A DERIVED EQUIVALENCE OF THE LIBGOBER-TEITELBAUM AND THE BATYREV-BORISOV MIRROR CONSTRUCTIONS

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ABSTRACT. In this paper we study a particular mirror construction to the complete intersection of two cubics in $\mathbb{P}^5$, due to Libgober and Teitelbaum. Using variations of geometric invariant theory and methods of Favero and Kelly, we prove a derived equivalence of this mirror to the Batyrev-Borisov mirror of the complete intersection.

1. Introduction

Libgober and Teitelbaum [LT93] proposed a mirror to a Calabi-Yau complete intersection $V_{\lambda}$ of two cubics in $\mathbb{P}^5$ defined as the zero locus for the two polynomials

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2.$$ 

Their proposed mirror $W_{LT,\lambda}$ is a (minimal) resolution of singularities of the variety $V_{LT,\lambda}$ with defining equations $Q_{1,\lambda}, Q_{2,\lambda}$ but in the quotient space $\mathbb{P}^5/G_{81}$, where $G_{81}$ is a specified order 81 subgroup of $\text{PGL}(5, \mathbb{C})$. They showed topological evidence that $V_{\lambda}$ and $W_{LT,\lambda}$ are a mirror pair, proving on the level of Euler characteristics that $\chi(V_{\lambda}) = -\chi(W_{LT,\lambda})$. In [FR18], Filipazzi and Rota verify a state space isomorphism between the two Calabi-Yau varieties by providing an explicit mirror map.

Batyrev and Borisov in [BB96] introduced a mirror construction for Calabi-Yau intersections in Fano toric varieties using polytopes, showing mirror duality for $(1, q)$-Hodge numbers. This mirror construction agrees with constructions by Green-Plesser [GP90] and Berglund-Hübsch [BH92] for Fermat hypersurfaces. However, the Batyrev-Borisov mirror to two cubics in $\mathbb{P}^5$ differs from the one given above by Libgober and Teitelbaum.

In this paper, we establish a connection between the mirrors of Libgober-Teitelbaum and Batyrev-Borisov for two cubics in $\mathbb{P}^5$ in the context of Homological Mirror Symmetry, using variations of geometric invariant theory (VGIT). In particular, we show that the bounded derived category of coherent sheaves of the Libgober-Teitelbaum mirror is derived equivalent to that of a complete intersection $Z \subseteq X_\nabla$ in the Batyrev-Borisov mirror family. Note that there exists a toric stack $X_\nabla$ with coarse moduli space $X_\nabla$ (see [2.1] for the toric stack construction and [3.1] for the fan associated to this toric stack). On the level of stacks, we will prove the following result.

**Theorem 1.1.** Let $\lambda \in \mathbb{C}$ such that $\lambda^6 \neq 0, 1$. Consider the two polynomials

$$p_{1,\lambda} = x_0^3 x_6 + x_1^3 x_7 + x_2^3 x_8 - 3\lambda x_3 x_4 x_5 x_6 x_7 x_8,$$

$$p_{2,\lambda} = x_3^3 x_9 + x_4^3 x_{10} + x_5^3 x_{11} - 3\lambda x_0 x_1 x_2 x_9 x_{10} x_{11}.$$ 

Let $Z_\lambda = Z(p_{1,\lambda}, p_{2,\lambda}) \subseteq X_\nabla$ and $V_{LT,\lambda} = Z(Q_{1,\lambda}, Q_{2,\lambda}) \subseteq [\mathbb{P}^5/G_{81}]$. Then

$$\mathcal{D}^b(\text{coh } V_{LT,\lambda}) \simeq \mathcal{D}^b(\text{coh } Z_\lambda).$$
This result is expected in the context of Kontsevich’s Homological Mirror Symmetry Conjecture. As both $\mathcal{V}_{LT,\lambda}$ and $Z_\lambda$ are conjectured to be (homological) mirrors of the complete intersection of two cubics, we expect their corresponding derived categories to be equivalent to each other and to the Fukaya category of the zero locus $Z(Q_1,\lambda, Q_2,\lambda) \subseteq P^5$.

There has been work to unify various (toric) mirror constructions in the literature via derived equivalence.

In particular, this is the first application of partial compactifications in VGIT quotients to prove the equivalence of derived categories for complete intersections, and not hypersurfaces, for Calabi-Yau varieties.

We start by giving some background on the mathematical tools necessary to prove Theorem 1.1 in Section 2. This includes a short introduction to the relevant tools in toric geometry, the Batyrev-Borisov mirror construction, and VGIT quotients as outlined in [FK19]. In Section 4 we then study the link between the Batyrev-Borisov mirror construction and the mirror given by Libgober-Teitelbaum, proving Theorem 1.1.

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2. Background

In this section we give the necessary background on the Batyrev-Borisov mirror construction, the Libgober-Teitelbaum construction, and the tools used to connect those two. All the varieties considered in this paper will be defined over the complex numbers. More detailed expositions can for example be found in [BN08], [CLS11], [FK17] and [LT93].

2.1. The Cox construction for toric stacks. Let $M$ be a lattice of rank $d$ and $N$ its dual lattice, with the pairing $\langle , \rangle : M \times N \to \mathbb{Z}$.

We extend this to a pairing between $M_\mathbb{R} := M \otimes \mathbb{Z} \otimes \mathbb{R}$ and $N_\mathbb{R} := N \otimes \mathbb{Z} \otimes \mathbb{R}$ in the natural way.

To associate a variety $X_\Sigma$ to a fan $\Sigma$, we can use the Cox construction (see §5 of [CLS11]). Start by noting that each ray $\rho$ of the fan $\Sigma$ corresponds to a divisor $D_\rho$ on $X_\Sigma$ (see §4 of [CLS11]). Then we have the following exact sequence:

$$0 \to M \xrightarrow{\iota} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \to \text{coker } \iota \to 0,$$

where $\iota(m) := \text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$.

We will write $\mathbb{Z}^{\Sigma(1)} := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho$. Since $\mathbb{C}^*$ is a divisible group and hence an injective $\mathbb{Z}$-module, the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is exact, so applying it to (1) yields the exact sequence:

$$1 \to \text{Hom}_{\mathbb{Z}}(\text{coker } \iota, \mathbb{C}^*) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \to \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \to 1.$$ 

Define

$$G_\Sigma := \text{Hom}_{\mathbb{Z}}(\text{coker } \iota, \mathbb{C}^*).$$

(3)
Note that $\text{Hom}_\mathbb{Z}(\mathbb{Z}^\Sigma(1), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\Sigma(1)}$ and $\text{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \simeq T_N$, where $T_N$ is the torus of the variety. Hence we may rewrite (2) as
\[ 1 \to G_\Sigma \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 1. \quad (4) \]

When describing $G_\Sigma$ explicitly, the following lemma is useful.

**Lemma 2.1** (Lemma 5.1.1(c) in [CLS11]). Let $G_\Sigma \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ be as in (4). Given a basis $e_1, \ldots, e_n$ of $M$, we have
\[ G_\Sigma = \left\{ (t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \prod_{\rho} t_\rho^{(e_i, u_\rho)} = 1 \text{ for } 1 \leq i \leq n \right\}. \]

We now have both an affine space $\mathbb{C}^{\Sigma(1)}$ and a group $G_\Sigma$, which can be shown to be reductive, thus only further require an exceptional set $\check{Z}$ in order to construct the toric variety $X_\Sigma$ as a geometric quotient. For each ray $\rho \in \Sigma(1)$, introduce a variable $x_\rho$ and consider the total coordinate ring of $X_\Sigma$,
\[ S := \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]. \]

For each cone $\sigma \in \Sigma$, let $x^\check{\sigma} = \prod_{\rho \notin \sigma(1)} x_\rho$. We define the irrelevant ideal
\[ B(\Sigma) = \langle x^\check{\sigma} \mid \sigma \in \Sigma \rangle \subseteq S. \]

Since $\tau \leq \sigma$, we have that $x^\check{\tau}$ is a multiple of $x^\check{\sigma}$. Thus, we only need to consider maximal cones to generate the irrelevant ideal. Define $Z(\Sigma) = Z(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}$. We then have:

**Theorem 2.2** (Theorem 5.1.11 in [CLS11]). Let $X_\Sigma$ be a toric variety without torus factors, associated to a fan $\Sigma$. Then
\[ X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) \sslash G_\Sigma. \]

Most of the discussion to follow happens on the level of stacks, so we define the toric stacks relevant for us here.

**Definition 2.3.** Let $\Sigma$ be a fan. Define the Cox fan $\text{Cox}(\Sigma) \subseteq \mathbb{R}^{\Sigma(1)}$ to be
\[ \text{Cox}(\Sigma) := \{ \text{Cone}(e_\rho \mid \rho \in \sigma) \mid \sigma \in \Sigma \}. \]

Denote by $n$ the number of rays in the fan $\Sigma$. Then the Cox fan of $\Sigma$ is a subfan of the standard fan corresponding to the toric variety $\mathbb{A}^n$. Thus, $U_\Sigma := X_{\text{Cox}(\Sigma)}$ is an open subset of $\mathbb{A}^n$. Consider the group $G_\Sigma$ as defined in Equation (3).

**Definition 2.4.** We call $U_\Sigma$ the Cox open set associated to $\Sigma$ and define the Cox stack associated to $\Sigma$ to be
\[ X_\Sigma := [U_\Sigma/G_\Sigma] \]

In the smooth and orbifold cases, we have the following result relating $X_\Sigma$ to $X_\Sigma$.

**Theorem 2.5** ([FMN10]). If $\Sigma$ is simplicial, then $X_\Sigma$ is a smooth Deligne-Mumford stack with coarse moduli space $X_\Sigma$. When $\Sigma$ is smooth (or equivalently $X_\Sigma$ is smooth) $X_\Sigma \cong X_\Sigma$. 

2.2. Polytopes and the Batyrev-Borisov construction. We now define reflexive polytopes and nef partitions. We can use them to introduce the Batyrev-Borisov duality, following [BN08, BB96].

Definition 2.6. A polytope \( \Delta \) in \( \mathbb{M}_\mathbb{R} \) is a convex hull of a finite set of points in \( \mathbb{M}_\mathbb{R} \). If this finite set can be chosen to only consist of lattice points of \( \mathbb{M} \), we call \( \Delta \) a lattice polytope.

Definition 2.7. Let \( \Delta \) be a full dimensional lattice polytope in \( \mathbb{M}_\mathbb{R} \) with 0 an interior lattice point. Its dual polytope \( \Delta^\vee \) is given by
\[
\Delta^\vee := \{ n \in \mathbb{N}_\mathbb{R} \mid \langle m, n \rangle \geq -1 \ \forall \ m \in \Delta \}
\]
We call \( \Delta \) reflexive if the dual polytope is also a lattice polytope.

Given a lattice polytope \( \Delta \), we can associate a toric variety to it by considering its normal fan \( \Sigma_{\Delta} \) with its corresponding toric variety \( X_{\Sigma_{\Delta}} \).

The polytope \( \Delta \) corresponds to the anticanonical divisor of \( X_{\Sigma_{\Delta}} \) in that the lattice points of \( \Delta \) correspond to the global sections of the anticanonical divisor. This in turn allows one to construct a Calabi-Yau hypersurface in \( X_{\Sigma_{\Delta}} \) by considering the zero-section of the global section; however, we want to construct Calabi-Yau complete intersections. To do so, we must construct a nef partition of the polytope \( \Delta \).

Definition 2.8. Let \( \Delta \subseteq \mathbb{M}_\mathbb{R} \) be a reflexive lattice polytope. A nef partition of length \( r \) of \( \Delta \) is a Minkowski sum decomposition \( \Delta = \Delta_1 + \cdots + \Delta_r \) where \( \Delta_1, \ldots, \Delta_r \) are lattice polytopes with \( 0 \in \Delta_i \).

Consider a reflexive polytope \( \Delta \subseteq \mathbb{M}_\mathbb{R} \) with nef partition \( \Delta = \Delta_1 + \cdots + \Delta_r \). Then, for \( 1 \leq j \leq r \), we define
\[
\nabla_j := \{ n \in \mathbb{N}_\mathbb{R} \mid \langle m, n \rangle \geq -\delta_{ij} \ \forall \ m \in \Delta_i, \ \text{for} \ 1 \leq i \leq r \}.
\]
We note that these polytopes are all lattice polytopes, and define the polytope \( \nabla \) as their Minkowski sum \( \nabla := \nabla_1 + \cdots + \nabla_r \). We call \( \nabla_1, \ldots, \nabla_r \) the dual nef partition to \( \Delta_1, \ldots, \Delta_r \).

Remark 2.9. The generic complete intersection in the family associated to the dual nef partition \( \nabla_1, \ldots, \nabla_r \) may be singular.

In [Bat94], Batyrev formulates the original construction in a way that fixes this problem. In this case, one uses a maximal projective crepant partial desingularization (MPCP-desingularization), which reduces to a combinatorial manipulation of the normal fan to \( \nabla \).

For every maximal cone of the normal fan, we choose a regular triangulation of it. Therefore, all maximal cones should contain exactly the minimal number of rays dictated by the dimension, since a triangulation uses simplices. Doing this for all maximal cones gives exactly a maximal projective triangulation. When speaking of \( X_{\nabla} \) we will thus think of a
MPCP-desingularization of the variety associated to the normal fan of \( \nabla \), obtained in this way.

Batyrev and Borisov prove the following result, showing that their construction produces topological mirror duality for \((1,q)\)-Hodge numbers.

**Theorem 2.10** (Theorem 9.6 in [BB96]). Let \( V \) be a Calabi-Yau complete intersection of \( r \) hypersurfaces in \( \mathbb{P}^d \) and \( d-r \geq 3 \) and \( \tilde{W} \) be a MPCP-desingularization of the Calabi-Yau complete intersection \( W \subseteq X_\nabla \). Then
\[
h^q(O^1_{\tilde{W}}) = h^{d-r-q}(O^1_V) \text{ for } 0 \leq q \leq d-r.
\]

### 2.3. Toric vector bundles and GIT quotients

We first discuss how to construct toric vector bundles. Recall that a Cartier divisor \( D = \sum_{\rho} a_\rho D_\rho \) on a toric variety \( X_\Sigma \) corresponds to the line bundle \( \mathcal{L} = \mathcal{O}_{X_\Sigma}(D) \), which is the sheaf of sections of a rank 1 vector bundle \( \pi : V_\mathcal{L} \to X_\Sigma \). The variety \( V_\mathcal{L} \) is toric and \( \pi \) is a toric morphism. This is shown by directly constructing the fan of \( V_\mathcal{L} \) in terms of \( \Sigma \) and \( D \), which we do now. Given a cone \( \sigma \in \Sigma \), set
\[
\tilde{\sigma} = \text{Cone } ((0,1), (u_\rho, -a_\rho) | \rho \in \sigma(1)).
\]
Then \( \tilde{\sigma} \) is a strongly convex rational polyhedral cone in \( N_\mathbb{R} \times \mathbb{R} \) for all cones \( \sigma \in \Sigma \). Now let \( \Sigma \times D \) be the collection consisting of cones \( \tilde{\sigma} \) for \( \sigma \in \Sigma \) and their faces. This is a fan in \( N_\mathbb{R} \times \mathbb{R} \) and the projection \( \bar{\pi} : N \times \mathbb{Z} \to N \) is compatible with \( \Sigma \times D \) and \( \Sigma \), thus inducing a toric morphism
\[
\pi : X_{\Sigma \times D} \to X_\Sigma.
\]

**Proposition 2.11** (Proposition 7.3.1 in [CLS11]). \( \pi : X_{\Sigma \times D} \to X_\Sigma \) is a rank 1 vector bundle whose sheaf of sections is \( \mathcal{O}_{X_\Sigma}(D) \).

