On Boman’s Theorem On Partial Regularity Of Mappings

Tejinder S. Neelon

Abstract. Let \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) and \( k \) be a positive integer. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a locally bounded map such that for each \( (\xi, \eta) \in \Lambda \), the derivatives
\[
D^j_\xi f(x) := \left. \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \right|_{t=0}, \quad j = 1, 2, \ldots, k,
\]
exist and are continuous. In order to conclude that any such map \( f \) is necessarily of class \( C^k \) it is necessary and sufficient that \( \Lambda \) be not contained in the zero-set of a nonzero homogenous polynomial \( \Phi(\xi, \eta) \) which is linear in \( \eta = (\eta_1, \eta_2, \ldots, \eta_m) \) and homogeneous of degree \( k \) in \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \).

This generalizes a result of J. Boman for the case \( k = 1 \). The statement and the proof of a theorem of Boman for the case \( k = \infty \) is also extended to include the Carleman classes \( C\{M_k\} \) and the Beurling classes \( C(M_k) \).

A continuous function \( f : \mathbb{R}^n \to \mathbb{R} \) that is differentiable when restricted to arbitrary differentiable curves is not necessarily differentiable as a function of several variables \([12]\). Indeed, there are discontinuous functions \( f : \mathbb{R}^n \to \mathbb{R} \) whose restrictions to arbitrary analytic arcs are analytic \([2]\). But a \( C^\infty \) function \( f : \mathbb{R}^n \to \mathbb{R} \) whose restriction to every line segment is real analytic is necessarily real analytic \([13]\). In \([8, 9, 10]\) and \([11]\) this result was extended by considering restrictions to algebraic curves and surfaces of functions belonging to more general classes of infinitely differentiable functions. It is also well known that a function \( f : \mathbb{R}^n \to \mathbb{R} \) that is infinitely differentiable in each variable separately may be no better than measurable \([7]\). In \([4]\), the opposite problem is considered; for vector valued functions hypothesis is made on the source as well as the target space. In this note, Theorem 4 of \([4]\) is generalized to \( C^k \), \( k \geq 1 \), the class of functions that have continuous derivatives up to order \( k \).

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a locally bounded map. For \( (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m \), set
\[
D_\xi \langle f, \eta \rangle (x) := \left. \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \right|_{t=0}
\]
in the sense of distributions, where \( \langle \cdot, \cdot \rangle \) denotes the inner product on \( \mathbb{R}^m \). By the Leibniz Integral rule, we have
\[
\frac{d}{dt} \int \langle f(x + t\xi), \eta \rangle \, dx = \int \frac{d}{dt} \langle f(x + t\xi), \eta \rangle \, dx.
\]
Let \( k, 1 \leq k < \infty \), be fixed. For \( \xi \in \mathbb{R}^n \), denote by \( C^k_\xi (\mathbb{R}^n) \) the space of all continuous functions \( f : \mathbb{R}^n \to \mathbb{R} \) such that the derivatives \( D^j_\xi f(x) := \frac{\partial^j f(x + t\xi)}{\partial t^j} \bigg|_{t=0} \), \( j = 1, 2, \ldots, k \), exist and are continuous. Similarly, \( C^\infty_\xi (\mathbb{R}^n) := \cap_{k=0}^\infty C^k_\xi (\mathbb{R}^n) \).

We are interested in finding the necessary and sufficient conditions on a subset \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) to have the following property:

\[
\text{if } f : \mathbb{R}^n \to \mathbb{R}^m \text{ is locally bounded such that } \langle f, \eta \rangle \in C^k_\xi (\mathbb{R}^n), \forall (\xi, \eta) \in \Lambda, \text{ then } f \in C^k (\mathbb{R}^n). 
\]

The case \( k = 1 \) and \( k = \infty \) was dealt in [4].

Let \( Z^n_+ \) denote all \( n \)-tuples of nonnegative integers. For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z^n_+ \), set \( |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \). The set \( Z^n_+ \) of multi-indices is assumed to be ordered lexicographically i.e., for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in Z^n_+ \), define \( \alpha < \beta \) if there is \( 1 \leq i \leq n \), such that \( \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i \).

