Using Edge-induced and Vertex-induced Subhypergraph Polynomials

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Abstract

For a hypergraph $H$, we consider the edge-induced and vertex-induced subhypergraph polynomials and study their relation. We use this relation to prove that both polynomials are reconstructible, and to prove a theorem relating the Hilbert series of the Stanley-Reisner ring of the independent complex of $H$ and the edge-induced subhypergraph polynomial. We also consider reconstruction of some algebraic invariants of $H$.

Key words and phrases: edge-induced subhypergraph polynomial, vertex-induced subhypergraph polynomial, Stanley-Reisner ring, Hilbert Series, Reconstruction conjecture.

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1 Introduction

To every hypergraph $H$ one can associate several subhypergraph enumerating polynomials. In this note we consider two of these polynomials: the vertex-induced subhypergraph polynomial $P_H(x, y)$ enumerating vertex-induced subhypergraphs of $H$, and the edge-induced subhypergraph polynomial $S_H(x, y)$. Precise definitions will be given in §2. These and several other polynomials were extensively studied for graphs, see [1, 4, 5, 8] and their citations. The notion has been naturally generalized to hypergraphs, see White [14].

L. Borzacchini, et al. [5] studied the relation between these and other subgraph enumerating polynomials. He earlier proved that both are reconstructible, i.e. they can be derived from the subgraph enumerating polynomials of vertex-deleted subgraphs, see [3,4]. A. Goodarzi [9] used $S_H(x, y)$ to compute the Hilbert series of the Stanley-Reisner ring of the independent complex of $H$. More precisely, if $R$ is such a ring, then its Hilbert series $H_R(t)$ is given by

\begin{equation}
H_R(t) = \frac{S_H(t, -1)}{(1 - t)^n}
\end{equation}

where $n$ is the number of vertices in $H$.

In section 2, we define the polynomials, and then prove that

$S_H(x, y) = (1 - x)^n P_H\left(\frac{x}{1 - x}, 1 + y\right)$.
In section 3, we use this relation to give a short and elementary proof of (1.1). One may compare our proof with the technical proof in [9]. In section 4, generalizing Borzacchini’s results [3, 4], we prove that both polynomials are reconstructible for hypergraphs. We also prove the reconstruction problems of some algebraic invariants of the independent complex of \( \mathcal{H} \), where their graph counterpart is proven by Dalili, Faridi and Traves in [6]. That is, we consider reconstructibility of the Hilbert series, the \( f \)-vector, the (multi-)graded Betti numbers and some graded Betti tables of the independent complex of \( \mathcal{H} \).

2 Preliminaries

A hypergraph is a pair \( \mathcal{H} = (V, E) \) where \( V \) is a set of elements called vertices and \( E \subset 2^V \) is a set of distinct subsets of \( V \) called edges such that for any two edges \( \varepsilon_1, \varepsilon_2 \in E \), we have \( \varepsilon_1 \subset \varepsilon_2 \Rightarrow \varepsilon_1 = \varepsilon_2 \). A hypergraph \( \mathcal{H} \) is called finite if the vertex set \( V \) is finite. We say \( \mathcal{H} \) is a \( d \)-hypergraph if \( |\varepsilon| = d \) for each \( \varepsilon \in E \), where \( |\varepsilon| \) is the cardinality of \( \varepsilon \). A graph is a 2-hypergraph. In this note we always consider finite hypergraphs.

Let \( \mathcal{H} = (V, E) \) be hypergraph, \( W \subset V \) and \( L \subset E \). We say that \( L = (W, L) \) is an edge-induced subhypergraph of \( \mathcal{H} \) if \( W = \bigcup \varepsilon \in L \varepsilon \). We say that \( \mathcal{H}_W = (W, L) \) is a vertex-induced subhypergraph if \( L \) is the largest subset of \( E \) such that \( L \subset 2^W \).

Let \( \mathcal{H} \) be a hypergraph. The edge-induced subhypergraph polynomial \( S_{\mathcal{H}}(x, y) \) is defined by

\[
S_{\mathcal{H}}(x, y) = \sum_{i,j} \theta_{ij} x^i y^j,
\]

where \( \theta_{00} = 1 \) and for \( i, j \geq 0 \), \( \theta_{ij} \) is the number of edge induced subhypergraphs of \( \mathcal{H} \) with \( i \) vertices and \( j \) edges. Similarly, the vertex-induced subhypergraph polynomial \( P_{\mathcal{H}}(x, y) \) of \( \mathcal{H} \) is defined by

\[
P_{\mathcal{H}}(x, y) = \sum_{i,j} \beta_{ij} x^i y^j,
\]

where \( \beta_{00} = 1 \) and for \( i, j \geq 0 \), \( \beta_{ij} \) is the number of vertex induced subhypergraphs of \( \mathcal{H} \) with \( i \) vertices and \( j \) edges.

