ON ELKIES SUBGROUPS OF ℓ-TORSION POINTS IN ELLIPTIC CURVES DEFINED OVER A FINITE FIELD

par

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1. Introduction

Let $K$ be a finite field with $q$ elements and $E$ be an elliptic curve over $K$ given by a plane equation of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

(1.1)

where the coefficients $a_1$, $a_2$, $a_3$, $a_4$ and $a_6$ are elements of $K$. For any field $L$ such that $K \subset L$, we denote by $E(L)$ the set of $L$-points of $E$, i.e. the set of solutions in $L$ of Equation (1.1), plus the additional point at infinity $O$ with homogeneous coordinates $(0 : 1 : 0)$. The curve $E/K$ has a structure of commutative algebraic group with neutral element $O$, derived from the secant and tangent rules. Its order is equal to $q + 1 - t$ with $t \in \mathbb{Z}$ such that $|t| \leq 2\sqrt{q}$.

We are interested in the determination of $\ell$-torsion points of $E$, that is the set $E[\ell]$ of points $P$ of $E(K)$ such that $\ell P = O$ for prime integers $\ell$, distinct from $p$. This
group is isomorphic to \( \mathbb{Z}/\ell \mathbb{Z} \times \mathbb{Z}/\ell \mathbb{Z} \) (cf.\([\text{17}]\) p. 89), its cardinal is thus \( \ell^2 \). In fact, the multiplication by \( \ell \) is given by a rational transformation of \( \mathbb{P}^2(\mathbb{K}) \), of degree \( \ell^2 \), of the form \( (x : y : z) \mapsto (X_\ell(x, y, z) : Y_\ell(x, y, z) : Z_\ell(x, y, z)) \) where \( X_\ell, Y_\ell \) and \( Z_\ell \) are three homogeneous polynomials of degree \( \ell^2 \) and \( \ell \)-torsion points are explicitly given by \( Z_\ell(x, y, z) = 0 \). Excluding \( O \), this equation can be easily transformed into an equality of the form \( f_\ell(x) = 0 \) where \( f_\ell \) is an univariate polynomial of degree \( (\ell^2 - 1)/2 \), called the \( \ell \)-th division polynomial.

The improvements by Atkin and Elkies to Schoof’s algorithm for counting points on elliptic curve stem from the fact that when the principal ideal \( (\ell) \) splits in the imaginary quadratic field \( \mathbb{Q}(\sqrt{\ell^2 - 4q}) \), in half the cases thus, there exists two subgroups of degree \( \ell \) in \( E[\ell] \) defined in a degree \( \ell - 1 \) extension of \( \mathbb{K} \). Such an integer \( \ell \) is called an Elkies prime. In this work, we more precisely focus on algorithmic efficient ways to compute degree \( (\ell - 1)/2 \) polynomials the roots of which are abscissas of points contained in such subgroups. We call these subgroups, and these degree \( (\ell - 1)/2 \) polynomials over \( \mathbb{K} \), \( \ell \)-th Elkies subgroups, and \( \ell \)-th Elkies polynomials.

Our main result, where we classically denote by \( \phi_1(x_1, \ldots, x_k) = O(\phi_2(x_1, \ldots, x_k)) \) functions \( \phi_1 \) and \( \phi_2 \) such that there exists an integer \( k \) with \( \phi_1(x_1, \ldots, x_k) = O(\phi_2(x_1, \ldots, x_k) \log^k \phi_2(x_1, \ldots, x_k)) \), is as follows.

**Theorem 1. —** Let \( E \) be an elliptic curve defined over a finite field \( \mathbb{K} \) with \( q \) elements and \( \ell \) be an Elkies prime, distinct from the characteristic of \( \mathbb{K} \), then there exists an algorithm which computes an \( \ell \)-th Elkies polynomial at cost \( \tilde{O}(\ell \max(\ell, \log q)^2) \) bit operations and space.

This problem is closely related to the problem of computing separable isogenies of degree \( \ell \) between two elliptic curves since an application of Velu’s formulas \([\text{19}]\) with starting point such polynomials yields an isogeny. Especially, counting points on elliptic curves first raised interest for such computations. But isogenies now play a role in numerous other fields, for instance to protect elliptic curve cryptographic devices against physical side attacks \([\text{18}]\), to improve Weil descent to calculate elliptic discrete logarithms \([\text{10}]\), to decrease the complexity of computing discrete logarithms in some family of finite fields \([\text{7}]\), to exhibit normal basis in finite field extensions \([\text{6}]\), etc.

