Modeling Presymptomatic Spread in Epidemics via Mean-Field Games

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Abstract—This paper is concerned with developing mean-field game models for the evolution of epidemics. Specifically, an agent’s decision — to be socially active in the midst of an epidemic — is modeled as a mean-field game with health-related costs and activity-related rewards. By considering the fully and partially observed versions of this problem, the role of information in guiding an agent’s rational decision is highlighted. The main contributions of the paper are to derive the equations for the mean-field game in both fully and partially observed settings of the problem, to present a complete analysis of the fully observed case, and to present some analytical results for the partially observed case.

I. INTRODUCTION

It is often asked what is so hard about modeling the spread of COVID-19? Although modeling proved to be invaluable during the early stages of the pandemic (March 2020), primarily by persuading the reluctant politicians to adopt the harsh lockdown measures [1], subsequent evolution of the pandemic has shone a harsh spotlight on the simplistic SIR models [2]. This in turn has spurred much work on its enhancements by including additional compartments [3]–[9], considering the effects of heterogeneity [10], [11], and modifying the mean-field interaction terms [12].

There are three aspects to modeling the pandemic: (i) evolution of virus in a single agent who has been infected; (ii) social behavior of a single agent up to the time that she is infected; and (iii) the net effect (mean-field) due to the social behavior of the population. Our understanding of the aspect (i) has improved by leaps and bounds based on COVID-19 data [13]–[17]. Aspects (ii) and (iii) have proved to be nearly intractable and this has prompted resorting to large-scale agent-based simulations with its many assumptions and parameters [18]–[23].

The mathematical modeling and analysis of the aspects (ii) and (iii) is an extremely complex problem of immense societal importance. This paper is a modest effort in this direction based on the mean-field game (MFG) formalism.

Specifically, the agent’s social behavior (aspect (ii)) is modeled as an optimal control problem based on health-related costs and activity-related rewards. The collective effect of the population (aspect (iii)) is modeled in terms of two mean-field processes $\beta$ and $\alpha$. The instantaneous $\beta_t$ in particular models the effect of the active infected agents.

The main contributions of the paper are to derive the equations for MFG in both fully and partially observed settings of the problem, to present complete analysis of the fully observed case, and to present some analytical results for the partially observed case. Specifically, the MFG model is used to obtain the following two conclusions:

Rationality of a single agent: A susceptible agent will choose to be active if and only if the reward outweighs the risk (formalized by deriving a certain critical value $\beta_{\text{crit}}$ for $\beta$). In contrast, a rational infected agent who also perfectly knows her epidemiological status will choose to self-isolate (quarantine). The latter guided the testing regimens deployed by the U.S. universities during the fall and spring of 2020 [24], [25]. Unfortunately, the observed behavior of partying U.S. undergraduates served to highlight the limitations of rationality [26], [27].

Imperfect information: Under imperfect information — when an agent does not have perfect belief regarding her own epidemiological status — an infected agent may behave as a susceptible agent. Such a behavior may in turn drive the epidemic which is consistent with the reported role of presymptomatic and asymptomatic population in the COVID-19 spread [28]–[31].

The importance of individual decision-making affecting the evolution of the epidemic was recognized early on with contributions on MFG modeling of epidemics appearing in [32]–[39]. Related to our work are [34] where an agent’s decision variable is her rate of contact with others, and [36], [38] where an agent strives to follow a prescribed rate of contact based on government guidelines. The novelty of our paper comes from partially observed settings and differences in cost structures which are helpful to model and analyze the presymptomatic spread of epidemic.

The remainder of this paper is organized as follows: The problem formulation appears in Sec. III. Its solution for the fully observed and partially observed cases is described in Sec. III and Sec. IV respectively. The proofs appear in the Appendix.
II. PROBLEM FORMULATION: MODELING

A. Model for a single agent

**Dynamics:** For a single agent, the epidemiological state is modeled as a Markov process and denoted by $X := \{X_t \in \mathcal{X} : t \geq 0\}$ where the state-space is $\mathcal{X} := \{s, a, i, r, d\}$. Figure 1 depicts the transition graph and includes a description of the epidemiological meaning of each of the five states. There are two types of infected states: (i) presymptomatic, denoted as $a$; and (ii) symptomatic, denoted as $i$. In either of these states, an agent is infectious, i.e., able to infect other agents. The modeling distinction is that, in partially observed settings, a presymptomatic agent may not know her true state but a symptomatic agent does. Note that, for the present paper, the transition graph does not include an edge from $a$ to $r$ and thus one may interpret asymptomatic as presymptomatic whereby the agent discovers that she is infected upon appearance of the symptoms. Broadly, there are two types of transitions:

1) On the subset $\{a, i, r, d\}$, the transition rate depends only upon the agent attribute $\theta$, which here represents the age of the agent. For example, an older infected agent may risk a longer recovery time (smaller $\lambda^s$) than a younger agent.

2) The transition from $s \rightarrow a$ depends upon three factors: (i) the intrinsic infectivity of the virus; (ii) the agent behavior (level of social activity); and (iii) the behavior of the infected agents in the population. The following equation is used to model the effect of these three factors:

$$\text{rate}[s \rightarrow a] = \lambda^s \beta_t U_t$$

where $U = \{U_t \in [0, 1] : t \geq 0\}$ is the agent’s activity level. For $U_t = 0$ (resp., $U_t = 1$) the agent is completely isolated (resp., active) at time $t$. The rate also depends upon the process $\beta := \{\beta_t \in [0, 1] : t \geq 0\}$ which is used to model the net effect of the behavior (activity) of infected agents in the population. Its model is introduced in Sec. II-B together with the population model. Finally, the parameter $\lambda^s$ models the rate of spread of the virus in a given population in the absence of any mitigation measures.

**Optimal control objective:** In the following, $\alpha := \{\alpha_t \in [0, 1] : t \geq 0\}$ and $\beta$ are given deterministic processes. The control objective for a single agent is to choose her activity $U$ to minimize

$$J(U; \beta, \alpha) = E \left( \int_0^T e^{-\gamma t} c(X_t, U_t; \alpha_t) \, dt + e^{-\gamma T} \phi(X_T) \right)$$

where $T = T(\omega) = \inf\{t > 0 : X_t(\omega) \in \{d, r\}\}$ is the random stopping time when the agent either recovers ($X_T = r$) or the agent dies ($X_T = d$); by convention, $\inf \emptyset = \infty$. The cost function is of the following form:

$$c(x, u; \alpha_t) = c^b(x) + c^a(x)u - r(x; \alpha_t)u$$

where models for the health related costs $c^b$ and $c^a$, the activity related reward $r(x; \cdot)$, and the terminal cost $\phi(x)$ are tabulated in the Table included as part of Fig. 1(b).

