A Note on Decoupling Conditions for Generic Level $\hat{sl}(3)_k$
and Fusion Rules

A.Ch. Ganchev*, V.B. Petkova*† and G.M.T. Watts**

*Institute for Nuclear Research and Nuclear Energy,
Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

†Arnold Sommerfeld Institute for Mathematical Physics, TU Clausthal,
Leibnizstr. 10, 38678 Clausthal-Zellerfeld, Germany

**Mathematics Department, King's College London,
Strand, London WC2R 2LS, UK

Abstract
We find the solution of the $\hat{sl}(3)_k$ singular vector decoupling equations on 3–point functions for the particular case when one of the fields is of weight $w_0 \cdot k \Lambda_0$. The result is a function with non-trivial singularities in the flag variables, namely a linear combination of \(2F_1\) hypergeometric functions. This calculation fills in a gap in [I] and confirms the $\hat{sl}(3)_k$ fusion rules determined there both for generic $\kappa \not\in \mathbb{Q}$ and fractional levels.

We have also analysed the fusion in $\hat{sl}(3)_k$ using algebraic methods generalising those of Feigin and Fuchs and again find agreement with [I]. In the process we clarify some details of previous treatments of the fusion of $\hat{sl}(2)_k$ fractional level admissible representations.

ganchev@inrne.bas.bg, ptvp@pt.tu-clausthal.de, gmtw@mth.kcl.ac.uk
1. Introduction

A physicist’s motivation for studying fractional level WZW models is that by Drinfeld–Sokolov reduction one can obtain a huge class of $W$-algebra models.

Much of the simplicity of integrable WZW models rests in the fact that all the relevant representations can be induced from finite dimensional representations of the horizontal Lie algebra – conversely, all correlation functions can be reduced by the Ward identities to correlation functions of fields transforming in finite dimensional representations of the horizontal subalgebra. Finite dimensional representations can be described in terms of polynomials on the flag manifold, and so correlation functions of fields in integrable models also have polynomial dependence on the flag manifold coordinates. In fractional level WZW models where the representations are characterised by non integral weights [2], the correlation functions of any fields can again be reduced to functions of flag manifold variables, but these correlation functions can have singularities not only in the chiral (‘space-time’) variable but also singularities on nontrivial submanifolds of the flag manifold.

The case of $\mathfrak{sl}(2)$ has been much studied and worked out in some detail. An example are the correlation functions of fractional level $\hat{\mathfrak{sl}}(2)$ found in [3], [4]: the 4–point blocks are functions of the flag manifold (“isospin”) coordinate $x$ given by certain multiple contour integrals and it was shown that there exists a choice of the contours, depending nontrivially on $x$, such that the asymptotic behaviour of the 4-point blocks reproduces the fusion rules found by Awata and Yamada (AY) [5]. A different fusion rule was proposed somewhat earlier by Bernard and Felder (BF) [6], and confirmed by examples of 4–point correlators [7]. The BF rule, however, looks rather degenerate and leads to nilpotent fusion matrices. A more abstract analysis of the number of independent 3-point couplings determining the fusion rules was carried out in [8] and [9], who confirmed the fusions rules of AY and BF respectively, the set of couplings allowed by the BF fusion rule being a subset of those allowed by the AY rules. In section 3 we explain clearly the relation between the calculations of [8] and [9].

The situation for higher rank cases is much more complicated, even on the level of 3–point invariants. The decoupling of the Verma module singular vectors is central in any of the methods used to derive the fusion rules. The main obstacle comes from the fact that the expressions [10] for the singular vectors in Verma modules of non integral highest weights are too complicated to be analysed as systematically as can be done in the simplest $\mathfrak{sl}(2)$ case, or as can be done for the general integrable representations where (apart from
a simple additional “affine” vector) the problem reduces to that of the finite dimensional representations of the horizontal subalgebra.

Despite these difficulties, the fusion rules for the admissible \( \widehat{sl}(3)_k \) representations at level \( \kappa = k + 3 = 3/p \) were found in [1] using only partial information about the solutions of the decoupling equations. These rules, which can be easily extended to the general admissible level values \( \kappa = p'/p \), are expected to be the \( sl(3) \) analogue of the generic \( sl(2) \) AY fusion rules.

Moreover, just as the fusion rules at integral \( k \) are a truncation of the ring of tensor products of finite dimensional representations of \( \mathfrak{g} \), so the fusion rules at a rational level \( \kappa = 3/p \) are truncations of the fusion ring of a nonrational CFT \( (\kappa \not\in \mathbb{Q}) \); similarly, the singular vector vector decoupling equations for the representations arising in this nonrational CFT are a subset of those in the rational CFT. The fusion ring of this irrational CFT was described in [1] where it was explicitly realised by a novel extension of the ring of formal characters of finite dimensional representations of \( sl(3) \). It is generated by a simple current \( \gamma, \gamma^3 = 1 \) and three ‘fundamental’ representations \( f, f^* \) and \( h \), satisfying one relation. These representations have (horizontal) weights \( f = -\bar{\Lambda}_1\kappa, f^* = -\bar{\Lambda}_2\kappa \) and \( h = w_{121} \cdot (-\bar{\Lambda}_1 + \bar{\Lambda}_2)\kappa \) respectively, where \( \bar{\Lambda}_1, \bar{\Lambda}_2 \) are the \( sl(3) \) fundamental weights. The fusion of the fields \( f \) and \( f^* \) with a generic field is a sum of seven terms, each with multiplicity one, and in this way they are analogues of the \( sl(3) \) fundamental representations; the fusion of \( h \) however has three terms of multiplicity one and one of multiplicity two. In [1], the singular vector decoupling equations were used to examine the space of three point couplings, and the (generically) 7-terms fusion of \( f \) was found, leading also to explicit solutions for the 3–point invariants. However, for fields which admitted fusions with non-trivial multiplicities, the number of three-point couplings with a generic field found in [1] was less than that predicted by the fusion ring.

In this paper we reconsider this problem and analyse the decoupling conditions for the simplest nontrivial multiplicity examples by two different methods. In section 2 we use (as in [1]) the standard method of representing the generators and the singular vectors in terms of differential operators with respect to the flag variable coordinates \( X \). Whereas in [1] an explicit ansatz was used for the three-point coupling, here we consider an arbitrary function of \( X \); we find that for the representation \( h \) the null-vector decoupling conditions reduce to two hypergeometric equations. For the multiplicity two state we get two solutions spanned by appropriate \( _2F_1 \) hypergeometric functions. Similar analysis applied to a more
complicated example involving 4-order differential equations also confirms the fusion rule multiplicities of [1].

The second method we exploit in section 3 is purely algebraic, extending a method of Feigin and Fuchs [11] for the Virasoro algebra, and of Maliikov and Feigin [8] for \( \hat{sl}(2) \). It is based on the fact that, while the fields \( \Phi(X) \) transform under a differential operator realisation of \( \mathfrak{g} = sl(3) \) which is neither highest nor lowest weight representation, these fields can be also interpreted as highest weight states with respect to a choice of the Borel subalgebra \( \hat{b}(X) \) depending on the coordinates \( X \) of the flag manifold. The idea is to consider the space of states generated by the first field in a 3–point function and to use the fact that the two other fields are highest/lowest weight states after an appropriate rotation of the Borel subalgebra. Thus this space of states is effectively factored by an ideal depending on \( X \) and the truncation is so severe that we are left with a finite dimensional space describing the possible fusion rules. In section 3 we start first with the simpler case of \( sl(2) \) where we discuss the relation between the fusion rules in [5] and those in [6]. In the \( sl(3) \) case, the algebraic approach is applied to the couplings of the representation \( h \) for generic values of the coordinate \( X \). The two approaches to the decoupling problem are in full agreement and confirm the fusion rule of [1].

1.1. Fields and generators

In this paper we shall deal with representations of \( \mathfrak{g} = \hat{sl}(3)_k \) labelled by non-integral weights. When (as is typically the case) the horizontal projection of the non-integral weight is also non-integral, the associated field does not transform in a finite dimensional representation of the horizontal subalgebra \( \mathfrak{g} \).

The infinite dimensional representations which arise this way are described classically by induced representations of \( SL(3) \) on functions \( f : SL(3) \rightarrow \mathbb{C} \) satisfying the condition \( f(g'YH) = \chi_{\mu}(H)f(g') \), where \( \chi_{\mu}(H) \) is a character of the Cartan subgroup and \( Y \) is the subgroup of upper triangular matrices with units on the diagonal. The representation is given by the left multiplication of \( SL(3) \) on the space of functions, \((T(g)f)(g') = f(g^{-1}g')\).

Equivalently these representations are realised in terms of functions on the homogeneous \( G/(YH) \) space obtained by dividing by the Borel subgroup \( YH \). This space is the flag manifold \( \bigcup_{w \in W} \hat{w}X \), where \( W \) is the Weyl group with elements \( w \) represented by matrices \( \hat{w} \in SL(3) \). The generic 3-dimensional “big cell” \( X = X_1 \) consists of lower triangular matrices with units on the diagonal; we shall denote them \( X(x, y, z) \), or, simply \( X \), where \((x, y, z)\) are coordinates corresponding to the matrix entries 21, 32 and 31 respectively (see,
e.g., \[12\], or, for a broader discussion, \[13\]). The remaining cells $\hat{w} \hat{X}_w$, with $\hat{X}_w$ given by lower triangular matrices with some zeros, account for the “infinites” encountered in the (local) Gauss decomposition of the elements of $SL(3)$ and are thus needed to give meaning to the global left action of the group. Hence these remaining cells can also be thought of as the result of taking some of the coordinates in $X$ to infinity in certain prescribed ways.

