Non-Laplace type Operators on Manifolds with Boundary

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Non-Laplace type operators are second-order elliptic partial differential operators acting on sections of vector bundles over a manifold with boundary with a non-scalar leading symbol. Such operators appear, in particular, in models of non-commutative gravity theories, when instead of a Riemannian metric there is a matrix valued self-adjoint symmetric two-tensor that plays the role of a “non-commutative” metric. We construct the corresponding heat kernel and compute its first two spectral invariants.
1 Introduction

Elliptic differential operators on manifolds play a very important role in mathematical physics, geometric analysis, differential geometry and quantum theory. Of special interest are the resolvents and the spectral functions of elliptic operators; the most important spectral functions being the trace of the heat kernel and the zeta function, which determine, in particular, the functional determinants of differential operators (see [29, 19, 18, 6] and the reviews [5, 7, 43]).

In particular, in quantum field theory and statistical physics the resolvent determines the Green functions, the correlation functions and the propagators of quantum fields, and the functional determinant determines the effective action and the partition function (see, for example [6]). In spectral geometry, one is interested, following Kac [36], in the question: “Does the spectrum of the scalar Laplacian determine the geometry of a manifold” or, more generally, “To what extent does the spectrum of a differential operator on a manifold determine the geometry of the manifold?” Of course, the answer to Kac’s and other questions depends on the differential operator. Most of the studies in spectral geometry and spectral asymptotics are restricted to so-called Laplace type operators. These are second-order partial differential operators acting on sections of a vector bundle with a positive definite scalar leading symbol.

Since, in general, it is impossible to find the spectrum of a differential operator exactly, one studies the asymptotic properties of the spectrum, so-called spectral asymptotics, which are best described by the asymptotic expansion of the trace of the heat kernel. If \( L : C^\infty(V) \rightarrow C^\infty(V) \) is a self-adjoint elliptic second-order partial differential operator with a positive definite leading symbol acting on smooth sections of a vector bundle \( V \) over a compact \( n \)-dimensional manifold \( M \), then there is a small-time asymptotic expansion as \( t \rightarrow 0 \) \cite{33, 29}

\[
\text{Tr}_L \exp(-tL) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} A_k.
\]

The coefficients \( A_k \) are called the global heat invariants (in mathematical literature they are usually called the Minakshisundaram-Pleijel coefficients; in physics literature, they are also called HMDS (Hadamard-Minakshisundaram-De Witt-Seeley) coefficients, or Schwinger-De Witt coefficients).

The heat invariants are spectral invariants of the operator \( L \) that encode the information about the asymptotic properties of the spectrum. They are of great importance in spectral geometry and find extensive applications in physics, where
they describe renormalization and quantum corrections to the effective action in quantum field theory and the thermal corrections to the high-temperature expansion in statistical physics among many other things. They describe real physical effects and, therefore, the knowledge of these coefficients in explicit closed form is important in physics. One would like to have formulas for some lower-order coefficients to be able to study those effects.

The proof [33, 29] of the existence of such an asymptotic expansion (1.1) has been a great achievement in geometric analysis. Now it is a well known fact, at least in the smooth category for compact manifolds. This is not the subject of our interest. The main objective in the study of spectral asymptotics (in spectral geometry and quantum field theory) is, rather, the explicit calculation of the heat invariants \( A_k \) in invariant geometric terms.

The approach of Greiner and Seeley [33, 41] is a very powerful general analytical procedure for analyzing the structure of the asymptotic expansion based on the theory of pseudo-differential operators and the calculus of symbols of operators (we will call it symbolic approach for simplicity). This approach can be used for calculation of the heat invariants explicitly in terms of the jets of the symbol of the operator; it provides an iterative procedure for such a calculation. However, as far as we know, because of the technical complexity and, most importantly, lack of the manifest covariance, such analytical tools have never been used for the actual calculation of the explicit form of the heat invariants in an invariant geometric form. As a matter of fact, the symbolic method has only been used to prove the existence of the asymptotic expansion and the general structure of the heat invariants (like their dependence on the jets of the symbol of the operator) (see the review [7] and other articles in the same volume, and [43, 38]). To the best of our knowledge there is no explicit formula even for the low-order coefficients \( A_1 \) and \( A_2 \) for a general non-Laplace type operator.

The development of the analysis needed to discuss elliptic boundary value problems is beyond the scope of this paper. We shall simply use the well known results about the existence of the heat trace asymptotics of elliptic boundary value problems from the classical papers of Greiner [33] and Seeley [41] (see also the books [35, 19]). Our approach can be best described by Greiner’s own words [33], pp. 165–166: “the asymptotic expansion can be obtained by more classical methods. Namely, one constructs the Taylor expansion for the classical parametrix [of the heat equation] . . . and iterates it to obtain the Green’s operator. This yields, at least formally, the asymptotic expansion for [the trace of the heat kernel]”. This is the approach exploited in [39] for a Laplace type operator and it is this approach that we will use in the present paper for non-Laplace type operators. However,
contrary to [33, 41] we do not use any Riemannian metrics but, instead, work directly with densities, so that our final answers are automatically invariant. Greiner [33], pp. 166, also points out that “Of course, at the moment it is not clear which representation will yield more easily to geometric interpretation.”

In spectral geometry as well as in physics the motivation and the goals of the study of spectral asymptotics are quite different from those in analysis. The analytic works are primarily interested in the existence and the type of the asymptotic expansion, but not necessarily in the explicit form of the coefficients of the expansion. In spectral geometry one is interested in the explicit form of the spectral invariants and their relation to geometry. One considers various special cases when some invariant topological and geometrical constraints are imposed, say, on the Riemannian structure (or on the connection of a vector bundle). Some of these conditions are: positive (negative, or zero) scalar curvature, or positive (negative) sectional curvature, Ricci-flat metrics, Einstein spaces, symmetric spaces, Kaehler manifolds etc. Such conditions lead to very specific consequences for the heat invariants which are obvious in the geometric invariant form but which are hidden in a non-invariant symbolic formula obtained in local coordinates. For example, if the scalar curvature is zero, then for the Laplacian on a manifold without boundary $A_2 = 0$. Such a conclusion cannot be reached until one realizes that the integrand of $A_2$ is precisely the scalar curvature. There are, of course, many more examples like this.

Another property that does not become manifest at all in the symbolic approach is the behavior of the heat invariants under the conformal transformation of the Riemannian structure and the gauge transformations. This is a very important property that is heavily used in the functorial approach [22, 24], but which is not used at all in the symbolic approach. For conformally covariant operators the symbolic calculus is exactly the same as for non-conformally covariant ones with similar results because the conformal covariance only concerns the low-order terms of the symbol but not its leading symbol. However, the conformal invariance leads to profound consequences for the heat invariants, zeta-function and the functional determinant (see [20]).

*The calculation of the explicit form of the heat invariants is a separate important and complicated problem that requires special calculational techniques.* The systematic explicit calculation of heat kernel coefficients was initiated by Gilkey [28] (see [29, 43, 38, 7] and references therein). A review of various algorithms for calculation of the heat kernel coefficients is presented in [4]. The two most effective methods that have been successfully used for the actual calculation of the heat invariants are: 1) the functorial method of Gilkey and Branson [22, 24, 29].
which is based on the invariance theory, behavior of the heat trace under conformal transformations and some special case calculations, and 2) the method of local Taylor expansion in normal coordinates (which is essentially equivalent to the geometric covariant Taylor expansions of [21]). The results of both of these methods are directly obtained in an invariant geometric form. The symbolic calculus approach, despite being a powerful analytical tool, fails to provide such invariant results. It gives answers in local coordinates that are not invariant and cannot be made invariant directly. For high-order coefficients the problem of converting such results in a geometric invariant form is hopeless. One cannot even decide whether a particular coefficient is zero or not.

One of the main problems in the study of spectral asymptotics is to develop a procedure that respects all the invariance transformations (diffeomorphisms and gauge transformations in the physics language) of the differential operator. Symbolic calculus gives an answer in terms of jets of the symbol of the operator in some local coordinates. Thus there remains a very important problem of converting these local expressions to global geometric invariant structures, like polynomials in curvatures and their covariant derivatives. For a general coefficient this problem becomes unmanageable; it is simply exponentially bad in the order of the heat kernel coefficient. The number of the jets of the symbol is much greater than the number of invariant structures of given order. This problem is so bad that it is, in fact, much easier to compute the coefficients by some other methods that directly give an invariant answer than to use the results of the symbolic approach. To our knowledge, none of the results for the explicit form of the spectral invariants were obtained by using the symbolic calculus.

Every problem in geometric analysis has two aspects: an analytical aspect and a geometric aspect. In the study of spectral asymptotics of differential operators the analytic aspect has been successfully solved in the classical works of Greiner [33] and Seeley [41] and others [35, 19].

The geometric aspect of the problem for Laplace type operators is now also well understood due to the work of Gilkey [28] and many others (see [29, 43, 38, 18, 17]). The leading symbol of a Laplace type operator naturally defines a Riemannian metric on the manifold, which enables one to employ powerful methods of differential geometry. In other words, the Riemannian structure on a manifold is determined by a Laplace type operator. We take this fact seriously: geometry (Riemannian structure) is determined by analysis (differential operator). In some sense, analysis is primary and geometry is secondary. What kind of geometry is generated does, of course, depend on the differential operator. A *Laplace type differential operator generates the Riemannian geometry.*
As a result, much is known about the spectral asymptotics of Laplace type operators, both on manifolds without boundary and on manifolds with boundary, with various boundary conditions, such as Dirichlet, Neumann, Robin, mixed, oblique, Zaremba etc. On manifolds without boundary all odd coefficients vanish, $A_{2k+1} = 0$, and all even coefficients $A_{2k}$ up to $A_8$ have been computed in our PhD thesis [1], which was published later as a book [6] (see also [29, 2, 16, 11, 44], the reviews [5, 7, 43] and references therein). By using our method [2] Yajima et al. [44] computed the coefficient $A_{10}$ recently. Of course, this remarkable progress can only be achieved by employing modern computer algorithms (the authors of [44] used a Mathematica package). The main reason for this progress is that the heat kernel coefficients are polynomial in the jets of the symbol of the operator (which can be expressed in terms of curvatures and their covariant derivatives); it is essentially an algebraic problem.

On manifolds with boundary, the heat invariants depend on the boundary conditions. For the classical boundary conditions, like Dirichlet, Neumann, Robin, and mixed combination thereof on vector bundles, the coefficients $A_k$ have been explicitly computed up to $A_5$ (see, for example, [37, 22, 24, 3]).