**Information structure:** There are two settings of the problem: (i) the fully observed case; and (ii) the partially observed case. In the partially observed setting, the observation process $Y := \{Y_t \in \{0, 1\}^2 : t \geq 0\}$ is defined according to

$$Y_t = \begin{cases} 1[\{x = s\}] & 1[\{x = d\}] \end{cases}$$

B. Model for the mean-field

To fully specify the problem, we need to define models for the attribute $\theta$ and the two deterministic processes, $\beta$ and $\alpha$, henceforth referred to as the mean-field processes. The probability mass function of the attribute $\theta$ is denoted $p(\cdot)$. To specify the models for $\beta$ and $\alpha$, we denote $\rho_t(x, u; \theta)$ as the joint distribution of the state-action pair $(X_t, U_t)$ at time $t$, conditioned on the attribute $\theta$. Set

$$\beta_t = \sum_{\theta} p(\theta) \sum_{x \in \{a, i\}} \int_0^1 u \rho_t(x, u; \theta) \, du \quad (1a)$$

$$\alpha_t = \sum_{\theta} p(\theta) \rho_t(r; \theta) + \sum_{x \in \{s, a, i\}} \int_0^1 u \rho_t(x, u; \theta) \, du \quad (1b)$$

where $\rho_t(x; \theta)$ denotes the marginal, and $\beta_t$ (resp. $\alpha_t$) represent the average activity level of infected agents (resp. all agents).

**Assumption 1:** Both $\beta$ and $\alpha$ are deterministic processes. Furthermore, at each time $t$, $0 \leq \beta_t < 1$ and $0 < \alpha_t < 1$.

The marginal pmf $\rho_t(x; \theta)$ evolves according to

$$\frac{d\rho_t}{dt}(s; \theta) = -\lambda^s \beta_t \int_0^1 u \rho_t(s, u; \theta) \, du \quad (2a)$$

$$\frac{d\rho_t}{dt}(x; \theta) = (A_t \rho_t)(x; \theta), \quad x \in \{i, a, r, d\} \quad (2b)$$

**Fig. 1:** (a) Epidemiological states and the transition graph. (b) Cost function.
from a given initial condition $\rho_0(x;\theta)$; $A^1$ is the adjoint of the generator $A$ of the Markov process $X$. It is noted that control $u$ affects only the transition from $s \rightarrow a$.

**Remark 1:** The evolution \( \{\rho_t : t \geq 0\} \) is nonlinear because $\beta_t$ depends upon $\rho_t$. Apart from the terms on the righthand side arising due to the transition $[s \rightarrow a]$, the other terms are linear. These other terms depend only upon the transition rates of the Markov process which can themselves depend upon the attribute $\theta$. With $U_t \equiv 1$, \( \{\} \) is an example of the classical Kermack-McKendrick model.

The basic reproduction number $R_0 = \frac{\lambda}{\mu}$ where $\mu$ is the typical time until removal (i.e., recovery or death) and $T^\ast$ is the typical time between infectious contacts. Evaluating $T^\ast$ and $T^\ast$ for the Markov process with $U_t \equiv 1$ and $\beta_t \equiv \bar{\beta}$,

$$R_0 = \lambda S^0 \bar{\beta} \left( \lambda M^0 + \lambda d^0 \right) - 1$$

In this paper, the choice of $U$ is guided by an MFG formulation which is described next.

**C. Mean-field game problem**

**Function spaces:** The filtration of the Markov process $X$ is denoted $\mathcal{F} := \{\mathcal{F}_t : t \geq 0\}$ where $\mathcal{F}_t := \sigma(X_t)$ ($\sigma(\cdot)$ denotes the $\sigma$-algebra generated by a stochastic process). The filtration of the observation process $Y$ is denoted $\mathcal{Y} := \{\mathcal{Y}_t : t \geq 0\}$ where $\mathcal{Y}_t := \sigma(Y_s : 0 \leq s \leq t)$. In the two settings of the problem, the space of admissible control inputs, denoted by $U_t$, is as follows:

- (fully obsvd.) $U = L_+([0, \infty); [0, 1])$
- (part. obsvd.) $U = L_+([0, \infty); [0, 1])$

i.e., an admissible control input $U$ is a $[0, 1]$-valued stochastic process adapted to $\mathcal{F}$ in the fully observed case, and adapted to $\mathcal{Y}$ in the partially observed case; the use of the common notation $U$ should not cause any confusion because the two cases are treated in separate sections. Apart from control, the other process of interest is $(\beta, \alpha)$ whose function space is denoted $\mathcal{M} := L^\infty([0, \infty); [0, 1]^2)$.

On these function spaces, define two operators:

1) The operator $\Psi : \mathcal{M} \rightarrow U$ as

$$\Psi(\beta, \alpha) = \arg \min_{U \in \mathcal{U}} J(U; \beta, \alpha)$$

2) The operator $\Xi : U \rightarrow \mathcal{M}$ is according to (1) and (2).

Assuming the two operators are well-defined, we have:

**Definition 1:** A mean-field equilibrium (MFE) is any fixed point $(\beta, \alpha)$ such that $\Xi(\Psi(\beta, \alpha)) = (\beta, \alpha)$.

**Remark 2:** Although $\Psi$ is well-defined under rather mild conditions, it is difficult to justify $\Xi$ without additional assumptions on the form of the control input $U$. For this purpose, it is useful to note that, provided it is well-defined, the optimal control input, denoted $U_t^{\text{opt}} = \{U_t^{\text{opt}} : t \geq 0\}$, is obtained using a deterministic feedback control law

- (fully obsvd.) $U_t^{\text{opt}} = \psi_t(X_t)$
- (part. obsvd.) $U_t^{\text{opt}} = \psi_t(\pi_t)$

where $\pi_t = P(X_t | \mathcal{Y}_t)$ is the belief state; again the use of the common notation $\psi_t(\cdot)$ should not cause any confusion because the two cases are treated in separate sections.

