A SPECTRAL BERNSTEIN THEOREM

PEDRO FREITAS AND ISABEL SALAVESSA

Abstract: We study the spectrum of the Laplace operator of a complete minimal properly immersed hypersurface $M$ in $\mathbb{R}^{n+1}$. (1) Under a volume growth condition on extrinsic balls and a condition on the unit normal at infinity, we prove that $M$ has only essential spectrum consisting of the half line $[0, +\infty)$. This is the case when $\lim_{r \to +\infty} \tilde{r} \kappa_i = 0$, where $\tilde{r}$ is the extrinsic distance to a point of $M$ and $\kappa_i$ are the principal curvatures. (2) If the $\kappa_i$ satisfy the decay conditions $|\kappa_i| \leq 1/\tilde{r}$, and strict inequality is achieved at some point $y \in M$, then there are no eigenvalues. We apply these results to minimal graphic and multigraphic hypersurfaces.

1. INTRODUCTION

The graphic minimal equation in $\mathbb{R}^{n+1}$, for a function $f : \mathbb{R}^n \to \mathbb{R}$, is given by

$$\sum_i \left( \frac{D_i f}{\sqrt{1 + |Df|^2}} \right) = 0.$$ 

It is well known that entire solutions of this equation are linear if $n \leq 7$ (see [4, 18, 13, 2, 23]), and there are counterexamples for $n \geq 8$ given by Bombieri, De Giorgi and Giusti [6]. A natural question to ask is whether these submanifolds may be distinguished by their spectral properties or not.

The Laplace operator $-\Delta$ on a complete noncompact Riemannian manifold $M$ acting on $C_0^\infty(M)$ is essentially self-adjoint and can be uniquely extended as an unbounded self-adjoint operator to a subspace $\mathcal{D}$ of $L^2(M)$ of functions $u$ for which $\Delta u \in L^2(M)$ in the sense of distributions. In what follows, by spectrum of $M$ we mean the spectrum of $-\Delta$. This is a closed subset of $[0, +\infty)$, which can be decomposed as $\sigma(M) = \sigma_p(M) \cup \sigma_{ess}(M)$, where $\sigma_p(M)$ is the pure point spectrum, composed by isolated eigenvalues of finite multiplicity, and $\sigma_{ess}(M)$ is the essential spectrum, a closed subset of $[0, +\infty)$, which is the set of values of $\lambda$ for which there exists an $L^2$-orthonormal sequence $u_m$ with $(\Delta + \lambda I)u_m \to 0$, in $L^2(M)$. The essential spectrum also includes eigenvalues of infinite multiplicity, and those

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of finite multiplicity that are not isolated, if they exist. The bottom of the spectrum admits a variational formulation and may be determined via the minimization of the Rayleigh quotient, namely,

$$\lambda(M) = \inf_{u \in C_{0}^{\infty}(M)} \frac{\int_{M} \|
abla u\|^2 dV}{\int_{M} u^2 dV}.$$  

In the case where $M$ has infinite volume, the lowest point of the essential spectrum

$$\lambda_{\text{ess}} = \inf \sigma_{\text{ess}}(M) \geq \lambda(M),$$

can be estimated by Brooks’s inequality [8]

$$\lambda_{\text{ess}} \leq \frac{1}{4} \mu_{M}^2,$$

where $\mu_{M}$ is the exponential volume growth of $M$

$$\mu_{M} = \lim \sup_{r \to +\infty} \frac{1}{r} \log(V_{M}(B_{r}(x))),$$

and where $V_{M}(B_{r}(x))$ is the volume of the geodesic ball in $M$ of radius $r$ and center $x$ ($\mu_{M}$ does not depend on $x$). If the Ricci tensor of $M$ is bounded from below by a constant $K \in \mathbb{R}$ then, by the Bishop volume comparison theorem, $\mu_{M}$ is finite (and in particular $\sigma_{\text{ess}}(M)$ is nonempty), and is zero if $K \geq 0$. In general, minimal submanifolds of $\mathbb{R}^{n+k}$ do not have to satisfy this property, for it corresponds to bounded second fundamental form.

Recall that the Euclidean space $\mathbb{R}^{n}$ has only essential spectrum consisting of the whole half line $[0, +\infty)$ and there are no (embedded) eigenvalues. We may ask which minimal submanifolds of $\mathbb{R}^{n+k}$ have a trivial spectrum, that is, which minimal submanifolds have the same spectrum as $\mathbb{R}^{n}$. In the case where a minimal submanifold is properly immersed in a ball of $\mathbb{R}^{n+k}$, then it is known that there exists only pure point spectrum, as proved by Bessa, Jorge and Montenegro [5]. The situation for unbounded submanifolds will, however, be different in general, and one may ask this question for particular cases such as minimal graphs in $\mathbb{R}^{n+1}$. A first step towards answering this is to determine whether or not there exist minimal graphic hypersurfaces with the same trivial spectrum as $\mathbb{R}^{n}$ as described above. In this paper we present some results towards an answer to this question – see Theorems 1.1 and 1.4 below. As an example, we recover a result that may already be obtained from the work of Donnelly [15], that the catenoid surface in $\mathbb{R}^{3}$ has spectrum $[0, \infty)$, and further prove – Corollary 1.3 – that it has no embedded eigenvalues. Thus minimal multigraphic hypersurfaces cannot be distinguished by their spectra. It remains open whether or not this is also the case for minimal graphic hypersurfaces.
In what follows, \( F : M \to \mathbb{R}^{n+1} \) is a complete, oriented, properly immersed minimal hypersurface. On \( M \) we give the induced metric \( g_M \) and corresponding Levi-Civita connection \( \nabla \), and denote by \( A \in C^\infty(\otimes^2 T^* M \otimes NM) \) the second fundamental form of \( F \), \( A(X, Y) = DXY - \nabla_X Y \), where \( D \) stands for the flat connection on \( \mathbb{R}^{n+1} \), and \( NM \) is the normal bundle of \( M \). We denote by \( \tilde{r} \) the distance function in \( \mathbb{R}^{n+1} \) to a fixed point \( F(x) \), and by \( \tilde{B}_r(x) \cap M \), the pull back by \( F \) of the ball \( \tilde{B}_r(F(x)) \) on \( \mathbb{R}^{n+1} \), with center \( F(x) \) and radius \( r \), and call it the extrinsic ball at \( x \in M \). It contains the intrinsic ball \( B_r(x) \).