In particular, $\hat{w}_\theta = \hat{w}_\theta X(0,0,0)$ is the only point of the cell of lowest dimension of the flag manifold where $\hat{w}_\theta$ is the matrix realising the action of the longest Weyl group element $w_{121} = w_\theta$, coinciding in the $sl(3)$ case with the reflection with respect to the highest root $\theta = \alpha_1 + \alpha_2$.

The right action of the group, $(T^R(g)f)(g') = f(g' g)$, or rather its infinitesimal version, has a different meaning: because of the above invariance with respect to $YH$, any of the functions $f$ can be identified with a highest weight state of a highest weight representation built by the generators of (right) translations.

Following the classical analogy, the operator valued quantum fields (chiral vertex operators) depend on a pair of variables, $\Phi_\mu(X) \equiv \Phi_\mu(X,u)$, where $u$ is the usual space-time chiral coordinate. The complex number $u$ can be similarly interpreted as a coordinate on the big cell of the flag manifold of $SL(2)$, the action of the non-trivial Weyl group element $r$ is given by $\hat{r}(u) = -1/u$, and the point at infinity on the Riemann sphere is the only point $\hat{r}(0)$ on $\hat{r} \hat{X}_r$.

The commutation relations of the generators $T_n$ of $\hat{g}$ with these fields are

$$[\Phi_\mu(X,u), T_n] = u^n D^{(\mu)}(T) \Phi_\mu(X,u)$$

(1.1)

where $T \equiv T_0 \in \hat{g}$ and $D^{(\mu)}(T)$ are differential operators (the infinitesimal version of the above left action of the group). In our convention these are

$$D(f^1) = \partial_x, \quad D(f^2) = -(\partial_y + x \partial_z), \quad D(f^3) = -\partial_z,$$
$$D(h^1) = 2x \partial_x - r_1 - y \partial_y + z \partial_z, \quad D(h^2) = 2y \partial_y - r_2 - x \partial_x + z \partial_z,$$
$$D(e^1) = (x)^2 \partial_x - r_1 x + (z - xy) \partial_y + xz \partial_z, \quad D(e^2) = (y)^2 \partial_y - r_2 y - z \partial_x,$$
$$D(e^3) = (z)^2 \partial_z - r_2 (z - xy) - r_1 z + xz \partial_x + y(z - xy) \partial_y,$$

(1.2)

where for simplicity we have omitted the index $\mu$. Here $r_i = \langle \mu, \alpha_i \rangle = \mu(h^i)$ are the components of the weight $\mu$. This is a representation of $\hat{g}$ with $k = 0$. 

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Analogously, the dual state \( |\gamma\rangle \) satisfying \( \langle \gamma | \nu \rangle = \delta_{\nu \gamma} \) and
\[
\langle \gamma | f^i = 0, \quad \langle \gamma | h^i = \gamma(h^i) \langle \gamma |, \quad \langle \gamma | T_{-n} = 0, \quad n > 0.
\]

One can define states \( |\nu\rangle = \lim_{x,y,z,u \to 0} \Phi_{\nu}(X,u)|0\rangle \), which satisfy the usual conditions of a highest weight state
\[
e^i |\nu\rangle = 0, \quad h^i |\nu\rangle = -D(\nu)(h^i)|\nu\rangle = \nu(h^i) |\nu\rangle, \quad T_n |\nu\rangle = 0, \quad n > 0, \quad (1.3)
\]
where furthermore \( |\nu\rangle \) is assumed to be an eigenstate of the central charge operator with eigenvalue \( k \); we shall omit the explicit dependence on \( k \) of this highest weight state. Analogously, the dual state \( \langle \gamma | \) can be reproduced, with a proper normalisation of the fields, as
\[
\langle \gamma| = \lim_{x,y,z,u \to 0} \langle 0 | \Phi_{\gamma^*}(\hat{w}_\theta X(x,y,z), \hat{r}(u)) \]
\[
= \lim_{x,y,z,u \to 0} z^{\gamma(h^2)}(z-xy)^{\gamma(h^1)} u^{-2\Delta_{\gamma}} \langle 0 | \Phi_{\gamma^*}(X(\frac{x}{z}, \frac{y}{xy-z}, \frac{1}{z}), -\frac{1}{u}),
\]

where \( \Delta_{\gamma} \) is the Sugawara conformal weight and \( \gamma^* = -w_\theta(\gamma) \). Note that this way the limits of the chiral coordinate \( u \) and the flag coordinate \( X \) are treated on the same footing. Eqn. (1.4) follows from (1.5) using
\[
[ \Phi_{\gamma}(\hat{w}_\theta X, \hat{r}(u)), T_n ] = (-u)^{-n} D(\gamma)(\hat{w}_\theta(T)) \Phi_{\gamma^*}(\hat{w}_\theta X, \hat{r}(u)),
\]
where \( \hat{w}_\theta(T) \) is the adjoint action of \( \hat{w}_\theta \) on \( T \).

In section 4 we discuss a purely algebraic treatment of correlation functions which relies on the use of a set of generators which can be seen as the analogue of the right action generators discussed above. Namely we define \( \hat{T}_n \equiv T_n(X) \) by
\[
\hat{T}_n = U(X) T_n U(X)^{-1}, \quad (1.6)
\]
where \( U(X) = \exp(x f^1 + y f^2 + (z - \frac{x y}{2}) f^3) \) is the operator implementing the translations
\[
U(X') \Phi_\mu(X) U(X')^{-1} = \Phi_\mu(X'X).
\]

(The corresponding infinitesimal generators appear in the first line of (1.2).) More explicitly (1.6) read
\[
\begin{align*}
\hat{f}^1 &= f^1 + y f^3, & \hat{f}^2 &= f^2 - x f^3, & \hat{f}^3 &= f^3, \\
\hat{h}^1 &= h^1 + 2 x f^1 - y f^2 + (z + x y) f^3, \\
\hat{h}^2 &= h^2 + 2 x f^2 - y f^1 + (z - 2 x y) f^3, \\
\hat{e}^1 &= e^1 - x h^1 - x^2 f^1 + z f^2 - x z f^3, \\
\hat{e}^2 &= e^2 - y h^2 - y^2 f^2 - (z - x y) f^1 - y(z - x y) f^3, \\
\hat{e}^3 &= e^3 - (z - x y) h^1 - z h^2 - y e^1 + x e^2 - x(z - x y) f^1 - y z f^2 - z(z - x y) f^3.
\end{align*}
\]

(1.7)
If the generators in the r.h.s. of (1.7) are replaced by their differential operators counterparts in (1.2) then they reduce (up to a sign) to the classical generators of the right action of the group, i.e. \( \hat{f}^i \) turn into the generators of right translations, \( \hat{e}^i \) vanish identically while \( \hat{h}^i \) reduce to the numerical values \(-\mu(h^i)\). Thus (1.6) can be also looked on as a change of basis in the algebra which partially diagonalises the left action of \( \mathfrak{g} \). (Such an algebra \( \mathfrak{g}(X) \) attached to a point \( X \) of the flag manifold has been discussed in the \( sl(2) \) case in \([8]\).) Explicitly, choosing \( u = 1 \)
\[
[\Phi_\mu(X), \hat{h}_n^i] = -\mu(h^i) \Phi_\mu(X), \\
[\Phi_\mu(X), \hat{e}_n^j] = 0.
\]

1.2. Correlation functions and singular vectors

One can now address the problem of finding \( n \)-point functions invariant with respect to the action of \( \mathfrak{g} \) in (1.1), (1.2). While the full 3-point correlators
\[
\langle 0 | \Phi_\gamma(X_3, u_3) \Phi_\mu(X_2, u_2) \Phi_\nu(X_1, u_1) | 0 \rangle,
\]
are indeed invariant w.r.t. \( \mathfrak{g} \), it is simpler to deal with the correlator with fixed first and third coordinates
\[
C^\gamma_{\mu\nu}(X) = \langle \gamma | \Phi_\mu(X) | \nu \rangle,
\]
which are only restricted by the counterpart of the Ward identity with respect to the Cartan generators,
\[
\left( \nu(h^i) - \gamma(h^i) - D^{(\mu)}(h^i) \right) C^\gamma_{\mu\nu}(X) = 0, \quad i = 1, 2.
\]
Similarly the Ward identity with respect to the scale generator \( L_0 \) fixes the dependence on \( u \) (suppressed in (1.9)) to a power given by \( \Delta_\gamma - \Delta_\nu - \Delta_\mu \).

The requirement that singular vectors in the Verma module of highest weight \( \nu \) decouple from correlation functions imposes additional restrictions on (1.9), thus selecting a subset of the possible 3-point invariants. The dimension of the total (linear) space of solutions gives the multiplicity \( N^\gamma_{\mu\nu} \) of the representation \( \gamma \) occurring in the fusion \( \mu \otimes \nu \).

