Representations of automorphism groups on the homology of matroids

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Abstract

Given a group $G$ of automorphisms of a matroid $M$, we describe the representations of $G$ on the homology of the independence complex of the dual matroid $M^*$. These representations are related with the homology of the lattice of flats of $M$, and (when $M$ is realizable) with the top cohomology of a hyperplane arrangement. Finally we analyze in detail the case of the complete graph, which has applications to algebraic geometry.

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1 Introduction

Recently, de Cataldo, Heinloth and Migliorini studied the Cattani-Kaplan-Schmid complex relative to a family of completely reducible spectral curves for the Hitchin fibration of type $A_m$ ([6]). The dual graph of such a spectral curve is the complete graph, and the action of the symmetric group on the irreducible components of the curve yields an action on its vertices, hence on the independence complex of the dual matroid of the graph. A crucial step in
the analysis performed in [6] is then to determine the representations of the symmetric group on the homology of this independence complex.

Motivated by this geometrical application, we describe representations on the homology of matroids in greater generality. Namely, given a matroid \( M \) of rank \( r \) on \( n \) elements and a group \( G \) of automorphisms of \( M \), we consider the independence complex \( \text{IN}(M^*) \) of the matroid dual to \( M \). We prove that the reduced homology of this simplicial complex, up to a shift and to a sign, is isomorphic to the reduced homology of the non-spanning complex of \( M \), \( \text{NS}(M) \), and to the reduced homology of the lattice of flats of \( M \), \( \mathcal{L}(M) \):

**Theorem.** The following representations of \( G \) are isomorphic for every \( i \geq 0 \) (and nonzero only for \( i = r - 2 \)):

1. \( \tilde{H}_{n-3-i}(\text{IN}(M^*)) \otimes \text{sgn} \)
2. \( \tilde{H}_i(\text{NS}(M)) \)
3. \( \tilde{H}_i(\mathcal{L}(M)) \)

Here, the isomorphism between 1. and 2. holds more generally for any simplicial complex, being a consequence of Alexander duality (see Theorem 2). The proof that we give in Section 2 is inspired by that of Björner and Tancer (5), but keeps track of the action of the group \( G \).

Also the isomorphism between 2. and 3. is a consequence of a more general phenomenon: indeed, in Section 3 we develop an equivariant version of Folkman’s machinery of cross-cuts [7]: see in particular our Theorem 11.

In Section 4 we specialize our results to the case of matroids, obtaining the above-mentioned isomorphisms.

Furthermore, if the matroid \( M \) is realizable, then it is naturally associated with a hyperplane arrangement \( \mathcal{A} \). The cohomology of the complement \( \mathcal{C}(\mathcal{A}) \) of the arrangement admits a well-known presentation in terms of \( M \), due to Orlik and Solomon (Section 2, [13]). In Section 5 we show that the top-degree part of this cohomology is isomorphic as a representation of \( G \), up to a sign, to the reduced homology of the dual matroid \( M^*_\mathcal{A} \) associated to \( \mathcal{A} \):

\[
H^r(\mathcal{C}(\mathcal{A})) \simeq_G \tilde{H}_{n-r-1}(\text{IN}(M^*_\mathcal{A})) \otimes \text{sgn}
\]

(see Theorem 19). This statement is consequence of our main theorem and of results of Orlik and Solomon [13].

Finally, in Section 6 we specialize our results to the case in which \( M \) is the matroid of the complete graph \( K_m \), or equivalently of the root system of type \( A_{m-1} \), which is the case of interest in [6]. For this matroid, whose lattice of flats is the partition lattice \( \Pi_m \), a result of Stanley [17] allows explicit determination of the representations:

\[
\tilde{H}_{n-m}(\text{IN}(M^*(K_m))) \simeq_{\text{ind}\text{-}\text{rep}} \text{ind}_{\mathcal{C}_m}(e^{2\pi i/m})
\]
where \( n = \binom{m}{2} \) is the number of edges \( K_m \), and \( C_m \) is the subgroup generated by an \( m \)-cycle in \( \mathcal{S}_m \).

It is natural to wonder if a similar description can be provided for root systems of other types. This seems an hard question, though, being equivalent to the following conjecture of Lehrer and Solomon ([11, Conj. 1.6]):

\[
\text{H}^p(\mathcal{C}(\mathcal{A}W)) \simeq \bigoplus_c \text{Ind}^W_{Z(c)}(\xi_c) \quad p = 0, \ldots, \text{rank}(W).
\]

A different direction of research, that we hope to develop in future papers, is to explicitly describe the representation arising for different classes of graphs or matroids.

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## 2 Representations and Alexander duality

Let \( K \) be an abstract simplicial complex with vertex set \( V \).

For \( \sigma \in K \), let 
\[
\overline{\sigma} = V \setminus \sigma.
\]

**Definition 1.** The *Alexander dual* of \( K \) is the simplicial complex on the same vertex set defined by
\[
K^* = \{ \sigma \subseteq V \mid \overline{\sigma} \notin K \}.
\]

It is easy to see that \( K^{**} = K \).

Suppose that \( K \) is a simplicial complex with vertex set \( V \) with \( |V| = n \). Let \( 2^V \) be the full simplex with vertex set \( V \).

Let \( G \) be a finite group of automorphisms of the face poset of \( K \). Then \( G \) is a subgroup of the symmetric group \( \mathcal{S}_n \) on \( V \), made out of the vertex maps \( g : V \to V \) such that whenever the vertices \( j_1, \ldots, j_{i+1} \) span a \( i \)-simplex of \( K \), the points \( g(j_1), \ldots, g(j_{i+1}) \) span a \( i \)-simplex of \( K \). Therefore \( g \) induces a simplicial homeomorphism \( \tilde{g} \), and \( \tilde{g} \) induces a chain-isomorphism \( \tilde{g}_\# \) on the group of oriented \( i \)-chains in the following way:

\[
\tilde{g}_\# : \quad C_i(K, \mathbb{C}) \quad \to \quad C_i(K, \mathbb{C}),
\]

\[
[j_1, \ldots, j_{i+1}] \quad \mapsto \quad [g(j_1), \ldots, g(j_{i+1})].
\]

Moreover \( \tilde{g}_\# \) induces an isomorphism on the homology groups \( \tilde{H}_i(K) \) (and the cohomology groups \( \tilde{H}^i(K) \)):

\[
\rho_{i,g} : \quad \tilde{H}_i(K, \mathbb{C}) \quad \to \quad \tilde{H}_i(K, \mathbb{C}).
\]
This defines representations of $G$ on the $\mathbb{C}$-vector spaces $\tilde{H}_i(K, \mathbb{C})$, i.e., homomorphisms
\[
\rho_i : G \rightarrow \text{GL}(\tilde{H}_i(K, \mathbb{C})),
\]
\[
g \mapsto \rho_{i,g}.
\]
It follows from the definition of $K^*$ that $G$ is also a finite group of automorphisms of the face poset of $K^*$. Therefore we can also study the following representations:
\[
\rho^* : G \rightarrow \text{GL}(\tilde{H}_i^*(K^*, \mathbb{C})),
\]
\[
g \mapsto \rho^*_{g}.
\]

