Lindbladians for controlled stochastic Hamiltonians

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Abstract

We construct Lindbladians associated with controlled stochastic Hamiltonians in the weak coupling regime. This construction allows us to determine the power spectrum of the noise from measurements of dephasing rates. Moreover, by studying the derived equation it is possible to optimize the control as well as to test numerical algorithms that solve controlled stochastic Schrödinger equations. A few examples are worked out in detail.

1. The problem and the result

In various applications of quantum information theory one is interested in describing the behavior of controlled open systems on timescales that are large compared with the timescales of the controls. Fast controls can lead to an effective decoupling of the system from its environment thereby increasing the coherence properties of the system. This is known as ‘dynamical decoupling’. One is then interested in describing the resulting effective dynamics on the long time scale in terms of an appropriate Lindblad operator [2, 9, 12]. This article describes how to construct such Lindbladians for stochastic Hamiltonians when the stochastic terms are weak in an appropriate sense.

Lindbladians in the weak coupling limit have been rigorously studied in [1, 4–7, 10, 13, 16, 18] in the time independent setting. Controls aimed at extending the coherence of qubits have been suggested in [9, 12] and periodically controlled Lindbladians have been studied in [2, 20]. However, a careful derivation of the Lindbladians for the controlled stochastic evolutions and in particular equations (1.10), (3.8) are new.

Consider the stochastic controlled Hamiltonian 3,

\[ H = \left( \sum_{\alpha} \xi_{\alpha}(t) H_{\alpha} \right) + H_c(t), \]  

(1.1)

where \( H_c \) is a time-dependent (Hermitian) matrix, represents the control. It is convenient to reformulate the problem in the interaction picture. Let

\[ H^I_c(t) = \sum_{\alpha} \xi_{\alpha}(t) H^{I}_{\alpha}(t), \quad H^I_{\alpha}(t) = V^* (t) H_{\alpha} V(t) \]  

(1.3)

\( H_{\alpha} \) are fixed Hermitian matrices representing independent and in general non-commuting sources of noise. \( \xi_{\alpha} \) are stationary Gaussian random processes

\[ E\left( \xi_{\alpha}(t) \right) = 0, \quad E\left( \xi_{\alpha}(t) \xi_{\beta}(u) \right) = J_{\alpha\beta} |t-u| \]  

(1.2)

with \( J \) rapidly decreasing on a time scale \( r \). We shall sometimes assume, w.l.o.g., that \( J \) is a diagonal matrix, and denote its diagonal elements by \( J_{\alpha\alpha} \). (This may always be achieved by a redefinition of \( H_{\alpha} \).) A spin in a magnetic field which has a fixed direction but noisy amplitude, often a good approximation [9, 12], is represented by a single \( \alpha \) value. The case where the direction of the field is also stochastic is modeled by several \( \alpha \) values. The case where the direction of the field is also stochastic is modeled by several \( \alpha \) values. Such noise, which was not treated before, defines a problem with higher complexity. \( H_c, a time-dependent (Hermitian) matrix, represents the control.  

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3 Controlled stochastic adiabatic evolutions are studied in [8].
where $V(t)$ is the unitary generated by the control $H_c(t)$,
\[
H_c = i \dot{V}(t) V^*(t), \quad V(0) = 1.
\] (1.4)

The weak coupling parameter in the present context is defined by
\[
\epsilon^2 = \tau \| J \| \| H_a \|^2 \ll 1 \quad J(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) \, dt \geq 0
\] (1.5)
where $\| \cdot \|$ are suitable norms. Weak coupling, $\epsilon \ll 1$, says that the phase acquired by the wave function during one correlation time (in the absence of control) is small. There are several ways to think of weak coupling: if we think of $\| J \|$, $\tau = O(1)$ then weak coupling means what its name suggests: that the noise is weak, in the sense that $\| H_c \| = O(\epsilon)$. An alternative approach, which is also insightful, is to take $\| J \|$, $\| H_a \| = O(1)$ and then weak coupling means short correlation time $\tau = O(\epsilon^2)$.

The noise affects the (average) state on the coarse grained timescale
\[
s = \epsilon^2 t / \tau.
\] (1.6)

Control problems are characterized by the rate of rotation of $H_c^{\dagger}(t)$. For example, when the control $H_c$ is time-independent (constant control), $\omega = \| H_c \|$ while for periodic Bang-Bang, where $H_c(t)$ is a (periodic) sequence of delta pulses, $\omega_c$ is the frequency of the bangs. This gives rise to a second dimensionless parameter $\omega_c \tau$. Our analysis of the weak coupling limit holds independently of $\omega_c \tau$. Dynamical decoupling requires however $\omega_c \tau \gtrsim 1$ where the timescale of the control, $\delta t = O(1/\omega_c)$, is not resolved on the coarse grained timescale $s$.

By stationary controls we shall mean that $H_c^{\dagger}(t)$ has a finite number of Fourier coefficients. It is convenient to factor $\epsilon$ so that the Fourier coefficients $H_a(\omega)$ are
\[
H_c^{\dagger}(t) = \epsilon \sum_{\omega \in F} H_a(\omega) e^{i\omega t}, \quad H_a(\omega) = H_a^{\dagger}(\omega)
\] (1.7)
and constitute a finite set.

