About an extension of the Matsumoto-Yor property

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March 11, 2022

Abstract

If \( \alpha, \beta > 0 \) are distinct and if \( A \) and \( B \) are independent non-degenerate positive random variables such that

\[
S = \frac{1}{B} \frac{\beta A + B}{\alpha A + B} \quad \text{and} \quad T = \frac{1}{A} \frac{\beta A + B}{\alpha A + B}
\]

are independent, we prove that this happens if and only if the \( A \) and \( B \) have generalized inverse Gaussian distributions with suitable parameters. Essentially, this has already been proven in Bao and Noack (2021) with supplementary hypothesis on existence of smooth densities.

The sources of these questions are an observation about independence properties of the exponential Brownian motion due to Matsumoto and Yor (2001) and a recent work of Croydon and Sasada (2000) on random recursion models rooted in the discrete Korteweg - de Vries equation, where the above result was conjectured.

We also extend the direct result to random matrices proving that a matrix variate analogue of the above independence property is satisfied by independent matrix-variate GIG variables. The question of characterization of GIG random matrices through this independence property remains open.

Keywords: Bessel differential equation, GIG distribution, discrete Korteweg - de Vries equation, Matsumoto-Yor property, Matrix GIG distribution

1 Detailed balance equation for discrete KdV model

Croydon and Sasada (2020) introduced two so called detailed balance equations. For type I model we say that a pair of probability measures \( \mu \) and \( \nu \) on some spaces \( U \) and \( V \) satisfies the detailed balance equation for a map \( F = (F_1, F_2) : U \times V \to U \times V \) iff

\[
F(\mu \otimes \nu) = \mu \otimes \nu,
\]

where \( F(\mu \otimes \nu) \) means \((\mu \otimes \nu) \circ F^{-1}\).

For type II model we say that two pairs of probability measures \( \mu, \nu \) on \( U, V \) and \( \tilde{\mu}, \tilde{\nu} \) on \( \tilde{U}, \tilde{V} \) satisfy the detailed balance equation for a map \( F = (F_1, F_2) : U \times V \to \tilde{U} \times \tilde{V} \) iff

\[
F(\mu \otimes \nu) = \tilde{\mu} \otimes \tilde{\nu}.
\]

In other terms for the type II model, one considers independent random variables \( X \) and \( Y \) with distributions \( \mu \) and \( \nu \) such that random variables \( U = F_1(X, Y) \) and \( V = F_2(X, Y) \) are independent and have distributions \( \tilde{\mu} \) and \( \tilde{\nu} \), respectively. Type I model is more restrictive since it imposes the additional condition that \( X \overset{d}{=} U \) and \( Y \overset{d}{=} V \).

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As an example of type II model, if $\mathcal{U} = \mathcal{V} = \mathcal{U} = \mathcal{V} = (0, \infty)$ and $F(x, y) = (x + y, \frac{1}{x+y})$ Matsumoto and Yor (2001) have observed that if $X$ and $Y$ are respectively Gamma and generalized inverse Gaussian (GIG) distributed with suitable parameters and independent, the same is true for the pair $U, V$ with different parameters (a description of the GIG distribution and of its limiting cases appears in Section 2 and in Section 3 respectively). This phenomena has been called the Matsumoto-Yor (MY) property, a name coined by Stürzaker (2005), p. 43. In Letac and Wesolowski (2000) a characterisation by independence of $X, Y$ and independence of $U, V$ of these pairs Gamma, GIG is given.

In the present paper we investigate a type II model, again with $\mathcal{U} = \mathcal{V} = \mathcal{U} = \mathcal{V} = (0, \infty)$ which has been considered by Croydon and Sasada (2020), Sec. 3.2. They introduced the involutive map $F_{dK}^{(\alpha, \beta)} : (0, \infty)^2 \to (0, \infty)^2$ defined by

$$F_{dK}^{(\alpha, \beta)}(x, y) = \left(y^{\frac{\alpha x y + 1}{\alpha y + 1}}, x^{\frac{\alpha x y + 1}{\alpha y + 1}}\right)$$

(3)

for $\alpha, \beta \geq 0$ and $\alpha \neq \beta$ in connection with the modified discrete Korteweg - de Vries (mKdV) model. The discrete mKdV model can be described by the following dynamics: to each point $(n, t) \in \mathbb{Z}^2$ one associates a vector $(x_n^t, y_n^t) \in \mathbb{R}^2$ defined by $(x_n^{t-1}, y_n^{t-1})$ and by the formula

$$(x_n^t, y_n^t) = F_{dK}^{(\alpha, \beta)}(x_n^{t-1}, y_n^{t-1}).$$

For instance $(x_n^t, y_n^t)$ is known for all $n, t > 0$ if it is known along $(0, n)$ and $(t, 0)$ for all $n, t \geq 0$.

If $X$ and $Y$ are independent as well as $U$ and $V$ defined by $(U, V) = F_{dK}^{(\alpha, \beta)}(X, Y)$ the fact that $F_{dK}^{(\alpha, \beta)}$ is involutive permits to create a stationary measure on $\mathbb{Z}^2$ for the above dynamics by taking $(x_n^t, y_n^t) = (X, Y)$ or $(U, V)$ according to the fact that $t + n$ is even or odd. The straight discrete KdV model corresponds to the particular case $\beta = 0$. For details, see Sect. 2.2 in Croydon, Sasada and Tsujimoto (2020) with slightly different notations.

Croydon and Sasada (2020) observed that the choice of $\mu$ and $\nu$ as generalized inverse Gaussian (GIG) distributions with suitable parameters fits with the detailed balance equation of type II model. Actually, their Proposition 3.9 considers the type I model, but the proof translates immediately to the type II model with $\mu$ and $\nu$ being also GIG distributions. Furthermore, their Conjecture 8.6 predicts that these sets of GIG distributions are the only possible ones for the type II model. This has been proved by Bao and Noack (2021) under the technical assumptions that the involved distributions have strictly positive densities on $(0, \infty)$ with second derivatives. Their approach was based on expressing the independence condition through the functional equation for logarithms $r_X, r_Y, r_U$ and $r_V$ of densities of $X, Y, U$ and $V$. They proved that the Jacobian of the change of variable $(u, v) = F_{dK}^{(\alpha, \beta)}(x, y)$ is $-1$. Consequently, they identified the functional equation to be solved as

$$r_U \left(\frac{y(1+\beta xy)}{1+\alpha xy}\right) + r_V \left(\frac{x(1+\alpha xy)}{1+\beta xy}\right) = r_X(x) + r_Y(y), \quad x, y > 0.$$ 

The first main result of the present paper proves the Croydon and Sasada Conjecture 8.6 in full generality. The technique uses a extended Laplace transform $L_X$ of a positive random variable $X$ defined by

$$L_X(s, \sigma, \theta) = \mathbb{E} X^s e^{\sigma X + \frac{\theta}{X}}, \quad (s, \sigma, \theta) \in \mathbb{R} \times (0, \infty)^2$$

(4)

and leads to a related function $g_s$ which satisfies the classical Bessel differential equation, see e.g. Watson (1966), Sec. 3.7,

$$z^2 g''_s(z) + z g'_s(z) - (z^2 + \nu^2) g_s(z) = 0.$$ 

One can mention here that the main tool of the paper Letac and Wesolowski (2000) is the simpler function $(\sigma, \theta) \to \mathbb{E} e^{\sigma X + \frac{\theta}{X}}$.

To describe our second main result we note that the limiting cases for $\alpha = 0$ or $\beta = 0$ have been considered in Letac and Wesolowski (2000). In particular, that paper deals with the natural extension from the domain $(0, \infty)$ to the cone of positive definite matrices of order $r$, which we denote by $\Omega_r^+$. Following the matrix
variate context, here, for $\alpha, \beta > 0$ we prove that matrix variate GIG distributions, which we rather denote MGIG, satisfy the detailed balance equation for type II model with the function $F_{\Delta K}^{(\alpha, \beta)} : \Omega_+^2 \to \Omega_+^2$ defined by

$$F_{\Delta K}^{(\alpha, \beta)}(x, y) = (y(I + \alpha xy)^{-1}(I + \beta xy), x(I + \beta yx)^{-1}(I + \alpha yx)),$$

where $x$ and $y$ are positive definite matrices of order $r$ and $I$ is the identity matrix.

Section 2 describes the GIG distribution and states our first main result in Theorem 2.3. Section 3 examines the limiting cases of $\alpha$ or $\beta$ equal zero. The long Section 4 gives the proof of Theorem 2.3. Section 5 considers the symmetric matrices generalization and gives our second main result. Section 6 comments on references about the MY property.

