array} \right),$$
$$C_{4}^{new'} = C_{4}^{new},$$
$$G_{MN}^{(E)}' = G_{MN}^{(E)}, \quad (7.5)$$
First, let us see how a solution to the H-J equation corresponding to \((p, q)\) string is obtained. Noting that (7.1) implies that \(\rho^{(E)} = \rho - \phi\), one can rewrite (5.1) with \(p = 1\) in terms of the Einstein metric as follows:

\[
\left( \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta g_{\mu\nu}} \right)^2 - \frac{1}{8} \left( \frac{g_{\mu
u}^{(E)}}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta g_{\mu\nu}} \right)^2 + \frac{1}{14} \left( \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta \rho^{(E)}} \right)^2
\]

\[
- \frac{1}{2} g_{\mu\nu}^{(E)} \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta g_{\mu\nu}} \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta \rho^{(E)}} + (\text{Im } \tau)^2 \left( \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta \tau} \right)^2
\]

\[
+ g_{\mu\nu}^{(E)} g_{\nu\rho} M_{ij}^{-1} \frac{1}{\sqrt{-g^{(E)}}} \frac{\delta S^{(0)}}{\delta C_{\mu\nu}} + e^{3\rho^{(E)}} R^{(S7)}
\]

\[
= 0. \quad (7.6)
\]

One can verify that this H-J equation is invariant under (7.5), which implies \(g_{\mu\nu}^{(E)} = g_{\mu\nu}^{(E)}\) and \(\rho^{(E)'} = \rho^{(E)}\). Therefore, the \(SL(2, R)\) transformed \(S^{(0)}\) is also a solution to the H-J equation, which clearly reproduces the supergravity solution of \((p, q)\) string.

Second, let us see briefly how a solution to the H-J equation corresponding to \((p, q)\) 5-brane is obtained. We consider the \(p = 5\) case of the reduction in section 3 with the ansatz for \(H_3\) replaced by

\[
H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma
\]

\[
+ \frac{1}{3!} \frac{\tau}{7!} e^{2\rho + \frac{3\rho}{4} \rho^{(E)} - \rho^{(E)^2}} H^{\alpha_1 \cdots \alpha_7}(\xi) \varepsilon_{\theta_1 \theta_2 \theta_3} d\theta_{\theta_1} \wedge d\theta_{\theta_2} \wedge d\theta_{\theta_3}. \quad (7.7)
\]

Then, we obtain as a consistent truncation a seven-dimensional gravity, and rewrite it in terms of the Einstein metric and the new R-R 4 form:

\[
I_7^{(p,q)5} = \int d^7 \xi \sqrt{-h} e^{\frac{3\rho}{2}} \left( R^{(E)} + e^{-\frac{3}{2}\rho(E)} R^{(S3)} + \frac{3}{8} \partial_\alpha \rho^{(E)} \partial^\alpha \rho^{(E)} - \frac{1}{2} \partial_\alpha \tau \partial^\alpha \tau \right)
\]

\[
- \frac{1}{2} M_{ij} F_{ij} \cdot F_{ij} - \frac{1}{2} |\tilde{F}_5|^2 - \frac{1}{2} M_{ij} F_{ij} \cdot F_{ij} \right), \quad (7.9)
\]

where

\[
H_3 = dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2, \quad \tilde{F}_3 = F_3 + C_0 \wedge H_3,
\]

\[
\tilde{F}_5 = dC_4^{\text{new}} - \frac{1}{2} B_2 \wedge F_3 + \frac{1}{2} C_2 \wedge H_3,
\]

\[
\tilde{F}_7 = dC_6 + C_4^{\text{new}} \wedge H_3 - \frac{1}{2} C_2 \wedge B_2 \wedge H_3,
\]

25
\[ \tilde{H}_7 = dB_6 - C_0 \wedge \tilde{F}_7 - C_4^{new} \wedge F_3 + \frac{1}{4} C_2 \wedge C_2 \wedge H_3, \]
\[ F_i^7 = \left( e^{\phi} \tilde{H}_7 - e^{\phi} C_0 \tilde{H}_7 \right), \quad (7.10) \]

and \( \tau, M_{ij} \) and \( F_3^i \) are formally the same in (7.4). (7.9) is invariant under the transformation (7.5) with
\[ F_i^7 = \Lambda_{ij}^i F_j^7. \quad (7.11) \]

The transformation laws for \( B_6 \) and \( C_6 \) are determined by (7.11). Clearly, the H-J equation derived from (7.9) is invariant under this \( SL(2, R) \) transformation. Therefore, by rewriting (5.2) with \( p = 5 \) in terms of the Einstein metric and the new R-R 4-form and applying the \( SL(2, R) \) transformation to it, we obtain a solution to the H-J equation reproducing the supergravity solution of \((p, q)\) 5-brane.