A more general scheme, called oblique boundary value problem [34, 32, 31], which includes tangential derivatives along the boundary, was studied in [15, 17, 16, 26, 27]. This problem is not automatically elliptic like the classical boundary problems; there is a certain condition on the leading symbol of the boundary operator that ensures the strong ellipticity of the problem. As a result, the heat invariants are no longer polynomial in the jets of the boundary operator, which makes this problem much more difficult to handle. So far, in the general case only the coefficient $A_1$ is known [15, 17]. In a particular Abelian case the coefficients $A_2$ and $A_3$ have been computed in [27].

A discontinuous boundary value problem, the so-called Zaremba problem, which includes Dirichlet boundary conditions on one part of the boundary and Neumann boundary conditions on another part of the boundary, was studied recently in [11, 42, 25]. Because this problem is not smooth, the analysis becomes much more subtle (see [11, 42] and references therein). In particular, there is a singular subset of codimension 2 on which the boundary operator is discontinuous, and, one has to put an additional boundary condition that fixes the behavior at that set. Seeley [42] showed that there are no logarithmic terms in the asymptotic expansion of the trace of the heat kernel, which are possible on general grounds, and that the heat invariants do depend on the boundary condition at the singular set; the neglect of that simple fact lead to some controversy on the coefficient $A_2$ in the past until this question was finally settled in [42, 11].
Contrary to the Laplace type operators, there are no systematic effective methods for an explicit calculation of the heat invariants for second-order operators which are not of Laplace type. Such operators appear in so-called matrix geometry \[8, 9, 10, 12\], when instead of a single Riemannian metric there is a matrix-valued symmetric 2-tensor, which we call a “non-commutative metric”. Matrix geometry is motivated by the relativistic interpretation of gauge theories and is intimately related to Finsler geometry (rather a collection of Finsler geometries) (see \[8, 9, 10\]). For an introduction to Finsler geometry see \[40\].

Of course, the existence and the form of the asymptotic expansion of the heat kernel is well established for a very large class of operators, including all self-adjoint elliptic partial differential operators with positive definite leading symbol; it is essentially the same for all second-order operators, whether of Laplace type or not. However, a non-Laplace type operator does not induce a unique Riemannian metric on the manifold. Of course, one can pick any Riemannian metric and work with it, but this is not natural; it does not reflect the properties of the differential operator and its leading symbol. Therefore, it is useless to try to use a Riemannian structure to cast the heat invariants in an invariant form. Rather, a non-Laplace type operator defines a collection of Finsler geometries (a matrix geometry in the terminology of \[8, 9, 10, 12\]). Therefore, it is the matrix geometry that should be used to study the geometric structure of the spectral invariants of non-Laplace type operators. This fact complicates the calculation of spectral asymptotics significantly. Of course, the general classical algorithms described in \[33, 41\] still apply.

Three decades ago Greiner \[33\], p. 164, indicated that “the problem of interpreting these coefficients geometrically remains open”. There has not been much progress in this direction. In this sense, the study of geometric aspects of spectral asymptotics of non-Laplace type operators is just beginning and the corresponding methodology is still underdeveloped in comparison with the Laplace type theory. The only exception to this is the case of exterior $p$-forms, which is pretty simple and, therefore, is well understood now \[30, 23, 21\]. Thus, the geometric aspect of the spectral asymptotics of non-Laplace type operators remains an open problem.

A first step in this direction was made in our papers \[13, 14\]. We studied a subclass of so-called natural non-Laplace type operators on Riemannian manifolds, which appear, for example, in the study of spin-tensor quantum gauge fields. The natural non-Laplace type operators are a special case of non-Laplace type operators whose leading symbol is built in a universal, polynomial way, using tensor product and contraction from the Riemannian metric, its inverse, together with (if applicable) the volume form and/or the fundamental tensor-spinor. These opera-
tors act on sections of spin-tensor bundles. These bundles may be characterized as those appearing as direct summands of iterated tensor products of the tangent, the cotangent and the spinor bundles (see sect 2.1). Alternatively, they may be described abstractly as bundles associated to representations of the spin group. These are extremely interesting and important bundles, as they describe the fields in quantum field theory. The connection on the spin-tensor bundles is built in a canonical way from the Levi-Civita connection. The symbols of natural operators are constructed from the jets of the Riemannian metric, the leading symbols being constructed just from the metric. In this case, even if the leading symbol is not scalar, its determinant is a polynomial in $|\xi|^2 = g^{\mu\nu}(x)\xi_\mu\xi_\nu$, and, therefore, its eigenvalues are functions of $|\xi|$ only. This allows one to use the Riemannian geometry and simplifies the study of such operators significantly.

For non-Laplace operators on manifolds without boundary even the invariant $A_4$ is not known, in general (for some partial results see [13, 10, 12] and the review [14]). For natural non-Laplace type differential operators on manifolds without boundary the coefficients $A_0$ and $A_2$ were computed in [13]. For general non-Laplace type operators they were computed in [10, 12].

The primary goal of the present work is to generalize this study to general non-Laplace type operators on manifolds with boundary. We introduce a “non-commutative” Dirac operator as a first-order elliptic partial differential operator such that its square is a second-order self-adjoint elliptic operator with positive definite leading symbol (not necessarily of Laplace type) and study the spectral asymptotics of these operators with Dirichlet boundary conditions.

This paper is organized as follows. In sect. 2 we describe the construction of non-Laplace type operators. In sect. 2.1 we define natural non-Laplace type operators in the context of Stein-Weiss operators [21]. In sect. 2.2 we describe a class of non-Laplace type operators that appear in matrix geometry following [10, 12]; we develop what can be called the non-commutative exterior calculus and construct first-order and second-order invariant differential operators. In sect. 2.3 we describe the general setup of the Dirichlet boundary value problem for such an operator and introduce necessary tools for the analysis of the ellipticity condition. In sect. 3 we review the spectral asymptotics of elliptic operators both from the heat kernel and the resolvent point of view. In sect. 4 we develop a formal technique for calculation of the heat kernel asymptotic expansion. In sect. 4.1 the interior coefficients $A_0$ and $A_2$ are computed (which are the same as for the manifolds without boundary), and in the sect. 4.2 we compute the boundary coefficient $A_1$. 
2 Non-Laplace Type Operators

2.1 Natural non-Laplace type Operators

Natural non-Laplace type operators can be constructed as follows [21]. Let $M$ be a smooth compact orientable $n$-dimensional spin manifold (with or without boundary). Let $S$ be the spinor bundle over a spin manifold $M$ and

$$\mathcal{V} = TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \otimes S$$  \hspace{1cm} (2.1)

be a spin-tensor vector bundle corresponding to a representation of the spin group Spin($n$) and

$$\nabla : \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(T^*M \otimes \mathcal{V})$$  \hspace{1cm} (2.2)

be a connection on $\mathcal{V}$. Then the decomposition

$$T^*M \otimes \mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_s$$  \hspace{1cm} (2.3)

of the bundle $T^*M \otimes \mathcal{V}$ into its irreducible components $\mathcal{V}_1, \ldots, \mathcal{V}_s$ defines the projections

$$P_j : T^*M \otimes \mathcal{V} \to \mathcal{V}_j$$  \hspace{1cm} (2.4)

and the first-order differential operators

$$G_j = P_j\nabla : \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V}_j),$$  \hspace{1cm} (2.5)

called Stein-Weiss operators (or simply the gradients). The number $s$ of gradients is a representation-theoretic invariant of the bundle $\mathcal{V}$.

Then every first-order Spin($n$)-invariant differential operator

$$D : \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V})$$  \hspace{1cm} (2.6)

is a direct sum of the gradients

$$D = c_1G_1 + \cdots + c_sG_s = P\nabla,$$  \hspace{1cm} (2.7)

where $c_j$ are some real constants and

$$P = \sum_{j=1}^s c_jP_j,$$  \hspace{1cm} (2.8)
and the second-order operators
\[ L : C^\infty(V) \to C^\infty(V) \tag{2.9} \]
defined by
\[ L = D^*D = \nabla^*P^2\nabla = \sum_{j=1}^s c_j^2 G_j^* G_j \tag{2.10} \]
are natural non-Laplace type operators. If all \( c_j \neq 0 \), then \( L \) is elliptic and has a positive definite leading symbol.

### 2.2 Non-commutative Laplacian and Dirac Operator in Matrix Geometry

Let \( M \) be a smooth compact orientable \( n \)-dimensional spin manifold with smooth boundary \( \partial M \). We label the local coordinates \( x^\mu \) on the manifold \( M \) by Greek indices which run over 1, \ldots, \( n \), and the local coordinates \( \tilde{x}^i \) on the boundary \( \partial M \) by Latin indices which run over 1, \ldots, \( n - 1 \). We use the standard coordinate bases for the tangent and the cotangent bundles. The components of tensors over \( M \) in the coordinate basis will be labeled by Greek indices and the components of tensors over \( \partial M \) in the coordinate basis will be labeled by Latin indices. We also use the standard Einstein summation convention for repeated indices.

Let \( S \) be now an arbitrary \( N \)-dimensional complex vector bundle over \( M \) (non necessarily the spinor bundle) with a positive definite Hermitean inner product \( \langle \cdot, \cdot \rangle \). \( S^* \) be its dual bundle and \( \text{End}(S) \) be the bundle of linear endomorphisms of the vector bundle \( S \). Further, let \( \text{Aut}(S) \) be the group of automorphisms of the vector bundle \( S \) and \( G(S) \) be the group of unitary endomorphisms of the bundle \( S \). We will call the unitary endomorphisms of the bundle \( S \) simply gauge transformations.

Let \( TM \) and \( T^*M \) be the tangent and the contangent bundles. We introduce the following notation for the vector bundles of vector-valued and endomorphism-valued \( p \)-forms and \( p \)-vectors
\[ \Lambda_p = (\wedge^p T^*M) \otimes S, \quad \Lambda^p = (\wedge^p TM) \otimes S, \tag{2.11} \]
\[ E_p = (\wedge^p T^*M) \otimes \text{End}(S), \quad E^p = (\wedge^p TM) \otimes \text{End}(S). \tag{2.12} \]

We will also consider vector bundles of densities of different weights over the manifold \( M \). For each bundle we indicate the weight explicitly in the notation of the vector bundle; for example, \( S[w] \) is a vector bundle of densities of weight \( w \).
Since $M$ is orientable there is the standard volume form $\operatorname{vol} = dx = dx^1 \wedge \cdots \wedge dx^n$ given by the standard Lebesgue measure in a local chart. The components of the volume form in a local coordinate basis are given by the completely anti-symmetric Levi-Civita symbol $\varepsilon_{\mu_1 \cdots \mu_n}$. The $n$-vector dual to the volume form is a density of weight $(-1)$ and, hence, is a section of the bundle $E^*[1]$. Its components are given by the contravariant Levi-Civita symbol $\tilde{\varepsilon}^{\mu_1 \cdots \mu_n}$.

