A duality between pairs of split decompositions for a $Q$-polynomial distance-regular graph

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Abstract

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $D \geq 3$ and standard module $V$. Recently Ito and Terwilliger introduced four direct sum decompositions of $V$; we call these the $(\mu, \nu)$–split decompositions of $V$, where $\mu, \nu \in \{\downarrow, \uparrow\}$. In this paper we show that the $(\downarrow, \downarrow)$–split decomposition and the $(\uparrow, \uparrow)$–split decomposition are dual with respect to the standard Hermitian form on $V$. We also show that the $(\downarrow, \uparrow)$–split decomposition and the $(\uparrow, \downarrow)$–split decomposition are dual with respect to the standard Hermitian form on $V$.

Keywords. Distance-regular graph, tridiagonal pair, subconstituent algebra, split decomposition.

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1 Introduction

We consider a distance-regular graph $\Gamma$ with vertex set $X$ and diameter $D \geq 3$ (see Section 4 for formal definitions). We assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Let $V$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We call $V$ the standard module. We endow $V$ with the Hermitian form $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \overline{v}$ for $u, v \in V$. We call this form the standard Hermitian form on $V$. Recently Ito and Terwilliger introduced four direct sum decompositions of $V$ [16]; we call these the $(\mu, \nu)$–split decompositions of $V$, where $\mu, \nu \in \{\downarrow, \uparrow\}$. These are defined as follows. Fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ that represents the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$. For $-1 \leq i, j \leq D$ we define

\begin{align*}
V_{i,j}^{\downarrow\downarrow} &= (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V), \\
V_{i,j}^{\downarrow\uparrow} &= (E_0^* V + \cdots + E_D^* V) \cap (E_0 V + \cdots + E_j V), \\
V_{i,j}^{\uparrow\downarrow} &= (E_0 V + \cdots + E_i^* V) \cap (E_D V + \cdots + E_D^* V), \\
V_{i,j}^{\uparrow\uparrow} &= (E_D^* V + \cdots + E_D^* V) \cap (E_D V + \cdots + E_D^* V).
\end{align*}
For $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$ we have $V_{i-1,j}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$ and $V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$; therefore $V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$. Let $\tilde{V}_{i,j}^{\mu\nu}$ denote the orthogonal complement of $V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu}$ in $V_{i,j}^{\mu\nu}$ with respect to the standard Hermitian form. By [16, Lemma 10.3],

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \quad \text{(direct sum).}$$

We call the above sum the $(\mu, \nu)$–split decomposition of $V$ with respect to $x$. We show that with respect to the standard Hermitian form the $(\downarrow, \downarrow)$–split decomposition (resp. $(\downarrow, \uparrow)$–split decomposition) and the $(\uparrow, \uparrow)$–split decomposition (resp. $(\uparrow, \downarrow)$–split decomposition) are dual in the following sense.

**Theorem 1.1** With the above notation, the following (i), (ii) hold for $0 \leq i, j, r, s \leq D$.

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow}$ and $\tilde{V}_{r,s}^{\downarrow\downarrow}$ are orthogonal unless $i + r = D$ and $j + s = D$.

(ii) $\tilde{V}_{i,j}^{\downarrow\uparrow}$ and $\tilde{V}_{r,s}^{\downarrow\uparrow}$ are orthogonal unless $i + r = D$ and $j + s = D$.

To prove Theorem 1.1 we use a result about tridiagonal pairs (Theorem 3.6) which may be of independent interest. We also use some results about the subconstituent algebra of $\Gamma$.

## 2 Tridiagonal pairs

We recall the notion of a tridiagonal pair \cite{13}. We will use the following terms. Let $V$ denote a vector space over $\mathbb{C}$ with finite positive dimension. By a *linear transformation* on $V$ we mean a $\mathbb{C}$-linear map from $V$ to $V$. Let $A$ denote a linear transformation on $V$. By an *eigenspace* of $A$ we mean a nonzero subspace of $V$ of the form

$$\{v \in V \mid Av = \theta v\},$$

where $\theta \in \mathbb{C}$. We say that $A$ is diagonalizable whenever $V$ is spanned by the eigenspaces of $A$. In this case $V$ is the direct sum of the eigenspaces of $A$.

**Definition 2.1** \cite{13} Definition 1.1] Let $V$ denote a vector space over $\mathbb{C}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on $V$ we mean an ordered pair $A, A^*$ of linear transformations on $V$ that satisfy the following four conditions.

(i) $A$ and $A^*$ are both diagonalizable on $V$.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of $A$ such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$. 

