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Stationary distribution and extinction of a stochastic staged progression AIDS model with staged treatment and second-order perturbation

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\textbf{A B S T R A C T}

Focusing on deterministic AIDS model proposed by Hyman (2000) and the detailed data from the World Health Organization (WHO), there are three stages of AIDS process which are described as Acute infection period, Asymptomatic phase and AIDS stage. Our paper is therefore concerned with a stochastic staged progression AIDS model with staged treatment. In view of the complexity of random disturbances, we reasonably take second-order perturbation into consideration for realistic sense. By means of our creative transformation technique and stochastic Lyapunov method, a critical value $R_0^S > 1$ is firstly obtained for the existence and uniqueness of ergodic stationary distribution to the stochastic system. Not only does it respectively reveal the corresponding dynamical effects of the linear and second-order perturbations to the model, but the unified form of second-order and linear fluctuations is derived. Next, some sufficient conditions about extinction of stochastic system are established in view of the basic reproduction number $R_0^S$. Finally, some examples and numerical simulations are introduced to illustrate our analytical results. In addition, some advantages of our new method and theory are highlighted by comparison with other existing results at the end of this paper.

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1. Introduction

1.1. Research background

As we know, Acquired Immune Deficiency Syndrome (AIDS) is a serious global infectious disease caused by Human Immunodeficiency Virus (HIV) infection. According to epidemiology study and medical research, HIV mainly attacks the immune system of human body, especially for CD4$^+$T lymphocyte. There are two HIV types worldwide which are described as HIV-1 and HIV-2. HIV-1 is the current main HIV epidemic strain in the real world, which has a long incubation period and high fatality rate for patients. Worse still, based on the statistic reported by the World Health Organization (WHO), almost 2 million people died worldwide in 2009 and about 32 million human beings were killed by HIV at the end of 2018. Consequently, most researchers have established some suitable deterministic HIV/AIDS models and studied the relevant dynamics in the past few decades (See [3–6]). Based on the thought of functional response in biology population, Huang and Ma [3] developed a deterministic HIV-1 model with Beddington-DeAngelis infection rate. They proved global stability of two equilibria for the model. Considering healthy condition of the susceptible, Hyman et al. proposed a deterministic multi-group SIA (Susceptible-Infected-AIDS) epidemic model for the transmission of HIV/AIDS in Hyman et al. [5], which reflects age structure. Additionally, by personal infection pathology investigation of HIV/AIDS patients, many authors developed some HIV models with nonlinear incidence ([7],[8],[10]) and delay differential equations of HIV infection [11–13].

Moreover, AIDS infection progression includes HIV acute infection period, asymptomatic phase and AIDS stage by means of the corresponding epidemiology analysis of AIDS and detailed description of Hyman et al. [1]. Thus Cai and Fang [14] considered a staged progression HIV model with imperfect vaccination, they derived the disease threshold and verified the corresponding global asymptotic stability. Due to the complex retrovirus gene of HIV, sufferers have to take some great treatment measures to survive longer, such as Antiretroviral Therapy (ART) [2]. Therefore, some
realistic AIDS models with great treatment have been proposed in the past few decades [15,16,18]. In [15], Musekwa et al. analyzed local stability of two equilibria of a deterministic AIDS model with screened disease carriers. Additionally, they also obtained the relevant basic reproduction number. However, these existing mathematical models indicate that staged treatment is not taken into account for deterministic AIDS system. In view of the difficulty in obtaining the corresponding basic reproduction number, as a result, they neglected the process of alleviating HIV/AIDS infection through great therapy. It is difficult to completely describe the transmission of AIDS epidemic by means of these models, thus a more reasonable staged progression AIDS model with staged treatment shall be proposed and investigated.

2.1. Ordinary differential equation model and dynamical properties

Following the above thoughts and discussion, we assume that

$$S(t)$$

is the number of susceptible individuals at time $$t$$. Similarly, let $$I_1(t), I_2(t)$$ and $$I_3(t)$$ be the numbers of HIV acute infection individuals, asymptomatic case patients and the AIDS suffers without therapy, respectively. $$T_1(t), T_2(t)$$ and $$T_3(t)$$ are separately the numbers of compartments $$I_1(t), I_2(t)$$ and $$I_3(t)$$ under treatment. Our paper focuses on the deterministic staged progression AIDS model with staged treatment which is given by

$$\begin{align*}
\frac{dS(t)}{dt} &= \Lambda - \mu_1 S(t) - \beta_1 I_1(t) + \beta_2 I_2(t) + \beta_3 I_3(t) \\
\frac{dI_1(t)}{dt} &= \beta_1 I_1(t) + \beta_2 I_2(t) + \beta_3 I_3(t) - \gamma_1 I_1(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) S(t) \\
\frac{dI_2(t)}{dt} &= \beta_2 I_1(t) - \gamma_1 I_2(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) S(t) \\
\frac{dT_1(t)}{dt} &= \gamma_1 I_2(t) - \mu_2 T_1(t) + \alpha_1 T_2(t) + \alpha_2 T_3(t) S(t) \\
\frac{dT_2(t)}{dt} &= \gamma_2 I_2(t) - \mu_3 T_2(t) + \alpha_1 T_3(t) + \alpha_2 T_3(t) S(t) \\
\frac{dT_3(t)}{dt} &= \gamma_3 I_2(t) - \mu_4 T_3(t) + \alpha_1 T_3(t) + \alpha_2 T_3(t) S(t)
\end{align*}$$

(1)

where $$\Lambda$$ is the recruitment rate of the susceptible, $$\mu_1, (i = 1, 2, 3, 4, 5, 6, 7)$$ denote the average death rates of the classes $$S(t), I_1(t), I_2(t), I_3(t), T_1(t), T_2(t), T_3(t)$$ and $$S(t)$$, respectively. $$\beta_1, \beta_2, \beta_3, \alpha_1$$ and $$\alpha_2$$ depict the effective contact coefficients between the group $$S(t)$$ and $$I_1(t), I_2(t), I_3(t), T_1(t), T_2(t), T_3(t)$$, respectively. $$\gamma_1, \gamma_2$$ and $$\gamma_3$$ are the death rates at the which the compartment $$I_1(t)$$ develops into the groups $$T_1(t), T_2(t)$$ and $$T_3(t)$$, $$\rho_1$$ and $$\rho_2$$ are transfer rates at which the groups $$I_i(t)$$ flows into the classes $$T_j(t)$$ and $$I_k(t)$$, respectively. $$\gamma_3$$ denotes the transmission rate from $$T_2(t)$$ to $$I_3(t)$$, $$\gamma_2$$ reflects the transmission rate from $$T_2(t)$$ to $$I_2(t)$$, $$\gamma_1$$ denotes the treatment rate of the AIDS patients, $$\omega$$ is the effective treatment rate of the group $$T_2$$. The above parameters are assumed to be positive in biology.

In system (1), the invariant and attracting domain is defined as

$$\Omega = \{S, I_1, I_2, I_3, T_1, T_2, T_3\} \bigg| S \geq 0, I_k \geq 0, T_k \geq 0 \ (k = 1, 2, 3), S + \sum_{i=1}^{3} (I_k + T_k) \leq \frac{\Lambda}{\mu_1 + \mu_2 + \mu_3 - \mu_7}. \bigg\}.$$

Then, we define

$$\bar{\mu}_2 = \mu_2 + \delta_1 + \delta_2 + \delta_3, \quad \bar{\mu}_3 = \mu_3 + \gamma_1 + \gamma_2 + \gamma_3 + \rho_1 + \rho_2 + \rho_3,$$

Then the corresponding basic reproduction number which can guarantee the AIDS epidemic occurs or not is derived as follows,

$$R_0 = \frac{\Lambda (\beta_1 + \beta_2 \varphi_1 + \beta_3 \varphi_2 + \alpha_1 \varphi_3 + \alpha_2 \varphi_4)}{\mu_1 \mu_2}.$$

2.2. Stochastic differential equation model

In reality, the AIDS treated population $$T_2(t)$$ has no effect on the dynamics of the compartments $$S(t), I_1(t), I_2(t), I_3(t)$$ and $$I_2(t), T_2(t)$$ and $$I_3(t)$$. Thus the whole dynamical behaviors of AIDS epidemic is only determined by the first six equations of system (1). In addition, various bio-mathematical models are inevitably affected by the environmental variations, see [17]. Consequently, some interesting stochastic epidemic models have been proposed and investigated in the past few decades [19,21-23,25,26]. Similarly, some authors also analyzed the corresponding stochastic HIV/AIDS models with various incidence rate types [24,27,29]. For example, Wang and Jiang [27] proved the existence of a unique ergodic stationary distribution for an HIV system with general incidence rate. However, we easily notice that all assumptions of stochastic perturbations of their paper are linear noise condition, such as “$$dI(t) = [I - dT(t) - \beta T(t) G(V(t)) dt + \sigma_1 T(t) dB_1(t)$$”, which is described in Liu and Jiang [28], where $$\sigma_1 > 0$$ is the intensity of one-dimensional standard Brownian motion $$B_1(t)$$. For a more realistic situation of the spread and development process of epidemics, the second-order perturbation shall be taken into consideration for system (1) by drawing on the thoughts of Liu and Jiang [29]. In this paper, we assume that the perturbation results of $$-\mu_k (k = 1, 2, 3, 4, 5, 6)$$ are separately described by

$$-\mu_1 \rightarrow -\mu_1 + (\sigma_{11} S(t) + \sigma_{12}) \dot{B}_1(t), \quad -\mu_2 \rightarrow -\mu_2 + (\sigma_{21} I_1(t) + \sigma_{22}) \dot{B}_2(t), \quad -\mu_3 \rightarrow -\mu_3 + (\sigma_{31} T_1(t) + \sigma_{32}) \dot{B}_3(t), \quad -\mu_4 \rightarrow -\mu_4 + (\sigma_{41} I_2(t) + \sigma_{42}) \dot{B}_4(t), \quad -\mu_5 \rightarrow -\mu_5 + (\sigma_{51} T_2(t) + \sigma_{52}) \dot{B}_5(t), \quad -\mu_6 \rightarrow -\mu_6 + (\sigma_{61} I_3(t) + \sigma_{62}) \dot{B}_6(t),$$

where $$B_1(t), B_2(t), B_3(t), B_4(t), B_5(t)$$ and $$B_6(t)$$ are six independent standard Brownian motions, $$\sigma_{ij} > 0 \ (i = 1, 2, 3, 4, 5, 6, j = 1, 2)$$ reflect the intensity of white noises, respectively. In other words, the corresponding stochastic staged progression AIDS model with staged treatment and second-order perturbation is given by

$$dS(t) = \left[\Lambda - \mu_1 S(t) - \sum_{i=1}^{3} (\beta_i S(t) + \alpha_i I_i(t)) S(t) + (\alpha_{11} S(t) + \alpha_{12}) B_1(t) \right] dt + (\sigma_{11} S(t) + \sigma_{12}) dB_1(t),$$

$$dI_1(t) = \left[\beta_1 I_1(t) + \beta_2 I_2(t) + \beta_3 I_3(t) - \gamma_1 I_1(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) S(t) + (\alpha_{21} I_1(t) + \alpha_{22}) B_2(t) \right] dt + (\sigma_{21} I_1(t) + \sigma_{22}) dB_2(t),$$

$$dI_2(t) = \left[\beta_2 I_1(t) - \gamma_1 I_2(t) + \alpha_1 T_1(t) + \alpha_2 T_2(t) S(t) + (\alpha_{31} T_1(t) + \alpha_{32}) B_3(t) \right] dt + (\sigma_{31} T_1(t) + \sigma_{32}) dB_3(t),$$

$$dI_3(t) = \left[\beta_3 I_1(t) + \beta_2 I_2(t) + \gamma_2 I_2(t) - \mu_3 T_2(t) + (\sigma_{41} I_2(t) + \sigma_{42}) B_4(t) \right] dt + (\sigma_{41} I_2(t) + \sigma_{42}) dB_4(t),$$

$$dT_1(t) = \left[\gamma_1 I_2(t) - \mu_2 T_1(t) + \alpha_1 T_2(t) + \alpha_2 T_3(t) S(t) + (\alpha_{51} T_1(t) + \alpha_{52}) B_5(t) \right] dt + (\sigma_{51} T_1(t) + \sigma_{52}) dB_5(t),$$

$$dT_2(t) = \left[\gamma_2 I_2(t) - \mu_3 T_2(t) + \alpha_1 T_3(t) - \mu_6 + (\sigma_{61} I_3(t) + \sigma_{62}) B_6(t) \right] dt + (\sigma_{61} I_3(t) + \sigma_{62}) dB_6(t).$$

(2)

Obviously, the assumption of stochastic second-order perturbation implies that the environmental variation is dependent on square of compartments $$S(t), I_1(t), I_2(t), I_3(t), T_1(t), T_2(t)$$ and $$I_3(t)$$ at some extent.
The remaining content of this paper is structured as follows. Preliminaries about stochastic differential equation and necessary lemmas are described in Section 2. By means of stochastic Lyapunov technique and our new method, some sufficient conditions of the unique ergodic stationary distribution and positive recurrence of system (2) are obtained in Section 3. Furthermore, the forms of secondary high-order perturbation and linear fluctuation are unified by this method. In Section 4, we derive the corresponding extinction result of system (2) based on the reproduction number \( R_0 \). Section 5 shows some examples and numerical simulations to validate our theoretical results. Additionally, the relevant dynamical effects of second-order and linear perturbations are analyzed and discussed in Section 6. At last, our new method and theory go further to compare with other existing results at the end of this paper.

