RANDOM GENERATORS OF THE SYMMETRIC GROUP:
DIAMETER, MIXING TIME AND SPECTRAL GAP

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Abstract. Let \( g, h \) be a random pair of generators of \( G = \text{Sym}(n) \) or \( G = \text{Alt}(n) \). We show that, with probability tending to 1 as \( n \to \infty \), (a) the diameter of \( G \) with respect to \( S = \{g, h, g^{-1}, h^{-1}\} \) is at most \( O(n^2 (\log n)^c) \), and (b) the mixing time of \( G \) with respect to \( S \) is at most \( O(n^3 (\log n)^c) \). (Both \( c \) and the implied constants are absolute.)

These bounds are far lower than the strongest worst-case bounds known (in Helfgott–Seress, 2013); they roughly match the worst known examples. We also give an improved, though still non-constant, bound on the spectral gap.

Our results rest on a combination of the algorithm in (Babai–Beals–Seress, 2004) and the fact that the action of a pair of random permutations is almost certain to act as an expander on \( \ell \)-tuples, where \( \ell \) is an arbitrary constant (Friedman et al., 1998).

1. Introduction

1.1. Results. Let \( G \) be a finite group. Let \( S \) be a set of generators of \( G \); assume \( S = S^{-1} \). The (undirected) Cayley graph \( \Gamma(G, S) \) is the graph having the elements of \( G \) as its vertices and the pairs \( \{g, gs\} \) \((g \in G, s \in S)\) as its edges. The diameter of \( G \) with respect to \( S \) is the diameter \( \text{diam}(\Gamma(G, S)) \) of the Cayley graph \( \Gamma(G, S) \):

\[
\text{diam}(\Gamma(G, S)) = \max_{g_1, g_2 \in G} \min_{P \text{ a path from } g_1 \text{ to } g_2} \text{length}(P),
\]

where the length of a path is the number of edges it traverses. In other words, \( \text{diam}(\Gamma(G, S)) \) is the maximum, for \( g \in G \), of the length \( \ell \) of the shortest expression \( g = s_1 s_2 \ldots s_\ell \) with \( s_i \in S \).

Theorem 1.1. Let \( S = \{g, h, g^{-1}, h^{-1}\} \), where \( g, h \) are elements of \( \text{Sym}(n) \) taken at random, uniformly and independently. Let \( G = \langle S \rangle \). Then, with probability \( 1 - o(1) \), the diameter \( \text{diam}(\Gamma(G, S)) \) of \( G \) with respect to \( S \) is at most \( O(n^2 (\log n)^c) \), where \( c \) and the implied constant are absolute.

In the study of permutation groups, bounds are wanted not just for the diameter but also for two closely related quantities that give a finer description of the quality of a generating set \( S \). The spectral gap is the difference \( \lambda_0 - \lambda_1 \) between the two largest eigenvalues \( \lambda_0, \lambda_1 \) (where \( \lambda_0 = 1 \) and \( \lambda_0 \geq \lambda_1 \)) of the normalized adjacency matrix \( \mathcal{A} \) on \( \Gamma(G, S) \), seen as an operator on functions \( f : G \to \mathbb{C} \):

\[
\mathcal{A} f(g) := \frac{1}{|S|} \sum_{h \in S} f(gh).
\]
The other quantity is the mixing time. A lazy random walk on \( \Gamma(G, S) \) consists of taking \( x_1, x_2, \ldots \in G \) at random and independently with distribution

\[
\mu = \frac{1}{2} 1_A + \frac{1}{2|S|} 1_S,
\]

where \( 1_A(x) = 1 \) if \( x \in A \) and \( 1_A(x) = 0 \) if \( x \notin A \); the outcome of the lazy random walk of length \( k \) is \( x_1 x_2 \cdots x_k \). The \((\epsilon, d)\)-mixing time \( t_{\text{mix}, \epsilon, d} \) is the least \( k \) such that the distribution \( \mu^{(k)} = \mu * \mu * \cdots * \mu \) of the outcome of a lazy random walk of length \( k \) is very close to the uniform distribution \( 1_G/|G| \) on \( G \):

\[
d\left( \frac{1}{|G|} 1_G, \mu^{(k)} \right) \leq \epsilon,
\]

where \( \epsilon > 0 \) and \( d \) is a distance function on \( C^G \) (e.g., \( d = \ell^1, \ell^2, B^\infty \)). What may be called the strong \((\epsilon)\)-mixing time corresponds to the distance function \( |G| : \ell^\infty \), i.e., the \( \ell^\infty \) norm scaled by a factor of \( |G| \); in other words, the strong mixing time with respect to \( \epsilon \) equals \( t_{\text{mix}, \epsilon}/|G| : \ell^\infty \). The mixing time is defined in \[LPW09\] (and several other sources) with respect to the total variation distance TV, which is simply \( (|G|/2) \ell^1 \) (where \( \ell^1 \) is scaled so that \( \ell^1(1_G) = 1 \), and where \( |G| \), as usual, denotes the number of elements of \( G \)). Thus, the mixing time as in \[LPW09\] (which sets \( \epsilon = 1/4 \); see \[LPW09, (4.33)\]) equals \( t_{\text{mix}, 1/(2|G|), \ell^1} \), which is bounded by the strong \((1/2)\)-mixing time, i.e., \( t_{\text{mix}, 1/(2|G|), \ell^\infty} \) (since \( | \cdot |_1 \leq | \cdot |_\infty \) on a space of measure 1). It is easy to check that the strong \((1/2)^k\)-mixing time is bounded by \( k \) times the strong \((1/2)\)-mixing time. This motivates us to define strong mixing time to mean the strong \((1/2)\)-mixing time; thus, as we were saying, the mixing time (defined as usual, with respect to the TV norm and \( \epsilon = 1/4 \)) is bounded by the strong mixing time.

**Theorem 1.2.** Let \( S = \{g, h, g^{-1}, h^{-1}\} \), where \( g, h \) are elements of \( \text{Sym}(n) \) taken at random, uniformly and independently. Let \( G = \langle S \rangle \). Then, with probability \( 1 - o(1) \), the spectral gap is bounded from below by a constant times \( 1/(n^3(\log n)^c) \):

\[
\lambda_0 - \lambda_1 \geq \frac{1}{n^3(\log n)^c}
\]

and the strong mixing time is bounded from above by \( O(n^3(\log n)^c) \):

\[
t_{\text{mix}, 1/(2|G|), \ell^\infty} \ll n^3(\log n)^c,
\]

where \( c \) and the implied constants are absolute.

(Here, as is usual, “\( A \ll B \)” means “\(|A| \leq cB \)” for some constant \( c \). If we wish to emphasize that \( c \) depends on some quantity \( \delta \), we write “\( A \ll_\delta B \)”.) An absolute constant is one that depends on no quantity, i.e., a constant that is truly a constant. For us, \( A \ll B \) and \( A = O(B) \) are synonyms.

There is a bound on the spectral gap in terms of the diameter (see, e.g., \[DSC93, \S 3, Cor. 1\], or the sources \[Ald87, Bab91, Gan91, Moh91\] given in \[DSC93\]) and also a standard bound on the mixing time in terms of the spectral gap (as in, e.g., \[Lov96\, Thm. 5.1\], or, for the \( \ell^1 \)-norm, \[LPW09\, Thm. 12.3\]). By means of such bounds, Theorem 1.1 implies that \( \lambda_0 - \lambda_1 \geq 1/(n^4(\log n)^{2c}) \) and \( t_{\text{mix}, 1/(2|G|), |G|, \ell^\infty} \ll n^{4c} \).
Given a Schreier graph, the normalized adjacency matrix \( f \) functions of permutation groups, we write \( \subset \) as its set of vertices and \( \{ (x, x^s) : x \in X, s \in S \} \) as its set of vertices. (As is common in the study of permutation groups, we write \( x^s \) for the image \( s(x) \) of \( x \) under the action of \( s \).)

