Locating-chromatic number of the edge-amalgamation of trees

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Abstract

The investigation on the locating-chromatic number for graphs was initially studied by Chartrand et al. on 2002. This concept is in fact a special case of the partition dimension for graphs. Even though this topic has received much attention, the current progress is still far from satisfaction. We can define the locating-chromatic number of a graph $G$ as the smallest integer $k$ such that there exists a proper $k$-coloring on the vertex-set of $G$ such that all vertices have distinct coordinates (color codes) with respect to this coloring. Not like the metric dimension of any tree which is completely solved, the locating-chromatic number for most types of trees are still open. In this paper, we study the locating-chromatic number of trees. In particular, we give lower and upper bounds of the locating-chromatic number of trees formed by an edge-amalgamation of the collection of smaller trees. We also show that the bounds are tight.

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1. Introduction

The topic of locating-chromatic number of graphs was introduced by Chartrand et al. [5] on 2002. They determined the locating-chromatic numbers of some well-known classes of graphs,
i.e., paths, cycles, and double stars. They also characterized all graphs of order $n$ with locating-chromatic number $n$, i.e. multipartite complete graphs. This topic has received much attention. Inspired by Chartrand et al., other authors have determined the locating-chromatic numbers of some well-known classes of graphs. But the results are still limited. In particular for trees, the locating-chromatic number for most types of trees are still open. Some classes of trees with their locating-chromatic numbers known are amalgamations of stars and firecrackers by Asmiati et al. [1, 2], homogeneous lobsters and binary trees by Syofyan et al. [6, 7], and complete $n$-ary trees by Welyyanti et al. [9]. Furthermore, all trees on $n$ vertices with locating-chromatic number 3 or $n - t$ where $2 \leq t < \frac{n}{2}$ have been successfully characterized, see [4] and [8], respectively. In this paper, our aim is to determine the locating-chromatic number of the edge-amalgamation of trees. We then estimate the locating-chromatic numbers for some structures of trees obtained by the edge-amalgamation of trees.

Throughout this paper, we only deal with connected graphs. Let $G = (V, E)$ be a connected graph. For $u, v \in V(G)$, let $d(u, v)$ denote the distance between $u$ and $v$. A $k$-coloring of $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any two adjacent vertices $u$ and $v$. In other words, $c$ is a partition $\Pi$ of $V(G)$ into color classes $C_1, C_2, \ldots, C_k$, where the vertices of $C_i$ are colored by $i$ for $1 \leq i \leq k$. The color code of vertex $u$ in $G$, denoted by $c_{\Pi}(u)$, is defined to be the ordered $k$-tuple $(d(u, C_1), d(u, C_2), \ldots, d(u, C_k))$, where $d(u, C_i) = \min\{d(u, x) | x \in C_i\}$ for $1 \leq i \leq k$. If any two distinct vertices of $G$ have distinct color codes, then $c$ is called a locating $k$-coloring of $G$. Moreover, the least integer $k$ such that there is a locating-coloring in $G$ is called the locating-chromatic number of $G$, denoted by $\chi_L(G)$.

The following two results are natural consequences and showed in [5].

**Lemma 1.1.** Let $G$ be a connected non-trivial graph. Let $c$ be a locating coloring of $G$ and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for every $w \in V(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$.

**Corollary 1.1.** If $G$ is a connected graph containing a vertex adjacent to $k$ leaves of $G$, then $\chi_L(G) \geq k + 1$.

### 2. Main Results

For $i = 1, 2, \ldots, t$, let $T_i$ be a tree with a fixed edge $e_{o_i}$ called the terminal edge. The edge-amalgamation of all these trees $T_i$s, denoted by $\text{Edge-Amal}\{T_i; e_{o_i}\}$, is a tree formed by taking all these trees $T_i$s and identifying their terminal edges. In this section, we will derive the (lower and upper) bounds for the locating-chromatic number of the edge-amalgamation of trees.

Let $T$ be a tree. A stem is a vertex in $T$ that is adjacent to a leaf. A pendant edge is an edge in $T$ incident to a leaf in a tree. For any vertices $u$ and $v$ in $T$, we denote by $_uP_v$ the unique path connecting $u$ and $v$. Let $u \in V(T)$ and define $N(u) = \{x \in V(T) | d(u, x) = 1\}$. For a $k$-locating-coloring $c$ of $T$, we denote $c(N(u)) = \{c(v) | v \in N(u)\}$.

