Schrödinger quantum modes on the Taub-NUT background

Ion I. Cotăescu *
The West University of Timișoara,
V. Pârvan Ave. 4, RO-1900 Timișoara, Romania

Mihai Visinescu †
Department of Theoretical Physics,
National Institute for Physics and Nuclear Engineering,
P.O.Box M.G.-6, Magurele, Bucharest, Romania

September 2, 2018

Abstract

The Schrödinger equation is investigated in the Euclidean Taub-NUT geometry. The bound states are degenerate and an extra degeneracy is due to the conserved Runge-Lenz vector. The existence of the extra conserved quantities, quadratic in four-velocities implies the possibility of separating variables in two different coordinate systems. The eigenvalues and the eigenvectors are given in both cases in explicit, closed form.

Pacs 04.62.+v

---

*E-mail: cota@quasar.uvt.ro
†E-mail: mvisin@theor1.theory.nipne.ro
1 Introduction

The discovery of the self-dual instanton solutions to Euclidean Yang-Mills theory suggested the possibility that analogous solutions to the Euclidean Einstein equations might be important in quantum gravity. One example of a metric which satisfies the Euclidean Einstein equations with self-dual Riemann tensor is the self-dual Taub-NUT metric \[1\]. In this case Einstein’s equations are satisfied with zero cosmological constant and the manifold is \(\mathbb{R}^4\) with a boundary which is a twisted three-sphere \(S^3\) possessing a distorted metric.

On the other hand much attention has been paid to the Euclidean Taub-NUT metric because it is involved in many modern studies in physics. The Kaluza-Klein monopole of Gross and Perry \[2\] and of Sorkin \[3\] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Remarkably, the same object has re-emerged in the study of monopole scattering. In the long-distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space \[4, 5\].

It is well known that space-time isometries give rise to constants of motion along geodesics. However it is worth to mention that not all conserved quantities arose from isometries of the manifolds and associated Killing vector fields. Other integrals of motion are related to ”hidden” symmetries of the manifold, which manifest themselves as tensor of rank \(n > 1\), satisfying a generalized Killing equation, namely \(\nabla_{(\mu}K_{\nu_1...\nu_n)} = 0\) \[6\]. They are usually referred to as Stäckel-Killing tensors.

An illustration of the existence of extra conserved quantities is provided by the Taub-NUT geometry. Indeed, for the geodesic motions in the Taub-NUT space, there is a conserved vector, analogous to the Runge-Lenz vector of the Kepler type problem, whose existence is rather surprising in view of the complexity of the equations of motion \[7, 8, 9, 10\]. The Runge-Lenz vector is quadratic in four-velocities and its components are Stäckel-Killing tensors.

The Taub-NUT space is also of mathematical interest. In the Taub-NUT geometry there are known to exist four Killing-Yano tensors. The Killing-Yano tensor \[11\] here is a 2-form, \(f_{\mu\nu} = f_{[\mu\nu]}\), which satisfies the Penrose-Floyd equation \(\nabla_{(\mu}f_{\nu_1...\nu_n)} = 0\). Three of these Killing-Yano tensors are rather special: they are covariantly constant, mutually anti-commuting and square...
the minus unity. Thus they are complex structure realizing the quaternionic algebra and the Taub-NUT manifold is hyper-Kähler [8]. The fourth Killing-Yano tensor has a non-vanishing field strength, is not trivial and leads to new constants of motion. More precisely, the symmetrized product of the fourth Killing-Yano tensor with the previous three is connected with the Stäckel-Killing tensors, i.e. the components of the Runge-Lenz vector [8, 12, 13, 14].

The aim of this paper is to study the Schrödinger equation in Taub-NUT space. The conserved Runge-Lenz vector implies an extra degeneracy of the bound states. It is remarkably the fact that there appears to be a close relation between the existence of extra conserved quantities and the possibility of separating variables in two different coordinate systems. In this way one obtains two different orthonormal energy bases, called here the central and axial bases, for which the degeneracy of the energy levels is well defined by the eigenvalues of suitable sets of commuting operators.

