The invariant-comb approach and its relation to the balancedness of multipartite entangled states

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Abstract. The invariant-comb approach is a method to construct entanglement measures for multipartite systems of qubits. The essential step is the construction of an antilinear operator that we call \textit{comb} in reference to the \textit{hairy-ball theorem}. An appealing feature of this approach is that, for qubits (or spins 1/2), the combs are automatically invariant under \text{SL}(2, \mathbb{C})\textsuperscript{1}, which implies that the obtained invariants are entanglement monotones by construction. By asking which property of a state determines whether or not it is detected by a polynomial \text{SL}(2, \mathbb{C})\textsuperscript{1} invariant, we find that it is the presence of a \textit{balanced part} that persists under local unitary transformations. We present a detailed analysis for the maximally entangled states detected by such polynomial invariants, which leads to the concept of \textit{irreducibly balanced} states. The latter indicates a tight connection with stochastic local operations and classical communication (SLOCC) classifications of qubit entanglement. Combs may also help to define measures for multipartite entanglement of higher-dimensional subsystems. However, for higher spins there are many independent combs, such that it is nontrivial to find an invariant one. By restricting the allowed local operations to rotations of the coordinate system (i.e. again to the \text{SL}(2, \mathbb{C})\textsuperscript{1}), we manage to define a unique extension of the concurrence to general half-integer spin with an analytic convex-roof expression for mixed states.

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1. Introduction

Entanglement is one of the most counterintuitive features of quantum mechanics [1], of which there is only rather incomplete knowledge to date. We will define a quantum mechanical state of distinguishable particles as having no global entanglement with respect to a given partition \( \mathcal{P} \) of the system if and only if (iff) it can be written as a tensor product of the parts of some subpartition of it. A state of indistinguishable particles we call not globally entangled iff it can be written as the proper symmetrization due to the particles’ statistics of such a tensor product [2]. The many different ways of partitioning a physical system already imply that there are many families of entanglement in multipartite systems or even bipartite systems with many inner degrees of freedom. The concept of entanglement instead remains unaltered. Having agreed how to decompose the physical system such that every quantum state can be expressed as a superposition of tensor products of states of its parts, the entanglement of its components follows the definition given above.

In order to be more specific, let \( \mathcal{H}_i \) be the \( i \)th local Hilbert space of some partition of the total Hilbert space \( \mathcal{H} = \bigotimes_{i \in I} \mathcal{H}_i \). In this case, the partition would be \( \mathcal{P} := \{ \mathcal{H}_i : i \in I \} \); if \( I_1 \cup I_2 = I \), then \( \mathcal{P}_{\text{sub}} := \{ \bigotimes_{i \in I_1} \mathcal{H}_i, \bigotimes_{i \in I_2} \mathcal{H}_i \} \) is a two-elemental subpartition of \( \mathcal{P} \). We call an operator on \( \mathcal{H} \) \( \mathcal{P} \)-local iff it is a tensor product with respect to the partition \( \mathcal{P} \). When it is clear from the context what the partition is, we will just use the term local.

While for two qubits there is only one type of entanglement, it has been noted rather early that, starting from the three-qubit case, there is more than one class of entanglement [3]. That is, for more than two parties there are different classes of states that are not interconvertible using only stochastic local operations and classical communication (SLOCC) [3]–[5]. Due to this complication, a key question that (despite considerable effort) has not been answered yet in the frame of a general theory is (see, e.g., [3], [6]–[20]) how to classify, to detect and to quantify multipartite pure-state entanglement in a sensible and physically justified way. Nice overviews

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of the state of the art are given in [21, 22]. Further, an initial analysis of systems of two and three qubits can be found in [13, 23], and the entanglement sharing properties for qubits have been studied in [24]. An interesting account of activities with respect to higher local dimension is given in [25]. Several collective multipartite measures for entanglement have been proposed for pure states [26]–[32]. Since these approaches have no control over how various classes of entanglement will be weighted in such a measure, the decision in favor of one specific collective entanglement measure is arbitrary unless we gain a significantly better understanding of the structure of entanglement itself.

In order to get additional insight, class-specific entanglement measures provide one research line to pursue. As an example of such a measure for three qubits, the three-tangle has been derived [33] as the unique measure that discriminates the two distinct classes of entanglement in three-qubit systems. It separates W states from the genuine class of three-qubit entanglement represented by the Greenberger–Horne–Zeilinger (GHZ) state. As well as its two-qubit counterpart, the concurrence [34]–[36], the three-tangle is a polynomial $SL(2, C)$ invariant. A procedure for the construction of similar class-specific entanglement quantifiers has been developed in [37, 38]. It has been systematically analyzed for four- and five-qubit systems in [18].

These measures combine a variety of desirable properties of class-specific quantifiers of global entanglement. Given a specific partition $\mathcal{P}$, a measure for genuine $n$-tangle $E_g: \mathcal{H} \rightarrow [0, 1]$ should satisfy the following requirements:

(i) $E_g[\Pi] = 0$ for all pure product states $\Pi$, relative to the partition $\mathcal{P}$,
(ii) invariance under $\mathcal{P}$-local unitary transformations,
(iii) the entanglement monotone property [4], i.e. the measure must not increase (on average) under SLOCC [5] and
(iv) invariance under permutations of the $\mathcal{P}$-local Hilbert spaces [33].

Clearly, the most basic requirement appears to be condition (i). This is accomplished by what has been termed a filter in [18, 37, 38], that is, an operator that ‘filters out’ all product states, in the sense that it has zero expectation value for them. In other words, the filter image of any product state must be orthogonal to the original product state. The filters are built from so-called comb operators for one- and two-copy single-qubit states (see below).

This approach appears appealing since, interestingly, for qubits it automatically implies $SL$ invariance and thus the monotone property [12]. Consequently, also (ii) and (iii) are satisfied. Nevertheless, already for five qubits there is a large number of such measures (even after imposing condition (iv) [18]) such that one would like to understand their essence more deeply and to reduce their number on physical grounds.

In this paper, we analyze what is common to the states that are detected by the $SL(2, C)$ invariant operators. We find in general that only the balanced part of a state is measured, a property that has already been noted for the three-qubit case in [33], in terms of a geometrical interpretation. On the other hand, it is known that the modulus of a polynomial invariant assumes its maximum on the set of stochastic states [12]. The combination of these characteristics leads us to the concept of irreducibly balanced states. (We would like to mention that Klyachko et al have discussed maximum multipartite entanglement and its relation to quantification of multipartite entanglement from a different, very interesting perspective (see [20, 39] and references therein.).) We study various interesting properties of irreducibly balanced states as we
are convinced that their investigation might give further insight into the nature of multipartite entanglement.

Because the basis of all this discussion is the invariant-comb approach, it is interesting to ask whether there is any possibility to extend the method to systems with a local Hilbert space dimension larger than 2. We discuss some basic aspects of such an extension.

The structure of the paper is as follows. The invariant-comb approach for qubits [37, 38] is summarized in section 2. Section 2.1 introduces the main concepts, notations and the elements that eventually build up the (filter) invariants. In section 2.2, we exemplarily write down filters for up to six qubits and discuss some elementary properties of them. Section 2.3 is devoted to the notion of states with maximal n-qubit entanglement. Interestingly, a central prerequisite of such maximally entangled states can be connected with the concept of balancedness in section 3. In section 3.1, we show that it is precisely the balanced part that is detected by the SL(2, C) invariant (filter) operators. In section 3.2, this leads us to the definition of irreducibly balanced states and the investigation of their properties. Finally, we discuss the possibilities for an application of the invariant-comb approach to non-qubit systems and general partitions in section 4. Firstly, we focus on the entanglement of blocks of qubits (section 4.1). Then, in section 4.2, a measure for bipartite entanglement, subject to a certain restricted class of local operations, is derived for general half-integer spin pure and mixed states. In the last section, we present our conclusions.