**Theorem 2.** Let $K$ be an abstract simplicial complex and let $K^*$ be its Alexander dual. Let $G$ be a finite group of automorphisms of the face poset of $K$. Then:
\[
\rho_i \simeq \rho^* \otimes \text{sgn}
\]
where $n = |V|$ and sgn is the sign representation (restricted from $\mathfrak{S}_n$ to its subgroup $G$). Or equivalently we have the following isomorphism of $\mathbb{C}[G]$-modules:
\[
\tilde{H}_i(K, \mathbb{C}) \simeq \tilde{H}_i^*(K^*, \mathbb{C}) \otimes \text{sgn}.
\]

Our proof follows that by Björner and Tancer ([5]), but carefully records the action $G$. We introduce some notations: let us denote by $\{1, 2, \ldots, n\}$ the elements of $V$. For $j \in \sigma \in K$, we define the sign
\[
\text{sgn}(j, \sigma) = (-1)^{i-1}
\]
where $j$ is the $i$-th smallest element of the set $\sigma$, and
\[
p(\sigma) = \prod_{j \in \sigma} (-1)^{i-1}.
\]
For $\sigma \in K$ we write $e_{\sigma}$ to denote the oriented simplex associated to $\sigma$ considered with an increasing order of its elements:
\[
\sigma = \{j_1, \ldots, j_{i+1}\} \quad e_{\sigma} = [j_1, \ldots, j_{i+1}] \text{ where } j_1 < \cdots < j_{i+1}.
\]
For every $g \in G$, if $e_{\sigma} = [j_1, \ldots, j_{i+1}]$ we denote
\[
g.\sigma = \{g(j_1), \ldots, g(j_{i+1})\} \quad g.e_{\sigma} = [g(j_1), \ldots, g(j_{i+1})].
\]
Not necessarily the $g(j_1), \ldots, g(j_{i+1})$ are in ascending order: let $\tau \in \mathfrak{S}_{i+1} \subseteq \mathfrak{S}_n$ be the permutation that rearranges the elements in ascending order, and fixes the elements that are not in $g.\sigma$, so that $\tau.(g.e_{\sigma}) = e_{g.\sigma}$. We also define:
\[
c(g, \sigma) = \text{sgn}(\tau).
\]
Since $\tau^{-1}$ permutes the elements of $e_{g.\sigma}$ we obtain:
\[
g.e_{\sigma} = \tau^{-1}(e_{g.\sigma}) = \text{sgn}(\tau)e_{g.\sigma} = \text{sgn}(\tau)e_{g.\sigma} = c(g, \sigma)e_{g.\sigma}.
\]
Similarly, we define a permutation $\tau \in S_{n-1-i} \subseteq S_n$ which rearranges the elements of $g.e_{\sigma}$ in ascending order:

$$
\tau . (g.e_{\sigma}) = e_{\tau . \sigma} \quad g.e_{\sigma} = c(g, \sigma)e_{g, \sigma}.
$$

We can now formulate an important lemma that will result crucial for the proof of Theorem 4:

**Lemma 3.** Let $V = \{1, \ldots , n\}$ and let $\sigma \subseteq V$. Then, for every $g \in S_n$, we have the following:

$$
p(\sigma) \ sgn(g) \ c(g, \sigma) = c(g, \sigma) \ p(g.\sigma).
$$

(1)

**Proof.** For every $g \in S_n$, we define a permutation $g' = \tau . g$. First we apply the permutation $g$ to $\sigma$ and $\sigma$, then, applying $\tau$ and $\tau$, we rearrange in ascending order both $g.e_{\sigma}$ and $g.e_{\sigma'}$.

As we have defined it, $g'$ is a permutation of $S_n$ such that:

if $i, j \in \sigma$ with $i < j$ then $g'(i) < g'(j)$

if $i, j \in \sigma'$ with $i < j$ then $g'(i) < g'(j)$.

In particular, we have that:

$$
g'.e_{\sigma} = e_{g'.\sigma} \quad g'.e_{\sigma'} = e_{g'. \sigma'}.
$$

We can express $g$ in the following way $g = \tau^{-1} \circ \tau^{-1} \circ g'$. It easy to see that $p(g.\sigma) = p(g'.\sigma)$. Thus, Equation (1) becomes:

$$
p(\sigma) \ sgn(\tau^{-1} \tau^{-1} g') \ c(g, \sigma) = c(g, \sigma) \ p(g.\sigma)
$$

$$
p(\sigma) \ sgn(\tau^{-1})\ sgn(\tau^{-1})\ sgn(g') \ sgn(\tau) = sgn(\tau) \ p(g'.\sigma)
$$

$$
p(\sigma) \ sgn(g') = p(g'.\sigma)
$$

$$
P_{i \in \sigma} (-1)^{i-1} \ sgn(g') = \prod_{i \in \sigma} (-1)^{g'(i)-1}
$$

$$
P_{i \in \sigma} (-1)^{-g'(i)} \ sgn(g') = 1
$$

$$
P_{i \in \sigma} (-1)^{-g'(i)} = sgn(g').
$$

(2)

In order to prove Equation (2), let $i \in \sigma$ be the $k$-th element of $e_{\sigma}$ and we define:

$$
A_i = \{(i, j) \mid j \in \sigma, \ i < j, \ g'(i) > g'(j)\}
$$

$$
B_i = \{(j, i) \mid j \in \sigma, \ j < i, \ g'(j) > g'(i)\}
$$

We have that:

$$
sgn(g') = (-1)^{\sum_{i \in \sigma} (|A_i| + |B_i|)}.
$$
Let us assume that \( i < g'(i) \), it is easy to see that \( |B_i| = 0 \). Furthermore:

\[
|\{(i, j) \mid j \in \mathcal{P}, i < j\}| = (n - i) - (|\sigma| - k) = n - i - |\sigma| + k,
\]

\[
|\{(i, j) \mid j \in \mathcal{P}, i < j, g'(i) < g'(j)\}| = \frac{(n - g'(i)) - (|\sigma| - k) = n - g'(i) - |\sigma| + k.}
\]

Similarly, if \( i > g'(i) \) we have that \( |B_i| = i - g'(i) \) and \( |A_i| = 0 \). Therefore:

\[
|A_i| + |B_i| = |g'(i) - i|.
\]

