MONO: AN ALGEBRAIC STUDY OF TORUS CLOSURES

JUSTIN CHEN

ABSTRACT. Given an ideal $I$ in a polynomial ring, we consider the largest monomial subideal contained in $I$, denoted mono($I$). We study mono as an interesting operation in its own right, guided by questions that arise from comparing the Betti tables of $I$ and mono($I$). Many examples are given throughout to illustrate the phenomena that can occur.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ in $n$ variables. For any ideal $I \subseteq R$, let mono($I$) denote the largest monomial subideal of $I$, i.e. the ideal generated by all monomials contained in $I$. Geometrically, mono($I$) defines the smallest torus-invariant subscheme containing $V(I) \subseteq \text{Spec } R$ (the so-called torus-closure of $V(I)$).

The concept of mono has been relatively unexplored, despite the naturality of the definition. The existing work in the literature concerning mono has been essentially algorithmic and/or computational. For convenience, we summarize this in the following two theorems:

**Theorem 0.1** ([5], Algorithm 4.4.2). Let $I = (f_1, \ldots, f_r)$. Fix new variables $y_1, \ldots, y_n$, and let $\tilde{f}_i := f_i(x_1 y_1, \ldots, x_n y_n) \cdot \prod_{i=1}^n y_i^{\deg_{x_i}(f)}$ be the multi-homogenization of $f_i$ with respect to $y_i$. Let $>$ be an elimination term order on $k[x, y]$ satisfying $y_i > x_j$ for all $i, j$. If $\mathcal{G}$ is a reduced Gröbner basis for $(\tilde{f}_1, \ldots, \tilde{f}_r) : (\prod_{i=1}^n y_i)^\infty$ with respect to $>$, then the monomials in $\mathcal{G}$ generate mono($I$).

Cf. also [2] for a generalization computing the largest $A$-graded subideal of an ideal, for an integer matrix $A$ (mono being the special case when $A$ is the identity matrix). The next theorem gives an alternate description of mono for a particular class of ideals, involving the dual concept of Mono, which is the smallest monomial ideal containing a given ideal (notice that Mono($I$) is very simple to compute, being generated by all terms appearing in a generating set of $I$).

**Theorem 0.2** ([4], Lemma 3.2). Let $I$ be an unmixed ideal, and suppose there exists a regular sequence $\beta \subseteq I$ consisting of codim $I$ monomials. Then mono($I$) = $\langle \beta \rangle : \text{Mono}(\langle \beta \rangle : I)$.

However, it appears that no systematic study of mono as an operation on ideals has yet been made. It is the goal of this note to provide first steps in this direction; in particular exploring the relationship between $I$ and mono($I$). By way of understanding mono as an algebraic process, we consider the following questions:

1. When is mono($I$) = 0, or prime, or primary, or radical?
2. To what extent does taking mono depend on the ground field?
3. Which invariants are preserved by taking mono? For instance, do $I$ and mono($I$) have the same (Castelnuovo-Mumford) regularity?

2010 Mathematics Subject Classification. 13D02, 13C99, 13A02, 05E40.

1
(4) How do the Betti tables of $I$ and $\text{mono}(I)$ compare? Do they have the same shape?
(5) To what extent is $\text{mono}$ non-unique? E.g. which monomial ideals arise as $\text{mono}$ of a non-monomial ideal?
(6) What properties of $I$ are preserved by $\text{mono}(I)$, and conversely, what properties of $I$ are reflected by $\text{mono}(I)$?

1. Basic Properties

We first give some basic properties of $\text{mono}$, which describe how $\text{mono}$ interacts with various algebraic operations. As above, $R$ denotes a polynomial ring $k[x_1, \ldots, x_n]$, and $m = (x_1, \ldots, x_n)$ denotes the homogeneous maximal ideal of $R$.

**Proposition 1.1.** Let $I$ be an $R$-ideal.

(1) $\text{mono}$ is decreasing, inclusion-preserving, and idempotent.
(2) $\text{mono}$ commutes with intersections, i.e. $\text{mono}(\bigcap I_i) = \bigcap \text{mono}(I_i)$ for any ideals $I_i$.
(3) $\text{mono}(I_1) \subseteq \text{mono}(I_1 \cap I_2) \subseteq \text{mono}(I_1) \cap \text{mono}(I_2)$.

