**QUOTIENTS OF DEFINITE PERIODIC KNOTS ARE DEFINITE**

KEEGAN BOYLE

**Abstract.** A knot $K$ is definite if $|\sigma(K)| = 2g(K)$. We prove that the quotient of a definite periodic knot is definite by considering equivariant minimal genus Seifert surfaces.

1. **Introduction**

Let $K$ be a knot in $S^3$ with signature $\sigma(K)$ and genus $g(K)$. Then $K$ is definite if $|\sigma(K)| = 2g(K)$. This is a relatively small class of knots, but this condition has a nice geometric interpretation. Specifically, a knot is definite if and only if it has a Seifert surface with definite linking form.

A knot $K \subset S^3$ is periodic if it is fixed by a finite cyclic group acting on $S^3$ with fixed set an unknot disjoint from $K$. In this case we refer to the image of $K$ in $S^3/(\mathbb{Z}/p)$ as the quotient knot.

The goal of this paper is to investigate periodic definite knots, and in particular apply a result of Edmonds [Edm84, Theorem 4] to prove the following theorem.

**Theorem 1.** The quotient of a periodic definite knot is definite.

2. **Background**

**Definition 2.1.** A quadratic form $\langle -, - \rangle$ is positive (resp. negative) definite if $\langle x, x \rangle \geq 0$ (resp. $\leq 0$) for all $x \neq 0$.

We will also use the equivalent characterisation that a matrix is positive (resp. negative) definite if and only if all of its eigenvalues are positive (resp. negative).

**Definition 2.2.** A Seifert surface $S$ for $K$ is positive (resp. negative) definite if the (symmetrized) linking form $\text{lk}(-, -)$ on $H_1(S)$ as defined in [GL78, Section 2] is positive (resp. negative) definite. That is, the symmetrized Seifert matrix for $S$ is definite.

**Definition 2.3.** A knot is definite if it has a definite Seifert surface.

**Lemma 1.** Let $K \subset S^3$ be a knot. Then the following are equivalent.

1. $K$ is definite.
2. Every minimal genus Seifert surface for $K$ is definite.
3. $|\sigma(K)| = 2g(K)$, where $g(K)$ is the genus of $K$.

**Proof.** (2) implies (1) is obvious, and we will show that (1) implies (3) and (3) implies (2).

To see that (1) implies (3), suppose $K$ is definite with definite Seifert surface $S$ and corresponding symmetrized Seifert matrix $M \in M_n(\mathbb{Z})$. Since $M$ is definite, $\sigma(M) = \pm n = \sigma(K)$. In particular, $M$ is a minimal dimensional symmetrized Seifert matrix and so $S$ is a minimal genus Seifert surface. Hence $|\sigma(K)| = 2g(K)$.

On the other hand, suppose $|\sigma(K)| = 2g(K)$. Then taking any minimal genus Seifert surface $S$ with symmetrized Seifert matrix $M \in M_n(\mathbb{Z})$, we see that $|\sigma(K)| = |\sigma(M)| \leq \dim(M) = 2g(K)$, and hence $|\sigma(M)| = n$ so $M$ is definite. \qed
The following proposition gives a strong restriction on the Alexander polynomial of definite knots.

**Proposition 1.** Let $K \subset S^3$ be a definite knot. Then $|\Delta_K(t)| = |\sigma(K)| = 2g(K)$, where $|\Delta_K(t)|$ is the width of the Alexander polynomial.

**Proof.** Let $S$ be a definite Seifert surface for $K$ with Seifert matrix $M \in M_n(\mathbb{Z})$, and recall that $\Delta_K(t) = \det(M^T - tM)$. Multiplying both sides by $\det(M^{-1})$ makes it clear that the first and last terms of $\Delta_K(t)$ will be $\det(M)t^n$ and $\det(M)$ respectively. Since $M$ is definite, $\det(M) \neq 0$, and so the width of the Alexander polynomial is $n = |\sigma(M)| = |\sigma(K)|$. The second inequality is proved in Lemma 1. □

### 3. Periodic definite knots

**Theorem 1.** The quotient knot of a periodic definite knot is definite.

The proof of this theorem relies on the following theorem of Edmonds.

**Theorem 2.** [Edm84, Theorem 4] Let $\tilde{K}$ be a periodic knot. Then there exists a minimal genus Seifert surface $\tilde{S}$ for $\tilde{K}$ which is preserved by the periodic action. Furthermore, the image of $\tilde{S}$ in the quotient is a Seifert surface for the quotient knot $K$.

We will also need the following lemma.

**Lemma 2.** If the preimage of a Seifert surface $S$ under a $\mathbb{Z}/p$ rotation action in $S^3$ is a positive (resp. negative) definite Seifert surface $\tilde{S}$, then $S$ is positive (resp. negative) definite.

**Proof.** Consider a curve $C \subset S$ which is homologically non-trivial. Let $\tilde{C}$ be the (possibly disconnected) preimage of $C$ in $\tilde{S}$. Note that since $C$ is homologically non-trivial, so is $\tilde{C}$. Now suppose $\tilde{S}$ is positive definite so that $\text{lk}(\tilde{C}, \tilde{C}) > 0$. We claim that $\text{lk}(C, C) > 0$, so that $S$ is also positive definite. The linking number $\text{lk}(C, C)$ is the sum of (signed) intersection points between $C$ and the Seifert surface $\Sigma$ for a positive push-off of $C$. Let $\tilde{\Sigma}$ be the preimage of $\Sigma$ which is an equivariant Seifert surface for a positive push-off of $\tilde{C}$. Then each intersection point between $\tilde{C}$ and $\tilde{\Sigma}$ lifts to $p$ intersection points (with the same sign) between $\tilde{C}$ and $\tilde{\Sigma}$. Hence $\text{lk}(\tilde{C}, \tilde{C}) = p \cdot \text{lk}(C, C)$, and so $\text{lk}(C, C) > 0$. □

**Proof of Theorem 1.** By Theorem 2 any periodic knot $\tilde{K}$ has an equivariant minimal genus Seifert surface $\tilde{S}$ with quotient $S$. By Lemma 1, $\tilde{S}$ is definite, and so by Lemma 2 $S$ is as well. □

### References

[Edm84] Allan L. Edmonds. Least area Seifert surfaces and periodic knots. *Topology Appl.*, 18(2-3):109–113, 1984.

[GL78] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

*E-mail address: kboyle@uoregon.edu*