Exterior Dirichlet and Neumann Problems and the Linked Ergodic Inverse Problems in the Entire Space

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Abstract
The paper is mainly concerned with the interconnection of the boundary behaviour of the solutions of the exterior Dirichlet and Neumann problems of harmonic analysis for the unit disk in $\mathbb{R}^2$ and the unit ball in $\mathbb{R}^3$ with the corresponding behaviour of the associated ergodic inverse problems for the entire space. The basis is the theory of semigroups of linear operators mapping a Banach space $X$ into itself. The classical one-parameter theory for semigroups applies in the present particular applications, actually for $X = L^2_{2\pi}$ in case of the unit disk, and $X = L^2(S)$ in the three dimensional setting, $S$ being the unit sphere in $\mathbb{R}^3$. Another tool is a Drazin-like inverse operator $B$ for the infinitesimal generator $A$ of a semigroup that arises naturally in ergodic theory. This operator $B$ is a closed, not necessarily bounded, operator. It was introduced in a paper with Butzer and Westphal (Indiana Univ Math J 20:1163–1174, 1970/1971) and extended to a generalized setting with Butzer and Koliha (J Oper Theory 62:297–326, 2009).

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In memory of Einar Hille and Karl Butzer, both of whom inspired this thread of research.

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A short biography of Einar is to be found in Sect. 11 and as to Karl Butzer (1934–2016), Raymond Dickson Centennial Professor of Liberal Arts, University of Texas, Austin, the founder of Geoarchaeology, let us just refer to several obituary addresses in his honour [24, 30, 53, 84, 95].

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1 Introduction

The paper is organized as follows. Section 2 contains preliminary results on semigroups of operators $\mathcal{T}_A = \{ T_A(t) ; t \geq 0 \}$ and its infinitesimal generator $A$. Section 3 deals with their approximation behaviour for $t \to 0^+$ and the corresponding rates of convergence and their sharpness. Section 4 is devoted to their ergodic behaviour for $t \to \infty$ including best possible rates.

In Sect. 5 we present the core results connecting the semigroup $\mathcal{T}_A$ with the semigroup $\mathcal{T}_B$ generated by a closed operator $B$, a generalized inverse of $A$. In fact, $B$ is the a-Drazin inverse, introduced by Butzer and Westphal in [8].

In the following Sect. 6 these results are used to connect the approximation behaviour of $T_A(t)f$ towards $f$ for $t \to 0^+$, with the ergodic behaviour of $T_B(t)f$ towards a projection $Pf$ for $t \to \infty$. Corresponding results are deduced for the associated resolvent operators. Emphasis is again placed upon best possible rates of convergence. Theorem 6.1, the so-called interconnection theorem, is the most important result in regard to all applications.

In Sect. 7 our theoretical investigations are applied to the solution of the exterior Dirichlet problem for the unit disk and Sect. 8 concerns the analogous questions for the solution of the exterior Dirichlet problem for the unit ball in $\mathbb{R}^3$, extensive use of expansions in terms of spherical harmonics being made.\(^1\)

Section 9 deals with the exterior Neumann problem in $\mathbb{R}^2$ and $\mathbb{R}^3$. The main difference to the Dirichlet problems above is the fact that one now does not investigate the behaviour of the solution of the differential equation but the behaviour of the derivative of the solution. Nevertheless, our semigroup approach can be applied as well. The results for the Neumann problem seem to be completely new.

In Sect. 10, two further examples are sketched, and possible extensions from Hilbert to Banach spaces as well as to cosine operator functions are discussed.

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\(^1\) The outcome of this three-dimensional example could be regarded as a first attempt to solve a problem in astrophysics (see also the remarks at the end of Sect. 8 in this respect). In a colloquium talk at the RWTH Aachen in 1964/65, Prof. Rudolph Kurth\(^2\) spoke of some speculations/unsolved problems in the broad realm of the astrophysics he worked in. As far as I (PLB) recall, it was roughly concerned with the question whether there exist mathematical equations of astrophysics in entire space the behaviour of which is somehow connected with that of certain equations on Earth. Since my brother Karl had received his masters degree in meteorology under Kenneth Hare (1919–2002) at McGill University (in 1955) and his bachelor thesis under H. Zassenhaus (1912–1991) there dealt with non-Euclidean geometry, I often discussed Kurth’s problem with him and he regularly urged me to tackle it. Regarding my astronomical-historical interests, one may consult [51]. This “Biographical Encyclopedia of Astronomers”, edited by T. Hockey et al., is a collective work of about 430 authors and provides bibliographical information on c. 1700 astronomers from antiquity to modern times. I have contributed five biographical sketches of Alcuin of York, Dungal of Saint Denis (both co-authored with Walter Oberschelp and Kerstin Springsfeld), John of Muris[Murs], Wilhelm Schickard and Eduard Heis. In this respect see also [21]

\(^2\) Rudolph F. Kurth, born 07.12.1917 in Berlin, studied mathematics, physics, astronomy and philosophy from 1938–39 at Berlin, Heidelberg and Bern, Doctorate in stellar dynamics in 1948, habilitation in 1951. He pursued an academic research career in four continents, in astronomy up to 1960, thereafter as professor emeritus of mathematics, Southern Illinois University, Edwardsville, in 1981. He was a prolific writer in books on epistemology, philosophy, mathematics, physics and astronomy, e. g. [57–59]. The authors would like to thank Alfred Gautschy, ETH Zürich, for information on R. Kurth, including the titlepage of his dissertation and his own vita of 1948.
Section 11 contains a short biography of Einar Hille.

2 Preliminary Results

Let \( T = \{ T(t); t \geq 0 \} \) be a \( C_0 \)-semigroup of operators on a (complex) Banach space \( X \) (with norm \( \| \cdot \|_X \) ), that is a family of bounded, linear operators mapping \( X \) into itself, in notation \( T \subset [X] \), satisfying the semigroup property \( T(s + t) = T(s)T(t) \) for all real \( t, s \geq 0 \) with \( T(0) = I \) (=identity operator on \( X \)) together with the \( (C_0) \)-property

\[
\lim_{t \to 0^+} \| T(t)f - f \|_X = 0 \quad (f \in X). \tag{2.1}
\]

The (infinitesimal) generator \( A \) of \( T \) is defined by the strong limit

\[
Af = T'(0)f = \text{s-lim}_{t \to 0^+}^{-1}[T(t)f - f]
\]

on \( \mathcal{D}(A) \), the domain of \( A \), the set of all \( f \in X \) for which this limit exits. The operator \( A \) is closed, densely defined, \( \mathcal{D}(A) = X \). We often use the notation \( T_A \) in order to indicate that \( T \) is a semigroup with generator \( A \). For the theory of semigroups see, e.g., the textbooks [6, 23, 25, 35, 43, 48, 50, 67, 98].

A semigroup \( T \) is said to be holomorphic if it has a holomorphic extension \( \{ T(z); z \in \Sigma_\vartheta \} \), where the sector \( \Sigma_\vartheta \) with angle \( \vartheta \) is given by \( \Sigma_\vartheta := \{ z \in \mathbb{C}; | \arg z | < \vartheta \} \), \( \vartheta \leq \pi/2 \), i.e., \( T: \Sigma_\vartheta \to [X] \) is a holomorphic function satisfying \( T(z_1 + z_2) = T(z_1)T(z_2) \) in \( \Sigma_\vartheta \), with s-lim \( z \to 0, z \in \Sigma_{\vartheta - \varepsilon} T(z)f = f \) for all \( f \in X, \varepsilon \in (0, \vartheta) \); see, e.g. [43, p. 33].

Its generator is equal to the generator of the semigroup \( \{ T(t); t \geq 0 \} \). A holomorphic semigroup is said to be equibounded of type \( \vartheta \in (0, \pi/2) \) if for any \( \varepsilon \in (0, \vartheta) \) there exists an \( M_\varepsilon < \infty \) such that

\[
\| T(z) \|_X \leq M_\varepsilon, \quad \| zT'(z) \|_X \leq M_\varepsilon \quad (z \in \Sigma_{\vartheta - \varepsilon}). \tag{2.2}
\]

The set of all equibounded and holomorphic semigroups of arbitrary type is denoted by \( \mathcal{H} \). For further details see [41].

It is clear that a holomorphic semigroup is uniformly bounded and belongs to the class \( (C_0) \). The converse question, whether a uniformly bounded \( (C_0) \)-semigroup possesses a holomorphic extension is answered by (see [37, p. 177]): If \( T_A \) is a uniformly bounded \( (C_0) \)-semigroup on a complex Banach space \( X \) such that

\[
T_A(t)f \in D(A), \quad \| tT_A'(t) \|_X \leq N \quad (f \in X; t > 0),
\]

then \( T_A \) can be extended to an equibounded holomorphic semigroup on \( \Sigma_\vartheta \) where the angle \( \vartheta \) is given by \( \vartheta = \arcsin(\min\{1/(eN), 1\}) \).

Further, \( A \) is the generator of an equibounded holomorphic semigroup of type \( \vartheta \in (0, \pi/2) \) if and only if for any \( \varepsilon \in (0, \vartheta) \) there exists an \( M_\varepsilon < \infty \) such that
$\Sigma_{\theta + \pi/2 - \varepsilon} \subset \rho(A)$, the resolvent set of $A$, with
\[ \| \lambda R(\lambda; A) \|_X \leq M_{\varepsilon} \quad (\lambda \in \Sigma_{\theta + \pi/2 - \varepsilon}). \]  

(2.3)

For the definition of the resolvent set $\rho(A)$ and the resolvent $R(\lambda; A)$ see, e.g., [6, p. 30].

Apart from the limit of $T(t)f$ towards $f$, given by (2.1), this paper will also deal with the Cesàro means of order $\gamma > 0$ or $(C, \gamma)$-means
\[ H^\gamma_{TA}(t)f := \frac{\gamma}{t^\gamma} \int_0^t (t-u)^{\gamma-1} T(u)f \, du \quad (f \in X; \ t > 0) \]  

(2.4)

with $H_0^\gamma(t):=T(t)$, and the strong limit $s\text{-}\lim_{t \to 0^+} H^\gamma_{TA}(t)f$.

In addition, we treat the strong limit of the Abel means of $T_A$, thus the Laplace transform of $T_A(t)$, namely
\[ \lambda R(\lambda, A)f = \lambda \int_0^\infty e^{-\lambda u} T_A(u)f \, du \quad (f \in X; \Re\lambda > \omega) \]  

(2.5)

for $\lambda \to \infty$. Here $\omega$ is a number such that $\| T_A(t) \|_X \leq Me^{\omega t}$ hold for all $t \geq 0$ and $M$ independent of $t$ and $\omega$.

Recall that the convergence assertion (2.1) implies the convergence of the Cesàro and Abel means to the same limit, i.e., $s\text{-}\lim_{t \to 0^+} H^\gamma_{TA}(t)f = \lim_{\lambda \to \infty} \lambda R(\lambda, A)f = f$ in $X$-norm; see [50, p. 321], [32, vol. I, p. 687].

Now, let $\mathcal{N}(A) := \{ f \in \mathcal{D}(A); Af = 0 \}$ and $\mathcal{R}(A)$ be the kernel and the range of $A$, respectively, then $\mathcal{R}(A) \cap \mathcal{N}(A) = \{ 0 \}$, and the direct sum $X_0 := \mathcal{R}(A) \oplus \mathcal{N}(A)$ is well defined. Further, let $P$ be the bounded linear projection of $X_0$ onto $\mathcal{N}(A)$ parallel to $\mathcal{R}(A)$, i.e., $P^2 = P$ and $APf = 0$. Note that $X_0$ is a closed subspace of $X$, with $X_0 = X$ if $X$ is reflexive. See [17, 18] in this respect.

If $A_0 := A|_{X_0}$ with $\mathcal{D}(A_0) = \mathcal{D}(A) \cap X_0$, then the operator $B$ is defined by
\[
B : \mathcal{R}(A_0) \oplus \mathcal{N}(A) \to \mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{D}(A_0) \cap \mathcal{N}(P),
\]

\[
f := Ag \mapsto g, \quad P g = 0.
\]

\[
ABf = f - Pf \quad (f \in \mathcal{D}(B)), \quad BAf = f - Pf \quad (f \in \mathcal{D}(A_0)).
\]  

(2.6)

This can be summarized in a more compact form as
\[ (B - P)(A - P) = (A - P)(B - P) = BA + P = AB + P = I. \]  

(2.7)

Thus, $ABf = f$ for all $f \in \mathcal{D}(B) \cap \mathcal{N}(P) = \mathcal{R}(A_0)$, and $BAf = f$ for all $f \in \mathcal{D}(A_0) \cap \mathcal{N}(P) = \mathcal{R}(B)$. In other words $B = A^{-1}$ on $\mathcal{R}(A_0) \cap \mathcal{D}(A_0)$, and conversely; see [16, 17]. This operator $B$ is precisely the $a$-Drazin inverse operator $A^{ad}$ in the notation of Butzer–Koliha [18]; see [12, 16, 18, 31, 89].

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3 In PLB’s plenary lecture at the international conference on operator theory, Timișoara, Romania, June 26–July 1, 2000, the inverse operator $B$ employed there was that first introduced in Aachen 1971 in association.
Under suitable assumptions upon the semigroup $T_A$, the operator $B$ also generates a semigroup $T_B$. A chief aim is to treat the basic connection between the approximation behaviour of the semigroup $T_A(t)f$ for $t \to 0+$ with the ergodic behaviour of the Cesàro means $H^\gamma_{T_B}(t)f$ for $t \to \infty$. It will turn out that

$$\lim_{t \to \infty} H^\gamma_{T_B}(t)f = Pf \quad (f \in X_0; 1 \leq \gamma < \infty).$$

3 Approximation Behaviour for $t \to 0+$

First to the rate of convergence for $T_A(t)f \to f$ for $t \to 0+$ and the associated convergence of the Cesàro and Abel means.

**Theorem 3.1** Let $T_A$ be a $(C_0)$-semigroup on a Banach space $X$ having generator $A$.

(a) The following four assertions are equivalent for any given $f \in X$ and any $\gamma > 0$:

(i) $\|T_A(t)f - f\|_X = o(t) \quad (t \to 0+),$
(ii) $\|H^\gamma_{T_A}(t)f - f\|_X = o(t) \quad (t \to 0+),$
(iii) $\|\lambda R(\lambda, A)f - f\|_X = o(\lambda^{-1}) \quad (\lambda \to \infty),$
(iv) $f \in \mathcal{N}(A), \text{i.e., } T_A(t)f = f \quad (t \geq 0).$

(b) The following assertions are equivalent for $0 < \alpha \leq 1$, any $f \in X$ and any $\gamma > 0$:

(i) $\|T_A(t)f - f\|_X = \mathcal{O}(t^\alpha) \quad (t \to 0+),$
(ii) $\|H^\gamma_{T_A}(t)f - f\|_X = \mathcal{O}(t^\alpha) \quad (t \to 0+),$
(iii) $\|\lambda R(\lambda, A)f - f\|_X = \mathcal{O}(\lambda^{-\alpha}) \quad (\lambda \to \infty),$
(iv) $K(t; f; X, \mathcal{D}(A)) = \mathcal{O}(t^\alpha) \quad (t \to 0+).$

(c) If, in particular, $\alpha = 1$, then assertions (i)–(iv) of Part (b) are also equivalent to

(v) $f \in \mathcal{D}(A)^{X}$.
(vi) $f \in \mathcal{D}(A)$, if $X$ is additionally reflexive.