**Proof.**

(1) Each property - $\text{mono}(I) \subseteq I$, $I_1 \subseteq I_2 \implies \text{mono}(I_1) \subseteq \text{mono}(I_2)$, $\text{mono}(\text{mono}(I)) = \text{mono}(I)$ - is clear from the definition.
(2) If $u \in \sqrt{\text{mono}(I)}$ is a monomial, say $u^m \in \text{mono}(I)$, then $u \in \sqrt{I} \implies u \in \text{mono}(\sqrt{I})$. Conversely, if $u \in \text{mono}(\sqrt{I})$ is monomial, then $u \in \sqrt{I}$, say $u^m \in I$ and hence $u^m \in \text{mono}(I) \implies u \in \sqrt{\text{mono}(I)}$.
(3) $\bigcap I_i \subseteq I_i \implies \text{mono}(\bigcap I_i) \subseteq \text{mono}(I_i)$, hence $\text{mono}(\bigcap I_i) \subseteq \bigcap \text{mono}(I_i)$. On the other hand, an arbitrary intersection of monomial ideals is monomial, and $\bigcap \text{mono}(I_i) \subseteq I_i$, hence $\bigcap \text{mono}(I_i) \subseteq \text{mono}(\bigcap I_i)$.
(4) $\text{mono}(I_1) \text{mono}(I_2) \subseteq I_1 I_2$, and a product of two monomial ideals is monomial, hence $\text{mono}(I_1) \text{mono}(I_2) \subseteq \text{mono}(I_1 I_2)$. The second containment follows from applying (1) and (3) to the containment $I_1 I_2 \subseteq I_1 \cap I_2$. □

Next, we consider how prime and primary ideals behave under taking $\text{mono}$:

**Proposition 1.2.** Let $I$ be an $R$-ideal.

(1) If $I$ is prime resp. primary, then so is $\text{mono}(I)$.
(2) $\text{Ass}(R/\text{mono}(I)) \subseteq \{\text{mono}(P) \mid P \in \text{Ass}(R/I)\}$.
(3) $\text{mono}(I)$ is prime iff $\text{mono}(I) = \text{mono}(P)$ for some minimal prime $P$ of $I$.
In particular, $\text{mono}(I) = 0$ iff $\text{mono}(P) = 0$ for some $P \in \text{Min}(I)$.

**Proof.**

(1) To check that $\text{mono}(I)$ is prime (resp. primary), it suffices to check that if $u, v$ are monomials with $uv \in \text{mono}(I)$, then $u \in \text{mono}(I)$ or $v \in \text{mono}(I)$ (resp. $v^m \in \text{mono}(I)$ for some $n$). But this holds, as $I$ is prime (resp. primary) and $u, v$ are monomials.
(2) If $I = Q_1 \cap \ldots \cap Q_r$ is a minimal primary decomposition of $I$, so that $\text{Ass}(R/I) = \{\sqrt{Q_i}\}$, then $\text{mono}(I) = \text{mono}(Q_1) \cap \ldots \cap \text{mono}(Q_r)$ is a primary decomposition of $\text{mono}(I)$, so every associated prime of $\text{mono}(I)$ is of the form $\sqrt{\text{mono}(Q_i)} = \text{mono}(\sqrt{Q_i})$ for some $i$.
(3) $\text{mono}(I) = \sqrt{\text{mono}(I)} = \text{mono}(\sqrt{I}) = \bigcap_{P \in \text{Min}(I)} \text{mono}(P)$. Since $\text{mono}(I)$ is prime and the intersection is finite, $\text{mono}(I)$ must equal one of the terms in the intersection. The converse follows from (1). □
We now examine the sharpness of various statements in Propositions \([1.1]\) and \([1.2]\).

**Example 1.3.** Let \( R = k[x, y] \), where \( k \) is an infinite field, and let \( I \) be an ideal generated by 2 random quadrics. Then \( R/I \) is an Artinian complete intersection of regularity 2, so \( m^3 \subseteq I \). By genericity, \( I \) does not contain any monomials in degrees \( \leq \text{reg}(R/I) \), so \( \text{mono}(I) = m^3 \). On the other hand, \( I^2 \) is a 3-generated perfect ideal of grade 2, so the Hilbert-Burch resolution of \( R/I^2 \) shows that \( \text{reg} R/I^2 = 4 \), hence \( m^5 \subseteq I^2 \subseteq m^4 \) as \( I^2 \) is generated by quartics (in fact, \( \text{mono}(I^2) = m^5 \)). Thus for such \( I = I_1 = I_2 \), both containments in Proposition \([1.1](4)\) are strict.