2. Polynomial SL(2, C) invariants for multipartite entangled states—the invariant comb approach

2.1. The concept of combs and filters

The fundamental concept of the method, and the basis of the construction of polynomial invariants, is the comb. A comb A is a local antilinear operator with zero expectation value for all states of the local Hilbert spaces, that is,

\[ \langle \psi | A_i | \psi \rangle = \langle \psi | L_i \psi \rangle = \langle \psi | L_i | \psi^* \rangle = 0 \quad \text{on } \mathcal{H}_i, \]

(1)

where \( \mathcal{C} \) is the complex conjugation in the computational basis

\[ | \psi^* \rangle := \mathcal{C} | \psi \rangle \equiv \mathcal{C} \sum_{j_1, \ldots, j_q = 1}^1 \psi_{j_1, \ldots, j_q} | j_1, \ldots, j_q \rangle \]

\[ = \sum_{j_1, \ldots, j_q = 1}^1 \psi_{j_1, \ldots, j_q}^* | j_1, \ldots, j_q \rangle. \]

We call \( L_i \) the linear operator associated with the comb \( A_i \). For simplicity, we abbreviate

\[ \langle \psi | L_i \psi \rangle =: \langle L_i \rangle \]

(2)

(throughout this paper, there will be no ambiguities whether we mean linear or antilinear expectation values). The requirement to have vanishing expectation values for an arbitrary single-qubit state clearly cannot be accomplished by any linear operator (it would be identically zero), but it is amenable to antilinear operators. The idea is to identify a sufficiently large set of combs in order to construct the desired filter operators that satisfy all the requirements (i)–(iv) listed above. It is worth noting that a filter constructed exclusively from combs is automatically
In particular, $\sigma$ from the corresponding unitary group, which is isomorphic to the special linear group. Since the latter mathematically represents the non-projective LOCC operations, the corresponding special linear group. Since the latter mathematically represents the non-projective LOCC operations, we can restrict ourselves to $SU(2)$ operations—authors’ remark). Then $M(|\psi\rangle \langle \psi|)$ is an entanglement monotone. (For the proof, cf [12].) It is evident that any polynomial invariant can be turned into a linearly homogeneous function of $|\psi\rangle \langle \psi|$ by applying the appropriate inverse power. In order to avoid misunderstandings, we emphasize that not every function of SL-invariants is a comb acting on a single qubit. We do not know whether combs acting on more than a single site might be needed to some extent. As yet there is no evidence that they need to be included in order to classify multipartite qubit entanglement. In what follows, comb operators are to be understood as acting on a single site only.

The main part of this work focuses on multipartite registers of $n$ qubits, i.e. $n$ spins $1/2$. Then, the local Hilbert space is $H_i = \mathbb{C}^2 = \mathfrak{h}$ for all $i \in \{1, \ldots, n\}$. We will need the Pauli matrices

$$\sigma_0 := \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(3)

Since here the local unitary group is $SU(2)^\otimes n$, we only need fully local combs and hence we can restrict ourselves to $SU(2)$ combs. We mention that any tensor product $f([\sigma_{\mu}]) := \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$ with an odd number $N_y$ of $\sigma_y$ is an $n$-qubit comb. This can be seen immediately from

$$\langle f ([\sigma_{\mu}]) \rangle = \langle \psi | \, f ([\sigma_{\mu}]) \, | \psi^* \rangle = \left( \langle \psi^* | \, f ([\sigma_{\mu}^*]) \, | \psi \rangle \right)^* = (-1)^{N_y} \langle \psi | \, f ([\sigma_{\mu}^*])^\dagger \, | \psi^* \rangle = (-1)^{N_y} \langle f ([\sigma_{\mu}]) \rangle. $$

In particular,

$$A_{1/2}^{(1)} := \sigma_y \mathcal{C}$$  

(4)

is a comb acting on a single qubit. We do not know whether combs acting on more than a single site might be needed to some extent. As yet there is no evidence that they need to be included in order to classify multipartite qubit entanglement. In what follows, comb operators are to be understood as acting on a single site only.

We will call $A^{(1)} : \mathfrak{h} \rightarrow \mathfrak{h}$ a comb of order 1. In general, we will call a (single-qubit) comb $A^{(n)} : \mathfrak{h}^{\otimes n} \rightarrow \mathfrak{h}^{\otimes n}$ to be of order $n$. It is worth noting that the $n$-fold tensor product $\mathfrak{h}^{\otimes n}$ on which an $n$th order comb acts symbolizes $n$-fold copies of one single-qubit state. In order to distinguish this merely technical introduction of a tensor product of copies of states from the physically motivated tensor product of different qubits, we will denote the tensor product of copies with the symbol $\bullet$, and hence write $A^{(n)} : \mathfrak{h}^{\bullet n} \rightarrow \mathfrak{h}^{\bullet n}$. When we say expectation value of $A^{(n)} = L^{(n)} \mathcal{C}$, we mean

$$\langle \psi | \, A^{(n)} \, | \psi^\bullet \rangle := \langle L^{(n)} \rangle.$$  

(5)

Strictly speaking, this is a linear combination of products of expectation values. If

$$A^{(n)} = \sum_j a_{j_1, \ldots, j_n} \sigma_{j_1} \bullet \cdots \bullet \sigma_{j_n} \mathcal{C},$$

we mean

$$\langle \psi | \, A^{(n)} \, | \psi^\bullet \rangle := \langle L^{(n)} \rangle.$$  

(5)
then the expectation value of $A^{(n)}$ would be

$$\langle L^{(n)} \rangle = \sum_j a_j \prod_{k=1}^n \langle \psi | \sigma^j_k | \psi^* \rangle = \sum_j a_j \prod_{k=1}^n (\langle \sigma^j_k \rangle).$$

There is a unique (up to rescaling) single-site comb of order 2, which is orthogonal to the trivial one $\sigma_y \bullet \sigma_y$ (with respect to the Hilbert–Schmidt scalar product). One can verify that, for an arbitrary single-qubit state,

$$0 = \langle \sigma_\mu \rangle \langle \sigma^\mu \rangle = \langle \psi | \sigma_\mu C | \psi \rangle \langle \psi | \sigma^\mu C | \psi \rangle := \sum_{\mu, \nu = 0}^3 \langle \psi | \sigma_\mu C | \psi \rangle g^{\mu, \nu} \langle \psi | \sigma_\nu C | \psi \rangle,$$

with

$$g^{\mu, \nu} = g_{\mu, \nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad g^{0,0} = -1$$

(note the similarity to the Minkowski metric). We will denote this second-order comb by

$$A^{(2)}_{1/2} := \sigma_\mu \bullet \sigma^\mu C = \sum_{\mu=0}^3 g^{\mu, \nu} \sigma_\mu \bullet \sigma_\nu C. \quad (6)$$

For both combs one can demonstrate that they are $\text{SL}(2, \mathbb{C})$ invariant [18, 38], and hence they satisfy the basic requirements for the construction of filter operators. Please note that any linear combination of combs is again a comb. The two combs $A^{(1)}_{1/2} \bullet A^{(1)}_{1/2}$ and $A^{(2)}_{1/2}$ have the above-mentioned additional important property of being mutually orthogonal with respect to the Hilbert–Schmidt scalar product. Filter invariants for $n$ qubits are obtained as antilinear expectation values of filter operators. The latter are constructed from combs so as to have vanishing expectation value for arbitrary product states. We will use the word ‘filter’ for both the filter operator and its filter invariant. For $n$-qubit filters we will use the symbol $\mathcal{F}^{(n)}$.

