Convergence rates in the strong law of large numbers for conditional expectations

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Abstract

This paper deals with rates of convergence in the strong law of large numbers, in the Baum-Katz form, for partial sums of Banach space valued random variables. The results are then applied to solve similar problems for weighted partial sums of conditional expectations. They are further used to treat partial sums of powers of a reversible Markov chain operator. The method of proof is based on martingale approximation. The conditions are expressed in terms of the moments of the individual summands.

Keywords: Strong law of large numbers, Markov chains, nonstationary sequences, maximal inequalities, smooth Banach spaces

Mathematical Subject Classification (2020): 60F15, 60E15, 60J05.

1 Introduction

Let \((X_j)_{j \geq 1}\) be a sequence random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) with values in a separable Banach space, adapted to \((\mathcal{F}^j)_{j \geq 1}\), a decreasing sequence of sub-sigma algebras of \(\mathcal{F}\). We are going to study the rates of convergence for quantities such as \(P(\max_{1 \leq k \leq n} |S_k| > n^\alpha)\), where \(S_n = \sum_{j=1}^n X_j\). These probabilities often appear in various problems in probability theory, especially when studying the almost sure convergence. For integrable \(X\), define the conditional expectation \(E^j X = E(X|\mathcal{F}^j)\). Special attention will be given to the situation when \(X_j = E(X|\mathcal{F}^j)\). Given a stationary and reversible Markov chain \((\xi_i)_{i \in \mathbb{Z}}\) with values in a measurable space \((S, S)\), for an integrable function \(f\) defined on \(S\) with values in a separable Banach space let \(X_j = f(\xi_j)\). We denote by \(Q^k f(\xi_0) = E(X_k|\xi_0)\), and also derive similar results for \(P(\max_{1 \leq k \leq n} |\sum_{j=1}^k Q^j f| > n^\alpha)\).

The motivation for this study comes from a remarkable result, Theorem 3.11 in Derriennic and Lin (2001). In the context of additive functionals of stationary reversible Markov chains, for \(f\) centered at expectation and square integrable, if \(n^{-1}E(S_n^2) \to \sigma^2\) then we have that \(n^{-1/2} \sum_{k=1}^n Q^k f \to 0\) a.s., which is a law of large numbers under the normalization \(\sqrt{n}\). Theorem 3.9 in the same paper has a similar result for Harris recurrent Markov operators and some special functions.
These results suggest that we can find a rate of convergence associated to the strong law of large numbers with normalizer $\sqrt{n}$, which was solved in Theorem 9 in our paper. We shall also obtain rates of convergence for the strong law of large numbers under other normalizations, namely $n^{1/p}$ for $1 < p \leq 2$. We shall express the rates of this convergence in the Baum-Katz form \( \sum_{k=1}^{n} E^k X / n^{1/p} \to 0 \) almost surely.

The problem of the rates of convergence in the Marcinkiewicz–Zygmund strong law of large numbers for i.i.d. random variables with moments of order $p$, $1 \leq p < 2$, has been solved for real-valued variables in a seminal work by Baum and Katz (1965). This result generated a very large number of papers. Among them, we would like to mention the contributions by de Acosta (1981) who considered the same problem for independent random variables with values in Banach spaces of type $p$. Extending a result by Woyczyński (1981), Dedecker and Merlevède (2008) obtained these types of rates for martingales and other weakly dependent sequences of random variables with values in a $p$-smooth separable Banach space.

As far as we know, the results of this type have not been previously considered for sums of conditional expectations or reversible Markov operators. Since martingale methods are very fruitful, we shall base our study on martingale decompositions.

As preliminary results, we also study the maximum inequalities as well as Baum-Katz rates for general $S_n$. The results can be easily applied to linear combinations such as $\sum_{k=1}^{n} a_k E^k X$ or $\sum_{k=1}^{n} a_k Q^k f$, with $a_k$ a sequence of constants.

As in Dedecker and Merlevède (2008) or Cuny (2009), whenever possible, we shall work with variables in a separable Banach space $B$. For $x \in B$ for simplicity we shall denote the norm by $|x|_B$.

We denote by $L_p$ the set of measurable functions $X$ defined on a probability space, with values in a separable Banach space such that $\|X\|_p = E|X|^p < \infty$. For random variables $X$ in $L_p$ the notation $\|X\|_p \ll b$ means that there is a constant $C_p$ such that $\|X\|_p \leq C_p b$. For a sequence of positive constants $(a_n)_n$, $(b_n)_n$, $a_n \ll b_n$ means that there is a constant $C$ such that $a_n \leq Cb_n$.

Sometimes, we shall also impose conditions of the type $E|X| \log(1 + |X|) < \infty$. For simplicity we introduce the following function

$$l(x) = x \log(1 + x) \text{ for } x \geq 0.$$  

For integrable $X$, recall the notation $E^j X = E(X|\mathcal{F}^j)$, $j \geq 1$. For these definitions we direct the reader to the book of Ledoux and Talagrand (1991). Now denote the reverse martingale difference adapted to $(\mathcal{F}^i)_{i \geq 1}$ by

$$P^i(X) = E^i X - E^{i+1} X.$$  

(1)

Sometimes we shall assume in addition that the Banach space is separable and $r$-smooth for an $r$ such that $1 < r \leq 2$. We shall use this property or rather
its consequence, for any sequence of $B$-valued martingale differences $(X_i)_{i \geq 1}$, if $B$ is separable and $r$-smooth then for some $D > 0$,

$$E|X_1 + X_2 + ... + X_n|^r \leq D(E|X_1|^r + E|X_2|^r + ... + E|X_n|^r).$$

(2)

(see Assouad, 1975).

We should mention that, according to our knowledge, our results are also new for real-valued random variables.