Similar to Proposition \([1.1](4)\), the containment in Proposition \([1.2](2)\) is also strict in general: take e.g. \( I = I' \cap m^N \) where \( \text{mono}(I') = 0 \) and \( N > 0 \) is such that \( I' \not\subseteq m^N \). However, combining these two statements yields:

**Corollary 1.4.** Let \( I \) be an \( R \)-ideal.

1. Nonzerodivisors on \( R/I \) are also nonzerodivisors on \( R/\text{mono}(I) \).
2. Let \( u \in R \) be a monomial that is a nonzerodivisor on \( R/I \). Then \( \text{mono}((u)I) = (u) \text{mono}(I) \).

**Proof.** (1) \( \bigcup_{P \in \text{Ass}(R/\text{mono}(I))} P \subseteq \bigcup_{P \in \text{Ass}(R/I)} \text{mono}(P) \subseteq \bigcup_{P \in \text{Ass}(R/I)} P. \)
\[ \text{(2) This follows from (1) and Proposition \([1.1](4)\).} \]

**Remark 1.5.** Since monomial prime ideals are generated by (sets of) variables, if \( P \subseteq m^2 \) is a nondegenerate prime, then \( \text{mono}(P) = 0 \). It thus follows from Proposition \([1.2](3)\) that “most” ideals \( I \) satisfy \( \text{mono}(I) = 0 \): namely, this is always the case unless each component of \( V(I) \) is contained in some coordinate hyperplane in \( \mathbb{A}^n = \text{Spec} R \). The case that \( \text{mono}(I) \) is prime is analogous: if \( \text{mono}(I) = (x_{i_1}, \ldots, x_{i_r}) \) for some \( \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\} \), then \( V(I) \) becomes nondegenerate upon restriction to the coordinate subspace \( V(x_{i_1}, \ldots, x_{i_r}) \cong \mathbb{A}^{r} \) (i.e. \( \text{mono}(\overline{I}) = \overline{0} \) where \( \overline{\cdot} \) denotes passage to the quotient \( R/(x_{i_1}, \ldots, x_{i_r}).) \)

In contrast to the simple picture when \( \text{mono}(I) \) is prime, the case where \( \text{mono}(I) \) is primary is much more interesting, due to nonreducedness issues. The foremost instance of this case is when \( \text{mono}(I) \) is \( m \)-primary, i.e. \( \text{mono}(I) \) is Artinian. A first indication that this case is interesting is that under this assumption, \( \text{mono}(I) \) is guaranteed not to be 0. For this and other reasons soon to appear, we will henceforth deal primarily with this case – the reader should assume from now on that \( I \) is an Artinian ideal.

2. Dependence on scalars

We now briefly turn to Question \([2]\) to what extent does taking \( \text{mono} \) depend on the ground field \( k \)? To make sense of this, let \( S = \mathbb{Z}[x_1, \ldots, x_n] \) be a polynomial ring over \( \mathbb{Z} \). Then for any field \( k \), the universal map \( \mathbb{Z} \to k \) induces a ring map \( S \to S_k := S \otimes_{\mathbb{Z}} k = k[x_1, \ldots, x_n] \). Given an ideal \( I \subseteq S \), one can consider the extended ideal \( IS_k \). The question is then: as the field \( k \) varies, how does \( \text{mono}(IS_k) \) change?

