New Applications of Resummation in Non-Abelian Gauge Theories: QED⊗QCD Exponentiation for LHC Physics, IR-Improved DGLAP Theory and Resummed Quantum Gravity

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We present the elements of three applications of resummation methods in non-Abelian gauge theories: (1), QED⊗QCD exponentiation and shower/ME matching for LHC physics; (2), IR improvement of DGLAP theory; (3), resummed quantum gravity and the final state of Hawking radiation. In all cases, the extension of the YFS approach, originally introduced for Abelian gauge theory, to non-Abelian gauge theories, QCD and quantum general relativity, leads to new results and solutions which we briefly summarize.

1. Introduction

Multiple gluon (n(G)) effects are already needed in many high energy collider physics scenarios, such as \( t \bar{t} \) production at FNAL, polarized pp processes at RHIC, b\( \bar{b} \) production at FNAL, · · · , and contribute a large part of the current uncertainty on \( m_t \), ~ 2-3 GeV [1]. For the LHC, and any TeV scale linear collider, the rather more demanding requirements make exponentiated soft n(G) results, exact through \( O(\alpha_s^2) \), realized by MC methods in the presence of parton showers without double counting on an event-by-event basis with exact phase space, an essential part of the necessary theory.

For such precision QCD results, ~ 1%, may QED higher order corrections also be relevant? The standard treatments of resummation in the two theories treat them separately, where we have considerable literature on resummation in QCD [2–5] and on exponentiation in QED [6–9]. We use the words resummation and exponentiation interchangeably here because in all cases a leading exponential factor is identified in the resummations which we discuss in this paper.

Results from Refs. [10–14] show a few per mille level QED effect from structure function evolution when QED and QCD DGLAP [15] kernels are treated simultaneously. After reviewing the YFS theory and our extension of it to QCD in the next Section, in Section 3 we combine [16] the respective two exponential algebras at the level of the amplitudes to treat QCD and QED exponentiation simultaneously, QED⊗QCD exponentiation, to be able to isolate, on an event-by-event basis, the possible interplay between the large effects from soft gluons and soft photons. We also show that the QCD exponentiation extends naturally to quantum gravity and allows us to develop UV finite resummed quantum gravity [17]. We find surprisingly that our extension of the ideas of Ref. [7] to Feynman’s formulation [18,19] of Einstein’s theory, the UV behavior of the respective theory becomes convergent.

The size of the QED⊗QCD threshold effects is illustrated in Sect. 4 wherein we present as well a new approach to shower/ME matching [20] for exact fixed-higher-order results. We also present IR-improved DGLAP theory [21] in this Section. In Sect. 5 we discuss the final state of Hawking [22] radiation for an originally very massive black hole in resummed
quantum gravity.

2. Review of YFS Theory and Its Extension to QCD

We consider first the QED case presented in Refs. [23] and realized by MC methods, for which, as we illustrate here for $e^q(p_1)e^{-q}(q_1) \rightarrow f(p_2)f(q_2) + n(\gamma)(k_1, \ldots, k_n)$, renormalization group improved YFS theory [24] gives

$$d\sigma_{exp} = e^{2\alpha} Re B + 2\alpha B \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^d k_j}{k_j^3} \int \frac{d^4 y}{(2\pi)^4} e^{iy(p_1+q_1-p_2-q_2-\sum k_j)+D}$$

$$\beta_n(k_1, \ldots, k_n) \frac{d^3 p_2 d^3 q_2}{P_2^0 Q_2^0}$$

where the YFS real and virtual infrared functions $B$, $D$, $B$ are known [7, 23]. Examples of the YFS for quarks $q, \bar{q}$, and anti-quarks $\bar{q}$, $q$ are given in the fourth paper in Refs. [23] for the MC BHLUMI 4.04. In Ref. [4] we have extended this YFS theory to QCD:

$$d\sigma_{exp} = \sum_n d\sigma^n$$

$$= e^{\text{SUM}_{IR}(QCD)} \sum_{n=0}^{\infty} \int \prod_{j=1}^{n} \frac{d^d k_j}{k_j^3} \int \frac{d^4 y}{(2\pi)^4} e^{iy(p_1+q_1-p_2-q_2-\sum k_j)+D_{QCD}}$$

$$\beta_n(k_1, \ldots, k_n) \frac{d^3 p_2 d^3 q_2}{P_2^0 Q_2^0}$$

where now the hard gluon residuals $\tilde{\beta}_n(k_1, \ldots, k_n)$ represented by

$$\tilde{\beta}_n(k_1, \ldots, k_n) = \sum_{l=0}^{\infty} \tilde{\beta}_n^{(l)}(k_1, \ldots, k_n)$$

are free of all infrared divergences to all orders in $\alpha_s(Q)$. The functions $\text{SUM}_{IR}(QCD)$, $D_{QCD}$, $\beta_n(k_1, \ldots, k_n)$ are the QCD analogs of the QED functions $2\alpha Re B + 2\alpha B$, D, $\beta_n(k_1, \ldots, k_n)$ and are defined in Ref. [4, 5].