The plan of the paper is as follows. In Section 2 we summarize the relevant features of the Euclidean Taub-NUT geometry. In Section 3 we analyze the Schrödinger equation in this space looking for the complete sets of commuting operators that define the mentioned bases. In the next two sections the Schrödinger equation is investigated in spherical and parabolic coordinates. We show that the central modes can be solved in spherical coordinates while for the axial modes we need to use parabolic coordinates. In both cases we obtain sets of regular modes as well as specific sets of irregular modes arising from the special boundary conditions of the Taub-NUT geometry. Our concluding remarks are presented in the last section. Finally, the Appendix is devoted to \(SO(3) \otimes U(1)\) harmonics.

2 Taub-NUT geometry

The Euclidean Taub-NUT manifold is a particular case of static Euclidean Kaluza-Klein space-time whose metric, in a static chart of coordinates, \(t\) and \(x^\mu\), \((\mu, \nu, ... = 1, 2, 3, 5)\), is given by the line element

\[
ds^2 = -dt^2 + g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \frac{1}{V} dl^2 + V(dx^5 + A_i dx^i)^2, \tag{1}
\]

where

\[
dl^2 = (d \vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{2}
\]
is the usual Euclidean 3-dimensional line element involving the Cartesian space coordinates \( x^i (i,j,... = 1,2,3) \) which cover the domain \( D \). The other coordinates are the time, \( t \), and the Cartesian Kaluza-Klein extra-coordinate, \( x^5 \in D_5 \). We suppose that \( V \) and \( A_i \) are static functions depending only on \( \vec{x} \). It is clear that in the absence of the scalar potential we have

\[
\text{div} \, \vec{A} = 0, \quad \text{rot} \, \vec{A} = \vec{B}.
\]  

From (1) we obtain the covariant components of the metric tensor

\[
g_{ij} = \frac{1}{V} \delta_{ij} + VA_i A_j, \quad g_{i5} = VA_i, \quad g_{55} = V, \tag{4}
\]

and

\[
g = \det(g_{\mu\nu}) = \frac{1}{V^2}. \tag{5}
\]

The corresponding contravariant components are

\[
g^{ij} = V \delta_{ij}, \quad g^{i5} = -VA_i, \quad g^{55} = \frac{1}{V} + VA^2. \tag{6}
\]

The Euclidean Taub-NUT manifold has the line element of the form (1) with

\[
V^{-1} = 1 + \frac{\mu}{r}, \quad A_1 = -\frac{\mu}{r^2} \frac{x^2}{r^2 + x^3}, \quad A_2 = \frac{\mu}{r^2} \frac{x^1}{r^2 + x^3}, \quad A_3 = 0, \tag{7}
\]

where \( r = |\vec{x}| \) and \( \mu \) is a real number. Hereby, it results the magnetic field with central symmetry

\[
\vec{B} = \mu \frac{\vec{x}}{r^3}. \tag{8}
\]

In fact, the geometry defined by (1) and (7) has the global symmetry of the group \( SO(3) \otimes U_5(1) \otimes T_t(1) \). This means that the line element is invariant under global rotations of the Cartesian space coordinates and \( x^5 \) and \( t \) translations of the Abelian groups \( U_5(1) \) and \( T_t(1) \) respectively. The generators of these groups are just the five Killing vectors corresponding to this global symmetry. Moreover, there is a three-component Killing tensor \( k_{\mu\nu} \) which satisfies \( \nabla_{(\sigma} k_{\mu\nu)} = 0 \) (where \( \nabla \) is the covariant derivative and \( (\ ) \) denotes the symmetrisation).
3 Observables

The quantum mechanics in the Taub-NUT geometry [10] is based on the Schrödinger equation

\[ H \psi = i \partial_t \psi \] (9)

involving the Hamiltonian operator

\[ H = -\frac{1}{2} \nabla_\mu g^{\mu\nu} \nabla_\nu , \] (10)

written in natural units with \( \hbar = c = 1 \). In any static chart, Eq.(8) has the particular solutions

\[ \psi_E(x) = U_E(x, x^5) e^{-iEt} , \] (11)

where \( U_E \) are energy eigenfunctions,

\[ HU_E = EU_E . \] (12)

The solutions (11) may be either square integrable functions or tempered distributions on \( D \times D_5 \). In both cases they must be orthonormal (in usual or generalized sense) with respect to the scalar product

\[ \langle \psi, \psi' \rangle = \int_{D \times D_5} d^4x \, \sqrt{g} \, \psi^* \psi' . \] (13)

We denote by \( L^2 \) the Hilbert space of the square integrable functions with respect to this scalar product.

