6. \(\Phi\)-\(\Gamma\)-modules and Galois cohomology

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6.0. Introduction

Let \(G\) be a profinite group and \(p\) a prime number.

**Definition.** A finitely generated \(\mathbb{Z}_p\)-module \(V\) endowed with a continuous \(G\)-action is called a \(\mathbb{Z}_p\)-adic representation of \(G\). Such representations form a category denoted by \(\text{Rep}_{\mathbb{Z}_p}(G)\); its subcategory \(\text{Rep}_{\mathbb{F}_p}(G)\) (respectively \(\text{Rep}_{p\text{-tor}}(G)\)) of mod \(p\) representations (respectively \(p\)-torsion representations) consists of the \(V\) annihilated by \(p\) (respectively a power of \(p\)).

**Problem.** To calculate in a simple explicit way the cohomology groups \(H^i(G, V)\) of the representation \(V\).

A method to solve it for \(G = G_K\) (\(K\) is a local field) is to use Fontaine’s theory of \(\Phi\)-\(\Gamma\)-modules and pass to a simpler Galois representation, paying the price of enlarging \(\mathbb{Z}_p\) to the ring of integers of a two-dimensional local field. In doing this we have to replace linear with semi-linear actions.

In this paper we give an overview of the applications of such techniques in different situations. We begin with a simple example.

6.1. The case of a field of positive characteristic

Let \(E\) be a field of characteristic \(p\), \(G = G_E\) and \(\sigma: E^{\text{sep}} \to E^{\text{sep}}, \sigma(x) = x^p\) the absolute Frobenius map.

**Definition.** For \(V \in \text{Rep}_{\mathbb{F}_p}(G_E)\) put \(D(V) := (E^{\text{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}; \sigma\) acts on \(D(V)\) by acting on \(E^{\text{sep}}\).

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Properties.

(1) \( \dim_E D(V) = \dim \mathfrak{g}_p V \);

(2) the “Frobenius” map \( \varphi: D(V) \to D(V) \) induced by \( \sigma \otimes \text{id}_V \) satisfies:
   a) \( \varphi(\lambda x) = \sigma(\lambda)\varphi(x) \) for all \( \lambda \in E, \ x \in D(V) \) (so \( \varphi \) is \( \sigma \)-semilinear);
   b) \( \varphi(D(V)) \) generates \( D(V) \) as an \( E \)-vector space.

Definition. A finite dimensional vector space \( M \) over \( E \) is called an \( \text{étale} \ \Phi\text{-module} \) over \( E \) if there is a \( \sigma \)-semilinear map \( \varphi: M \to M \) such that \( \varphi(M) \) generates \( M \) as an \( E \)-vector space.

\( \Phi\text{-modules} \) form an abelian category \( \Phi M_E^{et} \) (the morphisms are the linear maps commuting with the Frobenius \( \varphi \)).

Theorem 1 (Fontaine, [F]). The functor \( V \to D(V) \) is an equivalence of the categories \( \text{Rep}_{\mathbb{F}_p}(G_E) \) and \( \Phi M_E^{et} \).

We see immediately that \( H^0(G_E, V) = V^G_E \simeq D(V)\).

So in order to obtain an explicit description of the Galois cohomology of mod \( p \) representations of \( G_E \), we should try to derive in a simple manner the functor associating to an \( \text{étale} \ \Phi\text{-module} \) the group of points fixed under \( \varphi \). This is indeed a much simpler problem because there is only one operator acting.

For \( (M, \varphi) \in \Phi M_E^{et} \) define the following complex of abelian groups:

\[
C_1(M) : \quad 0 \to M \xrightarrow{\varphi^{-1}} M \to 0
\]

(\( M \) stands at degree 0 and 1).

This is a functorial construction, so by taking the cohomology of the complex, we obtain a cohomological functor \( (\mathcal{H}^i := H^i \circ C_1)_{i \in \mathbb{N}} \) from \( \Phi M_E^{et} \) to the category of abelian groups.

Theorem 2. The cohomological functor \( (\mathcal{H}^i \circ D)_{i \in \mathbb{N}} \) can be identified with the Galois cohomology functor \( (H^i(G_E, \cdot))_{i \in \mathbb{N}} \) for the category \( \text{Rep}_{\mathbb{F}_p}(G_E) \). So, if \( M = D(V) \) then \( \mathcal{H}^i(M) \) provides a simple explicit description of \( H^i(G_E, V) \).