We start by writing down some filters for two qubits:

$$\mathcal{F}^{(2)}_1 = \langle (\sigma_y \otimes \sigma_y) \rangle, \quad \text{(7)}$$

$$\mathcal{F}^{(2)}_2 = \frac{1}{2} \langle ([\sigma_\mu \otimes \sigma_\nu] \bullet [\sigma^\mu \otimes \sigma^\nu]) \rangle =: \frac{1}{2} \langle (\sigma_\mu \sigma_\nu \bullet \sigma^\mu \sigma^\nu) \rangle. \quad \text{(8)}$$

Both forms are explicitly permutation invariant, and they are filters. Indeed, if the state is a product, the combs lead to a vanishing expectation value. We obtain the pure state concurrence from them in two different, equivalent forms:

$$C = \left| \mathcal{F}^{(2)}_1 \right|, \quad \text{(9)}$$

$$C^2 = \left| \mathcal{F}^{(2)}_2 \right|. \quad \text{(10)}$$

Now we go ahead to three qubits and write down a selection of three-qubit filters,

$$\mathcal{F}^{(3)}_1 = \langle (\sigma_\mu \sigma_\nu \sigma_y \bullet \sigma^\mu \sigma_y \sigma_y) \rangle, \quad \text{(11)}$$

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\[ \mathcal{F}_2^{(3)} = \frac{1}{3}((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda)), \quad (12) \]

\[ \mathcal{F}_3^{(3)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\gamma \sigma_\delta \sigma_\epsilon \cdot \sigma_\gamma \sigma_\delta \sigma_\epsilon \cdot \sigma_\gamma \sigma_\delta \sigma_\epsilon \cdot \sigma_\gamma \sigma_\delta \sigma_\epsilon)), \quad (13) \]

The last two are evidently permutation invariant, but also the first filter is invariant under permutations. All coincide with the three-tangle [33] (or powers thereof)

\[ \tau_3 = |\mathcal{F}_1^{(3)}| = |\mathcal{F}_2^{(3)}|, \quad (14) \]

\[ \tau_3^2 = |\mathcal{F}_3^{(3)}|. \quad (15) \]

Interestingly, all two-qubit filters are homogeneous polynomials of the concurrence and in the same way all three-qubit filters coincide with polynomials of the three-tangle. This is due to the fact that both concurrence and three-tangle generate the whole algebra of polynomial \( SL(2, \mathbb{C}) \otimes^2 \) and \( SL(2, \mathbb{C}) \otimes^3 \) invariants, respectively (see, e.g., [18] and references therein).

### 2.2. Filters for four and more qubits

In this section, we will present explicitly a list of filters for systems of four and five qubits. By means of the six-qubit example, we sketch a straightforward procedure to construct filters for larger systems. In order to get compact formulae, the tensor product symbol \( \otimes \) will be omitted, as in equation (8).

For four qubits, the whole filter ideal in the ring of polynomial SL invariants is generated by [18]

\[ \mathcal{F}_{1}^{(4)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi)), \quad (16) \]

\[ \mathcal{F}_{2}^{(4)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi), \quad (17) \]

\[ \mathcal{F}_{3}^{(4)} = \frac{1}{2}(((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi). \quad (18) \]

For five qubits, we mention

\[ \mathcal{F}_{8:1}^{(5)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi), \quad (19) \]

\[ \mathcal{F}_{8:2}^{(5)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi), \quad (20) \]

\[ \mathcal{F}_{8:3}^{(5)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi), \quad (21) \]

\[ \mathcal{F}_{12:1}^{(5)} = (((\sigma_\mu \sigma_\nu \sigma_\lambda \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \cdot \sigma_\sigma \sigma \sigma_\xi \cdot \sigma_\sigma \sigma \sigma_\xi), \quad (22) \]
Construct filters for higher qubit numbers: filter invariants for five qubits up to polynomial degree 12. Fan antisymmetric invariant\(\langle\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\rangle\) as defined in [18]. Together with\(^6 P^2 - \sum_{j=1}^5 D_j^3\) and the square of an antisymmetric invariant \(F\) that is constructed from a \(\Omega\)-process (see [40]), they generate the filter invariants for five qubits up to polynomial degree 12.

The following examples for six-qubit filters provide a way to construct filters for higher qubit numbers:

\[
\mathcal{F}_{12;2}^{(5)} = \left\{ \langle \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \rangle \right\}
\]

\[
\mathcal{F}_{12;3}^{(5)} = \left\{ \langle \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \rangle \right\}
\]

\[
\mathcal{F}_{12;4}^{(5)} = \left\{ \langle \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \rangle \right\}
\]

\[
\mathcal{F}_{12}^{(6)} = \left\{ \langle \sigma_{\mu_1} \sigma_{\mu_2} \sigma_3 \sigma_3 \sigma_3 \rangle \right\}
\]

\[
\mathcal{F}_{i}^{(6)} = \left\{ \langle \sigma_{\mu_1} \sigma_{\mu_2} \sigma_3 \sigma_3 \sigma_3 \rangle \right\}
\]

where in the latter formula all the \(\mu\) are to be contracted properly. In the \(\sigma\) the ‘\(\bullet\)’ have to be substituted either by \(\sigma_j\) or by indices, which then have to be contracted properly. This suggests that, for an \(n\)-qubit system, the filter property requires at least \(6^{n(n-1)}\), leading to a polynomial degree of at least \(2(n-1)\) for the corresponding polynomial invariant.

We emphasize again that every filter is an SL invariant because the local elements it is constructed from (i.e. the combs) are SL invariant. It is clear that linear combinations and, in fact, any function of invariants is an invariant (but not necessarily an entanglement monotone, cf [12]). By noting the consequences of including global phases of the states, we see that only homogeneous functions of the same degree can be combined linearly.

\(^6\) Here, \(P = \sum_j D_j \) and \(D_j\) are those polynomial SL invariants of degree 4 with a single \(\sigma_{\mu} \bullet \sigma^\mu\) contraction on qubit number \(j\) as defined in [18], e.g. \(D_1 = (\sigma_1 \sigma_2 \sigma_2 \sigma_2 \bullet \sigma_1 \sigma_2 \sigma_2 \sigma_2)\); they coincide with some invariants proposed in [27].
2.3. Maximally entangled states

We will now define our notion of a multipartite state with maximal genuine multipartite entanglement.

**Definition 2.1.** A pure $q$-qubit state $|\psi_q\rangle$ has maximal multipartite entanglement, i.e. $q$-tangle, iff

1. the state is not a product, i.e. the minimal rank of any reduced density matrix of $|\psi_q\rangle$ is 2;
2. all reduced density matrices of $|\psi_q\rangle$ with rank 2 (this includes all $(q-1)$-site and single-site ones) are maximally mixed within their range.

Further, there is a list of desirable features for maximally multipartite entangled states:

1. All $p$-site reduced density matrices of $|\psi_q\rangle$ have zero $p$-tangle; $1 < p < q$.

This is clearly an implicit requirement in that a check of it would require the knowledge of convex-roof extensions of the relevant multipartite entanglement measures. Furthermore, it is not even clear which $q$-qubit entanglement families possess a representative for which all tangles for less than $q$ qubits vanish. Note that the $q$-qubit GHZ state is an example that satisfies condition (ii).

1. There is a canonical form of any maximally $q$-tangled state, for which properties (ia) and (ib) are unaffected by relative phases in the amplitudes, i.e. their quality of being maximally entangled is phase insensitive.

All the above requirements are invariant under local SL transformations. We briefly discuss the implications of each single requirement. Condition (ia) excludes product states, which are certainly not even globally $q$-tangled, and (ib) implies maximal gain of information when a bit of information is read out. This condition contains the definition of stochastic states in [12], where it is also proved that every entanglement monotone assumes its maximum on the set of stochastic states. An even more stringent condition has been imposed in [41], where all reduced density matrices are required to be maximally mixed within their range. Requirements (ia) and (ib) are therefore well established. Constraint (ii) is intriguing by itself: it excludes hybrids of various types of entanglement and thereby follows the idea of entanglement as a resource whose total amount has to be distributed among the possibly different types of entanglement (see, e.g., [33, 42]). To our knowledge, it is not clear whether this condition can be regarded as being fundamental, since to date no extended monogamy relation is found that would substantiate the idea of entanglement distribution (see, e.g., [42, 43]). We have no striking argument in favor of (iii), except that maximally entangled states for two and three qubits have such a canonical form. We mention, however, that according to [3], entangled states have a representation with a minimal number of product components. It appears that in this representation the entanglement is not ‘sensitive’ to changes in the relative phases between the components (consider, e.g., the GHZ state). It could be promising to analyze a possible connection to the concept of envariance put forward in [44].

