The Reachable Space of the Heat Equation for a Finite Rod as a Reproducing Kernel Hilbert Space

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Abstract. We use some results from the theory of reproducing kernel Hilbert spaces to show that the reachable space of the heat equation for a finite rod with either one or two Dirichlet boundary controls is a RKHS of analytic functions on a square, and we compute its reproducing kernel as an infinite double series. We also show that the null reachable space of the heat equation for the half line with Dirichlet boundary data is a RKHS of analytic functions on a sector, whose reproducing kernel is (essentially) the sum of pullbacks of the Bergman and Hardy kernels on the half plane C+.

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1. Introduction

Let $T > 0$ fixed. Consider the following control system

$$
\begin{align*}
\partial_t w - \partial_{xx} w &= 0, \quad 0 < x < 1, \quad 0 < t < T, \\
 w(0,t) &= u_\ell(t), \quad w(1,t) = u_r(t), \quad 0 < t < T, \\
 w(x,0) &= 0, \quad 0 < x < 1,
\end{align*}
$$

(1)

which models the temperature propagation in a rod with length 1.

In control theory is an important issue to describe the so-called null reachable space, at time $T > 0$, defined as follows

$$
\mathcal{R}_T := \{ w(\cdot,T) : w \text{ is solution of system (1) with controls } u_\ell, u_r \in L^2_C(0,T) \}.
$$

Using the null controllability of the system (1) in any positive time (see [5, Theorem 3.3]), one can show that the set of states $w(\cdot,T)$ reached by solutions of system (1) from any initial datum $w(x,0) \in L^2(0,1)$ coincides with $\mathcal{R}_T$. 
The null controllability also implies that $\mathcal{R}_T$ does not depend on $T > 0$ (see [7] and [9, Proposition 3.1]), thus $\mathcal{R}$ will denote this space.

The problem is to identify the space of all analytic extensions of the functions in (some subspace of) $\mathcal{R}$ in terms of spaces of analytic functions with some structure.

From [3, Theorem 1.1] and [10, Theorem 2.1] we have that

$$\text{hol}(\overline{Q}) \subset \mathcal{R} \subset \text{hol}(Q),$$

where $Q = \{(x, y) \in \mathbb{R}^2 : |y| < x, |y| < 1 - x\}$ and $\text{hol}(\overline{Q})$ is the set of all analytic functions on a neighborhood of $\overline{Q}$. Hence these results established the domain of analyticity to deal with.

In [7] it was proved that

$$E^2(Q) \subset \mathcal{R} \subset A^2(Q),$$

where $E^2(Q)$ is the Hardy–Smirnov space on $Q$ and $A^2(Q)$ is the (unweighted) Bergman space on $Q$. Thus, in this work well-known spaces of analytic functions with some structure appeared for the first time.

In [11] the author proves that the null reachable space $\mathcal{R}$ is the sum of two Bergman spaces on sectors (whose intersection is $Q$), i.e.

$$\mathcal{R} = A^2(\Delta) + A^2(1 - \Delta),$$

where $\Delta := \{z \in \mathbb{C} : |\arg z| < \pi/4\}$. As the author remarked, given any function $f \in \text{hol}(Q)$, how can we write $f$ as a sum of two functions in those different Bergman spaces?

Recently, in [8] the authors solve a separation of singularities problem in the setting of Bergman spaces. They show that if $\mathcal{P}$ is a polygon which is the intersection of $n$ half planes, then the Bergman space on $\mathcal{P}$ decomposes into the sum of the Bergman spaces on these half planes. As a consequence they get

$$A^2(Q) = A^2(\Delta) + A^2(1 - \Delta),$$

therefore the null reachable space coincides with the Bergman space on $Q$.

In [9] the authors improve (2), they decompose $\mathcal{R}$ as a sum of weighted Bergman spaces

$$\mathcal{R} = A^2(\Delta, \omega_{0,\delta}) + A^2(1 - \Delta, \omega_{1,\delta})$$

for all $\delta > 0$,

where

$$\omega_{0,\delta}(s) = \delta^{-1}e^{s^2/(2t)}, \quad s \in \Delta, \quad \omega_{1,\delta}(s) = \delta^{-1}e^{(1-s)^2/(2t)}, \quad s \in 1 - \Delta.$$  

They also get a characterization of the null reachable space when considering smooth controls.

In this work our main result (Theorem 4) shows that the null reachable space $\mathcal{R}$ is the sum of two RKHSs on the same domain $Q$, therefore $\mathcal{R}$ is a RKHS on $Q$ (see Theorem 2.16 in [13, page 93]). Unfortunately, our approach yields a reproducing kernel $\mathcal{K}_Q$ for $\mathcal{R}$ which is written as an infinite double series, see (3) and (5).
In the final section we write the Bergman kernel \( B_Q \) on \( A^2(Q) \) and pose an open question about the connection between \( K_Q \) and \( B_Q \). A positive answer would provide another proof that the null reachable space is \( A^2(Q) \).

### 2. Statement of the Results

To get the characterization of \( \mathcal{R} \) as a RKHS on \( Q \), we proceed in several steps. We characterize the subspaces in \( \mathcal{R} \) corresponding to the cases either \( u_r = 0 \) or \( u_\ell = 0 \) or \( u_r = -u_\ell \) or \( u_r = u_\ell \) as RKHSs of analytic functions on a square.

In [3] the authors ask for a characterization of the null reachable space, at time \( T > 0 \), with just one Dirichlet boundary control. So we consider the null reachable space, at time \( T > 0 \),

\[
\mathcal{R}^\ell_T := \{ w(\cdot, T) : w \text{ is solution of system (1) with } u_\ell \equiv 0, u_r \in L^2_C(0, T), u_r \neq 0 \}
\]

with one Dirichlet boundary control on the left. As before (using [5, Theorem 3.3] and [9, Proposition 3.1]), we can see that the space \( \mathcal{R}^\ell_T \) does not depend on \( T > 0 \), so \( \mathcal{R}^\ell \) will denote this space.

