Orthogonal foliations on riemannian manifolds

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Abstract

In this work, we find an equation that relates the Ricci curvature of a riemannian manifold $M$ and the second fundamental forms of two orthogonal foliations of complementary dimensions, $\mathcal{F}$ and $\mathcal{F}^{\perp}$, defined on $M$. Using this equation, we show a sufficient condition for the manifold $M$ to be locally a riemannian product of the leaves of $\mathcal{F}$ and $\mathcal{F}^{\perp}$, if one of the foliations is totally umbilical. We also prove an integral formula for such foliations.

Keywords: Totally umbilical foliation, mean curvature vector, integral formula

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1. Introduction

A great motivation for this work is a result of Brito and Walczak\textsuperscript{[1]} about a pair of two orthogonal foliations of complementary dimensions, $\mathcal{F}$ and $\mathcal{F}^{\perp}$, defined on a complete Riemannian manifold $M$. These authors shown that if the foliation $\mathcal{F}^{\perp}$ is totally geodesic (i.e. each leaf of $\mathcal{F}^{\perp}$ is a totally geodesic submanifold of $M$) and the Ricci curvature of the ambient manifold $M$ is not negative, then $M$ is a locally riemannian product of the leaves of $\mathcal{F}$ and $\mathcal{F}^{\perp}$. This generalizes an analogous result proved by Abe\textsuperscript{[2]} using the additional hypothesis of local symmetry of the ambient space $M$. The Brito-Walczak’s result was proved using an equation that relates the Ricci curvature of the riemannian manifold $M$ and the second fundamental form of the foliation $\mathcal{F}$.

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In this work, we find a general equation that relates the Ricci curvature of a riemannian manifold $M$ and the second fundamental forms of two orthogonal foliations of complementary dimensions, $\mathcal{F}$ and $\mathcal{F}^\perp$, defined on $M$. Using this equation, we generalize a theorem of Almeida, Brito and Colares [3] about such foliations in which all leaves of $\mathcal{F}^\perp$ are totally umbilical submanifolds of $M$ (the foliation $\mathcal{F}^\perp$ is then said a `totally umbilical foliation'). Using the same equation, we also find a more general integral formula than that obtained by Walczak [4].

The structure of this paper is as follows. In section 2 we present the main definitions and notations used in rest of the paper. In section 3 we prove the general equation cited above. In section 4 we analyze the case when the foliation $\mathcal{F}^\perp$ is totally umbilical and we prove a generalization of the theorem 1 of [3]. Finally, in section 5 we establish the general integral formula.

2. The basic terminology

Let $M$ be a riemannian manifold of dimension $n+p$. The symbols $\langle \ , \ \rangle$, $\nabla$ and $R$ denote, respectively, the riemannian metric, the riemannian connexion and the curvature tensor of $M$. Let $\mathcal{F}$ be a $C^\infty$-foliation of codimension $p$ on the manifold $M$ and let $\mathcal{F}^\perp$ be a $C^\infty$-foliation of codimension $n$ on the manifold $M$ and orthogonal to $\mathcal{F}$. Let $x \in M$ and $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}\}$ be an orthonormal adapted frame (i.e. $e_1, \ldots, e_n$ are tangent to $\mathcal{F}$ and $e_{n+1}, \ldots, e_{n+p}$ are tangent to $\mathcal{F}^\perp$) in a neighborhood of $x$. We shall make use of the following convention on the range of indices:

$$
1 \leq A, B, \ldots \leq n + p
$$

$$
1 \leq i, j, \ldots \leq n
$$

$$
n + 1 \leq \alpha, \beta, \ldots \leq n + p
$$

We define the second fundamental form of $\mathcal{F}$ in the direction of $e_\alpha$ by

$$
H^\alpha_{\mathcal{F}}(e_i, e_j) = \langle -\nabla_{e_i} e_\alpha, e_j \rangle
$$

and we define the second fundamental form of $\mathcal{F}^\perp$ in the direction of $e_i$ by

$$
H^{i,\perp}_{\mathcal{F}}(e_\alpha, e_\beta) = \langle -\nabla_{e_\alpha} e_i, e_\beta \rangle
$$
Let $X$ be a smooth vector field defined on the manifold $M$. We denote by $X^\perp$ and $X^\top$, respectively, the following smooth vector fields

$$X^\perp = \sum_\alpha \langle X, e_\alpha \rangle e_\alpha$$
$$X^\top = \sum_i \langle X, e_i \rangle e_i$$

The Weingarten operators of $H^\alpha_F$ and $H^i_{F^\perp}$ are given, respectively, by

$$A_{e_\alpha}(e_i) = - (\nabla_{e_\alpha} e_i)^\top \quad \text{and} \quad A_{e_i}(e_\alpha) = - (\nabla_{e_\alpha} e_i)^\perp$$

We define the norm of the second fundamental form $H^\alpha_F$ by

$$\|H^\alpha_F\| = \left( \sum_{i,j} \langle -\nabla_{e_i} e_\alpha, e_j \rangle^2 \right)^{1/2}$$

and, analogously, we define the norm of the second fundamental form $H^i_{F^\perp}$ by

$$\|H^i_{F^\perp}\| = \left( \sum_{\alpha,\beta} \langle -\nabla_{e_\alpha} e_i, e_\beta \rangle^2 \right)^{1/2}$$

