PERIODIC ORBITS FOR DOUBLE REGULARIZATION OF PIECEWISE SMOOTH SYSTEMS WITH A SWITCHING MANIFOLD OF CODIMENSION TWO

DINGHENG PI
Fujian Province University Key Laboratory of Computational Science
School of Mathematical Sciences, Huaqiao University
Fujian 362021, China

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Abstract. In this paper we consider an $n$ dimensional piecewise smooth dynamical system. This system has a co-dimension 2 switching manifold $\Sigma$ which is an intersection of two hyperplanes $\Sigma_1$ and $\Sigma_2$. We investigate the relation between periodic orbit of PWS system and periodic orbit of its double regularized system. If this PWS system has an asymptotically stable sliding periodic orbit (including type I and type II), we establish conditions to ensure that also a double regularization of the given system has a unique, asymptotically stable, periodic orbit in a neighbourhood of $\gamma$, converging to $\gamma$ as both of the two regularization parameters go to 0 by applying implicit function theorem and geometric singular perturbation theory.

1. Introduction. Many people study piecewise smooth (PWS) systems because they are widely used in economy, control theory, mechanical systems with dry frictions, Coulomb friction model, pest control models and so on. Another important source that leads people to consider PWS systems especially their bifurcation problems is the famous Hilbert’s 16th Problem. In the past few years, PWS systems have been studied by many authors from theoretical analysis, numerical simulations and so on. See [4], [19], [25], [36], [38] and the references therein.

The study of periodic orbits of PWS systems has attracted considerable attention. Periodic orbits of planar PWS systems have been discussed in [10], [11] and [13]. Regularization method is very useful when people study PWS systems. The first authors to formally introduce this technique were Sotomayor and Teixeira, see [35]. In 1995, Sotomayor and Teixeira proposed a regularization of a planar PWS system. They replaced switching manifold by a boundary layer of width $2\epsilon$, where $\epsilon > 0$ is a small parameter. Outside the boundary layer, the regularization is the same as the
PWS vector filed. Actually, the regularized system is a continuous approximation of the former PWS system.

Regularization method has been used to study singularities, Filippov sliding vector fields, limit cycles, dynamical behavior near sliding regions of the PWS vector field and two-fold bifurcations of planar PWS systems by applying qualitative theory of ordinary differential equations and geometric singular perturbation theory. See e.g. [24], [27], [31], [5], [6] and [33], etc.

In recent years, some authors have studied the persistence of limit cycles for regularized planar vector fields and structural stability of discontinuous vector fields. They show that if \( \gamma \) is a hyperbolic periodic orbit of planar PWS systems in \( \mathbb{R}^2 \), then, under suitable assumptions, the regularized vector field has a hyperbolic limit cycle \( \gamma_\epsilon \), converging to \( \gamma \) as \( \epsilon \to 0 \). In particular, they apply Poincaré Bendixson Theorem to obtain the existence of limit cycles. See. e.g. [8] and [34]. Although we have Poincaré Bendixson Theorem for planar PWS systems, we don’t have Poincaré Bendixson Theorem for general higher dimensional PWS systems. See e.g. [7]. Then many authors consider the regularization of PWS system in \( \mathbb{R}^3 \) and \( \mathbb{R}^n \). See e.g. [26], [27] and [28]. In our recent work [16], we apply Brouwer fixed Theorem to obtain the existence of limit cycles and get similar results in more general n-dimensional PWS system with a discontinuity hyperplane.

Here we recall some results in [16]. Let \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \) and \( H(x) : \mathbb{R}^n \to \mathbb{R} \). Define \( R_+ = \{ q \in \mathbb{R}^n : H(q) > 0 \} \), \( R_- = \{ q \in \mathbb{R}^n : H(q) < 0 \} \) and \( S = \{ q \in \mathbb{R}^n : H(q) = 0 \} \). \( S \) is called switching manifold. Assume that the gradient \( \nabla H(q) \neq 0 \). Consider the following system:

\[
\dot{x} = F_0(x) = \begin{cases} F_+(x), & \text{if } x \in R_+ \\ F_-(x), & \text{if } x \in R_- \end{cases} \tag{1}
\]

where \( F_- \) and \( F_+ \) are \( C^r \) functions, where \( r \geq 1 \), which we assume to be well defined in \( R_\pm \), on \( S \), and in a neighborhood of \( S \). Write \( F_0 = (F_-, F_+) \). A smooth function \( \phi : \mathbb{R} \to \mathbb{R} \) is a transition function if \( \phi(x) = -1 \) for \( x \leq -1 \), \( \phi(x) = 1 \) for \( x \geq 1 \) and \( \phi'(x) > 0 \) if \( x \in (-1, 1) \). In this paper, we always consider monotone transition functions.

The \( \phi \)-regularization of \( F_0 = (F_-, F_+) \) is a 1-parameter family of vector fields \( F_\epsilon \) and giving the following regularized system for (1):

\[
\dot{x} = F_\epsilon(x) = \frac{1}{2} \left( 1 - \phi\left( \frac{H(x)}{\epsilon} \right) \right) F_-(x) + \frac{1}{2} \left( 1 + \phi\left( \frac{H(x)}{\epsilon} \right) \right) F_+(x). \tag{2}
\]

For brevity, we use the notation \( \phi_\epsilon(z) = \phi\left( \frac{z}{\epsilon} \right) \). Here \( \epsilon \) is a small positive parameter. The vector field \( F_\epsilon \) is an average of \( F_- \) and \( F_+ \) inside the boundary layer \( \{ x \in \mathbb{R}^n : -\epsilon < H(x) < \epsilon \} \), while it is equal to either \( F_- \) or \( F_+ \) outside the boundary layer.

In [16], under appropriate assumptions, we have proved that, if the original PWS system has an asymptotically stable (crossing or sliding) periodic orbit, then so will the regularized system. In fact, this is not a trivial generalization. The first difficulty is that for higher dimensional piecewise smooth systems we do not have a Poincaré-Bendixson Theorem to help us obtain existence of the limit cycle (e.g. [7], [8], [34]); extensions of the Poincaré-Bendixson Theorem for systems in \( \mathbb{R}^n \), see [32], [39], require special type of systems (competitive or monotone systems), which do not fit our type of problem. We have used Brouwer’s fixed point theorem to solve this problem. Another difficulty is to study the stability of the limit cycle...
of the regularized problem. We have done this in our former work by using the
monodromy matrices of the discontinuous and regularized problems, and showing
that the latter converges (as $\epsilon \to 0$) to the former.

Awrejcewicz et al. [3] consider a continuous approximation of $n$-dimensional PWS
system with a codimension one switching manifold. By using Tikhonov’s Theorem
[37] for singularly perturbed systems, they study the relation between periodic
orbits of discontinuous differential equations and periodic orbits of their continuous
approximations. The proof of their main results rely on implicit function theorem.
In this paper, we will also prove our main results by using implicit function theorem
and Tikhonov’s Theorem.

If PWS system has a switching manifold of codimension 2, then the sliding vector
field on the codimension two switching manifold is very complicated. There are
many ways to define the sliding vector field. See e.g. [1], [2] and so on. Recently,
Dieci and Gugliemi consider the regularization of PWS systems with a switching
manifold of codimension 2 when this switching manifold is nodally attractive, see
[18]. However, the relation of periodic orbit of PWS system and periodic orbit of its
regularized system is not clear. We will discuss this relation in the present paper.

Regularization of two special intersecting switching manifold was considered by
Llibre, da Silva and Teixeira in [28]. In our recent paper [30], we consider the relation
between periodic orbit of PWS systems with a switching manifold of codimension
two and limit cycles of its regularization. We consider the following system:

$$\dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, 2, 3, 4$$

(3)

$R_i \in \mathbb{R}^n$ are open, disjoint and connected sets, and we may as well think that
$\mathbb{R}^n = \bigcup R_i$. Each $f_i$ is $C^k$, $k \geq 1$ on $R_i$, $i = 1, 2, 3, 4$. We will assume that each $f_i$
is actually $C^k$ in an open neighborhood of each $R_i$. Write $x = (x_1, x_2, x_3, \cdots, x_n)^T$, where
$x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, y = (x_3, \cdots, x_n)^T \in \mathbb{R}^{n-2}$. $f_i(x) = (f_{i1}(x), f_{i2}(x), f_{i3}(x))^T$,
where $f_{ij}(x) : \mathbb{R}^n \to \mathbb{R}, \; j = 1, 2$ and $f_{i3}(x) : \mathbb{R}^n \to \mathbb{R}^{n-2}, \; i = 1, 2, 3, 4$. $R'_i$s are locally
separated by two intersecting hyperplanes of codimension 1, $\Sigma_1 = \{x \in \mathbb{R}^n : x_1 = 0\}$
and $\Sigma_2 = \{x \in \mathbb{R}^n : x_2 = 0\}, \Sigma = \Sigma_1 \cap \Sigma_2$. Without loss of generality, we label the
four regions $R'_i$s and related vector fields as follows:

$$R_1 : f_1 \quad \text{when} \quad x_1 < 0, x_2 < 0, \quad R_2 : f_2 \quad \text{when} \quad x_1 < 0, x_2 > 0,
R_3 : f_3 \quad \text{when} \quad x_1 > 0, x_2 > 0, \quad R_4 : f_4 \quad \text{when} \quad x_1 > 0, x_2 < 0.$$  

(4)

Take $\epsilon > 0$ small, the regularization of (3) is the following differential equation

$$\dot{x} = f_\epsilon(x) = \frac{1}{2} - \frac{1}{2} \frac{\phi_r(x_1)}{1 - \phi_r(x_2)} f_1(x) + \frac{1}{2} \frac{1 - \phi_r(x_1)}{1 + \phi_r(x_2)} f_2(x) + \frac{1 + \phi_r(x_1)}{2} \frac{1 + \phi_r(x_2)}{2} f_3(x) + \frac{1}{2} \frac{1 - \phi_r(x_1)}{1 - \phi_r(x_2)} f_4(x).$$

(5)

Under suitable assumptions, we have proved if the original PWS system (3) has
an asymptotically stable periodic orbit, then so will the regularized system (6). This
work can be seen in [30].