The variety \( X_{\Sigma \times D} \) is sometimes also denoted by \( X_{\Sigma,D} \).

For decomposable vector bundles of rank higher than 1, we can repeatedly apply Proposition 2.11 to construct the total space of the vector bundle, following [FK18]. Taking \( r \) torus-invariant Weil divisors \( D_i = \sum_{\rho \in \Sigma} a_{i\rho} D_\rho \), we define
\[
\sigma_{D_1,\ldots,D_r} = \text{Cone } \{ \{ u_\rho - a_{1\rho} e_1 - \cdots - a_r e_r \mid \rho \in \sigma(1) \} \cup \{ e_i \mid i \in \{1,\ldots,r\} \} \} \subseteq N_\mathbb{R} \oplus \mathbb{R}^r.
\]
Let \( \Sigma_{D_1,\ldots,D_r} \) be the fan generated by the cones \( \sigma_{D_1,\ldots,D_r} \) and their proper faces, and call \( X_{\Sigma,D_1,\ldots,D_r} \) the associated stack. We obtain the following result.

**Proposition 2.12** (Proposition 4.13 in [FK18]). Let \( D_1,\ldots,D_r \) be divisors on \( X_\Sigma \). There is an isomorphism of stacks
\[
X_{\Sigma,D_1,\ldots,D_r} \cong \text{tot } (\mathcal{O}_{X_\Sigma}(D_i)).
\]

Geometric invariant theory (GIT), developed by Mumford, is a powerful tool in modern algebraic geometry. We will here discuss the toric version of it, following §14 of [CLS11]. Roughly speaking, GIT deals with ways to take almost geometric quotients of spaces by some reductive groups acting on them. As a model for this, recall the Cox construction in §2.1. It gives a toric variety as almost geometric quotient \( X_\Sigma \cong (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)) \sslash G_\Sigma \). Fundamentally, we start with \( \mathbb{C}^{\Sigma(1)} \) and remove a special Zariski closed subset in order to obtain an almost geometric quotient. GIT provides the machinery to do so, but the way is often not unique. The subsets that are removed depend on a choice of stability parameterised by a choice of line bundle. The different choices can give different quotients which are birational.
In GIT, deciding which points are removed is done by a lifting of the $G$-action on $\mathbb{C}^r$ to the rank 1 trivial vector bundle $\mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$. Define the character group of $G$ to be

$$\widehat{G} = \{ \chi : G \to \mathbb{C}^* | \chi \text{ is a homomorphism of algebraic groups} \}.$$ 

A character $\chi \in \widehat{G}$ then gives the action of $G$ on $\mathbb{C}^r \times \mathbb{C}$ defined by

$$g \cdot (p, t) = (g \cdot p, \chi(g)t), \quad g \in G, \quad (p, t) \in \mathbb{C}^r \times \mathbb{C}.$$ 

This lifts the $G$-action on $\mathbb{C}^r$ and furthermore all possible liftings arise this way.

Let $L_\chi$ or $O(\chi)$ denote the sheaf of sections of $\mathbb{C}^r \times \mathbb{C}$ with this $G$-action. It is called the linearised line bundle with character $\chi$. For $d \in \mathbb{Z}$, the tensor product $O(\chi)^{\otimes d}$ is the linearised line bundle with character $\chi^d$. Note that, if one forgets the $G$-action, then $O(\chi) \simeq O_{\mathbb{C}^r}$ as line bundles on $\mathbb{C}^r$. Thus, a global section $s \in \Gamma(\mathbb{C}^r, O(\chi))$ can be written as

$$s : \mathbb{C}^r \to \mathbb{C}^r \times \mathbb{C}$$

$$p \mapsto (p, F_s(p)),$$

for some unique $F_s \in \mathbb{C}[x_1, \ldots, x_r]$.

**Definition 2.13.** Fix $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, with linearised line bundle $O(\chi)$. Given a global section $s$ of $O(\chi)$, we denote

$$(\mathbb{C}^r)_s := \{ p \in \mathbb{C}^r \mid s(p) \neq 0 \}$$

This is an affine open subset of $\mathbb{C}^r$, as $s(p) \neq 0$ means $F_s(p) \neq 0$. Furthermore, $G$ acts on $(\mathbb{C}^r)_s$ when $s$ is $G$-invariant. We define:

(A) $p \in \mathbb{C}^r$ is **semistable** with respect to $\chi$ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, O(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$.

(B) $p \in \mathbb{C}^r$ is **stable** with respect to $\chi$ if there exist $d > 0$ and $s \in \Gamma(\mathbb{C}^r, O(\chi^d))^G$ such that $p \in (\mathbb{C}^r)_s$, the isotropy subgroup $G_p$ is finite, and all $G$-orbits in $(\mathbb{C}^r)_s$ are closed in $(\mathbb{C}^r)_s$.

(C) The set of all semistable (resp. stable) points with respect to $\chi$ is denoted $(\mathbb{C}^r)_\chi^{ss}$ (resp. $(\mathbb{C}^r)_\chi^s$).

Given a group $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, we next need to define the GIT quotient $\mathbb{C}^r \sslash \chi G$. Consider the graded ring $R_\chi = \bigoplus_{d=0}^\infty \Gamma(\mathbb{C}^r, O(\chi^d))^G$.

**Definition 2.14.** For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r \sslash \chi G$ is

$$\mathbb{C}^r \sslash \chi G = \text{Proj}(R_\chi).$$

An important property of GIT quotients is that in principle, this is the same as taking the quotient of $(\mathbb{C}^r)_\chi^{ss}$ under the action of $G$.

**Proposition 2.15** (Proposition 14.1.12.c) in [CLS11]. For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the GIT quotient $\mathbb{C}^r \sslash \chi G$ is a good categorical quotient of $(\mathbb{C}^r)_\chi^{ss}$ under the action of $G$, i.e. $\mathbb{C}^r \sslash \chi G \simeq (\mathbb{C}^r)_\chi^{ss} \sslash \chi G$.

Theorem 14.2.13 of [CLS11] shows, using a polyhedron associated to the character $\chi$, that the GIT quotient $\mathbb{C}^r \sslash \chi G$ is a toric variety.
2.4. **GKZ Fans.** Let $G \subseteq (\mathbb{C}^*)^r$. Studying the GIT quotient $\mathbb{C}^r / \chi G$ as $\chi$ varies gives rise to the GKZ fan of a toric variety, which has the structure of a generalised fan.

**Definition 2.16.** A generalised fan $\Sigma$ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

(A) Every $\sigma \in \Sigma$ is a rational polyhedral cone.

(B) For all $\sigma \in \Sigma$, each face of $\sigma$ is also in $\Sigma$.

(C) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

This agrees with the usual definition of a fan, with the exception that cones are not necessarily strongly convex. Consider the cone $\Sigma_0 = \bigcap_{\sigma \in \Sigma} \sigma$. It has no proper faces and is thus a subspace of $N_\mathbb{R}$. We consider the lattice $\overline{N} = N/(\sigma_0 \cap N)$. To associate a toric variety for the generalised fan $\Sigma$, one constructs the fan $\overline{\Sigma}$ where each cone comes from a cone of $\Sigma$ quotiented by $\sigma_0$. This is a fan in the usual sense, and hence we can associate a toric variety to it as usual. Then $X_\Sigma := X_{\overline{\Sigma}}$.

We will now discuss the notion of a GKZ fan, following both [CLS11] and [FK19]. Consider a toric variety $X$. It can be written as a GIT quotient $(\mathbb{C}^r \setminus Z) / \chi G$. Recall the character group $\widehat{G}$ of $G$. Each choice of character $\chi \in \widehat{G}$ determines an open subset $U_\chi := (\mathbb{C}^r)^{ss}_\chi$, the semi-stable locus of $X$ with respect to $\chi$. Several different characters can give the same semi-stable locus. Thinking of the vector space $\mathrm{Hom}(\widehat{G}, T_N) \otimes_{\mathbb{Z}} \mathbb{Q}$ as parameter space for linearisations, we investigate where the semi-stable locus $U_\psi$ is the same as $U_\chi$ for a given character $\chi$. It turns out that dividing the vector space into chambers where $U_\chi$ remains the same gives the space a natural fan structure. This fan-structure $\Sigma_{GKZ}$ is called the GKZ fan. Maximal cones are called chambers and codimension one cones are called walls.

Consider an arbitrary fan $\Sigma$, we can construct the GKZ fan as follows. Take the group $G = G_\Sigma \subseteq (\mathbb{C}^*)^r$ acting on $X_\Sigma$ to be the group in Equation (3). There is a well-known bijection between chambers of GKZ fans and regular triangulations of a certain set of points, constructed as follows. In the general setting, apply $\mathrm{Hom}( -, \mathbb{C}^*)$ to the sequence

$$0 \rightarrow G \xrightarrow{i_G} (\mathbb{C}^*)^r \xrightarrow{\text{proj}} \mathrm{coker}(i_G) \rightarrow 0$$

to obtain the sequence

$$\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*) \xrightarrow{\text{proj}} \mathbb{Z}^r \xrightarrow{i_G} \mathrm{Hom}(G, \mathbb{C}^*) \rightarrow 0.$$  

Let $\nu_i(G)$ be the element of $\mathrm{Hom}(\mathrm{coker}(i_G), \mathbb{C}^*)$ given by composing $\text{proj}$ with the projection of $\mathbb{Z}^r$ onto its $i^{th}$ factor. Compare this sequence with the sequence (3). We in fact reversed the process of obtaining (4) from (1). Starting with the correct group acting on the space, we thus recover the map corresponding to $\nu$ as $\text{proj}$. Hence, the $\nu(G)$ correspond to the primitive generators $u_\rho$ of the rays of $\Sigma$. Then the set we will triangulate is the convex hull of the set $\nu(G) = \{\nu_1(G), \ldots, \nu_r(G)\}$.

**Theorem 2.17** (Proposition 15.2.9 in [CLS11]). There is a bijection between chambers of the GKZ fan for the action of $G$ on $\mathbb{C}^r$ and regular triangulations of the set $\text{Conv}(\nu(G))$. In particular, there are only finitely many chambers of the GKZ fan.

Thus we can enumerate the chambers of the GKZ fan, say by $\sigma_1, \ldots, \sigma_k$. For any of those chambers, we can choose a character in its interior and consider the semi-stable locus with respect to it. As this locus does not depend on the choice of character, but solely on the choice of chamber, denote the open affine associated to chamber $\sigma_p$ by $U_p$. By the above theorem, it will also correspond to a specific triangulation $T_p$ of $\text{Conv}(\{\nu_1(G), \ldots, \nu_r(G)\})$. 


2.5. Categories of singularities and some results on the equivalence of derived categories. In this section, we introduce the categories of singularities (as outlined in [Ols09]) and their equivalences to derived categories through VGIT, reviewing §4 of [FK19].

Let $X$ be a variety and $G$ an algebraic group acting on $X$ (on the left).

**Definition 2.18.** An object of $\text{D}^b(\text{coh}[X/G])$ is called perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles. We denote the full subcategory of perfect objects by $\text{Perf}([X/G])$. The Verdier quotient of $\text{D}^b(\text{coh}[X/G])$ by $\text{Perf}([X/G])$ is called the category of singularities and denoted

$$\text{D}_{\text{sg}}([X/G]) := \text{D}^b(\text{coh}[X/G])/\text{Perf}([X/G]).$$

By the following observation of Orlov’s, the category can be viewed as studying the geometry of the singular locus.

**Proposition 2.19 (Orlov, [Ols09]).** Assume that $\text{coh}[X/G]$ has enough locally free sheaves. Let $i: U \to X$ be a $G$-equivariant open immersion such that the singular locus of $X$ is contained in $i(U)$. Then the restriction,

$$i^*: \text{D}_{\text{sg}}([X/G]) \to \text{D}_{\text{sg}}([U/G]),$$

is an equivalence of categories.

Next, consider a $G$-equivariant vector bundle $\mathcal{E}$ on $X$. Denote by $Z$ the zero locus of a $G$-invariant section $s \in \mathcal{H}^0(X, \mathcal{E})$. Then $\langle - , s \rangle$ induces a global function on $\text{tot}\mathcal{E}^\vee$. Let $Y$ be the zero-section of this pairing and consider the fibrewise dilation action on the torus $\mathbb{G}_m$. Then we have the following result.

**Theorem 2.20 (Isik [Ist13], Shipman [Shi12], Hirano [Hir17]).** Suppose the Koszul complex on $s$ is exact. Then there is an equivalence of categories

$$\text{D}_{\text{sg}}([Y/(G \times \mathbb{G}_m)]) \cong \text{D}^b(\text{coh}[Z/G]).$$

Combining the previous two results gives the following.

**Corollary 2.21 (Corollary 3.4 in [FK19]).** Let $V$ be an algebraic variety with a $G \times \mathbb{G}_m$ action. Suppose there is an open subset $U \subseteq V$ such that $U$ is $G \times \mathbb{G}_m$ equivariantly isomorphic to $Y$ as above and that $U$ contains the singular locus of $X$. Then

$$\text{D}_{\text{sg}}([V/(G \times \mathbb{G}_m)]) \cong \text{D}^b(\text{coh}[Z/G]).$$

We will move towards making these results applicable to the objects studied in this paper, adapting [FK19]. Consider an affine space $X := \mathbb{A}^{n+t}$ with coordinates $x_i, u_j$ for $1 \leq i \leq n, 1 \leq j \leq t$. Let $T$ denote the standard open torus $\mathbb{G}_m^{n+t}$ and consider a subgroup $S \subseteq T$, with $S$ the connected component that contains the identity.

Recall the notion of GKZ fans from §2.4. We adjust the notation so that $S$ above corresponds to the group $G$ from §2.4. We will now explain how to construct varieties corresponding to the chambers of the GKZ fan, and the goal of this setup is to apply Corollary 2.21 and VGIT to provide equivalences between derived categories.

**Definition 2.22.** Let $G$ be a group acting on a space $X$ and $f$ a global function on $X$. $f$ is said to be semi-invariant with respect to a character $\chi$ if, for any $g \in G$, $f(g \cdot x) = \chi(g)f(x)$. 