**Theorem 1.1.** Assume \( M \) is a complete oriented properly immersed minimal hypersurface of \( \mathbb{R}^{n+1} \), \( x \in M \), and that there exists a positive constant \( C_n > 0 \) such that, for all \( r > 0 \) sufficiently large

\[
V_M(\tilde{B}_r(x) \cap M) \leq C_n r^n,
\]

where \( V_M \) is the volume with respect to the induced metric \( g_M \) of \( M \). Then \( \mu_M = 0 \). Furthermore, if

\[
|d\tilde{r}(\nu)| \to \xi, \quad \text{when} \quad \tilde{r} \to +\infty,
\]

where \( 0 \leq \xi < 1 \) is a constant, and \( \nu \) is the unit normal to \( M \), then \( \sigma(M) = \sigma_{\text{ess}}(M) = [0, +\infty) \).

Condition (1) implies the number \( \kappa(M) \) of ends of \( M \) must be less than or equal to \( C_n/\omega_n \) where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \) (see [10]). The first part of next proposition is obtained from Lemma 2.4 and Theorem 2.2 of the work of Q. Chen [10] (see Remark 3 in Section 3):

**Proposition 1.1.** If \( \lim_{\tilde{r}(F(y)) \to +\infty} \tilde{r}(F(y)) ||A(y)|| = 0 \), then (1) is satisfied with \( C_n = \kappa(M) \omega_n \), and (2) is also satisfied with \( \xi = 0 \), for \( n \geq 3 \) or, for \( n = 2 \), provided \( M \) has embedded ends. In particular, \( \sigma(M) = \sigma_{\text{ess}}(M) = [0, +\infty) \).

Note that the decay condition on \( A \) in this proposition is not satisfied by the examples of non-linear minimal graphs of Bombieri, de Giorgi and Giusti.

The next theorem is well known for stable varifolds (see for instance [22], Theorem 17.7 [1]) and shows a reverse inequality to (1):

**Theorem 1.2.** (Volume monotonicity formula) Let \( M \) be a properly immersed minimal hypersurface of \( \mathbb{R}^{n+1} \). Then for each \( x \in M \) and for any \( r > \varepsilon > 0 \)

\[
\frac{V_M(\tilde{B}_r(x) \cap M)}{r^n} \geq \frac{V_M(\tilde{B}_\varepsilon(x) \cap M)}{\varepsilon^n} =: F_n(x, \varepsilon).
\]

\(^1\)We are indebted to Brian White for calling our attention to this result.
Furthermore, $\lim_{\varepsilon \to 0} F_n(x, \varepsilon) = k \omega_n$, where $k$ is the number of self-intersections of $F$ at $x$, and $\omega_n$ is the volume of the unit ball of $\mathbb{R}^n$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote the graphic hypersurface by $M = \Gamma_f = \{(p, f(p)) : p \in \mathbb{R}^n\}$. If $\Gamma_f$ is minimal then inequality (1) is satisfied by a classical result due to Miranda:

**Theorem 1.3 ([20]).** If $\Gamma_f$ is an entire minimal graphic hypersurface, then, for each $p \in \mathbb{R}^n$ and $r > 0$

$$\int_{\{q \in \mathbb{R}^n : (q, f(q)) \in B_r(p, f(p)) \cap \Gamma_f\}} (1 + |Df|^2)^{1/2} dV \leq \frac{(n+1)^2}{2} \omega_{n+1} r^n,$$

where $dV$ is the Euclidean volume element of $\mathbb{R}^n$. Furthermore

$$\int_{\{q \in \mathbb{R}^n : (q, f(q)) \in \tilde{B}_r(p, f(p)) \cap \Gamma_f\}} \|A\|^2 dV_M \leq k(n) r^{n-2},$$

where $k(n)$ is a constant depending on $n$.

The left-hand-side of the first inequality is just the volume of the extrinsic ball $V_M(\tilde{B}_r(x) \cap M)$. Thus, we obtain as a corollary of the main theorem:

**Corollary 1.1.** If $M = \Gamma_f$ is a minimal graphic hypersurface, then $\mu_M = 0$. Furthermore, if there is $x = (p, f(p))$ satisfying (2), then $\sigma(M) = \sigma_{\text{ess}}(M) = [0, +\infty)$

Condition (2) holds in several different situations. For instance, we will see in Lemma 3.1 that if $M = \Gamma_f$ for a function $f : \mathbb{R}^n \to \mathbb{R}$, and if there exists a unit vector $p_0 \in \mathbb{R}^n$ such that $\gamma(t) = f(tp_0 + p)$ has bounded derivative, then $\liminf_{r \to +\infty} |D\tilde{\nu}(v)| = 0$. If $|Df|$ is bounded and $\Gamma_f$ is minimal, then Moser in ([21]) proved that for each $k$, $u = \frac{\partial f}{\partial x_k}$ is solution of a uniformly elliptic second order differential equation in the selfadjoint form. Deriving a suitable Harnack theorem that allows for the estimation of the growth of the oscillation of $u$, he concludes that $\lim_{|q| \to +\infty} Df_q$ exists, and further that $f$ is a linear affine function. In fact from the existence of the previous limit, we can show that (2) holds with $\xi = 0$ (see Lemma 3.1).

Linear maps $f(q) = l(q) + b$, satisfy $d\tilde{\nu}(v) = 0$. More generally, (2) holds with $\xi = 0$ if at infinity, for $q$ within open sets, $f(q)$ is of the form $C|q|^{2\alpha}$, $C\log|q|$, or $Ce^{-|q|^{2\alpha}}$, where $C, B, \alpha$ are any reals. On the other hand, since $\Gamma_f$ is a minimal graph, then $f$ is a harmonic function for the graph metric, which implies that $f$ satisfies maximum and minimum principles. In particular, $f$ cannot be globally of the form $\phi(|q|)$ in some ball $B_\alpha(0)$ of $\mathbb{R}^n$, for some fixed function $\phi$, unless it is constant. For a given minimal graphic hypersurface, there could exist more than one limit point of $|D\tilde{\nu}(v)|$, when $r \to +\infty$, where $\nu$ is the unit normal to the hypersurface. It is not clear to us at this point if this may have implications on the spectral behaviour.
To prove Theorem 1.1, for each $\lambda > 0$ we build a sequence $u_m$ spanning an infinite dimensional subspace of $L^2(M)$ and such that $(\Delta + \lambda)u_m \to 0$. This is achieved by using test functions supported in annuli of extrinsic balls and using the volume growth estimates. This is a similar construction given by J. Li [19] where intrinsic balls were used.

Next we give a decay condition on the second fundamental form $A$ of $M$ that implies the non-existence of eigenvalues.