The main operators on \( L^2 \) can be introduced by using the geometric quantization [10]. In this way, one obtains the non-hermitian momentum operators in the coordinate representation,

\[ P_i = -i(\partial_i - A_i \partial_5) , \quad P_5 = -i\partial_5 , \] (14)

which obey the commutation rules

\[ [P_i, P_j] = i\epsilon_{ijk} B_k P_5 , \quad [P_i, P_5] = 0 . \] (15)

With their help we can write

\[ H = \frac{1}{2} \left( V \vec{P}^2 + \frac{1}{V} P_5^2 \right) . \] (16)
Hereby we see that the $U_5(1)$ generator, $P_5$, is conserved since it commutes with $H$. This is natural since $P_5$ is up to the factor $-i$ just a Killing vector. In the following it is convenient to replace it by $Q = -\mu P_5$. Furthermore, one can verify that other three Killing vectors are the components of the angular momentum operator

$$\vec{L} = \vec{x} \times \vec{P} + \frac{\vec{x}}{r} Q.$$  

These are conserved and satisfy the canonical commutation rules. Moreover, their commutators with the coordinates and the momentum operators are the usual ones. Other three important conserved operators are those defined by the three-component Killing tensor $\vec{k}_{\mu\nu}$ as

$$K = -\frac{1}{2} \nabla_\mu k^{\mu\nu} \nabla_\nu = \frac{1}{2} \left( \vec{P} \times \vec{L} - \vec{L} \times \vec{P} \right) - \mu \frac{\vec{x}}{r} \left( H - \frac{1}{\mu^2} Q^2 \right).$$  

Thus we obtain a vector operator which play the same role as the Runge-Lenz vector in the usual quantum mechanical Kepler problem. This transforms as a vector under space rotations since its components obey

$$[L_i, K_j] = i\epsilon_{ijk}K_k.$$  

Now we have to chose the suitable complete sets of commuting operators which should define usual or generalized bases for the Hilbert space $L^2$. Here two options are interesting, namely (I) that of the central basis given by the set $\{H, \vec{L}, L_3, Q\}$ or (II) the axial basis formed by the common eigenfunctions of the set $\{H, K_3, L_3, Q\}$. In both cases we must work in suitable coordinate systems which should allow us to separate the variables in all the differential equations corresponding to the eigenvalue problems.
4 Central modes

In order to study the central modes it is convenient to choose the local chart with spherical coordinates, \( r, \theta, \phi \), commonly related to the Cartesian ones and the new coordinate \( \chi \) defined as

\[
\chi = -\frac{1}{\mu} x^5 - \arctan \frac{x^2}{x^1}.
\]  

(20)

Here the Taub-NUT line element is

\[
ds^2 = -dt^2 + \frac{1}{V}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \mu^2 V(d\chi + \cos \theta d\phi)^2,
\]  

(21)

since

\[
A_r = A_\theta = 0, \quad A_\phi = \mu(1 - \cos \theta).
\]  

(22)

Moreover, we consider that in this chart \( r \in D_r = \{ r | V(r) > 0 \} \) (i.e., \( r > 0 \) if \( \mu > 0 \) or \( r > |\mu| \) if \( \mu < 0 \)), the angular coordinates \( \theta, \phi \) cover the sphere \( S^2 \) while \( \chi \in D_\chi = [0, 4\pi) \).

The main observables are

\[
Q = -i \partial_\chi
\]  

(23)

and the components of the angular momentum \([17]\) in the canonical basis (with \( L_\pm = L_1 \pm iL_2 \)),

\[
L_3 = -i \partial_\phi, \quad L_\pm = e^{\pm i\phi} \left[ \mp \partial_\theta + i \left( \cot \theta \partial_\phi - \frac{1}{\sin \theta} \partial_\chi \right) \right].
\]  

(24)

(25)

The \( SO(3) \) Casimir operator

\[
\hat{L}^2 = L_+ L_- + L_3^2 - L_3
\]  

