Deformations of infinite-dimensional Lie algebras, exotic cohomology and integrable nonlinear partial differential equations. II

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Abstract. We consider the four-dimensional reduced quasi-classical self-dual Yang–Mills equation and show that non-triviality of the second exotic cohomology group of its symmetry algebra implies existence of a two-component integrable generalization of this equation. The sequence of natural extensions of this symmetry algebra generate an integrable hierarchy of multi-dimensional nonlinear PDEs. We write out the first three elements of this hierarchy.

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1. Introduction

The purpose of this paper is to provide new examples of application of the method of \cite{15, 16} to the problem of finding a Lax representation for a PDE. Lax representations, also known as zero-curvature representations, Wahlquist–Estabrook prolongation structures, inverse scattering transformations, or differential coverings, are of great importance in the theory of integrable systems, \cite{13, 21, 12}. They are a key feature of integrable PDEs and a starting setting for a number of techniques of studying them such as Bäcklund transformations, Darboux transformations, recursion operators, nonlocal symmetries, and nonlocal conservation laws. Although these structures are of crucial significance in the theory of integrable systems, up to now the problem of finding conditions for a PDE to admit a Lax representation is unsolved. In \cite{15, 16} we propose a method for solving this problem in internal terms of the PDE under the study. The method uses the second exotic cohomology group of the symmetry algebra of the PDE and allows one to get rid of apriori assumptions about the form of the covering equations of a Lax representation.

In the overwhelming majority of known examples, the symmetry algebras of integrable multi-dimensional PDEs are infinite-dimensional. Computation of exotic cohomology groups for infinite-dimensional Lie algebras is a complicated problem, while in some cases its solution is possible owing to the specific graded structures of the algebras, \cite{16}. In this paper we show that existence of a Lax representation can be derived from non-triviality of the second exotic cohomology group of a finite-dimensional subalgebra of the symmetry algebra of the PDE.

Another important problem in the theory of integrable equations is concerned with the integrable hierarchies associated with the PDE under the study. In \cite{16} we demonstrate that in some cases the graded structure of the symmetry algebra allows one to find such hierarchies.

In this paper we consider equation

\[ u_{yz} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \quad (1) \]

which arise as a reduction of the quasi-classical self-dual Yang–Mills equation, \cite{10, 11, 1, 3},

\[ u_{yz} = u_{tx} + u_x u_{zs} - u_z u_{xs}, \quad (2) \]

Equation (1) has a number of remarkable properties. In \cite{5} the Lax representation

\[
\begin{cases}
q_t = \lambda q_y - u_y q_x, \\
q_z = (\lambda - u_x) q_x
\end{cases}
\]

with the nonremovable parameter $\lambda$ for equation (1) was presented. In \cite{8} a Bäcklund transformation between (1) and another reduction

\[ v_y = v_z v_{xy} - v_y v_{xx} \]

of equation (2) was found. The symmetry algebra of equation (1), as an abstract Lie algebra, is the semi-direct sum $\mathfrak{s}_2 = \mathfrak{s}_o \ltimes \mathfrak{s}_{2,\infty}$ of the finite-dimensional subalgebra $\mathfrak{s}_o$.
and the infinite-dimensional ideal \( s_{2,\infty} = \mathbb{R}_2[h] \otimes \mathfrak{q}, \) which is the tensor product of the (associative commutative unital) algebra \( \mathbb{R}_2[h] = \mathbb{R}[h]/\langle h^3 = 0 \rangle \) of truncated polynomials of degree less than 3 and the Lie algebra \( \mathfrak{q} \) of the vector fields of the form \( A(t, z) \partial_z \) on \( \mathbb{R}^2. \) The second exotic cohomology group of the finite-dimensional Lie algebra \( s_o \) is one-dimensional. The generating cocycle of this cohomology group defines a one-dimensional extension \( \tilde{s}_o \) of \( s_o. \) This extension provides a Lax representation \( (18) \) for a two-component generalization \( (19) \) of equation \( (11), \) while \( (3) \) and \( (1) \) are their evident reductions.

The symmetry algebra \( s_2 \) admits a series of natural extensions \( s_m = s_o \ltimes s_{m,\infty} = s_o \ltimes (\mathbb{R}_m[h] \otimes \mathfrak{q}), \) \( m \geq 3, \) which preserve the finite-dimensional part \( s_o \) as well as its extension \( \tilde{s}_o. \) The Maurer–Cartan of the series of Lie algebras \( \tilde{s}_o \ltimes s_{m,\infty} \) provide a series of systems of PDEs with their Lax representations. We write out the first three elements of this hierarchy of integrable equations.

2. Preliminaries

All considerations in this paper are local. All functions are assumed to be real-analytic.

2.1. Symmetries and differential coverings of PDEs

The relevant geometric formulation of Lax representations is based on the concept of differential covering of a PDE [6, 7]. In this subsection we closely follow [7, 2] to present the basic notions of the theory of differential coverings.

Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \) \( \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \) be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \( (x^i, u^\alpha, u^\alpha_I), \) where \( I = (i_1, \ldots, i_n) \) are multi-indices, and for every local section \( f: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \) of \( \pi \) the corresponding infinite jet \( j_\infty(f) \) is a section \( j_\infty(f): \mathbb{R}^n \to J^\infty(\pi) \) such that \( u^\alpha_I(f_\infty) = \frac{\partial^{|I|} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\ldots+in} f^\alpha}{(\partial x_1)^{i_1}\ldots(\partial x_n)^{i_n}}. \) We put \( u^\alpha = u^\alpha_{(0,\ldots, 0)}. \) Also, in the case of \( m = 1 \) and, e.g., \( n = 4 \) we denote \( x^1 = t, \) \( x^2 = x, \) \( x^3 = y, \) \( x^4 = z, \) and \( u^1_{(i,j,k,l)} = u_{i \ldots t \ldots y \ldots z} \) with \( i \) times \( t, \) \( j \) times \( x, \) \( k \) times \( y, \) and \( l \) times \( z. \)

The vector fields

\[
D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{|I| \geq 0} \sum_{\sigma = 1}^m u^\alpha_{I+1_k} \frac{\partial}{\partial u^\alpha_I}, \quad k \in \{1, \ldots, n\},
\]

with \( I + 1_k = (i_1, \ldots, i_k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n) \) are referred to as total derivatives. They commute everywhere on \( J^\infty(\pi): [D_{x^i}, D_{x^j}] = 0. \)

A system of PDEs \( F_r(x^i, u^\alpha_I) = 0, \#I \leq s, r \in \{1, \ldots, \sigma\}, \) of the order \( s \geq 1 \) with \( \sigma \geq 1 \) defines the submanifold \( \mathcal{E} = \{(x^i, u^\alpha_I) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^\alpha_I)) = 0, \#K \geq 0\} \) in \( J^\infty(\pi). \)
The evolutionary derivation associated to an arbitrary smooth function \( \varphi: J^\infty(\pi) \to \mathbb{R}^m \) is the vector field

\[
E_\varphi = \sum_{\#I \geq 0} \sum_{\alpha=1}^{m} D_I(\varphi^\alpha) \frac{\partial}{\partial u_\alpha^I}
\]

with \( D_I = D_{i_1 \ldots i_n} = D_{x_1}^{i_1} \circ \ldots \circ D_{x_n}^{i_n} \).

A function \( \varphi: E \to \mathbb{R}^m \) is called a (generator of an infinitesimal) symmetry of \( E \) when \( E_\varphi(F_\pi) = 0 \) on \( E \). The symmetry \( \varphi \) is a solution to the defining system

\[
\ell_E(\varphi) = 0,
\]

where \( \ell_E = \ell_F|_E \) with the matrix differential operator

\[
\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_a}{\partial u_\alpha^I} D_I \right).
\]

Solutions to (5) constitute the Lie algebra \( \text{Sym}(E) \) with respect to the Jacobi bracket \( [\varphi, \psi] = E_\varphi(\psi) - E_\psi(\varphi) \). The subalgebra of contact symmetries of \( E \) is \( \text{Sym}_1(E) = \text{Sym}(E) \cap C^\infty(J^1(\pi), \mathbb{R}^m) \). In its turn this subalgebra contains the subalgebra \( \text{Sym}_0(E) \) of point symmetries, whose generators have the form \( \varphi = (\varphi^1, \ldots, \varphi^m) \) with

\[
\varphi^\alpha = \eta^\alpha - \sum_{j=1}^{n} \xi^j u_\alpha^j,
\]

where \( \eta^\alpha, \xi^i \) are functions of \( J^0(\pi) \). Such generators are in one-to-one correspondence with the vector fields

\[
\dot{\varphi} = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{m} \eta^\alpha \frac{\partial}{\partial u^\alpha}
\]

on \( J^0(\pi) \).

Denote \( \mathcal{W} = \mathbb{R}^\infty \) with coordinates \( w^a, a \in \mathbb{N} \cup \{0\} \). Locally, an (infinite-dimensional) differential covering over \( E \) is a trivial bundle \( \tau: J^\infty(\pi) \times \mathcal{W} \to J^\infty(\pi) \) equipped with the extended total derivatives

\[
\dot{D}_{x^k} = D_{x^k} + \sum_{a=0}^{\infty} T_k^a(x^i, u_\alpha^I, w^b) \frac{\partial}{\partial w^a}
\]

such that \( [\dot{D}_{x^i}, \dot{D}_{x^j}] = 0 \) for all \( i \neq j \) whenever \( (x^i, u_\alpha^I) \in E \). For the partial derivatives of \( u^a \) which are defined as \( w_\alpha^a = \dot{D}_{x^k}(u^a) \) we have the system of covering equations

\[
w_\alpha^a = T_k^a(x^i, u_\alpha^I, w^b).
\]

This over-determined system of PDEs is compatible whenever \( (x^i, u_\alpha^I) \in E \).

Dually the covering with extended total derivatives (7) is defined by the differential ideal generated by the Wahlquist–Estabrook forms, \([12] \text{ p. 81}]\),

\[
\varpi^a = dw^a - \sum_{k=1}^{n} T_k^a(x^i, u_\alpha^I, w^b) \, dx^k.
\]
This ideal is integrable on $E$, that is,
\[
d\varpi^{a} \equiv \sum_{b} \eta^{a}_{b} \wedge \varpi^{b} \mod (\partial^{b}_{I}),
\]
where $\eta^{a}_{b}$ are some 1-forms on $E \times W$ and $\partial^{b}_{I} = (du^{b}_{I} - \sum_{k} u^{b}_{I+k} dx^{k})|_{E}$.

