A LARGE DATA THEORY FOR NONLINEAR WAVE ON THE SCHWARZSCHILD BACKGROUND

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Abstract. We study the solution to the semilinear wave equation satisfying the null condition on the Schwarzschild background. The initial data is given by the short pulse data on the past null infinity and is trivial on the past event horizon. We prove the uniqueness and global existence for the smooth solution in the entire exterior region and show that most of the wave packet is reflected to the future event horizon, while little is transmitted to the future null infinity. Moreover, when restricted in a null strip, the solution is large, and the degenerate energy decays at an arbitrarily polynomial decay rate of $u$, while the non-degenerate energy is bounded near the horizon. Our theorems also conclude both of the scattering theory (vanishing on the future event horizon or past event horizon) and the global Cauchy development of a semilinear wave equation with large data.

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1. Introduction

We are concerned with the semilinear wave equation in the exterior region of the Schwarzschild spacetime, of the form
\[ \Box g \varphi = Q(\partial \varphi, \partial \varphi), \tag{1.1} \]
where \( \Box g \) is the Laplace–Beltrami operator for the Schwarzschild metric, and \( Q \) denotes non-linear term that is quadratic in the first order derivatives of the field \( \varphi \) and satisfies the null condition (see Definition 1.1). The data that we will consider for (1.1) will be some specific large data.

The small data theory for (1.1) has been well studied in the Minkowski spacetime \( \mathbb{R}^{1+n} \). In dimension \( n \geq 4 \), the sufficiently fast decay rate of the linear wave allows one to prove the global existence for the nonlinear wave equations with any quadratic nonlinearity for sufficiently small data \cite{25}. However, in \( 1+3 \) dimension, F. John \cite{24} constructed a blowup example of nonlinear wave equations with certain quadratic nonlinearity. Nevertheless, if the quadratic nonlinearity satisfies the null condition, it has been proved independently by Christodoulou \cite{6} and Klainerman \cite{26} that small data lead to solutions that are global in time. There has been an extensive literature on its applications \cite{36, 37, 45, 46}. A far-reaching application of the idea of null condition in general relativity is the proof of nonlinear stability of the Minkowski spacetime \cite{8}, see also \cite{29, 30}.

Based on the structure of the null condition, Christodoulou initiated a large data theory. In \cite{7}, Christodoulou introduced the short pulse data, which is large in one certain null direction, and proved the formation of black holes due to the focusing of gravitational waves. This work has been generalized and significantly simplified by Klainerman and Rodnianski \cite{27}. In addition, the ideas used in \cite{7} and \cite{27} have been adapted to the wave equation (1.1) and the membrane equation in the Minkowski spacetime, \cite{38, 50, 51, 52}.

We briefly recall some work on the linear and nonlinear wave equations in the asymptotically flat black hole spacetimes. The decay rate of linear wave has received intensive attention; see \cite{2, 4, 11, 12, 13, 14, 17, 18, 19, 31, 32, 47, 48}. Closely related to this, there are quite a lot of results on the linearized gravity (related to Regge-Wheeler equation, Teukolsky equation, etc.) \cite{11, 8, 9, 22, 23, 34, 42, 43}. For the nonlinear wave, the global existence with power nonlinearity has been studied in \cite{5, 10, 28, 35, 39, 40, 49}; the small data global existence of the solution to (1.1) satisfying the null condition has been shown in \cite{33} in the slowly rotating Kerr spacetime. We also mention some works on the scattering of waves in black hole spacetimes \cite{15, 16, 20, 21, 41}, etc.

In the current work, we study the global-in-time behavior of smooth solutions to (1.1) with the short pulse data in the Schwarzschild spacetime. We prove the global existence in the full exterior region. Namely, starting form the data imposed on the past event horizon and the past null infinity, the nonlinear wave will propagate up to the future event horizon and the future null infinity. And in the evolution, the wave profile will preserve the short pulse type, i.e. the energy of wave is focusing along the incoming null direction. Moreover, this global result entails the scattering theorem 1.2 and the Cauchy development theorem 1.3. Compared to \cite{52}, our results cover the global existence in the future development (including and up to the future horizon) of any constant \( t \) hypersurface (see Theorem 1.3 or 1.4), whereas in \cite{52}, this part is excluded.

1.1. Main results. To state the main theorem, we introduce some necessary concepts and notations on the Schwarzschild geometry. The Schwarzschild spacetime is an \( 1+3 \)-dimensional Lorentzian manifold with the Lorentz metric taking the
following form in the Boyer-Lindquist coordinates \((x^a) = (t, r, \theta, \phi)\),
\[
    g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 + \frac{2m}{r}\right)^{-1}dr^2 + r^2d\sigma_{S^2},
\]
where \(d\sigma_{S^2}\) is always the standard metric on the unit 2-sphere \(S^2\). We consider the exterior region, which is given by \(\cal M = \mathbb{R} \times (2m, \infty) \times S^2\). For notational convenience, we set
\[
    \mu = \frac{2m}{r}, \quad \eta = 1 - \mu. \tag{1.3}
\]
Let \(r^*\) be the Regge-Wheeler tortoise coordinate
\[
    r^* = r + 2m \log(r - 2m) - 3m - 2m \log m, \tag{1.4}
\]
deﬁne the null coordinates \(u = \frac{1}{2}(t - r^*), \quad \tilde{u} = \frac{1}{2}(t + r^*)\). The future null inﬁnity \(\cal I^+\) of \(\cal M\) can be parametrized by \(\{u = +\infty\}\). For any \(c \in \mathbb{R}\), \(C_c\) is used to denote the level surface \(\{u = c\}\); Similarly, \(\cal C_u\) denotes a level set of \(u\). The intersection \(C_u \cap \cal C_u\) is a 2-sphere denoted by \(S_u\), and \(\Sigma_t\) is the constant \(t\) hypersurface.

Define \(L, L\) and \(Y\) by
\[
    L = \partial_u = \partial_t + \partial_r^*, \quad L = \partial_u = \partial_t - \partial_r^*, \quad Y = \eta^{-1}L.
\]
Then \(\{L, Y\}\) is a normalized null frame. Let \(\nabla\) be the induced covariant derivatives on \(S_u\). We can now deﬁne the “good” (\(\cal D\)) and “bad” (\(\cal L\)) derivatives.

\[
    \cal D = \{Y, \nabla\}, \quad \cal L = \{Y, L, \nabla\}.
\]
Besides, let \(\{\Omega_i\}_{i=1}^3\) be a basis of the killing vectors spanning the Lie algebra \(so(3)\). These are angular derivatives. We shall use the notation: For any given function \(\psi, \Omega \psi = \Omega_i \psi, \Omega^2 \psi = \Omega_i \Omega_j \psi, i, j \in \{1, 2, 3\}\), etc.

Near the horizon, we also use the Eddington-Finkelstein coordinates \((r, \tilde{u}, \theta, \phi)\), in which the metric reads
\[
    g_{\mu\nu}dx^\mu dx^\nu = -\eta du^2 + 2d\tilde{u}dr + r^2d\sigma_{S^2}, \tag{3}
\]
and extends across the event horizon.

We now deﬁne the null condition for a quadratic form \(Q\).

**Deﬁnition 1.1.** Consider the quadratic form \(Q(D\psi_1, D\psi_2)\). We say that \(Q\) satisﬁes the null condition if
\[
    Q = \Lambda_1(u, \tilde{u}, \theta, \phi)D\psi_1D\psi_2 + \Lambda_2(u, \tilde{u}, \theta, \phi)D\psi_2D\psi_1,
\]
and
\[
    |\partial^j_tY^i\Omega^j\Omega^i\Lambda_i| \lesssim t^{-i_1-r^{-i_2}}, \quad i = 1, 2.
\]

Now we are ready to state our ﬁrst main theorem concerning the scattering problem. The asymptotic characteristic data will be given on the future null inﬁnity \(\cal I^+\) and the future event horizon \(\cal H^+\). Let \(\delta > 0\) and let \(\varphi_{+\infty} : \cal I^+ \to \mathbb{R}\) be such that
\[
    \varphi_{+\infty}(u, \theta, \phi) = \begin{cases} 0, \quad \text{if } u > 0 \text{ or } u < -\delta, \\ \delta^3 \psi_0 \left(\frac{u}{\delta}, \theta, \phi\right), \quad \text{if } -\delta \leq u \leq 0, \end{cases} \tag{1.5}
\]
where \(\psi_0 : [-1, 0] \times S^2 \to \mathbb{R}\) is a smooth, compactly supported function deﬁned on \(\cal I^+\). We recall that \(\cal D^+(\Sigma)(\cal D^-\Sigma)\) is the future (past) Cauchy development of \(\Sigma\).

**Theorem 1.2** (Scattering Theorem). Consider on the Schwarzschild background the scattering problem (without contribution from \(\cal H^+\)) for the semilinear wave equation \(\cal L\) with \(Q\) satisfying the null condition. The asymptotic characteristic data are given by: ﬁx any number \(\alpha \geq 0\),
\[
    \lim_{u \to +\infty} u^{\alpha+1} \varphi(u, \tilde{u}, \theta, \phi) = \varphi_{+\infty}(u, \theta, \phi), \quad \lim_{u \to +\infty} \varphi(u, \tilde{u}, \theta, \phi) \equiv 0,
\]
where $\varphi_{+\infty} \in C^\infty(I^+)$ is defined in (1.5). If $\delta$ is small enough, then (1.1) has a unique and globally smooth solution in the null strip $N_1 := D^-(I^+) \cap D^-(H^+) \cap D^+(\Sigma_0) \cap \{-\delta \leq u \leq 0\}$ whose radiation field is exactly $\varphi_{+\infty}$ restricted to $u \in [-\delta, 0]$.

If $\alpha \geq 1$, and $\delta$ is small enough, (1.1) has a unique and globally smooth solution in $D^-(I^+) \cap D^-(H^+) \cap D^+(\Sigma_0)$ and the radiation field is exactly $\varphi_{+\infty}$. And most of the energy is concentrated in $N_1$, while little is dispersing out of the null strip (see Figure 2).

On the other hand, the global result for the Cauchy problem of the semilinear wave with large data is proved.

**Theorem 1.3** (Cauchy development). Consider the Cauchy problem for the semilinear equation (1.1) with $Q$ satisfying the null condition and the data $(\varphi|_{\Sigma_0}, \partial_t \varphi|_{\Sigma_0}) = (\psi_0, \psi_1)$. If $\delta$ is small enough, there is a smooth initial data set $(\psi_0, \psi_1)$ verifying

$$E_k(\psi_0, \psi_1) \sim \delta^{-k+1}, \quad \forall k \in \mathbb{N},$$

where $E_k(\psi_0, \psi_1) = \int_{\Sigma_0} (|D^k \psi_0|^2 + |D^{k-1} \psi_1|^2)dx^3$, so that a unique and globally smooth solution $\varphi$ exists in $D^+(\Sigma_0) \cap D^-(H^+) \cap D^-(I^+)$ (see Figure 2).

Theorems 1.2 and 1.3 follow from the theorem below. Define

$$\varphi_{-\infty}(u, \theta, \phi) = \begin{cases} 0, & \text{if } u < 0 \text{ or } u > \delta, \\ \delta^{\frac{2}{k}} \psi_0 \left(\frac{u}{\delta}, \theta, \phi\right), & \text{if } 0 \leq u \leq \delta, \end{cases} \quad (1.6)$$

where $\psi_0 : [0, 1] \times S^2 \to \mathbb{R}$ is a smooth, compactly supported function defined on $I^-$. Let $r_{NH}$ be close to $2m$, satisfying $2m < r_{NH} < 1.2r_{NH} < 3m$.
Theorem 1.4. Consider on the Schwarzschild background the semilinear wave equation \((1.1)\) with \(Q\) satisfying the null condition, where the asymptotic characteristic data are given by: fix any number \(\alpha \geq 0,\)

\[
|u|^{\alpha+1}\varphi(u,\theta,\phi)|_{I^+} = \varphi_-\infty(u,\theta), \quad \varphi(u,\theta,\phi)|_{H^+} \equiv 0,
\]

where \(\varphi_-\infty \in C^\infty(I^-)\) is defined in (1.6). There exists a constant \(\delta_0\) such that if \(\delta < \delta_0,\) (1.1) has a unique and globally smooth solution \(\varphi\) in the null strip \(R_1 \cup R_2 := D^+(I^-) \cap D^-(H^+) \cap \{0 \leq u \leq \delta\},\) with \(R_1 := D^+(I^-) \cap D^-(\Sigma_1) \cap \{0 \leq u \leq \delta\}\) and \(R_2 := D^+(\Sigma_1) \cap D^-(H^+) \cap \{0 \leq u \leq \delta\}\) (see Figure 3). Furthermore, the solution \(\varphi\) obeys the following estimates:

\[
|L^{1+k}L^\Omega\varphi| \lesssim \delta^{-1-k}|t|^{-\alpha-1},\quad |\bar{D}L^{1+k}L^\Omega\varphi| \lesssim \delta^\frac{1}{2}k|t|^{-\alpha-\frac{1}{2}}, \quad \text{in } R_1,
\]

and for any \(\beta\) independent of \(\alpha,\) and \(\beta > \frac{1}{2},\)

\[
|\eta^2 L^{1+k}L^\Omega\varphi| \lesssim \delta^{-\frac{1}{2}k}|u|^{-\beta},\quad |\eta^2 \bar{D}L^{1+k}L^\Omega\varphi| \lesssim \delta^{\frac{1}{2}k}|u|^{-\beta},\quad \text{in } R_2,
\]

\[
|L^{1+k}Y^\Omega\varphi| \lesssim \delta^{\frac{1}{2}k},\quad |\bar{D}L^{1+k}Y^\Omega\varphi| \lesssim \delta^{\frac{1}{2}k},\quad \text{in } R_2 \cap \{r < r_{NH}\}.
\]

If \(\alpha \geq 1,\) \(\beta > \frac{1}{2}\) and \(\delta < \delta_0,\) then the solution to (1.1) globally exists in the full exterior region, namely, \(D^+(I^-) \cap D^-(H^-) \cap D^-(H^+) \cap D^-(I^-)\), and the radiation field on \(I^-\) is exactly \(\varphi_-\infty\). Moreover, the wave packet is mostly transmitted to the future event horizon \(H^+\), and little is propagated to the future null infinity \(I^+\).

We remark that near the future event horizon, the non-degenerate energy in fact decays in terms of \(u\) (or \(t\), combining with the small data theory in \(D^+(\Sigma_\delta) \cap D^+(I^-)\).

In contrast to [52], our results are able to cover both of the past region \(D^-(\Sigma_1) \cap D^+(H^-) \cap D^+(I^-)\) and the future region \(D^+(\Sigma_1) \cap D^-(H^+)\), whereas by virtue of the main body of proof in [52], the global existence of (1.1) is merely obtained in the past region \(D^+(I^-) \cap D^-(\Sigma_{-1})\) (with Minkowski background) in [52]. Besides, here we show in \(R_1\) the decay rate of \(|u|^{-\alpha-1} \sim |t|^{-\alpha-1}\), as long as it holds true initially; and in \(R_2 \cap \{r \geq r_{NH}\}\), arbitrarily polynomial decay rate of \(|u| \sim t\).
As we can see, the exterior region is divided into three parts: \( I := \mathcal{R}_1 \cup \mathcal{R}_2 \), \( II := \mathcal{D}^+(\mathcal{I}^-) \cap \mathcal{D}^+(\mathcal{H}^-) \cap \mathcal{D}^+(\mathcal{C}_0) \) and \( III := \mathcal{D}^+(\mathcal{I}^-) \cap \mathcal{D}^+(\mathcal{H}^-) \cap \mathcal{D}^+(\mathcal{C}_2) \) (see Figure 3). Our main estimates are derived in region I, where the energy is large, while the solution is trivial in region II, and it is shown to be small in region III.

Roughly speaking, to carry out the energy argument in region I, we employ the following multipliers:

\[
\begin{align*}
\xi_1 &= r^{2\alpha} \eta L + \delta^{-1} u^{2\alpha} L, \quad \text{in } \mathcal{R}_1, \\
\xi_2 &= \eta L + \delta^{-1} (1 + \mu) L, \quad \text{in } \mathcal{R}_2, \\
\xi_3 &= (1 + y_2(r^*)) L + \delta^{-1} y_1(r^*) Y, \quad \text{in } \mathcal{R}_2 \cap \{r < r_{NH}\}.
\end{align*}
\]

The multiplier \( \xi_1 \), see Section 3.3.1, is used to show the desired energy decay rate \(|u|^{-\alpha}\) in \( \mathcal{R}_1 \). We remark that the decay rate in \( \mathcal{R}_1 \) is already given by the asymptotic characteristic data (or radiation fields on the past null infinity). However, in \( \mathcal{R}_2 \), one of the main difficulties is to figure out the quantitative decay rates for both of the degenerate and non-degenerate energies. The idea beyond the choice of \( \xi_2 \) as a multiplier in \( \mathcal{R}_2 \) lies in the facts that, upon using \( \xi_2 \), there is a positive (i.e. with a favorable sign) contribution from the spacetime integral

\[
\iint \left( \delta^{-1} |L\phi|^2 + \delta^{-1} |\nabla \phi|^2 + |L\phi|^2 \right) \eta r^2 u du d\sigma_{S^2},
\]

in the energy estimate, noting that \( r \) is always finite in \( \mathcal{R}_2 \), see Section 4.3.1. With this positive spacetime integral, a pigeon-hole argument can be applied to achieve the decay of energy (degenerate on the horizon) in \( \mathcal{R}_2 \). The non-degenerate energy will be retrieved by using \( \xi_3 \), which is actually the red-shift vector field \( Y \), and is now well adapt to the sizes of the profiles \( L\phi \) and \( Y\phi \), see Section 4.4.1.

Finally, the global existence in region III is actually a small data problem. This comes from the facts that the solution in region I is small on the last incoming cone \( C_3 \), as proved in Sections 3.5 and 4.6 without any lose of decay. As a consequence, we can make use of the global result for small data in \( \mathcal{R}_3 \) in region III, which holds true if the decay on \( C_3 \) is strong enough. For this purpose, we require \( \alpha \geq 1 \) and \( \beta > \frac{5}{2} \), and take \( uL \) as one of the commutators in \( \mathcal{R}_1 \), see Section 3.3.6.

The paper is organized as follows. In Section 2, we introduce several notations and the method of vector field in the Schwarzschild spacetime. In Section 3 we show the global existence of the scattering problem from the past event horizon and the past null infinity. In Section 4 the global existence up to the future event horizon and the future null infinity is stated. More background knowledge is collected in Appendix A.

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2. Preliminaries

2.1. Notations. We clarify the measures: \( d\mu_D = r^2 u du d\sigma_{S^2} \), \( d\mu_{C\alpha} = r^2 u du d\sigma_{S^2} \), \( d\mu_{C_\alpha} = r^2 u du d\sigma_{S^2} \), \( d\mu_{C_\alpha^2} = r^2 \eta u du d\sigma_{S^2} \). In \( (r,u) \) coordinates, \( d\mu_{C_\alpha^2} = r^2 dr d\sigma_{S^2} \).
We denote $∥ \cdot ∥_{L^2(C_1)}$, $∥ \cdot ∥_{L^2(C_2)}$ and $∥ \cdot ∥_{L^2(C^{N+1})}$ the $L^2$ norm with the corresponding volume form respectively.

Define the following truncated cone: $C_{u_1,u_2}^{[u_1,u_2]} := \{ u_1 \} \times [u_1, u_2] \times S^2$, $C_0^{[u_1,u_2]} := [u_1, u_2] \times \{ u_1 \} \times S^2$. The spacetime domain bounded by $C_{u_1,u_2}, C_0^{[u_1,u_2]}$, and $C_{b_1}^{[u_1,u_2]}$ is denoted by $D_{u_1,u_2}^{[u_1,u_2]}$.

Define the degenerate and non-degenerate null vector fields: $W \in \{ L, L \}$, $Z \in \{ L, Y \}$. We shall introduce the following simplifications: $W_{p,q}^n := L^p L^q$, $p + q = n$, $W_{p,q}^n \in \{ L^k L^l | k + l = n, k \leq p, l \leq q \}$ and $Z_{p,q}^n := L^p Y^q$, $p + q = n$, $Z_{p,q}^n \in \{ L^k Y^l | k + l = i, k \leq p, l \leq q \}$.

We use the notation $C, c$ to denote positive numerical constants that are free to vary from line to line. We allow $C, c$ to depend on the amount of Sobolev regularity that we assume on the initial data, but we always choose these constants so that they are independent of the solution. The notation $x \lesssim y$ means $x \leq cy$ for a universal constant $c$, and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. We always use the notation $(x) = \sqrt{1 + x^2}$.

Throughout this paper, we set
\[ \varphi_i = \Omega^i \varphi, \quad |\varphi_k|^2 = \sum_{i \leq k} |\Omega^i \varphi|^2. \] (2.1)

**2.2. Energy estimates scheme.** We would like to briefly review the vector field method. In the case of wave equation on the Schwarzschild background, the energy momentum tensor associated to the wave equation for $\psi$ is defined to be
\[ T_{\alpha\beta}(\psi) = D_{\alpha} \psi D_{\beta} \psi - \frac{1}{2} g_{\alpha\beta} D^\gamma \psi D_\gamma \psi, \] (2.2)
where $D$ denotes the covariant derivative corresponding to the spacetime metric $g$. We note that $T_{\alpha\beta}$ is symmetric and there is the divergence identity for the energy-momentum tensor,
\[ D^\alpha T_{\alpha\beta}(\psi) = \Box g \psi \cdot D_\beta \psi. \] (2.3)

Given a vector field $\xi$, which is usually called a multiplier vector field, the associated energy currents are defined as follows
\[ P_{\xi}^\alpha(\psi) = T_{\alpha\beta}(\psi) \cdot \xi^\beta, \quad K^\xi(\psi) = T^{\mu\nu}(\psi) \xi_{\mu\nu}, \]
where $\xi_{\mu\nu}$ is the deformation tensor defined by
\[ \xi_{\mu\nu} = \frac{1}{2} \mathcal{L}_\xi g_{\mu\nu} = \frac{1}{2} (D_\mu \xi_\nu - D_\nu \xi_\mu). \]
Due to (2.3), we have
\[ D^\alpha P_{\xi}^\alpha(\psi) = K^\xi(\psi) + \Box g \psi \cdot \xi \psi. \] (2.4)

Integrating (2.4) on the spacetime domain, we then derive the energy identity,
\[ \int_{\mathcal{C}_{[u_1,u_2]}} T_{\partial k \xi}(\psi) r^2 d\u d\sigma S^2 + \int_{\mathcal{C}_{[u_1,u_2]}} T_{\partial k \xi}(\psi) r^2 d\u d\sigma S^2 \]
\[ = \int_{\mathcal{C}_{[u_1,u_2]}} T_{\partial k \xi}(\psi) r^2 d\u d\sigma S^2 + \int_{\mathcal{C}_{[u_1,u_2]}} T_{\partial k \xi}(\psi) r^2 d\u d\sigma S^2 \]
\[ + \int_{\mathcal{D}_{[u_1,u_2]}} (-2 K^\xi(\psi) - 2 \Box g \psi \cdot \xi \psi) \eta r^2 d\u d\sigma S^2. \] (2.5)
2.3. **Vector fields.** In terms of the null frame \(\{e_A, e_B, L, L\}\), where \(\{e_A, e_B\}\) is an orthonormal basis on \(S^2_{\mathbb{R}^n}\), the energy-momentum tensor \((2.2)\) reads \(\mathcal{T}^\text{m}(\psi) = |L\psi|^2, \mathcal{T}^\text{u}(\psi) = \eta|\nabla\psi|^2, \mathcal{T}^\text{u}(\psi) = |L\psi|^2\). The corresponding deformation tensor for \(L\) is computed as

\[
\begin{align*}
\mathcal{L}_\pi & = 0, \quad \mathcal{L}_\pi^u = 0, \\
\mathcal{L}_\pi^u & = -\frac{\mu\eta}{r}, \quad \mathcal{L}_\pi^A = \frac{\eta g_{AB}}{r},
\end{align*}
\]  

and \(L_{\pi\alpha\beta} = -L_{\pi\alpha\beta}\).

2.3.1. **The multiplier vector fields.** We employ the multiplier of the following type

\[
X = f_1(u, \psi)\mathcal{L} + f_2(u, \psi)\mathcal{L},
\]

with \(f_i(u, \psi), i = 1, 2\) being some functions to be determined. The current is now calculated as

\[
K^X(\psi) = \partial_r f_1 g^{\mu u} |L\psi|^2 - \frac{1}{2} (\partial_r f_1 + \mu f_1) |\nabla\psi|^2 - \frac{2\eta}{r} f_1 g^{\mu u} L\psi L\psi
\]

\[
+ \partial_r f_2 g^{\mu u} |L\psi|^2 - \frac{1}{2} (\partial_r f_2 - \mu f_2) |\nabla\psi|^2 + \frac{2\eta}{r} f_2 g^{\mu u} L\psi L\psi.
\]

\[
(2.7)
\]

2.3.2. **The commutators.** For most of the computations throughout this paper, we will need several commutator formulae. Here, we collect all of them as follows,

\[
\begin{align*}
[\Omega, \nabla] & = 0, \quad [L, \nabla] = -\frac{\eta}{r} \nabla, \quad [L, \nabla] = \frac{\eta}{r} \nabla, \\
[\Box_g, \Omega] & = 0, \quad [L, \Omega] = 0, \quad [L, \Omega] = 0, \quad (2.8)
\end{align*}
\]

and the commutator with the wave operator:

\[
\begin{align*}
[\Box_g, Y] & = \frac{2m}{r^2} Y^2 - \frac{2}{r^2} \Delta + \frac{1}{r^2} Y - \frac{1}{r^2} L, \\
[\Box_g, L] & = \frac{n - \mu}{r^2} (L - L) + \frac{2n - \mu}{r^2} \Delta + \mu \Delta, \\
[\Box_g, L] & = \eta \frac{n - \mu}{r^2} (L - L) - \frac{2n - \mu}{r^2} \Delta - \mu \Delta \Box_g.
\end{align*}
\]

(2.9)

We also present the derived commutator,

\[
[\Box_g, u\mathcal{L}] = \left(1 - \frac{\mu u}{r}\right) \Box_g + \left(\frac{\mu - 2\eta}{r} u - 1\right) \Delta - \frac{1}{r^2} L + \frac{(n - \mu)u}{r^2} (L - L),
\]

which will be used in the region \(\mathcal{R}_1\). We know that, \(r \sim |u|\) in \(\mathcal{R}_1\), and hence,

\[
[\Box_g, u\mathcal{L}] \sim \Box_g \pm \Delta \pm \frac{1}{r} (L - L), \quad \text{in } \mathcal{R}_1.
\]

(2.10)

In general, we conclude the following lemma.