Let \( k_n = \binom{k+n-1}{k} \) denote the number of monomials of degree \( k \) in \( n \) variables. Then for any \( \varphi \in C^\infty_\xi (\mathbb{R}^n) \), we have

\[
\int D^j_\xi \langle f, \eta \rangle(x) \varphi(x) dx = \frac{d}{dt} \bigg|_{t=0} \langle f(x + t\xi), \eta \rangle \varphi(x) dx \\
= \frac{d}{dt} \bigg|_{t=0} \langle f(x) \varphi(x - t\xi)dx, \eta \rangle = \langle f(x) \frac{d}{dt} \varphi(x - t\xi)dx, \eta \rangle \\
= -\sum_i \xi_i \frac{d}{dt} \int f(x) \varphi(x - t\xi)dx, \eta \rangle \bigg|_{t=0} = \sum_{i,j} \xi_i \eta_j \int \partial_i f_j(x) \varphi(x) dx.
\]

By iteration, we obtain the formula for higher-order distributional derivatives:

\[
D^p_\xi \langle f, \eta \rangle(x) = \sum_{|\alpha|=p} \sum_{j=1}^m \xi^\alpha \eta_j \partial^\alpha f_j(x).
\]

Let

\[
B_k := \left\{ \Phi(\xi, \eta) = \sum_{j=1}^m \sum_{|\alpha|=k} \varphi_{\alpha j} \xi^\alpha \eta_j : \varphi_{\alpha j} \in \mathbb{R}, \alpha \in Z^n_+, j \in \mathbb{Z}_+ \right\}.
\]

For any function \( \Phi(\xi, \eta) \), set \( \| \Phi \| := \max_{\|\xi\| \leq 1, \|\eta\| \leq 1} |\Phi(\xi, \eta)| \). For a subset \( K \subset \subset \Lambda \), \( \subset \subset \Lambda \) denotes the compact inclusion) put \( \| \Phi \|_K := \max_{(\xi, \eta) \in K} |\Phi(\xi, \eta)| \).

**Theorem 1.** Let \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) be a subset and \( k \) be a positive integer. The following conditions are equivalent.

(i) \( \Lambda \) is not contained in an algebraic hypersurface defined by an element of \( B_k \) i.e., \( \Phi \in B_k, \Phi|_{\Lambda} \equiv 0 \Rightarrow \Phi \equiv 0 \);

(ii) there exists a set consisting of \( m \cdot k_n \) points \( (\xi^*, \eta^*) = \left\{ (\xi^{(p)}, \eta^{(p)}) \in \Lambda, p = 1, 2, \ldots, m k_n \right\} \) such that \( \det (\xi^*, \eta^*) \neq 0 \), where

\[
\Delta(\xi^*, \eta^*) := \left[ \begin{array}{c} \alpha \xi^{(p)} \\ \eta^{(p)} \end{array} \right]_{|\alpha|=k,1 \leq j \leq m,1 \leq p \leq m k_n};
\]

(iii) if \( f : \mathbb{R}^n \to \mathbb{R}^m \) is locally bounded and \( \langle f, \eta \rangle \in C^k_\xi (\mathbb{R}^n), \forall (\xi, \eta) \in \Lambda \), then \( f \in C^k (\mathbb{R}^n, \mathbb{R}^m) \).
If any one of the above equivalent conditions is satisfied, then there exists a constant\( B \) depending only on \( \Lambda \) such that the following inequality holds for all locally bounded maps \( f: \mathbb{R}^n \to \mathbb{R}^m \):

\[
\max_{1 \leq j \leq m} \max_{|\alpha| = k} |\partial^\alpha f_j(x)| \leq B \cdot \sup_{(\xi, \eta) \in \Lambda} |D^k_\xi \langle f, \eta \rangle(x)|, \forall x \in \mathbb{R}^n.
\]