(26)

\[
= -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} (\partial_\phi^2 + \partial_\chi^2 + 2 \cos \theta \partial_\phi \partial_\chi)
\]

is related to the momentum operators through

\[
-\hat{P}^2 = \partial_r^2 + \frac{2}{r} \partial_r - \frac{1}{r^2} \hat{L}^2 + \frac{1}{r^2} Q^2.
\]  

(27)
Looking for the common eigenfunctions of the set (I), we observe that the angular momentum as well as $Q$ act only upon functions of $\theta$, $\phi$, and $\chi$ which must be square integrable since the domain $S^2 \times D\chi$ is compact. These form the Hilbert space $L^2(S^2 \times D\chi)$ where the complete set of commuting operators $\{\vec{L}^2, L_3, Q\}$ determines a suitable basis. Their common eigenfunctions, $Y^q_{l,m}$, which satisfy the eigenvalue problems

$$\vec{L}^2 Y^q_{l,m} = l(l+1)Y^q_{l,m}, \quad (28)$$
$$L_3 Y^q_{l,m} = m Y^q_{l,m}, \quad (29)$$
$$Q Y^q_{l,m} = q Y^q_{l,m}, \quad (30)$$

and the orthonormalization condition

$$\langle Y^q_{l,m}, Y'^q_{l',m'} \rangle = \int_{S^2} d(\cos \theta) d\phi \int_0^{4\pi} d\chi Y^q_{l,m}(\theta, \phi, \chi)^* Y'^q_{l',m'}(\theta, \phi, \chi) = \delta_{l,l'}\delta_{m,m'}\delta_{q,q'}, \quad (31)$$

will be called $SO(3) \otimes U(1)$ harmonics. Notice that the boundary conditions on $S^2 \times D\chi$ require $l$ and $m$ to be integer numbers while $q = 0, \pm 1/2, \pm 1, ...$ [10]. The form of these harmonics is given in Appendix.

Thus the problem of the angular motion is completely solved and it is clear that the angular coordinates can be separated from the radial one if we take the common eigenfunctions of the set (I) of the form

$$U^q_{E,l,m}(r, \theta, \phi, \chi) = \frac{1}{r} R^q_{E,l}(r) Y^q_{l,m}(\theta, \phi, \chi). \quad (32)$$

Then from Eq.(12) we obtain the familiar radial equation,

$$\left[ -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{\alpha}{r} \right] R^q_{l,m}(r) = \beta R^q_{l,m}(r) \quad (33)$$

where we have denoted

$$\alpha = 2\mu \left[ E - \frac{q^2}{\mu^2} \right], \quad \beta = 2E - \frac{q^2}{\mu^2}. \quad (34)$$

The radial wave functions must be orthonormal with respect to the radial scalar product

$$\langle R^q_{E,l}, R'^q_{E',l} \rangle = \int_{D_r} dr \left( 1 + \frac{\mu}{r} \right) R^q_{E,l}(r)^* R'^q_{E',l}(r). \quad (35)$$
This is a Kepler-like problem similar to that of the non-relativistic quantum mechanics. The general solution of Eq. (33) can be written in terms of the confluent hypergeometric function as

\[ R_{E,l}^q(r) = N \rho^s e^{-\rho} F(a, 2s, \rho), \]

where

\[ \rho = 2r\sqrt{-\beta}, \quad a = s - \frac{\alpha}{2\sqrt{-\beta}}, \]

and \( N \) is the normalization constant. The parameter \( s \) is a solution of the equation \( s(s - 1) = l(l + 1) \). The modes with \( s = l + 1 \) are similar to those of the usual non-relativistic case. For this reason we say that these are regular modes. We shall see that in the Taub-NUT geometry there is a conjecture in which irregular modes with \( s = -l \) are also allowed. In the following we shall briefly discuss the corresponding energy spectra assuming that \( E \geq 0 \).

