Affine Geometric Crystal of type $G_2^{(1)}$

Toshiki NAKASHIMA

In honor of professor James Lepowsky and professor Robert L. Wilson.

Abstract. We shall realize certain affine geometric crystal of type $G_2^{(1)}$ explicitly in the fundamental representation $W(\varpi_1)$. Its explicit form is rather complicated but still keeps a positive structure.

1. Introduction

Geometric crystal is an object defined over certain algebraic (or ind-)variety which holds an analogous structure to Kashiwara’s crystal ([1], [13]). Precisely, for a fixed Cartan data $(A, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})$, a geometric crystal consists of an ind-variety $X$ over the complex number $\mathbb{C}$, a rational $\mathbb{C} \times$-action $\epsilon_i : \mathbb{C} \times \times X \rightarrow X$ and rational functions $\gamma_i, \epsilon_i : X \rightarrow \mathbb{C}$ $(i \in I)$, which satisfy certain conditions (see Definition 2.1). It has many similarity to the theory of crystals, e.g., some product structure, Weyl group actions, R-matrices, etc. Furthermore, there is a more direct correspondence between geometric crystals and crystals, called tropicalization/ultra-discretization procedure (see §2).

In [8], we presented certain conjecture. In order to mention it precisely, we need to prepare the following: Let $G$ (resp. $g$) be the affine Kac-Moody group (resp. algebra) associated with a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $B^\pm$ be the Borel subgroup and $T$ the maximal torus. Set $y_i(c) := \exp(c f_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \rightarrow T$ induced by the simple coroot $\alpha_i^\vee$ as in [21]. We set $Y_i(c) := y_i(c^{-1}) \alpha_i^\vee(c) = \alpha_i^\vee(c) y_i(c)$. Let $W$ (resp. $\tilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $g$. The Schubert cell $X_w := BwB/B$ ($w = s_{i_1} \cdots s_{i_k} \in W$) is birationally isomorphic to the variety $B_w^- := \{Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) | x_1, \cdots, x_k \in \mathbb{C}^\times \} \subset B^-$, and $X_w$ has a natural geometric crystal structure ([1], [13]).

We choose $0 \in I$ as in [2], and let $\{\varpi_i\}_{i \in I \setminus \{0\}}$ be the set of level 0 fundamental weights. Let $W(\varpi_1)$ be the fundamental representation of $U_q(g)$ with $\varpi_1$ as an
extremal weight \((\mathfrak{g})\). Let us denote its reduction at \(q = 1\) by the same notation \(W(\varpi_i)\). It is a finite-dimensional \(\mathfrak{g}\)-module. Note that though the representation \(W(\varpi_i)\) is irreducible over \(U_q(\mathfrak{g})\), the module \(W(\varpi_i)\) at \(q = 1\) for \(i \neq 1\) is not necessarily an irreducible \(\mathfrak{g}\)-module. We set \(\mathbb{P}(\varpi_i) := (W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times\).

For any \(i \in I\), define

\[
(1.1) 
\end{equation}
\]

Then the translation \(t(c_i^\nu)\) belongs to \(\widetilde{W}\) (see \[8\]). For a subset \(J\) of \(I\), let us denote by \(\mathfrak{g}_J\) the subalgebra of \(\mathfrak{g}\) generated by \(\{e_i, f_i\}_{i \in J}\). For an integral weight \(\mu\), define \(I(\mu) := \{j \in I | \langle h_j, \mu \rangle \geq 0\}\).

**Conjecture 1.1** \((\ref{8})\). For any \(i \in I\), there exist a unique variety \(X\) endowed with a positive \(\mathfrak{g}\)-geometric crystal structure and a rational mapping \(\pi: X \rightarrow \mathbb{P}(\varpi_i)\) satisfying the following property:

(i) for an arbitrary extremal vector \(u \in W(\varpi_i)_\mu\), writing the translation \(t(c_i^\nu)\mu\) as \(\xi u\) in \(\widetilde{W}\) with a Dynkin diagram automorphism \(\iota\) and \(w = s_{i_1} \cdots s_{i_k}\), there exists a birational mapping \(\xi: B_w \rightarrow X\) such that \(\xi\) is a morphism of \(\mathfrak{g}_{I(\mu)}\)-geometric crystals and that the composition \(\pi \circ \xi: B_w \rightarrow \mathbb{P}(\varpi_i)\) coincides with \(Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)_\varpi\), where \(\varpi\) is the line including \(u\),

(ii) the ultra-discretization of \(X\) is isomorphic to the crystal \(B_{\infty}(\varpi_i)\) of the Langlands dual \(\mathfrak{g}^L\).

In \[8\], we constructed a positive geometric crystal \(\mathcal{V}(\mathfrak{g})\) associated with the fundamental representation \(W(\varpi_1)\) for affine Lie algebras \(\mathfrak{g} = A_{n+1}^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}\) with this conjecture as a guide. In that article, we also show that the ultra-discretization limit of \(\mathcal{V}(\mathfrak{g})\) is isomorphic to the limit of certain coherent family of perfect crystals for \(\mathfrak{g}^L\) the Langlands dual of \(\mathfrak{g}\).

In this article, we shall construct such geometric crystal for \(\mathfrak{g} = G_2^{(1)}\). Its explicit form is given in \(\S 5\), which is rather complicated but we shall see that it is positive, which implies that the former half of our conjecture is affirmative for \(G_2^{(1)}\) and the \(i = 1\)-case. Then we obtain its ultra-discretization limit and we expect that it is isomorphic to the limit of certain coherent family of perfect crystals of type \(D_4^{(3)}\) \[7\].

**2. Geometric crystals**

In this section, we review Kac-Moody groups and geometric crystals following \[11, 12\].

**2.1. Kac-Moody algebras and Kac-Moody groups.** Fix a symmetrizable generalized Cartan matrix \(A = (a_{ij})_{i,j \in I}\) with a finite index set \(I\). Let \((\iota, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})\) be the associated root data, where \(\iota\) is a vector space over \(\mathbb{C}\) and \(\{\alpha_i\}_{i \in I} \subset \iota^*\) and \(\{\alpha_i^\vee\}_{i \in I} \subset \iota\) are linearly independent satisfying \(\alpha_j(\alpha_i^\vee) = a_{ij}\).

