RISK MEASURING UNDER MODEL UNCERTAINTY

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The framework of this paper is that of risk measuring under uncertainty which is when no reference probability measure is given. To every regular convex risk measure on $C_b(\Omega)$, we associate a unique equivalence class of probability measures on Borel sets, characterizing the riskless nonpositive elements of $C_b(\Omega)$. We prove that the convex risk measure has a dual representation with a countable set of probability measures absolutely continuous with respect to a certain probability measure in this class. To get these results we study the topological properties of the dual of the Banach space $L^1(c)$ associated to a capacity $c$.

As application, we obtain that every $G$-expectation $\mathbb{E}$ has a representation with a countable set of probability measures absolutely continuous with respect to a probability measure $P$ such that $P(|f|) = 0$ if and only if $\mathbb{E}(|f|) = 0$. We also apply our results to the case of uncertain volatility.

1. Introduction. The purpose of this paper is to introduce a very general framework enabling the study of risk measures and dynamic risk measures in a context of model uncertainty, which is when no reference probability measure is given.

In order to quantify the risk in finance, Artzner et al. [1] have introduced the notion of coherent (i.e., sublinear) risk measure in the context of finite probability spaces. This notion has been extended to general probability spaces [12] and then to the convex case [21] and [22]. The notion of conditional risk measure has been considered in [6] and [17]. Dynamic risk measures have then been studied in many papers, among them [7, 8, 11, 13, 25, 30]. For the particular case of dynamic risk measures on a Brownian filtration, one can cite [3, 14, 26]. Notice that in all these papers on dynamic risk measures, a reference probability space is fixed. This framework is rich enough to study models with stochastic volatility or models with jumps, but not to deal with model uncertainty.

What does uncertainty mean? Usually in mathematical finance, in order to compute the risk or the price associated to financial assets, one assumes that a reference family of liquid assets is given and that the dynamics of these reference assets is known. However, in a context of model uncertainty, the dynamics of the liquid reference assets is only assumed to belong to a certain class of models. A simple
example is given, within the Brownian framework, by a class of models with uncertain volatility. That is, one considers a family of possible models of the form
\[ dX^\sigma_t = b_t X^\sigma_t dt + \sigma_t X^\sigma_t dW_t \]
where \( \sigma_t \) is allowed to vary inside an interval \([\sigma, \bar{\sigma}]\).
When \( \sigma \) describes the set of predictable processes varying inside this interval, the laws of the processes \( X^\sigma_t \) are not all absolutely continuous with respect to some probability measure. Avellaneda, Levy and Paras [2], Denis and Martini [16] and Denis, Hu and Peng [15] have considered the problem of pricing for this family of models. Only a few papers study convex risk measures in a context of uncertainty. Föllmer and Schied [21] have studied static risk measures defined on the vector space of all bounded measurable maps. This has been extended by Bion-Nadal to the conditional case in [6]. Kervarec [24] has studied static risk measures when model uncertainty is specified by a nondominated weakly compact set of probability measures.

In this paper, motivated by the general context of model uncertainty, we study regular convex risk measures defined on \( C_b(\Omega) \), the set of continuous bounded functions on a Polish space \( \Omega \). Regularity is here equivalent to continuity with respect to a certain capacity \( c \). Considering the completion \( L^1(c) \) of \( C_b(\Omega) \) with respect to the capacity \( c \), this means that we study convex risk measures on the Banach space \( L^1(c) \). Our main result is that for every regular convex risk measure on \( C_b(\Omega) \), there is a unique equivalence class of probability measures characterizing the riskless nonpositive elements of \( C_b(\Omega) \) and that the convex risk measure has a dual representation with a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this equivalence class. The tools of the proof are the capacities, topological properties of the dual of the Banach space \( L^1(c) \) associated to a capacity \( c \) and convex duality for locally convex spaces.

The paper is organized as follows. First, in Section 2 we study the topological properties of the dual of \( L^1(c) \). We prove that the nonnegative part of the dual ball of \( L^1(c) \) is metric compact for the weak* topology \( \sigma(L^1(c)^*, L^1(c)) \).

Section 3 deals with convex risk measures on \( L^1(c) \). We prove that they satisfy the representation formula
\[ \rho(X) = \sup_{Q \in \mathcal{P}'} (E_Q[-X] - \alpha(Q)), \]
where \( \mathcal{P}' \) is a set of probability measures belonging to the dual of \( L^1(c) \). There are two important results in this section. The first one is the characterization of convex risk measures on \( L^1(c) \) admitting a representation of the form (1.1) having a compact set \( \mathcal{P}' \) of probability measures [for the weak* topology \( \sigma(L^1(c)^*, L^1(c)) \)]. In this case, the supremum in (1.1) is a maximum. Moreover, making use of the topological results of Section 2, we prove that every convex risk measure on \( L^1(c) \) has a dual representation of the form (1.1) with a countable set of probability measures.

In Section 4 we assume that the capacity is defined on \( C_b(\Omega) \) by \( c_{p, \mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p} \) for some weakly relatively compact set \( \mathcal{P} \) of probability
measures. We prove that the capacity $c_{p,P}$ is equal to the capacity $c_{p,Q}$ defined using a certain countable subset $Q$ of $P$. We introduce a new equivalence relation on the set of nonnegative measures belonging to the dual of $L^1(c_{p,P})$. When $P$ is a singleton, it coincides with the usual equivalence relation on nonnegative measures. The main result of Section 4 is the existence of an equivalence class of probability measures characterizing the null elements of $L^1(c_{p,P})_+$, that is, $P$ belongs to this equivalence class if and only if for all $f$ in $L^1(c_{p,P})$, $(E_P(|f|) = 0) \iff (c_{p,P}(|f|) = 0)$.

Section 5 deals with uniformly regular convex risk measures on $C_b(\Omega)$. We prove that every such risk measure on $C_b(\Omega)$ extends into a convex risk measure on $L^1(c)$ for a certain capacity $c$ associated to a weakly compact set $P$ of probability measures: $c(f) = \sup_{P \in P} E_P(f)$. Therefore, we can make use of the results obtained in Sections 4 and 3 in order to get the main result of the paper in Theorem 5.1: to every uniformly regular convex risk measure $\rho$ on $C_b(\Omega)$, one can associate a unique equivalence class of probability measures defined on the Borel sets, called $c_{\rho}$-class, characterizing the nonpositive elements of $C_b(\Omega)$ with risk 0. The convex risk measure has then a dual representation with a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this $c_{\rho}$-class.

Section 6 deals with two examples. The first one is $G$-expectations introduced by Peng [27]. The capacity associated to a $G$-expectation $E$ is $c(f) = E(|f|)$. As application of our results, we obtain that there is a unique equivalence class of probability measures characterizing the nonnegative elements $f$ of $C_b(\Omega)$ such that $E(f) = 0$. The $G$-expectation $E$ has then a representation in terms of a countable set of probability measures all absolutely continuous with respect to a certain probability measure belonging to this class

$$E(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(X).$$

The second example, for which all our results apply, is the case where model uncertainty is characterized by a relatively weakly compact set of probability measures $P$.

2. Topological properties of the dual space of $L^1(c)$.

2.1. The ordered space $L^1(c)$. Let $\Omega$ be a metrizable and separable space. One classical example, furthermore a Polish space, is $\Omega = C_0([0, \infty[, \mathbb{R}^d)$ endowed with the topology of uniform convergence on compact subspaces. $B(\Omega)$ denotes the Borel $\sigma$-algebra on $\Omega$. Denote $M(\Omega)$ the set of all bounded signed measures on $(\Omega, B(\Omega))$ and $M_+(\Omega)$ the subset of nonnegative finite measures.

In the following, $L$ denotes a linear vector subspace of $C_b(\Omega)$ containing the constants, generating the topology of $\Omega$ and which is a vector lattice. Recall the following definition of a capacity.
DEFINITION 2.1. A capacity on $\mathcal{L}$ is a semi-norm $c$ defined on $\mathcal{L}$ satisfying the following properties:

1. monotonicity: $\forall \ f, g \in \mathcal{L}$ such that $|f| \leq |g|$, $c(f) \leq c(g)$;
2. regularity along sequences: for every sequence $f_n \in \mathcal{L}$ decreasing to 0, $\inf c(f_n) = 0$.

The semi-norm $c$ is extended as in [20], Section 2, to all real functions defined on $\Omega$,

\[ (2.1) \quad \forall f \ \text{l.s.c.} \ f \geq 0 \quad c(f) = \sup \{ c(\phi) | 0 \leq \phi \leq f, \phi \in \mathcal{L} \}, \]

\[ (2.2) \quad \forall g \ c(g) = \inf \{ c(f) | f \geq |g|, f \ \text{l.s.c.} \}, \]

where l.s.c. means lower semi-continuous. $\mathcal{L}^1(c)$ denotes the closure of $\mathcal{L}$ in the set $\{ g | c(g) < \infty \}$. From Proposition 10 of [20], $\mathcal{L}^1(c)$ contains $C_b(\Omega)$. Let $\mathcal{L}(c)$ be the quotient of $\mathcal{L}^1(c)$ by the $c$ null elements. It is a Banach space. The following result shows that $c(1_A)$ can be expressed as the limit of a monotone sequence $c(f_n)$ for continuous functions $f_n$ with limit $1_A$, as soon as $A$ is either an open subset or a closed subset of $\Omega$.

PROPOSITION 2.1. Let $V$ be an open subset of $\Omega$. There is an increasing sequence of nonnegative continuous functions $h_n$ on $\Omega$ such that $1_V = \lim_{n \to \infty} h_n$ and $c(1_V) = \lim_{n \to \infty} c(h_n)$.

Let $F$ be a closed subset of $\Omega$. There is a decreasing sequence of continuous functions $g_n \leq 1$ on $\Omega$ such that $1_F = \lim_{n \to \infty} g_n$ and $c(1_F) = \lim_{n \to \infty} c(g_n)$.