**Theorem 1.4.** If $F : M \to \mathbb{R}^{n+1}$ is a complete properly immersed minimal hypersurface such that the second fundamental form satisfies $\|A(X,X)\| \leq 1/\tilde{r}$, for any unit tangent vector $X$, and strict inequality is achieved at some point of $M$, then $M$ has no eigenvalues.

**Remark 1.** If $M$ has finite total scalar curvature, that is $\int_M \|A\|^n dV_M < +\infty$ (this condition is sufficient to ensure a complete minimal immersed submanifold in a Euclidean space is properly immersed, see [3]), then Q. Chen in [10] proved that $\lim_{r \to +\infty} V_M(\tilde{B}_r(x) \cap M)/\omega_n r^n$ is just $\kappa(M)$. We also note that, under the assumption of finite total scalar curvature, Anderson [3] concluded that $\|A\| \leq c/\tilde{r}^n$, for some constant $c$, and that for $r$ sufficiently large one has $\sup_{\partial B_r(x)} \|A\| \leq \mu(r)/r$ where $\mu(r) \to 0$ when $r \to +\infty$. These inequalities with respect to $\|A\|$ also show that the assumption in Theorem 1.4 above is quite natural.

The condition on $A$ is equivalent to a similar condition on the principal curvatures $\kappa_i$ of $M$. The proof of theorem 1.4 consists on a similar construction in [17], using now the extrinsic distance function instead of the intrinsic one.

**Corollary 1.2.** If $\Gamma_f$ is a minimal graphic hypersurface defined by a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(0) = 0$ and

$$|D^2 f(x)(X,X)|^2 \leq (1 + |Df(x)|^2)/(|x|^2 + f^2(x)),$$

for any $|X| \leq 1$, with strict inequality at some $x$, then $\Gamma_f$ has no eigenvalues. Furthermore, if (2) is satisfied, then $\Gamma_f$ has trivial spectrum.

For a multigraph we have the example of the catenoid. It is not difficult to see that conditions (1) and (2) of Theorem 1.1 are satisfied, and that the principal curvatures $\kappa_i$ satisfy the conditions in Proposition 1.1 and in Theorem 1.4. To prove all this we only have to recall that $t \leq \sinh t \cosh t$ and $\sinh t \leq t \cosh t$, for any $t \geq 0$, with equality only at $t = 0$, as we can verify using an infinite Taylor expansion of the hyperbolic sine and cosine.

**Corollary 1.3.** The catenoid surface in $\mathbb{R}^3$ has trivial spectrum.

Minimal hypersurfaces of Euclidean spaces have nonpositive Ricci tensor, and by the generalization of the Hilbert-Efimov theorem given by Smith
and Xavier [24], \(\inf\|A\| = \sup\mathrm{Ricci}^M = 0\). No further information on the curvature is given. These results are insufficient to allow us to apply known results relating the spectrum (of a minimal hypersurface) to curvature (see e.g. [11, 14, 15, 16]), for the case \(n > 2\). For \(n = 2\), Donnelly in Theorem 6.3 of [15] proved that complete noncompact simply connected surfaces with nonpositive curvature that converges to zero at infinity, have essential spectrum \([0, +\infty)\). For surfaces with finite fundamental group he obtained only \(\lambda_{\text{ess}} = 0\) with no further conclusion on the nonexistence of eigenvalues. This last result may be applied to the catenoid, but Corollary 1.3 above characterizes the whole spectrum. We note that, for an arbitrary noncompact Riemannian manifold, it is sufficient that \(\mathrm{Ricci}^M\) converges to zero at infinity in order to have \(\lambda_{\text{ess}} = 0\). This can be derived from an argument used in [15], applying Cheng’s eigenvalue comparison inequality [12], as was used in [9] (see also Proposition 3.2).

2. PROOF OF THEOREM 1.2

We will give a proof of Theorem 1.2 with no need of using currents, closely following the proof of a recent result due to Alencar, Waley and Zhou:

**Theorem 2.1 ([1]).** If \(M\) is a minimal immersed hypersurface of \(\mathbb{R}^{n+1}\) with induced metric \(g_M\), then for each \(x \in M\) and \(r > 0\), away from cut locus distance of \(x\),

\[V_M(B_r(x)) \geq \alpha_n r^n.\]

By using extrinsic balls, we avoid the assumption on the cut locus. We denote by \(\tilde{g}\) the Euclidean metric of \(\mathbb{R}^{n+1}\), and by \(g_M\) the induced metric on \(M\). For \(z \in \mathbb{R}^{n+1}\), \(\tilde{r}(z) = |z - F(x)| = \sqrt{g(z - F(x), z - F(x))}\). Restricting \(\tilde{r}\) to \(M\), \(\tilde{r}(y) = \tilde{r}(F(y))\), we set \(h = \frac{1}{2}r^2 : M \to \mathbb{R}\). A standard computation shows that, for \(X \in T_M\), \(d\tilde{r}_y(X) = \tilde{g}(X, F(y) - F(x))/\tilde{r}(y)\) and

\(3\) \quad \text{Hess} h_y(X, X) = \tilde{g}(X, X) + \tilde{g}(A(X, X), F(y) - F(x)).

Therefore, if \(M\) is a minimal hypersurface with mean curvature \(H\), \(nH = \text{tr} A = 0\), and so,

\[\Delta h = n.\]

Integration of the previous equation on a normal domain \(D\) of \(M\) with boundary \(\partial D\) and Stokes’s Theorem yields

\(4\) \quad \int_{\partial D} \tilde{r} g_M(\nabla\tilde{r}, v_{\partial D}) dS = n V_M(D),

where \(v_{\partial D}\) is the unit normal to \(\partial D\) and \(dS\) is the volume element, and \(\nabla\tilde{r}\) the gradient in \(M\). Taking \(D = \tilde{B}_s(x) \cap M\), where \(s\) is a regular value of \(\tilde{r}_M\),
define
\[ V_M(s) := V_M(\bar{B}_s(x) \cap M). \]
Using (4) with \( v_{\partial \bar{B}_s} = \nabla \bar{r}/\|\nabla \bar{r}\| \), and the co-area formula,
\[ \int_{\partial \bar{B}_s(x) \cap M} \bar{r}/\|\nabla \bar{r}\| \; dS = n V_M(s) = n \int_0^s \frac{1}{t} \rho(t) \; dt, \]
where \( \rho(t) = \int_{\partial \bar{B}_s(x) \cap M} \bar{r}/\|\nabla \bar{r}\| \; dS \). Since \( \|\nabla \bar{r}\|^{-1} \geq 1 \geq \|\nabla \bar{r}\|, \) by (5),
\[ \rho(s) \geq \int_{\partial \bar{B}_s(x) \cap M} \bar{r}/\|\nabla \bar{r}\| \; dS = n \int_0^s \frac{1}{t} \rho(t) \; dt. \]
Therefore \( s V'_M(s) \geq n V_M(s) \) which implies \( \frac{d}{ds} \ln V_M(s) \geq \frac{d}{ds} \ln s^n \). Integration along \([\varepsilon,s]\), where \( 0 < \varepsilon < s \), leads to \( V_M(s)/s^n \geq V_M(\varepsilon)/\varepsilon^n \), and the inequality of the theorem is proved.