Motivated by the idea in [7] of writing the solution of system (1) in terms of integral operators having well known heat kernels (see (9)), and by using the characterization of the image of a linear mapping as a RKHS (see [13, page 134]) we have obtained the characterization of \( \mathcal{R}^\ell \) as a RKHS on a square.

First, we introduce some notation and definitions. Consider the square \( D := \{(x, y) \in \mathbb{R}^2 : |y| < x, |y| < 2 - x \} \), the open set \( D^\ell_\infty := \bigcup_{n \in \mathbb{Z}} (2n + D) \) and the following positive definite function on the sector \( \Delta \), see (23) below,

\[
K_0(z, w; T) := \frac{z \bar{w}}{\pi} e^{-\frac{z^2 + w^2}{4T}} \left( \frac{4}{(z^2 + \bar{w}^2)^2} + \frac{1}{T(z^2 + \bar{w}^2)} \right), \quad z, w \in \Delta.
\]

**Theorem 1.** For each \( T > 0 \) fixed, we have that

\[
\mathcal{R}^\ell = \{ f \in \text{hol}(D^\ell_\infty) : f(z + 2) = f(z) = -f(-z), f|_D \in \mathcal{H}^\ell_T(D) \}
\]

where \( \mathcal{H}^\ell_T(D) \) is the RKHS of analytic functions on \( D \) with reproducing kernel

\[
K_\ell(z, w; T) := \sum_{m, n \in \mathbb{Z}} K_0(z + 2n, w + 2m; T), \quad z, w \in D.
\]  

The space \( \mathcal{H}^\ell_T(D) \) is endowed with the norm given in (17).

Notice that the properties of 2-periodicity, oddness and analyticity domain of \( f \in \mathcal{R}^\ell \) are inherited from those of the analytic extension of the heat kernel \( (\partial_x \theta)(x, t) \), see Remarks 10 and 11.

We also have the corresponding result for the null reachable space with one Dirichlet boundary control on the right, defined as follows

\[
\mathcal{R}^r_T := \{ w(\cdot, T) : w \text{ is solution of system (1) with } u_\ell \equiv 0, u_r \in L^2_C(0, T) \}.
\]

As \( \mathcal{R}^r_T \), we have that \( \mathcal{R}^r_T \) does not depend on \( T > 0 \), so we will write \( \mathcal{R}^r \).
Theorem 2. Let $D^\infty_\infty := -1 + D^\ell_\infty$. For each $T > 0$ fixed, we have that
\[
\mathcal{R}^r = \{ f \in \text{hol}(D^\infty_\infty) : f(z + 2) = f(z) = -f(-z), f|_{-1+D} \in \mathcal{H}_T^r(-1 + D) \}
\]
where $\mathcal{H}_T^r(-1 + D)$ is the RKHS of analytic functions on $-1 + D$ with reproducing kernel
\[
\mathcal{K}_r(z, w; T) := \sum_{m,n \in \mathbb{Z}} \mathcal{K}_0(z + 2n + 1, w + 2m + 1; T), \quad z, w \in -1 + D. \tag{4}
\]
The space $\mathcal{H}_T^r(-1 + D)$ is endowed with the norm given in (18).

By (9) and Theorem 1 we also have
\[
\mathcal{R}^r = \{ f \in \text{hol}(D^\infty_\infty) : f(z + 2) = f(z) = -f(-z), f(\cdot - 1)|_{D} \in \mathcal{H}_T^r(D) \}.
\]
In [8, Corollary 3.3] the authors give a characterization of $\mathcal{R}^r$ in terms of the Bergman space on $-1 + D$.

Our approach also provides the characterization of the following subspaces in $\mathcal{R}$,
\[
\mathcal{R}_+^r := \{ w(\cdot, T) : w \text{ is solution of system (1) with } u_\ell \in L^2_n(0, T), u_r = -u_\ell \}, \quad \text{and } \mathcal{R}_-^r := \{ w(\cdot, T) : w \text{ is solution of system (1) with } u_\ell \in L^2_n(0, T), u_r = u_\ell \}.
\]

Once again, (2.14), (2.15) and Theorem 3.3 in [5] imply the null controllability of system (1) with initial datum $w(x, 0) \in L^2(0, 1)$ and the controls $u_\ell, u_r \in L^2_n(0, T)$ satisfying either $u_r = -u_\ell$ or $u_r = u_\ell$. Thus, the spaces $\mathcal{R}_+^r, \mathcal{R}_-^r$ do not depend on the initial data and the time $T > 0$.

Theorem 3. Let $Q_\infty := \bigcup_{n \in \mathbb{Z}} (n + Q)$. For each $T > 0$ fixed, we have that
1. $\mathcal{R}^+ = \{ f \in \text{hol}(Q_\infty) : f(z + 1) = f(z) = -f(-z), f|_Q \in \mathcal{H}_T^+(Q) \}$ where $\mathcal{H}_T^+(Q)$ is the RKHS on $Q$ with reproducing kernel
\[
\mathcal{K}_+(z, w; T) := \sum_{m,n \in \mathbb{Z}} \mathcal{K}_0(z + n, w + m; T)
\]
\[
= \mathcal{K}_\ell(z, w; T) + \mathcal{K}_\ell(z + 1, w + 1; T) + \mathcal{K}_\ell(z + 1, w; T) + \mathcal{K}_\ell(z, w + 1; T);
\]
for $z, w \in Q$. The space $\mathcal{H}_T^+(Q)$ is endowed with the norm given in (19).

2. $\mathcal{R}^- = \{ f \in \text{hol}(Q_\infty) : -f(z + 1) = f(z) = -f(-z), f|_Q \in \mathcal{H}_T^-(Q) \}$ where $\mathcal{H}_T^-(Q)$ is the RKHS on $Q$ with reproducing kernel
\[
\mathcal{K}_-(z, w; T) := \mathcal{K}_\ell(z, w; T) + \mathcal{K}_\ell(z + 1, w + 1; T)
\]
\[
- \mathcal{K}_\ell(z + 1, w; T) - \mathcal{K}_\ell(z, w + 1; T);
\]
for $z, w \in Q$. The space $\mathcal{H}_T^-(Q)$ is endowed with the norm given in (20).

As a consequence, we get a description of the null reachable space $\mathcal{R}$.

