The existence and uniqueness of viscosity solutions to generalized Hamilton-Jacobi-Bellman equations

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Abstract. In this paper, we study the existence and uniqueness of viscosity solutions to generalized Hamilton-Jacobi-Bellman (HJB) equations combined with algebra equations. This generalized HJB equation is related to a stochastic optimal control problem for which the state equation is described by a fully coupled forward-backward stochastic differential equation (FBSDE). By extending Peng’s backward semi-group approach to this problem, we obtain the dynamic programming principle (DPP) and show that the value function is a viscosity solution to this generalized HJB equation. As for the proof of the uniqueness of viscosity solution, the analysis method in Barles, Buckdahn and Pardoux usually does not work for this fully coupled case. With the help of the uniqueness of the solution to FBSDEs, we propose a novel probabilistic approach to study the uniqueness of the solution to this generalized HJB equation. We obtain that the value function is the minimum viscosity solution to this generalized HJB equation. Especially, when the coefficients are independent of the control variable or the solution is smooth, the value function is the unique viscosity solution.

Key words. Hamilton-Jacobi-Bellman equation, viscosity solution, fully coupled forward-backward stochastic differential equations, dynamic programming principle

AMS subject classifications. 93E20, 60H10, 35K15

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1 Introduction

In this paper, we study the existence and uniqueness of the viscosity solution to the following generalized HJB equation combined with an algebra equation,

$$\begin{cases}
\partial_t W(t, x) + \inf_{u \in U} H(t, x, W(t, x), DW(t, x), D^2W(t, x), u) = 0, \\
W(T, x) = \phi(x),
\end{cases}$$

(1.1)

where

$$H(t, x, v, p, A, u) = \frac{1}{2} \text{tr}[\sigma^T(t, x, v, V(t, x, v, p, u), u)A] + p^T b(t, x, v, V(t, x, v, p, u), u) + g(t, x, v, V(t, x, v, p, u), u),$$

(1.2)

This kind of problem has the following stochastic optimal control interpretation. The controlled system is described by the following fully coupled forward-backward stochastic differential equation (FBSDE):

$$\begin{cases}
\begin{aligned}
dX^{t,x,u}_s &= b(t, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)ds + \sigma(s, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)dB_s, \\
dY^{t,x,u}_s &= -g(t, X^{t,x,u}_s, Y^{t,x,u}_s, Z^{t,x,u}_s, u_s)ds + Z^{t,x,u}_s dB_s,
\end{aligned} \\
X^{t,x,u}_t = x, \; Y^{t,x,u}_T = \phi(X^{t,x,u}_T), \; s \in [t, T],
\end{cases}$$

(1.3)

where $B = (B_s)_{s \in [t, T]}$ is a standard $d$-dimensional Brownian motion and $u \in U[t, T]$ is an admissible control. The cost functional is defined by the solution to the backward stochastic differential equation (BSDE) at time $t$ in (1.3) and the value function of our control problem is

$$W(t, x) = \text{ess\ inf}_{u \in U[t, T]} Y^{t,x,u}_t.$$ 

(1.4)

When the coefficients $b$ and $\sigma$ of (1.3) are independent of the variables $y$ and $z$, Peng [16, 18] first obtained that the above defined $W$ is a viscosity solution to (1.1). For this case, the uniqueness of the viscosity solution to (1.1) can be obtained by applying the method in Barles, Buckdahn and Pardoux [2] (see Theorem 5.3 in [3] for details).

When $b$ and $\sigma$ depend on $y$ and $z$ in (1.3), the control system (1.3) becomes a fully coupled FBSDE and the corresponding HJB equation (1.1) becomes a fully nonlinear parabolic partial differential equation (PDE) combined with an algebra equation which leads to the solvability and uniqueness of (1.1) being extraordinarily difficult. Note that when (1.3) is independent of the control variable $u$, the HJB equation (1.1) degenerates to a semilinear parabolic PDE. In fact, even for this extreme case, the well-posedness of (1.1) is still an open problem which is proposed by Peng [17]. Recently, Li and Wei [10], Li [9] proved that $W$ is a viscosity solution to (1.1) under the monotonicity conditions for $b$, $\sigma$, $g$ and $\phi$. As for the uniqueness of the viscosity solution, there are only a few results for some special cases. For instance, when $b$, $\sigma$, $g$ are independent
of control variable \( u \), and \( \sigma \) is independent of \( y \) and \( z \), by applying the method in Barles, Buckdahn and Pardoux [2], Wu and Yu [21] proved that \( W \) is the unique viscosity solution to (1.1) under the monotonicity conditions for \( b \), \( \sigma \), \( g \) and \( \phi \) in the space of all continuous functions which are Lipschitz continuous in \( x \).

So the existence and uniqueness of the viscosity solution to (1.1) is an interesting and challenging problem. In this paper, we apply the probabilistic approach to deal with this problem. In more details, with the help of the value function of the above stochastic optimal control problem and the existence and uniqueness of the solution to the controlled system (1.3), we attacks this difficult problem, especially the uniqueness part. Before we focus on the uniqueness results, we need to deal with the following problems.

The first problem is the well-posedness of the fully coupled forward backward controlled system (1.3). There are many literatures on the well-posedness of fully coupled FBSDEs. When the coefficients of a fully coupled FBSDE are deterministic and the diffusion coefficient of the forward equation is nondegenerate, Ma, Protter and Yong [11] proposed the four-step scheme approach. Under some monotonicity conditions, Hu and Peng [8] first obtained an existence and uniqueness result which was generalized by Peng and Wu [19]. Yong [22, 23] developed this approach and called it the method of continuation. The fixed point approach is due to Antonelli [1], Pardoux and Tang [15]. The readers may refer to Ma and Yong [13], Cvitanic and Zhang [5], Ma, Wu, Zhang and Zhang [12], Yong and Zhou [24] for the FBSDE theory. In this paper, we adopt the fixed point approach which is given in Hu et. al. [7].

The second problem is the existence of the viscosity solution to (1.1). We obtain that value function \( W \) defined in (1.4) is a viscosity solution to the HJB equation (1.1) under the assumption in Hu et. al. [7]. As pointed out in Remark 3.17, our proofs still hold under monotonicity conditions. Comparing with Li and Wei [10], both of us develop Peng’s stochastic backward semigroup approach in [16, 18]. But our main proofs are different from the ones in Li and Wei [10]. To establish the DPP, the stochastic backward semigroup is defined by a fully coupled FBSDE. It is important to note that the change of \((Y, Z)\) will not affect \( X \) for the decoupled forward backward controlled system. In contrast, \((Y, Z)\) did affect \( X \) for the fully coupled one. Thus, the approach of establishing the DPP for the decoupled forward-backward controlled system does not work any more. By constructing two new auxiliary FBSDEs (see (3.22) and (3.25)), we introduce a new approach to prove the DPP for the fully coupled forward backward controlled system. We also simplify some proofs as follows. To prove the continuity property of the value function \( W(t, x) \) in \( t \), we build a FBSDE (see (3.30)) which makes the proof easier. For the combined algebra equation, we construct a simple contraction mapping to prove the existence and uniqueness, and obtain some properties of this algebra equation.

Then, we study the uniqueness of viscosity solution to the HJB equation (1.1) in four cases. The first case is that \( \sigma \) is independent of \( y \) and \( z \). By using the method in [2], we prove the uniqueness of viscosity solution to (1.1) in the space of all continuous functions which are Lipschitz continuous in \( x \). But, when \( \sigma \) depends on \( y \) and \( z \), the method in [2] does not work as pointed out in Remark 5.4. The second case is that \( \sigma \) is independent of \( z \). Different from the analysis method in [2], for this case, we propose a novel probabilistic approach to prove the uniqueness. In more details, we construct a new fully coupled forward-backward stochastic control system (4.19) in which \( \sigma \) only depends on the variables \( t \), \( x \) and \( u \). By the uniqueness result in the first case, the value function for this new control system is the unique viscosity solution to the HJB equation (4.18). Thanks to Proposition 4.2, we prove that \( W \) defined in (1.4) is the minimum viscosity solution to the HJB equation (1.1) in the space of all continuous functions which are Lipschitz continuous.
Our paper is organized as follows. In section 2, we formulate our problem and a related stochastic optimal control problem. In section 3, we prove that the value function of the related stochastic control problem is a viscosity solution to the HJB equation by establishing the DPP and the properties of the value function. The uniqueness results are obtained in section 4.

2 The problem formulation

Denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{R}^{k \times n}$ the set of $k \times n$ real matrices and $\mathbb{S}^n$ the set of $n \times n$ symmetric matrix. Let $U$ be a nonempty and compact subset in $\mathbb{R}^k$. Let $(\cdot, \cdot)$ (resp. $\| \cdot \|$) denote the usual scalar product (resp. usual norm) of $\mathbb{R}^n$ and $\mathbb{R}^{k \times n}$. The scalar product (resp. norm) of $M = (m_{ij})$, $N = (n_{ij}) \in \mathbb{R}^{k \times n}$ is denoted by $(M, N) = \operatorname{tr} \{ MN^T \}$ (resp. $\|M\| = \sqrt{MM^T}$), where the superscript $^\top$ denotes the transpose of vectors or matrices.

We will study the existence and uniqueness of viscosity solution to the following generalized HJB equation combined with an algebra equation

$$
\begin{cases}
\partial_t W(t, x) + \inf_{u \in U} H(t, x, W(t, x), DW(t, x), D^2W(t, x), u) = 0, \\
W(T, x) = \phi(x).
\end{cases}
$$

(2.1)

Here $H(\cdot)$ is defined as follows

$$
H(t, x, v, p, A, u) = \frac{1}{2} \text{tr} [\sigma \sigma^T(t, x, v, V(t, x, v, p, u), u) A] + p^\top b(t, x, v, V(t, x, v, p, u), u) + g(t, x, v, V(t, x, v, p, u), u),
$$

(2.2)

$(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U$,

$V(t, x, v, p, u)$ is the solution to the following algebra equation

$$
V(t, x, v, p, u) = p^\top \sigma(t, x, v, V(t, x, v, p, u), u),
$$

where

$$
\begin{align*}
b &: [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times U \to \mathbb{R}^n, \\
\sigma &: [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times U \to \mathbb{R}^{n \times d}, \\
g &: [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d} \times U \to \mathbb{R},
\end{align*}
$$

$\mathbb{R}^{n \times d}$ and $\mathbb{R}$ in the above equations denote the set of $n \times d$ real matrices and the set of real numbers, respectively.
\[ \phi : \mathbb{R}^n \to \mathbb{R}. \]

We impose the following assumption on these functions.

**Assumption 2.1** (i) \( b, \sigma, g, \phi \) are continuous with respect to \( s, x, y, z, u \), and there exist constants \( L_i > 0 \), \( i = 1, 2, 3 \) such that

\[
\begin{align*}
|b(s, x_1, y_1, z_1, u) - b(s, x_2, y_2, z_2, u)| &\leq L_1|x_1 - x_2| + L_2(|y_1 - y_2| + |z_1 - z_2|), \\
|\sigma(s, x_1, y_1, z_1, u) - \sigma(s, x_2, y_2, z_2, u)| &\leq L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|, \\
|g(s, x_1, y_1, z_1, u) - g(s, x_2, y_2, z_2, u)| &\leq L_1(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), \\
|\phi(x_1) - \phi(x_2)| &\leq L_1|x_1 - x_2|,
\end{align*}
\]

for all \( s \in [0, T], x_i, y_i, z_i \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{1 \times d}, \ i = 1, 2, u \in U \).

(ii) \( \Lambda := 8C_2(1 + T^2)c_1^2 < 1 \), where \( c_1 = \max\{L_2, L_3\} \), \( C_2 \) is defined in Lemma 5.1 in [2].

**Remark 2.2** Since \( U \) is compact, from the above assumption (i) we obtain that

\[
|\psi(s, x, y, z, u)| \leq L(1 + |x| + |y| + |z|),
\]

where \( L > 0 \) is a constant and \( \psi = b, \sigma, g \) and \( \phi \).

As pointed out in the introduction, this kind of problem has a stochastic optimal control interpretation. Now we formulate this related stochastic optimal control problem.

Let \( B = (B_1^t, B_2^t, ..., B_d^t)_{0 \leq t \leq T} \) be a standard \( d \)-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\) over \([0, T]\). Denote by \( \mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\} \) the natural filtration of \( B \), where \( \mathcal{F}_0 \) contains all \( P \)-null sets of \( \mathcal{F} \). Given \( t \in [0, T] \), denote by \( \mathcal{U}[t, T] \) the set of all \( \mathcal{F} \)-adapted \( U \)-valued processes on \([t, T]\). For each given \( p \geq 1 \), we introduce the following spaces.

\( L^p(\mathcal{F}_t; \mathbb{R}^n) \): the space of \( \mathcal{F}_t \)-measurable \( \mathbb{R}^n \)-valued random vectors \( \zeta \) such that \( \mathbb{E}[|\zeta|^p] < \infty \);

\( L^\infty(\mathcal{F}_t; \mathbb{R}^n) \): the space of \( \mathcal{F}_t \)-measurable \( \mathbb{R}^n \)-valued random vectors \( \zeta \) such that

\[
||\zeta||_\infty = \text{ess sup}_{\omega \in \Omega} |\zeta(\omega)| < \infty;
\]

\( L^p_F(t, T; \mathbb{R}^n) \): the space of \( \mathcal{F} \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([t, T]\) such that

\[
\mathbb{E}\left[ \int_t^T |f(r)|^p \, dr \right] < \infty;
\]

\( L^\infty_F(t, T; \mathbb{R}^n) \): the space of \( \mathcal{F} \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([t, T]\) such that

\[
||f(\cdot)||_\infty = \text{ess sup}_{(r, \omega) \in [t, T] \times \Omega} |f(r, \omega)| < \infty;
\]

\( L^{p,q}_F(t, T; \mathbb{R}^n) \): the space of \( \mathcal{F} \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([t, T]\) such that

\[
||f(\cdot)||_{p,q} = \left\{ \mathbb{E}\left[ \int_t^T |f(r)|^p \, dr \right]^{\frac{1}{p}} \right\}^q < \infty;
\]
$L^p_F(\Omega; C([t,T]; \mathbb{R}^n))$: the space of $\mathbb{F}$-adapted $\mathbb{R}^n$-valued continuous stochastic processes on $[t,T]$ such that

$$\mathbb{E}[\sup_{t \leq r \leq T}|f(r)|^p] < \infty.$$ 

Let $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t; \mathbb{R}^n)$ and an admissible control $u(\cdot) \in \mathcal{U}[t,T]$. Consider the following controlled fully coupled FBSDE:

$$
\begin{align*}
    dX^t_{s,i} &= b(s, X^t_{s,i}, Y^t_{s,i}, Z^t_{s,i}, u_s)ds + \sigma(s, X^t_{s,i}, Y^t_{s,i}, Z^t_{s,i}, u_s)dB_s, \\
    dY^t_{s,i} &= -g(s, X^t_{s,i}, Y^t_{s,i}, Z^t_{s,i}, u_s)ds + Z^t_{s,i}dB_s, \quad s \in [t,T], \\
    X^t_{t,i} &= \xi, \quad Y^t_{t,i} = \phi(X^t_{t,i}).
\end{align*}
$$

By Theorem 2.2 in [7], the equation (2.3) has a unique solution $(X^t_{s,i}, Y^t_{s,i}, Z^t_{s,i}) \in L^2_F(\Omega; C([t,T]; \mathbb{R}^n)) \times L^2(\Omega; C([t,T]; \mathbb{R})) \times L^2(\Omega; C([t,T]; \mathbb{R}^{1 \times d}))$. For each given $(t, x) \in [0,T] \times \mathbb{R}^n$, define the value function

$$W(t, x) = \inf_{u \in \mathcal{U}[t,T]} Y^t_{t,i}.$$  

3 The existence of viscosity solutions

In order to prove the existence of the viscosity solution, we need to study the above fully coupled stochastic optimal control problem. It is well-known that DPP is an important approach to solving stochastic optimal control problems (see [24-26]). So in the first subsection, the DPP for the stochastic control problem (2.3) is established. Then we prove the value function is a viscosity solution to the generalized HJB equation (2.4) in the second subsection.

3.1 The dynamic programming principle

Define $\mathcal{U}[t,T]$ the space of all $U$-valued $\{\mathcal{F}_s\}_{t \leq s \leq T}$-adapted processes on $[t,T]$, where $\{\mathcal{F}_s\}_{t \leq s \leq T}$ is the $P$-augmentation of the natural filtration of $(B_s - B_t)_{t \leq s \leq T}$. For each $u \in \mathcal{U}[t,T]$, it is easy to verify that the solution $(X^t_{s,i}, Y^t_{s,i}, Z^t_{s,i})_{s \in [t,T]}$ to equation (2.3) is $\{\mathcal{F}_s\}_{t \leq s \leq T}$-adapted, which implies that $Y^t_{t,i} \in \mathbb{R}$.

Definition 3.1 For each fixed $t \in [0,T]$, $\{A_i\}_{i=1}^m \subseteq \mathcal{F}_t$ is called a partition of $(\Omega, \mathcal{F}_t)$, if $\cup_{i=1}^m A_i = \Omega$, and $A_i \in \mathcal{F}_t$, $i = 1, 2, \ldots, m$; $A_i \cap A_j = \emptyset$ for $i \neq j$.

It should note that in this paper, the constant $C$ will change from line to line in the following proofs.

