The spreading fronts in a mutualistic model with delay*

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Abstract. This article is concerned with a system of semilinear parabolic equations with two free boundaries describing the spreading fronts of the invasive species in a mutualistic ecological model. The local existence and uniqueness of a classical solution are obtained and the asymptotic behavior of the free boundary problem is studied. Our results indicate that two free boundaries tend monotonically to finite or infinite at the same time, and the free boundary problem admits a global slow solution with unbounded free boundaries if the inter-specific competitions are strong, while if the inter-specific competitions are weak, there exist the blowup solution and global fast solution.

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1 Introduction

In this paper, we consider the following parabolic system with moving boundaries:

\[
\begin{cases}
    u_t - d_1 u_{xx} = u(a_1 - b_1 u + c_1 v(t - \tau_1, x)), & t > 0, \ g(t) < x < h(t), \\
    v_t - d_2 v_{xx} = v(a_2 + b_2 u(t - \tau_2, x) - c_2 v), & t > 0, \ -\infty < x < \infty, \\
    u(t, x) = 0, & t \geq 0, \ x < g(t) \text{ or } h(t) < x, \\
    v(t, x) = 0, & t \geq 0, \ x = h(t), \\
    u(0, x) = u_0(x) \geq 0, & -b \leq x \leq b, -\tau_2 \leq t \leq 0, \\
    v(0, x) = v_0(x) \geq 0, & -\infty \leq x \leq \infty, -\tau_1 \leq t \leq 0,
\end{cases}
\]

where \( x = h(t) \) and \( x = g(t) \) are the moving boundaries to be determined. Here \( a_1, b_1, c_1, c_2 \) \((i = 1, 2)\) are positive constants. \( u_0(x) \) and \( v_0(x) \) are initial functions satisfying

\[
\begin{cases}
    u_0 \in C^1(-b, b) \cap C^2(-b, b), \ v_0 \in L^\infty(-\infty, \infty) \cap C^2(-\infty, \infty), \\
    u(-b) = u_0(-b) > 0, \ u_0'(-b) > 0, \ u_0(b) < 0, \\
    u_0 > 0, \ v_0 > 0 \text{ in } (-b, b).
\end{cases}
\]

This system describes the cooperating two-species Lotka-Volterra model, where the native species \((v)\) migrates in the habitat \((-\infty, \infty)\) and the invasive species \((u)\) is initially limited in a special part and disperses through random diffusion only in \(g(t) < x < h(t)\). In biological terms, the unknowns \(u(x, t)\) and \(v(x, t)\) represent the spatial densities of the species at time \(t\) and location \(x\), \(a_i\) is its respective net birth rate and the constant \(d_i > 0\) is the diffusion coefficient. The coefficients \(b_1\) and \(c_2\) measure the intra-specific competitions whereas \(b_2\) and \(c_1\) represent inter-specific cooperation.

The corresponding problem on a fixed domain transforms into a Lotka-Volterra mutualistic model:

\[
\begin{cases}
    u_t = d_1 \Delta u + u(a_1 - b_1 u + c_1 v) & \text{for } t > 0, \ x \in \Omega, \\
    v_t = d_2 \Delta v + v(a_2 + b_2 u - c_2 v) & \text{for } t > 0, \ x \in \Omega,
\end{cases}
\]

which can be interpreted in biological terms that the presence of one species encourages the growth of the other species. Pao \cite{Pao1975} displayed that the solution of (1.3) under Dirichlet boundary condition with any initial data is unique and global when \(b_2c_1 < b_1c_2\), while the blowup solutions are possible when the two species are strongly mutualistic \((b_2c_1 > b_1c_2)\), which means that the geometric mean of the interaction coefficients exceeds that of population regulation coefficients.
The conditions on the free boundaries are \( h'(t) = -\mu u_x(t, h(t)) \) and \( g'(t) = -\mu u_x(t, g(t)) \), which are called the Stefan conditions. Here it means that the amount of the species flowing across the free boundary is increasing with respect to the moving length, see [13] in detail, and \( \mu \) is a positive constant.

Recently Kim and Lin [11] studied the corresponding system of semilinear parabolic with a free boundary

\[
\begin{align*}
\frac{u_t}{d_1} - u_{xx} &= u(a_1 - b_1 u + c_1 v), & t > 0, & 0 < x < h(t), \\
v_t - d_2 v_{xx} &= v(a_2 + b_2 u - c_2 v), & t > 0, & 0 < x < \infty, \\
u(t, x) &= 0, & i \geq 0, & h(t) < x < \infty, \\
u = 0, & h'(t) = -\mu \frac{\partial u}{\partial x}, & t > 0, & x = h(t), \\
\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 0) = 0, & t > 0, \\
h(0) = b, & (0 < b < \infty), \\
u(0, x) = u_0(x) \geq 0, & 0 \leq x \leq b, \\
v(0, x) = v_0(x) \geq 0, & 0 \leq x \leq \infty,
\end{align*}
\]

the blowup solution and global fast solution are given.

In the absence of \( v \) and the nonlinear reaction term for \( u \), problem (1.1) is reduced to one phase Stefan problem, which accounts for phase transitions between solid and fluid states such as the melting of ice in contact with water [22]. Stefan problem has been studied by many authors, see [3, 4, 5, 6, 12, 14, 18, 20, 22, 23].

As to the one-phase Stefan problem for the heat equation with a superlinear reaction term

\[
\begin{align*}
\frac{u_t}{d_1} - u_{xx} &= u^{1+p}, & t > 0, & 0 < x < h(t), \\
h'(t) &= -\frac{\partial u}{\partial x}, & t > 0, & x = h(t), \\
\frac{\partial u}{\partial x}(t, 0) &= u(0, h(t)) = 0, & t > 0, \\
u(0, x) &= u_0(x) \geq 0, & 0 \leq x \leq b, & h(0) = b,
\end{align*}
\]

it was shown in [7, 8] that all global solutions are bounded and decay uniformly to 0 as \( t \to \infty \) if the initial data is small, while if the initial date is big, the solution will blow up in a finite time. Moreover they showed that there exist global solutions with slow decay and unbounded free boundary.