Given a Schreier graph, the normalized adjacency matrix \( A \) is the operator on functions \( f : X \to \mathbb{C} \) given by

\[
A f(x) := \frac{1}{|S|} \sum_{h \in S} f(x^h).
\]

Just as for a Cayley graph, the spectral gap of \( \Gamma(G \to X, S) \) is the difference \( \lambda_0 - \lambda_1 \) between the two largest eigenvalues \( \lambda_0, \lambda_1 \) of \( A \). (Since \( S = S^{-1} \), the spectrum is real.)

What was proven in \[FJR+98\] is that, for every \( \ell \), there is a \( \delta > 0 \) such that the probability that \( \lambda_0 - \lambda_1 \geq \delta \) (for the largest eigenvalues \( \lambda_0, \lambda_1 \) of the Schreier graph \( \Gamma(G \to X, S) \), where \( X \) is the set of all \( \ell \)-tuples of distinct elements of \( \{1, 2, \ldots, n\} \)) tends to 1 as \( n \to \infty \). Here, as we said before, \( S = \{g, h, g^{-1}, h^{-1}\} \) and \( G = \langle S \rangle \), where \( g, h \) is a pair of random elements of \( \text{Sym}(n) \). We will use this for \( \ell = 3 \).

Lastly, let us remark that most of the arguments in this paper have an algorithmic flavor, in part inherited from \[BSB04\]. This motivates the following question: can theorem \[1.1\] be made fully algorithmic? That is, given random elements \( g, h \) of \( \text{Sym}(n) \), is it true that, with probability \( 1 - o(1) \), for every \( x \in \langle g, h \rangle \), we can quickly find a word \( w \) of length \( O(n^2 (\log n)^6) \) such that \( w(g, h) = x \)? We don’t attempt to answer this question fully; we sketch some ideas in Appendix \[B\].

1.2. Relation to the previous literature. The best bounds known for the problem addressed by Thm. \[1.1\] were, successively,

\[
O(n^{(1/2+o(1)) \log n}), \quad O(n^{8+o(1)}), \quad O(n^3 \log n) \quad \text{[SPT12].}
\]

The best worst-case bound known (i.e., the best bound holding for all generating sets \( S \)) is \( O(n^{O((\log n)^3 \log \log n)}) \) \[HHS\]. Back in \[SC04\], the kind of question addressed by both Thm. \[1.1\] and Thm. \[1.2\] had been described as “wide open” (see \[SC04\] Problem 8.11) and the remarks immediately following.

The bounds in \[1.1\] and \[1.2\] are close to the actual diameter and mixing times for at least some pairs \( g, h \) (called “slow shuffles” in \[SC04\] pp. 284–285]). Take, for instance, \( g = (12 \ldots n), h = (12) \). Let \( G = \text{Sym}(n) \) and \( S = \{g, h, g^{-1}, h^{-1}\} \). Then

\[
n^5 (\log n)^{2c}.
\]

We obtain the stronger bounds in Thm. \[1.2\] by using most of the proof of Thm. \[1.1\].

Both Thm. \[1.2\] and Thm. \[1.1\] rest in part on ideas from \[BBS04\] and \[BH05\] and in part on the fact that, for \( S = \{g, h, g^{-1}, h^{-1}\} \), where \( g, h \) are random elements of \( \text{Sym}(n) \), the Schreier graph associated to the action of \( G = \langle S \rangle \) on \( \ell \)-tuples (\( \ell \) constant) is almost certainly an expander. This fact was proven in \[FJR+98\] Thm. 2.1. We thank B. Tsaban for pointing this out to us; an earlier version of the present paper contained a proof close to the one that can be found in \[FJR+98\]. We had been inspired by the proof for the case \( \ell = 1 \) given in \[BSB07\].
diam$(G, S)$ is in the order of $n^2$ (see, e.g., [BKL89]) and the mixing time $t_{\text{mix}}$ is in the order of $n^3 \log n$ ([SC04], p. 337), [Wil03, Thm. 1]). The same bounds hold for related choices of $g$ and $h$, e.g., $g = (12 \ldots n)$, $h = (12 \ldots n - 1)$ (the Rudvalis shuffle; see [Wil03]).

An algorithmic approach is proposed in [KTT12]; the algorithm given there is shown to produce words of length $O(n^2 \log n)$ conditionally on a statement that remains unproven (the “Minimal Cycle Conjecture”).

In [Dia], Diaconis mentions “an old conjecture [of his]”, which states that the random walk on $\text{Alt}(n)$ for any generating set $\{g, h\}$ with two elements “gets random in at most $n^3 \log n$ steps” (i.e., has mixing time $O(n^3 \log n)$).

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2. Construction of small cycles

We will construct 2- and 3-cycles as short words on $g$ and $h$. The procedure goes back to [BBS04]; we get better results because our input — namely, the fact that the Schreier graph on $\ell$-tuples is almost certainly an expander — is much stronger than the result [BBS04, “Fact 2.1”] used in [BBS04].

**Proposition 2.1.** Let $g, h$ be two random elements of $\Sym(n)$. Then, with probability $1 - o(1)$, every 3-cycle in $\Sym(n)$ can be written as a word of length $O(n(\log n)^c)$ in $g$ and $h$, where $c$ and the implied constant are absolute.

**Proof.** We will first show that there is at least one 2- or 3-cycle that can be written as a word of length $O(n(\log n)^2)$ in $g$ and $h$. We can assume that the Schreier graph corresponding to the action of $\{g, h, g^{-1}, h^{-1}\}$ on the set $X$ of $3$-tuples of distinct elements of $[1, n]$ has a spectral gap of size at least $\delta$, since [FJR+98, Thm. 2.1] states that this is the case with probability $1 - o(1)$. Here $\delta > 0$ is an absolute constant.

Just as in the case of a Cayley graph, the existence of a constant spectral gap means that the Schreier graph has $(\epsilon, |X|^{-1/2})$-mixing time $\ll_\delta \log |X|/\epsilon \leq \log n^3/\epsilon = O(\log n/\epsilon)$. (This is classical; see the expositions in the proofs of e.g., [Lov96, Thm. 5.1] (undirected graphs) and [HS, Lem. 4.1].) In other words, there is an absolute constant $C$ such that, for every $\epsilon > 0$ and every $k \geq C \log n/\epsilon$, the probability $\Prob(\sigma^k = w)$ that the outcome $\sigma$ of a lazy random walk of length $k$ on $g, h$ will take a given $\vec{v} \in X$ to a given $\vec{w} \in X$ satisfies

\[
\frac{1 - \epsilon}{|X|} \leq \Prob(\sigma^k = \vec{w}) \leq \frac{1 + \epsilon}{|X|}.
\]
This implies that, for any two triples of distinct elements \((x, y_1, y_2), (x', y'_1, y'_2) \in X\),

\[
(2.2) \quad \frac{1 - O(\epsilon)}{n - 2} \leq \text{Prob}(x'' = x') \leq \frac{1 + O(\epsilon)}{n - 2},
\]

where the implied constants are absolute.