It follows that:

\[
\text{sgn}(g') = (-1)^{\sum_{\sigma \in \sigma} |g'(i) - i|} = \prod_{i \in \sigma} (-1)^{|g'(i) - i|} = \prod_{i \in \sigma} (-1)^{|g'(i) - i|}.
\]

Lemma 4. Let \( K \) be a simplicial complex with ground set \( V \) of size \( n \). Then

\[
\tilde{H}_{i+1}(2^V, K) \cong \tilde{H}^{n-i-3}(K^*).
\]

Furthermore, if we consider the representations of the group \( G \) on \((2^V, K)\) and \( K^*\):

\[
\alpha_{i+1} : G \rightarrow \text{GL}(\tilde{H}_{i+1}(2^V, K), \mathbb{C}))
\]

\[
\rho^{n-i-3} : G \rightarrow \text{GL}(\tilde{H}^{n-i-3}(K^*), \mathbb{C}))
\]

we have that

\[
\alpha_{i+1} \simeq \rho^{n-i-3} \otimes \text{sgn}
\]

or equivalently

\[
\tilde{H}_{i+1}(2^V, K) \cong G \tilde{H}^{n-i-3}(K^*) \otimes \text{sgn}.
\]

Proof. The chain complex for reduced homology of the pair \((2^V, K)\) is the complex:

\[
\cdots \xrightarrow{d_{i+1}} \mathcal{R}_i \xrightarrow{d_i} \mathcal{R}_{i+1} \xrightarrow{d_{i-1}} \cdots, \quad i \in \mathbb{Z}
\]

where \( \mathcal{R}_i = \langle e_\sigma \mid \sigma \subseteq V, \sigma \notin K, \dim(\sigma) = i \rangle \), and the \( d_i \)'s are the unique homomorphisms satisfying:

\[
d_i(e_\sigma) = \sum_{k \in \sigma \atop \sigma \setminus k \notin K} \text{sgn}(k, \sigma) e_{\sigma \setminus k}.
\]
The cochain complex for reduced cohomology of $K^*$ is the complex:

$$
\cdots \xrightarrow{\delta_{i-1}} C^{i-1} \xrightarrow{\delta_i} C^i \xrightarrow{\delta_{i+1}} \cdots, \quad i \in \mathbb{Z}
$$

where

$$
C^i = \langle e_{\sigma}^* \mid \sigma \subseteq V, \ dim(\sigma) = i, \sigma \in K^* \rangle = \\
= \langle e_{\sigma}^* \mid \sigma \subseteq V, \ dim(\overline{\sigma}) = n - i - 2, \overline{\sigma} \notin K \rangle
$$

and the $\delta$'s are the unique homomorphisms satisfying:

$$
\delta_i(e_{\sigma}^*) = \sum_{k \in \sigma \cup k, k \in K^*} \text{sgn}(k, \sigma \cup k) e_{\sigma \cup k} = \sum_{k \in \sigma \setminus k, k \in K} \text{sgn}(k, \sigma \cup k) e_{\sigma \setminus k}.
$$

Let $\phi_i$ be the following isomorphism:

$$
\phi_i : R_i \longrightarrow C^{n-2-i} \quad e_{\sigma} \mapsto p(\sigma) e_{\overline{\sigma}}, \quad \text{for } \sigma \notin K \text{ with } \dim(\sigma) = i. \quad (3)
$$

We then have the following diagram:

$$
\begin{array}{cccc}
\cdots & \xrightarrow{d_{i+1}} & R_i & \xrightarrow{d_i} & R_{i-1} & \xrightarrow{d_{i-1}} & \cdots \\
& \phi_i \downarrow & & \phi_{i-1} \downarrow & & & \\
\delta_{n-2-i} & \xrightarrow{\delta_{n-1-i-2}} & C^{n-2-i} & \xrightarrow{\delta_{n-1-i-1}} & C^{n-1-i} & \xrightarrow{\delta_{n-i}} & 
\end{array}
$$

We know from the proof of Lemma 4.2 of [5] that

$$
\phi_{i-1} \circ d_i = \delta_{n-i-1} \circ \phi_i. \quad (4)
$$

Thus, we have that

$$
\tilde{H}_{i+1}(2^V, K) \simeq \tilde{H}^{n-i-3}(K^*).
$$

From now on we denote $R_i = R_i((2^V, K), \mathbb{C})$ and $C^i = C^i(K^*, \mathbb{C})$. We now study the following two representations:

$$
\rho_1 : G \longrightarrow \text{GL}(R_i) \\
\rho_2 : G \longrightarrow \text{GL}(C^{n-2-i} \otimes \mathbb{C})
$$

for $\sigma \notin K$ with $\dim(\sigma) = i$. We want to show that this two representations are isomorphic. We extend the isomorphism [49]:

$$
\tilde{\phi}_i : R_i \longrightarrow C^{n-2-i} \otimes \mathbb{C} \\
e_{\sigma} \mapsto p(\sigma) e_{\overline{\sigma}} \otimes 1 \quad \text{for } \sigma \notin K \text{ with } \dim(\sigma) = i.
$$
To prove that $\rho_1 \simeq \rho_2$ we have to show that the following diagram commutes for every $g \in G$:

$$
\begin{array}{ccc}
R_i & \xrightarrow{\rho^1_g} & R_{i-1} \\
\downarrow \phi_i & & \downarrow \phi_i \\
C^{n-i-2} \otimes \mathbb{C} & \xrightarrow{\rho^2_g} & C^{n-i-1} \otimes \mathbb{C}
\end{array}
$$

We have to prove that the following equation holds:

$$
\rho^2_g \circ \tilde{\phi}_i = \tilde{\phi}_i \circ \rho^1_g.
$$

(5)

Since $g.\bar{\sigma} = \bar{g.\sigma}$ and from Lemma 3 we have that Equation (5) holds. We consider now the following diagram:

$$
\begin{array}{ccc}
d_{i+1} & \xrightarrow{} & R_i \\
\downarrow \phi_{i+1} & & \downarrow \phi_i \\
C^{n-i-2} \otimes \mathbb{C} & \xrightarrow{\delta_{n-i-2}} & C^{n-i-1} \otimes \mathbb{C}
\end{array}
$$

and we define the $\tilde{\delta}_i$’s as an extension of the homomorphisms $\delta_i$:

$$
\tilde{\delta}_i(e^*_\sigma \otimes 1) = \sum_{k \in \sigma \cup k \in K^*} \text{sgn}(k, \sigma \cup k)e^*_\sigma \cup k \otimes 1.
$$