2.2. Exotic cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\rho: \mathfrak{g} \to \text{End}(V)$ be its representation. Let $C^{k}(\mathfrak{g}, V) = \text{Hom}(\Lambda^{k}(\mathfrak{g}), V)$, $k \geq 1$, be the space of all $k$-linear skew-symmetric mappings from $\mathfrak{g}$ to $V$. Then the Chevalley–Eilenberg differential complex

\[
\begin{align*}
V = C^{0}(\mathfrak{g}, V) \xrightarrow{d} C^{1}(\mathfrak{g}, V) \xrightarrow{d} \ldots \xrightarrow{d} C^{k}(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \ldots
\end{align*}
\]

is generated by the differential defined as

\[
d\theta(X_{1}, \ldots, X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_{q}) (\theta(X_{1}, \ldots, \hat{X}_{q}, \ldots, X_{k+1})) + \sum_{1 \leq p < q \leq k+1} (-1)^{p+q} \theta([X_{p}, X_{q}], X_{1}, \ldots, \hat{X}_{p}, \ldots, \hat{X}_{q}, \ldots, X_{k+1}). \tag{8}
\]

The cohomology groups of the complex $(C^{*}(\mathfrak{g}, V), d)$ are referred to as the cohomology groups of the Lie algebra $\mathfrak{g}$ with coefficients in the representation $\rho$. For the trivial representation $\rho_{0}: \mathfrak{g} \to \mathbb{R}$, $\rho_{0}: X \mapsto 0$, the complex and its cohomology are denoted by $C^{*}(\mathfrak{g})$ and $H^{*}(\mathfrak{g})$, respectively.

Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ with non-trivial first cohomology group $H^{1}(\mathfrak{g})$ and take a closed 1-form $\alpha$ on $\mathfrak{g}$ non-cohomologous to 0. Then for any $\lambda \in \mathbb{R}$ define new differential $d_{\lambda}\alpha: C^{k}(\mathfrak{g}, \mathbb{R}) \to C^{k+1}(\mathfrak{g}, \mathbb{R})$ by the formula

\[
d_{\lambda}\alpha \theta = d\theta + \lambda \alpha \wedge \theta.
\]

From $d\alpha = 0$ it follows that

\[
d_{\lambda}^{2}\alpha = 0. \tag{9}
\]

The cohomology groups of the complex

\[
C^{1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda}\alpha} \ldots \xrightarrow{d_{\lambda}\alpha} C^{k}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda}\alpha} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{\lambda}\alpha} \ldots
\]

are referred to as the exotic cohomology groups of $\mathfrak{g}$ and denoted by $H^{*}_{\lambda}(\mathfrak{g})$. [17].

**Remark 1.** Cohomology $H^{*}_{\lambda}(\mathfrak{g})$ coincides with cohomology of $\mathfrak{g}$ with coefficients in the one-dimensional representation $\rho_{\lambda}: \mathfrak{g} \to \mathbb{R}$, $\rho_{\lambda}: X \mapsto \lambda \alpha(X)$. In particular, when $\lambda = 0$, cohomology $H^{*}_{0}(\mathfrak{g})$ coincides with $H^{*}(\mathfrak{g})$. \diamond

**Remark 2.** In all the cases considered in this paper $H^{1}(\mathfrak{g}) = Z^{1}(\mathfrak{g})$ due to $C^{0}(\mathfrak{g}) = \{0\}$ and $B^{1}(\mathfrak{g}) = \{0\}$, so a closed 1-form $\alpha$ can be identified with its cohomology class. \diamond
3. Reduced quasi-classical self-dual Yang–Mills equation and its integrable generalization

3.1. Symmetry algebra of rqsdYM

The symmetry algebra \( s_2 \) of equation (1) admits generators

\[
\begin{align*}
W_0(A) &= -(A_z x + A_t y) u_x - A u_z + A u_x + A^2 x^2 + A t x y, \\
W_1(A) &= -A u_x + A x + A t y, \\
W_2(A) &= A,
\end{align*}
\]

\[
X = -x u_x - y u_y + 2 u,
\]

\[
Y_0 = -u_t, \\
Y_1 = -t u_t + \frac{1}{2} (x u_x - y u_y) - u, \\
Y_2 = -\frac{1}{2} (t^2 u_t + t x u_x - t y u_y - x y) - t u, \\
Z_0 = -u_y, \\
Z_1 = -t u_y - x,
\]

where \( A = A(t, z) \) are arbitrary functions. These generators correspond to point symmetries \( \hat{W}_i(A), \hat{X}, \hat{Y}_j, \hat{Z}_k \) defined in accordance with (6). The commutator table of \( s_2 \) is given by equations

\[
\begin{align*}
[W_i(A), W_j(B)] &= \begin{cases} 
W_{i+j}(A B_z - B A_z), & i + j \leq 2 \\
0, & i + j > 2,
\end{cases} \\
[X, W_k(A)] &= -k W_k(A), \\
[Y_0, W_k(A)] &= W_k(A_t), \\
[Y_1, W_k(A)] &= W_k(t A_t + \frac{1}{2} k A), \\
[Y_2, W_k(A)] &= \frac{1}{2} W_k(t^2 A_t + k t A), \\
[Z_0, W_k(A)] &= \begin{cases} 
W_{k+1}(A_t), & k \leq 1 \\
0, & k = 2
\end{cases} \\
[Z_1, W_k(A)] &= \begin{cases} 
W_{k+1}(t A_t + k A), & k \leq 1 \\
0, & k = 2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
[X, Y_m] &= 0, & [X, Z_m] &= -Z_m, & [Z_0, Z_1] &= 0, \\
[Y_0, Y_1] &= Y_0, & [Y_0, Y_2] &= Y_1, & [Y_1, Y_2] &= Y_2, \\
[Y_0, Z_0] &= 0, & [Y_1, Z_0] &= -\frac{1}{2} Z_0, & [Y_2, Z_0] &= -\frac{1}{2} Z_1, \\
[Y_0, Z_1] &= Z_0, & [Y_1, Z_1] &= \frac{1}{2} Z_1, & [Y_2, Z_1] &= 0.
\end{align*}
\]