As to the sharpness of the rates of the processes, one has

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**Footnote 3 continued**

with a sequence $(T_n)_{n \in \mathbb{N}} \subset [X]$; see [11]. In the instance of semigroup operators $(T(t))_{t \geq 0} \subset [X]$ with a continuous parameter $t$, it was studied in [8, 14] Here $B$ turns out to be a generalized inverse of the infinitesimal generator $A$, which arises naturally in ergodic theory. The essential point is that it is in general a closed operator, but not necessarily a bounded one.

In the discussion following this lecture, Jerry Kolihia, Melbourne, Australia, raised the question whether the speaker was familiar with the extension of an inverse introduced by M. P. Drazin [31] in 1958 in ring theory to the instance of closed but bounded linear operators by Nashed and Zhao [64], González and Kolihia [46, 47], and Kolihia and Tran [54]. The speaker, however, was not aware of this “restricted” inverse operator. In fact, Butzer and Kolihia wrote a joint paper in 2009 [18] in which they treated the role of this inverse in the instance of semigroups and cosine operators in great detail; the Drazin inverse being denoted by $A^d$ and the Butzer–Westphal inverse by $A^{ad}$ (so coined by Jerry), ‘a’ referring to the Aachen school. This concept has since also been employed later by other researchers in the field, notably by Lin [61, 62], and Shaw and his collaborators [86–90], has turned out to be especially applicable in operator semigroups and cosine operator functions in connection with PDE’s. The operator $B$ in the present paper is precisely this inverse operator $A^{ad}$. 
(d) If the operator $A$ is unbounded, then for each $\alpha \in (0, 1]$ and any $\gamma > 0$ there exist elements $f_\alpha, f^*_\alpha, f^{**}_\alpha \in X$, such that

(i) $\| T_A(t) f_\alpha - f_\alpha \|_X = O(t^\alpha) \neq o(t^\alpha) \ (t \to 0+),$

(ii) $\| H_T^\gamma (t) f^*_\alpha - f^*_\alpha \|_X = O(t^\alpha) \neq o(t^\alpha) \ (t \to 0+),$

(iii) $\| \lambda R(\lambda, A) f^{**}_\alpha - f^{**}_\alpha \|_X = O(\lambda^{-\alpha}) \neq o(\lambda^{-\alpha}) \ (\lambda \to \infty).$

(e) If $\alpha = 1$, part (d) is valid even if $A$ is bounded but not the null operator.

Regarding the proofs see [16] and the literature cited there.

Above, $K(t, f) = K(t, f; X, U) := \inf_{g \in U} \{ \| f - g \|_X + t|g|_U \}$ is the $K$-functional of $f \in X$, defined for all $t \geq 0$, where $U \subset X$ is a submanifold with seminorm $|\cdot|_U$. It is a measure of smoothness in Banach spaces, covering the $r$th modulus of continuity in concrete spaces. There holds: $K(t, f)$ is a bounded, continuous, monotone increasing and subadditive functional of $t$ for each $f \in X$, with $\lim_{t \to 0+} K(t, f) = 0$ if and only if $f \in \overline{U}^X$. Further, if $Y \subset X$ is a Banach space with $\| f \|_X \leq \| f \|_Y$ for all $f \in Y$, then

$\tilde{Y}^X := \{ f \in X; \text{there exist } \{ f_n \}_{n \in \mathbb{N}} \subset Y \text{ with } \| f_n \|_Y \leq M < \infty, \lim_{n \to \infty} f_n = f \}$

is the completion of $Y$ relative to $X$. If $X$ is reflexive, then $\tilde{Y}^X = X$. See [1, 40] and [4].

The equivalence a) (i)$\Leftrightarrow$(ii) is the small-$o$ part of a saturation theorem for the approximation process $T_A(t)$. Likewise, (b) (i) for $\alpha = 1 \Leftrightarrow$ (c) (v) or (vi) is the corresponding large-$O$ part, stating that $\tilde{D}(A)^X$, respectively $D(A)$, ist the so-called saturation or Favard class. For a general, functional calculus approach to approximation of $C_0$-semigroups on Banach spaces by bounded completely monotone functions of their generators see [45].

4 While still stationed at McGill University Montreal, I (PLB) took interest in a small-$o$ approximation theorem for the Abel and Fejér means, which I found in Einar Hille’s famous book on semigroup theory (see [48, p. 352 ff.]). Later it turned out to be an essential part of saturation theory, a field which was installed by Jean Favard in 1955. During the preparation of the first course on semigroups of linear operators at Aachen for 1962/63 (see [6]), I first studied the role of Huygens’ principle, the basis of semigroups, which stands in close connection with Hille’s formulation of the Abstract Cauchy Problem and the semigroup solution of Dirichlet’s problem for the unit disk.
4 Ergodic Behaviour for $t \to \infty$

We need some further preliminary results.

**Lemma 4.1** Let $\mathcal{T}_A = \{T_A(t); t \geq 0\}$ be semigroup on $X$. For the classes $\mathcal{T}_M^\gamma$, given by

\[
\mathcal{T}_M^0 := \{T_A; \|H^\gamma_{T_A}(t)\| \leq M \} (0 \leq \gamma < \infty), \\
\mathcal{T}_M^\infty := \{T_A; (0, \infty) \subset \rho(A), \|\lambda R(\cdot; A)\| \leq M \text{ for all } \lambda > 0\}
\]

there holds

\[
\mathcal{T}_M^0 \subset \mathcal{T}_M^\delta \subset \mathcal{T}_M^\gamma \subset \mathcal{T}_M^\infty (0 \leq \delta \leq \gamma < \infty).
\]

The first inclusion is clear. For the second and third one see [41, p. 25].

**Proposition 4.2** For $H^\gamma_{T_A}(t)f$, $\gamma \geq 0$ defined by (2.4) there holds: If

\[
\lim_{t \to \infty} [H^\gamma_{T_A}(t)f - Pf]
\]

exists, then $f \in D(B)$ and the previous limit equals $Bf$.

**Proposition 4.3** For $R(\cdot; A)$ represented by (2.5) there holds: If

\[
\lim_{\lambda \to 0^+} [R(\cdot; A)f - \lambda^{-1} Pf]
\]

exists, then $f \in D(B)$ and this limit equals $-Bf$.

Conversely, one has for $f \in X_0$ that this strong limit exists, if and only if $f \in D(B)$. In this event the limit is $-Bf$.

For the proofs see [13].

Now to the basic mean ergodic convergence theorem for the Cesàro and Abel means of $T(t)f$.

**Proposition 4.4** Let $\mathcal{T}_A = \{T_A(t); t \geq 0\} \in \mathcal{T}_M^\gamma$ for $\gamma \geq 1$ acting on a Banach space $X$, with $P$ and $X_0$ as given above. The following assertions are equivalent for any $f \in X$ and any $\gamma \geq 1$:

(i) $s-lim_{t \to \infty} H^\gamma_{T_A}(t)f = g_1$,  
(ii) $s-lim_{\lambda \to 0^+} \lambda R(\cdot; A)f = s-lim_{\lambda \to 0^+} \int_0^\infty e^{-\lambda u}T_A(u)f \, du = g_2$,  
(iii) $f \in X_0$.

In both instances the strong limits are equal, with $g_1 = g_2 = Pf$. If $X$ is reflexive, then $X_0 = X$, and all convergence assertions are valid on all of $X$.

Proposition 4.4 in case of the Cesàro ($C, 1$) means (thus $\gamma = 1$) and Abel means, it was first established for equibounded sequences of linear operators (see e.g. [50,
Chapter XVII] and [14], also [13]) and later for the class \( T^1 \) by Dunford–Schwartz [32, vol. I, p. 687] and [15]. For the more general case \( \gamma \geq 1 \) see, e. g., [41, p. 29].

Next we state the mean ergodic theorem with rates for \( T_A \).

**Theorem 4.5** Let \( T_A \in T^\gamma \) for some \( \gamma \geq 0 \).

(a) The following assertions are equivalent for \( f \in X_0 \) and \( \alpha \in (0, 1] \):

(i)

\[
\| H_{T_A}^{\gamma+1} (t) f - Pf \|_X = \begin{cases} 
  o(t^{-1}) & (t \to \infty), \\
  \mathcal{O}(t^{-\alpha}) & \text{else}.
\end{cases}
\]

(ii)

\[
\| \lambda R(\lambda; A) f - Pf \|_X = \begin{cases} 
  o(\lambda) & (\lambda \to 0+), \\
  \mathcal{O}(\lambda^{-\alpha}) & \text{else},
\end{cases}
\]

(iii)

\[
K(t^{-1}, f; X, \mathcal{D}(B)) = \begin{cases} 
  o(t^{-1}) & (t \to \infty), \\
  \mathcal{O}(t^{-\alpha}) & \text{else}.
\end{cases}
\]

(iv)

\[
\begin{cases} 
  f \in \mathcal{N}(B) = \mathcal{N}(A), \ i.\ e., \ Pf = f & \text{if } \alpha = 1, \\
  f \in \mathcal{D}(B) \setminus X_0 & \text{if } \alpha > 0,
\end{cases}
\]

(b) If \( B \) is unbounded, then for each \( \alpha \in (0, 1] \) and any \( \gamma \geq 0 \) there exist elements \( f_\alpha, f_\alpha^* \in X_0 \), such that

(i)

\[
\| H_{T_A}^{\gamma+1} (t) f_\alpha - Pf_\alpha \|_X = \begin{cases} 
  \mathcal{O}(t^{-\alpha}) & (t \to \infty), \\
  \not\in o(t^{-\alpha}) & \text{else},
\end{cases}
\]

(ii)

\[
\| \lambda R(\lambda; A) f_\alpha^* - Pf_\alpha^* \|_X = \begin{cases} 
  \mathcal{O}(\lambda^{-\alpha}) & (\lambda \to 0+), \\
  \not\in o(\lambda^{-\alpha}) & \text{else}.
\end{cases}
\]

(c) If \( \alpha = 1 \), part (b) is valid even if \( B \) is bounded but not the null operator.

Concerning any (mean) ergodic theorem (with rates), the first paper ever to deal with the problem of rates is Butzer–Westphal [8] 1970/71, which treats sequences \( (T_n)_{n \in \mathbb{N}} \subset X \) with \( \| T_n \| \leq M \); see Krengel [56, p. 84]. Theorem 4.5 part (a)(i) for
\( \gamma = 0 \) and rate \( O(t^{-1}) \), i.e. \( \alpha = 1 \), is due to Goldstein et al. [44]. Theorem 4.5 part (a) in case \( \gamma = 0 \) and arbitrary \( \alpha > 0 \) was proved by Butzer–Dickmeis [14] (with the exception of (ii)) and Butzer–Gessinger [16]; for arbitrary \( \gamma \geq 0 \) see [41, p. 30]. Parts b) and c) on the sharpness of saturated and non-saturated approximation of the processes \( H_T \) and \( \lambda R(\lambda; A) \) are to be found in [13], see also [65]. These results were established via a general theorem of A. V. Davidov [26, 27], who solved a conjecture of Butzer–Dickmeis [15] on the sharpness of non-saturated approximation of semigroup operators using deep results on the uniform boundedness principle with rates due to Dickmeis–Nessel–van Wickeren [29].

5 Core Results Connecting the Semigroups \( T_A \) and \( T_B \) and Their Resolvent Operators

Observing the striking similarity between the characterisation of the approximation assertions of the resolvent in terms of the \( K \)-functional given by Theorem 3.1, and the ergodic behaviour of the resolvent given by Theorem 4.5, namely,

\[
K(t, f; X, \mathcal{D}(A)) = O(t^\alpha) \quad (t \to 0+) \iff \|\lambda R(\lambda; A) f - f\|_X = O(\lambda^{-\alpha}) \\
\lambda \to \infty,
\]

\[
K(t^{-1}, f; X, \mathcal{D}(B)) = O(t^{-\alpha}) \quad (t \to \infty) \iff \|\lambda R(\lambda; A) f - P f\|_X = O(\lambda^\alpha) \\
\lambda \to 0+,
\]

the question arises whether the roles of \( A \) and \( B \) can be exchanged, in the sense that there hold analogous assertions of the form

\[
K(t, f; X, \mathcal{D}(B)) = O(t^\alpha) \quad (t \to 0+) \iff \|\lambda R(\lambda; B) f - f\|_X = O(\lambda^{-\alpha}) \\
\lambda \to \infty,
\]

\[
K(t^{-1}, f; X, \mathcal{D}(A)) = O(t^{-\alpha}) \quad (t \to \infty) \iff \|\lambda R(\lambda; B) f - P f\|_X = O(\lambda^\alpha) \\
\lambda \to 0+.
\]

(5.1)

Of course, one has to assure that \( R(\lambda; B) \) exists. If \( 0 \in \rho(A) \), the latter equivalences are trivially true. For then \( B = A^{-1} \in [X] \), so that \( B \) generates again a \((C_0)\)-semigroup. Thus we can restrict ourselves to the case \( 0 \notin \rho(A) \).

The basic bond between the resolvents of the operators \( A \) and \( B \) and their resolvent sets \( \rho(A) \) and \( \rho(B) \), respectively, as they occur above reads:

**Theorem 5.1** Let \( T_A \in T_M^\infty \) on a Banach space \( X \), and let \( P, B, X_0 \) be given as above. Then

(a) \( \rho(B) = [\lambda; \lambda^{-1} \in \rho(A)] \) holds together with

\[
\lambda R(\lambda; B) f = [P + I - \lambda^{-1} R(\lambda^{-1}; A)] f \quad (\lambda^{-1} \in \rho(A); \ f \in X_0), \quad (5.2)
\]
or
\[ \| \lambda R(\lambda; B) f - P f \|_{[X]} = \| \lambda^{-1} R(\lambda^{-1}; A) f - f \|_{[X]}, \] (5.3)

(b) For \( T_A \in \mathcal{T}_M^0 \) there holds moreover for \( n \in \mathbb{N}_0 \),
\[ \lambda^{n+1} [R(\lambda; B)]^{n+1} f = P f + I - \int_0^\infty \lambda^{-1} e^{-u/\lambda} L_n^{(1)}(\lambda^{-1} u) T_A(u) f \, du \] (5.4)
\[ = P f + I - \int_0^\infty e^{-u} L_n^{(1)}(u) T_A(\lambda u) f \, du \] (5.5)
where \( L_n^{(1)}(z) = \sum_{k=0}^n \binom{n+1}{n-k} (-z)^k / k! \), \( n \in \mathbb{N}_0 \), \( z \in \mathbb{C} \) is the Laguerre polynomial.