**Lemma 2.1.** Let \(W \in \{L, \mathcal{L}\}, W^{n}_{p,q} := L^p \mathcal{L}^q, p + q = n\). Then

\[
[\Box_g W^{n}_{p,q} \varphi_k = W^{n-1}_{p,q} \Box_g \varphi \pm W^{n-1}_{p,q} (\varphi_k),
\]

where

\[
W^{n-1}_{p,q} (\varphi_k) = \sum_{i=0}^{n-1} \frac{1}{r^{n+1}} (L W^i_{p,q} \varphi_k \pm L W^i_{p,q} \varphi_k \pm r \Delta W^i_{p,q} \varphi_k),
\]

and \(W^i_{p,q} \in \{L^k \mathcal{L}^l | k + l = i, k \leq p, l \leq q\}\).
Denote $Z^{n}_{\mu,\nu} := L^{p}Y^{q}$, $p + q = n$ and $Z \in \{L, Y\}$. We have
\[
\Box_g Z^{n}_{\mu,\nu} \varphi_k = Z^{n}_{\mu,\nu} \Omega^k \Box_g \varphi + Z_n(\varphi_k) \pm Z_{\leq n-1}(\varphi_k),
\]
where
\[
Z_n(\varphi_k) = (q - p) \frac{2m}{r^2} Y Z^{n}_{\mu,\nu} \varphi_k,
\]
\[
Z_{\leq n-1}(\varphi_k) = \sum_{i \leq n-1} \frac{1}{r^{n+1-i}} \left( Z^{i}_{\mu,\nu} Y \varphi_k \pm Z^{i}_{\mu,\nu} L \varphi_k \pm r \partial Z^{i}_{\mu,\nu} \varphi_k \right) + l.o.t.,
\]
and $Z^{i}_{\mu,\nu} \in \{L^{k}Y^{l} | k + l = i, k \leq p, l \leq q\}$. Here l.o.t. denotes lower order terms in terms of $r$ weight and derivatives.

2.4. Null condition. The null condition is defined in Definition 1.1. There are several obvious examples of quadratic null forms: $Q_0 = g^\mu\nu \partial_n \psi \partial_r \psi; Q_{\mu\nu} = \partial_n \psi \partial_r \psi - \partial_r \psi \partial_n \psi$. Without confusion, we denote these null forms by $Q$ as well. Given any vector field $X$, let $Q \circ X(D\psi_1, D\psi_2) = Q(DX\psi_1, D\psi_2) + Q(D\psi_1, DX\psi_2)$ and $\{Q, X\} = XQ - Q \circ X$. One has then
\[
||Q,\varphi||_{(D\psi_1, D\psi_2)} \leq |D\psi_1 D\psi_2| + |D\psi_5 D\psi_1|, \quad (2.11)
\]
\[
||D,\varphi||_{(D\psi_1, D\psi_2)} \leq r^{-1} (|D\psi_1 D\psi_2| + |D\psi_2 D\psi_1|). \quad (2.12)
\]
Implied by the second line in (2.11),
\[
||uL,\varphi||_{(D\psi_1, D\psi_2)} \leq |u| r^{-1} (|D\psi_1 D\psi_2| + |D\psi_2 D\psi_1|).
\]
Hence
\[
||uL,\varphi||_{(D\psi_1, D\psi_2)} \leq |D\psi_1 D\psi_2| + |D\psi_2 D\psi_1|, \quad \text{in } \mathcal{R}_1, \quad (2.12)
\]
since $r \sim |u|$ in $\mathcal{R}_1$.

By the formula (2.8), we can calculate that for a general $Q$ satisfying the Definition 1.1 (2.11) and (2.12) are always valid. In order to do the main estimates in $\mathcal{R}_1 \cup \mathcal{R}_2$, we only need these inequalities (2.12), (2.11).

3. Solution from the past event horizon and the past null infinity

Let $\mathcal{R}_2$ be the null strip $\mathcal{D}^+(\Sigma^-) \cap \mathcal{D}^-(\Sigma_1) \cap \{0 \leq u \leq \delta\}$. Then in $\mathcal{R}_1, u \leq 1, r^* = u - u > -1$, and hence $\langle r^* \rangle \sim r$. Moreover, $t \sim \langle u \rangle \sim r$ in $\mathcal{R}_1$. We remind ourselves that $\langle u \rangle = \sqrt{|u|^2 + 1}$. In this section, we will prove that the solution exists from the past event horizon and the past null infinity up to any finite $u = u_1 \sim 1$. Without loss of generality, we assume that $u_1 = 1$ in the following discussion. And we shall simplify the notation $C^0_{u_0}$ by $C_u$, $C^0_{\{0, u\}}$ by $C^\infty_2$, where $-\infty \leq u_0 \leq 0$.

3.1. Initial data in $\mathcal{R}_1$. We refer to [7] for the short pulse data, and also refer to Section 3 in [52] for such data in the setting of wave equation.

Let $-\infty \leq u_0 \leq 0$ and $C_{u_0} = \{u = u_0\}$ be the initial outgoing light-cone. And $\mathcal{H}^- = \{u = -\infty\}$ denotes the past event horizon. Our data will be prescribed on $\mathcal{H}^- \cup C_{u_0}$. First of all, we require that the data of (1.1) verify:
\[
\varphi \equiv 0, \quad \text{on } \mathcal{H}^- \cup C_{[\infty, 0]}^{\infty, 0}. \quad (3.1)
\]
Consequently, according to the weak Huygens principle, the solution to (1.1) is trivial in the region $\mathcal{D}^+(\mathcal{H}^-) \cap \mathcal{D}^+(C_{-\infty, 0}^{\infty, 0})$, i.e. $\varphi \equiv 0$ in $\{u \leq 0, u \geq u_0\}$. Secondly, we set
\[
\varphi \big|_{C_{u_0}^{[0, 0], \alpha}} = \frac{\delta^\times}{|u_0|^{\alpha+1}} \psi_0 \frac{u}{\delta, \theta, \phi}, \quad \alpha \geq 0. \quad (3.2)
\]
where $\psi_0 : [0, 1] \times S^2 \to \mathbb{R}$ is a smooth, compactly supported function. We remark that when $\alpha = 0$, the factor $\frac{1}{|u_0|}$ manifests the decay of linear wave.
The data \( (3.2) \) immediately entail that for all \( l, k \in \mathbb{N} \),
\[
|u_0|^{\alpha + 1} \delta^\frac{1}{2} \| L^{l+1} \Omega^k \varphi \|_{L^\infty(C_{t_0})} + |u_0|^{\alpha} \| L^{l+1} \Omega^k \varphi \|_{L^2(C_{t_0})} \lesssim \delta^{-l},
\]
\[
|u_0|^{\alpha + 2} \| \nabla L^l \Omega^k \varphi \|_{L^\infty(C_{t_0})} + |u_0|^{\alpha + 1} \delta^{-\frac{1}{2}} \| \nabla L^l \Omega^k \varphi \|_{L^2(C_{t_0})} \lesssim \delta^{-l}. \tag{3.3}
\]

Following \([52]\), we commute \((1.1)\) with \( \Omega^k \), rewrite it as an ODE for \( L^l \Omega^k \varphi \) and integrate along \( L \) to derive
\[
\| L^l \Omega^k \varphi \|_{L^\infty(C_{t_0})} \lesssim \delta^\frac{1}{2} |u_0|^{-\alpha - 2}.
\]

We expect that these initial informations will be preserved during the evolution of the wave equation. For this purpose, we should relax \( \nabla \) a little bit, namely, we only expect that the estimate \( |u_0|^{\alpha} \| \nabla L^l \Omega^k \varphi \|_{L^2(C_{t_0})} \lesssim \delta^\frac{1}{2} \), rather than the originally corresponding one in \((3.3), \) propagates along the flow of \((1.1)\). This will be reflected in the definitions of energies \( E_k(u, u) \) \((3.5)\) and \( LF_{1+k}(u, u) \) \((3.6)\).

### 3.2. Bootstrap assumptions in \( \mathcal{R}_1 \)

To conduct the energy argument in \( \mathcal{R}_1 \), we need the commutators: \( L, \Omega \) and \( S \), where
\[
\tilde{S} := (u) L.
\]

Then a family of energy norms are defined as follows. Given any fixed number \( N \in \mathbb{N}, N \geq 6, \) and \( \alpha \geq 0, \) we define
\[
E_l(u, u) = \langle u \rangle^\alpha \| L^l \varphi \|_2 \| L^l (C_t) \| + \delta^{-\frac{1}{2}} \langle u \rangle^\alpha \| \nabla L^l \varphi \|_2 \| L^l (C_t) \|, \tag{3.5}
\]
for \( 0 \leq l \leq N \). And for \( 0 \leq k \leq N - 1, \)
\[
LF_{1+k}(u, u) = \delta \langle u \rangle^\alpha \| L^k \varphi_k \|_2 \| L^k (C_t) \| + \delta^\frac{1}{2} \langle u \rangle^\alpha \| \nabla L^k \varphi_k \|_2 \| L^k (C_t) \|,
\]
\[
LF_{2+k}(u, u) = \delta \langle u \rangle^\alpha \| L^k \varphi_k \|_2 \| L^k (C_t) \| + \delta^\frac{1}{2} \langle u \rangle^\alpha \| \nabla L^k \varphi_k \|_2 \| L^k (C_t) \|,
\]
\[
\tilde{S} F_{1+k}(u, u) = \delta^{-\frac{1}{2}} \langle u \rangle^\alpha \| \nabla \tilde{S} \varphi_k \|_2 \| L^k (C_t) \|,
\]
\[
\tilde{S} F_{2+k}(u, u) = \delta^{-\frac{1}{2}} \langle u \rangle^\alpha \| \nabla \tilde{S} \varphi_k \|_2 \| L^k (C_t) \|,
\]
and
\[
F_{1+k}(u, u) = LF_{1+k}(u, u) + \tilde{S} F_{1+k}(u, u),
\]
\[
F_{2+k}(u, u) = LF_{2+k}(u, u) + \tilde{S} F_{2+k}(u, u),
\]
and
\[
\tilde{t} E_k(u, u) = \delta^{-1} \langle u \rangle^{\alpha + 1} \| L \varphi_k \|_2 \| L \varphi_k \|_2 \| L^k (C_t) \| + \langle u \rangle^{\alpha + 1} \| L \varphi_k \|_2 \| L^k (C_t) \|. \tag{3.9}
\]

Finally, we also define for \( 0 \leq j \leq N - 2 \)
\[
\tilde{t} E_{1+j}(u, u) = \delta^{-1} \langle u \rangle^{\alpha + 1} \| L \tilde{S} \varphi_k \|_2 \| L^k (C_t) \|. \tag{3.10}
\]

With these definitions of energy norms, the data \((3.1)\) and \((3.2)\) satisfy
\[
E_l(u_0, \delta) + F_{k+1}(u_0, \delta) + \tilde{t} E_k(u_0, \delta) + \tilde{t} E_{1+j}(u_0, \delta) \leq I_{N+1}, \tag{3.11}
\]
where the first line has relaxed the initial bound \((3.3), \) and \( 0 \leq l \leq N, \) \( 0 \leq k \leq N - 1, \) \( 0 \leq j \leq N - 2. \) Here \( I_{N+1} \in \mathbb{R}^+ \) is a universal constant manifesting the initial norm. The subindex \( N + 1 \) in \( I_{N+1} \) denotes the number of derivatives used in the energy norms.

The energy estimates in \( \mathcal{R}_3 \) will be based on a standard bootstrap argument. Fix \( u_0 \leq u^* \leq 1 \) and \( 0 \leq u^* \leq \delta. \) We assume that there is a large constant \( M \) to
be determined, such that the solution of \[1.1\] defined on the domain \(\mathcal{D}_{0,\tilde{u}}^{0,\tilde{u}^*} \subset \mathcal{R}_1\) enjoys the estimate
\[
E_i(u', u') + E_i(u', u') + F_{1+k}(u', u') + F_{1+k}(u', u') + iE_k(u', u')
+ iE_{1+j}(u', u') \leq M, \tag{3.12}
\]
for all \(u' \in [u_0, u]\) and \(u' \in [0, u]\), where \(l \leq N, k \leq N - 1, j \leq N - 2, \) and \(u \leq u^*\) and all \(0 \leq u \leq u^*\). At the end of the current section, we aim to show that the \(M\) in \(3.12\) can be actually replaced by \(\frac{M}{\alpha}\), and the choice of \(M\) depends only on the norm of the initial data but not the wave profile \(\varphi\). Then the bootstrap argument will be closed and it yields the following estimates: There is a constant \(C(I_{N+1})\) depending only on \(I_{N+1}\) (in particular, not on \(\delta\) and \(u_0\)), so that for all \(u \leq 1\) and all \(0 \leq u \leq \delta\), we have
\[
E_i(u, u) + E_i(u, u) + F_{1+k}(u, u) + F_{1+k}(u, u) + iE_k(u, u)
+ iE_{1+j}(u, u) \leq C(I_{N+1}). \tag{3.13}
\]
Via the Sobolev inequalities, some preliminary estimates follow from the bootstrap assumption \(3.12\).

**Proposition 3.1.** Suppose that \(\alpha \geq 0\). In \(\mathcal{R}_1\), under the bootstrap assumption \(3.12\), we have for \(0 \leq k \leq N - 2\),
\[
\delta^{\frac{1}{2}} \langle u \rangle^{\alpha+1} \|L\varphi_k\|_{L^\infty(\mathcal{R}_1)} + \delta^{-\frac{1}{2}} \langle u \rangle^{\alpha+\frac{1}{2}} \|D\varphi_k\|_{L^\infty(\mathcal{R}_1)} \lesssim M,
\]
and for \(0 \leq k \leq N - 1\),
\[
\delta^{\frac{1}{2}} \langle u \rangle^{\alpha+\frac{1}{2}} \|L\varphi_k\|_{L^1(S_{\alpha,u})} + \delta^{-\frac{1}{2}} \langle u \rangle^{\alpha+1} \|D\varphi_k\|_{L^1(\mathcal{R}_1)} \lesssim M.
\]

**Remark 3.2.** To prove Proposition 3.1, we make use of the Sobolev inequalities on \(C_u\) and \(S_{\alpha,u}\) \([5.4, 5.6]\). In fact we have stronger estimates for the lower order derivatives of \(L\varphi_k\). Namely, when \(0 \leq k_1 \leq N - 3\) and \(0 \leq k_2 \leq N - 2\),
\[
\delta^{\frac{1}{2}} \langle u \rangle^{\alpha+2} \|L\varphi_{k_1}\|_{L^\infty(\mathcal{R}_1)} + \delta^{-\frac{1}{2}} \langle u \rangle^{\alpha+\frac{1}{2}} \|L\varphi_{k_2}\|_{L^1(S_{\alpha,u})} \lesssim M.
\]

It is worthwhile to mention that, the lose in the power of \(\delta\) for the top order derivative \(\|L\varphi_N\|_{L^1(\mathcal{R}_1)}\) is due to the weaker bootstrap assumption for \(\|L\varphi_N\|_{L^2(C_u)}\), see \(\delta F_{1+k} \tag{3.7}\): \(\langle u \rangle^{\alpha} \|\nabla L\varphi_k\|_{L^2(C_u)} \lesssim \delta^2 M, k \leq N - 1\), equivalently, \(\langle u \rangle^{\alpha} \|L\varphi_k\|_{L^2(C_u)} \lesssim \delta^2 M, k \leq N\). It will be more obvious if we compare this estimate with the bootstrap assumption for lower order cases \(iE_{1+k}(u, u)\): \(\langle u \rangle^{\alpha} \|L\varphi_k\|_{L^2(C_u)} \lesssim \delta M(u^{-1}), k \leq N - 1\). We can see that, there is an extra \(\delta^2 \langle u \rangle^{-1}\) in the estimates for the lower order cases.

### 3.3. Energy estimates in \(\mathcal{R}_1\).

#### 3.3.1. The multiplier in \(\mathcal{R}_1\).
Consider the multiplier \(\xi_1 := r^{2\alpha} \eta L + \delta^{-1} u^{2\alpha} L\), \(\alpha \geq 0\). That is, choose \(f_1 = r^{2\alpha} \eta, f_2 = \delta^{-1} u^{2\alpha}\). In view of \(2.7\), we have,
\[
\partial_u f_1 \eta^{2\alpha} \|L\psi\|^2 = \frac{1}{2} \left(2\alpha r^{2\alpha-1} \eta + m r^{2\alpha-2}\right) \|L\psi\|^2 > 0, \tag{3.14}
\]
\[
\frac{1}{2} \left(\partial_u f_2 - \frac{\rho f_2}{r}\right) \|\nabla \psi\|^2 = \delta^{-1} \left(\frac{u^{2\alpha}}{r} - \frac{m}{r} - \alpha u^{2\alpha-1}\right) \|\nabla \psi\|^2. \tag{3.15}
\]
Note that if \(u \leq -1\), then \(-u^{2\alpha-1} = |u|^{2\alpha-1}\), \(3.15\) is positive; if \(-1 < u \leq 1\), we choose \(\alpha = 0\), \(3.15\) is positive as well. In other words, if \(u \leq -1\), we take the multiplier \(\xi = r^{2\alpha} \eta L + \delta^{-1} u^{2\alpha} L\), \(\alpha \geq 0\); if \(-1 < u \leq 1\), we choose \(\xi = \eta L + \delta^{-1} L\).
Combining these two cases, we apply the scheme in Section 2.2 to the wave equation for \( \psi \), the energy identity (2.5), (2.7), yield that, for \( \alpha \geq 0 \), and \( u_0 \leq u \leq 1 \),

\[
\int_{C_u} \langle u \rangle^{2\alpha} (|L\psi|^2 + \delta^{-1} |\nabla\psi|^2) d\mu_{C_u} + \int_{L_u} \langle u \rangle^{2\alpha} (|\nabla\psi|^2 + \delta^{-1} |L\psi|^2) d\mu_{L_u}
\]

\[
+ \int_{D_0^1} \alpha \langle u \rangle^{2\alpha} (|L\psi|^2 + \delta^{-1} |\nabla\psi|^2) d\mu_{D}
\]

\[
\lesssim I_1^2(\psi) + C_1^k(\psi) + C_2^k(\psi) + F_k^L(\psi) + F_k^L(\psi),
\]

where we used the facts \( r \sim \langle u \rangle \), \( \eta \sim 1 \) in \( R_1 \) and \( 1 \sim \langle u \rangle^\alpha \) when \( -1 < u \leq 1 \), and here \( I_1^2(\psi) \) denotes the initial energy of \( \psi \), and

\[
C_1^k(\psi) = \int_{D_0^1} \langle u \rangle^{2\alpha-1} |\nabla\psi|^2 d\mu_{D}, \quad C_2^k(\psi) = \int_{D_0^1} \delta^{-1} \langle u \rangle^{2\alpha} |L\psi| d\mu_{D},
\]

\[
F_k^L(\psi) = \int_{D_0^1} \langle u \rangle^{2\alpha} |L\psi| d\mu_{D}, \quad F_k^L(\psi) = \int_{D_0^1} \delta^{-1} \langle u \rangle^{2\alpha} |L\psi| d\mu_{D}.
\]

For \( C_1^k(\psi) \), \( C_2^k(\psi) \),

\[
C_1^k(\psi) \lesssim \int_{C_u}^{w} \langle u \rangle^{2\alpha} |\nabla\psi|^2 d\mu_{C_u},
\]

\[
C_2^k(\psi) \lesssim \int_{C_u}^{w} \langle u \rangle^{-2} d\mu \int_{C_u}^{w} \langle u \rangle^\alpha L \psi^2 d\mu_{C_u},
\]

\[
+ \int_{C_u}^{w} \delta^{-1} d\mu \int_{C_u}^{w} \delta^{-1} \langle u \rangle^{2\alpha} |L\psi|^2 d\mu_{C_u},
\]

where both of them can be handled by the Gronwall’s inequality. For \( F_k^L(\psi) \), \( F_k^L(\psi) \),

\[
F_k^L(\psi) \lesssim \int_{D_0^1} \langle u \rangle^{2\alpha-\frac{1}{2}} |\nabla\psi|^2 d\mu_{D} + \int_{u_0}^{w} \langle u \rangle^{-\frac{1}{2}} d\mu \int_{C_u}^{w} \langle u \rangle^\alpha |L\psi|^2 d\mu_{C_u},
\]

\[
F_k^L(\psi) \lesssim \int_{D_0^1} \langle u \rangle^{2\alpha} |\nabla\psi|^2 d\mu_{D} + \int_{0}^{u} \delta^{-1} d\mu \int_{C_u}^{w} \delta^{-1} \langle u \rangle^{2\alpha} |L\psi|^2 d\mu_{C_u}.
\]

Hence, after the Gronwall’s inequality, we derive the energy inequality: for \( \alpha \geq 0 \), and \( u_0 \leq u \leq 1 \), \( 0 \leq u \leq \delta \),

\[
\int_{C_u} \langle u \rangle^{2\alpha} (|L\psi|^2 + \delta^{-1} |\nabla\psi|^2) d\mu_{C_u} + \int_{L_u} \langle u \rangle^{2\alpha} (|\nabla\psi|^2 + \delta^{-1} |L\psi|^2) d\mu_{L_u}
\]

\[
\lesssim I_1^2(\psi) + \int_{D_0^1} \langle u \rangle^{2\alpha+\frac{1}{2}} |\nabla\psi|^2 d\mu_{D},
\]

(3.17)

We should always note that \( \eta \sim 1 \) in \( R_1 \).

3.3.2. Energy Estimates for \( E_k(u, \psi), E_k(u, \varphi_k) \), \( k \leq N - 1 \). We take \( \psi = \varphi_k, k \leq N - 1 \) in (3.17), then

\[
E_k^2(u, \psi) + E_k^2(u, \varphi_k) \lesssim I_1^2 + \int_{D_0^1} \langle u \rangle^{2\alpha+\frac{1}{2}} |\nabla\varphi_k|^2 d\mu_{D},
\]

(3.18)
where by the null condition, the spacetime integral can be decomposed as: \( \sum_{i=k} H^k_i \), where for \( k_1 + k_2 \leq k, k_1 \leq k_2 \),

\[
\begin{align*}
H^k_1 &= \int_{D_{0,k}^0} (u)^{\frac{3}{2}} |(u) D \varphi_{k_1}|^2 |\mathcal{L} \varphi_{k_2}|^2 d\mu_D; \\
H^k_2 &= \int_{D_{0,k}^0} (u)^{\frac{3}{2}} |(u) D \varphi_{k_1}|^2 |\mathcal{L} \varphi_{k_2}|^2 d\mu_D; \\
H^k_3 &= \int_{D_{0,k}^0} (u)^{\frac{3}{2}} |(u) D \varphi_{k_1}|^2 |\nabla \varphi_{k_2}|^2 d\mu_D; \\
H^k_4 &= \int_{D_{0,k}^0} (u)^{\frac{3}{2}} |(u) L \varphi_{k_1}|^2 |\nabla \varphi_{k_2}|^2 d\mu_D. \\
\end{align*}
\]

(3.19)

Noting that, \( N \geq 6, k_1 \leq \left[ \frac{N}{2} \right] \leq N - 3 \), we can apply \( L^\infty \) to \( D \varphi_{k_1} \), see Proposition [3.1]. Then by the bootstrap assumptions,

\[
H^k_1 \lesssim \int_0^1 (u)^{\frac{3}{2}} \left( \delta^{-\frac{5}{2}} (u)^{-\alpha - 1} M \right)^2 d\mu' \int_{D^0} |(u) L \varphi_{k_2}|^2 d\mu D \\
\lesssim \delta (u)^{-2\alpha - \frac{1}{2}} M^4.
\]

In an analogous fashion, there is

\[
H^k_2 + H^k_3 \lesssim \delta^{\frac{3}{2}} (u)^{-2\alpha - \frac{1}{2}} M^4.
\]

For the last term \( H^k_4 \), we should notice that, \( k \leq N - 1 \). Thus, we are allowed to manipulate \( L^4, L^3, L^2 \) on the four factors and hence we gain some positive power of \( \delta \),

\[
H^k_4 \lesssim \int_{D_{0,k}^0} \int_0^1 (u)^{\frac{3}{2}} |L \varphi_{k_1}|^2 |L^* (S_{L^*} u')|^2 |(u) \varphi_{k_2}|^2 |\nabla \varphi_{k_2}|^2 d\mu' d\mu' \\
\lesssim \delta^{\frac{3}{2}} (u)^{-2\alpha - \frac{1}{2}} M^4.
\]

(3.20)

We remark that the estimate (3.20) is not valid in the top order case \( k = N \). These results are summarized as,

\[
\int_{D_{0,k}^0} |(u)^{2\alpha + \frac{3}{2}} |\square g \varphi_k|^2 d\mu_D \lesssim \delta^{\frac{3}{2}} (u)^{-2\alpha - \frac{1}{2}} M^4.
\]

(3.21)

That is, we can infer that

\[
E^k_{L} (u, w) + E^k_{2} (u, w) \lesssim I^2_N + \delta^{\frac{3}{2}} (u)^{-2\alpha - \frac{1}{2}} M^4, \quad k \leq N - 1.
\]

(3.22)

3.3.3. Energy estimates for \( L^{1+k} \), \( L^{2+k} \), \( \psi = \delta L \varphi_{k_1} \), \( k \leq N - 1 \). In this section, we take \( \psi = \delta L \varphi_{k_1} \), \( k \leq N - 1 \) in (3.17) to obtain the following energy inequality,

\[
L^{1+k} (u, w) + L^{2+k} (u, w) \lesssim I^2_N + \int_{D_{0,k}^0} \delta^2 (u)^{2\alpha + \frac{3}{2}} |\square g L \varphi_k|^2 d\mu_D.
\]

(3.23)
where the source term is split as: \( \int_{D_{u_0}^n} \delta^2(u)^{2\alpha+\frac{1}{2}} |\Box_y \varphi_k|^2 d\mu_D = L^k G^k + L^k \mathcal{H}^k + L^k \mathcal{J}^k + L^k W^k \) with \( k_1 + k_2 \leq k, k_1 \leq k_2 \) and

\[
L^k G^k = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha+\frac{1}{2}} |Q(\partial L \varphi_{k_1}, \partial L \varphi_{k_2})|^2 d\mu_D, \\
L^k \mathcal{H}^k = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha+\frac{1}{2}} |Q(\partial \varphi_{k_1}, \partial L \varphi_{k_2})|^2 d\mu_D, \\
L^k \mathcal{J}^k = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha+\frac{1}{2}} |Q(\partial \varphi_{k_1}, \partial \varphi_{k_2})|^2 d\mu_D, \\
L^k W^k = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha+\frac{1}{2}} |[\Box_y, L] \varphi_k|^2 d\mu_D.
\]

In the following we will estimate these four terms.

Taking (3.21) intoaccount, \( L^k \mathcal{J}^k \leq \delta^{2+\frac{1}{2}}(u)^{-2\alpha-\frac{1}{2}} M^4 \).

