Silting objects and torsion pairs in comma categories∗

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Abstract

In this paper, we first give a characterization of silting objects in the comma category $(F, \text{Mod}_S)$. Let $U$ is an $(S, R)$-bimodule. Take $F = U \otimes_R -$, then the comma category $(F, \text{Mod}_S)$ is isomorphic to the category $T$-Mod, where $T = \begin{pmatrix} R & 0 \\ U & S \end{pmatrix}$. Assume that $C_1$ and $C_2$ are two subcategories of left $R$-modules, $D_1$ and $D_2$ be two subcategories of left $S$-modules. We mainly prove that $(C_1, C_2)$ and $(D_1, D_2)$ are torsion pairs if and only if $\langle \mathfrak{B}^{C_1}_{D_1}, \mathbf{J}^{D_2}_{C_2} \rangle$ and $\langle \mathfrak{B}^{C_1}_{D_1}, \mathbf{J}^{D_2}_{C_2} \rangle$ are torsion pairs under certain conditions.

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1 Introduction and Preliminaries

The tilting theory is well known, and plays an important role in the representation theory of Artin algebra. The classical notion of tilting modules was first considered in the case of finite dimensional algebras by Brenner and Butler [4] and by Happel and Ringel [8]. As a generalization of tilting modules, silting modules over arbitrary rings were introduced by L. Angeleri-Hügel, F. Marks and J. Vitória, and they proved that silting modules are in bijection with 2-term silting complexes and with certain t-structures and co-t-structures in the derived module category.

Let $R$ be a ring. Given an subcategory $\mathcal{X}$ of $R$-module, write $\mathcal{X}^{\perp_0} = \{ Y : \text{Hom}_R(X, Y) = 0 \}$ for all $X \in \mathcal{X}$ and $\perp_0 \mathcal{X} = \{ Y : \text{Hom}_R(Y, X) = 0 \}$ for all $X \in \mathcal{X}$. Recall that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ is called a torsion pair [3] if the following conditions are satisfied: (1) $\text{Hom}_A(\mathcal{X}, \mathcal{Y}) = 0$; (2) for any object $M \in \mathcal{A}$, there exists an exact sequence $0 \to X \to M \to Y \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. It is equivalent to that $\mathcal{X}^{\perp_0} = \mathcal{Y}$ and $\perp_0 \mathcal{Y} = \mathcal{X}$. As we all known, for any silting (resp. tilting) module $T$, it can induce a torsion pair $(\text{Gen}T, T^{\perp_0})$.

Recall that for two Abelian categories $\mathcal{A}$ and $\mathcal{B}$ and a right exact functor $F : \mathcal{A} \to \mathcal{B}$. We can define the left comma category $(F, \mathcal{B})$ [9], which is also an Abelian category. The examples of comma categories include but are not limited to the category of modules or complexes over a triangular matrix ring, the morphism category of an abelian category and so on [9, 12, 13]. For the study of comma categories, you can also refer to [3, 12, 13].

This paper is organized as follows. In Section 2, we first introduce some definitions and properties, and give a characterization of silting objects in the comma category $(F, \text{Mod}_S)$, see

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the theorem \[.2\] In section 3, we introduce three functors \((p, q, h)\) and three classes, and mainly prove the theorem \[.2\]

**Theorem 1.1.** Let \(A \in \text{Mod-}R\) and \(B \in \text{Mod-}S\) and the functor \(F\) be commutes with direct sums. Then \((\begin{array}{c} A \\ FA \end{array}) \oplus \begin{array}{c} 0 \\ B \end{array}\) is silting with respect to \(\sigma\) if and only if

1. \(A\) is is silting with respect to \(\sigma_A\);
2. \(B\) is is silting with respect to \(\sigma_B\);
3. \(FA \in \text{Gen}B\).

**Theorem 1.2.** Let \(\mathcal{C}_1\) and \(\mathcal{C}_2\) be two subcategories of left \(R\)-modules, \(\mathcal{D}_1\) and \(\mathcal{D}_2\) be two subcategories of left \(S\)-modules.

1. If \(S^+ \in \mathcal{D}_2\), then \((\mathcal{C}_1, \mathcal{C}_2)\) and \((\mathcal{D}_1, \mathcal{D}_2)\) are torsion pairs if and only if \((\Omega^\mathcal{C}_1, \Omega^\mathcal{C}_2)\) is a torsion pair.
2. If \(R \in \mathcal{C}_1\), then \((\mathcal{C}_1, \mathcal{C}_2)\) and \((\mathcal{D}_1, \mathcal{D}_2)\) are torsion pairs if and only if \((\Omega^\mathcal{C}_1, \Omega^\mathcal{C}_2)\) is a torsion pair.

## 2 Silting objects

I this section, we first recall some definitions and notions. Let \(\mathcal{A}\) be an abelian category. Recall that a suncategory \(\mathcal{C} \subseteq \mathcal{A}\) is called a torsion class, if it is closed under images, direct sums and extensions (c.f. \[3\] Chapter VI). Given a subcategory \(\mathcal{X} \subseteq \mathcal{A}\), recalled that a left \(\mathcal{X}\)-approximation of \(T \in \mathcal{A}\) is a morphism \(\phi : T \to X\) such that \(\text{Hom}(\phi, X')\) is epic for any \(X' \in \mathcal{X}\).

For a morphism \(f\) in \(\text{Proj}\mathcal{A}\) consisting of all projective objects of \(\mathcal{A}\), we consider the following class of objects

\[\mathcal{D}_f = \{X \in \text{Mod}R|\text{Hom}_\mathcal{A}(f, X)\text{ is surjective}\}\].

**Definition 2.1.** \[2\] Definition 3.7] Let \(\mathcal{A}\) be an abelian category with enough projective objects and \(T\) be an object of \(\mathcal{A}\).

1. \(T\) is called partial silting if there exists a projective presentation \(\sigma\) of \(T\) such that \(\mathcal{D}_\sigma\) is a torsion class and \(T \in \mathcal{D}_\sigma\).
2. \(T\) is called silting if there exists a projective presentation \(\sigma\) of \(T\) such that \(\mathcal{D}_\sigma = \text{Gen}T\), where \(\text{Gen}T = \{X|\text{there is a surjective morphism }T(f) \to X\}\).

Sometimes, we also say that \(T\) is (partial) silting with respect to \(\sigma\).

By \[2\] Lemma 3.6], \(\mathcal{D}_\sigma\) is always closed under images and extensions. Hence \(\mathcal{D}_\sigma\) is a torsion class if and only if it is colsed under direct sums.