Next we prove \( \lim_{\varepsilon \to 0} V_M(\varepsilon)/\varepsilon^n = k \omega_n \). We take \( \gamma(t) \) a curve in \( M \) starting at \( x \). Thus, \( \bar{\gamma}(t) := F(\gamma(t)) = F(x) + t \gamma'(0) + o(t), \) with \( \gamma'(0) \) non zero.

Since \( \nabla \bar{r}(\bar{\gamma}(t)) = \bar{\gamma}'(t)/|\bar{\gamma}'(t)| > 0 \), we have
\[ \bar{g}(\nabla \bar{r}(\bar{\gamma}(t)), \nabla \gamma(t)) = \frac{t \bar{g}(\gamma'(0), \gamma(t)) + o(t)}{|t \gamma'(0) + o(t)|} = \frac{\bar{g}(\gamma'(0), \gamma(t)) + o(t)}{|\gamma'(0)| + o(t)|} \]
and this converges to 0 when \( t \to 0 \). For \( y \in M, \ 1 = |\nabla \bar{r}|^2 = |\nabla \bar{r}|^2 + |d\bar{r}(v)|^2, \) and so \( \lim_{y \to x, y \in M} \|\nabla \bar{r}\| = 1 \). Therefore,
\[ \lim_{\varepsilon \to 0} \frac{1}{V_M(\varepsilon)/\varepsilon^n} \int_{\partial \bar{B}_\varepsilon(x) \cap M} \frac{1}{\|\nabla \bar{r}\|} \; dS = 1. \]
Making \( \varepsilon \to 0 \), using l’Hôpital’s Rule, (5) and (6), we have
\[ \lim_{\varepsilon \to 0} \frac{V_M(\varepsilon)}{\varepsilon^n} = \lim_{\varepsilon \to 0} \frac{\rho(\varepsilon)}{n \varepsilon^n} = \lim_{\varepsilon \to 0} \frac{1}{n \varepsilon^{n-1}} \int_{\partial \bar{B}_\varepsilon(x) \cap M} \frac{1}{\|\nabla \bar{r}\|} \; dS \]
\[ = \lim_{\varepsilon \to 0} \frac{V_S(\partial \bar{B}_\varepsilon(x) \cap M)}{n \varepsilon^{n-1}} = \lim_{\varepsilon \to 0} \frac{V_S(\partial \bar{B}_\varepsilon(x) \cap M)}{V_{n-1}(S_{\varepsilon}^{n-1})} \omega_n = k \omega_n, \]
where \( k \) is the number of self-intersections of \( F(M) \) at \( F(x) \), and \( S_{\varepsilon}^{n-1} \) is the sphere of \( \mathbb{R}^n \) of radius \( \varepsilon \). In the last equality we used that, when \( \varepsilon \to 0 \), \( F(M) \cap \bar{B}_\varepsilon(F(x)) \) can be identified with \( k \) copies of \( T_xM = \mathbb{R}^n \).

3. PROOF OF THE THEOREM 1.1 AND COROLLARY 1.1

Since \( |F(x) - F(y)| \leq d(x,y) \) for any \( x,y \in M \), where \( d \) is the intrinsic distance on \( M \), then \( \bar{B}_r(x) \subset \bar{B}_r(x) \cap M \). In particular
\[ \mu_M \leq \lim_{r \to +\infty} \frac{1}{r} \ln \left( \int_{\bar{B}_r(x) \cap M} dV_M \right). \]
For the case \( M = \Gamma_f \), we denote by \( g \) and \( \bar{g} = g + dt^2 \) the Euclidean metrics of \( \mathbb{R}^n \) and \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \), respectively. We often identify \( \Gamma_f \), a subset of \( \mathbb{R}^{n+1} \), with \( \mathbb{R}^n \) endowed with the graph metric \( g_M = g + f^* dt^2 \), that is, we may see \( \Gamma_f \) as an immersion \( \Gamma_f : \mathbb{R}^n \to \mathbb{R}^{n+1} \), \( \Gamma_f(p) = (p, f(p)) \) and give to \( \mathbb{R}^n \) the pull back metric by \( \Gamma_f \). The graph metric is always complete.

Moreover, if there exists a direction \( p \) of \( \mathbb{R}^n \), we may see that the pull back metric by \( \Gamma_f \) is a ball with respect to the graph metric \( g_M \). The volume element of \( \mathbb{R}^n \) with respect to the graph metric is given by

\[
dV_M = \sqrt{1 + |Df|^2} dV,
\]

where \( Df \) denotes the \( g \)-gradient of \( f \), and the unit normal to the graph is

\[
\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}.
\]

We have, \( B_r^0(p) \subset B_r(p) \), where \( r = r' \sqrt{1 + F(p, r')^2} \) with \( F(p, r') = \sup_{q \in B_r(p)} |Df| \), and \( B_r(p) \subset B_r(p, f(p)) \cap \Gamma_f \subset B_r^0(p) \).

From (2), (8) and Miranda’s Theorem 1.3, we conclude:

**Proposition 3.1.** If \( \Gamma_f \) is minimal, then it has zero exponential volume growth, \( \mu_M = 0 \). In particular \( 0 \in \sigma_{ess}(M) \), and it is not an eigenvalue.

**Proof.** By a result due to Yau [25], there are no \( L^2 \) harmonic functions on noncompact complete manifolds of infinite volume (see Theorem 1.2).

**Remark 2.** From Miranda’s inequality and the volume monotonicity formula we see that \( \frac{\alpha_{n+1}}{\alpha_n} \geq \frac{2}{(n+1)^2} \). There is a sharper lower bound, \( \frac{\alpha_{n+1}}{\alpha_n} \geq \frac{\sqrt{2\pi}}{\sqrt{n+2}} \), [7].