### 7.2 NS 5-brane in type IIA supergravity

In this subsection, using the relation between 11-d supergravity and type IIA supergravity, we see that the NS 5-brane effective action is a solution to the H-J equation of supergravity and it reproduces the supergravity solution of NS 5-brane. In order to see the relation to type IIA supergravity, we consider a reduction of 11-d supergravity on \( S^3 \times S^1 \), which is different from the one done in section 3:

\[ ds_{11}^2 = G_{MND} dX^M dX^N \]
\[ F_4^{(M)} = \frac{1}{4!} F_{\alpha_1 \ldots \alpha_4}^M d\xi^{\alpha_1} \wedge \ldots \wedge d\xi^{\alpha_4} \]
\[ + \frac{1}{3! \cdot 7!} e^{\frac{1}{2} \rho_1 + \frac{3}{2} \rho_2} \varepsilon_{\alpha_1 \ldots \alpha_7} \tilde{F}_{\alpha_1 \ldots \alpha_7} \varepsilon_{\theta_1 \theta_2 \theta_3} d\theta_{i_1} \wedge d\theta_{i_2} \wedge d\theta_{i_3} \wedge dX^{10}, \quad (7.13) \]

where \( \alpha, \beta \) run from 0 to 6, and \( X^{10} \) parametrizes \( S^1 \). Then, we obtain as a consistent truncation a seven-dimensional gravity

\[ I_7^{(M)} = \int d^4 \xi \sqrt{-h} e^{\frac{1}{2} \rho_1 + \frac{3}{2} \rho_2} \left( R_h + e^{-\frac{1}{2} \rho_2} R(S^3) + \frac{3}{8} \partial_{\alpha} \rho_1 \partial^\alpha \rho_2 + \frac{3}{8} \partial_{\alpha} \rho_2 \partial^\alpha \rho_2 \right. \]
\[ - \frac{1}{2} \left| F_4^{(M)} \right|^2 - \frac{1}{2} \left| \tilde{F}_7^{(M)} \right|^2 \right), \quad (7.14) \]
where $\tilde{F}^{(M)}_7 = dA_6 - \frac{1}{2}A_3 \wedge F_4^{(M)}$. Let us consider the form

$$S^{(0)}_{\text{M5}}' = S^c_{\text{M5}}' + S^{BI}_{\text{M5}} + S^{WZ}_{\text{M5}} + \sigma_{\text{M5}},$$

(7.15)

with

$$S^c_{\text{M5}}' = \alpha_{\text{M5}}' \int d^6 x \sqrt{-g} e^{\frac{1}{4} \rho_1 + \frac{1}{2} \rho_2},$$

(7.16)

and $S^{BI}_{\text{M5}}$ and $S^{WZ}_{\text{M5}}$ the same in (6.9). One can verify that (7.15) satisfies the H-J equation of $I^{(M)}_7'$ under the assumption that the fields are constant on the fixed-time surface, up to the constraint (6.10), if

$$\alpha_{\text{M5}}' = 6R^{(S^3)} = 36, \quad \text{and} \quad \beta^2_{\text{M5}} = \gamma^2_{\text{M5}}.$$  

(7.17)

Following the relation between 11-d supergravity and type IIA supergravity [26], we define the fields in type IIA supergravity in terms of the fields in 11-d supergravity as follows.

$$h^{(M5)}_{\alpha \beta} = e^{-\frac{4}{3} \phi} h^{(IIA)}_{\alpha \beta},$$

$$\rho_1 = \frac{8}{3} \phi, \quad \rho_2 = \rho - \frac{4}{3} \phi,$$

$$A_3 = -C_3, \quad A_6 = -B_6.$$  

(7.18)

