Sparse graphs with bounded induced cycle packing number have logarithmic treewidth

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Abstract

A graph is $O_k$-free if it does not contain $k$ pairwise vertex-disjoint and non-adjacent cycles. We prove that "sparse" (here, not containing large complete bipartite graphs as subgraphs) $O_k$-free graphs have treewidth (even, feedback vertex set number) at most logarithmic in the number of vertices. This is optimal, as there is an infinite family of $O_2$-free graphs without $K_{2,3}$ as a subgraph and whose treewidth is (at least) logarithmic.

Using our result, we show that Maximum Independent Set and 3-Coloring in $O_k$-free graphs can be solved in quasi-polynomial time. Other consequences include that most of the central NP-complete problems (such as Maximum Independent Set, Minimum Vertex Cover, Minimum Dominating Set, Minimum Coloring) can be solved in polynomial time in sparse $O_k$-free graphs, and that deciding the $O_k$-freeness of sparse graphs is polynomial time solvable.

1 Introduction

Two vertex-disjoint subgraphs $H$ and $H'$ in a graph $G$ are independent if there is no edge between $H$ and $H'$ in $G$. Independent cycles are simply vertex-disjoint cycles that are pairwise independent. Let $O_k$ denote the family of all graphs consisting of the disjoint union of $k$ cycles.

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We say that a graph is $O_k$-free if it does not contain any graph of $O_k$ as an induced subgraph. Equivalently, a graph $G$ is $O_k$-free if it does not contain $k$ independent induced cycles, or (equivalently), if $G$ does not contain $k$ independent cycles. These graphs can equivalently be defined in terms of forbidden induced subdivisions. Letting $T_k$ be the disjoint union of $k$ triangles, a graph is $O_k$-free if and only if it does not contain an induced subdivision of $T_k$.

A feedback vertex set is a set of vertices whose removal yields a forest. Our main technical contribution is the following.

**Theorem 1.1.** Every $O_k$-free graph on $n$ vertices that does not contain $K_{t,t}$ as a subgraph has a feedback vertex set of size $O_{t,k}(\log n)$.

Since a graph with a feedback vertex set of size $k$ has treewidth at most $k + 1$, this implies a corresponding result on treewidth.

**Corollary 1.2.** Every $O_k$-free graph on $n$ vertices that does not contain $K_{t,t}$ as a subgraph has treewidth $O_{t,k}(\log n)$.

Corollary 1.2 implies that a number of fundamental problems, such as Maximum Independent Set, Minimum Vertex Cover, Minimum Dominating Set, Minimum Coloring, can be solved in polynomial time in "sparse" $O_k$-free graphs. Before we elaborate on the algorithmic consequences of our results, we mention that our work is related to an ongoing project devoted to unraveling an induced version of the grid minor theorem of Robertson and Seymour [RS86]. This theorem implies that every graph not containing a subdivision of a $k \times k$ wall as a subgraph has treewidth at most $f(k)$, for some function $f$. This result had a deep impact in algorithmic graph theory since many natural problems are tractable in graphs of bounded treewidth.

Now, what are the forbidden induced subgraphs in graphs of small treewidth? It is clear that large cliques, complete bipartite graphs, subdivided walls, and line graphs of subdivided walls shall be excluded. It was actually suggested that in graphs with no $K_{t,t}$ subgraphs, the absence of induced subdivisions of large walls and their line graphs might imply bounded treewidth, but counterexamples were found [ST21, Dav22, Tro22]. However, Korhonen recently showed that this absence suffices within bounded-degree graphs [Kor23]. Abrishami et al. [AAC+22] proved that a vertex with at least two neighbors on a hole (i.e., an induced cycle of length at least four) is also necessary in a counterexample. Echoing our main result, it was proven that (triangle,theta)-free graphs have logarithmic treewidth [ACHS22], where a theta is made of three paths each on at least two edges between the same pair of vertices. The interested reader is referred to [ACHS23a, ACHS23b] for more recent development on the ongoing project.

As we shall see, the class of $O_2$-free graphs that do not contain $K_{3,3}$ as a subgraph has unbounded treewidth. Since these graphs do not contain as an induced subgraph a subdivision of a large wall or its line graph, they constitute yet another family of counterexamples.

We leave as an open question whether $O_k$-free graphs that do not contain $K_{t,t}$ as a subgraph have bounded twin-width, that is, if there is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that their twin-width is at most $f(t, k)$, and refer the reader to [BKTW22] for a definition of twin-width.

**Algorithmic motivations and consequences**

A natural approach to tackle NP-hard graph problems is to consider them on restricted classes. A simple example is the case of forests, that is, graphs without cycles, on which most hard
problems become tractable. The celebrated Courcelle’s theorem [Cou90] generalizes that phenomenon to graphs of bounded treewidth and problems expressible in monadic second-order logic.

For the particular yet central Maximum Independent Set (MIS, for short), the mere absence of odd cycles makes the problem solvable in polynomial time. Denoting by $\text{ocp}(G)$ (for odd cycle packing) the maximum cardinality of a collection of vertex-disjoint odd cycles in $G$, the classical result that MIS is polytime solvable in bipartite graphs corresponds to the $\text{ocp}(G) = 0$ case. Artmann et al. [AWZ17] extended the tractability of MIS to graphs $G$ satisfying $\text{ocp}(G) \leq 1$. One could think that such graphs are close to being bipartite, in the sense that the removal of a few vertices destroys all odd cycles. This is not necessarily true: Adding to an $n \times n$ grid the edges between $(1, i)$ and $(n, n + 1 - i)$, for every $i = 1, \ldots, n$, yields a graph $G$ with $\text{ocp}(G) = 1$ such that no removal of less than $n$ vertices make $G$ bipartite; also see the Escher wall in [Ree99].

It was believed that Artmann et al.’s result could even be lifted to graphs with bounded odd cycle packing number. Conforti et al. [CFH+20] proved it on graphs further assumed to have bounded genus, and Fiorini et al. [FJWY21] confirmed the general conjecture for graphs with bounded odd cycle packing number. A polynomial time approximation scheme (PTAS), due to Bock et al. [BFMR14], was known for MIS in the (much) more general case of $n$-vertex graphs $G$ such that $\text{ocp}(G) = o(n/\log n)$.

Similarly let us denote by $\text{cp}(G)$, $\text{icp}(G)$, $\text{iocp}(G)$ the maximum cardinality of a collection of vertex-disjoint cycles in $G$ that are unconstrained, independent, and independent and of odd length, respectively (for cycle packing, induced cycle packing, and induced odd cycle packing). The Erdős-Pósa theorem [EP65] states that graphs $G$ with $\text{cp}(G) = k$ admit a feedback vertex set (i.e., a subset of vertices whose removal yields a forest) of size $O(k \log k)$, hence have treewidth $O(k \log k)$. Thus, graphs with bounded cycle packing number allow polynomial time algorithms for a wide range of problems.

However, graphs with bounded feedback vertex set are very restricted. This is a motivation to consider the larger classes for which solely the induced variants $\text{icp}$ and $\text{iocp}$ are bounded. Graphs with $\text{iocp} \leq 1$ have their significance since they contain all the complements of disk graphs [BGK+18] and all the complements of unit ball graphs [BBB+18]. Concretely, the existence of a polynomial time algorithm for MIS on graphs with $\text{iocp} \leq 1$—an intriguing open question—would solve the long-standing open problems of whether Maximum Clique is in P for disk graphs and unit ball graphs. Currently only efficient PTASes are known [BBB+21], even when only assuming that $\text{iocp} \leq 1$ and that the solution size is a positive fraction of the total number of vertices [DP20]. Let us mention that recognizing the class of graphs $G$ satisfying $\text{iocp}(G) \leq 1$ is NP-complete [GKPT12].

We have seen that graphs with bounded $\text{cp}$, $\text{ocp}$, $\text{iocp}$ have been studied in close connection with solving MIS (or a broader class of problems), respectively forming the Erdős-Pósa theory, establishing a far-reaching generalization of total unimodularity, and improving the approximation algorithms for Maximum Clique on some geometric intersection graph classes. Relatively less attention has been given to $\text{icp}$. As a graph $G$ satisfies $\text{icp}(G) < k$ if and only if it is $O_k$-free, our results (and their algorithmic consequences) are precisely about graphs with bounded induced cycle packing, with a particular focus on the sparse case.

So, what can be said about the complexity of classical optimization problems for $O_k$-free graphs? Even the class of $O_2$-free graphs is rather complex. Observe indeed that complements of graphs without $K_{3,3}$ subgraph are $O_2$-free. As MIS remains NP-hard in graphs with girth at least 5 (hence without $K_{3,3}$ subgraph) [Ale82], Maximum Clique is NP-hard in $O_2$-free graphs. Nonetheless MIS could be tractable in $O_k$-free graphs, as is the case in graphs of bounded $\text{ocp}$:
Conjecture 1.3. **Maximum Independent Set** is solvable in polynomial time in $O_k$-free graphs.

As far as we can tell, MIS could even be tractable in graphs with bounded iocp. This would be a surprising and formidable generalization of Conjecture 1.3 and of the same result for bounded ocp [FJWY21].

We note that Corollary 1.2 implies Conjecture 1.3 in the sparse case. We come short of proving Conjecture 1.3 in general, but not by much. We obtain a quasipolynomial time algorithm for MIS in general $O_k$-free graphs, excluding that this problem is NP-complete without any complexity-theoretic collapse (and making it quite likely that the conjecture indeed holds).

**Theorem 1.4.** There exists a function $f$ such that for every positive integer $k$, **Maximum Independent Set** can be solved in quasipolynomial time $n^{O(k^2 \log n+f(k))}$ in $n$-vertex $O_k$-free graphs.

This is in sharp contrast with what is deemed possible in general graphs. Indeed, any exact algorithm for MIS requires time $2^{\Omega(n)}$ unless the Exponential Time Hypothesis (asserting that solving $n$-variable 3-SAT requires time $2^{\Omega(n)}$) fails [IPZ01].

It should be noted that Conjecture 1.3 is a special case of an intriguing and very general question by Dallard, Milanić, and Štorgel [DMŠ21] of whether there are planar graphs $H$ for which MIS is NP-complete on graphs excluding $H$ as an induced minor. In the same paper, the authors show that MIS is in fact polytime solvable when $H$ is $W_4$ (the 4-vertex cycle with a fifth universal vertex), or $K_5^-$ (the 5-vertex clique minus an edge), or $K_{2,1}$. Gartland et al. [GLP+21] (at least partially) answered that question when $H$ is a path, or even a cycle, by presenting in that case a quasi-polynomial algorithm for MIS. As we will mention again later, Korhonen [Kor23] showed that bounded-degree graphs excluding a fixed planar graph $H$ as an induced minor have bounded treewidth, thereby fully settling the question of Dallard et al. when the degree is bounded. He also derived an algorithm running in time $2^{O(n/\log^{1/6} n)} = 2^{o(n)}$, in the general (non bounded-degree) case.

Theorem 1.4 now adds a quasi-polynomial time algorithm when $H$ is the disjoint union of triangles. This is an orthogonal generalization of the trivial case when $H$ is a triangle (hence the graphs are forests) to that of Gartland et al. We increase the number of triangles, while the latter authors increase the length of the cycle. Our proofs are very different, yet they share a common feature, that of measuring the progress of the usual branching on a vertex by the remaining amount of relevant (semi-)induced subgraphs.

A natural related problem is the complexity of deciding $O_k$-freeness. A simple consequence of Corollary 1.2 is that one can test whether a graph without $K_{t,t}$ subgraph is $O_k$-free in polynomial time. For $k = 2$, when no complete bipartite is excluded as a subgraph, Le [Le17] conjectured the following, which had been raised as an open question earlier by Raymond [Ray15].

**Conjecture 1.5** (Le [Le17]). There is a constant $c$ such that every $O_2$-free $n$-vertex graph has at most $n^c$ distinct induced paths.

Conjecture 1.5 was recently solved by Nguyen, Scott, and Seymour [NSS24] in the more general $O_k$-free case using our Theorem 1.1. This implies in particular that the number of induced cycles in $O_k$-free graphs is polynomial (since this number cannot be more than $n$ times the number of induced paths), and thus testing $O_k$-freeness can be done in polynomial time by enumerating all induced cycles and testing, for every $k$ cycles in this collection, whether they are pairwise independent.
Organization of the paper. In Section 2, we prove that Theorem 1.1 and Corollary 1.2 are tight already for \( k = 2 \) and \( t = 3 \). Section 3 solves MIS in \( O_k \)-free graphs in quasi-polynomial time, among other algorithmic applications of Corollary 1.2.