**Lemma 3.1.** (i) If \( M \) is a minimal hypersurface of \( \mathbb{R}^{n+1} \), then at any \( y \in M \)

\[
\frac{n - 1}{\rho} \leq \Delta \tilde{r} \leq \frac{n}{\rho}, \quad 1 - \|\nabla \tilde{r}\|^2 = \bar{g}(\tilde{\nabla} \tilde{r}, \nu)^2.
\]

(ii) In the case \( M = \Gamma_f \), \( x = (p, f(p)) \), at \( y = (q, f(q)) \),

\[
\bar{g}(\tilde{\nabla} \tilde{r}, \nu) = \frac{1}{\rho} \frac{(-g(q-p, Df_q) + (f(q) - f(p)))}{\sqrt{1 + |Df|^2}}.
\]

Moreover, if there exists a direction \( p_0 \in \mathbb{R}^n \), \( |p_0| = 1 \), such that \( \gamma(t) = f(tp_0 + p) \) is a curve such that \( \exists \lim_{t \to +\infty} \gamma'(t) \) and is bounded then

\[
\lim_{t \to +\infty} \bar{g}(\tilde{\nabla} \tilde{r}, \nu_{\gamma'(\gamma(t))})^2 = 0.
\]

In particular, if \( \exists \lim_{t \to +\infty} Df \) then (2) is satisfied with \( \xi = 0 \).
Proof. Let $e_i$ be a $g_M$ o.n. basis of $T_xM$. We denote by $(u)^\top$ and $(u)^\perp$ the orthogonal projections of a vector $u \in \mathbb{R}^{n+1}$ onto $T_xM$ and $NM_x$, respectively. For any $y \in M$, from $|\nabla r|^2 = 1$ and that $(\nabla r)^\top = \nabla r$, $(\nabla r)^\perp = \tilde{g}(\nabla r, v)v$, we obtain the second equality. Since the mean curvature $H$ of $M$ vanishes, a standard computation gives at $y$

$$\Delta\tilde{r} = \sum_i \tilde{D}dF(e_i, dF(e_i)) + \tilde{g}(nH, \nabla r) = \sum_i \tilde{D}dF(e_i, dF(e_i))$$

$$= \frac{n}{\tilde{r}} - \sum_i \tilde{g}(F(y) - F(x), e_i)^2 = \frac{n}{\tilde{r}} - |(F(y) - F(x))^\top|^2,$$

Now $|(F(y) - F(x))^\top|^2 \leq |F(y) - F(x)|^2 \leq \tilde{r}^2$, and we obtain the bounds of $\Delta\tilde{r}$. For $M = \Gamma_x$, $x = (p, f(p))$, $y = (q, f(q))$, (9) is expressed as (10). Now we assume $\exists \lim_{t \to +\infty} \gamma(t)$ and is bounded. Let $q(t) = tp_0 + p$. Along $(q(t), \gamma(t) = f(q(t)))$, by (10), for all $t$ sufficiently large,

$$|\tilde{g}(\nabla r, v)| = \frac{1}{\tilde{r}} \left| \frac{(-r\gamma(t) + \gamma(t) - f(p))}{t^2} \right|.$$

Therefore, multiplying by $1/t$, we have

$$\left| \frac{(-r\gamma(t) + \gamma(t) - f(p))}{t} \right| = |\tilde{g}(\nabla r, v)| \frac{\tilde{r}}{t} \sqrt{1 + |Df_{q(t)}|^2}$$

$$\geq |\tilde{g}(\nabla r, v)| \frac{\tilde{r}}{t} = |\tilde{g}(\nabla r, v)| \sqrt{1 + |(\gamma(t) - f(p))^2|} \geq |\tilde{g}(\nabla r, v)|.$$

By using l’Hôpital’s Rule, $\lim_{t \to +\infty} \frac{\gamma(t)}{t} = \lim_{t \to +\infty} \gamma'(t)$, and we conclude that $\lim_{t \to +\infty} |\tilde{g}(\nabla r, v)| = 0$. \hfill \Box

Set for each $r > 0$, $V_M(r) = V_M(\tilde{B}_r(x) \cap M)$.

**Lemma 3.2.** Let $C_n$ and $F_n$ be positive constants such that for $r$ sufficiently large,

$$F_n r^n \leq V_M(r) \leq C_n r^n.$$

Let $\tau > 0$ be a small constant and, $0 < a_m < b_m < d_m$ be constants such that

$$\frac{F_n}{C_n} \left( \frac{b_m}{d_m} \right)^n - \left( \frac{a_m}{d_m} \right)^n \geq \tau > 0.$$

Then, for any $\varepsilon > 0$, $\frac{V_M(\frac{\tilde{b}_m}{\varepsilon}) - V_M(\frac{\tilde{d}_m}{\varepsilon})}{V_M(\frac{\tilde{d}_m}{\varepsilon})} \geq \tau$. 

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Note that it is sufficient to take $b_m$ close to $d_m$ and $a_m$ sufficiently smaller than $b_m$ to obtain such a lower bound $\tau$. For example, set

\[ \theta^n = \frac{C_n}{F_n} \geq 1, \quad b_m = \frac{d_m}{2}, \quad a_m = \frac{d_m}{(2^\alpha \theta)} < b_m, \]

where $\alpha > 1$ and $\tau = \theta^{-n}(2^{-n} - 2^{-\alpha n}) > 0$.

**Proof.**

\[ \frac{V_M(b_m) - V_M(a_m)}{V_M(a_m)} \geq F_n \left( \frac{b_m}{d_m} \right)^n - \left( \frac{a_m}{d_m} \right)^n \]

Now we follow a similar construction as in J.Li [19]. For each $m$ we consider positive constants $0 < c_m < a_m < b_m < d_m$ that we will define later, and a smooth function $\psi_m(t)$ such that $|\psi_m| \leq 1$ and

\[ \psi_m(t) = \begin{cases} 1 & \text{if } a_m \leq t \leq b_m, \\ 0 & \text{if } t < c_m \text{ or } t > d_m. \end{cases} \]

with $|\psi'_m(t)| \leq C_m$, and $|\psi''_m(t)| \leq C_m$, where $C_m > 0$ is a positive constant that depends on $a_m, b_m, c_m$ and $d_m$, that can be given of the form

\[ C_m = E \left( \frac{1}{a_m - c_m} + \frac{1}{d_m - b_m} + \frac{1}{(a_m - c_m)^2} + \frac{1}{(d_m - b_m)^2} \right), \]

where $E > 0$ is a constant. We fix a decreasing sequence $\varepsilon_k \to 0$, of positive reals, and take

\[ \eta_{k,m} := \frac{1}{V_M(D(c_m, d_m))}, \]

where $D(c,d) = \{ y \in M : c \leq \bar{r}(y) \leq d \}$. Now we fix $\lambda > 0$ and consider the function

\[ u_{k,m}(y) = \sqrt{\eta_{k,m}} \psi_m(\varepsilon_k \bar{r}(y))e^{i\sqrt{\lambda} \bar{r}(y)}. \]

