A construction of diffusion processes associated with sublaplacian on CR manifolds and its applications

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Abstract
A diffusion process associated with the real sublaplacian $\Delta_b$, the real part of the complex Kohn-Spencer laplacian $\Box_b$, on a strictly pseudoconvex CR manifold is constructed via “rolling the manifold $M$” along the Brownian path on $\mathbb{C}^n$ by taking advantage of the metric connection due to Tanaka-Webster. Using the diffusion process and the Malliavin calculus, the heat kernel and the Dirichlet problem for $\Delta_b$ are studied in a probabilistic manner. Moreover, distributions of stochastic line integrals along the diffusion process will be investigated.

Introduction
Let $M$ be an oriented strictly pseudoconvex CR manifold of dimension $2n + 1$, and $\Delta_b$ be the real sublaplacian on $M$, i.e. the real part of the Kohn-Spencer laplacian $\Box_b$. For definitions, see Section 1. In this paper, we will construct a diffusion process generated by $-\Delta_b/2$ in a stochastic differential geometrical way by “rolling $M$ along the Brownian path on $\mathbb{C}^n$” with the help of the metric connection due to Tanaka [16] and Webster [19]. Making use of the diffusion process and the Malliavin calculus, we will study the heat kernel and the Dirichlet problem associated with $\Delta_b$ in a probabilistic manner. Moreover, distributions of stochastic line integrals along the diffusion process will be also investigated by the partial hypoelliptic argument.

In the case when $M$ is a boundary of a bounded domain in $\mathbb{C}^n$, Malliavin [12] gave a sufficient condition for the fundamental solution of the heat equation associated with $-\Delta_b/2$ to possess a smooth density function. In [6], Gaveau studied the Dirichlet problem when $M$ is a Heisenberg group. In this paper, we extend their attempts to general abstract CR manifolds. Furthermore, one may remind that the diffusion processes on sub-Riemannian manifolds, which include CR manifolds, are deeply studied by Baudoin et al. from the point of view of subelliptic laplacians and sub-Riemannian geometry, and interesting properties

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are achieved by them. For example, see [1, 2, 3, 4] and references therein. It then should be emphasized that one of the aims of this paper is constructing diffusion processes in a probabilistic manner via Itô calculus, i.e., following the original idea of Itô ([9]).

Our first problem is to construct a diffusion process \( X = \{(X(t))_{t \geq 0}, P_x; x \in M\} \) on \( M \) generated by \(-\Delta_b/2\) so that the Malliavin calculus is applicable to \( X \). Since \( \Delta_b \) is of locally Hörmander type but not globally, we cannot construct \( X \) directly via stochastic differential equations (SDE in abbreviation) governed by vector fields on \( M \). To go around this obstacle, we recall that Brownian motions on Riemannian manifolds are constructed via SDE on associated bundles of orthonormal frames, which realizes “rolling the Riemannian manifold along the Brownian path on the Euclidean space” ([8]). In this case the Riemannian connection plays an essential role. For a CR manifold \( M \), it has been seen by Tanaka [16] and Webster [19] that there exists a metric connection on a subbundle \( T_{1,0} \) of the complexified tangent bundle \( CTM \). It is this connection which plays the same role for \( X \) as the Riemannian connection does for the Brownian motion on it. We will obtain \( X \) by taking advantage of an SDE which realizes the stochastic parallel translation associated with the metric connection. See Section 2.

By using the partial hypoelliplicity argument in the Malliavin calculus, we will obtain the heat kernel related to this diffusion process, that is, we will show the transition probability function of \( X \) has a smooth density function \( p(t, x, y) \). Moreover, we will investigate stochastic line integrals of 1-forms on \( M \) along the diffusion process \( X \) and give a sufficient condition for distributions of stochastic line integrals to have smooth density functions. See Section 3.

We finally consider the Dirichlet problem associated with \( \Delta_b \). Let \( G \) be a relatively compact open set in \( M \) with \( C^3 \)-boundary. We shall show that, for each \( f \in C(\partial G) \), there is a \( u \in C(\bar{G}) \) such that

\[
\Delta_b u = 0 \quad \text{on } G \quad \text{in the weak sense, and } u = f \quad \text{on } \partial G.
\]

See Theorem 4.1. As will be seen in Remark 4.8 together with hypoelliplicity of \( \Delta_b \), this \( u \) is a classical solution to the Dirichlet problem. In the proof, a key role is played by the local representation of the sublaplacian \( \Delta_b \) so that, on every sufficiently small coordinate neighborhood \( U \), there are \( a^\alpha \in \mathbb{C} \) and \( C^\infty \)-vector fields \( Z_\alpha \) with \( \mathbb{C} \)-valued coefficients on \( U \) so that

\[
\Delta_b = -\sum_{\alpha=1}^n (Z_\alpha Z_\alpha + Z_{\alpha \bar{\alpha}}Z_{\alpha \bar{\alpha}}) + \sum_{\alpha=1}^n (a^\alpha Z_\alpha + a^{\bar{\alpha}} Z_{\bar{\alpha}}),
\]

and

\[
\text{span}_\mathbb{C} \{(Z_\alpha)_x, (Z_{\alpha \bar{\alpha}})_x, [Z_\alpha, Z_{\bar{\alpha}}]_x; 1 \leq \alpha \leq n\} = CT_xM, \quad x \in U,
\]

where \( [\cdot, \cdot] \) denotes the Lie bracket product, \( T_xM \) is the tangent space of \( M \) at \( x \), and we have used super and subscripts \( \bar{\alpha} \)'s to indicate that complex conjugates are taken; \( b_{\bar{\alpha}} = b_\alpha \), and \( c_\bar{\alpha} = c^\alpha \).

In Section 1, we shall give a brief review on CR geometry. In the same section, we shall construct vector fields on a complex vector bundle over \( M \), which are associated with the metric connection due to Tanaka-Webster. These vector fields will be used in Section 2 to construct a diffusion process \( X \) generated by \(-\Delta_b/2\). The heat kernel and distributions of stochastic line integrals along \( X \) are studied in Section 3. Section 4 will be devoted to the study of Dirichlet problems associated with \( \Delta_b \).
1. CR geometry

1.1. CR manifolds

We begin this section with listing the results on CR manifolds which we shall use later, following Lee [11] and Dragomir-Tomassini [5].

A CR manifold $M$ is a real differentiable manifold together with a complex subbundle $T_{1,0}$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T_{1,0} \cap T_{0,1} = \{0\}$ and $[T_{1,0}, T_{1,0}] \subset T_{1,0}$, where $T_{0,1} = \overline{T_{1,0}}$. We consider the case that $M$ is orientable and of real dimension $2n + 1$ with $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $T_{1,0}$ is of complex dimension $n$, i.e. the CR codimension is 1.

