Polynomially singular damping gives polynomial decay on the torus

Perry Kleinhenz and Ruoyu P. T. Wang

Abstract
We prove polynomial energy decay for polynomially controlled singular damping on the torus. This decay rate extends a result for bounded damping. We construct a semigroup that provides energy decay information in the singular case and use it to reduce the problem to resolvent estimates for the stationary damped wave equation. We then prove sufficiently good resolvent estimates using a version of the Morawetz multiplier method. We also establish exponential energy decay for such dampings on the circle, demonstrating that overdamping does not occur.

1 Introduction
In this paper we study the damped wave equation. Let \((M, g)\) be a Riemannian manifold and let \(W\) be a non-negative measurable function on \(M\). Then the viscous damped wave equation is
\[
\begin{cases}
(\partial_t^2 - \Delta + W \partial_t) u = 0 \\
(u, \partial_t u)_{t=0} = (u_0, u_1) \in H^2 \times H^1.
\end{cases}
\] (1.1)
The primary object of study in this paper is the energy
\[
E(u, t) = \frac{1}{2} \int |\nabla u|^2 + |\partial_t u|^2 \, dx.
\]
When \(W\) is continuous it is classical that uniform stabilization is equivalent to geometric control by the positive set of the damping. That is
\[
E(u, t) \leq Cr(t)E(u, 0),
\]
with \(r(t) \to 0\) as \(t \to \infty\), if and only if there exists \(L\), such that all geodesics of length at least \(L\) intersect \(\{W > 0\}\). Furthermore, due to basic semigroup theory, when such decay occurs \(r(t)\) can always be taken exponentially decaying in \(t\).

When the geometric control condition does not hold we instead look for \(r(t)\) such that
\[
E(u, t)^{1/2} \leq Cr(t) \left( \|u_0\|_{H^2} + \|u_1\|_{H^1} \right). \tag{1.2}
\]
Then the optimal $r(t)$ depends on the geometry of $M$ and $\{W > 0\}$, as well as properties of $W$ in a neighborhood of $W = 0$. Specifically on the torus we expect polynomial decay, so for some $\alpha$ we have
\[ E(u, t)^{1/2} \leq \frac{C}{t^\alpha} (\|u_0\|_{H^2} + \|u_1\|_{H^1}). \] (1.3)

Typically only $W \in L^\infty(M)$ is considered, in this paper we study $W \in L^{1-0}(M)$ with some additional structure.

**Definition 1.** Fix $\theta \in S^1$. Parametrized $S^1$ by $[-\pi, \pi]$ where $\theta$ is identified by 0, with ends identified periodically. We define, $X^{\beta}_0$, the space of polynomially controlled functions on $S^1$ as the space of measurable functions $f$ on $S^1$ such that there are $C_0, \sigma > 0$ such that $C_0^{-1}V^{\beta}(x) \leq W(x) \leq C_0V^{\beta}(x)$, where
\[ V^{\beta}(x) = \begin{cases} 0, & x \in [-\sigma, \sigma], \\ (|x| - \sigma)^\beta, & |x| \in (\sigma, \pi). \end{cases} \]

We will consider $\beta \in [-1, 0)$, so $W$ is unbounded. Because of the unboundedness of the damping we must adjust the space our initial data is in. For $s \geq 0$, let $H^s_W = \{u \in H^s(M); Wu \in L^2(M)\}$ and equip it with the norm
\[ \|u\|^2_{H^s_W} = \|u\|^2_{H^s} + \|W u\|^2_{L^2}. \]

Now we can state our results. First on $M = T^2$, $y$-invariant and unbounded polynomially controlled damping functions produce polynomial decay at a rate depending on the degree of the polynomial control.
Theorem 1.1. Let $M = T^2$ and fix $n \in \mathbb{N}$ and $\beta \in [-1, \infty)$. Suppose $W(x, y) = \sum_{j=1}^{n} W_j(x)$, with $W_j \in X^\beta_{\theta_j}$. Then there exists $C$, such that

$$E(u, t) \leq C t^{\frac{1}{\frac{1}{2^{\beta}}} (\|u_0\|_{H^2} + \|u_1\|_{H^1}),$$

for all solutions $u$ of (1.1) with initial data $(u_0, u_1) \in H^2_W \times H^1_W$.

Remark 1. Note for $\beta \in [0, \infty)$, this result was already shown by [DK19] and [Sta17], so we only address the case $\beta \in [-1, 0)$.

For $M = S^1$, we prove that exponential decay occurs when $W$ is a finite sum of polynomially controlled functions and bounded functions.

Theorem 1.2. Suppose $M = S^1$ and for $1 \leq j \leq n$, $W_j \in L^\infty$ or $W_j \in X^\beta_{\theta_j}$ with $\beta_j \in [-1, 0)$. If $W = \sum_{j=1}^{n} W_j$, then there exist $C, c > 0$ such that

$$E(u, t) \leq C e^{-ct} E(u, 0),$$

for all solutions $u$ of (1.1) with initial data $(u_0, u_1) \in H^1_W \times L^2_W$.

Remark 2. Note, the proof of Theorem 1.2 suffices for $W_j = 1_{[\pi, \sigma]}(x) V_j$ or $1_{[\sigma, \pi]}(x) V_j$ for $V_j \in X^\beta_{\theta_j}$. However for ease of notation we only work with the symmetric definition of polynomial control. See Remark 3 for further details.

This theorem states that, in dimension 1, the GCC implies exponential decay even if there are some singularities, as long as the singularities are not too large.

1.1 Literature Review

The equivalence of uniform stabilization and geometric control for continuous damping functions was proved by Ralston [Ral69], and Rauch and Taylor [RT75] (see also [BLR88], [BLR92] and [BG97], where $M$ is also allowed to have a boundary). For finer results concerning discontinuous damping functions, see Burq and Gérard [BG18].

Decay rates of the form (1.2) go back to Lebeau [Leb96]. If we assume only that $W \in C(M)$ is nonnegative and not identically 0, then the best general result is that $r(t) = 1/\log(2+t)$ in (1.2) [Bur98, Leb96]. Furthermore, this is optimal on spheres and some other surfaces of revolution [Leb96]. At the other extreme, if $M$ is a negatively curved (or Anosov) surface, $W \in C^\infty(M)$, and $W \neq 0$, then $r(t)$ may be chosen exponentially decaying [DJN19].

When $M$ is a torus, these extremes are avoided and the best bounds are polynomially decaying as in (1.3). Anantharaman and Léautaud [AL14] show (1.3) holds with $\alpha = \frac{1}{2}$ when $W \in L^\infty$, and $W > 0$ on some open set, as a consequence of Schrödinger observability/control [Ja90, Mac10] [BZ12]. The more recent result of Burq and Zworski on Schrödinger observability and control [BZ19] weakens the final requirement to merely $W \neq 0$. Anantharaman and Léautaud [AL14] further show that if supp $W$ does not satisfy the geometric control condition then (1.2) cannot hold for $\alpha > 1$. They also show if there exists $C > 0$, such that $W$ satisfies $|V W| \leq CW^{1-\varepsilon}$ for $\varepsilon < 1/29$, and $W \in W^{k_0, \infty}$ for $k_0 \geq 8$, then (1.3) holds with $\alpha = 1/(1 + 4\varepsilon)$.
Sharp decay results have been obtained on the torus when the damping is taken to be polynomially controlled, bounded and $y$-invariant. In particular [Kle19] and [DKT19] together show that for such dampings \( (1.3) \) holds with $\alpha = \frac{\beta+2}{\beta+3}$ and there are some solutions decaying no faster than this rate. See also [AL14] and [Sta17] for the original proof of the case $\beta = 0$. For improved decay rates under different geometric assumptions on the support of the damping see [LL17] and [Sun22]. In [Wan21], the second author showed that \( (1.3) \) holds with $\alpha = \frac{1}{2}$ when there is boundary damping. The boundary damping has a singularity structure similar to $\delta(x)$, which is in the Besov space $B^{-1}_{\infty, \infty}$, hinting that the decay rate $\alpha = \frac{\beta+2}{\beta+3}$ still holds when $\beta = -1$. We then conjectured that $\alpha = \frac{\beta+2}{\beta+3}$ still holds for all $\beta \in [-1, \infty)$. In this paper, we prove that our conjecture is correct.

