ANALYSIS OF AN INTERIOR PENALTY DG METHOD FOR THE QUAD-CURL PROBLEM

GANG CHEN, WEIFENG QIU, AND LIWEI XU

Abstract. The quad-curl term is an essential part of the resistive magnetohydrodynamic (MHD) equation and the fourth order inverse electromagnetic scattering problem, which are both of great significance in science and engineering. It is desirable to develop efficient and practical numerical methods for the quad-curl problem. In this paper, we first present some new regularity results for the quad-curl problem on Lipschitz polyhedron domains and then propose a mixed finite element method for solving the quad-curl problem. With a novel discrete Sobolev imbedding inequality for the piecewise polynomials, we obtain stability results and derive error estimates based on a relatively low regularity assumption of the exact solution.

1. Introduction

Let \( \Omega \) be a bounded simply connected Lipschitz polyhedron in \( \mathbb{R}^3 \) with connected boundary \( \partial \Omega \). We consider the following quad-curl (fourth order) problem: find the vector \( u \) and the Lagrange multiplier \( p \) such that

\[
\begin{align*}
\nabla \times (\nabla \times (\nabla \times (\nabla \times u))) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\nabla \times u &= 0 \quad \text{on } \partial \Omega, \\
\nabla \times (\nabla \times u) &= 0 \quad \text{on } \partial \Omega, \\
p &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Here \( f \in [L^2(\Omega)]^3 \), the vector \( n \) denotes the unit outer normal on \( \partial \Omega \). This model problem arises in many different applications, such as in the resistive magnetohydrodynamics (MHD) and the Maxwells transmission eigenvalue theory.

The resistive MHD system reads (\[15,32\]): find the velocity \( u \), the pressure \( p \) and the magnetic induction field \( B \) such that

\[
\begin{align*}
\rho(u_t + (u \cdot \nabla)u) + \nabla p &= \frac{1}{\mu_0}(\nabla \times B) + \mu \Delta u \quad \text{in } \Omega, \\
B_t - \nabla \times (u \times B) &= -\frac{\eta}{\mu_0}(\nabla \times (\nabla \times B)) \\
&\quad - \frac{d_i}{\mu_0} \nabla \times ((\nabla \times B) \times B) \\
&\quad - \frac{\eta_2}{\mu_0} \nabla \times (\nabla \times (\nabla \times (\nabla \times B))) \quad \text{in } \Omega,
\end{align*}
\]
\( \nabla \cdot \mathbf{u} = 0 \) in \( \Omega \),
\( \nabla \cdot \mathbf{B} = 0 \) in \( \Omega \),

with some proper boundary conditions. Here, \( \rho \) is the mass density, \( \eta \) is the resistivity, \( \eta_2 \) is the hyper-resistivity, \( \mu_0 \) is the magnetic permeability of free space, and \( \mu \) is the viscosity.

In the inverse electromagnetic scattering theory, the transmission eigenvalue problem for the anisotropic Maxwell equations can be formulated in the following fourth order problem (\cite{22}):

\[
(\nabla \times (\nabla \times) - k^2 N)(N - I)^{-1}(\nabla \times (\nabla \times \mathbf{u}) - k^2 \mathbf{u}) = 0 \quad \text{in } \Omega,
\]
\[
\mathbf{n} \times \mathbf{u} = 0 \quad \text{on } \partial \Omega,
\]
\[
\mathbf{n} \times (\nabla \times \mathbf{u}) = 0 \quad \text{on } \partial \Omega,
\]

where \( N \) is a given real matrix field and \( I \) is the identity matrix. The leading term in both (1.2) and (1.3) is \( \nabla \times (\nabla \times (\nabla \times (\nabla \times \mathbf{u}))) \).

There are vast literatures on numerical methods solving the MHD model without the quad-curl term \( \nabla \times (\nabla \times (\nabla \times (\nabla \times \mathbf{u}))) \), see \cite{13,19,25} and references therein for detailed information. However, when the quad-curl term \( \nabla \times (\nabla \times (\nabla \times (\nabla \times \mathbf{u}))) \) is present, the design and analysis of numerical methods for the MHD model becomes more difficult and challenging. Therefore, it is worth devising accurate and efficient numerical methods for the quad-curl problem, providing substantial tools for the solution of the resistive MHD system and the fourth order inverse electromagnetic scattering problem.

It is known that there is a strong correlation between the regularity of exact solutions and the extent of smoothness on the computational domain on which the quad-curl problem is imposed. At the continuous level of differential equations, the author proved in \cite{24} that: when the domain has no point and edge singularities, it holds that \( \mathbf{u} \in [H^4(\Omega)]^3 \); when the domain has a point or edge singularities, \( \mathbf{u} \) does not belong to \([H^3(\Omega)]^3\) in general. In \cite{30}, the author proved that on convex polyhedral domains, if \( \nabla \cdot \mathbf{f} = 0 \), there hold

\[
\mathbf{u} \in [H^2(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^2(\Omega)]^3, \quad p = 0.
\]

These results imply that a reasonable assumption on the regularity of the exact solution of the quad-curl problem, from which the stability and convergence results of numerics are derived, is highly desirable for designing practical numerical methods. This is indeed one of our motivations for this work.

There are already many works devoted to the numerical study on the quad-curl problem in the past decades. In \cite{32}, a nonconforming finite element method was studied under the regularity assumption

\[
\mathbf{u} \in [H^4(\Omega)]^3.
\]

A discontinuous Galerkin (DG) method using \( H(\text{curl}) \) conforming elements for the quad-curl model problem was investigated in \cite{15}, where the following regularity requirements were assumed:

\[
\mathbf{u} \in [H^2(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^2(\Omega)]^3.
\]
An interior penalty DG finite element method for the quad-curl eigenvalue problem was introduced and analyzed in \[26\] given that the following set of regularities
\[ u \in [H^3(\Omega)]^3, \quad \nabla \times u \in [H^3(\Omega)]^3 \]
holds. Instead of solving the quad-curl problem directly, through introducing extra unknowns, a mixed finite element method was proposed and analysed in \[31\] based on a Helmholtz decomposition. The author proved the well-posedness and stability of the method. Also, the convergence rate in the energy-norm is also proved on the convex domain. A finite element method for the quad-curl problem in two dimensions was studied in \[4\] based on the Hodge decomposition. Concerning conforming finite element methods, since the curl-curl conforming elements in three dimensions are still unknown (see \[29\] for curl-curl conforming elements in two dimensions), it would be complicated and far from being obvious (since the conforming elements for the biharmonic problem are quite complicated even in two dimensions, see \[9\] for example) if the curl-curl conforming elements in three dimensions are considered. We refer to \[28\] for more a mixed method for the quad-curl problem and to \[3, 27\] for the multigrid methods on the quad-curl problem.

In this paper, we firstly present several regularity results of the quad-curl problem on general Lipschitz domains (might be non-convex), which have not been documented in literature yet. Then, we introduce an interior penalty DG finite element method solving the quad-curl model problem (1.1). Even though our numerical scheme shares some features with that proposed in \[15\], the authors of \[15\] dealt with a quad-curl problem with a reaction term which makes their theoretical analysis different from ours. Finally, we prove the corresponding stability and convergence results of the numerical solution of \( u \) under a relatively low regularity compared to that in existing works, i.e.

\[ u \in [H^{r_{u0}}(\Omega)]^3, \quad \nabla \times u \in [H^{r_{u1}}(\Omega)]^3, \quad \nabla \times (\nabla \times u) \in [H^{r_{u2}}(\Omega)]^3, \quad p \in H^{r_{p}}(\Omega), \]

where \( r_{u0} > \frac{1}{2}, \; r_{u1} \geq 1, \; r_{u2} > \frac{1}{2}, \) and \( r_{p} > \frac{3}{2} \). In addition, we establish a **novel** discrete Sobolev imbedding inequality in the following piecewise \( H^1 \) norm (see Theorem 4.1):

\[
\sum_{K \in \mathcal{T}_h} \| v_h \|_{1,K} \leq C \left[ \sum_{K \in \mathcal{T}_h} (\| \nabla \times v_h \|_{0,K}^2 + \| \nabla \cdot v_h \|_{0,K}^2) + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| [v_h] \|_{0,F}^2 \right],
\]

where \( v_h \) is a piecewise polynomial of a fixed order. This inequality provides us with a fundamental tool to further achieve the discrete \( H^1 \) stability and \( H^1 \) error estimate for the approximation of \( \nabla \times u \). Moreover, turning to the inequality, we could apply our mixed method to solve eigenvalue problems and carry out the numerical analysis of nonlinear problems and high order problems in a low regularity region via a posteriori analysis techniques (\[11, 12\]), and we will consider it in the future works.

The rest of this paper is organized as follows. We present some regularity results for the partial differential equations (PDEs) with the quad-curl term in Section 2. A mixed method for the quad-curl problem is introduced in Section 3. We obtain a novel discrete Sobolev inequality and stability results for the underlying mixed method in Section 4. The convergence result is proved through an energy argument in Section 5. We give estimates in \( H(\text{curl}) \) norm through dual arguments in Section 6.
2. Regularity for PDEs

For any bounded domain $\Lambda \subset \mathbb{R}^s$ ($s = 2,3$), and any two functions $u, v \in L^2(\Lambda)$, we denote the $L^2(\Lambda)$ inner product and its norm by

$$(u, v)_{\Lambda} := \int_{\Lambda} uv \, dx, \quad \|u\|_{0, \Lambda} := (u, u)_{\Lambda}^{\frac{1}{2}},$$

and when $\Lambda = \Omega$, we set

$$(u, v) := (u, v)_{\Omega}, \quad \|u\|_0 := \|u\|_{0, \Omega},$$

for simplicity. We denote the Sobolev spaces defined on $\Lambda$ by $W^{m,p}(\Lambda)$ and $W_0^{m,p}(\Lambda)$, and denote its semi-norm and norm by $|v|_{m,p,\Lambda}$, $\|v\|_{m,p,\Lambda}$, respectively. When $p = 2$ we omit $p$ in $|v|_{m,p,\Lambda}$ and $\|v\|_{m,p,\Lambda}$; when $\Lambda = \Omega$, we omit $\Lambda$ in $|v|_{m,p,\Lambda}$ and $\|v\|_{m,p,\Lambda}$. For conventional notations, we denote

$$H^m(\Lambda) := W^{m,2}(\Lambda), \quad H_0^m(\Lambda) := W_0^{m,2}(\Lambda).$$

In particular, when $\Lambda \in \mathbb{R}^2$, we use $\langle \cdot, \cdot \rangle_{\Lambda}$ to replace $(\cdot, \cdot)_{\Lambda}$ for distinction. The bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. We define the following function spaces

- $H^{(\text{div}; \Omega)} := \{v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega)\},$
- $H^{(\text{curl}; \Omega)} := \{v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3\},$
- $H^s(\text{curl}; \Omega) := \{v \in [H^s(\Omega)]^3 : \nabla \times v \in [H^s(\Omega)]^3\}$ with $s \geq 0$,
- $H(\text{curl}^2; \Omega) := \{v \in [L^2(\Omega)]^3 : \nabla \times v \in H(\text{curl}; \Omega)\}$

equipped with the graph norms

$${\|v\|}_{H^{(\text{div}; \Omega)}} := (\|\nabla \cdot v\|_0^2 + \|\nabla \times v\|_0^2)^{\frac{1}{2}}, \quad {\|v\|}_{H^{(\text{curl}; \Omega)}} := (\|\nabla \times v\|_0^2 + \|\nabla \times v\|_0^2)^{\frac{1}{2}},$$

$${\|v\|}_{H^s(\text{curl}; \Omega)} := (\|\nabla \times v\|_s^2 + \|\nabla \times v\|_0^2)^{\frac{1}{2}}, \quad {\|v\|}_{H(\text{curl}^2; \Omega)} := (\|\nabla \times v\|_0^2 + \|\nabla \times v\|_{H(\text{curl}; \Omega)}^2)^{\frac{1}{2}},$$

respectively. Furthermore, we define

- $H_0^{(\text{div}; \Omega)} := \{v \in H(\text{curl}; \Omega) : \nabla \cdot v|_{\partial \Omega} = 0\},$
- $H_0^{(\text{curl}; \Omega)} := \{v \in H(\text{curl}; \Omega) : \nabla \times v|_{\partial \Omega} = 0\},$
- $H_0^s(\text{curl}; \Omega) := \{v \in H^s(\text{curl}; \Omega) : \nabla \times v|_{\partial \Omega} = 0\},$
- $H_0(\text{curl}^2; \Omega) := \{v \in H(\text{curl}^2; \Omega) : \nabla \times v|_{\partial \Omega} = \nabla \times (\nabla \times v)|_{\partial \Omega} = 0\}.$

Throughout this paper, we use $C$ to denote a positive constant independent of mesh size, not necessarily the same at its each occurrence. The following imbedding theory is standard but useful in the analysis of $H(\text{curl})$ space.