In what follows, we introduce double regularization of system (3). Take $\epsilon > 0$ and $\eta > 0$ small, the double regularization of system (3) is given by

$$\dot{x} = f_{\epsilon, \eta}(x) = \frac{1}{2} - \frac{1}{2} \frac{\phi_r(x_1)}{1 - \psi_r(x_2)} f_1(x) + \frac{1}{2} \frac{1 - \phi_r(x_1)}{1 + \psi_r(x_2)} f_2(x) + \frac{1}{2} \frac{1 + \phi_r(x_1)}{2} \frac{1 + \psi_r(x_2)}{2} f_3(x) + \frac{1}{2} \frac{1 - \phi_r(x_1)}{1 - \psi_r(x_2)} f_4(x).$$

(6)
Here, $\phi$ and $\psi$ are two monotone transition functions. Actually, the double regularization (6) is a generalization of (5). (5) was first seen in [1] and then studied by many authors, see e.g. [12]. Panazzolo and da Silva considered a multi-parameter regularization in [29], where they applied two monotonic transition functions. Then they discussed sliding dynamics and sliding region by considering limit dynamics with respect to the given regularization. However, they didn’t consider relation between periodic orbit of PWS system and periodic orbit of the regularized system. This work will be done in this paper. The present work can be seen as a follow-up of [30].

The study of the double regularized system (6) is important because it replaces the PWS system with a codimension two switching manifold by a continuous system. Since system (6) is of singular perturbation type, now geometric singular perturbation theory can be applied to study (6). Moreover, the solution of (6) is unique as long as it exists. Let $\epsilon \rightarrow 0$ and $\eta \rightarrow 0$, system (6) will reduce to system (3).

The remainder of this paper is organized as follows. Filippov sliding vector fields and some definitions are introduced in Section 2. In Section 3, we will state our main results. Their proofs will be given in Section 4. Our conclusions will be given in Section 5.

2. Filippov sliding vector fields.

2.1. Codimension 1 case. For system (1), if a solution intersects $S$ at an attractive sliding point $x$ then it must remain on $S$ and slides on $S$. However the sliding vector field needs to be defined. We follow Filippov (see [19]) and for each $x \in S$ we define the sliding vector field as

$$F_S(x) = \frac{1}{2} [(1 - \phi^*)F_- + (1 + \phi^*)F_+] (x), \quad \phi^*(x) = \frac{\nabla H^T(F_- + F_+)}{\nabla H^T(F_- - F_+)} (x),$$

(7)

where the value of $\phi^*(x)$ in (7) is such that $\nabla H^T F_S(x) = 0$. This equation (7) ensures that the sliding vector field $F_S(x)$ is tangent to $S$.

We assume that $f_i$ are $C^r$, $i = 1, 2, 3, 4$, $r \geq 1$, and in a neighborhood of $\Sigma_1, \Sigma_2$ and $\Sigma$. We denote the flow of (3) as $\varphi_i^t(x)$ and the flow of (5) as $\varphi_i^r(x)$.

The case when $\Sigma$ is of codimension one discontinuity surface is well understood. For the completeness, we recall it briefly. Let

$$\Sigma_1^+ = \{ x \in \Sigma_1 : x_2 > 0 \}, \quad \Sigma_1^- = \{ x \in \Sigma_1 : x_2 < 0 \},$$
$$\Sigma_2^+ = \{ x \in \Sigma_2 : x_1 > 0 \}, \quad \Sigma_2^- = \{ x \in \Sigma_2 : x_1 < 0 \}.$$

Note that the sliding vector field is actually defined on the codimension one switching manifold and it is tangent to this switching manifold, i.e.

$$x \in \Sigma_1^- \cap \Sigma_2^+ : f_{\Sigma_1^-} = (1 - \alpha^-)f_1 + \alpha^- f_4, 0 \leq \alpha^- \leq 1; n_1^T f_{\Sigma_1^-} = 0;$$
(8)
$$x \in \Sigma_1^+ \cap \Sigma_2^- : f_{\Sigma_1^+} = (1 - \alpha^+)f_2 + \alpha^+ f_3, 0 \leq \alpha^+ \leq 1; n_1^T f_{\Sigma_1^+} = 0;$$
(9)
$$x \in \Sigma_2^- \cap \Sigma_1^+ : f_{\Sigma_2^-} = (1 - \beta^-)f_1 + \beta^- f_2, 0 \leq \beta^- \leq 1; n_2^T f_{\Sigma_2^-} = 0;$$
(10)
$$x \in \Sigma_2^+ \cap \Sigma_1^- : f_{\Sigma_2^+} = (1 - \beta^+)f_4 + \beta^+ f_3, 0 \leq \beta^+ \leq 1; n_2^T f_{\Sigma_2^+} = 0.$$
(11)

Where $n_1^T = (1, 0, \cdots, 0)$, $n_2^T = (0, 1, 0, \cdots, 0)$.

**Definition 1.** If a solution of (3) reaches the switching manifold $\Sigma_1$ or $\Sigma_2$ or $\Sigma$ at an attractive sliding point $x$. Then $x$ is said to be a transversal entry point.
2.2. **Codimension 2 case.** Following [14] and [19], we call Filippov sliding vector filed (on $\Sigma$) any vector field of the form

$$f_\Sigma(x) = \sum_{i=1}^{4} \lambda_i(x) f_i(x), \quad \text{where} \quad \lambda_i(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{4} \lambda_i(x) = 1, x \in \Sigma, \quad (12)$$

subject to the constraint that $f_\Sigma(x)$ lies in the tangent plane to $\Sigma$ at $x$, i.e.

$$n_j^T f_\Sigma(x) = 0, \quad \text{for} \quad j = 1, 2. \quad (13)$$

Since we have 4 unknowns in 3 equations, it’s easy to know we will fail to select uniquely the coefficients $\lambda_i, i = 1, 2, 3, 4$.

There are many methods to choose Filippov sliding vector filed in codimension two switching manifold. For instance, bilinear interpolation method, moments method and limit method in [2]. However, we are interested in the bilinear interpolation method. To our knowledge, it was first seen in [1] and then it has been discussed thoroughly in [9], [14] and so on. This method selects the Filippov sliding vector field as follows.

$$f_\Sigma := (1 - \alpha)(1 - \beta)f_1 + (1 - \alpha)\beta f_2 + \alpha(1 - \beta)f_3 + \alpha\beta f_4, \quad (14)$$

where $\alpha, \beta \in (0, 1)$, $\lambda_B = \begin{pmatrix} (1 - \alpha)(1 - \beta) \\ \beta(1 - \beta) \\ \alpha(1 - \beta) \\ \alpha \beta \end{pmatrix}$, $W = \begin{pmatrix} w_1^1 & w_1^2 & w_1^3 & w_1^4 \\ w_2^1 & w_2^2 & w_2^3 & w_2^4 \\ w_3^1 & w_3^2 & w_3^3 & w_3^4 \\ w_4^1 & w_4^2 & w_4^3 & w_4^4 \end{pmatrix}$ such that $W\lambda_B = 0$. $w_j^i = n_j^T f_i, i = 1, 2, j = 1, 2, 3, 4$.

Assume that trajectories of the PWS system (3) exist in a neighborhood $U$ of $\Sigma$. This must be understood to imply that in case the value of $x_0$ is on either $\Sigma_1$ or $\Sigma_2$ (but not on $\Sigma$), there may be sliding motion on $\Sigma_1$ or $\Sigma_2$ according to Filippov’s first order theory. However, we also remark that motion in a neighborhood of $\Sigma$ may not be uniquely defined, such as when $x_0 \in \Sigma_1$ the sliding motion on $\Sigma_1$ is repulsive. For convenience, we will still write $\phi^t(x_0)$ to indicate a continuous Filippov trajectory of the system. The general characterization of attractive $\Sigma$ will require that $\Sigma$ be stable (with respect to the initial conditions) and approached by trajectories of the system.