2 Rates of convergence of Baum-Katz type

Definition 1 Let $(Y_n)_{n \geq 1}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a Banach space $B$, such that for some $1 \leq p \leq 2$ and every $n$, $Y_n$ is in $L^p$. We say that $(Y_n)_{n \geq 1}$ satisfies the strong law of large numbers with Baum-Katz rates if for any $\alpha$ with $1/p \leq \alpha \leq 1$ and every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{p\alpha - 2} P(\max_{1 \leq j \leq n} |Y_j| > \varepsilon n^\alpha) < \infty. \quad (3)$$

As explained by Dedecker and Merlevède (2008) in their Remark 1 and Remark 2, the Baum-Katz type of summability (3) implies the following results:

Remark 2 Relation (3) implies that

$$n^{-1/p} Y_n \rightarrow 0 \text{ a.s.}$$

and also, in case $\alpha p > 1$

$$P(\sup_{k > n} k^{-\alpha} |Y_k| \geq \varepsilon) = o(n^{1-p\alpha}) \text{ as } n \rightarrow \infty.$$

All our results are based on martingale decomposition. For this reason we present first the situation for martingales.

Theorem 3 Let $1 < p < 2$ and $r > p$. Let $(D_n)_{n \geq 1}$ be a sequence of reverse martingale differences defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a $r$-smooth separable Banach space, adapted to a sequence of decreasing sub-sigma fields of $\mathcal{F}$, $(\mathcal{F}^k)_{k \geq 1}$ and such that $\sup_n ||D_n||_p = K < \infty$. Denote $M_n = \sum_{j=1}^{n} D_j$. Then $(M_n)$ satisfies (3). If $\sup_n E(|D_n|) = K' < \infty$ then $(M_n)$ satisfies (3) with $p = 1$ and $\alpha = 1$.

There is a difference between Theorem 3 and Theorem 2 in Dedecker and Merlevède (2008). We replace in its statement condition $\sup_n P(|D_n| > x) \leq P(|D| > x)$ for a random variable $D$ satisfying $E|D|^p < \infty$ with $\sup_n E|D_n|^p < \infty$, which is a weaker assumption. We also assume $\sup_n E(|D_n|) < \infty$ instead of $E(|D|) < \infty$. We have just to notice that their proof, and also the original proof in Baum-Katz (1968), work with small changes. This fact was also noticed by
Our first result shows that Theorem 3 can be extended to a class of nonstationary generalized martingales.

**Theorem 4** Let $(X_j)_{j \geq 1}$ be a sequence of random variables with values in a separable Banach space, adapted to a sequence of decreasing sub-sigma fields of $\mathcal{F}$, $(\mathcal{F}_k)_{k \geq 1}$. For some $1 < p < 2$ and each $j$, $|X_j|$ is in $L_p$. Assume that (2) is satisfied for some $r > p$ and

$$\sup_n E |E^n(S_n)|^p < \infty. \quad (4)$$

Then $(S_n)_{n \geq 1}$ satisfies (3).

This theorem has connections with Theorem 6 in Dedecker and Merlevède (2008), who treated the stationary case under martingale-like conditions. Condition (4) is a type of generalized reverse martingale condition, which is also related to that one used by Gordin (1969) to obtain a martingale decomposition in the stationary case. The difference is that we do not assume stationarity but we use the conditioning with respect to a decreasing family of sigma algebras. To me more precise, in the stationary case, $(X_j)_{j \in \mathbb{Z}}$, the condition used in the literature to obtain a martingale coboundary decomposition, are moments conditions imposed to $E(S_n | \mathcal{F}_0)$, where $\mathcal{F}_0$ is the sigma algebra generated by $(X_j)_{j \leq 0}$. These conditions are difficult to extend to a nonstationary setting. An attempt to generalize Gordin’s condition to the nonstationary sequence is given in Theorem 4.1 in Merlevède et al. (2019). The difficulty of using that generalized martingale condition is that the martingale difference in the decomposition (4.1) in Merlevède et al. (2019) is constructed using limits and it is not easily tractable. So, in our paper, conditioning with respect to a decreasing sequence of sigma algebra is an important feature, which makes possible Theorem 4 and the other results presented in this paper.

It is also interesting to note that Theorem 4 is a generalization of Theorem 3 for the situation $1 < p < 2$, since for a reversed martingale condition (4) becomes $\sup_n \|X_n\|_p < \infty$. In the case $p = 2$ it is well-known that for a stationary i.i.d. sequence of random variables we have instead the law of the iterated logarithm. For the case $p = 2$ and variables with values in a 2-smooth Banach space, the law of the iterated logarithm is stated in Corollary 8.8 in Ledoux and Talagrand (1988). When $p = 1$, (3) holds if we impose as an additional assumption (17) of Proposition 20. Furthermore, this additional assumption cannot be removed (see Elton (1981)).

Now we restrict the class of the variables $(X_j)_{j \geq 1}$ and take them of the form $X_j = a_j E^jX$, where $X$ is an integrable random variable and $a_j$ a sequence of constants. We take $a_j$ real, but complex valued constants can be treated in the same way. We shall assume that $\sum_{i=1}^\infty |a_i| = \infty$ since otherwise, because $\max_{1 \leq k \leq n} |\sum_{j=1}^k a_j E^jX| \leq \max_{1 \leq k \leq n} |E^kX|$, by Doob’s maximal inequality, relation (4) trivially holds as soon as $X \in L_p$. 

Stoica (2011), who consider real valued martingale differences. For completeness we provide a short argument in Appendix.
We shall also use the following notations
\[ s_j = \sum_{i=1}^{j} a_i, \quad s_0 = 0, \quad s_k^* = \max_{1 \leq j \leq k} \left| \sum_{i=1}^{j} a_i \right|, \]
and
\[ b_k = \max \left( k^{-1} (s_{4k}^*)^2, s_k^2 - s_{k-1}^2 \right). \]  \( \text{(5)} \)

For this class of random variables we can also obtain rates of convergence of the type \( \text{(3)} \) for the case \( p = 2 \). Actually, this case is very important. Next theorem summarizes all the situations for \( 1 \leq p \leq 2 \).

