UNIFORMIZABLE SINGULAR PROJECTIVE STRUCTURES ON Riemann SURFACE ORBIFOLDS

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Abstract. This paper is devoted to characterizing complex projective structures defined on Riemann surface orbifolds and giving rise to injective developing maps defined on the monodromy covering of the surface (orbifold) in question. The relevance of these structures stems from several problems involving vector fields with uniform solutions as well as from problems about “simultaneous uniformization” for leaves of foliations by Riemann surfaces. In this paper, we first describe the local structure of the mentioned projective structures showing, in particular, that they are locally bounded. In the case of Riemann surface orbifolds of finite type, the previous result will then allow us to provide a detailed global picture of these projective structures by exploiting their connection with the class of “bounded covering projective structures”.

1. Introduction

It is fair to say that the notion of (singular) uniformizable projective structures on Riemann surfaces was first put forward in [11] where the author conducted a very complete study of classical Halphen systems by exploiting their connection with the Lie algebra $\mathfrak{sl}_2 \mathbb{C}$. Whereas the issue was already implicit in the classical literature, in particular in the work of Schwarz about triangle functions and solutions of Gauss hypergeometric equation, it is only in [11] that the relevance of the underlying projective structure is made explicit and effectively used to clarify the rather non-trivial dynamics of vector fields with single-valued solutions (aka semicomplete vector fields, see [13]) as those in Halphen systems. Similar issues also arise for more general automorphic functions as it will be seen.

Basically, a singular projective structure $P$ on a Riemann surface orbifold is said to be uniformizable if the developing map associated with the monodromy covering is one-to-one (see Section 2 for accurate definitions). The interest of these projective structures is further emphasized in some recent papers. For example, in [13], a crucial part of the argument relies on the existence of uniformizable affine structures on the polar divisor of the vector fields in question. Also, the systematic use of uniformizable projective structures allows for important generalizations of Halphen vector fields, see [12], [4] and [5]. Additional (potential) applications of these structures to the general problem of “simultaneous uniformization” for foliations will be indicated later. In any event, to have a feeling for the interest of this notion, we may simply consider an action of $\text{SL}(2, \mathbb{C})$ and a three-dimensional orbit $\mathcal{L}$ of it. Consider now the standard one-parameter subgroups $g, h_-, \text{ and } h_+$ of of $\text{SL}(2, \mathbb{C})$ represented respectively by the “geodesic flow”, the “negative horocyclic flow”, and the “positive horocyclic flow”. The semi-direct product of $g$ and $h_-$ yields an action of the affine
group $\text{Aff}(\mathbb{C})$ on $\mathcal{L}$ whose orbits are transverse to $h_+$ in $\mathcal{L}$. In particular, $\mathcal{L}$ is equipped with a foliation $\mathcal{F}$ whose leaves are the orbits of $\text{Aff}(\mathbb{C})$. Under these conditions, and wild as it may be, the leaf space of $\mathcal{F}$ is naturally endowed with a projective structure arising from the flow of $h_+$ which, in fact, is uniformizable since the action of $\text{SL}(2, \mathbb{C})$ is globally defined. Here, it is worth mentioning that similar projective structures have also been exploited in previous works by E. Ghys involving Anosov flows, see for example [8] and [9].

For the time being, it suffices to think that what precedes serves as motivation for us to try and describe singular uniformizable projective structures on Riemann surfaces, or more generally, on Riemann surface orbifolds. This problem is subtle already in the case of compact Riemann surfaces (orbifolds) $\mathcal{L}$ since Bers simultaneous uniformization theorem ensures that the space of uniformizable projective structures contains a copy of a suitable Teichmuller space. This is in stark contrast with the special case of (singular) uniformizable affine structures as exploited in [13]: on compact Riemann surface orbifolds there are only a handful of the latter structures.

In terms of classifying (singular) uniformizable projective structures, the main results obtained in this paper - ordered from local to global - are listed in the sequel. We also note that Theorem A below can naturally be viewed as an extension of the so-called “Fundamental Lemma” in [13] from the setting of “vector fields/uniformizable affine structures” to the context of “uniformizable projective structures”.

**Theorem A.** Let $P$ be a uniformizable projective structure on a punctured neighborhood of $0 \in \mathbb{C}$ and denote by $\xi(z)dz^2$ the corresponding Schwarzian differential. Then one of the following holds:

- The local monodromy of $P$ around $0 \in \mathbb{C}$ is an elliptic element of finite order $k \in \mathbb{Z}$. Moreover, the lift of $P$ to the ramified $k$-sheet covering $\widetilde{\mathbb{D}}_\mu$ of $(\mathbb{C}, 0)$ extends holomorphically to all of $\mathbb{D}_\mu$.
- The local monodromy of $P$ around $0 \in \mathbb{C}$ is a parabolic element. Then $\xi(z)$ has a simple pole at $0 \in \mathbb{C}$. Moreover, if the monodromy is given by $z \mapsto z + c$, then the residue of $\xi$ at $0 \in \mathbb{C}$ (coefficient of the term $z^{-1}$) in $\xi$ is given by $-c/(a_{-1}\pi i)$ where $a_{-1}$ is the coefficient of $z^{-1}$ in the Laurent extension of the holomorphic function $g$ on $\mathbb{D}^*$ cf. below.

As to the first item of Theorem A, we note that it is easy to obtain a “ramified” local expression for $\xi$ out of the fact that its lift to the coordinate $y, z = y^h$, is holomorphic at the origin. A useful consequence of Theorem A is Corollary B below. To state this result, recall that every projective structure on a Riemann surface (orbifold) $\mathcal{L}$ gives rise to a holomorphic quadratic differential $\omega = \xi(z)dz^2$ on $\mathcal{L}$ called its Schwarzian differential, see Section [2]. In turn, if $\omega = \xi(z)dz^2$ is a quadratic differential on $\mathcal{L}$, we define the $L^\infty$-(hyperbolic) norm of $\xi(z)dz^2$ as follows. Letting $\tilde{\omega} = \tilde{\xi}d\tilde{z}^2$ denote the lift of $\omega$ to the unit disc $\mathbb{D}$ viewed as the universal covering of $\mathcal{L}$, we pose

$$\|\omega\|_\infty = \|\tilde{\omega}\|_\infty = \frac{1}{4} \sup_{z \in \mathbb{D}} |\tilde{\xi}(z)| \left(1 - |\tilde{z}|^2\right)^2.$$ 

A projective structure is said to be bounded if it has finite $L^\infty$-norm. Another standard norm associated with projective structures is $L^1$-norm $\|\omega\|_1$ of $\omega$. The $L^1$-norm of $\omega$ also has a simple geometric interpretation as the area of $\mathcal{L}$ with respect to the singular Euclidean metric.
\[ |\xi(z)||dz|^2 \]. The $L^1$-norm is widely used in Teichmüller theory because of its natural behavior under conformal maps though the its intrinsic meaning is less clear than the finiteness of the $L^\infty$-norm. However, uniformizable projective structures verify the following:

**Corollary B.** Assume that $\mathcal{L}$ has finite hyperbolic area (e.g. it is a compact Riemann surface with finitely many punctures). If $P$ is a uniformizable projective structure on $\mathcal{L}$ then we have

\[
\max \{ \|\omega\|_\infty, \|\omega\|_1 \} < \infty,
\]

where $\omega$ stands for the Schwarzian differential associated with $P$. In other words, $P$ is both bounded and of finite Euclidean area.

Finally, the bounded character of $P$ allows us to resort to previously known results on covering projective structures to obtain further insight on uniformizable ones, provided that we are dealing with a compact Riemann surface (orbifold). More precisely results in [20] and [16] imply the following:

**Theorem C.** Let $\mathcal{L}$ denote a Riemann surface with finite area and denote by $U(\mathcal{L})$ the set of uniformizable projective structures on $\mathcal{L}$. Then the following holds:

1. The interior of $U(\mathcal{L})$ coincides with quasi-conformal deformations of the canonical projective structure on $\mathcal{L}$. Equivalently, it coincides with the Bers embedding, centered at $\mathcal{L}$ (with classes of elliptic elements fixed), of the Teichmüller space of $\mathcal{L}$.
2. A projective structure $P \in U(\mathcal{L})$ is an isolated point of $U(\mathcal{L})$ if and only if the monodromy group of $P$ contains no accidental parabolics and either the discontinuity domain of this monodromy group is connected or the quotients of the remaining components represent only thrice punctured spheres.

To close this introduction, it is now convenient to explain the structure of our approach to the above theorems together with their connections to some previous works. Since the definition of uniformizable projective structure involves one-to-one - i.e. injective - developing maps, it is probably convenient to begin by recalling that a classical result of Kraus [17] states that a complex projective structure on a hyperbolic Riemann surface giving rise to an injective (i.e. one-to-one) developing map on the disc must have $L^\infty$-norm bounded by $3/2$ (see below for details). As a partial converse, Nehari [18] showed that any projective structure whose $L^\infty$-norm is bounded by $1/2$ gives rise to an injective developing map from the disc to $\mathbb{C}$. Naturally both theorems essentially belong to the theory of univalent functions on the unit disc $\mathbb{D}$.

Compared to [17], [18], there is, however, a crucial difference which stems from the fact that our developing maps are assumed to be injective on the so-called monodromy covering rather than on the universal covering. As will soon be clear, when dealing with injective developing maps, it is the monodromy covering, rather than the universal one, that stands out as the most natural domain of definition. Replacing the universal covering by the monodromy one, however, makes the general theory of univalent functions on the disc a less effective tool since the monodromy covering may be a very non-trivial quotient of the disc. Incidentally, the bound of $3/2$ found by Kraus is no longer valid for uniformizable projective structures. In fact, it is easy to adapt the construction in Section 6 to obtain examples of uniformizable projective structures with arbitrarily large $L^\infty$ norm.
Along similar lines, there is also a considerable gap between the two sets of projective, namely uniformizable projective structures and projective structures whose developing map is injective on the universal covering. Indeed, the latter immediately rules out the existence of elliptic elements in the monodromy group hence imposing serious constraints on the projective structures in question, cf. \cite{7}. Ruling out elliptic monodromy also excludes several cases of interest, beginning with the simplest possible case which is provided by triangular groups and Schwarz automorphic functions. Clearly a triangular group is the image of the fundamental group of the thrice punctured sphere by a homomorphism with rather large kernel. In particular, the (automorphic) developing maps arising from the naturally associated projective structure are injective on the monodromy covering but very far from this if considered on the universal covering.

In view of the above mentioned issues, our approach to Theorems A and C will differ considerably from the methods in \cite{17}, \cite{18}.

