Superconvergence of the Direct Discontinuous Galerkin Method for Two-Dimensional Nonlinear Convection-Diffusion Equations

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Abstract

This paper is concerned with superconvergence properties of the direct discontinuous Galerkin (DDG) method for two-dimensional nonlinear convection-diffusion equations. By using the idea of correction function, we prove that, for any piecewise tensor-product polynomials of degree \( k \geq 2 \), the DDG solution is superconvergent at nodes and Lobatto points, with an order of \( O(h^{2k}) \) and \( O(h^{k+2}) \), respectively. Moreover, superconvergence properties for the derivative approximation are also studied and the superconvergence points are identified at Gauss points, with an order of \( O(h^{k+1}) \). Numerical experiments are presented to confirm the sharpness of all the theoretical findings.

Keywords: nonlinear convection-diffusion equations; direct discontinuous Galerkin method; superconvergence

1. Introduction

In this paper, we investigate the superconvergence behavior of the DDG method for the following two-dimensional nonlinear convection-diffusion equations

\[
\begin{align*}
  u_t + \nabla \cdot f(u) &= \Delta u, & (x, y, t) &\in [0, 2\pi] \times [0, 2\pi] \times (0, T], \\
  u(x, y, 0) &= u_0(x, y), & (x, y) &\in [0, 2\pi] \times [0, 2\pi],
\end{align*}
\]

where \( u_0, f(u) = (f_1(u), f_2(u)) \) are smooth functions. For simplicity, we only consider the periodic boundary condition.

The discontinuous Galerkin (DG) method is a class of finite element method using a completely discontinuous polynomial space for the numerical solution and the test functions. Due to its flexibilities such as the allowance of hanging nodes, \( p \)-adaptivity, extremely local data structure, high parallel efficiency, the DG method has been successfully applied to solve first order PDEs such as hyperbolic conservation laws (see, e.g., [21, 22, 23, 24]).
However, for equations containing higher order spatial derivatives, such as the convection-diffusion equation and the KdV equation, the DG method cannot be directly applied due to the discontinuous solution space at the element interfaces, which is not regular enough to handle higher derivatives. Several DG methods have been suggested in the literature to solve this problem, including the local discontinuous Galerkin (LDG) method in [15, 25, 27], the compact DG (CDG) method [34]; the interior penalty (IP) methods, [2, 3, 38]; the direct discontinuous Galerkin (DDG) method [31, 37, 39], as well as other DG schemes designed from a practical perspective.

The DDG method was first introduced by Liu and Yan in 2009 for convection-diffusion problems [30]. The basic idea of this method is to directly force the weak solution formulation of the PDE into the DG function space for both the numerical solution and test functions, without introducing any auxiliary variables or rewriting the original equation into a larger first-order system. One main advantage of the DDG method lies in that: it is compact in the sense that only the degrees of freedom belonging to neighboring elements are connected in the discretization, which means lower storage requirements and higher computational performance than noncompact schemes. Compared with LDG methods, DDG methods need to choose special numerical fluxes for the solution derivative at cell interfaces to ensure its stability, which gives DDG methods extra flexibility and advantage. It was proved in [19] that the DDG method satisfies strict maximum principle with at least third order of accuracy, while only second order can be obtained for IPDG and LDG methods [41]. For lower order piecewise constant or linear approximations, the DDG method may degenerate to the IPDG method and shares the same desired properties of the IPDG solution. While for higher order $P_k (k \geq 2)$ approximation, DDG method have quite a few advantages over the IPD method in some aspects, e.g., superconvergence approximations (see [33]).

During past decades, superconvergence analysis has attracted many researchers’ attention and significant progress has been. For an incomplete list of references, we refer to [3, 6, 28, 29, 36] for finite element methods (FEM), [7, 8, 9] for finite volume methods (FVM), and [1, 12, 14, 20, 40] for DG methods, and [13, 33] for DDG methods. In [13], the authors studied superconvergence phenomenon of DDG methods for the linear convection-diffusion equation in the one-dimensional setting. Under a special choice of the numerical flux (especially the choice of the parameter in the DDG numerical flux), a $(k + 2)$-th order superconvergence of the numerical solution at Lobatto points, $(2k)$-th order at nodes, and a $(k + 1)$-th order of the derivative approximation at Gauss points, are obtained.

The main purpose of the current work is to study and reveal the superconvergence phenomenon of the DDG method for two-dimensional nonlinear convection diffusion equation. To our best knowledge, no superconvergence results of DDG methods applied to nonlinear convection-diffusion equations are available in the literature. The main difficulties in the superconvergence analysis for nonlinear problems are how to deal with the nonlinear terms in the error equation and how to improve the error accuracy. To deal with the two issues, we first use Taylor expansion to linearize the error equation and then reduce the superconvergence analysis for nonlinear problems into two parts: a lower order linear term and high order nonlinear term. Then we use the idea of correction function to deal with the linear part to obtain higher-order accuracy and finally achieve our superconvergence goal. To be
more precise, we establish superconvergence result for the DDG approximation at nodes, Lobatto points and Gauss points (derivative approximation), whose convergence rates are $2k, k+2,$ and $k+1,$ respectively. As we may recall, all these superconvergence results are similar to these for the one-dimensional linear problems in [13]. It should be emphasized that the convergence analysis for nonlinear problems are much more sophisticated than that for the linear case.

The rest of the paper is organized as follows. In section 2, we present the DDG schemes and then discuss the choice of numerical fluxes and stability results for the equation (1). Section 3 and 4 are the main part of this paper, where we construct a series of special correction functions, by identify out the lower order terms in the scheme. With suitable initial discretizations and careful choice of numberical fluxes, superconvergence results for Lobatto, Gauss points and nodes are proved respectively. In Section 5, we present several numerical examples to validate our theoretical results. Some concluding remarks are provided in Section 6.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on subdomain $D \subset \Omega$ equipped with the norm $\| \cdot \|_{m,p,D}$ and semi-norm $| \cdot |_{m,p,D}$. When $D = \Omega$, we omit the index $D$. If $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\| \cdot \|_{m,p,D} = \| \cdot \|_{m,D}$, and $| \cdot |_{m,p,D} = | \cdot |_{m,D}$. We use the notation $A \lesssim B$ to indicate that $A$ can be bounded by $B$ multiplied by a constant independent of the mesh size $h$.

2. DDG schemes

Let $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x+\frac{1}{2}} = 2\pi$ and $0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y+\frac{1}{2}} = 2\pi$. For any positive integer $r$, we define $Z_r = \{1, 2, \cdots, r\}$, and denote by $\mathcal{T}_h$ the rectangular partition of $\Omega := [0, 2\pi] \times [0, 2\pi]$. That is

$$\mathcal{T}_h = \{\tau_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] : (i, j) \in Z_{N_x} \times Z_{N_y}\}.$$

Denote by $\tau_i^x = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $\tau_j^y = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, and $h = \max_{\tau \in \mathcal{T}_h} h_\tau$ the meshsize of $\mathcal{T}_h$, where $h_\tau$ is the diameter of the element $\tau \in \mathcal{T}_h$. We assume that the partition $\mathcal{T}_h$ is regular.

Define the discontinuous finite element space

$$V_h^k := \{v \in L^2(\Omega) : v|_\tau \in \mathbb{Q}_k(x,y) = \mathbb{P}_k(x) \times \mathbb{P}_k(y), \forall \tau \in \mathcal{T}_h\},$$

where $\mathbb{P}_k$ denotes the space of polynomials of degree at most $k$ with coefficients as functions of $t$. Let $\tau_1$ and $\tau_2$ be two neighboring cells with a common edge $e$, we denote by, for any function $w$, $\{w\}$ and $[w]$ the average and the jump of $w$, respectively. That is,

$$\{w\}|_e = \frac{1}{2}(w_1 + w_2), \quad [w]|_e = w_2 - w_1.$$

Here $w_i = w|_{\partial \tau_i}, i = 1, 2,$ and the jump term is calculated as a forward difference along the normal direction, which is defined to be oriented from $\tau_1$ to $\tau_2$. 