For \( L^k G^k \), we make the splitting: \( L^k G^k = L^k G^k_1 + L^k G^k_2 + L^k G^k_3 + L^k G^k_4 \), where for \( k_1 + k_2 \leq k, k_1 \leq k_2 \),

\[
L^k G^k_1 = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha \frac{1}{2}} |(u)^\alpha D \varphi_{k_2}| |L L \varphi_{k_1}|^2 d\mu_D; \\
L^k G^k_2 = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha \frac{1}{2}} |(u)^\alpha D \varphi_{k_2}| |L^2 \varphi_{k_1}|^2 d\mu_D; \\
L^k G^k_3 = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha \frac{1}{2}} |(u)^\alpha D \varphi_{k_2}| |\nabla L \varphi_{k_1}|^2 d\mu_D; \\
L^k G^k_4 = \int_{D_{u_0}^n} \delta^2(u)^{2\alpha \frac{1}{2}} |(u)^\alpha L \varphi_{k_2}|^2 |\nabla L \varphi_{k_1}|^2 d\mu_D.
\]

Since \( k_1 \leq \left[\frac{N}{2}\right] \leq N - 3, k_2 \leq N - 1 \), we can apply \( L^1 \) to all of the four factors in \( L^k G^k_1 \). By Proposition 3.1 \( \| (u)^\alpha D \varphi_{k_2} \|_{L^4(S_{u_0})} \leq \delta^{-\frac{1}{2}} M(u)^{-\frac{1}{2}} \), then

\[
L^k G^k_1 \lesssim \int_{u_0}^u \int_0^u \delta M^2(u)^{-1+\frac{1}{2}} \cdot \| LL \varphi_{k_1} \|_{L^4(S_{u'} \setminus S_{u_0})}^2 du' du \\
\lesssim \int_{u_0}^u \delta M^2(u)^{-1} \sum_{k_1 \leq k_1+1} (u)^{-\frac{1}{2}} \| LL \varphi_i \|_{L^2(S_{u'})}^2 du' \leq \delta (u)^{-2\alpha - \frac{1}{2}} M^4, \quad k_1 \leq N - 3. \tag{3.24}
\]

Here we have used the Sobolev inequality on the sphere \( S_{u_0} \):

\[
\| \phi \|_{L^4(S_{u_0})} \lesssim r^{-\frac{1}{2}} \| \phi \|_{L^2(S_{u_0})} + r^{\frac{1}{2}} \| \nabla \phi \|_{L^2(S_{u_0})}, \tag{3.25}
\]

Similarly, we can obtain

\[
L^k G^k_2 + L^k G^k_3 + L^k G^k_4 \lesssim \delta^{\frac{1}{2}} (u)^{-2\alpha - \frac{1}{2}} M^4.
\]
For \( L^4 \), it can be split into the following terms: for \( k_1 + k_2 \leq k_1 \leq k_2 \),

\[
L^4 = \int_{D^0_{\infty} u} \delta^2 (u)^2 \left| u \alpha D\varphi_{k_1} \right|^2 |L \varphi_{k_1}|^2 \, d\mu_D;
\]

\[
L^4 = \int_{D^0_{\infty} u} \delta^2 (u)^2 \left| u \alpha D\varphi_{k_1} \right|^2 |L^2 \varphi_{k_1}|^2 \, d\mu_D;
\]

\[
L^4 = \int_{D^0_{\infty} u} \delta^2 (u)^2 \left| u \alpha D\varphi_{k_1} \right|^2 |\nabla L \varphi_{k_1}|^2 \, d\mu_D;
\]

\[
L^4 = \int_{D^0_{\infty} u} \delta^2 (u)^2 \left| u \alpha L \varphi_{k_1} \right|^2 |\nabla L \varphi_{k_1}|^2 \, d\mu_D.
\]

Knowing that \( k_1 \leq \left\lfloor \frac{N}{2} \right\rfloor \leq N - 3 \), \( k_2 \leq N - 1 \), we can apply \( L^\infty, L^\infty, L^2, L^2 \) to the four factors in \( L^4 \). By Proposition 3.1 and the bootstrap assumption on \( E^{1, k}_1 \), we proceed in an analogous way to conclude

\[
L^4 \lesssim \int_{D^0_{\infty} u} \delta^2 (u)^2 \left\| u \alpha D\varphi_{k_1} \right\|^2 |L \varphi_{k_1}|^2 \, d\mu_D \lesssim \int_{D^0_{\infty} u} \delta^2 (u)^2 \left\| u \alpha D\varphi_{k_1} \right\|^2 |\nabla L \varphi_{k_1}|^2 \, d\mu_D \lesssim \delta^2 (u)^{2-\frac{1}{2}} M^4. \tag{3.27}
\]

For \( L^2_k, L^2_k, L^2_k \), we proceed in an analogous way to conclude

\[
L^2_k + L^2_k + L^2_k \lesssim \delta^2 (u)^{-2-\frac{1}{2}} M^4.
\]

Next, we turn to \( L^2 \). In view of \( (2.37) \) \( \Delta_g L \varphi_k \sim \Delta (L \varphi_k - L \varphi_k) + \frac{1}{r} D_\varphi_k + \frac{1}{r} \nabla \varphi_k \), then

\[
L^2 \lesssim \int_{D^0_{\infty} u} \delta^2 (u)^{2-4} \left( |L \varphi_k|^2 + |L \varphi_k|^2 + |\nabla \varphi_{k+1}|^2 \right) \, d\mu_D \tag{3.28}
\]

By the improved result \( 3.22 \), there is for \( k \leq N - 1 \),

\[
\int_{D^0_{\infty} u} \delta^2 (u)^{2-4} \left( |L \varphi_k|^2 + |L \varphi_k|^2 + |\nabla \varphi_{k+1}|^2 \right) \, d\mu_D \lesssim \delta^2 I^2_k + \delta^2 (u)^{-2-\frac{1}{2}} M^4.
\]

For the last term associated to \( \nabla \varphi_{k+1} \), we should note that \( k+1 \leq N \), and hence it hits the top order derivative. By using the bootstrap assumption on \( E_N(u, u) \),

\[
\int_{D^0_{\infty} u} \delta^2 (u)^{2+4} |\nabla \varphi_{k+1}|^2 \, d\mu_D \lesssim \delta^2 (u)^{-3} M^2, \quad k \leq N - 1.
\]

For the last term on the right hand of \( 3.28 \), we appeal to \( 3.21 \), thus,

\[
\int_{D^0_{\infty} u} (u)^{2-2\frac{1}{2}} |\nabla \varphi_k|^2 \, d\mu_D \lesssim \delta^2 (u)^{-2-\frac{1}{2}} M^4, \quad k \leq N - 1.
\]

Now, we conclude that

\[
L^1 F^1_{1+k} (u, u) + L^1 E^1_{1+k} (u, u) \lesssim I^2_{k+1} + \delta^2 (u)^{-2-\frac{1}{2}} M^4, \quad k \leq N - 1. \tag{3.29}
\]

Combining with the improved results obtained in \( 3.22 \) and \( 3.29 \), we can improve the \( L^\infty \) and \( L^4 \) estimates, which are collected as below.

Define

\[
I^2_{N+1} + \delta^2 (u)^{2-4} M^4 := I^2_{N+1} \tag{3.30}
\]

We will finally choose \( \delta \) to be small enough so that \( I_{N+1} \lesssim I_{N+1} \).
Proposition 3.3. In the region $R_1$, when $0 \leq k \leq N - 3$, we have

$$\delta^2 \langle u \rangle^{\alpha+\frac{1}{2}} \| L \varphi_k \|_{L^\infty(R_1)} + \delta^{-\frac{1}{2}} \langle u \rangle^{\alpha+\frac{1}{2}} \| \nabla \varphi_k \|_{L^\infty(R_1)} \lesssim \| u \|,$$

and when $0 \leq k \leq N - 2$,

$$\delta^2 \langle u \rangle^{\alpha+\frac{1}{2}} \| L \varphi_k \|_{L^4(S_{u,u})} + \delta^{-\frac{1}{2}} \langle u \rangle^{\alpha+1} \| \nabla \varphi_k \|_{L^4(S_{u,u})} \lesssim I_N.$$

3.3.4. Energy estimates for $E_N(u,u), E_N(u,u)$. For the top order case $k = N$, we can proceed along the line of Section 3.3.2 except that (3.20) is not valid for $H^N$, due to the restriction of regularity. Alternatively, we take advantage of the improved result of Proposition 3.3. Then

$$H^N \lesssim \sum_{k_1 + k_2 \leq k \leq k_2} \int_{\mathcal{D}_{u,u}} \langle u \rangle^{\frac{3}{2}} |L \varphi_{k_1} |^2 \langle u \rangle^\alpha \| \nabla \varphi_{k_2} \|^2 d\mu_D$$

$$\lesssim \int_0^\infty \delta^{-1} \langle u \rangle^{-2\alpha - \frac{1}{2}} \| \nabla \varphi \|^2 d\mu'_r \int_{\mathcal{C}_u} \langle u \rangle^\alpha \| \nabla \varphi \|^2 d\mu_{C_u},$$

where the Gronwall’s inequality works for $\| u \| \lesssim I_N$. Finally, we conclude

$$E_N^2(u,u) + E_N^2(u,u) \lesssim I_{N+1}^2.$$  \hspace{1cm} (3.31)

3.3.5. Energy estimates for $E_N(u,u), k \leq N - 1$. In this section, we will retrieve the estimates for $\| L \varphi_k \|_{L^4(C_u)}, \| L^2 \varphi_k \|_{L^4(C_u)}$, $k \leq N - 1$. For convenience, we let

$$\langle \nabla^2 \psi \rangle(u,u) = \int_{S_{u,u}} \| L \psi \|^2 r^2 d\sigma_{S^2}. \hspace{1cm} (3.32)$$

Then by virtue of the wave operator,

$$\partial_u \nabla^2 \psi = \int_{S_{u,u}} 2L \psi (L \psi + \frac{\eta}{r} L \psi) r^2 d\sigma_{S^2}$$

$$= \int_{S_{u,u}} 2L \psi \left( \frac{\eta}{r} L \psi + \eta \Delta \psi - \eta \Box g \psi \right) r^2 d\sigma_{S^2}$$

$$\lesssim \delta^{-1} \langle \nabla^2 \psi \rangle + \delta \int_{S_{u,u}} \langle u \rangle^{-2} \| L \psi \|^2 + |\Delta \psi|^2 + |\Box g \psi|^2 \rangle r^2 d\sigma_{S^2}.$$  \hspace{1cm} (3.33)

Suppose that $\psi \equiv 0$ on $C_0$. We then integrate along $\partial_u$ to derive,

$$\langle \nabla^2 \psi \rangle \lesssim \int_0^\infty \delta^{-1} \langle \nabla^2 \psi \rangle d\mu'_r + \delta \int_{C_n} \langle u \rangle^{-2} |L \varphi_k|^2 + |\Delta \varphi_k|^2 + |\Box g \varphi_k|^2 \rangle d\mu_{C_n}. \hspace{1cm} (3.33)$$

Taking $\psi = \varphi_k, k \leq N - 1$ in (3.33), since $\varphi \equiv 0$ on $C_0$, we have

$$\langle \nabla^2 \varphi_k \rangle \lesssim \int_0^\infty \delta^{-1} \langle \nabla^2 \varphi_k \rangle d\mu'_r + \delta \int_{C_n} \langle u \rangle^{-2} |L \varphi_k|^2 + |\Delta \varphi_k|^2 + |\Box g \varphi_k|^2 \rangle d\mu_{C_n}. \hspace{1cm} (3.34)$$

Using the results for $E_l(u,u), l \leq N$, we obtain

$$\delta \int_{C_n} \langle u \rangle^{-2} |L \varphi_k|^2 + |\Delta \varphi_k|^2 d\mu_{C_n} \lesssim \langle u \rangle^{-2\alpha - 2} I_{N+1}^2, k \leq N - 1.$$  \hspace{1cm} (3.35)
For the term $\delta \int_{C_u} |\square_y \varphi_k|^2 d\mu_{C_u}$, we make the splitting: for $k_1 + k_2 \leq k$, $k_1 \leq k_2$,

\begin{align*}
S^k_1 &= \delta \int_{C_u} |D\varphi_{k_1}|^2 |L\varphi_{k_2}|^2 d\mu_{C_u}, \\
S^k_2 &= \delta \int_{C_u} |D\varphi_{k_2}|^2 |L\varphi_{k_1}|^2 d\mu_{C_u}, \\
S^k_3 &= \delta \int_{C_u} |D\varphi_{k_1}|^2 |\nabla \varphi_{k_2}|^2 d\mu_{C_u}, \\
S^k_4 &= \delta \int_{C_u} |L\varphi_{k_2}|^2 |\nabla \varphi_{k_1}|^2 d\mu_{C_u}.
\end{align*}

We estimate the error terms one by one. In views of the improved $L^\infty$ estimate for $\|L\varphi_{k_1}\|_{L^\infty(S_{u,\infty})}$ (Proposition 3.3) and $\|D\varphi_{k_1}\|_{L^\infty(S_{u,\infty})} \lesssim \delta^4 M\langle u \rangle^{-\frac{\alpha-2}{2}}$, $k_1 \leq \lfloor \frac{k}{2} \rfloor \leq N-3$,

\begin{align*}
\delta |S^k_1|^2 &\lesssim \|\mathcal{I}_N\langle u \rangle^{-2\alpha-2} \int_0^u \chi^2(\varphi_k(u, u')) du', \\
\delta (|S^k_2|^2 + |S^k_3|^2) &\lesssim \delta^2 M\langle u \rangle^{-2\alpha-3} \left( \|L\varphi_{k_2}\|_{L^2(C_u)}^2 + \|\nabla \varphi_{k_1}\|_{L^2(C_u)}^2 \right), \\
\delta |S^k_4|^2 &\lesssim \|\mathcal{I}_N\langle u \rangle^{-2\alpha-2} \|\nabla \varphi_{k_2}\|_{L^2(C_u)}^2.
\end{align*}

Therefore, for $k \leq N-1$,

\begin{equation}
\delta \int_{C_u} |\square_y \varphi_k|^2 d\mu_{C_u} \lesssim \|\mathcal{I}_N\langle u \rangle^{-2\alpha-2} \int_0^u \chi^2(\varphi_k(u, u')) du' + \|\mathcal{I}_N\langle u \rangle^{-4\alpha-2} \right. 
\end{equation}

Based on (3.34)-(3.36), and the Gronwall’s inequality,

\begin{equation}
\|L\varphi_k\|_{L^2(S_{u,\infty})} \lesssim \langle u \rangle^{-\alpha-2} \mathcal{I}_N^{k+1}, \quad k \leq N-1.
\end{equation}

Integrating along $\partial_u$, we arrive at

\begin{equation}
\delta^{-2} \langle u \rangle^{2\alpha+2} \|L\varphi_k\|_{L^2(C_u)}^2 \lesssim \mathcal{I}_N^{k+1}, \quad k \leq N-1.
\end{equation}

As for $\|LL\varphi_k\|_{L^2(C_u)}$, we turn to the wave operator:

\begin{equation}
\eta^{-1} L L \varphi_k = \hat{\delta} \varphi_k + \frac{\hat{\eta} \varphi_k}{\hat{\eta}} - \frac{\hat{L} \varphi_k}{\hat{\eta}} - \square_y \varphi_k. \tag{3.37}
\end{equation}

Viewing (3.35)-(3.38), we deduce that

\begin{equation}
\|LL\varphi_k\|_{L^2(C_u)} \lesssim \langle u \rangle^{-\alpha-1} \mathcal{I}_N^{k+1}, \quad k \leq N-1.
\end{equation}

Noting that, $|L L \varphi_k| = \langle u \rangle^{-1} |L \hat{\varphi}_k|$, hence

\begin{equation}
\|L \hat{\varphi}_k\|_{L^2(C_u)} \lesssim \langle u \rangle^{-\alpha} \mathcal{I}_N^{k+1}, \quad k \leq N-1.
\end{equation}

3.3.6. Energy estimates for $\tilde{S} F_{1+k}(u, u)$, $\tilde{S} \mathcal{E}_{1+k}(u, u)$, $k \leq N-1$. In this section, we will take the multiplier $\xi = \delta^{-1} \langle u \rangle^{2\alpha+2}$ and $\psi = \hat{\varphi}_k$. Noting that, there is no $L$ part in this multiplier. Following the proof leading to (3.17), we have,

\begin{align}
\tilde{S} F_{1+k}(u, u) + \tilde{S} \mathcal{E}_{1+k}(u, u) \lesssim \mathcal{I}_N^{k+1} + \int_{T_{u,\infty}} \langle u \rangle^{2\alpha} |\square_y \hat{\varphi_k}|^2 d\mu_D,
\end{align}

where the last term is split as $\tilde{S} \mathcal{G}^k + \tilde{S} \mathcal{H}^k + \tilde{S} \mathcal{W}^k$, with $k_1 + k_2 \leq k$, $k_1 \leq k_2$,

\begin{align*}
\tilde{S} \mathcal{G}^k &= \int_{T_{u,\infty}} \langle u \rangle^{2\alpha} |Q(\partial \hat{\varphi}_{k_1}, \partial \hat{\varphi}_{k_2})|^2 d\mu_D, \\
\tilde{S} \mathcal{H}^k &= \int_{T_{u,\infty}} \sum_{k_1 \leq k_2} \langle u \rangle^{2\alpha} |Q(\partial \hat{\varphi}_{k_1}, \partial \hat{\varphi}_{k_2})|^2 d\mu_D, \\
\tilde{S} \mathcal{W}^k &= \int_{T_{u,\infty}} \langle u \rangle^{2\alpha} |\square_y \hat{\varphi}_k|^2 d\mu_D.
\end{align*}
$\tilde{S}G^k, \tilde{S}\mathcal{H}^k$ can be bounded in the same way as $L^kG^k, L^k\mathcal{H}^k$ respectively, 

$$\tilde{S}\mathcal{H}^k + \tilde{S}G^k \lesssim \delta^2 \langle u \rangle^{-2\alpha-2} M^4.$$ 

For $\tilde{S}W^k$, we recall (2.10), (3.21) and derive

$$\tilde{S}W^k \lesssim \int_{\mathbb{R}^n} \langle u \rangle^{2\alpha-2} (|L\varphi_k|^2 + |L_y\varphi_k|^2 + |\nabla \varphi_{k+1}|^2) + \langle u \rangle^{2\alpha} |\Box_y \varphi_k|^2 d\mu_d$$

$$\lesssim \langle u \rangle^{-1} \delta^{\frac{1}{2}} + \delta^2 \langle u \rangle^{-2\alpha-2} M^4.$$ 

We summarize the above estimates as

$$\tilde{S}F_{1+k}(u, u) + \tilde{S}F_{1+k}(u, u) \lesssim I^2_{N+1} + \delta^2 \langle u \rangle^{-2\alpha-2} M^4, \quad k \leq N - 1. \quad (3.41)$$

Now, we turn to retrieve the estimate for $\|L\tilde{S}\varphi_k\|_{L^2(C_u)}$, $k \leq N - 2$.

3.3.7. Energy estimates for $\tilde{E}_{1+k}(u, u)$, $k \leq N - 2$. We take $\psi = \tilde{S}\varphi_k$ in (3.33) to derive

$$\chi^2[\tilde{S}\varphi_k](u, u) \lesssim \int_0^u \delta^{-1} \chi^2[\tilde{S}\varphi_k](u, u') du'$$

$$+ \delta \int_{C_u} \left( \langle u \rangle^{-2} |L\tilde{S}\varphi_k|^2 + |\partial_u \tilde{S}\varphi_k|^2 + |\Box_y \tilde{S}\varphi_k|^2 \right) d\mu_{C_u}.$$ 

By the improved results (3.39) and (3.41), we obtain that

$$\delta \int_{C_u} \langle u \rangle^{-2} |L\tilde{S}\varphi_k|^2 + |\partial_u \tilde{S}\varphi_k|^2 d\mu_{C_u} \lesssim \delta \langle u \rangle^{-2\alpha-2} \|\tilde{S}\varphi_k\|_{N+1}^2, \quad k \leq N - 2.$$ 

For the term $\delta \int_{C_u} |\Box_y \tilde{S}\varphi_k|^2 d\mu_{C_u}$, it can be split into: $= \tilde{S}U^k + \tilde{S}V^k + \tilde{S}W^k$, where for $k_1 + k_2 \leq k \leq N - 2$, $k_1 \leq k_2$,

$$\tilde{S}U^k = \delta \int_{C_u} [Q(\tilde{S}\varphi_{k_1}, \partial \varphi_{k_2})]^2 d\mu_{C_u},$$

$$\tilde{S}V^k = \delta \int_{C_u} \sum_{i < k} [Q(\partial_u \varphi_{k_1}, \varphi_{k_2})]^2 d\mu_{C_u},$$

$$\tilde{S}W^k = \delta \int_{C_u} |\Box_y \tilde{S}\varphi_k|^2 d\mu_{C_u}.$$ 

To estimate $\tilde{S}U^k$, we adjust the argument for $L^kG^k$, namely, we apply $L^4(S_{\tilde{y}, u})$ on the four factors in $\tilde{S}U^k$, perform the $L^4 - L^2$ type of Sobolev inequality (3.25) on $\|\partial_u \tilde{S}\varphi_k\|_{L^4(S_{\tilde{y}, u})}$ and further integrate along $\partial_u \tilde{S}\varphi_k$ to generate $\|\partial_u \tilde{S}\varphi_k\|_{L^2(C_u)}^2$. For $\tilde{S}V^k$, we follow the argument for $\delta \int_{C_u} |\Box_y \varphi_k|^2 d\mu_{C_u}$, cf. Section 3.3.5. In brief,

$$\tilde{S}U^k + \tilde{S}V^k \lesssim I^2_N \langle u \rangle^{-2\alpha-2} \int_0^u \sum_{i \leq k} \int_{C_u} \langle u \rangle^{-2} |\tilde{S}\varphi_k|(u, u') du' + \delta \|\tilde{S}\varphi_k\|_{N+1}^2.$$ 

For $\tilde{S}W^k$, we remind ourself the improvement (3.22), (3.31), (3.36) and (3.38), then

$$\tilde{S}W^k \lesssim \delta \int_{C_u} \langle u \rangle^{-2} (|L\varphi_k|^2 + |L_y \varphi_k|^2 + |\nabla \varphi_{k+1}|^2 + |\Box_y \varphi_k|^2 d\mu_{C_u})$$

$$\lesssim \delta \langle u \rangle^{-2\alpha-2} \|\tilde{S}\varphi_k\|_{N+1}^2, \quad k \leq N - 2.$$ 

As shown in Section 3.3.5 we can achieve

$$\|L\tilde{S}\varphi_k\|_{L^2(S_{\tilde{y}, u})} \lesssim \delta \|\tilde{S}\varphi_k\|_{N+1}^2, \quad k \leq N - 2,$$

$$\|L\tilde{S}\varphi_k\|_{L^2(C_u)}^2 \lesssim \delta^2 \|\tilde{S}\varphi_k\|_{N+1}^2, \quad k \leq N - 2.$$
For the sake of clarity, we assemble these results with regard to the transversal derivative $L$ on $C_0$ in Sections 3.3.5 and 3.3.7 in the following Proposition.

**Proposition 3.4.** For $i + k ≤ N - 1$, and $i ≤ 1$,

$$
(u)^{α+1} \left( δ^{−\frac{1}{2}}∥L\tilde{S}^{t}φ_k∥_{L^2(S_{t=0})} + δ^{−1}∥L\tilde{S}^{t}φ_k∥_{L^2(C_0)} \right) \lesssim I_{N+1}.
$$

(3.43)

For $k ≤ N - 1$,

$$
(u)^{α+1}∥Lφ_k∥_{L^2(C_0)} + (u)^{α}∥L\tilde{S}φ_k∥_{L^2(C_0)} \lesssim I_{N+1}.
$$

(3.44)

3.3.8. **End of the bootstrap argument in the region $R_1$.** Putting the estimates (3.22), (3.29), (3.31), (3.41) and Proposition 3.4 together, we have arrived at

$$
E_t(u, u) + \sum_{p + q = l} δ^p(u)^{α}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)} + δ^{p−\frac{1}{2}}(u)^{α}∥∇L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)},
$$

$$
E_{i+1+k}(u, u) = \sum_{p + q = l} δ^p(u)^{α}∥∇L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)} + δ^{p−\frac{1}{2}}(u)^{α}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)},
$$

and for $i = 0$, and $i + l + k ≤ N - 1$,

$$
^tE_{i+1+k}(u, u) = \sum_{p + q = l} δ^{p−1}(u)^{α+1}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)}.
$$

The energy estimate (3.13) can be extended to general energy norms.

**Theorem 3.5.** In $R_1$, letting $N ≥ 6$ and $u_0 ≤ u ≤ 1$, $0 ≤ u ≤ δ$, we have

$$
E_{i+1+k}(u, u) + E_{i+1+k}(u, u) ≤ I_{N+1}, \quad i = 0, 1, \quad i + l + k ≤ N,
$$

$$
E_{i+1+k}(u, u) ≤ I_{N+1}, \quad i = 0, 1, \quad i + l + k ≤ N - 1,
$$

provided that the initial energy is bounded by $I_{N+1}$.

This theorem can be easily proved by an inductive argument on $l$, i.e. the numbers of $W$ derivative and thus we will omit the details here. In the proof, the following $L^∞$ and $L^1$ estimates can be inferred as well:

$$
δ^{p+1}(u)^{α+1}∥LW^l_{p,q}φ_k∥_{L^∞(R_1)} + δ^{p−\frac{1}{2}}(u)^{α+\frac{1}{2}}∥∇W^l_{p,q}φ_k∥_{L^∞(R_1)} \lesssim I_{N+1}, \quad l + k ≤ N - 2,
$$

$$
δ^{p+\frac{1}{2}}(u)^{α+1}∥LW^l_{p,q}φ_k∥_{L^1(S_{t=0})} + δ^{p−\frac{1}{2}}(u)^{α+1}∥∇W^l_{p,q}φ_k∥_{L^1(S_{t=0})} \lesssim I_{N+1}, \quad l + k ≤ N - 1.
$$

Besides, an analogous version of Proposition 3.3 can be derived as follows.

**Proposition 3.6.** For $i + l + k ≤ N - 1$, $i ≤ 1$,

$$
δ^p(u)^{α+1} \left( δ^{−\frac{1}{2}}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(S_{t=0})} + δ^{−1}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)} \right) \lesssim I_{N+1}.
$$

For $l + k ≤ N - 1$,

$$
δ^p(u)^{α+1}∥LW^l_{p,q}φ_k∥_{L^2(C_0)} + δ^p(u)^{α}∥L\tilde{S}^tW^l_{p,q}φ_k∥_{L^2(C_0)} \lesssim I_{N+1}.
$$
3.5. Smallness on the last cone in \( \mathcal{R}_1 \).

**Theorem 3.7.** In the region \( \mathcal{R}_1 \), for any fixed \( N \geq 6 \) and \( \alpha \geq 0 \), we have on the last cone \( \mathcal{C}_\delta \),

\[
\| \langle u \rangle^\alpha \bar{D}_i \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{3}{2}}, \quad i + 2l + k \leq N - 1, \; i \leq 1;
\]
\[
\| \bar{D}_i \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-1}, \quad i + 2l + k \leq N - 1, \; i \leq 1;
\]
\[
\| \bar{D}_i \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^\infty(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-2}, \quad i + 2l + k \leq N - 3, \; i \leq 1.
\]

And

\[
\| \langle u \rangle^\alpha \bar{L} \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{3}{2}}, \quad i + 2l + k \leq N - 2, \; i \leq 1;
\]
\[
\| \bar{L} \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-1}, \quad i + 2l + k \leq N - 2, \; i \leq 1;
\]
\[
\| \bar{L} \bar{S}^i \bar{W}_{p,q}^i \varphi_k \|_{L^\infty(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-2}, \quad i + 2l + k \leq N - 4, \; i \leq 1.
\]

For the proof, we begin with the cases involving merely good derivatives.