On the other hand, another much studied class of (singular) projective structures on compact Riemann surfaces are the so-called bounded covering projective structures, see for example \cite{15}, \cite{16}, and \cite{20}. Whereas it is easy to check that a uniformizable projective structure is of covering type (cf. Lemma 6.1), it is less clear whether or not the initial structure must be bounded. It is this boundedness issue that really prevents us from obtaining Theorem C straightly out of the results obtained by I. Kra and H. Shiga. We are then led to first work out the general discussion leading to Theorem A and Corollary B.

Finally in the proof of Theorem A, the local type of analysis needed to conclude, for example, the bounded nature of the projective structure in question will take us close to an interesting conjecture put forward by B. Elsner in \cite{6}, cf. Proposition 4.2. As a matter of fact, Proposition 4.2 is more general than what is strictly necessary for this paper and, in fact, its use in the Theorem A can be avoided. However, the ideas introduced to establish Proposition 4.2 seem fit to deal with more general situations. Indeed, whereas we have not tried to fully push forward the method to find out how much insight it can provide in Elsner’s question, we believe that a serious attempt in this direction would be worth a shot. Also, as another example of application of our ideas, we state Proposition 5.7 which appears naturally in some related questions and whose proof can straightforwardly be obtained from our discussion.

As to the structure of this paper, it should be said that some effort was made to make it as self-contained as possible. In particular, we have included proofs of some lemmas that may be regarded as well-known to experts in projective structures and related Teichmüller theory. The present paper, however, is likely to be of interest for experts in differential equations and integral systems by virtue of the connection between uniformizable projective structures and (generalized) Halphen systems, to mention just one example. A significant part of the latter community being less familiar with general Teichmüller theory and with the role of quadratic differentials in it, however, we feel that our choice might be justified.

In closing this introduction, let us mention that another (potential) application of these ideas appears in the context of foliated projective structures and it has some points of contact with the recent preprint \cite{2}. Whereas the paper \cite{2} discusses the existence of foliated projective structures on a (singular) holomorphic foliation on a complex surface, it would also be interesting to investigate how far leaves (not necessarily all of them) of (singular) holomorphic foliations can be equipped with singular uniformizable projective structures. This
question was pointed out by B. Deroin to the second author longtime ago. It might be viewed as a natural generalization of the study of semicomplete vector fields, i.e., of vector fields with single-valued solutions. Since the existence of uniformizable projective structures on Riemann surfaces is a much more common phenomenon than the existence of semicomplete vector fields, it is reasonable to wonder that the space of foliations admitting the singular uniformizable foliated projective structures is significantly larger than the set of foliations tangent to a semicomplete vector field (see [13] for the classification of the latter on Kähler surfaces). The upshot here is that (singular) uniformizable foliated projective structures are still capable of providing some fine control on the Riemann mapping uniformizing individual leaves. In other words, the set of foliations admitting singular uniformizable foliated projective structures may contain numerous interesting examples of foliations for which “simultaneous uniformization problems” can be discussed in depth. Clearly, the problem of “simultaneous uniformization for foliations” has basic intrinsic importance and, among their many applications, we single out the Ergodic theory of foliations as developed by Sibony, Fornaess, Dinh, and Nguyễn, see for example the nice survey [19] and references therein.

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2. Basic constructions and uniformizable projective structures

In what follows, by a Fuchsian group (resp. Kleinian group) it is meant a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ (resp. $\text{PSL}(2, \mathbb{C})$). In other words, both Fuchsian and Kleinian groups are allowed to contain elliptic elements of finite order.

The quotient of the hyperbolic disc by a Fuchsian group $\Gamma$ is not necessarily a (hyperbolic) Riemann surface but rather a Riemann surface orbifold. Naturally a point of the disc $\mathbb{D}$ that is fixed by a non-trivial elliptic element of $\Gamma$, necessarily of finite order, gives rise to a singular point in the quotient space $\mathbb{D}/\Gamma$. In particular, the singular points of a Riemann surface orbifold must form a discrete set. Moreover, if $p \in \mathbb{D}/\Gamma$ is one of the singular points, then the local structure of $\mathbb{D}/\Gamma$ around $p$ is equivalent to the quotient of a small disc in $\mathbb{C}$ by a finite group of rotations.

Unless otherwise mentioned, all Riemann surfaces and/or Riemann surface orbifolds considered in this paper are of hyperbolic nature, namely given by the quotient of $\mathbb{D}$ by a Fuchsian group. Let then $\mathcal{L}$ stand for a Riemann surface orbifold. A singular projective structure on $\mathcal{L}$ consists of the following data

(1) A discrete set $\mathcal{Y} \subseteq \mathcal{L}$ containing, in particular, all singular points of $\mathcal{L}$.
(2) A projective structure on the Riemann surface $\mathcal{L} \setminus \mathcal{Y}$. In other words, the Riemann surface $\mathcal{L} \setminus \mathcal{Y}$ is equipped with an atlas all of whose change of coordinates coincide with suitable restrictions of Möbius automorphisms of $\mathbb{C}P^1$. 

The points of $\Upsilon$ are thought of as the singular points of the projective structure in question. In the sequel, we will also use the phrase *projective surface* to refer to a Riemann surface equipped with a non-singular projective structure.

A singular projective structure on $\mathcal{L}$ gives rise to a homomorphism (monodromy representation) $\mu$ from the fundamental group $\pi_1(\mathcal{L} \setminus \Upsilon)$ to $\text{PSL}(2, \mathbb{C})$ along with a developing map $D$ from the universal covering of $\mathcal{L}$ to $\mathbb{C}P^1$. Furthermore, the pair $(D, \mu)$ satisfies the following equivariance relation
\begin{equation}
D([c].p) = \mu(c).D(p),
\end{equation}
for every $p$ in the universal covering of $\mathcal{L} \setminus \Upsilon$, $c \in \pi_1(\mathcal{L} \setminus \Upsilon)$ and where $[c]$ stands for the covering automorphism induced by $c$. However, the homomorphism $\mu$ need not be injective and, in fact, there is basically no restriction on $\mu$ as shown in [7]. Hence the kernel $\text{Ker}(\mu)$ is in general far from trivial. Nonetheless, a direct inspection in the standard construction of the developing map, makes it clear that such a map can be defined not only on the universal covering of $\mathcal{L}$ but on *every covering* $\tilde{\mathcal{L}}$ such that $\pi_1(\tilde{\mathcal{L}}) \subset \text{Ker}(\mu)$. Furthermore the resulting developing map still satisfies Equation (1). In particular, the regular covering $\mathcal{L}_\mu$ associated with $\text{Ker}(\mu)$ is the *smallest covering* of $\mathcal{L}$ on which a developing map is defined. The covering $\mathcal{L}_\mu$ will be called the *monodromy covering* of $\mathcal{L}$ and the corresponding developing map $D_\mu : \mathcal{L}_\mu \to \mathbb{C}P^1$ will be called the *monodromy-developing map*.

**Definition 2.1.** With the above notations, the singular projective structure $P$ on $\mathcal{L}$ is said to be uniformizable if the developing map $D_\mu$ is injective on $\mathcal{L}_\mu$.

It follows from Definition 2.1 that, if $P$ is a uniformizable singular projective structure on $\mathcal{L}$, then the restriction of $P$ to any open subset $U$ of $\mathcal{L}$ is still uniformizable on $U$. The lemma below provides an alternative characterization of uniformizable singular projective structures.

**Lemma 2.2.** The singular projective structure $P$ is uniformizable if and only if the projective surface $\mathcal{L} \setminus \Upsilon$ is isomorphic to the quotient of an open (invariant) subset $U$ of $\mathbb{C}P^1$ by a Kleinian group.

**Proof.** Owing to the equivariance relation (1), the image $U = D_\mu(\mathcal{L}_\mu) \subseteq \mathbb{C}P^1$ of $\mathcal{L}_\mu$ by $D_\mu$ is an open set of $\mathbb{C}P^1$ which is invariant by $\mu(\pi_1(\mathcal{L} \setminus \Upsilon))$, the subgroup of $\text{PSL}(2, \mathbb{C})$ obtained as image of the monodromy representation. Now, if $D_\mu : \mathcal{L}_\mu \to \mathbb{C}P^1$ is injective, then $U$ is diffeomorphic to $\mathcal{L}_\mu$ which, in turn, induces a diffeomorphism between the quotients of $\mathcal{L}_\mu$ and of $U$ by the corresponding actions of $\pi_1(\mathcal{L} \setminus \Upsilon)$. Hence, $\mathcal{L}$ is diffeomorphic to the quotient of $U$ by $\mu(\pi_1(\mathcal{L} \setminus \Upsilon))$. In particular, $\mu(\pi_1(\mathcal{L} \setminus \Upsilon))$ must be a discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Also, the projective structure $P$ coincides with the evident projective structure on the quotient of $U$.

The converse, follows directly from the standard construction of developing maps. \[\square\]

Since our purpose is to understand uniformizable singular projective structures, it is convenient to parameterize the space of all projective structures on a given Riemann surface orbifold. A standard parameterization of these structures can be obtained by fixing a particular projective structure and then comparing all the other projective structures with the fixed one. This comparison between projective structures is quantified by the Schwarzian
differential whose definition is summarized as follows. First recall that the Schwarzian derivative of a holomorphic function \( f \) defined on an open set of \( \mathbb{C} \) is given by

\[
S_z(f) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.
\]

It is straightforward to check that the Schwarzian derivative satisfies the following invariance properties:

\[
S_z(f \circ \gamma) = S_{\gamma(z)}(f) \gamma'(z)^2 \quad \text{and} \quad S_z(\gamma \circ f) = S_z(f)
\]

for every Möbius transformation \( \gamma \) and every holomorphic function \( f \).

Let then \( P_0 \) be a fixed singular projective structure on \( \mathcal{L} \) which is given by the atlas \( \mathcal{A}_0 = \{(U_i, \psi_i)\} \). Consider also a second singular projective structure \( P \) on \( \mathcal{L} \) whose atlas is denoted by \( \mathcal{A} = \{(V_j, \varphi_j)\} \). For every pair \((i, j)\) for which \( U_i \cap V_j \neq \emptyset \), we consider the composition map

\[
\varphi_j \circ \psi_i^{-1} : \psi_i(U_i \cap V_j) \subset \mathbb{C} \rightarrow \mathbb{C}
\]

which corresponds to a holomorphic function defined on an open set of \( \mathbb{C} \) with standard coordinate \( z \). The Schwarzian derivative of the function \( \varphi_j \circ \psi_i^{-1} \) will be denoted by \( \xi_{ij} \) so that we have \( \xi_{ij} = S_z(\varphi_j \circ \psi_i^{-1}) \). Owing to the invariance properties of the Schwarzian derivative ([3]), the resulting collection of locally defined functions \( \xi_{ij} \) patch together as a global holomorphic quadratic differential on \( \mathcal{L} \setminus \Upsilon \) (or more generally on the complement of the union of the singular sets of \( P_0 \) and \( P \)). This quadratic differential will be denoted by \( \omega = \xi(z)dz^2 \) and referred to as the Schwarzian differential.

