YONEDA EXTENSIONS OF ABELIAN QUOTIENT CATEGORIES

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Abstract. Let \( \mathcal{A} \) be a essentially small abelian category and \( \mathcal{C} \) be a Serre subcategory of \( \mathcal{A} \). Consider the quotient functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \). For an object \( A \in \mathcal{A} \) and a non-negative integer \( k \) we investigate when the natural map \( q_{X,A}^i : \text{Ext}^i_{\mathcal{A}}(X,A) \to \text{Ext}^i_{\mathcal{A}/\mathcal{C}}(q(X),q(A)) \) is invertible for every \( X \in \mathcal{A} \) and every \( i \in \{0, 1, \ldots, k\} \). In the end we give an application of the main theorem.

1. Introduction

Let \( \mathcal{A} \) be an abelian category. A Serre subcategory \( \mathcal{C} \subseteq \mathcal{A} \) is a full subcategory that is closed under subobjects, quotient objects, and extensions. In this case there exist an abelian category \( \mathcal{A}/\mathcal{C} \) and an exact functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) that is universal among exact functors from \( \mathcal{A} \) to an abelian category that annihilate \( \mathcal{C} \). The quotient category \( \mathcal{A}/\mathcal{C} \) was defined by Grothendieck and studied extensively by Gabriel [6].

The quotient functor \( q \) is not full or faithful, but by a result of Gabriel for each object \( A \) in the left perpendicular subcategory of \( \mathcal{C} \), the natural map \( q_{X,A}^i : \text{Hom}_{\mathcal{A}}(X,A) \to \text{Hom}_{\mathcal{A}/\mathcal{C}}(q(X),q(A)) \) is invertible, for every \( X \in \mathcal{A} \).

In many situations \( \mathcal{A} \) is a known and well behaved category like a module category or a functor category and we can compute Ext-groups for any two objects in \( \mathcal{A} \). But the quotient category \( \mathcal{A}/\mathcal{C} \) has a complicated structure, for instance if \( \mathcal{C} \) is a localizing subcategory (i.e. \( q \) has a right adjoint) then \( \mathcal{A}/\mathcal{C} \) is equivalent to a subcategory of \( \mathcal{A} \), but not an exact subcategory, because the right adjoint is fully faithful and left exact, but not exact in general [11, Lemma 2.2.10]. Thus it is hard to relate the Ext-groups of these two categories.

The main theorem of this paper is the following result.

Theorem 1.1. Let \( \mathcal{A} \) be a essentially small abelian category, \( \mathcal{C} \) a Serre subcategory of \( \mathcal{A} \) and \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) be the quotient functor. For an object \( A \in \mathcal{A} \) and a non-negative integer \( k \) the following statements are equivalent.

(i) \( A \in \mathcal{C}^\perp_{k+1} \), i.e. \( \text{Ext}^i_{\mathcal{A}}(\mathcal{C},A) = 0 \) for every \( i \in \{0, 1, \ldots, k+1\} \).

(ii) The natural map \( q_{X,A}^i : \text{Ext}^i_{\mathcal{A}}(X,A) \to \text{Ext}^i_{\mathcal{A}/\mathcal{C}}(q(X),q(A)) \) is invertible, for every \( X \in \mathcal{A} \) and every \( i \in \{0, 1, \ldots, k\} \).

Note that C. Psaroudakis proves this result for a recollement of abelian categories with enough projective and injective objects [19]. In other word we prove that the existence

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of left and right adjoint for the quotient functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) and also the existence of enough injective and enough projective objects are not necessary.

2. PRELIMINARIES

2.1. Localization of abelian categories. We recall below the basic theory of localization of abelian categories. The proofs can be found in Gabriel’s thesis [6] or in standard textbooks like Faith’s book [4] or the recent book of Krause [11].

Let \( \mathcal{A} \) be an abelian category. A full subcategory \( \mathcal{C} \) of \( \mathcal{A} \) is called a Serre subcategory if for any short exact sequence

\[
0 \to A_1 \to A_2 \to A_3 \to 0
\]

we have that \( A_2 \in \mathcal{C} \) if and only if \( A_1, A_3 \in \mathcal{C} \). In this case we have the quotient category \( \mathcal{A}/\mathcal{C} \) that is by definition localization of \( \mathcal{A} \) with respect to the class of all morphisms \( f : X \to Y \) with \( \text{Ker}(f), \text{Coker}(f) \in \mathcal{C} \). The quotient category \( \mathcal{A}/\mathcal{C} \) is an abelian category and there is a universal exact functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) such that \( q(X) = 0 \) if and only if \( X \in \mathcal{C} \). Furthermore any other exact functor \( F : \mathcal{A} \to \mathcal{D} \) annihilating \( \mathcal{C} \) where \( \mathcal{D} \) is an abelian category factor uniquely through \( q \).

A Serre subcategory \( \mathcal{C} \subseteq \mathcal{A} \) is called a localizing subcategory if the quotient functor \( q : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) admits a right adjoint \( r : \mathcal{A}/\mathcal{C} \to \mathcal{A} \). The right adjoint \( r \), called the section functor, is fully faithful [11, Lemma 2.2.10]. A Serre subcategory of a Grothendieck category is localizing if and only if it is closed under coproducts [11, Corollary 2.2.17].

Another useful property of Grothendieck categories is the fact that every object \( A \) admits an injective envelope \( A \hookrightarrow I \) [11, Corollary 2.5.4]. This means that \( A \) is a subobject of an injective object \( I \), and \( I \) is an essential extension of \( A \) (i.e. the intersection of \( A \) with every non-zero subobject of \( I \) is non-zero).