For $i = 1, 2, \ldots, t$, let $T_i$ be a tree with a chosen terminal edge $e_{o_i} = s_i l_i$, where $s_i$ is a stem and $l_i$ is a leaf. For any stem $z$ of a tree $T_i$ we denote $N_p(z)$ is the set of pendant vertices adjacent to stem $z$. Let $m_i$ be the number of pendant edges adjacent to stem $s_i$ and $r_i = \max\{|N_p(z)| z \text{ is a stem of } T_i\}$. Next, in $\text{Edge-Amal}\{T_i; e_{o_i}\}$, we denote $s = s_i$ and $l = l_i$. 

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Theorem 2.1. Let Edge-Amal\{T\_i; e\_o\} be an edge-amalgamation of t disjoint trees T\_i. Then, $\max\{r_i + 1, 2 + \sum_{i=1}^{t}(m_i - 1)\} \leq \chi_L(Edge-Amal\{T\_i; e\_o\}) \leq 2 + \sum_{i=1}^{t}(k_i - 2)$.

Proof. For $i = 1, 2, \ldots, t$, let $\chi_L(T\_i) = k_i$. Let $c_i$ be a $k_i$-locating coloring of $T\_i$ such that $c_i(s\_i) = 1$ and $c_i(l\_i) = 2$. Define $A = \{v \in V(T\_i) | c_i(v) = 1, \forall i \in [1, t]\}$ and $B = \{v \in V(T\_i) | c_i(v) = 2, \forall i \in [1, t]\}$. Now, define $c : V(Edge-Amal\{T\_i; e\_o\}) \to \{1, 2, \ldots, 2 + \sum_{i=1}^{t}(k_i - 2)\}$ as follows:

$$c(x) = \begin{cases} 
1, & \text{if } x \in A \\
2, & \text{if } x \in B \\
c_1(x), & \text{if } x \in V(T\_i) \\
(c_1(x) + \sum_{j=2}^{i}(k_j - 2)), & \text{if } x \in V(T\_i) \setminus (A \cup B), \text{ for all } i > 1.
\end{cases}$$

Since the coloring $c$ preserves the locating coloring in every tree $T\_1, T\_2, \ldots, T\_t$, two vertices $u$ and $v$ where $c(u) = c(v)$ and $c(N(u)) = c(N(v))$ only occur for two cases below.

1. $u, v \in V(T\_i)$ for some $i$.
   Then, their color codes are distinguished by the $k_i$-locating coloring $c_i$ of $T\_i$. Therefore, these vertices are also distinguished by $c$.

2. $u \in V(T\_i)$ and $v \in V(T\_j)$ for some $i \neq j$.
   Let $c(u) = c(v) = 1$. Since $c_i$ is a $k_i$-locating coloring and by the definition of the coloring $c$, there exists integer $p \neq 1, 2$ such that $c(x) = p$ for some $x \in N^2(s)$ and $x \in T\_i$. Thus, we have:

$$d_T(u, C_p) \leq d_T(u, s), \quad (1)$$

and

$$d_T(v, s) + 1 \leq d_T(v, C_p) \leq d_T(v, s) + 2. \quad (2)$$

Similarly, consider the subtree $T\_j$. Since $c_j$ is a $k_j$-locating coloring and by the definition of the coloring $c$, there exists integer $q \neq 1, 2$ and $q \neq p$ such that $c(y) = q$ for some $y \in N^2(s)$ and $y \in T\_j$. Thus, we have:

$$d_T(v, C_q) \leq d_T(v, s), \quad (3)$$

and

$$d_T(u, s) + 1 \leq d_T(u, C_q) \leq d_T(u, s) + 2. \quad (4)$$

Now, if $d_T(u, C_p) = d_T(v, C_p)$ then from Eqs (1), (2), (3) and (4), we have that:

$$d_T(v, C_q) < d_T(v, s) + 1 \leq d_T(v, C_p) = d_T(u, C_p) \leq d_T(u, s) < d_T(u, C_q). \quad (5)$$

Thus, we have that $d_T(u, C_q) \neq d_T(v, C_q)$. Therefore, the color codes of $u$ and $v$ are different. A similar argument holds for the case $c(u) = c(v) = 2$.