**Lemma 2.1** ([2 Proposition 3.7]). If the domain $\Omega$ is a Lipschitz polyhedron, then $X_T(\Omega)$ and $X_N(\Omega)$ are continuously imbedded in $[H^\alpha(\Omega)]^3$ for a real number $\alpha \in (\frac{1}{2}, 1]$, where the spaces $X_N(\Omega)$ and $X_T(\Omega)$ are defined as

$$X_N := H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \quad X_T := H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega).$$
**Definition 2.2.** We define the weak formulation of (1.1) as:

Find \((u, p) \in H_0^1(\Omega) \times H_0^1(\Omega)\) such that

\[
(\nabla \times (\nabla \times u), \nabla \times (\nabla \times v)) + (\nabla p, v) = (f, v),
\]

(2.1a)

\[
(u, \nabla q) = 0
\]

(2.1b)

for all \((v, q) \in H_0^1(\Omega) \times H_0^1(\Omega)\).

**Theorem 2.3.** The space \(C_0^\infty(\Omega)^3\) is dense in the space \(H_0^1(\Omega)\).

*Proof.* For any \(v \in H_0^1(\Omega)\), we define

\[
\tilde{v} = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3/\Omega. \end{cases}
\]

Since \(C_0^\infty(\Omega)^3\) is dense in \(H_0^1(\Omega)\), we have \(\tilde{v} \in H(\text{curl}; \mathbb{R}^3)\), thus \(\nabla \times \tilde{v} \in L^2(\mathbb{R}^3)\) and

\[
\nabla \times \tilde{v} = \begin{cases} \nabla \times v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^3/\Omega. \end{cases}
\]

Again, due to the fact that \(C_0^\infty(\Omega)^3\) is dense in \(H_0^1(\Omega)\), we have \(\nabla \times \tilde{v} \in H(\text{curl}; \mathbb{R}^3)\). Then we obtain that the space \(C_0^\infty(\Omega)^3\) is dense in space \(H_0^1(\Omega)\) by following the proof of the part (ii) of [20] Theorem 3.29. \(\square\)

Looking at the test function as \(C_0^\infty(\Omega)\) in (2.1), we know that the solution \((u, p) \in H_0^1(\Omega) \times H_0^1(\Omega)\) of (2.1) satisfies (1.1) in the distribution sense. On the other hand, the solution \((u, p)\) of (1.1) with the low regularity \(u \in [L^2(\Omega)]^3\), \(\nabla \times (\nabla \times u) \in [L^2(\Omega)]^3\) and \(p \in H_0^1(\Omega)\) satisfies (2.1). Now we are ready to prove the following regularity on Lipschitz polyhedron domains for the weak solution \((u, p) \in H_0^1(\Omega) \times H_0^1(\Omega)\).

**Theorem 2.4.** Let \(\Omega\) be a simply connected Lipschitz domain in \(\mathbb{R}^3\), for any given \(f \in [L^2(\Omega)]^3\), the problem (1.1) has a unique weak solution \((u, p) \in H_0^1(\Omega) \times H_0^1(\Omega)\). In addition, the following regularity estimate holds true:

\[
||u||_{r_{u_0}} + ||\nabla \times u||_1 + ||\nabla \times (\nabla \times u)||_0 + ||\nabla \times (\nabla \times (\nabla \times u))||_0 + ||\nabla p||_0 \leq C||f||_0,
\]

where the regularity index \(r_{u_0} \in (\frac{1}{2}, 1]\). Furthermore, if \(\nabla \times (\nabla \times u) \in [H^{r_{u_1}-1}(\Omega)]^3\), we have the following regularity

\[
||\nabla \times u||_{r_{u_1}} \leq C||\nabla \times (\nabla \times u)||_{r_{u_1}-1},
\]

where \(r_{u_1} \in [1, \frac{3}{2})\), and it is close to \(\frac{3}{2}\).

*Proof.* The following inf-sup condition is followed by Theorem 2.3

\[
\sup_{0 \neq v \in H_0^1(\mathbb{R}^3)} \frac{(\nabla p, v)}{||v||_{H^1(\mathbb{R}^3)}} \geq C||\nabla p||_0.
\]

In addition, we have the following V-elliptic property

\[
||u||_0 \leq C||\nabla \times u||_0 \leq C||\nabla \times (\nabla \times u)||_0,
\]

for \(u \in H_0^1(\mathbb{R}^3)\), and \((u, \nabla q) = 0\) for all \(q \in H_0^1(\Omega)\). The existence of a unique solution in \(H_0^1(\mathbb{R}^3) \times H_0^1(\Omega)\) of (2.1) is followed immediately. First, we present our proof of the regularity estimate in following several steps:
• Proof of $\|\nabla \times (\nabla \times u)\|_0 \leq C\|f\|_0$: Taking $v = u$ in (2.1a) and $q = -p$ in (2.1b) and adding them together, we have
$$\|\nabla \times (\nabla \times u)\|_0^2 = (f, u).$$

Considering the facts that $(u, \nabla q) = 0$ for all $q \in H^1_0(\Omega)$, $(\nabla \times u, \nabla q) = 0$ for all $q \in H^1_0(\Omega)$, and Lemma 2.1 or [21, Corollary 4.8], we obtain
$$\|\nabla \times (\nabla \times u)\|_0^2 \leq \|f\|_0 \|u\|_0 \leq C\|f\|_0 \|\nabla \times u\|_0 \leq C\|f\|_0 \|\nabla \times (\nabla \times u)\|_0,$$
which leads to
$$\|\nabla \times (\nabla \times u)\|_0 \leq C\|f\|_0.$$

• Proof of $\|u\|_{r_{u_0}} \leq C\|f\|_0$: It can be obtained directly from Lemma 2.1 and the fact that $\|\nabla \times (\nabla \times u)\|_0 \leq C\|f\|_0$.

• Proof of $\|\nabla \times u\|_1 \leq C\|f\|_0$: We define $\sigma := \nabla \times u$. From $n \times u|_{\partial \Omega} = 0$ we have,
$$n \cdot \sigma|_{\partial \Omega} = n \cdot (\nabla \times u)|_{\partial \Omega} = \nabla \cdot (n \times u) = 0,$$
which, together with $n \times \sigma|_{\partial \Omega} = 0$ in terms of (1.1d), leads to
$$\sigma = 0 \text{ on } \partial \Omega.\quad (2.2)$$

Meanwhile, $\nabla \cdot \sigma = 0$, gives
$$\Delta \sigma = \nabla(\nabla \cdot \sigma) - \nabla \times (\nabla \times \sigma) = -\nabla \times (\nabla \times \sigma) \text{ in } \Omega.\quad (2.3)$$

The combination of (2.2), (2.3), and the regularity for the elliptic problem in [7] yields
$$\|\nabla \times u\|_1 = \|\sigma\|_1 \leq C\|\nabla \times (\nabla \times \sigma)\|_1 \leq C\|\nabla \times (\nabla \times u)\|_0 \leq C\|f\|_0.$$

• Proof of $\|\nabla \times u\|_{r_{u_1}} \leq C\|\nabla \times (\nabla \times u)\|_{r_{u_1}-2}$: The combination of (2.2), (2.3) and the regularity for the elliptic problem in [17, Theorem 1.1], for some constant $r_{u_1} \in [1, \frac{3}{2})$ and it is close to $\frac{3}{2}$, yields
$$\|\nabla \times u\|_{r_{u_1}} = \|\sigma\|_{r_{u_1}} \leq C\|\nabla \times (\nabla \times \sigma)\|_{r_{u_1}-2} \leq C\|\nabla \times (\nabla \times u)\|_{r_{u_1}-2} \leq C\|\nabla \times (\nabla \times u)\|_{r_{u_1}-1}.$$

• Proof of $\|\nabla p\|_0 \leq \|f\|_0$: Letting $v = \nabla p$, it holds that $\nabla \times v = 0$ and $v \in H_0(\text{curl}^2; \Omega)$. We take $v = \nabla p$ in (2.1a) to get
$$\|\nabla p\|_0^2 = (f, \nabla p) \leq \|f\|_0 \|\nabla p\|_0.$$

Then $\|\nabla p\|_0 \leq \|f\|_0$ follows immediately.

• Proof of $\|\nabla \times (\nabla \times (\nabla \times u))\|_0 \leq C\|f\|_0$: It follows from (2.1a) that $\nabla \times (\nabla \times (\nabla \times u)) = f - \nabla p \in [L^2(\Omega)]^3$ in the distribution sense, therefore,
$$\|\nabla \times (\nabla \times (\nabla \times u))\|_0 = \|f - \nabla p\|_0 \leq \|f\|_0 + \|\nabla p\|_0 \leq 2\|f\|_0.$$

Finally, the uniqueness of the solution is followed from the regularity estimate, and thus the existence of the solution follows immediately.

□
3. an interior penalty DG finite element method

Let \( \mathcal{T}_h \) be a quasi-uniform partition of the domain \( \Omega \) consisting of tetrahedrons. For any element \( K \in \mathcal{T}_h \), let \( h_K \) be the infimum of the diameters of spheres containing \( K \) and denote the mesh size \( h := \max_{K \in \mathcal{T}_h} h_K \). Let \( \mathcal{E}_h \) be the set of all faces of the mesh \( \mathcal{T}_h \). For any element \( K \in \mathcal{T}_h \) and face \( F \in \mathcal{E}_h \), we denote by \( n_K \) and \( n_F \) the unit outward normal vector to \( \partial K \) and face \( F \), respectively. Let \( F = \partial K \cap \partial K' \) be an interior face shared by element \( K \) and element \( K' \), and \( n_F \) be pointing from \( K \) to \( K' \). Let \( \phi \) be a piecewise smooth function. We define the average and jump of any element \( K \) and face \( F \) as

\[
\llbracket \phi \rrbracket := \phi_K - \phi_{K'}, \quad \llbracket \phi \rrbracket_K := \phi_K - \phi_{K'}.
\]

On a boundary face \( F \subset \partial K \cap \partial \Omega \), we set \( \llbracket \phi \rrbracket := \phi \) and \( n_F = n_K \). We denote by \( \mathcal{P}_\ell(\Lambda) \) the set of all polynomials with order at most \( \ell \) on bounded domains \( \Lambda \), and denote by \( \mathcal{P}_\ell(\mathcal{T}_h) \) the set of all piecewise polynomials with order at most \( \ell \) with respect to the decomposition \( \mathcal{T}_h \).

3.1. Interior penalty DG methods. For any integer \( k \geq 1 \), we define the finite element spaces

\[
E_h := H^1(\text{curl}; \Omega) \cap [\mathcal{P}_{k+1}(\mathcal{T}_h)]^3, \quad Q_h := H^1(\Omega) \cap [\mathcal{P}_{k+2}(\mathcal{T}_h)]^3.
\]

Then our interior penalty DG method reads:

Find \( u_h \in E_h \) and \( p_h \in Q_h \) such that

\[
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times u_h), \nabla \times (\nabla \times v_h)) = \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} (n \times [\nabla \times u_h], n \times [\nabla \times v_h])_F + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} (n \times [\nabla \times u_h], n \times [\nabla \times v_h])_F = (f, v_h),
\]

(3.1a)

\[
(u_h, \nabla q_h) = 0.
\]

(3.1b)

hold for all \((v_h, q_h) \in E_h \times Q_h\). The stabilization parameter \( \tau > 0 \) is independent of the mesh size. In the following text, we consider the analysis only for the symmetry case (i.e., we replace ‘\( \mp \)’ by ‘\( - \)’ in (3.1a)), since the proof of the non-symmetry case is similar to the symmetry case.