Alternatively, the Schwarzian differential can be defined as follows. Denote by \( \tilde{\mathcal{L}} \) the universal covering of \( \mathcal{L} \setminus \Upsilon \) and let \( D \) be the resulting developing map of \( P \). The atlas \( \mathcal{A}_0 = \{(U_i, \psi_i)\} \) for \( P_0 \) naturally induces an atlas \( \tilde{\mathcal{A}}_0 = \{\tilde{U}_i, \tilde{\psi}_i\} \) for \( \tilde{\mathcal{L}} \). Thus, for every open set \( \tilde{U}_i \) of \( \tilde{\mathcal{A}}_0 \) we may consider the map \( D \circ \tilde{\psi}_i^{-1} : \tilde{\psi}_i(U_i) \subset \mathbb{C} \rightarrow \mathbb{C} \). Note that this map is holomorphic and defined on an open set of \( \mathbb{C} \). Also, it does not depend on the choice of the lift of \( U_i \). The Schwarzian derivatives of these locally defined maps can thus be taken and the resulting collection of locally defined functions again patch together as a quadratic differential on \( \mathcal{L} \setminus \Upsilon \) coinciding with the Schwarzian differential \( \omega \) as previously defined.

An immediate consequence of the above construction is that the Schwarzian differential \( \omega \) vanishes identically if and only if the projective structures \( P_0 \) and \( P \) coincide. A converse also holds, if \( \omega' \) is a quadratic differential on \( \mathcal{L} \setminus \Upsilon \), then by considering local solutions for the Schwarzian differential equations in the coordinates of \( P_0 \) (see for example Section [4]), a new singular projective structure \( P' \) on \( \mathcal{L} \) is defined. In addition, \( \omega' \) coincides with the Schwarzian differential of \( P' \). Summarizing, the choice of the initial \( P_0 \) allows us to identify the set of (singular) projective structures on \( \mathcal{L} \) with the space of quadratic differentials on \( \mathcal{L} \) (with suitable singular sets that the reader can easily work out).

To begin the approach to Theorem A, let us consider the behavior of a uniformizable singular projective structure \( P \) on \( \mathcal{L} \) around a singular point \( p \in \Upsilon \subset \mathcal{L} \). For the time being we make no distinction between the cases where the point \( p \in \mathcal{L} \) is a regular (smooth) point or an orbifold type singular point. Consider a small neighborhood \( U \subset \mathcal{L} \) of \( p \). The restriction \( P|_U \) of \( P \) to \( U \) is therefore uniformizable and we can assume without loss of generality that the monodromy group associated to \( P|_U \) has a unique generator \( \gamma \) arising from a small loop
around $p$. Since $\mu(\pi_1(U \setminus \{p\}))$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$ (cf. Lemma 2.2), $\gamma$ must be of one of the following types

1. Hyperbolic.
2. Elliptic with finite order $k$.
3. Parabolic.

However, the proof of Lemma 2.2 shows a bit more. In fact, it shows that for $p$ to be an orbifold point, it has to be identified with a fixed point of an elliptic element $h$ of finite order. Therefore, $U \setminus \{p\}$ must be given as the quotient of $\mathbb{D}$ by $h$. Furthermore, since our projective structure is uniformizable, it follows from the proof of Lemma 2.2 that $U \setminus \{p\}$ is also the quotient of an open set of $\mathbb{CP}^1$, or equivalently of $\mathbb{D}$, by the monodromy group $\mu(\pi_1(U \setminus \{p\}))$.

In particular, $\gamma$ has a fixed point at $p$. More precisely, $\gamma$ is the element defining the orbifold so that it is also elliptic of order $k$. Conversely, assume that the generator $\gamma$ of the monodromy group is not of elliptic type. It then follows that the action of $\mu(\pi_1(U \setminus \{p\})) \mathbb{D}$ is properly discontinuous so that $U$ must be smooth at $p$. We have then proved the following.

**Lemma 2.3.** The point $p$ is an orbifold point if and only if the local monodromy map around $p$ is elliptic of finite order.

Let us close this section by showing that the monodromy map around a singular point cannot be of hyperbolic type. This result is probably known to experts and the argument given below is a minor adaptation from [10].

**Lemma 2.4.** Let $\Gamma = \langle \gamma \rangle \subset \text{PSL}(2, \mathbb{C})$ be the monodromy group of a uniformizable projective structure $P$ on the punctured disc $\mathbb{D}^*$. Then $\gamma$ is either an elliptic or a parabolic transformation.

**Proof.** Assume aiming at a contradiction that $\gamma$ is hyperbolic. In particular the monodromy covering of $\mathbb{D}^*$ coincides with its universal covering $\mathbb{D}$ since $\gamma$ has infinite order. Since $P$ is uniformizable, there follows that the developing map $D : \mathbb{D} \rightarrow \mathbb{CP}^1$ is injective. Thus, $D(\mathbb{D}) \subset \mathbb{CP}^1$ is conformally equivalent to the unit disc. Moreover, since $\Gamma$ acts freely on $\mathbb{D}$ (as deck-transformations), the equivariance of the developing map implies that the action of $\Gamma$ on $\mathbb{CP}^1$ (given by the dynamics of $\gamma$) leaves invariant $D(\mathbb{D})$ and, in addition, is free when restricted to the $\Gamma$-invariant set $D(\mathbb{D})$. Hence, the two fixed points of the hyperbolic element $\gamma$ must lie in the boundary of $D(\mathbb{D})$.

Let us denote by $C$ the quotient of $D(\mathbb{D})$ by $\Gamma$, $D(\mathbb{D})/\Gamma$, which is a hyperbolic cylinder. Since $D$ is injective and equivariant with respect to the two actions of $\Gamma$, it induces a diffeomorphism $\Upsilon : \mathbb{D}^* \rightarrow C$. In particular, if $c$ is a loop representing the generator $[c]$ of the fundamental group of $\mathbb{D}^*$, then $\Upsilon(c)$ is a loop in $C$ representing the generator $[\Upsilon(c)]$ of the fundamental group of $C$.

Let $\mathbb{D}^*$ be equipped with the Poincaré metric $m_\mathbb{D}$ induced from its universal covering $\mathbb{D}$. Clearly, we can choose loops $c \in \mathbb{D}^*$ winding once around $0 \in \mathbb{D}$ and having arbitrarily small length with respect to $m_\mathbb{D}$. Similarly, let $C$ be endowed with its Poincaré metric $m_C$ obtained by taking the quotient of the disc by the action of the hyperbolic map $\gamma$. The length of $\Upsilon(c)$ with respect to $m_C$ must be bounded from below by a strictly positive number $\varepsilon > 0$ since $C$ is a hyperbolic cylinder. However, since $\Upsilon$ is a holomorphic map, the pull-back $\Upsilon^* m_C$ of the Poincaré metric $m_C$ on $C$ by $\Upsilon$ is a (hyperbolic) metric on $\mathbb{D}^*$ that is bounded by the previously introduced Poincaré metric $m_\mathbb{D}$ on $\mathbb{D}^*$ (cf. [1]). The desired contradiction follows.
at once from observing that the length of \( c \) with respect to \( m_D \) can be made arbitrarily small whereas its length with respect to \( \Upsilon^* m_C \) is bounded from below by \( \varepsilon > 0 \). □

3. Uniformizable projective structures with trivial monodromy

According to Lemma 2.4, the monodromy map \( \gamma \) of a uniformizable projective structure \( P \) around a singular point \( p \in \mathcal{L} \) is either an elliptic or a parabolic transformation. These two cases will be discussed separately. This section is devoted to the elliptic case.

For \( P \) and \( p \) as above, we assume in the sequel that the monodromy map \( \gamma \) is elliptic (necessarily having finite order \( k \)). Lemma 2.3 then ensures that \( p \) is an orbifold point and that a neighborhood \( U \subset \mathcal{L} \) of \( p \), is isomorphic to the quotient of \((\mathbb{C}, 0)\) by a rotation of angle \( 2\pi/k \). Let us first point out that this general situation can straightforwardly be reduced to the case of trivial monodromy and smooth point. Indeed, note that the ramified \( k \)-sheet covering of a neighborhood of \( p \) in \( \mathcal{L} \) is identified with a neighborhood \( U \subset \mathbb{C} \). In particular, the singular projective structure \( P \) can be lifted to a projective structure \( P^{(k)} \) on \( U \) which is regular away from \( 0_\mathbb{C} \). Also, by construction, the monodromy of \( P^{(k)} \) is trivial so that the monodromy-developing map of \( P^{(k)} \) is defined on \( U \setminus \{0\} \) and hence coincides with the monodromy-developing map of \( P \) on a punctured neighborhood of \( p \in \mathcal{L} \).

In view of the preceding, let us focus on uniformizable (singular) projective structures on the disc \( D \) which are regular on \( D^* \) and whose monodromy is trivial. We choose as initial projective structure \( P_0 \) the (regular, uniformizable) projective structure induced by the standard coordinate \( z \in \mathbb{C} \) (making use of the evident inclusion \( D^* \subset \mathbb{C} \)).

Given a regular uniformizable projective structure \( P \) on \( D^* \) with trivial monodromy, the coefficient \( \xi \) of the Schwarzian differential coincides with the Schwarzian derivative of the monodromy developing map \( D_\mu \) of \( P \) written in the coordinate \( z \). Clearly \( D_\mu \) is well defined on \( D^* \) and satisfies

\[
\xi(z) = S_z D_\mu = \left( \frac{D_\mu''(z)}{D_\mu'(z)} \right)' - \frac{1}{2} \left( \frac{D_\mu''(z)}{D_\mu'(z)} \right)^2.
\]

Moreover, we have the following:

**Lemma 3.1.** Let \( P \) be a uniformizable regular projective structure on \( D^* \) and assume that its monodromy is trivial. Then \( 0 \in \mathbb{D} \) is not an essential singularity for \( D_\mu \). More precisely, one of the following holds:

(1) \( D_\mu \) is meromorphic on \( \mathbb{D} \) with a simple pole at \( 0 \in \mathbb{D} \).