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The DDG method for (1) is to find a $u_h \in V_h^k$ such that for any $v_h \in V_h^k$,

$$(\partial_t u_h, v_h)_\tau = (f - \nabla u, \nabla v_h)_\tau - \int_{\partial \tau} v_h (\hat{f}(u_h) - \nabla u_h) \cdot \mathbf{n} ds - B, \ \forall \tau \in T_h,$$

where $(u, v)_\tau = \int_\tau uvdxdy$, $\mathbf{n}$ denotes the unit outward vector of $\tau$, $\hat{f}(u_h), \nabla u_h$ are numerical fluxes, and $B$ is the interface correction defined as

$$B := \frac{1}{2} \int_{\partial \tau} [u_h] \nabla v_h \cdot \mathbf{n} ds.$$

To ensure the stability as well as for the accuracy of the DDG method, the choice of numerical fluxes is of great importance. For the convection flux, we follow [32] and take the entropy flux (or E-flux) as our numerical flux. That is, at any $e \in \tau_1 \cap \tau_2$, let

$$u_i = u|_{\partial \tau_i}, \ i = 1, 2,$$

and the normal direction be oriented from $\tau_1$ to $\tau_2$, then $\hat{f}(u_1, u_2) = (\hat{f}_1(u_1, u_2), \hat{f}_2(u_1, u_2))$ satisfies the following condition

$$\text{sign}(u_2 - u_1)(\hat{f}_i(u_1, u_2) - f_i(u)) \leq 0, \ i = 1, 2,$$

for all $u$ between $u_1$ and $u_2$. For the diffusion term, we take the follows numerical fluxes:

$$\hat{\nabla} u := (\hat{u}_x, \hat{u}_y), \ \hat{u}_x := \beta_0 h^{-1} [u] + \{u_x\} + \beta_1 h [u_{xx}], \ \hat{u}_y := \beta_0 h^{-1} [u] + \{u_y\} + \beta_1 h [u_{yy}],$$

(4)

where $(\beta_0, \beta_1)$ satisfy the following stability condition

$$\beta_0 \geq \Gamma(\beta_1),$$

with

$$\Gamma(\beta_1) = \sup_{v \in P_k^{h-1}[-1, 1]} \frac{2(v(1) - 2\beta_1 \partial_\xi v(1))^2}{\int_{-1}^1 v^2(\xi)d\xi}.$$ 

Summing up all elements $\tau$ in (2) and using the periodic boundary condition, we have

$$(\partial_t u_h, v_h) + A(u_h, v_h) = F(u_h, v_h), \ \forall v_h \in V_h^k,$$

(5)

where $(u, v) = \sum_{\tau \in T_h} (u, v)_\tau$, and

$$A(u, v) = (\nabla u, \nabla v) + \sum_{e \in \mathcal{E}_h} \int_e ([v] \hat{\nabla} u + [u] \{\nabla v\}) \cdot \mathbf{n} ds,$$

(6)

$$F(u, v) = (f(u), \nabla v) + \sum_{e \in \mathcal{E}_h} \int_e [v] \hat{f}(u) \cdot \mathbf{n} ds.$$ 

(7)

Here $\mathcal{E}_h$ denotes the set of all edges of $T_h$.

Due to the special choice of numerical fluxes in (3)-(4), it was proved in [32] that the
DDG scheme (5) is stable and has Optimal error estimate in $L^2$ norm. Moreover, there holds for any function $v \in V_h^k$ that

$$F(v, v) \leq 0, \quad A(v, v) \geq \gamma \|v\|_E^2,$$

where $\gamma \in (0, 1)$ is some positive constant, and

$$\|v\|_E^2 = (\nabla v, \nabla v) + \frac{\beta_0}{h} \sum_{e \in \mathcal{E}_h} \int_e [v]^2 ds.$$

3. Construction of a special projection of the exact solution.

In this section, we will discuss the construction of a special projection $u_I$ of the exact solution, which is superconvergent towards the numerical solution $u_h$ in the $L^2$ norm. To this end, we begin with the error equation of the DDG scheme, which is the basis of the construction of $u_I$.

3.1. Error equations

In the rest of this paper, we will use the following notation

$$\xi = u_h - u_I, \quad \eta = u - u_I, \quad e_h = u - u_h.$$  \hfill (10)

Note that the exact solution $u$ of (1) also satisfies (5). Then for all $v_h \in V_h^k$,

$$(\partial_t u, v_h) + A(u, v_h) = F(u, v_h).$$  \hfill (11)

Subtracting (5) from (11) yields the following error equation

$$(\partial_t e_h, v_h) + A(e_h, v_h) = F(u, v_h) - F(u_h, v_h).$$  \hfill (12)

Taking $v_h = u_h - u_I$ in (12), we have

$$(\partial_t \xi, \xi) + A(\xi, \xi) = (\partial_t \eta, \xi) + A(\eta, \xi) + H, \quad H := -F(u, v_h) + F(u_h, v_h).$$  \hfill (13)

In light of (8), we get

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_0^2 + \|\xi\|_E^2 \leq (\partial_t \eta, \xi) + A(\eta, \xi) + H.$$  \hfill (14)

On the other hand, we have from the Taylor expansion that

$$f(u_h) - f(u) = -f'(u)e_h + \frac{f''(u)}{2}(e_h)^2, \quad f(\{u_h\}) - f(u) = -f'(u)(e_h) + \frac{f''(u)}{2}(e_h)^2.$$  \hfill (14)
where $f'' = f''(\theta_1 u + (1 - \theta_1)u_h)$ and $\tilde{f}'' = \tilde{f}''(\theta_2 u + (1 - \theta_2)u_h)$ with $0 < \theta_1, \theta_2 < 1$. Consequently,

$$H = (f(u_h) - f(u), \nabla \xi) + \sum_{e \in \mathcal{E}_h} \int_e [\xi] (\tilde{f}(u_h) - f(u_h) + f(\{u_h\}) - f(u)) \cdot n ds$$

$$= (-f'e_h + \frac{f''}{2} |e_h|^2, \nabla \xi) + \sum_{e \in \mathcal{E}_h} \int_e [\xi] (\tilde{f}(u_h) - f(u_h) + f'(u)e_h + \frac{f''}{2} \{e_h\}^2) \cdot n ds$$

$$= H_1 + H_2 + H_3 + H_4,$$

where

$$H_1 = \sum_{\tau \in \mathcal{T}_h} (f'(u)\xi, \nabla \xi)_\tau + \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} f'(u)\{\xi\}[\xi] \cdot n ds,$$

$$H_2 = -\sum_{\tau \in \mathcal{T}_h} (f'(u)\eta, \nabla \xi)_\tau - \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} f'(u)\{\eta\}[\xi] \cdot n ds,$$

$$H_3 = \frac{1}{2} \sum_{\tau \in \mathcal{T}_h} (f''|e_h|^2, \nabla \xi)_\tau + \frac{1}{2} \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} \tilde{f}'' \{e_h\}^2[\xi] \cdot n ds,$$

$$H_4 = \sum_{e \in \mathcal{E}_h} \int_e [\xi] (\tilde{f}(u_h) - f(\{u_h\})) \cdot n ds$$

We next estimate $H_i, i \leq 4$, respectively. As for $H_1$, a simply integration by parts yields

$$H_1 = -\frac{1}{2} \sum_{\tau \in \mathcal{T}_h} (\nabla f'(u), \xi^2)_\tau - \frac{1}{2} \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} f'(u)[\xi]^2 \cdot n ds + \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} f'(u)\{\xi\}[\xi] \cdot n ds$$

$$\leq -\frac{1}{2} \sum_{\tau \in \mathcal{T}_h} (\nabla f'(u), \xi^2)_\tau \leq C\|\xi\|_0^2.$$

To estimate $H_3$, we use the Cauchy-Schwarz inequality to derive that

$$H_3 \leq C\|e_h\|_{0,\infty}\|e_h\|_0\|\nabla \xi\|_0 + \frac{\gamma}{4} \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau} \frac{\beta_0}{h} |\xi|^2 ds + C\|e_h\|_{0,\infty}^4$$

$$\leq \frac{\gamma}{4}\|\xi\|_E^2 + C\|e_h\|_{0,\infty}^4.$$

As it was proved in [32] that $\|e_h\|_0 \lesssim h^{k+1}\|u\|_{k+1}$, we have $\|e_h\|_{0,\infty} \lesssim h^{k+\frac{1}{2}}\|u\|_{k+1}$, and thus

$$H_3 \leq \frac{\gamma}{4}\|\xi\|_E^2 + Ch^{k+\frac{1}{2}}\|u\|_{k+1}^2.$$
To estimate $H_4$, we define
\[ \alpha(\hat{f}, \zeta) := [u]^{-1}(\hat{f}(u^-, u^+) - f(\zeta)) := (\alpha_1, \alpha_2) \] (15)
for any function $u$ and any point $\zeta$ between $u^-$ and $u^+$. Then
\[ H_4 = \sum_{e \in \mathcal{E}_h} \int_{[\xi]} [u_h] \alpha(\hat{f}, \{u_h\}) \cdot \mathbf{n} ds \leq -\sum_{e \in \mathcal{E}_h} \int_{[\xi]} [\eta] \alpha(\hat{f}, \{u_h\}) \cdot \mathbf{n} ds, \]
where in the last step, we have used the identity $[u_h] = [\xi] - [\eta]$. Substituting the estimates of $H_i$ into the formula of $H$ yields
\[ H \leq -\mathcal{F}(\eta, \xi) + \frac{\gamma}{4} \| \xi \|^2_E + C h^{2(k+1)} \| u \|^2_{k+1} + C \| \xi \|^2_0, \] (16)
where for any $\varphi, v,$
\[ \mathcal{F}(\varphi, v) := (\varphi f'(u), \nabla v) + \sum_{e \in \mathcal{E}_h} (\int_e [\varphi] f'(u) \cdot \mathbf{n} ds + \int_e [\varphi] \alpha f(u_h), \{u_h\}) \cdot \mathbf{n} ds. \]
Plugging (16) into (13), we have
\[ \frac{1}{2} \frac{d}{dt} \| \xi \|^2_0 + \frac{3\gamma}{4} \| \xi \|^2_E \leq (\partial_t \eta, \xi) + A(\eta, \xi) + \mathcal{F}(\eta, \xi) + C h^{2(k+1)} \| u \|^2_{k+1} + C \| \xi \|^2_0. \] (17)
Define
\[ a(\varphi, v) := (\partial_t \varphi, v) + A(\varphi, v) - \mathcal{F}(\varphi, v), \quad \forall \varphi, v \in V^k_h. \] (18)
The error inequality in (17) indicates that the error $\| u_h - u_l \|_0$ depends on the error bound $a(u - u_l, \xi)$. In other words, to achieve our superconvergence goal, the function $u_l$ should be specially designed such that $a(u - u_l, v)$ is of high order for any function $v \in V^k_h$.

