Colorful combinatorics and Macdonald polynomials

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Abstract

The non-negative integer cocharge statistic on words was introduced in the 1970’s by Lascoux and Schützenberger to combinatorially characterize the Hall-Littlewood polynomials. Cocharge has since been used to explain phenomena ranging from the graded decomposition of Garsia-Procesi modules to the cohomology structure of the Grassman variety. Although its application to contemporary variations of these problems had been deemed intractable, we prove that the two-parameter, symmetric Macdonald polynomials are generating functions of a distinguished family of colored words. Cocharge adorns one parameter and the second measure its deviation from cocharge on words without color. We use the same framework to expand the plactic monoid, apply Kashiwara’s crystal theory to various Garsia-Haiman modules, and to address problems in K-theoretic Schubert calculus.

Dedicated to Alain Lascoux

1 Introduction

Kostka-Foulkes polynomials, $K_{\lambda \mu}(t) \in \mathbb{N}[t]$, describe the connections between characters of $GL_n(\mathbb{C})$ and the Hall-Steinitz algebra $[\text{Gre55}]$, give characters of cohomology rings of Springer fibers for $GL_n$ $[\text{Spr78}, \text{HS77}]$, and are graded multiplicities of modules for the general linear group obtained by twisting functions on the nullcone by a line bundle $[\text{Bry89}]$. Lusztig $[\text{Lus83}, \text{Lus84}, \text{Lus83}]$ showed they are the $t$-analog of the weight multiplicities in the irreducible representations of the classical Lie algebras,

$$K_{\lambda \mu}(t) = \sum_{\sigma \in \mathcal{W}} (-1)^{\ell(\sigma)} P_\lambda(\sigma(\lambda + \rho)) - (\mu + \rho),$$

obtained from a $t$-deformation of Kostant’s partition function $P$ defined by

$$\prod_{\alpha \text{ positive roots}} \frac{1}{(1-t\alpha)} = \sum_\beta P_\beta x^\beta.$$

Algebraically, Kostka-Foulkes polynomials are the entries in transition matrices between the Schur and the Hall-Littlewood $[H_\mu(x; t)]$ bases for the algebra $\Lambda$ of symmetric functions in variables $x = x_1, x_2, \ldots$ over the field $\mathbb{Q}(t)$. In fact, this reflects the graded decomposition of a simple quotient of the coinvariant ring viewed as

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an $S_n$-module $[GP92]$; each irreducible submodule of polynomials with homogeneous degree $r$ corresponds to a Schur function with coefficient $t^r$, and the sum over all irreducibles corresponds to a Hall-Littlewood polynomial.

Kostka-Foulkes polynomials are wrapped in the most fundamental combinatorial ideas. Namely, the set of words forms a monoid under the operations of RSK-insertion and jeu-de-taquin. The monoid structure was first motivated by Schützenberger [Sch77] in his proof that the Schubert structure constants for the cohomology of the Grassmann variety are enumerated by Young tableaux with a distinguished Yamanouchi property. The structure is compatible with the assignment of each word to a non-negative integer (statistic) called cocharge. Lascoux and Schützenberger [LS78] proved that the generating function for tableaux weighted by this statistic is precisely $K_{\lambda\mu}(t)$. Consequently, the spectrum of topics surrounding Kostka-Foulkes polynomials is accessible from a purely combinatorial study of cocharge. For example,

$$H_\mu(x; t) = \sum t^{\text{cocharge}(T)} s_{\text{shape}(T)}(x),$$

summing over all Young tableaux $T$ with $\mu_1$ ones, $\mu_2$ twos, etc.

In the 1980’s, Macdonald introduced a basis for $\Lambda$ over the field $\mathbb{Q}(q, t)$ to unify the Hall-Littlewood and the Jack polynomials (wave equations of the Calogero-Sutherland-Moser model [For92]). Ensuant studies of the basis have impacted an impressive range of areas including the representation theory of quantum groups [EAK94], double affine Hecke algebras [Che95], the shuffle algebra and diagonal harmonics [HHL+05], the geometry of Hilbert schemes [Hai01], affine Schubert calculus [LLM03], the elliptic Hall algebra of Schiffmann-Vasserot [SV11], and extensions of HOMFLY polynomials for knot invariants [ORS12].

Early characterizations of Macdonald polynomials were oblique, revealing little more than that they are elements of $\mathbb{Q}[q, t][x_1, x_2, \ldots]$. Nevertheless, using brute force to compute examples, Macdonald conjectured that the entries $K_{\lambda\mu}(q, t)$ of certain of their transition matrices lie in $\mathbb{N}[q, t]$. Garsia modified Macdonald’s polynomials so that $K_{\lambda\mu}(q, t)$ appeared as Schur expansion coefficients of the resulting polynomials $\tilde{H}_\mu(x; q, t)$, thus piquing the interest of representation theorists for whom Schur functions are synonymous with irreducible $S_n$-modules. The $q, t$-Kostka coefficients in

$$\tilde{H}_\mu(x; q, t) = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda(x),$$

have since been a matter of great interest.

Rich theories were born from the compelling feature that the $q, t$-Kostka coefficients reduce to the Kostka-Foulkes polynomials at $q = 0$. In [GH93], Garsia and Haiman introduced $S_n$-modules $R_\mu$, for $\mu$ a partition of $n$, given by the space of polynomials in variables $x_1, \ldots, x_n; y_1, \ldots, y_n$ spanned by all derivatives of a certain simple determinant $\Delta_\mu$. They conjectured that the dimension of $R_\mu$ equals $n!$, and that the modules provide a representation theoretic framework for (2). Their interpretation was designed to imply the Macdonald positivity conjecture. Haiman spent years putting together algebraic geometric tools which ultimately led him to prove the conjectures in [Hai01].

Formula (1) set the gold standard for defining Macdonald polynomials, but cocharge was abandoned after efforts to give a manifestly positive formula for generic $\tilde{H}_\mu(x; q, t)$ led no further than the most basic examples. In 2004, an explicit formula for Macdonald polynomials was established by Haglund-Haiman-Loehr. Rather than using Young tableaux and cocharge, the formula involves the major index and an intricate inversion-like statistic:

$$\tilde{H}_\mu(x; q, t) = \sum_F q^{\text{inv}(F)} t^{\text{maj}(F)} \prod_i x_{F(i)},$$

over all $\mathbb{Z}_+$-valued functions (fillings) $F$ on the partition $\lambda$. The Schur expansion was expected to come shortly behind this breakthrough, but it took another decade even to recover the Hall-Littlewood case. In [Rob17], Austin
Roberts converted the $q = 0$ case of (3) into a new Schur expansion formula:

$$\tilde{H}_\mu(x;0,t) = \sum_{\lambda \in \mathcal{U}} t^{\text{maj}(F)} s_{\text{weight}(F)}(x),$$

over a mysterious subset $\mathcal{U}$ of fillings (see §4.3). Roberts’ questioning of the comparison of his formula with the earlier formulation (1) sparked our interest and led us to revive the study of cocharge.

We discovered that the classical combinatorics of cocharge supports Macdonald polynomials as naturally as it does the less intricate setting surrounding Hall-Littlewood polynomials. The key idea is a broadening of the plactic monoid [LS81] [LLT02] whereby each letter in a word is colored. Of particular importance is the subset of tabloids, words with an increasing condition used by Young to define (Specht) modules. We prove that Macdonald polynomials are colored tabloid generating functions, weighted by cocharge and a betrayal statistic which measures the variation of cocharge on colored words from its value on usual words.

**Theorem.** For any partition $\mu$,

$$\tilde{H}_\mu(x; q, t) = \sum_T q^{\text{betrayal}(T)} t^{\text{cocharge}(T)} x^{\text{shape}(T)},$$

over colored tabloids with $\mu_1$ ones, $\mu_2$ twos, and so forth.

Further applications of colored words are geometrically inspired. The classical example in Schubert calculus addresses the cohomology of the Grassmann variety where the structure constants $c^\nu_{\lambda \mu}$ count Yamanouchi tableaux. Schubert calculus vastly expanded with efforts to characterize the structure of $K$-theory and (quantum) cohomology of other varieties; the problems are a combinatorial search for alternative, or more refined notions, of Yamanouchi. Thus, the combinatorial ideas surrounding the plactic monoid are often revisited in Schubert calculus. In fact, $c^\nu_{\lambda \mu}$ can be viewed as the number of skew tableaux with zero cocharge and the broader scope of colored words fits in well.

We extend Van Leeuwen’s approach [vL01] to the Yamanouchi condition using Young tableaux companions. We show that colored tabloids serve as companions for the generic $\mathbb{Z}_+$-valued functions used in the Macdonald polynomials (3). From this point of view, a super-Yamanouchi condition arises and is applicable to $K$-theoretic Schubert calculus problems as well as Kostka-Foulkes polynomials. The companion map $\epsilon$ simultaneously gives relations between

- the formulas (1) and (4) for $q = 0$ Macdonald polynomials,
- genomic tableaux of Pechenik-Yong [PY17] and set-valued tableaux of [Buc02], introduced to study $K$-theoretic problems in Schubert calculus, and
- cocharge and the Lenart-Schilling statistic [LS13] for computing the (negative of the) energy function on affine crystals.

Colored words also support equivariant $K$-theory of Grassmannians and Lagrangians, but details are sequestered in a forthcoming paper.

We investigate representation theoretic lines with the theory of crystal bases, introduced by Kashiwara [Kas90] [Kas91] in an investigation of quantized enveloping algebras $U_q(g)$ associated to a symmetrizable Kac–Moody Lie algebra $g$. Integrable modules for quantum groups play a central role in two-dimensional solvable lattice models. When the absolute temperature is zero ($q = 0$), there is a distinguished crystal basis with many striking features. The most remarkable is that the internal structure of an integrable representation can be combinatorially realized by associating the basis to a colored oriented graph whose arrows are imposed by the Kashiwara (modified root)
operators. From the crystal graph, characters can be computed by enumerating elements with a given weight, and the tensor product decomposition into irreducible submodules is encoded by the disjoint union of connected components. Hence, progress in the field comes from having a natural realization of crystal graphs.

A double crystal structure on colored tabloids using only the type-A crystal operators and jeu-de-taquin provides a lens giving clarity to problems in Macdonald theory and in Schubert calculus. Several crystal graphs arise simultaneously through different colored tabloid manifestations of tabloids. From these, we deduce Schur expansion formulas for dual Grothendieck polynomials and the $q$-Macdonald’s conjecture as one about Schur positivity: the $q=0$ case perfectly mimics the classical formula (1) for $q=0$. In particular,

**Theorem.** For any partition $\mu$,

$$\tilde{H}_\mu(x; 1, t) = \sum_{T} t^{\text{cocharge}(T)} s_{\text{shape}(T)},$$

over colored tabloids with column increasing entries.

## 2 Preliminaries

### 2.1 Garsia-Haiman modules

The algebra $\Lambda$ of symmetric functions in infinitely many indeterminants $x_1, x_2, \ldots$, over the field $\mathbb{Q}(q, t)$ has bases indexed by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell > 0) \in \mathbb{Z}^\ell$. The monomial basis is defined by elements $m_\lambda = \sum a^\lambda x^a$ taken over all distinct rearrangements $\alpha$ of $(\lambda_1, \ldots, \lambda_\ell, 0, \ldots)$. The homogeneous basis has elements $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, where $h_\lambda(x) = \sum_{\lambda_1 \leq \cdots \leq \lambda_\ell} x_{\lambda_1} \cdots x_{\lambda_\ell}$, and the power basis has elements $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, with $p_\lambda(x) = \sum_i x_{\lambda_i}^d$. A basis element indexed by partition $\lambda$ of degree $d = \sum_i \lambda_i$ (denoted by $\lambda + d$) is a sum of monomials of degree $d$.

