Some comparison theorems for Kähler manifolds

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Abstract. In this work, we will verify some comparison results on Kähler manifolds. They are: complex Hessian comparison for the distance function from a closed complex submanifold of a Kähler manifold with holomorphic bisectional curvature bounded below by a constant, eigenvalue comparison and volume comparison in terms of scalar curvature. This work is motivated by comparison results of Li and Wang (J Differ Geom 69(1):43–47, 2005).

1. Introduction

In this work, we will study some comparison theorems on Kähler manifolds. There is a well-known Hessian comparison for the distance function on Riemannian manifolds in terms of lower bound of sectional curvature. It is expected that for Kähler manifolds the lower bound of sectional curvature can be replaced by the lower bound of bisectional curvature to obtain a complex Hessian comparison for the distance function. In fact, in [12], Li and Wang gave a sharp upper estimate for the Laplacian of the distance function from a point. They also gave a sharp upper estimate for the complex Hessian for the distance function in the case that the lower bound $K$ of the bisectional curvature is zero, i.e., on Kähler manifolds with non-negative holomorphic bisectional curvature. This last result was also proved by Cao and Ni [5] using Li–Yau–Hamilton Harnack type inequality for the heat equation. In this note, we shall verify this complex Hessian comparison for general lower bound $K$ and show that the complex Hessian of the distance function is bounded above by that in the complex space forms. See Theorem 2.1 and 2.2 for more details.

Our next result is motivated by the well-known Licherowicz–Obata theorem on Riemannian manifolds, which says that if $(M^m, g)$ is a compact Riemannian manifold with Ricci curvature bounded below by $(m - 1)K$, where $K > 0$ is a constant, then the first nonzero eigenvalue $\lambda_1$ satisfies $\lambda_1 \geq mK$ and equality holds only if $(M, g)$ is isometric to the standard sphere of radius $1/\sqrt{K}$. It is well-known that if $(M^n, g)$ is a Kähler manifold such that Ricci curvature is such that $R_{\alpha\bar{\beta}} \geq kg_{\alpha\bar{\beta}}$ for
some constant $k > 0$, then the first nonzero eigenvalue $\lambda_1$ of the complex Laplacian is at least $k$ (see [9]). Our next result is the following:

Let $(M^n, g)$ be as above. Suppose the Kähler form is in the first Chern class. If $\lambda_1 = k$, then $M$ is Kähler–Einstein.

As a corollary, when $(M^n, g)$ is a Kähler manifold with positive holomorphic bisectional curvature such that the Ricci curvature is bounded below by $n + 1$, then $\lambda_1 \geq n + 1$ and equality holds if and only if $(M^n, g)$ is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini–Study metric (of constant holomorphic sectional curvature 2). As an application we obtain a partial result on the equality case for the diameter estimate of Li and Wang [12] on compact Kähler manifolds with holomorphic bisectional curvature bounded below by 1. See Corollary 3.2.

In [12], it was proved that if a compact Kähler manifold has bisectional curvature bounded below by 2 (see Definition 2.1), then the volume of the manifold is less than or equal to the volume of $\mathbb{CP}^n$ with the Fubini–Study metric. Our last result is an observation to relax their conditions by replacing the lower bound of the bisectional curvature by the lower bound of the scalar curvature, with the assumption that the bisectional curvature is positive. In fact, we prove:

Let $(M^n, g)$ be a compact manifold with positive holomorphic bisectional curvature. Suppose the scalar curvature $k_2 \geq R \geq k_1 > 0$ for some constants $k_1$ and $k_2$. Then the volume $V(M, g)$ of $M$ satisfies $V(\mathbb{CP}^n, h_{k_2}) \leq V(M, g) \leq V(\mathbb{CP}^n, h_{k_1})$. Moreover, if one of the inequalities is an equality, then $(M, g)$ is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini–Study metric.

The result is a consequence of a more general result, which is related to a conjecture of Schoen [17] which states: If $(M^n, h)$ is a closed hyperbolic manifold and $g$ is another metric on $M$ with scalar curvature $R(g) \geq R(h)$, then $V(g) \geq V(h)$. The conjecture was proved for $n = 3$ by Perelmann [15, 16]. We will compare the volume of a compact Kähler manifold with the volume of a related Kähler-Einstein metric in terms of upper and lower bounds of the scalar curvature. See Proposition 4.1.

The paper is organized as follows: In Sect. 2, we will study comparison of the complex Hessian for the distance function. In Sect. 3, we will study eigenvalue comparison and in Sect. 4, we will study volume comparison.

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2. Complex Hessian comparisons

Let $M^n$ be a complex manifold with complex dimension $n$ and let $J$ be the complex structure. Suppose $g$ is a Hermitian metric such that $(M, J, g)$ is Kähler. Suppose $\{e_1, e_2, \ldots, e_n\}$ is a frame on $T^{(1,0)}(M)$, let $g_{\alpha\bar{\beta}} := g(e_\alpha, \bar{e}_\beta)$. We also write $g(X, \bar{Y})$ as $\langle X, \bar{Y} \rangle$ and $||X||^2 = g(X, \bar{X})$ for $X, Y \in T^{(1,0)}(M)$. Following [12], we have the following definition.

Definition 2.1. Let $(M, J, g)$ be a Kähler manifold. We say that the holomorphic bisectional curvature of $M$ is bounded below by a constant $K$, denoted by $BK_M \geq K$, if