Let \( \mathcal{C} \) be a Serre subcategory of an abelian category \( \mathcal{A} \). Recall that an object \( A \in \mathcal{A} \) is called \( \mathcal{C} \)-closed if for every morphism \( f : X \to Y \) with \( \text{Ker}(f), \text{Coker}(f) \in \mathcal{C} \) we have that \( \text{Hom}_\mathcal{A}(f, A) \) is bijective. Denote by \( \mathcal{C}^\perp \) the full subcategory of all \( \mathcal{C} \)-closed objects, the following result is well known. See for instance the fundamental paper of Geigle and Lenzing [7] or the recent book of Krause [11].

**Theorem 2.1.** Let \( \mathcal{C} \) be a Serre subcategory of an abelian category \( \mathcal{A} \). The following statements hold:

(i) We have

\[
\mathcal{C}^\perp = \{ A \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(\mathcal{C}, A) = 0 = \text{Ext}_\mathcal{A}^1(\mathcal{C}, A) \}.
\]

(ii) The natural map \( q_{X,A} : \text{Hom}_\mathcal{A}(X, A) \to \text{Hom}_\mathcal{A/\mathcal{C}}(q(X), q(A)) \) is invertible for every \( X \in \mathcal{A} \) if and only \( A \in \mathcal{C}^\perp \).

(iii) If \( \mathcal{C} \) is a localizing subcategory, the restriction \( q : \mathcal{C}^\perp \to \mathcal{A}/\mathcal{C} \) is an equivalence of categories. And it’s quasi-inverse is induced by the section functor \( r : \mathcal{A}/\mathcal{C} \to \mathcal{C}^\perp \subseteq \mathcal{A} \).

(iv) If \( \mathcal{C} \) is localizing and \( \mathcal{A} \) has injective envelopes, then \( \mathcal{C}^\perp \) has injective envelopes and the inclusion functor \( \mathcal{C}^\perp \hookrightarrow \mathcal{A} \) preserves injective envelopes.

The following definition is borrowed from [19].
Definition 2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ be a subcategory of $\mathcal{A}$. For a positive integer $k$ we denote by $\mathcal{X}^\perp_k$ the full subcategory of $\mathcal{A}$ defined by $\mathcal{X}^\perp_k = \{ A \in \mathcal{A} | \text{Ext}^{0,...,k}(\mathcal{X}, A) = 0 \}$. $^\perp_k \mathcal{X}$ is defined similarly [19].

Note that by Theorem 2.1 for a Serre subcategory $C$ of $\mathcal{A}$ we have $C^{\perp_1} = C^\perp$.

Let $\mathcal{B}$ be an abelian category and $S$ be a localizing subcategory of $\mathcal{B}$. Denote the localization functor and the section functor with $e$ and $r$ respectively, we have the following diagram of functors.

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{e} & \mathcal{B}/S \\
& \searrow r & \\
\end{array}
\]

The following proposition can be seen as a variation of Proposition 3.4 and Theorem 3.10 from [19]. The Hom vanishing is the crucial step of the proof of Proposition 3.3 in Psaroudakis paper (the dual result - Proposition 3.4). We will prove this condition using the existence of injective envelopes in Grothendieck categories.

Proposition 2.3. Let $\mathcal{B}$ be a Grothendieck category and $S$ be a localizing subcategory of $\mathcal{B}$. For an object $A \in \mathcal{B}$ and a non-negative integer $k$ the following statements are equivalent.

(i) $A \in S^{\perp_{k+1}}$.

(ii) There exists an injective coresolution

\[
0 \rightarrow A \rightarrow r(I^0) \rightarrow r(I^1) \rightarrow \cdots \rightarrow r(I^{k+1})
\]

for $A$, where $I^i$’s are injective objects in $\mathcal{B}/S$.

(iii) The natural map $e^i_{X,A} : \text{Ext}_\mathcal{B}^i(X, A) \rightarrow \text{Ext}_\mathcal{B}^i(e(X), e(A))$ is invertible, for every $X \in \mathcal{B}$ and every $i \in \{0, 1, \cdots, k\}$.

Proof. (i) $\Rightarrow$ (ii) Let $A \in S^{\perp_{k+1}}$, and $A \hookrightarrow I^0$ be the injective envelope of $A$ in $\mathcal{B}$. Let $S \in S$ and $f \in \text{Hom}_\mathcal{B}(S, I^0)$ be a non-zero morphism, because $I^0$ is an essential extension of $A$ we have that $\text{Im}(f) \cap A \neq 0$. By the definition of Serre subcategory $\text{Im}(f) \cap A \in S$, and this contradicts the assumption $\text{Hom}_\mathcal{B}(S, A) = 0$. Thus $I^0 \in S^\perp$ because $I^0$ is injective.

Now by applying the functor $\text{Hom}_\mathcal{B}(S, -)$ for an arbitrary object $S \in S$, to the short exact sequence

\[0 \rightarrow A \rightarrow I^0 \rightarrow \Omega^{-1}A \rightarrow 0\]

we obtain the long exact sequence

\[0 \rightarrow \text{Hom}_\mathcal{B}(S, A) \rightarrow \text{Hom}_\mathcal{B}(S, I^0) \rightarrow \text{Hom}_\mathcal{B}(S, \Omega^{-1}A) \rightarrow \text{Ext}_\mathcal{B}^1(S, A) \rightarrow \text{Ext}_\mathcal{B}^1(S, I^0) \rightarrow \text{Ext}_\mathcal{B}^1(S, \Omega^{-1}A) \rightarrow \cdots\]

By assumption $\text{Hom}_\mathcal{B}(S, A) = \text{Ext}_\mathcal{B}^1(S, A) = \cdots = \text{Ext}_\mathcal{B}^{k+1}(S, A) = 0$, and $\text{Ext}_\mathcal{B}^1(S, I^0) = 0$, because $I^0$ is injective. Using these vanishing conditions and the above long exact sequence we see that $\Omega^{-1}A \in S^{\perp_k}$.

