Kinetic theory of two-dimensional point vortices with collective effects

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Received 14 November 2011
Accepted 24 January 2012
Published 27 February 2012

Abstract. We develop a kinetic theory of point vortices in two-dimensional hydrodynamics taking collective effects into account. We first recall the quasilinear theory of Dubin and O’Neil (1988 Phys. Rev. Lett. 60 1286) based on the Klimontovich equation and leading to a Lenard–Balescu-type kinetic equation for axisymmetric flows. When collective effects are neglected, it reduces to the Landau-type kinetic equation obtained independently in our previous papers (Chavanis 2001 Phys. Rev. E 64 026309; 2008 Physica A 387 1123) for more general flows. We also consider the relaxation of a test vortex in a ‘sea’ (bath) of field vortices. Its stochastic motion is described in terms of a Fokker–Planck equation. We determine the diffusion coefficient and the drift term by explicitly calculating the first-and second-order moments of the radial displacement of the test vortex from its equations of motion, taking collective effects into account. This generalizes the expressions obtained in our previous papers. We discuss the scaling with $N$ of the relaxation time for the system as a whole and for a test vortex in a bath.

Keywords: transport processes/heat transfer (theory), stochastic processes (theory), vortex matter (theory), kinetic theory of gases and liquids
1. Introduction

The point vortex model [1] is interesting not only because it provides a simplified model of two-dimensional (2D) hydrodynamics with potential applications to large-scale geophysical and astrophysical flows, but also because it constitutes a fundamental example of systems with long-range interactions. Systems with long-range interactions are numerous in nature (self-gravitating systems, non-neutral plasmas, 2D vortices, etc) and...
they present striking features that are absent in systems with short-range interactions [2]–[4]. Their dynamics and thermodynamics are actively studied at present [5]. The statistical mechanics of the point vortex gas was first considered by Onsager [6] who discovered that negative temperature states are possible for this system\(^1\). At negative temperatures (corresponding to high energies), point vortices of the same sign have the tendency to cluster into ‘supervortices’, similar to the large-scale vortices (e.g. Jupiter’s Great Red Spot) observed in the atmosphere of giant planets. The qualitative arguments of Onsager [6] were developed more quantitatively in a mean-field approximation by Joyce and Montgomery [8,9], Kida [10] and Pointin and Lundgren [11,12], and by Onsager himself in unpublished notes [13]. The statistical theory predicts that the point vortex gas should relax towards an equilibrium state described by the Boltzmann distribution. Specifically, the equilibrium stream function is the solution of a Boltzmann–Poisson equation. At positive temperatures, the Boltzmann–Poisson equation is similar to the one appearing in the theory of electrolytes in plasma physics (like-sign vortices ‘repel’ each other) [14]. At negative temperatures, it is similar to the one appearing in the statistical mechanics of stellar systems (like-sign vortices ‘attract’ each other) [15,16].

Several mathematical works [17]–[21] have shown how a proper thermodynamic limit could be rigorously defined for the point vortex gas (in the Onsager picture). It is shown that the mean-field approximation becomes exact in the limit \( N \to +\infty \) with \( \gamma \sim 1/N \) (where \( N \) is the number of point vortices and \( \gamma \) is the circulation of a point vortex). However, the equilibrium statistical theory is based on some assumptions of ergodicity (or efficient mixing) that may not be fulfilled [22]. Therefore, it is not clearly established that the point vortex gas will relax towards the Boltzmann distribution for \( t \to +\infty \). To settle this issue, and to determine the relaxation time (in particular its scaling with \( N \)), we must develop a kinetic theory of point vortices.

A kinetic theory has been developed by Dubin and O’Neil [23] in the context of non-neutral plasmas under a strong magnetic field, a system isomorphic to the point vortex gas. They consider axisymmetric mean flows and take collective effects into account. Their approach is formal (but powerful) and uses the methods developed in plasma physics. They start from the Klimontovich equation, make a quasilinear approximation and solve the resulting equations by using Fourier–Laplace transforms in the complex plane. They obtain a kinetic equation (43) that is the counterpart of the Lenard–Balescu [24,25] equation in plasma physics. Their work has been continued in [26]–[29] in different directions.

We independently developed a kinetic theory of point vortices [30]–[34] by using an analogy with the kinetic theory of stellar systems [35]–[39]. The originality of our approach is to remain in physical space so as to treat flows that are not necessarily axisymmetric. We obtained a generalized Landau equation (46) that applies to arbitrary mean flows. We derived this kinetic equation by different methods such as the projection operator formalism [31], the BBGKY hierarchy [32] and the quasilinear theory [32]. Interestingly, the structure of this kinetic equation bears a clear physical meaning in terms of generalized Kubo relations. This equation is valid at the order of \( 1/N \) and, for \( N \to +\infty \), it reduces to the (smooth) 2D Euler equation (13) that describes the collisionless evolution of the point vortex gas. For axisymmetric mean flows, we obtained a simpler and more explicit kinetic

\(^1\) His argument for the existence of negative temperatures was given two years before Purcell and Pound [7] reported the presence of negative ‘spin temperatures’ in an experiment on nuclear spin systems.
equation (45) that is the counterpart of the Landau [40] equation in plasma physics. Our formalism [32,34] is simple and clearly shows the structure of the kinetic equation and the physical origin of its different components. Furthermore, it avoids the use of Fourier–Laplace transforms and provides a direct and transparent derivation of the kinetic equation without going into the complex plane (which, somehow, hides the basic physics). The derived kinetic equation conserves the circulation, the energy and the angular momentum. It also increases monotonically the Boltzmann entropy ($H$ theorem). The collisional evolution is due to a condition of resonance between the orbits of distant vortices rotating with the same angular velocity and the relaxation stops when the profile of angular velocity becomes monotonic, even if the system has not reached the Boltzmann distribution of statistical equilibrium [41]. Since the kinetic equation is valid at the order of $1/N$, this ‘kinetic blocking’ implies that the relaxation time for an axisymmetric distribution of point vortices is larger than $N t_D$, where $t_D$ is the dynamical time. This is at variance with the case of 3D plasmas and 3D stellar systems where the Landau or Lenard–Balescu equations relax towards the Boltzmann distribution on a timescale $(N/\ln N) t_D$ (in plasma physics, $N$ represents the number of electrons in the Debye sphere) [42,43]. In fact, there is no proof so far from the kinetic theory that the system of point vortices relaxes towards the Boltzmann statistical equilibrium for $t \to +\infty$. To settle this issue, we need to develop the kinetic theory at higher orders (taking into account more complex three-body, four-body, etc, correlations between vortices). In contrast, for a non-axisymmetric evolution, there are potentially more resonances and the relaxation time can be shorter, of the order of the natural timescale $N t_D$ [34]. We also considered the stochastic motion of a test vortex in a ‘sea’ (bath) of field vortices [30]–[34]. It can be described in terms of a Fokker–Planck equation involving a diffusion term and a drift term. For a thermal bath, they are connected to each other by an Einstein relation. The diffusion coefficient (resp. drift term) is proportional to the vorticity distribution (resp. to the gradient of the vorticity distribution) of the field vortices and inversely proportional to the local shear, a feature first noted in [30]. The distribution of the test vortex relaxes towards the distribution of the field vortices (bath) on a timescale $(N/\ln N) t_D$.

A formal limitation of our approach is that it neglects collective effects. This is not so crucial as long as we are just interested in the main properties of the kinetic equation that do not sensibly depend on collective effects (as we shall see, collective effects essentially amount to replacing the ‘bare’ potential of interaction by a ‘dressed’ potential of interaction, without altering the overall structure of the kinetic equation). However, collective effects may change the results quantitatively. In this paper, we generalize our kinetic theory so as to take collective effects into account. We also extend the kinetic theory to arbitrary potentials of interaction between point vortices (in particular, they may include screening effects that are important in geophysics [44]). The present kinetic theory is, however, restricted to axisymmetric flows. In section 2, we recall basic features of the point vortex model and set the notations. We also distinguish the 2D Euler equation (similar to the Klimontovich equation in plasma physics) satisfied by the discrete vorticity field and the 2D Euler equation (similar to the Vlasov equation in plasma physics) satisfied by the smooth vorticity field in the collisionless regime. In section 3, we consider the collisional evolution of the system as a whole and re-derive the Lenard–Balescu-type kinetic equation obtained in [23]. This is done for the sake of completeness and in order to obtain intermediate results that will be useful in the following. We also show
the connection with the Landau-type kinetic equation derived in [30]–[34] with another method when collective effects are neglected\(^2\). In section 4, we consider the relaxation of a test vortex in a bath of field vortices and derive a Fokker–Planck equation. We determine the diffusion coefficient and the drift term by explicitly calculating the first- and second-order moments of the radial displacement of the test vortex from its equations of motion, taking collective effects into account. This generalizes the expressions obtained in our previous works [30]–[34] where collective effects are neglected. Finally, we show the connection between the Fokker–Planck equation describing the relaxation of a test vortex in a bath and the kinetic equation describing the evolution of the system as a whole and compare the relaxation time in these two situations.

There exist many analogies between systems with long-range interactions although they are of a very different nature. We have already mentioned analogies between plasmas, stellar systems and point vortices (the analogy between the statistical mechanics and the kinetic theory of stellar systems and 2D vortices is developed in [34, 46]–[48]). There also exist analogies with a toy model of systems with long-range interactions, called the Hamiltonian mean-field (HMF) model [49]. The kinetic theory of this model is developed in [5, 34, 50]–[56] and it presents features that are similar to those observed earlier for point vortices [23, 30, 31]. In particular, for spatially homogeneous systems, the Lenard–Balescu collision term vanishes in one dimension due to the absence of resonances (a result known long ago for 1D plasmas [57, 58] and rediscovered in the context of the HMF model [52, 53]) so that the relaxation time of the spatially homogeneous HMF model is larger than \(t_\text{D}N\) (it can be reduced to \(Nt_\text{D}\) if the system is spatially inhomogeneous due to additional resonances [34, 59, 60]). On the other hand, the relaxation of a test particle in a bath is described by a Fokker–Planck equation involving a term of diffusion and a term of friction [52, 53]. This is similar to the Fokker–Planck equation derived by Chandrasekhar [36, 37] in stellar dynamics and by Chavanis [30]–[34] for the point vortex system. This is also a particular case (in one dimension and for a potential of interaction truncated to one Fourier mode) of the general Fokker–Planck equation derived by Hubbard [61] in plasma physics. In order to stress the analogies between these different topics, we shall use a presentation that is similar to the one given in the review of Campa et al [5] on long-range interacting systems or in our pedagogical paper [62] on kinetic theories.

2. The point vortex model

Let us consider a system of point vortices with individual circulation \(\gamma\) moving on the infinite plane. Their dynamics is described by the Hamilton–Kirchhoff [1, 63] equations

\[
\gamma \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \gamma \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i},
\]

\(^2\) Recently, Sano [45] confirmed our generalized Landau-type kinetic equation (46) by using a BBGKY approach similar to the one developed in [52]. He also proposed a Lenard–Balescu-type equation, taking into account collective effects, by using the theory of the Fredholm integral equation. However, he argues that this equation reduces to the Landau-type equation for large \(N\) although our present work shows that both equations (Landau and Lenard–Balescu-type) are valid at the same order \(1/N\). The difference is that the Landau-type equation is an approximation of the Lenard–Balescu-type equation since it ignores collective effects.
with the Hamiltonian
\[ H = \gamma^2 \sum_{i<j} u(|\mathbf{r}_i - \mathbf{r}_j|). \]  

We note that the coordinates \( x \) and \( y \) of the point vortices are canonically conjugate. The standard potential of interaction between point vortices is a solution of the Poisson equation \( \Delta u = -\delta \). In an infinite domain, it is given by \( u(|\mathbf{r} - \mathbf{r}'|) = -(1/2\pi) \ln |\mathbf{r} - \mathbf{r}'| \). However, we shall let the function \( u \) be arbitrary in order to treat more general situations. For example, in the quasigeostrophic (QG) approximation of geophysical fluid dynamics, the Poisson equation is replaced by the screened Poisson equation \( \Delta u - k_R^2 u = -\delta \), where \( k_R^{-1} \) is the Rossby radius [44]. In an infinite domain, the potential of interaction is given by \( u(|\mathbf{r} - \mathbf{r}'|) = (1/2\pi) K_0(k_R|\mathbf{r} - \mathbf{r}'|) \). Equation (1) can be rewritten in vectorial form:
\[ \gamma \frac{d\mathbf{r}_i}{dt} = -\mathbf{z} \times \nabla_i H, \]  

where \( \mathbf{z} \) is a unit vector normal to the plane on which the vortices move. In an infinite domain, the point vortex system conserves the total circulation \( \Gamma = N\gamma \), the energy \( E = H \) and the angular momentum \( L = \sum \gamma r_i^2 \) (the centre of mass \( \mathbf{R} = \sum \gamma \mathbf{r}_i \) is also conserved and taken as the origin \( O \) of the system of coordinates). We introduce the discrete vorticity field
\[ \omega_d(\mathbf{r}, t) = \gamma \sum_i \delta(\mathbf{r} - \mathbf{r}_i(t)). \]  

Differentiating this expression with respect to time and using the equation of motion (3), we get
\[ \frac{\partial \omega_d}{\partial t} = \sum_i (\mathbf{z} \times \nabla_i H) \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_i(t)). \]  

This equation can be rewritten as
\[ \frac{\partial \omega_d}{\partial t} = \nabla \cdot \sum_i (\mathbf{z} \times \nabla_i H) \delta(\mathbf{r} - \mathbf{r}_i(t)) = \gamma \nabla \cdot \sum_i (\mathbf{z} \times \nabla \psi_d) \delta(\mathbf{r} - \mathbf{r}_i(t)), \]  

where
\[ \psi_d(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|) \omega_d(\mathbf{r}', t) \, d\mathbf{r}', \]  

is the discrete stream function induced by the discrete vorticity field (we have used the \( \delta \) function to replace \( \nabla \psi_d(\mathbf{r}_i(t), t) \) by \( \nabla \psi_d(\mathbf{r}, t) \) in the last equality of equation (6)). Introducing the discrete velocity field \( \mathbf{u}_d = -\mathbf{z} \times \nabla \psi_d \), we obtain
\[ \frac{\partial \omega_d}{\partial t} + \nabla(\omega_d \mathbf{u}_d) = 0. \]  

Finally, using the incompressibility condition \( \nabla \cdot \mathbf{u}_d = 0 \), we can rewrite this equation in the form
\[ \frac{\partial \omega_d}{\partial t} + \mathbf{u}_d \cdot \nabla \omega_d = 0. \]
This equation is exact and contains the same information as the Hamiltonian system (1)–(2). This is the counterpart of the Klimontovich equation in plasma physics. This is also the 2D Euler equation from which the point vortex model is issued.