3.2. Interpolations. For integer \( \ell \geq 1 \), we denote by \( \Pi^{\text{curl}}_{h,\ell} \) the standard \( H(\text{curl}) \)-conforming interpolation of the second kind from \( H^{s}(\text{curl}; \Omega) \) to \( H(\text{curl}; \Omega) \cap [\mathcal{P}_{\ell}(\mathcal{T}_h)]^3 \) with \( s > \frac{1}{2} \), and thus also from \( H^{s}(\text{curl}; \Omega) \cap H_0(\text{curl}; \Omega) \) to \( H_0(\text{curl}; \Omega) \cap [\mathcal{P}_{\ell}(\mathcal{T}_h)]^3 \) with \( s > \frac{1}{2} \). The following approximation properties hold (\cite{[1],21,23}):

\[
\|u - \Pi^{\text{curl}}_{h,\ell} u\|_{0,K} \leq Ch^\ell_K (\|u\|_{t,K} + h_K \|\nabla \times u\|_{t,K}),
\]

(3.2a)

\[
\|\nabla \times (u - \Pi^{\text{curl}}_{h,\ell} u)\|_{0,K} \leq Ch^\ell_K \|\nabla \times u\|_{t,K},
\]

(3.2b)
where $u \in H^t(\text{curl}; \Omega)$, and real number $t \in (\frac{1}{2}, \ell]$, and
\begin{equation}
\|u - \Pi_h^{\text{curl}} u\|_{0,K} \leq Ch^m_K \|u\|_{m,K},
\end{equation}
where $u \in H^m(\Omega)$, and real number $m \in (1, \ell + 1]$. We define the $L^2$-projection from $L^2(\Omega)$ onto $Q_h$ as: for any $p \in L^2(\Omega)$, find $\Pi_h^0 p \in Q_h$ such that
\begin{equation}
(\Pi_h^0 p, q) = (p, q) \quad \forall q \in Q_h.
\end{equation}

The following approximation result holds
\begin{equation}
|p - \Pi_h^0 p|_1 \leq Ch^{j-1} \|p\|_j
\end{equation}
for real number $j \in [1, k+3]$ and $p \in H^j(\Omega)$. Next, we introduce an interpolation (6.25): for any $v \in H^s(\text{curl}; \Omega)$ with $s > \frac{1}{2}$, we define $\Pi_h^E v \in E_h$ such that
\begin{equation}
\Pi_h^E v = \Pi_{h,k+1}^{\text{curl}} v + \nabla q_h,
\end{equation}
where $q_h \in Q_h$ is the solution of the well-posed elliptic problem:
\begin{equation}
(\nabla q_h, \nabla q_h) = (v - \Pi_{h,k+1}^{\text{curl}} v, \nabla q_h) \quad \forall q_h \in Q_h.
\end{equation}

Utilizing (3.3) and (3.4), we get the following result immediately.

**Lemma 3.1.** We have the following orthogonality
\begin{equation}
(v - \Pi_h^E v, \nabla q_h) = 0
\end{equation}
hold for all $v \in H^s(\text{curl}; \Omega)$ with $s > \frac{1}{2}$ and $q_h \in Q_h$. In addition, there holds the approximation property
\begin{equation}
\|v - \Pi_h^E v\|_0 \leq 2\|v - \Pi_{h,k+1}^{\text{curl}} v\|_0.
\end{equation}

## 4. Existence of a unique solution and stability for the mixed method

In this section, we will derive stability results for the underlying mixed method. To this purpose, we firstly develop a novel discrete Sobolev imbedding inequality which is also an efficient tool for numerical analysis of nonlinear problems, and we will report it in future papers. All results presented in Sections 4.1, 4.3 and 4.4 are valid as $\Omega \in \mathbb{R}^3$ is a bounded multi-connected Lipschitz polyhedron even though we only consider in our presentation $\Omega \in \mathbb{R}^3$ to be a bounded simply connected Lipschitz polyhedron. The $L^3$ and $L^6$ stability provided in Sections 4.3 and 4.4 allows a better bound of $u_h$ than the $L^2$ stability does.

### 4.1. A novel discrete Sobolev imbedding inequality

We present a novel discrete Sobolev inequality in the following subsection, which is the first to be reported in literatures.

**Theorem 4.1.** Assume that $\Omega$ is a bounded Lipschitz polyhedron (not necessarily simply-connected) in $\mathbb{R}^3$, then there exists a positive constant $C > 0$ such that
\begin{align*}
\sum_{K \in \mathcal{T}_h} \|v_h\|_{1,K}^2 & \leq C \left[ \sum_{K \in \mathcal{T}_h} (\|\nabla \times v_h\|_{0,K}^2 + \|\nabla \cdot v_h\|_{0,K}^2) + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|v_h\|_{0,F}^2 \right], \\
\|v_h\|_{0,F}^2 & \leq C \left[ \sum_{K \in \mathcal{T}_h} (\|\nabla \times v_h\|_{0,K}^2 + \|\nabla \cdot v_h\|_{0,K}^2) + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|v_h\|_{0,F}^2 \right]
\end{align*}
hold for all $v_h \in [P_\ell(\mathcal{T}_h)]^3$ with $\ell \geq 1$ being an integer.
We present in the next lemma concerning a continuous Sobolev imbedding inequality before the proof of Theorem 4.1.

**Lemma 4.2.** There exists a positive constant $C$ such that

$$\|v\|_1 \leq C (\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0)$$

holds for any $v \in [H^1_0(\Omega)]^3$.

**Proof.** Since $v \in [H^1_0(\Omega)]^3$, it holds $\Delta v \in [H^{-1}(\Omega)]^3$. The Poincaré’s inequality and integration by parts give that

$$\|v\|_1^2 \leq C (\nabla v, \nabla v) = C (\Delta v, v) \leq C \|\Delta v\|_{-1} \|v\|_1,$$

and hence

$$\|v\|_1 \leq C \|\Delta v\|_{-1}.$$  \hspace{1cm} (4.1)

Meanwhile, utilizing integration by parts and the Cauthy-Schwarz inequality, for any $w \in [H^1_0(\Omega)]^3$, we have

$$-(\Delta v, w) = (\nabla \times (\nabla \times v), w) - (\nabla (\nabla \cdot v), w)$$

$$= (\nabla \times v, \nabla \times w) + (\nabla \cdot v, \nabla \cdot w)$$

$$\leq C (\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0) \|w\|_1.$$

Then we arrive at

$$\|\Delta v\|_{-1} = \sup_{0 \neq w \in [H^1_0(\Omega)]^3} \frac{(\Delta v, w)}{\|w\|_1} \leq C (\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0).$$  \hspace{1cm} (4.2)

The proof can be obtained immediately from the combination of both (4.1) and (4.2). \hfill \Box

Now, we are in the position to prove Theorem 4.1.

**Proof of Theorem 4.1.** By [8, Theorem 2.2], for any $v_h \in [P_\ell(\mathcal{T}_h)]^3$ with $\ell \geq 1$, there exists an interpolation $\mathcal{J}_{h,\ell}^c v_h$ such that $\mathcal{J}_{h,\ell}^c v_h \in [H^1_0(\Omega)]^3$, and

$$\|\mathcal{J}_{h,\ell}^c v_h - v_h\|_0 + h \left( \sum_{K \in \mathcal{T}_h} \|\nabla (\mathcal{J}_{h,\ell}^c v_h - v_h)\|_{0,K}^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{F \in \mathcal{E}_h} h_F \|v_h\|_{0,F}^2 \right)^{\frac{1}{2}}.$$  \hspace{1cm} (4.3)

Using Lemma 4.2, the triangle inequality and the estimate in (4.3), we obtain that

$$\|\mathcal{J}_{h,\ell}^c v_h\|_1^2 \leq C \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times \mathcal{J}_{h,\ell}^c v_h\|_{0,K}^2 + \|\nabla \cdot \mathcal{J}_{h,\ell}^c v_h\|_{0,K}^2 \right)$$

$$\leq C \sum_{K \in \mathcal{T}_h} \left( \|\nabla \times v_h\|_{0,K}^2 + \|\nabla \cdot v_h\|_{0,K}^2 \right) + \sum_{F \in \mathcal{E}_h} \left( h_F^{-1} \|v_h\|_{0,F}^2 \right).$$

Due to the above estimate, the triangle inequality and the estimate in (4.3), the proof of the first inequality could be obtained immediately. The second inequality could be obtained through the combination of the discrete Sobolev imbedding inequality in [8, Theorem 2.1] and the first inequality. \hfill \Box
Lemma 4.3 ([25] Theorem 3.1). For any \( \mathbf{v}_h \in [P_{k+1}(\mathcal{T}_h)]^3 \) satisfying 
\( (\mathbf{v}_h, \nabla q_h) = 0, \quad \forall q_h \in Q_h = H_0^1(\Omega) \cap P_{k+2}(\mathcal{T}_h), \)
there exists a positive constant \( C \) such that
\[
\| \mathbf{v}_h \|_{0,3} \leq C \left( \sum_{K \in \mathcal{T}_h} \| \nabla \times \mathbf{v}_h \|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| n \times [\mathbf{v}_h] \|_{0,F}^2 \right)^{\frac{1}{2}}.
\]

With the above lemma, we can derive discrete Sobolev inequalities for \( H(\text{curl}) \)-conforming functions.

Lemma 4.4. For any \( \mathbf{v}_h \in E_h = H_0^1(\text{curl}; \Omega) \cap [P_{k+1}(\mathcal{T}_h)]^3 \), if there holds
\[
(\mathbf{v}_h, \nabla q_h) = 0, \quad \forall q_h \in Q_h = H_0^1(\Omega) \cap P_{k+2}(\mathcal{T}_h),
\]
then we have
\[
\| \mathbf{v}_h \|_{0,3} \leq C \| \nabla \times \mathbf{v}_h \|_0,
\]
and
\[
\| \nabla \times \mathbf{v}_h \|_{0,3} \leq C \left( \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times \mathbf{v}_h) \|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \| n \times [\nabla \times \mathbf{v}_h] \|_{0,E}^2 \right)^{\frac{1}{2}}.
\]

Proof. The result (4.5) follows from (4.4), Lemma 4.3 and the fact that \( \mathbf{v}_h \in H_0^1(\text{curl}; \Omega) \). Since
\[
(\nabla \times \mathbf{v}_h, \nabla q_h) = - (\nabla \cdot (\nabla \times \mathbf{v}_h), q_h) = 0
\]
from which the result (4.6) follows immediately because of Lemma 4.3 \( \square \)

4.2. Existence of a unique solution. In this subsection, we state the existence of a unique solution of (3.1).

Theorem 4.5. The variational equation (3.1) admits a unique solution \( (\mathbf{u}_h, p_h) \in E_h \times Q_h \) as \( \tau \) is sufficiently large.

Proof. We define a mesh-dependent norm as
\[
\| \| \mathbf{u}_h \| : = \| \mathbf{u}_h \|_0^2 + \| \nabla \times \mathbf{u}_h \|_0^2 + \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times \mathbf{u}_h) \|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times \mathbf{u}_h] \|_{0,F}^2.
\]
For all \( \mathbf{u}_h \in E_h \) satisfying
\[
(\mathbf{u}_h, \nabla q_h) = 0, \quad \forall q_h \in Q_h,
\]
there holds, due to Lemma 4.4
\[
\| \| \mathbf{u}_h \| \leq C \left( \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times \mathbf{u}_h) \|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times \mathbf{u}_h] \|_{0,F}^2 \right),
\]
and
\[
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times \mathbf{u}_h), \nabla \times (\nabla \times \mathbf{u}_h))_K
\]
\[
- 2 \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times \mathbf{u}_h), n \times [\nabla \times \mathbf{u}_h] \rangle_F
\]

Lemma 4.6. Let $L$ for the mixed problem [5, Theorem 1.1].