**Definition 2.2.** \[3\] Section 1] Let \(\mathcal{A}\) be an abelian category and \(F : \mathcal{A} \to \mathcal{A}\) an additive endofunctor. The right trivial extension of \(\mathcal{A}\) by \(F\), denoted by \(\mathcal{A} \times F\), is defined as follows. An object in \(\mathcal{A} \times F\) is a morphism \(\alpha : FA \to A\) for an object \(A\) in \(\mathcal{A}\) such that \(\alpha \cdot F(\alpha) = 0\); and a morphism in \(\mathcal{A} \times F\) is a pair \((F\beta, \beta)\) of morphisms in \(\mathcal{A}\) such that the following diagram

\[
\begin{array}{ccc}
FA & \xrightarrow{\alpha} & A \\
F\beta \downarrow & & \downarrow \beta \\
FA' & \xrightarrow{\alpha'} & A'
\end{array}
\]

is commutative.

**Definition 2.3.** \[3\] Section 1] Let \(\mathcal{A}\) and \(\mathcal{B}\) be abelian categories and \(F : \mathcal{A} \to \mathcal{B}\) an additive functor. We define the left comma category \((F, \mathcal{B})\) as follows. The objects of the category are \((\begin{array}{c} A \\ B \end{array})_\phi\) (sometimes, \(\phi\) is omitted) with \(A \in \mathcal{A}\), \(B \in \mathcal{B}\) and \(\phi \in \text{Hom}(FA, B)\); and the morphisms
Remark 2.4. Let \( A \) and \( B \) be abelian categories and \( F: A \to B \) an additive functor.

1. The functor induces a functor \( G: A \times B \to A \times B \) by \( G(A, B) = (0, FA) \) and \( G(\alpha, \beta) = (0, F\alpha) \). It is easy to verify that \( (F, B) \cong (A \times B) \times G \), by mapping the object \( (A)_{\phi} \) to the object \( (0, \phi): G(A, B) \to (A, B) \) in \( (A \times B) \times G \).

2. Suppose that \( F \) is right exact. It is easy to see that the functor \( G \) above is also right exact. So \( (A \times B) \times G \) is an abelian category by \[9\] Proposition 1.1, and then the comma category \( (F, B) \) is also abelian by (1).

3. The sequence \( 0 \to (A_1)_{\phi_1} \to (A_2)_{\phi_2} \to (A_3)_{\phi_3} \to 0 \) is exact in \( (F, B) \) if and only if the sequence \( 0 \to (0, \phi_1) \to (0, \phi_2) \to (0, \phi_3) \to 0 \) is exact in \( (A \times B) \times G \) by (1) if and only if the sequence \( 0 \to (A_1) \to (A_2) \to (A_3) \to 0 \) is exact in \( A \times B \) by \[9\] Corollary 1.2 if and only if the following two sequences \( 0 \to A_1 \to A_2 \to A_3 \to 0 \) in \( A \) and \( 0 \to B_1 \to B_2 \to B_3 \to 0 \) in \( B \) are exact.

4. The projective object in \( (F, B) \) is of the form \((0, P) \oplus (\overset{P}{A})\) with \( P \) projective in \( A \) and \( Q \) projective in \( B \) by \[17\] Lemma 3.1.

From now on, \( R \) and \( S \) are rings, denoted \( R \)-Mod by the category consisting of all left \( R \)-modules. The functor \( F: R \)-Mod \to S-\text{Mod} is right exact and covariant. In this subsection, we mainly consider the left comma category \( C := (F, S)-\text{Mod} \).

Let \( A \in R \)-Mod and \( B \in S \)-Mod, and the following sequences

\[
P_1 \xrightarrow{\sigma_A} P_0 \xrightarrow{\sigma} A \to 0
\]

and

\[
Q_1 \xrightarrow{\sigma_B} Q_0 \xrightarrow{\sigma} B \to 0
\]

are projective presentations of \( A \) and \( B \), respectively, with \( P_0, P_1 \) being projective \( R \)-modules and \( Q_0, Q_1 \) being projective \( S \)-modules. Since \( F \) is right exact and covariant, the sequence

\[
FP_1 \xrightarrow{F\sigma_A} FP_0 \xrightarrow{\sigma} FA \to 0
\]

is also exact. Hence, by Remark 2.4(4), we can obtain a projective presentation of \((0, P_1) \oplus (\overset{P_1}{A})\), denoted by \( \sigma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \):

\[
(0, P_1) \oplus (\overset{P_1}{A}) \xrightarrow{\sigma} (0, Q_0) \oplus (\overset{Q_0}{F}) \to (0, B) \oplus (\overset{B}{F}) \to 0,
\]
where \( a = (0_{\sigma_B}) \), \( b = (\sigma_A_{F\sigma_A}) \).

Next, we will give the characterization of \( D_\sigma \) in comma categories.

**Lemma 2.5.** \((\frac{M}{N})_h \in D_\sigma \) if and only if \( M \in D_{\sigma_A} \) and \( N \in D_{\sigma_B} \).

**Proof.** (\( \Rightarrow \)) For any morphisms \( f \in \text{Hom}_R(P_1, M) \) and \( m \in \text{Hom}_S(Q_1, N) \), we only need to find two morphisms such that the following diagram is commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & P_0 \\
\downarrow & & \downarrow \\
P_1 & \xrightarrow{\sigma} & \sigma_A \end{array}
\]

\[
\begin{array}{ccc}
N & \xrightarrow{m} & Q_0 \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{\sigma_B} & \sigma_B \\
\end{array}
\]

Set \( g = h \cdot Ff \), then the following diagram is commutative.

\[
\begin{array}{ccc}
FP_1 & \xrightarrow{1} & FP_1 \\
\downarrow & & \downarrow \\
FM & \xrightarrow{h} & N \\
\end{array}
\]

i.e., \((\frac{0}{m}), \frac{f}{g}\) \( \in \text{Hom}_C((\frac{0}{Q_1}) \oplus (\frac{P_1}{FP_1}), (\frac{M}{N})) \). By the assumption, there is a morphism \((\frac{0}{m}), \frac{f}{g}\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
\frac{M}{N} \xrightarrow{h} & (\frac{0}{Q_1}) \oplus (\frac{P_1}{FP_1}) \xrightarrow{\sigma} (\frac{0}{Q_0}) \oplus (\frac{P_0}{FP_0}) \\
\downarrow & & \downarrow \\
(\frac{0}{m}), \frac{f}{g} & & \frac{0}{Q_0} \oplus (\frac{P_0}{FP_0}) \\
\end{array}
\]

By the definition of morphisms in comma category, it is easy to prove that \( y = h \cdot Fx \). Since the above diagram is commutative, we have the following two equations \((\frac{0}{m}) = (\frac{0}{m}_{\sigma_B}) \) and \((\frac{f}{g}) = (\frac{f}{g}_{\sigma_A})  \). Thus \( f = x_{\sigma_A} \) and \( m = i_{\sigma_B} \). i.e., \( M \in D_{\sigma_A} \) and \( N \in D_{\sigma_B} \).