Set $H = \Re(T_{1,0} \oplus T_{0,1})$, which is called the Levi distribution of $(M, T_{1,0})$. There exists a pseudo-Hermitian structure, that is, a real non-vanishing 1-form $\theta$ on $M$ which annihilates $H$. For such $\theta$, the Levi form $L_\theta$ of $\theta$ is defined by

$$L_\theta(Z, W) = -\sqrt{-1} d\theta(Z, W), \quad Z, W \in \Gamma^{\infty}(T_{1,0} \oplus T_{0,1}),$$

where $\Gamma^{\infty}(V)$ stands for the space of $C^{\infty}$ cross sections of a vector bundle $V$. Throughout the paper, we assume that $M$ is strictly pseudoconvex, that is, the Levi form $L_\theta$ is positive definite. Then $T_{1,0}$ is an Hermitian fiber bundle with Hermitian fiber metric $L_\theta$. Let $T$ be the characteristic direction, that is, the unique real vector field on $M$ transverse to $H$, defined by

$$(1.1) \quad T|d\theta = 0, \quad T|\theta = 1,$$

where $T|\omega$ is the interior product: $T|\omega(X_1, \ldots, X_{p-1}) = \omega(T, X_1, \ldots, X_{p-1})$ for a $p$-form $\omega$.

As $M$ is strictly pseudoconvex, the $(2n + 1)$-form $\psi = \theta \wedge (d\theta)^n$ on $M$ determines a volume form, where we have chosen the orientation of $M$ so that $\psi$ is a positive form. Then it induces the $L^2$-inner product on functions:

$$\langle u, v \rangle_\theta = \int_M u\overline{v}\psi, \quad u, v \in C^0_\infty(M; \mathbb{C}) \equiv \{ f + \sqrt{-1}g; f, g \in C^\infty_0(M)\}.$$  

The Levi form induces a metric on $H$ (denoted by $L_\theta$ again), and the dual metric $L^*_\theta$ on $H^*$. Then the $L^2$-inner product on sections of $H^*$ is given by

$$\langle \omega, \eta \rangle_\theta = \int_M L^*_\theta(\omega, \eta)\psi, \quad \omega, \eta \in \Gamma^{\infty}(H^*).$$

Denoting by $r: T^*M \to H^*$ the natural restriction mapping, we define a section $d_\theta u$ for $u \in C^\infty(M)$ by $d_\theta u = r \circ du$. The real sublaplacian $\Delta_\theta$ on functions is given by

$$\langle \Delta_\theta u, v \rangle_\theta = \langle d_\theta^* d_\theta u, d_\theta v \rangle_\theta, \quad v \in C^\infty_0(M).$$

Similarly, denoting by $\overline{\partial}_b u$ the projection of $du$ onto $T^*_{0,1}$ for $u \in C^\infty(M; \mathbb{C})$, we introduce the Kohn-Spencer laplacian $\Box_b$ defined by

$$\langle \Box_b u, v \rangle_\theta = \langle \overline{\partial}_b u, \overline{\partial}_b v \rangle_\theta, \quad v \in C^\infty_0(M; \mathbb{C}).$$

These two operator are related to each other by

$$\Box_b = \Delta_\theta + \sqrt{-1} nT \quad \text{on} \quad C^\infty(M; \mathbb{C}).$$
1.2. Tanaka-Webster connection

We now review a connection due to Tanaka [16] and Webster [19].

Let $J: H \to H$ be the complex structure related to $(M,T_{1,0})$; that is, the $\mathbb{C}$-linear extension of $J$ is the multiplication by $\sqrt{-1}$ on $T_{1,0}$ and $-\sqrt{-1}$ on $T_{0,1}$, where we have used the fact that $H \otimes_\mathbb{R} \mathbb{C} = T_{1,0} \oplus T_{0,1}$. Moreover, we extend $J$ linearly to $TM$ by $J(T) = 0$.

Since $TM = H \oplus \mathbb{R}T = \{X + aT \mid X \in H, a \in \mathbb{R}\}$, there exists the unique Riemannian metric $g_\theta$ on $M$ satisfying that

$$g_\theta(X,Y) = d\theta(X,JY), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1$$

for $X, Y \in H$. $g_\theta$ is called the Webster metric. We extend $g_\theta$ to $\mathbb{C}TM$ $\mathbb{C}$-bilinearly.

The Tanaka-Webster connection is the unique linear connection $\nabla$ on $M$ satisfying that

\begin{align}
(1.2) & \quad \nabla_X Y \in \Gamma^\infty(H), \quad X \in \Gamma^\infty(TM), Y \in \Gamma^\infty(H), \\
(1.3) & \quad \nabla J = 0, \quad \nabla g_\theta = 0, \\
(1.4) & \quad T_\nabla(Z,W) = 0, \quad Z, W \in \Gamma^\infty(T_{1,0}), \\
(1.5) & \quad T_\nabla(Z,W) = 2\sqrt{-1} L_\theta(Z,W)T, \quad Z \in \Gamma^\infty(T_{1,0}), \quad W \in \Gamma^\infty(T_{0,1}), \\
(1.6) & \quad T_\nabla(T,J(X)) + J(T_\nabla(T,X)) = 0, \quad X \in \Gamma^\infty(TM),
\end{align}

where $\nabla_X$ is the covariant derivative in the direction of $X$ and $T_\nabla$ is the torsion tensor field of $\nabla$: $T_\nabla(Z,W) = \nabla_ZW - \nabla_WZ - [Z,W]$.

Let $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$, where $\langle n \rangle = \{1, \ldots, n\}$, be a local orthonormal frame for $T_{1,0}$ on an open set $U$, that is, $Z_\alpha$ is a $T_{1,0}$-valued section defined on $U$ and $g_\theta(Z_\alpha, Z_\beta) = \delta_{\alpha\beta}$, where $Z_\beta = \overline{Z_\beta}$. If we set $\langle \langle n \rangle \rangle = \{0, 1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$ and $Z_0 = T$, then $\{Z_A\}_{A \in \langle \langle n \rangle \rangle}$ is a local frame for $\mathbb{C}TM$. We define Christoffel symbols $\Gamma^C_{AB}$ for $A, B, C \in \langle \langle n \rangle \rangle$ by

$$\nabla_{Z_A} Z_B = \sum_{C \in \langle \langle n \rangle \rangle} \Gamma^C_{AB} Z_C.$$

Note that $\Gamma^C_{AB} = 0$ unless $(B,C) \in \{(\beta,\gamma), (\overline{\beta},\overline{\gamma}) \mid \beta, \gamma \in \langle n \rangle\}$, because $\nabla_X(\Gamma^\infty(T_{1,0})) \subset \Gamma^\infty(T_{1,0})$ and $\nabla T = 0$ by the conditions (1.2) and (1.3). We also have that

$$\Gamma^\gamma_{A\beta} + \Gamma^\beta_{A\gamma} = 0, \quad \beta, \gamma \in \langle n \rangle, A \in \langle \langle n \rangle \rangle$$

by the condition (1.3).