Now, define $\mathcal{U} \subset U$ as the subset of all control inputs that are obtained according to some deterministic feedback control law ($u_t = \psi_t(x)$ or $u_t = \psi_t(\pi)$ in the two cases). An MFE is then defined by restricting the domain of $\Xi$ to $\mathcal{U}$. For the fully observed settings, such a restriction is standard. For the partially observed settings, we will describe an explicit construction of the operator $\Xi$ in Sec. IV.

**Notation:** The bar is used to denote stationary (i.e., time-independent) quantities. For example, $\psi_t = \bar{\psi}$ means the control law is stationary, and $\beta_t = \bar{\beta}$ means the value of the process is a constant $\bar{\beta}$.

**III. Optimality equations:** Fully obsvd. case

**A. Solution for the single agent problem**

For each $x \in \mathcal{X}$ and $t \geq 0$, the value function

$$v_t(x) := \min_{U \in L^\infty_+} \mathbb{E} \left( \int_t^T e^{-\gamma(s-t)} c(X_s, U_s; \alpha_s) \, ds + e^{-\gamma T} \phi(X_T) \right) \bigg| X_t = x$$

(3)

For $x = d$ and $x = r$, the value function is $v_t(d) = \bar{\phi}(d)$ and $v_t(r) = \bar{\phi}(r)$. The remaining states are $\{i, a, s\}$. For the state $x = i$, the value function solves the HJB equation

$$- \frac{dv_t(i)}{dt} + (\gamma + \lambda^d + \lambda^s) v_t(i) = c^d(i) + \lambda^a \bar{\phi}(r) + \lambda^d \bar{\phi}(d) + \min_{u \in [0,1]} \{c^d(i) - \alpha_t\} u$$

Now because the altruistic cost $c^d(i) = 1$ and because $\alpha_t < 1$ (Assumption 1), the optimal action for an infected agent is to use $U_t^{\text{opt}} = \psi_t(i) = 0$, and

$$- \frac{dv_t(i)}{dt} + (\gamma + \lambda^d + \lambda^s) v_t(i) = c^d(i) + \lambda^a \bar{\phi}(r) + \lambda^d \bar{\phi}(d)$$

whose solution is stationary (i.e., time-independent) and given by

$$v_t(i) = \bar{\phi}(i) := \frac{c^d(i)}{(\gamma + \lambda^d + \lambda^s)} + \frac{\lambda^a \bar{\phi}(r) + \lambda^d \bar{\phi}(d)}{(\gamma + \lambda^a + \lambda^d)}$$

In the remainder of the paper, we make the following assumption whose justification is provided as part of the remark after the value function is fully described.

**Assumption 2:** The value $\bar{\phi}(i)$ is positive.

For the presymptomatic state $x = a$, the value function solves

$$- \frac{dv_t(a)}{dt} + (\gamma + \lambda^a) v_t(a) = c^d(a) + \lambda^d \bar{\phi}(i) + \min_{u \in [0,1]} \{c^d(a) - \alpha_t\} u$$
and once again because of the nature on the altruistic cost, \( c^0(a) = 1 \), the optimal action for an asymptomatic agent is to use \( U_t^{\text{opt}} = \psi_i(X_t) \). With the health cost \( c^0(a) = 0 \),
\[
v_t(a) = \tilde{\phi}(a) := \frac{\lambda^u}{\gamma + \lambda^u} \tilde{\phi}(\ell)
\]
for the fully observed problem.

It remains to obtain \( v_t(s) \). The HJB equation is
\[
-\frac{d v_t}{d t}(s) + \gamma v_t(s) = \min_{u \in [0,1]} \left( \lambda^{sa} \beta_i(\tilde{\phi}(a) - v_t(s)) - \alpha_t \right) u
\]
In the following, it is assumed that a unique solution exists and \( U_t^{\text{opt}} = \psi_i(s) \) is obtained as a feedback control law.

One may obtain additional insights by considering the stationary case whose solution is described in the following proposition with proof in the Appendix A.

**Proposition 1 (Stationary solution):** Suppose \( \gamma > 0 \), \( \tilde{\phi}(\ell) > 0 \), \( \beta_i = \tilde{\beta} \) and \( \alpha_t = \bar{\alpha} \) are both constants, and \( \bar{\alpha} < 1 \). Then the optimal control for a susceptible agent is stationary and described by the following cases:
1. If \( \tilde{\beta} < \frac{\bar{\alpha}}{\lambda^{sa} \bar{\phi}(a)} \) then the optimal control \( U_t^{\text{opt}} = 1 \) and the optimal value \( v_t(s) = \frac{\lambda^{sa} \tilde{\phi}(a) - \bar{\alpha}}{\lambda^{sa} + \gamma} \).
2. If \( \tilde{\beta} \geq \frac{\bar{\alpha}}{\lambda^{sa} \bar{\phi}(a)} \) then the optimal control \( U_t^{\text{opt}} = 0 \) and the optimal value \( v_t(s) = 0 \).

**Remark 3:** The \( \gamma = 0 \) case is ill-posed for the following two reasons:
1. Suppose \( \tilde{\beta} < \frac{\bar{\alpha}}{\lambda^{sa} \bar{\phi}(a)} \). Then the optimal control \( U_t^{\text{opt}} \) is not uniquely defined. In fact, any non-zero choice of \( U_t^{\text{opt}} \) yields the same value \( v_t(s) = \frac{\lambda^{sa} \tilde{\phi}(a) - \bar{\alpha}}{\lambda^{sa} + \gamma} \).
2. Suppose \( \tilde{\beta} > \frac{\bar{\alpha}}{\lambda^{sa} \bar{\phi}(a)} \). Then the optimal control \( U_t^{\text{opt}} = 0 \) with value \( v_t(s) = 0 \). This is directly verified from using the definition of the value function. However, the HJB equation is not useful in this regard because, using zero control, the terminal time \( T = \infty \); cf., [40].

Therefore, \( \gamma \) serves as a regularization parameter. Another choice is to modify rate \( \gamma \rightarrow a = \lambda^\gamma (\gamma + \beta_i U_t) \) which will also serve to regularize the problem (precluding \( T = \infty \) for all choices of control).

