Nonorientable four-ball genus can be arbitrarily large

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Abstract. The nonorientable four-ball genus $\gamma_4(K)$ of a knot $K \subset S^3$ is the smallest first Betti number of any smoothly embedded, nonorientable surface $F \subset B^4$ bounding $K$. In contrast to the orientable four-ball genus, which is bounded below by the invariants $\sigma, \tau,$ and $s$, the best lower bound in the literature on $\gamma_4(K)$ for any $K$ is 3. We prove that

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_1(K)),$$

where the first term is half the knot signature, and the second is the Heegaard-Floer $d$-invariant of the integer homology sphere given by $-1$ surgery on $K$. In particular, we show that $\gamma_4(T_{2k,2k-1}) = k - 1$.

1. Introduction

One measure of the complexity of a knot $K \subset S^3$ is the complexity, as codified by genus, of the simplest surface which bounds it. For example, the only knot which bounds a genus zero surface embedded in $S^3$ is the unknot. This definition of complexity depends dramatically on the class of surfaces allowed: orientable or nonorientable, embedded in $S^3$ or $B^4$, and for surfaces in $B^4$, whether or not the embedding is smooth or locally flat. (The genus of a nonorientable surface with boundary is defined to be its first Betti number $b_1$.) For example, the nonalternating knot $11_{31}$ from Thistlethwaite’s table bounds an orientable surface of genus 3 in $S^3$, a smooth orientable surface of genus 2 in $B^4$, and a locally flat orientable surface of genus 1 in $B^4$. Certifying the minimality of these surfaces requires a variety of modern and classical knot invariants: the Alexander polynomial $\Delta$ has degree 3, Ozsvath-Szabo’s $\tau$-invariant is equal to 2, and the Murasugi signature $\sigma$ is equal to 1; to construct the final surface, Stominiew found a genus one concordance to a knot with Alexander polynomial 1, which according to a result of Freedman bounds a locally flat disk.

We know that orientable techniques cannot apply verbatim to obstruct nonorientable surfaces because of a simple example: the $2k + 1$-twist torus knot $T_{2,2k+1}$ bounds a Mobius band in $S^3$, yet the genus $k$ Seifert surface in Figure 1.1 actually has minimal genus even among orientable, locally flat surfaces embedded in $B^4$ bounding the knot. The global property of orientability, perhaps recast as the existence of a top homology class or a complex structure, is somehow critical to both the proof and truth of the bounds involving $\Delta$, $\tau$, and $\sigma$. While some obstructions have been found to particular knots bounding Mobius bands or punctured Klein bottles in $B^4$ (see GL1 especially for a comprehensive survey), the following question remained open:

Question. Does every knot $K$ bound a punctured $\#^3\mathbb{RP}^2$ smoothly embedded in $B^4$?

The answer is, perhaps unsurprisingly, “no.”

1http://www.indiana.edu/~knotinfo/
2Gilmer and Livingston [GL1] use Casson-Gordon invariants to construct a family of knots $K_n$ such that $K_n$ does not bound a nonorientable ribbon surface in $B^4$ of genus less than $n$. 

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Theorem 1. Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then

$$b_1(F) \geq \frac{\sigma(K)}{2} - d(S^3_1(K)),$$

where $\sigma$ denotes the Murasugi signature and $d$ the Heegaard-Floer $d$-invariant of the integer homology sphere given by $-1$ surgery on $K$.

In particular, we show

Theorem 2. Any smoothly embedded surface $F \subset B^4$ bounding the torus knot $T_{2k,3k-1}$ has $b_1(F) \geq k - 1$.

The equivalent question in the topological category remains open.