Let us consider first the regular modes (\( s = l + 1 \)). It is easy to show that for \( \mu > 0 \) there is only a continuous energy spectrum, covering the domain \( E \geq q^2/2\mu^2 \), the levels of which are infinite degenerated as it result from the selection rules (68). Notice that these rules are in accordance with the behavior of the classical trajectories which are open and cannot reach the center. In the case of \( \mu < 0 \) we have the same continuous energy spectrum with infinite degenerate levels and, in addition, a discrete spectrum in the domain \([0, q^2/2\mu^2]\) where \( \alpha > 0 \) and \( \beta < 0 \). Here we can impose the quantization condition \( a = -n_r, n_r = 0, 1, 2, ... \) which gives

\[ \frac{\alpha}{\sqrt{-\beta}} = 2n \]

where \( n = n_r + l + 1 \) is the main quantum number of the regular modes. Hereby we find the energy levels

\[ E_n = \frac{1}{\mu^2} \left[ n\sqrt{n^2 - q^2} - (n^2 - q^2) \right] \]

for all \( n > |q| > 0 \). These levels are finite degenerated since for given \( q \) and \( n \) the angular quantum numbers can take all the values selected by the conditions \( |q| - 1 < l \leq n - 1 \) and (68). Thus one obtains a countable energy spectrum with the property

\[ \lim_{n \to \infty} E_n = \frac{q^2}{2\mu^2}. \]
The irregular modes arise only as discrete modes when \( \mu < 0 \) since the radial functions (36) remain square integrable on \( D_r = [|\mu|, \infty) \) for \( s = -l \) if \( a = -n_r \) and \( |q| + l < n_r < 2l \). The main quantum number of these modes, \( n = n_r - l \), obeys \( |q| < n < l \) and gives the same quantization rule (38) and energy levels (39). Therefore, the discrete spectrum of irregular modes coincides to that of the regular ones, spanning just the domain of energies where the classical motion has closed trajectories. The difference is that in the case of irregular modes the energy levels are infinite degenerated (since \( l > n \)) and there is no continuous energy spectrum.

5 Axial modes

Let us consider now the axial modes given by the eigenvalue problem of the set (II). This can be solved in a local chart with parabolic coordinates, \( \xi \in D_\xi, \eta \in D_\eta \) and \( \phi \in [0, 2\pi) \), defined by

\[
    x^1 = \sqrt{\xi\eta}\cos\phi, \quad x^2 = \sqrt{\xi\eta}\sin\phi, \quad x^3 = \frac{1}{2}(\xi - \eta),
\]

such that

\[
    r = \frac{1}{2}(\xi + \eta).
\]

Here the domains \( D_\xi \) and \( D_\eta \) corresponding to \( D_r \) are either \( \xi, \eta > 0 \) for \( \mu > 0 \) or \( \xi, \eta > |\mu| \) if \( \mu < 0 \). The line element reads

\[
    ds^2 = -dt^2 + \frac{1}{V}dl^2 + \mu^2V\left(d\chi + \frac{\xi - \eta}{\xi + \eta}d\phi\right)^2
\]

where

\[
    dl^2 = \frac{\xi + \eta}{4\xi}d\xi^2 + \frac{\xi + \eta}{4\eta}d\eta^2 + \xi\eta d\phi^2.
\]

Following the same procedure like in the case of the central modes we calculate how look the operators of the set (II) in these coordinates. It is not difficult to show that

\[
    H = \frac{1}{2\mu^2}Q^2 - \frac{2}{\xi + \eta + 2\mu}(X + Y)
\]
and
\[ K_3 = \frac{2}{\xi + \eta + 2\mu} \left[ (\eta + \mu)X - (\xi + \mu)Y \right] \] (46)
where we have denoted
\[ X = \partial_\xi (\xi \partial_\xi) - \frac{1}{4\xi} (L_3 + Q)^2 - \frac{1}{4\mu} Q^2, \] (47)
\[ Y = \partial_\eta (\eta \partial_\eta) - \frac{1}{4\eta} (L_3 - Q)^2 - \frac{1}{4\mu} Q^2. \] (48)

The whole eigenvalue problem of axial modes is
\[ HU_{E,\kappa,m}^q = EU_{E,\kappa,m}^q, \] (49)
\[ K_3 U_{E,\kappa,m}^q = \kappa U_{E,\kappa,m}^q, \] (50)
\[ L_3 U_{E,\kappa,m}^q = m U_{E,\kappa,m}^q, \] (51)
\[ QU_{E,\kappa,m}^q = q U_{E,\kappa,m}^q. \] (52)