There is a singular vector (denoted by \( S_\beta | \nu \rangle \)) in the Verma module with highest weight \( \nu \) whenever there is a real positive root \( \beta \in \Delta^* = \bar{\Delta}_+ \cup (\bar{\Delta} + \mathbb{Z}_{>0} \delta) \) satisfying the Kac-Kazhdan reducibility condition
\[
\langle \nu + \bar{\rho} + \kappa \Lambda_0, \beta \rangle \in \mathbb{Z}_{>0}.
\]
Here \( \delta = \alpha_0 + \alpha_1 + \alpha_2; \alpha_j, j=0,1,2, \) are the three simple roots of \( \mathfrak{g}; \) \( \mathfrak{g} = \tilde{\mathfrak{g}} = \tilde{\Lambda}_1 + \tilde{\Lambda}_2 \) is the Weyl vector of \( \tilde{\mathfrak{g}}; \) \( \kappa = k + 3 \) is the shifted value of the central charge; and \( \Lambda_0 \) is the fundamental weight of \( \mathfrak{g}^\vee \) dual to the affine root \( \alpha_0 \) satisfying \( \langle \Lambda_0, \alpha_j \rangle = \delta_{j0}. \) (for details see, e.g., [14]). \( S_\beta |\nu\rangle \) is a highest weight vector with weight \( w_\beta \cdot \nu \) given by the shifted action of the Kac-Kazhdan reflection \( w_\beta \) in the affine Weyl group \( W \) on \( \nu. \)

The element \( S_\beta \) is of a fixed grade \( n < 0 \) in the universal envelope of \( \mathfrak{g}_- \) (\( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \) being the triangular decomposition of \( \mathfrak{g} \)). Using this fact, eqns. (1.1) and (1.4), commuting \( S_\beta \) through the field we find

\[
\langle \gamma | \Phi_\mu(X,u) S_\beta |\nu\rangle = u^{-n} D^{(\mu)}(\tilde{S}_\beta) \langle \gamma | \Phi_\mu(X,u) |\nu\rangle , \tag{1.12}
\]

where \( \tilde{S}_\beta \) is the projection of \( S_\beta \) to the horizontal subalgebra. Imposing the vanishing of the matrix elements of \( S_\beta |\nu\rangle \) leads to the differential equation

\[
D^{(\mu)}(\tilde{S}_\beta) C^\gamma_{\mu\nu}(X) = 0. \tag{1.13}
\]

We turn to the solutions of these equations in section 2.

1.3. The model

We now turn to our problem in which we shall essentially deal with a non-rational counterpart of the admissible CFT [2].

We shall call “pre-admissible” the infinite set \( \mathcal{P}_+ \) of highest weights defined for generic \( (k \notin \mathbb{Q}) \) values of the central charge considered in [1],

\[
\mathcal{P}_+ = \{ \Lambda \equiv wt - \lambda \cdot (\lambda' + k\Lambda_0) \mid w \in \mathbb{N}, \lambda, \lambda' \in P_+, \langle \lambda, \alpha_i \rangle \delta + w(\alpha_i) \in \Delta^\text{re}_+, i = 1,2 \}. \tag{1.14}
\]

Here \( t_P \) is the group of translations in the weight lattice \( P = \oplus_i \mathbb{Z} \tilde{\Lambda}_i \) of the horizontal subalgebra \( \bar{\mathfrak{g}}. \) For such weights the Kac-Kazhdan roots \( \beta_i, i=1,2 \) are explicitly

\[
\beta_i = \langle \lambda, \alpha_i \rangle \delta + w(\alpha_i) , \tag{1.15}
\]

and the reducibility pattern of the \( \mathfrak{g} \) Verma modules with highest weights in this set parallels that of the Verma modules of \( \bar{\mathfrak{g}} \) of dominant integral highest weights; in particular, the maximal submodule of any such \( \mathfrak{g} \) Verma module is a union of the Verma submodules generated by the two singular vectors \( S_{\beta_i} |\Lambda\rangle \) with \( \beta_i \) as in (1.15). In what follows we shall mostly use the horizontal projections \( \bar{\Lambda} = w \cdot (\lambda' - \lambda \kappa) \) of the weights \( \Lambda. \)
If in (1.14) we take \( w = 1, \lambda = 0 \), we recover the weights of a non-rational CFT (which might be called “pre-integrable”) for which the fusion rules are given by the standard \( sl(3) \) tensor product rules for \( \lambda' \).

Here (as in [1]) we shall mostly concentrate on the other generic subset of (1.14) obtained by taking \( \lambda' = 0 \). The set of such weights can be looked on as the set of “integral dominant” weights of the “weight lattice” \( \tilde{W} = W \ltimes \mathfrak{p} \) – the extended affine Weyl group of \( g \). Each of these weights has associated to it a generalised weight diagram parametrised by a finite subset of the affine Weyl group \( \tilde{W} = W \ltimes \mathfrak{q} \) (where \( \mathfrak{q} \) is the root lattice of \( \mathfrak{g} \)), and a generalised formal character \( \chi_\lambda \). The characters close under multiplication and the structure constants of the resulting (commutative) ring serve as the fusion rule multiplicities of the corresponding non-rational CFT. The ring is an extension of the ring of characters of finite dimensional representations of \( sl(3) \). It is generated by three “fundamental” characters and a “simple current” character with highest weights given by

\[
\begin{align*}
f &= -\bar{\Lambda}_1 \kappa, & \quad f^* &= -\bar{\Lambda}_2 \kappa, & \quad h &= w_0 \cdot (k\Lambda_0) = w_{121} \cdot (-\bar{\rho} \kappa) = \bar{\rho}(\kappa - 2), & \quad \gamma &= w_{12} \cdot (-\bar{\Lambda}_2 \kappa),
\end{align*}
\]

and which satisfy

\[
\begin{align*}
\chi_\gamma^3 &= 1, & \quad \chi_h \chi_h &= 2 \chi_h + 1 + \chi_\gamma \chi_f + \chi_{\gamma^2} \chi_{f^*} \cdot
\end{align*}
\]

(1.16)

The singular vectors in the Verma modules of highest weights (1.16) are determined by the corresponding Kac-Kazhdan reflections \( w_{\beta_i}, i = 1, 2 \) and can be recovered from the general formulae of [10]. For the simple current \( \gamma \) these are especially simple \((\beta_1, \beta_2) = (\alpha_2, \delta - \alpha_1 - \alpha_2)\), i.e. the two singular vectors are given by monomials of \( f^2 \) and \( e_3^1 \), and the rules for the fusion of \( \gamma \) with an arbitrary weight \( \mu \) are correspondingly simple: \( \gamma \otimes \mu = w_{12} \cdot (-\bar{\Lambda}_2 \kappa + \mu) \). The fusion rules and the generalised formal characters of the remaining representations in (1.16) are described in [1]. In particular the representation denoted \( h \) provides the simplest example of a non-trivial multiplicity and its fusion with a generic \( \mu \) reads

\[
h \otimes \mu = 2\mu \oplus w_{121} \cdot (-\bar{\rho} \kappa + \mu) \oplus w_1 \cdot (\mu) \oplus w_2 \cdot (\mu).
\]

(1.18)

In the next section we shall recover this generic fusion rule from the decoupling of the singular vectors in the Verma module of highest weight \( h = \bar{\rho}(\kappa - 2) \).
2. Differential equation approach

We start by recalling the result in [1] for the first two of the fundamental representations in (1.16). In [1] the decoupling equations corresponding to the real positive roots $\beta_i, i = 1, 2$ were investigated using the ansatz

$$x^a(z - xy)^c y^b z^d.$$  \hfill (2.1)

We recall that the solution of (1.13) in the integrable case is given by such monomials with nonnegative integer powers $a, b, c, d$ restricted by $0 \leq a + c \leq \langle \nu, \alpha_1 \rangle, 0 \leq b + d \leq \langle \nu, \alpha_2 \rangle$, a basis being selected, e.g., by the subset $\{ (a, 0, c, d) \} \cup \{ (0, b, c, d), b \neq 0 \}$ (see e.g., [12]). Choosing $\mu$ (or $\gamma$) to be the identity representation, these monomials and the corresponding values of $\gamma = \nu - a\alpha_1 - b\alpha_2 - (c + d)(\alpha_1 + \alpha_2)$ (or $\mu$) are in one to one correspondence with the weight diagram of the finite dimensional representation of highest weight $\nu$ (or $\nu^*$ respectively) and the number of different monomials producing a given value of $\gamma$ coincides with the multiplicity of this weight in the weight diagram.

The solutions found in [1] involved monomials (2.1) with nonintegral powers. However it is not necessary to assume such an ansatz which is too restrictive in general and we present here an alternative derivation.