The Kac-Moody Lie algebra \(\mathfrak{g} = \mathfrak{g}(A)\) associated with \(A\) is the Lie algebra over \(\mathbb{C}\) generated by \(\iota\), the Chevalley generators \(e_i, f_i\) \((i \in I)\) with the usual defining relations \([10, 11]\). There is the root space decomposition \(\mathfrak{g} = \bigoplus_{\alpha \in \iota^*} \mathfrak{g}_\alpha\). Denote the set of roots by \(\Delta := \{\alpha \in \iota^* | \alpha \neq 0, \; \mathfrak{g}_\alpha \neq (0)\}\). Set \(Q = \sum_i \mathbb{Z} \alpha_i, \; Q^+ = \sum_i \mathbb{Z}_{\geq 0} \alpha_i, \; Q^\vee := \sum_i \mathbb{Z} \alpha_i^\vee\) and \(\Delta_+ := \Delta \cap Q^+\). An element of \(\Delta_+\) is called a
positive root. Let \( P \subset t^* \) be a weight lattice such that \( \mathbb{C} \otimes P = t^* \), whose element is called a weight.

Define simple reflections \( s_i \in \text{Aut}(t) \ (i \in I) \) by \( s_i(h) := h - \alpha_i(h)\alpha_i^\vee \), which generate the Weyl group \( W \). It induces the action of \( W \) on \( t^* \) by \( s_i(\lambda) := \lambda - \alpha(\alpha^{\vee})\alpha_i \). Set \( \Delta^e := \{ w(\alpha_i)|w \in W, \ i \in I \} \), whose element is called a real root.

Let \( g' \) be the derived Lie algebra of \( g \) and let \( G \) be the Kac-Moody group associated with \( g'(1) \). Let \( U_{\alpha} := \text{exp}\{\alpha \in \Delta^e\} \) be the one-parameter subgroup of \( G \). The group \( G \) is generated by \( U_{\alpha} \ (\alpha \in \Delta^e) \). Let \( U^\pm \) be the subgroup generated by \( U_{\pm\alpha} \ (\alpha \in \Delta^e = \Delta^e \cap Q_+) \), i.e., \( U^\pm := \langle U_{\pm\alpha}|\alpha \in \Delta^e \rangle \).

For any \( i \in I \), there exists a unique homomorphism; \( \phi_i : SL_2(\mathbb{C}) \rightarrow G \) such that
\[
\phi_i \left( \begin{array}{cc} c & 0 \\ 0 & c^{-1} \end{array} \right) = e^{\alpha_i^\vee}, \quad \phi_i \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) = \text{exp}(te_i), \quad \phi_i \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right) = \text{exp}(tf_i),
\]
where \( c \in \mathbb{C}^\times \) and \( t \in \mathbb{C} \). Set \( \alpha_i^\vee(c) := e^{\alpha_i^\vee}, \ x_i(t) := \text{exp}(te_i), \ y_i(t) := \text{exp}(tf_i), \ G_i := \phi_i(SL_2(\mathbb{C})), \ T_i := \phi_i(\{\text{diag}(c,c^{-1})|c \in \mathbb{C}^\times\}) \) and \( N_i := N_{G_i}(T_i) \).
Let \( T \) (resp. \( N \)) be the subgroup of \( G \) with the Lie algebra \( t \) (resp. generated by the \( N_i \)'s), which is called a maximal torus in \( G \), and let \( B^\pm = U^\pm T \) be the Borel subgroup of \( G \). We have the isomorphism \( \phi : W \sim \rightarrow N/T \) defined by \( \phi(s_i) = N_iT/T \). An element \( \bar{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i \left( \begin{array}{cc} 0 & \pm 1 \\ \mp 1 & 0 \end{array} \right) \) is in \( N_G(T) \), which is a representative of \( s_i \in W = N_G(T)/T \).

2.2. Geometric crystals. Let \( W \) be the Weyl group associated with \( g \). Define \( R(w) \) for \( w \in W \) by
\[
R(w) := \{ (i_1, i_2, \cdots, i_l) \in I^l | w = s_{i_1}s_{i_2}\cdots s_{i_l} \},
\]
where \( l \) is the length of \( w \). Then \( R(w) \) is the set of reduced words of \( w \).

Let \( X \) be an ind-variety, \( \gamma_i : X \rightarrow \mathbb{C} \) and \( \varepsilon_i : X \rightarrow \mathbb{C} \ (i \in I) \) rational functions on \( X \), and \( e_i : \mathbb{C}^\times \times X \rightarrow X \ (\langle c, x \rangle \mapsto e_i(x)) \) a rational \( \mathbb{C}^\times \)-action.

For a word \( i = (i_1, \cdots, i_l) \in R(w) \ (w \in W) \), set \( \alpha^{(j)} := s_{i_l}\cdots s_{i_{j+1}}(\alpha_{i_j}) \) \((1 \leq j \leq l)\) and
\[
e_i : \ T \times X \rightarrow X \ 
(t, x) \mapsto e_i^j(x) := e_i^{(1)}(t)e_i^{(2)}(t) \cdots e_i^{(l)}(t)(x).
\]

**Definition 2.1.** A quadruple \( (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}) \) is a \( G \) (or \( g \))-geometric crystal if

(i) \( \{1\} \times X \subset \text{dom}(e_i) \) for any \( i \in I \).

(ii) \( \gamma_j(e_i^j(x)) = e^{\alpha_i\gamma_j}(x) \).

(iii) \( e_i = e_t \) for any \( w \in W, \ i, j \in R(w) \).

(iv) \( \varepsilon_i(e_i^j(x)) = c^{-1}\varepsilon_i(x) \).

Note that the condition (iii) as above is equivalent to the following so-called Verma relations:
\[
e_{i_1}e_{i_2} = e_{i_2}e_{i_1} \quad \text{if } a_{ij} = a_{ji} = 0, 
\]
\[
e_{i_1}^ae_{i_2} = e_{i_2}e_{i_1}^{a_{ij}} \quad \text{if } a_{ij} = a_{ji} = -1, 
\]
\[
e_{i_1}e_{i_2}e_{i_1} = e_{i_2}e_{i_1}e_{i_2} \quad \text{if } a_{ij} = -2, a_{ji} = -1, 
\]
\[
e_{i_1}e_{i_2}e_{i_1}e_{i_2} = e_{i_2}e_{i_1}e_{i_2}e_{i_1} \quad \text{if } a_{ij} = -3, a_{ji} = -1, 
\]
Note that the last formula is different from the one in [1, 13, 14] which seems to be incorrect. The formula here may be correct.