A moment for notation: the minimal genus of any surface bounding $K \subset S^3$ will be written $g_3(K)$, $g_4(K)$, $g_4^{\text{top}}(K)$, $\gamma_3(K)$, $\gamma_4(K)$, $\gamma_4^{\text{top}}(K)$ depending on whether we allow orientable or nonorientable surfaces ($g$ vs. $\gamma$) embedded in $S^3$ or $B^4$ (3 vs. 4) smoothly or topologically (no superscript vs. $\text{top}$). Thus for $K = 11_{31}$, we have

$$2g_3(K) = 6 \quad 2g_4(K) = 4 \quad 2g_4^{\text{top}}(K) = 2 \quad \gamma_3(K) = 3 \quad \gamma_4(K) = ? \quad \gamma_4^{\text{top}}(K) = ?$$

This definite value for $\gamma_3(K)$, also called the crosscap genus, is due to Burton and Ozlen, who use integer programming and normal surface theory to construct a triangulation of the knot complement and certify minimal surfaces in it. Geometric techniques can also be used to exactly compute the nonorientable 3-genus of a torus knot $T_{p,q}$ using Gordon and Litherland. Let $$(F, \partial F) \hookrightarrow (B^4, S^3)$$ be an embedded surface bounding a knot $K$. The normal bundle $\nu(F)$ always admits a nonvanishing section $s$. On the boundary, $s|_{\partial F}$ provides a framing of $K$, which we use to define the normal Euler number of $F$:

$$e(F) := -\text{lk}(K, s(K)).$$
Gluing an orientable Seifert surface $\Sigma$ for $K$ to $F$ gives a closed surface in $B^4$ with self-intersection $e(F)$. If $F$ is orientable, then $F \cup \Sigma$ represents an integral homology class and self-intersection can be computed algebro-topologically; since $H_2(B^4,\mathbb{Z}) \cong 0$, $e(F)$ must be zero. If $F$ is nonorientable, then we must compute self-intersection geometrically. Take a transverse pushoff of $F$, and choose arbitrary orientations in the neighborhood of each intersection point. Together with the orientation of $B^4$, this allows us to assign signs to each intersection; the sum turns out to be independent of the choice of pushoff and local orientation. It must be even, since we may compute self-intersection algebraically over $\mathbb{Z}/2$, but it needn’t be zero. For example, the Mobius band bounding $T_{2,n}$ has normal Euler number $-2n$.

Let $W(F)$ denote the double cover of $B^4$ branched over $F$. Gordon and Litherland [GL2] use the $G$-signature theorem to show that the quantity

$$\sigma(W(F)) + \frac{e(F)}{2}$$

is independent of the choice of surface $F$ bounding $K$, and equal to the knot signature $\sigma(K)$. For any such $F$, then,

$$\left|\sigma(K) - \frac{e(F)}{2}\right| = |\sigma(W(F))| \leq b_2(W(F)) = b_1(F),$$

where the final equality can be proved using elementary algebraic topology.

This inequality is tight for both of the surfaces bounding $T_{2,2k+1}$ in Figure 1.1. The Seifert surface has $e(F) = 0$ and $b_1(F) = 2k$, the Mobius band has $e(F) = -2(2k+1)$ and $b_1(F) = 1$, and $\sigma(T_{2k,2k+1}) = -2k$. In light of the important role played by $e(F)$, it may be clarifying to sort surfaces based on the framing they induce on the knot, and try to compute

$$\gamma_4(K,n) := \min \left\{ \gamma(F) \mid (F,\partial F) \hookrightarrow (B^4,K) \text{ and } e(F) = 2n \right\}.$$ 

The signature inequality, in this notation, is $\gamma_4(K,n) \geq |\sigma(K) - n|$.

The strategy of this paper is as follows. First, we replace our nonorientable surface in $B^4$ with an orientable surface in another manifold:

**Proposition 3.** Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd $b_1$ bounding a knot $K \subset S^3$. Then there exists an orientable surface $F' \subset S^2 \times S^2 \setminus B^4$ which still bounds $K$, and has $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.

The construction is similar to one in [Yas].

We then attach a $-1$-framed 2-handle along $K$ to get a four-manifold $W$, with boundary $S^2_1(K)$. There is a closed, orientable surface $\Sigma$ in $W$, formed by union of $F'$ and the core of the 2-handle. By excising a neighborhood of $\Sigma$ from $W$, we get a negative semi-definite cobordism from a circle bundle over $\Sigma$ to $S^3_1(K)$. The definiteness of $W$ gives us an inequality between the Heegaard-Floer $d$-invariants of its two boundaries, ultimately yielding:

**Theorem 4.** Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then

$$\frac{e(F)}{2} \leq 2d(S^2_1(K)) + b_1(F).$$

That is, $\gamma_4(K,n) \geq n - 2d(S^3_1(K))$.