Since the ratio $\zeta = z/(xy)$ is invariant with respect to the global $SL(3)$ scale transformations $(x, y, z) \rightarrow (\rho_1 x, \rho_2 y, \rho_1 \rho_2 z)$, (i.e. it is annihilated by the Cartan generators $D^{(\mu)}(h^i)$ in (1.2) for $r_i = \mu (h^i) = 0$) the general solution to the Cartan subalgebra Ward identities (1.10) is

$$C(x, y, z) = x^A y^B F\left(\frac{z}{xy}\right),$$  \hfill (2.2)

where $F(\zeta)$ is an arbitrary function and $A$ and $B$ are determined by (1.10):

$$\langle \nu + \mu - A\alpha_1 - B\alpha_2 - \gamma, \alpha_i \rangle = 0, \quad i = 1, 2.$$  \hfill (2.3)

When $\nu$ is one of the analogues of the symmetric representations $\nu = -l\Lambda_1 \kappa$ or $\nu = -l\Lambda_2 \kappa$ one of the differential operators is very simple $-D(f^2)$ or $D(f^1)$. This restricts $F(\zeta)$ to a monomial $F(\zeta) = (\zeta - 1)^B$ or $F(\zeta) = (\zeta)^A$, respectively, and hence the solutions are indeed monomials of the form (2.1). It then remains to use the equation corresponding to the other singular vector to determine the possible values of $\gamma$ as a function of $\nu, \mu$.

This has been done in [1] for $l = 1$ and $l = 2$. For the representation $f$ given by (1.16) there turn out to be 7 solutions for $\gamma \in \mathcal{P}_+$ in (1.10), i.e. 7 terms in the fusion $f \otimes \mu$ for
generic values of $\mu \in \mathcal{P}_+$. For completeness we write down here the zero mode projection of the nontrivial singular vector for $l = 1$, i.e. for the first example in (1.14)

$$\bar{S}_{\beta_1} = (f^1)^{1+\kappa} (e^3)^{2-\kappa} f^2 (e^3)^{\kappa-1} (f^1)^{1-\kappa}$$

$$= (e^3 f^1 + (1+\kappa) e^2) (f^2 f^1 - \kappa f^3) + (1-\kappa) (e^1 f^1 - (1+\kappa) h^1 - \kappa (1+\kappa)) f^1.$$

We turn now to the third “fundamental” weight $h$ in (1.16), for which the two singular vectors correspond to the roots $\beta_i = \alpha_0 + \alpha_i$, $i = 1, 2$. Their horizontal projections are

$$\bar{S}_{\beta_1} = e^3 f^1 + (2-\kappa) e^2,$$

$$\bar{S}_{\beta_2} = e^3 f^2 - (2-\kappa) e^1.$$ (2.4)

Inserting the differential operator representation (1.2) one obtains two second order differential equations for the unknown function $F$ in (2.2)

$$\left[ \zeta^2 (1-\zeta) \frac{d^2}{d\zeta^2} - \zeta \left( (1+A+B-\kappa-r_2) + \zeta (\kappa-2A-B+r_1+r_2) \right) \frac{d}{d\zeta} + \left( (\kappa-2-A)(r_2-B) - A\zeta(1+A+B-\kappa-r_2-r_1) \right) \right] F(\zeta) = 0,$$ (2.5)

$$\left[ \zeta (1-\zeta)^2 \frac{d^2}{d\zeta^2} - \left( B-1-r_2+\zeta(1-A-3B+r_1+2r_2+\kappa) + \zeta^2(A+2B-r_1-r_2-\kappa) \right) \frac{d}{d\zeta} + \left( B(1+B-r_2-\kappa)+(A-r_1)(\kappa-2) - B\zeta(1+A+B-\kappa-r_2-r_1) \right) \right] F(\zeta) = 0.$$ (2.5)

For generic values of $A, B$ the two equations in (2.3) are different and their order can be reduced exploiting Euclid’s algorithm – for generic values of $\mu$ the second order terms can be eliminated between the two equations giving a first order equation; differentiating this equation again we can then eliminate the second order term from one of the two original equations – this whole process yielding two first order differential equations. For generic values of $\mu$ the resulting system of two first order equations is consistent for three particular values of the pair of parameters, $(A, B) = (r_1 + r_2, r_1 + r_2)$, $(r_1 + \kappa - 1, \kappa - 2)$, $(\kappa - 2, r_2 + \kappa - 1)$, and accordingly one obtains three monomial solutions $F(\zeta) = (\zeta - 1)^c \zeta^d$ with $(c, d) = (r_2, r_1)$, $(0, \kappa - 2 - r_2)$, $(\kappa - 2 - r_1, r_1)$, respectively. Inserting the values of $(A, B)$ in (2.3) we obtain for a given generic $\mu$ three values of $\gamma$ which precisely recover the remaining multiplicity 1 weights in the fusion $h \otimes \mu$ as predicted in [1].

However, for $A = B = \kappa - 2$ the two equations (2.5) become identical, and the resulting equation is a hypergeometric equation. For such $A, B$ and $r_2 \neq \kappa - 2$ the solution of (2.3) is a linear combination of two hypergeometric functions

$$F_1(\zeta) = \left. 2F_1(2 - \kappa, 3 + r_1 + r_2 - \kappa; 3 + r_2 - \kappa; \zeta) \middle| \right.$$ (2.6)

$$F_2(\zeta) = (\zeta)^{\kappa-2-r_2} 2F_1(-r_2, 1 + r_1; \kappa - 1 - r_2; \zeta).$$
For $r_2 = \kappa - 2$ these hypergeometric series formally coincide so one of the solutions is instead logarithmic,

$$
F_1(\zeta) = 2 F_1(2 - \kappa, 1 + r_1; 1; \zeta),
$$

$$
F_2(\zeta) = \ln(\zeta) \ 2 F_1(2 - \kappa, 1 + r_1; 1; \zeta) + \ldots
$$

To find the representation $\gamma$ to which this fusion corresponds we insert $A = B = \kappa - 2$ in (2.3) to find $\gamma = \mu$, i.e. the weight $\mu$ appears twice in the fusion $h \otimes \mu$. This is the missing multiplicity two solution predicted in [1] which is now seen to correspond to a novel hypergeometric function expression of the matrix element $C(X)$ (2.2). In fact a multiplicity 2 was noticed in the previous calculations but only for the particular choice $\mu = 0$. We can reconcile this with the treatment here by noticing that for $r_1 = 0$ the first hypergeometric series in (2.3) simplifies to a geometric series, i.e. $F_1(\zeta) = (1 - \zeta)^{\kappa - 2} = (xy)^{2 - \kappa} (xy - z)^{\kappa - 2}$, and similarly for $r_1 = r_2 = 0$ the second solution in (2.6) becomes $F_2(\zeta) = \zeta^{\kappa - 2} = (xy)^{2 - \kappa} z^{\kappa - 2}$, i.e. both solutions degenerate to monomial solutions of the type discussed above.

As another example we have checked by Mathematica the case with $\nu = -\rho \kappa$, the analogue of the adjoint representation of $sl(3)$. It involves a system of two fourth order linear differential equations for the function $F(\zeta)$ and reveals a relation between the order of the equation and the multiplicity of the corresponding value of $\gamma$ similar to the one encountered in the above simpler example. Namely for the particular values of the parameters $A = B = -\kappa$ the initial system degenerates to one fourth order differential equation possessing 4 linearly independent solutions and thus leading to a multiplicity 4 of the weight $\gamma = \mu$; If $\gamma \neq \mu$ these two fourth-order equations can be reduced to two third order equations but there are three different sets of values of $A$ and $B$ for which these third order equations become identical, leading to three solutions of multiplicity three; at the next step there are 9 possibilities of degeneration to one second order equation (again a hypergeometric equation) leading to multiplicity 2 solutions, and finally there are 12 monomial solutions of first order equations corresponding to multiplicity 1 values of $\gamma$. The final result is in perfect agreement with the prediction in [1].

One can expect that a similar mechanism holds in general. In particular the maximal order of the differential operator corresponding to a singular vector $S_{\beta_i}$ (with $\beta_i$ as in (1.13)) can be computed from the expression in [10] to be $\langle 3\lambda + w(\overline{\rho}), \alpha_i \rangle$. The minimal of these two numbers coincides precisely with the maximal multiplicity in the generalised weight diagram associated to $\overline{\lambda} = w \cdot (-\lambda \kappa)$. As discussed in [1], this multiplicity is also the maximal multiplicity in the standard weight diagram of the finite dimensional representation of $sl(3)$ of highest weight $3\lambda + w(\overline{\rho}) - \overline{\rho}$. 
3. Quotient space method

In this section we shall treat the decoupling problem and the appearance of finite dimensional solution spaces in an alternative way. This method is purely algebraic, and first appeared in conformal field theory in [11] where Feigin and Fuchs used it to study the fusion in Virasoro minimal models. Since then it has been applied to other models [15], and developed extensively in one direction by Zhu [16], but we shall stick to the spirit of [11].

To explain how this method works we shall first reconsider the case of \( sl(2) \) because \( sl(2) \) has fewer generators than \( sl(3) \), and hence the expressions are simpler. This case has already been treated by Feigin and Malikov [8] and by Dong et al [9], with apparently contradictory results, and we think it will be helpful to explain how the various different results fit together (n.b. the \( \hat{sl}(2) \) calculations here are essentially all contained in [8] and [9].)

The algebraic method relies on the observation that the inner product of a highest weight state, a primary field corresponding to the vertex operator of a highest weight state, and an arbitrary state in a highest weight representation, satisfies various identities.

