Hyperbolic positive energy theorems

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(joint work with Erwann Delay [9] and Gregory Galloway [10])

It is convenient to start this report with a few definitions. We say that a Riemannian manifold \((M, g)\) is *conformally compact* if there exists a compact manifold with boundary \(\hat{M}\) such that the following holds: First, we allow \(M\) to have a boundary, which is then necessarily compact. Next, \(M\) is the interior of \(\hat{M}\), whose boundary is the union of the boundary of \(M\) and of a number of new boundary components, at least one, which form the *conformal boundary at infinity*. Further, there exists on \(\hat{M}\) a smooth function \(\Omega \geq 0\) which is positive on \(M\), and which vanishes precisely on the new boundary components of \(\hat{M}\), with \(d\Omega\) nowhere vanishing there. Finally, the tensor field \(\Omega^2 g\) extends to a smooth metric on \(\hat{M}\).

We will say that a conformally compact manifold \((M, g)\) is *asymptotically locally hyperbolic* (ALH) if all sectional curvatures approach minus one as the conformal boundary at infinity is approached. An ALH metric has an *asymptotically hyperbolic* (AH) component of its boundary at infinity, or an AH end, if the conformal metric on that component of the boundary is conformal to a round sphere.

A useful global invariant of an ALH-but-not-AH end is its mass \(m\), while for AH ends we have an energy-momentum vector \(m \equiv (m_{\mu})\) [12, 15, 33] (compare [1, 16]). For this one considers metrics \(g\) which asymptote, at a suitable rate, to a background metric \(\hat{g}\). It is assumed that \(\hat{g}\) admits nontrivial *static potentials* which, in dimension \(n\), are defined as solutions of the overdetermined system of equations

\[
\tilde{D}_i \tilde{D}_j V = (\tilde{R}_{ij} - \frac{\tilde{R}}{n-1} \tilde{g}_{ij}) V.
\]

Here \(\tilde{D}\) is the covariant derivative of the background metric \(\tilde{g}\), while \(\tilde{R}_{ij}\) is its Ricci tensor and \(\tilde{R}\) is the trace \(\tilde{g}^{ij} \tilde{R}_{ij}\). To every static potential \(V\) and asymptotic end \(\partial M\) one associates a mass \(m = m(V, \partial M)\) by the formula [22] (compare [4, Equation (IV.40)])

\[
m(V, \partial M) = -\lim_{x \to 0} \int_{\{x\} \times \partial M} D_i V (\tilde{R}^i_j - \frac{\tilde{R}}{n} \delta^i_j) \, d\sigma_i,
\]

where \(\tilde{R}_{ij}\) is the Ricci tensor of the metric \(\tilde{g}\), \(\tilde{R}\) its trace, and we have ignored an overall dimension-dependent positive multiplicative factor which is often used in the physics literature. Here \(\partial M\) is a component of conformal infinity, and \(x\) is a coordinate near \(\partial M\) so that \(\partial M\) is given by the equation \(\{x = 0\}\).

The difference between AH ends and general ALH ends arises from the dimension of the space of static potentials. Indeed, the AH case is the only one where this dimension is larger than one. Then \(\hat{g}\) is taken to be the hyperbolic metric, which can be written as the following metric on \(\mathbb{R}^n\):

\[
\hat{g} = \frac{dr^2}{1 + r^2} + r^2 d\Omega^2_{n-1},
\]
where $d\Omega^2_{n-1}$ is the unit round metric on $S^{n-1}$. In this coordinate system a basis of the space of static potentials is provided by the functions
\[ V_0 = \sqrt{r^2 + 1}, \quad V_i = x^i. \]
One defines the components $m_\mu$ of the energy-momentum vector $\mathbf{m}$ as
\[ m_\mu := m(V_\mu). \]
One checks that $\mathbf{m}$ transforms as a Lorentz vector under conformal transformations of $S^{n-1}$, so that its Lorentzian norm is a geometric invariant.

For all remaining ALH ends, the mass is directly an invariant [15].

Strictly speaking, a rescaling of $V$ by a constant is always possible, and a preferred scale can be set as follows: In AH ends the standard normalisation is the one just described. In all remaining cases, in any chosen ALH end we can write $\hat{g}$ as
\[ \hat{g} = x^{-2}(dx^2 + \hat{h}), \quad \hat{h}(\partial_x, \cdot) = 0, \]
with the volume of $\partial M$, calculated in the metric $\hat{h}|_{x=0}$, normalised to one. One then normalises $V$ so that $\lim_{x \to 0} x V = 1$.

There is a closely related definition of energy-momentum for asymptotically flat general relativistic initial data sets $(M, g, K)$ which, perhaps somewhat unexpectedly, turns out to be relevant for the asymptotically hyperbolic problem at hand, and which is invoked in Theorem 1 below, we refer the reader to [3, 7] for details.