$\Sigma$ is stable if for any $x_0 \in \Sigma$, and for any $\epsilon > 0$ sufficiently small, there is $\delta > 0$ such that if $u \in B_\delta(x_0)/(\Sigma \cap B_\delta(x_0)$, then the distance between $\phi^t(u)$ and $\Sigma$ satisfies $d(\phi^t(u) - \Sigma) \leq \epsilon$ for all $t \geq 0$. However, in many situations of interest, we will want a more restrictive condition requiring not only that $\Sigma$ is attractive, but it is also approached in finite time. See the following definition in [14].

**Definition 2.** $\Sigma$ attracts in finite time trajectories of (3), if: (i) $\Sigma$ is stable; (ii) for any $x_0 \in U/(\Sigma \cap U)$, where $U$ is a neighborhood of $\Sigma$, there exists a first (finite) time $\tau(x_0) \geq 0$ such that $\phi^{\tau(x_0)}(x_0) \in \Sigma$.

**Definition 3.** Assume that the trajectory of system (3) slides on $\Sigma$. When the trajectory reaches a point $x_1 \in \Sigma$, where one-and only one- of the following four conditions satisfied.

(i) Exiting on $\Sigma^-_2$ or $\Sigma^+_2$:

(a) $n_1(\bar{x})^T f_{\Sigma^-_2}(\bar{x}) = 0$, or

(b) $n_1(\bar{x})^T f_{\Sigma^+_2}(\bar{x}) = 0$.

(ii) Exiting on $\Sigma^-_1$ or $\Sigma^+_1$:
3.1. Type I sliding periodic orbit.

(a) \( n_2(\bar{x})^T f_{\Sigma^-_2}(\bar{x}) = 0 \), or
(b) \( n_2(\bar{x})^T f_{\Sigma^+_2}(\bar{x}) = 0 \).

Then \( \bar{x} \) is called a (first order) generic tangential exit point. Here \( f_{\Sigma^-_2}, f_{\Sigma^+_2}, f_{\Sigma^-_1}, f_{\Sigma^+_1} \) are called exit vector fields, respectively. See e.g. [15].

When PWS systems have only codimension one switching manifold, flows can only exit from the switching manifold at a tangency point. However, if PWS systems have a codimension two switching manifold, we know from [23] that the flows exit from codimension two sliding modes may have many different cases. Along an intersection of two switching surfaces, exit can occur at a tangency with a lower codimension sliding flow or by a spiralling of the flow that exhibits geometric divergence and at an intersection point of two switching surfaces. In this paper, we will consider the cases of Figures 1(ii) and 1(iii) in [23] and related periodic orbits.

When we consider the stability of periodic orbit of high dimensional smooth dynamical systems, we usually study the Floquet multipliers. When we consider the stability of periodic orbit \( \gamma \) in PWS system, the jumps in the derivatives of the solution should be considered, and the monodromy matrix along \( \gamma \) is defined with the aid of suitable saltation matrices. We also study stability properties of a periodic orbit \( \gamma \) of PWS system via the eigenvalues of the monodromy matrix, i.e. the Floquet multipliers. See e.g. [15] and [17].

**Definition 4.** Let \( \gamma \) be a periodic orbit of (3). Let \( \mu_1, \mu_2, \cdots, \mu_n \) be the corresponding Floquet multipliers. We say that \( \gamma \) is asymptotically stable if one of the multipliers is 1, say \( \mu_1 = 1 \), and all other \( \mu_i \)'s are less than 1 in modulus.

3. Statement of main results. In this section, we will first introduce two types of sliding periodic orbits that we will consider later.

3.1. **Type I sliding periodic orbit.** Assume that the following conditions hold.

(A1) Crossing part. There is a solution \( p(t) \) of equation (3) defined on \([0, T_2] \) such that the \( x_2 \)-coordinate \( p_2(t) \) satisfy \( p_2(t) > 0 \) for \( t \in (0, T_2) \) and \( p_2(0) = p_2(T_2) = 0 \). At time \( t = T_1 \), the trajectory crosses \( \Sigma^+_1 \). When \( t \in [0, T_1] \) the trajectory is induced by equation \( \dot{x} = f_3(x) \). If \( t \in (T_1, T_2) \), the trajectory is controlled by equation \( \dot{x} = f_2(x) \).

On the other hand, we need to consider the sliding part on \( \Sigma^-_2 \). We consider

\[
\dot{x} = \bar{f}(x) = \begin{cases} 
  f_2(x), & \text{if } x_1 < 0, x_2 > 0 \\
  f_1(x), & \text{if } x_1 < 0, x_2 < 0
\end{cases}
\]  
(15)

A continuous approximation of (15) is given by \( \dot{x} = \bar{f}(x) \) for \( |x_2| \geq \eta \) and if \( |x_2| < \eta \):

\[
\dot{x} = \frac{1 + \psi(x_2)}{\eta} f_2(x) + \frac{1 - \psi(x_2)}{2} f_1(x).
\]  
(16)

Set \( \frac{x_2}{\eta} = z, |z| \leq 1 \), then we have

\[
\dot{z}_1 = \frac{1 + \psi(z)}{2} f_{21}(x) + \frac{1 - \psi(z)}{2} f_{11}(x),
\]
\[
\eta \dot{z} = \frac{1 + \psi(z)}{2} f_{22}(x) + \frac{1 - \psi(z)}{2} f_{12}(x),
\]  
(17)
\[
\dot{y} = \frac{1 + \psi(z)}{2} f_{23}(x) + \frac{1 - \psi(z)}{2} f_{13}(x).
\]
The reduced system of (17) is
\[
\begin{align*}
\dot{x}_1 &= \frac{1 + \psi(z)}{2} f_{21}(x) + \frac{1 - \psi(z)}{2} f_{11}(x), \\
\dot{y} &= \frac{1 + \psi(z)}{2} f_{23}(x) + \frac{1 - \psi(z)}{2} f_{13}(x),
\end{align*}
\tag{18}
\]
where \( \psi(z) = \frac{f_{23}(x_1,0,y) + f_{13}(x_1,0,y)}{f_{13}(x_1,0,y) - f_{23}(x_1,0,y)} := V(x_1,y). \)

(A2) Sliding part. Assume that \((p_1(t),p_3(t))\) is the \((x_1,y)\)-coordinate of \(p(t)\), then solution \((x_1(t),y(t))\) of equation (18) with initial condition \((x_1(0),y(0)) = (p_1(T_2),p_3(T_2))\) passes through the point \((p_1(0),p_3(0))\) at time \(T_2 + T_3\). For \(t \in [0,T_3]\), \(f_{22}(x_1(t),0,y(t)) < 0\) and \(f_{12}(x_1(t),0,y(t)) > 0\).

(A3) Assume that \(\eta = K(\epsilon) > 0\) is a continuous function of \(\epsilon\), \(\lim_{\epsilon \to 0} K(\epsilon) = 0\).

Under above assumptions, there exists a sliding periodic solution \(p_0(t)\). It consists of crossing part and sliding part. \(p_0(t)\) is given by
\[
p_0(t) = \begin{cases} 
    (p(t), & \text{if } t \in [0,T_2], \\
    (x_1(t-T_2),0,y(t-T_2)), & \text{if } t \in (T_2,T_2+T_3)
\end{cases}.
\tag{19}
\]

Where \(p_0(t)\) crosses \(\Sigma_1^+\) at time \(T_1\). It starts to slide on \(\Sigma_2^-\) at time \(T_2\).

A periodic orbit is called type I sliding periodic orbit if it starts from some initial point on \(\Sigma\), then it moves under the action of flow \(\varphi^t_1\) of \(f_1\) until it crosses \(\Sigma_1^-\). Then it moves under the action of flow \(\varphi^t_2\) of \(f_2\) until it reaches \(\Sigma_2^-\). Finally it slides on \(\Sigma_2^-\) until it returns the starting point \((p_1(0),0,p_3(0))\) on \(\Sigma\), \(p_1(0) = 0\). See the following Figure 1.

For the sliding periodic orbit of type I, we have the following

**Theorem 5.** Assume that system (3) has an asymptotically stable type I sliding periodic orbit \(\gamma\). Assume moreover that the entrance point \(x_2\) in \(\Sigma_2^-\) is a transversal entry point and that the exit point from \(\Sigma\) is an intersection point of \(\Sigma_1^+\) and \(\Sigma_2^-\). Then, for \(\epsilon > 0\) and \(\eta > 0\) sufficiently small, there exists a unique limit cycle \(\gamma_{\epsilon\eta}\) of system (6) in a neighborhood of \(\gamma\). Moreover \(\gamma_{\epsilon\eta}\) is asymptotically stable and \(\lim_{\epsilon \to 0,\eta \to 0} \gamma_{\epsilon\eta} = \gamma\).
Remark. In Theorem 5, we have a codimension one sliding motion and this sliding motion will exit from $\Sigma_2$ at an intersection point of $\Sigma_1$ and $\Sigma_2$. This point is not a tangential point. Then the trajectory continues as the flow of $f_3$. Actually, we will have a classical solution. This case is different from what we have known for the codimension one sliding motion. Moreover, the type I sliding periodic orbit is different from the sliding periodic orbit we have considered in [16]. From case 2 of Theorem 6.4 in [21], we know that this phenomenon indeed exists.

