Abstract: The mixing of a neutral unstable particle and its antiparticle has been discussed in the framework of quantum mechanics in the literature. In this paper, a method to study particle-antiparticle mixing fully in the quantum field theory is developed, and it is shown that quantum mechanics is not the proper non-relativistic limit of the quantum field theory in the presence of heavy particle-antiparticle mixing. Moreover, it is also shown that a discrepancy indeed exists between the results from quantum mechanics and the quantum field theory, which could sometimes make big differences in theoretical predictions of observables. This implies that the mixing of neutral mesons should be reanalyzed with the method developed in this paper especially when anomalies have been observed in the decays of mesons, although the quantum mechanical approach has been very successful so far.
1 Introduction

In a neutral meson system, the meson and antimeson mix with each other, which causes CP violation in the decay of the meson or antimeson. Such CP violation effects have been experimentally observed in various decay channels of various mesons for decades, and many of the measured values of CP asymmetries are found to be consistent with the values obtained by theoretical analyses. In the presence of such a mixing effect of unstable particles, the standard way to calculate the decay widths and CP asymmetries is finding the effective Hamiltonian of the neutral meson system and solving the associated effective Schrödinger equation.
This is because there seems to be a difficulty in handling the mixing effect in the quantum field theory. The mass eigenstates of neutral mesons, e.g., $K^0_L$, are thought to be some linear combinations of basis states, e.g., $K^0$, $\bar{K}^0$, and the associated mixing matrix is non-unitary. Since non-unitary mixing makes the mixed states non-orthogonal to each other, they cannot be treated like free states even in the free theory. More specifically, for two states $|a\rangle$, $|b\rangle$ which are non-orthogonal to each other, i.e., $\langle b|a \rangle \neq 0$, there exists a transition $a \rightarrow b$ even without any interactions in the theory, and thus they can never be regarded as asymptotically free states of the quantum field theory. In other words, there exists a fundamental difference between the mass eigenstates and the basis mesons, and the former should be regarded as the degrees of freedom which are generated by interactions. Since such particles cannot be considered to be excitations of external fields in any calculations, we cannot use the textbook approach of the quantum field theory to calculate, for example, the decay widths of mass eigenstates which require calculating matrix elements such as $\langle \pi^+ \pi^- | S | K^0_L \rangle$. Moreover, the decay widths of the basis states are often problematic as well, since the straightforward calculation using the matrix element such as $\langle \pi^+ \pi^- | S | K^0 \rangle$ is valid only when the effect of particle-antiparticle mixing can be safely neglected. The details will be discussed later in this paper.

In quantum mechanics, however, everything looks simpler and straightforward, as long as each mass eigenstate follows the time evolution of a plane wave with a damping factor which describes its decay. We can easily write down an effective Schrödinger equation whose solution is the time-dependent mass eigenstate. From that equation and the mixing matrix, it seems straightforward to find the equation for the basis states, as is conventionally done. Obtaining a time-dependent state, e.g., $|K^0(t)\rangle$ or $|K^0_L(t)\rangle$, as a solution and calculating the matrix element such as $\langle \pi^+ \pi^- | S | K^0(t) \rangle$ or $\langle \pi^+ \pi^- | S | K^0_L(t) \rangle$, we can find the decay widths of mesons, $\Gamma(K^0 \rightarrow \pi^+ \pi^-)$ or $\Gamma(K^0_L \rightarrow \pi^+ \pi^-)$, by integrating over time its textbook formula using that matrix element. Since the decay width is a quantity defined in the rest frame of the decaying particle and quantum mechanics is the non-relativistic limit of the quantum field theory, this approach based on quantum mechanics is supposed to work well.

However, a careful study of mixing in the quantum field theory shows that the situation is not so simple. It turns out that we cannot relate a single linear combination of basis states to the degree of freedom corresponding to the damped plane wave solution, and such a degree of freedom should therefore be regarded as a quasiparticle, i.e., an emergent particle dynamically generated by interactions. This is inconsistent with the standard picture in quantum mechanics, where it is always assumed that a single linear combination follows the time evolution of a decoupled plane wave with a damping factor. This in turn implies that quantum mechanics is not the proper non-relativistic limit of the quantum field theory in the presence of heavy particle-antiparticle mixing, and we should therefore find a way outside quantum mechanics in order to correctly study the physics of mixing and the properties of quasiparticles.

In this paper, we develop a method to investigate the phenomenology of heavy particle-antiparticle mixing fully in the quantum field theory. To be specific, we discuss how to calculate the decay widths of neutral mesons and the CP asymmetries in their decays, properly considering the mixing effect in the framework of the quantum field theory. The basic strategy is to read the decay widths from scattering mediated by on-shell quasiparticles. In other words, mesons are always considered as intermediate states in this formalism. To simplify the discussion, a toy model that would imitate the effective theory of neutral scalar mesons will be introduced. Using that model, we will discuss multiple ways of deriving decay widths, and show that they are all mutually consistent with each other. The result will be compared with the expressions from the conventional approach based on quantum mechanics, and we will see a discrepancy which sometimes could cause a big difference in the values of decay
widths. Note that the discrepancy is unnoticeable in many cases, which explains why the experiments and theoretical analyses have been usually consistent. Even the decay width of a basis meson will also be found from the on-shell quasiparticles, which means that the basis meson is slightly off-shell even when we consider its decay. Furthermore, it will be shown that it is not generally possible to separately define the partial decay width of each quasiparticle to a specific channel, and only their sum can always be well-defined. In addition, it will be clarified how the partial decay widths of a basis meson or a quasiparticle are related to the total decay width of a quasiparticle that can be found in the imaginary part of a physical pole of a resummed propagator. In this paper, all the derivations of decay widths will be done up to the leading order in perturbation, the meaning of which will be explained later.

The description of mixing in the quantum field theory has been an interesting research topic in the literature. For example, in reference [1], the physics of mixed fields was discussed in terms of their quantization. In that paper, however, the mixed fields were assumed to be obtained by a unitary transformation from the basis fields, although the mixing is actually non-unitary for unstable particles. Moreover, the mixed fields which are mutually non-orthogonal cannot be canonically quantized, since it is inconsistent with the canonical quantization of the basis fields. In contrast, this paper introduces a new approach to the phenomenology of mixing by interpreting the mixed states as quasiparticles.

This paper is organized as follows: in section 2, the renormalizable toy model that could imitate a meson system is introduced, and the expression of the renormalized self-energy is derived; in section 3, we discuss how to diagonalize the resummed propagator matrix of mesons; in section 4, the degree of freedom that propagates like a free particle until it decays is interpreted as a quasiparticle, and the limitations of quantum mechanics is discussed; in section 5, the decay widths of mesons are derived in multiple ways in the quantum field theory; in section 6, the results are compared with those obtained by conventional method based on quantum mechanics, and it is shown that a discrepancy exists; in section 7, two numerical examples that imitate real meson systems are provided to test the formalism and also to help understand the physics behind it.

2 Toy model and renormalization of its Lagrangian

A real meson system is very complicated, not only because many particles contribute to even the simplest type of interactions, but also because the non-perturbative interactions of quantum chromodynamics are involved in many processes. However, the physics of particle-antiparticle mixing can be understood in a much simpler way, since what is really important is the dynamics at the level of an effective field theory of mesons. In this section, we develop a toy model that would imitate such an effective field theory. By choosing model parameters, we can indeed make the model mimic real neutral meson systems quite closely, as we will see in section 7.

The toy model consists of four different kinds of fields, \( \Phi, \chi_i, \xi, \psi_i \), which interact with each other through

\[
\mathcal{L}_{\text{int}} = - \sum_i f_i \bar{\psi}_i \psi_i \Phi - \sum_i h_i \chi_i \xi \Phi + \text{H.c.} 
\]  

(2.1)

The roles of the fields are as follows:

1. \( \Phi \) : a neutral scalar field corresponding to a spin-0 meson such as \( B^0 \) or \( K^0 \). 

The description of mixing in the quantum field theory has been an interesting research topic in the literature. For example, in reference [1], the physics of mixed fields was discussed in terms of their quantization. In that paper, however, the mixed fields were assumed to be obtained by a unitary transformation from the basis fields, although the mixing is actually non-unitary for unstable particles. Moreover, the mixed fields which are mutually non-orthogonal cannot be canonically quantized, since it is inconsistent with the canonical quantization of the basis fields. In contrast, this paper introduces a new approach to the phenomenology of mixing by interpreting the mixed states as quasiparticles.