It is easy to see that if \( k_1 \) and \( k_2 \) have the same characteristic, then \( \text{mono}(IS_{k_1}) \) and \( \text{mono}(IS_{k_2}) \) have identical minimal generating sets. Thus it suffices to consider prime fields \( \mathbb{Q} \) and \( \mathbb{F}_p \), for \( p \in \mathbb{Z} \) prime. Another moment's thought shows that \( \text{mono} \) can certainly change in passing between different characteristics; e.g. if all but one
of the coefficients of some generator of $I$ is divisible by a prime $p$. However, even excluding simple examples like this, by requiring that the generators of $I$ all have unit coefficients, mono still exhibits dependence on characteristic. We illustrate this with a few examples:

**Example 2.1.** Let $S = \mathbb{Z}[x, y, z]$ be a polynomial ring in 3 variables.

(1) Set $I := (x^3, y^3, z^3, xyz(x + y + z))$. Then $xyz^2 \in \text{mono}(IS_k)$ iff $\text{char} k = 2$ (consider $xyz(x + y + z)^2 \in I$). Notice that $I$ is equi-generated, i.e. all minimal generators of $I$ have the same degree.

(2) For a prime $p \in \mathbb{Z}$, set $I_p := (x^p, y^p, x + y + z)$. Then $z^p \in \text{mono}(I_p S_k)$ iff $\text{char} k = p$ (consider $(x + y + z)^p \in I_p$). If the presence of the linear form is objectionable, one may increase the degrees, e.g. $(x^{2p}, y^{2p}, x^2 + y^2 + z^2)$.

From these examples we see that mono is highly sensitive to characteristic in general. However, this is not the whole story: cf. Remark 4.2 for one situation where taking mono is independent of characteristic.

3. **Betti tables**

We now consider how invariants of $I$ behave when passing to $\text{mono}(I)$. As mentioned in Remark 4.3, although $I$ and $\text{mono}(I)$ are typically quite different, for Artinian graded ideals there is a much closer relationship:

**Proposition 3.1.** Let $I$ be a graded $R$-ideal. Then $I$ is Artinian iff $\text{mono}(I)$ is Artinian. In this case, $\text{reg}(R/I) = \text{reg}(R/\text{mono}(I))$.

**Proof.** Since $I \subseteq m$ is graded, $I$ is Artinian iff $m^s \subseteq I$ for some $s > 0$. This occurs iff $m^s \subseteq \text{mono}(I)$ for some $s > 0$ iff $\text{mono}(I)$ is Artinian.

Next, recall that if $M = \bigoplus M_i$ is Artinian graded, then the regularity of $M$ is $\text{reg}(M) = \max\{i \mid M_i \neq 0\}$. The inclusion $\text{mono}(I) \subseteq I$ induces a (graded) surjection $R/\text{mono}(I) \twoheadrightarrow R/I$, which shows that $\text{reg}(R/\text{mono}(I)) \geq \text{reg}(R/I)$. Now if $u \in R$ is a standard monomial of $\text{mono}(I)$ of top degree ($= \text{reg}(R/\text{mono}(I))$), then $u \not\in \text{mono}(I) \implies u \not\in I$, hence $\text{reg}(R/I) \geq \text{deg} u = \text{reg}(R/\text{mono}(I))$. □

A restatement of Proposition 3.1 is that for any Artinian graded ideal $I$, the graded Betti tables of $I$ and $\text{mono}(I)$ have the same number of rows and columns (since any Artinian ideal has projective dimension $n = \dim R$). However, it is not true (even in the Artinian case) that the Betti tables of $I$ and $\text{mono}(I)$ have the same shape (= (non)zero pattern) – e.g. take an ideal $I'$ with $\text{mono}(I') = 0$, and consider $I := I' + m^N$ for $N \gg 0$. Despite this, there is one positive result in this direction:

**Proposition 3.2.** Let $I$ be an Artinian graded $R$-ideal. Then $\beta_{n,j}(R/\text{mono}(I)) \neq 0 \implies \beta_{n,j}(R/I) \neq 0$, for any $j$.

**Proof.** Notice that $\beta_{n,j}(R/I) \neq 0$ iff the socle of $R/I$ contains a nonzero form of degree $j$. Let $m \in R$ be a monomial with $\overline{0} \neq \overline{m} \in \text{soc}(R/\text{mono}(I))$ and $\text{deg} m = j$. Then $m \in (\text{mono}(I) :_R m) \setminus \text{mono}(I)$, hence $m \in (I :_R m) \setminus I$ as well, i.e. $\overline{0} \neq \overline{m} \in \text{soc}(R/I)$. □

**Corollary 3.3.** Let $I$ be an Artinian graded level $R$-ideal (i.e. $\text{soc}(R/I)$ is nonzero in only one degree). Then $\text{mono}(I)$ is also level, with the same socle degree as $I$.