From this table it follows that the symmetry algebra of equation (1) is the semi-direct sum \( s_2 = s_\infty \ltimes s_2,\infty \) of the finite-dimensional Lie algebra \( s_\infty \) generated by \( X, Y_i, Z_j \), and the infinite-dimensional ideal \( s_2,\infty \) generated by \( W_0(A), W_1(A), W_2(A) \). We have
$s_o = a \ltimes (\mathfrak{sl}_2(\mathbb{R}) \ltimes b)$, where $a = \langle X \rangle$, $\mathfrak{sl}_2(\mathbb{R}) = \langle Y_0, Y_1, Y_2 \rangle$, and $b = \langle Z_0, Z_1 \rangle$ is a two-dimensional Abelian Lie algebra, while $s_{2,\infty}$ is isomorphic to the tensor product $\mathfrak{q} \otimes \mathbb{R}_2[h]$ of the Lie algebra of $\mathfrak{q}$ of the vector fields of the form $A(t, z) \partial_z$ on $\mathbb{R}^2$ and the associative commutative unital algebra of truncated polynomials $\mathbb{R}_2[h] = \mathbb{R}[h]/(h^3)$ of order less than 3 in the (formal) variable $h$.

3.2. Maurer–Cartan forms and non-triviality of the second exotic cohomology group of $s_2$

Consider the Maurer–Cartan forms $\alpha, \beta_i, i \in \{0, 1, 2\}$, $\gamma_l, l \in \{0, 1\}$, $\theta_{k,m,n}, k \in \{0, 1, 2\}$, $m, n \in \mathbb{N} \cup \{0\}$, of the Lie algebra $s_2$, that are dual to the basis $X, Y_0, Z_l, W_k(t^m z^n)$ of $s_2$; in other words, take 1-forms such that there hold $\alpha(X) = 1$, $\beta_i(Y_i) = \delta_{i,\nu}$, $\gamma_l(\bar{Z}_l) = \delta_{l,\mu}$, $\theta_{k,m,n}(W_k(t^m z^n)) = \delta_{k,k'} \delta_{m,m'} \delta_{n,n'}$, while all the other values of these 1-forms on the elements of the basis are equal to zero. Denote

$$B = \beta_0 + h_1 \beta_1 + \frac{1}{2} h_1^2 \beta_2, \quad \Gamma = \gamma_0 + h_1 \gamma_1,$$

and consider the formal series of 1-forms

$$\Theta_k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{h_1^m h_2^n}{m! n!} \theta_{k,m,n},$$

where $h_1$ and $h_2$ are formal parameters such that $dh_1 = dh_2 = 0$. Then (8) and the commutator relations of $s_2$ yield Cartan’s structure equations

\begin{align*}
d\alpha &= 0, \quad \tag{10} \\
dB &= \nabla_1(B) \wedge B, \quad \tag{11} \\
d\Gamma &= \alpha \wedge \Gamma + \nabla_1(\Gamma) \wedge B + \frac{1}{2} \nabla_1(B) \wedge \Gamma, \quad \tag{12} \\
d\Theta_k &= k \left( \alpha - \frac{1}{2} \nabla_1(B) \right) \wedge \Theta_k + \sum_{m=0}^{k} \nabla_2(\Theta_{k-m}) \wedge \Theta_m + \nabla_1(\Theta_k) \wedge B + \nabla_1(\Theta_{k-1}) \wedge \Gamma \nonumber \\
& \quad \quad - (k-1) \nabla_1(\Gamma) \wedge \Theta_{k-1}, \quad \tag{13}
\end{align*}

where $k \in \{0, 1, 2\}$, $\Theta_{-1} = 0$, and $\nabla_l = \partial \partial_{h_l}$.

From the structure equations (10), (11), (12) of the Lie algebra $s_o$ we have the following theorem, which can be proved by direct computations:

**Theorem 1.** $H^1(s_2) = \mathbb{R}[\alpha]$,

$$H^2_{\lambda}(s_o) = \begin{cases} \mathbb{R}[\gamma_0 \wedge \gamma_1], & \lambda = -2, \\ \{0\}, & \lambda \neq -2. \end{cases}$$

**Corollary.** Equation

$$d\sigma = 2 \alpha \wedge \sigma + \gamma_0 \wedge \gamma_1 \quad \tag{14}$$

with unknown 1-form $\sigma$ is compatible with the structure equations (10), (11), (12), (13) of the Lie algebra $s_2$. 
We can find all the Maurer–Cartan forms $\alpha, \beta_i, \gamma, \theta_{k,m,n}$, and the additional form $\sigma$ by integration of the structure equations (10), (11), (12), (13), (14). For the purposes of this paper we need the forms $\alpha, \beta_i, \gamma, \theta_{k,0,0}, \sigma$ only. We have consequently