# GIG laws and the characterization

Denote by $\text{GIG}(\lambda, a, b)$, $\lambda \in \mathbb{R}$, $a, b > 0$, the generalized inverse Gaussian distribution. It is defined by the density

$$f(x) = \left(\frac{\lambda}{a - \sigma}\right)^{\frac{a + b}{2}} \frac{1}{2K_\lambda(2\sqrt{ab})} x^{a - 1} \exp\left(-\frac{ax}{2} - \frac{b}{2}\right) \frac{I_{\nu}(0, \infty)(x)}{I_{\nu}(0, \infty)}(x),$$

where $K_\nu$ is the modified Bessel function of the third kind, see (31).

The following reciprocity property of the GIG distribution is well known and easily seen: if a random variable $X$ has the $\text{GIG}(\lambda, a, b)$ distribution, then the distribution of $X^{-1}$ is $\text{GIG}(-\lambda, b, a)$. A less known property of GIG is the form of the extended Laplace transform defined in (4), which has the form

$$L_X(s, \sigma, \theta) = \left(\frac{b - \theta}{a - \sigma}\right)^{\frac{a + b}{2}} K_\nu(2\sqrt{(a - \theta)(b - \sigma)}), \quad \theta < a, \sigma < b$$

for $s \in \mathbb{R}, \sigma < a$ and $\theta < b$. Note that $L_X(0, \sigma, \theta)$ as above with $s = 0$ uniquely determines the law $X \sim \text{GIG}(\lambda, a, b)$. Actually $L(s, \theta, \sigma)$ of the above form for any fixed $s$ determines the GIG distribution as shown by the next proposition.

**Proposition 2.1.** Assume that there exists $s \in \mathbb{R}$ such that

$$L_X(s, \theta, \sigma) = c \left(\frac{b - \sigma}{a - \sigma}\right)^{\frac{a + b}{2}} K_\nu(2\sqrt{(a - \theta)(b - \sigma)}), \quad \theta < a, \sigma < b$$

for some constants $a, b, c > 0$. Then $X \sim \text{GIG}(\nu - s, a, b)$.

**Proof.** Take $\tilde{a} < a$ and $\tilde{b} < b$. Consider a random variable $Y$ with the distribution

$$P_Y(dx) = \frac{x^\nu e^{\frac{a}{x} + \frac{b}{2}} P_X(dx)}{L_X(s, \tilde{a}, \tilde{b})}.$$

Then for $\theta < a - \tilde{a}$ and $\sigma < b - \tilde{b}$ we have

$$L_Y(0, \theta, \sigma) = \frac{L_X(s, \tilde{a} + \theta, \tilde{b} + \sigma)}{L_X(s, \tilde{a}, \tilde{b})} = \left(\frac{(b - \tilde{b} - \sigma)(a - \tilde{a})}{(a - \tilde{a})(b - \sigma)}\right)^{\nu/2} K_\nu(2\sqrt{(a - \tilde{a})(b - \sigma)}).$$

Comparing this form with (5) we conclude that $Y \sim \text{GIG}(\nu - s, a - \tilde{a}, b - \tilde{b})$. Consequently,

$$P_X(dx) \propto x^{\nu - s - 1} \exp\left(-ax - \frac{b}{2}\right).$$

Thus, $P_X(dx) \propto x^{\nu - s - 1} \exp\left(-ax - \frac{b}{2}\right)$ and the result follows. \(\square\)
The result below gives the solution of the Croydon-Sasada conjecture for $\alpha, \beta > 0$ and $F_{dK}^{(\alpha, \beta)}$ defined by (3). As said before, it has already been proved by Bao and Noack (2021) with extra hypothesis and different techniques.

**Theorem 2.2.** Let $X$ and $Y$ be non-Dirac, non-negative independent random variables. Let $\alpha > 0$ and $\beta > 0$ be distinct real numbers. Let

$$ (U, V) = F_{dK}^{(\alpha, \beta)}(X, Y). $$

If $U$ and $V$ are independent then there exist $c_1, c_2 > 0$ and $\lambda \in \mathbb{R}$ such that

$$ X \sim \text{GIG}(-\lambda, \alpha c_1, c_2) \quad \text{and} \quad Y \sim \text{GIG}(-\lambda, \beta c_2, c_1) $$

(6)

and

$$ U \sim \text{GIG}(-\lambda, \alpha c_2, c_1) \quad \text{and} \quad V \sim \text{GIG}(-\lambda, \beta c_1, c_2). $$

(7)

We now prove that Theorem 2.2 is equivalent to the following theorem.

**Theorem 2.3.** Let $A$ and $B$ be non-Dirac, non-negative independent random variables. Let $\alpha > 0$ and $\beta > 0$ be distinct real numbers. Let

$$ S = \frac{1}{B} \frac{\beta A + B}{\alpha A + B} \quad \text{and} \quad T = \frac{1}{A} \frac{\beta A + B}{\alpha A + B}. $$

If $S$ and $T$ are independent then there exist $c_1, c_2 > 0$ and $\lambda \in \mathbb{R}$ such that

$$ A \sim \text{GIG}(-\lambda, \alpha c_1, c_2) \quad \text{and} \quad B \sim \text{GIG}(\lambda, c_1, \beta c_2) $$

(8)

and

$$ S \sim \text{GIG}(-\lambda, \alpha c_2, c_1) \quad \text{and} \quad T \sim \text{GIG}(\lambda, c_2, \beta c_1). $$

(9)

To see this equivalence it will be convenient to introduce a modification of $F_{dK}^{(\alpha, \beta)}$ of the form

$$ \psi^{(\alpha, \beta)} = I_2^{-1} \circ F_{dK}^{(\alpha, \beta)} \circ I_2, $$

where $I_2 : (0, \infty)^2 \to (0, \infty)^2$ is defined by $I_2(x, y) = \left( x, \frac{1}{y} \right)$. Since $F_{dK}^{(\alpha, \beta)}$ is involutive it is clear that $\psi^{(\alpha, \beta)}$ is also an involution on $(0, \infty)^2$. Note that

$$ \psi^{(\alpha, \beta)}(x, y) = \left( 1, \frac{\beta x + y}{y} \right), \frac{\beta x + y}{x} \right) = \left( \frac{\beta}{\alpha y} + \frac{\alpha - \beta}{\alpha (x + y)}, \frac{1}{x} - \frac{\alpha - \beta}{\alpha x + y} \right). $$

(10)

For a probability measure $\nu$ of $(0, \infty)$ denote by $\nu^{(-1)}$ a measure defined by $\nu^{(-1)}(D) = \nu(1/D)$ for all Borel sets $D \subset (0, \infty)$ and $1/D = \{1/x, \ x \in D\}$. It is obvious that $\mu, \nu$ and $\tilde{\mu}, \tilde{\nu}$ satisfy the detailed balance equation (2) for $F_{dK}^{(\alpha, \beta)}$ iff $\mu, \nu^{(-1)}$ and $\tilde{\mu}, \tilde{\nu}^{(-1)}$ satisfy (2) for $\psi^{(\alpha, \beta)}$. Since

$$ (S, 1/T) = I_2(S, T) = F_{dK}^{(\alpha, \beta)} \circ I_2(A, B) = F_{dK}^{(\alpha, \beta)} (A, 1/B) = (U, V) $$

the equivalence follows immediately from (6), (7) and (8), (9) in view of the reciprocity property of the GIG.

3 The limiting cases of $\alpha = 0$ or $\beta = 0$ and the MY property

Note that inserting $\beta = 0$ in (3) or (10) gives

$$ F_{dK}^{(\alpha, 0)}(x, y) = \left( \frac{x}{\alpha x y + 1}, x (\alpha x y + 1) \right) $$

(11)

and

$$ \psi^{(\alpha, 0)}(x, y) = \left( \frac{1}{\alpha x + y}, \frac{1}{x} - \frac{\alpha}{\alpha x + y} \right). $$

(12)
One can expect that the problem of detailed balance equation of type II model for $F_{d\kappa}^{(0,\alpha)}$ or similarly for $F_{d\kappa}^{(0,\beta)}$ can be derived by taking the limit as $\alpha \to 0$ or $\beta \to 0$ in the Theorem 2.2 or in its equivalent Theorem 2.3.