PROOF. $1_V$ is a nonnegative bounded l.s.c. function. Thus, it is the limit of an increasing sequence of nonnegative continuous functions $f_n$. On the other hand, from definition of $c(1_V)$ [equation (2.1)], there is a sequence of continuous functions $g_n \leq 1_V$ such that $c(1_V) = \lim c(g_n)$. Let $h_1 = g_1$ and for every $n$, $h_{n+1} = \sup(h_n, f_n, g_n)$. $h_n$ is an increasing sequence of continuous functions with limit $1_V$ and such that $c(1_V) = \lim c(h_n)$.

Let $F$ be a closed subset of $\Omega$. By definition of the capacity, $c(1_F) = \inf_{\{ \psi \ \text{l.s.c.}, 1_F \leq \psi \}} c(\psi)$. The infimum of two l.s.c. functions is also l.s.c., thus there is a decreasing sequence $\psi_n$ greater or equal to $1_F$ such that $c(1_F) = \lim c(\psi_n)$. Thus, there is a strictly increasing sequence $k(n)$ such that for all $n$, $c(\psi_{k(n)}) \leq c(1_F) + \frac{1}{n}$. Let $\varepsilon_n > 0$ such that $(\frac{1}{1 - \varepsilon_n})(c(1_F) + \frac{1}{n}) \leq c(1_F) + \frac{1}{n}$. Let $V_n = \{ x | \psi_{k(n)}(x) > 1 - \varepsilon_n \} \cap \{ x \in \Omega; \dist(x, F) < \frac{1}{n} \}$. As $\psi_{k(n)}$ is l.s.c., $V_n$ is an open set; furthermore, $F = \bigcap_{n \in \mathbb{N}} V_n$. For every $n$, there is a continuous function $f_n$ such that $F \leq f_n \leq V_n$. One can thus construct a decreasing sequence of continuous functions $g_n$ such that $1_F \leq g_n \leq 1_{V_n}$. Thus the sequence $g_n$ is decreasing to $1_F$. As $c(1_{V_n}) \leq \frac{1}{1 - \varepsilon_n} c(\psi_{k(n)}) \leq c(1_F) + \frac{1}{n}$, it follows that $c(1_F) \leq c(g_n) \leq c(1_F) + \frac{1}{n}$. \[ \Box \]
Further definitions and results on capacities are recalled in the Appendix. We refer also to [20].

**Partial order on** $L^1(c)$. 

**Definition 2.2.** Let $X \in L^1(c)$. We say that $X \geq 0$ if there is a sequence $(f_n)_{n \in \mathbb{N}}, f_n \in \mathcal{L}, f_n \geq 0$ such that for every $g \in L^1(c)$ of class $X$, $\lim_{n \to \infty} c(g - f_n) = 0$.

**Lemma 2.1.** Let $X, Y \in L^1(c)$. If $X \geq 0$ and $Y \geq 0$, then $X + Y \geq 0$.

**Proof.** The first part of the lemma is trivial. The second point follows from the inequality

$$c(|f| - |f_n|) \leq c(f - f_n).$$

Thus, as $f = |f|$, $c(f - |f_n|) \leq c(f - f_n)$. One can deduce from (2.3) that for all $X \in L^1(c)$, $|X| \in L^1(c)$. From point 2, $|X| - X \geq 0$. Thanks to (2.3) and the inequality $c(|f| - f) \leq c(|f| - f_n) + c(f - f_n)$, it follows that $X \geq 0$ if and only if $X = |X|$ in $L^1(c)$.

**Proposition 2.2.** The relation $X \leq Y$ defined by $Y - X \geq 0$ defines a partial order on $L^1(c)$.

**Proof.** (1) Reflexivity is trivial; take $f_n = 0$ for all $n$.

(2) Antisymmetry. Let $X \geq Y$ and $Y \geq X$. Let $h$ be in the class of $X - Y$. By definition, there are two sequences $f_n$ and $g_n$ of nonnegative functions in $\mathcal{L}$ such that $\lim_{n \to \infty} c(f_n - h) = 0$ and $\lim_{n \to \infty} c(g_n + h) = 0$. It follows that $\lim_{n \to \infty} c(f_n + g_n) = 0$. As $0 \leq |f_n - g_n| \leq f_n + g_n$, it follows that $\lim_{n \to \infty} c(|f_n - g_n|) = 0$. However, $\lim_{n \to \infty} c(f_n - g_n - 2h) = 0$. Thus, $X - Y$, the class of $h$ is equal to 0.

(3) Transitivity follows from the first part of Lemma 2.1.

**2.2. Topological properties of the nonnegative part of the unit ball of** $L^1(c)^*$. For the definition of a Prokhorov capacity, see the Appendix.

**Proposition 2.3.** Let $c$ be a Prokhorov capacity on a metrizable and separable space $\Omega$. Every continuous linear form $L$ on $L^1(c)$ admits a representation,

$$L(f) = \int f \, d\mu \quad \forall f \in L^1(c),$$

where $\mu$ is a regular bounded signed measure defined on a $\sigma$-algebra containing the Borel $\sigma$-algebra of $\Omega$.

If $L$ is a nonnegative linear form, the regular measure $\mu$ is nonnegative finite.
Following [5], a bounded signed measure \( \mu \) is called regular if for all Borel set \( A \), for all \( \varepsilon > 0 \), there is a closed set \( F \) and an open set \( G \) such that \( F \subset A \subset G \) and \( |\mu|(G - F) < \varepsilon \).

Notice that in [20], the existence of a bounded measure \( \mu \) satisfying (2.4) is proved. However, the statement of Proposition 11 of [20] does not give information on the \( \sigma \) algebra on which the measure \( \mu \) is defined. Therefore, we have to go inside the proof.

**PROOF OF PROPOSITION 2.3.** A metrizable space is completely regular and \( c \) is a Prokhorov capacity so Proposition 11 of [20] gives the existence of a measure \( \mu \) satisfying (2.4). We want to now prove that \( \mu \) is defined on the Borel \( \sigma \) algebra.

As in the proof of Proposition 11 of [20], let \( Z \) be a compactification of \( \Omega \) and \( c' \) the capacity defined on \( Z \) by \( c'(g) = c(g|_{\Omega}) \). As \( c \) is a Prokhorov capacity, from Proposition 11 of [20], \( c'(1_{Z - \Omega}) = 0 \) and \( L^1(c) = L^1(c') \).

As \( Z \) is a compact space, it follows from Theorem 3 of [19] that every nonnegative linear form on \( L^1(c') \) can be represented by a nonnegative measure obtained from the Riesz representation theorem applied to \( C(Z) \). Therefore, this measure is defined on a \( \sigma \)-algebra containing the Borel sets of \( Z \). From Theorem 6 of [19] every continuous linear form on \( L^1(c) \) is the difference of two nonnegative linear forms, thus, the bounded measure \( \mu \) satisfying (2.4) is defined on a \( \sigma \)-algebra \( B \) containing the Borel \( \sigma \)-algebra of \( Z \).

We want to prove that \( \mu \) is defined on the Borel \( \sigma \)-algebra of \( \Omega \). \( \mu \) is defined on the \( \sigma \)-algebra \( F \) obtained by completion of \( B \) with the \( \mu \)-null sets. Notice that from Theorem 3 of [19], every \( c' \)-negligible set [i.e., \( c'(1_A) = 0 \)] is also \( \mu \)-negligible. This is, in particular, the case for \( Z - \Omega \) which is therefore, \( \mu \)-measurable. Every open set \( V \) of \( \Omega \) can be written \( V = U \cap \Omega \) for some open set \( U \) of \( Z \). Therefore, \( V \) belongs to \( F \). It follows that the measure \( \mu \) defined on \( F \) is thus defined on the Borel \( \sigma \)-algebra of \( \Omega \). As \( \Omega \) is a metric space and \( \mu \) is defined on the Borel \( \sigma \)-algebra of \( \Omega \), \( \mu \) is regular from Theorem 1.1 of [5]. \( \square \)

Recall that the weak topology on \( M_+(\Omega) \), the set of nonnegative finite measures on \( (\Omega, B(\Omega)) \), is the coarsest topology for which the mappings

\[
\mu \in M_+(\Omega) \to \int f \, d\mu
\]

are continuous for every given \( f \) in \( C_b(\Omega) \).

**PROPOSITION 2.4.** Let \( c \) be a Prokhorov capacity on a metrizable separable space. The set of nonnegative linear forms on the Banach space \( L^1(c) \) is a subset of \( M_+(\Omega) \). The weak* topology [i.e., the \( \sigma (L^1(c)^*, L^1(c)) \) topology] on the nonnegative part \( K_+ \) of the unit ball of \( L^1(c)^* \) coincides with the restriction to \( K_+ \) of the weak topology on \( M_+(\Omega) \).
PROOF. From Proposition 2.3, every nonnegative linear form on $L^1(\mathcal{C})$ belongs to $\mathcal{M}_+(\Omega)$. Let $\mu \in K_+$. As $\mathcal{C}_b(\Omega)$ is dense in the Banach space $L^1(\mathcal{C})$, the open sets

$$V_{f_1,f_2,...,f_n,\varepsilon}(\mu) = \{ \nu \in K_+ | \forall i \in \{1,\ldots,n\}, |\mu(f_i) - \nu(f_i)| < \varepsilon \}$$

with $f_i \in \mathcal{C}_b(\Omega)$ form a basis of neighborhoods of $\mu$ in $K_+$ for the weak* topology. Thus, the weak* topology on $K_+$ coincides with the weak topology. □

PROPOSITION 2.5. Let $c$ be a Prokhorov capacity on a metrizable separable space $\Omega$. The set $K_+$ is compact metrizable for the weak* topology [i.e., the $\sigma(L^1(\mathcal{C})^*,L^1(\mathcal{C}))$ topology] as well as for the weak topology.

PROOF. Prove first that $K_+$ is metrizable for the weak* topology. From Proposition 2.4, the weak* topology on $K_+$ coincides with the restriction to $K_+$ of the weak topology on $\mathcal{M}_+(\Omega)$. As $\Omega$ is metrizable and separable, $\mathcal{M}_+(\Omega)$ is also metrizable and separable for the weak topology from [10], Section 5. Thus, $K_+$ is metrizable for the weak* topology.