**Remark 4:** We next justify Assumption 2 (\( \tilde{\phi}(\ell) > 0 \)). A susceptible agent always has an option to stay isolated (choose \( U_t = 0 \) for all \( t \geq 0 \)) and obtain the associated possibly sub-optimal value \( J(0) = 0 \). Assumption 2 says that the cumulative cost of being infected is greater than cost of staying isolated. Without such an assumption, an agent may wish to become active for the purposes of getting infected and thereby lowering their value. Let \( \beta_{\text{crit}} := \frac{\bar{\alpha}}{\lambda^{sa} \bar{\phi}(a)} \). It is the critical value of \( \tilde{\beta} \) when the cost of getting infected balances off the reward of being active, i.e.,
\[
\lambda^{sa} \beta_{\text{crit}} \tilde{\phi}(a) = \bar{\alpha}
\]

**B. Mean-field game**

In the fully observed version of the MFG, each agent uses the optimal control \( U_t^{\text{opt}} = \psi_i(X_t) \). For a population with heterogeneous agents, notation \( \psi_i(\cdot; \theta) \) is used to denote the dependence on the attribute \( \theta \). Using the optimal control,
\[
\frac{d \rho_t}{d t}(s; \theta) = -\lambda^{sa} \beta_i \psi_i(s; \theta) \rho_t(s; \theta)
\]
and because the optimal control \( \psi_i(x) = 0 \) for \( x \in \{a, i\} \)
\[
\beta_t := \sum_0 \rho(\theta) \psi_i(s; \theta) \rho_t(s; \theta)
\]
\[
\alpha_t := \sum_0 \rho(\theta) (\rho_t(r; \theta) + \psi_i(s; \theta) \rho_t(s; \theta))
\]
The main result is the following proposition whose straightforward proof is omitted on account of space.

**Proposition 2:** Suppose \( \rho_0(x; \theta) \) is the initial pmf for the agents. The solution for the fully observed MFG problem is:

1. For a single agent, the optimal control is stationary
\[
\psi_i(x) := \begin{cases} 0 & x \in \{a, i\} \\ 1 & x = s \end{cases}
\]
2. For the population, the distribution evolves as
\[
\rho_t(s; \theta) = \rho_0(s; \theta)
\]
\[
\frac{d \rho_t}{d t}(x; \theta) = (A_t \rho_t)(x; \theta), \quad x \in \{i, a, r, d\}
\]
3. The consistent mean-field terms are as follows:
\[
\beta_t = 0, \quad \alpha_t = \sum_0 \rho_t(\rho_i(r; \theta) + \rho_t(s; \theta))
\]

**Remark 5:** The conclusions of the theorem are not very practical. It indicates that on the ideal planet (of vulcan) where agents are perfectly rational and have perfect information, both the presymptomatic and symptomatic agents will self-isolate, and therefore \( R_0 = 0 \). It is not true that agents can continue to party without consequence. The main utility of the fully observed case is to set up the problem whereby the effects of some of the underlying assumptions – perfect rationality and perfect information – can be investigated. In the following, we consider the partially observed problem where an agent is rational but does not have perfect information regarding her epidemiological state.

**IV. Optimality equation: Partially Obsvd. Case**

**A. Solution for the single agent**

The partially observed problem is converted to a fully observed one by introducing the belief state which at time \( t \) is denoted by
\[
\pi_t := [\pi_t(s) \pi_t(a) \pi_t(i) \pi_t(r) \pi_t(d)]
\]
where \( \pi_t(x) := \mathbb{P}(X_t = x & Y_t) \) for \( x \in X \). Since the events \( [X_t = i] \) and \( [X_t = d] \) are both contained in \( Y_t \), \( \pi_t \) is not an arbitrary element of the probability simplex in \( \mathbb{R}^5 \). Let \( \mathcal{P}^1 \) denote the set of pmf-s on \( \{s, a\} \) and let \( \mathcal{P}^2 = \{\delta_t, \delta_r, \delta_d\} \).
Then the state-space for the belief is $\mathcal{P}^1 \cup \mathcal{P}^2$. For $t \geq 0$ and $\mu \in \mathcal{P}^1 \cup \mathcal{P}^2$, the value function

$$v_t(\mu) := \min_{U \in \mathcal{H}_T^\mu} \mathbb{E} \left( \int_t^T e^{-\gamma(s-t)} c(X_s, U_s; \alpha_s) ds + e^{-\gamma T} \phi(X_T) \mid \pi_t = \mu \right)$$

There are two cases to consider: (i) when $\mu \in \mathcal{P}^2$; and (ii) when $\mu \in \mathcal{P}^1$. In the first case, when $\mu \in \mathcal{P}^2$, the problem reduces to the fully-observed settings, and the value function is given by

$$v_t(\delta_t) = \tilde{\phi}(d), \quad v_t(\delta_t) = \tilde{\phi}(d), \quad v_t(\delta_t) = \tilde{\phi}(d)$$

The optimal control for the agent in the infected state ($\pi_t = \delta_t$) is $U_t^{opt} = \psi_t(\delta_t) = 0$.

For the second case, when $\mu \in \mathcal{P}^1$, a nonlinear filter is used to obtain the evolution of the belief. For this purpose, consider first the random variable $\tau(\omega) = \inf \{ t > 0 : X_t(\omega) = i \}$. Now, $\tau$ is a $\mathcal{Y}_t$-stopping time and

$$\pi_t = \begin{bmatrix} \pi_t(s) & \pi_t(a) & 0 & 0 & 0 \end{bmatrix} \quad \text{for } t < \tau$$

Let $A_t := \pi_t(a)$ for $t < \tau$. Then the stochastic process $\{A_t \in [0, 1] : 0 \leq t < \tau \}$ evolves according to the nonlinear filter (derived from the general form given in [41]):

$$\frac{dA_t}{dt} = (1 - A_t)(\lambda^\alpha \beta U_t - A_t \lambda^\alpha), \quad 0 \leq t < \tau, \quad A_0 = \pi_0(a)$$

We identify $\mathcal{P}^1$ with the interval $[0, 1]$ with $a$ serving as its coordinate ($a$ is the value of $A_t$). For an arbitrary element $\mu = [1 - a, a, 0, 0, 0]$ in $\mathcal{P}^1$, we denote the value function with respect to the $a$-coordinate as

$$\phi_t(a) := v_t(\mu), \quad 0 \leq a \leq 1, \quad t \geq 0$$

The process $\{\phi_t(a) \in \mathbb{R} : 0 \leq a \leq 1, t \geq 0 \}$ solves the HJB equation whose derivation appears in Appendix B:

$$-\frac{\partial \phi_t}{\partial t}(a) + \gamma \phi_t(a) = -\lambda^\alpha a(1 - a) \frac{\partial \phi_t}{\partial a}(a) + \lambda^\alpha(\tilde{\phi}(i) - \phi_t(1)) + \min_{u \in [0,1]} \left( \lambda^\alpha \beta(1 - a) \frac{\partial \phi_t}{\partial a}(a) + (a - \alpha_t) u \right) u$$