\[
\begin{align*}
\alpha &= \frac{da_0}{a_0}, \\
\beta_0 &= a_1^2 dt, \\
\beta_1 &= 2 \frac{da_1}{a_1} + a_2 dt, \\
\beta_2 &= \frac{1}{a_1^2} \left( da_2 + \frac{a_2^2}{2} dt \right), \\
\gamma_0 &= a_0 a_1 \left( dy + a_3 dt \right), \\
\gamma_1 &= a_0 a_1 \left( da_3 + \frac{1}{2} a_2 (dy + a_3 dt) \right), \\
\sigma &= a_0^2 \left( dv - a_3 dy - \frac{1}{2} a_5^2 dt \right), \\
\theta_{0,0,0} &= b_0 dz + b_1 dt, \\
\theta_{1,0,0} &= \frac{a_0 b_0}{a_1} \left( dx + \frac{b_1}{b_0} dy + b_2 dz + b_3 dt \right), \\
\theta_{2,0,0} &= \frac{a_0^2 b_0}{a_1^2} \left( du + (b_2 - a_3) dx + \left( b_3 - \frac{a_3 b_1}{a_0} \right) dy + b_4 dz + b_5 dt \right),
\end{align*}
\]

where $a_0, \ldots, a_3, b_0, \ldots, b_5, t, x, y, z, u, v$ are parameters (“constants of integration”) such that $a_0 \neq 0$, $a_1 \neq 0$, and $b_0 \neq 0$. Then we rename parameters in the form $\theta_{2,0,0}$ to make it to be a contact form

\[
\theta_{2,0,0} = \frac{a_0^2 b_0}{a_1^2} (du - u_t dt - u_x dx - u_y dy - u_z dz),
\]

of the order 0 on the bundle $J^1(\pi), \pi: \mathbb{R}^4 \to \mathbb{R}^3$, $\pi: (t, x, y, z, u) \mapsto (t, x, y, z)$, that is, we put $b_2 = a_3 - u_x, b_3 = -u_y + a_3 b_1 a_0^{-1}, b_4 = -u_z, b_5 = -u_t$. Then we have

\[
\sigma - \theta_{1,0,0} = a_0^2 \left( dv - \frac{a_3 (a_0^2 a_1 a_3 + 2 b_0 b_1)}{2 a_0 a_1} dt - \frac{b_0}{a_0 a_1} dx - \frac{a_1 a_3 + b_1}{a_0 a_1} dy + \frac{b_0 (u_x - a_3)}{a_0 a_1} dz \right).
\]

We introduce new parameters $v_x, v_y, w$ such that

\[
b_0 = a_0 a_1 v_x, \quad b_1 = a_1 (a_0 v_y - w), \quad a_3 = w.
\]

This gives

\[
\sigma - \theta_{1,0,0} = a_0^2 \left( dv - (w v_y - u_y v_x - \frac{1}{2} w^2) dt - v_x dx - v_y dy - (w - u_x) v_x dz \right). \tag{16}
\]

3.3. Lax representation of a generalization of rqsdYM

The 1-form (16) is equal to zero whenever the following over-determined system holds:

\[
\begin{align*}
vt &= w v_y - u_y v_x - \frac{1}{2} w^2, \\
vz &= (w - u_x) v_x.
\end{align*} \tag{17}
\]

In the case of $w = \lambda = \text{const}$ system (17) after the change of variable $v = q - \frac{1}{2} \lambda^2 t$ coincides with system (3). In the general case the compatibility condition $(v_t)_z = (v_z)_t$ of system (17) gives an over-determined system

\[
\begin{align*}
w_t &= w w_y - u_y w_x - (u_{yz} - u_{tx} - u_y u_{xx} + u_x u_{xy}), \\
w_z &= (w - u_x) w_x.
\end{align*} \tag{18}
\]

* we put $\alpha = da_0/a_0$ instead of the natural choice $\alpha = da_0$ and $\beta_0 = a_1^2 dt$, instead of $\beta_0 = a_1 dt$ to simplify the further computations.
In its turn this system is compatible whenever
\[
\begin{align*}
(u_{yz} - u_{tx} - u_y u_{xx} + u_x u_{xy})_x &= 0, \\
(u_{yz} - u_{tx} - u_y u_{xx} + u_x u_{xy})_z &= 0.
\end{align*}
\]
In other words, system (18) defines a covering over the two-component generalization
\[
\begin{align*}
u_{yz} &= u_{tx} + u_y u_{xx} - u_x u_{xy} + s, \\
}\frac{ds}{dx} &= 0, \\
}\frac{ds}{dz} &= 0.
\end{align*}
\]
(19)
of equation (1) and can be written in the form
\[
\begin{align*}
w_t &= w w_y - u_y w_x - s, \\
w_z &= (w - u_x) w_x.
\end{align*}
\]
(20)
In the particular case \(s = 0\) we have the covering over (1) defined by system
\[
\begin{align*}
w_t &= w w_y - u_y w_x, \\
w_z &= (w - u_x) w_x.
\end{align*}
\]
This system is related with system (3) by the following transformation, cf. [19]: suppose that a solution \(q\) to (3) is defined implicitly as \(W(t, x, y, z, q(t, x, y, z)) = \lambda = \text{const}\), then \(W\) is a solution to (20).

**Remark 3.** Function \(s\) can not be excluded from system (19) by a contact transformation, therefore equations (1) and (19) are not equivalent.