The Hall-inner product, $\langle \cdot, \cdot \rangle$, is defined on $\Lambda$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda \mu} z_\lambda \quad \text{where} \quad z_\lambda = \prod_i a_\lambda^i t^{p_\lambda^i} \quad \text{for} \quad \lambda = (\cdots, 2^\ell, 1^a),$$

and $\delta_{\lambda \mu}$ evaluates to 0 when $\lambda \neq \mu$ and is otherwise 1. In fact, the basis of Schur functions, $s_\lambda$, can be defined as the unique orthonormal basis which is unitriangularly related to the monomial basis; for each $\lambda \vdash d$,

$$s_\lambda = m_\lambda + \sum_{\mu \vdash d} a_{\lambda \mu} m_\mu,$$

where dominance order $\mu \triangleleft \lambda$ is defined by $\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$ for all $k$.

Macdonald [Mac95] proved the existence of another basis of polynomials, $P_\lambda(x; q, t)$, also unitriangularly related to the monomials, but orthogonal with respect to the $q,t$-deformation of $\langle \cdot, \cdot \rangle$,

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda \mu} z_\lambda \prod_j \frac{1 - q^{1-j}}{1 - t^{1-j}}.$$

Of interest to combinatorialists, but not apparent from the definition, he conjectured that $P_\lambda(x; q, t)$ have certain transition coefficients lying in $\mathbb{Z}_{\geq 0}[q, t]$. Garsia modified $P_\lambda(x; q, t)$ into polynomials $\tilde{H}_\lambda(x; q, t)$ to rephrase Macdonald’s conjecture as one about Schur positivity: the $q,t$-Kostka coefficients in

$$\tilde{H}_\mu(x; q, t) = \sum_\lambda K_{\lambda \mu}(q, t) s_\lambda(x)$$

(6)
lie in \( \mathbb{Z}_{\geq 0} [q, t] \). See (3) for a precise definition of \( \tilde{H}_\mu(x; q, t) \).

Garsia’s approach appealed to a broader audience. Namely, results of Frobenius dictate that a positive sum of Schur functions models the decomposition of an \( S_n \)-representation into its irreducible submodules. Namely, for \( \sigma \in S_n \) and \( \lambda + n \), the value of the irreducible character \( \chi^\lambda \) of \( S_n \) at \( \sigma \) arises in

\[
s_\lambda = \frac{1}{n!} \sum_{\tau \in S_n} \chi^\lambda(\sigma) p_{\tau(\sigma)},
\]

where \( \tau(\sigma) \) is the cycle-type of \( \sigma \). Define the linear Frobenius map from class functions on \( S_n \) to symmetric functions of degree \( n \) by

\[
F_x = \frac{1}{n!} \sum_{\tau \in S_n} \chi^\lambda p_{\tau(\sigma)},
\]

and consider the Frobenius image of a doubly-graded \( S_n \)-module \( M = \bigoplus_{r,s} M_{r,s} \),

\[
F_{\text{char}(M)}(x; q, t) = \sum_{r,s} t^r q^s F_{\text{char}(M_{r,s})}.
\]

The function \( F_{\text{char}(M)}(x; q, t) \) is thus a positive sum of Schur functions with coefficients in \( \mathbb{Z}_{\geq 0} [q, t] \) by (7).

So launched the search for a bi-graded module \( M \) for which \( \tilde{H}_\mu(x; q, t) \) is the Frobenius image. Garsia and Procesi settled the \( q = 0 \) case and gave the perfect guide \cite{GP92}. In particular, they gave an algebraic approach to Hotta and Springer’s result that \( K_{\lambda \mu}(0, t) \) describes the multiplicities of \( S_n \) characters \( \chi^\lambda \) in the graded character of the cohomology ring of a Springer fiber, \( B_\mu \). The cohomology ring \( H^*(B_\mu) \) can be defined by a particular quotient,

\[
R_\mu = \mathbb{C}[y_1, \ldots, y_n]/I_\mu,
\]

of the coinvariant ring \( R_\mu(y) = \mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n) \). \( R_\mu(y) \) is the Garsia-Procesi module under the natural \( S_n \)-action permuting variables; they proved the ideal \( I_\mu \) is generated by Tanisaki generators, defined to be the elementary symmetric functions \( e_k(S) \) in the variables \( S = \{y_1, \ldots, y_n\} \subset \{y_1, \ldots, y_n\} \) when \( r > k > |S| - \# \) cells of \( \mu \) weakly east of column \( r \).

The simplicity of Garsia and Procesi’s definition led them to an algebraic proof that \( K_{\lambda \mu}(0, t) \in \mathbb{N}[t] \) and offered an attack on the \( q, t \)-Kostka polynomials. Given that the Frobenius image of \( R_\mu(y) \) is \( \tilde{H}_\mu(x; 0, t) \), the task was to define an \( S_n \)-module

\[
R_\mu(x; y) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I_\mu,
\]

under the diagonal \( S_n \)-action, simultaneously permuting the \( x \) and \( y \) variables, so that

\[
\tilde{H}_\mu(x; q, t) = F_{\text{char}(R_\mu(x; y))}(x; q, t).
\]

Garsia and Haiman found just the candidate; it is the ideal

\[
J_\mu = \left\{ f : f \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right) \Delta_\mu = 0 \right\},
\]

where \( \Delta_\mu \) is a generalization of the Vandermonde defined using a graphical depiction of \( \mu \). A lattice square \((i, j) \) lies in the \( i \)th row and \( j \)th column of \( \mathbb{N} \times \mathbb{N} \). The (Ferrers) shape of a composition \( \alpha = (x_1, \ldots, x_\ell) \in \mathbb{Z}_{\geq 0}^\ell \) is the subset of \( \mathbb{N} \times \mathbb{N} \) made up of lattice squares left-justified in the \( i \)th row, for \( 1 \leq i \leq \ell \). A lattice square inside a shape \( \alpha \) is called a cell. Given \( \mu + n \), the cells \((i, c_1), \ldots, (n, c_n)\) in \( \mu \) define

\[
\Delta_\mu = \det \begin{bmatrix}
\begin{array}{cccc}
1 & x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & \cdots & 1
\end{array}
\end{bmatrix}.
\]
Although the construction of the modules $R_d(x; y)$ is quite simple, the proof of \cite{Hai01} required sophisticated geometric techniques developed by Haiman.

## 2.2 Cocharge

How $R_d(x; y)$ decomposes into irreducible submodules remains an open problem. It is particularly intriguing in light of the perfect description for decomposing $R_\mu(y)$ in terms of the following statistic on words. Given a word $w$ in the alphabet $\mathcal{A}$, $w_B$ is the subword of $w$ restricted to letters of $B \subset \mathcal{A}$. When $B = \{i\}$, we use simply $w_i = w_B$.

The weight of a word $w$ is the composition $\alpha$, where $\alpha_i$ is the number of times $i$ appears in $w$. A word with weight $(1, \ldots, 1)$ is called standard. The cocharge of a standard word $w \in S_n$ is defined by writing $w$ counter-clockwise on a circle with a $\star$ between $w_1$ and $w_n$, attaching a label to each letter, and summing these labels. The labels are determined iteratively starting by labeling 1 with a zero. Letter $i$ is then given the same label as $i - 1$ as long as $\star$ lies between $i - 1$ and $i$ (reading clockwise) and it is otherwise incremented by 1.

The cocharge of a word $w$ with weight $\mu + n$ is defined by writing $w$ counter-clockwise on a circle and computing the cocharge of $\mu_1$ standard subwords of $w$. Letters of the $i^{th}$ standard subword are adorned with a subscript $i$ and this subword is determined iteratively from $i = 1$ as follows: clockwise from $\star$, choose the first occurrence of letter 1 and proceed on to the first occurrence of letter 2. Continue in this manner until $\mu_i$ has been given the index 1. Start again at $\star$ with $i + 1$, repeating the process on letters without a subscript. The cocharge of $w$ is the sum of the cocharge of each standard subword. The charge of a word $w$ of weight $\mu$ is

$$\text{charge}(w) = n(\mu) - \text{cocharge}(w),$$

where $n(\mu) = \sum_i (i - 1) \mu_i$.

**Example 1.** The words $w_1 = 6714235$ and $w_2 = 64534223511123$ written counter-clockwise on circles

\[ \begin{array}{ccc} 6 & \star & 5 \\ 7 & \circ & 3 \\ 1 & \circ & 2 \end{array} \quad \Rightarrow \quad \text{cocharge is 6.} \]

\[ \begin{array}{ccc} 2 & 3 & \star \\ 1 & 6 & 4 \\ 5 & 1 & 3 \end{array} \quad \Rightarrow \quad \text{cocharge is 15.} \]

Kostka-Foulkes polynomials require only words coming from Young tableaux. Use $\alpha \models n$ to denote that $\alpha$ is a composition of degree $n = |\alpha| = \alpha_1 + \alpha_2 + \cdots$. For compositions $\alpha$ and $\beta$ where $\alpha_i \leq \beta_i$ for all $i$, we say $\alpha \leq \beta$.

The skew shape of $\alpha \subseteq \beta$ is $\beta/\alpha$, defined by the set theoretic difference of their cells and of degree $|\beta| - |\alpha|$. A (semi-standard) tableau is the filling of a skew shape with positive integers which increase up columns and are not decreasing along rows (from west to east).

**Definition 2.** The reading order of any collection $S \subset \mathbb{N} \times \mathbb{N}$ is the total ordering on elements in $S$ defined by saying that lattice squares decrease from left to right, starting in the highest row and moving downward.

Given a tableau $T$, the reading word $w = \text{word}(T)$ is defined by taking $w_i$ to be the letter in the $i^{th}$ cell of $T$, where cells are read in decreasing reading order. The weight of tableau $T$ is the weight of its reading word and $T$ is called standard when word$(T)$ is standard. For skew shape $\lambda/\mu$ of degree $n$ and $\gamma \models n$, the set of tableaux of shape $\lambda/\mu$ and weight $\gamma$ is denoted by $\text{SSYT}(\lambda/\mu, \gamma)$. Lascoux and Schützenberger \cite{LS78} proved, for partitions $\lambda$ and $\mu$ of the same degree,

$$K_{\lambda\mu}(0, t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} t^{\text{cocharge}(T)}, \quad (9)$$
where the cocharge of a tableau $T$ is defined by $\text{cocharge}(\text{word}(T))$.

A similarly beautiful formula for the $q, t$-Kostka polynomials has been actively pursued for decades. Because $K_{\lambda\mu}(1, 1) = |SSYT(\lambda, 1^n)|$, the endgame is to establish a formula for $K_{\lambda\mu}(q, t)$ by attaching a $q$ and a $t$ weight to each standard tableau.

### 2.3 Macdonald polynomials

Although the Schur expansion of Macdonald polynomial still eludes us, Jim Haglund made a breakthrough in 2004 by proposing a combinatorial formula for $\tilde{H}_{\mu}(x; q, t)$. Rather than using semi-standard tableaux and cocharge, different statistics are associated to arbitrary fillings.