Then the existence of a unique solution is followed by (4.7), (4.8), (4.9), and the theory (4.9) Combining (4.10) and (4.11) yields

$$\begin{align*}
&\leq \sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} - \sum_{F \in \mathcal{E}_h} \left( \sum_{q \in \mathcal{Q}_h} \langle \{ \nabla \times (\nabla \times u_h) \}, n \times [\nabla \times u_h] \rangle_F \right) \\
&\quad + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F} \\
&\quad \leq C \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} \right) + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F} \\
&\quad \leq C(\mathbf{f}, \mathbf{u}_h).
\end{align*}$$

(4.11)

Here, $\tau$ is a sufficiently large constant. By taking $v_h = \nabla q_h \in E_h$ and noticing $\nabla \times (\nabla q_h) = 0$, we get

$$\sup_{\theta \neq v_h \in E_h} \frac{\langle v_h, \nabla q_h \rangle}{\| v_h \|} \geq \| \nabla q_h \|_0.$$  

Then the existence of a unique solution is followed by (4.7), (4.8), (4.9), and the theory for the mixed problem [5, Theorem 1.1].

4.3. $L^3$ stability of $u_h$ and $\nabla \times u_h$.

Lemma 4.6. Let $(u_h, p_h) \in E_h \times Q_h$ be the solution of (3.1), then when the constant $\tau > 0$ is sufficient large, we have

$$\begin{align*}
&\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F} \\
&\quad \leq C(f, u_h).
\end{align*}$$

Then the existence of a unique solution is followed by (4.7), (4.8), (4.9), and the theory for the mixed problem [5, Theorem 1.1].

Proof. We take $v_h = u_h \in E_h$ in (3.1a), and $q_h = p_h \in Q_h$ in (3.1b), and combine the corresponding equalities to obtain

$$\begin{align*}
&\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} - 2 \sum_{F \in \mathcal{E}_h} \langle \{ \nabla \times (\nabla \times u_h) \}, n \times [\nabla \times u_h] \rangle_F \\
&\quad + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F} = (f, u_h).
\end{align*}$$

(4.10)

Since $\tau > 0$ is a sufficient large constant, we have

$$\begin{align*}
&\left| 2 \sum_{F \in \mathcal{E}_h} \langle \{ \nabla \times (\nabla \times u_h) \}, n \times [\nabla \times u_h] \rangle_F \right| \\
&\quad \leq 2 \sum_{F \in \mathcal{E}_h} \| \{ \nabla \times (\nabla \times u_h) \}_F \|_{0, F} \| n \times [\nabla \times u_h] \|_{0, F} \\
&\quad \leq \frac{1}{2} \sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} + \frac{1}{2} \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F}.
\end{align*}$$

(4.11)

Combining (4.10) and (4.11) yields

$$\begin{align*}
&\sum_{K \in T_h} \| \nabla \times (\nabla \times u_h) \|^2_{0, K} + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \| n \times [\nabla \times u_h] \|^2_{0, F} \\
&\quad \leq C(f, u_h).
\end{align*}$$

□
With the above lemma, we are ready to prove the following stability result.

**Theorem 4.7.** Let \((u_h, p_h) \in E_h \times Q_h\) be the solution of (3.1), then when the constant \(\tau > 0\) is sufficient large, we have the following stability result
\[
\|u_h\|_{0,3} + \|\nabla \times u_h\|_{0,3}
\]
\[
+ \left( \sum_{K \in T_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} \tau h_F^{-1}\|n \times [\nabla \times u_h]\|_{0,F}^2 \right)^{\frac{1}{2}} \leq C\|f\|_{0,\frac{3}{2}}.
\]

**Proof.** Using Lemma 4.4, one obtains
\[
\|u_h\|_{0,3} \leq C\|\nabla \times u_h\|_{0,3},
\]
and
\[
\|\nabla \times u_h\|_{0,3} \leq C \left( \sum_{K \in T_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} \tau h_F^{-1}\|n \times [\nabla \times u_h]\|_{0,F}^2 \right)^{\frac{1}{2}}.
\]

Therefore, applying (4.12), (4.13), Lemma 4.6 and the Cauthy-Schwarz inequality, we arrive at
\[
\|u_h\|_{0,3} + \|\nabla \times u_h\|_{0,3} + \left( \sum_{K \in T_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} \tau h_F^{-1}\|n \times [\nabla \times u_h]\|_{0,F}^2 \right)^{\frac{1}{2}}
\]
\[
\leq C\|f\|_{0,\frac{3}{2}}.
\]

\(\square\)

4.4. **\(L^6\) and discrete \(H^1\) stability of \(\nabla \times u_h\).**

**Theorem 4.8.** Let \((u_h, p_h) \in E_h \times Q_h\) be the solution of (3.1), when \(\tau > 0\) is a sufficient large constant, then we have the following stability result
\[
\|\nabla \times u_h\|_{0,3}^2 + \sum_{K \in T_h} \|\nabla (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} h_F^{-1}\|\nabla \times u_h\|_{0,F}^2
\]
\[
+ \sum_{K \in T_h} \|\nabla \times (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} h_F^{-1}\|n \times [\nabla \times u_h]\|_{0,F}^2 \leq C\|f\|_{0,\frac{3}{2}}^2.
\]

**Proof.** Since \(u_h \in H_0(\mathrm{curl}; \Omega)\), there holds that \(\nabla \times u_h \in H(\mathrm{div}; \Omega)\), and \(\nabla \cdot (\nabla \times u_h) = 0\). Using the triangle inequality, one arrives at
\[
\|\nabla \times u_h\|_{0,F} = \|(n \times [\nabla \times u_h]) \times n \|_{0,F}
\]
\[
\leq \|(n \times [\nabla \times u_h]) \times n\|_{0,F} + \|(n \cdot [\nabla \times u_h]) n\|_{0,F}
\]
\[
\leq C\|n \times [\nabla \times u_h]\|_{0,F}.
\]

Making use of the above estimate, Theorem 4.4 with \(\mathbf{v}_h = \nabla \times u_h\), and Lemma 4.6, we obtain
\[
\|\nabla \times u_h\|_{0,3}^2 + \sum_{K \in T_h} \|\nabla (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} h_F^{-1}\|\nabla \times u_h\|_{0,F}^2
\]
\[
\leq \sum_{K \in T_h} \|\nabla (\nabla \times u_h)\|_{0,K}^2 + \sum_{F \in E_h} h_F^{-1}\|\nabla \times u_h\|_{0,F}^2.
\]
Proof.

\[ \leq C \left[ \sum_{K \in \mathcal{T}_h} \left( \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \| \nabla \cdot (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| \nabla \times u_h \|^2_{0,F} \right) \right] = C \left[ \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| \nabla \times u_h \|^2_{0,F} \right] \leq C(\mathbf{f}, u_h) \leq C\| \mathbf{f} \|_{0,3} \| u_h \|_{0,3} . \]

Meanwhile, it holds from Lemma 4.4:

\[ \| u_h \|_{0,3} \leq C\| \nabla \times u_h \|_{0} \leq C\| \nabla \times u_h \|_{0,3} \]

\[ \leq C \left( \sum_{K \in \mathcal{T}_h} \| \nabla (\nabla \times u_h) \|^2_{0,K} + \sum_{F \in \mathcal{E}_h} h_F^{-1} \| \nabla \times u_h \|^2_{0,F} \right)^{\frac{1}{2}} . \]

The combination of the above two inequities completes the proof of the theorem.

\[ \Box \]

5. Main error estimates

We first give formulas for integration by parts on low regularity functions.

**Lemma 5.1 (\cite{10, Theorem 1.4.4.6}).** If \( \phi \in H^{1/2+\delta}(\Omega) \) with \( \delta \in (0, 1/2] \), then \( \frac{\partial \phi}{\partial x_i} \in H^{-1/2+\delta}(\Omega) \) for \( i = 1, 2, 3 \), and there exists a constant \( C > 0 \) such that

\[ \| \frac{\partial \phi}{\partial x_i} \|_{-1/2+\delta} \leq C\| \phi \|_{1/2+\delta} . \]

**Theorem 5.2.** If \( \nabla \times (\nabla \times \mathbf{u}) \in [H^{1/2+\delta}(\Omega)]^3 \) with \( \delta \in (0, 1/2] \), \( \nabla \times (\nabla \times (\nabla \times \mathbf{u})) \in [L^2(\Omega)]^3 \), and \( \mathbf{v}_h \in \mathbf{E}_h \), then it holds

\[ (\nabla \times (\nabla \times (\nabla \times \mathbf{u})), \mathbf{v}_h) = (\nabla \times (\nabla \times (\nabla \times \mathbf{u})), \nabla \times \mathbf{v}_h) , \]

where \( (\nabla \times (\nabla \times \mathbf{u})), \nabla \times \mathbf{v}_h) \) is regarded as a duality pairing between \( H^{-1/2+\delta}(\Omega) \) and \( H^{1/2-\delta}_0(\Omega) \).

**Proof.** For all \( \mathbf{v}_h \in \mathbf{E}_h \), it is easy to see that \( \mathbf{v}_h \in [H^{1/2-\delta}_0(\Omega)]^3 \), and \( \nabla \times \mathbf{v}_h \in [H^{1/2-\delta}(\Omega)]^3 \). Since \( \nabla \times (\nabla \times \mathbf{u}) \in [H^{1/2+\delta}(\Omega)]^3 \), according to Lemma 5.1 it holds \( \nabla \times (\nabla \times (\nabla \times \mathbf{u})) \in [H^{-1/2+\delta}(\Omega)]^3 \).

Moreover, since \( H^{1/2-\delta}(\Omega) = H^{1/2-\delta}_0(\Omega) \) for \( \delta \in (0, 1/2] \) (\cite{10, Theorem 1.4.2.4} or \cite{20, Theorem 3.40}), \( (\nabla \times (\nabla \times (\nabla \times \mathbf{u})), \nabla \times \mathbf{v}_h) \) can be regarded as a duality pairing between \([H^{-1/2+\delta}(\Omega)]^3 \) and \([H^{1/2-\delta}_0(\Omega)]^3 \).
Finally, the equation (5.1) follows immediately from the standard duality argument and the fact that
\[(\nabla \times (\nabla \times (\nabla \times u)), \nu) = ((\nabla \times (\nabla \times (\nabla \times u)), \nabla \times \nu) \quad \forall \nu \in [C^\infty_0(\Omega)]^3.\]

\[\square\]

**Theorem 5.3.** For all \(\phi \in H^{\frac{1}{2}+\delta}(\Omega)\) with \(\delta \in (0, \frac{1}{2}]\), and \(\psi_h \in \mathcal{P}_\ell(\mathcal{J}_h)\) with integer \(\ell \geq 0\), we have
\[(5.2) \quad \left(\frac{\partial \phi}{\partial x_i}, \psi_h \right) = -\sum_{K \in \mathcal{T}_h} \left(\phi, \frac{\partial \psi_h}{\partial x_i} \right)_K + \sum_{K \in \mathcal{T}_h} \langle \phi, \psi_h n_i \rangle_{\partial K}\]
for \(i = 1, 2, 3\), where \(\left(\frac{\partial \phi}{\partial x_i}, \psi_h \right)\) denotes a duality pairing between \(H^{-\frac{1}{2}+\delta}(\Omega)\) and \(H^\frac{1}{2}-\delta(\Omega)\).