To express these identities, we need some notation. The generators of \( \hat{sl}(2) \) are denoted by \( e_m, f_m \) and \( h_m \) and have commutation relations

\[
\begin{align*}
[ h_m , h_n ] &= 2 k m \delta_{m+n} , \\
[ h_m , e_n ] &= 2 e_{m+n} , \\
[ h_m , f_n ] &= -2 f_{m+n} , \\
[ e_m , f_n ] &= k m \delta_{m+n} + h_{m+n} .
\end{align*}
\]

We denote highest weight Verma modules with highest weight \(|r\rangle\) by \( M_r \) and the corresponding irreducible module by \( L_r \), where the states \(|r\rangle\) and \langle r|\) satisfy

\[
\begin{align*}
& f_m |r\rangle = 0 , \ m > 0 , \ e_m |r\rangle = 0 , \ m \geq 0 , \ h_m |r\rangle = r \delta_{m0} |r\rangle , \ m \geq 0 , \\
& \langle r| e_m = 0 , \ m < 0 , \ \langle r| f_m = 0 , \ m \leq 0 , \ \langle r| h_m = r \delta_{m0} \langle r| , \ m \leq 0 .
\end{align*}
\]

We are interested in working out the number of independent couplings of the form

\[
\langle \phi_\alpha | \phi_\beta (z) | \psi \rangle ,
\]

(3.3)
where $|\psi\rangle$ is some general state in a highest-weight representation of $\hat{sl}(2)$; $\langle \phi_\beta |$ is annihilated by all $T_m$, $m < 0$ (and hence is in some representation of the ‘horizontal’ $sl(2)$ subalgebra generated by $T \equiv T_0$). Finally, the fields $\phi_\beta(z)$ transform as

$$\left[ \phi_\beta(z), T_m \right] = z^m D_{\beta\gamma}(T) \phi_\gamma(z),$$

where $D(T)$ is some representation of $sl(2)$. A consequence of this definition is that

$$\left[ T_m - z T_{m-1}, \phi_\beta(z) \right] = 0,$$

so that for any generator $T$

$$\langle \phi_\alpha | \phi_\beta(z) (T_m - z^{m-1} T_{-1}) = 0, \quad m \leq -2$$

Consequently (taking $z = 1$) for any state $|\psi\rangle$, the state

$$\langle T_m - T_{-1} | \psi \rangle, \quad m \leq -2,$$

has zero inner product with $\langle \phi_\alpha | \phi_\beta(1)$. This means that to find the space of independent couplings from an irreducible highest weight representation $L_r$ of $\hat{sl}(2)$ to states of the form $\langle \phi_\alpha | \phi_\beta(z)$ one need only consider the quotient space

$$A(L_r) = L_r / \langle T_m - T_{-1} \rangle_L$$

Zhu’s algebra $A(L_0)$ is exactly the space defined in equation (3.7). He showed that the space $A(L_0)$ can itself be given an algebra structure, and that the space of irreducible representations of the vertex algebra $L_0$ are in one-to-one correspondence with the irreducible representations of the algebra $A(L_0)$. The main problem with trying to follow Zhu’s analysis for admissible but non-integrable representations of $\hat{sl}(2)$ is that typically $A(L_r)$ is infinite-dimensional for any $r$, and the analysis of this space correspondingly harder than for the integrable case (although it may be carried through – see e.g. [17])

However, in the spirit of [11], it is not necessary to consider the full space $A(L_r)$ to find the allowed fusions with $L_r$. For the integrable case $k$ one can use the Ward identities to express the general three-point function

$$\langle \phi_\alpha | \phi_\beta(z) | \psi \rangle,$$
in terms of some three point function

\[ \langle r'' | \phi_{r'}(x; z) | \psi \rangle , \tag{3.8} \]

where \( \langle r'' | \) is a highest weight state of \( \mathfrak{g} \) and \( \phi_{r'}(x; z) \) is a highest weight state for some Borel subalgebra \( \mathfrak{b}(x) \) of \( \mathfrak{g} \) parametrised by the coordinate \( x \) on the flag manifold.

For the integrable case the choice of \( x \) is not important for the reason that \( \phi_{r'}(x; z) \) can be expanded as a polynomial in \( x \). The leading coefficient \( \phi_{r'}(0; z) \) then turns out to be a highest weight for \( \mathfrak{b} \equiv \mathfrak{b}(0) \) and the space of couplings turns out to be independent of \( x \).

However, for an infinite-dimensional representation of \( sl(2) \), assuming that \( \phi_{r'}(x; z) \) can be expanded as an integer power series in \( x \) requires that \( \phi_{r'}(0; z) \) is a highest weight for \( \mathfrak{b}(0) \) and leads exactly to the restricted fusion rules found by BF. To find the full set of fusion rules one must accept that one cannot necessarily expand \( \phi_{r'}(x; z) \) about \( x = 0 \) or \( x = \infty \) and one must take (3.8) as a starting point. We shall see that different (non-generic) choices of the Borel subalgebra \( \mathfrak{b}(x) \) of \( \mathfrak{g} \) used to define \( \phi_{r'}(x; z) \) may lead for nonintegral \( r' \) to different (degenerate) results for the space of fusions.

Given our ‘standard’ splitting into \( e_m, f_m \) and \( h_m \), as for \( sl(3) \), with \( \mathfrak{b} \) the standard Borel subalgebra of \( sl(2) \) generated by \( e_0 \) and \( h_0 \), we can require fields to be highest weight states for any conjugate subalgebra

\[ \hat{\mathfrak{b}} = U \mathfrak{b} U^{-1} , \tag{3.9} \]

of our standard raising operators by some constant group element \( U \). If we consider conjugation by

\[ U = \exp( xf_0 ) , \tag{3.10} \]

for which

\[ \hat{e}_m = \exp( xf_0 ) e_m \exp( -xf_0 ) = e_m - xh_m - x^2 f_m , \]

\[ \hat{h}_m = \exp( xf_0 ) h_m \exp( -xf_0 ) = h_m + 2xf_m , \tag{3.11} \]

then the fields \( \phi_{r'}(x; z) \) being highest weight states for these generators implies

\[ \begin{bmatrix} e_m - xh_m - x^2 f_m , \phi_{r'}(x; z) \end{bmatrix} = 0 , \]

\[ \begin{bmatrix} h_m + 2xf_m , \phi_{r'}(x; z) \end{bmatrix} = z^m r' \phi_{r'}(x; z) . \tag{3.12} \]

If we allow \( x \) to take all values including \( x = \infty \) (which, suitably interpreted, corresponds to \( U = w \), the Weyl group element of \( SL(2) \)) then this covers all highest weight fields. One can
of course find a representation of $e_m, h_m, f_m$ on the fields $\phi_{r'}(x; z)$ in terms of differential operators, in a manner analogous to that for $sl(3)$, such that the relations (3.3) and (3.12) hold; however it is not necessary to consider the differential operator representation explicitly since (3.5) and (3.12) are the only relations needed in the algebraic treatment.

So, given these relations, we find that

\[
\langle r'' | \phi_{r'}(x; 1) (T_{-m} - T_{-1}) = 0, \quad m > 1, \\
\langle r'' | \phi_{r'}(x; 1) (e_{-1} - x h_{-1} - x^2 f_{-1}) = 0, \\
\langle r'' | \phi_{r'}(x; 1) (f_0 - f_{-1}) = 0, \\
\langle r'' | \phi_{r'}(x; 1) (h_0 - h_{-1}) = r'' \langle r'' | \phi_{r'}(x; 1), \\
\langle r'' | \phi_{r'}(x; 1) (h_{-1} + 2 xf_{-1}) = -r' \langle r'' | \phi_{r'}(x; 1).
\]

Consequently we define the new quotient space

\[
A_x^\leq(L_r) = L_r / J_x L_r,
\]

where $J_x$ is the linear span of the elements of $\mathfrak{g}_-$

\[
T_m - T_{-1}, m < -1, \quad e_{-1} - x h_{-1} - x^2 f_{-1}, \quad \text{and} \quad f_0 - f_{-1}.
\]

The space $A_x^\leq(L_r)$ then carries an action of $U(1) \oplus U(1)$ with generators

\[
H(\infty) = h_0 - h_{-1}, \quad \text{and} \quad H = -(h_{-1} + 2xf_{-1}),
\]

which take values $r''$ and $r'$ on $\langle r'' | \phi_{r'}(x; 1)$, respectively. The space of allowed fusions to the representation $L_r$ is now the space $A_x^\leq(L_r)$ viewed as a $\mathbb{C}[H(\infty), H]$ module – that is $(H(\infty), H)$ are restricted to lie on some curves and points in the $(H(\infty), H)$ plane, possibly with multiplicities. Although $H(\infty)$ and $H$ may not be strictly diagonalisable, we shall often refer to their allowed values as their ‘spectrum’; in the case of $L_r$ admissible it turns out that $H(\infty)$ and $H$ are genuinely diagonalisable, with $A_x^\leq(L_r)$ being a direct sum of eigenspaces of $H(\infty)$ and $H$. The dimension of the factor-space of fixed eigenvalues $A_x^\leq(L_r)^{(r', r'')} \leq$ describes the fusion rule multiplicities,

\[
\dim A_x^\leq(L_r)^{(r', r'')} = N^{r''}_{r', r'}.
\]

\footnote{The generators $H(\infty)$ and $H$ which commute up to an element in $J_x$ are analogues of the linear combinations of Virasoro generators in (1), $L_0 - 2zL_{-1} + z^2L_{-2}$ and $L_{-1} - zL_{-2}$ respectively, while the combinations (3.13) are analogues of $L_{-n} - 2zL_{-n-1} + z^2L_{-n-2}, n > 0.$}
A natural spanning set for $L_r/J_x L_r$ are the states

$$(h_{-1})^m (f_0)^n |r\rangle .$$

(3.18)