These objects naturally define the maps
\[ \varepsilon : \Lambda^p[w] \to \Lambda_{n-p}[w + 1], \quad \tilde{\varepsilon} : \Lambda_p[w] \to \Lambda^{n-p}[w - 1]. \tag{2.13} \]

It is not difficult to see that
\[ \varepsilon \tilde{\varepsilon} = \tilde{\varepsilon} \varepsilon = (-1)^{p(n-p)} \operatorname{Id}. \tag{2.14} \]

Further, we define the diffeomorphism-invariant $L^2$-inner product on the space $C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right)$ of smooth endomorphism-valued $p$-form densities of weight $\frac{1}{2}$ by
\[ (\psi, \varphi) = \int_M dx \langle \psi(x), \varphi(x) \rangle. \tag{2.15} \]

The completion of $C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right)$ in this norm defines the Hilbert space $L^2 \left( \Lambda_p \left[ \frac{1}{2} \right] \right)$.

Suppose we are given a map
\[ \Gamma : T^*M \to \operatorname{End}(S) \tag{2.16} \]
determined by a self-adjoint endomorphism-valued vector $\Gamma \in C^\infty (TM \otimes \operatorname{End}(S)[0])$, which is described locally by a matrix-valued vector $\Gamma^\mu$. Let us define an endomorphism-valued tensor $a \in C^\infty (TM \otimes TM \otimes \operatorname{End}(S)[0])$ by
\[ a(\xi_1, \xi_2) = \frac{1}{2} \left[ \Gamma(\xi_1) \Gamma(\xi_2) + \Gamma(\xi_2) \Gamma(\xi_1) \right], \tag{2.17} \]
where $\xi_1, \xi_2 \in T^*M$. Then $a$ is self-adjoint and symmetric
\[ a(\xi_1, \xi_2) = a(\xi_2, \xi_1), \quad a(\xi_1, \xi_2) = a(\xi_2, \xi_1). \tag{2.18} \]

One of our main assumptions about the tensor $a$ is that it defines an isomorphism
\[ a : T^*M \otimes S \to TM \otimes S. \tag{2.19} \]
Let us consider the endomorphism
\[ H(x, \xi) = a(\xi, \xi) = [\Gamma(\xi)]^2, \]
with \( x \in M \), and \( \xi \in T^*_x M \) being a cotangent vector. Our second assumption is that this endomorphism is positive definite, i.e.
\[ H(x, \xi) > 0 \]
for any point \( x \) of the manifold \( M \) and \( \xi \neq 0 \). This endomorphism is self-adjoint and, therefore, all its eigenvalues are real and positive for \( \xi \neq 0 \). We call the endomorphism-valued tensor \( a \) the non-commutative metric and the components \( \Gamma(\xi) \) of the endomorphism-valued vector \( \Gamma \) the non-commutative Dirac matrices.

This construction determines a collection of Finsler geometries \[9, 12\]. Assume, for simplicity, that the matrix \( H(x, \xi) = a(\xi, \xi) \) has distinct eigenvalues: \( h(a)(x, \xi), a = 1, \ldots, N \). Each eigenvalue defines a Hamilton-Jacobi equation
\[ h(a)(x, \partial S) = m^2(a), \]
where \( m(a) \) are some constants, a Hamiltonian system
\[ \frac{dx^\mu}{dt} = \frac{1}{2} \frac{\partial}{\partial \xi_\mu} h(a)(x, \xi), \]
\[ \frac{d\xi_\mu}{dt} = -\frac{1}{2} \frac{\partial}{\partial x^\mu} h(a)(x, \xi), \]
(the coefficient \( 1/2 \) is introduced here for convenience) and a positive definite Finsler metric
\[ g^{\mu \nu}(x, \xi) = \frac{1}{2} \frac{\partial^2 h(a)}{\partial \xi_\mu \partial \xi_\nu}. \]

Moreover, each eigenvalue is a positive homogeneous function of \( \xi \) of degree 2 and, therefore, the Finsler metric is a homogeneous function of \( \xi \) of degree 0. This leads to a number of identities, in particular,
\[ h(a)(x, \xi) = g^{\mu \nu}(x, \xi) \xi_\mu \xi_\nu, \]
and
\[ \dot{x}^\mu = g^{\mu \nu}(x, \xi) \xi_\nu. \]
Next, one defines the inverse (covariant) Finsler metrics
\[ g_{(a) \mu \nu}(x, \dot{x}) g^{(a) \mu \nu}(x, \xi) = \delta^\alpha_\mu, \]
the interval
\[ ds^2_{(a)} = g_{(a)\mu\nu}(x, \dot{x}) \, dx^\mu dx^\nu, \] (2.29)
connections, curvatures etc (for details, see [40]). Thus, a non-Laplace type operator generates a collection of Finsler geometries.

The isomorphism \( a \) naturally defines a map
\[ A : \Lambda_p \to \Lambda^p, \] (2.30)
by
\[ (A\varphi)^{\mu_1\ldots\mu_p} = A^{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} \varphi_{\nu_1\ldots\nu_p}, \] (2.31)
where
\[ A^{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} = \delta_{\alpha_1}^{[\mu_1} \cdots \delta_{\alpha_p}^{\mu_p]} \delta_{\beta_1}^{\nu_1} \cdots \delta_{\beta_p}^{\nu_p} \, a_{\alpha_1\beta_1} \cdots a_{\alpha_p\beta_p}, \] (2.32)
and the square brackets denote the complete antisymmetrization over the indices included. We will assume that these maps are isomorphisms as well. Then the inverse operator
\[ A^{-1} : \Lambda^p \to \Lambda_{p}, \] (2.33)
is defined by
\[ (A^{-1}\varphi)_{\mu_1\ldots\mu_p} = (A^{-1})_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} \varphi^{\nu_1\ldots\nu_p}, \] (2.34)
where \( A^{-1} \) is determined by the equation
\[ (A^{-1})_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} A^{\nu_1\ldots\nu_p\alpha_1\ldots\alpha_p} = \delta_{[\alpha_1}^{\mu_1} \cdots \delta_{\alpha_p]}^{\mu_p}. \] (2.35)

This can be used further to define the natural inner product on the space of \( p \)-forms \( \Lambda_p \) via
\[ \langle \psi, \varphi \rangle = \frac{1}{p!} \tilde{\psi}_{\mu_1\ldots\mu_p} A^{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} \varphi_{\nu_1\ldots\nu_p}. \] (2.36)

Let \( d \) be the exterior derivative on \( p \)-form densities of weight 0
\[ d : C^\infty(\Lambda_p[0]) \to C^\infty(\Lambda_{p+1}[0]) \] (2.37)
and \( \tilde{d} \) be the coderivative on \( p \)-vector densities of weight 1
\[ \tilde{d} = (-1)^{p+1} \varepsilon d \varepsilon : C^\infty(\Lambda^p[1]) \to C^\infty(\Lambda^{p-1}[1]). \] (2.38)
These operators are invariant differential operators defined without a Riemannian metric. They take the following form in local coordinates
\[ (d\varphi)_{\mu_1\ldots\mu_{p+1}} = (p+1) \partial_{[\mu_1} \varphi_{\mu_2\ldots\mu_p]}, \] (2.39)
\[ (\tilde{d}\varphi)^{\mu_1\ldots\mu_{p-1}} = \partial_\rho \varphi^{\mu_1\ldots\mu_{p-1}}. \] (2.40)
Now, let $\mathcal{B} \in C^\infty (T^*M \otimes \text{End}(S)[0])$ be a smooth anti-self-adjoint endomorphism-valued connection 1-form on the bundle $S$, defined by the matrix-valued covector $\mathcal{B}_\mu$. Such a section naturally defines the maps:

$$\mathcal{B} : \Lambda_p \left( \frac{1}{2} \right) \to \Lambda_{p+1} \left( \frac{1}{2} \right)$$

and

$$\tilde{\mathcal{B}} = (-1)^{np+1} \varepsilon \mathcal{B} \varepsilon : \Lambda^p \left( \frac{1}{2} \right) \to \Lambda^{p-1} \left( \frac{1}{2} \right)$$

given locally by

$$(\mathcal{B}\varphi)_{\mu_1\cdots\mu_{p+1}} = (p + 1)\mathcal{B}_{[\mu_1} \varphi_{\mu_2\cdots\mu_{p+1}]} , \quad (2.43)$$

$$(\tilde{\mathcal{B}}\varphi)_{\mu_1\cdots\mu_{p-1}} = \mathcal{B}_{\mu} \varphi_{\mu_{1}\cdots\mu_{p-1}} . \quad (2.44)$$

Finally, we introduce a self-adjoint non-degenerate endomorphism-valued density $\rho \in C^\infty \left( \text{End}(S) \left[ \frac{1}{2} \right] \right)$ of weight $\frac{1}{2}$. Then $\rho^2$ has weight 1 and plays the role of a non-commutative measure.

This enables us to define the covariant exterior derivative of $p$-form densities of weight $\frac{1}{2}$

$$\mathcal{D} : C^\infty \left( \Lambda_p \left( \frac{1}{2} \right) \right) \to C^\infty \left( \Lambda_{p+1} \left( \frac{1}{2} \right) \right) . \quad (2.45)$$

and the covariant coderivative of $p$-vector densities of weight $\frac{1}{2}$

$$\tilde{\mathcal{D}} = (-1)^{np+1} \varepsilon \mathcal{D} \varepsilon : C^\infty \left( \Lambda^p \left( \frac{1}{2} \right) \right) \to C^\infty \left( \Lambda^{p-1} \left( \frac{1}{2} \right) \right) , \quad (2.46)$$

by

$$\mathcal{D} = \rho (d + \mathcal{B}) \rho^{-1} , \quad (2.47)$$

$$\tilde{\mathcal{D}} = \rho^{-1} (\tilde{d} + \tilde{\mathcal{B}}) \rho . \quad (2.48)$$

These operators transform covariantly under both the diffeomorphisms and the gauge transformations.