**Proposition 3.8.** We have in the region \( \mathcal{R}_1 \), for any fixed \( N \geq 6 \), \( \alpha \geq 0 \),

\[
\| \langle u \rangle^\alpha \bar{D}_i \bar{S}^i \bar{L}_i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{3}{2}}, \quad i + l + k \leq N - 1, \; i \leq 1;
\]
\[
\| \bar{D}_i \bar{S}^i \bar{L}_i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-1}, \quad i + l + k \leq N - 1, \; i \leq 1;
\]
\[
\| \bar{D}_i \bar{S}^i \bar{L}_i \varphi_k \|_{L^\infty(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-2}, \quad i + l + k \leq N - 3, \; i \leq 1.
\]

**Proof.** First of all, considering \( \bar{D} \) to be \( \nabla \), we define

\[
\omega^2[\psi](u, u) = \int_{S_{\omega,u}} |\nabla \psi|^2 r^2 d\sigma_{S^2}. \tag{3.46}
\]

We take \( \psi = \bar{S}^i \bar{L}_i \varphi_k, (i + l + k \leq N - 1, \; i \leq 1) \) and derive the transport equation

\[
\partial_{\omega} \omega^2[\bar{S}^i \bar{L}_i \varphi_k] = \int_{S_{\omega,u}} 2\nabla \bar{S}^i \bar{L}_i \varphi_k \cdot \nabla \bar{S}^i \bar{L}_i \varphi_k r^2 d\sigma_{S^2}
\]
\[
\lesssim \delta^{-1} \omega^2[\bar{S}^i \bar{L}_i \varphi_k] + \delta \int_{S_{\omega,u}} \langle u \rangle^{-2} |\bar{L} \bar{S}^i \bar{L}_i \varphi_{k+1}|^2 r^2 d\sigma_{S^2},
\]

where in the last inequality, \( |\nabla \bar{L} \bar{S}^i \bar{L}_i \varphi_k|^2 \sim \langle u \rangle^{-2} |\bar{L} \bar{S}^i \bar{L}_i \varphi_{k+1}|^2 \) in \( \mathcal{R}_1 \) is used. Now that \( \nabla \bar{S}^i \bar{L}_i \varphi_k \equiv 0 \) on the incoming cone \( \mathcal{C}_\delta \), by the Gronwall’s inequality,

\[
\omega^2[\bar{S}^i \bar{L}_i \varphi_k](u, u) \lesssim \delta \| \langle u \rangle^{-1} \bar{L} \bar{S}^i \bar{L}_i \varphi_{k+1} \|_{L^2(S_{\delta,u})} \lesssim \delta \langle u \rangle^{-2}. \]

Integrate along \( \partial_{\omega} \),

\[
\| \langle u \rangle^\alpha \bar{S}^i \bar{L}_i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta.
\]

The \( L^\infty \) estimate follows from the Sobolev inequality.

Secondly, when \( \bar{D} \) is taken as \( \bar{L} \), the smallness follows straightforwardly as a consequence of Theorem 3.5 and Proposition 3.8.

For \( \bar{L} \bar{S}^i \bar{L}_i \varphi_k, i \leq 1 \), the smallness will take place on the last incoming cone \( \mathcal{C}_\delta \).

**Proposition 3.9.** On \( \mathcal{C}_\delta \cap \mathcal{R}_1 \), we have, for any fixed \( N \geq 6 \), \( \alpha \geq 0 \),

\[
\| \langle u \rangle^\alpha \bar{L} \bar{S}^i \bar{L}_i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{3}{2}}, \quad i + l + k \leq N - 2, \; i \leq 1;
\]
\[
\| \bar{L} \bar{S}^i \bar{L}_i \varphi_k \|_{L^2(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-1}, \quad i + l + k \leq N - 2, \; i \leq 1;
\]
\[
\| \bar{L} \bar{S}^i \bar{L}_i \varphi_k \|_{L^\infty(S_{\delta,u})} \lesssim \delta^{\frac{1}{2}} \langle u \rangle^{-2}, \quad i + l + k \leq N - 4, \; i \leq 1.
\]
Proof. We should note that \( \langle u \rangle \sim r \) in \( R_1 \). Define
\[
\chi^2[\psi](u, \bar{u}) = \int_{S_{\omega}} r^{2\alpha+1} |L\psi|^2 r^2 d\sigma_{S^2}, \quad \alpha \geq 0. \tag{3.47}
\]
To illustrate the idea, we will carry out the estimates for \( l = 0 \) in detail. Take \( \psi = \tilde{S}^i \varphi_k \), with \( i + k \leq N - 2, i \leq 1 \). There is the transport equation
\[
\partial_u \chi^2[\tilde{S}^i \varphi_k] = \int_{S_{\omega}} 2r^{2\alpha+1} L\tilde{S}^i \varphi_k \left( \frac{L L \tilde{S}^i \varphi_k}{r^{2\alpha+1}} - (\alpha + \frac{3}{2}) \frac{\eta}{r} L^2 \tilde{S}^i \varphi_k \right) r^2 d\sigma_{S^2}
\]
\[
= \int_{S_{\omega}} 2r^{2\alpha+1} L\tilde{S}^i \varphi_k \eta \left( \Delta \tilde{S}^i \varphi_k - \Box_y \tilde{S}^i \varphi_k - \frac{1}{r} L \tilde{S}^i \varphi_k \right) r^2 d\sigma_{S^2}
\]
\[
- \int_{S_{\omega}} (2\alpha + 1) r^{2\alpha} |L\tilde{S}^i \varphi_k|^2 r^2 d\sigma_{S^2}.
\]
That is,
\[
\partial_u \chi^2[\tilde{S}^i \varphi_k] + \int_{S_{\omega}} (2\alpha + 1) r^{2\alpha} |L\tilde{S}^i \varphi_k|^2 r^2 d\sigma_{S^2}
\]
\[
= \int_{S_{\omega}} 2r^{2\alpha+1} L\tilde{S}^i \varphi_k \eta \left( \Delta \tilde{S}^i \varphi_k - \Box_y \tilde{S}^i \varphi_k - \frac{1}{r} L \tilde{S}^i \varphi_k \right) r^2 d\sigma_{S^2}. \tag{3.48}
\]
Integrating from \( u_0 \) to \( u \), using the Cauchy-Schwartz inequality and absorbing terms which can be bounded by the positive term on the right hand side of (3.48) after a small change in constant, we derive
\[
\chi^2[\tilde{S}^i \varphi_k](u, \bar{u}) + \int_{C_{\eta}} \langle u \rangle^{2\alpha} |L\tilde{S}^i \varphi_k|^2 d\mu_{C_{\eta}}
\]
\[
\leq \chi^2[\tilde{S}^i \varphi_k](u_0, \bar{u}) + \int_{C_{\eta}} \langle u \rangle^{2\alpha} \left( |L \tilde{S}^i \varphi_k|^2 + |\nabla \tilde{S}^i \varphi_{k+1}|^2 \right) d\mu_{C_{\eta}} \tag{3.49}
\]
\[
+ \int_{C_{\eta}} \langle u \rangle^{2\alpha+2} |\Box_y \tilde{S}^i \varphi_k|^2 d\mu_{C_{\eta}}.
\]
Indicated by Proposition 3.8,
\[
\int_{C_{\eta}} \langle u \rangle^{2\alpha} \left( |L \tilde{S}^i \varphi_k|^2 + |\nabla \tilde{S}^i \varphi_{k+1}|^2 \right) d\mu_{C_{\eta}} \leq \delta, \quad i + k \leq N - 2.
\]
Therefore, we are left with
\[
\chi^2[\tilde{S}^i \varphi_k](u, \bar{u}) + \int_{C_{\eta}} \langle u \rangle^{2\alpha} |L\tilde{S}^i \varphi_k|^2 d\mu_{C_{\eta}}
\]
\[
\leq \chi^2[\tilde{S}^i \varphi_k](u_0, \bar{u}) + \delta + \int_{C_{\eta}} \langle u \rangle^{2\alpha+2} |\Box_y \tilde{S}^i \varphi_k|^2 d\mu_{C_{\eta}}. \tag{3.50}
\]
For the remaining error term \( \int_{C_{\eta}} \langle u \rangle^{2\alpha+2} |\Box_y \tilde{S}^i \varphi_k|^2 d\mu_{C_{\eta}} \), we should propose a hierarchy of estimate: We will first justify the smallness for \( \chi^2[\varphi_k](u, \bar{u}) \) by estimating \( \int_{C_{\eta}} \langle u \rangle^{2\alpha+2} |\Box_y \varphi_k|^2 d\mu_{C_{\eta}} \). After that, we should recur to the result for \( \chi^2[\varphi_k](u, \bar{u}) \) to complete the proof for \( \chi^2[\tilde{S}^i \varphi_k](u, \bar{u}) \).

Considering \( i = 0 \), as we know, by the null condition,
\[
\langle u \rangle^{2\alpha+2} |\Box_y \varphi_k|^2 \lesssim \sum_{k_1 + k_2 \leq k, k_1 \leq k_2} \langle u \rangle^{2\alpha+2} |\Box \varphi_{k_1} \Box \varphi_{k_2} |^2 \tag{3.51}
\]
\[
+ \langle u \rangle^{2\alpha+2} |\Box \varphi_{k_1} L \varphi_{k_2} |^2 + \langle u \rangle^{2\alpha+2} |L \varphi_{k_1} \Box \varphi_{k_2} |^2.
\]
We label these three terms by $E_{R_1}$, $E_{R_2}$, $E_{R_3}$ orderly. Using Proposition 3.8, it is easy to check that $E_{R_1}$ can be bounded by
\[
\int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} |D\varphi_k, D\varphi_k|^2 \lesssim \delta^2.
\]
Similarly, for $E_{R_2}$, we have, in views of Proposition 3.8,
\[
\int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} |D\varphi_k, L\varphi_k|^2 \lesssim \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} ||D\varphi_k||_{L,\infty}^2 |L\varphi_k|^2 \lesssim \int_{C_{\bar{u}}} \delta(u)^{-2} |L\varphi_k|^2,
\]
which can be absorbed by the left hand side of (3.50). At last for $E_{R_3}$, recalling that $k_1 \leq k_2 \leq k \leq N - 2$, we perform $L^4$ on all the four factors in $E_{R_3}$. And analogous to $L^2$, there is,
\[
\int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} |L\varphi_k, D\varphi_k|^2 \lesssim \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} ||L\varphi_k||_{L,\infty}^2 ||D\varphi_k||_{L,\infty}^2 |L\varphi_k| d\mu' \lesssim \int_{C_{\bar{u}}} \delta(u)^{-2} \sum_{j \leq k+1} ||L\varphi_j||_{L,\infty}^2 |L\varphi_k| d\mu',
\]
which can be absorbed by the right hand side of (3.50) as well. In a word, we deduce that
\[
\chi^2[\varphi_k](u, \bar{u}) + \int_{C_{\bar{u}}} \langle u \rangle^\alpha |L\varphi_k|^2 d\mu_{C_{\bar{u}}} \lesssim \chi^2[\varphi_k](u_0, \bar{u}) + \delta.
\]
Letting $\bar{u} = \delta$, and knowing that $\chi^2[\varphi_k](u_0, \delta) = 0$ for the data are compactly supported in $C_{u_0, \delta}$, we have
\[
\chi^2[\varphi_k](u, \delta) + ||\langle u \rangle^\alpha L\varphi_k||_{L,\infty}^2 \lesssim \delta. \quad (3.51)
\]
The $L^\infty$ estimate is implied by the Sobolev inequality.

Secondly, we turn to the $i = 1$ case. In regard of the error term, there is
\[
\int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} |\square_y \tilde{S}\varphi_k|^2 d\mu_{C_{\bar{u}}} \lesssim \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha} ||L\varphi_k||^2 + ||\nabla\varphi_{k+1}||^2 + ||L\varphi_k||^2 d\mu_{C_{\bar{u}}} + \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} ||\square_y \varphi_k||^2 d\mu_{C_{\bar{u}}}.
\]
Thus in views of the inequality (3.50) and the proof leading to (3.51), we obtain
\[
\chi^2[\tilde{S}\varphi_k](u, \bar{u}) + \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha} |L\tilde{S}\varphi_k|^2 d\mu_{C_{\bar{u}}} \lesssim \chi^2[\tilde{S}\varphi_k](u_0, \bar{u}) + \delta + \int_{C_{\bar{u}}} \langle u \rangle^{2\alpha+2} |\tilde{S}\square_y \varphi_k|^2 d\mu_{C_{\bar{u}}}.
\]
An analogous argument for $\langle u \rangle^{2\alpha+2} |\square_y \varphi_k|^2$ and $\langle u \rangle^{2\alpha+2} |\tilde{S}\square_y \varphi_k|^2, k \leq N - 3$. Then, we accomplish the proof for the $l = 0$ case, while the argument for $l > 0$ is similar and hence omitted.

**Proof of Theorem 3.7** We will prove this theorem by an inductive argument on $l$. Theorem 3.7 with $l = 0$ has been verified by Proposition 3.8 and 3.9. Suppose Theorem 3.7 holds for $l \leq n$ case, we wish to continue to the case $l = n + 1$. That
is, we shall prove the smallness for $L \tilde{S}^i W_{p,q}^{n+1} \varphi_k$, $i + 2(n + 1) + k \leq N - 2$, $i \leq 1$ and $\tilde{D} \tilde{S}^i W_{p,q}^{n+1} \varphi_k$, $i + 2(n + 1) + k \leq N - 1$, $i \leq 1$.

In the case of $p = 0$, the smallness holds by virtue of Proposition $3.8$ and $3.9$.

In the case of $p \geq 1$, we first consider \textbf{Step I:} $\tilde{D} \tilde{S}^i W_{p,q}^{n+1} \varphi_k$, with $1 \leq p$, and $i + 2(n + 1) + k \leq N - 1$, $i \leq 1$. We note that

$$\nabla \tilde{S}^i W_{p,q}^{n+1} \varphi_k \sim r^{-1} L \tilde{S}^i W_{p-1,q}^{n+1} \varphi_{k+1}, \quad i + 2n + k + 1 \leq N - 2, \quad i \leq 1,$$

which reduces to the $l = n$ case. The smallness holds by the inductive assumption.

We next consider $L W_{p,q}^{n+1} \varphi_k$ with $1 \leq p$, and $2(n + 1) + k \leq N - 1$. By an analogous idea, $L W_{p,q}^{n+1} \varphi_k = L L W_{p-1,q}^{n} \varphi_k$ and

$$L L W_{p-1,q}^{n} \varphi_k = \eta \left( -\Box W_{p-1,q}^{n} \varphi_k + \Delta W_{p-1,q}^{n} \varphi_k + \frac{1}{r} L W_{p-1,q}^{n} \varphi_k - \frac{1}{r} L W_{p-1,q}^{n} \varphi_k \right),$$

where $|\Box W_{p-1,q}^{n} \varphi_k| \leq |W_{p-1,q}^{n} \varphi_k| + |W_{<n-1} \varphi_k|$. All of them are of lower order in the derivative of $W$, and can be reduced to the $l \leq n$ case, noting that $2n + k \leq N - 3$. Hence, the smallness for $L W_{p,q}^{n+1} \varphi_k$ with $1 \leq p$, and $2(n + 1) + k \leq N - 1$ follows by the inductive assumption. Similarly, we can confirm the smallness for $L \tilde{S}^i W_{p,q}^{n+1} \varphi_k$ with $1 \leq p$, and $i + 2(n + 1) + k \leq N - 1$, $i \leq 1$.

Consider \textbf{Step II:} $\tilde{S}^i W_{p,q}^{n+1} \varphi_k$, with $1 \leq p$, and $i + 2(n + 1) + k \leq N - 2$, $i \leq 1$.

Recall the definition \textbf{3.47}: $\chi^2 [\tilde{S}^i W_{p,q}^{n+1} \varphi_k] (u, \underline{w}) = \int_{S^{n-1}} \tau^{2\alpha+1} |L \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 d\sigma_{S^2}$.

Following the proof of Proposition $3.9$, we achieve a general version of \textbf{3.49}:

\begin{equation}
\chi^2 [\tilde{S}^i W_{p,q}^{n+1} \varphi_k] (u, \underline{w}) + \int_{\mathcal{C}_L} \langle u \rangle^{2\alpha} |L \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 d\mu_{\mathcal{C}_L} \\
\lesssim \chi^2 [\tilde{S}^i W_{p,q}^{n+1} \varphi_k] (u, \underline{w}) + \int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |\Box \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 d\mu_{\mathcal{C}_L} \\
+ \int_{\mathcal{C}_L} \langle u \rangle^{2\alpha} \left( |L \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 + |\nabla \tilde{S}^i W_{p,q}^{n+1} \varphi_{k+1}|^2 \right) d\mu_{\mathcal{C}_L}.
\end{equation}

For the last line, we have, by the results in Step I,

$$\int_{\mathcal{C}_L} \langle u \rangle^{2\alpha} \left( |L \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 + |\nabla \tilde{S}^i W_{p,q}^{n+1} \varphi_{k+1}|^2 \right) d\mu_{\mathcal{C}_L} \lesssim \delta,$$

noticing that, for the second term above, $i + 2(n + 1) + k + 1 \leq N - 1$. Now we turn to the error term $\int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |\Box \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 d\mu_{\mathcal{C}_L}$.

As in Proposition $3.9$, we give priority to the $i = 0$ case. Recalling that $|\Box \tilde{S}^i W_{p,q}^{n+1} \varphi_k| \lesssim |W_{p,q}^{n+1} \varphi_k| + |W_{<n} \varphi_k|$, then $\int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |\Box \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2 d\mu_{\mathcal{C}_L}$ can be bounded by $\mathcal{F}_{L_1}^{p,q,k} + \mathcal{F}_{L_2}^{p,q,k}$, where

$$\mathcal{F}_{L_1}^{p,q,k} = \int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |W_{<n} \varphi_k|^2, \quad \mathcal{F}_{L_2}^{p,q,k} = \int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |\Box \tilde{S}^i W_{p,q}^{n+1} \varphi_k|^2.$$

By the inductive assumption, $\mathcal{F}_{L_2}^{p,q,k} \lesssim \delta$. Similarly, combined the inductive assumption and the results in Step I, $\mathcal{F}_{L_2}^{p,q,k}$ can be estimated in an analogous way as $\int_{\mathcal{C}_L} \langle u \rangle^{2\alpha+2} |\tilde{S} \Box \varphi_k|^2 d\mu_{\mathcal{C}_L}$, cf. Proposition $3.9$. Thus we conclude that

$$\|u\|^{2\alpha+1} L W_{p,q}^{n+1} \varphi_k \|_{L^2(S^2 \cup \mathcal{C}_L)} + \|u\|^{2\alpha} L W_{p,q}^{n+1} \varphi_k \|_{L^2(S^2 \cup \mathcal{C}_L)} \lesssim \delta, \quad 2(n + 1) + k \leq N - 2.$$

The $L^\infty$ estimate then follows by the Sobolev inequality.

For the case of $i = 1$, we note that

\begin{align*}
\tilde{\Box} W_{p,q}^{n+1} \varphi_k & \sim \tilde{\Box} W_{p,q}^{n+1} \varphi_k + \tilde{\Box} W_{p,q}^{n+1} \varphi_k \pm \tilde{\Box} W_{p,q}^{n+1} \varphi_k \\
& \pm \frac{1}{r} \left( L W_{p,q}^{n+1} \varphi_k - L W_{p,q}^{n+1} \varphi_k \right), \quad \text{in } \mathcal{R}_1,
\end{align*}
and $|\mathring{S}_{\gamma}W_{p,q}^n\phi_k| \leq |\mathring{S}_{\gamma}W_{p,q}^{n+1}\phi_k| + |\mathring{S}_{\gamma}W_{p,q}^n(\phi_k)|$. Combining with the results of Step I and the $i = 0$ case, we can prove the case of $i = 1$ in a similar way. □

In view of Theorem 3.7, the existence of a solution for (1.1) in $\{ t \leq 1 \}$ of

region III, is reduced to a small data problem with characteristic data prescribed on $C_{u_0}^{[\beta, +\infty]}$ and $\mathcal{C}_T$. Here the data on $C_{u_0}^{[\beta, +\infty]}$ is simply a zero extension of the short

pulse datum prescribed on $C_{u_0}$. For the data on $\mathcal{C}_T$, we take $\alpha = 1$ in Theorem 3.7 and noting that $r \sim \langle u \rangle$ in $\mathcal{K}_1$, then

\[
\begin{align*}
\| \langle u \rangle D S_{p,q}^t \phi_k \|_{L^2(C_T \cap \mathcal{K}_1)} & \lesssim \delta, \quad i + 2l + k \leq N - 2, i \leq 1, \\
\| r D S_{p,q}^t \phi_k \|_{L^\infty(s_{k-\delta}, \cap \mathcal{K}_1)} & \lesssim \delta, \quad i + 2l + k \leq N - 4, i \leq 1.
\end{align*}
\]

(3.52)

Now using Rendall’s local existence theorem for semilinear wave equation with characteristic data [44], we can change into a small Cauchy data problem and apply

Luk’s theorem (Theorem 5.1), noting that both of the regions $\mathcal{R}_1$ and $\{ t \leq 1 \}$ in III are away from the past event horizon, then the global existence and uniqueness in $\{ t \leq 1 \} \cap \mathcal{D}^+(C_{u_0}) \cap \mathcal{D}^+(\mathcal{C}_T)$ follow.

3.6. Scattering theorem. By Rendall’s local existence theorem [44] and the Arzela-

Ascoli Lemma, we can show the global existence and uniqueness in the region $\{ t \leq 1 \} \cap \mathcal{D}^+(\mathcal{I}^-) \cap \mathcal{D}^+(\mathcal{H}^-)$, i.e. from the past null infinity and past event horizon up to $t = 1$, see Sections 5.1 and 5.3 in [52].

Theorem 3.10. Consider on the Schwarzschild background the scattering problem for the semilinear wave equation (1.1) with null condition, where the asymptotic characteristic data are given by: fix any number $\alpha \geq 0$,

\[
\varphi^{\alpha+1}(u, \theta, \phi)|_{\mathcal{I}^-} = \varphi_{-\infty}(u, \theta, \phi), \quad \varphi(u, \theta, \phi)|_{\mathcal{H}^-} = 0,
\]

where $\varphi_{-\infty} \in C^\infty(\mathcal{I}^-)$ is defined in (1.6). If $\delta$ is small enough, (1.1) has a unique and globally smooth solution $\varphi$ in the null strip $\mathcal{R}_1 = \mathcal{D}^-(\Sigma_1) \cap \mathcal{D}^+(\mathcal{I}^-) \cap \mathcal{D}^+(\mathcal{H}^-) \cap \{ 0 \leq u \leq \delta \}$. If $\alpha \geq 1$, then the unique solution exists in $\mathcal{D}^+(\mathcal{I}^-) \cap \mathcal{D}^+(\mathcal{H}^-) \cap \mathcal{D}^-(\Sigma_1)$. Furthermore, restricted in $\mathcal{R}_1$ (see Figure 3), the solution admits the decay estimates: $|L^k L^{j} \varphi| \lesssim \delta^{-k-\frac{j}{2}}|u|^{\alpha-1}, |L^k L^{j} \varphi| \lesssim \delta^{-k-\frac{j}{2}}|u|^{\alpha-\frac{j}{2}}$.

Reversing the time $t$, we achieve the scattering theorem, i.e. Theorem 1.2.

Remark 3.11. If we reverse the time function $t$ to be $-t$, the multiplier is replaced by $\xi = \delta^{-1} \mu^2 L + r^{2\alpha} \eta L$. That is, taking $f_1 = \delta^{-1} \mu^2 f_2 = r^{2\alpha} \eta$. In views of (2.7),

\[
-\frac{1}{2} (\partial_t f_1 + \frac{\mu f_1}{r}) |\nabla \psi|^2 = \frac{1}{2} (2\alpha \mu^2 - \mu \frac{\mu}{r}) |\nabla \psi|^2,
\]

\[
\partial_t f_2 g^{u}\mu|L^{j} \psi|^2 = \frac{1}{2} (2\alpha \mu^2 - \mu r^{2\alpha - 1}) |L^{j} \psi|^2.
\]

Define the corresponding energy $E_k(u) := \int_{C_{n=1}} (\delta^{-1} \mu^2 |L \varphi|^2 + r^{2\alpha} \eta |\nabla \varphi|^2) \, d\mu_{C_{n=1}}$ and $E_k(u) := \int_{C_{n=-1}} (\delta^{-1} \mu^2 |L \varphi|^2 + r^{2\alpha} \eta |\nabla \varphi|^2) \, d\mu_{C_{n=-1}}$. Let $-\delta \leq u_1 \leq u_2 \leq 0$ and $1 \leq u_1 \leq u_2 \leq \infty$ (thus $r \sim \langle u \rangle$ in this region). If $1 \leq u_1 \leq u_2 \leq \infty$, we set $\alpha \geq 0$; if $-1 \leq u_1 \leq u_2 \leq 1$, we let $\alpha = 0$. Then,

\[
\begin{align*}
E_k(u_2) + E_k(u_1) - \int_{D_{n=1}^{(u_1, u_2)}} (\alpha + \frac{\mu}{2}) |\mu^{2\alpha - 1} (\delta^{-1} |\nabla \varphi|^2 + \alpha |L \varphi|^2) \eta d\mu_D \\
\lesssim E_k(u_1) + E_k(u_2) + \int_{D_{n=1}^{(u_1, u_2)}} (\mu^{2\alpha - 1} (\delta^{-1} |L \varphi|^2 + |\nabla \varphi|^2) \eta d\mu_D \\
+ \int_{D_{n=1}^{(u_1, u_2)}} |\nabla \varphi|^2 \langle \mu \rangle^{2\alpha - 1} (\delta^{-1} |L \varphi|^2 + |\nabla \varphi|^2) \eta d\mu_D,
\end{align*}
\]
where the current on the left hand side has the wrong sign. But if we consider the scattering problem: given the data on $\mathcal{I}^+(\mathbb{R}^4) \cap \mathcal{H}^+(u \to +\infty)$ with $\varphi|_{\mathcal{H}^+} \equiv 0$, the above formula turns out to be

$$E_k(u) + \mathcal{E}_k(u) + \int_{\mathcal{D}^+_{u,\mathbb{R}^+}} (\alpha + \frac{k}{2}) (\eta^2) (\delta^{-1}|\nabla \varphi_k|^2 + |\Delta \varphi_k|^2) \eta \, d\mu$$

$$\leq E_k(u) + \mathcal{E}_k(u) + \int_{\mathcal{D}^+_{u,\mathbb{R}^+}} (\eta^2) (\delta^{-1}|\nabla \varphi_k|^2 + |\Delta \varphi_k|^2) \eta \, d\mu$$

$$+ \int_{\mathcal{D}^+_{u,\mathbb{R}^+}} |\nabla \varphi_k| \cdot (\eta^2) (\delta^{-1}|\nabla \varphi_k|^2 + |\Delta \varphi_k|^2) \eta \, d\mu.$$  

We can prove along the line of Section 3 to show the global existence and the uniqueness of solution to the scattering problem.

### 4. Energy decay up to the future horizon and the future null infinity

Let $\mathcal{R}_2$ be the null strip $\mathcal{D}^+(\mathbb{R}^4) \cap \mathcal{D}^-(\mathbb{R}^4) \cap \mathcal{D}^+(-\mathbb{R}^4) \cap \{0 \leq u \leq \delta\}$. In $\mathcal{R}_2$, $u \geq 1-\delta$, $u \sim t$ and $r$ is finite. For notational convenience, we denote $u_0 := 1 - \delta$. Before the energy argument, we will introduce several notations.