(\( \Leftarrow \)) For any \((\frac{0}{m}), \frac{0}{\sigma_d}\) \( \in \text{Hom}_C((\frac{0}{Q_1}) \oplus (\frac{P_1}{FP_1}), (\frac{M}{N})_h) \). By the definition of morphisms in comma category, it is easy to verify that \( d = h \cdot Fc \). Now, we need to find a morphism such that the following diagram is commutative.

\[
\begin{array}{ccc}
\frac{M}{N} \xrightarrow{h} & (\frac{0}{Q_1}) \oplus (\frac{P_1}{FP_1}) \xrightarrow{\sigma} (\frac{0}{Q_0}) \oplus (\frac{P_0}{FP_0}) \\
\downarrow & & \downarrow \\
(\frac{0}{m}), \frac{0}{\sigma_d} & & \frac{0}{Q_0} \oplus (\frac{P_0}{FP_0}) \\
\end{array}
\]

Since \( M \in D_{\sigma_A} \) and \( N \in D_{\sigma_B} \), there are two morphisms \( j_1 \) and \( c_1 \) satisfying the following two diagrams is commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{c} & P_0 \\
\downarrow & & \downarrow \\
P_1 & \xrightarrow{\sigma} & \sigma_A \\
\end{array}
\]

\[
\begin{array}{ccc}
N & \xrightarrow{j} & Q_0 \\
\downarrow & & \downarrow \\
Q_1 & \xrightarrow{\sigma_B} & \sigma_B \\
\end{array}
\]

Set \( d_1 = h \cdot Fc_1 \), then we have the following commutative diagram.

\[
\begin{array}{ccc}
FP_0 & \xrightarrow{Fc_1} & FM \\
\downarrow & & \downarrow \\
FP_0 & \xrightarrow{d_1} & N \\
\end{array}
\]
We claim that the morphism \( ((0_{j1}), (c_{i2})') \in \text{Hom}_C((0_{Q0}) \oplus (P_{F^h}), (M_{N})_h) \) is exactly what we want. Thus we need to verify that the following two equations hold.

\[
\begin{pmatrix}
0 \\
0 \\
c \\
d
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
c_1 \\
c_2
\end{pmatrix}
\begin{pmatrix}
\sigma_A \\
F\sigma_A
\end{pmatrix}
\]

Note that \( d = h \cdot Fc = h \cdot F(c_1) = (h \cdot F(c_1)) \cdot F(\sigma_A) = d_1 \cdot F(\sigma_A) \). Thus the second equation holds. The first is obvious. So we completed this proof. \( \square \)

By the above conclusion, we obtain the following corollary immediately.

**Corollary 2.6.** Let \( M \in R\text{-Mod} \) and \( N \in S\text{-Mod} \), then the following statements hold.

1. If \( M \in D_{\sigma_A} \), then \( (M_0) \in D_{\sigma} \).
2. If \( N \in D_{\sigma_B} \), then \( (N_0) \in D_{\sigma} \).
3. If \( M \in D_{\sigma_A} \) and \( FM \in D_{\sigma_B} \), then \( (M_{FM}) \in D_{\sigma} \).

**Lemma 2.7.** Let \( A \in R\text{-Mod} \) and \( B \in S\text{-Mod} \) and the functor \( F \) be commutes with direct sums. Then \( D_{\sigma} \) is a torsion class if and only if both \( D_{\sigma_A} \) and \( D_{\sigma_B} \) are torsion class.

**Proof.** By [2, Lemma 3.6], \( D_{\sigma} \) is always closed under images and extensions. Hence \( D_{\sigma} \) is a torsion class if and only if it is closed under direct sums.

(\( \Rightarrow \)) For any \( M_i \in D_{\sigma_A} \), then \( (M_i) \in D_{\sigma} \) by Corollary 2.6. Since \( D_{\sigma} \) is a torsion class and \( F \) is commutes with direct sums, \( \bigoplus_{i \in I} M_i \approx \bigoplus_{i \in I} (M_i) \in D_{\sigma} \) for any set \( I \). By Lemma 2.5, \( \bigoplus_{i \in I} M_i \in D_{\sigma_A} \) for any set \( I \), i.e., \( D_{\sigma_A} \) is a torsion class. Similarly, we can prove that \( D_{\sigma_B} \) is a torsion class.

(\( \Leftarrow \)) For any \( (X_i)_{f_i} \in D_{\sigma} \), then \( X_i \in D_{\sigma_A} \) and \( Y_i \in D_{\sigma_B} \) by Lemma 2.5. Since the functor \( F \) be commutes with direct sums and \( D_{\sigma} \) is a torsion class, we have that \( \bigoplus (X_i)_{f_i} \approx \bigoplus (X_i)_{f_i} \). Since both \( D_{\sigma_A} \) and \( D_{\sigma_B} \) are torsion class, \( \bigoplus X_i \in D_{\sigma_A} \) and \( \bigoplus Y_i \in D_{\sigma_B} \). By Lemma 2.5, \( \bigoplus (X_i)_{f_i} \approx \bigoplus (X_i)_{f_i} \in D_{\sigma} \), i.e., \( D_{\sigma} \) is a torsion class. \( \square \)

**Proposition 2.8.** Let \( A \in R\text{-Mod} \) and \( B \in S\text{-Mod} \) and the functor \( F \) be commutes with direct sums. Then \( (A_{FA} \oplus B_{FB})_{(1)} \) is partial silting with respect to \( \sigma \) if and only if

1. \( A \) is is partial silting with respect to \( \sigma_A \);
2. \( B \) is is partial silting with respect to \( \sigma_B \);
3. \( FA \in D_{\sigma_B} \).

**Proof.** By the lemma 2.5, corollary 2.6 and lemma 2.7 it is obvious. \( \square \)

**Lemma 2.9.** [2, Proposition 3.11] Let \( T \in R\text{-Mod} \) with a projective presentation \( \sigma \). Then \( T \) is silting with respect to \( \sigma \) if and only if \( T \) is partial silting with respect to \( \sigma \) and there exists an exact sequence

\[
\begin{array}{cccccc}
R & \rightarrow & T_0 & \rightarrow & T_1 & \rightarrow & 0
\end{array}
\]

in \( \text{Mod}R \) with \( T_0, T_1 \in \text{Add}T \) and \( \alpha \) a left \( D_{\sigma} \)-approximation.

Next, we will give the main result of this section.

**Theorem 2.10.** Let \( A \in R\text{-Mod} \) and \( B \in S\text{-Mod} \) and the functor \( F \) be commutes with direct sums. Then \( (A_{FA} \oplus B_{FB})_{(1)} \) is silting with respect to \( \sigma \) if and only if

1. \( A \) is is silting with respect to \( \sigma_A \);
2. \( B \) is is silting with respect to \( \sigma_B \);
3. \( FA \in \text{Gen}B \).
Proof. Note that \((A_{FA} \oplus B)_{(i)} \cong (A_{FA}) \oplus (0_B)\).

