Research of the electro-vortex flows in the liquid metals at different currents

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Abstract. The electro-vortex flow is a very important phenomenon, taking place in the different processes in metallurgy, such as electrical welding or electrical melting. It appears in the conducting medium under the action of the electromagnetic force due to interaction of the non-uniform electrical current (which passes through this medium) and self-magnetic field. Here we present the study of the electro-vortex flow for the hemispherical geometry. The equations were solved in variables “vorticity – vector potential of velocity”. We considered the problem using Stokes approximation and non-linear case with a finite size of the central electrode, and compared the results obtained by different methods of solving the equations.

1. Introduction

Electro-vortex flows occur in liquid conducting media. They are caused by the interaction of an electric current with a non-uniform density and a self-magnetic field. Such processes are very important for electrotechnical metallurgy: electric smelting and electric welding of metals are mainly based on electro-vortex flows. Electro-vortex flows can also be interesting when studying the generation of a magnetic field in a conducting medium (the so-called dynamo mechanism). Such processes are well known in astrophysics and describe the origin of the magnetic field of the Sun and galaxies [1]. It may be very interesting to study the possibility of implementing this mechanism under terrestrial conditions and to study the role of electro-vortex flows. It can be assumed that the electro-vortex flow in a certain configuration with large Reynolds numbers can describe generation of a magnetic field.

From the historical point of view, the electro-vortex flows have been studied during a large period since 1970s. The analytical solution for the infinitely small central electrode was obtained by Sozou and Pickering [2]. They have also obtained the analytical solution for the plane electrode with a finite size [3]. It is important to mention the researches at the Institute of Physics of University of Latvia, which were carried since 1970s [4]. Nowadays, the electro-vortex flows are studied at the Joint Institute for High Temperature of RAS (Moscow, Russia) [5], G.I.Nosov Magnitogorsk State Technical University (Russia) [6], Institute of Continuous Medium Mechanics of the Ural Branch of RAS (Perm, Russia) [7], Helmholtz-Zentrum Dresden-Rossendorf (Germany), Leoben Zentrum (Austria) [8] and other institutes. In this work, we continue these researches and return to the early formulation of this problem.

Here we study the electro-vortex flow in a hemispherical container from the theoretical point of view. We describe the problem using the variables of vorticity and vector potential of the velocity. The equations are solved using pseudo-transient method, which allows us to change the elliptic equation by...
the parabolic one. After that we take Stokes’ approximation and neglect the convective part of the Navier – Stokes equation.

2. Basic equations

The movement of the incompressible conducting medium can be described by the Navier – Stokes equation:

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v}, \text{grad}) \vec{v} \right) = -\text{grad}(p) + \eta \rho \Delta \vec{v} + \vec{f};$$

(1)

where \( \rho \) is the density of the liquid, \( \vec{v} \) is the velocity, \( p \) is the pressure, \( \eta \) is the kinematic viscosity coefficient and \( \vec{f} \) is the volume density of the Ampere’s force, which can be write as:

$$\vec{f} = [j, \vec{B}]$$

(2)

where \( j \) is the electrical current density and \( \vec{B} \) is the magnetic field.

For the hemispherical container [9], where the current \( I \) passes from the inner hemispherical electrode of radius \( a \), the current density is as follows (we solve the equations in spherical coordinates \( r – \theta – \varphi \)):

$$j = \frac{I}{2\pi r^2} \vec{e}_r.$$

(3)

The magnetic field can be obtained as a solution of the Maxwell’s equation:

$$\text{curl}(\vec{B}) = \mu_0 \vec{j};$$

(4)

$$\text{curl}(\vec{B}) = \frac{\mu_0 I}{2\pi r^2} \vec{e}_r.$$

(5)

In the axisymmetric case:

$$\frac{1}{r \sin \theta} \left( \frac{\partial (B_\varphi \sin \theta)}{\partial \theta} \right) = \frac{I}{2\pi r^2}.$$  

(6)

The magnetic field will be the following [10]:

$$B_\varphi = \frac{\mu_0 I (1 - \cos \theta)}{2\pi \sin \theta};$$

(7)

and for the force we have [9]:

$$\vec{f} = -\frac{\mu_0 I^2 (1 - \cos \theta)}{4\pi r^3 \sin \theta} \vec{e}_\theta.$$  

(8)

It is quite convenient to use the vorticity function \( \omega = \text{curl}(\vec{v}) \).