**Proof.** We will prove \((i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)\).

Suppose \( \det \Delta (\xi^*, \eta^*) = 0 \) for every set of \( mk_n \) elements \((\xi^*, \eta^*) = \left\{ \left( \xi^{(p)}, \eta^{(p)} \right) \right\}_{1 \leq p \leq mk_n} \in \Lambda. \) Fix one such set \((\xi^*, \eta^*)\) so that the rank \( l := \text{rank } \Delta (\xi^*, \eta^*) \) is positive. Let \( \Delta^{(l)} \) denote some \( l \times l \) submatrix of \( \Delta (\xi^*, \eta^*) \) such that the minor \( \det \Delta^{(l)} \) is nonzero. Let \( \Delta^{(l+1)} \) be a \((l+1) \times (l+1)\) submatrix of \( \Delta (\xi^*, \eta^*) \) that contains \( \Delta^{(l)} \) as a submatrix. Replace the point \( (\xi^{(p)}, \eta^{(p)}) \) in \( \Delta^{(l+1)} \) which does not appear in \( \Delta^{(l)} \) by variables \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m.\) By expanding \( \Delta^{(l+1)} \) along the row where the replacement took place we obtain an element

\[
\Phi(\xi, \eta) = \sum_{\alpha,j} \varphi_{\alpha j} \xi^\alpha \eta_j,
\]

of \( \mathcal{B}_k \) which is nonzero since one of its coefficients coincides with \( \det \Delta^{(l)} \) up to a sign.

Since \( \Delta (\xi^*, \eta^*) \) has rank \( l \), we find that \( \Phi(\xi, \eta) = 0 \) for all \((\xi, \eta) \in (\xi^*, \eta^*)\).

If \( \Phi(\xi, \eta) = 0 \) for all \((\xi, \eta) \in \Lambda \), we are done. Otherwise, choose a point \((\tilde{\xi}, \tilde{\eta}) \in \Lambda \setminus (\xi^*, \eta^*) \) with \( \Phi(\tilde{\xi}, \tilde{\eta}) \neq 0.\)

Let \((\tilde{\xi}, \tilde{\eta}) \) be the set which is obtained from \((\xi^*, \eta^*)\) by replacing the point \((\xi^{(p)}, \eta^{(p)}) \) by \((\tilde{\xi}, \tilde{\eta}).\) Then, the rank \( \Delta (\tilde{\xi}, \tilde{\eta}) \geq l + 1.\) By repeating above procedure, we find a sequence of subsets \((\xi^*, \eta^*)^{(i)} \subset \Lambda, \ i = 1, 2, 3, \ldots, \) each with \( mk_n \) elements such that the rank \( \Delta (\xi^*, \eta^*)^{(j)} \) is a strictly increasing sequence of nonnegative integers. After finitely many steps we obtain a nonzero element of \( \mathcal{B}_k \) which vanishes on the entire \( \Lambda \).

\((ii) \Rightarrow (iii)\).

Let \((\xi^*, \eta^*)\) be \( \left\{ \left( \xi^{(p)}, \eta^{(p)} \right) \right\}_{1 \leq p \leq mk_n} \in \Lambda \). be a set of points such that

\[ \det \Delta (\xi^*, \eta^*) \neq 0. \]

By applying Cramer’s rule to \((\Pi)\), we get

\[
\partial^\alpha f_j(x) = \sum_{p=1}^{mk_n} \frac{\det \Delta_{\alpha j}^{(p)}}{\det \Delta} D^k_\xi \langle f, \eta^{(p)} \rangle(x) \]

in the distributional sense, where \( \Delta^{-1}_{\alpha j} \) denotes the cofactor obtained by deleting the \((\alpha, j)-th \) row and the \( p-th \) column. Since \( D^k_\xi \langle f, \eta \rangle \in C^0 \) for all \((\xi, \eta) \in \Lambda,\) we have

\[
\partial^\alpha f_j(x) = \sum_{p=1}^{mk_n} \frac{\det \Delta_{\alpha j}^{(p)}}{\det \Delta} D^k_\xi \langle f, \eta^{(p)} \rangle(x) \in C^0.
\]

Furthermore, there exists a constant \( B = B(k, f, \Lambda) \) such that

\[
|\partial^\alpha f_j(x)| \leq \sum_{p=1}^{mk_n} \left| \frac{\det \Delta_{\alpha j}^{(p)}}{\det \Delta} \right| \left| D^k_\xi \langle f, \eta^{(p)} \rangle(x) \right| \leq B \cdot \sup_{(\xi, \eta) \in \Lambda} |D^k_\xi \langle f, \eta \rangle(x)|,
\]
for all $\alpha$ with $|\alpha| = k$, and all $j = 1, 2, \ldots, m$.