2. Preliminaries and necessary lemmas

Throughout this paper, the relevant stochastic differential equation theories are established on the complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^\infty, \mathbb{P}) \) with a filtration \( \{\mathcal{F}_t\}_{t=0}^\infty \) unless otherwise specified, and \( \{\mathcal{F}_t\}_{t=0}^\infty \) is increasing and right continuous when \( \mathcal{F}_0 \) contains all \( \mathcal{F} \)-null sets [20]. In addition, some notations shall be defined in the first place. Let \( \mathbb{R}^n \) be an \( n \)-dimensional standard Euclidean space. Denote

\[
\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \quad \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \quad \mathbb{R}_+^k = \{(x_1, \ldots, x_k) | x_i > 0, 1 \leq i \leq k\}.
\]

If \( f(t) \) is an integral function on \( [0, \infty) \), let \( f^n = \sup_{t \in [0, \infty)} f(t) \), \( f^\prime = \inf_{t \in [0, \infty)} f(t) \).

For the following dynamical investigation of system (2), we shall firstly introduce some important lemmas.

**Lemma 1.** Assume that \( x \geq 0 \), then the following two inequalities holds

(i) \( x^3 \geq \left( x - \frac{1}{2} \right)(x^2 + 1) \), 
(ii) \( x^4 \geq \left( \frac{3}{4}x^2 - \frac{1}{4} \right)(x^2 + 1) \). 

The validation of Lemma 1 can refer to the subsection (II) in Appendix A.

Next, the existence and uniqueness of the global positive solution of system (2) is described by the following Lemma 2. The corresponding proof is in accordance with the general theory of Theorem 2.4 of Chapter 4 in Mao [20], thus we only propose it here.

**Lemma 2.** For any initial value \( \{S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)\} \in \mathbb{R}_+^6 \), there then exists a unique solution \( \{S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t)\} \) of the system (2) on \( t \geq 0 \) and the solution will remain in \( \mathbb{R}_+^6 \) with probability one (a.s.).

In view of systematic description of Has’minskii [9], let \( X(t) \) be a homogeneous Markov process in \( \mathbb{R}^n \), which follows the stochastic differential equation

\[
\begin{aligned}
\,dX(t) &= g(X(t))dt + \sum_{i=1}^n h_i(X)dB_i(t),
\end{aligned}
\]

where the diffusion matrix \( A(x) = (a_{ij}(x)) \), and \( a_{ij}(x) = \sum_{k=1}^n h_k^i(x)h_k^j(x) \). Moreover, the sufficient conditions of the existence of a unique ergodic stationary distribution is given by the following Lemma 3.

**Lemma 3.** (Has’minskii [9]) If there exists a bounded domain \( D \subseteq \mathbb{R}^n \) with a regular boundary \( \Gamma \), which follows

(C1). There is a positive number \( M_0 \) such that \( \sum_{j=1}^n a_{ij}(x)\theta_j \geq M_0 |\theta|^2 \), \( \forall x \in D, \theta \in \mathbb{R}^n \).

(C2). There is a non-negative \( C^2 \)-function \( V \) such that \( CV \) is negative for any \( \mathbb{R}^n \setminus \Gamma = \mathbb{R}^n \).

Then the Markov process \( X(t) \) has a unique ergodic stationary distribution \( \pi(\cdot) \). Let \( g(\cdot) \) be an integral function with respect to the measure \( \pi \), then it satisfies

\[
\begin{aligned}
\mathbb{P}\left( \lim_{t \to \infty} \frac{1}{t} \int_0^t g(X(t))dt = \int_{\mathbb{R}^n} g(x)\pi(dx) \right) = 1.
\end{aligned}
\]

Finally, a stochastic comparison theorem with respect to the compartment \( S(t) \) of system (2) is given by

\[
\begin{aligned}
d\tilde{S}(t) &= (\Lambda - \mu_1\tilde{S}(t))dt + (\sigma_{11}\tilde{S}(t) + \sigma_{12})d\tilde{B}_1(t),
\end{aligned}
\]

with the same initial value \( \tilde{S}(0) = S(0) > 0 \). In view of the stochastic comparison theorem [18], the relevant results are described by the following Lemma 4.

**Lemma 4.** If \( \tilde{S}(t) \) is the solution of system (4), then \( \tilde{S}(t) \) is ergodic. Assume that \( \pi(x)|x > 0 \) is the weakly convergent invariant density of \( \tilde{S}(t) \), by means of ergodicity property, one has

\[
\begin{aligned}
(i) \, S(t) \leq \tilde{S}(t), \quad \text{a.s.,} \quad (ii) \, \lim_{t \to \infty} \frac{1}{t} \int_0^t \tilde{S}(\tau)\,d\tau = \int_0^\infty x\pi(x)\,dx, \quad \text{a.s.,}
\end{aligned}
\]

where

\[
\pi(x) = C_1x^{-2(1+c_0)}(\sigma_{11}x + \sigma_{12})^{-2(1-c_0)}\exp\left(-\frac{2(\Lambda + \sigma_{12}^2c_0x)}{\sigma_{12}^2(\sigma_{11}x + \sigma_{12})}\right), \quad (x > 0)
\]

with \( c_0 = \frac{2\Lambda + \mu_1 + \mu_2 c_1}{\sigma_{12}^2} \), and \( C_1 \) is a positive constant which satisfies \( \int_0^\infty \pi(x)\,dx = 1 \).

The detailed proof of Lemma 4 is mostly similar to the corresponding analysis in [29]. We therefore omit it here.
3. Stationary distribution and ergodicity property of system (2)

In this section, we aim to obtain the sufficient conditions of a unique ergodic stationary distribution of system (2). To make the later description and proof clear and simple, some constants need to be defined as follows

\[ \bar{\mu}_i = \bar{\mu}_i + \frac{\sigma_{\alpha i}^2}{2} \]  \hspace{1em} (i = 2, 3, \ldots, 6), \quad \psi_4 = \frac{\rho_1 (\bar{\mu}_3 \beta_2 + \gamma \delta_1)}{\mu_3 \mu_4} \quad \psi_3 = \frac{\bar{\mu}_4 \mu_3 \beta_1 + \delta_2 \rho_1}{\mu_3 \mu_4} \quad \psi_1 = \frac{\bar{\mu}_5 \psi_4}{\mu_4} \quad \psi_2 = \frac{\delta_3 + \rho_2 \psi_1 + \gamma \psi_4}{\mu_6}.

Denote

\[ \mathbb{R}_0^H = \frac{\Lambda (\beta_1 + \beta_2 \psi_1 + \beta_3 \psi_2 + \alpha_1 \psi_3 + \alpha_2 \psi_4)}{\mu_1 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} + 2 \sqrt{\Lambda \sigma_{i 1} \sigma_{1 i}}} \left( \mu_2 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} \right). \]

**Theorem 1.** For any initial value \((S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)) \in \mathbb{R}^6_1\). Assume that \(\mathbb{R}_0^H > 1\), then the solution \((S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t))\) of system (2) has a unique stationary distribution \(\pi(\cdot)\) and it has ergodicity property.

**Proof.** According to the results of Lemmas 2 and 3, a pair of suitable bounded closed domain \(D\) and non-negative Lyapunov function \(V(S, I_1, T_1, I_2, T_2, I_3)\) shall be constructed to validate the conditions \((C_1)\) and \((C_2)\) of Lemma 3. Next we will prove Theorem 1 by the following five steps.

Step 1. The corresponding diffusion matrix of system (2) is given by

\[
A = \begin{pmatrix}
\sigma_{11} S_1 + \sigma_{12} S_2 & 0 & 0 & 0 & 0
\sigma_{21} I_1 + \sigma_{22} I_2 & \sigma_{31} \bar{T}_1 + \sigma_{32} \bar{T}_2 & 0 & 0 & 0
\sigma_{41} I_1 + \sigma_{42} I_2 & 0 & \sigma_{51} \bar{T}_1 + \sigma_{52} \bar{T}_2 & 0 & 0
0 & 0 & 0 & \sigma_{61} I_1 + \sigma_{62} I_2 & 0
0 & 0 & 0 & 0 & \sigma_{61} I_1 + \sigma_{62} I_2
\end{pmatrix}.
\]

If \((S(0), I_1(0), I_2(0), I_3(0)) = \bar{U}_0 = (\frac{1}{2}, n) \times (\frac{1}{2}, n) \times (\frac{1}{2}, n) \times (\frac{1}{2}, n) \times (\frac{1}{2}, n) \times (\frac{1}{2}, n)\), we can always choose a constant \(M_0 := \inf_{I,S,I_1,T_1,I_2,T_2}((\sigma_{11} S_1 + \sigma_{12} S_2)^2 + (\sigma_{21} I_1 + \sigma_{22} I_2)^2 + (\sigma_{31} \bar{T}_1 + \sigma_{32} \bar{T}_2)^2 + (\sigma_{41} I_1 + \sigma_{42} I_2)^2 + (\sigma_{51} \bar{T}_1 + \sigma_{52} \bar{T}_2)^2 + (\sigma_{61} I_1 + \sigma_{62} I_2)^2)^2 > 0\), such that

\[
0 \leq M_0 \| \Theta \|^2
\]

for any \(\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \in \mathbb{R}^6\). Hence the assumption \((C_1)\) in Lemma 3 holds.

Step 2. \((\epsilon_0, \text{ threshold theory})\) Let the variable \(\epsilon \in (0, 1)\), define the functions \(\bar{\mu}_i = \bar{\mu}_i + \frac{\sigma_{\alpha i}^2}{2} + \frac{\alpha_1^2}{6}\) \((i = 3, 4, 5, 6)\). and \(R_0^H(\epsilon)\) is constructed by

\[
R_0^H(\epsilon) = \frac{\Lambda (\beta_1 + \beta_2 \phi_1 + \beta_3 \phi_2 + \alpha_1 \phi_3 + \alpha_2 \phi_4)}{\mu_1 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} + 2 \sqrt{\Lambda \sigma_{i 1} \sigma_{1 i}}} \left( \mu_2 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} \right),
\]

where

\[
\phi_4 = \frac{\rho_1 (\bar{\mu}_3 \beta_2 + \gamma \delta_1)}{\mu_3 \mu_4} \quad \phi_3 = \frac{\bar{\mu}_4 \mu_3 \beta_1 + \delta_2 \rho_1}{\mu_3 \mu_4} \quad \phi_1 = \frac{\bar{\mu}_5 \psi_4}{\mu_4} \quad \phi_2 = \frac{\delta_3 + \rho_2 \phi_1 + \gamma \phi_4}{\mu_6}.
\]

For purpose of convenience of the following description, we still need to define

\[
\bar{\mu}_1 = \mu_1 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} + 2 \sqrt{\Lambda \sigma_{i 1} \sigma_{1 i}} \quad \bar{\mu}_2 = \mu_2 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2}.
\]

\[
\bar{\mu}_1 = \mu_1 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2} + 2 \sqrt{\Lambda \sigma_{i 1} \sigma_{1 i}} \quad \bar{\mu}_2 = \mu_2 + \frac{\sigma_{\alpha i}^2}{2} + 2 \sqrt{\Lambda^2 \sigma_{i i}^2}.
\]

Clearly,

(i). \(R_0^H = \frac{\Lambda \beta_1}{\mu_1 \mu_2} + \frac{\Lambda \beta_2 \psi_1}{\mu_1 \mu_2} + \frac{\Lambda \beta_3 \psi_2}{\mu_1 \mu_2} + \frac{\Lambda \phi_1}{\mu_1 \mu_2} + \frac{\Lambda \phi_2}{\mu_1 \mu_2} := H_0 + H_1 + H_2 + H_3 + H_4.
\]

