ON THE HIGHER-LOOP CONTRIBUTIONS TO
THE AXIAL ANOMALY

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Abstract

The problem of the higher-loop contributions to the axial anomaly is reexam-
ined by a new method. We demonstrate that these contributions depend on the
order of the calculations. If the divergence of the axial current by nonperturba-
tive Fujikawa method is calculated first and then average it over the photon field
in the presence of an external photon source, a nonzero contribution is obtained.
However perturbative Feynman diagram method has an uncertainty. De-
pending on the order of the calculations above mentioned or zero results are
obtained.

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1 Introduction

The problem of the axial anomaly [1, 2] is one of the attractive problems of the quantum field theory. In the framework of QED it was first discussed in [1, 3], using Feynman diagram method, with the conclusion that two and higher-loop contribution to the divergence of the axial current is zero.

On other hand in [4] this anomaly was calculated by a nonperturbative technique, based on the observation that the anomaly is due to the noninvariance of the fermion measure under axial transformations of the fermion fields.

In a recent work [5], the 3-loop contribution to the divergence of the axial current was shown to be nonzero. This conclusion was supported in [6] based on the investigations of the renormalization properties of axial current and its divergence.

Given the controversy we would like to discuss this problem by using yet another method [7]. This method is based on an old formula of DeWitt [8] that connects the vacuum expectation of a quantum functional with the classical ones (and its derivatives with respect to the classical fields). Using this method we have previously calculated the two-loop effective action for scalar $\lambda\phi^4$ theory and for spinor electrodynamics in [7, 8].

In this paper we discuss the divergence of the axial current in the simplest model which describes interacting massless fermion $\psi$ and photon fields $A^\mu$ via axial-vector coupling. The Lagrangian of this model is invariant under local axial transformations and have the form (with gauge fixing):

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} \left[ i \left( \gamma^5 \right) \right] \psi - \frac{\alpha}{2} \left( f(\partial^2)\frac{\partial^2}{\partial \mu A^\mu} \right)^2.$$  \hspace{1cm} (1)

The regularization is carried out by adding to the Lagrangian (1) the higher derivative term [7]:

$$\frac{1}{4}F_{\mu\nu}\frac{\partial^2}{\partial^2}F^{\mu\nu},$$  \hspace{1cm} (2)

which regularizes the free photon propagator. For consistency we choose the function $f(\partial^2)$ as

$$f(\partial^2) = \partial^2 + \kappa^2,$$

where $\kappa^2$ is an arbitrary parameter. Then the final form of the model we consider is:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}F_{\mu\nu}\frac{\partial^2}{\partial^2}F^{\mu\nu} + \bar{\psi} \left[ i \left( \gamma^5 \right) \right] \psi - \frac{1}{2\alpha} \left( f(\partial^2)\frac{\partial^2}{\partial \mu A^\mu} \right)^2.$$  \hspace{1cm} (3)

2 DeWitt Method

In this section we would like to review briefly the DeWitt method which we will need in the following discussion.
DeWitt's formula \[8\] relates the vacuum expectation value of any quantum functional \(Q[\hat{\varphi}_i]\), in the presence of the source \(J_i\), to the classical one \(Q[\varphi]\):

\[
\langle Q[\hat{\varphi}] \rangle =: \exp\left(\frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} G^{n_{1\cdots n}} \frac{\delta^n}{\delta \varphi^{i_1} \cdots \delta \varphi^{i_n}}\right) : Q[\varphi],
\]

where \(\hat{\varphi}_i = \hat{\varphi}(x_i)\) denotes any quantum field and \(\varphi_i = \delta W / \delta J_i\) is its vacuum expectation value, \(W = -i \log Z[J]\) is the generating functional of the connected Green's functions, and

\[
G^{i_1\cdots i_n} = \frac{\delta^n W}{\delta J_{i_1} \cdots \delta J_{i_n}}
\]
is the connected \(n\)-point Green's functions. Note that repeated indices imply a summation over discrete indices and integration over the continuous ones. The colons in Eq.(4) mean that derivatives in exponent acts only on functional \(Q[\varphi]\).

Thus the vacuum expectation value of any quantum functional is defined by the corresponding classical functional and the connected total Green's functions.