From the Banach–Alaoglu theorem ([18], Theorem V 4.2), the closed unit ball of the dual space of a Banach space is always compact for the weak* topology. As $K_+$ is a closed subset of this unit ball for the weak* topology, it is also compact. This proves the result for the weak* topology. From Proposition 2.4, $K_+$ is also metrizable compact for the weak topology. □

COROLLARY 2.1. Assume that $\Omega$ is a Polish space. For every capacity $c$ on $\Omega$, the set $K_+$ is compact metrizable for the weak* topology.

PROOF. From [20] (see also the Appendix), every capacity on a Polish space is a Prokhorov capacity and thus the result follows from Proposition 2.5. □

In the particular case of a compact metrizable space, we obtain the following stronger result.

PROPOSITION 2.6. Let $\Omega$ be a metrizable compact space. Let $c$ be a capacity on $\Omega$. Then the Banach space $L^1(\mathcal{C})$ is separable and the unit ball of $L^1(\mathcal{C})^*$ is metrizable compact for the weak* topology.

PROOF. As $\Omega$ is a metrizable compact space, $\mathcal{C}(\Omega)$ is separable from Theorem 1, Section 3, of [9]. Thus, for every capacity $c$ on $\Omega$, $L^1(\mathcal{C})$ is also separable. Then from [18], Theorem V 51, the unit ball of $L^1(\mathcal{C})^*$ (and not only its nonnegative part) is metrizable compact for the weak* topology. □
3. Representation of a convex risk measure on $L^1(c)$. In this section, $c$ denotes a Prokhorov capacity on a metrizable separable space $\Omega$. Recall that a partial order has been defined on $L^1(c)$ in Section 2.1. We can define convex risk measures in the usual way as follows.

**Definition 3.1.** Let $\rho : L^1(c) \to \mathbb{R}$.

- $\rho$ is monotonic if $\rho(X) \geq \rho(Y)$ for every $X, Y \in L^1(c)$, such that $X \leq Y$.
- $\rho$ is convex if for every $X, Y \in L^1(c)$, for every $0 \leq \lambda \leq 1$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$.
- $\rho$ is translation invariant if $\rho(X + a) = \rho(X) - a$ for every $X \in L^1(c)$ and $a \in \mathbb{R}$.
- $\rho$ is a convex risk measure if it satisfies all these conditions.

**3.1. Representation for convex risk measures.** Duality results for risk measures are well known in other settings. A duality result was first proved in the case of risk measures on $L^\infty$ spaces assuming, furthermore, continuity from below. Duality results are based on the Fenchel–Legendre duality, generalized to the context of locally convex topological spaces by Rockafellar [29]. This is the generalized version that we need here. No additional hypothesis is needed in order to prove the dual representation result. The important and new discussion will be developed in Section 3.2 using the topological results proved in Section 2.2.

**Theorem 3.1.** Let $\rho$ be a convex risk measure on $L^1(c)$. Then, $\rho$ is continuous and admits a representation of the form

\[
\forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in \mathcal{P}'} (E_Q[-X] - \alpha(Q)),
\]

where

\[
\alpha(Q) = \sup_{X \in L^1(c)} (E_Q[-X] - \rho(X)).
\]

$\mathcal{P}'$ is the set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ belonging to $L^1(c)^*$.

**Proof.** The continuity of $\rho$ follows from Theorem 1 of [4]. We call $\alpha$ the function on $L^1(c)^*$ defined by

\[
\forall \mu \in L^1(c)^* \quad \alpha(\mu) = \sup_{X \in L^1(c)} (\mu(X) - \rho(X)).
\]

As the dual of $L^1(c)^*$ (with the weak* topology) is $L^1(c)$, the locally convex topological spaces $L^1(c)$ and $L^1(c)^*$ are paired in the sense of [29]. $\rho$ is continuous; we can thus apply Theorem 5 in Rockafellar [29]. We get the following equality:

\[
\forall X \in L^1(c) \quad \rho(X) = \sup_{\mu \in L^1(c)^*} (\mu(X) - \alpha(\mu)).
\]
In the supremum above, we can obviously restrict to the elements $\mu$ of $L^1(c)^*$ such that $\alpha(\mu) < +\infty$.

Let $\mu_0 \in L^1(c)^*$ such that $\alpha(\mu_0) < +\infty$, we first prove that $-\mu_0$ is a positive linear form. Let $X \in L^1(c)$ such that $X \geq 0$. For all $\lambda > 0$, using the monotonicity of $\rho$, $\rho(\lambda X) \leq \rho(0)$, which implies that

$$\lambda \mu_0(X) - \alpha(\mu_0) \leq \rho(0).$$

$\rho(0)$ and $\alpha(\mu_0)$ are finite and the above inequality is satisfied for all $\lambda > 0$, thus, $\mu_0(X) \leq 0$.

From Proposition 2.3, $-\mu_0$ is represented by a finite nonnegative measure defined on $(\Omega, \mathcal{B}(\Omega))$. Thanks to the translation invariance of $\rho$, for all $\lambda \in \mathbb{R}$, $\rho(\lambda) = \rho(0) - \lambda$, which means that

$$\rho(0) = \lambda + \sup_{\mu \in L^1(c)^*} (\lambda \mu(1) - \alpha(\mu)) \geq \lambda (1 + \mu_0(1)) - \alpha(\mu_0).$$

We conclude as above that $1 + \mu_0(1) = 0$. Thus, $-\mu_0$ is a probability measure on $(\Omega, \mathcal{B}(\Omega))$ and $-\mu_0 \in L^1(c)^*$. □

3.2. Risk measures represented by a weakly relatively compact set of probability measures. In this section we want to characterize risk measures $\rho$ on $L^1(c)$ admitting a dual representation with a relatively compact set of probability measures for the weak* topology.

DEFINITION 3.2. A convex risk measure $\rho$ on $L^1(c)$ is normalized if $\rho(0) = 0$.

PROPOSITION 3.1. Let $\rho : L^1(c) \to \mathbb{R}$ be a normalized convex risk measure. The following conditions are equivalent:

1. $\rho$ is majorized by a sublinear risk measure;
2. $\forall X \in L^1(c)$, $\sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda} < \infty$;
3. there exits $K > 0$ such that $\forall X \in L^1(c)$, $|\rho(X)| \leq K c(X)$;
4. $\rho$ is represented by a set $Q$ of probability measures in $L^1(c)^*$ relatively compact for the weak* topology, that is,

$$\forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in Q} (E_Q[-X] - \alpha(Q)).$$

Before giving the proof of the proposition, we prove the following lemma.

LEMMA 3.1. Let $Q$ be a set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that $Q \subset L^1(c)^*$. Assume that $Q$ is relatively compact for the weak* topology $\sigma(L^1(c)^*, L^1(c))$. Then $Q$ is contained in some closed ball of $L^1(c)^*$ and the weak* closure of $Q$ is also compact for the weak topology.
PROOF. Denote $\overline{Q}$ the closure of $Q$ for the weak* topology. $\overline{Q}$ is compact. Let $X \in L^1(c)$. The map $Q \rightarrow E_Q(X)$ is continuous for the weak* topology, thus, $\sup_{Q \in \overline{Q}} |E_Q(X)| < \infty$. From Banach–Steinhauss theorem (cf. [31]) it follows that $\overline{Q}$ is contained in some closed ball of $L^1(c)^*$ and thus in the nonnegative part of this closed ball. From Proposition 2.4, $\overline{Q}$ is weakly compact. □

We can now give the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. Consider the dual representation of $\rho$ given by (3.1). Denote $Q = \{Q \in \mathcal{P}^0|\alpha(Q) < \infty\}$. Then

\begin{equation}
\forall X \in L^1(c) \quad \rho(X) = \sup_{Q \in Q} (E_Q(-X) - \alpha(Q)).
\end{equation}

(1) implies (2). Let $\rho_1$ be a sublinear risk measure majorizing $\rho$. Then for every $\lambda \in \mathbb{R}^*_+$, $\rho(\lambda X) \leq \lambda \rho_1(X)$. Thus, $\sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda} \leq \rho_1(X)$ and (2) is proved.

(2) implies (3). For every $X \in L^1(c)$, denote $\beta_X = \sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda}$. From the dual representation (3.4), applied with $\lambda X$ for every $\lambda > 0$, it follows that $\forall Q \in Q, E_Q(-X) \leq \beta_X$ and thus, $\sup_{Q \in Q} E_Q(-X) \leq \beta_X < \infty$ for every $X \in L^1(c)$. With $X = -|Y|$, we get that

\begin{equation}
\forall Y \in L^1(c) \quad \sup_{Q \in Q} |E_Q(Y)| < \infty.
\end{equation}

$L^1(c)$ is a Banach space and from Theorem 3.1, every $E_Q$ is a continuous linear form on $L^1(c)$. Denote $\|E_Q\|$ its norm. From Banach–Steinhauss theorem, equation (3.5) implies the existence of $K > 0$ such that $\sup_{Q \in Q} \|E_Q\| \leq K$. Notice that from the normalization condition $[\rho(0) = 0]$, it follows from (3.2) that for every $Q$, $\alpha(Q) \geq 0$. Thus, from the representation (3.4), for every $X \in L^1(c)$,

\begin{equation}
\rho(X) \leq K c(X).
\end{equation}

From the convexity, the monotonicity of $\rho$ and $\rho(0) = 0$, it follows that

\begin{equation}
-\rho(X) \leq \rho(-X) \leq \rho(-|X|) \leq K c(-|X|) = K c(X).
\end{equation}

Thus, from (3.6) and (3.7), for every $X \in L^1(c)$,

\begin{equation}
|\rho(X)| \leq K c(X).
\end{equation}

This proves (3).

(3) implies (4). From the representation of $\rho$, equation (3.4) applied with $-\lambda |X|$ for every $\lambda > 0$, it follows from hypothesis (3) that for every $Q \in Q \|E_Q\| \leq K$. This means that $Q$ is contained in a closed ball of the dual of $L^1(c)$. Every such closed ball is compact for the weak* topology (Banach–Alaoglu theorem). Thus, $Q$ is relatively compact for the weak* topology.