Another motivation for the study of unbounded damping to understand their over-damping behavior. Overdamping is an intuitive term without a precise definition that broadly refers to cases where “more” damping leads to slower decay. At a basic level this can be observed on $M = [0,1]$ with $W = C$, a constant. Such a damping satisfies the GCC and so experiences exponential decay, but as shown in [CZ94] the exponential rate is not monotone in $C$. The decay becomes faster as $C$ increases from 0, up to a point, and then becomes slower as $C$ goes to infinity. Because of this, one might expect that unbounded dampings would exhibit slower decay than bounded analogs. However, when $M = [0,1]$ and $W = \frac{2}{x}$, [CC01] show that all solutions are identically 0 for $t > 2$. The case where $W = \frac{2\alpha}{x}$ on $[0,1]$ for $\alpha$ a constant was studied in [FHS20]. The authors showed that although the finite extinction time behavior is unique to $\alpha = 1$, in general all solutions decay exponentially and for $\alpha \in \mathbb{N}$ most solutions have a finite extinction time. Unbounded damping has also been studied on non-compact manifolds in [FST18], [Ger22] and [Arn22].

On the torus Theorem 1.1 indicates that unbounded damping could exhibit the overdamping behavior present in bounded damping. In particular, decreasing $\beta$ corresponds to a larger value of the damping at the boundary of its support, for which only slower decay is guaranteed. This can be thought of as the boundary of the damping acting as an interface across which some energy is transmitted and some is reflected, and the sharper the turn on of the damping the more energy is reflected. A lower bound on the energy decay is necessary to make this intuitive picture precise in the unbounded case.

Theorem 1.2 begins to address some of the questions on unbounded damping. As the Theorem shows location, scale, and constant factors of singularities do not interfere with exponential decay when the GCC is satisfied. So overdamping does not occur for many unbounded dampings in 1 dimension. It remains to generalize these results to manifolds and see how adjusting these features can guarantee or prohibit finite extinction time of solutions.

### 1.2 Paper Outline

Theorems 1.1 and 1.2 are proved via resolvent estimates for the stationary equation. First we define the stationary operator

$$P_\lambda = -\Delta - i\lambda W - \lambda^2,$$
which can be thought of as the damped wave operator after a partial Fourier transform in the $t$ variable. We will later make an assumption that on $H^1_W$, the $H^1$-norm is equivalent to $H^1_W$-norm, in Assumption 1. With Assumption 1 we have two distinct results to characterize polynomial and exponential decay by resolvent estimates of $P_\lambda$ on $L^2$.

**Proposition 1.3.** Fix $\alpha > 0$. Under Assumption 1 there exists $C > 0$ such that

$$E(u, t)^{1/2} \leq \frac{C}{\langle t \rangle^{\alpha}} \left( \|u_0\|_{H^2} + \|u_1\|_{H^1} \right),$$

for all solutions $u$ with initial data $(u_0, u_1) \in H^2_W \times H^1_W$, if and only if there exists $C > 0$ such that for all $\lambda \in \mathbb{R}$ and $\lambda \neq 0$ we have

$$\|P_\lambda^{-1}\|_{L^2 \to L^2} \leq C|\lambda|^{1/\alpha - 1}. \tag{1.4}$$

The exponential decay resolvent estimate result can be thought of as a refinement of the polynomial result when $\alpha = \infty$.

**Proposition 1.4.** Under Assumption 1 there exist $C, c > 0$ such that

$$E(u, t) \leq Ce^{-ct}E(u, 0),$$

for all solutions $u$ with initial data $(u_0, u_1) \in H^1_W \times L^2$, if and only if there exist $C > 0$ such that for all $\lambda \in \mathbb{R}$ and $\lambda \neq 0$ we have

$$\|P_\lambda^{-1}\|_{L^2 \to L^2} \leq \frac{C}{|\lambda|}. \tag{1.5}$$

When $W$ is bounded these propositions are classical consequences of [BT10] and [Gea78], [Pru84], [Hua85] respectively. In our case, since $W$ is unbounded, additional care is required, but the same semigroup results can be used. Propositions 1.3 and 1.4 are proved in Section 2. We do this by proving solutions of (1.1) still form a semigroup and showing that estimates on $P_\lambda^{-1}$ are equivalent to resolvent estimates for the semigroup generator, which allows us to apply the semigroup decay results. The requirement that $H^1$ and $H^1_W$ have equivalent norms is a key assumption, which is automatically satisfied when $W$ is bounded. As we will show in the next section, this assumption holds for all $W \in X^\beta_{\theta}$ with $\beta \in [-1, 0)$ and Lemma 2.1 provides a more general condition.

Once the above equivalences are shown it remains to show sufficiently good resolvent estimates to establish the desired decay rates. In Section 3 the necessary resolvent estimates on the stationary operator are proved using a multiplier method.

## 2 Semigroups Generated by Unbounded Damping

In this section we provide a general framework for the damped wave semigroup with unbounded damping, which we then use to prove Propositions 1.3 and 1.4.

Let $M$ be a compact smooth manifold without boundary. Recall that $H^1_W(M) = \{u \in H^1(M) : Wu \in L^2(M)\}$. In this section we will impose only one assumption on the damping $W$:
**Assumption 1.** We assume there exists $C > 0$ such that for any $u \in H^1_W$ we have

$$
\|Wu\|_{L^2} \leq C \|u\|_{H^1}.
$$

(2.1)

In other words, the $H^1_W$-norm is equivalent to the $H^1$-norm on $H^1_W$.

To provide a class of examples and show that $W \in X^3_\theta$ satisfy Assumption 1 we provide a general framework for the damped wave semigroup with damping $W$ unbounded near a closed hypersurface. Let $M$ be a compact smooth manifold without boundary. Let $N$ be a connected closed and orientable hypersurface in $M$ with an orientable normal bundle. Take a normal neighborhood $U = U_+ \sqcup N \sqcup U_-$ divided into two components $U_\pm$ by $N$. Denote $\rho \in C^\infty(U)$ by

$$
\rho(p) = \begin{cases} 
\pm \text{dist}(p, N), & p \in U_\pm \\
0, & p \in N.
\end{cases}
$$

There exists some small $\delta > 0$ such that $N_\delta = \rho^{-1}((-\delta, \delta))$ is compactly embedded in $U$ and $\rho^{-1}(s)$ is diffeomorphic to $N$ for all $s \in (-\delta, \delta)$. We can then identify $N_\delta$ by $N \times (-\delta, \delta)$. We show that under mild assumptions on the unboundedness of $W$ near $N$, Assumption 1 holds.

**Lemma 2.1** (Polynomial Singularities). Consider a damping function $W \geq 0$ on $M$ which is $L^\infty$ on $M \setminus N_\delta$. For $\rho \in (-\delta, \delta)$, define

$$
W_\pm(\rho) = \pm \operatorname{esssup}\{\pm W(q, \rho) : q \in N\}.
$$

Suppose either of the following holds:

1. $W_+ \in |\rho|^{-\frac{1}{2}} L^2(-\delta, \delta)$, or
2. $W_+ \in |\rho|^{-1} L^\infty(-\delta, \delta)$ and $W_- \notin L^2(I)$ for all open intervals $I$ containing 0.

Then Assumption 1 is satisfied.

**Proof.** Since $W \in L^\infty(M \setminus N_\delta)$, it suffices to prove

$$
\|Wu\|_{L^2(N_\delta)} \leq C \|u\|_{H^1(N_\delta)}.
$$

1. Assume $W_+ \in |\rho|^{-\frac{1}{2}} L^2(-\delta, \delta)$. By the Sobolev embedding we have

$$
u \in H^1(N_\delta) \hookrightarrow C^{0, \frac{1}{2}}((-\delta, \delta), L^2(N)).
$$

Therefore

$$
\|Wu\|_{L^2(N_\delta)}^2 \leq C \int_{-\delta}^{\delta} |\rho| W_+^2(\rho) |\rho|^{-1} \int_N |u(q, \rho)|^2 d\rho \leq C \|W_+\|_{L^2(-\delta, \delta)}^2 \|u\|_{H^1(N_\delta)}^2,
$$

Since $|\rho|^{\frac{1}{2}} W_+ \in L^2(-\delta, \delta)$, this is the desired conclusion.
2. When \( W_- \notin L^2(I) \) for any \( I \) containing 0, we claim \( H^1_W(M) \subset \{ u \in H^1(M) : u|_N = 0 \} \). Let \( u \in H^1_W(M) \) and \( u_N \) be its trace on \( N \), which is in \( H^{\frac{1}{2}}(N) \). Again by the Sobolev embedding \( u \in C^{0, \frac{1}{2}}((-\delta, \delta)_p, L^2(N)) \) and so

\[
\left| \|u(q, \rho)\|_{L^2}^2 - \|u_0(q)\|_{L^2}^2 \right| \leq C|\rho|.
\]

Thus there exists \( \delta' > 0 \) such that \( \|u(q, \rho)\|_{L^2}^2 \geq \frac{1}{2}\|u_0(q)\|_{L^2}^2 \) for \( \rho \in (-\delta', \delta') \). Then

\[
\infty > \|Wu\|_{L^2(N_\delta)}^2 \geq \int_{-\delta'}^{\delta'} \|u(q, \rho)\|_{L^2}^2 \, dq \, d\rho \geq \frac{1}{2}\|W_-(\delta', -\delta')\|\|u_N\|_{L^2}^2.
\]