We now introduce a smooth vorticity field \( \omega(r, t) = \langle \omega_d(r, t) \rangle \) corresponding to an average of \( \omega_d(r, t) \) over a large number of initial conditions. We then write \( \omega_d = \omega + \delta \omega \), where \( \delta \omega \) denotes the fluctuations around the smooth distribution (i.e. the deviation of the random function \( \omega_d \) from its mean). Similarly, we write \( \psi_d = \psi + \delta \psi \) and \( u_d = u + \delta u \). Substituting these decompositions in equation (9), we obtain

\[
\frac{\partial \omega}{\partial t} + \nabla \cdot \omega + u \cdot \nabla \omega + u \cdot \nabla \delta \omega + \delta u \cdot \nabla \omega + \delta u \cdot \nabla \delta \omega = 0.
\]

Taking the average of this equation over the initial conditions, we get

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nabla \cdot \langle \delta \omega \delta u \rangle,
\]

where the right-hand side can be interpreted as a ‘collision’ term. Of course, it does not correspond to direct collisions between vortices but to distant collisions, or correlations, due to finite \( N \) effects. Subtracting this expression from equation (10), we obtain

\[
\frac{\partial \delta \omega}{\partial t} + u \cdot \nabla \delta \omega + \delta u \cdot \nabla \omega = \nabla \cdot \langle \delta \omega \delta u \rangle - \nabla \cdot \langle \delta \omega \delta u \rangle.
\]

These equations are still exact since no approximation has been made so far. We now consider the thermodynamic limit \( N \to +\infty \) with \( \gamma \sim 1/N \). In this limit, we have \( \omega \sim 1 \), \( \delta \omega \sim 1/\sqrt{N} \) and \( \delta \psi \sim 1/\sqrt{N} \). We can therefore consider an expansion of the equations of the problem in terms of the small parameter \( 1/N \), which is a measure of the ‘graininess’ of the distribution (as \( 1/N \) approaches zero, collisional effects disappear).

Considering equation (11), we see that the collision term scales like \( 1/N \). Therefore, for \( N \to +\infty \), equation (11) reduces to the (smooth) 2D Euler equation:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0, \quad u = -z \times \nabla \psi, \quad \psi(r, t) = \int u(|r - r'|)\omega(r', t) \, dr'.
\]

The 2D Euler equation is obtained when ‘collisions’ (more properly, granular effects, discreteness effects or finite \( N \) corrections) between point vortices are neglected. The 2D Euler equation can also be obtained from the BBGKY hierarchy by neglecting correlations between point vortices. This is valid, for a fixed interval of time, when \( N \to +\infty \). In that case, the mean-field approximation becomes exact and the \( N \)-body distribution function can be factorized in a product of \( N \) one-body distributions, resulting in equation (13). The 2D Euler equation (13) is the counterpart of the Vlasov equation used in plasma physics, in stellar dynamics and for the HMF model. Starting from an unsteady or unstable initial condition, the 2D Euler equation is known to develop a complicated mixing process leading to the formation of a quasistationary state (QSS) on a coarse-grained scale. This QSS has the form of a large-scale vortex. This process takes place on a very short timescale, of the order of the dynamical time \( t_D \sim \omega^{-1} \), where \( \omega \sim \Gamma/R^2 \) is the typical vorticity (\( R \) is the system size). This process, which is purely collisionless and driven by mean-field effects, is known as violent relaxation. A statistical theory of the 2D Euler equation has been developed by Miller and Robert and Sommeria to predict the QSS that results from violent relaxation assuming ergodicity.

\[\text{doi:10.1088/1742-5468/2012/02/P02019} \]
This is the counterpart of the statistical theory of the Vlasov equation introduced by Lynden-Bell [69] in astrophysics to describe the violent relaxation of collisionless stellar systems and the formation of elliptical galaxies [43]. The process of violent relaxation towards a non-Boltzmannian QSS has also been extensively studied for the HMF model [5]. In some cases, violent relaxation is incomplete due to a lack of efficient mixing (see appendix A).

On longer timescales, ‘collisions’ (more properly, correlations) between point vortices develop and the system deviates from the 2D Euler dynamics. For $t \to +\infty$, we expect that the system will reach a statistical equilibrium state. In the thermodynamic limit $N \to +\infty$ with $\gamma \sim 1/N$, the mean-field approximation is exact [17]–[21]. The statistical equilibrium state in the microcanonical ensemble is obtained by maximizing the Boltzmann entropy $S = -\int (\omega/\gamma) \ln(\omega/\gamma) \, dr$ at fixed circulation $\Gamma = \int \omega \, dr$, energy $E = (1/2) \int \omega \psi \, dr$ and angular momentum $L = \int \omega r^2 \, dr$ [8, 41]. This variational principle determines the most probable macrostate $\omega(r)$. Writing the first-order variations as $\delta S - \beta \delta E - \alpha \delta \Gamma - (1/2) \beta \Omega_L \delta L = 0$, where $\beta = 1/T$, $\alpha$ and $\Omega_L$ are Lagrange multipliers, we obtain the Boltzmann distribution

$$
\omega = Ae^{-\beta \psi^*}, \quad \psi(r) = \int u(|r-r'|)\omega(r') \, dr',
$$

where $\psi^* = \psi + (1/2)\Omega_L r^2$ is the relative stream function taking into account the invariance by rotation of the system. The flow is steady in the frame rotating with angular velocity $\Omega_L$. For the ordinary interaction between point vortices, the second part of equation (14) can be written $\Delta \psi = -\omega$ and the equilibrium stream function is a solution of the Boltzmann–Poisson equation $\Delta \psi = -Ae^{-\beta \psi^*}$ with the gauge condition $\psi + \Gamma/2\pi \ln r \to 0$ for $r \to +\infty$. The temperature of the point vortices can be positive or negative [6]. At positive temperatures $\beta > 0$, the interaction between like-sign vortices is ‘repulsive’ like between electric charges in an electrolyte [14]. In that case, the spreading of the point vortices is prevented by the conservation of the angular momentum. At negative temperatures $\beta < 0$, the interaction between like-sign vortices is ‘attractive’ like between stars in a stellar system [15, 16]. In that case, the system forms an ordered structure (a large-scale vortex) similar to globular clusters in astrophysics [43]. We must remember, however, that the Boltzmann statistical theory is based on an assumption of ergodicity (or efficient mixing) and on the postulate that all the accessible microstates are equiprobable. In fact, the point vortex gas is non-ergodic in a strict sense [22] and there is no guarantee that it will mix sufficiently well to justify the postulate of equiprobability of the accessible microstates. If we want to prove that the point vortex gas truly reaches the Boltzmann distribution (14), and if we want to determine the relaxation time (in particular its scaling with the number $N$ of point vortices), we must develop a kinetic theory of point vortices. Similar kinetic theories have been developed for other systems with long-range interactions such as stellar systems, Coulombian plasmas and the HMF model (for recent reviews see, e.g., [5, 34, 56, 62]).

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3] Equation (14), or the Boltzmann–Poisson equation, determines all the critical points of constrained entropy, i.e. the distributions that cancel the first-order variations. Of course, among these solutions, only entropy maxima are physically relevant. We must therefore consider the sign of the second-order variations of entropy and discard entropy minima and saddle points. If several entropy maxima are found for the same values of the constraints, we must distinguish fully stable states (global entropy maxima) and metastable states (local entropy maxima).
3. Collisional evolution of the system as a whole

3.1. Lenard–Balescu and Landau-type kinetic equations

As we have previously indicated, the ‘collisions’ between point vortices can be neglected for times much shorter than $Nt_D$, where $t_D$ is the dynamical time. If we want to describe the evolution of the system on a longer timescale, we must take finite $N$ corrections into account. At the order of $1/N$, we can neglect the quadratic term on the right-hand side of equation (12). Indeed, the left-hand side is of order $1/\sqrt{N}$ and the right-hand side is of order $1/N \ll 1/\sqrt{N}$. We therefore obtain the set of coupled equations:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nabla \cdot \langle \delta \omega \delta u \rangle,
\]

\[
\frac{\partial \delta \omega}{\partial t} + u \cdot \nabla \delta \omega + \delta u \cdot \nabla \omega = 0.
\]

They form the starting point of the quasilinear theory. It can be shown that these equations describe the evolution of the system under the effect of two-body collisions (higher order correlations are neglected). If we restrict ourselves to axisymmetric mean flows, we can write $u(r,t) = u(r,t) \hat{e}_\theta$ with $u(r,t) = -\partial \psi/\partial r(r,t) = \Omega(r,t)r$, where $\Omega(r,t) = r^{-2} \int_0^r \omega(r',t)r'dr'$ is the angular velocity field (see section 6.1 of [41]). In that case, equations (15)–(16) reduce to the coupled equations:

\[
\frac{\partial \omega}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \omega}{\partial \theta} \frac{\partial \psi}{\partial \theta} \right),
\]

\[
\frac{\partial \delta \omega}{\partial t} + \Omega \frac{\partial \delta \omega}{\partial \theta} + \frac{1}{r} \frac{\partial \delta \psi}{\partial \theta} \frac{\partial \omega}{\partial r} = 0.
\]

These equations are valid as long as the axisymmetric distribution is Euler-stable so that it evolves under the sole effect of ‘collisions’ and not because of dynamical instabilities. We shall assume that the fluctuations evolve rapidly compared to the transport timescale, so that time variations of $\omega, \psi$ and $\Omega$ can be neglected in the calculation of the collision term (Bogoliubov ansatz). Therefore, for the purpose of solving equation (18) and obtaining the correlation function $\langle \delta \omega \delta \psi \rangle$, we shall regard $\omega(r)$ as constant in time. Actually, the time dependence of $\omega$ is given by equation (17). Only the fluctuation term $\langle \delta \omega \delta \psi \rangle$ drives $\omega$ and then its evolution is much slower than the evolution of $\delta \omega$. Indeed, $\omega$ changes on a timescale of the order of $Nt_D$ (or larger) while $\langle \delta \omega \delta \psi \rangle$ relaxes, by a form of Landau damping, to its asymptotic value on a much shorter timescale, of the order of the dynamical time $t_D$. After the correlation function has been obtained as a functional of $\omega$, the time dependence of $\omega$ may be reinserted in the kinetic equation. This is an adiabatic hypothesis that is valid for $N \gg 1$. With this approximation, equation (18) can be solved with the aid of Fourier–Laplace transforms and the collision term in equation (17) can be explicitly calculated. This is the kinetic approach developed by Dubin and O’Neil [23]. We shall recall the main steps of their analysis by using a close parallel with the derivation of the Lenard–Balescu equation in plasma physics [5, 15, 42, 62, 64, 70, 71]. This will allow us to set the notations and derive useful intermediate results that will be needed in the following.

doi:10.1088/1742-5468/2012/02/P02019

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The Fourier–Laplace transform of the fluctuations of the vorticity field $\delta \omega$ is defined by

$$
\delta \tilde{\omega}(n, r, \sigma) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{+\infty} dt \, e^{-i(n\theta - \sigma t)} \delta \omega(\theta, r, t).
$$

(19)

This expression for the Laplace transform is valid for $\text{Im}(\sigma)$ sufficiently large. For the remaining part of the complex $\sigma$ plane, it is defined by an analytic continuation. The inverse transform is

$$
\delta \omega(\theta, r, t) = \sum_{n=-\infty}^{+\infty} \int_{\mathcal{C}} \frac{d\sigma}{2\pi} e^{i(n\theta - \sigma t)} \delta \tilde{\omega}(n, r, \sigma),
$$

(20)

where the Laplace contour $\mathcal{C}$ in the complex $\sigma$ plane must pass above all poles of the integrand. Similar expressions hold for the fluctuations of the stream function $\delta \psi$. If we take the Fourier–Laplace transform of equation (18), we find that

$$
-\delta \tilde{\omega}(n, r, 0) - i\sigma \delta \tilde{\omega}(n, r, \sigma) + i\Omega \delta \tilde{\omega}(n, r, \sigma) + \frac{1}{i(n\Omega - \sigma)} \delta \psi(n, r, \sigma) = 0,
$$

(21)

where the first term is the spatial Fourier transform of the initial value:

$$
\delta \tilde{\omega}(n, r, 0) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta} \delta \omega(\theta, r, 0).
$$

(22)

The foregoing equation can be rewritten as

$$
\delta \tilde{\omega}(n, r, \sigma) = -\frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \delta \tilde{\omega}(n, r, \sigma) + \frac{\delta \tilde{\omega}(n, r, 0)}{i(n\Omega - \sigma)},
$$

(23)

where the first term on the right-hand side corresponds to ‘collective effects’ and the second term is related to the initial condition. The fluctuations of the stream function are related to the fluctuations of the vorticity by an equation of the form

$$
L \delta \psi = -\delta \omega,
$$

(24)

equivalent to $\delta \psi = u \ast \delta \omega$ (see appendix B). The linear operator $L$, satisfying $Lu(|r-r'|) = -\delta(|r-r'|)$, can be interpreted as a generalized Laplacian. For example, if $L = \Delta - k_R^2$, one has in polar coordinates $L = (1/r)(\partial/\partial r)(r \partial/\partial r) + (1/r^2)(\partial^2/\partial \theta^2) - k_R^2$. Taking the Fourier–Laplace transform of equation (24), we obtain

$$
\mathcal{L} \delta \tilde{\psi}(n, r, \sigma) = -\delta \tilde{\omega}(n, r, \sigma),
$$

(25)

where $\mathcal{L}$ is the Fourier transform of $L$. For example, if $L = \Delta - k_R^2$, then $\mathcal{L} = (1/r)(\partial/\partial r)(r \partial/\partial r) - n^2/r^2 - k_R^2$. Using equation (23), we find that

$$
\left[ \mathcal{L} - \frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \right] \delta \tilde{\omega}(n, r, \sigma) = -\frac{\delta \tilde{\omega}(n, r, 0)}{i(n\Omega - \sigma)}.
$$

(26)

Therefore, the Fourier–Laplace transform of the fluctuations of the stream function is related to the initial condition by

$$
\delta \tilde{\psi}(n, r, \sigma) = \int_0^{+\infty} 2\pi r' dr' \, G(n, r, r', \sigma) \frac{\delta \tilde{\omega}(n, r', 0)}{i(n\Omega - \sigma)},
$$

(27)

This equation with the right-hand side equal to zero is the Rayleigh equation determining the dispersion relation associated with the linearized 2D Euler equation [72].

doi:10.1088/1742-5468/2012/02/P02019
where the Green function $G(n, r, r', \sigma)$ is defined by
\[
\left[ \mathcal{L} - \frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \right] G(n, r, r', \sigma) = -\frac{\delta(r - r')}{2\pi r}.
\] (28)

The Fourier–Laplace transform of the fluctuations of the vorticity is then given by equation (23) with equation (27). In these equations, we have noted $\Omega$ for $\Omega(r)$ and $\Omega'$ for $\Omega'(r')$. Similarly, we shall note $\omega$ for $\omega(r)$ and $\omega'$ for $\omega(r')$. To avoid confusion, the derivatives with respect to $r$ will be denoted $\partial/\partial r$.