**Theorem 5** Let \( X \) be in \( L^p \) with \( 1 \leq p \leq 2 \) and assume that
\[ \sum_{k \geq 1} k^{-2} E|s_{4k}^* X|^p < \infty. \]
Then, \( \left( \sum_{j=1}^{k} a_j E^j X \right)_{k \geq 1} \) satisfies \( \text{(3)} \) under any one of the following additional conditions:

(A) \( 1 < p < 2 \), \( \text{(2)} \) holds for some \( r > p \), and \( \sup_n E|s_n P^n X|^p < \infty. \)

(B) \( p = 1 \) and \( \sup_n E|s_n P^n X| \)

(C) \( p = 2 \), (2) holds with \( r = 2 \), and \( \sum_{n=1}^{\infty} n^{-1} E|s_n P^n X|^2 < \infty. \)

By taking particular values of constants we obtain various particular results. For instance, if \( (s_n) \) is bounded we can remove \( s_n \) and \( s_n^* \) in all the conditions of Theorem 5. For the situation for \( a_i = 1 \) for all \( i \in N \) we have \( s_n = s_n^* = n \) for all \( n \in N \). When \( a_i = i^{-1/2} \) for all for all \( i \in N \) we have \( s_n = s_n^* = n^{1/2} \) for all \( n \in N \), and so on.

When the variables have values in a separable Hilbert space the case \( p = 2 \) simplifies. For Hilbert spaces condition \( \text{(2)} \) always holds with \( p = 2 \).

**Theorem 6** Assume that \( X \) has values in a separable Hilbert space, \( E|X|^2 < \infty \), and
\[ \sum_{n=1}^{\infty} n^{-1/2} E|E^n X|^2 < \infty. \]
Then
\[ \sum_{n=1}^{\infty} n^{-2} E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 < \infty, \]
and therefore \( \text{(3)} \) holds with \( p = 2 \) and any \( \alpha > 0 \).

In particular:

**Corollary 7** Assume that \( X \) has values in a separable Hilbert space, \( E|X|^2 < \infty \), and
\[ \sum_{j=1}^{\infty} E|E^j X|^2 < \infty. \]
Then
\[ \sum_{n=1}^{\infty} n^{-2} E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^j X \right|^2 < \infty, \]
and therefore, \( \text{(3)} \) holds with \( p = 2 \) and any \( \alpha > 0 \).
3 Results for reversible Markov chains

All the results presented so far can be adapted to the context of stationary reversible Markov chains \((\xi_i)_{i \in \mathbb{Z}}\) defined on \((\Omega, \mathcal{F}, P)\) with values in a general measurable space \((S, \mathcal{S}, \pi)\) with stationary transitions \(Q(x, A)\). Reversible, means that the distribution of \((\xi_i, \xi_{i+1})\) is the same as the distribution of \((\xi_{i+1}, \xi_i)\).

For \(f\) defined on \(S\) with values in a separable Banach space \(B\), integrable, for \(n \in \mathbb{N}\) denote by \(Q^n f(\xi_0) = E(f(\xi_n)|\xi_0)\).

We also denote by \(Q\) the operator defined on integrable \(f\) by

\[
Qf(x) = \int_S f(y)Q(x, dy).
\]

We denote the invariant distribution by \(\pi\), which is a measure on \(S\). The integral with respect to \(\pi\) is denoted by \(E_\pi\).

First, we give two theorems for \(p = 2\), which shad light on some familiar results. We shall use below the same notations as in the previous sections.

**Theorem 8** For \(f\) with values in a separable Hilbert space with \(E_\pi |f|^2 < \infty\), we have

\[
E_\pi \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} Q^j f \right|^2 \ll \sum_{j=1}^{n} j E_\pi |Q^j f|^2.
\]

This maximal inequality gives the following Baum-Katz type result.

**Theorem 9** Let \(f\) be as in Theorem 8. Assume that we have

\[
\sum_{n=1}^{\infty} E_\pi |Q^n f|^2 < \infty.
\]

Then,

\[
\sum_{n=1}^{\infty} n^{-2} E_\pi \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} Q^j f \right|^2 < \infty.
\]

and therefore (6) holds for \(\left( \sum_{j=1}^{k} Q^j f \right)_{k \geq 1}\) with \(p = 2\) and any \(\alpha > 0\).

As mentioned in Remark 3 this result implies that

\[
\frac{1}{\sqrt{n}} \sum_{j=1}^{n} Q^j f \to 0 \quad \pi - a.s.
\]

and so (3) gives additional information on the rate of convergence in the law of large numbers given in Theorem 3.11 in Derriennic and Lin (2001).

Next, we give the Baum-Katz type result for reversible Markov chains for other values of \(p\).

**Theorem 10** Assume \(f\) has values in a separable Banach space, and for \(1 < p < 2\), \(E_\pi |f|^p < \infty\). Assume that (2) holds for an \(r\) with \(r < p\) and \(1/p \leq \alpha \leq 1\). Also assume that \(\sup_{n} \|k P^k f\|_p < \infty\) and \(\sum_{n=1}^{\infty} n^{p-2} \|Q^n f\|_p^p < \infty\). Then (5) holds for \(\left( \sum_{j=1}^{k} Q^j f \right)_{k \geq 1}\).
Remark 11 Under the conditions of Theorem 10 we have
\[ n^{-1/p} \sum_{j=1}^{n} Q^j f \to 0 \text{ } \pi \text{ - a.s.} \]
and also
\[ \pi(\sup_{k>n} k^{-\alpha} | \sum_{j=1}^{k} Q^j f | \geq \varepsilon) = o(n^{1-p\alpha}) \text{ as } n \to \infty. \]

Remark 12 If \( f \) is real valued, mean 0 and has finite second moment, the conclusion of Theorem 9 also holds with condition (6) replaced by one of the following equivalent conditions

(a) \( \sum_{k=1}^{n} E_\pi(fQ^k f) \) is bounded in \( L_1 \).

(b) \( \sup_{n} E(S_n)^2 / n < C. \)

(c) \( \lim_{n \to \infty} E(S_n)^2 / n = \sigma^2. \)

(d) \( \int_{-1}^{1} \frac{1}{1-t} d\rho_f < \infty. \)

(e) \( f \in \sqrt{1-QL_2^0}. \)

Above, \( \rho_f \) denotes the spectral measure of \( f \) associated with the self-adjoint operator \( Q \) and function \( f \) on \( L_2(S,\pi) \). Also \( L_2^0 \) is the set of functions which are square integrable and have mean 0.