Thus, all vertices of the Edge-Amal\{T\_i; e\_o\} have distinct color codes. We conclude that

$$\chi_L(Edge-Amal\{T\_i; e\_o\}) \leq 2 + \sum_{i=1}^{t}(k_i - 2).$$
Next, since there is a stem adjacent to \( \max \{ r_i, 1 + \sum_{i=1}^{t} (m_i - 1) \} \) leaves, by Corollary 1.1

\[
\chi_L(\text{Edge-Amal}\{T_i; e_{o_i}\}) \geq \max \{ r_i + 1, 2 + \sum_{i=1}^{t} (m_i - 1) \}.
\]

The following two theorems show the existence of trees formed by an edge-amalgamation operation with the locating-chromatic number equals to the lower or upper bounds of Theorem 2.1. Furthermore, in Theorem 2.4, we give the example of trees formed by an edge-amalgation operation with the locating-chromatic number lies in between upper and lower bounds of Theorem 2.1.

**Theorem 2.2.** If \( \chi_L(T_i) = k_i \) and \( m_i = k_i - 1 \) for any \( i \), then \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t} (\chi_L(T_i) - 2) \).

**Proof.** By using the locating-coloring \( c \) in proof Theorem 2.1, we have \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \leq 2 + \sum_{i=1}^{t} (k_i - 2) \).

Next, since there are \( 1 + \sum_{i=1}^{t} (k_i - 2) \) leaves adjacent to a stem in \( \text{Edge-Amal}(T_i; e_{o_i}) \), by Lemma 1.1 \( \chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq 2 + \sum_{i=1}^{t} (k_i - 2) \). So, we conclude that

\[
\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = 2 + \sum_{i=1}^{t} (k_i - 2).
\]

Let \( G_{w_i} \) be a tree having a pendant \( e_{o_i} \) as depicted in Figure 1, where \( w_i \geq 2 \).

![Figure 1](image)

**Figure 1.** A tree \( G_{w_i} \), where \( w_i \geq 2 \).

**Theorem 2.3.** For \( i = 1, 2, \ldots, t \), let \( T_i = G_{w_i} \). If \( t \leq \max \{ w_i \mid i \in [1, t] \} \), then

\[
\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = \max \{ w_i + 1 \mid i \in [1, t] \}.
\]
Proof. Let $r = \max\{w_i \mid i \in [1, t]\}$. Since there are $r$ leaves adjacent to a stem in $\text{Edge-Amal}(T_i; e_{o_i})$, by Lemma 1.1 $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq r + 1$.

Now, let $T_i = G_{w_i}$ such that $w_1 \leq w_2 \leq \ldots \leq w_t$. We denote $x_i, y_i, z_{ij}$ the non stem vertex, the stem adjacent to $w_i$ leaves, and all leaves adjacent to $y_i$, respectively.

Define a coloring $c : V(\text{Edge-Amal}\{T_i; e_{o_i}\}) \rightarrow \{1, 2, \ldots, r + 1\}$ as follows:

$$c(u) = \begin{cases} 
1, & \text{if } u = s \\
2, & \text{if } u = l \text{ or } u = x_i \text{ for } 1 \leq i \leq t - 1 \text{ and } i \neq 2 \\
3, & \text{if } u = x_2 \\
i, & \text{if } u = y_i \\
j, & \text{if } u = z_{ij} \text{ and } i \neq j \\
r + 1, & \text{if } u = z_{ij} \text{ and } i = j.
\end{cases}$$

By this coloring, any two vertices $u$ and $v$ satisfying $c(u) = c(v)$ and $c(N(u)) = c(N(v))$ only occur for the pair of vertices $s$ and $y_i$ for $w_1 = 2$, and the pair of vertices $l$ and $x_1$. Their color codes are distinguished by the last ordinate (their distances to a vertex in the color class $r + 1$). Hence, all vertices have distinct color codes. So, $\chi_L(\text{Edge-Amal}\{T_i; e_{o_i}\}) \leq \max\{r_i + 1\}$. 