The proof of our main structural result, Theorem 1.1, spans from Section 4 to Section 8. After some preliminary results (Section 4), we show in Section 5 that it suffices to prove Theorem 1.1 when the graph \( G \) has a simple structure: a cycle \( C \), its neighborhood \( N \) (an independent set), and the remaining vertices \( R \) (inducing a forest). Instead of directly exhibiting a logarithmic-size feedback vertex set, we rather prove that every such graph contains a vertex of degree linear in the so-called “cycle rank” (or first Betti number) of the graph. For sparse \( O_k \)-free graphs, the cycle rank is at most linear in the number of vertices and decreases by a constant fraction when deleting a vertex of linear degree. We then derive the desired theorem by induction, using as a base case that if the cycle rank is small, we only need to remove a small number of vertices to obtain a tree. To obtain the existence of a linear-degree vertex in this simplified setting, we argue in Section 6 that we may focus on the case where the forest \( G[R] \) contains only paths or only large “well-behaving” subdivided stars. In Section 7, we discuss how the \( O_k \)-freeness restricts the adjacencies between these stars/paths and \( N \). Finally, in Section 8, we argue that the restrictions yield a simple enough picture, and derive our main result.

2 Sparse \( O_2 \)-free graphs with unbounded treewidth

In this section, we show the following.

Theorem 2.1. For every natural \( k \), there is an \( O_2 \)-free graph with \( 2^k + k - 1 \) vertices, which does not contain \( K_{3,3} \) as a subgraph and has treewidth \( k \).

In particular, for infinitely many values of \( n \), there is an \( O_2 \)-free \( n \)-vertex graph which does not contain \( K_{3,3} \) as a subgraph and has treewidth at least \( \log_2 n - 1 \).

Construction of \( G_k \). To build \( G_k \), we first define a word \( w_k \) of length \( 2^k - 1 \) on the alphabet \([k]\). We set \( w_1 = 1 \), and for every integer \( i > 1 \), \( w_i = i w_{i-1}[1] i w_{i-1}[2] i \ldots i w_{i-1}[2^{i-1} - 2] i w_{i-1}[2^{i-1} - 1] i \). It is worth noting that equivalently \( w_i = \text{incr}(w_{i-1}) 1 \text{incr}(w_{i-1}), \) where incr adds 1 to every letter of the word. Let \( \Pi_k \) be the \((2^k - 1)\)-path where the \( \ell \)-th vertex of the path (say, from left to right) is denoted by \( \Pi_k[\ell] \).

The graph \( G_k \) is obtained by adding to \( \Pi_k \) an independent set of \( k \) vertices \( v_1, v_2, \ldots, v_k \), and linking by an edge every pair \( v_i, \Pi_k[\ell] \) such that \( i \in [k] \) and \( w_k[\ell] = i \).

Observe that we can also define the graph \( G_k \) directly, rather than iteratively: it is the union of a path \( u_1, \ldots, u_{2^k-1} \) and an independent set \( \{v_0, \ldots, v_{k-1}\} \), with an edge between \( v_i \) and \( u_j \) if and only if \( i \) is the 2-order of \( j \) (the maximum \( k \) such that \( 2^k \) divides \( j \)).

See Fig. 1 for an illustration.

\( G_k \) is \( O_2 \)-free and has no \( K_{3,3} \) subgraph. The absence of \( K_{3,3} \) (even \( K_{2,3} \)) as a subgraph is easy to check. At least one vertex of the \( K_{3,3} \) has to be some \( v_i \), for \( i \in [k] \). It forces that its three neighbors \( x, y, z \) are in \( \Pi_k \). In turn, this implies that a common neighbor of \( x, y, z \) (other than \( v_i \)) is some \( v_i' \neq v_i \); a contradiction since distinct vertices of the independent set have disjoint neighborhoods.

We now show that \( G_k \) is \( O_2 \)-free. Assume towards a contradiction that \( G_k[C_1 \cup C_2] \) is isomorphic to the disjoint union of two cycles \( G_k[C_1] \) and \( G_k[C_2] \). As \( C_1 \) and \( C_2 \) each induce
Figure 1: The graph $G_k$ for $k = 5$: an $O_2$-free graph without $K_{3,3}$ subgraph, $k + 2^k - 1$ vertices, and treewidth $k$.

a cycle, they each have to intersect $\{v_1, \ldots, v_k\}$. Assume without loss of generality that $C_1$ contains $v_i$, and $C_2$ is disjoint from $\{v_i, v_{i+1}, \ldots, v_k\}$. Consider a subpath $S$ of $C_2$ with both endpoints in $\{v_1, \ldots, v_k\}$, thus in $\{v_1, \ldots, v_{i-1}\}$, and all the other vertices of $S$ form a set $S' \subseteq V(\Pi_k)$. It can be that the endpoints are in fact the same vertex $v_{i'}$, and in that case $S$ is the entire $C_2$.

Let $v_{i'}, v_{i''}$ be the two (possibly equal) endpoints. Observe that $S'$ is a subpath of $\Pi_k$ whose two endpoints have label $i', i'' < i$. In particular there is a vertex labeled $i$ somewhere along $S'$. This makes an edge between $v_i \in C_1$ and $C_2$, which is a contradiction.

$G_k$ has treewidth $k$. Since $\{v_2, \ldots v_k\}$ is a feedback vertex set, the treewidth of $G_k$ is at most $k$, so it is enough to prove that $G_k$ has treewidth at least $k$. We do this by proving that $G_k$ contains the complete graph $K_{k+1}$ as a minor (we thank an anonymous reviewer for suggesting the argument below, our initial argument only gave a $K_k$-minor in $G_k$). The minor $K_{k+1}$ is constructed as follows: for each $i \in [k]$, we denote by $V_i$ the subpath of $\Pi_k$ whose right endpoint is the leftmost vertex of $\Pi_k$ labeled $i$, and which is maximal with the property that it does not contain any vertex labeled $i + 1$. Note that each set $V_i$ contains vertices labeled $i, i + 2, i + 3, \ldots, k$, and is adjacent to a vertex labeled $i + 1$. For each $i \in [k]$, we let $V_i'$ be the union of $V_i$ and the vertex $v_i$ (this set induces a connected subgraph of $G_k$), and we define $V_{k+1}'$ as the set of vertices of $\Pi_k$ lying to the right of the unique vertex of $\Pi_k$ labeled 1. Note that the sets $V_i', i \in [k + 1]$, form a partition of $V(k + 1)$. By definition there is an edge between any two sets $V_i', V_j'$ in $G_k$ for $1 \leq i < j \leq k + 1$, and thus $G_k$ contains $K_{k+1}$ as a minor, as desired.

The twin-width of $G_k$, however, can be shown to be at most a constant independent of $k$.

3 Algorithmic applications

This section presents algorithms on $O_k$-free graphs based on our main result, specifically using the treewidth bound.

Corollary 1.2. Every $O_k$-free graph on $n$ vertices that does not contain $K_{t,t}$ as a subgraph has treewidth $O_{t,k}(\log n)$.
Single-exponential parameterized $O(1)$-approximation algorithms exist for treewidth. Already in 1995, Robertson and Seymour [RS95] present a $2^{O(tw)n^2}$-time algorithm yielding a tree-decomposition of width $4(tw + 1)$ for any input $n$-vertex graph of treewidth $tw$. Run on $n$-vertex graphs of logarithmic treewidth, this algorithm outputs tree-decompositions of width $O(\log n)$ in polynomial time. We thus obtain the following.

**Corollary 3.1.** Maximum Independent Set, Hamiltonian Cycle, Minimum Vertex Cover, Minimum Dominating Set, Minimum Feedback Vertex Set, and Minimum Coloring can be solved in polynomial time on $O_k$-free graphs with no $K_{t,t}$ subgraph, for some function $g$.

**Proof.** Let $h(t, k)$ be the implicit function in Corollary 1.2 such that every $O_k$-free $n$-vertex graph with no $K_{t,t}$ subgraph has treewidth at most $h(t, k) \log n$.

Algorithms running in time $2^{O(tw)n^{O(1)}} = 2^{h(t, k) \log n n^{O(1)}} = n^{h(t, k) + O(1)} = n^{o(t, k)}$ exist for all these problems but for Minimum Coloring. They are based on dynamic programming over a tree-decomposition, which by Corollary 1.2 has logarithmic width and by [RS95] can be computed in polynomial time. For Maximum Independent Set, Minimum Vertex Cover, Minimum Dominating Set, and $q$-Coloring (for a fixed integer $q$) see for instance the textbook [CFK*15, Chapter 7.3]. For Hamiltonian Cycle and Minimum Feedback Vertex Set, deterministic parameterized single-exponential algorithms require the so-called rank-based approach; see [CFK*15, Chapter 11.2].

By Corollary 4.6, $O_k$-free graphs with no $K_{t,t}$ subgraph have bounded chromatic number. Thus a polynomial time algorithm for Minimum Coloring is implied by the one for $q$-Coloring.

In a scaled-down refinement of Courcelle’s theorem [Cou90], Pilipczuk showed that any problem expressible in Existential Counting Modal Logic (ECML) admits a single-exponential fixed-parameter algorithm in treewidth [Pil11]. In particular:

**Theorem 3.2** ([Pil11]). ECML model checking can be solved in polynomial time on any class with logarithmic treewidth.

In a nutshell, this logic allows existential quantifications over vertex and edge sets followed by a counting modal formula that should be satisfied from every vertex $v$. Counting modal formulas enrich quantifier-free Boolean formulas with $\Diamond S \phi$, whose semantics is that the current vertex $v$ has a number of neighbors satisfying $\phi$ in the ultimately periodic set $S$ of non-negative integers. Another consequence of Corollary 1.2 (and Theorem 3.2) is that testing if a graph is $O_k$-free can be done in polynomial time among sparse graphs, further indicating that the general case could be tractable.

**Corollary 3.3.** For any fixed $k$ and $t$, deciding whether a graph with no $K_{t,t}$ subgraph is $O_k$-free can be done in polynomial time.

**Proof.** One can observe that $O_k$-freeness is definable in ECML. Indeed, one can write

$$
\varphi = \exists X_1 \exists X_2 \ldots \exists X_k \left( \bigwedge_{1 \leq i \leq k} X_i \rightarrow \Diamond^{(2)} X_i \right) \land \left( \bigwedge_{1 \leq i < j \leq k} \neg (X_i \land X_j) \land (X_i \rightarrow \Diamond^{(0)} X_j) \right).
$$

Formula $\varphi$ asserts that there are $k$ sets of vertices $X_1, X_2, \ldots, X_k$ such that every vertex has exactly two neighbors in $X_i$ if it is itself in $X_i$, the sets are pairwise disjoint, and every vertex has no neighbor in $X_j$ if it is in some distinct $X_i$ (with $i < j$). Thus $G$ is $O_k$-free if and only if $\varphi$ does not hold in $G$. 

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We now show the main algorithmic consequence of our structural result. This holds for any (possibly dense) $O_k$-free graph, and uses the sparse case (Corollary 3.1) at the basis of an induction on the size of a largest collection of independent 4-vertex cycles. It should be noted that this result (as well as the previous result on MIS above) also works for the weighted version of the problem, with minor modifications.

**Theorem 1.4.** There exists a function $f$ such that for every positive integer $k$, Maximum Independent Set can be solved in quasipolynomial time $n^{O(k^2 \log n + f(k))}$ in $n$-vertex $O_k$-free graphs.

**Proof.** Let $G$ be our $n$-vertex $O_k$-free input. Let $q$ be the maximum integer such that $G$ admits $q$ independent 4-vertex cycles (the cycles themselves need not be induced). Clearly $q < k$.

We show the theorem by induction on $q$, namely that MIS can be Turing-reduced in time $n^{c(q+1)^2 \log n}$ for some constant $c$ (specified later) to smaller instances with no $K_{2,2}$ subgraphs (hence such that $q = 0$). We first examine what happens with the latter instances. Let $f(k) = h(2, k)$ with $h(t, k)$ the hidden dependence of Corollary 1.2. If $q = 0$, $G$ does not contains $K_{2,2}$ as a subgraph, so we can solve MIS in polynomial time $n^{f(k) + O(1)}$ by Corollary 3.1.