For any smooth function $v$ and any interval $\tau^i$, we define a function $P(x)v \in \mathbb{P}^k(\tau^i)$ satisfying the following conditions:
\[ \int_{\tau^i} (P(x)v - v) \partial_x^2 w dx = 0, \quad \forall w \in \mathbb{P}^k(\tau^i), \quad i \in \mathbb{Z}_N, \] (19a)
\[ \partial_x (P(x)v)|_{x_i + \frac{1}{2}} := \beta_0 (h_i^x)^{-1}[P(x)v] + \{\partial_x (P(x)v)\} + \beta_1 h_i^x [\partial_x^2 (P(x)v)]|_{x_i + \frac{1}{2}} = \partial_x v|_{x_i + \frac{1}{2}}, \] (19b)
\[ \{P(x)v\}|_{x_i + \frac{1}{2}} = \{v\}|_{x_i + \frac{1}{2}}, \quad i \in \mathbb{Z}_N. \] (19c)
Here $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$. Note that when $k = 1$, $P(x)v$ only needs to satisfy the conditions (19b)-(19c). Similarly, we can define $P(y)v$ along the $y$-direction. Define $\Pi_h v \in V^k_h$ of $v$ by
\[ \Pi_h v = P(x) \otimes P(y)v. \] (20)
It is shown in [32] that the global projection $\Pi_h$ is uniquely defined, and
\[
\|v - P(x)v\|_0 + \|v - P(y)v\|_0 + \|v - \Pi_h v\|_0 \lesssim h^{l+1}\|v\|_{l+1}, \quad 0 \leq l \leq k. \tag{21}
\]
Moreover, a straightforward analysis using the approximation in (21) yields
\[
a(u - \Pi_h u, v_h) \lesssim h^{k+1}\|v_h\|_0, \quad \forall v_h \in V_h^k.
\]
In other words, if we choose $u_I = \Pi_h u$, then optimal error estimates can be obtained, just as proved in [32]. Based on this optimal error estimates, we construct a special correction function $\omega$ to correct the term such that
\[
a(u - u_I, v_h) = a(u - \Pi_h u + \omega, v_h) \lesssim h^{k+1+l}\|v_h\|_0, \quad \forall v_h \in V_h^k
\]
for some positive $l$. In light of (20), we have
\[
u - \Pi_h u = E_x u + E_y u - E_x E_y u, \tag{22}
\]
where
\[
E_x u = u - P(x)u, \quad E_y u = u - P(y)u,
\]
\[
E_x E_y u = P(y)(P(x)u - u) - (P(x)u - u). \tag{23}
\]
Then
\[
a(u - \Pi_h u, v) = a(E_x u, v) + a(E_y u, v) - a(E_x E_y u, v).
\]
In the following, we will construct functions $\omega$ and $\bar{\omega}$ to separately correct the error $a(E_x u, v)$ and $a(E_y u, v)$ such that $\omega = \omega + \bar{\omega}$, and $a(E_x u + \omega, v), a(E_y u + \bar{\omega}, v)$ are both of high orders.

3.2. Correction function for $a(E_x u, v)$

We begin with some preliminaries. First, let
\[
D_x^{-1}v(s) = \int_{x, i - \frac{1}{2}}^{x, i + \frac{1}{2}} v(s')ds', \quad x \in \tau_i^x.
\]
It is obviously that
\[
\|D_x^{-1}v\|_{0,0,\tau_i^x} \leq h_i^x\|v\|_{0,\infty,\tau_i^x}. \tag{24}
\]
Secondly, we denote by $Q_h^x v$ the Gauss-Lobatto projection of any function of $v$ along the $x$-direction. That is, $Q_h^x v|_{\tau_i^x} \in P^k(x)$ is a function satisfying
\[
Q_h^x v(x_{i - \frac{1}{2}}^+) = v(x_{i - \frac{1}{2}}^-), \quad Q_h^x v(x_{i + \frac{1}{2}}^-) = v(x_{i + \frac{1}{2}}^+), \tag{25a}
\]
\[
\int_{x, i - \frac{1}{2}}^{x, i + \frac{1}{2}} (v - Q_h^x v)\phi dx, \quad \forall \phi \in P^{k-2}(\tau_i^x), \quad k \geq 2. \tag{25b}
\]
Theres hold the following error estimates (see, e.g., [9, 18])

\[ \|v - Q_h^r v\|_{r,q} \lesssim h^{m-r+\frac{2}{q}} \|\partial^m x v\|_{0}, \quad 0 \leq r \leq m \leq k. \]  

(26)

Similarly, we can define \( Q_h^r \) along the \( y \)-direction, and the above approximation property also holds true for \( Q_h^r \).

Thirdly, recalling the definition of \( A(\cdot, \cdot) \) in (13) and using the integration by parts and the equation

\[ [uv] = [u]\{v\} + \{u\}[v], \]  

(27)

we get

\[
A(E^x u, \xi) = - \sum_{\tau_{i,j}} (E^x u, \xi_{\tau_{i,j}})_{\tau_{i,j}} + \sum_{\tau_{i,j}} \int_{\tau_{i,j}} (\partial_x E^x u)[\xi] - \{E^x u\}[\xi]|_{x+\frac{1}{2}1} dy + \sum_{\tau_{i,j}} \int_{\tau_{i,j}} (\partial_y E^x u)[\xi] + \{E^x u\}[\xi]|_{y+\frac{1}{2}1} dx.
\]

(28)

Using the definition of \( P^{(x)} u \) in (13) and the integration again yields

\[
A(E^x u, \xi) = - \sum_{\tau_{i,j}} (\partial_y E^x u, \xi_{\tau_{i,j}}) - \sum_{\tau_{i,j}} \int_{\tau_{i,j}} [\partial_y (E^x u)] \xi|_{y+\frac{1}{2}1} dx + \sum_{\tau_{i,j}} \int_{\tau_{i,j}} (\partial_y E^x u)[\xi]|_{y+\frac{1}{2}1} dx
\]

\[ = - \sum_{\tau_{i,j}} (\partial_y E^x u, \xi_{\tau_{i,j}}), \]

where in the second step we have used the fact that \( \partial_y E^x u, r \geq 0 \) are continuous about \( y \).

Similarly, we recall the definition of \( a(\cdot, \cdot) \) in (13)

\[
a(E^x u, v) = ((\partial_t - \partial_y^2) E^x u, v) + (E^x u \nabla f(u), \nabla v) + \sum_{e \in E_h} (\int_e [v] E^x u f'(u) \cdot nds + \int_e [v][E^x u] \alpha \cdot nds)
\]

\[ = ((\partial_t - \partial_y^2) E^x u, v) - (f'_1(u) E^x u, v_x) + (f'_2(u) E^x u, v_y) - \sum_{\tau_{i,j}} \int_{\tau_{i,j}} \alpha_1 E^x u[v]|_{x+\frac{1}{2}1} dy. \]

(29)

Here \( \alpha, \alpha_1 \) are given in (13), and in the last step, we have used integration by parts and the identities \( \{E^x u\}|_{x+\frac{1}{2}1} = 0 \), \( [E^x u](\cdot, y_{j+\frac{1}{2}}) = 0 \).

Now we are ready to construct the correction function corresponding to the term \( a(E^x u, v) \).

Let \( \omega_0 = E^x u \) and we define \( \omega_l|_{x_i} \in \mathbb{P}_k(x), 1 \leq l \leq k - 1 \) by

\[
\langle \omega_l, \partial^2_y v \rangle_{\tau^i} = \mathcal{B}(\omega_{l-1}, v)_{\tau^i}, \quad \forall v \in \mathbb{P}_k(\tau^i) \setminus \mathbb{P}^1(\tau^i), \quad i \in \mathbb{Z}_{N_x}, \]

(30a)

\[
\omega|_{x_i+\frac{1}{2}} := \beta_0(h)^{-1}[\omega] + \{\partial_x(\omega)\}, \quad \beta_1 h[\partial_x^2(\omega)]|_{x_i+\frac{1}{2}1} = \alpha_1[\omega_{l-1}]|_{x_i+\frac{1}{2}1}, \]

(30b)

\[
\{\omega_l\}|_{x_i+\frac{1}{2}1} = 0, \quad i \in \mathbb{Z}_{N_x}, \]

(30c)
where $\alpha_1$ is defined in (13), $(w,v)_{x^\tau} = \int_{x^\tau} (uv) (x,\cdot) \, dx$, and

$$B(\omega,v)_{x^\tau} = \langle (\partial_t - \partial_y^2) \omega, v \rangle_{x^\tau} - \langle f'_1(u)\omega, \partial_x v \rangle_{x^\tau} + \langle (f'_2(u)\omega)_y, v \rangle_{x^\tau}.$$  

The properties of the proposed correction function $\omega_l$, $1 \leq l \leq k - 1$ in the following theorem are essential to the proof of superconvergence; see Lemma 3.1 below.