The mean curvature vector of $\mathcal{F}$ is defined by

$$h = \sum_i (\nabla_{e_i} e_i)^\perp$$

and the mean curvature vector of $\mathcal{F}^\perp$ is defined by

$$h^\perp = \sum_\alpha (\nabla_{e_\alpha} e_\alpha)^\top$$

For each fixed $\alpha$, we denote by $(K^\alpha_{ij})$ the $n \times n$ matrix with entries given by $R(e_\alpha, e_i, e_j, e_\alpha)$. The trace of the matrix $(K^\alpha_{ij})$ is then given by

$$\text{Tr}(K^\alpha_{ij}) = \sum_i R(e_\alpha, e_i, e_i, e_\alpha)$$

For all smooth vector field $X$ defined on $M$ we have the following definitions

$$\text{div}_\mathcal{F}(X) = \sum_i \langle e_i, \nabla_{e_i} X \rangle$$
$$\text{div}_{\mathcal{F}^\perp}(X) = \sum_\alpha \langle e_\alpha, \nabla_{e_\alpha} X \rangle$$
3. The main equation

**Theorem 1.** Let \( M \) be a riemannian manifold and denote by \( \mathcal{F} \) and \( \mathcal{F}^\perp \) two orthogonal foliations of complementary dimensions on \( M \). Then we have

\[
e_\alpha \langle h, e_\alpha \rangle - \| H^\alpha_\mathcal{F} \|^2 - \text{Tr}(K^\alpha_{ij}) = \sum_{i=1}^{n} H^\alpha_{\mathcal{F}^\perp}(e_\alpha, \nabla^\perp e_i) - [e_\alpha, e_i]^\perp - \text{div}_\mathcal{F}(\nabla e_\alpha e_\alpha)
\]

**Proof.** Let us take \( \alpha \). We observe that

\[
e_\alpha \langle h, e_\alpha \rangle - \| H^\alpha_\mathcal{F} \|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha)
\]

\[
= e_\alpha \left( \sum_{i=1}^{n} \nabla e_i e_i, e_\alpha \right) - \| H^\alpha_\mathcal{F} \|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha)
\]

\[
= \sum_{i=1}^{n} \left( \langle \nabla e_i e_i, e_\alpha \rangle + \langle \nabla e_i e_i, \nabla e_\alpha e_\alpha \rangle \right) - \| H^\alpha_\mathcal{F} \|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha)
\]

\[
= \sum_{i=1}^{n} \left( \langle \nabla e_i e_i, e_\alpha \rangle + \langle \nabla e_i e_i, \nabla e_\alpha e_\alpha \rangle \right) - \| H^\alpha_\mathcal{F} \|^2
\]

\[
- \sum_{i=1}^{n} \left( \langle \nabla e_i e_i - \nabla e_i e_i - [e_\alpha, e_i] e_i, e_\alpha \rangle \right)
\]

\[
= \sum_{i=1}^{n} \langle \nabla e_i e_i, e_\alpha \rangle - \| H^\alpha_\mathcal{F} \|^2 + \sum_{i=1}^{n} \langle \nabla e_i e_i, e_\alpha \rangle
\]

\[
+ \sum_{i=1}^{n} \langle \nabla e_\alpha e_i, e_\alpha \rangle
\]

On the other hand

\[
\nabla e_\alpha e_i = \sum_{j=1}^{n} b_{ij} e_j + \sum_{\beta} b_{i\beta} e_\beta,
\]

where \( b_{iA} = \langle \nabla e_\alpha e_i, e_A \rangle \). We must have \( b_{ii} = 0 \), since

\[
0 = e_\alpha \langle e_i, e_i \rangle = 2 \langle \nabla e_\alpha e_i, e_i \rangle = 2b_{ii}
\]

We assume that the basis \( \{ e_1, \ldots, e_n \} \) diagonalizes the operator \( H^\alpha_\mathcal{F} \). Then

\[
\nabla e_i e_\alpha = -\lambda^\alpha_i e_i + \sum_{\beta=n+1}^{n+p} c_\beta e_\beta,
\]

where \( c_B = \langle \nabla e_i e_\alpha, e_B \rangle \), and we have

\[
\langle \nabla e_i e_\alpha, e_j \rangle = -H^\alpha_\mathcal{F}(e_i, e_j) = -\lambda^\alpha_i \delta_{ij}
\]
Using (2), (3) and (4)

\[ \langle \nabla_{e_\alpha e_\beta} e_i, e_\alpha \rangle = \langle \nabla_{e_\alpha} e_i, e_\alpha \rangle - \langle \nabla_{e_\beta e_i} e_\alpha, e_\alpha \rangle = \langle \nabla_{e_\beta e_\alpha e_i} e_\alpha, e_\alpha \rangle - \langle \nabla_{e_\alpha e_i} e_\beta e_\alpha, e_\alpha \rangle + \lambda^i \langle e_\alpha e_i, e_\alpha \rangle \]

\[ = \sum_j b_{ij} \langle \nabla_{e_j} e_i, e_\alpha \rangle + \sum_\beta b_{i\beta} \langle \nabla_{e_\beta} e_i, e_\alpha \rangle - c_\beta \langle \nabla_{e_\beta} e_i, e_\alpha \rangle + (\lambda^i)^2. \]

