A LEARNING-ENHANCED PROJECTION METHOD FOR
SOLVING CONVEX FEASIBILITY PROBLEMS

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Abstract. We propose a generalization of the method of cyclic projections, which uses the lengths of projection steps carried out in the past to learn about the geometry of the problem and decides on this basis which projections to carry out in the future. We prove the convergence of this algorithm and illustrate its behavior in a first numerical study.

1. Introduction. The method of cyclic projections, originally proposed in [7], is an established numerical algorithm, which computes a point in the intersection of finitely many closed convex subsets of a Hilbert space when this intersection is nonempty. A broad overview over convergence properties of this method as well as the underlying theory is given in [3], [4], [9] and the references therein. The method is frequently used in imaging, see e.g. [1], [10] and [15] for applications in computerized tomography and [14] and [19] for applications in phase retrieval.

Estimates for the speed of convergence of the method of cyclic projections are well-known in the case when the sets are affine linear subspaces. For this situation, accelerated variants of the original scheme, which are often based on line-search ideas, have been developed, see e.g. [5], [13] and [18]. Recently, a first result on the speed of convergence of the method of cyclic projections has been given in the case of semi-algebraic sets, see [6]. In general, however, the method can be arbitrarily slow, see [12] for a pathological example.

When the sets are affine linear subspaces with codimension 1, the method of cyclic projections reduces to the Kaczmarz method, see [16], which has gained popularity in the context of very large, but sparse consistent linear systems, see [8]. A probabilistic version of this algorithm, which converges exponentially in expectation, has been introduced in [17], and an accelerated version of this method has been proposed in [2].

The numerical method presented in this paper is supposed to accelerate the method of cyclic projections in settings where the above-mentioned refined algorithms for subspaces are not applicable. The guiding idea behind the method is to gather as much information on the relative geometry of the closed convex sets from

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the lengths of the projection steps carried out in the past. This is motivated by the convergence proof in [7], which reveals that the performance of the algorithm is in worst case determined by the lengths of the projection steps carried out.

We prove that our method converges, using techniques which are common in the dynamical systems community. The main challenge is to guarantee convergence for a reasonably broad class of strategies our basic algorithm can be equipped with. As it seems very hard to quantify a speed of convergence even in the subspace case, we provide several numerical studies performed on a toy example, which provide some insight as to why and how our method can outperform the standard methods of cyclic and random projections.

2. The algorithm. Given closed convex sets $C_1, \ldots, C_N \subset \mathbb{R}^d$ with 

$$C := \cap_{j=1}^N C_j \neq \emptyset,$$

we wish to find a point $x^* \in C$. We first present two common projection algorithms for solving this problem in Section 2.1. Then we propose a new projection algorithm in Section 2.2, which learns the geometry of the problem to some extent from the lengths of the projection steps carried out in the past and uses this knowledge to select favourable projections in the future.

The notation used in this paper is mostly standard. Given a point $x \in \mathbb{R}^d$ and a closed convex set $C \subset \mathbb{R}^d$, it is well-known that the projection

$$\text{proj}(x, C) := \arg\min_{z \in C} \|x - z\|$$

of $x$ to $C$ exists and is a unique point. By randperm$(1, \ldots, N)$ we denote a permutation of the numbers $1, \ldots, N$ which is sampled uniformly from the set of all such permutations. By urs$(I)$, we denote a uniform random sample from an index set $I \subset \{1, \ldots, N\}$, which means that urs$(I)$ is a randomly selected element of $I$, and all elements of $I$ are selected with the same probability.

2.1. The benchmark: MCP and MRP. The now classical method of cyclic projections, which was originally published in [7], approximates a point $x^* \in C$ by iteratively projecting to the sets $C_1, \ldots, C_N$ in a cyclic fashion, see Algorithm 1.

**Algorithm 1**: Method of cyclic projections (MCP)

```
Input: $C_1, \ldots, C_N \subset \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$

1 for $k \leftarrow 0$ to $\infty$ do
2 $x_{k+1} \leftarrow \text{proj}(x_k, C_{\text{mod}(k,N)+1});$
3 end
```

Algorithm 1 may converge very slowly when many of the projection steps are small. This behavior may originate from an unfavorable ordering of the convex sets $C_1, \ldots, C_N$, which can be helped by randomly shuffling the order of the sets in every cycle, see Algorithm 2.