Theorem 4. We have that
\[
\mathcal{R} = \mathcal{R}^+ + \mathcal{R}^- \quad \text{and } \mathcal{R}^+ \cap \mathcal{R}^- = \{0\}.
\]
Moreover, $\mathcal{R}|_Q := \{ f|_Q : f \in \mathcal{R} \} = \mathcal{H}_T^+(Q) + \mathcal{H}_T^-(Q)$ with reproducing kernel
\[
\mathcal{K}_Q(z, w; T) := 2\mathcal{K}_\ell(z, w; T) + 2\mathcal{K}_\ell(z + 1, w + 1; T), \quad z, w \in Q. \tag{5}
\]
The space $\mathcal{R}|_Q$ is endowed with the norm given in (21).
Clearly, the condition $R^+ \cap R^- = \{0\}$ follows from the functional equations in the last result. Considering the definition of $K_\ell$, we have that for all $T > 0$ the function $K_Q(z, w; T)$ satisfies the functional equation

$$F(z + 1, w + 1) = F(z, w) = F(-z, -w), \quad \text{for all } z, w \in Q_\infty. \quad (6)$$

Next we consider the case with Neumann boundary data.

$$\partial_t v - \partial_{xx} v = 0, \quad 0 < x < \infty, \quad 0 < t < T,$$

$$(\partial_x v)(t, 0) = u_\ell(t), \quad (\partial_x v)(t, 1) = u_r(t), \quad 0 < t < T,$$

$$v(0, x) = 0, \quad 0 < x < \infty. \quad (7)$$

We set

$$R^N_T := \{v(\cdot, T) : v \text{ is solution of system (7) with } u_\ell, u_r \in L^2_C(0, T)\}.$$ 

**Corollary 5.** We have that $R^N_T$ does not depend on $T > 0$, and

$$R^N = \{f \in hol(Q_\infty) : f' \in R\}.$$ 

As in [8], we have another situation where the null reachable space at time $T > 0$ of the heat equation, with a suitable boundary condition, can be described in terms of well known analytic functions spaces. Consider the heat equation for the half line,

$$\partial_t v - \partial_{xx} v = 0, \quad 0 < x < \infty, \quad 0 < t < T,$$

$$v(0, t) = u(t), \quad 0 < t < T,$$

$$v(x, 0) = 0, \quad 0 < x < \infty,$$

$$\lim_{x \to \infty} v(x, t) = 0, \quad 0 < t < T. \quad (8)$$

Its corresponding null reachable space at time $T > 0$ is given by

$$R^q_T := \{v(\cdot, T) : v \text{ is solution of system (8) with } u \in L^2_C(0, T)\}.$$ 

As usual, $\Re z, \Im z$ denote the real and the imaginary parts of $z \in \mathbb{C}$. Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re z > 0\}$ be the positive half plane. The following result characterizes the null reachable space $R^q_T$ as a RKHS whose reproducing kernel is (essentially) a sum of pullbacks of the Bergman and Hardy kernels on $\mathbb{C}^+$.

**Theorem 6.** Let $T > 0$ fixed. We have that

$$R^q_T = e^{-\frac{z^2}{4T}} A^2(\Delta) + ze^{-\frac{z^2}{4T}} \{f \circ \varphi | f \in H^2(\mathbb{C}^+)\},$$

where $\varphi(z) = z^2, z \in \Delta, A^2(\Delta)$ is the unweighted Bergman space on $\Delta$, and $H^2(\mathbb{C}^+)$ is the Hardy space on $\mathbb{C}^+$. The space $R^q_T$ is endowed with the norm $\| \cdot \|_*$ given in (24), and has the reproducing kernel given in (23). Moreover, $(R^q_T, \| \cdot \|_*)$ is isometrically isomorphic to $(L^2_C(0, T), \| \cdot \|_2)$.

This paper is organized as follows. In the next section we introduce notation, give some results about RKHSs, and make the computations needed to prove the results. In Sect. 4 we provide the proofs of the theorems.
3. Preliminaries

In this section we use some results about the one-dimensional heat equation that can be found in [2]. First, consider the heat kernel on the upper half plane $\mathbb{R}^2_+ := \{(x, t) \in \mathbb{R}^2 : t > 0\}$ given as follows

$$K(x, t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad (x, t) \in \mathbb{R}^2_+.$$ 

In order to describe the solution $w(x, t)$ of system (1) we introduce the so-called theta function

$$\theta(x, t) := \sum_{n \in \mathbb{Z}} K(x + 2n, t), \quad (x, t) \in \mathbb{R}^2_+,$$

so we have the system (1) admits a unique weak solution $w \in C([0, T], W^{-1,2}(0, 1))$ given by (see [2, Theorem 6.3.1] [7])

$$w(x, t) = -2 \int_0^t (\partial_x \theta)(x, t - \tau) u_\ell(\tau) d\tau + 2 \int_0^t (\partial_x \theta)(x - 1, t - \tau) u_r(\tau) d\tau. \quad (9)$$

For an open set $\Omega \subset \mathbb{C}$, the (unweighted) Bergman space on $\Omega$ is the vector space

$$A^2(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ analytic on } \Omega \text{ and } f \in L^2(\Omega)\}$$

endowed with the inner product

$$\langle f, g \rangle_{A^2(\Omega)} := \int_{\Omega} f(z) \overline{g(z)} dxdy.$$ 

We also consider the Hardy space on the half space $\mathbb{C}^+$,

$$H^2(\mathbb{C}^+) := \left\{ f : \mathbb{C}^+ \to \mathbb{C} \mid f \text{ analytic on } \mathbb{C}^+ \text{ and } \sup_{x > 0} \int_{-\infty}^{\infty} |f(x + iy)|^2 dy < \infty \right\}$$

endowed with the inner product

$$\langle f, g \rangle_{H^2(\mathbb{C}^+)} := \sup_{x > 0} \int_{-\infty}^{\infty} f(x + iy) \overline{g(x + iy)} dy.$$ 

Consider the following positive definite functions on $\Delta$

$$K_1(z, w) := \frac{4zw}{\pi(z^2 + \overline{w}^2)^2}, \quad K_2(z, w) := \frac{1}{z^2 + \overline{w}^2}, \quad z, w \in \Delta,$$

and the biholomorphism $\varphi(z) = z^2$ from $\Delta$ onto $\mathbb{C}^+$.