(4) implies (1). $\rho$ is represented by a set of probability measures $Q \subset L^1(c)^*$ relatively compact for the weak* topology. From Lemma 3.1, $Q$ is contained in
some closed ball of $L^1(c)^*$. Define $\rho_1$ by $\rho_1(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X)$. As $\mathcal{Q}$ is bounded, $\rho_1(X)$ is finite for every $X$ in $L^1(c)$. It is easy to verify that $\rho_1$ is a sublinear risk measure and that $\rho$ is majorized by $\rho_1$. □

**Theorem 3.2.** Let $\rho$ be a convex risk measure on $L^1(c)$. Assume that $\rho$ is represented by

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q(-X) - \alpha(Q)),$$

where $\mathcal{Q}$ is a set of probability measures in $L^1(c)^*$ relatively compact for the weak* topology. Let $\overline{\mathcal{Q}}$ be the closure of $\mathcal{Q}$ for the weak* topology. Then $\overline{\mathcal{Q}}$ is metrizable compact both for the weak* topology and the weak topology.

For every $X \in L^1(c)$, there is a probability measure $Q_X \in \overline{\mathcal{Q}}$ such that

$$\rho(X) = E_{Q_X}(-X) - \alpha(Q_X). \tag{3.8}$$

**Proof.** From Lemma 3.1, $\overline{\mathcal{Q}}$ is contained in a closed ball of $L^1(c)^*$ and is compact both for the weak and the weak* topology. From Proposition 2.5 it is metrizable compact. Let $X \in L^1(c)$. Let $Q_n$ be a sequence of elements in $\mathcal{Q}$ such that for every $n$,

$$\rho(X) - \frac{1}{n} < E_{Q_n}(-X) - \alpha(Q_n) \leq \rho(X). \tag{3.9}$$

As $\overline{\mathcal{Q}}$ is metrizable compact for the weak* topology, there is a subsequence $Q_{\phi(n)}$ converging to $\tilde{Q} \in \overline{\mathcal{Q}}$, satisfying the inequality

$$E_{\tilde{Q}}(-X) - \frac{1}{n} < E_{Q_{\phi(n)}}(-X) < E_{\tilde{Q}}(-X) + \frac{1}{n}. \tag{3.10}$$

From inequality (3.9), applied with $Q_{\phi(n)}$, inequality (3.10) and the inequality $\phi(n) \geq n$, it follows that

$$E_{\tilde{Q}}(-X) - \rho(X) - \frac{1}{n} < \alpha(Q_{\phi(n)}) < E_{\tilde{Q}}(-X) - \rho(X) + \frac{2}{n}. \tag{3.11}$$

Let $Y \in L^1(c)$. Let $\varepsilon > 0$. There is $N(Y)$ such that for every $n > N(Y)$, $E_{\tilde{Q}}(-Y) < E_{Q_{\phi(n)}}(-Y) + \varepsilon$. $N(Y)$ can be chosen such that $N(Y) \geq \frac{1}{\varepsilon}$. Then for $n \geq N(Y)$,

$$E_{\tilde{Q}}(-Y) - \rho(Y) \leq \alpha(Q_{\phi(n)}) + \varepsilon \leq E_{\tilde{Q}}(-X) - \rho(X) + \frac{2}{n} + \varepsilon \leq E_{\tilde{Q}}(-X) - \rho(X) + 3\varepsilon. \tag{3.12}$$

As the inequality is satisfied for every $Y$ and every $\varepsilon > 0$, it follows that

$$\alpha(\tilde{Q}) = \sup_{Y \in L^1(c)} (E_{\tilde{Q}}(-Y) - \rho(Y)) \leq E_{\tilde{Q}}(-X) - \rho(X).$$
and thus, 
\[
\rho(X) = E_{\hat{Q}}(-X) - \alpha(\hat{Q}).
\]

**Proposition 3.2.** Let \( \rho \) be a normalized convex risk measure on \( L^1(c) \) majorized by a sublinear risk measure. There is a countable set \( \{ R_n, n \in \mathbb{N} \} \) of probability measures belonging to \( L^1(c)^* \) which is relatively compact for the weak* topology of \( L^1(c)^* \) and also for the weak topology and such that 
\[
\forall X \in L^1(c) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{R_n}[-X] - \alpha(R_n)),
\]
where
\[
\alpha(R) = \sup_{X \in L^1(c)} (E_R[-X] - \rho(X)).
\]

**Proof.** From Proposition 3.1, there is a set \( Q \) of probability measures in \( L^1(c)^* \), relatively compact for the weak* topology, such that equation (3.3) is satisfied. From Lemma 3.1, \( Q \) is contained in \( mK_+ \), the nonnegative part of a certain closed ball of \( L^1(c)^* \). From Proposition 2.6, \( mK_+ \) is metrizable compact for the weak* topology. There is thus a countable dense set \( \{ Q_n \} \) in \( mK_+ \) satisfying the required condition. \( \square \)

**Theorem 3.3.** Every convex risk measure on \( L^1(c) \) can be represented by a countable set of probability measures \( \{ R_n, n \in \mathbb{N} \} \) belonging to \( L^1(c)^* \).
\[
\forall X \in L^1(c) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{R_n}(-X) - \alpha(R_n)),
\]
where \( \alpha(R) \) is given by (3.14).

**Proof.** From Theorem 3.1, \( \rho \) has a dual representation given by (3.1). Denote then \( \rho_m(X) = \sup_{Q \in mK_+} (E_Q(-X) - \alpha(Q)) \). Even if \( \rho_m \) is not necessarily normalized, all the arguments of the proof of Proposition 3.2 apply as \( mK_+ \) is metrizable compact for the weak* topology and \( \alpha \) is l.s.c. Thus, \( \rho_m \) has a representation with a countable set of probability measures. As \( \rho = \sup_{m \in \mathbb{N}} \rho_m \), this gives the result. \( \square \)
4. Equivalence class of probability measures associated to a nondominated set of probability measures. Let $\Omega$ be a metrizable and separable space. In this section we study a capacity defined from a weakly relatively compact set of probability measures $\mathcal{P}$ possibly nondominated.

**Definition 4.1.** Let $\mathcal{P}$ be a weakly relatively compact set of probability measures on $(\Omega, \mathcal{B}(\Omega))$. Let $1 \leq p < \infty$. The capacity $c_{p, \mathcal{P}}$ is defined on $C_b(\Omega)$ by

$$c_{p, \mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p}$$

and extended to every function on $\Omega$ as explained in Section 2.1, equations (2.1) and (2.2).

Notice that as $\mathcal{P}$ is a weakly relatively compact set of probability measures, $c_{p, \mathcal{P}}$ is a capacity (see Proposition I.3 of [24] or the Appendix). The Banach space associated to the capacity $c_{p, \mathcal{P}}$ is denoted $L^1(c_{p, \mathcal{P}})$. When there is no ambiguity on the set $\mathcal{P}$ we simply write $c_p$ for $c_{p, \mathcal{P}}$.

When $\mathcal{P} = \{\mu_0\}$, $L^1(c_{p, \mu_0}) = L^1(\Omega, \mathcal{B}(\Omega), \mu_0)$. A nonnegative measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$ belongs to the (usual) equivalence class of the probability measure $\mu_0$ if and only if $\forall A \in \mathcal{B}(\Omega), \mu(A) = 0 \iff \mu_0(A) = 0$.

Equivalently, for $\mu$ in the dual of $L^1(\Omega, \mathcal{B}(\Omega), \mu_0)$,

$$\mu \sim \mu_0 \iff \left[ \forall X \in L^1(\Omega, \mathcal{B}(\Omega), \mu_0)_+ , X = 0 \iff \int X d\mu = 0 \right].$$

We address the following question: when $\mathcal{P}$ is weakly relatively compact, can one associate a probability measure $P$ to $L^1(c_{p, \mathcal{P}})$ characterizing the null elements in the cone $L^1(c_{p, \mathcal{P}})_+$, that is, such that $\forall X \in L^1(c_{p, \mathcal{P}})_+, X = 0 \iff E_P(X) = \int X dP = 0$? If yes, can one define a natural equivalence relation so that one gets a unique equivalence class of such probability measures? Notice that when $\mathcal{P}$ is not finite, characteristic functions of Borelian sets are not all in $L^1(c_{p, \mathcal{P}})$.

4.1. Properties of the capacity.

**Lemma 4.1.** For all $X$ in $L^1(c_{p, \mathcal{P}})$, $c_{p, \mathcal{P}}(X) = \sup_{Q \in \mathcal{P}} E_Q(|X|^p)^{1/p}$.

**Proof.** Denote $c_p = c_{p, \mathcal{P}}$. For all $f, g$ in $C_b(\Omega)$, for all $Q \in \mathcal{P}$,

$$|E_Q(|f|^p)^{1/p} - E_Q(|g|^p)^{1/p}| \leq E_Q(|f - g|^p)^{1/p} \leq c_p(|f - g|).$$

As $C_b(\Omega)$ is dense in $L^1(c_p)$ for the $c_p$ norm, it follows that for every $X \in L^1(c_p)$, $g \in C_b(\Omega)$ and $Q \in \mathcal{P}$,

$$|E_Q(|X|^p)^{1/p} - E_Q(|g|^p)^{1/p}| \leq c_p(|X - g|).$$

(4.2)
From (4.2) it follows that
\[ E_Q(|X|^p)^{1/p} \leq c_p(X) \quad \forall Q \in \mathcal{P}. \]

For every \( X \in L^1(c_p) \), for every \( \varepsilon > 0 \) there is \( g \in C_b(\Omega) \) such that
\[ c_p(X - g) \leq \varepsilon. \]

From Definition 4.1, there is \( Q_0 \in \mathcal{P} \) such that
\[ c_p(g) \leq E_{Q_0}(|g|^p)^{1/p} + \varepsilon. \]

As \( c_p(X) \leq c_p(g) + \varepsilon \), it follows from (4.2), (4.4) and (4.5) that \( c_p(X) \leq \sup_{Q \in \mathcal{P}} E_Q(|X|^p)^{1/p} \). The result follows from (4.3).