**Proof** Sketch of proof (For a detailed proof see [41, pp. 37 ff.])

(a) Let \( \lambda^{-1} \in \rho(A) \) and \( f \in X_0 \). Then
\[ \lambda^{-1} R(\lambda^{-1}; A) + P + I \rightarrow = -AR(\lambda^{-1}; A) f + P f \in \mathcal{R}(A|X_0) \oplus \mathcal{N}(A|X_0) = D(B). \]

This readily yields, by the definition of \( B \) and (2.7),
\[ (\lambda - B)[-\lambda^{-1} R(\lambda^{-1}; A) + P + I] f = \lambda f. \]

On the other hand, for \( f \in D(B) \),
\[ \lambda^{-1} R(\lambda^{-1}; A) + P + I \lambda^{n+1} R(\lambda^{-1}; A) f = \lambda f - \lambda[R[\lambda^{-1}; A] A P f] = \lambda f \]

Hence the operator on the right side of (5.2) is a right and left inverse of \( (\lambda - B) \), apart from the multiple \( \lambda \). This yields the identity of part (a). and, in particular, \( \rho(B) \supset \{ \lambda; \lambda^{-1} \in \rho(A) \} \). Interchanging the roles of \( A \) and \( B \), yields the equality of the sets.

(b) Observing (2.4) and part (a),
\[ \lambda R(\lambda; B) f = P f + f - \lambda^{-1} \int_0^\infty e^{-u/\lambda} T_A(u) f \, du \] (5.6)

This is the desired result for \( n = 0 \) since \( L_0^{(1)}(z) = 1 \). For \( n \geq 1 \), note that by induction, recalling the differential rule for Laguerre polynomials (see e.g. [91, (5.1.14)]),
\[ \frac{d^n}{\lambda^n} (\lambda^{-2} e^{-u/\lambda}) = \frac{(-1)^n n!} {\lambda^{n+2}} L_n^{(1)}(\lambda^{-1} u) \quad (\lambda \neq 0). \]
Collecting the results this readily yields (see [41, p. 38]),

\[
\lambda^{n+1}[R(\lambda; B)]^{n+1} f = (-1)^n \frac{\lambda^{n+1}}{n!} \left( \frac{d}{d\lambda} \right)^n R(\lambda; B) f
\]

\[
= (-1)^n \frac{\lambda^{n+1}}{n!} \left( \frac{d}{d\lambda} \right)^n \left\{ -\lambda^{-2} \int_0^\infty e^{-u/\lambda} T_A(u) f \, du + \lambda^{-1}(P + I) f \right\}
\]

\[
= -\int_0^\infty \lambda^{-1} e^{-u/\lambda} L_n^{(1)}(\lambda^{-1}u) T_A(u) f \, du + (P + I) f.
\]

The second equation of b) by the substitution $\lambda \mapsto \lambda u$. □

A first implication of this theorem are the assertions of (5.1).

Our major result concerning the interconnection between the approximation behaviour of the semigroup $T_A$ at zero and the ergodic behaviour of $T_B$ at infinity is given by

**Theorem 5.2** Let $T_A \in T_0^0 M$ be an equibounded semigroup of type $\theta$ on a complex Banach space $X$. Then the operator $B$ generates an equibounded holomorphic semigroup $T_B$ on $X_0$ of the same type $\theta$, given by

\[
T_B(t) f = f + Pf - \sqrt{t} \int_0^\infty J_1(2\sqrt{tu}) T_A(u) f \frac{du}{\sqrt{u}} \quad (f \in X_0) \quad (5.6)
\]

with $T_B(0) f := f$, where

\[
J_\alpha(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu! \Gamma(\nu + \alpha + 1)} \left( \frac{z}{2} \right)^{\alpha + 2\nu} \quad (z \in \mathbb{C})
\]

is the Bessel function of order $\alpha \in \mathbb{R}$.

**Remark 5.3** Notice that \( \sqrt{t} \int_0^\infty J_1(2\sqrt{tu}) u^{-1/2} \, du = \int_0^\infty J_1(u) \, du = 1 \), but $J_1 \notin L^1(0, \infty)$, so that the integral in (5.6) is not absolutely convergent. But since $T_A(t)$ is differentiable and $-J_1' = J_1$, partial integration for $1 \leq \beta_1 < \beta_2 < \infty$, $t > 0$ gives

\[
\sqrt{t} \int_{\beta_1}^{\beta_2} J_1(2\sqrt{tu}) T_A(u) f \frac{du}{\sqrt{u}} = -J_0(2\sqrt{tu}) T_A(u) f \bigg|_{u=\beta_2}^{\beta_1} + \int_{\beta_1}^{\beta_2} J_0(2\sqrt{tu}) T_A'(u) f \, du.
\]

Now both terms on the right vanish in the norm as $\beta_1, \beta_2 \to \infty$, since $|J_\nu(x)| = \mathcal{O}(x^{-1/2})$ for $x \to \infty$ and by (2.2).

Further, (5.6) may be interpreted formally: In view of (5.4) we have

\[
\exp[-u(P - A)] = \exp(uA) + (e^{-u} - 1) P - T_A(u) + (e^{-u} - 1) P.
\]
Noting the counterpart for \( \exp[-u(P - A)] \), and a Laplace transform result needed, namely,

\[
\int_0^\infty \sqrt{tu}^{-1} J_1(2\sqrt{tu}) e^{-\lambda u} \, du = 1 - e^{-t/\lambda} \quad (k \neq 0; t > 0)
\]  \hspace{1cm} (5.7)

(see [36, (36) p. 245]), one has, with the formal setting \( \lambda = A - P \),

\[
\exp(tB) = I + P - \sqrt{t} \int_0^\infty J_1(2\sqrt{tu}) \exp(uA) \frac{du}{\sqrt{u}}.
\]

This is nothing but (5.6).

The latter two theorems lead to the so-called general interconnection theorem below, a principle achievement of Gessinger’s doctoral thesis [41, 42].

### 6 General Interconnection Theorem

Since we know that the operator \( B \) can also generate in the non-trivial case \( B \notin [X] \) a \((C_0)\)-semigroup we can interconnect Theorems 3.1 and 4.5 via (5.3) as follows

**Theorem 6.1** Let \( T_A = \{T_A(t); t \geq 0\} \in \mathcal{T}_M^\gamma \) for \( \gamma_1 \geq 0 \) on a Banach space \( X \), with \( B, P \) and \( X_0 \) being defined as above. If the operator \( B \) generates a \((C_0)\)-semigroup \( T_B = \{T_B(t); t \geq 0\} \) on \( X_0 \) (which is the case if in addition \( T_A \in \mathcal{H} \)) then one has for \( f \in X_0, \gamma_2 \geq 0 \):

(a) The following assertions are equivalent:

(i) \( \| H_{T_A}^{\gamma_1 + 1}(t) f - P f \|_X = o(t^{-1}) \quad (t \to \infty) \),

(ii) \( \| H_{T_B}^{\gamma_2}(\tau) f - f \|_X = o(\tau) \quad (\tau \to 0+) \),

(iii) \( \| \lambda R(\lambda; A) f - P f \|_X = \| \lambda^{-1} R(\lambda^{-1}; B) f - f \|_X = o(\lambda) \quad (\lambda \to 0+) \),

(iv) \( K(\tau, f; X, D(B)) = o(\tau) \quad (\tau \to 0+) \),

(v) \( f \in \mathcal{N}(A) = \mathcal{N}(B) \).

(b) The following five assertions are equivalent for \( \alpha \in (0, 1] \):

(i) \( \| H_{T_A}^{\gamma_1 + 1}(t) f - P f \|_X = O(t^{-\alpha}) \quad (t \to \infty) \),

(ii) \( \| H_{T_B}^{\gamma_2}(\tau) f - f \|_X = O(\tau^\alpha) \quad (\tau \to 0+) \),

(iii) \( \| \lambda R(\lambda; A) f - P f \|_X = \| \lambda^{-1} R(\lambda^{-1}; B) f - f \|_X = O(\lambda^\alpha) \quad (\lambda \to 0+) \),

(iv) \( K(\tau, f; X, D(B)) = O(\tau^\alpha) \quad (\tau \to 0+) \),

(v) \( f \in \overset{\sim}{D}(B)_{X_0} \) if \( \alpha = 1 \).

The foregoing theorem in which the core result of our paper, namely formula (5.3) plays the basic role, is perhaps our most important theorem in regard to all of our applications. In fact, Eq. (5.3), which comes into play via assertion (a)(iii) and (b)(iii), is the link between the approximation behaviour of the two semigroups generated by \( A \) and \( B \), respectively.
As to the applications, not only Dirichlet’s problem in two and three dimensions (Sects. 7 and 8), but also the examples in Sect. 9 are consequences of the foregoing interconnection theorem.

7 Exterior Dirichlet Problem for the Unit Disk

This section is devoted to the following particular Dirichlet problem: If \( f \) is a given function defined on the unit circle, more precisely, \( f \in L^2_{2\pi} \), determine a function \( w(x, r) \) which is \( 2\pi \)-periodic with respect to \( x \in \mathbb{R} \) for each \( r > 1 \), twice continuously differentiable with respect to \( x \) and \( r \) and satisfies the Laplace equation (in polar coordinates)

\[
\frac{\partial^2}{\partial r^2} w(x, r) + \frac{1}{r} \frac{\partial}{\partial r} w(x, r) + \frac{1}{r^2} \frac{\partial^2}{\partial x^2} w(x, r) = 0, \quad (x \in \mathbb{R}; \ r > 1),
\]

(7.1)

\[
\lim_{r \to 1^+} \|w(\cdot, r) - f(\cdot)\|_{L^2_{2\pi}} = 0.
\]

(7.2)

The most general single-valued solution of (7.1) is

\[
w(x, r) = A_0 + B_0 \log r + \sum_{k=1}^{\infty} \left( C_k r^k + D_k r^{-k} \right) (A_k \cos kx + B_k \sin kx)
\]

(7.3)

For the exterior problem the solution is not unique unless some restriction is placed on the behaviour of \( w(x, r) \) as \( r \to \infty \). It is unique provided \( w(x, r) \) is bounded as \( r \to \infty \), so that we can discard the logarithmic term \( \log r \) and the positive powers of \( r \) from the general solution (7.3). Hence the general solution now reads

\[
w(x, r) = \frac{A_0'}{2} + \sum_{k=1}^{\infty} r^{-k} (A'_k \cos kx + B'_k \sin kx) \quad (r > 1),
\]

(7.4)

the boundary condition at \( r = 1 \) to determine the constants now being

\[
f(x) = \frac{A_0'}{2} + \sum_{k=1}^{\infty} (A'_k \cos kx + B'_k \sin kx).
\]

(7.4)

Hence

\[
A'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos ku \, du, \quad B'_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \sin ku \, du.
\]

The convergence of trigonometric series is always understood in \( L^2_{2\pi} \)-Norm.
Thus the solution in the real form is (formally)

\[
\begin{align*}
    w(x, r) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \left[ \frac{1}{2} + \sum_{k=1}^{\infty} r^{-k} \cos k(x - u) \right] du \\
    &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - u) \frac{r^2 - 1}{1 - 2r \cos u + r^2} du,
\end{align*}
\]

the Abel-Poisson convolution integral. The factor \( r^2 - 1 \) in the solution of the corresponding interior Dirichlet problem (see [9, p. 286]) is replaced by \( 1 - r^2 \).

The solution in complex form is

\[
    w(x, r) = \sum_{k=-\infty}^{\infty} r^{-|k|} \hat{f}(k)e^{ikx} \quad (r > 1),
\]

where

\[
    \hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)e^{-iku} du, \quad f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}
\]

denote the (complex) Fourier coefficients and the Fourier series of \( f \), respectively.

Now \( V(t)f(x) := w(x, r) \) with \( r = e^t \) forms a contraction semigroup of class \( (C_0) \) in \([L^2_{2\pi}]\),

\[
    V(t)f(x) = \sum_{k=-\infty}^{\infty} e^{-|k|t} \hat{f}(k)e^{ikx} \quad (t > 0; x \in \mathbb{R}).
\]

Its generator will turn out to be \( Af = -(\tilde{f})' \), where \( \tilde{f} \) is the Hilbert transform or conjugate function of \( f \), defined by

\[
    \tilde{f}(x) := \lim_{\varepsilon \to 0^+} -\frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \left[ f(x + u) - f(x - u) \right] \frac{1}{2} \cot \frac{u}{2} du
\]

with the Fourier series

\[
    \tilde{f}(x) = \sum_{k=-\infty}^{\infty} (-i\text{sgn}k)\hat{f}(k)e^{ikx}.
\]

Thus \( Af \) has the Fourier expansion

\[
    Af(x) = -(\tilde{f})'(x) = -\sum_{k=-\infty}^{\infty} |k|\hat{f}(k)e^{ikx}
\]
and its domain can shown to be

\[ \mathcal{D}(A) = \left\{ f \in L^2_{2\pi} : \tilde{f} \in AC_{2\pi} \text{ and } \tilde{f}' \in L^2_{2\pi} \right\} = \left\{ f \in L^2_{2\pi} : (k\hat{f}(k))_{k \in \mathbb{Z}} \in L^2(\mathbb{Z}) \right\}. \]  

(7.9)

Further, \( V_A(t) f \) is a holomorphic and equibounded semigroup satisfying \( \|AV_A(t)\| \leq 2/t \) for \( t > 0 \). See e.g. [6, pp. 71, 118, 122].

This completes our results concerning the facts that \( w(x, r) \) of (7.5) is the unique solution of the exterior Dirichlet problem for the unit disk (7.1), under the conditions (7.2). Further, the solution \( w_A(x, r) := w(x, r) = V(t)f(x) \) with \( t := \log r \) generates a holomorphic contraction semigroup of class \( (C_0) \) in \( L^2_{2\pi} \) having generator \( Af = -(\tilde{f})' \) with Fourier expansion (7.8) and domain (7.9).

As to the resolvent \( R(\lambda; A) \), it is given by

\[
R(\lambda; A) f(x) = \int_0^\infty e^{-\lambda u} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} e^{-|k|u} \, du
= \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{\lambda + |k|} e^{ikx} \quad (x \in \mathbb{R}; \lambda \in \mathbb{C}, \Re \lambda > 0).
\]  

(7.10)

By Fejér’s theorem, for \( f \in L^2_{2\pi} \),

\[
f(x) = \text{s-lim}_{n \to \infty} \sum_{k=-n, k \neq 0}^{n} \left( 1 - \frac{|k|}{n + 1} \right) \hat{f}(k) e^{ikx} + \hat{f}(0).
\]

Since \( \hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) \, du \) belongs to the kernel \( \mathcal{N}(A) \), the sum belongs to the range \( \mathcal{R}(A) \), it follows that \( L^2_{2\pi} = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A) \). Thus \( X_0 = L^2_{2\pi} \) and \( Pf = \hat{f}(0) \).

Since \( V_A(\cdot) \) is equibounded, it is a \( T_0 \)-semigroup, and its unbounded generator \( A \) possesses the \( a \)-Drazin inverse \( B = A^a \), which can be evaluated by Proposition 4.2 or 4.3 to

\[
Bf(x) = -\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|} \hat{f}(k) e^{ikx}.
\]

It follows that \( B \) is a bounded operator on \( \mathcal{D}(B) = X_0 = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A) = L^2_{2\pi} \), i.e. \( B \in [L^2_{2\pi}] \), with \( \|Bf\|_{L^2_{2\pi}} \leq 2\|f\|_{L^2_{2\pi}} \).