Proof of Theorem 2. We need to check that the cohomological functor \( (\mathcal{H}^i)_{i \in \mathbb{N}} \) is universal; therefore it suffices to verify that for every \( i \geq 1 \) the functor \( \mathcal{H}^i \) is effaceable: this means that for every \( (M, \varphi_M) \in \Phi M_E^{et} \) and every \( x \in \mathcal{H}^i(M) \) there exists an embedding \( u \) of \( (M, \varphi_M) \) in \( (N, \varphi_N) \in \Phi M_E^{et} \) such that \( \mathcal{H}^i(u)(x) \) is zero in \( \mathcal{H}^i(N) \). But this is easy: it is trivial for \( i \geq 2 \); for \( i = 1 \) choose an element \( m \) belonging to the class \( x \in M/(\varphi - 1)(M) \), put \( N := M \oplus Et \) and extend \( \varphi_M \) to the \( \sigma \)-semi-linear map \( \varphi_N \) determined by \( \varphi_N(t) := t + m \).

\[ \square \]
6.2. Φ-Γ-modules and \( \mathbb{Z}_p \)-adic representations

**Definition.** Recall that a Cohen ring is an absolutely unramified complete discrete valuation ring of mixed characteristic \((0, p > 0)\), so its maximal ideal is generated by \( p \).

We describe a general formalism, explained by Fontaine in [F], which lifts the equivalence of categories of Theorem 1 in characteristic 0 and relates the \( \mathbb{Z}_p \)-adic representations of \( G \) to a category of modules over a Cohen ring, endowed with a “Frobenius” map and a group action.

Let \( R \) be an algebraically closed complete valuation (of rank 1) field of characteristic \( p \) and let \( H \) be a normal closed subgroup of \( G \). Suppose that \( G \) acts continuously on \( R \) by ring automorphisms. Then \( F := R^H \) is a perfect closed subfield of \( R \).

For every integer \( n \geq 1 \), the ring \( W_n(R) \) of Witt vectors of length \( n \) is endowed with the product of the topology on \( R \) defined by the valuation and then \( W(R) \) with the inverse limit topology. Then the componentwise action of the group \( G \) is continuous and commutes with the natural Frobenius \( \sigma \) on \( W(R) \). We also have \( W(R)^H = W(F) \).

Let \( E \) be a closed subfield of \( F \) such that \( F \) is the completion of the \( p \)-radical closure of \( E \) in \( R \). Suppose there exists a Cohen subring \( \mathcal{O}_E \) of \( W(R) \) with residue field \( E \) and which is stable under the actions of \( \sigma \) and of \( G \). Denote by \( \mathcal{O}_E^{ur} \) the completion of the integral closure of \( \mathcal{O}_E \) in \( W(R) \): it is a Cohen ring which is stable by \( \sigma \) and \( G \), its residue field is the separable closure of \( E \) in \( R \) and \( (\mathcal{O}_E^{ur})^H = \mathcal{O}_E \).

The natural map from \( H \) to \( G_E \) is an isomorphism if and only if the action of \( H \) on \( R \) induces an isomorphism from \( H \) to \( G_F \). We suppose that this is the case.

**Definition.** Let \( \Gamma \) be the quotient group \( G/H \). An étale \( \Phi \)-\( \Gamma \)-module over \( \mathcal{O}_E \) is a finitely generated \( \mathcal{O}_E \)-module endowed with a \( \sigma \)-semi-linear Frobenius map \( \phi: M \to M \) and a continuous \( \Gamma \)-semi-linear action of \( \Gamma \) commuting with \( \phi \) such that the image of \( \phi \) generates the module \( M \).

Étale \( \Phi \)-\( \Gamma \)-modules over \( \mathcal{O}_E \) form an abelian category \( \Phi \Gamma M_{\mathcal{O}_E}^{\text{ét}} \) (the morphisms are the linear maps commuting with \( \phi \)). There is a tensor product of \( \Phi \)-\( \Gamma \)-modules, the natural one. For two objects \( M \) and \( N \) of \( \Phi \Gamma M_{\mathcal{O}_E}^{\text{ét}} \), the \( \mathcal{O}_E \)-module \( \text{Hom}_{\mathcal{O}_E}(M, N) \) can be endowed with an étale \( \Phi \)-\( \Gamma \)-module structure (see [F]).