Since \( W_- \notin L^2(-\delta', \delta') \), we must have \( u_N = 0 \) almost everywhere on \( N \). To complete the proof of (2.1) in this case, note by the Hardy inequality

\[
\|Wu\|_{L^2(\rho^{-1}(0, \delta/2))}^2 \leq C\int_0^{\rho^{-1}} \int_N |\rho|^{-2} |u|^2 \, dq \, d\rho \leq C\int_0^{\delta/2} \int_N |\partial_\rho u|^2 \, dq \, d\rho \leq C\|u\|_{H^1(N_\delta/2)}^2,
\]

for any \( u \in C_c^\infty(N_\delta) \). Similar estimates hold on \((-\delta/2, 0)\). Since in this case we have \( W \in L^\infty(M \setminus N_\delta/2) \), extending by density produces the desired result. \( \square \)

Next we show some mapping properties of the resolvent of the stationary damped wave operator with unbounded damping. For \( s \geq 0 \), let \( H^s_W(M) \) be the space of bounded linear functional mapping \( H^s_W(M) \) to \( C \). Note that \( H^s_W \subset H^s \subset H^{-s} \subset H^{-s}_W \) for \( s \geq 0 \).

**Lemma 2.2.** For \( \lambda \in \mathbb{C} \), consider \( P_\lambda = -\Delta - i\lambda W - \lambda^2 \) as a bounded operator from \( H^1_W \) to \( H^{-1}_W \). For \( \text{Im} \lambda \geq 0 \) and \( \lambda \neq 0 \), \( P_\lambda^{-1} : H^{-1}_W \to H^1_W \) is bijective, and restricts to a bijective map from \( L^2 \) to \( H^1_W \).

**Proof.** 1. First we will show that \( P_\lambda \) is Fredholm with index 0 for all \( \lambda \in \mathbb{C} \). Note

\[
\langle P_\lambda u, v \rangle = \langle (-\Delta + 1 + W) u, v \rangle,
\]

is a coercive form on \( H^1_W \). Indeed from (2.1),

\[
\langle P_\lambda u, u \rangle = \|u\|_{H^1}^2 + \|\sqrt{W} u\|^2 \geq \frac{1}{2}\|u\|_{H^1_W}^2.
\]

By the Lax-Milgram theorem, we have

\[
P_\lambda^{-1} : H^{-1}_W \to H^1_W,
\]

is bounded. Then \( P_\lambda \) is Fredholm with index 0. Now note

\[
P_\lambda = P_\lambda - (1 + \lambda^2) - (1 + i\lambda) W,
\]

and that \((1 + \lambda^2) + (1 + i\lambda) W : H^1_W \to L^2 \to H^{-1} \subset H^{-1}_W \) compactly. Thus \( P_\lambda \) is Fredholm with index 0 for all \( \lambda \in \mathbb{C} \).
2. We show that \( P_\lambda^* : H_W \to H_W^{-1} \) has trivial kernel. For \( \text{Im} \lambda \geq 0, \lambda \neq 0 \) let \( \lambda = \alpha + i\beta \). Consider
\[
P_\lambda^* u = (-\Delta + (\beta^2 + \beta W - \alpha^2) + i\alpha(W + 2\beta)) u = 0.
\]
Pair it with \( u \) to see
\[
(P_\lambda^* u, u) = \|\nabla u\|^2 + (\beta^2 - \alpha^2)\|u\|^2 + \beta\|\sqrt{W} u\|^2 + i\left(\alpha\|\sqrt{W} u\|^2 + 2\alpha\beta\|u\|^2\right) = 0.
\]
Suppose \( \beta > 0 \). When \( \alpha \neq 0 \), the imaginary part of (2.2) implies \( \|u\| = 0 \). When \( \alpha = 0 \), the real part of (2.2) implies \( \|u\| = 0 \). When \( \beta = 0 \) and \( \alpha \neq 0 \), (2.2) is reduced to
\[
\|\nabla u\|^2 - \alpha^2\|u\|^2 - i\alpha\|\sqrt{W} u\|^2 = 0,
\]
and \( \|\sqrt{W} u\| = 0 \). From the unique continuation we know \( u = 0 \) almost everywhere. Now since Ker \( P_\lambda^* \) is trivial and \( P_\lambda \) has index 0, we know CoKer \( P_\lambda \) is trivial and \( P_\lambda \) is invertible.

3. To see \( P_\lambda^{-1} \) maps \( L^2 \) to \( H_W^2 \) consider \( f \in L^2 \) and \( u = P_\lambda^{-1} f \in H_W^1 \). We have
\[
(-\Delta - \lambda^2) u = f + i\lambda Wu,
\]
where \( Wu \in L^2 \). Since \( -\Delta - \lambda^2 \) is classically elliptic, we have
\[
\|u\|_{H^2} \leq C\|u\|_{H^1} + C\|f\|_{L^2} + \|Wu\|_{L^2} \leq C\|f\|_{L^2},
\]
as desired. \( \square \)

Now we consider the semigroup. Let \( \mathcal{H} = \{(u, v) \in H_W^1(M) \oplus L^2(M)\} \) with norm \( \|(u, v)\|_{\mathcal{H}} = \|u\|_{H^1}^2 + \|v\|_{L^2}^2 \). Define
\[
A = \begin{pmatrix} 0 & \text{Id} \\ -\Delta & -W \end{pmatrix} : D(A) \to \mathcal{H},
\]
with \( D(A) = (H^2 \oplus H_W^1) \cap \mathcal{H} = H_W^2 \oplus H_W^1 \). The operator \( A \) is closed and densely defined on \( \mathcal{H} \), and hence generates a strongly continuous semigroup \( e^{tA} : \mathcal{H} \to \mathcal{H} \). Solutions of (1.1) are equivalent to solutions of
\[
\begin{cases}
\partial_t U(t) = AU(t) \\
U(0) = (u_0, u_1)^t,
\end{cases}
\]
where \( U(t) = e^{tA} U(0) \). Note that the embeddings \( H^2 \cap H^1_W \to H^1_W \to L^2 \) are compact so \( A \) has compact resolvent. In turn this means that the spectrum of \( A \) contains only isolated eigenvalues.

We would like to apply the following results to \( A \), if possible.

**Proposition 2.3** (Borichev-Tomilov, [BT10]). Let \( e^{tA} \) be a strongly continuous semigroup on a Hilbert space \( \mathcal{H} \), generated by \( A \). If \( i\mathbb{R} \cap \text{Spec}(A) = \emptyset \), then the following conditions are equivalent:
\[
\|e^{tA} A^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(t^{-\alpha}) \quad \text{as } t \to \infty,
\]
\[
\|(i\lambda \text{Id} - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} = O(|\lambda|^{1/\alpha}) \quad \text{as } \lambda \to \infty.
\]
Proposition 2.4 (Gearhart-Prüss-Huang, [Gea78, Pru84, Hua85]). Let $e^{tA}$ be a strongly continuous semigroup on a Hilbert space $\mathcal{H}$ and assume that there exists a positive constant $M > 0$ such that $\|e^{tA}\| \leq M$ for all $t \geq 0$. Then there exist $C, c > 0$ such that for all $t > 0$

$$\|e^{tA}\|_{L(\mathcal{H})} \leq Ce^{-ct},$$

if and only if $i\mathbb{R} \cap \text{Spec}(A) = \emptyset$ and

$$\sup_{\lambda \in \mathbb{R}} \| (A - i\lambda \text{Id})^{-1} \|_{L(X)} < \infty.$$

An issue with applying these is that $A$ can have spectrum at 0. The outline for the remainder of the section is to define a semigroup generator $\dot{A}$ with no spectrum at 0 and which provides energy decay information for $e^{tA}$. We then establish an equivalence of resolvent estimates for $\dot{A}$ and $P_A$, which we use to prove Propositions 1.3 and 1.4 using Propositions 2.3 and 2.4 respectively.

We follow the strategy of [AL14] to separate the zero-frequency modes from others. Define the spectral projector of $A$ onto its kernel by

$$\Pi_0 = \frac{1}{2\pi i} \int_{\gamma} (z \text{Id} - A)^{-1} dz,$$

where $\gamma$ is a circle around 0 in $\mathbb{C}$ of radius small enough that 0 is the only eigenvalue of $A$ contained in its interior. Then set $\dot{\mathcal{H}} = (\text{Id} - \Pi_0)\mathcal{H}$, and equip it with the norm

$$\|(u, v)\|_{\dot{\mathcal{H}}^2} := \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2,$$

and its associated inner product. This is indeed a norm on $\dot{\mathcal{H}}$ since $\|(u, v)\|_{\dot{\mathcal{H}}} = 0$ is equivalent to $(u_0, u_1) \in \text{Ker}(\Delta) \oplus \{0\} = \Pi_0\mathcal{H}$. Also set $\dot{A} = A|_{\dot{\mathcal{H}}}$ with domain $D(\dot{A}) = D(A) \cap \dot{\mathcal{H}}$. Note that $\text{Spec}(\dot{A}) = \text{Spec}(A) \setminus \{0\}$.