We now assume that $q \geq 1$, $n \geq 4$, and that the case $q - 1$ of the induction has been established (or $q = 1$). Let $C$ be a 4-vertex cycle part of a $4q$-vertex subset consisting of $q$ independent 4-vertex cycles. Let $S$ be the set of all $4q$-vertex subsets consisting of $q$ independent 4-vertex cycles in the current graph (at this point, $G$), and $s = |S|$. Thus $1 \leq s \leq n^{4q}$. By assumption, the closed neighborhood of $C$, $N[C]$, intersects every subset in $S$. In particular, there is one of the four vertices of $C$, say, $v$, such that $N[v]$ intersects at least $s/4$ subsets of $S$.

We branch on two options: either we put $v$ in (an initially empty set) $I$, and remove its closed neighborhood from $G$, or we remove $v$ from $G$ (without adding it to $I$). With the former choice, the size of $S$ drops by at least $s/4$, whereas with the latter, it drops by at least 1.

Even if fully expanded while $s > 0$, this binary branching tree has at most

$$
\sum_{0 \leq i \leq 4q \log_{4/3} n} \binom{n}{i} = n^{O(q \log n)} \text{ leaves,}
$$

since including a vertex in $I$ can be done at most $4q \log_{4/3} n$ times within the same branch; thus, leaves can be uniquely described as binary words of length $n$ with at most $4q \log_{4/3} n$ occurrences of $0$, say, 1.

We retrospectively set $c \geq 1$ such that the number of leaves is at most $n^{cq \log n}$, running the algorithm thus far (when $q \geq 1$) takes at most time $n^{c+cq \log n}$. At each leaf of the branching, $s = 0$ holds, which means that the current graph does not admit $q$ independent 4-vertex cycles. By the induction hypothesis, we can Turing-reduce each such instance in time $n^{cq^2 \log n}$. Thus the overall running time is

$$
n^{c+cq \log n} + n^{cq \log n} \cdot n^{cq^2 \log n} = n^{c+cq \log n} \cdot (n^{cq^2 \log n} + 1) \leq n^{c(q+1)^2 \log n - c \log n - c \log n + c + \frac{1}{\log n}}.
$$

Note that $n^{cq^2 \log n} \geq 1$ thus we could upper-bound $n^{cq^2 \log n} + 1$ by $2n^{cq^2 \log n} = n^{cq^2 \log n + \frac{1}{\log n}}$. Since $c, q \geq 1$ and $\log n \geq 1$, it holds that $-cq \log n - c \log n + c + \frac{1}{\log n} \leq -2c + c + 1 < 0$. Hence we get the claimed running time of $n^{c(q+1)^2 \log n + f(k)}$ for the reduction to $q = 0$, and the overall running time of $n^{c(q+1)^2 \log n + f(k) + O(1)} = n^{O(k^2 \log n + f(k))}$. \qed

One may wonder if some other problems beside MIS become (much) easier on $O_k$-free graphs than in general. As $2K_2$-free graphs are $O_2$-free, one cannot expect a quasi-polynomial...
time algorithm for Minimum Dominating Set [Ber84, CP84], Hamiltonian Cycle [Gol04], Maximum Clique [Pol74], and Minimum Coloring [KKTW01] since these problems remain NP-complete on $2K_2$-free graphs. Nevertheless we give a quasi-polynomial time algorithm for 3-Coloring.

**Theorem 3.4.** There exists a function $f$ such that for every positive integer $k$, 3-Coloring can be solved in quasi-polynomial time $n^{O(k^2 \log n + f(k))}$ in $n$-vertex $O_k$-free graphs.

**Proof.** We solve the more general List 3-Coloring problem, where, in addition, every vertex $v$ is given a list $L(v) \subseteq \{1, 2, 3\}$ from which one has to choose its color. Note that when $L(v) = \emptyset$ for some vertex $v$, one can report that the instance is negative, and when $|L(v)| = 1$, $v$ has to be colored with the unique color in its list, and this color has to be deleted from the lists of its neighbors (once this is done, $v$ might as well be removed from the graph). These reduction rules are performed as long as they apply, so we always assume that the current instance has only lists of size 2 and 3.

We follow the previous proof, and simply adapt the branching rule, and the value of $s$.

Now $s$ is defined as the sum taken over all vertex sets $X$ consisting of $q$ independent 4-vertex cycles (the cycles themselves need not be induced), of the sum of the list sizes of the vertices of $X$. Hence $8 \leq s \leq 12 \cdot n^4$. There is a vertex $v \in C$ and a color $c \in L(v)$ such that $c$ appears in at least $\frac{1}{2} \cdot \frac{1}{12} \cdot \frac{s}{n^2} = \frac{s}{24n}$ of the lists of its neighbors. This is because all the lists have size at least 2, and are subsets of $\{1, 2, 3\}$, thus pairwise intersect. (Note that this simple yet crucial fact already breaks down for 4-Coloring.)

We branch on two options: either we color $v$ with $c$, hence we remove color $c$ from the lists of its neighbors or we commit to not color $v$ by $c$, and simply remove $c$ from the list of $v$. With the former choice, the size of $S$ drops by at least $s/96$, whereas with the latter, it drops by at least 1. The rest of the proof is similar with a possibly larger constant $c$.

---

### 4 Preliminary results

An important property of graphs which do not contain the complete bipartite graph $K_{t,t}$ as a subgraph is that they are not dense (in the sense that they have a subquadratic number of edges).

**Theorem 4.1** (Kővári, Sós, and Turán [KST54]). For every integer $t \geq 2$ there is a constant $c_t$ such that any $n$-vertex graph with no $K_{t,t}$ subgraph has at most $c_t n^{2-1/t}$ edges.

The following lemma shows that for $O_k$-free graphs, excluding $K_{t,t}$ as a subgraph is equivalent to a much stronger ‘large girth’ condition, up to the removal of a bounded number of vertices.

**Lemma 4.2.** There is a function $f$ such that for any integer $\ell$ and any $O_k$-free graph $G$ with no $K_{\ell,t}$ subgraph, the maximum number of vertex-disjoint cycles of length at most $\ell$ in $G$ is at most $f(\ell, t, k)$.

**Proof.** If $\ell \leq 2$, we define $f(\ell, t, k) = 0$ for any integers $t$ and $k$, and we observe that since $G$ does not contain any cycle of length at most $\ell$, the statement of the lemma holds trivially.

Assume now that $\ell \geq 3$, and define $f(\ell, t, k) := (2c_t k \ell^2)^t$, where $c_t$ is the constant of Theorem 4.1.

Assume for the sake of contradiction that $G$ contains $N := f(\ell, t, k)$ vertex-disjoint cycles of length at most $\ell$, which we denote by $C_1, \ldots, C_N$. Let $H$ be the graph with vertex set
with an edge between \( v_i \) and \( v_j \) in \( H \) if and only if there is an edge between \( C_i \) and \( C_j \) in \( G \). Since \( G \) is \( O_k \)-free, \( H \) has no independent set of size \( k \). By Turán’s theorem [Tur41], \( H \) contains at least \( \frac{N^2}{2k-2} - \frac{N}{2} \geq \frac{N^2}{2k} - \frac{N}{2} \) edges.

Consider the subgraph \( G' \) of \( G \) induced by the vertex set \( \bigcup_{i=1}^{N} C_i \). The graph \( G' \) has \( n \leq \ell N \) vertices, and \( m \geq 3N + \frac{N^2}{2k} - \frac{N}{2} > \frac{N^2}{2k} \) edges. Note that by the definition of \( N \), we have

\[
m > \frac{N^2}{2k} = \frac{1}{2k} \cdot N^{2-1/\ell} \cdot N^{1/\ell} \geq \frac{1}{2k \ell^{2-1/\ell}} \cdot n^{2-1/\ell} \cdot 2c_k \ell^2 \geq c_t \cdot n^{2-1/\ell},
\]

which contradicts Theorem 4.1, since \( G' \) (as an induced subgraph of \( G \)) does not contain \( K_{1,t} \) as a subgraph.

The **girth** of a graph \( G \) is the minimum length of a cycle in \( G \) (if \( G \) is acyclic, its girth is set to be infinite). We obtain the following immediate corollary of Lemma 4.2.

**Corollary 4.3.** There is a function \( g \) such that for any integer \( \ell \geq 3 \), any \( O_k \)-free graph \( G \) with no \( K_{1,t} \) subgraph contains a set \( X \) of at most \( g(\ell, t, k) \) vertices such that \( G - X \) has girth at least \( \ell \).

**Proof.** Let \( f \) be the function of Lemma 4.2, and let \( g(\ell, t, k) := (\ell - 1) \cdot f(\ell - 1, t, k) \). Consider a maximum collection of disjoint cycles of length at most \( \ell - 1 \) in \( G \). Let \( X \) be the union of the vertex sets of all these cycles. By Lemma 4.2, \( |X| \leq (\ell - 1)f(\ell - 1, t, k) = g(\ell, t, k) \), and by definition of \( X \), the graph \( G - X \) does not contain any cycle of length at most \( \ell - 1 \), as desired.

We now state a simple consequence of Corollary 4.3, which will be particularly useful at the end of the proof of our main result. A **banana** in a graph \( G \) is a pair of vertices joined by at least 2 disjoint paths whose internal vertices all have degree 2 in \( G \).

**Corollary 4.4.** There is a function \( f' \) such that any \( O_k \)-free graph \( G \) with no \( K_{1,t} \) subgraph contains a set \( X \) of at most \( f'(t, k) = O_t(k^t) \) vertices such that all bananas of \( G \) intersect \( X \).

**Proof.** Let \( G' \) be the graph obtained from \( G \) by replacing each maximal path whose internal vertices have degree 2 in \( G \) by a path on two edges (with a single internal vertex, of degree 2). Note that each banana in \( G \) is replaced by a copy of some graph \( K_{2,s} \) in \( G' \), with \( s \geq 2 \). In particular, every set \( X' \in V(G') \) intersecting all 4-cycles in \( G' \) intersects all copies of graphs \( K_{2,s} \) with \( s \geq 2 \). Moreover, any such set \( X' \) in \( G' \) can be lifted to a set \( X \in V(G) \) of the same size that intersects all bananas of \( G \). The result then follows from the application of Corollary 4.3 to \( G' \) with \( \ell = 5 \).

In all the applications of Corollary 4.4, \( t \) will be a small constant (2 or 3).

The **average degree** of a graph \( G = (V, E) \), denoted by \( \text{ad}(G) \), is defined as \( 2|E|/|V| \). Let us now prove that \( O_k \)-free graphs with no \( K_{1,t} \) subgraph have bounded average degree. This can also be deduced from the main result of [KO04], but we include a short proof for the sake of completeness. Moreover, the decomposition used in the proof will be used again in the proof of our main result.

**Lemma 4.5.** Every \( O_k \)-free graph \( G \) of girth at least 11 has average degree at most \( 2k \).

**Proof.** We proceed by induction on \( k \). When \( k = 1 \), \( G \) is a forest, with average degree less than 2. Otherwise, let \( C \) be a cycle of minimal length in \( G \). Let \( N \) be the neighborhood of \( C \), let \( S \) the second neighborhood of \( C \), and let \( R = V(G) \setminus (C \cup N) \). Thus \( V(G) \) is partitioned
into \( C, N, R \), and we have \( S \subseteq R \). Observe that there are no edges between \( C \) and \( R \) in \( G \), so it follows that \( G[R] \) is \( \mathcal{O}_{k-1} \)-free, and thus \( \text{ad}(G[R]) \leq 2k - 2 \) by induction. Observe also that since \( G \) has girth at least 11 and \( C \) is a minimum cycle, the two sets \( N \) and \( S \) are both independent sets. Moreover each vertex of \( N \) has a unique neighbor in \( C \), and each vertex in \( S \) has a unique neighbor in \( N \). Indeed, in any other case we obtain a path of length at most 5 between two vertices of \( C \), contradicting the minimality of \( C \). It follows that \( C \) is the only cycle in \( G[C \cup N \cup S] \), hence this graph has average degree at most 2. As a consequence, \( G \) has a partition of its edges into two subgraphs of average degree at most \( 2k - 2 \) and at most \( 2 \), respectively, and thus \( \text{ad}(G) \leq 2k - 2 + 2 = 2k \), as desired.