(2) \( D_\mu \) is holomorphic on \( \mathbb{D} \) and \( D_\mu'(0) \neq 0 \).

**Proof.** Since the monodromy is trivial, \( D_\mu \) is by construction well defined and holomorphic on \( D^* \). Furthermore, \( D_\mu : \mathbb{D}^* \to \mathbb{C} \) is injective since \( P \) is uniformizable. In particular, it follows from Picard Theorem that \( 0 \in \mathbb{D} \) is not an essential singularity of \( D_\mu \). Hence, \( D_\mu \) is (at worst) meromorphic on \( \mathbb{D} \) and can be written as

\[
D_\mu(z) = az^l + \text{h.o.t.}
\]

for some \( l \in \mathbb{Z} \) and \( a \in \mathbb{C}^* \). We can assume that \( l \neq 0 \), otherwise the constant term “a” can be eliminated by adding a translation to \( D_\mu \) which would yield an equivalent developing map with strictly positive order at \( 0 \in \mathbb{C} \). Next, we have:
Claim: The map \( z \mapsto az^l, l \neq 0 \), is injective.

Proof of the claim. Consider the map \( \Lambda(z) = \lambda z \) for some \( \lambda \in \mathbb{C}^* \), which is clearly injective. Since, \( D_{\mu} \) is injective on \( \mathbb{D}^* \), the conjugate \( \Lambda^{-1}D_{\mu}\Lambda \) is also injective on \( \mathbb{D}^* \) so long \( |\lambda| \leq 1 \). However, we have

\[
\Lambda^{-1}D_{\mu}\Lambda(z) = \lambda^{-1}D_{\mu}(\lambda z) = \lambda^{-1+l} \left[ az^l (1 + \lambda O(z)) \right].
\]

Hence for \( \lambda \in \mathbb{C}^* \) with \( |\lambda| \leq 1 \), all the maps \( z \mapsto az^l(1 + \lambda O(z)) \) are injective on \( \mathbb{D}^* \). Hence so is the map \( z \mapsto az^l \) as a non-constant uniform limit of injective maps. The claim is proved. \( \square \)

Since \( l \neq 0 \), there follows from the above claim that \( l \) is either 1 or \(-1\). When \( l = -1 \), \( D_{\mu} \) is meromorphic with a simple pole at \( 0 \in \mathbb{D} \). In turn, if \( l = 1 \), then \( D_{\mu} \) is holomorphic and \( D_{\mu}(0) \neq 0 \). The lemma is proved. \( \square \)

Proposition 3.2 is a useful consequence of Lemma 3.1.

Proposition 3.2. Let \( P \) be a projective structure as in Lemma 3.1. Then the coefficient \( \xi \) of its Schwarzian differential possesses a holomorphic extension to the entire disc \( \mathbb{D} \).

Proof. Owing to Lemma 3.1 and up to adding a suitable translation, the monodromy-developing map \( D_{\mu} \) of \( P \) can be written in Laurent series as

\[
D_{\mu}(z) = a_{-1}z^{-1} + a_1 z + a_2 z^2 + \cdots.
\]

for some constants \( a_i \in \mathbb{C} \) where at least one between \( a_{-1} \) and \( a_1 \) is non-zero. In turn, the coefficient \( \xi \) of the Schwarzian differential is related to \( D_{\mu} \) by means of Formula (4). In particular, when \( D_{\mu} \) is holomorphic at \( 0 \in \mathbb{C} \), it follows at once from Formula (1) that \( \xi \) is holomorphic on all of \( \mathbb{D} \) as well. It is therefore sufficient to discuss the case where \( a_{-1} \neq 0 \). Then a direct computation provides

\[
\frac{D''_{\mu}(z)}{D'_{\mu}(z)} = \frac{2a_{-1}z^{-3}(1 + z^3A_2(z))}{-a_{-1}z^{-1}(1 + z^2A_1(z))} = \frac{2}{z}(1 + z^2A(z))
\]

for some holomorphic functions \( A, A_1 \) and \( A_2 \). It then follows that

\[
\left( \frac{D''_{\mu}(z)}{D'_{\mu}(z)} \right)' = \frac{2}{z^2} + G(z) \quad \text{and} \quad \left( \frac{D''_{\mu}(z)}{D'_{\mu}(z)} \right)^2 = \frac{4}{z^2} + H(z)
\]

for some holomorphic functions \( G \) and \( H \). Now, Formula (1) ensures that \( \xi \) is holomorphic at the origin. \( \square \)

To close this section, let us return to the initial problem of describing a uniformizable singular projective structure \( P \) on a neighborhood of the orbifold point \( p \in \mathcal{L} \). Recall that the local structure of \( \mathcal{L} \) around \( p \) is isomorphic to the quotient of the disc \( \mathbb{D} \) by a rotation of angle \( 2\pi/k \), \( k \geq 2 \). In particular, \( \mathcal{L} \) can be equipped with the local “orbifold-type” coordinate \( y \) defined by \( y = \sqrt[k]{z} = \eta(z) \) where \( z \) is the standard coordinate on \( \mathbb{D} \subset \mathbb{C} \).

As previously seen, \( P \) lifts to a uniformizable (regular) projective structure \( P^{(k)} \) on \( \mathbb{D}^* \) with trivial monodromy and whose Schwarzian differential will be denoted by \( \xi^{(k)}(y)dy^2 \) (Proposition 3.2). Denoting by \( \xi_{\text{orb}}(y)dy^2 \) the Schwarzian differential of \( P \) in the singular (orbifold-type) coordinate \( y \), it follows that

\[
\xi_{\text{orb}}(y) = \xi^{(k)}(\eta^{-1}(y)) \left[ (\eta^{-1}(y))' \right]^2 = k^2y^{2k-2}\xi^{(k)}(y^k).
\]
To summarize the discussion in this section, we state:

**Proposition 3.3.** Consider a uniformizable singular projective structure $P$ on the orbifold given by the quotient of $\mathbb{D}$ by a rotation of angle $2\pi/k$, $k \geq 2$. Then all of the following hold:

- The monodromy of $P$ is generated by an elliptic element of order $k$.
- In the orbifold coordinate $y = \sqrt[3]{z}$, the Schwarzian differential of $P$ is given by Formula (4), where $\xi^{(k)}$ is holomorphic on all of $\mathbb{D}$.
- Both the $L^\infty$ and the $L^1$ norms of $P$ are locally bounded around the singular point.

\[ \square \]

4. The parabolic case

It remains to discuss the nature of a uniformizable projective structure $P$ around a singular point $p \in \mathcal{L}$ in the case where the local monodromy is a parabolic map. Since Lemma 2.3 ensures that $\mathcal{L}$ is smooth at $p$, we can once and for all assume that $P$ induces a uniformizable regular projective structure - still denoted by $P$ - on the punctured disc $\mathbb{D}^* \subset \mathbb{C}$ equipped with the standard coordinate $z \in \mathbb{C}$. The monodromy of $P$ on $\mathbb{D}^*$ is a parabolic map denoted by $\gamma$.

Taking advantage of the coordinate $z$ defined on all of $\mathbb{D}$, the Schwarzian differential can be written as $\xi(z)dz^2$. Also, the monodromy covering of $\mathbb{D}^*$ coincides with its universal covering and the corresponding developing map $D_\mu$ locally satisfies the differential equation

\[ \xi(z) = S_zD_\mu. \]

It is well known (cf. [14]) that two solutions of Equation (6) differ by post-composition with a Möbius transformation. Furthermore, any solution of this equation is given as the quotient of two suitable linearly independent solutions of the linear ordinary differential equation

\[ u''(z) + \frac{1}{2}\xi(z)u(z) = 0. \]

Being a linear equation, the local solutions of (7) can be continued along paths. In particular, the same holds for the solutions of Equation (6). Thus, every solution of (6) is globally defined on $\mathbb{D} \setminus \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the set of non-negative reals. Also, given $x_0 \in \mathbb{D} \cap \mathbb{R}_+$ and a local solution $\phi$ of (6), consider the continuations of $\phi$ along the path $t \mapsto x_0e^{it}$, $t \in (0, 2\pi)$. This gives us two local solutions around $x_0$, denoted respectively by $\phi_0$ and $\phi_1$ and, by construction, we have $\phi_1 = \gamma \circ \phi_0$. However, since $\gamma$ is parabolic, we can assume without loss of generality that $\gamma(z) = z + c$ for some $c \in \mathbb{C}^*$. Thus, we conclude that $\lim_{t \to 0^+} \phi(x_0e^{it}) = \lim_{t \to 2\pi^-} \phi(x_0e^{it}) = c$. Similar conclusions hold if $\mathbb{R}_+$ is replaced by any semi-line issued from $0 \in \mathbb{C}$. More precisely, given $\theta \in [0, 2\pi]$, let $l^\theta$ be defined by

\[ l^\theta = \{ z \in \mathbb{C} : z = 0 \text{ or } z = e^{a+i\theta} \text{ with } a \in \mathbb{R} \}. \]

Clearly for every $\theta$ as above, the solutions of (6) are globally defined on $\mathbb{D} \setminus l^\theta$. Furthermore, if $\phi$ is a local solution continued along a path $t \mapsto x_0e^{it}$, $t \in (\theta, \theta + 2\pi)$, then we still have

\[ \lim_{t \to \theta^+} \phi(x_0e^{it}) = \lim_{t \to (\theta+2\pi)^-} \phi(x_0e^{it}) = c. \]

Equation (7) is well known and, in general, it is not easy (possible) to find explicit solutions. Yet, the standard method of “variation of parameters” yields the following:
Lemma 4.1. Assume that \( h \colon U \subset \mathbb{C} \rightarrow \mathbb{C} \) is a (non-zero) solution of (7), where \( U \subset \mathbb{C} \) is a simply connected domain. Then \( \phi \colon U \rightarrow \mathbb{C} \) defined by

\[
\phi(z) = \int_{z_0}^{z} \frac{1}{h^2(s)} \, ds + \text{const}
\]

is a solution of Equation (6), where \( z_0 \in U \) is a fixed base point and \( \text{const} \in \mathbb{C} \).