In view of Theorem 5.2, the operator \( B \) itself also generates a semigroup \( V_B(t) f \) on the whole of \( L^2_{2\pi} \) having the representation,
\[
V_B(t) f(x) = f(x) + \hat{f}(0) - \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{ut})}{\sqrt{u}} \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-|k|u} \hat{f}(k) e^{ikx} \, du
\]
\[
= f(x) + \hat{f}(0) - \sum_{k \in \mathbb{Z} \setminus \{0\}} (1 - e^{-t/|k|}) \hat{f}(k) e^{ikx}
\]
\[
= \hat{f}(0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-t/|k|} \hat{f}(k) e^{ikx}
\]
in the exponential form, in view of (5.7) and the convergence of the Fourier series of \( f \) towards \( f \) in \( L^2_{2\pi} \).

Now to the behaviour of the function

\[
w_B(x, r) := V_B(t) f(x) = \hat{f}(0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} r^{-1/|k|} \hat{f}(k) e^{ikx}.
\] (7.11)

It will turn out to be the unique solution of the hilbertian integro-differential equation

\[
\int_1^r u(x, \rho) \frac{d\rho}{\rho} + \frac{\partial}{\partial x} \tilde{u}(x, r) = \hat{f}(0) \log r + (\tilde{f})'(x)
\] (7.12)
under the conditions \( s\text{-lim}_{r \to 1^+} u(x, r) = f(x) \) for \( f \in D(A) \). Note that Eq. (7.12) can be rewritten in terms of the generator \( A \) as

\[
\int_1^r u(x, \rho) \frac{d\rho}{\rho} - Au(x, r) = \hat{f}(0) \log r - Af(x).
\] (7.13)

Let us restate the integro-differential equation in a more detailed form.

**Proposition 7.1** Let \( f \in D(A) \). We are looking for a complex valued function \( u(x, r) \in C(\mathbb{R} \times (1, \infty)) \), which is \( 2\pi \)-periodic with respect to \( x \) and satisfies the following four conditions:

(i) the integral on the left of (7.12) exists in the sense
\[
\int_1^r u(x, \rho) \frac{d\rho}{\rho} := \text{s-lim}_{\varepsilon \to 1^+} \int_\varepsilon^r u(x, \rho) \frac{d\rho}{\rho},
\]
(ii) \( \frac{\partial}{\partial x} \tilde{u}(x, r) \) exists on \( \mathbb{R} \times (0, \infty) \),
(iii) \( u \) satisfies Eq. (7.12),
(iv) \( s\text{-lim}_{r \to 1^+} u(x, r) = f(x) \).

This problem has the unique solution \( u(x, r) = w_B(x, r) \) given by (7.11).
\textbf{Proof} Assume that \( u \) is a solution of the problem. Then one has by (7.8) for its Fourier coefficients of \( \hat{u}(k, r) \),
\begin{align*}
\int_{1}^{r} \hat{u}(0, \rho) \frac{d\rho}{\rho} &= \log r \hat{f}(0), \\
\int_{1}^{r} \hat{u}(k, \rho) \frac{d\rho}{\rho} + |k| \hat{w}(k, \rho) &= |k| \hat{f}(k) \quad (k \in \mathbb{Z} \setminus \{0\}),
\end{align*}
(7.14)
(7.15)

Note that \( \left[ \int_{1}^{r} u(\cdot, \rho) \frac{d\rho}{\rho} \right](k) = \int_{1}^{r} \hat{u}(k, \rho) \frac{d\rho}{\rho} \) in view of Fubini’s theorem and the strong convergence of the integral.

Differentiating these equations with respect to \( r \) yields
\begin{align*}
\hat{u}(0, r)r^{-1} &= r^{-1} \hat{f}(0), \\
\hat{u}(k, r)r^{-1} + |k| \frac{d}{dr} \hat{u}(k, r) &= 0 \quad (k \in \mathbb{Z} \setminus \{0\}).
\end{align*}
(7.16)
(7.17)

Now, (7.17) is a homogeneous linear differential equation for \( \hat{u}(k, r) \) having the general solution
\[ \hat{u}(k, r) = c_{k} r^{-1/|k|} \quad (k \in \mathbb{Z} \setminus \{0\}). \]

Further, it follows from the initial condition (iv) that \( c_{k} = \hat{f}(k) \). Thus, if a solution exists, it is uniquely given by \( w_{B}(x, r) \) of (7.11).

It remains to show that \( w_{B}(x, r) \) is actually a solution. Indeed, the infinite series defining \( w_{B}(x, r) \) is uniformly convergent on \([1 + \varepsilon, \infty) \times r\) for each \( \varepsilon > 0 \), giving the continuity of \( w_{B}(x, r) \). Further,
\begin{align*}
\int_{1}^{r} w_{B}(x, \rho) \frac{d\rho}{\rho} &= \hat{f}(0) \log r + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\{ \int_{1}^{r} \rho^{-1/|k| - 1} d\rho \hat{f}(k) e^{ikx} \right\} \\
&= \hat{f}(0) \log r - \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|r^{-1/|k|} \hat{f}(k) e^{ikx} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k| \hat{f}(k) e^{ikx} \\
&= \hat{f}(0) \log r - \frac{\partial}{\partial x} \hat{w}_{B}(x, r) + (\hat{f})'(x).
\end{align*}

Further, \( w_{B}(x, r) \) satisfies the initial condition in view of the \((C_{0})\)-property of the semigroup \( w_{B}(x, r) := V_{B}(t) f(x), \ t = \log r \).

Finally, to the resolvents \( r_{A}(x; \lambda) := \lambda R(\lambda; A) f(x) \) and \( r_{B}(x; \lambda) := \lambda R(\lambda; B) f(x) \).

\textbf{Proposition 7.2} Let \( f \in L_{2\pi}^{2} \). Then \( r_{A}(x; \lambda) \) is the unique solution of the hilbertian differential equation
\[ \frac{d}{dx} \tilde{u}(x; \lambda) = -\lambda u(x; \lambda) + \lambda f(x) \quad (x \in \mathbb{R}; \lambda \in \mathbb{C}, \Re \lambda > 0), \]
(7.18)

where \( u \) is a function which belongs to \( L_{2\pi}^{2} \) as a function of \( x \) for each \( \lambda \in \mathbb{C}, \Re \lambda > 0 \).
Proof Noting (7.8), it follows that the Fourier coefficients \( \hat{u}(k; \lambda) \) of a solution of (7.18) satisfy

\[
|k| \hat{u}(k; \lambda) = -\lambda \hat{u}(k; \lambda) + \lambda \hat{f}(k) \quad (k \in \mathbb{Z}),
\]

and hence by (7.10),

\[
\hat{u}(k; \lambda) = \frac{\lambda}{\lambda + |k|} \hat{f}(k) = \hat{r}_A(k; \lambda) \quad (k \in \mathbb{Z}).
\]

This gives the uniqueness of the solution.

On the other hand, by (7.10) and (7.8)

\[
\frac{d}{dx} \hat{r}_A(x; \lambda) = \sum_{k \in \mathbb{Z}} \frac{\lambda |k|}{\lambda + |k|} \hat{f}(k) e^{ikx}
\]

\[
= -\lambda \sum_{k \in \mathbb{Z}} \frac{1}{\lambda + |k|} \hat{f}(k) e^{ikx} + \lambda \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}
\]

\[
= -\lambda r_A(x; \lambda) + \lambda f(x),
\]

all series being convergent in the norm of \( L^2_{2\pi} \). This shows that \( r_A(x; \lambda) \) is actually a solution of (7.18). \( \square \)

The resolvent \( R(\lambda; B) \) is given, in view of (5.2) and (7.10), by

\[
\lambda R(\lambda; B) f(x) = \hat{f}(0) + f(x) - \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{1 + \lambda |k|} e^{ikx}
\]

\[
= \hat{f}(0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) \frac{\lambda |k|}{1 + \lambda |k|} e^{ikx},
\]

(7.20)

where representation (7.20) follows from (7.19) by replacing \( f \) by its Fourier series.

The resolvent \( r_B(x; \lambda):=\lambda R(\lambda; B) f(x) \) for \( B = A^{\text{ad}} \) satisfies the ‘inverse’ hilbertian integral equation

\[
\int_0^x \hat{u}(v; \lambda) dv = [\lambda u(v; \lambda) - \lambda f(v)]_0^x
\]

under the initial condition

\[
\text{s-lim}_{\lambda \to 0+} u(x; \lambda) = \hat{f}(0).
\]

More precisely,

\( \mathbb{B} \) Birkhäuser
Proposition 7.3 Let \( f \in L^2_{2\pi} \). Then each solution \( u(x; \lambda) \) of (7.21), (7.22) which belongs to \( L^2_{2\pi} \) as a function of \( x \) for each \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \), is of the form \( u(x; \lambda) = r_B(x; \lambda) + \varphi(\lambda) \), where \( \varphi(\lambda) \) is defined for \( \Re \lambda > 0 \) and satisfies \( \lim_{\lambda \to 0} \varphi(\lambda) = 0 \).

Conversely, each function \( u \) of the form \( u(x; \lambda) = r_B(x; \lambda) + \varphi(\lambda) \) with \( \varphi \) as above is a solution of (7.21), (7.22).

Proof Indeed, assume \( u \) is a solution of (7.21). Setting \( g(x):=u(x; \lambda) - f(x) \), then \( g \in AC_{2\pi} \) with \( g' \in L^2_{2\pi} \) and \( \hat{g}(k) = ik[\hat{u}(k; \lambda) - \hat{f}(k)] \). Hence, one has for the Fourier coefficients of the differentiated equation (7.21),

\[
(-i \text{sgn} k)\hat{u}(k; \lambda) = \lambda ik[\hat{u}(k; \lambda) - \hat{f}(k)] \quad (k \in \mathbb{Z}).
\]

This yields

\[
\hat{u}(k; \lambda) = \frac{\lambda |k|}{1 + \lambda |k|} \hat{f}(k) \quad (k \in \mathbb{Z} \setminus \{0\})
\]

As to \( \hat{u}(0; \lambda) \), it follows from (7.22) that \( \hat{u}(0; \lambda) = \hat{f}(0) + \varphi(\lambda) \) with \( \varphi \) as specified above. This proves the first part of the proposition.

Concerning the second part, it is enough to consider the case \( \varphi = 0 \), since another function \( \varphi \) does not change the left-hand side of nor the right-hand side of (7.21). In order to show that \( r_B(x; \lambda) \) is actually a solution, one has only to note that a Fourier series can be integrated term by term, i.e.,

\[
\int_0^x \tilde{r}_B(v; \lambda) \, dv = \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_0^x (-i \text{sgn} k) \hat{f}(k) \frac{\lambda |k|}{1 + \lambda |k|} e^{ikv} \, dv
\]

\[
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) \frac{-\lambda}{1 + \lambda |k|} (e^{ikx} - 1).
\]

By (7.19), the latter series equals \( \lambda [r_B(x; \lambda) - f(x)] - \lambda [r_B(0; \lambda) - f(0)] \). This shows that \( r_B(x; \lambda) \) is a solution of (7.21), which obviously satisfies (7.22).

Hilbertian differential or integral equations such as (7.18) and (7.21), or hilbertian integro-differential equations such as (7.12) have not been considered in the literature as yet.

Now we apply our general theorems of Sects. 2–6 to the concrete operators above.

Theorem 7.4 Let \( f \in L^2_{2\pi}, \) \( w_A(x, r):=V_A(t) f(x), \) \( w_B(x, r):=V_B(t) f(x) \) as well as \( w_A^\gamma(x, r):=H_A^\gamma(t) f(x) \) and \( w_B^\gamma(x, r):=H_B^\gamma(t) f(x) \) for \( \gamma \geq 0 \) with \( r = e^t \).

(a) For any \( \gamma_1, \gamma_2 \geq 0 \) and \( \alpha \in (0, 1) \), the following seven assertions are equivalent:

(i)

\[
\|w_A(x, r) - f(x)\|_{L^2_{2\pi}} = \left\| \sum_{k \in \mathbb{Z}} r^{-|k|} \hat{f}(k)e^{ikx} - f(x) \right\|_{L^2_{2\pi}} = \begin{cases} o(\log r) \\ O((\log r)^\alpha) \end{cases} \quad (r \to 1+),
\]
or, more generally, for the case $\gamma_1 > 0$ (instead of $\gamma_1 = 0$ above),

(ii)\[ \| w^\gamma_A (x, r) - f(x) \|_{L^2_{2\pi}} = \begin{cases} o\left( (\log r) \right) & (r \to 1+) , \\ \mathcal{O}\left( (\log r)^{\alpha} \right) & \end{cases} \]

(iii)\[ \| r_A (x; \lambda) - f(x) \|_{L^2_{2\pi}} = \begin{cases} o\left( \lambda^{-1} \right) & (\lambda \to \infty) , \\ \mathcal{O}\left( \lambda^{-\alpha} \right) \end{cases} \]

(iv)\[ \left\| \frac{1}{\log r} \int_1^r w_B (x, \rho) \frac{d\rho}{\rho} - \hat{f}(0) \right\|_{L^2_{2\pi}} \\
= \left\| \frac{1}{\log r} \int_1^r \left\{ \hat{f}(0) + \sum_{k \in \mathbb{Z}\setminus\{0\}} \rho^{-1/|k|} \hat{f}(k) e^{ikx} \left\} \frac{d\rho}{\rho} - \hat{f}(0) \right\|_{L^2_{2\pi}} \\
= \begin{cases} o\left( (\log r)^{-1} \right) \\ \mathcal{O}\left( (\log r)^{-\alpha} \right) \end{cases} \quad (r \to \infty) , \]

or, more generally, for the case $\gamma_2 > 0$ (instead of $\gamma_2 = 0$ above),

(v)\[ \| w^{\gamma_2+1}_B (x, r) - \hat{f}(0) \|_{L^2_{2\pi}} = \begin{cases} o\left( (\log r)^{-1} \right) \\ \mathcal{O}\left( (\log r)^{-\alpha} \right) \end{cases} \quad (r \to \infty) , \]

(vi)\[ \| r_B (x; \lambda) - \hat{f}(0) \|_{L^2_{2\pi}} = \begin{cases} o\left( \lambda \right) \\ \mathcal{O}\left( \lambda^{\alpha} \right) \end{cases} \quad (\lambda \to 0+) , \]

(vii)\[ K(t, f; L^2_{2\pi}, \mathcal{D}(A)) = \begin{cases} o(t) \\ \mathcal{O}(t^{\alpha}) \end{cases} \quad (t \to 0+) , \]

(viii)\[ \| f(x + h) - f(x) \|_{L^2_{2\pi}} = \begin{cases} o(h) \\ \mathcal{O}(h^{\alpha}) \end{cases} \quad (h \to 0+) . \]
(ix) \[
\begin{cases}
  f(x) = \hat{f}(0) \text{ a.e.,} \\
  f \in \mathcal{D}(A) \text{ if } \alpha = 1.
\end{cases}
\]

(b) For any \( \gamma \geq 0 \) there exists an element \( f_{\alpha} \in L^2_{2\pi} \), such that

(i) \[
\|w^\gamma_A(x, r) - f_{\alpha}(x)\|_{L^2_{2\pi}} = \mathcal{O}((\log r)^{\alpha}) \quad (r \to 1+),
\]

(ii) \[
\|w^{\gamma+1}_B(x, r) - \hat{f}_{\alpha}(0)\|_{L^2_{2\pi}} = \mathcal{O}((\log r)^{-\alpha}) \quad (r \to \infty),
\]

(iii) \[
\|r_A(x; \lambda^{-1}) - f_{\alpha}(x)\|_{L^2_{2\pi}} = \|r_B(x; \lambda) - \hat{f}_{\alpha}(0)\|_{L^2_{2\pi}} = \mathcal{O}((\log r)^{-1}) \quad (r \to \infty),
\]

Concerning the proof of a), one applies Theorem 6.1 to the semigroup generated by \( B \), i.e., one interchanges the roles of \( A \) and \( B \). This gives everything apart from the equivalence with assertion (vi).