The random Kaczmarz method proposed in [17] is a prominent variant of Algorithm 2 in the framework of row-action methods for solving linear systems, which is known to converge in expectation. Since MRP slightly outperformed the random Kaczmarz method in all examples we have studied, we use MRP as the benchmark for randomized algorithms.
Algorithm 2: Method of randomized projections (MRP)

Input: $C_1, \ldots, C_N \subset \mathbb{R}^d$, $x^{(0)}_0 \in \mathbb{R}^d$

1. for $k \leftarrow 0$ to $\infty$ do
   2. $\pi \leftarrow \text{randperm}(1, \ldots, N)$;
   3. for $j \leftarrow 0$ to $N - 1$ do
      4. $x^{(k)}_{j+1} \leftarrow \text{proj}(x^{(k)}_j, C_{\pi(j+1)})$;
   5. end
   6. $x^{(k+1)}_0 \leftarrow x^{(k)}_N$;
7. end

2.2. A projection algorithm with learning ability. The idea behind Algorithm 3 (PAM) is to keep a record of the lengths of projection steps performed in the past and to give preference to operations that have lead to large projection steps. This enables our algorithm to learn to some extent the geometry of the problem with manageable additional computational cost.

From a formalistic point of view, our approach resembles to some extent the techniques of loping and flagging introduced in [10] in the setting of row-action methods. These techniques suppress the effect of noise in the data on MCP by ignoring projections which had very small residuals in previous cycles. From a phenomenological perspective, however, these modifications of MCP do not have much in common with PAM.

In the following, we give an intuitive description how some of the individual components interact in Algorithm 3. They will be treated with proper mathematical rigour in the next section.

i) The sequence of matrices $(D_k^k)_{k \in \mathbb{N}}$ records – up to the impact of the function $\varphi$ – the length of the $k$-th projection step from set $C_{j_k}$ to $C_{j_{k+1}}$ in the component $D_{j_k, j_{k+1}}^{k+1}$.

ii) The input $D^0$ has three distinct effects.
   a) If $D^0_{m,n} = 0$, a transition from $C_m$ to $C_n$ will not occur during the entire runtime of the algorithm, see Lemma 3.3. Thus, by choosing a sparse $D^0$ as in Example 1(i), one can limit the amount of information that needs to be stored and processed at runtime. For a graphic illustration, see Example 5, and for the impact on performance in the context of a toy model, see Example 6.
   b) The choice of $D^0$ can incorporate a priori knowledge: The more likely a transition from $C_m$ to $C_n$ is to be beneficial, the larger the entry $D^0_{m,n}$ should be chosen, see Example 1(ii).
   c) If the entries of $D^0$ are small relative to the first several lengths $\|x_{k+1} - x_k\|$ of steps to be carried out, the algorithm will not perform well in an initial stage, see Example 4(ii). If they are larger, the algorithm will initially behave like MRP, see Example 4(i).

iii) The function $\varphi$ modifies the step length $\|x_{k+1} - x_k\|$ before it is recorded in the matrix $D^{k+1}$.
   a) It ensures that only strictly positive values are written into $D^{k+1}$. This property is needed to ensure the recurrence of PAM, see Proposition 2, and hence its convergence, see Proposition 1.
b) It determines at what level of overall performance a transition from $C_m$ to $C_n$ will get reactivated after it generated a short step.

**Algorithm 3:** Projection algorithm with memory (PAM)

**Input:** $C_1, \ldots, C_N \subset \mathbb{R}^d$, $x_0 \in C_1$, $D^0 \in \mathbb{R}^{N \times N}_{\geq 0}$, $\varphi : \{1, \ldots, N\} \times \mathbb{R}^{N \times N}_{\geq 0} \to \mathbb{R}_{\geq 0}$

1. $j_0 \leftarrow 1$
2. **for** $k \leftarrow 0$ **to** $\infty$ **do**
   
   /* carry out most promising admissible projection */

3. $j_{k+1} \leftarrow \text{urs}(\text{argmax}_{\ell \in \{1, \ldots, N\} \setminus \{j_k\}} D^k_{j_k, \ell})$;

4. $x_{k+1} \leftarrow \text{proj}(x_k, C_{j_{k+1}})$;

/* update distance matrix */

5. $D^{k+1} \leftarrow D^k$;

6. $D^{k+1}_{j_k, j_{k+1}} \leftarrow \max\{\|x_{k+1} - x_k\|, \varphi(j_k, D^k)\}$;

**end**

Finally, we would like to mention that we represent the recorded step-lengths in a matrix $D^0$ to keep the notation manageable. Depending on the size of the problem, the sparsity pattern of $D^0$ and the policy $\varphi$, it can be beneficial to use a different data structure – such as one red-black tree per set $C_j$ – to store and search this data with moderate on-cost compared to MCP and MRP.