1.3. Canonical vector fields

To construct diffusion processes on CR manifolds in the next section, we introduce suitable vector bundles and principal bundles on them. The method employed here is a modification of the one used to construct the Brownian motion on a Riemannian manifold by using the orthonormal frame bundle [13].

Let $p: [a, b] \to M$, where $a < b$, be a smooth curve. We say that a smooth curve $W: [a, b] \to T_{1,0}$ is a parallel section along $p$ if $W(t) \in (T_{1,0})_{p(t)}$ and $\nabla_p W = 0$, where the dot always means the differentiation in $t$. 
For any \( v \in (T_{1,0})_{p(a)} \), there exists a unique parallel section \( W \) with \( W(a) = v \). This can be seen by the localization argument as follows: Let \( \{Z_{\alpha}\}_{\alpha \in \langle n \rangle} \) be a local orthonormal frame for \( T_{1,0} \) on \( U \) and suppose that \( p([a,b]) \subset U \). Then for a smooth curve \( W: [a,b] \to T_{1,0} \) satisfying \( W(t) \in (T_{1,0})_{p(t)} \), it holds that

\[
W(t) = \sum_{\alpha \in \langle n \rangle} c^\alpha(t)(Z_\alpha)_{p(t)},
\]

where \( c^\alpha(t) = g_\theta(W(t), (Z_{\alpha})_{p(t)}) \) for \( \alpha \in \langle n \rangle \). By the very definition of the covariant derivative,

\[
\nabla_\beta W(t) = \sum_{\alpha \in \langle n \rangle} (\dot{c}^\alpha(t)(Z_\alpha)_{p(t)} + c^\alpha(t)\nabla_\beta Z_\alpha(t)) = \sum_{\alpha \in \langle n \rangle} \dot{c}^\alpha(t)(Z_\alpha)_{p(t)} + \sum_{A \in \langle n \rangle} c^\alpha(t)g_\theta(\dot{p}(t), (Z_{\alpha})_{p(t)})\Gamma^\beta_{\alpha A}(p(t))Z_\beta_{p(t)},
\]

where we have used the convention that \( \overline{0} = 0 \). Therefore \( \nabla_\beta W = 0 \) if and only if

\[
\dot{c}^\beta(t) + \sum_{A \in \langle n \rangle} \sum_{\alpha \in \langle n \rangle} c^\alpha(t)g_\theta(\dot{p}(t), (Z_{\alpha})_{p(t)})\Gamma^\beta_{\alpha A}(p(t)) = 0
\]

for each \( \beta \in \langle n \rangle \). Now, as an application of the theory of ordinary differential equations, given \( v \in (T_{1,0})_{p(a)} \) there exists a unique parallel section \( W \) along \( p \) such that \( W(a) = v \).

Parallel sections can be represented locally as follows:

**Lemma 1.8.** Let \( \{Z_{\alpha}\}_{\alpha \in \langle n \rangle} \) be a local orthonormal frame for \( T_{1,0} \) on \( U \) and suppose that \( p([a,b]) \subset U \). Then there exists a unique \( \Lambda_p: [a,b] \to U(n) \), where \( U(n) \) is the group of \( n \times n \) unitary matrices, such that \( \Lambda_p(a) = I_n \), the identity matrix, and

\[
(1.9) \quad \dot{\Lambda}_p(t)^\gamma_\beta + \sum_{A \in \langle n \rangle} \sum_{\delta \in \langle n \rangle} \Lambda_p(t)^\delta_\beta g_\theta(\dot{p}(t), (Z_{A})_{p(t)})\Gamma^\gamma_{\delta A}(p(t)) = 0,
\]

where \( \dot{\Lambda}_p(t)^\gamma_\beta \) holds for each \( \beta, \gamma \in \langle n \rangle \). Moreover, given \( v \in (T_{1,0})_{p(a)} \),

\[
W(t) = \sum_{\beta, \gamma \in \langle n \rangle} \Lambda_p(t)^\gamma_\beta g_\theta(v, (Z_{\beta})_{p(a)})Z_\gamma_{p(t)}
\]

is a parallel section along \( p \) and satisfies \( W(a) = v \).

**Proof.** It is clear that the condition for \( \Lambda_p \) defines a unique curve on \( M_n(\mathbb{C}) \), the group of \( n \times n \) complex matrices. By (1.7) and (1.9) it is easy to check that

\[
\frac{d}{dt} \left( \sum_{\gamma \in \langle n \rangle} \Lambda_p(t)^\gamma_\alpha \dot{\Lambda}_p(t)^\beta_\gamma \right) = 0
\]

holds for \( \alpha, \beta \in \langle n \rangle \), which in conjunction with \( \Lambda_p(a) = I_n \) implies that \( \Lambda_p(t) \in U(n) \).
We next show the second assertion. Recall that

$$\nabla_p W(t) = \sum_{\beta, \gamma \in \langle n \rangle} g_0(v,(Z_\beta)_{p(a)})(\hat{\Lambda}_p(t)\hat{\gamma}_\beta(Z_\gamma)_{p(t)} + \Lambda_\beta(t)\hat{\gamma}_\beta\nabla_p Z_\gamma(t)).$$

Plugging (1.9) and the identity

$$\nabla_p Z_\gamma(t) = \sum_{A \in \langle n \rangle} g_0(\hat{p}(t),(Z_\alpha)_{p(t)})(\nabla_{Z_A} Z_\gamma)_{p(t)}$$

into this, we obtain the desired equality $\nabla_p W = 0$. \hfill \Box

Now we introduce the bundles over $M$ given by

$$L(T_{1,0}) = \prod_{x \in M} \{ r : \mathbb{C}^n \to (T_{1,0})_x ; r \text{ is a non-singular linear map} \},$$

$$U(T_{1,0}) = \{ r \in L(T_{1,0}) ; r \text{ is isometric} \}.$$

For $r \in L(T_{1,0})$ with $r : \mathbb{C}^n \to (T_{1,0})_x$, let $\pi(r) = x$. We write $r\xi$ for the image of $\xi \in \mathbb{C}^n$ by $r \in L(T_{1,0})$.