**Lemma 3.1.** The function $\omega_l$, $1 \leq l \leq k - 1$ defined in (30) is uniquely determined. Moreover, if $u \in H^{k+l+2}(\Omega)$ and $f(u) \in C^2(\Omega)$, then

$$\|\partial^r \omega_l\| \lesssim h^{k+l+1} \|u\|_{k+l+r}, \quad \|\partial^r \omega_l\| \lesssim h^{k+l+1} \|u\|_{k+l+r}, \quad \forall r \geq 0. \quad (31)$$

**Proof.** Since $\omega_l|_{x^\tau} \in \mathbb{P}_k$, we suppose for any $y \in [0,2\pi]$ that

$$\omega_l(x,y) := \sum_{m=0}^{k} c_{i,m}^l(y,t)L_{i,m}(x), \quad \forall x \in x^\tau,$$

where $L_{i,m}(x)$ is the Legendre polynomial of degree $m$ in interval $x^\tau$. For any $v \in \mathbb{P}^{k-2}(x^\tau)$, noticing that $\partial_y^2 v \in \mathbb{P}^{k-2}(x^\tau)$, then we choose $\partial_y^2 v = L_{i,m}$, $m \in \mathbb{Z}_{k-2}$ in (30a) to get

$$c_{i,m}^l = \frac{2m+1}{2h_i^2} \left( \langle (\partial_t - \partial_y^2) \omega_{l-1} + (f'_2(u)\omega_{l-1})_y, D_x^{-1}D_x^{-1}L_{i,m} \rangle_{x^\tau} - \langle f'_1(u)\omega_{l-1} + D_x^{-1}L_{i,m} \rangle_{x^\tau} \right). \quad (32)$$

It remains to determine $c_{i,k-1}, c_{i,k}$. In light of the conditions in (30b)-(30c), we get

$$\sum_{m=k-1}^{k} (L_m(1)c_{i,m}^l + L_m(-1)c_{i+1,m}^l) = b_1^l, \quad b_1^l = -\sum_{m=0}^{k-2} (L_m(1)c_{i,m}^l + L_m(-1)c_{i+1,m}^l),$$

$$\sum_{m=k-1}^{k} (g_0(m)c_{i,m}^l + g_1(m)c_{i+1,m}^l) = b_2^l, \quad b_2^l = h \alpha_1 [\omega_1^{-1}]_{x^\tau} - \sum_{m=0}^{k-2} (g_0(m)c_{i,m}^l + g_1(m)c_{i+1,m}^l),$$

where

$$g_0(m) = -\beta_0 L_m(1) + L'_m(1) - 4\beta_1 L''_m(1), \quad g_1(m) = \beta_0 L_m(-1) + L'_m(-1) + 4\beta_1 L''_m(-1).$$

Set an $N_x \times N_x$ block circulant matrix called $M$, with the first row $[A \ B \ 0 \ \cdots \ 0]$ and the last row $[B \ 0 \ \cdots \ 0 \ A]$, where $O$ is an $2 \times 2$ zero matrix, and

$$A = \begin{pmatrix} L_{k-1}(1) & L_k(1) \\ g_0(k-1) & g_0(k) \end{pmatrix}, \quad B = \begin{pmatrix} L_{k-1}(-1) & L_k(-1) \\ g_1(k-1) & g_1(k) \end{pmatrix}. \quad (33)$$

Then the coefficients $c = (c_1, \cdots, c_{N_x})^T$ with $c_i = (c_{i,k-1}^l, c_{i,k}^l)$ satisfy the linear system

$$M c = b, \quad b = (b_1, \cdots, b_{N_x})^T, \quad b_i = (b_1^l, b_2^l).$$
It is proved in \[32\] that \(M\) is non-singular when \(\beta_0 \geq \Gamma(\beta_1)\). Therefore, \(c^l_{i,k-1}, c^l_{i,k}\) are uniquely determined and thus the uniqueness of \(\omega_l\) follows.

We next estimate the coefficients \(c^l_{i,m}\). By a direct calculation from \[32\] and \[24\], we get

\[
|c^l_{i,m}| \leq h\left(\|(\partial_t - \partial^2_y)\omega_{l-1}\|_{0,1,\tau^x} + \|\partial_y \omega_{l-1}\|_{0,1,\tau^x}\right) + \|\omega_{l-1}\|_{0,1,\tau^x}, \ m \leq k - 2.
\]

Consequently, for all \(\tau_{i,j} = \tau^x_i \times \tau^y_j\),

\[
\|c^l_{i,m}\|_{0,\tau^y_j} \leq h^{3/2}\left(\|(\partial_t - \partial^2_y)\omega_{l-1}\|_{0,\tau_{i,j}} + \|\partial_y \omega_{l-1}\|_{0,\tau_{i,j}}\right) + h^{3/2}\|\omega_{l-1}\|_{0,\tau_{i,j}}, \ m \leq k - 2. \tag{34}
\]

On the other hand, since \(M\) is a circulant matrix, we have from \[33\] that

\[
\|c\|_0^2 := \sum_{i=1}^{N_x} \left(\|c^l_{i,k-1}\|_0^2 + |c^l_{i,k}|^2\right) \leq \|M^{-1}\|_0^2 \|b\|_0^2 \leq \sum_{i=1}^{N_x} \left(\|b_1^l\|_0^2 + \|b_2^l\|_0^2\right),
\]

which yields

\[
\sum_{i=1}^{N_x} \left(\|c^l_{i,k-1}\|_{0,\tau^y_j}^2 + |c^l_{i,k}|_{0,\tau^y_j}^2\right) \leq \sum_{i=1}^{N_x} \left(\|b_1^l\|_{0,\tau^y_j}^2 + \|b_2^l\|_{0,\tau^y_j}^2\right).
\]

Note that

\[
\sum_{i=1}^{N_x} \left(\|b_1^l\|_{0,\tau^y_j}^2 + \|b_2^l\|_{0,\tau^y_j}^2\right) \leq \sum_{i=1}^{N_x} \sum_{m=0}^{k-2} \|c^l_{i,m}\|_{0,\tau^y_j}^2 + h^2(\omega_{l-1})_{\hat{x}_{i+\frac{1}{2}}}, (\omega_{l-1})_{\hat{x}_{i+\frac{1}{2}}})_{0,\tau^y_j}
\]

\[
\leq \sum_{i=1}^{N_x} \sum_{m=0}^{k-2} \|c^l_{i,m}\|_{0,\tau^y_j}^2 + h\|\omega_{l-1}\|_{0,\tau_{i,j}}^2.
\]

Then

\[
\sum_{i=1}^{N_x} \left(\|c^l_{i,k-1}\|_{0,\tau^y_j}^2 + |c^l_{i,k}|_{0,\tau^y_j}^2\right) \leq \sum_{i=1}^{N_x} \sum_{m=0}^{k-2} \|c^l_{i,m}\|_{0,\tau^y_j}^2 + h\|\omega_{l-1}\|_{0,\tau_{i,j}}^2,
\]

which yields, together with \[34\] that

\[
\sum_{i=1}^{N_x} \sum_{m=0}^{k-2} \|c^l_{i,m}\|_{0,\tau^y_j}^2 \leq h \sum_{i=1}^{N_x} \left(\|\omega_{l-1}\|_{0,\tau_{i,j}}^2 + h^2\|(\partial_t - \partial^2_y)\omega_{l-1}\|_{0,\tau_{i,j}}^2 + h^2\|\partial_y \omega_{l-1}\|_{0,\tau_{i,j}}^2\right).
\]

Consequently,

\[
\|\omega_l\|_0^2 \leq \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{m=0}^{k} h_{\hat{x}}^2 \sum_{m=0}^{k} \|c^l_{i,m}\|_{0,\tau^y_j}^2 \leq h^2(\|\omega_{l-1}\|_0^2 + \|\partial_t - \partial^2_y\|_{0,\tau_{i,j}}^2 + \|\partial_y \omega_{l-1}\|_{0,\tau_{i,j}}^2).
\]