From Equation (4) follows that:

$$
\tilde{\phi}_{i-1} \circ d_i = \tilde{\delta}_{n-i-1} \circ \tilde{\phi}_i.
$$

Thus, we have that:

$$
\alpha_{i+1} \simeq \rho^{n-i-3} \otimes \text{sgn}.
$$

**Lemma 5.** Let $K$ be a simplicial complex with ground set $V$ of size $n$. Then:

$$
\tilde{H}_i(K) \simeq \tilde{H}_{i+1}((2^V, K), \mathbb{C}).
$$

Furthermore if we consider the representations of the group $G$ on $K$ and $(2^V, K)$

$$
\rho_i : G \rightarrow \text{GL}(\tilde{H}_i(K, \mathbb{C})) \quad \alpha_{i+1} : G \rightarrow \text{GL}(\tilde{H}_{i+1}((2^V, K), \mathbb{C}))
$$

we have that

$$
\rho_i \simeq \alpha_{i+1}.
$$
Proof. The isomorphism follows from Theorem 23.3 of [12]: we have the long exact sequence of the pair \((2^V, K)\):
\[
\cdots \rightarrow \tilde{H}_{i+1}(2^V) \rightarrow \tilde{H}_{i+1}(2^V, K) \rightarrow \tilde{H}_i(K) \rightarrow \tilde{H}_i(2^V) \rightarrow \cdots
\]
Since \(2^V\) is the full simplex the spaces \(\tilde{H}_{i+1}(2^V)\) and \(\tilde{H}_i(2^V)\) are zero. Hence, the sequence becomes:
\[
\cdots \rightarrow 0 \rightarrow \tilde{H}_{i+1}(2^V, K) \rightarrow \tilde{H}_i(K) \rightarrow 0 \rightarrow \cdots
\]
It follows that the groups \(\tilde{H}_{i+1}(2^V, K)\) and \(\tilde{H}_i(K)\) are isomorphic.
We now consider the following diagram:
\[
\begin{array}{ccc}
\tilde{H}_{i+1}(2^V, K) & \xrightarrow{\partial^*} & \tilde{H}_i(K) \\
\downarrow{\alpha_{i+1, g}} & & \downarrow{\rho_{i, g}} \\
\tilde{H}_{i+1}(2^V, K) & \xrightarrow{\partial^*} & \tilde{H}_i(K)
\end{array}
\]
where \(\partial^*\) is the homology boundary isomorphism (See [12], Lemma 24.1):
\[
\begin{array}{ccc}
C_{i+1}(2^V) & \xrightarrow{\pi_{\#}} & \mathcal{R}_{i+1}(2^V, K) \\
\downarrow{\partial^i_{\#}} & & \\
C_i(K) & \xrightarrow{i_{\#}} & C_i(2^V)
\end{array}
\]
The isomorphism \(\partial^*\) is defined by a certain zig-zag process: pull back via \(\pi_{\#}\), apply \(\partial^i_{\#}\), pull back via \(i_{\#}\). For each \(g \in G\) we consider the action on the chain groups of the full simplex, of \(K\) and of \((2^V, K)\):
\[
\begin{align*}
g^V_{\#, i} : & \quad C_i(2^V) \rightarrow C_i(2^V) \\
& \quad [j_1, \ldots, j_{i+1}] \mapsto [g(j_1), \ldots, g(j_{i+1})] \\
\end{align*}
\]
\[
\begin{align*}
g_{\#, i} : & \quad C_i(K) \rightarrow C_i(K) \\
& \quad [j_1, \ldots, j_{i+1}] \mapsto [g(j_1), \ldots, g(j_{i+1})] \\
\end{align*}
\]
\[
\begin{align*}
g^{V,K}_{\#, i} : & \quad \mathcal{R}_i(2^V, K) \rightarrow \mathcal{R}_i(2^V, K) \\
& \quad [j_1, \ldots, j_{i+1}] \mapsto [g(j_1), \ldots, g(j_{i+1})] \\
\end{align*}
\]
We have that:
\[
\begin{align*}
g^V_{\#, i} \mid_{C_i(K)} &= \tilde{g}_{\#, i} \\
g^V_{\#, i} \mid_{\mathcal{R}_i(2^V, K)} &= \tilde{g}^{V,K}_{\#, i}
\end{align*}
\]
We also know that each boundary operator commutes with \(\tilde{g}_{\#, i}, \tilde{g}^V_{\#, i}\) and \(\tilde{g}^{V,K}_{\#, i+1}\) from [12], Lemma 12.1. Let \(b \in \tilde{H}_{i+1}(2^V, K)\), there exists an \(a \in \mathcal{R}_{i+1}(2^V, K)\) such that \(b = a + \text{Im}(d_{i+2})\). Therefore:
\[
\rho_{i, g}(\partial^*(b)) = \rho_{i, g}(\partial^i_{i+1}(a) + \text{Im}(d_{i+2})) = \tilde{g}_{\#, i}(\partial^V_{i+1}(a)) + \text{Im}(d_{i+2}) =
\]

\[ \partial^* (\alpha_{i+1,g}(b)) = \partial^* (\tilde{g}^V_{#.,i+1}(a) + \text{Im}(d_{i+2})) = \partial^* (\tilde{g}^V_{#.,i+1}(a)) + \text{Im}(d_{i+2}) = \partial^* (\tilde{g}^V_{#.,i+1}(a)) + \text{Im}(d_{i+2}). \]

Thus, we have that
\[ \partial^* \circ \alpha_{i+1,g} = \rho_{i,g} \circ \partial^* \quad \text{for every } g \in G \]
and this implies that \( \rho_i \simeq \alpha_{i+1} \).

Combining the results of Lemma 5 and Lemma 4 we obtain the proof of Theorem 2.

**Remark.** From the Alexander’s duality we know that for every simplicial complex \( K \) with vertex set \( V \) such that \( V \notin K \), with \( n = |V| \):
\[ \tilde{H}_i(K) \simeq \tilde{H}_{n-3-i}(K^*). \]

In fact, working with complex coefficients the reduced cohomology group \( \tilde{H}^j(K) \) is the dual vector space of the reduced homology group \( \tilde{H}_j(K) \), so that \( \tilde{H}_j(K) \simeq \tilde{H}^j(K) \). Combining the two results we obtain:
\[ \tilde{H}_i(K) \simeq \tilde{H}_{n-3-i}(K^*). \quad (6) \]

## 3 Equivariant cross-cut theory

Let \( L \) be a lattice with maximal and minimal element \( \hat{1} \) and \( \hat{0} \) respectively. We recall the following definition from [7]:

**Definition 6.** If \( L \) is a lattice with \( \hat{0} \) and \( \hat{1} \), a **cross-cut** of \( L \) is a set \( C \subseteq L \) such that:

i) \( \hat{0}, \hat{1} \notin C \).

ii) If \( x, y \in C \) then \( x \not\leq y \) and \( y \not\leq x \). (\( x \) and \( y \) are incomparable)

iii) Any finite chain \( x_1 < x_2 < \cdots < x_n \) in \( L \) can be extended to a chain which contains an element of \( C \).