The formal adjoint of the operator $\mathcal{D}$

$$\tilde{\mathcal{D}} : C^\infty \left( \Lambda_p \left( \frac{1}{2} \right) \right) \to C^\infty \left( \Lambda_{p-1} \left( \frac{1}{2} \right) \right) , \quad (2.49)$$

has the form

$$\tilde{\mathcal{D}} = -A^{-1} \rho^{-1} (\tilde{d} + \tilde{\mathcal{B}}) \rho A , \quad (2.50)$$
By making use of these operators we define a second-order operator (that can be called the \textit{non-commutative Laplacian})

$$\Delta : C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right),$$

by

$$\Delta = -\bar{D} D - D \bar{D}.$$ (2.52)

In the special case $p = 0$ the non-commutative Laplacian $\Delta$ reads

$$\Delta = \rho^{-1} (\tilde{d} + \tilde{B}) \rho (d + B) \rho^{-1},$$

which in local coordinates has the form

$$\Delta = \rho^{-1} (\partial_\mu + B_\mu) \rho (\partial_\nu + B_\nu) \rho^{-1}.$$ (2.54)

Next, notice that the endomorphism-valued vector $\Gamma$ introduced above naturally defines the maps

$$\Gamma : C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda_{p+1} \left[ \frac{1}{2} \right] \right)$$ (2.55)

and

$$\tilde{\Gamma} = (-1)^{np+1} \varepsilon \tilde{\varepsilon} : C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda_{p-1} \left[ \frac{1}{2} \right] \right)$$ (2.56)

given locally by

$$(\Gamma \varphi)_{\mu_1 \ldots \mu_p+1} = (p+1) \Gamma^{\mu_1 \mu_2 \ldots \mu_{p+1}} \varphi$$

and

$$(\tilde{\Gamma} \varphi)_{\mu_1 \ldots \mu_p-1} = \Gamma^\mu \varphi_{\mu \mu_1 \ldots \mu_{p-1}}.$$ (2.58)

Therefore, we can define a first-order invariant differential operator (that can be called the \textit{non-commutative Dirac operator})

$$D : C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right) \to C^\infty \left( \Lambda_p \left[ \frac{1}{2} \right] \right)$$

by

$$D = i\tilde{\Gamma} \bar{D} = i\Gamma \rho (d + B) \rho^{-1},$$

where, of course, $i = \sqrt{-1}$. The formal adjoint of this operator is

$$\bar{D} = iA^{-1} \bar{D} \Gamma A = iA^{-1} \rho^{-1} (\tilde{d} + \tilde{B}) \rho \Gamma A.$$ (2.61)

These operators can be used to define second order differential operators $D\bar{D}$ and $\bar{D}D$. 
In the case \( p = 0 \) these operators have the following form in local coordinates
\[
D = i \Gamma^\mu \rho (\partial_\mu + \mathcal{B}_\mu) \rho^{-1},
\]
(2.62)
\[
\bar{D} = i \rho^{-1} (\partial_\nu + \mathcal{B}_\nu) \rho \Gamma^\nu,
\]
(2.63)
and, therefore, the second-order operators \( D \bar{D} \) and \( \bar{D} D \) read
\[
D \bar{D} = - \Gamma^\mu \rho (\partial_\mu + \mathcal{B}_\mu) \rho^{-2} (\partial_\nu + \mathcal{B}_\nu) \rho \Gamma^\nu,
\]
(2.64)
\[
\bar{D} D = - \rho^{-1} (\partial_\nu + \mathcal{B}_\nu) \rho \Gamma^\nu \rho (\partial_\mu + \mathcal{B}_\mu) \rho^{-1}.
\]
(2.65)

In the present paper we will primarily study the second-order operators \( \Delta, \bar{D} D \) and \( D \bar{D} \) in the case \( p = 0 \), that is,
\[
\Delta, \bar{D} D, D \bar{D} : C^\infty \left(S \left[ \frac{1}{2} \right]\right) \to C^\infty \left(S \left[ \frac{1}{2} \right]\right).
\]

These are all formally self-adjoint operators by construction. This means that they are symmetric on smooth sections of the bundle \( S \left[ \frac{1}{2} \right] \) with compact support in the interior of \( M \) (that is, sections that vanish together with all their derivatives on the boundary \( \partial M \)).

The leading symbols of all these operators are equal to the matrix \( H(x, \xi) = a(\xi, \xi) \), i.e.
\[
\sigma_L(\Delta; x, \xi) = \sigma_L(\bar{D} D; x, \xi) = \sigma_L(D \bar{D}; x, \xi) = H(x, \xi) = a(\xi, \xi),
\]
(2.66)
where \( \xi \in T^*_x M \). By our main assumption about the non-commuting metric the leading symbol is self-adjoint and positive definite in the interior of the manifold. Therefore, the leading symbol is invertible (or elliptic) in the interior of \( M \). Notice that the leading symbol is non-scalar, in general. That is why such operators are called non-Laplace type operators.

### 2.3 Elliptic Boundary Value Problem

Let us consider a neighborhood of the boundary \( \partial M \) in \( M \). Let \( x = (x^\mu) \) be the local coordinates in this neighborhood. The boundary is a smooth hypersurface without boundary. Therefore, there must exist a local diffeomorphism
\[
r = r(x) \quad \hat{x}^i = \hat{x}^i(x), \quad i = 1, \ldots, n - 1,
\]
(2.67)
and the inverse diffeomorphism
\[ x^\mu = x^\mu(r, \hat{x}), \quad \mu = 1, \ldots, n, \] (2.68)
such that
\[ r(x) = 0 \quad \text{for any } x \in \partial M, \] (2.69)
\[ r(x) > 0 \quad \text{for any } x \notin \partial M, \] (2.70)
and the vector \( \partial_r = \partial/\partial r \) is transversal (nowhere tangent) to the boundary \( \partial M \). Then the coordinates \( \hat{x}^i \) are local coordinates on the boundary \( \partial M \).

Let \( \delta > 0 \). We define a disjoint decomposition of the manifold
\[ M = M_{\text{int}} \cup M_{\text{bnd}}, \] (2.71)
where
\[ M_{\text{bnd}} = \{ x \in M \mid r(x) < \delta \} \] (2.72)
is a \( \delta \)-neighborhood of the boundary and
\[ M_{\text{int}} = M \setminus M_{\text{bnd}} \] (2.73)
is the part of the interior of the manifold on a finite distance from the boundary.

For \( r = 0 \), that is, \( x \in \partial M \), the vectors \( \{ \partial_i = \partial/\partial \hat{x}^i \} \) are tangent to the boundary and give the local coordinate basis for the tangent space \( T_x \partial M \). The set of vectors \( \{ \partial_r, \partial_i \} \) gives the local coordinate basis for the tangent space \( T_x M \) in \( M_{\text{bnd}} \). Similarly, the 1-forms \( d\hat{x}^i \) determine the local coordinate basis for the cotangent space \( T^*_x \partial M \), and the 1-forms \( dr, d\hat{x}^i \) give the local coordinate basis for the cotangent space \( T^*_x M \) in \( M_{\text{bnd}} \).

We fix the orientation of the boundary by requiring the Jacobian of this diffeomorphism to be positive, in other words, for any \( x \in M_{\text{bnd}} \)
\[ J(x) = \text{vol}(\partial_r, \partial_1, \ldots, \partial_{n-1}) > 0. \] (2.74)

Let \( \varphi \in C^\infty(TM[1]) \) be a smooth vector density of weight 1. Then Stokes’ Theorem has the form
\[ \int_M dx \, \tilde{\varphi} = \int_{\partial M} d\hat{x} \, N(\varphi), \] (2.75)
where $N$ is a 1-form defined by

$$ N(\varphi) = \text{vol}(\varphi, \hat{\partial}_1, \ldots, \hat{\partial}_{n-1}) = \frac{1}{J} dr(\varphi) $$

$$ = \varepsilon_{\mu_1 \ldots \mu_{n-1}} \frac{\partial x^{\mu_1}}{\partial \hat{x}_1} \cdots \frac{\partial x^{\mu_{n-1}}}{\partial \hat{x}_{n-1}} \varphi^\mu = \frac{1}{J} \frac{\partial r}{\partial \hat{x}^\mu} \varphi^\mu. \quad (2.76) $$

Notice that this formula is valid for densities, and there is no need for a Riemannian metric.

We will study in the present paper, for simplicity, the Dirichlet boundary conditions

$$ \varphi|_{\partial M} = 0. \quad (2.77) $$

By integration by parts it is not difficult to see that all operators $\Delta, D \bar{D}$ and $\bar{D} D$ are symmetric on smooth sections of the bundle $S^{\frac{1}{2}}$ satisfying the boundary conditions. One can show that these operators are essentially self-adjoint, that is, their closure is self-adjoint and, hence, they have unique self-adjoint extensions to $L^2(S^{\frac{1}{2}})$.

Let $L$ be one of the operators $\bar{D} D, D \bar{D}, \Delta$ with the Dirichlet boundary conditions. Our primary interest in this paper is the study of elliptic boundary value problems. Ellipticity means invertibility up to a compact operator in appropriate functional spaces (see, for example, \cite{19, 31, 29}). This is, roughly speaking, a condition that implies local invertibility. For a boundary value problem it has two components: i) in the interior of the manifold, and ii) at the boundary.

An operator $L$ is elliptic in the interior of the manifold if for any interior point $x \in M$ and for any nonzero cotangent vector $\xi \in T^*_x M, \xi \neq 0$, its leading symbol $\sigma_L(L; x, \xi)$ is invertible. Since all operators $D \bar{D}, \bar{D} D$ and $\Delta$ all have positive leading symbols, namely $H(x, \xi)$, they are elliptic in the interior of the manifold.

At the boundary $\partial M$ of the manifold we use the coordinates $(r, \hat{x})$ and define a split of the cotangent bundle $T^*_x M = \mathbb{R} \oplus T^*_x \partial M$, so that $\xi = (\xi_\mu) = (\omega, \hat{\xi}) \in T^*_x \partial M$, where $\omega \in \mathbb{R}$ and $\hat{\xi} = (\xi_j) \in T^*_x \partial M$.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ be a complex number that does not lie on the positive real axis and $H(r, \hat{x}, \omega, \hat{\xi})$ be the leading symbol of the operator $L$. We substitute $r = 0$ and $\omega \mapsto -i \partial_r$ and consider the following second-order ordinary differential equation on the half-line, i.e. $r \in \mathbb{R}_+$,

$$ \left[ H(0, \hat{x}, -i \partial_r, \hat{\xi}) - \lambda \right] \varphi = 0, \quad (2.78) $$

with an asymptotic condition

$$ \lim_{r \to \infty} \varphi = 0. \quad (2.79) $$
Let \( \hat{S} = S|_{\partial M} \) be the restriction of the vector bundle \( S \) to the boundary. The operator \( L \) with Dirichlet boundary conditions is elliptic if for each boundary point \( \hat{x} \in \partial M \), each \( \hat{\xi} \in T^*_{\hat{x}} \partial M \), each \( \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \), such that \( \hat{\xi} \) and \( \lambda \) are not both zero, and each \( f \in C^\infty(\hat{S}[\frac{1}{2}]) \) there is a unique solution \( \varphi(\lambda, r, \hat{\xi}) \) to the equation (2.78) subject to the asymptotic condition (2.79) at infinity and the boundary condition at \( r = 0 \)

\[ \varphi(\lambda, 0, \hat{\xi}) = f. \]  