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(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of $A^*$ such that
\[ AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \]
where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There is no subspace $W$ of $V$ such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

Note 2.2 According to a common notational convention $A^*$ denotes the conjugate-transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

With reference to Definition 2.1 we have $d = \delta$ [13, Lemma 4.5]; we call this common value the diameter of $A, A^*$. See [13, 14, 15] for more information on tridiagonal pairs.

With reference to Definition 2.1 by the construction we have the direct sum decompositions $V = \sum_{i=0}^d V_i$ and $V = \sum_{i=0}^d V_i^*$. We now recall four more direct sum decompositions of $V$ called the split decompositions.

Lemma 2.3 [15, Lemma 4.2] With reference to Definition 2.1 for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have
\[ V = \sum_{i=0}^d U_{i}^{\mu\nu} \] (direct sum),
where
\[
\begin{align*}
U_{i}^{\downarrow\downarrow} & = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}), \\
U_{i}^{\uparrow\downarrow} & = (V_{d-i}^* + \cdots + V_d^*) \cap (V_0 + \cdots + V_{d-i}), \\
U_{i}^{\downarrow\uparrow} & = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \\
U_{i}^{\uparrow\uparrow} & = (V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_{d-i}).
\end{align*}
\]

3 Hermitian forms

In this section we consider a tridiagonal pair for which the underlying vector space supports a certain Hermitian form. We start with the definition of a Hermitian form. Throughout this section $V$ denotes a vector space over $\mathbb{C}$ with finite positive dimension. For $\alpha \in \mathbb{C}$ let $\overline{\alpha}$ denote the complex conjugate of $\alpha$.

Definition 3.1 By a Hermitian form on $V$ we mean a function $(\cdot, \cdot) : V \times V \to \mathbb{C}$ such that for all $u, v, w$ in $V$ and all $\alpha \in \mathbb{C}$,
\[
\begin{align*}
(i) \quad (u + v, w) & = (u, w) + (v, w), \\
(ii) \quad (\alpha u, v) & = \alpha (u, v), \\
(iii) \quad (v, u) & = (u, v).
\end{align*}
\]
Definition 3.2 Let \((, )\) denote a Hermitian form on \(V\). By Definition 3.1(iii) we have \((v, v) \in \mathbb{R}\) for \(v \in V\). We say that \((, )\) is positive definite whenever \((v, v) > 0\) for all nonzero \(v \in V\).

**Lemma 3.3** Let \((, )\) denote a positive definite Hermitian form on \(V\). Suppose that we are given a linear transformation \(A : V \rightarrow V\) satisfying

\[
(Au, v) = (u, Av), \quad u, v \in V.
\]

Then all the eigenvalues of \(A\) are in \(\mathbb{R}\).

**Proof:** Let \(\lambda\) denote an eigenvalue of \(A\). We show that \(\lambda \in \mathbb{R}\). Since \(\mathbb{C}\) is algebraically closed there exists a nonzero \(v \in V\) such that \(Av = \lambda v\). By (1) \((Av, v) = (v, Av)\). Evaluating this using Definition 3.1(ii),(iii) we have \((\lambda - \overline{\lambda})(v, v) = 0\). But \((v, v) \neq 0\) since \((, )\) is positive definite so \(\lambda = \overline{\lambda}\). Therefore \(\lambda \in \mathbb{R}\). 

\(\square\)

**Assumption 3.4** Let \(A, A^*\) denote a tridiagonal pair on \(V\) as in Definition 2.1. For \(0 \leq i \leq d\) let \(\theta_i\) (resp. \(\theta_i^*\)) denote the eigenvalue of \(A\) (resp. \(A^*\)) associated with \(V_i\) (resp. \(V_i^*\)). We remark that \(\theta_0, \theta_1, \ldots, \theta_d\) are mutually distinct and \(\theta_0^*, \theta_1^*, \ldots, \theta_d^*\) are mutually distinct. We assume that there exists a positive definite Hermitian form \((, )\) on \(V\) satisfying

\[
(Au, v) = (u, Av), \quad u, v \in V, \quad (A^*u, v) = (u, A^*v), \quad u, v \in V.
\]

**Lemma 3.5** With reference to Assumption 3.4 the following (i), (ii) hold.

(i) The eigenspaces \(V_0, V_1, \ldots, V_d\) are mutually orthogonal with respect to \((, )\).

(ii) The eigenspaces \(V_0^*, V_1^*, \ldots, V_d^*\) are mutually orthogonal with respect to \((, )\).