Combining this theorem with the signature inequality yields Theorem 1, which can be written as

$$\gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_1(K)).$$
The $d$-invariants of integer homology spheres are in general somewhat difficult to compute, though $d(S^3_1(K))$ can in general be calculated from the filtered Heegaard Floer knot complex $CFK^\infty(K)$ [Pet]. When $K$ admits a lens space surgery, however, these $d$-invariants can be read off from the Alexander polynomial of $K$. Using a recursive formula of Murasugi to calculate the signature of torus knots, we are able to prove Theorem 2.1 that $\gamma_4(T_{2k,2k-1}) \geq k - 1$. In fact, we can construct a surface $F_{2k,2k-1}$ for which equality holds.

**Proposition 5.** The torus knot $T_{2k,2k-1}$ has $\gamma_4(T_{2k,2k-1}) = k - 1$. That is, $T_{2k,2k-1}$ does not bound a punctured #\$k - 2\$RP$^2$ smoothly embedded in $B^4$, and does bound a punctured #\$k - 1\$RP$^2$.

The surface $F_{2k,2k-1}$ is an example of a more general construction. For each relatively prime $p$ and $q$, we find a nonorientable surface $F_{p,q}$ in $B^4$ bounding $T_{p,q}$, whose first Betti number satisfies the recursion $b_1(F_{p,q}) = b_1(F_{p-2t,q-2h}) + 1$ where $t$ and $h$ are the minimal nonnegative representatives of $-q^{-1}$ modulo $p$ and $p^{-1}$ modulo $q$, respectively. We conjecture that these surfaces always have minimal genus, ie, that $b_1(F_{p,q}) = \gamma_4(T_{p,q})$.

In contrast to the orientable case, where the so-called Milnor conjecture $g_4(T_{p,q}) = g_3(T_{p,q})$ holds, we show that $\gamma_4(T_{p,q}) = 1$ while Teregaito has computed that $\gamma_3(T_{p,q}) = 2$ [Ter].

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# 2. Constructing an orientable replacement

In this section, we prove

**Proposition 5.** Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd $b_1$ bounding a knot $K \subset S^3$. Then there exists an orientable surface $F' \subset S^2 \times S^2 \setminus B^4$ which still bounds $K$, and has $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.

**Proof.** We break the proof into four steps.

**Step 1:** There is an embedded disk $D \subset B^4$, with boundary contained in $F$, such that $F \setminus \partial D$ is orientable.

Since $F$ has odd $b_1$, it is diffeomorphic to a punctured orientable surface boundary-connect summed with a Mobius band (Figure 2.1). Let $C \subset F$ be the core of the Mobius band; note that $F \setminus C$ is orientable. After an ambient isotopy, we may arrange that $C$ lies in the sphere of radius $1/2$, $S^3_{1/2} \subset B^4$, and that $F$ intersects $S^3_{1/2}$ transversely. Think of $C$ as a knot: it bounds some immersed disk $D^2$ in $S^3_{1/2}$, with clasp and ribbon singularities (Figure 2.2). We may remove the ribbon singularities by pushing the inner immersed segment in towards the centre of the 4-ball. To remove the clasp singularities, we push both immersed segments of the disk off the 1/2-level, one in towards the centre, and the other out towards the boundary. (The ability to push the surface both inwards and outwards is crucial, since a knot on the boundary of the $B^4$ bounds an embedded disk only if it is slice.) By a small isotopy, we may arrange that this embedded disk $D$ bounding $C$ intersects $F'$ transversely on its interior.

Let $N$ be a small regular neighborhood of $D$.

**Step 2:** The intersection $\partial N \cap F$ is the link $L$ shown in Figure 2.3.

$N$ is diffeomorphic to $D \times D^2$, and intersects our surface $F$ in a Mobius band (in the neighborhood of $\partial D = C$) and a collection of disks $pt \times D^2$ (neighborhoods of the transverse intersections of $F$ with the interior of $D$). If we draw $S^3 = \partial N$ with its standard decomposition into solid tori $S^3 = S^1 \times D^2 \cup_{T_3} D^2 \times S^1$, we see $F \cap \partial N$ as the link $L$ consisting of a $(2k+1)$-cable of the core of the first factor, together with a collection of $l$ longitudes for the second. By construction, $L$ bounds a Mobius band disjoint union a collection of $l$ disks in $N \cong B^4$. 