The free boundary problems associated with the ecological models have attracted considerable research attention in the past due to their relevance in applications, see for example, [9, 13, 15, 16, 17] and the references therein.

Motivated by Kim and Lin [11], we are interested asymptotic behaviors of the solution for two free boundaries problem (1.1), especially a more detailed category about the global solutions. We will show that if \( b_1 c_2 > b_2 c_1 \), there exists a global slow solution of (1.1), while if \( b_1 c_2 < b_2 c_1 \) there exist a blowup solution and global fast solution of (1.1).
Throughout this paper, a solution \((u, v, g)\) of (1.1) is said to be classical if
\(u \in C([0, T] \times [g(t), h(t)]) \cap C^{1,2}((0, T) \times (g(t), h(t)), v \in C([0, T] \times (-\infty, \infty)) \cap C^{1,2}((0, T) \times (-\infty, \infty)) \cap C([0, T] \times L^\infty(-\infty, \infty))\) and \(h, g \in C^1[0, T]\) with \(T_{\text{max}} \leq +\infty\) and satisfies (1.1), where \(T_{\text{max}}\) denotes the maximal existing time of solution. If \(T_{\text{max}} = +\infty\), we say the solution exists globally whereas if the solution ceases to exist for some finite time, that is, \(T_{\text{max}} < +\infty\) and \(\lim_{t \to T_{\text{max}}} (\|u(t, x)\|_{L^\infty(g(t), h(t))} + \|v(t, x)\|_{L^\infty(-\infty, +\infty)}) \to +\infty\), we say that the solution blows up. If \(T_{\text{max}} = \infty\) and \(h_\infty := \lim_{t \to \infty} h(t) < \infty\), \(g_\infty := \lim_{t \to \infty} g(t) > -\infty\), the solution is called global fast solution since the solution decays uniformly to 0 at an exponential rate, while if \(T_{\text{max}} = \infty\) and \(h_\infty = \infty\), \(g_\infty = -\infty\), it is called global slow solution, whose decay rate is at most polynomial, see [7, 8].

We now briefly give an outline of the paper. In Section 2, local existence and uniqueness of two free boundaries problem (1.1) are obtained by using Schauder fixed point theorem. Results pertaining to global solution for the case \(b_1c_2 > b_2c_1\) are presented in Section 3, and in Section 4, results regarding nonglobal solutions and global fast solution for the case \(b_1c_2 < b_2c_1\) are established.

## 2 Local existence and uniqueness

We first prove the following local existence and uniqueness results of the solution to (1.1) by virtue of the Schauder fixed point theorem:

**Theorem 2.1** There exists a \(T > 0\) such that problem (1.1) admits a unique solution
\[
(u, v, h, g) \in C^{1+\alpha,(1+\alpha)/2}(\overline{D}_1,T) \times C^{1+\alpha,(1+\alpha)/2}(\overline{D}_2,T) \times C^{1+\alpha/2}[0, T] \times C^{1+\alpha/2}[0, T],
\]

furthermore
\[
\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_1,T)} + \|v\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{D}_2,T)} + \|h\|_{C^{1+\alpha/2}[0, T]} + \|g\|_{C^{1+\alpha/2}[0, T]} \leq C, \quad (2.1)
\]

where \(D_1,T = (0, T] \times (g(t), h(t)), D_2,T = (0, T] \times (-\infty, +\infty), 0 < \alpha < 1,\) and \(C, T\) only depend on \(b, \|u_0\|_{C^2[-b,b]}\) and \(\|v_0\|_{C^2(-\infty, +\infty)}\).

**Proof** As in [2] and [4], we first straighten the double free boundaries. Let \(\zeta(y)\) be a function in \(C^3(-\infty, +\infty)\) satisfying
\[
\zeta(y) = 1 \quad \text{if} \quad |y - b| < \frac{b}{8},
\]
\[ \zeta(y) = 0 \quad \text{if } |y-b| > \frac{b}{2}, \quad |\zeta'(y)| < \frac{6}{b}, \]
\[ \xi(y) = \zeta(-y). \]

Let a transformation be
\[ (t, x) \to (t, y), \] where \( x = y + \xi(y)(g(t) + b) + \zeta(y)(h(t) - b), \quad -\infty < y < \infty. \]

As long as
\[ \max\{|g(t) + b|, |h(t) - b|\} \leq \frac{b}{8}, \]
the above transformation is a diffeomorphism from \((-\infty, +\infty)\) onto \((-\infty, +\infty)\).

Moreover, the free boundary \( x = h(t), \) \( x = g(t) \) becomes the lines \( y = b, \) \( y = -b \) respectively. Now, a straightforward computation yields
\[ \frac{\partial y}{\partial x} = \frac{1}{1+\xi'(y)(g(t)+b)+\zeta'(y)(h(t)-b)} \equiv \sqrt{A(g(t), h(t), y)} \equiv C(g(t), h(t), y), \]
\[ \frac{\partial^2 y}{\partial x^2} = -\frac{\xi''(y)(g(t)+b)+\zeta''(y)(h(t)-b)}{(1+\xi'(y)(g(t)+b)+\zeta'(y)(h(t)-b))^2} \equiv B(g(t), h(t), y), \]
\[ -\frac{\partial y}{\partial t} = \frac{\xi(y)g'(t)+\zeta(y)h'(t)}{1+\xi'(y)(g(t)+b)+\zeta'(y)(h(t)-b)} \equiv C(g(t), h(t), y)[\xi(y)g'(t) + \zeta(y)h'(t)]. \]