These equations can be solved in parabolic coordinates if we put
\[ U_{E,\kappa,m}^q(\xi, \eta, \phi, \chi) = N f_{E,\kappa,m}^q(\xi) h_{E,\kappa,m}^q(\eta) e^{im\phi} e^{iq\chi}, \] (53)
where \( N \) is the normalization constant calculated with the help of the scalar product (13) rewritten in parabolic coordinates as
\[ \langle U, U' \rangle = \frac{1}{4} \int_{D_\xi} d\xi \int_{D_\eta} d\eta (\xi + \eta + 2\mu) \times \int_0^{2\pi} d\phi \int_{-4\pi}^{4\pi} d\chi U^*(\xi, \eta, \phi, \chi) U' (\xi, \eta, \phi, \chi). \] (54)

By taking into account that the operators of Eqs. (49) and (50) have the form (15) and (16) respectively, we find that \( f \) and \( h \) satisfy
\[ \left[ \partial_\xi (\xi \partial_\xi) - \frac{(m + q)^2}{4\xi} + \frac{\alpha}{4} + \frac{\beta}{4\xi} \right] f_{E,\kappa,m}^q(\xi) = \frac{\kappa}{2} f_{E,\kappa,m}^q(\xi), \] (55)
\[ \left[ \partial_\eta (\eta \partial_\eta) - \frac{(m - q)^2}{4\eta} + \frac{\alpha}{4} + \frac{\beta}{4\eta} \right] h_{E,\kappa,m}^q(\eta) = -\frac{\kappa}{2} h_{E,\kappa,m}^q(\eta), \] (56)
where \( \alpha \) and \( \beta \) are the functions of \( E \) and \( q \) defined by Eqs. (34). The last step is to solve these equations. Defining the new variables
\[ x = \xi \sqrt{-\beta}, \quad y = \eta \sqrt{-\beta}, \] (57)
we obtain the general form of the solutions,

\[ f^E_{q,\kappa,m}(\xi) = x^{s_1} e^{-x/2} F(a_1, 2s_1 + 1, x), \quad (58) \]
\[ h^E_{q,\kappa,m}(\eta) = y^{s_2} e^{-y/2} F(a_2, 2s_2 + 1, y), \quad (59) \]

where

\[ a_1 = s_1 + \frac{1}{2} + \frac{2\kappa - \alpha}{4\sqrt{-\beta}}, \quad (60) \]
\[ a_2 = s_2 + \frac{1}{2} - \frac{2\kappa + \alpha}{4\sqrt{-\beta}}, \quad (61) \]

depend on the parameters \( s_1 = \pm(\vert m \vert + q)/2 \) and \( s_2 = \pm(\vert m \vert - q)/2 \).

We define the regular axial modes for \( s_1 = (\vert m \vert + q)/2 > -1/2 \) and \( s_2 = (\vert m \vert - q)/2 > -1/2 \) which means that \( \vert m \vert > \vert q \vert - 1 \). These correspond to the regular central modes we have discussed in the previous section. The discrete energy spectrum appears in the same conditions \((\alpha > 0 \text{ and } \beta < 0)\) but now the quantization rules are

\[ a_1 = -n_1, \quad a_2 = -n_2, \quad n_1, n_2 = 0, 1, 2, \ldots. \quad (62) \]

Hereby we recover the quantization rule \( (38) \) with the main quantum number of the regular modes \( n = n_1 + n_2 + |m| + 1 \) and the same formula of the energy levels \( (39) \). Moreover, we find that

\[ \kappa = \sqrt{|\beta|} (n_2 - n_1 - q). \quad (63) \]

Thus the regular discrete axial modes are labeled by the quantum numbers \( n_1, n_2, m \) and \( q \). Notice that \( \alpha \) and \( \beta \) depend only on \( q \) and \( n \) as it results from Eqs. \((34)\) and \((39)\).