To apply Corollary 2.21 we will add a $\mathbb{G}_m$-action which is $S$-invariant and $\mathbb{G}_m$-semi-invariant, acting with weight 0 on the $x_i$ and 1 on the $u_j$. We refer to this action as R-charge. Consider the action of $S$ on the scheme $\text{Spec} \mathbb{C}[u_j]$. It corresponds to a character $\gamma_j$ of $S$. Let $f_1, \ldots, f_t$ be a collection of $S$-semi-invariant functions in the $x_i$ with respect to $\gamma_j^{-1}$. Then define a function, called superpotential, by

$$w := \sum_{j=1}^t u_j f_j.$$  

The superpotential $w$ is $S$-invariant and $\chi$-semi-invariant with respect to the projection character $\chi : S \times \mathbb{G}_m \to \mathbb{G}_m$, hence $w$ is homogeneous of degree 0 with respect to the $S$-action and of degree 1 with respect to the R-charge. Let $Z(w) \subseteq X$ be its zero-locus and define $Y_p := Z(w) \cap U_p$. Then we have the following result.

**Theorem 2.23** (Theorem 3 in [HW12]). If $S$ is quasi-Calabi-Yau, there is an equivalence of categories

$$\mathbb{D}_{\mathbb{S}}([Y_p/S \times \mathbb{G}_m]) \cong \mathbb{D}_{\mathbb{S}}([Y_q/S \times \mathbb{G}_m])$$

for all $1 \leq p, q \leq k$, where $k$ is the number of chambers in the GKZ fan.

We will use this result to show a useful equivalence of derived categories. We start by explicitly describing the open sets $U_p$ corresponding to a chamber $\sigma_p$ of the GKZ fan, defined in 2.24. For $1 \leq p \leq k$ we associate an irrelevant ideal $\mathcal{I}_p$ to $\sigma_p$ by considering the (regular) triangulation $\mathcal{T}_p$ that the chamber corresponds to. So, let

$$\mathcal{I}_p := \left\langle \prod_{i \in I} x_i \prod_{j \notin J} u_j \mid \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j \in J} \nu_{n+j}(S) \right\rangle \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

Then $U_p = X \setminus Z(\mathcal{I}_p)$. Another ideal we will need is a subideal of $\mathcal{I}_p$, given similarly to $\mathcal{I}_p$ by requiring $J$ to be the full set $\{1, \ldots, t\}$, i.e.,

$$\mathcal{J}_p := \left\langle \prod_{i \in I} x_i \mid \bigcup_{i \in I} \nu_i(S) \cup \bigcup_{j=1}^t \nu_{n+j}(S) \right\rangle \text{ is the set of vertices of a simplex in } \mathcal{T}_p \right\rangle.$$

This ideal is therefore generated by those simplices whose sets of vertices contain all $\nu_{n+j}$ for $1 \leq j \leq t$. Using this subideal, we get a new open set $V_p := X \setminus Z(\mathcal{J}_p) \subseteq U_p$. Since $\mathcal{J}_p$ has no $u_j$ in its generators, we can see it as a toric stack $X_p := [V_p^x/S]$. Now suppose $\mathcal{J}_p$ is non-zero. Then the last two quantities defined are nonempty, and one can show $[V_p/S]$ is a vector bundle over $X_p$, with the inclusion of rings $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_t]$ restricting to a $S$-equivariant morphism

$$[V_p/S] \to [V_p^x/S] = X_p.$$  

This morphism gives the following proposition.

**Proposition 2.24** (Proposition 4.6 in [FK19]). Suppose $\mathcal{J}_p$ is non-zero. The morphism $[V_p/S] \to X_p$ realizes $[V_p/S]$ as the total space of a vector bundle

$$[V_p/S] \cong \text{tot} \bigoplus_{j=1}^t \mathcal{O}(\gamma_j).$$
Furthermore, the \( R \)-charge action of \( \mathbb{G}_m \) is the dilation action along fibers. Finally, for each \( j \), the function \( f_j \) gives a section of \( \mathcal{O}(\gamma_j^{-1}) \) and the superpotential \( w = \sum u_j f_j \) restricts to the pairing with the section \( \partial w \).

In particular, from this we can view the function \( \oplus f_j \) as a section of \( V_p \) which defines, for all \( p \), a complete intersection \( Z_p := \mathcal{Z}(\oplus f_j) \subseteq X_p \). Finally, we introduce the Jacobian ideal \( \partial w \), generated by the partial derivatives of \( w \) with respect to the coordinates \( x_i, u_j \).

**Proposition 2.25** (Proposition 4.7 in [FK19]). Suppose \( J_p \) is non-zero. If \( \mathcal{I}_p \subseteq \sqrt{\partial w}, J_p \), then

\[
D_{sg}(\mathcal{Y}_p/S \times \mathbb{G}_m) \cong D^b(\text{coh } Z_p).
\]

This finally leads us to the following result, which we will use in \([4]\).

**Corollary 2.26.** Assume \( S \) satisfies the quasi-Calabi-Yau condition and that \( J_p \) and \( J_q \) are non-zero. If \( \mathcal{I}_p \subseteq \sqrt{\partial w}, J_p \) and \( \mathcal{I}_q \subseteq \sqrt{\partial w}, J_q \) for some \( 1 \leq p, q \leq r \), then

\[
D^b(\text{coh } Z_p) \cong D^b(\text{coh } Z_q).
\]

### 3. The Libgober-Teitelbaum and the Batyrev-Borisov constructions

#### 3.1. The Batyrev-Borisov construction in \( \mathbb{P}^5 \)

We now construct a Batyrev-Borisov mirror to a complete intersection of two cubics in \( \mathbb{P}^5 \). We will do this by giving a nef partition of the anticanonical polytope of \( \mathbb{P}^5 \) which corresponds to a complete intersection. Then we will apply the Batyrev-Borisov construction to that nef partition, obtaining a polytope \( \nabla \) corresponding to the mirror. Fix the lattice \( M \cong \mathbb{Z}^5 \) and its dual lattice \( N \).

**Remark 3.1.** Due to the way we will derive certain fans in this section via methods inspired by mirror symmetry (see \( \S \text{3.3} \)) our first fan lives in \( M_\mathbb{R} \) and not in the conventional \( N_\mathbb{R} \).

Define the rays \( r_0, \ldots, r_{11} \) in \( M_\mathbb{R} \oplus \mathbb{R}^2 \) with primitive generators

\[
\begin{align*}
&u_{r_0} = (3, 0, 0, -1, -1, 0, 1), & u_{r_6} = (2, -1, -1, 0, 0, 1, 0),
&u_{r_1} = (0, 3, 0, -1, -1, 0, 1), & u_{r_7} = (-1, 2, -1, 0, 0, 1, 0),
&u_{r_2} = (0, 0, 3, -1, -1, 0, 1), & u_{r_8} = (-1, -1, 2, 0, 0, 1, 0),
&u_{r_3} = (-1, -1, 3, 0, 0, 1, 0), & u_{r_9} = (0, 0, 0, 2, -1, 0, 1),
&u_{r_4} = (-1, -1, -1, 0, 0, 1, 0), & u_{r_{10}} = (0, 0, 0, -1, 2, 0, 1),
&u_{r_5} = (-1, -1, 0, 0, 0, 1, 0), & u_{r_11} = (0, 0, 0, 0, 0, 0, 1),
&u_{r_7} = (0, 0, 0, 0, 0, 0, 1).
\end{align*}
\]

**Notation 3.2.** For \( 0 \leq j \leq 11 \), we denote by \( u_{\rho_j} \) the lattice point in \( M \) obtained from \( u_{r_7} \) by projecting onto the first \( 5 \) coordinates. Denote by \( \rho_j \) the ray generated by \( u_{\rho_j} \) in \( M_\mathbb{R} \).

**Proposition 3.3.** Consider the fan \( \Sigma_\nabla \) with rays \( \rho_0, \ldots, \rho_{11} \) defined above and maximal cones listed in Table 1 (page 12). Then a general complete intersection in the toric variety \( X_\nabla \) corresponding to the fan \( \Sigma_\nabla \) is a Batyrev-Borisov mirror to a complete intersection of two cubics in \( \mathbb{P}^5 \).

**Proof.** The anticanonical sheaf of \( \mathbb{P}^5 \) is \( \mathcal{O}_{\mathbb{P}^5}(6) \), corresponding to the divisor class

\[
-K_{\mathbb{P}^5} = T_0 + \cdots + T_5 = (T_0 + T_1 + T_2) + (T_3 + T_4 + T_5).
\]

The anticanonical polytope for \( \mathbb{P}^5 \) is given by

\[
\Delta_{-K_{\mathbb{P}^5}} = \{ m \in M_\mathbb{R} | \langle m, u_\rho \rangle \geq -1 \text{ for } \rho \in \Sigma_{\mathbb{P}^5}(1) \} \subseteq M_\mathbb{R},
\]
Next, we shall compute the dual nef partition, as defined in the variety associated to the above fan. To do this, we subdivide each of the maximal cones procedure, the Table 1 (see below) gives the 42 maximal cones in the fan corresponding to $\Delta_{-K_{X'}}$. These polytopes are

$$\Delta_1 = \text{Conv}((2, -1, -1, 0, 0), (-1, 2, -1, 0, 0), (-1, -1, 2, 0, 0),$$
$$(-1, -1, -1, 3, 0), (-1, -1, -1, 0, 3), (-1, -1, -1, 0, 0)),$$

$$\Delta_2 = \text{Conv}((0, 0, 0, -1, 2), (0, 0, 0, 2, -1), (0, 0, 3, -1, -1),$$
$$(0, 3, 0, -1, -1), (3, 0, 0, -1, -1), (0, 0, 0, -1, -1)).$$

Next, we shall compute the dual nef partition, as defined in [2]. We have:

$$\nabla_1 = \text{Conv}((1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 0))$$

$$\nabla_2 = \text{Conv}((0, 0, 0, 0, 1), (0, 0, 0, 1, 0), (0, 0, 0, 0, 0), (-1, -1, -1, -1, -1)).$$

Their Minkowski sum $\nabla \subseteq N_\mathbb{R}$ is then the convex hull of the 15 points

$$(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0),$$
$$(0, 0, 0, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 0, 0),$$
$$(0, -1, -1, -1, -1), (-1, 0, -1, -1, -1), (-1, 0, -1, -1, -1).$$

A SAGE computation shows the normal fan of $\nabla$, $\Sigma_\nabla \subseteq M_\mathbb{R}$, has rays $\rho_0, \ldots, \rho_{11}$ from Notation 3.2. The maximal-dimensional cones are the following 15 cones:

$$\rho_0\rho_1\rho_2\rho_9\rho_{10}, \quad \rho_0\rho_1\rho_3\rho_4\rho_6\rho_7\rho_9\rho_{10}, \quad \rho_0\rho_2\rho_3\rho_4\rho_6\rho_8\rho_9\rho_{10},$$
$$\rho_1\rho_2\rho_4\rho_5\rho_7\rho_{10}\rho_{11}, \quad \rho_0\rho_1\rho_2\rho_9\rho_{11}, \quad \rho_0\rho_1\rho_2\rho_{10}\rho_{11},$$
$$\rho_0\rho_1\rho_3\rho_5\rho_6\rho_7\rho_{10}\rho_{11}, \quad \rho_0\rho_1\rho_2\rho_5\rho_6\rho_7\rho_{10}\rho_{11}, \quad \rho_3\rho_4\rho_5\rho_6\rho_8,$$
$$\rho_0\rho_2\rho_3\rho_5\rho_6\rho_8\rho_{10}\rho_{11}, \quad \rho_3\rho_4\rho_5\rho_7\rho_8\rho_{10}\rho_{11}.$$

We listed the cones by giving the rays generating them. For instance, $\rho_0\rho_1\rho_2\rho_9\rho_{10}$ stands for the cone $\text{Cone}(\rho_0, \rho_1, \rho_2, \rho_9, \rho_{10})$. Note here that some of these maximal cones contain more rays than the others. So, as described in Remark 2.9, we want a MPCP-resolution of the variety associated to the above fan. To do this, we subdivide each of the maximal cones which has more than 5 rays. This procedure involves choice, as each cone can be subdivided in 24 ways (being a total of 49 possible choices!). However, all these choices are related by GIT, so any choice gives us a mirror family, all of which are birational. Following this procedure, the Table 1 (see below) gives the 42 maximal cones in the fan corresponding to a MPCP-resolution of the variety associated to the fan $\Sigma_\nabla$. Define the fan $\Sigma_\nblacksquare$ to be the fan consisting of those 42 5-dimensional cones and all of their faces. Determining the variety $X_\nblacksquare$ explicitly is not straightforward, but also not necessary for our purposes, so long as we have the fan $\Sigma_\nblacksquare$.

For $i = 0, \ldots, 11$, call $D'_i$ the torus-invariant divisor on $X_\nblacksquare$ corresponding to the ray $\rho_i$ of $\Sigma_\Delta$. Let $D'_a = D'_0 + D'_1 + D'_2 + D'_3 + D'_4 + D'_5 + D'_6 + D'_7 + D'_8$.

**Corollary 3.4.** Let $\Sigma_\nblacksquare, D'_a, D'_b$ be the fan with rays $\rho_0, \ldots, \rho_{11}, \tau_1, \tau_2$, and cones over those rays inherited from $\Sigma_\nblacksquare$. Then $\Sigma_\nblacksquare, D'_a, D'_b$ is a fan corresponding to $\text{tot}(\mathcal{O}_{X_\nblacksquare}(-D'_b) \oplus \mathcal{O}_{X_\nblacksquare}(-D'_a))$.

**Proof.** Apply Proposition 2.11 twice to get the result (recalling that we can do this by Proposition 2.12).
3.2. Libgober and Teitelbaum’s Mirror. We now recall the family Libgober and Teitelbaum give as a mirror to the generic complete intersection of two cubics in $\mathbb{P}^5$. To start, define $V_\lambda \subseteq \mathbb{P}^5$ to be the vanishing set of the following two polynomials: 

$$Q_{1,\lambda} = x_0^3 + x_1^3 + x_2^3 - 3\lambda x_3 x_4 x_5, \quad Q_{2,\lambda} = x_3^3 + x_4^3 + x_5^3 - 3\lambda x_0 x_1 x_2. \quad (5)$$

For generic $\lambda$, this gives a smooth complete intersection in $\mathbb{P}^5$ which is a Calabi-Yau threefold.

Let $\zeta_n$ denote a primitive $n$-th root of unity. Let $\alpha, \beta, \delta, \epsilon, \mu \in \mathbb{Z}$ (mod 3) and $\mu \in \mathbb{Z}$ (mod 9) with $3\mu = \alpha + \beta = \delta + \epsilon$. Define the diagonal matrix

$$g_{\alpha,\beta,\delta,\epsilon,\mu} := \text{diag} \left( \zeta_3^\alpha \zeta_9^\mu, \zeta_3^\beta \zeta_9^\mu, \zeta_9^\mu, \zeta_3^{-\delta} \zeta_9^{-\mu}, \zeta_3^{-\epsilon} \zeta_9^{-\mu}, \zeta_9^{-\mu} \right)$$

and let $G_{81} \subseteq \text{PGL}(5, \mathbb{C})$ denote the order 81 group generated by the $g_{\alpha,\beta,\delta,\epsilon,\mu}$. Note that $G_{81}$ acts on $\mathbb{P}^5$ by restricting the natural action of $\text{PGL}(5, \mathbb{C})$ on $\mathbb{P}^5$. The polynomials $Q_{1,\lambda}, Q_{2,\lambda}$ are invariant with respect to the action of $G_{81}$, hence $G_{81}$ acts on $V_\lambda$.