(iii) $\Rightarrow$ (i).

Suppose (i) does not hold. Let $\Phi \in \mathcal{B}_k$ be such that $\Phi|_\Lambda \equiv 0$. We can write $\Phi(\xi, \eta) = \langle \varphi_1(\xi), \varphi_2(\xi), \ldots, \varphi_m(\xi) \rangle$ and $\varphi_j(\xi) = \sum_{|\alpha| = k} \varphi_{\alpha j} \xi^\alpha$, $j = 1, 2, \ldots, m$, homogeneous polynomials of degree $k$.

Define the map

$$f(x) := \begin{cases} (\ln |x||) \varphi_1(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}.$$ 

Clearly $f \notin C^k$ and $f$ is $C^\infty$ in $\{x \in \mathbb{R}^n : 0 < |x| < 1\}$. We will prove that $D^k_\xi \langle f(x), \eta \rangle$ exists at $x = 0$, for all $(\xi, \eta) \in \Lambda$. It is easy to see that here are constants $C_\alpha$ such that

$$\left| \partial^\alpha \ln |x|| \right| \leq \frac{C_\alpha}{|x|^{|\alpha|} \ln |x|}, \forall \alpha, |\alpha| \geq 1.$$ 

Since the $\varphi_j(x)$’s are homogeneous polynomials of degree $k$, when the Leibniz’s formula is applied to the products $(\ln |x||) \varphi_j(x)$, it is clear that all terms in $D^p_\xi \langle f(x), \eta \rangle$, $1 \leq p \leq k$, except possibly

$$(\ln |x||) \langle D^p_\xi \varphi_j(x), \eta \rangle$$

tend to 0 as $x \to 0$. We only need to prove that the function in (3) also tends to 0 as $x \to 0$. By expanding $(x_1 + t\xi_1)^{\alpha_1} (x_2 + t\xi_2)^{\alpha_2} \cdots (x_n + t\xi_n)^{\alpha_n}$ binomially, we can write

$$\varphi_j(x + t\xi) := \varphi_j(x) + P(x, \xi, t) + \varphi_j(\xi)t^k.$$ 

But since $(\xi, \eta) \in \Lambda$,

$$\langle D^p_\xi \varphi_j(x), \eta \rangle = k! \langle \varphi_j(\xi), \eta \rangle = 0.$$ 

It follows that $\left| D^p_\xi \langle f(0), \eta \rangle \right| = 0$ for $p \leq k$. Thus, $f \in C^k_\xi$ for all $(\xi, \eta) \in \Lambda$, but $f \notin C^k$.

\[ \square \]

Remark 1. (cf. [6]) Suppose (i) is satisfied for all $k \geq 0$. It would be of interest to know whether there exists a constant $\rho = \rho(\Lambda)$, depending only on some appropriate notion of capacity of $\Lambda$, so that (2) is satisfied with $B = (\rho(\Lambda))^{-k}$ for all $f$ and all $k$.

Remark 2. Suppose $\Lambda$ satisfies (i) or (ii). The proof of Theorem 4 shows that if $f$ is continuous and $D^k_\xi \langle f, \eta \rangle = 0, \forall (\xi, \eta) \in \Lambda$, then $f$ is a polynomial. The assumption of continuity of $f$ is not necessary but our proof is valid only if $f$ is continuous. See [4].

Remark 3. If $\Lambda$ satisfies (i), then $\Lambda$ contains at least $mk_\Lambda$ elements. Furthermore, if (i) holds for $k$ then (i) also holds for all $j \leq k$. Suppose there exists $\Phi \in B_j, j < k$ such that $\Phi|_\Lambda \equiv 0$ but $\Phi \neq 0$. Then, $\xi_1^{k-j}\Phi \xi_1^{k-j}\Phi|_\Lambda \equiv 0$ but this is a contradiction.