(2.80)

We have

\[
H(0, \hat{x}, \omega, \hat{\xi}) = [A(\hat{x})\omega + C(\hat{x}, \hat{\xi})]^2 = A^2(\hat{x})\omega^2 + B(\hat{x}, \hat{\xi})\omega + C^2(\hat{x}, \hat{\xi}),
\]

(2.81)

where \( A, B, \) and \( C \) are self-adjoint matrices defined by

\[
A(\hat{x}) = \Gamma(dr), \quad C(\hat{x}, \hat{\xi}) = \Gamma(d\hat{x}^j\hat{\xi}_j),
\]

(2.82)

\[
B(\hat{x}, \hat{\xi}) = A(\hat{x})C(\hat{x}, \hat{\xi}) + C(\hat{x}, \hat{\xi})A(\hat{x}).
\]

(2.83)

Then the differential equation (2.78) has the form

\[
\left(-A^2 \partial^2_r - iB \partial_r + C^2 - \lambda I \right) \varphi = 0.
\]

(2.84)

We notice that the matrix \( [(A\omega + C)^2 - \lambda I] \) is non-degenerate when \( \omega \) is real and \( \lambda \) and \( \hat{\xi} \) are not both zero, i.e. \( (\lambda, \hat{\xi}) \neq (0, 0) \). Moreover, when \( \lambda \) is a negative real number, then this matrix is self-adjoint and positive definite for real \( \omega \). Therefore, we can define

\[
\Phi(\lambda, y, \hat{\xi}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega y} R_{\lambda}(\omega, \hat{\xi}),
\]

(2.85)

where

\[
R_{\lambda}(\omega, \hat{\xi}) = \left[(A(\hat{x})\omega + C(\hat{x}, \hat{\xi}))^2 - \lambda I \right]^{-1}.
\]

(2.86)

The matrix \( \Phi(\lambda, y, \hat{\xi}) \) is well defined for any \( y \in \mathbb{R} \). It: i) vanishes at infinity,

\[
\lim_{y \to \pm \infty} \Phi(\lambda, y, \hat{\xi}) = 0,
\]

(2.87)
ii) satisfies the symmetry relations
\[
\Phi(\lambda, y, \hat{\xi}) = \Phi(\bar{\lambda}, -y, \hat{\xi}), \quad \Phi(\lambda, y, -\hat{\xi}) = \Phi(\lambda, -y, \hat{\xi}),
\]
\begin{equation}
(2.88)
\end{equation}

iii) is homogeneous, i.e. for any \( t > 0 \),
\[
\Phi\left(\frac{\lambda}{t}, \sqrt{t} y, \frac{\hat{\xi}}{\sqrt{t}}\right) = t^{1/2} \Phi(\lambda, y, \hat{\xi}),
\]
\begin{equation}
(2.89)
\end{equation}

iv) is continuous at zero with a well defined value at \( y = 0 \)
\[
\Phi_0(\lambda, \hat{\xi}) = \Phi(\lambda, 0, \hat{\xi}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} R_\lambda(\omega, \hat{\xi}),
\]
\begin{equation}
(2.90)
\end{equation}

v) has a discontinuous derivative \( \partial_y \Phi(\lambda, y, \hat{\xi}) \) at \( y = 0 \) with a finite jump.

We also notice that the matrix \( \Phi_0(\lambda, \hat{\xi}) \) is an even function of \( \hat{\xi} \) and is self-adjoint for real \( \lambda \), i.e.
\[
\Phi_0(\lambda, -\hat{\xi}) = \Phi_0(\bar{\lambda}, \hat{\xi}), \quad \Phi_0(\lambda, \hat{\xi}) = \Phi_0(\bar{\lambda}, \hat{\xi}).
\]
\begin{equation}
(2.91)
\end{equation}

Moreover, for real negative \( \lambda \) the matrix \( \Phi_0 \) is positive and, therefore, non-degenerate. More generally, it is non-degenerate for \( \text{Re} \lambda < w \), where \( w \) is a sufficiently large negative constant.

In an important particular case, when \( B = AC + CA = 0 \), one can compute explicitly
\[
\Phi(\lambda, y, \hat{\xi}) = \frac{1}{2} A^{-1} \mu^{-1} e^{-\mu|y|} A^{-1}, \quad \Phi_0(\lambda, \hat{\xi}) = \frac{1}{2} A^{-1} \mu^{-1} A^{-1},
\]
\begin{equation}
(2.92)
\end{equation}
where \( \mu = \sqrt{A^{-1}(C^2 - \lambda I)A^{-1}} \), defined as an analytical continuation in \( \lambda \) of a positive square root of a self-adjoint matrix when \( \lambda \in \mathbb{R}_{-} \).

One can prove now that the eq. \( (2.84) \) with initial condition \( (2.80) \) and the asymptotic condition at infinity \( (2.79) \) has a unique solution given by
\[
\varphi(\lambda, r, \hat{\xi}) = \Phi(\lambda, r, \hat{\xi})[\Phi_0(\lambda, \hat{\xi})]^{-1} f.
\]
\begin{equation}
(2.93)
\end{equation}
Thus, the Dirichlet boundary value problem for our operator is elliptic.
3 Spectral Asymptotics

3.1 Heat Kernel

Let $L$ be a self-adjoint elliptic second-order partial differential operator acting on smooth sections of the bundle $S\left[\frac{1}{2}\right]$ over a compact manifold $M$ with boundary $\partial M$ with positive definite leading symbol and with some boundary conditions

$$B\varphi|_{\partial M} = 0, \quad (3.1)$$

with some boundary operator $B$. It is well known that such an operator has a discrete real spectrum $\{\lambda_k\}_{k=1}^{\infty}$ bounded from below $[29]$, i.e.,

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots. \quad (3.2)$$

Furthermore: i) each eigenspace is finite-dimensional, ii) the eigensections are smooth sections of the bundle $S\left[\frac{1}{2}\right]$, and iii) the set of eigensections $\{\varphi_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $L^2\left(S\left[\frac{1}{2}\right]\right)$.

For $t > 0$ the heat semigroup

$$\exp(-tL) : L^2\left(S\left[\frac{1}{2}\right]\right) \to L^2\left(S\left[\frac{1}{2}\right]\right) \quad (3.3)$$

is a bounded operator. The integral kernel of this operator, called the heat kernel, is given by

$$U(t; x, x') = \sum_{k=1}^{\infty} e^{-t\lambda_k} \varphi_k \otimes \overline{\varphi}_k(x'), \quad (3.4)$$

where each eigenvalue is counted with its multiplicity. The heat kernel satisfies the heat equation

$$(\partial_t + L)U(t; x, x') = 0 \quad (3.5)$$

with the initial condition

$$U(0^+; x, x') = \delta(x, x'), \quad (3.6)$$

where $\delta(x, x')$ is the Dirac distribution, as well as the boundary conditions

$$B_x U(t; x, x') \bigg|_{x \in \partial M} = 0, \quad (3.7)$$

and the self-adjointness condition

$$\overline{U(t; x, x')} = U(t; x', x). \quad (3.8)$$
The heat kernel $U(t) = \exp(-tL)$ is intimately related to the resolvent $G(\lambda) = (L - \lambda)^{-1}$. Let $\lambda$ be a complex number with a sufficiently large negative real part, $\text{Re}\lambda << 0$. Then the resolvent and the heat kernel are related by the Laplace transform

$$G(\lambda) = \int_0^\infty dt \ e^{\lambda t} U(t), \quad (3.9)$$

$$U(t) = \frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} d\lambda \ e^{-t\lambda} G(\lambda), \quad (3.10)$$

where $w$ is a sufficiently large negative number, $w << 0$.

The resolvent satisfies the equation

$$(L - \lambda I)G(\lambda; x, x') = \delta(x, x') \quad (3.11)$$

with the boundary condition

$$B_x G(\lambda; x, x') \big|_{x \in \partial M} = 0, \quad (3.12)$$

and the self-adjointness condition

$$\overline{G(\lambda; x', x)} = G(\bar{\lambda}; x', x). \quad (3.13)$$

The integral kernel of the resolvent reads

$$G(\lambda; x, x') = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - \lambda} \varphi_k \otimes \bar{\varphi}_k(x'), \quad (3.14)$$

where each eigenvalue is counted with its multiplicity.

For $t > 0$ the heat kernel $U(t; x, x')$ is a smooth section of the exterior tensor product bundle $S \left[ \frac{1}{2} \right] \otimes S^* \left[ \frac{1}{2} \right]$; that is, it is a two-point density of weight $\frac{1}{2}$ at each point. In particular, it is a smooth function near the diagonal of $M \times M$ and has a well defined diagonal value $U(t; x, x)$. The diagonal is, of course, a smooth section of the bundle $S[1]$, a density of weight 1.

Moreover, the heat semigroup is a trace-class operator with a well defined $L^2$-trace

$$\text{Tr}_{L^2} \exp(-tL) = \int_M dx \text{tr}_S U(t; x, x), \quad (3.15)$$
where $\text{tr}_S$ is the trace over the fiber vector space $S$ of the vector bundle $S$. The trace of the heat kernel is a spectral invariant of the operator $L$ since

$$\text{Tr}_{L^2} \exp(-tL) = \sum_{k=1}^{\infty} e^{-tk}.$$ (3.16)

Since the diagonal is a density of weight 1 the trace $\text{Tr}_{L^2} \exp(-tL)$ is invariant under diffeomorphisms.