For a vector field $V$, let $\int_{\mathcal{D}} |V\varphi_k|^2 \eta \, d\mu = V^k S^k + V^k G^k + L^k$ where

$$V^k S^k = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |Q(\partial V \varphi_q, \partial \varphi_p)|^2 \eta \, d\mu,$$

$$V^k G^k = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |Q(\partial V \varphi_p, \partial \varphi_q)|^2 \eta \, d\mu,$$

and the lower order term $L^k$ takes the form of (4.1) with $V = 1$, and $V^k S^k = V^k S^k_1 + \cdots + V^k S^k_4$, with

$$V^k S^k_1 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_p|^2 |Y V \varphi_q|^2 \eta \, d\mu,$$

$$V^k S^k_2 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_p|^2 |L V \varphi_q|^2 \eta \, d\mu,$$

$$V^k S^k_3 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_p|^2 |\nabla V \varphi_q|^2 \eta \, d\mu,$$

$$V^k S^k_4 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |L \varphi_p|^2 |\nabla V \varphi_q|^2 \eta \, d\mu,$$

and $V^k G^k = V^k G^k_1 + \cdots + V^k G^k_4$, where for $p + q \leq k$, $p < q$,

$$V^k G^k_1 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_q|^2 |Y V \varphi_p|^2 \eta \, d\mu,$$

$$V^k G^k_2 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_q|^2 |L V \varphi_p|^2 \eta \, d\mu,$$

$$V^k G^k_3 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |D \varphi_q|^2 |\nabla V \varphi_p|^2 \eta \, d\mu,$$

$$V^k G^k_4 = \int_{\mathcal{D}} \sum_{p+q \leq k, p \leq q} |L \varphi_q|^2 |\nabla V \varphi_p|^2 \eta \, d\mu.$$


Letting $u_0^* \leq u_1 < u$, $0 \leq u_1 \leq u \leq \delta$, we define the degenerate energy for $\psi$,

$$E^{\text{deg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} \eta \left( |L\psi|^2 + \delta^{-1} |\nabla\psi|^2 \right) \, d\mu_{C^u},$$

(4.5)

$$tE^{\text{deg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} \eta \left( |Y\psi|^2 + \delta^{-2} |Y\psi|^2 \right) \, d\mu_{C^u},$$

(4.6)

$$L E^{\text{deg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} \left( \delta^{-1} |L\psi|^2 + \eta^2 |\nabla\psi|^2 \right) \, d\mu_{C^u}.$$  

(4.7)

Fix $N \in \mathbb{N}$, $N \geq 6$. We set: for $k \leq N$,

$$E_k^{\text{deg}}(u; [u_1, u]) := E^{\text{deg}}[\varphi_k](u; [u_1, u]),$$

(4.8)

$$F_k^{\text{deg}}(u; [u_1, u]) := F^{\text{deg}}[\varphi_k](u; [u_1, u]),$$

(4.9)

The flux is denoted by: for $k \leq N - 1$,

$$L F_k^{\text{deg}}(u; [u_1, u]) := E^{\text{deg}}[\delta\varphi_k](u; [u_1, u]),$$

$$t F_k^{\text{deg}}(u; [u_1, u]) := t E^{\text{deg}}[\varphi_k](u; [u_1, u]),$$

(4.10)

$$L_{k+1}^{\text{deg}}(u; [u_1, u]) := E^{\text{deg}}[\delta\varphi_k](u; [u_1, u]),$$

(4.11)

At the same time, we define the non-degenerate energy,

$$E^{\text{ndeg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} \left( |L\psi|^2 + \delta^{-1} |\nabla\psi|^2 \right) \, d\mu_{C^u},$$

$$t E^{\text{ndeg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} |Y\psi|^2 + \delta^{-1} |Y\psi|^2 \, d\mu_{C^u},$$

(4.12)

$$L E^{\text{ndeg}}[\psi](u; [u_1, u]) := \int_{C^{[u]} \setminus \omega} \left( \delta^{-1} \eta^{-1} |L\psi|^2 + \eta |\nabla\psi|^2 \right) \, d\mu_{C^u}.$$  

Then we set for $k \leq N$,

$$E_k^{\text{ndeg}}(u; [u_1, u]) = E^{\text{ndeg}}[\varphi_k](u; [u_1, u]),$$

(4.13)

$$F_k^{\text{ndeg}}(u; [u_1, u]) = F^{\text{ndeg}}[\varphi_k](u; [u_1, u]),$$

and the non-degenerate flux is denoted by: for $k \leq N - 1$,

$$L F_k^{\text{ndeg}}(u; [u_1, u]) := E^{\text{ndeg}}[\delta\varphi_k](u; [u_1, u]),$$

$$t F_k^{\text{ndeg}}(u; [u_1, u]) := t E^{\text{ndeg}}[\varphi_k](u; [u_1, u]),$$

(4.14)

$$F_{k+1}^{\text{ndeg}}(u; [u_1, u]) := E^{\text{ndeg}}[\delta\varphi_k](u; [u_1, u]),$$

(4.15)

Define the spacetime integral

$$1^{\text{deg}}[\psi](D) = \int_D \left( \delta^{-1} |L\psi|^2 + \delta^{-1} |\nabla\psi|^2 + |L\psi|^2 \right) \eta \, d\mu_D,$$

(4.16)

$$2^{\text{deg}}[\psi](D) = \int_D \left( \delta^{-2} \eta^{-1} |L\psi|^2 + \delta^{-1} |\nabla\psi|^2 + |L\psi|^2 \right) \eta \, d\mu_D.$$  

(4.17)
We denote for \( l \leq N, \ k \leq N - 1, \) and \( i = 1, 2, \)
\[
\begin{align*}
    &iS^{\text{deg}}_l(D) := iS^{\text{deg}}_l[\varphi](D), \quad 2S^{\text{deg}}(D) := 2S^{\text{deg}}[\varphi](D), \\
    &L_i S^{\text{deg}}_{k+1}(D) := iS^{\text{deg}}_l[\delta L \varphi_k](D), \quad L_i S^{\text{deg}}_{k+1}(D) := iS^{\text{deg}}_l[L \varphi_k](D).
\end{align*}
\]

And define the non-degenerate integrated energy:
\[
    S^{\text{ndeg}}[\varphi](D) = \int_D (\delta^{-1} |Y \varphi|^2 + \delta^{-1} |\nabla \varphi|^2 + |L \varphi|^2) \eta d\mu_D. \quad (4.18)
\]

We denote for \( l \leq N, \ k \leq N - 1, \)
\[
    \begin{align*}
        S^{\text{ndeg}}_l(D) := & iS^{\text{deg}}_l[\varphi](D), \\
        L S^{\text{ndeg}}_l(D) := & S^{\text{ndeg}}_l[\delta L \varphi_k](D), \\
        L_2 S^{\text{ndeg}}_l(D) := & S^{\text{ndeg}}_l[L \varphi_k](D).
    \end{align*}
\]

In this section, we set
\[
    I^2_k = (I^2_k + \delta^4 M^4). \quad (4.19)
\]

We will first choose \( M \) (which depends on the initial data) large enough and \( \delta \) small enough such that \( \delta^4 M^2 \ll 1. \) For any \( u_0 \leq u_1 \leq u, \) \( 0 \leq u_1 \leq u, \) we shall also use the short cut \( C_u \) for \( C^{[u, u]}_l \) and \( C_{u_0} \) for \( C^{[u_1, u]}_{u_0}. \)

4.1. Initial data in \( \mathcal{R}_2. \) We restrict the solution obtained in Section 3 on the \( \Sigma_1 = \{ t = 1 \} \) hypersurface, namely \( \psi_0 = \varphi|_{\Sigma_1}, \psi_1 = \partial_t \varphi|_{\Sigma_1}. \) Now, \( (\varphi_0, \psi_1) \) serve as the initial data for the Cauchy problem of \((1.1)\) considered in \( \mathcal{R}_2. \) As shown in Section 3 if we consider the Cauchy problem of \((1.1)\) with data \((\varphi_0, \varphi_1)|_{0 \leq u \leq \delta}, \) there is a unique solution \( \varphi, \) which is exactly the one obtained in Section 3 existing at least in \( \{ t \geq 1 \} \cap \{ u \leq 1 \} \cap \{ 0 \leq u \leq \delta \}. \) And it admits the following estimates: recalling that \( u_0 = 1, \) and \( p + q = l, \)
\[
    \begin{align*}
        &E^{\text{deg}}[\delta^p W_{p,q}^l \varphi_k](u_0', [0, \delta]) \lesssim I^2_{l+k, k}, \quad \text{on} \ C_{u_0}, \\
        &\delta^{-1} \|D W_{p,q}^l \varphi_k\|^2_{L^2(S_{u_0, \delta})} \lesssim I^2_{l+k, k}, \quad \text{on} \ S_{u_0, \delta}.
    \end{align*}
\]

In addition, we know that \( \varphi \equiv 0 \) in \( \{ u < 0 \} \cap \{ t \geq 1 \}. \)

In this section, we will finally prove the decay estimates for the degenerate energy and the energy bound for the non-degenerate energy. Note that, the degenerate energy vanishes on the future horizon.

**Theorem 4.1.** Suppose \( \beta > \frac{1}{2}, \) then there are the decay estimates in \( \mathcal{R}_2:\)
\[
\begin{align*}
    iS^{\text{deg}}_l(D \cap u, u, +\infty) + E^{\text{deg}}[\psi_0; [0, u]] + E^{\text{deg}}[\psi_1; [u, +\infty]] \lesssim I^2_{t_N+1} |u|^{-2\beta}, & \quad l \leq N, \\
    L_i F^{\text{deg}}_{k+1}(u; [0, u]) + L_i F^{\text{deg}}_{k+1}(u; [u, +\infty]) \lesssim I^2_{t_N+1} |u|^{-2\beta}, & \quad k \leq N - 1, \\
    t F^{\text{deg}}_{k+1}(u; [0, u]) + t F^{\text{deg}}_{k+1}(u; [u, +\infty]) \lesssim I^2_{t_N+1} |u|^{-2\beta}, & \quad k \leq N - 1,
\end{align*}
\]
where \( i = 1, 2, \) and
\[
L_i S^{\text{deg}}_{k+1}(D \cap u, u, +\infty) + L_i S^{\text{deg}}_{k+1}(D \cap u, u, +\infty) \lesssim I^2_{t_N+1} |u|^{-2\beta}, \quad k \leq N - 1.
\]

Near the horizon, that is, in the region \( \mathcal{R}_2^{NH} \) := \( \mathcal{R}_2 \cap \{ 2m \leq r \leq r_{NH} \}, \) we have
\[
\begin{align*}
    E^{\text{deg}}(u; [0, u]) + E^{\text{deg}}[\psi_0; [0, u]] + E^{\text{deg}}[\psi_1; [u, +\infty]] \lesssim I^2_{t_N+1}, & \quad l \leq N, \\
    L F^{\text{deg}}_{k+1}(u; [0, u]) + L F^{\text{deg}}_{k+1}(u; [u, +\infty]) \lesssim I^2_{t_N+1}, & \quad k \leq N - 1, \\
    t F^{\text{deg}}_{k+1}(u; [0, u]) + t F^{\text{deg}}_{k+1}(u; [u, +\infty]) \lesssim I^2_{t_N+1}, & \quad k \leq N - 1,
\end{align*}
\]
where \( u_{NH} = u - r_{NH} \) is the \( u \) value of the intersecting sphere \( C_{u_0} \cap \{ r = r_{NH} \}. \)
4.2. Bootstrap assumptions in $R_2$. We now address the bootstrap assumptions. Given any number $\beta > \frac{1}{2}$ and $N \in \mathbb{N}$, $N \geq 6$ such that $\frac{N}{2} + 3 \leq N$, we assume that, there is a large constant $M$ to be determined, such that in $R_2$,

The degenerate case: $\|\eta^\frac{1}{2} L \varphi_l \|_{L^2(C_u)} + \delta^{-\frac{1}{2}} \|\eta^\frac{1}{2} Y \varphi_l \|_{L^2(C_u)} \lesssim M |u|^{-\beta}$, $l \leq N$;

(4.20)

And the flux $L F_{k+1}^{\text{deq}}(u; [0, u]) + t F_{k+1}^{\text{deq}}(u; [0, u]) + L F_{k+1}^{\text{deq}}(u; [0, u]) \lesssim M^2 |u|^{-2\beta}$, $k \leq N - 1$, i.e.

(4.21)

And the spacetime integral: for $l \leq N$, $k \leq N - 1$, $i = 1, 2$,

(4.22)

The non-degenerate case: $E_n^{\text{deq}}(u; [0, u]) \lesssim M^2$, $l \leq N$; i.e.

(4.23)

And the flux $L F_{k+1}^{\text{deq}}(u; [0, u]) + t F_{k+1}^{\text{deq}}(u; [0, u]) + t F_{k+1}^{\text{deq}}(u; [0, u]) \lesssim M^2$, $k \leq N - 1$, i.e.

(4.24)

(4.25)

(4.26)

With these bootstrap assumptions, we deduce some preliminary estimates as follows.

Proposition 4.2. We have the non-degenerate estimates:

$\delta^\frac{1}{2} \|L \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} + \delta^{-\frac{1}{2}} \|D \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} \lesssim M$, $k \leq N - 1$,

$\delta^\frac{1}{2} \|L \varphi_k\|_{L^\infty(R_2)} + \delta^{-\frac{1}{2}} \|D \varphi_k\|_{L^\infty(R_2)} \lesssim M$, $k \leq N - 2$,

and the degenerate decay estimates:

$\delta^\frac{1}{2} \|\eta^\frac{1}{2} L \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} + \delta^{-\frac{1}{2}} \|\eta^\frac{1}{2} D \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} \lesssim |u|^{-\beta} M$, $k \leq N - 1$,

$\delta^\frac{1}{2} \|L \varphi_k\|_{L^\infty(R_2)} + \delta^{-\frac{1}{2}} \|D \varphi_k\|_{L^\infty(R_2)} \lesssim |u|^{-\beta} M$, $k \leq N - 2$.

Remark 4.3. This Proposition can be proved by the Sobolev inequality [5.4]–[5.6].

We also remark that, there are the following improved estimates for lower order derivatives of $Y \varphi_k$ or $L \varphi_k$:

$\|u|^{\beta} \eta^\frac{1}{2} L \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} + \|Y \varphi_k\|_{L^2(S_{u^{\frac{1}{2}}})} \lesssim \delta^2 M$, $k \leq N - 2$,

$\|u|^{\beta} \eta^\frac{1}{2} L \varphi_k\|_{L^\infty(R_2)} + \|Y \varphi_k\|_{L^\infty(R_2)} \lesssim \delta^2 M$, $k \leq N - 3$.

The $\delta^\frac{1}{2}$ lose in the estimates for the top order of $Y \varphi_k$, is due to the weaker assumption for the top order energy $\|Y \varphi_N\|_{L^2(C_u)}$ and $\|\eta^\frac{1}{2} L \varphi_N\|_{L^2(C_u)}$, or equivalently $\|\nabla Y \varphi_k\|_{L^2(C_u)}$ and $\|\eta^\frac{1}{2} \nabla \varphi_k\|_{L^2(C_u)}$, $k = N - 1$, see [4.25] and [4.21].

Namely, compared to the lower order bootstrap assumption $\|u|^{\beta} \eta^\frac{1}{2} Y \varphi_k\|_{L^2(C_u)}$ and $\|Y \varphi_k\|_{L^2(C_u)} \lesssim \delta M$, $k \leq N - 1$, the one for the top order case $\|u|^{\beta} \eta^\frac{1}{2} \nabla L \varphi_k\|_{L^2(C_u)} + \|Y \varphi_k\|_{L^2(C_u)} \lesssim \delta^2 M$, $k = N - 1$ is weaker.

The Sobolev inequality on $C_u$ will not be used for it does not give decay rate of $|u|$.
4.3. Degenerate energy in $\mathcal{R}_2$. At the first stage, we devote to itself the degenerate energy estimates.

**Theorem 4.4.** Suppose $\beta > \frac{1}{2}$ and $u_0 \leq u \leq +\infty$, $0 \leq u \leq \delta$. There are the decay estimates for the degenerate energies

\[
E_{k+1}^{\deg}(u; [0, u]) \leq \|L^{\deg} \phi_k\|_{L^2}^2 + \|L^{\deg} \phi_k\|_{L^2} + \delta^{-1} \|L^\beta \psi\|_{L^2}^2 + \delta^{-\frac{m}{2}} \|L^\beta \psi\|_{L^2}^2 > 0,
\]

\[
\frac{1}{2} \left( \partial_2 f_2 - \frac{\mu f_2}{r} \right) \|\nabla \psi\|_{L^2}^2 = \delta^{-1} \|\nabla \psi\|_{L^2}^2 > 0.
\]

Therefore, by virtue of (2.7) and the energy identity (2.5), we additionally get some positive spacetime integrals and the energy inequality takes the following form (where we have ignored irrelevant constants),

\[
\int_{C_{t_1 \leq t \leq t_2}} (|L\psi|^2 + \delta^{-1} |\nabla \psi|^2) \eta \, d\mu_{\mathcal{C}_u} + \int_{C_{t_1 \leq t \leq t_2}} (\eta^2 |\nabla \psi|^2 + \delta^{-1} |L\psi|^2) \, d\mu_{\mathcal{C}_u} + \int_{C_{t_1 \leq t \leq t_2}} (\eta^2 |\nabla \psi|^2 + \delta^{-1} |L\psi|^2) \, d\mu_{\mathcal{C}_u} + C(\psi) + \mathcal{F}(\psi),
\]

where the current $C(\psi)$ is given by

\[
C(\psi) = \int_{C_{t_1 \leq t \leq t_2}} (|\nabla \psi|^2 + \delta^{-1} |L\psi| L\psi) \eta \, d\mu_{\mathcal{D}}.
\]

and the nonlinear error term $\mathcal{F}(\psi)$ is given as below,

\[
\mathcal{F}(\psi) = \int_{C_{t_1 \leq t \leq t_2}} (|\nabla \psi|^2 + \delta^{-1} |L\psi| \eta) \, d\mu_{\mathcal{D}}.
\]
For the current $C(\psi)$,
\[
\int_{D_{2t-1}^{u_1, u_3}} |\nabla \psi|^2 \eta d\mu_D \leq \delta \int_{D_{2t}^{u_1, u_3}} \delta^{-1} |\nabla \psi|^2 \eta d\mu_D ,
\]
which can be absorbed by the spacetime integral on the left hand side of (4.32):
\[
\int_{D_{2t-1}^{u_1, u_3}} \delta^{-1} |L\psi L\psi| \eta d\mu_D \\
\lesssim c \int_{D_{2t}^{u_1, u_3}} |L\psi|^2 \eta^3 d\mu_D + \int_{2t}^{2t} c^{-1} \eta^0 d\eta \int_{C_{2}^{u_1, u_3}} \delta^{-1} |L\psi|^2 d\mu_{C_{2}^{u_1, u_3}}.
\]
Here $c$ is a constant to be determined. Meanwhile, we estimate $F(\psi)$ by
\[
|F(\psi)| \lesssim \int_{D_{2t}^{u_1, u_3}} c^{-1} \Box g \eta^3 d\mu_D + c \int_{D_{2t}^{u_1, u_3}} |L\psi|^2 \eta d\mu_D \\
+ c \int_{D_{2t}^{u_1, u_3}} |\Box g \psi|^2 \eta^2 d\mu_D + \int_{2t}^{2t} c^{-1} \eta^0 d\eta \int_{C_{2}^{u_1, u_3}} \delta^{-1} |L\psi|^2 d\mu_{C_{2}^{u_1, u_3}}.
\]
We chose $c \ll 1$ so that $c \int_{D_{2t}^{u_1, u_3}} |L\psi|^2 \eta d\mu_D$ can be absorbed by the positive integral on the left hand side of (4.32), while the last terms in (4.36) and (4.37) can be handled by the Gronwall’s inequality. As a consequence, we deduce
\[
E_{\text{deg}}[\psi](u_2; [u_1, u_2]) + E_{\text{deg}}[\psi](u_2; [u_1, u_3]) + 1S_{\text{deg}}[\psi](D_{2t}^{u_1, u_3}) \\
\lesssim E_{\text{deg}}[\psi](u_1; [u_1, u_2]) + E_{\text{deg}}[\psi](u_1, [u_1, u_2]) + F_1(\psi) + F_2(\psi),
\]
where $c \ll 1$ is a constant to be determined and
\[
F_1(\psi) \lesssim \int_{D_{2t}^{u_1, u_3}} c^{-1} \Box g \eta^2 d\mu_D, \quad F_2(\psi) \lesssim c \int_{D_{2t}^{u_1, u_3}} |\Box g \eta|^2 \eta^2 d\mu_D.
\]
Such energy inequality (4.38)-(4.39) will come into play in the energy estimate for the top order case, see Section 4.3.4. Alternatively, without lost of generality, we have as well
\[
E_{\text{deg}}[\psi](u_2; [u_1, u_2]) + E_{\text{deg}}[\psi](u_2; [u_1, u_3]) + 1S_{\text{deg}}[\psi](D_{2t}^{u_1, u_3}) \\
\lesssim E_{\text{deg}}[\psi](u_1; [u_1, u_2]) + E_{\text{deg}}[\psi](u_1, [u_1, u_2]) + \int_{D_{2t}^{u_1, u_3}} |\Box g \eta|^2 \eta^2 d\mu_D.
\]

### 4.3.2. Estimates for $E_k^{\text{deg}}(u; [0, u])$ and $E_k^{\text{deg}}(u; [u, +\infty])$, $k \leq N - 1$. Taking $\psi = \phi_k$, $k \leq N - 1$ in (4.40), we obtain the energy inequality,
\[
E_k^{\text{deg}}(u_2; [u_1, u_2]) + E_k^{\text{deg}}(u_2; [u_1, u_2]) \\
+ \int_{D_{2t}^{u_1, u_3}} \left(\delta^{-1} |L\phi_k|^2 + \delta^{-1} |\nabla \phi_k|^2 + |L\phi_k|^2\right) \eta d\mu_D \\
\lesssim E_k^{\text{deg}}(u_1; [u_1, u_2]) + E_k^{\text{deg}}(u_1, [u_1, u_2]) + \int_{D_{2t}^{u_1, u_3}} |\Box g \phi_k|^2 \eta^2 d\mu_D.
\]

The last term in (4.41), denoted by $F_k$, is split as $F_k = S_k + \cdots + S_k$, where $S_k^j, j = 1:4$ are defined as (4.3) with $D = D_{2t}^{u_1, u_3}$, $V = 1$, $i = 2$, i.e. for $p + q \leq$
In particular, letting $u_1 = 0$, we have for any $u_0 \leq u_1 < u_2$,

$$E_{k}^{\text{deg}}(u_1; [u_1, u_2]) = \int_{u_1}^{u_2} E_{k}^{\text{deg}}(u; [u_1, u_2]) du \lesssim E_{k}^{\text{deg}}(u_1; [u_1, u_2]) + \delta^2 M |u_1|^{-4\beta + 1}, \quad k \leq N - 1. \quad (4.45)$$

We consider $\beta > \frac{1}{2}$, so that $-4\beta + 1 < -2\beta$. By the pigeon-hole principle (see Lemma 5.2), we achieve that for any $\beta > \frac{1}{2}$ and $u_0 \leq u$, $0 \leq u_0 \leq \delta$,

$$E_{k}^{\text{deg}}(u; [u_1, u_2]) \lesssim \|u\|_{k}^2 |u|^{-2\beta}, \quad k \leq N - 1. \quad (4.46)$$

Letting $u_1 = u$, $u_2 \to +\infty$ and $u_1 = 0$, $0 < u_2 = u \leq \delta$ in (4.44),

$$E_{k}^{\text{deg}}(u; [u, +\infty]) + i S_{k}^{\text{deg}}(D_{u}^{n+\infty}) \lesssim E_{k}^{\text{deg}}(u; [0, u]) + \delta^2 M |u|^{-4\beta + 1}, \quad k \leq N - 1.$$
Substituting (4.46) into the above formula, we deduce
\[ L_{k+1}^{deg}([u; u_{0}]); u_{0}) + L_{k}^{deg}([u^{(i)}]; u_{0}) \leq \frac{1}{2} L_{k} M^{2}|u_{1}|^{-4\delta+1}, \quad k \leq N - 1. \] (4.47)

At last, we can make use of (4.46) and (4.47) and follow the proof leading to (4.41) to derive
\[ \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} |\psi|^{2} \eta^{2} \, d\mu \leq \frac{1}{2} L_{k} M^{2}|u_{1}|^{-4\delta+1}, \quad k \leq N - 1. \] (4.48)

### 4.3.3. Estimates for \( L_{k+1}^{deg}([u; u_{0}]), k \leq N - 1 \)

We take \( \psi = \delta L\varphi_{k}, k \leq N - 1 \) in (4.46) to derive
\[ L_{k+1}^{deg}([u_{1}; u_{2}]) + L_{k}^{deg}([u_{1}; u_{2}]) \]
\[ + \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} (\delta L_{k})^{2} + \delta \nabla L_{k}^{2} + \delta^{2} L_{k}^{2} \) \( \eta \, d\mu \)
\[ \leq L_{k+1}^{deg}([u_{1}; u_{2}]) + L_{k}^{deg}([u_{1}; u_{2}]) \]
\[ + \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} |\nabla L_{k}|^{2} \eta^{2} \, d\mu. \] (4.49)

The last term, denoted \( \mathcal{F}(\delta L\varphi_{k}) \) can be split as: \( \mathcal{F}(\delta L\varphi_{k}) = L S_{k} + L G_{k} + \delta L^{k} + W_{k} \), where \( L S_{k}, L G_{k} \) take the forms of (4.11) and (4.2), with \( D = D_{u_{1}, u_{2}}^{1, u_{2}}; V = \delta L, \)
\( i = 2; \delta L^{k} \) is defined as (4.1) with \( D = D_{u_{1}, u_{2}}^{1, u_{2}}, V = \delta, i = 2 \); And \( L W^{k} \) associated to \( |\nabla\psi, \delta L| \varphi_{k} \) is given by
\[ L W_{k} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |L_{k}^{2} + |L_{k}^{2} + |L_{k}^{2} |L_{k}^{2} \right) \eta^{2} \, d\mu. \] (4.50)

First of all, (4.48) leads to \( \delta L^{k} \leq \frac{1}{2} L_{k} M^{2}|u_{1}|^{-4\delta+1}, k \leq N - 1. \)

For the error terms \( L S_{j}^{k} \), we make the further splitting: \( L S_{j}^{k} = L S_{j}^{1} + \cdots + L S_{j}^{4} \), where \( L S_{j}^{k}, j = 1 : 4 \) are defined as (4.3) with \( D = D_{u_{1}, u_{2}}^{1, u_{2}}; V = \delta L, i = 2 \). The estimates for the \( L S_{j}^{k}, j = 1 : 3 \) are the same as that for \( S_{j}^{k}, j = 1 : 3 \) (4.42), except that \( \varphi_{q} \) therein is replaced now by \( \delta L\varphi_{q} \). For the remaining one \( L S_{j}^{4} \), it reads
\[ L S_{j}^{4} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |L_{k}^{2} |L_{k}^{2} \right) \eta^{2} \, d\mu, \quad k \leq N - 1. \]

By the bootstrap assumption (4.20), noting that \( p + q \leq N, \) and the \( L^{\infty} \) estimate
\( \eta^{2} |L_{p}| \leq \frac{1}{2} M|u_{1}|^{-\beta}, q \leq N/2 \leq N - 3, \)
\[ L S_{j}^{4} \leq \int_{u_{1}}^{u_{2}} \delta^{2} \left( |L_{p}^{2} |L_{p}^{2} \right) \eta^{2} M|u_{1}|^{-4\beta + 1}, \quad k \leq N - 1. \]

For \( L G_{j}^{k} \), we make the following splitting: \( L G_{j}^{k} = L G_{j}^{1} + \cdots + L G_{j}^{4} \), where \( L G_{j}^{k}, j = 1 : 4 \) are defined as (4.4) with \( D = D_{u_{1}, u_{2}}^{1, u_{2}}; V = \delta L, i = 2 \), i.e. for \( p + q \leq k, p < q, \)
\[ L G_{j}^{1} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |D_{p}^{2} |Y_{p}^{2} \right) \eta^{2} \, d\mu, \]
\[ L G_{j}^{2} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |D_{p}^{2} |L_{2}^{2} \right) \eta^{2} \, d\mu, \]
\[ L G_{j}^{3} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |D_{p}^{2} |L_{2}^{2} \right) \eta^{2} \, d\mu, \]
\[ L G_{j}^{4} = \int_{\mathbb{R}^{4} \times \mathbb{R}^{2}} \delta^{2} \left( |L_{p} |L_{p} \right) \eta^{2} \, d\mu. \]
We note that \( p + q \leq k \leq N - 1, p < q, \) then \( q \leq N - 1, p \leq N - 2. \) We can always perform \( L^4 \) norm to the four factors in each term above.