**Proof.** Follows immediately from Proposition 3.2. □
We illustrate these statements with some examples of how the Betti tables of \( R/I \) and \( R/\text{mono}(I) \) can differ:

**Example 3.4.** Let \( R = k[x,y,z,w] \), \( J = I_2 \binom{x^2 y \quad yw \quad z}{y \quad xz \quad z^2} \) the ideal of the rational quartic curve in \( \mathbb{P}^3 \), and \( I = J + (x^2, y^4, z^4, w^4) \). Then the Betti tables of \( R/I \) and \( R/\text{mono}(I) \) respectively in Macaulay2 format are:

|   | 0  | 1  | 2  | 3  | 4  |
|---|----|----|----|----|----|
| total: | 1 | 7 | 15 | 13 | 4 |
| 0: | 1 | . | . | . | . |
| 1: | . | 2 | . | . | . |
| 2: | . | 3 | 5 | 1 | . |
| 3: | . | 2 | 5 | 4 | 1 |
| 4: | . | 4 | 5 | 1 | . |
| 5: | . | 1 | 3 | 2 | . |

|   | 0  | 1  | 2  | 3  | 4  |
|---|----|----|----|----|----|
| total: | 1 | 11 | 28 | 26 | 8 |
| 0: | 1 | . | . | . | . |
| 1: | . | 1 | . | . | . |
| 2: | . | 2 | 1 | . | . |
| 3: | . | 6 | 10 | 5 | 1 |
| 4: | . | 2 | 14 | 14 | 3 |
| 5: | . | 3 | 7 | 4 | . |

Notice that \( \beta_{1,5}(R/\text{mono}(I)) = 2 \neq 0 = \beta_{3,5}(R/I) \), and likewise \( \beta_{3,5}(R/I) = 1 \neq 0 = \beta_{3,5}(R/\text{mono}(I)) \). In addition, \( \beta_{1,2}, \beta_{1,3}, \beta_{2,4} \) for \( R/\text{mono}(I) \) are all strictly smaller than their counterpart for \( R/I \) (and still nonzero).

**Example 3.5.** Let \( R = k[x,y,z] \), \( \ell \in R_1 \) a general linear form (e.g. \( \ell = x+y+z \)), and \( I = (x\ell, y\ell, z\ell) + (x, y, z)^3 \). Then \( \text{mono}(I) = (x, y, z)^3 \), and the Betti tables of \( R/I \) and \( R/\text{mono}(I) \) respectively are:

|   | 0  | 1  | 2  | 3  | 4  |
|---|----|----|----|----|----|
| total: | 1 | 7 | 10 | 4 | . |
| 0: | 1 | . | . | . | . |
| 1: | . | 3 | 3 | 1 | . |
| 2: | . | 4 | 7 | 3 | . |

|   | 0  | 1  | 2  | 3  |
|---|----|----|----|----|
| total: | 1 | 10 | 15 | 6 |
| 0: | 1 | . | . | . |
| 1: | . | . | . | . |
| 2: | . | 10 | 15 | 6 |

Here \( \text{mono}(I) \) is level, but \( I \) is not. This shows that the converse to Corollary 3.3 is not true in general.

**Example 3.6.** We revisit Example 1.3. Since \( \text{mono}(I) \) is a power of the maximal ideal, \( R/\text{mono}(I) \) has a linear resolution, whereas \( R/I \) has a Koszul resolution (with no linear forms), so in view of Proposition 3.2 the Betti tables of \( R/I \) and \( R/\text{mono}(I) \) have as disjoint shapes as possible. Thus no analogue of Proposition 3.2 can hold for \( \beta_i, i < n \), in general.

From the examples above, one can see that the earlier propositions on Betti tables are fairly sharp. Another interesting pattern observed above is that even when the ideal-theoretic description of \( \text{mono}(I) \) became simpler than that of \( I \), the Betti table often grew worse (e.g. had larger numbers on the whole). This leads to some natural refinements of Questions 4 and 6:

1. Are the total Betti numbers of \( \text{mono}(I) \) always at least those of \( I \)?
2. Does \( \text{mono}(I) \) Gorenstein imply \( I \) Gorenstein?