A filling $F$ of shape $\beta/\alpha$ and weight $\gamma$ is any placement of letters from a word with weight $\gamma$ into shape $\beta/\alpha$. The entry in row $r$ and column $c$ of $F$ is denoted by $F(r, c)$, and the set of fillings of shape $\beta/\alpha$ and weight $\gamma$ is $F(\beta/\alpha, \gamma)$. Immediate from the definition is $|F(\alpha/\beta, \gamma)| = (|\gamma|, |\gamma_1, \gamma_2, ..., \gamma_{l(\gamma)}|)$. (10)

For a filling $F$ of partition shape $\lambda$, an inversion triple is a triple of entries $(r, t, s)$ which are arranged in a collection of cells in $F$ of the form $\small{\begin{array}{c} r \\ s \end{array}}$, and meeting the criteria that $r = s \neq t$ or some cycle of $(r, t, s)$ is decreasing, i.e. $r > s > t, s > t > r, \text{or } t > r > s$. If the cells containing $r$ and $t$ are in the first row we envision that $s = 0$. The inversion statistic is the number $\text{inv}(F)$ of inversion triples in $F$. The major index of $F$ is 

$$\text{maj}(F) = \sum_{F(\alpha) > F(r, c)} (\lambda'_c - r + 1)$$

where $\lambda'_c$ is the number of cells in column $c$ of $\lambda$. Every partition $\lambda$ has a conjugate $\lambda'$ given by reflecting shape $\lambda$ about $y = x$. Alternatively, a filling $F$ has a descent at cell $(r, c)$ when $F(r, c) > F(r-1, c)$ and $\text{maj}(F)$ is the number of descents of $F$, each weighted by the number of cells appearing weakly above it in $F$.

**Example 3.** The following filling $F \in F((333), (342))$ has $\text{maj}(F) = 1 + 2 + 1 = 4$ and $\text{inv}(F) = 4$.

**Theorem 4.** 

For any partition $\lambda$,

$$\tilde{H}_{\lambda}(x; q, t) = \sum_{F \in F(\lambda')} q^{\text{inv}(F)} t^{\text{maj}(F)} x^{\text{weight}(F)}.$$ (11)

### 3 Frobenius image of Garsia-Haiman modules using cocharge

We introduce a new combinatorial structure and prove that the Macdonald polynomials are generating functions attached to cocharge and a second statistic called betrayal.
3.1 Colored words and circloids

A colored letter \( x_i \) is a letter \( x \) in an alphabet \( A \) adorned with a subscript (its color) \( i \) from \( A \). A colored word \( w \) is a string of distinct colored letters. The weight of \( w \) is a skew composition recording the colors which adorn each letter; weight(\( w \)) = \( \alpha/\beta \) where \( \{\beta_1 + 1, \ldots, \alpha_1\} \) are the colors attached to letter \( x \) in \( w \). When \( \beta \geq 0 \), we simply say the weight of \( w \) is \( \alpha \).

Colored words also come equipped with shapes which are assigned using the prismatic order on colored letters: \( u_v > r_c \) when \( r < u \) or \( (r = u \text{ and } c > v) \). Equivalently,

\[
u_v > r_c \iff \text{cell (} u, v \text{) precedes (} r, c \text{) in reading order.}
\]

A strict composition (one without zero entries) \( \gamma \models n \) is a shape admitted by a colored word \( w = w_n \cdots w_1 \) if \( w_1 > \cdots > w_{y_1}, w_{y_1+1} > \cdots > w_{y_1+y_2}, \) and so forth. A weak composition \( \gamma' \models n \) is a shape admitted by \( w \) when the strict composition obtained by removing the zeroes from \( \gamma' \) is a shape admitted by \( w \).

The set of all shapes admitted by a colored word \( w = w_1 \cdots w_n \) corresponds multisets that contain

\[
\text{Des}(w) = \{ p : w_p < w_{p+1} \},
\]

under the bijection sending compositions of degree \( n \) to sub-multisets of \( \{1, \ldots, n-1\} \) defined by

\[
(\gamma_1, \ldots, \gamma_t) \mapsto \text{set}(\gamma) = \{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\cdots+\gamma_{(\gamma_t)-1}\}. \text{A}\text{shape}\text{isa}\text{strict}\text{composition}\text{ifandonlyif}\text{itcorrespondsto}\text{a}\text{true}\text{set.}
\]

**Example 5.** The colored word \( w = 3_21_31_31_22_21_23_23_3 \) has weight \((3,3,3)\) and \( \text{Des}(w) = \{1,3,6\} \). It thus admits shape \( \alpha = (3,3,2,1) \) and, for example, \( \beta = (3,1,2,2,1) \).

A circular representation of colored words is convenient when attaching statistics. We write a colored word \( w \) counter-clockwise on a circle and separate its letters into sectors to give a concept of shape.

**Definition 6.** A circloid \( C \) of shape \( \gamma \models n \) is a placement of \( n \) distinct colored letters on the perimeter of a subdivided circle such that, reading clockwise from a distinguished point \( \bigstar \), \( \gamma_x \) colored letters lie in decreasing prismatic order in sector \( x \), for \( x = 1, \ldots, \ell(\gamma) \).

Each circloid \( C \) is uniquely associated to a colored word \( w \) of the same shape by reading the letters of \( C \) in counter-clockwise order. The weight of \( C \) is defined to be weight(\( w \)). The set of circloids of weight \( \alpha/\beta \) and shape \( \gamma \) is denoted by \( C(\gamma, \alpha/\beta) \). Note that letter \( b \) appears in a circloid \( C \in C(\gamma, \alpha/\beta) \) exactly \( \alpha_b - \beta_b \) times since there are \( \alpha_b - \beta_b \) colors needed to adorn the set of \( b \)'s.

**Example 7.** Circloids \( C_1 \in C((3,3,2,1),(3,3,3)) \) and \( C_2 \in C((3,1,2,2,1),(3,3,3)) \) with underlying \( w = 3_21_31_31_22_21_23_23_3 \) are

\[
C_1 = \begin{array}{c}
1 \\
3 \\
1 \\
2 \\
1 \\
3 \\
2 \\
3 \\
2 \\
1 \\
3 \\
\end{array}
\]

\[
C_2 = \begin{array}{c}
1 \\
3 \\
1 \\
2 \\
1 \\
3 \\
2 \\
3 \\
2 \\
1 \\
3 \\
\end{array}
\]

If unspecified, entries and positions of a circloid are always taken clockwise. For example, \( 3_2 \) and \( 3_3 \) lie between \( 1_3 \) to \( 2_3 \) in \( C_1 \) since \( 3_2 \) and \( 3_3 \) are passed when reading clockwise from \( 1_3 \) to \( 2_3 \). We consider the following two restrictions of a circloid \( C \),

\[
C_i = \{ x_j \in C : j = i \} \quad \text{and} \quad C^{\geq j} = \{ x_j \in C : x \geq j \}.
\]
For the traditionalists, we interpret circloids as fillings of shapes. For compositions $\gamma$ and $\alpha/\beta$, a colored tabloid $T \in CT(\gamma, \alpha/\beta)$ is a filling of shape $\gamma$ with colored letters so that row entries are increasing from west to east under the prismatic order and the colors adorning letter $x$ in $T$ are $\{\beta_x + 1, \ldots, \alpha_x\}$. As expected, $\alpha/\beta$ is called the weight of $T$. It is straightforward to see that a bijection is given by the map 

$$\iota : C(\gamma, \alpha/\beta) \longrightarrow CT(\gamma, \alpha/\beta),$$

defined by putting the colored letters of sector $r$ from circloid $C$ into row $r$ of shape $\gamma$ so that the row is colored increasing from west to east, for each $r = 1, \ldots, \ell(\gamma)$.

### 3.2 Circloid statistics

The cocharge statistic on words naturally extends to circloids. Macdonald polynomials turn out to be generating functions of circloids, weighted by cocharge and a second statistic which measures the variation of cocharge from the Lascoux-Schützenberger statistic.

For a circloid $C \in C(\gamma, \mu)$ of partition weight $\mu$, the cocharge of $C$ is defined by

$$\text{cocharge}(C) = \sum_{i=1}^{\ell(\mu)} \text{cocharge}(C_i).$$

**Remark 8.** Any word $w$ with partition weight $\mu$ can be uniquely identified with a circloid $C \in C(1^n, \mu)$ of the same cocharge. Place the letters of $w$ counter-clockwise on a circle and color according to the labeling for standard subwords. That is, moving clockwise from $\star$, label the first 1 with $i = 1$. By iteration, the first $x + 1$ encountered in the clockwise reading from $x_1$ is colored 1. Once $\mu'_1$ letters have been labeled by 1, repeat with 2 on uncolored letters, and so on.

The second statistic measures how different a coloring is from standard subwords. When choosing which letter $x$ to color $j$, each candidate passed over in clockwise order increases the statistic by 1. Precisely,

$$\text{betrayal}(C) = \sum_{i \geq 1} \sum_j s_{i,j},$$

where $s_{i,j}$ is the number of $i_j$ with $j > j$ lying between $(i-1)_j$ and $i_j$ in $C$, with the understanding that $0_j = \star$ for all $j = 1, \ldots, \mu_1$.

**Example 9.** The circloids in the previous example have a betrayal of 2 and cocharge of 4.

It is through the lens of circloids that we can prove cocharge is as fundamental to the $q,t$-Macdonald setting as it is to Kostka-Foulkes and Hall-Littlewoods polynomials. We show that a Macdonald polynomial is none other than the shape generating function of circloids weighted by cocharge and betrayal. Moreover, the result follows straightforwardly from a correspondence between circloids and skew fillings.

**Definition 10.** The map $\dagger$ acts on a circloid $C$ of shape $\gamma$ by placing entry $x$ in cell $(r, c)$, for each colored letter $r_c$ in sector $x$, moving through sectors $x = 1, \ldots, \ell(\gamma)$.

**Example 11.** The action of $\dagger$ on two circloids:
Theorem 12. For any partition $\lambda$,
\[
\hat{H}_t(x; q, t) = \sum_{C \in \mathbb{C}(\lambda)} C^{\text{betrayal}(C)} \cdot C^{\text{cocharge}(C)} \cdot x^{\text{shape}(C)}. \tag{12}
\]

Proof. Consider any compositions $\gamma$ and $\beta \subseteq \alpha$ where $|\gamma| = |\alpha| - |\beta|$. We first establish that $\dagger$ is a bijection where
\[
\dagger : C(\gamma, \alpha/\beta) \to \mathcal{F}(\alpha/\beta, \gamma). \tag{13}
\]

Given a circloid $C$, let $f = \dagger(C)$. Each letter $r$ in sector $x$ of $C$ corresponds to an entry $x$ in row $r$ of $f$ implying that $C \in C(\gamma, \alpha/\beta)$ if and only if $f \in \mathcal{F}(\alpha/\beta, \gamma)$.

Note similarly that any other circloid $D$ where $\dagger(C) = \dagger(D)$ lies in $C(\gamma, \alpha/\beta)$. Consider the set of colored letters $\{r_{(\ell)}^{(1)}, \ldots, r_{(\ell)}^{(y)}\}$ in sector $x$ of $C$. By definition of $\dagger$, $f_{(\ell), (\ell)} = x$ for $\ell = 1, \ldots, y$. Therefore, $\{r_{(\ell)}^{(1)}, \ldots, r_{(\ell)}^{(y)}\}$ is also the set of colored letters in sector $x$ of $D$. Since colored letters lie in unique decreasing prismatic order within sectors, every sector of $C$ and $D$ is the same and we see that $C = D$. That $\dagger$ is bijective then follows by noting that the number of fillings given in (10) matches the number of circloids $C \in C(\gamma, \cdot)$. That is, again viewing letters from sector $x$ of $C$ as a subset $\{r_{(\ell)}^{(1)}, \ldots, r_{(\ell)}^{(y)}\}$ of the distinct $|\gamma|$ colored letters in $C$, we see that
\[
|C(\gamma, \alpha/\beta)| = \binom{|\gamma|}{\gamma_1, \gamma_2, \ldots, \gamma_{(\ell)}}. \tag{14}
\]

We next restrict our attention to circloid $C \in C(\gamma, \lambda)$ for $\lambda \vdash n$ and claim that
\[
\text{cocharge}(C) = \text{maj}(f) \quad \text{and} \quad \text{betrayal}(C) = \text{inv}(f), \tag{15}
\]
for $f = \dagger(C)$. Since maj is computed on columns, we need only verify that cocharge($C$) equals the maj of column $i$ in $f$ to prove that maj($f$) = cocharge($C$). A slight reinterpretation of the cocharge definition gives cocharge($C_i$) = $\sum_r L_r$ where $L_r = \lambda' - r + 1$ when $r_i$ occurs (clockwise) between $(r-1)_i$ and $\dagger$ in $C$ and otherwise $L_r = 0$. In fact, since $r_i > (r-1)_i$ and sectors are prismatic order decreasing, $L_r \neq 0$ if and only if $r_i$ lies in a sector $y$ strictly larger than the sector $x$ containing $(r-1)_i$. On the other hand, the action of $\dagger$ dictates that $r_i$ is in sector $y$ and $(r-1)_i$, is in sector $x$ of $C$ precisely when $y$ lies in cell $(r, i)$ above $x$ in cell $(r-1, i)$ of $f$.