**Proof:** (i) For distinct \(i, j\) \((0 \leq i, j \leq d)\) and for \(u \in V_i, v \in V_j\) we show that \((u, v) = 0\). By (2) \((Au, v) = (u, Av)\). Evaluating this using Definition 3.1(ii),(iii) we find \((\theta_i - \theta_j)(u, v) = 0\). But \(\theta_j = \theta_j\) by Lemma 3.3 and \(\theta_i \neq \theta_j\) so \((u, v) = 0\).

(ii) Similar to the proof of (i). 

\(\square\)

**Theorem 3.6** With reference to Lemma 2.3 and Assumption 3.4 the following (i), (ii) hold for \(0 \leq i, j \leq d\) such that \(i + j \neq d\).

(i) The subspaces \(U_i^\perp\) and \(U_j^\perp\) are orthogonal with respect to \((, )\).

(ii) The subspaces \(U_i^\perp\) and \(U_j^\perp\) are orthogonal with respect to \((, )\).

**Proof:** (i) We consider two cases: \(i + j < d\) and \(i + j > d\). First suppose that \(i + j < d\). By Lemma 2.3 \(U_i^\perp \subseteq V_0^* + \cdots + V_i^*\) and \(U_j^\perp \subseteq V_{d-j}^* + \cdots + V_d^*\). Observe that \(V_0^* + \cdots + V_i^*\) is orthogonal to \(V_{d-j}^* + \cdots + V_d^*\) by Lemma 3.5(ii) and since \(i < d - j\). Therefore \(U_i^\perp\) is orthogonal to \(U_j^\perp\). Next suppose that \(i + j > d\). By Lemma 2.3 \(U_i^\perp \subseteq V_0 + \cdots + V_{d-i}\) and \(U_j^\perp \subseteq V_j + \cdots + V_d\). Observe that \(V_0 + \cdots + V_{d-i}\) is orthogonal to \(V_j + \cdots + V_d\) by Lemma 3.5(i) and since \(d - i < j\). Therefore \(U_i^\perp\) is orthogonal to \(U_j^\perp\).

(ii) Similar to the proof of (i). 

\(\square\)
4 Distance-regular graphs

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1], [3], [11] and [20].

Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe that $\text{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the standard module. We endow $V$ with the Hermitian form $\langle \cdot , \cdot \rangle$ that satisfies $\langle u, v \rangle = u^\top \overline{\tau} v$ for $u, v \in V$, where $t$ denotes transpose. Observe that $\langle \cdot , \cdot \rangle$ is positive definite. We call this form the standard Hermitian form on $V$. Observe that for $B \in \text{Mat}_X(\mathbb{C})$,

$$\langle Bu, v \rangle = \langle u, \overline{B} v \rangle \quad \quad u, v \in V. \quad (4)$$

For all $y \in X$, let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. Observe that $\{ \hat{y} \mid y \in X \}$ is an orthonormal basis for $V$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D := \max \{ \partial(x, y) \mid x, y \in X \}$. We call $D$ the diameter of $\Gamma$. We say that $\Gamma$ is distance-regular whenever for all integers $h, i, j \ (0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = | \{ z \in X \mid \partial(x, z) = i, \partial(z, y) = j \} |$$

is independent of $x$ and $y$. The $p_{ij}^h$ are called the intersection numbers of $\Gamma$.

For the rest of this paper we assume that $\Gamma$ is distance-regular with diameter $D \geq 3$.

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $xy$ entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. We abbreviate $A := A_1$ and call this the adjacency matrix of $\Gamma$. We observe that (i) $A_0 = \mathbf{I}$; (ii) $\sum_{i=0}^D A_i = \mathbf{J}$; (iii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (iv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (v) $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$ $(0 \leq i, j \leq D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1’s matrix) in $\text{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$. We call $M$ the Bose-Mesner algebra of $\Gamma$. It turns out that $A$ generates $M$ [11, p. 190]. By [4] and since $A$ is real symmetric,

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \quad u, v \in V. \quad (5)$$

By [31, p. 45], $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1} J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $E_i^t = E_i$ $(0 \leq i \leq D)$; (iv) $E_i^t = E_i$ $(0 \leq i \leq D)$; (v) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$. We call $E_0, E_1, \ldots, E_D$ the primitive idempotents of $\Gamma$.

We recall the eigenvalues of $\Gamma$. Since $E_0, E_1, \ldots, E_D$ form a basis for $M$ there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^D \theta_i E_i$. Observe that $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. 