As to assertion (vi), we first note that \( \mathcal{D}(A) \) as given in (7.9) can be rewritten as

\[
\mathcal{D}(A) = \left\{ f \in L^2_{2\pi}; (ik\hat{f}(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\} = \left\{ f \in L^2_{2\pi}; f \in AC_{2\pi} \text{ and } f' \in L^2_{2\pi} \right\},
\]

i.e., \( \mathcal{D}(A) \) is the Sobolev space \( W^1_{L^2_{2\pi}} \); cf. [9, p. 173]. The equivalence of (vi) and (v) now follows from the equivalence of the modulus of continuity and the \( K \)-functional in the sense that for two constants \( c_1, c_2 > 0 \) and all \( t > 0 \),

\[
c_1 \sup_{0 < h \leq t} \|f(x + h) - f(x)\|_{L^2_{2\pi}} \leq K(t, f; L^2_{2\pi}, \mathcal{D}(A)) \leq c_2 \sup_{0 < h \leq t} \|f(x + h) - f(x)\|_{L^2_{2\pi}};
\]

see e.g. [28, p. 177].

The existence of the function \( f_{\alpha} \) in Part (b)(i), follows from Theorem 3.1. That the same function \( f_{\alpha} \) also satisfies (ii) and (iii) then follows from the equivalence (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) in Part (a), and (5.3).

Actually \( w^\gamma_A(x, r) = w^\gamma_A(f; x, r) \), indicating the dependence upon the initial value \( f \), and similarly \( r_A(x; \lambda) = r_A(f; x; \lambda) \).

Some essential aspects of the preceding theorem are summarised in
Conclusion 7.5 The exterior Dirichlet problem for the unit disk calls for a solution $V(t)f(x) = w(x, r)$ with $r = e^t$ satisfying the Laplace equation (7.1) with boundary condition (7.2). The solution is a holomorphic contraction semigroup on $L^2_{2\pi}$ with generator $Af = -(\tilde{f})'$. The unbounded operator $A$ has an a-Drazin inverse $B = A^{ad}$, which is a bounded operator on the whole of $L^2_{2\pi}$. $B$ itself also generates a semigroup $V_B(t)f$ on $L^2_{2\pi}$, which in turn is the unique solution of the hilbertian integro-differential equation (7.12) or (7.13) under the boundary condition $s\text{-}\lim_{r \to 1^+} V_B(\log r) f(x) = f(x)$ for $f \in D(A)$ in view of the $(C_0)$-property of the semigroup $V_B(t)$.

Concerning Theorem 7.4, assertion (a)(i), revealing that the solution of Dirichlet’s problem, $w_A(x, r)$ tends to the boundary value $f$ from the outside of the unit disk with a certain rate $O((\log r)^\alpha)$, can be interpreted in the sense that $w_A(x, r)$ is $(C, 0)$-summable to $f$ or, more generally in case that $\gamma_1 > 0$ it is $(C, \gamma_1)$-summable to $f$, both with the same order.

Assertion (ii) is the corresponding result for Abel summability to $f$ with order $O(\lambda^{-\alpha})$ for $\lambda \to \infty$.

As to assertion (iii), it states that the integral means or the $(C, \gamma_2 + 1)$-means of the solution $w_B(x, r)$ of the hilbertian integro-differential equation (7.12) tend to the integral mean of the boundary value, namely $\tilde{f}(0)$, with order $O((\log r)^{-\alpha})$ for $r \to \infty$, and assertion (iv) is the corresponding Abel summability result.

Furthermore, all of these assertions are equivalent to another if and only if the boundary value $f$ satisfies the Lipschitz condition $\|f(x + h) - f(x)\|_{L^2_{2\pi}} = O(h^\alpha)$ for $h \to 0$; see assertion (vi).

The best possible order which can be achieved, unless $f = \text{const.}$, is given for $\alpha = 1$. This is the so-called saturation order in the sense of J. Favard. The set of functions for which this order is achieved, i.e., the saturation class, is given by $D(A)$, the domain of the generator $A$; see assertion (vii).

8 Dirichlet’s Problem for the Three Dimensional Unit Ball and Its Interconnections with the Associated Inverse Problem in Unlimited Space

In order to apply the material of the present paper to a true world situation, we need to extend the results of Sect. 7 to three dimensions. The background and literature to Dirichlet’s problem goes back for some two hundred years, to Laplace, to the three dimensional Laplace equation

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in which $x = (x, y, z)$ represent rectangular coordinates in $\mathbb{R}^3$, its solutions being known as harmonic functions and its theory as potential theory.\(^6\)

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\(^6\) Good examples of books on harmonic analysis, potential theory or PDEs, also treating Dirichlet’s problem for the unit ball, are [3, 60, 77, 96], [34, Chapters VI, VIII], [92, §§ 31, 32].
Dirichlet’s exterior boundary value problem for the ball amounts to solving

\[ \Delta u = 0 \quad (x^2 + y^2 + z^2 > 1), \tag{8.1} \]
\[ u(x, y, z) = f(x, y, z) \quad (x^2 + y^2 + z^2 = 1), \tag{8.2} \]

\( f \) prescribing the values of \( u \) on the bounding unit sphere.

In order to solve this problem, one writes equation (8.1) in spherical coordinates \( r, \phi, \theta \),

\[
\begin{align*}
  x &= r \sin \phi \cos \theta, \\
  y &= r \sin \phi \sin \theta, \\
  z &= r \cos \phi,
\end{align*}
\]

where \(-\pi < \theta \leq \pi\) is the longitude, while \(0 \leq \phi \leq \pi\) is the latitude on the sphere of radius \(r = \sqrt{x^2 + y^2 + z^2}\).

The exact formulation of the above problem now reads: Determine a function \( w(\theta, \phi, r) \), defined \((1, \infty) \times [0, \pi] \times [-\pi, \pi]\), which is twice continuously differentiable on its domain and satisfies Laplace’s equation (in spherical coordinates)

\[ \Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{\cos \phi}{r^2 \sin \phi} \frac{\partial w}{\partial \phi} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 w}{\partial \theta^2} = 0 \tag{8.3} \]

together with the boundary conditions

\[ w(r, \phi, -\pi) = w(r, \phi, \pi) \quad (r > 1, \phi \in [0, \pi]) \tag{8.4} \]
\[ \lim_{r \to 1^+} \|w(r, \phi, \theta) - h(\phi, \theta)\|_{L^2(S)} = 0, \tag{8.5} \]
\[ \lim_{r \to \infty} w(r, \phi, \theta) = 0 \quad (\phi \in [0, \pi], \theta \in [-\pi, \pi], \theta) \tag{8.6} \]

where \( h(\phi, \theta) = f(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \) for a given \( f \in L^2(S) \).

To construct separable solutions of Laplace’s equation of the form (8.3) we begin by separating the radial part of the solution, writing

\[ w(r, \phi, \theta) = G(r)H(\phi, \theta) \tag{8.7} \]

and substituting this into (8.3) leads to the two differential equations

\[ r^2 G'' + 2r G' - \mu G = 0, \tag{8.8} \]
\[ \frac{\partial^2 H}{\partial \phi^2} + \frac{\cos \phi}{\sin \phi} \frac{\partial H}{\partial \phi} + \frac{1}{\sin^2 \phi} \frac{\partial^2 H}{\partial \theta^2} + \mu H = 0, \tag{8.9} \]

\( \mu \) being the separation constant.

Let us first turn to the second equation, also known as spherical Helmholtz equation. A further separation ansatz,

\[ H(\phi, \theta) = p(\phi)q(\theta) \tag{8.10} \]
splits the Helmholtz equation into a pair of ordinary differential equations,

\[
\sin^2 \varphi \frac{dp}{d\varphi} \cos \varphi \sin \varphi \frac{dp}{d\varphi} + (\mu \sin^2 \varphi - \nu)p = 0, \quad \frac{d^2 q}{d\theta^2} + \nu q = 0 \tag{8.11}
\]

with another separation constant \( \nu \).

Since the solutions of the second equation must satisfy (8.4), it has non-trivial solutions only for \( \nu = m^2, m = 0, 1, \ldots \), given by

\[ q_m(\theta) = e^{im\theta} \quad (m = 0, \pm 1, \pm 2 \ldots). \]

As to the first equation in (8.11), the substitution \( \cos \varphi = t \) together with \( p(\varphi) = P(\cos \varphi) = P(t) \) leads to the boundary value problem

\[
(1 - t^2)^2 \frac{d^2 P}{dt^2} - 2t(1 - t^2) \frac{dP}{dt} + [\mu(1 - t^2) - m^2]P = 0, \tag{8.12}
\]

\[ |P(-1)| < \infty, \quad |P(1)| < \infty. \tag{8.13} \]

This problem has eigenvalues \( \mu_k = k(k + 1), k = 0, 1, 2, \ldots \), and the associated eigenfunctions are

\[
P^m_k(t) = (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_k(t) = \frac{(-1)^k(1 - t^2)^{m/2}}{2^k k!} \frac{d^{k+m}}{dt^{k+m}} (1 - t^2)^k \]

\[ (m = 0, 1, \ldots, k). \tag{8.14} \]

Above \( P_k = P^0_k \) are the Legendre polynomials defined by

\[ P_k(t) := \frac{(-1)^k}{2^k k!} \frac{d^k}{dt^k} (1 - t^2)^k. \]

The \( P^m_k \) are known as associated Legendre functions. They are polynomials for \( m \) even, and polynomials multiplied by a factor \( (1 - t)^{1/2} \) for \( m \) odd.

As to Eq. (8.9), the ansatz (8.10) gives the normalized solutions

\[
Y^m_k(\varphi, \theta) := \sqrt{\frac{(2k + 1)(k - |m|)!}{4\pi(k + |m|)!}} P^{|m|}_k(\cos \varphi) e^{im\theta} \]

\[ (k = 0, 1, 2, \ldots; m = 0, \pm 1, \ldots, \pm k) \]

which are the so-called (complex) spherical harmonics.

It is well known that spherical harmonics form a complete orthonormal system in the Hilbert space \( L^2(S) \), \( S \) being the unit sphere in \( \mathbb{R}^3 \), with respect to the scalar
product and the norm
\[
< g_1, g_2 > := \int_S g_1 \overline{g_2} \, dS = \int_{-\pi}^\pi \int_0^\pi g_1(\varphi, \theta) \overline{g_2(\varphi, \theta)} \sin \varphi \, d\varphi \, d\theta,
\]
\[
\| g \|_{L^2(S)} := \sqrt{< g, g >},
\]
\[dS = \sin \varphi \, d\varphi \, d\theta \text{ being the surface area element of the unit sphere. In particular, there holds}
\]
\[
\langle Y^m_k, Y^l_j \rangle = \delta_{k, l} \delta_{m, l} \quad (k = 0, 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots, \pm k).
\]
It follows that every function \( g \in L^2(S) \) can be expanded into a Fourier series with respect to the complex spherical harmonics,
\[
g(\varphi, \theta) = \sum_{k=0}^\infty \sum_{m=-k}^k \hat{g}(m, k) Y^m_k(\varphi, \theta),
\]
the series being convergent in \( L^2(S) \)-norm. The Fourier coefficients \( \hat{g}(m, k) \) are given by
\[
\hat{g}(m, k) := \langle g, Y^m_k \rangle = \int_{-\pi}^\pi \int_0^\pi g(\varphi, \theta) \overline{Y^m_k(\varphi, \theta)} \sin \varphi \, d\varphi \, d\theta.
\]
In order to obtain a solution of Laplace’s equation (8.3), it remains to solve (8.8). Since we already know that the separation constant must be \( k(k + 1) \), \( k = 0, 1, 2, \ldots \), the ansatz \( G(r) = r^a \) gives the solutions
\[
G(r) = r^k, \quad G(r) = r^{-k-1} \quad (k = 0, 1, 2, \ldots).
\]
In view of (8.6) we are interested only in solutions which vanish for \( r \to \infty \). Hence the general solution of (8.3) is given by
\[
w(r, \varphi, \theta) = \sum_{k=0}^\infty r^{-(k+1)} \sum_{m=-k}^k c_{m,k} Y^m_k(\varphi, \theta).
\]
The coefficients \( c_{m,k} \) are uniquely prescribed by the boundary condition (8.5), namely,
\[
c_{m,k} := \hat{h}(m, k) = \int_{-\pi}^\pi \int_0^\pi h(\varphi, \theta) Y^m_k(\varphi, \theta) \sin \varphi \, d\varphi \, d\theta.
\]
This finally yields that

\[ w(r, \varphi, \theta) = \sum_{k=0}^{\infty} r^{-(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) \]

\[ (r > 1, \varphi \in [0, \pi], \theta \in [-\pi, \pi]). \quad (8.15) \]

The fact that (8.15) is actually a solution follows by exactly the same arguments as for the inner three-dimensional Dirichlet problem in [85, pp. 289 ff.]. Further, the solution is unique, as is shown in [97, p. 52].