We rewrite $I^{(M)}_7'$ in terms of these new fields and obtain

$$I^{(NS5)}_7 = \int d^7 \xi \sqrt{-h} \left[ e^{-2 \phi + \frac{4}{3} \rho} \left( R_h + e^{-\frac{4}{3} \phi} R^{(S^3)} + 4 \partial_\alpha \phi \partial^\alpha \phi + \frac{3}{8} \partial_\alpha \rho \partial^\alpha \rho - 3 \partial_\alpha \phi \partial^\alpha \rho \right) 
\right. 
- \frac{1}{2} e^{\frac{4}{3} \rho} |F_4|^2
\left. - \frac{1}{2} e^{2 \phi + \frac{4}{3} \rho} |\tilde{H}_7|^2 \right],$$

(7.19)

where $F_4 = dC_3$ and $\tilde{H}_7 = dB_6 + \frac{1}{2} C_3 \wedge F_4$. This action is actually given by a consistent truncation of type IIA supergravity in which the ansatzes for the fields are

$$ds^2_{10} = h_{\alpha \beta}(\xi) d\xi^\alpha d\xi^\beta + e^{\frac{4}{3} \rho(\xi)} d\Omega_3,$$

$$\phi = \phi(\xi),$$

$$H_3 = -\frac{1}{3! \frac{7}{7}} e^{2 \phi + \frac{4}{3} \rho} \varepsilon_{\alpha_1 \cdots \alpha_7} \tilde{H}^{\alpha_1 \cdots \alpha_7}(\xi) \varepsilon_{\theta_1 \theta_2 \theta_3} d\theta_{i_1} \wedge d\theta_{i_2} \wedge d\theta_{i_3},$$

$$F_4 = \frac{1}{4!} F_{\alpha_1 \cdots \alpha_4}(\xi) d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_4}.$$  

(7.20)
Thus, by rewriting (7.15) in terms of the fields in type IIA supergravity, we obtain the NS 5-brane effective action (plus the cosmological term) that is a solution to the H-J equation of $I_7^{(NS5)}$, a reduction of type IIA supergravity. Clearly, this solution reproduces the supergravity solution of NS 5-brane.

8 Discussion and perspective

In this paper, we found that $S_{p+1}^{(0)}$ is a solution to the H-J equation of type IIA(IIB) supergravity and it reproduces the supergravity solution representing Dp-branes in a constant $B_2$ field. $S_{p+1}^{(0)}$ is not a complete solution to the H-J equation though it has some arbitrary constants. In other words, it should be obtained by making some of the arbitrary constants in a certain complete solution take specific values. It is interesting to see if $S_{p+1}^{(0)}$ can be generalized so that it includes more arbitrary constants and to look for a complete solution. It is relevant to investigate what class of supergravity solutions $S_{p+1}^{(0)}$ can reproduce. We verified that $S_{4}^{(0)}$ does not reproduce the black 3-brane solution, which preserves no supersymmetry, or the solution of the D3-brane with the wave, which preserves 8 supersymmetries, and that $S_{6}^{(0)}$ does not reproduce the solution of the D1-D5 bound state, which preserves 8 supersymmetries.

As is clear from the general argument in section 2, the quantities,

$$\frac{\delta S_{p+1}^{(0)}}{\delta F_{\mu\nu}^{(p+1)}} = \sqrt{-g} \pi_{B}^{\mu\nu} \quad \text{and} \quad \frac{\delta S_{p+1}^{(0)}}{\delta \beta_{p+1}} = S_{p+1}^{B} + S_{p+1}^{WZ},$$

(8.1)

are constant with respect to the time, where we take the sign in (5.9) such that $\beta_{p+1} = \gamma_{p+1}$. We verified this statement by an explicit calculation. We also obtain other conserved quantities from the $SL(2, R)$ transformed $S_{p+1}^{(0)}$ in type IIB supergravity, since it includes the continuous parameters of $SL(2, R)$. As is discussed in section 2, we can reduce the equations of motion to a set of the first order differential equations by using $S_{p+1}^{(0)}$. We may simplify these first order equations by using the conserved quantities so that we can answer the above question and/or find a new solution of supergravity. As we discussed in the introduction, it is also interesting to consider a reduction more complicated than that on higher dimensional sphere and obtain another solution to the H-J equation of supergravity, which should be
relevant to the gauge/gravity correspondence with less supersymmetries.