**Proof.** According to part (i) of [20] Theorem 3.29, \(\mathcal{D}(\bar{\Omega})\) is dense in \(H^{\frac{1}{2}+\delta}(\Omega)\), where
\[\mathcal{D}(\bar{\Omega}) = \{v : v = \tilde{v}|_\Omega \text{ for some } \tilde{v} \in C^\infty(\mathbb{R}^3)\}.\]
We choose \(\{\phi_n\}_{n=1}^\infty \subset \mathcal{D}(\bar{\Omega})\) such that
\[(5.3) \quad \|\phi_n - \phi\|_{\frac{1}{2}+\delta} \to 0 \text{ as } n \to \infty.\]
According to Lemma 5.1, we have
\[
\left\|\frac{\partial \phi_n}{\partial x_i} - \frac{\partial \phi}{\partial x_i}\right\|_{-\frac{1}{2}+\delta} \to 0 \text{ as } n \to \infty.
\]
For any positive integer \(n\), we have
\[
\left(\frac{\partial \phi_n}{\partial x_i}, \psi_h \right) = -\sum_{K \in \mathcal{T}_h} \left(\phi_n, \frac{\partial \psi_h}{\partial x_i} \right)_K + \sum_{K \in \mathcal{T}_h} \langle \phi_n, \psi_h n_i \rangle_{\partial K}.
\]
Note that (5.3) implies that
\[
\sum_{K \in \mathcal{T}_h} \|\phi_n - \phi\|^2_{L^2(\partial K)} \to 0 \text{ as } n \to \infty,
\]
\(\psi_h \in \mathcal{P}_\ell(\mathcal{J}_h)\) and \(\delta \in (0, \frac{1}{2}]\), we have \(\psi_h \in H^{\frac{1}{2}-\delta}(\Omega) = H^\frac{1}{2}-\delta(\Omega)\), so \(\left(\frac{\partial \phi}{\partial x_i}, \psi_h \right)\) can be regarded as a duality pairing between \(H^{-\frac{1}{2}+\delta}(\Omega)\) and \(H^\frac{1}{2}-\delta(\Omega)\). Then it holds
\[
\left|\left(\frac{\partial \phi}{\partial x_i}, \psi_h \right) - \left(\frac{\partial \phi_n}{\partial x_i}, \psi_h \right)\right| \leq C \left\|\frac{\partial \phi}{\partial x_i} - \frac{\partial \phi_n}{\partial x_i}\right\|_{-\frac{1}{2}+\delta} \|\psi_h\|_{\frac{1}{2}-\delta},
\]
and (5.2) follows immediately. \(\square\)

**Assumption 5.4.** In the rest part of this paper, we will assume that the following regularities hold true for the weak solution \((u, p) \in H_{0}(\text{curl}^2; \Omega) \times H^{\delta}_0(\Omega)\) of (1.1):
\[u \in [H^{r_0u}(\Omega)]^3, \quad \nabla \times u \in [H^{r_1u}(\Omega)]^3, \quad \nabla \times (\nabla \times u) \in [H^{r_2u}(\Omega)]^3, \quad p \in H^{r_p}(\Omega),\]
where \(r_0u \in (\frac{1}{2}, \infty), r_1u \in [1, \infty), r_2u \in (\frac{1}{3}, \infty),\) and \(r_p \in (\frac{3}{2}, \infty).\) We also assume that \(r_0u \leq r_{u_1} \leq r_{u_0} + 1\) in order to simplify the notations in the error analysis.

We notice that the above assumption holds with \(r_0u = r_{u_2} = r_p = 2\) when \(\Omega\) is convex.
Lemma 5.5. Let \((u, p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) be the weak solution of (1.1) and Assumption 5.4 holds true, then

\[
\sum_{K \in \mathcal{T}_h} \langle \nabla \times (\nabla \times u), \nabla \times (\nabla \times v)_h \rangle_K + \langle \nabla p, v_h \rangle - \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times u)], n \times [\nabla \times v_h] \rangle_F \\
- \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times v_h)], n \times [\nabla \times u] \rangle_F \\
+ \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \langle n \times [\nabla \times u], n \times [\nabla \times v_h] \rangle_F = (f, v_h)
\]  

(5.4a) 

\((u, \nabla q_h) = 0\) 

are true for all \((v_h, q_h) \in \mathbf{E}_h \times Q_h\).

Proof. For any \(v_h \in \mathbf{E}_h\), since the weak solution \((u, p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) satisfies (1.1) in the distribution sense, by Theorem 2.4, we have \(\nabla \times (\nabla \times (\nabla \times u)) \in [L^2(\Omega)]^3\), which further leads to, together with the fact that \(\nabla p, f \in [L^2(\Omega)]^3\),

\((\nabla \times (\nabla \times (\nabla \times u)), v_h) + (\nabla p, v_h) = (f, v_h)\).

From Theorem 5.2 we have

\((\nabla \times (\nabla \times (\nabla \times u)), \nabla \times v_h) + (\nabla p, v_h) = (f, v_h)\).

According to Theorem 5.3, we have

\[
\sum_{K \in \mathcal{T}_h} \langle \nabla \times (\nabla \times u), \nabla \times (\nabla \times v_h) \rangle_K \\
- \sum_{F \in \mathcal{E}_h} \langle [\nabla \times (\nabla \times u)], n \times [\nabla \times v_h] \rangle_F + \langle \nabla p, v_h \rangle = (f, v_h),
\]

where we have utilized the fact \(\nabla \times (\nabla \times u) = [\nabla \times (\nabla \times u)] \) because of \(\nabla \times (\nabla \times u) \in [H^s_{u_2}(\Omega)]^3\) with \(s_{u_2} > \frac{3}{2}\). Moreover, since \(n \times [\nabla \times u] = 0\) because of \(\nabla \times u \in [H^s_{u_1}(\Omega)]^3\) with \(s_{u_1} \geq 1\) and \(n \times (\nabla \times u)|_{\partial \Omega} = 0\), (5.4a) follows immediately. Finally, (5.4b) can be obtained due to the fact \(Q_h \in H_0^1(\Omega)\).

\[\text{Theorem 5.6.} \quad \text{Let} \ (u, p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega) \text{ be the weak solution of (1.1) and Assumption 5.4 holds true; let} \ (u_h, p_h) \in \mathbf{E}_h \times Q_h \text{ be the solution of (3.1). Then we have the following error estimates}
\]

\[
\|u - u_h\|_0 + \|\nabla \times (u - u_h)\|_0 \\
+ \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times (u - u_h))\|_{0, K}^2 + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \|n \times [\nabla \times (u - u_h)]\|_{0, F}^2 \right)^{\frac{1}{2}} \\
\leq C(h^{s_{u_0}}\|u\|_{s_{u_0}} + h^{s_{u_1} - 1}\|\nabla \times u\|_{s_{u_1}} + h^{s_{u_2}}\|\nabla \times (\nabla \times u)\|_{s_{u_2}} + h^{s_p - 1}\|p\|_{s_p}),
\]

\[
\|\nabla (p - p_h)\|_0 \leq C h^{s_p - 1}\|p\|_{s_p},
\]
and the discrete $H^1$ norm error estimate for $\nabla \times (u - u_h)$

\[
\left(\sum_{K \in T_h} \|\nabla (\nabla \times (u - u_h))\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \|\nabla \times (u - u_h)\|_{0,F}^2\right)^{1/2} \\
\leq C(h^{s_{u_0}}\|u\|_{s_{u_0}} + h^{s_{u_1}-1}\|\nabla \times u\|_{s_{u_1}} + h^{s_{u_2}}\|\nabla \times (\nabla \times u)\|_{s_{u_2}} + h^{s_{p}-1}\|p\|_{s_p}),
\]

where $s_{u_0} \in (\frac{1}{2}, \min(r_{u_0}, k + 2)]$, $s_{u_1} \in [1, \min(r_{u_1}, k + 1)]$, $s_{u_2} \in [1, \min(r_{u_2}, k + 1)]$, and $s_p \in (\frac{3}{2}, \min(r_p, k + 3)]$.

**Proof.** We subtract (5.3.1) from (5.4) to get: for all $(v_h, q_h) \in E_h \times Q_h$, there hold

\[
\sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times v_h))_K + (\nabla (p - p_h), v_h) \\
- \sum_{F \in \mathcal{E}_h} \langle\{\{\nabla \times (\nabla \times (u - u_h))\}\}, n \times [\nabla \times v_h]\rangle_F \\
- \sum_{F \in \mathcal{E}_h} \langle\{\{\nabla \times (\nabla \times v_h)\}\}, n \times [\nabla \times (u - u_h)]\rangle_F \\
+ \sum_{F \in \mathcal{E}_h} \tau h_F^{-1}\langle n \times [\nabla \times (u - u_h)], n \times [\nabla \times v_h]\rangle_F = 0,
\]

(5.5a)

(5.5b)

\[
(u - u_h, \nabla q_h) = 0.
\]

To simplify the notation, we define

\[
e^u_h := \Pi^E_h u - u_h, \quad e^p_h := \Pi^Q_h p - p_h.
\]

By taking $v_h = e^u_h \in E_h$ in (5.5a) and $q_h = e^p_h \in Q_h$ in (5.5b), we can get

\[
\sum_{K \in T_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times e^u_h))_K + (\nabla (p - p_h), e^u_h) \\
- \sum_{F \in \mathcal{E}_h} \langle\{\{\nabla \times (\nabla \times (u - u_h))\}\}, n \times [\nabla \times e^u_h]\rangle_F \\
- \sum_{F \in \mathcal{E}_h} \langle\{\{\nabla \times (\nabla \times e^u_h)\}\}, n \times [\nabla \times (u - u_h)]\rangle_F \\
+ \sum_{F \in \mathcal{E}_h} \tau h_F^{-1}\langle n \times [\nabla \times (u - u_h)], n \times [\nabla \times e^u_h]\rangle_F = 0,
\]

(5.6a)

(5.6b)

\[
(u - u_h, \nabla e^p_h) = 0.
\]

Reformulating (5.6) and noticing that $(\Pi^E_h u - u, \nabla e^p_h) = 0$ from (3.6), we obtain that