Now consider the injective envelope $\Omega^{-1}A \hookrightarrow I^1$ for $\Omega^{-1}A$ and the induced short exact sequence
0 \rightarrow \Omega^{-1}A \rightarrow I^1 \rightarrow \Omega^{-2}A \rightarrow 0.

Applying the above argument for this short exact sequence we see that $I^1 \in \mathcal{S}^1$ and $\Omega^{-2}A \in \mathcal{S}^{1+k-1}$. By repeating this argument and using the dimension shifting argument we obtain an injective coresolution as (2.1) inductively.

(ii) \Rightarrow (iii) Apply the functor \( \text{Hom}_B(X,-) \) to the injective coresolution (2.1) and use the fact that \((e,r)\) is an adjoint pair.

(iii) \Rightarrow (i) By assumption for every \( S \in \mathcal{S} \) and every \( i \in \{0,1,\cdots,k\} \), we have

\[
\text{Ext}^i_B(S,A) \cong \text{Ext}^i_{B/\mathcal{S}}(e(S),e(A)) = \text{Ext}^i_{B/\mathcal{S}}(0,e(A)) = 0.
\]

This means that \( A \in \mathcal{S}^{1+k} \), so it remains to show that \( \text{Ext}^{k+1}_B(S,A) = 0 \). We prove the later by induction. For \( k = 0 \) the result follows from Theorem 2.1 Assume that \( k \geq 1 \) be a positive integer, and the claim is true for \( k-1 \). Consider exact sequence

\[
0 \rightarrow A \rightarrow I \rightarrow B \rightarrow 0
\]

Where \( I \) is the injective envelope of \( A \). We need to show that \( \text{Ext}^k_B(S,B) = 0 \).

As we saw in above, \( I \in \mathcal{S}^1 \) and \( B \in \mathcal{S}^{1+k-1} \) because \( A \in \mathcal{S}^{1+k} \). Thus \( \text{Ext}^k_B(S,B) = 0 \) if and only if \( B \in \mathcal{S}^{1+k} \) and this is by induction hypothesis, equivalent to the condition that natural map

\[
e^i_{X,B} : \text{Ext}^i_B(X,B) \rightarrow \text{Ext}^i_{B/\mathcal{S}}(e(X),e(B))
\]

be invertible, for every \( X \in \mathcal{B} \) and every \( i \in \{0,1,\cdots,k-1\} \). Because \( B \in \mathcal{S}^{1+k-1} \), \( e^i_{X,B} \) is invertible for every \( i \in \{0,1,\cdots,k-2\} \). Thus we only need to show that \( e^{k-1}_{X,B} \) is invertible. Applying the functors \( \text{Hom}_B(X,-) \) and \( \text{Hom}_{B/\mathcal{S}}(e(X),-) \) to short exact sequences \( 0 \rightarrow A \rightarrow I \rightarrow B \rightarrow 0 \) and \( 0 \rightarrow e(A) \rightarrow e(I) \rightarrow (B) \rightarrow 0 \) respectively we obtain the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
\text{Ext}^{k-1}_B(X,A) & \rightarrow & \text{Ext}^{k-1}_B(X,I) & \rightarrow & \text{Ext}^{k-1}_B(X,B) & \rightarrow & \text{Ext}_B^k(X,A) \rightarrow 0 \\
\downarrow e^{k-1}_{X,A} & & \downarrow e^{k-1}_{X,I} & & \downarrow e^{k-1}_{X,B} & & \downarrow e^k_{X,A} \\
\text{Ext}^{k-1}(e(X),e(A)) & \rightarrow & \text{Ext}^{k-1}(e(X),e(I)) & \rightarrow & \text{Ext}^{k-1}(e(X),e(B)) & \rightarrow & \text{Ext}^k(e(X),e(A)) \rightarrow 0 \\
\end{array}
\]

By assumption \( e^{k-1}_{X,A} \) and \( e^k_{X,A} \) are invertible, and because \( I \in \mathcal{S}^{1+k} \), \( e^{k-1}_{X,I} \) is also invertible. Thus the five Lemma yields that \( e^{k-1}_{X,B} \) is an isomorphism.

Let \( \mathcal{A} \) be a essentially small additive category. Recall that \( \text{Mod} \mathcal{A} \) is the category of all additive contravariant functors from \( \mathcal{A} \) to the category of all abelian groups. It is an abelian category with all limits and colimits, which are defined point-wise. A functor \( F \in \text{Mod} \mathcal{A} \) is called finitely presented (or coherent) if there exists an exact sequence

\[
\text{Hom}(-,X) \rightarrow \text{Hom}(-,Y) \rightarrow F \rightarrow 0
\]

in \( \text{Mod} \mathcal{A} \). We denote by \( \text{mod} \mathcal{A} \) the full subcategory of \( \text{Mod} \mathcal{A} \) consists of all finitely presented functors. It is a well known result that \( \text{mod} \mathcal{A} \) is an abelian category and the inclusion \( \text{mod} \mathcal{A} \hookrightarrow \text{Mod} \mathcal{A} \) is an exact functor if and only if \( \mathcal{A} \) have weak kernels \[\text{[21]} \text{Lemma 2.1.6]}. In particular this is the case when \( \mathcal{A} \) is an abelian category.
By the Yoneda lemma, representable functors are projective and the direct sum of all representable functors \( \bigoplus_{X \in \mathcal{M}} \text{Hom}(-, X) \), is a generator for \( \text{Mod} \mathcal{A} \). Thus \( \text{Mod} \mathcal{A} \) is a Grothendieck category \([5, \text{Proposition 5.21}]}\).