In particular, axiom iii) implies that every maximal chain contains an element of \( C \).

Let \( L \) be a lattice with \( \hat{0} \) and \( \hat{1} \) and let \( C \) be a **cross-cut** of \( L \).

**Definition 7.** A finite subset \( \{x_1, \ldots, x_n\} \subseteq C \) **spans** if and only if
\[ x_1 \land x_2 \land \cdots \land x_n = \hat{0} \quad \text{and} \quad x_1 \lor x_2 \lor \cdots \lor x_n = \hat{1} \]
Here $x \wedge y$ denotes the largest element $\leq x$ and $\leq y$, and $x \vee y$ denotes the smallest element $\geq x$ and $\geq y$.

Let $K(C)$ be the abstract simplicial complex whose vertices are the elements of $C$ and whose simplexes are all finite subsets of $C$ which do not ‘span’. We define $\tilde{H}_i(C) = \tilde{H}_i(K(C))$. Let $K(L)$ be the order complex of the lattice $L$ and define $\tilde{H}_i(L) = \tilde{H}_i(K(L))$. The following result was proved in [7], Theorem 3.1:

**Theorem 8.** Let $L$ be a lattice and let $C$ be a cross-cut of $L$, then:

$$\tilde{H}_i(C) \simeq \tilde{H}_i(L).$$

In order to see that the previous isomorphism is also a $\mathbb{C}[G]$-module isomorphism we need the following result:

**Lemma 9.** Let $K$ be an abstract simplicial complex and let $K'$ be its first barycentric subdivision. Let also $G$ be a finite group of automorphisms of the face poset of $K$. Then we have the following isomorphism of $\mathbb{C}[G]$-modules:

$$\tilde{H}_i(K) \simeq G \tilde{H}_i(K').$$

**Proof.** First, we need to describe the action of $G$ on $K'$. Let $\mathcal{L}(K)$ be the face poset of $K$, it is clear that the order complex of $\mathcal{L}(K)$ is the barycentric subdivision of $K$. Thus, we have a straightforward $G$-action on the order complex of $\mathcal{L}(K)$ and its homology spaces. We have to show that the following two representations are isomorphic:

$$\tilde{\rho}_i : G \longrightarrow \text{GL}(H_i(K)) \quad \rho'_i : G \longrightarrow \text{GL}(H_i(K'))$$

Let $w * K$ be a cone. If $e_{\sigma} = [a_0, \ldots, a_i]$ is an oriented simplex of $K$, let

$$[w, e_{\sigma}] = [w, a_0, \ldots, a_i]$$

denote an oriented simplex of $w * K$. This operation is well defined and is called the bracket operation (See [12], Section §8).

If $\sigma = \{a_0, \ldots, a_i\}$ is a simplex, let $\hat{\sigma}$ denote the barycenter of $\sigma$. The complex $K'$ equals the collection of all simplices of the form

$$[\hat{\sigma}_1, \ldots, \hat{\sigma}_n] \quad \text{where} \quad \sigma_1 \supset \cdots \supset \sigma_n.$$  

We know from [12], Section §17 that there is a unique augmentation-preserving chain map $\text{sd} : C_i(K) \longrightarrow C_i(K')$ called the barycentric subdivision operator that induces an isomorphism of homology spaces. There is an inductive formula for the operator $\text{sd}$. It is the following:

$\text{sd}(v) = \hat{v} = v \quad \forall v \in K^0$

$$\text{sd}(e_{\sigma}) = [\hat{\sigma}, \text{sd}(\partial_i(e_{\sigma}))] \quad \text{for} \ \sigma \in K \ \text{with} \ \dim(\sigma) = i.$$
Now we consider the following two representations:

\[ \rho_i : G \to GL(C_i(K)) \quad \rho'_i : G \to GL(C_i(K')) \]

We want to show that the following diagram

\[
\begin{array}{ccc}
C_i(K) & \xrightarrow{sd} & C_i(K') \\
\downarrow{\rho_{i,g}} & & \downarrow{\rho'_{i,g}} \\
C_i(K) & \xrightarrow{sd} & C_i(K')
\end{array}
\]

commutes for every \( g \in G \). We proceed by induction on \( i \):

- Suppose \( i = 0 \). It follows from the action of \( G \) on the vertices of \( K \) and \( K' \) that \( \rho_{0,g}(v) = \rho'_{0,g}(v) \) for every \( v \in K^0 \). Thus:

\[
\rho'_{0,g}(sd(v)) = \rho_{0,g}(v) = \rho_{0,g}(v) = sd(\rho_{0,g}(v)).
\]

- We now suppose the diagram commutes for \( i = n \) and we prove it for \( i = n + 1 \). Let \( g, \sigma = \tau \), thus:

\[
\rho'_{i+1,g}(sd(e_\sigma)) = \rho'_{i+1,g}(\left[\hat{\sigma}, sd(\partial_{i+1}(e_\sigma))\right]) = \left[\hat{\tau}, \rho'_{i+1,g}(sd(\partial_{i+1}(e_\sigma)))\right] =
\]

\[
= \left[\hat{\tau}, sd(\partial_{i+1}(g.e_\sigma))\right] = sd(g.e_\sigma) = sd(\rho_{i+1,g}(e_\sigma)).
\]

Since the diagram (7) commutes and both \( sd, \rho_{i,g}, \rho'_{i,g} \) commute with the border operator \( \partial \) we have that the following diagram commutes and consequently the Lemma is proved:

\[
\begin{array}{ccc}
\tilde{H}_i(K) & \xrightarrow{sd^*} & \tilde{H}_i(K') \\
\downarrow{\tilde{\rho}_{i,g}} & & \downarrow{\tilde{\rho}'_{i,g}} \\
\tilde{H}_i(K) & \xrightarrow{sd^*} & \tilde{H}_i(K')
\end{array}
\]

**Definition 10.** Let \( L \) be a lattice and \( G \) a group of automorphism of \( L \). A crosscut \( C \) of \( L \) is \( G \)-stable if \( G.C = C \), i.e., if \( C \) is union of \( G \)-orbits.