When $\epsilon \ll 1$ we shall show that the evolution is governed by (complete) positivity preserving Lindbladian
\[
\frac{d\rho}{ds} = \mathcal{L}_\epsilon \rho
\] (1.8)
where $\rho = \mathbb{E}(\rho_t)$, meaning averaging over the noise configuration, and $\rho$ is defined in the interaction picture. Moreover, we shall show that, in the case of stationary control, $\mathcal{L}_\epsilon$ has a limit as $\epsilon \to 0$ given by:
\[
\mathcal{L} = \sum_{\alpha} \mathcal{L}_\alpha, \quad \mathcal{L}_\alpha = \mathcal{H}_\alpha - \mathcal{D}_\alpha
\] (1.9)
with
\[
\mathcal{H}_\alpha = \frac{i\tau}{4} \sum_{\omega \in F} K_\omega(\omega) \left[ H_a(\omega), H_a^{\dagger}(\omega) \right], \\
\mathcal{D}_\alpha = \tau \sum_{\omega \in F} I_\omega(\omega) \left[ H_a(\omega), H_a^{\dagger}(\omega), \rho \right]
\] (1.10)

$J$ denotes the Fourier transform and $K$ is the anti-symmetric partner of $J$:
\[
K_\omega(u) = -i \text{ sgn}(u) J_\omega(u),
\] (1.11)
Note that $K(\omega)$ is real and $K(0) = 0$. $\mathcal{H}_\alpha$ is a generator of unitary evolution since $[H_a(\omega), H_a^{\dagger}(\omega)]$ is Hermitian. Since $J(\omega) \geq 0$, $\mathcal{D}_\alpha \geq 0$ is a contraction, generating decoherence.

From the special form of $\mathcal{L}_\alpha$ it can be seen that stochastic noise inflicts unitary evolutions, i.e., the fully mixed state $\rho \propto I$ is stationary.

One potential application of our theory is to combine the structural information on the evolution given by the Lindbladian with the quantum theory of ‘parameter estimations’ to optimize the measurement protocols of physically interesting parameters such as $J(\omega)$, the correlation times, $\tau$, and the various dephasing rates, $\gamma$
\[3, 14, 15, 19].

\[4\] In contrast to evolution of the (unaveraged) state where timescales $O(\tau / \epsilon)$ lead to effects of $O(1)$.

\[5\] In the general case one needs to add conditions so that the series in equation (1.10) converge.

\[6\] This is related to the procedure of ‘adiabatic elimination’ [11].
2. Some exact results

The Hamiltonian $H_I$ generates a stochastic unitary evolution $U_\xi$ given by

$$U_\xi(t) = \left(e^{-i\int_0^t H_I(u) \, du}\right)_T = \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_n < \ldots < t_{n+1} \leq t} H_I^\dagger(t_n) \, dt_n \ldots H_I^\dagger(t) \, dt_1$$

(2.1)

The time ordering, denoted by the subscript $T$ in the first line is defined explicitly in the second. More crucial to us is the super-operator $\mathcal{U}_\xi$ acting on states

$$\rho_0 \mapsto \rho_\xi(t) = U_\xi(t) \rho_0 U_\xi^\dagger(t).$$

(2.2)

The super-operator can be written similarly

$$U_\xi(t) = \left(e^{-i\int_0^t H_I^\dagger(u) \, du}\right)_T = \sum_{n=0}^{\infty} (-i)^n \int_{0 \leq t_n < \ldots < t_{n+1} \leq t} H_I^\dagger(t_n) \, dt_n \ldots H_I^\dagger(t) \, dt_1$$

(2.3)

where the super-operators $H$ acts by the adjoint action

$$H \rho = (\text{ad} [H]) \rho \equiv [H, \rho]$$

(2.4)

Note that $\text{ad} [H_1 \text{ad} [H_2]] \neq \text{ad} [H_1 H_2]$. Rather,

$$\left(\text{ad} [H_1] \text{ad} [H_2]\right) (\rho) = \left(H_1^\dagger H_2^\dagger\right) (\rho) = H_1 \left(H_2 \rho\right) = \left[H_1, \left[H_2, \rho\right]\right].$$

(2.5)

We also need the fact that

$$\text{ad} [\left[A, B\right]] = \left[\text{ad} [A], \text{ad} [B]\right]$$

(2.6)

which follows from Jacobi’s identity.

The key object of this study is the (stochastic) averaged evolution

$$\rho_0 \mapsto \mathbb{E}\left(\rho_\xi(t)\right) = (U^\dagger(t)) \rho_0$$

(2.7)

The super-operator $U^\dagger(t)$ is trace preserving, (completely) positivity preserving and unital (i.e. $U^\dagger 1 = 1$), but, in general, not unitary or Markovian.

Recall that for Gaussian averages

$$\mathbb{E}\left(e^{\phi}\right) = e^{-\mathbb{E}\left(\phi^2\right)/2}$$

(2.8)

It follows that for $\xi$ a stationary Gaussian process,

$$U^\dagger(t) = \left\{\exp\left(-\frac{i}{2} \int_0^t \int_0^u \, dv \, \mathcal{K}(u, v) \right)\right\}_T$$

(2.9)

where

$$\mathcal{K}(u, v) = \mathbb{E}\left(H_\xi^\dagger(u) H_\xi^\dagger(v)\right) = \sum_{\alpha \beta} \lambda_{\alpha \beta} (u - v) H_\alpha^\dagger(u) H_\beta^\dagger(v)$$

(2.10)

So far, no approximation has been made. However, the time ordering remains a major complication. For its precise meaning one can either go back to equation (2.3), or alternatively, see the discussion and graphical representation in appendix A. There is no issue with time ordering in two cases: when $\xi$ is white noise and when the (interaction picture) Hamiltonian commutes at different times. We examine these cases first.

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7 We shall use script characters to denote super-operators.

8 $U^\dagger$ may be viewed as the grand canonical partition function of a 1-D quantum gas with short range interaction.
2.1. White noise

White noise is the limit \( \tau \to 0 \) with \( \tau J = O(1) \). By equation (1.5) this corresponds to \( \varepsilon \propto \sqrt{\tau} \to 0 \). Not surprisingly, for white noise, the reduction to the Lindblad evolution is exact. Since \( J_{\alpha \beta}(t) = J_{\alpha \beta} \delta(t) \) we have

\[
\mathcal{U}(t) = \exp\left( \int_0^t du \mathcal{L}(u) \right)
\]

with

\[
\mathcal{L}(t) = -\frac{1}{2} \sum_{\alpha \beta} J_{\alpha \beta} H_\alpha^I(t) H_\beta^I(t).
\]

Since \( \mathcal{L} \) and \( H^I \) have the same time argument \( t \), we may use the definition of time-ordering in equation (2.3) with \( H \to \mathcal{L} \), to conclude that \( \mathcal{L} \) is the generator of \( \mathcal{U} \).