\((\Rightarrow)\) Since \((A_{FA}) \oplus (0_B)\) is silting, \(D_{\sigma} = \text{Gen}((A_{FA}) \oplus (0_B)) = \text{Gen}(A_{FA}) \oplus \text{Gen}(0_B)\). By the proposition 2.8, \(A\) is partial silting, and then \(\text{Gen}A \subseteq D_{\sigma_A}\). Next, we will prove that \(D_{\sigma_A} \subseteq \text{Gen}A\). For any \(X \in D_{\sigma_A}\), by the corollary 2.6, we have that \((\bar{X})_0 \in D_{\sigma} = \text{Gen}(A_{FA}) \oplus \text{Gen}(0_B)\). We claim that \((\bar{X})_0 \in \text{Gen}(A_{FA})\). If \((\bar{X})_0\) has a direct summand \((\bar{X}_1)_0 \in \text{Gen}(0_B)\), then there is an exact sequence \((0_{B(i)}) \rightarrow (\bar{X}_1)_0 \rightarrow 0\), i.e., \(0 \rightarrow X_1 \rightarrow 0\) is exact by Remark 2.4 and then \(X_1 = 0\). So we have that \((\bar{X})_0 \in \text{Gen}(A_{FA})\). By Remark 2.4 again, we have that \(X \in \text{Gen}A\), and then \(D_{\sigma_A} \subseteq \text{Gen}A\). Consequently, \(A\) is silting with respect to \(\sigma_A\). Similarly, we can prove that \(B\) is silting with respect to \(\sigma_B\). Note that \((A_{FA}) \oplus (0_B)\) is partial silting by Proposition 2.8 and then \(\text{Gen}((A_{FA}) \oplus (0_B)) \subseteq D_{\sigma}\) by Remark 3.8 in [2]. So we only need to prove that \(D_{\sigma} \subseteq \text{Gen}((A_{FA}) \oplus (0_B))\).

Since both \(A\) and \(B\) are silting in \(\text{Mod}R\) and \(\text{Mod}S\), respectively. Thus there are two exact sequences
\[
\begin{align*}
R & \xrightarrow{\phi} A_0 \rightarrow A_1 \rightarrow 0 \\
S & \xrightarrow{\varphi} B_0 \rightarrow B_1 \rightarrow 0
\end{align*}
\]
with \(A_i \in \text{Add}A\) and \(B_i \in \text{Add}B\) for \(i = 0, 1\), where \(\phi\) and \(\varphi\) is left \(D_{\sigma_A}\)-approximation and \(D_{\sigma_B}\)-approximation. Since \(F\) is right exact, then there are the following two exact sequences in comma category.
\[
\begin{align*}
(R_{FR}) & \rightarrow (A_0)_{FA_{0}} \rightarrow (A_1)_{FA_{1}} \rightarrow 0 \\
0_{(S)} & \rightarrow (0_{B_0}) \rightarrow (0_{B_1}) \rightarrow 0
\end{align*}
\]
We consider the following exact sequence in comma category
\[
(R_{FR}) \oplus (0_{S}) \xrightarrow{a} (A_0)_{FA_{0}} \oplus (0_{B_0}) \rightarrow (A_1)_{FA_{1}} \oplus (0_{B_1}) \rightarrow 0
\]
where \(a = \begin{pmatrix} (\phi) & 0 \\ 0 & (\varphi) \end{pmatrix}\). We claim that the morphism \(a\) is left \(D_{\sigma}\)-approximation. For any \((\bar{X}, X) \in D_{\sigma}\) and the morphism \((f, y) \in \text{Hom}((R_{FR}) \oplus (0_{S}), (\bar{X}, X))\), we need to find a morphism \(\xi\) such that the following diagram is commutative.
\[
\begin{array}{ccc}
(R_{FR}) \oplus (0_{S}) & \xrightarrow{a} & (A_0)_{FA_{0}} \oplus (0_{B_0}) \rightarrow (A_1)_{FA_{1}} \oplus (0_{B_1}) \rightarrow 0 \\
\xi
\end{array}
\]
By the lemma 2.3, \(X \in D_{\sigma_A}\) and \(Y \in D_{\sigma_B}\), and then there are two morphisms \(f_1\) and \(x_1\) such that the following two diagrams is commutative.
\[
\begin{array}{ccc}
X & \xrightarrow{x} & Y \\
R \xrightarrow{\phi} A_0 \rightarrow A_1 \rightarrow 0 & \xrightarrow{\varphi} B_0 \rightarrow B_1 \rightarrow 0
\end{array}
\]
We conclude that this morphism $\xi = ((f_1), (0))$ is what we want, where $y = h \cdot Ff_1$. In fact, we only need to prove that the following two diagrams is commutative, where the first diagram is obvious.

Note that $g = h \cdot Ff$ by the definition of $(f)$. Since $y \cdot F\phi = (h \cdot Ff_1) \cdot F\phi = h \cdot (Ff_1 \cdot F\phi) = h \cdot Ff = g$, the second is also commutative. Note that there exists a surjective morphism $b: (R^{X'}) \oplus (S^{X'}) \to (X')_h$, for some set $X$. According to the above discussion, we can find a morphism $c$ satisfying $b = cd(X)$, and then $c$ is surjective. i.e., $(X')_h \in \text{Gen}((A) \oplus (B))$. Then $(A) \oplus (B)$ is silting with respect to $\sigma$.

Let the functor $G$: Mod-$S \to$ Mod-$R$ be right exact and covariant. Similarly, we have the following result in the right comma category $(G, \text{Mod-} R)$.

**Theorem 2.11.** Let $X \in \text{Mod-} R$ and $Y \in \text{Mod-} S$ and the functor $G$ be commutes with direct sums. Then $(GY \oplus X, Y)_y$ is silting if and only if $X$ is silting, $Y$ is silting and $GY \in \text{Gen}X$.

By the proposition 2.4 and the theorem 2.10 we can get the following corollary immediately.

**Corollary 2.12.** Let $A \in R$-Mod and $B \in S$-Mod and the functor $F$ be commutes with direct sums. We have that the following corollary hold.

1. Then $(A) \oplus (B)$ is silting (or, partial silting) if and only if $A$ is silting (or, partial silting).
2. $(B)$ is is silting (or, partial silting) if and only if $B$ is silting (or, partial silting).

### 3 Torsion pairs

Let $U$ is an $(S, R)$-bimodule. Take $F = U \otimes_R -$, then the comma category $(F, S$-Mod) is isomorphic to the category $T$-Mod [10] p.1923-1924, where $T = \left( \begin{array}{c} R \\ U & S \end{array} \right)$. In this subsection, we mainly consider the special comma category $(U \otimes_R - , S$-Mod) $\cong T$-Mod. In this case, for any object $(A)$ is in $(U \otimes_R - , S$-Mod), we have that $A \in R$-Mod and $B \in S$-Mod.