**Definition 2.4.** (i) A functor \( F \in \text{Mod} \mathcal{A} \) is called **weakly effaceable**, if for each object \( X \in \mathcal{A} \) and \( x \in F(X) \) there exists an epimorphism \( f : Y \to X \) such that \( F(f)(x) = 0 \) \([21, \text{Page 28}]}\). We denote by \( \text{Eff}(\mathcal{A}) \) the full subcategory of all weakly effaceable functors.

(ii) A functor \( F \in \text{mod} \mathcal{A} \) is called **effaceable**, if there exists an exact sequence
\[
\text{Hom}(-, Y) \to \text{Hom}(-, X) \to F \to 0
\]
such that \( Y \to X \) is an epimorphism. We denote by \( \text{eff}(\mathcal{A}) \) the full subcategory of all effaceable functors.

(iii) A functor \( F \in \text{Mod} \mathcal{A} \) is called a **left exact functor**, if for each short exact sequence
\[
0 \to X \to Y \to Z \to 0
\]
in \( \mathcal{A} \), the sequence of abelian groups
\[
0 \to F(Z) \to F(Y) \to F(X)
\]
is exact. We denote by \( \mathcal{L}(\mathcal{A}) \) the full subcategory of all left exact functors.

Note that Krause in his recent book \([11]\), has used the terminology **locally effaceable** instead of weakly effaceable. And this is reasonable from the viewpoint of locally finitely presented categories (see the next subsection), because we can show that objects in \( \text{Eff}(\mathcal{A}) \) are exactly direct limits of objects in \( \text{eff}(\mathcal{A}) \) \([10, \text{Page 672}]}\).

**Proposition 2.5.** Let \( \mathcal{A} \) be a essentially small abelian category.

(i) \( \text{Eff}(\mathcal{A}) \) is a localizing subcategory of \( \text{Mod} \mathcal{A} \).

(ii) We have
\[
\mathcal{L}(\mathcal{A}) = \text{Eff}(\mathcal{A})^\perp \simeq (\text{Mod} \mathcal{A}) / (\text{Eff}(\mathcal{A})),
\]
and the inclusion \( \mathcal{L}(\mathcal{A}) \hookrightarrow \text{Mod} \mathcal{A} \) is the right adjoint of the localization functor \( e : \text{Mod} \mathcal{A} \to \mathcal{L}(\mathcal{A}) \).

(iii) \( e(G) = 0 \) if and only if \( G \in \text{Eff}(\mathcal{A}) \).

**Proof.** This is a standard result proved by Gabriel \([6, \text{II.2]}\) (see also \([11, \text{Proposition 2.3.7}]}\)). \(\square\)

We denote by \( i : \mathcal{A} \to \mathcal{L}(\mathcal{A}) \) the composition of the Yoneda functor \( \mathcal{A} \to \text{Mod} \mathcal{A} \) and the localization functor \( \text{Mod} \mathcal{A} \to (\text{Mod} \mathcal{A}) / (\text{Eff}(\mathcal{A})) \simeq \text{Eff}(\mathcal{A})^\perp = \mathcal{L}(\mathcal{A}) \). The canonical functor \( i \) is exact \([11, \text{Proposition 2.3.7}]}\). For an object \( A \in \mathcal{A} \) sometimes we denote \( i(A) = \text{Hom}_{\mathcal{A}}(-, A) \) by \( H_A \). The following easy lemma will be used throughout the paper.

**Lemma 2.6.** Let \( A \in \mathcal{A} \) and \( \alpha : F \to H_A \) be an epimorphism in \( \mathcal{L}(\mathcal{A}) \). Then the cokernel of \( \alpha \) computed in \( \text{Mod} \mathcal{A} \) is weakly effaceable. In particular, there exists an epimorphism \( B \to A \) inducing the following commutative diagram with exact rows in \( \mathcal{L}(\mathcal{A}) \).

\[
\begin{array}{ccc}
H_B & \longrightarrow & H_A \\
\downarrow & & \downarrow \\
F & \longrightarrow & H_A \\
\end{array}
\]

\[
0 \longrightarrow 0
\]
Proof. Denote by $G$ the cokernel of $\alpha$ in $\text{Mod} \mathcal{A}$. So we have the exact sequence $F \to H_A \to G \to 0$ in $\text{Mod} \mathcal{A}$. By applying the localization functor $e : \text{Mod} \mathcal{A} \to \mathcal{L}(A)$ to this exact sequence and using Proposition 2.5 we conclude that $G$ is weakly effaceable. Now consider the image of $1_A$, the identity morphism on $A$, under the map $\text{Hom}_A(A, A) \to G(A)$ and denote it by $a$. Since $G$ is weakly effaceable, by definition there exists an epimorphism $f : B \to A$ such that $G(f)(a) = 0$. This means that $H_B \to H_A$ factor through $\alpha$, and so we have the desire diagram (cf. [9, Page 409]).

2.2. Locally coherent categories. In this subsection we recall some of the basic properties of Locally finitely presented categories, for details and more information the reader is referred to [2]. Recall that a non-empty category $I$ is said to be filtered provided that for each pair of objects $\lambda_1, \lambda_2 \in I$ there are morphisms $\varphi_i : \lambda_i \to \mu$ for some $\mu \in I$, and for each pair of morphisms $\varphi_1, \varphi_2 : \lambda \to \mu$ there is a morphism $\psi : \mu \to \nu$ with $\psi\varphi_1 = \psi\varphi_2$.

Let $A$ be an additive category, $I$ a small filtered category and $X : I \to A$ be an additive functor. As usual we use the term direct limit for colimit of $X$ when $I$ is filtered. And we denote it by $\text{Lim}_{\leftarrow \rightarrow}^I X_i$.