Note that $G_{81}$ is of isomorphism type $(\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/9\mathbb{Z})$ and can be generated by $(\zeta_3, \zeta_3^{-1}, 1, 1, 1, 1)$, $(1, 1, 1, \zeta_3^{-1}, \zeta_3, 1)$ and $(\zeta_9, \zeta_9^4, \zeta_9, \zeta_9^{-1}, \zeta_9^{-4}, \zeta_9^{-1})$.

Let $V_{LT,\lambda}$ be the quotient of $V_\lambda$ by the action of $G_{81}$ and let $W_{LT,\lambda}$ be a minimal resolution of singularities of $V_{LT,\lambda}$ which is a Calabi-Yau manifold.

3.3. Expressing Libgober-Teitelbaum torically. In the following, we aim to give a toric description of $V_{LT,\lambda}$. First we give a fan for the toric variety $X_{LT} := \mathbb{P}^5/G_{81}$ and then employ methods of §7.3 of [CLS11] to construct a vector bundle over $X_{LT}$ that has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

**Proposition 3.5.** Consider the 1-dimensional cones $\rho_0, \ldots, \rho_5$ with corresponding primitive generators

$$u_{\rho_0} = (3, 0, 0, -1, -1), \quad u_{\rho_1} = (0, 3, 0, -1, -1), \quad u_{\rho_2} = (0, 0, 3, -1, -1),$$

$$u_{\rho_3} = (-1, -1, -1, 3, 0), \quad u_{\rho_4} = (-1, -1, -1, 0, 3), \quad u_{\rho_5} = (-1, -1, -1, 0, 0).$$

Consider the collection $\mathcal{C}$ of sets of the form

$$\{ \rho_i \mid i \in I, I \subseteq \{0, \ldots, 5\}, |I| = 5 \}.$$ 

Let $\Sigma_{LT} \subseteq M_\mathbb{R}$ be the fan consisting of maximal cones

$$\{ \text{Cone}(C) \mid C \in \mathcal{C} \}$$

and all their faces.
Then the toric stack associated to $\Sigma_{LT}$ is the stack corresponding to the Libgober-Teitelbaum construction, $X_{LT} = [\mathbb{C}^6 \setminus \{0\}/(\mathbb{C}^* \times G_{81})]$, with $\mathbb{C}^*$ acting by $(\lambda x_0, \ldots, \lambda x_5) \sim (x_0, \ldots, x_5)$ and $G_{81}$ acting as described above in §3.2.

Proof. We use the Cox construction described in §2.1. By Lemma 2.1, we obtain the following system of equations characterising elements of $G := G_\Sigma$:

\[
\begin{align*}
t_3t_4t_5 &= t_0^3, \\
t_3t_4t_5 &= t_1^3, \\
t_3t_4t_5 &= t_2^3, \\
t_0t_1t_2 &= t_3^3, \\
t_0t_1t_2 &= t_4^3. 
\end{align*}
\]

First we note that we have a copy of $\mathbb{C}^*$ in $G$, given by \( \{ t \cdot (1, 1, 1, 1, 1, 1) \mid t \in \mathbb{C}^* \} \), so to compute $G$ we consider the group $H$ of cosets of $\mathbb{C}^*$. We will explicitly describe $H$ and subsequently use the direct product theorem to compute $G$. Consider an element $(t_0, \ldots, t_5) \in G$. By an appropriate choice of coset representative of $(t_0, t_1, t_2, t_3, t_4, t_5) \cdot \mathbb{C}^*$, we may assume $\prod_{i=0}^5 t_i = 1$.

Using equations (6), (7) and (8), we have $t_0^3 = t_1^3 = t_3$, and thus $t_0 = \zeta_3^\alpha t_2, t_1 = \zeta_3^\beta t_2$ for some $\alpha, \beta \in \mathbb{Z}_3$. Using equations (9) - (11), we have that $t_2^3t_5^3 = t_0^3t_1^3t_2^3 = t_3^3t_4^3$, which implies

\[
t_2^3t_5^3 = t_3^3 = t_4^3.
\]

Hence, similarly to above, we obtain $t_3 = \zeta_3^{-\delta}t_5, t_4 = \zeta_3^{-\varepsilon}t_5$ for some $\delta, \varepsilon \in \mathbb{Z}_3$.

By combining (8), (9) and (11) we obtain

\[
t_2^3t_5^3 = t_0t_1t_2t_3t_4t_5 = 1.
\]

Equation (12) implies $t_2^3 = (t_2^{-1})^3$, thus $t_5 = \zeta_3^{-\gamma}t_2^{-1}$ for some $\gamma \in \mathbb{Z}_3$. Using $t_3 = \zeta_3^{-\delta}t_5$ and $t_4 = \zeta_3^{-\varepsilon}t_5$ and equation (8), we obtain

\[
t_2^3 = t_3t_4t_5 = t_3^3\zeta_3^{-(\delta + \varepsilon)} = t_2^{-3}\zeta_3^{-(\delta + \varepsilon)}.
\]

Hence $t_2^{18} = 1$. So we can write $t_2 = \zeta_3^l$ for some $l \in \mathbb{Z}_{18}$.

We now claim that $t_2$ can be assumed to be a ninth root of unity and $t_5$ to be its inverse, i.e. $t_2 = \zeta_9^\mu, t_5 = \zeta_9^{-\mu}$ for some $\mu \in \mathbb{Z}_9$. Indeed, note that $(\zeta_9, \ldots, \zeta_9) \in (1, 1, 1, 1, 1, 1) \cdot \mathbb{C}^* \subseteq G$, so we can scale an element $(t_0, \ldots, t_5) \in G$ by sixth roots of unity, leaving the product $\prod_{i=1}^6 t_i$ invariant. The claim follows by multiplication with an appropriate sixth root of unity.

Expressing all the $t_i$ in terms of $t_2$, the assumption $1 = \prod_{i=1}^6 t_i$ implies $1 = \zeta_3^{\alpha + \beta - \delta - \varepsilon}$, or, equivalently,

\[
\alpha + \beta = \delta + \varepsilon \pmod{3}.
\]

Finally, using (8) gives $\zeta_3^{3\mu} = \zeta_3^{-\delta + \varepsilon} \zeta_9^{-3\mu}$ and therefore $\zeta_9^{3\mu} = \zeta_3^{\delta + \varepsilon}$. Thus $H$, the group of cosets of $\mathbb{C}^*$, is isomorphic to $G_{81}$, where $G_{81}$ has the same group described in §2. In particular, all elements of $G$ are of the form $g \cdot \lambda$ with $g \in G_{81}$, $\lambda \in (1, 1, 1, 1, 1, 1) \cdot \mathbb{C}^*$ and $G_{81} \cap \{(1, 1, 1, 1, 1, 1) \cdot \lambda \mid \lambda \in \mathbb{C}^*\} = \{(1, 1, 1, 1, 1, 1)\}$. Hence, by the direct product theorem, $G \cong \mathbb{C}^* \times G_{81}$.

The Cox fan of $\Sigma_{LT}$ can be described as follows. It has six rays $e_{\rho_0}, \ldots, e_{\rho_5}$. It is straightforward to see that the maximal cones are all 5-dimensional cones generated by any 5 of the
rays above. Therefore, we obtain $U_{\Sigma_{LT}} = \mathbb{A}^6 \setminus \{0\}$. Thus, the Cox stack associated to $\Sigma_{LT}$ is

$$\mathcal{X}_{LT} = [U_{\Sigma_{LT}}/G] = [(\mathbb{C}^6 \setminus \{0\})/(\mathbb{C}^* \times G_{S1})],$$

with the prescribed action, as required. \hfill \Box

**Remark 3.6.** We note that by Theorem 2.5 the coarse moduli space of the stack $\mathcal{X}_{LT}$ is $X_{LT}$, since $\Sigma_{LT}$ is simplicial.

Starting with the fan $\Sigma_{LT}$ of $X_{LT}$, we apply Proposition 2.11 twice to construct a vector bundle. Let $D_i$ be the Weil divisor corresponding to the ray $\rho_i$ in $\Sigma_{LT}$. Let $D_a = D_0 + D_1 + D_2$ and $D_b = D_3 + D_4 + D_5$.

**Corollary 3.7.** Denote by the rays $\overline{\rho}_0, \ldots, \overline{\rho}_5$, $\tau_1$ and $\tau_2$ the rays generated by the primitive generators:

- $u_{\overline{\rho}_0} = (3, 0, 0, -1, -1, 0, 1)$
- $u_{\overline{\rho}_1} = (0, 3, 0, -1, -1, 0, 1)$
- $u_{\overline{\rho}_2} = (0, 0, 3, -1, -1, 0, 1)$
- $u_{\overline{\rho}_3} = (-1, -1, -1, 3, 0, 1, 0)$
- $u_{\overline{\rho}_4} = (-1, -1, -1, 0, 3, 1, 0)$
- $u_{\overline{\rho}_5} = (-1, -1, -1, 0, 0, 1, 0)$
- $u_{\tau_1} = (0, 0, 0, 0, 1, 0)$
- $u_{\tau_2} = (0, 0, 0, 0, 0, 1)$

Consider the collection $\mathcal{S}$ of sets of the form

$$\{\overline{\rho}_i | i \in I, I \subseteq \{0, \ldots, 5\}, |I| = 5\} \cup \{\tau_1, \tau_2\}.$$

Let $\Sigma_{LT,D_a,D_b}$ be the fan in $M_{\mathbb{R}} \oplus \mathbb{R}^2$ consisting of the maximal cones

$$\{\text{Cone}(S)|S \in \mathcal{S}\}$$

and all their faces. Then:

(a) $\Sigma_{LT,D_a,D_b}$ is a fan corresponding to $\text{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$;

(b) The vector bundle $\mathcal{O}_{X_{LT}}(D_b) \oplus \mathcal{O}_{X_{LT}}(D_a)$ has the global section $Q_{1,\lambda} \oplus Q_{2,\lambda}$.

**Proof.** Applying Proposition 2.11 twice yields (a).

We now turn to (b) and show that $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$ and $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$. We start by noting that on $X_{LT}$ we have $\text{div}(x_i^3) = 3D_i$, so $\text{div}(x_i^3) - 3D_i \geq 0$, i.e. $x_i^3 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(3D_i))$. Similarly, $x_0x_1x_2 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $x_3x_4x_5 \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$.

To show the linear equivalence of two divisors, it suffices to consider their difference and show it is principal. We recall that $\text{div}(\chi^n) = \sum_{\rho \in \Sigma(1)}\langle u_\rho, n \rangle D_\rho$, corresponding to the map $\nu$ in the exact sequence (1). So, for instance $3D_1 - 3D_0 = \text{div}(x_0^{-3}x_1^3)$ which is the character associated to the lattice point $(-1, 1, 0, 0, 0)$. Hence $3D_1 - 3D_0 = 0$ in $\text{Cl}(X_{LT})$, i.e. $3D_0 \sim 3D_1$. Similarly $3D_1 \sim 3D_2$ and $3D_3 \sim 3D_4 \sim 3D_5$. Using the lattice points $(-1, 0, 0, 0, 0)$ and $(0, 0, 0, -1, 0)$ respectively, we also see that $3D_0 \sim D_b$ and $3D_3 \sim D_a$.

Thus

$$\mathcal{O}_{X_{LT}}(3D_a) \simeq \mathcal{O}_{X_{LT}}(3D_1) \simeq \mathcal{O}_{X_{LT}}(3D_2) \simeq \mathcal{O}_{X_{LT}}(D_b)$$

and

$$\mathcal{O}_{X_{LT}}(3D_4) \simeq \mathcal{O}_{X_{LT}}(3D_1) \simeq \mathcal{O}_{X_{LT}}(3D_2) \simeq \mathcal{O}_{X_{LT}}(D_a),$$

implying $Q_{2,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_a))$ and $Q_{1,\lambda} \in \Gamma(X_{LT}, \mathcal{O}_{X_{LT}}(D_b))$, as required. \hfill \Box

\footnote{These are the same as on page 10}
3.3.1. Intuition for constructing $X_{LT}$ torically. We now explain how we found an explicit description for the fan $\Sigma_{LT}$. We start by considering the standard fan $\Sigma_{p^5} \subseteq \mathbb{R}^5$ in the standard basis. It is the fan consisting of the cones generated by any proper subset of the six rays $\nu_0, \ldots, \nu_5$ with primitive generators

$$u_{\nu_0} = (1, 0, 0, 0, 0), \quad u_{\nu_1} = (0, 1, 0, 0, 0), \quad u_{\nu_2} = (0, 0, 1, 0, 0), \quad u_{\nu_3} = (0, 0, 0, 1, 0), \quad u_{\nu_4} = (0, 0, 0, 0, 1), \quad u_{\nu_5} = (-1, -1, -1, -1, -1).$$

Denote by $T_0, \ldots, T_5$ the six primitive Weil divisors corresponding to the rays $u_{\nu_0}, \ldots, u_{\nu_5}$ respectively. Then

$$O(\langle T_0 + T_1 + T_2 \rangle) = O(\langle T_3 + T_4 + T_5 \rangle) = O(-3),$$

and we can use the methods of §7.3 of [CLS1] again to construct a fan of $\text{tot}(O_{p^5}(-3) \oplus O_{p^5}(-3))$. This yields the fan $\Sigma_{p^5,T_0,T_b}$ in $\mathbb{R}^5 \oplus \mathbb{R}^2$ with the 8 rays $\nu_0, \ldots, \nu_5$, $\tau_1$ and $\tau_2$ having primitive ray generators

$$\begin{align*}
u_{\tau_1} &= (1, 0, 0, 0, 1, 0), & \nu_{\tau_2} &= (0, 0, 0, 1, 0, 1), \\
u_{\tau_3} &= (0, 1, 0, 0, 1, 0), & \nu_{\tau_4} &= (-1, -1, -1, -1, 0, 1), \\
u_{\tau_5} &= (0, 0, 1, 0, 0, 1), & \nu_{\tau_6} &= (0, 0, 0, 0, 0, 1).
\end{align*}$$

The fan $\Sigma_{p^5,T_0,T_b}$ is the star subdivision of $\text{Cone}(\nu_{\tau_0}, \ldots, \nu_{\tau_6}, \nu_{\tau_1}, \nu_{\tau_2})$ along $\nu_{\tau_1}$ and $\nu_{\tau_2}$ (noting the abuse of notation by which $u_{\tau_1}$ represent the same vector in both lattices $M, N$). The dual cone to $\Sigma_{p^5,T_0,T_b}$ in $\mathbb{R}^5 \oplus \mathbb{R}^2$ is spanned by the 12 rays $\overrightarrow{\nu_0}, \ldots, \overrightarrow{\nu_T}$ defined in §3.1 (page 10).