By [BH05] Lem. 6.2, since \(g\) and \(h\) are random, then, with probability \(1 - o(1)\),
there is a \(j \leq 10 \log n\) such that either \(v = h^j\) or \(v = gh^j\) is an element
with a cycle of length \(l \geq 3n/4\). The argument in [BH05], §6, proof of Thm. 2.2 gives us,
moreover, that \(v' \neq e\) with probability \(1 - O((\log n)/n^{1/4})\). Set \(s = v'\).

We can now follow the argument in [BBS04] §2, only with much shorter walks. Let us give it in full
for the sake of completeness.

Given \(x \in \operatorname{Sym}(n)\), write \(\text{supp}(x)\) for the support of \(x\), i.e., the set of elements
of \([1, n]\) moved by \(x\). Choose \(y_1, y'_1 \in \text{supp}(s)\), \(y_2 \notin \text{supp}(s)\), \(y'_2 = (y'_1)^{s^{-1}}\). We define \([x, y]\) (the commutator) to mean \(x^{-1}y^{-1}xy\). (This definition is standard for
permutation groups). Let \(\sigma \in \operatorname{Sym}(n)\) be such that \(y''_i = y'_i\) for \(i = 1, 2\). Let \(\tau = \sigma^{-1}s\sigma\).
The idea here is that

\[
(y'_1)^{s^{-1}r^{-1}sr} = (y_2)^{r^{-1}sr} = (y'_2)^{sr} = (y'_1)^r = y''_1 = y'_1,
\]

and this assures us that \([s, \tau] = e\) cannot be the identity. (This was the entire purpose
of our definitions of \(y_i, y'_i\) for \(i = 1, 2\), and of the condition \(y''_i = y'_i\).)

We wish to show that \([s, \tau] = e\) has support much smaller than that of \(g\). Let us see
- \(s\) is fixed and \(\tau = \sigma^{-1}s\sigma\). How shall we choose \(\sigma\)? Let \(\sigma\) be the outcome of a
lazy random walk on \(g\) and \(h\) of length \(k \geq C \log n/\epsilon\). Let us impose the condition
that \(y''_i = y'_i\) for \(i = 1, 2\); it is easy to see from \((2.1)\) that this happens with positive
probability (provided that \(\epsilon > 0\) is smaller than an absolute constant).

Let \(S = \text{supp}(s)\). A brief case-by-case analysis (as in [BBS04] §2, or as in the
proof of Prop. 5.3 in the survey article [Hel]) gives us that

\[
\text{supp}([s, \tau]) \subset (S \cap S^\sigma) \cup (S \cap S^\sigma)^s \cup (S \cap S^\sigma)^r
\]

and so \(|\text{supp}([s, \tau])| \leq 3|S \cap S^\sigma|\). Since expected values are additive,

\[
\mathbb{E}(|S \cap S^\sigma||y''_i = y'_i|) = \sum_{x' \in S} \text{Prob}(x' \in S^\sigma|y''_i = y'_i)
= 1 + \sum_{x' \in S} \sum_{\substack{x \in S \setminus \{y_1, y_2\} \atop x \neq y_i}} \text{Prob}(x'' = x'|y''_i = y'_i)
= 1 + \sum_{x' \in S} \sum_{\substack{x \in S \setminus \{y_1, y_2\} \atop x \neq y_i}} \frac{1 + O(\epsilon)}{n - 2} = 1 + \frac{(1 + O(\epsilon))|S|}{n - 2} |S|,
\]

where \(i = 1, 2\). (We are using the assumptions that \(y_1, y'_1 \in S\) and \(y_2 \notin S\).) We set \(\epsilon\) small enough so \((1 + O(\epsilon))/(n - 2)\) here is less than \(7n/6\) (say), assuming (as we
may) that \(n\) is larger than an absolute constant.

We conclude that there exists a \(\sigma\) given as a word of length at most \(k\) on \(g\) and \(h\)
such that \(y''_i = y'_i\) for \(i = 1, 2\) and \(|S \cap S^\sigma| \leq 1 + (7/6)|S|^2/n\). As we have seen, this
implies that \([s, \tau]\) is a non-identity element such that
\[
|\text{supp}([s, \tau])| \leq 3 \left(1 + \frac{7|S|}{6n}|S|\right) \leq 3 + \frac{7|S|}{2n}|S| = 3 + \frac{7}{8} |\text{supp}(s)|.
\]

Notice that \([s, \tau]\) is given by a word on \(g\) and \(h\) whose length is at most 4 times the length of the word giving \(s\), plus 4\(k\).

We define \(s_1 = [s, \tau]\) and iterate, constructing \(s_2, s_3, \ldots\) of decreasing support. We have \(|\text{supp}(s_{j+1})| \leq 3 + (7/2) |\text{supp}(s_j)|^2/|S|\), and so, already for some \(j_0 \ll \log \log n\), we have \(|\text{supp}(s_{j_0})| \leq 3\), where the implied constants are absolute and \(s_{j_0}\) is not the identity. It is easy to check that \(s_{j_0}\) is given as a word of length at most \(O(n(\log n)^c)\) on \(g\) and \(h\), where \(c\) and the implied constant are absolute.

Again, a random walk of length \(k\) takes any 3-tuple in \(X\) to any other 3-tuple in \(X\) with positive probability. Hence, we can express any 3-cycle (or 2-cycle, if \(|\text{supp}(s_{j_0})| = 2\)) as \(rs_{j_0}r^{-1}\), where \(r\) is a word of length at most \(k\) on \(g\) and \(h\). If \(|\text{supp}(s_{j_0})| = 2\), note that every 3-cycle can be expressed as a product of two 2-cycles. Hence, in general, we conclude that every 3-cycle in \(\text{Sym}(n)\) can be expressed as a word of length at most \(O(n(\log n)^c)\) on \(g\) and \(h\), where \(c\) and the implied constant are absolute.

3. Diameter, mixing times and spectral gaps

It is easy to see that Proposition 2.1 implies a bound on the diameter of the Cayley graph.

**Proof of theorem 1.1.** The diameter of \(\text{Alt}(n)\) with respect to the set of 3-cycles is \(O(n)\). Hence, Proposition 2.1 implies that, with probability 1, every element of \(\text{Alt}(n)\) can be written as a word of length at most \(O(n^2(\log n)^c)\) on \(g\) and \(h\), where \(c\) and the implied constant are absolute.

This implies, in particular, that \(g\) and \(h\) generate either \(\text{Alt}(n)\) or \(\text{Sym}(n)\). If \(g\) and \(h\) generate \(\text{Alt}(n)\), we are done. If they generate \(\text{Sym}(n)\), then either \(g\) or \(h\) is in \(\text{Sym}(n) \setminus \text{Alt}(n)\). Then every element of \(\text{Sym}(n)\) can be written as a word of length at most \(O(n^2(\log n)^c) + 1\) on \(g\) and \(h\), and so we are done, too.

We will now see how Proposition 2.1 implies upper bounds on the mixing time and spectral gap for the Cayley graph \(\Gamma(G, \{g, h, g^{-1}, h^{-1}\})\). As we discussed in the introduction, these bounds are better than what one would obtain by proceeding from the final result, Thm. 1.1 via comparison methods.