We first recall in Section 2 the complexity of the algorithms known to solve this problem. In Section 3 we focus on the fastest algorithm in finite fields of large characteristic published so far, due to Bostan, Morain, Salvy and Schost \([\text{2}]\). We then show in Section 4 how we can combine this algorithm with the \( p \)-adic approach introduced by Joux and Lercier in \([\text{11}]\) to get a fast algorithm in any finite field and we clarify that we need a \( p \)-adic precision of only \( O(\log^2 \ell / \log p) \). A detailed example is given in Section 5.
2. Related work

For the sake of simplicity, we restrict ourself to finite fields \( K \) of characteristic larger than three, and to prime integers \( \ell > 2 \). In this case, an elliptic curve is simply given by a plane equation of the form \( y^2 = x^3 + ax + b \). Its discriminant, always non zero, is equal to \( \Delta_E = -16(4a_4^3 + 27a_6^2) \) and its \( j \)-invariant is equal to \( j_E = -12^3(4a_4)^3/\Delta_E \).

2.1. Naive approach. — \( \ell \)-th Elkies polynomials are factors of the \( \ell \)-th division polynomial \( f_\ell \). Therefore, a naive approach consists in computing \( f_\ell \), which can be done at cost \( O(\ell^2 \log q) \) elementary operations thanks to a “Square and Multiply” method \( [17] \), and then in factorizing it with cost \( O(\ell^{1.815 \times 2} \log^2 q) \) \( [16] \). This algorithm needs a total of \( O(\ell^4 \log^2 q) \) bit operations.

2.2. Schoof-Elkies-Atkin framework. — Let \( \pi_E \) be the Frobenius endomorphism of \( E \). Its restriction to \( E[\ell] \), seen as a \( \mathbb{F}_\ell \)-vector space of dimension two, is still an endomorphism. When \( \ell \) is an Elkies prime, its eigenspaces correspond to \( \ell \)-th Elkies subgroups \( C \) of \( E[\ell] \) and from each \( C \) one can construct an isogeny of degree \( \ell \) between \( E \) and the elliptic curve \( E' = E/C \), defined over \( K \).

The following algorithm takes advantage of these facts.

**Step 1** : Compute the modular polynomial of degree \( \ell \), \( \Phi_\ell (X, Y) \), equation of the modular curve \( X_0(\ell) \). This is a bivariate symmetric polynomial, of degree \( \ell + 1 \) in \( X \) and \( Y \), whose coefficients are integers of \( \hat{O}(\ell) \) bits (cf. \([4]\)). \( j \)-invariants of \( \ell \)-isogenous elliptic curves are roots of \( \Phi_\ell (X, Y) \).

**Step 2** : Compute roots \( j_1 \) and \( j_2 \) of \( \Phi_\ell (X, j_\ell) \).

**Step 3** : Compute a normalized Weierstrass equation for elliptic curves of \( j \)-invariants \( j_1 \) and \( j_2 \), and the sum \( p_1 \) of the abscissas of points in the kernel of the isogeny, using the polynomials \( \Phi_\ell, \partial \Phi_\ell / \partial X, \partial \Phi_\ell / \partial Y, \partial^2 \Phi_\ell / \partial X^2, \partial^2 \Phi_\ell / \partial X \partial Y, \partial^2 \Phi_\ell / \partial Y^2 \) (cf. \([15]\)).

**Step 4** : Compute from each isogenous curve, a \( \ell \)-th Elkies polynomial thanks to the kernel of the corresponding isogeny.

The complexity of the method comes now.

**Step 1** : The modular polynomial \( \Phi_\ell (X, Y) \) has \( O(\ell^3) \) coefficients, each with about \( \hat{O}(\ell) \) bits. There exists methods to compute this polynomial at cost quasi-linear in its size, i.e. \( \hat{O}(\ell^3) \) bit operations (cf. \([9]\)). We need to reduce this polynomial modulo \( p \), that is \( \hat{O}(\ell^3) \) bit operations too. The result is then of size \( O(\ell^2 \log p) \) bits.

**Step 2** : With the help of Horner’s method, the evaluation of \( \Phi_\ell (X, Y) \) at \( j_E \) costs \( \hat{O}(\ell^2 \log q) \) bit operations. In order to compute roots of the resulting degree \( \ell + 1 \) polynomial, we have first to compute its gcd with \( X^q - X \), that is \( \hat{O}(\ell \log^2 q) \) bit operations (cf. \([12]\)). We obtain a degree 2 polynomial whose roots can then be found with negligible cost.

**Step 3** : The computations of the derivatives of \( \Phi_\ell \) and their evaluations can be done at cost \( \hat{O}(\ell^2 \log q) \) bit operations.
Step 4: Here, we have to distinguish several cases.

– In finite fields of large characteristic, the best algorithm known so far to compute isogenies is due to Bostan et al. \[2\] and takes time $\tilde{O} (\ell \log q)$ bit operations.

– In finite fields of small but fixed characteristic, the best algorithm known is due to Couveignes \[5\] and needs $\tilde{O} (\ell^2 \log q)$ bit operations (but the contribution of $p$ in the $\tilde{O}$ complexity constant is exponential in $\log p$).