(4)

In the following, it is assumed that a unique solution exists and yields a well-posed optimal control law, denoted as $\psi_t(a)$ for $a \in [0, 1]$ and $t \geq 0$. The optimal control is

$$U_t^{opt} = \psi_t(A_t), \quad 0 \leq t < \tau$$

Because $A_t \equiv 1$ is an equilibrium of the filter, the HJB equation for the point $a = 1$ reduces to an ordinary differential equation whereby

$$-\frac{d\phi_t}{dt}(1) + \gamma \phi_t(1) = \lambda^\alpha(\tilde{\phi}(i) - \phi_t(1)) + \min_{u \in [0,1]} (1 - \alpha) u$$

and because $\delta < 1$ (Assumption 3), the optimal control law is $\psi_t(1) = 0$. The optimal value function solves the ODE

$$-\frac{d\phi_t}{dt}(1) + \gamma \phi_t(1) = \lambda^\alpha(\tilde{\phi}(i) - \phi_t(1))$$

whose solution is easily obtained as

$$\phi_t(1) = \frac{\lambda^\alpha}{\gamma + \lambda^\alpha} \tilde{\phi}(i)$$

It is noted that the righthand side is the value $\tilde{\phi}(a)$ for the fully observed case.

In summary, the optimal value is $\phi_t(1) = \tilde{\phi}(a)$ and the optimal control law is $\psi_t(1) = 0$. That is, an agent who has a perfect belief that she is asymptomatic will act the same way (isolate) and will have the same value as her fully observed counterpart.

B. Partially observed mean-field game

To setup an MFG, consider the space of probability distributions on the belief space $\mathcal{P}^1 \cup \mathcal{P}^2$. The random variable $A_t$ is well-defined on the set $[t < \tau]$, and we denote by $p_t(a)$ as its density for $0 \leq a \leq 1$:

$$P([a < A_t < a + da] \cap [t < \tau]) = p_t(a) da, \quad t \geq 0$$

Assumption 3: The density $p_t(0) = 0$ for all $t \geq 0$.

Under Assumption 3, that an agent uses the optimal control $U_t = U_t^{opt} = \psi_t(A_t)$ for $0 \leq t < \tau$, the density process $\{p_t(a) \in [0, \infty] : 0 \leq a \leq 1, t \geq 0 \}$ solves the FPK equation whose derivation appears in Appendix B:

$$\frac{d p_t}{dt}(a) = -\frac{\partial}{\partial a}(\lambda^\alpha \beta \psi_t(a) - a \lambda^\alpha p_t(a)) - a \lambda^\alpha p_t(a)$$

(5)

where $p_0(a)$ is the initial density (assumed given).

By using the tower property,

$$\rho_t(a) = P([X_t = a]) = \mathbb{E}(\pi_t(a)) = \mathbb{E}(A_t 1_{t < \tau}) = \int_0^1 \alpha p_t(a) \ da$$

and therefore we have

$$\frac{d \rho_t}{dt}(i) = \lambda^\alpha \int_0^1 \alpha p_t(a) \ da - (\lambda^\alpha + \lambda^\alpha) \rho_t(i)$$

(6a)

$$\frac{d \rho_t}{dt}(x) = (A^\dagger \rho_t)(x), \quad x \in \mathbb{R}, \ d \}$$

(6b)

where expression for $A^\dagger$ is obtained from the transition graph.

With a heterogenous population, the notation $p_t(a; \theta)$ is used to denote the density conditioned on the attribute $\theta$ and $\psi_t(a; \theta)$ is the optimal control law. The mean-field processes are then consistently obtained as

$$\beta_t = \sum_{\theta} \rho_t(\theta) \int_0^1 \alpha \psi_t(a; \theta) p_t(a; \theta) \ da$$

(7a)

$$\alpha_t = \sum_{\theta} \rho_t(\theta) \left( \psi_t(x; \theta) p_t(x; \theta) \right) \ da$$

(7b)

This completes the derivation of the system of equations for the partially observed MFG: Eq. 5-6 is the forward FPK equation, Eq. (4) is the backward HJB equation. Eq. 7 defines the consistency relationship that links the two equations. Its solution is an MFE (satisfies Defn. 1).
The analytical and numerical study of the mean-field equations is a subject of continuing work. In the following, we discuss some preliminary analytical results.

C. Some special cases

To gain further insights into the model, we consider stationary solutions of the HJB equation. For this purpose, in this subsection, we assume that $\beta = \bar{\beta}$ and $\alpha_t = \bar{\alpha}$ are both constants. The stationary HJB equation is then

$$\gamma \bar{\phi}(a) = -\lambda^u a(1 - a) \frac{d\bar{\phi}}{da}(a) + \lambda^u a(\bar{\phi}(i) - \bar{\phi}(a))$$

$$+ \min_{\alpha \in [0,1]} \left( \lambda^s \bar{\beta}(1 - a) \frac{d\bar{\phi}}{da}(a) + (a - \bar{\alpha}) \right) u$$ (8)

The stationary optimal control law, obtained upon evaluating the minimizer of the HJB equation, is denoted by $u = \bar{\psi}(a)$ for $a \in [0,1]$. As already described for the general non-stationary case, at $a = 1$, the optimal value is $\bar{\phi}(1) = \frac{\lambda^u}{\gamma + \lambda^u} \phi(i)$ and the associated optimal control is $\bar{\psi}(1) = 0$.