**Theorem 4.1.** Assume that \( \Omega \) is a Polish space. There is a countable subset \( Q \) of \( \mathcal{P} \), \( Q = \{ P_n, n \in \mathbb{N} \} \), such that for every \( X \in L^1(c_p, \mathcal{P}) \), for every \( p \in [1, \infty[, \)
\[ c_{p, \mathcal{P}}(X) = \sup_{n \in \mathbb{N}} E_{P_n}(|X|^p)^{1/p}. \]

The capacities \( c_{p, \mathcal{P}} \) and \( c_{p, Q} \) defined on \( C_b(\Omega) \) by (4.1) and extended to real functions using formulas (2.1) and (2.2) are equal. The associated Banach spaces are equal: \( L^1(c_{p, \mathcal{P}}) = L^1(c_{p, Q}). \)

**Proof.** From the previous lemma, applied with \( p = 1 \), it follows that the set \( \mathcal{P} \) is contained in \( K_+ \), the nonnegative part of the unit ball of the dual of \( L^1(c_{1, \mathcal{P}}) \). \( \Omega \) is a Polish space, so from Corollary 2.1, \( K_+ \) is metrizable compact for the weak* topology. Thus \( \overline{\mathcal{P}} \), the closure of \( \mathcal{P} \) for the weak* topology, is metrizable compact. There is then in \( \mathcal{P} \) a countable set \( (P_n)_{n \in \mathbb{N}} \) dense in \( \overline{\mathcal{P}} \) for the weak* topology. It follows that for every \( X \in L^1(c_{1, \mathcal{P}}) \), \( \sup_{Q \in \mathcal{P}} E_Q(|X|^p)^{1/p} = \sup_{n \in \mathbb{N}} E_{P_n}(|X|^p)^{1/p} \). The equation (4.6) follows for every \( p \geq 1 \) for every \( X \in C_b(\Omega) \).

The two capacities
\[ c_{p, \mathcal{P}}(f) = \sup_{P \in \mathcal{P}} E_P(|f|^p)^{1/p} \quad \text{and} \quad c_{p, Q} = \sup_{Q \in \mathcal{Q}} E_Q(|f|^p)^{1/p} \]

coincide on \( C_b(\Omega) \). By definition of the extension of a capacity to the set of all functions on \( \Omega \), these extensions are the same. Therefore, \( L^1(c_{p, \mathcal{P}}) = L^1(c_{p, Q}) \).

In the following proposition we study possible extensions of (4.1).

**Proposition 4.1.** Let \( c_p = c_{p, \mathcal{P}} \). For every nonnegative bounded lower semi-continuous map \( g \),
\[ c_p(g) = \sup_{Q \in \mathcal{P}} E_Q(g^p)^{1/p}. \]

For every Borelian map \( f \),
\[ \sup_{Q \in \mathcal{P}} E_Q(|f|^p)^{1/p} \leq c_p(f). \]
Proof. The proof of the first part of Proposition 2.1, which was given for the characteristic function of an open set, applies without any change to every nonnegative bounded l.s.c. function \( g \). Thus, there is an increasing sequence of continuous functions \( h_n \) with limit \( g \) and such that \( c_p(g) = \lim c_p(h_n) \). As \( g \) is bounded, \( c_p(g) \) is finite. Let \( \epsilon > 0 \). There is \( n \) such that \( c_p(g) - \epsilon \leq c_p(h_n) \leq c_p(g) \). By definition of \( c_p \) on \( C_b(\Omega) \), there is \( Q_n \) in \( \mathcal{P} \) such that \( c_p(h_n) - \epsilon \leq E_{Q_n}(h_n) \leq c_p(h_n) \). Thus,

\[
E_{Q_n}(g^p)^{1/p} \geq c_p(g) - 2\epsilon.
\]

On the other hand, for all \( Q \) in \( \mathcal{P} \), \( E_Q(h_n) \leq c_p(h_n) \leq c_p(g) \). From the monotone convergence theorem it follows that

\[
\forall Q \in \mathcal{P} \quad E_Q(g^p)^{1/p} \leq c_p(g).
\]

Thus, from (4.9) and (4.10) we get that

\[
(4.11) \quad c_p(g) = \sup_{Q \in \mathcal{P}} E_Q(g^p)^{1/p}.
\]

Let \( f \) be a Borelian map. If \( c_p(f) = +\infty \), the result is trivial. Assume that \( c_p(f) < \infty \). Let \( \epsilon > 0 \). By definition of \( c_p(f) \), equation (2.2), there is \( g \) l.s.c., \( g \geq |f| \) such that \( c_p(g) < c_p(f) + \epsilon \). As \( g \) is l.s.c., we already know that \( \sup_{Q \in \mathcal{P}} E_Q(|g|^p)^{1/p} = c_p(g) \). As \( f \) is Borel measurable, for all \( Q \in \mathcal{P} \), \( E_Q(|f|^p)^{1/p} \) is defined. As \( g \geq |f| \) it follows that \( E_Q(|f|^p)^{1/p} \leq c_p(f) + \epsilon \). This inequality is true for every \( \epsilon \) and every \( Q \in \mathcal{P} \). This proves the announced result for every \( f \) Borel measurable. \( \square \)

Remark 1. For every open subset \( V \) of \( \Omega \), \( 1_V \) is lower semi-continuous, so from Proposition 4.1, \( c_p(1_V) = \sup_{Q \in \mathcal{P}} Q(V)^{1/p} \). However, there are Borelian subsets of \( \Omega \) for which the equality \( c_p(1_A) = \sup_{Q \in \mathcal{P}} Q(A)^{1/p} \) is not satisfied.

For example, let \( \Omega = [0, 1] \). Let \( x_n \in [0, 1] \) be a sequence converging to 0. Let \( A = [0, 1] - \{ x_n, n \in \mathbb{N} \} \). Let \( Q_n = \delta_{x_n} \). Let \( \mathcal{P} = \{ Q_n, n \in \mathbb{N} \} \). \( \mathcal{P} \) is weakly relatively compact. Let \( f \) l.s.c. such that \( 1_A \leq f \leq 1 \). For every \( \eta > 0 \), \( V = \{ x| f(x) > 1 - \eta \} \) is an open set containing \( A \). As \( 0 \in A \), there is \( \epsilon > 0 \) such that \( [0, \epsilon] \subset V \). So there is \( N \in \mathbb{N} \) such that \( x_n \in V \forall n \geq N \). So \( E_{Q_n}(f^p) = (f(x_n))^p > (1 - \eta)^p \).

From (4.7), \( 1 \geq c_p(f) = \sup_{n \in \mathbb{N}} (E_{Q_n}(f^p))^{1/p} \). From every \( \eta > 0 \). Thus, \( c_p(f) = 1 \). It follows that \( c_p(1_A) = 1 \). On the other hand, \( Q_n(1_A) = 0 \) for all \( n \in \mathbb{N} \). Therefore, \( \sup_{Q \in \mathcal{P}} Q(A)^{1/p} = 0 \). This gives a counterexample.

4.2. Canonical equivalence class of nonnegative measures associated to \( c_p \).

In all this section, we assume that \( \Omega \) is a Polish space. We denote \( c_p \) the capacity defined on \( C_b(\Omega) \) by \( c_p(f) = \sup_{Q \in \mathcal{P}} E_Q(|f|^p)^{1/p} \).

Definition 4.2. \( \mathcal{M}^+(c_p) \) is the set of nonnegative finite measures on \( (\Omega, \mathcal{B}(\Omega)) \) defining an element of \( L^1(c_p)^* \).
In the following, we identify an element $\mu$ of $\mathcal{M}^+(c_p)$ with its associated linear form on $L^1(c_p)$.

**Remark 2.** A nonnegative finite measure $\mu$ on $(\Omega, \mathcal{B}(\Omega))$ belongs to $\mathcal{M}^+(c_p)$ if and only if there is a constant $K > 0$ such that $\forall f \in C_b(\Omega), |\mu(f)| \leq Kc_p(f)$. It follows easily that every element in the weak closure of the convex hull of $\mathcal{P}$ defines an element of $\mathcal{M}^+(c_p)$.

**Definition 4.3.** Define on $\mathcal{M}^+(c_p)$ the relation $R_{c_p}$ by

$$\mu R_{c_p} \nu \iff \{ X \in L^1(c_p), X \geq 0 \mid \mu(X) = 0 \} = \{ X \in L^1(c_p), X \geq 0 \mid \nu(X) = 0 \}. $$

(4.12)

The following lemma is trivial.

**Lemma 4.2.** $R_{c_p}$ defines an equivalence relation on $\mathcal{M}^+(c_p)$.

**Definition 4.4.** Let $\mu \in \mathcal{M}^+(c_p)$. The $c_p$-class of $\mu$ is the equivalence class of $\mu$ for the equivalence relation $R_{c_p}$.

**Theorem 4.2.** To every weakly relatively compact set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$, possibly nondominated, can be associated canonically a $c_p$-class of nonnegative measures on $(\Omega, \mathcal{B}(\Omega))$ such that an element $\mu$ of $\mathcal{M}^+(c_p)$ belongs to this class if and only if $\forall X \in L^1(c_p), X \geq 0 \{ \mu(X) = 0 \} \iff \{ X = 0 \text{ in } L^1(c_p) \}$.

This class is referred to as the canonical $c_p$-class.

For every set $\{ Q_n, n \in \mathbb{N} \}$ of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that equality (4.6) is satisfied for all $X \in L^1(c_p)$, for $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n = 1$, the probability measure $\sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical $c_p$-class.

**Proof.** Let $p \in [1, \infty[$. Let $\{ Q_n \}$ be a countable set of probability measures such that equality (4.6) is satisfied. Let $Q = \{ Q_n, n \in \mathbb{N} \}$. Let $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$. Let $X \in L^1(c_p), X \geq 0$, that is, from Lemma 2.1, $X = |X|$. $E_P(X) = 0$ if and only if $E_{Q_n}(|X|) = 0$ for all $n \in \mathbb{N}$.

From (4.6), it follows that for $X \geq 0$, $E_P(X) = 0$ if and only if $c_p(X) = 0$, if and only if $X = 0$ in $L^1(c_p)$.