(ii). \(R_0^H(\epsilon) = \frac{\Lambda \beta_1}{\mu_1 \mu_2} + \frac{\Lambda \beta_2 \phi_1}{\mu_1 \mu_2} + \frac{\Lambda \beta_3 \phi_2}{\mu_1 \mu_2} + \frac{\Lambda \phi_1}{\mu_1 \mu_2} + \frac{\Lambda \phi_2}{\mu_1 \mu_2} := h_0(\epsilon) + h_1(\epsilon) + h_2(\epsilon) + h_3(\epsilon) + h_4(\epsilon).
\]

Furthermore, we can easily notice that \(\bar{\mu}_i \) \((i = 1, 2, 3, 4, 5, 6)\) are all monotonically decreasing functions of the variable \(\epsilon\), which follow

\[
\inf_{\epsilon \in (0, 1)} \bar{\mu}_i = \lim_{\epsilon \to 0^+} \bar{\mu}_i \quad (i = 1, 2, 3, 4, 5, 6).
\]

Consequently, it indicates that \(h_\epsilon(\epsilon) \) \((k = 0, 1, 2, 3, 4)\) are all monotonically increasing functions of the variable \(\epsilon\). They still satisfy

\[
\sup_{\epsilon \in (0, 1)} h_\epsilon(\epsilon) = \lim_{\epsilon \to 0^+} h_\epsilon(\epsilon) = h_k \quad (k = 0, 1, 2, 3, 4).
\]
Therefore, $R_0^H(\epsilon)$ is monotonically increasing function with respect to the variable $\epsilon \in (0, 1)$, and it follows $\lim_{\epsilon \to 0^+} R_0^H(\epsilon) = R_0^H$. That is to say, we can always select a constant $\epsilon_0 \in (0, 1)$ to make $R_0^H(\epsilon_0) = 1$ if $R_0^H > 1$. Throughout the following proof of this section, unless otherwise specified, we will fix a value $\epsilon < \epsilon_0$ which guarantees $R_0(\epsilon) > 1$. Similarly, it means that $\bar{\mu}_k$ ($k = 1, 2, \ldots, 6$) are all positive constants.

According to all proofs about other existing stationary distribution of stochastic infectious disease models, we clearly conclude that the effect of a linear perturbation $\sigma > 0$ only reflects by $\frac{\sigma}{\epsilon}$, such as [23,30,31]. Nevertheless, the previous method has a difficulty applying to the second-order perturbation condition. More precisely, it can not eliminate the influence of square terms of variables $S$, $I_1$, $T_1$, $I_2$, $T_2$ and $T_3$, that is $S^2$, $I_1^2$, $T_1^2$, $I_2^2$, $T_2^2$ and $T_3^2$. In view of $\epsilon$-threshold theory in Step 2, a kind of new Lyapunov function used for eliminating second-order noise is introduced in Step 3 in detail, which refers to $W_k$ ($k = 1, 2, \ldots, 6$).

Step 3. Some important $C^2$-stochastic Lyapunov functions are given by

$$W_1 = \sum_{i=1}^{2} \frac{v_i (\bar{S} + u_i) \epsilon}{\epsilon}, \quad W_2 = \varrho_0 S + \frac{v_1 (I_1 + u_3) \epsilon}{\epsilon}, \quad W_3 = \frac{v_4 (T_1 + \epsilon) \epsilon}{\epsilon},$$

$$W_4 = \frac{v_4 (I_2 + \epsilon) \epsilon}{\epsilon}, \quad W_5 = \frac{v_4 (I_2 + \epsilon) \epsilon}{\epsilon}, \quad W_6 = \frac{v_4 (I_3 + \epsilon) \epsilon}{\epsilon},$$

where the parameters $u_i, v_j$ ($i = 1, 2, 3; j = 1, 2, 3, 4$) and $\varrho_0$ are determined below.

By means of Lemma 1, we can derive the following results by Itô’s formula described in Appendix B

$$\mathcal{L}W_1 = \sum_{k=1}^{2} \frac{v_k (S + u_k) \epsilon}{\epsilon} \left[ - \frac{2}{\epsilon} \frac{\lambda - \mu_k S}{(1 - \epsilon) \alpha_{ij}^2} \left( \sum_{i=1}^{2} \beta_i I_i + \sum_{j=1}^{2} \alpha_{ij} T_j \right) \right] - \frac{2}{\epsilon} \frac{(1 - \epsilon)^2 v_k}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \sum_{k=1}^{2} \frac{\lambda v_k}{\alpha_{ij}^2} \left[ \frac{1}{\epsilon} \frac{v_k}{(1 - \epsilon) \alpha_{ij}^2} \right] - \sum_{k=1}^{2} \frac{(1 - \epsilon)^2 v_k}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \sum_{k=1}^{2} \frac{\lambda v_k}{\alpha_{ij}^2} \left[ \frac{1}{\epsilon} \frac{v_k}{(1 - \epsilon) \alpha_{ij}^2} \right] - \sum_{k=1}^{2} \frac{(1 - \epsilon)^2 v_k}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \sum_{k=1}^{2} \frac{\lambda v_k}{\alpha_{ij}^2} \left[ \frac{1}{\epsilon} \frac{v_k}{(1 - \epsilon) \alpha_{ij}^2} \right] - \sum_{k=1}^{2} \frac{(1 - \epsilon)^2 v_k}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

Choose

$$v_1 = \frac{8}{3(1 - \epsilon) \alpha_{ij}^2}, \quad v_2 = 2 \sqrt{\frac{\lambda}{(1 - \epsilon) \alpha_{ij}^2}}, \quad v_3 = 2 \sqrt{\frac{\lambda}{(1 - \epsilon) \alpha_{ij}^2}}, \quad u_2 = 2 \sqrt{\frac{\lambda}{(1 - \epsilon) \alpha_{ij}^2}}.$$

where the values of $u_1$ and $u_2$ depend on the average inequalities $a_1 + a_2 + a_3 \geq 3 \frac{1}{\alpha_{ij}^2} \sum_{i=1}^{3} \alpha_{ij}$ and $b_1 b_2 \geq 2 \sqrt{b_1 b_2}$, respectively. In addition, their signs separately hold if and only if the positive variable $a_1 = a_2 = a_3$ and $b_1 = b_2$. Hence we have

$$\mathcal{L}W_1 \leq 2 \sqrt{\frac{\lambda^2 \alpha_{ij}^2}{(1 - \epsilon)^2}} + 2 \sqrt{\frac{\lambda \alpha_{11} \alpha_{12}}{1 - \epsilon} - \frac{\sigma_{11}^2 S^2}{2}} - \sigma_{11} \alpha_{12} S.$$ (8)

Similarly, we derive

$$\mathcal{L}W_2 = \varrho_0 \left[ - \frac{3}{\epsilon} \frac{\lambda - \mu_k S}{(1 - \epsilon) \alpha_{ij}^2} \left( \sum_{i=1}^{2} \beta_i I_i + \sum_{j=1}^{2} \alpha_{ij} T_j \right) \right] - \frac{3}{\epsilon} \frac{(1 - \epsilon)^2 v_3}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \varrho_0 \lambda - \frac{v_3 (I_1 + u_3) \epsilon}{\epsilon} \left( \sum_{i=1}^{2} \beta_i I_i + \sum_{j=1}^{2} \alpha_{ij} T_j \right) S - \frac{(1 - \epsilon)^2 v_3}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \varrho_0 \lambda - \frac{v_3 (I_1 + u_3) \epsilon}{\epsilon} \left( \sum_{i=1}^{2} \beta_i I_i + \sum_{j=1}^{2} \alpha_{ij} T_j \right) S - \frac{(1 - \epsilon)^2 v_3}{(1 - \epsilon) \alpha_{ij}^2} (\sigma_{11}^2 + \sigma_{12}^2)^2$$

$$\leq \left[ \left( \varrho_0 \lambda - \frac{(1 - \epsilon)^2 v_3}{16} \right) - \left( \varrho_0 - v_3 u_3 \right) \right] \left( \sum_{i=1}^{2} \beta_i I_i + \sum_{j=1}^{2} \alpha_{ij} T_j \right) S - \frac{3(1 - \epsilon)^2 v_3 \sigma_{11}^2 T_i}{16}.$$ (9)
By the similar method for parameters \( u_1 \) and \( v_1 \) of function \( W_1 \), we choose
\[
\varrho_0 = v_3 u_3^{-1}, \quad v_3 = \frac{8}{3(1 - \epsilon) u_3^2}, \quad u_3 = 2 \sqrt{\frac{\lambda}{(1 - \epsilon) \sigma_3^2}}.
\]
then we have
\[
\mathcal{L}W_2 \leq 2 \sqrt{\frac{\lambda^2 \sigma_2^2}{(1 - \epsilon)^2} - \frac{\sigma_2^2 R}{2}}.
\]
Employing Itô's formula to function \( W_3 \), one has
\[
\mathcal{L}W_3 = \nu_3 (T_1 + \epsilon) e^{-\frac{\delta_3 T_1}{(1 - \epsilon)^2}} (\delta_3 T_1 + \omega T_2 - \tilde{\mu}_3 T_1) - \frac{(1 - \epsilon) \nu_3}{2} (T_1 + \epsilon) e^{-\frac{\delta_3 T_1}{(1 - \epsilon)^2}} (\sigma_{31} T_1^2 + \sigma_{32} T_1)^2
\]
\[
\leq \nu_3 \delta_3 T_1 e^{1 - \epsilon} + \nu_3 \omega T_2 e^{1 - \epsilon} - \frac{(1 - \epsilon) \nu_3}{2} \delta_3 T_1^2 e^{1 - \epsilon}
\]
\[
\leq \nu_3 \delta_3 T_1 e^{1 - \epsilon} + \nu_3 \omega T_2 e^{1 - \epsilon} - \frac{(1 - \epsilon) \nu_3}{2} \delta_3 T_1^2 e^{1 - \epsilon}
\]
\[
\leq \nu_3 \delta_3 T_1 e^{1 - \epsilon} + \nu_3 \omega T_2 e^{1 - \epsilon} + \frac{(1 - \epsilon) \nu_3}{6} \sigma_{32}^2 T_1^2 e^{1 - \epsilon}.
\]
Let \( \nu_4 = \frac{8}{3(1 - \epsilon)^2} \), one has
\[
\mathcal{L}W_3 \leq \frac{8 \delta_3 T_1}{3(1 - \epsilon)} + \frac{8 \omega T_2}{3(1 - \epsilon)} + \frac{\sigma_{32}^2}{6} - \frac{\sigma_{31}^2 T_1^2}{2}.
\]
In view of the methods described in (11) and (12), we similarly get
\[
\mathcal{L}W_4 \leq \frac{8 \delta_3 T_1}{3(1 - \epsilon)} + \frac{8 \gamma_1 T_1}{3(1 - \epsilon)} + \frac{\sigma_{31}^2}{6} - \frac{\sigma_{32}^2 T_1^2}{2}.
\]
\[
\mathcal{L}W_5 \leq \frac{8 \rho_1 T_2}{3(1 - \epsilon)} + \frac{8 \rho_2 T_2}{3(1 - \epsilon)} + \frac{8 \gamma_2 T_2}{3(1 - \epsilon)} + \frac{\sigma_{32}^2}{6} - \frac{\sigma_{31}^2 T_1^2}{2}.
\]
\[
\mathcal{L}W_6 \leq \frac{8 \delta_3 T_1}{3(1 - \epsilon)} + \frac{8 \omega T_2}{3(1 - \epsilon)} + \frac{8 \gamma_1 T_1}{3(1 - \epsilon)} + \frac{\sigma_{31}^2}{6} - \frac{\sigma_{32}^2 T_1^2}{2}.
\]
From the corresponding expressions of (8), (10), (12)–(15), we derive all negative square terms of the variables in system (2) by Steps 2 and 3. Now our new method together with previous technique can completely eliminate the effect of second-order fluctuation which is difficulty solved by the existing theory. Next, some suitable C2-stochastic Lyapunov functions are constructed as
\[
V_1 = -\ln S + W_1, \quad V_2 = -\ln I_1 + W_3, \quad V_3 = -\ln T_1 + W_5, \quad V_4 = -\ln I_2 + W_4, \quad V_5 = -\ln T_2 + W_5,
\]
\[
V_6 = -\ln I_3 + W_6, \quad W_7 = -\ln S - \sum_{i=1}^{2} (\ln I_{i+1} + \ln T_i). \quad W_8 = 3 \sqrt{S + I_1 + T_1 + I_2 + T_2 + I_3 + 1}.
\]
In view of (8), (10), (12)–(15), we can obtain by applying Itô's formula to \( V_i \) (i = 1, 2, 3, 4, 5, 6)
\[
\mathcal{L}V_1 \leq \left[ \frac{\lambda S}{S} + \mu_1 + \sum_{i=1}^{3} \beta_i I_i + \sum_{j=1}^{2} \alpha_j T_j + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} \right] + 2 \sqrt{\frac{\lambda \sigma_{31}^2}{(1 - \epsilon)^2}} + 2 \sqrt{\frac{\lambda \sigma_{31} T_1}{(1 - \epsilon)^2} - \frac{\sigma_{32}^2 T_1^2}{2} - \sigma_{31} T_1}
\]
\[
= - \frac{\lambda S}{S} + \mu_1 + \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \alpha_1 T_1 + \alpha_2 T_2.
\]
\[
\mathcal{L}V_2 \leq \left[ - \left( \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \alpha_1 T_1 + \alpha_2 T_2 \right) \right] S + \tilde{\mu} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} + 2 \sqrt{\frac{\lambda \sigma_{31}^2}{(1 - \epsilon)^2}} - \frac{\sigma_{31}^2}{2}
\]
\[
= - \left( \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \alpha_1 T_1 + \alpha_2 T_2 \right) S + \tilde{\mu} + \sigma_{31} T_1 + \sigma_{32} T_1.
\]
\[
\mathcal{L}V_3 \leq \left[ \left( \frac{\delta_1 T_1}{T_1} - \frac{\omega T_1}{T_1} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} \right) \right] S + \tilde{\mu} + \frac{\sigma_{31}^2}{6} + \frac{\sigma_{32}^2}{6} + \frac{\sigma_{31}^2 T_1^2}{2}
\]
\[
= - \left( \frac{\delta_1 T_1}{T_1} + \frac{\omega T_1}{T_1} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} \right) S + \tilde{\mu} + \sigma_{31} T_1 + \sigma_{32} T_1.
\]
\[
\mathcal{L}V_4 \leq \left[ \left( \frac{\delta_2 I_2}{I_2} - \frac{\gamma I_1}{I_2} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} \right) \right] S + \tilde{\mu} + \frac{\sigma_{31}^2}{6} + \frac{\sigma_{32}^2}{6} + \frac{\sigma_{31}^2 T_1^2}{2}
\]
\[
= - \left( \frac{\delta_2 I_2}{I_2} - \frac{\gamma I_1}{I_2} + \frac{\sigma_{31}^2}{2} + \frac{\sigma_{32}^2}{2} + \frac{\sigma_{31}^2 T_1^2}{2} \right) S + \tilde{\mu} + \sigma_{31} T_1 + \sigma_{32} T_1.
\]
\[ \mathcal{L}V_3 \leq \left( -\frac{\rho_1 I_2}{T_2} + \mu_5 + \frac{\sigma_1^2}{2} + \sigma_1 \sigma_2 T_2 + \frac{\sigma_1^2}{2} T_2^2 \right) + \frac{8 \rho_1 I_2}{3 \epsilon (1 - \epsilon)} + \frac{\sigma_1^2 \epsilon^2}{6} - \frac{\sigma_1^2 T_2^2}{2} \\
= -\frac{\rho_1 I_2}{T_2} + \mu_5 + \sigma_1 \sigma_2 T_2 + \frac{8 \rho_1 I_2}{3 \epsilon (1 - \epsilon)}. \]