– In between, that is finite fields of small but non-fixed characteristic, the best algorithm is due to Joux and Lercier \[11\] and needs $\tilde{O} ((1 + \ell/p) \ell^2 \log q)$ bit operations.

The best total complexity is thus equal to $\tilde{O} (\ell \max(\ell, \log q)^2)$, achieved in finite fields of large characteristic. But, in finite fields of small characteristic, the complexity can be as large as $\tilde{O} (\ell^3 \log q)$ bit operations when $\ell \gg p$.

This work yields an algorithm of same complexity as in the large characteristic case without any limitation on the characteristic or the degree of the base field $K$.

3. The large characteristic case

In order to get an algorithm with good complexity in finite fields of small characteristic too, we first reformulate the algorithm of Bostan et al. in such a way that its extension in the $p$-adics is more easily studiable. The general strategy is the same except that we take into account some specificities of the involved differential equation in the resolution. As a result, we obtain a precise and compact algorithm (cf. Algorithm 1).

3.1. Differential equation. — In a field $K$ of characteristic larger than three, an isogeny between two elliptic curves, $E : y^2 = x^3 + a_4 x + a_6$ and $E' : y^2 = x^3 + a'_4 x + a'_6$, can be given by

\[
I(x, y) = \left( \frac{N(x)}{D(x)} \right)^{1/2} \left( \frac{N(x)}{D(x)} \right) \left( \frac{N(x)}{D(x)} \right)' ,
\]

where $N$ and $D$ are unitary polynomials of degree $\ell$ and $\ell - 1$. When $c$ is equal to one, the isogeny is said to be normalized. This is in particular the case in the Schoof-Elkies-Atkin framework.

If we now state that the image of a point of $E$ by $I$ is on $E'$, we get the following differential equation

\[
(x^3 + a_4 x + a_6) \left( \frac{N(x)}{D(x)} \right)^2 = \left( \frac{N(x)}{D(x)} \right) \left( \frac{N(x)}{D(x)} \right)' = \left( \frac{N(x)}{D(x)} \right)^3 + a'_4 \left( \frac{N(x)}{D(x)} \right) + a'_6.
\]

This equation can be solved with a Taylor series expansion of $N(x)/D(x) - x$ in $1/x$ for $1/x$ close to 0. The relations obtained thanks to Equation (3.1) enable to compute by recurrence each coefficient in turn, if the first coefficients are known. It is then possible to recover $N$ and $D$ with the help of Berlekamp-Massey’s algorithm, or one.
of its optimized variant. In [2], one takes advantage of a Newton algorithm so that the number of coefficients computed at each iteration doubles.

More precisely, let $S$ be defined by

$$S(x) = \sqrt{D(1/x^2)/N(1/x^2)}, \quad \text{or equivalently} \quad \frac{N(x)}{D(x)} = \frac{1}{S(1/\sqrt{x})^2}.$$  

Equation (5.1) becomes

$$(a_6 x^6 + a_4 x^4 + 1)S'(x)^2 = 1 + a'_4 S(x)^4 + a_6'' S(x)^6.$$  

At the infinity, $N(x)/D(x)$ has a series expansion of the form $x + O(1)$. We thus have $S(x) = x + O(x^3)$ and this knowledge is finally enough to completely recover $N(x)/D(x)$.

### 3.2. Resolution

We consider more generally equations of the form $S'^2 = G \cdot (H \circ S)$. In Equation (3.1), we have for instance $H(z) = a'_6 z^6 + a'_4 z^4 + 1$ and $G(x) = 1/(a_6 x^6 + a_4 x^4 + 1)$. We now look for a solution modulo $x^\mu$, where $\mu$ is any integer given in input. The way to solve this equation is first to assume that we know the solution modulo $x^d$ and then, thanks to a Newton iteration, to obtain a solution modulo $x^{2d}$. After roughly $\log \mu$ such iterations, one gets the full solution.

We now present a compact algorithm for this task. Its complexity can be easily established, it is equal to $O(\mu \log q)$ bit operations. Its correctness is slightly more difficult to prove and we delay it to Appendix A.