Indeed, for $\lambda > 0$

$$GIG(\lambda, a, b) \xrightarrow{w} G(\lambda, a)$$

for $b \to 0$,

where $\xrightarrow{w}$ denotes the weak convergence of probability measures and $G(\lambda, a)$ is the Gamma distribution defined by the density

$$f(x) = \frac{e^{-ax}}{\Gamma(\lambda)} x^{\lambda-1} I_{(0,\infty)}(x),$$

Similarly, for $\lambda > 0$

$$GIG(-\lambda, a, b) \xrightarrow{w} \text{InvG}(\lambda, b)$$

for $a \to 0$,

where $\text{InvG}(\lambda, b)$ is the inverse Gamma distribution defined by the density

$$f(x) = \frac{b^{-\lambda}}{\Gamma(\lambda)} x^{-\lambda-1} e^{-b/x} I_{(0,\infty)}(x),$$

Note that without any loss of generality regarding the independence property we can assume that $\alpha = 1$ in (11) or (12). Actually, we would rather consider $\psi^{(1,0)}$ which assumes the form

$$\psi^{(1,0)}(x, y) = \left( \frac{1}{x+y} \frac{1}{x} - \frac{1}{x+y} \right),$$

since it is directly related to the MY property. More specifically, this property says:

Let $A \sim GIG(-\lambda, c_1, c_2)$ and $B \sim G(\lambda, c_1)$ be independent and $(S, T) = \psi^{(1,0)}(A, B)$. Then $S$ and $T$ are independent, $S \sim GIG(-\lambda, c_2, c_1)$ and $T \sim G(\lambda, c_2)$. Matsumoto and Yor (2001) discovered this property while studying the conditional structure of the exponential Brownian motion. Moreover, another paper, Matsumoto and Yor (2003), represents this property through hitting times of Brownian motion with drift (the case $\lambda = 1/2$).

In Proposition 2.9 of Croydon and Sasada (2020) the authors identified some GIG probability measures $\mu$ and $\nu$, which satisfy the detailed balance equation (1), therefore for Type I model, with $F = F_{d\kappa}^{(\alpha,\beta)}$ as follows:

Let $c > 0$ and either $\lambda \in \mathbb{R}$ and $\alpha \beta > 0$ or $\lambda > 0$ and $\alpha \beta = 0$ then

$$\mu \otimes \nu = GIG(-\lambda, \alpha c, c) \otimes GIG(-\lambda, \beta c, c)$$

(13)

satisfies (1) with $F = F_{d\kappa}^{(\alpha,\beta)}$, with a suitable interpretation in terms of Gamma distributions of the above GIG laws when $\alpha \beta = 0$. Moreover, they observed that the characterization by the MY property given in Letac and Wesołowski (2000) provides uniqueness of the invariant measures in case $\alpha \beta = 0$. This fact was crucial for their analysis of the discrete KdV model in this special case. Searching for uniqueness of invariant measures in the generalized discrete KdV model, Croydon and Sasada (2020) conjecture our Theorem 2.2 above for $\alpha, \beta > 0$, extending what was indeed already known for $\alpha \beta = 0$.

4 Proof of Theorem 2.3

4.1 Useful identities and extended Laplace transforms

Observe that for $(s, t) = \psi^{(\alpha,\beta)}(a, b)$ with $a, b > 0$ we have

$$\frac{s}{t} = \frac{a}{b},$$

(14)

$$t + \alpha s = \frac{1}{s} + \frac{b}{t},$$

(15)

$$b + \alpha a = \frac{1}{s} + \frac{a}{t}.$$
If \( s \in \mathbb{R} \) and \( \sigma, \theta \in (-\infty, 0) \) then \((0, \infty) \ni x \mapsto x^s e^{\sigma x + \frac{\theta}{x}} \) is a bounded function. Therefore, the following functions are well defined

\[
  x_s(\sigma, \theta) = \mathbb{E} A^s e^{\sigma A + \frac{\theta}{A}}, \quad y_s(\sigma, \theta) = \mathbb{E} B^s e^{\sigma B + \frac{\theta}{B}},
\]

\[
  u_s(\sigma, \theta) = \mathbb{E} S^s e^{\sigma S + \frac{\theta}{S}}, \quad v_s(\sigma, \theta) = \mathbb{E} T^s e^{\sigma T + \frac{\theta}{T}},
\]

for \((s, \sigma, \theta) \in \mathbb{R} \times (-\infty, 0)^2\). Thus identities (14), (15) and (16) give the equality

\[
  \mathbb{E} \left[ A^{-s} B^s e^{\sigma(B+\alpha A)+\theta(T+\alpha B)} \right] = \mathbb{E} \left[ S^{-s} T^s e^{\sigma(S+\beta S)+\theta(T+\beta S)} \right], \quad (s, \sigma, \theta) \in \mathbb{R} \times (-\infty, 0)^2.
\]

The independence assumptions of Theorem 2.3 together with the notation introduced above lead to the equation

\[
  x_s y_s = u_s v_s,
\]

which after taking logarithms assumes the form

\[
  \log x_s + \log y_s = \log u_s + \log v_s. \tag{17}
\]

We now apply \( \frac{\partial^2}{\partial \theta \partial \sigma} \) to (18). To this aim we observe that

\[
  \frac{\partial^2}{\partial \theta \partial \sigma} \log v_s = \beta(1 - \bar{v}_s), \quad \frac{\partial^2}{\partial \theta \partial \sigma} \log y_s = \beta(1 - \bar{y}_s), \tag{19}
\]

\[
  \frac{\partial^2}{\partial \theta \partial \sigma} \log u_s = \alpha(1 - \bar{u}_s), \quad \frac{\partial^2}{\partial \theta \partial \sigma} \log x_s = \alpha(1 - \bar{x}_s), \tag{20}
\]

where we denoted

\[
  \bar{x}_s = \frac{x_{s-1} x_{s+1}}{x_s^2}, \quad \bar{y}_s = \frac{y_{s-1} y_{s+1}}{y_s^2}, \quad \bar{u}_s = \frac{u_{s-1} u_{s+1}}{u_s^2}, \quad \bar{v}_s = \frac{v_{s-1} v_{s+1}}{v_s^2}.
\]

Consequently, (18) yields

\[
  \alpha \bar{x}_s + \beta \bar{y}_s = \alpha \bar{u}_s + \beta \bar{v}_s. \tag{21}
\]

Now observe that from (17) we have

\[
  \bar{x}_s \bar{y}_s = \bar{u}_s \bar{v}_s. \tag{22}
\]

Combining (21) and (22) by eliminating either \( \bar{x}_s \) or \( \bar{y}_s \) we get

\[
  (\alpha \bar{u}_s - \beta \bar{y}_s)(\bar{v}_s - \bar{y}_s) = (\alpha \bar{x}_s - \beta \bar{u}_s)(\bar{x}_s - \bar{u}_s) = 0. \tag{23}
\]

### 4.2 Equality \( \alpha \bar{u}_s = \beta \bar{y}_s \) is impossible

In this section we consider a domain \( D \subset \mathbb{C}^3 \) defined as the set of \( z = (z_1, z_2, z_3) \in \mathbb{C}^3 \) such that \( \Re z_2 \) and \( \Re z_3 \) are strictly negative.

Note that functions \((s, \sigma, \theta) \mapsto \bar{u}_s(\sigma, \theta) \) and \((s, \sigma, \theta) \mapsto \bar{y}_s(\sigma, \theta) \) are quotients of Laplace transforms on \( \mathbb{R}^3 \). These Laplace transforms are extendable to \( D \) as holomorphic functions. Therefore \((s, \sigma, \theta) \mapsto f(s, \sigma, \theta) = \alpha u_s(\sigma, \theta) - \beta y_s(\sigma, \theta) \) is extendable to \( D \) as a meromorphic function on \( D \) (here a meromorphic function on \( D \) is defined as the quotient of two holomorphic functions on \( D \)). Similarly, \((s, \sigma, \theta) \mapsto g(s, \sigma, \theta) = \bar{v}_s(\sigma, \theta) - \bar{y}_s(\sigma, \theta) \) is extendable to a meromorphic function on \( D \). Furthermore, (23) yields \( f(s, \sigma, \theta) g(s, \sigma, \theta) = 0 \) and therefore \( fg = 0 \) in the field of meromorphic functions on \( D \). This implies that either \( f(s, \sigma, \theta) = 0 \) for all \((s, \sigma, \theta) \) or \( g(s, \sigma, \theta) = 0 \) for all \((s, \sigma, \theta) \).

It is worthwhile to recall the short proof of the fact that if \( f(z)g(z) = 0 \) in \( D \) then either \( f(z) = 0 \) for all \( z \in D \) or \( g(z) = 0 \) for all \( z \in D \). We thank A. Zeriahi for it. Indeed if \( f = f_1/f_2, \ g = g_1/g_2 \) where
Suppose now that there exists \( z_0 \in D \) such that \( f_1(z_0) \neq 0 \) and consider the set
\[
D_0 = \left\{ z \in D; g_1^{(k)}(z) = 0 \text{ for all } k \in \mathbb{N}^3 \right\}.
\]
The set \( D_0 \) is an open subset of \( D \), since for all \( a \in D_0 \) the set
\[
\left\{ z \in D; g_1(z) = \sum_{k \in \mathbb{N}^3} (z - a)^{(k)} \frac{g^{(k)}(a)}{k!} \right\}
\]
is a subset of \( D_0 \) containing a neighborhood of \( a \). Also \( D_0 \) is a closed subset of \( D \) and \( D_0 \) is not empty since it contains \( z_0 \). Since \( D \) is connected we have \( D = D_0 \). The same reasoning will show that if \( g(z_0) \neq 0 \) then \( f \equiv 0 \).