Case 1. Limit $\lambda^u \uparrow \infty$: Evaluate the stationary HJB equation at $a = 0$:

$$\gamma \bar{\phi}(0) = \min_{\alpha \in [0,1]} \left( \lambda^s \bar{\beta}(0) - \bar{\alpha} \right) u$$

Assume now that $\phi$ is continuously differentiable at $a = 0$. Then the value of $\frac{d\bar{\phi}}{da}(0)$ can be obtained using the dominant balance in the limit as $\lambda^u \to \infty$ and $a > 0$ but small:

$$-\lambda^u a(1 - a) \frac{d\bar{\phi}}{da}(0) + \lambda^u a(\bar{\phi}(i) - \bar{\phi}(a)) = 0$$

which yields $\frac{d\bar{\phi}}{da}(0) = \bar{\phi}(i) - \bar{\phi}(a)$. Therefore, in the limit as $\lambda^u \uparrow \infty$, the stationary HJB equation for $a = 0$ is given by

$$\gamma \bar{\phi}(0) = \min_{\alpha \in [0,1]} \left( \lambda^s \bar{\beta}(0) - \bar{\alpha} \right) u$$

which is identical to the stationary HJB equation (11) for the fully observed problem. The optimal control law is

$$\bar{\psi}(0) = \begin{cases} 1 & \text{if } \lambda^s \bar{\beta}(0) < \bar{\alpha} \\ 0 & \text{o.w.} \end{cases}$$

which is also the same as the optimal control for a susceptible agent in the fully observed case. This suggests that if $a = 0$ (agent has certain belief that she is susceptible), she will be active if $\bar{\beta} < \bar{\beta}_{\text{crit}}$.

To obtain insights for arbitrary values of $a \neq 1$, consider the nonlinear filter using the stationary control law $\bar{\psi}$:

$$\frac{dA_t}{dt} = (1 - A_t)(\lambda^s \bar{\beta}(\bar{\psi}(A_t) - A_t \lambda^u)), \quad 0 < t < \tau, \quad A_0 \neq 1$$ (9)

where we note that $0 \leq \bar{\psi}(A_t) \leq 1$ and $\bar{\psi}(1) = 0$. In the asymptotic limit as $\lambda^u \to \infty$, its solution is given by

$$A_t = \lambda^s \bar{\beta}(\bar{\psi}(A_t)) + o\left(\frac{1}{\lambda^u}\right) \approx 0, \quad 0 < t < \tau$$

and therefore, the optimal control $U_t^{\text{opt}} = \bar{\psi}(A_t) \approx \bar{\psi}(0)$ whenever $A_0 \neq 1$.

In summary, an agent who has a perfect belief that she is asymptomatic ($a = 1$) will isolate ($\bar{\psi}(1) = 0$). For all other values ($a \neq 1$), the agent will behave as a susceptible agent in the fully observed settings of the problem: isolate if $\bar{\beta} > \bar{\beta}_{\text{crit}}$ and fully active if $\bar{\beta} < \bar{\beta}_{\text{crit}}$.

Case 2. $\lambda^u$ large: The limit ($\lambda^u = \infty$) represents the case when there is no uncertainty in belief. As one deviates away from the limit, the uncertainty increases and an agent is no longer perfectly sure of her epidemiological state. The following proposition shows that for sufficiently large values of the parameter $\lambda^u$, the stationary optimal control law is of threshold type.

Proposition 3: Suppose $\bar{\beta} = \bar{\beta}$, $\alpha_t = \bar{\alpha}$. Then

1. If $\bar{\beta} \geq \bar{\beta}_{\text{crit}}$ then the optimal control law is $\bar{\psi}(a) = 0$ for all $0 \leq a \leq 1$.

2. For each fixed $\bar{\beta} \in \left(\frac{\alpha}{\lambda^u(\bar{\psi}(a))} < \bar{\beta}_{\text{crit}}\right)$ there exists a $\lambda^u = \lambda^u(\bar{\beta})$ such that for all $\lambda^u > \lambda^u$, the optimal control law is of threshold type:

$$\bar{\psi}(a) = \begin{cases} 1 & \text{if } 0 \leq a < a_{\text{thresh}} \\ 0 & \text{if } a_{\text{thresh}} < a \leq 1 \end{cases}$$ (10)

where the threshold $a_{\text{thresh}} \in (0,1)$. The function $\lambda^u(\bar{\beta})$ is monotonic in its argument and $\lim_{\bar{\beta} \to \bar{\beta}_{\text{crit}}} \lambda^u(\bar{\beta}) = 0$.

The exact formulae for the function $\lambda^u(\bar{\beta})$ and the threshold $a_{\text{thresh}}$ are fairly complicated. These formulae appear along with the proof of Prop. 3 in the Appendix. Apart from these special cases, we do not yet have a complete understanding of the stationary solutions of (5). This remains a subject of continuing work.

V. CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

In this paper, we proposed a partially observed MFG model for epidemics. The main contribution is derivation of the forward-backward equations. The analytical and numerical study of these equations is a topic of continuing research. A major simplifying assumption in the model is that we ignored the transition from $a$ to $r$. With the transition present, an agent will maintain a belief over a three-dimensional state $(s, a, r)$ leading to a loss of total order on beliefs. Although the extension of the forward-backward equations should be easily possible, the analysis will be much more complicated. Including the effects of testing and vaccination are other directions to extend the basic model.

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APPENDIX

A. Proof of Prop. 7

With \( \beta_t = \bar{\beta} \) and \( \alpha_t = \bar{\alpha} \), the HJB equation for \( v_t(s) \) is

\[
- \frac{dv_t}{dt}(s) + \gamma v_t(s) = \min_{u \in [0,1]} \left( \lambda s^a \bar{\beta}(\delta(a) - \bar{\phi}(s)) - \bar{\alpha} \right) u
\]

We investigate its stationary solutions \( v_t(s) = \bar{\phi}(s) \) in which case the stationary HJB equation is

\[
\gamma \bar{\phi}(s) = \min_{u \in [0,1]} \left( \lambda s^a \bar{\beta}(\bar{\phi}(a) - \bar{\phi}(s)) - \bar{\alpha} \right) u
\]

We have the following two cases:

- If \( \bar{\beta} < \frac{\bar{\beta}}{\lambda s^a \bar{\beta}(a)} \) then \( \bar{\phi}(s) = \frac{\lambda s^a \bar{\beta}(a) - \bar{\beta}}{\lambda s^a \bar{\beta} + \gamma} \) solves the HJB equation with the minimizing choice of \( u = 1 \) because
  \[
  M = \frac{\gamma}{\lambda s^a \bar{\beta} + \gamma} \left( \lambda s^a \bar{\beta} \bar{\phi}(a) - \bar{\alpha} \right) < 0
  \]

- If \( \bar{\beta} > \frac{\bar{\beta}}{\lambda s^a \bar{\beta}(a)} \) then \( \bar{\phi}(s) = 0 \) solves the HJB equation with the minimizing choice of \( u = 0 \) because
  \[
  M = \lambda s^a \bar{\beta} \bar{\phi}(a) - \bar{\alpha} > 0
  \]

B. Derivations

Let \( \mathcal{P} := \mathcal{P}^1 \cup \mathcal{P}^2 \). The belief process \( \pi := \{ \pi_t \in \mathcal{P} : t \geq 0 \} \) is a Markov process [42, Theorem 1.7]. The HJB equations and FPK equations are easily derived once we obtain the infinitesimal generator of the process.