2.3. Geometric crystal on Schubert cell. Let \( w \in W \) be a Weyl group element and take a reduced expression \( w = s_{i_1} \cdots s_{i_l} \). Let \( X := G/B \) be the flag variety, which is an ind-variety and \( X_w \subset X \) the Schubert cell associated with \( w \), which has a natural geometric crystal structure ([11], [13]). For \( i := (i_1, \cdots, i_k) \), set
\[
B_i^{-} := \{ Y_i(c_1, \cdots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) \mid c_1, \cdots, c_k \in \mathbb{C}^x \} \subset B^-,
\]
which has a geometric crystal structure ([13]) isomorphic to \( X_w \). The explicit forms of the action \( e^f_i \), the rational function \( \epsilon_i \) and \( \gamma_i \) on \( B_i^{-} \) are given by
\[
e^f_i(Y_{i_1}(c_1) \cdots Y_{i_k}(c_k)) = Y_{i_1}(c_1) \cdots Y_{i_k}(c_k),
\]
where
\[
C_j := c_j \cdot \sum_{1 \leq m \leq j, t_m = j} \frac{c}{c_1 \cdots c_{m-1} c_m},
\]
\[
\epsilon_i(Y_{i_1}(c_1) \cdots Y_{i_k}(c_k)) = \sum_{1 \leq m \leq k, t_m = i} \frac{1}{c_1 \cdots c_{m-1} c_m},
\]
\[
\gamma_i(Y_{i_1}(c_1) \cdots Y_{i_k}(c_k)) = c_1 \cdots c_{a_1-1} c_{a_2-1} \cdots c_{a_k-1}.
\]

2.4. Positive structure, Ultra-discretizations and Tropicalizations. Let us recall the notions of positive structure, ultra-discretization and tropicalization.

The setting below is same as [8]. Let \( T = (\mathbb{C}^x)^l \) be an algebraic torus over \( \mathbb{C} \) and \( X^*(T) := \text{Hom}(T, \mathbb{C}^x) \cong \mathbb{Z}^l \) (resp. \( X_*(T) := \text{Hom}(\mathbb{C}^x, T) \cong \mathbb{Z}^l \)) be the lattice of characters (resp. co-characters) of \( T \). Set \( R := \mathbb{C}(c) \) and define
\[
v : \ R \setminus \{ 0 \} \longrightarrow \mathbb{Z},
\]
\[
f(c) \mapsto \deg(f(c)),
\]
where \( \deg \) is the degree of poles at \( c = \infty \). Here note that for \( f_1, f_2 \in R \setminus \{ 0 \} \), we have
\[
v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2)
\]
A non-zero rational function on an algebraic torus \( T \) is called positive if it is written as \( g/h \) where \( g \) and \( h \) are a positive linear combination of characters of \( T \).

Definition 2.2. Let \( f : T \rightarrow T' \) be a rational morphism between two algebraic tori \( T \) and \( T' \). We say that \( f \) is positive, if \( \chi \circ f \) is positive for any character \( \chi : T' \rightarrow \mathbb{C} \).

Denote by \( \text{Mor}^+(T, T') \) the set of positive rational morphisms from \( T \) to \( T' \).

Lemma 2.3 ([1]). For any \( f \in \text{Mor}^+(T_1, T_2) \) and \( g \in \text{Mor}^+(T_2, T_3) \), the composition \( g \circ f \) is well-defined and belongs to \( \text{Mor}^+(T_1, T_3) \).

By Lemma 2.3 we can define a category \( T_\circ \) whose objects are algebraic tori over \( \mathbb{C} \) and arrows are positive rational morphisms.
Let $f : T \to T'$ be a positive rational morphism of algebraic tori $T$ and $T'$. We define a map $\hat{f} : X_*(T) \to X_*(T')$ by
$$\langle \chi, \hat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$
where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

**Lemma 2.4**. For any algebraic tori $T_1, T_2, T_3$, and positive rational morphisms $f \in \text{Mor}^+(T_1,T_2)$, $g \in \text{Mor}^+(T_2,T_3)$, we have $g \circ \hat{f} = \hat{g} \circ f$.

By this lemma, we obtain a functor
$$UD : \mathcal{T}_+ \to \textit{Gr}$$
$$T \mapsto X_*(T)$$
$$(f : T \to T') \mapsto (\hat{f} : X_*(T) \to X_*(T'))$$

**Definition 2.5**. Let $\chi = (X,\{\varepsilon_i\}_{i \in I},\{\omega_i\}_{i \in I})$ be a geometric crystal, $T'$ an algebraic torus and $\theta : T' \to X$ a birational isomorphism. The isomorphism $\theta$ is called a **positive structure** on $\chi$ if it satisfies

(i) for any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive.

(ii) For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C} \times T' \to T'$ defined by $e_{i,\theta}(c,t) := \theta^{-1} \circ e_i^\theta \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X,\{\varepsilon_i\}_{i \in I},\{\omega_i\}_{i \in I})$. Applying the functor $UD$ to positive rational morphisms $\varepsilon_{i,\theta}$ : $\mathbb{C} \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain
$$\hat{\varepsilon}_i := UD(\varepsilon_{i,\theta}) : \mathbb{Z} \times X_*(T) \to X_*(T)$$
$$\varepsilon_i := UD(\varepsilon_{i,\theta}) : X_*(T') \to \mathbb{Z}.$$

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X,\{\varepsilon_i\}_{i \in I},\{\omega_i\}_{i \in I})$, we associate the quadruple $(X_*(T'),\{\hat{\varepsilon}_i\}_{i \in I},\{\varepsilon_i\}_{i \in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $UD_{\theta,T'}(\chi)$. We have the following theorem:

**Theorem 2.6** ([1], [13]). For any geometric crystal $\chi = (X,\{\varepsilon_i\}_{i \in I},\{\omega_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $UD_{\theta,T'}(\chi) = (X_*(T'),\{\varepsilon_i\}_{i \in I},\{\omega_i\}_{i \in I})$ is a crystal (see [1, 2.2])

Now, let $GC^+$ be a category whose object is a triplet $(\chi,T',\theta)$ where $\chi = (X,\{\varepsilon_i\},\{\omega_i\},\{\gamma_i\})$ is a geometric crystal and $\theta : T' \to X$ is a positive structure on $\chi$, and morphism $f : (\chi_1,T_1',\theta_1) \to (\chi_2,T_2',\theta_2)$ is given by a morphism $\varphi : X_1 \to X_2$ ($\chi_i = (X_i,\cdot,\cdot)$) such that
$$f := \theta_2^{-1} \circ \varphi \circ \theta_1 : T_1' \to T_2,'$$
is a positive rational morphism. Let $CR$ be a category of crystals. Then by the theorem above, we have