For every \( \mathbb{Z}_p \)-adic representation \( V \) of \( G \), let \( D_H(V) \) be the \( \mathcal{O}_E \)-module \( (\mathcal{O}_E^{ur} \otimes \mathbb{Z}_p V)^H \). It is naturally an étale \( \Phi \)-\( \Gamma \)-module, with \( \phi \) induced by the map \( \sigma \otimes \text{id}_V \) and \( \Gamma \) acting on both sides of the tensor product. From Theorem 2 one deduces the following fundamental result:

**Theorem 3** (Fontaine, [F]). The functor \( V \to D_H(V) \) is an equivalence of the categories \( \text{Rep}_{\mathbb{Z}_p}(G) \) and \( \Phi \Gamma M_{\mathcal{O}_E}^{\text{ét}} \).

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Remark. If $E$ is a field of positive characteristic, $\mathcal{O}_E$ is a Cohen ring with residue field $E$ endowed with a Frobenius $\sigma$, then we can easily extend the results of the whole subsection 6.1 to $\mathbb{Z}_p$-adic representations of $G$ by using Theorem 3 for $G = G_E$ and $H = \{1\}$.

6.3. A brief survey of the theory of the field of norms

For the details we refer to [W], [FV] or [F].

Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p$. Put $G = G_K = \text{Gal}(K^{\text{sep}}/K)$.

Let $\mathbb{C}$ be the completion of $K^{\text{sep}}$, denote the extension of the discrete valuation $v_K$ of $K$ to $\mathbb{C}$ by $v_K$. Let $R^* = \varprojlim \mathbb{C}_n^*$ where $\mathbb{C}_n = \mathbb{C}$ and the morphism from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$ is raising to the $p$th power. Put $R := R^* \cup \{0\}$ and define $v_R((x_n)) = v_K(x_0)$. For $(x_n), (y_n) \in R$ define

$$(x_n) + (y_n) = (z_n) \quad \text{where} \quad z_n = \lim_m (x_{n+m} + y_{n+m})^{p^m}.$$ 

Then $R$ is an algebraically closed field of characteristic $p$ complete with respect to $v_R$ (cf. [W]). Its residue field is isomorphic to the algebraic closure of $k$ and there is a natural continuous action of $G$ on $R$. (Note that Fontaine described this field by $\text{Fr} R$ in [F]).

Let $L$ be a Galois extension of $K$ in $K^{\text{sep}}$. Recall that one can always define the ramification filtration on $\text{Gal}(L/K)$ in the upper numbering. Roughly speaking, $L/K$ is an arithmetically profinite extension if one can define the lower ramification subgroups of $G$ so that the classical relations between the two filtrations for finite extensions are preserved. This is in particular possible if $\text{Gal}(L/K)$ is a $p$-adic Lie group with finite residue field extension.

The field $R$ contains in a natural way the field of norms $N(L/K)$ of every arithmetically profinite extension $L$ of $K$ and the restriction of $v$ to $N(L/K)$ is a discrete valuation. The residue field of $N(L/K)$ is isomorphic to that of $L$ and $N(L/K)$ is stable under the action of $G$. The construction is functorial: if $L'$ is a finite extension of $L$ contained in $K^{\text{sep}}$, then $L'/K$ is still arithmetically profinite and $N(L'/K)$ is a separable extension of $N(L/K)$. The direct limit of the fields $N(L'/K)$ where $L'$ goes through all the finite extensions of $L$ contained in $K^{\text{sep}}$ is the separable closure $E^{\text{sep}}$ of $E = N(L/K)$. It is stable under the action of $G$ and the subgroup $G_L$ identifies with $G_E$. The field $E^{\text{sep}}$ is dense in $R$.