Let $\{M_k\}_{k=0}^\infty$ be a sequence of nonnegative numbers. For $h > 0$ and $K \subset \subset \mathbb{R}^n$ define the seminorm on $C^\infty(\mathbb{R}^n)$,

$$p_{h,K}(f) = \sup_{\alpha \in \mathbb{Z}_+^n} \sup_{x \in K} \frac{|\partial^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$
The spaces

\[ C \{M_k\} = \{ f \in C^\infty(\mathbb{R}^n) : \forall K \subset \subset \mathbb{R}^n, \exists h > 0, \text{ s.t. } p_{h,K}(f) < \infty \} \]

and

\[ C(M_k) = \{ f \in C^\infty(\mathbb{R}^n) : p_{h,K}(f) < \infty, \forall K \subset \subset \mathbb{R}^n, \forall h > 0 \} \]

are called the Carleman and Beurling classes, respectively. The classes \( C \{(k!)^\nu\} \), \( \nu > 1 \), known as Gevrey classes, are especially important in partial differential equations and harmonic analysis. The class \( C \{k!\} \) is precisely the class of real analytic functions.

We assume that

\[ M_0 = 1 \text{ and } M_k \geq k!, \forall k; \]

\[ M_k^{1/k} \text{ is strictly increasing; } \]

\[ \exists C > 0 \text{ such that } M_{k+1} \leq C^k M_k, \ \forall k. \]

These conditions insure that the classes \( C \{M_k\} \) and \( C(M_k) \) are nontrivial and are closed under product and differentiation of functions. For more properties of these spaces, see [5], [11] and references there.

It is well known that \( f \in C^\infty(\mathbb{R}^n) \) if and only if \( \sup_{\xi \in \mathbb{R}^n} |\xi|^j |\hat{f}(\xi)| < \infty, \forall \chi \in C_c^\infty(\mathbb{R}^n), j \geq 1 \). A similar characterization is also available for \( C \{M_k\} \) (see [5]) a routine modification of which yields an analogous characterization of \( C(M_k) \).

Let \( r > 0 \). Choose a sequence of cut-off functions \( \chi_j \in C_c^\infty, j = 1, 2, ..., \) such that \( \chi_j(x) = 1 \text{ if } |x - x_0| < r, \chi_j(x) = 0 \text{ if } |x - x_0| > 3r \) and

\[ |\partial^\alpha \chi_j(x)| \leq (C_1 j)^{|\alpha|}, \forall j, \forall |\alpha| \leq j, \forall x, \]

where the constant \( C_1 \) is independent of \( j \).

Then \( f \in C \{M_k\} \) (resp. \( C(M_k) \)) in a neighborhood of \( x_0 \in \mathbb{R}^n \) if and only if there exists a constant \( h > 0 \) (resp. for every \( h > 0 \))

\[ \sup_{\xi \in \mathbb{R}^n} \sup_{j \geq 1} h^{-j} M_j^{-1} |\xi|^j |\hat{f}(\chi_j)(\xi)| < \infty. \]

Call a subset \( \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \) a determining set for bilinear forms of rank 1 if there is no nonzero bilinear form \( \varphi(\xi, \eta), \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^m \) of rank 1 such that \( \varphi(\xi, \eta) = 0 \) for all \( (\xi, \eta) \in \Lambda \).

Clearly \( \Lambda \) is a determining set for bilinear forms of rank 1 if and only if

\[ \langle u, \xi \rangle \langle v, \eta \rangle = 0, \forall (\xi, \eta) \in \Lambda \Rightarrow |u||v| = 0 \]

(here \( \langle u, \xi \rangle \) and \( \langle v, \eta \rangle \) are dot products on \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively), or equivalently,

\[ \bigcap_{(\xi, \eta) \in \Lambda} \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : u, \xi > v, \eta = 0 \} = (\mathbb{R}^n \times 0) \cup (0 \times \mathbb{R}^m). \]