We next claim that betrayal($C$) = inv($f$) using the observation that betrayal($C$) = $\sum_{j \geq 1} (|I_{j, i}| + \cdots + |I_{j, i'}|)$, where

$I_j = \{ j > j : i_j \text{ lies between } (i-1)_j \text{ and } i_j \}$,

with the convention that $0_j = \dagger$ for all $j$. For any $j > 1$, $i_j$ is in sector $x$ of $C$ if and only if entry $x$ lies in $(1, j)$ of $f$. Note that $j \in I_j$ implies $i_j$ lies in sector $y < x$ since $i_j < i_j$ and therefore $j \in I_j$ corresponds uniquely to an inversion of $x$ with the entry $y$ in $(i, j)$ of $f$. When $i > 1$, for each pair of $(i-1)_j$ in sector $x$ and $i_j$ in sector $y$ of $C$, one of the following relations concerning the sector $z$ with $j \in I_j$ must be true: $x = y$ and $z \neq x$, $x < z < y$, $y < x < z$, or $z < y < x$. Correspondingly, entries $x, y$, and $z$ in cells $(i-1, j), (i, j)$, and $(i, j)$ of $f$ respectively, form a triple inversion.

We can extend the definitions of cocharge and betrayal to colored tabloid,
\[
\text{cocharge}(T) = \text{cocharge}(\iota^{-1}T) \quad \text{and} \quad \text{betrayal}(T) = \text{betrayal}(\iota^{-1}T).
\]

Immediately following from Theorem 12 is an expression using charge and one using fixed weight colored tabloid.

Corollary 13. For any partition $\mu$,
\[
\hat{H}_\mu(X; q, 1/t)^{\mu} = \sum_{C \in \mathbb{C}(\mu)} q^{\text{betrayal}(C)} \cdot C^{\text{charge}(C)} \cdot x^{\text{shape}(C)} = \sum_{T \in \mathcal{C}(\mu)} q^{\text{betrayal}(T)} \cdot C^{\text{charge}(T)} \cdot x^{\text{shape}(T)}.
\]
4 Colorful companions

The subset of Young tableaux with an additional Yamanouchi condition is of particular importance; its cardinality gives tensor product multiplicities of \( GL_n \), the Schur expansion coefficients in a product of Schur functions, and the Schubert structure constants in the cohomology of the Grassmannian \( Gr(k,n) \) of \( k \)-dimensional subspaces of \( \mathbb{C}^n \).

For partition \( \lambda \), \( T_\lambda \) denotes the unique tableau of shape and weight \( \lambda \). A word \( w \) is \( \lambda \)-Yamanouchi when \( w \cdot \text{word}(T_\lambda) = b_\lambda \cdots b_2 b_1 \) has the property that the weight of each suffix \( b_1 \cdots b_2 b_1 \) is a partition. A filling is \( \lambda \)-Yamanouchi when its reading word is \( \lambda \)-Yamanouchi and a circloid is \( \lambda \)-Yamanouchi when the counter-clockwise reading of its letters is \( \lambda \)-Yamanouchi. A \( \emptyset \)-Yamanouchi object is simply called Yamanouchi.

Remark 14. Since a word of weight \( \mu \) has zero charge only when every standard subword is the maximal length permutation, zero charge matches the Yamanouchi condition.

Because many open problems in representation theory, geometry, and symmetric function theory involve a search for contemporary notions of Yamanouchi and tableaux to characterize mysterious invariants, the Yamanouchi condition has been revisited often from different viewpoints. The combinatorics of circloids naturally captures several of these simultaneously.

4.1 Companions and the Yamanouchi condition

Van Leeuwen addresses the classical Littlewood-Richardson rule by rephrasing the Yamanouchi condition on skew tableaux \( P \) in terms of companion tableaux. A companion of \( P \) is any skew tableau \( Q \) such that the entries in row \( x \) match the row positions of letters \( x \in P \) and are aligned to meet the condition that entries increase up columns. He proves that a Yamanouchi tableau \( P \) always has a companion tableau of (straight) partition shape \( \mu \).

We forsake the column increasing condition and instead view a companion as the tabloid where rows are uniquely aligned into a straight shape. Such a companion of semi-standard tableau \( P \) is precisely the tabloid obtained by ignoring colors of \( \iota \circ \iota^{-1}(P) \). This approach opens the door to a more inclusive study allowing for companions of arbitrary fillings.

Definition 15. The companion map is the bijection,\[ \iota = \iota \circ \iota^{-1} : \mathcal{F}(\nu/\lambda,\mu) \rightarrow \mathcal{CT}(\mu,\nu/\lambda). \]

The companion of a filling \( F \in \mathcal{F}(\nu/\lambda,\mu) \) is the unique colored tabloid \( \iota(F) \).

Following directly from the definition of \( \iota \) and \( \iota \), the action of \( \iota \) on a filling \( F \) takes entry \( e \) in cell \((r,c)\) to the colored letter \( r \), placed in row \( e \) of \( T \), arranged so that each row of \( T \) is colored increasing. Companions give a valuable mechanism to study Yamanouchi related problems.

Definition 16. A filling \( F \) is super-Yamanouchi when the non-decreasing rearrangement of entries within each row is a Yamanouchi tabloid.

Proposition 17. Given partitions \( \mu \) and \( \nu/\lambda \), consider a filling \( F \in \mathcal{F}(\nu/\lambda,\mu) \) and its companion \( T = \iota(F) \).

1. \( F \) is Yamanouchi if and only if entries of \( T \) are prismatic increasing in columns,
2. \( F \) is a super-Yamanouchi filling if and only if letters of \( T \) increase in columns,
3. letters of \( F \) increase in columns if and only if \( T \) is \( \lambda \)-Yamanouchi.
Proof. (1) If \( F \) is Yamanouchi if and only if the letter \( x \) in a cell \((r,c)\) of \( F \) can be paired uniquely with an \( x - 1 \) in some cell \((\hat{r},\hat{c})\) occurring after \((r,c)\) in reading order. Equivalently, each entry \( r_c \) in row \( x \) of \( T \) pairs uniquely with an entry \( \hat{r}_c \) in row \( x - 1 \) (in prismatic order). Since row entries in a colored tabloid lie in increasing prismatic order, such a pairing can occur exactly when the entry immediately below \( r_c \) is smaller in prismatic order.

(2) Consider a colored tabloid \( T \) where letters do not strictly increase up some column. If columns of \( T \) are not prismatic increasing, \( F \) is not Yamanouchi by (1). Otherwise, we can choose \( b \) to be the rightmost column of \( T \) with an \( r_c \) in row \( x \) and an \( r_{\hat{c}} \) in row \( x - 1 \) where \( c < \hat{c} \). Correspondingly, \( F \) has an \( x \) in cell \((r,c)\) and an \( x - 1 \) in \((r,\hat{c})\). Since entries in rows of \( T \) are prismatic non-decreasing and \( r_c \) and \( r_{\hat{c}} \) lie in column \( b \), the subset of cells in \( F \) weakly smaller than \((r,\hat{c})\) in reading order contain \( b x - 1 \)'s and \( b x \)'s. However, the filling \( \hat{F} \) obtained by rearranging letters in row \( r \) of \( F \) into weakly increasing order is not Yamanouchi since the \( x \) in column \( c < \hat{c} \) of \( F \) moves to the east of all \( x - 1 \)'s in that row.

On the other hand, a colored tabloid \( T \) with letters increasing up columns has prismatic increasing columns and therefore \( F \) is Yamanouchi by (1). Suppose that \( F \) has an \( x \) and an \( x - 1 \) in cells \((r,c)\) and \((r,\hat{c})\), respectively, such that when letters in row \( r \) are put into weakly increasing order, the resulting filling is not Yamanouchi. Since \( F \) is Yamanouchi, this can only happen if \( c > \hat{c} \) and there is an equal number of \( x - 1 \)'s and \( x \)'s in the subset of cells of \( F \) occurring weakly after \((r,\hat{c})\) in the reading order of cells. However, under the \( \hat{\imath} \)-correspondence, \( T \) has an \( r_c \) in row \( x \) and an \( r_{\hat{c}} \) in row \( x - 1 \) lying in the same column. The violation of increasing columns establishes the claim.

(3) Given \( F \in \mathcal{F}(\nu/\lambda, \mu) \) is column increasing, construct the unique filling \( \hat{F} \in \mathcal{F}(\nu, (\lambda, \mu)) \) by replacing each \( i \in F \) with \( i + |\lambda| \) and putting \( 1, 2, \ldots, |\lambda| \) into cells of \( \lambda \) so the reading word taken from these cells is \( |\lambda| \cdots 21 \). Note that \( \hat{C} = \hat{\imath}^{-1}(\hat{F}) \) differs from \( \imath^{-1}(F) \) by the deletion of the first \(|\lambda| \) sectors.

A letter in an arbitrary cell \((r,c)\) of \( \hat{F} \) is larger than the letter in cell \((r - 1, c)\) if and only if entry \( r_c \) occurs in a later sector than \( r - 1_c \) of \( \hat{C} = \hat{\imath}^{-1}(\hat{F}) \). This is equivalent to the Yamanouchi condition on \( \hat{C} \); \( r \) can be paired with a letter \( r - 1 \) which occurs earlier than it, for each letter \( r \) in \( \hat{C} \). The claim follows by noting that the counter-clockwise reading of letters from the first \(|\lambda| \) sectors of \( \hat{C} \) is word(T\( \lambda \)). \( \square \)

### 4.2 Reverse companions

The initial study of companions involved only the subset of fillings which are semi-standard tableaux. Proposition 17 pinpoints that dropping the row condition and requiring only that letters increase in columns of a filling imposes the \( \lambda \)-Yamanouchi condition on its companion circloid (or colored tabloid). On the other hand, it is also natural to examine the subset of fillings which are tabloids, that is, fillings which are non-decreasing in rows from west to east. Let \( \mathcal{F}(\alpha, \beta) \) be the set of tabloids of shape \( \alpha \) and weight \( \beta \).

A distinguished coloring on circloids comes to light under these conditions. A circloid is reverse colored when the colors adorning letter \( x \) increase clockwise from \( \bullet \), for each fixed letter \( x \). A tabloid \( T \) is reverse colored if \( \imath^{-1}(T) \) is a reverse colored circloid.

Remark 18. Since the reverse coloring uniquely assigns a color to each letter of a tabloid, reverse colored circloids are a manifestation of tabloids.

**Proposition 19.** Given compositions \( \gamma \) and \( \beta \leq \alpha \), the companion \( T \) of a filling \( F \in \mathcal{F}(\alpha/\beta, \gamma) \) is reverse colored if and only if \( F \) is a tabloid.