The Lie group $U(n)$ acts on $U(T_{1,0})$; for each $\Lambda \in U(n)$ we have the map $R_\Lambda : U(T_{1,0}) \to U(T_{1,0})$ defined by

$$(R_\Lambda r)(\xi) = r\Lambda \xi, \quad r \in U(T_{1,0}), \xi \in \mathbb{C}^n.$$

Moreover, if $\Lambda : [a, b] \to U(n)$ is a smooth curve with $\Lambda(a) = I_n$ and $r \in U(T_{1,0})$, then $\dot{\Lambda}(a)$ is a skew Hermitian matrix and $\frac{d}{dt} \bigg|_{t=a} R_{\Lambda(t)} r = \lambda(\dot{\Lambda}(a))r$, where $\lambda$ is given by

$$\lambda(u)_r = \frac{d}{ds} \bigg|_{s=0} R_{\exp(su)} r.$$

For smooth curves $p : [a, b] \to M$ and $\hat{p} : [a, b] \to U(T_{1,0})$, we say that $\hat{p}$ is a horizontal lift of $p$ to $U(T_{1,0})$ if $\pi \circ \hat{p} = p$ and $\hat{p}(t)e_{\alpha}$ is a parallel section for any $\alpha \in \langle n \rangle$, where $\{e_{\alpha}\}_{\alpha \in \langle n \rangle}$ is the standard coordinate of $\mathbb{C}^n$. For $v \in T_zM$, $r \in \pi^{-1}(x)$ and $\eta \in T_{\pi(U(T_{1,0}))}$, we say that $\eta$ is a horizontal lift of $v$ if there exist a smooth curve $p$ on $M$ and a smooth curve $\hat{p}$ on $U(T_{1,0})$ which is a horizontal lift of $p$, satisfying $\hat{p}(0) = r$, $\dot{\hat{p}}(0) = \eta$ and $\pi \circ \eta = v$.

For each $v \in T_zM$, there exists a unique horizontal lift $\eta_p(r) \in T_{\pi(U(T_{1,0}))}$. Namely, let $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$ be a local orthonormal frame for $T_{1,0}$ on $U$ and suppose the curve $p$ is contained in $U$. Let $Z : U \to U(T_{1,0})$ be the section determined by $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$, i.e. $Z(x)e_{\alpha} = (Z_\alpha)_x$. By virtue of Lemma 11.8 $\hat{p} : [a, b] \to U(T_{1,0})$ is a horizontal lift of $p$ if and only if $\pi(\hat{p}(a)) = p(a)$ and

$$\hat{p}(t) = R_{Z(p(a))^{-1} \circ \hat{p}(a)} \circ R_{\Lambda_p(t)} Z(p(t)),$$

where $Z(p(a))^{-1} \circ \hat{p}(a) : \mathbb{C}^n \to \mathbb{C}^n$ is regarded as an element of $U(n)$ and $\Lambda_p$ is the curve defined in Lemma 11.8.

Under the identification of $Z(x) \in U(T_{1,0})$ with $((Z_1)_x, \ldots, (Z_n)_x) \in (T_{1,0})_x^n$, we have $R_{\Lambda} Z(x) = Z(x)\Lambda$ for $\Lambda \in U(n)$. Then we can calculate as

$$\frac{d}{dt}(R_{\Lambda_p(t)} Z(p(t))) = \frac{d}{dt}(Z(p(t))\Lambda_p(t)) = Z_\ast(\dot{p}(t))\Lambda_p(t) + Z(p(t))\dot{\Lambda}_p(t).$$
By differentiating (1.10) and substituting the above identity, we arrive at the unique horizontal lift of \( v \):

\[
\eta_r(v) = (R_{Z(x)}^{-1})_*(Z_*(v) - \lambda(\Phi(v))) 
\in T_r U(T_{1,0}),
\]

where \( \Phi : T_x M \to \mathfrak{u}(n) \), \( \mathfrak{u}(n) \) being the set of \( n \times n \) skew Hermitian matrices, is defined by

\[
\Phi(v) = \left( \sum_{A \in \langle \alpha \rangle} g_\theta(v, (Z_\mathcal{A})_x) \Gamma^\gamma_{A\beta}(x) \right)_{\beta, \gamma \in \langle n \rangle}.
\]

For each \( r \in U(T_{1,0}) \), the horizontal subspace at \( r \) is defined by

\[
\text{Hor}_r U(T_{1,0}) = \{ \eta_r(v) ; v \in T_r M \} \subset T_r U(T_{1,0}).
\]

If we set \( \text{Ver}_r U(T_{1,0}) = \text{Ker}(\pi_* : T_r U(T_{1,0}) \to T_{\pi(r)} M) \), the vertical subspace, then

\[
T_r U(T_{1,0}) = \text{Ver}_r U(T_{1,0}) \oplus \text{Hor}_r U(T_{1,0})
\]

holds.

\( \eta_r \) extends naturally to a \( \mathbb{C} \)-linear map from \( T_x M \otimes \mathbb{R} \mathbb{C} \) to \( \text{Hor}_r U(T_{1,0}) \otimes \mathbb{R} \mathbb{C} \). Then for each \( \xi \in \mathbb{C}^n \), we can define the canonical vector field \( L(\xi) \) by \( L(\xi)_r = \eta_r(r\xi) \) for \( r \in U(T_{1,0}) \). We set

\[
L_\alpha = L(e_\alpha), \quad \alpha \in \langle n \rangle
\]

and call \( \{ L_\alpha \}_{\alpha \in \langle n \rangle} \) the canonical vector fields.

Let \( \{ Z_\alpha \}_{\alpha \in \langle n \rangle} \) be a local orthonormal frame for \( T_{1,0} \) on \( U \). Define \( \{ e_\beta^\alpha(r) \} \in \mathbb{C}^n \otimes \mathbb{C}^n \) for \( r \in L(T_{1,0}) \) with \( \pi(r) \in U \) by \( r(e_\alpha) = \sum_{\beta \in \langle n \rangle} e_\beta^\alpha(r)(Z_\beta)_{\pi(r)} \). We can then introduce a local coordinate system \( \{(x^k, e_\alpha^k)\} \) of \( L(T_{1,0}) \), \((x^k)_{1 \leq k \leq 2n+1}\) being a local coordinate system of \( M \). With respect to this coordinate we represent the canonical vector field \( L_\alpha, \alpha \in \langle n \rangle \) as follows.