This paper is organized as follows: in section 2, the renormalizable toy model that could imitate a meson system is introduced, and the expression of the renormalized self-energy is derived; in section 3, we discuss how to diagonalize the resummed propagator matrix of mesons; in section 4, the degree of freedom that propagates like a free particle until it decays is interpreted as a quasiparticle, and the limitations of quantum mechanics is discussed; in section 5, the decay widths of mesons are derived in multiple ways in the quantum field theory; in section 6, the results are compared with those obtained by conventional method based on quantum mechanics, and it is shown that a discrepancy exists; in section 7, two numerical examples that imitate real meson systems are provided to test the formalism and also to help understand the physics behind it.

2 Toy model and renormalization of its Lagrangian

A real meson system is very complicated, not only because many particles contribute to even the simplest type of interactions, but also because the non-perturbative interactions of quantum chromodynamics are involved in many processes. However, the physics of particle-antiparticle mixing can be understood in a much simpler way, since what is really important is the dynamics at the level of an effective field theory of mesons. In this section, we develop a toy model that would imitate such an effective field theory. By choosing model parameters, we can indeed make the model mimic real neutral meson systems quite closely, as we will see in section 7.

The toy model consists of four different kinds of fields, \( \Phi, \chi_i, \xi, \psi_i \), which interact with each other through

\[
\mathcal{L}_{\text{int}} = - \sum_i f_i \bar{\psi}_i \psi_i \Phi - \sum_i h_i \chi_i \xi \Phi + \text{H.c.} 
\]  

(2.1)

The roles of the fields are as follows:

1. \( \Phi \) : a neutral scalar field corresponding to a spin-0 meson such as \( B^0 \) or \( K^0 \). 

The description of mixing in the quantum field theory has been an interesting research topic in the literature. For example, in reference [1], the physics of mixed fields was discussed in terms of their quantization. In that paper, however, the mixed fields were assumed to be obtained by a unitary transformation from the basis fields, although the mixing is actually non-unitary for unstable particles. Moreover, the mixed fields which are mutually non-orthogonal cannot be canonically quantized, since it is inconsistent with the canonical quantization of the basis fields. In contrast, this paper introduces a new approach to the phenomenology of mixing by interpreting the mixed states as quasiparticles.

This paper is organized as follows: in section 2, the renormalizable toy model that could imitate a meson system is introduced, and the expression of the renormalized self-energy is derived; in section 3, we discuss how to diagonalize the resummed propagator matrix of mesons; in section 4, the degree of freedom that propagates like a free particle until it decays is interpreted as a quasiparticle, and the limitations of quantum mechanics is discussed; in section 5, the decay widths of mesons are derived in multiple ways in the quantum field theory; in section 6, the results are compared with those obtained by conventional method based on quantum mechanics, and it is shown that a discrepancy exists; in section 7, two numerical examples that imitate real meson systems are provided to test the formalism and also to help understand the physics behind it.
2. \( \psi_i \): a light fermion that allows particle-antiparticle mixing. The decay \( \Phi \to \psi_i \psi_i^c \) could mimic the decay of mesons into CP eigenstates. The index \( i \) denotes a flavor of \( \psi \), and \( \psi_i^c \) is the charge conjugate of \( \psi_i \).

3. \( \chi_i, \xi \): light fermions used to tag the initial meson states. The decays \( \Phi \to \chi_i \xi^c \) and \( \Phi \to \chi_i \xi \) could imitate the semileptonic decays of mesons. The index \( i \) denotes a flavor of \( \chi \).

The light particles are assumed to be fermions rather than scalars so that (i) the toy model is a renormalizable theory, and (ii) the lowest-order loop corrections to the self-energy are the one-loop diagrams. In a non-renormalizable theory, there could be an ambiguity in the choice of the renormalized self-energy, which is never desirable in the discussion at the level of a toy model. Moreover, we assume that \( \psi_i \) and \( \xi \) are massless for simplicity, since introducing non-zero masses to those fermions only complicates the analysis with no practical advantage.

In order to carefully derive the renormalized self-energy of \( \Phi \), we begin with the bare Lagrangian of the toy model and renormalize it step-by-step. The bare Lagrangian involving meson \( \Phi \) is given by

\[
\mathcal{L} = \partial^\mu \Phi_0^\dagger \partial_\mu \Phi_0 - m_0^2 \Phi_0^\dagger \Phi_0 - \sum_i f_{i0} \bar{\psi}_{i0} \psi_i \Phi_0 - \sum_i f_{i0} \bar{\psi}_{i0} \psi_i \Phi_0^\dagger - \sum_i h_{i0} \bar{\chi}_{i0} \xi_0 \Phi_0 - \sum_i h_{i0} \bar{\xi}_{i0} \chi_0 \Phi_0^\dagger
- \delta_f \partial^\mu \Phi_0 \partial_\mu \Phi_0 - \delta_h \partial^\mu \Phi_0^\dagger \partial_\mu \Phi_0^\dagger - \delta m_0^2 \Phi_0 \Phi_0^\dagger - (\delta m_0^2)^* \Phi_0^\dagger \Phi_0^\dagger.
\]

(2.2)

The field-strength renormalization factors are defined by

\[
\Phi_0 = Z_\Phi^1 \Phi; \quad \psi_{i0} = Z_\psi^1 \psi_i; \quad \chi_{i0} = Z_\chi^1 \chi_i; \quad \xi_0 = Z_\xi^1 \xi,
\]

(2.3)

and we also define the counterterm \( \delta_f \) associated with \( Z_\Phi \) as

\[
Z_\Phi^0 = 1 + \frac{1}{2} \delta_f.
\]

(2.4)

The renormalized mass \( m_\Phi \), mass renormalization factor matrix \( Z_M \), renormalized Yukawa coupling \( f_i \), and vertex counterterms \( \delta f_i, \delta h_i \) are defined by

\[
m_0 = Z_M m_\Phi, \quad \delta m_0^2 := m_0^2 Z_M^2 |Z_\Phi| - m_\Phi^2,
\]

(2.5)

\[
f_{i0} Z_\psi^1 Z_\Phi^0 Z_\chi^0 Z_\xi^0 =: h_i + \delta h_i.
\]

(2.6)

The bare Lagrangian can now be rewritten in terms of renormalized fields and couplings as

\[
\mathcal{L} = \partial^\mu \Phi_0^\dagger \partial_\mu \Phi_0 - m_0^2 \Phi_0^\dagger \Phi_0 - \sum_i f_{i0} \bar{\psi}_{i0} \psi_i \Phi_0 - \sum_i f_{i0} \bar{\psi}_{i0} \psi_i \Phi_0^\dagger - \sum_i h_{i0} \bar{\chi}_{i0} \xi_0 \Phi_0 - \sum_i h_{i0} \bar{\xi}_{i0} \chi_0 \Phi_0^\dagger
+ \delta_f \partial^\mu \Phi_0 \partial_\mu \Phi_0 + \delta_h \partial^\mu \Phi_0^\dagger \partial_\mu \Phi_0^\dagger + \delta m_0^2 \Phi_0 \Phi_0^\dagger + (\delta m_0^2)^* \Phi_0^\dagger \Phi_0^\dagger,
\]

\[
= |Z_M| \partial^\mu \Phi^\dagger \partial_\mu \Phi - m_\Phi^2 Z_M^2 |Z_\Phi| \Phi^\dagger \Phi - \sum_i f_{i0} Z_\psi Z_\Phi^0 Z_{\psi_i} \psi_i \Phi - \sum_i f_{i0} Z_\psi Z_\Phi^0 Z_{\psi_i} \psi_i \Phi^\dagger
- \sum_i h_{i0} Z_\chi Z_\xi Z_\Phi^0 Z_{\chi_i} \Phi - \sum_i h_{i0} Z_\chi Z_\xi Z_\Phi^0 Z_{\chi_i} \Phi^\dagger
- \delta_f Z_M \partial^\mu \Phi \partial_\mu \Phi - \delta_h Z_M \partial^\mu \Phi^\dagger \partial_\mu \Phi^\dagger - \delta m_\Phi^2 Z_M \Phi \Phi^\dagger - (\delta m_\Phi^2)^* Z_M \Phi^\dagger \Phi^\dagger,
\]

\[
= \partial^\mu \Phi^\dagger \partial_\mu \Phi - m_\Phi^2 \Phi^\dagger \Phi - \sum_i f_i \bar{\psi}_i \psi_i \Phi - \sum_i f_i \bar{\psi}_i \psi_i \Phi^\dagger - \sum_i h_i \bar{\chi}_i \xi \Phi - \sum_i h_i \bar{\chi}_i \xi \Phi^\dagger
\]