\[ \mathcal{L}V_6 \leq \left( -\frac{\delta_1 I_1}{I_3} - \frac{\rho_2 I_2}{I_3} - \frac{\gamma_2 T_3}{I_3} + \mu_6 + \frac{\sigma_2^2}{2} + \sigma_1 \sigma_2 I_3 + \frac{\sigma_2^2 I_3^2}{2} \right) + \frac{8 \rho_2 T_2}{3 \epsilon (1 - \epsilon)} + \frac{\sigma_2^2 \epsilon^2}{6} - \frac{\sigma_2^2 I_3^2}{2} \\
+ \frac{8 \rho_2 I_2}{3 \epsilon (1 - \epsilon)} + \frac{8 \gamma_2 T_3}{I_3} + \frac{\sigma_1^2 \epsilon^2}{6} - \frac{\sigma_1^2 I_3^2}{2} \\
= -\frac{\delta_1 I_1}{I_3} - \frac{\rho_2 I_2}{I_3} - \frac{\gamma_2 T_3}{I_3} + \mu_6 + \sigma_1 \sigma_2 I_3 + \frac{8 \delta_1 I_1}{3 \epsilon (1 - \epsilon)} + \frac{8 \rho_2 I_2}{3 \epsilon (1 - \epsilon)} + \frac{8 \gamma_2 T_3}{3 \epsilon (1 - \epsilon)}. \]

By means of Itô’s formula again, we similarly have

\[ \mathcal{L}W_7 = -\frac{A S}{S} + \mu + \frac{\sigma_1^2}{2} + \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \alpha_1 T_1 + \alpha_2 T_2 + \frac{\sigma_1^2 \epsilon^2}{2} + \sigma_1 \sigma_2 S \\
- \frac{\delta_1 I_1}{T_1} - \omega T_2 + \mu_3 + \frac{\sigma_1^2}{2} + \frac{\sigma_1}{2} T_1^2 + \sigma_1 \sigma_2 T_1 - \frac{\delta_1 I_1}{I_3} - \frac{\gamma_1 T_1}{I_3} + \mu_4 + \frac{\sigma_1^2}{2} + \sigma_1 \sigma_2 I_3 \\
- \frac{\rho_1 I_2}{I_3} - \mu_5 + \frac{\sigma_1^2}{2} + \sigma_1 \sigma_2 I_3 - \frac{\delta_1 I_1}{I_3} - \frac{\gamma_1 T_1}{I_3} + \mu_6 + \sigma_1 \sigma_2 I_3 \\
\leq -\frac{A S}{S} - \frac{\delta_1 I_1}{I_3} - \frac{\delta_1 I_1}{I_3} - \frac{\delta_1 I_1}{I_3} - \frac{\rho_1 I_2}{I_3} + \mu + \frac{\sigma_1^2}{2} + \sum_{i=3}^{6} \left( \mu_i + \frac{\sigma_i^2}{2} \right) \\
+ \left( \sigma_1 \sigma_2 S + \frac{\sigma_1^2 \epsilon^2}{2} \right) - \beta_1 I_1 + \left[ \left( \beta_2 + \sigma_1 \sigma_2 I_3 \right) I_3 + \frac{\sigma_1^2}{2} T_1^2 \right] \\
+ \left( \left( \alpha_1 + \sigma_1 \sigma_2 I_3 \right) T_1 + \frac{\sigma_1^2}{2} I_3 \right) \left( \left( \alpha_2 + \sigma_1 \sigma_2 I_3 \right) T_2 + \frac{\sigma_1^2}{2} I_3 \right). \]

Choosing \( \lambda_1 = (\sigma_1^2 \sigma_1 \sigma_2^2 \sigma_1 \sigma_4^2 \sigma_1 \sigma_5^2 \sigma_1 \sigma_6^2) > 0 \), \( \lambda_2 = (\sigma_1^2 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6) > 0 \), \( \lambda_3 = \frac{1}{84} (\lambda_1 + 6 \lambda_2) > 0 \),

then we can obtain

\[ \mathcal{L}W_8 = \left( S + I_1 + T_1 + I_2 + T_2 + I_3 + 1 \right)^{-\frac{1}{2}} (A - \mu_1 S - \mu_2 I_1 - \mu_3 T_1 - \mu_4 I_2 - \mu_5 I_3 - \mu_6) \\
- \frac{1}{3} \left( S + I_1 + I_2 + I_3 + 1 \right)^{-\frac{1}{2}} \left[ (\sigma_1 S^2 + \sigma_2 S^2 I_3 + (\sigma_1 T_1 + \sigma_2 I_3)^2 + (\sigma_1 T_1^2 + \sigma_2 I_3^2) \\
+ (\sigma_1 T_1 + \sigma_2 I_3) + (\sigma_1 \sigma_2 T_1 + \sigma_2 I_3)^2 + (\sigma_1 \sigma_2 T_1^2 + \sigma_2 I_3^2) \right] \\
\leq A - \frac{\left( \sigma_1^2 S^2 + \sigma_1^2 T_1^2 + \sigma_1^2 T_1^3 + \sigma_1^2 T_1^4 + \sigma_1^2 T_1^5 + \sigma_1^2 T_1^6 + \sigma_1^2 T_1^7 + \sigma_1^2 \sigma_2 T_1^2 + \sigma_1^2 \sigma_2 T_1^3 + \sigma_1^2 \sigma_2 T_1^4 + \sigma_1^2 \sigma_2 T_1^5 + \sigma_1^2 \sigma_2 T_1^6 + \sigma_1^2 \sigma_2 T_1^7 \right)}{3(S + I_1 + I_2 + I_3 + 1)^2} \\
\leq A - \frac{\left( \lambda_1 (S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7) + 6 \lambda_2 \left( S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7 \right) \right)}{21(S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7 + T_1^7) \\
\leq \frac{\lambda_1 (S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7)}{84(S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7 + T_1^7)} \\
\leq A - \lambda_3 (S^2 + T_1^2 + T_1^4 + T_1^5 + T_1^6 + T_1^7). \]

Step 4. By similar method of deriving the reproduction number \( r_0 \) of deterministic system (1), we consider the following equations

\[
\begin{align*}
\delta_1 T_1 - \mu_1 T_1 + \omega T_2 = 0, \\
\delta_2 T_3 - \mu_2 T_3 - \omega T_2 = 0, \\
\rho_1 I_2 - \mu_3 I_2 = 0, \\
\delta_3 T_1 + \rho_2 I_2 + \rho_3 I_3 - \mu_6 = 0.
\end{align*}
\]

Let \( \tilde{T}_1 = 1 \), then Eq. (24) have a unique solution \( (\tilde{I}_1, \tilde{T}_1, \tilde{I}_2, \tilde{T}_2, \tilde{I}_3, \tilde{T}_4) \) as \( (1, \phi_1, \phi_2, \phi_3, \phi_4) \). Clearly, the positive constant \( \phi_i \) (\( i = 1, 2, 3, 4 \)) are the same as the value in \( R^0(\epsilon) \). Next, we are in position to construct the stochastic \( \epsilon_0 \)-threshold \( R^0(\epsilon) \). The following compartmental proportional transformations \( (S, \tilde{I}_1, \tilde{I}_2, \tilde{T}_1, \tilde{T}_2) \) is established as follows

\[
S = \frac{A}{\mu_1} \tilde{I}_2, \quad I_2 = \phi_1 \tilde{I}_2, \quad I_3 = \phi_3 \tilde{I}_2, \quad T_1 = \phi_3 \tilde{I}_2, \quad T_2 = \phi_4 \tilde{I}_2.
\]

For simplicity and clarity of the later presentations, \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) equivalently follows