It can easily be deduced from this result that every \( \mathcal{O}_k \)-free graph with no \( K_{t,t} \) subgraph has average degree at most \( h(t, k) \), for some function \( h \) (and thus chromatic number at most \( h(t, k) + 1 \)).

**Corollary 4.6.** There is a function \( h \) such that every \( \mathcal{O}_k \)-free graph with no \( K_{t,t} \) subgraph has average degree at most \( h(t, k) \), and chromatic number at most \( h(t, k) + 1 \).

**Proof.** Let \( G \) be an \( \mathcal{O}_k \)-free graph that does not contain \( K_{t,t} \) as a subgraph. By Corollary 4.3, \( G \) has a set \( X \) of at most \( g(11, t, k) \) vertices such that \( G - X \) has girth at least 11. Note that \( \text{ad}(G) \leq \text{ad}(G - X) + |X| \leq \text{ad}(G - X) + g(11, t, k) \leq 2k + g(11, t, k) \), where the last inequality follows from Lemma 4.5.

Let \( h(t, k) = 2k + g(11, t, k) \). As the class of \( \mathcal{O}_k \)-free graphs with no \( K_{t,t} \) subgraph is closed under taking induced subgraphs, it follows that any graph in this class is \( h(t, k) \)-degenerate, and thus \( \chi(h(t, k) + 1) \)-colorable.

We would like to note that using a result of [Dvo18], extending earlier results of [KO04], it can be proved that the class of \( \mathcal{O}_k \)-free graphs with no \( K_{t,t} \) subgraph actually has bounded expansion, which is significantly stronger than having bounded average degree. This will not be needed in our proofs, and it can also be deduced from our main result, as it implies that sparse \( \mathcal{O}_k \)-free graphs have logarithmic separators, and thus polynomial expansion.

A feedback vertex set (FVS) \( X \) in a graph \( G \) is a set of vertices of \( G \) such that \( G - X \) is acyclic. The minimum size of a feedback vertex set in \( G \) is denoted by \( \text{fvs}(G) \). The classical Erdős-Pósa theorem [EP65] states that graphs with few vertex-disjoint cycles have small feedback vertex sets.

**Theorem 4.7** (Erdős and Pósa [EP65]). There is a constant \( c > 0 \) such that if a multigraph \( G \) contains less than \( k \) vertex-disjoint cycles, then \( \text{fvs}(G) \leq ck \log k \).

We use this result to deduce the following useful lemma.

**Lemma 4.8.** There is a constant \( c > 0 \) such that the following holds. Let \( G \) consist of a cycle \( C \), together with \( \ell \) paths \( P_1, \ldots, P_\ell \) on at least 2 edges

- whose endpoints are in \( C \), and
- whose internal vertices are disjoint from \( C \), and
- such that the internal vertices of each pair of different paths \( P_i, P_j \) are pairwise distinct and non-adjacent.

Suppose moreover that \( G \) is \( \mathcal{O}_k \)-free (with \( k \geq 2 \)) and has maximum degree at most \( d + 2 \). Then

\[
\ell \leq cdk \log k.
\]
Proof. Observe that each path $P_i$ intersects or is adjacent to at most $2(d - 1) + 4d < 6d$ other paths $P_j$: indeed, if $P_i$ has endpoints $x, y$ in $C$, then there are at most $2(d - 1)$ paths $P_j$ which intersect $P_i$ by sharing $x$ or $y$ as endpoint, and at most $4d$ paths $P_j$ which are adjacent to $P_i$ because some endpoint of $P_j$ is adjacent to either $x$ or $y$. It follows that there exist $s \geq \frac{L}{6d}$ of these paths, say $P_1, \ldots, P_s$ without loss of generality, that are pairwise non-intersecting and non adjacent.

Consider the subgraph $G'$ of $G$ induced by the union of $C$ and the vertex sets of the paths $P_1, \ldots, P_s$. Since the paths $P_i, 1 \leq i \leq s$, are pairwise independent, and since $G'$ does not contain $k$ independent cycles, the graph $G'$ does not contain $k$ vertex-disjoint cycles. Let $G''$ be the multigraph obtained from $G'$ by suppressing all vertices of degree 2 (i.e., replacing all maximal paths whose internal vertices have degree 2 by single edges). Observe that since $G'$ does not contain $k$ vertex-disjoint cycles, the graph $G''$ does not contains $k$ vertex-disjoint cycles either. Observe also that $G''$ is cubic and contains $2s$ vertices. It was proved by Jaeger [Jae74] that any cubic multigraph $H$ on $n$ vertices satisfies $fvs(H) \geq \frac{n^2}{4}$. As a consequence, it follows from Theorem 4.7 that $\frac{2s + 2}{d} \leq fvs(G'') \leq c'k \log k$ (for some constant $c'$), and thus $\ell \leq 12dc'k \log k = cdk \log k$ (for $c = 12c'$), as desired.

A strict subdivision of a graph is a subdivision where each edge is subdivided at least once.

Lemma 4.9. There is a constant $c > 0$ such that for any integer $k \geq 2$, any strict subdivision of a graph of average degree at least $ck \log k$ contains a graph of the family $O_k$ as an induced subgraph.

Proof. Note that if a graph $G$ contains $k$ vertex-disjoint cycles, then any strict subdivision of $G$ contains an induced $O_k$. Hence, it suffices to prove that any graph with less than $k$ vertex-disjoint cycles has average degree at most $ck \log k$, for some constant $c$. By Theorem 4.7, there is a constant $c'$ such that any graph $G$ with less than $k$ vertex-disjoint cycles contains a set $X$ of at most $c'k \log k$ vertices such that $G - X$ is acyclic. In this case $G - X$ has average degree at most 2, and thus $G$ has average degree at most $c'k \log k + 2 \leq ck \log k$ (for some constant $c$), as desired.

5 Logarithmic treewidth of sparse $O_k$-free graphs

Recall our main result.

Theorem 1.1. Every $O_k$-free graph on $n$ vertices that does not contain $K_{t,t}$ as a subgraph has a feedback vertex set of size $O_{t,k}(\log n)$.

The proof of Theorem 1.1 relies on the cycle rank, which is defined as $r(G) = |E(G)| - |V(G)| + |C(G)|$ where $C(G)$ denotes the set of connected components of $G$. The cycle rank is exactly the number of edges of $G$ which must be deleted to make $G$ a forest, hence it is a trivial upper bound on the size of a minimum feedback vertex set. Remark the following simple properties.

Lemma 5.1. The cycle rank is invariant under the following operations:

1. Deleting a vertex of degree 1.

2. Deleting a connected component which is a tree (and in particular, deleting a vertex of degree 0).
We call reduction the operation of iteratively deleting vertices of degree 0 or 1, which preserves cycle rank by the above lemma. A graph is reduced if it has minimum degree at least 2, and the core of a graph \( G \) is the reduced graph obtained by applying reductions to \( G \) as long as possible. The inclusion-wise minimal FVS of \( G \) and of its core are exactly the same.

In a graph \( G \), a vertex \( x \) is called \( \varepsilon \)-rich if \( d(x) \geq \varepsilon \cdot r(G) \). Our strategy to prove Theorem 1.1 is to iteratively reduce the graph, find an \( \varepsilon \)-rich vertex, add it to the FVS and delete it from the graph. The following lemma shows that the cycle rank decreases by a constant factor each iteration, implying that the process terminates in logarithmically many steps.

**Lemma 5.2.** In a reduced \( \mathcal{O}_k \)-free graph, deleting a vertex of degree \( d \) decreases the cycle rank by at least \( \frac{d - k + 1}{2} \).

**Proof.** In any graph \( G \), deleting a vertex \( x \) of degree \( d \) decreases the cycle rank by \( d - c \), where \( c \) is the number of connected components of \( G - x \) which contain a neighbor of \( x \). If \( G \) is \( \mathcal{O}_k \)-free, then all but at most \( k - 1 \) components of \( G - x \) are trees. Furthermore, if \( T \) is a connected component of \( G - x \) which is a tree, then \( T \) must be connected to \( x \) by at least two edges, as otherwise \( T \) must contain a vertex of degree 1 in \( G \), which should have been deleted during reduction. Thus we have

\[
2c - (k - 1) \leq d. \tag{1}
\]

Therefore the cycle rank decreases by at least \( d - \frac{d + k - 1}{2} = \frac{d - k + 1}{2} \) as desired. \( \square \)

The existence of rich vertices is given by the following result.

**Theorem 5.3.** For any \( k \), there is some \( \varepsilon_k > 0 \) such that any \( \mathcal{O}_k \)-free graph with girth at least 11 has an \( \varepsilon_k \)-rich vertex.

Let us first prove Theorem 1.1 using Theorem 5.3.

**Proof of Theorem 1.1.** Fix \( k \) and \( t \). Given a graph \( G \) which is \( \mathcal{O}_k \)-free and does not contain \( K_{t,t} \) as a subgraph, we apply Lemma 4.2 to obtain a set \( X \) of size at most \( f(11, t, k) \) such that \( G' \equiv G - X \) has girth at least 11. Thus, it suffices to prove the result for \( G' \), and finally add \( X \) to the resulting FVS of \( G' \). Since \( \log r(G') \leq \log \left( \frac{|V(G')|}{2} \right) \leq 2 \log |V(G')| \), we have reduced the problem to the following.

**Claim 5.4.** For any \( k \), there is a constant \( c_k \) such that if \( G \) is an \( \mathcal{O}_k \)-free graph with girth at least 11, then \( \text{fvs}(G) \leq c_k \cdot \log r(G) \).

Let us now assume that \( G \) is as in the claim, and consider its core \( H \), for which \( r(H) = r(G) \) and \( \text{fvs}(H) = \text{fvs}(G) \). Consider an \( \varepsilon_k \)-rich vertex \( x \) in \( H \) with \( \varepsilon_k \) as in Theorem 5.3. If \( r(G) \geq 2k \cdot \varepsilon_k^{-1} \), then \( d(x) \geq 2k \), hence by Lemma 5.2, deleting \( x \) decreases the cycle rank of \( G \) by at least

\[
\frac{d(x) - k + 1}{2} \geq \frac{d(x)}{4} \geq \frac{\varepsilon_k}{4} r(G). \tag{2}
\]

Thus, as long as the cycle rank is more than \( 2k \cdot \varepsilon_k^{-1} \), we can find a vertex whose deletion decreases the cycle rank by a constant multiplicative factor. After logarithmically many steps, we have \( \text{fvs}(G) \leq r(G) \leq 2k \cdot \varepsilon_k^{-1} \). In the end, the feedback vertex set consists of at most \( f(11, t, k) \) vertices in \( X \), logarithmically many rich vertices deleted in the induction, and at most \( 2k \cdot \varepsilon_k^{-1} \) vertices for the final graph. \( \square \)
Figure 2: Subgraph of an $O_4$-free graph $G$. $V(G)$ is partitioned into three sets $C$, $N$, $R$, where $C$ is a shortest cycle, $N$ is an independent set and first neighborhood of $C$, and $R$ is $O_3$-free. $S$ is the second neighborhood of $N$. Gray lines correspond to induced paths where all internal vertices have degree 2.

We now focus on proving Theorem 5.3. Let $G$ be an $O_k$-free graph with girth at least 11. Consider $C$ a shortest cycle of $G$, $N$ the neighborhood of $C$, and $R \equiv G - (C \cup N)$ the rest of the graph (see Figure 2). Remark that there is no edge between $C$ and $R$, hence $G[R]$ is an $O_{k-1}$-free graph. As a special case, if $k = 2$, then $G[R]$ is a forest. We will show that in general, it remains possible to reduce the problem to the case where $G[R]$ is a forest, which is our main technical theorem.

**Theorem 5.5.** For any $k$, there is some $\delta_k > 0$ such that if $G$ is a connected $O_k$-free graph with girth at least 11, and furthermore $G[R]$ is a forest where $R$ is as in the decomposition described above, then $G$ has a $\delta_k$-rich vertex.

Theorem 5.5 will be proved in Section 8. In the remainder of this section, we assume Theorem 5.5 and explain how Theorem 5.3 can be deduced from it.