Taking time derivative or \(y\)-direction derivative on both side of the above equation and...
following the same argument, we have for all \( m, r \geq 0 \)

\[
\| (\partial_t^r + \partial_y^m) \omega_t \|_0 \lesssim h (\| (\partial_t^r + \partial_y^m) \omega_{t-1} \|_0 + h^2 (\| (\partial_t^r + \partial_y^m) (\partial_t - \partial_y^0) \omega_{t-1} \|_0 + h^2 \| \partial_y (\partial_t^r + \partial_y^m) \omega_{t-1} \|_0).
\]

By choosing \( l = 1 \) and using the estimate of the projection \( P(x) \), we obtain

\[
\begin{align*}
\| \omega_t \|_0 & \lesssim h (\| E^x u \|_0 + h^2 (\| \partial_t - \partial_y^0 \| E^x u \|_0 + h^2 \| \partial_y E^x u \|_0) \lesssim h^{k+2} \| u \|_{k+1}, \\
\| \partial_y \omega_t \|_0 & \lesssim h^{k+2} \| \partial_t u \|_{k+1} \lesssim h^{k+2} \| u \|_{k+3}, \quad \| \partial_y^r \omega_t \|_0 \lesssim h^{k+2} \| u \|_{k+1+r}, \quad r \leq 2.
\end{align*}
\]

Then (31) follows from the method of recursion. \( \Box \)

**Theorem 3.1.** Suppose all the conditions of Lemma 3.1 hold. Let \( u \in H^{k+p+2} (\Omega) \) be the solution of (1), \( u_h \) be the numerical solution of (2), and \( \omega_l, l \leq p \leq k - 1 \) be defined in (30). Then for all \( v \in V^k_h \),

\[
| a(E^x u + \sum_{l=1}^p Q^y_h \omega_l, v) | \leq C h^{2(k+p+1)} \| u \|_{k+p+2}^2 + \frac{\gamma}{4} \| v \|_E^2 + \| v \|_0^2.
\] (35)

**Proof.** First, we observe that

\[
a(E^x u + \sum_{l=1}^p Q^y_h \omega_l, v) = a(E^x u + \sum_{l=1}^p \omega_l, v) - \sum_{l=1}^p a(\omega_l - Q^y_h \omega_l, v).
\] (36)

We next estimate the two terms appeared in the right side of the above equation.

**Step 1:**

Let

\[
b(w, v) = \sum_{\tau \in T_h} ( (\partial_t - \partial_y^0) w, v \tau ) - (f_1'(u) w, v \tau ) + ((f_2'(u) w) y, v \tau ) - \sum_{\tau_{i,j}} \int_{y} \alpha _{i} [w][v]_{x+i,y+j} dy.
\]

Using the Cauchy-Schwarz inequality and the inverse inequality yields

\[
|b(w, v)| \lesssim C (\| \partial_t w \|_0^2 + \sum_{r=0}^2 \| \partial_y^r w \|_0^2 + \| v \|_0^2 + h \sum_{\tau_{i,j}} \int_{y} [w]^2_{x+i,y+j} dy + \frac{\gamma}{8} \| v \|_E^2
\]

\[
\lesssim C (\| \partial_t w \|_0^2 + \sum_{r=0}^2 \| \partial_y^r w \|_0^2 + \| v \|_0^2) + \frac{\gamma}{8} \| v \|_E^2.
\]

On the other hand, by (31) and (29), we derive

\[
|a(E^x u + \sum_{l=1}^p \omega_l, v)| = |b(\omega_p, v)| \lesssim C (\| \partial_t \omega_p \|_0^2 + \sum_{r=0}^2 \| \partial_y^r \omega_p \|_0^2 + \| v \|_0^2) + \frac{\gamma}{8} \| v \|_E^2.
\]

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In light of (31), we obtain
\[ |a(E^x u + \sum_{i=1}^{p} \omega_l, v)| \leq C h^{2(k+p+1)} \|u\|_{k+p+2}^2 + C \|v\|_0^2 + \frac{\gamma}{8} \|v\|_{E^y}^2. \]

**Step 2:** Recalling the definition of \( a(\cdot, \cdot) \), we have
\[ a(\omega_l - Q_h^y \omega_l, v) = (\partial_l (\omega_l - Q_h^y \omega_l), v) + A(\omega_l - Q_h^y \omega_l, v) + \mathcal{F}(\omega_l - Q_h^y \omega_l, v). \quad (37) \]

In light of (26) and (31), we easily get for all \( m \leq k + 1 \) that
\[ |(\partial_l (\omega_l - Q_h^y \omega_l), v)| \lesssim h^m \|\partial_l \omega_l\|_0 \|v\|_0 \lesssim h^{k+l+m+1} \|u\|_{k+l+m+2} \|v\|_0. \quad (38) \]

To estimate the term \( A(\omega_l - Q_h^y \omega_l, v) \), we first note that \( Q_h^y \) is the Lobatto projection along \( y \)-direction, which yields
\[ \{Q_h^y \omega_l\}(x_{i+\frac{1}{2}}, y) = \{\omega_l\}(x_{i+\frac{1}{2}}, y), \quad (\hat{\omega}_l T)(x_{i+\frac{1}{2}}, y) = (Q_h^y \omega_l)(x_{i+\frac{1}{2}}, y). \]

For convenience, we adopt the following notation
\[
\|v\|_{E_x}^2 = (v_x, v_x) + \sum_{\tau, j \in T_h} \int_{\tau} \beta_0 \frac{h}{h} [v]^2_{x_{i+\frac{1}{2}}} dy, \quad \|v\|_{E_y}^2 = (v_y, v_y) + \sum_{\tau, j \in T_h} \int_{\tau} \beta_0 \frac{h}{h} [v]^2_{y_{j+\frac{1}{2}}} dx.
\]

which implies that \( \|v\|_{E}^2 = \|v\|_{E_x}^2 + \|v\|_{E_y}^2 \).

Then, we use (36) and the integration by parts to derive
\[ A(\omega_l - Q_h^y \omega_l, v) = -(\omega_l - Q_h^y \omega_l, v_{xx}) + (\partial_y (\omega_l - Q_h^y \omega_l), v_y) + \sum_{\tau, j} \int_{\tau} \partial_y (\omega_l - Q_h^y \omega_l)[v]_{y_{j+\frac{1}{2}}} dx. \]

Then, a direct calculation from the Cauchy-Schwarz and the inverse inequalities yields
\[
|A(\omega_l - Q_h^y \omega_l, v)| \leq C(h^{-4} \|\omega_l - Q_h^y \omega_l\|_0^2 + \|v\|_0^2 + h \sum_{\tau, j} \int_{\tau} \partial_y (\omega_l - Q_h^y \omega_l)_{y_{j+\frac{1}{2}}} dx) + \frac{\gamma}{8} \|v\|_{E_y}^2 \leq C(h^{-4} \|\omega_l - Q_h^y \omega_l\|_0^2 + \|v\|_0^2 + \sum_{r=0}^{2r+1} h^{2r-1} \|\partial_y (\omega_l - Q_h^y \omega_l)\|_{0, \infty}^2) + \frac{\gamma}{8} \|v\|_{E_y}^2 \leq C h^{2(k+l+m-1)} \|u\|_{k+l+m}^2 + \frac{\gamma}{8} \|v\|_{E_y}^2 + C \|v\|_0^2, \quad m \leq k. \quad (39) \]

Here in the last step, we have used the estimates (31) and (26) with \( Q_h^x \) replaced by \( Q_h^y \).
Choosing function \( \alpha \) where

\[
\mathcal{F}(\omega_l - Q^\mu_h \omega_l, v) = -(f_1^l(u)(\omega_l - Q^\mu_h \omega_l), v_x) + ((f_2^l(u)(\omega_l - Q^\mu_h \omega_l), v_y)
\]

Then (35) follows from (36).

3.3. Correction function for \( a \)

Similarly, as for the last term \( F \), we use the properties of \( Q^\mu_h \) to obtain

\[
\mathcal{F}(\omega_l - Q^\mu_h \omega_l, v) = -(f_1^l(u)(\omega_l - Q^\mu_h \omega_l), v_x) + ((f_2^l(u)(\omega_l - Q^\mu_h \omega_l), v_y)
\]

\[
- \sum_{i,j} \int_{\Omega} \alpha_1 [\omega_l - Q^\mu_h \omega_l] [v] |x_1 + \frac{1}{4} dy.
\]

Again we use (31), the Cauchy-Schwarz and the inverse inequalities and then derive

\[
|\mathcal{F}(\omega_l - Q^\mu_h \omega_l, v)| \leq \|\omega_l - Q^\mu_h \omega_l\|_0 |v_x|_{E_2} + \|\partial_y (\omega_l - Q^\mu_h \omega_l)\|_0 |v|_0
\]

\[
\leq C h^{2(k+l+m-1)} |u|_{k+l+m}^2 + \frac{\gamma \epsilon_i}{8} |v|_{E}^2 + C |v|_0^2, \quad m \leq k.
\]

Substituting (38)-(40) into (37) and taking \( l + m = p + 2 \) yields

\[
|a(\omega_l - Q^\mu_h \omega_l, v)| \leq C h^{2(k+p+1)} |u|_{k+p+2}^2 + \frac{\gamma \epsilon_i}{8} |v|_{E}^2 + C |v|_0^2.
\]

Choosing \( \sum_{l=1}^p \epsilon_l = 1 \), there holds

\[
\sum_{l=1}^p |a(\omega_l - Q^\mu_h \omega_l, v)| \leq C h^{2(k+p+1)} |u|_{k+p+2}^2 + \frac{\gamma \epsilon_i}{8} |v|_{E}^2 + C |v|_0^2.
\]

Then (35) follows from (36).