This enables one to define other spectral functions by integral transforms of the trace of the heat kernel. In particular, the zeta function, $\zeta(L; s, \lambda)$, is defined as follows. Let $\lambda$ be a complex parameter with $\text{Re} \lambda < \lambda_1$, so that the operator $(L - \lambda)$ is positive. Then for any $s \in \mathbb{C}$ such that $\text{Re} s > n/2$ the trace of the operator $(L - \lambda)^{-s}$ is well defined and determines the zeta function,

$$\zeta(L; s, \lambda) = \text{Tr}_{L^2} (L - \lambda)^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \ t^{s-1} e^{t\lambda} \text{Tr}_{L^2} \exp(-tL).$$ (3.17)

The zeta function enables one to define further the regularized determinant of the operator $(L - \lambda)$ by

$$\frac{\partial}{\partial s} \zeta(L; s, \lambda) \Big|_{s=0} = -\log \text{Det} (L - \lambda).$$ (3.18)

There is an asymptotic expansion as $t \to 0$ of the trace of the heat kernel $[29]$ (for a review, see also $[2, 5, 6, 7, 43]$)

$$\text{Tr}_{L^2} \exp(-tL) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} A_k(L).$$ (3.19)

The coefficients $A_k(L)$, called the global heat invariants, are spectral invariants determined by the integrals over the manifold $M$ and the boundary $\partial M$ of some scalar densities $a_k(L; x)$ and $b_k(L; \hat{x})$, called local heat invariants, viz.

$$A_k(L) = \int_M dx \ a_k(L; x) + \int_{\partial M} d\hat{x} \ b_k(L; \hat{x}).$$ (3.20)

The local heat invariants $a_k(L; x)$ and $b_k(L; \hat{x})$ are constructed polynomially from the jets of the symbol of the operator $L$; the boundary coefficients $b_k$ depend, of course, on the boundary conditions and the geometry of the boundary as well.
Contrary to the heat kernel, the resolvent is singular at the diagonal and does not have a well defined trace. However, the derivatives of the resolvent do. Let \( m \geq n/2 \). Then the trace \( \text{Tr}_{L^2}(\partial_\lambda)^m G(\lambda) \) is well defined and has the asymptotic expansion as \( \lambda \to -\infty \)

\[
\text{Tr}_{L^2} \frac{\partial^m}{\partial \lambda^m} G(\lambda) \sim (4\pi)^{-n/2} \sum_{k=0}^\infty \Gamma \left( (k - n + 2m + 2)/2 \right) (-\lambda)^{(n-k-2m-2)/2} A_k(L).
\]

Therefore, one can use either the heat kernel or the resolvent to compute the coefficients \( A_k \).

### 3.2 Index of Noncommutative Dirac Operator

Notice that the operator \( \Delta \) can have a finite number of negative eigenvalues, whereas the spectrum of the operators \( \bar{D}D \) and \( D\bar{D} \) is non-negative. Moreover, one can easily show that all non-zero eigenvalues of the operators \( \bar{D}D \) and \( D\bar{D} \) are equal

\[
\lambda_k(\bar{D}D) = \lambda_k(D\bar{D}) \quad \text{if} \quad \lambda_k(\bar{D}D) > 0. \tag{3.22}
\]

Therefore, there is a well defined index

\[
\text{Ind}(D) = \dim \text{Ker}(\bar{D}D) - \dim \text{Ker}(D\bar{D}), \tag{3.23}
\]

which is equal to the difference of the number of zero modes of the operators \( \bar{D}D \) and \( D\bar{D} \).

This leads to the fact that the difference of the heat traces for the operators \( \bar{D}D \) and \( D\bar{D} \) determines the index

\[
\text{Tr}_{L^2} \exp(-t \bar{D}D) - \text{Tr}_{L^2} \exp(-t D\bar{D}) = \text{Ind}(D). \tag{3.24}
\]

This means that the spectral invariants of the operators \( \bar{D}D \) and \( D\bar{D} \) are equal except for the invariant \( A_n \) which determines the index

\[
A_k(\bar{D}D) = A_k(D\bar{D}) \quad \text{for} \quad k \neq n, \tag{3.25}
\]

and

\[
A_n(\bar{D}D) - A_n(D\bar{D}) = (4\pi)^{n/2} \text{Ind}(D). \tag{3.26}
\]

Thus, for \( n > 2 \) the spectral invariants \( A_0, A_1 \) and \( A_2 \) of the operators \( \bar{D}D \) and \( D\bar{D} \) are equal. Therefore, we can pick any of these operators to compute these
invariants. Of course, the spectral invariants of the noncommutative Laplacian $\Delta$ are, in general, different. However, since the operators $\bar{D}D$ and $D\bar{D}$ have the same leading symbol as the operator $\Delta$ there must exist a corresponding Lichnerowicz-Weitzenbock formula (for the spinor bundle see, for example, [18]), which means that the spectral invariants of these operators must be related.

4 Heat Invariants

4.1 Interior Coefficients

The heat kernel in the interior part is constructed as follows. We fix a point $x_0 \in M_{\text{int}}$ in the interior of the manifold and consider a neighborhood of $x_0$ disjoint from the boundary layer $M_{\text{bnd}}$ covered by a single patch of local coordinates. We introduce a scaling parameter $\varepsilon > 0$ and scale the variables according to

$$x^\mu \mapsto x_0^\mu + \varepsilon(x^\mu - x_0^\mu), \quad x'^\mu \mapsto x_0'^\mu + \varepsilon(x'^\mu - x_0'^\mu), \quad t \mapsto \varepsilon^2 t,$$

(4.1)

so that

$$\partial_\mu \mapsto \frac{1}{\varepsilon} \partial_\mu, \quad \partial_t \mapsto \frac{1}{\varepsilon^2} \partial_t.$$

(4.2)

Then the differential operator $L(\hat{x}, \hat{\partial})$ is scaled according to

$$L \mapsto L_\varepsilon \sim \sum_{k=0}^\infty \varepsilon^{-2n+k} L_{\text{int}}^k,$$

(4.3)

where $L_{\text{int}}^k$ are second-order differential operators with homogeneous symbols.

Next, we expand the scaled heat kernel in $M_{\text{int}}$, which we denote by $U_{\varepsilon, \text{int}}$ in a power series in $\varepsilon$

$$U_{\varepsilon, \text{int}} \sim \sum_{k=0}^\infty \varepsilon^{2n+k} U_{\text{int}}^k,$$

(4.4)

and substitute into the scaled version of the heat equation. By equating the like powers of $\varepsilon$ we get an infinite set of recursive differential equations determining all the coefficients $U_{\text{int}}^k$.

The leading order operator $L_{\text{int}}^0$ is an operator with constant coefficients determined by the leading symbol

$$L_{\text{int}}^0 = H(x_0, -i\partial).$$

(4.5)
The leading-order heat kernel $U^\text{int}_0$ can be easily obtained by the Fourier transform

$$U^\text{int}_0(t; x, x') = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} e^{i\xi(x-x')-tH(x_0, \xi)},$$

where $\xi(x-x') = \xi_\mu(x^\mu - x'^\mu)$.

The higher-order coefficients $U^\text{int}_k$, $k \geq 1$, are determined from the recursive equations

$$(\partial_t + L^\text{int}_0)U^\text{int}_k = - \sum_{j=1}^k L^\text{int}_j U^\text{int}_{k-j},$$

with the initial condition

$$U^\text{int}_k(0; x, x') = 0.$$

This expansion is nothing but the decomposition of the heat kernel into the homogeneous parts with respect to the variables $(x-x_0), (x'-x_0)$, and $\sqrt{t}$. That is,

$$U^\text{int}_k(t; x, x') = t^{(k-n)/2} U^\text{int}_k \left(1; x_0 + \frac{(x-x_0)}{\sqrt{t}}, x_0 + \frac{(x'-x_0)}{\sqrt{t}}\right).$$

In particular, the heat kernel diagonal at the point $x_0$ scales by

$$U^\text{int}_k(t; x_0, x_0) = t^{(k-n)/2} U^\text{int}_k (1; x_0, x_0).$$

To compute the contribution of these coefficients to the trace of the heat kernel we need to compute the integral of the diagonal of the heat kernel $U^\text{int}(t; x, x)$ over the interior part of the manifold $M^\text{int}$. By using the homogeneity property we obtain

$$\int_{M^\text{int}} dx \text{ tr}_S U^\text{int}(t; x, x) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} \int_{M^\text{int}} dx \text{ tr}_S U^\text{int}_k (1; x, x).$$

Next, we take the limit as $\delta \to 0$. Then the integrals over the interior part $M^\text{int}$ become the integrals over the whole manifold $M$ and give all the interior coefficients $a_k(L)$ in the global heat kernel coefficients $A_k(L)$.

Instead of this rigorous procedure, we present below a pragmatic formal approach that enables one to compute all interior coefficients in a much easier and compact form. Of course, both approaches are equivalent and give the same answers.
First, we present the heat kernel diagonal for the operator $L = \bar{D} D$ in the form

$$U^{\text{int}}(t; x, x) = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} e^{-i\xi x} \exp(-t \bar{D} D) e^{i\xi x},$$  \hspace{1cm} (4.12)

where $\xi x = \xi \mu x^\mu$, which can be transformed to

$$U^{\text{int}}(t; x, x) = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \exp\left[-t\left( H + K + \bar{D} D \right) \right] \cdot I,$$  \hspace{1cm} (4.13)

where $H = [\Gamma(\xi)]^2$ is the leading symbol of the operator $\bar{D} D$, and $K$ is a first-order self-adjoint operator defined by

$$K = -\Gamma(\xi)D - \bar{D} \Gamma(\xi).$$  \hspace{1cm} (4.14)

Here the operators in the exponent act on the unity matrix $I$ from the left.

By changing the integration variable $\xi \rightarrow t^{1/2} \xi$ we obtain

$$U^{\text{int}}(t; x, x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{n/2}} \exp\left(-H - \sqrt{t} K - t \bar{D} D \right) \cdot I.$$  \hspace{1cm} (4.15)

Now, the coefficients of the asymptotic expansion of this integral in powers of $t^{1/2}$ as $t \to 0$ determine the interior heat kernel coefficients $a_k(L)$ via

$$\text{tr}_S U^{\text{int}}(t; x, x) \sim (4\pi)^{-n/2} \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(L).$$  \hspace{1cm} (4.16)

By using the Volterra series

$$\exp(A + B) = e^A + \sum_{k=1}^{\infty} \int_0^1 d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 \times$$

$$\times e^{(1-\tau_k)A} Be^{(\tau_k-\tau_{k-1})A} \cdots e^{(\tau_2-\tau_1)A} Be^{\tau_1A},$$  \hspace{1cm} (4.17)
we get

\[
\exp(-H - \sqrt{t} K - t \bar{D} D) = e^{-H} - t^{1/2} \int_{0}^{1} d\tau_{1} e^{-(1-\tau_{1})H} K e^{-\tau_{1}H} \\
+ t \left[ \int_{0}^{1} d\tau_{2} \int_{0}^{\tau_{2}} d\tau_{1} e^{-(1-\tau_{2})H} K e^{-(\tau_{2}-\tau_{1})H} K e^{-\tau_{1}H} - \right. \\
- \left. \int_{0}^{1} d\tau_{1} e^{-(1-\tau_{1})H} \bar{D} De^{-\tau_{1}H} \right] + O(t^{2}) . \tag{4.18}
\]

Now, since \( K \) is linear in \( \xi \) the term proportional to \( t^{1/2} \) vanishes after integration over \( \xi \). Thus, we obtain the first three interior coefficients of the asymptotic expansion of the heat kernel diagonal in the form

\[
a_{0}(L) = \int_{\mathbb{R}^{n}} \frac{d\xi}{\pi^{n/2}} \text{tr} \, S \, e^{-H} , \tag{4.19}
\]

\[
a_{1}(L) = 0 , \tag{4.20}
\]

\[
a_{2}(L) = \int_{\mathbb{R}^{n}} \frac{d\xi}{\pi^{n/2}} \text{tr} \left[ \int_{0}^{1} d\tau_{2} \int_{0}^{\tau_{2}} d\tau_{1} e^{-(1-\tau_{2})H} K e^{-(\tau_{2}-\tau_{1})H} K e^{-\tau_{1}H} - \right. \\
- \left. \int_{0}^{1} d\tau_{1} e^{-(1-\tau_{1})H} \bar{D} De^{-\tau_{1}H} \right] . \tag{4.21}
\]

### 4.2 Boundary Coefficients

On manifolds with boundary, as far as we know, the coefficients \( A_{k} \) have not been studied at all, so, even \( A_{1} \) is not known. In the present paper we are going to compute the coefficient \( A_{1} \) on manifolds with boundary for the operators \( \bar{D}D \) and \( D\bar{D} \). The coefficient \( A_{0} \) is, of course, the same as for the manifolds without boundary. We will follow the general framework for computation of the heat kernel asymptotics outlined in [11, 16].