If we set
\[ u(t, x) = u(t, y + \xi(y)(g(t) + b) + \zeta(y)(h(t) - b)) = w(t, y), \]
\[ v(t, x) = v(t, y + \xi(y)(g(t) + b) + \zeta(y)(h(t) - b)) = z(t, y), \]
then
\[ u_t = w_t - \xi(y)g'(t) + \zeta(y)h'(t)C(g(t), h(t), y)w_y, \]
\[ v_t = z_t - \xi(y)g'(t) + \zeta(y)h'(t)C(g(t), h(t), y)z_y, \]
\[ u_x = C(g(t), h(t), y)w_y, \quad v_x = C(g(t), h(t), y)z_y, \]
\[ u_{xx} = A(g(t), h(t), y)w_{yy} + B(g(t), h(t), y)w_y, \]
\[ v_{xx} = A(g(t), h(t), y)z_{yy} + B(g(t), h(t), y)z_y \]
and problem (1.1) turns into
\[
\begin{cases}
\begin{aligned}
w_t - Ad_1w_{yy} - [Bd_1 + (\xi(y)g'(t) + \zeta(y)h'(t))C]w_y &= w(a_1 - b_1w + c_1(z - t - \tau_1, y)), & t > 0, \quad -b < y < b, \\
z_t - Ad_2z_{yy} - [Bd_2 + (\xi(y)g'(t) + \zeta(y)h'(t))C]z_y &= z(a_2 - b_2w(t - \tau_2, y) - c_2z), & t > 0, \quad -\infty < y < \infty, \\
w(t, y) &= 0, & t \geq 0, \quad -\infty < y < -b, \\
w(t, y) &= 0, & t \geq 0, \quad b < y < \infty, \quad (2.2) \\
w = 0, & t > 0, \quad y = b, \\
0, & t > 0, \quad y = -b, \\
h(0) = -g(0) = b, & 0 < b < \infty. \\
w(t, y) = u_0(y) \geq 0, & -b \leq y \leq b, -\tau_2 \leq t \leq 0 \\
z(t, y) = v_0(y) \geq 0, & -\infty \leq y < \infty, -\tau_1 \leq t \leq 0,
\end{aligned}
\end{cases}
\]
where $A = A(g(t), h(t), y), B = B(g(t), h(t), y), C = C(g(t), h(t), y), u_0(y) \in C^1[-b, b] \cap C^2(-b, b)$ and $v_0(y) \in L^\infty(-\infty, \infty) \cap C^2(-\infty, \infty)$.

Let $g_1 = -\mu u'_0(-b), h_1 = -\mu u'_0(b), D_{1,T} = (0, T) \times (-b, b)$ and $0 < T < \min(\frac{b}{8(1+g_1)}, \frac{b}{8(1+h_1)})$, choosing

\[ D_1 = \{ w(t, y) \in C(\overline{D}_{1,T}) : w(t, y) = u_0(y) \}, \]
\[ D_{1T} = \{ w \in D_1 : \sup_{0 \leq t \leq T, -b \leq y \leq b} |w(t, y) - u_0(y)| \leq 1 \}, \]
\[ D_2 = \{ z(t, y) \in C(\overline{D}_{2,T}) : z(t, y) = v_0 \}, \]
\[ D_{2T} = \{ z \in D_2 : \sup_{0 \leq t \leq T, -\infty < y < \infty} |z(t, y) - v_0(y)| \leq 1 \}, \]
\[ D_3 = \{ g(t) \in C^1[0, T] : g(0) = -b, g'(0) = g_1 \}, \]
\[ D_{3T} = \{ g(t) \in D_3 : \sup_{0 \leq t \leq T} |g'(t) - g_1| \leq 1 \}, \]
\[ D_4 = \{ h(t) \in C^1[0, T] : h(0) = b, h'(0) = h_1 \}, \]
\[ D_{4T} = \{ h(t) \in D_4 : \sup_{0 \leq t \leq T} |h'(t) - h_1| \leq 1 \}. \]

It’s well known that $D_{1T} \times D_{2T} \times D_{3T} \times D_{4T}$ is a closed convex set in $C(\overline{D}_{1,T}) \times C(\overline{D}_{1,T}) \times C^1[0, T] \times C^1[0, T]$.

Next, we can obtain the existence and uniqueness by using the contraction mapping theorem as in [2, 4] with some obvious adaptation. For brief, we omit it here. \hfill \Box

**Theorem 2.2** The double free boundaries in problem (1.1) are sterkly monotone, namely, for any solution on $[0, T]$ we have

\[ h'(t) > 0 \text{ and } g'(t) < 0 \text{ for } 0 \leq t \leq T. \]

**Proof** Using the Hopf Lemma to the system of (1.1), we deduce that

\[ u_x(t, h(t)) < 0, \quad u_x(t, g(t)) > 0 \text{ for } 0 \leq t \leq T. \]

Then, combining the above two inequalities with the Stefan conditions in (1.1), the result can be deduced. \hfill \Box

Furthermore, the double free boundaries $g(t)$ and $h(t)$ have another notable properties which will be showed below.

**Theorem 2.3** Let $(u, v, g, h)$ be a solution of system (1.1) in $[0, T_{\text{max}}] \times [g(t), h(t)]$. Then $g(t)$ and $h(t)$ satisfy

\[ -2b < g(t) + h(t) < 2b, \quad t \in [0, T_{\text{max}}]. \]
Proof It follows from continuity that \( g(t) + h(t) < 2b \) for small \( t > 0 \). Define

\[
T := \sup \{ s : g(t) + h(t) < 2b, \ t \in [0, s) \}.
\]

We can deduce that \( T = T_{\text{max}} \) in the following proof by contradiction. Suppose that \( T < T_{\text{max}} \), we then have

\[
g(t) + h(t) < 2b, \ t \in [0, T), \quad g(T) + h(T) = 2b.
\]

Hence

\[
g'(T) + h'(T) \geq 0. \tag{2.3}
\]

In order to obtain a contradiction, we define the function \( F(t, x) := u(t, x) - u(t, -x + 2b) \) on the region

\[
\Omega' = \{(t, x) : 0 \leq t \leq T, \ b \leq x \leq h(t)\}.
\]