**Corollary 2.7.** $UD_{\theta,T'}$: as above defines a functor
$$UD : GC^+ \to CR,$$
$$(\chi_1,T',\theta_1) \mapsto X_*(T'),$$
$$(f : (\chi_1,T_1',\theta_1) \to (\chi_2,T_2',\theta_2)) \mapsto (\hat{f} : X_*(T_1') \to X_*(T_2')).$$
We call the functor \( U \mathcal{D} \) “ultra-discretization” as \([13],[14]\) instead of “tropicalization” as in \([1]\). And for a crystal \( B \), if there exists a geometric crystal \( \chi \) and a positive structure \( \theta : T^* \rightarrow X \) on \( \chi \) such that \( U\mathcal{D}(\chi,T',\theta) \cong B \) as crystals, we call an object \((\chi,T',\theta)\) in \( \mathcal{G}C^+ \) a tropicalization of \( B \), where it is not known that this correspondence is a functor.

3. Limit of perfect crystals

We review limit of perfect crystals following \([4]\). (See also \([5],[6]\)).

3.1. Crystals. First we review the theory of crystals, which is the notion obtained by abstracting the combinatorial properties of crystal bases.

**Definition 3.1.** A crystal \( B \) is a set endowed with the following maps:

\[
\begin{align*}
\varepsilon_i & : B \rightarrow \mathbb{Z} \cup \{-\infty\}, \quad \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\} \quad \text{for} \quad i \in I, \\
e_i & : \mathbb{B} \rightarrow \mathbb{Z} \cup \{0\} \quad \rightarrow B \cup \{0\}, \quad f_i : B \cup \{0\} \rightarrow B \cup \{0\} \quad \text{for} \quad i \in I, \\
e_i(0) &= f_i(0) = 0.
\end{align*}
\]

those maps satisfy the following axioms: for all \( b, b_1, b_2 \in B \), we have

\[
\begin{align*}
\varphi_i(b) &= \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle, \\
\text{wt}(e_i b) &= \text{wt}(b) + \alpha_i \text{ if } e_i b \in B, \\
\text{wt}(f_i b) &= \text{wt}(b) - \alpha_i \text{ if } f_i b \in B, \\
e_i b_2 &= b_1 \iff f_i b_1 = b_2 \quad (b_1, b_2 \in B), \\
\varepsilon_i(b) &= -\infty \iff e_i b = f_i b = 0.
\end{align*}
\]

The following tensor product structure is one of the most crucial properties of crystals.

**Theorem 3.2.** Let \( B_1 \) and \( B_2 \) be crystals. Set \( B_1 \otimes B_2 := \{b_1 \otimes b_2; \ b_j \in B_j \ (j = 1, 2)\} \). Then we have

(i) \( B_1 \otimes B_2 \) is a crystal.

(ii) For \( b_1 \in B_1 \) and \( b_2 \in B_2 \), we have

\[
\begin{align*}
f_i(b_1 \otimes b_2) &= \begin{cases} 
f_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases} \\
e_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes e_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\
\tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \end{cases}
\end{align*}
\]

**Definition 3.3.** Let \( B_1 \) and \( B_2 \) be crystals. A strict morphism of crystals \( \psi : B_1 \rightarrow B_2 \) is a map \( \psi : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\} \) satisfying: \( \psi(0) = 0, \ \psi(B_1) \subset B_2 \), \( \psi \) commutes with all \( e_i \) and \( f_i \) and

\[
\text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \quad \text{for any} \quad b \in B_1.
\]

In particular, a bijective strict morphism is called an isomorphism of crystals.

**Example 3.4.** If \((L, \mathcal{B})\) is a crystal base, then \( \mathcal{B} \) is a crystal. Hence, for the crystal base \((L(\infty), \mathcal{B}(\infty))\) of the nilpotent subalgebra \( U_{\mathcal{Q}}^- \mathfrak{g} \) of the quantum algebra \( U_{\mathcal{Q}}(\mathfrak{g}) \), \( B(\infty) \) is a crystal.
Example 3.5. For $\lambda \in P$, let $T_\lambda := \{t_\lambda\}$. We define a crystal structure on $T_\lambda$ by

$$
\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \wt(t_\lambda) = \lambda.
$$

**Definition 3.6.** For a crystal $B$, a colored oriented graph structure is associated with $B$ by

$$b_1 \xrightarrow{\delta} b_2 \iff \tilde{f}_1 b_1 = b_2.$$

We call this graph a *crystal graph* of $B$.

**3.2. Affine weights.** Let $g$ be an affine Lie algebra. The sets $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ be as in [2.1]. We take $\dim t = \frac{1}{2}I + 1$. Let $\delta \in Q_+$ be the unique element satisfying $\{\lambda \in Q | \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = Z\delta$ and $c \in g$ be the canonical central element satisfying $\{h \in Q^\vee | \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = Zc$. We write \([9/6.1]\)

$$
c = \sum_i a_i \alpha_i^\vee, \quad \delta = \sum_i a_i \alpha_i.
$$

Let $(\ , \ )$ be the non-degenerate $W$-invariant symmetric bilinear form on $t^*$ normalized by $(\delta, \lambda) = \langle c, \lambda \rangle$ for $\lambda \in t^*$. Let us set $t^*_0 := t^* / \langle \delta \rangle$ and let $cl : t^* \rightarrow t^*_0$ be the canonical projection. Here we have $t^*_0 \cong \oplus_i \langle \alpha_i^\vee \rangle^*$. Set $t^*_0 := \{\lambda \in t^* | \langle c, \lambda \rangle = 0\}$, $(t^*_0)_0 := cl(t^*_0)$. Since $(\delta, \delta) = 0$, we have a positive-definite symmetric form on $t^*_0$ induced by the one on $t^*$. Let $\Lambda_i \in t^*_0$ ($i \in I$) be a classical weight such that $\langle \alpha_i^\vee, \Lambda_i \rangle = \delta_{i,j}$, which is called a fundamental weight. We choose $P$ so that $P_\cl := cl(P)$ coincides with $\oplus_{i \in I} Z\Lambda_i$ and we call $P_\cl$ a *classical weight lattice*.