The procedures for the resolvent and the heat kernel are very similar. One can, of course, use either of them. We will describe below the construction of the heat kernel.
The main idea can be described as follows. Recall that we decomposed the manifold into a neighborhood of the boundary \( M_{\text{bnd}} \) and the interior part \( M_{\text{int}} \). We can use now different approximations for the heat kernel in different domains. Strictly speaking one has to use ‘smooth characteristic functions’ of those domains (partition of unity) to glue them together in a smooth way. Then, one has to control the order of the remainder terms in the limit \( t \to 0^+ \) and their dependence on \( \delta \) (the size of the boundary layer). However, since we are only interested in the trace of the heat kernel, this is not needed here and we will not worry about such subtle details. We can compute the asymptotic expansion as \( t \to 0 \) of the corresponding integrals in each domain and then take the limit \( \delta \to 0 \).

The origin of the boundary terms in the heat trace asymptotics can be explained as follows. The heat kernel of an elliptic boundary value problem in \( M_{\text{bnd}} \) has exponentially small terms like \( \exp(-r^2/t) \) as \( t \to 0 \). These terms do not contribute to the asymptotic expansion of the diagonal of the heat kernel as \( t \to 0 \). However, they behave like distributions near the boundary (recall that \( r > 0 \) inside the manifold and \( r = 0 \) on the boundary). Therefore, the integral over \( M_{\text{bnd}} \), more precisely, the limit \( \lim_{\delta \to 0} \int_{\partial M} d\hat{x} \int_0^\delta dr (\ldots) \) does contribute to the asymptotic expansion of the trace of the heat kernel with coefficients in form of integrals over the boundary. It is this phenomenon that leads to the boundary terms in the global heat invariants.

The heat kernel in the boundary layer \( M_{\text{bnd}} \) is constructed as follows. We fix a point \( \hat{x}_0 \in \partial M \) on the boundary and choose coordinates as described above in section 2.2. Let \( \varepsilon > 0 \) be a positive real parameter. We use it as a scaling parameter; at the very end of the calculation it will be set to 1. Now we scale the coordinates according to

\[
\hat{x}^j \mapsto \hat{x}_0^j + \varepsilon (\hat{x}^j - \hat{x}_0^j), \quad \hat{x}'^j \mapsto \hat{x}_0^j + \varepsilon (\hat{x}'^j - \hat{x}_0^j),
\]

\[
r \mapsto \varepsilon r, \quad r' \mapsto \varepsilon r', \quad t \mapsto \varepsilon^2 t.
\]

The differential operators are scaled correspondingly by

\[
\partial_j \mapsto \frac{1}{\varepsilon} \partial_j, \quad \partial_r \mapsto \frac{1}{\varepsilon} \partial_r, \quad \partial_t \mapsto \frac{1}{\varepsilon^2} \partial_t.
\]

Let \( L(r, \hat{x}, \partial_r, \hat{\partial}) \) be the operator under consideration. The scaled operator, which we denoted by \( L_\varepsilon \), has the following formal power series expansion in \( \varepsilon \)
\begin{align}
L \mapsto L_\varepsilon & \sim \sum_{k=0}^{\infty} \varepsilon^{k-2} L_k^{\text{bnd}},
\end{align}

(4.25)

where $L_k$ are second-order differential operators with homogeneous symbols. The leading order operator is determined by the leading symbol

\begin{align}
L_0^{\text{bnd}} & = H(0, \hat{x}_0, -i\partial_r, -i\hat{\partial}).
\end{align}

(4.26)

This is a differential operator with constant coefficients.

Next, we expand the scaled heat kernel in $M_{\text{bnd}}$, which we denote by $U_\varepsilon^{\text{bnd}}$ in a power series in $\varepsilon$

\begin{align}
U_\varepsilon^{\text{bnd}} & \sim \sum_{k=0}^{\infty} \varepsilon^{2-n+k} U_k^{\text{bnd}},
\end{align}

(4.27)

and substitute into the scaled version of the heat equation and the boundary conditions. By equating the like powers of $\varepsilon$ we get an infinite set of recursive differential equations determining all the coefficients $U_k^{\text{bnd}}$.

The leading-order heat kernel $U_0^{\text{bnd}}$ is determined by the equation

\begin{align}
(\partial_t + L_0^{\text{bnd}}) U_0^{\text{bnd}} & = 0
\end{align}

(4.28)

with the initial condition

\begin{align}
U_0^{\text{bnd}}(0; r, \hat{x}, r', \hat{x}') & = \delta(r - r')\delta(\hat{x}, \hat{x}'),
\end{align}

(4.29)

the boundary condition

\begin{align}
U_0^{\text{bnd}}(t; 0, \hat{x}, r', \hat{x}') & = U_0^{\text{bnd}}(t; r, \hat{x}, 0, \hat{x}') = 0,
\end{align}

(4.30)

and the asymptotic condition

\begin{align}
\lim_{r \to \infty} U_0^{\text{bnd}}(t; r, \hat{x}, r', \hat{x}') & = \lim_{r' \to \infty} U_0^{\text{bnd}}(t; r, \hat{x}, r', \hat{x}') = 0.
\end{align}

(4.31)

The higher-order coefficients $U_k^{\text{bnd}}$, $k \geq 1$, are determined from the recursive equations

\begin{align}
(\partial_t + L_0^{\text{bnd}}) U_k^{\text{bnd}} & = - \sum_{j=1}^{k} L_j^{\text{bnd}} U_{k-j}^{\text{bnd}},
\end{align}

(4.32)

with the initial condition

\begin{align}
U_k^{\text{bnd}}(0; r, \hat{x}, r', \hat{x}') & = 0,
\end{align}

(4.33)
the boundary condition

\[ U_{k}^{\text{bnd}}(t; 0, \hat{x}, r', \hat{x}') = U_{k}^{\text{bnd}}(t; \hat{x}, 0, \hat{x}') = 0, \quad (4.34) \]

and the asymptotic condition

\[ \lim_{r \to \infty} U_{0}^{\text{bnd}}(t; r, \hat{x}, r', \hat{x}') = \lim_{r' \to \infty} U_{0}^{\text{bnd}}(t; r, \hat{x}, r', \hat{x}') = 0. \quad (4.35) \]

This expansion is nothing but the decomposition of the heat kernel into the homogeneous parts with respect to the variables \((\hat{x} - \hat{x}_0), (\hat{x}' - \hat{x}_0), r, r'\) and \(\sqrt{t}\). That is,

\[ U_{k}^{\text{bnd}}(t; r, \hat{x}, r', \hat{x}') = t^{(k-n)/2} U_{k}^{\text{bnd}} \left( 1; \frac{r}{\sqrt{t}}, \hat{x}_0, \frac{r'}{\sqrt{t}}, \hat{x}_0 \right). \quad (4.36) \]

In particular, the heat kernel diagonal at the point \((r, \hat{x}_0)\) scales by

\[ U_{k}^{\text{bnd}}(t; r, \hat{x}_0, r, \hat{x}_0) = t^{(k-n)/2} U_{k}^{\text{bnd}} \left( 1; \frac{r}{\sqrt{t}}, \hat{x}_0, \frac{r}{\sqrt{t}}, \hat{x}_0 \right). \quad (4.37) \]

To compute the contribution of these coefficients to the trace of the heat kernel we need to compute the integral of the diagonal of the heat kernel \(U_{k}^{\text{bnd}}(t; r, \hat{x}, r, \hat{x})\) over the boundary layer \(M_{\text{bnd}}\). This heat kernel diagonal can be decomposed as the sum of two terms, the first coming from the standard interior heat kernel on manifolds without boundary (that does not satisfy the boundary conditions) and the second ‘compensating’ part, which is the crucial boundary part and whose role is to make the heat kernel to satisfy the boundary conditions (for more details see [11]). The integral of the ‘boundary’ part over the boundary layer in the limit when the size of the boundary layer goes to zero produces the boundary contributions \(b_k(L)\) to the global heat kernel coefficients \(A_k(L)\).

By using the homogeneity property \((4.37)\) we obtain

\[
\int_{M_{\text{bnd}}} dx \text{ tr}_S U_{k}^{\text{bnd}}(t; x, x) = \int_{\partial M} d\hat{x} \int_0^\delta d\hat{\chi} \int_0^\delta dr \text{ tr}_S U_{k}^{\text{bnd}}(t; r, \hat{x}, r, \hat{x})
\]

\[
\sim \sum_{k=0}^\infty t^{(k-n)/2} \int_{\partial M} d\hat{x} \int_0^\delta dr \text{ tr}_S U_{k}^{\text{bnd}} \left( 1; \frac{r}{\sqrt{t}}, \hat{x}, \frac{r}{\sqrt{t}}, \hat{x} \right)
\]

\[
\sim \sum_{k=0}^\infty t^{(k-n+1)/2} \int_{\partial M} d\hat{x} \int_0^{\delta/\sqrt{t}} du \text{ tr}_S U_{k}^{\text{bnd}}(1; u, \hat{x}, u, \hat{x})
\]

\( (4.38) \)
where \( u = r / \sqrt{t} \). Notice the appearance of the extra power of \( \sqrt{t} \) in the asymptotic expansion. Of course, if one takes the limit \( \lim_{\delta \to 0} \) for a finite \( t \), then all these integrals vanish. However, if one takes the limit \( \lim_{\delta \to 0} \) first for a finite \( \delta \), and then the limit \( \lim_{\delta \to 0} \), then one gets finite answers for the boundary coefficients \( b_k(L) \).