Proposition 3.2 Suppose Assumption (2.7) holds. Then

$$W(t, x) = \inf_{u \in \mathcal{U}[t,T]} Y^t_{t,i}. \tag{3.1}$$

Proof. Since $\mathcal{U}[t,T] \supset \mathcal{U}[t,T]$, we obtain $W(t, x) \leq \inf_{u \in \mathcal{U}[t,T]} Y^t_{t,i}$ by the definition of $W(t, x)$. On the other hand, for each given $u \in \mathcal{U}[t,T]$, by Lemma 13 in [7], there exists a sequence $(u^m)$ in $\mathcal{U}[t,T]$ such that
Note that \( T \) by Remark 2.2, where

\[
\text{Proof.}
\]

\[
\begin{align*}
\int_t^T |u_t^m - u_s|^2 \, ds & \to 0 \text{ as } m \to \infty. \quad \text{Moreover, we can take } u_t^m = \sum_{i=1}^m v_i^m I_{A_i}, \ s \in [t, T], \text{ where } \{A_i\}_{i=1}^m \text{ is a partition of } (\Omega, \mathcal{F}_t) \text{ and } v_i^m \in \mathcal{U}^t, t. \text{ By Theorem 2.2 in [a], we get}
\int_t^T |u_t^m - Y_t^{t,x,u}|^2 \, ds & \leq C \int_t^T \left[ |b(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s^m) - b(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s)|^2 \right. \\
& \quad + C \left. \int_t^T |g(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s^m) - g(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s)|^2 \right. \\
& \quad + C \left. \int_t^T |\sigma(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s^m) - \sigma(s, X_s^{t,x,u}, Y_s^{t,x,u}; Z_s^{t,x,u}, u_s)|^2 \right] ds.
\end{align*}
\]

(3.2)

By Remark 2.2

\[
|\psi(s, X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u)| \leq L(1 + |X_s^{t,x,u}| + |Y_s^{t,x,u}| + |Z_s^{t,x,u}|),
\]

where \( \psi = b, \sigma, g \) and \( s \in [t, T], \ u \in U \). Applying the dominated convergence theorem to (3.2), we obtain

\[
\int_t^T |Y_t^{t,x,u} - Y_t^{t,x,u}|^2 \, ds \to 0 \text{ as } m \to \infty.
\]

(3.3)

Note that

\[
\left( X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u} \right)_{s \in [t, T]} = \left( \sum_{i=1}^m X_s^{t,x,u} I_{A_i}, \sum_{i=1}^m Y_s^{t,x,u} I_{A_i}, \sum_{i=1}^m Z_s^{t,x,u} I_{A_i} \right)_{s \in [t, T]}.
\]

Then,

\[
Y_t^{t,x,u} = \sum_{i=1}^m Y_t^{t,x,u} I_{A_i} \geq \inf_{v \in \mathcal{U}^t} Y_t^{t,x,v}, \text{ P.a.s.}
\]

(3.4)

Combining (3.3) and (3.4), we obtain

\[
Y_t^{t,x,u} \geq \inf_{v \in \mathcal{U}^t} Y_t^{t,x,v}, \text{ P.a.s.}
\]

which yields that \( W(t, x) \geq \inf_{v \in \mathcal{U}^t} Y_t^{t,x,v} \). Thus \( W(t, x) = \inf_{v \in \mathcal{U}^t} Y_t^{t,x,v} \).

The above Proposition shows that \( W(t, x) \) is a deterministic function. To prove \( W(t, \xi) = \text{ess inf}_{u \in \mathcal{U}^t} Y_t^{t,x,u} \), we need the following two lemmas.

**Lemma 3.3** Suppose Assumption 2.1 holds. Then there exists a constant \( C \) depending on \( L_1, L_2, L_3 \) and \( T \) such that, for each \( u \in \mathcal{U}^t \) and \( \xi, \xi' \in L^2(F_t; \mathbb{R}^n) \),

\[
E \left[ \sup_{t \leq s \leq T} \left( |X_s^{t,x,u} - X_s^{t,x,u}|^2 + |Y_s^{t,x,u} - Y_s^{t,x,u}|^2 + \int_t^T |Z_s^{t,x,u} - Z_s^{t,x,u}|^2 \, ds \right) \right] \leq C \| \xi - \xi' \|^2 ;
\]

(3.5)

\[
E \left[ \sup_{t \leq s \leq T} (|X_s^{t,x,u}|^2 + |Y_s^{t,x,u}|^2 + \int_t^T |Z_s^{t,x,u}|^2 \, ds \right] \leq C \left( 1 + |\xi|^2 \right) .
\]

(3.6)

**Proof.** Without loss of generality, we only prove the case \( d = 1 \). Set \( \hat{X} = X^{t,x,u} - X^{t,x,u}, \ \hat{Y} = Y^{t,x,u} - Y^{t,x,u}, \ \hat{Z} = Z^{t,x,u} - Z^{t,x,u} \). Then \( \left( \hat{X}, \hat{Y}, \hat{Z} \right) \) satisfies the following FBSDE,

\[
\begin{align*}
\hat{X}_t &= \left[ b^1(s) \hat{X}_s + b^2(s) \hat{Y}_s + b^3(s) \hat{Z}_s \right] ds + \left[ \sigma^1(s) \hat{X}_s + \sigma^2(s) \hat{Y}_s + \sigma^3(s) \hat{Z}_s \right] dB_s, \\
\hat{Y}_t &= \left[ -g^1(s) \hat{X}_s + g^2(s) \hat{Y}_s + g^3(s) \hat{Z}_s \right] ds + \hat{Z}_s dB_s, s \in [t, T], \\
\hat{X}_t &= \xi - \xi', \ \hat{Y}_t = \phi(T)\hat{X}_T, \\
\end{align*}
\]

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Proof. It is clear that there exists a sequence of random vectors 

\[
(\mathbf{X}_s^{t,\xi,m,u}, \mathbf{Y}_s^{t,\xi,m,u}, \mathbf{Z}_s^{t,\xi,m,u})_{s \in [t, T]} = \left( \sum_{i=1}^{m} \mathbf{X}_s^{t,x_i^m,u} I_{A_i^m}, \sum_{i=1}^{m} \mathbf{Y}_s^{t,x_i^m,u} I_{A_i^m}, \sum_{i=1}^{m} \mathbf{Z}_s^{t,x_i^m,u} I_{A_i^m} \right)_{s \in [t, T]},
\]

where \(b^i(s) = \left\{ \frac{b(s, X_s^{t,\xi,m}, Y_s^{t,\xi,m}, Z_s^{t,\xi,m,u})}{X_s^{t,\xi,m} - \overline{X}_s} \mathbf{X}_s \neq 0, \quad 0, \quad \mathbf{X}_s = 0 \right\} \), 

and \(b^i, \sigma^i, g^i, \phi^i \) are defined similarly, \(i = 1, 2, 3\). By Assumption \(2.1\) \(b^i, \sigma^i, g^i\) and \(\phi^i\) are bounded for \(i = 1, 2, 3\). Then \(\text{(3.5)}\) holds by Theorem 2.2 in \(\cite{7}\).

Since \(|\psi(s,0,0,0,u)| \leq L \) for \(s \in [t,T]\) and \(u \in U\) where \(\psi = b, \sigma, g\), by Theorem 2.2 in \(\cite{7}\), we obtain

\[
E \left[ \sup_{t \leq s \leq T} \left( |X_s^{t,\xi,u}|^2 + |Y_s^{t,\xi,u}|^2 \right) + \int_t^T |Z_s^{t,\xi,u}|^2 \, ds \right] \leq C \left( |\xi|^2 + \left( \int_t^T (|b| + |g|) (s,0,0,0,u) \, ds \right)^2 + \int_t^T |\sigma (s,0,0,0,u)|^2 \, ds \right) \bigg| \mathcal{F}_t \bigg] \leq C \left( 1 + |\xi|^2 \right).
\]

This completes the proof. \(\Box\)

Lemma 3.4 Suppose Assumption \(2.1\) holds. Then there exist two constants \(C\) and \(C'\) depending on \(L_1, L_2, L_3\) and \(T\) such that, for each \(t \in [0,T]\) and \(x,x' \in \mathbb{R}^n\),

\[
|W(t,x) - W(t,x')| \leq C|x - x'| \text{ and } |W(t,x)| \leq C'(1 + |x|),
\]

where \(C = \sqrt{C_2} \left(1 - \sqrt{\Lambda} \right)^{-1}\).

Proof. By Proposition \(3.2\), Lemma \(3.3\) and Theorem 2.2 in \(\cite{7}\), we obtain

\[
|W(t,x) - W(t,x')| = \left| \inf_{v \in U'(t,T)} Y_t^{t,x,v} - \inf_{v \in U'(t,T)} Y_t^{t,x',v} \right|
\leq \sup_{v \in U'(t,T)} \left| Y_t^{t,x,v} - Y_t^{t,x',v} \right|
\leq \sup_{v \in U'(t,T)} \left\{ E \left[ \sup_{t \leq s \leq T} \left| Y_s^{t,x,v} - Y_s^{t,x',v} \right|^2 \bigg| \mathcal{F}_t \right] \right\}^{\frac{1}{2}}
\leq C|x - x'|,
\]

where \(C = \sqrt{C_2} \left(1 - \sqrt{\Lambda} \right)^{-1}\). The second inequality can be proved similarly. \(\Box\)

Proposition 3.5 Suppose Assumption \(2.1\) holds. Then, for each \(\xi \in L^2(\mathcal{F}_T, \mathbb{R}^n)\), we have

\[
W(t,\xi) = \inf_{u \in U(t,T)} Y_t^{t,\xi,u}.
\]

Proof. It is clear that there exists a sequence of random vectors \(\xi^m = \sum_{i=1}^{m} x_i^m I_{A_i^m}\) such that \(E \left[ |\xi^m - \xi|^2 \right] \to 0\) as \(m \to \infty\), where \(\{A_i^m\}_{i=1}^{m}\) is a partition of \((\Omega, \mathcal{F}_1)\) and \(x_i^m \in \mathbb{R}^n\). Similar to the proof of Proposition \(3.2\), we have

\[
(\mathbf{X}_s^{t,\xi,m,u}, \mathbf{Y}_s^{t,\xi,m,u}, \mathbf{Z}_s^{t,\xi,m,u})_{s \in [t, T]} = \left( \sum_{i=1}^{m} \mathbf{X}_s^{t,x_i^m,u} I_{A_i^m}, \sum_{i=1}^{m} \mathbf{Y}_s^{t,x_i^m,u} I_{A_i^m}, \sum_{i=1}^{m} \mathbf{Z}_s^{t,x_i^m,u} I_{A_i^m} \right)_{s \in [t, T]},
\]
Thus

\[
\begin{aligned}
\underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} Y_t^{t,z_m,u} &= \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} \sum_{i=1}^{m} Y_t^{t,x_{i,m}^{u},u} I_{A_{i}^{n}} \\
&= \sum_{i=1}^{m} \left( \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} Y_t^{t,x_{i,m}^{u},u} \right) I_{A_{i}^{n}} \\
&= \sum_{i=1}^{m} W(t,x_{i,m}) I_{A_{i}^{n}} \\
&= W(t,x^{m}).
\end{aligned}
\]

(3.8)

On the other hand, by Lemmas 3.3 and 3.4, we have

\[
\left| \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} Y_t^{t,z_m,u} - \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} Y_t^{t,z_{i,m}^{u},u} \right| \leq \text{ess sup}_{u \in \mathcal{U}[t,T]} |Y_t^{t,z_m,u} - Y_t^{t,z_{i,m}^{u},u}| \leq C |\xi^{m} - \xi|,
\]

(3.9)

\[
|W(t,x^{m}) - W(t,\xi)| \leq C |\xi^{m} - \xi|.
\]

(3.10)

Combining (3.8) and (3.9), we get

\[
\left| \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} Y_t^{t,z_m,u} - W(t,\xi) \right| \leq 2C |\xi^{m} - \xi|.
\]

Thus we obtain the desired result by letting \( m \to \infty \).

Before studying the DPP, we introduce the notion of backward semigroup, which was first introduced by Peng in [18]. For each given \((t,x) \in [0,T) \times \mathbb{R}^n, \delta \in (0,T-t)\) and \(u \in \mathcal{U}[t,t+\delta]\), define \(G_{t,t+\delta}^{t,x,u} : \text{Lip}(\mathbb{R}^n) \to L^2(F_t;\mathbb{R})\) as

\[
G_{t,t+\delta}^{t,x,u} \left[ \psi(X_t^{t,x,u}) \right] = \tilde{Y}_t^{t,x,u} \text{ for each } \psi \in \text{Lip}(\mathbb{R}^n),
\]

where \(\text{Lip}(\mathbb{R}^n)\) denotes the spaces of all Lipschitz functions on \(\mathbb{R}^n\) and \((X_t^{t,x,u},\tilde{Y}_t^{t,x,u},\tilde{Z}_t^{t,x,u})\) is the solution to the following FBSDE on \([t,t+\delta]\),

\[
\begin{cases}
\frac{dX_t^{t,x,u}}{dt} = b(s,X_t^{t,x,u},\tilde{Y}_t^{t,x,u},\tilde{Z}_t^{t,x,u},u_s)ds + \sigma(s,X_t^{t,x,u},\tilde{Y}_t^{t,x,u},\tilde{Z}_t^{t,x,u},u_s)dB_s, \\
\frac{d\tilde{Y}_t^{t,x,u}}{dt} = -g(s,X_t^{t,x,u},\tilde{Y}_t^{t,x,u},\tilde{Z}_t^{t,x,u},u_s)ds + \tilde{Z}_t^{t,x,u}dB_s, \\ X_{t+\delta}^{t,x,u} = x, \tilde{Y}_{t+\delta}^{t,x,u} = \psi(X_{t+\delta}^{t,x,u}).
\end{cases}
\]

(3.11)

Since the coefficients of (3.11) satisfy Assumption 2.1, there exists a unique solution \((X_t^{t,x,u},\tilde{Y}_t^{t,x,u},\tilde{Z}_t^{t,x,u})\) to (3.11). Thus \(G_{t,t+\delta}^{t,x,u}\) is well-defined.

Now we prove the DPP for the control system (2.3).

**Theorem 3.6** Suppose Assumption 2.1 holds. Then for each \((t,x) \in [0,T) \times \mathbb{R}^n\) and \(\delta \in (0,T-t)\), we have

\[
W(t,x) = \underset{u \in \mathcal{U}[t,T]}{\text{ess inf}} G_{t,t+\delta}^{t,x,u} \left[ W(t+\delta,X_{t+\delta}^{t,x,u}) \right] = \inf_{u \in \mathcal{U}[t,T]} G_{t,t+\delta}^{t,x,u} \left[ W(t+\delta,X_{t+\delta}^{t,x,u}) \right].
\]

**Proof.** We first prove

\[
W(t,x) \geq \inf_{u \in \mathcal{U}[t,t+\delta]} G_{t,t+\delta}^{t,x,u} \left[ W(t+\delta,X_{t+\delta}^{t,x,u}) \right].
\]

(3.12)
For each given \( v \in \mathcal{U}[t, T] \), noting that
\[
(X^{t,x,v}_s, Y^{t,x,v}_s, Z^{t,x,v}_s)_{s \in [t, T]} = \left( X^{t+\delta, X^{t,x,v}_s, Y^{t+\delta, X^{t,x,v}_s, v}_s, Z^{t+\delta, X^{t,x,v}_s, v}_s}_s, Z^{t+\delta, X^{t,x,v}_s, v}_s \right)_{s \in [t+\delta, T]},
\]
we have
\[
\begin{align*}
\frac{dX^{t,x,v}_s}{dx} &= b(s, X^{t,x,v}_s, Y^{t,x,v}_s, Z^{t,x,v}_s, v, u_s)dt + \sigma(s, X^{t,x,v}_s, Y^{t,x,v}_s, Z^{t,x,v}_s, v, u_s)dz_s, \\
\frac{dY^{t,x,v}_s}{dx} &= -g(s, X^{t,x,v}_s, Y^{t,x,v}_s, Z^{t,x,v}_s, v, u_s)dt + Z^{t,x,v}_s \cdot dW_s, \\
X^{t,x,v}_t &= x, \ Y^{t,x,v}_t = Y^{t, X^{t,x,v}_t, v}_t.
\end{align*}
\]
By Proposition 3.3 we get
\[
W(t+\delta, X^{t,x,v}_{t+\delta}) \leq Y^{t+\delta, X^{t,x,v}_{t+\delta}}_{t+\delta}.
\]
Taking \( u = v \) and \( \psi(\cdot) = W(t+\delta, \cdot) \) in (3.11), by Theorem 5.1 in Appendix, we get
\[
Y^{t,x,v}_t \geq \bar{Y}^{t,x,v}_t.
\]
By Proposition 3.2 we obtain
\[
W(t, x) \geq \inf_{v \in \mathcal{U}[t, t+\delta]} C^{t,x,v}_{t, t+\delta} \left( W(t+\delta, \bar{X}^{t,x,v}_{t+\delta}) \right).
\]
Next, we prove
\[
W(t, x) \leq \inf_{u \in \mathcal{U}[t, t+\delta]} C^{t,x,u}_{t, t+\delta} \left( W(t+\delta, \bar{X}^{t,x,u}_{t+\delta}) \right). \tag{3.13}
\]
It is obvious that we only need to prove
\[
C^{t,x,u}_{t, t+\delta} \left( W(t+\delta, \bar{X}^{t,x,u}_{t+\delta}) \right) \geq W(t, x), \ P\text{-a.s.} \tag{3.14}
\]
for each \( u \in \mathcal{U}[t, t+\delta] \). The proof for (3.14) is divided into four steps.