Let us define the generating functional

\[
Z[J_\mu] = \exp\{iW[J_\mu]\} = \int D\hat{A}_\mu D\hat{\psi} D\hat{\bar{\psi}} \exp\left(i \int \left( L + J_\mu \hat{A}_\mu \right) dx \right).
\]

The integrations over spinor variables in Eq.(5) leads to:

\[
Z[J_\mu] = \int D\hat{A}_\mu \exp\left(i S_{\text{eff}} + i \int J_\mu \hat{A}_\mu dx \right),
\]

where

\[
S_{\text{eff}} = \frac{1}{2} \hat{A}_\mu K^{-1}_{\mu\nu} A_\nu + i\hbar \text{Tr} \log \hat{K},
\]

and

\[
K^{\mu\nu} = \frac{1}{\partial^4 (1 - \partial^2 \Lambda - 2)} \left( g^{\mu\nu} \partial^2 - \partial^{(\mu} \partial^{\nu)} \right) + \frac{\alpha}{f^2} \frac{\partial^{(\mu} \partial^{\nu)}}{\partial^4}, \quad \hat{K}^{-1} = i\hat{\partial} + e\hat{A}_5.
\]

We next define the effective action

\[
\Gamma[A_\mu] = W[J_\mu] - J_\mu A_\mu.
\]

According to \[8\] and \[4\]:

\[
\frac{\delta \Gamma}{\delta A_\mu} = \langle \frac{\delta S_{\text{eff}}}{\delta A_\mu} \rangle,
\]

and from Eq.(4) we have the following equation:

\[
\frac{\delta \Gamma}{\delta A_\mu(x)} =: \exp(\hat{G}) \frac{\delta S_{\text{eff}}}{\delta A_\mu(x)}.
\]
where
\[
\hat{G} = \frac{-i\hbar}{2} D^{\mu\nu}(y, z) \frac{\delta^2}{\delta A^\mu(y) \delta A^\nu(z)} - \frac{\hbar^2}{6} D^{\mu\nu\lambda}(y, z, t) \frac{\delta^3}{\delta A^\mu(y) \delta A^\nu(z) \delta A^\lambda(t)} + \ldots,
\]
and
\[
D^{\mu\nu\ldots\lambda}(x, y, \ldots, t) = \frac{\delta^n W}{\delta A^\mu(y) \delta A^\nu(z) \ldots \delta A^\lambda(t)}
\]
are connected Green’s functions of the vector fields \(A^\mu\).

3 The Divergence of the Axial Current

Using the formalism developed in Sec. 2 it is easy to show that
\[
\langle e \hat{\psi}(x) \gamma^\mu \gamma_5 \hat{\psi}(x) \rangle_J = \delta \Gamma / \delta A^\mu(x) - K^{-1\mu\nu}(x, y) A^\nu(y) =
\]
\[
= -i\hbar \epsilon \text{Tr} \left( \hat{K}(x, x) \gamma^\mu \gamma_5 \right) - \frac{i\hbar}{2} e D^{\alpha\beta}(y, z) \frac{\delta^2}{\delta A^\alpha(y) \delta A^\beta(z)} \text{Tr} \left( \hat{K}(x, x) \gamma^\mu \gamma_5 \right) + \ldots.
\]

The dots denote the terms with higher order derivatives over \(A^\mu\). The left side of this equation is vacuum expectation of the axial current \(j_5^\mu\) in the presence of the source \(J\).

The vacuum expectation of the divergence of the axial current in the presence of the source \(J\) is:
\[
\langle \partial^\mu j_5^\mu \rangle_J = -i\hbar e \text{Tr} \left( \hat{K}(x, x) \gamma^\mu \gamma_5 \right) -
\]
\[
- \frac{i\hbar}{2} e D^{\alpha\beta}(y, z) \frac{\delta^2}{\delta A^\alpha(y) \delta A^\beta(z)} \text{Tr} \left( \hat{K}(x, x) \gamma^\mu \gamma_5 \right) + \ldots.
\]