We call \( \theta_i \) the \textit{eigenvalue} of \( \Gamma \) associated with \( E_i \) (\( 0 \leq i \leq D \)). By Lemma 3.3 and (5) the eigenvalues \( \theta_0, \theta_1, \ldots, \theta_D \) are in \( \mathbb{R} \). Observe that \( \theta_0, \theta_1, \ldots, \theta_D \) are mutually distinct since \( A \) generates \( M \). Observe that

\[
V = E_0V + E_1V + \cdots + E_DV
\]

(orthogonal direct sum).

For \( 0 \leq i \leq D \) the space \( E_iV \) is the eigenspace of \( A \) associated with \( \theta_i \).

We now recall the Krein parameters. Let \( \circ \) denote the entrywise product in \( \text{Mat}_X(\mathbb{C}) \). Observe that \( A_i \circ A_j = \delta_{ij}A_i \) for \( 0 \leq i, j \leq D \), so \( M \) is closed under \( \circ \). Thus there exist complex scalars \( q_{ij}^h \) \((0 \leq h, i, j \leq D)\) such that

\[
E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D).
\]

By [2, p. 170], \( q_{ij}^h \) is real and nonnegative for \( 0 \leq h, i, j \leq D \). The \( q_{ij}^h \) are called the \textit{Krein parameters}. The graph \( \Gamma \) is said to be \textit{Q-polynomial} (with respect to the given ordering \( E_0, E_1, \ldots, E_D \) of the primitive idempotents) whenever for \( 0 \leq h, i, j \leq D \), \( q_{ij}^h = 0 \) (resp. \( q_{ij}^h \neq 0 \)) whenever one of \( h, i, j \) is greater than (resp. equal to) the sum of the other two [3, p. 59]. See [1, 4, 5, 7, 13] for more information on the \( Q \)-polynomial property. From now on we assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E_0, E_1, \ldots, E_D \).

We recall the dual Bose-Mesner algebra of \( \Gamma \). Fix a vertex \( x \in X \). We view \( x \) as a “base vertex.” For \( 0 \leq i \leq D \) let \( E_i^* = E_i^*(x) \) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{C}) \) with \( yy \) entry

\[
(E_i^*)_{yy} = \begin{cases} 
1, & \text{if } \partial(x, y) = i \\
0, & \text{if } \partial(x, y) \neq i
\end{cases} \quad (y \in X).
\]

(6)

We call \( E_i^* \) the \( i \)th \textit{dual idempotent} of \( \Gamma \) with respect to \( x \) [20, p. 378]. We observe that (i) \( \sum_{i=0}^{D} E_i^* = I \); (ii) \( \overline{E_i^*} = E_i^* \) \((0 \leq i \leq D)\); (iii) \( E_i^{*t} = E_i^* \) \((0 \leq i \leq D)\); (iv) \( E_i^* E_j^* = \delta_{ij} E_i^* \) \((0 \leq i, j \leq D)\). By these facts \( E_0^*, E_1^*, \ldots, E_D^* \) form a basis for a commutative subalgebra \( M^* = M^*(x) \) of \( \text{Mat}_X(\mathbb{C}) \). We call \( M^* \) the \textit{dual Bose-Mesner algebra} of \( \Gamma \) with respect to \( x \) [20, p. 378]. For \( 0 \leq i \leq D \) let \( A_i^* = A_i^*(x) \) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{C}) \) with \( yy \) entry \( (A_i^*)_{yy} = |X|(E_i)_{yy} \) for \( y \in X \). Then \( A_0^*, A_1^*, \ldots, A_D^* \) is a basis for \( M^* \) [20, p. 379]. Moreover (i) \( A_0^* = I \); (ii) \( \overline{A_i^*} = A_i^* \) \((0 \leq i \leq D)\); (iii) \( A_i^{*t} = A_i^* \) \((0 \leq i \leq D)\); (iv) \( A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \) \((0 \leq i, j \leq D)\) [20, p. 379]. We call \( A_0^*, A_1^*, \ldots, A_D^* \) the \textit{dual distance matrices} of \( \Gamma \) with respect to \( x \). The matrix \( A^* \) generates \( M^* \) [20, Lemma 3.11]. By (4) and since \( A^* \) is real symmetric,

\[
\langle A^* u, v \rangle = \langle u, A^* v \rangle \quad u, v \in V.
\]

(7)