First, we introduce the following three functors on the product categories and the comma categories and three classes, which is very useful.

**Definition 3.1.** (1) $p$: R-Mod $\times$ S-Mod $\to$ T-Mod is defined as follows: for each objects $(A, B)$ of R-Mod $\times$ S-Mod, let $p(A, B) = (A \oplus B, (L))$ with the obvious map and for any morphism $(f, g)$ in R-Mod $\times$ S-Mod, let $p(f, g) = (f \oplus g)$. It is easy to see that $p(A, B) = p(A, 0) \oplus p(0, B)$.

(2) $q$: T-Mod $\to$ R-Mod $\times$ S-Mod is defined as follows: for each objects $(A, B)$ of T-Mod, let $q(A, B) = (A, B)$ with the obvious map and for any morphism $(f, g)$ in T-Mod, let $q(f, g) = (f, g)$.

(3) $h$: R-Mod $\times$ S-Mod $\to$ T-Mod is defined as follows: for each objects $(A, B)$ of R-Mod $\times$ S-Mod, let $h(A, B) = (A \oplus \text{Hom}_{S}(U, B))$ with the obvious map and for any morphism $(f, g)$ in R-Mod $\times$ S-Mod, let $h(f, g) = (f \oplus \text{Hom}_{S}(U, g))$.
It is easy to verify that \( p \) is a left adjoint of \( q \) and \( q \) is a left adjoint of \( h \).

The functor \( p \) was introduced by Mitchell \[15\], which is a particular case of the additive left Kan extension functor. Recently, it was used by Enochs \[7\] and Hu \[10\]. The functors \( q \) and \( h \) was used by Mao \[13\].

Let \( C \) be a class of \( R\Mod \) and \( D \) be a class of \( S\Mod \). We will denote by \( \Upsilon_C^D \) the class of objects \( \{ (C)_{\phi}^\varphi; C \in C \text{ and } D \in D \} \). Denoted by \( \mathcal{B}_D^C \) the class of objects \( \{ (C)_{\varphi}^\psi; \ker \varphi \in C \text{ and } D/\text{Im}(\varphi) \in D, \varphi \text{ is a monomorphism} \} \). Denoted by \( \mathcal{J}_C^D \) the class of objects \( \{ (C)_{\varphi}^\psi; \ker \varphi \in C \text{ and } D \in D, \varphi \text{ is an epimorphism} \} \), where \( \varphi \) is the morphism from \( A \) to \( \text{Hom}_S(U, B) \) given by \( \varphi(x)(u) = \varphi(u \otimes x) \) for each \( u \in U \) and \( x \in A \).

Next, we will give some isomorphisms on \( \text{Hom} \) set in \( T\Mod \).

**Lemma 3.2.** Let \((A)_{B}^C\) and \((C)_{D}^\varphi\) be in \( T\Mod \). Then the following statements hold.

1. \( \text{Hom}_T((A)_{B}^C), (C)_{D}^\varphi ) \cong \text{Hom}_R(A, C) \).
2. \( \text{Hom}_T((A)_{B}^C), (C)_{D}^\varphi ) \cong \text{Hom}_S(B, D) \).
3. \( \text{Hom}_T((A)_{B}^C), (C)_{D}^\varphi ) \cong \text{Hom}_S(A, C) \).
4. \( \text{Hom}_T((A)_{B}^C), (U \otimes R_A) ) \cong \text{Hom}_S(B, D) \).
5. \( \text{Hom}_T((A)_{B}^C), (U \otimes R_A) ) \cong \text{ker} \varphi \).

**Proof.** (1) Since the pair \((q, h)\) is an adjoint pair, we have that \( \text{Hom}_T((A)_{B}^C), (C)_{D}^\varphi ) \cong \text{Hom}_T((A)_{B}^C), h(C, 0) \)
\( \cong \text{Hom}_{R\Mod} \times S\Mod(q(A)_{B}^C), (C, 0)) \cong \text{Hom}_{R\Mod} \times S\Mod((A, B), (C, 0)) \cong \text{Hom}_R(A, C) \).
Since \((p, q), (q, h)\) are adjoint pairs, the proof of (2)–(4) is similar to (1).
(5) holds by Lemma 4.1(2) in \[13\].

Analogously, we have the category \( \text{Mod,} T \) of right \( T \)-modules whose objects are triples \((X, Y)_\phi\), where 
\( X \in \text{Mod-} R, Y \in \text{Mod-} S \) and \( \phi: Y \otimes B U \to X \) a \( R \)-module morphism, and 
whose morphisms from \((X_1, Y_1)_\phi\) to \((X_2, Y_2)_\phi\) are pairs \((g_1, g_2)\) such that \( g_1 \in \text{Hom}_R(X_1, X_2) \)
\( g_2 \in \text{Hom}_S(Y_1, Y_2) \) and \( \phi_2(g_2 \otimes 1) = g_1 \phi_1 \).
By \[11\] Proposition 3.6.1 there is an isomorphism of abelian groups
\[(X, Y)_\phi \otimes_T \left( \frac{A}{B} \right)_\varphi \cong \left( \text{Hom}_R(A, C) \right) \otimes \left( \text{Hom}_S(B, D) \right) \]
where the subgroup \( H \) is generated by all elements of the form \((\phi(y \otimes u)) \otimes a - y \otimes \varphi(u \otimes a)\) with \( y \in Y \), \( u \in U \) and \( a \in A \). It follows from the definition that the following conclusion holds.

**Proposition 3.3.** Let \((A, B)\) be a right \( T \)-module and \((C)_{D}^\varphi\) be a left \( T \)-module. Then the following statements hold.

1. \((A, 0) \otimes_T (C)_{D}^\varphi \cong A \otimes_R C \).
2. \((A, B) \otimes_T (0)_{D}^\varphi \cong B \otimes_S D \).
3. \((A, B) \otimes_T (U \otimes_R C) \cong A \otimes_R C \).
4. \((B) \otimes_S (B, D) \otimes_T (C)_{D}^\varphi \cong B \otimes_S D \).
5. \[13\] Lemma 4.1(1) \((0, B) \otimes_T (C)_{D}^\varphi \cong D/\text{Im} \varphi \).

The following proposition given a very important property of the two classes \( \Upsilon_C^D \) and \( \mathcal{B}_D^C \).

**Proposition 3.4.** Let \( C \) be a class of left \( R \)-modules and \( D \) be a class of left \( S \)-modules. Then the following statements hold.