**Step 1.** Let \( (\bar{X}^{t,x,u}, \bar{Y}^{t,x,u}, \bar{Z}^{t,x,u}) \) be the solution to the following FBSDE:
\[
\begin{align*}
\frac{d\bar{X}^{t,x,u}_s}{dx} &= b(s, \bar{X}^{t,x,u}_s, \bar{Y}^{t,x,u}_s, \bar{Z}^{t,x,u}_s, u_s)dt + \sigma(s, \bar{X}^{t,x,u}_s, \bar{Y}^{t,x,u}_s, \bar{Z}^{t,x,u}_s, u_s)dz_s, \\
\frac{d\bar{Y}^{t,x,u}_s}{dx} &= -g(s, \bar{X}^{t,x,u}_s, \bar{Y}^{t,x,u}_s, \bar{Z}^{t,x,u}_s, u_s)dt + \bar{Z}^{t,x,u}_s \cdot dW_s, \\
\bar{X}^{t,x,u}_t &= x, \ \bar{Y}^{t,x,u}_t = W(t+\delta, \bar{X}^{t,x,u}_{t+\delta}).
\end{align*}
\]
Since \( \bar{X}^{t,x,u}_{t+\delta} \in L^2(\mathcal{F}_{t+\delta}; \mathbb{R}^n) \), for each integer \( m \), we can choose a partition \( \{A^m_i : i = 1, \ldots, m\} \) of \( \mathcal{F}_{t+\delta} \) and \( x^m_i \in \mathbb{R}^n \) such that
\[
\mathbb{E} \left[ \left( \xi - \bar{X}^{t,x,u}_{t+\delta} \right)^2 \right] \to 0 \text{ as } m \to \infty,
\]
where \( \xi = \sum_{i=1}^m x^m_i I_{A^m_i} \). For each given \( x^m_i \), by Proposition 3.2 we can find a \( v^{i,m} \in \mathcal{U}[t+\delta, t+\delta] \) such that
\[
W(t+\delta, x^m_i) \leq Y^{t+\delta, x^{i,m}}_{t+\delta} \leq W(t+\delta, x^m_i) + \frac{1}{m}. \tag{3.16}
\]
Set
\[
u^{i,m}_s = u_s I_{[t, t+\delta]}(s) + \left( \sum_{i=1}^m v^{i,m}_s I_{A^m_i} \right) \left( I_{(t+\delta, T]}(s) \right).
\]
Step 2. By (3.15) in Lemma 3.3, we get
\[
E \left[ \left| Y_{t+\delta}^{t+\delta,\xi^m;u^m} - Y_{t+\delta}^{t+\delta,\xi^m;u^m} \right|^2 \right] \leq C E \left[ \left| \tilde{X}_{t+\delta} - \xi^m \right|^2 \right] \to 0 \text{ as } m \to \infty.
\] (3.17)

Similar to the proof of Proposition 3.2, one can check that
\[
\left( X_s^{t+\delta,\xi^m;u^m}, Y_s^{t+\delta,\xi^m;u^m}, Z_s^{t+\delta,\xi^m;u^m} \right)_{s \in [t+\delta,T]}
= \left( \sum_{i=1}^m X_s^{t+\delta,x_i^m;u_i^m} I_{A^n_i}, \sum_{i=1}^m Y_s^{t+\delta,x_i^m;u_i^m} I_{A^n_i}, \sum_{i=1}^m Z_s^{t+\delta,x_i^m;u_i^m} I_{A^n_i} \right)_{s \in [t+\delta,T]}.
\] (3.18)

Combining (3.16) and (3.18), we obtain
\[
W(t+\delta,\xi^m) \leq Y_{t+\delta}^{t+\delta,\xi^m;u^m} \leq W(t+\delta,\xi^m) + \frac{1}{m}.
\] (3.19)

Thus, by Lemma 3.1,
\[
E \left[ \left| Y_{t+\delta}^{t+\delta,\xi^m;u^m} - W(t+\delta,\tilde{X}_{t+\delta}^{t,x;u}) \right|^2 \right] \to 0 \text{ as } m \to \infty.
\] (3.20)

By (3.17) and (3.20), we get
\[
E \left[ \left| Y_{t+\delta}^{t+\delta,\tilde{X}_{t+\delta}^{t,x;u},u^m;\xi^m} - W(t+\delta,\tilde{X}_{t+\delta}^{t,x;u}) \right|^2 \right] \to 0 \text{ as } m \to \infty.
\] (3.21)

Consider the following decoupled FBSDE:
\[
\begin{aligned}
&d\tilde{X}_s^m = b(s, \tilde{X}_s^m, \tilde{Y}_s^m, \tilde{Z}_s^m, u_s)ds + \sigma(s, \tilde{X}_s^m, \tilde{Y}_s^m, \tilde{Z}_s^m, u_s)dB_s, \\
&d\tilde{Y}_s^m = -g(s, \tilde{X}_s^m, \tilde{Y}_s^m, \tilde{Z}_s^m, u_s)ds + dB_s, \\
&\tilde{X}_t^m = x, \tilde{Y}_t^m = Y_{t+\delta}^{t+\delta,\tilde{X}_{t+\delta}^{t,x;u},u^m}.
\end{aligned}
\] (3.22)

By (3.15), we know that \((\tilde{X}_s^{t,x;u})_{s \in [t,t+\delta]}\) satisfies the SDE in (3.22), which implies \(\tilde{X}_s^m = \tilde{X}_s^{t,x;u}\) on \([t,t+\delta]\).

Thus, by the estimate of BSDE, we get
\[
E \left[ \sup_{s \in [t,t+\delta]} \left| \tilde{Y}_s^{t,x;u} - \tilde{Y}_s^m \right|^2 + \int_t^{t+\delta} \left| \tilde{Z}_s^{t,x;u} - \tilde{Z}_s^m \right|^2 ds \right] \leq C E \left[ Y_{t+\delta}^{t+\delta,\tilde{X}_{t+\delta}^{t,x;u},u^m} - W(t+\delta,\tilde{X}_{t+\delta}^{t,x;u}) \right]^2.
\] (3.23)

It follows from (3.21) and (3.23) that
\[
E \left[ \sup_{s \in [t,t+\delta]} \left| \tilde{Y}_s^{t,x;u} - \tilde{Y}_s^m \right|^2 + \int_t^{t+\delta} \left| \tilde{Z}_s^{t,x;u} - \tilde{Z}_s^m \right|^2 ds \right] \to 0
\] (3.24)

as \(m \to \infty\).

Step 3. Define \((X_s^m, Y_s^m, Z_s^m)_{s \in [t,T]}\) as follows
\[
(X_s^m, Y_s^m, Z_s^m)_{s \in [t,T]} = (\tilde{X}_s^m, \tilde{Y}_s^m, \tilde{Z}_s^m)_{s \in [t,T]},
\]
\[
(X_s^{t+\delta,\xi^m;u^m}, Y_s^{t+\delta,\xi^m;u^m}, Z_s^{t+\delta,\xi^m;u^m})_{s \in [t^{t+\delta},T]},
\]

as \(m \to \infty\).
It is easy to verify that \((X^m_s, Y^m_s, Z^m_s)_{s \in [t,T]}\) satisfies the following FBSDE:

\[
\begin{aligned}
    dX^m_s &= [b(s, X^m_s, Y^m_s, Z^m_s, u^m_s) + l^1_s]ds + \sigma(s, X^m_s, Y^m_s, Z^m_s, u^m_s) + l^2_s dB_s, \\
    dY^m_s &= -g(s, X^m_s, Y^m_s, Z^m_s, u^m_s)ds + Z^m_s dB_s, \\
    X^m_t &= x, \ Y^m_T = \phi(X^m_T),
\end{aligned}
\]  

(3.25)

where

\[
\begin{aligned}
    l^1_s &= b(s, X^m_s, Y^m_s, Z^m_s, u^m_s) - b(s, X^m_s, Y^m_s, Z^m_s, u^m_s), \ s \in [t, t+\delta]; \ l^1_s = 0, s \in (t+\delta, T]; \\
    l^2_s &= \sigma(s, X^m_s, Y^m_s, Z^m_s, u^m_s) - \sigma(s, X^m_s, Y^m_s, Z^m_s, u^m_s), \ s \in [t, t+\delta]; \ l^2_s = 0, s \in (t+\delta, T].
\end{aligned}
\]

By Theorem 2.2 in \([\text{1}]\), we obtain

\[
\mathbb{E} \left[ \int_t^{t+\delta} \left( |Y^m_t - Y^m_s|^2 + |Z^m_t - Z^m_s|^2 \right) ds \right] \leq C \mathbb{E} \left[ \left( |Y^m_t - Y^m_s|^2 + |Z^m_t - Z^m_s|^2 \right) \right].
\]  

(3.26)

Step 4. By (3.24) and (3.26), we get

\[
\mathbb{E} \left[ |Y^m_{t+\delta} - Y^m_t|^2 \right] \leq 2 \left( \mathbb{E} \left[ |Y^m_{t+\delta} - Y^m_t|^2 \right] + \mathbb{E} \left[ |Y^m_t - Y^m_s|^2 \right] \right) \rightarrow 0 \text{ as } m \rightarrow \infty.
\]  

(3.27)

By the definition of \(W(t,x)\), we know that \(Y^m_{t+\delta} \geq W(t,x)\) P-a.s. for \(m \geq 1\). Thus we obtain (3.14) by (3.26).

Finally, since

\[
\inf_{v \in \mathcal{U}([t,T])} \mathcal{C}^{t,x,v}_{t,t+\delta} \left[ W(t+\delta, \tilde{X}^m_{t+\delta}) \right] \geq \text{ess inf}_{u \in \mathcal{U}([t,T])} \mathcal{C}^{t,x,u}_{t,t+\delta} \left[ W(t+\delta, \tilde{X}^m_{t+\delta}) \right],
\]

we obtain the desired result by (3.12) and (3.13). \(\blacksquare\)

**Remark 3.7** It is important to note that \((Y^m_{t+\delta}, X^m_{t+\delta})_{s \in [t+\delta]}\) varies with \(u^m\), which leads to the change of \((X^m_{t+\delta})_{s \in [t+\delta]}\) in the fully coupled case. This is different from the decoupled case, and the approach of establishing the DPP for the decoupled case does not work now. To overcome this difficulty, we introduce two auxiliary FBSDEs (3.22) and (3.25) to prove the DPP.

Now we prove the continuity property of \(W(t,x)\) in \(t\).

**Lemma 3.8** Suppose Assumption 2.1 holds. Then the value function \(W(t,x)\) is \(\frac{1}{2}\) Hölder continuous in \(t\).

**Proof.** For each \((t, x) \in [0, T) \times \mathbb{R}^n\) and \(\delta \in (0, T - t]\), by Theorem 3.6 we have

\[
W(t, x) = \inf_{v \in \mathcal{U}([t,T])} \mathcal{C}^{t,x,v}_{t,t+\delta} \left[ W(t+\delta, \tilde{X}^m_{t+\delta}) \right].
\]

Thus

\[
|W(t, x) - W(t + \delta, x)| \leq \sup_{v \in \mathcal{U}([t,T])} \left| \mathcal{C}^{t,x,v}_{t,t+\delta} \left[ W(t+\delta, \tilde{X}^m_{t+\delta}) \right] - W(t + \delta, x) \right|.
\]

(3.28)
For each $v \in \mathcal{U}[t, t + \delta]$, by the definition of $G_{t, t + \delta}^{t, x, u}$, we have
\[
G_{t, t + \delta}^{t, x, u} \left[ W(t + \delta, \hat{X}_{t + \delta}) \right] = E \left[ W(t + \delta, \hat{X}_{t + \delta}) + \int_t^{t + \delta} g(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s, v_s) \, ds \right].
\]
Thus, by Lemma 3.3,
\[
\left| G_{t, t + \delta}^{t, x, u} \left[ W(t + \delta, \hat{X}_{t + \delta}) \right] - W(t + \delta, x) \right| \\
\leq E \left[ W(t + \delta, \hat{X}_{t + \delta}) - W(t + \delta, x) + \int_t^{t + \delta} g(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s, v_s) \, ds \right] \\
\leq CE \left[ \hat{X}_{t + \delta} - x + \int_t^{t + \delta} \left( 1 + \left| \hat{X}_s \right| + \left| \hat{Y}_s \right| + \left| \hat{Z}_s \right| \right) \, ds \right].
\] (3.28)

It follows from Theorem 2.2 in [4] that
\[
E \left[ \sup_{t \leq s \leq t + \delta} \left( \left| \hat{X}_s \right|^2 + \left| \hat{Y}_s \right|^2 + \left| \hat{Z}_s \right|^2 \right) + \int_t^{t + \delta} \left| \hat{Z}_s \right|^2 \, ds \right] \leq C \left( 1 + |x|^2 \right),
\]
which implies that
\[
E \left[ \int_t^{t + \delta} \left( 1 + \left| \hat{X}_s \right| + \left| \hat{Y}_s \right| + \left| \hat{Z}_s \right| \right) \, ds \right] \leq C (1 + |x|) \delta^{\frac{3}{4}}. \tag{3.29}
\]

By (3.28) and (3.29), we just need to estimate $E \left[ \hat{X}_{t + \delta} - x \right]$. Denote $\hat{X}_s = \hat{X}_s^{t, x, u} - x$, $\hat{Y}_s = \hat{Y}_s^{t, x, u} - W(t + \delta, x)$, $\hat{Z}_s = \hat{Z}_s^{t, x, u}$, then $(\hat{X}, \hat{Y}, \hat{Z})$ satisfies the following FBSDE:

\[
\begin{aligned}
   d\hat{X}_s &= b(s, \hat{X}_s + x, \hat{Y}_s + W(t + \delta, x), \hat{Z}_s, v_s) \, ds + \sigma(s, \hat{X}_s + x, \hat{Y}_s + W(t + \delta, x), \hat{Z}_s, v_s) \, dB_s, \\
   d\hat{Y}_s &= -g(s, \hat{X}_s + x, \hat{Y}_s + W(t + \delta, x), \hat{Z}_s, v_s) \, ds + \hat{Z}_s \, dB_s, \quad s \in [t, t + \delta], \\
   \hat{X}_t &= 0, \quad \hat{Y}_{t + \delta} = W(t + \delta, \hat{X}_{t + \delta} + x) - W(t + \delta, x).
\end{aligned}
\] (3.30)

By Theorem 2.2 in [4] and Lemma 3.3 we get
\[
E \left[ \sup_{t \leq s \leq t + \delta} \left| \hat{X}_s \right|^2 \right] \leq CE \left[ \left( \int_t^{t + \delta} \left| b + \left| g \right| \right|(s, x, W(t + \delta, x), 0, v_s) \, ds \right)^2 \right] \\
+ CE \left[ \int_t^{t + \delta} \left| \sigma(s, x, W(t + \delta, x), 0, v_s) \right|^2 \, ds \right] \\
\leq C \left( 1 + |x|^2 \right) \delta^{\frac{3}{4}},
\]
which yields $E \left[ \left| \hat{X}_{t + \delta}^{t, x, u} - x \right| \right] \leq C (1 + |x|) \delta^{\frac{3}{4}}$. Noting that the above constant $C$ does not depend on $u$, then
\[
E \left[ \left| W(t, x) - W(t + \delta, x) \right| \right] \leq C (1 + |x|) \delta^{\frac{3}{4}}.
\]

This completes the proof. $\blacksquare$

**Remark 3.9** In order to prove $E \left[ \left| \hat{X}_{t + \delta}^{t, x, u} - x \right| \right] \leq C (1 + |x|) \delta^{\frac{3}{4}}$, we construct another FBSDE, which different from the proof in [10]. Specially, we do not need additional assumption on $L_3$ as in [10].
3.2 The value function and the HJB equation

In this subsection, we show that the value function $W(t, x)$ defined in (2.4) is a viscosity solution to the following HJB equation

$$
\begin{cases}
\partial_t W(t, x) + \inf_{u \in U} H(t, x, W(t, x), DW(t, x), D^2W(t, x), u) = 0, \\
W(T, x) = \phi(x),
\end{cases}
$$

where

$$H(t, x, v, p, A, u) = \frac{1}{2} \text{tr} [\sigma \sigma^T (t, x, v, V(t, x, v, p, u), u) + p^T b(t, x, v, V(t, x, v, p, u), u) + g(t, x, v, V(t, x, v, p, u), u),$$

$$V(t, x, v, p, u) = p^T \sigma (t, x, v, V(t, x, v, p, u), u),$$

$$(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U.$$  

(3.31)

We first give the definition of viscosity solution (see [4]).

**Definition 3.10**

(i) A real-valued continuous function $W(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity subsolution (resp. supersolution) to (3.31) if $W(T, x) \leq \phi(x)$ (resp. $W(T, x) \geq \phi(x)$) for all $x \in \mathbb{R}^n$ and if for all $\varphi \in C^{2,3}_b([0, T] \times \mathbb{R}^n)$ such that $W(t, x) = \varphi(t, x)$ and $W - \varphi$ attains a local maximum (resp. minimum) at $(t, x) \in (0, T) \times \mathbb{R}^n$, we have

$$
\begin{cases}
\partial_t \varphi(t, x) + \inf_{u \in U} H(t, x, \varphi(t, x), D\varphi(t, x), D^2\varphi(t, x), u) \geq 0 \\
(\text{resp. } \varphi(t, x) + \inf_{u \in U} H(t, x, \varphi(t, x), D\varphi(t, x), D^2\varphi(t, x), u) \leq 0).
\end{cases}
$$

(ii) A real-valued continuous function $W(\cdot, \cdot) \in C([0, T] \times \mathbb{R}^n)$ is called a viscosity solution to (3.31), if it is both a viscosity subsolution and viscosity supersolution.

In order to prove that $W(t, x)$ is a viscosity solution to the HJB equation (3.31), we need the following assumption.

**Assumption 3.11** $L_3 L_W < 1$ and $8C_4 L_3^4 < 1$, where $L_W = \sqrt{C_2} \left(1 - \sqrt{A}\right)^{-1}$ is the Lipschitz constant of value function $W$ with respect to $x$, $C_2$ and $C_4$ are defined in Lemma 5.1 in [7].

**Theorem 3.12** Suppose Assumptions 2.1 and 3.11 hold. Then the value function $W(t, x)$ is the viscosity solution to the HJB equation (3.31).

We first prove the following lemmas.