Step 3: \( L \) bounds \( l + 1 \) disjoint embedded disks in \( S^2 \times S^2 \setminus B^4 \)

A handle decomposition for \( S^2 \times S^2 \setminus B^4 \) consists of two zero-framed 2-handles \( H_1 \) and \( H_2 \) attached along a Hopf link in the boundary \( S^3 \), together with a 4-handle. To construct the slice disks for \( L \), we begin with \( |k| + l \) parallel copies of the core of \( H_2 \) and 2 parallel copies of the core of \( H_1 \)—their boundaries form a multi-Hopf link, with components \( U_1, \ldots, U_{|k|+l}, L_1, L_2 \), as in the first frame of Figure 2.4. For each \( 1 \leq i \leq |k| \), connect \( U_i \) to \( L_1 \) with a twisted strip, and with one additional twisted strip, connect \( V_1 \) to \( V_2 \). Call the surface consisting of the parallel cores and the strips \( E \), and note that the boundary of \( E \) is isotopic to \( L \). Since each strip connects a distinct disk to \( L_1 \), \( E \), or rather a slightly isotoped copy of \( E \), is a collection of \( l + 1 \) disjoint embedded disks with boundary \( L \).

Step 4: Construct \( F' \), and compute \( b_1(F'') \) and \( e(F') \).

If we excize \( N \) from \( B^4 \), we are left with an orientable surface \( F'' \subset S^3 \times [0,1] \), with boundary \( K \) in \( S^3 \times \{0\} \) and \( L \) in \( S^3 \times \{1\} \). Attach \( S^2 \times S^2 \setminus B^4 \) along \( S^3 \times \{1\} \) to form a larger manifold, still diffeomorphic to \( S^2 \times S^2 \setminus B^4 \). The slice disks \( E \) for \( L \) combine with \( F'' \) to form an orientable surface \( F' \), whose only remaining boundary is the original knot \( K \).
For $k = -7$, $l = 3$, we have drawn the multihopf link bounding a collection of parallel disks, the strips which join them to form $E$, and the boundary of $E$, which is isotopic to $L$.

Since we have removed $l$ disks and an annulus from $F$, and replaced them with $l + 1$ disks, $b_1(F') = b_1(F) - 1$. It remains to compare the normal Euler numbers. The remaining $l$ unknots, $U_1, \ldots, U_l$ have the same framing induced by $E$ and $F \cap N$. The torus knot component of $L$ is bounded by a Mobius band in $F \cap N$, and by an interesting disk in $E$. We invite the reader to verify that the induced framings differ by 2, due to the difference between the vertical twisted strip connecting $V_1$ to $V_2$ in $E$ and the horizontal one in Mobius band. That is, $e(E) = e(F \cap N) + 2$. Since Euler number, like any self-intersection, is additive, $e(F') = e(F) + 2$. \hfill $\Box$

For future reference, we note that the homology class $[F'] \in H_2(S^2 \times S^2 \setminus B^4)$ is $(2, m)$, in the basis given by $H_1$ and $H_2$, with $m = |k| + l$. Since $F'$ is orientable, its algebraic self-intersection number, $4m$, must be equal to its geometric self-intersection number, $e(F')$.

3. $d$-invariants

Heegaard Floer homology associates to a 3-manifold $Y$ equipped with a Spin$^c$ structure $t$ a suite of $\mathbb{Z}[U]$-modules which fit into a long exact sequence:

$$\cdots \rightarrow HF^-(Y, t) \xrightarrow{i} HF^\infty(Y, t) \xrightarrow{\pi} HF^+(Y, t) \xrightarrow{\delta} HF^-(Y, t) \rightarrow \cdots$$

If $c_1(t)$ is torsion (in which case we also say that $t$ is torsion), then there is a $\mathbb{Q}$-grading $gr$ on the each of these groups which is respected by $i$ and $\pi$. The action of $U$ decreases grading by 2. If $Y$ is a rational homology sphere, then $HF^\infty(Y, t) \cong \mathbb{Z}[U, U^{-1}]$, and every Spin$^c$ structure is torsion. In that case, the $d$-invariant (or correction term) $d(Y, t)$ is the minimal grading of a non-$\mathbb{Z}$-torsion element of $HF^+(Y, t)$ in the image of $\pi$. 