For reference, the respective process exponentiated in [2] can be taken to be $q + q' \rightarrow q'' + q''' + n(G)$ for quarks $q, q'$ and anti-quarks $\bar{q}, \bar{q}'$. Our exponential factor corresponds to the $N=1$ term in the exponent in Gatherall’s formula [25] for the general exponentiation of the eikonal cross sections for non-Abelian gauge theory; his result is an approximate one in which everything that does not eikonalize and exponentiate is dropped whereas our result [4] is exact.

3. Extension to QED⊗QCD and Quantum Gravity

Simultaneous exponentiation of QED and QCD higher order effects [16] gives

$$B_{QCD}^{nls} \rightarrow B_{QCD}^{nls} + B_{QCD}^{nls} \equiv B_{QCD}^{nls},$$

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which leads to

$$d\sigma_{exp} = e^{\text{SUM}_{IR}(QCD)} \sum_{n,m=0}^{\infty} \int \prod_{j=1}^{n} \frac{d^d k_{j_1}}{k_{j_1}} \int \prod_{j=2}^{m} \frac{d^d k_{j_2}}{k_{j_2}}$$

$$\tilde{\beta}_n(k_1, \ldots, k_n; k_1', \ldots, k_m') \frac{d^3 p_2 d^3 q_2}{P_2^0 Q_2^0}$$

where the new YFS residuals $\tilde{\beta}_n(k_1, \ldots, k_n; k_1', \ldots, k_m')$, with $n$ hard gluons and $m$ hard photons, represent the successive application of the YFS expansion first for QCD and subsequently for QED.

The infrared functions are now

$$\text{SUM}_{IR}(QCD) = 2\alpha_s R_B^{QCD} + 2\alpha_s \tilde{B}^{QCD}$$

$$D_{QCD} = D_{QCD} + D.$$
for
\[ B_\rho''(k) = -2i\kappa^2k^4 \int \frac{d^4 \ell}{16\pi^2} \frac{\ell}{\ell^2 - \lambda^2 + i\epsilon} \frac{1}{(\ell^2 + 2\ell k + \Delta + i\epsilon)^2} \] \tag{7}

This is the basic result.

Note the following: \( \Sigma'_\epsilon \) starts in \( \mathcal{O}(\kappa^2) \), so we may drop it in calculating one-loop effects; explicit evaluation gives, for the deep UV regime, \( B_\rho''(k) = \frac{\kappa^2|k|^2}{8\pi^2} \ln \left( \frac{m^2}{m^2 + |k|^2} \right) \), so that \( \Sigma'_\epsilon \) falls faster than any power of \( |k|^2 \); and, if \( m \) vanishes, using the usual \(-\mu^2\) normalization point we get \( B_\rho''(k) = \frac{\kappa^2|k|^2}{8\pi^2} \ln \left( \frac{\mu^2}{|k|^2} \right) \)

so that \( \Sigma'_\epsilon \) again vanishes faster than any power of \( |k|^2 \). These observations mean [17] that, when the respective analoga of \( \Sigma'_\epsilon \) are used, one-loop corrections are finite. Indeed, all quantum gravity loops are UV finite [17]. We refer to this approach to quantum general relativity as resummed quantum gravity (RQG).

4. QED\(\otimes\)QCD Threshold Corrections, Shower/ME Matching and IR-Improved DGLAP Theory at the LHC

We shall apply the new simultaneous QED\(\otimes\)QCD exponentiation calculus to the single Z production with leptonic decay at the LHC (and at FNAL) to focus on the ISR alone, for definiteness. See also the work of Refs. [26–28] for exact \( \mathcal{O}(\alpha) \) results and Refs. [29–31] for exact \( \mathcal{O}(\alpha_s^2) \) results.

For the basic formula
\[ d\sigma_{exp}(pp \rightarrow V + X \rightarrow \ell' + X') = \sum_{i,j} \int dx_idx_j F_i(x_i)F_j(x_j) d\sigma_{exp}(x_i,x_j,s), \] \tag{8}

we use the result in [3] here with semi-analytical methods and structure functions from Ref. [32]. A MC realization will appear elsewhere [33].

Here we make the following observations.