3. **Admissible input.** For Algorithm 3 to converge, we require the inputs to have certain properties. The matrix $D^0$ is required to be irreducible in the following sense.

**Definition 3.1** (admissible matrix). A matrix $D \in \mathbb{R}^{N \times N}_{\geq 0}$ is called admissible if it satisfies

i) $D_{m, m} = 0$ for all $m \in \{1, \ldots, N\}$, and 

ii) for any indices $m, n \in \{1, \ldots, N\}$ with $m \neq n$, there exist some $\ell \in \mathbb{N}$ and indices $i_1, \ldots, i_\ell \in \{1, \ldots, N\}$ such that

\[ i_1 = m, \quad i_\ell = n, \quad \text{and} \quad D_{i_s, i_{s+1}} > 0 \quad \forall s \in \{1, \ldots, \ell - 1\} \]

We give a few examples how the matrix $D^0$ can be chosen.

**Example 1** (some admissible matrices). For a good performance of Algorithm 3, it is helpful to multiply the matrices proposed below by a positive scalar to ensure that their respective nonzero entries are – at least on average and for small $k$ – similar to or larger than the length $\|x_{k+1} - x_k\|$ of the $k$-th projection step from set $C_{j_k}$ to $C_{j_{k+1}}$.

i) To limit the effective size of the matrices $D^k$, one can choose $D^0$ to be a sparse matrix such as the banded matrices $D^{\leftrightarrow} \in \mathbb{R}^{N \times N}$ given by

\[ D^{\leftrightarrow}_{m, n} = \begin{cases} 1, & 0 < |n - m| \leq \omega \text{ or } N + m - n \leq \omega \text{ or } N + n - m \leq \omega, \\ 0, & \text{otherwise} \end{cases} \]

and $D^{\rightarrow} \in \mathbb{R}^{N \times N}$ given by

\[ D^{\rightarrow}_{m, n} = \begin{cases} 1, & 0 < n - m \leq \omega \text{ or } N + n - m \leq \omega, \\ 0, & \text{otherwise} \end{cases} \]
with some $\omega \in \mathbb{N}$ with $1 \leq \omega \ll N$.

ii) In scenarios where the concept of an angle makes sense, it is reasonable to work with a matrix $D^{\angle} \in \mathbb{R}^{N \times N}$ given by

$$D_{m,n}^{\angle} = \begin{cases} 
\gg 1, & \angle(C_m, C_n) \text{ known to be large}, \\
1, & \angle(C_m, C_n) \text{ unknown}, \\
\ll 1, & \angle(C_m, C_n) \text{ known to be small}, \\
0, & m = n
\end{cases}$$

to introduce a bias in favour of transitions with large angles, which are more likely to result in large step-lengths.

It is easy to check that the above matrices are admissible. Please note that MCP is a special case of Algorithm 3, which can be realized by choosing $D^0$ to be the matrix $D^\theta$ with $\omega = 1$.

The function $\varphi$ must be strictly positive on all meaningful input to ensure the recurrence of PAM. It also must not generate unreasonably large numbers, which would enter the matrix $D^{k+1}$ in line 6 of Algorithm 3 instead of the actual step-length $\|x_{k+1} - x_k\|$.

**Definition 3.2 (admissible policies).** A function

$$\varphi : \{1, \ldots, N\} \times \mathbb{R}_{\geq 0}^{N \times N} \to \mathbb{R}_{\geq 0}$$

is called an admissible policy if there exists $\beta \in (0, 1)$ such that

i) $\varphi(m, D) > 0$ holds for all $m \in \{1, \ldots, N\}$ and $D \in \mathbb{R}_{\geq 0}^{N \times N}$ which satisfy that

$$\max_n D_{m,n} \neq 0,$$

and

ii) $\varphi(m, D) \leq \beta \max_n D_{m,n}$ for all $m \in \{1, \ldots, N\}$ and $D \in \mathbb{R}_{\geq 0}^{N \times N}$.