Finally, we make a remark on the case in which we perform a reduction on $T^{8-p}$ ($R^{8-p}$). Let us consider an ansatz for the metric

$$ds_{10}^2 = h_{\alpha\beta}(\xi)d\xi^\alpha d\xi^\beta + e^{\frac{1}{2}p(\xi)}dy^i dy^j,$$

(8.2)

where $\alpha$, $\beta$ run from 0 to $p + 1$, $i$ runs from 1 to $8 - p$ and the $y^i$ parametrize $T^{8-p}$ or $R^{8-p}$. We make ansatzes for the other fields similar to the ones in the reduction on $S^{8-p}$, and obtain a $(p + 2)$-dimensional gravity as a consistent truncation. It follows that (5.2) with $\alpha_{p+1} = 0$ is a solution to the H-J equation of this $(p + 2)$-dimensional gravity. This fact seems to imply that the vacuum to vacuum amplitude vanishes in the reduction on the flat manifolds.

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**Appendix A: Equations of motion and Bianchi identities**

In this appendix, we list explicitly the equations of motion and the Bianchi identity for type IIA(IIB) and 11-d supergravities. The equations of motion for type IIA supergravity are

$$R_{MN} + 2D_M D_N \phi - \frac{1}{4}H_{ML1L2}H_{N1L2} - \frac{1}{2}e^{2\phi}F_{ML}F_N^L - \frac{1}{12}e^{2\phi}\tilde{F}_{ML1L2L3}\tilde{F}_{N1L2L3}$$

$$+ \frac{1}{4}e^{2\phi}G_{MN}(|F_2|^2 + |\tilde{F}_4|^2) = 0,$$

(A.1)

$$R + 4D_M D^M \phi - 4\partial_M \phi \partial^M \phi - \frac{1}{2}|H_3|^2 = 0,$$

(A.2)
where $D_M$ represents the covariant derivative in ten dimensions. The Bianchi identities for type IIA supergravity are

\[ dH_3 = 0, \quad (A.6) \]
\[ dF_2 = 0, \quad (A.7) \]
\[ d\tilde{F}_4 + F_2 \wedge H_3 = 0. \quad (A.8) \]

The equations of motion for type IIB supergravity are

\[ R_{MN} = 2D_M D_N \phi - \frac{1}{4} H_{MLL} L_2 L_3^2 - \frac{1}{2} e^{2\phi} F_M F_N - \frac{1}{4} e^{2\phi} \tilde{F}_M L_2 L_3 \tilde{F}_N L_2 L_3 - \frac{1}{4} 2 \phi G_{MN}(|F_4|^2 + |\tilde{F}_3|^2) = 0, \quad (A.9) \]
\[ R + 4D_M D^M \phi - 4 \partial_M \phi \partial^M \phi - \frac{1}{2} |H_3|^2 = 0, \quad (A.10) \]
\[ DL(e^{-2\phi} H_{LMN}) + F_4 L \tilde{F}_{LMN} + \frac{1}{6} \tilde{F}_{L_1L_2L_3} \tilde{F}^{MN} L_1 L_2 L_3 = 0, \quad (A.11) \]
\[ DL F_L - \frac{1}{6} H_{L_1L_2L_3} L_1 L_2 L_3 \tilde{F}^{LMN} L_1 L_2 L_3 = 0, \quad (A.12) \]
\[ DL \tilde{F}^{LMN} - \frac{1}{6} H_{L_1L_2L_3} \tilde{F}^{MN} L_1 L_2 L_3 = 0, \quad (A.13) \]
\[ \tilde{F}^{LMN} L_1 L_2 L_3 \tilde{F}^{MN} L_1 L_2 L_3 = 0, \quad (A.14) \]

where $D_M$ represents the covariant derivative in ten dimensions again. The Bianchi identities for type IIB supergravity are

\[ dH_3 = 0, \quad (A.15) \]
\[ dF_1 = 0, \quad (A.16) \]
\[ d\tilde{F}_3 - F_1 \wedge H_3 = 0, \quad (A.17) \]
\[ d\tilde{F}_5 - \tilde{F}_3 \wedge H_3 = 0. \quad (A.18) \]