3.1 Random walks on orbits of probability preserving transformation

We give here an example of a family of reversible Markov chains for which the conclusion of Theorem 9 holds. This family of Markov chains was considered in Derriennic and Lin (2007).

Let \( \tau \) be an invertible measure preserving transformation on \( (S,A,\pi) \), and denote by \( U \), the unitary operator induced by \( \tau \) on \( L^2(\pi) \). Given a probability \( \nu = \{p_k : k \in \mathbb{Z}\} \) on \( \mathbb{Z} \), we consider the Markov operator \( Q \) with invariant measure \( \pi \), defined by

\[ Qf = \sum_{k \in \mathbb{Z}} p_k f \circ \tau^k \text{ for every } f \text{ with } E_\pi f < \infty. \]

This operator is associated to the transition probability

\[ Q(s,A) = \sum_{k \in \mathbb{Z}} p_k 1_A(\tau^k s), \quad s \in S, A \in A. \quad (8) \]

We assume that \( \nu \) is symmetric and therefore the operator \( Q \) is symmetric.
Denote by $\Gamma$ the unit circle. For every $\lambda \in \Gamma$ define the Fourier transform of $\nu$ by $\phi(\lambda) = \sum_{k \in \mathbb{Z}} p_k \lambda^k$. Since $\nu$ is symmetric, $\phi(\lambda) \in [-1, 1]$, and if $\mu_f$ denotes the spectral measure (on $\Gamma$) of $f \in L^2(\pi)$, relative to the unitary operator $U$, then, the spectral measure $\rho_f$ (on $[-1, 1]$) of $f$, relative to the symmetric operator $Q$ is given by

$$\int_{-1}^{1} \psi(s) \rho_f(ds) = \int_{\Gamma} \psi(\phi(\lambda)) \mu_f(d\lambda),$$

for every positive Borel function $\psi$ on $[-1, 1]$.

Verifying condition (d) in Remark 12, we easily obtain:

**Theorem 13** Let $\tau$ be an ergodic invertible measure preserving transformation on the probability space $(S, \mathcal{A}, \pi)$. Let $\nu = \{p_k : k \in \mathbb{Z}\}$ be a symmetric probability on $\mathbb{Z}$. Let $Q$ be the transition probability defined in (8). Let $f \in L^2(\pi)$ be such that

$$\int_{\Gamma} \frac{1}{1 - \phi(\lambda)} \mu_f(d\lambda) < \infty. \quad (9)$$

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} E \max_{1 \leq k \leq n} |\sum_{j=1}^{k} Q^j f|^2 < \infty.$$

Let $a \in \mathbb{R} - \mathbb{Q}$, and let $\tau$ be the rotation by $a$ on $\mathbb{R}/\mathbb{Z}$. Define a measure $\sigma$ on $\mathbb{R}/\mathbb{Z}$ by $\sigma = \sum_{k \in \mathbb{Z}} p_k \delta_{ka}$. For that $\tau$, the canonical Markov chain associated to $Q$ is the random walk on $\mathbb{R}/\mathbb{Z}$ of law $\sigma$.

In this setting, if $(c_n(f))$ denotes the Fourier coefficients of a function $f \in L^2(\mathbb{R}/\mathbb{Z})$, condition (9) reads as

$$\sum_{n \in \mathbb{Z}} \frac{|c_n(f)|^2}{1 - \phi(e^{2i\pi na})} < \infty.$$

4 Proofs

4.1 Proofs of maximal inequalities.

**Proposition 14** Let $p > 1$ and let $(X_j)_{j \geq 1}$ be a sequence of Banach space valued random variables in $L_p$ adapted to a sequence of decreasing sub-sigma fields of $\mathcal{F}$, $(F_k)_{k \geq 1}$. Then

$$E \max_{1 \leq k \leq n} |S_k|^p \ll E \max_{1 \leq k \leq n} |E^k S_k|^p + E |S_n|^p.$$

For $1 \leq p \leq 2$ and if $B$ is separable and $p$-smooth, we also have

$$E \max_{1 \leq k \leq n} |S_k|^p \ll E \max_{1 \leq k \leq n} |E^k S_k|^p + \sum_{i=1}^{n-1} E |P^i(S_i)|^p.$$
Proof
Assume \( S_0 = 0 \). It is easy to see that

\[
S_n = E^n S_n + \sum_{i=1}^{n-1} (E^i S_i - E^{i+1} S_i).
\]  

(10)

By taking the maximum

\[
\max_{1 \leq k \leq n} |S_k| \leq \max_{1 \leq k \leq n} |E^k S_k| + \max_{2 \leq k \leq n} \left| \sum_{i=1}^{k-1} P^i(S_i) \right|,
\]  

(11)

whence, by Minkowski’s type inequality for norms in \( L_p \) we obtain

\[
E \max_{1 \leq k \leq n} |S_k|^p \ll E \max_{1 \leq k \leq n} |E^k S_k|^p + E \max_{2 \leq k \leq n} \left| \sum_{i=1}^{k-1} P^i(S_i) \right|^p.
\]

By writing

\[
\left| \sum_{i=1}^{k-1} P^i(S_i) \right|^p \ll \left| \sum_{i=1}^{n-1} P^i(S_i) \right|^p + \left| \sum_{i=k}^{n-1} P^i(S_i) \right|^p,
\]

we deduce that

\[
E \max_{2 \leq k \leq n} \left| \sum_{i=1}^{k-1} P^i(S_i) \right|^p \ll E \left| \sum_{i=1}^{n-1} P^i(S_i) \right|^p
\]  

(12)

\[+ E \max_{1 \leq k \leq n-1} \left| \sum_{i=k}^{n-1} P^i(S_i) \right|^p.
\]

Since \( P^k(S_k) = E^k S_k - E^{k+1} S_k \) is a reverse martingale difference adapted to the decreasing sequence of sigma algebras \( (\mathcal{F}_k)_{k \geq 1} \), by Doob’s maximal inequality for submartingales, for \( p > 1 \)

\[
E \max_{1 \leq k \leq n-1} \left| \sum_{i=k}^{n-1} P^i(S_i) \right|^p \ll E \left| \sum_{i=1}^{n-1} P^i(S_i) \right|^p.
\]  

(13)

By relation (10),

\[
E \left| \sum_{i=1}^{n-1} P^i(S_i) \right|^p \ll E |E^n S_n|^p + E |S_n|^p,
\]

and the first inequality in this proposition follows.