The irregular discrete modes can be derived for \( \mu < 0 \) in the same manner like in the case of the central modes, by looking for the conditions in which the wave functions \( (58) \) remain square integrable on the domains \( \xi, \eta > |\mu| \) for \( s_1, s_2 < 0 \). We observe that this happens only when \( a_1 = -n_1, a_2 = -n_2, s_1 = -(|m| + q)/2 \) and \( s_2 = -(|m| - q)/2 \) such that \( n_1 < |m| + q - 1 \) and \( n_2 < |m| - q - 1 \). Hereby it results that the main quantum number of these modes, \( n = n_1 + n_2 - |m| + 1 \), accomplishes the selection rules \( |q| < n < |m| - 1 \) and \( \kappa = \sqrt{|\beta|} (n_2 - n_1 + q) \).
6 Concluding remarks

The study of the Schrödinger equation in Euclidean Taub-NUT space is well motivated. The Killing tensors of the Taub-NUT geometry imply the existence of extra conserved quantities, quadratic in four-velocities. The Taub-NUT case is analogous to the Coulomb problem where an extra degeneracy is present and there is the possibility of separating variables in two different coordinate systems. This analogy is not perfect since in the Taub-NUT geometry the selection rules of the angular quantum numbers depend on \( q \) and the boundary conditions allow irregular modes. However, this has nothing surprising in general relativity where we know already the irregular modes of spinless \[16\] or spin-half \[17\] particles in anti-de Sitter space-times.

In the last time, Iwai and Katayama \[18, 19, 20, 21\] extended the Taub-NUT metric so that it still admits a Kepler-type symmetry. This class of metrics, of course, includes the original Taub-NUT metric. In general the Killing tensors involved in the Runge-Lenz vector cannot be expressed as a contracted product of Killing-Yano tensors. The only exception is the original Taub-NUT space \[22\]. It will be interested to investigate the Schrödinger equation in the more complicated case of the generalized Taub-NUT spaces. The extension of the results for this class of space-times will be discussed elsewhere \[23\].

A \( SO(3) \otimes U(1) \) harmonics

The \( SO(3) \otimes U(1) \) harmonics defined by Eqs.(28)-(31) cannot be expressed in terms of usual spherical harmonics. These are new harmonics that must be separately studied by solving the system of eigenvalue equations. We start with

\[
Y_{\ell,m}^{q}(\theta, \phi, \chi) = \frac{1}{4\pi} \Theta_{\ell,m}^{q}(\cos \theta) e^{im\phi} e^{iq\chi} \tag{64}
\]

since then Eqs.(29) and (30) are satisfied. It remains to solve only Eq.(28) and calculate the normalization factor of \( \Theta_{\ell,m}^{q} \) according to the condition

\[
\int_{-1}^{1} d(\cos \theta) |\Theta_{\ell,m}^{q}(\cos \theta)|^2 = 2. \tag{65}
\]
To this end, we introduce the new variable \( z = \sin^2 \theta / 2 \) in Eq. (28) where \( \vec{L}^2 \) is given by (26). Thus we obtain

\[
\left[ z(1 - z) \frac{d^2}{dz^2} + (1 - 2z) \frac{d}{dz} + l(l + 1) - \frac{m^2 + q^2 - 2(1 - 2z)mq}{4z(1 - z)} \right] \Theta^q_{l,m}(z) = 0.
\]

This equation has solutions of the form

\[
\Theta^q_{l,m}(z) \sim z^p(1 - z)^k F(p + k - l, p + k + l + 1, 2p + 1, z),
\]

where the Gauss hypergeometric functions, \( F \), depend on the real parameters \( p \) and \( k \) which satisfy \( p^2 = (m - q)^2 / 4 \) and \( k^2 = (m + q)^2 / 4 \).

First of all we observe that the solutions (67) are square integrable only when \( F \) are polynomials selected by a quantization condition since otherwise \( F \) becomes strongly divergent for \( z \to 1 \). This means that \( l - p - k \) must be a non-negative integer number. If we replace the functions \( F \) of (67) by Jacobi polynomials \([24]\), we observe that the solutions of Eq. (66) remain square integrable if \( 2p > -1 \) and \( 2k > -1 \). Like in the case of the usual spherical harmonics, the good choice is \( p = (|m| - q) / 2 \) and \( k = (|m| + q) / 2 \) where

\[
|m| - 1 < |m| \leq l.
\]

