Three-orbifolds with positive scalar curvature

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Abstract

We prove the following result: Let \((\mathcal{O}, g_0)\) be a complete, connected 3-orbifold with uniformly positive scalar curvature, with bounded geometry, and containing no bad 2-suborbifolds. Then there is a finite collection \(\mathcal{F}\) of spherical 3-orbifolds, such that \(\mathcal{O}\) is diffeomorphic to a (possibly infinite) orbifold connected sum of copies of members in \(\mathcal{F}\). This extends work of Perelman and Bessières-Besson-Maillot. The proof uses Ricci flow with surgery on complete 3-orbifolds, and are along the lines of the author’s previous work on 4-orbifolds with positive isotropic curvature.

Key words: Ricci flow with surgery, three-orbifolds, positive scalar curvature

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1 Introduction

In Perelman [P1, P2, P3] Perelman used Hamilton’s Ricci flow to attack Poincaré and geometrization conjectures; cf. also [BBB⁺], [CaZ], [KL1], [MT] and [Z]. In particular he completed the previous work of Gromov-Lawson (see for example [GL]) and Schoen-Yau [SY] on the classification of compact, oriented 3-manifolds with positive scalar curvature. In [BBM], Bessières, Besson, and Maillot extended Perelman’s work to get a classification of open, oriented 3-manifolds with uniformly positive scalar curvature and with bounded geometry. In this short note we will go further to extend the above work to the orbifold case. More precisely we will prove the following

Theorem 1.1. Let \((\mathcal{O}, g_0)\) be a complete, connected 3-orbifold with uniformly positive scalar curvature, with bounded geometry, and containing no bad 2-suborbifolds. Then there is a finite collection \(\mathcal{F}\) of spherical 3-orbifolds, such that \(\mathcal{O}\) is diffeomorphic to a (possibly infinite) orbifold connected sum of copies of members in \(\mathcal{F}\).

Note that the result in the compact, oriented case is implicit in [KL2]. For an introduction to orbifold see [T]. Recall that an orbifold is good if it has a covering which is a manifold, otherwise it is bad. A Riemannian orbifold \((\mathcal{O}, g)\) is said to have uniformly positive scalar curvature if there exists a constant \(c > 0\) such that the scalar curvature satisfies \(R \geq c\). Also recall that a complete Riemannian
orbifold $O$ is said to have bounded geometry if the sectional curvature is bounded (in both sides) and the volume of any unit ball in $O$ is uniformly bounded away from zero. The notion of a (possibly infinite) orbifold connected sum will be given later in this section; cf. [Hu2].

By extending the construction in [GL] to the orbifold case and utilizing a result in Dinkelbach-Leeb (see Proposition 2.2 in [DL]), one can show that the converse of Theorem 1.1 is also true: Any 3-orbifold as in the conclusion of the theorem admits a complete metric with uniformly positive scalar curvature and with bounded geometry.

Now we explain the notion of (possibly infinite) orbifold connected sum (see [Hu2]), which extends several related notions in the literature (see for example [CTZ] and [KL2]). Let $O_i$ ($i = 1, 2$) be two $n$-orbifolds, and let $D_i \subset O_i$ be two suborbifolds-with-boundary, both diffeomorphic to some quotient orbifold $D^n/\Gamma$, where $D^n$ is the closed unit $n$-ball, and $\Gamma$ is a finite subgroup of $O(n)$. Choose a diffeomorphism $f : \partial D_1 \to \partial D_2$, and use it to glue together $O_1 \setminus \text{int}(D_1)$ and $O_2 \setminus \text{int}(D_2)$. The result is called the orbifold connected sum of $O_1$ and $O_2$ by gluing map $f$, and is denoted by $O_1 \sharp_f O_2$. If $D_i$ ($i = 1, 2$) are disjoint suborbifolds-with-boundary (both diffeomorphic to some quotient orbifold $D^n/\Gamma$) in the same $n$-orbifold $O$, the result of similar process as above is called the orbifold connected sum of $O$ to itself, and is denoted by $O_\sharp_f$.