**Remark 7.** Notice that $K_1(z, w) = B_{\mathbb{C}^+}(\varphi(z), \varphi(w))\varphi'(z)\overline{\varphi'(w)}$ where $B_{\mathbb{C}^+}(z, w)$ is the reproducing kernel for the Bergman space $A^2(\mathbb{C}^+)$, so that $K_1(z, w)$ is the reproducing kernel for the Bergman space $A^2(\Delta)$ (see [4, page 12]).

The following result shows that $K_2(z, w)$ is the reproducing kernel for the pullback space, induced by the function $\varphi$, of the Hardy space $H^2(\mathbb{C}^+)$. 

**Lemma 8.** $K_2(z, w)$ is the reproducing kernel for the RKHS $\mathcal{H}_\varphi(\Delta) := \{f \circ \varphi : f \in H^2(\mathbb{C}^+)\}$. 

Proof. Here $\mathcal{K}_H$ stands for the reproducing kernel for $H^2(C^+)$. Let $ev_z : H^2(C^+) \to \mathbb{C}$ be the functional evaluation at $z \in \mathbb{C}$. If $g \in \bigcap_{p \in \Delta} \ker(ev_{\varphi(p)})$ then $g \equiv 0$, so Theorem 2.9 in [13, page 81] implies that the RKHS with reproducing kernel $\mathcal{K}_H(\varphi(z), \varphi(w)) = \mathcal{K}_2(z, w)$ is the space $\mathcal{H}_\varphi(\Delta)$ equipped with the inner product

$$\langle f \circ \varphi, g \circ \varphi \rangle_{\mathcal{H}_\varphi(\Delta)} = \langle f, g \rangle_{H^2(C^+)}.$$  

(10)

For each $t > 0$ fixed, consider the entire function

$$(\partial_x K)(z, t) := -\frac{z}{4\sqrt{n}t^{3/2}}e^{-\frac{z^2}{4t}}, \quad z \in \mathbb{C},$$

which is the analytic extension of the function $(\partial_x K)(x, t)$, $x \in \mathbb{R}$.

Lemma 9. i. For each $t > 0$ fixed, the function

$$(\partial_x \theta)(z, t) := \sum_{n \in \mathbb{Z}} (\partial_x K)(z + 2n, t)$$  

(11)

is holomorphic on $D$ and continuous on $\overline{D}$, the closure of $D$.

ii. For each compact set $\mathcal{F} \subset D$ and $t > 0$, there exists a constant $C_{\mathcal{F}, t} > 0$ such that

$$\sum_{n \in \mathbb{Z}} \left( \int_0^t |(\partial_x K)(z + 2n, t - \tau)|^2 d\tau \right)^{1/2} \leq C_{\mathcal{F}, t} \quad \text{for all } z \in \mathcal{F}. \quad (12)$$

Proof. For $n \leq -2$ we have that $n^2 \geq -2nx$ if $0 \leq x \leq 1$, and $3n^2 \geq 4 - 4x - 4nx$ if $1 \leq x \leq 2$; therefore

$$|z + 2n| e^{-\frac{(z + 2n)^2}{4t}} \leq \begin{cases} 4ne^{-\frac{n^2}{4t}}, & n \geq 1, \quad z \in \overline{D}, \\ 2e^{-\frac{\Re(z^2)}{4t}}, & n = 0, \quad z \in D, \\ 2e^{-\frac{\Re((z + 2)^2)}{2t}}, & n = -1, \quad z \in D, \\ 4|n|e^{-\frac{n^2}{4t}}, & n \leq -2, \quad z \in \overline{D}. \end{cases}$$

(13)

Since $e^{-s} \leq C_s s^{-\sigma}$ for all $s, \sigma > 0$, together the Weierstrass M-test imply the series in (11) converges absolutely and uniformly on $\overline{D}$, and the result i) follows.

For $n \in \mathbb{Z}\setminus\{-1, 0\}$, $z \in \overline{D}$, we have

$$\int_0^t |(\partial_x K)(z + 2n, t - \tau)|^2 d\tau \leq \frac{4}{\pi n^2} \int_{n^2/(2t)}^\infty \rho e^{-\rho} d\rho \leq \frac{4}{\pi} \left(1 + \frac{1}{t}\right) e^{-\frac{n^2}{2t}}.$$ 

Let $z_0 \in \mathcal{F}$ be such that $\Re(z_0^2) = \min_{z \in \mathcal{F}} \Re(z^2)$, therefore

$$\int_0^t |(\partial_x K)(z, t - \tau)|^2 d\tau \leq \frac{1}{\pi (\Re(z_0^2))^2} \int_{\Re(z_0^2)/(2t)}^\infty \rho e^{-\rho} d\rho$$

$$= \frac{1}{\pi (\Re(z_0^2))^2} \left(1 + \frac{\Re(z_0^2)}{2t}\right) e^{-\frac{\Re(z_0^2)}{2t}}$$

for all $z \in \mathcal{F}$.
In a similar way, let \( z_1 \in \mathcal{F} \) be such that \( \Re((z_1 - 2)^2) = \min_{z \in \mathcal{F}} \Re((z - 2)^2) \), therefore
\[
\int_0^t |(\partial_x K)(z - 2, t - \tau)|^2 d\tau \\
\leq \frac{1}{\pi(\Re((z_1 - 2)^2))^2} \left(1 + \frac{\Re((z_1 - 2)^2)}{2t}\right) e^{-\frac{\Re((z_1 - 2)^2)}{2t}}
\]
for all \( z \in \mathcal{F} \), and the result \( \text{ii}) \) follows from the last inequalities. \( \square \)

**Remark 10.** In fact, by making an easy modification in the last proof we get that \( (\partial_x \theta) (\cdot, t) \in \text{hol}(D_{\infty}^t) \). Clearly, for each \( t > 0 \) fixed the function \( (\partial_x \theta) (\cdot, t) \) fulfills the following functional equations
\[
f(z + 2) = f(z) = -f(-z), \quad z \in D_{\infty}^t.
\]

**Remark 11.** Let \( t > 0 \) fixed and \( u \in L^2_c(0, t) \). Lemma 9 together Morera and Fubini's theorems imply that the continuous function (by (13) and the dominated convergence theorem)
\[
z \mapsto \int_0^t u(\tau)(\partial_x \theta)(z, t - \tau) d\tau
\]
is holomorphic on \( D_{\infty}^t \) and satisfies the functional equations in (14).