Infinitesimal generator: Consider a smooth test function \( v : \mathcal{P} \to \mathbb{R} \). Let \( \mu \in \mathcal{P} \). The infinitesimal generator (for the general time-inhomogeneous case) is

\[
(A^\mu_t v)(\mu) = \lim_{\delta t \to 0} \frac{E(v(\pi_{t+\delta t})|\pi_t = \mu) - v(\mu)}{\delta t}
\]

There are two cases to consider:

- If \( \mu \in \mathcal{P}^2 \), upon identifying the measures \( \{ \delta_t, \delta_r, \delta_d \} \) with the states \( \{ i, r, d \} \), the generator is the same as the generator for the Markov process.

- If \( \mu \in \mathcal{P}^1 \) then using the coordinate \( a \) for \( \mathcal{P}^1, \mu = [1 - a, a, 0, 0, 0] \) for \( a \in [0,1] \). With \( \pi_t = \mu \), in the asymptotic limit as \( \delta t \to 0 \),

\[
\pi_{t+\delta t} = \begin{cases} \mu + e f(a, t) \delta t + o(\delta t) & \text{w.p. } (1 - a \lambda s^a \delta t) + o(\delta t) \\ \delta_i & \text{w.p. } a \lambda s^a \delta t + o(\delta t) \end{cases}
\]

where \( f(a, t) = (1 - a)(\lambda s^a \beta u - a \lambda s^a u) \) and \( e = [-1, 1, 0, 0, 0] \). Denoting \( v(\mu) = \phi(a) \), the generator is then easily calculated to be

\[
(A^\mu_t v)(\mu) = f(a, t) \frac{\partial \phi}{\partial a}(a) + a \lambda s^a \phi(\delta u) - \phi(a)
\]

where the superscript \( u \) denotes the fact that \( f(a, t) \), and therefore also the generator, depends also upon \( u \). The subscript \( t \) denotes the fact that the generator is for a time-inhomogeneous Markov process (because \( \beta \) may depend upon time).

Derivation of the HJB equation: The HJB equation is

\[
- \frac{\partial v_t}{\partial t}(\mu) + \gamma v_t(\mu) = \min_{u \in [0,1]} \left( A^u_t v_t(\mu) + \mu(c(\cdot, u; \alpha_t)) \right)
\]

For \( \mu = [1 - a, a, 0, 0, 0] \) and \( v_t(\mu) = \phi_t(a) \), the HJB equation \ref{eq:1} is obtained because

\[
\mu(c(\cdot, u; \alpha_t)) = (1 - a)(-\alpha t u) + a(1 - \alpha t)u = (a - \alpha t)u
\]

Derivation of the FPK equation: We derive the adjoint of the generator \( A^u \) where dependence on \( t \) is suppressed for notational ease. Let \( \rho \) be a measure on \( \mathcal{P} \). On \( \mathcal{P}^1 \), \( \rho \) has density \( p(a) \). Consider \( \rho(A^u v) = \int A^u v(\mu) \rho(d\mu) = \int_0^1 \phi(a) \left( \frac{\partial a}{\partial a} (a f(a)) + a \lambda s^a p(a) \right) da + p(a) \phi(a) f(a) \left|_0^1 \right| + v(\delta_t) \lambda s^a \rho(\delta_t) + v(\delta_d) \lambda s^a \rho(\delta_d)
\]

\[
+ v(\delta_t) \int_0^1 a \lambda s^a p(a) da - \rho(\delta_t)(\lambda s^a + \lambda s^a)^\prime
\]

where boundary terms vanish because \( f(1) = 0 \) and \( p(0) = 0 \) (Assumption \ref{assumption:3}).

C. Proof of Prop. 3

We are interested in solutions of \ref{eq:8} repeated below

\[
\gamma \bar{\phi}(\alpha(a) = - \lambda s^a (1 - a) \frac{d \bar{\phi}}{da}(a) + \lambda s^a \bar{\phi}(\bar{\phi}(a) - \bar{\phi}(a))
\]

\[
+ \min_{u \in [0,1]} \left( \lambda s^a \bar{\beta}(1 - a) \frac{d \bar{\phi}}{da}(a) + (a - \bar{\alpha}) \right) u
\]

\[
:= M(a)
\]

Proof of Part 1): If \( \bar{\beta} \geq \beta_{\text{thresh}} \) then \( \bar{\phi}(a) = 0 \) solves \ref{eq:12} with \( u = 0 \). To show that \( u = 0 \) is a minimizer, we need to verify that \( M(\alpha) > 0 \) for all \( 0 \leq \alpha \leq 1 \). This is true because

\[
M(\alpha) = \lambda s^a \bar{\beta}(1 - a) \bar{\phi}(a) + (a - \bar{\alpha})
\]

is an affine function with \( M(1) = 1 - \bar{\alpha} > 0 \) (Assumption \ref{assumption:1}) and \( M(0) = \lambda s^a \bar{\beta}(a) - \bar{\alpha} > 0 \) (because \( \bar{\beta} \geq \beta_{\text{thresh}} \)).

Proof of Part 2): With \( U_t \equiv 1 \) the point \( \bar{a} := \frac{\lambda s^a \beta}{\lambda s^a \beta + \gamma} \) is a stable equilibrium of the filter \ref{eq:9}. Now let

\[
a_{\text{thresh}} := \frac{\bar{a} - \lambda s^a \beta}{1 - \lambda s^a \beta}
\]

where \( k := \frac{\lambda s^a \beta + 1 - \lambda s^a u}{\lambda s^a \beta + \gamma} \) and \( y := \frac{\lambda s^a \beta + 1 - \bar{a}}{\lambda s^a \beta + \gamma} \).