If $x$ is generic, i.e. $x \neq 0, \infty$, then we have

$$h_{-1} = h_0 - H(\infty), \quad f_0 = (f_0 - f_{-1}) + \frac{1}{2x}(H_0 - H(\infty) - h_0).$$

(3.19)

and so an equally good spanning set for $A^\leq_x (L_r)$ is

$$(H(\infty))^m (H)^n |r\rangle .$$

(3.20)

However this is not a good choice if $x = 0$ (unless $r = 0$) or $x = \infty$ (unless $r = k$). If $x = 0$, then we have $e_{-1} = e_{-1}$, and more importantly, $h_{-1} = -H$ and $H(\infty) - H = h_0$, so that $H(\infty)$ and $H$ are simultaneously diagonalised on the weight spaces, and we can take as a spanning set of $A^\leq_0 (L_r)$

$$(H)^m (f_0)^n |r\rangle .$$

(3.21)

If, conversely, $x = \infty$, then we find $f_0$ should be included in $J$, that $h_{-1} = -H$, $H(\infty) + H = h_0$, so that $H(\infty)$ and $H$ are again diagonalised on the weight spaces, but that $e_{-1}$ in unconstrained, so that we can take as a spanning set of $A^\leq_0 (L_r)$

$$(H)^m (e_{-1})^n |r\rangle .$$

(3.22)

If $M_r$ is irreducible then in each case the states (3.20), (3.21) and (3.22) are linearly independent, but if $M_r$ contains null vectors, there will be linear relations among these states. If $x$ is generic, $A^\leq_x (L_r)$ naturally takes the form of a simple quotient of the polynomial ring $\mathbb{C}[H(\infty), H]$, since singular vectors in the Verma module $M_r$ lead to polynomial constraints in $A^\leq_x (M_r)$. However in the other two cases the structures of $A^\leq_0 (L_r)$ and $A^\leq_\infty (L_r)$ may be more complicated.

Physically, taking $x = 0$ or $x = \infty$ puts strong constraints on the allowed fusions. For representations for which $r, r'$ or $r''$ are not non-negative integers,

$$\langle r'' | \phi_{r'} (x; z) | r\rangle ,$$

may have singular expansions around $x = 0$ and $x = \infty$. One would hope that the algebraic method would only find the fusions for which the three point functions are regular, that the dimension of $A^\leq_0 (L_r)$ and $A^\leq_\infty (L_r)$ would be smaller than the generic result $A^\leq_x (L_r)$,
excluding precisely those fusions for which the three-point function does not have a power-series expansion at \( x = 0 \) and \( x = \infty \) respectively. This is exactly what has been found.

Feigin and Malikov calculated \( \mathcal{A}^<_{x}(L_r) \) for the generic values \( x = 1 \) and found the ‘nice’ fusion rules of Awata & Yamada \([5]\); The fusion rules were also investigated in \([8]\) through the construction of a space ‘\( \mathcal{A}(L) \)’ – which is nothing but \( \mathcal{A}^<_{x}(L_r) \) with \( x = 0 \) – and instead of the fusion rules of \([8]\), they found the fusion rules of Bernard & Felder \([6]\); these fusion rules are a ‘subset’ of the rules of \([8]\) in the sense that all couplings allowed by \([6]\) are allowed by \([8]\), but not all the \([8]\) couplings are allowed by \([6]\).

Taking \( x = 0 \) is not ‘wrong’ in a mathematical sense, but rather the mathematics gives the right answer to what may be the wrong physical question.

We illustrate these points in the simplest non-trivial cases. If we define \( \kappa = k+2 \), then the simplest representations containing singular vectors which are not of the form \( f_0^n |r\rangle \) or \( e_{n-1} |r\rangle \) are \( r = -\kappa \) and \( r = 2\kappa - 2 \), with singular vectors

\[
|1\rangle = \left( f_0 f_0 e_{-1} + (1-\kappa) f_0 h_{-1} + \kappa(1-\kappa) f_{-1} \right) | -\kappa \rangle , \tag{3.23}
\]

and

\[
|2\rangle = \left( e_{-1} e_{-1} f_0 - (1-\kappa) e_{-1} h_{-1} + \kappa(1-\kappa) e_{-2} \right) | 2\kappa - 2 \rangle , \tag{3.24}
\]

respectively. These are the two simplest non-trivial representations in the set of ‘pre-admissible’ representations, of spins

\[
r \in \{ n' - n\kappa , \ n, n' = 0, 1, 2, \ldots ; \ -n' + (n + 1)\kappa , \ n, n' = 0, 1, 2, \ldots \} . \tag{3.25}
\]

It is expected that these representations form a closed subalgebra for generic \( \kappa \neq 0 \).

The admissible representations have \( \kappa \) certain rational numbers and spins \( r \) a subset of the ‘pre-admissible’ representations,

\[
\kappa = p'/p , \quad r \in \{ n' - n\kappa = -(p' - n') + (p - n)\kappa , \ 0 \leq n' \leq p' - 2 , \ 0 \leq n \leq p - 1 \} . \tag{3.26}
\]

The simplest non-trivial admissible representation is at level \( k = -4/3 \), \( \kappa = 2/3 \), of spin \( r = -\kappa = 2\kappa - 2 = -2/3 \); hence both \((3.23)\) and \((3.24)\) are in \( M_{-\kappa} \) in this case, and are the linearly independent generators of the maximal submodule of \( M_{-\kappa} \). For this level, there are only three admissible representations, \( r \in \{ 0, -\kappa, -2\kappa \} = \{ 0, -2/3, -4/3 \} \), and we again expect the fusion algebra to be closed on this set.
3.1. The case of $x$ generic

Let us first consider the case $x \neq 0, \infty$. Using equations (3.19) it is straightforward to show that in $A_x^\leq(L_r)$ the singular vectors $|1\rangle$ and $|2\rangle$ are equivalent to

$$|1\rangle \cong -\frac{1}{8x} \left( H - H(\infty) - \kappa \right) \left( H - H(\infty) + \kappa \right) \left( H + H(\infty) - (\kappa - 2) \right) |\kappa\rangle \quad (3.27)$$

$$|2\rangle \cong -\frac{x}{8} \left( H + H(\infty) + 2(1 - \kappa) \right) \left( H + H(\infty) + 2 \right) \left( H - H(\infty) \right) |2\kappa - 2\rangle \quad (3.28)$$

This means that these singular vectors each lead to a single polynomial constraint between $H$ and $H(\infty)$. If $H$ is in the ‘pre-admissible set’, then so is $H(\infty)$, except for the cases of $H$ being on the ‘edge’ of the ‘pre-admissible set’, that is one of $n$ or $n'$ being zero, in which case the simple constraints arising from the null-vectors in the representation $H$ must also be taken into account. In each case, the fusion of these two representations with representations $r'$ in the pre-admissible set yields fields $r''$ in the pre-admissible set.

The values of $(H(\infty), H)$ which are allowed to couple to the representation $|1\rangle$ are shown in figure 1.

In the case that $\kappa$ takes the admissible value $\kappa = 2/3$, we then have $-\kappa = 2\kappa - 2$, and so $H$ and $H(\infty)$ must satisfy the two simultaneous equations

$$\left( H - H(\infty) - \frac{2}{3} \right) \left( H - H(\infty) + \frac{3}{2} \right) \left( H + H(\infty) + \frac{4}{3} \right) = 0 \quad (3.29a)$$

$$\left( H + H(\infty) + \frac{2}{3} \right) \left( H + H(\infty) + 2 \right) \left( H - H(\infty) \right) = 0 \quad (3.29b)$$

The solutions to these simultaneous equations are

$$(H(\infty), H) \in \left\{ \left( -\frac{2}{3}, -\frac{4}{3} \right), \left( -\frac{4}{3}, -\frac{2}{3} \right), \left( -\frac{2}{3}, -\frac{2}{3} \right), \left( 0, -\frac{2}{3} \right), \left( -\frac{2}{3}, 0 \right) \right\}, \quad (3.30)$$

all of which lie in the set of admissible representations with no further constraints necessary. This corresponds to a fusion matrix for the fundamental field $f = (-\kappa)$ of the form

$$\mathcal{N}_f = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \quad (3.31)$$

\[1 \quad \text{In other words, as stressed in \cite{8}, one assigns distinct Borel subalgebras to the three fields in the correlator, or, choosing } x = 1, \text{ we have } \bar{\mathfrak{b}}_0 = e \oplus h, \bar{\mathfrak{b}}_\infty = f \oplus h, \bar{\mathfrak{b}}_1 = (e - h - f) \oplus (h + 2f).\]
3.2. The case \( x = 0 \)

In this case we have to consider the constraints arising from the vanishing of all \( f_0 \)-descendents of the singular vectors. Taking \(|1\rangle\) first, it is easy to show that the only relations among the spanning set (3.21) arise from

\[
(f_0)^p |1\rangle = \left( e_{-1} (f_0)^2 - (\kappa + p + 1) h_{-1} f_0 - (\kappa + p + 1)(\kappa + p) f_{-1} \right) (f_0)^p |\kappa\rangle
\]

\[
(f_0)^p |1\rangle \cong (\kappa + p + 1) \left( H - \kappa - p \right) (f_0)^{p+1} |\kappa\rangle.
\]  \( (3.32) \)

This makes for quite a complicated structure. For generic \( \kappa \),

\[
\mathcal{A}_0^<(L_{-\kappa}) = \mathbb{C}[H]|-\kappa\rangle \oplus \left( \oplus_{p>0} \mathbb{C} (f_0)^p |-\kappa\rangle \right).
\]  \( (3.33) \)

On the first summand \( H_{(\infty)} \) and \( H \) are only restricted by \( H_{(\infty)} - H = -\kappa \), but on each of the discrete eigenvectors \( (f_0)^p |-\kappa\rangle \) they take values \( \{ H_{(\infty)} = -p - 1, \ H = \kappa + p - 1 \} \). This is exactly the subset of the spectrum at generic \( x \) which satisfies \( H - H_{(\infty)} - \kappa = 0, 2, 4, \ldots \), and is also shown on figure 1. The algebraic method exactly reproduces the physically intuitive result - putting \( x = 0 \), one restricts to the subset for which the three-point functions have a regular expansion around \( x = 0 \). (This is essentially the same restriction as that imposed by [3].)