Fontaine described how to lift these constructions in characteristic 0 when $L$ is the cyclotomic $\mathbb{Z}_p$-extension $K_\infty$ of $K$. Consider the ring of Witt vectors $W(R)$ endowed with the Frobenius map $\sigma$ and the natural componentwise action of $G$. Define the topology of $W(R)$ as the product of the topology defined by the valuation on $R$. Then one can construct a Cohen ring $\mathcal{O}_{E^{\text{ur}}}$ with residue field $E^{\text{sep}}$ ($E = N(L/K)$) such that:

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(i) $\mathcal{O}_{\hat{E}_{\text{ur}}}$ is stable by $\sigma$ and the action of $G$,
(ii) for every finite extension $L$ of $K_\infty$ the ring $(\mathcal{O}_{\hat{E}_{\text{ur}}}^G)_L$ is a Cohen ring with residue field $E$.

Denote by $\mathcal{O}_{\hat{E}(K)}$ the ring $(\mathcal{O}_{\hat{E}_{\text{ur}}}^G)_K$. It is stable by $\sigma$ and the quotient $\Gamma = G/G_{K_\infty}$ acts continuously on $\mathcal{O}_{\hat{E}(K)}$ with respect to the induced topology. Fix a topological generator $\gamma$ of $\Gamma$: it is a continuous ring automorphism commuting with $\sigma$. The fraction field of $\mathcal{O}_{\hat{E}(K)}$ is a two-dimensional standard local field (as defined in section 1 of Part I). If $\pi$ is a lifting of a prime element of $N(K_\infty/K)$ in $\mathcal{O}_{\hat{E}(K)}$ then the elements of $\mathcal{O}_{\hat{E}(K)}$ are the series $\sum_{i \in \mathbb{Z}} a_i \pi^i$, where the coefficients $a_i$ are in $W(k_{K_\infty})$ and converge $p$-adically to 0 when $i \to -\infty$.

6.4. Application of $\mathbb{Z}_p$-adic representations of $G$ to the Galois cohomology

If we put together Fontaine’s construction and the general formalism of subsection 6.2 we obtain the following important result:

Theorem 3' (Fontaine, [F]). The functor $V \to D(V) := (\mathcal{O}_{\hat{E}_{\text{ur}}} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$ defines an equivalence of the categories $\text{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi\Gamma M_{\text{ét}}^{\hat{\text{O}}_{\hat{E}(K)}}$.

Since for every $\mathbb{Z}_p$-adic representation of $G$ we have $H^0(G, V) = V^G \simeq D(V)^\sigma$, we want now, as in paragraph 6.1, compute explicitly the cohomology of the representation using the $\Phi$-$\Gamma$-module associated to $V$.

For every étale $\Phi$-$\Gamma$-module $(M, \varphi)$ define the following complex of abelian groups:

$C_2(M) : 0 \to M \xrightarrow{\alpha} M \oplus M \xrightarrow{\beta} M \to 0$

where $M$ stands at degree 0 and 2,

$$\alpha(x) = ((\varphi - 1)x, (\gamma - 1)x), \quad \beta((y, z)) = (\gamma - 1)y - (\varphi - 1)z.$$  

By functoriality, we obtain a cohomological functor $(\mathcal{H}^i := H^i \circ C_2)_{i \in \mathbb{N}}$ from $\Phi\Gamma M_{\text{ét}}^{\hat{\text{O}}_{\hat{E}(K)}}$ to the category of abelian groups.

Theorem 4 (Herr, [H]). The cohomological functor $(\mathcal{H}^i \circ D)_{i \in \mathbb{N}}$ can be identified with the Galois cohomology functor $(H^i(G, .))_{i \in \mathbb{N}}$ for the category $\text{Rep}_{\text{p-tors}}(G)$. So, if $M = D(V)$ then $\mathcal{H}^0(M)$ provides a simple explicit description of $H^0(G, V)$ in the $p$-torsion case.

Idea of the proof of Theorem 4. We have to check that for every $i \geq 1$ the functor $\mathcal{H}^i$ is effaceable. For every $p$-torsion object $(M, \varphi_M) \in \Phi\Gamma M_{\text{ét}}^{\hat{\text{O}}_{\hat{E}(K)}}$ and every $x \in \mathcal{H}^i(M)$ we construct an explicit embedding $u$ of $(M, \varphi_M)$ in a certain $(N, \varphi_N) \in \Phi\Gamma M_{\text{ét}}^{\hat{\text{O}}_{\hat{E}(K)}}$. 