Given a collection $F$ of $n$-orbifolds, we say an $n$-orbifold $O$ is a (possibly infinite) orbifold connected sum of members of $F$ if there exist a graph $G$ (in which we allow an edge to connect some vertex to itself), a map $v \mapsto F_v$ which associates to each vertex of $G$ a copy of some orbifold in $F$, and a map $e \mapsto f_e$ which associates to each edge of $G$ a self-diffeomorphism of some $(n-1)$-dimensional spherical orbifold, such that if we do an orbifold connected sum of $F_v$'s along each edge $e$ using the gluing map $f_e$, we obtain an $n$-orbifold diffeomorphic to $O$.

The proof of the theorem is along the lines of [Hu1, Hu2], which in turn are based on [P2] and which incorporate ideas from [BBB+], [BBM], [CZ], [CTZ] and [H97]. Meanwhile we will establish the short time existence of Ricci flow on complete orbifolds with bounded curvature, which extends Shi [S]. This result was taken for granted in the literature without a proof (see for example [Hu2]). We will construct Ricci flow with surgery with initial data a complete 3-orbifold with bounded geometry and with no bad 2-suborbifolds, which extends [P2], [BBM] and [KL2]. In contrast to the oriented case considered in [P2], [BBM] and [KL2], new orbifold singularities may be introduced in the process of surgery in the nonoriented case. Following [CTZ] and [Hu2], we deal with this situation using Hamilton’s canonical parametrization for a neck region ([H97]), which also applies to the 3-dimensional case.

In Section 2 we establish the short time existence of Ricci flow on complete orbifolds with bounded curvature, and describe the canonical neighborhood structure of ancient $\kappa$-solutions on 3-orbifolds. In Section 3 we construct Ricci flow with surgery with initial data a complete 3-orbifold with bounded geometry and containing no bad 2-suborbifolds, then Theorem 1.1 follows immediately.
2 Ancient $\kappa$-solutions on 3-orbifolds

First we establish short time existence for Ricci flow on complete orbifolds (not necessarily of 3-dimensional) with bounded curvature. Let $(O, g_0)$ be a complete $n$-orbifold with $|Rm| \leq K$ for a constant $K$. Consider the Ricci flow ([H82])

$$\frac{\partial g}{\partial t} = -2 \text{Ric}, \quad g|_{t=0} = g_0.$$ 

We will extend Shi’s result in [S] to get the following

**Theorem 2.1.** Let $(O, g_0)$ be a complete $n$-orbifold with $|Rm| \leq K$. Then the Ricci flow with initial data $(O, g_0)$ has a short time solution.

**Proof** The proof of Shi [S] in the manifold case can be adapted to the orbifold case with some minor modifications. We only indicate the necessary changes. First one can write the complete $n$-orbifold $O$ as a union of a sequence of compact $n$-suborbifolds with boundary $(n-1)$-suborbifolds. (This is possible, as is seen by smoothing the distance function from a given smooth point via heat equation, and using Sard theorem and preimage theorem (see [BB]).) Also note that Stokes theorem holds in a bounded domain (with boundary a suborbifold) in a Riemannian orbifold. (In fact it holds for a slightly more general domain, see [C].) In Lemma 3.1 (on p. 244) of [S], the dependence on the injectivity radius of $D$ can be replaced by that on the Sobolev constant in some Sobolev inequality which holds in a bounded domain (with boundary a suborbifold) in a Riemannian orbifold. (For Sobolev inequalities on compact manifolds with boundary, one can see [A] and [He]; for Sobolev inequalities on closed orbifold, see [N] and [F]. The extension to the case of compact orbifolds with boundary is routine.) In pages 260 and 286 of [S], one can pull back the solution to (a suitable ball in) $T_{\tilde{x}_0} \tilde{U}$ via $exp_{\tilde{x}_0} \circ \pi_*$, where $(\tilde{U}, G, \pi)$ is a orbifold chart for some $U$ around $x_0$. $\square$

By extending the proof of [Ko] to the orbifold case, one also has uniqueness for the Ricci flow (with a given initial data) in the category of bounded curvature solutions (even in a slightly larger category).