We recall that each lattice point in the interior of the dual cone corresponds to a global function of $X_{\Sigma_{p^5,T_0,T_b}}$ by associating $m$ to the monomial

$$x^m := \prod_{\rho \in \Sigma_{p^5,T_1,T_2}} x_{\rho}^{(m,u_{\rho})}.$$

Now a section $s_1 \oplus s_2 \in \Gamma(\mathbb{P}^5, O(3) \oplus O(3))$ will correspond to a global function on $\text{tot}(O(-3) \oplus O(-3))$ of the form $u_1s_1 + u_2s_2$, where $u_i$ is the variable corresponding to $u_{\tau_i}$. Recalling the polynomials $Q_{1, \lambda}, Q_{2, \lambda}$ from §3.2 we would like to express the global function $F := u_2Q_{1, \lambda} + u_1Q_{2, \lambda}$ as a linear combination of global functions of the form $x^m$. We do this by finding the lattice points in the dual cone corresponding to each monomial in $F$.

By splitting it up into its monomials, $u_2Q_{1, \lambda}$ corresponds to the 4 points $(3, 0, 0, -1, -1, 0, 1), (0, 3, 0, -1, -1, 0, 1)$, $u_1Q_{2, \lambda}$ corresponds to the points $(-1, -1, -1, 3, 0, 1, 0), (-1, -1, -1, 0, 3, 1, 0), (-1, -1, -1, 0, 1, 0)$ and $(0, 0, 0, 0, 1, 0)$.

We find that these 8 points are the primitive generators for the rays of $\Sigma_{LT,D_a,D_b}$ (see Corollary 3.7).

Quotienting $M_\mathbb{R} \oplus \mathbb{R}^2$ by the rays associated to the bundle coordinates (i.e. the lattice points that are the elements of the dual basis dual to $u_{\tau_1}$ and $u_{\tau_2}$) corresponds to a toric morphism $X_{\Sigma_{LT,D_a,D_b}} \to X_{\Sigma_{LT}}$. We emphasize that the dual cone to $\text{Cone}(\Sigma_{p^5,T_0,T_b}(1))$ is given by $\text{Conv}(u_{\overrightarrow{\nu_0}}, \ldots, u_{\overrightarrow{\nu_T}})$. Here, we take a subcone generated by a subset of $\{u_{\overrightarrow{\nu_0}}, \ldots, u_{\overrightarrow{\nu_T}}\}$. 
3.3.2. Expressing the zero locus of $Q_{1,\lambda}, Q_{2,\lambda}$. We remark that the cone $|\Sigma_{LT,D_a,D_b}|$ is not a reflexive Gorenstein cone, hence the Batyrev-Borisov construction does not apply to it.

The variety $V_{LT,\lambda} \subseteq X_{LT}$ is the zero-locus of the polynomials $Q_{1,\lambda}, Q_{2,\lambda}$, where $Q_{1,\lambda} \oplus Q_{2,\lambda}$ is a section of the vector bundle constructed above in Corollary 3.7. Proceeding in the same way as in §3.3.1, we consider lattice points on the cone $|\Sigma_{LT,D_a,D_b}| \subseteq \mathbb{N}_R \oplus \mathbb{R}^2$ to get global functions of $X_{LT,D_a,D_b}$. The cone $|\Sigma_{LT,D_a,D_b}| \cdot \nu$ is the cone over the convex hull of the following 12 points:

\[
\begin{align*}
(1,0,0,0,0,1,0), & \quad (0,1,0,0,0,1,0), & \quad (0,0,1,0,0,1,0), \\
(0,0,0,1,0,0,1), & \quad (0,0,0,0,1,0,1), & \quad (2, -1, -1, 0, 0, 0, 3), \\
(-1,2, -1, 0, 0, 0, 3), & \quad (-1, -1, 2, 0, 0, 0, 3), & \quad (1, 1, 3, 0, 3, 0), \\
(1,1,1,0,3,3,0), & \quad (-1, -1, -1, -1, -1, 0, 1), & \quad (-2, -2, -2, -3, -3, 3, 0).
\end{align*}
\]

The points corresponding to the monomials in $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$, and hence to the section $Q_{1,\lambda} \oplus Q_{2,\lambda}$, are the lattice points $u_{\mathcal{P}}$ and $u_{\mathcal{P}'}$ in (13). Later on, describing $V_{LT}$ by these 8 points will allow us to work with $D^b(coh V_{LT})$, using results in [FK19].

**Remark 3.8.** In their recent work [Ros21a, Ros21b], Rossi proposes a generalisation of the Batyrev-Borisov mirror construction, called **framed duality** (f-duality). f-duality provides an algorithm to obtain mirror candidates of hypersurfaces and complete intersections in toric varieties. Applying f-duality to $V_{LT} \subset \mathbb{P}^5 / G_{81}$ produces $V_{\lambda} \subset \mathbb{P}^5$, which in turn gives the same mirror as the Batyrev-Borisov construction when applying f-duality to it. Theorem 1.1 suggests that different mirror candidates obtained via f-duality may be derived equivalent and prompts the question under what conditions this is the case.

4. A derived equivalence between the constructions by Libgober-Teitelbaum and Batyrev-Borisov

Here we will prove the main result, Theorem 1.1.

4.1. Picking a partial compactification. Looking at the dual of the fan $\Sigma_{LT,D_a,D_b}$ as in Corollary 3.7, we recall from §3.3.2 that the global function $u_1 Q_{1,\lambda} + u_2 Q_{2,\lambda}$ corresponds to the points

\[
\begin{align*}
(1,0,0,0,0,1,0), & \quad (0,1,0,0,0,1,0), & \quad (0,0,1,0,0,1,0), \\
(0,0,0,1,0,0,1), & \quad (0,0,0,0,1,0,1), & \quad (-1, -1, -1, -1, -1, 0, 1), \\
(0,0,0,0,0,0,1), & \quad (0,0,0,0,0,1,0), & \quad (0,0,0,0,0,1,0).
\end{align*}
\]

Consider the GKZ fan of $\text{tot}(\mathcal{O}_{X_{\mathcal{P}}}(D'_{\mathcal{P}}) \oplus \mathcal{O}_{X_{\mathcal{P}}}(D'_{\mathcal{P}}))$. We note that the chambers of this GKZ fan correspond to regular triangulations of the polytope $\mathcal{P} = \text{Conv}(\mathcal{C})$, where $\mathcal{C}$ is the collection of the following 14 points:

\[
\begin{align*}
P_0 & = (3,0,0, -1, -1, 0, 1), & \quad P_6 & = (2, -1, -1, 0, 0, 1, 0), \\
P_1 & = (0,3,0, -1, -1, 0, 1), & \quad P_7 & = (-1,2, -1, 0, 0, 1, 0), \\
P_2 & = (0,0,3, -1, -1, 0, 1), & \quad P_8 & = (-1, -1, 2, 0, 0, 1, 0), \\
P_3 & = (-1, -1, -1, 3, 0, 1, 0), & \quad P_9 & = (0,0,0,2, -1, 0, 1), \\
P_4 & = (-1, -1, -1, 0, 3, 1, 0), & \quad P_{10} & = (0,0,0, -1, 2, 0, 1), \\
P_5 & = (-1, -1, -1, 0, 0, 1, 0), & \quad P_{11} & = (0,0,0, -1, -1, 0, 1), \\
S_1 & = (0,0,0,0,0,1,0), & \quad S_2 & = (0,0,0,0,0,0,1).
\end{align*}
\]
In the (regular) triangulations of $\Psi$, we look for a subtriangulation corresponding to $\Sigma_{LT,D_a,D_b}$, as then we obtain a partial compactification of $\text{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ from Corollary 3.7.

**Proposition 4.1.** There exists a chamber $\sigma_{LT}$ in the GKZ fan of $\text{tot}(\mathcal{O}_{X_{\mathcal{C}}}(-D'_b) \oplus \mathcal{O}_{X_{\mathcal{C}}}(-D'_a))$ (from Corollary 3.4) so that the triangulation $\mathcal{T}$ corresponding to the chamber $\sigma_{LT}$ (in the sense of 2.11) has the following properties:

- $\mathcal{T}$ contains the following set of simplices, listed via their vertices:
  $$\mathcal{T}_0 := \{\{P_i, S_1, S_2 \mid i \in I\} \mid I \subset \{0, 2, \ldots, 5\}, |I| = 5\}.$$

- Any simplex $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills either of the following conditions:
  
  (A) $S_1, P_0, P_7, P_8 \notin T$ and $\exists 3 \leq j \leq 6$ such that $P_j, P_{6+j} \notin T$.
  
  (B) $S_2, P_0, P_{10}, P_{11} \notin T$ and $\exists 0 \leq j \leq 2$ such that $P_j, P_{6+j} \notin T$.

Moreover, the toric variety $X_{\Sigma}$ corresponding to the chamber $\sigma_{LT}$ is a partial compactification of the variety $\text{tot}(\mathcal{O}_{X_{LT}}(-D_b) \oplus \mathcal{O}_{X_{LT}}(-D_a))$ from Corollary 3.7.

The first property of $\mathcal{T}$ means that the associated variety $X_{\Sigma}$ is a partial compactification of $X_{\mathcal{P}}$, so proving the existence of the triangulation $\mathcal{T}$ is sufficient to prove the Proposition. The second property of $\mathcal{T}$ is not a natural one to consider, but will become necessary to apply results from $\S$ 2.5.

The proposition can be checked via a simple SAGE program [The21] using the TOPCOM package [Ram02], however, we include an explicit proof on how such a triangulation can be constructed.

To prove the proposition, we break the statement up into 3 steps.

**Step 1:** We start by defining an explicit regular polyhedral subdivision $\mathcal{S}$ of $\Psi$ containing $\mathcal{T}_0$.

**Step 2:** We prove that the polyhedral subdivision $\mathcal{S}$ can be refined to a regular triangulation $\mathcal{T}$ of $\Psi$ containing $\mathcal{T}_0$.

**Step 3:** We show that any regular triangulation obtained this way fulfills the conditions outlined in the Proposition.

4.1.1. **Step 1:** We note that $\mathcal{T}_0$ is a regular triangulation of the set of points $P_0, \ldots, P_5, S_1, S_2$. It is in fact a star subdivision with respect to $S_1$, $S_2$ of the convex hull $\text{Conv}(P_0, \ldots, P_5, S_1, S_2)$. Indeed, an example of an explicit weight function $w$ giving the triangulation $\mathcal{T}_0$ is $w(S_1) = w(S_2) = 1$, $w(P_i) = 2$ for $0 \leq i \leq 5$. To complete Step 1, we extend this weight function to all 14 points of $\mathcal{C}$.

Consider the weight function $w(P_i) = 2$ for $0 \leq i \leq 5$, $w(S_1) = w(S_2) = 1$ ($j = 1, 2$) and $w(P_j) = 5$ for $6 \leq j \leq 11$. The convex hull of the points

$$Z_i = (P_i, w(P_i)), \quad R_j = (S_j, w(S_j)), \quad (0 \leq i \leq 11, j = 1, 2)$$

then forms a polyhedron $Q$ in $\mathbb{R}^8$. To obtain the regular subdivision of $\Psi$ corresponding to the weight function $w$, we need to project the lower facets of the polyhedron $Q$ down to $\mathbb{R}^7$ along the last coordinate. A lower facet is defined to be a facet of $Q$ where the inward pointing normal has a positive last coordinate.

We claim that there are exactly 12 lower facets of $Q$. We write each lower facet $F_i$ in the form $u_i \cdot x + a_i = 0$ where $u_i$ is the inward pointing normal of the $i^{th}$ facet. Take $H_i$ to be the halfspace corresponding to the lower facet $F_i$, i.e. the halfspace given by $u_i \cdot x + a \geq 0$. The normals and additive constants are:

- $H_0 : (5, -1, -1, 0, 0, 0, 0, 3)x - 3 \geq 0$
This is a direct consequence of the fact that \( \hat{F}_i \) obtained by projecting the facet \( F_i \) down to \( \mathbb{R}^7 \) along the last coordinate. Denoting by \( \hat{F}_i \) the polyhedron obtained by projecting the facet \( F_i \), we obtain the set of 12 polyhedra given in Table 3. We note here that when projecting, all points that lied on the facet \( F_i \) lie in the polyhedron \( \hat{F}_i \), by convexity of the polyhedron \( \mathcal{Q} \) in \( \mathbb{R}^8 \).

\[
\begin{align*}
\mathcal{F}_0 &= \text{Conv}(P_1, \ldots, P_5, S_1, S_2) \\
\vdots & \quad \vdots \\
\hat{\mathcal{F}}_5 &= \text{Conv}(P_0, \ldots, P_4, S_1, S_2) \\
\mathcal{F}_6 &= \text{Conv}(P_1, \ldots, P_5, S_1, P_7, P_8) \\
\hat{\mathcal{F}}_7 &= \text{Conv}(P_0, P_2, \ldots, P_5, S_1, P_6, P_8) \\
\hat{\mathcal{F}}_8 &= \text{Conv}(P_0, P_1, P_3, P_4, P_5, S_1, P_6, P_7) \\
\mathcal{F}_9 &= \text{Conv}(P_0, P_1, P_2, P_4, P_5, S_2, P_{10}, P_{11}) \\
\hat{\mathcal{F}}_{10} &= \text{Conv}(P_0, \ldots, P_3, P_5, S_2, P_9, P_{11}) \\
\hat{\mathcal{F}}_{11} &= \text{Conv}(P_0, \ldots, P_4, S_2, P_9, P_{10}).
\end{align*}
\]

Table 3. Polyhedra in the regular subdivision.

| Facet | contains |
|-------|----------|
| \( F_0 \) | \( Z_1, \ldots, Z_5, R_1, R_2 \) |
| \( \vdots \) | \( \vdots \) |
| \( F_5 \) | \( Z_0, \ldots, Z_4, R_1, R_2 \) |
| \( F_6 \) | \( Z_1, \ldots, Z_5, R_1, Z_7, Z_8 \) |
| \( F_7 \) | \( Z_0, Z_2, \ldots, Z_5, R_1, Z_6, Z_8 \) |
| \( F_8 \) | \( Z_0, Z_1, Z_3, Z_4, Z_5, R_1, Z_6, Z_7 \) |
| \( F_9 \) | \( Z_0, Z_1, Z_2, Z_4, Z_5, R_2, Z_{10}, Z_{11} \) |
| \( F_{10} \) | \( Z_0, Z_1, Z_2, Z_3, Z_5, R_2, Z_9, Z_{11} \) |
| \( F_{11} \) | \( Z_0, \ldots, Z_4, R_2, Z_9, Z_{10} \) |

Table 2. Dictionary of points contained in each lower facet of \( \mathcal{Q} \).