If $b_1(Y) > 0$, then there is an additional action of $H := H_1(Y)/\text{Tors}$ on the $HF$ groups, which decreases grading by 1. If for every torsion $t \in \text{Spin}^c(Y)$, $HF^\infty(Y, t) \cong \mathbb{Z}[U, U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H$, then we say that $Y$ has standard $HF^\infty$. In that case, there are many correction terms, one for each generator of $\Lambda^* H$. We will be concerned with the bottom-most correction term, $d_t(Y, t)$, defined to be the minimal grading of a non-torsion element of $HF^+(Y, t)$ in the image of $\pi$ and in the kernel of the $H$-action. The $d$-invariants terms will be useful to us because of their relationship to definite cobordisms.

**Proposition 6.** [OS1] Let $Y$ be a closed oriented 3-manifold (not necessarily connected) with standard $HF^\infty$, endowed with a torsion $\text{Spin}^c$ structure $t$. If $X$ is a negative semi-definite four-manifold bounding $Y$ such that the restriction map $H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$ is trivial, and $s$ is a $\text{Spin}^c$ structure on $X$ restricting to $t$ on $Y$, then

$$c_1(s)^2 + b_2(X) \leq 4d_t(Y, t) + 2b_1(Y).$$

In the previous section, we constructed an orientable surface $F' \subset S^2 \times S^3 \setminus B^4$ with boundary $K \subset S^3$. Attach a $-1$-framed 2-handle along $K$ to form a 4-manifold $\overline{W}$ with boundary $S^3_1(K)$ and intersection form

$$Q_{\overline{W}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We may cap off $F'$ with the core of the 2-handle to form a closed surface $\Sigma$ with genus $g = (b_1(F) - 1)/2$, homology class $(1, 2, m)$, and self-intersection

$$n := 4m - 1 = e(F) + 1 > 0.$$

If we decompose $\overline{W} = \nu(\Sigma) \cup W$, then $W$ will be a negative semi-definite cobordism from $Y_{g,n}$, the Euler number $n$ circle bundle over $\Sigma$, to $S^3_1(K)$. Alternatively, we can view $W$ as a negative semi-definite four-manifold with disconnected boundary $Y_{g,n} \amalg S^3_1(K)$. To apply the above proposition, and so prove Theorem 4, we need to understand the homology, $HF^\infty$, and $d$-invariants of $Y_{g,n}$ and $S^3_1(K)$, and the intersection form on $W$.

The Gysin sequence for the disk bundle $\nu(\Sigma)$ gives

$$0 \to H^1(\nu(\Sigma)) \xrightarrow{\nu(\Sigma)} H^1(Y_{g,n}) \to H^2(\nu(\Sigma), Y_{g,n}) \xrightarrow{\nu(\Sigma)} H^2(Y_{g,n}) \to H^1(\Sigma) \to 0$$

where $e \in H^2(\nu(\Sigma)) \cong \mathbb{Z}$ is $n$ times the generator. Thus $H^2(Y_{g,n}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n$. Note that the restriction of $H^1(\nu(\Sigma))$ to $H^1(Y_{g,n})$ is an isomorphism. Since $H^1(\overline{W}) = 0$ (no 1-handles were used in its construction), the Mayer-Vietoris sequence

$$0 \to H^1(\overline{W}) \to H^1(\nu(\Sigma)) \oplus H^1(W) \to H^1(Y_{g,n}) \to H^2(\overline{W}) \to H^2(\nu(\Sigma)) \oplus H^2(W) \to H^2(Y_{g,n})$$