We present the proof in three steps. In step 1, we show that if \( 0 < \bar{a} < a_{\text{thresh}} < 1 \), then the optimal control obtained as a solution to the HJB equation is of threshold type \ref{eq:10}. In step 2, we present a tight bound to attain \( 0 < \bar{a} < a_{\text{thresh}} < 1 \), which is implied by the inequality \( \lambda s^a > \lambda s^a (\bar{\beta}) \) derived in the final step 3.
Step 1: Assume $0 < \bar{a} < a_{\text{thresh}} < 1$. We want to show that the solution to (12) is the following:
\[ \tilde{\phi}(a) = \begin{cases} 
  k(a - 1) + y & \text{if } a < a_{\text{thresh}} \\
  \tilde{\phi}(a)a + c(1 - a)^{1+b}a^{-b} & \text{if } a \geq a_{\text{thresh}}
\end{cases} \]
where $b := \frac{\bar{a}}{\bar{\lambda}}$ and
\[ c := \frac{\tilde{\phi}(a) - k}{(1 - a_{\text{thresh}})^{b(1-a_{\text{thresh}})} - b - 1(b + a_{\text{thresh}})} \]
and the optimal policy is (10).

For all $a < a_{\text{thresh}}$, we have
\[ (1 - a)\lambda \beta k + a - \bar{a} < 0 \]
and then $\tilde{\phi}(a) = k(a - 1) + y$ solves the HJB equation with $u = 1$. Note that this affine formula uniquely solves (12) with $u = 1$, since the homogeneous solution blows up at $\bar{a}$.

With $u = 0$, we already know that $\tilde{\phi}(a)a$ solves (12) and $c(1 - a)^{1+b}a^{-b}$ is the homogeneous solution for $a_{\text{thresh}}$. Next we want to show that $\mathcal{M}(a) > 0$ for all $a > a_{\text{thresh}}$. For $a > a_{\text{thresh}}$, $\mathcal{M}(a)$ can be expressed as
\[ \mathcal{M}(a) = \lambda \beta(1 - a) \left( \tilde{\phi}(a) - c(1 - a)^{b}a^{-b} \left( 1 + \frac{b}{a} \right) \right) + (a - \bar{a}) \]
We readily observe that
\[ \frac{d^2 \mathcal{M}}{da^2}(a) = \lambda \beta \frac{-b}{a} \left[ 1 + b + (b + 2) \left( 1 - \frac{a}{a_{\text{thresh}}} \right) \right] > 0 \]
where $\bar{\phi}(a) := (1 - a)^{1+b}a^{-b}$. Noting that $\mathcal{M}(a_{\text{thresh}}) = 0$ and $\mathcal{M}(a)$ is convex, it remains to show that $\lim_{a \uparrow a_{\text{thresh}}} \frac{d \mathcal{M}}{da}(a) > 0$.

\[ \lim_{a \uparrow a_{\text{thresh}}} \frac{d \mathcal{M}}{da}(a) = 1 - \lambda \beta k - \lambda \beta \frac{k - \tilde{\phi}(a)}{a_{\text{thresh}}(b + a_{\text{thresh}})}(b + 1) \]
\[ > 1 - \lambda \beta k - \frac{k - \tilde{\phi}(a)}{b + a_{\text{thresh}}} \gamma(b + 1) \]
where the last inequality follows from $a_{\text{thresh}} > \bar{a}$ which implies $-b > -\frac{a_{\text{thresh}}}{\lambda \beta}$. Therefore it remains to show
\[ (b + a_{\text{thresh}})(1 - \lambda \beta k) - (k - \tilde{\phi}(a)) \gamma(b + 1) > 0 \]
The lefthand side of the equation above turns out to be exactly 0. This can be shown as
\[ (b + a_{\text{thresh}})(1 - \lambda \beta k) - (k - \tilde{\phi}(a)) \gamma(b + 1) = b(1 - \lambda \beta k) + (\bar{a} - \lambda \beta k) - (k - \tilde{\phi}(a)) \gamma(b + 1) = \bar{a} + b - (b + 1) \left( \lambda \beta k + \gamma \tilde{\phi}(a) \right) = \bar{a} + b - (b + 1) \left( 1 - \frac{\alpha}{\bar{\lambda} \beta + \gamma} \right) = \bar{a} + b - b - 1 + 1 - \bar{a} = 0 \]
and thus $\mathcal{M}(a) > 0$ for all $a > a_{\text{thresh}}$.

Step 2: We now show if $\lambda a > \lambda \beta$, and
\[ \frac{1 - \bar{a}}{\lambda a + \gamma} + \frac{\lambda \beta + \gamma}{(\lambda a - \lambda \beta)\gamma} < \frac{\alpha}{\lambda \beta + \gamma} \]
then $0 < \bar{a} < a_{\text{thresh}} < 1$.

Equation (13) could be rearranged as
\[ \frac{\lambda a \tilde{\phi}(a) + 1 - \bar{a}}{\lambda a + \gamma} < \frac{\bar{a}}{\lambda \beta + \gamma} \]
Writing $k$ as
\[ k = \frac{1}{\lambda a + \gamma} \left( \gamma \lambda a \tilde{\phi}(a) + 1 - \bar{a} + \bar{a} \right) \]
and applying (14) gives
\[ \lambda \beta k < \bar{a} - \frac{\lambda \beta}{\lambda a + \gamma} \]
Since the second term is positive we obtain $\bar{a} > \lambda \beta k$ which implies $0 < a_{\text{thresh}} < 1$. Moreover, the former inequality implies
\[ (\lambda a - \lambda \beta)\lambda \beta k < \lambda a \alpha - \lambda \beta \alpha \]
which gives $\bar{a} < a_{\text{thresh}}$.

Step 3: It remains to derive a sufficient condition that $\lambda a > \lambda \beta$ implies (13) where
\[ \Delta a(\beta) = \max \left( \lambda \beta, \frac{(\lambda \beta + \gamma)(2 - \alpha)\bar{\lambda} a + \lambda \beta \gamma}{\lambda a + \gamma}, \frac{\lambda a \beta \gamma}{\alpha - \lambda \beta \phi(\bar{\alpha})} \right) \]
Note that
\[ \frac{1 - \bar{a}}{\lambda a + \gamma} + \frac{\lambda a \beta + \gamma}{(\lambda a - \lambda \beta)\gamma} < \frac{\bar{a}}{\lambda \beta + \gamma} \]
implies (13) and doing simple manipulations gives
\[ \lambda a > \frac{\lambda a \beta \gamma}{(2 - \alpha)\gamma + \lambda a \beta} + \frac{\lambda a \beta \gamma}{\alpha - \lambda \beta \phi(\bar{\alpha})} \]
Therefore $\lambda a > \Delta a(\beta)$ implies (13). Note that the function $\Delta a(\beta)$ is monotonic in its argument and $\lim_{\beta \to 0} \Delta a(\beta) = 0$. 