The second inequality follows from (11) and (13) combined with (2). □

Remark 15 Note that if \( p = 2 \) and if the variables have values in a separable Hilbert space, then, for all \( n \geq 1 \),

\[
E \left| \sum_{i=1}^{n} P^i(S_i) \right|^2 = \sum_{i=1}^{n} \left( E|E^i S_i|^2 - E|E^{i+1} S_i|^2 \right).
\]

The next corollary follows from Proposition 14 applied to \( X_i = a_i E^i X \).
Corollary 16 Let $X$ be a random variable with values in a separable Banach space and, for any $p > 1$, $E|X|^p < \infty$. Then

$$E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^p \ll E \max_{1 \leq k \leq n} |s_k E X|^p + E \left| \sum_{j=1}^{n} a_j E^j X \right|^p.$$  \hspace{1cm} (14)

For $1 \leq p \leq 2$ and if $B$ is $p$-smooth, we also have

$$E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^p \ll E \max_{1 \leq k \leq n} |s_k E X|^p + \sum_{i=1}^{n-1} E|s_i P^i(X)|^p.$$  \hspace{1cm} (15)

Remark 17 For $p \geq 1$, an estimate of $E \max_{1 \leq k \leq n} |s_k E X|^p$ is

$$E \max_{1 \leq k \leq n} |s_k E X|^p \ll \sum_{k=1}^{n} k^{-1} (s_{2k})^p E|E X|^p.$$  

Proof of Remark 17

By Doob’s maximal inequality

$$E \max_{2^r \leq k \leq 2^{r+1}} |E^k X|^p \ll E|E^{2^r} X|^p.$$  

Note that

$$E \max_{1 \leq k \leq 2^r} |s_k E X|^p \leq \sum_{i=0}^{r-1} E \max_{2^{i} \leq k \leq 2^{i+1}} |s_k E X|^p \leq \sum_{i=0}^{r-1} (s_{2^{i+1}})^p E \max_{2^{i} \leq k \leq 2^{i+1}} |E X|^p \ll \sum_{i=0}^{r-1} (s_{2^{i+1}})^p E|E^{2^i} X|^p.$$  

Using the fact that $(s_{n})_{n \geq 1}$ is increasing and $(E|E^n X|^p)_{n \geq 1}$ is decreasing, we easily obtain

$$E \max_{1 \leq k \leq 2^r} |s_k E X|^p \ll \sum_{i=0}^{r-1} (s_{2^{i+1}})^p E|E^{2^i} X|^p \ll \sum_{i=1}^{2^{r-1}} i^{-1} (s_{4^i})^p E|E^{i} X|^p \ll \sum_{i=1}^{2^{r-1}} i^{-1} (s_{4^i})^p E|E^{i} X|^p.$$  

Now if $2^{r-1} < n \leq 2^r$, clearly

$$E \max_{1 \leq k \leq n} |s_k E X|^p \leq E \max_{1 \leq k \leq 2^r} |s_k E X|^p \ll \sum_{k=1}^{n} k^{-1} (s_{4^{k}})^p E|E X|^p.$$  

\[\square\]

For $p = 2$ we also have the following result:

Corollary 18 Assume that $X$ has values in a separable Hilbert space and $E|X|^2 < \infty$. Then

$$E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 \ll \sum_{k=1}^{n} b_k E|E^k X|^2.$$  

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Remark 19 In particular, under the conditions of Corollary 18, if $a_j = j^{-1/2}$ for all $j$, then
\[
E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} j^{-1/2} E^j X \right|^2 \ll \sum_{j=1}^{n} E|E^j X|^2,
\]
and if $a_j = 1$ for all $j$, then
\[
E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^j X \right|^2 \ll \sum_{j=1}^{n} j E|E^j X|^2.
\]

Proof of Corollary 18

In a Hilbert space $P^i(X)$ and $P^j(X)$ are orthogonal for $i \neq j$. By the properties of the conditional expectations and Remark 15, ($s_0 = 0, n \geq 2$)
\[
E \left| \sum_{i=1}^{n-1} s_i P^i(X) \right|^2 = \sum_{i=1}^{n-1} s_i^2 E|P^i(X)|^2
= \sum_{i=1}^{n-1} s_i^2 (E|E^i X|^2 - E|E^{i+1} X|^2) \leq \sum_{j=1}^{n-1} (s_j^2 - s_{j-1}^2) E|E^j X|^2.
\]
Combining the latter inequality with Proposition 14 and Remark 17, we get
\[
E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 \ll \sum_{k=1}^{n} (s_{4k}^2)^2 E|E^k X|^2
+ \sum_{k=1}^{n-1} (s_k^2 - s_{k-1}^2) E|E^k X|^2
\leq \sum_{k=1}^{n} b_k E|E^k X|^2.
\]
where $b_k$ is given in (5).

□

4.2 Proof of Baum-Katz rates

Based on Theorem 3 and martingale decomposition in (10) we establish first a technical result that can be exploited in different directions.

Proposition 20 Let $(X_j)_{j \geq 1}$ be a sequence of random variables with values in a separable Banach space adapted to a sequence of decreasing sub-sigma fields of $\mathcal{F}$, $(\mathcal{F}^k)_{k \geq 1}$. Assume that for some $1 \leq p \leq 2$, $(E^n S_n)_{n \geq 1}$ satisfies (3). If in addition one of (A'), (B') or (C') below is satisfied, then $(S_n)_{n \geq 1}$ satisfies (3).

