Comment on Complex Extension of Quantum Mechanics

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Abstract

In their Erratum [Phys. Rev. Lett. 92, 119902 (2004), quant-ph/0208076], written in reaction to quant-ph/0310164, Bender, Brody and Jones propose a revised definition for a physical observable in $\mathcal{PT}$-symmetric quantum mechanics. We show that although this definition avoids the dynamical inconsistency revealed in quant-ph/0310164, it is still not a physically viable definition. In particular, we point out that a general proof that this definition is consistent with the requirements of the quantum measurement theory is lacking, give such a proof for a class of $\mathcal{PT}$-symmetric systems by establishing the fact that this definition implies that the observables are pseudo-Hermitian operators, and show that for all the cases that this definition is consistent with the requirements of measurement theory it reduces to a special case of a more general definition given in quant-ph/0310164. The latter is the unique physically viable definition of observables in $\mathcal{PT}$-symmetric quantum mechanics.

Bender, Brody and Jones [Phys. Rev. Lett. 92, 119902 (2004), quant-ph/0208076] have recently proposed the following definition of a physical observable in $\mathcal{PT}$-symmetric quantum mechanics. **Def. 1:** A linear operator $A$ is called an observable if it satisfies $A^T = \mathcal{CPT} A \mathcal{CPT}$. (1)

This definition avoids the incompatibility of their initial definition [2] with the dynamical aspects of the theory [3]. The purpose of this comment letter is to use the requirements of the quantum measurement theory to provide a critical assessment of the viability of Def. 1. In particular, we point out that (a) a general proof that Def. 1 is consistent with these requirements is lacking, (b) give such a proof for a class of $\mathcal{PT}$-symmetric systems by establishing the fact that (1) implies that $A$ is a pseudo-Hermitian operator [4], and (c) show that for all the cases that Def. 1 is consistent with these requirements it reduces to a more general definition [5].

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namely **Def. 2**: A linear operator \( A \) is called an observable if it is Hermitian with respect to the \( \mathcal{CPT} \)-inner product \( \langle \cdot | \cdot \rangle \), i.e., \( \langle \cdot | A \cdot \rangle = \langle A \cdot | \cdot \rangle \).

Standard quantum measurement theory imposes the following conditions on any linear operator \( A \) that is to be identified with a physical observable. (i) the eigenvalues of \( A \) must be real; (ii) \( A \) has a complete set of eigenvectors that are mutually orthogonal with respect to the defining inner product \( \langle \cdot | \cdot \rangle \) of the Hilbert space \( \mathcal{H} \).

It is a well-known result of linear algebra that (i) and (ii) are the necessary and sufficient conditions for the Hermiticity of an operator \( A \), i.e., \( \langle \cdot | A \cdot \rangle = \langle A \cdot | \cdot \rangle \). In \( \mathcal{PT} \)-symmetric QM, \( \langle \cdot | \cdot \rangle \) is the \( \mathcal{CPT} \)-inner product [2]. This shows that the most general definition that is compatible with (i) and (ii) is Def. 2. As a result, Def. 1 would be a physically viable definition, only if it turns out to be a special case of Def. 2. It is not equivalent to Def. 2, for it puts the additional restriction that the Hamiltonian \( H \) be not only \( \mathcal{PT} \)-symmetric but also symmetric (\( H^T = H \)); it cannot for example be used to determine the observables for the \( \mathcal{PT} \)-symmetric system defined by the Hamiltonian \( H = (p + ix)^2 + x^2 \), [6].

Next, we note that one can use the identities \([\mathcal{P}, \mathcal{T}] = [\mathcal{C}, \mathcal{PT}] = 0 \) and \( \mathcal{C}^2 = \mathcal{P}^2 = 1 \) to show that Eq. (1) implies

\[
A^\dagger = \eta_+^{-1} A \eta_+ ,
\]

where \( A^\dagger = \mathcal{T} A^T \mathcal{T} \) is the usual adjoint of \( A \) and \( \eta_+ := \mathcal{PC} \). Eq. [2] is the defining relation for a pseudo-Hermitian operator [4]. It is equivalent to the condition that \( A \) be Hermitian with respect to the inner product \( \langle \cdot, \cdot \rangle_{\eta_+} := \langle \cdot, \eta_+ \cdot \rangle \) where \( \langle \cdot, \cdot \rangle \) is the ordinary \( L^2 \)-inner product. Therefore, Def. 1 implies that the observables \( A \) are Hermitian operators with respect to \( \langle \cdot, \cdot \rangle_{\eta_+} \), i.e., \( \langle \cdot, A \cdot \rangle_{\eta_+} = \langle A \cdot, \cdot \rangle_{\eta_+} \). For \( \mathcal{PT} \)-symmetric theories defined on the real line, one can show by a direct computation [7] that \( \langle \cdot, \cdot \rangle_{\eta_+} \) coincides with the \( \mathcal{CPT} \)-inner product. This proves that for these theories Def. 1 does indeed adhere to the requirements (i) and (ii) above. For \( \mathcal{PT} \)-symmetric theories defined using a complex contour, such a proof is lacking.

This is a serious shortcoming. In effect it means that in order to employ Def. 1 one must not only establish the reality of the eigenvalues of an observable \( A \) but also prove that (1) implies the completeness of the eigenvectors of \( A \) and their orthogonality. Moreover, Def. 1 does not provide any practical means to construct the observables of the theories to which it applies. As argued in [8] the situation is different if one adopts Def. 2. One then would just compute the matrix elements \( A_{mn} = \langle \phi_m | A | \phi_n \rangle \) in the energy eigenbasis \( \{ \phi_n \} \) and check whether \( A^*_{mn} = A_{nm} \).

In conclusion, there is no logical reason why one should adopt Def. 1 while there is already an alternative, namely Def. 2, that avoids all the above-mentioned problems. A conceptual consequence of adapting Def. 2 is that the only structural difference between conventional QM and \( \mathcal{PT} \)-symmetric QM is that in the latter one defines the Hilbert space using the eigenvalue
problem of a differential operator. As explained in [5], the fact that there is (up to unitary equivalence) a single separable Hilbert space shows that this difference does not have any fundamental ramifications. This in turn suggests that the $\mathcal{PT}$-symmetric QM should be viewed as a framework for dealing with phenomenological and effective theories.

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References

[1] According to [2], given a linear operator $A$ represented by the infinite matrix $A(x, x')$ in the standard position representation, the transpose $A^T$ of $A$ is the operator that is represented by $A^T(x, x') := A(x', x)$. $A$ is called symmetric if $A^T = A$.

[2] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002); See also ibid Am. J. Phys. 71, 1095 (2003).

[3] This incompatibility was initially revealed in an attempt to construct the observables for a finite-dimensional toy model. It was then generally established in [5].

[4] A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).

[5] A. Mostafazadeh, preprint: quant-ph/0310164.

[6] For other examples of nonsymmetric $\mathcal{PT}$-symmetric Hamiltonians see Eq. (15) of [2] and A. Mostafazadeh, J. Phys. A 36, 7081 (2003).

[7] A. Mostafazadeh, J. Math. Phys. 44, 974 (2003).