3.2. Type II sliding periodic orbit. If $\gamma$ is a type II sliding periodic orbit. Without loss of generality, we assume that it looks like Figure 2. This kind of sliding periodic orbit has been studied in [15] for 3D case. Let $\bar{x} \in \Sigma$ be a tangential exit point that satisfies condition (i)(a) in definition 3. Take the initial condition as $x(0) = \bar{x}$. This periodic orbit starts sliding on $\Sigma_2^+$ with the sliding vector filed $f_{\Sigma_2^+}$. Then at $x = \hat{x}_1 \in \Sigma_2^+$, the trajectory exits $\Sigma_2^+$ smoothly and enters in $R_3$. Then at $x = \hat{x}_2 \in \Sigma_1^+$, the trajectory reaches $\Sigma_1^+$ transversally and slides on it. At $x = \hat{x}_3 \in \Sigma$, the trajectory reaches $\Sigma$ transversally and slides on it with the sliding vector field $f_{\Sigma}$ until it reaches the exit point $\bar{x}$. Assume that system (3) has an asymptotically stable sliding periodic orbit $\gamma$ of type II given as the following Figure 2.

![Figure 2. Sliding periodic orbit of type II](image)

For codimension one sliding modes, we will use Lemma 18 in [16] to deal with their convergence. It follows from Lemma 18, when we enter the codimension one switching manifold at a entry point, the solution of the regularized differential equation (2) converges uniformly to the Filippov solution of (1).

For codimension two sliding modes, in order to consider the relationship between solutions of PWS systems and solutions of regularized differential equations, we apply the following results. See e.g. [22]. Recall that $x = (x_1, x_2, x_3, \ldots, x_n)$, $y = (x_3, \ldots, x_n)$, double regularization of system (3) can be written as
\[
\dot{\phi} = \frac{d\phi}{dx_1} e_1^T f_{\epsilon, \eta}(\phi, \psi, y)
\]
\[
\dot{\psi} = \frac{d\psi}{dx_2} e_2^T f_{\epsilon, \eta}(\phi, \psi, y)
\]
\[
\dot{y} = g(\phi, \psi, y).
\]

(20)

where

\[
f_{\epsilon, \eta}(\phi, \psi, y) = \frac{1 - \phi(\frac{x_1}{\epsilon})}{2} - \frac{1 - \psi(\frac{x_2}{\eta})}{2} f_1(x) + \frac{1 - \phi(\frac{x_1}{\epsilon})}{2} \frac{1 + \psi(\frac{x_2}{\eta})}{2} f_2(x) + \frac{1 + \phi(\frac{x_1}{\epsilon})}{2} \frac{1 + \psi(\frac{x_2}{\eta})}{2} f_3(x) + \frac{1 + \phi(\frac{x_1}{\epsilon})}{2} \frac{1 - \psi(\frac{x_2}{\eta})}{2} f_4(x).
\]

Where we still use \( f_i, i = 1, 2, 3, 4 \) for convenience. Here \( e_i \) is the standard \( i \)-th unit vector in \( \mathbb{R}^n \), \( i = 1, 2, \ldots \), \( f_{\epsilon, \eta}(x) = (g_1(x), g_2(x), g(x))^T \). Note that if \( x \in \{x_1, x_2 : -\epsilon \leq x_1 \leq \epsilon, -\eta \leq x_2 \leq \eta \} \), then \( \phi \) and \( \psi \) can be seen as the fast variables and \( y \) can be seen as the slow variable. Set \( u = \frac{x_1}{\epsilon} \) and \( v = \frac{x_2}{\eta} \), we have

\[
\frac{d\phi}{du} = \frac{\phi}{\epsilon} \quad \frac{d\phi}{dv} = \frac{\phi}{\eta}
\]

Analogously, \( \frac{d\psi}{dv} = \frac{\psi}{\eta} \).

Since \( \eta = k\epsilon \), we conclude from (20) that

\[
\epsilon \dot{\phi} = \frac{d\phi}{du} e_1^T f_{\epsilon, \eta}(\phi, \psi, y) := \frac{d\phi}{du} g_1(\phi, \psi, y)
\]
\[
\epsilon \dot{\psi} = \frac{d\psi}{dv} e_2^T f_{\epsilon, \eta}(\phi, \psi, y) := \frac{d\psi}{dv} g_2(\phi, \psi, y)
\]
\[
\dot{y} = g(\phi, \psi, y).
\]

(21)

We denote by \((\phi_\epsilon(.), \psi_\epsilon(.), y_\epsilon(.))\) the solution of (20). Solutions of (6) and (21) are the same if \( \epsilon \neq 0 \) and their initial values are consistent.

Let \( \epsilon = 0 \), we have

\[
0 = g_1(\phi, \psi, y)
\]
\[
0 = g_2(\phi, \psi, y)
\]
\[
\dot{y} = g(\phi, \psi, y).
\]

(22)

The fast system is given by

\[
\phi' = \frac{d\phi}{du} e_1^T f_{\epsilon, \eta}(x) := \frac{d\phi}{du} g_1(\phi, \psi, y)
\]
\[
\psi' = \frac{1}{k} \frac{d\psi}{dv} e_2^T f_{\epsilon, \eta}(x) := \frac{1}{k} \frac{d\psi}{dv} g_2(\phi, \psi, y).
\]

(23)

We now consider the relationship between solutions of PWS systems (3) and solutions of regularized differential equations (6). To do so, we refer to the following results. See e.g [12] and [22]. Notice that solutions of (22) are sliding solutions on \( \Sigma \) with bilinear vector field. In our case, we need to answer the following two questions:

1. Whether solutions of (21) converge to solutions of (22) as \( \epsilon \to 0 \) or not?
2. What happens if we consider the behavior of solutions of (21) in the neighborhood of a first order tangential exit point?

For the second question, the codimension two sliding vector field of the sliding periodic orbit (Type II) will exit from a tangential point and enter \( \Sigma_2^+ \), then it will continue sliding on \( \Sigma_2^+ \) as a codimension one sliding mode.

For the first question, let \((\phi^*(y), \psi^*(y))\) be the solution of
We apply the following results. See e.g. Theorem 2.3 in [12].

Lemma 6. Assume that the following conditions hold.

(i) The solution \((\phi^*(y), \psi^*(y))\) of (24) is continuous in \(y\).
(ii) \((\phi^*(y), \psi^*(y))\) is a locally asymptotically stable equilibrium of system (23).
(iii) The initial condition \((\phi_0, \psi_0, y_0)\) of system (20) is such that the \(\omega\)-limit set of \((\phi(t, y_0), \psi(t, y_0))\) is \((\phi^*(y_0), \psi^*(y_0))\).
(iv) The equation \(\dot{y} = g(\phi^*(y), \psi^*(y), y)\), \(y(0) = y_0\) has a unique solution, denote it by \(y_0(t)\).

Then as \(\epsilon \to 0\), the solution of equation (20) with initial condition \((\phi_0, \psi_0, y_0)\) satisfies the following two results.

(1) \(y(.)\) converges to \(y_0(.)\), uniformly in time on intervals of the form \([0, T]\).
(2) \((\phi(t), \psi(t))\) converges to \((\phi^*(y_0(t)), \psi^*(y_0(t)))\) uniformly in time on intervals of the form \([\delta, T], \delta > 0\).

From Lemma 6, we know that if the above four conditions hold, the solution of the regularized system converges uniformly to the sliding solution of bilinear vector field. The stability of solutions in this system is determined by the two-dimensional matrix

\[
G(\phi, \psi, y) = \begin{pmatrix}
\frac{d\phi}{dx} \partial_\phi g_1(\phi, \psi, y) & \frac{d\phi}{dx} \frac{1}{k} \partial_\psi g_1(\phi, \psi, y) \\
\frac{d\psi}{dx} \partial_\psi g_2(\phi, \psi, y) & \frac{d\psi}{dx} \frac{1}{k} \partial_\psi g_2(\phi, \psi, y)
\end{pmatrix}.
\]

(25)

Easy calculation shows that

\[
G(\phi, \psi, y) = \begin{pmatrix}
\frac{d\phi}{dx} & 0 \\
0 & \frac{d\psi}{dx}
\end{pmatrix} \cdot \begin{pmatrix}
\partial_\phi g_1(\phi, \psi, y) & \frac{1}{k} \partial_\psi g_1(\phi, \psi, y) \\
\partial_\psi g_2(\phi, \psi, y) & \frac{1}{k} \partial_\psi g_2(\phi, \psi, y)
\end{pmatrix}.
\]

Denote by

\[
G^*(\phi, \psi, y) = \begin{pmatrix}
\partial_\phi g_1(\phi, \psi, y) & \frac{1}{k} \partial_\psi g_1(\phi, \psi, y) \\
\frac{1}{k} \partial_\psi g_2(\phi, \psi, y) & \frac{1}{k} \partial_\psi g_2(\phi, \psi, y)
\end{pmatrix}.
\]

(26)

If both eigenvalues \(\lambda_j = \lambda_j(\phi, \psi, y), j = 1, 2\) of matrix \(G^*(\phi, \psi, y)\) of (26) satisfy \(\lambda_j \leq \mu < 0\) on a \(\epsilon\)-independent neighborhood \(U\) of \(\{(\phi_0(t), \psi_0(t), y_0(t)) : t \in [0, T]\}\), then from Theorem 1 in [22] and proof of Proposition 2.7 in [12] we know the robustness of codimension two sliding modes, i.e. the solution of the regularized problem will remain close to the codimension two sliding mode.