Next we show that \( \alpha \tilde{u}_{-s} = \beta \tilde{y}_{s} \) is impossible. Indeed, in that case from (19) and (20) for \( s = 0 \) we get
\[
\alpha - \frac{\partial^2}{\partial \theta \partial \sigma} \log u_0 = \beta - \frac{\partial^2}{\partial \theta \partial \sigma} \log y_0,
\]
which gives
\[
e^{-\theta \sigma + H(\theta)} \mathbb{E} e^{\sigma B + \frac{\beta \theta}{B}} = e^{\alpha \theta \sigma + G(\sigma)} \mathbb{E} e^{\alpha \theta S + \frac{\sigma}{S}}, \quad (\sigma, \theta) \in (-\infty, 0)^2, \tag{24}
\]
for some functions \( G \) and \( H \). Plugging \((\theta, 0), (0, \sigma)\) and \((0, 0)\) in (24) we get
\[
e^{G(0)} \mathbb{E} e^{\alpha \theta S} = e^{H(\theta)} \mathbb{E} e^{\frac{\beta \theta}{B}}, \tag{25}
\]
\[
e^{G(\sigma)} \mathbb{E} e^{\frac{\sigma}{S}} = e^{H(0)} \mathbb{E} e^{\sigma B}, \tag{26}
\]
\[
e^{H(0)} = e^{G(0)}. \tag{27}
\]
Multiplying (24)-(27) side-wise and taking \( \sigma = \theta \) we arrive at
\[
e^{\beta \theta} \mathbb{E} \mathbb{E} e^{\theta \left( B + \frac{\beta}{B} \right)} e^{\theta S} e^{\frac{\theta}{S}} = e^{\alpha \theta^2} \mathbb{E} e^{\theta S} e^{\alpha \frac{\theta}{S} + \frac{1}{S} \theta S} e^{\frac{\beta \theta}{B}} e^{\theta B}. \tag{28}
\]
Now introduce \( S_1 \overset{d}{=} S_2 \overset{d}{=} S \) such that \( S_1, S_2 \) and \( B \) are independent, and \( B_1 \overset{d}{=} B_2 \overset{d}{=} B \) such that \( B_1, B_2 \) and \( S \) are independent. Consider positive random variables \( P_1 = B + \frac{\beta}{B} + \alpha S_1 + \frac{1}{S_1} \) and \( P_2 = \alpha S + \frac{1}{S} + \frac{\beta}{B} + B_2 \). From (28) we get
\[
e^{\beta \theta} \mathbb{E} e^{\theta P_1} = e^{\alpha \theta^2} \mathbb{E} e^{\theta P_2}. \tag{29}
\]
Now assume \( \alpha < \beta \) and consider a Gaussian random variable \( Z \sim N(0, 2(\alpha - \beta)) \) such that \( Z \) and \( P_2 \) are independent. Then we get \( \mathbb{E} e^{\theta P_1} = \mathbb{E} e^{\theta(P_1 + Z)}, \theta < 0 \), which implies \( P_1 \overset{d}{=} P_2 + Z \), which is a contradiction since the support of the distribution of \( P_2 + Z \) is \( \mathbb{R} \). A similar contradiction is obtained for \( \beta < \alpha \).

\section*{4.3 Exploiting the connection between \( v_s \) and \( y_s \)}

So, we necessarily have
\[
\tilde{v}_s = \tilde{y}_s. \tag{29}
\]
Combining this equality with (19) we obtain for all \( s \in \mathbb{R} \) that
\[
\frac{\partial^2}{\partial \theta \partial \sigma}(\log y_s - \log v_s) = 0.
\]
Thus, there exist functions $H_s$ and $G_s$ satisfying
\[
\log v_s(\sigma, \theta) = H_s(\theta) - G_s(\sigma) + \log y_s(\sigma, \theta).
\] (30)

Since $v_s$ and $y_s$ are analytic functions of $\sigma$ and $\theta$ it follows that $G_s$ and $H_s$ are also analytic. Thus we can take derivatives of (30) with respect to $\theta$ and $\sigma$; whence we get
\[
\frac{\beta v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} = -G_s'(\sigma) + \frac{y_{s+1}(\sigma, \theta)}{y_s(\sigma, \theta)} 
\] (31)
\[
\frac{v_{s+1}(\sigma, \theta)}{v_s(\sigma, \theta)} = H_s'(\theta) + \beta \frac{y_{s-1}(\sigma, \theta)}{y_s(\theta, \sigma)}. 
\] (32)

Side-wise multiplication of equations (31) and (32) yields
\[
\beta v_s(\theta, \sigma) = -H_s'(\theta)G_s'(\sigma) - \beta G_s'(\sigma) \frac{y_{s-1}(\sigma, \theta)}{y_s(\sigma, \theta)} + H_s'(\theta) \frac{y_{s+1}(\sigma, \theta)}{y_s(\sigma, \theta)} + \beta y_s(\sigma, \theta), 
\]
which, in view of (29) yields
\[
H_s'(\theta)G_s'(\sigma) = H_s'(\theta) \frac{y_{s+1}(\sigma, \theta)}{y_s(\sigma, \theta)} - \beta G_s'(\sigma) \frac{y_{s-1}(\sigma, \theta)}{y_s(\sigma, \theta)}. 
\] (33)

The fact that $G_s'$ and $H_s'$ can be zero or not is crucial in the sequel. Observe that in view of (33) if $G_s'(\sigma_1) = 0$ for some $\sigma_1 < 0$ then $H_s'(\theta) = 0$ for every $\theta < 0$. By symmetry we conclude that for any $s \in \mathbb{R}$ either $G_s'$ and $H_s'$ are identically zero or they are both non-zero for any value of arguments. Therefore
\[
\Lambda := \{s \in \mathbb{R} : G_s' \equiv 0 \equiv H_s'\} = \{s \in \mathbb{R} : G_s'H_s' \equiv 0\}. 
\]

### 4.4 Properties of the set $\Lambda$

**Proposition 4.1.**

(i) If $s \in \Lambda$ then there exists a constant $C(s) > 0$ such that $v_s = C(s)y_s$.

(ii) There is no $s \in \mathbb{R}$ such that $s, s - 1 \in \Lambda$.

(iii) There is no $s \in \mathbb{R}$ such that $s \notin \Lambda$ and $s - 1, s + 1 \in \Lambda$.

(iv) There exists $s \in \mathbb{R}$ such that $s, s - 1 \notin \Lambda$.

**Proof.** Ad(i) Follows immediately for (30).

Ad(ii) From (32), in view of (i) we have
\[
\frac{\beta C(s-1)y_{s-1}}{C(s)y_s} = \frac{y_{s+1}}{y_s}
\]
which implies $y_{s+1} = ky_{s-1}$, where $k = \frac{\beta C(s-1)}{C(s)}$. This gives $\frac{\partial y_s}{\partial \theta} = k \frac{\partial y_s}{\partial \sigma}$. Consequently, there exists a function $f$ defined on $(-\infty, 0)$ such that $y_s(\sigma, \theta) = f(k\theta + \sigma)$. Denoting $t = k\theta + \sigma$ we get for all $t < 0$ and all $\theta \in \left(\frac{t}{k}, 0\right)$
\[
y_s(\sigma, \theta) = E B^s \exp \left(tB + \theta \left(\frac{\beta}{B} - kB\right)\right) = f(t).
\]
Therefore for $t$ fixed $\theta \mapsto y_s(\theta, t - k\theta)$ is constant on $\left(\frac{t}{k}, 0\right)$ and thus we conclude that $\frac{\beta}{B} - kB$ is constant. Since $B > 0$ and non Dirac we get a contradiction.