From now on we will restrict to 3-dimensional orbifolds. The notion of ancient $\kappa$-solutions can be extended obviously from manifolds to orbifolds. (See [KL2].) We will investigate their structure. First we define necks and caps. (See also [KL2].) Let $\Gamma$ be a finite subgroup of $O(3)$. A (topological) neck is an orbifold which is diffeomorphic to $S^2//\Gamma \times \mathbb{R}$, where $S^2//\Gamma$ denotes the quotient orbifold. To define caps, let $\Gamma'$ be a finite subgroup of $O(3)$ such that $S^2//\Gamma'$ admits an isometric involution $\sigma$ (i.e. a nontrivial isometry $\sigma$ with $\sigma^2 = 1$), and consider the quotient $(S^2//\Gamma' \times \mathbb{R})//\{1, \hat{\sigma}\}$, where $\hat{\sigma}$ is the reflection on the orbifold $S^2//\Gamma' \times \mathbb{R}$ defined by $\hat{\sigma}(x, s) = (\sigma(x), -s)$ for $x \in S^2//\Gamma'$ and $s \in \mathbb{R}$. We denote this orbifold by $S^2//\Gamma' \times_{Z_2} \mathbb{R}$. (By the way, note that $\Gamma'$ and $\hat{\sigma}$ may act isometrically on $S^3$ in a natural way.) We define a (topological) cap to be an orbifold diffeomorphic to either $S^2//\Gamma' \times_{Z_2} \mathbb{R}$ as above, or $\mathbb{R}^3//\Gamma'$, where $\Gamma'$ is a finite subgroup of $O(3)$. 

By extending the proof of [Ko] to the orbifold case, one also has uniqueness for the Ricci flow (with a given initial data) in the category of bounded curvature solutions (even in a slightly larger category).
It turns out that any neck or cap can be written as an infinite orbifold connected sum of spherical 3-orbifolds. Let $\Gamma$ be a finite subgroup of $O(3)$, then $\Gamma$ acts on $S^3$ by suspension. Using a somewhat ambiguous notation, $S^2//\Gamma \times \mathbb{R}^1 \approx \cdots S^3//\Gamma \times S^3//\Gamma \times \cdots$, $\mathbb{R}^3//\Gamma \approx S^3//\Gamma \times Z$, and $S^2//\Gamma' \times Z_\mathbb{R} \approx S^3//\Gamma' \times S^3//\Gamma' \times \cdots$. Note that the orbifold connected sums appeared in these three examples are actually the operation of performing 0-surgery as defined in [KL2] (which can be extended to the non-oriented case), and we have omitted the $f$'s in the notation. Also note that for a diffeomorphism $f: S^2//\Gamma \rightarrow S^2//\Gamma$ the mapping torus $S^2//\Gamma \times f S^1 \approx S^3//\Gamma \times f$. By a result in [DL] (see Proposition 2.2 there), $f$ is isotopic to an isometry, so $S^2//\Gamma \times f S^1$ admits a $S^2 \times \mathbb{R}$-geometry.

The following proposition is an analog of Proposition 2.4 in [Hu2].

**Proposition 2.2.** There exist a universal positive constant $\eta$ and a universal positive function $\omega : [0, \infty) \rightarrow (0, \infty)$ such that for any 3-dimensional ancient $\kappa$-solution $(O, g(t))$, we have

\[(i) \quad R(x, t) \leq R(y, t)\omega(R(y, t)d_e(x, y)^2)\]

for any $x, y \in O$, $t \in (-\infty, 0]$, and

\[(ii) \quad |\nabla R|(x, t) \leq \eta R^2(x, t), \quad |\frac{\partial R}{\partial t}|(x, t) < \eta R^2(x, t)\]

for any $x \in O$, $t \in (-\infty, 0]$.

**Proof** We can prove along the lines of [Hu2], using the corresponding result in the manifold case (see [P1, P2] and [CaZ]), and using an orbifold version of Gromoll-Meyer theorem in [Hu2]. \qed

Now we define $\varepsilon$-neck, $\varepsilon$-cap, and strong $\varepsilon$-neck. As in Definition 2.15 in [KL2], we do not require the map in the definition of $\varepsilon$-closeness of two pointed orbifolds to be precisely basepoint-preserving. Given a Riemannian 3-orbifold $(O, g)$, an open subset $U$, and a point $x_0 \in U$. $U$ is an $\varepsilon$-neck centered at $x_0$ if there is a diffeomorphism $\psi : (S^2//\Gamma) \times \mathbb{I} \rightarrow U$ such that the pulled back metric $\psi^*g$, scaling with some factor $Q$, is $\varepsilon$-close (in $C^{[\varepsilon^{-1}]}$ topology) to the standard metric on $(S^2//\Gamma) \times \mathbb{I}$ with scalar curvature 1 and $\mathbb{I} = (-\varepsilon^{-1}, \varepsilon^{-1})$, and the distance $d(x_0, |\psi((S^2//\Gamma \times \{0\}))|) < \varepsilon/\sqrt{Q}$. (Here $\Gamma$ is a finite subgroup of isometries of $S^2$.)