We can use these expressions to compute the collision term appearing on the right-hand side of equation (17). One has
\[
\left\langle \frac{\delta \omega \partial \delta \psi}{\partial \theta} \right\rangle = \sum_n \sum_{n'} \int_C \frac{d\sigma'}{2\pi} \int_C \frac{d\sigma'}{2\pi} \text{in} \times e^{i(n'\theta - \sigma t)} \left( \delta \hat{\omega}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \right).
\] (29)

Using equation (23), we find that
\[
\langle \delta \hat{\omega}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \rangle = -\frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \langle \delta \hat{\psi}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \rangle + \frac{\langle \delta \hat{\omega}(n, r, 0) \delta \hat{\psi}(n', r, \sigma') \rangle}{i(n\Omega - \sigma)}.
\] (30)

The first term corresponds to the self-correlation of the stream function, while the second term corresponds to the correlations between the fluctuations of the stream function and of the vorticity at time $t = 0$. Let us consider these two terms separately.

From equation (27), we obtain
\[
\langle \delta \hat{\psi}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \rangle = -\int_0^{+\infty} 2\pi r' dr' \int_0^{+\infty} 2\pi r'' dr'' G(n, r', \sigma) G(n', r'', \sigma')
\times \frac{\langle \delta \hat{\omega}(n, r', 0) \delta \hat{\omega}(n', r'', 0) \rangle}{(n\Omega' - \sigma)(n'\Omega'' - \sigma')}. \tag{31}
\]

Using the expression of the autocorrelation of the fluctuations at $t = 0$ (see appendix D) given by
\[
\langle \delta \hat{\omega}(n, r', 0) \delta \hat{\omega}(n', r'', 0) \rangle = \gamma \delta_{n,-n'} \frac{\delta(r' - r'')}{2\pi r'} \omega(r'), \tag{32}
\]
we find that
\[
\langle \delta \hat{\psi}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \rangle = 2\pi \gamma \delta_{n,-n'} \int_0^{+\infty} r' dr' G(n, r, r', \sigma) G(-n, r, r', \sigma')
\times \frac{\omega(r')}{(n\Omega' - \sigma)(n\Omega' + \sigma')}. \tag{33}
\]
Considering only the contributions that do not decay in time, it can be shown [5, 42] that $[\gamma \Omega' - \sigma](-\Omega' + \sigma')^{-1}$ can be substituted by $(2\pi)^2 \delta(\sigma + \sigma') \delta(\sigma - n\Omega')$. Then, using the property $G(\gamma n, \sigma, r', -\sigma) = G(n, r', \sigma)$, one finds that the correlations of the fluctuations of the stream function are given by
\[
\langle \delta \hat{\psi}(n, r, \sigma) \delta \hat{\psi}(n', r, \sigma') \rangle = (2\pi)^3 \gamma \delta_{n,-n'} \delta(\sigma + \sigma') \int_0^{+\infty} r' dr' G(n, r, r', \sigma)^2 \delta(\sigma - n\Omega') \omega(r'). \tag{34}
\]

\[\text{doi}:10.1088/1742-5468/2012/02/P02019\]
Similarly, one finds that the second term on the right-hand side of equation (30) is given by

$$
\langle \delta \omega(n, r, 0) \delta \psi(n', r, \sigma') \rangle = (2\pi)^2 \gamma \delta_{n, -n'} \delta(\sigma + \sigma') G(-n, r, -r) \delta(\sigma - n\Omega) \omega(r).
$$

From equation (34), we get the contribution to (29) of the first term of equation (30). It is given by

$$
\left\langle \frac{\delta \omega}{\partial \theta} \right\rangle = i (2\pi)^2 \gamma \sum_n \int_{\mathbb{C}} \frac{d\sigma}{2\pi} \int_0^{+\infty} r' dr' \frac{n^2(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \times |G(n, r, r', \sigma)|^2 \delta(\sigma - n\Omega) \omega(r').
$$

Using the Landau prescription $\sigma \to \sigma + i0^+$ and the Plemelj formula

$$
\frac{1}{x \pm i0^+} = P \frac{1}{x} \mp i\pi \delta(x),
$$

where $P$ denotes the principal value, we can replace $1/(n\Omega - \sigma - i0^+)$ by $+i\pi \delta(n\Omega - \sigma)$. Then, integrating over $\sigma$ and using the identity

$$
\delta(\lambda x) = \frac{1}{|\lambda|} \delta(x),
$$

we obtain

$$
\left\langle \frac{\delta \omega}{\partial \theta} \right\rangle = -2\pi^2 \gamma \sum_n \int_0^{+\infty} r' dr' n|G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \omega(r') \frac{1}{r} \frac{\partial \omega'}{\partial r'}.
$$

From equation (35), we get the contribution to (29) of the second term of equation (30). It is given by

$$
\left\langle \frac{\delta \omega}{\partial \theta} \right\rangle = -2\pi^2 \gamma \sum_n \int_{\mathbb{C}} \frac{d\sigma}{2\pi} n \Im G(n, r, r, \sigma) \delta(\sigma - n\Omega) \omega(r).
$$

To determine $\Im G(n, r, r, \sigma)$, we multiply equation (28) by $2\pi r G(-n, r', r', -\sigma)$, integrate over $r$ from 0 to $+\infty$ and take the imaginary part of the resulting expression using the Plemelj formula (37). This yields

$$
\Im G(n, r, r, \sigma) = -2\pi^2 \int_0^{+\infty} dr' n|G(n, r, r', \sigma)|^2 \delta(\sigma - n\Omega') \frac{1}{r'} \frac{\partial \omega'}{\partial r'}.
$$

Substituting this expression in equation (40) and using equation (38), we obtain

$$
\left\langle \frac{\delta \omega}{\partial \theta} \right\rangle = 2\pi^2 \gamma \sum_n \int_0^{+\infty} r' dr' n|G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \omega(r') \frac{1}{r} \frac{\partial \omega'}{\partial r'}.
$$

Finally, regrouping equations (17), (39) and (42), we end up with the kinetic equation

$$
\frac{\partial \omega}{\partial t} = 2\pi^2 \gamma \frac{1}{r} \frac{\partial}{\partial r} \sum_n \int_0^{+\infty} r' dr' n|G(n, r, r', \sigma)|^2 \delta(\Omega - \Omega') \left( \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r'} \frac{\partial \omega'}{\partial r'} \right).
$$

doi:10.1088/1742-5468/2012/02/P02019
This equation, taking collective effects into account, was derived by Dubin and O’Neil [23]. It is the counterpart of the Lenard–Balescu equation in plasma physics. When collective effects are neglected, we independently obtained in [31]–[34] a kinetic equation of the form

$$\frac{\partial \omega}{\partial t} = 2\pi^2 \gamma \frac{1}{r} \frac{\partial}{\partial r} \sum_n \int_0^{+\infty} r' dr' |n| \tilde{u}_n(r, r')^2 \delta(\Omega - \Omega') \left( \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r'} \frac{\partial \omega'}{\partial r'} \right).$$  \hfill (44)

For the usual potential of interaction, the sum over $n$ can be done explicitly (see appendix B) and the foregoing equation reduces to

$$\frac{\partial \omega}{\partial t} = -\gamma \frac{1}{4} \frac{\partial}{\partial r} \int_0^{+\infty} r' dr' \ln \left[ 1 - \left( \frac{r_c}{r} \right)^2 \right] \delta(\Omega - \Omega') \left( \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r'} \frac{\partial \omega'}{\partial r'} \right),$$ \hfill (45)

where $r_c$ (resp. $r_>$) is the min (resp. max) of $r$ and $r'$. Equation (44) is the counterpart of the Landau equation in plasma physics. It was derived in [31]–[34] by a method that does not use Fourier–Laplace transforms. It arises from the generalized kinetic equation [31, 32]:

$$\frac{\partial \omega}{\partial t} + \frac{N - 1}{N} \mathbf{u} \cdot \nabla \omega = \frac{\partial}{\partial \tau} \int_0^t d\tau \int d\mathbf{r}_1 V^\nu(1 \to 0) \mathcal{G}(t, t - \tau) \times \left[ V^\nu(1 \to 0) \frac{\partial}{\partial \tau} + V^\nu(0 \to 1) \frac{\partial}{\partial \tau} \right] \omega(\mathbf{r}, t - \tau) \frac{\omega}{\gamma}(\mathbf{r}_1, t - \tau),$$ \hfill (46)

which is valid for flows that are not necessarily axisymmetric and not necessarily Markovian\(^5\). If we consider axisymmetric flows and make a Markovian approximation, equation (46) directly leads to equation (44) after simple calculations [32, 34]. Equation (46) has the structure of a Fokker–Planck equation in which the coefficients of diffusion and drift are given by generalized Kubo formulae. The connection between the Lenard–Balescu-type equation (43) and the Landau-type equation (44) is clear. If we neglect collective effects, equation (28) reduces to $t \mathcal{L}_G\text{bare}(n, r, r') = -\delta(r - r')/2\pi r$ (it does not depend on $\sigma$ anymore) so that $\mathcal{G}_\text{bare}(n, r, r')$ is just the Fourier transform of the ‘bare’ potential of interaction $u$ (i.e. the Green function of $L$). We thus have $\mathcal{G}_\text{bare}(n, r, r') = \hat{u}_n(r, r')$ (see appendix B). When collective effects are taken into account, the ‘bare’ potential $\mathcal{G}_\text{bare}(n, r, r') = \hat{u}_n(r, r')$ is replaced by a ‘dressed’ potential $G(n, r, r', \sigma)$ without changing the overall structure of the kinetic equation. This is similar to the case of plasma physics where the bare potential $\hat{u}(k)$ in the Landau equation is replaced by the dressed\(^6\) potential $\hat{u}(k)/|\epsilon(k, k \cdot v)|$, including the dielectric function, in the Lenard–Balescu equation. In plasma physics, collective effects are important because they account for screening effects and regularize, at the scale of the Debye length, the logarithmic divergence that occurs in the Landau equation. For 2D point vortices, there is no divergence in equation (45) so that collective effects may be less crucial than in plasma physics. So far, their influence is not clearly understood.

\(^{5}\) The Markovian approximation may not be justified in every situation since it has been found numerically that point vortices can exhibit long jumps (Lévy flights) and strong correlations [73, 74].

\(^{6}\) This terminology comes from the fact that the Coulombian potential created by a particle is ‘dressed’ by its polarization cloud, accounting for shielding effects.

doi:10.1088/1742-5468/2012/02/P02019
Finally, restoring the time variable, the kinetic equation (43) can be rewritten as
\[
\frac{\partial \omega}{\partial t} = 2\pi^2 \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} r' \, dr' \, \chi(r, r', t) \delta(\Omega(r, t) - \Omega(r', t)) \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r'} \frac{\partial}{\partial r'} \right) \omega(r, t) \omega(r', t),
\]
with the notation
\[
\chi(r, r', t) = \sum_n |n| |G(n, r, r', n\Omega(r, t))|^2.
\]
When collective effects are neglected \(\chi(r, r', t)\) is replaced by \(\chi_{\text{bare}}(r, r') = \sum_n |n| \hat{u}_n(r, r')^2\).

For the usual potential of interaction (see appendix B), it reduces to
\[
\chi_{\text{bare}}(r, r') = -(1/8\pi^2) \ln[1 - (r_\text{<}/r_\text{>})^2].
\]

3.2. The relaxation time of the system as a whole

The kinetic equation (47) is valid at the order of 1/N so it describes the ‘collisional’ evolution of the system on a timescale \(\sim Nt_D\). This kinetic equation conserves the total circulation \(\Gamma\), the energy \(E\) and the angular momentum \(L\). It also monotonically increases the Boltzmann entropy \(S\) (\(H\) theorem). The Boltzmann distribution (14) is always a steady state of this kinetic equation, but it is not the only one: any vorticity distribution that is associated with a monotonic profile of angular velocity is a steady state of equation (47). The kinetic equation admits therefore an infinite number of steady states! As explained in previous works, the collisional evolution of the point vortex gas (at the order of 1/N) is due to a condition of resonance between distant orbits of the point vortices. For axisymmetric systems, the condition of resonance, encapsulated in the \(\delta\) function, corresponds to \(\Omega(r', t) = \Omega(r, t)\) with \(r' \neq r\) (the self-interaction at \(r' = r\) does not produce transport since the term in parentheses in equation (47) vanishes identically). The vorticity profile \(\omega(r, t)\) at position \(r\) changes under the effects of ‘collisions’ if there exist point vortices at \(r' \neq r\) that rotate with the same angular velocity as point vortices located at \(r\). This is possible only if the profile of angular velocity is non-monotonic\(^7\).

Therefore, the evolution is truly due to distant collisions between point vortices (this is strikingly different from the case of plasmas and stellar systems where the collisions are more ‘local’). The evolution stops when the profile of angular velocity becomes monotonic (so that there is no resonance) even if the system has not reached the statistical equilibrium state given by the Boltzmann distribution. This ‘kinetic blocking’ has been illustrated numerically in [41]. Distant collisions between point vortices have the tendency to create a monotonic profile of angular velocity. If the initial condition already has a monotonic profile of angular velocity, the collision term vanishes (at the order of 1/N) because there is no resonance. The kinetic equation reduces to
\[
\frac{\partial \omega}{\partial t} = 0,
\]
so that the vorticity does not evolve at all on a timescale \(\sim Nt_D\). Therefore, the kinetic theory predicts no thermalization to a Boltzmannian at first order in 1/N. For an

\(^7\) We recall that the system must be dynamically stable for the kinetic theory to be valid. It is experimentally observed that there exist vorticity profiles with a non-monotonic profile of angular velocity that are Euler stable [23].
axisymmetric distribution of point vortices, the relaxation time is larger than $Nt_D$ and consequently a very slow process. The relaxation time satisfies

$$t_R > Nt_D \quad \text{(axisymmetric flows)}.$$  \hfill (50)

Since the relaxation process is due to more complex correlations between point vortices, we have to develop the kinetic theory at higher orders (taking into account three-body, four-body, etc, correlation functions) in order to obtain the relaxation time. If the collision term does not vanish at the next order of the expansion, the kinetic theory would imply a relaxation time of the order of $N^2t_D$ \cite{34}. However, the problem could be more complicated and yield a larger relaxation time like $e^{Nt_D}$. In fact, it is not even granted that the system will ever relax towards statistical equilibrium (the point vortex gas is non-ergodic \cite{22} and it may not mix sufficiently well to justify the establishment of the Boltzmann distribution).