**Proof.** By definition of \( \hat{\imath} \), a filling \( F \) has the property that the letter in cell \((r,c)\) is not smaller than the letter in \((r, c - 1)\) if and only if \( r_c \) does not occur before \( r_{c-1} \) in \( \imath^{-1}(F) \). \( \square \)
Example 20.

\[\begin{array}{cccc}
1 & 1 \\
1 & 2 \\
1 & 2 & 4 \\
2 & 3 & 3 & 4 & 5
\end{array}\]

\[\rightarrow\]

\[\begin{array}{cccc}
1 & 1 & & 4 \\
1 & & & 2 \\
2 & & & 1 \\
1 & \downarrow / & 2 & 1 \\
1 & 1 & & 4 \\
1 & & & 2 \\
1 & 1 & & 3 \\
2 & & & 4 \\
\end{array}\]

\[\equiv\]

\[\begin{array}{cccc}
1 & 1 & & 4 \\
1 & & & 2 \\
1 & 1 & & 3 \\
2 & & & 4 \\
3 & 2 & & 1 \\
1 & 3 & & 2 \\
2 & & & 3 \\
\end{array}\]

In particular, the companion \(Q\) of a semi-standard tableau filling \(P\) is a reverse colored \(\lambda\)-Yamanouchi circloid, or equivalently, is a manifestation of a \(\lambda\)-Yamanouchi tableau by Remark 18. Furthermore, when \(F\) is both Yamanouchi and a semi-standard tableau, we recover the classical result that Yamanouchi tableaux of shape \(\nu/\lambda\) and weight \(\mu\) and \(\lambda\)-Yamanouchi tableaux of shape \(\mu\) and weight \(\nu - \lambda\) are equinumerous.

Corollary 21. For partitions \(\nu/\lambda\) and \(\mu\), \(P \in \text{SSYT}(\nu/\lambda, \mu)\) if and only if its companion is a \(\lambda\)-Yamanouchi tabloid, and \(P\) is Yamanouchi if and only if its companion is a \(\lambda\)-Yamanouchi tableau \(Q \in \text{SSYT}(\mu, \nu - \lambda)\).

In the combinatorial theory of \(K\)-theoretic Schubert calculus, tableaux are replaced by more intricate combinatorial objects such as reverse plane partitions, set-valued tableaux, and genomic tableaux. The later were introduced recently by Pechunik and Yong [PY17] to solve a difficult problem concerning equivariant \(K\)-theory of the Grassmannian. We have discovered that reverse-colored companions are closely related to genomic tableaux and carry out the details separately. A glimpse of this application is given in §6.3 where a crystal structure on reverse colored circloids is used to study the representatives for \(K\)-homology classes of the Grassmannian.

4.3 Faithful companions

Another useful manifestation of tabloids arises from a second distinguished circloid coloring. A circloid \(C\) is faithfully colored when, for each \(i \geq 1\), if entries of color \(j < i\) are ignored, the closest 1 to \(\star\) (moving clockwise) has color 1 and the closest \(x + 1\) to \(x\) has color \(i\), for \(x \geq 1\). A colored tabloid is defined to be faithfully colored if it is the \(\iota\)-image of a faithfully colored circloid.

Example 22. A faithfully colored circloid and its corresponding (faithfully colored) tabloid:

\[C = \begin{array}{cccc}
\star & / & 3_3 \\
1_3 & 2_2 & 3_1 \\
1_2 & 2_3 & 1_1
\end{array}\]

\[\iota(C) = \begin{array}{cccc}
3_2 & 1_3 & 3_1 \\
1_2 & 2_3 & 1_1 \\
1_1 & 2_2 & 3_2
\end{array}\]

Proposition 23. For composition \(\alpha\) and partition \(\lambda\) where \(|\alpha| = |\lambda|\), the image of the companion map \(\iota\) on

\[F^\dagger(\lambda, \alpha) = \{F \in F(\lambda, \alpha) : \text{inv}(F) = 0\}\]

is the subset of faithfully colored tabloid in \(\text{CT}(\alpha, \lambda)\).

Proof. The number of inversion triples in a filling \(F\) matches the betrayal of \(\tilde{f}(F)\) by [15]. It thus suffices to note, by definition, that restricting the set of circloids to those with zero betrayal gives the subset of faithfully colored elements.

Theorem 24. For any partition \(\mu\),

\[\tilde{H}_\mu(x; 0, t) = \sum_{F \in F^\dagger(\mu)} \tilde{f}(\text{maj}(F)) s_{\text{weight}(F)} t.\]
Proof. Consider an inversionless filling $F \in \mathcal{F}^{\ast}(\mu, \cdot)$ which is super-Yamanouchi. Note that the weight of $F$ must be a partition $\lambda + [\mu]$ since $F$ is Yamanouchi. Propositions 17 and 23 give that the $\varepsilon$-image of $F$ is a faithfully colored tabloid with letters which increase up columns. Since each tabloid has a unique faithful coloring, ignoring colors gives the bijection between

$$\{F \in \mathcal{F}(\mu, \lambda) : \text{inv}(F) = 0 \text{ and } F \text{ super } - \text{Yamanouchi} \} \leftrightarrow \mathcal{SSYT}(\lambda, \mu).$$

(16)

The major $(F) = \text{cocharge}(\varepsilon(F))$ by (15), and any faithfully colored circoloid $C \in \mathcal{C}(\lambda, \mu)$ has the same cocharge as that of its manifest tabloid $T$ by Remark 8. The result then follows from (9). □

The comparison of Theorem 24 to (4) suggests that the set $\mathcal{U}$ defined by Roberts is related to the super-Yamanouchi condition. Roberts’ formula requires inversionless, Yamanouchi fillings with an additional property imposed upon entries in a pistol configuration, one that is made up of cells in row $r$ lying in columns $1, \ldots, c$ and cells of row $r + 1$ lying in columns $c, \ldots, \mu_{r+1}$, for any fixed $r, c$. In our language, a filling is jammed if its reverse coloring results in a pistol containing both $x_i$ and $(x + 1)_{r+1}$ for some letter $x$ with color $y$. When a filling is not jammed, we say it is jamless.

Lemma 25. The set of inversionless, super-Yamanouchi fillings is the same as the set of inversionless, jamless, Yamanouchi fillings.

Proof. Suppose an inversionless, super-Yamanouchi filling $F$ is jammed and for convenience, consider its reverse coloring. Since $F$ is jammed, it has rows $r$ and $r + 1$ with a pistol containing $x_i$ and $(x + 1)_{r+1}$. If $x_i$ does not lie in a lower row than $(x + 1)_j$ of a super-Yamanouchi filling, then $j < i$. Therefore, $x_i$ must lie in cell $(r, c)$ of $F$ and $(x + 1)_{r+1}$ lies in cell $(r + 1, \hat{c})$, for some $\hat{c} \geq c$. Moreover, $x_{r+1}$ lies in row $r$ of $F$. Consider the minimal color $i$ adorning $x$ in the set of rows higher than row $r$. Since $x_{r+1}, x_{r-1}, \ldots, x_1$ lie west of column $\hat{c}$ in row $r$, and $F$ is inversionless, an $x_i$ lies above each of these $x$'s. Therefore, $(x + 1)$ lies in row $r + 1$ contradicting that $F$ is super-Yamanouchi.

On the other hand, suppose that a filling $F$ is jamless, inversionless, and Yamanouchi but not super-Yamanouchi. Then there is some row $r$ and letter $x + 1$ in $F$ where the number $y$ of $x + 1$'s weakly below row $r$ is greater than the number of $x$'s below row $r$. In particular, $y$ is the color adorning the leftmost $x + 1$ in row $r$ in the reverse coloring of $F$. Further, the $x$ colored $y$ must lie after this $(x + 1)_y$ in reading order since $F$ is Yamanouchi. However, it is not super-Yamanouchi and therefore $x_y$ lies in row $r$. Since $F$ is inversionless and has $(x + 1)_y$ west of $x_y$ in row $r$, $x$ must lie below $(x + 1)_y$. When reverse colored, this $x$ has color $z < y$. In turn, $z < y$ implies $(x + 1)_{r+1}$ must lie weakly after $(x + 1)_z$ (in reading order). However, the Yamanouchi condition requires that $(x + 1)_{r+1}$ lies before $x_z$. Therefore, the pistol based at $x_z$ contains $(x + 1)_{r+1}$ contradicting that $F$ is not jammed. □

Corollary 26. Roberts’ formula (4) and the Lascoux-Schützenberger formula (5) for $q = 0$ Macdonald polynomials are related by the companion map $\varepsilon$.

5 Crystals

The quantum enveloping algebra $U_q(sl_{n+1})$ is the $\mathbb{Q}(q)$-algebra generated by elements $e_i, f_i, t_i, r_i^{-1}$, for $1 \leq i \leq n$, subject to certain relations. For a $U_q(sl_{n+1})$-module $M$ and $\lambda \in \mathbb{Z}^{n+1}$, the weight vectors (of weight $\lambda$) are elements of the set $M_\lambda = \{u \in M : t_i u = q^{1 - \lambda_i} u \}$. A weight vector is said to be primitive if it is annihilated by the $e_i$’s. A highest weight $U_q(sl_{n+1})$-module is a module $M$ containing a primitive vector $v$ such that $M = U_q(sl_{n+1})v$. The irreducible highest weight module with highest weight $\lambda$ is denoted $V_\lambda$.

Kashiwara [Kas90, Kas91] introduced a powerful theory whereby combinatorial graphs are used to understand finite-dimensional integrable $U_q(sl_n)$-modules $M$. The crystal of $M$ is a set $B$ equipped with a weight function $w t : B \rightarrow \{a^\alpha : \alpha \in \mathbb{Z}_{\geq 0}^n\}$ and operators $\tilde{e}, \tilde{f} : B \rightarrow B \cup \{0\}$ satisfying the properties, for $a, b \in B$, 

\[
\begin{align*}
\tilde{e} \cdot a + 1 & \rightarrow a, \\
\tilde{f} \cdot a - 1 & \rightarrow a, \\
\tilde{e} \cdot a & \rightarrow 0, \quad a > 0, \\
\tilde{f} \cdot a & \rightarrow 0, \quad a < 0.
\end{align*}
\]
• $\tilde{e}_i a = b \iff \tilde{f}_i b = a$
• $\tilde{e}_i a = b \implies \text{wt}(b) = \frac{1}{k+1} \text{wt}(a)$.

The crystal graph associated to $B$ is a directed, colored graph with vertices from $B$ and edges $E = \{(a, b) : b = \tilde{e}_i(a) \text{ for some } i \}$ labeled by color $i \in I = \{1, \ldots, n-1\}$. Irreducible submodules are in correspondence with highest weight vectors $b \in B$ where $\tilde{e}_i(b) = 0$ for all $i$. These are the vertices of the crystal graph with no incoming edges; each connected component of $B$ represents an irreducible and contains a single highest weight vector. The subset of highest weight vectors $b \in B$ where $\text{wt}(b) = x^\gamma$ is denoted by $\mathcal{Y}(B, \gamma)$.

The tensor product crystal graph $B_1 \otimes \cdots \otimes B_k$ has vertices in the Cartesian product $(b_1, \ldots, b_k) \in B_1 \times \cdots \times B_k$ which are denoted $b = b_1 \otimes \cdots \otimes b_k$. Its weight function is defined by

$$\text{wt}(b) = \prod_{j=1}^{n} \text{wt}_{B_j}(b_j).$$

A morphism $\Phi : B \to B'$ is a map on crystal graphs where $\Phi(0) = 0$ and otherwise, for $b \in B$,

• $\Phi(\tilde{f}_i(b)) = \tilde{f}_i(\Phi(b))$
• $\Phi(\tilde{e}_i(b)) = \tilde{e}_i(\Phi(b))$
• $\text{wt}(\Phi(b)) = \text{wt}(b)$.