**Algorithm 1** Solving equation $S'^2 = G \cdot (H \circ S)$, $S(0) = \alpha$ and $S'(0) = \beta$.

**Input:** $\mu \in \mathbb{N}, (\alpha, \beta) \in K^2$, $H \in K[z], G \in K[x]$  

**Output:** $S \in K[x]$, a solution of the differential equation modulo $x^\mu$

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\begin{align*}
d &\leftarrow 2, \quad U &\leftarrow 1/\beta, \quad J &\leftarrow 1, \quad V &\leftarrow 1 \\
S &\leftarrow \alpha + \beta x + \left[(G'(0) + H'(\alpha) \beta^3)/(4\beta)\right] x^2 \\
\text{while} (d < \mu - 1) \text{ do} \\
&U \leftarrow U \cdot (2 - S' \cdot U) \mod x^d \\
&V \leftarrow (V + J \cdot (H \circ S) \cdot (2 - V \cdot J)) / 2 \mod x^d \\
&J \leftarrow J \cdot (2 - V \cdot J) \mod x^d \\
&S \leftarrow S + V \cdot \int (G \cdot (H \circ S) - S'^2) \cdot (U \cdot J / 2) \, dx \mod x^{\min(2d+1, \mu)} \\
d &\leftarrow 2d \\
\text{end while} \\
\text{return } S
\end{align*}
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**Proposition 3.1.** Let $(\alpha, \beta) \in K^2$ where $K$ is a finite field of characteristic $p$, let $G$ be a formal series defined over $K$, let $H$ be a polynomial defined over $K$ such that $H(\alpha) = 1$ and $\beta^2 = G(0) \neq 0$. Let $\mu \in \{1, \ldots, p\}$, then Algorithm 1 computes a Taylor series (modulo $x^\mu$) of the solution $S$ of the differential equation

$$S'(x)^2 = G(x) H(S(x)), \quad S(0) = \alpha, \quad S'(0) = \beta.$$
3.3. Full algorithm. — We first compute $G(x) = 1/(a_6 x^6 + a_4 x^4 + 1)$ modulo $x^{2\ell-1}$ thanks to the classical iterative following formula, $G_1(x) = 1$, $G_{2d}(x) = G_d(x) \left(2 - G_d(x) \cdot (a_6 x^5 + a_4 x^4 + 1)\right) \mod x^{2d}$. We then apply Algorithm 1 to $G(x)$ and $H(z) = a_6' z^5 + a_4' z^4 + 1$ with $\mu = 4\ell$, $\alpha = 0$ and $\beta = 1$.

The obtained solution $S$ is odd, we define from it

$$T(x) = \sum_{i=0}^{2\ell-1} t_i x^i, \text{ where } \forall i \in \{0, \ldots, 2\ell - 1\}, \ t_i = s_{2i+1}.$$

We denote by $R(x)$ the inverse of the square of $T(x)$, modulo $x^{2\ell}$, with the same inverse formulas as those used for $G$. We then have

$$\frac{N(x)}{D(x)} = x R\left(\frac{1}{x}\right), \text{ i.e. } R(x) = \frac{x^{\ell} N(1/x)}{x^{\ell-1} D(1/x)}.$$

Applying Berlekamp-Massey algorithm [1, 13, 8] or one of its optimized variant [3, 14] to $R$ yields $D$ and the searched $\ell$-th Elkies polynomial is equal to the square root of $D$.

4. Extension to any finite field

To extend the Schoof-Elkies-Atkin framework in any characteristic, the techniques developed in [11] give the general idea: to use the $p$-adics to authorize divisions by the characteristic $p$ of the field. These divisions make it possible to use in any finite field algorithms primarily designed in large characteristic. There exists one main obstacle with this approach. Calculations in the $p$-adics imply losses of precision at the time of divisions by $p$. It is thus necessary to anticipate a sufficient precision, which results in an increase in the size of the handled objects.

One could hope to perform this lift in the $p$-adics only in the last stage of the algorithm, i.e. for the calculation of the isogeny. It is actually not possible because fast algorithms for computing isogenies need normalized models for the isogenous curves.

It is thus necessary to lift in the $p$-adics from the very beginning of the algorithm. It is exactly what is done in [11], with a $p$-adic precision linear in $\ell$. Instead, we consider here the techniques of [2], and one shows that the necessary $p$-adic precision can be brought back to only $O(\log^2 \ell / \log p)$. The total complexity of the algorithm is then similar to the one of the large characteristic case, that is $\tilde{O}(\ell \max(\ell, \log q)^2)$.

4.1. Lifting curves and isogenies. — One starts by lifting arbitrarily the curve $E$ in the $p$-adics. Any coefficient $\bar{a}_4$ and $\bar{a}_6$ such that $\bar{a}_4 = a_4 \mod p$ and $\bar{a}_6 = a_6 \mod p$ is appropriate and one works on the elliptic curve $\bar{E}/\mathbb{Q}_q$ with model $y^2 = x^3 + \bar{a}_4 x + \bar{a}_6$.

The computation of the $j$-invariant $\bar{j}_E$ of the curve $\bar{E}$, of the solutions $\bar{j}_1$ and $\bar{j}_2$ of the equation $\phi(x, \bar{j}_E) = 0$, as well as Weierstrass models of the corresponding curves $\bar{E}_1$ and $\bar{E}_2$, proceeds exactly as in the SEA framework. The curves $\bar{E}_1$ and $\bar{E}_2$ are $\ell$-isogenous with the curve $\bar{E}$, and the isogenies can be calculated as in the large characteristic case.
Projection $E_1$ of the curve $E$ on the base field $K$ is $\ell$-isogenous with $E$, and the connecting isogeny is the projection on the base field of the isogeny connecting $E$ to $E_1$. It is the same for $E_2$. It is thus enough to project the denominators of the isogenies on $K$ to identify the required factors of the $\ell$-th division polynomial of $E$.