The Lindbladian reduces to:

\[
\mathcal{L}_t \rho = -\frac{1}{2} \sum_{\alpha \beta} J_{\alpha \beta} \left[ H_\alpha^I(t), \left[ H_\beta^I(t), \rho \right] \right].
\]

This result is exact. Since \( H_\alpha^I(t) \) are unitarily related for different \( t \) it follows that the family \( \mathcal{L}_t \) is unitarily related and hence isospectral. In particular, the instantaneous dephasing rates are independent of the control \( V(t) \). This could be anticipated since to affect the dephasing rates, the control must be at least as fast as the noise correlations.

2.2. Commutative case

In general, it is difficult to extract a generator of the evolution from equation (2.9) because of the time ordering. In the commutative case this not an issue and the generator of the evolution follows from equation (2.10). Let us denote

\[
\mathcal{G}(t) = -\int_0^t du \mathcal{K}(t, u) = -\sum_\alpha \int_0^t du J_{\alpha \delta}(t-u) \left[ H_\alpha^I(t), H_\delta^I(u) \right].
\]

We then have

\[
\mathcal{U}(t) = \exp\left( \int_0^t du \mathcal{G}(u) \right)
\]

Of course, in the commutative case the index \( T \) is redundant. Now although \( \mathcal{G} \) is an exact generator, it is not in general of Lindblad form. More precisely, it may fail to satisfy positivity at short times as the following example shows.

**Example 2.1.** (Commutative case)

The commutative case arises, for example, when the (interaction picture) noise has a stochastic amplitude but a fixed ‘direction’, i.e. when

\[
H_\alpha^I(t) = \xi(t) H_\alpha.
\]

Since \( H_\alpha^I(t) \) is a commuting family, the generator of the evolution can be exactly calculated from equation (2.15) to be:

\[
\mathcal{G}(t) \rho = -\frac{\gamma(t)}{2} H_\delta H_\alpha \rho = -\frac{\gamma(t)}{2} \left[ H_\delta, \left[ H_\alpha, \rho \right] \right].
\]

The ‘dephasing rate’ \( \gamma(t) \) is given by

\[
\gamma(t) = 2 \int_0^t du J(t-u) = 2 \int_0^t du J(u).
\]

Although \( \gamma(0) \geq 0 \) for very short times (since \( J(0) > 0 \)), \( \gamma(t) \) may be negative \(^9\) for \( t = O(\tau) \) as in figure 1. In these cases \( \mathcal{G}(t) \) does not generate a contraction for all times. This reflects the fact that the evolution is not strictly Markovian. At longer times, \( t \gg \tau \), one always has \( \gamma(t) > 0 \) (since \( J(0) \geq 0 \)).

Positivity is regained in the weak coupling limit. Here it is convenient to consider the limit in the sense of short correlation time, so \( \tau = \varepsilon^2 \). We get from equations (1.6), (1.8)

\(^9\) Take e.g. \( J(\omega) \propto \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \).
In the limit $\epsilon \to 0$, we get for $s > 0$,
\[ \mathcal{L}_\rho = -\frac{j(0)}{2} \left[ H_0, \left[ H_0, \rho \right] \right], \]  
(2.19)

with time-independent positive dephasing rate:
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L}_\rho = -\epsilon \int_{0}^{\infty} \int_{0}^{\infty} J(\tau) d\tau d\rho. \]  
(2.20)

### 3. Weak coupling

Our aim is to obtain an approximate generator that is valid for small nonzero $\epsilon$. This involves two steps. The first step, shown in figure 3, involves a boundary term and leads to an error $O(\epsilon^2)$ uniformly in $t$. The second step is an approximate time ordering and leads to an error $O(\epsilon^{-1} \tau)$. For time scales $t = O(\tau/\epsilon^2)$ the two errors are comparable.

Consider first the issue of time ordering. We shall show that $\mathcal{G}$ defined in equation (2.14) remains an approximate generator in the noncommutative case. More precisely, moving the time ordering $T$ in the exact formula equation (2.9) into the exponential comes with the penalty:
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L}_\rho = -\epsilon \int_{0}^{\infty} \int_{0}^{\infty} J(\tau) d\tau d\rho. \]  
(3.1)

Here $T_u$ denotes time ordering with respect to the integration variable $u$. This differs from the usual time ordering defined with respect to the argument of the Hamiltonian. The error term results from the inequivalence of the two types of ordering. It is proportional to $t$ and hence is cumulative. It reflects the non-commutativity of the Hamiltonian at different times (there is no error in the commutative case as we have seen in section 2.2). In particular, on the coarse grained timescale $s = O(1) \Rightarrow t = O(\tau/\epsilon^2)$ the error is $O(\epsilon^2)$. We justify this estimate in appendix A.

As we have seen (again in section 2.2) $\mathcal{G}$ may not be a Lindbladian. Our next step is to show that within the framework of weak coupling, $\mathcal{G}$ is close to a generator that is of the Lindbladian form up to an $O(\epsilon^2)$ error.

To show this we introduce a useful representation of the noise $\xi$ in terms of white noises $\omega$:
\[ \xi_u(t) = \sum_{\beta} \int_{-\infty}^{\infty} du \ j_{\alpha \beta}(t - u) W^\alpha(u) \]  
(3.2)

where
\[ \mathbb{E}\left( W^\alpha_u(t) \right) = 0, \quad \mathbb{E}\left( W^\alpha_u(t) W^\beta_v(u) \right) = \delta_{\alpha \beta} \delta(t - u). \]  
(3.3)

There is freedom in defining $j$, which allows us to assume, w.l.o.g., that its Fourier transform is non-negative, $j(\omega) \geq 0$. $J$ is then the convolution of $j$ with itself:
\[ J_{\alpha \beta}(u - v) = \sum_{\gamma} \int_{-\infty}^{\infty} dw \ j_{\alpha \gamma}(u - w) j_{\beta \gamma}(v - w), \]  
(3.4)

an example is shown in figure 2.
With this notation in place we first note the identity

\[
\int_0^T \int_0^T \text{d}u \text{d}v \, \mathcal{K}(u, v) = \sum_{\alpha \beta} \left( \int_0^T \int_0^T \text{d}u \text{d}v \, (u - v) H_{\alpha}^I(u) H_{\beta}^I(v) \right) = \sum_{\tau} \int_{-\infty}^{\infty} \text{d}w \left( \sum_{\alpha} \int_0^T \text{d}u \, j_{\alpha}(w - u) H_{\alpha}^I(u) \right)^2
\]

which follows from equation (3.4).