Lascoux and Schützenberger anticipated the necessary ingredients for the Kashiwara type-A crystal in their development of the plactic monoid on words [LS81] [LLT02]. It is given by the set $B(1)^n = B(1) \otimes \cdots \otimes B(1)$ of words in the alphabet $B(1) = [n]$; the crystal action $\tilde{e}_i, \tilde{f}_i$ is defined on $b \in B(1)^n$ by changing a single $i$ (or $i + 1$) to an $i + 1$ (or $i$) in the restriction of $b$ to the subword $w_{i(i+1)}$. Regarding each letter as a parenthesis, $i + 1$ as a left and $i$ as a right, adjacent pairs of parentheses "()" are matched and declared to be invisible until no more matching can be done. It is a letter in the remaining subword, $z = p^i(i + 1)^q$ for some $p, q \in \mathbb{Z}_{\geq 0}$, which is changed. Precisely, $\tilde{e}_i(b) = 0$ when $q = 0$, $\tilde{f}_i(b) = 0$ when $p = 0$, and otherwise $\tilde{e}_i(b)$ is the word formed from $w$ by replacing the subword $z$ with $p^i(i + 1)^q$ and $\tilde{f}_i(b)$ is formed by replacing $z$ with $p^{i+1}(i + 1)^{q-1}$.

Remark 27. Parentheses pairing of any $b \in B(1)^n$ has the property that every adjacent $(i + 1) i$ is paired, and the first $i$ in any adjacent pair $ii$ is never the rightmost unpaired entry. Therefore, descents of such pairs are preserved by the action of $\tilde{e}_i$, $\tilde{f}_i$ and $\text{Des}(\tilde{e}_i(b)) = \text{Des}(\tilde{f}_i(b)) = \text{Des}(b)$ when $b$ is not annihilated.

For $\mu \vdash n$, since $\tilde{e}_i$ annihilates only the Yamanouchi words, the set of highest weights of $B = B(1)^n$ with $\text{wt}(b) = x^\mu$ is

$$\mathcal{Y}(B, \mu) = \{b \in B : b \text{ is Yamanouchi of weight } \mu\}. \quad (17)$$

As dictated by Kashiwara’s theory, the crystal graph $B(\mu)$ of the irreducible submodule $V_\mu$ is isomorphic to a connected subgraph of $B(1)^n$ which contains a Yamanouchi word of weight $\mu$, and

$$B \cong \bigoplus_{\mu \vdash n} B(\mu) \times \mathcal{Y}(B, \mu). \quad (18)$$

The crystal graph $B(m)$ is isomorphic to the subgraph of elements $b \in B(1)^n$ with no descents since $b = (1, \ldots, 1)$ is the only element in $\mathcal{Y}(B(1)^n, (m))$. Therefore, for any $\gamma \vdash n$ of length $\ell$, the tensor product crystal

$$B = B(\gamma_1) \otimes B(\gamma_2) \otimes B(\gamma_\ell)$$

has highest weight elements given by Yamanouchi words which are non-decreasing in the first $\gamma_1$ positions, in the next $\gamma_2$ positions, and so forth.
5.1 Singly graded Garsia-Haiman modules

A crystal structure on circloids leads us to a characterization for the singly graded decomposition of Garsia-Haiman modules which preserves the spirit of the Garsia-Procesi module decomposition given by (9).

We first refine the decomposition of $B = B(1) \otimes \cdots \otimes B(1)$. For any $D \subset \{1, \ldots, n-1\}$, define the induced subposet $B(D)$ of $B$ by restriction to vertex set $\{b \in B : \text{Des}(b) = D\}$.

**Theorem 28.** For $\lambda \vdash n$,

$$\tilde{H}_i(x; 1, t) = \sum_{D \subset \{1\ldots n\-1\}} t^{\text{maj}_e(D)} \sum_{\text{highest weight } b \in B(D)} s_{\text{wt}(b)}(x),$$

where

$$\text{maj}_e(D) = \sum_{i=1}^{f(\nu)} \sum_{|\nu_i| = |\nu_d|} |\nu_i| - d.$$

**Proof.** For any $D \subset \{n-1\}$, Remark 27 implies that $B(D)$ is a crystal graph made up of the disjoint union of connected components in $B(1)^n$. Therefore, the Frobenius image of the module associated to $B(D)$ is

$$\sum_{\text{highest weight } b \in B(D)} s_{\text{wt}(b)}(x).$$

The highest weights are

$$\mathcal{Y}(B(D, \mu) = \{b \in B(1)^n : \text{Des}(b) = D \text{ and } b \text{ is Yamanouchi of weight } \mu\}.$$ (20)

More generally, the right hand side of (19) reflects the graph decomposition of the crystal $B(1)^n$ into $B(D)$, graded by maj$_e(D)$.

On the other hand, Macdonald polynomials at $q = 1$ are presented in (11) as weight generating functions of $\lambda$-shaped fillings graded by maj. Each filling $f \in \mathcal{F}(\lambda, \cdot)$ can be uniquely identified with a vertex $b \in B(1)^n$ by reading the columns of $f$ from top to bottom (choosing any fixed column order). Consider the filling $f_b$ identified by vertex $b$. Since the computation of maj$_e(f_b)$ relies only on descents in columns of $f_b$, precisely the subset of descents involved in the computation of maj$_e(\text{Des}(b))$, we have that maj$_e(\text{Des}(b)) = \text{maj}(f_b)$. By definition, maj$_e(\text{Des}(b))$ is constant over all elements $b \in B(D)$ and therefore maj(f) is constant on all fillings $f$ associated to $b \in B(D)$.

**Remark 29.** Although each vertex $b \in B(1)^n$ could be uniquely identified with the filling $f$ of shape $\lambda \vdash n$ whose reading word is $b$, maj is not constant on all fillings in the same connected component under this correspondence.

The interaction of crystals with the cocharge statistic comes out of a directed, colored graph $\mathcal{B}(\gamma)$ whose vertices are circloids of weight $\gamma$. An $i$-colored edge between circloids is imposed by operators, $\tilde{e}_i$ and $\tilde{f}_i$, which move an entry from sector $i + 1$ to sector $i$ or vice versa using a method of pairing colored letters.

Pairing is a process which iterates over each entry in a given sector. Entries are considered from smallest to largest with respect to the co-prismatic order, defined on colored letters by

$$x_{y'} > u \iff y > v \quad \text{or} \quad y = v \text{ and } x > u.$$

Pairing is done by writing the entries from sectors $i$ and $i + 1$ in co-prismatic decreasing order, assigning every entry from sector $i + 1$ a left parenthesis and every entry from sector $i$ a right parenthesis. Entries are then paired as per the Lascoux and Schützenberger rule for parenthesis.
Proof. Given \( y \models n \), let \( \phi_1 \) act on \( \mathcal{F}(y, \cdot) \) by the induced action \( \phi_1 = \mathcal{T} \circ \mathcal{E}_1 \circ \mathcal{T}^{-1} \). When an entry \( x_i \) in circloid \( C \) is paired with \( u < x_i \) by the action of \( \mathcal{E}_1 \), an \( i + 1 \) in cell \( (x, y) \) of \( \mathcal{F}(\mathcal{E}_1^{-1}(C)) \) is paired with an \( i \) in cell \( (u, v) \) where either \( y > v \) or \( y = v \) and \( x > u \). Consider the graph on \( \mathcal{F}(y', \cdot) \) where an \( i \)-colored edge connects \( f \) and \( \hat{f} \) when \( \hat{f} = \phi_1(f') \); when each filling is replaced by its reading word \( b \), this is the crystal graph \( B(1)^n \). In particular, a crystal morphism \( \Phi : \mathcal{B}(\gamma) \to B(1)^n \) is given by
\[
\Phi(C) = b,
\]
where \( b \) is the reading word obtained by reading down the columns of the filling \( \mathcal{T}(C) \) from right to left. That is, cell \( (x, y) \) of \( \mathcal{T}(C) \) is read before cell \( (u, v) \) if \( y > v \) or \( y = v \) and \( x > u \). This is equivalent to \( x_i > u \). The weight function on \( \mathcal{B}(\gamma) \) maps circloid \( C \) to its shape by \([\mathcal{F}] \).

A highest weight \( C \in \mathcal{B}(\gamma) \) satisfies \( \mathcal{E}_1(C) = 0 \) for all \( i \) if and only if each entry \( x_i \) in row \( i + 1 \) of \( T = \iota(C) \in C\mathcal{T}(\mu, \cdot) \) pairs with an entry \( u < x_i \) in row \( i \). By rearranging the rows of \( T \) so that they are co-prismatically non-decreasing this will result in co-prismatic increasing columns. Note that \( \mu \) must be a partition, for if a sector \( i + 1 \) of \( C \) has more entries than sector \( i \), then \( C \) has an unpaired entry and is not anihilated.

The circloid crystal captures a formula for the \( q = 1 \) Macdonald polynomials which is perfectly aligned with the long-standing formula for \( q = 0 \) given by Lascoux-Schützenberger.

Corollary 33. For any partition \( \mu \),
\[
\tilde{H}_\mu(x; 1, t) = \sum_{T \in \mathcal{SSCT}(\cdot, \mu)} t^{\text{cocharge}(T)} s_{\text{shape}(T)}(x).
\]
Proof. We have seen that each \( b \in B(D) \) corresponds to a filling \( f_b \) with \( \text{maj}_D(D) = \text{maj}(f_b) \). Theorem 28 gives that
\[
\tilde{H}_\mu(x; 1, t) = \sum_{D \subseteq [n-1]} \sum_{\nu \models n} \sum_{b \in \mathcal{Y}(B(D), \mu)} t^{\text{maj}(f_b)} s_\nu(x) = \sum_{\nu \models n} \sum_{b \in \mathcal{Y}(B(1)^n, \nu)} t^{\text{maj}(f_b)} s_\nu(x).
\]
The claim follows by recalling that \( \text{maj}(f_b) = \text{cocharge}(\mathcal{T}(f_b)) \) and that \( \Phi \) is a morphism of crystals.

From this, it is not difficult to rederive Macdonald’s formula taken over standard tableaux.

Corollary 34. [Mac95] For any partition \( \mu + n \),
\[
\tilde{H}_\mu(x; 1, t) = \sum_{T \in \mathcal{SSYT}(\cdot, 1^*)} \prod_{i=1}^{\mu_n-1} t^{\text{cocharge}(T_i)} s_{\text{shape}(T)}(x)
\]
where \( T_i \) is the subtableau of \( T \) restricted to letters in \( [\mu'_{i} + 1, \mu'_{i+1}] \).
Proof. Give a prismatic column increasing circloid $C \in C(\cdot, \mu)$, replace each entry $i_\ell$ of $C$ with letter $i + \sum_{j < \ell} \mu_j'$. The condition on $C$ that $x < u$ or $x = u$ and $y < v$ for any $x$, above $u$, implies that letters are strictly increasing in columns of the tabloid $T$. Since the computation of cocharge on a circloid independently calculates cocharge on standard subwords of a given color, $\mu - \text{cocharge}(T) = \text{cocharge}(C)$. \hfill \Box

5.2 Double crystal structure

Characterization of the doubly graded irreducible decomposition of Garsia-Haiman modules presents major obstacles. Although the identification of fillings with elements of $B(1)^n$ given by column reading yields connected components constant on the maj-statistic, it is incompatible with the inversion triples. Even the subset of vertices with zero inversion triples is not a connected component. The crystal cannot be applied to $B(1)$, even when $q = 0$, to gain insight on bi-graded decomposition of $R_q(x, y)$ into its irreducible components. However, a double crystal structure using dual Knuth relations (jeu-de-taquin) and $\tilde{e}_i$, $\tilde{f}_i$ operators on colored tabloids can be applied to the Garsia-Procesi modules. Double crystal structures on $B(\mu_1) \otimes \cdots \otimes B(\mu_r)$ have been studied in various contexts [VL01] [Shi] [Las03], but without regard to graded modules.