**Proof of Theorem 5.3.** The proof is by induction on $k$. Let $\delta_k > 0$ be as in Theorem 5.5, and let $\varepsilon_{k-1} > 0$ be as in Theorem 5.3, obtained by induction hypothesis. We fix

$$
\varepsilon_k \equiv \min \left\{ \frac{\varepsilon_{k-1}}{20}, \frac{\delta_k}{20}, \frac{\delta_k}{5(k + 1)}, \frac{1}{30(k - 2)} \right\},
$$

(3)

Let $G$ be any $O_k$-free graph with girth at least 11. Reductions preserve all the hypotheses of the claim, and the value of $r(G)$, hence we can assume $G$ to be reduced. Consider the decomposition $C$, $N$, $R$ as previously described. We construct a subset $F \subset R$ inducing a rooted forest in $G$ such that the only edges from $F$ to $R \setminus F$ are incident to roots of $F$, and each root of $F$ is incident to at most one such edge.

**Claim 5.6.** If $F \subset R$ has the former property and $F' \subset R \setminus F$ induces a forest in $G$, then $F \cup F'$ induces a forest in $G$.

**Proof.** Each connected component of $G[F]$ has a single root, which is the only vertex which can be connected to $F'$.
We construct \( F \) inductively, starting with an empty forest, and applying the following rule as long as it applies: if \( x \in R \setminus F \) is adjacent to at most one vertex in \( R \setminus F \), we add \( x \) to \( F \), and make it the new root of its connected component in \( F \). The condition on \( F \) obviously holds for \( F = \emptyset \). When adding \( x \), by Claim 5.6, \( F \cup \{ x \} \) is still a forest. Furthermore, if \( y \in F \cup \{ x \} \) is adjacent to \( R \setminus (F \cup \{ x \}) \), then either \( y = x \) or \( y \) was a root before the addition of \( x \), and is not adjacent to \( x \), and therefore \( x \) and \( y \) are in distinct connected components of \( F \cup \{ x \} \). In either case, \( y \) is a root of \( F \cup \{ x \} \) as required.

We now denote by \( F \) the forest obtained when the previous rule no longer applies, and let \( R' = R \setminus F \). As observed by a reviewer, \( R' \) and \( F \) can be defined equivalently by saying that \( R' \) is the core of \( G[R] \) and \( F \) is equal to \( R \setminus R' \) (however the procedure described above will be useful in order to prove the next claims). Remark that it might be the case that \( F = R \), meaning that \( G[R] \) is a forest (and we fall in the case of Theorem 5.5), or \( F = \emptyset \), which means that \( G[R] \) has minimum degree at least 2.

**Claim 5.7.** All vertices in \( G[R'] \) have degree at least 2.

**Proof.** A vertex of degree less than 2 in \( G[R'] \) should have been added to \( F \).

**Claim 5.8.** The graph \( G[C \cup N \cup F] \) is connected.

**Proof.** It suffices to show that each connected component \( T \) of \( G[F] \) is connected to \( N \). Each such component \( T \) is a tree. If \( T \) consists of a single vertex \( v \), then \( v \) is the root of \( T \) and has at most one neighbor in \( R' \) by definition. Since \( G \) is reduced, \( v \) has degree at least 2 in \( G \), hence it must be connected to \( N \).

If \( T \) contains at least two vertices, then it contains at least 2 leaves, and in particular at least one leaf \( v \) which is not the root of \( T \). The vertex \( v \) has a single neighbor in \( R \) (its parent in \( T \)), and thus similarly as above it must have a neighbor in \( N \).

Define \( B \) as the set of vertices of \( R' \) adjacent to \( N \cup F \), and let \( A \) be the set of edges between \( N \cup F \) and \( B \).

**Claim 5.9.** If \( |A| \leq \frac{9}{10} r(G) \), then \( G \) has an \( \varepsilon_k \)-rich vertex.

**Proof.** Deleting \( A \) from \( G \) decreases the cycle rank by at most \( |A| \), hence \( r(G - A) \geq r(G)/10 \). Since \( G[C \cup N \cup F] \) and \( G[R'] \) are unions of connected components of \( G - A \), we have

\[
 r(G - A) = r(G[C \cup N \cup F]) + r(G[R']),
\]

Thus either \( G[C \cup N \cup F] \) or \( G[R'] \) has cycle rank at least \( r(G)/20 \). If it is \( G[C \cup N \cup F] \), then we can apply Theorem 5.5 to find a \((\delta_k/20)\)-rich vertex, and if it is \( G[R'] \), then we can apply the induction hypothesis to find an \((\varepsilon_{k-1}/20)\)-rich vertex. In either case, this gives an \( \varepsilon_k \)-rich vertex.

Thus we can now assume that \( |A| \geq \frac{9}{10} r(G) \).

Let \( B_1 \), resp. \( B_2 \), be the set of vertices of \( B \) incident to exactly one, resp. at least two edges of \( A \), and let \( A_1, A_2 \subseteq A \) be the set of edges of \( A \) incident to \( B_1, B_2 \) respectively. Remark that \( A_1, A_2 \) and \( B_1, B_2 \) partition \( A \) and \( B \) respectively, and \( |A_1| = |B_1| \).

**Claim 5.10.** If \( |A_2| \geq \frac{4}{9} |A| \), then \( G \) has an \( \varepsilon_k \)-rich vertex.
Proof. Assume that \(|A_2| \geq \frac{4}{3} |A|\), and thus \(|A_2| \geq \frac{2}{3} r(G)\). By Lemma 4.5, \(G\) is 2\(k\)-degenerate, hence it can be vertex-partitioned into \(k + 1\) forests. Consider this partition restricted to \(B_2\), and choose \(B_3 \subseteq B_2\) which induces a forest and maximizes the set \(A_3 \subseteq A_2\) of edges incident to \(B_3\). Thus \(|A_3| \geq |A_2|/(k + 1) \geq \frac{2}{3(k+1)} r(G)\). By Claim 5.6, \(F \cup B_3\) is a forest, hence Theorem 5.5 applies to \(G[C \cup N \cup F \cup B_3]\).

By Claim 5.8, \(G[C \cup N \cup F]\) is connected, thus adding the vertices \(B_3\) and the edges \(A_3\) increases the cycle rank by \(|A_3| - |B_3|\). This quantity is at least \(|A_3|/2\) since any vertex of \(B_3\) is incident to at least two edges of \(A_3\), and each edge of \(A_3\) is incident to exactly one vertex of \(B_3\). Thus Theorem 5.5 yields the existence of a vertex of degree at least

\[
\delta_k \cdot r(G[C \cup N \cup F \cup B_3]) \geq \frac{|A_3|}{2} \delta_k \geq \frac{1}{5(k+1)} \delta_k \cdot r(G) \geq \varepsilon_k \cdot r(G) \tag{4}
\]

as desired. \(\square\)

Thus we can now assume that \(|A_1| \geq \frac{5}{6} |A|\), and thus \(|B| \geq \frac{5}{6} |A| \geq \frac{1}{2} r(G)\).

Let \(X\), resp. \(Y\), be the set of vertices of \(B\) with degree at least 3, resp. exactly 2, in \(G[R']\). By Claim 5.7, this is a partition of \(B\).

**Claim 5.11.** If \(|X| \geq |B|/5\), then \(G\) has an \(\varepsilon_k\)-rich vertex.

**Proof.** Assume that \(|X| \geq |B|/5\), and thus \(|X| \geq \frac{1}{10} r(G)\).

The cycle rank is lower-bounded by the following sum:

\[
r(G[R']) \geq |E(G[R'])| - |R'| = \frac{1}{2} \sum_{x \in R'} (d_{G[R']}(x) - 2). \tag{5}
\]

By Claim 5.7, every term in the sum is non-negative, and each \(x \in X\) contributes by at least \(1/2\) to the sum. Thus \(r(G[R']) \geq |X|/2 \geq \frac{1}{20} r(G)\), and the induction hypothesis applied to \(G[R']\) (which is \(O_{k-1}\)-free) yields an \((\varepsilon_{k-1}/20)\)-rich vertex, which is also \(\varepsilon_k\)-rich. \(\square\)

Thus we can now assume that \(|Y| \geq \frac{2}{3} |B| \geq \frac{2}{5} r(G)\).

Let \(Z\) be the set of vertices of \(R'\) that either are in \(Y\) or have degree at least 3 in \(G[R']\). Remark that \(Z\) is exactly the set of vertices of \(R'\) with degree at least 3 in \(G\). In \(G[R']\), a **direct path** is a path whose endpoints are in \(Z\), and whose internal vertices are not in \(Z\). In particular, internal vertices of a direct path have degree 2. A direct path need not be induced, as its endpoints may be adjacent. As a degenerate case, we consider a cycle that contains a single vertex of \(Z\) to be a direct path whose two endpoints are equal. One can naturally construct a multigraph \(G_Z\) with vertex set \(Z\) and whose edges correspond to direct paths in \(G[R']\).

Remark that vertices of \(Z\) have the same degree in \(G_Z\) and in \(G[R']\).

Any \(y \in Y\) has two neighbors \(x_1, x_2\) in \(G_Z\). In degenerate cases, it may be that \(x_1 = x_2 \neq y\) (multi-edge in \(G_Z\)), in which case \(G[R']\) contains a banana between \(y\) and \(x_1\), or that \(x_1 = x_2 = y\) (loop in \(G_Z\)), in which case \(y\) is the only vertex of \(Z\) in \(C_y\). We partition \(Y\) into \(Y_1, Y_2\) as follows: for \(y, x_1, x_2\) as above, if \(x_1, x_2 \in Y\), then we place \(y\) in \(Y_1\), and otherwise \((x_1 \text{ or } x_2 \text{ is in } Z \setminus Y)\) we place \(y\) in \(Y_2\).

**Claim 5.12.** If \(|Y_2| \geq \frac{5}{6} |Y|\), then \(G\) has an \(\varepsilon_k\)-rich vertex.
Proof. Assume $|Y_e| \geq \frac{3}{4} |Y|$, and thus $|Y_e| \geq \frac{3}{10} r(G)$.

By definition, any vertex of $Y_e$ is adjacent in $G_Z$ to some vertex of $Z \setminus Y$. Thus, using that $d_{G_Z}(z) = d_{G[R']}_{Z}(z)$ for any $z \in Z$, we obtain

$$\sum_{z \in Z \setminus Y} d_{G[R']}_{Z}(z) \geq |Y_e|.$$  

(6)

Recall inequality (5) on cycle rank:

$$r(G[R']) \geq \frac{1}{2} \sum_{x \in R'} (d_{G[R']}_{x} - 2).$$  

(7)

By Claim 5.7, the terms of this sum are non-negative. Thus, restricting it to $Z \setminus Y$, we have

$$r(G[R']) \geq \frac{1}{2} \sum_{z \in Z \setminus Y} (d_{G[R']}_{z} - 2).$$  

(8)

By definition of $Z$, vertices of $Z \setminus Y$ have degree at least 3 in $G[R']$. Thus, each term of the previous sum satisfies $d_{G[R']}_{z} - 2 \geq d_{G[R']}_{z}/3$. It follows using (6) that

$$r(G[R']) \geq \frac{1}{2} \sum_{z \in Z \setminus Y} \frac{d_{G[R']}_{z}}{3} \geq \frac{|Y_e|}{6} \geq \frac{1}{20} r(G).$$  

(9)

Thus the induction hypothesis applied to $G[R']$ (which is $O_{k-1}$-free) yields an $(\varepsilon_{k-1}/20)$-rich vertex, which is also $\varepsilon_k$-rich.

Thus we can now assume that $|Y_e| \geq \frac{1}{2} |Y| \geq \frac{1}{10} r(G)$.

We now consider the induced subgraph $H$ of $G[R']$ consisting of $Y$, and direct paths joining vertices of $Y$. Thus $H$ has maximum degree 2, and since $G[R']$ is $O_{k-1}$-free, at most $k - 2$ components of $H$ are cycles, the rest being paths. Remark that the endpoints of paths in $H$ correspond exactly to $Y_e$. Also, each connected component of $H$ must contain at least one vertex of $Y$.

We perform the following cleaning operations in order:

- In each cycle of $H$, pick an arbitrary vertex and delete it, so that all connected components are paths.

- Iteratively delete a vertex of degree 0 or 1 which is not in $Y$, so that the endpoints of paths are all in $Y$.

- Delete all isolated vertex.

Let $H'$ be the subgraph of $H$ obtained after these steps.

Claim 5.13. All but $3(k - 2)$ vertices of $Y_i$ are internal vertices of paths of $H'$.