shows that $H^1(W) = 0$, trivially satisfying the restriction hypothesis of Proposition 6. Since $H^2(\overline{W}) \cong \mathbb{Z}^3$ has no 2-torsion, a $\text{Spin}^c$ structure on $\overline{W}$ is determined by its first Chern class. Any $\text{Spin}^c$ structure on $W$ will give us some inequality between $d$-invariants, but we will only need to consider a certain $\text{Spin}^c$ structure $s_i$ with $PD(c_1(s_i)) = (\pm 1, 2, 2a)$, where

$$a = \frac{2(m - g) - 1 \pm 1}{4}$$

and the sign is chosen so as to make $a$ an integer. The given vector is characteristic for $Q_{\overline{W}}$, so does correspond to a $\text{Spin}^c$ structure. Crucially for our later use, $c_1(s_i)$ evaluates to $n - 2g$ on $\Sigma$. 
To compute the $c_1^2$ term in the proposition, we decompose the intersection form of $W$ in terms of the $\mathbb{Q}$-valued intersection forms on $\nu(\Sigma)$ and $W$; if $c \in H^2(W)$, then

$$Q_W(c) = Q_{\nu(\Sigma)}(c|_{\nu(\Sigma)}) + Q_W(c|_W).$$

A generator of $H^2(\nu(\Sigma), Y_{g,n})$ maps to $n$ times the generator of $H^2(\nu(\Sigma))$ in the gysin sequence above, so $Q_{\nu(\Sigma)} = \left(\frac{1}{n}\right)$. The value of $c|_{\nu(\Sigma)} \in H^2(\nu(\Sigma))$ is determined by integrating it over $\Sigma$, giving

$$(3.1) \quad Q_W(c) = \frac{c(\Sigma)}{n} + Q_W(c|_W).$$

In our case,

$$c_1(s_t|_W)^2 = Q_W(c_1(s_t)) = \frac{c_1(s_t), [\Sigma]}{n} = -1 + 8a - \frac{(n - 2g)^2}{n} = -2 \pm 2 - \frac{4g^2}{n}.$$  \hspace{1cm}

The relevant $d$-invariant of $Y_{g,-n}$ is computed in section 9 of [OS1], for use in their proof of the Thom conjecture. If $n > 2g$, then

$$d_b(Y_{g,-n}, s_t|_{Y_{g,-n}}) = \frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}.$$  \hspace{1cm}

That calculation uses the integer surgeries exact sequence associated to the Borromean knot in $K \subset \#^{2g}S^1 \times S^2$: the $-n$ surgery on $K$ gives $Y_{g,-n}$. Since $\#^{2g}S^1 \times S^2$ has standard $HF^\infty$, so does $Y_{g,-n}$ (cf Proposition 9.4 of [OS1]). Finally, since $S^3(K)$ is an integer homology sphere, it also has standard $HF^\infty$.

We are now ready to prove Theorem 4. By Proposition 6, we have

$$c_1(s_t)^2 + b_2(W) \leq 4d_b(Y_{g,-n}, t) + 4d(S^3_{-1}(K)) + 2b_1(Y_{g,-n}) + 2b_1(S^3_{-1}(K)).$$

After substituting all the values computed above, this reduces to

$$\left(-2 \pm 2 - \frac{4g^2}{n}\right) + 2 \leq 4\left(\frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}\right) + 4d(S^3_{-1}(K)) + 2(2g).$$

If we take the unfavorable sign on $\pm$, and recall that $b_1(F) = 2g + 1$ and $e(F) + 1 = n$, we get the inequality

$$(3.2) \quad \frac{e(F)}{2} \leq 2d(S^3_{-1}(K)) + b_1(F).$$

This argument relied on a value for $d_b(Y_{g,-n})$ only valid if $n > 2g$, ie, $e(F) + 2 \geq b_1(F)$. Proposition 6 applied to the surgery cobordism $S^3 \to S^3_{-1}(K)$, guarantees that $d(S^3_{-1}(K)) \geq 0$, so if $e(F) + 2 < b_1(F)$, the above inequality is trivially satisfied.