In the case of spherical conformal infinity, it has been known that $\mathbf{m}$ is timelike future pointing under a spin condition [12, 13, 21, 28, 33], or under restrictive hypotheses [2, 11]. In my talk in Oberwolfach, summarised here, I reported on results presented in [9, 10] where it is shown how to remove these hypotheses.

The starting point of the analysis in [9] is the following result:

**Theorem 1.** Let $(M, g)$ be an asymptotically Euclidean Riemannian manifold, where $M$ is the union of a compact set and of an asymptotically flat region, of dimension $n \geq 3$. Suppose that the general relativistic initial data set $(M, g, K)$ possesses a well defined energy-momentum vector $\mathbf{m}$. If the dominant energy condition holds, then $\mathbf{m}$ is timelike future pointing or vanishes. Furthermore, in the last case $(M, g, K)$ arises from a hypersurface in Minkowski spacetime.

A published proof of Theorem 1 in dimensions less than or equal to seven can be found in [17, 19, 25], building upon [29, 30, 31]. A proof covering all dimensions is available in preprint form in [27], with the borderline cases covered in [5, 14, 25]. Conjecturally, this result also follows in all dimensions basing on the preprint [32].

In [9] it is shown how Theorem 1 together with the perturbation results in [11] and the gluing constructions of [8], can be used to remove all unnatural restrictions in the proof of positivity of asymptotically hyperbolic mass:

**Theorem 2.** Let $(M, g)$ denote an $n$-dimensional Riemannian manifold which is the union of a compact set and an AH end. If the scalar curvature $R(g)$ satisfies $R(g) \geq -n(n-1)$, then the energy-momentum vector of $(M, g)$ is causal future pointing, or vanishes.
The impossibility of a null future pointing energy-momentum vector, under the hypotheses above, has been established in [24].

Theorem 2 has been generalised in [10] to allow manifolds with several ends, and with boundaries satisfying an optimal mean-curvature condition:

**Theorem 3.** Let \((M, g)\) be a conformally compact \(n\)-dimensional, \(3 \leq n \leq 7\), asymptotically locally hyperbolic manifold with boundary. Assume that the scalar curvature of \(M\) satisfies \(R(g) \geq -n(n-1)\), and that the boundary has mean curvature \(H \leq n-1\) with respect to the normal pointing into \(M\). Then, the energy-momentum vector \(m\) of every spherical component of the conformal boundary at infinity of \((M, g)\) is future causal.

In this theorem neither the boundary \(\partial M\), nor the conformal boundary at infinity of \(M\), need to be connected. The proof relies heavily on the results of [18], which assume \(3 \leq n \leq 7\).

The above theorems concern AH ends, and one is led to wonder about properties of mass for ALH-but-not-AH ends. Here the following is known: First, positivity is known on manifolds with suitable spin structure [33], or under restrictive conditions [10] [11], but such \((M, g)\) are scarce. Next, boundaryless conformally compact examples with negative mass and toroidal infinity are due to Horowitz and Myers [23]; nontrivial quotients of spheres at infinity with, again, negative mass have been constructed by Chen and Zhang [6]. Finally, a natural negative lower bound, together with a Penrose-type inequality (compare [16] [20]), has been established by Lee and Neves in [26] for a class of three dimensional models with higher genus conformal infinity.

**References**

1. L.F. Abbott and S. Deser, *Stability of gravity with a cosmological constant*, Nucl. Phys. B195 (1982), 76–96.
2. L. Andersson, M. Cai, and G.J. Galloway, *Rigidity and positivity of mass for asymptotically hyperbolic manifolds*, Ann. H. Poincaré 9 (2008), 1–33, arXiv:math.dg/0703259. MR MR2389888 (2009e :53054)
3. R. Bartnik, *The mass of an asymptotically flat manifold*, Commun. Pure Appl. Math. 39 (1986), 661–693. MR 849427 (88b:58144)
4. H. Barzegar, P.T. Chruściel, and M. Hörzinger, *Energy in higher-dimensional spacetimes*, Phys. Rev. D 96 (2017), 124002, 25 pp., arXiv:1708.03122 [gr-qc].
5. R. Beig and P.T. Chruściel, *Killing vectors in asymptotically flat spacetimes: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, Jour. Math. Phys. 37 (1996), 1939–1961, arXiv:gr-qc/9510015.
6. J. Chen and X. Zhang, *Metrics of Eguchi-Hanson types with the negative constant scalar curvature*, Jour. Geom. Phys. 161 (2021), 104010, 10, arXiv:2007.15964 [math.DG]. MR 4180104
7. P.T. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological Properties and Global Structure of Space–Time (P. Bergmann and V. de Sabbata, eds.), Plenum Press, New York, 1986, pp. 49–59, arXiv:1312.0251 [gr-qc].
8. P.T. Chruściel and E. Delay, *Exotic hyperbolic gluings*, Jour. Diff. Geom. 108 (2018), 243–293, arXiv:1511.07858 [gr-qc].
9. P.T. Chruściel and E. Delay, *The hyperbolic positive energy theorem*, (2019), arXiv:1901.05263v1 [math.DG].
10. P.T. Chruściel and G.J. Galloway, *Positive mass theorems for asymptotically hyperbolic Riemannian manifolds with boundary*, Class. Quantum Grav. 38 (2021), 237001, arXiv:2107.05603 [gr-qc].
11. P.T. Chruściel, G.J. Galloway, L. Nguyen, and T.-T. Paetz, On the mass aspect function and positive energy theorems for asymptotically hyperbolic manifolds, Class. Quantum Grav. **35** (2018), 115015, arXiv:1801.03442 [gr-qc].