- \( H_1 : (-1, 5, -1, 0, 0, 0, 0, 3)x - 3 \geq 0 \)
- \( H_2 : (-1, -1, 5, 0, 0, 0, 0, 3)x - 3 \geq 0 \)
- \( H_3 : (1, 1, 1, 6, 0, 0, 0, 3)x - 3 \geq 0 \)
- \( H_4 : (1, 1, 1, 0, 6, 0, 0, 3)x - 3 \geq 0 \)
- \( H_5 : (-5, -5, -5, -6, -6, 0, 0, 3)x - 3 \geq 0 \)
- \( H_6 : (3, -1, -1, 0, 0, 0, 2, 1)x - 1 \geq 0 \)
- \( H_7 : (-1, -3, -1, 0, 0, 0, 2, 1)x - 1 \geq 0 \)
- \( H_8 : (-1, -1, 3, 0, 0, 0, 2, 1)x - 1 \geq 0 \)
- \( H_9 : (1, 1, 1, 4, 0, 0, -2, 1)x + 1 \geq 0 \)
- \( H_{10} : (1, 1, 1, 0, 4, 0, -2, 1)x + 1 \geq 0 \)
- \( H_{11} : (-3, -3, -3, -4, -4, 0, -2, 1)x + 1 \geq 0 \).

An easy computation shows that all 14 points lie in the intersection of the relevant half-spaces. This is a direct consequence of the fact that \( \mathcal{Q} \subseteq H_i \) for \( i = 0, \ldots, 11 \). Table 2 shows which points lie on each lower facet.
It remains to show that the above collection $F_i$ contains all the lower facets of $Q$. Showing that there is no other lower facet of $Q$ apart from $F_0, \ldots, F_{11}$ is equivalent to showing that $\bigcup F_i + \{(0, \ldots, 0, 1)\}_{\mathbb{R}_{\geq 0}}$ contains the entire polyhedron $Q$. Since all vertices of $Q$ lie inside each half-space $H_i$, it suffices to show that the union of the projections $\hat{F}_i$ contains the convex hull of $P_0, \ldots, P_{11}, S_1, S_2$, i.e. contains $\mathfrak{P}$. This is equivalent to saying that they give a polyhedral subdivision (regularity is given by construction).

So we aim to prove the following claim.

**Lemma 4.2.** For $\hat{F}_i$ and $\mathfrak{P}$ as above, we have $\bigcup_{i=0}^{11} \hat{F}_i = \mathfrak{P}$.

To prove Lemma 4.2, we will need the following result.

**Lemma 4.3.** Suppose we are given a set of $m$ inequalities $L_j \leq R_j$ with $\sum_{j=1}^{m} L_j \leq C \leq \sum_{j=1}^{m} R_j$, then there exists an $m$-tuple of real numbers $a_j$ such that $L_j \leq a_j \leq R_j$ and $\sum_{j=1}^{m} a_j = C$.

**Proof.** To show that the claim holds, we define $a_j(x) = L_j + x(R_j - L_j)$. This is a linear function such that, for all $x \in [0, 1]$, $L_j \leq a_j(x) \leq R_j$. Define $f(x) = \sum a_j(x)$. $f$ is itself linear and thus continuous in $x$, with $f(0) = \sum_{j=1}^{m} L_j \leq C \leq \sum_{j=1}^{m} R_j = f(1)$. By the intermediate value theorem, there is an $x_C \in [0, 1]$ such that $f(x) = \sum_{j=1}^{m} a_j(x_C) = C$. Setting $a_j = a_j(x_C)$ gives the $m$-tuple, proving the claim. \Box

**Proof of Lemma 4.2.** The first thing to note is that

$$\mathfrak{P} = \text{Conv}(P_0, \ldots, P_{11}, S_1, S_2) = \text{Conv}(P_0, \ldots, P_{11}).$$

So we will show that $\bigcup_{i=0}^{11} \hat{F}_i = \text{Conv}(P_0, \ldots, P_{11})$.

We start by showing that $\bigcup_{i=0}^{5} \hat{F}_i = \text{Conv}(P_0, \ldots, P_5, S_1, S_2)$, which is equivalent to saying that $\hat{F}_0, \ldots, \hat{F}_5$ form a polyhedral subdivision of $\text{Conv}(P_0, \ldots, P_5, S_1, S_2)$.

The inclusion $\subseteq$ is immediate from Table 3, so it remains to check the opposite inclusion. Any point $X \in \text{Conv}(P_0, \ldots, P_5, S_1, S_2)$ can be written as $X = \sum_{i=0}^{5} \lambda_i P_i + \mu_1 S_1 + \mu_2 S_2$ for some $\lambda_i, \mu_j \in \mathbb{R}_{\geq 0}$ with $\sum \lambda_i + \mu_1 + \mu_2 = 1$. Note also that $\sum_{i=0}^{5} P_i = 3(S_1 + S_2)$. Now define $j$ such that $\lambda_j = \min_{0 \leq i \leq 5} \{\lambda_i\}$. Then

$$X = \sum_{i=0}^{5} (\lambda_i - \lambda_j)P_i + (3\lambda_j + \mu_1)S_1 + (3\lambda_j + \mu_2)S_2 = \sum_{0 \leq i \leq 5, i \neq j} (\lambda_i - \lambda_j)P_i + (3\lambda_j + \mu_1)S_1 + (3\lambda_j + \mu_2)S_2.$$

Since $\lambda_j = \min_{0 \leq i \leq 5} \{\lambda_i\} \leq \lambda_i$ for $0 \leq i \leq 5$, we have that $(\lambda_i - \lambda_j) \geq 0$ for $0 \leq i \leq 5$. As $\lambda_i, \mu_1, \mu_2 \geq 0$, we also have $3\lambda_j + \mu_1, 3\lambda_j + \mu_2 \geq 0$. Also,

$$\sum_{0 \leq i \leq 5, i \neq j} (\lambda_i - \lambda_j) + (3\lambda_j + \mu_1) + (3\lambda_j + \mu_2) = \sum_{i=0}^{5} \lambda_i + \mu_1 + \mu_2 = 1,$$

and thus $X \in \hat{F}_j$. This shows $\bigcup_{i=0}^{5} \hat{F}_i = \text{Conv}(P_0, \ldots, P_5, S_1, S_2)$.

To show $\bigcup_{i=0}^{11} \hat{F}_i = \mathfrak{P}$, we note again that the inclusion $\subseteq$ is immediate. For the opposite inclusion $\supseteq$, take a general point $X$ in $\mathfrak{P}$. Then $X$ can be written as $X = \sum_{i=0}^{11} \lambda_i P_i$ with $\lambda_i \geq 0$ for $0 \leq i \leq 11$ and $\sum_{i=0}^{11} \lambda_i = 1$.

Without loss of generality, assume that $(\lambda_6 + \lambda_7 + \lambda_8) \geq (\lambda_9 + \lambda_{10} + \lambda_{11})$ (the case where the inequality is reversed is analogous). We will now show that if $X \not\in \bigcup_{i=0}^{5} \hat{F}_i =$
Conv\((P_0, \ldots, P_5, S_1, S_2)\), then \(X \in \bigcup_{i=6}^{8} \hat{F}_i\) (if the inequality had been reversed, then \(X\) would be in \(\bigcup_{i=9}^{11} \hat{F}_i\)).

Let
\[
\nu_i = \lambda_i + \lambda_{6+i} - \frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})) \quad \text{for } 0 \leq i \leq 2,
\]
\[
\nu_i = \lambda_i + \lambda_{6+i} \quad \text{for } 3 \leq i \leq 5,
\]
\[
\mu_1 = ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})),
\]
\[
\mu_2 = 0.
\]

Then
\[
\sum_{i=0}^{5} \nu_i P_i + \mu_1 S_1 + \mu_2 S_2 = \sum_{i=0}^{11} \lambda_i P_i
\]
and
\[
\sum_{i=0}^{5} \nu_i + \mu_1 + \mu_2 = \sum_{i=0}^{11} \lambda_i = 1.
\]

Note \(\mu_1 \geq 0\) by assumption and \(\mu_2 = 0\). Thus, if \(\nu_i \geq 0\) for \(0 \leq i \leq 5\), \(X\) is expressed as an element of \(\text{Conv}(P_0, \ldots, P_5, S_1, S_2) = \bigcup_{i=0}^{5} \hat{F}_i\) using the above equations. Otherwise, we will claim that \(X \in \bigcup_{i=0}^{5} \hat{F}_i\). For \(3 \leq i \leq 5\), we have \(\nu_i \geq 0\) as both \(\lambda_i\) and \(\lambda_{6+i}\) are \(\geq 0\). We turn our attention to the \(\nu_i\) for \(i = 0, 1, 2\).

For \(0 \leq i \leq 2\), \(\nu_i \geq 0\) is equivalent to
\[
\frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})) \leq \lambda_i + \lambda_{6+i},
\]
so the condition that all \(\nu_i\) are non-negative is equivalent to
\[
\frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})) \leq \min_{0 \leq i \leq 2} \{\lambda_i + \lambda_{6+i}\}.
\]

Therefore, \(X \in \bigcup_{i=0}^{5} \hat{F}_i = \text{Conv}(P_0, \ldots, P_5, S_1, S_2)\) if \(\frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})) \leq \min_{0 \leq i \leq 2} \{\lambda_i + \lambda_{6+i}\}\). Suppose this condition does not hold, i.e.
\[
\min_{1 \leq i \leq 3} \{\lambda_i + \lambda_{6+i}\} < \frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_9 + \lambda_{10} + \lambda_{11})). \tag{14}
\]

Without loss of generality, we may assume that \(\lambda_0 + \lambda_6 = \min_{0 \leq i \leq 2} \{\lambda_i + \lambda_{6+i}\}\) (by symmetry, the other cases are analogous). We will show that \(X \in \hat{F}_6\). Any point \(Y\) in \(\hat{F}_6 = \text{Conv}(P_1, \ldots, P_5, S_1, P_7, P_8)\) can be written as
\[
Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1.
\]
If we find \(\nu_i, \mu_1\) such that this sum is equal to \(\sum_{i=0}^{11} \lambda_i P_i = X\), we are done as we will have expressed \(X\) as an element of \(\hat{F}_6\).

Given a choice of real numbers \(\alpha_1, \alpha_2\) with \(\alpha_1 + \alpha_2 = 1\), define
\[
\nu_i = \lambda_i + \alpha_1 (3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})) - (\lambda_0 + \lambda_6) \quad \text{for } 1 \leq i \leq 2,
\]
\[
\nu_i = \lambda_i + \lambda_{6+i} \quad \text{for } 3 \leq i \leq 5,
\]
\[
\mu_1 = 3\lambda_0 + 3\lambda_6,
\]
\[
\nu_{6+i} = \lambda_{6+i} + \alpha_i (-3\lambda_0 - 2\lambda_6 - (\lambda_9 + \lambda_{10} + \lambda_{11})) \quad \text{for } 1 \leq i \leq 2.
\]
Substituting these values into the expression for $Y$ gives

$$Y = \sum_{i=1}^{5} \nu_i P_i + \sum_{i=7}^{8} \nu_i P_i + \mu_1 S_1 = \sum_{i=0}^{11} \lambda_i P_i = X,$$

as well as

$$\sum_{i=0}^{11} \nu_i + \mu_1 = \sum_{i=0}^{11} \lambda_i = 1.$$

For this choice of $\nu_i$’s and $\mu_1$ to define an element $Y \in \widehat{F}_6$, we require $\nu_i \geq 0$ for all $i$ and $\mu_1 \geq 0$. We note that, as $\lambda_0, \lambda_6 \geq 0$, we have $\mu_1 \geq 0$.

Therefore, what remains to prove is that there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$ such that $\nu_i \geq 0$ for $i \in \{1, \ldots, 5, 7, 8\}$. For $i = 1, 2$, we can arrange the inequalities $\nu_i \geq 0$ and $\nu_{6+i} \geq 0$ to give

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq \alpha_i \leq \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})}. \quad (15)$$

This works provided $3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11}) \neq 0$ but if that term was zero, then by non-negativity of the $\lambda_i$ we would have $\lambda_0 = \lambda_6 = \lambda_9 = \cdots = \lambda_{11} = 0$ and thus $X \in \widehat{F}_6$. So if there exists a pair $(\alpha_1, \alpha_2)$ with (15) holding for $i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$, then $X \in \widehat{F}_6$.

Note that for all $i$,

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})},$$

as $\lambda_0 + \lambda_6 = \min_{0 \leq i \leq 2} \{\lambda_i + \lambda_{6+i}\}$.

Furthermore,

$$0 \leq \frac{2\lambda_0 + 2\lambda_6}{3\lambda_0 + \lambda_1 + \lambda_2 + \lambda_9 + \lambda_{10} + \lambda_{11}} \quad \equiv \quad \sum_{i=1}^{2} \frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq 1.$$

Lastly, we are given that $\lambda_0 + \lambda_6 = \min_{0 \leq i \leq 2} \{\lambda_i + \lambda_{6+i}\} \leq \frac{1}{3} ((\lambda_6 + \lambda_7 + \lambda_8) - (\lambda_6 + \lambda_{10} + \lambda_{11}))$.

This leads to the following sequence of implications:

$$\begin{align*}
& \quad 3\lambda_0 + 2\lambda_6 + \lambda_9 + \lambda_{10} + \lambda_{11} \leq \lambda_7 + \lambda_8 \\
& \quad 1 \leq \sum_{i=1}^{2} \frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})}.
\end{align*}$$

In summary, we have shown that for $i = 1, 2$, we have

$$\frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})},$$

and that

$$\sum_{i=1}^{2} \frac{\lambda_0 + \lambda_6 - \lambda_i}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})} \leq 1 \leq \sum_{i=1}^{2} \frac{\lambda_{6+i}}{3\lambda_0 + 2\lambda_6 + (\lambda_9 + \lambda_{10} + \lambda_{11})}.$$

Applying Lemma 4.3 gives us the existence of a pair $\alpha_1, \alpha_2$ as required, concluding the proof to Lemma 4.2. □
The Lemma 4.2 shows that we have indeed found all lower facets of the polyhedron $Q$, meaning that the collection $\hat{F}_1, \ldots, \hat{F}_{11}$ gives a regular polyhedral subdivision $S$ of $\mathfrak{Q}$, thus concluding Step 1.

4.1.2. Step 2: This is true by general convex geometry (using the poset of refinements and the secondary polytope). By Theorem 2.4 in Chapter 7 of [GKZ94], the poset of (non-empty) faces of the secondary polytope $\Sigma(\mathfrak{Q})$ is isomorphic to the poset of all regular subdivisions of $\mathfrak{Q}$, partially ordered by refinement (see also Theorem 16.4.1 in [GOT18]). The vertices of $\Sigma(\mathfrak{Q})$ correspond to regular triangulations. Thus, our regular subdivision obtained by projection must correspond to some face of $\Sigma(\mathfrak{Q})$ and any vertex of that face will correspond to a regular triangulation refining it.