Notice that the truth of Question 7 implies the truth of Question 8. As it turns out, the answer to these will follow from the answer to Question 5.

4. **Uniqueness and the Gorenstein property**

**Lemma 4.1.** Let \( M \) be a monomial ideal, and \( u_1 \neq u_2 \) standard monomials of \( M \). Then \( \text{mono}(M + (u_1 + u_2)) = M \) iff \( M : u_1 = M : u_2 \).
Proof. \(\Rightarrow\): By symmetry, it suffices to show that \(M: u_1 \subseteq M: u_2\). Let \(m\) be a monomial in \(M: u_1\). Then \(mu_2 = m(u_1 + u_2) - mu_1 \in \text{mono}(M + (u_1 + u_2)) = M\), i.e. \(m \in M: u_2\).

\(\Leftarrow\): Passing to \(R/M\), it suffices to show that \((\bar{u}_1 + \bar{u}_2)\) contains no monomials in \(R/M\). Let \(g \in R\) be such that \(g(\bar{u}_1 + \bar{u}_2) \neq \emptyset \in R/M\), and write \(g = g_1 + \ldots + g_r\) as a sum of monomials. By assumption, \(g_i u_1 \in M\) iff \(g_i u_2 \in M\), so after removing some terms of \(g\) we may assume there exists \(g_i\) of top degree in \(g\) such that \(g_i u_1, g_i u_2 \notin M\). But then \(g_i \bar{u}_1\) and \(g_i \bar{u}_2\) both appear as distinct terms in \(g(\bar{u}_1 + \bar{u}_2)\), so \(g(\bar{u}_1 + \bar{u}_2)\) is not a monomial in \(R/M\).

Remark 4.2. Since colons of monomial ideals are characteristic-independent, the second condition in Lemma 4.1 is independent of the ground field \(k\). Thus if \(I\) is an ideal defined over \(\mathbb{Z}\) which is “nearly” monomial (i.e. is generated by monomials and a single binomial), and \(\text{mono}(I)\) is as small as possible in one characteristic, then \(\text{mono}(I)\) is the same in all characteristics.

Remark 4.3. For any polynomial \(f \in R\), it is easy to see that
\[
\text{mono}(M + (f)) \supseteq M + \sum_{u \in \text{terms}(f)} \text{mono}(M : f - u)u
\]
If \(f = u_1 + u_2\) is a binomial, then this simplifies to the statement that \(\text{mono}(M + (u_1 + u_2)) \supseteq M + (M : u_2)u_1 + (M : u_1)u_2\). However, equality need not hold: e.g. \(M = (x^2, y^2, x^2y^2), u_1 = x^2, u_2 = xy^2\) (or even \(M = (x^3, y^2), u_1 = x, u_2 = y\) if one allows linear forms).

Theorem 4.4. The following are equivalent for a monomial ideal \(M\):

1. There exists a graded non-monomial ideal \(I\) such that \(\text{mono}(I) = M\).
2. There exist \(t \geq 2\) monomials \(u_1, \ldots, u_t\) not contained in \(M\) and of the same degree, such that \(M : u_i = M : u_j\) for all \(i, j\).
3. There exist monomials \(u_1 \neq u_2\) with \(u_1, u_2 \notin M\), \(\deg u_1 = \deg u_2\) and \(M : u_i = M : u_j\) for all \(i, j\).

Proof. (1) \(\implies\) (2): Fix \(f \in I \setminus M\) graded of minimal support size \(t\) (so \(t \geq 2\)), and write \(f = u_1 + \ldots + u_t\) where \(u_i\) are standard monomials of \(M\) of the same degree. Fix \(1 \leq i \leq t\), and pick a monomial \(m \in M : u_i\). Then \(m(f - u_i) \in I\) has support size \(< t\), so minimality of \(t\) gives \(m(f - u_i) = \sum_{j \neq i} m u_j \in M\). Since \(M\) is monomial, \(mu_j \in M\) for each \(j \neq i\), i.e. \(m \in M : u_j\) for all \(j\). By symmetry, \(M : u_i = M : u_j\) for all \(i, j\).

(2) \(\implies\) (3): Clear.

(3) \(\implies\) (1): Set \(I := M + (u_1 + u_2)\), and apply Lemma 4.1. Notice that \(I\) is not monomial: if it were, then \(u_1 + u_2 \in I \implies u_1 \in I \implies u_1 \in \text{mono}(I) = M\), contradiction.