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such that $\mathcal{H}^i(u)(x)$ is zero in $\mathcal{H}^i(N)$. For details see [H]. The key point is of topological nature: we prove, following an idea of Fontaine in [F], that there exists an open neighbourhood of 0 in $M$ on which $(\varphi - 1)$ is bijective and use then the continuity of the action of $\Gamma$.

As an application of theorem 4 we can prove the following result (due to Tate):

**Theorem 5.** Assume that $k_K$ is finite and let $V$ be in $\text{Rep}_p\text{-tor}(G)$. Without using class field theory the previous theorem implies that $H^i(G, V)$ are finite, $H^i(G, V) = 0$ for $i \geq 3$ and

$$\sum_{i=0}^{2} l(H^i(G, V)) = -|K: \mathbb{Q}_p| l(V),$$

where $l(\cdot)$ denotes the length over $\mathbb{Z}_p$.

See [H].

**Remark.** Because the finiteness results imply that the Mittag–Leffler conditions are satisfied, it is possible to generalize the explicit construction of the cohomology and to prove analogous results for $\mathbb{Z}_p$ (or $\mathbb{Q}_p$)-adic representations by passing to the inverse limits.

### 6.5. A new approach to local class field theory

The results of the preceding paragraph allow us to prove without using class field theory the following:

**Theorem 6** (Tate’s local duality). Let $V$ be in $\text{Rep}_p\text{-tor}(G)$ and $n \in \mathbb{N}$ such that $p^nV = 0$. Put $V^*(1) := \text{Hom}(V, \mu_{p^n})$. Then there is a canonical isomorphism from $H^2(G, \mu_{p^n})$ to $\mathbb{Z}/p^n$ and the cup product

$$H^i(G, V) \times H^{2-i}(G, V^*(1)) \xrightarrow{\cup} H^2(G, \mu_{p^n}) \simeq \mathbb{Z}/p^n$$

is a perfect pairing.

It is well known that a proof of the local duality theorem of Tate without using class field theory gives a construction of the reciprocity map. For every $n \geq 1$ we have by duality a functorial isomorphism between the finite groups $\text{Hom}(G, \mathbb{Z}/p^n) = H^1(G, \mathbb{Z}/p^n)$ and $H^1(G, \mu_{p^n})$ which is isomorphic to $K^*/(K^*)^p$ by Kummer theory. Taking the inverse limits gives us the $p$-part of the reciprocity map, the most difficult part.
Sketch of the proof of Theorem 6. ([H2]).

a) Introduction of differentials:

Let us denote by $\Omega^1$ the $\mathcal{O}_{E(K)}$-module of continuous differential forms of $\mathcal{O}_E$ over $W(k_{K_{\infty}})$. If $\pi$ is a fixed lifting of a prime element of $E(K_{\infty}/K)$ in $\mathcal{O}_{E(K)}$, then this module is free and generated by $d\pi$. Define the residue map from $\Omega^1_c$ to $W(k_{K_{\infty}})$ by $\text{res} \ (\sum_{i \in \mathbb{Z}} a_i \pi^i d\pi) := a_{-1}$; it is independent of the choice of $\pi$.

b) Calculation of some $\Phi$-$\Gamma$-modules:

The $\mathcal{O}_{E(K)}$-module $\Omega^1_c$ is endowed with an étale $\Phi$-$\Gamma$-module structure by the following formulas: for every $\lambda \in \mathcal{O}_{E(K)}$ we put:

$$p \varphi(\lambda d\pi) = \sigma(\lambda) d(\sigma(\pi)) \quad \gamma(\lambda d\pi) = \gamma(\lambda) d(\gamma(\pi)).$$

The fundamental fact is that there is a natural isomorphism of $\Phi$-$\Gamma$-modules over $\mathcal{O}_{E(K)}$ between $D(p^n)$ and the reduction $\Omega^1_{c,n}$ of $\Omega^1_c$ modulo $p^n$.