- \delta_f \partial^\mu \Phi \partial_\mu \Phi - \delta_h \partial^\mu \Phi^\dagger \partial_\mu \Phi^\dagger - \delta m_\Phi^2 \Phi \Phi^\dagger - (\delta m_\Phi^2)^* \Phi^\dagger \Phi^\dagger,
\[ + \frac{1}{2} (\delta_\phi + \delta_\Phi + \cdots) \partial^\mu \Phi^\dagger \partial_\mu \Phi - \delta m_\Phi^2 \Phi^3 - \sum_i \delta f_i \bar{\psi}_i \psi_i \Phi - \sum_i \delta f_i^* \bar{\psi}_i \psi_i \Phi^\dagger - \sum_i \delta h_i \bar{\chi}_i \xi \Phi - \sum_i \delta h_i^* \bar{\xi}_i \chi_i \Phi^\dagger \]
\[ - \delta_\Phi(1 + \cdots) \partial^\mu \Phi \partial_\mu \Phi - \delta_\Phi^* (1 + \cdots) \partial^\mu \Phi^\dagger \partial_\mu \Phi^\dagger - \delta m_\Phi^2 (1 + \cdots) \Phi \Phi - (\delta m_\Phi^2)^* (1 + \cdots) \Phi^\dagger \Phi^\dagger, \]

where the counterterms are explicitly shown only up to \( O(f^2/4\pi) \) and \( O(h^2/4\pi) \). We further assume \( |f_i| \gtrsim |h_j| \) so that \( O(f^2/4\pi) \) can be considered as the typical magnitude of perturbative corrections in the theory. The Feynman diagrams of tree-level interactions in this toy model are given in figure 1. Note that \( \Phi^* := \Phi^\dagger \) as usual.

![Figure 1](image1.png)

These interactions generate self-energy diagrams given by figures 2 and 3. For convenience, we define

![Figure 2](image2.png)

![Figure 3](image3.png)

and write the self-energy matrix of \( \Phi_\alpha \) as \( \Sigma_0 (p^2) := p^2 \Sigma_0^\prime (p^2) \), which is explicitly written as

\[ (\Sigma^\prime_0)_{\beta\alpha} (p^2) = \sum_i \left[ (\Sigma_0^\psi \psi_i^\dagger)_{\beta\alpha} (p^2) + \delta_{\beta\alpha} \Sigma_0^{\chi_i \xi} (p^2) \right], \]
where
\[
\Sigma_{0\chi,\xi}(p^2) := \frac{|h_i|^2}{16\pi^2} \left( 1 - \frac{m_{\chi,\xi}^2}{p^2} \right) \left[ \frac{m_{\chi,\xi}^2}{p^2} \left( 1 + \frac{m_{\chi,\xi}^2}{p^2} \right)^{-1} \log \left( \frac{m_{\chi,\xi}^2}{m_{\chi,\xi}^2} \right) - \log \left( \frac{p^2 - m_{\chi,\xi}^2}{\Lambda^2} \right) \right] + i \pi, \quad (2.10)
\]
and
\[
\Sigma_{0\psi,\psi}(p^2) := \frac{1}{16\pi^2} \left( \frac{|f_i|^2}{f_i^2} \right) \left[ -\log \left( \frac{p^2}{\Lambda^2} \right) + i \pi \right]. \quad (2.11)
\]
The derivation of these expressions are given in appendix A. Using the counterterms in equation 2.7, we can write the diagonal components of the renormalized self-energy \( \Sigma(p^2) \) up to \( \mathcal{O}(f^2/4\pi) \) as
\[
\Sigma_{11}(p^2) = p^2 \Sigma_{11}'(p^2) = p^2 \left[ (\Sigma_0')_{11}(p^2) + \frac{1}{2}(\delta^*_\Phi + \delta_\Phi) \right] - \delta m^2_\Phi, \quad (2.12)
\]
\[
\Sigma_{22}(p^2) = p^2 \Sigma_{22}'(p^2) = p^2 \left[ (\Sigma_0')_{22}(p^2) + \frac{1}{2}(\delta^*_\Phi + \delta_\Phi) \right] - \delta m^2_\Phi = \Sigma_{11}(p^2). \quad (2.13)
\]
In addition, the off-diagonal components of \( \Sigma(p^2) \) up to the same precision are written as
\[
\Sigma_{12}(p^2) = p^2 \Sigma_{12}'(p^2) = p^2 \left[ (\Sigma_0')_{12}(p^2) + \delta^*_\Phi \right] - (\delta m^2_\Phi)^*, \quad (2.14)
\]
\[
\Sigma_{21}(p^2) = p^2 \Sigma_{21}'(p^2) = p^2 \left[ (\Sigma_0')_{21}(p^2) + \delta^*_\Phi \right] - \delta m^2_\Phi. \quad (2.15)
\]
In order to have a UV-finite self-energy \( \Sigma(p^2) \) for any \( p \), the counterterms \( \delta m^2_\Phi \) and \( \delta m^2_\Phi \) must be finite, which implies we may set \( \delta m^2_\Phi = \delta m^2_\Phi = 0 \), i.e., \( Z^2 = |Z_\Phi| \), without loss of generality. Moreover, we may choose a real number for \( \delta_\Phi \) since its imaginary part has no role in renormalization. Hence, the bare Lagrangian can be rewritten as
\[
\mathcal{L} = \partial^\mu \Phi^0_0 \partial_\mu \Phi_0 - m_0^2 \Phi^0_0 \Phi^0_0 - \sum_i f_{i0} \overline{\psi}_{i0} \psi_{i0} \Phi_0 - \sum_i h_{i0} \overline{\chi}_{i0} \xi_{i0} \Phi_0 - \sum_i h^*_i \overline{\chi}_i \xi_i \Phi^0_0
\]
\[
+ \delta^*_\Phi \partial^\mu \Phi_0 \partial_\mu \Phi_0 + \delta^*_\Phi \partial^\mu \Phi^0_0 \partial_\mu \Phi^0_0
\]
\[
= \partial^\mu \Phi^0 \partial_\mu \Phi - m_\Phi^2 \Phi^0 \Phi - \sum_i f_i \overline{\psi}_i \psi_i \Phi - \sum_i h_i \overline{\chi}_i \xi_i \Phi - \sum_i h^*_i \overline{\chi}_i \xi_i \Phi^0 + \cdot \cdot \cdot \quad (2.16)
\]
In addition, the renormalized self-energy is now given by
\[
\Sigma'(p^2) = \Sigma_0'(p^2) + \delta \Sigma', \quad (2.17)
\]
where \( \delta \Sigma \) is a counterterm matrix defined by
\[
\delta \Sigma' := \begin{pmatrix} \delta_\Phi & \delta^*_\Phi \\ \delta^*_\Phi & \delta_\Phi \end{pmatrix}. \quad (2.18)
\]
Note that $\delta \Sigma'$ is a Hermitian matrix, which implies the skew-Hermitian part of $\Sigma(p^2)$ should be identical to that of $\Sigma_0(p^2)$. In other words, we have some freedom in choosing the dispersive (or Hermitian) part of the self-energy by renormalization, while the absorptive (or skew-Hermitian) part is fixed under the choices of counterterms, which is a constraint on the renormalization condition for the self-energy. To obtain the expression of the renormalized self-energy, we may choose counterterms

$$\delta \Phi = \frac{1}{16\pi^2} \sum_i \left\{ |h_i|^2 \left(1 - \frac{m^4_{\chi_i}}{m^2_{\Phi}}\right) \left[ \frac{m^2_{\chi_i}}{m^2_{\Phi}} \left(1 + \frac{m^2_{\chi_i}}{m^2_{\Phi}}\right)^{-1} \log \left( \frac{m^2_{\chi_i}}{\Lambda^2} \right) + \log \left( \frac{m^2_{\Phi} - m^2_{\chi_i}}{\Lambda^2} \right) \right] \right\},$$

$$\delta' \Phi = \frac{1}{16\pi^2} \sum_i f_i^2 \log \left( \frac{m^2_i}{\Lambda^2} \right),$$

such that

$$\Sigma'_{\beta\alpha}(p^2) = \sum_i \left( \Sigma'_{\beta\alpha}(p^2) + \delta'_{\beta\alpha} \Sigma'_{\chi\epsilon}(p^2) \right),$$

where

$$\Sigma'_{\psi\psi}(m^2_{\Phi}) = \frac{1}{16\pi^2} \left( \frac{|f_i|^2}{f_i^2} \sum_i \left[ -\log \left( \frac{m^2_{\Phi}}{h_i^2 + i\pi} \right) - \log \left( \frac{m^2_{\Phi}}{h_i^2} \right) \right] \right), \quad \Sigma'_{\chi\epsilon}(m^2_{\Phi}) = i \frac{|h_i|^2}{16\pi} \left(1 - \frac{m^4_{\chi_i}}{m^2_{\Phi}}\right).$$

This expression of the renormalized self-energy and the choice of the counterterms should be consistent with the observed values of pole masses and decay widths of quasiparticles, which will be theoretically calculated in the next section.