While getting a good bound on the spectral gap (and hence on the mixing time) is slightly subtler than bounding the diameter, the basic strategy is similar:

1) Solve the problem for the set of generators \(A = \{3\text{-cycles}\}\) of \(\text{Alt}(n)\) or the set of generators
\[
A = \{3\text{-cycles}\} \cup \{\text{one element of } \text{Sym}(n) \setminus \text{Alt}(n)\}
\]
of \(\text{Sym}(n)\). This, as we have seen, is trivial when the problem consists in bounding the diameter. For the problem of bounding the spectral gap, the solution for \(G = \text{Alt}(n)\) and \(A = \{3\text{-cycles}\}\) is a computation that we leave for Appendix A here, we will show how to deduce from it the spectral gap for \(G = \text{Sym}(n)\) and \(A\) as in (3.1).
2.1) To give a bound for the spectral gap with respect to the generating set $L$ acting on $\text{Alt}(n)$, we also denote by $M$ convolution with $p'$. (In other words, we will use a known comparison technique (as in [DSC93]; see [LPW09, §13.5] for the previous history of the method).

We will show in Appendix A that the spectral gap in $\text{Alt}(n)$ with respect to the set $C$ of all 3-cycles is $\lambda_0 - \lambda_1 = 3/(n - 1) > 1/n$ (Proposition A.3). If $g$ and $h$ generate $\text{Sym}(n)$ rather than $\text{Alt}(n)$ it is not enough to (a) prove a spectral gap for $\text{Alt}(n)$ with respect to the set $C$ of 3-cycles; we must actually (b) prove a spectral gap for $\text{Sym}(n)$ with respect to the set $A = C \cup \{g\}$, where we assume without loss of generality that $g \notin \text{Alt}(n)$. Let us see how (a) leads to (b).

Here and in what follows, given a finite set $X$, we write $L^2(X)$ for the space of functions $f : X \to \mathbb{C}$, equipped with the unnormalized $L^2$-norm

$$\|f\|_2 = \sqrt{\sum_{x \in X} |f(x)|^2}.$$ 

Let $M$ be the operator on $L^2(\text{Alt}(n))$ defined by convolution with the probability measure $p'$ that is uniformly distributed on the set $C \subset \text{Alt}(n)$ of all 3-cycles. By abuse of language, we also denote by $M$ the operator on $L^2(\text{Sym}(n))$ given by convolution with $p'$. (In other words, $MF = p' \ast F$ for $F \in L^2(\text{Alt}(n))$ and also for $F \in L^2(\text{Sym}(n))$, with the convolution being taken in $L^2(\text{Alt}(n))$ and $L^2(\text{Sym}(n))$, respectively.)

For $g \in \text{Sym}(n) \setminus \text{Alt}(n)$, consider the operator

$$\tilde{M} = \frac{1}{2}(g + g^{-1})M$$

acting on $L^2(\text{Sym}(n))$. (Here $hM$ is defined by $((hM)(F))(x) = MF(h^{-1}x)$; in other words, $hM$ is the composition of (1) the convolution with the point measure $\mu_h$ at $h$ and (2) the operator $M$.)

**Proposition 3.1.** The spectral gap of $\tilde{M}$ on $L^2(\text{Sym}(n))$ is at least as large as the spectral gap of $M$ on $L^2(\text{Alt}(n))$.

Since Prop. A.3 states that the spectral gap of $M$ on $L^2(\text{Alt}(n))$ is $3/(n - 1)$, Prop. 3.1 implies that the spectral gap of $\tilde{M}$ on $L^2(\text{Sym}(n))$ is at least $3/(n - 1)$.

**Proof.** The operator $\tilde{M}$ is a convolution with a symmetric probability measure, namely, the average of the uniform probability measure on $gC$ and the uniform probability measure on $g^{-1}C = Cg^{-1}$. (Here $g^{-1}C = Cg^{-1}$ because $C = gCg^{-1}$.) In particular, $\tilde{M}$ is self-adjoint.

Let us first examine the action of $\tilde{M}$ on eigenfunctions $f$ of $M$ with eigenvalue 1. Any such function must be constant on $\text{Alt}(n)$ (say equal to $a$) and constant on $\text{Sym}(n) \setminus \text{Alt}(n)$ (say equal to $b$). Since $M$ and $\tilde{M} = (1/2)(g + g^{-1})M$ commute (thanks to $gCg^{-1} = C = g^{-1}Cg$), $f$ must also be an eigenfunction of $\tilde{M}$ with eigenvalue $\lambda$, say. Then $\lambda a = b$ and $\lambda b = a$; hence $\lambda^2 = 1$, and so either $\lambda = 1$ or $\lambda = -1$. If $\lambda = 1$, then $a = b$, and so $f$ is just a constant function on $\text{Sym}(n)$. 
(If $\lambda = -1$, then $a = -b$. Obviously, we need not worry about $\lambda = -1$, since then $1 - \lambda = 1 - (-1) \geq 2$, and $2$ is certainly at least as large as the spectral gap of $M$ on $L^2(\text{Alt}(n))$.)

Now consider the action of $\tilde{M}$ on the space $H$ of functions $f$ on $\text{Sym}(n)$ such that $f|_{\text{Alt}(n)}$ is orthogonal to constant functions on $\text{Alt}(n)$ and $f|_{\text{Sym}(n) \setminus \text{Alt}(n)}$ is orthogonal to constant functions on $\text{Sym}(n) \setminus \text{Alt}(n)$. Then, for every $f \in H$, we know that $\|Mf\|_2 \leq (1 - \delta)\|f\|_2$, where $\delta$ is the spectral gap of $M$ on $L^2(\text{Alt}(n))$. Now, convolution with $(1/(2\mu_g + 1))$ is an operator of norm at most $1$, simply because convolution with any probability measure is an operator of norm at most $1$. Hence

$$\|\tilde{M}f\|_2 = \left\| \frac{1}{2}(\mu_g + 1/Mf) \right\|_2 \leq \|Mf\|_2 \leq (1 - \delta)\|f\|_2,$$

proving our statement.

Let us now examine the mixing time. We now work with the normalized $\ell^p$-norm:

$$|f|_p = \left( \frac{1}{|G|} \sum_x |f(x)| \right)^{1/p}.$$  
(3.2)

(This is consistent with the normalization of the $\ell^1$-norm in the introduction.) We know from (3.6) that $t_{\text{mix},\epsilon/|G|,\ell^2} \ll \epsilon^n \log n$ for $G = \text{Alt}(n)$ and $S$ equal to $C$, the set of all 3-cycles. Consider now $G = \text{Sym}(n)$ and a random walk with respect to the distribution

$$\mu'(x) = \frac{1}{2} \mu(x) + \frac{1}{2} \mu(g^{-1}x),$$

where $g \in \text{Sym}(n) \setminus \text{Alt}(n)$ and $\mu = 1_{\{e\}}/2 + 1_C/(2|C|)$. The class $C$ is invariant under conjugation by $g$ (or by any other element). Hence, a random walk of length $k$ with respect to $\mu'$ gives the result $g^r x$, where $x$ is the outcome of a random walk of length $k$ with respect to $\mu$ (i.e., a lazy random walk of length $k$ with respect to $C$) and $r$ is a random integer that is both independent of the random walk and of equidistributed parity. In other words,

$$(\mu')^{(k)} = \sum_{r=0}^{k} p_r \mu^{(k)}(g^{-r}x),$$

where $\sum_{r \text{ odd}} p_r = \sum_{r \text{ even}} p_r = 1/2$. It is easy to check that this implies that, for any left-invariant distance function $d$ (such as $d = \ell^2$),

$$d \left( (\mu')^{(k)}, \frac{1}{|G|} \right) \leq \frac{1}{2} d \left( (\mu)^{(k)}, \frac{1}{|\text{Alt}(n)|} \right).$$

(3.4)

(We are using the fact that $g^r \text{ Alt}(n) = \text{Alt}(n)$ for $r$ even, and $g^r \text{ Alt}(n) = \text{Sym}(n) \setminus \text{Alt}(n)$ for $r$ odd.)