**Proof.** The standard method of “variation of parameters” indicates the existence of a solution \( u(z) \) for Equation (7) having the form \( u(z) = \phi(z)h(z) \) for some non-constant function \( \phi \). Up to finding such a solution \( u(z) \), it follows from the preceding that \( \phi \) is a solution of Equation (6). On the other hand, \( u = \phi h \) is solution of (7) if and only if \( \phi \) verifies \( h \phi'' + 2h'\phi' = 0 \) which is a linear first order equation on \( \phi' \). From this, it promptly follows that

\[
\phi'(z) = \frac{1}{h^2(z)}.
\]

Equation (10) follows at once. \( \square \)

Since any solution \( \phi \) of (6) is obtained as the quotient of two suitable solutions of (7), we can assume that \( \phi \) as in (10) also satisfies (9). In other words, the function

\[
g(z) = \int_{z_0}^{z} \frac{1}{h^2(s)} \, ds + \text{const} - \frac{c}{2\pi i} \int_{z_0}^{z} \frac{1}{s} \, ds - \frac{c}{2\pi i} \ln(z/z_0)
\]

is holomorphic on the punctured disc \( D^* \). In fact, the multivalued character of \( \phi \) represented by Equation (9) cancels the multivaluedness of the logarithm to ensure that the analytic extension of \( g \) over the loop \( t \mapsto x_0 e^{it}, t \in [0, 2\pi] \) is well defined.

Now recalling that \( P \) is uniformizable on \( D^* \), we conclude that the extension \( D_\mu \) of \( \phi \) given by (10) on the universal covering \( \hat{D} \) of \( D^* \) gives an injective map from \( \hat{D} \) to \( \mathbb{C} \mathbb{P}^1 \). Here the reader is reminded that the monodromy covering of \( D^* \) coincides with its universal covering since the monodromy representation of \( P \) arises from a parabolic Möbius transformation.

The remainder of this section is devoted to deriving specific information on the behavior of the Schwarzian differential \( \omega = \xi dz^2 \) from the fact that \( D_\mu : \hat{D} \rightarrow \mathbb{C} \mathbb{P}^1 \) is an injective map. As a first evident consequence of the fact that \( D_\mu \) is injective, we see that the restriction of \( \phi \) to any domain of the form \( \hat{D} \setminus I^\mu \) is injective as well. Nonetheless, we have the following:

**Proposition 4.2.** Let \( g \) and \( \phi \) be as above. If the restrictions of \( \phi \) to domains of the form \( \hat{D} \setminus I^\mu \) are injective then \( g \) cannot have an essential singular point at \( 0 \in \mathbb{C} \).

The proof of Proposition 4.2 will be deferred to Section 5. The remainder of the present section is devoted to providing a detailed description of the Schwarzian differential \( \omega = \xi dz^2 \) assuming throughout the sequel that \( P \) is uniformizable on \( D^* \). First we have:

**Lemma 4.3.** Let \( P \) be a uniformizable projective structure on \( D^* \) with parabolic monodromy. Then, its Schwarzian differential \( \omega = \xi dz^2 \) cannot have an essential singularity at \( 0 \in \mathbb{C} \).

**Proof.** Keeping the preceding notation, consider the function \( g \) defined by (11). In view of Proposition 4.2 \( g \) is meromorphic (possibly holomorphic) on \( D \). In particular, its derivative \( g' \) is meromorphic on \( D \) what, in turn, informs us that \( h^2 \) is meromorphic as well. Though
$h$ does not need to be meromorphic as the square root of a meromorphic function, it does have a well-defined order at $0 \in \mathbb{C}$. More precisely, there is $\kappa \in \mathbb{R}$ such that

$$\lim_{z \to 0} |h(z)| = O(|z|^\kappa),$$

where $O(|z|^\kappa)$ means that the left side is bounded - from below and by above - by suitable constant multiples of $|z|^\kappa$. Also, $\kappa$ is such that $\kappa - [\kappa] \in \{0,1/2\}$, where the brackets $[\cdots]$ stand for the integral part, i.e., $\kappa$ is either an integer or half an integer. The same applies to its second derivative $h''$ which satisfies

$$\lim_{z \to 0} |h''(z)| = O(|z|^\kappa')$$

for a suitable $\kappa' \in \mathbb{R}$. Equation (7) then implies that

$$\lim_{z \to 0} |\xi(z)| = O(|z|^\kappa' - \kappa).$$

In view of Picard theorem (actually the more elementary Casorati-Weierstrass theorem), the last limit implies that $\xi$ cannot have an essential singular point at $0 \in \mathbb{C}$. The lemma is proved. \hfill \Box

Next, let us investigate what can be said about the order of $\phi$ at $0 \in \mathbb{C}$. For this, we let $\phi(z) = g(z) + c \ln(z/z_0)/2\pi i$. Since $g$ is meromorphic, we can set

$$g(z) = \sum_{k=k_0}^{\infty} a_k z^k$$

for some $k_0 \in \mathbb{Z}$ and $a_{k_0} \neq 0$. Now, similar to Lemma 3.1, the following holds:

**Lemma 4.4.** We have $k_0 \geq -1$ provided that $P$ is uniformizable.

**Proof.** Assume for a contradiction that $k_0 \leq -2$ and choose a determination of $\ln$ on the set $\mathbb{D} \setminus \{3\pi/2\}$. We will show that $\phi$ cannot be injective on $\mathbb{D} \setminus \{3\pi/2\}$ hence contradicting the condition that $P$ is uniformizable.

To begin, we have

$$\phi(z) = z^{k_0} \left( (a_{k_0} + a_{k_0+1} z + \text{h.o.t.}) + \frac{c}{2\pi i} z^{-k_0} \ln(z/z_0) \right)$$

$$= a_{k_0} z^{k_0} + z^{k_0+1} \left( (a_{k_0+1} + \text{h.o.t.}) + \frac{c}{2\pi i} z^{-k_0-1} \ln(z/z_0) \right)$$

$$= a_{k_0} z^{k_0} + z^{k_0+1} \Psi_1(z),$$

where $\Psi_1$ is holomorphic on $\mathbb{D} \setminus \{3\pi/2\}$. Since $-k_0 - 1 \geq 1$ and $\lim_{z \to 0} |z \ln z| = 0$, there follows that the function $z \mapsto |\Psi_1(z)|$ is uniformly bounded by a constant $C_1$ on $\mathbb{D} \setminus \{3\pi/2\}$. On the other hand, $\phi'(z) = g'(z) + c/2\pi iz$ so that we similarly obtain

$$\phi'(z) = k_0 a_{k_0} z^{k_0-1} + z^{k_0} \Psi_2(z)$$

where $\Psi_2$ is, in fact, holomorphic on all of $\mathbb{D}$. In particular $\Psi_2$ is bounded by some constant $C_2$. Also, both $|\phi'(\epsilon)|$ and $|\phi'(\lambda \epsilon)|$ are $O(|\epsilon^{k_0-1}|)$, where $\lambda$ is the $|k_0|$-root of the unity with smallest real part while $\epsilon > 0$ is sufficiently small. Let $B_1$ and $B_2$ denote the balls of (same) radius $\epsilon/2$ respectively centered at $\epsilon$ and at $\lambda \epsilon$. Clearly these balls are disjoint and contained in $\mathbb{D} \setminus \{3\pi/2\}$. Since $\phi$ is injective on $\mathbb{D} \setminus \{3\pi/2\}$, the contradiction proving Lemma 4.4 arises at once from the following claim:
Claim. The intersection $\phi(B_1) \cap \phi(B_2)$ is not empty.

Proof of the Claim. First we note that the distance between $\phi(\epsilon)$ and $\phi(\lambda \epsilon)$ satisfies

$$|\phi(\epsilon) - \phi(\lambda \epsilon)| \leq |\epsilon^{k_0+1}\Psi_1(\epsilon) - (\lambda \epsilon)^{k_0+1}\Psi_1(\lambda \epsilon)| \leq 2C_1|\epsilon^{k_0+1}|.$$

On the other hand, Koebe’s quarter theorem [3] implies that $\phi(B_1)$ contains a ball centered at $\phi(\epsilon)$ of radius

$$\frac{1}{4} \frac{|\epsilon|}{2} |\phi'(\epsilon)| = \frac{1}{8} |\epsilon| O(|\epsilon^{k_0-1}|) = O(|\epsilon^{k_0}|).$$

Analogously we check that $\phi(B_2)$ also contains a ball centered at $\phi(\lambda \epsilon)$ of radius $O(|\epsilon^{k_0}|)$. Since the distance from $\phi(\epsilon)$ to $\phi(\lambda \epsilon)$ is bounded by a constant times $|\epsilon^{k_0+1}|$ ($k_0$ being strictly negative), there follows that $\phi(B_1) \cap \phi(B_2) \neq \emptyset$ provided that $\epsilon$ is sufficiently small. This establishes the claim and completes the proof of Lemma 4.4. \qed

Next, recalling that $\phi$ locally coincides with the developing map $D_{\mu}$, there follows that the equation $\xi = S_\phi \phi$ holds on $D \setminus i\mathbb{R}/2$. Furthermore $\phi$ is given by $c \ln(z/z_0)/2\pi i + g(z) = c \ln(z/z_0)/2\pi i + \sum_{k=1}^\infty a_k z^k$. Thus, we obtain:

Lemma 4.5. The order of $\xi$ at $0 \in \mathbb{C}$ is greater than or equal to $-1$.