In order to illustrate the above conditions and to check the existence of such states, we give some examples. The Bell states $(|\sigma, \sigma'\rangle \pm |\bar{\sigma}, \bar{\sigma}'\rangle)/\sqrt{2}$ are the canonical form of maximally two-tangled states. By tracing out one qubit, one obtains $\frac{1}{2} \mathbb{1}$ as the reduced density matrix of the remaining qubit. The two-tangle is indeed robust against multiplication of the components.
with arbitrary phases: \((|\sigma, \sigma'| + e^{i\phi} |\bar{\sigma}, \bar{\sigma}'|)/\sqrt{2}\) is maximally entangled for arbitrary real \(\phi\). Condition (ii) is meaningless here. For two qubits, these are all maximally entangled states, and the class of maximally entangled states is represented by \(|\text{GHZ}_2\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle)\), which is like the GHZ state but for two qubits. Also, the generalized GHZ state for \(q\) qubits, \(\frac{1}{\sqrt{2}}(|11\ldots 1\rangle + |00\ldots 0\rangle)\), satisfies all the above requirements. It is straightforward to see that the GHZ state is detected by every filter constructed in the way described in the preceding sections. Having a pure state of three qubits, there are two other classes of entangled states: the class represented by \(|\text{GHZ}_3\rangle\), which thus have zero concurrence. However, this state is unitarily equivalent to a GHZ state by the transformation \(H_2 \otimes H_2 \otimes H_2\), where \(H_2\) is the Hadamard transformation \(\frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle)\).

An apparently different class of maximally three-tangled states instead can be read off directly from the coordinate expression for the three-tangle [33]. Its representative is

\[
|X_3\rangle = \frac{1}{\sqrt{2}}(|111\rangle + |100\rangle + |010\rangle + |001\rangle) \tag{28}
\]

and satisfies all the above conditions for a maximally three-tangled state. It is interesting to note that all its reduced two-site density matrices are an equal mixture of two orthogonal Bell states, which thus have zero concurrence. However, this state is unitarily equivalent to a GHZ state by the transformation \(H_2 \otimes H_2 \otimes H_2\), where \(H_2\) is the Hadamard transformation \(\frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle)\).

Summarizing the above examples, we conclude that the set of states that satisfy the conditions in definition (2.1) is not empty for any number \(q\) of qubits and we have one two-tangled and three-tangled representatives (actually two, which are equivalent for three qubits, though). In what follows, we analyze the above conditions and prove that there are at least \(q - 1\) inequivalent \(q\)-tangled representatives.

3. Polynomial \(\text{SL}(2,\mathbb{C})\) invariants and maximally entangled states

3.1. The states measured by filters

In order to understand what the filters do, it is convenient to consider subnormalized states

\[
|\psi\rangle =: \sum_{j=0}^{1} \sum_{(k,\bar{k})} \psi_{j,k,j} |j\rangle \otimes |k,j\rangle =: \sum_{j=0}^{1} |j\rangle \otimes |\phi_j\rangle
\]

and then to study the action of the combs on this state. Defining \(\tilde{0} := 1\) and \(\tilde{1} := 0\), we obtain

\[
\langle \psi | \sigma_2 \otimes . |\psi^*\rangle = \sum_{j} (-1)^{i+1} i \psi_{j,k,j} |j\rangle |\phi_j\rangle \langle \psi | \sigma_1 \otimes . |\psi^*\rangle = \sum_{i,j} |\phi_j\rangle |\phi^*_i\rangle \langle \phi_i | . |\phi^*_j\rangle
\]

for the first-order comb. The second-order comb is the sum of the following outcomes:

\[
- \langle \psi | \sigma_0 \otimes . |\psi^*\rangle \langle \psi | \sigma_0 \otimes . |\psi^*\rangle = - \sum_{i,j} \langle \phi_j | . |\phi^*_i\rangle \langle \phi_i | . |\phi^*_j\rangle,
\]

\[
\langle \psi | \sigma_1 \otimes . |\psi^*\rangle \langle \psi | \sigma_1 \otimes . |\psi^*\rangle = \sum_{i,j} \langle \phi_j | . |\phi^*_i\rangle \langle \phi_i | . |\phi^*_j\rangle,
\]

\[
\langle \psi | \sigma_3 \otimes . |\psi^*\rangle \langle \psi | \sigma_3 \otimes . |\psi^*\rangle = \sum_{i,j} (-1)^{|i+j|} \langle \phi_j | . |\phi^*_i\rangle \langle \phi_i | . |\phi^*_j\rangle.
\]
The remaining indices of the quantum state are kept fixed for the moment. Summing up these three terms and performing the sum over \( j \), we obtain

\[
\sum_i \left[ \langle \phi_i | . | \phi_i^* \rangle + \langle \phi_i | . | \phi_i^* \rangle \right] - 2 \langle \phi_i | . | \phi_i^* \rangle \langle \phi_i | . | \phi_i^* \rangle .
\]

(30)

It is seen from this result that, in order to give a nonzero outcome, every component \( i \) must come with the flipped component \( \bar{i} \). This is what we will call a balanced qubit component. Since the above consideration has to be extended to all qubits, we conclude that the filter has contributions only from the balanced part of a state. We can say even more: the homogeneous degree of the filter must fit the length of the balanced parts in the state, i.e. the number of product states in the computational basis needed for this balanced part. As a consequence, the way the filter is constructed already implies valuable information about which type of state the filter can possibly detect. This further underpins the relevance of polynomial SL(2, \( \mathbb{C}^{\otimes q} \)) invariants as far as entanglement classification and quantification of multipartite qubit states are concerned. In particular, we see that a state that can be locally transformed into a normal form without balanced part has zero expectation value for all filter operators (the \(|W\rangle\) state is a prominent example).

This analysis suggests in-depth investigation of states with balanced parts in their pure-state decomposition into the computational (product) basis. It is worth noting that it is not conclusive to look at some given pure state and see whether it has a balanced part or not. In fact, every pure state has a balanced part after a suitable choice of local basis. The concept becomes useful only modulo local unitary transformations. Then, two qualitatively different classes of states occur:

- states that are unitarily equivalent to a form without a balanced part and
- states for which arbitrary local unitaries lead to a state with a balanced part.

The latter case naturally splits up into two sub-classes. One is characterized as the reducibly balanced case in the sense that distinct smaller balanced parts always exist. The complementary situation is the irreducibly balanced case.

It is clear that maximal entanglement as measured by some polynomial SL invariant is achieved when no unbalanced residue is present, i.e. when the state is balanced as a whole. Indeed, we will show that stochasticity of a state implies balancedness. Against the background of the finding in [12] that every entanglement monotone assumes its maximum on the set of stochastic states, this underpins a tight connection between balanced states and the notion of maximal (multipartite) entanglement.

### 3.2. Irreducibly balanced states

To analyze the first two conditions (ia) and (ib) in definition 2.1, it is convenient to express a pure state \( \sum_i w_i |i\rangle \) as an array. The first row of this array contains the amplitudes \( w_i, p_i := |w_i|^2 \). The column below each amplitude is the binary sequence of the corresponding product basis state. For example,

\[
\frac{1}{2} (|111\rangle + |100\rangle + |010\rangle + |001\rangle) \leftrightarrow \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\]
For the moment, we will not pay much attention to the weights $p_i := |w_i|^2$; they will become important later on (cf theorem 3.3). In the following, we define two types of matrices that are based on this array representation of a state. It turns out that these matrices are quite helpful in the discussion of the properties of balanced states. The proofs of some of the theorems will be rather straightforward by using this representation.