Ad(iii) Using (29) and part (i) we get $v_s^2 = (s-1)C(s+1)y_s^2$. Since $s \notin \Lambda$ it follows from (30) that $v_s = e^{H_s(-\theta) - G_s(\sigma)}y_s$, which contradicts the fact that for $s \notin \Lambda$ the function $H_s(\theta) - G_s(\sigma)$ is not constant.
Ad(iv) It follows from (ii) that there exists \( s \not\in \Lambda \). If \( s - 1, s + 1 \in \Lambda \) it contradicts (iii). Therefore at least one of \( s \pm 1 \) is not in \( \Lambda \).

\[ \square \]

**Proposition 4.2.** (i) If \( s \not\in \Lambda \) then there exist \( r_s \neq 0 \), \( \sigma_0(s) \geq 0 \), \( \theta_0(s) \geq 0 \) such that for \( \theta < 0 \) and \( \sigma < 0 \) we have

\[
G_s'(\sigma) = \frac{r_s}{\sigma_0(s) - \sigma} \quad \text{and} \quad H'(\theta) = \frac{r_s}{\theta_0(s) - \theta}.
\]

Furthermore, there exist constants \( C_G(s) \) and \( C_H(s) \) such that for all \( \theta < 0 \) and \( \sigma < 0 \)

\[
G_s'(\sigma) = -r_s \log(\sigma_0(s) - \sigma) + C_G(s), \quad \text{and} \quad H_s(\theta) = -r_s \log(\theta_0(s) - \theta) + C_H(s).
\]

(ii) If \( s, s - 1 \not\in \Lambda \) then \( r_s = r_{s-1} + 1, \sigma_0(s-1) = \sigma_0(s), \theta_0(s-1) = \theta_0(s) \).

**Proof.** Ad(i) Since \( s \not\in \Lambda \) we can divide by \( H'_s G'_s \) in (33) getting

\[
1 = \frac{1}{G'_s(\sigma)} \frac{v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} - \frac{\beta}{H'_s(\theta)} \frac{y_{s-1}(\sigma, \theta)}{y_s(\sigma, \theta)}
\]

Comparing (31) and (36) we obtain

\[
\frac{1}{G'_s(\sigma)} \frac{v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} = \frac{1}{H'_s(\theta)} \frac{y_{s-1}(\sigma, \theta)}{y_s(\sigma, \theta)}
\]

We take derivative of (38) with respect to \( \sigma \)

\[
\frac{v_{s-1}(\sigma, \theta) G''_s(\sigma)}{v_s(\sigma, \theta) G'_s(\sigma)} - \frac{1}{G'_s(\sigma)} \frac{\partial}{\partial \sigma} \left( \frac{v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} \right) + \frac{1}{H'_s(\theta)}(1 - \bar{y}_s(\theta, \sigma)) = 0.
\]

Since

\[
\frac{\partial}{\partial \sigma} \left( \frac{v_{s-1}}{v_s} \right) = -\beta \left( \frac{v_{s-1}}{v_s} \right)^2 (1 - \bar{v}_{s-1}),
\]

using this last equality and (38), after cancelling by \( \frac{v_{s-1}}{v_s} \), we get

\[
\frac{G''_s(\sigma)}{(G'_s(\sigma))^2} + \frac{1}{G'_s(\sigma)} \left( \frac{\beta v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} - \frac{y_{s-1}(\sigma, \theta)}{y_s(\sigma, \theta)} \right) - \frac{1}{G'_s(\sigma)} \left( \frac{\beta v_{s-2}(\sigma, \theta)}{v_{s-1}(\sigma, \theta)} - \frac{y_s(\sigma, \theta)}{y_{s-1}(\sigma, \theta)} \right) = 0
\]

Finally, using (31) we obtain

\[
\frac{G''_s}{G'_s^2} = \frac{G'_s - G'_{s-1}}{G'_s} = 0
\]

A similar calculation starting with comparison (32) and (37), and taking the derivative with respect to \( \theta \) gives the same thing with \( H \):

\[
\frac{H''_s}{H'_s^2} - \frac{H'_s - H'_{s-1}}{H'_s} = 0
\]

Now reactivating (38) we get and

\[
\frac{G'_s(\sigma)}{H'_s(\theta)} = \frac{v_{s-1}(\sigma, \theta)}{v_s(\sigma, \theta)} \frac{y_s(\sigma, \theta)}{y_{s-1}(\sigma, \theta)} \quad \text{and} \quad \frac{H'_{s-1}(\theta)}{G'_{s-1}(\sigma)} = \frac{v_{s-1}(\sigma, \theta) y_{s-2}(\sigma, \theta)}{v_{s-2}(\sigma, \theta) y_{s-1}(\sigma, \theta)}.
\]
Consequently,
\[ \frac{G'_s(\sigma)}{H'_s(\theta)} \frac{H'_{s-1}(\theta)}{G'_{s-1}(\sigma)} = \frac{\bar{y}_{s-1}(\sigma, \theta)}{\bar{r}_{s-1}(\sigma, \theta)} = 1. \]

Combining this result with (40) and (41) we land on a separation of variables equation
\[ \frac{G''_s(\sigma)}{G'^2_s(\sigma)} = \frac{H''_s(\theta)}{H'^2_s(\theta)} = \frac{1}{r_s} \]

where \( r_s \) is a non-zero constant with respect to \( \sigma \) and \( \theta \). Integrating these two simple differential equations we arrive at (34).

The formula (35) is an immediate consequence of (34).

Ad(ii) From (40), in view of (34), we get
\[ \frac{1}{r_s} = 1 - \frac{r_{s-1}}{r_s} \frac{\sigma_0(s)-\sigma}{\sigma_0(s-1)-\sigma}, \quad \sigma < 0. \]

Therefore,
\[ (r_s - 1 - r_{s-1})\sigma + r_{s-1}\sigma_0(s) - (r_s - 1)\sigma_0(s-1) = 0. \]

Thus, looking at the coefficient of \( \sigma \) we get \( r_s = r_{s-1} + 1 \). Thus, the constant term gives \( \sigma_0(s) = \sigma_0(s-1) \). Similarly, using (41) we get \( \theta_0(s) = \theta_0(s-1) \).

\[ \square \]

4.5 Computing \( y_s \) and \( v_s \)

Let us introduce an auxiliary function
\[ \bar{y}_s(\sigma, \theta) = \log y_s(\sigma, \theta) - G_s(\sigma) + H_s(\theta). \]

Thus, in view of (35) we get
\[ y_s(\sigma, \theta) = C(s)e^{\bar{y}_s(\sigma, \theta)} \left( \frac{\theta_0 - \theta}{\sigma_0 - \sigma} \right)^{r_s/2}. \]

Rewrite now (36) as
\[ 1 - \frac{1}{G'_s(\sigma)} \frac{\partial \log y_s(\sigma, \theta)}{\partial \sigma} + \frac{1}{H'_s(\theta)} \frac{\partial \log y_s(\sigma, \theta)}{\partial \theta} = 0. \]

Note that (43) yields
\[ \frac{\partial \log y_s(\sigma, \theta)}{\partial \sigma} = \frac{\partial \bar{y}_s(\sigma, \theta)}{\partial \sigma} + \frac{r_s}{2(\sigma_0 - \sigma)} \quad \text{and} \quad \frac{\partial \log y_s(\sigma, \theta)}{\partial \theta} = \frac{\partial \bar{y}_s(\sigma, \theta)}{\partial \theta} - \frac{r_s}{2(\theta_0 - \theta)}. \]

Plugging these two expressions into (44), in view of (34), we get
\[ (\theta_0 - \theta) \frac{\partial \bar{y}_s(\sigma, \theta)}{\partial \theta} - (\sigma_0 - \sigma) \frac{\partial \bar{y}_s(\sigma, \theta)}{\partial \sigma} = 0. \]

For
\[ p = \sqrt{(\sigma_0 - \sigma)(\theta_0 - \theta)} \quad \text{and} \quad q = \sqrt{\sigma_0 - \sigma}/\sqrt{\theta_0 - \theta}, \]

we introduce temporarily a function \( h_s \) defined by \( h_s(p, q) = \bar{y}_s(\theta, \sigma) \). Since
\[ h_s(p, q) = \bar{y}_s \left( \theta_0 - \frac{q}{p}, \sigma_0 - pq \right) \]

we arrive at
\[ \frac{\partial h_s(p, q)}{\partial q} = \frac{p}{q^2} \frac{\partial \bar{y}_s}{\partial \theta} - p \frac{\partial \bar{y}_s}{\partial \sigma} = \frac{1}{q} \left( (\theta_0 - \theta) \frac{\partial \bar{y}_s}{\partial \theta} - (\sigma_0 - \sigma) \frac{\partial \bar{y}_s}{\partial \sigma} \right). \]
Thus, (45) implies that \( h_s(p, q) = \ell_s(2p) \) for some function \( \ell_s \). Consequently,

\[
\overline{y}_s(\sigma, \theta) = \ell_s \left( 2\sqrt{(\sigma_0 - \sigma)(\theta_0 - \theta)} \right).
\]

(47)