Corollary 4.5. Let \(I\) be an Artinian graded \(R\)-ideal. Then \(\text{mono}(I)\) is a complete intersection iff \(\text{mono}(I)\) is Gorenstein iff \(I = m^b := (x_1^{b_1}, \ldots, x_n^{b_n})\) for some \(b \in \mathbb{N}^n\).

Proof. Any Artinian Gorenstein monomial ideal is irreducible, hence is of the form \(m^b\), which is a complete intersection. By Theorem 4.4, it suffices to show that for \(M := m^b\), no distinct standard monomials of \(M\) satisfy \(M : u_1 = M : u_2\). To see this, note that since \(u_1 \neq u_2\), there exists \(j \in [n]\) such that \(x_j\) appears to different powers \(a_1 \neq a_2\) in \(u_1\) and \(u_2\), respectively. Taking \(a_1 < a_2\) WLOG gives \(x_j^{a_1 - a_2} \in (M : u_2) \setminus (M : u_1)\).
Combining the proofs above shows that an Artinian monomial ideal is not expressible as mono of any non-monomial ideal iff it is Gorenstein:

**Corollary 4.6.** Let \( M \) be an Artinian monomial ideal. Then there exists a non-monomial \( R \)-ideal \( I \) with \( \text{mono}(I) = M \) iff \( M \) is not Gorenstein (iff \( M \) is not of the form \( m^b \) for \( b \in \mathbb{N}^n \)).

**Proof.** \( \Rightarrow \): If \( M = \text{mono}(I) \) were Gorenstein, then by Corollary 4.5 \( I \) is necessarily of the form \( m^b \), contradicting the hypothesis that \( I \) is non-monomial.

\( \Leftarrow \): Since \( M \) is not Gorenstein, there exist monomials \( u_1 \neq u_2 \) in the socle of \( R/M \). Then \( u_1 \in M : m \Rightarrow m \subseteq M : u_1 \Rightarrow m = M : u_1 \), and similarly \( m = M : u_2 \). By Lemma 4.1 \( I := M + (u_1 + u_2) \) is a non-monomial ideal with \( \text{mono}(I) = M \).

As evidenced by Remark 4.3, finding \( \text{mono}(M + (f)) \) can be subtle, for arbitrary \( f \in R \). There is one situation however which can be determined completely:

**Theorem 4.7.** Let \( M \) be a monomial ideal, and let \( u_1, \ldots, u_r \) be the socle monomials of \( R/M \). Let \( f_j := \sum_{i=1}^r a_{ij} u_i \), \( 1 \leq j \leq s \), be \( k \)-linear combinations of the \( u_i \). Then \( \text{mono}(M + (f_1, \ldots, f_s)) = M \) iff no standard basis vector \( e_i \) is in the column span of the matrix \( (a_{ij}) \) over \( k \).

**Proof.** Let \( v \in \text{mono}(M + (f_1, \ldots, f_s)) \) be a monomial. Pick \( g_i \in R \) and \( m \in M \) such that \( v = m + \sum_{j=1}^s g_j f_j \). Write \( g_j = b_j + g'_j \) where \( b_j \in k \) and \( g'_j \in m \).

Since \( f_j \in \text{soc}(R/M) \), this is the same as saying \( v = \sum_{j=1}^s b_j f_j \) in \( R/M \). Since \( v \) is a monomial, it must appear as one of the terms in the sum, hence \( v \) must be a socle monomial of \( M \). Then \( v = u_i \) for some \( i \), so \( v \) corresponds to a standard basis vector \( e_i \), and then writing \( v \) as a \( k \)-linear combination of \( f_j \) is equivalent to writing \( e_i \) as a \( k \)-linear combination of the columns of \( (a_{ij}) \).

**Example 4.8.** Let \( R = k[x, y, z] \), and set \( M := (x^2, xy, xz, y^2, z^2) \). Then \( R/M \) has a 2-dimensional socle \( k(x, yz) \), so \( \text{mono}(M + (x + yz)) = M \) by Lemma 4.1 or Theorem 4.7. However, the only standard monomials of \( M \) of the same degrees are \( x, y, z \), which have distinct colons into \( M \). Thus by Theorem 4.4 there is no graded non-monomial \( I \) with \( \text{mono}(I) = M \).