Taking now the case of \(|2\rangle\), we have

\[
(f_0)^p |2\rangle \sim -p(\kappa - p) \left( H - (p - 1) \right) \left( H + \kappa - p + 1 \right) (f_0)^{p-1} |2\kappa - 2\rangle,
\]  \( (3.34) \)

so that

\[
\mathcal{A}_0^<(L_{2\kappa - 2}) = \oplus_{p \geq 0} \left( \mathbb{C} (f_0)^p |2\kappa - 2\rangle \oplus \mathbb{C} H (f_0)^p |2\kappa - 2\rangle \right).
\]  \( (3.35) \)

On each of the two dimensional spaces in the sum in (3.35), \( H_{(\infty)} \) and \( H \) are diagonalisable with joint eigenvalues \( (2\kappa - p - 2, p) \) and \( (\kappa - p - 2, -\kappa + p) \). This is now the subset of the spectrum at generic \( x \) which satisfies \( H - H_{(\infty)} + 2\kappa - 2 = 0, 2, 4, \ldots \)

In the admissible case \( \kappa = 2/3 \), both (3.32) and (3.34) must be set to zero in \( \mathcal{A}_0^<(L_{-2/3}) \). This reduces the space to two dimensions,

\[
\mathbb{C} |-2/3\rangle \oplus \mathbb{C} H |-2/3\rangle,
\]  \( (3.36) \)

on which the eigenvalues are

\[
(H_{(\infty)}, H) \in \left\{ \left( -\frac{4}{3}, -\frac{2}{3} \right), \left( -\frac{2}{3}, 0 \right) \right\},
\]  \( (3.37) \)
a subset of (3.30), agreeing with the results of Bernard & Felder. This would correspond to a fusion matrix for the fundamental field of the form

\[
N_f = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

(3.38)

which is rather degenerate – for example, it is nilpotent, and there is no conjugate field \( f^* \) such that the identity would appear in the fusion of \( f \) and \( f^* \).

3.3. The case \( x = \infty \)

The case of \( x = \infty \) is analogous to that of \( x = 0 \), but since the representation \( \langle r'' \rangle \) can also be thought of as having \( x = \infty \), the results are now symmetric in \( H \) and \( H(\infty) \). In each case we find that \( \mathcal{A}_\infty^<(L_r) \) consists of the subset of the spectrum for \( x \) generic satisfying \( H + H(\infty) - r = 0, 2, 4, \ldots \).

For \( \mathcal{A}_\infty^<(L_{-\kappa}) \) this is now a set of discrete points while (as shown in figure 1) and for \( \mathcal{A}_\infty^<(L_{2\kappa-2}) \) this a line and a set of points.

In the rational case \( \kappa = 2/3 \), the \( \mathcal{A}_\infty^<(L_{-\kappa}) \) is again two-dimensional, the spectrum this time consisting of \( (-\kappa, 0) \) and \( (0, -\kappa) \), leading to a ‘fusion’ matrix

\[
N_f = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(3.39)

Although the matrix (3.39) is symmetric (unlike (3.38)) and thus there is a conjugate \( f^* = f \), it is however decomposable, i.e. as for \( x = 0 \) the result for \( x = \infty \) does not satisfy the usual requirements for a fusion algebra.

\[\text{3} \] This case appears implicitly in [8], where it is mentioned that both lowest and highest weight chiral vertex operators are needed to have in particular nonvanishing 2-point functions; the 4-point blocks construction in [6] provides explicit examples of “mixed” correlators. They can be interpreted as proper limits with coordinates \( x_i \) taken at 0 or \( \infty \) of a subset of the generic conformal blocks in [8], [5].
These three lines are the joint spectrum of $H_{(s)}$ and $H$ for the representation $(-\kappa)$ at $x=0$, a subset of the spectrum at generic $x$.

Figure 1: The ‘spectrum’ of $H_{(s)}$ and $H$ on $A_{x}^{s}(L_{-\kappa})$ for generic $\kappa$, showing the results for generic $x$, $x=0$ and $x=\infty$.

3.4. The case of $sl(3)$

All the preceding discussion passes over to the case of $\hat{sl}(3)$, with some minor modifications and some unfortunate complications. We recall that the fields $\Phi_{\mu}(X)$ we introduced earlier are highest weight primary fields for the conjugate algebra generators, satisfying

$$
\left[ T_{m} - T_{m-1} , \Phi_{\mu}(X) \right] = 0 ,
$$

$$
\left[ \hat{e}^{i}_{m} , \Phi_{\mu}(X) \right] = 0 ,
$$

$$
\left[ \hat{h}^{i}_{m} , \Phi_{\mu}(X) \right] = \mu(h^{i}) \Phi_{\mu}(X) .
$$

We are interested in the space of three-point functions

$$
\langle \gamma | \Phi_{\mu}(X) | \psi \rangle ,
$$

(3.40)
for some highest weight state $\langle \gamma |$ and $| \psi \rangle$ an arbitrary state in some irreducible highest weight representation $L_\nu$. By analogy to the case of $\hat{sl}(2)$ we shall define $A_X^\leq (L_\nu)$ to be the quotient

$$A_X^\leq (L_\nu) = L_\nu / J L_\nu ,$$

where for finite $X \mathcal{J}$ is the linear span of the elements

$$T_{-n} - T_{-1}, \quad n > 2, \quad f^{j}_{1} - f^{j}_{0}, \quad \hat{e}^{-j}_{1} .$$

As before, we define

$$H^i = h^i_0 - h^i_{-1} , \quad H^i = -\hat{e}^i_{-1} ,$$

which commute on $A_X^\leq (L_\nu)$, and the space of fusions to $L_\nu$ is given by the ‘joint spectrum’ on $A_X^\leq (L_\nu)$ of these operators.

From (3.41) and definition of $H^i$, it is clear that for finite $X$, $A_X^\leq (M_\nu)$ is spanned by the states

$$(H_1^1)^{b_1} (H_2^2)^{b_2} (f^1_{0} )^{a_1} (f^2_{0} )^{a_2} (f^3_{0} )^{a_3} | \nu \rangle , \quad a_i , b_j = 0, 1, \ldots$$

For generic $X$ it is further possible to show that the space $A_X^\leq (M_\nu)$ is also spanned by the states

$$(H_1^{(\infty)})^{a_1} (H_2^{(\infty)})^{a_2} (H^1)^{a_3} (H^2)^{a_4} (f^3_{0} )^{a_5} | \nu \rangle , \quad a_i = 0, 1, \ldots .$$

The values of $X$ on which this inversion is not possible are similar to the points $x = 0, \infty$ for $\hat{sl}(2)$, but are no longer just points but subspaces.

The main technical difficulty in computing $A_X^\leq (L_\nu)$, even for generic $X$, is that since we cannot assume $a_5 = 0$ in (3.45), $A_X^\leq (M_\nu)$ is not just $\mathbb{C}[H_1^{(\infty)}, H^i]$, singular vectors in $M_\nu$ do not just lead to polynomial constraints on $H^i_{(\infty)}$ and $H^i$, and $A_X^\leq (L_\nu)$ is not just automatically a quotient of $\mathbb{C}[H^{(\infty)}, H^i]$. This fact is intimately connected with the possible existence of Verlinde fusion numbers greater than 1, as we shall see. Rather, (as for the case $x = 0$ in $\hat{sl}(2)$) we must also consider descendents of the highest-weight singular vectors, and the explicit construction of the space $A_X^\leq (L_\nu)$ is rather messy.