**Proposition 12.** Let \( t > 0 \) fixed. The series introduced in (3) converges absolutely and uniformly on \( \overline{D} \times D \) (or \( D \times \overline{D} \)). Thus \( \mathcal{K}_t (\cdot, w; t) \) is an analytic function on \( D \), and \( \mathcal{K}_t (z, \cdot; t) \) is an anti-analytic function on \( D \).

**Proof.** For \( |n|, |m| \geq 3 \) and \( z, w \in \overline{D} \) we have that
\[
|(z + 2n)^2 + (w + 2m)^2| \geq 4(n^2 + m^2 - 2|n| - 2|m| - 2) \geq 16.
\]
By using (13) and the last inequality we have the series defining \( \mathcal{K}_t (\cdot, \cdot; t) \) converges absolutely and uniformly on \( \overline{D} \times D \) whenever we sum over all the indexes satisfying \( |n|, |m| \geq 3 \).

Now suppose that there exist \( z \in \overline{D}, w \in D \) such that \( (z + 2n)^2 + (w + 2m)^2 = 0 \). Then \( z + 2n = \pm i(w + 2m) \), thus \( |2n + \Re z| = |3w| \leq 1 \), so \( n = 0 \) or \( n = -1 \). By symmetry, we also have \( m = 0 \) or \( m = -1 \). If \( n = m = 0 \) then \( z = \pm i\overline{w} \), which is a contradiction because \( \overline{D} \cap (\Re D) = \emptyset \). In any case, we get a contradiction because \( \overline{(D - 2)} \cap (\pm i(D - 2)) = \emptyset \). This completes the proof. \( \square \)

Let \( \mathcal{F}(E) \) be the vector space consisting of all complex-valued functions on a set \( E \), and let \( (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) be a Hilbert space. For a mapping \( h : E \to \mathcal{H} \), consider the induced linear mapping \( L : \mathcal{H} \to \mathcal{F}(E) \) defined by
\[
L(h) = \langle f, h(p) \rangle_{\mathcal{H}}.
\]
The vector space \( \mathcal{R}(L) := \{ Lf : f \in \mathcal{H} \} \) is endowed with the norm
\[
\|f\|_{\mathcal{R}(L)} = \inf\{\|f\|_{\mathcal{H}} : f \in \mathcal{H}, f = L(f)\}.
\]
A fundamental problem about the linear mapping \( L \) is to characterize the vector space \( \mathcal{R}(L) \). The following result summarizes Theorems 2.36, 2.37 in
(see also the seminal work [12]) and provides an answer to the last question.

**Theorem A.** 1. \((\mathcal{R}(L), \| \cdot \|_{\mathcal{R}(L)})\) is a RKHS with reproducing kernel
\[
K(p, q) = \langle h(q), h(p) \rangle_H, \quad p, q \in E.
\]
2. The linear mapping \(L : H \to \mathcal{R}(L)\) is a surjective bounded operator with operator norm less than 1.
3. The linear mapping \(L : H \to \mathcal{R}(L)\) is an isometric isomorphism iff the set \(\{h(p) : p \in E\}\) is complete in \(H\).

**4. Proofs of the Results**

**Proof of Theorem 1.** By (9) and Remark 11 we have that \(w(\cdot, T) \in \text{hol}(D^\infty_\ell)\) and fulfills the functional equations in (14).

Lemma 9-ii) implies that the function \(h : D \to L^2_{\mathcal{C}}(0, T)\) given by
\[
h_z(t) = -2(\partial_x \theta)(z, T-t), \quad t \in (0, T),
\]
makes sense, and Remark 11 implies that the linear mapping \(\mathcal{L}^\ell_T : L^2_{\mathcal{C}}(0, T) \to \text{hol}(D)\) given by
\[
(\mathcal{L}^\ell_T u)(z) = \langle u, h_z \rangle_{L^2_{\mathcal{C}}(0, T)}, \quad z \in D,
\]
is well defined. By (9) we have
\[
w(z, T) = (\mathcal{L}^\ell_T u_\ell)(z), \quad z \in D, \quad u_\ell \in L^2_{\mathcal{C}}(0, T).
\]

Theorem A implies that \(\mathcal{H}^\ell_T(D) := \mathcal{R}(\mathcal{L}^\ell_T)\) is a RKHS on \(D\) with reproducing kernel
\[
K^*(z, w; T) = \langle h_w, h_z \rangle_{L^2_{\mathcal{C}}(0, T)}.
\]
The inequality in (12), the dominated convergence theorem and Proposition 12 allow us to compute
\[
K^*(z, w; T) = \lim_{N, M \to \infty} \sum_{|n|, |m| \leq N} \frac{(z + 2n)(w + 2m)}{4\pi} \int_0^T e^{-\frac{(z + 2n)^2}{4(T-t)}} \frac{(w + 2m)^2}{4(T-t)} dt
\]
\[
= \lim_{N, M \to \infty} \sum_{|n|, |m| \leq N} K_0(z + 2n, w + 2m; T) = K_\ell(z, w; T). \quad (16)
\]
We also have
\[
\|w(\cdot, T)\|_{\mathcal{H}^\ell_T(D)} = \inf \left\{ \|u\|_{L^2_\mathcal{C}(0, T)} : w(\cdot, T) = \mathcal{L}^\ell_T u, u \in L^2_{\mathcal{C}}(0, T) \right\}. \quad (17)
\]