The étale $\Phi$-$\Gamma$-module associated to the representation $V^r(1)$ is $\tilde{M} := \text{Hom}(M, \Omega^1_{c,n})$, where $M = D(V)$. By composing the residue with the trace we obtain a surjective and continuous map $\text{Tr}_n$ from $M$ to $\mathbb{Z}/p^n$. For every $f \in \tilde{M}$, the map $\text{Tr}_n \circ f$ is an element of the group $M^\vee$ of continuous group homomorphisms from $M$ to $\mathbb{Z}/p^n$. This gives in fact a group isomorphism from $\tilde{M}$ to $M^\vee$ and we can therefore transfer the $\Phi$-$\Gamma$-module structure from $\tilde{M}$ to $M^\vee$. But, since $k$ is finite, $M$ is locally compact and $M^\vee$ is in fact the Pontryagin dual of $M$.

c) Pontryagin duality implies local duality:

We simply dualize the complex $C_2(M)$ using Pontryagin duality (all arrows are strict morphisms in the category of topological groups) and obtain a complex:

$$C_2(M)^\vee : \quad 0 \to M^\vee \xrightarrow{\beta^\vee} M^\vee \oplus M^\vee \xrightarrow{\alpha^\vee} M^\vee \to 0,$$

where the two $M^\vee$ are in degrees 0 and 2. Since we can construct an explicit quasi-isomorphism between $C_2(M^\vee)$ and $C_2(M)^\vee$, we easily obtain a duality between $\mathcal{H}^i(M)$ and $\mathcal{H}^{2-i}(M^\vee)$ for every $i \in \{0, 1, 2\}$.

d) The canonical isomorphism from $\mathcal{H}^2(\Omega^1_{c,n})$ to $\mathbb{Z}/p^n$:

The map $\text{Tr}_n$ from $\Omega^1_{c,n}$ to $\mathbb{Z}/p^n$ factors through the group $\mathcal{H}^2(\Omega^1_{c,n})$ and this gives an isomorphism. But it is not canonical! In fact the construction of the complex $C_2(M)$ depends on the choice of $\gamma$. Fortunately, if we take another $\gamma$, we get a quasi-isomorphic complex and if we normalize the map $\text{Tr}_n$ by multiplying it by the unit $\gamma = p^{v_p(\log \chi(\gamma))}/\log \chi(\gamma)$ of $\mathbb{Z}_p$, where $\gamma$ is the $p$-adic logarithm, $\chi$ the cyclotomic character and $v_p = v_{Q_p}$, then everything is compatible with the change of $\gamma$.

e) The duality is given by the cup product:

We can construct explicit formulas for the cup product:

$$\mathcal{H}^i(M) \times \mathcal{H}^{2-i}(M^\vee) \xrightarrow{\cup} \mathcal{H}^2(\Omega^1_{c,n})$$
associated with the cohomological functor $(\mathcal{H}^i)_{i \in \mathbb{N}}$ and we compose them with the preceding normalized isomorphism from $\mathcal{H}^2(\Omega^1_{C,n})$ to $\mathbb{Z}/p^n$. Since everything is explicit, we can compare with the pairing obtained in c) and verify that it is the same up to a unit of $\mathbb{Z}_p$.

**Remark.** Benois, using the previous theorem, deduced an explicit formula of Coleman’s type for the Hilbert symbol and proved Perrin-Riou’s formula for crystalline representations ([B]).

### 6.6. Explicit formulas for the generalized Hilbert symbol on formal groups

Let $K_0$ be the fraction field of the ring $W_0$ of Witt vectors with coefficients in a finite field of characteristic $p > 2$ and $\mathcal{F}$ a commutative formal group of finite height $h$ defined over $W_0$.

For every integer $n \geq 1$, denote by $\mathcal{F}[p^n]$ the $p^n$-torsion points in $\mathcal{F}(\mathcal{M}_C)$, where $\mathcal{M}_C$ is the maximal ideal of the completion $C$ of an algebraic closure of $K_0$. The group $\mathcal{F}[p^n]$ is isomorphic to $(\mathbb{Z}/p^n \mathbb{Z})^h$.

Let $K$ be a finite extension of $K_0$ contained in $K^{\text{sep}}$ and assume that the points of $\mathcal{F}[p^n]$ are defined over $K$. We then have a bilinear pairing:

$$(\ , )_{\mathcal{F},n} : G^{\text{ab}}_K \times \mathcal{F}(\mathcal{M}_K) \to \mathcal{F}[p^n]$$

(see section 8 of Part I).

When the field $K$ contains a primitive $p^n$th root of unity $\zeta_{p^n}$, Abrashkin gives an explicit description for this pairing generalizing the classical Brückner–Vostokov formula for the Hilbert symbol ([A]). In his paper he notices that the formula makes sense even if $K$ does not contain $\zeta_{p^n}$ and he asks whether it holds without this assumption. In a recent unpublished work, Benois proves that this is true.