Note that $W(t, x)$ is Lipschitz continuous with respect to $x$. For each given $\varphi \in C^{2,3}_b([0, T] \times \mathbb{R}^n)$, by $L_3 L_W < 1$, we can assume $L_3 \sup_{(t, x) \in [0, T] \times \mathbb{R}^n} |D\varphi(t, x)| < 1$ by the definition of viscosity solution. Consider the following FBSDE and BSDE: $\forall s \in [t, t + \delta]$,
\[
\begin{aligned}
\begin{cases}
    dX_s^u &= b(s, X_s^u, Y_s^u, Z_s^u, u_s)ds + \sigma(s, X_s^u, Y_s^u, Z_s^u, u_s)dB_s, \\
    dY_s^u &= -g(s, X_s^u, Y_s^u, Z_s^u, u_s)ds + Z_s^udB_s, \\
    X_t^u &= x, Y_t^{u, \delta} = \varphi(t + \delta, X_t^{u, \delta})
\end{cases}
\end{aligned}
\]
and
\[
\begin{aligned}
\begin{cases}
    dY_1^{u, x} &= -F_1(s, X_s^u, Y_s^u, Z_s^u, u_s)ds + Z_1^{u, x}dB_s, \\
    Y_1^{u, x} &= 0,
\end{cases}
\end{aligned}
\]
where
\[
F_1(s, x, y, z, u) = \partial_t\varphi(s, x) + (D\varphi(s, x))^T b(s, x, y + \varphi(s, x), h(s, x, y, z, u), u)
+ \frac{4}{3} \text{tr}\left[\sigma\sigma^T(s, x, y + \varphi(s, x), h(s, x, y, z, u), u)D^2\varphi(s, x)\right]
+ q(s, x, y + \varphi(s, x), h(s, x, y, z, u), u),
\]
\[
h(s, x, y, z, u) = z + D\varphi(s, x)^T \sigma(s, x, y + \varphi(s, x), h(s, x, y, z, u), u).
\]

**Lemma 3.13** Suppose Assumptions 2.1 and 3.11 hold. Then there exists a unique function \(h(s, x, y, z, u)\) satisfying (3.33) for each \(s \in [0, T]\), \(x, \bar{x} \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R}^{1 \times d}\) and \(u \in U\). Furthermore, for each given \(s \in [0, T]\), \(x \in \mathbb{R}^n, y, \bar{y} \in \mathbb{R}, z, \bar{z} \in \mathbb{R}^{1 \times d}\) and \(u \in U\),

\[
|h(s, x, y, z, u)| \leq C(1 + |x| + |y| + |z|),
\]
\[
|h(s, x, y, z, u) - h(s, \bar{x}, \bar{y}, \bar{z}, u)| \leq C(1 + |x| + |y| + |z|) |x - \bar{x}| + C(|y - \bar{y}| + |z - \bar{z}|),
\]
and \(h(\cdot)\) is continuous with respect to \(s, x, y, z, u\).

**Proof.** For each given \(s \in [0, T]\), \(x \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R}^{1 \times d}\) and \(u \in U\), we define a mapping \(\Gamma : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^{1 \times d}\) as follows

\[
\Gamma z' := z + D\varphi(s, x)^T \sigma(s, x, y + \varphi(s, x), z', u) \text{ for } z' \in \mathbb{R}^{1 \times d}.
\]
For each \(z_1, z_2 \in \mathbb{R}^{1 \times d}\), we have

\[
|\Gamma z_1 - \Gamma z_2| = |\sigma(s, x, y + \varphi(s, x), z_1, u)^T D\varphi(s, x) - \sigma(s, x, y + \varphi(s, x), z_2, u)^T D\varphi(s, x)|
\leq L_3 \sup_{(t, x) \in [0, T] \times \mathbb{R}^n} |D\varphi(t, x)| |z_1 - z_2|,
\]
which implies that \(\Gamma\) is a contraction mapping. Thus there exists a unique \(z' \in \mathbb{R}^{1 \times d}\) such that \(\Gamma z' = z'\).

Define \(h(s, x, y, z, u) = z'\), then \(h(\cdot)\) satisfies (3.35).

For each \(s \in [0, T]\), \(x, \bar{x} \in \mathbb{R}^n, y, \bar{y} \in \mathbb{R}, z, \bar{z} \in \mathbb{R}^{1 \times d}\) and \(u \in U\),

\[
|h(s, x, y, z, u)|
\leq |z| + L_3 |D\varphi|_{\infty} |h(s, x, y, z, u)| + |D\varphi|_{\infty} |\sigma(s, x, y + \varphi(s, x), u)|
\leq L_3 |D\varphi|_{\infty} |h(s, x, y, z, u)| + C(1 + |x| + |y| + |z|),
\]

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Proof. Lemma 3.14

For each \( s \), we have

\[
|h(s, x, y, z, u) - h(s, \bar{x}, \bar{y}, \bar{z}, u)|
\leq |z - \bar{z}| + \|D\varphi\|_\infty \| \sigma(s, x, y + \varphi(s, x), h(s, x, y, z, u), u) - \sigma(s, \bar{x}, \bar{y} + \varphi(s, \bar{x}), h(s, \bar{x}, \bar{y}, \bar{z}, u), u) \|
\]

\[
+ |D\varphi(s, x) - D\varphi(s, \bar{x})| \| \sigma(s, x, y + \varphi(s, x), h(s, x, y, z, u), u) \|
\]

\[
\leq |z - \bar{z}| + \|D\varphi\|_\infty \{ L_3 |h(s, x, y, z, u) - h(s, \bar{x}, \bar{y}, \bar{z}, u)| + C (|x - \bar{x}| + |y - \bar{y}|) \}
\]

\[
+ C |x - \bar{x}| (1 + |x| + |y| + |h(s, x, y, z, u)|),
\]

which implies (3.36) and (3.37). Now we prove that \( h(\cdot) \) is continuous. For each \((s_m, x_m, y_m, z_m, u_m) \rightarrow (s, x, y, z, u)\),

\[
|h(s_m, x_m, y_m, z_m, u_m) - h(s, x, y, z, u)|
\leq |z_m - z| + L_3 \|D\varphi\|_\infty |h(s_m, x_m, y_m, z_m, u_m) - h(s, x, y, z, u)|
\]

\[
+ |D\varphi(s_m, x_m)^\top \sigma(s_m, x_m, y_m + \varphi(s_m, x_m), h(s, x, y, z, u), u_m)
- D\varphi(s, x)^\top \sigma(s, x, y + \varphi(s, x), h(s, x, y, z, u), u)|,
\]

which implies that \( h(\cdot) \) is continuous with respect to \( s, x, y, z, u \).  

Lemma 3.14 For each \( s \in [t, t + \delta] \), we have

\[
Y^{1.\text{u}}_s = Y^\text{u}_s - \varphi(s, X^\text{u}_s),
\]

\[
Z^{1.\text{u}}_s = Z^\text{u}_s - D\varphi(s, X^\text{u}_s)^\top \sigma(s, X^\text{u}_s, Y^\text{u}_s, Z^\text{u}_s, u_s).
\]

Proof. Applying Itô’s formula to \( Y^\text{u}_s - \varphi(s, X^\text{u}_s) \), we can obtain the desired result.  

Under Assumption 3.11 we can choose a \( \delta_0 > 0 \) such that \( 8C_4 \left[ L_2^4 (\delta_0^2 + \delta_0^4) + L_4^4 \right] < 1 \). By Theorem 5.2 in Appendix, then for each \( \delta < \delta_0 \), we have

\[
\mathbb{E} \left[ \sup_{t \leq s \leq t + \delta} (|X^\text{u}_s|^4 + |Y^\text{u}_s|^4) + \left( \int_t^{t + \delta} |Z^\text{u}_s|^2 ds \right)^2 \right]
\leq C \left( |x|^4 + \mathbb{E} \left[ \left( \int_t^{t + \delta} (|b(s, 0, 0, 0, u_s)| + |g(s, 0, 0, 0, u_s)|) ds \right)^4 + \left( \int_t^{t + \delta} |\sigma(s, 0, 0, 0, u_s)|^2 ds \right)^2 \right) \right]
\leq C \left( 1 + |x|^4 \right),
\]

(3.39)

where \( C \) is a constant independent of \( u \) and \( \delta \). Consider the following BSDE: \( \forall s \in [t, t + \delta] \),

\[
\begin{cases}
  dY^{2.\text{u}}_s &= -F_1(s, x, 0, 0, u_s) ds + Z^{2.\text{u}}_s dB_s, \\
  Y^{2.\text{u}}_{t+\delta} &= 0.
\end{cases}
\]

(3.40)

We have the following estimate.
Lemma 3.15 For each given $v \in \mathcal{U}^t [t, t + \delta]$, we have
\[
|Y^{1,v}_t - Y^{2,v}_t| \leq C\delta^{1/2},
\]
where $C$ is a positive constant depend on $x$ and independent of $v, \delta$.

**Proof.** Since
\[
Y^{1,v}_t = \mathbb{E} \left[ \int_t^{t+\delta} F_1(s, X^v_s, Y^{1,v}_s, Z^{1,v}_s, v_s) ds \right],
\]
\[
Y^{2,v}_t = \mathbb{E} \left[ \int_t^{t+\delta} F_1(s, x, 0, v_s) ds \right],
\]
we obtain
\[
|Y^{1,v}_t - Y^{2,v}_t| \leq \mathbb{E} \left[ \int_t^{t+\delta} \tilde{F}_s ds \right],
\]
where
\[
\tilde{F}_s = |F_1(s, X^v_s, Y^{1,v}_s, Z^{1,v}_s, v_s) - F_1(s, x, 0, v_s)|.
\]
Note that $\varphi \in C^{2,3}_b([0, T] \times \mathbb{R}^n)$. By Lemmas 3.13 and 3.14 it is easy to check that
\[
\tilde{F}_s \leq C \left( |X^v_s - x| + |Y^{1,v}_s| + |Z^{1,v}_s| + |X^v_s - x|^2 + |Y^{1,v}_s|^2 + |Z^{1,v}_s|^2 \right).
\]
Set $\tilde{X}^v_s = X^v_s - x$, $\tilde{Y}^v_s = Y^{1,v}_s - \tilde{X}^v_s$, $\tilde{Z}^v_s = Z^{1,v}_s$. Then $(\tilde{X}^v_s, \tilde{Y}^v_s, \tilde{Z}^v_s)$ satisfies the following FBSDE:
\[
\begin{align*}
\frac{d\tilde{X}^v_s}{ds} &= b(\tilde{X}^v_s + x, \tilde{Y}^v_s + \varphi(s, \tilde{X}^v_s), \tilde{Z}^v_s, v_s) ds + \sigma(s, \tilde{X}^v_s + x, \tilde{Y}^v_s + \varphi(s, \tilde{X}^v_s), \tilde{Z}^v_s, v_s) dB_s, \\
\frac{d\tilde{Y}^v_s}{ds} &= -g(s, \tilde{X}^v_s + x, \tilde{Y}^v_s + \varphi(s, \tilde{X}^v_s), \tilde{Z}^v_s, v_s) ds + \tilde{Z}^v_s dB_s, \\
\tilde{X}^v_t &= 0, \quad \tilde{Y}^v_{t+\delta} = 0.
\end{align*}
\]
By (3.39) and Theorem (3.42) in Appendix, we have
\[
\mathbb{E} \left[ \sup_{t \leq s \leq t+\delta} \left( |\tilde{X}^v_s|^p + |\tilde{Y}^v_s|^p \right) + \left( \int_t^{t+\delta} |\tilde{Z}^v_s|^2 ds \right)^2 \right] \leq C \left( 1 + \mathbb{E} \left[ \sup_{t \leq s \leq t+\delta} |X^v_s|^p \right] \right) \delta^{1/2} \leq C \delta^{1/2},
\]
where $p \in [2, 4]$. Thus
\[
\mathbb{E} \left[ \int_t^{t+\delta} \left( |X^v_s - x| + |Y^{1,v}_s| + |X^v_s - x|^2 + |Y^{1,v}_s|^2 \right) ds \right] \leq C \delta^{1/2}.
\]
On the other hand, by (3.31) and (3.42), we have
\[
\mathbb{E} \left[ \int_t^{t+\delta} |Z^{1,v}_s|^2 ds \right] = |Y^{1,v}_t|^2 + \mathbb{E} \left[ \left( \int_t^{t+\delta} F_1(s, X^v_s, Y^{1,v}_s, Z^{1,v}_s, v_s) ds \right)^2 \right] \leq 2 \mathbb{E} \left[ \left( \int_t^{t+\delta} |F_1(s, X^v_s, Y^{1,v}_s, Z^{1,v}_s, v_s)| ds \right)^2 \right].
\]
It is easy to check that
\[
|F_1(s, X^v_s, Y^{1,v}_s, Z^{1,v}_s, u_s)| \leq C(1 + |X^v_s|^2 + |Y^v_s|^2 + |Z^v_s|^2).
\]
Thus, by (3.39) and (3.43), we obtain
\[
E \left[ \int_t^{t+\delta} |Z_s^{1,v}|^2 ds \right] \leq C \left( \delta^2 + E \left[ \left( \int_t^{t+\delta} |Z_s^{1,v}|^2 ds \right)^{\frac{3}{2}} \right] \right) \leq C\delta^2.
\]
Since
\[
E \left[ \int_t^{t+\delta} |Z_s^{1,v}| ds \right] \leq \left( E \left[ \int_t^{t+\delta} |Z_s^{1,v}|^2 ds \right] \right)^{\frac{3}{2}} \delta^{\frac{1}{2}} \leq C\delta^2,
\]
we obtain the desired result. ■

Now we compute \[ \inf_{v \in U^\delta[t,t+\delta]} Y_{t}^{2,v}. \]

**Lemma 3.16** We have
\[
Y_t^0 = \inf_{v \in U^\delta[t,t+\delta]} Y_{t}^{2,v},
\]
where \( Y_t^0 \) is the solution to the following ordinary differential equation:
\[
\begin{aligned}
\frac{dY_s^0}{ds} &= -F_0(s,x) ds, \\
Y_{t+\delta}^0 &= 0, \quad s \in [t,t+\delta],
\end{aligned}
\]
and
\[
F_0(s,x) = \inf_{u \in U} F_1(s,x,0,0,u).
\]

**Proof.** For each given \( v \in U^\delta[t,t+\delta], F_1(s,x,0,0,u) \geq F_0(s,x) \), by comparison theorem of BSDE, we get \( Y_{t}^{2,v} \geq Y_{t}^{0} \). On the other hand, we can choose a deterministic control \( \mu \) in \( U^\delta[t,t+\delta] \) such that
\[
F_0(s,x) = F_1(s,x,0,0,\mu).
\]
It is clear that \( Y_{t}^{2,\mu} = Y_{t}^{0} \). Thus we obtain the desired result. ■

Finally, we give the proof of Theorem 3.12.

**Proof.** Obviously, \( W(T,x) = \phi(x), x \in \mathbb{R}^n \). We first prove that \( W \) is a viscosity subsolution. For each given \( (t,x) \in [0,T] \times \mathbb{R}^n \), suppose \( \varphi(\cdot) \in C^2_{\delta}([0,T] \times \mathbb{R}^n) \) such that \( \varphi(t,x) = W(t,x), \varphi \geq W \) on \([0,T] \times \mathbb{R}^n\) and \( ||D\varphi||_{\infty} \leq L_W + 1 \), where \( L_W \) is the Lipschitz constant of \( W \). By Theorem 3.6 we have
\[
W(t,x) = \inf_{v \in U^\delta[t,t+\delta]} C_{t,t+\delta}^{t,x,v} \left[ W(t+\delta, \tilde{X}_{t+\delta}^{t,x,v}) \right].
\]
Since \( \varphi(t+\delta,\cdot) \geq W(t+\delta,\cdot) \), by Theorem 5.1 in Appendix, we get \( Y_t^\varepsilon \geq W(t,x) \) for each \( v \in U^\delta[t,t+\delta] \), which implies
\[
\inf_{v \in U^\delta[t,t+\delta]} |Y_t^\varepsilon - \varphi(t,x)| = \inf_{v \in U^\delta[t,t+\delta]} Y_{t+\delta}^1 \geq 0.
\]
By (3.31), we deduce
\[
\inf_{v \in U^\delta[t,t+\delta]} Y_{t}^{2,v} \geq -C\delta^2.
\]
It yields that \( Y_{t}^{0} \geq -C\delta^2 \) by Lemma 3.16. Thus
\[
-C\delta^2 \leq \int_t^{t+\delta} F_0(s,x) ds.
\]
Letting \( \delta \to 0 \), we get \( F_0(t,x) \geq 0 \), which implies that \( W \) is a viscosity subsolution. By the same method, we can prove that \( W \) is a viscosity supersolution. Thus \( W \) is a viscosity solution. ■
Remark 3.17 Note that Assumption 2.1 (ii) is only used to guarantees the well-posedness of our fully coupled forward backward controlled system. In fact, following our approach, the readers may verify that all the results in Section 3 still hold under Assumptions 2.1 (i), 3.11 and the following monotonicity conditions.

Given a nonzero $G \in \mathbb{R}^{1 \times n}$, define

$$
\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A(t, \lambda, u) = \begin{pmatrix} -G^T g \\ Gb \\ G\sigma \end{pmatrix} (t, \lambda, u).
$$

Assumption 3.18 (Monotonicity conditions) (i) $\langle A(t, \lambda, u) - A(t, \bar{\lambda}, u), \lambda - \bar{\lambda} \rangle \leq -\beta_1 |G\hat{x}|^2 - \beta_2 \left( |G^T \hat{y}|^2 + |G^T \hat{z}|^2 \right)$, for $u \in U$;

(ii) $\langle \phi(x) - \phi(\bar{x}), G\hat{x} \rangle \geq \mu_1 |G\hat{x}|^2$, where $\hat{x} = x - \bar{x}$, $\hat{y} = y - \bar{y}$, $\hat{z} = z - \bar{z}$, $\beta_1$, $\beta_2$, $\mu_1$ are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover, $\beta_2 > 0$ when $n > 1$.

4 The uniqueness of viscosity solutions

In this section, we study the uniqueness of the viscosity solution to the HJB equation (3.31).

4.1 $\sigma$ independent of $y$ and $z$

In this case, the corresponding HJB equation becomes

$$
\begin{align*}
\frac{\partial W(t, x)}{\partial t} + \inf_{u \in U} \left( H(t, x, W(t, x), DW(t, x), D^2W(t, x), u) \right) &= 0, \\
W(T, x) &= \phi(x),
\end{align*}
$$

where

$$
H(t, x, v, p, A, u) = \frac{1}{2} \text{tr} [\sigma \sigma^T (t, x, u) A] + p^T b(t, x, v, p^T \sigma(t, x, u), u) + g(t, x, v, p^T \sigma(t, x, u), u),
$$

$(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U$.