We now show that this $\dot{A}$ indeed generates a semigroup that provides energy decay information about $e^{tA}$.

Proposition 2.5. On $\dot{\mathcal{H}}$, the $\dot{\mathcal{H}}$-norm and $\mathcal{H}$-norm are equivalent. The generator $\dot{A} : D(\dot{A}) \to \dot{\mathcal{H}}$ is maximally dissipative and thus generates a contraction semigroup on $\dot{\mathcal{H}}$. Moreover, the strongly continuous semigroup generated by $A$ on $\mathcal{H}$ can be decomposed as

$$e^{tA} = \Pi_0 + e^{t\dot{A}}(\text{Id} - \Pi_0).$$

Proof. 1. Note that $\|(u, v)\|_{\dot{\mathcal{H}}} \leq \|(u, v)\|_{\mathcal{H}}$ and $(\mathcal{H}, \|(\cdot)\|_{\dot{\mathcal{H}}})$ is continuously embedded in $(\dot{\mathcal{H}}, \|(\cdot)\|_{\dot{\mathcal{H}}})$. Since the embedding is a bijective continuous map, it is further an open map and admits a continuous inverse. This implies that the norms are equivalent.

2. To see $\dot{A}$ generates a contraction semigroup we will show $\text{Id} - \dot{A}$ is bijective and $\dot{A}$ is dissipative, so $\dot{A}$ is maximally dissipative, which gives the desired conclusion by the Lumer-Phillips Theorem.

We first show $\text{Id} - \dot{A}$ is bijective from $D(\dot{A})$ to $\dot{\mathcal{H}}$. It suffices to show that for any $(\tilde{u}, \tilde{v}) \in \dot{\mathcal{H}}$, there exists a unique $(u, v) \in D(\dot{A})$ such that

$$(\text{Id} - \dot{A}) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ -\Delta u + (1 + W)v \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$
Take $v = u - \tilde{u}$, and we have
\[ P_i u = (-\Delta + 1 + W) u = \tilde{v} + (1 + W) \tilde{u} \in L^2. \]

From Lemma 2.2, we know there is an unique $u = P_i^{-1}(\tilde{v} + (1 + W) \tilde{u}) \in H^2 \cap H^1_W$, and $v = u - \tilde{u} \in H^1_W$. So $(u, v) \in D(\tilde{A})$ and $\tilde{A}(u, v) = (\tilde{u}, \tilde{v})$, and thus $\tilde{A}$ is surjective. To see $\tilde{A}$ is injective, it suffices to see when $\tilde{u}, \tilde{v} = 0$, $u = P_i^{-1}(0) = 0$ and $v = u - \tilde{u} = 0$.

3. It is straightforward to compute
\[ \Re \langle \tilde{A}(u, v), (u, v) \rangle_{\tilde{H}} = -\int_M W |v|^2 \leq 0, \]

which demonstrates the dissipative nature of $\tilde{A}$, and so by the Lumer-Phillips theorem, $\tilde{A}$ generates a contraction semigroup.

4. Consider $e^{t\tilde{A}} = e^{t\tilde{A}}(\text{Id} - \Pi_0) + e^{t\tilde{A}} \Pi_0$. As $\tilde{H} = (\text{Id} - \Pi_0)\mathcal{H}$ and $D(\tilde{A}) = (\text{Id} - \Pi_0)D(\tilde{A})$, we have $e^{t\tilde{A}}(\text{Id} - \Pi_0) = e^{t\tilde{A}}(\text{Id} - \Pi_0)$ on $\mathcal{H}$. It remains to show that $e^{t\tilde{A}} = \text{Id}$ on $\Pi_0\mathcal{H}$, which follows immediately from the fact that $\tilde{A}$ generates a unique strongly continuous semigroup on $\Pi_0\mathcal{H}$. \hfill \Box

Before showing resolvent estimates for $\tilde{A}$ are equivalent to resolvent estimates for $P_\lambda$, we prove a lemma that allows us to compare different operator norms of $P_\lambda^{-1}$.

**Lemma 2.6.** There exists $C > 0$ such that for all $\lambda \in \mathbb{R}$ we have
\[ \|u\|_{H^1} \leq C(\lambda)\|u\|_{L^2} + C(\lambda)^{-1}\|P_\lambda u\|_{L^2}, \tag{2.3} \]

and
\begin{align*}
\|P_\lambda^{-1}\|_{H^1_W \to L^2} &\leq C(\lambda)\|P_\lambda^{-1}\|_{L^2 \to L^2} + C(\lambda)^{-1}, \\
\|P_\lambda^{-1}\|_{H^1_W \to H^{-1}_W} &\leq C(\lambda)^2\|P_\lambda^{-1}\|_{L^2 \to L^2} + C.
\end{align*}

**Proof.**
1. Pair
\[ P_\lambda u = (-\Delta - i\lambda W - \lambda^2) u \]

with $u$ and take the real part to see
\[ \|\nabla u\|^2 - \lambda^2 \|u\|^2 = \Re(P_\lambda u, u). \]

Apply the Cauchy-Schwartz inequality to the right hand side to obtain (2.3) and
\[ \|u\|_{H^1} \leq C(\lambda)\|u\|_{L^2} + C\|P_\lambda u\|_{H^{-1}}, \tag{2.4} \]

which we will use later.

2. Note that $P_\lambda^* : H^1_W \to H^{-1}_W$ is given by
\[ P_\lambda^* = -\Delta + i\lambda W - \lambda^2 = P_{-\lambda}. \]

Consider the map $P_\lambda^{-1} : H^1_W \to L^2$ and its adjoint $(P_\lambda^{-1})^* : L^2 \to H^1_W$ and note
\[ \|P_\lambda^{-1}\|_{H^1_W \to L^2} = \left\| (P_\lambda^{-1})^* \right\|_{L^2 \to H^1_W} = \left\| (P_\lambda^*)^{-1} \right\|_{L^2 \to H^1_W} = \|P_{-\lambda}\|_{L^2 \to H^1_W}. \tag{2.5} \]
from the duality. Apply (2.3) to obtain
\[
\|P^{-1}_\lambda\|_{H^{-1}_W \to L^2} = \|P^{-1}_\lambda\|_{L^2 \to H^{-1}_W} \leq C(\lambda)\|P^{-1}_\lambda\|_{L^2 \to L^2} + C(\lambda)^{-1}.
\] (2.6)

Let \( u \in L^2, f \in H^{-1}_W \) and \( P_\lambda u = f \). From (2.5) and (2.6) we have
\[
\|u\|_{L^2} \leq (C(\lambda)\|P^{-1}_\lambda\|_{L^2 \to L^2} + C(\lambda)^{-1}) \|f\|_{H^{-1}}.
\]
Apply (2.4) to see
\[
\|u\|_{H^1} \leq (C(\lambda)^2\|P^{-1}_\lambda\|_{H^{-1}_W \to L^2} + C) \|f\|_{H^{-1}},
\]
as desired. \( \square \)

We now show that resolvent estimates for \( P_\lambda \) are equivalent to resolvent estimates for \( \hat{A} \).