We recall some simple properties of these polynomials. In what follows, \( F_{\mathcal{H}}(x, y) \) refers to any one of the two polynomials.

1. If the hypergraph has connected components \( \mathcal{H}_1, \ldots, \mathcal{H}_m \), we have \( F_{\mathcal{H}}(x, y) = F_{\mathcal{H}_1}(x, y) \cdots F_{\mathcal{H}_m}(x, y) \). We also have \( F(0, y) = 1 \). If \( E = \emptyset \), then \( F_{\mathcal{H}}(x, y) = (1 + x)^m \).

2. \( \sum_i \beta_{ij} = \binom{n}{j} \) and \( \sum_i \theta_{ij} = \binom{m}{j} \) where \( m \) is the number of edges in \( \mathcal{H} \).

3. \( S_{\mathcal{H}}(x, 0) \) is a subgraph polynomial of the 0-subhypergraphs, i.e. isolated vertices. \( P_{\mathcal{H}}(x, 0) \) the polynomial of the independent subsets, i.e. sets of vertices having no edges in common.

4. If \( \mathcal{H} = K_n \) is the complete graph, then \( P_{\mathcal{H}}(x, y) = \sum_{i=0}^n \binom{n}{i} x^i y^{(i)} \) and if \( \mathcal{H} \) is a star with \( m \) edges, then \( S_{\mathcal{H}}(x, y) = \sum_{j=0}^m \binom{m}{j} x^{j+1} y^j \).
The following Proposition is a generalization of Borzacchini [3]. Even though he considered graphs, the proofs can easily be generalized to hypergraphs.

**Proposition 2.1.** Let \( H \) be a hypergraph on \( n \) vertices. Then

\[
S_H(x, y) = (1 - x)^n P_H(\frac{x}{1 - x}, 1 + y)
\]

**Proof.** To every vertex induced subhypergraph with \( i \) vertices and \( l \) edges there are \( \binom{i}{j} \) hypergraphs with \( j \) vertices and \( j \) edges. Moreover, those obtained from different vertex induced subhypergraphs are different since they contain different vertex sets. On the other hand, to every edge induced subhypergraph with \( l \) vertices and \( j \) edges we can construct \( \binom{n-l}{i-j} \) hypergraphs with \( i \) vertices and \( j \) edges. So

\[
(2.3) \quad \sum_{l=0}^{j} \beta_{i,j+l}(\binom{j+l}{j}) = \sum_{l=0}^{i} \theta_{i-l,j}(\binom{n-l}{l})
\]

Setting \( r = j+l \) and \( s = i-l \), substituting this in (2.3) and multiplying it by \( x^i y^j \), we obtain:

\[
\sum_{i,j} x^i y^j \left[ \sum_{l=0}^{j} \beta_{i,j+l}(\binom{j+l}{j}) \right] = \sum_{i,j} x^i y^j \left[ \sum_{l=0}^{i} \theta_{i-l,j}(\binom{n-l}{l}) \right].
\]

By change of variable, we obtain \( S_H(x, y) = (1 - x)^n P_H(\frac{x}{1 - x}, 1 + y) \). \( \square \)

**Corollary 2.2.** Let \( H \) be a hypergraph on \( n \) vertices. Then

\[
P_H(x, y) = (1 + x)^n S_H(\frac{x}{1 + x}, y - 1).
\]

3 \( P_H(x, y) \) and \( S_H(x, y) \) in Algebra

A simplicial complex \( \Delta \) on a vertex set \( V = \{v_1, \ldots, v_n\} \) is a set of subsets of \( V \), called faces or simplices such that \( \{v_i\} \in \Delta \) for each \( i \) and every subset of a face is itself a face. If \( B \subset V \), the restriction of \( \Delta \) to \( B \) is a simplicial complex defined by \( \Delta(B) = \{\delta \in \Delta \mid \delta \subset B\} \). The dimension of a face \( \delta \in \Delta \) is \( |\delta| - 1 \). Let \( f_i = f_i(\Delta) \) denote the number of faces of \( \Delta \) of dimension \( i \). Setting \( f_{-1} = 1 \), the sequence \( f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \) is called the \( f \)-vector of \( \Delta \).
Let $A = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and $\Delta$ be a simplicial complex over $n$ vertices $V = \{v_1, \ldots, v_n\}$. The Stanley-Reisner ideal of $\Delta$ is the ideal $I(\Delta) \subset A$ generated by those square free monomials $x_{i_1} \cdots x_{i_m}$ where $\{v_{i_1}, \ldots, v_{i_m}\} \notin \Delta$.