4.2. $p$-adic computations. — From now on, we are interested in the $p$-adic precision of the lift of the elliptic curve $E$. This precision must be large enough so that at the end of the resolution of the differential equation with Algorithm $[\text{4}]$, the result $S$ can be reduced in $K$.

To this purpose, we need first some definitions.

**Definition.** — For any positive integer $r$, one defines $\text{PDiv}(p, r)$ by the largest power of $p$ which divides $r$, $\text{PDiv}(p, r) = \max \{k \in \mathbb{N} | p^k \text{ divides } r \}$.

We denote by $\text{Loss}(p, \ell)$ the sum $\sum_{1 \leq i < \log_2(4\ell-1)} \text{LpLoss}(p, \ell, i)$, where

$$\text{LpLoss}(p, \ell, i) = \max \{ \text{PDiv}(p, r) | 2^i + 1 \leq r \leq \min(2^{i+1}, 4\ell - 1) \}.$$  

The following lemma relates the precision needed to the function $\text{Loss}$.

**Lemma 4.1.** — Let $\mu$ be the $p$-adic precision of the coefficients $\bar{a}_4$ and $\bar{a}_6$, then when $\mu > \text{Loss}(p, \ell)$ the polynomials $U$, $V$, $J$ and $S$ computed in Algorithm $[\text{4}]$ have $p$-adic integer coefficients. Furthermore the precision of the result $S$ is at least equal to $(\mu - \text{Loss}(p, \ell))$.

**Démonstration.** — One proves this theorem by recurrence on $j$, the number of iterations of the loop “while” in Algorithm $[\text{4}]$. We assume that at rank $j$, $0 \leq j < \log_2(4\ell - 1)$, the polynomials $U$, $V$, $J$ and $S$ have $p$-adic integer coefficients and that their precision is at least equal to $\mu - \sum_{1 \leq i \leq j} \text{LpLoss}(p, \ell, i)$.

**Initialization.** In input of the algorithm, we have $\alpha = 0$, $\beta = 1$, $H(z) = \bar{a}_6' z^6 + \bar{a}_4' z^4 + 1$ and $G(x) = 1/(\bar{a}_6 x^6 + \bar{a}_4 x^4 + 1)$. The elements $\bar{a}_4$, $\bar{a}_6$, $\bar{a}_4'$ and $\bar{a}_6'$ are integers of precision $\mu$ and thus $G$ and $H$ are of precision $\mu$ too (no division by $p$ occurs in the computation of $G$). The same is true for $U$, $V$, $J$ and $S$.

**Heredity.** Let $j < \log_3(4\ell - 1)$, we suppose the assumption true at rank $j - 1$. At the $j$th iteration, polynomials $U$, $V$ and $J$ are updated via additions, multiplications, derivations and compositions of the values of $U$, $V$, $J$ and $S$ before the entry in the loop. All these operations preserve the precision, polynomials $U$, $V$ and $J$ have thus $p$-adic integer coefficients with precision at least equal to $\mu - \sum_{1 \leq i \leq j - 1} \text{LpLoss}(p, \ell, i)$.

For $S$, except the integral operation, the calculations preserve the precision. Coefficients of the series after the integral operation are inverses of degrees between $2^j + 1$ and $\min(2^{j+1}, 4\ell - 1)$. The largest power of $p$ by which we carry out a division is thus $\text{LpLoss}(p, \ell, j)$. The absolute precision of the coefficients of $S$ is thus higher or equal to $\mu - \sum_{1 \leq i \leq j - 1} \text{LpLoss}(p, \ell, i)$. Furthermore, since this precision is positive, each coefficient of $S$ is a lift of the coefficient of the series deduced from the isogeny over $K$, and these coefficients are $p$-adic integers.
To minimize the loss of precision, we may use the additional fact that \( S \) is odd. We thus have to consider only coefficients of odd degree in the algorithm and the loss of precision in the loop of the algorithm becomes

\[
L_p^\ell(i, \ell) = \max \left\{ \text{PDiv}(p, 2r + 1) / 2^r \leq r \leq \min(2^i - 1, 2\ell - 1) \right\}.
\]

Lemma 4.2 yields a clear asymptotic bound on the loss of precision stated in Lemma 4.1.

**Lemma 4.2.** — We have \( \text{Loss}(p, \ell) = O \left( \log \frac{2^\ell}{\log p} \right) \).

**Démonstration.** — For all \( i < \log_2(4\ell - 1) \), \( L_p^\ell(p, \ell, i) \) is the largest power of \( p \) which divides a range of integers, at most equal to \( 2^i + 1 \), we have therefore \( L_p^\ell(p, \ell, i) \leq \log_2(2^i + 1) \), and

\[
\text{Loss}(p, \ell) \leq \log_2 \left( \frac{\sum_{1 \leq i < \log_2(4\ell - 1)} (i + 1)}{\log_2(4\ell - 1)(\log_2(4\ell - 1) + 1)} \right) \\
\leq \frac{(\log_2(4\ell - 1) + 1)^2}{\log_2 p}.
\]

We finally can state our main result.