For any composition $\gamma$ of length $\ell$, we consider a crystal $B^i(\gamma)$ on vertices $CT(\cdot, \gamma)$ which is dual to $B(\gamma)$. An $i$-colored edge is prescribed by a sliding operation defined on an inflation of rows $i$ and $i + 1$ in a colored tabloid. The $i$-inflation of a vertex $b \in B^i(\gamma)$ is defined by spacing out the colored letters in row $i$ of $b$ while preserving their relative order as follows: entries $e$ are taken from west to east from row $i$ of $b$ and placed in the leftmost empty cell of row $i$ without an entry $e' < e$ directly above it. The operator $e_i^1$ on $b \in B^i$ is then defined by a jeu-de-taquin sliding action whereby the largest entry of row $i + 1$ in the $i$-inflation of $b$ which lies immediately above an empty cell is swapped with this empty cell, after which all empty cells are removed. When no empty cell lies in row $i$, $e_i^1(b) = 0$.

It is convenient to define the inflation a vertex $b \in B^i(\gamma)$ as the punctured colored tabloid obtained by inflating rows of $b$ in succession from top to bottom. Note that the inflation of $b$ has entries (prismatic order) increasing up columns.

Example 35. The 2-inflation of $T = \begin{array}{llll}
1 & 1 & 2 & 3 \\
2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3
\end{array}$ is $\begin{array}{llll}
1 & 2 & 3 & 3 \\
2 & 2 & 2 & 3 \\
3 & 3 & 3 & 3
\end{array}$. The inflation of $T$ is $\begin{array}{llll}
1 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3
\end{array}$.

Definition 36. For any $\gamma \vdash n$, let $B^i(\gamma)$ be the graph on vertices $CT(\cdot, \gamma)$ with a directed, $i$-colored edge from $b \rightarrow b'$ when $e_i^1(b) = b'$.

We will establish that $B^i(\gamma)$ is a crystal graph doubly related to $B(1)^n = B(1) \otimes \cdots \otimes B(1)$ by the companion map. For each $\gamma \vdash n$, define the map $c_\gamma$ on $B(1)^n$ by

$$c_\gamma(b) = c(f_b),$$

where $f_b$ is the unique filling of shape $\gamma$ whose reading word is $b$.

Theorem 37. For each $\gamma \vdash n$, $c_\gamma$ is a crystal isomorphism

$$B(1) \otimes \cdots \otimes B(1) \cong B^i(\gamma).$$

The highest weights in $B^i(\gamma)$ of weight $\mu \vdash n$ are

$$\mathcal{W}(B^i(\gamma), \mu) = \{T \in CT(\mu, \gamma) : \text{entries of } T \text{ prismatically increase up columns}\}.$$
Proof. Since the image of the companion bijection \( \phi \) on \( \mathcal{F}(x, y) \) is the set of vertices in \( B^1(\gamma) \), the map \( \phi_x \) is a bijection between \( B(1) \otimes \cdots \otimes B(1) \) and the vertices of \( B^1(\gamma) \). To check that edges match, we need to prove that \( \tilde{e}_i(b) = b' \) if and only if \( \tilde{e}_i(\phi_x(b)) = \phi_x(b') \), for each \( b \in B(1) \otimes \cdots \otimes B(1) \).

We will show that an entry \( x_i \) in row \( i + 1 \) of the \( i \)-inflation of \( \phi_x(b) \) has an empty cell under it if and only if the corresponding \( i + 1 \) in cell \((x, y)\) of \( b \) is unpaired. The proof then follows because sliding the rightmost \( x_i \), down to \( i \) is the equivalent of changing the leftmost unpaired \( i + 1 \) to an \( i \) in \( b \).

Suppose that \( x_i \) lies above an empty cell in the \( i \)-inflation of \( \phi_x(b) \). Then for each \( u_v < x_i \) in row \( i \) the entry \( u_v' \) immediately above in row \( i + 1 \) satisfies \( u_v < u_v' < x_i \). The entry \( x_v \) corresponds to an \( i + 1 \) in cell \((x, y)\) of \( b \), and for every \( i \) appearing afterward, there is a distinct \( i + 1 \) appearing between it and cell \((x, y)\). Therefore the \( i + 1 \) in cell \((x, y)\) will be unpaired.

Suppose that an \( i + 1 \) in cell \((x, y)\) of \( b \) is unpaired. Then every \( i \) appearing afterward is paired with an \( i + 1 \) that appears between it and cell \((x, y)\). Therefore, for each \( u_v \) in row \( i \) of \( \phi_x(b) \) with \( u_v < x_i \), there is a unique \( u_v' \) in row \( i + 1 \) such that \( u_v < u_v' < x_i \). Thus we are guaranteed that there is an empty cell under \( x_i \) when we \( i \)-inflate \( \phi_x(b) \).

The highest weights of \( B(1)^n \) are defined by the Yamanouchi property and thus their companions are prismatic column increasing by Proposition 17. Alternatively, \( e_i^1 \) annihilates a colored tabloid \( T \) when there is no entry in row \( i + 1 \) of the inflation of \( T \) above an empty square in row \( i \). In particular, \( T \) is its own inflation and thus has prismatic increasing columns. \( \Box \)

5.3 Garsia-Procesi modules

As a first application, we show how the graded irreducible decomposition of a Garsia-Procesi module is readily apparent in the crystal \( B^1 \). For this, we need only the induced subposet \( B^1_0(\gamma) \) on the restricted set of reverse-colored vertices in the crystal graph \( B^1(\gamma) \).

Proposition 38. For a composition \( \gamma \) of length \( \ell \), the companion map is a crystal isomorphism

\[
B^1_0(\gamma) \cong B(\gamma_1) \otimes \cdots \otimes B(\gamma_\ell)
\]

The highest weights of \( B^1_0(\gamma) \) are (reverse-colored) semi-standard tableaux of weight \( \gamma \).

Proof. Given \( b \in B^1_0 \), consider \( b' = e_i^1(b) \) if \( b' \neq 0 \), there is a (largest) unpaired \( x_i \) in row \( i + 1 \) of the inflation \( b \) above an empty cell. The definition of inflation thus implies \( x_{i-1} \) cannot lie in row \( i \). Therefore, the image of a reverse colored tabloid under the crystal action remains as such since the action merely slides \( x_i \) into row \( i \). That is, each connected component in \( B^1_0 \) is a connected component of \( B^1 \). We then note that the set of vertices \( B^1_0 \) is in bijection with \( B(\gamma_1) \times \cdots \times B(\gamma_\ell) \) since reverse-colored tableaux of weight \( \gamma \) are the companion images of tableaux with shape \( \gamma \) by Proposition 19. In turn, \( B(\gamma_1) \times \cdots \times B(\gamma_\ell) \) are defined as induced subposets of \( B(1)^n \) allowing us to apply Theorem 17 to establish the isomorphism.

The highest weights of \( B^1(\gamma) \) are the colored tableaux with prismatically increasing columns by Theorem 37. Thus, a highest weight element \( b \) has entries \( x_i \) above \( x_{i'} \) in the same column only when \( y < y' \). However, if \( b \) is also reverse colored, then \( y > y' \) and therefore its letters increase up columns.

Define the faithful recoloring of \( b \in B^1 \) to be the colored tableau \( b_0 \) obtained by stripping \( b \) of its colors and then faithfully coloring its letters.

Lemma 39. For \( \mu + n \) and \( b \in B^1(\mu) \) with the property that letters increase up columns of the inflation of \( b \),

\[
\text{cocharge}(b_0) = \text{cocharge}((b')_0) \quad \text{for} \quad b' = e_i^1(b) \neq 0.
\]

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Proof. Consider $b \in B^l$ such that letters increase up columns in its inflation $b$. If $b' = \tilde{\epsilon}_i^j(b) \neq 0$, then there is a largest entry $x_i$ in row $i + 1$ of $b$ which lies above an empty cell and it is moved into row $i$ to obtain $b'$ from $b$. Since letters must increase up columns of $b$ and an $x$ in row $i + 1$ lies above an empty cell, there must be fewer $x - 1$’s in row $i$ than there are $x$’s in row $i$ of $b$. For the number $n'_i$ of letters smaller than $x$ in row $i$, $n'_i > n'^{-1}_{i-1}$.

Let $d$ be the smallest color adorning an $x$ in row $i + 1$ of $b_0$ and note that there can be no $(x - 1)_d$ with $d' < d$ in row $i$ of $b_0$ by definition of faithful coloring. If $(x - 1)_d$ does not lie in row $i$ of $b_0$, then $b_0$ and $(b')_0$ are the same with the exception of one entry: $x_d$ lies in row $i + 1$ of $b_0$ and in row $i$ of $(b')_0$. Therefore, their cocharges equal since $(x - 1)_d$ is not in row $i$ of either element.

Otherwise, $d \in D \cap E$ for the set $D$ of colors adorning letter $x$ in row $i + 1$ and the set $E$ of colors attached to $x - 1$ in row $i$ of $b_0$. In this case, $b_0$ and $(b')_0$ again differ only by one entry; $x_e$ lies in row $i$ of $b_0$ and in row $i - 1$ of $(b')_0$ for the smallest element $e \in D \cap E$. Therefore, their cocharges match since $(x - 1)_e$ is not in row $i$.

Proposition 40. For $\mu \vdash n$, the graded decomposition of the Garsia-Procesi module $R_\mu(y)$ into its irreducible components is encoded in the crystal graph $B^l_\mu$ with cocharge($\lambda_0$) attached to each vertex $b$.

Proof. The Frobenius image of the module $R_\mu(y)$ is obtained by setting $q = 0$ in our formula (12). We then apply $\iota$ to obtain

$$\tilde{H}_\mu(x; 0, t) = \sum_T t^{\text{cocharge}(T)} x^{\text{shape}(T)},$$

over faithfully colored tabloids $T$ of weight $\mu$. The map $b \mapsto b_0 = T$ is a bijection between $B^l_\mu(\mu)$ and $CT(\mu_\iota)$ since reverse colored and faithfully colored tabloids are both manifestations of uncolored tabloids. This allows us to convert the previous identity to

$$\tilde{H}_\mu(x; 0, t) = \sum_{b \in \tilde{B}^l_\mu(\mu)} t^{\text{cocharge}(b)} x^{\text{shape}(b)} = \sum_{b \in \tilde{B}^l_\mu(\mu)} t^{\text{cocharge}(b)} x^{\text{wt}(b)},$$

recalling that the weight function on the crystal $\tilde{B}^l_\mu(\mu)$ sends $b \mapsto \text{shape}(b)$. Now, letters increase up columns in the inflation of a reverse colored tabloid $b$; if $x_i$ lies in a higher row than $x_e$, then $x_i < x_e$ and thus these entries cannot lie in the same column of the inflation of $b$. This given, Lemma 59 implies that attaching each vertex $b$ to cocharge($\lambda_0$) is a statistic which is constant on the connected components of $\tilde{B}^l_\mu(\mu)$. Therefore,

$$\tilde{H}_\mu(x; 0, t) = \sum_{b \in \tilde{B}^l_\mu(\mu), \lambda} t^{\text{cocharge}(b)} s_{\lambda}(x) = \sum_{T \in SSYT(\lambda, \mu)} t^{\text{cocharge}(T)} s_{\lambda}(x),$$

since Proposition 38 characterizes the highest weights by (reverse colored) semi-standard tableaux.