Let $\mathcal{H} = (V, E)$ be a hypergraph with $n$ vertices $V = \{v_1, \ldots, v_n\}$. An independent set of $\mathcal{H}$ is a subset $W \subset V$ such that $\varepsilon \notin W$ for all $\varepsilon \in E$. The collection of $\Delta_H$ of independent sets forms a simplicial complex, called the independent complex. Thus the Stanley-Reisner ideal of $\Delta_H$ is the edge ideal of $\mathcal{H}$. More precisely, $I(\Delta_H) = I(\mathcal{H}) \subset A$ is the ideal generated by the squarefree monomials $\prod_{\varepsilon \in \varepsilon} x$ where $\varepsilon \in E$. Conversely, every squarefree monomial ideal $I \subset A$ can be associated with a hypergraph $\mathcal{H}_I = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ and $\varepsilon \in E$ if $\prod_{\varepsilon \in \varepsilon} x$ is in the minimal generating set of $I$. So one has $I(\Delta_{\mathcal{H}_I}) = I$. We have the following lemma.

**Lemma 3.1.** Let $(f_0, f_1, \ldots, f_d)$ be the $f$-vector of the independent complex of a hypergraph $\mathcal{H}$. Then $P_{\mathcal{H}}(t, 0) = \sum_{i=0}^d f_i t^i$.

Let $R = \oplus_{i \in \mathbb{N}} R_i$ be a finitely generated graded $\mathbb{K}$-algebra, where $R_0 = \mathbb{K}$ is a field. The Hilbert series of $R$ is the generating function defined by $H_R(t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{K}}(R_i) t^i$. If $I \subset A$ is a monomial ideal, the Hilbert series of the monomial ring $R = A/I$ is the rational function $H_R(t) = \frac{K(R, t)}{1-\sum_{i=0}^d f_i t^i}$, where $K(R, t) \in \mathbb{Z}[t]$. P. Renteln [13], and also D. Ferrarello and R. Fröberg [7] used the subgraph induced polynomial $S_G(x, y)$ of a graph $G$ to compute the Hilbert series of the Stanley-Reisner ring $R$ of the independent complex of $G$, namely:

$$H_R(t) = \frac{S_G(t, -1)}{(1-t)^n}.$$

Recently A. Goodarzi [9] generalized it for any squarefree monomial ideal by using the combinatorial Alexander duality and Hochster’s formula. Below is a very short and direct proof of this result.

**Theorem 3.2.** Let $\mathcal{H}$ be a hypergraph on $n$ vertices, $I_{\mathcal{H}} \subset A = \mathbb{K}[x_1, \ldots, x_n]$ be its associated squarefree monomial ideal, and $R = A/I_{\mathcal{H}}$. Then

$$H_R(t) = \frac{S_{\mathcal{H}}(t, -1)}{(1-t)^n}.$$

**Proof.** We know by Lemma 3.1 that $P_{\mathcal{H}}(t, 0) = \sum_{i=0}^d f_i t^i$ is the polynomial of the $f$-vectors of the independent complex of $\mathcal{H}$. It follows that by [12] Proposition 51.3 that $H_R(t) = P_{\mathcal{H}}(\frac{t}{1-t}, 0)$ and by Theorem 2.1 we have

$$S_{\mathcal{H}}(t, -1) = (1-t)^n P_{\mathcal{H}}(\frac{t}{1-t}, 0) = H_R(t)(1-t)^n.$$

**Remark 3.3.** Let $\mathcal{H}$ be a hypergraph and $R = A/I_{\mathcal{H}}$. It then follows by Lemma 3.1 and [12] Proposition 51.2 that $P_{\mathcal{H}}(t, 0)$ is the Hilbert polynomial of the exterior algebra $R/(x_1^2, \ldots, x_n^2)$. 