**Proposition 4.1.** — A \( p \)-adic precision of \( O(\log^2 \ell / \log p) \) is asymptotically enough to compute a \( \ell \)-th Elkies polynomial. The total computation needs \( \tilde{O}(\ell \max(\ell, \log q)^2) \) bit operations.

**Démonstration.** — Computations performed in the Schoof-Elkies-Atkin framework, especially calls to Algorithm 1, are realized in the \( p \)-adics with precision at most \( O(\log^2 \ell / \log p) \). This precision does not modify the \( \tilde{O} \) complexities of the large characteristic case and we still have in the \( p \)-adic case a total complexity equal to \( \tilde{O}(\ell \max(\ell, \log q)^2) \) bit operations.

## 5. Experiments

We have implemented this algorithm in the computer algebra system MAGMA. Thanks to it, we were able to observe that the bound on the precision stated in Proposition 4.1 is tight. We can illustrate the method with an example too.

### 5.1. \( p \)-adic precision.

Figure 1 shows the evolution of the precision when \( p \) and \( \ell \) vary. The “The(oretical)” bound mentioned corresponds to \( \text{Loss}(p, \ell) \) calculations. The “Obs(erved)” bound is what seems necessary at the time of calculations (checked on some examples).

It turns out that the precision observed in practice is near the theoretical bound. For many values of \( \ell \), a gap between the theoretical bound and the observed bound appears, but this difference remains quite small.
5.2. Example. — Let $E : y^2 = x^3 + x + 4$ be defined over $\mathbb{F}_5$ and $\ell = 11$.

We first need to compute an upper bound for the 5-adic precision,

$$\text{LpLoss}(5, 11, 1) = 0, \quad \text{LpLoss}(5, 11, 2) = 1, \quad \text{LpLoss}(5, 11, 3) = 1,$$

$$\text{LpLoss}(5, 11, 4) = 2, \quad \text{LpLoss}(5, 11, 5) = 1.$$ We find $\text{Loss}(5, 11) = 5$ and the 5-adic precision is thus 6.

A 5-adic lift of the curve is $y^2 = x^3 + x + 4$. With the help of the classical 5-th modular polynomial $\Phi_{11}$, we find that a 11-isogenous curve is given by $y^2 = x^3 - 7329x - 3934 + O(5^6)$.

We can now compute the series $\tilde{G}(x)$ modulo $x^{44}$.

$$\tilde{G}(x) = 4374x^{42} + 4298x^{40} - 2331x^{38} - 4417x^{36} + 3936x^{34} + 3505x^{32}$$

$$+ 228x^{30} - 1041x^{28} - 616x^{26} + 97x^{24} + 236x^{22} + 95x^{20} - 48x^{18}$$

$$- 47x^{16} - 12x^{14} + 15x^{12} + 8x^{10} + x^8 - 4x^6 - x^4 + 1 + O(5^6) \mod x^{43}.$$ A solution of the differential equation based on $\tilde{G}(x)$ and $\tilde{H}(z) = \tilde{a}_0 z^6 + \tilde{a}_1 z^4 + 1$ is then given modulo $x^{44}$ by

$$S(x) = - (2 + O(5)) x^{43} + (2 + O(5)) x^{41} - (1 + O(5)) x^{39} + (8 + O(5^2)) x^{37}$$

$$- (1 + O(5)) x^{35} + (O(5^2)) x^{33} + (O(5^2)) x^{31} - (10 + O(5^2)) x^{29} - (7 + O(5^2)) x^{27}$$

$$- (1 + O(5)) x^{25} + (192 + O(5^2)) x^{23} + (125 + O(5^4)) x^{21} + (293 + O(5^4)) x^{19}$$

$$+ (4 + O(5^5)) x^{17} - (161 + O(5^5)) x^{15} - (611 + O(5^5)) x^{13} + (211 + O(5^5)) x^{11}$$

$$- (1494 + O(5^5)) x^9 + (1058 + O(5^5)) x^7 - (733 + O(5^5)) x^5 + (O(5^5)) x^3 + (1 + O(5^6)) x.$$
and modulo 5, we find

\[ T(x) = 3x^{21} + 2x^{20} + 4x^{19} + 3x^{18} + 4x^{17} + 3x^{16} + 3x^{15} + 4x^{12} + 2x^{11} \\
+ 3x^9 + 4x^8 + 4x^7 + 4x^6 + x^5 + x^4 + 3x^3 + 2x^2 + 1 \mod x^{22}. \]