We can also take the vector potential \( \vec{\psi} \) as:

$$\vec{v} = \text{curl}(\vec{\psi}).$$

If we take the curl of both parts of the Navier – Stokes equation, it can be represented in variables “vector potential – vorticity” [11]:

$$\frac{\partial}{\partial t} \vec{\omega} + (\text{div} \vec{v}) \vec{\omega} + (\vec{v}, \text{grad}) \vec{\omega} - \vec{\omega} \text{grad} \vec{v} = \eta \rho \Delta \vec{\omega} + \text{curl} \vec{f}.$$

(9)

For incompressible fluid, \( \text{div} \vec{v} = 0 \). Also, it is interesting to describe the stationary solutions, which do not depend on time. So the equation becomes simpler:

$$\text{curl} \vec{\psi} \vec{\omega} - (\vec{\omega}, \text{grad}) \text{curl} \vec{\psi} = \eta \rho \Delta \vec{\omega} + \text{curl} \vec{f}.$$

(10)
\[ \psi = \psi \varphi. \] (11)

So we can neglect the second term in the left part of equation
\[ \left( \text{curl} \psi, \nabla \right) \phi = \eta_\rho \Delta \phi + \text{curl} \phi; \] (12)
\[ \left( \text{curl} \phi, \nabla \phi \right) = \eta_\rho \Delta \phi + \frac{\mu_0 I^2 (1 - \cos \theta)}{2 \pi^2 r^4 \sin \theta} \varphi. \]

Using dimensionless units (the distances are measured in radius \( b \) of the outer electrode, velocities are measured in units \( \frac{I}{b \rho} \)), we can rewrite the equations in the following form [9]:
\[ 2\pi^2 A \left( \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \cot \theta \right) \frac{\partial \omega}{\partial r} - \left( \frac{\psi}{r} + \frac{\partial \omega}{\partial r} \right) \frac{1}{r} \frac{\partial \omega}{\partial \theta} = \Delta \omega - \frac{\omega}{r^2 \sin^2 \theta} + \frac{A (1 - \cos \theta)}{r^4 \sin \theta}; \] (13)
\[ \Delta \psi - \frac{\psi}{r^2 \sin \theta} = -\omega, \] (14)

where \( A = \frac{I}{2 \pi^2 \eta \rho} \) is the parameter characterizing the current.

As for the boundary condition, the simplest case is connected with the Dirichlet problem:
\[ \omega \bigg|_{r=a} = \omega \bigg|_{\theta=\pi/2} = \psi \bigg|_{r=a} = \psi \bigg|_{\theta=\pi/2} = 0. \] (15)

We solve the equations numerically, and obtain the solutions, which can be associated with the electro-vortex flow. To obtain the solutions, we use pseudo-transient method [12], which is quite popular for parabolic problem. We take 100x95 grid points, using \( \Delta r = 0.01 \) and \( \Delta \theta = 0.0157 \). For time we take the typical step as \( 7 \times 10^{-7} \), which is quite sufficient for the stability of the scheme (it is necessary to have \( \Delta t < \min\{\Delta r^2, \Delta \theta^2\} \)). As for the approximation of the solution, it has the order of \( O(\Delta t + \Delta r^2 + \Delta \theta^2) \), so the typical error is of order of \( 10^{-4} \). The most interesting thing is to study the dependence of the velocities on the parameter \( A \). In figure 1, we show the ratio between the velocity on the axis and this parameter. It can be seen that the velocity grows nonlinearly: for large \( A \) the nonlinear terms become sufficient, and the field growth saturates.