By contrast, for non-axisymmetric flows, since there are potentially more resonances between point vortices (see the complicated kinetic equation (46)), the relaxation time can be reduced and may approach the natural scaling:

$$t_R \sim Nt_D \quad \text{(non-axisymmetric flows)}$$  \hfill (51)

predicted by the first-order kinetic theory. This linear scaling has been observed for 2D point vortices with non-axisymmetric distribution in \cite{75}. However, very little is known concerning the properties of equation (46) and its convergence (or not) towards the Boltzmann distribution. It could approach the Boltzmann distribution (since entropy increases), without reaching it exactly.

Similar results have been found for 1D plasmas, 1D stellar systems and for the HMF model (see section 5 for more details). For spatially homogeneous 1D systems, the Lenard–Balescu collision term vanishes (no resonance) so that the relaxation time is larger than $Nt_D$. For spatially inhomogeneous 1D systems, since there are potentially more resonances (this can be seen by using angle-action variables \cite{34,59,60}), the relaxation time can be reduced and approaches the natural scaling $Nt_D$. For 2D and 3D plasmas and stellar systems, there are always resonances since the condition is $k \cdot v = k' \cdot v'$. The Lenard–Balescu collision term vanishes only for the Boltzmann distribution and the relaxation time is $t_R \sim Nt_D$ (or $t_R \sim (N/\ln N)t_D$ for 3D plasmas and 3D stellar systems) corresponding to the first order of the kinetic theory. We recall that, for plasmas, $N$ represents the number of particles in the Debye sphere.

4. Stochastic process of a test vortex: diffusion and drift

4.1. The Fokker–Planck equation

In section 3, we have studied the collisional evolution of a system of point vortices as a whole. We now consider the relaxation of a test vortex in a ‘sea’ of field vortices with a steady axisymmetric vorticity profile $\omega(r)$. The field vortices play the role of a bath. We assume that the field vortices are either at statistical equilibrium with the Boltzmann distribution (thermal bath), in which case their distribution does not change at all, or in a stable axisymmetric steady state of the 2D Euler equation with a monotonic profile of angular velocity (as we have just seen, this profile does not change on a timescale of order $Nt_D$). We assume that the test vortex is initially located at a radial distance $r_0$ and we study how it progressively acquires the distribution of the field vortices due to distant
‘collisions’. As we shall see, the test vortex has a stochastic motion and the evolution of the distribution function $P(r, t)$, the probability density of finding the test vortex at radial position $r$ at time $t$, is governed by a Fokker–Planck equation involving a diffusion term and a drift term that can be analytically obtained. The Fokker–Planck equation may then be solved with the initial condition $P(r, t_0) = \delta(r - r_0)/2\pi r$ to yield $P(r, t)$. This problem has been investigated in our previous papers [30]–[34], but we shall give here a direct and more rigorous derivation of the coefficients or diffusion and drift, taking collective effects into account.

The equations of motion of the test vortex are

$$
\frac{dr}{dt} = \frac{1}{r} \frac{\partial \delta \psi}{\partial \theta}, \quad \frac{d\theta}{dt} = \Omega(r) - \frac{1}{r} \frac{\partial \delta \psi}{\partial r}.
$$

(52)

They include the effect of the mean field $\Omega(r)$ that produces a zeroth-order net rotation plus a stochastic component $\delta \psi$ of order $1/\sqrt{N}$ (fluctuations) that takes into account the deviations from the mean field. They can be formally integrated into

$$
r(t) = r + \int_0^t dt' \frac{1}{r(t')} \frac{\partial \delta \psi}{\partial \theta}(r(t'), \theta(t'), t'),
$$

(53)

$$
\theta(t) = \theta + \int_0^t dt' \Omega(r(t')) - \int_0^t dt' \frac{1}{r(t')} \frac{\partial \delta \psi}{\partial r}(r(t'), \theta(t'), t'),
$$

(54)

where we have assumed that, initially, the test vortex is at $(r, \theta)$. Note that the ‘initial’ time considered here does not necessarily coincide with the original time $t_0$ mentioned above. Since the fluctuation $\delta \psi$ of the stream function is a small quantity, the foregoing equations can be solved perturbatively. At the order of $1/N$, which corresponds to quadratic order in $\delta \psi$, we get for the radial distance

$$
r(t) = r + \int_0^t dt' \frac{1}{r} \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t', t')
$$

$$
+ \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \frac{1}{r} \frac{\partial^2 \delta \psi}{\partial \theta^2}(r, \theta + \Omega t', t') \frac{\partial \delta \psi}{\partial r}(r, \theta + \Omega t'', t''')
$$

$$
- \int_0^t dt' \int_0^{t'} dt'' \frac{1}{r} \frac{\partial^2 \delta \psi}{\partial \theta^2}(r, \theta + \Omega t', t') \frac{\partial \delta \psi}{\partial r}(r, \theta + \Omega t'', t'')
$$

$$
+ \int_0^t dt' \int_0^{t'} dt'' \frac{1}{r} \frac{\partial^2 \delta \psi}{\partial \theta^2}(r, \theta + \Omega t', t') \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t'', t'')
$$

$$
- \int_0^t dt' \int_0^{t'} dt'' \frac{1}{r} \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t', t') \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t'', t'').
$$

(55)

As the changes in the radial distance are small, and the fluctuation time short, the dynamics of the test vortex can be represented by a stochastic process governed by a Fokker–Planck equation [76]. If we denote by $P(r, t)$ the probability density of finding the test vortex at radial distance $r$ at time $t$, normalized such that $\int_0^{+\infty} P(r, t)2\pi r \, dr = 1$, the general form of this equation is

$$
\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} DP \right) - \frac{1}{r} \frac{\partial}{\partial r}(r PA).
$$

(56)

doi:10.1088/1742-5468/2012/02/P02019
The diffusion coefficient and the drift term are defined by

$$D(r) = \lim_{t \to +\infty} \frac{1}{2t} \langle (r(t) - r)^2 \rangle,$$

(57)

$$A(r) = \lim_{t \to +\infty} \frac{1}{t} \langle r(t) - r \rangle.$$

(58)

In writing these limits, we have implicitly assumed that the time $t$ is long with respect to the fluctuation time but short with respect to the relaxation time of the test vortex (of order $Nt_D$), so that the expression (55) can be used to evaluate equations (57) and (58). As shown in our previous papers [30]–[34], it is relevant to rewrite the Fokker–Planck equation in the alternative form

$$\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \left( D \frac{\partial P}{\partial r} - PV_{\text{pol}} \right) \right].$$

(59)

The total drift term is

$$A = V_{\text{pol}} + \frac{dD}{dr},$$

(60)

where $V_{\text{pol}}$ is the part of the drift term due to the polarization, while the second term is due to the spatial variations of the diffusion coefficient. As we shall see, this decomposition arises naturally in the following analysis. The two expressions (56) and (59) have their own interest. The expression (56), where the diffusion coefficient is placed after the second derivative $\partial^2(DP)$, involves the total drift $A$ and the expression (59), where the diffusion coefficient is placed between the derivatives $\partial D/\partial P$, isolates the part of the drift $V_{\text{pol}}$ due to the polarization. We shall see in section 4.5 that this second form is directly related to the kinetic equation (47). It has therefore a clear physical interpretation.

We shall now calculate the diffusion coefficient and the drift term from equations (57) and (58), using the results of section 3.1 that allow us to take collective effects into account. Note that a similar calculation of these terms, directly from the equations of motion of a test vortex, has been made in appendix C of [41], neglecting collective effects. The diffusion coefficient can also be obtained from the Kubo formula and the drift term from a linear response theory [30]–[32].

### 4.2. The diffusion coefficient

We first compute the diffusion coefficient defined by equation (57). Using equation (55), we see that it is given, at the order of $1/N$, by

$$D = \frac{1}{2r^2 t} \int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t', t') \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega t'', t'') \right\rangle.$$

(61)
By the inverse Fourier–Laplace transform, we have

$$\left\langle \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega', t') \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega'', t'') \right\rangle = -\sum_n \sum_{n'} \int_C \frac{d\sigma}{2\pi} \int_C \frac{d\sigma'}{2\pi} \times n' n e^{i(n+\Omega')s} e^{-i\Omega s} e^{i(n' + \Omega')s} \langle \delta \tilde{\psi}(n, r, \sigma) \delta \tilde{\psi}(n', r, \sigma') \rangle. \quad (62)$$

Substituting equation (34) into equation (62), and carrying out the summation over $n'$ and $\sigma'$ and $\sigma$, we end up with the result

$$\left\langle \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega', t') \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega'', t'') \right\rangle = 2\pi \gamma \sum_n \int_0^{+\infty} r' \, dr' \times n^2 e^{i(n-\Omega)(t'-t'')} \langle G(n, r, r', n\Omega') \rangle^2 \omega(r'). \quad (63)$$

This expression shows that the correlation function appearing in equation (61) is an even function of $t' - t''$. Using the identity

$$\int_0^t dt' \int_0^{t''} dt'' f(t' - t'') = 2 \int_0^t dt' \int_0^{t''} dt'' f(t' - t'') = 2 \int_0^t ds \, (t - s) f(s), \quad (64)$$

we find, for $t \to +\infty$, that

$$D = \frac{1}{r^2} \int_0^{+\infty} ds \left\langle \left\langle \frac{\partial \delta \psi}{\partial \theta}(r, \theta, 0) \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega s, s) \right\rangle \right\rangle. \quad (65)$$

This is the Kubo formula for our problem. Replacing the correlation function by its expression (63), we get

$$D = 2\pi \gamma \frac{1}{r^2} \int_0^{+\infty} ds \sum_n \int_0^{+\infty} r' \, dr' n^2 \langle G(n, r, r', n\Omega') \rangle^2 \omega(r'). \quad (66)$$

Making the change of variables $s \to -s$ and $n \to -n$, we see that we can replace $\int_0^{+\infty} ds$ by $(1/2) \int_{-\infty}^{+\infty} ds$ in equation (66). Then, using the identity

$$\delta(\sigma) = \int_{-\infty}^{+\infty} e^{i\sigma t} \frac{dt}{2\pi}, \quad (67)$$

and equation (38), we obtain the final expression:

$$D = 2\pi^2 \gamma \frac{1}{r^2} \sum_n \int_0^{+\infty} r' \, dr' \langle n \rangle \langle G(n, r, n\Omega) \rangle^2 \delta(\Omega - \Omega') \omega(r'). \quad (68)$$

### 4.3. The part of the drift term due to the polarization

We now compute the drift term defined by equation (58). We need to keep terms up to order $1/N$. From equation (55), the first term to compute is

$$A_I = \frac{1}{rt} \int_0^t dt' \left\langle \frac{\partial \delta \psi}{\partial \theta}(r, \theta + \Omega', t') \right\rangle. \quad (69)$$
By the inverse Fourier–Laplace transform, we have
\[
\left\langle \frac{\partial \delta \hat{\psi}}{\partial \theta} (r, \theta + \Omega t', t') \right\rangle = i \sum_n \int_C \frac{d\sigma}{2\pi} n e^{i(n\Omega - \sigma)\theta} e^{-i\sigma t'} \langle \delta \hat{\psi}(n, r, \sigma) \rangle. \tag{70}
\]

Using equation (27), we find that
\[
\langle \delta \hat{\psi}(n, r, \sigma) \rangle = \int_0^{+\infty} 2\pi r' \, dr' \, \frac{\langle \delta \tilde{\omega}(n, r', 0) \rangle}{i(n\Omega - \sigma)} G(n, r, r', \sigma). \tag{71}
\]

Now, using the fact that the test vortex is initially located in \((r, \theta)\), so that \(\langle \delta \omega(\theta', r', 0) \rangle = \gamma \delta(\theta' - \theta) \delta(r' - r)/r\), we obtain from equation (22) the result
\[
\langle \delta \tilde{\omega}(n, r', 0) \rangle = \gamma e^{-i\sigma} \delta(r' - r) \tag{72}
\]
Substituting these expressions in equation (70), we get
\[
\left\langle \frac{\partial \delta \hat{\psi}}{\partial \theta} (r, \theta + \Omega t', t') \right\rangle = \gamma \sum_n \int_C \frac{d\sigma}{2\pi} n e^{i(n\Omega - \sigma)\theta} G(n, r, r, \sigma) \frac{1}{n\Omega - \sigma}. \tag{73}
\]

Therefore, the drift term (69) is given by
\[
A_1 = \gamma \frac{1}{rt} \int_0^t dt' \sum_n \int_C \frac{d\sigma}{2\pi} n e^{i(n\Omega - \sigma)\theta} G(n, r, r, \sigma) \frac{1}{n\Omega - \sigma}. \tag{74}
\]

We now use the Landau prescription \(\sigma \rightarrow \sigma + i0^+\) and the Plemelj formula (37) to evaluate the integral over \(\sigma\). The term corresponding to the imaginary part in the Plemelj formula is
\[
A_1^{(a)} = i\pi \gamma \frac{1}{rt} \int_0^t dt' \sum_n \int_C \frac{d\sigma}{2\pi} n e^{i(n\Omega - \sigma)\theta} G(n, r, r, \sigma) \delta(n\Omega - \sigma). \tag{75}
\]

Integrating over \(\sigma\) and \(t'\), we obtain
\[
A_1^{(a)} = -\frac{\gamma}{2r} \sum_n n \, \text{Im} \, G(n, r, r, n\Omega). \tag{76}
\]

The term corresponding to the real part in the Plemelj formula is
\[
A_1^{(b)} = \gamma \frac{1}{rt} \int_0^t dt' \sum_n P \int_C \frac{d\sigma}{2\pi} n e^{i(n\Omega - \sigma)\theta} G(n, r, r, \sigma) \frac{1}{n\Omega - \sigma}. \tag{77}
\]

Integrating over \(t'\), we can convert this expression to the form
\[
A_1^{(b)} = -\frac{i\gamma}{r} \sum_n P \int_C \frac{d\sigma}{2\pi} n G(n, r, r, \sigma) \frac{1}{(n\Omega - \sigma)^2} \frac{1}{t} \times \{i \sin (n\Omega - \sigma)t + \cos (n\Omega - \sigma)t - 1\}. \tag{78}
\]

For \(t \rightarrow +\infty\), using the identity (as in Appendix C of [41])
\[
\lim_{t \rightarrow +\infty} \frac{1 - \cos(tx)}{tx^2} = \pi \delta(x), \tag{79}
\]

\[\text{doi:10.1088/1742-5468/2012/02/P02019}\]
and integrating over \( \sigma \), we find that \( A_1^{(b)} \) is given by equation (76) just like \( A_1^{(a)} \). Therefore, writing \( A_1 = A_1^{(a)} + A_1^{(b)} = 2A_1^{(a)} \), we obtain
\[
A_1 = -\frac{\gamma}{r} \sum_n n \text{Im} G(n, r, r, n\Omega).
\] (80)

Finally, using equation (41), we find that
\[
A_1 = 2\gamma^2 \sum_n \int_0^{r \rightarrow \infty} \text{d}r' |n||G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \frac{\text{d}\omega'}{\text{d}r'}.
\] (81)

As we shall see, this term corresponds to the part of the drift due to the polarization denoted \( V_{\text{pol}} \) in equation (60).