**Definition 3.1 (alternating and binary matrix).** We call binary matrix $B_{|\psi\rangle}$ of the state $|\psi\rangle$ the matrix of binary sequences below the amplitude vector and equivalently we call alternating matrix $A_{|\psi\rangle}$ of the state $|\psi\rangle$ the matrix obtained from its binary matrix, when all zeros are replaced by $-1$. It will be useful to allow for multiple repetition of certain columns. This means of course that the alternating and the binary matrix will not be unique. The minimal form without repetitions is unique modulo permutations of the columns and qubits.

We define the length of a state as the minimal number of elements of the standard product basis that occur in the state (without repetition of columns), i.e. the number of columns of the minimal form.

In the above example we have

$$B_{|X_3\rangle} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \quad A_{|X_3\rangle} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and the length of this state is 4. $A_{|\psi\rangle}$ and $B_{|\psi\rangle}$ are $q \times L$ matrices, where $L$ is the number of basis states required for its representation.

**Definition 3.2 (irreducibly balanced states).**

1. We call a pure state $|\psi\rangle$ (entirely) balanced iff in each row of $B_{|\psi\rangle}$ there are as many ones as zeros (allowing for multiple occurrence of some of its columns), or equivalently, iff the elements of each row of $A_{|\psi\rangle}$ sum up to zero (including multiplicities as for the binary matrix). This can be expressed as

$$\exists n_1, \ldots, n_L \in \mathbb{N}(n_j > 0) \supseteq \sum_{j=1}^{L} n_j (A_{|\psi\rangle})_{ij} = 0 \forall i \in \{1, \ldots, q\}, \quad (31)$$

where $A_{|\psi\rangle} \in \mathbb{Z}^{q \times L}$, i.e. qubit number $q$ and length $L$.

We furthermore call a balanced state irreducible or irreducibly balanced, iff it cannot be split into different smaller balanced parts (i.e. iff there is no subset of less than $L$ columns that is already balanced).

In contrast, a balanced state that can be split into different smaller balanced parts will be called reducible.

2. We call a pure state $|\psi\rangle$ partly balanced (i.e. it has a balanced part) if (31) is satisfied for $n_j \in \mathbb{N}$ only if some $n_j = 0$ (but not all of them).

A partly balanced state is called reducible/irreducible iff its balanced part is reducible/irreducible.
As an example, we give the $B_{\{\psi\}}$ matrix for a reducibly balanced three-qubit state

$$B_{\text{reducible}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$ 

**Definition 3.3 (completely unbalanced states).** We call a state completely unbalanced if it is locally unitarily equivalent to a state without balanced part.

Please note that the maximally $q$-tangled states for $q = 2$ and $3$ are irreducibly balanced, and it can be straightforwardly verified that they are the only ones for these cases. The W-states are completely unbalanced. Further examples of completely unbalanced states are fully factorizing states. Therefore, it is clear that complete unbalancedness can occur for both globally entangled states and completely disentangled states; it is therefore an indicator only as far as genuine multipartite entanglement (i.e. $q$-qubit entanglement in $q$-qubit states) is concerned.

**Theorem 3.1.** Product states are not irreducibly balanced.

**Proof.** First we observe that a product state is balanced iff its factors are. Let the state be $|\Phi\rangle \otimes |\Psi\rangle$, which have $n$ and $m$ components, respectively, i.e. $|\Phi\rangle = \sum_{i=1}^{n} v_i |\Phi_i\rangle$ and $|\Psi\rangle = \sum_{i=1}^{n} w_i |\Psi_i\rangle$. Then

$$B_{|\Phi\rangle \otimes |\Psi\rangle} = \begin{pmatrix} B_{|\Phi_1\rangle} & \cdots & B_{|\Phi_i\rangle} & \cdots & B_{|\Phi_n\rangle} \\ B_{|\Psi_1\rangle} & \cdots & B_{|\Psi_i\rangle} & \cdots & B_{|\Psi_n\rangle} \end{pmatrix}$$

with length $mn$ divided into $n$ blocks, $n > m$ without loss of generality. Note that $m$ and $n$ are even because $|\Phi\rangle$ and $|\Psi\rangle$ are assumed to be balanced. Consequently, the smallest common multiple of $m$ and $n$ is smaller than or equal to $mn/2$. That is, there do exist $f, g \in \mathbb{N}$ relatively prime such that $f m = gn$ and $g \leq m/2$, $f \leq n/2$. Now we choose from each of the $n$ blocks (corresponding to the state $|\Phi_i\rangle \otimes |\Psi\rangle$, $i = 1, \ldots, n$) $g$ states such that from the first $f$ blocks we choose the first state, from blocks $f + 1$ to $2f$ we choose the 2nd state, ..., from blocks $(m - 1)f + 1$ modulo $n$ to $m \cdot f$ modulo $n$ we choose the $m$th state. This state is balanced and has length $m \cdot f \leq mn/2$. This proves that any product state—if balanced—is reducible.

It is important to emphasize that every state can be made balanced by local unitary transformations, raising in general the number of components in the state.

**Theorem 3.2.** Every balanced $q$-qubit state with length larger than $q + 1$ is reducible.

**Proof.** Balancedness of the state implies the existence of $n_1, \ldots, n_L$ such that $\sum_{j=1}^{L} n_j (A_{|\psi\rangle})_{ij} = 0$. Irreducibility implies that no subset $K$ of $\mathcal{L} := \{1, \ldots, L\}$ exists with $K \cap \mathcal{L} \neq \mathcal{L}$ such that $\sum_{j \in K} m_j (A_{|\psi\rangle})_{ij} = 0$ for some positive integers $m_j$. Without loss of generality, $A_{|\psi\rangle}$ has rank $q$. In order to fix the idea of the proof, we insert a vertical cut in the matrix $A_{|\psi\rangle}$ such that both parts contain at least $(q + 1)/2$ columns. This means to introduce two disjunct non-empty sets $\mathcal{K}$ and $\mathcal{K}' := \mathcal{L} \setminus \mathcal{K}$ with $|\mathcal{K}|, |\mathcal{K}'| \geq \frac{q + 1}{2}$. We define $\bar{\alpha}^K =: (\alpha^K_1, \ldots, \alpha^K_q)$ such that $\alpha^K_i = \sum_{j \in \mathcal{K}} n_j (A_{|\psi\rangle})_{ij}$. Irreducibility implies $\bar{\alpha}^K \neq 0$. We now split the sets $\kappa \subset \mathcal{K}$ and

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κ′ ⊂ K′ off the subsets K and K′ and define $\tilde{K} := (K \setminus \kappa) \cup \kappa′$. Then, including arbitrary non-negative integers $m_j$, $j \in \kappa′$, and defining $m_j = n_j$ for $j \in K$, we find

$$\sum_{j \in K} m_j (A_{|\psi\rangle})_{ij} = \alpha^K_i - \sum_{j \in \kappa} m_j (A_{|\psi\rangle})_{ij} + \sum_{j \in \kappa′} m_j (A_{|\psi\rangle})_{ij}.$$ 

Irreducibility then implies that for all such subsets $K$ and $\kappa$ no integer numbers $\tilde{m}_j$ ($\tilde{m}_j$ can be also negative or zero) do exist such that $\tilde{m}_j (A_{|\psi\rangle})_{ij} = \alpha^K_i$ for all $i \in \{1, \ldots, q\}$. Without loss of generality we can assume that $(A_{|\psi\rangle})_{i \in \{1, \ldots, q\}; j \in \kappa\cup\kappa′}$ has rank $q$ (a suitable choice of $K$ and $\kappa$ guarantees that). This implies that the condition can be satisfied for every integer vector $\tilde{\alpha}^K$ even in $\mathbb{Z}^q$, hence contradicting our assumption of irreducibility.