1. \( \Upsilon_C^D \) is a left adjoint of \( \mathcal{B}_D^C \).
2. \( \mathcal{B}_D^C \) is a left adjoint of \( \Upsilon_C^D \). The converses hold if \( S^+ \in D \), where \( S^+ := \text{Hom}(S, Q/Z) \).
Proof. (1) For any $(\hat{A}) \in (\mathcal{B}_D^{j^0})^{-1}$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Since $(\hat{C}) \in \mathcal{B}_D^{C}$, we have that $(\hat{C}) \in \text{Hom}_R(C, A) \cong \text{Hom}((\hat{C}) \in \mathcal{B}_D^{C}) = 0$ by Lemma 3.2(3). Since $(\hat{0}) \in \mathcal{B}_D^{C}$, we have that $\text{Hom}_S(D, B) \cong \text{Hom}((\hat{0}) \in \mathcal{B}_D^{C}) = 0$ by Lemma 3.2(2). i.e., $(\mathcal{B}_D^{C})^{-1} \subseteq \mathcal{C}_D^{j^0}$.

On the contrary, For any $(\hat{M}) \in \mathcal{B}_D^{j^0}$ and $(\hat{Y}) \in \mathcal{B}_D^{C}$. We can obtain the following exact sequence in the comma category by the definition of $\mathcal{B}_D^{C}$.

$$0 \rightarrow (U \otimes_R X) \rightarrow (\hat{Y}) \rightarrow (Y/\otimes_R X) \rightarrow 0.$$ 

Applying the functor $\text{Hom}_T(\hat{C}, (\hat{M}))$ to the above exact sequence, we can get the following exact sequence

$$0 \rightarrow \text{Hom}_T((Y/\otimes_R X), (\hat{M})) \rightarrow \text{Hom}_T((\hat{Y}), (\hat{M})) \rightarrow \text{Hom}_T((U \otimes_R X), (\hat{M})).$$

Note that $K \in \mathcal{D}$ and $\otimes_R X \in \mathcal{D}$ by the definition of $\mathcal{B}_D^{C}$. We have that $\text{Hom}_T((Y/\otimes_R X), (\hat{M}))

\cong \text{Hom}_S(Y/\otimes_R X, N) = 0$ and $\text{Hom}_T((U \otimes_R X), (\hat{M})) \cong \text{Hom}_R(X, M)$ by the lemma 3.2 (2) and (3). So we get that $\text{Hom}_T((\hat{Y}), (\hat{M})) = 0$ from the above exact sequence. i.e., $\mathcal{D}^{j^0} \subseteq (\mathcal{B}_D^{C})^{-1}$.

(2) For any $(\hat{G}) \in \mathcal{B}_D^{j^0}, P \in \mathcal{C}$ and $Q \in \mathcal{D}$. We can obtain the following exact sequence in the comma category by the definition of $\mathcal{B}_D^{j^0}$.

$$0 \rightarrow (U \otimes_R G) \rightarrow (\hat{G}) \rightarrow (H/\otimes_R G) \rightarrow 0$$

with $G \in j^0 \mathcal{C}$ and $H/\otimes_R G \in j^0 \mathcal{D}$. Applying the functor $\text{Hom}_T(\hat{G}, (\hat{P}))$ to the above exact sequence, we can obtain the following exact sequence.

$$0 \rightarrow \text{Hom}_T((H/\otimes_R G), (\hat{P})) \rightarrow \text{Hom}_T((\hat{G}), (\hat{P})) \rightarrow \text{Hom}_T((U \otimes_R G), (\hat{P})).$$

Note that $\text{Hom}_T((H/\otimes_R G), (\hat{P})) \cong \text{Hom}_S(H/\otimes_R G, Q) = 0$ and $\text{Hom}_T((U \otimes_R G), (\hat{P})) \cong \text{Hom}_R(G, P) = 0$ by the lemma 3.2 (2) and (3). Consequently, we get that $\text{Hom}_T(\hat{G}, (\hat{P}), (\hat{Q})) = 0$ from the above exact sequence. i.e., $\mathcal{D}^{j^0} \subseteq (\mathcal{B}_D^{C})^{-1}$.

Conversely, $(\hat{I}) \in \mathcal{D}^{j^0}$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$. By the lemma 3.2 (1), we have that $\text{Hom}_R(I, C) \cong \text{Hom}_T(I, (\hat{C})) = 0$, and then $I \in j^0 \mathcal{C}$.

Note that $((0, S) \otimes_T (0, S), (0, S), (0, S))$ is an adjoint pair. From the following isomorphisms, we have that $J/(U \otimes_R I) \in j^0 \mathcal{D}$.

$$\text{Hom}_S(J/(U \otimes_R I), D) \cong \text{Hom}_S((0, S) \otimes_T (\hat{I}), D),$$

by the definition of the tensor products.

$$\cong \text{Hom}_T((\hat{I}), \text{Hom}_S((0, S), D)),$$

by the definition of the adjoint pairs.

$$\cong \text{Hom}_T((\hat{I}), (\hat{D})),

= 0$$

We consider the following exact sequence in category Mod-T.

$$0 \rightarrow (U, 0) \rightarrow (U, S) \rightarrow (0, S) \rightarrow 0.$$
Applying the functor $- \otimes_T (f')$ to the above exact sequence, we can obtain the following exact sequence.

$$\text{Tor}_1((0, S) \otimes_T (f')) \longrightarrow (U, 0) \otimes_T (f') \longrightarrow (U, S) \otimes_T (f') \longrightarrow (0, S) \otimes_T (f') \longrightarrow 0.$$  

By the proposition $3.3$ (1) and (4), we can get that $(U, 0) \otimes_T (f') \cong U \otimes_R I$ and $(U, S) \otimes_T (f') \cong (S \otimes_S U, S) \otimes_T (f') \cong S \otimes_S J \cong J$. Note $\text{Tor}_1((0, S), (f'))^+ \cong \text{Ext}_1(f', (S_s^+))$. Since $S^+$ is injective, the injective dimension of $(S_s^+)$ is at most by the corollary 3.4 (2) in [12]. It is easy to verify from $\text{Hom}_T((f'), (S_s^+)) = 0$ that $\text{Ext}_1(f', (S_s^+)) = 0$, i.e., $\text{Tor}_1((0, S) \otimes_T (f')) = 0$. So the morphism $U \otimes_R I \longrightarrow J$ is an monomorphism. i.e., $\varphi_0(\mathcal{H}^C_2) \subseteq \mathcal{B}_2$.

Next, we will give the theorem $1.2$ (1) and its proof.

**Theorem 3.5.** Let $C_1$ and $C_2$ be two subcategories of left $R$-modules and $D_1$ and $D_2$ be two subcategories of left $S$-modules. If $S^+ \in D_2$, then $(C_1, C_2)$ and $(D_1, D_2)$ are torsion pairs if and only if $(\mathcal{B}^{C_1}_{D_1}, \mathcal{B}^{C_2}_{D_2})$ is a torsion pair.