In the framework of the model Eq. (11) the divergence of the axial current was calculated by the nonperturbative method in [4], and by perturbative one in [9] (for other models see e.g. [1], [3]). Let us first consider the nonperturbative method of Fujikawa [4] to calculate the divergence of the axial current. Following this method we have as a result of the noninvariance of the fermion measure in Eq. (5) (we keep the conservation of the vector current):
\[
\langle \partial_\mu j_5^\mu \rangle_{NP} = \int D\hat{A}_\mu \hat{D}\hat{\psi} \hat{D}\hat{\psi} \hat{\psi}(x) \gamma^\mu \gamma_5 \hat{\psi}(x) \exp \left( i \int (L + J^\mu \hat{A}_\mu) \, dx \right)
\]
\[
= \int D\hat{A}_\mu \hbar e^2 \frac{\epsilon^\mu\nu\lambda\sigma}{16\pi^2} \hat{F}_{\mu\nu}(x) \hat{F}_{\lambda\sigma}(x) \exp \left( iS_{eff} + i \int J^\mu \hat{A}_\mu \, dx \right).
\]

Application of the DeWitt’s formula (4) to the right hand side of the last equation gives
\[
\langle \partial_\mu j_5^\mu (x) \rangle_{NP} = \hbar e^2 \frac{\epsilon^\mu\nu\lambda\sigma}{16\pi^2} F_{\mu\nu}(x) F_{\lambda\sigma}(x) - i\hbar^2 e^3 \frac{\epsilon^\mu\nu\lambda\sigma}{2\pi^2} \partial_\nu \partial_\lambda D^{\nu\sigma}(y, z) \big|_{y=z=x}.
\]
This equation is the main result of this paper. Now implication of this result will be discussed.

We would like to stress that, to find this result we first calculate the path integral over fermions with further formal calculations of the path integral over bosons. This prescription completely coincides with the suggestion of F. Berezin [11]. He gave the order of calculations of the path integral over fermions and bosons as a definition: first integrate over fermions, and next over bosons.

By comparison with Eq.(14) we get

\[ -ie\hbar\partial_\mu \text{Tr} \left( \hat{K}(x,x)\gamma^\mu\gamma^5 \right) = \hbar\frac{e^2}{16\pi^2}e^{\mu\nu\lambda\sigma}F_{\mu\nu}(x)F_{\lambda\sigma}(x). \]  

(17)

Now we discuss the calculation [3], [9] of the same quantity by perturbative method. The left side of Eq.(17) in fact is the one-loop fermion contribution. In order to calculate this contribution we need additional (to Eq. (2)) regularization of this contribution. We can choose Pauli–Villars method for this one to preserve the conservation of vector currents, and get the same answer as the right hand side of Eq.(17).

As we can see from Eq.(14) to compute the higher-loop contributions we must perform three operations: the divergence, the functional derivatives, and the trace over discrete and continuous coordinates. The final results depend on the order of these calculations.

Calculated the variational derivatives on \( A_\mu \) first and then the divergence and the trace we get for the second term on the r.h.s of Eq.(14):

\[ -\frac{\hbar^2}{2}e\text{Tr} D^{\alpha\beta}(y,z)\frac{\delta^2}{\delta A_\alpha(y)\delta A_\beta(z)} \left( \hat{K}(x,x)\gamma^\mu\gamma^5 \right) = \]

\[ -\frac{e^3\hbar^2}{2}\text{Tr} D^{\alpha\beta}(y,z) \left( \gamma^\mu\gamma^5 \left( \hat{K}(x,y)\gamma_\alpha\gamma_\beta\hat{K}(y,z)\gamma_\beta\gamma_\alpha\hat{K}(z,x) + (y,\alpha \leftrightarrow z,\beta) \right) \right). \]  

(18)

Note that in this case the Pauli-Villars procedure is not needed since the convergence of the integrals in Eq.(18) is insured by the regularization prescription (see Eq.(2)) and the traces can be computed. Acting with the operator \( \partial_\mu \) on this expression we get for the trace part of this term:

\[ \text{Tr} \left[ \gamma^5\delta(y-x) \left( \hat{K}(z,y)\gamma_\beta\hat{K}(x,z)\gamma_\alpha - \hat{K}(y,z)\gamma_\alpha\hat{K}(z,x)\gamma_\beta \right) \right] + \]

\[ \text{Tr} \left[ \gamma^5\delta(x-z) \left( \hat{K}(y,z)\gamma_\alpha\hat{K}(x,y)\gamma_\beta - \hat{K}(z,y)\gamma_\beta\hat{K}(y,x)\gamma_\alpha \right) \right] = 0. \]  

(19)

That is, this contribution to the anomaly vanishes.