We propose some particular policies $\varphi$.

**Example 2 (some admissible policies).** It is easy to check that both policies proposed below are indeed admissible for every $\beta \in (0, 1)$.

i) The function

$$\varphi_{\min}(m, D) := \beta \min_{\{n : D_{m,n} > 0\}} D_{m,n}$$

ensures that the number which is written into the distance matrix $D^{k+1}$ in line 7 of Algorithm 3 is at least $\beta$ times the minimal previously recorded step-length from $C_m$ to another $C_n$.

ii) The function

$$\varphi_{av}(m, D) := \frac{\beta}{\#\{n : D_{m,n} > 0\}} \sum_{\{n : D_{m,n} > 0\}} D_{m,n}$$

ensures that the number which is written into $D^{k+1}$ in line 7 of Algorithm 3 is at least $\beta$ times the average of the previously recorded step-lengths from $C_m$ to another $C_n$.

Note that the value of the minimal nonzero entry in a row as well as the average of the nonzero entries in a row can be updated with negligible computational cost in every step.

The following lemma ensures that the irreducibility of the initial matrix $D^0$ is inherited by all matrices $D^k$ generated by Algorithm 3.
Lemma 3.3 (preservation of sparsity pattern). Let both \( D^0 \in \mathbb{R}^{N \times N}_{\geq 0} \) as well as \( \varphi : \{1, \ldots, N\} \times \mathbb{R}^{N \times N}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be admissible, and let \((D^k)_{k \in \mathbb{N}} \in (\mathbb{R}^{N \times N}_{\geq 0})^N\) be the matrices generated by Algorithm 3 with arbitrary initial value \( x \in \mathbb{R}^d \). Then for any \( k \in \mathbb{N} \) and \( m, n \in \{1, \ldots, N\} \), we have
\[
D_{m,n}^k > 0 \quad \text{if and only if} \quad D_{m,n}^0 > 0,
\]
and, in particular, the matrices \( D^k \) are admissible for all \( k \in \mathbb{N} \).

Proof. If \( D_{m,n}^0 > 0 \) and \( D_{m,n}^k = 0 \), then there exists \( \ell \in \{0, \ldots, k - 1\} \) such that \( D_{m,n}^\ell > 0 \) and \( D_{m,n}^{\ell + 1} = 0 \). According to lines 6 and 7 of Algorithm 3, and by property ii) of Definition 3.2, this is impossible, so \( D_{m,n}^0 > 0 \) implies \( D_{m,n}^k > 0 \). In particular, the matrices \( D^k \) are admissible for all \( k \in \mathbb{N} \).

If \( D_{m,n}^0 = 0 \) and \( D_{m,n}^k > 0 \), then there exists \( \ell \in \{0, \ldots, k - 1\} \) such that \( D_{m,n}^\ell = 0 \) and \( D_{m,n}^{\ell + 1} > 0 \). Since \( D^\ell \) is admissible, line 3 of Algorithm 3 implies \( j_{\ell + 1} \neq n \). Now lines 6 and 7 of Algorithm 3 yield the contradiction \( D_{m,n}^\ell = D_{m,n}^{\ell + 1} \). \(\square\)

4. Convergence analysis. We first prove a general principle for projection algorithms in Section 4.1. Then we show in Section 4.2 that Algorithm 3 satisfies the assumptions of this statement.

4.1. Recurrence implies convergence. We restate a slightly modified version of Corollaries 1 and 2 from [7].

Lemma 4.1 (projections reduce error). Let \( C_1, \ldots, C_N \subseteq \mathbb{R}^d \) be closed convex sets and \( z \in \bigcap_{j=1}^NC_j \), and let the sequences \((j_k)_{k \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N}\) and \((x_k)_{k \in \mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}\) satisfy
\[
x_{k+1} = \text{proj}(x_k, C_{j_k}) \quad \forall k \in \mathbb{N}.
\]
Then we have
\[
\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - \|x_{k+1} - x_k\|^2 \quad \forall k \in \mathbb{N},
\]
(1)
\[
\|x_{k+1} - z\| \leq \|x_k - z\| \leq \|x_0 - z\| \quad \forall k \in \mathbb{N},
\]
(2)

It is well-known, see e.g. Theorem 1 in [11], that every projection algorithm, which projects to every set \( C_j \) infinitely often, generates a sequence that converges to a point in \( C \).