We shall not attempt any further general discussion, but simply outline the results of an explicit calculation in the simplest ‘pre-admissible’ representation which has fusions with non-trivial multiplicities.
3.5. The representation \( h \)

The representation \( 'h' \) has weight \( \tilde{\rho}(\kappa - 2) \) and we shall denote the highest weight state by \( |h\rangle \). The irreducible representation \( L_h \) is the quotient of the Verma module \( M_h \) by its maximal submodule which is generated by the two independent singular vectors

\[
|1\rangle = \left( e_{-1}^3 f_0^2 - (2 - \kappa) e_{-1}^1 \right) |h\rangle ,
\]
\[
|2\rangle = \left( e_{-1}^3 f_0^1 + (2 - \kappa) e_{-1}^2 \right) |h\rangle .
\]

Using the relations generated by (3.42), the singular vectors (3.46) are equivalent in \( M_h / \mathcal{J} M_h \) to

\[
|1\rangle \cong \left( \frac{4}{3} x H^1 + \frac{4}{3} x^2 f_0^1 + \left( H^2(xy - z) - H^1 z - \frac{4}{3}(xy - z) \right) f_0^2 + \frac{4}{3} x^2 y f_0^3 \\
- xz f_0^1 f_0^2 + y(xy - z) f_0^3 f_0^2 + (x^2 y^2 + z(xy - z)) f_0^3 f_0^2 \right) |h\rangle \tag{3.47a}
\]
\[
|2\rangle \cong \left( -\frac{4}{3} y H^2 - \frac{4}{3} y^2 f_0^2 + \left( H^2(xy - z) - H^1 z + \frac{4}{3} z \right) f_0^1 + \left( \frac{4}{3} xy^2 + y(xy - z) \right) f_0^3 \\
- xz f_0^1 f_0^2 + y(xy - z) f_0^3 f_0^2 - (x^2 y^2 - z(xy - z)) f_0^3 f_0^2 \right) |h\rangle . \tag{3.47b}
\]

As is obvious, just considering the constraints in \( \mathcal{A}_X^\leq (L_h) \) from these two singular vectors does not lead to any restrictions on \( H^1(\infty) \) and \( H^2(\infty) \). At the moment we do not have an analytic method to analyse \( \mathcal{A}_X^\leq (L_h) \), but using Mathematica we have investigated explicitly the constraints arising from many (up to 56) \( f_0^i \)-descendents of the singular vectors \( |1\rangle \) and \( |2\rangle \), and have found that \( \mathcal{A}_X^\leq (L_h) \) is (at largest) a direct sum

\[
\mathcal{A}_X^\leq (L_h) = \oplus_{i=1}^5 \mathbb{C}[H^1, H^2] \cdot v_i ,
\]

where one choice of \( v_i \) is:

\[
v_1 = (H^i(\infty) - H^1(\infty))(H^1(\infty)-H^1-H^2-1)(H^1(\infty)+H^2+2-\kappa) |h\rangle \\
v_2 = (H^i(\infty)-H^1)(H^1(\infty)-H^1-H^2-1)(H^1(\infty)+H^1+2) |h\rangle \\
v_3 = (H^i(\infty)-H^1)(H^1(\infty)+H^1+2)(H^1(\infty)+H^2+2-\kappa) |h\rangle \\
v_4 = (H^i(\infty)+H^1+2)(H^1(\infty)-H^1-H^2-1)(H^1(\infty)+H^2+2-\kappa) |h\rangle \\
v_5 = (H^i(\infty)+H^1+2)(H^1(\infty)-H^1-H^2-1)(H^1(\infty)+H^2+2-\kappa) f_0^3 |h\rangle
\]

Of course this representation of the \( v_i \) is not unique, but it has one advantage in that it is independent of \( X \). On each of these vectors the actions of \( H^1(\infty) \) and \( H^2(\infty) \) are given in terms of \( H^1 \) and \( H^2 \) by

\[
(H^1(\infty), H^2(\infty)) = \begin{cases} 
    v_1 : & (-H^1 - 2, H^1 + H^2 + 1) , \\
    v_2 : & (-H^2 - 2 + \kappa, -H^1 - 2 + \kappa) , \\
    v_3 : & (H^1 + 1 + H^2, -H^2 - 2) , \\
    v_4 , v_5 : & (H^1, H^2) .
\end{cases}
\]

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This is exactly the same space of allowed fusions as we found in section 3, with the same multiplicities. Note that the representation of \( v_4 \) presented in (3.49) uses \( f_0^3 \) – this is essential to provide the multiplicity two for the eigenvalues \( H^i(\infty) = H^i \).

Viewed as a representation of \( \mathbb{C}[H^1, H^2] \), \( A_X^\leq (L_h) \) is five-dimensional. We can also choose to fix the value of \( H^i \) to some generic weight \( \mu \) by extending the set (3.42) generating the right ideal by the combinations \( H^i - \mu(h^i) \). This leads to the smaller space \( A_X^\leq (L_h)^\mu \) which is genuinely a five-dimensional vector space over \( \mathbb{C} \). There is no canonical choice for the representatives of a basis of \( A_X^\leq (L_h)^\mu \), but we note that one choice of vectors spanning the space is the set

\[
\{ |h\rangle, f^1 f^2 |h\rangle, f^2 f^1 |h\rangle, f^1 f^3 |h\rangle, f^2 f^3 |h\rangle \}.
\]

Although the action of \( h_0^i \) on this space has no immediate connection with the actions of \( H^i(\infty) \) or \( H^i \), this choice does exhibit clearly a connection with the subset of the weight diagram of the \( sl(3) \) adjoint representation of highest weight \( \bar{\rho} = \iota(w_0) \) – the image of \( h = w_0 \cdot k\Lambda_0 \) under the map \( \iota \) described in [1]. For \( H^i = 0 \) the set of eigenvalues of \( H^i(\infty) \) in (3.50) together with their multiplicities recovers the generalised weight diagram associated with the representation \( h \). Similarly for any ‘preadmissible’ weight \( \nu \) the eigenvalues of \( H^i(\infty) \) in the space \( A_X^\leq (L_\nu)^0 \) define a generalised weight diagram. While for the ‘pre-integrable’ sub-series of weights in (1.14) this coincides with the standard weight diagram \( \Gamma_\nu \) of the finite dimensional representation of \( sl(3) \) of highest weight \( \nu \), we expect that the generalised weight diagrams introduced in [1] will be reproduced for the sub-series of (1.14) with \( \lambda' = 0 \).

We have also repeated the same analysis for the representation \( f \) in (1.16) and this time find \( A_X^\leq (L_f) \) to be a \( \mathbb{C}[H^1, H^2] \) module as (3.48) with (3.43) replaced by a seven dimensional space. Each set of eigenvalues have multiplicity one in this case, agreeing with the results of [1].

For non-generic \( X \), e.g., any of \( x, y \) or \( z \) being 0 or \( \infty \), or satisfying \( xy - z = 0 \), the analysis leading to these results breaks down and the spaces \( A_X^\leq (L_\nu) \) are expected to become more complicated.

It would be nice to find a general method to treat the case of \( \widehat{sl}(3) \) rather than have to use explicit calculations in each case, and this is a problem to which we hope to return in the future.
4. Conclusions

In this paper we have employed two methods of dealing with the null vectors decoupling constraints on some 3–point $sl(3)$ invariants. Both lead to the same result and are in full agreement with the fusion rules determined in [I]. We have concentrated here on “preadmissible” representations characterised by a generic value of the level, the additional singular vectors arising at rational level values $k + 3 = p'/p$ can be similarly analysed.

The case of non integral (dominant) highest weights reveals a new phenomena, namely a dependence of the fusion rules on the coordinates of the flag manifold. Taking $x = 0$ or $x = \infty$ in the $\hat{sl}(2)$ 3–point decoupling equations leads to a subset of the generic fusions of $\hat{sl}(2)$ – the rule in [G], which gives three-point functions with regular power expansions around these points. The extension of this analysis to $\hat{sl}(3)$, with the fusion rules in [I] corresponding to generic $X$, is possible and reasonably straightforward, if messy.

We have shown how the approach of Feigin and Malikov produces the correct results for $sl(3)$ as well as for $sl(2)$. In [H], Dong et al. found the same results for the space of fields $A^\prec_0(L_0) = A^\prec_X(L_0)$ as Feigin and Malikov, and called this space a ‘$\mathbb{Q}$–graded Zhu’s algebra’. This has the calculational advantage that it is finite dimensional for all admissible models, whereas Zhu’s algebra itself is only finite dimensional for unitary models, but the disadvantage that it is too “small” and does not agree with Zhu’s algebra in any of the latter cases except the trivial case of $k = 0$. There is clearly some point in extending Zhu’s methods to cover general admissible models, but supposing that [H] is along the right route, it will certainly be necessary to consider more general spaces $A^\prec_X(L_\nu)$ for arbitrary $\nu$ rather than simply $A^\prec_0(L_\nu)$ as in [H] to recover the full non-degenerate fusion rules.

The computations here are still not sufficient for a full proof that the fractional level fusion rule multiplicities in [I] are precisely the ones resulting from the solutions of the singular vectors decoupling equations at generic $X$. However they strongly support this expectation in confirming the basic rules for all “fundamental” representations generating the fusion ring.

The algebraic and differential equation methods are clearly equivalent, and for the model presented here the differential equation method is much easier to understand and faster to analyse. However, there are algebras, such as the $W^{(2)}_3$ algebra of Bershadsky–Polyakov for which there is no known action of the algebra on the fields in terms of differential operators, but for which a naive application of the algebraic method has so far only produced fusion rules akin to those of [I] for $sl(2)$, with similar problems (nilpotency
etc). We expect that the insight gained from these calculations, and the way in which degenerate fusion rules can be seen as coming from an incomplete parametrisation of the space of primary fields, will lead to progress on such problems.

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