Proof. The proof is a direct computation similar to the proof of Lemma 3.2. Owing to Lemma 4.4, $g(z) = \sum_{k=1}^\infty a_k z^k$. Since $\phi = c \ln(z/z_0)/2\pi i + \sum_{k=1}^\infty a_k z^k$, it follows that

$$\phi'(z) = \frac{c}{2\pi iz} + \sum_{k=1}^\infty k a_k z^{k-1} = -\frac{a_{-1}}{z^2} + \frac{c}{2\pi iz} + a_1 + \sum_{k=2}^\infty k a_k z^{k-1}$$

and

$$\phi''(z) = \frac{2a_{-1}}{z^3} - \frac{c}{2\pi iz^2} + \sum_{k=2}^\infty k(k-1) a_k z^{k-2}.$$

It suffices to treat the case $a_{-1} \neq 0$ since the computations become actually simpler for $a_{-1} = 0$. For $a_{-1} \neq 0$, we have

$$\phi'(z) = -\frac{a_{-1}}{z^2} \left(1 - \frac{cz}{2a_{-1} \pi i} + z^2 h_1(z) \right)$$

and

$$\phi''(z) = \frac{2a_{-1}}{z^3} \left(1 - \frac{cz}{4a_{-1} \pi i} + z^3 h_2(z) \right)$$

for suitable holomorphic functions $h_1$ and $h_2$. Also, there exists $h_3$ holomorphic such that

$$(1 - \frac{cz}{2a_{-1} \pi i} + z^2 h_1(z))^{-1} = (1 + \frac{cz}{2a_{-1} \pi i} + z^2 h_3(z)).$$

Hence,

$$\frac{\phi''(z)}{\phi'(z)} = -\frac{2}{z} \left(1 - \frac{cz}{4a_{-1} \pi i} + z^2 h_2(z) \right) \left(1 + \frac{cz}{2a_{-1} \pi i} + z^2 h_3(z) \right).$$

Finally, from Equation (12), we conclude that the quotient $\phi''(z)/\phi'(z)$ equals $-2/z + h_4(z)$, where $h_4$ is holomorphic. Hence

$$\left(\frac{\phi''(z)}{\phi'(z)} \right)' = \frac{2}{z^2} + h'_4(z).$$
where $h'_4$ is holomorphic. In turn,

$$
\left( \frac{\phi''(z)}{\phi'(z)} \right)^2 = 4 \frac{z^2}{z^2} \left( 1 - \frac{cz}{2a_1 \pi i} + z^2 h_5(z) \right) \left( 1 + \frac{cz}{a_1 \pi i} + z^2 h_6(z) \right)
$$

$$
= \frac{4}{z^2} + \frac{2c}{a_1 \pi iz} + h_7(z),
$$

where $h_5, h_6, h_7$ are all holomorphic functions. Thus, in view of Formula (4), we conclude that

$$
\xi(z) = S_z \phi = -\frac{c}{a_1 \pi iz} + h_8(z)
$$

for a suitable holomorphic function $h_8$. This proves the lemma. □

We are now ready to derive Theorem A and Corollary B.

**Proof of Theorem A and of Corollary B.** Consider a uniformizable singular projective structure $P$ on a Riemann surface orbifold $\mathcal{L}$. Let $p \in \mathcal{L}$ be a singular point of $P$ and denote by $\gamma \in \text{PSL}(2, \mathbb{R})$ the local monodromy of $P$ arising from a small loop around $p$. According to Lemma 2.4, $\gamma$ cannot be hyperbolic. On the other hand, if $\gamma$ is elliptic, it must be of finite order. In this case, and only in this case, $p$ is a singular point of orbifold type. The local structure of $P$ around $p$ is described by Proposition 3.3. Finally, if $\gamma$ is parabolic, the corresponding statement in Theorem A follows from Lemma 4.5 along with its proof.

As to Corollary B. The local finiteness of the norms $L^1$ and $L^\infty$ follows again from Proposition 3.3 in the case of elliptic monodromy. In turn, when the monodromy is parabolic, the statement is a consequence of Lemma 4.5. This completes the proof of Corollary B. □

5. PROOF OF PROPOSITION 4.2

The method to be employed in this section to prove Proposition 4.2 seems to be rather flexible and can be used to handle other similar situations. As an example, we state Proposition 5.7 (see Remark 5.6) that also appears in problems about vector fields having univalued solutions, see for example [13] and its references. Also, let us point out that Proposition 4.2 is a special case of a nice conjecture put forward in [6]: it might be interesting to check how far our method can be pushed to provide insight in Elsner’s conjecture.

We fix a domain $\mathbb{D} \setminus l^\theta \subset \mathbb{C}$ where a branch of logarithm is defined. Also we have a holomorphic function $g$ defined on $\mathbb{D}^*$ and a multivalued function $\phi$ on $\mathbb{D}^*$ which is defined by

$$
\phi = g + \frac{c}{2\pi i} \ln(z/z_0)
$$

for some $c \in \mathbb{C}$ and $z_0 \in \mathbb{D}^*$. Upon restriction to $\mathbb{D} \setminus l^\theta$, all these functions become well defined and $\phi$ is assumed to be injective for every fixed value of $\theta$. Clearly, as $\theta$ varies, both $\phi$ and the branch of the logarithm can be continued in such way that Equation (13) will still hold for every value of $\theta$. In other words, for every value of $\theta$ the map $\phi$ on $\mathbb{D} \setminus l^\theta$ defined by a suitable choice of a branch of logarithm satisfies (9) and is injective on $\mathbb{D} \setminus l^\theta$.

Proposition 4.2 will follow from the following proposition:
Proposition 5.1. Let $\phi$ and $g$ be as above and assume that $\phi$ is injective on domains of the form $\mathbb{D} \setminus t^0$ (cf. Proposition 4.2). Then there exists the limit
\[
\lim_{z \to 0} \phi(z)
\]
with $z \in \mathbb{D} \setminus t^0$.

Proof of Proposition 4.2. From Equation (13), we have $2\pi i \phi / c = 2\pi i g + \ln(z/z_0)$. By taking the exponential on both sides, we conclude that
\[
\exp \left( \frac{2\pi i \phi(z)}{c} \right) = \frac{z}{z_0} \exp \left( \frac{2\pi i g(z)}{c} \right).
\]
In particular, $\exp(2\pi i \phi(z)/c)$ is holomorphic on a punctured neighborhood of $0 \in \mathbb{C}$. However, Proposition 5.1 ensures that $ze^{2\pi i g(z)/c}$ has a limit as $z \to 0$ so that Casorati-Weierstrass theorem shows that $g$ cannot have an essential singularity at $0 \in \mathbb{C}$. \hfill \Box

The remainder of this section is devoted to the proof of Proposition 5.1. Given real numbers $r, R$, such that $0 < r < R < 1$, the annulus of radii $r$ and $R$ around $0 \in \mathbb{C}$ will be denoted by $A_{(r,R)}$.

Next, we have that $\phi$ is injective on $\mathbb{D} \setminus t^0$ and this still holds as $\theta$ varies and $\phi$ is suitably continued, as previously indicated. Let now $\theta$ be fixed and choose $t > 0$ arbitrarily small. Set
\[
A_{i(r,R)} = \left\{ \rho e^{i\alpha} : \rho, \alpha \in \mathbb{R}, r < \rho < R, \theta + t < \alpha < \theta + 2\pi - t \right\}.
\]
In other words, $A_{i(r,R)}$ consists of those points in $A_{(r,R)}$ whose argument does not lie in $[\theta - t, \theta + t]$ (modulo $2\pi$). In particular, for $R$ small enough, we have $A_{i(r,R)} \subset \mathbb{D} \setminus t^0$.

Now let $\Lambda : \mathbb{C} \to \mathbb{C}$ denote the map defined by $\Lambda(z) = rz/R$. The positive iterates of $\Lambda$ form the family $\{\Lambda^{k}\}_{k \in \mathbb{N}}$ and it is clear that $\Lambda(A_{i(r,R)}) = A_{i(r^2/R,r,R)}$. Although $A_{i(r,R)}$ and $\Lambda(A_{i(r,R)})$ are disjoint sets, their closures have non-empty intersection. More precisely $\overline{A_{i(r,R)}} \cap \overline{\Lambda(A_{i(r,R)})}$ consists of those complex numbers having modulus $r$ and argument in the interval $[\theta + t, \theta + 2\pi - t]$. Furthermore
\[
\overline{A_{i(r,R)}} \cup \overline{\Lambda(A_{i(r,R)})} = \overline{A_{i(r^2/R,r,R)}}.
\]

More generally, one has $\Lambda^{k}(A_{i(r,R)}) = A_{i(r^{k+1}/R^k,r^{k+1}/R^{k-1},\theta)}$. Summarizing, the family of pairwise disjoint open sets $\{A_{i(r^{k+1}/R^k,r^{k+1}/R^{k-1},\theta)}\}_{k \in \mathbb{N}}$ is such that the union of their closures $\{\overline{A_{i(r^{k+1}/R^k,r^{k+1}/R^{k-1},\theta)}}\}_{k \in \mathbb{N}_0}$ yields a sector of angle $2\pi - 2t$ with vertex at $0 \in \mathbb{C}$. In fact, the following holds:
\[
\bigcup_{k \geq 0} \overline{\Lambda^{k}(A_{i(r,R)})} = \overline{B_0(R)} \setminus \left\{ \rho e^{i\alpha} : \rho, \alpha \in \mathbb{R}, 0 < \rho \leq R, \theta - t < \alpha < \theta + t \right\}
\]
where $\overline{B_0(R)}$ stands for the closure of the disc $B_0(R)$ of radius $R$ around $0 \in \mathbb{C}$. The interior of the set $\overline{B_0(R)} \setminus \left\{ \rho e^{i\alpha} : \rho, \alpha \in \mathbb{R}, 0 < \rho \leq R, \theta - t < \alpha < \theta + t \right\}$ will be denoted by $B_0^0(R,\theta)$.

From now on, let $R$ be such that $B_0(2R)$ is contained in $\mathbb{D}$, where $B_0(2R)$ is the disc of radius $2R$ around $0 \in \mathbb{C}$. In particular, $\phi$ is injective on $B_0(2R) \setminus t^0$. 


Consider the sequence of maps $\{\xi^k\}$ defined on $A^t_{(r,R,\theta)}$ by

$$\xi^k = \phi \circ \Lambda^{ok^k}.$$ 

The maps $\xi^k$ are viewed as taking values in $\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\}$. We have:

**Lemma 5.2.** The maps $\{\xi^k\}$ (taking values in $\mathbb{CP}^1$) form a normal family for the compact open topology.