The second step is the claim that we can interchange the limits of the \( \text{d}w \) and \( \text{d}u \) integration in equation (3.5) up to a small error, i.e.

\[
\int_{-\infty}^{\infty} \text{d}w \left( \sum_{\alpha} \int_0^T \text{d}u \, j_{\alpha}(w - u) H_{\alpha}^I(u) \right)^2 = \int_0^T \text{d}w \left( D_f(w) \right)^2 + O(\varepsilon^2)
\]

where

\[
D_f(w) = \sum_{\alpha} \int_{-\infty}^{\infty} \text{d}u \, j_{\alpha}(w - u) H_{\alpha}^I(u).
\]

This follows from the fact that \( j(u) \) is localized near the origin on a timescale \( O(\tau) \). It is clear that the main contribution to the integral comes from the region where both \( w, u \) are in \([0, T]\). The error corresponds to contributions where \( w, u \) are in an \( O(\tau) \) neighborhood of the interval endpoints (see figure 3). As this region has volume \( O(\tau^3) \) and the integrand is \( O(j^2H^2) \) we conclude that the error is of magnitude \( \tau^3 j^2 H^2 \sim \tau^3 j H^2 \sim \varepsilon^2 \), whereas the first term is of the order \( \tau^2 j^2 H^2 \sim \tau j H^2 \sim \varepsilon^2/\tau \sim s \) which dominates the error. This proves equation (3.6) with an error uniform in time.

It follows that for \( \varepsilon \) small the (time-dependent) super-operator

\[
\mathcal{G}_c = -\frac{1}{2} \sum_{\alpha} \left( D_{\alpha}(t) \right)^2
\]

generates a CP map which is \( O(\varepsilon^2) \) close to \( \mathcal{U}^c \).

To rephrase \( \mathcal{G}_c \) in terms of operators, rather than super-operators, use

\[
2 \langle \mathcal{H}(u) \mathcal{H}(v) \rangle_T = \{ \mathcal{H}(u), \mathcal{H}(v) \} + \text{sgn}(u - v) \mathcal{H}(u) \mathcal{H}(v)
\]
and the dictionary in equation (2.5), (2.6) gives a time dependent generator
\[ G, \rho = i \left[ H_{\text{ren}}^\text{(t)}, \rho \right] - \frac{1}{2} \sum_a \left[ D_\alpha(t), \left[ D_\alpha(t), \rho \right] \right] \] (3.10)
\[ H_{\text{ren}}^\text{(t)} = \frac{i}{4} \sum_a \int_{-\infty}^{\infty} j_\alpha(u) j_\alpha(v) \text{sgn}(u - v) \left[ H_{\alpha}^\text{d}(t + u), H_{\alpha}^\text{d}(t + v) \right] du dv \]
where for simplicity we assumed (w.l.o.g.) that \( J, j \) are diagonal. Since the operators \( D_\alpha(t) \) and \( H_{\text{ren}}^\text{(t)} \) are self-adjoint \( \varepsilon \) is a bona fide time dependent generator of a CP map.

3.1. Coarse graining: the Lindbladian in the \( \varepsilon \to 0 \) limit
So far we kept \( \varepsilon \) small but finite and allowed arbitrary time dependence of \( H^\text{I}(t) \). This gives the time-dependent generator of the previous section. To properly define the limit \( \varepsilon \to 0 \), one should also specify the limiting behavior of the dimensionless parameter \( \alpha_\varepsilon \tau \). If \( \alpha_\varepsilon \tau \to 0 \) then \( \tau \) is the smallest timescale and \( \xi \) becomes effectively equivalent to white noise discussed in section 2.1. The interesting case and the one relevant to dynamic decoupling is when \( \alpha_\varepsilon \tau \geq O(1) \) (implying \( \alpha_\varepsilon \tau / \varepsilon^2 \to \infty \)). In this limit equation (3.10) reduces to equation (1.10). To see this note:

- Weak coupling may be interpreted as \( J, \tau = O(1) \) while \( \| H_\alpha \| = O(\varepsilon) \), equation (1.7) then implies that \( H_\alpha(\omega) = O(1) \).
- The ansatz equation (1.7) says that the integral in equation (3.7) reduces to the sum:
\[ D_\alpha(t) = \varepsilon \sum_{\omega \in F} \tilde{j}_\alpha(\omega) e^{i\omega t} d(\bar{H}_\alpha(\omega)) \] (3.11)
- The limit \( \varepsilon \to 0 \) means that \( \omega = \omega = 0 \) in the sense of distributions.
- The limiting Lindbladian generates the evolution on the time scale \( s = \varepsilon^2 t / \tau \) and it is related to \( G_\varepsilon \) by \( L = \tau e^{-2} G_\varepsilon \).