A comment is in order here. It must be stressed that the integers entering the balancedness condition must be positive. Therefore, linear dependence of the column vectors does not imply balancedness. In fact, not every $q$-qubit state with more than $q + 1$ product state components is balanced. The reason is that the set of positive integers is not a field. Our proof, however, nicely highlights that the balancedness condition itself bridges this gap and provides a mapping onto a set of linear equations over the field $\mathbb{Z}$. Thus, for balanced states the argument of linear independence can indeed be used. The state being irreducibly balanced thus implies that the rank of its corresponding alternating $(q \times L)$-matrix ($q$ rows and $L$ columns) is equal to $L - 1$. Since its maximal rank is $\min\{q, L\}$, this implies $L \leq q + 1$. A canonical form of such an irreducibly balanced state thus becomes

$$(32)$$

From this canonical form, further such states (except the GHZ state) can be generated by duplicating rows, NOT-operations on certain bits, and permutation of rows, i.e. of bits. We mention that in [42], the procedure of duplicating rows has been termed telescoping; it was used to generate certain multipartite entangled states that obey a monogamy relation.

**Theorem 3.3.** Stochastic states (i.e. states satisfying (ia) and (ib)) are balanced. In other words: stochasticity implies balancedness.

**Proof.** For the proof, let us consider an arbitrary $q$-qubit state $|\psi\rangle$ that satisfies the conditions (ia), (ib)—and maybe (iii)—in definition (2.1) and trace out everything but the first qubit. We can write the array of the state $|\psi\rangle$ as

$$(33)$$

Now assume that some of the states $|\Phi_i\rangle$ coincide with some of the $|\Phi'_i\rangle$, and call $|\Psi\rangle$ their superposition with corresponding weights of the right-hand side; the corresponding
superposition of the $\Phi'_i$ can be written as $|\tilde{\Psi}\rangle = \alpha |\Psi\rangle + \beta |\Psi'_\perp\rangle$, $|\alpha|^2 + |\beta|^2 = 1$; $\langle \tilde{\Psi} | \tilde{\Psi}\rangle := \tilde{x}$. Note that these states are not normalized to one. Then, the whole state can be written as $(|0\rangle + \alpha |1\rangle) \otimes |\Psi\rangle + \beta |1\rangle \otimes |\Psi'_\perp\rangle + |0\rangle \otimes |\Psi'\rangle + |1\rangle \otimes |\Psi''\rangle$ with pairwise orthogonal states $|\Psi\rangle$, $|\Psi'_\perp\rangle$, $|\Psi'\rangle$ and $|\Psi''\rangle$ and squared norms $x$, $x'_\perp$, $y$ and $z$, respectively. The one-site reduced density matrix then is $(\begin{smallmatrix} i + z & i x \\ i x & i + y \end{smallmatrix})$. Since the only $2 \times 2$ density matrix with two eigenvalues $\frac{1}{2}$ is $\frac{1}{2} \mathbb{1}$, the condition (ib) of definition 2.1 implies: $(x = 0$, $y = \frac{1}{2})$ or $(\alpha = 0$, $y = \frac{1}{2} - x)$. The latter relation means that $|\Psi\rangle \perp |\tilde{\Psi}\rangle$, and gives a phase-relation for the amplitudes $w_i$, contradicting the requirement (iii). The former condition means that no two $|\Phi_i\rangle$, $|\Phi'_i\rangle$ are equal and therefore orthogonal. In both cases (irrespective of the phase insensitivity) we find

$$\rho^{(1)} = \begin{pmatrix} \sum_{i \in I^j_1} p_i & 0 \\ 0 & \sum_{i \in I^j_0} p_i \end{pmatrix} \equiv \frac{1}{2} \mathbb{1} \iff \sum_{i \in I^j_1} p_i - \sum_{i \in I^j_0} p_i = 0, \quad (34)$$

where $I^j_1$ and $I^j_0$ are the set of column numbers, where the $j$th qubit has values 1 and 0, respectively. The above applies to each single qubit. Equation (34) can be written in a more compact form in terms of the corresponding alternating matrix and the vector $\tilde{p} := (p_1, \ldots, p_L)$ of weights

$$A_{|\psi\rangle} \tilde{p} = \tilde{0}. \quad (35)$$

Note that balancedness means

$$A_{|\psi\rangle} \tilde{1} = \tilde{0}, \quad (36)$$

where $\tilde{1} := (1, \ldots, 1)$. Equation (35) has a unique solution with all weights equal iff the state is irreducible\(^7\). Otherwise the state is reducible and all states in each irreducible block $B_0$ have the same weight $p_0$. This corresponds to a superposition of irreducibly balanced states. Phase insensitivity, however, turns out to be incompatible with more than one block except when all the states $|\Phi_i\rangle$, $|\Phi'_i\rangle$ were perpendicular to each other. This means that the superposed irreducibly balanced states must be orthogonal to each other. Tracing out only one qubit (including possible telescope copies of it) gives exactly the same condition (34). \hfill \Box

It is worth noting at this point that, with local operations on $q$ qubits, the maximum number of free phases is $q + 1$ (including a global phase), which coincides with the maximum length of an irreducibly balanced block. Therefore the only remaining reducible states that could be maximally entangled by virtue of the demanded phase insensitivity are superpositions of irreducible ones with a total length not larger than that of the irreducible state of maximal length.

We want to note that we can—without loss of generality—shrink all states such that no telescope bits occur; the shrinking does not affect the reducibility.

The above observations eventually lead to the following set of maximally entangled $q$-qubit states of maximal length,

$$|X_q\rangle = \sqrt{q + 2} |1 \ldots 1\rangle + \sum_i |i\rangle, \quad (37)$$

\(^7\) Allowing that one state can occur more than once, corresponding to a greater weight.
where \( |i \rangle \) denotes the state with all bits zero except the \( i \)th, which is one. The maximally entangled state of minimal length is always the GHZ state. States of intermediate length are obtained from those with maximal length for \( p \) qubits \( (p < q) \) by means of telescoping\(^8\).

It is worth mentioning that irreducibility does not trivially imply the form (32). An example of such a state of five qubits is

\[
B_x = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.
\]  (38)

**Theorem 3.4.** Every irreducibly balanced state is equivalent under local filtering operations \( SL^{\otimes q} \) (LFO) to a stochastic state.

**Proof.** The proof goes by construction. Let \( a_j, \ j = 1, \ldots, L, \) be the amplitudes of the product state written in the \( i \)th column of \( B_{|\psi\rangle} \), and let us consider the LFOs

\[
\mathcal{T}_{\text{LFO}}^{(i)} = \begin{pmatrix}
t_i & 0 \\
0 & t_i^{-1} \\
\end{pmatrix},
\]  (39)

with \( t_i = \sqrt{z_i} \) for some real positive \( t \) and complex \( z_i; \ i = 1, \ldots, q \). Without loss of generality, let the multiplicities \( n_j = 1 \) for all \( (at \ most \ q+1) \) \( j \) (differing multiplicities could be absorbed in the \( p_i \)). We must then show that, after suitable LFOs, all amplitudes are equal.

After this transformation, the amplitude of the product states (i.e. of the column vectors) would become

\[
a_j t \sum_{i=1}^{q} z_i (A_{|\psi\rangle})_{ij} =: a_j t_j.
\]  (40)

Balancedness implies \( \prod_{j=1}^{L} t_j = 1 \). Without loss of generality, let \( (B_{|\psi\rangle})_{i,1} = 0 \) for all \( i = 1, \ldots, q \). Dividing by \( t_1 \) the amplitudes become \( a_j t \sum_{i=1}^{q} 2z_i (B_{|\psi\rangle})_{ij} \). Stochasticity requires that all amplitudes have to be equal and leads to the set of linear equations

\[
\sum_{i=1}^{q} 2z_i (B_{|\psi\rangle})_{ij} = \log \frac{a_1}{a_j}; \ j = 2, \ldots, L.
\]

Since \( L \leq q + 1 \) and \( B_{|\psi\rangle} \) has rank \( L-1 \), a solution vector \((z_1, \ldots, z_q)\) exists for arbitrary \( a_j \neq 0 \). The resulting pure state is stochastic. \( \square \)

The fact that every irreducibly balanced state is SL-equivalent to a stochastic state, in combination with the negation of theorem 3.3, leads to the following:

**Corollary 3.1.** An irreducibly balanced state is locally unitarily inequivalent to every state without balanced part. In other words, irreducibly balanced states are not completely unbalanced.