(i). \( \delta_1 + \phi_4 = \phi_4 \tilde{I}_2 \), (ii). \( \delta_2 + \gamma_1 \phi_1 = \phi_1 \tilde{I}_2 \), (iii). \( \rho_1 \phi_1 = \phi_4 \tilde{I}_2 \), (iv). \( \delta_3 + \rho_2 \phi_1 + \gamma_2 \phi_4 = \phi_2 \tilde{I}_2 \).
In view of the inequality \( u - 1 - \ln u \geq 0 \) \((u > 0)\), we can derive the following results:

\[
\mathcal{L}\left(\frac{1}{\mu_1} V_1\right) \leq \frac{-1}{\mu_1} \left( -\frac{A}{S} + \tilde{\mu}_1 + \beta_1 I_1 + \beta_2 I_2 + \alpha_1 T_1 + \alpha_2 T_2 \right) = \left( \frac{1}{S} - 1 \right) + \frac{\beta_1}{\mu_1} I_1 + \frac{\beta_2}{\mu_1} I_2 + \frac{\beta_3}{\mu_1} I_3 + \frac{\alpha_1}{\mu_1} T_1 + \frac{\alpha_2}{\mu_1} T_2.
\]

\[
\mathcal{L}\left(\frac{\tilde{\mu}_1}{A} V_2\right) = -\tilde{\mu}_1 \left( \beta_1 I_1 + \beta_2 I_2 + \alpha_1 T_1 + \alpha_2 T_2 \right) + \frac{\tilde{\mu}_1}{A} \left( \frac{\beta_1}{\mu_1} I_1 + \frac{\beta_2}{\mu_1} I_2 + \frac{\beta_3}{\mu_1} I_3 + \frac{\alpha_1}{\mu_1} T_1 + \frac{\alpha_2}{\mu_1} T_2 \right).
\]

\[
\mathcal{L}(\phi_1 V_3) = \phi_1 \phi_2 I_1 + \phi_4 \sigma_{12} T_1 + \frac{8 \phi_1 \sigma_{11} T_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_4 \sigma_{12} T_1}{3(1 - \epsilon)}
\]

\[
\mathcal{L}(\phi_2 V_4) = -\phi_1 \phi_2 I_1 + \phi_4 \sigma_{12} T_1 + \frac{8 \phi_1 \sigma_{11} T_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_4 \sigma_{12} T_1}{3(1 - \epsilon)}
\]

\[
\mathcal{L}(\phi_3 V_5) = -\phi_1 \phi_2 I_1 + \phi_4 \sigma_{12} T_1 + \frac{8 \phi_1 \sigma_{11} T_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_4 \sigma_{12} T_1}{3(1 - \epsilon)}
\]

\[
\mathcal{L}(\phi_4 V_6) = -\phi_1 \phi_2 I_1 + \phi_4 \sigma_{12} T_1 + \frac{8 \phi_1 \sigma_{11} T_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_4 \sigma_{12} T_1}{3(1 - \epsilon)}
\]

\[
\mathcal{L}(\phi_5 V_7) = -\phi_1 \phi_2 I_1 + \phi_4 \sigma_{12} T_1 + \frac{8 \phi_1 \sigma_{11} T_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \sigma_{12} T_1}{3(1 - \epsilon)} + \frac{8 \phi_4 \sigma_{12} T_1}{3(1 - \epsilon)}
\]

In view of the inequality \( u - 1 - \ln u \geq 0 \) \((u > 0)\), we can derive the following results:

\[
\mathcal{L}(\frac{1}{\mu_1} V_1) \leq \frac{-1}{\mu_1} \left( -\frac{A}{S} + \tilde{\mu}_1 + \beta_1 I_1 + \beta_2 I_2 + \alpha_1 T_1 + \alpha_2 T_2 \right) = \left( \frac{1}{S} - 1 \right) + \frac{\beta_1}{\mu_1} I_1 + \frac{\beta_2}{\mu_1} I_2 + \frac{\beta_3}{\mu_1} I_3 + \frac{\alpha_1}{\mu_1} T_1 + \frac{\alpha_2}{\mu_1} T_2.
\]

\[
\mathcal{L}(\frac{\tilde{\mu}_1}{A} V_2) = -\tilde{\mu}_1 \left( \beta_1 I_1 + \beta_2 I_2 + \alpha_1 T_1 + \alpha_2 T_2 \right) + \frac{\tilde{\mu}_1}{A} \left( \frac{\beta_1}{\mu_1} I_1 + \frac{\beta_2}{\mu_1} I_2 + \frac{\beta_3}{\mu_1} I_3 + \frac{\alpha_1}{\mu_1} T_1 + \frac{\alpha_2}{\mu_1} T_2 \right).
\]

Step 5. Finally, we will construct a pair of suitable non-negative \( \mathcal{L}^2 \)-stochastic Lyapunov function \( \mathcal{V}(S, I_1, I_2, T_1, T_2) \) and bounded domain \( D \) to prove the assumption \((C_2)\).

Let

\[
\lambda_0 = \left( \sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \right) > 0.
\]

then a stochastic Lyapunov function \( \mathcal{V}(S, I_1, I_2, T_1, T_2) \) is given by

\[
\mathcal{V} = M_0 \left( \frac{\tilde{\mu}_1}{A} V_2 + a_1 \phi_3 V_3 + b_1 \phi_4 V_4 + a_2 \phi_5 V_5 + b_2 \phi_6 V_6 + c_1 \phi_1 V_1 + p_1 T_1 + p_2 T_2 + q_1 T_1 + q_2 T_2 \right) + W_0 + \frac{\lambda_0}{\lambda_3} W_0,
\]

where the parameters \( M_0 > 0, a_i > 0, b_i > 0, p_i > 0, q_i > 0 \) \((i = 1, 2)\) and \( c_i \) are determined in \((36)\) and \((33)\) and \((34)\), respectively. More importantly, the values of \( a_1, a_2, b_1, b_2 \) and \( c_1 \) can eliminate terms of \( \ln S, \ln I_1, \ln T_1, \ln I_2, \ln T_2 \) and \( \ln T_3 \). Moreover, the simplicity of \( \mathcal{L} \mathcal{V} \) is derived by the values of \( p_1, p_2, q_1 \) and \( q_2 \).

In view of \( \mathcal{V}(S, I_1, I_2, T_1, T_2) \) is a continuous function which follows

\[
\lim_{n \to \infty} \inf_{(S, I_1, I_2, T_1, T_2, T_3) \in \mathbb{R}^5 \setminus \mathbb{W}_s} \mathcal{V}(S, I_1, I_2, T_1, T_2) = +\infty.
\]
We can therefore construct a suitable non-negative $C^2$-function $V(S, I_1, I_2, T_1, T_2)$:

$$V(S, I_1, T_1, I_2, T_2) = \tilde{V}(S, I_1, I_2, T_1, T_2, I_3) - V(S^0, \bar{R}_1, \bar{T}_1, \bar{R}_2, \bar{T}_2),$$

in which $(S^0, \bar{R}_1, \bar{T}_1, \bar{R}_2, \bar{T}_2)$ is the minimum value point. According to the results in (22) and (23), we have

$$\mathcal{L}(W_2 + \frac{\lambda_0}{\lambda_3}W_3) \leq -\frac{\Delta}{S} - \frac{\delta I_1}{T_1} - \frac{\delta I_2}{T_2} - \frac{\delta I_3}{T_3} + \frac{\lambda_0}{\lambda_3} \mu_1 + \frac{\sigma_{12}^2}{2} + \frac{\sigma_{12}^2}{2} + (\beta_1 I_1 - \sigma_{12}^2 I_1^2)$$

$$+ (\sigma_{11} I_1 S - \frac{\sigma_{11}^2}{2} S^2) + \left[ (\beta_2 + \sigma_{41} I_2) I_2 - \frac{\sigma_{12}^2}{2} I_2^2 \right] + \left[ (\beta_3 + \sigma_{61} I_3) I_3 - \frac{\sigma_{12}^2}{2} I_3^2 \right].$$

(31)

For simplicity and clarity of the following proof, some positive constants are still defined as follows

$$m_1 = \frac{c_1 \beta_1}{\mu_1} + \frac{\mu_1 \sigma_{12} S}{A} + \frac{8 \alpha_1 \phi_1 \delta_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \phi_1 \delta_1}{3(1 - \epsilon)} + \frac{8 \phi_3 \phi_1 \delta_1}{3(1 - \epsilon)}, \quad m_2 = \frac{c_1 \beta_2}{\mu_1} + b_1 \phi_1 \sigma_{41} I_2 + \frac{8 \alpha_1 \phi_4 \phi_1}{3(1 - \epsilon)} + \frac{8 \phi_2 \phi_2 \phi_1}{3(1 - \epsilon)}, \quad m_3 = \frac{c_1 \beta_3}{\mu_1} + b_2 \phi_2 \sigma_{61} I_2, \quad m_4 = \frac{c_1 \alpha_2}{\mu_1} + a_1 \phi_3 \sigma_{31} I_2 + \frac{8 \phi_2 \phi_1 \gamma_1}{3(1 - \epsilon)}, \quad m_5 = \frac{c_1 \alpha_2}{\mu_1} + a_2 \phi_2 \sigma_{51} I_2 + \frac{8 \alpha_1 \phi_1 \omega}{3(1 - \epsilon)} + \frac{8 \phi_2 \phi_2 \phi_1}{3(1 - \epsilon)}.$$

Therefore, by (25)–(31), we have

$$\mathcal{LV} \leq M_0 \left[ -\frac{\mu_1 \sigma_{12} S}{A} \left( \frac{\sigma_{11}^2}{2} S^2 \right) - \frac{\Delta}{S} - \frac{\delta I_1}{T_1} - \frac{\delta I_2}{T_2} - \frac{\delta I_3}{T_3} + \frac{\lambda_0}{\lambda_3} \mu_1 + \frac{\sigma_{12}^2}{2} + \frac{\sigma_{12}^2}{2} + (\beta_1 I_1 - \sigma_{12}^2 I_1^2)$$

$$+ (\sigma_{11} I_1 S - \frac{\sigma_{11}^2}{2} S^2) + \left[ (\beta_2 + \sigma_{41} I_2) I_2 - \frac{\sigma_{12}^2}{2} I_2^2 \right] + \left[ (\beta_3 + \sigma_{61} I_3) I_3 - \frac{\sigma_{12}^2}{2} I_3^2 \right] + [ \left( \alpha_1 + \sigma_{31} I_2 \right) I_2 - \frac{\sigma_{12}^2}{2} I_2^2 ] + [ \left( \alpha_2 + \sigma_{51} I_2 \right) I_2 - \frac{\sigma_{12}^2}{2} I_2^2 ] \right] \tag{32}$$

Let the parameters $a_1, a_2, b_1, b_2$ and $c_1$ be the unique solution of the following equations

$$(33)$$

By detailed calculation, we can derive based on (24) and (33)

$$c_1 = \beta_1 + \beta_2 \phi_1 + \beta_2 \phi_2 + \alpha_1 \phi_3 + \alpha_2 \phi_4 = 0, \quad b_2 = \frac{\beta_3}{\mu_6} > 0, \quad a_1 = \frac{\alpha_1 \mu_4 \mu_5 + \gamma_1 \phi_1 (\phi_2 + \phi_2 \phi_1)}{\mu_3 \mu_4 \mu_5 - \gamma_1 \omega \phi_1} > 0,$$

$$b_1 = \frac{\alpha_1 \phi_1 \omega + \gamma_1 \phi_1 (\phi_2 + \phi_2 \phi_1)}{\mu_3 \mu_4 \mu_5 - \gamma_1 \omega \phi_1} > 0, \quad a_2 = \frac{\alpha_1 \omega \phi_4 + \gamma_1 \phi_1 (\phi_2 + \phi_2 \phi_1)}{\mu_3 \mu_4 \mu_5 - \gamma_1 \omega \phi_1} > 0.$$

Similarly, we assume that the other parameters $p_1, p_2, q_1$ and $q_2$ are the unique solution of the following equations

$$(34)$$

More precisely, we can also obtain

$$q_2 = \frac{m_1}{\mu_6} > 0, \quad p_1 = \frac{m_4 \mu_4 \mu_5 + \gamma_1 \phi_1 (m_5 + \gamma_1 \phi_2) \phi_1}{\mu_3 \mu_4 \mu_5 - \gamma_1 \omega \phi_1}, \quad q_1 = \frac{m_4 \mu_4 \mu_5 + \gamma_1 \phi_1 (m_5 + \gamma_1 \phi_2) \phi_1}{\mu_3 \mu_4 \mu_5 - \gamma_1 \omega \phi_1} > 0.$$

By (32)–(34), we therefore get
\[ L V \leq - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) - \frac{A}{S} - \frac{\delta_1 I_1}{I_1} - \frac{\delta_2 I_2}{I_2} - \frac{\delta_3 I_3}{I_3} - \frac{\rho I_2}{I_2} + f_1(S) + f_2(l_1) + f_3(l_2) + f_4(l_3) + f_5(T_1) + f_6(T_2). \] (35)

Where

\[ f_1(S) = \frac{\lambda_0 A}{A_3} + \mu_1 + \frac{\sigma_1^2}{2} + \frac{6}{\sum_{i=3}^{6} (\mu_i + \frac{\sigma_i^2}{2}) + \sigma_{12}S - \frac{\sigma_1^2}{2} s^2}. \]

\[ f_2(l_1) = \left[ M_h (m_1 + p_1 \delta_1 + p_2 \delta_2 + q_2 \delta_3) + \beta_1 \right] I_1 - \frac{\sigma_{21}^2}{2} I_1^2. \]

\[ f_3(l_2) = (\beta_2 + \sigma_4 \alpha_{42}) I_2 - \frac{\sigma_{21}^2}{2} I_2^2. \]

\[ f_4(l_3) = (\beta_3 + \sigma_5 \alpha_{52}) I_3 - \frac{\sigma_{21}^2}{2} I_3^2. \]

\[ f_5(T_1) = (\alpha_1 + \sigma_{12} \alpha_{12}) T_1 - \frac{\sigma_{21}^2}{2} T_1^2. \]

\[ f_6(T_2) = (\alpha_2 + \sigma_{12} \alpha_{12}) T_2 - \frac{\sigma_{21}^2}{2} T_2^2. \]