It follows that for the second term in equation (3.10) we get
\[ \frac{\tau}{2\varepsilon^2} D_\alpha^2(t) \to \frac{\tau}{2} \sum_{\omega \in F} \tilde{j}_\alpha(\omega) \tilde{j}_\alpha(-\omega) d\left( \bar{H}_\alpha(\omega) \right) d\left( \bar{H}_\alpha(-\omega) \right) \]
which is \( D_\alpha \) of Equation (1.10). Similarly, for the first term in equation (3.10) we get
\[ \frac{i}{\varepsilon^2} d\left( H_{\text{ren}} \right) \to \frac{\tau}{4} \sum_{\alpha, \omega} \int \tilde{j}_\alpha(u) \tilde{j}_\alpha(v) \text{sgn}(v - u) e^{i\omega(u-v)} d\left( \left[ H_\alpha(\omega), H_\alpha^*(\omega) \right] \right) du dv \]
The \( u, v \) integration can be carried out explicitly to give
\[ \int \tilde{j}_\alpha(u) \tilde{j}_\alpha(v) \text{sgn}(v - u) e^{i\omega(u-v)} du dv = \int \tilde{j}_\alpha(u) \text{sgn}(u) e^{-iu} = i K_\alpha(\omega) \].

4. Examples
Since the formulas for the Lindbladian simplify when weak coupling is interpreted as \( \tau = \varepsilon^2 \), we make this choice throughout the chapter.

4.1. No control
Consider the (commutative) stochastic Hamiltonian for spin S
\[ H = \sum_\alpha \varepsilon H_\alpha = \xi S_z. \] (4.1)
equation (2.19) gives the dephasing Lindbladian

$$\mathcal{L}\rho = -\frac{i}{2} \left[ S_x, \left[ S_z, \rho \right] \right], \quad \gamma = J(0) \quad (4.2)$$

with a single rate parameter $\gamma$. The decoherence can be computed using the spectral properties of the superoperator of angular momentum given in appendix B

$$\text{Spectrum} \left( \left( \text{ad} \left( S_z \right) \right)^2 \right) = \left\{ m^2 | m \in \{ 0, \ldots, 2S \} \right\}. \quad (4.3)$$

The coherence decreases quadratically with the polarization $m$: $|m\rangle \langle m| \rightarrow |m\rangle \langle m| - m_1^2 / (m_1^2 + m_2^2)$.  

### 4.2. Bang-bang

'Bang-bang' makes $\gamma$ smaller and improves the coherence: a sequence of rapid $\pi$ rotations about an axis perpendicular to the magnetic field self-average the noise. The $\pi$ rotations are given by the unitary:

$$\omega \pi \pi \omega \pi \omega \pi = \begin{cases} e^{i\pi/2}, & \text{if } \theta \text{ mod } 2\pi \in (-\pi, 0) \\ 1, & \text{if } \theta \text{ mod } 2\pi \in (0, \pi) \end{cases}$$

The controlled stochastic Hamiltonian corresponding to equation (4.1) then takes the form:

$$H^2_{\xi}(t) = \xi(t) S_x w(\theta t), \quad (4.4)$$

where $w$ is the square wave

$$w(t) = \begin{cases} -1, & \text{if } \theta \text{ mod } 2\pi \in (-\pi, 0) \\ 1, & \text{if } \theta \text{ mod } 2\pi \in (0, \pi). \end{cases}$$

Using the Fourier expansion

$$w(t) = -\frac{2i}{\pi} \sum_{n \geq 0} \frac{e^{i(2n+1)\theta t}}{2n+1}$$

and equation (1.10), we obtain the functional form of the dephasing Lindbladian of equation (4.2) but with a renormalized $\gamma \rightarrow \gamma_0(\omega)$:

$$\gamma_0(\omega) = \frac{8}{\pi^2} \sum_{n \geq 0} \frac{\tilde{f}(2n+1)\omega}{(2n+1)^2} \quad (4.5)$$

Since $\tilde{f}(\omega) \rightarrow 0$ as $\omega \rightarrow \infty$, in this limit we have $\gamma_0(\omega) \rightarrow 0$, and there is no loss of coherence.

### 4.3. Constant control

Consider the stochastic Hamiltonian with time-independent control

$$H = \omega \xi S_x + \xi S_x \quad (4.6)$$

The control is effective in the sense of appendix C. In the interaction picture the stochastic Hamiltonian has the form

$$H^I_{\xi}(t) = \xi(t) \left( S_x \cos \omega_\xi t + S_y \sin \omega_\xi t \right) \quad (4.7)$$

The frequency set $F$ in equation (1.7) has two elements, $F = \{ \pm \omega_\xi \}$ and

$$\tilde{H}(\pm \omega_\xi) = \frac{1}{2} \left( S_x \mp iS_y \right)$$

we find, from equations (1.10) the Lindbladian

$$\mathcal{L}\rho = -\frac{i\tilde{H}(\omega_\xi)}{4} \left[ S_x, \rho \right] + \frac{\tilde{f}(\omega_\xi)}{4} \sum_{j=x,y} \left[ S_j, \left[ S_j, \rho \right] \right]. \quad (4.8)$$

The two terms in $\mathcal{L}$ commute. This follows from the fact that $\tilde{f}_j \equiv \text{ad} \left( S_j \right)$, $i = x, y, z$ give an $SU(2)$ representation. As in any such representation $\tilde{f}_x^2 + \tilde{f}_y^2$ is invariant under rotation around the z-axis, one has $[\tilde{f}_x, \tilde{f}_x^2 + \tilde{f}_y^2] = 0$. This may also be verified directly by calculating the commutators.

It follows that the first term in $\mathcal{L}$ determines the imaginary part of the eigenvalues while the second term determines the real part. The decoherence rate is then determined by

$^{10}$ For a monotonically decreasing $\tilde{f}(\omega)$ one has $\gamma_0(\omega) < \gamma_0(0) = \gamma$.
and the index denotes multiplicities. In the case of general $S$ the spectrum is $\{ j \mid j \in \mathbb{Z}, |j| \leq 2S \}$ as computed in appendix B. In particular using Schur’s lemma implies that $0$ is always a simple eigenvalue. It follows that the Lindbladian is depolarizing: the unique equilibrium state is the fully mixed state. Numerical examples are shown in figures 4 and 5.