Since \( \mathbb{R}[u, v] \) is a Noetherian ring, \( \Lambda \) contains a finite subset \( \Lambda' \) such that the sets \( \{ u, \xi > v, \eta > 0 : (\xi, \eta) \in \Lambda \} \) and \( \{ u, \xi > v, \eta > 0 : (\xi, \eta) \in \Lambda \} \) generate the same ideal in \( \mathbb{R}[u, v] \) and thus define the same varieties:

\[ \bigcap_{(\xi, \eta) \in \Lambda} \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : u, \xi > v, \eta = 0 \} = \bigcap_{(\xi, \eta) \in \Lambda'} \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^m : u, \xi > v, \eta = 0 \}. \]
Thus, any determining set for bilinear forms of rank 1 contains a finite determining set for bilinear forms of rank 1.

Let $C \{M_k\} (\xi)$ (resp. $C \{M_k\} (\xi)$) denote the set of all $f \in C_\infty^\alpha (\mathbb{R}^n)$ such that for every subset $K \subset \subset \mathbb{R}^n$, sup$_{j,x \in K} \left| D^j_k f(x) \right| h^{-j} M_j^{-1} < \infty$, $\forall j$, for some $h > 0$ (resp. for every $h > 0$).

**Theorem 2.** Let $\{M_k\}_{k=0}^\infty$ be a sequence of nonnegative numbers satisfying the conditions (4), (5) and (6). The following statements are equivalent.

(i) $\Lambda$ is a determining set for bilinear forms of rank 1;

(ii) for any locally bounded map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$,

\[ \langle \eta, f \rangle \in C \{M_k\} (\xi), \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C \{M_k\}; \]

(iii) for any locally bounded map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$,

\[ \langle \eta, f \rangle \in C \{M_k\} (\xi), \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C \{M_k\}; \]

(iv) for any locally bounded map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$,

\[ \langle \eta, f \rangle \in C^\infty (\xi), \forall (\eta, \xi) \in \Lambda \Rightarrow f \in C^\infty. \]

**Proof.** (cf. Theorem 4 in [4]) Assume (i) holds. By the remark above, by replacing $\Lambda$ by a subset, if necessary, we may assume $\Lambda$ is finite. Suppose for every $(\eta, \xi) \in \Lambda$, $(\eta, f) \in C \{M_k\} (\xi)$ (resp. $(\eta, f) \in C \{M_k\} (\xi)$). Now for a suitable function $f$,

\[ \langle \xi, z \rangle = \xi \langle \eta, f \rangle (z) = \xi \eta \langle \eta, f \rangle (z) = \xi \eta \left( \eta, i \int \left[ \langle \xi, \partial_x \rangle e^{-i(x,z)} f(x) dx \right] \right) \]

\[ = \xi \eta \left( \eta, -i \int e^{-i(x,z)} \langle \xi, \partial_x \rangle f(x) dx \right) = \xi \eta \left( \eta, -i \int e^{-i(x,z)} D \xi f(x) dx \right). \]

Let $g(\xi) := f \chi(\xi) \in C \{M_k\}$ near a fixed point $x_0$. Assume, without loss of generality, $x_0 = 0$. By assumption, for all $(\xi, \eta) \in \Lambda$ there exist constants $C = C_{\xi, \eta}$ and $h = h_{\xi, \eta} > 0$ (resp. for all $(\xi, \eta) \in \Lambda$ and for all $h > 0$ there exists a constant $C = C_{\xi, \eta}$ such that

\[ \left| \langle \xi, g(\xi) \rangle \right| \langle (\xi, \eta) \rangle = \left| \langle \xi, g(\xi) \rangle \right| \left| \langle (\xi, \eta) \rangle \right| \leq Ch^l M_j, \forall (\xi, \eta) \in \Lambda, \xi \in \mathbb{R}^n, j \in \mathbb{Z}. \]