An open subset $U$ is an $\varepsilon$-cap centered at $x_0$ if $U$ is diffeomorphic to $\mathbb{R}^3//\Gamma$ or $S^2//\Gamma' \times Z_\mathbb{R}$, and there is an open set $V$ with compact closure such that $x_0 \in V \subset \overline{V} \subset U$, and $U \setminus \overline{V}$ is an $\varepsilon$-neck. Given a 3-dimensional Ricci flow $(O, g(t))$, an open subset $U$, and a point $x_0 \in U$. $U$ is a strong $\varepsilon$-neck centered at $(x_0, t_0)$ if there is a diffeomorphism $\psi : (S^2//\Gamma) \times \mathbb{I} \rightarrow U$ such that, the pulled back solution $\psi^*g(\cdot, \cdot)$ on the parabolic region $\{(x, t) | x \in U, t \in [t_0 - Q^{-1}, t_0]\}$ (for some $Q > 0$), parabolically rescaled with factor $Q$, is $\varepsilon$-close (in $C^{[\varepsilon^{-1}]}$ topology) to the subset $(S^2//\Gamma \times \mathbb{I}) \times [-1, 0]$ of the evolving round cylinder $S^2//\Gamma \times \mathbb{R}$, with scalar curvature
1 and length $2\varepsilon^{-1}$ to $I$ at time zero, and the distance at time $t_0, d_{t_0}(x_0, |\psi|(S^2//\Gamma \times \{0\})) < \varepsilon/\sqrt{Q}$.

The following proposition is analogous to Proposition 2.5 in [Hu2].

**Proposition 2.3.** For every $\varepsilon > 0$ there exist constants $C_1 = C_1(\varepsilon)$ and $C_2 = C_2(\varepsilon)$, such that for every 3-dimensional ancient $\kappa$-solution $(O, g(\cdot))$ containing no bad 2-suborbifolds, for each space-time point $(x, t)$, there is a radius $r$, $\frac{1}{C_1}(R(x, t))^{-\frac{1}{2}} < r < C_1(R(x, t))^{-\frac{1}{2}}$, and an open neighborhood $B, \overline{B}(x, r) \subset B \subset B(x, 2r)$, which falls into one of the following categories:

(a) $B$ is a strong $\varepsilon$-neck centered at $(x, t)$,
(b) $B$ is an $\varepsilon$-cap,
(c) $B$ is diffeomorphic to a spherical orbifold $S^3//\Gamma$ (for a finite subgroup $\Gamma$ of $O(4)$).

Moreover, the scalar curvature in $B$ in cases (a) and (b) at time $t$ is between $C_2^{-1}R(x, t)$ and $C_2R(x, t)$.

**Proof** The proof is similar to that of Proposition 2.5 in [Hu2], using (an extension of) Hamilton’s canonical parametrization for a neck region ([H97]). \qed

One can easily establish 3-dimensional analogues of Proposition 3.2 and Lemma 3.4 in [Hu2] on overlapping $\varepsilon$-necks. We also have the following

**Proposition 2.4.** Let $\varepsilon$ be sufficiently small. Let $(O, g)$ be a complete, connected 3-orbifold. If each point of $O$ is the center of an $\varepsilon$-neck or an $\varepsilon$-cap, then $O$ is diffeomorphic to a spherical orbifold, or a neck, or a cap, or an orbifold connected sum of at most two spherical 3-orbifolds.

**Proof.** The proof is along the lines of that of Proposition 3.3 in [Hu2]. \qed

## 3 Ricci flow with surgery on 3-orbifolds

For the notion of evolving Riemannian orbifold $\{(O(t), g(t))_{t \in I}\}$ see [Hu2] (also see [BBM] for the manifold case).