4. **Integrable hierarchy associated to rqsdYM**

The Lie algebra \(\mathfrak{s}_2\) admits a sequence of natural extensions \(\mathfrak{s}_m = \mathfrak{s}_2 \ltimes \mathfrak{s}_{m,\infty}, m \geq 3\), where
\[
\mathfrak{s}_{m,\infty} = \{A(t, z) \partial_z\} \otimes \mathbb{R}[h]/\langle h^{m+1}\rangle,
\]
with the structure equations of the form (10), (11), (12), (13) such that \(k \in \{0, \ldots, m\}\) in (13). Since the finite-dimensional part in all the algebras \(\mathfrak{s}_m\) is the same, every Lie algebra \(\mathfrak{s}_m\) satisfies Theorem 1 and its Corollary. Therefore we can find forms \(\theta_{k,0,0}, k \leq m\) by integration of the structure equations in the same way as in Subsection 3.2. Then we consider \(\theta_{m,0,0}\) as a multiple of the contact form \(du - u_{x_1} dx_1 - \ldots - u_{x_{m+2}} dx_{m+2}\) of order 0 on the jet bundle \(J^1(\pi)\) for \(\pi: \mathbb{R}^{m+2} \times \mathbb{R} \to \mathbb{R}^{m+2}, \pi: (x_1, \ldots, x_{m+2}, u) \mapsto (x_1, \ldots, x_{m+2})\), take the 1-form \(\sigma - \theta_{m-1,0,0}\) as the Wahlquist-Estabrook form of a covering and write out the compatibility conditions explicitly. In this Section we consider the cases \(m = 3, m = 4,\) and \(m = 5\). Below we alter notation as follows: \(t \mapsto x_1, x \mapsto x_2, y \mapsto x_3, z \mapsto x_4\).
4.1. Case $k \in \{0, \ldots, 3\}$.

While the 1-forms $\alpha, \beta_i, \gamma_l, \sigma, \theta_{0,0,0}, \theta_{1,0,0}$ are the same as in Subsection 3.2 instead of (15) we have now

$$\theta_{2,0,0} = \frac{a_2^3 b_0}{a_1^2} \left( dx_5 + (b_2 - a_3) dx_2 + \left( b_3 - \frac{a_3 b_1}{b_0} \right) dx_3 + b_4 dx_4 + b_5 dx_1 \right).$$

Then we put

$$b_2 = 2 a_3 - u_{x_5}, \quad b_4 = a_3^2 - a_3 u_{x_5} - u_{x_2}, \quad b_5 = a_3 b_3 - \frac{a_3^3 b_1}{b_0} - u_{x_3}$$

and obtain

$$\theta_{3,0,0} = \frac{a_0^3 b_0}{a_3^2} \left( du - \sum_{i=1}^{5} u_{x_i} dx_i \right).$$

Further we rename $b_0 = a_1^2 v_{x_5}, a_3 = v_{x_2} v_{x_5}^{-1} + u_{x_5}, b_3 = (a_1^2 (v_{x_5} (v_{x_3} - u_{x_5}) - v_{x_1}) + b_1 (v_{x_2} + u_{x_5} v_{x_5})) a_1^{-2} v_{x_5}^{-3}$. This yields

$$\sigma - \theta_{2,0,0} = a_0^2 \left( dv - v_{x_2} dx_2 - v_{x_3} dx_3 - v_{x_5} dx_5 - \frac{v_{x_2}^2 + u_{x_5} v_{x_2} v_{x_5} - u_{x_2} v_{x_5}^2}{v_{x_5}} dx_4 \right.$$  

$$\left. - \left( u_{x_5} v_{x_3} + \frac{v_{x_3} - u_{x_5}}{v_{x_5}} v_{x_2} - u_{x_3} v_{x_5} - \frac{u_{x_2} v_{x_5}^2 - v_{x_2}^2}{2 v_{x_5}^2} \right) dx_1 \right).$$

This 1-form is equal to zero whenever there holds the over-determined system

$$\begin{cases} u_{x_1} = u_{x_5} v_{x_3} + \frac{(v_{x_3} - u_{x_5}) v_{x_2}}{v_{x_5}} - u_{x_3} v_{x_5} - \frac{u_{x_2} v_{x_5}^2 - v_{x_2}^2}{2 v_{x_5}^2}, \\ v_{x_4} = \frac{v_{x_2}^2 + u_{x_5} v_{x_2} v_{x_5} - u_{x_2} v_{x_5}^2}{v_{x_5}}, \end{cases}$$

The compatibility condition $(v_{x_1})_{x_4} = (v_{x_4})_{x_1}$ of this system gives three equations for the function $u$:

$$u_{x_4 x_5} = u_{x_2 x_2} - u_{x_2} u_{x_5} + u_{x_5} u_{x_5}, \quad \quad (21)$$

$$u_{x_1 x_5} = u_{x_2 x_3} - u_{x_3} u_{x_5} + u_{x_5} u_{x_3}, \quad \quad (22)$$

$$u_{x_3 x_4} = u_{x_1 x_2} + u_{x_3} u_{x_2} - u_{x_2} u_{x_3}, \quad \quad (23)$$

This system is compatible. Equations (22) and (23) differ from equations (11) and (2) only by notation, while equation (21) was introduced in [9] and is known to have a covering with non-removable parameter, see [14, 18, 4].

4.2. Case $k \in \{0, \ldots, 4\}$.

In this case we get

$$\theta_{3,0,0} = \frac{a_0^3 b_0}{a_1^3} \left( dx_6 + (b_3 - 2 a_3) dx_5 + (b_4 - a_3 b_2 + a_3^2) dx_2 + (b_5 - a_3 b_3 + b_1 a_3^2) dx_3 + b_6 dx_4 + b_7 dx_1 \right),$$
then substituting for \( b_2 = 3 a_3 - u_{x_6} \), \( b_4 = -u_{x_5} - 2 a_3 u_{x_6} + 3 a^2_3 \), \( b_7 = a^3_0 - u_{x_2} - a^2_0 u_{x_6} - a_3 u_{x_5} \) into \( \theta_{4,0,0} \) yields \( \theta_{4,0,0} = a^3_0 b_0 a^{-4}_1 (du - u_{x_1} dx_1 - \ldots - u_{x_6} dx_6) \). Further we introduce new parameters \( v_{x_3}, v_{x_5}, v_{x_6} \) such that \( b_0 = a^3 a_0^{-1} v_{x_6}, a_3 = v_{x_5} v_{x_6}^{-1} + u_{x_6}, b_5 = (v_{x_6} (v_{x_5} u_{x_6} + v_{x_5}) b_3 - (v_{x_6} u_{x_6} + v_{x_5})^2 b_1 - v_{x_5} - v_{x_6} (u_{x_6} - u_{x_3}) v_{x_6}^{-2} \). This gives 