This proves that the canonical $c_p$-class is well defined (as it is not empty) and that $\sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical $c_p$-class. □

**Lemma 4.3.** Let $P$ be a probability measure belonging to the canonical $c_p$-class. Let $X$ be an element of $L^1(c_p)$. Then $X \geq 0$ [for the order in $L^1(c_p)$] if and only $X \geq 0$ $P$ a.s.
PROOF. For every $X \in L^1(c_p)$, $|X| - X \geq 0$. From Lemma 2.1 $X \geq 0$ if and only if $|X| - X = 0$ in $L^1(c_p)$. By definition of the canonical $c_p$-class, this is equivalent to $|X| - X = 0$ $P$ a.s., that is, $X \geq 0$ $P$ a.s. □

REMARK 3. When $\mathcal{P} = \{P\}$, the canonical $c_p$-class is the restriction to $\mathcal{M}^+(c_p)$ of the usual equivalence class of the probability measure $P$.

When $\mathcal{P}$ is a finite set, $\mathcal{P} = \{P_1, \ldots, P_n\}$ the canonical $c_p$-class is the restriction to $\mathcal{M}^+(c_p)$ of the equivalence class (in the usual sense) of the probability measure $P = \frac{1}{n} \sum_{i=1}^{n} P_i$.

Our next goal is to give a description of $L^1(c_p)^*$.

**THEOREM 4.3.** There is a regular probability measure $P$ belonging to the canonical $c_p$-class and a countable subset $\mathcal{D} = \{L_n, n \in \mathbb{N}\}$ of the set $L^1(c_p)^+_*$ of nonnegative continuous linear forms on $L^1(c_p)$ such that:

- $\{L_n, n \in \mathbb{N}\}$ is dense in $L^1(c_p)^+_* = \mathcal{M}^+(c_p)$ for the weak* topology.
- Every $L_n$ is represented by a nonnegative measure on $(\Omega, \mathcal{B}(\Omega))$ absolutely continuous with respect to $P$.

Every continuous linear form $\Phi$ on $L^1(c_p)$ is the weak* limit of a sequence $\Phi_n$ where every $\Phi_n$ is the difference of two elements of $\mathcal{D}$.

Furthermore, for every $X \geq 0$ in $L^1(c_p)$, $X = 0$ iff $P(X) = 0$, iff $L_n(X) = 0$ for all $n \in \mathbb{N}$.

**PROOF.** Denote $nK_+ = \{L \in L^1(c_p)^*, L \geq 0$ and $\|L\| \leq n\}$. From Corollary 2.1, every $nK_+$ is metrizable compact for the weak* topology. There is then in $nK_+$ a dense countable set $\mathcal{D}_n$. Thus, $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$ is countable and dense in $L^1(c_p)^+_*$ for the weak* topology. Enumerate the elements of $\mathcal{D}$, $\mathcal{D} = \{L_n, n \in \mathbb{N}\}$. From Proposition 2.3, every $L_n$ is represented by a nonnegative finite measure $\mu_n$ on $(\Omega, \mathcal{B}(\Omega))$. Let $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n \|L_n\| < \infty$. Then $\tilde{L} = \sum_{n \in \mathbb{N}} \alpha_n L_n \in L^1(c_p)^+_*$. From Proposition 2.3, $\tilde{L}$ is represented by a nonnegative finite measure $\mu$. Denote $P$ the probability measure $P = \frac{\mu}{\mu(\Omega)}$. $P$ is a probability measure on $(\Omega, \mathcal{B}(\Omega))$, $P \in \mathcal{M}^+(c_p)$. Furthermore, every $\mu_n$ is absolutely continuous with respect to $P$ and $P$ is regular from Theorem 1.1 of [5].

We prove now that $P$ belongs to the canonical $c_p$-class. Every $L_n$ belongs to $L^1(c_p)^*$. Thus, for every $X$ in $L^1(c_p)$ such that $X = 0$ in $L^1(c_p)$, $L_n(X) = 0$ and thus $\tilde{L}(X) = 0$. It follows that $P(X) = 0$. Conversely let $X \geq 0$ in $L^1(c_p)$ such that $P(X) = 0$. It follows that $\tilde{L}(X) = 0$. Every $L_n$ belongs to $L^1(c_p)^+_*$ and $X \geq 0$, thus, $L_n(X) \geq 0$ for all $n$. From the equality $\tilde{L}(X) = 0$, it follows that $L_n(X) = 0 \forall n \in \mathbb{N}$. $\{L_n, N \in \mathbb{N}\}$ is dense in $L^1(c_p)^+_*$ for the weak* topology, therefore, $L(X) = 0$ for all $L \in L^1(c_p)^+_*$. From the representation result of continuous linear forms on $L^1(c_p)$ (Proposition 2.3) and the Jordan decomposition
of bounded signed measures on \((\Omega, \mathcal{B}(\Omega))\), it follows that every \(\Phi \in L^1(c_p)^*\) is represented by a bounded measure \(\mu = \mu^+ - \mu^-\). There is a Borelian set \(A\) such that \(\int f \, d\mu^+ = \int f \chi_A \, d\mu\) for every \(f \in C_b(\Omega)\). \(|\mu| = \mu^+ + \mu^-\) is defined on \((\Omega, \mathcal{B}(\Omega))\) and is thus regular from Theorem 1.1 of [5].

\[
(4.13) \quad \forall \varepsilon > 0, \exists V \text{ open, } A \subset V \quad \text{such that} \quad |\mu|(1_A - 1_A) \leq \varepsilon.
\]

\(1_V\) is lower semi-continuous so it is the increasing limit of a sequence of continuous functions \(h_n\). From the monotone convergence theorem and equation (4.13), it follows that

\[
(4.14) \quad \forall \varepsilon > 0, \exists h \in C_b(\Omega), 0 \leq h \leq 1_V \quad \text{such that} \quad \int |1_A - h| \, d|\mu| < \varepsilon.
\]

Thus,

\[
(4.15) \quad \left| \int f \chi_A \, d\mu - \int f h \, d\mu \right| < \|f\|\infty \varepsilon.
\]

By definition of \(\mu\),

\[
(4.16) \quad \forall f \in C_b(\Omega) \quad \left| \int f h \, d\mu \right| < \|\Phi\|c_p(f h) \leq \|\Phi\|c_p(f).
\]

From (4.15) and (4.16), we get \(\int f \, d\mu^+ = \int f \chi_A \, d\mu \leq \|\Phi\|c_p(f)\). It follows that \(\mu^+\) defines an element of \(L^1(c_p)^*\). It is the same for \(\mu^-\). Thus, for every \(\Phi \in L^1(c_p)^*\), \(\Phi(X) = 0\). From Hahn–Banach theorem, it follows that \(X = 0\) in \(L^1(c_p)\). This proves that \(P\) belongs to the canonical \(c_p\)-class.

We have proved that every \(\Phi \in L^1(c_p)^*\) can be written \(\Phi = \Phi^+ - \Phi^-\), \(\Phi^+, \Phi^- \in L^1(c_p)^*\). The result follows then from the density of \(\mathcal{D}\) in \(L^1(c_p)^*\).

The results of the previous section on convex risk measures on \(L^1(c)\) can be specified when the capacity is \(c_p\).

**Proposition 4.2.** Let \(\rho\) be a convex risk measure on \(L^1(c_p)\). There is a probability measure \(Q\) in the canonical \(c_p\)-class and a countable set \(\{Q_n, n \in \mathbb{N}\}\) of probability measures all absolutely continuous with respect to \(Q\) such that

\[
(4.17) \quad \rho(X) = \sup_{n \in \mathbb{N}} [E_{Q_n}(-X) - \alpha(Q_n)] \quad \forall X \in L^1(c_p).
\]

**Proof.** From Theorem 3.3, there is a countable set \(\{Q_n, n \in \mathbb{N}\}\) of probability measures such that equation (4.17) is satisfied. From Theorem 4.2 there is a probability measure \(P\) in the canonical \(c_p\)-class. Let \(Q = \frac{P}{2} + \sum_{n \in \mathbb{N}} \frac{Q_n}{2^n + 2}\). It is easy to verify that \(Q\) satisfies the required conditions. \(\square\)
Remark 4. Even if the capacity $c_p$ is defined from a weakly relatively compact set of probability measures, the set of probability measures $\{Q_n, n \in \mathbb{N}\}$ in the above dual representation (4.17) of a convex risk measure $\rho$ on $L^1(c_p)$ is not always relatively compact for the weak* topology. From Proposition 3.3, $\{Q_n, n \in \mathbb{N}\}$ is relatively compact iff $\rho$ is majorized by a sublinear risk measure.

5. Regular risk measures on $C_b(\Omega)$.

5.1. Regularity. Notice that in a context of uncertainty, which is when no reference probability measure is given, it is natural to consider risk measures defined on the space $C_b(\Omega)$ or more generally on a lattice vector subspace of $C_b(\Omega)$. As in Section 2.1, $L$ denotes a linear vector subspace of $C_b(\Omega)$ containing the constants, generating the topology of $\Omega$ and which is a vector lattice.

Definition 5.1. $\rho : L \rightarrow \mathbb{R}$ is a convex risk measure on $L$ if it satisfies the axioms of Definition 3.1, replacing everywhere $L^1(c)$ by $L$. It is normalized if $\rho(0) = 0$.

A sublinear risk measure $\rho$ on $L$ is regular if for every decreasing sequence $X_n$ of elements of $L$ with limit 0, $\rho(-X_n)$ tends to 0. A normalized convex risk measure is uniformly regular if for all $X \sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda} < \infty$ and for every decreasing sequence $X_n$ of elements of $L$ with limit 0, $\frac{\rho(-X_n)}{\lambda}$ converges to 0 uniformly in $\lambda$.

Remark 5. For sublinear risk measures, the two notions of regularity and uniform regularity are equivalent.

From now on in this section $\rho$ is a normalized convex risk measure on $L$.

Lemma 5.1. Assume that $\rho$ is uniformly regular. $\rho_{\text{min}}(X) = \sup_{\lambda > 0} \frac{\rho(\lambda X)}{\lambda}$ defines a regular sublinear risk measure on $L$. It is the minimal sublinear risk measure on $L$ majorizing $\rho$.