Proposition 1 (recurrence implies convergence). Let \( C_1, \ldots, C_N \subseteq \mathbb{R}^d \) be closed convex sets with \( \bigcap_{j=1}^NC_j \neq \emptyset \), and let \((j_k)_{k \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N}\) and \((x_k)_{k \in \mathbb{N}} \in (\mathbb{R}^d)^\mathbb{N}\) be sequences which satisfy
\[
x_{k+1} = \text{proj}(x_k, C_{j_k}) \quad \forall k \in \mathbb{N}
\]
as well as the recurrence condition
\[
\# \{k \in \mathbb{N} : j_k = j\} = \infty \quad \forall j \in \{1, \ldots, N\}.
\]
(3)
Then there exists \( x^* \in \bigcap_{j=1}^NC_j \) such that \( \lim_{k \to \infty} x_k = x^* \).

It remains to show that PAM satisfies the recurrence condition (3).
4.2. Convergence of PAM. We check that Algorithm 3 satisfies the assumptions of Proposition 1, whenever the matrix \( D^0 \) and the policy \( \varphi \) are admissible.

**Proposition 2** (PAM is recurrent). Let \( C_1, \ldots, C_N \subset \mathbb{R}^d \) be closed convex sets which satisfy \( \bigcap_{j=1}^N C_j \neq \emptyset \), and assume that both the matrix \( D^0 \in \mathbb{R}^{N \times N} \) and the policy \( \varphi : \{1, \ldots, N\} \times \mathbb{R}^{N \times N} \to \mathbb{R}_{\geq 0} \) are admissible. Then for any initial point \( x_0 \in C_1 \), the sequences \( (j_k)_{k \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} \) and \( (D^k)_{k \in \mathbb{N}} \in (\mathbb{R}^{N \times N})^\mathbb{N} \) generated by Algorithm 3 satisfy

\[
\lim_{k \to \infty} \max_{m, n \in \{1, \ldots, N\}} D^k_{m, n} = 0, \quad \text{(4)}
\]

\[
\# \{ k \in \mathbb{N} : j_k = m \} = \infty \quad \forall m \in \{1, \ldots, N\}. \quad \text{(5)}
\]

**Proof.** Let \( z \in \bigcap_{j=1}^N C_j \). Applying inequality (1) from Lemma 4.1 multiple times yields

\[
0 \leq \|x_k - z\|^2 \leq \|x_0 - z\|^2 - \sum_{j=0}^{k-1} \|x_{j+1} - x_j\|^2 \quad \forall k \in \mathbb{N},
\]

which forces

\[
\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0. \quad \text{(6)}
\]

Let us denote

\[
J_\infty := \{ m \in \{1, \ldots, N\} : \# \{ k \in \mathbb{N} : j_k = m \} = \infty \}.
\]

Obviously, we have \( J_\infty \neq \emptyset \). Let \( m \in J_\infty \), and let \( (k_\ell)_{\ell \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \) be the maximal strictly increasing sequence with \( j_{k_\ell} = m \) for all \( \ell \in \mathbb{N} \). Let \( \varepsilon > 0 \). By statement (6), there exists \( k^* \in \mathbb{N} \) such that

\[
\|x_{k+1} - x_k\| \leq \varepsilon \quad \text{for all} \quad k \geq k^*. \quad \text{(7)}
\]

Because of lines 5 and 6 of Algorithm 3, and since \( \varphi \) is admissible with a decay rate \( \beta \in (0, 1) \), we know that for all \( k \geq k^* \), the matrices \( D^{k+1} \) and \( D^k \) differ only in the index \((j_k, j_{k+1})\), and

\[
D^k_{j_k, j_{k+1}} = \max\{\|x_{k+1} - x_k\|, \varphi(j_k, D^k)\} \leq \max\{\varepsilon, \beta \max_n D^k_{j_k, n}\} \leq \max\{\varepsilon, \max_n D^k_{j_k, n}\}.
\]