Proof. If $y \in Y_i$ belongs to a path of $H$, then it must be an internal vertex of this path, and the path is unaffected by the cleaning operations. Thus it suffices to prove that in each cycle of $H$, at most 3 vertices of $Y_i$ are deleted or become endpoints of paths during the clean up.

Let $C'$ be a cycle of $H$. If $C'$ contains no more than 2 vertices of $Y_i$, there is nothing to prove. Remark in this case that $C'$ is entirely deleted by the clean up. Otherwise, let $x$ be the vertex deleted from $H$ (which may be in $Y_i$), and let $y_1, y_2$ be the first vertices of $Y_i$ strictly
before and after \( x \) in the cyclic order of \( C' \). Since \( C' \) has at least 3 vertices of \( Y_i, x, y_1, y_2 \) are all distinct. Then, it is clear that the cleaning operations transform \( C' \) into a path with endpoints \( y_1, y_2 \), such that any \( y \in Y_i \cap C' \), distinct from \( x, y_1, y_2 \) is an internal vertex of this path.

We now add \( H' \) to \( F \), which yields a forest by Claim 5.6. Recall that vertices of \( Y \) are adjacent to \( N \cup F \), and all endpoints of paths of \( H' \) are in \( Y \). Thus, in \( G[C \cup N \cup F \cup H'] \), every vertex of \( H' \) has degree at least 2, and vertices of \( Y_i \) in the interior of paths of \( H' \) have degree at least 3. Since \( G[C \cup N \cup F] \) is connected by Claim 5.8, the addition of \( H' \) does not change the number of connected components. Using Claim 5.13, this implies that

\[
r(G[C \cup N \cup F \cup H']) \geq |Y_i| - 3(k - 2).
\]  

(10)

We finally apply Theorem 5.5 to \( G[C \cup N \cup F \cup H'] \) to obtain a vertex with degree at least

\[
d_k \cdot (|Y_i| - 3(k - 2)).
\]

Since \( G \) contains vertices of degree at least 2, we can always assume that \( \varepsilon_k \cdot r(G) \geq 2 \), and thus

\[
|Y_i| \geq \frac{1}{10} \cdot 2\varepsilon_k^{-1} \geq \frac{1}{5} \cdot 30(k - 2) = 6(k - 2).
\]

(11)

It follows that \( |Y_i| - 3(k - 2) \geq |Y_i|/2 \), and the previous argument yields a vertex of degree at least \( \frac{1}{2} |Y_i| \geq \frac{6k}{20} r(G) \), which is an \( \varepsilon_k \)-rich vertex.

\[\square\]

6 Cutting trees into stars and paths

Recall the statement of Theorem 5.5: we start with an \( O_k \)-free graph \( G \) of large girth, and divide its vertex set into some shortest cycle \( C \), its neighborhood \( N \), and the remainder of the vertex set \( R \) (including the second neighborhood \( S \) of \( C \)). Moreover we assume that \( R \) induces a forest. Since \( C \) is a shortest cycle, it is not difficult to check that every vertex of \( S \) must have exactly one neighbor in \( N \). Moreover, up to reducing the graph under consideration, we can assume that all the leaves of \( G[R] \) lie in \( S \).

Our goal in this section will be to simplify \( G[R] \) by only keeping a linear number (in \( |S| \)) of subdivided stars or paths with endpoints in \( S \). To this end it will be convenient to leave \( C \) aside and only consider \( N \) and \( R \) for now (or more precisely what remains of \( R \) after the graph has been reduced). Curious readers are invited to have a quick look at Figures 4, 5 and 6 to have an idea of how the results of this section will be used in the proof of Theorem 5.5.

The paragraphs above motivate the following definitions. A forest \( H \) is said to be \((S \subseteq F)\)-decorated if \( V(H) = F \), every leaf of \( H \) lies in \( S \subseteq F \), and every connected component of \( F \) contains at least 2 vertices. A graph \( H \) is said to be \((N, S \subseteq F)\)-divided if its vertex set is partitioned into two sets \( F \) and \( N \), such that

- \( F \) induces a forest and \( N \) is an independent set,
- the neighborhood of \( N \) in \( F \) is a subset \( S \subseteq V(F) \) containing all the leaves of \( F \),
- each vertex of \( S \) has a unique neighbor in \( N \), and
- every connected component of \( H[F] \) contains at least two vertices.
Note that the second and fourth conditions imply that $H[F]$ is $(S \subseteq F)$-decorated. It can be deduced from the definition that if $H$ is $(N, S \subseteq F)$-divided, then it does not contain $K_{3,3}$ as a subgraph.

A subdivided star is a graph with at least two vertices, which is a subdivision of a star (a graph obtained from a star by replacing its edges by paths of arbitrary length). We insist on the fact that we do not consider singleton vertices as subdivided stars. A path (on at least two vertices) is a special case of subdivided star. The center of a subdivided star is the vertex of degree at least 3, if any. If none, the subdivided star is a path, and its center is a vertex of degree 2 that belongs to $S$, if any, and an arbitrary vertex otherwise. We say that a forest $F' \subseteq F$ is $S'$-clean, for some $S' \subseteq S$, if $V(F') \cap S' = L(F')$, where $L(F')$ denotes the set of leaves of $F'$. We define being quasi-$S'$-clean for a subdivided star as intersecting $S'$ at exactly its set of leaves, plus possibly its center. Formally, a subdivided star $T$ is quasi-$S'$-clean if $L(T) \subseteq V(T) \cap S' \subseteq L(T) \cup \{c\}$ where $c$ is the center of $T$. The degree of a subdivided star is the degree of its center. A forest $F' \subseteq F$ of subdivided stars is quasi-$S'$-clean, for some $S' \subseteq S$, if all its connected components are quasi-$S'$-clean (subdivided stars).

Our approach in this section is summarized in Figure 3. We start with our forest $F$ and a subset $S$ of vertices including all the leaves of $F$ (the vertices of $S$ are depicted in white, while the vertices of $F - S$ are depicted in black). We first extract quasi-$S$-clean subdivided stars (Lemma 6.1). We then extract quasi-$S$-clean subdivided stars of large degree, or $S$-clean paths (Lemma 6.3). Finally we extract $S$-clean subdivided stars of large degree or paths (Corollary 6.7). At each step the number of vertices of $S$ involved in the induced subgraph of $F$ we consider is linear in $|S|$.

**Lemma 6.1.** Let $H$ be an $(S \subseteq F)$-decorated forest. Then there is a subset $F^* \subseteq F$ containing at least $\frac{1}{2}|S|$ vertices of $S$ such that each connected component of $H[F^*]$ is a quasi-$S$-clean subdivided star.

**Proof.** We first use the following claim.

**Claim 6.2.** There is a set of edges $X \subseteq E(H)$ such that every connected component of $H \setminus X$ is either a quasi-$S$-clean subdivided star or a single vertex that does not belong to $S$.

**Proof.** We proceed greedily, starting with $X = \emptyset$. While $H \setminus X$ contains a component $T$ and an edge $e \in T$ such that each of the two components of $T - e$ contains either no vertex of $S$ or at least two vertices of $S$, we add $e$ to $X$. 

---

Figure 3: A visual summary of Section 6.
Observe that in $H$, every connected component contains at least 2 vertices of $S$. Throughout the process of defining $X$, every connected component of $H \setminus X$ contains either 0 or at least 2 vertices of $S$.

At the end of the process, for any connected component $T$ of $H \setminus X$ with at least one edge, all the leaves of $T$ belong to $S$. Otherwise, the edge incident to the leaf of $T$ that is not in $S$ can be added to $X$.

Thus, $H \setminus X$ does not contain any component with more than one vertex of degree at least 3, since otherwise any edge on the path between these two vertices would have been added to $X$, yielding two components containing at least 2 leaves, and thus at least 2 vertices of $S$.

Observe also that if $H \setminus X$ contains a component $T$ with a vertex $v \in S$ that has degree 2 in $T$, then $T$ is a path containing exactly 3 vertices of $S$, and thus $T$ is a subdivided star whose center and leaves are in $S$, and whose other internal vertices are not in $S$.

To conclude, we need to select connected components of $H \setminus X$ with at least two vertices of $S$ and that are pairwise independent in $H$. Consider the minor $G_H$ of $H$ obtained by contracting each connected component of $H \setminus X$ into a single vertex and deleting those that are a single vertex not in $S$. Since $H$ is a forest, the graph $G_H$ is a forest. We weigh each vertex of $G_H$ by the number of elements of $S$ that the corresponding connected component of $H \setminus X$ contains. Since $G_H$ is a forest, there is an independent set $\{u_1, u_2, \ldots, u_p\}$ that contains at least half the total weight. The connected components corresponding to $u_1, u_2, \ldots, u_p$ together form a forest $H[F^x]$ with the required properties.

We observe that subdivided stars of small degree can be transformed into paths for a low price, as follows. A subdivided star forest is a forest whose components are subdivided stars (possibly paths).

**Lemma 6.3.** Let $H$ be an $(S \subseteq F)$-decorated forest. For every $S' \subseteq S$, every quasi-$S'$-clean subdivided star forest $F' \subseteq F$, and every integer $D \geq 2$, there is a subdivided star forest $F'' \subseteq F'$ such that every connected component of $H[F'']$ is either an $S'$-clean path or a quasi-$S'$-clean subdivided star of degree at least $D$. Additionally, $F''$ contains at least $\frac{2|S' \cap F'|}{D}$ vertices of $S'$.

**Proof.** We define $F''$ from $F'$ as follows. Consider a connected component $T$ of $H[F']$. If the center of $T$ has degree at least $D$, we add $T$ to $F''$. Consider now the case where $T$ is a quasi-$S'$-clean subdivided star whose center $c$ has degree less than $D$. If $c \in S'$, we select a non-edgeless path $P \subseteq T$ between $c$ and $S'$, and add $P$ to $F''$. If $c \notin S'$, we select two internally-disjoint paths $P_1, P_2 \subseteq T$ between $c$ and $S'$, and add $P_1 \cup P_2$ to $F''$. Note that $P_1 \cup P_2$ yields an $S'$-clean path.

To see that $F''$ contains at least $\frac{2|S' \cap F'|}{D}$ vertices of $S'$, we simply observe that in the second case, out of a maximum of $(D - 1) + 1$ vertices of $S'$ in a component $T$, we keep at least 2 in $F'$. This adds up to $\frac{2|S'|}{D}$ vertices of $S'$ since connected components of $H[F']$ are disjoint by definition.

**Lemma 6.4.** Let $H$ be an $O_k$-free graph which is $(N, S \subseteq F)$-divided. If each vertex of $N$ has degree less than $\frac{1}{6k} |S|$, then one of the following holds.

- there is a subset $S'$ of $S$ and a subset $F_2$ of $F$ such that $F_2$ contains $\frac{1}{32} |S|$ vertices of $S'$, and each connected component of $H[F_2]$ is an $S'$-clean subdivided star.

- there is a subset $F_3$ of $F$ such that every connected component of $F_3$ is a quasi-$S$-clean subdivided star of degree at most 4 and $F_3$ contains at least $\frac{1}{8} |S|$ vertices of $S$.
Proof. Let $F^* \subseteq F$ be the forest obtained from Lemma 6.1, applied to the $(S \subseteq F)$-decorated forest $H[F]$. Then $F^*$ contains at least $\frac{1}{4}|S|$ vertices of $S$, and each component of $H[F^*]$ is a quasi-$S$-clean subdivided star or an $S$-clean path. We define the label of a vertex of $S$ to be its only neighbor in $N$.

**Claim 6.5.** There is a subset $F_1$ of $F^*$ containing at least $\frac{1}{3}|S|$ vertices of $S$, such that no subdivided star of $F_1$ has its center and one of its endpoints sharing the same label.

**Proof.** Let $\ell$ be the maximum integer such that there exist $\ell$ subdivided stars $S_1, S_2, \ldots, S_\ell$ in $H[F^*]$ and $\ell$ different labels $v_1, \ldots, v_\ell \in N$, such that for any $1 \leq i \leq \ell$, $S_i$ has its center and at least one of its endpoint labeled $v_i$. Note that in this case $G$ contains $\ell$ independent cycles, and thus $\ell < k$ by assumption.