Proof. The convexity, monotonicity and translation invariance of $\rho_{\text{min}}$ follow easily from the same properties of $\rho$. The homogeneity of $\rho_{\text{min}}$ follows from its definition. Thus, $\rho_{\text{min}}$ is a sublinear risk measure on $L$ majorizing $\rho$. The regularity of $\rho_{\text{min}}$ follows from the uniform regularity of $\rho$. For every sublinear risk measure $\rho_1$ majorizing $\rho$, for every $X \in L$, $\rho_{\text{min}}(X) \leq \rho_1(X)$. Thus, $\rho_{\text{min}}$ is minimal. □

Lemma 5.2. For every $Y$ in $L$, for every sequence $\lambda_n$ of real numbers decreasing to 1, the sequence $\rho(\lambda_n Y)$ converges to the limit $\rho(Y)$.

Proof. As $\lambda_n$ is a decreasing sequence with limit 1, one can assume that $2 > \lambda_n \geq 1$. Write $\lambda_n = 1 + \varepsilon_n$, $0 \leq \varepsilon_n < 1$. From the convexity of $\rho$ and $\rho(0) = 0$, it follows that

\[\rho((1 + \varepsilon_n)Y) \geq (1 + \varepsilon_n)\rho(Y),\]
\((1 + \varepsilon_n)Y = (1 - \varepsilon_n)Y + \varepsilon_n(2Y)\). Using the convexity of \(\rho\), it follows that
\[
(5.2) \quad \rho((1 + \varepsilon_n)Y) \leq (1 - \varepsilon_n)\rho(Y) + \varepsilon_n\rho(2Y).
\]
From (5.1) and (5.2),
\[
(5.3) \quad (1 + \varepsilon_n)\rho(Y) \leq \rho((1 + \varepsilon_n)Y) \leq (1 - \varepsilon_n)\rho(Y) + \varepsilon_n\rho(2Y).
\]
Passing now to the limit in inequality (5.3), it follows that the sequence \(\rho((1 + \varepsilon_n)Y)\) has a limit equal to \(\rho(Y)\). □

Using the preceding lemma, we prove now that every normalized uniformly regular convex risk measure can be extended into a convex risk measure on \(L^1(c)\) for some capacity \(c\). Therefore, we will be able to apply the representation results of Section 3.

**Lemma 5.3.** Assume that \(\rho\) is uniformly regular. Denote \(\rho_1\) a regular sublinear risk measure on \(\mathcal{L}\) such that \(\rho \leq \rho_1\).

- \(c(X) = \rho_1(-|X|)\) defines a capacity on \(\mathcal{L}\).
- \(\rho_1\) has a unique continuous extension into a sublinear risk measure \(\overline{\rho}_1\) on \(L^1(c)\).
- \(\rho\) has a unique continuous extension into a normalized convex risk measure \(\overline{\rho}\) on \(L^1(c)\) majorized by \(\overline{\rho}_1\).

**Proof.** The sublinearity, monotonicity and regularity of \(\rho_1\) imply that \(c\) is a capacity on \(\mathcal{L}\). As usual, this leads to the Banach space \(L^1(c)\). As \(\rho_1\) is sublinear, for every \(X, Y \in \mathcal{L}\), \(\rho_1(X) \leq \rho_1(Y) + \rho_1(X - Y)\).

Exchanging \(X\) and \(Y\) and using the monotonicity of \(\rho_1\) and the definition of \(c\), it follows that \(|\rho_1(X) - \rho_1(Y)| \leq c(X - Y)\). Thus, \(\rho_1\) is uniformly continuous on \(\mathcal{L}\) for the \(c\) semi-norm. It extends uniquely into a continuous function \(\overline{\rho}_1\) on \(L^1(c)\). \(\overline{\rho}_1\) is a sublinear risk measure. Let \(\varepsilon_n > 0\) decreasing to 0:
\[
X = \frac{1}{1 + \varepsilon_n}[(1 + \varepsilon_n)Y] + \frac{\varepsilon_n}{1 + \varepsilon_n}\left[\frac{1 + \varepsilon_n}{\varepsilon_n}(X - Y)\right].
\]
From the convexity of \(\rho\), the majoration of \(\rho\) by \(\rho_1\) and the homogeneity of \(\rho_1\) (cf. \(\rho_1\) is sublinear), it follows that
\[
(5.4) \quad \rho(X) \leq \frac{1}{1 + \varepsilon_n}\rho((1 + \varepsilon_n)Y) + \rho_1(X - Y).
\]
From inequality (5.4) and Lemma 5.2 applied with \((1 + \varepsilon_n)Y\), passing to the limit, it follows then that \(\rho(X) - \rho(Y) \leq \rho_1(X - Y) \leq c(X - Y)\). Exchanging \(X\) and \(Y\), this proves the uniform continuity of \(\rho\) for the \(c\) semi-norm. \(\rho\) extends then uniquely into a continuous function \(\overline{\rho}\) on \(L^1(c)\). \(\overline{\rho}\) is a convex risk measure on \(L^1(c)\) majorized by \(\overline{\rho}_1\). □

**Definition 5.2.** Let \(\rho\) be a normalized uniformly regular convex risk measure on \(\mathcal{L}\). The capacity \(c_\rho\) defined as \(c_\rho(X) = \rho_{\min}(-|X|)\) is called the capacity canonically associated with \(\rho\).
5.2. Representation of uniformly regular convex risk measures. In this section, we assume that $\Omega$ is a Polish space. Taking into account the liquidity risk in a financial market, we introduce the following definition for a riskless asset, which means that all investment in this asset is risk-free.

**Definition 5.3.** A nonpositive element $X$ of $C_b(\Omega)$ is riskless if for all $\lambda > 0$, $\rho(\lambda X) = 0$ [or equivalently for all $\lambda > 0$, $\rho(\lambda X) \leq 0$].

**Theorem 5.1.** Let $\rho$ be a normalized uniformly regular convex risk measure on $L$. Then $\rho$ extends uniquely to $C_b(\Omega)$ and admits the following representation,

\[
\forall X \in C_b(\Omega) \quad \rho(X) = \sup_{n \in \mathbb{N}} (E_{Q_n}(-X) - \alpha(Q_n))
\]

(5.5)

for a certain weakly relatively compact set $\{Q_n, n \in \mathbb{N}\}$ of probability measures. Furthermore, for $\alpha_n > 0$ such that $\sum_{n \in \mathbb{N}} \alpha_n = 1$, the probability measure $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$ characterizes the riskless nonnegative elements of $C_b(\Omega)$, that is, $X \leq 0$ is riskless iff $X = 0$ $P$ a.s.

For every $X \in C_b(\Omega)$ there is a probability measure $Q_X$ in the weak closure of $\{Q_n, n \in \mathbb{N}\}$, such that

\[
\rho(X) = E_{Q_X}(-X) - \alpha(Q_X).
\]

(5.6)

**Proof.** Let $c_\rho(X) = \rho_{\min}(-|X|)$ be the capacity canonically associated with $\rho$ (Definition 5.2). As $\Omega$ is a Polish space, every capacity is a Prokhorov capacity. Denote $\rho$ (resp., $\rho_{\min}$) the extensions of $\rho$ (resp., $\rho_{\min}$) to $L^1(c_\rho)$ given by Lemma 5.3.

As $\rho$ is majorized by $\rho_{\min}$, the representation result with a countable weakly relatively compact set $Q = \{Q_n\}$ follows from Proposition 3.2. We can, of course, restrict to $Q_n$ such that $\alpha(Q_n) < \infty$. Then $c_\rho(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(|X|)$, that is, $c_\rho = c_{1, Q}$. From Theorem 4.2 the probability measure $P = \sum_{n \in \mathbb{N}} \alpha_n Q_n$ belongs to the canonical $c_{\rho}$-class. Let $X \leq 0$ in $C_b(\Omega)$, $X$ is riskless iff $\rho(\lambda X) = 0 \forall \lambda > 0$, iff $c_\rho(-X) = 0$, iff $X = 0$ $P$ a.s. The existence of $Q_X$ follows from Theorem 3.2. $\square$

6. Examples.

6.1. $G$-expectations. In this section, $\Omega = C_0([0, \infty[, \mathbb{R}^d)$, the set of continuous functions $f$ defined on $[0, \infty[$ with values in $\mathbb{R}^d$ such that $f(0) = 0$, $C_0([0, \infty[, \mathbb{R}^d)$ endowed with the topology of uniform convergence on compact spaces is a Polish space.

Peng introduced the notion of sublinear expectation and of $G$-expectations [27, 28] defined on a vector lattice $\mathcal{H}$ of real functions containing 1 and included
in $C_b(\Omega)$. For the definition of a sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ we refer to [15], Section 3. $G$-expectations are defined from solutions of P.D.E. in [27] and [28]. A $G$-expectation is up to a minus sign a sublinear risk measure.

It is proved in [15] and [23] that every $G$-expectation $\mathbb{E}$ has a representation with respect to a weakly relatively compact set of probability measures $\mathcal{P}: \mathbb{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$ for all $f \in \mathcal{H}$. $\mathbb{E}$ extends naturally to $C_b(\Omega)$,

$$\mathbb{E}(f) = \sup_{P \in \mathcal{P}} E_P(f) \quad \forall f \in C_b(\Omega). \quad (6.1)$$

As $\mathcal{P}$ is weakly relatively compact, $\rho(f) = \mathbb{E}(-f)$ is a sublinear regular risk measure on $C_b(\Omega)$. Denote $c_\mathbb{E} = c_\rho$ the corresponding capacity $c_\mathbb{E}(X) = \mathbb{E}(|X|)$ $\forall X \in C_b(\Omega)$.

Notice that, alternatively, regularity could be proved directly for $G$-expectations. Theorem 5.1 would thus give the representation result [equation (6.1)].