\[
\sum_{K \in T_h} (\nabla \times (\nabla \times e^u_h), \nabla \times (\nabla \times e^u_h))_K + (\nabla e^p_h, e^u_h) \\
- 2 \sum_{F \in \mathcal{E}_h} \langle\{\{\nabla \times (\nabla \times e^u_h)\}\}, n \times [\nabla \times e^u_h]\rangle_F \\
+ \sum_{F \in \mathcal{E}_h} \tau h_F^{-1}\langle n \times [\nabla \times e^u_h], n \times [\nabla \times e^u_h]\rangle_F
\]
Now, we make estimates
\begin{equation}
\frac{1}{2} \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times (\Pi_h \mathbf{u} - \mathbf{u})) \|_{0,K}^2 + \frac{1}{2} \sum_{F \in \mathcal{E}_h} \tau F^{-1} \| n \times [\nabla \times e_h^K] \|_{0,F}^2
\end{equation}
Utilizing (5.7) and arguments similar to those in the proof of Lemma 4.6, we arrive at
\begin{equation}
\frac{1}{2} \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times (\Pi_h \mathbf{u} - \mathbf{u})) \|_{0,K}^2 + \frac{1}{2} \sum_{F \in \mathcal{E}_h} \tau F^{-1} \| n \times [\nabla \times e_h^K] \|_{0,F}^2
\end{equation}
Now, we make estimates \( \{R_i\}_{i=1}^5 \) individually. Let \( \Pi_{h,k} \) be the \( L^2 \)-projection from \( L^2(\Omega) \) to \( \mathbb{P}_k(\mathcal{T}_h) \). Then, by the triangle inequality, the inverse inequality and the estimate (3.2b), there holds
\begin{align}
\| \nabla \times (\nabla \times \Pi_{h,k+1} \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \|_{0,K} & \leq \| \nabla \times (\nabla \times \Pi_{h,k+1} \mathbf{u} - \Pi_{h,k} (\nabla \times \mathbf{u})) \|_{0,K} + \| \nabla \times (\Pi_{h,k} (\nabla \times \mathbf{u}) - \nabla \times \mathbf{u}) \|_{0,K} \\
& \leq Ch_{k+1}^{-1}\| \nabla \times \Pi_{h,k+1} \mathbf{u} - \Pi_{h,k} (\nabla \times \mathbf{u}) \|_{0,K} + \| \Pi_{h,k} (\nabla \times \mathbf{u}) - \nabla \times \mathbf{u} \|_{1,K} \\
& \leq Ch_{k+1}^{-1}\| \nabla \times \mathbf{u} \|_{s_{K+1},K}. 
\end{align}
We use the approximation property for $\Pi_h^2$ in (3.3) and Lemma 4.4 to obtain
\[
R_2 \leq Ch^{s_p-1} ||p||_{sp} ||e_h^u||_0 \\
\leq Ch^{s_p-1} ||p||_{sp} ||e_h^u||_{0.3} \\
\leq Ch^{s_p-1} ||p||_{sp} \|\nabla \times e_h^u\|_0 \\
\leq Ch^{s_p-1} ||p||_{sp} \|\nabla \times e_h^u\|_{0.3} \\
(5.11) \leq Ch^{s_p-1} ||p||_{sp} \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times e_h^u)\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \|\mathbf{n} \times [\nabla \times e_h^u]\|_{0,F}^2 \right)^{\frac{1}{2}}.
\]
Using the triangle inequality, the inverse inequality, and the approximation property of $\Pi_{h,k+1}^{\text{curl}}$ in (3.2b), we have
\[
\sum_{F \in \mathcal{E}_h} h_F \left\{ \|\nabla \times (\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u))\|\right\}_{0,F}^2 \\
\leq 2 \sum_{F \in \mathcal{E}_h} h_F \left\{ \|\nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \Pi_{h,k} \nabla \times (\nabla \times u)\|\right\}_{0,F}^2 \\
+ 2 \sum_{F \in \mathcal{E}_h} h_F \left\{ \|\Pi_{h,k} \nabla \times (\nabla \times u)\| - \nabla \times (\nabla \times u)\|\right\}_{0,F}^2 \\
\leq C \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times \Pi_{h,k+1}^{\text{curl}} u) - \Pi_{h,k} \nabla \times (\nabla \times u)\|_{0,K}^2 \\
+ Ch^{2s_u} \|\nabla \times (\nabla \times u)\|_{s_u}^2 \\
\leq C \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u) - \Pi_{h,k} \nabla \times (\nabla \times u)\|_{0,K}^2 \\
+ C \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times u) - \Pi_{h,k} \nabla \times (\nabla \times u)\|_{0,K}^2 + Ch^{2s_u} \|\nabla \times (\nabla \times u)\|_{s_u}^2 \\
\leq C \left( h^{s_u-1} \|\nabla \times u\|_{s_u} + h^{s_u} \|\nabla \times (\nabla \times u)\|_{s_u} \right)^2.
(5.12)
\]
We use the above estimate to get
\[
|R_3| = \left| \sum_{F \in \mathcal{E}_h} \left\{ \|\nabla \times (\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u))\|, \mathbf{n} \times [\nabla \times e_h^u]\right\}_F \right| \\
\leq C \left( h^{s_u-1} \|\nabla \times u\|_{s_u} + h^{s_u} \|\nabla \times (\nabla \times u)\|_{s_u} \right) \\
\times \left( \sum_{F \in \mathcal{E}_h} \tau h_F^{-1} \|\mathbf{n} \times [\nabla \times e_h^u]\|_{0,F}^2 \right)^{\frac{1}{2}}.
(5.13)
\]
Again, by the triangle inequality, the approximation property of $\Pi_{h,k+1}^{\text{curl}}$ in (3.2b), we have
\[
\sum_{F \in \mathcal{E}_h} h_F^{-1} \|\mathbf{n} \times [\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u)]\|_{0,F}^2
\]
\[\leq 2 \sum_{F \in E_h} h_F^{-1} \| n \times [\nabla \times \Pi_{h,k+1}^{\text{curl}} u - \Pi_{h,k} \nabla \times u]\|^2_{0,F} + 2 \sum_{F \in E_h} h_F^{-1} \| n \times [\Pi_{h,k} \nabla \times u - \nabla \times u]\|^2_{0,F}\]
\[\leq C \sum_{K \in T_h} h_K^{-2} \| \nabla \times \Pi_{h,k+1}^{\text{curl}} u - \Pi_{h,k} \nabla \times u\|^2_{0,K} + Ch^{2(su_1 - 1)} \| \nabla \times u\|^2_{su_1}\]
\[\leq C \sum_{K \in T_h} h_K^{-2} \| \nabla \times u - \Pi_{h,k} \nabla \times u\|^2_{0,K} + C \sum_{K \in T_h} h_K^{-2} \| \nabla \times u - \Pi_{h,k} \nabla \times u\|^2_{0,K} + Ch^{2(su_1 - 1)} \| \nabla \times u\|^2_{su_1}\]
\[\leq Ch^{2(su_1 - 1)} \| \nabla \times u\|^2_{su_1}.\]

Using the above estimate and the inverse inequality, we arrive at

\[|R_4| = \left| \sum_{F \in E_h} \langle [\nabla \times (\nabla \times e_h^u)], n \times [\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u)] \rangle_F \right|\]
\[\leq Ch^{su_1 - 1} \| \nabla \times u\|_{su_1} \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times e_h^u)\|_0 \right)^\frac{1}{2},\]
\[|R_5| = \left| \sum_{F \in E_h} \tau h_F^{-1} \langle n \times [\nabla \times (\Pi_{h,k+1}^{\text{curl}} u - u)], n \times [\nabla \times e_h^u] \rangle_F \right|\]
\[\leq Ch^{su_1 - 1} \| \nabla \times u\|_{su_1} \left( \sum_{F \in E_h} \tau h_F^{-1} \| n \times [\nabla \times e_h^u]\|_{0,F}^2 \right)^\frac{1}{2}.\]

From (5.8), (5.10), (5.11), (5.13) and (5.14), it gives us that
\[\sum_{K \in T_h} \| \nabla \times (\nabla \times e_h^u)\|^2_{0,K} + \sum_{F \in E_h} \tau h_F^{-1} \| n \times [\nabla \times e_h^u]\|^2_{0,F}\]
\[\leq C(h^{su_1 - 1} \| \nabla \times u\|_{su_1} + h^{su_2} \| \nabla \times (\nabla \times u)\|_{su_2} + h^{sp-1} \| p\|_{sp}).\]

The result for the estimates of \( u - u_h \) can be concluded as follows
\[\| e_h^u \|_0 + \| \nabla \times e_h^u \|_0\]
\[\leq C \left( \sum_{K \in T_h} \| \nabla \times (\nabla \times e_h^u)\|^2_{0,K} + \sum_{F \in E_h} \tau h_F^{-1} \| n \times [\nabla \times e_h^u]\|^2_{0,F} \right)^\frac{1}{2},\]
according to Lemma 4.4 and the triangle inequality. Using (5.5a) with \( v_h = -\nabla e_h^p \in E_h \) yields
\[\| \nabla e_h^p \|^2_0 = -\langle \nabla (p - \Pi_h^Q p), \nabla e_h^p \rangle \leq Ch^{sp-1} \| p\|_{sp} \| \nabla e_h^p \|_0,\]
which further provides us with completeness of the proof of the estimate for \( \nabla (p - p_h) \) by using the triangle inequality.
6. \(H(\text{curl})\) ERROR ESTIMATE

To derive the \(H(\text{curl})\) error estimate, we need the following dual problem: Find \((\Phi, \Psi) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) such that

\[
\begin{align*}
(6.1a) & \quad \nabla \times (\nabla \times (\nabla \times \Phi)) + \nabla \Psi = \Theta & \text{in } \Omega, \\
(6.1b) & \quad \nabla \cdot \Phi = 0 & \text{in } \Omega, \\
(6.1c) & \quad n \times \Phi = 0 & \text{on } \partial \Omega, \\
(6.1d) & \quad n \times (\nabla \times \Phi) = 0 & \text{on } \partial \Omega, \\
(6.1e) & \quad \Psi = 0 & \text{on } \partial \Omega.
\end{align*}
\]

We notice that \(\Psi = 0\) when \(\nabla \cdot \Theta = 0\).

**Assumption 6.1.** We assume that the following regularity holds for the weak solution \((\Phi, \Psi) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) of \((6.1)\):

\[
(6.2) \quad \|\Phi\|_{\beta} + \|
abla \times (\nabla \times \Phi)\|_{\beta} + \|
abla \times \Phi\|_{1+\gamma} \leq C_{\text{reg}}\|\Theta\|_0,
\]

where \(\beta \in (\frac{1}{2}, 1]\), \(\gamma \in [0, 1]\), \(\gamma \leq \beta\), and \(C_{\text{reg}}\) is a constant independent of mesh size.

We notice that when \(\Omega\) is convex, \((6.2)\) holds as \(\gamma = 1\), \(\beta = 1\) from the regularity result in \([30]\) Theorem 11.

**Lemma 6.2.** Let \((u, p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) be the weak solution of \((1.1)\), and let \((u, p_h) \in E_h \times Q_h\) be the solution of \((6.1)\), and let \(\nabla \cdot \Theta = 0\). Then we have the following error estimates

\[
(\Theta, u - u_h) \leq Ch^\sigma(h^{s_{u_0}}\|u\|_{s_{u_0}} + h^{s_{u_1}}\|\nabla \times u\|_{s_{u_1}} + h^{s_{u_2}}\|\nabla \times (\nabla \times u)\|_{s_{u_2}} + h^{s_p-1}\|p\|_{s_p})\|\Theta\|_0,
\]

where \(\sigma = \min(\beta, \gamma)\) with \(\beta, \gamma\) being defined in \((6.2)\).

**Proof.** By arguments similar to those in the proof of **Lemma 5.5**, we have the following equations hold for the weak solution \((\Phi, \Psi) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) of \((6.1)\):

\[
\begin{align*}
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times \Phi), \nabla \times (\nabla \times (u - u_h)))_K + (\nabla \Psi, (u - u_h)) & - \sum_{F \in \mathcal{E}_h} \langle \left\{ \left\{ \nabla \times (\nabla \times \Phi) \right\} \right\}, n \times \left[ \nabla \times (u - u_h) \right] \rangle_F \\
& - \sum_{F \in \mathcal{E}_h} \langle \left\{ \left\{ \nabla \times (u - u_h) \right\} \right\}, n \times \left[ \nabla \times \Phi \right] \rangle_F \\
& + \sum_{F \in \mathcal{E}_h} \tau h^{-1}_F (n \times \left[ \nabla \times \Phi \right], n \times \left[ \nabla \times (u - u_h) \right])_F = (\Theta, (u - u_h)), \\
(6.3a) & \quad (\Phi, \nabla q) = 0.
\end{align*}
\]

We use the fact \(\Psi = 0\) to get

\[
(\Theta, u - u_h) = \sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times (u - u_h)), \nabla \times (\nabla \times \Phi))_K - \sum_{F \in \mathcal{E}_h} \langle \left\{ \left\{ \nabla \times (\nabla \times \Phi) \right\} \right\}, n \times \left[ \nabla \times (u - u_h) \right] \rangle_F
\]
Using Theorem 5.6, and the fact $\Psi = 0$, we can get the error estimate:

$$
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times (\mathbf{u} - \mathbf{u}_h)) \}, \mathbf{n} \times [\nabla \times \Phi_h] \rangle_F
$$

(6.4)

$$
+ \sum_{F \in \mathcal{T}_h} \tau h_F^{-1} \langle \mathbf{n} \times [\nabla \times \Phi_h], \mathbf{n} \times [\nabla \times (\mathbf{u} - \mathbf{u}_h)] \rangle_F.
$$

Let $(\Phi_h, \Psi_h) \in \mathbf{E}_h \times Q_h$ be the solution of the following system: for $\forall (\mathbf{v}_h, q_h) \in \mathbf{E}_h \times Q_h$

$$
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times \Phi_h), \nabla \times (\nabla \times \mathbf{v}_h))_K + (\nabla \Psi_h, \mathbf{v}_h)
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times \Phi_h) \}, \mathbf{n} \times [\nabla \times \Phi_h] \rangle_F
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times \Psi_h) \}, \mathbf{n} \times [\nabla \times \Phi_h] \rangle_F
$$

(6.5a)

$$
+ \sum_{F \in \mathcal{T}_h} \tau h_F^{-1} \langle \mathbf{n} \times [\nabla \times \Phi_h], \mathbf{n} \times [\nabla \times \mathbf{v}_h] \rangle_F = (\mathbf{f}, \mathbf{v}_h),
$$

(6.5b) \quad (\Phi_h, \nabla q_h) = 0

Using [Theorem 5.6] and the fact $\Psi = 0$, we can get the error estimate:

$$
\| \Phi - \Phi_h \|_0 + \| \nabla \times (\Phi - \Phi_h) \|_0 + \| \nabla (\Psi - \Psi_h) \|_0
+ \sum_{K \in \mathcal{T}_h} \| \nabla \times (\nabla \times (\Phi - \Phi_h)) \|_{0, \mathbf{K}}^2
+ \sum_{F \in \mathcal{T}_h} \tau h_F^{-1} \| \mathbf{n} \times [\nabla \times (\Phi - \Phi_h)] \|_{0, F}^2
$$

(6.6)

$$
\leq C \left( h^\beta \| \Phi \|_\beta + h^\beta \| \nabla \times (\nabla \times \Phi) \|_\beta + h^{\gamma} \| \nabla \times \Phi \|_{1+\gamma} \right).
$$

From (5.5), it follows that

$$
\sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times (\mathbf{u} - \mathbf{u}_h)), \nabla \times (\nabla \times \Phi_h))_K + (\nabla (p - p_h), \Phi_h)
$$

$$
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times (\mathbf{u} - \mathbf{u}_h)) \}, \mathbf{n} \times [\nabla \times \Phi_h] \rangle_F
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times \Phi_h) \}, \mathbf{n} \times [\nabla \times (\mathbf{u} - \mathbf{u}_h)] \rangle_F
$$

(6.7a)

$$
+ \sum_{F \in \mathcal{T}_h} \tau h_F^{-1} \langle \mathbf{n} \times [\nabla \times (\mathbf{u} - \mathbf{u}_h)], \mathbf{n} \times [\nabla \times \Phi_h] \rangle_F = 0,
$$

(6.7b) \quad (\mathbf{u} - \mathbf{u}_h, \nabla \Psi_h) = 0.