### 3 Diagonalization of the propagator matrix

The resummed propagator is the propagator in which all the quantum corrections are considered, and it is obtained by a geometric summation at a matrix level as follows:

$$i\Delta(p^2) = i(p^2 - M^2_{\Phi})^{-1} + i(p^2 - M^2_{\Phi})^{-1}[i\Sigma(p^2)]i(p^2 - M^2_{\Phi})^{-1} + i(p^2 - M^2_{\Phi})^{-1}[i\Sigma(p^2)]i(p^2 - M^2_{\Phi})^{-1}[i\Sigma(p^2)]i(p^2 - M^2_{\Phi})^{-1} + \cdots$$

$$= \sum_{n=0}^{\infty} i(p^2 - M^2_{\Phi})^{-1}\left\{[i\Sigma(p^2)]i(p^2 - M^2_{\Phi})^{-1}\right\}^n$$

$$= i\left\{[1 + \Sigma'(p^2)]p^2 - M^2_{\Phi}\right\}^{-1}.$$ (3.1)

Here, we have defined $M_{\Phi} := m_{\Phi} I_2$ where $I_2(= 1)$ is the $2 \times 2$ identity matrix. The components of the resummed propagator are depicted in figure 4. Note that $i\Delta_{\beta\alpha}(p^2)$ is associated with $\Phi_\alpha \rightarrow \Phi_\beta$.

Now we discuss how the propagator matrix can be diagonalized. The pole of $\Delta(p^2)$ can be defined as a solution of

$$\det[\Delta^{-1}(p^2)] = 0,$$ (3.3)
which implies that a similarity transformation would be involved in the diagonalization. First we diagonalize the self-energy matrix as follows:

\[ \hat{\Sigma}'(p^2) := C(p^2)\Sigma'(p^2)C^{-1}(p^2), \]

where \( C(p^2) \) is a momentum-dependent mixing matrix. Then, we can write

\[ P^2(p^2) := m^2_{\Phi}C(p^2)[1 + \Sigma'(p^2)]^{-1}C^{-1}(p^2) = m^2_{\Phi}[1 + \hat{\Sigma}'(p^2)]^{-1}. \]

For simplicity, we introduce a shorthand notation for each diagonal component of \( \hat{\Sigma}'(p^2) \) and \( P^2(p^2) \):

\[ (\hat{\Sigma}')_{\beta \alpha}(p^2) =: \delta_{\beta \alpha}(\hat{\Sigma}')_{\alpha}(p^2) \quad \text{and} \quad P^2_{\beta \alpha}(p^2) =: \delta_{\beta \alpha}P^2_{\alpha}(p^2). \]

It is straightforward to show that the matrix \( C(p^2) \) which diagonalize the self-energy matrix also diagonalizes the propagator matrix, and the diagonalized propagator \( \hat{\Delta}(p^2) \) can be written as

\[ \hat{\Delta}(p^2) := C(p^2)\Delta(p^2)C^{-1}(p^2) = m^{-2}_\Phi P^2(p^2)[p^2 - P^2(p^2)]^{-1}. \]

\[ i.e., \]

\[ \hat{\Delta}_{\alpha}(p^2) = \frac{P^2_{\alpha}(p^2)}{m^2_{\Phi}} \frac{1}{p^2 - P^2_{\alpha}(p^2)}, \]

where \( \hat{\Delta}_{\alpha} := \hat{\Delta}_{\beta \alpha} \). Let us denote the degree of freedom related to the diagonal component \( \hat{\Delta}_{\alpha} \) by \( \Phi_{\alpha} \). The scattering cross section of a process mediated by mesons have the Breit-Wigner resonance pattern, and accordingly the complex pole of the propagator should be in the form of

\[ p^2_{\Phi_{\alpha}} = m^2_{\Phi_{\alpha}} - im_{\Phi_{\alpha}}\Gamma_{\Phi_{\alpha}}, \]

where \( m_{\Phi_{\alpha}} \) and \( \Gamma_{\Phi_{\alpha}} \) are the pole mass and total decay width of \( \Phi_{\alpha} \). The complex mass \( p_{\Phi_{\alpha}} \) is a solution of the equation

\[ p^2 = P^2_{\alpha}(p^2) = m^2_{\Phi}[1 + \hat{\Sigma}'_{\alpha}(p^2)]^{-1}, \]

in accordance with the definition of the pole given by equation \( 3.3 \). Up to \( \mathcal{O}(f^2/4\pi) \), we can write

\[ P^2_{\alpha}(p^2) = m^2_{\Phi}[1 - \hat{\Sigma}'_{\alpha}(p^2)], \]

and thus

\[ \Re[\hat{\Sigma}'_{\alpha}(p^2_{\Phi_{\alpha}})] = \frac{m^2_{\Phi_{\alpha}}}{m_{\Phi}} - 1, \quad \Im[\hat{\Sigma}'_{\alpha}(p^2_{\Phi_{\alpha}})] = \frac{\Gamma_{\Phi_{\alpha}}}{m_{\Phi}}. \]

Expanding the component of the diagonalized propagator around its complex pole and taking the leading part, we can obtain the form of the propagator that is frequently used in the practical calculations. Since

\[ \lim_{p^2 \to p^2_{\Phi_{\alpha}}} \frac{p^2 - P^2_{\alpha}(p^2)}{p^2 - p^2_{\Phi_{\alpha}}} = 1 - \frac{dP^2_{\alpha}}{dp^2}(p^2_{\Phi_{\alpha}}), \]

\[ \frac{dP^2_{\alpha}}{dp^2}(p^2_{\Phi_{\alpha}}) = \frac{1}{2m_{\Phi}} \]
the residue of the pole defined by

$$\lim_{p^2 \to p^2_{\Phi_\alpha}} (p^2 - p^2_{\Phi_\alpha}) \tilde{\Delta}_\alpha(p^2) = R_{\Phi_\alpha}$$

is

$$R_{\Phi_\alpha} := \frac{p^2_{\Phi_\alpha}}{m^2_\Phi} \left[ 1 - \frac{dP^2_{\Phi}}{dp^2(p^2_{\Phi_\alpha})} \right]^{-1}.$$ (3.13)

Note that $R_{\Phi_\alpha}$ is a complex number which can be written as $R_{\Phi_\alpha} = 1 + \mathcal{O}(f^2/4\pi)$, and it cannot be set to unity in general. Now each component of the diagonalized propagator is given by

$$i \tilde{\Delta}_\alpha(p^2) = \frac{i R_{\Phi_\alpha}}{p^2 - p^2_{\Phi_\alpha}} + \cdots.$$ (3.14)

Defining $C_{\Phi_\alpha} := C(p^2_{\Phi_\alpha})$, we can also write the component of the non-diagonal propagator as

$$i \Delta_{\beta \alpha}(p^2) = \sum_\gamma (C_{\Phi_\gamma}^{-1})_{\beta \gamma} \frac{i R_{\Phi_\gamma}}{p^2 - p^2_{\Phi_\gamma}} (C_{\Phi_\gamma})_{\gamma \alpha} + \cdots.$$ (3.15)

In the calculation of decay widths up to the leading order in perturbation, we can use the leading-order expressions such as $\Sigma'(m^2_\Phi) = \Sigma'(p^2_{\Phi_\alpha})$, $C_{\Phi} := C(m^2_\Phi) = C(p^2_{\Phi_\alpha})$, and $R_{\Phi_\alpha} = 1$. The resummed propagator up to that precision is written as

$$i \Delta_{\beta \alpha}(p^2) = \sum_\gamma (C_{\Phi_\gamma}^{-1})_{\beta \gamma} \frac{i R_{\Phi_\gamma}}{p^2 - p^2_{\Phi_\gamma}} (C_{\Phi_\gamma})_{\gamma \alpha} + \cdots,$$ (4.16)

which is the expression of the resummed propagator that will be used throughout the paper.