We adopt the approach in Barles, Buckdahn and Pardoux (see also Wu and Yu) to prove the uniqueness of the viscosity solution to (4.1) in the following theorem. Note that applying the approach in Barles, Buckdahn and Li studied a decoupled case and Wu and Yu obtained the uniqueness result for coefficients which are independent of $u$.

Theorem 4.1 Suppose Assumption 2.1 (i) holds. Then there exists at most one viscosity solution to (4.1) in the class of continuous functions which are Lipschitz continuous with respect to $x$.

For the reader’s convenience, we give the detailed proofs in the Appendix.
4.2 $\sigma$ depends on $y$ and $z$

Wu and Yu [2] studied a PDE system for which the coefficient $\sigma$ of the corresponding FBSDE satisfies $\sigma(t,x,y,0) = 0$. Under this assumption, the fully coupled FBSDE degenerates to a forward-backward ordinary differential equation and the PDE system degenerates to a first order PDE. Thus, for this case, the uniqueness result is implied by Theorem 4.1.

In this subsection, we study the HJB equation in which $\sigma$ is dependent on $y$ and $z$. As pointed out in Remark 4.3 the method in [1] does not work. We first give the following proposition.

**Proposition 4.2** Suppose $\sigma$ is independent of $y$ and $z$; and one of the following two conditions holds true:

(i) Assumption 2.1(i) holds;

(ii) Assumptions 2.1 (i) and 3.18 hold.

Let $W$ be the value function. Then, for each $(t,x) \in [0,T) \times \mathbb{R}^n$, we can find a sequence $u^m \in U^t[t,T]$ such that

$$E \left[ \int_t^T \left| Y^{t,x,u^m}_s - W \left( s, X^{t,x,u^m}_s \right) \right|^2 ds \right] + \left| Y^{t,x,u^m}_t - W(t,x) \right| \to 0 \text{ as } m \to \infty.$$ 

**Proof.** We only prove the first case (the condition (i) holds). The proof for the second case is similar. The proof is divided into three steps.

**Step 1.** For each given integer $m \geq 1$, set $t^m_i = t + i(T-t)m^{-1}$ for $i = 0, \ldots, m$. We want to choose a $u^{i,m} \in U^t[t^m_i, t^m_{i+1}]$ such that

$$|Y^{i,m}_{t^m_i} - W \left( t^m_i, X^{i-1,m}_{t^m_i} \right)| \leq \frac{1}{m^2(2C + 1)^m} \text{ for } i = 0, \ldots, m - 1,$n

(4.2)

where $C$ is given in Lemma 3.3. $X^{i-1,m}_{t^m_i} = x$ and $(X^{i,m}, Y^{i,m}, Z^{i,m})$ is the solution to following FBSDE:

$$\begin{cases}
  dX^{i,m}_s = b(s, X^{i,m}_s, Y^{i,m}_s, Z^{i,m}_s, u^{i,m}_s) ds + \sigma(s, X^{i,m}_s, u^{i,m}_s) dB_s, \\
  dY^{i,m}_s = -g(s, X^{i,m}_s, Y^{i,m}_s, Z^{i,m}_s, u^{i,m}_s) ds + Z^{i,m}_s dB_s, \quad s \in [t^m_i, t^m_{i+1}], \\
  X^{i,m}_{t^m_i} = X^{i-1,m}_{t^m_i}, \quad Y^{i,m}_{t^m_{i+1}} = W(t^m_{i+1}, X^{i,m}_{t^m_{i+1}}).
\end{cases}$$

(4.3)

Precisely, on the interval $[t^m_i, t^m_{i+1}]$, by Theorem 3.6 we can choose a $u^{0,m} \in U^t[t^m_0, t^m_1]$ such that

$$\left| Y^{0,m}_{t^m_0} - W \left( t^m_0, x \right) \right| \leq \frac{1}{m^2(2C + 1)^m},$$

where $(X^{0,m}, Y^{0,m}, Z^{0,m})$ is the solution to FBSDE (4.3) for $i = 0$. On the interval $[t^m_1, t^m_2]$, we first choose a partition $\{ A^m_j : j \geq 1 \}$ of $\mathbb{R}^n$ and $x^m_j \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{\infty} x^m_j I_{A^m_j} \left( X^{0,m}_{t^m_1} - X^{0,m}_{t^m_0} \right) \leq \frac{1}{m^2(2C + 1)^m}.$$n

By Theorem 3.6 for each $x^m_j$, we can choose a $u^{1,j,m} \in U^t[t^m_1, t^m_2]$ such that

$$\left| Y^{1,j,m}_{t^m_1} - W \left( t^m_1, x^m_j \right) \right| \leq \frac{1}{m^2(2C + 1)^m}.$$n
where \((X^{1:j,m}, Y^{1:j,m}, Z^{1:j,m})\) is the solution to the following FBSDE:

\[
\begin{align*}
\begin{cases}
    dX^{1:j,m}_s = b(s, X^{1:j,m}_s, Y^{1:j,m}_s, Z^{1:j,m}_s, u^{1:j,m}_s)ds + \sigma(s, X^{1:j,m}_s, u^{1:j,m}_s)dB_s, \\
    dY^{1:j,m}_s = -g(s, X^{1:j,m}_s, Y^{1:j,m}_s, Z^{1:j,m}_s, u^{1:j,m}_s)ds + Z^{1:j,m}_s dB_s, \\
    X^{1:j,m}_{t_1} = x^m, \quad Y^{1:j,m}_{t_2} = W(t_2, X^{1:j,m}_{t_2}).
\end{cases}
\end{align*}
\]

Set

\[
\begin{align*}
    u^{1:m}_s &= \sum_{j=1}^{\infty} I_{A^j}^m(X^{0:m}_{t_1})u_s^{1:j,m} \in U^f[t_1^{m}, t_2^{m}] \quad \text{for } s \in [t_1^{m}, t_2^{m}].
\end{align*}
\]

By Theorem 2.2 in \cite{Reference} for (4.3) and (4.4),

\[
\begin{align*}
    |Y^{1:m}_{t_1} - Y^{1:j,m}_{t_1}| I_{A^j}^m(X^{0:m}_{t_1}) \leq C |X^{0:m}_{t_1} - x^m| I_{A^j}^m(X^{0:m}_{t_1}).
\end{align*}
\]

Thus, by Lemma 3.4

\[
\begin{align*}
    &|Y^{1:m}_{t_1} - W(t_1^{m}, X^{0:m}_{t_1})| \\
    \leq & |Y^{1:m}_{t_1} - \sum_{j=1}^{\infty} Y^{1:j,m}_{t_1} I_{A^j}^m(X^{0:m}_{t_1})| + \left| \sum_{j=1}^{\infty} Y^{1:j,m}_{t_1} I_{A^j}^m(X^{0:m}_{t_1}) - W(t_1^{m}, X^{0:m}_{t_1}) \right| \\
    \leq & C \left| X^{0:m}_{t_1} - \sum_{j=1}^{\infty} x^m_j I_{A^j}^m(X^{0:m}_{t_1}) \right| + \frac{1}{m^{2(2^c+1)m+1}} + \left| W(t_1^{m}, \sum_{j=1}^{\infty} x^m_j I_{A^j}^m(X^{0:m}_{t_1})) - W(t_1^{m}, X^{0:m}_{t_1}) \right| \\
    \leq & \frac{1}{m^{2(2^c+1)m}}.
\end{align*}
\]

Similarly, we can choose the desired \(u^{i:m} \in U^f[t_1^{m}, t_{i+1}^{m}]\) for \(i = 2, \ldots, m - 1\).

**Step 2.** Set

\[
\begin{align*}
    u^{i:m}_s &= \sum_{i=0}^{m-1} u^{i:m}_s I_{t_1^{m}, t_{i+1}^{m}}(s), \quad X^{i:m}_s = \sum_{i=0}^{m-1} X^{i:m}_s I_{t_1^{m}, t_{i+1}^{m}}(s), \\
    Y^{i:m}_s &= \sum_{i=0}^{m-1} Y^{i:m}_s I_{t_1^{m}, t_{i+1}^{m}}(s), \quad Z^{i:m}_s = \sum_{i=0}^{m-1} Z^{i:m}_s I_{t_1^{m}, t_{i+1}^{m}}(s),
\end{align*}
\]

where \(u^{i:m}, X^{i:m}, Y^{i:m}\) and \(Z^{i:m}\) are given in Step 1. It is easy to check that \(X^m\) satisfies the following SDE:

\[
\begin{align*}
\begin{cases}
    dX^m_s = b(s, X^m_s, Y^m_s, Z^m_s, u^m_s)ds + \sigma(s, X^m_s, u^m_s)dB_s, \\
    X^m_t = x, \quad s \in [t, T].
\end{cases}
\end{align*}
\]

Let \((X^m, \tilde{Y}^m, \tilde{Z}^m)\) be the solution to the following decoupled FBSDE:

\[
\begin{align*}
\begin{cases}
    dX^m_s = b(s, X^m_s, Y^m_s, Z^m_s, u^m_s)ds + \sigma(s, X^m_s, u^m_s)dB_s, \\
    d\tilde{Y}^m_s = -g(s, X^m_s, \tilde{Y}^m_s, \tilde{Z}^m_s, u^m_s)ds + \tilde{Z}^m_s dB_s, \\
    X^m_t = x, \quad \tilde{Y}^m_T = \phi(X^m_T).
\end{cases}
\end{align*}
\]

\[
\text{(4.6)}
\]
In the followings, we want to prove
\[
E \left[ \int_t^T \left( |\tilde{Y}^m_s - W(s, X^m_s)|^2 + |\tilde{Y}^m_s - Y^m_s|^2 + |\tilde{Z}^m_s - Z^m_s|^2 \right) ds \right] + |\tilde{Y}^m_t - W(t, x)| \to 0 \text{ as } m \to \infty. \quad (4.7)
\]

Note that \((X^m, Y^m, \tilde{Z}^m)\) satisfies the following FBSDE on \([t^m_i, t^m_{i+1}]:\)
\[
\begin{cases}
    dX^m_s = b(s, X^m_s, Y^m_s, Z^m_s, u^m_s)ds + \sigma(s, X^m_s, u^m_s)dB_s, \\
    d\tilde{Y}^m_s = -g(s, X^m_s, \tilde{Y}^m_s, \tilde{Z}^m_s, u^m_s)ds + \tilde{Z}^m_sdB_s, \ s \in [t^m_i, t^m_{i+1}], \\
    X^m_{t^m_i} = X^{i-1,m}_{t^m_{i+1}}, \ \tilde{Y}^m_{t^m_i} = Y^m_{t^m_{i+1}}.
\end{cases} \quad (4.8)
\]

Then by Theorem 2.2 in [2] for FBSDEs (4.3) and (1.5),
\[
E \left[ \sup_{t^m_i \leq s \leq t^m_{i+1}} |\tilde{Y}^m_s - Y^m_s|^2 + \int_{t^m_i}^{t^m_{i+1}} |\tilde{Z}^m_s - Z^m_s|^2 ds \right] \leq C^2 \mathbb{E} \left[ \left| W(t^m_{i+1}, X^m_{t^m_{i+1}}) - \tilde{Y}^m_{t^m_{i+1}} \right|^2 \mathcal{F}_{t^m_i} \right], \quad (4.9)
\]
where \(C\) is the same as in Step 1. It yields that
\[
|\tilde{Y}^m_{t^m_i} - W(t^m_i, X^m_{t^m_i})| \leq \left| Y^m_{t^m_i} - W(t^m_i, X^m_{t^m_i}) \right| + C \left\{ \mathbb{E} \left[ \left| W(t^m_{i+1}, X^m_{t^m_{i+1}}) - \tilde{Y}^m_{t^m_{i+1}} \right|^2 \mathcal{F}_{t^m_i} \right] \right\}^{\frac{1}{2}}. \quad (4.10)
\]
Note that (1.2), (1.10), \(W(t^m_i, X^m_{t^m_i}) = \phi(X^m_i)\) and Lemma 3.3. By doing estimate recursively, we obtain
\[
|\tilde{Y}^m_{t^m_i} - W(t^m_i, X^m_{t^m_i})| \leq \frac{1 + C + \cdots + C^{m-1}}{m^2(2C + 1)^m} \leq \frac{1}{m}. \quad (4.11)
\]
Combining (4.3) and (4.11), we get
\[
E \left[ \int_t^T \left( |\tilde{Y}^m_s - Y^m_s|^2 + |\tilde{Z}^m_s - Z^m_s|^2 \right) ds \right] \leq C^2 \sum_{i=0}^{m-1} E \left[ \left| W(t^m_{i+1}, X^m_{t^m_{i+1}}) - \tilde{Y}^m_{t^m_{i+1}} \right|^2 \right] \leq \frac{C^2}{m}. \quad (4.12)
\]
By Theorem 2.2 in [2] for FBSDE (4.6) and (1.12), we obtain that
\[
E \left[ \sup_{t \leq s \leq T} |X^m_s|^2 + \int_t^T \left( |\tilde{Y}^m_s|^2 + |\tilde{Z}^m_s|^2 + |\tilde{Y}^m_s|^2 + |\tilde{Z}^m_s|^2 \right) ds \right] \leq C'(1 + |x|^2), \quad (4.13)
\]
where \(C'\) is a constant which is independent of \(m\). For \(s \in [t^m_i, t^m_{i+1}]\), by (4.13) and Lemma 3.3, we have
\[
E \left[ \tilde{Y}^m_s - Y^m_s \right]^2 = E \left[ E \left[ \int_t^{t^m_i} g(r, X^m_r, Y^m_r, \tilde{Z}^m_r, u^m_r)dr \right| \mathcal{F}_t \right]^2 \right] \leq \frac{T}{m} E \left[ \int_{t^m_i}^{t^m_{i+1}} |g(r, X^m_r, \tilde{Y}^m_r, \tilde{Z}^m_r, u^m_r)|^2 dr \right] \leq C(1 + |x|^2) \frac{1}{m},
\]
\[
E \left[ \tilde{X}^m_s - X^m_s \right]^2 \leq 2E \left[ \left( \int_{t^m_i}^{t^m_{i+1}} |b(r, X^m_r, Y^m_r, \tilde{Z}^m_r, u^m_r)|dr \right)^2 + \int_{t^m_i}^{t^m_{i+1}} |\sigma(r, X^m_r, u^m_r)|^2 dr \right] \leq \frac{2T}{m} E \left[ \int_{t^m_i}^{t^m_{i+1}} |b(r, X^m_r, Y^m_r, \tilde{Z}^m_r, u^m_r)|^2 dr \right] + \frac{2T}{m} E \left[ \left( L + L_1 \sup_{t^m_i \leq r \leq t^m_{i+1}} |X^m_r| \right)^2 \right] \leq C(1 + |x|^2) \frac{1}{m},
\]
\[ E \left[ |\tilde{Y}_m^s - W(s, X^m_s)|^2 \right] \leq C \left[ |\tilde{Y}_m^s - \tilde{Y}_m^s| + |\tilde{Y}_m^s - W(t^m_{i+1}, X^m_{t^m_{i+1}})| + W(t^m_{i+1}, X^m_{t^m_{i+1}}) - W(s, X^m_s)|^2 \right] \leq \tilde{C}(1 + |x|^2) \frac{1}{m}, \]

where \( \tilde{C} \) is a constant which is independent of \( m \). Thus

\[ E \left[ \int_t^T |\tilde{Y}_m^s - W(s, X^m_s)|^2 \, ds \right] \leq \sum_{i=0}^{m-1} E \left[ \left| \tilde{Y}_m^s - W(s, X^m_s) \right|^2 \right] ds \leq \tilde{C}(1 + |x|^2) \frac{T}{m}. \tag{4.14} \]

Then we obtain (4.14) by (4.11), (4.12) and (4.14).

**Step 3.** Note that \((X^{t,x,u^m}, Y^{t,x,u^m}, Z^{t,x,u^m})\) satisfies the following FBSDE:

\[
\begin{align*}
    &dX^t,x,u^m = b(s, X^{t,x,u^m}_s, Y^{t,x,u^m}_s, Z^{t,x,u^m}_s, u^m_s)ds + \sigma(s, X^{t,x,u^m}_s, u^m_s)dB_s, \\
    &dY^t,x,u^m = -g(s, X^{t,x,u^m}_s, Y^{t,x,u^m}_s, Z^{t,x,u^m}_s, u^m_s)ds + Z^{t,x,u^m}_s dB_s, \quad s \in [t,T],
\end{align*}
\tag{4.15}
\]

Then, by Theorem 2.2 in [3] for FBSDEs (4.16) and (4.15), we obtain

\[
\begin{align*}
    &E \left[ \sup_{t \leq s \leq T} \left| X^s - X^t,x,u^m \right|^2 + |\tilde{Y}_m^s - Y^{t,x,u^m}_s|^2 \right] + \int_t^T |\tilde{Z}_s^m - Z^{t,x,u^m}_s|^2 \, ds \\
    &\leq \tilde{C} E \left[ \int_t^T \left| \tilde{Y}_m^s - Y^s|^2 + |\tilde{Z}_s - Z^s|^2 \right| ds \right], \tag{4.16}
\end{align*}
\]

where \( \tilde{C} \) is a constant which is independent of \( m \). Due to

\[
\left| Y^{t,x,u^m}_s - W(s, X^{t,x,u^m}_s) \right| \leq \left| Y^{t,x,u^m}_s - \tilde{Y}_m^s \right| + \left| \tilde{Y}_m^s - W(s, X^m_s) \right| + \left| W(s, X^m_s) - W(s, X^{t,x,u^m}_s) \right|,
\]

then, by (4.17), (4.16) and Lemma 3.3 we get

\[
E \left[ \int_t^T \left| Y^{t,x,u^m}_s - W(s, X^{t,x,u^m}_s) \right|^2 \, ds \right] + \left| Y^{t,x,u^m}_t - W(t,x) \right| \to 0 \text{ as } m \to \infty. \tag{4.17}
\]

This completes the proof. \( \blacksquare \)

We first give a uniqueness result when \( \sigma \) is independent of \( z \).