**3.3. Definitions of perfect crystal and its limit.** Let $g$ be an affine Lie algebra, $P_\cl$ be a classical weight lattice as above and set $(P_\cl)^+ := \{\lambda \in P_\cl | \langle c, \lambda \rangle = l, \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$ ($l \in \mathbb{Z}_{>0}$).

**Definition 3.7.** A crystal $B$ is a *perfect* of level $l$ if

(i) $B \otimes B$ is connected as a crystal graph.

(ii) There exists $\lambda_0 \in P_\cl$ such that

$$
\wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} cl(\alpha_i), \quad \sharp B_{\lambda_0} = 1
$$

(iii) There exists a finite-dimensional $U'_q(g)$-module $V$ with a crystal pseudo-base $B_{ps}$ such that $B \cong B_{ps} / \pm 1$

(iv) The maps $\varepsilon, \varphi : B_{min} := \{b \in B | \langle c, \varepsilon(b) \rangle = l\} \rightarrow (P_\cl)^+_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b) \Lambda_i$.

Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals of level $l$ and set $J := \{(l, b) | l > 0, b \in B^l_{min}\}$.

**Definition 3.8.** A crystal $B_\infty$ with an element $b_\infty$ is called a *limit* of $\{B_l\}_{l \geq 1}$ if

(i) $\wt(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.

(ii) For any $(l, b) \in J$, there exists an embedding of crystals:

$$
f_{(l, b)} : T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty
$$

$$
t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \hookrightarrow b_\infty
$$

(iii) $B_\infty = \bigcup_{(l, b) \in J} \text{Im} f_{(l, b)}$.  


As for the crystal $T_\lambda$, see Example 3.3. If a limit exists for a family $\{B_l\}$, we say that $\{B_l\}$ is a coherent family of perfect crystals.

The following is one of the most important properties of limit of perfect crystals.

**Proposition 3.9.** Let $B(\infty)$ be the crystal as in Example 3.4. Then we have the following isomorphism of crystals:

$$B(\infty) \otimes B_\infty \sim \rightarrow B(\infty).$$

4. Fundamental Representations

4.1. Fundamental representation $W(\varpi_1)$. Let $c = \sum a_i^h a_i^v$ be the canonical central element in an affine Lie algebra $\mathfrak{g}$ (see [9, 6.1]), $\{\Lambda_i | i \in I\}$ the set of fundamental weight as in the previous section and $\varpi_1 := \Lambda_1 - a_1^h \Lambda_0$ the (level 0)fundamental weight.

Let $V(\varpi_1)$ be the extremal weight module of $U_q(\mathfrak{g})$ associated with $\varpi_1$ ([2]) and $W(\varpi_1) \cong V(\varpi_1)/(z_1 - 1)V(\varpi_1)$ the fundamental representation of $U_q'(\mathfrak{g})$ where $z_1$ is a $U_q'(\mathfrak{g})$-linear automorphism on $V(\varpi_1)$ (see [2] Sect 5.1).

By [2] Theorem 5.17, $W(\varpi_1)$ is an finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $\mathfrak{g}$-module $W(\varpi_1)$, which we call a fundamental representation of $\mathfrak{g}$ and use the same notation as above.

We shall present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = G_2^{(1)}$.

4.2. $W(\varpi_1)$ for $G_2^{(1)}$. The Cartan matrix $A = (a_{i,j})_{i,j=0,1,2}$ of type $G_2^{(1)}$ is as follows:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}.$$ 

Then the simple roots are

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2, \quad \alpha_2 = -\Lambda_1 + 2\Lambda_2,$$

and the Dynkin diagram is:

```
0 1 2
```

The $\mathfrak{g}$-module $W(\varpi_1)$ is a 15 dimensional module with the basis,

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}.$$

The explicit form of $W(\varpi_1)$ is given in [15], which slightly differs from our description below.

$$\begin{align*}
\text{wt}(1) &= \Lambda_1 - 2\Lambda_0, \\
\text{wt}(2) &= -\Lambda_0 - \Lambda_1 + 3\Lambda_2, \\
\text{wt}(3) &= -\Lambda_0 + \Lambda_1 - 2\Lambda_2, \\
\text{wt}(4) &= -\Lambda_0 + \Lambda_1 - \Lambda_2, \\
\text{wt}(5) &= -\Lambda_1 + 2\Lambda_2, \\
\text{wt}(6) &= -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2, \\
\text{wt}(7) &= -\Lambda_1 + \Lambda_2, \\
\text{wt}(8) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(9) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(10) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(11) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(12) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(13) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(14) &= -\Lambda_0 + \Lambda_1, \\
\text{wt}(15) &= -\Lambda_0 + \Lambda_1.
\end{align*}$$
The actions of $e_i$ and $f_i$ on these basis vectors are given as follows:

\[
\begin{align*}
    f_0 \begin{pmatrix} 0_2 & 1 & 1 & 1 & 3 & 6 & 1 & 0 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 0 & 2 & 1 \end{pmatrix}, \\
    e_0 \begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 0_2 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 6 & 1 & 2 & 2 \end{pmatrix}, \\
    f_1 \begin{pmatrix} 1 & 4 & 6 & 0_1 & 0_2 & 1 & 3 & 0 \end{pmatrix} &= \begin{pmatrix} 2 & 5 & 0_2 & 3 & 2 & 1 & 1 & 3 & 6 \end{pmatrix}, \\
    e_1 \begin{pmatrix} 2 & 5 & 0_1 & 0_2 & 1 & 3 & 6 & 0_2 \end{pmatrix} &= \begin{pmatrix} 4 & 3 & 6 & 2 & 6 & 0_2 & 1 & 3 & 6 \end{pmatrix}, \\
    f_2 \begin{pmatrix} 2 & 3 & 4 & 6 & 0_1 & 0_2 & 1 & 3 & 6 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 4 & 3 & 6 & 0_1 & 2 & 1 & 3 \end{pmatrix}, \\
    e_2 \begin{pmatrix} 1 & 4 & 6 & 0_1 & 0_2 & 1 & 3 & 6 & 0_2 \end{pmatrix} &= \begin{pmatrix} 2 & 2 & 4 & 3 & 6 & 0_1 & 2 & 1 & 3 \end{pmatrix},
\end{align*}
\]

where we only give non-trivial actions and the other actions are trivial. We can easily check that these define the module $W(\varpi_1)$ by direct calculations.