Definition 2.7. Let $B$ be an additive category with direct limits.

(i) An object $X \in B$ is called finitely presented (finitely generated) provided that for every direct limit $\text{Lim}_{\leftarrow \rightarrow}^I X_i$ in $B$ the natural morphism

$$\text{Lim}_{\leftarrow \rightarrow}^I \text{Hom}_B(X, Y_i) \to \text{Hom}_B(X, \text{Lim}_{\leftarrow \rightarrow}^I Y_i)$$

is an isomorphism (a monomorphism). The full subcategory of finitely presented objects of $\mathcal{A}$ is denoted by $\text{fp}(\mathcal{A})$.

(ii) $B$ is called a locally finitely presented category if $\text{fp}(B)$ is essentially small and every object in $B$ is a direct limit of objects in $\text{fp}(B)$.

(iii) Let $B$ be a locally finitely presented abelian category. $B$ is said to be locally coherent provided that finitely generated subobjects of finitely presented objects are finitely presented.

We recall a concrete description of locally finitely presented additive categories due to Crawley-Boevey [2]. Let $C$ be a essentially small additive category, recall that there is a tensor product bifunctor

$$\text{Mod} C \otimes_C \text{Mod} C^{\text{op}} \to \text{Ab}.$$ 

$$F, G \mapsto F \otimes_C G$$

A functor $F \in \text{Mod} C$ is said to be flat provided that $F \otimes_C -$ is an exact functor. We denote by $\text{Flat}(C)$ the full subcategory of $\text{Mod} C$ consist of flat functors. A theorem of Lazard state that an $R$-module is flat if and only if it is a direct limit of finitely generated free modules. Lazard theorem has been generalized to functors by Oberst and Rohrl [15]. Indeed a functor $F \in \text{Mod} C$ is flat if and only if $F$ is a direct limit of representable functors. Using this description of flat functors we have:

Lemma 2.8. Let $\mathcal{A}$ be a essentially small abelian category and $F \in \text{Mod} \mathcal{A}$. The following statements are equivalent.

(a) $F$ is a flat functor.
(b) $F$ is a left exact functor.
In other words \( \text{Flat}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) \).

**Proof.** See [11, Lemma 11.1.14]. \( \square \)

**Theorem 2.9.** ([2, Theorem 1.4])

(a) If \( \mathcal{C} \) is an essentially small additive category, then \( \text{Flat}(\mathcal{C}) \) is a locally finitely presented category, and \( \text{fp}(\text{Flat}(\mathcal{C})) \) consists of the direct summands of representable functors. If \( \mathcal{C} \) has split idempotents then the Yoneda functor \( h : \mathcal{C} \to \text{fp}(\text{Flat}(\mathcal{C})) \) is an equivalence.

(b) If \( \mathcal{B} \) is a locally finitely presented category then \( \text{fp}(\mathcal{B}) \) is an essentially small additive category with split idempotents, and the functor

\[
g : \mathcal{B} \to \text{Flat}(\text{fp}(\mathcal{B}))
\]

\[
M \mapsto \text{Hom}( -, M)|_{\text{fp}(\mathcal{B})}
\]

is an equivalence.

**Proposition 2.10.**

(i) Every locally finitely presented abelian category is a Grothendieck category.

(ii) A locally finitely presented abelian category \( \mathcal{B} \) is locally coherent if and only if \( \text{fp}(\mathcal{B}) \) is an abelian category.

**Proof.** We refer the reader to [2] for the statement (i) and to [20] for (ii). \( \square \)

In the proof of the main theorem we need the following proposition.

3. PROOF OF THE MAIN THEOREM

Let \( \mathcal{A} \) be an essentially small abelian category and \( \mathcal{C} \) be a Serre subcategory of \( \mathcal{A} \). By Theorem 2.9, \( \mathcal{L}(\mathcal{A}) \simeq (\text{Mod}\, \mathcal{A})/(\text{Eff}(\mathcal{A})) \) is a locally coherent category and the essential image of the canonical functor

\[
i : \mathcal{A} \to \mathcal{L}(\mathcal{A})
\]

denoted by \( i(\mathcal{A}) \) is the subcategory of finitely presented objects. Thus by [12, Theorem 2.8] \( i(\mathcal{C}) \), the subcategory of \( \mathcal{L}(\mathcal{A}) \) consists of all direct limits of objects in \( i(\mathcal{C}) \) is a localizing subcategory of \( \mathcal{L}(\mathcal{A}) \). Also by [13, Proposition A5] \( (\mathcal{L}(\mathcal{A}))/\overline{(i(\mathcal{C}))} \) is a locally coherent category and its subcategory of finitely presented objects is equivalent to \( \mathcal{A}/\mathcal{C} \). Thus we have the commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{q} & \mathcal{A}/\mathcal{C} \\
i & & \downarrow j \\
\mathcal{L}(\mathcal{A}) & \xrightarrow{e} & (\mathcal{L}(\mathcal{A}))/\overline{(i(\mathcal{C}))} \cong \mathcal{L}(\mathcal{A}/\mathcal{C}) \\
\end{array}
\]

(3.1)
satisfies

(i) \( q \) and \( e \) are quotient functors.

(ii) \((e,r)\) is an adjoint pair and \( r \) is fully faithful.

(iii) Both of vertical functors \( i \) and \( j \) are the canonical functors from an abelian category to the abelian category of left exact functors.

The following lemma due to Mitchell \cite{16} is one of the crucial key steps for proving the main result of the paper. For the convenient of the reader we write a self contained proof.