### 4.4. Non-commutative noise

The simplest case of non-commutative noise is ‘planar’ noise

$$\sum_{\alpha=x,y} \xi_\alpha S_\alpha,$$

equation (1.10) gives the depolarizing Lindbladian

$$\mathcal{L}\rho = -\frac{1}{2} \sum_{j\in x,y} \gamma_j [S_j, [S_j, \rho]], \quad \gamma_j = |\tilde{J}_j(0)|$$

(4.9)

$H_c = \omega S_x$ is an effective control. Moreover, the results of sections 4.2, 4.3 carry over to this case, in a straightforward way: Bang-Bang leads to equation (4.9) with $\gamma \mapsto \gamma_0$ as in equation (4.5). Constant control leads to an equation similar to equation (4.8) up to an extra factor of 2.
4.5. Isotropic noise

Isotropic noise is represented by the Hamiltonian

$$\sum_{\alpha=1}^{3} \xi_{\alpha} S_{\alpha}, \quad J_{1} = J_{2} = J_{3}$$

leading to the isotropic depolarizing Lindbladian

$$\mathcal{L}\rho = -\frac{j(0)}{2} \sum_{j=1}^{3} [S_{j}, [S_{j}, \rho]].$$

For $S = 1/2$ constant control is not effective\(^{11}\). One can, however, find an effective Bang-bang.\(^{12}\)

The simplest version of Bang-bang about all three axes is associated with the unitary $V$

$$V(t) = \begin{cases} 
\sigma_{1} \text{ oot } \mod 2\pi \in [0, \pi/2] \\
\sigma_{2} \text{ oot } \mod 2\pi \in [\pi/2, \pi] \\
\sigma_{3} \text{ oot } \mod 2\pi \in [3\pi/2, 2\pi] \\
1 \text{ oot } \mod 2\pi \in [3\pi/2, 2\pi].
\end{cases} \quad (4.10)$$

This control self-averages the Hamiltonian in the interaction picture to zero but leads to a non-isotropic Lindblad equation (with $\gamma_{1} \neq \gamma_{2}$). In order to retain isotropy, we choose a somewhat more complicated $V(t)$ corresponding to dividing $[0, 2\pi]$ into 12 equal parts$^{13}$. We demand $V(t) = V_{j}$ for $\text{ oot } \mod 2\pi \in [2\pi j/12, 2\pi (j + 1)/12]$ where

$$\{V_{j}\}_{j=1}^{12} = \{\sigma_{1}, \sigma_{2}, \sigma_{3}, 1, \sigma_{2}, \sigma_{3}, \sigma_{1}, 1, \sigma_{3}, \sigma_{1}, \sigma_{2}, 1\}. \quad (4.11)$$

This gives the stochastic Hamiltonian

$$H_{t} = \sum_{\alpha=1}^{3} \xi_{\alpha} S_{\alpha} w_{\alpha}(\text{ oot }) \quad (4.12)$$

where $w_{1}(t) = w_{2}(t + 2\pi/3) = w_{3}(t - 2\pi/3) = \pm 1$ takes on $[2\pi j/12, 2\pi (j + 1)/12] \mod 2\pi$ the values

$$w_{1}(t) \Leftrightarrow \{+1, -1, -1, +1, -1, -1, +1, +1, +1, -1, +1, -1\}, \quad (4.13)$$

as shown in figure 6.

From symmetry considerations, $H_{t} = 0$ (since $[H_{t} (\omega), H_{t}^{*} (\omega)] \propto [S_{\alpha}, S_{\alpha}] = 0$) and the depolarizing Lindblad has renormalized rates:

$$\mathcal{L}\rho = -\frac{\gamma(\omega)}{2} \sum_{j=1}^{3} [S_{j}, [S_{j}, \rho]].$$

$\gamma(\omega)$ is a more complicated version of equation (4.5)

$$\gamma(\omega) = \frac{8}{\pi^{2}} \sum_{n \neq 0} \frac{J(\omega n)}{n^{4}} \sin^{4} \left(\frac{n\pi}{12}\right) p(n) \quad (4.14)$$

\(^{11}\) For $S = 1$ a possible effective constant control is $H_{t} = \sum_{n} n S_{n}^{2}$.

\(^{12}\) The generalization to arbitrary spin is quite simple and only requires replacing the $\sigma_{i}$ matrices in Equations (4.10), (4.11) with the appropriate rotation operator $R_{j} = \exp (i\omega S_{j})$.

\(^{13}\) We thank Ori Hirschberg for this suggestion.
\[ p(n) = 5 + 4 \cos \left( \frac{n\pi}{6} \right) + 2 \cos \left( \frac{4n\pi}{3} \right) + (-1)^n \left( 1 + 4 \cos \left( \frac{n\pi}{2} \right) + 2 \cos \left( \frac{2n\pi}{3} \right) \right). \]

### 4.6. Stochastic harmonic oscillator

The stochastic harmonic oscillator provides a good model for trapped atoms, mechanical oscillators and trapped ions [17]. Since \( |1\rangle \) is not a state in an infinite dimensional Hilbert space, the Lindbladian associated with stochastic evolution may have no stationary state.

There are various types of noise one may consider. The first is \( \omega_\xi \xi = + + \xi \xi \), \( H_p x p x \frac{1}{2} \) \( H_p x \cos \sin \cos \sin \). \( I_p p c c c \). The Lindbladian is:

\[ -2\mathcal{L} = \Gamma_x \textrm{ad}(x) \textrm{ad}(x) + \Gamma_p \textrm{ad}(p) \textrm{ad}(p) + 2\Gamma_{xp} \left\{ \textrm{ad}(x), \textrm{ad}(p) \right\} \]

with matrix \( \Gamma \propto \tilde{f}(\omega) \) at the oscillator frequency. Since \( \textrm{ad}(x) \) and \( \textrm{ad}(p) \) commute, and \( \textrm{spect}(\textrm{ad}(x)) = \psi(\textrm{ad}(p)) = (0, \infty) \) and \( \Gamma \) is a positive matrix, we have

\[ \psi(\mathcal{L}) = \psi(\eta, \Gamma \eta) \eta \in \mathbb{R}^2 = (0, \infty). \]

0 is in the spectrum but is not associated with an eigenvalue: there is no stationary equilibrium state.