**Proof.** $(\Rightarrow)$ Since $(C_1, C_2)$ and $(D_1, D_2)$ are torsion pairs, we have that $(\mathcal{B}^{C_1}_{D_1})^{\perp_0} = \mathcal{U}^{1+0}_{D_1} = \mathcal{U}^{1+0}_{D_2}$ by the proposition $3.3$ (1). And $\varphi_0(\mathcal{B}^{C_2}_{D_2}) = \varphi_0(\mathcal{B}^{C_2}_{D_1}) = \mathcal{B}^{C_2}_{D_1}$ by the proposition $3.3$ (2).

$(\Leftarrow)$ (1) For any $A \in C_1$ and $B \in C_2$, then $(A, (0)) \in \mathcal{B}^{C_1}_{D_1}$ and $(B, (0)) \in \mathcal{B}^{C_2}_{D_2}$. Since $(\mathcal{B}^{C_1}_{D_1}, \mathcal{B}^{C_2}_{D_2})$ is a torsion pair, we have that $\text{Hom}_R(A, B) \cong \text{Hom}_T((A, (0)), (B, (0))) = 0$ by the lemma $3.2$ (3). i.e., $\text{Hom}_R(C_1, C_2) = 0$.

For any $X \in C_1^{\perp_0}$, then $(X, 0) \in \mathcal{B}^{C_1}_{D_1}$, $\varphi_0(\mathcal{B}^{C_1}_{D_1}) = \mathcal{B}^{C_2}_{D_1}$, i.e., $X \in C_2$.

For any $Y \in \varphi_0^{-1}C_2$, then $(U, (0)) \in \mathcal{B}^{C_2}_{D_2}$, $\varphi_0(\mathcal{B}^{C_2}_{D_2}) = \mathcal{B}^{C_2}_{D_1}$, i.e., $Y \in C_1$. So $(C_1, C_2)$ is a torsion pair.

(2) For any $M \in D_1$ and $N \in D_2$, then $(0, (0) \in \mathcal{B}^{C_1}_{D_1}$ and $(0, N) \in \mathcal{B}^{C_2}_{D_2}$. By the lemma $3.2$ (2), we have that $\text{Hom}_S(M, N) \cong \text{Hom}_T((0, (0)), (0, N)) = 0$ since $(\mathcal{B}^{C_1}_{D_1}, \mathcal{B}^{C_2}_{D_2})$ is a torsion pair.

For any $P \in D_1^{\perp_0}$, then $(0, P) \in \mathcal{B}^{C_1}_{D_1}$, $\varphi_0(\mathcal{B}^{C_1}_{D_1}) = \mathcal{B}^{C_2}_{D_2}$, i.e., $P \in D_2$.

For any $Q \in D_2^{\perp_0}$, then $(0, Q) \in \mathcal{B}^{C_2}_{D_2}$, $\varphi_0(\mathcal{B}^{C_2}_{D_2}) = \mathcal{B}^{C_1}_{D_1}$, i.e., $Q \in D_1$. So $(D_1, D_2)$ is a torsion pair.

**Proposition 3.6.** Let $C$ be a class of left $R$-modules and $D$ be a class of left $S$-modules. Then the following statements hold.

1. $\varphi_0(\mathcal{C}^C_D) = \mathcal{U}^{1+0}_D$.
2. $\mathcal{C}^C_D \subseteq \mathcal{U}^{1+0}_D$. The converses hold if $R \in C$.

**Proof.** (1) $(\Rightarrow)$ For any $(X, Y) \in \mathcal{B}^{1+0}_D$, $C \in C$ and $D \in D$. Note that $(Y, 0) \in \mathcal{C}^C_D$. By the lemma $3.2$ (1), we have that $\text{Hom}_R(X, C) \cong \text{Hom}_T((X, (0)), (C, 0)) = 0$, i.e., $X \in \varphi_0^{-1}C$.

Note that $(\text{Hom}_S(U, D)) \in \mathcal{C}^C_D$. By the lemma $3.2$ (4), we have that $\text{Hom}_S(Y, D) \cong \text{Hom}_T((Y, 0), (\text{Hom}_S(U, D))) = 0$, i.e., $Y \in \perp_0^1D$. So $\perp_0(\mathcal{C}^C_D) \subseteq \mathcal{U}^{1+0}_D$.

$(\Leftarrow)$ For any $(M, N) \in \mathcal{U}^{1+0}_D$, $(A, B) \in \mathcal{C}^C_D$. By the definition of $\mathcal{C}^C_D$, we can obtain the following exact sequence.

Applying the functor $\text{Hom}_T((M, N), \cdot)$ to the above exact sequence, we can obtain the following exact sequence.

$$0 \longrightarrow \text{Hom}_T((M, N), (\ker \varphi_0)) \longrightarrow \text{Hom}_T((M, N), (A, B)) \longrightarrow \text{Hom}_T((M, N), (\text{Hom}_S(U, B))) \longrightarrow 0.$$
By the lemma 3.2 (1) and (4), we have that Hom$_T((M), (\ker \tilde{\varphi})) \cong \text{Hom}_R(M, \ker \tilde{\varphi}) = 0$ and Hom$_T((N), (\text{Hom}_S(U, B))) \cong \text{Hom}_S(N, B) = 0$. From the above sequence, Hom$_T((\frac{M}{N}), (\frac{A}{B})) = 0$. i.e., $(\frac{M}{N}) \in \mathcal{D}_1^\perp$.  

(2) For any $(\frac{G}{H}) \in \mathcal{D}_2^\perp$, $(\frac{I}{J}) \in \mathcal{D}_1^\perp$. We consider the following exact sequence.

\[
0 \rightarrow (\ker \tilde{\varphi}) \rightarrow (\frac{G}{H}) \rightarrow (\text{Hom}_S(U, H)) \rightarrow 0
\]

Applying the functor Hom$_T((\frac{I}{J}), -)$ to the above exact sequence, we can obtain the following exact sequence.

\[
0 \rightarrow \text{Hom}_T((\frac{I}{J}), (\ker \tilde{\varphi})) \rightarrow \text{Hom}_T((\frac{I}{J}), (\frac{G}{H})) \rightarrow \text{Hom}_T((\frac{I}{J}), (\text{Hom}_S(U, H)))
\]

By the lemma 3.2 (1) and (4), we have that Hom$_T((\frac{I}{J}), (\ker \tilde{\varphi})) \cong \text{Hom}_R(I, \ker \tilde{\varphi}) = 0$ and Hom$_T((\frac{I}{J}), (\text{Hom}_S(U, H))) \cong \text{Hom}_S(J, H) = 0$. From the above sequence, Hom$_T((\frac{I}{J}), (\frac{G}{H})) = 0$. So we can obtain that $\mathcal{D}_1^\perp \subseteq (\mathcal{U}_D^\perp)^1_0$.