We now observe that

\[ \sum_j b_{ij} \lambda^i \delta_{ij} = 0 \quad (\text{because } b_{ii} = 0) \]

\[ b_{i\beta} - c_\beta = \langle e_\alpha, e_i \rangle, e_\beta \rangle \]

\[ H^i_{\perp} (e_\beta, e_\alpha) = -\langle \nabla_{e_\beta} e_i, e_\alpha \rangle \]

and then

\[ \langle \nabla_{e_\alpha e_\beta} e_i, e_\alpha \rangle = -\sum_\beta \langle e_\alpha, e_i \rangle, e_\beta \rangle H^i_{\perp} (e_\beta, e_\alpha) + (\lambda^i)^2 \]

\[ = -\sum_\beta H^i_{\perp} (\langle e_\alpha, e_i \rangle, e_\beta \rangle, e_\alpha \rangle + (\lambda^i)^2 \]

\[ = -H^i_{\perp} (\sum_\beta \langle e_\alpha, e_i \rangle, e_\beta \rangle, e_\alpha \rangle + (\lambda^i)^2 \]

\[ = -H^i_{\perp} (\langle e_\alpha, e_i \rangle, e_\alpha \rangle + (\lambda^i)^2. \]

By (1) and (5) we have

\[ e_\alpha \langle h, e_\alpha \rangle - \| H^a \|^2 = \sum_{i=1}^n R(e_\alpha, e_i, e_i, e_\alpha) \]

\[ = \sum_{i=1}^n \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \sum_{i=1}^n H^i_{\perp} (\langle e_\alpha, e_i \rangle, e_\alpha \rangle + \sum_{i=1}^n \langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha \rangle \]

(6)

The equality \( e_\alpha \langle e_i, e_\alpha \rangle = 0 \) implies \( \langle \nabla_{e_\alpha} e_i, e_\alpha \rangle + \langle e_i, \nabla_{e_\alpha} e_\alpha \rangle = 0 \) and then

\[ \langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha \rangle = -\langle \nabla_{e_\alpha} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \langle e_i, \nabla_{e_i} \nabla_{e_\alpha} e_\alpha \rangle \]

(7)
Finally, using (6) and (7)

\[ e_\alpha \langle h, e_\alpha \rangle - \|H^\alpha_F\|^2 - \sum_i R(e_\alpha, e_i, e_i, e_\alpha) \]

\[ = \sum_{i=1}^n \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i \langle \nabla_{e_i} e_i, e_\alpha \rangle \]

\[ = \sum_{i=1}^n \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) \]

\[ - \sum_i \left( \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle + \langle \nabla_{e_\alpha} e_i, \nabla_{e_\alpha} e_\alpha \rangle + \langle e_i, \nabla_{e_i} e_\alpha \rangle \right) \]

\[ = \sum_{i=1}^n \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) - \sum_{i=1}^n \langle \nabla_{e_i} e_i, \nabla_{e_\alpha} e_\alpha \rangle \]

\[ - \sum_i \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \sum_i \langle e_i, \nabla_{e_i} e_\alpha \rangle \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) - \sum_i \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) - \sum_i \sum_{\beta} \langle \nabla_{e_\alpha} e_i, e_\beta \rangle \langle \nabla_{e_i} e_\alpha, e_\beta \rangle - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i \sum_{\beta} H^{i+}_F (e_\alpha, e_\beta) \langle \nabla_{e_i} e_\alpha, e_\beta \rangle - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i \sum_{\beta} H^{i+}_F (e_\alpha, \nabla_{e_i} e_\alpha, e_\beta) - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i \sum_{\beta} H^{i+}_F (e_\alpha, \nabla_{e_i} e_\alpha, e_\beta) - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i H^{i+}_F (e_\alpha, \nabla_{e_i} e_\alpha) - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = - \sum_i H^{i+}_F ([e_\alpha, e_i]^+, e_\alpha) + \sum_i H^{i+}_F (e_\alpha, e_\alpha) - \text{div}_F (\nabla_{e_\alpha} e_\alpha) \]

\[ = \sum_i H^{i+}_F (e_\alpha, e_i) - [e_\alpha, e_i]^+ \] // End of proof

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4. The totally umbilical case

**Proposition 1.** Let $\mathcal{F}$ and $\mathcal{F}^\perp$ be two orthogonal foliations of complementary dimensions over a riemannian manifold $M$. If we suppose that $\mathcal{F}^\perp$ is totally umbilical, then there exists a basis $\{e_{n+1}, \ldots, e_{n+p}\}$ of the bundle $\mathcal{T}\mathcal{F}^\perp$ that simultaneously diagonalizes all symmetric operators $A_{e_i}$ for $i \in \{1, \ldots, n\}$.