3.3. Correction function for \( a(E^y u, v) \)

By the same argument as what we did for \( a(E^x u, v) \) in (29), we can define the correction function \( \bar{w}_l, l \leq k - 1 \) for the term \( a(E^y u, v) \). Denote \( \bar{w}_0 = E^y u \) and \( \bar{w}_l \) is defined as

\[
(\bar{w}_l, \partial^2 y v)_{\tau^y_j} = \bar{B}(\bar{w}_{l-1}, v)_{\tau^y_j}, \quad \forall v \in \mathbb{P}^k(\tau^y_j) \setminus \mathbb{P}^1(\tau^y_j), j \in \mathbb{Z}_{N_y},
\]

\[
(\bar{w}_l)|_{y_j + \frac{1}{2}} := \beta_0(h)^{-1} [\bar{w}_l] + \{ \partial_y \bar{w}_l \} + \beta_1 h [\partial^2_y \bar{w}_l]|_{y_j + \frac{1}{2}} = \alpha_2 [\omega_l^{-1}]|_{y_j + \frac{1}{2}},
\]

\[
\{\bar{w}_l\}|_{y_j + \frac{1}{2}} = 0,
\]

where \( \alpha_2 \) is given in (15) and

\[
\bar{B}(w, v)_{\tau^y_j} = \langle \{ \partial_x - \partial_x^2 \} w, v \rangle_{\tau^y_j} - \langle f_2^l(u)w, \partial_y v \rangle_{\tau^y_j} + \langle \partial_x (f_1^l(u)w), v \rangle_{\tau^y_j}.
\]

Theorem 3.2. Suppose all the conditions of Lemma 3.1 hold. Let \( u \in H^{k+p+2}(\Omega) \) be the solution of (1), \( u_h \) be the numerical solution of (2), and \( \bar{w}_l, l \leq p \leq k - 1 \) be defined in
Then for all $l \leq p \leq k - 1$,
\[
\|\partial^l \omega_i\| \lesssim h^{k+l+1}\|u\|_{k+l+2}, \quad \|\partial^l \omega_j\| \lesssim h^{k+l+1}\|u\|_{k+l+r}, \quad \forall r \geq 0.
\] (42)

Moreover, there holds for all $v \in V_h^k$
\[
|a(E^y u + \sum_{l=1}^p \bar\omega_l, v)| \leq Ch^{2(k+p+1)}\|u\|^2_{k+p+2} + \frac{\gamma}{4}\|v\|_E^2 + \|v\|_0^2.
\] (43)

Here we omit the proof since it is similar to that for $a(E^x u + \sum_{l=1}^p \omega_l, v)$.

3.4. Estimate

Define the final correction function
\[
\omega := \omega^p = \sum_{l=1}^p (Q^y_l \bar\omega_l + Q^x_l \omega_l),
\] (44)

where $\omega_l$ and $\bar\omega_l$ are defined in (30) and (41), respectively.

**Theorem 3.3.** Assume that $u \in H^{k+p+3}$, $1 \leq p \leq k - 1$ is the solution of (1) and $\Pi_h u$ is the projection of $u$ defined in (20). Suppose that $f(u) \in C^2$. Then
\[
|a(u - \Pi_h u + \omega^p, v)| \leq Ch^{2(k+p+1)}\|u\|^2_{k+p+3} + \frac{3\gamma}{4}\|v\|_E^2 + C\|v\|_0^2, \quad 1 \leq p \leq k - 1.
\] (45)

**Proof.** As a direct consequence of (22), (35) and (43), we have for all $v \in V_h^k$
\[
|a(u - \Pi_h u + \omega^p, v)| \leq Ch^{2(k+p+1)}\|u\|^2_{k+p+2} + \frac{\gamma}{2}\|v\|_E^2 + 2\|v\|_0^2 + |a(E^x E^y u, v)|.
\] (46)

To estimate $a(E^x E^y u, v)$, we use the properties of $E^x$ and $E^y$ and the integration by parts to derive
\[
A(E^x E^y u, v) = 0, \quad \mathcal{F}(E^x E^y u, v) = (f_1'(u) E^x E^y u, v_x) + (f_2'(u) E^x E^y u, v_y),
\]
and thus
\[
|a(E^x E^y u, v)| = |(\partial_t E^x E^y u, v) - (f_1'(u) E^x E^y u, v_x) - (f_2'(u) E^x E^y u, v_y)| \\
\leq \frac{\gamma}{4}\|v\|_E^2 + \|v\|_0^2 + C(\|E^x E^y u\|^2_0 + \|\partial^2 E^x E^y u\|_0).
\]

Here $C$ is a constant independent of $h$. Using the estimates of $E^x$, $E^y$, we get
\[
\|E^y E^x u\|_0 \lesssim h^m \|\partial^m E^x u\|_0 \lesssim h^{m+r} \|\partial^m \partial^r E^x u\|_0, \quad 0 \leq m, r \leq k + 1.
\]
Similarly, there holds
\[ \|E^y E^x u_t\|_0 \lesssim h^{m+r} \|\partial_y \partial_x u_t\|_0 \lesssim h^{m+r} \|u\|_{m+r+2}, \quad 0 \leq m, r \leq k+1. \]
Consequently,
\[ |a(E^x E^y u, v)| \leq \frac{\gamma}{4} \|v\|_0^2 + C \|v\|_0^2 + C h^{2(k+1+p)} \|u\|_{k+p+3}^2. \] (47)
Substituting (47) into (46) yields the desired result (46).

4. Superconvergence

Define
\[ u_I := u_I = \Pi_h u - \omega^p. \] (48)
To study the superconvergence behavior of the DDG solution, we begin with the analysis of the supercloseness between the projection function \( u_I \) and \( u_h \).

**Theorem 4.1.** Let \( u \in H^{k+p+3} \) with \( 1 \leq p \leq k-1 \) be the solution of (1), and \( u_I \) the special projection of \( u \) defined in (48). Assume that \( u_h \) is the solution of the DDG scheme (2) with the initial solution chosen such that
\[ \|u - u_I\|(0) \lesssim h^{k+p+1} \|u_0\|_{k+p+3}. \] (49)
Then
\[ \|u_t - u_h\|_0(t) \lesssim h^{k+p+1} \|u\|_{k+p+3}, \quad t > 0. \] (50)
**Proof.** As a direct consequence of (17)-(18) and (45),
\[ \frac{1}{2} \frac{d}{dt} \|\xi\|_0^2 \lesssim C \|\xi\|_0^2 + C h^{2(k+1+p)} \|u\|_{k+p+3}^2. \] (51)
Then the desired result follows from the Gronwall inequality and (49).

4.1. Superconvergence of numerical fluxes at nodes and for the cell averages

**Theorem 4.2.** Let \( u \in H^{k+p+3} \) with \( 0 \leq p \leq k-1 \) be the solution of (1), and \( u_I \) the special projection of \( u \) defined in (48). Assume that \( u_h \) is the solution of the DDG scheme (2) with the initial solution chosen such that (49) holds with \( p = k-1 \). Then
\[ e_n := \left( \frac{1}{N_x N_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (u - \{u_h\})^2(x_{i+1/2}, y_{j+1/2}, t) \right)^{1/2} \lesssim h^{2k} \|u\|_{2k+2}. \] (52)
Proof. On the one hand, by choosing $p = k - 1$ in (50), we obtain
\[ \|u^k - u_h\|(t)_0 \lesssim h^{2k}\|u\|_{2k+2}. \] (53)

On the one hand, we directly obtain from (30c) and (41c)
\[ \{Q_h^{k}\omega_1\}(x_i+\frac{1}{2}, y) = \{\omega_1\}(x_i+\frac{1}{2}, y) = 0, \{Q_h^{k}\bar{\omega}_1\}(x, y_{i+\frac{1}{2}}) = \{\bar{\omega}_1\}(x, y_{j+\frac{1}{2}}, y) = 0. \]

Consequently,
\[ \{\omega^p\}(x_i+\frac{1}{2}, y_{j+\frac{1}{2}}) = 0, \forall p \leq k - 1, \]
and thus
\[ \{u\}(x_i+\frac{1}{2}, y_{j+\frac{1}{2}}) = \{\Pi_h u\}(x_i+\frac{1}{2}, y_{j+\frac{1}{2}}) = \{u^k\}(x_i+\frac{1}{2}, y_{j+\frac{1}{2}}). \]

Then a direct calculation from the inverse inequality yields that
\[ e_n = \left( \frac{1}{N_xN_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\{u^{k-1}\} - \{u_h\})^2(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t) \right)^{\frac{1}{2}} \lesssim \|u^{k-1} - u_h\|_0. \]