Remark. We can obtain equation (81) in a slightly more direct manner from equation (73) by using the contour of integration shown in figure 9 of [42]. In that case, the integral over \( \sigma \) is just \(-2\pi\imath\) times the sum of the residues at the poles of the integrand in equation (73). The poles of the function \( G(n, r, r, \sigma) \) give a contribution that rapidly decays with time since \( \text{Im}(\sigma) < 0 \) (the system is Euler-stable). Keeping only the contribution of the pole \( \sigma = n\Omega \) that does not decay in time we obtain
\[
\langle \partial \partial \psi (r, \theta + \Omega', t') \rangle = i\gamma \sum_n nG(n, r, r, n\Omega).
\] (82)

Substituting this result in equation (69), we get equation (80) and then equation (81).

4.4. The part of the drift term due to the spatial inhomogeneity of the diffusion coefficient

In the evaluation of the total drift, at the order of \( 1/N \), the second term to compute is
\[
A_{II} = \frac{1}{t} \int_0^t \text{d}t' \int_0^t \text{d}t'' \int_0^{t''} \text{d}t''' \frac{1}{r^2} \text{d}r \left( \frac{\partial^2 \delta \psi}{\partial t^2} (r, \theta + \Omega', t') \frac{\partial \delta \psi}{\partial \theta} (r, \theta + \Omega t''' + t''') \right).
\] (83)

By the inverse Fourier–Laplace transform, we have
\[
\langle \frac{\partial^2 \delta \psi}{\partial t^2} (r, \theta + \Omega', t') \frac{\partial \delta \psi}{\partial \theta} (r, \theta + \Omega t''' + t''') \rangle = -i \sum_n \sum_n \int_0^{2\pi} \text{d}\sigma \int_0^{2\pi} \text{d}\sigma' \int_{-\infty}^{\infty} n^2 n' e^{i n (\theta + \Omega t')} e^{-i n' (\theta + \Omega t''')} e^{-i \sigma' t'''} \langle \delta \tilde{\psi}(n, r, \sigma) \delta \tilde{\psi}(n', r, \sigma') \rangle.
\] (84)

Substituting equation (34) in equation (84), and carrying out the summation over \( n' \) and the integrals over \( \sigma' \) and \( \sigma \), we end up with the result
\[
\langle \frac{\partial^2 \delta \psi}{\partial t^2} (r, \theta + \Omega', t') \frac{\partial \delta \psi}{\partial \theta} (r, \theta + \Omega t''' + t''') \rangle = i 2\pi \gamma \sum_n \int_0^{r \rightarrow \infty} \text{d}r' \int_0^{r'} \text{d}r''' \frac{n^3 \text{e}^{i n (\Omega - \Omega')(t' - t''')}}{|G(n, r, r', n\Omega)|^2 \text{d}t'}
\] (85)

This expression shows that the correlation function appearing in equation (83) is an odd function of \( t' - t''' \). Using the identity
\[
\int_0^{t'} \text{d}t'' \int_0^{t''} \text{d}t''' f(t' - t''') = \int_0^{t'} \text{d}t'' (t' - t'') f(t' - t''),
\] (86)

\text{doi:10.1088/1742-5468/2012/02/P02019}
we find that

\[ A_{II} = i2\gamma \frac{1}{t} \int_0^t dt' \int_0^{t'} dt'' \sum_n \int_0^{+\infty} r' dr' (t' - t'') n^2 \frac{1}{r^2} \frac{d\Omega}{dr} e^{in(\Omega - \Omega')(t' - t'')} \times |G(n, r, r', n\Omega')|^2 \omega(r'). \] (87)

This can be rewritten as

\[ A_{II} = 2\pi \gamma \frac{1}{t} \int_0^t dt' \int_0^{t'} dt'' \sum_n \int_0^{+\infty} r' dr' n^2 \frac{1}{r^2} \frac{\partial}{\partial r} (e^{in(\Omega - \Omega')(t' - t'')}) \times |G(n, r, r', n\Omega')|^2 \omega(r'), \] (88)

or, equivalently,

\[ A_{II} = 2\pi \gamma \frac{1}{t} \frac{\partial}{\partial r} \int_0^t dt' \int_0^{t'} dt'' \sum_n \int_0^{+\infty} r' dr' n^2 \frac{1}{r^2} e^{in(\Omega - \Omega')(t' - t'')} |G(n, r, r', n\Omega')|^2 \omega(r') \]

\[ - 2\pi \gamma \frac{1}{t} \int_0^t dt' \int_0^{t'} dt'' \sum_n \int_0^{+\infty} r' dr' n^2 e^{in(\Omega - \Omega')(t' - t'')} \times \frac{\partial}{\partial r} \left( \frac{1}{r^2} |G(n, r, r', n\Omega')|^2 \right) \omega(r'). \] (89)

Since the integrand only depends on \( t' - t'' \), using the identity (64), we obtain for \( t \to +\infty \):

\[ A_{II} = 2\pi \gamma \frac{\partial}{\partial r} \int_0^{+\infty} ds \sum_n \int_0^{+\infty} r' dr' n^2 \frac{1}{r^2} e^{in(\Omega - \Omega')s} |G(n, r, r', n\Omega')|^2 \omega(r') \]

\[ - 2\pi \gamma \int_0^{+\infty} ds \sum_n \int_0^{+\infty} r' dr' n^2 e^{in(\Omega - \Omega')s} \frac{\partial}{\partial r} \left( \frac{1}{r^2} |G(n, r, r', n\Omega')|^2 \right) \omega(r'). \] (90)

Making the change of variables \( s \to -s \) and \( n \to -n \), we see that we can replace \( \int_0^{+\infty} ds \) by \( (1/2) \int_{-\infty}^{+\infty} ds \). Then, using the identities (67) and (38), we obtain the expression

\[ A_{II} = 2\pi^2 \gamma \frac{\partial}{\partial r} \sum_n \int_0^{+\infty} r' dr' |n\frac{1}{r^2} \delta(\Omega - \Omega')|G(n, r, r', n\Omega')|^2 \omega(r') \]

\[ - 2\pi^2 \gamma \sum_n \int_0^{+\infty} r' dr' |n\delta(\Omega - \Omega') \frac{\partial}{\partial r} \left( \frac{1}{r^2} |G(n, r, r', n\Omega')|^2 \right) \omega(r'). \] (91)

In the first term, we recover the diffusion coefficient (68) so that finally

\[ A_{II} = \frac{dD}{dr} - 2\pi^2 \gamma \sum_n \int_0^{+\infty} r' dr' |n\delta(\Omega - \Omega') \frac{\partial}{\partial r} \left( \frac{1}{r^2} |G(n, r, r', n\Omega')|^2 \right) \omega(r'). \] (92)
The third term to compute is

\[ A_{III} = -\frac{1}{r^3 t} \int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial^2 \tilde{\psi}}{\partial r^2} (r, \theta + \Omega t', t') \frac{\partial \tilde{\psi}}{\partial \theta} (r, \theta + \Omega t'', t'') \right\rangle. \tag{93} \]

This term is just proportional to the diffusion coefficient (61) so we get

\[ A_{III} = -\frac{2}{r} D = -4\pi^2 \gamma \int_0^\infty r' dr' |n| |G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \omega(r'). \tag{94} \]

Finally, the fourth and fifth terms to compute are

\[ A_{IV} = -\frac{1}{r^2 t} \int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial^2 \tilde{\psi}}{\partial r \partial \theta} (r, \theta + \Omega t', t') \frac{\partial \tilde{\psi}}{\partial \theta} (r, \theta + \Omega t'', t'') \right\rangle, \tag{95} \]

and

\[ A_{V} = \frac{1}{r^2 t} \int_0^t dt' \int_0^t dt'' \left\langle \frac{\partial^2 \tilde{\psi}}{\partial r^2} (r, \theta + \Omega t', t') \frac{\partial \tilde{\psi}}{\partial \theta} (r, \theta + \Omega t'', t'') \right\rangle. \tag{96} \]

Substituting the inverse Fourier–Laplace transform of the fluctuations of the stream function in these equations, and summing the resulting expressions using the fact that the correlation function of the fluctuations of the stream function is proportional to \( \delta_{n,-n'} \), we obtain

\[ A_{IV} + A_{V} = \frac{1}{r^2 t} \int_0^t dt' \int_0^t dt'' \sum_{n} \sum_{n'} \int_0^\infty \int_0^\infty \frac{d\sigma}{2\pi} \frac{d\sigma'}{2\pi} \times n^2 e^{in(\theta+\Omega t')} e^{-in'(\theta+\Omega t'')} e^{-is' t''} \partial r \langle \tilde{\psi}(n, r, \sigma) \tilde{\psi}(n', r, \sigma') \rangle. \tag{97} \]

Substituting equation (34) in equation (97), and carrying out the summation over \( n' \) and the integrals over \( \sigma' \) and \( \sigma \), we end up with the result

\[ A_{IV} + A_{V} = 2\pi\gamma \frac{1}{r^2 t} \int_0^t dt' \int_0^t dt'' \sum_{n} \int_0^\infty \int_0^\infty r' dr' n^2 e^{in(\Omega - \Omega')(t' - t'')} dG(n, r, r', n\Omega)^2) \omega(r'). \tag{98} \]

This expression shows that the correlation function appearing under the integral sign is an even function of \( t' - t'' \). Using the identity (64) we find, for \( t \to +\infty \), that

\[ A_{IV} + A_{V} = 2\pi\gamma \frac{1}{r^2} \int_0^{+\infty} ds \sum_{n} \int_0^{+\infty} r' dr' n^2 e^{in(\Omega - \Omega')s} \partial r (dG(n, r, r', n\Omega)^2) \omega(r'). \tag{99} \]

Making the change of variables \( s \to -s \) and \( n \to -n \), we see that we can replace \( \int_0^{+\infty} ds \) by \( (1/2) \int_{-\infty}^{+\infty} ds \). Then, using the identities (67) and (38), we obtain the expression

\[ A_{IV} + A_{V} = 2\pi^2 \gamma \frac{1}{r^2} \sum_{n} \int_0^{+\infty} r' dr' |n| \partial r (dG(n, r, r', n\Omega)^2) \delta(\Omega - \Omega') \omega(r'). \tag{100} \]
Finally, summing equations (92), (94) and (100), we get

$$A_{II} + A_{III} + A_{IV} + A_{V} = \frac{dD}{dr}. \quad (101)$$

Therefore, recalling equation (81), the complete expression of the drift term is

$$A = 2\pi^2 \gamma \frac{1}{r^2} \sum_n \int_0^{+\infty} dr' |n| |G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \frac{d\omega'}{dr'} + \frac{dD}{dr}. \quad (102)$$

### 4.5. Connection between the kinetic equation (47) and the Fokker–Planck equation (59)

We have established that the diffusion coefficient and the drift term are given by equations (68) and (102). Introducing the notation (48), they can be written

$$D = 2\pi^2 \gamma \frac{1}{r^2} \int_0^{+\infty} r' dr' \chi(r, r') \delta(\Omega - \Omega') \omega(r'), \quad (103)$$

and

$$A = 2\pi^2 \gamma \frac{1}{r} \int_0^{+\infty} dr' \chi(r, r') \delta(\Omega - \Omega') \frac{d\omega}{dr}(r') + \frac{dD}{dr}. \quad (104)$$

Comparing equation (104) with equation (60), we see that the part of the drift term due to the polarization is

$$V_{pol} = 2\pi^2 \gamma \frac{1}{r} \int_0^{+\infty} dr' \chi(r, r') \delta(\Omega - \Omega') \frac{d\omega}{dr}(r'). \quad (105)$$

On the other hand, using an integration by part in the first term of equation (104), the total drift can be written as

$$A = 2\pi^2 \gamma \int_0^{+\infty} r r' dr' \omega(r') \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r'} \frac{\partial}{\partial r'} \right) \chi(r, r') \delta(\Omega - \Omega') \frac{1}{r^2}. \quad (106)$$

Finally, using equations (103) and (105), we find that the Fokker–Planck equation (59) becomes

$$\frac{\partial P}{\partial t} = 2\pi^2 \gamma \frac{1}{r} \frac{\partial}{\partial r} \int_0^{+\infty} r' dr' \chi(r, r') \delta(\Omega - \Omega') \left( \frac{1}{r} \omega' \frac{\partial P}{\partial r} - \frac{1}{r'} \frac{\partial P}{\partial r'} \frac{d\omega'}{dr'} \right). \quad (107)$$

When collective effects are neglected, we recover the results obtained in [30]–[34] by a different method. As observed in our previous works, we note that the form of equation (107) is very similar to the form of equation (47). This shows that the Fokker–Planck equation (107), with the diffusion coefficient (103) and the drift term (105), can be directly obtained from the kinetic equation (47) by replacing the time-dependent distribution \(\omega(r', t)\) by the static distribution \(\omega(r')\) of the bath. This procedure transforms an integro-differential equation (47) into a differential equation (107) [32]. Although natural, the rigorous justification of this procedure requires the detailed calculation that we have given here. In fact, we can understand this result in the following manner. Equations (47) and (107) govern the evolution of the distribution function of a test vortex (described by the variable \(r\)) interacting with field vortices (described by the running variable \(r'\)). In equation (47), all the vortices are equivalent so that the distribution of
the field vortices \( \omega(r',t) \) changes with time exactly like the distribution of the test vortex \( \omega(r,t) \). In equation (107), the test vortex and the field vortices are not equivalent since the field vortices form a ‘bath’. The field vortices have a steady (given) distribution \( \omega(r') \) while the distribution of the test vortex \( \omega(r,t) = N\gamma P(r,t) \) changes with time. This distinction is particularly clear when the kinetic and Fokker–Planck equations are derived from projection operator methods [31] or from the BBGKY hierarchy [32].