- In [3] we do not attempt at this time to replace Herwig [34] and/or Pythia [35] – we intend to combine our exact YFS calculus, \( d\sigma_{exp}(x_i,x_j,s) \), with Herwig and/or Pythia by using them/it “in lieu” of \( \{F_i\} \) as follows: A: Use a Herwig/Pythia shower for \( p_T \leq \mu \), and YFS \( nG \) radiation for \( p_T > \mu \).
- B. Or, expand the Herwig/Pythia shower formula \( \otimes d\sigma_{exp} \) and adjust \( \hat{\beta}_{n,m} \) to exactness for the desired order with new \( \hat{\beta}_{n,m} \).

In either A or B, we first use \( \{F_i\} \) to pick \((x_1,x_2)\); make an event with \( d\sigma_{exp} \); then shower event using Herwig/Pythia via Les Houches recipe [36].

- This combination of theoretical constructs can be systematically improved with exact results order-by-order in \( \alpha_s, \alpha, \) with exact phase space.
- Possible new parton showers such as one based on the recent alternative parton evolution algorithm in Refs. [37] can also be used.
- Due to its lack of color coherence [38] Isajet [39] is not considered here.

With this said, we compute \( r_{exp} \), with and without QED, the ratio \( r_{exp} = \sigma_{exp}/\sigma_{Born} \) to get the results (We stress that we do not use the narrow resonance approximation here.)

\[ r_{exp} = \begin{cases} 1.1901, & \text{QCED } \equiv \text{QCD+QED, LHC} \\ 1.1872, & \text{QCD, LHC} \\ 1.1911, & \text{QCED } \equiv \text{QCD+QED, Tevatron} \\ 1.1879, & \text{QCD, Tevatron} \end{cases} \] \tag{9}

We note that QED is at .3% at both LHC and FNAL, this is stable under scale variations, we agree with the results in Refs. [26,27,29,30], and the QED effect similar in size to the structure function results [10–14]. DGLAP synthesisization [40] has not compromised the normalization.

With the precision tag needed for the LHC in mind, we apply QCD exponentiation theory to DGLAP kernels [21]: we get

\[ P_{q\bar{q}}(z) = C_F^2 F_{YFS}(\gamma_q) e^{\delta_q} \left[ \frac{1 + z^2}{1 - z} (1 - z)^{\gamma_q} - f_q(\gamma_q) \delta(1 - z) \right] \] \tag{10}

where \( f_q(\gamma_q) = \frac{2}{\gamma_q} - \frac{2}{\gamma_q + 1} + \frac{1}{\gamma_q + 2} \), \( \gamma_q = C_F \frac{\mu^2}{\mu^2} \), \( \delta_q = \frac{\gamma_q}{2} + \frac{\gamma_q - 1}{2} \) and \( F_{YFS}(\gamma_q) = \frac{4C_F}{\Gamma(1 + \gamma_q)} \). Similar results hold for \( P_{Gq}, P_{qG}, P_{G\bar{q}}, \) so
that we have:

\[ P_{\text{NS}}(z) = C_F F_{\text{FS}}(\gamma_\alpha) e^{\frac{n}{2}} \frac{1}{z} (1 - z)^2 \gamma_\alpha, \] (11)

\[ P_{\text{G}(z)} = 2C_F F_{\text{FS}}(\gamma_\alpha) e^{\frac{n}{2}} \left( \frac{1}{z} (1 - z)^2 \gamma_\alpha + \frac{1}{2} (1 + \gamma_\alpha (1 - z) + z(1 - z)^1 \gamma_\alpha) - f_G(\gamma_\alpha) \delta(1 - z) \right), \] (12)

\[ P_{\gamma_G}(z) = F_{\text{FS}}(\gamma_\alpha) e^{\frac{n}{2}} \left( \frac{1}{2} (z^2 (1 - z)^2 \gamma_\alpha + (1 - z)^2 \gamma_\alpha) \right), \] (13)

where \( \gamma_\alpha = C_G \frac{M^2}{\pi} \), \( \delta_\alpha = \frac{2}{\pi} + \frac{\alpha_2}{2} \left( \frac{z^2}{\beta} \right) \), and

\[ f_G(\gamma_\alpha) \equiv \frac{1}{C_G} \frac{1}{(2 + \gamma_\alpha)(4 + \gamma_\alpha) (3 + \gamma_\alpha) \gamma_\alpha (1 + \gamma_\alpha) (2 + \gamma_\alpha) \gamma_\alpha} \left( \frac{1}{(2 + \gamma_\alpha)(3 + \gamma_\alpha)(4 + \gamma_\alpha)} \right). \] (14)