**Theorem 4.3** Suppose \( \sigma \) is independent of \( z \), \( \tilde{W} \) is a viscosity solution to HJB equation (3.37); and one of the following two conditions holds true:

(i) Assumption 2.1 holds;

(ii) Assumptions 2.1 (i) and 3.18 hold. Moreover,

\[
\hat{A}(t, x, y, z, u) = \begin{pmatrix}
    -G^\top g(t, x, y, z, u) \\
    Gb(t, x, y, z, u) \\
    G\sigma(t, x, \tilde{W}(t, x), u)
\end{pmatrix}
\]

satisfies Assumption 3.18.
Let $W$ be the value function. Furthermore, we assume that $\tilde{W}$ is Lipschitz continuous in $x$. Then $W \leq \tilde{W}$.

**Proof.** We only prove the first case (the condition (i) holds). The proof for the second case is similar. Consider the following HJB equation:
\[
\begin{align*}
\partial_t F(t, x) + \inf_{u \in U} H(t, x, F(t, x), DF(t, x), D^2F(t, x), u) &= 0, \\
F(T, x) &= \phi(x),
\end{align*}
\]
(4.18)

where
\[
H(t, x, v, p, A, u) \\
= \frac{1}{2}tr[\sigma\sigma^T(t, x, \tilde{W}(t, x), u)A] + p^Tb(t, x, v, p^T\sigma(t, x, \tilde{W}(t, x), u), u) + g(t, x, v, p^T\sigma(t, x, \tilde{W}(t, x), u), u),
\]
\[(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \times U
\]

and
\[
F(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}.
\]

Note that
\[
\tilde{\sigma}(t, x, u) := \sigma(t, x, \tilde{W}(t, x), u)
\]
is Lipschitz continuous in $x$. By the definition of viscosity solution, it is easy to verify that $\tilde{W}$ is also a viscosity solution to HJB equation (4.18). Since $\tilde{\sigma}$ is independent of $(y, z)$, by Theorems 3.12 and 3.1, $\tilde{W}$ is the value function of the following optimization problem:
\[
\inf_{u(\cdot) \in U([t, T])} \mathbb{E} \left[ \int_t^T Y^{t,x,u}_{s} d s \right],
\]
where the controlled system is
\[
\begin{align*}
&\begin{cases}
\frac{dX^{t,x,u}_{s}}{ds} = b(s, X^{t,x,u}_{s}, Y^{t,x,u}_{s}, Z^{t,x,u}_{s}, u_s)ds + \sigma(s, X^{t,x,u}_{s}, \tilde{W}(s, X^{t,x,u}_{s}), u_s)dB_s, \\
\frac{dY^{t,x,u}_{s}}{ds} = -g(s, X^{t,x,u}_{s}, Y^{t,x,u}_{s}, Z^{t,x,u}_{s}, u_s)ds + Z^{t,x,u}_{s}dB_s, \quad s \in [t, T], \\
X^{t,x,u}_{t} = x, \quad Y^{t,x,u}_{T} = \phi(X^{t,x,u}_{T}).
\end{cases}
\end{align*}
\]
(4.19)

For each fixed $(t, x) \in [0, T) \times \mathbb{R}^n$, by Proposition 4.12 we can find a sequence $u^m \in U([t, T])$ such that
\[
\mathbb{E} \left[ \int_t^T \left| Y^{t,x,u^m}_{s} - \tilde{W}(s, X^{t,x,u^m}_{s}) \right|^2 ds \right] + \left| Y^{t,x,u^m}_{T} - \tilde{W}(t, x) \right| \to 0 \text{ as } m \to \infty.
\]
(4.20)

Consider the following FBSDE:
\[
\begin{align*}
&\begin{cases}
\frac{dX^{t,x,u^m}_{s}}{ds} = b(s, X^{t,x,u^m}_{s}, Y^{t,x,u^m}_{s}, Z^{t,x,u^m}_{s}, u^m_s)ds + \sigma(s, X^{t,x,u^m}_{s}, Y^{t,x,u^m}_{s}, Z^{t,x,u^m}_{s}, u^m_s)dB_s, \\
\frac{dY^{t,x,u^m}_{s}}{ds} = -g(s, X^{t,x,u^m}_{s}, Y^{t,x,u^m}_{s}, Z^{t,x,u^m}_{s}, u^m_s)ds + Z^{t,x,u^m}_{s}dB_s, \quad s \in [t, T], \\
X^{t,x,u^m}_{t} = x, \quad Y^{t,x,u^m}_{T} = \phi(X^{t,x,u^m}_{T}).
\end{cases}
\end{align*}
\]
(4.21)
By Theorem 2.2 in [1] for FBSDEs (4.19) and (4.21),
\[
|Y_t^{t,x;um} - \bar{Y}_t^{t,x;um}| \leq C \left\{ \mathbb{E} \left[ \int_t^T \left| \sigma(s, X_s^{t,x;um}, Y_s^{t,x;um}, u_s^m) - \sigma(s, X_s^{t,x;um}, \bar{W}(s, X_s^{t,x;um}), u_s^m) \right|^2 ds \right] \right\}^{\frac{1}{2}}
\]
\[
\leq C \left\{ \mathbb{E} \left[ \int_t^T \left| Y_s^{t,x;um} - \bar{W}(s, X_s^{t,x;um}) \right|^2 ds \right] \right\}^{\frac{1}{2}}.
\]
(4.22)
Since $Y_t^{t,x;um} \geq W(t, x)$ for any $m \geq 1$, we get $W(t, x) \leq \tilde{W}(t, x)$ by (4.20) and (4.22).

Now we study the case in which $\sigma$ is dependent on $y$ and $z$.

Theorem 4.4 Suppose one of the following two conditions holds true:

(i) Assumptions 2.1 and 3.11 hold;

(ii) Assumptions 2.1 (i), 3.11 and 3.18 hold.

Let $W$ be the value function and $\tilde{W}$ be a viscosity solution to HJB equation (3.31). Furthermore, we assume that $\tilde{W}$ is Lipschitz continuous in $(t, x)$, $D\tilde{W}$ is Lipschitz continuous in $x$ and $||D\tilde{W}||_{L_3} < 1$. Then $W \leq \tilde{W}$.

Proof. We only prove the first case (the condition (i) holds). The proof for the second case is similar. Consider the following HJB equation:

\[
\begin{aligned}
\partial_t F(t, x) + \inf_{u \in U} H(t, x, F(t, x), DF(t, x), D^2 F(t, x), u) &= 0, \\
F(T, x) &= \phi(x),
\end{aligned}
\]
(4.23)

where
\[
H(t, x, v, p, A, u) = \frac{1}{2} \text{tr} [\sigma \sigma^T(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u) A] + p^T b(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u) + g(t, x, v, p^T \sigma(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u), u),
\]
\[
\tilde{V}(t, x, u) = D\tilde{W}(t, x)^T \sigma(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u),
\]
\[
(t, x, v, p, A, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U
\]

and
\[
F(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}.
\]
Recall that for given $\tilde{W}$, there exists a unique solution $\tilde{V}(t, x, u)$ to the above algebra equation by Lemma [3].

Note that
\[
\tilde{b}(t, x, u) := b(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u),
\]
\[
\tilde{\sigma}(t, x, u) := \sigma(t, x, \tilde{W}(t, x), \tilde{V}(t, x, u), u)
\]
satisfy the following conditions:

\[
\begin{aligned}
|\bar{b}(t, x, u)| + |\bar{\sigma}(t, x, u)| &\leq C(1 + |x|), \\
|\bar{b}(t, x, u) - \bar{b}(t, x', u)| + |\bar{\sigma}(t, x, u) - \bar{\sigma}(t, x', u)| &\leq C(1 + |x| + |x'|)|x - x'|.
\end{aligned}
\]  
(4.24)

By the definition of viscosity solution, \( \hat{W} \) is also a viscosity solution to HJB equation (4.23). Consider the following controlled system:

\[
\begin{aligned}
\begin{cases}
dX^t_{s} = & b(s, \hat{X}^t_{s}; u) dt + \sigma(s, \hat{X}^t_{s}; u) dW^t_s, \\
\hat{V}^t_{s} = & \bar{b}(s, \hat{X}^t_{s}; u) dt + \bar{\sigma}(s, \hat{X}^t_{s}; u) dW^t_s,
\end{cases}
\end{aligned}
\]  
(4.25)

By Proposition 3.28 in [14], the FBSDE (4.25) has a unique solution \((\hat{X}^t_{s}, \hat{V}^t_{s})\) satisfying the following conditions:

\[
\begin{aligned}
F \to 0, \quad \text{where}
\end{aligned}
\]

(4.26)

Let \( \vartheta(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) be a non-negative smooth function such that its support is included in the unit ball and \( \int_{\mathbb{R} \times \mathbb{R}^n} \vartheta(t, x) \, dx \, dt = 1 \). For Lipschitz functions \( \tilde{W} : [0, T] \times \mathbb{R}^n \to \mathbb{R} \), we set

\[
\tilde{W}_\varepsilon(t, x) = \varepsilon^{-(n+1)} \int_{\mathbb{R} \times \mathbb{R}^n} \tilde{W}((t - t') \vee 0 \wedge T, x - x') \vartheta(\varepsilon^{-1}t', \varepsilon^{-1}x') \, dx \, dt', \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad \varepsilon > 0.
\]

Then, it is easily to verify that

\[
\begin{aligned}
\partial_t \tilde{W}_\varepsilon(t, x) = & \varepsilon^{-(n+1)} \int_{\mathbb{R} \times \mathbb{R}^n} \partial_t \tilde{W}((t - t') \vee 0 \wedge T, x - x') \vartheta(\varepsilon^{-1}t', \varepsilon^{-1}x') \, dx \, dt', \\
D \tilde{W}_\varepsilon(t, x) = & \varepsilon^{-(n+1)} \int_{\mathbb{R} \times \mathbb{R}^n} D \tilde{W}((t - t') \vee 0 \wedge T, x - x') \vartheta(\varepsilon^{-1}t', \varepsilon^{-1}x') \, dx \, dt',
\end{aligned}
\]

where \( \partial_t \tilde{W}_\varepsilon \) is defined almost everywhere.

Thus we obtain that \( \|\partial_t \tilde{W}_\varepsilon\|_\infty \leq L_{\tilde{W}}, \|D \tilde{W}_\varepsilon\|_\infty \leq L_{D \tilde{W}}, \tilde{W}_\varepsilon \to \tilde{W} \) and \( D \tilde{W}_\varepsilon \to D \tilde{W} \) pointwisely as \( \varepsilon \to 0 \), where \( L_{\tilde{W}} \) is the Lipschitz constant of \( \tilde{W} \) with respect to \( t \) and \( L_{D \tilde{W}} \) is the Lipschitz constant of \( D \tilde{W} \) with respect to \( x \).

Applying Itô’s formula to \( \tilde{W}_\varepsilon(s, \hat{X}^t_{s}; u) \) on \([t, T] \), we have

\[
\begin{aligned}
\begin{cases}
dY^\varepsilon_s = & [\partial_t \tilde{W}_\varepsilon(s, \hat{X}^t_{s}; u) + \left(D \tilde{W}_\varepsilon(s, \hat{X}^t_{s}; u)\right)^T b(s, \hat{X}^t_{s}; u) + \bar{\sigma}(s, \hat{X}^t_{s}; u) \sigma(s, \hat{X}^t_{s}; u)] dt + \bar{\sigma}(s, \hat{X}^t_{s}; u) \sigma(s, \hat{X}^t_{s}; u) \bar{\sigma}(s, \hat{X}^t_{s}; u) \sigma(s, \hat{X}^t_{s}; u) dW^t_s, \\
Y^\varepsilon_T = & \phi(X^t_T), \quad s \in [t, T].
\end{cases}
\end{aligned}
\]  
(4.27)
Applying Itô’s formula to $|Y^t_{s,x,u,m} - Y^t_s|^2$ on $[t, T]$,

$$
\mathbb{E} \left[ \int_t^T |Z^t_{s,x,u,m} - Z^t_s|^2 ds \right] \leq 2 \mathbb{E} \left[ \int_t^T |Y^t_{s,x,u,m} - Y^t_s|^2 I^m_s ds \right] \\
\leq 2 \left\{ \mathbb{E} \left[ \int_t^T |Y^t_{s,x,u,m} - Y^t_s|^2 ds \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \int_t^T |I^m_s|^2 ds \right] \right\}^{\frac{1}{2}},
$$

(4.28)

where

$$I^m_s = \left[ g(s, X^t_{s,x,u,m}, Y^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s) + \partial_t \bar{W}_c(s, X^t_{s,x,u,m}) \right] \\
+ \left( D\bar{W}_c(s, X^t_{s,x,u,m}) \right)^T b(s, X^t_{s,x,u,m}, \bar{W}(s, X^t_{s,x,u,m}), \bar{V}(s, X^t_{s,x,u,m}, u^m_s), u^m_s) \\
+ \frac{1}{2} \text{tr} \left( \sigma^T(s, X^t_{s,x,u,m}, \bar{W}(s, X^t_{s,x,u,m}), \bar{V}(s, X^t_{s,x,u,m}, u^m_s), u^m_s) D^2 \bar{W}_c(s, X^t_{s,x,u,m}) \right) \\
\leq C \left( 1 + |X^t_{s,x,u,m}|^2 + |Y^t_{s,x,u,m}|^2 + |Z^t_{s,x,u,m}| \right).
$$

(4.29)

By standard estimate for decoupled FBSDE (4.25), we obtain

$$\sup_{m \geq 1} \mathbb{E} \left[ \int_t^T |I^m_s|^2 ds \right] \leq C \left( 1 + |x|^4 \right).$$

By dominate convergence theorem, we have

$$\mathbb{E} \left[ \int_t^T \left( |Y^t_{s,x,u,m} - \bar{W}(s, X^t_{s,x,u,m})|^2 + |Z^t_{s,x,u,m} - \bar{V}(s, X^t_{s,x,u,m}, u^m_s)|^2 \right) ds \right] \to 0$$

as $\epsilon \to 0$. Then we deduce

$$\mathbb{E} \left[ \int_t^T |Z^t_{s,x,u,m} - \bar{V}(s, X^t_{s,x,u,m}, u^m_s)|^2 ds \right] \leq C \left( 1 + |x|^2 \right) \left\{ \mathbb{E} \left[ \int_t^T |Y^t_{s,x,u,m} - \bar{W}(s, X^t_{s,x,u,m})|^2 ds \right] \right\}^{\frac{1}{2}},
$$

(4.30)

by taking $\epsilon \to 0$ in (4.28). Consider the following FBSDE:

$$
\begin{cases}
    dX^t_{s,x,u,m} = b(s, X^t_{s,x,u,m}, Y^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s)ds + \sigma(s, X^t_{s,x,u,m}, Y^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s)dB_s, \\
    dY^t_{s,x,u,m} = -g(s, X^t_{s,x,u,m}, Y^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s)ds + Z^t_{s,x,u,m}dB_s, \\
    X^t_{t,x,u,m} = x, \ Y^t_{t,x,u,m} = \phi(X^t_{t,x,u,m}).
\end{cases}
$$

(4.31)

By Theorem 2.2 in [1] for FBSDEs (4.25) and (4.31), we obtain

$$|Y^t_{s,x,u,m} - \tilde{Y}^t_{s,x,u,m}|$$

$$\leq C \left\{ \mathbb{E} \left[ \int_t^T \left( b(s, X^t_{s,x,u,m}, \tilde{Y}^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s) - b(s, X^t_{s,x,u,m}, \bar{W}(s, X^t_{s,x,u,m}), \bar{V}(s, X^t_{s,x,u,m}, u^m_s), u^m_s) \right)^2 ds \right] \right\}^{\frac{1}{2}}$$

$$+ \int_t^T \sigma(s, X^t_{s,x,u,m}, \tilde{Y}^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s) - \sigma(s, X^t_{s,x,u,m}, \bar{W}(s, X^t_{s,x,u,m}), \bar{V}(s, X^t_{s,x,u,m}, u^m_s), u^m_s) \sigma(s, X^t_{s,x,u,m}, \tilde{Y}^t_{s,x,u,m}, Z^t_{s,x,u,m}, u^m_s) \right)^2 ds \right\}^{\frac{1}{2}}$$

$$\leq C \left\{ \mathbb{E} \left[ \int_t^T \left( Y^t_{s,x,u,m} - \bar{W}(s, X^t_{s,x,u,m}) \right)^2 + Z^t_{s,x,u,m} - \bar{V}(s, X^t_{s,x,u,m}, u^m_s) \right)^2 ds \right\}^{\frac{1}{2}}$$

(4.32)

Since $Y^t_{s,x,u,m} \geq W(t, x)$ for any $m \geq 1$, we get $W(t, x) \leq \bar{W}(t, x)$ by (4.29), (4.30) and (4.32).
Remark 4.5 Note that $\tilde{b}$ and $\tilde{\sigma}$ are only local Lipschitz continuous in $x$. Under this condition (4.23), we can still prove that the value function of the controlled system (4.25) is the viscosity solution to HJB equation (4.23) by using the method in [18]. The uniqueness of the solution to HJB equation (4.23) can be still obtained similarly as in [2, 3].

Remark 4.6 If $b$ and $\sigma$ are independent of $z$, we only need to suppose that $\tilde{W}$ is Lipschitz continuous in $x$ in Theorem 4.4.

Remark 4.7 In Theorem 4.4 if $\partial_t \tilde{W} \in C([0,T] \times \mathbb{R}^n)$, then we do not need the assumption that $\tilde{W}$ is Lipschitz continuous in $t$. In this case, by the definition of viscosity solution, we can deduce that

$$\left| \partial_t \tilde{W}(t,x) \right| \leq C(1 + |x|^2).$$

Thus the inequality (4.29) still holds. Following the same steps as in the above proof, we can obtain the same result.

Now we study the case in which the coefficients of the controlled system $b, \sigma$ and $g$ are independent of control variable $u$. It is obviously that for this case the corresponding HJB equation (3.31) degenerates to a semilinear parabolic equation with an algebra equation.