As in [BBM], a time $t \in I$ is regular if $t$ has a neighborhood in $I$ where $O(\cdot)$ is constant and $g(\cdot)$ is $C^1$-smooth. Otherwise it is singular. We also denote by $f_{\text{max}}$ and $f_{\text{min}}$ the supremum and infimum of a function $f$, respectively, as in [BBM].

**Definition** (Compare [BBM], [Hu2]) A piecewise $C^1$-smooth evolving Riemannian 3-orbifold $\{(O(t), g(t))_{t \in I}\}$ is a surgical solution of the Ricci flow if it has the following properties.

i. The equation $\frac{\partial g}{\partial t} = -2Ric$ is satisfied at all regular times;
ii. For each singular time $t_0$ one has $R_{\text{min}}(g_+(t_0)) \geq R_{\text{min}}(g(t_0));$
iii. For each singular time $t_0$ there is a locally finite collection $S$ of disjoint embedded $S^2//\Gamma$’s in $O(t_0)$ (where $\Gamma$’s are finite subgroups of $O(3)$), and an orbifold $O'$ such that
(a) $O'$ is obtained from $O(t_0) \setminus S$ by gluing back $B^3/\Gamma$'s;
(b) $O_+(t_0)$ is a union of some connected components of $O'$ and $g_+(t_0) = g(t_0)$ on $O_+(t_0) \cap O(t_0)$;
(c) Each component of $O' \setminus O_+(t_0)$ is diffeomorphic to a spherical 3-orbifold, or a neck, or a cap, or an orbifold connected sum of at most two spherical 3-orbifolds.

Now we introduce the notion of canonical neighborhood following [P2].

**Definition** Let $\varepsilon$ and $C$ be positive constants. A point $(x,t)$ in a surgical solution to the Ricci flow is said to have an $(\varepsilon,C)$-canonical neighborhood if there is an open neighborhood $U$, $B_t(x,\sigma) \subset U \subset B_t(x,2\sigma)$ with $C^{-1}R(x,t)^{-\frac{1}{2}} < \sigma < CR(x,t)^{-\frac{1}{2}}$, which falls into one of the following four types:
(a) $U$ is a strong $\varepsilon$-neck with center $(x,t)$,
(b) $U$ is an $\varepsilon$-cap with center $x$ for $g(t)$,
(c) at time $t$, $U$ is diffeomorphic to a closed spherical orbifold $S^3/\Gamma$,
and if moreover, the scalar curvature in $U$ at time $t$ satisfies the derivative estimates
$$|\nabla R| < CR^2 \quad \text{and} \quad \left|\frac{\partial R}{\partial t}\right| < CR^2,$$
and, for cases (a) and (b), the scalar curvature in $U$ at time $t$ is between $C^{-1}R(x,t)$ and $CR(x,t)$, and for case (c), the curvature operator of $U$ is positive, and the infimal sectional curvature of $U$ is greater than $C^{-1}R(x,t)$.

We choose constants $\varepsilon_0 > 0$ and $C_0$ similarly as in [Hu2].

Now we consider some a priori assumptions, which consist of the pinching assumption and the canonical neighborhood assumption.

Following [MT] and [BBB+], a 3-dimensional surgical solution $(O(t),g(t))$ to the Ricci flow has curvature pinched toward positive at all space-time points if for any $x \in O(t)$ there holds
$$R(x,t) \geq -\frac{6}{4t+1},$$
$$R(x,t) \geq 2(-\nu(x,t))(\ln(-\nu(x,t)) + \ln(1+t) - 3) \quad \text{when} \quad \nu(x,t) < 0,$$
where $\nu$ is the least eigenvalue of the curvature operator.

**Pinching assumption**: A 3-dimensional surgical solution $(O(t),g(t))$ to the Ricci flow satisfies the pinching assumption if it has curvature pinched toward positive at all space-time points.