We have

\[ \nabla u_{k,m} = \sqrt{\eta_{k,m}} \left( \varepsilon_k \psi_m(\varepsilon_k \bar{r}(y)) + i \sqrt{\lambda} \psi_m(\varepsilon_k \bar{r}(y)) \right)e^{i\sqrt{\lambda} \bar{r}(y)} \nabla \bar{r}. \]

Set $\zeta^2 = 1 - \xi^2$. Then

\[ \Delta u_{k,m} + \zeta^2 \lambda u_{k,m} = \]

\[ = \varepsilon_k \sqrt{\eta_{k,m}} e^{i\sqrt{\lambda} \bar{r}} \left( \varepsilon_k \psi_m''(\varepsilon_k \bar{r}) + 2i \sqrt{\lambda} \psi_m'(\varepsilon_k \bar{r}) \right) ||\nabla \bar{r}||^2 \]

\[ + \sqrt{\eta_{k,m}} e^{i\sqrt{\lambda} \bar{r}} \left( \varepsilon_k \psi_m'(\varepsilon_k \bar{r}) + i \sqrt{\lambda} \psi_m(\varepsilon_k \bar{r}) \right) \Delta \bar{r} \]

\[ + \lambda (\zeta^2 - ||\nabla \bar{r}||^2) u_{k,m}. \]
Now we prove that (11) and (12) tend to 0 in $L^2$, when we chose $a_m, b_m, c_m, d_m$ and $\varepsilon_k$ in a suitable way and let $m, k$ go to $+\infty$.

$$\int_M |(11)|^2 d\nu \leq \eta_{k,m} \varepsilon_k^2 \int_{D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k})} (2 \varepsilon_k^2 C_m^2 + 4 \lambda C_m^2) d\nu$$

(14)

and using Lemma 3.1

$$\int_M |(12)|^2 d\nu \leq \eta_{k,m} \int_{D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k})} 2(\varepsilon_k^2 C_m + \lambda) \frac{n}{\epsilon^2} d\nu$$

(15)

Now we consider (13). We have $\xi^2 - ||\nabla\tilde{r}||^2_{L^2} = -\xi^2 + \tilde{g}(\nabla \tilde{r}, \nu)^2$. Then,

$$\int_M |(13)|^2 d\nu \leq \int_{D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k})} \lambda^2 \eta_{k,m} (\xi^2 - ||\nabla\tilde{r}||^2)^2$$

(16)

Thus, for any choice of $a_m, b_m, c_m, d_m \rightarrow +\infty$, and taking $k = k_m \rightarrow +\infty$, such that $\varepsilon_k \eta_k C_m \rightarrow 0$, we may assume (16) $\rightarrow 0$, and so, we obtain a sequence $u_{k,m} \in L^2$ s.t. $\Delta u_{k,m} + \xi^2 \lambda u_{k,m} \rightarrow 0$. On the other hand, we have

$$\int_M u_{k,m}^2 d\nu \geq \eta_{k,m} \int_{D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k})} d\nu = \frac{V_M(D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k}))}{V_M(D(\epsilon_k \eta_k, \frac{d_m}{\epsilon_k}))} \geq \frac{V_M(b_m \epsilon_k)}{V_M(a_m \epsilon_k)}.$$ 

Choosing $a_m, b_m, c_m, d_m$ satisfying Lemma 3.2, we have $\int_M u_{k,m}^2 \geq \tau > 0$, and we may chose $a_m, b_m, c_m, d_m \rightarrow +\infty$ to obtain a sequence $u_{k,m}$ that spans an infinite dimensional subspace of $L^2$. Thus $\xi^2 \lambda$ belongs to the essential spectrum, for any $\lambda > 0$. This proves Theorem 1.1.

Remark 3. Both in Lemma 2.4 and Theorem 2.2 of [10] it is only required that $\lim_{t \rightarrow +\infty} \sup_{F(y), ||A|| = 1} \tilde{r}(F(y)) = 0$ holds in order to obtain (1) with $C = \kappa(M)\omega_\nu$ and $\lim_{t \rightarrow +\infty} \inf_{\tilde{r} \geq t} ||\nabla\tilde{r}||^2 = 1$. We note that our assumption in the first limit in Proposition 1.1 is equivalent to the above one, for, if such lim on $\sup_{\tilde{r} \geq t} ||A||$ is zero, then $\lim_{\tilde{r} \rightarrow +\infty} \tilde{r}|A|$ exists and is zero as well, and vice versa. Similarly, $\lim_{t \rightarrow +\infty} \inf_{\tilde{r} \geq t} ||\nabla\tilde{r}|| = 1$ is equivalent to $\lim_{\tilde{r} \rightarrow +\infty} ||\nabla\tilde{r}||^2 = 1$, for $||\nabla\tilde{r}|| \leq 1$. The later is equivalent to (2) with $\xi = 0$.

The next proposition is obtained from an argument used in the proof of Theorem 3.1 of [15], of which we give here a proof for the sake of completeness:
**Proposition 3.2.** If $M$ is a complete noncompact Riemannian manifold with $\text{Ricci}^M \to -(n-1)c$, when $r(y) \to +\infty$, where $c \geq 0$ is a constant, and $r(y)$ is the intrinsic distance in $M$ to a fixed point $x \in M$, then $\lambda_{\text{ess}} \leq \frac{(n-1)^2}{4}c$.