**Proof.** Recall that $\phi$ is injective on $B_0(2R) \setminus l^0$. Similarly, the maps $\{\xi^k\}$ are also injective on $A^t_{(r,R,\theta)}$. In particular, the sets $\xi^{k_1}(A^t_{(r,R,\theta)})$ and $\xi^{k_2}(A^t_{(r,R,\theta)})$ are disjoint provided that $k_1 \neq k_2$. Next let $U$ be a small disc contained in $A_{(r,2R)} \cap B_0(2R) \setminus l^0$ and set $V = \phi(U)$. Since $\phi$ is injective on $B_0(2R) \setminus l^0$, it follows that $\xi^k(A^t_{(r,R)})$ is disjoint from $V$ for every $k \in \mathbb{N}_0$. In fact, the latter sets are all contained in $\phi(B_0(R) \setminus l^0)$ which is itself disjoint from $V$. Up to choosing an appropriate coordinate on $\mathbb{CP}^1$, we can assume without loss of generality that $\phi(B_0(R) \setminus l^0)$ is contained in some disc $\Omega \subset \mathbb{C}$. Thus the images of all the maps $\xi^k : A^t_{(r,R,\theta)} \rightarrow \mathbb{CP}^1$ are actually contained in $\Omega$. The lemma now follows from Montel theorem. \[\Box\]

Next, we also have:

**Lemma 5.3.** Every convergent subsequence of $\{\xi^k\}$ converges to a constant map.

**Proof.** Since $\{\xi^k\}$ is a sequence of injective (holomorphic) maps, the limit of any uniformly convergent subsequence is either a constant map or another (injective) holomorphic map (see, for example, page 5 of [3]). Let us then choose a convergent subsequence $\{\xi^{k_n}\}$ of $\{\xi^k\}$ and assume for a contradiction that its limit $\xi^\infty$ is not constant. In this case, the image of $A^t_{(r,R,\theta)}$ under $\xi^\infty$ must contain some open ball $B$ which, in turn, satisfies $B = \xi^\infty(U)$ for some open set $U \subset A^t_{(r,R,\theta)}$. Now, the fact that the sequence $\{\xi^{k_n}\}$ is uniformly convergent on $U$ and that the corresponding images converge to $B$ implies that $\xi^{k_{n_1}}(U) \cap \xi^{k_{n_2}}(U) \neq \emptyset$ for $n_1$ and $n_2$ sufficiently large. This is, however, impossible since the images of $A^t_{(r,R,\theta)}$ by the iterates of $\xi$ are pairwise disjoint. The lemma is proved. \[\Box\]

Now, denote by $\mathcal{P}_0$ the set of points $p \in \mathbb{C} \cup \{\infty\}$ satisfying the following condition: $p = \lim \phi(z_n)$ for some sequence $\{z_n\}$ in $B_0(R) \setminus l^0$ converging to $0 \in \mathbb{C}$. Furthermore, for every sequence $\{z_n\} \subset B_0^c(R, \theta)$ converging to $0 \in \mathbb{C}$, we define a sequence $\{z'_n\} \subset A^t_{(r,R,\theta)}$ along with a sequence of positive integers $\{k(n)\}$ by means of the equation

$$z_n = \Lambda^{ok(n)}(z'_n).$$

Naturally, up to passing to a subsequence, we can assume without loss of generality that the sequence $\{z'_n\}$ converges to some point in the closure of $A^t_{(r,R,\theta)}$. Let us denote by $\mathcal{P}'_0$ the subset of $\mathcal{P}_0$ consisting of the accumulation points of $\phi$ arising from sequences $\{z_n\}$ satisfying the following condition: the sequence $\{z'_n\}$ is convergent and its limit does not lie in the boundary $\partial A^t_{(r,R,\theta)}$ of $A^t_{(r,R,\theta)}$. This condition ensures that $\{z'_n\}$ will be contained in some compact subset of $A^t_{(r,R,\theta)}$.

**Lemma 5.4.** $\mathcal{P}'_0$ is a countable set.
Proof. Let $p$ be a point in $P'_0$ and $\{z_n\}$ a sequence as above such that $\lim \phi(z_n) = p$. Clearly, we have $\phi(z_n) = \phi(\Lambda^{k(n)}(z'_n)) = \xi^{k(n)}(z'_n)$. Owing to Montel theorem, up to passing to suitable subsequences, we can assume that the sequence of maps $\{\xi^{k(n)}\}$ converges on some compact subset of $A'_{(r,R,\theta)}$ containing all the points in the sequence $\{z'_n\}$. Since Lemma 5.3 states that the limit of $\{\xi^{k(n)}\}$ on a compact set is constant, there follows that $p$ is, in fact, the limit of some subsequence of $\{\xi^k\}$ on a compact subset of $A'_{(r,R,\theta)}$. Now, since there are only countably many subsequences of $\{\xi^k\}$, and since we can fix a (countable) exhaustion of $A'_{(r,R,\theta)}$ by compact sets, we conclude that only countably many of these accumulation points can exist. This proves the lemma. 

Now Lemma 5.4 can be strengthened to encompass the entire set of accumulation points $P_0$ of $\phi$.

Lemma 5.5. $P_0$ is a countable set.

Proof. Clearly, we only need to consider those sequences $\{z_n\} \in B_0(R) \setminus I^\theta$ converging to $0 \in \mathbb{C}$ that fall short of verifying the condition of Lemma 5.4. In other words, the limit of the corresponding sequence $\{z'_n\} \in A'_{(r,R,\theta)}$ lies in the boundary of $A'_{(r,R,\theta)}$. To handle this situation, it suffices to note that the sequence in question is contained in the compact set given as the closure of $A'_{(r,R,\theta)}$. Clearly the closure of $A'_{(r,R,\theta)}$ is contained in some open set $A'_{(r',R',\theta)}$ for some real constants $t', r', R'$ such that $0 < t' < t$, $0 < r' < r$ and $R < R' < 2R$ so that $\phi$ is still injective on the relevant sets. The result follows by applying the previous lemma with respect to the open set $A'_{(r',R',\theta)}$. 

Let us now consider the set $\mathcal{A}C$ of accumulation points $p = \lim \phi(z_n)$ where $\{z_n\} \to 0 \in \mathbb{C}$ and where $\phi$ can be changed by changing the branch of logarithm $\ln$. We claim that $\mathcal{A}C$ still is a countable set. More precisely, consider two distinct angles $\theta_1$ and $\theta_2$ and $t > 0$. For $i = 1, 2$, define the sectors $V_1$, $V_2$ by

$$V_i = \{\rho e^{i\alpha} : 0 < \rho < R \text{ and } \theta_i + t < \alpha < \theta_i + 2\pi - t\}.$$ 

Now, up to reducing $t$, there follows that $V_1 \cup V_2$ contains a punctured neighborhood of $0 \in \mathbb{C}$. The previous lemma shows that the set of points $p = \lim \phi(z_n)$ with $z_n \to 0$ and $\{z_n\} \subset V_i$ $(i = 1, 2)$ is countable. Clearly, if $p = \lim \phi(z_n)$ for some sequence $\{z_n\} \to 0 \in \mathbb{C}$, then we still have $p = \lim \phi(z_{n(k)})$ for a subsequence $\{z_{n(k)}\}$ fully contained in either $V_1$ or $V_2$. Thus, $\mathcal{A}C$ is contained in the union of accumulation points obtained from sequences contained in either $V_1$ or $V_2$, at least up to the choice of a determination for $\ln$ (which, in turn, gives rise to a determination of $\phi$ itself). Since there are only countably many possible choices of branches of logarithm, two of them differing by a translation, there follows that $\mathcal{A}C$ is a countable set as desired.

We are now ready to prove Proposition 5.7.

Proof of Proposition 5.7. The proof amounts to showing that $\mathcal{A}C$, in fact, consists of a single point once a branch of logarithm $\ln$ is chosen. In other words, let $\theta$ be fixed and let $\ln$ denote a branch of logarithm on $\mathbb{D} \setminus I^\theta$. Now, consider a (small) $t > 0$, and set

$$V = \{\rho e^{i\alpha} : 0 < \rho < R \text{ and } \theta + t < \alpha < \theta + 2\pi - t\}.$$
To prove the proposition it suffices to ensure the existence of a single point $p \in \mathbb{C}P^1$ such that $p = \lim \phi(z_n)$ where $z_n \to 0$ with $\{z_n\} \subset V$.

Without loss of generality, we can assume $\theta = 0$. Denote by $\mathcal{P}_0 \subseteq \mathcal{AC}$, the set of points $p \in \mathbb{C}P^1$ such that $p = \lim \phi(z_n)$ for some sequence $z_n \to 0$ with $\{z_n\} \subset V$. The set $\mathcal{P}_0$ is countable since so is $\mathcal{AC}$. Assume for a contradiction that $\mathcal{P}_0$ contains two distinct points $p, q \in \mathbb{C}P^1$ and choose sequences $\{x_n\}$ and $\{y_n\}$ satisfying the following conditions:

- Both $\{x_n\}$ and $\{y_n\}$ converge to $0 \in \mathbb{C}$ and we have $\{x_n\} \subset V$ and $\{y_n\} \subset V$.
- $p = \lim \phi(x_n)$ and $q = \lim \phi(y_n)$.

Next, let $l_n$ be a path joining $x_n$ to $y_n$ and entirely contained in $V$. Moreover, since $\{x_n\}$ and $\{y_n\}$ converge to $0 \in \mathbb{C}$, we can find a decreasing sequence $\{\delta_n\}$, $\delta_n \in \mathbb{R}_+$, such that the following holds:

1. For every $n$, the path $l_n$ is entirely contained in the disc $B_0(\delta_n)$ of radius $\delta_n$ around $0 \in \mathbb{C}$.
2. The decreasing sequence $\{\delta_n\}$ converges to $0$.

Since $p$ and $q$ are distinct points, there exists $\varepsilon > 0$ such that $q$ does not belong to the closed ball $B_p(\varepsilon)$ of radius $\varepsilon$ around $p$. We also fix $\varepsilon > 0$ such that the closed disc $B_q(\varepsilon)$ of radius $\varepsilon$ around $q$ lies entirely in the complement of $B_p(\varepsilon)$ inside $\mathbb{C}P^1$.

Now, since $p = \lim \phi(x_n)$ and $q = \lim \phi(y_n)$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, the following holds:

$$\phi(x_n) \in B_p(\varepsilon) \quad \text{and} \quad \phi(y_n) \in B_q(\varepsilon).$$

Thus, for every $n$ large enough, there is a point $w_n \in \phi(l_n)$ such that $w_n \in \partial B_q(\varepsilon)$. The sequence $\{w_n\}$ is contained in a compact set so that it possesses at least one accumulation point in $\partial B_q(\varepsilon)$. Let $w_\varepsilon \in \partial B_q(\varepsilon)$ be one accumulation point of the sequence $\{w_n\}$. Clearly, for every $n$, $w_n = \phi(z_n)$ for some $z_n \in l_n$. By construction, the sequence $\{z_n\}$ is contained in $V$ and converges to $0 \in \mathbb{C}$ so that $w_\varepsilon$ lies in $\mathcal{P}_0$.

To complete the proof, just let $\varepsilon$ vary on an arbitrarily small interval. The preceding construction then yields a continuum of points in $\mathcal{P}_0$ hence contradicting the fact that $\mathcal{P}_0$ is countable. This ends the proof of Proposition 5.7. \qed

**Remark 5.6.** Let us close this section by pointing out another result (Proposition 5.7) that can also be proved by following essentially the same argument provided above for Propositions 4.2 and 5.1. With this notation, Proposition 5.7 reads:

Consider a holomorphic function $\mathcal{H}$ defined on a punctured neighborhood of $0 \in \mathbb{C}$. Given $\alpha \in \mathbb{C}$, let $f$ be the multivalued function defined by $f(z) = \mathcal{H}(z) z^\alpha$. In accurate terms, $f$ is a well-defined holomorphic function on domains of the form $\mathbb{D} \setminus l^\theta$ up to re-scaling coordinates and this applies to every choice of $\theta \in [0, 2\pi]$.