Denote \( f_s = C(s)e^{t\sigma} \) with \( C(s) \) from (43). Referring to (44) and (45) we get

\[
y_s(\sigma, \theta) = \left( \frac{\theta_0 - \theta}{\sigma_0 - \sigma} \right)^{r_s/2} f_s \left( 2\sqrt{(\theta_0 - \theta)(\sigma_0 - \sigma)} \right) = q^{-r_s} f_s(2p).
\]

(48)

To identify \( f_s \) we will derive an ordinary second order differential equation satisfied by this function. We first show that

\[
\beta y_s(\sigma, \theta) - \frac{r_{s-1}}{\sigma_0 - \sigma} \frac{\partial y_s(\sigma, \theta)}{\partial \theta} - \frac{\theta_0 - \theta}{\sigma_0 - \sigma} \frac{\partial^2 y_s(\sigma, \theta)}{\partial \theta^2} = 0.
\]

(49)

To this end we use (34) and rewrite (36) with \( s \) changed into \( s - 1 \) as

\[
r_{s-1}y_{s-1}(\sigma, \theta) = (\sigma_0 - \sigma) y_s(\sigma, \theta) - \beta(\theta_0 - \theta)y_{s-2}(\sigma, \theta).
\]

Now, (49) follows from the last equality since \( \frac{\partial y_s}{\partial \sigma} = \beta y_s - 1 \) and \( \frac{\partial^2 y_s}{\partial \sigma^2} = \beta^2 y_{s-2} \). Then we insert \( y_s \) as obtained in (48) into (49). Recalling (46) we note that

\[
\frac{1}{q} \frac{\partial y}{\partial \sigma} = \frac{1}{2(\theta_0 - \theta)} \quad \text{and} \quad \frac{1}{p} \frac{\partial y}{\partial \sigma} = -\frac{1}{2(\theta_0 - \theta)}.
\]

(50)

Using (50), after careful calculation, we get

\[
\frac{\partial y_s(\sigma, \theta)}{\partial \sigma} = -\frac{q^{-r_s}}{\theta_0 - \theta} \left[ \frac{r_s}{2} f_s(2p) + f_s'(2p) \right]
\]

and

\[
\frac{\partial^2 y_s(\sigma, \theta)}{\partial \sigma^2} = \frac{q^{-r_s}}{\theta_0 - \theta} \left[ -\frac{r_s}{p} \left( 1 - \frac{r_s}{2} \right) f_s(2p) + (r_s - \frac{1}{2}) f_s'(2p) + p^2 f_s'' \right].
\]

Plugging these two expressions into (49) we get

\[
f_s''(2p) - \left( r_{s-1} - r_s + \frac{1}{2} \right) \frac{f_s'(2p)}{p} - \left( \beta p^2 + \frac{r_{s-1}r_s}{2} + \frac{r_s^2}{4} \right) \frac{f_s(2p)}{p^2} = 0.
\]

Using the fact that \( r_{s-1} = r_s - 1 \) the above equation can be rewritten as

\[
f_s''(2p) + \frac{1}{2} \left( \beta p^2 + \frac{r_s^2}{4} \right) \frac{f_s(2p)}{p^2} = 0.
\]

Denoting \( t = 2p \) we get

\[
f_s''(t) + \frac{1}{4} \left( \beta t^2 + \frac{r_s^2}{4} \right) \frac{f_s(t)}{t^2} = 0.
\]

Thus, for a function \( g_s \) defined by \( g_s(z) = f_s(z/\sqrt{3}) \) we obtain the classical Bessel equation (see e.g. Watson (1966))

\[
g_s''(z) + \left( \frac{z^2}{2} + \frac{r_s^2}{4} \right) \frac{g_s(z)}{z^2} = 0.
\]

Since, \( g_s = aI_{r_s} + bK_{r_s} \), where \( I_{r_s} \) and \( K_{r_s} \) are the modified Bessel functions of the first and third type, defined by

\[
I_{r_s}(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{r_s+2m}}{m!(r_s+m+1)!},
\]

e.g. Watson (1966), p.77 (2), and

\[
K_{r_s}(z) = \left( \frac{z}{4} \right)^r \int_0^\infty \tau^{r_s-1} e^{-\tau} \frac{z^2}{4\tau} d\tau,
\]

(51)

e.g. Watson (1966), p. 183 (15). We note that \( g_s \) is bounded at infinity. Indeed, it suffices to show that \( f_s \) is bounded at infinity. Note that for a fixed \( s \) the function \( y_s \) is bounded on \( (-\infty, -\epsilon)^2 \) for a fixed \( \epsilon > 0 \). Inserting
\(\theta_0 - \theta = \sigma_0 - \sigma = t/2\) in (48) for \((\sigma, \theta) \in (-\infty, -\epsilon)^2\) we get \(y_s(\theta, \sigma) = f_s(t)\). Thus \(f_s\) is bounded at infinity. But \(I_0(z) \to z_{2 \to \infty}\). Therefore, \(y_s = bK_{r_s}\) for some real \(b\) depending on \(s\). Consequently, for \(\sigma < 0\) and \(\theta < 0\) we have

\[
y_s(\sigma, \theta) = q^{-r_s} g_s(2\sqrt{\beta}p) = b(s) \left(\frac{\theta_0 - \theta}{\sigma_0 - \sigma}\right)^{r_s/2} K_{r_s} \left(2\sqrt{\beta(\theta_0 - \theta)/(\sigma_0 - \sigma)}\right).
\]

(52)

Repeating for \(v_s\) the argument we used for \(y_s\) while using (57) instead of (56) requires only flipping \(\theta_0 - \theta\) with \(\sigma_0 - \sigma\). Therefore,

\[
v_s(\sigma, \theta) = t(s) \left(\frac{\sigma_0 - \sigma}{\theta_0 - \theta}\right)^{r_s/2} K_{r_s} \left(2\sqrt{\beta(\theta_0 - \theta)/(\sigma_0 - \sigma)}\right).
\]

(53)

Note that in both formulas (52) and (53) we have \(\theta_0 \geq 0\) and \(\sigma_0 \geq 0\).

### 4.6 Conclusion of the proof

Let us first show that \(\theta_0 > 0\) and \(\sigma_0 > 0\). For this consider a possibly unbounded positive measure

\[
r_B(dx) = x^{r_s-s-1} e^{-\sigma_0 x - \frac{\beta_0}{x}} I_{(0,\infty)}(x) \, dx
\]

Then for \(\sigma < 0\) and \(\theta < 0\) we have

\[
\int_{0}^{\infty} x^s e^{\sigma x + \frac{\beta_0}{x}} r_B(dx) = \left(\frac{\theta_0 - \theta}{\sigma_0 - \sigma}\right)^{r_s/2} K_{r_s} \left(2\sqrt{\beta(\sigma_0 - \sigma)(\theta_0 - \theta)}\right) = \frac{\sigma_0 - \sigma}{\theta_0 - \theta} y_s(\sigma, \theta),
\]

where the last equality follows by (52). Consequently, we get

\[
\int_{0}^{\infty} x^s e^{\sigma x + \frac{\beta_0}{x}} r_B(dx) = \frac{\sigma_0 - \sigma}{\theta_0 - \theta} \int_{0}^{\infty} x^s e^{\sigma x + \frac{\beta_0}{x}} \theta_B(dx).
\]

Therefore,

\[
\theta_B(dx) \propto x^{r_s-s-1} e^{-\sigma_0 x - \frac{\beta_0}{x}} I_{(0,\infty)}(x) \, dx
\]

(54)

Similarly, using (53), we see that

\[
\theta_T(dx) \propto x^{r_s-s-1} e^{-\sigma_0 x - \frac{\beta_0}{x}} I_{(0,\infty)}(x) \, dx.
\]

(55)

Note that (54) or (55) imply that \(\theta_0 = \sigma_0 = 0\) is impossible. Note also that when \(\theta_0 > 0\) and \(\sigma_0 = 0\), then, according to (54), the measure \(\theta_B\) is finite only if \(r_s - s < 0\). But then the measure at the right hand side of (55) is not finite, which is impossible. By the similar argument we exclude the case \(\sigma_0 > 0\) and \(\theta_0 = 0\). Thus, both these situations are not possible. Finally, we conclude that \(\sigma_0 > 0\), \(\theta_0 > 0\) and \(B \sim \text{GIG}(\lambda, \theta_0, \beta \sigma_0)\), \(T \sim \text{GIG}(\lambda, \theta_0, \beta \sigma_0)\), with \(\lambda = r_s - s\).