Then (52) follows from (53). \( \square \)

4.2. Superconvergence at Lobatto and Gauss points

Denote by $L_\mu$ the Legendre polynomial of degree $\mu$, and $\{\varphi_\mu\}_{\mu=0}^{\infty}$ the series of Lobatto polynomials, on the interval $[-1, 1]$. That is, $\varphi_0 = \frac{1-s}{2}$, $\varphi_1 = \frac{1+s}{2}$, and
\[ \varphi_{\mu+1} = \int_{-1}^{s} L_\mu(s')ds', \mu \geq 1, s \in [-1, 1]. \]

Let $L_{i,\mu}$ and $\varphi_{i,\mu}$ be the Legendre and Lobatto polynomials of degree $\mu$ on the interval $\tau_i^x$ respectively. That is,
\[ L_{i,\mu}(x) = L_\mu(\xi), \varphi_{i,\mu}(x) = \varphi_\mu(\xi), \xi = \frac{2x - x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}{2}, \mu \geq 0. \]

Zeros of $L_{i,k}$ and $\varphi_{i,k+1}$ are called the Gauss points of degree $k$ and Lobatto points of degree $k + 1$ on $\tau_i^x$, respectively. By tensor product, we can obtain the $k^2$ Gauss points $G_\tau$ and $(k + 1)^2$ Lobatto points $L_\tau$ on each interval $\tau \in T_h$. We denote by $G$ and $L$ the set of Gauss and Lobatto points on the whole domain, respectively.

In each element $\tau_i^x$, denote by $I(x)v \in P_k(x)$ the Gauss-Lobatto projection of $v$ along the $x$-direction satisfying
\[ I(x)v|_{\tau_i^x} = \sum_{\mu=0}^{k} v_{i,\mu} \varphi_{i,\mu}(x), \] (54)

where
\[ v_{i,0} = v(x_{i-\frac{1}{2}}), v_{i,1} = v(x_{i+\frac{1}{2}}), \quad v_{i,\mu} = \frac{2\mu - 1}{2} \langle v_x, L_{i,\mu-1}\rangle_{\tau_i^x}, \mu \geq 2. \]
Similarly, we can define the Gauss-Lobatto projection $I^{(y)}$ of $v$ along the $y$-direction. Define

$$I_h v = I^{(x)} \otimes I^{(y)} v.$$  \hfill (55)

**Lemma 4.1.** For any function $v \in H^{k+2}$, if $\beta_1 = \frac{1}{2k(k+1)}$, then

$$||\Pi_h v - I_h v||_0 \lesssim h^{k+2}||v||_{k+2}.$$  \hfill (56)

**Proof.** When $\beta = \frac{1}{2k(k+1)}$, there holds for all $s \leq k$ that (see, e.g., [13])

$$||P^{(x)}v - I^{(x)}v||_{0,\tau_x} \lesssim h^{s+2}||\partial_{x}^{s+2}v||_{0,\tau_x}, \quad ||P^{(y)}v - I^{(y)}v||_{0,\tau_y} \lesssim h^{s+2}||\partial_{y}^{s+2}v||_{0,\tau_y}.$$

Then

$$||\Pi_h v - I_h v||_0 = ||P^{(x)}(P^{(y)}v) - I^{(x)}(I^{(y)}v)||_0 \leq ||P^{(x)}(P^{(y)}v - I^{(y)}v)||_0 + ||I^{(y)}(P^{(x)}v - I^{(x)}v)||_0 \lesssim h^{k+2}||v||_{k+2}.$$  \hfill (57)

Here in the last step, we have used the fact both the Gauss-Lobatto projection $I^{(y)}$ and the projection $P^{(x)}$ are bounded. The proof is complete.  \hfill \square

Now we are ready to present the superconvergence of DDG solution at Gauss points and Lobatto points.

**Theorem 4.3.** Let $u \in H^{k+4}$ be the solution of (1), and $u_h$ be the solution of (2) with the initial value $u_h(\cdot, \cdot, 0)$ chosen such that (49) holds with $p = 1$. If $\beta_1 = \frac{1}{2k(k+1)}$, then for any fixed $t \in (0, T]$

$$e_g : = \left( \frac{1}{N_x N_y k^2} \sum_{\tau \in T_h} \sum_{z \in G_{\tau}} |\nabla (u - u_h)(z, t)| \right)^{\frac{1}{2}} \lesssim h^{k+1}||u||_{k+4},$$  \hfill (57)

$$e_l : = \left( \frac{1}{N_x N_y (k+1)^2} \sum_{\tau \in T_h} \sum_{z \in L_{\tau}} (u - u_h)(z, t) \right)^{\frac{1}{2}} \lesssim h^{k+2}||u||_{k+4}.$$  \hfill (58)

**Proof.** First, we choose $p = 1$ in (50) and use the triangle inequality to derive

$$||P_h u - u_h||_0 \lesssim ||u^1 - u_h||_0 + ||\omega^1||_0 \lesssim h^{k+2}||u||_{k+4},$$

which yields, together with (4.1) that

$$||I_h u - u_h||_0 \lesssim h^{k+2}||u||_{k+4}.$$  \hfill (59)
Using the inverse inequality, we have
\[ \| I_h u - u_h \|_1 \lesssim h^{k+1} \| u \|_{k+4}. \] (60)

On the other hand, there holds (see, e.g., [17]) that
\[ |(u - I_{h} u)(z_0)| \lesssim h^{k+2} |u|_{k+2, \infty}, \quad |\nabla (u - I_{h} u)(z_1)| \lesssim h^{k+1} |u|_{k+2, \infty}, \quad \forall z_0 \in L, z_1 \in G. \]

Then
\[ e_l \lesssim \left( \frac{1}{N_x N_y} \sum_{\tau \in T_h} \sum_{z \in L} (I_{h} u - u_h)^2(z, t) \right)^{\frac{1}{2}} + \max_{z \in L} |(u - I_{h} u)(z, t)| \]
\[ \lesssim \left( h^{-2} \sum_{\tau \in T_h} \| I_{h} u - u_h \|_{0, 0, \tau}^2 \right)^{\frac{1}{2}} + h^{k+2} \| u \|_{k+2, \infty} \lesssim \| I_{h} u - u_h \|_1 + h^{k+2} \| u \|_{k+2, \infty}. \]

Here in the last step, we have used the inverse inequality. Similarly, there holds
\[ e_g \lesssim \left( \frac{1}{N_x N_y} \sum_{\tau \in T_h} \sum_{z \in G} \nabla (I_{h} u - u_h)^2(z, t) \right)^{\frac{1}{2}} + \max_{z \in G} |\nabla (u - I_{h} u)(z, t)| \]
\[ \lesssim \| I_{h} u - u_h \|_1 + h^{k+2} \| u \|_{k+2, \infty}. \]

Then the desired result (57) follows from (59) and (60).

**Remark 4.1.** Same as the one-dimensional case, the choice of $\beta_1$ in (1) has influence on the superconvergence phenomenon of the DDG solution $u_h$ at Gauss and Lobatto points. However, it does not affect the superconvergence rate of $u_h$ at nodes. Our numerical examples will demonstrate this fact.

To end with this section, we would like to demonstrate how to obtain the initial solution. To ensure the validity of (59), a nature method for the initial discretization is to choose
\[ u_h(x, y, 0) = \Pi_h u_p(x, y, 0), \quad p \leq k - 1. \]

Then we divide the initial discretization into the following steps:
1. According to the definition of $\Pi_h u$, compute $\omega_l, \bar{\omega}_l, 1 \leq l \leq p$ by (40) and (41).
2. Calculate $Q_h^{\omega_l}, Q_h^{\bar{\omega}_l}, 1 \leq l \leq p$.
3. Compute $\omega^p = \sum_{l=1}^{p} (Q_h^{\omega_l} + Q_h^{\bar{\omega}_l})$.
4. Figure out $u_h(x, y, 0) = \Pi_h u - \omega^p$. 

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5. Numerical tests

In this section, we provide numerical examples to validate our theoretical results. In our experiments, we adopt the DDG method as spatial discretization and the classic fourth order Runge-Kutta method as our time discretization method. The convection flux is chosen as Godnov numerical flux. The CFL condition is taken as \( \tau = \mathcal{O}(h^2) \), where \( \tau \) is the temporal step size. We obtain our meshes by equally dividing \([0, 2\pi] \times [0, 2\pi]\) into \(N \times N\) subintervals with \(N = 4, \ldots, 32\).

Example 5.1. Consider the following equation with the periodic boundary condition:

\[
\begin{align*}
  u_t + \left( \frac{u^2}{2} \right)_x + \left( \frac{u^2}{2} \right)_y &= u_{xx} + u_{yy} + g(x, y, t), & (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \times (0, 1) \\
  u(x, y, 0) &= \sin(x + y), & (x, y, t) \in [0, 2\pi] \times [0, 2\pi],
\end{align*}
\]

where the source term \( g(x, t) \) is chosen such that the exact solution is \( u = e^{(-2t)} \sin(x + y) \).