6 Further applications of the double crystal

6.1 Energy function on affine crystals

Lenart and Schilling [LS13] connect the $q = 0$ Macdonald polynomials to a tensor product of Kashiwara-Nakashima single column crystals, producing a new statistic for computing the (negative of the) energy function on affine crystals. The double crystal reveals that their statistic is precisely the companion of cocharge.

For a partition $\lambda$ of length $\ell$ and $b \in B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$, we define

$$\text{zmaj}(b) = \sum_{w \in \text{maj}(b)} \text{maj}(w),$$

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where $Z(b)$ is a set of $\lambda_i$ words extracted from the letters of $b = (b_1, \ldots, b_\ell)$. The first word $w = w_{b_1} \cdots w_1$ is constructed by selecting $w_1$ to be the smallest entry in $b_1$. Iteratively, $w_i$ is selected to be the smallest entry larger than $w_{i-1}$ in $b_i$ (breaking ties by taking the easternmost). If there is no larger entry available, the smallest entry in $b_i$ is selected instead. The first word $w$ is fully constructed after a letter has been selected from $b_\ell$. The remaining words of $Z(b)$ are constructed by the same process, ignoring previously selected letters of $b$.

**Example 41.** For $b = (223, 123, 12) \in B(3) \otimes B(3) \otimes B(2)$, the words in $Z(b)$ are $Z(b) = (132, 212, 23)$ implying that $\text{zmaj}(b) = 2 + 1 + 0$. 

**Proposition 42.** For partition $\lambda$ of length $\ell$ and $b \in B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$,

$$\text{zmaj}(b) = \text{cocharge}(\epsilon(f_b)_0),$$

where $f_b$ is the unique $\lambda$-shaped tabloid with reading word $b$.

**Proof.** For $b \in B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$, consider first the case that the tabloid $f_b$ contains an $x$ in row $i$ and letter $z > x$ in row $i + 1$. For the colored tabloid $T = \epsilon(f_b)$, there is an $i$ in row $x$ of $T$ and $z$ is the lowest row above $x$ with an $i + 1$. If $f_b$ does not have any $z > x$ in row $i + 1$, instead take $z \leq x$ to be the minimal entry in row $i + 1$ and note that $z$ is the lowest row in $T$ containing an $i + 1$. More generally, for the $i$th word $w = w_{j'1} \cdots w_{11}$ in $Z(b)$, $w_j$ records the row containing $j_i$ in $T$. Since each $w_{j_i1} > w_j$ contributes $\lambda_j' - j$ to the maj and each $j + 1$ higher than $j$ contributes the same to cocharge($T$) the claim follows. □

Lemmas 39 and Proposition 42 imply that connected components of the crystal graph $B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$ are constant on $\text{zmaj}$.

**Corollary 43.** For a partition $\lambda$ of length $\ell$ and $b \in B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$,

$$\text{zmaj}(b) = \text{zmaj}(\tilde{\epsilon}(b)) \quad \text{and} \quad \text{zmaj}(b) = \text{zmaj}(\tilde{f}_i(b)), $$

for any $i \in [1, \ldots, n - 1]$.

**Theorem 44.** For any partition $\lambda$, 

$$\tilde{H}_\lambda(x; 0, t) = \sum_{b \in B(\lambda) \otimes \cdots \otimes B(\lambda)} t^{\text{zmaj}(b)} \text{weight}(b) \cdot x.$$ 

**Proof.** The expansion (21) of $\tilde{H}_\lambda(x; 0, t)$ over elements of $B^\dagger$ can be converted to one involving $B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$ by Proposition 38 As reviewed in (17), the highest weights of $B = B(\lambda_1) \otimes \cdots \otimes B(\lambda_\ell)$ are characterized by the Yamanouchi condition. Corollary 43 implies that $\text{zmaj}$ is constant on connected components of $B$, which are in correspondence with Schur functions indexed by the the highest weights. □

### 6.2 Zero inversion map on Macdonald fillings

The faithful recoloring of vertices in $B_\lambda^\dagger$ not only gives a formula for the energy function, it exposes an identification between inversionless fillings and tabloids used in [HHL05a]. Define 

$$s : \{ F \in \mathcal{F}(\lambda, \alpha) : \text{inv}(F) = 0 \} \rightarrow \mathcal{F}(\lambda, \alpha)$$

on a filling $F$ simply by rearranging entries in each row into non-decreasing order from west to east. The inverse of $s$ is defined in [HHL05a](Proof of Proposition 7.1) by uniquely constructing an inversionless filling from a
collection of multisets, \( m = \{m_1, m_2, \ldots, m_k\} \). The unique placement of entries from \( m_i \) into row \( i \) of \( f \) so that \( \text{inv}(f) = 0 \) requires first that entries of \( m_1 \) are put into the bottom row in non-decreasing order, from west to east. Proceeding to the next row \( r = 2 \), letters of \( m_r \) are placed in columns \( c \) from west to east as follows: an empty cell \((r, c)\) is filled with the smallest value that is larger than the entry in \((r-1, c)\). If there are no values remaining that are larger than that in \((r-1, c)\), the smallest available value is chosen. The filling \( f \) arises from iteration on rows and by construction, \( f \) has no inversion triples.

In fact, \( s^{-1}(f) \) is none other than the companion preimage of the faithful recoloring of \( f \)'s companion. That is, the companions of \( f \) and \( s(f) \) are both manifestations of the same tabloid, one is reverse colored and the other is faithful.

**Proposition 45.** For any tabloid \( f \in \mathcal{T}(\gamma, \cdot) \),

\[
\epsilon(s^{-1}(f)) = \epsilon(f)_0.
\]

**Proof.** For any tabloid \( f, s^{-1}(f) \) and \( f \) differ only by the rearrangement of entries within rows. Thus, by definition of companions, \( T' = \epsilon(s^{-1}(f)) \) and \( T = \epsilon(f) \) differ only by their colorings. Since \( f \) is a tabloid, \( T \) is reverse colored by Proposition 19 and since \( s^{-1}(f) \) is inversionless, \( T' \) is faithfully colored by Proposition 23 Each of these colorings is uniquely defined on the manifest tabloid and the claim follows. \( \square \)

### 6.3 \( K \)-theoretic implications

To give a flavor of how circloid crystals fit into \( K \)-theoretic Schubert calculus, consider tabloids with the property that their conjugate is also a tabloid. Such a filling is called a reverse plane partition. If the weight of a reverse plane partition is defined to be the vector \( \alpha \) where \( \alpha_i \) records the number of columns containing an \( i \), the weight generating functions are representatives for \( K \)-homology classes of the Grassmannian: for skew partition \( \nu/\lambda \),

\[
g_{\nu/\lambda}(x) = \sum_{r \in \RPP(\nu/\lambda)} x^{\text{weight}(r)}.
\]

From this respect, repeated entries in a column of the reverse plane partition \( r \) are superfluous motivating us to instead identify \( r \) with the tabloid \( f \) obtained by deleting any letter that is not the topmost in its column and then left-justifying letters in each row. The inflated shape of tabloid \( f \) is defined to be the shape of its inflation. This recovers the shape of the reverse plane partition \( r \) from whence \( f \) came.

**Proposition 46.** For skew partition \( \nu/\lambda \),

\[
g_{\nu/\lambda}(x) = \sum_{\text{Yamanouchi tabloid } T \text{ in inflated shape } \nu/\lambda} s_{\text{weight}(f)}(x).
\]

**Proof.** For any composition \( \gamma \), since \( B = B(\gamma_1) \otimes \cdots \otimes B(\gamma_r) \) is a crystal graph under \( \tilde{e}_i \), it suffices to show that the inflated shape of \( f \in \mathcal{T}(\gamma, \cdot) \) is the same as the inflated shape of \( \tilde{e}_i(f) \). From this, the induced subposet of \( B \) on vertices with fixed inflated shape is also a crystal and the Schur expansion comes from the highest weights.

Consider a filling \( f \) and \( f' = \tilde{e}_i(f) \) differing by only one letter \( i + 1 \) changed to an \( i \) in some row \( r \). The shape of the inflation of \( f \) can differ from that of \( f' \) only if there is an \( i + 1 \) in row \( r + 1 \) lying above an empty cell. However, since the leftmost unpaired \( i + 1 \) in \( f \) lies in row \( r \), the process of pairing ensures that every \( i + 1 \) in row \( r + 1 \) is paired with an \( i \) in row \( r \). Therefore, there are more \( i \)'s in row \( r \) of \( f \) than there are \( i + 1 \)'s in row \( r + 1 \) and the inflation of \( f \) must have an entry smaller than \( i + 1 \) below the rightmost \( i + 1 \) in row \( r + 1 \). \( \square \)

An expression for the Schur expansion of \( g_{\nu/\lambda} \) over a sum of semi-standard tableaux arises as a corollary of Proposition 46 by applying the companion map to (23) and using Corollary 21. Such an expression opens up the study of problems in \( K \)-theoretic Schubert calculus to the classical theory of tableaux. For example, a simple bijective proof of the \( K \)-theoretic Littlewood-Richardson was given in [LMS17] using this approach.
7 Quasi-symmetric expansion

It is not difficult to use dual equivalence graphs instead of crystals to deduce our previous results. For this, we formulate Macdonald polynomials using colored words, without mention of shape, in terms of Gessel’s fundamental quasisymmetric function. Defined for any $S \subseteq [n]$, let

$$Q_S(x) = \sum_{\substack{i_1 \leq \cdots \leq i_n \in S \cap [n] \text{ if } j \in S \setminus i_j}} x_{i_1}^{\beta_i} \cdots x_{i_n}^{\beta_n}.$$ 

Betrayal and cocharge are defined on a colored word $w$ by computing these statistics on the circloid $C_{\text{ob}}$ obtained by writing entries of $w$ counter-clockwise on a circle. Since the shape of a circloid has no bearing on the statistics, it suffices to write each entry in its own sector so that $C$ has shape $1^{\ell(w)}$.

**Theorem 47.** For partition $\mu \vdash n$, 

$$\tilde{H}_\mu(X; q, t) = \sum_{\text{colored word } w \text{ of weight } \mu} q^{\text{betrayal}(w)} t^{\text{cocharge}(w)} Q_{\text{Des}_n(w)}(x),$$

where $\text{Des}_n(w) = \{n - d : d \in \text{Des}(w)\}$.

**Proof.** Given a fixed colored word $w$ of weight $\mu$, consider the set of weak compositions $\beta$ for which shape($w$) = $\beta$. For each such $\beta$, $\text{Des}(w) \subseteq \text{set}(\beta)$. Therefore, a unique circloid $C \in C(\beta, \mu)$ is obtained by counter-clockwise inscribing the entries of $w$ on a circle, separated into sectors of sizes $\beta_\ell, \ldots, \beta_1$. Since the computation of cocharge and betrayal of circloid $C$ does not involve its shape, every $C$ arising in this way satisfies cocharge($C$) = cocharge($w$) and betrayal($C$) = betrayal($w$). Theorem 12 can thus be rewritten as

$$\tilde{H}_\mu(X; q, t) = \sum_{C(\beta, \mu)} q^{\text{betrayal}(C)} t^{\text{cocharge}(C)} x^{\text{sh}(C)} = \sum_{\text{colored word } w \text{ of weight } \mu} \sum_{\beta : \text{shape}(w) = \beta} q^{\text{betrayal}(w)} t^{\text{cocharge}(w)} x^{\beta}.$$ 

The claim follows by noting that

$$\sum_{\beta : \text{shape}(w) = \beta} x^{\beta} = \sum_{\gamma \vdash \ell(w)} \sum_{\substack{n - \text{Des}(w) \subseteq \text{set}(\gamma) \cap \gamma \vdash \ell(w)}} x^{\gamma_1} \cdots x^{\gamma_\ell(w)}.$$ 

□

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