**Proof.** In fact, there exists a basis of $\mathcal{T}\mathcal{F}^\perp$ that diagonalizes the operator $A_{e_1}$ and then, by hypothesis, $A_{e_1}$ is a multiple of the identity matrix. Thus, in this basis, the operator $A_{e_1}$ commutes with the operator $A_{e_2}$. By a well known theorem of Linear Algebra, there exists a basis of $\mathcal{T}\mathcal{F}^\perp$ that simultaneously diagonalizes the operators $A_{e_1}$ and $A_{e_2}$ and then, by hypothesis, both operators are multiples of the identity matrix. By induction, assume that the operators $A_{e_1}, \ldots, A_{e_{n-1}}$ are multiples of the identity matrix and use the same argument to conclude that there exists a basis of $\mathcal{T}\mathcal{F}^\perp$ with the desired property $\square$

**Remark 1.** By proposition 1, for each $i$, there exists a basis $(e_\alpha)$ of the bundle $\mathcal{T}\mathcal{F}^\perp$ and a common eigenvalue $\lambda^i$ such that

$$A_{e_\alpha}(e_\alpha) = \lambda^i e_\alpha$$

and

$$H^i_{\mathcal{F}^\perp}(e_\alpha, e_\beta) = \lambda^i \delta_{\alpha\beta}$$

for all $\alpha, \beta$.

**Theorem 2.** If $\mathcal{F}^\perp$ is a totally umbilical foliation and $e_{n+1}, \ldots, e_{n+p}$ is the basis of the proposition 1, then

$$e_\alpha \langle h, e_\alpha \rangle - \left\|H^i_\mathcal{F} \right\|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha) = \sum_{i=1}^{n} (\lambda^i)^2 - \text{div}_\mathcal{F}(\nabla_{e_\alpha} e_\alpha)$$

**Proof.** It is sufficient to show that

$$\sum_i H^i_{\mathcal{F}^\perp}(e_\alpha, \nabla_{e_i} e_\alpha - [e_\alpha, e_i]^\perp) = \sum_i (\lambda^i)^2$$

Then, by theorem 1, we conclude that

$$e_\alpha \langle h, e_\alpha \rangle - \left\|H^i_\mathcal{F} \right\|^2 - \sum_{i=1}^{n} R(e_\alpha, e_i, e_i, e_\alpha) = \sum_{i=1}^{n} (\lambda^i)^2 - \text{div}_\mathcal{F}(\nabla_{e_\alpha} e_\alpha)$$
The proof of (8) is an easy computation

\[ \sum_i H_{\mathcal{F}}^\perp (e_\alpha, \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp) = \sum_i (-\nabla_{e_\alpha} e_i, \sum_\beta \langle \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp, e_\beta \rangle e_\beta) \]

\[ = \sum_i \sum_\beta \langle \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp, e_\beta \rangle \langle -\nabla_{e_\alpha} e_i, e_\beta \rangle \]

\[ = \sum_i \sum_\beta \langle \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp, e_\beta \rangle \langle A_{e_i} (e_\alpha), e_\beta \rangle \]

\[ = \sum_i \sum_\beta \langle \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp, e_\beta \rangle \langle \lambda^i e_\alpha, e_\beta \rangle \]

\[ = \sum_i \lambda^i \langle \nabla_{e_i}^\perp e_\alpha - [e_\alpha, e_i]^\perp, e_\alpha \rangle \]

\[ = \sum_i \lambda^i \langle \nabla_{e_i}^\perp e_\alpha, e_\alpha \rangle - \sum_i \lambda^i \langle [e_\alpha, e_i]^\perp, e_\alpha \rangle \]

\[ = \sum_i \lambda^i \langle \nabla_{e_i}^\perp e_\alpha, e_\alpha \rangle - \sum_i \lambda^i \langle \nabla_{e_i}^\perp e_i - \nabla_{e_i} e_\alpha, e_\alpha \rangle \]

\[ = 2 \sum_i \lambda^i \langle \nabla_{e_i}^\perp e_\alpha, e_\alpha \rangle - \sum_i \lambda^i \langle \nabla_{e_i} e_i, e_\alpha \rangle \]

\[ = 2 \sum_i \lambda^i \langle \nabla_{e_i} e_\alpha, e_\alpha \rangle - \sum_i \lambda^i \langle \nabla_{e_i} e_i, e_\alpha \rangle \]

\[ = 0 + \sum_i \lambda^i \langle -\nabla_{e_\alpha} e_i, e_\alpha \rangle \]

\[ = \sum_i \lambda^i \langle A_{e_i} (e_\alpha), e_\alpha \rangle \]

\[ = \sum_i \lambda^i \langle \lambda^i e_\alpha, e_\alpha \rangle \]

\[ = \sum_i (\lambda^i)^2 \quad \square \]

In this section we suppose that \( h \) and \( h^\perp \) are normalized, that is, we have the following equalities

\[ h = \frac{1}{n} \sum_i (\nabla_{e_i} e_i)^\perp \]

\[ h^\perp = \frac{1}{p} \sum_\alpha (\nabla_{e_\alpha} e_\alpha)^\top \]