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4 \( P_\mathcal{H}(x, y) \) and \( S_\mathcal{H}(x, y) \) in reconstruction conjecture

For a graph \( G = (V, E) \) on a vertex set \( V = \{v_1, \ldots, v_n\} \), the deck of \( G \) is the collection \( \mathcal{D}(G) = \{G_1, \ldots, G_n\} \) where \( G_l = G - v_l, v_l \in V \) is the vertex deleted subgraph of \( G \). An element of \( \mathcal{D}(G) \) is called a card. The long standing graph reconstruction conjecture posed by Kelly and Ulam says that every simple graph on \( n \geq 3 \) vertices is uniquely determined, up to isomorphism, by its deck. Numerous unsuccessful attempts have been made to prove the conjecture, nevertheless, a significant amount of work has been made. The reader may see Bondy [2] for a survey on the subject. Reconstruction of hypergraphs is defined similarly to graphs. Kocay [10] and Kocay and Lui [11] have constructed a family of non-reconstructible 3-hypergraphs.

Remark 4.1. Another obvious example of non-constructible hypergraphs are the 0-hypergraph containing no edges, and the \( n \)-hypergraph containing one edge with \( n \)-elements. So all the hypergraphs under consideration in this section are neither of these two.

In recent years questions has been asked if a graph invariant is reconstructible, that is, if it can be obtained from the its deck. Borzachini in [3][4] proved that both \( S_G(x, y) \) and \( P_G(x, y) \) are reconstructible. In fact, he proved that if \( F_G(x, y) \) is any one of the subgraph polynomials and \( F_{G_l}(x, y) \) is a subgraph polynomial of the card \( G_l \), then

\[
(4.1) \quad \quad nF_G(x, y) = x \frac{\partial F_G(x, y)}{\partial x} + \sum_{l=1}^{n} F_{G_l}(x, y).
\]

It is natural to extend this reconstructibility question to hypergraphs. Below we obtain a similar result.

**Proposition 4.2.** Let \( \mathcal{H} \) be a hypergraph on \( n \geq 3 \) vertices. Then both \( S_\mathcal{H}(x, y) \) and \( P_\mathcal{H}(x, y) \) are reconstructible.

**Proof.** We prove the proposition for \( S_\mathcal{H}(x, y) \) since the other will follow by Proposition 2.1. Let \( S_\mathcal{H}(x, y) = \sum_{ij} \theta_{ij} x^i y^j \) and \( S_{\mathcal{H}_l}(x, y) = \sum_{ij} \theta_{ij}^{(l)} x^i y^j \) for \( l = 1, \ldots, n \). By direct calculation we have

\[
\quad \quad nS_\mathcal{H}(x, y) - x \frac{\partial (S_\mathcal{H}(x, y))}{\partial x} = n + \sum_{l=1}^{n} \sum_{ij} (n - j)\theta_{ij} x^i y^j.
\]

Now if \( j < n \), then any edge induced subhypergraph with \( i \) vertices and \( j \) edges is an edge induced subhypergraph for \( n - j \) cards. It follows that \( \sum_{l=1}^{n} \theta_{ij}^{(l)} = (n - j)\theta_{ij} \). Putting this in the equation and recalling that \( n = \sum_{l=1}^{n} \theta_{00}^{(l)} \) we obtain

\[
(4.2) \quad \quad nS_\mathcal{H}(x, y) = x \frac{\partial S_\mathcal{H}(x, y)}{\partial x} + \sum_{i=1}^{n} S_{\mathcal{H}_i}(x, y).
\]

\[ \square \]
4.1 Hilbert series and \( f \)-vector

The authors in [6] studied reconstructibility of some algebraic invariants of the edge ideal of a graph \( G \) such as the Krull dimension, the Hilbert series, and the graded Betti numbers \( b_{i,j} \), where \( j < n \). We extend these results to hypergraphs.

**Proposition 4.3.** Let \( H \) be a hypergraph on \( n \geq 3 \) vertices. The Hilbert function of \( R = A/I_H \) is reconstructible.