Moreover, \( M_h \) is assumed to satisfy the following inequality

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p \leq -2. \] (36)

Next a suitable compact subset \( D \) is constructed as follows,

\[ D = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| \kappa \leq S \leq \frac{1}{\kappa}, \ k \leq I_1 \leq \frac{1}{k}, \ k^2 \leq T_1 \leq \frac{1}{k^2}, \ k^3 \leq I_2 \leq \frac{1}{k^3}, \ k^6 \leq I_3 \leq \frac{1}{k^6} \right. \right\} \]

where \( \kappa > 0 \) is a sufficient small number satisfying the following inequalities

\[ -2 + J_2 - \frac{J_1}{\kappa} \leq -1. \] (37)

\[ -2 + J_2 - \frac{\sigma_1^2}{2k^2} \leq -1. \] (38)

\[ -2 + [M_h (m_1 + p_1 \delta_1 + p_2 \delta_2 + q_2 \delta_3) + \beta_1] \kappa \leq -1. \] (39)

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p - \frac{\sigma_{11}^2}{4k^4} \leq -1. \] (40)

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p - \frac{\sigma_{11}^2}{4k^4} \leq -1. \] (41)

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p - \frac{\sigma_{11}^2}{4k^4} \leq -1. \] (42)

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p - \frac{\sigma_{11}^2}{4k^4} \leq -1. \] (43)

\[ - \frac{M_h \tilde{\mu}_1 \tilde{\mu}_2}{A} \left( R_0^h(\epsilon) - 1 \right) + f_1^p + f_2^p + f_3^p + f_4^p + f_5^p + f_6^p - \frac{\sigma_{11}^2}{4k^4} \leq -1. \] (44)

where the constants \( J_i \ (i = 1, 2, 3, 4, 5, 6, 7) \) are determined later.

Clearly, the bounded set \( D^c = R^6 \setminus D \) can be divided into the following twelve subsets:

\[ D_1 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| S > \frac{1}{\kappa} \right. \right\}, \]

\[ D_2 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 < \frac{1}{\kappa} \right. \right\}. \]

\[ D_3 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| S < \frac{1}{\kappa} \right. \right\}, \]

\[ D_4 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 < \kappa \right. \right\}. \]

\[ D_5 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 > \frac{1}{k^2} \right. \right\}, \]

\[ D_6 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 > \frac{1}{k^2} \right. \right\}. \]

\[ D_7 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 > \frac{1}{k^2} \right. \right\}, \]

\[ D_8 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 > \frac{1}{k^2} \right. \right\}. \]

\[ D_9 = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 \geq \frac{k}{k^2} I_2 < k^2 \right. \right\}, \]

\[ D_{10} = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 \geq \frac{k}{k^2} I_2 < k^2 \right. \right\}. \]

\[ D_{11} = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 \geq \frac{k}{k^2} I_2 < k^2 \right. \right\}, \]

\[ D_{12} = \left\{ (S, I_1, T_1, I_2, T_2, I_3) \in R^6 \left| I_1 \geq k^2, I_2 < k^2 \right. \right\}. \]

In other words, \( D^c = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \cup D_7 \cup D_8 \cup D_9 \cup D_{10} \cup D_{11} \cup D_{12}. \) Next we are devoted to prove

\[ L V \leq -1, \] for any \( (S, I_1, T_1, I_2, T_2, I_3) \in D_i \ (i = 1, 2, \ldots, 12). \)
Case 1. If \((S, I_1, I_2, T_2, I_3) \in D'_5 \cup D'_3 \cup D'_0 \cup D'_7 \cup D'_2\), we have by (36) and (37)

\[
\begin{align*}
\mathcal{L}V & \leq -\frac{M_0 \mu_1 \mu_2}{A} \left( \sum_{i=1}^{6} f_i^3 - \left( \frac{\Lambda}{k} \right) \right) + \left( \sum_{i=3}^{6} f_i^3 + f_1^3 + f_2^3 + f_3^3 + f_4^3 + f_5^3 + f_6^3 \right) - \frac{\Lambda}{S} \left( \delta_1 I_1 + \delta_3 I_3 - \rho_1 I_2 \right) \\
& \leq -2 + f_2^3 - \left( \frac{\Lambda}{k} \right) - \frac{\delta_1}{\delta_3} - \left( \frac{\delta_3}{\delta_2} \right) - \frac{\rho_1}{\delta_2} - \frac{1}{K} \\
& \leq -2 + f_2^3 - \frac{\delta_1}{\delta_3} - \left( \frac{\delta_3}{\delta_2} \right) - \frac{\rho_1}{\delta_2} - \frac{1}{K} \leq -1,
\end{align*}
\]

where \(J_1 = \left( \Lambda \setminus \delta_1 \setminus \delta_3 \setminus \delta_2 \setminus \rho_1 \right) > 0\).

Case 2. For any \((S, I_1, I_2, T_2, I_3) \in D'_5\), it follows by (36) and (38)

\[
\begin{align*}
\mathcal{L}V & \leq -\frac{M_0 \mu_1 \mu_2}{A} \left( \sum_{i=1}^{6} f_i^3 - \left( \frac{\Lambda}{k} \right) \right) + \left( \sum_{i=3}^{6} f_i^3 + f_1^3 + f_2^3 + f_3^3 + f_4^3 + f_5^3 + f_6^3 \right) - \frac{\Lambda}{S} \left( \delta_1 I_1 + \delta_3 I_3 - \rho_1 I_2 \right) \\
& \leq -2 + f_2^3 - \frac{\delta_1}{\delta_3} - \left( \frac{\delta_3}{\delta_2} \right) - \frac{\rho_1}{\delta_2} - \frac{1}{K} \leq -1,
\end{align*}
\]

where \(J_2 = \sup_{(S, I_1, I_2, T_2, I_3) \in D'_5} \left\{ \left( \sum_{i=3}^{6} f_i^3 + f_1^3 + f_2^3 + f_3^3 + f_4^3 + f_5^3 + f_6^3 \right) - \frac{\Lambda}{S} \left( \delta_1 I_1 + \delta_3 I_3 - \rho_1 I_2 \right) \right\} \).
Case 8. For any \((S, I_1, I_2, T_2, I_3) \in D_8^g\), we obtain by \((44)\)

\[
\mathcal{L}V \leq -\frac{M_\beta \bar{\mu}_1}{\bar{\Lambda}} \left( R_0^0(e) - 1 \right) + f_1^3 + f_2^3 + f_3^1 + f_4^1 + f_5^1 + f_6^1 + \left[ (\alpha_2 + \sigma_1 \sigma_{52}) T_2 - \frac{\sigma_2^2}{4} T_2^2 \right] - \frac{\sigma_1^2}{4} T_2^2
\]

\[
\leq -\frac{M_\beta \bar{\mu}_1}{\bar{\Lambda}} \left( R_0^0(e) - 1 \right) + f_1^3 + f_2^3 + f_3^1 + f_4^1 + f_5^1 + f_6^1 + f_7 - \frac{\sigma_1^2}{4} T_2^2
\]

\[
\leq -1.
\]

where \(J_2 = \sup_{(S, I_1, I_2, T_2, I_3) \in \bar{D}_8^g} \left\{ (\alpha_2 + \sigma_1 \sigma_{52}) T_2 - \frac{\sigma_2^2}{4} T_2^2 \right\} \).

By the above analysis, we can equivalently obtain

\[
\mathcal{L}V \leq -1. \quad \text{for any } (S, I_1, I_2, T_2, I_3) \in D^g.
\]

Consequently, the condition \((C_2)\) in Lemma 3 also holds. If \(R_0^0 > 1\), that is \(R_0^0(e) > 1\), we can derive that the solution \((S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t))\) of system \((2)\) has a unique ergodic stationary distribution \(\mathcal{O}(\cdot)\) from Steps 2 to 5. Thus the proof is confirmed. \(\square\)

Remark 1. Our new method is mainly reflected in Steps 2–4. In Step 2, we construct a \(\epsilon_0\)-threshold \(R_0^0(e)\) which is similar to \(R_0^0\), then we prove that \(R_0^0(e) > 1\) when \(\epsilon < \epsilon_0\) and \(R_0^0 > 1\). Step 3 is devoted to eliminate the influence of second-order fluctuation by new Lyapunov function type \(W_k\) \((k = 1, 2, \ldots, 6)\). By means of a proportional transformation between \((S, I_1, T_1, I_2, T_2, I_3)\) and \((\bar{S}, \bar{I}_1, \bar{T}_1, \bar{I}_2, \bar{T}_2, \bar{I}_3)\), we can completely verify the condition \((C_2)\) if \(R_0^0(e) > 1\). By Theorem 1, we can derive the corresponding behavior of the susceptible people and HIV acute infection individuals play a significant role in persistence of system \((2)\). In fact, epidemiology study and AIDS epidemic statistics reported by WHO exactly prove this conclusion. As we know, the persistence of system \((2)\) is similar to the endemic equilibrium \(P^*\) of deterministic model \((1)\). We still notice that \(R_0^0 = R_0\) when all environmental perturbations \(\sigma_1 = 0\) \((i = 1, 2, 3, 4, 5, 6; j = 1, 2)\). Thus we can derive the unified criterion of the persistence of systems \((1)\) and \((2)\). Moreover, the corresponding threshold value of system \((1)\) with linear perturbation is described as follows

\[
R_0^0 = \frac{\Lambda(\beta_1 + \beta_2 \psi_1 + \beta_3 \psi_2 + \alpha_1 \psi_3 + \alpha_2 \psi_4)}{\bar{\mu}_1 + \frac{\sigma_2^2}{4}}
\]

where \(\psi_k\) \((k = 1, 2, 3, 4)\) are the same as the above. Similarly, the result of linear noise condition can be validated by the previous technique, see \([23,30]\).

4. Extinction of system \((2)\)

If \(R_0 \leq 1\), the global asymptotic stability of \(P_0\) means that AIDS epidemic of deterministic system \((1)\) will go to extinction. Based on the value of \(R_0\), we will introduce the corresponding extinction result of system \((2)\) in this section.

Define

\[
\mathcal{R}_0 = \int_0^\infty \left| \frac{x - \frac{\Lambda}{\bar{\mu}_1}}{\mu_1} \right| \pi(x) dx + \frac{\Lambda(R_0 - 1)}{\bar{\mu}_1} \left( \eta_1 I_{(R_0 > 1)} + \eta_2 I_{(R_0 \leq 1)} \right) - \frac{\sigma_{12}^2 + \sigma_{22}^2 + \sigma_{42}^2 + \sigma_{52}^2 + \sigma_{62}^2}{10},
\]

where \(I_A\) is the indicator function with respect to set \(A\). \(\eta_1\) and \(\eta_2\) are given by

\[
\eta_1 = \frac{\beta_1 \beta_2 \beta_3 \alpha_1 \alpha_2}{\bar{\theta}_0 \bar{\theta}_1 \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4}, \quad \eta_2 = \frac{\beta_1 \beta_2 \beta_3 \alpha_1 \alpha_2}{\bar{\theta}_0 \bar{\theta}_1 \bar{\theta}_2 \bar{\theta}_3 \bar{\theta}_4} > 0.
\]

Theorem 2. For any initial value \((S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)) \in \mathbb{R}_+^6\), if \(\mathcal{R}_0 \leq 0\), then the solution \((S(t), I_1(t), T_1(t), I_2(t), T_2(t), I_3(t))\) of system \((2)\) follows

\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left( \vartheta_1 I_1(t) + \vartheta_1 T_1(t) + \vartheta_2 I_2(t) + \vartheta_2 T_2(t) + \vartheta_4 I_3(t) \right) \leq \mathcal{R}_0 \leq 0, \quad \text{a.s.}
\]

where the positive constants \(\vartheta_i\) \((i = 0, 1, 2, 3, 4)\) are determined in \((49)\)–\((52)\). It is equivalent to the following result

\[
\lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} T_1(t) = 0, \quad \text{a.s.} \quad (i = 1, 2, 3; j = 1, 2)
\]

which implies that AIDS epidemic will be exponentially eradicated with probability one (a.s.).