If the foliation \( \mathcal{F}^\perp \) is totally umbilical, then

\[ h^\perp = \sum_{i=1}^n \lambda(x, e_i) e_i = \sum_{i=1}^n \lambda^i e_i \quad (9) \]

where \( \lambda^i \) is the eingevalue of the Weingarten operator \( A_{e_i} \).
Proposition 2. Let $\mathcal{F}$ and $\mathcal{F}^\perp$ be two orthogonal foliations of complementary dimensions over a Riemannian manifold $M$. If we suppose that $\mathcal{F}^\perp$ is totally umbilical, then
\[
n (\nabla_{e_\alpha} h, e_\alpha) - \left\| H^\perp_x \right\|^2 - \sum_{i=1}^n R(e_\alpha, e_i, e_i, e_\alpha) - \| h^\perp \|^2 + \operatorname{div}_\mathcal{F}(h^\perp) = 0.
\]

Proof. By (8) and (9) we have
\[
\sum_i H^\perp_i (e_\alpha, \nabla_{e_i} e_\alpha - [e_\alpha, e_i]^\perp) = \| h^\perp \|^2 \tag{10}
\]
We also have the following equation
\[
\operatorname{div}_\mathcal{F}(\nabla_{e_\alpha} e_\alpha) = \operatorname{div}_\mathcal{F}(h^\perp) - n (\nabla_{e_\alpha} e_\alpha, h). \tag{11}
\]
In fact, by (9)
\[
h^\perp = \sum_{i=1}^n \lambda^i e_i = \sum_{i=1}^n (\nabla_{e_\alpha} e_\alpha, e_i) e_i = (\nabla_{e_\alpha} e_\alpha)^T
\]
and then
\[
\operatorname{div}_\mathcal{F}(\nabla_{e_\alpha} e_\alpha) = \sum_{i=1}^n (\nabla_{e_i} \nabla_{e_\alpha} e_\alpha, e_i)
= \sum_{i=1}^n e_i (\nabla_{e_\alpha} e_\alpha, e_i) - \sum_{i=1}^n (\nabla_{e_\alpha} e_\alpha, \nabla_{e_i} e_i)
= \sum_{i=1}^n e_i (\nabla_{e_\alpha} e_\alpha, e_i) - n (\nabla_{e_\alpha} e_\alpha, h) - \sum_{i=1}^n (\nabla_{e_\alpha} e_\alpha, \nabla_{e_i}^T e_i)
= \sum_{i=1}^n e_i (h^\perp, e_i) - n (\nabla_{e_\alpha} e_\alpha, h) - \sum_{i=1}^n (h^\perp, \nabla_{e_i}^T e_i)
= \sum_{i=1}^n (\nabla_{e_i} h^\perp, e_i) - n (\nabla_{e_\alpha} e_\alpha, h)
= \operatorname{div}_\mathcal{F}(h^\perp) - n (\nabla_{e_\alpha} e_\alpha, h)
\]
We conclude the proof using (10), (11) and the theorem 2 \qed

Using the proposition 2 we obtain the following theorem
Theorem 3. Let $M$ be a complete riemannian manifold and denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ two orthogonal foliations of complementary dimensions on $M$. Suppose that

1. $\mathcal{F}^\perp$ is totally umbilical for $p \geq 2$ or a foliation by round circles
2. $h^\perp$ has bounded length and $\|h^\perp\|$ attains a maximum at a leaf $L^\perp$ of $\mathcal{F}^\perp$
3. $\text{Tr}(K^\alpha_{ij}) \geq 0$
4. $\text{div}_x(h^\perp)(x) \leq \text{Tr}(K^\alpha_{ij})(x)$ for all $x \in L^\perp$

Then $\text{Tr}(K^\alpha_{ij}) = 0$, $\mathcal{F}$ and $\mathcal{F}^\perp$ are totally geodesic and $M$ is locally a riemannian product of a leaf of $\mathcal{F}$ and a leaf of $\mathcal{F}^\perp$.

Then, as a corollary, we have the following theorem (proved in [3])

Theorem 4. Let $M$ be a complete riemannian manifold of constant curvature $c \geq 0$ and denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ two orthogonal foliations of complementary dimensions on $M$. Suppose that

1. $\mathcal{F}^\perp$ is totally umbilical for $p \geq 2$ or a foliation by round circles
2. $h^\perp$ has bounded length and $\|h^\perp\|$ attains a maximum at a leaf $L^\perp$ of $\mathcal{F}^\perp$
3. $\text{div}_x(h^\perp)(x) \leq nc$ for all $x \in L^\perp$

Then $c = 0$, $\mathcal{F}$ and $\mathcal{F}^\perp$ are totally geodesic and $M$ is locally a riemannian product of a leaf of $\mathcal{F}$ and a leaf of $\mathcal{F}^\perp$.

Our proof of the theorem 3 is very similar to the proof of theorem 4, but we obtain a more general result by applying the proposition 2. The following lemma is an example of application of the proposition 2, it is a little bit more general than the lemma 4.1 of [3].

Lemma 1. $h$ vanishes along a leaf $L^\perp$ of $\mathcal{F}^\perp$ where $\|h^\perp\|$ attains a maximum. Furthermore, the mean curvature $h^\perp$ of each leaf of $\mathcal{F}^\perp$ vanishes, $\text{Tr}(K^\alpha_{ij}) = 0$ and $h = 0$.