Suppose for simplicity that $K$ contains only $\zeta_p$. Abrashkin considers in his paper the extension $\widetilde{K} := K(\pi^{p^{-r}}, r \geq 1)$, where $\pi$ is a fixed prime element of $K$. It is not a Galois extension of $K$ but is arithmetically profinite, so by [W] one can consider the field of norms for it. In order not to loose information given by the roots of unity of order a power of $p$, Benois uses the composite Galois extension $L := K^{\infty} \widetilde{K}/K$ which is arithmetically profinite. There are several problems with the field of norms $N(L/K)$, especially it is not clear that one can lift it in characteristic 0 with its Galois action. So, Benois simply considers the completion $F$ of the $p$-radical closure of $E = N(L/K)$ and its separable closure $F^{\text{sep}}$ in $R$. If we apply what was explained in subsection 6.2 for $\Gamma = \text{Gal}(L/K)$, we get:

**Theorem 7.** The functor $V \to D(V) := (W(F^{\text{sep}}) \otimes_{\mathbb{Z}_p} V)^{G_L}$ defines an equivalence of the categories $\text{Rep}_{\mathbb{Z}_p}(G)$ and $\Phi \Gamma M_{W(F)}^{\text{et}}$. 

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Choose a topological generator $\gamma'$ of $\text{Gal}(L/K_{\infty})$ and lift $\gamma$ to an element of $\text{Gal}(L/\tilde{K})$. Then $\Gamma$ is topologically generated by $\gamma$ and $\gamma'$, with the relation $\gamma \gamma' = (\gamma')^a \gamma$, where $a = \chi(\gamma)$ ($\chi$ is the cyclotomic character). For $(M, \varphi) \in \Phi \Gamma M_{\text{ét}}^\text{ét}(F)$, the continuous action of $\text{Gal}(L/K_{\infty})$ on $M$ makes it a module over the Iwasawa algebra $\mathbb{Z}_p[[\gamma' - 1]]$. So we can define the following complex of abelian groups:

$C_3(M) : 0 \to M_0 \xrightarrow{\alpha_{A_0}} M_1 \xrightarrow{\alpha_{A_1}} M_2 \xrightarrow{\alpha_{A_2}} M_3 \to 0$

where $M_0$ is in degree 0, $M_0 = M_3 = M$, $M_1 = M_2 = M^3$,

$A_0 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \gamma' - 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \gamma' - 1 & 0 & 1 - \varphi \\ 0 & \gamma'^a - 1 & \delta - \gamma \end{pmatrix}$, $A_2 = ((\gamma')^a - 1 \delta - \gamma \varphi - 1)$

and $\delta = ((\gamma')^a - 1)(\gamma' - 1)^{-1} \in \mathbb{Z}_p[[\gamma' - 1]]$.

As usual, by taking the cohomology of this complex, one defines a cohomological functor $(\mathcal{H}^i)_i \in \mathbb{N}$ from $\Phi \Gamma M_{\text{ét}}^\text{ét}(F)$ in the category of abelian groups. Benois proves only that the cohomology of a $p$-torsion representation $V$ of $G$ injects in the groups $\mathcal{H}^i(D(V))$ which is enough to get the explicit formula. But in fact a stronger statement is true:

**Theorem 8.** The cohomological functor $(\mathcal{H}^i \circ D)_i \in \mathbb{N}$ can be identified with the Galois cohomology functor $(H^i(G,))_i \in \mathbb{N}$ for the category $\text{Rep}_{p\text{-tor}}(G)$.

**Idea of the proof.** Use the same method as in the proof of Theorem 4. It is only more technically complicated because of the structure of $\Gamma$. 

Finally, one can explicitly construct the cup products in terms of the groups $\mathcal{H}^i$ and, as in [B], Benois uses them to calculate the Hilbert symbol.

**Remark.** Analogous constructions (equivalence of category, explicit construction of the cohomology by a complex) seem to work for higher dimensional local fields. In particular, in the two-dimensional case, the formalism is similar to that of this paragraph; the group $\Gamma$ acting on the $\Phi \Gamma$-modules has the same structure as here and thus the complex is of the same form. This work is still in progress.

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