(A') $1 < p < 2$, $(2)$ is satisfied for some $r > p$ and
\[
\sup_n E|P^n (S_n)|^p < \infty.
\]

(B') $p = 1$, $\alpha = 1$ and
\[
\sup_n E\ell(\{|P^n (S_n)|\}) < \infty.
\]
(C') \( p = 2 \), (2) is satisfied for \( p = 2 \), and

\[
\sum_{n=1}^{\infty} n^{-1} E |P^n(S_n)|^2 < \infty. \tag{18}
\]

**Proof of Proposition 20**

Let us first consider the case given in (A'). The starting point is the inequality (11), which shows that we can prove (3) by showing that each of the sequences \((E^n S_n)_{n \geq 1}\) and \((\sum_{i=1}^{n} P^i(S_i))_{n \geq 1}\) satisfy (3).

Since \( P^k(S_k) \) is a difference of a reverse martingale adapted to \((F^k)_{k \geq 1}\), by Theorem 3, the condition for the validity of (3) is

\[
\sup_n E |P^n(S_n)|^p < \infty.
\]

The point (B') of this proposition is proven in the same way.

For proving point (C'), we cannot use Theorem 3 for the martingale part, since, in general, it does not hold for \( p = 2 \). However we can use directly Doob’s maximal inequality combined with (2) with \( p = 2 \). We obtain,

\[
\sum_{n=1}^{\infty} n^{2-\alpha} P \left( \max_{1 \leq k \leq n} |P^i(S_i)| > \varepsilon n^{1/\alpha} \right) \ll \sum_{n=1}^{\infty} n^{-2} E \sum_{i=1}^{n} |P^i(S_i)|^2.
\]

By changing the order of summation

\[
\sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} E|P^i(S_i)|^2 \ll \sum_{i=1}^{\infty} i^{-1} E|P^i(S_i)|^2.
\]

\( \square \)

**Proof of Theorem 4**

By Proposition 20 we have to show that \( \sup_n E |P^n(S_n)|^p = C < \infty \) and \((E^n S_n)_{n \geq 1}\) satisfies (3). Note that by properties of conditional expectation (see page 43 in Ledoux and Talagrand (1991)),

\[
\sup_n E |P^n(S_n)|^p \ll \sup_n E |E^n S_n|^p < \infty.
\]

In order to establish (3) for the sequence \((E^n S_n)_{n \geq 1}\) we use a truncation argument.

\[
E^k S_k = Y^\prime_k + Y^{\prime\prime}_k \quad \text{where}
\]

\[
Y^\prime_k = E^k S_k I(|E^k S_k| \leq \varepsilon n^\alpha/2)
\]

\[
Y^{\prime\prime}_k = E^k S_k I(|E^k S_k| > \varepsilon n^\alpha/2).
\]

So

\[
P(\max_{1 \leq k \leq n} |E^k S_k| > \varepsilon n^\alpha) \leq P(|Y^\prime_k| > \varepsilon n^\alpha/2).
\]
By applying the Markov inequality, and taking into account the previous inequality,
\[
\sum_{n=1}^{\infty} n^{p \alpha - 2} P\left( \max_{1 \leq k \leq n} |E^k S_k| > \varepsilon n^\alpha \right) \ll \sum_{n=1}^{\infty} n^{p \alpha - 2 - \alpha} \sum_{k=1}^{n} E|Y_k''|
\leq \sup_{k \geq 1} \sum_{n=1}^{\infty} n^{(p-1)\alpha - 1} E|Y_k''|.
\]

For \( k \) fixed and \( u \in N \), denote \( b_{k,u} = P(u - 1 < |E^k S_k| \leq u) \). By changing the order of summation, for \( p > 1 \) we obtain
\[
\sum_{n=1}^{\infty} n^{(p-1)\alpha - 1} E|Y_k''| \ll \sum_{n=1}^{\infty} n^{(p-1)\alpha - 1} \sum_{u > n^\alpha} u b_{k,u}
\ll \sum_{u=1}^{\infty} u b_{k,u} \sum_{n < u^{1/\alpha}} n^{(p-1)\alpha - 1} \ll \sum_{u=1}^{\infty} u^p b_{k,u} \ll E|E^k S_k|^p.
\]
Therefore relation (3) is satisfied if we assume (4), which completes the proof of this Theorem.

Note that in case \( p = 1 \), we have
\[
\sum_{n=1}^{\infty} n^{p \alpha - 2} P\left( \max_{1 \leq k \leq n} |E^k S_k| > \varepsilon n^\alpha \right) \ll \sup_{k \geq 1} \sum_{u=1}^{\infty} u b_{k,u} \sum_{n < u^{1/\alpha}} n^{-1}
\ll \sup_{k \geq 1} (E|E^k S_k|),
\]
and we also have to impose (17).

\[ \square \]

**Proof of Theorem 5**

To prove (A) of this theorem we use Proposition 20 and verify the conditions there. In the context of this theorem, they are satisfied if
\[
\sup_n E|P^n (s_n X)|^p < \infty 
\quad \text{and} \quad
\sum_{n=1}^{\infty} n^{-2} E\left( \max_{1 \leq k \leq n} |s_k E^k X|^p \right) < \infty.
\]
By Remark 17 for any \( p \geq 1 \)
\[
E\left( \max_{1 \leq k \leq n} |s_k E^k X|^p \right) \ll \sum_{k=1}^{n} k^{-1} E|s^*_k E^k X|^p.
\]
and so
\[
\sum_{n=1}^{\infty} n^{-2} E\left( \max_{1 \leq k \leq n} |s_k E^k X|^p \right) \leq \sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{n} k^{-1} E|s^*_k E^k X|^p
\ll \sum_{k=1}^{\infty} k^{-2} E|s^*_k E^k X|^p < \infty.
\]
The proof of (B) is similar, with the difference that we use the suitable condition for the reversed martingale differences \( (s_j P^j (X)) \).
For proving (C) we proceed directly. By Corollary 16

\[ E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 \ll E \left( \max_{1 \leq k \leq n} \left| s_k E^k X \right|^2 \right) + \sum_{j=1}^{n} E \left| P^j (s_j X) \right|^2. \]