For the sliding periodic orbit case II, set \(\eta = k\epsilon, k > 0\). We have the following results.

Theorem 7. Assume that system (3) has an asymptotically stable type II sliding periodic orbit \(\gamma\) and both eigenvalues \(\lambda_j = \lambda_j(\phi, \psi, y), j = 1, 2\) of matrix \(G^*(\phi, \psi, y)\) satisfy \(\lambda_j \leq \mu < 0\) on a \(\epsilon\)-independent neighborhood \(U\) of \(\{(\phi_0(t), \psi_0(t), y_0(t)) : t \in [0, T]\}\). Assume moreover that the entrance point \(x_2\) in \(\Sigma^+_1\) is a transversal entry point and that both the exit points \(\bar{x}\) from \(\Sigma\) and \(x_1\) from \(\Sigma^+_2\) are first order tangential exit points. Then, for \(\epsilon > 0\) sufficiently small and \(\eta = k\epsilon\), there exists a unique periodic orbit \(\gamma_\epsilon\) of system (6) in a neighborhood of \(\gamma\). Moreover \(\gamma_\epsilon\) is asymptotically stable and \(\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma\).
Remark. In Theorem 7, we have a codimension two sliding motion and this sliding motion will exit from $\Sigma$ at a first order tangential exit point and continue sliding on $\Sigma^1_\epsilon$ as a codimension one sliding motion. From case 2 of Theorem 6.5 in [21], we know that there indeed exist this phenomenon.

4. Proofs of main results. In this section, we prove our main results. We will treat the sliding periodic orbit when it slides only on codimension one switching manifold (type I) and it slides on codimension two switching manifold (type II) separately.

4.1. Proof of Theorem 5. Consider

$$\dot{x} = f_0(x) = \begin{cases} f_2(x), & \text{if } x_1 < 0, x_2 > 0 \\ f_3(x), & \text{if } x_1 > 0, x_2 > 0. \end{cases} \quad (27)$$

A continuous approximation of (27) is given by $\dot{x} = f_0(x)$ for $|x_1| > \epsilon$ and if $|x_1| \leq \epsilon$:

$$\dot{x} = f_\epsilon(x) = \frac{1 + \phi(\frac{t_2}{\epsilon})}{2} f_3(x) + \frac{1 - \phi(\frac{t_2}{\epsilon})}{2} f_2(x). \quad (28)$$

Denote by $\varphi^t_0$ and $\varphi^t_\epsilon$ the flows of (27) and (28) respectively.

Lemma 8. For each $x_0 \in \overline{B_\delta(p(0))}$ the following equation

$$\lim_{t \to 0} \varphi^t_\epsilon(x_0) = \varphi^t_0(x_0),$$

holds uniformly for $t$ in a compact interval.

Proof. $\varphi^t_0$ is $\varphi^t_2$ which is flow of $f_2$ when $x_1 < 0, x_2 > 0$, $\varphi^t_0$ is $\varphi^t_3$ which is flow of $f_3$ when $x_1 > 0, x_2 > 0$. Together with $\Sigma_1$, consider also the hyperplanes $\Sigma_{1,\epsilon} = \{x \in \mathbb{R}^n \mid h_1(x) = x_1 = \epsilon\}$ and $\Sigma_{1,\epsilon'} = \{x \in \mathbb{R}^n \mid h_1(x) = x_1 = -\epsilon\}$. In what follows, for $x_0 \in \Sigma_{1,\epsilon}$, we want to estimate the distance between $\varphi^t_0(x_0)$ and $\varphi^t_\epsilon(x_0)$ at their intersection points with $\Sigma_1$ and $\Sigma_{1,\epsilon}$. Without loss of generality assume that $x_0 \in \Sigma_{1,\epsilon}$. Then for $\epsilon > 0$ sufficiently small $\nabla h_i^T f_\epsilon(x_0) < 0$, $\nabla h_i^T f_2(x_0) < 0$ and $\nabla h_i^T f_3(x_0) < 0$. Let $t_1$ be such that $\varphi^{t_1}_0(x_0) \in \Sigma_{1,\epsilon}$ and similarly, let $\varphi^{t_1}_\epsilon(x_0) \in \Sigma_{1,\epsilon}$. Let $t_2$ be such that $x_1 = \varphi^{t_2}_\epsilon(x_0) \in \Sigma_1$ and similarly, let $x_1 = \varphi^{t_2}_0(x_0) \in \Sigma_1$. We want to bound $\|\varphi^{t_2}_0(x_0) - \varphi^{t_2}_\epsilon(x_0)\|$ and show that it goes to zero when $\epsilon \to 0$. To fix ideas, assume $t_2 > t_1$. Let $L_1 = \max (\max_{x \in [0,t_2]} Df_2(\varphi^t_0(x_0)), \max_{x \in [0,t_2]} Df_3(\varphi^t_\epsilon(x_0)))$ and $M_i = \max_{x \in [0,t_2]} \|f_i(\varphi^t_\epsilon(x_0))\|, i = 2, 3$. Then we have

$$\|\varphi^{t_2}_0(x_0) - \varphi^{t_2}_\epsilon(x_0)\| = \|\varphi^{t_2-t_1}_i(\varphi^{t_1}_3(x_0)) - \varphi^{t_2-t_1}_i(\varphi^{t_1}_3(x_0))\|.$$

We will consider $\|\varphi^{t_2}_3(x_0) - \varphi^{t_2}_\epsilon(x_0)\|$, the following inequalities hold.

$$\|\varphi^{t_2}_3(x_0) - \varphi^{t_2}_\epsilon(x_0)\| = \|\varphi^{t_2}_3(x_0) - \varphi^{t_2}_\epsilon(x_0) + \varphi^{t_2}_\epsilon(x_0) - \varphi^{t_2}_\epsilon(x_0)\|
\leq \|\varphi^{t_2}_3(x_0) - \varphi^{t_2}_\epsilon(x_0)\| + \|\varphi^{t_2}_\epsilon(x_0) - \varphi^{t_2}_\epsilon(x_0)\|.$$

Easy calculation shows that

$$\|\varphi^{t_2}_3(x_0) - \varphi^{t_2}_\epsilon(x_0)\| = \| \int_0^{t_2} f_3(\varphi^s_3(x_0))ds - \int_0^{t_2} f_\epsilon(\varphi^s_\epsilon(x_0))ds \|
\leq \| \int_0^{t_2} f_3(\varphi^s_3(x_0))ds - f_3(\varphi^s_\epsilon(x_0))ds \|
+ \| \int_0^{t_2} f_\epsilon(\varphi^s_\epsilon(x_0))ds - f_2(\varphi^s_\epsilon(x_0))ds \|. $$
It follows that
\[
\|\varphi^2_t(x_0) - \varphi^1_t(x_0)\| \leq L_1 \int_0^{t_2} \|\varphi^3_s(x_0) - \varphi^1_s(x_0)\| ds + t_2(M_1 + M_2)
\]
\[
\leq t_2(M_1 + M_2)e^{t_2L_1},
\]
where the last inequality follows from Gronwall’s inequality.

Using \(h_1(\varphi^3_0(x_0)) = 0\), if we consider the Taylor polynomial in Lagrange form at the point \(t_2 = 0\), we obtain
\[
t_2 = \frac{\nabla h_1(\varphi^3_0(x_0))}{\|
abla h_1(\varphi^3_0(x_0))\|^2 f_3(\varphi^3_0(x_0))}, \quad \xi \in (0, t_2).
\]
In particular \(\lim_{\epsilon \to 0} t_2 = 0\) and, in a similar way, \(\lim_{\epsilon \to 0} t'_2 \to 0\). This, together with the bound for \(\|\varphi^3_0(x_0) - \varphi^1_t(x_0)\|\), implies \(\lim_{\epsilon \to 0} \varphi^3_t(x_0) = \varphi^3_t(x_0)\). It follows that \(\varphi^3_t(x_0) = \varphi^3_t(x_0) + \alpha(\epsilon)\), where \(\lim_{\epsilon \to 0} \alpha(\epsilon) = 0\).