For \( L^k G^1_k \), by the a priori estimate \( \| D \varphi \|_{L^k(S_{\varphi, u})} \lesssim \delta^{\frac{1}{2}} M, \) \( q \leq N - 1, \)

\[
|L^k G^1_k| \lesssim \int_{u_1}^{u_2} \int_{\varphi_1}^{\varphi_2} \delta^{-1} M^2 \cdot \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 d\varphi du \\
\lesssim \delta M^2 \int_{u_1}^{u_2} \int_{\varphi_1}^{\varphi_2} \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 d\varphi du \\
\lesssim \int_{u_1}^{u_2} M^2 d\varphi \int_{C_\varphi} \sum_{p \leq \varphi + 1} \delta^2 \|L \varphi\|_{S_{\varphi, u}}^2 d\mu_{C_\varphi}, \quad p \leq N - 2,
\]

where we had used the Sobolev inequality \( (3.25) \) and the fact that \( r \) is finite in the region \( \mathcal{R}_2 \) in the second inequality. Hence we can apply the Gronwall’s inequality.

For \( L^k G^1_k, L^k G^3_k \), knowing that \( \| D \varphi \|_{L^k(S_{\varphi, u})} \lesssim \delta^{\frac{1}{2}} M, q \leq N - 1, \) and \( p \leq N - 2, \) we have similarly,

\[
|L^k G^2_k| + |L^k G^3_k| \lesssim \int_{u_1}^{u_2} \int_{\varphi_1}^{\varphi_2} \delta^2 \|L^2 \varphi\|_{L^2(S_{\varphi, u})}^2 \eta^2 d\varphi du \\
\lesssim \delta^{\frac{1}{2}} \int_{u_1}^{u_2} \sum_{k \leq N - 1} \delta^2 \|L^2 \varphi\|_{L^2(S_{\varphi, u})}^2 d\mu_D,
\]

which can be absorbed by the left hand side of (4.40) if \( \delta \) is small enough.

For the last one, by the a priori estimate \( \eta \|L \varphi\|_{L^k(S_{\varphi, u})} \lesssim \delta^{-1} M^2 |u|^{-2\beta}, q \leq N - 1, \) and \( p \leq N - 2, \)

\[
|L^k G^4_k| \lesssim \int_{u_1}^{u_2} \int_{\varphi_1}^{\varphi_2} \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 \eta^2 \|\nabla L \varphi\|_{L^2(S_{\varphi, u})}^2 d\varphi du \\
\lesssim \int_{u_1}^{u_2} \int_{\varphi_1}^{\varphi_2} \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 \eta^2 \|\nabla L \varphi\|_{L^2(S_{\varphi, u})}^2 d\varphi du \\
\lesssim \int_{u_1}^{u_2} \delta^2 \sum_{k \leq N - 1} \sum_{p \leq \varphi + 1} |L \varphi|^2 \eta^2 d\mu_{C_\varphi} \lesssim \delta M^2 |u|^{-4\beta - 1},
\]

where in the last inequality, the bootstrap assumption (4.20) is used.

Finally, noting that \( k \leq N - 1, \)

\[
|L^k W| \lesssim \delta^2 \int_{D^i_{1, \varphi}} \int_{D^i_{2, \varphi}} \eta^2 \|L \varphi\|^2 + |L \varphi|^2 + |\varphi \varphi|^2 + |\varphi \varphi|^2 |d\mu_D \\
\lesssim \delta^2 M^2 |u|^{-2\beta} + \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 |u|^{-4\beta - 1},
\]

where we used the bootstrap assumption for \( \mathcal{S}^{\text{deg}_{\infty}}(D^i_{0, \varphi}), l \leq N \) (4.23) and the improvement (4.48).

In summary, we have proved that for \( k \leq N - 1, 0 \leq u_1 < u_2 < \delta, \) \( u_0 < u_1 < u_2 < +\infty, \) and \( \beta > \frac{1}{2}, \)

\[
L^k F_{i+k+1}(u_1; [u_1, u_2]) + L^k F_{k+i+1}(u_1; [u_1, u_2]) + \frac{1}{2} \delta \mathcal{S}^{\text{deg}}(D^i_{1, \varphi}) \\
\lesssim L^k F_{k+i+1}(u_1; [u_1, u_2]) + L^k F_{k+i+1}(u_1; [u_1, u_2]) + \delta^2 \|L \varphi\|_{L^2(S_{\varphi, u})}^2 |u|^{-2\beta}.
\]

Following the argument for (4.46), (4.47), we can achieve (4.28) and (4.30).

4.3.4. Estimates for \( L^k \varphi (u; [0, u]) \) and \( L^k \varphi (u; [u, +\infty]). \) We should note that, \( (4.28) \) and (4.46) yield the improved \( L^\infty \) estimate,

\[
\|\eta^2 L \varphi\|_{L^\infty} \lesssim \delta^{-\frac{1}{2}} \|L \varphi\|_{L^2(S_{\varphi, u})}^2 |u|^{-2\beta}, \quad k \leq N - 3.
\]
As explained before, the estimate for $S^N_k$, $k \leq N - 1 \ (4.43)$ is not allowed when $k = N$. We will combine the improvement (4.51) and the refined energy inequality (4.38)-(4.39) to linearize $S^N_k$. We take $\psi = \varphi_N$ in (4.38)-(4.39), then the error terms are

$$F_1(\varphi_N) \lesssim \int_{D^N_{I_1:|u_2|}} c^{-1}|\nabla \varphi_N|^2 \eta^2 d\mu_D, \quad F_2(\varphi_N) \lesssim c \int_{D^N_{I_1:|u_2|}} |\nabla \varphi_N|^2 \eta^2 d\mu_D,$$

where $c \ll 1$ is a constant to be determined. As in the case of $k \leq N - 1$, there is the splitting (4.42) for $F_1(\varphi_N), F_2(\varphi_N)$. And $S^N_k$, $i = 1 : 3$, can be treated in the same way as previously, while $S^N_4$ taking the form of

$$S^N_4 = \int_{D^N_{I_1:|u_2|}} \sum_{p+q \leq N, p \leq q} c^{-1}|\eta^j L \varphi_p|^2 |\nabla \varphi_q|^2 \eta^2 d\mu_D$$

$$+ \int_{D^N_{I_1:|u_2|}} \sum_{p+q \leq N, p \leq q} c|\eta^j L \varphi_p|^2 |\nabla \varphi_q|^2 \eta d\mu_D,$$

should be linearized. Note that $\eta^j$ in $F_1(\varphi_N)$ is crucial here. In views of the improved estimate (4.51),

$$|S^N_4| \lesssim \int_{\Omega_1} \delta^{-1} \| \nabla \varphi_N \|^2 \eta^2 d\mu_{\Omega_1} + \int_{D^N_{I_1:|u_2|}} c^{-1}|\eta^j L \varphi_p|^2 |\nabla \varphi_q|^2 \eta d\mu_D.$$ 

We choose $c \ll 1$ so that the second term can be absorbed, see (4.52), and the first term can be handled by the Gronwall’s inequality. We here recall (4.19) for the definition of $I^N_1$.

After these estimates, we arrive at

$$L^2_{k+1}((u_2;[u_1,u_2]) + E_{k}^{\text{deg}}((u_2;[u_1,u_2])) + iS^N_2(D^N_{[0,u_2]} \ (4.52)$$

Proceeding in an analogous way as in Section 4.3.2 and combining with the previous results, we finally achieve (4.27) and (4.29). Thus, we prove Theorem 4.4. As a result, (4.51) is improved as: in $R_2$, for $k \leq N - 1, j \leq N - 2$, there is

$$\|\eta^j L \varphi_p\|_{L^\infty(R_2)} \| \nabla \varphi_q\|_{L^\infty(S_\infty)} \lesssim \delta^{-1} \| N_{u_1}^{N_1} |u_1|^{-2\beta_j} \| (4.53)$$

4.3.5. Estimates for $L_{k+1}^{\text{deg}}((u;[0,u]))$ and $L_{k+1}^{\text{deg}}((u;[u,\infty]))$, $k \leq N - 1$. In this section, we will make use of Theorem 4.4 to prove that

**Theorem 4.6.** We have in $R_2$, for any $\beta > \frac{1}{2}$,

$$L^2_{k+1}((u;[0,u])) + L^2_{k+1}(D^N_{[0,u]}) \lesssim \| N_{u_1}^{N_1} |u_1|^{-2\beta_j} \| (4.54)$$

And integrating (4.55) along $\partial_u$, we have the spacetime decay estimate

$$\int_{D^N_{[0,u]}} \delta^{-1} L^2 \varphi_k^2 d\mu_D \lesssim \| N_{u_1}^{N_1} |u_1|^{-2\beta_j} \| (4.56)$$

**Remark 4.7.** Combining Theorem 4.4 and Theorem 4.6, we achieve

$$2 \delta_{k+1}(D^N_{[0,u]}) + 2 S_{k+1}(D^N_{[0,u]}) + 2 S_{k+1}(D^N_{[0,u]}) \lesssim \| N_{u_1}^{N_1} |u_1|^{-2\beta_j} \| \quad l \leq N.$$ 

**Proof.** We take $\psi = L \varphi_k, k \leq N - 1$ in (4.40) to derive,

$$L_{k+1}^{\text{deg}}((u;[u_1,u_2]) + L_{k+1}^{\text{deg}}((u_2;[u_1,u_2]))$$

$$+ \int_{D^N_{I_1:|u_2|}} (\delta^{-1} L^2 \varphi_k^2 + \delta^{-1} |\nabla L \varphi_k|)^2 + |L L \varphi_k^2| \eta d\mu_D \lesssim L_{k+1}^{\text{deg}}((u;[u_1,u_2]) + L_{k+1}^{\text{deg}}((u_2;[u_1,u_2])) + L f_k + L y_k,$$
where
\[ L\mathcal{F}^k = \int_{D^{u_1, u_2}} |\square_g \varphi_k|^2 \eta^2 d\mu_D, \]
and \( L\mathcal{W}^k \) is associated to \( |\square_g L\varphi_k|^2 \).

\[ L\mathcal{W}^k = \int_{D^{u_1, u_2}} \left( |L\varphi_k|^2 + |L_\nu \varphi_k|^2 + |\Delta \varphi_k|^2 + |\square_g \varphi_k|^2 \right) \eta^2 d\mu_D. \]

Moreover, \( L\mathcal{F}^k \) can be split into: \( L\mathcal{F}^k = L\mathcal{S}^k + L\mathcal{G}^k + L\mathcal{L}^k \), where \( L\mathcal{S}^k \), \( L\mathcal{G}^k \) take the forms of (4.1)-(4.2) with \( D = D_{u_1, u_2}, V = L, i = 2, \) and \( L\mathcal{L}^k \) is defined as (4.1) with \( D = D_{u_1, u_2}, V = 1, i = 2. \)

Appealing to (4.48), there is \( L\mathcal{S}^k \leq \delta^2 \|h_2\| L^2 |u_1|^{-4\beta+1}, k \leq N - 1. \)

We next turn to estimate \( L\mathcal{S}^k \), \( L\mathcal{G}^k \). \( L\mathcal{S}^k \) can be split as: \( L\mathcal{S}^k = L\mathcal{S}_S^k + \cdots + L\mathcal{S}_L^k \), where \( L\mathcal{S}_S^k, j = 1 : 4 \) are defined as (4.3) with \( D = D_{u_1, u_2}, V = L, i = 2. \) The estimates for \( L\mathcal{S}_S^k, j = 1 : 3 \), are similar to that for \( S^j, j = 1 : 3 \), with only \( \varphi_\xi \) therein being replaced by \( L\varphi_\xi \). We are left with \( L\mathcal{S}_L^k \), which reads, noting that \( p < q \leq k \leq N - 1, p \leq N - 2, \)
\[ L\mathcal{S}_L^k = \int_{D^{u_1, u_2}} \sum_{p+q \leq k, p < q} |L\varphi_p|^2 |\nabla L\varphi_q|^2 \eta^2 d\mu_D. \]

By the improved estimates for \( ||\eta^2 L\varphi_p||_{L^2} \leq (4.51) \) and \( \eta^{S}_{deg}(D^{u_\infty, u_\infty}), l \leq N (4.27), \)
\[ L\mathcal{S}_L^k \leq \int_{D^{u_1, u_2}} \sum_{p \leq k, p \leq N} \delta^{-1} \|h_2\| u_1^{-2\beta} |L\varphi_p|^2 \eta^2 d\mu_D \leq \|h_2\| u_1^{-4\beta}. \]

For \( L\mathcal{G}^k \), we make the following splitting: \( L\mathcal{G}^k = L\mathcal{G}_S^k + \cdots + L\mathcal{G}_L^k \), where \( L\mathcal{G}_j^k, j = 1 : 4 \) is defined as (4.4) with \( D = D_{u_1, u_2}, V = L, i = 2. \) The estimates for \( L\mathcal{G}_j^k, j = 1 : 3 \) are similar to that for \( G^j, j = 1 : 3. \) For \( L\mathcal{G}_4^k \), we take advantage of the enhanced \( L^4 \) estimate (4.53) and (4.27) to deduce (4.58)
\[ L\mathcal{G}_4^k \leq \int_{D^{u_1, u_2}} \delta^{-1} \|h_2\| u_1^{-2\beta} \sum_{p \leq k, p \leq N} |L\varphi_p|^2 \eta^2 d\mu_D \leq \|h_2\| u_1^{-4\beta}. \]

Finally, we achieve that for \( k \leq N - 1, \beta > \frac{1}{2}, \)
\[ L\mathcal{F}_{k+1}^{deg}(u_2; [u_1, u_2]) + L\mathcal{F}_{k+1}^{deg}(u_2; [u_1, u_2]) + L\mathcal{S}_{k+1}(D_{u_1, u_2}) \leq \|h_2\| u_1^{-2\beta}. \]

In the same way as the argument in Section 4.3.2 we prove Theorem 4.6 \( \square \)

4.4. Non-degenerate energy near the future horizon. We denote the region near the future horizon by \( R_{2NH}^2 := R_2 \cap \{2m \leq r \leq r_{NH}\} \), where \( r_{NH} \) satisfying \( 2m < r_{NH} < 1.2r_{NH} < 3m \), is close to \( 2m \).

Consider the region \( r \leq 1.2r_{NH} \), and take \( 0 \leq u_1 \leq u_2 \leq \delta. \) Let \( u_i^H \) be the \( u \) value of the intersecting sphere \( \{r = 1.2r_{NH}\} \cap C_{u_i} \), and \( u_i^{NH} \) be the \( u \) value of the intersecting sphere \( \{r = r_{NH}\} \cap C_{u_i}. \) That is, \( u_i^H = u_i^{H} = u_{i+1}, i = 1, 2. \) In the domain of \( \{r \leq 1.2r_{NH}\} \cap R_2 \), i.e. \( u_i^H < u_{i+1}^{NH} \leq u \leq +\infty, \) \( 0 \leq u_1 < u_2 \leq \delta, \) we define the following exterior and interior region
\[ D^e := \{r_{NH} < r \leq 1.2r_{NH}\} \cap \{u_1 < u < u_2\}, \]
\[ D^h := \{r \leq r_{NH}\} \cap \{u_1 < u < u_2\}, \]
\[ C_{u_i}^{NH} := C_{u_i} \cap D^h, \quad C_{u_i}^e := C_{u_i} \cap D^e. \]
We will also use the notation: \( C^N_H = C_u \cap \{ r \leq r_H \} = C_u^{[u,N_H, + \infty)} \), where \( u_{N_H} := u - r_N^H \), and \( C_u^{\infty} = C_u \cap \{ r_N < r \leq 2r_N \} \), if there is no room for confusion.

We will prove the following energy decay estimates near the horizon.

**Theorem 4.8.** In \( \mathcal{R}^N_H \), \( 0 - r_N^H < u \leq +\infty, \) \( 0 \leq u \leq \delta \), we have

\[
E^e_i(u; [0, u]) + S^{indeg}(D^h) \lesssim I^{N+1}_N, \quad i \leq N, \tag{4.59}
\]

\[
L^e_i(D^h(u; [0, u]) + L^{indeg}(D^h) \lesssim I^{N+1}_N, \quad k \leq N - 1. \tag{4.60}
\]

And letting \( u_{N_H} = u - r_N^H \),

\[
E^e_i(u; [u_{N_H}, +\infty]) \lesssim I^{N+1}_N, \quad l \leq N, \tag{4.61}
\]

\[
L^e_i(u; [u_{N_H}, +\infty]) \lesssim I^{N+1}_N, \quad k \leq N - 1. \tag{4.62}
\]

The proof of this theorem will be given in Sections 4.4.2, 4.4.3, and 4.4.4.

### 4.4.1. The multiplier near the horizon

We choose \( y_1(r^*) > 0, y_2(r^*) > 0 \) that are supported in \( r < 2r_N \), with \( y_1|_{\mathcal{H}+} = 0, y_2|_{\mathcal{H}+} = 0, \), and \( \partial_r y_1 > 0, \partial_r y_2 > 0 \) if \( 2m < r \leq r_N \). An example is given by [13] (we notice that \( |r^*| = -r^* \) near the horizon)

\[
y_1 = \xi_{r,r_H}(r^*) (1 + |r^*|^{-\epsilon}), \quad y_2 = c \xi_{r,r_H}(r^*) |r^*|^{-1 - \epsilon},
\]

where \( \epsilon \) is a small positive constant, \( \xi_{r,r_H} \) is a cutoff function such that \( \xi_{r,r_H} = 1 \) for \( r \leq r_N \) and \( \xi_{r,r_H} = 0 \) for \( r \geq 2r_N \). One has then \( y_2|_{\mathcal{H}+} = 0, \partial_r y_2|_{\mathcal{H}+} = 0, \) \( y_1|_{\mathcal{H}+} = 1, \partial_r y_1|_{\mathcal{H}+} = 0. \) To carry out the estimates near horizon, we will consider the following vector field

\[
N^h = (1 + y_2(r^*)) L + \delta^{-1} y_1(r^*) Y. \tag{4.63}
\]

We take the multiplier \( \xi = N^h \) and apply the energy identity to the wave equation for \( \psi \). In addition, we split up the integrals into the exterior and interior parts and obtain

\[
E^e[u; [u_1, u_2]] + E^e[u; [u_2, [u_1, u]]] + \int_{D^h \setminus \mathcal{D}} \left( \delta^{-1} \eta^2 L^2 \psi^2 + \delta^{-1} \eta \psi \right) d\mu_D \lesssim E^e[u; [u_1, u_2]] + E^e[u; [u_2, [u_1, u]]] + hC(\psi) + hF(\psi) + eC(\psi) + eF(\psi), \tag{4.64}
\]

where \( hC(\psi), eC(\psi) \) are the exterior and interior currents respectively,

\[
hC(\psi) = \int_{\mathcal{D}^h} \left( \eta \psi^2 + \delta^{-1} L^2 \psi \right) d\mu_D, \tag{4.65}
\]

\[
eC(\psi) = \int_{\mathcal{D}^h} \left( \delta^{-1} L^2 \psi^2 + \delta^{-1} \psi \right) d\mu_D,
\]

and \( hF(\psi), eF(\psi) \) are the exterior and interior source terms,

\[
hF(\psi) = \int_{\mathcal{D}^h} \left( \delta \psi + \delta^{-1} V \psi \right) \eta d\mu_D, \tag{4.66}
\]

\[
eF(\psi) = \int_{\mathcal{D}^h} \left( \delta \psi + \delta^{-1} V \psi \right) d\mu_D.
\]

The interior current \( hC(\psi) \) can be estimated in the same way as \( C(\psi) \) : The first term in \( hC(\psi) \) can be absorbed while the second term is bounded by

\[
\lesssim \int_{\mathcal{D}^h} c_L^2 \eta d\mu_D + c^{-1} \int_{\mathcal{D}^h} \delta^{-1} d\mu_D \int_{\mathcal{D}^h} \delta^{-1} \eta \psi^2 d\mu_D.
\]
As (4.42), we apply $L^Dh$erate case. Estimates for inequality, there is

Similarly, as (4.40) in Section 4.3.1, we choose $D^k$.

For $h \leq k$, $V_i$, we note that $E^k \leq p,q$ and $\delta \leq \eta|\nabla\varphi_k|^2 + \eta|L\varphi_k|^2 \leq \eta|L\varphi_k|^2) \, d\mu_D$

As (4.40) in Section 4.3.1, we choose $c \ll 1$, and after applying the Gronwall’s inequality, there is

$$E^{ndeg}[\psi](u_2; [u_1, u_2]) + E^{ndeg}[\psi](u_1; [u_1, u_2]) + S^{ndeg}[\psi](D^k)$$

$$\lesssim E^{ndeg}[\psi](u_1^i; [u_1, u_2]) + E^{ndeg}[\psi](u_1_i; [u_1, u_2])$$

$$+ \int_{D^k} \left| \Box_y \psi \right|^2 \eta \, d\mu_D + \epsilon C(\psi) + \epsilon F(\psi).$$

4.4.2. Estimates for $E^{ndeg}_k(u; [0, u])$ and $E^{ndeg}_k(u; [u, +\infty])$, $k \leq N - 1$. We take $\psi = \varphi_k$, $k \leq N - 1$ in (4.67) to derive

$$E^{ndeg}_k(u; [u_1, u_2]) + E^{ndeg}_k(u_1; [u_1^i, u])$$

$$+ \int_{D^k} \left( \delta^{-1} \eta^{-1} |L\varphi_k|^2 + \delta^{-1} \eta |\nabla\varphi_k|^2 + \eta |L\varphi_k|^2 \right) \, d\mu_D$$

$$\lesssim E^{ndeg}_k(u_1^i; [u_1, u_2]) + E^{ndeg}_k(u_1_i; [u_1, u])$$

$$+ \int_{D^k} \left| \Box_y \varphi_k \right|^2 \eta \, d\mu_D + \epsilon C(\varphi_k) + \epsilon F(\varphi_k).$$

The estimates for $\int_{D^k} \left| \Box_y \varphi_k \right|^2 \eta \, d\mu_D$, denoted by $h^F_k$, is analogous to the degenerate case. $h^F = h^S_1 + \cdots + h^S_4$, where $h^S_j$, $j = 1 : 4$ are defined as (4.43) with $D = D^k, V = 1, i = 1$, i.e. for $p + q \leq k, p \leq 4$

$$h^{S_1} = \int_{D^k} |D\varphi|^2 |\nabla\varphi|^2 \eta \, d\mu_D,$$

$$h^{S_2} = \int_{D^k} |D\varphi|^2 |L\varphi|^2 \eta \, d\mu_D,$$

$$h^{S_3} = \int_{D^k} |D\varphi|^2 |\nabla\varphi|^2 \eta \, d\mu_D,$$

$$h^{S_4} = \int_{D^k} |L\varphi|^2 |\nabla\varphi|^2 \eta \, d\mu_D.$$

As (4.42), we apply $L^\infty, L^\infty, L^2, L^2$ to the four factors in each of $h^S_k, j = 1 : 4$

$$|h^{S_1}| \lesssim \int_{D^k} M^2 \, d\mu_D \int_{\mathbb{R}^N} \delta^{-1} \eta^{-1} |L\varphi|^2 \, d\mu_D,$$

$$|h^{S_2}| + |h^{S_3}| \lesssim \delta^{\frac{1}{2}} M^2 \int_{D^k} \left( |L\varphi|^2 + |\nabla\varphi|^2 \right) \, \eta \, d\mu_D,$$

where the first one can be treated by the Gronwall’s inequality, while the second one can be absorbed by the left hand side of (4.68).