Directly calculating \( F \) shows that it satisfies

\[
F_t = F_{xx} + c(t, x)F, \ 0 < t \leq T, \ b < x < h(t),
\]

with some \( c(t, x) \in L^\infty(\Omega') \) and

\[
F(t, b) = 0, \ F(t, h(t)) < 0, \ 0 < t < T.
\]

Moreover,

\[
F(T, h(T)) = u(T, h(T)) - u(T, -h(T) + 2b) = u(T, h(T)) - u(T, g(T)) = 0.
\]

Then we have

\[
F(t, x) < 0, \ (t, x) \in (0, T] \times (b, h(t)),
\]

and

\[
F_x(T, h(T)) < 0
\]

by applying the strong maximum principle and the Hopf lemma. However

\[
F_x(T, h(T)) = u_x(T, h(T)) + u_x(T, g(T)) = -[g'(T) + h'(T)]/\mu,
\]

namely

\[
g'(T) + h'(T) > 0,
\]

which contradicts to (2.3). Therefore we claim that \( g(t) + h(t) < 2b \) for all \( 0 < t < T_{\text{max}} \). Similarly we can prove \( g(t) + h(t) > -2b \) for all \( 0 < t < T_{\text{max}} \). □
Theorem 2.1 implies that there exists a $T$ such that the solution exists in time interval $[0, T]$, and the solution can be further extended to $[0, T_{\text{max}}]$ with $T_{\text{max}} \leq +\infty$ by Zorn’s lemma. The maximal exist time of the solution $T_{\text{max}}$ depends on a prior estimate with respect to $\|u\|_{L^{\infty}}, \|v\|_{L^{\infty}}$ and $g'(t), h'(t)$.

Next we will give that if $\|u\|_{L^{\infty}} < \infty$, the solution is global. For this purpose we first provide the following lemma:

**Lemma 2.4** Suppose that $M \equiv \|u\|_{L^{\infty}([0, T] \times [g(t), h(t)])} < \infty$. Then the solution of the free boundary problem (1.1) satisfies

$$0 \leq v \leq M^2 \quad \text{for} \quad 0 \leq t \leq T, \quad -\infty \leq x < \infty,$$

$$0 < -g'(t) \leq M_3(M) \quad \text{for} \quad 0 \leq t \leq T,$$

$$0 < h'(t) \leq M_4(M) \quad \text{for} \quad 0 \leq t \leq T,$$

where $M_2, M_3$ and $M_4$ are independent of $T$.

**Proof** Since that $u_t - d_2 v_{xx} \leq v(a_2 + b_2 M - c_2 v)$ for $0 < t \leq T, -\infty < x < \infty$, the estimate for $v$ is directly from the Phragman-Lindelof principle.

Set

$$\Omega = \{(t, x) : 0 < t \leq T, \ g(t) < x < g(t) + \frac{1}{M}\}$$

and constitute an auxiliary function

$$w(t, x) = M[2M(x - g(t)) - M^2(x - g(t))^2].$$

In the following proof, we will choose $M$ such that $w(t, x)$ is the supersolution of $u(t, x)$ in $\Omega$.

Tedious but fairly straightforward computation show that

$$w_t = 2M(-g'(t))(1 - M(x - g(t))) \geq 0,$$

$$-w_{xx} = 2M^2,$$

$$u(a_1 - b_1 u + c_1 v) \leq M(a_1 + c_1 M_2).$$

It follows that

$$w_t - d_1 w_{xx} \geq M(a_1 + c_1 M_2) \geq u(a_1 - b_1 u + c_1 v)$$

if $M^2 \geq \frac{a_1 + c_1 M_2}{d_1}$. On the other hand,

$$w(t, g(t) + \frac{1}{M}) = M \geq u(t, g(t) + \frac{1}{M}),$$
Recalling that $u_0(-b) = 0$ and $u'_0(-b) = -g_1/\mu$ gives that there exists $0 < \delta < b$ such that $u_0(x) \le \frac{3}{2}M$ and $|u'_0(x)| \le |b/\mu| + 1$ for $x \in [-b, b + \frac{\delta}{4}]$, we then have $w(0, x) \ge u_0(x)$ in $[-b, b + \frac{1}{M}]$ if $M \ge \max\{\frac{1}{2}, \frac{|g_1|/\mu + 1}{M_1}\}$. Making use of the comparison principle yields $u(t, x) \le w(t, x)$ in $\Omega$. Noticing that $u(t, g(t)) = w(t, g(t)) = 0$, we have

$$u_x(t, g(t)) \le w_x(t, g(t)) = 2M M.$$ 

Recollecting the free boundary condition in (1.1) deduces

$$0 < -g'(t) \le 2\mu M M \triangleq M_3, \quad 0 < t \le T,$$

where $M_3$ is independent of $T$. Analogously, we can define

$$w(t, x) = M[2M(h(t) - x) - M^2(h(t) - x)^2].$$

over the region

$$\Omega' = \{(t, x) : 0 < t \le T, \ h(t) - \frac{1}{M} < x < h(t)\}$$

get that

$$0 < h'(t) \le M_4, \quad 0 < t \le T,$$

where $M_4$ is independent of $T$. \hfill \Box

**Theorem 2.5** The solution of problem (1.1) exists and is unique, and it can be extended to $[0, T_{\text{max}}]$ with $T_{\text{max}} \le \infty$. Moreover, if $T_{\text{max}} < \infty$, we have

$$\limsup_{t \to T_{\text{max}}} ||u||_{L^\infty([g(t), h(t)] \times [0, T])} = \infty.$$ 