Conversely, For any $(\frac{P}{Q}) \in (\mathcal{U}_D^\perp)^1_0$, $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

By the lemma 3.2 (5), we get that Hom$_R(C, \ker \tilde{\psi}) \cong \text{Hom}_R(C, \text{Hom}_T((\frac{R}{0}), (\frac{P}{Q}))) \cong \text{Hom}_T((\frac{R}{0}) \otimes_R C, (\frac{P}{Q})) \cong \text{Hom}_T((\frac{R}{0}), (\frac{P}{Q})) = 0$ since $(\frac{C}{0}) \in \mathcal{U}_D^\perp$. i.e., ker $\tilde{\psi} \in \mathcal{C}_{\perp}^1$.  

Note that $(\frac{0}{D}) \in \mathcal{U}_D^\perp$, and then Hom$_S(D, Q) \cong \text{Hom}_T((\frac{0}{D}), (\frac{P}{Q})) = 0$. i.e., $Q \in D_1^\perp$.

We consider the following exact sequence.

\[
0 \rightarrow (\frac{0}{D}) \rightarrow (\frac{R}{0}) \rightarrow (\frac{P}{Q}) \rightarrow 0
\]

Applying the functor Hom$_T(-, (\frac{P}{Q}))$ to the above exact sequence, we can obtain the following exact sequence.

\[
0 \rightarrow \text{Hom}_T((\frac{R}{0}), (\frac{P}{Q})) \rightarrow \text{Hom}_T((\frac{R}{0}), (\frac{P}{Q})) \rightarrow \text{Hom}_T((\frac{0}{D}), (\frac{P}{Q})) \rightarrow \text{Ext}_1((\frac{R}{0}), (\frac{P}{Q}))
\]

The projective dimension of $(\frac{R}{0})$ is at most 1 by the corollary 3.4 (1) in [13]. It is easy to verify from Hom$_T((\frac{0}{D}), (\frac{P}{Q})) = 0$ that Ext$_1((\frac{R}{0}), (\frac{P}{Q})) = 0$. Note that Hom$_T((\frac{R}{0}), (\frac{P}{Q})) \cong \text{Hom}_T((\frac{R}{0} \otimes_R R), (\frac{P}{Q})) \cong \text{Hom}_R(R, P) \cong P$ and Hom$_T((\frac{0}{D}), (\frac{P}{Q})) \cong \text{Hom}_S(U, Q)$ by the lemma 3.2 (2) and (3). i.e., $(\frac{P}{Q}) \in (\mathcal{U}_D^\perp)^1_0$. So we have that $(\mathcal{U}_D^\perp)^1_0 \subseteq \mathcal{D}_1^\perp$. \hfill \Box

Finally, we will give the theorem 3.2 (2) and its proof.

**Theorem 3.7.** Let $C_1$ and $C_2$ be two subcategories of left $R$-modules, $D_1$ and $D_2$ be two subcategories of left $S$-modules. If $R \in C_1$, then $(C_1, C_2)$ and $(D_1, D_2)$ are torsion pairs if and only if $(\mathcal{U}_{D_1}^\perp, \mathcal{C}_{D_2}^\perp)$ is a torsion pair.

**Proof.** ($\Rightarrow$) Since $(C_1, C_2)$ and $(D_1, D_2)$ are torsion pairs, we have that $(\mathcal{U}_{D_1}^\perp)^1_0 = \mathcal{C}_{D_2}^\perp$ by the proposition 3.6 (2). And $\mathcal{C}_{D_2}^\perp = \mathcal{U}_{D_2}^{1_0} = \mathcal{D}_{D_1}$ by the proposition 3.6 (1). So we have that $(\mathcal{U}_{D_1}^\perp, \mathcal{C}_{D_2}^\perp)$ is a torsion pair.

($\Leftarrow$) (1) For any $X \in C_1$ and $Y \in C_2$, then $(\frac{X}{0}) \in \mathcal{U}_{D_1}^\perp$ and $(\frac{Y}{0}) \in \mathcal{C}_{D_2}^\perp$. Since $(\mathcal{U}_{D_1}^\perp, \mathcal{C}_{D_2}^\perp)$ is a torsion pair, we have that Hom$_R(X, Y) \cong \text{Hom}_T((\frac{X}{0}), (\frac{Y}{0})) = 0$ by the lemma 3.2 (1). i.e., Hom$_R(C_1, C_2) = 0$.

For any $M \in C_1^\perp$, then $(\frac{M}{0}) \in \mathcal{C}_{D_1}^\perp = (\mathcal{U}_{D_1}^\perp)^1_0 = \mathcal{C}_{D_2}^\perp$ by the proposition 3.6 (2). i.e., $M \in C_2$. 11
For any $N \in \gamma^{\perp} \mathcal{C}_2$, then $(N) \in \Omega^{\perp} \mathcal{C}_2 = \Omega^{\perp} \mathcal{C}_2$ by the proposition 3.6 (1). i.e., $N \in \mathcal{C}_1$. So $(\mathcal{C}_1, \mathcal{C}_2)$ is a torsion pair.

(2) For any $I \in \mathcal{D}_1$ and $J \in \mathcal{D}_2$, then $(0, I) \in \Omega^{\perp} \mathcal{D}_1$ and $(\text{Hom}_{\mathcal{S}}(U, J)) \in \mathcal{C}_2$. By the lemma 3.2, we have that \( \text{Hom}_{\mathcal{S}}(I, J) \cong \text{Hom}_{\mathcal{F}}((0, I), (\text{Hom}_{\mathcal{S}}(U, J))) = 0 \) since $(\Omega^{\perp} \mathcal{D}_1, \mathcal{C}_2)$ is a torsion pair. i.e., \( \text{Hom}_{\mathcal{S}}(\mathcal{D}_1, \mathcal{D}_2) = 0 \).

For any $P \in \mathcal{D}_1^{\perp}$, then $(\text{Hom}_{\mathcal{S}}(U, P)) \in \mathcal{C}_2^{\perp}$ by the proposition 3.4 (2). i.e., $P \in \mathcal{D}_2$.

For any $Q \in \mathcal{D}_2^{\perp}$, then $(0, Q) \in \Omega^{\perp} \mathcal{D}_2 = \Omega^{\perp} \mathcal{C}_2$ by the proposition 3.6 (2). i.e., $Q \in \mathcal{D}_1$. So $(\mathcal{D}_1, \mathcal{D}_2)$ is a torsion pair.

By the proposition 2.8, theorem 2.10, theorem 3.5, theorem 3.7 and [2, Lemma 2.3], we can get the following corollary immediately.

**Corollary 3.8.** Let $(\mathcal{A}_1 \mathcal{B}_1) \oplus (\mathcal{A}_2 \mathcal{B}_2)$ be silting (or, partial silting) in $T$-Mod. Then the following conclusions hold.

1. If $S^+ \in \mathcal{B}_1$, then $(\mathcal{A}_1 \mathcal{B}_2)$ is a torsion pair.
2. If $R \in \mathcal{A}_2$, then $(\mathcal{A}_2 \mathcal{B}_1)$ is a torsion pair.

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