5. Affine Geometric Crystal $\mathcal{V}_1(G_2^{(1)})$

We shall construct the affine geometric crystal $\mathcal{V}(G_2^{(1)})$ in $W(\varpi_1)$ explicitly. For $\xi \in \{t_0, t_1, \ldots, t_5\}$, let $t(\xi)$ be the shift as in [2, Sect 4]. Then we have

\[
    t(\varpi_1) = s_0 s_1 s_2 s_1 s_2 s_1 =: w_1, \\
    t(\varpi_2) = s_2 s_1 s_2 s_1 s_0 s_1 =: w_2,
\]

Associated with these Weyl group elements $w_1$ and $w_2$, we define algebraic varieties $\mathcal{V}_1 = \mathcal{V}_1(G_2^{(1)})$ and $\mathcal{V}_2 = \mathcal{V}_2(G_2^{(1)}) \subset W(\varpi_1)$ respectively:

\[
\begin{align*}
    \mathcal{V}_1 &= \{ v_1(x) := Y_0(x_0)Y_1(x_1)Y_2(x_2)Y_3(x_3)Y_4(x_4)Y_5(x_5) \mid x_i \in \mathbb{C}^\times, (0 \leq i \leq 5) \}, \\
    \mathcal{V}_2 &= \{ v_2(y) := Y_2(y_2)Y_3(y_3)Y_4(y_4)Y_5(y_5)Y_6(y_6) \mid y_i \in \mathbb{C}^\times, (0 \leq i \leq 5) \}.
\end{align*}
\]

Due to the explicit forms of $f_i$'s on $W(\varpi_1)$ as above, we have $f_0^3 = 0$, $f_1^3 = 0$ and $f_2^3 = 0$ and then

\[
    (5.1)\ Y_i(c) = (1 + \frac{f_i}{c} + \frac{f_i^2}{2c^2})\alpha_i^\vee(c) \quad (i = 0, 1), \quad Y_2(c) = (1 + \frac{f_2}{c} + \frac{f_2^2}{2c^2} + \frac{f_2^3}{6c^3})\alpha_2^\vee(c).
\]

Thus, we can get explicit forms of $v_1(x) \in \mathcal{V}_1$ and $v_2(y) \in \mathcal{V}_2$. Set

\[
\begin{align*}
    v_1(x) &= \sum_{1 \leq i \leq 6} \left( X_i [1] + X_5 [7] \right) + X_0 [0] + X_2 [0] + X_3 [0], \\
    v_2(y) &= \sum_{1 \leq i \leq 6} \left( Y_i [1] + Y_5 [7] \right) + Y_0 [0] + Y_2 [0] + Y_3 [0].
\end{align*}
\]

Then by direct calculations, we have
Lemma 5.1. The rational function $X_1, X_2, \ldots$, and $Y_1, Y_2, \ldots$ are given as:

$$X_1 = 1 + \frac{x_3}{x_0} + \frac{x_1 x_3^2}{x_0 x_3} + \frac{3 x_1 x_3 x_4}{x_0 x_2} + \frac{3 x_1 x_4^2}{x_0 x_3} + \frac{x_1 x_4^3}{x_0} + \left( \frac{x_1}{x_0} + \frac{x_1 x_3}{x_0^2} \right) x_5,$$

$$X_2 = \frac{x_2^3}{x_1^2} + \frac{x_2^3}{x_2^2} + \frac{3 x_3 x_4}{x_2^2} + \frac{3 x_4^2}{x_2} + \frac{x_4^3}{x_3} + x_5 + \frac{x_3 x_5}{x_0},$$

$$+ \frac{x_0 x_3}{x_0} \left( 2 x_3 + 3 x_2 x_4 \right) + x_2^3 \left( x_4^3 + x_3 x_5 \right),$$

$$X_3 = \frac{x_2^3}{x_1} + \frac{x_3}{x_2} + \frac{x_4}{x_0} + \frac{x_2 x_4^3}{x_0 x_3} + \frac{x_2^2 x_4^3}{x_0},$$

$$X_4 = x_2 + \frac{x_1 x_3}{x_0} + \frac{2 x_1 x_4^2}{x_0} + \frac{x_1 x_2 x_4^3}{x_0} + \frac{x_1 x_2 x_5}{x_0},$$

$$X_5 = \left( \frac{x_2^2}{x_1} + \frac{x_3}{x_2} \right) x_4 + 2 x_4^2 + \frac{x_2 x_4^3}{x_3} + x_2 x_5,$$

$$X_6 = x_1 + \frac{x_1^3}{x_0} + \frac{3 x_3 x_4}{x_0} + \frac{3 x_4^2}{x_0 x_2} + \frac{x_4^3}{x_0 x_3} + \frac{2 x_3 + 3 x_2 x_4}{x_0},$$

$$X_{0_1} = x_2 x_4, \quad X_{0_2} = x_3 + \frac{x_1 x_3^2}{x_0} + \frac{x_3 x_4}{x_0 x_2} + \frac{3 x_1 x_3 x_4}{x_0 x_2} + \frac{3 x_1 x_4^2}{x_0 x_3} + \frac{x_1 x_4^3}{x_0} + x_1 x_5,$$

$$X_{0_0} = x_0 x_1, \quad X_{\tau} = x_0^2, \quad X_0 = x_0,$$

$$Y_1 = y_1 y_3, \quad Y_2 = \frac{y_2^3}{y_1} \left( y_1 y_3 + y_4^3 \right), \quad Y_3 = y_2^2 y_3 + y_2 y_4^2 + \frac{y_2^2 y_4^3}{y_1},$$

$$Y_4 = y_2 y_3 + \frac{y_1 y_4}{y_2} + 2 y_1^2 + \frac{y_2 y_4^3}{y_1}, \quad Y_5 = y_2^2 y_4,$$

$$Y_6 = y_3 + \frac{3 y_3 y_4}{y_2^2} + \frac{3 y_4^2}{y_2} + \frac{y_1^2}{y_2} \left( y_3^3 + y_3 y_4^2 y_5 \right),$$

$$Y_{0_1} = y_2 y_4, \quad Y_{0_2} = y_1 + \left( y_3 + y_1 y_3^2 \right) y_5,$$

$$Y_{0_0} = y_2^2 y_4 + y_0 \left( \frac{y_2^3}{y_1} + \frac{y_2^2 y_4^3}{y_1 y_3} \right) + \left( \frac{2 y_2^3 y_3}{y_1} + \frac{y_3^3 y_4^2}{y_1} + \frac{y_2^3 y_4^3}{y_1} \right),$$