### 4.2.1 Leading-Order Heat Kernel

To compute the coefficient \( A_1 \) we just need the leading-order heat kernel \( U_0^{\text{bnd}} \). We will, in fact, be working in the tangent space \( \mathbb{R}_+ \times T_{\hat{x}_0} \partial M \) at a point \( \hat{x}_0 \) on the boundary and reduce our problem to a problem on the half-line. The operator \( L_0^{\text{bnd}} \) acts on square integrable sections of the vector bundle \( S[\frac{1}{2}] \) in a neighborhood of the point \( \hat{x}_0 \). We extend the operator appropriately to the space \( L^2(S[\frac{1}{2}], \mathbb{R}_+, \mathbb{R}^{n-1}, dr d\hat{x}) \) so that it coincides with the initial operator in the neighborhood of the point \( \hat{x}_0 \). When computing the trace below we set \( \hat{x}_0 = \hat{x} = \hat{x}' \).

By using the Laplace transform in the variable \( t \) and the Fourier transform in the boundary coordinates \( \hat{x} \)

\[
U_0^{\text{bnd}}(t; \hat{x}, \hat{x}', \hat{x}') = \frac{1}{2\pi i} \int_{\mathbb{C}_-} \int \frac{d\hat{\xi}}{(2\pi)^{n-1}} e^{-t\lambda + i\hat{\xi}(\hat{x} - \hat{x}')} F(\lambda, r, r', \hat{\xi}),
\]

we obtain an ordinary differential equation

\[
\left( -A^2 \partial_r^2 - iB \partial_r + C^2 - \lambda \right) F(\lambda, r, r', \hat{\xi}) = \mathbb{I} \delta(r - r')
\]

where the matrices \( A, B \) and \( C \) are defined in (2.82), (2.83), and are frozen at the point \( \hat{x}_0 \) (they are constant for the purpose of this calculation), with the boundary condition

\[
F(\lambda, 0, r', \hat{\xi}) = F(\lambda, r, 0, \hat{\xi}) = 0
\]

the asymptotic condition

\[
\lim_{r \to \infty} F(\lambda, r, r', \hat{\xi}) = \lim_{r' \to \infty} F(\lambda, r, r', \hat{\xi}) = 0,
\]

and the self-adjointness condition

\[
F(\lambda, r, r', \hat{\xi}) = F(\lambda, r', r, \hat{\xi}).
\]

It is easy to see that \( F \) is a homogeneous function

\[
F \left( \frac{\lambda}{t}, \sqrt{t} r, \sqrt{t} r', \frac{\hat{\xi}}{\sqrt{t}} \right) = t^{1/2} F \left( \lambda, r, r', \frac{\hat{\xi}}{\sqrt{t}} \right).
\]
We decompose the Green function in two parts,

\[ F = F_\infty + F_B, \quad (4.45) \]

where \( F_\infty \) is the part that is valid for the whole real line and \( F_B \) is the compensating term. The part \( F_\infty \) can be easily obtained by the Fourier transform; it has the form

\[ F_\infty(\lambda, r, r', \hat{\xi}) = \Phi(\lambda, r - r', \hat{\xi}), \quad (4.46) \]

where \( \Phi(\lambda, r, \hat{\xi}) \) is defined in (2.85). It is not smooth at the diagonal \( r = r' \) and is responsible for the appearance of the delta-function \( \delta(r - r') \) on the right-hand side of the eq. (4.40).

The corresponding part of the leading heat kernel is then easily computed to be

\[ U_{0,\infty}^{\text{bnd}}(t; x, x') = \int_{\mathbb{R}^n} \frac{d\xi}{2\pi} e^{i\hat{\xi}(x - x') - tH(x_0, \xi)}, \quad (4.47) \]

where \( x_0 = (0, \hat{x}_0) \). This part does not contribute to the asymptotics of the trace of the heat kernel in the limit \( \delta \to 0 \). By rescaling \( \xi \mapsto \xi/\sqrt{t} \) we obtain

\[ \int_{M_{\text{bnd}}} dx \, \text{tr}_S U_{0,\infty}^{\text{bnd}}(t; x, x) = (4\pi t)^{-n/2} \int_{M_{\text{bnd}}} dx \int_{\mathbb{R}^n} d\hat{\xi} \frac{\pi^{n/2}}{\pi^n} \text{tr}_S e^{-H(x, \hat{\xi})}, \quad (4.48) \]

and in the limit \( \delta \to 0 \) this integral vanishes.

However, \( F_\infty \) does not satisfy the boundary conditions. The role of the boundary part, \( F_B \), is exactly to guarantee that \( F \) satisfies the boundary conditions. The function \( F_B \) is smooth at the diagonal \( r = r' \). It can be presented in the following form

\[ F_B(\lambda, r, r', \hat{\xi}) = -\Phi(\lambda, r, \hat{\xi})[\Phi_0(\lambda, \hat{\xi})]^{-1}\Phi(\lambda, -r', \hat{\xi}). \quad (4.49) \]

### 4.2.2 The Coefficient \( A_1 \)

The coefficient \( A_1 \) is a pure boundary coefficient that is computed by integrating the boundary part \( U_{0,B}^{\text{bnd}} \) of the heat kernel. We have

\[ \int_{M_{\text{bnd}}} dx \, \text{tr}_S U_{0,B}^{\text{bnd}}(t; x, x) \quad (4.50) \]

\[ = \int_{\delta M} d\hat{x} \int_0^\delta dr \int_{\mathbb{R}^{n-1}} \frac{d\hat{\xi}}{(2\pi)^{n-1}} \int_{w^{-i\infty}}^{w+i\infty} \frac{d\lambda}{2\pi i} e^{-t\lambda} \text{tr}_S F_B(\lambda, r, \hat{\xi}). \]
Now, by rescaling the variables
\[ \lambda \mapsto \frac{\lambda}{t}, \quad r \mapsto \sqrt{tr}, \quad \hat{\xi} \mapsto \frac{\hat{\xi}}{\sqrt{t}} \] (4.51)
and using the homogeneity property (4.44) we obtain
\[ \int_{M_{\text{bnd}}} dx \, \text{tr} \, S_{U_{0,B}}^{\text{bnd}}(t; x, x) \] (4.52)
\[ = t^{(1-n)/2} \int_{\partial M} d\hat{x} \int_{\mathbb{R}^{n-1}} \frac{d\hat{\xi}}{(2\pi)^{n-1}} \int_0^{\delta/\sqrt{t}} dr \int_{w-\infty}^{w+\infty} d\lambda \frac{e^{-\lambda \text{tr} \, S}}{2\pi i} \Phi_B(\lambda, r, \hat{\xi}) . \]

Therefore, the coefficient \( A_1 \) is given by
\[ A_1 = 2 \sqrt{\pi} \int_{\partial M} d\hat{x} \int_{\mathbb{R}^{n-1}} \frac{d\hat{\xi}}{(2\pi)^{n-1/2}} \int_0^{w+\infty} dr \int_{w-\infty}^{w+\infty} d\lambda \frac{e^{-\lambda \text{tr} \, S}}{2\pi i} \Phi_B(\lambda, r, \hat{\xi}) . \] (4.53)

Thus, finally, by using eq. (4.49), eliminating the odd functions of \( \hat{\xi} \) (since the integrals of them vanish), using the property (2.88) of the function \( \Phi \) and extending the integration over \( r \) from \(-\infty\) to \(+\infty\) (since the integrand is an even function) we obtain
\[ A_1 = \sqrt{\pi} \int_{\partial M} d\hat{x} \int_{\mathbb{R}^{n-1}} \frac{d\hat{\xi}}{(2\pi)^{n-1/2}} \Psi_1(\hat{\xi}) \] (4.54)
where
\[ \Psi_1(\hat{\xi}) = -\sqrt{\pi} \int_{-\infty}^{w+\infty} dr \int_{w-\infty}^{w+\infty} d\lambda \frac{e^{-\lambda \text{tr} \, S}}{2\pi i} \Phi_0(\lambda, \hat{\xi}) - 1 \]
\[ \times \left\{ \Phi(\lambda, r, \hat{\xi}) \Phi(\lambda, -r, \hat{\xi}) + \Phi(\lambda, -r, \hat{\xi}) \Phi(\lambda, r, \hat{\xi}) \right\} . \] (4.55)

Recall that \( w \) is a sufficiently large negative constant.

Now, using eq. (2.90) and integrating over \( r \) we obtain finally
\[ \Psi_1(\hat{\xi}) = -\sqrt{\pi} \int_{w-\infty}^{w+\infty} d\lambda \frac{e^{-\lambda \text{tr} \, S}}{2\pi i} \left[ \Phi_0(\lambda, \hat{\xi}) \right]^{-1} \frac{\partial}{\partial \lambda} \Phi_0(\lambda, \hat{\xi}) \]
\[ = -\sqrt{\pi} \int_{w-\infty}^{w+\infty} d\lambda \frac{e^{-\lambda \text{tr} \, S}}{2\pi i} \frac{\partial}{\partial \lambda} \log \det \left[ \Phi_0(\lambda, \hat{\xi}) \right] . \] (4.56)
Thus, the problem is now reduced to the computation of the integral over $\lambda$. This is not at all trivial because of the presence of two non-commuting matrices, essentially, $A^{-1}(AC + CA)A^{-1}$ and $A^{-1}(C^2 - \lambda I)A^{-1}$, where the matrices $A = \Gamma'(\hat{x})$ and $C = \Gamma^j(\hat{x})\hat{\xi}_j$ are defined by (2.82). We will report on this problem in a future work. Here let us just mention that in the particular case when $B = AC + CA = 0$ (for example, this is so in the case of the original Dirac operator) we get

$$\text{tr}_S [\Phi_0(\lambda, \hat{\xi})]^{-1} \frac{\partial}{\partial \lambda} \Phi_0(\lambda, \hat{\xi}) = \frac{1}{2} \text{tr}_S (C^2 - \lambda I)^{-1}, \quad (4.57)$$

and, therefore, one can compute the integral over $\lambda$ to obtain

$$A_1 = -\frac{\sqrt{\pi}}{2} \int_{\partial M} d\hat{x} \int_{\mathbb{R}^{n-1}} d\hat{\xi} \frac{1}{\pi^{(n-1)/2}} \text{tr}_S e^{-(C(\hat{x},\hat{\xi}))^2}. \quad (4.58)$$

Of course, for Laplace type operators, when $[C(\hat{x},\hat{\xi})]^2 = g^{ij}(\hat{x})\hat{\xi}_i\hat{\xi}_j$, the integral can be computed explicitly, which gives the induced Riemannian volume of the boundary, $A_1 = -(\sqrt{\pi}/2) N \text{vol}(\partial M)$, and coincides with the standard result for Dirichlet Laplacian [29].

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