By the arguments used in the proof of point (A),

\[ \sum_{n=1}^{\infty} n^{-2} E \left( \max_{1 \leq k \leq n} \left| s_k E^k X \right|^2 \right) \ll \sum_{k=1}^{\infty} k^{-2} E \left| s_k E^k X \right|^2. \]

Now

\[ \sum_{n=1}^{\infty} n^{-2} \sum_{j=1}^{n} E \left| P^j (s_j X) \right|^2 \ll \sum_{j=1}^{\infty} j^{-1} E \left| P^j (s_j X) \right|^2, \]

and so

\[ \sum_{n=1}^{\infty} n^{-2} E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 < \infty. \]

Therefore (3) follows by the Markov inequality.

\[ \square \]

**Proof of Theorem 6**

By Corollary 18

\[ E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 \ll \sum_{k=1}^{n} b_k E \left| E^k X \right|^2. \]

Now

\[ \sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{n} b_k E \left| E^k X \right|^2 = \sum_{k=1}^{\infty} b_k E \left| E^k X \right|^2 \sum_{n \geq k} n^{-2} \ll \sum_{k=1}^{\infty} k^{-1} b_k E \left| E^k X \right|^2. \]

and so

\[ \sum_{n=1}^{\infty} n^{-2} E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} a_j E^j X \right|^2 < \infty. \]

\[ \square \]

### 4.3 Proofs for reversible Markov chains.

In this section we take \( \mathcal{F}_n = \sigma(\xi_i, i \leq n) \) and \( \mathcal{F}^n = \sigma(\xi_i, i \geq n) \). First, we shall establish the following lemma.

**Lemma 21** For \( n \geq 1 \)

\[ \sum_{j=1}^{2n} (Q^j f)(\xi_0) = E_0 \left( \sum_{j=1}^{n} E^j f (\xi_0) + \sum_{j=0}^{n-1} E^{j+1} f(\xi_1) \right). \]

**Proof of Lemma 21** We estimate \( E(X_{2j}|\mathcal{F}_0) = Q^{2j} f(\xi_0) \). By the Markov property and reversibility

\[ Q^{2j} f(\xi_0) = E(f(\xi_{2j})|\mathcal{F}_0) = E_0 E(f(\xi_{2j})|\xi_j) \]

\[ = E_0 E(f(\xi_0)|\mathcal{F}^j) = E_0 E^j f (\xi_0). \]

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By the above identity
\[ \sum_{j=1}^{n} Q^{2j} f(\xi_0) = E_0 \sum_{j=1}^{n} E^j f(\xi_0). \]  
(19)

Similarly
\[ (Q^{2j+1} f)(\xi_0) = E(f(\xi_{2j+1})|\xi_0) = E_0 E(f(\xi_{2j+1})|\xi_{j+1}) = E_0 E^{j+1} f(\xi_1), \]
and so
\[ \sum_{j=0}^{n-1} (Q^{2j+1} f)(\xi_0) = E_0 \sum_{j=0}^{n-1} E^{j+1} f(\xi_1). \]  
(20)

Overall, by (19) and (20), we have the result of this lemma.
\[ \square \]

**Remark 22** A similar result as in Lemma 21 holds for odd sums. We easily deduce
\[ \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} (Q^j f)(\xi_0) \right| \leq E_0 \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^j f(\xi_0) \right| + E_0 \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^{j} f(\xi_1) \right|. \]

**Proof of Theorem 8**

From Lemma 21 we expect that our results concerning the conditional expectations with respect to a decreasing sequence of sigma algebras will also hold for \( Q^n f \).

Directly from Remark 22 by stationarity,
\[ E \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} Q^j f(\xi_0) \right|^2 \leq 2 E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^j f(\xi_0) \right|^2, \]
and by the second part of Remark 19
\[ E \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} E^j f(\xi_0) \right|^2 \ll \sum_{j=1}^{n} j E |E^j f(\xi_0)|^2. \]
The result follows.
\[ \square \]

**Proof of Theorem 9**

By Theorem 8 we get
\[ \sum_{n=1}^{\infty} n^{-2} E_\pi \left( \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} Q^j f \right| \right) \ll \sum_{n=1}^{\infty} n^{-2} \sum_{j=1}^{n} j E_\pi |Q^j f|^2. \]
By changing the order of summation the result follows.
Here we sketch the proof of Theorem 10.

**Proof of Theorem 10**

From Lemma 21, we get

$$\sum_{j=1}^{2n} (Q^j f)(\xi_0) = E_0 \left( \sum_{j=1}^{n} E^j f(\xi_0) + \sum_{j=0}^{n-1} E^{j+1} f(\xi_1) \right).$$

Note that

$$\sum_{n=1}^{\infty} n^{p_\alpha - 2} P \left( \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} Q^j f(\xi_0) \right| > \varepsilon n^\alpha \right) \leq \sum_{n=1}^{\infty} n^{p_\alpha - 2} P \left( E_0 \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^j f(\xi_0) \right| > \varepsilon n^\alpha / 2 \right) + \sum_{n=1}^{\infty} n^{p_\alpha - 2} P \left( E_0 \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} E^{j+1} f(\xi_1) \right| > \varepsilon n^\alpha / 2 \right).$$

Now we have to carefully repeat the proof of Theorem 5 (A) and notice that in all the steps we use Markov inequality, which makes $E_0$ to void. We leave this task to the reader. As in Theorem 5 (A) with $a_i = 1$ for all $i$, we have to impose

$$\sup_n n E \left| P^n (f(\xi_0)) \right|^p < \infty$$

and

$$\sum_{n=1}^{\infty} n^{p-2} E \left| E^n f(\xi_0) \right|^p = \sum_{n=1}^{\infty} n^{p-2} E \left| Q^n f \right|^p < \infty.$$

**Proof of Remark 12**

First we find a more flexible maximal inequality which has interest in itself. From Theorem 8 with $f$ replaced by $f + Qf$, we get

$$E_\pi \max_{1 \leq k \leq 2n} \left( \sum_{j=1}^{k} (Q^j f + Q^{j+1} f) \right)^2 \ll \sum_{j=1}^{n} j E_\pi (Q^j f + Q^{j+1} f)^2.$$