For \(t > t_2\), \(\varphi^3_0(x_0)\) moves away from \(\Sigma_1\) and it will meet \(\Sigma_{1,-}\). Let \(t_1 - t_2\) be such that \(\varphi^1_{t_1-t_2}(x_1) \in \Sigma_{1,-}\) and \(t'_1 - t'_2\) be such that \(x'_2 = \varphi^1_{t_1-t_2}(x'_1) \in \Sigma_{1,-}\). To fix ideas, we assume \(t'_1 - t'_2 > t_1 - t_2\). Let \(N_1 = \max_{t \in [0,t_1-t_2]} \|f_s(\varphi^3_s(x'_1))\|\), \(i = 2, 3\), and \(L = \max_{t \in [0,t_1-t_2]} \|f_s(\varphi^3_s(x'_1))\|\). From (29) \(\varphi^3_t(x_0) = \varphi^3_t(x_0)\).

Outside the band \(\{x \in \mathbb{R}^n : |x_1| \leq \epsilon\}\), notice that \(x'_1 \to x_1\) and \(f_2 = f_2\) for \(h_1(x) = x_1 < -\epsilon\) and \(h_2(x) = x_2 > \eta\). Then we have \(\lim_{\epsilon \to 0} \varphi^3_t(x'_1) = \varphi^3_t(x_1)\). In a similar way we can show that \(\|\varphi^3_0(x_0) - \varphi^3_t(x_0)\| \to 0\) up to their first return time to \(B_{\delta}(p(T_2))\). This proves the Lemma.

In what follows we construct Poincaré map \(P\) for the regularized system near \(p_0(t)\) as follows. Let \(B_\delta(p_1(0), p_3(0))\) be a small ball in \(\mathbb{R}^{n-1}\) centered at \((p_1(0), p_3(0))\) with a small radius \(\delta > 0\). Take a solution \(x(t)\) of (27) starting from \((x_1, \eta, y)\), \((x_1, y) \in B_\delta(p_1(0), p_3(0))\). This solution hits the surface \(x_1 = \epsilon\) near \((\epsilon, p_2(T_1), p_3(T_1))\) at a certain point. Then the solution will hit the surface \(x_1 = -\epsilon\) at a certain point under the action of flow \(\varphi^1_t\), which is the flow of (28). After that the solution will hit the surface \(x_2 = \eta\) again near \((p_1(T_2), \eta, p_3(T_2))\) at the point \((x_1(\tilde{t}), \eta, y(\tilde{t}))\) under the action of flow of (27).

Define a map
\[
\Phi(x_1, \eta, y) := (x_1(\tilde{t}), \eta, y(\tilde{t})).
\]
Let us consider the solution \((x_1(t), z(t), y(t))\) of (17) starting from the point \((x_1, 1, y)\), where \((x_1, y) \in B_\delta(p_1(T_2), p_3(T_2))\) for a small \(\delta_1 > 0\). This solution will hit the surface \(z = 1\) at a time \(\tilde{t}\) near \((p_1(0), 1, p_3(0))\). Tikhonov’s Theorem implies that
\[
\begin{align*}
\tilde{z}(\tilde{t}) &= V(x_1(\tilde{t}), y(\tilde{t})) + O(\eta) = 1 \\
x_1(\tilde{t}) &= \bar{x}_1(\tilde{t}) + O(\eta) \\
y(\tilde{t}) &= \bar{y}(\tilde{t}) + O(\eta),
\end{align*}
\]
where \((\bar{x}_1(t), \bar{y}(t))\) is the solution of the reduced system (18) with the initial condition \((\bar{x}_1(0), \bar{y}(0)) = (x_1, y)\). Since \((x_1, y)\) is near to \((p_1(T_2), p_3(T_2))\) and \(1 = \tilde{z}(\tilde{t}) = V(x_1(\tilde{t}), y(\tilde{t})) + O(\eta)\), it follows that \((\bar{x}_1(\tilde{t}), \bar{y}(\tilde{t}))\) is \(O(\eta)\) near to the first and third components of the point \(\tilde{x}\). We mention that Tikhonov’s Theorem can be found in Section 39 of [37].
Define a map
\[ \Psi(x_1, \eta, y) := (x_1(\hat{t}), \eta, y(\hat{t})), \]
where \((x_1, y) \in B_{\delta_0}(p_1(T_2), p_3(T_2))\). We define the Poincaré map as
\[ P(x_1, \eta, y) = \Psi(\Phi(x_1, \eta, y)), \quad (x_1, y) \in B_\delta(p_1(0), p_3(0)). \]

We conclude from Lemma 8 and (30) that
\[ \lim_{\epsilon, \eta \to 0} P(x_1, \eta, y) = \Psi(\Phi(x_1, 0, y)) := P_0(x_1, 0, y) \]
\[ \lim_{\epsilon, \eta \to 0} DP(x_1, \eta, y) = D\Psi(\Phi(x_1, 0, y)) := DP_0(x_1, 0, y). \]
\[ \lim_{\epsilon, \eta \to 0} \]
Hence there exists a function \(\Delta(\epsilon, \eta)\) such that \(P(x_1, \eta, y) = \Psi(\Phi(x_1, 0, y)) + \Delta(\epsilon, \eta)\), where \(\Delta(\epsilon, \eta)\) goes to 0 as \(\epsilon \to 0, \eta \to 0\).

In order to find a periodic orbit near the sliding periodic solution \(p_0(t)\), we need to find the solution of the equation
\[ P(x_1, \eta, y) = (x_1, \eta, y). \]
(32)

Since we have a type I sliding periodic orbit, \(\Psi(\Phi(0, 0, p_3(0))) = (0, 0, p_3(0))\). Note that the sliding periodic orbit is asymptotically stable; the Floquet multipliers are inside the unit circle. However, one Floquet multiplier is 0 due to we have a codimension one sliding motion on \(\Sigma_1\). Hence the linearization \(I - DP_0(0, 0, p_3(0))\) is singular, we cannot use implicit function theorem at once.

We take a tubular neighborhood \(M \times W\) of \(M\) in \(\mathbb{R}^{n-1}\) near the starting point \(p_0(0)\), where \(W \subset \mathbb{R}\) is an open neighborhood of \(0 \in \mathbb{R}\). In order to use implicit function theorem to solve equation (32), we define the following projections. Denote by \(\Delta(\epsilon, \eta) = (\Delta_1(\epsilon, \eta), \Delta_2(\epsilon, \eta), \Delta_3(\epsilon, \eta))\). We have
\[ x_1 = \Gamma_1P(x_1, x_2, y), \]
\[ x_2 = \Gamma_2P(x_1, x_2, y), \]
\[ y = \Gamma_3P(x_1, x_2, y), \]
where \((x_1, y) \in M, x_2 \in W\). Notice that we have
\[ x_2 = \Gamma_2P(x_1, x_2, y) = \delta_2(\epsilon, \eta). \]

Where \(\Delta_2(\epsilon, \eta) \to 0\) as \(\epsilon \to 0, \eta \to 0\).

For brevity, we write
\[ (\Gamma_1P(x_1, x_2, y), \Gamma_3P(x_1, x_2, y)) = (\Gamma_1\Psi(\Phi(x_1, 0, y)), \Gamma_3\Psi(\Phi(x_1, 0, y))) \]
\[ + (\Delta_1(\epsilon, \eta), \Delta_3(\epsilon, \eta)) \]
\[ := \Omega(x_1, 0, y) + (\Delta_1(\epsilon, \eta), \Delta_3(\epsilon, \eta)). \]
(33)

The map \(\Omega(x_1, 0, y)\) is a Poincaré map (without the second component) for the type I sliding periodic orbit. Therefore, \(\Omega(0, 0, p_3(0)) = (0, p_3(0))\). Notice that \(I - D\Omega(0, 0, p_3(0))\) is nonsingular, we can solve equation (32) near \((0, 0, p_3(0))\) by applying implicit function theorem. It follows that \(P\) has a unique fixed point in a neighborhood of \((0, 0, p_3(0))\), denote it by \(x_{\epsilon\eta}\) (the second component is \(\eta\)). Therefore, system (6) has a unique periodic orbit \(\gamma_{\epsilon\eta} = \{x \in \mathbb{R}^n | x = \phi_{\epsilon\eta}(x_{\epsilon\eta}), t \in \mathbb{R}\}\) in a neighborhood of \(\gamma\). It is also asymptotically stable.