The initial construction of an orientable replacement required that $b_1(F)$ be odd. Luckily, both sides of Equation 3.2 change by the same amount under a positive real 'blow-up.' More precisely, if we connect sum $F \subset B^4$ with the standard embedding of $\mathbb{R}P^2 \subset S^4$ with Euler number $+2$, then both $b_1$ and $e/2$ increase by 1. One way to construct this $\mathbb{R}P^2$ is to glue together the Mobius band and disk bounding $T_{2,-1}$ (cf the mirror of Figure 1.1 at $k = 0$), then push them off into opposite sides of $S^3 \subset S^4$. If $b_1(F)$ is even, we may apply Equation 3.2 to $F\# \mathbb{R}P^2$, and so deduce it for $F$.

This completes the proof of Theorem 4 and hence of Theorem 1.

Remark. Our final lower bound on $\gamma_4$ is the gap $\frac{\sigma(K)}{4} - d(S^3_{-1}(K))$. For alternating knots, this quantity is nonpositive—in [OS2], Ozsváth and Szabó show that

$$d(S^3_{-1}(K)) = \max\left(0, 2\left[\frac{\sigma(K)}{4}\right]\right).$$
For nonalternating knots, $\frac{\sigma(-)}{2}$ and $d(S^3_{-1}(-))$ can diverge widely, though both invariants satisfy a crossing-change inequality \cite{PET}:

$$\eta(K_+) \leq \eta(K_-) \leq \eta(K_+) + 2.$$  

If $K$ becomes alternating after $c$ crossing changes, then $\frac{\sigma(K)}{2} - d(S^3_1(K))$ can be as large as $2c$.

### 4. Torus Knots

Signatures of torus knots satisfy a recursion relation \cite{MK}. If $\sigma(p, q) := \sigma(T_{-p,q})$, then

$$\sigma(p, q) = \begin{cases} 
\sigma(q, p) & \text{if } q > p \\
\sigma(p - 2q, q) + q^2 (-1) & \text{if } 2q < p \ (q \text{ odd}) \\
-\sigma(2q - p, p) + q^2 - 2 (+1) & \text{if } 2q > p \ (q \text{ odd}) \\
p - 1 & \text{if } q = 2 \\
0 & \text{if } q = 1
\end{cases}$$

Let $\sigma_k := \sigma(T_{-2k, 2k-1}) = \sigma(2k, 2k - 1)$. Applying the first and third conditions twice, we arrive at the recursion

$$\sigma_k = 4k - 2 + \sigma_{k-1},$$

whence $\sigma_k = 2k^2 - 2$.

The $d$-invariants of torus knots are also simple to compute, since they admit lens space surgeries.

**Proposition 7.** \cite{OS1} Let $K$ be a knot admitting a positive lens space surgery. Then

$$d_{-1/2}(S^3_0(K)) = -\frac{1}{2} \quad \text{and} \quad d_{1/2}(S^3_0(K)) = \frac{1}{2} - 2t_0$$

where if

$$\Delta_K(T) = a_0 + \sum_{j=1}^{d} a_j (T^j + T^{-j})$$

then

$$t_0 = \sum_{j=1}^{d} ja_j.$$  

The $d$-invariants of zero-surgery are related to those of ±1-surgery via Proposition 4.12 of \cite{OS1}:

$$d \left( S^3_{-1}(K) \right) = d_{-1/2} \left( S^3_0(K) \right) + \frac{1}{2} \quad \text{and} \quad d \left( S^3_1(K) \right) = d_{1/2} \left( S^3_0(K) \right) - \frac{1}{2}.$$  

Since $T_{p,q}$ admits a positive lens space surgery, we have

$$d \left( S^3_{-1}(T_{p,q}) \right) = -d \left( S^3_1(T_{p,q}) \right) = -\left( d_{1/2} \left( S^3_0(T_{p,q}) \right) - \frac{1}{2} \right) = 2t_0.$$  

The Alexander polynomial of $T_{p,q}$ is

$$\Delta_{T_{p,q}}(T) = T^{-(p-1)(q-1)/2} \left( \frac{1 - T}{1 - T^p} \right)^{1/2} \left( \frac{1 - T^q}{1 - T^q} \right)^{1/2}.$$  