\(^8\) See the discussion after (32).
In light of the fact that the minimal number of orthogonal product states in which a pure quantum state can be represented is invariant under $\mathrm{SL}(2, \mathbb{C})$ transformations [3], the following property of irreducibly balanced states is important.

**Theorem 3.5.** For $q > 3$ qubits, irreducibly balanced states descending from equation (37) are minimal in the sense that an irreducibly balanced state of length $L$ cannot be represented as a superposition of less than $L$ states of a computational basis (i.e. elements of a completely factorized basis).

**Proof.** The irreducibly balanced state $|X_q\rangle$ of $q$ qubits (cf equation (37)) has length $q + 1$ and its $(q - 1)$ qubit reduced density matrix is spanned by a generalized $(q - 1)$ qubit GHZ and W state; for $q > 3$ it has no product state in its range. The minimal lengths of the $(q - 1)$ qubit GHZ and W are 2 and $(q - 1)$, respectively, and hence they are different for $q \neq 3$ (this implies that they are SLOCC inequivalent [3]). It can be shown that the possibility to express $|X_q\rangle$ as a superposition of less than $q + 1$ computational basis states implies that there must be a product state in the range of the $(q - 1)$ qubit reduced density matrix, which leads to a contradiction. This inductively proves the minimality of all irreducibly balanced states as defined before. \(\square\)

It is an important step now to realize the following:

**Theorem 3.6.** All irreducibly balanced states belong to the SLOCC nonzero class, i.e. they are robust against infinitely many LFOs $\mathrm{SL}^{\otimes q}$ and therefore possess a finite normal form [12]. As a consequence, they are maximally entangled states according to definition 2.1 (also in the sense of [12]).

**Proof.** Those transformations that go beyond $\mathrm{SU}(2)$ are essentially the LFOs of the form

$$T_{\text{LFO}}^{(i)} = \begin{pmatrix} t_i & 0 \\ 0 & t_i^{-1} \end{pmatrix}$$

(41)

when expressed in a suitable local basis for the $i$th qubit. Now assume the existence of LFOs that scale the state down to zero after infinitely many applications—we will call this the zero-class assumption. Defining a set of real numbers $p_i \in \mathbb{R}$, $i = 1, \ldots, q$ such that $t_i = t^{p_i}$ with $t > 1$ (without loss of generality), the action of this single LFO rescales the weight of the $j$th column of the alternating matrix $A_{\psi}$ by the factor

$$t^{s_j} = t \sum_{i=1}^q p_i A_{ij}$$

(42)

such that $s_j < 0$ for all $j \in \{1, \ldots, L\}$, where $L$ is the length of the state of $q$ qubits and the negativity of all the $s_j$ expresses the zero-class assumption\(^9\). This is equivalent to

$$0 > w_j = \sum_{i=1}^q p_i A_{ij}$$

(43)

for all $j \in \{1, \ldots, L\}$. Now we make use of the balancedness of the state, meaning that $A_{i,L} = -\sum_{j=1}^{L-1} A_{i,j}$ for all $i \in \{1, \ldots, q\}$ and that (43) must apply to all $j \in \{1, \ldots, L - 1\}$

\(^9\) It is important to realize in this context that every finite succession of LFOs of the type (41) can be replaced by a single LFO with a suitably chosen set of real numbers $\tilde{p}_i$. 

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by virtue of our zero-class assumption. Consequently,

\[ w_L = \sum_{i=1}^{q} p_i A_{iL} \]

\[ = - \sum_{j=1}^{L-1} \sum_{i=1}^{q} p_i A_{i,j} \]

\[ = - \sum_{j=1}^{L-1} w_j > 0. \]

Thus, at least one positive scaling exponent must exist. This contradicts our initial assumption.

Now it is crucial that, for irreducibly balanced states, no basis exists in which the state has no balanced part (theorem 3.1). This completes the proof. \[\square\]

The same applies to \(q\)-qubit states, which are superpositions of orthogonal irreducibly balanced states with length smaller than \(q + 2\).

As a consequence, there must exist independent entanglement monotones which attribute to each of these states a nonzero value and which can distinguish between them. Equivalently, all completely unbalanced states belong to the SLOCC zero-class. An example is the class of \(W\) states for an arbitrary number of qubits, but also products of states where at least one of the factors is part of the corresponding SLOCC zero-class. Therefore, every \(\text{SL}(2, \mathbb{C})^\otimes q\) invariant entanglement monotone when applied to such states gives zero.

We now briefly discuss the requirements in definition 2.1 for irreducibly balanced states with a particular emphasis on condition (ii) of vanishing sub-tangles. The GHZ state satisfies all conditions; the subtangles are all trivially zero, because the reduced density matrices are mixtures of product states. Therefore, they are maximally \(q\)-tangled. The states of maximal length behave analogously, except that tracing out the first qubit, a mixture of a generalized GHZ state and a \(W\) state is obtained. For three qubits, the resulting \(W\) state is a GHZ (or Bell-) state and the mixture has zero two-tangle. Also for four qubits, one can construct a decomposition of the density matrix whose elements all have zero three-tangle. For growing numbers of qubits, the GHZ weight monotonically decreases to zero. However, it can be shown that it contains a subtangle that is detected by certain factorized filters.

It is straightforward to show that the GHZ state is detected by all simple filters (i.e. those \(\text{SL}(2, \mathbb{C})\) invariants that are directly created from invariant combs, but not, e.g., linear combinations of such invariants).

4. Toward a generalization beyond qubits

4.1. Compound entanglement or block entanglement

The invariant-comb approach also provides suggestions on how to possibly extend such an ansatz toward entanglement measures for blocks of spins of variable size. To this end, we exploit the fact that each operator with an odd number of \(\sigma_y\) is a comb. Furthermore, if two \(q\)-qubit filters are identical for pure \(q\)-qubit states, but not identical as operators, their difference is a comb. Examples are \(\sigma_i \sigma_j \sigma_i \sigma_j\) for \(i \neq j\), \(\sigma_i \sigma_j \sigma_i \sigma_j - 3 \sigma_i \sigma_j \sigma_i \sigma_j\) for \(i \neq j\), \(\sigma_i \sigma_j \sigma_i \sigma_j - 3 \sigma_i \sigma_j \sigma_i \sigma_j\) for...
two qubits and $\sigma_\mu \sigma_\nu \sigma_\tau - 3 \sigma_\mu \sigma_\nu \sigma_\nu$ for three qubits. However, this constitutes just a starting point as this will typically lead to a set of combs on which the local unitary group acts in a nontrivial way. In order to construct a filter that is an entanglement monotone, we need an invariant comb. Clearly, abandoning the requirement for the monotone property would open up a large variety of possible ‘measures’ or ‘indicators’ for entanglement. This, however, is not what we have in mind.

In order to be invariant, it is necessary that the combs are regular and all their eigenvalues must have equal modulus. This is a clear criterion for designing an approach we have in mind. The approach pursued, e.g., in [45] has some overlap with concurrence vector approaches (see [25, 46]), which for bipartite systems coincide with the universal state inversion (see [25] and [45]). The local antilinear operators used there are not regular and therefore cannot be invariant under local SL operations in higher local dimensions.

This opens up a rich and promising field for future investigation. Some insight into the intricacies and consequences involved with this requirement is given in the next section on general spin $S$. It is worth noting that the concept of balancedness introduced above is tailor-made for qubit systems; it is not appropriate for higher local dimension. The notion of maximal entanglement would have to be modified correspondingly once such invariant combs have been found.