**Proof of Theorem 2.** We only give a sketch. Let \(\tilde{h} : -1 + D \to L^2_{\mathcal{C}}(0, T)\) given by
\[
\tilde{h}_z(t) = 2(\partial_x \theta)(z + 1, T-t), \quad t \in (0, T),
\]
and the linear mapping \(\mathcal{L}^\ell_T : L^2_{\mathcal{C}}(0, T) \to \text{hol}(-1 + D)\) given by
\[
(\mathcal{L}^\ell_T u)(z) = \langle u, \tilde{h}_z \rangle_{L^2_{\mathcal{C}}(0, T)}, \quad z \in -1 + D.
\]
By (9) and Remark 10 we have
\[ w(z, T) = (\mathcal{L}^+ T u_r)(z), \quad z \in -1 + D, \; u_r \in L^2_C(0, T). \]

Theorem A implies that \( \mathcal{H}^+_T(-1 + D) := \mathcal{R}(\mathcal{L}^+_T) \) is a RKHS on \(-1 + D\) with reproducing kernel
\[ \langle \tilde{h}_w, \tilde{h}_z \rangle_{L^2_C(0, T)} = \mathcal{K}_r(z, w; T). \]

We also have
\[ \|w(\cdot, T)\|_{\mathcal{H}^+_T(-1+D)} = \inf \left\{ \|u\|_{L^2_C(0, T)} : w(\cdot, T) = \mathcal{L}^+_T u, u \in L^2_C(0, T) \right\}. \tag{18} \]

**Remark 13.** (1) Since \( \mathcal{K}_r(\cdot, w; T) \in \mathcal{H}^+_T(D) \) for all \( w \in D \) (see [13, Proposition 2.1, page 71]), we get that the function \( \mathcal{K}_r(\cdot, y; 1) : (0, 2) \to \mathbb{R} \) is in \( \mathcal{R}^\ell \) for all \( y \in (0, 2) \).

(2) Since \( \mathcal{K}_r(\cdot, w; T) \in \mathcal{H}^+_T(D) \) for all \( w \in D \), we get that the function \( \mathcal{K}_r(\cdot, y; 1) : (0, 2) \to \mathbb{R} \) is in \( \mathcal{R}^\ell \) for all \( y \in (0, 2) \).

**Proof of Theorem 3.** (1) We set \( u_r = -u_\ell \) in (9) to get
\[
w(x, T) = -2 \int_0^T \left[ (\partial_x \theta)(x, T - \tau) + (\partial_x \theta)(x - 1, T - \tau) \right] u_\ell(\tau) d\tau
= -2 \int_0^T (\partial_x \tilde{\theta})(x, T - \tau) u_\ell(\tau) d\tau
\]
where
\[
\tilde{\theta}(x, t) = \sum_{n \in \mathbb{Z}} K(x + n, t), \quad (x, t) \in \mathbb{R}^2_+.
\]

For \( t > 0 \) fixed, clearly the function \( (\partial_x \tilde{\theta})(z, t) \) has similar properties to the analytic theta function \( (\partial_x \theta)(z, t) \) in Lemma 9, and also satisfies the following functional equations,
\[
f(z + 1) = f(z) = -f(-z), \quad z \in Q_\infty.
\]

Therefore, \( w(\cdot, T) \in hol(Q_\infty) \) and fulfills the last functional equations.

Now we proceed as in the proof of Theorem 1: consider the function \( h^+ : Q \to L^2_C(0, T) \) given by
\[
h^+_z(t) = -2(\partial_x \tilde{\theta})(z, T - t)
= -2(\partial_x \theta)(z, T - \tau) - 2(\partial_x \theta)(z + 1, T - \tau), \quad t \in (0, T),
\]
and the linear mapping \( \mathcal{L}^+_T : L^2_C(0, T) \to hol(Q) \) given by
\[
(\mathcal{L}^+_T u)(z) = \langle u, h^+_z \rangle_{L^2_C(0, T)}, \quad z \in Q.
\]

By (9) we have
\[
w(z, T) = (\mathcal{L}^+_T u_\ell)(z), \quad z \in Q, \; u_\ell \in L^2_C(0, T).
\]

Theorem A implies that \( \mathcal{H}^+_T(Q) := \mathcal{R}(\mathcal{L}^+_T) \) is a RKHS on \( Q \) with reproducing kernel (the computation is similar to (16))
\[
\langle h^+_w, h^+_z \rangle_{L^2_C(0, T)} = \mathcal{K}_+(z, w; T).
\]
By the other hand,
\[
\langle h_w^+, h_z^+ \rangle_{L^2_c(0,T)} = \langle h_w, h_z \rangle_{L^2_c(0,T)} + \langle h_{w+1}, h_{z+1} \rangle_{L^2_c(0,T)}
\]

\[
+ \langle h_{w+1}, h_z \rangle_{L^2_c(0,T)} + \langle h_w, h_{z+1} \rangle_{L^2_c(0,T)}
\]

\[
= \mathcal{K}_\ell(z, w; T) + \mathcal{K}_\ell(z + 1, w + 1; T)
\]

\[
+ \mathcal{K}_\ell(z, w + 1; T) + \mathcal{K}_\ell(z + 1, w; T),
\]

for \( z, w \in Q \), where \( h \) is the function in (15).

We also have
\[
\left\| w(\cdot, T) \right\|_{\mathcal{H}^{1+}_T(Q)} = \inf \left\{ \left\| u \right\|_{L^2_c(0,T)} : w(\cdot, T) = \mathcal{L}_{T}^+ u, u \in L^2_c(0,T) \right\}.
\]

(2) We set \( u_r = u_\ell \) in (9) to get
\[
w(x, T) = -2 \int_{0}^{T} \left[ (\partial_x \theta)(x, T - \tau) - (\partial_x \theta)(x - 1, T - \tau) \right] u_\ell(\tau) d\tau
\]

\[
= -2 \int_{0}^{T} \left[ (\partial_x \theta)(x, T - \tau) - (\partial_x \theta)(x + 1, T - \tau) \right] u_\ell(\tau) d\tau
\]

By Lemma 9 and Remark 10 we have
\[
(\partial_x \theta)(\cdot, t) - (\partial_x \theta)(\cdot + 1, t) \in hol(Q_\infty), \quad \text{for all } t > 0,
\]

and satisfies the functional equations
\[
-f(z + 1) = f(z) = -f(-z), \quad z \in Q_\infty.
\]

Therefore, \( w(\cdot, T) \in hol(Q_\infty) \) and fulfills the last functional equations.