4.6. Monotonic profile of angular velocity

If the profile of angular velocity \( \Omega(r) \) of the field vortices is monotonic then, using the identity \( \delta(\Omega(r) - \Omega(r')) = \delta(r-r')/|\Omega'(r)| \), we find that the expressions of the diffusion coefficient (103) and of the drift (105) simplify into

\[
D(r) = 2\pi^2 \gamma \frac{\chi(r,r)}{|\Sigma(r)|} \omega(r),
\]

and

\[
V_{\text{pol}}(r) = 2\pi^2 \gamma \frac{\chi(r,r) \frac{d\omega}{dr}(r)}{|\Sigma(r)|} dr(r),
\]

where \( \Sigma(r) = r\Omega'(r) \) is the local shear created by the field vortices. Comparing equations (108) and (109), we obtain

\[
V_{\text{pol}} = D(r) \frac{d\ln\omega}{dr}.
\]

This relation is valid for an arbitrary Euler-stable distribution of the field vortices with a monotonic profile of angular momentum. Such distributions are steady on a timescale \( \sim Nt_D \) (see section 3.2) and they form therefore a bath. Equation (110) can be viewed as a generalization of the Einstein relation (see section 4.7) for a bath that is out of equilibrium [31]. Using equation (110), the Fokker–Planck equation (59) can be written as

\[
\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ rD(r) \left( \frac{\partial P}{\partial r} - P \frac{d\ln\omega}{dr} \right) \right].
\]

This is a drift–diffusion equation describing the evolution of the distribution \( P(r,t) \) of the test vortex in an ‘effective potential’ \( U_{\text{eff}} = -\ln\omega(r) \) produced by the field vortices. From this equation, we find that, for \( t \to +\infty \), the distribution of the test vortex becomes equal to the distribution of the bath: \( P_e(r) = \omega(r)/N\gamma \) at equilibrium.

If we neglect collective effects, we recover the results obtained in [30]–[34]. For the usual potential of interaction, the function \( \chi_{\text{bare}}(r,r) \) is given by (see appendix B)

\[
\chi_{\text{bare}}(r,r) = \frac{1}{8\pi^2} \ln \Lambda,
\]

where \( \ln \Lambda = \sum_{m=1}^{+\infty} 1/m \) is a logarithmically diverging Coulomb factor that has to be regularized appropriately (see below). Combining equations (108), (109) and (112), the diffusion coefficient and the drift term can be rewritten as

\[
D(r) = \frac{\gamma}{4} \frac{\ln \Lambda}{|\Sigma(r)|} \omega(r), \quad V_{\text{pol}}(r) = \frac{\gamma}{4} \frac{\ln \Lambda}{|\Sigma(r)|} \frac{d\omega}{dr}(r).
\]

doi:10.1088/1742-5468/2012/02/P02019
These expressions were derived for a thermal bath in [30] and extended to an arbitrary (steady Euler-stable) distribution of the field vortices in [31]. They also appear in the works [26]–[29] with a different interpretation (see section 5 for more details). An important feature of these expressions is that the diffusion coefficient and the drift term are inversely proportional to the local shear [30]. Furthermore, the diffusion coefficient is proportional to the vorticity and the drift term is proportional to the gradient of the vorticity. Since \( V_{\text{pol}} = \frac{(D/\omega)}{\nabla \omega} \) in vectorial form, we find that, due to the drift, the test vortex ascends the vorticity gradient. This is a purely deterministic effect that appears when the distribution of the field vortices is spatially inhomogeneous (in the absence of vorticity gradient, there is no drift). As shown in [30,31] and [26,27], the systematic drift of the test vortex is due to a polarization process: the test vortex modifies the distribution of the field vortices and, in response, the retroaction of this perturbation causes the drift of the test vortex. In the absence of fluctuation, the test vortex would reach a maximum of background vorticity where \( \nabla \omega = 0 \). In fact, this systematic effect is counterbalanced by the diffusion term that tends to disperse the test vortex. Finally, an equilibrium state results from these two antagonistic effects in which the distribution of the test vortex coincides with the distribution of the bath. This equilibration process is governed by the Fokker–Planck equation (111).

The logarithmic divergence of the Coulomb factor \( \ln \Lambda \) (that persists if collective effects are included) was noted in [30]. It is due to the failure of the kinetic theory at small scales where collisions between vortices are strong. Phenomenologically, the logarithmic divergence can be regularized by introducing cutoffs so that \( \ln \Lambda = \ln(r/d) \). A detailed calculation of the lower cutoff \( d \) has been made by Dubin and Jin [28,29]. They propose to take \( d = \text{Max}(\delta, l) \), where \( l \) is the trapping distance \( l = (\gamma/4\pi|\Sigma|)^{1/2} \) and \( \delta \) is the diffusion-limited maximum separation \( \delta = (4D/|\Sigma|)^{1/2} \), where \( D \) is the diffusion coefficient given by equation (113). Orders of magnitude indicate that \( r/l \sim R(\Sigma/\gamma)^{1/2} \sim R(\Gamma/\gamma R^2)^{1/2} \sim N^{1/2} \) (where \( R \) is the system size), \( \delta/l \sim (D/\gamma)^{1/2} \sim (\ln \Lambda)^{1/2} \sim [\ln(\ln N)]^{1/2} \) and \( r/\delta \sim N^{1/2}/[\ln(\ln N)]^{1/2} \). Therefore, at leading order, the Coulombian factor scales with \( N \) like [41]

\[
\ln \Lambda \sim \frac{1}{2} \ln N. 
\] (114)

4.7. Thermal bath: Boltzmann distribution

For a thermal bath, the field vortices have the Boltzmann distribution of statistical equilibrium

\[
\omega(r) = Ae^{-\beta \gamma \psi_*}, \tag{115}
\]

where \( \psi_* = \psi + (\Omega_L/2)r^2 \) is the relative stream function taking into account the invariance by rotation of the system (see section 2). We note the identity

\[
\frac{d\omega}{dr} = -\beta \gamma \omega \frac{d\psi_*}{dr} = \beta \gamma \omega (\Omega - \Omega_L)r, \tag{116}
\]

where we have used \( -d\psi/dr = \Omega r \). Substituting this relation in equation (105), we obtain

\[
V_{\text{pol}} = 2\pi^2 \gamma^2 \beta \frac{1}{r} \int_0^{+\infty} dr' \chi(r, r') \delta(\Omega - \Omega') \omega(\Omega' - \Omega_L)r'. \tag{117}
\]

A similar logarithmic divergence at small scales arises in 3D Coulombian plasmas and stellar systems. In that case, the Coulombian factor has to be regularized at the Landau length.
Using the δ function to replace Ω′ by Ω, then using Ω − Ω_L = −(1/r) dψ_*/dr, and comparing the resulting expression with equation (103), we finally find that

\[ V_{pol} = -Dβγ \frac{dψ_*}{dr}. \]  

(118)

In vectorial form, this can be written as \( \mathbf{V}_{pol} = -Dβγ \nabla ψ_* \) [30]. The drift is perpendicular to the relative mean-field velocity \( \mathbf{u}_* = -\mathbf{z} \times \nabla ψ_* \) and the drift coefficient (mobility) satisfies an Einstein relation

\[ ξ = Dβγ. \]  

(119)

We note that the drift coefficient ξ and the diffusion coefficient \( D \) depend on the position \( r \) of the test vortex and we recall that the inverse temperature \( β \) is negative in cases of physical interest. We stress that the Einstein relation is valid for the drift \( V_{pol} \) due to the polarization, not for the total drift \( A \) that has a more complex expression due to the term \( dD/dr \). We do not have this subtlety for the usual Brownian motion where the diffusion coefficient is constant. For a thermal bath, using equation (118), the Fokker–Planck equation (59) takes the form

\[ \frac{∂P}{∂t} = \frac{1}{r} \frac{∂}{∂r} \left[ r D(r) \left( \frac{∂P}{∂r} + βγP \frac{dψ_*}{dr} \right) \right], \]  

(120)

where \( D(r) \) is given by equation (103) with equation (115). This equation has a form similar to the familiar Smoluchowski equation in Brownian theory. This is a drift–diffusion equation describing the evolution of the distribution \( P(r,t) \) of the test vortex in an ‘effective potential’ \( U_{eff} = ψ_* \) produced by the field vortices. For \( t \to +\infty \), the distribution of the test vortex relaxes towards the Boltzmann distribution: \( P_e(r) = (A/Nγ)e^{-βγψ(r)} \) at equilibrium. Of course, if the profile of angular velocity of the Boltzmann distribution is monotonic, we find that equation (109) with equation (115) returns equation (118) with a diffusion coefficient given by equation (108) with equation (115). However, equations (118) and (103) are valid even if the profile of angular velocity in the Boltzmann distribution is non-monotonic. Finally, we note that the systematic drift \( \mathbf{V}_{pol} = -D(r)βγ \nabla ψ_* \) of a point vortex [30] is the counterpart of the dynamical friction \( \mathbf{F}_{pol} = -D(r)βm \mathbf{v} \) of a star in a cluster [36], and the Smoluchowski-type Fokker–Planck equation (120) is the counterpart of the Kramers–Chandrasekhar equation (see [34,48] for a development of the analogy between 2D vortices and stellar systems).

### 4.8. Gaussian vortex

For illustration, we consider the particular case of a Gaussian distribution of field vortices:

\[ ω(r) = \frac{Γγλ}{2π} e^{-λγr^2/2}. \]  

(121)

It can be viewed as a particular statistical equilibrium state of the form (115) corresponding to \( β \to 0 \) and \( Ω_L \to +\infty \) in such a way that the product \( λ ≡ βΩ_L \) remains finite [12]. It is also similar to the familiar Maxwellian distribution of velocities in a gas at inverse temperature \( λ \) (provided that \( r \) is replaced by \( v \) and \( γ \) by \( m \)). The Fokker–Planck equation (111) or (120) describing the relaxation of a test vortex in a Gaussian bath is

\[ \frac{∂P}{∂t} = \frac{1}{r} \frac{∂}{∂r} \left[ r D(r) \left( \frac{∂P}{∂r} + λγPr \right) \right], \]  

(122)
where $D(r)$ is given by equation (108) with equation (121). This equation has a form similar to the familiar Kramers equation in Brownian theory (provided that $r$ is replaced by $v$ and $\gamma$ by $m$). This is a drift–diffusion equation describing the evolution of the distribution $P(r,t)$ of the test vortex in an ‘effective potential’ $U_{\text{eff}} = r^2/2$ (quadratic) produced by the field vortices. For $t \to +\infty$, the distribution of the test vortex relaxes towards the Maxwellian distribution of the bath: $P_e(r) = (\lambda \gamma / 2 \pi) e^{-\lambda \gamma r^2/2}$ at equilibrium.

Using $\Omega(r) = r^{-2} \int_0^r \omega(r') r' \, dr'$ (see section 6.1 of [41]), the profile of angular velocity of the bath is

$$\Omega(r) = \frac{\Gamma}{2\pi r^2} (1 - e^{-\lambda \gamma r^2/2}).$$

If we neglect collective effects, the diffusion coefficient is given by equation (113). Using equations (121) and (123), we obtain the expression

$$D(r) = \frac{\gamma^2 \lambda \ln \Lambda r^2}{4(2e^{\lambda \gamma r^2/2} - \lambda \gamma r^2 - 2)}.$$  

We first note that the diffusion coefficient diverges like $D(r) \sim \ln \Lambda / (\lambda r^2)$ for $r \to 0$ due to the vanishing of the shear in the core of the vortex: $\Sigma(r) \sim -(\lambda^2 \gamma^2 \Gamma / 8\pi)r^2$ as $r \to 0$. This indicates a failure of the kinetic theory for $r \to 0$. In fact, the kinetic theory developed in this paper is valid for sufficiently large shears. In the absence of shear, the expression of the diffusion coefficient is different (and finite) as discussed in [48, 77]. Therefore, the expression (124) of the diffusion coefficient is valid for sufficiently large $r$. We note that the diffusion coefficient decays very rapidly at large distances since $D(r) \sim (\gamma^2 \lambda \ln \Lambda / 8)r^2 e^{-\lambda \gamma r^2/2}$ for $r \to +\infty$. More generally, using equation (108) and the results of section 6.1 of [41], we have $D(r) \sim (2\pi^3 \gamma / \Gamma) \chi(r, r) \omega(r)r^2$ for $r \to +\infty$ for an arbitrary distribution $\omega(r)$ of the field vortices.

The Fokker–Planck equation (122) with the diffusion coefficient (124) has been studied in [41] by applying results previously obtained in the context of the HMF model. For this model, the equivalent of the Fokker–Planck equation (122) with equation (124), where the position $r$ is replaced by the velocity $v$, has been studied in [52, 54]. This Fokker–Planck equation presents unusual features because the diffusion coefficient decreases very rapidly with distance. This leads to ‘anomalies’ with respect to the usual Brownian motion. By applying the approach of Bouchet and Dauxois [52], one finds [41] that the autocorrelation function of the position of the test vortex $\langle r(0)r(t) \rangle$ decays algebraically like $\ln t/t$ (this algebraic decay was first obtained by Marksteiner et al [78] for the logarithmic Fokker–Planck equation to which equation (122) can be mapped). On the other hand, by applying the approach of Chavanis and Lemou [54], one finds [41] that the normalized distribution $u(r,t) = P(r,t)/P_e(r)$ has a front structure and that the front evolves very slowly with time, scaling like $r_f(t) \propto (\ln t)^{1/2}$ for $t \to +\infty$. These results show that the relaxation of $P(r,t)$ towards the equilibrium distribution $P_e(r)$ is not exponential. This is intrinsically due to the absence of a gap [41, 52, 78] in the spectrum of the Fokker–Planck equation (122) with diffusion coefficient (124).

4.9. The relaxation time of a test vortex in a bath

The derivation of the Fokker–Planck equation (107), relying on a bath approximation, assumes that the distribution of the field vortices is ‘frozen’ so that their vorticity profile
ω(r) does not evolve in time. This is always true for a thermal bath (115), corresponding to a distribution at statistical equilibrium (Boltzmann), because it does not evolve at all. For a stable steady state of the 2D Euler equation with a monotonic profile of angular velocity, this is true only on a timescale shorter than the relaxation time \( t_R \) of the system as a whole. However, as we have indicated in section 3.2, this timescale is very long because the relaxation time \( t_R \) of the system as a whole is larger than \( N t_D \).

Recalling that \( \gamma \sim 1/N \), the Fokker–Planck operator in equations (111) and (120) scales like \( \ln N/N \) (see equation (114) for the logarithmic correction). Therefore, the distribution \( P(r,t) \) of the test vortex relaxes towards the distribution \( \omega(r) \) of the bath on a typical time

\[
t_R^{\text{bath}} \sim \frac{N}{\ln N} t_D. \tag{125}
\]

This is the timescale controlling the relaxation of the test vortex, i.e. the time needed by the test vortex to acquire the distribution of the bath (see appendix F for a more precise estimate). It should not be confused with the timescale (50) controlling the relaxation of the system as a whole. Since the timescale \( t_R^{\text{bath}} \) is shorter than the timescale \( t_R \) on which \( \omega(r,t) \) changes due to collisions (in the case where \( \omega(r,t) \) has a monotonic profile of angular velocity), we can consider that the distribution of the field vortices is ‘frozen’ on the timescale (125). Therefore, our bath approximation is justified on this timescale and our approach is self-consistent.