4.1.3. Step 3: Consider a regular triangulation $\mathcal{T}$ obtained by refining $S$. By definition, it is a regular triangulation of $\mathfrak{Q}$. Recall Table 3 Denote by $C_i$ the collection of points to define the polyhedron $\hat{F}_i$ in the table. Note that $\hat{F}_0, \ldots, \hat{F}_5$ are the simplices in $\mathcal{T}_0$, and therefore any simplices in $\mathcal{T} \setminus \mathcal{T}_0$ do not originate from refining any of $\hat{F}_0, \ldots, \hat{F}_5$.

Thus the last step of the proof reduces to showing that none of the polyhedra $\hat{F}_i$, $0 \leq i \leq 11$, contain any of the points we did not define it by, i.e. $\hat{F}_i \cap \mathcal{C} = C_i$. Indeed, in that case we note that, by consulting Table 3 the polyhedra $\hat{F}_i$ each fulfill at least one of the conditions $A$ or $B$ in the proposition. If $\hat{F}_i \cap \mathcal{C} = C_i$, then all simplices in a refinement of $\hat{F}_i$ are defined as the convex hull of a subset of $C_i$ (as there is no interior point to refine upon), thus inheriting the properties $A$ or $B$ from $\hat{F}_i$.

Showing that $\hat{F}_i \cap \mathcal{C} = C_i$ for $6 \leq i \leq 11$ reduces to a simple computation. We shall do the computation for $\hat{F}_6$, as the remaining cases are analogous by symmetry.

We need to show that $P_0, P_6, P_9, P_{10}, P_{11}, S_1 \not\in \hat{F}_6$. Any point $X$ in $\hat{F}_6$ can be written as

$$
\begin{align*}
\lambda_1 P_1 + \cdots + \lambda_5 P_5 + \mu_1 S_1 + \lambda_7 P_7 + \lambda_8 P_8 &= (-\lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 - \lambda_8, \\
3\lambda_1 - \lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_7 - \lambda_8, 3\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_7 + 2\lambda_8, -\lambda_1 - \lambda_2 + 2\lambda_3, \\
\lambda_1 + \lambda_2 + 2\lambda_4, \lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8, \lambda_1 + \lambda_2),
\end{align*}
$$

with $\lambda_1, \mu_1 \geq 0$ and $\sum \lambda_i + \mu_1 = 1$. We note that the last two coordinates of $X$ are $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8$ and $\lambda_1 + \lambda_2$, respectively. Assume $P_0 \in \hat{F}_6$ and had an expression as in Equation (16). Then, as $\lambda_1, \mu_1 \geq 0$, we can see by looking at the last two coordinates that $\lambda_3 = \lambda_4 = \lambda_5 = \mu_1 = \lambda_7 = \lambda_8 = 0$ and $\lambda_1 + \lambda_2 = 1$. But then the first coordinate is $\lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0 \neq 2$, hence we get a contradiction and $P_0 \not\in \hat{F}_6$. By an analogous reasoning, for $S_2, P_9, P_{10}, P_{11}$ we obtain that all but $\lambda_1, \lambda_2$ would need to be 0 again and the sum of these two would need to be 1, which means that not both the second and third coordinate (being $3\lambda_1, 3\lambda_2$) can be 0. Hence $S_2, P_9, P_{10}, P_{11} \not\in \hat{F}_6$.

Finally, we need to show $P_6 \not\in \hat{F}_6$. Assume we had an expression for $P_6$ as in Equation (16). Since $\lambda_1 \geq 0$, considering the last two coordinates gives $\lambda_1 = \lambda_2 = 0$ (since $\lambda_1 \geq 0$) and $\lambda_3 + \lambda_4 + \lambda_5 + \mu_1 + \lambda_7 + \lambda_8 = 1$. But then the first coordinate is $-(\lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8) \leq 0 < 2$, a contradiction. Thus $P_6 \not\in \hat{F}_6$, and thus $\hat{F}_6 \cap \mathcal{C} = C_6$ as claimed.

The other cases are analogous by symmetry. Thus we finished Step 3, hence proving Proposition 4.11.

4.2. The ideals associated to the partial compactification. Recalling the notation from §2.3 we denote by $x_i$ the variable in $\mathbb{C}[x_0, \ldots, x_{11}, u_1, u_2]$ corresponding to the point
Consider the group \(G\) \(\times 8\) points correspond to on \(X\). Recall that there is a correspondence between points in the dual cone \(C\) with this correspondence with the \(8\) points above, one obtains the superpotential \(c\) Teitelbaum, we choose \(I \subseteq \sqrt{10}\).

**Lemma 4.4.** There exists a global function on \(X\) \(\times\) \(D_0, D_6\) that has the form

\[
w = u_1(c_0x_0^3x_6^3 + c_1x_1^3x_7^3 + c_2x_2^3x_8^3 - 3\lambda_3x_3x_4x_5x_6x_7x_8) + u_2(c_3x_3^3x_9^3 + c_4x_4^3x_{10}^3 + c_5x_5^3x_{11}^3 - 3\lambda_2x_0x_1x_2x_9x_{10}x_{11}),
\]

for some \(c_i, \lambda_j \in \mathbb{C}\).

**Proof.** Consider the hyperplane \(H := \{(m, t_1, t_2) \in M_{\mathbb{R}} \oplus \mathbb{R}^2 \mid t_1 + t_2 = 1\}\) in \(M_{\mathbb{R}} \oplus \mathbb{R}^2\). The cone \(|\Sigma_{V, T_0, D_0}^1|\) is given by the cone the over the convex hull of the following \(8\) points on \(H\):

\[
(0, 0, 0, 1, 0, 1), \quad (1, -1, 1, -1, -1, 0, 1), \quad (0, 0, 0, 0, 1, 0, 1), \quad (0, 0, 1, 0, 0, 1, 0),
\]

\[
(0, 1, 0, 0, 0, 0, 1), \quad (0, 0, 0, 0, 0, 0, 1), \quad (1, 0, 0, 0, 0, 1, 0), \quad (0, 0, 0, 0, 0, 1, 0).
\]

Recall that there is a correspondence between points in the dual cone \(|\Sigma_{LT, D_0, D_6}^1|\) and global functions on \(X_{LT, D_0, D_6}\). Note that, when one constructs a superpotential for \(X_{LT, D_0, D_6}\) using this correspondence with the \(8\) points above, one obtains the superpotential \(w = u_1Q_{1,\lambda} + u_2Q_{2,\lambda}\). To see the global function on \(X_{V, T_0, D_0}^1\), it suffices to compute what monomials these \(8\) points correspond to on \(X_{V, T_0, D_0}^1\). This gives the \(8\) monomials in (17).

For our purposes of comparing the Batyrev-Borisov construction with the one by Libgober-Teitelbaum, we choose \(c_i = 1\) and \(\lambda_1 = \lambda_2 =: \lambda\).

We fix a triangulation \(T\) fulfilling the properties of Proposition 4.1. Let \(X = \mathbb{C}^{14}\) and consider the group \(G_X\) corresponding to the fan \(\Sigma_{V, T_0, D_0}^1\) with its action on \(X\). From the triangulation \(T\) we obtain the ideals:

\[
I := \left\langle \prod_{i \notin I} x_i \prod_{j \notin J} u_j \middle| \bigcup_{i \in I} u_{\overline{p}_i} \cup \bigcup_{j \in J} u_{\tau_j} \text{ give the set of vertices of a simplex in } T \right\rangle,
\]

\[
J := \left\langle \prod_{i \notin I} x_i \bigcup_{i \in I} u_{\overline{p}_i} \bigcup_{j = 1}^2 u_{\tau_j} \text{ give the set of vertices of a simplex in } T \right\rangle.
\]

Before we can apply Proposition 2.25 and Corollary 2.26, we need to ensure the condition \(I \subseteq \sqrt{\partial w, J}\) holds.

**Lemma 4.5.** For any triangulation \(T\) as in Proposition 4.1, defining \(I, J\) and \(w\) as above with \(\lambda^6 \neq 0, 1\), we have \(I \subseteq \sqrt{J, \partial w}\). Therefore, this choice of superpotential fulfills the condition of Proposition 2.25.

**Proof.** To show the containment \(I \subseteq \sqrt{\partial w, J}\), we prove that all the generators of \(I\) are in \(\sqrt{\partial w, J}\). The ideal \(I\) is, by definition, generated by the monomials which correspond to the simplices in the triangulation \(T\). For a simplex \(T \in T_0\), both \(S_1\) and \(S_2\) are vertices. Thus, by definition, the monomial associated to \(T\) is in \(J\) and hence in \(\sqrt{\partial w, J}\).

For any simplex \(T \in T \setminus T_0\), either condition \((A)\) or \((B)\) of Proposition 4.1 holds. We claim that the monomial associated to a simplex \(T\) fulfilling either of those two conditions is an element of \(\sqrt{\partial w}\) and therefore an element of \(\sqrt{\partial w, J}\).
We note that if $T \in \mathcal{T} \setminus \mathcal{T}_0$ fulfills condition (A), i.e. does not contain any of the points $S_1, P_6, P_7, P_8$ and there is a pair of points of the form $P_j, P_{6+j}$ with $3 \leq j \leq 5$ also not contained, then by definition $u_1 x_j x_{6+j} x_6 x_7 x_8$ divides the monomial associated to $T$. Similarly, if $T$ fulfilled condition (B) instead, $u_2 x_j x_{6+j} x_9 x_{10} x_{11}$ (for some $0 \leq j \leq 2$) would divide the monomial generator of $\mathcal{I}$ associated to $T$.

To show that any monomial associated to a simplex in $\mathcal{T} \setminus \mathcal{T}_0$ is in $\sqrt{\partial w} \subseteq \sqrt{\partial w, \mathcal{J}}$, it is thus sufficient to prove that the six monomials $u_2 x_6 x_7 x_8 x_{10} x_{11}, u_2 x_7 x_8 x_9 x_{10} x_{11}, u_2 x_8 x_9 x_{10} x_{11}, u_1 x_3 x_6 x_7 x_8, u_1 x_4 x_6 x_7 x_8$ and $u_1 x_5 x_{10} x_6 x_7 x_8$ are elements of $\sqrt{\partial w}$.

By symmetry of the $x_i$ in $w$, we note that it is sufficient to show that $u_2 x_6 x_7 x_8 x_{10} x_{11} \in \sqrt{\partial w}$. Start by explicitly writing down the ideal $\langle \partial w \rangle$, i.e. the ideal generated by the partial derivatives of $w$.

$$\langle \partial w \rangle = \langle 3 u_1 x_6^3 x_6^3 - 3 \lambda u_2 x_1 x_2 x_9 x_{10} x_{11}, 3 u_1 x_7^3 x_7^3 - 3 \lambda u_2 x_0 x_2 x_9 x_{10} x_{11}, 3 u_1 x_8^3 x_8^3 - 3 \lambda u_2 x_0 x_8 x_9 x_{10} x_{11}, 3 u_2 x_9^3 x_9^3 - 3 \lambda u_1 x_3 x_6 x_7 x_8, 3 u_2 x_{10}^3 x_{10}^3 - 3 \lambda u_1 x_3 x_6 x_7 x_8, 3 u_2 x_{11}^3 x_{11}^3 - 3 \lambda u_1 x_3 x_6 x_7 x_8, 3 u_1 x_0 x_6 x_7 x_8, 3 u_1 x_0 x_7 x_8 x_9, 3 u_1 x_0 x_8 x_9 x_{10} x_{11}, 3 u_2 x_0 x_6 x_7 x_8, 3 u_2 x_0 x_7 x_8 x_9, 3 u_2 x_0 x_8 x_9 x_{10} x_{11}, 3 u_1 x_0 x_6 x_7 x_8, 3 u_1 x_0 x_7 x_8 x_9, 3 u_1 x_0 x_8 x_9 x_{10} x_{11} \rangle.$$

We see that $3 u_1 x_i^3 x_{6+i}^3 - 3 \lambda u_2 x_i^3 x_{6+i}^3 x_{6+i} x_9 x_{10} x_{11} \in \langle \partial w \rangle$ for $0 \leq i \leq 2$. Notice that since $ac - bd = c(a - b) + b(c - d)$, if $a - b, c - d$ are elements in an ideal, then so is $ac - bd$. Hence by iterating this we obtain that

$$27 u_1 x_0^3 x_6^3 x_7^3 x_8^3 - 27 \lambda x_0^3 x_6^3 x_7^3 x_8^3 x_{10} x_{11} \in \langle \partial w \rangle.$$

Similarly,

$$27 u_2 x_6^3 x_7^3 x_8^3 x_{10} x_{11}^3 - 27 \lambda x_3 x_4 x_5 x_6^3 x_7^3 x_8^3 \in \langle \partial w \rangle.$$

Therefore,

$$\begin{align*}
(27)^2 u_i^3 x_6^3 x_7^3 x_8^3 x_{10} x_{11} - (27)^2 \lambda x_0^3 x_6^3 x_7^3 x_8^3 x_{10} x_{11} &\in \langle \partial w \rangle \\
\Rightarrow 27^2 (1 - \lambda^3) u_i^3 x_6^3 x_7^3 x_8^3 x_{10} x_{11} &\in \langle \partial w \rangle \\
\Rightarrow u_i^3 x_6^3 x_7^3 x_8^3 x_{10} x_{11} &\in \langle \partial w \rangle \\
\Rightarrow (u_i u_2 x_0 - \lambda x_0) x_i^3 &\in \langle \partial w \rangle.
\end{align*}
\tag{19}$$

Consider $\frac{\partial w}{\partial x_i}$, giving

$$x_0^3 x_6^3 + x_1^3 x_7^3 + x_2^3 x_8^3 - 3 \lambda x_0 x_4 x_5 x_6 x_7 x_8 \in \langle \partial w \rangle. \tag{20}$$

Furthermore, we note that $\sum_{i=0}^{2} x_i \frac{\partial w}{\partial x_i} \in \langle \partial w \rangle$, and thus

$$\sum_{i=0}^{2} (3 u_1 x_i^3 x_{6+i}^3 - 3 \lambda u_2 x_0 x_1 x_2 x_9 x_{10} x_{11}) \in \langle \partial w \rangle. \tag{21}$$

By (20) we have that $x_3 x_4 x_5 x_6 x_7 x_8 \in \langle \partial w \rangle = \frac{1}{3 \lambda} (x_0^3 x_6^3 + x_7^3 x_7^3 + x_2^3 x_8^3) + \langle \partial w \rangle$. We use this to substitute into (19) to obtain that

$$\frac{1}{27 \lambda^3} u_0 x_0^3 x_1^3 x_2^3 x_3^3 x_4 x_5 x_6 x_7 x_8 (u_1 (x_0^3 x_6^3 + x_7^3 x_7^3 + x_2^3 x_8^3))^3 \in \langle \partial w \rangle$$

Performing the same style of substitution with (21), we obtain
Thus $u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt[\partial w]{}$.