Let $H_m$ be a tree having a pendant $e_{o_i}$ as depicted in Figure 2, where $m \geq 3$.

![Figure 2. A tree $H_m$ where $m \geq 3$.](image)

**Theorem 2.4.** For $i = 1, 2, \ldots, t$, let $T_i = H_m$. We have that $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 2$, if $2 \leq t \leq m$.

**Proof.** Let $t \in [2, m]$. Then, there are $tm$ stems and each is adjacent to $m$ leaves in graph $\text{Edge-Amal}(T_i; e_{o_i})$. We suppose that $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 1$. Then, there are $m + 1$ possibilities to coloring all stems and their neighbors in $\text{Edge-Amal}(T_i; e_{o_i})$. Since $t \geq 2$, there are at least two stems having the same color. Therefore, the color codes of these stems are the same, a contradiction to $\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) = m + 1$. So,

$$\chi_L(\text{Edge-Amal}(T_i; e_{o_i})) \geq m + 2.$$ 

Next, we define a coloring $c : V(\text{Edge-Amal}\{T_i; e_{o_i}\}) \rightarrow \{1, 2, \ldots, m + 2\}$ as follows:
Note that all the colors above in modulo $m$. We will show that $\chi_L(\text{Edge-Amal}(T; e_o)) \leq m + 2$. Let $u$ and $v$ be any two vertices with $c(u) = c(v)$. Then, by the coloring $c$, $c(N(u)) \neq c(N(v))$ because the $m - 1$ neighbors colors of $u$ are permutation of $m - 1$ neighbors colors of $v$ in modulo $m + 2$. Hence, all vertices in $\text{Edge-Amal}(T; e_o)$ have distinct color codes. So, $\chi_L(\text{Edge-Amal}(T; e_o)) \leq m + 2$.

From Theorem 2.3, we shows the exact value of locating-chromatic number for some classes of trees. First, we give definition of some classes of trees and their locating-chromatic number, i.e. double stars, homogeneous caterpillars, and homogeneous lobsters. A double star, denoted by $S_{m,n}$ where $n \geq m \geq 1$, is the graph consisting of two stars $K_{1,n}$ and $K_{1,m}$ together with an edge joining their centers. Chartrand et al. [5] have proved $\chi_L(S_{m,n}) = n + 1$. The homogeneous caterpillar $C(m, n)$ is the graph consisting of $m$ stars $K_{1,n}$ by linking the centers from each stars. Asmiati et al. [3] showed that the locating-chromatic number of homogeneous caterpillar is $n + 1$ for $1 \leq m \leq n + 1$, and $n + 2$ for $m > n + 1$. The homogeneous lobster $Lb(m, n)$ is the graph obtained by attaching the centers of stars $K_{1,n}$ to each leaf of $C(m, n)$. Syofyan et al. [6] showed that the locating-chromatic number of the homogeneous lobster is $n + 1$ if $m = 1$, $n + 2$ for $2 \leq m \leq 3(n - 2) + 1$, or $n + 3$ for $m > 3(n - 2) + 1$.

Based on Theorem 2.3 and the locating-chromatic numbers of double stars, homogeneous caterpillars, and homogeneous lobsters, we have the locating-chromatic number of edge-amalgamation of these trees as follows. The terminal edge in each tree is chosen from the edges incident to a stem having maximum leaves.

**Corollary 2.1.** For $i = 1, 2, \ldots, t$, let $T_i = S_{m,n}$. Then, $\chi_L(\text{Edge-Amal}(T_i; e_o)) = t(n - 1) + 1$, if $n \geq m \geq 1$.

**Corollary 2.2.** For $i = 1, 2, \ldots, t$, let $T_i = C(m, n)$. If $1 \leq m \leq n + 1$, then $\chi_L(\text{Edge-Amal}(T_i; e_o)) = t(n - 1) + 1$.

**Corollary 2.3.** For $i = 1, 2, \ldots, t$, let $T_i = Lb(m, n)$. If $m = 1$, then $\chi_L(\text{Edge-Amal}(T_i; e_o)) = t(n - 1) + 1$.

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