To compute the distributions of \(A\) and \(S\) we follow the arguments we used for \(B\) and \(T\), but we start with the second equality in (23), instead of the first one, which we used while computing \(v_s\) and \(y_s\). This equality yields \(\tilde{u}_s = \tilde{x}_s\) for all \(\sigma < 0\) and \(\theta < 0\). As in the case \(\tilde{v}_s = \tilde{y}_s\) we obtain \(A \sim \text{GIG}(\lambda', \alpha \sigma'_0, \theta'_0)\) and \(S \sim \text{GIG}(\lambda', \alpha \sigma'_0, \theta'_0)\) for some \(\sigma'_0, \theta'_0 > 0\) and \(\lambda' \in \mathbb{R}\).

Consequently, there exist constants \(C_i, i = 1, 2, 3, 4\), which may depend on \(s\) such that

\[
x_{-s} = C_1 \left(\frac{\sigma_0 - \sigma}{\theta_0 - \theta}\right)^{(\lambda' - s)/2} K_{\lambda' - s} \left(2\sqrt{\alpha(\sigma_0' - \sigma)(\theta_0' - \theta)}\right),
\]

\[
y_s = C_2 \left(\frac{\sigma_0 - \sigma}{\theta_0 - \theta}\right)^{(\lambda + s)/2} K_{\lambda + s} \left(2\sqrt{\beta(\sigma_0 - \sigma)(\theta_0 - \theta)}\right),
\]

\[
u_{-s} = C_3 \left(\frac{\sigma_0 - \sigma}{\theta_0 - \theta}\right)^{(\lambda' - s)/2} K_{\lambda' - s} \left(2\sqrt{\alpha(\sigma_0' - \sigma)(\theta_0' - \theta)}\right),
\]

\[
u_s = C_4 \left(\frac{\sigma_0 - \sigma}{\theta_0 - \theta}\right)^{(\lambda + s)/2} K_{\lambda + s} \left(2\sqrt{\beta(\sigma_0 - \sigma)(\theta_0 - \theta)}\right).
\]
Thus (17) implies
\[
C_1 C_2 \left( \frac{\theta_0 - \theta}{\sigma_0 - \sigma} \right)^{(\lambda' - s)/2} \left( \frac{\theta_0 - \theta}{\sigma_0 - \sigma} \right)^{(\lambda + s)/2} = C_3 C_4 \left( \frac{\sigma_0 - \sigma}{\theta_0 - \theta} \right)^{(\lambda' - s)/2} \left( \frac{\sigma_0 - \sigma}{\theta_0 - \theta} \right)^{(\lambda + s)/2},
\]
for all \( \sigma < 0, \theta < 0 \). Consequently, \( \lambda + s = -(\lambda' - s) \), i.e. \( \lambda = -\lambda' \). Moreover, \( \theta_0 = \theta'_0 := c_2 \) and \( \sigma_0 = \sigma'_0 := c_1 \). Thus the result follows.

5 GMY property for GIG matrices

Let \( \Omega \) be the Euclidean space of symmetric \( r \times r \) matrices with real entries and the inner product defined by \( \langle x, y \rangle = \text{tr}(xy) \), \( x, y \in \Omega \). Denote by \( \Omega_+ \) the cone of symmetric positive definite \( r \times r \) matrices. We say that an \( \Omega_+ \)-valued random matrix \( X \) has a matrix GIG law, MGIG\((p, a, b)\), for \( p \in \mathbb{R} \) and \( a, b \in \Omega_+ \) if it has the density with respect to the Lebesgue measure \( dx \) in \( \Omega \) of the form
\[
\mu_{p,a,b}(x) = \frac{1}{K_{p}(a,b)} (\det x)^{p-\frac{r+1}{2}} \exp \left( -\frac{\text{tr}(ax) + \text{tr}(bx)}{2} \right) I_{\Omega_+}(x),
\]
see e.g. Sec. 2 of Letac and Wesolowski (2000).

For \( \alpha, \beta \geq 0 \) we recall the map \( F_{dK}^{(\alpha, \beta)} \) on \( \Omega_+ \times \Omega_+ \) defined in the introduction by the formula
\[
F_{dK}^{(\alpha, \beta)}(x, y) = \left( y(I + \alpha xy)^{-1}(I + \beta xy), x(I + \beta yx)^{-1}(I + \alpha yx) \right),
\]
where \( I \) denotes the identity matrix. Note that for \( \alpha = 0, \beta > 0 \) and \( \alpha > 0, \beta = 0 \) the problem we discuss reduces to the matrix variate version of the original Matsumoto-Yor property given in Letac and Wesolowski (2000). Moreover, \( F_{dK}^{(\alpha, \alpha)} \) is just the identity. Therefore, we are not interested in these cases in the sequel.

**Proposition 5.1.** For every distinct \( \alpha, \beta > 0 \) the map \( F_{dK}^{(\alpha, \beta)} \) is a differentiable involution on \( \Omega_+ \times \Omega_+ \). Moreover, its Jacobian equals 1.

**Proof.** For distinct \( \alpha, \beta > 0 \) we denote \( F := F_{dK}^{(\alpha, \beta)} \). For \( x, y \in \Omega_+ \) set \( (u, v) = F(x, y) \). Note that
\[
y(I + \alpha xy)^{-1} = (I + \alpha yx)^{-1}y \quad \text{and} \quad y(I + \beta xy) = (I + \beta yx)y.
\]
Combining these two identities we get
\[
u = y(I + \alpha xy)^{-1}(I + \beta xy) = (I + \alpha yx)^{-1}y(I + \beta xy) = (I + \alpha yx)^{-1}(I + \beta yx)y.
\]
But the right hand side above equals \( u^T \), i.e. \( u \) is symmetric. Similarly, we conclude that \( v \) is symmetric. Since \( u \) and \( v \) are products of positive definite matrices, they are also positive definite. Consequently, \( F(\Omega_+ \times \Omega_+) \subset \Omega_+ \times \Omega_+ \). Note that \( u^T = (I + \alpha xy)(I + \beta yx)^{-1}x = (I + \beta yx)^{-1}(I + \alpha xy)x \). Since \( v = u^T \) we get
\[
u v = yx.
\]
Plugging (57) or its transpose to definitions of \( u \) and \( v \) we get
\[
u = y(I + \alpha vu)^{-1}(I + \beta vu) \quad \text{and} \quad v = x(I + \beta vu)^{-1}(I + \alpha vu)
\]
whence
\[
u x = v(I + \alpha vu)^{-1}(I + \beta vu) \quad \text{and} \quad y = u(I + \beta vu)^{-1}(I + \alpha vu).
\]
Thus \( F \) is an involution, i.e. \( F = F^{-1} \). Differentiability on \( \Omega_+ \times \Omega_+ \) is clear.
To compute the Jacobian $J_F$ of $F$ we first note that $F = \phi^{-1} \circ \psi \circ \phi$, where $\phi(x,y) = (x, y^{-1})$ and

$$\psi(x,y) = (y^{-1}(y + \beta x)(y + \alpha x)^{-1}, \ x^{-1}(y + \beta x)(y + \alpha x)^{-1}) =: (u', v').$$

Thus

$$J_F(x, y) = J_{\phi^{-1}}(x, y)J_\psi(x, y^{-1})J_\phi(u', v').$$

Note that

$$u' = \frac{\beta}{\alpha} y^{-1} + \frac{\alpha - \beta}{\alpha} (y + \alpha x)^{-1} \quad \text{and} \quad v' = x^{-1} - (\alpha - \beta)(y + \alpha x)^{-1}.$$

Recall that the derivative $D_x(x^{-1})$ is $-P_{x^{-1}}$, where $P_a$ is the endomorphism on $\Omega$ defined by $P_a(h) = aha$, $h \in \Omega$. Consequently, the derivatives $D_x$ and $D_y$ of $u'$ are

$$D_x(u') = -\alpha - \beta)P_{(y + \alpha x)^{-1}} \quad \text{and} \quad D_y(u') = -\frac{\beta}{\alpha} P_{y^{-1}} - \frac{\alpha - \beta}{\alpha} P_{(y + \alpha x)^{-1}}.$$

Similarly, the derivatives of $v'$ are

$$D_x(v') = -P_{x^{-1}} + (\alpha - \beta)\alpha P_{(y + \alpha x)^{-1}} \quad \text{and} \quad D_y(v') = (\alpha - \beta)P_{(y + \alpha x)^{-1}}.$$

For clarity we write det $x$ for the determinant of $x \in \Omega$ and Det $M$ for the determinant of the endomorphism $M$ on $\Omega$ or $\Omega^2$. For example, in the sequel we will need the formula

$$\text{Det } P_x = (\text{det } x)^{r+1},$$

see e.g. Faraut and Koranyi (1994), p. 52. Thus

$$J_\psi(x, y) = \text{Det} \left[ \begin{array}{cc} -\alpha - \beta)P_{(y + \alpha x)^{-1}} & -\frac{\beta}{\alpha} P_{y^{-1}} - \frac{\alpha - \beta}{\alpha} P_{(y + \alpha x)^{-1}} \\ -P_{x^{-1}} + (\alpha - \beta)\alpha P_{(y + \alpha x)^{-1}} & (\alpha - \beta)P_{(y + \alpha x)^{-1}} \end{array} \right]$$