**Proof** It follows from the uniqueness that there is a number $T_{\text{max}}$ such that $[0, T_{\text{max}})$ is the maximal time interval in which the solution exists. In order to prove the present theorem, it suffices to show that, when $T_{\text{max}} < \infty$, \text{lim sup}_{t \to T_{\text{max}}} ||u||_{L^\infty([g(t), h(t)] \times [0, T])} = \infty$. In what follows we use the contradiction argument. Assume that $T_{\text{max}} < \infty$ and $||u||_{L^\infty([0, T_{\text{max}}] \times [g(t), h(t)])} < \infty$. Since $v \le M_2(M)$ in $[g(t), h(t)] \times [0, T_{\text{max}}]$ and $0 < -g'(t) \le M_3, 0 < h'(t) \le M_4$ in $[0, T_{\text{max}}]$ by Lemma 2.3, using a bootstrap argument and Schauder’s estimate yields a priori bound of $||u(t, x)||_{C^{1+\alpha}(g(t), h(t))} + ||v(t, x)||_{C^{1+\alpha}(-\infty, \infty)}$ for all $t \in [0, T_{\text{max}})$. Let the bound be $M_5$. It follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending only on $M$, $M_2$, $M_3$, $M_4$ and $M_5$ such that the solution of problem (1.1) with the initial time $T_{\text{max}} - \tau/2$ can be extended uniquely to the time $T_{\text{max}} - \tau/2 + \tau$ that contradicts the assumption. Thus the proof is complete. \hfill \Box

9
3 Global solution for the case \( b_1c_2 > b_2c_1 \)

To obtain the global existence, we first derive a prior estimate for the solution of (1.1).

**Lemma 3.1** If \( b_1c_2 > b_2c_1 \), then the solution of the free boundary problem (1.1) satisfies

\[
0 < u(t, x) \leq K_1 \quad \text{for} \quad 0 \leq t \leq T, \quad g(t) < x < h(t),
\]

\[
0 \leq v(x, t) \leq K_2 \quad \text{for} \quad 0 \leq t \leq T, \quad -\infty < x < \infty,
\]

where \( K_i \) is independent of \( T \) for \( i = 1, 2 \).

**Proof** Firstly we have that \( u > 0 \) in \([g(t), h(t)] \times [0, T]\) and \( v \geq 0 \) in \((-\infty, \infty) \times [0, T]\) provided that solution exists.

Since the solution is classical in \([0, T]\), there exists a \( \tilde{K}(T) \) such that \( u(t, x) \leq c_1 \tilde{K} \) and \( v(t, x) \leq \tilde{K} \). Next we give the proof for \( u(t, x) \leq K_1 \) and \( v(t, x) \leq K_2 \), where

\[
K_1 := m \frac{a_1c_2 + a_2c_1}{b_1c_2 - b_2c_1} > \max_{[-b, b]} u_0(x), \quad K_2 := m \frac{a_1b_2 + a_2b_1}{b_1c_2 - b_2c_1} > \|v_0\|_{L^\infty(-\infty, \infty)}
\]

for some \( m > 1 \).

Because the interval \((-\infty, \infty)\) is unbounded, maximum principle becomes invalid, next we prove that for any \( t > b \),

\[
u(t, x) \leq K_1 + \frac{(1 + b_1)c_1 \tilde{K}(x^2 + 2\tilde{d}t)}{b_1}, \quad v(t, x) \leq K_2 + (1 + b_1) \frac{\tilde{K}(x^2 + 2\tilde{d}t)}{l^2}
\]

for \( 0 \leq t \leq T, -l \leq x \leq l \), where \( \tilde{d} = \max(d_1, d_2) \). Setting

\[
\overline{u}(t, x) = K_1 + \frac{(1 + b_1)c_1 \tilde{K}(x^2 + 2\tilde{d}t)}{b_1},
\]

\[
\overline{v}(t, x) = K_2 + (1 + b_1) \frac{\tilde{K}(x^2 + 2\tilde{d}t)}{l^2},
\]

then \((\overline{u}, \overline{v})\) satisfies

\[
\begin{cases}
\overline{u}_t - d_1\overline{u}_{xx} \geq \overline{u}(a_1 - b_1 \overline{u} + c_1 \overline{u}(t - \tau_1, x)), & 0 < t \leq T, \quad -l < x < l, \\
\overline{v}_t - d_2\overline{v}_{xx} \geq \overline{v}(a_2 + b_2 \overline{v}(t - \tau_2, x) - c_2 \overline{v}), & 0 < t \leq T, \quad -l < x < l, \\
\overline{u} \geq K_1 + \frac{(1 + b_1)c_1 \tilde{K}}{b_1} > u, \quad \overline{v} \geq K_2 + (1 + b_1) \tilde{K} > v, & 0 < t \leq T, \quad x = \pm l, \\
\overline{u}(t, x) \geq K_1 > u_0(x), & -\tau_2 \leq t \leq 0, \quad -l \leq x \leq l \\
\overline{v}(t, x) \geq K_2 > v_0(x), & -\tau_1 \leq t \leq 0, \quad -l \leq x \leq l.
\end{cases}
\]
It follows that \( u \leq \bar{u} \) and \( v \leq \bar{v} \) by using the maximum principle on \([0, T] \times [-l, l] \).

Now for any fixed \((t_0, x_0) \in [0, T] \times (-\infty, \infty)\), let \( l \) sufficiently large so that \((t_0, x_0) \in [0, T] \times [-l, l]\), we deduce from the above proof that

\[
\begin{align*}
u(t_0, x_0) & \leq \bar{v}(t_0, x_0) = K_2 + (1 + b_1) \frac{\tilde{K}(x_0^2 + 2 \tilde{d} t_0)}{l^2}, \\
u(t_0, x_0) & \leq \bar{v}(t_0, x_0) = K_2 + (1 + b_1) \frac{\tilde{K}(x_0^2 + 2 \tilde{d} t_0)}{l^2}.
\end{align*}
\]

Taking \( l \to \infty \) gives the desired estimates. \( \square \)

Combing Theorem 2.4 with Lemma 3.1 yields the following global existence:

**Theorem 3.2** If parameters in double free boundaries problem (1.1) satisfy 

\( b_1 c_2 > b_2 c_1 \), then (1.1) admits a unique global solution.