12. P.T. Chruściel and M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, Pacific Jour. Math. **212** (2003), 231–264, arXiv:math/0110035 [math.DG]. MR MR2038048 (2005d:53052)

13. P.T. Chruściel, J. Jezierski, and S. Leski, The Trautman-Bondi mass of hyperboloidal initial data sets, Adv. Theor. Math. Phys. **8** (2004), 83–139, arXiv:gr-qc/0307109. MR MR2086675 (2005j:83027)

14. P.T. Chruściel and D. Maerten, Killing vectors in asymptotically flat spacetimes: II. Asymptotically translational Killing vectors and the rigid positive energy theorem in higher dimensions, Jour. Math. Phys. **47** (2006), 022502, arXiv:gr-qc/0512042. MR MR2208148 (2007b:83054)

15. P.T. Chruściel and G. Nagy, The mass of spacelike hypersurfaces in asymptotically anti–de Sitter spacetimes, Adv. Theor. Math. Phys. **5** (2001), 697–754, arXiv:gr-qc/0110014.

16. P.T. Chruściel and W. Simon, Towards the classification of static vacuum spacetimes with negative cosmological constant, Jour. Math. Phys. **42** (2001), 1779–1817, arXiv:gr-qc/0004032.

17. M. Eichmair, The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight, Commun. Math. Phys. **319** (2013), 575–593, arXiv:1206.2553 [math.dg]. MR 3040369

18. M. Eichmair, G.J. Galloway, and A. Mendes, Initial data rigidity results, (2020), arXiv:2009.00527 [gr-qc].

19. M. Eichmair, L.-H. Huang, D.A. Lee, and R. Schoen, The spacetime positive mass theorem in dimensions less than eight, Jour. Eur. Math. Soc. (JEMS) **18** (2016), 83–121, arXiv:1110.2087 [math.DG]. MR 3438380

20. G.W. Gibbons, Some comments on gravitational entropy and the inverse mean curvature flow, Class. Quantum Grav. **16** (1999), 1677–1687, arXiv:hep-th/9809167. MR 1697098

21. G.W. Gibbons, S.W. Hawking, G.T. Horowitz, and M.J. Perry, Positive mass theorem for black holes, Commun. Math. Phys. **88** (1983), 295–308.

22. M. Herzlich, Computing asymptotic invariants with the Ricci tensor on asymptotically flat and asymptotically hyperbolic manifolds, Ann. Henri Poincaré **17** (2016), 3605–3617, arXiv:1503.00508 [math.DG]. MR 3568027

23. G.T. Horowitz and R.C. Myers, The $AdS/CFT$ correspondence and a new positive energy conjecture for general relativity, Phys. Rev. D **59** (1998), 026005, arXiv:hep-th/9808079.

24. L.-H. Huang, H.C. Jang, and D. Martin, Mass rigidity for hyperbolic manifolds, Commun. Math. Phys. (2019), 1–21, arXiv:1904.12010 [math.DG].

25. L.-H. Huang and D.A. Lee, Equality in the spacetime Positive Mass Theorem, Commun. Math. Phys. **376** (2020), 2379–2407, arXiv:1706.03732 [math.DG]. MR 4104553

26. D.A. Lee and A. Neves, The Penrose inequality for asymptotically locally hyperbolic spaces with non-positive mass, Commun. Math. Phys. **339** (2015), 327–352. MR 3370607

27. J. Lohkamp, The Higher Dimensional Positive Mass Theorem II, (2016), arXiv:1612.07505 [math.DG].

28. D. Maerten, Positive energy-momentum theorem in asymptotically anti-de Sitter spacetimes, Ann. H.Poincaré **7** (2006), 975–1011, arXiv:math.DG/0506061. MR 22254577 (2007d:83016)

29. R. Schoen and S.-T. Yau, Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity, Proc. Nat. AcadSci. U.S.A. **76** (1979), 1024–1025. MR 524327 (80k:58034)

30. , On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. **65** (1979), 45–76. MR 526976 (80j:83024)

31. , Proof of the positive mass theorem. II, Commun. Math. Phys. **79** (1981), 231–260. MR 612249

32. , Positive Scalar Curvature and Minimal Hypersurface Singularities, (2017), arXiv:1704.05490 [math.DG].

33. X. Wang, Mass for asymptotically hyperbolic manifolds, Jour. Diff. Geom. **57** (2001), 273–299. MR 1879228 (2003c:53044)