**Proof.** By assumption, for any $\delta > 0$, $\exists r_0 > 0$, such that $\text{Ricci}^M + (n-1)c \geq -\delta$, for all $y \in M$ with $r(y) \geq r_0$. Fix a sequence $r_i \to +\infty$. We can find $y_1 \in M$ sufficiently far away from $x$ such that on $B_{r_i}(y_1)$, $\text{Ricci}^M \geq -(n-1)(c + \frac{1}{r_i})$. Next, we take $y_2$ sufficiently far away from $x$ and such that $B_{r_2}(y_2) \subset M \setminus B_{r_i}(y_1)$ and on $B_{r_2}(y_2)$, $\text{Ricci}^M \geq -(n-1)(c + \frac{1}{r_2})$. By induction we construct a sequence $y_i$, and balls $B_{r_i}(y_i) \subset M \setminus \bigcup_{s=1}^{i-1} B_{r_s}(y_s)$ where $\text{Ricci}^M \geq -(n-1)(c + \frac{1}{r_i})$. From Cheng’s eigenvalue comparison inequality, on each ball $B_{r_i}(y_i)$, $\lambda_1(r_i) := \lambda_1(B_{r_i}(y_i)) \leq \lambda_1(D_{r_i})$, where $D_{r_i}$ is the disk of radius $r_i$ of the $n$-dimensional space form of constant sectional curvature $-c_1 = -(c + \frac{1}{r_i})$. Then $\lambda_1(r_i) \leq (n-1)^2c_1/4 + \psi(r_i)$ where $\psi(r_i) \to 0$ when $r_i \to +\infty$. We consider $u_i \in L^2(M)$ the solution of the Dirichlet problem $\Delta u_i + \lambda_1(r_i) u_i = 0$, $u_i = 0$ on $\partial B_{r_i}(y_i)$ ( $u_i$ extended to zero on $M \setminus B_{r_i}(y_i)$), and $\int_M u_i^2 = 1$. Let $\lambda_1$ be an accumulation point of $\lambda_1(r_i)$. Then for a subsequence, $\Delta u_i + \lambda_1 u_i \to 0$ in $L^2$, what shows that $\lambda_1 \in \sigma_{\text{ess}}(M)$. This proves that $\lambda_{\text{ess}} \leq (n-1)^2c/4$. 

**Remark 4.** In the case of a graphic minimal hypersurface, we have

$$A(X, Y) = (0, D^2 f(X, Y)) \perp \frac{1}{\sqrt{1 + |Df|^2}} D^2 f(X, Y),$$

$$\text{Ricci}^M (X, X) = -\sum_i |A(X, e_i)|^2 \leq 0, \quad s^M = -\|A\|^2,$$

and $\Delta f = 0$, that is, $\Delta^0 f - 2\frac{|Df|^4}{(1+|Df|^2)^2} D^2 f(Df, Df) = 0$, where $\Delta^0$ is the Euclidean Laplacian.

We recall the example of Bombieri, de Giorgi and Giusti [6], of nonlinear minimal graphic hypersurface $\Gamma_f$, with $f : \mathbb{R}^{2m} \to \mathbb{R}$, $m \geq 4$, is obtained as a limit of solutions $f^R$ defined on balls $B_R$ satisfying the minimal surface equation with $f^R = f_1$ on $\partial B_R$, and it satisfies $|f_1(x)| \leq |f(x)| \leq |f_2(x)|$ where $f_1$ and $f_2$ are defined as follows. Set $\alpha = (2p + 1 - \sqrt{\delta})/4 > 1$, $\delta = 4p^2 - 12p + 1$, $p = m - 1$, $\lambda$ such that, $\alpha((2p + 1)/(2p + 2)) < \lambda < \min\{\alpha, p/\alpha^2\}$, $D = D(\lambda, p)$, $B = B(\lambda, p)$ sufficiently large positive constants, and

$$P(z) = \int_0^z \exp \left( B \int_{|w|}^\infty t^{\lambda-2}(1 + t^{2\alpha(\lambda-1)})^{-1} dt \right) dw.$$

Then,

$$f_1(x) = (u^2 - v^2)(u^2 + v^2)^{-1}$$

$$f_2 = P \left( (u^2 - v^2) + f_1[1 + D|u^2 - v^2]/(u^2 + v^2)^{\lambda-1}] \right),$$
where $u = (x_1^2 + \ldots + x_m)^{1/2}$, and $v = (x_{m+1}^2 + \ldots + x_{2m}^2)^{1/2}$. For $x \in B_h(0)$ the following gradient estimate holds:

$$|Df(x)| \leq c_1 \exp\left(\frac{c_2}{2h} \sup_{B_{2h}(0)} |f_2|\right) = C(h),$$

and one has

$$\lim_{|x| \to +\infty} \sup_{|x|} \frac{|f(x)|}{|x|^{2\alpha}} \geq \lim_{|x| \to +\infty} \sup_{|x|} \frac{|f_1(x)|}{|x|^{2\alpha}} = 1.$$

This does not allow us to conclude anything about $\lim_{r \to +\infty} d\tilde{r}(\nu)$. This is an unknown limit as can be seen from the behaviour of the second fundamental form of $\Gamma_f$ given in remark 5 in section 4. The study of the spectrum of such examples seems to require the understanding of the behaviour of $d\tilde{r}(\nu^R)$ at $B_R$ of the solution $f^R$, using methods from partial differential equations, a study that is outside the scope of this paper.

4. PROOF OF THEOREM 1.4

We recall the following lemma of Escobar-Freire ([17], Lemma 3.1) (here we change the sign of $\lambda$ and denote their $f$ by $h$):

Lemma 4.1. ([17]) Let $D$ be a bounded domain with $C^2$ boundary in a Riemannian manifold $M$. Let $u$ and $h$ be functions in $C^1(\bar{D})$. Then for any $\lambda \in \mathbb{R}$

$$\int_D (\|\nabla u\|^2 - \lambda u^2) \Delta h - 2 \int_D Hessh(\nabla u, \nabla u) - 2 \int_D (\Delta u + \lambda u) g_M(\nabla h, \nabla u)$$

$$= \int_{\partial D} (\|\nabla u\|^2 - \lambda u^2) \frac{\partial h}{\partial n} - 2 \int_{\partial D} g_M(\nabla h, \nabla u) \frac{\partial u}{\partial n}.$$

We are considering $F : M \to \mathbb{R}^{n+1}$ a complete minimal properly immersed hypersurface. Now we follow [17] closely, but use extrinsic distance instead of intrinsic. We take the function $h = \frac{1}{2} r^2$ restricted to $M$. Then $\|\nabla h\| \leq \tilde{r}$ and $\Delta h = n$. We assume $A$ satisfies the conditions of Theorem 1.4. Then by (3), Hess$h(X,X)$ is bounded, and for any $X \in T_yM$,

$$\text{Hess} h(X,X) \geq 0,$$

with Hess$h_{y_0} > 0$ at some point $y_0 \in M$. Let us assume that $\lambda$ is an eigenvalue. We take $u \in \mathcal{D}(\Delta)$ non zero, such that $\Delta u + \lambda u = 0$. By the unique continuation property, $u$ (or $\nabla u$) cannot identically vanish in any open set. On each extrinsic ball $\tilde{B}_r$ (that means $\tilde{B}_r(x) \cap M$), we have

$$\int_{\tilde{B}_r} (\|\nabla u\|^2 - \lambda u^2) = \int_{\tilde{B}_r} \|\nabla u\|^2 + u \Delta u = \int_{\tilde{B}_r} \frac{1}{2} \Delta u^2 = \int_{\partial \tilde{B}_r} \frac{1}{2} \frac{\partial u^2}{\partial n}.$$
Then applying Lemma 4.1 and the above equality, we have

\[
\int_{\tilde{B}_r} \text{Hess} h(\nabla u, \nabla u) = \frac{n}{2} \int_{\partial \tilde{B}_r} \frac{\partial u^2}{\partial n} - \int_{\partial \tilde{B}_r} (\|\nabla u\|^2 - \lambda u^2) \frac{\partial h}{\partial n} + 2g_M(\nabla h, \nabla u) \frac{\partial u}{\partial n}.
\]  