Recall that \( U(T_{1,0}) \) can be identified with \( M \times U(n) \) locally, and under this identification \( R_\Lambda((x,e)) = (x,e\Lambda) \) for \((x,e) \in M \times U(n) \). Therefore it holds that

\[
(R_{Z(x)}^{-1} \circ r)_* Z_*(v) = v, \quad v \in T_x M \otimes \mathbb{R} \mathbb{C},
\]

where \( Z : U \to U(T_{1,0}) \) is the section determined by \( \{ Z_\alpha \}_{\alpha \in \langle n \rangle} \) as before. Since

\[
\lambda(\Phi \left( \text{Re} \sum_{\beta} e_\alpha^\beta Z_\beta \right)) = \frac{1}{2} \sum_{\beta, \gamma, \delta \in \langle n \rangle} \left( e_\alpha^\beta \Gamma^\gamma_{\beta\delta} + e_\alpha^\beta \bar{\Gamma}^\gamma_{\beta\delta} \right) \frac{\partial}{\partial e_\delta} + \left( e_\alpha^\beta \bar{\Gamma}^\gamma_{\beta\delta} + e_\alpha^\beta \Gamma^\gamma_{\beta\delta} \right) \frac{\partial}{\partial e_\delta},
\]

\[
\lambda(\Phi \left( \text{Im} \sum_{\beta} e_\alpha^\beta Z_\beta \right)) = \frac{1}{2\sqrt{-1}} \sum_{\beta, \gamma, \delta \in \langle n \rangle} \left( e_\alpha^\beta \Gamma^\gamma_{\beta\delta} - e_\alpha^\beta \bar{\Gamma}^\gamma_{\beta\delta} \right) \frac{\partial}{\partial e_\delta} - \left( e_\alpha^\beta \bar{\Gamma}^\gamma_{\beta\delta} - e_\alpha^\beta \Gamma^\gamma_{\beta\delta} \right) \frac{\partial}{\partial e_\delta},
\]

we have from (1.11) that

\[
(L_\alpha)_r = \sum_{\beta \in \langle n \rangle} e_\alpha^\beta Z_\beta - \sum_{\beta, \gamma, \delta, \varepsilon \in \langle n \rangle} \Gamma^\gamma_{\beta\delta} e_\varepsilon^\beta e_\alpha^\delta \frac{\partial}{\partial e_\varepsilon} - \sum_{\beta, \gamma, \delta, \varepsilon \in \langle n \rangle} \bar{\Gamma}^\gamma_{\beta\delta} e_\varepsilon^\beta e_\alpha^\delta \frac{\partial}{\partial e_\varepsilon}.
\]
2. Construction of a diffusion process

In this section, we construct a diffusion process \( X = \{\{X(t)\} \}_{t \geq 0}, P_x \); \( x \in M \) generated by \(-\Delta_b/2\).

Let \( \{L_\alpha\}_{\alpha \in \langle \kappa \rangle} \) be the canonical vector fields on \( U(T_{1,0}) \) constructed in the previous section. Take a \( \mathbb{C}^\kappa \)-valued Brownian motion \( \{B(t) = (B^1(t), \ldots, B^\kappa(t))\}_{t \geq 0} \), that is, \( \{B(t)\}_{t \geq 0} \) is a \( \mathbb{C}^\kappa \)-valued continuous martingale with \( \langle B^\alpha, B^\beta \rangle \) \( t = 0 \) and \( \langle B^\alpha, B^\gamma \rangle \) \( t = \delta_\alpha^\beta t \), where \( \langle M, N \rangle(t) \) denotes the quadratic variation of continuous martingales \( \{M(t)\}_{t \geq 0} \) and \( \{N(t)\}_{t \geq 0} \). Let \( \{r(t) = r(t, r, B)\}_{t \geq 0} \) be the unique solution to an SDE on \( U(T_{1,0}) \):

\[
\begin{align*}
2.1 & \quad dr(t) = \sum_{\alpha \in \langle \kappa \rangle} (L_\alpha(r(t)) \circ dB^\alpha(t) + L_\alpha^\top(r(t)) \circ dB^\alpha(t)), \quad r(0) = r \in U(T_{1,0}),
\end{align*}
\]

or equivalently

\[
\begin{align*}
2.2 & \quad dr(t) = \sum_{\alpha \in \langle \kappa \rangle} (\sqrt{2} \Re L_\alpha(r(t)) \circ d\xi^\alpha(t) + \sqrt{2} \Im L_\alpha(r(t)) \circ d\eta^\alpha(t)), \quad r(0) = r \in U(T_{1,0}),
\end{align*}
\]

where \( \Re L_\alpha = (L_\alpha + L_\alpha^\top)/2 \), \( \Im L_\alpha = (L_\alpha - L_\alpha^\top)/2\sqrt{-1} \), \( \xi^\alpha(t) = \sqrt{2} \Re B^\alpha(t) \) and \( \eta^\alpha(t) = \sqrt{2} \Im B^\alpha(t) \). The process \( r(t) \) may explode. Note that \( (\xi^1(t), \eta^1(t), \ldots, \xi^\kappa(t), \eta^\kappa(t)) \) is an \( \mathbb{R}^{2\kappa} \)-valued Brownian motion.

Let \( \{Z_\alpha\}_{\alpha \in \langle \kappa \rangle} \) be a local orthonormal frame for \( T_{1,0} \) and \( (x^k, e^\beta_k) \) be the associated local coordinate of \( L(T_{1,0}) \) as in the previous section. Then \( 2.1 \) can be rewritten locally as

\[
\begin{align*}
\begin{cases}
\quad \quad \quad dx(t) = \sum_{\alpha, \beta \in \langle \kappa \rangle} (e^\beta_\alpha(t)Z_\beta(x(t)) \circ dB^\alpha(t) + e^\beta_\alpha(t)Z^\beta(x(t)) \circ dB^\alpha(t)), \\
\quad \quad \quad de^\gamma_\xi(t) = - \sum_{\alpha, \beta, \delta \in \langle \kappa \rangle} (\Gamma^\gamma_\beta_\delta(x(t))e^\delta_\alpha(t)e^\beta_\alpha(t) \circ dB^\alpha(t) + \Gamma^\gamma_\beta_\delta(x(t))e^\delta_\alpha(t)e^\beta_\alpha(t) \circ dB^\alpha(t)).
\end{cases}
\end{align*}
\]

Hence it follows from the uniqueness of \( \{r(t, r, B)\}_{t \geq 0} \) that \( r(t, r \Lambda, \bar{X}B) = r(t, r, B) \) for every unitary matrix \( \Lambda \). Denoting by \( \bar{M} \) a one-point compactification of \( M \), we have that the induced measures \( Q_r \) of \( \pi(r(\cdot, r, B)) \) on \( C([0, \infty); \bar{M}) \), the space of \( \bar{M} \)-valued continuous functions defined on \( [0, \infty) \), coincide for all \( r \in \pi^{-1}(x) \). Put

\[
P_x = Q_r, \quad r \in \pi^{-1}(x).
\]