In general, even if there are \( u_1, u_2 \) of the same degree with \( M : u_1 = M : u_2 \), there may not be any such in top degree: e.g. \( (x^2, y^2)m + (z^3) \) is equi-generated with symmetric Hilbert function \( 1, 3, 6, 3, 1 \), but is not level (hence not Gorenstein).

**Example 4.9.** For an exponent vector \( b \in \mathbb{N}^n \) with \( b_j \geq 2 \) for all \( i \), the irreducible ideal \( m^b \) has a unique socle element \( x^{b-1} := x_1^{b_1-1} \ldots x_n^{b_n-1} \). Let \( M := m^b + (x^{b-1}) \), which is Artinian level with \( n \)-dimensional socle \( \langle \frac{x_i}{x_i} | 1 \leq i \leq n \rangle =: \langle s_1, \ldots, s_n \rangle \).

By setting all socle elements of \( M \) equal to each other we obtain a graded ideal \( I := M + (s_1 - s_i | 2 \leq i \leq n) \). As all the non-monomial generators are in the socle of \( R/M \), we may apply Theorem 4.7 the coefficient matrix \( (a_{ij}) \) is given by

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
-1 & 0 & \ldots & 0 \\
0 & -1 & \ldots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -1
\end{pmatrix}
\]
so by Theorem 4.7, \( \text{mono}(I) = M. \) For \( b = (2, 2, 3, 3) \), the Betti tables of \( R/I \) and \( R/\text{mono}(I) \) respectively are:

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
\text{total:} & 1 & 7 & 17 & 16 & 5
\end{array}
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
\text{total:} & 1 & 5 & 10 & 10 & 4
\end{array}
\begin{array}{cccc}
0: & 1 & \ldots & \ldots & \\
1: & 2 & \ldots & \ldots & \\
2: & 2 & 1 & \ldots & \\
3: & \ldots & 4 & \ldots & \\
4: & 3 & 12 & 16 & 4
\end{array}
\begin{array}{cccc}
4: & \ldots & 1 & 2 & \\
5: & \ldots & 1 & 4 & 8 & 4
\end{array}
\]

Notice that the (total) Betti numbers of \( R/I \) are strictly greater than those of \( R/\text{mono}(I) \). This shows that the answer to Question 7 is false in general.

Finally, we include a criterion for recognizing when a monomial subideal of \( I \) is equal to \( \text{mono}(I) \), in terms of its socle monomials:

**Proposition 4.10.** Let \( I \) be an \( R \)-ideal and \( M \subseteq I \) an Artinian monomial ideal. Then the following are equivalent:

(a) \( M = \text{mono}(I) \)
(b) \( I \) contains no socle monomials of \( M \)
(c) \( (M : m) \cap \text{mono}(I) \subseteq M \)

**Proof.** (a) \( \implies \) (b), (a) \( \implies \) (c): Clear.

(b) \( \iff \) (c): Notice that (b) is equivalent to: any monomial \( u \in (M : m) \setminus M \) is not in \( I \); or alternatively, any monomial in both \( M : m \) and \( I \) is also in \( M \); i.e. \( \text{mono}((M : m) \cap I) \subseteq M \). Now apply Proposition 1.1(3). \( \square \)

**References**

[1] Grayson, Daniel R.; Stillman, Michael E. *Macaulay2, a software system for research in algebraic geometry.* Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/)

[2] Miller, Ezra. Finding all monomials in a polynomial ideal. [arXiv:1605.08791](https://arxiv.org/abs/1605.08791)

[3] Miller, Ezra; Sturmfels, Bernd. Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

[4] Polini, Claudia; Ulrich, Bernd; Vitulli, Marie A. The core of zero-dimensional monomial ideals. *Adv. Math.* 211 (2007), no. 1, 72–93.

[5] Saito, Mutsumi; Sturmfels, Bernd; Takayama, Nobuki. Gröbner deformations of hypergeometric differential equations. Algorithms and Computation in Mathematics, 6. Springer-Verlag, Berlin, 2000.

Department of Mathematics, University of California, Berkeley, California, 94720 U.S.A

E-mail address: jchen@math.berkeley.edu