Using the above statements recursively, we obtain

\[
\max_n D^{k'}_{m, n} \leq \max\{\varepsilon, \max_n D^k_{m, n}\} \quad \text{whenever} \quad k^* \leq k \leq k'. \quad \text{(8)}
\]

Now let \( \ell \in \mathbb{N} \) be such that \( k_\ell \geq k^* \). We wish to show that

\[
\max_n D^{k_\ell + N}_{m, n} \leq \max\{\varepsilon, \beta \max_n D^{k_\ell}_{m, n}\}. \quad \text{(9)}
\]

To this end, we introduce the quantity

\[
\nu(p) := \# \{ n : D^{k_\ell + p}_{m, n} > \max\{\varepsilon, \beta \max_n D^{k_\ell}_{m, n'}\} \}
\]

and prove the statement

\[
\nu(p) \leq N - p \quad \text{for} \quad p \in \{0, \ldots, N\} \quad \text{(10)}
\]

by induction. Statement (10) is trivial for \( p = 0 \). Assume that statement (10) holds for some \( p \in \{0, \ldots, N - 1\} \). We distinguish two cases:
• **Case 1:** If \( \nu(p) = 0 \), then

\[
\max_n D_{m,n}^{k_{i+1}} \leq \max\{\varepsilon, \beta \max_n D_{m,n}^{k_i}\}.
\]

Statement (8) implies that

\[
\max_n D_{m,n}^{k_{i+1}+1} \leq \max\{\varepsilon, \max_n D_{m,n}^{k_{i+1}}\} \leq \max\{\varepsilon, \beta \max_n D_{m,n}^{k_i}\},
\]

and hence that \( \nu(p+1) = 0 \), so the induction hypothesis (10) holds for \( p+1 \) as well.

• **Case 2:** If \( \nu(p) > 0 \), then line 3 of Algorithm 3 selects an index

\[
j_{k_{i+1}+1} \in \arg\max_{q \in \{1, \ldots, N\} \backslash \{m\}} D_{m,q}^{k_{i+1}}
\]

that satisfies

\[
D_{m,j_{k_{i+1}+1}}^{k_{i+1}} > \max\{\varepsilon, \beta \max_{n'} D_{m,n'}^{k_i}\},
\]

By line 6 of Algorithm 3, by statement (7), since \( \varphi \) is admissible, and by statement (8), we have

\[
D_{m,j_{k_{i+1}+1}}^{k_{i+1}+1} = \max\{\|x_{k_{i+1}+1} - x_{k_i}\|, \varphi(m, D_{m,n}^{k_i})\} \leq \max\{\varepsilon, \beta \max_n D_{m,n}^{k_i}\} \leq \max\{\varepsilon, \beta \max_n D_{m,n}^{k_i}\}.
\]

By construction of the sequence \((k_\ell)_{\ell \in \mathbb{N}}\), it follows that

\[
D_{m,n}^{k_{i+1}+1} = D_{m,n}^{k_{i+1}} \quad \forall n \in \{1, \ldots, N\}.
\]

Now \( \nu(p+1) = \nu(p) - 1 \) follows from statements (11), (12) and (13), and hence statement (10) holds for \( p+1 \).

This completes the induction. Hence inequality (10) holds for all \( p \). In particular, inequality (10) implies that \( \nu(N) = 0 \), and consequently, statement (9) is verified.

Since \( \beta < 1 \), statements (8) and (9) imply that there exists \( k^* \in \mathbb{N} \) such that \( \max_n D_{m,n}^{k_{i}} \leq \varepsilon \) for all \( k \geq k^* \). Since \( m \in J_\infty \) and \( \varepsilon > 0 \) were arbitrary, we have shown that

\[
\lim_{k \to \infty} \max_n D_{m,n}^{k} \to 0 \quad \forall m \in J_\infty.
\]

By Lemma 3.3, we have

\[
D_{m,n}^{k} > 0 \quad \text{if and only if} \quad D_{m,n}^{0} > 0,
\]

so statement (14) implies that

\[
n \in J_\infty \quad \text{whenever} \quad m \in J_\infty \text{ and } D_{m,n}^{0} > 0.
\]

Since \( D^0 \) satisfies part ii) of Definition 3.1, a simple recursion on statement (15) yields \( J_\infty = \{1, \ldots, N\} \), which is statement (5). Consequently, statement (14) implies (4).