$$Y_{\tau} = \frac{y_1}{y_2} + y_4 + \left( y_3 \left( \frac{1}{y_2} + \frac{y_1}{y_2 y_4} \right) + \frac{y_1 y_3^2}{y_2 y_4^3} \right) y_5,$$

$$Y_{\tau} = y_2^2 + y_0 \left( \frac{y_2^2}{y_1} + \frac{y_2 y_4^2}{y_1 y_3} + \frac{y_2^2 y_4^3}{y_1 y_3} \right) + \left( \frac{y_3}{y_1} + \frac{2 y_2}{y_4} + \frac{y_2 y_3^2}{y_1} + \frac{y_4}{y_2} + \frac{2 y_4^2 y_3}{y_1} \right) y_5,$$

$$Y_0 = y_2 + y_0 \left( \frac{y_2}{y_1} + \frac{y_4}{y_2 y_3} + \frac{2 y_4^2}{y_3} + \frac{y_2 y_4^3}{y_1 y_3} \right) + \left( \frac{y_3}{y_1} + \frac{2 y_2}{y_4} + \frac{y_2 y_3^2}{y_1} + \frac{y_4}{y_2} + \frac{2 y_4^2}{y_1} + \frac{y_2 y_4^3}{y_1} \right) y_5.$$
Then we have the rational function

\[
Y_7 = 1 + y_6 \left( \frac{1}{y_1 + \frac{y_1}{y_2^3 y_3}} + \frac{y_4}{y_2^3 y_3} + \frac{3 y_4^2}{y_1 y_2 y_3} + \frac{y_4^3}{y_1^2 y_3} \right) \\
+ \left( \frac{y_1}{y_2^3} + y_3 \left( \frac{2}{y_1 + \frac{3}{y_2 y_4}} \right) + \frac{y_0 y_1 y_3}{y_2^3 y_4^3} + \frac{y_3^2}{y_1^2 y_4^2} + \frac{3 y_4}{y_1 y_2} + \frac{3 y_4^2}{y_2^3 y_4^2} \right) y_5,
\]

\[
Y_{10} = \frac{y_0^2}{y_1 y_3} + y_5 + y_0 \left( \frac{1}{y_3} + \frac{y_5}{y_1} + \frac{y_3 y_5}{y_4^3} \right), \quad Y_0 = y_6.
\]

Now we solve the equation

\[
v_2(y) = a(x)v_1(x),
\]

where \(a(x)\) is a rational function in \(x = (x_0, \cdots, x_6)\). Though this equation is over-determined, we can solve it and obtain the explicit form of the unique solution as follows:

**Proposition 5.2.** Set

\[
M := \frac{x_4 x_5 x_2^2}{x_0 x_2} + \frac{3 x_4^3 x_5}{x_0 x_2} + \frac{3 x_4^2 x_5}{x_0 x_4} + x_2 \left( \frac{3 x_4^4}{x_0 x_3} + \frac{3 x_4 x_5}{x_0} \right) \\
+ x_2 \left( \frac{x_4^2}{x_0 x_1} + \frac{x_4^2}{x_1 x_3} + \frac{x_4^5}{x_1 x_4} + \frac{x_5}{x_1 x_4} + \frac{2 x_4^2 x_5}{x_0 x_3} + \frac{x_5^2}{x_0 x_4} \right),
\]

\[
N := \frac{3 x_1 x_3}{x_2 x_4} + \frac{x_2 x_4}{x_1 x_4} + \frac{2 x_5^2}{x_2 x_4} + \frac{x_1 x_3^3}{x_2 x_4} + \frac{3 x_3 x_5}{x_2 x_4} + \frac{3 x_1 x_3^2}{x_2 x_4} \\
+ \frac{x_1 x_4}{x_2^2 x_4} + \frac{x_2 x_4}{x_0} + \frac{x_1 x_3 x_5}{x_2^4 x_4^2} + \frac{x_2 x_3 x_5}{x_0 x_4} + \frac{x_1 x_3 x_5}{x_0 x_2^2 x_4^2}.
\]

Then we have the rational function \(a(x)\) and the unique solution of (5.2):

\[
a(x) = \frac{M}{(x_2 x_4)^2}, \quad y_2 = \frac{x_2}{x_1} + \frac{x_3}{x_2^2} + \frac{2 x_4}{x_2} + \frac{x_5}{x_4}, \quad y_4 = \frac{M}{y_2 x_2 x_4},
\]

\[
y_0 = a(x)x_0, \quad y_1 = \frac{y_3^2 (a(x)X_1 + y_3^2)}{a(x)X_2}, \quad y_3 = \frac{a(x)X_1}{y_1}, \quad y_5 = \frac{y_3^2 (a(x)X_2 y_3 + y_3^2 y_4^2)}{a(x)X_2 (y_3 + \frac{y_3 y_5^2}{y_4^2})},
\]

where \(X_1\) and \(X_2\) are as in **LEMMA 5.1** Furthermore, the morphism given by (5.3)

\[
\overline{\sigma} : \quad \mathcal{V}_1 \rightarrow \mathcal{V}_2, \\
age_{x_0, \cdots, x_5} \mapsto (y_0, \cdots, y_5).
\]

is a bi-positive birational isomorphism, that is, there exists the inverse birational isomorphism \(\overline{\sigma}^{-1}\) and it is also positive:

\[
x_0 = \frac{Y_7}{y_0}, \quad x_1 = \frac{Y_7}{y_0}, \quad x_2 = \frac{Y_7}{y_0}, \quad x_4 = \frac{y_2 y_4 Y_7}{y_0 Y_7},
\]

\[
x_3 = \frac{PY_7}{y_0 Y_7}, \quad x_5 = \frac{y_5 Y_7 \left( 1 + \frac{2 y_1}{y_0} + \frac{3 y_2}{y_0} + \frac{3 y_3}{y_0} + \frac{y_1 y_5^2 y_4}{y_0^2 y_1 y_3} \right)}{y_0 x_1 x_3}
\]
where $Y_\tau, Y_{\tau'}, Y_{\tau''}$ are as in Lemma 5.1 and

$$
P = y_0 + y_1 + \frac{y_0 y_1 y_5}{y_2 y_4} + 2 y_3 y_5 + \frac{2 y_0 y_3 y_5}{y_1} + \frac{y_0 y_3^2 y_5}{y_4} + \frac{2 y_1 y_3^2 y_5}{y_4^3} + \frac{3 y_0 y_3 y_5}{y_2 y_4 y_1} + \frac{3 y_0 y_4 y_5}{y_2^2} + \frac{3 y_0 y_4^2 y_5}{y_1 y_2 y_3} + \frac{3 y_0 y_4^3 y_5}{y_1^2} + \frac{3 y_1 y_3 y_5^2}{y_2^3} + \frac{6 y_3 y_5^2}{y_2 y_4} + \frac{3 y_3 y_4 y_5^2}{y_1 y_2} + \frac{3 y_3 y_4^2 y_5^2}{y_1^2}.
$$

**Proof.** By the direct calculations, we obtain the results. Indeed, certain computer softwares are useful to the calculations.