## 4 Generation of quasiparticles and limitations of quantum mechanics

Now we identify the degree of freedom associated with each component of the diagonalized propagator. The correlation function corresponding to the component of the non-diagonal resummed propagator can be compactly written as

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} i \Delta_{\beta \alpha}(p^2) = \langle \Omega | \Phi_\beta(x) \Phi_\alpha^\dagger(y) | \Omega \rangle, \ (x^0 > y^0),$$ (4.1)

where $\Phi_1 = \Phi$, $\Phi_2 = \Phi^\dagger(= \Phi^*)$. The two-point function on the right-hand side should be already time-ordered since $\Delta_{\beta \alpha}(p^2)$ has a direction in the energy transfer: $\Delta_{\beta \alpha}(p^2) \neq \Delta_{\alpha \beta}(p^2)$ in general for $\beta \neq \alpha$. From equations 3.16 and 4.1, we can obtain

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{p^2 - p^2_{\Phi_\alpha}} + \cdots = \langle \Omega | \hat{\Phi}_\beta^\dagger(x) \hat{\Phi}_\alpha^\dagger(y) | \Omega \rangle, \ (x^0 > y^0),$$ (4.2)

where

$$\hat{\Phi}_\beta^\dagger = \sum_\beta (C_{\Phi}^{-1})_{\beta \gamma} \hat{\Phi}_\gamma, \quad \hat{\Phi}_\alpha = \sum_\beta (C_{\Phi})_{\gamma \beta} \hat{\Phi}_{\gamma}.$$ (4.3)
Calculating the Fourier transform in the rest frame and neglecting the subdominant part in equation 4.2, we can write

\[ e^{-i p_{\Phi_\alpha}(x^0 - y^0)} \propto \langle \Omega | \hat{\Phi}_{\alpha}^f(x) \hat{\Phi}_{\alpha}^{\dagger}(y) | \Omega \rangle, \quad (x^0 > y^0). \]  

(4.4)

Note that \( \hat{\Phi}_{\alpha}^f \neq \hat{\Phi}_{\alpha}^i \) since \( C_\Phi \) is non-unitary. In other words, we cannot relate each component of the diagonalized propagator to a single linear combination of \( \Phi \) and \( \Phi^* \). Since that component is the degree of freedom which propagates like a free particle until it decays in the presence of interactions, we can interpret the result as follows: the particle of \( \hat{\Phi}_{\alpha} \) in the interacting theory is the degree of freedom that emerges as an excitation of \( \hat{\Phi}_{\alpha}^i \) and ends as an excitation of \( \hat{\Phi}_{\alpha}^f \) (\( \hat{\Phi}_{\alpha}^f \neq \hat{\Phi}_{\alpha}^i \)).

The difference between \( \hat{\Phi}_{\alpha}^f \) and \( \hat{\Phi}_{\alpha}^i \) is determined by the deviation of \( C_\Phi \) from the unitarity, which is of the order of the CP asymmetry due to \( \Phi \cdot \Phi^* \) mixing. On account of the generic non-perturbative effect in an on-shell unstable particle, this CP asymmetry is usually much larger than the typical perturbative correction \( O(f^2/4\pi) \). The non-perturbative effect can be understood from figure 5. When \( \hat{\Phi}_{\alpha} \) is on-shell, we have \( p^2 \sim m_\Phi^2 \) and the factor \( \Sigma_{\beta\alpha}(p^2)/(p^2 - m_\Phi^2) \) can be hugely enhanced

![Figure 5](image.png)

Figure 5. A non-perturbative effect is generated when \( \hat{\Phi}_{\alpha} \) is on-shell.

to generate an effect beyond the typical perturbative correction of the theory. Note that an on-shell stable particle does not have such a non-perturbative effect. It can be better understood in the on-shell renormalization scheme, in which \( \Sigma_{\beta\alpha}(m_\Phi^2) = 0 \) and \( \lim_{p^2 \to m_\Phi^2} |\Sigma_{\beta\alpha}(p^2)/(p^2 - m_\Phi^2)| \sim O(f^2/4\pi) \). On the other hand, the absorptive part in the self-energy of an unstable particle cannot be affected by renormalization, i.e., \( \Sigma_{\beta\alpha}(p^2) \sim \Sigma_{\beta\alpha}(m_\Phi^2) \sim O(f^2/4\pi) \) in any renormalization scheme, and the deviation of the quasiparticle from a unitary combination of \( \Phi \) and \( \Phi^* \) can be hugely enhanced compared with the typical perturbative correction. In real meson systems, it is sometimes called a non-perturbative effect in the weak interaction, since mixing is caused by the exchange of the weak gauge boson. Since the phenomenon implied by equation 4.4 is never a small effect, it should not be neglected in general in a theoretical analysis of particle-antiparticle mixing when we are dealing with on-shell unstable particles.

It is an emergent phenomenon dynamically generated by interactions, and the degree of freedom associated with \( \hat{\Phi}_{\alpha} \) should be interpreted as a quasiparticle. We had better not simply call it a mass eigenstate, not only because it has a specific total decay width as well as mass given by equation 3.8, but also because the physics of a quasiparticle is much different from that of a mass eigenstate. It should be emphasized that this emergent phenomenon has never been appropriately considered in the literature, where a mass eigenstate has always been thought to be a specific linear combination of a particle and its antiparticle which follows the time evolution of a plane wave with a damping factor, i.e.,

\[ e^{-ip_{\Phi_\alpha}(t-t_0)} = e^{-ip_{\frac{m_\Phi}{2}}(t-t_0)} e^{-i(t \cdot \frac{p_{\Phi_\alpha}}{2})(t-t_0)} \]  
in the rest frame. In other words, the conventional approach of solving the effective Schrödinger equation to obtain a time evolution cannot work, since the equation of motion we would obtain after diagonalizing the effective Hamiltonian is always \( i(d/dt)|\hat{\Phi}_{\alpha}(t)\rangle = p_{\Phi_\alpha}|\hat{\Phi}_{\alpha}(t)\rangle \) and its solution is nothing but \( |\hat{\Phi}_{\alpha}(t)\rangle = e^{-ip_{\Phi_\alpha}(t-t_0)}|\hat{\Phi}_{\alpha}(t_0)\rangle \), where \( |\hat{\Phi}_{\alpha}(t_0)\rangle \) is the mass eigenstate which is a fixed linear combination of the basis states.
We therefore conclude that quantum mechanics is not the proper non-relativistic limit of the quantum field theory in the presence of heavy particle-antiparticle mixing. A similar conclusion can be made about the mixing of multiple flavors of unstable particles especially with small mass differences, which will be studied in follow-up papers. In consequence, we must find a way outside quantum mechanics to deal with this emergent phenomenon.

5 Derivations of decay widths in the quantum field theory

We have seen that the properties of quasiparticles should be investigated in the quantum field theory, not in quantum mechanics. Since the quasiparticle, as a degree of freedom dynamically generated by interactions, cannot be treated like an external state, it can be legitimately dealt with only when it appears as an intermediate state, i.e., on an internal propagator. In this section, we define and calculate some transition rates associated with scattering processes involving quasiparticles of mesons in various ways, and relate them to the decay widths of mesons.

In advance of explicit derivations, we try to calculate them in a straightforward but simple-minded way in the quantum field theory. For simplicity, we only consider the loop corrections causing particle-antiparticle mixing, neglecting any vertex loop corrections. We can naively calculate the partial decay widths such as $\Gamma(\Phi \rightarrow \chi_i \xi^c)$, treating $\Phi$ as the initial state and summing over all the diagrams in figure 6. Note however that this method fails to work if we consider only a finite number of diagrams in figure 6, since the internal propagators of $\Phi$ diverge as $\Phi$ goes on-shell. The standard solution to this problem is summing over all the diagrams in figure 6 for off-shell $\Phi$, and taking an analytic continuation of the resulting expression to the on-shell region. Since the final expression is convergent for on-shell $\Phi$, this procedure should work. Summing over all the contributions given in figure 6, we write

$$i \mathcal{M}(\Phi \rightarrow \chi_i \xi^c) = \sum_{\chi_i} \delta(p_{\chi_i} - p_{\xi} - \xi^c) \left[ 1 + \frac{i}{p^2 - m_\Phi^2} i \Sigma_{1\alpha}(p^2) + \sum_\alpha \sum_{\Sigma_{1\alpha}(p^2)} \frac{i}{p^2 - m_\Phi^2} i \Sigma_{1\alpha}(p^2) + \cdots \right]. \tag{5.1}$$

Using equation 3.1 and $p^2 = m_\Phi^2$ for on-shell $\Phi$, we can rewrite the expression in the parentheses as

$$1 + \sum_\alpha i \Delta_{1\alpha}(p^2) i \Sigma_{1\alpha}(p^2) = 1 - \sum_\alpha \Delta_{1\alpha}(p^2) \left\{ \delta_{\alpha 1} + \Sigma'_{\alpha 1}(p^2) [p^2 - m_\Phi^2] \right\}$$

$$= 1 - \sum_\alpha \Delta_{1\alpha}(p^2) (\Delta^{-1})_{\alpha 1}(p^2) = 0. \tag{5.2}$$

Hence, we conclude $\mathcal{M}(\Phi \rightarrow \chi_i \xi^c) = 0$, from which we might want to deduce $\Gamma(\Phi \rightarrow \chi_i \xi^c) = 0$. 