**Proof.** By Proposition 3.2 and (4.2) we have

\[
nH_R(t) = \frac{nS_H(t, -1)}{(1-t)^n} = \frac{t dS_H(t, -1)}{(1-t)^n} + \sum_{i=1}^{n} S_H(t, -1)
\]

Since \( \frac{dH_R(t)}{dt} = \frac{d}{dt}\left( \frac{S_H(t, -1)}{(1-t)^n} \right) = \frac{1}{1-t} \frac{t dS_H(t, -1)}{dt} + \frac{n}{1-t} H_R(t) \), substituting this into the above, we obtain a first order ordinary linear differential equation

\[
\frac{n}{1-t} H_R(t) = t \frac{dH_R(t)}{dt} - \frac{1}{1-t} \sum_{i=1}^{n} H_R(t).
\]

**Proposition 4.4.** Let \( H \) be a hypergraph on \( n \geq 3 \) vertices. The \( f \)-vector of \( \Delta_H \) is reconstructible.

**Proof.** This, in fact, follows from Proposition 4.3 but we give an independent proof. Let \( f(\Delta_H) = (f_0, \ldots, f_{d-1}) \). If \( d < n \), by (4.2) when \( F = P_H \) and Lemma 3.1 we have \( nf_{i-1} = if_{i-1} + \sum_{i=1}^{n} f_{i-1} \) for all \( i \leq d \). If \( d = n \), then \( H \) has no edges so \( f_{d-1} = 1 \).

Let \( \Delta_H \) be the independent complex of a hypergraph \( H \). We can compute other invariants of \( \Delta_H \) from its \( f \)-vector \( f(\Delta_H) = (f_0, \ldots, f_{d-1}) \). Recall, for example, that the \( h \)-vector \( h(\Delta_H) = (h_0, \ldots, h_d) \) is defined by the formula \( \sum_{i=0}^{d} f_{i-1}(1-t)^{d-i} = \sum_{i=0}^{d} h_i t^i \). We can also obtain the multiplicity of the \( R = A/I_H \), namely \( e(R) = f_{d-1} \). The following are consequences of Propositions 4.3 and 4.4.

**Corollary 4.5.** Let \( H \) be a hypergraph on \( n \geq 3 \) vertices. The \( h \)-vector of \( \Delta_H \) is reconstructible.

**Corollary 4.6.** Let \( H \) be a hypergraph on \( n \geq 3 \) vertices. Then the Krull dimension and the multiplicity of \( R = A/I_H \) are reconstructible.

4.2 Multi-graded Betti numbers

In this subsection we assume that \( \text{char}\ K = 0 \). Let \( I \subset A = \mathbb{K}[x_1, \ldots, x_n] \) be a monomial ideal and consider the \( \mathbb{Z}^n \)-graded minimal free resolution of the \( A \)-module \( R = A/I \):

\[
\cdots \rightarrow \oplus_j A(-b_j)_{b_j}^{b_j} \rightarrow \cdots \rightarrow \oplus_j A(-b_j)_{b_j}^{b_j} \rightarrow \oplus_j A(-b_j)_{b_j}^{b_j} \rightarrow A \rightarrow A/I \rightarrow 0
\]
where \( b \in \mathbb{Z}^n \) and the modules \( A(-b) \) are the shifts of \( A \) to make the multi-graded differentials degree zero maps. The numbers \( b_{i,b} \) are multi-graded Betti numbers and \( b_{ij} = \sum_{b_i = j} b_{i,b} \), where \( |b| = b_1 + \cdots + b_n \), are the graded Betti numbers of \( R \). In particular, the \( b_{in} \)'s are the extremal graded Betti numbers. The importance of the assumption that char \( K = 0 \) is that these numbers depend on the characteristic of the ground field, see eg. [12, Example 12.4]. If \( I = I_{K} \) is the edge ideal of a hypergraph \( \mathcal{H} \), then each \( b \in \{0,1\}^n \), see for example [12, Corollary 26.10]. One can use graded Betti numbers to compute the Hilbert series of \( R = A/I_{K} \). So by Theorem 3.2 we have

\[
(4.3) \quad S_{\mathcal{H}}(t, -1) = \sum_{i=0}^{n} \sum_{j} (-1)^{i} b_{ij} t^j.
\]

We generalize [6, Theorem 5.1] with a similar proof.

**Proposition 4.7.** Let \( \mathcal{H} \) be a hypergraph on with a vertex set \( V = \{v_1, \ldots, v_n\} \) and \( n \geq 3 \). Then the multi-graded Betti numbers \( b_{ij} \) of the Stanley Reisner ring \( R = A/I_{\mathcal{H}} \) are reconstructible for all \( j < n \).