4.2. Spin $S > \frac{1}{2}$

The operator $S_y = -i(S^+ - S^-)C$ is a comb for arbitrary spin $S$. The crucial difference compared with the spin-1/2 case is that there are more first-degree combs for $S > 1/2$ due to the fact that there are nontrivial powers of spin operators up to order $2S > 1$ – since $(S^+)^{2S+1} = 0$. It turns out that, for $S = 1$, there is a three-parameter variety of first-degree combs

$$A_1[a, b, c] := (a S_y + (b S_x S_y + c S_y S_z + h.c.) ) C \quad \text{(44)}$$

and a six-parameter variety for spin-3/2

$$A_{3/2}[a, b, c] := S_y (a 1 + b S_x + c S_z + d S_x S_y + e S_y^2 + f S_z^2 + h.c.) C. \quad \text{(45)}$$

As in the qubit case, every product of spin operators containing an odd number of $S_y$ (plus its Hermitian conjugate) is a comb. A generalization to general spin $S$ is therefore straightforward: we have $S(2S + 1)$ independent off-diagonal (pure imaginary) entries, which is the dimension of the variety. The corresponding operators are those appearing in $(S_x + S_z)^m; \ m = 0, \ldots, 2S - 1$.

Unfortunately, for integer spin, i.e. for $2S + 1$ odd, these combs are not regular. This follows from the hairy ball theorem, stating that in order to have a continuous map from the surface of a $d$-dimensional sphere onto itself, $d$ has to be even. In our case, the surface corresponds to the real part of the normalized Hilbert space (due to the antilinearity, every comb on the real Hilbert space is a comb on all Hilbert spaces). Therefore, for integer spin, one has to look out for a comb of higher order. Unfortunately, also for half-integer spin, no first-order SL($2S + 1, C$) invariant combs exist.

In order to make a first step towards higher spins in the spirit of the invariant-comb approach, let us first consider a simplified scenario, where only local rotations are accessible in the laboratory. Then, the group of local operations is the complex extension of the $2S + 1$ dimensional representation of SU(2), hence still SL($2, \mathbb{C}$). We want to stress that this situation
differs considerably from that of an arbitrary $2S + 1$ level system, where the most general local operations are out of the complexified \( SU(2S + 1) \), which is the \( SL(2S + 1, \mathbb{C}) \). For half-integer spin \( S \), the \( SL(2, \mathbb{C}) \)-invariant comb is obtained as

\[
A_S = i \text{antidiag}\{(-1)^m; \ m = 1, \ldots, 2S + 1\} C; \quad 2S \quad \text{odd}
\]

with associated linear operator

\[
L_S = i \text{antidiag}\{(-1)^m; \ m = 1, \ldots, 2S + 1\}.
\]

Here, \( \text{antidiag}\{\lambda_1, \ldots, \lambda_n\} \) indicates the \( n \times n \) matrix with \( \lambda_1, \ldots, \lambda_n \) on the antidiagonal, e.g. \( \sigma_y = \text{antidiag}\{-i, i\} \). With these combs, we can immediately construct an analogue for the concurrence for arbitrary half-integer spin

\[
C_S[\Psi] = |\langle \Psi | L_S \otimes L_S | \Psi^* \rangle|
\]

for which the convex-roof extension procedure from \([47]\) can be applied, and hence the \( SL(2, \mathbb{C}) \)-concurrence for general half-integer spin \( S \) is

\[
C_S = \max\{0, 2\lambda_1 - \text{tr} \ R\},
\]

\[
R = \sqrt{\rho L_S \otimes L_S \rho^* L_S \otimes L_S \rho}.
\]

It must be stressed that this concurrence is a measure of entanglement under restricted local operations, namely to local rotations of the Cartesian axis. The notion of SLOCC is modified correspondingly. Each restricted entanglement class will be subdivided into classes with respect to the full group of local transformations \( SL(2S + 1, \mathbb{C}) \). We are therefore confident that an analysis of the \( SL(2, \mathbb{C}) \) invariant concurrence (47) will nevertheless give interesting insight into the entanglement classes for higher local dimensions.

It would be interesting to compare these combs with further existing proposals, such as the universal state inversion \([48]\), which however are constructed for general \( d \)-state systems. We reserve this topic for our future studies.

5. Conclusions

In the recent literature, an efficient procedure for the construction of local \( SL(2, \mathbb{C}) \)-\( \otimes \) invariant operators for \( q \)-qubit wavefunctions has emerged out of the simple requirement to create entanglement indicators that should vanish for all product states (a minimal requirement for a quantity to detect only global entanglement) \([37, 38]\). We call this procedure the invariant-comb approach, because the local building blocks already are \( SL(2, \mathbb{C}) \) invariant. It is interesting that some definitely globally entangled states as the W state are not detected by any of these polynomial invariants. This motivates the concept of genuine multipartite entanglement in order to distinguish globally entangled quantum states detected by some nonzero polynomial \( SL \) invariant from others. The fact that those invariants automatically lead to entanglement monotones has motivated our detailed analysis of the properties of many-qubit states that are detected by the entanglement measures created by invariant combs. We have chosen an approach from two different points of view, with significantly overlapping results. On the one hand,
we find that a necessary requirement for a pure quantum state of many qubits to have finite genuine multipartite entanglement is that the state has a balanced part. This balancedness constitutes a continuation of the curious geometric picture of the three-tangle as highlighted in [33] to a higher number of qubits. On the other hand, also basic necessary requirements for maximal pure-state entanglement, namely that the state has to be stochastic [12, 41], are demonstrated here to readily imply balancedness. This curious coincidence justifies a systematic analysis of balanced states. We have extracted the locally SU(2) invariant ‘nucleus’ of balancedness, which is the irreducible balancedness. It is shown that irreducible balanced states are locally SL invariant to stochastic states, a prerequisite for being maximally entangled. Irreducible balancedness is also shown to exclude the existence of a completely unbalanced form (just as, e.g., the W state has). This result is essential in that it demonstrates that irreducibly balancedness is a well-defined and valuable concept. Furthermore, we could prove that irreducibly balanced states belong to the nonzero SLOCC class of states. Hence, they have a nontrivial normal form after maybe infinitely many local filtering operations [12].

A canonical form for a family of irreducibly balanced states has been found, and this family has the minimal number of components in a fully factorized basis. This minimal ‘length’ is a non-polynomial SL invariant [3, 17], which according to our analysis has a tight connection with an entanglement classification using polynomial SL invariant entanglement measures. This connection consists in that the homogeneous degree of the polynomial invariant has to fit the length of the balanced part of the minimal form. From the latter, we can read off (up to a normalization factor) the value of the polynomial SL invariant.

Precise sufficient conditions have been singled out for reducibly balanced states in order to be maximally entangled. Such states clearly exist, possibly even without an irreducibly balanced form. However, irreducibly balanced states provide a generating ‘basis’ (not claiming completeness) for the construction of such states, in the sense that reducibly balanced states are superpositions of irreducibly balanced ones.

It is worth making reference to a collective entanglement measure proposed in [26]. For qubits, it is equivalent to the averaged one-tangle \( \tau_1 = 4/n \sum_j \det \rho_j \) (see, e.g., [28]), where \( \rho_j \) is the reduced density matrix of qubit number \( j \). These measures are only sensitive to the requirement (ia) of definition 2.1. They assume their global maxima for all those maximally entangled states presented here (satisfying the condition (ib)). This also includes arbitrary tensor products of such maximally entangled states. So, this measure is an indicator of stochasticity of a pure state, but cannot discriminate any of the SLOCC entanglement classes present in that state. This shortcoming might be overcome to some extent by looking at maxima of suitable functions of, e.g., von Neumann entropies of certain reduced density matrices. Such an analysis has been pursued among others in [32] and has singled out the four-qubit ‘X state’ in equation (37): an irreducibly balanced state in the canonical form presented here (and before in [37, 38]).

An additional advantage of the invariant-comb approach is that it suggests possible generalization to general subsystems. We have discussed to some extent generic complications encountered with such an extension. A specific analysis for bipartite entanglement of general half-integer spins is added. With a restriction of the local operations to just local rotations in the laboratory, an analogue to the concurrence is presented explicitly and its exact convexroof extension has been constructed using a result of [47]. Its comparison with other existing proposals remains to be further investigated.
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