Proceeding similarly we will have zero contribution to the divergence of the axial current from the other terms in Eq.(14) as well.
Next, we would like to change the order of operations. Acting by $\partial_\mu$ first and then calculating the trace over fermions (which would require Pauli-Villars regularization), and noting that r.h.s. of Eq. (17) is quadratic in $A_\alpha(x)$, our series (14) can be reduced to the same final form of the nonperturbative method given in Eq. (16). Since $D^{\nu\sigma}(y, z)$ is the full connected Green’s function of the photon, the second term of this expression contains all the higher-loop radiative corrections.

It is clear that the second term in Eq. (16) is at least third order in $\bar{\hbar}$, with the substitution of the Dyson equation for $D^{\nu\sigma}(y, z)$ into Eq. (16) (see Appendix). Indeed for the second term in Eq. (16) we have

$$i\hbar^2 \frac{e^3}{2\pi^2} \epsilon^{\mu\nu\lambda\sigma} \partial_\mu \partial_\nu \left[ -K^{\nu\sigma}(y, z) - i\hbar \text{Tr} \left( \gamma_\mu \gamma^5 G^\nu(t, y, t) K^{\nu\sigma}(t, z) \right) \right] =$$

$$= \frac{e^4 \hbar^3}{2\pi^2} \epsilon^{\mu\nu\lambda\sigma} \text{Tr} \left( \gamma_\nu \gamma^5 \partial_y \gamma^\sigma(t, y, t) \right) \frac{1}{\partial^2} \partial_\mu \delta(t - z).$$

(20)

Here we used the fact that the free Green’s function of photon $K^{\mu\nu}(x, y)$ is symmetric in $\mu$ and $\nu$. After some algebra, for three-loop contribution we get

$$\frac{e^5 \hbar^3}{2\pi^2} \epsilon^{\mu\nu\lambda\sigma} \text{Tr} \left( \gamma_\nu \hat{K}(t, v) \gamma^\sigma \hat{K}(v, t) \right) \frac{1}{\partial^2 (1 - \partial^2/\Lambda^2)} \partial_\mu \delta(t - z).$$

(21)

We can conclude that the nonperturbative method of Fujikawa provides us with a well defined procedure:

- At first, we calculate the divergence of the axial current in the presence of the photon field by regularization of the measure of the path integral over fermions to get Eq. (17) (We preserve the conservation of the vector current). Then the result of this calculation is exact. (It is important that this result coincides with the result of the perturbative calculations of Feynman fermion triangle diagram via Pauli-Villars regularization prescription);

- Next, we calculate the functional derivatives over the photon fields, in Eq. (14) to get our final result Eq. (16).

On the other hand, perturbative calculations of Feynman diagrams, in principle, permit a change of the order of the calculations over fermion and boson variables in higher-loop diagrams as in Eq. (21). As a result we can have zero contribution as is demonstrated in Eq. (19).
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Appendix

In [7] the system of equations for the effective action in QED was derived. In the present case the first equation of this system is (we will present it in the form where $\eta = \bar{\eta} = 0$):

$$\frac{\delta \Gamma}{\delta A_\mu} = K^{-1}^{\mu\nu} A_\nu + i e \hbar \text{Tr} \left( \gamma^{\mu} \gamma^{5} G(x,x) \right).$$

Differentiating this equation with respect to $A_\nu$ and using the identity

$$D_{\mu\nu} \frac{\delta^2 \Gamma}{\delta A_\nu \delta A_\lambda} = -\delta_{\mu}^{\lambda}$$

we get the following Dyson equation for total Green’s function $D_{\mu\nu}$:

$$D_{\mu\nu}(x,y) = -K_{\mu\nu}(x,y) - i e \hbar \text{Tr} \left( \gamma_{\lambda} \gamma^{5} G^{\mu}(z,x,z) \right) K^{\lambda\nu}(z,y),$$

where $G^{\mu}(z,x,z)$ is three-point connected Green’s function which is related to the three-point total vertex function $\Gamma^{\mu}(x,y,z)$ as:

$$\hat{G}^{\mu}(x,y,z) = \hat{G}(x,x_1) \Gamma^{\lambda}(x_1,y_1,z_1) \hat{G}(z_1,z) D^{\nu\lambda}(y_1,y),$$

where $\hat{G}(x,y)$ is the full Green’s function of the fermion.

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