Proof. Suppose that $h(p) \neq 0$ for some point $p \in L^\perp$. With this assumption, let $\gamma : I \rightarrow L^\perp$ be a maximal integral curve of $h$ with $I = [0, b)$ and $\gamma(0) = p$. 

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In a small neighborhood of \( p \) we define \( e_\alpha = h / \| h \| \). The proposition 2 and the hypothesis implies the following inequality

\[
ne_\alpha(\| h \|) = n \langle \nabla e_\alpha h, e_\alpha \rangle = \| H^\alpha_F \|^2 + \text{Tr}(K^\alpha_{ij}) + \| h^\perp \|^2 - \text{div}_F(h^\perp) \geq \| H^\alpha_F \|^2
\]

and by the Cauchy-Schwarz inequality we have

\[
ne_\alpha(\| h \|) \geq \| H^\alpha_F \|^2 \geq \| h \|^2 / n
\]

therefore

\[
\| h \circ \gamma \|^3 = (\| h \circ \gamma \|') = \gamma'(\| h \|) = \| h \| e_\alpha(\| h \|) \geq \| h \circ \gamma \|^3 / n^2
\]

Note that \( \gamma \) has infinite length because \( \| h \| \) is increasing along the curve \( \gamma \). Then, we can now consider the function \( g : [0, +\infty[ \rightarrow \mathbb{R} \) given by

\[
g = -\frac{1}{\| h \circ \gamma_1 \|}
\]

where \( \gamma_1 \) is a reparametrization of \( \gamma \) by arc length. Note that

\[
g'(\| h \circ \gamma_1 \|') / \| h \circ \gamma \|^2 \geq \| h \circ \gamma_1 \| / n^2 \geq \| h(p) \| / n^2
\]

Fixing a positive number \( a \), we have

\[
\frac{1}{\| h \circ \gamma_1(a) \|} \geq -\frac{1}{\| h \circ \gamma_2(s) \|} + \frac{1}{\| h \circ \gamma_1(a) \|} = g(s) - g(a) \geq \frac{\| h(p) \|}{n^2} (s - a)
\]

for all \( s > a \) and this is impossible. The conclusion is that \( h = 0 \) on \( L^\perp \). As a consequence, the mean curvature \( h^\perp \) of each leaf of \( F^\perp \) vanishes, \( \text{Tr}(K^\alpha_{ij}) = 0 \) and \( h = 0 \). In fact, again by proposition 2, on the leaf \( L^\perp \) we have

\[
0 = n \langle \nabla e_\alpha h, e_\alpha \rangle = \| H^\alpha_F \|^2 + \text{Tr}(K^\alpha_{ij}) + \| h^\perp \|^2 - \text{div}_F(h^\perp) \geq \| h^\perp \|^2
\]

It follows that \( \| h^\perp \| = 0 \) on the leaf \( L^\perp \) where \( \| h^\perp \| \) attains a maximum. Then \( h^\perp = 0 \) on \( M \). Since \( \text{div}(h^\perp) = 0 \), it also follows that \( \text{Tr}(K_{ij}) = 0 \). Finally, let \( q \) any point on \( M \) and let \( L^\perp_q \) be a leaf of \( F^\perp \) passing through the point \( q \). We already know that \( h = 0 \) on the leaf \( L^\perp_q \) and then, in particular, \( h(q) = 0 \) \( \Box \)

The proof of the theorem 3 now follows from the fact that \( H^\alpha_F = 0 \) (by proposition 2 and the lemma 1). Then, as a consequence, \( F \) is a totally geodesic foliation. As \( F^\perp \) is a totally umbilical foliation and \( h^\perp = 0 \), we have that \( F^\perp \) is also a totally geodesic foliation.
5. The integral formula

**Theorem 5.** Let $M$ be a closed oriented riemannian manifold and denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ two orthogonal foliations of complementary dimensions on $M$. Then

$$\int_M \left( e_\alpha \langle h, e_\alpha \rangle - \|H^\alpha\|^2 - \text{Tr}(K^\alpha_{ij}) - \sum_{i=1}^n H^i_{\perp,\alpha} - \text{div}_{\mathcal{F}^\perp}(\nabla e_\alpha e_\alpha) \right) d\nu = 0$$

where $H^i_{\perp,\alpha} = H^i_{\perp}(e_\alpha, \nabla_{e_i} e_\alpha - [e_\alpha, e_i]_{\perp})$.

**Proof.** The result is a consequence of

$$\text{div}_{\mathcal{F}}(\nabla e_\alpha e_\alpha) = \text{div}(\nabla e_\alpha e_\alpha) - \text{div}_{\mathcal{F}^\perp}(\nabla e_\alpha e_\alpha)$$

and of theorem 1 □

We will now use the following notations and definitions of the reference [4]

$$B_F(e_i, e_j) = (\nabla e_i e_j)^\perp$$

$$B_{\mathcal{F}^\perp}(e_\alpha, e_\beta) = (\nabla e_\alpha e_\beta)^\top$$

$$K(\mathcal{F}, \mathcal{F}^\perp) = \sum_{i,\alpha} R(e_\alpha, e_i, e_i, e_\alpha)$$

As a corollary of theorem 5, we have the following theorem (proved in [4])