**Proposition 2.7.** Let \( \alpha \geq -1 \) and \( \lambda_0 > 0 \). The following are equivalent:

1. There exists \( C > 0 \) such that for all \( \lambda \in \mathbb{R} \) with \( |\lambda| \geq \lambda_0 \) we have
\[
\|P^{-1}_\lambda\|_{L^2 \to L^2} \leq C(\lambda)^{\alpha}.
\] (2.7)

2. There exists \( C > 0 \) such that for all \( \lambda \in \mathbb{R} \) with \( |\lambda| \geq \lambda_0 \) we have
\[
\|(\hat{A} + i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C(\lambda)^{\alpha+1}.
\] (2.8)

**Proof.** 1. Assume (2.7). Let \( (f, g) \in \mathcal{H} \) and \( (u, v) = (\hat{A} + i\lambda)^{-1}(f, g) \in \mathcal{D}(\hat{A}) \). This indicates
\[
P_\lambda u = (-\Delta - i\lambda W - \lambda^2) u = -g - (W - i\lambda)f,
\]
\[
v = f - i\lambda u.
\] (2.9)
(2.10)

Note that \( (W - i\lambda)f = \frac{1}{i\lambda}P_\lambda f + \frac{1}{i\lambda}\Delta f \) and reorganize (2.9) into
\[
P_\lambda \left( u + \frac{1}{i\lambda}f \right) = -g - \frac{1}{i\lambda}\Delta f.
\]
By Lemma 2.2 there exists a unique \( u_1 \in H^2_W \) such that
\[
(-\Delta - i\lambda W - \lambda^2) u_1 = -g.
\]
Furthermore, by (2.3) we have
\[
\|u_1\|_{H^1} \leq C \left( \langle \lambda \rangle\|P^{-1}_\lambda\|_{L^2 \to L^2} + \langle \lambda \rangle^{-1} \right) \|g\|_{L^2} \leq C(\lambda)^{\alpha+1}\|g\|_{L^2},
\]
\[
\|u_1\|_{L^2} \leq \|P^{-1}_\lambda\|_{L^2 \to L^2}\|g\|_{L^2} \leq C(\lambda)^{\alpha}\|g\|_{L^2}.
\] (2.11)

Meanwhile we have
\[
(-\Delta - i\lambda W - \lambda^2) \left( u + \frac{1}{i\lambda}f - u_1 \right) = -\frac{1}{i\lambda}\Delta f.
\]
Apply Lemma 2.6 to see
\[ \|u + \frac{1}{i\lambda} f - u_1 \|_{H^1} \leq C \| F^{-1}_\lambda \|_{H^1_{u_1} \to H^1_{u}} \| - \frac{1}{i\lambda} \Delta f \|_{H^{-1}} \leq (C(\lambda)^2 \| P^{-1}_\lambda \|_{L^2 \to L^2} + C) |\lambda|^{-1} \| f \|_{H^1} \leq (C(\lambda)^{\alpha+1} + C(\lambda)^{-1}) \| f \|_{H^1} \leq C(\lambda)^{\alpha+1} \| f \|_{H^1}. \] (2.12)

and
\[ \| u + \frac{1}{i\lambda} f - u_1 \|_{L^2} \leq C(\lambda)^{\alpha} \| f \|_{H^1}. \]

Combine (2.11) and (2.12) to observe
\[ \| u \|_{H^1} \leq C(\lambda)^{\alpha+1} (\| f \|_{H^1} + \| g \|_{L^2}). \]

Similarly we have
\[ \| u \|_{L^2} \leq C(\lambda)^{\alpha} (\| f \|_{H^1} + \| g \|_{L^2}) \]
and from (2.10) we have
\[ \| v \|_{L^2} \leq C \| f \|_{L^2} + C(\lambda)^{\alpha+1} (\| f \|_{H^1} + \| g \|_{L^2}) = C(\lambda)^{\alpha+1} (\| f \|_{H^1} + \| g \|_{L^2}). \]

Thus \[ \| (\hat{A} + i\lambda)^{-1}(f, g) \|_{\hat{H}} \leq C(\lambda)^{\alpha+1} \| (f, g) \|_{\hat{H}}. \] Since \( \hat{H} \)-norm is equivalent to \( \hat{H} \)-norm on \( \hat{H} \), we have (2.8).

2. Conversely assume (2.8). Consider
\[(\hat{A} + i\lambda)(u, -i\lambda u) = (0, P\lambda u).\]

Apply (2.8) to see
\[ | - i\lambda u \|_{L^2} \leq C(\lambda)^{\alpha+1} \| P\lambda u \|_{L^2}. \]
Divide both sides by \( \langle \lambda \rangle \) to observe (2.7).

As a final piece of our proof of Proposition 1.4 we must obtain a low-frequency resolvent estimate for \( \hat{A} \).

Lemma 2.8. For any \( \lambda_0 > 0 \),
\[ \sup_{\lambda \in [-\lambda_0, \lambda_0]} \| (\hat{A} + i\lambda)^{-1} \|_{L(\hat{H})} < \infty. \]

In other words, \( \text{Spec}(\hat{A}) \cap i\mathbb{R} = \emptyset. \)

Proof. Assume not, so there exists \( (u_n, v_n) \in \hat{H} \) such that \( (\hat{A} + i\lambda_n)(u_n, v_n) = (f_n, g_n) \in L(\hat{H}) \) for \( \lambda_n \to \lambda \in [-\lambda_0, \lambda_0] \setminus \{0\} \), and
\[ \| (u_n, v_n) \|_{\hat{H}} \equiv 1, \| (f_n, g_n) \|_{\hat{H}} \to 0. \]

Here we used the fact that \( \text{Spec} \hat{A} = \text{Spec} A \setminus \{0\} \). This implies
\[ P\lambda_n u_n = (-\Delta - i\lambda_n W - \lambda_n^2) u_n = -g_n - (W - i\lambda_n)f_n, \] (2.13)
\[ v_n = f_n - i\lambda_n u_n. \] (2.14)
Since \( \{u_n\} \) is bounded in \( H^1 \), there exists \( u \in H^1 \) such that \( u_n \to u \) in \( L^2 \) when passed to a subsequence, since \( L^2 \) is compactly embedded in \( H^1 \). The equation (2.13) implies
\[
(-\Delta - i\lambda W - \lambda^2) u = 0.
\] (2.15)

We claim that \( u = 0 \) almost everywhere. Pair (2.15) with \( u \) and take the imaginary part to see \( \langle Wu, u \rangle = 0 \) which implies \( u \equiv 0 \) on supp \( W \), since \( \lambda \neq 0 \). Taking the real part of (2.15) gives \( (-\Delta - \lambda^2) u = 0 \), but \( u \equiv 0 \) on supp \( W \) which contains an open set. By the unique continuation principle we know \( u \equiv 0 \) and \( \|u_n\|_{L^2} \to 0 \). Note \( \|v_n\|_{L^2} \to 0 \) due to (2.14). Pair (2.13) with \( u_n \) and take the real part to see
\[
\|\nabla u_n\|_{L^2} = |\lambda_n| \|u_n\|_{L^2} + o_n(1) = o_n(1).
\]

Thus \( \|(u_n, v_n)\|_{\dot{H}}^2 = \|u_n\|_{H^1}^2 + \|v_n\|_{L^2}^2 \to 0 \). This contradicts our assumption.

We now give full proofs to Propositions 1.3 and 1.4, that resolvent estimates (1.4) and (1.5) of \( P_\lambda \) imply energy decay.

**Proof of Proposition 1.3.** In Proposition 1.3, we assume \( P_\lambda^{-1} \), for \( |\lambda| \geq \lambda_0 \)
\[
\|P_\lambda^{-1}\|_{L^2 \to L^2} \leq C |\lambda|^{1/\alpha - 1},
\]
by Proposition 2.7 this is equivalent to
\[
\|((\dot{A} + i\lambda)^{-1})\|_{\mathcal{L}(\dot{H})} \leq C(\lambda)^{1/\alpha},
\]
for \( |\lambda| \geq \lambda_0 \). By Proposition 2.3 this implies
\[
\|e^{t\dot{A}}\|_{\mathcal{L}(\dot{H})} = O(t^\alpha).
\]

Therefore the energy of solution \( u \) to the damped wave equation (1.1) is bounded by
\[
E(u, t)^{1/2} \leq C \left\| e^{t\dot{A}}(u_0, u_1) \right\|_{\dot{H}} = C \left\| e^{t\dot{A}}(\text{Id} - \Pi_0) (u_0, u_1) \right\|_{\dot{H}} \leq \left\| e^{t\dot{A}}\dot{A}^{-1} \right\|_{\mathcal{L}(\dot{H})} \left\| \dot{A}(u_0, u_1) \right\|_{\dot{H}} \leq CT^\alpha (\|u_0\|_{H^2} + \|u_1\|_{H^1}),
\]
as desired.

**Proof of Proposition 1.4.** Given an estimate on \( P_\lambda^{-1} \), for \( |\lambda| \geq \lambda_0 \)
\[
\|P_\lambda^{-1}\|_{L^2 \to L^2} \leq C \frac{1}{|\lambda|}
\]
by Proposition 2.7 this is equivalent to
\[
\|((\dot{A} + i\lambda)^{-1})\|_{\mathcal{L}(\dot{H})} \leq C,
\]
for \( |\lambda| \geq \lambda_0 \). This along with Lemma 2.8 allows us to apply Proposition 2.4 and obtain
\[
\|e^{t\dot{A}}\|_{\mathcal{L}(\dot{H})} = Ce^{-ct}.
\]
Therefore the energy of solution \( u \) to the damped wave equation (1.1) is bounded by

\[
E(u,t) \leq C \left \| e^{\mathcal{A} t} (u_0, u_1) \right \|_{\mathcal{H}}^2 \leq C \left \| e^{\mathcal{A} t} (\text{Id} - \Pi_0) (u_0, u_1) \right \|_{\mathcal{H}}^2 \leq C e^{-ct} E(u,0),
\]

as desired. \( \square \)

## 3 Resolvent Estimates in One Dimension

Consider the equation

\[
P_\lambda u = (-\Delta - i\lambda W - \lambda^2) u = f.
\]

To show \( \| P_\lambda^{-1} \|_{L^2 \to L^2} \leq \frac{C}{g(\lambda)^2} \), so that Propositions 1.3 and 1.4 can be applied, it is enough to show that there exist \( C, \lambda_0 \geq 0 \) such that for any \( f \in L^2(M) \) and any \( |\lambda| \geq \lambda_0 \), if \( u \in H^2(M) \) solves (3.1), then

\[
\| u \|_{L^2}^2 \leq C g(\lambda)^2 \| f \|_{L^2}^2.
\]

To show Theorem 1.1 we must show (3.2) holds with \( g(\lambda)^2 = |\lambda|^2 \). To show Theorem 1.2 we must show (3.2) holds with \( g(\lambda)^2 = \frac{1}{|\lambda|^2} \).