Now we summarize the above in the main theoretical result of this paper.

**Theorem 4.2 (convergence of PAM).** Let \( C_1, \ldots, C_N \subset \mathbb{R}^d \) be closed convex sets which satisfy \( \cap_{j=1}^N C_j \neq \emptyset \), and assume that both, the matrix \( D^0 \in \mathbb{R}_{\geq 0}^{N \times N} \) and the policy \( \varphi : \{1, \ldots, N\} \times \mathbb{R}_{\geq 0}^{N \times N} \to \mathbb{R}_{\geq 0} \), are admissible. Then there exists a point \( x^* \in \cap_{j=1}^N C_j \) such that the sequence \((x_k)_{k \in \mathbb{N}} \in (\mathbb{R}^d)^N \) generated by Algorithm 3 satisfies

\[
\lim_{k \to \infty} x_k = x^*.
\]
**Proof.** Proposition 2 verifies that Algorithm 3 satisfies the assumptions of Proposition 1, so PAM is indeed convergent.

5. **An instructive toy example.** We explore the performance of PAM with different matrices $D^0$ and policies $\varphi$ in a very simple toy example, and compare its behavior with MCP and MRP. We are fully aware that this example has many unrealistic features, but it allows us to illustrate key features of our algorithm in a nice graphic way. To keep things simple, we measure the computational cost of all three algorithms in the number of iterations, which is the number of projection steps carried out.

Throughout this section, we consider the one-dimensional subspaces

$$C_j := \left\{ s \begin{pmatrix} r \cos\left(\frac{j\pi}{N}\right) \\ r \sin\left(\frac{j\pi}{N}\right) \\ 1 \end{pmatrix} : s \in \mathbb{R}\right\}, \quad j = 1, \ldots, N,$$

with $\cap_{j=1}^{N} C_j = \{0\}$, and the initial point $x_0 = (\cos(\frac{\pi}{N}), \sin(\frac{\pi}{N}), 1)$. For aesthetical reasons, we choose $N = 9$ and $r = 0.05$ in most illustrations.

Let us first compare MCP, MRP and PAM without going into too much technical detail.

**Example 3** (benchmark versus PAM). In Figure 1, we see at a glance how the strategies behind MCP, MRP and PAM impact their behavior and performance when applied to the toy model. The fixed order of projections in MCP can result in significant underperformance, while the random order of the projections in MRP guarantees that the average of the achievable progress is realized.

![Figure 1](image.png)

**Figure 1.** Methods MCP, MRP and PAM applied to toy problem. Top row: Iterates red, subspaces blue. Bottom row: Frequencies (yellow=high, blue=low) of transitions from set $C_m$ to set $C_n$. 

Let us first compare MCP, MRP and PAM without going into too much technical detail.

**Example 3** (benchmark versus PAM). In Figure 1, we see at a glance how the strategies behind MCP, MRP and PAM impact their behavior and performance when applied to the toy model. The fixed order of projections in MCP can result in significant underperformance, while the random order of the projections in MRP guarantees that the average of the achievable progress is realized.
This motivates us to try and outperform the average by assigning a high probability to transitions, which performed better than average in previous iterations, in the new method PAM. The toy problem suggests that this is not a bad idea, when the matrix \( D^0 \) and the policy \( \varphi \) are chosen well for the problem at hand. For this showcase, we used \( D^0 \) and \( \varphi \) as in Example 4 part i), and carried out 315 iterations with each method.

In Example 4, we examine how a good performance of PAM can be achieved by a proper scaling of the initial matrix \( D^0 \). Please note that the intention of this example is not to discuss the size of numerical errors, but rather the qualitative behavior of PAM. The matrices and the iteration numbers are chosen in such a way that these characteristics become clearly visible.