**Theorem 6.** Let $M$ be a closed oriented riemannian manifold and denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ two orthogonal foliations of complementary dimensions on $M$. Then

$$\int_M \left( K(\mathcal{F}, \mathcal{F}^\perp) + \|B_F\|^2 + \|B_{\mathcal{F}^\perp}\|^2 - \|h\|^2 - \|h^\perp\|^2 \right) d\nu = 0.$$

To prove the theorem 6, we first observe that

$$B_F(e_i, e_j) = (\nabla e_i e_j)_{\perp} = \sum_\alpha \langle \nabla e_i e_j, e_\alpha \rangle e_\alpha = \sum_\alpha H^\alpha_F(e_i, e_j) e_\alpha$$

and then we obtain

$$\|B_F\|^2 = \sum_{i,j} \langle (\nabla e_i e_j)_{\perp}, (\nabla e_j e_i)_{\perp} \rangle = \sum_\alpha \|H^\alpha_F\|^2 = \sum_{i,\alpha} (\lambda^\alpha_i)^2$$

(12)
On the other hand, by equation (5) and the definition of $H^i_{\perp}$, we have

\[-H^i_{\perp}([e_\alpha, e_i], e_\alpha) = \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle - (\lambda^\alpha_i)^2\]

\[H^i_{\perp}(e_\alpha, \nabla^\perp_{e_i} e_\alpha) = -\langle \nabla_{e_\alpha} e_i, \nabla^\perp_{e_i} e_\alpha \rangle\]

Adding the above equations we obtain

\[H^i_{\perp}(e_\alpha, \nabla^\perp_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp}) = -(\langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle) - (\lambda^\alpha_i)^2\]

We observe now that

\[
\langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle = \langle \nabla_{e_\alpha} e_i, \nabla_{e_\alpha} e_i \rangle + \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle
\]

\[= \langle \nabla_{e_\alpha} e_i, \nabla_{e_\alpha} e_i \rangle - \langle \nabla_{e_i} e_\alpha, e_i \rangle + \langle \nabla_{e_\alpha} e_i, e_\alpha \rangle\]

\[= \sum_j \langle \nabla_{e_i} e_\alpha, e_j \rangle \langle \nabla_{e_\alpha} e_i, e_j \rangle + \sum_{j, \beta} \langle \nabla_{e_i} e_\alpha, e_\beta \rangle \langle \nabla_{e_\alpha} e_i, e_\beta \rangle
\]

\[= \sum_j \langle \nabla_{e_i} e_\alpha, e_j \rangle \langle \nabla_{e_\alpha} e_i, e_j \rangle - \sum_{j, \beta} \langle \nabla_{e_i} e_\alpha, e_\beta \rangle \langle \nabla_{e_\alpha} e_i, e_\beta \rangle
\]

\[+ \sum_j \langle \nabla_{e_\alpha} e_i, e_j \rangle \langle \nabla_{e_j} e_\alpha, e_i \rangle + \sum_{j, \beta} \langle \nabla_{e_\alpha} e_i, e_\beta \rangle \langle \nabla_{e_\beta} e_\alpha, e_i \rangle\]

and then

\[
\sum_{i, \alpha} H^i_{\perp}(e_\alpha, \nabla^\perp_{e_i} e_\alpha - [e_\alpha, e_i]^{\perp})
\]

\[= \sum_{i, \alpha} \left[ -\left( \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle \right) - (\lambda^\alpha_i)^2 \right]
\]

\[= -\sum_{i, j, \alpha} \left( \langle \nabla_{e_\alpha} e_i, e_j \rangle \langle \nabla_{e_i} e_\alpha, e_j \rangle + \langle \nabla_{e_\alpha} e_i, e_j \rangle \langle \nabla_{e_j} e_\alpha, e_i \rangle \right)
\]

\[-\langle e_\alpha, \nabla_{e_\alpha} e_j \rangle \langle e_\alpha, \nabla_{e_j} e_i \rangle
\]

\[+ \sum_{i, \alpha, \beta} \langle \nabla_{e_i} e_\alpha, e_\beta \rangle \langle \nabla_{e_\alpha} e_i, e_\beta \rangle + \langle \nabla_{e_i} e_\alpha, e_\beta \rangle \langle e_\alpha, \nabla_{e_\beta} e_i \rangle
\]

\[-\langle e_i, \nabla_{e_i} e_\beta \rangle \langle \nabla_{e_\beta} e_\alpha, e_\alpha \rangle \right] - \sum_{i, \alpha} (\lambda^\alpha_i)^2
\]

\[= \sum_{i, j, \alpha} \langle e_\alpha, \nabla_{e_i} e_j \rangle \langle e_\alpha, \nabla_{e_j} e_i \rangle + \sum_{i, \alpha, \beta} \langle e_i, \nabla_{e_\alpha} e_\beta \rangle \langle \nabla_{e_\beta} e_\alpha, e_i \rangle - \sum_{i, \alpha} (\lambda^\alpha_i)^2
\]

\[= \sum_{i, j} \langle \nabla_{e_i} e_j \rangle^{\perp} \langle \nabla_{e_j} e_i \rangle^{\perp} + \sum_{i, \alpha, \beta} \langle \nabla_{e_\alpha} e_\beta \rangle^{\top} \langle \nabla_{e_\beta} e_\alpha \rangle^{\top} - \sum_{i, \alpha} (\lambda^\alpha_i)^2
\]

\[= \|B\|_2^2 + \|B_{\perp}\|_2^2 - \sum_{i, \alpha} (\lambda^\alpha_i)^2
\]

\[= \|B\|_2^2 \quad (13)\]
Lemma 2. We have the following equality
\[ \sum_{\alpha} \text{div}_F(\nabla e_{\alpha} e_{\alpha}) = \text{div}_F(h^\perp) - \sum_{\alpha} \langle h, \nabla e_{\alpha} e_{\alpha} \rangle \]