Set

\[
\mathcal{L} = \frac{1}{2} \sum_{\alpha \in \langle \kappa \rangle} (L_\alpha L_\alpha + L_\alpha L_\alpha) |_M.
\]

It is easily seen that

\[
f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds
\]

is a martingale under \( P_x \) for every \( x \in M \) and \( f \in C_0^\infty(M) \), where \( X(t) \) denotes the position of \( X \in C([0, \infty); \bar{M}) \) at time \( t \). By a straightforward computation, we have a local representation of \( \mathcal{L} \) as follows:

\[
2.4 \quad \mathcal{L} = \frac{1}{2} \left( \sum_{\alpha \in \langle \kappa \rangle} (Z_\alpha Z_\alpha + Z_\alpha Z_\alpha) - \sum_{\alpha, \beta \in \langle \kappa \rangle} (\Gamma^\alpha_\beta_\gamma Z_\alpha + \Gamma^\alpha_\beta_\gamma Z_\alpha) \right).
\]
Recall, moreover, an identity that
\[ d_b f = \sum_{\alpha \in \langle n \rangle} (Z_\alpha f \theta^\alpha + Z_{\pi} f \bar{\theta}^\alpha) \]
and Greenleaf’s result \[7\] that
\[ \langle Z_\alpha f, g \rangle_\theta = \left\langle f, (- Z_{\pi} + \sum_{\beta} \Gamma_{\beta}^\alpha g) \right\rangle_\theta. \]
Plugging these into (2.4), we see that
\[ \mathcal{L} = -\frac{1}{2} \Delta_b. \]
Thus we have shown that

**Theorem 2.5.** There exists a diffusion process \( \mathbb{X} = \{(\{X(t)\}_{t \geq 0}, P_x); x \in M\} \) generated by \(-\Delta_b/2\) and which is obtained via the SDE (2.1).

**Example 2.6.** Let \( \mathbb{H}_n = \mathbb{C}^n \times \mathbb{R} \) be the \((2n+1)\)-dimensional Heisenberg group with a coordinate system \((z,t), z = (z^1, \ldots, z^n) \in \mathbb{C}^n, t \in \mathbb{R}\). Define
\[ \theta = \frac{1}{2} \left( dt - \sqrt{-1} \sum_{\alpha=1}^{n} (\bar{z}^\alpha dz^\alpha - z^\alpha d\bar{z}^\alpha) \right), \]
\[ T_{1,0} = \bigoplus_{\alpha=1}^{n} \mathbb{C}Z_\alpha, \quad \text{where} \quad Z_\alpha = \frac{\partial}{\partial z^\alpha} + \sqrt{-1} z^\alpha \frac{\partial}{\partial t}. \]
Then \( \mathbb{H}_n \) is a strictly pseudoconvex CR manifold, see \[5\]. Since \( \{Z_\alpha\}_{\alpha \in \langle n \rangle} \) is a global orthonormal frame for \( T_{1,0} \) and
\[ d\theta = \sqrt{-1} \sum_{\alpha=1}^{n} dz^\alpha \wedge d\bar{z}^\alpha, \quad d(dz^\alpha) = 0, \]
the associated covariant derivation is a null mapping. In particular, \( \Delta_b = -\sum_{\alpha} (Z_\alpha Z_{\pi} + Z_{\pi} Z_\alpha) \). The diffusion process described in Theorem 2.5 is exactly the same one as that studied by Gaveau in \[6\].

### 3. Heat kernel and stochastic line integral

In this section, we apply the result \[17\] on partial hypoellipticity to the diffusion process constructed in the previous section and stochastic line integrals along the diffusion process.

We first consider the heat equation
\[ \frac{\partial}{\partial t} u = -\frac{1}{2} \Delta_b u, \quad u(0, x) = f(x), f \in C_b^\infty(M), \]
via the diffusion process \( \mathbb{X} = \{(\{X(t)\}_{t \geq 0}, P_x); x \in M\} \) constructed in Theorem 2.5.

By Whitney’s embedding theorem, we may think of \( U(T_{1,0}) \) as a closed submanifold of \( \mathbb{R}^k \) for some \( k \). We further assume that
there exist $C^\infty$ vector fields $L'_\alpha$, $\alpha \in \langle n \rangle$, on $\mathbb{R}^k$ with $\mathbb{C}$-valued coefficients such that (i) $L_\alpha = L'_\alpha$ on $U(T_{0,1})$, and (ii) the coefficients of $L'_\alpha$ and their derivatives of all orders are bounded.

The hypothesis (H) implies that $r(t)$ does not explode. This hypothesis is fulfilled if $M$ is compact. We shall establish

**Theorem 3.2.** Assume that (H) holds. Then there is a $p \in C^\infty((0, \infty) \times M \times M)$ such that

$$P_x(X(t) \in dy) = p(t,x,y)\psi(dy).$$

**Proof.** Recall the expression (2.2). By virtue of [17, Theorem 3.1] and [18, Lemma 3.1], it suffices to show that

$$\text{span}_{\mathbb{R}}\{(\pi_\ast, \text{Re} L_\alpha), (\pi_\ast, \text{Im} L_\alpha), (\pi_\ast, [\text{Re} L_\alpha, \text{Im} L_\alpha]); \alpha \in \langle n \rangle\} = T_{\pi(r)}M$$

for every $r \in U(T_{1,0})$, where $\text{span}_{\mathbb{R}}$ stands for taking all real linear combinations. To see this, let $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$ be a local orthonormal frame for $T_{1,0}$, and $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ be the dual basis of $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. By (1.12), it holds that

$$\text{(3.4)} \quad (\pi_\ast)L_\alpha = \sum_{\beta \in \langle n \rangle} e_\alpha^\beta Z_\beta.$$

Moreover, observe that

$$\text{(3.5)} \quad (\pi_\ast)[L_\alpha, L_{\overline{\alpha}}] = -2\sqrt{-1} T \mod \{Z_\alpha, Z_{\overline{\alpha}}\}_\alpha,$$

where we have meant by “$A = B \mod \{Z_\alpha, Z_{\overline{\alpha}}\}_\alpha$” that $A = B + \sum_{\alpha \in \langle n \rangle} a^\alpha Z_\alpha + \sum_{\alpha \in \langle n \rangle} b^{\overline{\alpha}} Z_{\overline{\alpha}}$ for some $a^\alpha, b^{\overline{\alpha}} \in \mathbb{C}$. Namely, recall that

$$d\theta(Z, W) = \frac{1}{2}(Z(\theta(W)) - W(\theta(Z)) - \theta([Z, W])).$$

Since $\theta(T_{1,0} \oplus T_{0,1}) = 0$, it holds that

$$\text{(3.6)} \quad \theta([Z_\alpha, Z_{\overline{\beta}}]) = -2\sqrt{-1} L\theta(Z_\alpha, Z_{\overline{\beta}}) = -2\sqrt{-1} \delta_{\alpha\beta}.$$

Hence

$$\text{(3.7)} \quad [Z_\alpha, Z_{\overline{\beta}}] = -2\sqrt{-1} \delta_{\alpha\beta} T \mod \{Z_\alpha, Z_{\overline{\beta}}\}_\alpha,$$

which yields that (3.5) holds. 