The main estimate for this section is the following 1 dimensional resolvent estimate.

**Proposition 3.1** (1D Resolvent Estimate). Consider \( u \in H^2(S^1) \) that satisfies

\[
(-\partial_x^2 - i\lambda W - \mu^2) u(x) = f(x),
\]

then there is \( C, \lambda_0 > 0 \) such that for \( \mu^2 \leq \lambda^2 \) and \( |\lambda| \geq \lambda_0 \) we have

\[
\| u \|_{L^2}^2 + \langle \mu \rangle^{-1} \| \partial_x u \|_{L^2}^2 \leq C \langle \mu \rangle^{-1} \left( 1 + \langle \mu \rangle^{-1} |\lambda|^{\frac{2}{\theta}} \right) \| f \|_{L^2}^2.
\]

The proof of Proposition 3.1 will be delayed to the second part of this section. Note that Proposition 3.1 with \( \mu^2 = \lambda^2 \) and Proposition 1.4 together imply Theorem 1.2. Proposition 3.1 can also be used to show the following proposition, which along with Proposition 1.3 implies Theorem 1.1.

**Proposition 3.2** (Resolvent Estimate on Tori). Let \( u \in H^2(T^2) \) be the solution to

\[
P_\lambda u(x,y) = (-\Delta - i\lambda W - \lambda^2) u(x,y) = f(x,y).
\]

Then there exists \( C, \lambda_0 > 0 \) such that for \( |\lambda| > \lambda_0 \) we have

\[
\| u \|_{L^2}^2 + \lambda^{-2} \| \nabla u \|_{L^2}^2 \leq C \left( 1 + |\lambda|^{\frac{2}{\theta}} \right) \| f \|_{L^2}^2.
\]

**Proof.** Consider the eigenfunctions \( e_n(y) \) with

\[
-\partial_y^2 e_n(y) = \lambda_n^2 e_n(y), \quad \lambda_n^2 \to \infty.
\]
Decompose
\[ u(x, y) = \sum u_n(x) e_n(y), \quad f(x, y) = \sum f_n(x) e_n(y) \]
and the equation (3.4) is reduced to
\[(\partial^2_x - i\lambda W - \mu^2) u_n(x) = f_n(x), \quad \mu^2 = (\lambda^2 - \lambda_n^2). \]

Apply Proposition 3.1 to see uniformly in \(n, \lambda\) that for \(|\lambda| > \lambda_0\) we have
\[ \|u_n\|^2 L^2 + (\mu^2)^{-1} \|\nabla u_n\|^2 L^2 \leq C(\mu^2)^{-1} \left( 1 + (\mu^2)^{-1} |\lambda|^2 \right) \|f_n\|^2 L^2. \]
In particular uniformly in \(n, \lambda\)
\[ \|u_n\|^2 \leq C \left( 1 + |\lambda|^2 \right) \|f_n\|^2. \]

Apply the Parserval theorem to obtain
\[ \|u\|^2 \leq C \left( 1 + |\lambda|^2 \right) \|f\|^2. \] (3.5)

Pair (3.4) with \(u\) and take the real part to see
\[ \|\nabla u\|^2 - \lambda^2 ||u||^2 \leq C \lambda^{-2} ||f||^2 + \frac{1}{2}\lambda^2 ||u||^2. \]
This along with (3.5) produces the desired estimate.

The rest of this section is devoted to the proof of Proposition 3.1. In particular, we will show that there is a \(\mu_0^2 > 0\) such that for any real \(\lambda \geq \lambda_0\) and \(\mu^2 \leq \lambda^2\) and \(u \in H^2(S^1), f \in L^2(S^1)\) solving (3.3) then
\[ \int |u|^2 + |u'|^2 \leq C \int |f|^2 \quad \text{when } \mu^2 < \mu_0^2 \] (3.6)
\[ \int |u|^2 + \frac{1}{\mu^2} |u'|^2 \leq C \frac{\lambda^2 + 1}{\mu^2} \int |f|^2 \quad \text{when } \mu^2 \geq \mu_0^2. \] (3.7)

Here and below integrals are taken over \(S^1\). The general case of \(|\lambda| \geq \lambda_0\) follows by an identical argument, but we focus on \(\lambda > 0\) for ease of notation. Note that in the case \(\mu^2 < \mu_0^2\) we can actually have \(\mu^2 < 0\). However, the bulk of our argument is devoted to the proof of (3.7), where \(\mu^2\) is indeed a positive real number.

In our proof we use a version of the Morawetz multiplier method, which is arranged via the energy functional
\[ F(x) = |u'(x)|^2 + \mu^2|u(x)|^2. \]
This method was introduced by [Mor61]. It has been used in [CV02] and [CD17], we will follow its use in [DK19].

Following the proof of Lemma 1 from [DK19], with the modification that \(b\) must be chosen such that \(b' < 0\) on a neighborhood of each zero interval for \(W\), we obtain basic estimates on the size of \(u\) and \(u'\) on the damped region, and (3.6). The proof is otherwise identical, so we do not include the details.
Lemma 3.3. If $\lambda > 0, \mu^2 \in \mathbb{R}$ and $u, f$ solve (3.3) then
\[ \int |W|u|^2 \leq \frac{1}{\lambda} \int |fu|. \] (3.8)
Furthermore if for some $c > 0, \psi \in C_0^\infty$ is supported in $\{W > c\}$, then there exists $C > 0$ such that
\[ \int \psi |u'|^2 \leq C \left(1 + \frac{\mu^2}{\lambda}\right) \int |fu|. \] (3.9)
Finally there are positive constants $\mu_0^2$ and $C$ such that for any $\lambda > 0, \mu^2 \leq \mu_0^2$ and $u, f$ solving (3.3) we have (3.6)

We now set up a multiplier, which we call $b$. The multiplier method then provides the following estimate, which must be refined to obtain our desired resolvent estimates.

Lemma 3.4. Let $\delta_j > 0, j = 1, \ldots, n$ be fixed constants and let
\[ \phi = \phi(x) = \begin{cases} \lambda \delta_j & |x| \in [\sigma_j - \lambda^{-\delta_j}, \sigma_j + \lambda^{-\delta_j}] \\ 1 & \text{otherwise.} \end{cases} \]
If $\lambda, \mu^2 > 0$ and $u, f$ solve (3.3) then
\[ \int \phi |u'|^2 + \mu^2 \phi |u|^2 \leq \int |f|^2 + \sum_j \lambda \int W_j |uu'|. \]

Proof. Choose $c > 0$ small enough so that $\{W > c\}$ intersects the support of each $W_j$ and let $b$ be piecewise linear on $S^1$ with
\[ b'(x) = \begin{cases} \lambda \delta_j & |x| \in [\sigma_j - \lambda^{-\delta_j}, \sigma_j + \lambda^{-\delta_j}] \\ -M & \{W > c\} \text{ and } |x| \notin [\sigma_j - \lambda^{-\delta_j}, \sigma_j + \lambda^{-\delta_j}] \\ 1 & \text{elsewhere.} \end{cases} \]
It is possible to choose $M$ big enough so that $b$ is indeed periodic on $S^1$. So then let $F = |u'|^2 + \mu^2 |u|^2$ and compute
\[ 0 = \int_0^{2\pi} (bF)' = \int b'|u'|^2 + \mu^2 b'|u|^2 + 2\text{Re}[u''\bar{u}] + \mu^2 2\text{Re}[u'\bar{u}'] \\
= \int b'|u'|^2 + \mu^2 |u|^2 - 2b\text{Re}[f\bar{u}' - 2\lambda b\text{Re}[iWu\bar{u}']]. \]
Therefore
\[ \int b'|u'|^2 + \mu^2 |u|^2 \leq 2 \int b|fu'| + 2\lambda \int bW|uu'|. \]
Now adding a multiple of (3.8) and (3.9) to both sides gives
\[ \int \phi |u'|^2 + \phi \mu^2 |u|^2 \leq 2 \int |fu'| + 2\lambda \int W |uu'| + \lambda^{-1} \int |fu| + C \left(1 + \frac{\mu^2}{\lambda}\right) \int |fu|. \]
Applying Young’s inequality for products to the $|fu|$ terms on the right hand side, absorbing the resultant $|u|^2$ terms back into the left hand side and recalling $W = \sum W_j$ and $\mu^2 \leq \lambda^2$ gives the desired inequality. \qed
It now remains to estimate the $W_j$ terms. We begin with an estimate in the case $M = S^1$, so $\mu^2 = \lambda^2$ and consider $W_j \in L^\infty$. Because $W_j$ satisfies hypotheses of the classical geometric control argument, one expects this argument to be straightforward and it is.