Figure 6. Loop corrections to $\Phi \rightarrow \chi_i \xi^c$ associated with particle-antiparticle mixing.
This is, of course, a wrong result. In this calculation, Φ is assumed to be on-shell as an external field, which is actually inappropriate. This result only shows that the on-shell field Φ does not contribute to the unitarity cut of $\chi_i\xi^c \to \chi_i\xi^c$, as is well known since it was first proved by Veltman [2]. We will come back to this issue later. In the following discussion, we will derive the decay widths of mesons, assuming that what should be on-shell is not Φ but the quasiparticle of mesons, $\hat{\Phi}_\alpha$. Even the partial decay width such as $\Gamma(\Phi \to \chi_i\xi^c)$ will be calculated under the same assumption, i.e., with off-shell Φ.

5.1 From the self-energy

The expression of the total decay width of a quasiparticle has been obtained in the process of diagonalizing the resummed propagator, and it is given by

$$\Gamma_{\hat{\Phi}_\alpha} = m_\Phi \text{Im}[\hat{\Sigma}'_\alpha(m_\Phi^2)] = m_\Phi \text{Im}[(CS\Sigma^{-1})_{\hat{\alpha}\hat{\alpha}}(m_\Phi^2)].$$  (5.3)

However, it tells nothing about its partial decay width to any decay channel. Even though the total decay width is the sum of all the partial decay widths and it is apparent that the contribution of each loop to the self-energy, e.g., $\Sigma'_{\chi_i\xi^c}(m_\Phi^2)$, can be easily identified, it is still not possible to directly find the partial decay width from the self-energy. For example, we might try to replace $\Sigma'_{\chi_i\xi^c}(m_\Phi^2)$ with $\Sigma'_{\chi_i\xi^c}(m_\Phi^2)$ to find $\Gamma(\hat{\Phi}_\alpha \to \chi_i\xi^c)$, but it does not work since $(CS\Sigma^{-1} C^{-1})_{\hat{\alpha}\hat{\alpha}}(m_\Phi^2)$ is no longer diagonal. Only when the self-energy is diagonal, each component can be interpreted as a physical quantity that belongs to each quasiparticle. This occurs since the mixing matrix $C$ itself is also a complicated function of the self-energy, and thus there does not exist a simple way to identify the contribution of specific final states to $\Gamma_{\hat{\Phi}_\alpha}$. Now we discuss various methods of calculating the partial decay width in the quantum field theory.

5.2 From the on-shell contributions to scattering

5.2.1 Overview

The first method is to read the partial decay width out of scattering mediated by on-shell quasiparticles of mesons. This method is intuitive and practical, since it is exactly how the heavy particles are observed in an experiment. The strategy is as follows:

1. For the given final states of decay products, consider all the possible initial states that can produce those final states through s-channel scattering mediated by mesons.

2. Integrate over all the phase spaces of the initial and final states with delta functions appropriately inserted.

To explicitly calculate the decay width, let us consider the cases with the decay product $\chi_i\xi^c$. The associated initial states and Feynman diagrams are given in figures 7 and 8. The processes of figure 7 tells us how the phase space integration should be done to obtain the correct expression of the decay width. The correct choice is to apply the phase space integration of the final states:

$$\int d\Pi_{\chi_i} \int d\Pi_{\xi^c} (2\pi)^3\delta^3(p - p_{\chi_i} - p_{\xi^c})$$  (5.4)

to $|\mathcal{M}|^2$ integrated over the phase space of the initial state:

$$\int d\Pi_{\Phi_\alpha} (2\pi)^4\delta^4(p_{\Phi_\alpha} - p_{\chi_i} - p_{\xi^c}) |\mathcal{M}(\Phi_\alpha \to \chi_i\xi^c)|^2,$$  (5.5)
where

\[ d\Pi_X := \prod_{j \in X} \frac{d^3p_j}{(2\pi)^3 2E_j}. \]  

(5.6)

The straightforward calculation gives

\[ \frac{1}{2E_{\Phi_\alpha}} \int d\Pi_{\chi_i} \int d\Pi_{\xi^c} (2\pi)^4 \delta(E_{\Phi_\alpha} - E_{\chi_i} - E_{\xi^c}) \delta^3(p - p_{\chi_i} - p_{\xi^c}) |M[\Phi_\alpha(p) \rightarrow \chi_i \xi^c]|^2. \]  

(5.7)

Setting \( p = 0 \), we obtain the familiar expression of the partial decay width:

\[ \Gamma(\Phi_\alpha \rightarrow \chi_i \xi^c) = \frac{1}{2m_{\Phi}} \int d\Pi_{\chi_i} \int d\Pi_{\xi^c} (2\pi)^3 \delta^4(p_{\chi_i} + p_{\xi^c} - p_{\Phi_\alpha}) |M(\Phi_\alpha \rightarrow \chi_i \xi^c)|^2, \]  

(5.8)

and the expression for \( p \neq 0 \) given by equation 5.7 is nothing but the partial decay width in a boosted frame. Even though we have used the diagrams of figure 7 to choose the right phase space integration to obtain the decay width, it turns out that they actually *do not* contribute to the decay width of \( \Phi_\alpha \) when the loop corrections are fully considered, which we will soon see. This means that equation 5.7 is only correct for amputated diagrams such as the tree-level decay or the decay only with vertex-loop corrections, and it cannot properly incorporate loop corrections such as those in figure 7, which is one of the reasons why the quantum mechanical approach has been the standard in studying heavy particle-antiparticle mixing.

We therefore claim that the real contributions to the decay width come from the scattering processes given in figure 8. As for the scattering \( \chi_j \xi \rightarrow \chi_i \xi^c \) of figure 8d, we apply the integration

\[ \int d\Pi_{\chi_i} \int d\Pi_{\xi^c} (2\pi)^3 \delta^3(p - p_{\chi_i} - p_{\xi^c}) \]  

(5.9)

to

\[ \frac{1}{4} \int d\Pi_{\chi_j} \int d\Pi_{\xi} (2\pi)^4 \delta^4(p_{\chi_j} + p_{\xi} - p_{\chi_i} - p_{\xi^c}) |M(\chi_j \xi \rightarrow \chi_i \xi^c)|^2 \]  

(5.10)
to calculate its contribution to the associated partial decay width. The factor 1/4 is the spin average over the initial states.

Note that this phase space integration over all the initial states except for the spin average is equivalent to applying the unitarity cut to all the possible internal states of the scattering $\chi_i\xi^c \rightarrow \chi_i\xi^c$. The associated diagram is given in figure 9. It is well known that the cutting through $\Phi_\alpha$ do not contribute to this unitarity cut, and only the cutting through the internal propagators of light fields such as $\chi_i$, $\xi$, and $\psi_i$ matters, as proved by Veltman [2]. This is why the diagrams of figure 7 are irrelevant to our purpose as mentioned above. Only the cutting applied to the intermediate multiparticle states need to be considered, and the resulting diagrams are those of figure 8. The decay width should therefore be calculated from those diagrams.

In this method, the initial and final states $\chi_i\xi^c$ of figure 9 are the decay product corresponding to the partial decay width we want to calculate, and the unitarity cut gives the various initial states of the scattering processes of figure 8. This is a convenient choice since the focus is how to calculate the partial decay width to a specific channel. Note the difference from the usual approach to calculate the decay width using the optical theorem, in which the decaying particle is chosen as the initial and final state of the diagram before cutting and the unitarity cut gives the various decay products. So far it has been unclear how the diagrams of figure 8 can be related to the decay widths of mesons, as claimed above. In the following sections, we will obtain some transition rates associated with those diagrams applying the phase space integration discussed above, and show that they can be related to the decay widths of intermediate mesons in a specific way.