(18)

Since \(u\) and \(\nabla u\) are both in \(L^2(M)\), and \(\|\nabla \tilde{r}\| \leq 1\) then, by the co-area formula

\[
\int_{\varepsilon}^{+\infty} dt \int_{\partial \tilde{B}_t} (\|\nabla u\|^2 + u^2) dS = \int_{M \setminus \tilde{B}_\varepsilon} \|\nabla \tilde{r}\| (\|\nabla u\|^2 + u^2) < +\infty.
\]

Lemma 4.2. \((17)\) If \(\phi : [\varepsilon, +\infty) \to [0, +\infty)\) is a measurable function, with \(\varepsilon > 0\), and such that \(\int_{\varepsilon}^{+\infty} \phi(t) dt < +\infty\), then there exists \(t_i \to +\infty\), such that \(t_i \phi(t_i) \to 0\).

Proof. If such sequence \(t_i\) did not exist, then there exists a constant \(C > 0\) and a \(t_0 \geq \varepsilon\) such that for all \(t \geq t_0\), \(t \phi(t) \geq C\). But then \(\int_{t_0}^{+\infty} \phi(t) \geq C \log(t)_{t_0} = +\infty\), contradicting the assumption. \(\Box\)

Applying the lemma to \(\phi(t) = \int_{\partial \tilde{B}_t} (\|\nabla u\|^2 + u^2) dS\), we conclude for a sequence \(r_i \to +\infty\), \(r_i \int_{\partial \tilde{B}_{r_i}} (\|\nabla u\|^2 + u^2) dS \to 0\). Thus, (18) with \(r = r_i\) converges to zero when \(r_i \to +\infty\) (note that \(\|\frac{\partial h}{\partial n}\| \leq \|\nabla h\| \leq \tilde{r}\) and \(\frac{\partial u^2}{\partial n} = 2u \frac{\partial u}{\partial n}\)). Therefore we arrive at

\[
\int_M \text{Hess} h(\nabla u, \nabla u) dV_M = 0.
\]

Convexity of \(h(17)\) implies \(\text{Hess} h(\nabla u, \nabla u) = 0\) everywhere. Since \(h\) is strictly convex at \(y_0\), \(\nabla u\) must vanish on a neighbourhood of a point, yielding that \(u\) vanishes everywhere. This finishes the proof of Theorem 1.4. The corollaries are an immediate consequence of Theorem 1.4 and Remark 4. \(\Box\)

Remark 5. As mentioned in \([10]\), the examples of minimal graphs (hence with one end) of Bombieri, de Giorgi and Giusti for \(n \geq 8\) satisfy

\[
+\infty > \sup_r \frac{V_M(M \cap \tilde{B}_r(x))}{r^n} > \omega_n.
\]

By Proposition 1.1, this means that \(\lim_{\tilde{r}(F(y)) \to +\infty} \tilde{r}(F(y)) \|A_y\| = 0\) does not hold. Furthermore, we do not know if (2) is satisfied and, as a consequence, we cannot exclude the existence of minimal graphs with no trivial spectrum.
Remark 6. We cannot replace the condition \( \|A(X,X)\| \leq 1/\bar{r} \) by the weaker one \( \|A\| \leq c/\bar{r} \), \( c > 0 \) a constant, if we want to use the argument of Escobar-Freire. Indeed, if we try to rescale \( F \) by taking \( \hat{F} = cF \), \( \hat{F} \) defines an immersion \( \hat{M} \) with the induced metric \( \hat{g} = c^2 g_M \). Identify \( T_{cF}(y) \hat{F} (M) = T_{cF}(y) F(M) \), by identifying \( T_{cF}(y) R^{n+1} \) with \( T_{F}(y) R^{n+1} \), and \( \hat{\nu}_{\hat{F}}(x) = \nu_F(x) \), \( \hat{M} \) and \( M \) have the same Levi-Civita connections and \( \hat{B}_{\hat{F}}(x,y) = \frac{1}{c} B_F(y)(\hat{X},\hat{Y}) \), where \( \hat{X}_{cF(y)} = \hat{X}_{F(y)} \) is a tangent vector (under the above identifications). Moreover, \( \hat{\Delta} = \frac{1}{c^2} \Delta^M \), \( \hat{H} = \frac{1}{c^2} H = 0 \), and \( \hat{\lambda}(M) = c^{-2} \lambda(M) \), \( \hat{\lambda}_{\text{ess}} = c^{-2} \lambda_{\text{ess}} \), \( \sigma(M) = c^{-2} \sigma(M) \) and similar for the essential and the pure point spectrum. Unfortunately \( \bar{\rho}(\hat{F}(y)) = c\bar{\rho}(F(y)) \) which means that we have again \( \|\hat{A}(\hat{X},\hat{X})\| \leq c/\bar{\rho}(\hat{F}(y)) \).

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DEPARTMENT OF MATHEMATICS, FACULDADE DE MATRICIDADE HUMANA (TU LISBON) and GROUP OF MATHEMATICAL PHYSICS OF THE UNIVERSITY OF LISBON, COMPLEXO INTERDISCIPLINAR, AV. PROF. GAMA PINTO 2, P-1649-003 LISBOA, PORTUGAL

E-mail address: freitas@cii.fc.ul.pt

CENTRO DE FÍSICA DAS INTERACÇÕES FUNDAMENTAIS, INSTITUTO SUPERIOR TÉCNICO, TECHNICAL UNIVERSITY OF LISBON, EDIFÍCIO CIÊNCIA, PISO 3, AV. RODRIGO PAIS, P-1049-001 LISBOA, PORTUGAL

E-mail address: isabel.salavessa@ist.utl.pt