To study the long time behavior, note first that \( \mathcal{L} = \mathcal{L}^* \). One readily checks that \( \mathcal{L}^*(x^2) = \text{const} \), \( \mathcal{L}^*(p^2) = \text{const} \). It follows that the energy increases linearly with time. Heating rates have been analyzed in detail in the context of trapped ions [21].

In the case of noise in the frequency of the harmonic oscillator the Hamiltonian is:

\[ H_f = \frac{1}{2} \begin{pmatrix} \omega_c + 2\xi_p & p x \frac{1}{2} \\ p x \frac{1}{2} & \omega_c + 2\xi_x \end{pmatrix} = \frac{1}{2} \omega_c \left( p^2 + x^2 \right) + \xi_p p^2 + \xi_x x^2. \]

In the interaction picture one has

\[ H_i = \xi_p \left( p \cos \omega_c t - x \sin \omega_c t \right)^2 + \xi_x \left( x \cos \omega_c t + p \sin \omega_c t \right)^2. \]

Hence

\[ \mathcal{R}_d(0) = \begin{pmatrix} \mathcal{R}_o(-2\omega_c), \mathcal{R}_o(2\omega_c) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p^2 + x^2 \end{pmatrix}, \quad \mathcal{R}_d(\pm 2\omega_c) = \frac{1}{4} (1)^n (p \pm ix)^2 \]

and the Lindbladian is:

\[ \mathcal{L} = \mathcal{K}_2 \textrm{ad} \mathcal{H}(0) + \mathcal{I}_0 \left\{ \textrm{ad} \mathcal{H}(0) \right\}^2 \]

\[ + \mathcal{I}_d \left\{ \textrm{ad} \mathcal{H}(2\omega_c) \right\}^2 + \left\{ \textrm{ad} \mathcal{H}(-2\omega_c) \right\}^2 \]

(4.16)

where \( \mathcal{K}_2 \propto \tilde{K}(2\omega_c), \mathcal{I}_0 \propto \tilde{f}(0) \) and \( \mathcal{I}_d \propto \tilde{f}(2\omega_c) \). While all the terms \(|n\rangle\langle m|\) are eigenstates of the dephasing part with eigenvalues \((m - n)^2\) the parametric drive part does not have a steady state and drives the system towards infinite temperatures.
5. Comparison with stochastic evolutions

Numerical algorithms for solving stochastic evolution equations have two advantages: they can also work beyond weak coupling and evolve pure states rather than density matrices. They also have several disadvantages: they tend to be slow because of the necessity of accumulating enough statistics; they are prone to long time-drifts and can be adversely affected by a poor random number generator, and finally are prone to bugs. Our results on the Lindblad evolution can be used to test numerical algorithms for stochastic evolutions in those cases where both apply.

A comparison between Lindbladian evolutions of sections 4.1, 4.2, 4.3 and stochastic evolutions with the Ornstein–Uhlenbeck process is shown in figure 7. Three cases have been studied: no control, control by constant $H_0$ and Bang-bang. The weak coupling parameter is $\epsilon = 0.15$ and the control parameter is $\omega \tau = 0.5 \pi$. A time grid spacing $\Delta = \tau/20$ was used, and the stochastic averaging is done on an ensemble of 500 runs.

6. Summary

We derived the Lindbladian for controlled weakly stochastic evolutions both for small but finite $\epsilon$ and in the limit $\epsilon \to 0$ for stationary control. Our results can be used to measure the power spectrum of the noise and to test numerical algorithms for solving stochastic evolution. Several examples are studied in detail. Extending the theory from matrices to operators is an open problem that gives rise to analytical questions that we have avoided.

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Appendix A. Weak coupling expansion

The purpose of this appendix is to justify the estimate of equation (3.1). This requires a comparison of two different time orderings of the same exponent. Let us first ignore the ordering and consider

$$ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x = \int_0^t ds \mathcal{G}(s). $$

(A.1)

The Taylor series for the exponent $e^x$ is dominated by terms of order $n = O(x)$. In our case this gives $n \sim x \sim H^2 \pi t \sim e^{2t/\tau}$. Writing the $n$-th term in the expansion
we conclude that typically $s_{i+1} - s_i = O(t/n) = O(\tau/\varepsilon^2) \gg \tau$.

Next let us consider the possible time orderings. The naive $\tilde{T}$ time ordering with respect to the argument of $G(s)$ implied by equation (A.2) differs from the correct $T$-ordering because the relation between $G$ and $H$

$$G(s_j) = -\int_{u_t > v_t} du_t dv_{j_1} (u_t - s_{j_1}) f (v_t - s_t) \left( H (u_t), H (v_t) \right)$$

is non-local in time: the ordering of $s_j$ does not guarantee the ordering of $\{u_t, v_t\}$ (see figure A1). The fact that $j (u - s)$ is fast decaying implies, however, that the nonlocality in time is rather small $\sim \tau$. When $s_{i+1} - s_i \gg \tau$ the wrong ordering is almost the same as the correct one.

In order to estimate the error generated by using the $\tilde{T}$ ordering, consider more closely the two orderings. Each contribution to the exponent is given as in equation (A.2) by some choice of $j (u - s)$ fast decaying implies, however, that the nonlocality in time is rather small $\sim \tau$. When $s_{i+1} - s_i \gg \tau$ the wrong ordering is almost the same as the correct one.

In order to estimate the error generated by using the $\tilde{T}$ ordering, consider more closely the two orderings. Each contribution to the exponent is given as in equation (A.2) by some choice of $j (u - s)$ fast decaying implies, however, that the nonlocality in time is rather small $\sim \tau$. When $s_{i+1} - s_i \gg \tau$ the wrong ordering is almost the same as the correct one.

$$\frac{X^n}{n!} = \int_{0 \leq n_1 \leq \ldots \leq n_k} \prod_{j=1}^n G(s_j) \, ds_j$$

is non-local in time: the ordering of $s_j$ does not guarantee the ordering of $\{u_t, v_t\}$ (see figure A1). The fact that $j (u - s)$ is fast decaying implies, however, that the nonlocality in time is rather small $\sim \tau$.