Applying this new kernel set to the evolution of the parton distributions (Recall that moments of kernels \( \propto \) logarithmic exponents for evolution..), we have for the NS anomalous dimension \( A_{\text{NS}} \) the result

\[ A_{\text{NS}}^n = C_F F_{\text{FS}}(\gamma_\alpha) e^{\frac{n}{2}} \left[ B(n, \gamma_\alpha) + B(n + 2, \gamma_\alpha) - f_G(\gamma_\alpha) \right] \] (15)

where \( B(x, y) \) is the beta function given by \( B(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x + y) \). Compare the usual result

\[ A_{\text{NS}}^n \equiv C_F \left[ -\frac{1}{2} + \frac{1}{n(n + 1)} - 2 \sum_{j=2}^{n} \frac{1}{j} \right]. \] (16)

The IR-improved n-th moment goes for large n to a multiple of \(-f_G\), consistent with \( \lim_{n \to \infty} z^{n-1} = 0 \) for \( 0 \leq z < 1 \); the usual result diverges as \(-2C_F \ln n \). The two results differ for finite n as well: we get, for example, for \( \alpha_s \approx 0.118 \), \( A_{\text{NS}}^2 = C_F (-1.33), C_F (-0.966) \) for (13) and (14), respectively. For the n-th moment of the NS parton distribution itself, \( M_{\text{NS}}^n \), we have [21]

\[ M_{\text{NS}}^n(t) = M_{\text{NS}}^n(t_0) e^{\alpha(t)/\alpha(t_0) - \alpha(t)/\alpha(t_0)} \]

\[ \xrightarrow{t, t_0 \text{ large with } t > t_0} \quad M_{\text{NS}}^n(t_0) \left( \frac{\alpha(t_0)}{\alpha(t)} \right)^{a_n} \]

Figure 1. The graviton((a),(b)) and its ghost((c)) one-loop contributions to the graviton propagator. \( q \) is the 4-momentum of the graviton.

\[ \text{where } Ei(x) = \int_{-\infty}^{x} dy e^y / y \text{ is the exponential integral function,} \]

\[ \bar{a}_n = \frac{2C_F}{\beta_0} F_{\text{FS}}(\gamma_\alpha) e^{\frac{n}{2}} \left[ B(n, \gamma_\alpha) + B(n + 2, \gamma_\alpha) - f_G(\gamma_\alpha) \right] \]

\[ \bar{a}_n = \bar{a}_n \left( 1 + \frac{1}{2} (\alpha(t_0) - \alpha(t)) \right) \quad \text{with } \delta_1 = \frac{2}{\beta_0} \left( \frac{z^2}{\beta} \right). \]

This is to be compared with the un-IR-improved result where last line in eq. (17) holds exactly with \( a_n = 2A_{\text{NS}}^n / \beta_0 \). Comparison with the higher order exact results on the kernels in Refs. [41, 42] is done in Ref. [21].

5. Final State of Hawking Radiation

Consider the graviton propagator in the theory of gravity coupled to a massive scalar (Higgs) field, as studied by Feynman in Refs. [18, 19]. We have the graphs in Figs 1 and 2. Using the resummed theory, we get that the Newton potential becomes [17]

\[ \Phi_N(r) = -\frac{G_N M}{r} (1 - e^{-a r}), \] (19)

for \( a \equiv 0.210 M_{\text{Pl}} \). Our results imply

\[ G(k) = G_N / (1 + \frac{k^2}{a^2}) \]
so that we have fixed point behavior for \( k^2 \to \infty \), in agreement with the phenomenological asymptotic safety approach of Refs. [43–45]. We can also see [17] that our results imply that an elementary particle has no horizon which also agrees with the result of Ref. [44] that a black hole with a mass less than \( M_c \sim M_{Pl} \) has no horizon. The basic physics is the following: \( G(k) \) vanishes as \( 1/k^2 \) for \( k^2 \to \infty \).

There is a further agreement: final state of Hawking radiation of an originally very massive black hole near \( M_c \sim M_{Pl} \), quantum loops allow us to replace \( G(r) \) with \( G_N(1 - e^{-\alpha r}) \) in the lapse function for \( r < r_\ast \), the outermost solution of

\[
G(r) = G_N(1 - e^{-\alpha r}).
\]

In this way, we see that the inner horizon moves to negative \( r \) and the outer horizon moves to \( r = 0 \) at the new critical mass \( \sim 2.38 M_{Pl} \). We note that the results in Ref. [48] show that loop quantum gravity [49] concurs with this general conclusion.

We have arrived at the following prediction: there should energetic cosmic rays at \( E \sim M_{Pl} \) due the decay of such a remnant.

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