\[
\sigma - \theta_{3,0,0} = a^3_0 \left( dv - v_{x_3} dx_3 - v_{x_4} dx_4 - v_{x_6} dx_6 - \frac{v_{x_5}^2 + u_{x_6} v_{x_5} v_{x_6} - u_{x_5} v_{x_6}^2}{v_{x_6}} dx_2 - \left( u_{x_6} v_{x_5} - \frac{v_{x_5} (v_{x_5} + 2 (u_{x_6} - v_{x_3}) v_{x_6})}{2 v_{x_6}^2} - u_{x_3} v_{x_6} - \frac{u_{x_6}^2}{2} \right) dx_1 - \left( \frac{v_{x_5}^2 (v_{x_5} + 2 u_{x_6} v_{x_6})}{v_{x_6}} - (u_{x_5} u_{x_6} + u_{x_2}) v_{x_6} - (u_{x_3} - u_{x_5}^2) v_{x_5} \right) dx_4 \right).
\]

This 1-form defines an over-determined system

\[
\begin{align*}
v_{x_1} &= u_{x_6} v_{x_5} - \frac{v_{x_5} (v_{x_5} + 2 (u_{x_6} - v_{x_3}) v_{x_6})}{2 v_{x_6}^2} - u_{x_3} v_{x_6} - \frac{u_{x_6}^2}{2}, \\
v_{x_2} &= \frac{v_{x_5}^2 + u_{x_6} v_{x_5} v_{x_6} - u_{x_5} v_{x_6}^2}{v_{x_6}^2}, \\
v_{x_4} &= \frac{v_{x_5}^2 (v_{x_5} + 2 u_{x_6} v_{x_6})}{v_{x_6}^2} - (u_{x_5} u_{x_6} + u_{x_2}) v_{x_6} - (u_{x_3} - u_{x_5}^2) v_{x_5}.
\end{align*}
\]

System (24) is compatible by virtue of the following system of the second order equations for function \( u \):

\[
\begin{align*}
u_{x_5} x_5 &= u_{x_2} x_6 + u_{x_5} u_{x_6} x_6 - u_{x_6} u_{x_5} x_6, \\
u_{x_4} x_6 &= u_{x_2} x_5 + u_{x_6} u_{x_2} x_6 - u_{x_2} u_{x_6} x_6, \\
u_{x_5} x_5 &= u_{x_1} x_6 + u_{x_3} u_{x_6} x_6 - u_{x_6} u_{x_3} x_6, \\
u_{x_2} x_3 &= u_{x_1} x_5 + u_{x_3} u_{x_5} x_6 - u_{x_5} u_{x_3} x_6, \\
u_{x_3} x_4 &= u_{x_1} x_2 + u_{x_3} u_{x_2} x_6 - u_{x_2} u_{x_3} x_6, \\
u_{x_4} x_5 &= u_{x_2} x_2 + u_{x_5} u_{x_2} x_6 - u_{x_2} u_{x_5} x_6.
\end{align*}
\]

The last system is compatible. Equations (25), (26), (27), (28), (29) differ from equations (21), (11), (11), (24), (24), respectively, by notation. Equation (30) was introduced in (20), where a covering with a non-removable parameter for this equation was presented.

4.3. Case \( k \in \{0, \ldots, 5\} \).

In this case we have

\[
\theta_{4,0,0} = \frac{a^3_0 b_0 a^{-4}_1}{a^3_1} (dx_7 + b_10 dx_1 + b_0 dx_4 + (b_2 - 3 a_3) dx_6 + (b_4 - 2 a_3 b_2 + 3 a^2_3) dx_5 \\
+ (b_7 - a_3 b_4 + a^2_3 b_2 - a^3_3) dx_2 + (b_8 - a_3 b_5 + a^2_3 b_3 - a^3_3 b_1) dx_3) .
\]

Then after altering notation \( b_4 = 6 a^2 - 3 a_3 u_{x_7} - u_{x_6}, b_7 = 4 a^3 - 3 a^2 u_{x_7} - 2 a_3 u_{x_6} - u_{x_5}, b_9 = a^4 - a^3 u_{x_7} - u_{x_6} - a_3 u_{x_5} - u_{x_2}, b_{10} = -a^4 b_1 + a^3 b_3 - a^2 b_5 + a_3 b_8 - u_{x_3} \) we obtain \( \theta_{5,0,0} = a^5_0 b_0 a^{-5}_1 (du - u_{x_1} dx_1 - \ldots - u_{x_7} dx_7) \). Further we rename \( b_0 = v_{x_7}, a^4_{0} a^{-2}_1, \).
This 1-form produces the over-determined system