When $s_{i+1} - s_i \gg \tau$ the wrong ordering is almost the same as the correct one.

Consider for example the case where $u_{j+1} > u_j > v_{j+1} > v_j$ (see figure A1) while all other points are at typical positions. This will lead to an error term of the type

$$U_{n_0, \ldots, n_j} \times \left\{ \int_{s_{j+1} < s_j} ds_j ds_{j+1} \int_{u_{j+1} > u_j > v_{j+1} > v_j} du_j du_{j+1} dv_{j+1} H (u_{j+1}) \left[ H (u_j), H (v_{j+1}) \right] H (v_j) \right. \times j (s_j - u_j) j (s_j - v_j) j (s_{j+1} - u_{j+1}) j (s_{j+1} - v_{j+1}) \} \times U_{j-1, \ldots, 1}$$

Here $U_{n_0, \ldots, n_j} = \int_{u_{j+1} > u_j > v_{j+1} > v_j} du_{j+1} dv_{j+1} \prod_{j=1}^n G(s_j) \, ds_j$ correspond to the evolution before $t = v_j$ and after $t = u_{j+1}$. The integrand in equation (A.4) is clearly fast decaying whenever its six integration variables are at inter-distance large compared to $\tau$. It thus follows that the main contribution to the integral comes from a region of volume $\tau^6 t$. The integral is thus at most $14$ of the order of $\tau^4 t^4 \| H \|_4^4 \sim \tau^4 t^2 \| H \|_4^4 e^{-t/\tau}$. Other nontypical cases (e.g. $u_j > u_{j+1} > v_{j+1} > v_j$) lead to error terms of a similar general form which again scale as $e^{-t/\tau}$.

The error terms we found are of the form $\int_0^\infty ds \ U(t, s) \Delta G(s) U(s, 0)$ for some $\Delta G$ which is quartic in $H$. This suggests defining an improved generator as $G \mapsto G + \Delta G$. We however did not pursue this direction here.

**Appendix B. The spectrum of the super-operators of angular momenta**

The adjoint representation $ad(S)$ of a representation $S$ is constructed as the tensor product of $S$ with its dual (contragredient) representation $S^*$. Since $SU(2)$ has a single representations in each dimension, it is obvious that $S^* \simeq S$. It thus follows that

$$ad(S) = S \otimes S^* = S \otimes S = 0 \oplus 1 \oplus 2 \oplus \ldots \oplus (2S)$$

The spectrum (including multiplicities) of various operators such as $ad(S_j)$ and $\sum a_j (S_j) ad(S_j)$ is then easily deduced

$$\text{Spect } \left( ad(S_j) \right) = \bigcup_{j=0, \ldots, 2S} \left\{ m \left| m = -j, \ldots, j \right\} \right.$$
\[
\text{Spect} \left( \sum_{j=0}^{\infty} \left( \alpha \left( S_j \right)^2 \right) \right) = \bigcup_{j=0}^{\infty} \{ j(j+1) \mid m = -j, \ldots \}.
\]

\[
\text{Spect} \left( \sum_{j=0}^{\infty} \left( \alpha \left( S_j \right)^2 \right) \right) = \bigcup_{j=0}^{\infty} \{ j(j+1) - m^2 \mid m = -j, \ldots \}.
\]

In particular the eigenvalue zero appears in \( \text{Spect} \left( \alpha \left( S_j \right)^2 + \alpha \left( S_j \right)^2 \right) \) with trivial multiplicity 1. This last fact could also be deduced from Schur’s lemma since by positivity \( \alpha \left( S_j \right)^2 + \alpha \left( S_j \right)^2 \) \( \rho = 0 \) implies \( \alpha \left( S_j \right)^2 \rho = \alpha \left( S_j \right)^2 \rho = 0 \) and hence also \( \alpha \left( S_j \right)^2 = -i \left( \alpha \left( S_j \right), \alpha \left( S_j \right) \right) \rho = 0 \).

**Appendix C. Effective control**

In dynamical decoupling one is interested in making \( L \) small at the price of strong control, \( \omega \alpha \gg 1 \). Since \( \tilde{f}(\omega) \) is small for large arguments and since the terms \( \omega \neq 0 \) in equation (1.10) tend to be of the order \( \tilde{f}(\omega) \) the ‘bad term’ in \( L \) is the one with \( \omega = 0 \). We say that the control is ‘effective’ if \( \tilde{H}(\omega = 0) = 0 \). The notion is independent of \( J_0(\omega) \), which is often not known.

Consider first strong continuous controls. Let \( P_j(t) \) be the (instantaneous) spectral projections of \( H_z \):

\[
H_z(t) = \omega \alpha \sum j \epsilon_j(t) P_j(t)
\]

and suppose that \( P_j(t) \) vary smoothly with \( t \) and that the eigenvalues \( \epsilon_j(t) \) do not cross. Then, by the adiabatic theorem, for \( \omega \alpha \) large

\[
H_z^2(t) \approx \sum \epsilon_j(t) e^{i \int_{\omega(\alpha)} \omega(\alpha) d\alpha} P_j(t) H_0 P_i(t) \omega(\alpha) \rightarrow \sum_j P_j(t) H_0 P_j(t)
\]

(in the sense of distributions). It follows that the continuous control is effective if, for all \( t \),

\[
\sum_j P_j(t) H_0 P_j(t) = 0
\]

Bang–bang at times \( t_j \) is effective if \( \tilde{H}_d(\omega = 0) = 0 \), which is the case if \( H_z(t) \) has zero average, i.e.

\[
\forall \alpha, \sum_j \left( t_{j+1} - t_j \right) V^* \left( t_j \right) H_0 V \left( t_j \right) = 0
\]

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