**Canonical neighborhood assumption**: Let $\varepsilon_0$ and $C_0$ be given as above. Let $r : [0, +\infty) \to (0, +\infty)$ be a non-increasing function. An evolving Riemannian 3-orbifold $\{(O(t),g(t))\}_{t \in I}$ satisfies the canonical neighborhood assumption $(CN)$, if any space-time point $(x,t)$ with $R(x,t) \geq r^{-2}(t)$ has an $(\varepsilon_0, C_0)$-canonical neighborhood.
Bounded curvature at bounded distance is one of the key ideas in Perelman [P1], [P2]; compare [MT, Theorem 10.2], [BBB⁺, Theorem 6.1.1] and [BBM, Theorem 6.4]. 4-dimensional versions have appeared in [CZ2] and [Hu1, Hu2]. The following version is an extension of that in [BBB⁺] and [BBM] to the orbifold case.

**Proposition 3.1.** For each $A, C > 0$ and each $\varepsilon \in (0, 2\varepsilon_0]$, there exists $Q = Q(A, \varepsilon, C) > 0$ and $\Lambda = \Lambda(A, \varepsilon, C) > 0$ with the following property. Let $I \subset [0, \infty)$ and \{(O(t), g(t))\}$_{t \in I}$ be a 3-dimensional surgical solution with bounded curvature (at each time) and satisfying the pinching assumption. Let $(x_0, t_0)$ be a space-time point such that:

1. $(1 + t_0)R(x_0, t_0) \geq Q$;
2. For each point $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, if $R(y, t_0) \geq 4R(x_0, t_0)$, then $(y, t_0)$ has an $(\varepsilon, C)$-canonical neighborhood.

Then for any $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, we have

$$\frac{R(y, t_0)}{R(x_0, t_0)} \leq \Lambda.$$

**Proof** The proof is similar to that of Proposition 4.1 in [Hu2]. □

The following proposition is analogous to Proposition 4.2 in [Hu2]; compare [BBM, Theorem 6.5] and [BBB⁺, Theorem 6.2.1].

**Proposition 3.2.** For any $r, \delta > 0$, there exist $h \in (0, \delta r)$ and $D > 10$, such that if $(O(\cdot), g(\cdot))$ is a complete 3-dimensional surgical solution with bounded curvature, defined on a time interval $[a, b]$ and satisfying the pinching assumption and the canonical neighborhood assumption $(CN)_r$, then the following holds:

Let $t \in [a, b]$ and $x, y, z \in O(t)$ such that $R(x, t) \leq 2/r^2$, $R(y, t) = h^{-2}$ and $R(z, t) \geq D/h^2$. Assume there is a curve $\gamma$ in $O(t)$ connecting $x$ to $z$ via $y$, such that each point of $\gamma$ with scalar curvature in $[2C_0r^{-2}, C_0^{-1}Dh^{-2}]$ is the center of an $\varepsilon_0$-neck. Then $(y, t)$ is the center of a strong $\delta$-neck.

**Proof** The proof is along the lines of that of Proposition 4.2 in [Hu2], using Proposition 3.1. □

The metric surgery on a $\delta$-neck is similar as in [H97] and [P2]; for the orbifold case see also the 4-dimensional analog in [Hu2]. Usually we will be given two non-increasing step functions $r, \delta : [0, +\infty) \to (0, +\infty)$ as surgery parameters. Let $h(r, \delta), D(r, \delta)$ be the associated parameter as determined in Proposition 3.2, (h is also called the surgery scale,) and let $\Theta := 2Dh^{-2}$ be the curvature threshold for the surgery process (as in [BBB⁺], [BBM] and [Hu1, Hu2]), that is, we will do surgery when $R_{\max}(t)$ reaches $\Theta(t)$. Now we adapt two more definitions from [BBM] and [Hu1, Hu2].
Definition (compare [BBM], [Hu1, Hu2]) Given an interval $I \subset [0, +\infty)$, fix surgery parameter $r$, $\delta : I \to (0, +\infty)$ (two non-increasing functions) and let $h$, $D$, $\Theta = 2 Dh^{-2}$ be the associated cutoff parameters. Let $(\mathcal{O}(t), g(t)) (t \in I)$ be an evolving Riemannian 3-orbifolds. Let $t_0 \in I$ and $(\mathcal{O}_+, g_+)$ be a (possibly empty) Riemannian 3-orbifolds. We say that $(\mathcal{O}_+, g_+)$ is obtained from $(\mathcal{O}(-), g(-))$ by $(r, \delta)$-surgery at time $t_0$ if

i. $R_{\text{max}}(g(t_0)) = \Theta(t_0)$, and there is a locally finite collection $\mathcal{S}$ of disjoint embedded $\mathbb{S}^2/\Gamma$’s in $\mathcal{O}(t_0)$ which are in the middle of strong $\delta(t_0)$-necks with radius equal to the surgery scale $h(t_0)$, such that $\mathcal{O}_+$ is obtained from $\mathcal{O}(t_0)$ by doing surgery along these necks (where $\Gamma$’s are finite subgroups of $O(3)$), and removing the components each of which is diffeomorphic to