Next we mainly give the long-time behavior of the free boundary problem (1.1). Here, we first give the slow solution.

**Theorem 3.3** If \( b_1 c_2 > b_2 c_1 \) and \( a_1 > d_1 \left( \frac{\pi}{2b} \right)^2 \), the free boundaries of the problem (1.1) satisfy 

\( h_\infty = \infty \) and \( g_\infty = -\infty \).

**Proof** Combing Theorems 2.2 with Theorem 3.2, we know that the solution is global, \( x = g(t) \) is monotonic decreasing and \( x = h(t) \) is monotonic increasing. Assume that \( \lim_{t \to +\infty} g(t) = 0 \).

On the other hand, the condition \( a_1 > d_1 \left( \frac{\pi}{2b} \right)^2 \) implies that \( a > \lambda_1 \), where \( \lambda_1 \) denotes the first eigenvalue of the problem

\[-d_1 \phi'' = \lambda \phi \quad \text{in} \quad (-b, b), \quad \phi(\pm b) = 0.\]

Therefore for all small \( \delta > 0 \), the first eigenvalue \( \lambda_1^\delta \) of the problem

\[-d_1 \phi_{xx} - \delta \phi' = \lambda \phi \quad \text{in} \quad (-b, b), \quad \phi(\pm b) = 0\]

satisfies \( \lambda_1^\delta < a_1 \). Fix such an \( \delta > 0 \) and consider the problem

\[L_\delta \psi = a_1 \psi - b_1 \psi^2 \quad \text{in} \quad (-b, b), \quad \psi(\pm b) = 0, \quad \psi' \quad \text{in} \quad (-b, b), \quad \psi(\pm b) = 0, \quad \psi' \]

where \( L_\delta \psi = -d_1 \psi'' - \delta \psi' \). It is well known (Proposition 3.3 in [1]) that the problem (3.1) admits a unique positive solution \( \psi = \psi_\delta \). By the moving plane method one easily sees that \( \psi(x) \) is symmetric about \( x = 0 \) with \( \psi'(x) < 0 \) for
Moreover using the comparison principle, we have $\psi < \frac{a_1}{b_1}$ in $[-b, b]$. We now set

$$F(t, x) = \psi \left( \frac{b}{g(t)} x \right),$$

and directly compute

$$F_t - d_1 F_{xx} = \frac{-bx}{g^2(t)} g'(t) \psi' - \frac{b^2}{g^2(t)} \psi''$$

$$= \frac{b^2}{g^2(t)} [ -d_1 \psi'' + \frac{xg'(t)}{-b} \psi' ].$$

Note that $g'(t) \to 0$ as $t \to +\infty$, we can choose $T_0 > 0$ such that $g'(t) > \frac{\delta b}{g_{\infty}}$ for $t \geq T_0$ and hence for $t \geq T_0$ and $x \in [g(t), 0]$, we have $\frac{xg'(t)}{-b} \geq -\delta$. Therefore for such $t$ and $x$,

$$F_t - d_1 F_{xx} \leq \frac{b^2}{g^2(t)} (-d_1 \psi'' - \delta \psi')$$

$$= \frac{b^2}{g^2(t)} (a_1 \psi - b_1 \psi^2).$$

Because of $0 \leq \psi < \frac{a_1}{b_1}$, we have $a_1 \psi - b_1 \psi^2 \geq 0$ and hence from $-\frac{b}{g(t)} \leq 1$ we get

$$F_t - d_1 F_{xx} \leq a_1 \psi - b_1 \psi^2 = a_1 F - b_1 F^2 \quad \text{for} \quad t \geq T_0, \ x \in [g(t), 0].$$

Now we choose $\varepsilon \in (0, 1)$ sufficient small so that $\varepsilon F(T_0, x) \leq u(T_0, x)$. Then $\underline{u}(t, x) := \varepsilon F(t, x)$ satisfies

$$\begin{cases} u_t - d_1 u_{xx} \leq a_1 u - b_1 u^2, \quad t \geq T_0, \ x \in [g(t), 0], \\ u(t, g(t)) = 0, \ u(t, 0) = 0, \ t \geq T_0, \\ u(T_0, x) \leq u(T_0, x), \quad 0 \leq x \leq g(T_0). \end{cases}$$

So we can use the comparison principle to draw a conclusion

$$\underline{u}(t, x) \leq u(t, x) \quad \text{for} \quad t \geq T_0, \ x \in [g(t), 0].$$

It follows that

$$u_x(t, g(t)) \geq u_x(t, g(t)) = \delta \frac{b}{g(t)} \psi'(b) \to \delta \frac{b}{g_{\infty}} \psi'(b) > 0,$$

which means that $g'(t) \leq -\mu \delta \frac{b}{g_{\infty}} \psi'(b) < 0$. This is a contradiction to the fact that $g'(t) \to 0$ as $t \to \infty$. This contradiction implies that $g_{\infty} = -\infty$. Likewise, we can set

$$F(t, x) = \psi \left( \frac{b}{h(t)} x \right), \ x \in [0, h(t)]$$

to prove that $h_{\infty} = +\infty$. \qed
4 Global and nonglobal solutions

In this section, we discuss the asymptotic behavior of the solution for the case $b_1c_2 < b_2c_1$, which is more intricate than that for the case $b_1c_2 > b_2c_1$. First we present the blowup result.

**Theorem 4.1** If $b_1c_2 < b_2c_1$, then

(i) the solution of the free boundary problem (1.1) with any nontrivial nonnegative initial data blows up in case $a_i > d_i(\frac{\pi}{2b})^2$ for $i = 1, 2$.

(ii) the solution of the free boundary problem (1.1) blows up for any $a_i$ in case the initial data is sufficiently large.