For more information about Yoneda extension groups we refer the reader to \cite[Chapter VII]{17}. We just recall that for two objects \( A, B \in \mathcal{A} \), a \( k \)-fold extension of \( A \) by \( B \) is an exact sequence of the form

\[
\xi : 0 \to B \to X^k \to X^{k-1} \to \cdots \to X^1 \to A \to 0,
\]

and \( \text{Ext}_{\mathcal{A}}^k(A, B) \) is the set of all Yoneda equivalence classes of \( k \)-fold extensions of \( A \) by \( B \). For a morphism \( f : B \to B' \), by taking push out along \( f \) we obtain a \( k \)-fold extension of \( A \) by \( B' \) that we denote by \( f\xi \). In a similar way and using pull back we can define \( \xi g \) for a morphism \( g : A' \to A \).

Lemma 3.1. Let \( \mathcal{A} \) be a essentially small abelian category. The canonical functor \( i : \mathcal{A} \to \mathcal{L}(\mathcal{A}) \) is an \( \text{Ext} \)-preserving functor. i.e. for every two object \( A, B \in \mathcal{A} \) and every non-negative integer \( i \) the natural map

\[
i_{i,A,B}^i : \text{Ext}_{\mathcal{L}(\mathcal{A})}^i(i(A),i(B)) \to \text{Ext}_{\mathcal{A}}^i(A,B)
\]

is invertible.

Proof. By the Yoneda lemma \( i \) is fully faithful, so for \( i = 0 \) the claim follows. Now let \( i \geq 1 \) be a positive integer. First we prove by induction that \( i_{i,A,B}^i \) is surjective. Consider an element \( \xi \in \text{Ext}_{\mathcal{L}(\mathcal{A})}^i(i(A),i(B)) \) represented as

\[
0 \to i(B) \to F^i \to F^{i-1} \to \cdots \to F^1 \to i(A) \to 0.
\]

Decompose \( \xi \) as splicing of extensions

\[
\xi_{i-1} : 0 \to i(B) \to F^i \to F^{i-1} \to \cdots \to F^1 \to i(A) \to 0
\]

and

\[
\xi_1 : 0 \to G^2 \to F^1 \to i(A) \to 0.
\]

By Lemma 2.6 there is an exact sequence \( 0 \to Y^2 \to X^1 \to A \to 0 \) inducing the following commutative diagram with exact rows (\( \varphi \) is induced by the universal property of kernel).

\[
\begin{array}{cc}
\xi_1 & : & 0 & \longrightarrow & i(Y^2) & \longrightarrow & i(X^1) & \longrightarrow & i(A) & \longrightarrow & 0 \\
& & & \downarrow & \varphi & & \downarrow & & \downarrow & & \\
\xi_1 & : & 0 & \longrightarrow & G^2 & \longrightarrow & F^1 & \longrightarrow & i(A) & \longrightarrow & 0.
\end{array}
\]

So \( \xi_1 = \varphi \xi_1^i \). If \( i = 1 \) the above diagram is of the form
0 \longrightarrow i(Y^2) \to i(X^1) \to i(A) \longrightarrow 0 \]
\[\varphi \downarrow \quad \downarrow \quad \downarrow \]
\[0 \longrightarrow i(B) \longrightarrow F^1 \longrightarrow i(A) \longrightarrow 0.
\]
Since \( i \) is fully faithful, \( \varphi = i(f) \) for some \( f : Y^2 \to B \). Taking push out along \( f \) and then applying the exact functor \( i \) we obtain the following commutative diagram with exact rows
\[
0 \longrightarrow i(Y^2) \to i(X^1) \to i(A) \longrightarrow 0 \\
\varphi \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow i(B) \longrightarrow i(C) \to i(A) \longrightarrow 0.
\]
In both of above diagrams the left-hand squares are push out diagrams. Thus by the uniqueness of push out we see that short exact sequences \( 0 \to i(B) \to F^1 \to i(A) \to 0 \) and \( 0 \to i(B) \to i(C) \to i(A) \to 0 \) are Yoneda equivalent.

For \( i \geq 2 \) we have \( \xi = \xi_{i-1} \circ \xi_1 = \xi_{i-1} \circ (\varphi \xi_1') = (\xi_{i-1} \varphi) \circ \xi_1' \). Now by induction hypothesis \( \xi_{i-1} \varphi \in \text{Ext}_{i-1}^1(i(Y^{i-1}), i(B)) \) belongs to the image of \( i_{Y^{i-1}, B}^{-1} \) and \( \xi_1' \in \text{Ext}_{i}^1(i(A), i(Y^2)) \) belongs to the image of \( i_{A,Y^2}^1 \). And this complete the proof of the induction step. In other word we proved that there exists a commutative diagram with exact rows of the following form.
\[
0 \longrightarrow i(B) \longrightarrow i(X^1) \longrightarrow \cdots \longrightarrow i(X^1) \longrightarrow i(A) \longrightarrow 0 \\
\varphi \downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow \\
0 \longrightarrow i(B) \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^1 \longrightarrow i(A) \longrightarrow 0.
\]
(3.2)

Now we prove that \( i_{A,B}^i \) is injective. Let \( \eta \in \text{Ext}_A^1(X, Y) \) such that \( i(\eta) = 0 \). By [17, Theorem 4.2] we can find a commutative diagram with exact rows
\[
\xi : \quad 0 \longrightarrow i(B) \longrightarrow F^i \longrightarrow \cdots \longrightarrow F^1 \longrightarrow i(A) \longrightarrow 0 \\
\eta : \quad 0 \longrightarrow i(B) \longrightarrow i(Y^i) \longrightarrow \cdots \longrightarrow i(Y^1) \longrightarrow i(A) \longrightarrow 0
\]

where \( i(B) \to F^i \) is a split monomorphism. Forming the Diagram (3.2) for \( \xi \) we see that \( i(B) \to i(X^i) \) is also a split monomorphism, and so is \( B \to X^i \) because \( i \) is fully faithful. Again because \( i \) is fully faithful we conclude that \( \eta = 0 \).