The following Lemma will tell us the limit relationship between the periodic orbit of regularized system and the periodic orbit of original PWS system.
Lemma 9.

$$\lim_{\epsilon \to 0, \eta \to 0} x_{\epsilon \eta} = (0, 0, p_3(0)).$$

Proof. From the way we define Poincaré map $P$ and $\eta = K(\epsilon) > 0$, the double limit $\lim_{\epsilon \to 0, \eta \to 0} x_{\epsilon \eta}$ is actually a limit $\lim_{\epsilon \to 0} X_\epsilon$, where $X_\epsilon = x_{\epsilon K(\epsilon)}$. Denote by $X^k_\epsilon$ the $k$-th component of $X_\epsilon$, $k \geq 1$. The second component of $x_{\epsilon \eta}$ is $\eta$ and it goes to 0 as $\eta \to 0$. Since $x_{\epsilon \eta}$ is in a neighborhood of $(0, 0, p_3(0))$, it is bounded. Therefore $X_\epsilon$ is bounded. Let

$$\liminf_{\epsilon \to 0} X^1_\epsilon = \overline{X}^1, \quad \limsup_{\epsilon \to 0} X^1_\epsilon = \overline{X}^1$$

and let $X_{\epsilon_i}$ and $X_{\epsilon_s}$ be two sequences such that we have $\lim_{\epsilon_i \to 0} X^1_{\epsilon_i} = \overline{X}^1$ and $\lim_{\epsilon_s \to 0} X^1_{\epsilon_s} = \overline{X}^1$. From $X_{\epsilon_i}$ and $X_{\epsilon_s}$ we can extract convergent subsequences which we still denote as $X_{\epsilon_i}$ and $X_{\epsilon_s}$. Let

$$\lim_{\epsilon_i \to 0} X_{\epsilon_i} = \overline{X}, \quad \lim_{\epsilon_s \to 0} X_{\epsilon_s} = \overline{X}, \tag{34}$$

Notice that

$$\|P_0(\overline{X}) - \overline{X}\| \leq \|P_0(\overline{X}) - P(\overline{X})\| + \|P(\overline{X}) - P(X_\epsilon)\| + \|P(X_\epsilon) - \overline{X}\|.$$ 

By equation (31), the following equations

$$\lim_{\epsilon_i \to 0} P(X_{\epsilon_i}) = \lim_{\epsilon_s \to 0} X_{\epsilon_s} = \overline{X},$$

$$\lim_{\epsilon_i \to 0} P(X_{\epsilon_i}) = \lim_{\epsilon_s \to 0} X_{\epsilon_s} = \overline{X}$$

and the fact $P$ is continuous, it follows that $P(\overline{X}) = \overline{X}$. Analogously, $P(\overline{X}) = \overline{X}$. Hence $\overline{X} = \overline{X} = (0, 0, p_3(0))$. This proves convergence of the first component of $X_\epsilon$ to the first component of $(0, 0, p_3(0))$. The proof for the other components is done in a similar way. It completes the proof of this Lemma 9. \qed

We continue proving Theorem 5. Notice that

$$\|\phi_{\epsilon \eta}(x_{\epsilon \eta}) - \phi_0^t(0, 0, p_3(0))\| \leq \|\phi_{\epsilon \eta}(x_{\epsilon \eta}) - \phi_{\epsilon \eta}(0, 0, p_3(0))\| + \|\phi_{\epsilon \eta}(0, 0, p_3(0)) - \phi_0^t(0, 0, p_3(0))\|,$$

the first term of the righthand side goes to 0 as $\epsilon \to 0$ and $\eta \to 0$ due to the continuity of $\phi_{\epsilon \eta}$. Since type I sliding periodic orbit consists of two parts, Lemma 8 implies $\phi_{\epsilon \eta}(x_{\epsilon \eta}) \to \phi_0^t(0, 0, p_3(0))$ for the crossing part. For the sliding part, we get the convergence from equation (31). Consequently, $\gamma_{\epsilon \eta} \to \gamma$ as $\epsilon \to 0, \eta \to 0$. This completes the proof of Theorem 5.

4.2. Proof of Theorem 7. In [30], we have proved a similar theorem according to the following steps.

(1) Prove that (5) has at least one limit cycle. To do this, we define a Poincaré map $P_\epsilon$, and use Brouwer’s fixed point Theorem to show that it has a fixed point. This gives at least one limit cycle $\gamma_\epsilon$ of (5), and we show that $\gamma_\epsilon \to \gamma_0$ when $\epsilon \to 0$.

(2) Then, we show that $\gamma_\epsilon$ is asymptotically stable, so $\gamma_\epsilon$ is the unique limit cycle of (5).

In this paper, we will prove Theorem 7 by applying implicit function theorem. We will obtain the existence and uniqueness of periodic orbit for regularized system at a time.
Remark 1. We conclude from [12] that if $\Sigma$ is attractive in finite time upon sliding along $\Sigma_{1,2}^+$ and there is attractive sliding along these codimension one surfaces, then for $k = 1$, $(\phi^*(y), \psi^*(y))$ is exponentially asymptotically stable and hence asymptotically stable. See e.g. Proposition 2.7 in that paper. However, after we construct Poincaré map $P$ for the regularized system, we want to estimate the distance between $P$ and $P_1$. Hence we use Theorem 1 in [22] instead of Proposition 2.7 in [12]. We write it as the following

**Proposition 10.** If the bilinear sliding vector field does not have an equilibrium on $\Sigma$, and real part of both eigenvalues $\lambda_j = \lambda_j(\phi, \psi, y), j = 1, 2$ of matrix $G^*(\phi, \psi, y)$ of (26) satisfy $\lambda_j \leq \mu < 0$, then solution of (23) is asymptotically stable.

For the sliding periodic orbit type II, it will exit from $\Sigma$ at a first order tangential exit point, and then we slide on $\Sigma_2^+$ with a codimension one sliding vector field $f_{\Sigma_2^+}$. Although in some cases, even if $\Sigma$ is not locally attractive, the regularized solution will still remain close to $\Sigma$. However, there are many unexpected cases. Hence we use the conditions “the eigenvalues $\lambda_j \leq \mu < 0$ “ instead of “$\Sigma$ is attractive upon sliding”. When the codimension two sliding vector field exit $\Sigma$ at the first order tangential exit point $\bar{x}$, it will remain close to the codimension one sliding vector field $f_{\Sigma_2^+}$ in a neighborhood of $\bar{x}$.

We continue to prove Theorem 7. From the above assumptions, we will get a type II sliding periodic solution $q_0(t)$. For convenience, let $\bar{x}_1$ be the starting point. Therefore, $q_0(t)$ is given by

$$q_0(t) = \begin{cases} q_1(t), & \text{if } t \in [0, T_1], \\ q_2(t), & \text{if } t \in (T_1, T_2], \\ q_3(t), & \text{if } t \in (T_2, T_3], \\ q_4(t), & \text{if } t \in (T_3, T_4]. \end{cases}$$

(35)

$q_0(t)$ has two codimension one sliding modes and one codimension two sliding mode. $q_1(t)$ is the flow of sliding vector field $f_{\Sigma_2^+}$ on $\Sigma_2^+$ when $t \in [0, T_1]$, $q_2(t)$ is the flow of $f_3$. Then it starts to slide on $\Sigma_4^+$ when $t \in (T_2, T_3]$ with the sliding vector field $f_{\Sigma_4^+}$. Finally, it slides on $\Sigma$ when $t \in (T_3, T_4]$ with the sliding vector field $f_S$.

In what follows we construct Poincaré map $P_*$ for the regularized system near $q_0(t)$ as follows. Note that when it slides on $\Sigma_2^+$, the second component of the sliding motion is 0. When it slides on $\Sigma$, both the first and second components of sliding motion are 0. Let $B_8(q_13(0))(\text{here we omit the first and second components of } q_1(0))$ be a small ball in $\mathbb{R}^{n-2}$ centered at $q_13(0)$ with a small radius $\delta > 0$. Take a solution $x(t)$ of (15) starting from $(\epsilon, \eta, y), y \in B_8(q_13(0))$. This solution hits the surface $x_2 = \eta$ at a certain point $x_1^* \in \bar{x}_1$ which is the exit point of $f_{\Sigma_2^+}$ and it will hit $x_1 = \epsilon = \epsilon $ near $x_2^*$ which is the entry point of the sliding motion of $f_{\Sigma_2^+}$. Then this solution will hit the surface $x_1 = \epsilon = \epsilon $ again at $x_3^*$ near $\bar{y}$, and finally this solution will meet $x_1 = \epsilon, x_2 = \eta$ again at the point $(\epsilon, \eta, w_3(t))$ near $\bar{x}$.

Define a map

$$Q(\epsilon, \eta, y) := (\epsilon, \eta, w_3(t)).$$

Actually, this mapping $Q$ is composed by four mappings $\Phi_1^*, \Phi_2^*, \Phi_3^*$ and $\Phi_4^*$, where $\Phi_1^* : (\epsilon, \eta, y) \rightarrow x_1^*, \Phi_2^* : x_1^* \rightarrow x_2^*, \Phi_3^* : x_2^* \rightarrow x_3^*, \Phi_4^* : x_3^* \rightarrow (\epsilon, \eta, w_3(t))$, all of them are under the action of flow $\phi_{\epsilon \eta}^*$ which is the flow of the vector field $f_{\epsilon \eta}(x)$. 