Before we proceed, we should make a brief remark on how $\ell^p$-mixing times relate to each other for different $p$. It is well-known that, if $\ell^p$ norms are defined as in (3.2) and $p \leq q$, then $|\cdot|_p \leq |\cdot|_q$ for $p \leq q$; this is a special case of Jensen’s inequality (see, e.g., [Rud87, Thm. 3.3]). This allows us to use $\ell^q$-mixing times to bound $\ell^p$-mixing...
times for \( q \geq p \). There is a way (also well-known) to use \( \ell^2 \)-mixing times to bound \( \ell^\infty \)-mixing times: for any measure \( \mu \) on any finite group \( G \) and any \( x \in G \),

\[
\mu^{(2k)}(x) = \sum_{g \in G} \mu^{(k)}(g)\mu^{(k)}(g^{-1}x)
\]

\[
\geq \sum_{g \in G} \frac{1}{|G|} \mu^{(k)}(g^{-1}x) + \sum_{g \in G} \left( \mu^{(k)}(g) - \frac{1}{|G|} \right) \left( \mu^{(k)}(g^{-1}x) - \frac{1}{|G|} \right)
\]

\[
+ \sum_{g \in G} \left( \mu^{(k)}(g) - \frac{1}{|G|} \right)^2 \frac{1}{|G|} \leq \frac{1}{|G|} + \| \mu^{(k)} - \frac{1}{|G|} \|_2^2,
\]

i.e., \( |\mu^{(2k)}| - 1/|G|_\infty \leq |G|\|\mu^{(k)} - 1/|G|_2^2 \). This implies that

\[
t_{\text{mix}, \ell^2/|G|, \ell^\infty} \leq 2t_{\text{mix}, \ell^2/|G|, \ell^2}.
\]

**Proof of Theorem 1.2.** We would like to estimate the spectral gap for the random choice of generators by comparing it with the spectral gap for either \( M \) or \( \tilde{M} \). We will use a comparison technique from [DSC93].

Let \( S \) be a symmetric set of generators of \( G \). For \( y \in G \) and \( s \in S \) we define \( N(s, y) \) to be the number of times \( s \) occurs in a chosen expression for \( y \) as a product of elements of \( S \). Let \( p \) and \( p' \) be symmetric probability distributions on \( G \). Suppose that the support of \( p \) contains \( S \).

Consider the following quantity:

\[
A = \max_{s \in S} \frac{1}{p(s)} \sum_{y \in G} |y|N(s, y)p'(y).
\]

We can use \( A \) to compare the spectral gap \( \delta(p) = \lambda_0(p) - \lambda_1(p) \) for the convolution by \( p \) with the spectral gap \( \delta(p') = \lambda_0(p') - \lambda_1(p') \) for the convolution by \( p' \) [DSC93]:

\[
\delta(p) \geq \frac{1}{A} \delta(p').
\]

We set \( S = \{ g, h, g^{-1}, h^{-1} \} \), where \( g, h \in \text{Sym}(n) \) are chosen randomly. If \( g, h \in \text{Alt}(n) \), we let \( p' \) be uniformly supported on the set \( C \) of all 3-cycles; otherwise, we assume without loss of generality that \( g \notin \text{Alt}(n) \), and we let \( p' \) be the average of the uniform probability distribution on \( gC \) and the uniform probability distribution on \( g^{-1}C \). We set \( p = \mu \), with \( \mu \) given as in \([12]\). From Proposition 2.1 we get that, with probability \( 1 - o(1) \), \( A \ll (n(\log n)^c)^2 \), where \( c \) and the implied constant are absolute. Indeed, Prop. 2.1 assures us that \( G = \langle g, h \rangle \) is either \( \text{Alt}(n) \) or \( \text{Sym}(n) \); even more importantly, Prop. 2.1 tells us that the diameter of \( G \) with respect to \( S \) is \( O(n(\log n)^c) \) if \( G = \text{Alt}(n) \), and also that it is \( O(n(\log n)^c) + 1 = O(n(\log n)^c) \) if \( G = \text{Sym}(n) \). Since we can bound both \( |y| \) and \( N(s, y) \) by the diameter of \( G \) with respect to \( S \), this means that \( |y| \) and \( N(s, y) \) are both \( O(n(\log n)^c) \). Hence \( A \ll (n(\log n)^c)^2 \).

We know that

\[
\delta(p') \geq \frac{3}{n-1}
\]
by Prop. A.3 if $G = \text{Alt}(n)$, and by Prop. A.3 and Prop. 3.1. We conclude that
\[
\lambda_0(p) - \lambda_1(p) = \delta(p) \geq \frac{1}{A} \cdot \frac{3}{n - 1} \Rightarrow \frac{1}{(n \log n)^{c/2}} \cdot \frac{3}{n} > \frac{1}{n^3(\log n)^{2c}}.
\]

We could bound the mixing time by $O\left(n^4(\log n)^{O(1)}\right)$ using this spectral gap estimate. We will do better by working with mixing times directly.

Again, we work with $S = \{g, h, g^{-1}, h^{-1}\}$. If $g, h \in \text{Alt}(n)$, we let $p' = \mu$, where $\mu = 1_{e}/2 + 1_{C}/(2|C|)$, for $C$ the set $C$ of all 3-cycles; otherwise, we assume w.l.o.g. that $g \notin \text{Alt}(n)$, and we let $p' = \mu'$, where $\mu'$ is as in (3.3). We let $p = 1_{e}/2 + 1_{S}/(2|S|)$, just as before. By the same argument as above, the quantity $A$ defined in (3.6) is $\leq C(n(\log n)^c)^2$ (where $C, c$ are absolute constants) with probability $1 - o(1)$ (for $g$ and $h$ random). Now, a comparison result in [DSC93] allows us to finish the task. We will quote the result as stated in [SC04, Thm. 10.2 and 10.3]:
\[
|p^{(k)} - \frac{1}{|G|}|^2 \leq |G|e^{-k/2A} + \left|p'\right|(\frac{k}{2|A|}) - \frac{1}{|G|}^2.
\]
(Some of the terms in [SC04, Thm. 10.2] disappear due to the fact that the spectrum of $p$ is non-negative.) At the same time, by (A.6) and (3.4),
\[
\left|p'\left(k'\right) - \frac{1}{|G|}\right| \leq \frac{\epsilon}{|G|}
\]
for any $k' \geq C_\epsilon n \log n$, where $C_\epsilon$ depends only on $\epsilon$. We set
\[
k = [2|A| \cdot \max(4, C_\epsilon) n \log n],
\]
and we obtain that
\[
|p^{(k)} - \frac{1}{|G|}|^2 \leq \frac{1}{|G|^2} + \frac{\epsilon^2}{|G|^2}.
\]
Setting $\epsilon = 1/2$, we get that $|p^{(k)} - 1/|G|| \leq (2/3)|G|$ (say) for $n$ larger than a constant. Hence, by (3.5),
\[
t_{\text{mix}, 2|G|, \epsilon^2} \leq 2t_{\text{mix}, 2/3|G|, \epsilon^2} \leq 2k \ll n^3(\log n)^{2c+1}.
\]

**Appendix A. The spectral gap and the mixing time of $\text{Alt}(n)$ with respect to 3-cycles**

We need to bound the spectral gap of $\text{Alt}(n)$ with respect to the generating set consisting of all 3-cycles in $\text{Alt}(n)$. We will actually compute the spectral gap exactly.