**Proof** To prove this, it suffices to compare the free boundary problem with the corresponding problem in the fixed domain:

\[
\begin{align*}
  u_t - d_1 u_{xx} &= u(a_1 - b_1 u + c_1 v(t - \tau_1, x)), & t > 0, -b < x < b, \\
  v_t - d_2 v_{xx} &= v(a_2 + b_2 u(t - \tau_2, x) - c_2 v), & t > 0, -b < x < b, \\
  u(t, -b) &= v(t, -b) = 0, & t > 0, \\
  u(t, b) &= v(t, b) = 0, & t > 0, \\
  u(0, x) &= u_0(x) \geq 0, \quad -\tau_2 \leq t \leq 0, -b \leq x \leq b, \\
  v(0, x) &= v_0(x) \geq 0, \quad -\tau_1 \leq t \leq 0, -b \leq x \leq b.
\end{align*}
\]

(4.1)

It follows from [19] that the solution blows up if $a_i > d_i(\frac{\pi}{2b})^2$, $i = 1, 2$ or if the initial data is sufficiently large. We come to a conclusion by making use of maximum principle.

**Remark 4.1** The above theorem means that if the initial length $b$ is large enough or if the initial data is sufficiently large, the solution will blow up. The constant $\left(\frac{\pi}{2b}\right)^2$ is the first eigenvalue of $-\Delta$ in $[-b, b]$ with homogeneous Dirichlet boundary condition.

The comparison principle used above is for the stationary boundary. In the following we introduce a comparison principle for double free boundaries $x = h(t)$ and $x = g(t)$.

**Lemma 4.2** Suppose that $T \in (0, \infty)$, $\overline{h}, \overline{g} \in C^1([0, T])$, $\overline{u} \in C(\overline{D}_{1,T}) \cap C^{1,2}(D_{1,T})$ and $\overline{v} \in C(\overline{D}_{2,T}) \cap C^{1,2}(D_{2,T})$ with $D_{1,T} = (0, T] \times (\overline{g}(t), \overline{h}(t))$, $D_{2,T} = (0, T] \times (-\infty, +\infty)$, and

\[
\begin{align*}
  \overline{u}_t - d_1 \overline{u}_{xx} &\geq \overline{u}(a_1 - b_1 \overline{u} + c_1 \overline{v}(t - \tau_1, x)), & t > 0, \overline{g}(t) < x < \overline{h}(t), \\
  \overline{v}_t - d_2 \overline{v}_{xx} &\geq \overline{v}(a_2 + b_2 \overline{u}(t - \tau_2, x) - c_2 \overline{v}), & t > 0, -\infty < x < \infty, \\
  \overline{u}(t, x) &= 0, & t > 0, -\infty < x < \overline{g}(t), \\
  \overline{v}(t, x) &= 0, & t > 0, \overline{h}(t) < x < \infty, \\
  \overline{u} &= 0, & \overline{h}'(t) \geq -\mu \frac{\overline{u}}{\overline{v}}, & t > 0, x = \overline{h}(t), \\
  \overline{v} &= 0, & \overline{g}'(t) \leq -\mu \frac{\overline{v}}{\overline{u}}, & t > 0, x = \overline{g}(t).
\end{align*}
\]
If \(-b \geq \overline{g}(0), b \leq \overline{h}(0), u_0(x) \leq \overline{u}(t, x)\) in \([-b, b] \times [-\tau_2, 0]\) and \(v_0(x) \leq \overline{v}(t, x)\) in \((-\infty, +\infty) \times [-\tau_1, 0]\), then the solution \((u, v, g, h)\) of the free boundary problem (1.1) satisfies

\[
\begin{align*}
g(t) &\geq \overline{g}(t), \ h(t) \leq \overline{h}(t) \text{ in } (0, T], \\
u(t, x) &\leq \overline{u}(t, x) \text{ in } [0, T] \times (g(t), h(t)) \\
v(t, x) &\leq \overline{v}(t, x) \text{ in } [0, T] \times (-\infty, +\infty).
\end{align*}
\]

**Proof** We first suppose that \(g(0) > \overline{g}(0), h(0) < \overline{h}(0)\). In this case we assert that \(g(t) > \overline{g}(t)\) and \(h(t) < \overline{h}(t)\) for \(0 < t \leq T\) by using contradiction. If it is not true, then there exists \(t^*_1 \in (0, T)\) such that \(h(t) < \overline{h}(t)\) for \(t \in [0, t^*_1]\) and \(h(t^*_1) = \overline{h}(t^*_1)\). It follows that

\[h'(t^*_1) \geq \overline{h}'(t^*_1)\]

Because of the system of (1.1) is nondecreasing, and applying the strong maximum principle for the parabolic systems give that \(u(t, x) < \overline{u}(t, x)\) in \((0, t^*_1] \times (g(t), h(t))\) and

\[\frac{\partial}{\partial x}(u - \overline{u})(t^*_1, h(t^*_1)) > 0\]

by \(u(t^*_1, h(t^*_1)) = 0 = \overline{u}(t^*_1, \overline{h}(t^*_1))\), then

\[h'(t^*_1) = -\mu \frac{\partial u}{\partial x}(t^*_1, h(t^*_1)) < \overline{h}'(t^*_1)\]

This leads to a contradiction, which proves our assert that \(h(t) < \overline{h}(t)\) for \(0 < t \leq T\) if \(h(0) = b < \overline{h}(0)\). Analogously, we can prove that \(g(t) > \overline{g}(t)\) for \(0 < t \leq T\). Now we may draw a conclusion that \(u(t, x) \leq \overline{u}(t, x)\) in \([0, T] \times (g(t), h(t))\) and \(v(t, x) \leq \overline{v}(t, x)\) in \([0, T] \times (-\infty, +\infty)\) by approximation. \(\square\)

**Remark 4.2** The \((\overline{u}, \overline{v}, \overline{h}, \overline{g})\) in Lemma 4.2 is usually called an upper solution of the problem (1.1). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 4.2 for lower solutions.

In the following theorem, we show existence of a global fast solution.