The function

\[ (\mathbb{R}^n \times \mathbb{R}^m) \ni (u, v) \rightarrow \sum_{(\xi, \eta) \in \Lambda} |\langle \eta, v \rangle | \langle \xi, u \rangle |^l, \]

is homogeneous of degree 1 in $v$, of homogeneous degree $l$ in $u$. Since none of the terms $|\langle \eta, v \rangle | \langle (\xi, u) \rangle$ can vanish on all of $\Lambda$, the function in (7) has a positive minimum on the compact set $\{ (u, v) : |u| = 1, |v| = 1 \}$. Thus, there is an $\varepsilon > 0$ such that

\[ \sum_{(\xi, \eta) \in \Lambda} |\langle \eta, v \rangle | \langle \xi, u \rangle |^l \geq \varepsilon |v| |u| \]

(see Lemma 1 [4]). Applying this to $u = \xi, v = \xi \xi g(\xi)$, we get

\[ \left| g(\xi) \right| \xi \xi \leq \varepsilon^{-1} \sum_{(\xi, \eta) \in \Lambda} \left| \langle \eta, g(\xi) \rangle \right| \langle (\xi, \eta) \rangle \xi \xi \leq C h^l M_j, \]
where \( h = \max_{(\xi, \eta) \in \Lambda} h_{\xi \eta} \) (resp. for all \( h > 0 \)) and \( C = \varepsilon^{-1} \sum_{(\xi, \eta) \in \Lambda} C_{\xi \eta} \). Thus (ii) and (iii) hold. By setting \( h = 1 \) and \( M_j = 1, \forall j \), in the above argument, it is clear that (iii) holds as well.

Conversely if \( \Lambda \) is not a determinant set for bilinear forms of rank 1, there exist \( u \neq 0 \) and \( v \neq 0 \) such that
\[
\langle u, \xi \rangle \langle v, \eta \rangle = 0, \forall (\xi, \eta) \in \Lambda.
\]

Let \( h : \mathbb{R} \to \mathbb{R} \) be an arbitrary continuous function. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be defined as \( f(z) = h((u, z)) \cdot v \). Then
\[
\left. \frac{d}{dt} \langle \eta, f(z + t\xi) \rangle \right|_{t=0} = \langle \eta, v \rangle \langle u, \xi \rangle h'(\langle u, z + t\xi \rangle)|_{t=0} \equiv 0
\]
Thus \( \langle \eta, f \rangle \in C(M_k) \langle \xi \rangle \subset C\{M_k\} \langle \xi \rangle \subset C^\infty \langle \xi \rangle, \forall (\xi, \eta) \in \Lambda \) but \( f \) need not be even differentiable.

\[\square\]

References

[1] Agbor, D., and Boman, J., On modulus of continuity of mappings between Euclidean spaces, Mathematica Scandinavica. To appear

[2] Bierstone, E.; Milman, P. D.; Parusinski, A.: A function which is arc-analytic but not continuous. Proc. Amer. Math. Soc. 113 (1991) 419–423.

[3] Bochnak, J.: Analytic functions in Banach Spaces, Studia Mathematica, XXXV (1970) 273–292.

[4] Boman, J., Partial regularity of mappings between Euclidean spaces. Acta Math. 119. 1967. 1–25.

[5] Hormander, L., The analysis of linear partial differential operators I, Springer-Verlag.

[6] Korevaar, J., Applications of \( C^n \) capacities. Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), 105–118,

[7] Krantz S. G. and Parks H.R., A primer of real analytic functions. Second edition. Birkhauser, 2002.

[8] Neelon, T. S., On separate ultradifferentiability of functions. Acta Sci. Math. (Szeged) 64 (1998) 489–494.

[9] Neelon, T. S., Ultradifferentiable functions on lines in \( \mathbb{R}^n \). Proc. Amer. Math. Soc. 127 (1999) 2099–2104.

[10] Neelon, T. S., A Bernstein–Walsh type inequality and applications. Canad. Math. Bull. 49 (2006) 256–264.

[11] Neelon, T. S., Restrictions of power series and functions to algebraic surfaces. Analysis, Vol. 29, Issue 1 (2009), page 1-15.

[12] Rudin, W., Principles of Mathematical Analysis, 3rd ed. McGraw-Hill. 1976.

[13] Siciak, J.: A characterization of analytic functions of \( n \) real variables, Studia Mathematica, XXXV (1970) 292–297.