Consider the function \( h^- : Q \to L^2_c(0,T) \) given by
\[
h^-_z(t) = -2(\partial_x \theta)(z, T - t) + 2(\partial_x \theta)(z + 1, T - t), \quad t \in (0, T),
\]

and the linear mapping \( \mathcal{L}_{T}^- : L^2_c(0,T) \to H(Q) \) given by
\[
(\mathcal{L}_{T}^- u)(z) = (u, h^-_z)_{L^2_c(0,T)}, \quad z \in Q.
\]

Theorem A implies that \( \mathcal{H}_{T}(Q) := R(\mathcal{L}_{T}^-) \) is a RKHS on \( Q \) with reproducing kernel
\[
\langle h^-_w, h^-_z \rangle_{L^2_c(0,T)} = \langle h_w, h_z \rangle_{L^2_c(0,T)} + \langle h_{w+1}, h_{z+1} \rangle_{L^2_c(0,T)}
\]

\[
- \langle h_{w+1}, h_z \rangle_{L^2_c(0,T)} - \langle h_w, h_{z+1} \rangle_{L^2_c(0,T)}
\]

\[
= \mathcal{K}_\ell(z, w; T) + \mathcal{K}_\ell(z + 1, w + 1; T)
\]

\[
- \mathcal{K}_\ell(z, w + 1; T) - \mathcal{K}_\ell(z + 1, w; T),
\]

for \( z, w \in Q \), where \( h \) is the function in (15).

We also have
\[
\left\| w(\cdot, T) \right\|_{\mathcal{H}_{T}(Q)} = \inf \left\{ \left\| u \right\|_{L^2_c(0,T)} : w(\cdot, T) = \mathcal{L}_{T}^- u, u \in L^2_c(0,T) \right\}.
\]

Remark 14. (1) Since \( \mathcal{K}_+ (\cdot, w; T) \in \mathcal{H}_{T}(Q) \) for all \( w \in Q \), we get that the function \( \mathcal{K}_+ (\cdot, y; 1) : (0, 1) \to \mathbb{R} \) is in \( \mathcal{R}^+ \) for all \( y \in (0, 1) \).

(2) Since \( \mathcal{K}_- (\cdot, w; T) \in \mathcal{H}_{T}(Q) \) for all \( w \in Q \), we get that the function \( \mathcal{K}_- (\cdot, y; 1) : (0, 1) \to \mathbb{R} \) is in \( \mathcal{R}^- \) for all \( y \in (0, 1) \).
Proof of Theorem 4. Let \( u_\ell, u_r \in L^2_\mathbb{C}(0, T) \). By (9) we have
\[
2w(z, T) = \mathcal{L}_T^+[u_\ell - u_r](z) + \mathcal{L}_T^-[u_\ell + u_r](z), \quad z \in Q.
\]
Therefore \( R \mid Q \subset \mathcal{H}_T^+(Q) + \mathcal{H}_T^-(Q) \). Since \( \mathcal{H}_T^+(Q) \) and \( \mathcal{H}_T^-(Q) \) are RKHS on \( Q \) with reproducing kernels \( K_+(z, w; T) \) and \( K_-(z, w; T) \) respectively, it follows that \( \mathcal{H}_T^+(Q) + \mathcal{H}_T^-(Q) \) is a RKHS with reproducing kernel
\[
K_+(z, w; T) + K_-(z, w; T) = 2K_\ell(z, w; T) + 2K_\ell(z + 1, w + 1; T),
\]
z, \( w \in Q \), and is equipped with the norm (see [13, page 93])
\[
\|f\|^2 = \min \{\|f_1\|^2_{\mathcal{H}_T^+(Q)} + \|f_2\|^2_{\mathcal{H}_T^-(Q)} : f = f_1 + f_2, f_1 \in \mathcal{H}_T^+(Q), f_2 \in \mathcal{H}_T^-(Q)\}.
\]
(21)

Proof of Corollary 5. If \( v \in C([0, T]; L^2_\mathbb{C}(0, 1) \) is a solution of system (7), with \( u_\ell, u_r \in L^2_\mathbb{C}(0, T) \), then \( (\partial_x v)(x, t) \) is a solution of system (1), therefore \( (\partial_x v)(z, t) = \frac{d}{dx}(v(z, t)) \in R \), \( 0 < t < T \).

Proof of Theorem 6. The solution of the system (8) is given by (see [14, Page 181])
\[
v(x, T) = -2 \int_0^T (\partial_x K)(x, T - t)u(t)dt.
\]
(22)

As in the proof of (13, case \( n = 0 \)), we have that for any compact set \( F \subset \Delta \) and \( T > 0 \), there exist a constant \( C_{\mathcal{F}, T} > 0 \) such that
\[
\int_0^T |(\partial_x K)(z, T - t)|^2 dt \leq C_{\mathcal{F}, T}, \quad z \in \mathcal{F}.
\]

Fubini and Morera’s theorems imply that the continuous function
\[
z \mapsto -2 \int_0^T (\partial_x K)(z, T - t)u(t)dt
\]
is analytic on \( \Delta \) for all \( u \in L^2_\mathbb{C}(0, T) \).