Table 1: Errors and convergence rates for \( f_1(u) = f_2(u) = \frac{1}{2}u^2 \) and \( \beta_1 = \frac{1}{2k(k+1)} \).

| \( k \) | \( N \times N \) | \( e_t \) rate | \( e_n \) rate | \( e_{g_x} \) rate | \( \|e_h\|_0 \) rate |
|---|---|---|---|---|---|
| 1 | 4 \times 4 | 6.3e-03 — | 2.4e-03 — | 2.6e-02 — | 1.6e-01 — |
| | 8 \times 8 | 1.1e-03 2.5 | 4.6e-04 2.4 | 7.3e-03 1.8 | 4.4e-02 1.8 |
| | 16 \times 16 | 2.5e-04 2.2 | 1.2e-04 2.0 | 1.9e-03 2.0 | 1.1e-02 2.0 |
| | 32 \times 32 | 6.2e-05 2.0 | 3.0e-05 2.0 | 4.7e-04 2.0 | 2.8e-03 2.0 |
| 2 | 4 \times 4 | 4.7e-04 — | 1.7e-04 — | 2.4e-03 — | 1.4e-02 — |
| | 8 \times 8 | 2.1e-05 4.5 | 3.2e-06 5.7 | 2.9e-04 3.0 | 1.7e-03 3.1 |
| | 16 \times 16 | 1.2e-06 4.1 | 1.4e-07 4.5 | 3.6e-05 3.0 | 2.1e-04 3.0 |
| | 32 \times 32 | 7.6e-08 4.0 | 9.5e-09 3.9 | 4.4e-06 3.0 | 2.6e-05 3.0 |
| 3 | 4 \times 4 | 2.7e-05 — | 1.7e-05 — | 2.1e-04 — | 1.2e-03 — |
| | 8 \times 8 | 1.0e-06 4.8 | 3.1e-07 5.8 | 1.4e-05 3.9 | 7.5e-05 4.0 |
| | 16 \times 16 | 3.3e-08 4.9 | 4.9e-09 6.0 | 9.1e-07 4.0 | 4.7e-06 4.0 |
| | 32 \times 32 | 1.1e-09 5.0 | 7.7e-11 6.0 | 5.8e-08 4.0 | 3.0e-07 4.0 |
| 4 | 4 \times 4 | 1.9e-06 — | 2.3e-07 — | 1.7e-05 — | 9.2e-05 — |
| | 8 \times 8 | 2.1e-08 6.4 | 7.8e-10 8.2 | 4.9e-07 5.1 | 2.9e-06 5.0 |
| | 16 \times 16 | 2.9e-10 6.2 | 3.1e-12 8.0 | 1.5e-08 5.0 | 9.0e-08 5.0 |
| | 32 \times 32 | 4.4e-12 6.1 | 1.5e-14 7.7 | 4.7e-10 5.0 | 2.8e-09 5.0 |

Listed in Table 1 are various errors and the corresponding convergence rates for \( k = 1, 2, 3, 4 \), and \( T = 1 \), with the parameter \( (\beta_0, \beta_1) \) taken as \( (12, \frac{1}{4}), (12, \frac{1}{12}), (12, \frac{1}{24}), (12, \frac{1}{40}) \) for \( k = 1, 2, 3, 4 \), respectively. Note that \( \beta_1 = \frac{1}{2k(k+1)} \) in this case. From Table 1, we observe a optimal convergence rate of \((k+1)\)-th order for the \( L^2 \) error \( \|e_h\|_0 \), a superconvergence rates of \( 2k \) for the error \( e_n \) at nodes, \((k+2)\)-th order for the error \( e_l \) at Lobatto points.
Table 2: Errors and convergence rates for $f_1(u) = f_2(u) = \frac{1}{2}u^2$ and $\beta_1 \neq \frac{1}{2k(k+1)}$.

| $k$ | $N \times N$ | $e_l$ rate | $e_n$ rate | $e_{eg}$ rate | $\|e_h\|_0$ rate |
|-----|---------------|------------|------------|------------|----------------|
| 1   | 4 \times 4    | 6.3e-03    | 2.4e-03    | 2.6e-02    | 1.6e-01       |
|     | 8 \times 8    | 1.1e-03    | 4.6e-04    | 7.3e-03    | 1.8e-02       |
|     | 16 \times 16  | 2.5e-04    | 1.2e-04    | 1.9e-03    | 2.0e-02       |
|     | 32 \times 32  | 6.2e-05    | 3.0e-05    | 4.7e-04    | 2.8e-03       |
| 2   | 4 \times 4    | 6.7e-04    | 1.7e-04    | 1.9e-03    | 1.3e-02       |
|     | 8 \times 8    | 9.9e-05    | 3.0e-06    | 2.5e-04    | 2.9e-03       |
|     | 16 \times 16  | 1.3e-05    | 1.2e-07    | 4.6e-05    | 2.4e-04       |
|     | 32 \times 32  | 1.7e-06    | 3.0e-09    | 4.0e-05    | 2.2e-05       |
| 3   | 4 \times 4    | 8.4e-05    | 1.5e-05    | 2.9e-04    | 1.1e-03       |
|     | 8 \times 8    | 3.3e-06    | 4.7e-07    | 5.6e-05    | 3.7e-05       |
|     | 16 \times 16  | 1.1e-07    | 4.9e-09    | 5.9e-05    | 3.9e-05       |
|     | 32 \times 32  | 3.5e-09    | 5.0e-11    | 6.0e-05    | 4.0e-07       |
| 4   | 4 \times 4    | 1.5e-05    | 3.1e-07    | 5.7e-05    | 1.2e-04       |
|     | 8 \times 8    | 5.2e-07    | 1.3e-09    | 7.9e-06    | 4.0e-06       |
|     | 16 \times 16  | 1.7e-08    | 5.0e-12    | 7.9e-05    | 4.0e-07       |
|     | 32 \times 32  | 5.3e-10    | 2.3e-14    | 7.8e-08    | 4.0e-09       |

and $(k + 1)$-th order for the error $e_g$ at Gauss points. These results are consistent with the theoretical results given in Theorems 4.2 and 4.3, which indicates that superconvergence of the function value approximation at Lobatto points and nodes, and superconvergence of the derivative approximation at Gauss points exist.

To test the influence of $\beta_1$ on the superconvergence rate, we also test the case in which $\beta_1 \neq \frac{1}{2k(k+1)}$. Table 2 demonstrates the error and corresponding convergence rate for $k = 1, 2, 3, 4$, with $(\beta_0, \beta_1) = (12, \frac{1}{40}), (12, \frac{1}{4}), (12, \frac{1}{12}), (12, \frac{1}{24})$, respectively. As indicated by Table 2, the convergence rates of $e_l$ and $e_g$ in case for $k = 2, 4$ are separately $k + 1$ and $k$, which indicates that the superconvergence phenomena at Gauss and Lobatto points disappear. While we still observe a superconvergence rate of $2k$-th order for the error $e_n$. In other words, the choice of $\beta_1$ affects the superconvergence of $u_h$ at Gauss and Lobatto points while the superconvergence properties of $u_h$ at nodes are independent of the value of $\beta_1$ in the DDG schemes.

**Example 5.2.** Consider the following equation with the periodic boundary condition:

\[
\begin{align*}
    u_t + (\sin(u))_x + (\sin(u))_y &= u_{xx} + u_{yy} + g(x, y, t), & (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \times (0, 1] \\
    u(x, y, 0) &= \sin(x + y), & (x, y, t) \in [0, 2\pi] \times [0, 2\pi].
\end{align*}
\]

We choose a suitable $g(x, y, t)$ such that the exact solution is

\[ u = e^{(-2t)} \sin(x + y). \]
We list in Table 3 the approximation errors and corresponding convergence rates calculated by the DDG scheme for \( k = 1, 2, 3, 4 \), with the parameter chosen as \((\beta_0, \beta_1) = (12, \frac{1}{4}), (12, \frac{1}{12}), (12, \frac{1}{24}), (12, \frac{1}{40})\), respectively. As the same with Example 1, we observe the function value error is superconvergent at Lobatto points and the derivative error is superconvergent at Gauss points, with an order of \( k + 2 \) and \( k + 1 \), respectively. Moreover, we see the error of numerical fluxes at mesh nodes are superconvergent with an order of \( 2k \). All these numerical results are consistent with the theoretical findings in Theorems 4.2 and 4.3. In other words, the error bounds established in [42] and [47]-[58] are sharp.

6. Conclusion

In this paper, we investigate superconvergence properties of the DDG method for two-dimensional nonlinear convection-diffusion equations using tensor product meshes and tensor product polynomials of degree \( k \). We prove that, with suitable initial discretizations, the error between the DDG solution and the exact solution converges with an order of \( (k + 2) \) at Lobatto points, with \( (2k) \) at nodes, and the convergence rate of derivative approximation at Gauss points can achieve \( (k + 1) \). Numerical experiments demonstrate that all the established error bounds are optimal.
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