We recall the dual eigenvalues of \( \Gamma \). Since \( E_0^*, E_1^*, \ldots, E_D^* \) form a basis for \( M^* \) and since \( A^* \) is real, there exist real scalars \( \theta_0^*, \theta_1^*, \ldots, \theta_D^* \) such that \( A^* = \sum_{i=0}^{D} \theta_i^* E_i^* \). Observe that \( A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \) for \( 0 \leq i \leq D \). We call \( \theta_i^* \) the \textit{dual eigenvalue} of \( \Gamma \) associated with \( E_i^* \) \((0 \leq i \leq D)\). Observe that \( \theta_0^*, \theta_1^*, \ldots, \theta_D^* \) are mutually distinct since \( A^* \) generates \( M^* \).
We recall the subconstituents of \( \Gamma \). From (6) we find
\[
E^*_i V = \operatorname{span}\{ \hat{y} \mid y \in X, \quad \partial(x, y) = i \} \quad (0 \leq i \leq D).
\] (8)
By (8) and since \( \{ \hat{y} \mid y \in X \} \) is an orthonormal basis for \( V \) we find
\[
V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \quad \text{(orthogonal direct sum)}.
\]
For \( 0 \leq i \leq D \) the space \( E^*_i V \) is the eigenspace of \( A^* \) associated with \( \theta^*_i \). We call \( E^*_i V \) the \( i \)th subconstituent of \( \Gamma \) with respect to \( x \).

We recall the subconstituent algebra of \( \Gamma \). Let \( T = T(x) \) denote the subalgebra of \( \text{Mat}_X(\mathbb{C}) \) generated by \( M \) and \( M^* \). We call \( T \) the subconstituent algebra (or Terwilliger algebra) of \( \Gamma \) with respect to \( x \) \cite{20, Definition 3.3}. We observe that \( T \) is generated by \( A, A^* \). We observe that \( T \) has finite dimension. Moreover \( T \) is semi-simple since it is closed under the conjugate transpose map \cite{8, p. 157}. See \cite{6, 7, 9, 10, 12, 19, 20, 21, 22} for more information on the subconstituent algebra.

For the rest of this paper we adopt the following notational convention.

**Notation 4.1** We assume that \( \Gamma = (X, R) \) is a distance-regular graph with diameter \( D \geq 3 \). We assume that \( \Gamma \) is \( Q \)-polynomial with respect to the ordering \( E_0, E_1, \ldots, E_D \) of the primitive idempotents. We fix \( x \in X \) and write \( A^* = A^*(x), \ E^*_i = E^*_i(x) \ (0 \leq i \leq D), \ T = T(x) \). We abbreviate \( V = \mathbb{C}^X \). For notational convenience we define \( E_{-1} = 0, \ E_{D+1} = 0 \) and \( E^*_{-1} = 0, \ E^*_{D+1} = 0 \).

We finish this section with a comment.

**Lemma 4.2** \cite{20, Lemma 3.2} With reference to Notation 4.1 the following (i), (ii) hold for \( 0 \leq i \leq D \).

(i) \( A E^*_i V \subseteq E^*_{i-1} V + E^*_i V + E^*_{i+1} V \).

(ii) \( A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \).

5 The irreducible \( T \)-modules

In this section we recall some useful results on \( T \)-modules.

With reference to Notation 4.1 by a \( T \)-module we mean a subspace \( W \subseteq V \) such that \( BW \subseteq W \) for all \( B \in T \). Let \( W \) denote a \( T \)-module. Then \( W \) is said to be irreducible whenever \( W \) is nonzero and \( W \) contains no \( T \)-modules other than 0 and \( W \).

Let \( W \) denote a \( T \)-module and let \( W' \) denote a \( T \)-module contained in \( W \). Then the orthogonal complement of \( W' \) in \( W \) is a \( T \)-module \cite{10, p. 802}. It follows that each \( T \)-module is an orthogonal direct sum of irreducible \( T \)-modules. In particular \( V \) is an orthogonal direct sum of irreducible \( T \)-modules.

Let \( W \) denote an irreducible \( T \)-module. By the endpoint of \( W \) we mean \( \min\{i | 0 \leq i \leq D, \ E_i^* W \neq 0 \} \). By the diameter of \( W \) we mean \( |\{i | 0 \leq i \leq D, \ E_i^* W \neq 0 \}| - 1 \). By the dual
endpoint of $W$ we mean $\min\{i|0 \leq i \leq D, \ E_iW \neq 0\}$. By the dual diameter of $W$ we mean $|\{i|0 \leq i \leq D, \ E_iW \neq 0\}| - 1$. The diameter of $W$ is equal to the dual diameter of $W$ [17, Corollary 3.3].

**Lemma 5.1** [20, Lemma 3.4, Lemma 3.9, Lemma 3.12] With reference to Notation [4.1], let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then $\rho, \tau, d$ are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)–(iv) hold.

(i) $E_{\rho}^*W \neq 0$ if and only if $\rho \leq i \leq \rho + d$ \hspace{0.5cm} (0 \leq i \leq D).