(a) a spherical 3-orbifold, and either has sectional curvature bounded below by $\frac{100}{100}$ or is covered by $\varepsilon_0$-necks and $\varepsilon_0$-caps, or

(b) a neck, and is covered by $\varepsilon_0$-necks, or

(c) a cap, and is covered by $\varepsilon_0$-necks and an $\varepsilon_0$-cap, or

(d) an orbifold connected sum of at most two spherical 3-orbifolds, and is covered by $\varepsilon_0$-necks and $\varepsilon_0$-caps.

ii. If $\mathcal{O}_+ \neq \emptyset$, then $R_{\text{max}}(g_+) \leq \Theta(t_0)/2$.

Definition (cf. [BBM] and [Hu1, Hu2]) A surgical solution $(\mathcal{O}(-), g(-))$ defined on some time interval $I \subset [0, +\infty)$ is an $(r, \delta)$-surgical solution if it has the following properties:

i. It satisfies the pinching assumption, and $R(x, t) \leq \Theta(t)$ for all $(x, t)$;

ii. At each singular time $t_0 \in I$, $(\mathcal{O}_+(t_0), g_+(t_0))$ is obtained from $(\mathcal{O}(-), g(-))$ by $(r, \delta)$-surgery at time $t_0$;

iii. Condition (CN)$_r$ holds.

Recall that in our 3-dimensional case, $g(\cdot)$ is $\kappa$-noncollapsed (for some $\kappa > 0$) on the scale $r$ at time $t$ if at any point $x$, whenever $|Rm| \leq r^{-2}$ on $P(x, t, r, -r^2)$ we have $\text{vol}B(x, t, r) \geq \kappa r^3$. Let $\kappa : I \to (0, +\infty)$ be a function. We say $\{(\mathcal{O}(t), g(t))\}_{t \in I}$ has property (NC)$_\kappa$ if it is $\kappa(t)$-noncollapsed on all scales $\leq 1$ at any time $t \in I$. An $(r, \delta, \kappa)$-surgical solution which also satisfies condition (NC)$_\kappa$ is called an $(r, \delta, \kappa)$-surgical solution.

Now we establish the existence of Ricci flow with surgery with initial data a complete 3-orbifold with bounded geometry and containing no bad 2-orbifolds. (Note that by working equivariantly as in [DL], the conclusion in the following theorem should also hold true when the initial data has a covering (instead of itself) which has bounded geometry; but we will not use this more general version here.)

**Theorem 3.3.** Given $v_0 > 0$, there are surgery parameter sequences

$$K = \{\kappa_i\}_{i=1}^\infty, \Delta = \{\delta_i\}_{i=1}^\infty, \mathbf{r} = \{r_i\}_{i=1}^\infty$$
such that the following holds. Let \( r(t) = r_i \) and \( \delta(t) = \delta_i \) and \( \kappa(t) = \kappa_i \) on \([(i-1)2^{-5}, i\cdot2^{-5}) \), \( i = 1, 2, \ldots \). Let \((\mathcal{O}, g_0)\) be a complete 3-orbifold with \(|Rm| \leq 1\), with \( \text{vol} B(x, 1) \geq v_0 \) at any point \( x \), and containing no bad 2-suborbifolds. Then there exists an \((r, \delta, \kappa)\)-surgical solution defined on time interval \([0, \infty)\) with initial data \((\mathcal{O}, g_0)\).

**Proof** The proof is similar to that of Theorem 5.5 in [Hu2]. \( \square \)

Note that if we assume the initial metric has uniformly positive scalar curvature in addition, the surgical solution constructed in Theorem 3.3 will become extinct in finite time. Then Theorem 1.1 follows easily, using our procedure of surgery, Proposition 2.4 and the observations before Proposition 2.2.

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