PERIODIC ORBITS FOR REGULARIZED PWS SYSTEMS CODIMENSION TWO 15
For the type II sliding periodic orbit of PWS system, we also define a Poincaré map $Q_0$ which is composed by four mappings $\Phi_1, \Phi_2, \Phi_3$ and $\Phi_4$, where $\Phi_1 : q_1(0) = \bar{x} \rightarrow \bar{x}_1$ under the action of flow $\phi_{\Sigma_2^+}^t$ of $f_{\Sigma_2^+}$, $\Phi_2 : \bar{x}_1 \rightarrow \bar{x}_2$ under the action of flow of $f_3$, $\Phi_3 : \bar{x}_2 \rightarrow \bar{y}$ under the action of flow $\phi_{\Sigma_1^2}^q$ which is the flow of $f_{\Sigma_1^2}$, $\Phi_4 : \bar{x}_3 \rightarrow \bar{x}$ under the action of flow $\phi_2^t$ which is the flow of $f_2$.

In the sequel we will prove the following Lemma.

**Lemma 11.** Let $\eta = k \varepsilon$, $k > 0$. For each $y \in B_d(q_{13}(0))$ the following equation
\[
\lim_{\varepsilon \to 0} Q(\varepsilon, \eta, y) = Q_0(\bar{x}),
\]
holds uniformly for $t$ in a compact interval, where $q_{13}(0)$ is $y$-coordinate of $q_1(0)$.

**Proof.** In fact, for $\Phi_1$ and $\Phi_1^*$, we have an attractive sliding motion on $\Sigma_2^+$, from Lemma 18 of [16], we know that $\Phi_1^*(\varepsilon, \eta, y) \rightarrow \Phi_1(\bar{x})$ as $\varepsilon \to 0$. Analogously, we have $\Phi_1^*(x_2^*) \rightarrow \Phi_1(x_2)$ as $\varepsilon \to 0$.

By applying the solution is continuous with respect to the initial value, we have $\Phi_2^*(\bar{x}_1) \rightarrow \Phi_2(\bar{x}_1)$, as $\varepsilon \to 0$.

As for $\Phi_3$ and $\Phi_3^*$, we will have a codimension two sliding motion on $\Sigma$, under the assumptions, the solution of regularized equation converges to the sliding solution of the bilinear sliding vector field. Moreover, in a neighborhood of the first tangential exit point, it remains close to sliding solution of the codimension one sliding vector field. Therefore, we have proved this Lemma. \qed

Since the PWS system we consider is an $n$-dimensional system, we take a tubular neighborhood $M \times W$ of $M$ in $\mathbb{R}^{n-2}$ near the point $\bar{x}$, where $W \subset \mathbb{R}^2$ is an open neighborhood of $(0, 0) \in \mathbb{R}^2$. Since type II sliding periodic orbit has a codimension two sliding motion on $\Sigma$, from [15] and [17], we know that the Floquet multipliers of its Fundamental matrix solution have two zeros. This implies that the derivative of its Poincaré map will have two zeros and so derivative of its Poincaré map will be singular. Hence we cannot use implicit function theorem directly.

In order to use implicit function theorem to solve equation
\[
Q(\varepsilon, \eta, y) = (\varepsilon, \eta, y), \tag{36}
\]
we define the following projections
\[
\varepsilon = \Gamma_1 Q(\varepsilon, \eta, y), \\
\eta = \Gamma_2 Q(\varepsilon, \eta, y), \\
y = \Gamma_3 Q(\varepsilon, \eta, y),
\]
where $(\varepsilon, \eta) \in M$, $y \in W$. From Lemma 11, we have
\[
\Gamma_3 Q(\varepsilon, \eta, y) = \Gamma_3 Q_0(0, 0, y) + O(\varepsilon), \\
D\Gamma_3 Q(\varepsilon, \eta, y) = D\Gamma_3 Q_0(0, 0, y) + O(\varepsilon). \tag{37}
\]

Note that we have a type II sliding periodic orbit, we have $\Gamma_3 Q_0(\bar{x}) = \bar{x}$. In our case, since the sliding periodic orbit is asymptotically stable, the Floquet multipliers are inside the unit circle. Hence the linearization $I - D\Gamma_3 Q_0(\bar{x})$ is nonsingular, then we can solve equation (36) near $\bar{x}$ by implicit function theorem. It follows that in a neighborhood of $\bar{x}$, $\Gamma_3 Q(\varepsilon, \eta, y)$ has a unique fixed point. Therefore, system (6) has a unique periodic orbit $\gamma_\varepsilon$ in a neighborhood of $\gamma$. It is also asymptotically stable. By using similar proof as Lemma 14 in [16], we know that $\gamma_\varepsilon \rightarrow \gamma$ as $\varepsilon \to 0$. 
Remark 2. Example 4.1 in [12] has considered the case when Σ loses attractivity through a tangential exit point and then the flow slides on a codimension one switching manifold. Example 4.1 has shown that the solution of the regularized system will remain close to the solution of the original PWS systems near the switching manifold Σ. This case is very helpful to consider Type II sliding periodic orbit in our paper. Example 4.2 in [12] has studied the case when Σ loses attractivity through a non-tangential exit point and enters $R_1$. This example shows that the solution of the regularized system will remain close to the solution of the original PWS systems near the switching manifold even when we choose $\epsilon_\alpha = 10^{-6}, \epsilon_\beta = 10^{-5}$. The case is useful to consider Type I sliding periodic orbit in our paper. When we consider the relationship between the Type I or Type II sliding periodic orbit and the periodic orbit of the original PWS system, the most difficult point is to show the convergence near the switching manifold. These two examples have solved the most difficult part. Hence, we don’t offer new examples.

5. Conclusions. When PWS systems have only codimension one switching manifolds, it is well known that the sliding vector fields will exit from the tangential points at the boundary points of sliding regions. These points are also tangency points. Regularization process have been used to study the sliding vector fields, sliding region, singular points and the persistence of periodic orbit.

If PWS systems have a codimension two switching manifold Σ which is an intersection of two codimension one hyperplanes, the sliding vector field on this codimension two switching manifold is not unique. Even if where the sliding vector field will exit from the codimension two switching manifold is not clear. There are many complicated cases. Many authors have applied regularization process to bridge a gap between this PWS systems and smooth systems in order to analyze the codimension two sliding vector field and sliding region.

In this paper, we mainly investigate the relationship between the periodic orbit $\gamma$ of PWS systems with an intersection of two hyperplanes and periodic orbit of its double regularization system. Under suitable conditions, we show that its double regularization system will also have a periodic orbit $\gamma_{\epsilon \eta}$ in a neighborhood of $\gamma$ by applying implicit function theorem. Moreover, as $\epsilon \rightarrow 0, \eta \rightarrow 0$, we have $\gamma_{\epsilon \eta} \rightarrow \gamma$. Our results show that under suitable assumptions, PWS systems are structurally stable to perturbations that smooth out their discontinuities. The type I sliding periodic orbit is different from crossing and sliding periodic orbit (see e.g [16]) because it exits from $\Sigma$ at intersection point of $\Sigma_1$ and $\Sigma_2$. However, the crossing and sliding periodic orbit exits from the switching manifold at a tangency point. Moreover, implicit function theorem helps us to obtain the existence and uniqueness of periodic orbit at a time. Therefore, it simplifies the proof of our main results.

When the periodic orbit of the original PWS planar systems is not asymptotically stable but merely hyperbolic, this case has been considered by Dieci et.al. in [10] for planar PWS systems. When the periodic orbit of the original planar PWS systems is repelling, Reves and Seara also consider this case in their recent work [6], their results show that some bifurcation phenomena may occur under suitable assumptions, such as saddle node bifurcation, see Theorem 2.4. However, these authors just consider planar PWS systems with codimension one switching manifold. There are few results for higher dimensional PWS systems with a codimension two switching manifold. If the periodic orbit of the original PWS higher dimensional
systems is not asymptotically stable but merely hyperbolic, the scenario will be more complicated. We also expect that some bifurcation phenomena may occur.

Another case we want to mention is that when there is an unstable equilibrium on the switching manifold, the scenario will be more complicated. We have found sliding heteroclinic bifurcation in [31]. Recently, planar PWS systems with fold-fold singularity have been studied by some authors. When the fold-fold of PWS system is visible-invisible, the sliding vector field has a singularity, which is an unstable pseudo-saddle, see Proposition 3.3 in [24]. Consider the following PWS vector field

\[ Z_{\epsilon\mu}(x, y) = \begin{cases} 
(1 & -1) 
\begin{pmatrix} x + \mu \\ y - \mu \end{pmatrix}, & y > 0 \\
1 & 1 
\begin{pmatrix} x - \epsilon \\ y \end{pmatrix}, & y < 0 
\end{cases} \]

For \( \mu < 0 \), we conclude from P2002–2005 of [20] that the sliding vector field has a pseudo-node, which is attracting if \( \epsilon > 0 \) and repelling if \( \epsilon < 0 \). Moreover, system \( Z_{\epsilon\mu}(x, y) \) underdoes some codimension 1 bifurcations and some codimension two bifurcations. To our knowledge, there are few results studying bifurcation phenomena of the regularized system. These work will be done in the future.

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E-mail address: pidh@hqu.edu.cn