Then by using the normalization condition (65) we find the final result

\[
\Theta^q_{l,m}(\cos \theta) = \frac{\sqrt{2l + 1}}{2^{2|m|}} \left[ \frac{(l - |m|)! (l + |m|)!}{\Gamma(l - q + 1) \Gamma(l + q + 1)} \right]^\frac{1}{2} \times (1 - \cos \theta)^{\frac{|m| + q}{2}} (1 + \cos \theta)^{\frac{|m| - q}{2}} P_{l-|m|}(|m| - q, |m| + q)(\cos \theta).
\]

For \( m = |m| \) the \( SO(3) \otimes U(1) \) harmonics are given by (64) and (69) while for \( m < 0 \) we have to use the obvious formula

\[
Y_{l,-m}^q = (-1)^m \left( Y_{l,m}^{-q} \right)^*.
\]

When the boundary conditions allow half-integer quantum numbers \( l \) and \( m \) then we say that the functions defined by Eqs. (64) and (69) (up to a suitable factor) represent \( SU(2) \otimes U(1) \) harmonics.
Thus we have obtained a non-trivial generalization of the spherical harmonics of the same kind as the spin-weighted spherical harmonics \[25\]. Indeed, if \( l, m \) and \( q = m' \) are either integer or half-integer numbers then we have

\[ Y_{lm}'(\theta, \phi, \chi) = \sqrt{\frac{2l+1}{4\pi}} D^l_{m,m'}(\phi, \theta, \chi) \]

(71)

where \( D^l_{m,m'} \) are the matrix elements of the irreducible representation of weight \( l \) of the \( SU(2) \) group corresponding to the rotation of Euler angles \( (\phi, \theta, \chi) \). What is new in the case of our harmonics is that these are defined for any real number \( q \).

References

[1] S. W. Hawking, Phys. Lett. \textbf{60A} (1977) 81.
[2] D. J. Gross and M. J. Perry, Nucl. Phys. \textbf{B226} (1983) 29.
[3] R. D. Sorkin, Phys. Rev. Lett. \textbf{51} (1983) 87.
[4] N. S. Manton, Phys. Lett. \textbf{B110} (1985) 54; \textit{id}, \textbf{B154} (1985) 397; \textit{id}, (E) \textbf{B157} (1985) 475.
[5] M. F. Atiyah and N. Hitchin, Phys. Lett. \textbf{A107} (1985) 21.
[6] G. W. Gibbons and C. A. R. Herdeiro, \texttt{hep-th/9906093}.
[7] G. W. Gibbons and N. S. Manton, Nucl. Phys. \textbf{B274} (1986) 183.
[8] G. W. Gibbons and P. J. Ruback, Phys. Lett. \textbf{B188} (1987) 226; Commun. Math. Phys. \textbf{115} (1988) 267.
[9] L. Gy. Feher and P. A. Horvathy, Phys. Lett. \textbf{B182} (1987) 183; \textit{id}, (E) \textbf{B188} (1987) 512.
[10] B. Cordani, L. Gy. Feher and P. A. Horvathy, Phys. Lett. \textbf{B201} (1988) 481.
[11] K. Yano, Ann. Math. \textbf{55} (1952) 328.
[12] J. W. van Holten, Phys. Lett. \textbf{B342} (1995) 47.
[13] D. Vaman and M. Visinescu, *Phys. Rev.* **D57** (1998) 3790.

[14] D. Vaman and M. Visinescu, *Fortschr. Phys* **47** (1999) 493.

[15] B. Carter, *Phys. Rev.* **D16** (1977) 3395.

[16] S. J. Avis, C. J. Isham and D. Storey, *Phys. Rev.* **D10** (1978) 3565.

[17] I. I. Cotăescu, *Phys. Rev.* **D60** (to appear).

[18] T. Iwai and N. Katayama, *J. Geom. and Phys.* **12** (1993) 55.

[19] T. Iwai and N. Katayama, *J. Phys. A: Math. Gen.* **27** (1994) 3179.

[20] T. Iwai and N. Katayama, *J. Math. Phys.* **35** (1994) 2914.

[21] Y. Miyake, *Osaka J. Math.* **32** (1995) 659.

[22] M. Visinescu, *in preparation.*

[23] I. I. Cotăescu and M. Visinescu, *in preparation.*

[24] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964)

[25] E. Newman and R. Penrose, *J. Math. Phys.* **7** (1966) 863.