Using the Cholesky decomposition we thus get

$$J_\psi(x, y) = \text{Det} \left\{ (\alpha - \beta)P_{(y + \alpha x)^{-1}} \right\} \times \text{Det} \left\{ -\beta P_{y^{-1}} + \frac{1}{\alpha(\alpha - \beta)} P_{x^{-1}} (P_{(y + \alpha x)^{-1}})^{-1} (\beta P_{y^{-1}} + (\alpha - \beta)P_{(y + \alpha x)^{-1}}) \right\}.$$

Using the fact that $P_{a^{-1}} = P_a^{-1}$ we see that the operator under the second determinant has the form

$$-\beta P_{y^{-1}} + \frac{1}{\alpha(\alpha - \beta)} P_{x^{-1}} P_{y + \alpha x} \left( \beta P_{y^{-1}} + (\alpha - \beta)P_{y + \alpha x} \right)$$

$$= -\beta P_{y^{-1}} + \frac{1}{\alpha(\alpha - \beta)} P_{x^{-1}} P_{y + \alpha x} P_{y^{-1}} + \frac{1}{\alpha} P_{x^{-1}}$$

$$= \frac{1}{\alpha(\alpha - \beta)} P_{x^{-1}} \left[ -\alpha(\alpha - \beta)P_{x} + \beta P_{y + \alpha x} + (\alpha - \beta)P_{y} \right] P_{y^{-1}}.$$

For $x, y \in \Omega$ denote by $\mathbb{L}_{x,y}$ the endomorphism of $\Omega$ defined by $\mathbb{L}_{x,y}(h) = xhy + yhx$, $h \in \Omega$. Then

$$P_{y + \alpha x} = P_{y} + \alpha \mathbb{L}_{x,y} + \alpha^2 P_{x}$$

whence

$$-\alpha(\alpha - \beta)P_{x} + \beta P_{y + \alpha x} + (\alpha - \beta)P_{y} = -\alpha(\alpha - \beta)P_{x} + \beta(\alpha^2 P_{x} + \alpha \mathbb{L}_{x,y} + P_{y}) + (\alpha - \beta)P_{y} = \alpha P_{y + \beta x}.$$

Summing up, in view of (59), we obtain

$$J_\psi(x, y) = \frac{\text{Det } P_{y + \beta x}}{\text{Det } P_{y + \alpha x} \text{ Det } P_{x} \text{ Det } P_{y}} = \left( \frac{\det(y + \beta x)}{\det(y + \alpha x) \det(x) \det(y)} \right)^{r+1}.$$
Thus
\[
J_\psi(x, y^{-1}) = \left( \frac{\det(I + \beta xy) \det(y)}{\det(I + \alpha xy) \det(x)} \right)^{r + 1}
\]
Note that \( J_\psi(x, y) = \det \{-P_y - 1\} = \frac{(-1)^r}{(\det y)} \). Therefore,
\[
J_F(x, y) = \frac{1}{(\det y)^r+1} \left( \frac{\det(I + \beta xy) \det(y)}{\det(I + \alpha xy) \det(x)} \right)^{r + 1}
\]
From (58) we have \( \det(\frac{J_\psi(x, y^{-1})}{\det(x) \det(I + \alpha xy)}) = 1 \). Thus we conclude that \( |J_F(x, y)| = 1 \)

Now we are ready to formulate the main result of this section which gives a detailed balance equation of the generalized Matsumoto-Yor type satisfied by GIG random matrices.

**Theorem 5.2.** Let \((X, Y) \sim \text{MGIG}(\lambda, \alpha a, b) \otimes \text{MGIG}(\lambda, \beta b, a)\). Then
\[
(U, V) = F^{(\alpha, \beta)}_{dK}(X, Y) \sim \text{MGIG}(\lambda, \alpha b, a) \otimes \text{MGIG}(\lambda, \beta a, b).
\]

**Proof.** Consider the joint density \( f \) of \((U, V)\). Since the Jacobian of \( F^{(\alpha, \beta)}_{dK} \) is 1, it follows that
\[
f(u, v) = f_X(x(u, v)) f_Y(y(u, v))
\]
\[
\propto (\det(xy))^{\lambda - \frac{r + 1}{2}} \exp \left( -\alpha \text{tr}(ax) + \beta \text{tr}(by) + \alpha \text{tr}(ay^{-1}) \right)
\]
We now consider the exponent. Denoting \( \tilde{\alpha} = I + \alpha uv \in \Omega_+ \), \( \tilde{\beta} = I + \beta uv \in \Omega_+ \) and using the fact that
\[
\text{tr } a = \text{tr } a^T
\]
we get
\[
\alpha \text{tr}(ax) + \beta \text{tr}(by) + \alpha \text{tr}(ay^{-1})
\]
\[
= \alpha \text{tr} \left( (av\tilde{\alpha}^{-1}\tilde{\beta}) + (\tilde{\beta}^{-1}av^{-1}) + \beta \text{tr} \left( bu \left( \tilde{\beta}^T \right)^{-1} a^T \right) + \text{tr} \left( (\tilde{\alpha}^T)^{-1} \tilde{\beta}^Tu^{-1} \right) \right)
\]
Since
\[
u \left( \tilde{\beta}^T \right)^{-1} \tilde{\alpha}^T = \tilde{\beta}^{-1}av
\]
\[
(a^T)^{-1}\tilde{\beta}^Tu^{-1} = u^{-1}\tilde{\alpha}^{-1}\tilde{\beta}
\]
We get
\[
\alpha \text{tr}(ax) + \beta \text{tr}(by) + \alpha \text{tr}(ay^{-1})
\]
\[
= \text{tr} \left( a (u^{-1} + av) \tilde{\alpha}^{-1}\tilde{\beta} \right) + \text{tr} \left( b\tilde{\beta}^{-1}\tilde{\alpha} (v^{-1} + \beta u) \right)
\]
\[
= \text{tr} \left( au^{-1}(I + \alpha uv)\tilde{\alpha}^{-1}\tilde{\beta} \right) + \text{tr} \left( b\tilde{\beta}^{-1}\tilde{\alpha}(I + \beta uv)v^{-1} \right)
\]
\[
= \text{tr} \left( au^{-1}(I + \beta uv) \right) + \text{tr} \left( b(I + \alpha uv)v^{-1} \right)
\]
\[
= \text{tr}((au^{-1}) + \text{tr}(\beta av) + \text{tr}(bv^{-1}) + \text{tr}(abu).
\]
Combining this with (57) we get
\[
f(u, v) \propto (\det(u))^{\lambda - \frac{r + 1}{2}} \exp \left( -\frac{\text{tr}(abu) + \text{tr}(au^{-1})}{2} \right) (\det(v))^{\lambda - \frac{r + 1}{2}} \exp \left( -\frac{\text{tr}(\beta av) + \text{tr}(bv^{-1})}{2} \right),
\]
which ends the proof.
6 Bibliographical comments

Let us mention that the MY property and related characterization triggered a lot of further research developing in several directions: (1) more general algebraic structures as, a multivariate tree-generated version in Massam and Wesolowski (2004), matrix variate versions in Letac and Wesolowski (2000), Wesolowski (2002), Massam and Wesolowski (2006), a combination of the matrix variate and multivariate tree-generated setting in Bobecka (2015), symmetric cone variate in Kołodziejek (2017), a version in free probability in Szpojankowski (2017); (2) characterizations based on a weaker assumption of constancy of regressions of moments of $S$ given $T$ instead of the assumption of independence of $S$ and $T$ - in univariate case in Wesolowski (2002), Chou and Wang (2004) and in free probability in Szpojankowski (2017) and ´Swieca (2021); (3) a search of more general maps of the form $\psi_f(a,b) = (f(a+b), f(a) - f(a+b))$ and product measures $\mu \otimes \nu$, such that $\psi_f(\mu \otimes \nu)$ remains a product measure and characterization of respective $\mu$ and $\nu$ by the independence property in Koudou and Vallois (2012) with a genuine special case of $\mu$ and $\nu$ being the Kummer and gamma distributions, see also Koudou and Vallois (2011), Koudou (2015), Piliszek and Wesolowski (2016), Kołodziejek (2018). For a survey on characterizations of the GIG distribution and other references (up to 2014) see Koudou and Ley (2014).

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