**Theorem 4.3** If \(b_1c_2 < b_2c_1\), then the free boundary problem (1.1) admits a global fast solution provided that the initial data \(u_0\) and \(b\) are suitably small. Moreover, there exist constant \(C, \beta > 0\) depending on \(b, u_0\) and \(k\) such that

\[||u||_\infty \leq Ce^{-\beta t}, \ \ t \geq 0\]

for some \(k > 1\).
By making use of the maximum principle, we can get that $h(t) < \sigma(t)$, $g(t) > \lambda(t)$, and $u(t, x) < \pi(t, x)$, $v(t, x) < \pi(t, x)$ for $g(t) \leq x \leq h(t)$ provided $(u, v)$ exists. Particularly, it follows from Lemma 4.2 that $(u, v)$ exists globally and $g_{\infty} > -\infty$, $h_{\infty} < \infty$.  

**Proof** [21] was the main source of inspiration for its proof, we have only to the structure proper global supersolution. Define

$$\sigma(t) = 2b(k - e^{-\gamma t}), \lambda(t) = -\sigma(t), \ t > 0,$$  

and

$$\overline{u}(t, x) = \delta e^{-\beta t} W(x/\sigma(t)), \ t > 0, \ \lambda(t) \leq x \leq \sigma(t).$$

$\overline{v}(t, x) = \frac{\alpha_2}{c_2}$, $t > 0$, $-\infty \leq x \leq \infty$,

where $\gamma, \beta$ and $\delta > 0$ to be determined later.

Straightforward calculations yields

$$\overline{u}_t - d_1\overline{u}_{xx} - \overline{u}(a_1 - b_1\overline{u} + c_1\overline{v}(t - \tau_1, x))$$

$$= \delta e^{-\beta t}[ - \beta W - 2\sigma' \sigma^{-2} W' - d_1\sigma^{-2} W'' - W(a_1 - b_1\delta e^{-\beta t} W + c_1 \overline{v})]$$

$$\geq \delta e^{-\beta t} W[-\beta + \left(\frac{\pi}{2}\right)^2 \frac{d_1}{4k^2b^2} - 2 - a_1 - k\frac{c_1\alpha_2}{c_2}]$$

for all $t > 0$ and $\lambda(t) < x < \sigma(t)$ and

$$\overline{v}_t - d_2\overline{v}_{xx} - \overline{v}(a_2 + b_2\overline{u}(t - \tau_2, x) - c_2\overline{v})$$

$$= \frac{\alpha_2}{c_2}(-a_2 - b_2\delta e^{-\beta(t-\tau_2)}W + ka_2) \geq \frac{\alpha_2}{c_2}((k - 1)a_2 - b_2\delta e^{\beta \tau_2})$$

for all $t > 0$ and $-\infty < x < \infty$. On the other hand, we can easily deduce $\sigma'(t) = 2\gamma \delta e^{-\gamma t} > 0$, $-\pi_x(t, \sigma(t)) = \frac{\pi}{2}\delta \sigma^{-1}(t)e^{-\beta t}$ and $-\overline{u}_x(t, \lambda(t)) = \frac{\pi}{2}\delta \lambda^{-1}(t)e^{-\beta t}$.

Now we set $b_0$ such that

$$\frac{d_1}{8k^2b_0^2} \left(\frac{\pi}{2}\right)^2 = a_1 + \frac{k\alpha_2 c_1}{c_2},$$

if $0 < b \leq b_0$, setting

$$\delta = \min\{\frac{(k - 1)a_2}{2b_2 e^{\beta \tau_2}}, \frac{(k - 1)d_1\pi}{2k^2 \mu} \frac{\frac{b}{2b_0}}{\left(\frac{b}{2b_0}\right)^2}\}, \ \beta = \gamma = \left(\frac{\pi}{2}\right)^2 \frac{d_1}{16k^2b_0^2},$$

It follows that

$$\left\{ \begin{array}{l}
\overline{u}_t - d_1\overline{u}_{xx} \geq \overline{u}(a_1 - b_1\overline{u} + c_1\overline{v}(t - \tau_1, x)), \ t > 0, \ \lambda < x < \sigma(t), \\
\overline{v}_t - d_2\overline{v}_{xx} \geq \overline{v}(a_2 + b_2\overline{u}(t - \tau_2, x) - c_2\overline{v}), \ t > 0, \ -\infty < x < \infty, \\
\overline{u} = 0, \ \sigma'(t) > -\mu \frac{\sigma'}{\sigma''}, \ t > 0, \ x = \sigma(t), \\
\overline{v} = 0, \ \lambda'(t) < -\mu \frac{\lambda'}{\lambda''}, \ t > 0, \ x = \lambda(t), \\
\sigma(0) = 2b > b, \lambda(0) = -2b < -b.
\end{array} \right.$$

By making use of the maximum principle, we can get that $h(t) < \sigma(t)$, $g(t) > \lambda(t)$, and $u(t, x) < \pi(t, x)$, $v(t, x) < \pi(t, x)$ for $g(t) \leq x \leq h(t)$ provided $(u, v)$ exists. Particularly, it follows from Lemma 4.2 that $(u, v)$ exists globally and $g_{\infty} > -\infty$, $h_{\infty} < \infty$.  \[\square\]

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From the above proof, we have the following global existence result

**Theorem 4.4** If \( b_1 c_2 < b_2 c_1 \) and \( a_1 \leq 0, a_2 \leq 0 \), then the free boundary problem (1.1) admits a global fast solution provided \( u_0 \) is suitably small.

**Remark 4.3** If \( b_1 c_2 > b_2 c_1 \), Theorem 3.3 shows that the solution is slow for any initial data. If \( b_1 c_2 < b_2 c_1 \), Theorem 4.1 shows that the solution blows up for large initial data, and sufficient conditions for the global fast solution are given in Theorems 4.3 and 4.4, which implies that the global fast solution is possible if the initial data is suitably small.

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