Proof. By definition
\[
\text{div}_F(\nabla e_{\alpha} e_{\alpha}) = \sum_i \langle e_i, \nabla_{e_i} (\nabla e_{\alpha} e_{\alpha}) \rangle \\
= \sum_i \langle e_i, \nabla e_{\alpha} e_{\alpha} \rangle - \langle \nabla e_{\alpha} e_{i} \nabla e_{\alpha} e_{\alpha} \rangle \\
= \sum_i \langle e_i, \nabla e_{\alpha} e_{\alpha} \rangle - \sum_j \langle \nabla e_{\alpha} e_{\alpha} e_j \rangle - \sum_{\beta} \langle \nabla e_{\alpha} e_{i}, \nabla e_{\alpha} e_{\alpha} \rangle \\
= \sum_i \langle e_i, \nabla e_{\alpha} e_{\alpha} \rangle - \sum_j \langle \nabla e_{\alpha} e_{\alpha} e_j \rangle - \sum_{\beta} \langle \nabla e_{\alpha} e_{i}, \nabla e_{\alpha} e_{\alpha} \rangle \\
\]
and then
\[
\sum_{\alpha} \text{div}_F(\nabla e_{\alpha} e_{\alpha}) = \sum_{i, \alpha} \langle e_i, \nabla e_{\alpha} e_{\alpha} \rangle - \sum_{\alpha} \langle h, \nabla e_{\alpha} e_{\alpha} \rangle \quad (14)
\]

Analogously, we have
\[
\text{div}_F(h^\perp) = \sum_{i, \alpha} \langle e_i, \nabla e_{\alpha} e_{\alpha} \rangle - \sum_{\alpha} \langle h, \nabla e_{\alpha} e_{\alpha} \rangle \quad (15)
\]

Substituting (15) in (14) we obtain the desired equality □

Now, we proceed the proof of the theorem 6. The equation of theorem 1 is
\[
ee_{\alpha} \langle h, e_{\alpha} \rangle - \|H^\alpha_F\|^2 - \text{Tr}(K^\alpha) = \sum_{i=1}^{n} H_{F,\perp}^i (e_{\alpha}, \nabla e_{\alpha} e_{\alpha} - [e_{\alpha}, e_i] \perp) - \text{div}_F(\nabla e_{\alpha} e_{\alpha})
\]

Summing (in \(\alpha\)) all these equations, using the relations (12)-(13) and also the definition of \(K = K(F, F^\perp)\), we can write
\[
\sum_{\alpha} e_{\alpha} \langle h, e_{\alpha} \rangle - \|B_F\|^2 - K = \|B_F\|^2 - \sum_{\alpha} \text{div}_F(\nabla e_{\alpha} e_{\alpha})
\]

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Using now the following elementary equations (see [4])

\[
\text{div} F_\perp (h) = \text{div}(h) + \|h\|^2 \quad \text{and} \quad \text{div} F (h^\perp) = \text{div}(h^\perp) + \|h^\perp\|^2
\]

and using the lemma 2, we obtain

\[
\sum_\alpha e_\alpha \langle h, e_\alpha \rangle - \|B\|^2 - K = \sum_\alpha (h, \nabla_{e_\alpha} e_\alpha)
\]

\[
= \|B_\perp\|^2 - \text{div} F (h^\perp) + \sum_\alpha (h, \nabla_{e_\alpha} e_\alpha)
\]

\[
= \|B_\perp\|^2 - \text{div} F (h^\perp) + \sum_\alpha e_\alpha \langle h, e_\alpha \rangle - \sum_\alpha \langle e_\alpha, \nabla_{e_\alpha} h \rangle
\]

\[
= \|B_\perp\|^2 - \text{div} F (h^\perp) + \sum_\alpha e_\alpha \langle h, e_\alpha \rangle - \text{div} F (h) + \|h\|^2
\]

\[
= \|B_\perp\|^2 - \text{div} (h^\perp) - \|h^\perp\|^2 + \sum_\alpha e_\alpha \langle h, e_\alpha \rangle - \text{div} (h) - \|h\|^2
\]

and we conclude that

\[
- \|B\|^2 - K = \|B_\perp\|^2 - \text{div} (h^\perp) - \|h^\perp\|^2 - \text{div} (h) - \|h\|^2
\]

The theorem 6 will follow from the integration of the above equality on \( M \).

6. Final remarks

1. The theorem 6 was obtained first by Ranjan [5] and after generalized by Walczak [4] for the case of two distributions.

2. It is also possible to study the completeness of the foliations using the theorem 1. We will do it in a forthcoming paper.

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