But, by the fact that on $L_2$ the operator $Q$ is self-adjoint, with the notation $< f, g > = E_\pi fg$, $E_\pi (Q^j f + Q^{j+1} f)^2 = < f, Q^{2j} f > + 2 < f, Q^{2j+1} > + < f, Q^{2j+2} f >$. So, after some algebraic computation, using that $< f, Q^{2j} f > = \|Q^j f\|^2 \geq 0$ we obtain

$$\sum_{j=1}^{k} j E_\pi (Q^j f + Q^{j+1} f)^2 \ll \sum_{j=1}^{2k} j E_\pi (fQ^j f) + < f, Q^2 f >,$$
On the other hand, because

\[ E_{\pi} \max_{1 \leq k \leq 2n} \left( \sum_{j=1}^{k} (Q^j f + Q^{j+1} f) \right)^2 \ll \sum_{j=1}^{2n} j E_{\pi} (f Q^j f) + < f, Q^2 f >. \]

(21)

On the other hand, because

\[ \sum_{j=1}^{k} (Q^j f + Q^{j+1} f) = 2 \sum_{j=1}^{k} Q^j f - Qf + Q^{k+1} f, \]

we obtain

\[ \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} Q^j f \right| \ll \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} (Q^j f + Q^{j+1} f) \right| + \max_{1 \leq k \leq 2n} |Q^{k+1} f| + |Qf|. \]

To estimate the second moment of \( \max_{1 \leq k \leq n+1} |Q^k f| \) we use the Stein Theorem (see page 106 in Stein (1970) or Krengel (1895), page 190).

\[ E_{\pi} \max_{1 \leq k \leq 2n} |Q^{k+1} f|^2 \leq E_{\pi} (Q f)^2 = E_{\pi} (f Q^2 f). \]

By combining these results with the estimate in (21) we get

\[ E_{\pi} \max_{1 \leq k \leq 2n} \left| \sum_{j=1}^{k} Q^j f \right|^2 \ll \sum_{j=1}^{2n} j E_{\pi} (f Q^j f) + E_{\pi} (f Q^2 f), \]

which we combine now with Theorem 9.

The equivalences (a)-(c) are well-known results in the literature (see for instance pages 3 and 4 in Kipnis and Varadhan (1986)).

\[ \square \]

5 Appendix

Proof of Theorem 3 As usual, the proof is based on a truncation argument. Set

\[ M_k = M'_k + M''_k = \sum_{j=1}^{n} D'_j + \sum_{j=1}^{n} D''_j, \]

where

\[ D'_j = D'_{j,n} = D_j I(|D_j| \leq n^\alpha) - E^{j+1} D_j I(|D_j| \leq n^\alpha) \]

and

\[ D''_j = D''_{j,n} = D_j I(|D_j| > n^\alpha) - E^{j+1} D_j I(|D_j| > n^\alpha). \]

Now, by the Markov inequality,

\[ \sum_{n=1}^{\infty} n^{p\alpha-2} P(\max_{1 \leq k \leq n} |M''_k| > \varepsilon n^\alpha) \ll \sup_{1 \leq k \leq n} \sum_{n=1}^{\infty} n^{p\alpha-1-\alpha} E|D''_k| \ll \sup_{1 \leq k \leq n} \sum_{n=1}^{\infty} n^{(p-1)\alpha-1} E|D_k| I(|D_k| > n^\alpha). \]
For \( k \) fixed and \( u \) in \( N \), denote \( b_{k,u} = P(u - 1 < |D_k| \leq u) \). By changing the order of summation, for \( p > 1 \) we obtain

\[
\sum_{n=1}^{\infty} n^{(p-1)\alpha-1} E|D_k| I(|D_k| > n^\alpha) \ll \sum_{n=1}^{\infty} n^{(p-1)\alpha-1} \sum_{u>n^\alpha} u b_{k,u} \\
\ll \sum_{u=1}^{\infty} u b_{k,u} \sum_{n<n^{1/\alpha}} n^{(p-1)\alpha-1} \\
\ll \sum_{u=1}^{\infty} u^p b_{k,u} \ll E|D_k|^p < K.
\]

When \( p = 1 \)

\[
\sum_{n=1}^{\infty} n^{\alpha-2} P(\max_{1 \leq k \leq n} |M''_k| > \epsilon n^\alpha) \ll \sum_{u=1}^{\infty} (u \log u) b_{k,u} \\
\ll E(\|D_k\|) \leq K'.
\]

For the other part we apply Doob’s inequality for the reverse submartingale, together with the fact that \( B \) is \( r \)-smooth

\[
P(\max_{1 \leq k \leq n} |M'_k| > \epsilon n^\alpha) \ll E\|D_k\|^r \ll \sum_{k=1}^{n} E|D'_k|^r.
\]

But

\[
E|D'_k|^r \ll \sum_{1 \leq u \leq n^\alpha} u^r b_{k,u}.
\]

So

\[
\sum_{n=1}^{\infty} n^{\alpha-2} P(\max_{1 \leq k \leq n} |M'_k| > \epsilon n^\alpha) \leq \sup_{k} \sum_{n=1}^{\infty} n^{(p-r)\alpha-1} \sum_{1 \leq u \leq n^\alpha} u^r b_{k,u}.
\]

By changing the order of summation

\[
\sum_{n=1}^{\infty} n^{(p-r)\alpha-1} \sum_{1 \leq u \leq n^\alpha} u^r b_{k,u} \ll \sum_{u=1}^{\infty} u^r b_{k,u} \sum_{n>u^{1/\alpha}} n^{(p-r)\alpha-1} \\
\ll \sum_{u=1}^{\infty} u^p b_{k,u} \ll E|D_k|^p < K.
\]

The same computation works for \( p = 1 \) and the result follows.

\( \square \)

**Acknowledgement 23** This paper was partially supported by the NSF grant DMS-2054598.

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