There exists also a representation of (8.15) as a singular convolution integral, namely (cf., e.g. Berens–Butzer–Pawelke [5], also [97] and [85, p. 288])

\[ w(r, \varphi, \theta) = \frac{1}{4\pi} \int_{S^3} \frac{r^2 - 1}{(1 - 2r \langle x, y \rangle + r^2)^{3/2}} f(y) \, ds(y) \]

\[ (x = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)). \quad (8.16) \]

As in the two dimension case, we set \( r = e^t \), i.e.,

\[ w(\varphi, \theta, r) = w(\varphi, \theta, e^t) = \sum_{k=0}^{\infty} e^{-(k+1)t} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta). \quad (8.17) \]

Again \( T_A(t)h(\varphi, \theta) := w(e^t, \varphi, \theta) = w_A(e^t, \varphi, \theta) \) is a semigroup of class \((C_0)\), and its generator \( A \) is given by

\[ Ah(\varphi, \theta) = -\sum_{k=0}^{\infty} (k + 1) \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta). \quad (8.18) \]

Its domain \( D(A) \) can be characterized in terms of the Fourier coefficients, namely,

\[ D(A) = \left\{ g \in L^2(S); \sum_{k=0}^{\infty} (k + 1)^2 \sum_{m=-k}^{k} |\hat{g}(m, k)|^2 < \infty \right\}. \quad (8.19) \]

---

7 According to the monograph Kunyang Wang and Luoqing Li, Beijing [96], the H. Berens, Butzer and S. Pawelke paper [5] of 1968, is the first born dealing with approximation and saturation problems on the sphere, which became a very active field of research from 1980 onwards, with the publication of papers by Nikol’skii, Lizorkin, Kamzolov and others. Its basis is C. Müller’s “Spherical Harmonics” [63] of 1966 — Müller, Butzer, Berens and Pawelke all being stationed at RWTH Aachen at the time. Paper [5], 67 pp. long, written in German, was accepted for publication in Publ. RIMS, Kyoto Univ. Ser. A, Vol. 4 (1968), pp. 201–268 by Masuo Hukuhara (or Fukuhara) as its chief-editor, it being communicated by Minoru Urabe, both professors at Kyoto University, Japan’s top Imperial University of seven at the time; see also [22, Section 4.5]. As important as firstlings were also Butzer and H. Johnen’s [10], David Ragozin’s [82] and Pawelke’s [66] of 1971 and 1972. The extensive reference list of [96] includes two joint papers of Berens with Luoqing Li of 1993 and 1995, and ten further papers by Aachen authors, dating from 1972 to 1992.
whereas the nullspace $\mathcal{N}(A)$ is trivial, i.e., $A$ is injective and the operator $B$ of (2.6) is just $A^{-1}$. It is defined on all of $L^2(S)$, hence bounded, and has the Fourier series representation

$$Bh(\varphi, \theta) = A^{-1}h(\varphi, \theta) = -\sum_{k=0}^{\infty} (k+1)^{-1} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta).$$

(8.20)

Furthermore, the semigroup generated by $A^{-1}$ is given by

$$V_{A^{-1}}(t)h(\varphi, \theta) = w_{A^{-1}}(e^t, \varphi, \theta) = \sum_{k=0}^{\infty} e^{-t/(k+1)} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta).$$

(8.21)

Setting $r = e^t$, it can be shown that $w_{A^{-1}}(r, \varphi, \theta)$ is a solution of the equation

$$\int_{1}^{r} u(\rho, \varphi, \theta) \frac{d \rho}{\rho} - Au(r, \varphi, \theta) = -Ah(\varphi, \theta).$$

(8.22)

This corresponds exactly to the results in the two dimensional setting; see (7.12), (7.13). Recalling that the operator $A$ is the derivative of the semigroup $T_A(t)$ at zero, this is again an integro-differential equation. Concerning the uniqueness of its solution, it is conjectured that there also holds a counterpart of Proposition 7.1.

Now we apply our general Theorem 6.1 to the concrete operators above, where $B = A^{-1}$.

**Theorem 8.1** Let $h(\varphi, \theta) = f(\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$ for a given $f \in L^2(S)$, $w_{A^{-1}}(r, \varphi, \theta):=V_{A^{-1}}(t)h(\varphi, \theta)$, $w_{A^{-1}}(r, \varphi, \theta):=V_{A^{-1}}(t)h(\varphi, \theta)$ with $r = e^t$.

(a) For any $\gamma_1, \gamma_2 \geq 0$ and $\alpha \in (0, 1]$, the following assertions are equivalent:

(i) $$\|w_{A^{-1}}(r, \varphi, \theta) - h(\varphi, \theta)\|_{L^2(S)} = \begin{cases} O((\log r)^\alpha) & (r \to 1^+), \\ \mathcal{O}(\log r) & (r \to 1^+). \end{cases}$$

or, more generally, for the case $\gamma_1 > 0$ (instead of $\gamma_1 = 0$ above),

(ii) $$\|w_{A}(\gamma_1, \varphi, \theta) - h(\varphi, \theta)\|_{L^2(S)} = \begin{cases} O((\log r)^\alpha) & (r \to 1^+), \\ \mathcal{O}(\log r) & (r \to 1^+). \end{cases}$$

(iii) $$\|r_{A}(\varphi, \theta; \lambda) - h(\varphi, \theta)\|_{L^2(S)} = \begin{cases} O(\lambda^{\alpha-1}) & (\lambda \to \infty), \\ \mathcal{O}(\lambda^{-\alpha}) & (\lambda \to \infty). \end{cases}$$
(iv)\[
\left\| \frac{1}{\log r} \int_1^r w_{A^{-1}}(\rho, \varphi, \theta) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \begin{cases} o\left( (\log r)^{-1} \right) \\ \mathcal{O}\left( (\log r)^{-\alpha} \right) \end{cases} \quad (r \to \infty),
\]
or, more generally, for the case $\gamma_2 > 0$ (instead of $\gamma_2 = 0$ above), (v)\[
\left\| w_{A^{-1}}^{\gamma_2+1}(\rho, \varphi, \theta) \right\|_{L^2(S)} = \begin{cases} o\left( (\log r)^{-1} \right) \\ \mathcal{O}\left( (\log r)^{-\alpha} \right) \end{cases} \quad (r \to \infty),
\]
(vi)\[
\left\| r_{A^{-1}}(\varphi, \theta; \lambda) \right\|_{L^2(S)} = \begin{cases} o(\lambda) \\ \mathcal{O}(\lambda^\alpha) \end{cases} \quad (\lambda \to 0+),
\]
(vii)\[
K(t, h; L^2(S), \mathcal{D}(A)) = \begin{cases} o(t) \\ \mathcal{O}(t^\alpha) \end{cases} \quad (t \to 0+),
\]
(viii)\[
\begin{cases} h = 0 \ a.e., \\ h \in \mathcal{D}(A) \text{ if } \alpha = 1.
\end{cases}
\]
(b) For each $\alpha \in (0, 1]$ there exists an element $h_\alpha \in L^2(S)$, such that (i)\[
\left\| w_A(r, \varphi, \theta) - h_\alpha(\varphi, \theta) \right\|_{L^2(S)} = \begin{cases} \mathcal{O}(\log r)^\alpha \\ \neq o(\log r) \end{cases} \quad (r \to 1+),
\]
(ii)\[
\left\| \frac{1}{\log r} \int_1^r w_{A^{-1}}(\rho, \varphi, \theta) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \begin{cases} \mathcal{O}\left( (\log r)^{-\alpha} \right) \\ \neq o\left( (\log r)^{-1} \right) \end{cases} \quad (r \to \infty),
\]
(iii)\[
\left\| r_A(\varphi, \theta; \lambda^{-1}) - h_\alpha(\varphi, \theta) \right\|_{L^2(S)} = \left\| r_{A^{-1}}(\varphi, \theta; \lambda) \right\|_{L^2(S)} = \begin{cases} \mathcal{O}(\lambda^\alpha) \\ \neq o(\lambda^\alpha) \end{cases} \quad (\lambda \to 0+).
The proof follows along the same lines as that of Theorem 7.4. Again \( w_A(r, \varphi, \theta) = w_A(h; r, \varphi, \theta) \) etc., indicating the dependence upon the boundary value \( h \).

In Theorem 7.4 a)(v), (vi) the smoothness of the boundary function is described in terms of the \( K \)-functional as well as in terms of a modulus of continuity. Here, in contrast, we have only a characterisation by the \( K \)-functional (Theorem 8.1 a)(v)). What is missing is a modulus of continuity for functions defined on the surface of the sphere that is equivalent to the given \( K \)-functional. In this respect see [55, 68–76].

The main aspects of the preceding theorem are contained in

**Conclusion 8.2** The exterior Dirichlet problem for the unit sphere in three-dimensional space (8.3)–(8.6) calls for a solution \( w(r, \varphi, \theta) = w(h; r, \varphi, \theta) \) given by (8.15) or (8.16), \( h \in L^2(S) \) the boundary function in (8.5). Then, \( \mathbb{T}_A(t)h(\varphi, \theta) := w(h; e^t, \varphi, \theta) \) is a semigroup of class \((C_0)\) on \( L^2(S) \) with generator \( A \), defined by (8.18), its domain \( \mathbb{D}(A) \) being characterized by (8.19). The unbounded operator \( A \) has a bounded inverse \( A^{-1} : L^2(S) \rightarrow \mathbb{D}(A) \), i.e., the \( \alpha \)-Drazin inverse \( A^{\alpha} \) equals \( A^{-1} \) in this case.

The operator \( A^{-1} \) itself also generates a semigroup \( \mathbb{V}_A^{-1}(t) \) on \( L^2(S) \), given by (8.21), and \( \mathbb{V}_A^{-1}(t)h(\varphi, \theta) = w_A^{-1}(r, \varphi, \theta) \), \( r = e^t \), is the solution of the integro-differential equation (8.22).

Now let \( 0 < \alpha \leq 1 \), \( h(\varphi, \theta) = f(\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi) \) for a given \( f \in L^2(S) \), \( w_A(r, \varphi, \theta) := V_A(t)h(\varphi, \theta) \), \( w_A^{-1}(r, \varphi, \theta) := V_A^{-1}(t)h(\varphi, \theta) \), both with \( r = e^t \).

Concerning Theorem 8.1, assertion (a)(i) reveals that the solution of Dirichlet’s problem for the three dimensional unit ball, \( w_A(r, \varphi, \theta) \), tends to the boundary value \( h(\varphi, \theta) \) from the outside of the unit ball with a certain rate \( \mathcal{O}(\log r)^{\alpha} \). It can be regarded in the sense that \( w_A(r, \varphi, \theta) \) is \((C, 0)\)-summable to \( h \) or, more generally, is \((C, \gamma_1)\)-summable to \( h \) for each \( \gamma_1 \geq 0 \), with the same order. Assertion (ii) is the corresponding result for Abel summability to \( h \) with order \( \mathcal{O}(\log r)^{\alpha} \) for \( \lambda \rightarrow \infty \).

As to assertion (a)(iii), it states that the integral means over more generally, the \((C, \gamma_2 + 1)\)-means of \( w_A^{-1}(r, \varphi, \theta) \), which is the solution of the hilbertion integro-differential equation (8.22), tend to zero with order \( \mathcal{O}(\log r)^{-\alpha} \) as \( r \rightarrow \infty \). Assertion (a)(iv) is again the corresponding Abel summability result.

Furthermore, these equivalent assertions hold if and only if the \( K \)-functional \( K(t, h; L^2(S), \mathbb{D}(A)) \), describing the smoothness of the boundary value \( h \in L^2(S) \) behaves like \( \mathcal{O}(t^{\alpha}) \) for \( t \rightarrow 0+ \).

The best possible order which can be achieved, unless \( h = 0 \), is given for \( \alpha = 1 \). This is the saturation order and the saturation class is given by \( \mathbb{D}(A) \); see assertion a)(vi).

Taking the three dimensional unit ball as a model for the Earth, then one can regard our results as new observations in astrophysics. The essence is that the behaviour of the solution of Dirichlet’s problem \( w_A(r, \varphi, \theta) \) in the near vicinity of the Earth as well as the behaviour of the solution of the associated hilbertion integro-differential equation \( w_A^{-1}(r, \varphi, \theta) \) in entire space depend only on the smoothness of the boundary function \( h(\varphi, \theta) \) on the Earth’s surface.\(^8\)

The basis to this paper is Huygens’ principle which (in the form of the Chapman-Kolmogorov equation) is the semigroup property; this principle together with the law

\(^8\) Concerning the three dimensional exterior Dirichlet problem, I (PLB) had a very intensive 90 minute discussion with Gerhard Hensler, a professor for theoretical astronomy.
of inertia are universal properties revealing that the results of the paper are not only valid for our Earth but for any planet or star in the universe.

Of course, the authors are aware of the fact that one has to be careful to apply the Euclidean model to the entire universe we live in. According to A. Einstein we definitely live in a curved spacetime. For this reason the real life situation can be very different from what we have in Euclidean space. Thus at least our applications have to be treated in Non-Euclidean geometry.9

The immediate question: What basic properties upon which this paper is based do carry over to this general situation. The basis to semigroup theory, Huygens’ principle, and the closely connected Hille’s abstract Cauchy problem, should be satisfied. In fact, according to Yvonne Choquet-Bruhat (Fourès-Bruhat) [38], Einstein’s equations can be treated as a well-posed Cauchy problem.

The Cauchy formulation of general relativity splits the problem of solving Einstein’s equations, and studying the behaviour of these solutions, into two equally important tasks: First, one finds an initial data set—a “snapshot” of the gravitational field and its rate of change—which satisfies the Einstein constraint equations, which are essentially four of the ten Einstein field equations. Then, using the rest of the equations, one evolves the gravitational fields forward and backward in time, thereby obtaining the spacetime and its geometry; see [2, pp. 347 ff.].

The questions raised by R. Kurth (see Footnote 1) actually fit into the frame of this general extension.

9 Exterior Neumann’s Problem for the Unit Disk and Unit Ball

For a given \( f \in L^2_{2\pi} \) with \( \hat{f}(0) = 0 \), determine a function \( w(x, r) \) which is \( 2\pi \)-periodic with respect to \( x \in \mathbb{R} \) for each \( r > 1 \), twice continuously differentiable with respect to \( x \) and \( r \) and satisfies the Laplace equation (7.1) and the boundary conditions

\[
I_\text{tried to explain him the main properties of the two semigroups } w_A(r, \varphi, \theta) \text{ and } w_A^{-1}(r, \varphi, \theta), \text{ in particular, that the first one tends for } r \to 1^+ \text{ to the given boundary value } h(\varphi, \theta) \in L^2(S) \text{ from the outside of the unit ball, hence is the solution of the Dirichlet problem, whereas the integral means of the semigroup } w_A^{-1}(r, \varphi, \theta) \text{ tend to zero for } r \to \infty.
\]

Although he had received a copy of a previous version of the paper some months earlier, enabling him to look at it, he remarked at this stage, roughly, that he could not follow all of the mathematics involved, the issue seeming to be that a mathematical equation, having a function \( h \) defined on the surface of the Earth as boundary value, is connected in an ergodic sense with a different mathematical assertion in entire space, but associated with the same \( h \). Thus, an event on Earth is connected with the same event in space, and vice versa, but the mathematical equations associated with that event are quite different in both instances, a result indeed of interest in astrophysics. This statement reminded me strongly of the question raised by R. Kurth during his colloquium talk at Aachen (see Footnote 1).

9 In WS 1948/49, I (PLB) attended the course “Non-Euclidean Geometry” (and number theory) at U. of Toronto, held by Prof. H. S. M. Coxeter (1907–2003; born in London, Ph.D., U. Cambridge 1931, and at U. Toronto ever since 1936). He was remembered as the “Icon of Mathematics” of the past century, the “King of Infinite Space” (see [83]), “the geometry of our bestirring twentieth century” (as Buckminster Fuller wrote about him as he dedicated his book Synergetics, to Coxeter [39]). As fellow of the Canadian Royal Society, editor-in-chief of the “Canadian Journal of Mathematics” he accepted (apart from two other papers of 1952) our paper [20].
where $\phi$ belongs to the domain of the generator $A$ and apply Theorem 7.4 in order to determine the approximation error. Indeed, since $\parallel p. 254\rangle$.