5. Conclusion

In this paper, we have developed the kinetic theory of point vortices by taking collective effects into account. We have used the formalism of Dubin and O’Neil [23] valid for axisymmetric mean flows. This improves our previous works [30]–[34] where collective effects were neglected (but more general flows were considered). Collective effects amount to replacing the ‘bare’ potential of interaction by a ‘dressed’ potential, without altering the overall structure of the kinetic equation. In plasma physics, collective effects included in the Lenard–Balescu [24,25] equation are important because they take into account screening effects and regularize, at the scale of the Debye length, the logarithmic divergence at large scales that appears in the Landau [40] equation. In the case of point vortices, there is no divergence in the Landau-type kinetic equation (45) that ignores collective effects, so their influence may be less important than in plasma physics. We have also developed a test vortex approach and a Fokker–Planck theory. We have obtained the expressions of the diffusion coefficient and drift term directly from the equations of motion, taking collective effects into account. We have shown that they can also be obtained from the kinetic equation (47) by making a bath approximation leading to the Fokker–Planck equation (107). We have presented the results for axisymmetric flows, but similar results can be obtained for unidirectional flows [30,31].

The kinetic theory developed in this paper is valid at the order of \( 1/N \) so it describes the evolution of the system on a timescale \( N t_D \). This is sufficient to study the relaxation of a test vortex in a bath of field vortices since the corresponding relaxation time is of order \( (N/\ln N) t_D \) (see section 4.9). However, this is not sufficient to describe the evolution of the system as a whole towards the Boltzmann statistical equilibrium state because, for
axisymmetric flows (with a monotonic profile of angular velocity), the relaxation time is larger than $N t_D$ (see section 3.2). Therefore, we need to develop the kinetic theory at higher orders (taking into account three-body, four-body, etc, correlations). At present, this is not done and the scaling of the relaxation time with $N$ remains an open problem.

We can try to make speculations by using analogies with other systems with long-range interactions that present similar features [34]. The most natural scaling would be $N^2 t_D$ that corresponds to the next-order term in the expansion of the basic equations of the kinetic theory in powers of $1/N$ [32, 34, 79]. An $N^2$ scaling is indeed obtained numerically [80, 81] for spatially homogeneous 1D plasmas for which the Lenard–Balescu collision term vanishes at the order of $1/N$ [57, 58]. However, the scaling of the relaxation time may be more complex. For example, for the permanently spatially homogeneous HMF model (for which the Lenard–Balescu collision term also vanishes at the order of $1/N$ [52, 53]), Campa et al [82] report a relaxation time scaling like $e^{N t_D}$ (this timescale is, however, questioned in recent works [83] who find an $N^2 t_D$ scaling). It could also happen that the point vortex gas never reaches Boltzmann’s statistical equilibrium because the evolution is non-ergodic [22] and the system does not mix well enough under the effect of collisions. All these speculations [34] could be checked numerically by solving the N-vortex dynamics. The situation should be different for more general flows that are not axisymmetric. In that case, there are potentially more resonances so that the relaxation time could be reduced and achieve the natural scaling $N t_D$ corresponding to the first-order term in the kinetic theory [32, 34]. An $N t_D$ scaling has been numerically observed for the relaxation of a non-axisymmetric distribution of point vortices [75]. Similarly, 1D spatially inhomogeneous systems with long-range interactions are expected to relax towards statistical equilibrium on a timescale $N t_D$ due to additional resonances [34, 59, 60]. Such a scaling is indeed observed numerically for spatially inhomogeneous 1D stellar systems [84]–[87] and for the spatially inhomogeneous HMF model [88]. On the other hand, for the HMF model, if an initially spatially homogeneous distribution function becomes Vlasov-unstable during the collisional evolution, a dynamical phase transition from a non-magnetized to a magnetized state takes place (as theoretically studied in [89]) and the relaxation time should be intermediate between $N^2 t_D$ (permanently homogeneous) and $N t_D$ (permanently inhomogeneous). In that situation, Yamaguchi et al [90] find a relaxation time scaling like $N^\delta t_D$ with $\delta = 1.7$. The previous argument (leading to $1 < \delta < 2$) may provide a first step towards the explanation of this anomalous exponent. The same phenomenon (loss of Euler stability due to collisions and dynamical phase transition from an axisymmetric distribution to a non-axisymmetric distribution) could happen for the point vortex system.

Finally, we would like to conclude by briefly mentioning other works on the kinetic theory of point vortices that are related to our study. Schecter and Dubin [26, 27] have considered the motion of a point vortex in a background vorticity gradient (without fluctuation). They obtain an expression of the drift that coincides with the one obtained previously in [30, 31]. However, the point of view is different because they consider the evolution of a test vortex in an external vorticity field without fluctuation, while we consider the evolution of a test vortex in a bath of field vortices that creates a vorticity gradient but also contains fluctuations. The vorticity gradient leads to a (deterministic) drift as a result of a polarization process and the fluctuations lead to a diffusion. Therefore, the test vortex has a stochastic motion that can be modelled by a Fokker–Planck equation.
of a drift–diffusion type. There is no Fokker–Planck equation in the approach of Schecter and Dubin [26,27] since the evolution of the test vortex is purely deterministic. On the other hand, Dubin and Jin [28] consider the diffusion of point vortices in an external shear. They obtain an expression of the diffusion coefficient that coincides with the one obtained previously in [30,31] from the Kubo formula (in particular, they recover the shear reduction found in [30]). However, the point of view is different because they consider the effect of an external shear without vorticity gradient. Therefore, there is no drift. In contrast, in our approach, the test vortex moves in a bath of field vortices that presents a vorticity gradient. In addition to its diffusive motion, the test vortex modifies the distribution of the field vortices (as in a polarization process) and the retroaction of the field vortices causes the drift of the test vortex. In a sense, the works of Dubin and collaborators isolate the effects of drift [26,27] and diffusion [28] while they are taken into account simultaneously in our works [30]–[34]. The extension of the kinetic theory to the case of point vortices with different values of individual circulation $\gamma$ has been performed in [29,41,91]–[93]. The statistics of the velocity fluctuations arising from a random distribution of point vortices has been studied in [77,94]–[96] by analogy with the study of Chandrasekhar and von Neumann [97] on the statistics of the gravitational force arising from a random distribution of stars. This theory has been used in [77] to obtain an expression $D \propto \gamma (\ln N)^{1/2}$ of the diffusion coefficient of point vortices in the absence of shear (i.e. in the opposite limit to the one considered here). This result has been applied to the problem of 2D decaying turbulence in [98] in order to obtain the expression of the exponent of anomalous diffusion $\langle x^2 \rangle \sim t^{1+\xi/2}$, where $\xi$ is the exponent controlling the decay of the vortex number $N \sim t^{-\xi}$ (it is argued in [98,99] that $\xi = 1$ in the strict asymptotic scaling regime $t \to +\infty$ so that $\nu \equiv 1 + \xi/2 = 3/2$). The energy spectrum of the point vortex gas has been determined in [48,96,100]–[102]. Finally, a quasilinear theory of the 2D Euler equation has been developed in [103] in order to describe a phase of ‘gentle’ collisionless relaxation. For general references about point vortex dynamics, see [104,105].

Additional comment. In a recent paper [106], we managed to derive the Lenard–Balescu-type kinetic equation (43), taking collective effects into account, directly from the BBGKY hierarchy starting from the Liouville equation. As is well known in plasma physics [64,70,107], the approach based on the first two equations of the BBGKY hierarchy (neglecting three-body correlations) is equivalent to the approach starting from the Klimontovich equation and making a quasilinear approximation. More generally, the whole BBGKY hierarchy can be reconstructed from the Klimontovich equation (which is equivalent to the Liouville equation). However, the Klimontovich approach exposed here provides a technically simpler and more transparent derivation of the Lenard–Balescu-type equation than the one starting from the BBGKY hierarchy.

Appendix A. About incomplete relaxation

The statistical theory of violent relaxation developed by Lynden-Bell [69] for the Vlasov equation and by Miller [67] and Robert and Sommeria [68] for the 2D Euler equation is based on an assumption of ergodicity or efficient mixing. However, in many cases, it has been observed that this assumption is not fully realized and that relaxation is incomplete [108].

doi:10.1088/1742-5468/2012/02/P02019
Numerical simulations have been made in the ‘two-level’ case where the vorticity $\omega$ (resp. distribution function $f$) takes only two values $\sigma_0$ and 0 (resp. $\eta_0$ and 0) [65, 66, 109]. Generically, one observes the existence of a ‘mixing zone’ characterized by a linear relationship between $\ln(\bar{\omega}/(\sigma_0 - \bar{\omega}))$ and $\psi_*$ (resp. $\ln(f/(\eta_0 - \bar{f}))$ and $\epsilon = \nu^2/2 + \Phi$). This corresponds to a local Lynden-Bell distribution $\bar{\omega} = \sigma_0/(1 + \exp(\beta'\psi_* + \alpha'))$ (resp. $\bar{f} = \eta_0/(1 + \exp(\beta\epsilon + \alpha))$) with parameters $\beta'$ and $\alpha'$ different from the global ones $\beta$ and $\alpha$. On the other hand, violent relaxation is incomplete in the core and in the halo, resulting in deviations from this linear relationship for high and low $\bar{\omega}$ (resp. $\bar{f}$).

In particular, the peak vorticity (resp. distribution function) has the tendency to be conserved and the system has the tendency to remain confined. This can be explained by a kinetic theory of collisionless violent relaxation [32, 79]. The evolution of $\bar{\omega}$ (resp. $\bar{f}$) is governed by a kinetic equation of the Landau type with a term describing relaxation towards the Lynden-Bell distribution. One feature of this term is that the relaxation current involves the product $\bar{\omega}(\sigma_0 - \bar{\omega})$ (resp. $\bar{f}(\eta_0 - \bar{f})$) measuring the strength of the fluctuations of the vorticity (or distribution function). This implies that relaxation is slow in the core ($\bar{\omega} \rightarrow \sigma_0$, $\bar{f} \rightarrow \eta_0$) and in the tail ($\bar{\omega} \rightarrow 0$, $\bar{f} \rightarrow 0$) of the distribution where the fluctuations are weak. Since, in addition, the efficiency of relaxation decreases with time (through a prefactor $\epsilon(t) \rightarrow 0$ in the kinetic equation related to the correlation length of the fluctuations), the system will not have time to mix well in these regions. Therefore, the vorticity (resp. distribution function) will be higher in the core and lower in the tail as compared to the Lynden-Bell prediction. In the intermediate region, in contrast, the system will mix efficiently and the kinetic equation will rapidly converge towards a local Lynden-Bell distribution which cancels the relaxation current. This theoretically motivated scenario [32, 79] seems to be consistent with the numerical simulations of 2D turbulence (see figure 2 in [65] and figure 5 in [66]) and 1D self-gravitating systems (see figure 20 in [109]). Note that, in the case of incomplete relaxation (non-ergodicity), the QSS is sometimes found to be close to a polytropic (Tsallis) distribution [66, 110, 111] but this is not general.

**Appendix B. Bare and dressed potentials of interaction**

We assume that the stream function $\psi(r)$ is related to the vorticity $\omega(r)$ by an equation of the form $L\psi = -\omega$, where $L$ is a linear operator of the Laplacian type. The potential of interaction $u(r,r')$, which is the Green function of the operator $L$, is the solution of $Lu(r,r') = -\delta(r-r')$ with appropriate boundary conditions. For simplicity, we shall consider an infinite domain where $u = u(|r-r'|)$ only depends on the absolute distance between two points (our results can be extended to more general situations by using the Green functions of Lin [112]). In that case, the stream function is related to the vorticity field by an expression of the form

$$\psi(r,t) = \int u(|r-r'|)\omega(r',t) \, dr.$$  \hspace{1cm} (B.1)

It can be written as $\psi = u \ast \omega$, where $\ast$ denotes the product of convolution. For ordinary flows, the stream function is related to the vorticity by the Poisson equation $\Delta \psi = -\omega$. In that case $L = \Delta$ and the potential of interaction in an infinite domain is given by $u(|r-r'|) = -(1/2\pi) \ln |r-r'|$. This corresponds to a Newtonian (or Coulombian)
interaction in two dimensions. In the quasigeostrophic (QG) model describing geophysical flows [44], the stream function is related to the (potential) vorticity by the screened Poisson equation \( \Delta \psi - k_R^2 \psi = -\omega \), where \( k_R^{-1} \) is the Rossby radius. In that case \( L = \Delta - k_R^2 \) and the potential of interaction in an infinite domain is given by \( u(|r-r'|) = \frac{1}{2\pi} K_0(k_R|r-r'|) \), where \( K_0(x) \) is a modified Bessel function. The previous results are recovered for \( k_R \to 0 \).

Let us first consider the ordinary potential corresponding to \( L = \Delta \). Introducing polar coordinates, the Poisson equation \( \Delta \delta \psi = -\delta \omega \) for the fluctuations can be written as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \delta \psi}{\partial \theta^2} = -\delta \omega. \tag{B.2}
\]

Taking the Fourier–Laplace transform of this equation, we obtain

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{\psi}}{\partial r} \right) - \frac{n^2}{r^2} \right] \tilde{\psi}(n, r, \sigma) = -\delta \tilde{\omega}(n, r, \sigma). \tag{B.3}
\]

This is of the form \( \mathcal{L} \tilde{\psi} = -\delta \tilde{\omega} \) with \( \mathcal{L} = (1/r)(\partial/\partial r)r(\partial/\partial r) - (n^2/r^2) \). Substituting equation (23) in equation (B.3), we find that

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \hat{\psi}}{\partial r} \right) - \frac{n^2}{r^2} - \frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \right] \hat{\psi}(n, r, \sigma) = -\frac{\delta \hat{\omega}(n, r, 0)}{i(n\Omega - \sigma)}. \tag{B.4}
\]

Therefore \( \tilde{\psi}(n, r, \sigma) \) is related to \( \delta \omega(n, r, 0) \) by a relation of the form (27) where the Green function \( G(n, r, r', \sigma) \) is defined by

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) - \frac{n^2}{r^2} - \frac{n(1/r)(\partial \omega/\partial r)}{n\Omega - \sigma} \right] G(n, r, r', \sigma) = -\frac{\delta(r - r')}{2\pi r}. \tag{B.5}
\]

If we neglect collective effects in the foregoing equation, we obtain

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial G_{\text{bare}}}{\partial r} \right) - \frac{n^2}{r^2} \right] G_{\text{bare}}(n, r, r') = -\frac{\delta(r - r')}{2\pi r}. \tag{B.6}
\]

This shows that the bare Green function \( G_{\text{bare}}(n, r, r') = \hat{u}_n(r, r') \) is the Fourier transform of the potential of interaction \( u \) that is a solution of the Poisson equation \( \Delta u = -\delta \).