Theorem 4.8 Suppose $b, \sigma$ and $g$ are independent of control variable $u$, $\sigma$ is independent of $z$, $\tilde{W}$ is a viscosity solution to HJB equation (3.31); and one of the following two conditions holds true:

(i) Assumption 2.1 holds;

(ii) Assumptions 2.1 (i) and 3.18 hold. Moreover,

$$\tilde{A}(t,x,y,z) = \begin{pmatrix}
-G^\top g(t,x,y,z) \\
G_b(t,x,y,z) \\
G\tilde{\sigma}(t,x,\tilde{W}(t,x))
\end{pmatrix}$$

satisfies Assumption 3.18.

Let $W$ be the value function. Furthermore, we assume that $\tilde{W}$ is Lipschitz continuous in $x$. Then $W = \tilde{W}$.

Proof. We only prove the first case (the condition (i) holds). The proof for the second case is similar. Following the same steps in Theorem 4.3, $\tilde{W}$ is also a viscosity solution to PDE system

$$\begin{cases}
\partial_t F(t,x) + H(t,x,F(t,x),DF(t,x),D^2F(t,x)) = 0, \\
F(T,x) = \phi(x),
\end{cases}$$

where $H(\cdot)$ is the function in equation (4.18) without control variables. Since $\tilde{\sigma}$ is independent of $(y,z)$, by Theorems 3.12 and 1.1 we have $\tilde{W}(t,x) = \tilde{Y}_t^{t,x}$, where $\tilde{Y}_t^{t,x}$ is the solution to the following FBSDE at time
\begin{align}
  d\bar{X}^{t,x}_s &= b(s, \bar{X}^{t,x}_s, \bar{Y}^{t,x}_s, \bar{Z}^{t,x}_s)ds + \sigma(s, \bar{X}^{t,x}_s, \bar{Y}^{t,x}_s, \bar{Z}^{t,x}_s)dB_s, \\
  d\bar{Y}^{t,x}_s &= -g(s, \bar{X}^{t,x}_s, \bar{Y}^{t,x}_s, \bar{Z}^{t,x}_s)ds + \bar{Z}^{t,x}_s dB_s, \quad s \in [t, T], \\
  \bar{X}^{t,x}_t &= x, \quad \bar{Y}^{t,x}_T = \phi(\bar{X}^{t,x}_T).
\end{align} \tag{4.33}

For each fixed \((t, x) \in [0, T) \times \mathbb{R}^n\), by Proposition 4.2, we have
\begin{align}
  \bar{Y}^{t,x}_s &= \tilde{W}(s, \bar{X}^{t,x}_s) \quad \text{for} \quad s \in [t, T]. \tag{4.34}
\end{align}

Then \((\bar{X}^{t,x}, \bar{Y}^{t,x}, \bar{Z}^{t,x})\) satisfies the following fully coupled FBSDE:
\begin{align}
  dX^{t,x}_s &= b(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)ds + \sigma(s, X^{t,x}_s, Y^{t,x}_s)dB_s, \\
  dY^{t,x}_s &= -g(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s)ds + Z^{t,x}_s dB_s, \quad s \in [t, T], \\
  X^{t,x}_t &= x, \quad Y^{t,x}_T = \phi(X^{t,x}_T). \tag{4.35}
\end{align}

By the uniqueness of FBSDEs (4.33) and (4.35), one has
\begin{align}
  Y^{t,x}_s &= \tilde{W}(s, X^{t,x}_s) \quad \text{for} \quad s \in [t, T]. \tag{4.36}
\end{align}

Since \(Y^{t,x}_t = W(t, x)\), we get \(W(t, x) = \tilde{W}(t, x)\) by (4.34) and (4.36). \[\square\]

Similarly, we have the following theorem.

**Theorem 4.9** Suppose \(b, \sigma\) and \(g\) are independent of control variable \(u\); and one of the following two conditions holds true:

(i) Assumptions 2.1 and 3.11 hold;

(ii) Assumptions 2.1 (i), 3.11 and 3.18 hold.

Let \(W\) be the value function and \(\tilde{W}\) be a viscosity solution to HJB equation (3.31). Furthermore, we assume that \(\tilde{W}\) is Lipschitz continuous in \((t, x)\), \(D\tilde{W}\) is Lipschitz continuous in \(x\) and \(||D\tilde{W}||_\infty L^3 < 1\). Then \(W = \tilde{W}\).

**Remark 4.10** If \(b\) and \(\sigma\) are independent of \(z\), we only need to suppose that \(\tilde{W}\) is Lipschitz continuous in \(x\) in Theorem 4.9.

**Remark 4.11** In the above theorems, we assume that the Assumption 2.1 or the monotonicity conditions hold. It is well-known that there are other conditions which can guarantee the existence and uniqueness of the fully coupled controlled system (2.3). In fact, our approach can be generalized to deal with any fully coupled controlled system which is well-posed and the related \(L^2\)-estimates of the solution hold.
4.3 The smooth case

In this subsection, we assume that the solution of the HJB equation \( \tilde{W} \in C^{1,2}([0,T] \times \mathbb{R}^n) \). Then, we have the following theorem.

**Theorem 4.12** Suppose one of the following two conditions holds true:

(i) Assumptions 2.1(i), 3.11 and 3.18 hold;

(ii) Assumptions 2.1(ii), 3.11 and 3.15 hold.

Let \( W \) be the value function and \( \tilde{W} \in C^{1,2}([0,T] \times \mathbb{R}^n) \) be a solution to the HJB equation (3.37). Furthermore, we assume \( ||\sigma||_{\infty} < \infty \), \( ||D\tilde{W}||_{\infty} L_3 < 1 \) and \( ||D^2\tilde{W}||_{\infty} < \infty \). Then \( W = \tilde{W} \).

**Proof.** Without loss of generality, we only prove the case \( d = 1 \). For each given \( u \in \mathcal{U}^d[t,T] \), let \( (X^{t,x,u},Y^{t,x,u},Z^{t,x,u}) \) be the solution to FBSDE (2.23) with \( \xi = x \). Applying Itô’s formula to \( \tilde{W}(s,X^{t,x,u}_s) \), we obtain

\[
\begin{align*}
\frac{d\tilde{W}(s,X^{t,x,u}_s)}{\partial s} = & \left\{ \begin{array}{l}
\tilde{W}(s,X^{t,x,u}_s) \left[ b(s,X^{t,x,u}_s,Y^{t,x,u}_s,Z^{t,x,u}_s,u_s) \\
+ \frac{1}{2} \sigma\sigma^T s(X^{t,x,u}_s,Y^{t,x,u}_s,Z^{t,x,u}_s,u_s) D^2\tilde{W}(s,X^{t,x,u}_s) \right]
\end{array} \right\} ds \\
+ & \tilde{W}(s,X^{t,x,u}_s) \sigma(s,X^{t,x,u}_s,Y^{t,x,u}_s,Z^{t,x,u}_s,u_s) dB_s,
\end{align*}
\]

Then, \( s \in [t,T] \),

\[
\begin{align*}
\frac{d\tilde{Y}_s}{\partial s} = & - (\Pi_1(s) + \Pi_2(s)) ds + \tilde{Z}_s dB_s, \\
\tilde{Y}_T = & 0,
\end{align*}
\]

where

\[
\begin{align*}
\Pi_1(s) = & \quad H(s,X^{t,x,u}_s,\tilde{W}(s,X^{t,x,u}_s),D\tilde{W}(s,X^{t,x,u}_s),D^2\tilde{W}(s,X^{t,x,u}_s),u_s) + \partial_1 \tilde{W}(s,X^{t,x,u}_s) \geq 0, \\
\Pi_2(s) = & \quad \left( D\tilde{W}(s,X^{t,x,u}_s) \right)^T [b_1(s) - b_2(s)] + \frac{1}{2} \sigma\sigma^T \left[ \sigma_1^T(s) - \sigma_2^T(s) \right] D^2\tilde{W}(s,X^{t,x,u}_s) \\
& + g_1(s) - g_2(s), \\
b_1(s) = & b(s,X^{t,x,u}_s,Y^{t,x,u}_s,Z^{t,x,u}_s,u_s), \\
b_2(s) = & b(s,X^{t,x,u}_s,\tilde{W}(s,X^{t,x,u}_s),\tilde{V}(s,X^{t,x,u}_s,u_s),u_s), \\
\tilde{V}(t,x,u) = & D\tilde{W}(t,x) \sigma(t,x,\tilde{W}(t,x),\tilde{V}(t,x,u),u),
\end{align*}
\]

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and $\sigma_i, g_i$ are defined similarly for $i = 1, 2$. Let

$$b_1(s) - b_2(s) = \beta_1(s)\bar{Y}_s + \gamma_1(s)\left(Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s)\right)$$

and

$$Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s)$$

$$= \tilde{Z}_s + \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T(\sigma_1(s) - \sigma_2(s))$$

$$= \tilde{Z}_s + \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\left[\beta_2(s)\bar{Y}_s + \gamma_2(s)\left(Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s)\right)\right]$$

$$= \tilde{Z}_s + \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\beta_2(s)\bar{Y}_s + \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\gamma_2(s)\left(Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s)\right)$$

where

$$\beta_1(s) = \begin{cases} \frac{b_1(s) - b_2(s)}{Y_{s}^{t,x,u} - \bar{Y}_s}, & \text{if } Y_{s}^{t,x,u} - \bar{Y}_s \neq 0, \\ 0, & \text{if } Y_{s}^{t,x,u} - \bar{Y}_s = 0, \end{cases}$$

$$\gamma_1(s) = \begin{cases} \frac{b_1(s) - b_2(s)}{Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s)}, & \text{if } Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s) \neq 0, \\ 0, & \text{if } Z_{s}^{t,x,u} - \bar{V}(s, X_{s}^{t,x,u}, u_s) = 0, \end{cases}$$

$\beta_2(\cdot)$ and $\gamma_2(\cdot)$ are defined similarly. Set

$$b_1(s) - b_2(s) = \tilde{\beta}_1(s)\bar{Y}_s + \tilde{\gamma}_1(s)\tilde{Z}_s,$$

(4.38)

where

$$\tilde{\beta}_1(s) = \beta_1(s) + \left(1 - \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\gamma_2(s)\right)^{-1}\tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\beta_2(s)\gamma_1(s),$$

$$\tilde{\gamma}_1(s) = \left(1 - \tilde{D}\bar{W}(s, X_{s}^{t,x,u})^T\gamma_2(s)\right)^{-1}\gamma_1(s).$$

It is easy to check that $\beta_1, \gamma_1, \beta_2, \gamma_2, \tilde{\beta}_1$ and $\tilde{\gamma}_1$ are bounded. Note that $\sigma$ is bounded. Similar to the proof of (4.35), we have

$$\text{tr}[\sigma_1(\cdot)s^2 - \sigma_2(\cdot)s^2(\cdot)]D\bar{W}(s, X_{s}^{t,x,u}) = \tilde{\beta}_2(s)\bar{Y}(s) + \tilde{\gamma}_2(s)\tilde{Z}(s),$$

$$g_1(s) - g_2(s) = \tilde{\beta}_3(s)\bar{Y}(s) + G_3(s)\tilde{Z}(s),$$

where $\tilde{\beta}_2, \tilde{\gamma}_2, \tilde{\beta}_3, \tilde{\gamma}_3$ are defined similarly and bounded. Then, we can rewrite $\Pi_2(s)$ as

$$\Pi_2(s) = \beta(s)\bar{Y}(s) + \gamma(s)\tilde{Z}(s),$$

where $\beta, \gamma \in \mathbb{R}$ are bounded. By the comparison theorem of BSDEs, we get $\bar{Y}_t \geq 0$ which implies $\bar{W}(t, x) \leq Y_{t}^{t,x,u}$. Thus $\bar{W} \leq W$. On the other hand, by Theorem 5.11 in [10] we have $W \leq \bar{W}$. Thus, $W = \bar{W}$. □

5 Appendix

5.1 The comparison theorem for FBSDEs

Under Assumption [2.1] we deduce a generalized comparison theorem for FBSDEs. The proof is similar to that of Theorem 3.1 in [20] and Theorem 5.11 in [10]. For the reader’s convenience, we give a detailed proof.
Consider the following FBSDEs:
\[
\begin{align*}
    dX^i_t &= b(s, X^i_s, Y^i_s, Z^i_s)ds + \sigma(s, X^i_s, Y^i_s, Z^i_s)dB_s, \\
    dY^i_t &= -g(s, X^i_s, Y^i_s, Z^i_s)ds + Z^i_sdB_s, \\
    X^i_t &= \xi, \quad Y^i_{t+\delta} = \phi_i(X^i_{t+\delta}), \quad i = 1, 2.
\end{align*}
\] (5.1)

**Theorem 5.1** Suppose Assumption [2.1] holds. Then for \( \delta \in (0, T-t) \) and \( \xi \in L^2(F_t; \mathbb{R}^n) \), (5.1) has a unique solution \((X^1_t, Y^1_t, Z^1_t)\) associated with \((b, \sigma, g, \phi)\). If \( \phi_1(X^2_{t+\delta}) \geq \phi_2(X^2_{t+\delta}), \) P-a.s. \((\text{resp.} \phi_1(X^2_{t+\delta}) \geq \phi_2(X^2_{t+\delta}), \) P-a.s.), then we have \( Y^1_t \geq Y^2_t, \) P-a.s.

**Proof.** Without loss of generality, we only prove the case \( d = 1 \). Let \( \tilde{X} = X^1 - X^2, \tilde{Y} = Y^1 - Y^2, \tilde{Z} = Z^1 - Z^2. \) Then \((\tilde{X}, \tilde{Y}, \tilde{Z})\) satisfies the following FBSDE:
\[
\begin{align*}
    d\tilde{X}_s &= \left[ b^i(s)\tilde{X}_s + b^2(s)\tilde{Y}_s + b^3(s)\tilde{Z}_s \right] ds + \left[ \sigma^i(s)\tilde{X}_s + \sigma^2(s)\tilde{Y}_s + \sigma^3(s)\tilde{Z}_s \right] dB_s, \\
    d\tilde{Y}_s &= -\left[ g^1(s)\tilde{X}_s + g^2(s)\tilde{Y}_s + g^3(s)\tilde{Z}_s \right] ds + \tilde{Z}_s dB_s, \quad s \in [t, t+\delta], \\
    \tilde{X}_t &= 0, \quad \tilde{Y}_{t+\delta} = \phi^1(t+\delta)\tilde{X}_{t+\delta} + \phi_1(X^1_{t+\delta}) - \phi_2(X^2_{t+\delta}),
\end{align*}
\]
where \( b^i, \sigma^i, g^i, \phi^i, i = 1, 2, 3 \) are defined in the proof of Lemma 3.3. Introduce the adjoint equation for the above equation as follows
\[
\begin{align*}
    dh_s &= \left[ g^2(s)h_s + b^2(s)m_s + \sigma^2(s)n_s \right] ds + \left[ g^3(s)h_s + b^3(s)m_s + \sigma^3(s)n_s \right] dB_s, \\
    dm_s &= -\left[ g^1(s)h_s + b^1(s)m_s + \sigma^1(s)n_s \right] ds + n_s dB_s, \\
    h_t &= 1, \quad m_{t+\delta} = \phi^i(t+\delta)h_{t+\delta}.
\end{align*}
\] (5.2)

It is easy to check that (5.2) satisfies the assumptions of Theorem 2.2 in [7]. Consequently, it has a unique solution \((h, m, n) \in L^2_F(\Omega, C([t, t+\delta]; \mathbb{R})) \times L^2_F(\Omega, C([t, t+\delta]; \mathbb{R}^n)) \times L^2_F(\Omega, C([t, t+\delta]; \mathbb{R}^n \times d))\). Applying Itô’s formula to \(m\tilde{X} - h\tilde{Y}\), we get
\[
\tilde{Y}_t = \mathbb{E} \left[ (\phi_1(X^2_{t+\delta}) - \phi_2(X^2_{t+\delta})) h_{t+\delta} \big| F_t \right].
\]
Since \( \phi_1(X^2_{t+\delta}) \geq \phi_2(X^2_{t+\delta}), \) P-a.s., we only need to prove \( h_{t+\delta} \geq 0, \) P-a.s.. Define \( \tau = \inf \{ s > t : h_s = 0 \} \wedge (t+\delta) \) and consider the following FBSDE on \([\tau, t+\delta]\),
\[
\begin{align*}
    d\tilde{h}_s &= \left[ g^2(s)\tilde{h}_s + b^2(s)\tilde{m}_s + \sigma^2(s)\tilde{n}_s \right] ds + \left[ g^3(s)\tilde{h}_s + b^3(s)\tilde{m}_s + \sigma^3(s)\tilde{n}_s \right] dB_s, \\
    d\tilde{m}_s &= -\left[ g^1(s)\tilde{h}_s + b^1(s)\tilde{m}_s + \sigma^1(s)\tilde{n}_s \right] ds + \tilde{n}_s dB_s, \\
    \tilde{h}_\tau &= 0, \quad \tilde{m}_{t+\delta} = \phi^1(t+\delta)\tilde{h}_{t+\delta}.
\end{align*}
\] (5.3)

This FBSDE has a unique solution \((\tilde{h}, \tilde{m}, \tilde{n}) = (0, 0, 0)\). Set
\[
\tilde{h}_s = h_sI_{[t, \tau]}(s) + \tilde{h}_sI_{(\tau, t+\delta]}(s), \quad \tilde{m}_s = m_sI_{[t, \tau]}(s) + \tilde{m}_sI_{(\tau, t+\delta]}(s), \quad \tilde{n}_s = n_sI_{[t, \tau]}(s) + \tilde{n}_sI_{(\tau, t+\delta]}(s).
\]
It is clear that \((\tilde{h}, \tilde{m}, \tilde{n})\) is a solution to (5.2). The definition of \( \tau \) yields the desired result \( h_{t+\delta} \geq 0. \)
5.2 $L^p$ estimate of FBSDEs

Consider the following FBSDE:

\[
\begin{aligned}
    dX_t &= b(s, X_s, Y_s, Z_s)ds + \sigma(s, X_s, Y_s, Z_s)dB_s, \\
    dY_t &= -g(s, X_s, Y_s, Z_s)ds + Z_s dB_s, \\
    X_0 &= x, \quad Y_T = \phi(X_T),
\end{aligned}
\] (5.4)

where $T' \leq T$ for some fixed $T > 0$.