We have \( R(x) = 1 / T(x)^2 \mod x^{27} \), that is

\[ R(x) = 2x^{20} + 2x^{19} + 3x^{18} + x^{16} + 2x^{15} + 3x^{14} + x^{13} + 3x^{12} + 2x^{11} \\
+ x^{10} + x^8 + 3x^7 + x^6 + 4x^5 + 4x^3 + x^2 + 1 \mod x^{22}. \]

The rebuilding of the rational fraction corresponding to \( R \) gives

\[ R(x) = 3x^{11} + x^9 + x^8 + x^7 + x^6 + 3x^5 + 2x^4 + 3x^3 + 2x^2 + 2x + 1 \mod x^{22}. \]

One reverses the order of the coefficients of the denominator to obtain

\[ D(x) = x^{10} + 2x^9 + x^8 + 2x^7 + 3x^6 + 3x^5 + 3x^4 + x^3 + x^2 + x + 1. \]

The \( \ell \)-th Elkies polynomial is then

\[ \sqrt{D(x)} = x^5 + x^4 + x^2 + 3x + 1. \]

Références

[1] E.R. Berlekamp. *Algebraic Coding Theory*. McGraw-Hill, 1968.

[2] A. Bostan, F. Morain, B. Salvy, and É. Schost. Fast algorithms for computing isogenies between elliptic curves. *Mathematics of Computation*, 2008. To appear.

[3] R.P. Brent, F.G. Gustavson, and D.Y.Y. Yun. Fast solution of Toeplitz systems of equations and computation of Padé approximants. *Journal of Algorithms*, 1:259–295, 1980.

[4] H. Cohen. On the coefficients of the transformation polynomials for the elliptic modular function. *Math. Proc. Cambridge Philos. Soc.*, 95:389–402, 1984.

[5] J.-M. Couveignes. Computing \( l \)-isogenies with the \( p \)-torsion. In *Proc. of the 2nd Algorithmic Number Theory Symposium (ANTS-II)*, volume 1122, pages 59–65, 1996.

[6] J.-M. Couveignes and R. Lercier. Elliptic periods for finite fields. [arXiv:0802.0163](http://arxiv.org/abs/0802.0163), 2008.

[7] J.-M. Couveignes and R. Lercier. Galois invariant smoothness basis. *Series on number theory and its application*, 5:154–179, 2008.

[8] J.-L. Dornstetter. On the equivalence between Berlekamp’s and Euclid’s algorithms. *IEEE Transactions on Information Theory*, 33(3):428–431, 1987.

[9] A. Enge. Computing modular polynomials in quasi-linear time. [arXiv:0704.3177](http://arxiv.org/abs/0704.3177), 2007.

[10] S.D. Galbraith, F. Hess, and N.P. Smart. Extending the GHS Weil Descent Attack. In *Proc. of Advances in Cryptology – Eurocrypt’2002*, volume 2332, pages 29–44, 2002.

[11] A. Joux and R. Lercier. Counting points on elliptic curves in medium characteristic. *Cryptography ePrint Archive 2006/176*, 2006.

[12] R. Lidl and H. Niederreiter. *Finite Fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, 1983.

[13] J.L. Massey. Shift-register synthesis and BCH decoding. *IEEE Transactions on Information Theory*, 15(1):122–127, 1969.

[14] V.Y. Pan. New techniques for the computation of linear recurrence coefficients. *Finite Fields and Their Applications*, 6(1):93–118, 2000.
Appendix A

Proof of Proposition 3.1

Let \( d \) be a non-zero even integer, we assume that we know a solution of the differential equation modulo \( x^{d+1} \). We thus have

\[
S_d'{}^2 = G \cdot (H \circ S_d) \mod x^d, \quad S_d(0) = \alpha, \quad S_d'(0) = \beta.
\]

Let \( S_{2d} = S_d + A_{2d} \) be a solution modulo \( x^{2d+1} \), with \( x^{d+1} \) dividing \( A_{2d} \), therefore \((S_d' + A_{2d}')^2 = G \cdot (H \circ (S_d + A_{2d})) \mod x^{2d}\). This yields a linear differential equation in \( A_{2d} \).

\[
2S_d' \cdot A_{2d}' - G \cdot (H' \circ S_d) \cdot A_{2d} = G \cdot (H \circ S_d) - S_d'{}^2 \mod x^{2d}.
\]

With initial condition \( A_{2d}(0) = 0 \), a solution of this equation is

\[
A_{2d} = \frac{1}{J_{2d}} \int \frac{(G \cdot (H \circ S_d) - S_d'{}^2)}{2S_d'} \cdot J_{2d} \, dx \mod x^{2d+1},
\]

where \( J_{2d} = \exp \left( - \frac{1}{2} \int \frac{G \cdot (H' \circ S_d)}{S_d'} \, dx \right) \mod x^{2d+1} \).