(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^* W$ \hspace{0.5cm} (orthogonal direct sum).

(iii) $E_{\tau}^*W \neq 0$ if and only if $\tau \leq i \leq \tau + d$ \hspace{0.5cm} (0 \leq i \leq D).

(iv) $W = \sum_{h=0}^{d} E_{\tau+h}^* W$ \hspace{0.5cm} (orthogonal direct sum).

**Lemma 5.2** [23, Lemma 3.2] With reference to Notation [4.1], let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then the following (i), (ii) hold for $0 \leq i \leq d$.

(i) $AE_{\rho+i}^* W \subseteq E_{\rho+i-1}^* W + E_{\rho+i}^* W + E_{\rho+i+1}^* W$.

(ii) $A^* E_{\tau+i} W \subseteq E_{\tau+i-1} W + E_{\tau+i} W + E_{\tau+i+1} W$.

**Remark 5.3** With reference to Notation [4.1] let $W$ denote an irreducible $T$-module. Then $A$ and $A^*$ act on $W$ as a tridiagonal pair in the sense of Definition [2.1]. This follows from Lemma 5.1, Lemma 5.2, and since $A, A^*$ together generate $T$.

**Lemma 5.4** With reference to Notation [4.1], let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have

$$W = \sum_{h=0}^{d} W_{h}^{\mu\nu} \hspace{0.5cm} \text{(direct sum),}$$

where for $0 \leq h \leq d$,

\begin{align*}
W_{h}^{\downarrow\downarrow} &= (E_{\rho}^*W + \cdots + E_{\rho+h}^* W) \cap (E_{\tau}W + \cdots + E_{\tau+d-h}W), \\
W_{h}^{\downarrow\uparrow} &= (E_{\rho+d-h}^*W + \cdots + E_{\rho+d}^* W) \cap (E_{\tau}W + \cdots + E_{\tau+d-h}W), \\
W_{h}^{\uparrow\downarrow} &= (E_{\rho}^*W + \cdots + E_{\rho+h}^* W) \cap (E_{\tau+h}W + \cdots + E_{\tau+d}W), \\
W_{h}^{\uparrow\uparrow} &= (E_{\rho+d-h}^*W + \cdots + E_{\rho+d}^* W) \cap (E_{\tau+h}W + \cdots + E_{\tau+d}W).
\end{align*}

**Proof:** Immediate from Lemma [2.3] and Remark [5.3] \qqed

We remark that the sum (9) is not orthogonal in general. However we do have the following result.
Lemma 5.5 With reference to Notation 4.1, let $W$ denote an irreducible $T$-module with diameter $d$. Then the following (i), (ii) hold for $0 \leq h, \ell \leq d$ such that $h + \ell \neq d$.

(i) The subspaces $W_{h \downarrow \downarrow}$ and $W_{\ell \uparrow \uparrow}$ are orthogonal with respect to the standard Hermitian form.

(ii) The subspaces $W_{h \downarrow \uparrow}$ and $W_{\ell \uparrow \downarrow}$ are orthogonal with respect to the standard Hermitian form.

Proof: Combine Theorem 3.6, (5), (7), Remark 5.3, and Lemma 5.4. 

6 The split decompositions of the standard module

In this section we recall the four split decompositions for the standard module and discuss their basic properties.

Definition 6.1 [16, Definition 10.1] With reference to Notation 4.1, for $-1 \leq i, j \leq D$ we define

$$
V_{i,j}^{\downarrow \downarrow} = (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_j V), \\
V_{i,j}^{\downarrow \uparrow} = (E_D^* V + \cdots + E_{D-i}^* V) \cap (E_0 V + \cdots + E_j V), \\
V_{i,j}^{\uparrow \downarrow} = (E_0^* V + \cdots + E_i^* V) \cap (E_D V + \cdots + E_{D-j} V), \\
V_{i,j}^{\uparrow \uparrow} = (E_D^* V + \cdots + E_{D-i}^* V) \cap (E_D V + \cdots + E_{D-j} V).
$$

In each of the above four equations we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

Definition 6.2 [16, Definition 10.2] With reference to Notation 4.1 and Definition 6.1, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$ we have $V_{i-1,j}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}$ and $V_{i,j-1}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}$. Therefore

$$
V_{i-1,j}^{\mu \nu} + V_{i,j-1}^{\mu \nu} \subseteq V_{i,j}^{\mu \nu}.
$$

Referring to the above inclusion, we define $\tilde{V}_{i,j}^{\mu \nu}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is

$$
\tilde{V}_{i,j}^{\mu \nu} = (V_{i-1,j}^{\mu \nu} + V_{i,j-1}^{\mu \nu})^\perp \cap V_{i,j}^{\mu \nu}.
$$

Lemma 6.3 [16, Lemma 10.3] With reference to Notation 4.1 and Definition 6.2, the following holds for $\mu, \nu \in \{\downarrow, \uparrow\}$:

$$
V = \sum_{i=0}^D \sum_{j=0}^D \tilde{V}_{i,j}^{\mu \nu} \quad \text{(direct sum).} \quad (10)
$$

Definition 6.4 We call the sum (10) the $(\mu, \nu)$–split decomposition of $V$ with respect to $x$.