Let us now consider a general form of interaction between point vortices. The fluctuations of the stream function are related to the fluctuations of the vorticity by

\[
\delta \psi(r, t) = \int u(|r-r'|) \delta \omega(r', t) \, dr. \tag{B.7}
\]

The potential of interaction can be written as

\[
u(|r-r'|) = u \left( \sqrt{r^2 + r'^2 - 2rr'\cos(\theta - \theta')} \right) = u(r, r', \phi), \tag{B.8}
\]

where \( \phi = \theta - \theta' \). We note that \( u(r, r', \phi) = u(r', r, \phi) \) and \( u(r, r', -\phi) = u(r, r', \phi) \). Due to its \( \phi \) periodicity, it can be decomposed in Fourier series as

\[
\hat{u}_n(r, r') = \int_0^{2\pi} \frac{d\phi}{2\pi} u(r, r', \phi) \cos(n\phi). \tag{B.9}
\]
Taking the Fourier–Laplace transform of equation (B.7) and using the fact that the integral is a product of convolution, we get

\[ \delta \tilde{\psi}(n, r, \sigma) = 2\pi \int_0^{+\infty} r' \, dr' \, \hat{\mu}(r, r') \delta \tilde{\omega}(n, r', \sigma). \] (B.10)

If we introduce equation (23) in equation (B.10), we obtain a Fredholm integral equation:

\[ \delta \tilde{\psi}(n, r, \sigma) + 2\pi \int_0^{+\infty} r' \, dr' \, \hat{\mu}(r, r') \frac{n(1/r')(\partial \omega'/\partial r')}{n\Omega' - \sigma} \delta \tilde{\psi}(n, r', \sigma) = 2\pi \int_0^{+\infty} r' \, dr' \, \hat{\mu}(r, r') \frac{\delta \tilde{\omega}(n, r', 0)}{i(n\Omega' - \sigma)}. \] (B.11)

This equation relates the Fourier–Laplace transform of the fluctuations of the stream function to the Fourier transform of the initial fluctuations of the vorticity. It is therefore equivalent to equation (26). If we neglect collective effects, the foregoing equation reduces to

\[ \delta \tilde{\psi}(n, r, \sigma) = 2\pi \int_0^{+\infty} r' \, dr' \, \hat{\mu}(r, r') \frac{\delta \tilde{\omega}(n, r, 0)}{i(n\Omega - \sigma)}. \] (B.12)

Comparing equation (B.12) with equation (27), we see that \( G_{\text{bare}}(n, r, r') = \hat{\mu}(r, r') \) is just the Fourier transform of the ‘bare’ potential of interaction \( u \). This is equivalent to \( LG_{\text{bare}}(n, r, r') = -\delta(r - r')/2\pi r \). When collective effects are taken into account, equation (B.12) is replaced by equation (27) where, according to equation (B.11), the ‘dressed’ potential of interaction satisfies

\[ G(n, r, r', \sigma) + 2\pi \int_0^{+\infty} r'' \, dr'' \, \hat{\mu}(r, r'') \frac{n(1/r'')(\partial \omega''/\partial r'')}{n\Omega'' - \sigma} G(n, r'', r', \sigma) = \hat{\mu}(n, r'), \] (B.13)

which is equivalent to equation (28).

For the Newtonian interaction \( u(|r - r'|) = -(1/2\pi) \ln |r - r'| \), we have \( u(r, r', \phi) = -(1/4\pi) \ln (r^2 + r'^2 - 2rr' \cos \phi) \). The integrals in equation (B.9) can be performed analytically [34] and we obtain

\[ \hat{\mu}(r, r') = \frac{1}{4\pi |n|} \left( \frac{r_\sigma}{r_\pi} \right)^{|n|}, \quad \hat{\mu}_0(r, r') = -\frac{1}{2\pi} \ln r_\sigma. \] (B.14)

Therefore, the potential of interaction can be written as

\[ u(r, r', \phi) = -\frac{1}{2\pi} \ln r_\sigma + \frac{1}{4\pi} \sum_{n \neq 0} \frac{1}{|n|} \left( \frac{r_\sigma}{r_\pi} \right)^{|n|} e^{i n \phi}, \] (B.15)

which is just the Fourier decomposition of the logarithm in two dimensions. When collective effects are neglected, the function defined by equation (48) can be written [31] as

\[ \chi_{\text{bare}}(r, r') = \frac{1}{8\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n} \left( \frac{r_\pi}{r_\sigma} \right)^{2n} = -\frac{1}{8\pi^2} \ln \left[ 1 - \left( \frac{r_\pi}{r_\sigma} \right)^2 \right]. \] (B.16)
Taking \( r' = r \) in the foregoing expression, we obtain
\[
\chi_{\text{bare}}(r, r) = \frac{1}{8\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n} = \frac{1}{8\pi^2} \ln \Lambda, \tag{B.17}
\]
where \( \ln \Lambda \) is the Coulomb factor that has to be regularized appropriately (see section 4.6). Analogous expressions, valid in a circular domain of size \( R \), are given in [32].

**Appendix C. The resolvent**

Substituting equation \( (27) \) in equation \( (23) \), we obtain
\[
\delta \omega(n, r, \sigma) = -\frac{n(1/r)(\partial \omega/\partial r)}{\sigma - n\Omega(r)} \int_0^{+\infty} 2\pi r'' dr'' G(n, r, r'', \sigma) \frac{\delta \omega(n, r', 0)}{i(\sigma - n\Omega(r'))} \delta(r-r')/r', \tag{C.1}
\]
This equation relates the Fourier–Laplace transform of the fluctuations of the vorticity to the Fourier transform of the initial fluctuations of the vorticity. If we consider an initial condition of the form \( \delta \omega(r, \theta, 0) = \gamma \delta(\theta - \theta') \delta(r-r')/r, \) implying
\[
\delta \omega(n, r, 0) = \frac{\gamma}{2\pi r} e^{-i\theta'} \delta(r-r'), \tag{C.2}
\]
we find that
\[
\delta \omega(n, r, \sigma) = -\frac{n(1/r)(\partial \omega/\partial r)}{\sigma - n\Omega(r)} \gamma e^{-i\theta'} \frac{G(n, r, r', \sigma)}{i(\sigma - n\Omega(r'))} - \gamma \frac{e^{-i\theta'}}{2\pi r} \frac{\delta(r-r')}{i(\sigma - n\Omega(r'))}. \tag{C.3}
\]
This quantity, with \( \theta' = 0 \) and \( \gamma = 1 \), is called the propagator \( U_n(r, r', \sigma) \) of the linearized 2D Euler equation. This operator is also called the resolvent operator as it connects \( \delta \omega(n, r, \sigma) \) to its initial value. Indeed, equation \( (C.1) \) can be written as
\[
\delta \omega(n, r, \sigma) = \int_0^{+\infty} U_n(r, r'', \sigma) \delta \omega(n, r'', 0) 2\pi r'' dr'', \tag{C.4}
\]
with
\[
U_n(r, r', \sigma) = \frac{i}{\sigma - n\Omega(r)} \frac{\delta(r-r')}{2\pi r} + i \frac{G(n, r, r', \sigma)}{(\sigma - n\Omega(r))} \frac{1}{(\sigma - n\Omega(r'))} \frac{\partial \omega}{\partial r}. \tag{C.5}
\]
If we substitute equation \( (B.10) \) in equation \( (23) \), we obtain
\[
\delta \omega(n, r, \sigma) = -\frac{n(1/r)(\partial \omega/\partial r)}{n\Omega(r) - \sigma} \int_0^{+\infty} 2\pi r' dr' \dot{u}_n(r, r') \delta \omega(n, r', \sigma) + \frac{\delta \omega(n, r, 0)}{i(n\Omega(r) - \sigma)}. \tag{C.6}
\]
This equation relates the Fourier–Laplace transform of the fluctuations of the vorticity to the Fourier transform of the initial fluctuations of the vorticity, so it is equivalent to equation \( (C.1) \).
Appendix D. Autocorrelation of the fluctuations of the vorticity

According to equation (22), we have

\[
\langle \delta \omega(n, r, 0) \delta \omega(n', r', 0) \rangle = \int_{0}^{2\pi} \frac{d\theta}{2\pi} \int_{0}^{2\pi} \frac{d\theta'}{2\pi} e^{-i(n\theta+n'\theta')} \langle \delta \omega(\theta, r, 0) \delta \omega(\theta', r', 0) \rangle
\]

\[
= \int_{0}^{2\pi} \frac{d\theta}{2\pi} \int_{0}^{2\pi} \frac{d\theta'}{2\pi} e^{-i(n\theta+n'\theta')} \left[ \langle \omega_d(\theta, r, 0) \omega_d(\theta', r', 0) \rangle - \omega(r)\omega(r') \right].
\]

The expression (4) of the discrete vorticity distribution leads to

\[
\langle \omega_d(\theta, r, 0) \omega_d(\theta', r', 0) \rangle = \gamma^2 \sum_{i,j} \left\langle \delta(\theta - \theta_i) \frac{\delta(r - r_i)}{r} \delta(\theta' - \theta_j) \frac{\delta(r' - r_j)}{r'} \right\rangle
\]

\[
= \gamma^2 \sum_{i} \left\langle \delta(\theta - \theta_i) \frac{\delta(r - r_i)}{r} \delta(\theta' - \theta) \frac{\delta(r' - r)}{r'} \right\rangle
\]

\[
+ \gamma^2 \sum_{i \neq j} \left\langle \delta(\theta - \theta_i) \frac{\delta(r - r_i)}{r} \delta(\theta' - \theta_j) \frac{\delta(r' - r_j)}{r'} \right\rangle
\]

\[
= 2\pi \gamma \omega(r) \delta(\theta - \theta') \frac{\delta(r - r')}{2\pi r} + \omega(r)\omega(r'),
\]

where we have assumed that there is no correlation initially (if there are initial correlations, it can be shown that they are washed out rapidly [5, 42] so they have no effect on the final form of the collision term). Combining equations (D.1) and (D.2), we obtain

\[
\langle \delta \omega(n, r, 0) \delta \omega(n', r', 0) \rangle = \gamma \delta_{n,-n'} \frac{\delta(r - r')}{2\pi r} \omega(r).
\]

Appendix E. The function \( K(r, r', t) \)

The kinetic equation (47) can be rewritten in the more compact form

\[
\frac{\partial \omega}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} r \int_{0}^{+\infty} dr' K(r, r', t) \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r'} \frac{\partial}{\partial r'} \right) \omega(r, t)\omega(r', t),
\]

with

\[
K(r, r', t) = 2\pi^2 \gamma r' \chi(r, r', t) \delta(\Omega(r, t) - \Omega(r', t)).
\]

In the bath approximation, the Fokker–Planck equation (107) can be rewritten as

\[
\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} r \int_{0}^{+\infty} dr' K(r, r', t) \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r'} \frac{\partial}{\partial r'} \right) P(r, t)\omega(r').
\]

The diffusion coefficient and the drift terms are

\[
D(r) = \frac{1}{r} \int_{0}^{+\infty} dr' K(r, r')\omega(r'), \quad V_{pol}(r) = \int_{0}^{+\infty} \frac{dr'}{r'} K(r, r') \frac{d\omega}{dr}(r'),
\]

\[
A(r) = \int_{0}^{+\infty} r r' \omega(r') \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r'} \frac{\partial}{\partial r'} \right) K(r, r').
\]
Appendix F. Relaxation time of a test vortex in a Gaussian bath

When the field vortices have the Gaussian distribution (121), a more precise estimate of the relaxation time of the test vortex can be given. We first define the ‘vortex size’ $R$ by $R^2 = \langle r^2 \rangle = L/\Gamma$, where $L$ is the angular momentum. For a Gaussian distribution, $R = (2/\lambda \gamma)^{1/2}$. If we set $x = (\lambda \gamma/2)^{1/2}r$, the Fokker–Planck equation (122) can be rewritten as

$$\frac{\partial P}{\partial t} = \frac{1}{t_{\text{bath}}} \frac{1}{x} \frac{\partial}{\partial x} \left[ xG(x) \left( \frac{\partial P}{\partial x} + 2Px \right) \right], \quad (F.1)$$

with

$$G(x) = \frac{x^2}{e^{x^2} - x^2 - 1}, \quad t_{\text{bath}}^R = \frac{8}{\lambda \gamma^2 \ln \Lambda}. \quad (F.2)$$

The ‘reference time’ $t_{\text{bath}}^R$ gives an estimate of the relaxation time of the test vortex in a Gaussian bath. It we define the dynamical time by $t_D = R^2/\Gamma = L/\Gamma^2 = 2/\lambda \gamma \Gamma$, we obtain

$$t_{\text{bath}}^R = \frac{4\Gamma}{\gamma \ln \Lambda} \sim \frac{8N}{\ln N} t_D, \quad (F.3)$$

where we have used equation (114) to get the equivalent for $N \to +\infty$. We can also estimate the relaxation time as follows. The typical position of the test vortex increases like $\langle r^2 \rangle \sim 4D(R)t$. The relaxation time $t_{\text{r}}^\text{bath}$ is the time needed by the vortex to diffuse over the distance $R$. Taking $\langle r^2 \rangle = R^2$ in the foregoing formula, we obtain

$$t_{\text{r}}^\text{bath} = \frac{R^2}{4D(R)} = \frac{2}{\lambda \gamma^2 G(1) \ln \Lambda} = 0.18 t_{\text{r}}^\text{bath}, \quad (F.4)$$

where we have used $G(1) = 1/(e - 2) \approx 1.39221$. Finally, although the relaxation of the test vortex towards the Gaussian distribution is not exponential (see section 4.8), a measure of the ‘relaxation time’ is provided by $t_{\text{r}}^\text{bath} = \xi^{-1}$, where $\xi = D(R)\lambda \gamma$ is the drift coefficient (mobility) given by an Einstein formula. This yields

$$t_{\text{r}}^\text{bath} = \frac{1}{D(R)\lambda \gamma} = \frac{4}{\lambda \gamma^2 G(1) \ln \Lambda} = 2t_{\text{r}}^\text{bath}. \quad (F.5)$$

The same arguments can be extended to the Fokker–Planck equations (111) and (120), leading to the scaling (125) of the relaxation time of a test vortex in a bath, up to a numerical factor.

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