The equations of motion for 11-d supergravity are

\[ R_{MN} = \frac{1}{12} F^{(M)}_{M12L_3} F^{(N)}_{L_1L_2L_3} + G_{MN} \left( -\frac{1}{2} R + \frac{1}{4} |F^{(M)}_4|^2 \right) = 0, \quad (A.19) \]
\[ D_M F^{(M)}_{LM_1 M_2 M_3} - \frac{1}{2 (4!)^2} \varepsilon_{M_1 M_2 M_3} \varepsilon_{L_1 L_2 L_3 L_4 L_5 L_6 L_7 L_8} F^{(M)}_{L_1 L_2 L_3 L_4} F^{(M)}_{L_5 L_6 L_7 L_8} = 0, \]  
(A.20)

where \( D_M \) represents the covariant derivative in eleven dimensions. The Bianchi identities for 11-d supergravity is

\[ dF^{(M)}_4 = 0. \]  
(A.21)

### Appendix B: Ansatzes for the fields

In this appendix, we write down the ansatzes for the fields except the metric and the dilaton in the reduction of type IIA(IIB) supergravity on \( S^{8-p} \).

**\( p = 0 \)**

\[ F_2 = \frac{1}{2} F_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta. \]  
(B.1)

**\( p = 1 \)**

\[ H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \quad F_1 = F_\alpha(\xi) d\xi^\alpha, \quad \tilde{F}_3 = \frac{1}{3!} \tilde{F}_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma. \]  
(B.2)

**\( p = 2 \)**

\[ H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \quad F_2 = \frac{1}{2} F_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta, \quad \tilde{F}_4 = \frac{1}{4!} \tilde{F}_{\alpha_1 \ldots \alpha_4}(\xi) d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_4}. \]  
(B.3)

**\( p = 3 \)**

\[ H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \quad F_1 = F_\alpha(\xi) d\xi^\alpha, \quad \tilde{F}_3 = \frac{1}{3!} \tilde{F}_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \]
\[ \tilde{F}_5 = \frac{1}{5!} \tilde{F}_{\alpha_1 \ldots \alpha_5}(\xi) d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_5} \]
\[ \quad - \frac{1}{5! 5!} e^{4\rho/5} \varepsilon_{\alpha_1 \ldots \alpha_5} \tilde{F}_{\alpha_1 \ldots \alpha_5}(\xi) \varepsilon_{\theta_1 \ldots \theta_5} d\theta_{\theta_1} \wedge \cdots \wedge d\theta_{\theta_5}. \]  
(B.4)

**\( p = 4 \)**

\[ H_3 = \frac{1}{3!} H_{\alpha\beta\gamma}(\xi) d\xi^\alpha \wedge d\xi^\beta \wedge d\xi^\gamma, \quad F_2 = \frac{1}{2} F_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta, \]
\[ \tilde{F}_4 = \frac{1}{4!} \tilde{F}_{\alpha_1 \ldots \alpha_4}(\xi) d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_4} \]
\[ \quad + \frac{1}{4! 6!} e^{\rho/6} \varepsilon_{\alpha_1 \ldots \alpha_6} \tilde{F}_{\alpha_1 \ldots \alpha_6}(\xi) \varepsilon_{\theta_1 \ldots \theta_6} d\theta_{\theta_1} \wedge \cdots \wedge d\theta_{\theta_6}. \]  
(B.5)
\[ p = 5 \]

\[ H_3 = \frac{1}{3!} H_{\alpha \beta \gamma} (\xi) d \xi^\alpha \land d \xi^\beta \land d \xi^\gamma, \quad F_1 = F_\alpha (\xi) d \xi^\alpha, \]

\[ \tilde{F}_3 = \frac{1}{3!} \tilde{F}_{\alpha_1 \alpha_2 \alpha_3} (\xi) d \xi^{\alpha_1} \land d \xi^{\alpha_2} \land d \xi^{\alpha_3} \]

\[ + \frac{1}{3! \cdot 7!} \epsilon^{\beta_1 \beta_2 \beta_3 \gamma \alpha_1 \alpha_2 \alpha_3} \theta_{\alpha_1 \alpha_2 \alpha_3} (\xi) d \theta_{\beta_1} \land d \theta_{\beta_2} \land d \theta_{\beta_3}, \]