Following along the same lines as in Sect. 7, the unique solution of this problem turns out to be (cf. [9, Section 7.1.2]),

$$w(x, r) = -\sum_{k \neq 0}^r e^{-|k| |x|} \hat{f}(k)e^{ikx} = -2\sum_{k=1}^{\infty} r^{-k} k \hat{f}(k) \cos kx$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) \log \left[ \frac{r^2}{1 - 2r \cos u + r^2} \right] (x \in \mathbb{R}; \ r > 1).$$

Again,

$$U(t)f(x):=w(x, e^t) = -\sum_{k \neq 0} e^{-|k| t} \frac{1}{|k|} \hat{f}(k)e^{ikx} \quad (t > 0; \ x \in \mathbb{R}).$$

has the semigroup property $U(t_1 + t_2)f(x) = U(t_1)U(t_2)f(x)$, but the $(C_0)$-property, namely $\lim_{t \to 0^+} U(t)f = f$, is violated in view of (9.1). In fact, it is easy to see

$$\lim_{t \to 0^+} U(t)f(x) = -(f * \varphi_1)(x) = \int_{-\pi}^{\pi} \hat{f}(u) du + \int_{-\pi}^{\pi} u \hat{f}(u) du,$$

where $\varphi_1 \in L^1_{2\pi}$ is defined via its Fourier series $\varphi_1(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-1} e^{ikx}$ (cf. [9, p. 254]).

Nevertheless, one may ask for the behaviour of the approximation error $\parallel U(t)f - f_1 \parallel_{L^2_{2\pi}}$ for $t \to 0^+$, where $f_1 = -(f * \varphi_1)(x)$. Noting that $U(t)f(x) = V(t)f_1(x)$, where $V(t)$ is the semigroup of (7.6), we can rewrite

$$\parallel U(t)f - f_1 \parallel_{L^2_{2\pi}} = \parallel V(t)f_1 - f_1 \parallel_{L^2_{2\pi}}$$

and apply Theorem 7.4 in order to determine the approximation error. Indeed, since $f_1$ belongs to the domain of the generator $A$ of the semigroup $V(t)$ (cf. (7.8) and (7.9)), it follows that

$$\parallel U(t)f - f_1 \parallel_{L^2_{2\pi}} = \parallel V(t)f_1 - f_1 \parallel_{L^2_{2\pi}} = O(t) \quad (t \to 0^+)$$

for all $f \in L^2_{2\pi}$. Moreover, one has

$$\parallel U(t)f - f_1 \parallel_{L^2_{2\pi}} = o(t) \quad (t \to 0^+) \iff f = 0.$$

This means that the solution of the exterior Neumann’s problem tends to $f_1$ with the optimal rate $O(t)$ for all $f \in L^2_{2\pi}$, so far an unsolved problem.
On the other hand, one may be interested in the order of approximation in (9.1). Noting that
\[
\frac{\partial}{\partial r} w(x, r) = \sum_{k \in \mathbb{Z} \setminus \{0\}} r^{-|k|+1} \hat{f}(k) e^{ikx} \bigg|_{r=e^t} = \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-(|k|+1)t} \hat{f}(k) e^{ikx},
\]
one sees that the right-hand side defines a $C_0$-semigroup $S(t)$ on the Hilbert space $X := \{ f \in L^2_{2\pi} ; \hat{f}(0) = 0 \}$. The associated generator $A$ is given by
\[
Af(x) = -\sum_{k \in \mathbb{Z} \setminus \{0\}} (|k| + 1) \hat{f}(k) e^{ikx} = -\left((\hat{f})'(x) + f(x)\right)
\]
with domain
\[
\mathcal{D}(A) = \left\{ f \in X ; (|k| + 1) \hat{f}(k) \in l^2(\mathbb{Z}) \right\} = \left\{ f \in X ; \tilde{f} \in AC_{2\pi}, (\tilde{f})' \in L^2_{2\pi} \right\}.
\]
Since $A$ is injective, the a-Drazin inverse $B = A^{ad}$ is just $A^{-1}$ and is given by
\[
Bf(x) = A^{-1} f(x) = -\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k| + 1} \hat{f}(k) e^{ikx}.
\]
It is a bounded linear operator defined on all of $X$.

As to the semigroup $V_B(t)$, it is given by (cf. (5.6), (5.7))
\[
V_B(t) f(x) = f(x) - \sqrt{t} \int_0^\infty \frac{J_1(2\sqrt{ut})}{\sqrt{u}} \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-(|k|+1)u} \hat{f}(k) e^{ikx} du
\]
\[
= f(x) - \sum_{k \in \mathbb{Z} \setminus \{0\}} (1 - e^{-t/(|k|+1)}) \hat{f}(k) e^{ikx}
\]
\[
= \sum_{k \in \mathbb{Z} \setminus \{0\}} e^{-t/(|k|+1)} \hat{f}(k) e^{ikx},
\]
and the resolvents $R(\lambda; A)$ and $R(\lambda; B)$ turn out to be
\[
R(\lambda; A) f(x) = \int_0^\infty e^{-\lambda u} \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) e^{ikx} e^{-(|k|+1)u} du = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) \frac{1}{\lambda + |k| + 1} e^{ikx} \quad (x \in \mathbb{R}; \lambda \in \mathbb{C}, \Re \lambda > 0),
\]
\[
R(\lambda; B) f(x) = \int_0^\infty e^{-\lambda u} \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) e^{ikx} e^{-u/(|k|+1)} du
\]

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\[
\hat{f}(k) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|k| + 1}{\lambda (|k| + 1) + 1} e^{ikx} \quad (x \in \mathbb{R}; \lambda \in \mathbb{C}, \Re \lambda > 0).
\]

Now we can prove a complete counterpart of Theorem 7.4, which we present here in a shortened form.

**Theorem 9.1** Let \( f \in L^2_{2\pi} \), \( w_A(x, r); := V_A(t) f(x) \) and \( w_B(x, r); := V_B(t) f(x) \) with \( r = \log t \).

(a) The following seven assertions are equivalent:

(i) \[
\left\| \frac{\partial}{\partial r} w_A(x, r) - f(x) \right\|_{L^2_{2\pi}} = \begin{cases} o(\log r) & (r \to 1^+) \\ \mathcal{O}(\log r^\alpha) & \end{cases}
\]

(ii) \[
\|r_A(x; \lambda) - f\|_{L^2_{2\pi}} = \begin{cases} o(\lambda^{-1}) & (\lambda \to \infty) \\ \mathcal{O}(\lambda^{-\alpha}) & \end{cases}
\]

(iii) \[
\left\| \frac{1}{\log r} \int_1^r w_B(x, \rho) \frac{d\rho}{\rho} \right\|_{L^2_{2\pi}} = \begin{cases} o((\log r)^{-1}) & (r \to \infty) \\ \mathcal{O}((\log r)^{-\alpha}) & \end{cases}
\]

(iv) \[
\|r_B(x; \lambda)\|_{L^2_{2\pi}} = \begin{cases} o(\lambda) & (\lambda \to 0^+) \\ \mathcal{O}(\lambda^{-\alpha}) & \end{cases}
\]

(v) \[
K(t, f; L^2_{2\pi}, \mathcal{D}(A)) = \begin{cases} o(t) & (t \to 0^+) \\ \mathcal{O}(t^\alpha) & \end{cases}
\]

(vi) \[
\|f(x + h) - f(x)\|_{L^2_{2\pi}} = \mathcal{O}(h^\alpha) \quad (h \to 0),
\]

(vii) \[
\begin{cases} f = 0, \\ f \in \mathcal{D}(A) \text{ if } \alpha = 1.
\end{cases}
\]
(b) For any \( \alpha \in (0, 1] \) there exists an element \( f_\alpha \in L^2_{2\pi} \), such that

\[
\begin{align*}
(\text{i}) & \quad \| \frac{\partial}{\partial r} w_A(x, r) - f_\alpha(x) \|_{L^2_{2\pi}} = O((\log r)^\alpha) \quad (r \to 1+), \\
(\text{ii}) & \quad \| \frac{1}{\log r} \int_1^r w_B(x, \rho) \frac{d\rho}{\rho} \|_{L^2_{2\pi}} = O((\log r)^{-\alpha}) \quad (r \to \infty), \\
(\text{iii}) & \quad \| r_A(x; \lambda) - f_\alpha(x) \|_{L^2_{2\pi}} = \| r_B(x; \lambda^{-1}) \|_{L^2_{2\pi}} = O(\lambda^\alpha) \neq o(\lambda^\alpha) \quad (\lambda \to 0+).
\end{align*}
\]

Another problem of great interest would be to investigate the behaviour of the solution of the corresponding exterior Neumann problem for the unit ball in \( \mathbb{R}^3 \):

The problem is to determine a function \( w(\varphi, \theta, r) \), defined \( [0, \pi] \times \mathbb{R} \times (1, \infty) \), which is \( 2\pi \)-periodic with respect to \( \theta \), twice continuously differentiable on its domain, and satisfies Laplace’s equation (in spherical coordinates)

\[
\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{r^2} \frac{\partial w}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2} = 0 \quad (9.6)
\]

together with the boundary conditions

\[
\begin{align*}
(\text{9.7}) & \quad \lim_{r \to 1^+} \left\| \frac{\partial}{\partial r} w(\varphi, \theta, r) - h(\varphi, \theta) \right\|_{L^2(S)} = 0, \\
(\text{9.8}) & \quad \lim_{r \to \infty} w(\varphi, \theta, r) = 0 \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}),
\end{align*}
\]

where \( h(\varphi, \theta) \) is a given function on the unit sphere belonging to \( L^2(S) \). The solution of the problem is unique, and the condition \( \hat{h}(0,0) = 0 \) is necessary and sufficient for the existence of a solution; see [92, pp. 174 ff.].

As seen in Sect. 8, the general solution of Eq. (9.6) satisfying the boundary condition (9.8) is given by

\[
w(\varphi, \theta, r) = \sum_{k=0}^{\infty} r^{-(k+1)} \sum_{m=-k}^{k} c_{m,k} Y_k^m(\varphi, \theta).
\]

For the coefficients \( c_{m,k} \), one has by (9.7),

\[
\lim_{r \to 1^+} \frac{d}{dr} r^{-(k+1)} c_{m,k} = -(k + 1) c_{m,k} = \hat{h}(m, k)
\]
This yields that
\[
  w(\varphi, \theta, r) = - \sum_{k=0}^{\infty} \frac{r^{-(k+1)}}{k+1} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 1)
\]
(9.9)
is the solution of the boundary value problem (9.6)–(9.8).

Setting
\[
  U(t)h(\varphi, \theta) := w(\varphi, \theta, e^t),
\]
one has
\[
  \lim_{t \to 0^+} U(t)h(\varphi, \theta) = - \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) := h_1(\varphi, \theta).
\]

Similarly as above, the approximation error \( \|U(t)h - h_1\|_{L^2(S)} \) can be expressed in terms of the semigroup \( T_A \) of (8.17) with generator \( A \) of (8.18), namely,
\[
  \|U(t)h - h_1\|_{L^2(S)} = \|T_A(t)h_1 - h_1\|_{L^2(S)}.
\]

Since \( h_1 \in \mathcal{D}(A) \) (cf. (8.19)), it follows from Theorem 8.1 that \( \|U(t)h - h_1\|_{L^2(S)} = \mathcal{O}(t) \) for all \( h \in L^2(S) \), and \( \|U(t)h - h_1\|_{L^2(S)} = o(t) \iff h = 0, \text{ all for } t \to 0^+ \).

As to the order of approximation in (9.7), one has
\[
  \frac{\partial}{\partial r} w(\varphi, \theta, r) \bigg|_{r=e^t} = \sum_{k=0}^{\infty} e^{-(k+2)t} \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, r > 0).
\]

The right-hand side defines a \( C_0 \)-semigroup \( S(t) \) on \( L^2(S) \), and its generator \( A \) is given by
\[
  Ah(\varphi, \theta) = - \sum_{k=0}^{\infty} (k+2) \sum_{m=-k}^{k} \hat{h}(m, k) Y^m_k(\varphi, \theta)
\]
(9.10)
with domain
\[
  \mathcal{D}(A) = \left\{ g \in L^2(S); \sum_{k=0}^{\infty} (k+2)^2 \sum_{m=-k}^{k} |\hat{g}(m, k)|^2 < \infty \right\}.
\]
(9.11)
The associated resolvent operator is given by

\[ R(\lambda; A)h(\varphi, \theta) = \sum_{k=0}^{\infty} \frac{1}{\lambda + k + 2} \sum_{m=-k}^{k} \widehat{h}(m, k)Y_k^m(\varphi, \theta) \quad (\varphi \in [0, \pi], \theta \in \mathbb{R}, t > 0). \]

The generator \( A \) is injective and the operator \( B \) of (2.6) is just \( A^{-1} \). It is defined on all of \( L^2(S) \) and has the Fourier series representation

\[ A^{-1}h(\varphi, \theta) = -\sum_{k=0}^{\infty} \frac{1}{k + 2} \sum_{m=-k}^{k} \widehat{h}(m, k)Y_k^m(\varphi, \theta). \]

Furthermore, the semigroup generated by \( A^{-1} \) is given by

\[ V_{A^{-1}}(t)h(\varphi, \theta) = w_{A^{-1}}(\varphi, \theta, e^t) = \sum_{k=0}^{\infty} e^{-t/(k+2)} \sum_{m=-k}^{k} \widehat{h}(m, k)Y_k^m(\varphi, \theta), \]

(9.12)

having resolvent

\[ R(\lambda; A^{-1})h(\varphi, \theta) = \sum_{k=0}^{\infty} \frac{k + 2}{\lambda(k + 2) + 1} \sum_{m=-k}^{k} \widehat{h}(m, k)Y_k^m(\varphi, \theta) \]

\[ (\varphi \in [0, \pi], \theta \in \mathbb{R}, t > 0). \]

Again \( w_{A^{-1}}(\varphi, \theta, r), r = e^t \), is a solution of the integro-differential equation

\[ \int_{1}^{r} u(\varphi, \theta, \rho) \frac{d\rho}{\rho} - Au(\varphi, \theta, r) = -Ah(\varphi, \theta). \]

The three-dimensional counterpart of Theorem 9.1 now reads.

**Theorem 9.2** Let \( h \in L^2(S), w(\varphi, \theta, r) = w_A(\varphi, \theta, r) \) be defined by (9.9), and \( w_{A^{-1}}(\varphi, \theta, r) \) as in (9.12) with \( r = \log t \). Further, let \( r_A(\varphi, \theta; \lambda) = \lambda R(\lambda; A)h(\varphi, \theta) \), and \( r_{A^{-1}}(\varphi, \theta; \lambda) = \lambda R(\lambda; A^{-1})h(\varphi, \theta) \).