For $h^{S_4}$, we note that $p, q \leq N - 1$ and $\|\nabla\varphi\|_{L^4(S^\infty_{u+\infty})} \lesssim \delta^{\frac{1}{2}} M$. Moreover, by Theorem 4.4, $\eta^\beta |L\varphi|^2_{L^4(S^\infty_{u+\infty})} \lesssim \delta^{-\beta} u^{-\beta} \|\varphi\|_{L^4(S^\infty_{u+\infty})} d\mu_D$

$$|h^{S_4}| \lesssim \int_{u_1}^{u_2} \int_{D^k} \|\nabla\varphi\|_{L^4(S^\infty_{u+\infty})} \, d\mu_D,$$

$$\lesssim \delta^{\frac{1}{2}} M^2 \lesssim \delta^{\frac{1}{2}} M^2.$$
In the exterior region $\mathcal{D}^e$, $u_1^e < u \leq u_1^{NH}$, and $1 - \mu \sim 1$. Viewing the estimates (4.27) and (4.31) in Theorem 4.4

$$|c(\varphi_\mu)| \lesssim \int_{\mathcal{D}^e} (|L_{\varphi_\mu}|^2 + \delta^{-1}|\bar{D}_{\varphi_\mu}|^2 + \delta^{-2}|L_{\varphi_\mu}|^2) \ d\mu_{\mathcal{D}} \lesssim \mathcal{I}_{NH}^2, \quad k \leq N - 1.$$  

Besides, making use of Theorem 4.4 and following the proof leading to (4.44), we can also conclude

$$|\mathcal{F}(\varphi_\mu)| \lesssim \mathcal{I}_{NH}^2, \quad k \leq N - 1.$$  

As a summary, we arrive at: for any $u_1 < u_2$ and $u \geq u_1^{NH}$, $k \leq N - 1$,

$$E_{k}^{\text{indeg}}(u_1; [u_1, u_2]) + E_{k+1}^{\text{indeg}}(u_2; [u_1, u_2]) + S_k^{\text{indeg}}(\mathcal{D}^b) \lesssim E_{k}^{\text{indeg}}(u_1^e; [u_1, u_2]) + E_{k+1}^{\text{indeg}}(u_2^e; [u_1, u_2]) + \mathcal{I}_{NH}^2. \quad (4.70)$$

Noticing that, $\{u = u_1^e\} \cap \mathcal{R}_2$ is away form the horizon, hence by Theorem 4.4

$$E_{k}^{\text{indeg}}(u_1^e; [u_1, u_2]) \sim E_{k}^{\text{indeg}}(u_1^e; [u_1, u_2]) \lesssim \mathcal{I}_{NH}^2, \quad k \leq N - 1. \quad (4.71)$$

Substituting (4.71) into (4.70), letting $u_1 = 0$, we obtain that for all $u \geq u_0^{NH} > u_0$ where $u_0^{NH} := -r_{SH}^N$, $u_0^e := -(1.2r_{NH})^+$ and $0 \leq \mu \leq \delta$,

$$E_{k}^{\text{indeg}}(u; [u_0, u_2]) + E_{k+1}^{\text{indeg}}(u; [u_0, u_2]) \lesssim \mathcal{I}_{NH}^2, \quad k \leq N - 1. \quad (4.72)$$

Letting $u \to +\infty$, $u_1 = 0$ in (4.70), and combining with (4.71), we have

$$E_{k}^{\text{indeg}}(u; [u_0^{NH}, +\infty]) + S_k^{\text{indeg}}(\mathcal{D}_{NH}^{H, +\infty}) \lesssim \mathcal{I}_{NH}^2, \quad k \leq N - 1. \quad (4.73)$$

As a result, there is the enhanced estimate as well

$$\int_{\mathcal{D}^e} |\nabla_y \varphi_\mu|^2 \eta d\mu_{\mathcal{D}} \lesssim \delta^2 \mathcal{I}_{NH}^2 M^2, \quad k \leq N - 1. \quad (4.74)$$

4.4.3. Estimates for $L_k E_{k}^{\text{indeg}}(u; [0, +\infty])$, $k \leq N - 1$. We take $\psi = \delta L_{\varphi_\mu}$, $k \leq N - 1$ in (4.67), then

$$L_{k+1} E_{k+1}^{\text{indeg}}(u_1; [u_1, u_2]) + \int_{\mathcal{D}^e} (\delta \eta^{-1}|L_{\varphi_\mu}|^2 + \delta \eta|\nabla L_{\varphi_\mu}|^2 + \delta^2 \eta|L_{\varphi_\mu}^2|^2) \ d\mu_{\mathcal{D}} \lesssim L_{k+1} E_{k+1}^{\text{indeg}}(u_1^e; [u_1, u_2]) + \int_{\mathcal{D}^e} \delta^2 |\nabla_y L_{\varphi_\mu}|^2 \eta d\mu_{\mathcal{D}} + c(\delta L_{\varphi_\mu}) + c(\delta L_{\varphi_\mu}). \quad (4.75)$$

The source term $\int_{\mathcal{D}^e} \delta^2 |\nabla_{\mu} L_{\varphi_\mu}|^2 \eta d\mu_{\mathcal{D}}$ denoted by $h_L F_k$, can be split as: $h_L F_k = h_L S_k^H + h_L G_k + h_L L_k + h_L \mathcal{W}_k$, where $h_L S_k^H, h_L G_k$ are defined as (4.1)-(4.2) with $\mathcal{D} = \mathcal{D}^h$, $V = \delta L$, $i = 1$, $h_L L_k$ is defined as (4.11) with $\mathcal{D} = \mathcal{D}^b$, $V = \delta L$, $i = 1$, and $h_L \mathcal{W}_k$ is related to $\delta[\nabla_g, L] \varphi_\mu$.

$$h_L \mathcal{W}_k = \int_{\mathcal{D}^h} \delta^2 (|L_{\varphi_\mu}|^2 + |\nabla L_{\varphi_\mu}|^2 + |\nabla_y L_{\varphi_\mu}|^2 + |\nabla_y \varphi_\mu|^2) \eta d\mu_{\mathcal{D}}.$$  

We estimate these error terms one by one.

First of all, by (4.74), $h_L L_k \lesssim \delta^2 \mathcal{I}_{NH}^2 M^2, \quad k \leq N - 1$.

For $h_L S_k^H$, it is split into: $h_L S_k^H = h_L S_1^k + \cdots + h_L S_4^k$, where $h_L S_j^k$, $j = 1 : 4$ are defined as (4.13) with $\mathcal{D} = \mathcal{D}^b$, $V = \delta L$, $i = 1$. The estimates for $h_L S_j^k$, $j = 1 : 3$ are analogous to that for $h S_j^k$, $j = 1 : 3$; hence we will not give the details here. For $h_L S_4^k$, which reads

$$h_L S_4^k = \int_{\mathcal{D}^b} \sum_{p+q \leq k, p \leq q} \delta^2 |L_{\varphi_\mu}|^2 |\nabla L_{\varphi_\mu}|^2 \eta d\mu_{\mathcal{D}}, \quad k \leq N - 1.$$
Note that, $|L\varphi_p| \lesssim \delta^{-\frac{3}{2}} M$, $p \leq N/2 \leq N - 3$, and $q \leq N - 1$, hence,
\[
|^{hL}\mathcal{S}_k^2| \lesssim \delta M^2 \int_{D_h} |L\varphi_{q+1}|^2 \eta d\mu_D \lesssim \delta M^2 \eta^2_{N+1}, \tag{4.76}
\]
where the degenerate spacetime estimate (4.27) is used in the second inequality above.

For $^{hL}G^k$, we make the following splitting: $^{hL}G^k = ^{hL}G^k_1 + \cdots + ^{hL}G^k_4$, where

$^{hL}G^k_j, j = 1 : 4$ are defined as (4.4) with $D = D^h, V = \delta L, i = 1$, i.e. for

$p + q \leq k, p < q$,
\[
^{hL}G^k_1 = \int_{D_h} \delta^2 |D\varphi_q|^2 |Y L\varphi_p|^2 \eta d\mu_D,
\]
\[
^{hL}G^k_2 = \int_{D_h} \delta^2 |D\varphi_q|^2 |L^2 \varphi_p|^2 \eta d\mu_D,
\]
\[
^{hL}G^k_3 = \int_{D_h} \delta^2 |D\varphi_q|^2 |L^2 \varphi_p|^2 \eta d\mu_D,
\]
\[
^{hL}G^k_4 = \int_{D_h} \delta^2 |L\varphi_q|^2 |L^2 \varphi_p|^2 \eta d\mu_D.
\]

Note that $k \leq N - 1$, then $q \leq N - 1, p \leq N - 2$. The estimates are similar to those for $^{L}G^k_j, j = 1 : 4$. Hence, we will omit some detailed calculations.

\[
^{hL}G^k_1 \lesssim \int_{\Omega_1} \int_{\Omega_1} \delta^{-1} M^2 \delta^2 \eta^{-1} \|
\]

\[
= \int_{D_h} \delta^{-1} M^2 \delta^2 \eta^{-1} \|L^2 \varphi_p\|^2_{L^2(S_{u_1})} \eta d\mu_D,
\]

which can be handled by the Gronwall’s inequality. For $^{hL}G^k_2, ^{hL}G^k_3$,
\[
^{hL}G^k_2 + ^{hL}G^k_3 \lesssim \int_{\Omega_1} \int_{\Omega_1} \delta^2 M^2 \eta^{-1} \|L^2 \varphi_p\|^2_{L^2(S_{u_1})} \eta d\mu_D,
\]

\[
\lesssim \delta^2 M^2 \sum_{p \leq i \leq p+1} \delta^2 \|
\]

\[
\lesssim \delta^2 M^2 \sum_{p \leq i \leq p+1} \delta^2 \|
\]

\[
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\]

\[
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\[
\lesssim \delta^2 M^2 \eta^2_{N+1}, \quad p \leq N - 2,
\]

which can be absorbed by the left hand side of (4.75). Similarly for $^{hL}G^k_4$, we have by (4.27) in Theorem 4.4
\[
^{hL}G^k_4 \lesssim \int_{\Omega_1} \int_{\Omega_1} \delta^{-1} M^2 \delta^2 \eta^{-1} \|L^2 \varphi_p\|^2_{L^2(S_{u_1})} \eta d\mu_D,
\]

\[
\lesssim \delta^2 M^2 \sum_{p \leq i \leq p+1} \eta |L\varphi_p|^2 d\mu_D \lesssim \delta M^2 \eta^2_{N+1}, \quad p \leq N - 2.
\]

Again, by virtue of (4.27) and (4.74), $^{hL}W^k$ is bounded by,
\[
|^{hL}W^k| \lesssim \delta^2 \eta^2_{N+1} + \delta^2 \|L^2 M^2, \quad k \leq N - 1.
\]

Furthermore, making use of Theorem 4.4 we can also deduce $|^{C}(\delta L \varphi_k)| + |^{C}(\delta L \varphi_k)| \lesssim \|L^2\eta^2_{N+1}, k \leq N - 1$. Finally, we arrive at, for $k \leq N - 1$,
\[
L_{r \text{ndeg}}(u; [u_1^j, \bar{u}_2]) + L_{r \text{ndeg}}(u; [u_1^c, u]) + L_{s \text{ndeg}}(D^h) \lesssim \|L^2 \varphi_{k+1}^j(u_1^j, \bar{u}_2) + L^2 \varphi_{k+1}^c(u_1^c, u) + \|L^2 \varphi_{k+1}^r |^2, \quad k \leq N - 3.
\]

Combining the degenerate estimates of Theorem 4.4 with the result of (4.76), (4.77), (4.78), we can improve the non-degenerate $L^\infty$ estimate for $L \varphi_k, k \leq N - 3$. 


Proposition 4.9. In $\mathcal{R}_2$, there is the upgraded $L^\infty$ estimate,
\[ \|L\varphi_k\|_{L^\infty(\mathcal{R}_2)} \lesssim \delta^{-\frac{1}{2}}\|h S^N_k\|_{L^2}, \quad k \leq N - 3. \]  

4.4.4. Estimates for $E_N^{\text{endeg}}(u; [0, u])$ and $E_N^{\text{endeg}}(u; [u^{NH}, +\infty])$. As explained in the degenerate case, if $k = N$, the estimate for $h S^N_k$ (4.69) is illegal. However, the enhanced estimate (4.79) will help to linearize $h S^N_k$. We remind ourself that,
\[ h S^N_k = \int_{\mathcal{D}_h} \sum_{p+q \leq N, p \leq q} |L\varphi_p|^2 |\nabla\varphi_q|^2 \eta d\mu_D. \]

With the help of (4.79) (knowing that $p \leq \lfloor \frac{N}{2} \rfloor \leq N - 3$), and the spacetime estimate (4.27),
\[ |h S^N_k| \lesssim \int_{\mathcal{D}_h} \delta^{-1} \|\nabla\varphi_k\|^2 \eta d\mu_D \lesssim \|h S^N_k\|_{L^2}. \]

The other terms can be bounded in the same way as that in the lower order case. After that, we achieve Theorem 4.3. And as a consequence, there is, in $\mathcal{R}_2$, for $k \leq N - 1, j \leq N - 2$
\[ \|L\varphi_k\|_{L^\infty(\mathcal{R}_2)} + \|L\varphi_k\|_{L^1(\mathcal{S}_{u^*})} \lesssim \delta^{-\frac{1}{2}}\|h S^N_k\|_{L^2}. \]  

4.4.5. Estimates for $Y F_{k+1}^{\text{endeg}}(u; [0, u])$ and $Y F_{k+1}^{\text{endeg}}(u; [u^{NH}, +\infty])$, $k \leq N - 1$. We will prove the bound for the energy related to $Y$ near the horizon in this section.

Theorem 4.10. In $\mathcal{R}_2^{NH}, 2m \leq r \leq r_{NH}$, letting $u^{NH} = u - r_{NH}^*$, we have,
\[ Y F_{k+1}^{\text{endeg}}(u; [0, u]) \lesssim \|h S^N_k\|_{L^2}, \quad k \leq N - 1. \]  

Proof. For the proof, we take $\psi = Y \varphi_k, k \leq N - 1$ in (4.67), to derive
\[ Y F_{k+1}^{\text{endeg}}(u; [u_1, u_2]) + Y F_{k+1}^{\text{endeg}}(u_2; [u_1, u]) \]
\[ + \int_{\mathcal{D}_h} (\delta^{-1} \eta^{-1} Y \varphi_k)^2 + \eta |LY \varphi_k|^2 + \delta^{-1} \eta |\nabla Y \varphi_k|^2) d\mu_D \]
\[ \lesssim Y F_{k+1}^{\text{endeg}}(u_1; [u_1, u_2]) + Y F_{k+1}^{\text{endeg}}(u_2; [u_1, u]) \]
\[ + \int_{\mathcal{D}_h} \|\varphi_k\|^2 \eta d\mu_D + \mathcal{C}(Y \varphi_k) + \mathcal{F}(Y \varphi_k), \]  

where $\int_{\mathcal{D}_h} \|\varphi_k\|^2 \eta d\mu_D$ is split as: $= h Y S^k + h Y G^k + h Y L^k + h Y W^k$. And $h Y S^k, h Y G^k$ associated to $Y \varphi_k$ is defined as (4.1)-(4.2) with $D = D^h, V = Y, i = 1, h Y \mathcal{L}^k$ defined as (4.1) with $D = D^h, V = 1, i = 1, h Y \mathcal{W}^k$ related to $[D, Y] \varphi_k$ is given by,
\[ h Y \mathcal{W}^k = \int_{\mathcal{D}_h} (|Y^2 \varphi_k|^2 + |D \varphi_k|^2 + |Y \varphi_k|^2 + |L \varphi_k|^2) \eta d\mu_D. \]

We split $h Y S^k$ into: $h Y S^k = h Y S_{2}^k + \cdots + h Y S_{4}^k$, where $h Y S_{2}^k, j = 1 \leq 4$ are defined as (4.3) with $D = D^h, V = Y, i = 1$. The estimates for $h Y S_{2}^k, j = 1 \leq 3$ are similar to that for $h Y S_{2}^k, j = 1 \leq 3$. Hence, we will only focus on $h Y S_{2}^k$, which reads,
\[ h Y S_{2}^k = \int_{\mathcal{D}_h} \sum_{p+q \leq k, p \leq q} |L\varphi_p|^2 |\nabla Y \varphi_q|^2 \eta d\mu_D, \quad k \leq N - 1. \]

We make use of the improved $L^\infty$ estimates for $L\varphi_p, p \leq \frac{N}{2} \leq N - 3$ (4.79), then
\[ |h Y S_{2}^k| \lesssim \int_{\mathcal{D}_h} \sum_{p+q \leq k, p \leq q} |L\varphi_p|^2 |\nabla Y \varphi_q|^2 \eta d\mu_D, \]

can be handled by the Gronwall's inequality.
For $Y h G^k$, there is, $h Y G^k = h Y G^k + \cdots + h Y G^k$, where $h Y G^j$, $j = 1 : 4$ are defined as \([4.4]\) with $D = D^h$, $V = V$, $i = 1$. The estimates for $h Y G^j$, $j = 1 : 3$ are similar to that for $h Y G^j$, $j = 1 : 3$. We are left with $h Y G^4$, which takes

$$h Y G^4 = \int_{D^h} \sum_{p+q \leq k, p < q} |L \varphi_q|^2 |\nabla Y \varphi_p|^2 \eta d\mu_D, \quad k \leq N - 1.$$ 

Noticing that $q \leq N - 1$, $p \leq N - 2$ and referring to \([4.58]\), via the upgraded $L^4$ estimate in \([4.80]\) and the spacetime estimate \([4.59]\) in Theorem 4.8, we obtain

$$h Y G^k \lesssim \int_{D^h} \sum_{p \leq k + 2} \delta^{-1} I^{12}_{N+1} |Y \varphi_k|^2 \eta d\mu_D \lesssim I^{12}_{N+1}.$$ 

Finally, $h Y W^k$ can be estimated by, noting that $k \leq N - 1$,

$$|h Y W^k| \lesssim \int_{\mathcal{L}_+} \delta \int_{\mathcal{L}_+} \delta^{-1} \eta^{-1} |L Y \varphi_k|^2 d\mu_D + I^{12}_{N+1},$$

where we have used \([4.27]\), \([4.59]\), and the remaining term can be handled by the Gronwall’s inequality.

As before, there is $|C(Y \varphi_k)| + |F(Y \varphi_k)| \lesssim I^{12}_{N+1} + \delta^2 I^{12}_{N+1}$. Therefore, we arrive at that for $k \leq N - 1$,

$$Y F^{\text{endeg}}_{k+1}(u; [u_1^1, u_2]) + Y F^{\text{endeg}}_{k+1}(u; [u_1^1, u]) + Y F^{\text{endeg}}_{k+1}(D^h) \lesssim Y F^{\text{endeg}}_{k+1}(u_1^1; [u_1^1, u_2]) + Y F^{\text{endeg}}_{k+1}(u_1^1; [u_1^1, u]) + I^{12}_{N+1}, \tag{4.84}$$

which gives rise to Theorem 4.10 \(\square\)

In the following sections, we will make use of Theorem 4.4 and Theorem 4.8 to retrieve $t F^{\text{endeg}}_{k+1}(u; [0, u])$, $t F^{\text{endeg}}_{k+1}(u; [u^{NH}, +\infty])$, $k \leq N - 1$. The proof will be analogous to that in Section 4.3.5.

4.4.6. Estimates for $t F^{\text{endeg}}_{k+1}(u; [0, u])$, $k \leq N - 1$.

**Proposition 4.11.** In $R_2$, given any real number $\beta > \frac{1}{2}$ and $k \leq N - 1$,

$$\delta^{-1} \|\eta^2 Y \varphi_k\|^2_{L^2(S^u_\infty)} + \delta^{-2} \|\eta^2 Y \varphi_k\|^2_{L^2(C_u)} \lesssim I^{12}_{N+1} |u|^{-2\beta}, \tag{4.85}$$

$$\|\eta^2 Y L \varphi_k\|^2_{L^2(C_u)} \lesssim I^{12}_{N+1} |u|^{-2\beta}. \tag{4.86}$$

**Proof.** Define $\chi^2[\psi](u, u) = \int_{S^u_\infty} |Y \psi|^2 \eta r^2 d\sigma_{S^2}$. Take $\psi = \varphi_k$, $k \leq N - 1$,

$$\partial_u \chi^2[\varphi_k](u, u) + \int_{S^u_\infty} \eta^{-1} \mu_r |L \varphi_k|^2 d\sigma_{S^2}$$

$$= \int_{S^u_\infty} 2r^2 \eta^{-1} L \varphi_k \left(L L \varphi_k + \frac{\eta}{r} L \varphi_k\right) d\sigma_{S^2}.$$ 

Thanks to the wave equation and the Cauchy-Schwartz inequality, we integrate along $\partial_u$ to derive, cf. \([3.33]\)

$$\chi^2[\varphi_k](u, u) \lesssim \int_0^u \delta^{-1} \chi^2[\varphi_k](u, u') d\mu'$$

$$+ \int_{C_u} \delta \eta \left(|\nabla \varphi_k|^2 + |L \varphi_k|^2 + |\square \varphi_k|^2\right) d\mu_{C_u}.$$ \(4.87\)
We make the following splitting: \( \int_{C_u} |\nabla_y \varphi_k|^2 = \sum_{i=1}^4 |F_k^i|^2 \), with \( F_k^i \) defined as below: for all \( p + q \leq k \leq N - 1, \ p \leq q, \)

\[
|F_k^1|^2 := \int_{C_u} |D \varphi_p|^2 |Y \varphi_q|^2 \eta \mu_{C_u},
|F_k^2|^2 := \int_{C_u} |D \varphi_p|^2 |L \varphi_q|^2 \eta \mu_{C_u},
|F_k^3|^2 := \int_{C_u} |D \varphi_p|^2 |\nabla \varphi_q|^2 \eta \mu_{C_u},
\]

(4.88)

\[
|F_k^4|^2 := \int_{C_u} |L \varphi_p|^2 |\nabla \varphi_q|^2 \eta \mu_{C_u}.
\]

Then, we will estimate the error terms one by one. In views of the improved \( L^\infty \) estimate for \( \|L \varphi_p\|_{L^\infty(\mathcal{R}_2)} \) (4.80) and \( \|\bar{D} \varphi_p\|_{L^\infty(\mathcal{R}_2)} \lesssim \delta^{2} M, \ p \leq \frac{N}{2} \leq N - 3, \)

\[
\delta |F_k^1|^2 \lesssim \int_0^\mu \int_{\mathcal{N}} \left| \frac{1}{2} \frac{1}{\mu^2} \frac{1}{\mathcal{M}} \varphi_k (u, u') \right| d\mu',
\delta \left( |F_k^2|^2 + |F_k^3|^2 \right) \lesssim \delta^{2} M^2 \left( \|\eta^{\frac{1}{2}} L \varphi_q\|^2_{L^2(\mathcal{C}_u)} + \|\eta^{\frac{1}{2}} \nabla \varphi_q\|^2_{L^2(\mathcal{C}_u)} \right),
\delta |F_k^4|^2 \lesssim \int_{\mathcal{N}} \|\eta^{\frac{1}{2}} \nabla \varphi_q\|^2_{L^2(\mathcal{C}_u)}.
\]

Therefore, based on the result of Theorem 4.4

\[
\|\eta^{\frac{1}{2}} \nabla \varphi_k\|_{L^2(\mathcal{C}_u)} \lesssim \int_0^\mu \int_{\mathcal{N}} \left| \frac{1}{2} \frac{1}{\mu^2} \frac{1}{\mathcal{M}} \varphi_k (u, u') \right| d\mu' + \left( \delta \int_{\mathcal{N}} + \delta^{2} M^2 \right) \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]

(4.89)

And by the Gronwall’s inequality, (4.87) turns into

\[
\|\eta^{\frac{1}{2}} \nabla \varphi_k\|_{L^2(\mathcal{C}_u)} \lesssim \delta \int_{\mathcal{N}} + \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]

which yields

\[
\delta \int_{\mathcal{N}} \lesssim \delta \int_{\mathcal{N}} + \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]

(4.90)

Integrating (4.90) along \( \partial_u \), (4.85) follows.

Meanwhile, using the wave equation, we have, for \( k \leq N - 1, \)

\[
\|\eta^{\frac{1}{2}} L \varphi_k\|^2_{L^2(\mathcal{C}_u)} \lesssim \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)} + \|\eta^{\frac{1}{2}} L \varphi_k\|^2_{L^2(\mathcal{C}_u)} + \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)} + \|\eta^{\frac{1}{2}} \nabla \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]

With the aid of Theorem 4.4 and (4.89), (4.85), we prove (4.86).

4.4.7. Estimates for \( t^2 \Delta u(s; [u^{NH}, +\infty)), k \leq N - 1. \)

**Proposition 4.12.** Given any real number \( \beta > \frac{1}{2} \), in the region \( \mathcal{R}_2 \cap \{ r \leq r_{NH} \} \)

\[
\delta^{-1} \|Y \varphi_k\|^2_{L^2(\mathcal{S}_{2, u})} + \delta^{-2} \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} \lesssim \delta^{-1}_N + \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} \lesssim \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} \lesssim \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} + \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} + \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]

(4.91)

(4.92)

**Proof.** Define \( h \Delta [\varphi_k] (u, y) = \int_{\mathcal{S}_{2, u}} \frac{1}{y} L \varphi_k \varphi_k^2 d\sigma_{S^2}, k \leq N - 1. \)

Then,

\[
\partial_u h \Delta [\varphi_k] (u, y) = \int_{\mathcal{S}_{2, u}} 2 \eta^{-2} L \varphi_k \left( L \varphi_k + \frac{\eta}{r} L \varphi_k \right) r^2 d\sigma_{S^2} - \int_{\mathcal{S}_{2, u}} 2 \eta^{-2} \frac{2m}{r^2} |L \varphi_k|^2 r^2 d\sigma_{S^2}.
\]

Following the proof leading to Proposition 4.11, we have for \( k \leq N - 1, \)

\[
h \Delta [\varphi_k] (u, y) \lesssim \int_0^\mu \left( \delta^{-1}_N + \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)} \right) d\mu' + \delta \int_{\mathcal{N}} + \|Y \varphi_k\|^2_{L^2(\mathcal{C}_u)}.
\]
Then, with the help of the Gronwall’s inequality and Theorem 4.13, we obtain
\[ h \frac{1}{2} [\varphi_k]^2(u, \tilde{u}) \lesssim \delta I_N^{2} N + \frac{1}{4} I_N^{2} N + 1. \]
Integrating the above formula along \( \partial \mathcal{U} \), we have (4.91). Besides, the above estimates imply
\[ \| \Box \varphi_k \|^2_{L^2(S)} \lesssim I_N^{2} N + \frac{1}{4} I_N^{2} N + 1. \] (4.93)
Thus (4.92) follows from the wave equation and (4.91), (4.93).

4.4.8. Closing the bootstrap argument in \( \mathcal{R}_2 \). As in Section 3.3.8, we choose \( M \) (which depends on the initial data) large enough such that \( C I_{N+1}^{2} \leq \frac{M}{2} \), and \( \delta \) small enough such that \( \frac{\delta}{\beta} M^2 \ll 1 \), hence \( C I_{N+1}^{2} \leq \frac{M}{2} \), which closes the bootstrap argument and we prove Theorem 4.4.

4.5. Estimates for general derivatives in \( \mathcal{R}_2 \).

Define
\[ E^{\text{deg}}_{t+k}(u; [u], u) := \sum_{p+q=l} E^{\text{deg}}[\delta^p \mathcal{W}^{l}_{p,q} \varphi_k](u; [u], u), \]
\[ E^{\text{deg}}_{t+k}(u; [u], u) := \sum_{p+q=l} E^{\text{deg}}[\delta^p \mathcal{W}^{l}_{p,q} \varphi_k](u; [u], u), \]
\[ t E^{\text{deg}}_{t+k}(u; [u], u) := \sum_{p+q=l} t E^{\text{deg}}[\delta^p \mathcal{W}^{l}_{p,q} \varphi_k](u; [u], u). \]

And similar definition is made for \( E^{\text{deg}}_{t+k}(u; [u], u) \) and \( E^{\text{deg}}_{t+k}(u; [u], u) \), \( E^{\text{deg}}_{t+k}(u; [u], u) \), \( E^{\text{deg}}_{t+k}(u; [u], u) \), \( E^{\text{deg}}_{t+k}(u; [u], u) \).

\[ i S^{\text{deg}}_{t+k}(\mathcal{D}) := \sum_{p+q=l} i S^{\text{deg}}[\delta^p \mathcal{W}^{l}_{p,q} \varphi_k](\mathcal{D}), \quad i = 1, 2, \]
\[ S^{\text{deg}}_{t+k}(\mathcal{D}) := \sum_{p+q=l} S^{\text{deg}}[\delta^p \mathcal{W}^{l}_{p,q} \varphi_k](\mathcal{D}). \]

Theorem 4.13. Fix \( N \geq 6 \). In \( \mathcal{R}_2 \), for any \( \beta > \frac{1}{2} \) and \( l + k \leq N \), \( i = 1, 2, \)
\[ E^{\text{deg}}_{t+k}(u; [u], +\infty) + E^{\text{deg}}_{t+k}(u; [0], u) + i S^{\text{deg}}_{t+k}(\mathcal{D}_{u}, +\infty) \lesssim I_{N+1}^{2} |u|^{-2\beta}, \]
\[ E^{\text{deg}}_{t+k}(u; [u], +\infty) + E^{\text{deg}}_{t+k}(u; [0], u) + S^{\text{deg}}_{t+k}(\mathcal{D}) \lesssim I_{N+1}^{2}, \]
where \( u^{NH} \equiv u - r_{NH}^{N} \) and for \( l + k \leq N - 1, \)
\[ |u|^{-2\beta}, t E^{\text{deg}}_{t+k}(u; [0], u) + t E^{\text{deg}}_{t+k}(u; [0], u) \lesssim I_{N+1}^{2}. \]

Theorem 4.13 with \( l \leq 1, l + k \leq N \) has been verified by Theorem 4.1. The general case can be proved by an inductive argument on \( l \) and no new difficulty occurs. In the same way as the propositions 4.11 and 4.12 we can retrieve the estimates for energies related to \( Y \) by induction on \( l \).