Let us first review the literature briefly. It was computed in [DS81, p. 175] that, for $G = \text{Sym}(n)$ and $S$ equal to the set $\{(i,j) : 1 \leq i < j \leq n\} \cup \{e\}$, with the identity $e$ being given weight $1/n$, the eigenvalue gap $\lambda_0 - \lambda_1$ is $2/n$. We will also need a result for $S$ equal to the set of 3-cycles. For such an $S$, we could deduce a bound of $\lambda_0 - \lambda_1 \gg 1/(n \log n)$ from [BSZ11]. Here we will follow the approach of [DS81] to show that $\lambda_0 - \lambda_1 = 3/(n - 1)$.
We will now remind the reader of some basics in the representation theory of finite groups. A representation $\rho$ of a finite group $G$ is a homomorphism from $G$ to the group of invertible linear operators of vector space $V$. We write $d_\rho$ for the dimension of $V$; we will consider only the case of $V$ finite-dimensional.

A representation $\rho$ is said to be irreducible if there is no non-trivial $\rho$ invariant subspace of $V$. From now on, $V$ will be a vector space over $\mathbb{C}$. Schur’s lemma states that a linear operator from $V$ to $V$ which commutes with an irreducible representation is a multiple of the identity. Two representations $\rho, \rho'$ are equivalent if they are conjugates of each other, i.e., if there is an isomorphism $\phi : V \rightarrow W$ such that $\rho'(g) = \phi \circ \rho(g) \circ \phi^{-1}$.

Given a representation $\rho$ and a function $p$ on $G$, we define the Fourier transform of $p$ by

$$\rho(p) = \sum_{\gamma \in G} p(\gamma) \rho(\gamma),$$

which is an endomorphism from $V$ to $V$. As usual, the Fourier transform transforms a convolution into a multiplication: $\rho(p_1 * p_2) = \rho(p_1) \rho(p_2) = \rho(p_2) \circ \rho(p_1)$.

The character $\chi_\rho : G \rightarrow \mathbb{C}$ of a representation $\rho$ is defined by $\chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$. Characters are constant on conjugacy classes.

Now consider any finite group $G$ and any irreducible representation $\rho$ of $G$. Let $p$ be a function from $G$ to the complex numbers which is constant on each conjugacy class. Put the conjugacy classes in some arbitrary order. Let $p_i$ be the value of $p$ on the $i$-th conjugacy class, $n_i$ the cardinality of the $i$-th conjugacy class, and $\chi_{\rho,i}$ the value of $\chi_\rho$ on the $i$-th conjugacy class. Then

$$\rho(p) = \left( \frac{1}{d_\rho} \sum_i p_i n_i \chi_{\rho,i} \right) \text{Id.} \quad \text{(A.1)}$$

Indeed $\rho(p) = \sum_\gamma p(\gamma) \rho(\gamma)$ can be written as $\sum_i p_i M_i$ where $M_i$ is the sum of $\rho(\gamma)$ over the $i$-th conjugacy class. By the definition of the conjugacy class each matrix $M_i$ commutes with $\rho(\gamma)$. Since $\rho$ is irreducible, Schur’s lemma tells us that $M_i = c_i \text{Id}$. The trace of $M_i$ is equal both to $n_i \chi_{\rho,i}$ and to $c_i d_\rho$; this gives the above formula.

The left regular representation $\lambda$ is defined as follows. For $f \in \ell^2(G)$,

$$\lambda(\gamma)(f)(\gamma') = f(\gamma^{-1}\gamma').$$

The action of $\lambda(p)$ on $\ell^2(G)$ corresponds to a convolution by $p$. Indeed, for $f \in \ell^2(G)$

$$\lambda(p)f(\gamma) = \sum_{\nu \in G} \lambda(\nu)p(\nu)f(\gamma) = \sum_{\nu \in G} p(\nu)f(\nu^{-1}\gamma) = p * f(\gamma).$$

The regular representation (and more generally any representation) can be decomposed into irreducible representations – that is to say, it can be written as a direct sum of irreducible representations. Every irreducible representation appears in the decomposition of the regular representation. Therefore, it follows from (A.1) that, for $p$ constant on conjugacy classes (for instance, equidistributed on $k$ cycles),
the eigenvalues of the convolution operator $f \mapsto p * f$ on $\ell^2(G)$ are precisely the values of
\begin{equation}
\frac{1}{d_\rho} \sum_i p_i n_i \chi_{\rho,i}
\end{equation}
as $\rho$ ranges over the irreducible representations of $G$.

We will apply this theory to symmetric groups $G = \text{Sym}(n)$. We let $p$ be the uniform probability measure on the set $C$ of all 3-cycles. Since $C$ generates only the alternating group $\text{Alt}(n)$, which is a subgroup of index 2 in $\text{Sym}(n)$, and we are using the representation theory of $\text{Sym}(n)$, this will result in doubling the multiplicity of all eigenvalues. We can see this explicitly as follows. Let $f : \text{Alt}(n) \to \mathbb{C}$ be an eigenfunction of the action of $p$ within the left regular representation of $\text{Alt}(n)$ (i.e., $Cf = \lambda f$, where $Cf$ is defined by $Cf(h) = \sum g p(g) f(gh)$). Then $f$ defines two eigenfunctions of the action of $p$ within the left regular representation of $\text{Sym}(n)$, both of them restricting to $f$ on $\text{Alt}(n)$: given a fixed $s \in \text{Sym}(n) \setminus \text{Alt}(n)$, we let $f(gs) = f(g)$ for $g \in \text{Alt}(n)$ to define one of the eigenfunctions, and $f(gs) = -f(g)$ to define the other one.

(We can also see the doubling of the multiplicity of all eigenvalues more abstractly, by using a result such as [FH91, §5, Prop. 5.1].)

By a partition $\lambda = (\lambda_1, ..., \lambda_k)$ of $n$ we mean a non-increasing sequence of positive integers $\lambda_j$ with sum $n$.

It is a fundamental fact from the representation theory of $\text{Sym}(n)$ that the irreducible representations of $\text{Sym}(n)$ are in one to one correspondence with partitions of $n$.

Let $p$ be the uniform probability measure on the set $C \subset \text{Alt}(n)$ of all 3-cycles. Then, by (A.1),
\begin{equation}
\rho(p) = \frac{\chi_\rho(\sigma)}{d_\rho} \text{Id},
\end{equation}
where $\sigma$ is a 3-cycle. Therefore, the eigenvalues of $p$ are
\begin{equation}
\chi_\rho(\sigma)/d_\rho
\end{equation}
as $\rho$ ranges over all irreducible representations of $\text{Alt}(n)$.

By a computation of Frobenius (as in [Ing50 (5.2)]),
\begin{equation}
\frac{\chi_\rho(\sigma)}{d_\rho} = \frac{M_3}{2n(n-1)(n-2)} - \frac{3}{2(n-2)}
\end{equation}
for
\begin{equation}
M_3 = \sum_{j=1}^k (\lambda_j - j)(\lambda_j - j + 1)(2\lambda_j - 2j + 1) + j(j - 1)(2j - 1),
\end{equation}
where $(\lambda_1, ..., \lambda_k)$ is the partition corresponding to $\rho$.