From the physical point of view, the Thom’s boundary conditions [13] are more realistic and close to a solid wall:
\[ \omega \bigg|_{r=a} - \frac{2}{h^2} \psi \bigg|_{r=a+h} = \omega \bigg|_{r=a} - \frac{2}{h^2} \psi \bigg|_{r=a-h} = \psi \bigg|_{r=\pi/2} - \frac{2}{r^2 \theta} \psi \bigg|_{\theta=\pi/2 - \theta} = \psi \bigg|_{r=a} = \psi \bigg|_{\theta=\pi/2} = 0. \] (16)

3. Linear case

Analyzing the results presented in figures 1 – 2, we can conclude that for large values of the parameter \( A \) the velocity grows nonlinearly. However, for small values we can suppose that the parameter \( A \) will be small and we can assume that the left part of the equation (13) is smaller than the right one.

If we neglect the nonlinear terms in the equations, as it is done in the Stokes approximation, the equations will become the following:
\[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \omega}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \omega}{\partial \theta} \right) - \frac{\omega}{\sin \theta} = -\frac{A (1 - \cos \theta)}{r^2 \sin \theta}; \] (17)
The solution for different boundary conditions for \( A=1 \) is shown in figure 3. It can be seen that the difference between the linear and nonlinear solutions is quite small. It is important to say that for this case it is possible to construct the analytical solution [11]. Also we can say that for \( A<1 \) we can take the linear parts of the equations and describe the processes using Stokes approximation.

4. Conclusion
We have studied the electro-vortex flow using the Stokes’ approximation and the nonlinear model. The equations were solved using the pseudo-transient method. We can see that for small values of the parameter \( A \) the velocity grows proportionally to this parameter and the solution of the equation is quite close to the linearized problem. So for \( A<1 \), we can take the Stokes approximation which does not take into account the convective terms of the equations. Such equations have been used in previous works, such as in [9] where the authors took the Reynolds number, which is proportional to our parameter \( A \) with a factor of \( 2\pi^2 \).

From the physical point of view, we can conclude that if we take small currents of several \( A \), the convection is not very important and we can take into account only the main part, which is connected with the viscosity. It lets us take the simpler equations, which can be solved using the numerical schemes, which are not very complicated. Also for the linear problem, we can construct the analytical solution, which can be constructed as a sum of eigenfunctions [11].

\[
\frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - \frac{\psi}{\sin \theta} = -\omega r^2;
\]
Figure 2. Normalized velocity for different A (nonlinear case, \(a=0.05\), Thom’s boundary conditions). Dashed line shows \(A=1\), solid line shows \(A=10\), dot-dashed line – \(A=100\).

Figure 3. Velocity for \(A=1\), \(a=0.05\). Dashed line shows linear case and Dirichlet boundary conditions, solid line shows linear case and Thom’s boundary condition (nonlinear is the same), and dot-dashed line shows nonlinear case and Dirichlet boundary condition.

References
[1] Sokoloff D D, Stepanov R A, Frick P G 2014 Phys. Usp. 57 313-35.
[2] Sozou C, Pickering W M 1976 J.Fluid Mech. 73 4 641.
[3] Sozou C, Pickering W M 1978 Proc. R. Soc. Lond. A. 362 509.
[4] Boyarevich V, Freibergs Ja, Shilova E I, Shcherbinin E V 1989. Electrically induced vertical flows (Kluwer: Dordrecht)
[5] Ivochkin Yu P, Teplyakov I O, Vinogradov D A 2016 Magnetohydrodynamics 11 1-2 11.
[6] Yachikov I M, Portnova I V, Larina T P. 2018 Steel in Translation 48 1-6.
[7] Khripchenko S Yu Elektrovikhrevie techeniya v kanalakh MGD-ustroystv (Yekaterinburg: UrO RAN) [in Russian]
[8] Karimi-Sibaki E, Kharicha A, Wu M, Ludwig A, Bohacek J 2018 Journal of the Electrochemical Society 165 E604-15.
[9] Shatrov V, Gerbeth G 2012 Magnetohydrodynamics 48 469 – 84.
[10] Mikhailov E A, Teplyakov I O 2017 J.Phys.: Conf.Ser. 891 012060.
[11] Mikhailov E A, Teplyakov I O 2018 Moscow University Physics Bulletin 73 2 162-7.
[12] Kalitkin N N 1978 Chislennie metody [Numerical methods] (Moscow: Nauka)
[13] Weinan E, Liu J G 1996 Journal of Computational Physics 124 368 - 82.