**PROPOSITION 6.1.** There is a countable weakly relatively compact set $\{Q_n, n \in \mathbb{N}\}$ of probability measures, $Q_n \in \mathcal{P}$ such that

$$\forall X \in C_b(\Omega) \quad \mathbb{E}(X) = \sup_{n \in \mathbb{N}} E_{Q_n}(X). \quad (6.2)$$

Let $P = \sum_{n \in \mathbb{N}^*} \frac{Q_n}{2^n + 1}$. For all $f \geq 0 \in C_b(\Omega)$, $\mathbb{E}(f) = 0$ iff $f = 0$ $P$ a.s.

For every $X \in C_b(\Omega)$, there is a probability measure $Q_X$ in the weak closure of $\{Q_n, n \in \mathbb{N}^*\}$, such that $\mathbb{E}(X) = E_{Q_X}(X)$.

**PROOF.** The result follows from Theorem 5.1. $\Box$

6.2. **Risk measure in context of uncertain volatility.** We consider a framework introduced in [16]. Let $\Omega = C_0([0, T], \mathbb{R}^d)$ the space of continuous functions on $[0, T]$ null in zero. For every $t \leq T$, let $\Omega_t = C_0([0, t], \mathbb{R}^d)$. $\Omega_t$ is identified with the subset of $\Omega$ of elements which are constant on $[t, T]$. Let $\mathcal{B}_t$ be the $\sigma$-algebra on $\Omega$ generated by the open sets of $\Omega_t$. Denote $B_i$ the coordinate process. A probability measure $Q$ on $(\Omega, \mathcal{B}(\Omega))$ is called an orthogonal martingale measure if the coordinate process $(B_i)$ is a martingale with respect to $\mathcal{B}_t$ under $Q$ and if the martingales $(B_i)_1 \leq i \leq d$ are orthogonal in the sense that for all $i \neq j$, $\langle B_i, B_j \rangle^Q_t = 0$ $Q$ a.s. $\langle B_i, B_j \rangle^Q$ denotes the quadratic covariational process corresponding to $B^i$ and $B^j$, under $Q$ and $\langle B \rangle^Q$ the quadratic variation of $B$ under $Q$. Fix for all $i \in \{1, \ldots, d\}$ two finite deterministic Hölder-continuous measures $\mu_i$ and $\mu_i$ on $[0, T]$ and consider the set $\mathcal{P}$ of orthogonal martingale measures such that

$$\forall i \in \{1, \ldots, d\} \quad d\mu_{i,t} \leq d\langle B_i \rangle^Q_t \leq d\mu_{i,t}.$$ 

Kervarec has proved in [24], Lemma 1.3, that the set $\mathcal{P}$ is weakly relatively compact. Thus, $c_1(f) = \sup_{Q \in \mathcal{P}} E_Q(|f|)$ defines a capacity on $C_b(\Omega)$ (see the Appendix). As in Section 4, $L^1(c_1)$ denotes the corresponding Banach space containing $C_b(\Omega)$ as a dense subset. From Theorems 4.1 and 4.2, there is a countable set
(Pn)n∈N, Pn ∈ ℙ such that ∀X ∈ L²(c₁), c₁(X) = supn∈N EPn(|X|) and such that P = ∑n∈N Pn ²n belongs to the canonical c₁-class.

**Lemma 6.1.** For every probability measure R defining an element of L¹(c₁)∗,

∀i ∈ {1, …, d} dµi,t ≤ d(Bi)²t ≤ dµi,t.

Notice that a probability measure R in L¹(c)∗ does not necessarily belong to ℙ and therefore the result is not trivial.

**Proof of Lemma 6.1.** From [16], (Bi)²t ∈ L¹(c₁) for every t, thus, ∫₀ᵗ(Bi)s d(Bi)s can be defined as an element of L¹(c₁). We thus define the quadratic variation of B in L¹(c₁) by

(6.3) \langle Bi \rangle_c^t = (Bi)²t - 2 ∫₀ᵗ(Bi)s d(Bi)s.

This equation is satisfied in L¹(c₁) thus it is satisfied R a.s. for every probability measure R defining an element of L¹(c₁)∗. Let A = \{ω|⟨Bi⟩_t^c₁ - ⟨Bi⟩_s^c₁ > µi[s,t]| ∪ \{ω|⟨Bi⟩_t^c₁ - ⟨Bi⟩_s^c₁ < µi[s,t]|. By hypothesis Pn(A) = 0. Thus, P(A) = 0. The inequality

(6.4) µi[s,t] ≥ ⟨Bi⟩_t^c₁ - ⟨Bi⟩_s^c₁ ≥ µi[s,t]

is thus satisfied P a.s. From Lemma 4.3, inequality (6.4) is then satisfied in L¹(c₁) and then also R a.s. for every probability measure defining an element of L¹(c₁)∗.

□

**Proposition 6.2.** The set ℙ is convex metrizable compact for the weak* topology σ(L¹(c₁)∗, L¹(c₁)) and also for the weak topology.

**Proof.** The convexity of ℙ is obvious. Denote as in Section 2, K⁺ the non-negative part of the unit ball of L¹(c)∗. From the definition of c₁ it follows that ℙ ⊂ K⁺. Thus, the weak* closure \overline{ℙ} of ℙ is a subset of K⁺. From Lemma 6.1 it follows that every element Q ∈ \overline{ℙ} satisfies

∀i ∈ {1, …, d} dµi,t ≤ d⟨Bi⟩_t^Q ≤ dµi,t.

From Corollary 2.1, K⁺ is metrizable compact for the weak* topology thus, for every Q ∈ \overline{ℙ}, there is a sequence Qn, Qn ∈ ℙ converging to Q for the weak* topology.

From [16], |⟨Bi⟩_t|^k ∈ L¹(c₁) for k = 1 or 2, so (E_Qn - E_Q)(|⟨Bi⟩_t|^k) → 0. Passing to the limit, E_Q(|⟨Bi⟩_t|^k) ≤ c₁(|⟨Bi⟩_t|^k) and

(6.5) E_Q(|⟨Bi⟩_t|^2) ≤ c₁(|⟨Bi⟩_t|^2).
Let $g$ in $C_b(\Omega_s)$. $g$ can be identified with the element $\tilde{g}$ of $C_b(\Omega)$ defined by $\tilde{g}(x) = g(x_{|[0,s]})$. It follows from the inequality $c_1(Xg) \leq \|g\|_\infty c_1(|X|)$ that $\forall u \geq s$, $(B_t)ug \in L^1(c_1)$, so $\forall g \in C_b(\Omega_s) \forall \lambda \in \mathbb{R}$,

\begin{equation}
(E_{Q_n} - E_{Q})((B_t)_u (g + \lambda)) \to 0.
\end{equation}

$(B_t)_t$ is a martingale for $Q_n$, thus passing to the limit in (6.6), with $u = t$ and $u = s$, we obtain $\forall g \in C_b(\Omega_s) \forall \lambda \in \mathbb{R}$,

\begin{equation}
E_{Q}((B_t)_t (g + \lambda)) = E_{Q}((B_s)_s (g + \lambda)).
\end{equation}

From (6.5), $(B_t)_u \in L^2(\Omega, \mathcal{B}_u, Q)$ for $u = t, s$, and $\{g + \lambda, g \in C_b(\Omega_s), \lambda \in \mathbb{R}\}$ is dense in $L^2(\Omega, \mathcal{B}_s, Q)$ thus, equality (6.7) is satisfied for every $g \in L^2(\Omega, \mathcal{B}_s, Q)$.

This proves that $(B_t)_t$ is a martingale for $Q$. A very similar proof leads to the fact that the martingales $(B_i)_t$ and $(B_j)_t$ are mutually orthogonal for $i \neq j$. Thus, $\mathcal{P}$ is closed for the weak* topology. As $\mathcal{P} \subset \mathcal{K}_+$, $\mathcal{P}$ is metrizable compact for the weak* topology. The result follows from Proposition 2.4 for the weak topology. □

For every $P \in \mathcal{P}$ let $\beta(P) \geq 0$. Let $\rho$ be defined by

\begin{equation}
\forall X \in C_b(\Omega) \quad \rho(X) = \sup_{P \in \mathcal{P}} (E_P(\rho(-X) - \beta(P))).
\end{equation}

As $\mathcal{P}$ is metrizable compact for the weak topology, $\rho - \rho(0)$ is a uniformly regular convex risk measure. Thus, Theorem 5.1 applies.

The link between the two previous examples is studied in [15]. The convex weakly compact set characterizing the $G$-expectation $\mathcal{E}$ is in fact contained in the set $\mathcal{P}$ of orthogonal martingale measures introduced in [16] and considered in Section 6.2.

**APPENDIX**

Let $\Omega$ be a metrizable separable space and $\mathcal{L}$ as in Section 2 a lattice of continuous bounded functions, containing constants and generating the topology of $\Omega$. We now recall some definitions and propositions proved in Section 2 of [20]. A capacity is defined as in Definition 2.1, Section 2.

**DEFINITION A.1.** A capacity $c$ defined on $\mathcal{L}$ is regular if it satisfies the following:

For all decreasing net $f_\alpha \in \mathcal{L}$ converging to 0, $\inf_{P \in \mathcal{P}} c(f_\alpha) = 0$.

**DEFINITION A.2.** A capacity $c$ defined on $\mathcal{L}$ is a Prokhorov capacity if for all $\epsilon > 0$, there exists a compact set $K$ such that $c(f) \leq \epsilon$ for all $f \in \mathcal{L}$ such that $|f| \leq 1_{\Omega \setminus K}$.
PROPOSITION A.3. If $\Omega$ is a Lindelöf space, then every capacity is a regular capacity.

PROPOSITION A.4. If $\Omega$ is locally compact or a Polish space, then every regular capacity is a Prokhorov capacity.

REMARK 6. If $\Omega$ is a Polish space, then it is a Lindelöf space and thus every capacity is a Prokhorov capacity.

PROPOSITION A.5. If $\mathcal{P}$ is weakly relatively compact, $c$ defined on $\mathcal{C}_b(\Omega)$ by $c(f) = \sup_{P \in \mathcal{P}} (E_P[|f|^p])^{1/p}$ is a capacity.

The proof follows from Dini’s theorem (see [24], Proposition I.3, for more details).

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