Proof. An equivalent \(C^2\)-function \(P(t)\) is constructed by

\[
P(t) = \vartheta_1 I_1(t) + \vartheta_1 T_1(t) + \vartheta_2 I_2(t) + \vartheta_2 T_2(t) + \vartheta_4 I_3(t).
\]

Applying Itô’s formula to \(P(t)\), one has

\[
d \{ \ln P \} = \mathcal{L}(\ln P) dt + \frac{1}{2} \left[ \vartheta_1 (\sigma_{21} I_1^2 + \sigma_{22} I_1) dB_2(t) + \vartheta_1 (\sigma_{31} T_1^2 + \sigma_{32} T_1) dB_3(t) + \vartheta_2 (\sigma_{41} I_2^2 + \sigma_{42} I_2) dB_4(t)
\]

\[
+ \vartheta_3 (\sigma_{51} T_2^2 + \sigma_{52} T_2) dB_5(t) + \vartheta_4 (\sigma_{61} I_3^2 + \sigma_{62} I_3) dB_6(t) \right].
\]
where

\[ \mathcal{L}(\ln P) = \frac{1}{p} \left[ \phi_0 \left( \sum_{i=1}^{3} \beta_i l_i + \sum_{j=1}^{2} \alpha_j T_j \right) S - \phi_0 \mu_3 T_3 + \phi_1 (\delta_1 \mu_1 + \omega \bar{\mu}_3 T_3) + \phi_2 (\delta_2 \mu_2 + \gamma_1 T_1 - \bar{\mu}_4 T_2) ight] - \frac{\partial_0^2 (\mu_1 T_1^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_2^2 (\sigma_3 T_3^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_3^2 (\sigma_4 T_4^2 + \sigma_2 T_2)}{2p^2}.
\]

Let \( \phi_i \) (i = 0, 1, 2, 3, 4) be a solution of the following equations

\[ \phi_0 = \frac{\lambda_3}{\mu_1}, \quad \phi_1 = \frac{\lambda_1}{\mu_1}, \quad \phi_2 = \frac{\lambda_2}{\mu_1}, \quad \phi_3 = \frac{\lambda_2}{\mu_1}.
\]

In fact, the solution of (48) is unique. We can still calculate

\[ \phi_4 = \frac{\lambda_3}{\mu_1} > 0, \quad \phi_i = \frac{\lambda_i}{\mu_1} (\mu_1 \mu_4 \mu_5 - \gamma_1 \omega p_1) > 0, \quad i = 1, 2, 3.
\]

Therefore, by (48)-(51), we can obtain

\[ \phi_0 = \frac{1}{\mu_3} \left( \frac{\lambda_3}{\mu_1} + \phi_1 \delta_1 + \phi_2 \delta_2 + \phi_4 \delta_4 \right) = \frac{\lambda_3}{\mu_1}.
\]

By (49)-(52), then (47) can be rewritten as

\[ \mathcal{L}(\ln P) = \frac{1}{p} \left[ \phi_0 \left( \sum_{i=1}^{3} \beta_i l_i + \sum_{j=1}^{2} \alpha_j T_j \right) S - \frac{\lambda_3}{\mu_1} \left( \sum_{i=1}^{3} \beta_i l_i + \sum_{j=1}^{2} \alpha_j T_j \right) \right] - \frac{\partial_0^2 (\mu_1 T_1^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_2^2 (\sigma_3 T_3^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_3^2 (\sigma_4 T_4^2 + \sigma_2 T_2)}{2p^2}.
\]

In view of (5) and (45), one can get that

\[ \mathcal{L}(\ln P) = \frac{1}{p} \left[ \phi_0 \left( \sum_{i=1}^{3} \beta_i l_i + \sum_{j=1}^{2} \alpha_j T_j \right) S - \frac{\lambda_3}{\mu_1} \left( \sum_{i=1}^{3} \beta_i l_i + \sum_{j=1}^{2} \alpha_j T_j \right) \right] - \frac{\partial_0^2 (\mu_1 T_1^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_2^2 (\sigma_3 T_3^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_3^2 (\sigma_4 T_4^2 + \sigma_2 T_2)}{2p^2} - \frac{\partial_4^2 (\sigma_6 T_6^2 + \sigma_2 T_2)}{2p^2}.
\]
\[ \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \geq \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \]

By means of the inequality described in subsection (l) in Appendix A, let the variable \( \varepsilon \in (0, 1) \), by choosing the parameters \( \varepsilon = \alpha \), \( \beta = 2\ln \frac{n}{\varepsilon} \), then we obtain

\[ \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \geq \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \]

which means

\[ \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \geq \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \]

Moreover, by ergodicity property of the solution \( \hat{S}(t) \) of system (4), we obtain

\[ \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau = \frac{10 \ln n \nu}{n} \]

Taking the superior limit of \( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \) on both sides of (54), which means \( n \to +\infty \), then it follows from (55) and (56)

\[ \lim_{n \to \infty} \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau \geq \frac{10 \ln n \nu}{n} \]

In view of the arbitrariness of \( \varepsilon \in (0, 1) \), let \( \varepsilon \to 0^{+} \), one has

\[ \lim_{n \to \infty} \frac{1}{t} \int_{0}^{t} \frac{\partial_{\tau} \left( \sigma_{S\omega} T_{n} (\tau) + \sigma_{\omega} T_{n} (\tau) \right)^{2}}{P(\tau)} d\tau = \frac{10 \ln n \nu}{n} \]

which means \( \lim_{n \to \infty} P(t) = 0 \), a.s., that is to say,

\[ \lim_{t \to \infty} I_{1}(t) = 0 \text{, a.s.} \]

Therefore, AIDS epidemic of system (2) will exponentially go to extinction in a long term. The proof is completed.
Remark 2. Following the method of undetermined coefficients used in Step 5, we cheerfully construct the result \((R_0 - 1)\) by Eq. (48). By means of exponential martingale inequality and \(\varepsilon \to 0^+\), we still eliminate the influence of second-order perturbation. From the corresponding expression of \(R_0^E\), we easily get that \(R_0^E = \int_0^\infty |\lambda - \mu_1|^2 \rho(x) dx\), where \(\lambda = \frac{\Lambda_{\psi_1}(1-R_0)}{\mu_1} - \frac{\sigma_3^2 \sigma_0^2 \sigma_2^2 \sigma_2^2}{10}\). Assume that \(R_0 > 1\). It means that \(R_0^E = \int_0^\infty |\lambda - \mu_1|^2 \rho(x) dx\) if \(R_0 \leq 1\). Hence we can conclude that the condition \(R_0 \leq 1\) is more likely to lead to the disease extinction by comparison with \(R_0 > 1\). Furthermore, we derive that the extinction result mainly depends on the large linear perturbations \(\sigma_2\) \((k = 2, 3, 4, 5, 6)\) instead of second-order noises.

5. Examples and numerical simulations

In view of the higher-order method developed by Milstein [32], we will introduce some examples and numerical simulations to validate the above theoretical results in this section. The corresponding discretization equation of system (2) is given by

\[
S^{k+1} = S^k + \left[ \Lambda - \mu_1 S^k \right] \Delta t + \left( \beta_1 S^k + \beta_2 S^k + \beta_3 S^k + \alpha_1 T^k + \bar{\alpha}_2 T^k \right) S^k \Delta t + \left( \sigma_{12} S^k + \sigma_{12} T^k \right) \sqrt{\Delta t} \xi_{1,k} + \frac{\sigma_2^2}{2} \left( \sigma_2^2 S^k \right)^2 + 3 \sigma_2 \sigma_3 \sigma_5 + \left( \sigma_2^2 \right) \left( \Delta t \xi_{1,k} - 1 \right) \Delta t,
\]

\[
T^{k+1} = \left( \alpha_1 T^k + \alpha_2 T^k \right) \Delta t + \left( \sigma_{12} T^k + \sigma_{12} T^k \right) \sqrt{\Delta t} \xi_{1,k} + \frac{\sigma_2^2}{2} \left( \sigma_2^2 T^k \right)^2 + 3 \sigma_2 \sigma_3 \sigma_5 T^k + \left( \sigma_2^2 \right) \left( \Delta t \xi_{1,k} - 1 \right) \Delta t,
\]

where the time increment \(\Delta t > 0\), \(\xi_{i,k} (i = 1, 2, 3, 4, 5, 6)\) are separately six independent Gaussian random variables which follow the Gaussian distribution \(N(0, 1)\) for \(i = 1, 2, \ldots, n\). For sake of the following analysis, we assume that the initial value \((S(0), T(0), \psi(0), \varphi(0), \gamma(0), \beta(0), \varphi(0), \alpha(0)) = (0.5, 1, 2, 0.8, 0.2, 0.2, 0.2)\). In addition, the bio-mathematical parameters in system (1) are presented as follows.

\[
\Lambda = 0.5, \quad \mu_1 = 0.1, \quad \mu_2 = 0.15, \quad \mu_3 = 0.12, \quad \mu_4 = 0.2, \quad \mu_5 = 0.18, \quad \mu_6 = 0.28, \quad \beta_1 = 0.2, \quad \beta_2 = 0.22, \quad \beta_3 = 0.26, \quad \alpha_1 = 0.17,
\]

\[
\alpha_2 = 0.202, \quad \delta_1 = 0.2, \quad \delta_2 = 0.23, \quad \delta_3 = 0.08, \quad \rho_1 = 0.28, \quad \rho_2 = 0.25, \quad \gamma_1 = 0.18, \quad \gamma_2 = 0.21, \quad \eta = 0.34, \quad \omega = 0.14.
\]

In this section, we mainly focus on the following two results:

(i) The existence of the unique ergodic stationary distribution when \(R_0^E > 1\).

(ii) The dynamical behavior of the AIDS of system (2) if \(R_0^E < 0\).

Example 1. Let the second-order perturbations \((\sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)\) and the linear perturbations \((\sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}) = (0.1, 0.1, 0.1, 0.1, 0.1)\). Thus we can compute

\[
R_0 = \frac{\Lambda(1 + \beta_2 \psi_1 + \beta_3 \psi_2 + \alpha_1 \psi_3 + \alpha_2 \psi_4)}{\mu_1 \mu_2} = 4.63886 > 1, \quad \int_0^\infty \left| x - \frac{\Lambda}{\mu_1} \right|^2 \rho(x) dx = 2.448263,
\]

\[
\eta_1 = 2.256095, \quad \eta_0^E = \int_0^\infty \left| x - \frac{\Lambda}{\mu_1} \right|^2 \rho(x) dx + \frac{\Lambda_{\psi_1}(1-R_0 - 1)}{\mu_1} - \frac{\sigma_3^2 \sigma_0^2 \sigma_2^2 \sigma_2^2}{10} = -0.336642 < 0.
\]

It means that there exists a unique endemic equilibrium of determined model (1), which is globally asymptotically stable. In contrast, the AIDS epidemic of system (2) will be exponentially eradicated in a long term by Theorem 2. Fig. 1 can validate them.

Example 2. For the environmental noise intensities \((\sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}) = (0.01, 0.01, 0.01, 0.01, 0.01)\) and \((\sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}, \sigma_{12}) = (0.05, 0.05, 0.05, 0.05, 0.05)\), we still calculate

\[
R_0^E = \frac{\Lambda(1 + \beta_2 \psi_1 + \beta_3 \psi_2 + \alpha_1 \psi_3 + \alpha_2 \psi_4)}{\left( \mu_1 + \frac{\sigma_3^2}{2} + 2 \sqrt{\Lambda_1} \sigma_2^2 \right) \left( \mu_2 + \frac{\sigma_3^2}{2} + 2 \sqrt{\Lambda_2} \sigma_2^2 \right)} = 2.22788 > 1.
\]

Based on Theorem 1, we can obtain that there exists the unique stationary distribution \(\pi(\cdot)\) which has ergodicity property. It means that the AIDS epidemic of system (2) will be persistent. Fig. 2 and Fig. 3 can validate it.
Fig. 1. The left figure: The simulation of the number of groups $S(t), I_1(t), T_1(t), I_2(t), T_2(t)$ and $I_3(t)$ in system (1). The right figure: The simulations of the solution of system (2) with the initial value $(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)) = (0.5, 1.2, 0.8, 0.2, 0.2, 0.2)$ and the environmental noise intensities $(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{31}, \sigma_{41}, \sigma_{51}, \sigma_{61}, \sigma_{12}, \sigma_{22}, \sigma_{32}, \sigma_{42}, \sigma_{52}, \sigma_{62}) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1)$. The right column shows the relevant histogram of density functions of the classes $S(t), I_1(t)$ and $T_1(t)$. The left column reflects the simulation of number variations of $S(t), I_1(t)$ and $T_1(t)$ of system (2) with the initial value $(S(0), I_1(0), T_1(0), I_2(0), T_2(0), I_3(0)) = (0.5, 1.2, 0.8, 0.2, 0.2, 0.2)$ and the noise intensities given in Example 2. The right column reveals the relevant histogram of density functions of the classes $S(t), I_1(t)$ and $T_1(t)$.
From Example 1, we notice that these quadratic noises are all small and \( \sigma_{k1} = 70 \gg 1 \) \( (k = 2, 3, \ldots, 6) \). Then the result \( \mathcal{R}_0^H < 0 \) holds and the AIDS epidemic extinction of stochastic system (2) is obtained. Furthermore, by taking the small linear perturbation and second-order noise into consideration, we can derive the existence of a unique ergodic stationary distribution \( \mu(\cdot) \) when \( \mathcal{R}_0 > 1 \). The above numerical simulations show that the big white noise leads to the disease extinction while the small white noise guarantees the persistence of AIDS epidemic.