**Theorem 5.2** Suppose Assumption (2.1) (i) holds and $C_p2^{-p-1} [L^p_2(T^2 + T^p) + L^p_3] < 1$ for some $p \geq 2$, where $C_p$ is defined in Lemma 5.1 in [7]. Then FBSDE (5.4) admits a unique solution $(X, Y, Z) \in L^p_2(\Omega; C([0, T']; \mathbb{R}^n)) \times L^p_2(\Omega; C([0, T']; \mathbb{R}^n)) \times L^2_p(0, T'; \mathbb{R}^{1 \times d})$ and

\[
|| (X, Y, Z) ||^p_p = \mathbb{E} \left[ \sup_{t \in [0, T']} (|X_t|^p + |Y_t|^p) + \left( \int_0^{T'} |Z_t|^2 dt \right)^{\frac{p}{2}} \right] 
\]

\[
\leq C \mathbb{E} \left[ \left( \int_0^{T'} |b| + |g|((t, 0, 0, 0)) dt \right)^p + \left( \int_0^{T'} |\sigma(t, 0, 0, 0)|^2 dt \right)^{\frac{p}{2}} + |\phi(0)|^p + |x|^p \right],
\]

where $C$ depends only on $T$, $p$, $L_1$, $L_2$, $L_3$.

**Proof.** Let $\mathcal{L}$ denote the space of all $\mathbb{F}$-adapted processes $(Y, Z)$ such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T'} |Y_t|^p + \left( \int_0^{T'} |Z_t|^2 dt \right)^{\frac{p}{2}} \right] < \infty.
\]

For each given $(y, z) \in \mathcal{L}$, consider the following decoupled FBSDE:

\[
\begin{aligned}
    dX_t &= b(t, X_t, y_t, z_t)dt + \sigma(t, X_t, y_t, z_t)dB_t, \\
    dY_t &= -g(t, X_t, Y_t, Z_t)dt + Z_t dB_t, \\
    X_0 &= x, \quad Y_T = \phi(X_T).
\end{aligned}
\] (5.5)

Under Lipschitz conditions on $b$, $\sigma$, $g$ and $\phi$, it is easy to deduce that the solution $(Y, Z)$ to (5.5) belongs to $\mathcal{L}$. Denote the operator $(y, z) \rightarrow (Y, Z)$ by $\Gamma$. For two elements $(y^1, z^1) \in \mathcal{L}$, $i = 1, 2$, let $(X^i, Y^i, Z^i)$ be the corresponding solution to (5.5). Set

\[
\dot{y}_t = y^1_t - y^2_t, \quad \dot{z}_t = z^1_t - z^2_t, \quad \dot{X}_t = X^1_t - X^2_t, \quad \dot{Y}_t = Y^1_t - Y^2_t, \quad \dot{Z}_t = Z^1_t - Z^2_t.
\]

Due to Lemma 5.1 in [7], we obtain

\[
\begin{aligned}
\mathbb{E} \left[ \sup_{0 \leq t \leq T'} \left( |\dot{X}_t|^p + |\dot{Y}_t|^p \right) + \left( \int_0^{T'} |\dot{Z}_t|^2 dt \right)^{\frac{p}{2}} \right] 
\leq C_p \mathbb{E} \left[ \left( \int_0^{T'} |b(t, X^1_t, y^1_t, z^1_t) - b(t, X^2_t, y^2_t, z^2_t)| dt \right)^p + \left( \int_0^{T'} |\sigma(t, X^1_t, y^1_t, z^1_t) - \sigma(t, X^2_t, y^2_t, z^2_t)|^2 dt \right)^{\frac{p}{2}} \right] 
\leq C_p2^{-p-1} [L^p_2(T^2 + T^p) + L^p_3] \mathbb{E} \left[ \left( \int_0^{T'} |\dot{Y}_t|^p + \left( \int_0^{T'} |\dot{Z}_t|^2 dt \right)^{\frac{p}{2}} \right] \right. \\
\leq C_p2^{-p-1} [L^p_2(T^2 + T^p) + L^p_3] \mathbb{E} \left[ \sup_{0 \leq t \leq T'} |\dot{Y}_t|^p + \left( \int_0^{T'} |\dot{Z}_t|^2 dt \right)^{\frac{p}{2}} \right] 
\end{aligned}
\] (5.6)
Proof. Let \( \epsilon \) where \( \epsilon \) is a positive parameter which is devoted to tend to zero. By Lemma 3.7 in [2], there exists a \( w \in \mathbb{R}^n \) such that \( \psi \) be the solution to (5.5) with respect to the fixed point \( (\epsilon, \alpha) \). Furthermore, assume that \( W \) is a supersolution to (4.1). Suppose Assumption 2.1 (i) holds. Let \( \epsilon, \alpha \) be the solution to (5.5) with respect to the fixed point \( (\epsilon, \alpha) \). Following the same steps in Theorem 2.2 in [7], we can obtain the estimate. □

### 5.3 The proof of Theorem 4.1

In order to prove this theorem, we need the following lemmas.

**Lemma 5.3** Suppose Assumption 2.1 (i) holds. Let \( W_1 \) be a viscosity subsolution and \( W_2 \) be a viscosity supersolution to (4.1). Furthermore, assume that \( W_1 \) and \( W_2 \) are Lipschitz continuous with respect to \( t \). Then the function \( w := W_1 - W_2 \) is a viscosity subsolution to the following equation

\[
\begin{align*}
 w_t(t, x) + \sup_{u \in \mathbb{R}} \left\{ \frac{1}{2} \sigma \Gamma(t, x, u) D^2 w(t, x) + C(1 + |x|) |Dw(t, x)| + C|w(t, x)| \right\} &= 0, \\
w(T, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^n,
\end{align*}
\]

where \( C \) is a constant depending only on the Lipschitz constants of \( b, \sigma, g, W_1 \) and \( W_2 \).

**Proof.** Let \( \varphi \in C^{2,1}_b([0, T] \times \mathbb{R}^n) \) and let \( (t_0, x_0) \in (0, T) \times \mathbb{R}^n \) be a global maximum point of \( W_1 - W_2 - \varphi \) with \( w(t_0, x_0) = \varphi(t_0, x_0) \). Define the function

\[
\psi_{\epsilon, \alpha}(t, x, s, y) = W_1(t, x) - W_2(s, y) - \frac{|x - y|^2}{\epsilon^2} - \frac{|t - s|^2}{\alpha^2} - \varphi(t, x),
\]

with \( \epsilon, \alpha \) are positive parameters which are devoted to tend to zero. By Lemma 3.7 in [2], there exists a sequence \( (t_{\epsilon, \alpha}, s_{\epsilon, \alpha}, x_{\epsilon, \alpha}, y_{\epsilon, \alpha}) \) such that

(i) \( (t_{\epsilon, \alpha}, x_{\epsilon, \alpha}, s_{\epsilon, \alpha}, y_{\epsilon, \alpha}) \) is a global maximum point of \( \psi_{\epsilon, \alpha} \) in \( [0, T] \times B_R \times [0, T] \times B_R \), where \( B_R \) is a ball with a large radius \( R \);

(ii) \( (t_{\epsilon, \alpha}, x_{\epsilon, \alpha}, s_{\epsilon, \alpha}, y_{\epsilon, \alpha}) \to (t_0, x_0) \) as \( \epsilon, \alpha \to 0 \);

(iii) \( |x_{\epsilon, \alpha} - y_{\epsilon, \alpha}| \) and \( |s_{\epsilon, \alpha} - y_{\epsilon, \alpha}| \) are bounded and tend to zero when \( \epsilon, \alpha \to 0 \);

(iv) there exist \( X, Y \in \mathbb{R}^n \) such that

\[
\left( \frac{2(t_{\epsilon, \alpha} - s_{\epsilon, \alpha})}{\alpha^2} + \varphi_t(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}), \frac{2(x_{\epsilon, \alpha} - y_{\epsilon, \alpha})}{\epsilon^2} + D\varphi(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}), X \right) \in D^{2,+} W_1(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}),
\]

\[
\left( \frac{2(t_{\epsilon, \alpha} - s_{\epsilon, \alpha})}{\alpha^2} + \frac{2(x_{\epsilon, \alpha} - y_{\epsilon, \alpha})}{\epsilon^2} + D\varphi(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}), Y \right) \in D^{2,-} W_2(s_{\epsilon, \alpha}, y_{\epsilon, \alpha}),
\]

\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix} \leq \frac{1}{\epsilon} \begin{pmatrix}
I & -I \\
-I & I
\end{pmatrix} + \begin{pmatrix}
D^2\varphi(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}) & 0 \\
0 & 0
\end{pmatrix},
\]

where \( D^{2,+} \) and \( D^{2,-} \) can be found in [2]. Since \( W_1 \) and \( W_2 \) are sub and supersolution to (4.1), respectively, by (iv) we have

\[
\frac{2(t_{\epsilon, \alpha} - s_{\epsilon, \alpha})}{\alpha^2} + \varphi_t(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}) + \inf_{u \in \mathbb{R}} H(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}, W_1(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}), \frac{2(x_{\epsilon, \alpha} - y_{\epsilon, \alpha})}{\epsilon^2} + D\varphi(t_{\epsilon, \alpha}, x_{\epsilon, \alpha}), X, u) \geq 0,
\]

\[
\frac{2(t_{\epsilon, \alpha} - s_{\epsilon, \alpha})}{\alpha^2} + \inf_{u \in \mathbb{R}} H(s_{\epsilon, \alpha}, y_{\epsilon, \alpha}, W_2(s_{\epsilon, \alpha}, y_{\epsilon, \alpha}), \frac{2(x_{\epsilon, \alpha} - y_{\epsilon, \alpha})}{\epsilon^2}, Y, u) \leq 0.
\]
It follows from (5.8) and (5.9) that

\[
\varphi_t(t_{e,\alpha}, x_{e,\alpha}) + \sup_{u \in U} \left\{ H(t_{e,\alpha}, x_{e,\alpha}, W_1(t_{e,\alpha}, x_{e,\alpha}), \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} + D\varphi(t_{e,\alpha}, x_{e,\alpha}), X, u \right\} - H(s_{e,\alpha}, y_{e,\alpha}, W_2(s_{e,\alpha}, y_{e,\alpha}), \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2}, Y, u) \geq 0.
\] (5.10)

By (iv) and Lipschitz continuity of $\sigma$, $b$ and $W_i$, then

\[
\text{tr}[\sigma \sigma^T(t_{e,\alpha}, x_{e,\alpha}, u)X] - \text{tr}[\sigma \sigma^T(s_{e,\alpha}, y_{e,\alpha}, u)Y] \\
\leq \frac{1}{\epsilon^2} \text{tr}[(\sigma(t_{e,\alpha}, x_{e,\alpha}, u) - \sigma(s_{e,\alpha}, y_{e,\alpha}, u))(\sigma(t_{e,\alpha}, x_{e,\alpha}, u) - \sigma(s_{e,\alpha}, y_{e,\alpha}, u))^\top] \\
+ \text{tr}[\sigma \sigma^T(t_{e,\alpha}, x_{e,\alpha}, u)D^2 \varphi(t_{e,\alpha}, x_{e,\alpha})] \\
\leq \frac{1}{\epsilon^2} \rho_\epsilon(t_{e,\alpha} - s_{e,\alpha}) + C \frac{|x_{e,\alpha} - y_{e,\alpha}|^2}{\epsilon^2} + \text{tr}[\sigma \sigma^T(t_{e,\alpha}, x_{e,\alpha}, u)D^2 \varphi(t_{e,\alpha}, x_{e,\alpha})],
\]

\[
\left| \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} + D\varphi(t_{e,\alpha}, x_{e,\alpha}) \right| \leq L_{W_1}, \quad \left| \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} \right| \leq L_{W_2}
\]

and

\[
\left( \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} + D\varphi(t_{e,\alpha}, x_{e,\alpha}) \right)^\top b(t_{e,\alpha}, x_{e,\alpha}, W_1(t_{e,\alpha}, x_{e,\alpha}), \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} + D\varphi(t_{e,\alpha}, x_{e,\alpha}))^\top \sigma(t_{e,\alpha}, x_{e,\alpha}, u, u) \\
- \left( \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} \right)^\top b(s_{e,\alpha}, y_{e,\alpha}, W_2(s_{e,\alpha}, y_{e,\alpha}), \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2})^\top \sigma(s_{e,\alpha}, y_{e,\alpha}, u, u) \\
\leq C \left( \frac{2(x_{e,\alpha} - y_{e,\alpha})}{\epsilon^2} \rho_\epsilon(t_{e,\alpha} - s_{e,\alpha}) + |w(t_{e,\alpha}, x_{e,\alpha})| + (1 + |x_{e,\alpha}|) |D\varphi(t_{e,\alpha}, x_{e,\alpha})| + |x_{e,\alpha} - y_{e,\alpha}| + \frac{|x_{e,\alpha} - y_{e,\alpha}|^2}{\epsilon^2} \right),
\]

where $L_{W_i}$ is the Lipschitz constant for $W_i$ and $\rho_\epsilon(s) \to 0$ as $s \to 0^+$ for fixed $\epsilon$. We can do the same analysis for $g$. For (5.10) we first let $\alpha \to 0$. And then let $\epsilon \to 0$, we have

\[
\varphi_t(t_0, x_0) + \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}[\sigma \sigma^T(t_0, x_0)D^2 \varphi(t_0, x_0)] + C (1 + |x_0|) |D\varphi(t_0, x_0)| + C |w(t_0, x_0)| \right\} \geq 0.
\]

Therefore, $w$ is a subsolution to (5.11). \hfill \blacksquare

Remark 5.4 When $\sigma$ depends on $y$, the right hand side of (5.11) will include the term

\[
\frac{C}{\epsilon^4} |W_1(t_{e,\alpha}, x_{e,\alpha}) - W_2(t_{e,\alpha}, x_{e,\alpha})|^2,
\]

which tends to $\infty$ as $\epsilon \to 0$. Thus the above method does not work.

Set $\psi(x) = \left[ \log \left( \left( |x|^2 + 1 \right)^{\frac{1}{2}} \right) \right]^2$, $x \in \mathbb{R}^n$.

Lemma 5.5 Suppose Assumption (iv) (i) holds. Then, for any $A > 0$, there exists a constant $C_1 > 0$ such that the function $\chi(t, x) = \exp \{ (C_1 (T - t) + A) \psi(x) \}$ satisfies

\[
\chi(t, x) + \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}[\sigma \sigma^T(t, x)D^2 \chi(t, x)] + C (1 + |x|) |D\chi(t, x)| + C \chi(t, x) \right\} < 0
\]

in $[t_1, T] \times \mathbb{R}^n$, where $t_1 = T - \frac{A}{C_1}$.
It is easy to verify this lemma directly. So we omit the proof.

**Proof of Theorem 4.1** We only need to prove that for any $\alpha > 0$, $w$ satisfies

$$|w(t, x)| \leq \alpha \chi(t, x), \text{ in } [0, T] \times \mathbb{R}^n.$$ 

It is clear that for some $A > 0$,

$$\lim_{|x| \to \infty} |w(t, x)| \exp \left(-A \log \left(\left(|x|^2 + 1\right)^{\frac{1}{2}}\right)\right) = 0$$

uniformly for $t \in [0, T]$. This implies that $|w| - \alpha \chi$ is bounded from above in $[t_1, T] \times \mathbb{R}^n$ and that

$$M := \max_{[t_1, T] \times \mathbb{R}^n} (|w(t, x)| - \alpha \chi(t, x)) \exp (-C(T - t))$$

is achieved at some point $(t_0, x_0)$. Without loss of generality, we assume that $|w(t_0, x_0)| > 0$ and $w(t_0, x_0) > 0$.

Note that

$$w(t, x) - \alpha \chi(t, x) \leq (w(t_0, x_0) - \alpha \chi(t_0, x_0)) \exp (-C(t - t_0)).$$

Then, $(t_0, x_0)$ can be seen as a global maximum point for $w(t, x) - h(t, x)$ where

$$h(t, x) = \alpha \chi(t, x) + (w(t_0, x_0) - \alpha \chi(t_0, x_0)) \exp (-C(t - t_0)).$$

Since $w$ is a viscosity subsolution to (4.1), if $t_0 \in [t_1, T)$, then we have

$$h_t(t_0, x_0) + \sup_{u \in U} \left\{ \frac{1}{2} \text{tr} \left[ \sigma_r \sigma^\top(t_0, x_0, u) D^2 h(t_0, x_0) \right] + C(1 + |x_0|) |Dh(t_0, x_0)| + Cw(t_0, x_0) \right\} \geq 0.$$ 

That is

$$\chi_t(t_0, x_0) + \sup_{u \in U} \left\{ \frac{1}{2} \text{tr} \left[ \sigma_r \sigma^\top(t_0, x_0, u) D^2 \chi(t_0, x_0) \right] + C(1 + |x_0|) |D\chi(t_0, x_0)| + C\chi(t_0, x_0) \right\} \geq 0.$$ 

It is a contradiction to Lemma 5.5 Therefore $t_0 = T$. Since $|w(T, x)| = 0$, we obtain

$$|w(t, x)| \leq \alpha \chi(t, x), \text{ in } [t_1, T] \times \mathbb{R}^n.$$ 

Thus, let $\alpha \to 0$, we can obtain $|w| = 0$ in $[t_1, T] \times \mathbb{R}^n$. Applying successively the same argument on the interval $[t_2, t_1]$ where $t_2 = (t_1 - A/C_1)^+$ and then, if $t_2 > 0$ on $[t_3, t_2]$ where $t_3 = (t_2 - A/C_1)^+...$etc.. We finally obtain that $|w| = 0$ in $[0, T] \times \mathbb{R}^n$. This completes the proof. ■

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