**Theorem 11.** Let \( L \) be a lattice and \( G \) a group of automorphism of \( L \). Let \( C \) be a \( G \)-stable crosscut of \( L \). Then we have the following \( \mathbb{C}[G] \)-module isomorphism:

\[ \tilde{H}_i(L) \simeq_G \tilde{H}_i(C). \]
Proof. We briefly recall Folkman’s argument. Let \( K = K(L) \) be the order complex of \( L \) and let \( C = \{ \alpha_1, \ldots, \alpha_n \} \) be a crosscut of \( L \) fixed by \( G \). For each \( \alpha \in C \) let \( L_\alpha \) be the subcomplex of \( K \) consisting of all simplexes \( \{ y_1, \ldots, y_t \} \) such that the set \( \{ y_1, \ldots, y_t, \alpha \} \) is simply ordered. By the third property of a crosscut, the family \( \{ L_\alpha \}_{\alpha \in C} \) is a covering of \( K \). In the proof of Theorem 3.1 (\cite{2}) Folkman shows that \( L_{\alpha_1} \cap \cdots \cap L_{\alpha_n} \) has the homology of a point or is empty and shows also that

\[ K(C) = \mathcal{N}(\{ L_\alpha \}_{\alpha \in C}) \]  

(8)

where \( K(C) \) is the simplicial complex associated to the crosscut \( C \) and \( \mathcal{N} = \mathcal{N}(\{ L_\alpha \}_{\alpha \in C}) \) is the nerve of the covering \( \{ L_\alpha \}_{\alpha \in C} \). Thus, we can apply a nerve theorem. We follows the construction made by Björner in \cite{4} Theorem 10.6.

Let \( P(K) \) and \( P(N) \) be the face lattice associated to respectively \( K \) and \( N \). Björner defines the following order-reversing map of posets:

\[ \tilde{f} : P(K) \to P(N) \quad \sigma \mapsto \{ \alpha \in C \mid \sigma \in L_\alpha \} \]

This map \( \tilde{f} \) induces a simplicial map \( f \) between the respective order complex of \( P(K) \) and \( P(N) \) which are the first barycentric subdivision of \( K \) and \( N \):

\[ f : K' \to \mathcal{N}' \quad \{ \sigma_0, \ldots, \sigma_i \} \mapsto \{ \tilde{f}(\sigma_0), \ldots, \tilde{f}(\sigma_i) \} \]

where \( \{ \sigma_0, \ldots, \sigma_i \} \) is a simplex of \( K' \), then we have \( \sigma_0 \supseteq \cdots \supseteq \sigma_i \) with \( \sigma_j \) simplex of \( K \). Applying Theorem 10.6 of \cite{4} we get, in particular, that \( f \) induces a chain map \( f_\# \) between \( C_i(K') \) and \( C_i(N') \) in the following manner:

\[ f_\# ([\sigma_0, \ldots, \sigma_i]) = \begin{cases} [\tilde{f}(\sigma_0), \ldots, \tilde{f}(\sigma_i)], & \text{if } \tilde{f}(\sigma_0), \ldots, \tilde{f}(\sigma_i) \text{ are distinct} \\ 0, & \text{otherwise} \end{cases} \]

and moreover an isomorphism \( f_* \) on homology spaces:

\[ \tilde{H}_i(K') \cong H_i(\mathcal{N}'). \]

We need to describe the action of \( G \) on \( K' \) and \( \mathcal{N}' \); the \( G \)-action on \( L \) induces an action on \( K \) and therefore on \( K' \) (in the sense of \cite{2}). Since \( C \) is \( G \)-stable, every \( g \in G \) acts on \( C \) permuting its elements and, since \( g \) is an order automorphism of \( L \), acts on the covering \( \{ L_\alpha \}_{\alpha \in C} \) respecting the intersection relations. Therefore \( G \) yields an action on the nerve \( \mathcal{N} \) and therefore on \( \mathcal{N}' \). We want to show that the following two representations are isomorphic:

\[ \tilde{\rho}_1 : G \rightarrow GL(\tilde{H}_i(K')) \quad \tilde{\rho}_2 : G \rightarrow GL(\tilde{H}_i(\mathcal{N}')). \]

Let

\[ \rho_1 : G \rightarrow GL(C(K)) \quad \rho_2 : G \rightarrow GL(C(\mathcal{N}')). \]

\[ \rho_{1,g} : G \rightarrow GL(C(K)) \quad \rho_{2,g} : G \rightarrow GL(C(\mathcal{N}')). \]

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be the representations on the chain spaces. Since $g \in G$ is an order automorphism of $L$ we have the following: if $\tilde{f}(\sigma) = \{\alpha_{j_0}, \ldots, \alpha_{j_t}\} = \beta$ then $\tilde{f}(g.\sigma) = \{g.\alpha_{j_0}, \ldots, g.\alpha_{j_t}\} = g.\beta$. We explicitly describe the maps induced by $g \in G$ on the chain spaces:

$$\rho_1, g : C_i(K') \longrightarrow C_i(K')$$

$$[\sigma_0, \ldots, \sigma_i] \mapsto [g.\sigma_0, \ldots, g.\sigma_i]$$

$$\rho_2, g : C_i(N') \longrightarrow C_i(N')$$

$$[\beta_{j_0}, \ldots, \beta_{j_i}] \mapsto [g.\beta_{j_0}, \ldots, g.\beta_{j_i}]$$

where $\beta_j$ are simplexes of $N$ satisfying $\beta_{j_0} \subseteq \cdots \subseteq \beta_{j_i}$. We want to show that the following diagram commutes:

$$C_i(K') \xrightarrow{f\#} C_i(N')$$

$$\rho_1, g \downarrow$$

$$\rho_2, g \downarrow$$

$$C_i(K') \xrightarrow{\tilde{f}\#} C_i(N').$$

$$\rho_2, g(f\#([\sigma_0, \ldots, \sigma_i])) = [g.\beta_{j_0}, \ldots, g.\beta_{j_i}] = (f\#(\rho_1, g([\sigma_0, \ldots, \sigma_i]))).$$

Therefore the diagram commutes and since $f\#$ is a chain map we have that $\tilde{\rho}_1 \simeq \tilde{\rho}_2$, i.e., $\bar{H}_i(K') \simeq_G \bar{H}_i(N')$. Using the results of Lemma 9 and Equation 8 we have the following $C[G]$-module isomorphism

$$\bar{H}_i(K) \simeq_G \bar{H}_i(K(C)) = \bar{H}_i(C).$$

4 Applications to matroids

We now specialize the results of the previous two sections to matroids. For basic facts on matroids, the reader may refer to [14]. Let $M = (E, I)$ be a matroid with ground set $E$ and collection of independent sets $I$, which forms an abstract simplicial complex. Let $M^* = (E, I^*)$ be its dual. We recall that the rank of $A \subseteq E$ is the maximal cardinality of an element of $I$ contained in $A$. We say that $A \subseteq E$ is non-spanning in $M$ if $\text{rk}(A) < \text{rk}(E)$, i.e., $A$ does not contain any basis of $M$. Let

$$NS(M) = \{A \subseteq E \mid A \text{ is non-spanning in } M\}$$

It is easy to see that $NS(M)$ is an abstract simplicial complex.