**Example 4** (scaling \( D^0 \)). Here and elsewhere, the notation \( \varphi_{\min} \) refers to the admissible policy introduced in Example 3.2.

i) In Figure 2, we apply PAM with \( \varphi_{\min}, \beta = 0.01 \) and initial matrix \( D^0 \in \mathbb{R}^{N \times N}_{\geq 0} \) given by

\[
D^0_{m,n} = \begin{cases} 
1, & m \neq n, \\
0, & \text{else}.
\end{cases}
\]

While the entries of the matrix \( D^k \) are large compared to the actual step-sizes of the algorithm, we see a more or less uniform sampling of the transitions, similar to the behavior of the superior benchmark method MRP. Once the sizes of the entries of the matrix \( D^k \) are similar to the sizes of the steps carried out, PAM has learned the geometry of the problem and focusses with high probability on profitable transitions, which allows it to outperform MRP.

ii) In Figure 3, we apply PAM with \( \varphi_{\min}, \beta = 0.01 \) and initial matrix \( D^0 \in \mathbb{R}^{N \times N}_{\geq 0} \) given by

\[
D^0_{m,n} = \begin{cases} 
0.01, & m \neq n, \\
0, & \text{else}.
\end{cases}
\]

so the entries of the matrices \( D^k \) underestimate the actual step-sizes in the initial phase of the algorithm. This leads to unpredictable qualitative behaviour of PAM and incomplete exploration of the admissible transitions, and in many cases to an underperformance relative to MRP. Once the sizes of the entries of the matrix \( D^k \) are similar to the sizes of the steps carried out, the qualitative behavior will be as in part i) above.

The effect of choosing a sparse \( D^0 \) is not surprising.

**Example 5** (sparse \( D^0 \)). We apply PAM to the model problem with \( \varphi_{\min}, \beta = 0.01 \) and the matrix \( D^\rightarrow \) from Example 1 with parameters \( \omega = 2, 4, 6 \), and obtain the results shown in Figure 4 after 432 iterations. Note that the matrix \( D^\rightarrow \) overestimates the first step-lengths of the algorithm and therefore needs no scaling.

The algorithm behaves exactly as expected: After an initial learning phase, PAM focusses on the most profitable admissible transitions. A small bandwidth \( \omega \) results in a shorter initial learning phase, but small gain in long-term performance as compared to MCP. On the other hand, a large \( \omega \) results in a longer learning phase with a seizable long-term gain in performance.

Our toy model is not sophisticated enough to reveal a significant difference between the behavior induced by different policies \( \varphi \). We can, however, observe how
Figure 2. Trajectories and frequencies of PAM as in Example 4(i).

Figure 3. Trajectories and frequencies of PAM as in Example 4(ii).
the choice of the bandwidth of the matrix $D^→$ from Example 1 impacts the performance of the method in this particular example.

**Example 6** (first quantitative tests in toy example). We apply MCP, MRP and PAM with initial matrix $D^→$ from Example 1 and three different choices of the bandwidth $ω$ to our toy problem. There are a few interesting features of the results displayed in Figure 5 we wish to summarize:

i) The initial learning phase in which PAM explores the geometry of the problem is clearly visible in the error plot.

ii) When $ω = N$, i.e. when every transition from set $C_m$ to set $C_n$ with $m \neq n$ is admissible, PAM never performed worse than MRP.

iii) The harder the problem is to solve for MCP and MRP (in this example this is the case when $r > 0$ is small), the more clearly PAM (with large $ω$) outperforms both methods.

6. **Conclusion.** This paper introduces the idea of learning to the realm of algorithms for feasibility problems. The focus is on establishing a first feasible algorithm and proving its convergence for a range of admissible learning strategies. Since it was a major effort and achievement to quantify the speed of convergence for MCP and MRP in the setting of affine subspaces, it seems impossible to achieve something similar for PAM, which is, in a sense, path-dependent. For this reason, we believe that we completed the theoretical analysis of PAM in the present paper.

First experiments with PAM applied to computerized and seismic tomography data reveal that the performance of PAM varies between different types of problems. The choice of the matrix $D^0$ and the strategy $φ$ really seems to matter in a real-world context, which calls for a detailed computational investigation of the performance of PAM. As the numerical handling of these problems is a challenge in itself, and
Figure 5. Error plots of methods applied to toy model with varying parameters. Solid black line MCP, dashed black line MRP, solid red line PAM $\omega = N/4$, dashed red line PAM $\omega = N/2$, dash-dotted red line PAM $\omega = N$. More details given in Example 6.

unrelated to the key issue of the present paper, we postpone a detailed exploration of this issue to future work.

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