6. Discussions and main parameter analysis

6.1. Result discussions

In the real world, the spread and development of many infectious diseases have inevitably affected by the environmental fluctuation, such as Ebola, Cholera, and COVID-19. The nonlinear environmental variations have a great property to explain the realistic dynamical phenomenon of epidemics. Next, we reasonably take three pathological stages (i.e., \( I_1, I_2, I_3 \)) of AIDS patients into consideration. Additionally, the compartments \( I_1, I_2 \) and \( I_3 \) under treatment are introduced to keep in line with the actual situation. Thus our paper focuses on a stochastic staged progression AIDS model with the corresponding staged treatment and second-order perturbation. In contrast, by means of the previous theory and existing method, like [23, 25, 30, 31, 33], it is difficult to obtain the suitable results of stationary distribution and extinction of more realistic stochastic system (2). The main difficulties are described by

(i) Eliminating the influence of second-order perturbations.
(ii) Acquiring the stochastic threshold which is similar to \( \mathcal{R}_0 \) only if the linear perturbation is taken into account.
(iii) The criterion for judging whether the corresponding dynamical results are appropriate or inappropriate.

Thus these problems need to be handled in this paper. Considering the unique ergodic stationary distribution and positive recurrence, we creatively introduce a stochastic \( \epsilon_0 \)-threshold \( \mathcal{R}_0^H(\epsilon) \) of \( \epsilon_0 \)-threshold theory defined in Step 2. Next, we construct a kind of new Lyapunov function to eliminate the relevant square terms of all compartments of system (2). Then problem (i) is completely solved. In view of the proportional transformation described in Step 4, we ultimately prove the assumption \( (C_2) \) of Lemma 3 under \( \mathcal{R}_0^H(\epsilon) > 1 \). In other words, we indirectly acquire the existence and uniqueness of stationary distribution which has ergodicity property if \( \mathcal{R}_0^H > 1 \). Hence problem (ii) is also handled. More importantly, \( \mathcal{R}_0^H \) is a comprehensive result. Not only does it unified the forms of the reproduction number \( \mathcal{R}_0 \) linear and second-order perturbations, but it reveals that the high-order perturbation has no effect on linear noise condition. To further verify these properties of our results, the corresponding reproduction number and stochastic threshold of stationary distribution in Liu and Jiang [29] are respectively given by \( R_0 = \frac{\beta A}{\rho_1(\mu_2+\gamma+a)\gamma} \) and \( \mathcal{R}_0^H = \frac{\beta A}{\sigma_{k1}(\mu_1+\sigma_1^2)\gamma(\sigma_1^2+\gamma+a)+\sigma_1^2} \), where \( \sigma_{k1} (k = 1, 2) \) are linear perturbations and \( \sigma_{k2} (i = 1, 2) \) are second-order fluctuations. Obviously, the term \( \frac{2\lambda\sigma_{k1}}{\sigma_{k1}} \) indicates that \( \sigma_{k1} \neq 0 \). It reveals the imperfection of result. Moreover, if \( \sigma_{k2} = 0 \), we easily derive that \( \mathcal{R}_0^H = R_0 \). Consequently, the above threshold \( \mathcal{R}_0^H \) is unreasonable.

Focusing on the AIDS epidemic extinction of system (2), by the exponential martingale inequality introduced in Mao [20], we similarly eliminate the influence of second-order perturbation. In view of the method of undetermined coefficients used in Step 5 and value of \( \mathcal{R}_0 \), we obtain that the disease will exponentially go to extinction with probability one if \( \mathcal{R}_0^H < 0 \).

Finally, we shall state that our new method and relevant theory are general and universal for current stochastic epidemic models, such as [23, 30, 33, 35]. That is to say, the great dynamics of these complicated stochastic systems under second-order perturbation can be similarly obtained by means of our new theory.

6.2. The parameter analysis of \( \mathcal{R}_0^H \)

In view of Lemmas 2 and 3, we obtain the sufficient condition for the existence of the unique ergodic stationary distribution \( \mu(\cdot) \),
which is described by
\[
\mathcal{R}_0^D = \frac{\Lambda(\beta_1 + \beta_2 \psi_1 + \beta_3 \psi_2 + \alpha_1 \psi_3 + \alpha_2 \psi_4)}{\left( \mu_1 + \frac{\alpha_1}{2} + 2\sqrt{\Lambda^2 \sigma_{11}} + 2\sqrt{\Lambda^2 \sigma_{12}} \right) \left( \mu_2 + \frac{\alpha_2}{2} + 2\sqrt{\Lambda^2 \sigma_{21}} \right)} > 1.
\]

From the expression of \( \mathcal{R}_0^D \), we easily notice that the result of \( \mathcal{R}_0^D > 1 \) is derived if the linear noises \( \sigma_{ij} \) (\( k = 1, 2, \ldots, 6 \)) and the second-perturbations \( \sigma_{ij} \), \( \sigma_{21} \) are all small. Let \( \Lambda \to 0^+ \) or \( \Lambda \to \infty \), we still get that \( \mathcal{R}_0^D \to 0 \). Thus it means that persistence of AIDS will not be derived for the sufficient small or large recruitment rate in system (2). In addition, if we can take effective measures to reduce the movement of susceptible people and isolate those who are infected, then the disease will go to extinction in a certain term. In other words, the sufficient small \( \beta_j \) (\( j = 1, 2, 3 \)) and \( \sigma_{ij} (j = 1, 2) \) lead to the result of \( \mathcal{R}_0^D < 1 \). For example, reasonable joint prevention and control greatly stopped the spread of COVID-19 in 2020.

6.3. The parameter analysis of \( \mathcal{R}_0^D \)

By Theorem 2, we establish the sufficient condition for AIDS epidemic extinction of system (2), which is given by
\[
\mathcal{R}_0^D = \int_0^\infty x - \frac{\Lambda}{\mu_1} \mathcal{E}(\pi) dx + \frac{\Lambda(1 - R_0)}{\mu_1} \left( \eta_1[\eta_{0,1} + \eta_2[\eta_{0,1} \leq 1] \right) - \frac{\sigma_{21}^2 + \sigma_{23}^2 + \sigma_{24}^2 \sigma_{25}^2 + \sigma_{26}^2 \sigma_{27}^2}{10} < 0.
\]

Clearly, the result \( \mathcal{R}_0^D < 0 \) can be obtained by the large linear perturbations \( \eta_{0,1} \) (\( k = 2, 3, 4, 5, 6 \)). Moreover, we realize that \( \int_0^\infty x - \frac{\Lambda}{\mu_1} \mathcal{E}(\pi) dx \) is less than a given constant by a small stochastic fluctuation of \( \mathcal{S}(t) \). Thus the dynamical behavior of the susceptible individuals has a great impact on the disease eradication. In general, we can conclude the fact that large fluctuation leads to the disease eradication but small white noise brings about AIDS persistence. In view of the expressions of \( \mathcal{R}_0^D \) and \( \mathcal{R}_0^D \), not only do they reveal that the dynamical effect of the second-order noises is only reflected in the susceptible and HIV acute infection individuals, but also they show that it is difficult for us to eliminate AIDS without the effective vaccine.

At the end of this paper, several important viewpoints shall be mentioned. Theorems 1 and 2 present the relevant results of stationary distribution and extinction, respectively. However, considering the environmental regime which is effected by factors such as temperature and humidity, we shall consider the corresponding stochastic stage progression AIDS model with staged treatment and regime switching. Furthermore, there is a value gap between \( \mathcal{R}_0^D \) and \( \mathcal{R}_0^D \). Due to the limitation of high-dimensional epidemic model (i.e. system (2)) and our present theory, we have a difficulty obtaining a more accurate criterion of AIDS extinction. Focusing on the result of ergodic stationary distribution developed by our new method, we will be devoted to perfect the form \( \mathcal{R}_0^D \), which is regarded as our future work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Bingtao Han: Validation, Software, Formal analysis, Writing - original draft, Writing - review & editing. Daqing Jiang: Conceptualization, Investigation, Methodology, Writing - original draft, Writing - review & editing. Tasawar Hayat: Methodology, Investigation, Writing - original draft, Writing - review & editing. Ahmed Al-saedi: Conceptualization, Writing - original draft, Writing - review & editing. Bashir Ahmad: Investigation, Writing - original draft, Writing - review & editing.

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Appendix A

(1). (The exponential martingale inequality): Assume that \( g = (g_1, \ldots, g_n) \in L^2(\mathbb{R}^1, \mathbb{R}^{1 \times n}) \), and let \( T, \alpha, \beta \) be any positive numbers. \( B(t) \) denotes an \( n \)-dimensional standard Brownian motion defining on the complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \), then
\[
P\left( \sup_{0 \leq t \leq T} \left| \int_0^t g(s) dB(s) - \frac{\alpha}{2} \int_0^t |g(s)|^2 ds \right| > \beta \right) \leq e^{-\alpha \beta}.
\]

(II. (The proof of Lemma 1): (1)) If \( x > 0 \), we notice that
\[
x^2 - \left( x - \frac{1}{2} \right)(x^2 + 1) = \frac{1}{4} x^2 - x + \frac{1}{2} = \frac{1}{2} (x - 1)^2 > 0,
\]

in which the sign of the above inequality holds if and only if \( x = 1 \). For any \( x \geq 0 \), we have
\[
x^2 - 3 \left( \frac{1}{4} x^2 - \frac{1}{4} \right)(x^2 + 1) = \frac{1}{4} x^2 - 1 x^2 + \frac{1}{4} = \frac{1}{4} (x^2 - 1)^2 \geq 0,
\]

where the sign of the inequality (ii) also holds if and only if \( x = 1 \). Consequently, the results (i) and (ii) are obtained. This completes the proof.

Appendix B

For an \( n \)-dimensional stochastic differential equation
\[
dY(t) = f(Y(t), t) dt + g(Y(t), t) dB(t)
\]

with the initial value \( Y(0) = Y_0 \in \mathbb{R}^n \), let \( B(t) \) be an \( n \)-dimensional standard Brownian motion defining on complete probability space \( (\Omega, \Gamma, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). An important differential operator \( \mathcal{L} \) with respect to the solution \( Y(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \) is given by
\[
\mathcal{L} = \frac{\partial}{\partial t} + \sum_{k=1}^n f_k(Y(t), t) \frac{\partial}{\partial y_k} + \frac{1}{2} \sum_{i,j=1}^n \left[ g_{ij}(Y(t), t) g_{ij}(Y(t), t) \right] \frac{\partial^2}{\partial y_i \partial y_j}.
\]

If the operator \( \mathcal{L} \) act on a stochastic function \( V \in C^{2,1}(\mathbb{R}^n \times [0, \infty); \mathbb{R}^1) \), we have
\[
\mathcal{L} V(Y(t), t) = V_t(Y(t), t) + V_y(Y(t), t) f(Y(t), t) + \frac{1}{2} \text{trace} \left[ g(y(Y(t), t)) g(y(Y(t), t)) \right],
\]

where \( V_t = \frac{\partial V}{\partial t}, \quad V_y = (\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_n}), \quad V_{yy} = (\frac{\partial^2 V}{\partial y_i \partial y_j})_{n \times n} \). Let \( Y(t) \in \mathbb{R}^n \), one has
\[
dV(Y(t), t) = \mathcal{L} V(Y(t), t) dt + V_y(Y(t), t) g(Y(t), t) dB(t).
\]

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