Remark 6.5 The decomposition (10) is not orthogonal in general.
Lemma 6.6 With reference to Notation 4.1, let \( W \) denote an irreducible \( T \)-module with endpoint \( \rho \), dual endpoint \( \tau \), and diameter \( d \). Then for \( 0 \leq h \leq d \) and \( 0 \leq i, j \leq D \) the following (i)--(iv) hold.

(i) \( W_{i,j}^{11} \subseteq \tilde{V}_{i,j}^{11} \) if and only if \( i = \rho + h \) and \( j = \tau + d - h \).

(ii) \( W_{i,j}^{11} \subseteq \tilde{V}_{i,j}^{11} \) if and only if \( i = D - \rho - d + h \) and \( j = \tau + d - h \).

(iii) \( W_{i,j}^{11} \subseteq \tilde{V}_{i,j}^{11} \) if and only if \( i = \rho + h \) and \( j = D - \tau - h \).

(iv) \( W_{i,j}^{11} \subseteq \tilde{V}_{i,j}^{11} \) if and only if \( i = D - \rho - d + h \) and \( j = D - \tau - h \).

Proof: Immediate from [16, Lemma 11.4] and (10).

\[ \Box \]

Lemma 6.7 With reference to Notation 4.1, fix an orthogonal direct sum decomposition of the standard module \( V \) of \( \Gamma \) into irreducible \( T \)-modules:

\[ V = \sum_{w} W. \tag{11} \]

Then the following (i)--(iv) hold for \( 0 \leq i, j \leq D \).

(i) \( \tilde{V}_{i,j}^{11} = \sum W_{i,j}^{11} \), where the sum is over all ordered pairs \( (W, h) \) such that \( W \) is assumed in (11) with endpoint \( \rho \leq i \), dual endpoint \( \tau = i + j - \rho - d \), diameter \( d \geq i - \rho \), and \( h = i - \rho \).

(ii) \( \tilde{V}_{i,j}^{11} = \sum W_{i,j}^{11} \), where the sum is over all ordered pairs \( (W, h) \) such that \( W \) is assumed in (11) with endpoint \( \rho \leq D - i \), dual endpoint \( \tau = i + j + \rho - D \), diameter \( d \geq D - \rho - i \), and \( h = \rho + d - D + i \).

(iii) \( \tilde{V}_{i,j}^{11} = \sum W_{i,j}^{11} \), where the sum is over all ordered pairs \( (W, h) \) such that \( W \) is assumed in (11) with endpoint \( \rho \leq i \), dual endpoint \( \tau = \rho + D - i - j \), diameter \( d \geq i - \rho \), and \( h = i - \rho \).

(iv) \( \tilde{V}_{i,j}^{11} = \sum W_{i,j}^{11} \), where the sum is over all ordered pairs \( (W, h) \) such that \( W \) is assumed in (11) with endpoint \( \rho \leq D - i \), dual endpoint \( \tau = 2D - \rho - d - i - j \), diameter \( d \geq D - \rho - i \), and \( h = \rho + d - D + i \).

Proof: (i) For \( 0 \leq i, j \leq D \) define

\[ v_{i,j} = \sum W_{i,j}^{11}, \tag{12} \]

where the sum is over all ordered pairs \( (W, h) \) such that \( W \) is assumed in (11) with endpoint \( \rho \leq i \), dual endpoint \( \tau = i + j - \rho - d \), diameter \( d \geq i - \rho \), and \( h = i - \rho \). We show that \( \tilde{V}_{i,j}^{11} = v_{i,j} \). We first show that \( V_{i,j}^{11} \geq v_{i,j} \). Let \( W_{i,j}^{11} \) denote one of the terms in the sum on the right in (12). We show that \( W_{i,j}^{11} \) is contained in \( V_{i,j}^{11} \). Let \( \rho, \tau, d \) denote the endpoint, dual endpoint, and diameter of \( W \), respectively. By construction \( \tau = i + j - \rho - d \) and \( h = i - \rho \).
Subtracting the second equation from the first equation we find $j = \tau + d - h$. Now $W_{h}^{\downarrow\downarrow}$ is contained in $V_{i,j}^{\downarrow\downarrow}$ by Lemma 6.1(i). We have now shown that $V_{i,j}^{\downarrow\downarrow} \supseteq v_{i,j}$. We can now easily show that $V_{i,j}^{\downarrow\downarrow} = v_{i,j}$. Expanding the sum (11) using Lemma 5.4 we get