**Lemma 3.2.** Let \( \mathcal{B} \) be a locally coherent category and \( \mathcal{C} \) be a Serre subcategory of \( \mathcal{A} = \text{fp}(\mathcal{B}) \). Then \( \overset{\rightarrow}{\mathcal{C}} \), the full subcategory of \( \mathcal{B} \) consists of all direct limits of object in \( \mathcal{A} \), is a localizing subcategory of \( \mathcal{B} \) and as subcategories of \( \mathcal{B} \) we have \( \mathcal{C}^\perp = (\overset{\rightarrow}{\mathcal{C}})^\perp \).

**Proof.** See [12, Theorem 2.8 and Corollary 2.11].

\[\square\]
The following proposition is a generalization of the above result for arbitrary positive integer \( k \) (Krause in [12] proved this result for \( k=1 \)).

**Proposition 3.3.** Let \( \mathcal{B} \) be a locally coherent category, \( \mathcal{C} \) be a Serre subcategory of \( \mathcal{A} = \text{fp}(\mathcal{B}) \) and \( k \) be a positive integer. Then as subcategories of \( \mathcal{B} \) we have \( C^{\perp k} = (\overline{C})^{\perp k} \).

**Proof.** By Lemma 3.2 we have \( C^{\perp 1} = (\overline{C})^{\perp 1} \). Because \( C \subseteq \overline{C} \) it is clear that \( (\overline{C})^{\perp k} \subseteq C^{\perp k} \).

For the converse inclusion let \( X \in C^{\perp k} \). We most show that \( X \in (\overline{C})^{\perp k} \). Because by Lemma 3.2 \( \overline{C} \) is a localizing subcategory of \( \mathcal{B} \), by Proposition 2.3 it is enough to show that there is an injective coresolution

\[
0 \to X \to I^0 \to I^1 \to \cdots \to I^k
\]

such that for every \( i \in \{0, 1, \cdots, k\} \) we have \( I^i \in (\overline{C})^{\perp 1} = C^{\perp 1} \). Let \( X \hookrightarrow I^0 \) be the injective envelope of \( X \) in \( \mathcal{B} \). Let \( C \in \mathcal{C} \) and \( f \in \text{Hom}_\mathcal{B}(C, I^0) \) be a non-zero morphism, because \( I^0 \) is an essential extension of \( X \) we have that \( \text{Im}(f) \cap X \neq 0 \). By the definition of Serre subcategory \( \text{Im}(f) \cap X \subseteq C \), and this contradicts the assumption \( \text{Hom}_\mathcal{B}(C, X) = 0 \). Thus \( I^0 \subseteq C^\perp \) because \( I^0 \) is injective.

Now by applying the functor \( \text{Hom}_\mathcal{B}(C, -) \) for an arbitrary object \( C \in \mathcal{C} \), to the short exact sequence

\[
0 \to X \to I^0 \to \Omega^{-1}X \to 0
\]

we obtain the long exact sequence

\[
0 \to \text{Hom}_\mathcal{B}(C, X) \to \text{Hom}_\mathcal{B}(C, I^0) \to \text{Hom}_\mathcal{B}(C, \Omega^{-1}X)
\]

\[
\to \text{Ext}^1_\mathcal{B}(C, X) \to \text{Ext}^1_\mathcal{B}(C, I^0) \to \text{Ext}^1_\mathcal{B}(C, \Omega^{-1}X)
\]

\[
\to \text{Ext}^2_\mathcal{B}(S, X) \cdots .
\]

By assumption \( \text{Hom}_\mathcal{B}(C, X) = \text{Ext}^1_\mathcal{B}(C, X) = \cdots = \text{Ext}^k_\mathcal{B}(C, X) = 0 \), and \( \text{Ext}^1_\mathcal{B}(C, I^0) = 0 \), because \( I^0 \) is injective. Using these vanishing conditions and the above long exact sequence we see that \( \Omega^{-1}X \subseteq C^{\perp k-1} \).

Now consider the injective envelope \( \Omega^{-1}X \hookrightarrow I^1 \) for \( \Omega^{-1}X \) and the induced short exact sequence

\[
0 \to \Omega^{-1}X \to I^1 \to \Omega^{-2}X \to 0.
\]

Applying the above argument for this short exact sequence we see that \( I^1 \subseteq C^\perp \) and \( \Omega^{-2}X \subseteq C^{\perp k-2} \). By repeating this argument and using the dimension shifting argument we obtain the desire injective coresolution inductively. \( \square \)

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** (i) \( \Rightarrow \) (ii) Let \( A \in C^{\perp k+1} \) and \( X \) be an arbitrary object in \( \mathcal{A} \). Having the Diagram (3.1) in mind by Proposition 3.3, Proposition 2.3 and Lemma 3.1 for every \( i \in \{0, 1, \cdots, k\} \) we have

\[
\text{Ext}^i_\mathcal{A}(X, A) \cong \text{Ext}^i_{\mathcal{E}(A)}(i(X), i(A))
\]

\[
\cong \text{Ext}^i_{\mathcal{E}(A)}(e_i(X), e_i(A)).
\]
Because $i_{X,A}^j \circ q_{X,A}^i = e_{X,A}^i \circ i_{X,A}^j$ we have that $i_{X,A}^j \circ q_{X,A}^i$ is invertible, and since by Lemma 3.1 $i_{X,A}^j$ is invertible, $q_{X,A}^i$ is also invertible.