For any $1 \leq i \leq \ell$, remove all the leaves $u$ of $F^*$ that are labeled $v_i$, and also remove the maximal path of $H[F^*]$ ending in $u$. By assumption, there are at most $\frac{1}{\ell k}|S|$ such vertices $u$ for each $1 \leq i \leq \ell$, and thus we delete at most $k \cdot \frac{1}{\ell k}|S| \leq \frac{1}{3}|S|$ vertices of $S$ from $F^*$. We also delete the centers that have no leaves left (there are at most $k \cdot \frac{1}{8}|S|$ such deleted centers). Let $F_1$ be the resulting subset of $F^*$. Note that $F_1$ contains at least $|F^* \cap S| - 2 \cdot \frac{1}{3}|S| \geq \left(\frac{1}{2} - \frac{1}{3}\right)|S| = \frac{1}{6}|S|$ vertices of $S$.

We can assume that a subset $Y$ of at least $\frac{1}{3}|S|$ vertices of $S$ in the forest $F_1$ obtained from Claim 6.5 is involved in a quasi-$S$-clean subdivided star of degree at least 5. Indeed, otherwise at least $\frac{1}{8}|S|$ vertices of $S$ in the forest $F_1$ obtained from Claim 6.5 are involved in a quasi-$S$-clean subdivided star of degree at most 4 (note that an $S$-clean path is an $S$-clean subdivided star), and in this case the second outcome of Lemma 6.4 holds.

For each label $v \in N$, we choose uniformly at random with probability $\frac{1}{2}$ whether $v$ is a center label or a leaf label. We then delete all the subdivided stars of $F_1$ whose center is labeled with a leaf label, and all the leaves whose label is a center label. Moreover, we delete from $N$ all the vertices that are a center label, and let $S'$ be the set of vertices of $S$ whose neighbor in $N$ is not deleted.

Take a vertex $u$ of $Y$. If $u$ is a center of a subdivided star, then the probability that $u$ is not deleted is at least $\frac{1}{2}$. If $u$ is a leaf, $u$ is kept only if $u$ and the center of the subdivided star it belongs to (which has by construction a different label) are correctly labeled, so $u$ is kept with probability at least $\frac{1}{4}$. Overall, each vertex $u$ of $Y$ has probability at least $\frac{1}{4}$ to be kept. Thus the expectation of the fraction of vertices of $Y$ not deleted is at least $\frac{1}{4}$, thus we can find an assignment of the labels to leaf labels or center labels, such that a subset $Z \subseteq Y$ with $|Z| \geq \frac{1}{4}|Y|$ survives.

We then iteratively delete vertices of degree 1 that do not belong to $S'$ and all vertices of degree 0. Let $F_2$ be the resulting forest. Note that $S'$ contains only the endpoints of stars with a leaf label, thus the forest $F_2$ is $S'$-clean. It remains to argue that $F_2$ contains a significant fraction of vertices of $S$. Note that a connected component of $F_1$ is deleted if and only if it contains at most one element of $S$. Every such component has at least 4 elements in $Y \setminus Z$, hence there are at most $\frac{1}{4} + \frac{4}{16}|Y| = \frac{3}{16}|Y|$ such components. It follows that $F_2$ contains at least $|Z| - \frac{3}{16}|Y| \geq \frac{1}{4}|Y| - \frac{3}{16}|Y| \geq \frac{1}{16}|S|$ elements of $Z \subseteq S$. 

We now have all the ingredients to obtain the following two corollaries.

**Corollary 6.6.** Let $H$ be an $(S \subseteq F)$-decorated forest. For any $D \geq 2$, there is a subset $F^* \subseteq F$ containing at least $\frac{1}{2D^2}|S|$ vertices of $S$ such that each

1. $F^*$ induces a quasi-$S$-clean subdivided star forest whose components all have degree at least $D$, or
2. $F^*$ induces an $S$-clean path forest.

Corollary 6.6 follows from Lemma 6.1 by applying Lemma 6.3 and observing that one of the two outcomes contains half the corresponding vertices in $S$.

**Corollary 6.7.** Let $H$ be an $O_k$-free graph which is $(N, S \subseteq F)$-divided, and let $D \geq 2$. If each vertex of $N$ has degree less than $\frac{1}{8k} |S|$, then there are $F'' \subseteq F$, $S' \subseteq S$ such that $F''$ contains at least $\frac{1}{2D} |S|$ vertices of $S'$ and one of the following two cases apply.

1. $F''$ induces an $S'$-clean subdivided star forest whose components all have degree at least $D$, or
2. $F''$ induces an $S'$-clean path forest.

Similarly, Corollary 6.7 follows from Lemma 6.4 by applying Lemma 6.3 and observing that one of the two outcomes contains half the corresponding vertices in $S$.

## 7 Trees, stars, and paths

In the proof of Theorem 5.5, we will apply Corollaries 6.6 and 6.7 several times, and divide our graph into two parts: a union of subdivided stars on one side, and a union of subdivided stars or paths on the other side (see again Figures 4, 5 and 6 for an idea of what these two sides will correspond to in the final applications). We now explain how to find a rich vertex in this context.

We start with the case where subdivided stars appear on both sides.

**Lemma 7.1 (Star-star lemma).** Let $c > 0$ be the constant of Lemma 4.9. Let $H$ be an $O_k$-free graph whose vertex set is the union of two sets $L, R$, such that

- $S = L \cap R$ is an independent set,
- there are no edges between $L \setminus S$ and $R \setminus S$, and
- $L$ (resp. $R$) induces in $H$ a disjoint union of subdivided stars, whose centers have average degree at least $3ck \log k$, and whose set of leaves is precisely $S$.

Then $H$ contains a vertex of degree at least $\frac{1}{2f'(3,k)} |S| = \Omega(\frac{1}{f'} |S|)$, where $f'$ is the function of Corollary 4.4.

**Proof.** Note that $H$ does not contain $K_{3,3}$ as a subgraph (but might contain $K_{2,2}$ as a subgraph) and is $O_k$-free. By Corollary 4.4, there is a set $X$ of at most $f'(3,k)$ vertices of $H$ such that all bananas of $H$ intersect $X$. Since the centers of the subdivided stars are the only vertices of degree larger than 2 in $H$, we can assume that $X$ is a subset of the centers of the subdivided stars.

Assume first that less than $\frac{1}{2} |S|$ vertices of $S$ are leaves of subdivided stars centered in an element of $X$. Let $S' \subseteq S$ be the leaves of the subdivided stars whose center is not in $X$ (note that $|S'| \geq \frac{1}{2} |S|$), and remove from the subdivided stars of $H[L]$ and $H[R]$ all branches whose endpoint is not in $S'$ to get new sets of vertices $L', R'$. The centers of the resulting $S'$-clean subdivided stars now have average degree at least $\frac{1}{2} \cdot 3ck \log k > ck \log k$. We denote the resulting $S'$-clean subdivided stars of $H[L']$ by $S_1, S_2$, etc. and their centers by $s_1, s_2$, etc. Similarly, we denote the resulting $S'$-clean subdivided stars of $H[R']$ by $S'_1, S'_2$, etc. and their
centers by $s'_1, s'_2$, etc. Observe that by the definition of $X$, for any two centers $s_i, s'_j$, there is at most one vertex $u \in S'$ which is a common leaf of $S_i$ and $S'_j$.

Let $B$ be the bipartite graph with partite set $s_1, s_2, \ldots$ and $s'_1, s'_2, \ldots$, with an edge between $s_i$ and $s'_j$ if and only if some vertex of $S'$ is a common leaf of $S_i$ and $S'_j$. Note that $B$ has average degree more than $ck \log k$, and some induced subgraph of $H$ (which is $O_k$-free) contains a strict subdivision of $B$. This contradicts Lemma 4.9.

So we can assume that at least $\frac{1}{2} |S|$ vertices of $S$ are leaves of subdivided stars centered in an element of $X$. Then some vertex of $X$ has degree at least $\frac{1}{2} |S|$, as desired. \hfill \Box

We now consider the case where subdivided stars appear on one side, and paths on the other.

**Lemma 7.2** (Star-path lemma). Let $c > 0$ be the constant of Lemma 4.9. Let $H$ be an $O_k$-free graph whose vertex set is the union of two sets $L, R$, such that

- $S = L \cap R$ is an independent set,
- $L$ induces in $H$ a disjoint union of paths, whose set of endpoints is precisely $S$, and
- $R$ induces in $H$ a disjoint union of subdivided stars, whose centers have average degree at least $4ck \log k$, and whose set of leaves is precisely $S$.

Then $H$ contains a vertex of degree at least $\frac{1}{3f'(2,k)} |S| = \Omega(\frac{1}{k^2} |S|)$, where $f'$ is the function of Corollary 4.4.

**Proof.** Note that $H$ does not contain $K_{2,2}$ as a subgraph, and is $O_k$-free. By Corollary 4.4, there is a set $X$ of at most $f'(2,k)$ vertices of $H$ such that all bananas of $H$ intersect $X$. Since the centers of the subdivided stars are the only vertices of degree more than $2$ in $H$, we can assume that $X$ is a subset of the centers of the subdivided stars.

Assume first that less than $\frac{1}{10} |S|$ vertices of $S$ are leaves of subdivided stars centered in an element of $X$. Then there are at least $\frac{1}{5} |S|$ paths in $H[L]$ whose endpoints are not leaves of centers in $X$. Let $S' \subseteq S$ be the endpoints of these paths (note that $|S'| \geq \frac{1}{5} |S|$), and remove from the subdivided stars of $H[R]$ all branches whose endpoint is not in $S'$ to get $R'$. The centers of the resulting $S'$-clean subdivided stars in $H[R']$ now have average degree at least $\frac{1}{10} \cdot 4ck \log k > ck \log k$. We denote these subdivided stars by $S_1, \ldots, S_t$, and their centers by $s_1, \ldots, s_t$.

Given two centers $s_i, s_j$, we say that a pair $u_i, u_j \in S'$ is an \{i, j\}-route if $u_i$ is a leaf of $S_i$, $u_j$ is a leaf of $S_j$, and there is a path with endpoints $u_i, u_j$ in $H[L]$. Observe that by the definition of $X$, for every pair $s_i, s_j$, there is at most one \{i, j\}-route.

Let $G$ be the graph with vertex set $s_1, \ldots, s_t$, with an edge between $s_i$ and $s_j$ if and only if there is an \{i, j\}-route. Note that $G$ has average degree more than $ck \log k$, and some induced subgraph of $H$ (which is $O_k$-free) contains a strict subdivision of $G$. This contradicts Lemma 4.9.

So we can assume that at least $\frac{1}{3} |S|$ vertices of $S$ are leaves of subdivided stars centered in an element of $X$. Then some vertex of $X$ has degree at least $\frac{1}{3} |S|$, as desired. \hfill \Box

From the two previous lemmas and Lemma 6.1 we deduce the following.

**Lemma 7.3** (Star-tree lemma). There is a constant $c > 0$ such that the following holds. Let $H$ be an $O_k$-free graph which does not contain $K_{1,t}$ as a subgraph. Assume that the vertex set of $H$ is the union of two sets $L, R$, such that
\( S = L \cap R \) is an independent set partitioned into \( S_P, S_T \),
- there are no edges between \( L \setminus S \) and \( R \setminus S \),
- \( L \) induces in \( H \) a disjoint union of subdivided stars, whose centers have average degree at least \((8ck \log k)^2\), and whose set of leaves is equal to \( S \), and
- \( R \) induces in \( H \) the disjoint union of
  - paths on a vertex set \( R_P \), whose set of endpoints is equal to \( S_P \), and
  - a tree \( T \) on a vertex set \( R_T \) such that \( S_T \) is a subset of leaves of \( T \).

Then \( H \) contains a vertex of degree at least \( \Omega\left(\frac{1}{k^{4} \log k} |S| \right) \).

**Proof.** Let \( c > 0 \) be the constant of Lemma 4.9. Assume first that \( |S_T| \leq 1 \). Then since the subdivided stars of \( L \) have average degree at least \((8ck \log k)^2\), we have \( |S_P| = |S| - |S_T| \geq (8ck \log k)^2 - 1 \geq 1 \) and thus \( |S_P| \geq \frac{1}{2} |S| \). By removing the branch of a subdivided star of \( L \) that has an endpoint in \( S_T \) (if any), we obtain a set of \( S_P \)-clean subdivided stars of average degree at least \( \frac{1}{2} \cdot (8ck \log k)^2 \geq 4ck \log k \). By Lemma 7.2, we get a vertex of degree at least \( \Omega\left(\frac{1}{k^{2}} |S_P| \right) = \Omega\left(\frac{1}{k^{2}} |S| \right) \), as desired. So in the remainder we can assume that \( |S_T| \geq 2 \).