There is a following partial order on partitions of $n$. Let $\lambda = (\lambda_1, ..., \lambda_k)$ and $\lambda' = (\lambda'_1, ..., \lambda'_{k'})$ be partitions of $n$. We define $\lambda \geq \lambda'$ if $k \leq k'$ and $\lambda_j \geq \lambda'_j$, $\lambda_1 + \lambda_2 \geq \lambda'_1 + \lambda'_2$, ..., $\lambda_1 + ... + \lambda_k \geq \lambda'_1 + ... + \lambda'_{k'}$. 
We say that a partition $\lambda'$ is obtained from a partition $\lambda$ by a single switch if for some indices $a < b$, $\lambda_a = \lambda'_a + 1$, $\lambda_b = \lambda'_b - 1$, and $\lambda_j$ and $\lambda'_j$ coincide for all other indices. (If $\lambda'_1 = 1$, then the partition $\lambda$ simply ends at $k = k' - 1$; otherwise, $k = k'$.)

It is not difficult to see that for any partitions $\lambda \geq \lambda'$ of $n$ there is decreasing sequence of partitions obtained by a sequence of switches starting at $\lambda$ and ending at $\lambda'$.

**Lemma A.1.** Consider two partitions $\lambda > \lambda'$ which differ by a single switch, i.e. for some indices $a < b$, $\lambda_a = \lambda'_a + 1$, $\lambda_b = \lambda'_b - 1$, and for all other indices $\lambda_j$ and $\lambda'_j$ coincide. Then the value of $M_3$ for $\lambda$ minus the value of $M_3$ for $\lambda'$ equals

$$6((\lambda'_a + 1 - a)^2 - (\lambda'_b - b)^2).$$

**Proof.** Consider the expression (A.5) defining $M_3$. When one makes a switch there is a change in its value for two values for $j$, namely $a$ and $b$.

For $a$ the difference is

$$(\lambda'_a - a + 1)(\lambda'_a - a + 2)(2\lambda'_a - 2a + 3) - (\lambda'_a - a + 1)(2\lambda'_a - 2a + 1),$$

which equals $6(\lambda'_a - a + 1)^2$. In the same way, the difference for $b$ is equal to $6(\lambda'_b - b)^2$ (and, in particular, it is $6(1 - b)^2$ when $\lambda'_b = 1$). \hfill \Box$

To any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ one can associate a Young diagram, which consists of $n$ squares. The first column of the diagram consists of $\lambda_1$ squares, the second of $\lambda_2$ and so on. In the Young diagram correspond to the conjugate representation, the first row of the diagram consists of $\lambda_1$, the second of $\lambda_2$ and so on. In other words, the diagram is flipped so that columns become rows. It is easy to see from (A.1) that the eigenvalues corresponding to conjugate representations are the same.

**Lemma A.2.** Consider the action of the measure supported uniformly on 3-cycles in the regular representation of $\text{Alt}(n)$. The largest eigenvalue 1 corresponds to partitions $(n)$ and $(1, 1, \ldots, 1)$. The second largest eigenvalue corresponds to partitions $(n-1, 1)$ and $(2, 1, \ldots, 1)$.

**Proof of Lemma A.2.** For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ and its conjugate $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ we have $\lambda'_1 = k$ and $\lambda_1 = k'$.

From any partition we can obtain the partition $(n)$ by doing a sequence of inverse switches of the form

$$(\lambda'_1, \ldots, \lambda'_k) \rightarrow (\lambda'_1 + 1, \ldots, \lambda'_k - 1)$$

in case $\lambda'_k \geq 2$, or

$$(\lambda'_1, \ldots, \lambda'_{k-1}, 1) \rightarrow (\lambda'_1 + 1, \ldots, \lambda'_{k-1})$$

if $\lambda'_k = 1$.

It follows from Lemma A.1 that a single switch will increase the eigenvalue as soon as $\lambda'_1 \geq k$. This condition is satisfied either for a partition or its conjugate; moreover, it is preserved by the inverse switches described before. As conjugation does not change the value of the eigenvalue, by taking if necessary a conjugate, we
can suppose that we consider a partition for which there is sequence of switches ending up at the partition \((n)\), each of them increasing the eigenvalue. The one before the last partition in this sequence is \((n-1,1)\).

**Proposition A.3.** The spectral gap in the regular representation of \(\text{Alt}(n)\) for the measure supported uniformly on 3-cycles is \(3/(n-1)\).

This is, of course, the same as the spectral gap for the Cayley graph of \(\text{Alt}(n)\) with respect to the set of generators consisting of all 3-cycles.

**Proof.** By Lemma A.2 the spectral gap equals the difference in the eigenvalues corresponding to partitions \((n)\) and \((n-1,1)\). A simple computation starting from (A.4) shows that the eigenvalue associated to \((n-1,1)\) is \(1-3/(n-1)\), whereas the eigenvalue associated to \((n)\) is, naturally, 1. Thus, the spectral gap is \(3/(n-1)\). □

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**Algorithmic remarks**

Let \(g\) and \(h\) be two random elements of \(\text{Sym}(n)\). With probability \(1-o(1)\), the Schreier graph corresponding to the action of \(S = \{g, h, g^{-1}, h^{-1}\}\) on the set of 3-tuples of distinct elements of \([1, n]\) is an expander graph. Suppose from now on that this is the case.

We have shown that, in such a case, the diameter of \(G\) with respect to \(S\) is \(O(n^2(\log n)^c)\). In other words, for every \(\pi \in \langle S \rangle\), there is a word \(w\) of length \(O(n^2(\log n)^c)\) such that \(\pi = w(g, h)\). The question is: can such a word \(w\) always be found quickly?
First of all, we have to define our goals, i.e., what is meant by “quickly”. It would seem at first sight that we cannot hope for an algorithm taking less time than the length of $w$, since, in general, it takes time proportional to the length of $w$ to write down $w$. However, the words $w$ that the proof of Thm. 1.1 yields are of a very special sort, in that they involve high powers. To be precise, every word $w$ we find is of the form

$$w(g, h) = v(g, h, u(g, h)^l),$$

where $v$ is a word of length $O(n(\log n)^c)$, $u$ is a word of length $O(\log n)$ and $l < n$. Such a word can be written using $O(n(\log n)^{O(1)})$ symbols; thus, it is not ruled out \textit{a priori} that there may be an algorithm that finds the word in $O(n(\log n)^{O(1)})$ steps. This is so even if we assume – as we shall – that just multiplying two elements of $\text{Sym}(n)$ takes time $O(n)$.

Let us explain how the proof we have given strongly suggests a way to construct just such an algorithm. First of all, the algorithm in the [BBS04] (and hence that in the proof of Prop. 2.1) is in essence algorithmic. There seems to be only one problematic spot: while it would seem that most (and not just some) random walks in the proof of Prop. 2.1) is in essence algorithmic. There seems to be only one such an algorithm. First of all, the algorithm in the [BBS04] (and hence that in the proof of Prop. 2.1) is in essence algorithmic. There seems to be only one problematic spot: while it would seem that most (and not just some) random walks make the argument work, this stops being the case when we come to the point in the proof where we fix the condition $y'_2 = (y'_1)^{s^{-1}}$. If, on the other hand, we prefer not to impose this condition, we can no longer guarantee that $[s, \tau]$ have non-zero support.

It seems possible to do without the condition $y'_2 = (y'_1)^{s^{-1}}$ while also constructing elements with non-zero support: (a) it is possible to assume that the length $l$ of the long cycle is smaller than $(1 - \epsilon)l$ by an appropriate use of [BH05]; (b) we can choose to define $s_{i+1} = [s_i, \sigma^{-1}s_i, \sigma]$ for an appropriate $j_i \leq i$, rather than always use $j_i = i$. Thus modified, the proof of Prop. 2.1 should give a permutation $\pi$ of bounded support, as a word of length $O(n(\log n)^{O(1)})$, in time $O(n(\log n)^{O(1)})$.