Proposition 4.14. In \( \mathcal{R}_2 \), given any real number \( \beta > \frac{1}{2} \), \( l + k \leq N - 1, \)
\[ \delta^{-1+2\beta} \| Y Z^{l}_{p,q} \varphi_k \|^2_{L^2(S)} + \delta^{-2+2\beta} \| Y^{2} Z^{l}_{p,q} \varphi_k \|^2_{L^2(S)} \lesssim I_{N+1}^{2}, \]
\[ \delta^{-1+2\beta} \| Y^{2} W^{l}_{p,q} \varphi_k \|^2_{L^2(S)} + \delta^{-2+2\beta} \| Y^{2} W^{l}_{p,q} \varphi_k \|^2_{L^2(S)} \lesssim I_{N+1}^{2} |u|^{-2\beta}, \]
\[ |u|^{-2\beta} \| Y^{2} W^{l}_{p,q} \varphi_k \|^2_{L^2(S)} + \| Y^{2} Z^{l}_{p,q} \varphi_k \|^2_{L^2(S)} \lesssim I_{N+1}^{2}. \]
4.6. Smallness on the last cone in $\mathcal{R}_2$. We denote $S_{r,u}$ the sphere which is intersection of the hypersurfaces of constant $r$ and constant $u$ (in $(r, u)$ coordinate).

**Proposition 4.15.** We have in $\mathcal{R}_2$, for $\beta > \gamma > \frac{1}{2}$,

\[
\|u\|^\beta - \eta \frac{\partial u}{\partial u} \| DB \varphi_k \|_{L^2(C^H_{2}\Delta}} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^2(C^H_{2}\Delta)} \lesssim \delta^2, \quad l + k \leq N - 1,
\]

\[
|u|^\beta \| DB \varphi_k \|_{L^2(S_{r,u})} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^2(S_{r,u})} \lesssim \delta^2, \quad l + k \leq N - 1,
\]

\[
|u|^\beta \| DB \varphi_k \|_{L^\infty(S_{r,u})} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^\infty(S_{r,u})} \lesssim \delta^2, \quad l + k \leq N - 3.
\]

**Proof.** Define $\omega^2[\psi](u, u) = \int_{S_{r,u}} |\nabla \psi|^2 \eta r^2 d\sigma_{S^2}$. Take $\psi = L^i \varphi_k$, $l + k \leq N - 1$,

\[
\partial_u \omega^2[L^i \varphi_k](u, u) = \int_{S_{r,u}} \left(2\nabla L^i \varphi_k \nabla L^i \varphi_k \eta r^2 + \mu |\nabla L^i \varphi_k|^2 \eta r \right) d\sigma_{S^2}
\]

\[
\lesssim \delta^2 \|L^i \varphi_k\|_{L^2(S_{r,u})}^2 + \delta^{-1} \omega^2[L^i \varphi_k](u, u).
\]

Similarly, define $h \omega^2[\psi](u, u) = \int_{S_{r,u}} |\nabla \psi|^2 r^2 d\sigma_{S^2}$ and take $\psi = Y^i \varphi_k$, then

\[
\partial_u h \omega^2[Y^i \varphi_k](u, u) \lesssim \delta \|LY^i \varphi_k\|_{L^2(S_{r,u})}^2 + \delta^{-1} h \omega^2[Y^i \varphi_k](u, u).
\]

Then we obtain (since $\varphi \equiv 0$ on $C^H$) for $l + k \leq N - 1$,

\[
\omega^2[L^i \varphi_k](u, u) \lesssim \delta^2 \|L^i \varphi_k\|_{L^2(S_{r,u})}^2 \lesssim \delta, \quad l + k \leq N - 1,
\]

\[
\omega^2[Y^i \varphi_k](u, u) \lesssim \delta \|LY^i \varphi_k\|_{L^2(S_{r,u})}^2 \lesssim \delta.
\]

The bound for $\|u\|^\beta - \gamma \frac{\partial u}{\partial u} \| DB \varphi_k \|_{L^2(C^H_{2}\Delta)}$ is done by integrating $|u|^{2\beta - 2\gamma} \omega^2[L^i \varphi_k](u, u)$ along $\partial_u$, $u \in [u_0, +\infty)$, for any $\gamma > \frac{1}{2}$. On the other hand, we parametrize $C_{r,u}$ (in $(r, u)$ coordinate) by $\cup_{r \in [0, r_0]} S_{r,u}$, $h \omega^2[Y^i \varphi_k](u, u)$ by $h \omega^2[Y^i \varphi_k](r, u)$, and further integrate $h \omega^2[Y^i \varphi_k](r, u)$ with respect to the measure $dr$ on $C^H_{2\Delta}$ to achieve the bound for $\|DB \varphi_k \|_{L^2(C^H_{2\Delta})}$.

Finally, combining Proposition 4.14 and Theorem 4.10 [4.82], which automatically give the estimates for $\eta^2 Y^i \varphi_k$ and $L^i \varphi_k$, with the Sobolev theorem on the sphere, we complete the proof.

**Proposition 4.16.** We have on the last cone $\mathcal{R}_2 \cap C_0$, for any $\beta > \gamma > \frac{1}{2}$,

\[
\|u\|^\beta - \gamma \frac{\partial u}{\partial u} \| DB \varphi_k \|_{L^2(C^H_{2}\Delta)} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^2(C^H_{2}\Delta)} \lesssim \delta^2, \quad l + k \leq N - 2,
\]

\[
|u|^\beta \| DB \varphi_k \|_{L^2(S_{r,u})} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^2(S_{r,u})} \lesssim \delta^2, \quad l + k \leq N - 2,
\]

\[
|u|^\beta \| DB \varphi_k \|_{L^\infty(S_{r,u})} \lesssim \delta^2, \quad \|DY^i \varphi_k \|_{L^\infty(S_{r,u})} \lesssim \delta^2, \quad l + k \leq N - 4.
\]

**Proof.** Since the proof for general $l$ is similar to the case of $l = 0$, we will take $L \varphi_k$, $k \leq N - 2$ for instance here. The proof is analogous to Proposition 3.9

**Degenerate case:** Define $\chi^2[\psi](u, u) = \int_{S_{r,u}} |\nabla \psi|^2 \eta r^2 d\sigma_{S^2}$. Take $\psi = \varphi_k$, $k \leq N - 2$. Noting that $\partial_u \eta = -\frac{\mu}{r} \eta < 0$, we have, after using the wave equation,

\[
\partial_u \chi^2[\varphi_k](u, \delta) + \int_{S_{r,u}} \frac{\mu}{r} |L \varphi_k|^2 \eta r^2 d\sigma_{S^2}
\]

\[
\lesssim \int_{S_{r,u}} (\eta^2 \varphi_k - \eta^2 L \varphi_k - \eta^2 \Box \varphi_k) L \varphi_k r^2 d\sigma_{S^2}.
\]

Integrating along $\partial_u$, and applying the Cauchy-Schwartz inequality (the positive term $\int_{S[1, r]} \mu |L \varphi_k|^2 \eta r^2 d\sigma_{S^2}$ on the left hand side is needed and notice that $r$ is
finite in $\mathcal{R}_2$, we have
\begin{equation}
\chi^2[\varphi_k](u, \delta) + \int_{C_{\delta}(u)} |L_2\varphi_k|^2 \eta^2 d\mu_{C_{\delta}} \leq \chi^2[\varphi_k](u_1, \delta)
\end{equation}
(4.94)
By the result of Proposition 4.15 and $k \leq N - 2$,
\[\int_{C_{\delta}(u)} \eta^3 (|L_2\varphi_k|^2 + |\square_\varphi\varphi_k|^2) d\mu_{C_{\delta}} \leq \int_{u_1}^{u} \delta |u'|^{-23} du'.\]
Furthermore, the last term in (4.94), is split as $\int_{C_{\delta}(u)} |\square_\varphi\varphi_k|^2 \eta^3 d\mu_{C_{\delta}} = *F_1^k + \ldots + *F_3^k$, where $p + q \leq k \leq N - 2$, $p \leq q$, and
\[*F_1^k = \int_{C_{\delta}(u)} \eta^3 |\bar{D} \varphi_p|^2 \overline{|\bar{D} \varphi_q|^2} d\mu_{C_{\delta}} ,\]
\[*F_2^k = \int_{C_{\delta}(u)} \eta^3 |\bar{D} \varphi_p|^2 |\bar{L}_\varphi \varphi_q|^2 d\mu_{C_{\delta}} ,\]
\[*F_3^k = \int_{C_{\delta}(u)} \eta^3 |\bar{L}_\varphi \varphi_p|^2 |\bar{D} \varphi_q|^2 d\mu_{C_{\delta}} .\]
It is obvious to see that $*F_1^k \lesssim \int_{u_1}^{u} \delta^2 |u'|^{-43} du'$ and $*F_2^k \lesssim \delta \int_{C_{\delta}(u)} |\bar{L}_\varphi \varphi_k|^2 \eta^3 d\mu_{C_{\delta}}$. For $*F_3^k$, we apply $L^4$ to all the four factors, since $p \leq [\frac{N - 2}{2}] \leq N - 4$ and $q \leq k - 2$,
\[*F_3^k \lesssim \int_{u_1}^{u} \eta^3 |\bar{L}_\varphi \varphi_k|^2 |\bar{L}_\varphi \varphi_k|^2 d\mu_{C_{\delta}} \lesssim \delta \int_{C_{\delta}(u)} |\bar{L}_\varphi \varphi_k|^2 \eta^3 d\mu_{C_{\delta}} ,\]
where we have used the Sobolev inequalities on $S_{u, \omega}$. Hence both of $*F_2^k$ and $*F_3^k$ can be absorbed by the left hand side of (4.94).
In a word, we deduce that for any $u_0 \leq u \leq u, \delta \leq N - 2$,
\[\chi^2[\varphi_k](u, \delta) + \int_{u_1}^{u} \chi^2[\varphi_k](u', \delta) du' \leq \chi^2[\varphi_k](u_1, \delta) + \int_{u_1}^{u} \delta |u'|^{-23} du'.\]
Additionally, the smallness in Theorem 3.7 tells that $\chi^2[\varphi_k](u_0, \delta) \leq \delta$. By the pigeon-hole principle (see Lemma 5.3), we derive that for any $u_0 \leq u$
\[\chi^2[\varphi_k](u, \delta) \leq \delta |u|^{-23}, \quad k \leq N - 2.\]
(4.95)
And integrating (4.95) along $\partial u$, we obtain $\|u\|^{\beta - \gamma^{2}} \bar{L}_\varphi \varphi_k \|_{L^2(\mathcal{S}_u)} \lesssim \delta, \quad \gamma > \frac{1}{2}, \quad k \leq N - 2$.

Non-degenerate case: We work in $(r, u, \theta, \phi)$ coordinates. Define $h \chi^2[\phi](r, u) = \int_{S_{r, \omega}} |L\psi|^2 r^3 d\sigma_{S^2}$ and take $\psi = \phi_k, k \leq N - 2$. Noting that $\partial_r r^3 = 3r^2 > 0$,
\[\partial_r h \chi^2[\phi](r, u) = \int_{S_{r, \omega}} 3r^2 |L\phi_k|^2 d\sigma_{S^2} = \int_{S_{r, \omega}} 2L\phi_k \partial_r L\phi_k r^3 d\sigma_{S^2} .\]
Integrating on $C_{\delta}$ along $\partial_r$ within the interval $r \in [r, r_N]$, one derives,
\[h \chi^2[\phi_k](r, \delta) + \int_{C_{\delta}_{N}} 3|L\phi_k|^2 r^2 d\sigma_{S^2} = h \chi^2[\phi_k](r_N, \delta) + \int_{C_{\delta}_{N}} 2L\phi_k \partial_r L\phi_k r^3 d\sigma_{S^2} .\]
Theorem 4.17. In the region analogous fashion as that in Theorem 3.7.

We finish the proof. \[ \square \]

Based on Proposition \[4.15\] and \[4.16\] the smallness for the general derivatives on the last cone can be proved by an inductive argument which is actually in an analogous fashion as that in Theorem \[3.7\].

Theorem 4.17. In the region \( R_2 \), for any fixed \( N \geq 6 \) and \( \beta > \gamma > \frac{1}{2} \), we have on the last cone \( C_\delta \) defined as before,

\[
\|[u]^{\beta - \eta} \frac{1}{2} DW^{l}_{p,q} \varphi_k \|_{L^1(C_\delta)} + \| DZ^{l}_{p,q} \varphi_k \|_{L^2(C_\delta^H)} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 1,
\]

\[
[u]^{\beta} \| \frac{1}{2} DW^{l}_{p,q} \varphi_k \|_{L^2(S_{(k-1)u})} + \| DZ^{l}_{p,q} \varphi_k \|_{L^2(S_{(k-1)u})} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 1,
\]

\[
[u]^{\beta} \| \eta \frac{1}{2} DW^{l}_{p,q} \varphi_k \|_{L^\infty(S_{(k-1)u})} + \| DZ^{l}_{p,q} \varphi_k \|_{L^\infty(S_{(k-1)u})} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 3.
\]

and

\[
\|[u]^{\beta - \eta} \frac{1}{2} LW^{l}_{p,q} \varphi_k \|_{L^1(C_\delta)} + \| LZ^{l}_{p,q} \varphi_k \|_{L^2(C_\delta^H)} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 2,
\]

\[
[u]^{\beta} \| \eta \frac{1}{2} LW^{l}_{p,q} \varphi_k \|_{L^2(S_{(k-1)u})} + \| LZ^{l}_{p,q} \varphi_k \|_{L^2(S_{(k-1)u})} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 2,
\]

\[
[u]^{\beta} \| \eta \frac{1}{2} LW^{l}_{p,q} \varphi_k \|_{L^\infty(S_{(k-1)u})} + \| LZ^{l}_{p,q} \varphi_k \|_{L^\infty(S_{(k-1)u})} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 4.
\]

4.7. Global existence in \( R_2 \). We take \( \frac{1}{2} < \gamma < 1, \beta = \gamma + 2 \), and noting that \( r \) is finite in \( R_2 \), then for any \( N \geq 6 \),

\[
\|[u]^{2} DW^{l}_{p,q} \varphi_k \|_{L^2(C_\delta \cap (r \geq r_{NH}))} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 2,
\]

\[
DZ^{l}_{p,q} \varphi_k \|_{L^2(C_\delta \cap (r < r_{NH}))} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 2,
\]

\[
| \varphi_k \|_{L^\infty(S_{(k-1)u} \cap (r \geq r_{NH}))} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 4,
\]

\[
| \varphi_k \|_{L^\infty(S_{(k-1)u} \cap (r < r_{NH}))} \lesssim \delta^{\frac{1}{2}}, \quad 2l + k \leq N - 4.
\]

Making use of Rendall’s local existence with characteristic data, we can change into a small Cauchy data problem and apply Theorem \[5.1\] (in the smooth setting). Then the global existence in region \( \{ u \geq \delta \} \cap \{ t \geq 1 \} \) follows.

Theorem 4.18. Consider the Cauchy problem for the nonlinear equation \[1.1\] with \( Q \) satisfying the null condition and the Cauchy data \( (\varphi|_{\Sigma_1}, \partial_t \varphi|_{\Sigma_1}) = (\psi_0, \psi_1) \).
There exist some constant $\delta_0$ such that if $\delta < \delta_0$, we can construct a smooth initial data set $(\psi_0, \psi_1)$ verifying

$$E_k(\psi_0, \psi_1) \sim \delta^{-k+1}, \quad 34 \leq k \leq N,$$

where $E_k(\psi_0, \psi_1) = \int_{\Sigma_t} |D^k\psi_0|^2 + |D^{k-1}\psi_1|^2$, so that there is a unique and globally solution in $D^+(\Sigma_{t_0})$ (see Figure 2). Furthermore, restricted in the null strip $\mathcal{R}_2 = D^+(\Sigma_t) \cap \{0 \leq u \leq \delta\}$, the solution $\varphi$ obeys the following estimates: for any $\beta > 1/2$, $j + k + l \leq N - 2$,

$$|\eta_j^2 L^{1+j} \varphi| \lesssim \delta^{-2j-4} |u|^{-\beta}, \quad |\eta_j^2 D^j L^i \varphi| \lesssim \delta^{-2j} |u|^{-\beta}, \quad \text{in } \mathcal{R}_2,$$

$$|L^{1+j} \varphi| \lesssim \delta^{-2j}, \quad |D^j L^i \varphi| \lesssim \delta^{-2j}, \quad \text{in } \mathcal{R}_2 \cap \{r < r_{NH}\}.$$  

This accomplishes the proof of Theorem 1.3.

Putting the results of Theorem 3.10 and Theorem 1.3 together, we have Theorem 1.4.

5. Appendix

5.1. Small data theorem. Following the proof of Luk [33], we specify Theorem 1.4 of [33] on the Schwarzschild background within our notations.

Theorem 5.1 (Luk [33], 2013). Consider the nonlinear wave equation (1.1) with null quadratic form. There exists an $\epsilon$ such that if the initial data satisfy

$$\sum_{i+j+k+l \leq 16} \int_{\Sigma_m \cap \{r \geq r_{NH}\}} (u^2 + \bar{u}^2)|D^k \delta_i^{ij} \Omega_i^j \varphi|^2 r^2 d^* \sigma_s \lesssim \epsilon,$$

$$\sum_{i+j+k+l \leq 16} \int_{\Sigma_m \cap \{r < r_{NH}\}} |D^k \delta_i^{ij} \Omega_i^j \varphi|^2 r^2 d^* \sigma_s \lesssim \epsilon,$$

and

$$\sum_{l \leq 13} |rD^j \varphi| + |rD^j S \varphi| \lesssim \epsilon,$$

where $S = uL + \bar{u}L$ if $r \geq R$, for some large $R$; $S \sim 2u\partial_t + r\varphi \partial_r$, if $r < 2m$. Then $\varphi$ exists globally in time. Moreover, for all $\gamma > 0$, which we can take sufficiently small that the solution $\varphi$ obeys the decay estimate

$$|\varphi| \lesssim \epsilon r^{-1} |u|^{-\frac{3}{2}} \gamma, \quad |D \varphi| \lesssim \epsilon r^{-1} |u|^{-1} \gamma, \quad |\bar{D} \varphi| \lesssim \epsilon r^{-1} |u|^{-1} \gamma, \quad r \geq R > r_{NH},$$

$$|\varphi| \lesssim \epsilon r^{-1} |t|^{-\frac{3}{2}} \gamma, \quad |D \varphi| \lesssim \epsilon r^{-1} |t|^{-\frac{3}{2}} \gamma, \quad r_{NH} \leq r \leq \frac{1}{4},$$

$$|\varphi| \lesssim \epsilon r^{-1} (u)^{-\frac{3}{2}} \gamma, \quad |D \varphi| \lesssim \epsilon r^{-1} (u)^{-\frac{3}{2}} \gamma, \quad r < r_{NH}.$$  

5.2. Applications of the pigeon-hole principle.

Lemma 5.2. Suppose $f(t) > 0$ satisfies the following inequality: for any $t_2 > t_1$ and $\alpha > 0$,

$$f(t_2) + \int_{t_1}^{t_2} f(t) dt \lesssim f(t_1) + t_1^{-\alpha}, \quad (5.1)$$

then there exists a universal constant $A$ depending on the initial data $f(t_0)$, such that

$$f(t) \lesssim A t^{-\alpha}.$$
By the pigeon-hole principle, there exists \( \hat{\tau} > \) loss of generality that

Now, for any \( \tau \) we first take a dyadic sequence \( \{ \tau_i \} \), such that \( \tau_i = 1.1^i t_0 \). Apply (5.1) to the interval \([\tau_i, \tau_{i+1}]\),

\[
f(\tau_{i+1}) + \int_{\tau_i}^{\tau_{i+1}} f(t)dt \lesssim f(\tau_i) + \tau_i^{-\alpha}.
\]

By the pigeonhole principle, there exists a sequence \( \{ \tau'_i \} \) with \( \tau_i \leq \tau'_i \leq \tau_{i+1} \), such that

\[
f(\tau'_i) \lesssim \frac{f(\tau_i) + \tau_i^{-\alpha}}{\tau_{i+1} - \tau_i} \lesssim \frac{f(\tau_i) + \tau_i^{-\alpha}}{\tau_i}.
\]

(5.2)

Now, for any \( \tau \), there must exist one interval \([\tau'_i, \tau_{i+1}]\), such that \( \tau'_i \leq \tau \leq \tau_{i+1} \). Then, applying (5.1) to the interval \([\tau'_i, \tau]\), we have

\[
f(\tau) \lesssim f(\tau'_i) + \tau'^{-\alpha}.
\]

In views of (5.2) and \( \tau_i \leq \tau' \leq \tau_{i+1} \leq \tau_{i+2} = 1.1^2 \tau_i \), we have

\[
f(\tau) \lesssim \frac{f(\tau_i) + \tau_i^{-\alpha}}{\tau_i} + \tau'^{-\alpha} \lesssim \frac{f(\tau) + \tau^{-\alpha}}{\tau} + \tau^{-\alpha} \lesssim \frac{f(\tau_i) + \tau_i^{-\alpha} + \tau'^{-\alpha}}{\tau} + \tau^{-\alpha} \lesssim \tau^{-1} + \tau^{-\alpha}.
\]

This completes the first generation of iteration.

For any fixed integer \( k \in \mathbb{N} \), we can repeat this procedure \( k \) times to obtain

\[
f(\tau) \lesssim_k \tau^{-k} + \tau^{-\alpha}, \quad \text{for any fixed } k \in \mathbb{N}.
\]

There is another version of estimate derived from the pigeon-hole principle (cf. [31]).

**Lemma 5.3.** Suppose \( f(t) > 0 \) satisfies the following inequality: for any \( t_2 > t_1 \) and \( \alpha > 0 \),

\[
f(t_2) + \int_{t_1}^{t_2} f(t)dt \leq C f(t_1) + B \max\{t_2 - t_1, 1\} t_1^{-\alpha},
\]

(5.3)

where \( C \) and \( B \) are some universal constants. Then there exists a universal constant \( A \) depending on the initial data \( f(t_0) \), such that

\[
f(t) \lesssim A t^{-\alpha}.
\]

**Proof.** We first take \( C \) to be fixed from this point on, and assume \( C > 1 \). The proof is given by a bootstrap argument. Assume \( f(t) \leq A t^{-\alpha} \) for some large \( A \) that is to be determined. We want to show that \( f(t) \leq \frac{A}{4} t^{-\alpha} \).

Let \( \tau_1 = \tau - 8C^2 \). Since we are only concerned with \( \tau \) large, we assume without loss of generality that \( \tau > 8 \left(1 - 2^{-\frac{1}{2}}\right)^{-1} C^2 \) so that \( \tau < 2^\frac{1}{2} \tau_1 \). Then applying (5.3) on the interval \([\tau_1, \tau]\), we have

\[
f(\tau) + \int_{\tau - 8C^2}^{\tau} f(t)dt \leq C \left(A \tau_1^{-\alpha} + 8C^2 B \tau_1^{-\alpha} \right),
\]

\[
\leq 2C \left(A + 8C^2 B \right) \tau^{-\alpha}.
\]

By the pigeon-hole principle, there exists \( \hat{t} \) with \( \tau - 8C^2 \leq \hat{t} \leq \tau \) such that

\[
f(\hat{t}) \leq \frac{1}{8C^2} \int_{\tau - 8C^2}^{\tau} f(t)dt \leq \frac{(A + 8C^2 B)}{4C} \tau^{-\alpha}.
\]
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We can similarly prove that,

\[ f(\tau) + \int_0^\tau f(t)dt \leq C \left( f(0) + 8C^2 \hat{r}^{-\alpha} \right) \]

\[ \leq A/4\tau^{-\alpha} + 2C^2 B\tau^{-\alpha} + 16C^3 B\tau^{-\alpha}, \]

\[ \leq A/2\tau^{-\alpha}, \quad \text{if } A \geq 72C^3 B. \]

Of course to have \( f(\tau) \leq A\tau^{-\alpha} \) for all \( t \), we also need it to hold initially, namely \( A \geq f(\tau_0) \).

\[ \Box \]

5.3. Gronwall’s inequality. We recall another version of the Gronwall’s inequality [27], which will be useful in our proof.

**Lemma 5.4.** Let \( f(x, y), g(x, y) \) be positive functions defined in the rectangle, \( 0 \leq x \leq x_0, 0 \leq y \leq y_0 \) which verify the inequality,

\[ f(x, y) + g(x, y) \leq J + a \int_0^x f(x', y)dx' + b \int_0^y g(x, y')dy' \]

for some nonnegative constants \( a, b \) and \( J \). Then, for all \( 0 \leq x \leq x_0, 0 \leq y \leq y_0 \),

\[ f(x, y), g(x, y) \leq J e^{ax+by}. \]

5.4. Sobolev inequality. The Sobolev inequalities on \( S_{x,y} \),

\[ \|\psi\|_{L^\infty(S_{x,y})} \lesssim r^{-\frac{1}{2}}\|\psi\|_{L^1(S_{x,y})} + r^{\frac{1}{2}}\|\nabla \psi\|_{L^1(S_{x,y})}, \]

\[ \|\psi\|_{L^p(S_{x,y})} \lesssim r^{\frac{3}{2}} \left( r^{-1}\|\psi\|_{L^2(S_{x,y})} + \|\nabla \psi\|_{L^2(S_{x,y})} \right), \quad p \in \mathbb{N}. \]

Referring to [7], there is the Sobolev inequality on the outgoing cone: For any real function \( \psi \equiv 0 \) on \( \mathcal{C}_0 \),

\[ r^{\frac{3}{2}}\|\psi\|_{L^1(S_{x,y})} \lesssim \|L\psi\|_{L^2(C_{\mathcal{C}})} \left( \|\psi\|_{L^2(C_{\mathcal{C}})} + \|\nabla \psi\|_{L^2(C_{\mathcal{C}})} \right). \]

We can similarly prove that,

\[ r^{\frac{3}{2}}\|\nabla \psi\|_{L^1(S_{x,y})} \lesssim \|\nabla^\frac{1}{2}\psi\|_{L^2(C_{\mathcal{C}})} \left( \|\psi\|_{L^2(C_{\mathcal{C}})} + \|\nabla \psi\|_{L^2(C_{\mathcal{C}})} \right). \]

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