(3.3) follows from (3.1) and (3.5).

**Remark 3.8.** By Theorem 3.2, a bounded solution to the heat equation (3.1) can be written as

$$u(t,x) = E_x[f(X(t))] = \int_M f(y)p(t,x,y)\psi(dy).$$
We next investigate stochastic line integrals. Let $\Xi$ be a 1-form on $M$, which, under the imbedding made in the assumption (H), can be extended to a 1-form on $\mathbb{R}^k$ such that its derivatives of all orders are bounded.

Denote by $\int_{X[0,t]} \Xi$ the stochastic line integral of $\Xi$ along $\{X_t\}_{t \geq 0}$ from time 0 to $t$. For definition, see [8]. It is easily checked that

$$\int_{X[0,t]} \Xi = \sum_{A \in \langle \langle n \rangle \rangle \setminus \{0\}} \int_0^t (\pi^* \Xi)_{\pi(s)}(L_A) \circ dB^A(s),$$

where $\pi^* \Xi$ is the pull-back of $\Xi$ through $\pi: U(T_{1,0}) \to M$. Thus, $\{\tilde{\tau}(t) = (\tau(t), \int_{X[0,t]} \Xi)\}_{t \geq 0}$ obeys the SDE

$$d\tilde{\tau}(t) = \sum_{A \in \langle \langle n \rangle \rangle \setminus \{0\}} \tilde{L}_A(\tilde{\tau}(t)) \circ dB^A(t),$$

where $\tilde{L}_A$'s are vector fields on $U(T_{1,0}) \times \mathbb{R}$ defined by

$$\tilde{L}_A = L_A + (\pi^* \Xi)(L_A) \frac{\partial}{\partial \xi},$$

$\xi$ being the coordinate on $\mathbb{R}$.

For $x \in M$, take a local orthonormal frame $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$ for $T_{1,0}$ on $U$, and set $\Xi_A = \Xi(Z_A)$ for $A \in \langle \langle n \rangle \rangle \setminus \{0\}$. For $A_1, \ldots, A_m \in \langle \langle n \rangle \rangle \setminus \{0\}$, define $\Phi_{A_1,\ldots,A_m}(\Xi): U \to \mathbb{C}$ successively by

$$\Phi_{A_1}(\Xi) = \Xi_{A_1} \quad \text{and} \quad \Phi_{A_1,\ldots,A_m}(\Xi) = Z_{A_1} \Phi_{A_2,\ldots,A_m}(\Xi) - [Z_{A_2}, \ldots, [Z_{A_{m-1}}, Z_{A_m}] \ldots] \Xi_{A_1}.$$  

**Theorem 3.9.** Suppose that (H) holds and for each $x \in M$ there exists $A_1, \ldots, A_m \in \langle \langle n \rangle \rangle \setminus \{0\}$ such that $\Phi_{A_1,\ldots,A_m}(\Xi)(x) \neq 0$. Then the distribution of $\int_{X[0,t]} \Xi$ under $P_x$ admits a smooth density function with respect to the Lebesgue measure on $\mathbb{R}$ for every $x \in M$.

**Proof.** Under the same notation as used in (1.12), set $(f^\beta_\alpha)_{\alpha,\beta \in \langle n \rangle} = (e^\beta_\alpha)^{-1} \Xi_{\alpha,\beta \in \langle n \rangle}$ and define locally

$$\widehat{\tilde{L}}_\alpha = \sum_{\beta \in \langle n \rangle} f^\beta_\alpha \tilde{L}_\beta.$$

Then it is easily seen that

$$\text{span}_C \{(\widehat{\tilde{L}}_A)_r, ([\widehat{\tilde{L}}_{A_1}, \ldots, [\widehat{\tilde{L}}_{A_{m-1}}, \widehat{\tilde{L}}_{A_m}] \ldots)]_r); A, A_1, \ldots, A_m \in \langle \langle n \rangle \rangle \setminus \{0\}, m = 2, 3, \ldots \}$$

$$= \text{span}_C \{(\widehat{\tilde{L}}_A)_r, ([\widehat{\tilde{L}}_{A_1}, \ldots, [\widehat{\tilde{L}}_{A_{m-1}}, \widehat{\tilde{L}}_{A_m}] \ldots)]_r); A, A_1, \ldots, A_m \in \langle \langle n \rangle \rangle \setminus \{0\}, m = 2, 3, \ldots \},$$

and that

$$(\overline{\pi}_r)([\widehat{\tilde{L}}_{A_1}, \ldots, [\widehat{\tilde{L}}_{A_{m-1}}, \widehat{\tilde{L}}_{A_m}] \ldots])_r) = \Phi_{A_1,\ldots,A_m}(\Xi)(\pi(r)) \frac{\partial}{\partial \xi},$$

where $\overline{\pi}: U(T_{1,0}) \times \mathbb{R} \to \mathbb{R}$ is the natural projection. Hence, applying [17, Theorem 3.1], we obtain the desired result. \qed

**Remark 3.10.** Although $Z_A$’s in the definition of $\Phi_{A_1,\ldots,A_m}(\Xi)$ are all in $T_{1,0} \oplus T_{0,1}$, the direction $T$ appears in $\Phi_{A_1,\ldots,A_m}(\Xi)$’s because the expression $[Z_\alpha, Z_\beta]$ contains $T$-part by (3.7). Hence, for example, even if $\Xi_A(x) = 0$ for each $A \in \langle \langle n \rangle \rangle \setminus \{0\}$, the assumption $\Phi_{A_1,\ldots,A_m}(\Xi)(x) \neq 0$ may be satisfied.
4. Dirichlet problem

In this section, we study Dirichlet problems related to $\Delta_b$. For $f \in C(\partial G)$, we want to find $u_f \in C^2(G) \cap C(\overline{G})$ such that $\Delta_b u_f = 0$ and $u_f|_{\partial G} = f$. We first establish a weak solution in a probabilistic manner following Stroock and Varadhan [15]. As will be seen in Remark 4.8, we indeed obtain a classical solution stated above.