$$V = \sum_{W} W \quad \text{(direct sum)}$$

$$= \sum_{W} \sum_{h} W_{h}^{\downarrow\downarrow} \quad \text{(direct sum)},$$

where the second sum is over the integer $h$ from 0 to the diameter of $W$. In the above sum we change the order of summation to get

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} v_{i,j} \quad \text{(direct sum)},$$

where the third sum is over all ordered pairs $(W, h)$ such that $W$ is assumed in (11) with endpoint $\rho \leq i$, dual endpoint $\tau = i + j - \rho - d$, diameter $d \geq i - \rho$, and $h = i - \rho$. In other words,

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} v_{i,j} \quad \text{(direct sum).}$$

By this, (10), and since $V_{i,j}^{\downarrow\downarrow} \supseteq v_{i,j}$ for $0 \leq i, j \leq D$, we find $V_{i,j}^{\downarrow\downarrow} = v_{i,j}$ for $0 \leq i, j \leq D$. (ii), (iii), (iv) Similar to the proof of (i). \hfill \Box

Now we have the main result.

**Theorem 6.8** With reference to Notation 4.1 and Definition 6.2, the following (i), (ii) hold for $0 \leq i, j, r, s \leq D$.

(i) $V_{i,j}^{\downarrow\downarrow}$ and $V_{r,s}^{\uparrow\uparrow}$ are orthogonal unless $i + r = D$ and $j + s = D$.

(ii) $V_{i,j}^{\downarrow\downarrow}$ and $V_{r,s}^{\uparrow\downarrow}$ are orthogonal unless $i + r = D$ and $j + s = D$.

**Proof:** (i) Assume that $i + r \neq D$ or $j + s \neq D$. We show that $V_{i,j}^{\downarrow\downarrow}$ and $V_{r,s}^{\uparrow\uparrow}$ are orthogonal. To do this we will use Lemma 6.7(i),(iv). Let $W_{h}^{\downarrow\downarrow}$ (resp. $W_{h'}^{\uparrow\uparrow}$) denote one of the terms in the sum in Lemma 6.7(i) (resp. Lemma 6.7(iv)). We show that $W_{h}^{\downarrow\downarrow}$ and $W_{h'}^{\uparrow\uparrow}$ are orthogonal. There are two cases to consider. First assume that $W \neq W'$. Then $W$ and $W'$ are orthogonal so $W_{h}^{\downarrow\downarrow}$ and $W_{h'}^{\uparrow\uparrow}$ are orthogonal. Next assume that $W = W'$. Let $\rho, \tau, d$ denote the corresponding endpoint, dual endpoint, and diameter. By Lemma 6.7(i),

$$\tau = i + j - \rho - d, \quad h = i - \rho. \quad (13)$$

By Lemma 6.7(iv),

$$\tau = 2D - \rho - d - r - s, \quad h' = \rho + d - D + r. \quad (14)$$
Adding the equations on the right in (13), (14) we get
\[ i + r - D = h + h' - d. \] (15)
Subtracting the equation on the left in (13) from the equation on the left in (14) and evaluating the result using (15) we get
\[ j + s - D = d - h - h'. \] (16)
By (15), (16) and since \( i + r \not= D \) or \( j + s \not= D \) we find \( h + h' \not= d \). Now \( W_{h}^{\uparrow \uparrow} \) and \( W_{h'}^{\downarrow \downarrow} \) are orthogonal by Lemma 5.5(i).
(ii) Similar to the proof of (i).

**Corollary 6.9** With reference to Notation 4.4 and Definition 6.2, the following (i), (ii) hold for \( 0 \leq i, j \leq D \).

\[(i) \quad \dim \tilde{V}_{i,j}^{\downarrow \downarrow} = \dim \tilde{V}_{D-i,D-j}^{\uparrow \uparrow} \]
\[(ii) \quad \dim \tilde{V}_{i,j}^{\uparrow \uparrow} = \dim \tilde{V}_{D-i,D-j}^{\downarrow \downarrow} \]

**Proof:** Immediate from Theorem 6.8 and elementary linear algebra. \( \square \)

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