In terms of (6.4) and (6.7), and by the fact $(\nabla (p - p_h), \Phi_h) = (\nabla (p - p_h), \Phi_h - \Phi)$, we have

$$
(\Theta, \mathbf{u} - \mathbf{u}_h) = \sum_{K \in \mathcal{T}_h} (\nabla \times (\nabla \times (\mathbf{u} - \mathbf{u}_h)), \nabla \times (\nabla \times (\Phi - \Phi_h)))_K
$$

$$
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times (\Phi - \Phi_h)) \}, \mathbf{n} \times [\nabla \times (\mathbf{u} - \mathbf{u}_h)] \rangle_F
- \sum_{F \in \mathcal{T}_h} \langle \{ \nabla \times (\nabla \times (\mathbf{u} - \mathbf{u}_h)) \}, \mathbf{n} \times [\nabla \times (\Phi - \Phi_h)] \rangle_F
$$
Using the Cauchy-Schwartz inequality, the triangle inequality, the inverse inequality, the error estimates in (6.6), (5.6), and the regularity (6.2), we get

\[ \langle \nabla (p - p_h), \Phi - \Phi_h \rangle \]

(6.8) \[ =: T_1 + T_2 + T_3 + T_4 + T_5. \]

Now we make the estimate for \( \{T_i\}_{i=1}^5 \) individually. To simply the notation, we define

\[ M := (h^{s_0} \|u\|_{s_0} + h^{s_1} \|\nabla \times u\|_{s_1} + h^{s_2} \|\nabla \times (\nabla \times u)\|_{s_2} + h^{s_p-1} \|p\|_{s_p}). \]

We use the Cauchy-Schwartz inequality, the estimate in Theorem 5.6, the error estimate (6.6) and the regularity (6.2) to get

\[ |T_1| \leq \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times (u - u_h))\|_{0,K} \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\nabla \times (\Phi - \Phi_h))\|_{0,K} \right)^{\frac{1}{2}} \]

\[ \leq CM \left( h^\beta \|\Phi\|_\beta + h^\beta \|\nabla \times (\nabla \times \Phi)\|_\beta + h^\gamma \|\nabla \times \Phi\|_{1+\gamma} \right) \]

(6.9) \[ \leq Ch^\sigma M \|\Theta\|_0. \]

Using the Cauchy-Schwartz inequality, the triangle inequality, the inverse inequality, the error estimates in (6.6), (5.6), and the regularity (6.2), we get

\[ |T_2| \leq \sum_{F \in \mathcal{E}_h} h_F^2 \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} h_F^{-\frac{1}{2}} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ \leq \sum_{F \in \mathcal{E}_h} \|\| \nabla \times (\nabla \times (\Phi - \Pi_{h+1}^{\text{curl}}(\nabla \times \Phi))\|_{0,F} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ + \sum_{F \in \mathcal{E}_h} \|\| \nabla \times (\Pi_{h+1}^{\text{curl}}(\nabla \times \Phi) - \nabla \times \Phi_h)\|_{0,F} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ \leq Ch^\beta M \|\nabla \times (\nabla \times \Phi)\|_\beta + Ch^\gamma M \|\nabla \times \Phi\|_{1+\gamma} \]

\[ + CM \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\Pi_{h+1}^{\text{curl}}(\nabla \times (\Phi - \Phi_h))\|_{0,K} \right)^{\frac{1}{2}} \]

\[ \leq CM \left( h^\beta \|\Phi\|_\beta + h^\beta \|\nabla \times (\nabla \times \Phi)\|_\beta + h^\gamma \|\nabla \times \Phi\|_{1+\gamma} \right) \]

(6.10) \[ \leq Ch^\sigma M \|\Theta\|_0. \]

Similar to (6.10), one can get

\[ |T_3| \leq \sum_{F \in \mathcal{E}_h} \|\| \nabla \times (\nabla \times (u - u_h))\|_{0,F} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ \leq \sum_{F \in \mathcal{E}_h} \|\| \nabla \times (\nabla \times (\Phi - \Phi_h))\|_{0,F} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ + \sum_{F \in \mathcal{E}_h} \|\| \nabla \times (\Pi_{h+1}^{\text{curl}}(\nabla \times u) - \nabla \times u_h)\|_{0,F} \|\nabla \times (\nabla \times (u - u_h))\|_{0,F} \]

\[ \leq CM \left( h^\beta \|\Phi\|_\beta + h^\beta \|\nabla \times (\nabla \times \Phi)\|_\beta + h^\gamma \|\nabla \times \Phi\|_{1+\gamma} \right) \]

(6.11) \[ \leq Ch^\sigma M \|\Theta\|_0, \]

(6.12) \[ |T_4| \leq CM \left( h^\beta \|\Phi\|_\beta + h^\beta \|\nabla \times (\nabla \times \Phi)\|_\beta + h^\gamma \|\nabla \times \Phi\|_{1+\gamma} \right) \leq Ch^\sigma M \|\Theta\|_0. \]
Again, we use the Cauchy-Schwartz inequality, the estimate in Theorem 5.6, the error estimates in (6.6) and the regularity (6.2) to get
\[
|T_5| \leq \|\nabla p - \nabla p_h\|_0 \|\Phi - \Phi_h\|_0
\leq CM (h^\beta \|\Phi\|_\beta + h^\beta \|\nabla \times (\nabla \times \Phi)\|_\beta + h^\gamma \|\nabla \times \Phi\|_{1+\gamma})
\]
(6.13)
\[
\leq Ch^\sigma M \|\Theta\|_0.
\]
We thus complete the proof by applying (6.8), (6.9), (6.10), (6.11), (6.12), (6.13).

In the following two subsections, we will give estimates for \(\|u - u_h\|_0\) and \(\|\nabla \times (u - u_h)\|_0\), respectively, and then obtain the error estimate in \(H(\text{curl})\) norm.

### 6.1. \(L^2\) error estimate.

**Theorem 6.3.** Let \((u, p) \in H_0^1(\Omega)^2 \times H_0^1(\Omega)\) be the weak solution of (1.1), and let Assumption 5.4, Assumption 6.1 hold true; let \((u_h, p_h) \in E_h \times Q_h\) be the solution of (3.1). Then we have the following error estimates
\[
\|u - u_h\|_0 \leq Ch^\alpha (h^{s_{u_1}} - 1 \|\nabla \times u\|_{s_{u_1}} + h^{s_{u_2}} \|\nabla \times (\nabla \times u)\|_{s_{u_2}} + h^{s_{u_3}} \|p\|_{s_p}) + Ch^{s_{u_0}} \|u\|_{s_{u_0}},
\]
where \(\alpha = \min(\alpha, \beta, \gamma)\), \(\alpha\) is defined in Lemma 2.1, \(\beta\) and \(\gamma\) are defined in (6.2).

**Proof.** We take \(\Theta\) satisfying the following problem:

Find \(\Theta \in H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)\) such that
\[
\begin{align*}
\nabla \times \Theta &= \nabla \times (\Pi_h^E u - u_h) & \text{in } \Omega, \\
\nabla \cdot \Theta &= 0 & \text{in } \Omega, \\
n \times \Theta &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
It follows form (3.1a) and (3.6) that
\[
(\Pi_h^E u - u_h, \nabla q_h) = (u, \nabla q_h) = - (\nabla \cdot u, q_h) = 0
\]
for all \(q_h \in Q_h\). Due to the result in [14, Lemma 4.5], one has
\[
\|\Pi_h^E u - u_h - \Theta\|_0 \leq Ch^\alpha \|\nabla \times (\Pi_h^E u - u_h)\|_0,
\]
where \(\alpha\) is defined in Lemma 2.1. As a result, using the triangle inequality, the estimate (6.18), and \(\sigma \leq \alpha\), one can get
\[
\|\Theta\|_0 \leq \|\Pi_h^E u - u_h - \Theta\|_0 + \|\Pi_h^E u - u_h\|_0
\leq Ch^\sigma \|\nabla \times (\Pi_h^E u - u_h)\|_0 + \|u - u_h\|_0
\leq Ch^{s_{u_0}} (\|u\|_{s_{u_0}} + \delta(s_{u_0})h \|\nabla \times u\|_{s_{u_0}})
\]
(6.19)
\[
\|u - u_h\|_0 \leq Ch^\alpha M + Ch^{s_{u_0}} (\|u\|_{s_{u_0}} + \delta(s_{u_0})h \|\nabla \times u\|_{s_{u_0}}) + \|u - u_h\|_0.
\]
Due to the triangle inequality, (6.19), Lemma 6.2 and (6.18), one arrives at
\[
\|u - u_h\|_0^2 = (\Theta, u - u_h) + (\Pi_h^E u - u_h - \Theta, u - u_h) + (u - \Pi_h^E u, u - u_h)
\leq Ch^\sigma M \|\Theta\|_0 + Ch^\sigma \|\nabla \times (\Pi_h^E u - u_h)\|_0 \|u - u_h\|_0
\leq Ch^{s_{u_0}} (\|u\|_{s_{u_0}} + \delta(s_{u_0})h \|\nabla \times u\|_{s_{u_0}}) \|u - u_h\|_0.
\]
\[ \leq C h^{2\alpha} M^2 + C h^{2s_u} (\|u\|_{s_u} + \delta(s_u) h \|\nabla \times u\|_{s_u})^2 + \frac{1}{2} \|u - u_h\|^2, \]

where we have defined \(\delta(s_u) = 0\) when \(s_u > 1\) and \(\delta(s_u) = 1\) when \(s_u \in \left(\frac{1}{2}, 1\right]\). The above inequality further implies that
\[
\|u - u_h\|_0 \leq C \left( h^\sigma (h^{s_u-1} \|\nabla \times u\|_{s_u}) + h^s \|p\|_{s_p} + h^{s_u} (\|u\|_{s_u} + \delta(s_u) h \|\nabla \times u\|_{s_u}) \right) 
\leq C \left( h^\sigma (h^{s_u-1} \|\nabla \times u\|_{s_u}) + h^s \|p\|_{s_p} + h^{s_u} \|u\|_{s_u} \right),
\]

and we complete the proof.

6.2. Curl operator error estimate. We first introduce an interpolation denoted by \(\Pi_{h,\ell}^{\text{curl}}\). Form [16, Proposition 4.5], for any integer \(\ell \geq 1\), let \(v_h \in [P_{\ell}(J_h)]^3 \cap H_0(\text{curl};\Omega)\) such that
\[
\|\Pi_{h,\ell}^{\text{curl}} v_h - v_h\|_0 + h \left( \sum_{K \in \mathcal{T}_h} \|\nabla \times (\Pi_{h,\ell}^{\text{curl}} v_h - v_h)\|_{0,K}^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{F \in \mathcal{E}_h} h_F \|n \times [v_h]\|_{0,F}^2 \right)^{\frac{1}{2}}.
\]

Theorem 6.4. Let \((u,p) \in H_0(\text{curl}^2;\Omega) \times H_0^1(\Omega)\) be the weak solution of (1.1), and let Assumption 5.4 and Assumption 6.1 hold true; let \((u_h,p_h) \in E_h \times Q_h\) be the solution of (3.1). Then we have the following error estimates
\[
\|\nabla \times (u - u_h)\|_0 \leq C h^\sigma (h^{s_u-1} \|\nabla \times u\|_{s_u}) + h^{s_u} \|\nabla \times (\nabla \times u)\|_{s_u} + h^{s_p-1} \|p\|_{s_p}
\]
\[+ C h^{s_u} \|u\|_{s_u},\]

where \(\sigma = \min(\alpha, \beta, \gamma)\), \(\alpha\) is defined in Lemma 2.1, \(\beta\) and \(\gamma\) are defined in (6.2).