From Eq. \((A.1)\), we know that \((G \cdot (H \circ S_d) - S_d'{}^2)\) is divisible by \( x^d \). Moreover, \( S_d' \) has a non-zero constant coefficient. A factor \( x^d \) appears then in the integral and it’s enough to compute \( J_{2d} \) modulo \( x^d \). The inverse of \( J_{2d} \) is multiplied by the integral, it will thus be multiplied by \( x^{d+1} \), and it’s enough to evaluate this inverse modulo \( x^d \). The inverse of \( S_d' \) is needed in the computations of \( A_{2d} \) and \( J_{2d} \). In \( A_{2d} \), this inverse is multiplied by \( x^d \) and we then compute a primitive. In \( J_{2d} \), we compute only modulo \( x^d \). In both cases, the inverse of \( S_d' \) modulo \( x^d \) is enough. This inverse is provided by Eq. \((A.1)\):

\[
\frac{1}{S_d'} = \frac{S_d'}{G \cdot (H \circ S_d)} \mod x^d.
\]

We plug this expression in the computation of \( J_{2d} \) modulo \( x^d \), we find

\[
\int \frac{G \cdot (H' \circ S_d)}{2S_d'} \, dx = \int \frac{S_d' \cdot (H' \circ S_d)}{2(H \circ S_d)} \, dx \mod x^d
= \frac{\log(H \circ S_d)}{2} \mod x^d.
\]
We then find the following nice formulas for $J_{2d}$ and $1/J_{2d}$ modulo $x^d$,

$$J_{2d} = \frac{1}{\sqrt{H \circ S_d}} \mod x^d, \quad \frac{1}{J_{2d}} = \sqrt{H \circ S_d} \mod x^d.$$  

These formulas allow to efficiently compute $S_{2d}$ from $S_d$ and other known quantities.

- From the inverse of $S_{d/2}$ modulo $x^{d/2}$, denoted by $U_d$, we use a classical Newton iteration to compute $U_{2d}$. Since $S_d = S_{d/2} \mod x^{d/2+1}$, we have $U_{2d} = U_d \mod x^{d/2}$ and we compute the coefficients of $U_{2d}$ thanks to

$$U_{2d} = U_d \cdot (2 - S_d' \cdot U_d) \mod x^d.$$  

- From $\sqrt{H \circ S_{d/2}}$ modulo $x^{d/2}$, denoted by $V_d$, and the inverse of $V_d$ modulo $x^{d/2}$, denoted by $J_d$, we compute $V_{2d}$ and $J_{2d}$ as follows. Getting $V_{2d}$ consists in computing a solution of $v^2 - (H \circ S_d)(x) = 0$. We use

$$V_{2d} = \frac{1}{2} \left( V_d + \frac{H \circ S_d}{V_d} \right) \mod x^d.$$  

$J_d$ and $V_d$ are by definition inverses of each other modulo $x^{d/2}$. We obtain the inverse $W_{2d}$ of $V_d$ modulo $x^d$ thanks to Newton formulas too,

$$W_{2d} = J_d \cdot (2 - V_d \cdot J_d) \mod x^d.$$  

If we now plug this value in the $V_{2d}$ formula, we finally find

$$2V_{2d} = V_d + J_d \cdot (H \circ S_d) \cdot (2 - V_d \cdot J_d) \mod x^d.$$  

**Figure 2.** Computation of $U_{2d}$, $V_{2d}$, $J_{2d}$ and $S_{2d}$
Another use of Newton’s inversion formula yields $J_{2d}$.

$$J_{2d} = J_d \cdot (2 - J_d \cdot V_{2d}) \mod x^d.$$ 

Thanks to all these equations, we can compute $(U_{2d}, V_{2d}, J_{2d})$ from $(U_d, V_d, J_d, S_d)$. The quantity $S_{2d}$ is then obtained from Eq. (A.2),

$$S_{2d} = S_d + \frac{V_{2d}}{2} \int U_{2d} \cdot J_{2d} \cdot (G \cdot (H \circ S_d) - S_d^2) \, dx \mod x^{2d+1}.$$ 

We illustrate the corresponding computations in Fig. 2.

It remains to obtain initial values, for $d = 2$. Let $\gamma$ be defined by $S_2(x) = \alpha + \beta x + \gamma x^2 \mod x^3$. The series $S_2$ is solution of the differential equation modulo $x^2$ and thus $\beta^2 + 4 \beta \gamma x = G(x) H(\alpha + \beta x) \mod x^2$. Once derivated, and evaluated at $x = 0$, we obtain $\gamma$, and thus the value of $S_2$,

$$S_2(x) = \alpha + \beta x + \left( \frac{G'(0)}{4\beta} + \frac{\beta^2 H' (\alpha)}{4} \right) x^2 \mod x^3.$$ 

We deduce as well

$$U_2(x) = \frac{1}{\beta} \mod x, \quad V_2(x) = 1 \mod x \quad \text{and} \quad J_2(x) = 1 \mod x.$$ 

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