By comparing the elements $x_2 \frac{\partial w}{\partial x_2}, u_2x_0x_1x_2x_9x_{10}x_{11} \in \sqrt[\partial w]{}$, we obtain that $u_1x^3_2x^3_8 \in \sqrt[\partial w]{}$, implying $u_1x^2_8 \in \sqrt[\partial w]{}$. This, in turn, implies that $u_2x_0x_1x_9x_{10}x_{11} \in \sqrt[\partial w]{}$, by inspection of $\frac{\partial w}{\partial x_2} \in \sqrt[\partial w]{} \subseteq \sqrt[\partial w]{}$. Similarly, $u_2x_0x_2x_9x_{10}x_{11} \in \sqrt[\partial w]{}$. We also have $u_2x_5x_{11} \in \sqrt[\partial w]{}$ by an analogous computation. Finally, $\frac{\partial w}{\partial x_1} = x^3_0x^3_6 + x^3_1x^3_7 + x^3_2x^3_8 - 3\lambda x_3x_4x_5x_6x_7x_8 \in \sqrt[\partial w]{}$.

Therefore, one can intuit and then compute that

\[
\begin{align*}
(u_2x_0x_6x_9x_{10}x_{11})^4 &= -u^2_2x^2_6x^2_7x^3_9x^3_{10}x^{11}_1 \cdot (u_2x_0x_1x_9x_{10}x_{11}) \\
&- u^3_2x^2_5x^2_6x^3_8x^4_{10}x^{11}_1 \cdot (u_2x_0x_2x_9x_{10}x_{11}) \\
&+ 3\lambda u^3_2x_0x_3x_4x^2_6x^2_7x^3_8x^4_{10}x^{11}_1 \cdot (u_2x_5x_{11}) \\
&+ u^4_2x_0x_6x^4_{10}x^4_{11} \cdot (x^3_0x^3_6 + x^3_1x^3_7 + x^3_2x^3_8 - 3\lambda x_3x_4x_5x_6x_7x_8) \in \sqrt[\partial w]{}
\end{align*}
\]

Thus, we have shown that $u_2x_0x_6x_9x_{10}x_{11} \in \sqrt[\partial w]{}$. By symmetry, any simplex fulfilling properties (A) or (B) corresponds to a monomial in $\sqrt[\partial w, J]{}$. Hence any monomial associated to a simplex $T \in T \setminus T_0$ is an element of $\sqrt[\partial w, J]{}$, concluding the proof that $T \subseteq \sqrt[\partial w, J]{}$. \qed

**Corollary 4.6.** Consider the GKZ fan of tot $(O_{X_V}(D'_b) \oplus O_{X_V}(D'_a))$ and the group $G_\Sigma$ from above. There is a chamber $\sigma_p$ with affine open $U_p$ such that:

(i) $[U_p/G_\Sigma]$ is a partial compactification of tot $(O_{X_{LT}}(-D'_b) \oplus O_{X_{LT}}(-D'_a))$.

(ii) There is a superpotential corresponding to the eight points in $|\Sigma \nabla, D'_a, D'_b| \cap H$ taking the form $w = u_1(x^3_0x^3_6 + x^3_1x^3_7 + x^3_2x^3_8 - 3\lambda x_3x_4x_5x_6x_7x_8) + u_2(x^3_3x^3_9 + x^3_4x^3_{10} + x^3_5x^3_{11} - 3\lambda x_0x_1x_2x_9x_{10}x_{11})$.

(iii) With $I_p, J_p$ as defined in (2.3), we have $I_p \subseteq \sqrt[\partial w, J_p]{}$.

**Proof.** Proposition 4.1 proves (i), Lemma 4.4 proves (ii) and finally Lemma 4.5 shows (iii). \qed

### 4.3. Relating $X_V$ and $X_{LT}$. Recall that the partial compactification of the total space tot $(O_{X_{LT}}(-D'_b) \oplus O_{X_{LT}}(-D'_a))$ in Corollary 4.6 corresponds to a chamber $\sigma_p$ of the GKZ fan of tot $(O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a))$. We then know that it is birationally equivalent to tot $(O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a))$.

Thus we want to now explicitly find a triangulation of $\mathfrak{P}$ corresponding to the Batyrev-Borisov mirror family. There, the superpotential will take the form

\[
\begin{align*}
w &= u_1(x^3_0x^3_6 + x^3_1x^3_7 + x^3_2x^3_8 - 3\lambda x_3x_4x_5x_6x_7x_8) + u_2(x^3_3x^3_9 + x^3_4x^3_{10} + x^3_5x^3_{11} - 3\lambda x_0x_1x_2x_9x_{10}x_{11}).
\end{align*}
\]

Note that, by Lemma 4.4, this is the form the superpotential should take in the Batyrev-Borisov mirror. In other words, we need a chamber $\sigma_q$ in the GKZ fan corresponding to tot $(O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a))$, where a general section of $O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a)$ will yield a complete intersection in $X_V$, and thus a Batyrev-Borisov mirror.

**Lemma 4.7.** Consider the GKZ fan of tot $(O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a))$ and recall the group $G_\Sigma$ from above. There is a chamber $\sigma_q$ with affine open $U_q$ such that:

(i) $[U_q/G_\Sigma] = \text{tot} (O_{X_V}(-D'_b) \oplus O_{X_V}(-D'_a))$. 

(ii) A superpotential corresponding to the eight lattice points of $|\Sigma_{\Delta', D_a'}| \cap H$ is of the form
\[ w = u_1(x_0^3x_6^3 + x_3^3x_7^3 + x_2^3x_8^3 - 3\alpha x_3x_4x_5x_6x_7x_8) + u_2(x_3^3x_9^3 + x_4^3x_{10}^3 + x_5^3x_{11}^3 -
3\alpha x_0x_1x_2x_9x_{10}x_{11}).\]

(iii) For $I_q, J_q$ as defined in (2.5) $I_q \subseteq \sqrt{\partial w, J_q}$.

Proof. This proof will construct the triangulation $T_q$ corresponding to the chamber $\sigma_q$. We consider the 42 maximal cones from Table 1. For each of those cones $\sigma_i, 1 \leq i \leq 42$, we associate a simplex given as convex hull of the 5 vertices corresponding to the 5 rays of $\sigma_i$ plus the two vertices corresponding to the bundle coordinates, i.e. $(0, 0, 0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 0, 1, 0)$. So for example the first cone, with rays $\rho_0, \rho_1, \rho_2, \rho_9, \rho_{10}$, will correspond to the simplex with vertices $(3, 0, 0, -1, -1, 1, 0), (0, 3, 0, -1, -1, 1, 0), (0, 0, 3, -1, -1, 1, 0),$ $(0, 0, 0, 0, 0, 1, 0), (0, 0, 0, 2, -1, 1, 0), (0, 0, 0, -1, 2, 1, 0)$. Another way to formulate this is that we take the star subdivision of the cones from Table 1 on the two bundle points $S_1, S_2$.

Regularity of this triangulation of the 14 points is an easy consequence of its construction as a star subdivision, hence it corresponds to some chamber $\sigma_q$ in the GKZ-fan. Indeed, the star subdivision can be obtained by giving the points $S_1, S_2$ a weight of 1 and giving all other points the same weight of $w = 2$ and then refining the resulting regular polyhedral subdivision into a triangulation. Alternatively, one can check the regularity of this triangulation by using SAGE.

The third item follows from the fact that we do not partially compactify, hence $J_q = I_q$ and therefore $I_q \subseteq \sqrt{\partial w, J_q}$, as required.

We now have all the necessary tools to prove the main result of this paper, Theorem 1.1.

Proof of Theorem 1.1. Recall the chambers $\sigma_p$ and $\sigma_q$ in the GKZ fan of the toric variety $\text{tot} (\mathcal{O}_{XV}(-D'_b) \oplus \mathcal{O}_{XV}(-D'_a))$ given in Corollary 1.6 and Lemma 4.7. By applying Corollary 2.26 we have $\text{D}^b(\text{coh } Z_\lambda) \cong \text{D}^b(\text{coh } V_{LT, \lambda})$, as required.

We note that analogous computations to the ones displayed in this paper can yield the following result in lower dimension.

Theorem 4.8. Let $Q_1 = x_1^2 + x_2^2 - x_3x_4$, $Q_2 = x_3^2 + x_4^2 - x_1x_2$ and let $p_1 = x_1^2x_3^2 + x_2^2x_4^2 - x_3x_4x_5x_6$, $p_2 = x_3^2x_7^2 + x_4^2x_8^2 - x_1x_2x_7x_8$. We define the group $G_4 \subseteq \text{PGL}(3, \mathbb{C})$ given by the four automorphisms
\[ \text{diag}(1, 1, 1, 1), \text{diag}(\zeta_8, -\zeta_8, -\zeta_8^{-1}, \zeta_8^{-1}), \text{diag}(\zeta_4, \zeta_4, \zeta_4^{-1}, \zeta_4^{-1}), \text{diag}(\zeta_8^3, -\zeta_8^3, -\zeta_8^{-3}, \zeta_8^{-3}), \]
where $\zeta_k$ is a primitive $k^{th}$ root of unity.

The Batyrev-Borisov mirror to $Z(Q_1, Q_2) \subseteq \mathbb{P}^3$ can be computed to be a complete intersection $Z_2$ in a 3-dimensional toric stack $\mathcal{X}_{BB}$ given as the zero locus $Z_2 = Z(p_1, p_2) \subseteq \mathcal{X}_{BB}$. Take the stacky complete intersection $V_2 := Z(Q_1, Q_2) \subseteq (\mathbb{C}^4 \setminus \{0\})/(\mathbb{C}^* \times G_4)$. Then
\[ \text{D}^b(\text{coh } V_2) \cong \text{D}^b(\text{coh } Z_2). \]

Remark 4.9. One can aim to generalise this to higher dimensions by looking at the zero-set of the two polynomials
\[ Q_{1,n} = x_1^n + \cdots + x_n^n - x_{n+1} \cdots x_{2n} \quad \text{and} \quad Q_{2,n} = x_{n+1}^n + \cdots + x_{2n}^n - x_1 \cdots x_n \]
in $\mathbb{P}^{2n-1}$.

Unfortunately, $Z(Q_{1,n}, Q_{2,n}) \subseteq \mathbb{P}^{2n-1}$ is itself singular for $n \geq 4$, which poses problems for the required ideal containment condition $I \subseteq \sqrt{\partial w, J}$ to hold. However, using these
methods of VGIT is still interesting in the context of categorical resolutions. Indeed, the direct generalisation of the Libgober-Teitelbaum construction above can be categorically resolved. This technique and its generalisations are a subject of future work.

**Remark 4.10.** The notion of $f$-duality introduced by Rossi in [Ros21a] and [Ros21b] gives an efficient method of computing and extending the Batyrev-Borisov mirror construction. In particular, applying $f$-duality to the variety $V_{LT,\lambda} \subseteq \mathbb{P}^5 / G_{81}$ yields $V_\lambda \subseteq \mathbb{P}^5$.

The generalisations looked at in the Remark 4.9 were inspired by $f$-duality and it seems to be an interesting question when, in general, one can use the methods of variations of GIT employed in this paper to strengthen the notion of $f$-duality.

**References**

[ACG16] Michela Artebani, Paola Comparin, and Robin Guilbot. Families of Calabi-Yau hypersurfaces in Q-Fano toric varieties. *J. Math. Pures Appl. (9)*, 106(2):319–341, 2016.

[Bat94] Victor V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Algebraic Geom.*, 3(3):493–535, 1994.

[BB96] Victor V. Batyrev and Lev A. Borisov. On Calabi-Yau complete intersections in toric varieties. In *Higher-dimensional complex varieties (Trento, 1994)*, pages 39–65. de Gruyter, Berlin, 1996.

[BH92] Per Berghlund and Tristan Hübsch. A generalized construction of mirror manifolds. In *Essays on mirror manifolds*, pages 388–407. Int. Press, Hong Kong, 1992.

[BN08] Victor Batyrev and Benjamin Nill. Combinatorial aspects of mirror symmetry. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 35–66. Amer. Math. Soc., Providence, RI, 2008.

[Cla17] Patrick Clarke. Duality for toric Landau-Ginzburg models. *Adv. Theor. Math. Phys.*, 21(1):243–287, 2017.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[DFK18] Charles F. Doran, David Favero, and Tyler L. Kelly. Equivalences of families of stacky toric Calabi-Yau hypersurfaces. *Proc. Amer. Math. Soc.*, 146(11):4633–4647, 2018.

[FK17] David Favero and Tyler L. Kelly. Proof of a conjecture of Batyrev and Nill. *Amer. J. Math.*, 139(6):1493–1520, 2017.

[FK18] David Favero and Tyler L. Kelly. Fractional Calabi-Yau categories from Landau-Ginzburg models. *Algebr. Geom.*, 5(5):596–649, 2018.

[FK19] David Favero and Tyler L. Kelly. Derived categories of BHK mirrors. *Adv. Math.*, 352:943–980, 2019.

[FMN10] Barbara Fantechi, Etienne Mann, and Fabio Nironi. Smooth toric Deligne-Mumford stacks. *J. Reine Angew. Math.*, 648:201–244, 2010.

[FR18] Stefano Filipazzi and Franco Rota. An example of Berglund-Hübsch mirror symmetry for a Calabi-Yau complete intersection. *Matematiche (Catania)*, 73(1):191–209, 2018.

[GKZ94] I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[GOT18] Jacob E. Goodman, Joseph O’Rourke, and Csaba D. Tóth, editors. *Handbook of discrete and computational geometry*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018. Third edition of [MR1730156].

[GP90] B. R. Greene and M. R. Plesser. Duality in Calabi-Yau moduli space. *Nuclear Phys. B*, 338(1):15–37, 1990.

[Hir17] Yuki Hirano. Derived Knörrer periodicity and Orlov’s theorem for gauged Landau-Ginzburg models. *Compos. Math.*, 153(5):973–1007, 2017.

[HW12] Manfred Herbst and Johannes Walcher. On the unipotence of autoequivalences of toric complete intersection Calabi-Yau categories. *Math. Ann.*, 355(3):783–802, 2012.

[Isi13] Mehmet Umut Isik. Equivalence of the derived category of a variety with a singularity category. *Int. Math. Res. Not. IMRN*, (12):2787–2808, 2013.
A. Libgober and J. Teitelbaum. Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations. *Internat. Math. Res. Notices*, (1):29–39, 1993.

Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser Boston, Boston, MA, 2009.

Jörg Rambau. TOPCOM: triangulations of point configurations and oriented matroids. In *Mathematical software (Beijing, 2002)*, pages 330–340. World Sci. Publ., River Edge, NJ, 2002.

Michele Rossi. An extension of polar duality of toric varieties and its consequences in mirror symmetry, 2021.

Michele Rossi. Framed mirror symmetry for projective complete intersections of non-negative kodaira dimension, 2021.

Ian Shipman. A geometric approach to Orlov’s theorem. *Compos. Math.*, 148(5):1365–1389, 2012.

The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.4)*, 2021. [https://www.sagemath.org](https://www.sagemath.org).