5.2.2 Derivation of transition rates

Now we discuss how to derive the expressions of partial decay widths of mesons. In this paper, all the calculations will be done up to the leading order in perturbation. Here, the leading order implies the followings:

1. The one-loop order in the self-energy.

2. The tree level in the vertices.

The reason why these conditions correspond to the leading order will be clarified after we derive the first expression of transition rates, i.e., equation 5.32.
In order to calculate the unitarity cut, it is convenient to use the $T$-matrix element, which is the non-trivial part of the $S$-matrix:

$$S = 1 + iT. \quad (5.11)$$

For a transition $i \to f \ (i \neq f)$, the $T$-matrix element is written as

$$\langle f | T | i \rangle = (2\pi)^4 \delta^4(p_i - p_f)M(i \to f). \quad (5.12)$$

Denoting the one- and multi-particle state in the Fock space by $|X^a\rangle$ where $a$ denotes the internal degrees of freedom of $X$, we can write the completeness relation as

$$1 = \sum_{a,X} \int d\Pi_X |X^a\rangle \langle X^a|. \quad (5.13)$$

The unitarity of $S$ implies

$$1 = SS^\dagger = (1 + iT)(1 - iT^\dagger) = 1 + i(T - T^\dagger) + TT^\dagger \quad (5.14)$$

i.e.,

$$i(T^\dagger - T) = TT^\dagger. \quad (5.15)$$

In this section, we consider only the right-hand side of equation 5.15. Cutting through the intermediate states of $\chi_i\xi^c \to \chi_i\xi^c$ is equivalent to inserting the completeness relation given by equation 5.13. It is straightforward to obtain

$$\langle \chi_i, r, \xi^c | TT^\dagger | \chi_i, r, \xi^c \rangle = \sum_{X,a} \int d\Pi_X \langle \chi_i, r, \xi^c | T | X^a \rangle \langle X^a | T^\dagger | \chi_i, r, \xi^c \rangle$$

$$= (2\pi)^4 \delta^4(0) \left\{ \sum_{j,k,l} \int d\Pi_{\psi_j} \int d\Pi_{\psi^c_j} (2\pi)^4 \delta^4(p_{\psi_j} + p_{\psi_j} - p_{\chi_i} - p_{\xi^c}) \left| \mathcal{M}(\psi_j, k, \psi^c_j | \chi_i, r, \xi^c) \right|^2 
+ \sum_{j,k,l} \int d\Pi_{\chi_j} \int d\Pi_{\xi^c_j} (2\pi)^4 \delta^4(p_{\chi_j} + p_{\chi_j} - p_{\chi_i} - p_{\xi^c}) \left| \mathcal{M}(\chi_j, k, \xi^c_j | \chi_i, r, \xi^c) \right|^2 
+ \sum_{j,k,l} \int d\Pi_{\xi^c_j} \int d\Pi_{\chi_j} (2\pi)^4 \delta^4(p_{\chi_j} + p_{\chi_j} - p_{\chi_i} - p_{\xi^c}) \left| \mathcal{M}(\chi_j, k, \xi^c_j | \chi_i, r, \xi^c) \right|^2 \right\}$$

$$+ \cdots, \quad (5.16)$$

where $r, s, k, l$ are the spin indices of light fermions. The ellipsis denotes subdominant contributions from the multiparticles states other than those written above. Those multiparticle states come from cutting the self-energy diagrams of the intermediate meson beyond the one-loop order, and thus they can be neglected up to the working precision. As mentioned above, we integrate over the phase space of $\chi_i, r, s, k, l$ as follows:

$$\int d\Pi_{\chi_i} \int d\Pi_{\xi^c} (2\pi)^3 \delta^3(p - P_{\chi_i} - P_{\xi^c}) \sum_{r, s} \langle \chi_i, r, \xi^c | TT^\dagger | \chi_i, r, \xi^c \rangle. \quad (5.17)$$
First we consider the third term in equation (5.16) corresponding to the initial states $\chi_j^c \xi$. Rearranging the phase space integration and delta functions, we can write

$$
\frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi} \int d\Pi_{\chi_j} \int d\Pi_{\xi} (2\pi)^d \delta^4(p - p_{\chi_j} - p_{\xi})(2\pi)^d \delta^4(p_{\chi_j} + p - p_{\chi_j} - p_{\xi})
\sum_{r,s,k,l} |M(\chi_j^c k \xi_l \to \chi_i r \xi_s^c)|^2
= \frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi} (2\pi)^d \delta^4(p - p_{\chi_j} - p_{\xi}) \sigma'(\chi_j^c \xi \to \chi_i^c \xi),
$$

(5.18)

where we have defined

$$
\sigma'(\chi_j^c \xi \to \chi_i^c \xi) := \int d\Pi_{\chi_j} \int d\Pi_{\xi} (2\pi)^d \delta^4(p - p_{\chi_j} - p_{\xi}) \delta(E_{\chi_j} + E_{\xi} - E_{\chi_i} - E_{\xi}),
\sum_{r,s,k,l} |M(\chi_j^c k \xi_l \to \chi_i r \xi_s^c)|^2.
$$

(5.19)

Here, $\sigma'$ is a dimensionless quantity related to the scattering cross section $\sigma$ as follows:

$$
\sigma'(\chi_j^c \xi \to \chi_i^c \xi) = 2E_{\chi_j}^2 E_{\xi}^2 |v_{\chi_j} - v_{\xi}| \frac{\sigma(\chi_j^c \xi \to \chi_i^c \xi)}{4},
$$

(5.20)

and the factor 4 is relevant to the spin average of the initial states in $\sigma$.

The expression given by equation (5.18) is in fact divergent in the limit $E_{\chi_j} \to \infty$ in the center-of-momentum (CM) frame, and such a limit corresponds to a highly off-shell contribution which should be irrelevant to the decay widths of intermediate particles. We have to consider only the on-shell contributions to equation (5.18) to relate them to the decay widths, and will discuss how to do it soon.

The scattering amplitude for $\chi_j^c k \xi_l \to \chi_i r \xi_s^c$ is written as

$$
iM(\chi_j^c k \xi_l \to \chi_i r \xi_s^c) = \overline{u}_{\chi_j}(p_{\chi_j}) v_{\xi_s^c}(p_{\xi}) (-ih_{r}) [i\Delta_{12}(p^2)][(-ih_j) v_{\xi_r}^c(p_{\xi})] u_{\chi_i}^c(p_{\chi_i})
= -i\overline{u}_{\chi_j}(p_{\chi_j}) v_{\xi_s^c}(p_{\xi}) v_{\xi_r}^c(p_{\xi}) u_{\chi_i}^c(p_{\chi_i}) \sum_{\hat{\alpha}} Q_{\Phi_{\hat{\alpha}} \to \chi_i \xi} \cdot \cdot \cdot,
$$

(5.21)

where we have defined a coefficient

$$
Q_{\Phi_{\hat{\alpha}} \to \chi_i \xi} := h_{i} h_{j} (C_{\Phi}^{-1})_{1\hat{\alpha}} (C_{\Phi})_{\hat{\alpha}2}.
$$

(5.22)

A straightforward calculation in the CM frame gives

$$
\sigma'_{\text{CM}}(\chi_j^c \xi \to \chi_i^c \xi) = \frac{1}{2\pi} \sum_{\hat{\alpha},\hat{\beta}} Q_{\Phi_{\hat{\alpha}} \to \chi_i \xi} \cdot Q_{\Phi_{\hat{\beta}} \to \chi_j \xi} \frac{(E^2 - m_{\chi_j}^2)(E^2 + m_{\chi_i}^2)}{E^2(E^2 - p_{\Phi_{\hat{\beta}}}^2)(E^2 - p_{\Phi_{\hat{\alpha}}}^2)},
$$

(5.23)

where $E$ denotes the total energy. Defining

$$
m_{\chi,j} := \begin{cases} 
m_{\chi,i}, & m_{\chi,i} \geq m_{\chi,j}, \\
m_{\chi,j}, & m_{\chi,i} < m_{\chi,j}, \end{cases}
$$

(5.24)
we integrate the energy-dependent part over the phase space of the initial states to obtain

\[
\int \! d\Pi_{\chi^c_j} \int \! d\Pi_\xi (2\pi)^3 \delta^3(\mathbf{p}_{\chi^c_j} + \mathbf{p}_\xi) \frac{(E^2 - m^2_{\chi^c_j})(E^2 + m^2_{\chi^c_j})(E^2 - m^2_{\chi^c_j})}{E^2(E^2 - p_{\Phi_\beta}^2)(E^2 - p_{\Phi_\alpha}^2)} \left. \right|_{\text{OS}} = \frac{1}{16\pi^2} \int_{m_{\chi^c_j}}^{\infty} dE \frac{(E^2 + m^2_{\chi^c_j})(E^2 - m_{\chi^c_j})}{E^4(E^2 - p_{\Phi_\beta}^2)(E^2 - p_{\Phi_\alpha}^2)} \left. \right|_{\text{OS}}
\]

\[
= \frac{1}{16\pi^2(p_{\Phi_\beta}^2 - p_{\Phi_\alpha}^2)} \int_{m_{\chi^c_j}}^{\infty} dE \frac{1}{E^4}(E^2 + m^2_{\chi^c_j})(E^2 - m_{\chi^c_j})(E^2 + m^2_{\chi^c_j})(E^2 - m_{\chi^c_j}) \left[ \frac{E^2 - m^2_{\Phi_\alpha} - im_\Phi_\alpha \Gamma_{\Phi_\alpha}}{(E^2 - m^2_{\Phi_\alpha})^2 + (m_\Phi_\alpha \Gamma_{\Phi_\alpha})^2} - \frac{E^2 - m^2_{\Phi_\beta} + im_\Phi_\beta \Gamma_{\Phi_\beta}}{(E^2 - m^2_{\Phi_\beta})^2 + (m_\Phi_\beta \Gamma_{\Phi_\beta})^2} \right].
\]