**Proof.** Let \( \Delta = \Delta_{\mathcal{H}}, \Delta^{(i)} = \Delta_{\mathcal{H}_i} \), \( b \in \mathbb{Z}^n \), \( b_{i,b} \) be the multi-graded Betti numbers of \( \Delta \), and \( b^{(i)}_{ij} \) be the multi-graded Betti numbers of \( \Delta^{(i)} \). By Hochester’s formula, we have

\[
b_{i,b} = b_{i,B} = \bar{h}_{j-i-1}(\Delta(B)),
\]

where \( B = \{v_i \in V \mid b_i \neq 0\} \) and \( \bar{h}_{j-i-1}(\Delta(B)) = \dim_{K}(\bar{H}_{j-i-1}(\Delta(B); K)) \) is the reduced simplicial homology of the subcomplex \( \Delta(B) \). Since \( \Delta(B) = \Delta^{(i)}(B) \) whenever \( v_i \notin B \), it follows by Hochester’s formula that \( b_{i,b} = \bar{h}_{j-i-1}(\Delta^{(i)}(B)) = \tilde{h}^{(i)}_{i,b} \). So the result holds.

**Corollary 4.8.** Let \( \mathcal{H} \) be a hypergraph with a vertex set \( V = \{v_1, \ldots, v_n\} \) and \( n \geq 3 \). Then the graded Betti numbers \( b_{ij} \) of the Stanley Reisner ring \( R = A/I_{\mathcal{H}} \) are reconstructible for all \( j < n \).

**Proof.** \( b_{ij} = \sum_{|b| = j} b_{i,b} \) and multi-graded Betti numbers are reconstructible.

Reconstruction of the extremal graded Betti numbers seems a bit hard to determine. We know that the coefficient of \( t^n \) in \( S_{\mathcal{H}}(t, -1) \) is the alternating sum \( \sum_{i} (-1)^{i} b_{in} \). It follows that \( b_{in} \) is reconstructible if there is only one \( i \) such that \( b_{in} \neq 0 \). Fortunately, we have a good class of ideals with this property: for example, edge ideals of complements of chordal graphs, metroidal ideals, ideals with linear quotients and Cohen-Macaulay ideals. However, there are also edge ideals with more than one non-zero extremal graded Betti numbers [6, Example 5.3]. On the other hand, it is a useful invariant since it gives us information on many other invariants of \( I_{\mathcal{H}} \). The following extends [6, Proposition 5.4] to hypergraphs.

**Proposition 4.9.** Let \( \mathcal{H} \) be a hypergraph on \( n \geq 3 \) vertices. If the graded top degree Betti numbers \( b_{in} \) of \( I_{\mathcal{H}} \) are reconstructible, then the depth, projective dimension and regularity of \( I_{\mathcal{H}} \) are reconstructible.
We investigate if the Betti table of $I_H$ is reconstructible. Let $B = (b_{ij})$ be the Betti table of $I_H$ and $B_l = (b_{ij}^{(l)})$ be the Betti table of $I_{H_l}$. Then combining (4.2) and (4.3) and comparing the coefficients of $t^j$ we obtain

$$(n-j) \sum_i (-1)^i b_{ij} = \sum_i (-1)^i \sum_{l=1}^n b_{ij}^{(l)} \quad \text{for } j < n.$$

This equation shows it is difficult to determine each $b_{ij}$ only from the data $\{B_l\}_{l=1}^n$ since anti-diagonals of $B$ might contain more than one non-zero entry. We thus have the following which gives a partial answer to Question 5.6.

**Proposition 4.10.** Let $H$ be a hypergraph on $n \geq 3$ vertices. If each anti-diagonal of the Betti table of $I_H$ contains at most one non-zero entry, then the Betti table of $I_H$ is reconstructible.

In fact, in this case, we can compute the non-zero entries from the coefficients of $S_H(x, y)$.

**Proposition 4.11.** Let $H$ be a hypergraph on $n \geq 3$ vertices and $S_H(x, y) = \sum_{ij} \theta_{ij} x^i y^j$. Assume that each anti-diagonal of the Betti table contains at most one non-zero entry $b_0, b_1, \ldots, b_d$. Then $b_i = \sum_j \theta_{ij}$.

**Proof.** Since $S_H(t, -1) = \sum_{ij} \theta_{ij} (-1)^i t^j = \sum_{j=0}^n (-1)^j b_j t^j$, $b_i$ is the coefficient of $t^i$ in $S_H(t, -1)$.

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