Let \( T' \) be the subtree of \( T \) obtained by repeatedly removing leaves that are not in \( S_T \). Since \( |S_T| \geq 2 \), \( L(T') = S_T \). Observe that \( F' = T' \cup R_P \) is an \( S \)-clean forest (with \( L(F') = S \)), thus any \( S \)-quasi-clean subforest of \( F' \) is \( S \)-clean. It follows from Corollary 6.6 (applied to \( S \), \( F' \), and \( D = 4ck \log k \)) that \( F' \) contains a subset \( F^* \) containing at least \( \frac{1}{2 - \frac{4ck \log k}{k}} |S| \) vertices of \( S \), such that \( H[F^*] \) induces either (1) an \( S \)-clean forest of path, or (2) an \( S \)-clean forest of subdivided stars of degree at least \( 4ck \log k \).

We denote this intersection of \( S \) and \( F^* \) by \( S^* \), and we remove in the subdivided stars of \( H[L] \) all branches whose endpoint is not in \( S^* \) to get a new set of vertices \( L^* \subseteq L \). By assumption, the average degree of the subdivided stars in \( L^* \) is at least \( \frac{(8ck \log k)^2}{8ck \log k} = 8ck \log k \geq 4ck \log k \).

In case (1) above we can now apply Lemma 7.2, and in case (2) we can apply Lemma 7.1. In both cases we obtain a vertex of degree at least \( \Omega\left(\frac{1}{k^{4} \log k} |S^*| \right) = \Omega\left(\frac{1}{k^{4} \log k} |S| \right) \), as desired. \( \square \)

**8 Proof of Theorem 5.5**

We start with recalling the setting of Theorem 5.5. The graph \( G \) is a connected \( O_k \)-free graph of girth at least 11, and \( C \) is a shortest cycle in \( G \). The neighborhood of \( C \) is denoted by \( N \), and the vertex set \( V(G) \setminus (C \cup N) \) is denoted by \( R \). The subset of \( R \) consisting of the vertices adjacent to \( N \) is denoted by \( S \). Since \( C \) is a shortest cycle, of size at least 11, each vertex of \( S \) has a unique neighbor in \( N \), and a unique vertex at distance 2 in \( C \). Moreover \( N \) and \( S \) are independent sets. In the setting of Theorem 5.5, \( R \) is a forest.

Our goal is to prove that there is a vertex whose degree is linear in the cycle rank \( r(G) \). To this end, we assume that \( G \) has maximum degree at most \( \delta \cdot r(G) \), for some \( \delta > 0 \), and prove that this yields a contradiction if \( \delta \) is a small enough function of \( k \).

By Lemma 5.1, we can assume that \( G \) is reduced, i.e., contains no vertex of degree 0 or 1. If \( G \) consists only of the cycle \( C \), then \( r(G) = 1 \) and the theorem is immediate. Thus we can assume that \( N \) is non-empty, which in turn implies that \( S \) is non-empty since \( G \) is reduced. Since \( R \) does not contain any vertex of degree 0 or 1 in \( G \), we also have that \( G[R] \) does not contain any isolated vertex (all its components have size at least 2) and all the leaves of \( G[R] \)
lie in $S$. Using the terminology introduced in Section 6, $G[R]$ is an $(S \subseteq R)$-decorated forest, and $G \setminus V(C)$ is $(N, S \subseteq R)$-divided.

Using that $G$ is connected, remark that

$$r(G) = |E(G)| - |V(G)| + 1 = 1 + \frac{1}{2} \sum_{v \in V(G)} (d(v) - 2).$$  \hspace{1cm} (12)

We start with proving that the cardinality of $S$ is at least the cycle rank $r(G)$.

**Claim 8.1.** $|S| \geq r(G)$, and thus $G$ has maximum degree at most $\delta |S|$.

**Proof.** Observe that $\frac{1}{2} \sum_{v \in C \cup N} (d(v) - 2) = \frac{1}{2} |S|$. Furthermore $\frac{1}{2} \sum_{v \in R} (d(v) - 2)$ is equal to $\frac{1}{2} |S|$ minus the number of connected components of $G[R]$, as $R$ induces a forest and each vertex of $S$ has a unique neighbor outside of $R$. Since $R$ is non-empty, it follows from (12) that $r(G) \leq |S|$. We assumed that $G$ has maximum degree at most $\delta \cdot r(G)$ which is at most $\delta |S|$, as desired. \[\]

In the remainder of the proof, we let $c > 0$ be a sufficiently large constant such that Lemmas 4.8 and 7.3 both hold for this constant.

We consider $\delta < \frac{1}{8k}$, and use Claim 8.1 to apply Corollary 6.7 to the subgraph $H$ of $G$ induced by $N$ and $F = R$ (which is $O_k$-free), with $D = 2 \cdot (8ck \log k)^2$. We obtain subsets $N' \subseteq N$, $R'' \subseteq R$ such that if we define $S'$ as the subset of $S \cap R''$ with a neighbor in $N'$, we have $|S'| \geq \frac{1}{32D} |S|$ and at least one of the following two cases apply.

1. Each connected component of $H[R'']$ is an $S'$-clean subdivided star of degree at least $D$,

or

2. Each connected component of $H[R'']$ is an $S'$-clean path.

We first argue that the second scenario holds.

**Claim 8.2.** Each connected component of $H[R'']$ is an $S'$-clean path.

**Proof.** Assume for a contradiction that Case 2 does not apply, hence Case 1 applies.

Let $G_1$ be the subgraph of $G$ induced by $C \cup N' \cup R''$ (see Figure 4, left). Since $|C| \geq 11$ and vertices of $C$ have disjoint second neighborhoods in $S'$, there exists a vertex $v^* \in C$ that sees at most $\frac{1}{11} |S'|$ vertices of $S'$ in its second neighborhood. If we remove from $G_1$ the vertex $v^*$, its neighborhood $N(v^*) \subseteq N'$, its second neighborhood $N^2(v^*) \subseteq S'$, and the corresponding

![Figure 4: The graphs $G_1$ (left) and $G_2$ (right) in the proof of Claim 8.2.](image-url)
branches of the subdivided stars of \( R'' \), we obtain a graph \( G_2 \) whose vertex set is partitioned into a path \( P = C - v^* \), its neighborhood \( N_2 = N' - N(v^*) \), and the rest of the vertices \( R_2 \) (which includes the set \( S_2 = S' - N^2(v^*) \)), with the property that each component of \( G_2[R_2] \) is an \( S_2 \)-clean subdivided star (see Figure 4, right). More importantly,

\[
|S_2| \geq \frac{10}{11}|S'| \geq \frac{10}{11} \cdot \frac{1}{32D} |S| \geq \frac{1}{36D} |S|,
\]

and the average degree of the centers of the subdivided stars is at least \( \frac{10}{11} D \geq (8ck \log k)^2 \).

Observe that \( P \cup N_2 \cup S_2 \) induces a tree in \( G_2 \), such that all leaves of \( G_2[P \cup N_2 \cup S_2] \) except at most two (the two neighbors of \( v^* \) on \( C \)) lie in \( S_2 \), and non leaves of the tree are not in \( S_2 \). We can now apply Lemma 7.3 with \( P \) and \( L = R_2 \). It follows that \( G_2 \) contains a vertex of degree at least \( \Omega(\frac{1}{k \log^3 k} |S_2|) = \Omega(\frac{1}{k \log^3 k} |S|) > \delta |S| \). Since \( G_2 \) is an induced subgraph of \( G \), this contradicts Claim 8.1.

We denote the connected components of \( H[R'' \setminus S] \) by \( P_1, \ldots, P_\ell \), with \( \ell \geq \frac{1}{64D} |S| \).

**Claim 8.3.** There is a vertex \( u^* \) in \( C \) which has at least \( \frac{1}{16(8ck \log k)^2} |S| \) endpoints of the paths \( P_1, \ldots, P_\ell \) in its second neighborhood, where \( c > 0 \) is the constant of Lemma 4.8.

**Proof.** Assume for the sake of contradiction that each vertex of \( C \) has less than \( \frac{1}{16(8ck \log k)^2} |S| \) endpoints of the paths \( P_1, \ldots, P_\ell \) in its second neighborhood.

![Figure 5: The graphs \( G_3 \) (left) and \( G_4 \) (right) in the proof of Claim 8.3.](image)

Let \( G_3 \) be subgraph of \( G \) induced by \( C \cup N' \) and \( \bigcup_{i=1}^\ell V(P_i) \) (see Figure 5, left), and let \( G_4 \) be the graph obtained from \( G_3 \) by contracting each vertex of \( N' \) with its unique neighbor in \( C \) (i.e., \( G_4 \) is obtained from \( G_3 \) by contracting disjoint stars into single vertices), see Figure 5, right. Note that now \( G_3 \) is \( O_\ell \)-free, \( G_3 \) and \( G_4 \) are also \( O_\ell \)-free (from the structural properties of \( C, N, \) and \( S \), each cycle in \( G_4 \) can be canonically associated to a cycle in \( G_3 \), and for any set of independent cycles in \( G_4 \), the corresponding cycles in \( G_3 \) are also independent). By our assumption, each vertex of \( C \) in \( G_4 \) has degree at most \( \frac{1}{16(8ck \log k)^2} |S| + 2 \), and \( G_4 \) consists of the cycle \( C \) together with \( \ell \geq \frac{1}{64D} |S| \) paths whose endpoints are in \( C \) and whose internal vertices are pairwise disjoint and non-adjacent. By Lemma 4.8, it follows that

\[
\frac{1}{64D} |S| < \ell \leq c \cdot \frac{1}{16(8ck \log k)^2} |S| \cdot k \log k,
\]

and thus \( D > 2(8ck \log k)^2 \), which contradicts the definition of \( D = 2(8ck \log k)^2 \).

**Claim 8.4.** If the vertices in \( N[u^*] \) have average degree at least \( (8ck \log k)^2 \) in \( S' \), then \( G \) contains a vertex of degree at least \( \delta |S| \).
Proof. The key idea of the proof of the claim is to consider the neighbors of \( u^* \) as the centers of stars (L) in Claim 7.3. In order to do that, we consider the subgraph \( G_5 \) of \( G \) induced by
- the path \( C - u^* \),
- \( N(u^*) \) and the paths \( P_i \) (1 \( \leq i \leq \ell \)) with at least one endpoint in the second neighborhood \( N^2(u^*) \) of \( u^* \) (call these paths \( P'_1, \ldots, P'_t \)), and
- the neighbors of the endpoints of the paths \( P'_1, \ldots, P'_t \) in \( N \).

All the components of \( G_5 - N(u^*) \) are either paths \( P'_i \) with both endpoints in \( N^2(u^*) \), or a tree whose leaves are all in \( N^2(u^*) \) (except at most two leaves, which are the two neighbors of \( u^* \) in \( C \)). See Figure 6, right, for an illustration, where the vertices of \( N^2(u^*) \) are depicted with squares and the components of \( G_5 - N(u^*) \) are depicted with bold edges.

![Figure 6: The graphs \( G_3 \) with the vertex \( u^* \) (left) and the graph \( G_5 \) (right) in the proof of Claim 8.4.](image)

By considering the vertices of \( N(u^*) \) and their neighbors in \( S' \) as stars (whose centers, depicted in white in Figure 6, right, have average degree at least \( (8ck \log k)^2 \)) we can apply Lemma 7.3, and obtain a vertex of degree at least \( \Omega\left(\frac{1}{k^4 \log k} |S'| \right) \geq \Omega\left(\frac{1}{k^6 \log^3 k} |S| \right) \geq \delta |S| \) in \( G_5 \) (and thus in \( G \)), which contradicts Claim 8.1.

Observe that if the vertices of \( N(u^*) \) have average degree at most \( (8ck \log k)^2 \) in \( S' \), then \( u^* \) has degree at least \( \frac{1}{16(8ck \log k)^2} |S| \geq \delta |S| \). If not, by Claim 8.4, \( G \) also contains a vertex of degree at least \( \delta |S| \). Both cases contradict Claim 8.1, and this concludes the proof of Theorem 5.5. □

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