**Proposition 5.7.** Assume that the function $f(z) = \mathcal{H}(z) z^\alpha$ is injective on the sector $\mathbb{D} \setminus l^\theta$, for every $\theta \in [0, 2\pi)$. Then $\mathcal{H}(z)$ does not have an essential singularity at $0 \in \mathbb{C}$.

6. Covering projective structures and proof of Theorem C

In this final section, we shall derive Theorem C from well-known results in [20] and [16]. Throughout the section $\mathcal{L}$ stands for a Riemann surface (orbifold) of finite type: this means that $\mathcal{L}$ is isomorphic to a compact Riemann surface (orbifold) from which finitely many points were removed.
Recall that a (singular) bounded projective structure $P$ on $\mathcal{L}$ is said to be a covering projective structure if it satisfies the following conditions:

- The developing map $D$ (associated with the universal covering) is a covering of its image.
- The monodromy group of $P$ acts discontinuously on the image of $D$.

**Lemma 6.1.** A uniformizable projective structure $P$ on $\mathcal{L}$ is necessarily a bounded covering projective structure.

**Proof.** The monodromy developing map $D_\mu$ realizes a diffeomorphism between $\mathcal{L}_\mu$ and its image in $\mathbb{CP}^1$. If $\ker(\mu)$ is trivial then $\mathcal{L}_\mu$ is the universal covering and $D = D_\mu$ is, in particular, a covering map. Let us now assume that $\ker(\mu)$ is non-trivial, and consider the covering map $h : \mathbb{D} \rightarrow \mathcal{L}_\mu$. By construction, the (universal covering) developing map $D$ and the monodromy-developing map $D_\mu$ have the same image $U$ in $\mathbb{CP}^1$ since the monodromy of the lift of $P$ to $\mathcal{L}_\mu$ is trivial. In other words, the developing map $D$ can be factored as a composition $D = D_\mu \circ h$. In particular, $D$ is a composition of a diffeomorphism $(D_\mu : \mathcal{L}_\mu \rightarrow U)$ and a covering map $(h : \mathbb{D} \rightarrow \mathcal{L}_\mu)$. Thus $D$ is a covering map itself. Finally, this projective structure $P$ on $\mathcal{L}$ is necessarily bounded thanks to Corollary B. \qed

Strictly speaking the converse to Lemma 6.1 does not hold in general. The simplest example of a covering projective structure that is not uniformizable arises in $\mathbb{C}$ (or equivalently in $\mathbb{CP}^1$) by means of the singular projective structure $P_k$ underlining the translation structure induced by the vector field $z^k \partial/\partial z$, $k \geq 3$. Naturally, $\mathbb{C}$ can be identified to its own ramified $(k - 1)$-sheet covering $\mathbb{C}_{k - 1}$ by means of the map $z \mapsto z^k \in \mathbb{C}_{k - 1}$. With this notation, the projective structure $P_k$ is nothing but the lift to $\mathbb{C}_{k - 1} \simeq \mathbb{C}$ of the (translation) structure $P_2$ induced on $\mathbb{C}$ by the vector field $z^2 \partial/\partial z$. Whereas $P_2$ is uniformizable and has a singular point at $0 \in \mathbb{C}$, it has no periods on $\mathbb{C}^*$. Thus, the monodromy covering associated with $P_2$ is trivial and so is the monodromy covering associated with $P_k$, $k \geq 3$. However, for $k \geq 3$, the loop $c : [0, 1] \rightarrow \mathbb{C}^*$ given by $c(t) = e^{2\pi it}$ lifts to $\mathbb{C}_{k - 1} \simeq \mathbb{C}$ as an open path whose endpoints have, by construction, the same image under the monodromy-developing map. In other words, the monodromy developing map arising from $P_k$ is not injective so that $P_k$ is not uniformizable.

A partial converse to Lemma 6.1, however, is still possible on Riemann surfaces of finite type and this is the content of Lemma 6.2 below.

**Lemma 6.2.** Let $\mathcal{L}$ denote a Riemann surface (orbifold) of finite type and let $P$ be a (singular) bounded covering projective structure on $\mathcal{L}$. Then, there exists a finite quotient $\mathcal{L}^* = \mathcal{L}/\sim$ of $\mathcal{L}$ where $P$ induces a uniformizable projective structure. In other words, the pair $(\mathcal{L}, P)$ is obtained as a finite ramified covering of a pair $(\mathcal{L}^*, P_\sim)$ where $\mathcal{L}^*$ is a Riemann surface (orbifold) of finite type and where $P_\sim$ is a uniformizable projective structure on $\mathcal{L}^*$.

**Proof.** On the universal covering of $\mathcal{L}$, the developing map arising from $P$ is a covering map of its image. The same then applies to the monodromy-developing map $D_\mu$ from the monodromy covering $\mathcal{L}_\mu$ of $\mathcal{L}$ to its image in $\mathbb{CP}^1$. Denote by $\text{Im}(D_\mu) \subset \mathbb{CP}^1$ the image of $D_\mu$. By definition, it is an open set invariant by the monodromy group $\Gamma$ of $P$.

Next, we claim that the fibers of $D_\mu$ are finite. To check this claim, consider a point $q_t \in \mathcal{L}_\mu$ lying in a fundamental domain $\mathcal{L}_0 \subset \mathcal{L}_\mu$ with respect to the covering $\mathcal{L}_\mu \rightarrow \mathcal{L}$. Let $U_0 \subset \text{Im}(D_\mu)$ be given by $U_0 = D_\mu(\mathcal{L}_0)$. The equivariance relation verified by developing
maps (Equation (1)) implies that $U_q$ as a point $q$ naturally identified with the group $\Gamma$ combined with Equation (1) to ensure that another point $q_2 \in L_\mu$ satisfying $D_\mu(q_1) = D_\mu(q_2)$ must belong to the same fundamental domain $L_0$ as $q_1$.

Letting $z_0 = D_\mu(q_1)$, the fiber $D_\mu^{-1}(z_0)$ is therefore contained in $L_0$ which is identified with the original Riemann surface (orbifold) $L$. The fiber $D_\mu^{-1}(z_0)$ is also a discrete set since $D_\mu$ is a covering map. To conclude that $D_\mu^{-1}(z_0)$ is a finite set, we now proceed as follows.

Recall that $L_0 \simeq L$ is isomorphic to a compact Riemann surface (orbifold) $S$ minus finitely many points $p_1, \ldots, p_k$. The bounded nature of $P$, however, ensures that a small neighborhood of any of the punctures $p_i$ can contain only finitely many points of the fiber $D_\mu^{-1}(z_0)$: this follows from that $P$ is of bounded type and therefore singular points of $P$ are at worst poles of order 1 for the corresponding Schwarzian differential. Therefore, up to finitely many points, the fiber $D_\mu^{-1}(z_0)$ yields a discrete set of a compact part of $S$. The finiteness of $D_\mu^{-1}(z_0)$ follows at once.

Next, consider the following equivalence relation in $L_\mu$: two points in $L_\mu$ are identified if they have the same image under $D_\mu$. The preceding shows that equivalence classes are finite (and of same cardinality). Moreover the group of deck-transformations of the covering $L_\mu \to L$ sends equivalence classes to equivalence classes owing to Equation (1). In particular, we obtain an equivalence relation in $L$ itself: two points $p, q$ are equivalent if they have lifts $\tilde{p}, \tilde{q}$ in $L_\mu$ (necessarily belonging to a same fundamental domain) such that $D_\mu(\tilde{p}) = D_\mu(\tilde{q})$. Clearly, the quotient $L^*$ of $L$ by this equivalence relation still is a Riemann surface (orbifold) of finite type.

It only remains to show that $L^*$ is endowed with a uniformizable projective structure $P_\sim$ induced by $P$. For this, recall that a projective structure can also be defined by the pair consisting of a developing map and a monodromy group satisfying Equation (1). In the present case, we simply identify the points of $L_\mu$ having the same image under $D_\mu$ obtaining a new manifold $L^*_\mu$. By construction $D_\mu$ induces a developing map $D_{\mu, \sim}$ which is a diffeomorphism from $L^*_\mu$ to $\text{Im} (D_\mu)$. Similarly, what precedes ensures that the group of deck transformations of $L^*_\mu$ still acts on $L^*_\mu$ and that the quotient of this action is nothing but $L^*$. Finally, $D_{\mu, \sim}$ is still equivariant with respect to the action of the mentioned deck-transformation group on $L^*_\mu$ and the action of $\Gamma$ on $\text{Im} (D_\mu)$. It therefore defines a projective structure on $L^*$ which is uniformizable since $D_{\mu, \sim} : L^*_\mu \to \text{Im} (D_\mu)$ is a diffeomorphism. The lemma is proved.

We are now ready to derive Theorem C from the analogous results in [20] and [16] about bounded covering projective structures.

Proof of Theorem C. To prove the first assertion of Theorem C, let $S(L)$ denote the space of bounded covering projective structures on $L$. Similarly, let $K(L)$ be the set formed by bounded projective structures whose monodromy group is Kleinian with non-empty discontinuity set. Finally $U(L)$ denotes the set of uniformizable projective structure on $L$. Bers simultaneous uniformization theorem shows that quasi-conformal deformations of the canonical projective structure on $L$ is contained in the interior of $U(L)$. To show the converse inclusion, we need the result of [20] claiming that the interior of $K(L) \cap S(L)$ actually coincides with the mentioned space of quasi-conformal deformations. However, the interior
of $\mathcal{U}(\mathcal{L})$ is contained in the interior of $\mathcal{K}(\mathcal{L}) \cap \mathcal{S}(\mathcal{L})$. In fact, Lemma 6.1 yields $\mathcal{U}(\mathcal{L}) \subseteq \mathcal{S}(\mathcal{L})$ while we also have $\mathcal{U}(\mathcal{L}) \subseteq \mathcal{K}(\mathcal{L})$. For the latter assertion, recall from Lemma 2.2 that a Riemann surface with a uniformizable projective structure is the quotient of an open set of $\mathbb{CP}^1$ - namely the image of its monodromy developing map - by its monodromy group. In particular, the monodromy group must be discrete and with non-empty discontinuity set.

In turn, the second assertion of Theorem C is an immediate consequence of the analogous statement in [16] for bounded projective structures complemented by Lemma 6.2.

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