\[ \tilde{F}_5 = \frac{1}{5!} \tilde{F}_{\alpha_1 \cdots \alpha_5} (\xi) d \xi^{\alpha_1} \land \cdots \land d \xi^{\alpha_5} \]

\[ - \frac{1}{2! \cdot 3! \cdot 5!} \epsilon^{3 \beta_1 \beta_2 \beta_3 \alpha_1 \cdots \alpha_5} \tilde{F}_5 (\xi) d \xi^{\beta_1} \land d \xi^{\beta_2} \land d \theta_{\beta_1} \land d \theta_{\beta_2} \land d \theta_{\beta_3}. \quad (B.6) \]

\[ p = 6 \]

\[ H_3 = \frac{1}{3!} H_{\alpha \beta \gamma} (\xi) d \xi^\alpha \land d \xi^\beta \land d \xi^\gamma + \frac{1}{2!} d_\alpha (\xi) \epsilon_{\theta_{\alpha_1} \alpha_2} d \xi^\alpha \land d \theta_{\alpha_1} \land d \theta_{\alpha_2}, \]

\[ F_2 = \frac{1}{2!} \tilde{F}_{\alpha_1 \alpha_2} (\xi) d \xi^{\alpha_1} \land d \xi^{\alpha_2} - \frac{1}{2! \cdot 8!} \epsilon^{\beta_1 \beta_2 \beta_3 \gamma_1 \gamma_2 \alpha_1 \alpha_2} \tilde{F}_{\alpha_1 \alpha_2} (\xi) d \theta_{\beta_1} \land d \theta_{\beta_2} \land d \theta_{\beta_3}, \]

\[ \tilde{F}_4 = \frac{1}{4!} \tilde{F}_{\alpha_1 \cdots \alpha_4} (\xi) d \xi^{\alpha_1} \land \cdots \land d \xi^{\alpha_4} \]

\[ + \frac{1}{2! \cdot 2! \cdot 6!} \epsilon^{\beta_1 \beta_2 \beta_3 \alpha_1 \cdots \alpha_5 \beta_4} \tilde{F}_4 (\xi) d \xi^{\beta_1} \land d \xi^{\beta_2} \land d \theta_{\beta_1} \land d \theta_{\beta_2}. \quad (B.7) \]

\[ p = 7 \]

\[ H_3 = \frac{1}{3!} H_{\alpha \beta \gamma} (\xi) d \xi^\alpha \land d \xi^\beta \land d \xi^\gamma + \frac{1}{2!} d_{\alpha \beta \gamma} (\xi) d \xi^{\alpha_1} \land d \alpha^{\alpha_2} \land d \theta_1, \]

\[ \tilde{F}_1 = \tilde{F}_\alpha (\xi) d \xi^{\alpha_1} - \frac{1}{9!} \epsilon^{\beta_1 \beta_2 \beta_3 \alpha_1 \cdots \alpha_9} \tilde{F}_{\alpha_1 \cdots \alpha_9} (\xi) d \theta_1, \]

\[ \tilde{F}_3 = \frac{1}{3!} \tilde{F}_{\alpha_1 \alpha_2 \alpha_3} (\xi) d \xi^{\alpha_1} \land d \xi^{\alpha_2} \land d \xi^{\alpha_3} + \frac{1}{2! \cdot 7!} \epsilon^{\beta_1 \beta_2 \beta_3 \gamma_1 \gamma_2 \alpha_1 \cdots \alpha_7} \tilde{F}_4 (\xi) d \xi^{\beta_1} \land d \xi^{\beta_2} \land d \theta_1 \]

\[ \tilde{F}_5 = \frac{1}{5!} \tilde{F}_{\alpha_1 \cdots \alpha_5} (\xi) d \xi^{\alpha_1} \land \cdots \land d \xi^{\alpha_5} \]

\[ - \frac{1}{4! \cdot 5!} \epsilon^{\beta_1 \beta_2 \beta_3 \beta_4 \alpha_1 \cdots \alpha_5 \beta_5} \tilde{F}_5 (\xi) d \xi^{\beta_1} \land \cdots \land d \xi^{\beta_4} \land d \theta_1. \quad (B.8) \]

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