**Lemma 3.5.** When $W_j \in L^\infty$ and $M = S^1$ for any $\varepsilon > 0$ there exists $C > 0$ such that if $\lambda > 0$, $\mu^2 = \lambda^2$ and $u, f$ solve \((3.3)\)

$$\lambda \int W_j |u'| \leq C \int |f|^2 + \frac{\varepsilon \mu^2}{2} \int |u|^2 + \frac{\varepsilon}{2} \int |u'|^2.$$

**Proof.** Well, using that $W_j$ is bounded and \((3.8)\)

$$\lambda \int W_j |u'| \leq \frac{\lambda^2}{2\varepsilon} \int W_j^2 |u|^2 + \frac{\varepsilon}{2} \int |u'|^2 \leq \frac{C\lambda^2}{2\varepsilon} \int W_j |u|^2 + \frac{\varepsilon}{2} \int |u'|^2$$

$$\leq \frac{C\lambda^2}{\mu^2} \int |f|^2 + \frac{\varepsilon \mu^2}{2} \int |u|^2 + \frac{\varepsilon}{2} \int |u'|^2.$$

Finally, since $\mu^2 = \lambda^2$ this gives the desired inequality. \qed

We now turn to $W_j$ with polynomial type singularities. Following the structure of [DK19] we prove an intermediate result and reduce the problem to estimating $V_j \chi_j |f u|$. In this proof we specify $\delta_j = \frac{1}{2+\beta_j}$ in order to control the growth of a term. The proof of this lemma requires a change in technique from the proof in [DK19] in order to account for the singularity and the fact that $W_j'$ is larger than $W_j$ near its singularity.

Let $\chi \in C_0^\infty$ be supported on $|x| > 1/2$ and be identically 1 on $|x| > 1$.

**Lemma 3.6.** If $W_j \in X_{\beta_j}^\delta$, then for any $\bar{\varepsilon} > 0$, there exists $C > 0$ and $\varepsilon > 0$, such that if $\chi_j(x) = \chi \left( \frac{x - \sigma_j}{\varepsilon \lambda^{-\delta_j}} \right)$ and if $\lambda, \mu^2 > 0$ and $u, f$ solve \((3.3)\), then

$$\lambda \int W_j |u'| \leq C(\lambda^2 + \mu^2) \int |f u| + C\lambda^{1/2} \left( \int |f u| \right)^{1/2} \left( \int V_j \chi_j |f u| \right)^{1/2} + \bar{\varepsilon} \int \phi |u'|^2.$$

**Proof.** Recall throughout that $\beta_j \in [-1, 0)$. From $W_j \in X_{\beta_j}^\delta$ we have that there exists $C_j > 0$ and $V_j = (|x| - \sigma_j)^{\beta_j}$ such that $\frac{1}{C_j} V_j(x) \leq W(x) \leq C_j V_j(x)$.

To begin make a change of variables so that $\sigma_j = 0$ then $V_j = |x|^\delta_j$ and $\int W_j |u'| \leq C \int V_j |u'|$. The strategy is to split this integral into $|x| > \varepsilon \lambda^{-\delta_j}$ and $|x| < \varepsilon \lambda^{-\delta_j}$ where $\varepsilon > 0$ is to be chosen later.

Case 1: $|x| > \varepsilon \lambda^{-\delta_j}$. Note $\chi_j$ as defined above is supported on $|x| > \frac{\varepsilon \lambda^{-\delta_j}}{2}$ and is identically 1 on $|x| > \varepsilon \lambda^{-\delta_j}$. So applying Cauchy-Schwarz and \((3.8)\)

$$\int_{|x| > \varepsilon \lambda^{-\delta_j}} V_j |u'| \leq C \left( \int V_j |u|^2 \right)^{1/2} \left( \int V_j \chi_j |u'| \right)^{1/2} \leq \lambda^{-1/2} \left( \int |f u| \right)^{1/2} \left( \int V_j \chi_j |u'| \right)^{1/2}.$$
Using integration by parts
\[
\int V_j \chi_j |u'|^2 = -\text{Re} \int (V_j \chi_j)' u' \bar{u} - \text{Re} \int V_j \chi_j u'' \bar{u}. \tag{3.10}
\]

To control the first term note \(|x| \geq \frac{\lambda^{1/2}}{2} \) on supp \(\chi_j\) so \(\frac{1}{|x|} \leq \frac{2 \lambda^{1/2}}{\varepsilon}\) there, also \(|\chi_j'| \lesssim \frac{\lambda^{1/2}}{\varepsilon}\) and so \(|(V_j \chi_j)'| \leq C \frac{\lambda^{1/2}}{\varepsilon} \). Therefore
\[
\text{Re} \int (V_j)' u' \bar{u} \leq C \frac{\lambda^{1/2}}{\varepsilon} \int V_j |u'|. \tag{3.11}
\]

For the second term apply (3.3) and (3.8) to get
\[
\text{Re} \int V_j \chi u'' \bar{u} = \text{Re} \int V_j \chi (-i \lambda W u - \mu^2 u - f) \bar{u} \leq \text{Re} \int V_j \chi \mu^2 |u|^2 + V_j \chi |fu|
\leq \text{Re} \int \frac{\mu^2}{\lambda} |fu| + V_j \chi |fu|. \tag{3.12}
\]

Combining (3.11) and (3.12) with (3.10) gives
\[
\lambda \int_{|x| > \varepsilon \lambda^{-\delta_j}} V_j |uu'| \leq C \lambda^{1/2} \left( \int |fu| \right)^{1/2} \left( \frac{\lambda^{1/2}}{\varepsilon} \int V_j |uu'| + \frac{\mu^2}{\lambda} |fu| + V_j \chi |fu| \right)^{1/2}. \tag{3.13}
\]

Case 2: \(|x| < \varepsilon \lambda^{-\delta_j}\). To begin note that
\[
\phi \lambda^{-\delta_j} = 1 \text{ on } |x| < \varepsilon \lambda^{-\delta_j} < \lambda^{-\delta_j}. \tag{3.14}
\]

So applying Cauchy-Schwarz
\[
\lambda \int_{|x| < \varepsilon \lambda^{-\delta_j}} V_j |uu'| \leq \lambda^{-\frac{\delta_j}{2}} \left( \int_{|x| < \varepsilon \lambda^{-\delta_j}} V_j^2 |u|^2 \right)^{1/2} \left( \int \phi^2 |u'|^2 \right)^{1/2}. \tag{3.15}
\]

Now let \(\psi\) be a cutoff supported in \(|x| < 2 \lambda^{-\delta_j}\) and identically 1 on \(|x| < \lambda^{-\delta_j}\) then let \(\psi_e(x) = \psi(x/\varepsilon)\) and insert it into the below integral. Then rewriting \(|x|^{2\beta_j} = -C \partial_x |x|^{2\beta_j+1}\) and integrating by parts
\[
\int_{|x| < \varepsilon \lambda^{-\delta_j}} V_j^2 |u|^2 \leq \int |x|^{2\beta_j} \psi_e |u|^2 = -C \int \partial_x (|x|^{2\beta_j+1}) \psi_e |u|^2
\]
\[
= C \int |x|^{2\beta_j+1} \partial_x (\psi_e |u|^2) \leq C \int |x|^{2\beta_j+1} \psi_e' |u|^2 + C \int |x|^{2\beta_j+1} \psi_e |u| \tag{3.16}
\]

To control the first term of (3.16) note
\[
\int_{|x| < 2 \varepsilon \lambda^{-\delta_j}} |x|^{2\beta_j+1} \psi_e' |u|^2 = \int_{|x| < 2 \varepsilon \lambda^{-\delta_j}} |x|^{1+\beta_j} |x|^{-\beta_j} \varepsilon^{-1} \psi' |u|^2
\]
\[
\leq C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j} \int_{|x| < \varepsilon \lambda^{-\delta_j}} V_j |u|^2
\]
\[
\leq C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j} \int W |u|^2 \leq C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j-1} \int |fu|. \tag{3.17}
\]