**Proposition 12.** $A \subseteq E$ is not-spanning in $M^*$ if and only if $A^c$ is dependent in $M$.  

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Proof. If $A \subseteq E$ is not-spanning in $M^*$ we have:

$$\text{rk}^*(A) < \text{rk}^*(E)$$

This is equivalent to:

$$\text{rk}(A^c) + |A| - \text{rk}(E) < \text{rk}^*(E)$$

and therefore to

$$\text{rk}(A^c) < -|A| + \text{rk}(E) + \text{rk}^*(E) = |E| - |A| = |A^c| \iff A^c \notin I$$

For every $A \subseteq E$, we have $\text{rk}(A) \leq |A|$, thus $A$ is independent if and only if $\text{rk}(A) = |A|$.

**Proposition 13.** Let $\Delta = \text{IN}(M) = I$ be the abstract simplicial complex associated with the independents of the matroid $M = (E, I)$ and let $\Delta^*$ be its Alexander dual, then:

$$\Delta^* = \text{NS}(M^*)$$

**Proof.** Using the result shown in Proposition 12 we claim that:

$$\Delta^* = \{A \subseteq V : (V \smallsetminus A) \notin \Delta\} = \{A \subseteq E : A^c \notin I\} =$$

$$= \{A \subseteq E : A^c \text{ is dependent in } M = (E, I)\} =$$

$$= \{A \subseteq E : A \text{ is not spanning of } M^*\} = \text{NS}(M^*)$$.

The previous result, together with Equation (6), implies the following:

$$\tilde{H}_i(\text{NS}(M)) \simeq \tilde{H}_{n-3-i}(\text{IN}(M^*))$$

This is not only an isomorphism of vector spaces, but also of representations, up to a sign. Indeed, by applying Theorem 2 we obtain:

**Theorem 14.** Let $G$ be the automorphism group of a matroid. Then we have the following $\mathbb{C}[G]$-module isomorphism:

$$\tilde{H}_i(\text{NS}(M)) \simeq_G \tilde{H}_{n-3-i}(\text{IN}(M^*)) \otimes \text{sgn}.$$
C is a cross-cut of $\mathcal{L}$, indeed the three axioms of Definition 6 are trivially true.

We want to prove that:

$$K(C) = NS(M).$$

In the following proposition we perform a slight abuse of notation by identifying:

$$C = \{\{a_1\}, \{a_2\}, \ldots, \{a_m\}\} = \{a_1, a_2, \ldots, a_m\}$$

**Proposition 15.** $A \subseteq C$ does not ‘span’ (in the sense of Def 7) if and only if $A$ is a non-spanning set in $M = (E, I)$.

**Proof.** $\implies$) In $\mathcal{L}(M)$ we have:

$$\hat{0} = \emptyset \quad \hat{1} = E$$

Let $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ be a subset of $C$. If $A \subseteq C$ does not ‘span’:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} = D \neq \hat{1} \quad (9)$$

$D \in \mathcal{L}(M)$ and $D \neq \hat{1}$ implies that $D$ is a non-spanning subset of $E$ because the only spanning subset in $\mathcal{L}(M)$ is $E = \hat{1}$.

It follows from (9) that $A \subseteq D$; since $D$ is a non-spanning subset of $E$ then $A$ is a non-spanning subset of $E$.

$\implies$) In $NS(M)$ the bases are the maximal non-spanning subsets of $E$, (i.e, the subsets of $E$, such that if we add an element they become spanning set) so they are flats, in particular they correspond to the co-atoms of $(\mathcal{L}(M), \subseteq)$.

Let $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$ be a non-spanning subset of $E$, there exist a basis $\mathcal{B}$ of $NS(M)$ such that:

$$A \subseteq \mathcal{B} \quad \mathcal{B} \text{ is a flat } \implies \mathcal{B} \in \mathcal{L}(M)$$

This implies:

$$a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n} \subseteq \mathcal{B} \neq \hat{1}$$

then $A$ does not ‘span’.

Using the result of Proposition 15 we obtain:

$$K(C) = NS(M).$$

Since $C$ is a crosscut fixed by $G$, we can apply Theorem 11 to $\mathcal{L}(M)$ and $C$ itself:

**Theorem 16.** Let $G$ be the group of automorphism of the simple matroid $M$. Then we have the following $\mathbb{C}[G]$-module isomorphism:

$$\tilde{H}_i(\mathcal{L}(M)) \simeq_G \tilde{H}_i(C) = \tilde{H}_i(\mathcal{L}(M))$$

where $C$ is the cross-cut of $\mathcal{L}(M)$ composed of its atoms.
By combining Theorem 14 and Theorem 16 we get that, for every simple matroid, we have the following:

**Theorem 17.** Let $G$ be the group of automorphism of the simple matroid $M$. Then we have the following $\mathbb{C}[G]$-module isomorphism:

$$
\tilde{H}_{n-3-i}(IN(M^*)) \simeq_G \tilde{H}_i(L(M)) \otimes \text{sgn}
$$

where $n$ is the cardinality of the ground set of $M$.

## 5 Top cohomology of hyperplane arrangements

Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^r$ and let $L(\mathcal{A})$ be its intersection lattice. Let $M_{\mathcal{A}}$ be the matroid associated with $\mathcal{A}$; then the lattice of flats $L(M_{\mathcal{A}})$ of $M_{\mathcal{A}}$ is isomorphic to $L(\mathcal{A})$. We can assume that the arrangement is essential: then the rank of the matroid is $r$. We define the complement of the arrangement:

$$
C(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup_{H \in \mathcal{A}} H.
$$

Let $G$ be a subgroup of $GL(\mathbb{C}^r)$ that permutes the elements of $\mathcal{A}$; it is easy to see that $G$ is also a group of automorphism of the matroid $M_{\mathcal{A}}$. Let $\mathfrak{A}$ be the Orlik-Solomon algebra associated to $L(\mathcal{A})$, and let $\mathfrak{B}$ be the algebra defined by shuffle defined respectively in Section 2 and 3 of [13]. In Theorem 3.7 of the same paper, Orlik and Solomon provide a $G$-isomorphism of graded algebras:

$$
\theta: \mathfrak{A} \rightarrow \mathfrak{B}.
$$

Furthermore, we report Theorem 4.3:

**Theorem 18.** Let $L$ be a finite geometric lattice of rank $r > 1$. Then $\mathfrak{B}_1$ and $H_{r-2}(L)$ are isomorphic $\mathbb{C}[G]$-modules.