(a) The following assertions are equivalent:

(i)

\[ \left\| \frac{\partial}{\partial r} w_A(\varphi, \theta, r) - h(\varphi, \theta) \right\|_{L^2(S)} \]

\[ = \left\| \sum_{k=0}^{\infty} r^{-(k+2)} \sum_{m=-k}^{k} \widehat{h}(m, k)Y_k^m(\varphi, \theta) - h(\varphi, \theta) \right\|_{L^2(S)} \]

\[ \left\{ \begin{array}{ll}
\mathcal{O}(\log r) & (r \to 1+) \\
\mathcal{O}((\log r)^{\alpha}) & \end{array} \right. \]
(ii)
\[
\left\| r_A(\varphi, \theta; \lambda) - h(\varphi, \theta) \right\|_{L^2(S)} = \left\| \sum_{k=0}^{\infty} \frac{\lambda}{\lambda + k + 2} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) - h(\varphi, \theta) \right\|_{L^2(S)} = \left\{ \begin{array}{ll}
o(\lambda^{-1}) & (\lambda \to \infty), \\
o(\lambda) & (\lambda \to 0+), \\
o(\log r) & (\lambda \to 0^+). \\
\end{array} \right.
\]

(iii)
\[
\left\| \frac{1}{\log r} \int_1^{r} w_{A^{-1}}(\varphi, \theta, \rho) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \left\| \frac{1}{\log r} \int_1^{r} \sum_{k=0}^{\infty} \rho^{-\frac{1}{\alpha} - 1} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) d\rho \right\|_{L^2(S)} = \left\{ \begin{array}{ll}
o((\log r)^{-1}) & (r \to \infty), \\
o((\log r)^{-1}) & (r \to \infty). \\
\end{array} \right.
\]

(iv)
\[
\left\| r_{A^{-1}}(\varphi, \theta; \lambda) \right\|_{L^2(S)} = \left\| \sum_{k=0}^{\infty} \frac{\lambda}{\lambda + k + 2} \sum_{m=-k}^{k} \hat{h}(m, k) Y_k^m(\varphi, \theta) \right\|_{L^2(S)} = \left\{ \begin{array}{ll}
o(\lambda) & (\lambda \to \infty), \\
o(\lambda) & (\lambda \to 0^+), \\
o(\log r)^\alpha & (\lambda \to 0^+). \\
\end{array} \right.
\]

(v)
\[
K(t, h; L^2(S), D(A)) = \left\{ \begin{array}{ll}
o(t) & (t \to 0^+), \\
o(t^\alpha) & (t \to 0^+). \\
\end{array} \right.
\]

(b) For any $\alpha \in (0, 1]$ there exists an element $h_\alpha \in L^2(S)$, such that

(i)
\[
\left\| w_A(\varphi, \theta, r) - h_\alpha(\varphi, \theta) \right\|_{L^2(S)} = \left\{ \begin{array}{ll}
o((\log r)^\alpha) & (r \to 1^+), \\
o((\log r)^{\alpha-1}) & (r \to \infty). \\
\end{array} \right.
\]

(ii)
\[
\left\| \frac{1}{\log r} \int_1^{r} w_{A^{-1}}(\varphi, \theta, \rho) \frac{d\rho}{\rho} \right\|_{L^2(S)} = \left\{ \begin{array}{ll}
o((\log r)^{-\alpha}) & (r \to \infty), \\
o((\log r)^{-1}) & (r \to \infty). \\
\end{array} \right.
\]
Conclusion 9.3 Let \( w_A(\varphi, \theta, r) \) be the solution of the exterior Neumann problem. Then \( \frac{\partial}{\partial r} w_A(\varphi, \theta, r) \) tends in the norm to the boundary value \( h(\varphi, \theta) \) from the outside of the unit ball with maximal approximation rate \( O(\log r) \) if and only if the integral mean of the semigroup generated by \( A^{-1} \), namely \( \frac{1}{\log r} \int_1^r w_{A^{-1}}(\varphi, \theta, \rho) \, d\rho \), tends to zero with maximal rate \( O(1/\log r) \) for \( r \to \infty \). Furthermore, these two equivalent assertions hold if and only if the boundary value \( h \) belongs to the domain of the generator \( A \). In particular, the behaviour of the integral mean of \( w_{A^{-1}}(\varphi, \theta, r) \) at infinity (assertion (a)(iii)) depends only upon the properties of the function \( h \) on the surface of the unit ball (assertions (a)(v), (vi)).

As in Theorem 8.1, one would like to have a characterisation of the smoothness of the boundary function \( h \) in terms of a modulus of continuity which is equivalent to the \( K \)-functional in assertion (a)(v).

The volume by David Edmunds and William Evans [34] treats the Dirichlet and Neumann problem in their Chapters VI and VII in an abstract setting.

10 Further Examples and Possible Extensions

As another example, we sketch the Dirichlet problem for the upper half plane; see [6, Section 4.2.3]. It will turn out that the generalized inverse operator \( B \) is unbounded, unlike the examples discussed above. However, this is not a significant change, since the theoretical foundations from Sects. 2–6 explicitly allow \( B \) to be unbounded.

Given a function \( f \in L^2(\mathbb{R}) \), we call for a function \( u(x, y) \), defined on \( \mathbb{R}^2_+ := \{ (x, y) \in \mathbb{R}^2; y > 0 \} \) such that

\[
\Delta u = \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \quad ((x, y) \in \mathbb{R}^2_+),
\]

\[
\lim_{y \to 0^+} \| u(0, y) - f(0) \|_{L^2(\mathbb{R})} = 0.
\]

The unique solution of this problem is given by the singular integral of Cauchy-Poisson,

\[
u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y|v|} \hat{f}(v) e^{ixv} \, dv = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-s)}{y^2 + s^2} \, ds \quad ((x, y) \in \mathbb{R}^2_+),
\]

\[
\hat{f}(v) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixv} \, dx \text{ being the } L^2(\mathbb{R}) \text{-Fourier transform of } f.
\]
The singular integral of Cauchy-Poisson defines a holomorphic $C_0$-semigroup $V_A(y)$ on $L^2(\mathbb{R})$, i.e.,

$$V_A(y)f(x) := \begin{cases} \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-s)}{y^2+s^2} \, ds, & y > 0, \\ f(x), & y = 0. \end{cases}$$

Its infinitesimal generator $A$ is given by $Af = -\tilde{f}'$, where

$$\tilde{f}(x) := PV \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(u)}{x-u} \, du$$

is the Hilbert transform of $f$. For the domain and range of $A$ one has

$$\mathcal{D}(A) = \{ f \in L^2(\mathbb{R}); \tilde{f}' \in L^2(\mathbb{R}) \} = \{ f \in L^2(\mathbb{R}); \tilde{v}\tilde{f}(v) \in L^2(\mathbb{R}) \},$$

$$\mathcal{R}(A) = \{ g \in L^2(\mathbb{R}); v^{-1}\tilde{g}(v) \in L^2(\mathbb{R}) \}.$$ Since $\hat{(Af)}(v) = -|v|\hat{f}(v)$, it follows that $A$ is injective, hence $\mathcal{N}(A) = \{ 0 \}$, $X_0 = \mathcal{R}(A) = L^2(\mathbb{R})$ and $Pf = 0$.

As to the resolvent $R(\lambda; A)$, it is given by

$$R(\lambda; A)f(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda y} \left( \int_{-\infty}^{\infty} e^{-y|v|} \hat{f}(v)e^{-ixv} \, dv \right) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v) \frac{1}{\lambda + |v|} e^{ixv} \, dv \quad (x \in \mathbb{R}; \lambda \in \mathbb{C}, \Re \lambda > 0).$$

Since $V_A(y)$ is equibounded with $\|V_A(\cdot)\|_{L^2(\mathbb{R})} = 1$, the generator $A$ possesses a $a$-Drazin inverse $B = A^{ad}$, which can be evaluated by Proposition 4.3 to

$$Bf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |v|^{-1} \hat{f}(v)e^{-ixv} \, dv.$$ It follows that $B$ is an unbounded operator on $\mathcal{D}(B) = \mathcal{R}(A)$, which, in view of Theorem 5.2, generates a semigroup $V_B(t)f$ on the whole of $L^2(\mathbb{R})$. It has the representation,

$$V_B(t)f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t/|v|} \hat{f}(v)e^{ixv} \, dv \quad (x \in \mathbb{R}; t > 0). \quad (10.1)$$

For a proof of this representation formula one may proceed via (5.6) and (5.7), but this approach leads to the difficulty of justifying the interchange of the order of integration in the double integral involved. On the other hand, it is easy to show that the right-hand side of (10.1) actually defines a $C_0$-semigroup with generator $B$. Since there exists at most one such semigroup (cf. [6, p. 37]), it must be equal to $V_B(t)$. 

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Birchhauser
Now we are in a position to state a theorem quite analogous to Theorem 7.4. The results on the approximation behaviour of $V_A(y)$ towards $f$ are already contained in [6, Section 4.2.3], in particular the saturation result, i.e., the case $\alpha = 1$. The details are left to the reader.

Similarly, one may treat the so-called Fourier’s ring problem with initial distribution $f \in L^2_2\pi$. Its solution $w(x, t)$ is the singular convolution integral of Weierstraß having Jacobi’s theta function as kernel,

$$w(x, t) = \sum_{k=-\infty}^{\infty} e^{-k^2 t} \hat{f}(k)e^{ikx} \quad (x \in \mathbb{R}; \ t > 0);$$

see [6, Sections 1.5.2 and 2.4.3], [9, Section 7.1.1]. Again, $W(t)f(x) := w(x, t)$ defines a $C_0$-semigroup, its generator being $Af = f''$. Since the conjugate function is not involved, the treatment of this application is simpler; see [41, p. 54 ff.]. For the saturation class see also [9, Section 12.2.3] and [14].

The Gauß-Weierstraß singular integral on $\mathbb{R}^n$,

$$W(t)f(x) := \begin{cases} 
\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(x - u) \exp\left(\frac{-|u|^2}{4t}\right) du, & t > 0, \\
f(x), & t = 0 
\end{cases} \quad (x \in \mathbb{R}^n)$$

for a given function $f \in L^2(\mathbb{R}^n)$ is the solution of Cauchy’s problem for the heat conduction equation in $\mathbb{R}^n$,

$$\frac{\partial w(x, t)}{\partial t} = \Delta w(x, t) \quad (x \in \mathbb{R}^n),$$

$$\lim_{t \to 0^+} \|w(o, t) - f(o)\|_{L^2(\mathbb{R}^n)} = 0.$$
s-lim\(_{t \to 0} C(t)f = f\) for all \(f \in X\). The infinitesimal generator \(A'\) is then given by
\[A'f := s-lim_{h \to 0} 2h^{-2}[C(h)f - f]\]
for those elements \(f \in X\) for which this limit exists. A generalized inverses operator \(B'\) can also be defined in an appropriate way. Counterparts of Theorems 3.1, 4.5, 5.2, and 6.1 can be found in [13, 41].

As an example, one may consider the partial differential equation
\[v_{tt}(x, t) = v_{xx}(x, t) + v_x(x, 0) = f(x), v_t(x, 0) = 0, x, t \in \mathbb{R}.\]
Its solution \(v\) defines a cosine operator function
\[C(t)f(x) := v(x, t)\text{ on } X = \text{UCB}(\mathbb{R}),\]
the space of uniformly continuous and bounded function on \(\mathbb{R}\). For details see [17]. Further examples can be found in [13, 41].

### 11 A Short Biography of Einar Hille

Einar (Carl) Hille, born June 28, 1894 in New York City, both parents being immigrants from Sweden, grew up with his mother in Stockholm from 1896 onwards. He received his doctorate under Marcel Riesz at the University of Stockholm in 1918. Hille had the fortune to have had as teachers such exceptional persons as the chemist Hans von Euler-Chelpin, the mathematicians Ivar Bendixson, Helge von Koch, and the economist Gustaf Cassel. Hille returned to the USA in 1920 when he was 26 years old.

He first taught at Stockholm, spent the academic year 1920–21 at Harvard with Birkhoff, learning further from Kellogg, Coolidge and Osgood, then taught there, and left 1922 for Princeton, where he advanced from instructor to associate professor in 1927. In 1926 he received a fellowship, enabling him to visit Copenhagen, Stockholm and Göttingen, first working with Nörlund on pdes. At Copenhagen he also made friends with both Bohrs, Nielson, Bonnessen, Steffensen, and Molerup. At Göttingen, the mecca of mathematicians at the time, he met some 20 of them, from Germany, Russia, Hungary and Poland. He visited Göttingen again in 1932, 1953 and 1956. While at Princeton he spent the summer quarters of 1028 at Stanford, of 1931 at Chicago, and 1932 he spent his sabbatical year at Zürich, meeting Zygmund and Karamata, then Vienna, Budapest and Stockholm. His final position was an endowed professorship in the graduate school of Yale University in 1933.

It was most fortunate for Hille that he learned to know J. D. Tamarkin (1880–1945). Their close collaboration 1927–37 resulted in 23 joint papers. Hille began his twenty-year study of analytical semigroups at Yale in 1936. Nelson Dunford (1906–1986), a student of Tamarkin and teacher of J. T. Schwartz, expressed his thanks to Hille when he came as instructor to Yale in 1936. An outcome was Nelson’s monumental three volume treatise with Jack Schwartz dealing with functional analysis and operator theory [32].

In the period 1956–57 Einar Hille came as visiting professor to the university of Mainz together with his spouse Kirsti (sister of Øystein Ore) and their sons Harald and Bertil. While there he aroused great Enthusiasm for Analytical Semigroups and so conducted a well-attended seminar in the subject. Being in a somewhat similar position at Mainz for 1955–57, I (PLB) took active part in it, particularly since I had
already become familiar with his monograph on semigroups [48] while teaching at McGill University, Montreal.

Learning to know Hille, with his unusual kindness and readiness to help, I being just 28 at the time, enhanced my own standpoint that students be treated with equal respect as one’s colleagues, and that teaching is as important as research. Concerning my first course on semigroups of operators at the RWTH-Aachen in 1962, in its resulting monograph “Semi-groups of Operators and Approximation” (1967) with Hubert Berens [6], Chapters I and II depended heavily upon Hille’s monograph [48] of 1948. As one of the conductors of the Oberwolfach conference of 1971 [19], I was very happy that Hille accepted my invitation to present the opening lecture.

Hille’s contributions to mathematics covered a broad spectrum, a leitmotive being his fusion of classical analysis into modern operator theory. He published approximately 175 papers and 12 books. According to the mathematical genealogy project, Hille has 28 students and 1665 descendants.

Hille was a member of the National Academy of Sciences (1953), of the Royal Academy of Sciences of Stockholm and several others, a Knight of the Swedish Order of the North Star, a recipient of the John Ericsson Gold Medal by the American Society of Swedish Engineers. He retired in 1962 and died in La Jolla in 1980.

An appraisal of “Hille’s work in semigroup theory and approximation” by Walter Trebels and PLB, appeared in [49]. Concerning the history of semigroup theory, see [6, pp. 75–78]. For obituary addresses and personal recollections including a list of his publications and Ph.D. students see [33, 52, 81, 99, 100].

Research Legacy

This paper presents the research legacy of joint work in the specific fields of semigroup theory and harmonic analysis with several generations of our students born between 1936 and 1970, namely H. Berens, W. Köhnen [7], U. Westphal, W. Trebels [93, 94], S. Pawelke, R. L. Stens, W. Dickmeis, D. Pfeifer [78–80], M. Wehrens and A. Gessinger, who may loosely be regarded as academic grandsons of Einar Hille.

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