\[
\begin{align*}
\sigma - \theta_{4,0,0} &= a_0^2 (dv - v_x d\alpha_3 - v_x d\alpha_1 - v_x d\alpha_2 - v_x d\alpha_4) \\
&= (u_{x_3} v_{x_7} - u_{x_7} v_{x_3} + v_{x_6} (u_{x_3} - u_{x_7}) v_{x_7}^{-1} + \frac{1}{2} (u_{x_7} + v_{x_6} v_{x_7}^{-2})) dx_1 \\
&- (u_{x_3} v_{x_7} - u_{x_7} v_{x_3} + v_{x_6} (u_{x_7} - v_{x_3}) v_{x_7}^{-1} + \frac{1}{2} (u_{x_7} + v_{x_6} v_{x_7}^{-2})) dx_2 \\
&- ((v_{x_7} - u_{x_6}) v_{x_6} + 2 v_{x_6}^2 u_{x_7} v_{x_7}^{-1} + v_{x_6}^3 v_{x_7}^{-2} - (u_{x_7} u_{x_6} + u_{x_5}) v_{x_7}) dx_3 \\
&- ((u_{x_7}^2 - 2 u_{x_7} u_{x_6} - u_{x_3}) v_{x_6} + (3 u_{x_7}^2 - u_{x_6}) v_{x_6}^2 v_{x_7}^{-1} + 3 u_{x_7}^3 v_{x_6}^2 v_{x_7}^{-2} + v_{x_6}^4 v_{x_7}^{-3}) dx_4.
\end{align*}
\]

This 1-form produces the over-determined system

\[
\begin{align*}
\begin{aligned}
v_{x_1} &= u_{x_3} v_{x_7} - u_{x_7} v_{x_3} + v_{x_6} (u_{x_7} - u_{x_3}) v_{x_7}^{-1} + \frac{1}{2} (u_{x_7} + v_{x_6} v_{x_7}^{-2}), \\
v_{x_2} &= (u_{x_7}^2 - u_{x_6}) v_{x_6} + 2 u_{x_7} v_{x_6} v_{x_7}^{-1} + v_{x_6}^3 v_{x_7}^{-2} - (u_{x_7} u_{x_6} + u_{x_5}) v_{x_7}, \\
v_{x_3} &= (u_{x_7}^3 - 2 u_{x_7} u_{x_6} - u_{x_3}) v_{x_6} + (3 u_{x_7}^2 - u_{x_6}) v_{x_6}^2 v_{x_7}^{-1} + 3 u_{x_7}^3 v_{x_6}^2 v_{x_7}^{-2} + v_{x_6}^4 v_{x_7}^{-3}, \\
v_{x_4} &= -u_{x_7}^2 v_{x_6} + u_{x_7} u_{x_5} + u_{x_2}) v_{x_7}, \\
v_{x_5} &= u_{x_7} v_{x_6} - u_{x_6} v_{x_7} + v_{x_6} v_{x_7}^{-1}.
\end{aligned}
\end{align*}
\]

The compatibility conditions of this system yield the following equations of the second order for function \( u \):

\[
\begin{align*}
u_{x_6 x_6} &= u_{x_5 x_7} - u_{x_7 x_6 x_7} + u_{x_6 x_7 x_7}, \tag{31} \\
u_{x_1 x_7} &= u_{x_3 x_6} + u_{x_7 x_3 x_7} - u_{x_3} u_{x_7 x_7}, \tag{32} \\
u_{x_2 x_7} &= u_{x_5 x_6} + u_{x_7 x_5 x_7} - u_{x_5} u_{x_7 x_7}, \tag{33} \\
u_{x_1 x_5} &= u_{x_2 x_3} - u_{x_3} u_{x_5 x_7} + u_{x_5} u_{x_3 x_7}, \tag{34} \\
u_{x_1 x_6} &= u_{x_3 x_5} + u_{x_6} u_{x_3 x_7} - u_{x_3} u_{x_6 x_7}, \tag{35} \\
u_{x_2 x_6} &= u_{x_4 x_7} - u_{x_6} u_{x_5 x_7} - u_{x_7} u_{x_5 x_6} + (u_{x_5} u_{x_7} + u_{x_2}) u_{x_7 x_7}, \tag{36} \\
u_{x_4 x_5} &= u_{x_2 x_2} + u_{x_5} u_{x_3 x_6} + (u_{x_5} u_{x_7} - u_{x_2}) u_{x_5 x_7} - u_{x_2} u_{x_3 x_7}, \tag{37} \\
u_{x_4 x_6} &= u_{x_2 x_5} + u_{x_6} u_{x_5 x_6} + u_{x_6} u_{x_7} u_{x_5 x_7} - u_{x_2} u_{x_6 x_7} - u_{x_3} u_{x_6} u_{x_7 x_7}, \tag{38} \\
u_{x_5 x_5} &= u_{x_4 x_7} - u_{x_7} u_{x_5 x_6} + (u_{x_7}^2 + u_{x_6}) u_{x_5 x_7} + u_{x_5} u_{x_6 x_7} + (u_{x_5} u_{x_7} + u_{x_2}) u_{x_7 x_7}, \tag{39} \\
u_{x_3 x_4} &= u_{x_1 x_2} - u_{x_2} u_{x_3 x_7} + u_{x_3} u_{x_5 x_6} + u_{x_3} u_{x_7} u_{x_3 x_7} - u_{x_3} u_{x_5} u_{x_7 x_7}, \tag{40}
\end{align*}
\]

In its turn this system is compatible. Equations \((31), (32), (33), (34), (35)\) differ from equations \((21), (11), (11), (21), (22)\), respectively, by notation. We have not found equations \((36), (37), (38), (39)\) with five independent variables and equation \((40)\) with seven independent variables in the literature.

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