Here we obtain the positive birational isomorphism $\sigma : \mathcal{V}_1 \to \mathcal{V}_2 (v_1(x) \to v_2(y))$ and its inverse $\sigma^{-1}$ as above. The actions of $c_0^i$ on $v_2(y)$ (respectively $\gamma_0(v_2(y))$ and $\varepsilon_0(v_2(y))$) are induced from the ones on $Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)$ as an element of the geometric crystal $\mathcal{V}_2$ since $c_0^i \sigma = e_i \sigma = 0$. Now, we define the action $c_0^i$ on $v_1(x)$ by

$$
c_0^i v_1(x) = \sigma^{-1} \circ c_0^i \circ \sigma(v_1(x)). \tag{5.4}
$$

We also define $\gamma_0(v_1(x))$ and $\varepsilon_0(v_1(x))$ by

$$
\gamma_0(v_1(x)) = \gamma_0(\sigma(v_1(x))), \quad \varepsilon_0(v_1(x)) := \varepsilon_0(\sigma(v_1(x))). \tag{5.5}
$$

**Theorem 5.3.** Together with (5.4), (5.5) on $\mathcal{V}_1$, we obtain a positive affine geometric crystal $\chi := \{\mathcal{V}_i, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I} \ (I = \{0, 1, 2\})$, whose explicit form is as follows: first we have $e_i^c, \gamma_i$ and $\varepsilon_i$ for $i = 1, 2$ from the formula (2.2), (2.3) and (2.4).

$$
e_1^c(v_1(x)) = v_1(x_0, C_1 x_1, x_2, C_3 x_3, x_4, C_5 x_5), \quad e_2^c(v_1(x)) = v_1(x_0, x_1, C_2 x_2, x_3, C_4 x_4, x_5),
$$

where

$$C_1 = \frac{c x_0}{x_1} + \frac{x_0 x_3^3}{x_1^2 x_2} + \frac{x_0 x_3^2 x_4^3}{x_1^2 x_3 x_4}, \quad C_3 = \frac{c x_0}{x_1} + \frac{x_0 x_3^3}{x_1^3 x_2} + \frac{x_0 x_3^2 x_4^3}{x_1^2 x_3 x_4},
$$

$$C_5 = \frac{c x_0}{x_1} + \frac{x_0 x_3^3}{x_1^2 x_2} + \frac{x_0 x_3^2 x_4^3}{x_1 x_3 x_4}, \quad C_2 = \frac{c x_1}{x_2} + \frac{x_1 x_3}{x_2^2 x_4}, \quad C_4 = \frac{c x_1}{x_2} + \frac{x_1 x_3}{x_2 x_3},
$$

$$\varepsilon_1(v_1(x)) = x_0 + x_0 x_2^3 + x_0 x_2^2 x_3 x_4^3, \quad \varepsilon_2(v_1(x)) = \frac{x_1}{x_2} + \frac{x_1 x_3}{x_2^2 x_4},
$$

$$\gamma_1(v_1(x)) = \frac{x_0 x_2 x_4^2}{x_0 x_2}, \quad \gamma_2(v_1(x)) = \frac{x_0 x_2^2}{x_1 x_3 x_5}.$$
2.7 we obtain the ultra-discretization $U$ of perfect crystals of type $G$.

We also have $e_0^\varepsilon$, $\varepsilon_0$ and $\gamma_0$ on $v_1(x)$ as:

$$e_0^\varepsilon(v_1(x)) = v_1\left(\frac{D}{c \cdot E} x_0, \frac{F}{c \cdot E} x_1, \frac{G}{c \cdot E} x_2, \frac{D}{c \cdot G} x_3, \frac{D}{c \cdot H} x_3\right),$$

$$\varepsilon_0(v_1(x)) = \frac{E}{x_0^3 x_2 x_3}, \quad \gamma_0(v_1(x)) = \frac{x_0^2}{x_1 x_2 x_3},$$

where

$$D = c^2 x_0^2 x_2 x_3 + x_1 x_2^3 x_3^2 x_5 + c x_0 (x_1 x_3^3 + 3 x_1 x_2 x_3^2 x_4 + 3 x_1 x_2 x_3 x_5 + c x_0),$$

$$E = x_0^2 x_2 x_3 + x_1 x_2^3 x_3^2 x_5 + x_0 (x_1 x_3^3 + 3 x_1 x_2 x_3^2 x_4 + 3 x_1 x_2 x_3 x_5 + c x_0),$$

$$G = c x_0^2 x_2 x_3 + x_1 x_2^3 x_3^2 x_5 + x_0 (x_1 x_3^3 + (2 + c) x_1 x_2 x_3^2 x_4 + 3 x_1 x_2 x_3 x_5 + c x_0),$$

$$H = c x_0^2 x_2 x_3 + x_1 x_2^3 x_3^2 x_5 + x_0 (x_1 x_3^3 + 3 x_1 x_2 x_3^2 x_4 + 3 x_1 x_2 x_3 x_5 + c x_0).$$

Here we denote the positive structure on $\chi$ by $\theta : V_1 \rightarrow T$. Then by Corollary 2.1, we obtain the ultra-discretization $UD(\chi, T, \theta)$, which is a Kashiwara’s crystal.

In [8], we show that such crystal is isomorphic to the limit of certain perfect crystal for the Langlands dual algebra. So we present the following conjecture:

**Conjecture 5.4.** The crystal $UD(\chi, T, \theta)$ as above is the limit of coherent family of perfect crystals of type $D_4^{(3)}$ in [7].

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DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY, KIOicho 7-1, CHIYODA-KU, TOKYO 102-8554, JAPAN
E-mail address: toshiki@mm.sophia.ac.jp