What would remain to do would be to show how to express every permutation in $\text{Alt}(n)$ or $\text{Sym}(n)$ as a word of length $O(n(\log n)^{O(1)})$ in $g$, $h$ and $\pi$, to be found in time $O(n(\log n)^{O(1)})$. This is not completely trivial even when $\pi$ is a 3-cycle, in that the long cycle of the element $v$ (constructed in the proof of Prop. 2.1) has to be used in conjunction with the effect of a random walk on $g$ and $h$. For $\pi$ general, the matter is even less obvious.

\*

If all one aims for is running time $O(2^2(\log n)^{O(1)})$, the problem becomes considerably simpler. In brief – one can use the algorithm implicit in the proof of Prop. 2.1 essentially as it stands: at the problematic spot, we can choose $y'_1 \in \text{supp}(s)$ arbitrarily, and fix $y'_2 = (y'_1)^{s^{-1}}$ as before; then any choice of $y_1 \in \text{supp}(s)$, $y_2 \notin \text{supp}(s)$ is valid – and there are plenty of such choices. Summing over them, we see that the probability that $(y'_1)^{s^{-1}} \in \text{supp}(s)$, $(y'_2)^{s^{-1}} \notin \text{supp}(s)$ is $\gg 1/n$ even in the worst case (which is the case of $|\text{supp}(s)|$ bounded). Thus, we just need to keep generating $\sigma$ (at most $O(n(\log n)^{O(1)})$ times) until we succeed in finding an $\sigma$ satisfying $(y'_1)^{s^{-1}} \in \text{supp}(s)$, $(y'_2)^{s^{-1}} \notin \text{supp}(s)$ and $|S \cap S'| \geq (7/6)|S|^2/n$ (with probability $\geq 1 - 1/n^A$, $A$ arbitrary).
We follow the rest of the proof of Prop. 2.1 and we obtain a 3-cycle \( \kappa \) as a word of length \( O(n(\log n)^c) \), in time \( O(n^2(\log n)^O(1)) \), with probability \( \geq 1 - \epsilon \); we can bring the probability arbitrarily close to 1 by repeating the procedure. (If \( \langle \{g, h\} \rangle = \text{Sym}(n) \), it could happen that we actually construct a 2-cycle, rather than a 3-cycle; in that case; the procedure is still essentially what we are about to outline, only simpler.) Once we have the 3-cycle \( \kappa \), one issue remains: how do we use it to construct all the 3-cycles we need, as words of length \( O(n(\log n)^c) \), in time \( O(n^2(\log n)^c) \) (in total) or less?

It would not make sense to construct all 3-cycles, since there are \( \gg n^3 \) of them – meaning they could not be constructed in time less than \( O(n^3) \). Instead, we start by writing the permutation \( \pi \) we are given as a product of 3-cycles. (The way to do it is easy and well-known.) Our task is to express each one of those \( O(n) \) cycles as a word of length \( O(n(\log n)^c) \). Note it would not make sense to simply conjugate our 3-cycle \( \kappa \) by short random walks, and store the result every time it happens to be one of the 3-cycles we need: this would take \( \gg n^3 \) repetitions. Rather, let us see a way to generate any 3-cycle we need rapidly, after a little initial preparation. The way to do this is to use both our ability to scramble elements by means of random walks, and the fact that we can use the long cycle in \( v \) to shift 3-cycles around. Let us see how.

We can assume without loss of generality that the long cycle is labelled \( (1 \ldots l) \)

Recall that \( l \geq 3n/4 \). (Any bound of the form \( l \gg n \) would do.) We can also assume that \( \text{supp} \kappa \subset \{1, 2, \ldots, l\} \); if this is not the case, we are still fine, since, for the outcome \( \sigma \) of a short random walk, \( (\text{supp} \kappa)^\sigma \subset \{1, 2, \ldots, l\} \) with probability \( \geq (l/n)^3 - o(1) \), and so, if we take a small number \( (\ll \log n) \) of random walks, it is almost certain (probability \( 1 - O(n^{-C}) \)) that one of them will give us a \( \sigma \) such that

\[
(\text{supp} \kappa)^\sigma \subset \{1, 2, \ldots, l\}.
\]

Notice that \( \text{supp} \sigma^{-1} \kappa \sigma = (\text{supp} \kappa)^\sigma \). We redefine \( \kappa \) to be \( \sigma^{-1} \kappa \sigma \), and so we obtain \( \text{supp} \kappa \subset \{1, 2, \ldots, l\} \) after all. We can assume w.l.o.g. that \( \kappa = (1 \ a \ b) \), where \( 1 \leq a, b \leq l \).

First, let us see how to construct a 3-cycle of the form \( (1 \ 2 \ x) \). This is done as follows. Taking expected values and variances, it is easy to show that, for the result \( \gamma \) of a random walk of length \( C \log n \), the probability that there are \( 1 \leq r < l \), \( 0 \leq s < l \) such that

\[
1 + s \equiv r^\gamma \mod l \\
a + s \equiv (r + 1)^\gamma \mod l
\]

is positive and bounded from below \( (\geq (l/n)^2 - o(1)) \). Moreover, we can check in time \( O(n \log n) \) whether such \( (r, s) \) exist. Taking a small number of random walks, it is almost certain that we find a \( \gamma \) and \( (r, s) \) satisfying (B.1). Then the short word

\[
\gamma v^{-s}(1 \ a \ b) v^s \gamma^{-1}
\]

gives us a 3-cycle of the form \( (r \ r + 1 \ ?) \). Changing labels, we can write this as \( (1 \ 2 \ x) \) for some \( x \).
Now \( \phi := v(1 2 x)^{-1} \) fixes 1, and acts on \( 2, \ldots, l \) as follows: \((2 3 \ldots x - 1)(x x + 1 \ldots l)\). By conjugating \((1 2 x)\) by a power of \( \phi \), we can construct quickly an element of the form \((1 y ?)\) for any given \( y \in \{2, \ldots, l\} \). Conjugating such an element by a power of \( v \), we can construct quickly an element of the form \((r s ?)\) for any \( r, s \in \{1, 2, \ldots l\} \). Since
\[
(r t ?)^{-1}(r s ?')^{-1}(r t ?)(r s ?') = (r s t)
\]
for \( ?' \) distinct from \( t \) and \( ? \) distinct from \( s \), we see that we can construct quickly any specified 3-cycle of the form \((r s t)\) with \( 1 \leq r, s, t \leq l \).

In turn, this allows us to express any 3-cycle as a short enough word, in the time we want: we simply try out random walks until we find one that sends our 3-cycle to a 3-cycle with support within \( \{1, 2, \ldots, l\} \), and then we apply the above algorithm to that 3-cycle. (Since \( l \gg n \), we succeed with probability \( 1 - O(1/n^C) \), \( C \) arbitrary, after \( \ll C \log n \) tries.) In this way, we can construct all the 3-cycles we need - each in time \( O(n(\log n)^O(1)) \).

Notice that we have tacitly assumed that we are representing (that is, store) our permutations either as we usually write them down (products of disjoint cycles) or as maps from \( \{1, 2, \ldots, n\} \) to itself; we can go from one of these two forms of representation to the other one quickly. It may be more challenging to solve the problem without using heavily the particular way in which permutations are stored – or with constraints that forbid us to access directly the internal representation of a permutation, whatever that representation may be.

In the end, the importance of finding a solution in time \( O(n(\log n)^O(1)) \) to the algorithmic problem we have discussed here will depend on whether there are potential applications. There seem to be some: see [KTT12].

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