Combining the previous results we get the following $\mathbb{C}[G]$-module isomorphism:

$$
H^r(C(\mathcal{A})) \simeq_G \mathfrak{A}_1 \simeq_G \mathfrak{B}_1 \simeq_G H_{r-2}(L(\mathcal{A})).
$$

(10)

Applying Theorem 17 we obtain the following:

**Theorem 19.** Let $\mathcal{A}$ be a central essential hyperplanes arrangement of dimension $r$ and let $M_{\mathcal{A}}$ be the associated matroid with ground set of cardinality $n$. Then we have the following $\mathbb{C}[G]$-module isomorphism:

$$
H^r(C(\mathcal{A})) \simeq_G H_{n-r-1}(IN(M_{\mathcal{A}}^*)) \otimes \text{sgn}.
$$

In [11] Lehrer and Solomon conjecture that if $W$ is a Coxeter group and $\mathcal{A}_W$ is the hyperplane arrangement associated to $W$ then there is a $\mathbb{C}[G]$-module isomorphism

$$
H^p(C(\mathcal{A}_W)) \simeq_W \bigoplus_c \text{Ind}^W_{Z(c)}(\xi_c) \quad p = 0, \ldots, \text{rank}(W)
$$
where $c$ runs over a set of representatives for the conjugacy classes of $W$ with $\dim(\text{im}(c)) = p$ and $\xi_c$ is a suitable character of the centralizer $Z(c)$ of $c$ in $W$. They proved the conjecture for group of rank 2 and for $W = S_r$. In the case of the symmetric group $S_r$, the arrangement $A_{S_r}$ is the braid arrangement and the intersection lattice $\mathcal{L}(A)$ is the partition lattice $\Pi_r$, that is, the family of all partitions of the set $\{1, \ldots, r\}$ partially ordered by refinement. Stanley studied the representations on the homology of the partition lattice in $[17]$, using Equation 10 his result agrees with the conjecture of Lehrer and Solomon.

Theorem 19 allows to rewrite Lehrer and Solomon’ conjecture in the top cohomology case with the language of matroids:

$$H_{n-r-1}(\text{IN}(M^*_{A_{W}})) \simeq_{W} \bigoplus_{c} \text{Ind}_{Z(c)}^{W}(\xi_c) \otimes \text{sgn}.$$  

6 The case of the complete graph

We now consider the matroid $M^*(K_m)$ of the complete graph $K_m$, which has rank $r = m - 1$ and ground set of cardinality $n = \binom{m}{2}$. This matroid is isomorphic to the matroid $M(\Phi^+_{A_{m-1}})$ associated with the positive roots of the root system of type $A_{m-1}$. In fact, this is the case of interest in $[6]$.

We recall that the lattice of flats of this matroid is isomorphic to the partition lattice $\Pi_m$. In this case, Theorem 17 specializes to the following:

**Theorem 20.** $\tilde{H}_{n-3-i}(\text{IN}(M^*(K_m)))$ and $\tilde{H}_i(\Pi_m) \otimes \text{sgn}$ are isomorphic as $\mathfrak{S}_m$-modules for every $i \geq 0$.

Rephrased in terms of root system of type $A_{m-1}$ we get this $\mathbb{C}[G]$-module isomorphism:

$$\tilde{H}_{n-3-i}(\text{IN}(M^*(\Phi^+_{A_{m-1}}))) \simeq_{\mathfrak{S}_m} \tilde{H}_i(\Pi_m) \otimes \text{sgn} \label{11}$$

where

$$n = |\text{E}(M^*(\Phi^+_{A_{m-1}}))| = |\Phi^+(A_{m-1})| = \binom{m}{2} = \frac{m(m-1)}{2}.$$  

**Remark.** We can make a dimensional calculation to better understand the dimensional shift. The matroid $M(\Phi^+_{A_{m-1}}, I)$ has rank equal to $m - 1$, i.e. each basis has $m - 1$ elements. Therefore, the matroid $M^*(\Phi^+_{A_{m-1}}, I)$ has rank equal to:

$$n - (m - 1) = \frac{m(m-1)}{2} - (m - 1) = \frac{(m-1)(m-2)}{2}.$$  

Thus, the dimension of the top homology of $\text{IN}(M^*(\Phi^+_{A_{m-1}}))$ is one less than the number of the elements of a basis of $M^*(\Phi^+_{A_{m-1}}, I)$:

$$\frac{(m-1)(m-2)}{2} - 1.$$
By Equation 11 we have the following isomorphism of $\mathbb{C}$-vector spaces:

$$\tilde{H}_{n-3-i}(IN(M^\ast(\Phi_{A_{m-1}}^\ast))) \simeq \tilde{H}_i(\Pi_m).$$

We impose

$$n - 3 - i = \frac{(m - 1)(m - 2)}{2} - 1$$

and we get the $i$:

$$\frac{m(m-1)}{2} - 3 - i = \frac{(m - 1)(m - 2)}{2} - 1$$

$$m^2 - m - 6 - 2i = m^2 - 2m - m + 2 - 2 \implies i = m - 3$$

Indeed, $H_{m-3}(\Pi_m)$ is the only nonzero homology group of $\Pi_m$.

By Theorem 20 these two representations

$$\rho_{n-m} : \mathfrak{S}_m \rightarrow \text{GL}(\tilde{H}_{n-m}(IN(M^\ast(\Phi_{A_{m-1}}^\ast))))$$

and

$$\gamma_{m-3} : \mathfrak{S}_m \rightarrow \text{GL}(\tilde{H}_i(\Pi_m) \otimes \text{sgn})$$

are isomorphic. From a result due to Stanley ([17], Theorem 7.3) we know that the representations on the top homology of the partition lattice

$$\tilde{\gamma}_{m-3} : \mathfrak{S}_m \rightarrow \text{GL}(\tilde{H}_i(\Pi_m))$$

are the following

$$\tilde{\gamma}_{m-3} \simeq \text{sgn} \otimes \text{ind}_{\mathbb{C}[\mathfrak{S}_m]} \left(e^{2\pi i/m}\right).$$

Thus, we get

$$\rho_{n-m} \simeq \text{ind}_{\mathbb{C}[\mathfrak{S}_m]} \left(e^{2\pi i/m}\right)$$

or as $\mathbb{C}[\mathfrak{S}_m]$-modules:

$$\tilde{H}_{n-m}(IN(M^\ast(\Phi_{A_{m-1}}^\ast))) \simeq_{\mathfrak{S}_m} \text{ind}_{\mathbb{C}[\mathfrak{S}_m]} \left(e^{2\pi i/m}\right).$$
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