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Now let $\eta > 0$ be a constant to be specified and applying Young’s inequality for products to the second term of \[(3.16)\], then using that $\psi_{\varepsilon}$ is supported on $\{|x| < 2 \varepsilon \lambda^{\delta_j}\}$ and \[(3.14)\]

$$
\int |x|^{2 \delta_j + \lambda^2} \psi_{\varepsilon} |u u'| \leq \frac{\eta}{2} \int |x|^{2 \delta_j} \psi_{\varepsilon} |u|^2 + \frac{1}{2 \eta} \int |x|^{2 + 2 \beta_j} \phi |u'|^2 \\
\leq \frac{\eta}{2} \int |x|^{2 \delta_j} \psi_{\varepsilon} |u|^2 + \frac{C}{\eta} \varepsilon^{2 + 2 \beta_j} \lambda^{-2 \beta_j \delta_j - 3 \delta_j} \int \phi |u'|^2. \quad (3.18)
$$

So then combining \[(3.16)\], \[(3.17)\] and \[(3.18)\]

$$
\int \psi_{\varepsilon} V_j^2 |u|^2 \leq C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j + 1} \int |f u| + \eta C \int |x|^{2 \beta_j} \psi_{\varepsilon} |u|^2 + \frac{C}{\eta} \varepsilon^{2 + 2 \beta_j} \lambda^{-2 \beta_j \delta_j - 3 \delta_j} \int \phi |u'|^2 \\
\int \psi_{\varepsilon} V_j^2 |u|^2 \leq C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j + 1} \int |f u| + C \varepsilon^{2 + 2 \beta_j} \lambda^{-2 \beta_j \delta_j - 3 \delta_j} \int \phi |u'|^2.
$$

Where the second integral in the first inequality was absorbed back into the left hand side by choosing $\eta$ small enough. Combining this last inequality with \[(3.15)\], letting $\varepsilon_0 > 0$ be a constant to be specified and applying Young’s inequality for products

$$
\lambda \int_{|x| < \lambda^{-\delta_j} \varepsilon} V_j |u u'| \leq \lambda^{1-\delta_j/2} \left( C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j - 1} \int |f u| + C \varepsilon^{2 + 2 \beta_j} \lambda^{-2 \beta_j \delta_j - 3 \delta_j} \int \phi |u'|^2 \right)^{1/2} \left( \int \phi |u'|^2 \right)^{1/2} \\
\leq \frac{\lambda^{2-\delta_j}}{2 \varepsilon_0} \left( C \varepsilon^{\beta_j} \lambda^{-\beta_j \delta_j - 1} \int |f u| + C \varepsilon^{2 + 2 \beta_j} \lambda^{-2 \beta_j \delta_j - 3 \delta_j} \int \phi |u'|^2 \right) + \frac{\varepsilon_0}{2} \int \phi |u'|^2 \\
= C \varepsilon^{\beta_j} \lambda^{1-(\beta_j + 1) \delta_j} \int |f u| + C \frac{\varepsilon^{2 + 2 \beta_j}}{\varepsilon_0} \lambda^{-2 \beta_j \delta_j + 2} \int \phi |u'|^2 + \frac{\varepsilon_0}{2} \int \phi |u'|^2. \quad (3.19)
$$

Now combining \[(3.13)\] and \[(3.19)\] then applying Young’s inequality for products to absorb the $V_j |u u'|$ term and a $\lambda^{1/2}$ from the right hand side back into the left hand side.

$$
\lambda \int V_j |u u'| \leq C \lambda^{1/2} \left( \int |f u| \right)^{1/2} \left( \int \frac{\lambda^{\delta_j}}{\varepsilon} \int V_j |u u'| + \frac{\mu^2}{\lambda} |f u| + V_j \lambda |f u| \right)^{1/2} \\
\lambda \int V_j |u u'| \leq C \lambda^{1/2} \left( \int |f u| \right)^{1/2} \left( \int V_j \lambda |f u| \right) + C \frac{\varepsilon^{2 + 2 \beta_j}}{\varepsilon_0} + \frac{\varepsilon_0}{2} \int \phi |u'|^2.
$$

Where the $\lambda$ dependence on the second to last $\phi |u'|^2$ term is eliminated by setting $\delta_j = \frac{1}{\beta_j + 2}$, as then $-(2 \beta_j + 4) \delta_j + 2 = 0$. This also ensures that $1 - (\beta_j + 1) \delta_j = \delta_j$.

Choosing $\varepsilon_0$ small enough and then $\varepsilon$ small enough gives the desired inequality. \(\square\)
Remark 3. Note that when $M = S^1$, the estimate of $\int W_j |uu'|$ can be modified to have a third case without changing the result. That is for some small $c > 0$, consider $c > |x| > \varepsilon \lambda^{-\delta_j}$, $|x| < \varepsilon \lambda^{-\delta_j}$ and $|x| > c$. The first two cases are proved as normal and the case $c > |x|$ can be controlled by Lemma 3.5. This is what makes it possible to adress $W_j = 1_{[\pi, -\pi]}(x)V_j$ or $1_{[\pi, \pi]}(x)V_j$ for $V_j \in X_{\theta_j}^\beta$ when $M = S^1$.

To obtain the desired resolvent estimate it remains to control the $V_j \chi_j |fu|$ term.

Lemma 3.7. If $\lambda > 0, \mu^2 \in \mathbb{R}$ and $u, f$ solve (3.3) then

$$\lambda^{1/2} \left( \int |fu| \right)^{1/2} \left( \int V_j \chi_j |fu| \right)^{1/2} \leq C \int |f|^2.$$  

Proof. By linearity there are two cases

1. supp $f \subset (\text{supp } V_j)^c$
2. supp $f \subset \text{supp } V_j$

In case 1 the term $\int V_j \chi_j |fu|$ vanishes.

In case 2 note that $V_j \geq C$ on supp $V_j$ so

$$\int |fu| \leq C \int V_j^{1/2} |fu|.$$  

Then using Cauchy-Schwarz and 3.8

$$\int |fu| \leq C \left( \int |f|^2 \right)^{1/2} \left( \int V_j |u|^2 \right)^{1/2} \leq C \lambda^{1/2} \left( \int |f|^2 \right)^{1/2} \left( \int |fu| \right)^{1/2}.$$  

Therefore $\int |fu| \leq C \int |f|^2$. From this and Cauchy-Schwarz

$$\lambda^{1/2} \left( \int |fu| \right)^{1/2} \left( \int \chi_j V_j |fu| \right)^{1/2} \leq \left( \int |f|^2 \right)^{1/2} \left( \int |fu| \right)^{1/2} \left( \int \chi_j V_j^2 |u|^2 \right)^{1/4}.$$  

It remains to control the final term on the right hand side. Because $\chi_j$ is supported on $x > \varepsilon \lambda^{-\delta_j}$ and $V_j = |x|^{\beta_j}$.

$$\int \chi_j^2 |u|^2 \leq C \lambda^{-\delta_j} \int V_j |u|^2 \leq C \lambda^{-\delta_j} \int |fu| \leq C \int |f|^2 \leq C \int |f|^2.$$  

Where the final inequality holds because $\beta > -1 > -4/3$ and $\delta \beta = \frac{2}{\beta + 2} \leq 2$ when $\beta \leq 2\beta + 4$ so $-\delta \beta - 2 \leq 0$. Therefore in case 2

$$\lambda^{1/2} \left( \int |fu| \right)^{1/2} \left( \int V_j \chi_j |fu| \right)^{1/2} \leq C \int |f|^2.$$  

Combining Lemmas 3.6 and 3.7 and, when $M = S^1$, Lemma 3.5

$$\lambda \int W_j |uu'| \leq C (\lambda^{\delta_j} + \mu) \int |fu| + C \int |f|^2 + \tilde{C} \int \phi |u'|^2.$$  

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This along with Lemma 3.4

\[ \int \phi |u'|^2 + \mu^2 \phi |u|^2 \leq C \sum_j (\lambda_j \delta_j + \mu \|u\|) \int |fu| + C \int |f|^2 + \bar{\varepsilon} \int \phi |u'|^2. \]

Then since \( \bar{\varepsilon} \) can be taken small enough to allow the \( \phi |u'|^2 \) on the right hand side to be absorbed into the left hand side

\[ \int \phi |u'|^2 + \mu^2 \phi |u|^2 \leq C \left( \frac{\lambda_j \delta_j}{\mu^2} + \frac{1}{|\mu|} \right) \int |f|^2. \]

And so

\[ \int \frac{1}{\mu^2} |u'|^2 + \int |u|^2 \leq C \left( \frac{\lambda_j}{\mu^2} + 1 \right) \int |f|^2, \]

which is exactly (3.7).

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