Let $\mathbb{X} = \{(\{X(t)\}_{t \geq 0}, P_x); x \in M\}$ be the diffusion process obtained in Theorem 2.5. Let $G$ be a relatively compact connected open set in $M$ with $C^3$ boundary. Define

$$\tau' = \inf\{t \geq 0; X(t) \notin \overline{G}\}.$$  

We shall show that

**Theorem 4.1.** For $f \in C(\partial G)$, define $u_f(x) = E_x[f(X(\tau'))]$. Then $u_f \in C(\overline{G})$ and satisfies that

$$\langle u_f, \Delta_b v \rangle_{\theta} = 0 \quad \text{for any } v \in C^\infty_0(G), \quad \text{and} \quad u_f = f \quad \text{on } \partial G.$$  

Due to the result by Stroock and Varadhan [15], the theorem is verified once we have established the following two lemmas.

**Lemma 4.2.** It holds that

$$\sup_{x \in \overline{G}} E_x[\tau'] < \infty.$$  

**Lemma 4.3.** Every boundary point is $\tau'$-regular, that is,

$$P_x(\tau' = 0) = 1, \quad x \in \partial G.$$  

**Proof of Lemma 4.2.** On account of [15, Remark 5.2], it suffices to show that

$$P_x(\tau' < T) > 0, \quad x \in \overline{G} \text{ and } T > 0.$$  

To do this, take a family $\{U_j\}_{j=1}^N$ of coordinate neighborhoods of $M$ such that $\overline{G} \subset \bigcup_{j=1}^N U_j$. Let $\Lambda = \{j; U_j \cap \partial G \neq \emptyset\}$. Take $j \in \Lambda$ and a local orthonormal frame $\{Z_\alpha\}_{\alpha \in \langle n \rangle}$ for $T_{1,0}$ on $U_j$. Then, by virtue of (2.4), we may assume that the part of $\{X(t)\}_{t \geq 0}$ on $U_j$ is governed by an SDE

$$dX(t) = \sum_{\alpha \in \langle n \rangle} (\sqrt{2} \text{Re} Z_\alpha(X(t)) \circ d\xi^\alpha(t) + \sqrt{2} \text{Im} Z_\alpha(X(t)) \circ d\eta^\alpha(t)) + b(X(t)) dt,$$  

where $(\xi^1(t), \eta^1(t), \ldots, \xi^n(t), \eta^n(t))$ is an $\mathbb{R}^{2n}$-valued Brownian motion and

$$b = - \sum_{\alpha, \beta \in \langle n \rangle} (\Gamma^\alpha_{\beta\gamma} Z_\alpha + \Gamma^\alpha_{\beta\gamma} \overline{Z_\alpha}).$$  

Due to (3.7), applying the support theorem (cf. [10, Theorem 3.2]), we obtain that

$$P_x(\tau' < T) > 0, \quad x \in U_j, j \in \Lambda, T > 0.$$  

12
For $U_k$ such that $k \notin \Lambda$ and $U_k \cap U_j \neq \emptyset$ for some $j \in \Lambda$, by the same reasoning as above, applying the support theorem again, we have that
\[
P_x(X(t) \text{ hits } U_j \text{ before } T) > 0, \quad x \in U_k, T > 0.
\]
Combined with (4.6) and the strong Markov property, this yields that
\[
P_x(\tau' < T) > 0, \quad x \in U_k, T > 0.
\]
Repeating this argument successively, we can conclude (4.4).

**Proof of Lemma 4.3.** Let $x \in \partial G$ and $U$ be a coordinate neighborhood of $x$. For a local orthonormal frame $\{Z_{\alpha}\}_{\alpha \in \langle n \rangle}$ for $T_{1,0}$ defined on $U$, we may and will assume that the part of $\{X(t)\}_{t \geq 0}$ on $U$ obeys the SDE (4.5).

Let $\varphi$ be a local defining function of $G$ around $x$; there is an open set $V$ containing $x$ such that $\varphi \in C^3(V)$, $V \cap G = \{ y \in V; \varphi(y) < 0 \}$, and $d\varphi(y) \neq 0$ for $y \in \partial G \cap V$. If either $(\text{Re} Z_{\alpha})\varphi(x) \neq 0$ or $(\text{Im} Z_{\alpha})\varphi(x) \neq 0$, then by [14, Corollary 4], $x$ is $\tau'$-regular. Now we suppose that
\[
(4.7) \quad (\text{Re} Z_{\alpha})\varphi(x) = (\text{Im} Z_{\alpha})\varphi(x) = 0, \quad \alpha \in \langle n \rangle.
\]

Since $\{\text{Re} Z_{\alpha}, \text{Im} Z_{\alpha}, T\}_{\alpha \in \langle n \rangle}$ forms a local basis of $TM$ on $U$, this implies that $T\varphi(x) \neq 0$. Moreover, in conjunction with (3.7), (4.7) also implies that
\[
[\text{Re} Z_{\alpha}, \text{Im} Z_{\alpha}]\varphi(x) = T\varphi(x) \neq 0.
\]
Hence it follows that, for each $\alpha$, either $(\text{Re} Z_{\alpha})(\text{Im} Z_{\alpha})\varphi(x) \neq 0$ or $(\text{Im} Z_{\alpha})(\text{Re} Z_{\alpha})\varphi(x) \neq 0$ and that a matrix
\[
\begin{pmatrix}
(\text{Re} Z_{\alpha})(\text{Re} Z_{\beta})\varphi(x) & (\text{Re} Z_{\alpha})(\text{Im} Z_{\beta})\varphi(x) \\
(\text{Im} Z_{\alpha})(\text{Re} Z_{\beta})\varphi(x) & (\text{Im} Z_{\alpha})(\text{Im} Z_{\beta})\varphi(x)
\end{pmatrix}_{\alpha, \beta \in \langle n \rangle}
\]
is not symmetric. Applying [14 Corollary 7], we see that $x$ is $\tau'$-regular.

**Remark 4.8.** Since $\Delta_b$ is hypoelliptic ([3 Theorem 2.1]), that is, if $\Delta_b v = g$ and $g \in C^\infty(U)$ then $v \in C^\infty(U)$, $u_f$ is a classical solution to the Dirichlet problem, namely it holds that $u_f \in C^\infty(G) \cap C(\overline{G})$, $\Delta_b u_f = 0$ and $u_f|_{\partial G} = f$.

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