The proof of the theorem can be obtained through arguments similar to those in the proof of Theorem 6.3 and through considering the problem (6.15) with \(\nabla \times (\Pi_{h,k+1}(\nabla \times u) - \Pi_{h,k+1}(\nabla \times u_h))\) being the right hand side term, and we skip the details of the proof. Finally, we present the optimal error estimates in \(H(\text{curl})\)-norm as follows.

Proposition 6.5. Let \((u,p) \in H_0(\text{curl}^2;\Omega) \times H_0^1(\Omega)\) be the weak solution of (1.1), and \((u_h,p_h) \in E_h \times Q_h\) be the solution of (3.1). When \(\Omega\) is convex and the weak solution \((u,p) \in H_0(\text{curl}^2;\Omega) \times H_0^1(\Omega)\) of (1.1) is sufficiently smooth, then we have the following error estimates
\[
\|u - u_h\|_0 + \|\nabla \times (u - u_h)\|_0 \leq C h^{k+1} (\|u\|_{k+1} + \|\nabla \times u\|_{k+1} + \|p\|_{k+1}).
\]

7. Numerical experiments

We consider the nonhomogeneous problem:

\[
(7.1a) \quad \nabla \times (\nabla \times (\nabla \times u)) + \nabla p = f \quad \text{in } \Omega,
\]

\[
(7.1b) \quad \nabla \cdot u = 0 \quad \text{in } \Omega,
\]

\[
(7.1c) \quad n \times u = g_T \quad \text{on } \partial \Omega,
\]

\[
(7.1d) \quad n \times (\nabla \times u) = m_T \quad \text{on } \partial \Omega,
\]

\[
(7.1e) \quad p = 0 \quad \text{on } \partial \Omega,
\]
instead of (1.1). As in [6], we take \( \Omega = [0,1]^3 \) and the exact solution \( u = (u_1,u_2,u_3)^T \) and \( p \) read
\[
\begin{align*}
    u_1 &= \sin(\pi y) \sin(\pi z), \\
    u_2 &= \sin(\pi z) \sin(\pi x), \\
    u_3 &= \sin(\pi x) \sin(\pi y), \\
    p &= \sin(2\pi x) \sin(2\pi y) \sin(2\pi z).
\end{align*}
\]
The functions \( f, g_T \) and \( m_T \) are determined according to (7.1) and the above exact solutions. We modify the scheme (3.1) to approximate the nonhomogeneous problem (7.1) as follows:

Find \( u_h \in E_h^{g_T} \) and \( p_h \in Q_h \) such that
\[
\begin{align*}
    &\sum_{K \in T_h} (\nabla \times (\nabla \times u_h), \nabla \times (\nabla \times v_h))_K + (\nabla p_h, v_h) \\
    &- \sum_{F \in E_h} \langle \langle \nabla \times (\nabla \times u_h), \nabla \times (\nabla \times v_h) \rangle \rangle_F \\
    &\pm \sum_{F \in E_h} \langle \langle \nabla \times (\nabla \times v_h), \nabla \times (\nabla \times u_h) \rangle \rangle_F \\
    &+ \sum_{F \in E_h} \tau h_F^{-1} \langle \nabla \times (\nabla \times u_h), \nabla \times (\nabla \times v_h) \rangle_F \\
    &= (f, v_h) \pm \sum_{F \in E_h \cap \partial \Omega} \langle m_T, \nabla \times (\nabla \times v_h) \rangle_F \\
    &+ \sum_{F \in E_h \cap \partial \Omega} \tau h_F^{-1} \langle m_T, \nabla \times (\nabla \times v_h) \rangle_F,
\end{align*}
\]
(7.2a)
\[
(u_h, \nabla q_h) = 0
\]
(7.2b)
hold for all \((v_h, q_h) \in E_h \times Q_h\), where \( E_h^{g_T} \) is defined as
\[
E_h^{g_T} := \{ v \in [P_{k+1}(T_h)]^3 : \nabla \cdot v|_{\partial \Omega} = \Pi_{h}^{\text{div}} g_T \},
\]
and \( \Pi_{h}^{\text{div}} \) is some \( H(\text{div}) \)-projection onto \( k^{th} \) order piecewise polynomial spaces defined on \( E_h \cap \partial \Omega \).

We simply compute the symmetry case and take \( \tau = 10 \) for the numerical implementation. The convergence results for \( k = 0,1,2 \) are presented in Table 7.1 (although we have not performed analysis for the case of \( k = 0 \), we present the numerical results for a purpose of complete illustration). We can observe that the convergence rate for \( p_h \) is \( k + 3 \) which is optimal; the convergence rate for \( u_h \) is 0 when \( k = 0 \); the convergence rate for \( u_h \) is 2 which is not optimal when \( k = 1 \); the convergence rate for \( u_h \) is 4 which is optimal when \( k = 2 \). All these results are in a good agreement with the theoretical analysis results.

**Acknowledgments**

Gang Chen is supported by National Natural Science Foundation of China (NSFC) under grant no. 11801063, China Postdoctoral Science Foundation under grant no. 2018M633339 and 2019T120828. Weifeng Qiu is supported by Research Grants Council of the Hong Kong Special
Table 7.1. History of convergence for \( k = 0, 1, 2 \)

| \( k \) | \( h^{-1} \) | Error | Rate | Error | Rate | DOF |
|-------|---------|-------|------|-------|------|-----|
| 0     | 2       | 5.69E-01 |       | 2.60E-01 |       | 321 |
|       | 4       | 4.18E-01 | 0.44 | 5.05E-02 | 2.36 | 1937 |
|       | 8       | 3.27E-01 | 0.35 | 6.25E-03 | 3.01 | 13281 |
|       | 16      | 2.81E-01 | 0.22 | 7.37E-04 | 3.08 | 97985 |
|       | 24      | 2.66E-01 | 0.14 | 2.15E-04 | 3.04 | 321697 |
| 1     | 2       | 1.48E-01 |       | 9.72E-02 |       | 997 |
|       | 4       | 5.72E-02 | 1.37 | 8.77E-03 | 3.47 | 6601 |
|       | 8       | 1.96E-02 | 1.54 | 5.57E-04 | 3.98 | 47761 |
|       | 10      | 1.34E-02 | 1.72 | 2.23E-04 | 4.09 | 91381 |
|       | 12      | 9.65E-03 | 1.79 | 1.06E-04 | 4.09 | 155737 |
| 2     | 2       | 2.70E-02 |       | 3.18E-02 |       | 2273 |
|       | 4       | 2.61E-03 | 3.37 | 1.44E-03 | 4.46 | 15777 |
|       | 6       | 5.72E-04 | 3.51 | 1.99E-04 | 4.62 | 50689 |
|       | 8       | 1.88E-04 | 3.79 | 4.89E-05 | 4.88 | 117185 |

Administrative Region of China under grant no. CityU 11304017. Liwei Xu is supported by a Key Project of the Major Research Plan of NSFC under grant no. 91630205 and NSFC under grant no. 11771068.

REFERENCES

[1] A. Alonso and A. Valli, An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations, Math. Comp., 68 (1999), pp. 607–631.
[2] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci., 21 (1998), pp. 823–864.
[3] S. C. Brenner, J. Cui, and L.-y. Sung, Multigrid methods based on Hodge decomposition for a quad-curl problem, Comput. Methods Appl. Math., 19 (2019), pp. 215–232.
[4] S. C. Brenner, J. Sun, and L.-y. Sung, Hodge decomposition methods for a quad-curl problem on planar domains, J. Sci. Comput., 73 (2017), pp. 495–513.
[5] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.
[6] H. Chen, W. Qiu, and K. Shi, A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations, Comput. Methods Appl. Mech. Engrg., 333 (2018), pp. 287–310.
[7] M. Dauge, Elliptic boundary value problems on corner domains, vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1988. Smoothness and asymptotics of solutions.
[8] D. A. Di Pietro and A. Ern, Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations, Math. Comp., 79 (2010), pp. 1303–1330.
[9] J. Douglas, Jr., T. Dupont, P. Percell, and R. Scott, A family of \( C^1 \) finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems, RAIRO Anal. Numér., 13 (1979), pp. 227–255.
[10] P. Grisvard, Elliptic problems in nonsmooth domains, vol. 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.
[11] T. Gudi, A new error analysis for discontinuous finite element methods for linear elliptic problems, Math. Comp., 79 (2010), pp. 2169–2189.
[12] T. Gudi and M. Neilan, An interior penalty method for a sixth-order elliptic equation, IMA J. Numer. Anal., 31 (2011), pp. 1734–1753.
[13] J.-L. Guermond, R. Laguerre, J. Léorat, and C. Nore, An interior penalty Galerkin method for the MHD equations in heterogeneous domains, J. Comput. Phys., 221 (2007), pp. 349–369.
[14] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer., 11 (2002), pp. 237–339.
[15] Q. Hong, J. Hu, S. Shu, and J. Xu, A discontinuous Galerkin method for the fourth-order curl problem, J. Comput. Math., 30 (2012), pp. 565–578.
[16] P. Houston, I. Perugia, A. Schneebeli, and D. Schötzau, Interior penalty method for the indefinite time-harmonic Maxwell equations, Numer. Math., 100 (2005), pp. 485–518.
[17] D. Jerison and C. E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal., 130 (1995), pp. 161–219.
[18] O. A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, SIAM J. Numer. Anal., 41 (2003), pp. 2374–2399.
[19] B. Li, J. Wang, and L. Xu, A convergence linearized Lagrange finite element method for the magneto-hydrodynamic equations in 2D nonsmooth and nonconvex domains, SIAM Numer. Anal., (2020).
[20] W. McLean, Strongly elliptic systems and boundary integral equations, Cambridge University Press, Cambridge, 2000.
[21] P. Monk, Finite element methods for Maxwell’s equations, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.
[22] P. Monk and J. Sun, Finite element methods for Maxwell’s transmission eigenvalues, SIAM J. Sci. Comput., 34 (2012), pp. B247–B264.
[23] J.-C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, Numer. Math., 50 (1986), pp. 57–81.
[24] S. Nicaise, Singularities of the quad-curl problem, J. Differential Equations, 264 (2018), pp. 5025–5069.
[25] W. Qiu and K. Shi, A mixed DG method and an HDG method for incompressible magnetohydrodynamics, IMA J. Numer. Anal., (2019), pp. 1–34.
[26] J. Sun, A mixed FEM for the quad-curl eigenvalue problem, Numer. Math., 132 (2016), pp. 185–200.
[27] Z. Sun, J. Cui, F. Gao, and C. Wang, Multigrid methods for a quad-curl problem based on $C^0$ interior penalty method, Comput. Math. Appl., 76 (2018), pp. 2192–2211.
[28] C. Wang, Z. Sun, and J. Cui, A new error analysis of a mixed finite element method for the quad-curl problem, Appl. Math. Comput., 349 (2019), pp. 23–38.
[29] Q. Zhang, L. Wang, and Z. Zhang, $H(\text{curl}^2)$-conforming finite elements in 2 dimensions and applications to the quad-curl problem, SIAM J. Sci. Comput., 41 (2019), pp. A1527–A1547.
[30] S. Zhang, Mixed schemes for quad-curl equations, ESAIM Math. Model. Numer. Anal., 52 (2018), pp. 147–161.
[31] ______, Regular decomposition and a framework of order reduced methods for fourth order problems, Numer. Math., 138 (2018), pp. 241–271.
[32] B. Zheng, Q. Hu, and J. Xu, A nonconforming finite element method for fourth order curl equations in $\mathbb{R}^3$, Math. Comp., 80 (2011), pp. 1871–1886.

School of Mathematics, Sichuan University, Chengdu 610064, China, and School of Mathematical Sciences, University of Electronic Science and Technology of China, Sichuan 611731, China
E-mail address: cglwdm@scu.edu.cn

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Hong Kong, China
E-mail address: weifeqiu@cityu.edu.hk

School of Mathematical Sciences, University of Electronic Science and Technology of China, Sichuan 611731, China
E-mail address: xul@uestc.edu.cn