Consider the function \( h^q : \Delta \rightarrow L^2_\mathbb{C}(0, T) \) given by
\[
h^q(t) = -2(\partial_x K)(z, T - t), \quad t \in (0, T),
\]
and the linear mapping \( \mathcal{L}_T^q : L^2_\mathbb{C}(0, T) \rightarrow \text{hol}(\Delta) \) defined by
\[
(\mathcal{L}_T^q u)(z) = \langle u, h^q \rangle_{L^2_\mathbb{C}(0, T)}, \quad z \in \Delta.
\]
Hence \( \mathcal{R}_T^q = \mathcal{R}(\mathcal{L}_T^q) \). Theorem A implies that \( \mathcal{R}(\mathcal{L}_T^q) \) is a RKHS on \( \Delta \) with reproducing kernel
\[
K^q(z, w; T) = \langle h^q_w, h^q_z \rangle_{L^2_\mathbb{C}(0, T)} = \frac{z\overline{w}}{4\pi} \int_0^T e^{-\frac{z^2}{4(T-t)}} e^{-\frac{w^2}{4(T-t)}} dt
\]
\[
= \frac{1}{\pi} e^{-\frac{z^2}{4T}} e^{-\frac{w^2}{4T}} \left( \frac{4zw}{(z^2 + w^2)^{3/2}} + \frac{z\overline{w}}{T(z^2 + w^2)} \right).
\]
(23)

By Remark 7 and Corollary 2.5 in [13, page 107] we have that \( e^{-z^2/(4T)} A^2(\Delta) \) with the inner product
\[
\langle e^{-z^2/(4T)} f, e^{-z^2/(4T)} g \rangle_{e^{-z^2/(4T)} A^2(\Delta)} := \langle f, g \rangle_{A^2(\Delta)}
\]
is a RKHS with reproducing kernel
\[ e^{-\frac{z^2}{4T}} e^{-\frac{w^2}{4T}} \frac{4zw}{\pi(z^2 + w^2)^2}. \]

By Lemma 8 and (10) we have that
\[ ze^{-\frac{z^2}{4T}} H_\phi(\Delta) \]

with the inner product
\[ \langle ze^{-\frac{z^2}{4T}} f, ze^{-\frac{z^2}{4T}} g \rangle_{H_\phi(\Delta)} = \langle f, g \rangle_{H^2(\mathbb{C}^+)} \]

where \( f, g \in H^2(\mathbb{C}^+) \), is a RKHS with reproducing kernel
\[ e^{-\frac{z^2}{4T}} e^{-\frac{w^2}{4T}} \frac{zw}{z^2 + w^2}. \]

Therefore Theorem 2.16 in [13, page 93] implies that
\[ R_q T = e^{-\frac{z^2}{4T}} A^2(\Delta) + ze^{-\frac{z^2}{4T}} H_\phi(\Delta) \]

with the norm
\[ \| f \|^2 = \min \{ \| e^{\frac{z^2}{4T}} f_1 \|^2_{A^2(\Delta)} + \pi T \| z^{-1} e^{\frac{z^2}{4T}} f_2 \|^2_{H_\phi(\Delta)} : f = f_1 + f_2, f_1 \in e^{-\frac{z^2}{4T}} A^2(\Delta), f_2 \in ze^{-\frac{z^2}{4T}} H_\phi(\Delta) \} \]

is a RHKS with reproducing kernel \( K^q(z, w; T) \).

Now we claim that \( \{ h^q_z : z \in \Delta \} \) is a complete set in \( L^2_C(0, T) \). Assume that \( g \in L^2_C(0, T) \) satisfies
\[ \langle g, h^q_z \rangle_{L^2_C(0, T)} = 0 \text{ for all } z \in \Delta, \]

therefore
\[ \int_0^T (T-t)^{-3/2} e^{-(T-t)^{-1}} z g(t) dt = 0 \text{ for all } z \in \mathbb{C}^+. \]

Then we make the change of variable \( \rho = \rho(t) := (T-t)^{-1} - T^{-1} \) to get
\[ \int_0^\infty (\rho + T^{-1})^{-1/2} e^{-x\rho} e^{-iy\rho} g(t(\rho)) d\rho = 0 \]

for all \( x > 0, y \in \mathbb{R} \). Since
\[ \int_0^\infty |g(t(\rho))|^2 (\rho + T^{-1})^{-2} d\rho < \infty, \]

the factor in (25) multiplied by \( e^{-iy\rho} \) is in \( L^2(\mathbb{R}^+) \). The injectivity of the Fourier transform in \( L^2(\mathbb{R}) \) implies that \( g = 0 \) a.e. on \( (0, T) \).

By Theorem A-(3) we have that \( (R^q_T, \| \cdot \|_*) \) is isometrically isomorphic to \( (L^2_C(0, T), \| \cdot \|_2) \).

Remark 15. The integral operator in (22) is the operator \( \tilde{\Phi}_T \) in [7,11].
The Bergman Kernel $B_Q$ on the Square $Q$

In order to give a conformal mapping from the square $Q$ onto the open unit disk $\mathbb{D}$ we recall the Legendre elliptic integral of the first kind

$$ u = F(\phi, k) := \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - k^2 t^2)(1 - t^2)}} $$

where $0 < k^2 < 1$, and we consider the Jacobi elliptic function (see [1])

$$ cn(u, k) := \cos(\phi). $$

The Jacobi elliptic function $cn(\cdot, k)$ is an even function and periodic in $\kappa_1(k) := F(\pi/2, k)$ and $\kappa_2(k) := F(\pi/2, \sqrt{1 - k^2})$ as

$$ cn(u + 2m\kappa_1 + 2ni\kappa_2, k) = (-1)^{m+n} cn(u, k). \quad (26) $$

Let $\kappa_e := F(\pi/2, 1/\sqrt{2})$. Then the function

$$ \varphi(z) = cn \left( 2\kappa_e(z - 1), \frac{1}{\sqrt{2}} \right) $$

is a conformal mapping from $Q$ onto $\mathbb{D}$, see [6], therefore the Bergman kernel on $Q$ is

$$ B_Q(z, w) = \varphi'(z)\overline{\varphi'(w)} B_\mathbb{D}(\varphi(z), \varphi(w)), \quad (27) $$

where

$$ B_\mathbb{D}(z, w) = \frac{1}{\pi(1 - zw)} $$

is the Bergman kernel on $\mathbb{D}$.

When $k = 1/\sqrt{2}$ we have $\kappa_1 = \kappa_2 = \kappa_e$, and (26) implies that

$$ -\varphi(z + 1) = \varphi(z) = \varphi(-z), \quad z \in Q_\infty. $$

Therefore $B_Q(z, w)$ satisfies the functional equations (6), as the reproducing kernel $K_Q(z, w; T)$ does. Thus, we pose the following question:

**Open question** Does it exist $T > 0$ and a corresponding constant $C_T$ such that $K_Q(z, w; T) = C_T B_Q(z, w)$, $z, w \in Q$?

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