For torus knots $T_{2k, 2k-1}$, the Alexander polynomial has a simple form:

$$\Delta_{T_{2k, 2k-1}} = \sum_{j=1}^{k-1} T^{j(2k-1)} - T^{j(2k-1)-(k-j)} + T^{-(j(2k-1)-j)},$$

$$- T^{j(2k-1)+(k-j)}.$$
so

\[ t_0 = \sum_{j=1}^{k-1} j(2k-1) - (j(2k-1) - (k-j)) = \sum_{j=1}^{k-1} k - j = \frac{k^2 - k}{2}. \]

Hence

\[ d \left( S^3_1 (T_{-2k, 2k-1}) \right) = k^2 - k. \]

The relevant difference between signature and \( d \) is

\[ \frac{\sigma}{2} - d = k^2 - 1 - (k^2 - k) = k - 1. \]

Of course, the reflection of a surface bounding \( T_{2k, 2k-1} \) bounds \( T_{-2k, 2k-1} \).

**Corollary 8.** If \( F \subset B^4 \) is a smoothly embedded nonorientable surface bounding \( T_{2k, 2k-1} \subset S^3 \), then \( b_1(F) \geq k - 1 \).

We obtain this lower bound by the following construction. Consider \( T_{p,q} \) as actually lying in a standard torus, as in Figure 4.1. Take any two adjacent strands and join them with a strip, or, equivalently, perform...
an index 1 Morse move merging them. The resulting cobordism is nonorientable, since the strands were parallel; it is a punctured Mobius band. Since the resulting knot still lives on the torus, it must be $T_{r,s}$ for some $r$ and $s$. The values of $r$ and $s$ can be easily computed by orienting the resulting knot and counting the signed intersection with the horizontal and vertical generators of $H_1(T^2)$. A short calculation shows that

\[ r = p - 2t \quad s = q - 2h \]

where $t \equiv -q^{-1} \mod p$, with $0 \leq t < p$, and $h \equiv p^{-1} \mod q$, with $0 \leq h < q$. After an isotopy, $T_{r,s}$ will be in standard, taut form on the torus, and we can repeat the process. Eventually, we arrive at $T_{n,1}$ for some $n$, which is just an unknot. By concatenating all of these cobordisms, then capping off the final unknot with a disk, we have successfully found a surface $F_{p,q}$ in $B^4$ bounding $T_{p,q}$.

For example, if $p = 2k$ and $q = 2k - 1$, we have $t = -(-1)^{-1} = 1$ and $h = 1^{-1} = 1$, giving $r = 2k - 2$ and $s = 2k - 3$. Thus $T_{2k,2k-1}$ has a $\chi = -1$ cobordism to $T_{2(k-1),2(k-1)-1}$. Concatenate $k - 1$ of these, then cap off $T_{2,1}$ with a disk to get a closed surface $F_{2k,2k-1} \subset B^4$ bounding $T_{2k,2k-1}$, with $b_1(F_{2k,2k-1}) = k - 1$.

Since the isotopies and Morse moves take place inside of the torus, we can actually embed each of these cobordisms in a thickened torus $T^2 \times [-\epsilon,\epsilon]$ in $S^3$, where we view the $[-\epsilon,\epsilon]$ direction as a sort of time. The obstruction to embedding all of $F_{p,q}$ in $S^3$ is that the final disk bounding $T_{n,1}$ cuts through all of the previous layers, unless $n = 0$. To get a surface in $S^3$, we must continue with these within-torus cobordisms: $T_{n,1} \mapsto T_{n-2,1} \mapsto \cdots$. If $n$ is even, or, equivalently, if $pq$ was even to start, then we do get a surface in $S^3$. Teragaito has computed $\gamma(T_{p,q})$, and it agrees with $b_1(F)$ [Ter]. For example, $\gamma(T_{2k,2k-1}) = k$. If $n$ is odd, then this construction fails to give a surface in $S^3$, though a slight modification (cf. [Ter] Remark 4.9) will do.

We conjecture that the surfaces $F_{p,q}$ bounding $T_{p,q}$ are best possible, that $b_1(F_{p,q}) = \gamma_4(T_{p,q})$. Many pairs $(p,q)$ for which the conjecture holds can be certified using the $d$-invariant bounds of this paper. Similar invariants, derived by considering larger surgeries on the knot, give even more examples. These stronger bounds will be discussed in a forthcoming paper.

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