(ii) \( \Rightarrow \) (i) Assume that $q_{X,A}^i$ is invertible for every $X \in A$ and every $i \in \{0, 1, \cdots, k\}$.

Because $i_{X,A}^j \circ q_{X,A}^i = e_{X,A}^i \circ i_{X,A}^j$ we have that $e_{X,A}^i \circ i_{X,A}^j$ is invertible, and since by Lemma 3.1 $i_{X,A}^j$ is invertible, $e_{X,A}^i$ is also invertible for every $i \in \{0, 1, \cdots, k\}$. Thus by Proposition 2.3 and Proposition 3.3 $A \in (\mathcal{C})^{-k+1} = \mathcal{C}^{-k+1}$.

We state the dual of Theorem 1.1.

**Theorem 3.4.** Let $A$ be a essentially small abelian category, $C$ a Serre subcategory of $A$ and $q: A \to A/C$ be the quotient functor. For an object $A \in A$ and a non-negative integer $k$ the following statements are equivalent.

(i) $A \in \mathcal{F}_{k+1}^C$.

(ii) The natural map $q_{A,X}^i : \text{Ext}_{A}(A, X) \to \text{Ext}_{A/C}(q(A), q(X))$ is invertible, for every $X \in A$ and every $i \in \{0, 1, \cdots, k\}$.

**Proof.** Apply Theorem 1.1 to the opposite category $A^{op}$. \(\square\)

4. Applications

Let $n$ be a positive integer and $M$ be a essentially small $n$-abelian category in the sense of [8]. Denote by $\text{eff}(M)$ the full subcategory of $\text{mod } M$ consist of all effaceable functors, i.e. functors $F \in \text{mod } M$ with a projective presentation

\[(4.1) \quad \text{Hom}_M(-, Y^n) \xrightarrow{\sim, f} \text{Hom}_M(-, Y^{n+1}) \to F \to 0\]

for some epimorphism $f : Y^n \to Y^{n+1}$. $\text{eff}(M)$ is a Serre subcategory of $\text{mod } M$ (see [3, Proposition 3.5] or [14, Proposition 4.3]). The composition of functors

\[H : M \to \text{mod } M \to (\text{mod } M)/\text{eff}(M)\]

\[X \longmapsto \text{Hom}_M(-, X) \longmapsto \text{Hom}_M(-, X)\]

is fully faithful [14, Proposition 4.6], and it’s essential image is an $n$-cluster tilting subcategory of $(\text{mod } M)/(\text{eff}(M))$, see [3, Theorem 4.7] or [14, Theorem 7.3 and Proposition 7.5]. Because $(\text{mod } M)/(\text{eff}(M))$ has no injective or projective object in general, proving that the essential image is $n$-rigid (i.e. for every $i \in \{1, \cdots, n - 1\}$ and every $X, Y \in M$ we have $\text{Ext}_{(\text{mod } M)/(\text{eff}(M))}^{i}(H_X, H_Y) = 0$) is not easy, see [14, Section 6]. In the sequel we give a different proof using Theorem 1.1.

**Proposition 4.1.** For every $X \in M$, $H_X \in \text{eff}(M)^{ \bot n}$.

**Proof.** Let $F \in \text{eff}(M)$ and consider the projective presentation (11). Because $f$ is an epimorphism, by the axioms of $n$-abelian categories [8, Definition 3.1], $f$ sits into an $n$-exact sequence

\[Y^0 \to Y^1 \to \cdots \to Y^n \xrightarrow{f} Y^{n+1}.\]

By the definition of $n$-exact sequences we have the exact sequence

\[0 \to H_{Y^0} \to H_{Y^1} \to \cdots \to H_{Y^n} \to H_{Y^{n+1}} \to F \to 0\]
which is a projective resolution for $F$. Now let $X \in \mathcal{M}$. Applying $\operatorname{Hom}_{\mathcal{M}}(-, H_X)$ to the projective resolution of $F$, we get the complex

$$0 \to \operatorname{Hom}(H_{Y^n+1}, H_X) \to \operatorname{Hom}(H_{Y^n}, H_X) \to \cdots$$

$$\to \operatorname{Hom}(H_{Y1}, H_X) \to \operatorname{Hom}(H_{Y0}, H_X) \to 0$$

which is by Yoneda lemma isomorphic to the complex

$$0 \to \operatorname{Hom}(Y^{n+1}, X) \to \operatorname{Hom}(Y^n, X) \to \cdots \to \operatorname{Hom}(Y^1, X) \to \operatorname{Hom}(Y^0, X).$$

And by the definition of $n$-exact sequences this is an exact sequence of abelian groups. Thus we see that for every $i \in \{0, 1, \cdots, n\}$ we have $\operatorname{Ext}^i_{(\mathcal{M})/(\operatorname{eff}(\mathcal{M}))}(F, H_X) = 0$.

Because $F$ was an arbitrary effaceable functor, $H_X \in \operatorname{eff}(\mathcal{M})^+$. □

**Corollary 4.2.** The essential image of $H : \mathcal{M} \longrightarrow (\mathcal{M})/(\operatorname{eff}(\mathcal{M}))$ is an $n$-rigid subcategory.

**Proof.** Let $X, Y \in \mathcal{M}$. By Proposition 4.1 and Theorem 1.1 for every $i \in \{1, \cdots, n-1\}$ we have $\operatorname{Ext}^i_{(\mathcal{M})/(\operatorname{eff}(\mathcal{M}))}(H_X, H_Y) \cong \operatorname{Ext}^i_{\mathcal{M}}(H_X, H_Y) = 0$ because $H_X = \operatorname{Hom}_{\mathcal{M}}(-, X)$ is a projective object in $\mathcal{M}$.

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