Note that the contributions of \( E^2 - m^2_{\Phi_\beta} \) in the numerators diverge after the integration over \( E \), which makes \( \sigma' \) diverges as mentioned above. Those terms are, however, off-shell contributions for sure since they vanish when the quasiparticles are on-shell, i.e., when \( E^2 = m^2_{\Phi_\beta} \). We neglect those off-shell contributions, and call this procedure an on-shell prescription. Applying the narrow-width approximation to the remaining part:

\[
\lim_{m_\Phi_\beta \to 0} \frac{1}{m_{\Phi_\alpha} \Gamma_{\Phi_\alpha}} \delta(E^2 - m^2_{\Phi_\alpha}) = \frac{\pi m_{\Phi_\beta}}{\Gamma_{\Phi_\beta}} \delta(E^2 - m^2_{\Phi_\beta}),
\]

we obtain

\[
\int \! d\Pi_{\chi^c_j} \int \! d\Pi_\xi (2\pi)^3 \delta^3(\mathbf{p}_{\chi^c_j} + \mathbf{p}_\xi) \frac{(E^2 - m^2_{\chi^c_j})(E^2 + m^2_{\chi^c_j})(E^2 - m^2_{\chi^c_j})}{E^2(E^2 - p_{\Phi_\beta}^2)(E^2 - p_{\Phi_\alpha}^2)} \left. \right|_{\text{OS}} = -\frac{i}{32\pi(p_{\Phi_\beta}^2 - p_{\Phi_\alpha}^2)} \left[ m_{\Phi_\alpha}^2 \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi_\alpha}^2} \right) \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi_\beta}^2} \right) + m_{\Phi_\beta}^2 \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi_\beta}^2} \right) \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi_\alpha}^2} \right) \right]
\]

\[
= -\frac{i}{32\pi(p_{\Phi_\beta}^2 - p_{\Phi_\alpha}^2)} m_{\Phi}^2 \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi}^2} \right) \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi}^2} \right),
\]

where the subscript OS implies the on-shell prescription has been imposed, and the last equality is correct up to the leading order because \( m_{\Phi_\beta} \sim m_{\Phi}[1 + \mathcal{O}(f^2/4\pi)] \). Note also that the narrow-width approximation works well even for couplings which are quite large, e.g., \( f \sim \mathcal{O}(10^{-1}) \), and the derivation can therefore be considered to be generally valid since \( f \lesssim \mathcal{O}(10^{-5}) \) for mesons, which we will see in specific examples in section 7.

We have thus obtained an expression of a transition rate associated with the scattering \( \chi^c_j \xi \to \chi_i \xi^c \):

\[
\sum_{j,\hat{\alpha}} \Gamma_{\chi^c_j \xi \to \chi_i \xi^c} = \frac{1}{4} \sum_j \int \! d\Pi_{\chi^c_j} \int \! d\Pi_\xi (2\pi)^3 \delta^3(\mathbf{p}_{\chi^c_j} + \mathbf{p}_\xi) \sigma'_{\text{CM}}(\chi^c_j \xi \to \chi_i \xi^c) \left. \right|_{\text{OS}} = \frac{i}{2\pi^2} \sum_{j,\hat{\alpha},\hat{\beta}} \frac{Q_{\chi^c_j \to \chi_i \xi} Q_{\chi_{2\alpha} \to \chi_{2\beta} \xi}}{p_{\Phi_\beta}^2 - p_{\Phi_\alpha}^2} m_{\Phi}^2 \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi}^2} \right) \left( 1 - \frac{m_{\chi^c_j}}{m_{\Phi}^2} \right).
\]
This expression is correct up to the leading order in perturbation, the meaning of which needs some clarification. Let us first consider the denominator $p_{\Phi_a}^* - p_{\Phi_a}$. Since $p_{\Phi_a}^* - p_{\Phi_a} = \Gamma_{\Phi_a}$, up to the leading order and $\Gamma_{\Phi_a}$ is calculated from the self-energy, the leading contributions to $p_{\Phi_a}^* - p_{\Phi_a}$ come from the one-loop diagrams of the self-energy. In addition, the numerator containing $Q_{\chi_j \xi_j}$ has already been calculated up to the leading order, i.e., up to the tree-level vertices. In other words, the leading-order contributions to $\sum_j \Gamma_{\Phi_a \rightarrow \chi_j \xi_j}$ consist of the one-loop diagrams of the self-energy and the tree-level diagrams of the vertices. If we want to raise the precision of calculation to the next-leading order, the one-loop corrections to the vertices as well as the two-loop corrections to the self-energy must be taken into account for consistency in perturbative calculation.

We can proceed similar calculations for all the other initial and final states, and define

$$\sum_\alpha \Gamma_{\Phi_a \rightarrow \chi_i \xi_i} = \sum_\alpha \Gamma_{\Phi_a \rightarrow \chi_i \xi_i} + \sum_\alpha \Gamma_{\phi_\alpha \rightarrow \chi_i \xi_i} + \sum_\alpha \Gamma_{\psi_i \psi_i \rightarrow \Phi_a \rightarrow \chi_i \xi_i}$$

(5.33)

$$\sum_\alpha \Gamma_{\Phi_a \rightarrow \chi_j \xi_j} = \sum_\alpha \Gamma_{\Phi_a \rightarrow \chi_j \xi_j} + \sum_\alpha \Gamma_{\phi_\alpha \rightarrow \chi_j \xi_j} + \sum_\alpha \Gamma_{\psi_j \psi_j \rightarrow \Phi_a \rightarrow \chi_j \xi_j}$$

(5.34)

$$\sum_\alpha \Gamma_{\Phi_a \rightarrow \psi_i \psi_i} = \sum_\alpha \Gamma_{\Phi_a \rightarrow \psi_i \psi_i} + \sum_\alpha \Gamma_{\phi_\alpha \rightarrow \psi_i \psi_i} + \sum_\alpha \Gamma_{\eta_i \eta_i \rightarrow \Phi_a \rightarrow \psi_i \psi_i}$$

(5.35)

i.e.,

$$\sum_\alpha \Gamma_{\Phi_a \rightarrow Y} = \sum_\alpha \Gamma_{X \rightarrow \Phi_a \rightarrow Y}$$

(5.36)

The expression of each transition rate will be presented in section 5.2.3. It is tempting to move forward and remove $\sum_\alpha$ to obtain an expression associated with each quasiparticle $\Phi_a$. This is, however, not allowed in general, since $\Gamma_{\Phi_a \rightarrow Y}$ is complex-valued in many cases. This means that $\Gamma_{\Phi_a \rightarrow Y}$ cannot be regarded as a physical transition rate such as a partial decay width of $\Phi_a$ because it must be real-valued to allow such an interpretation.

### 5.2.3 Summary of $\Gamma_{X \rightarrow \Phi_a \rightarrow Y}$

Here, $\Gamma_{X \rightarrow \Phi_a \rightarrow Y}$ for all possible $X, Y$ are provided. Every expression is obtained up to the leading order in perturbation.

When the initial states are $\chi_j \xi_j$, we have

$$\sigma_{CM}[\chi_j(p_{\chi_j})\xi_j(p_{\xi_j}) \rightarrow \chi_i \xi_i] = \frac{1}{2\pi} \sum_{\alpha, \beta} Q_{\phi_\alpha \rightarrow \chi_i \xi_i} Q_{\phi_\beta \rightarrow \xi_j \xi_j} \frac{(E^2 - m_{\chi_j}^2)(E^2 - m_{\chi_i}^2)}{E^2(E^2 - p_{\Phi_a}^2)(E^2 - p_{\Phi_a}^2)}$$

(5.37)

where

$$Q_{\phi_\alpha \rightarrow \chi_i \xi_i} = h_1 h_2 (C_1 - 1) \delta_{\alpha \beta}$$

(5.38)

Hence,

$$\sum_{j, \alpha} \Gamma_{\chi_j \xi_j \rightarrow \Phi_a \rightarrow \chi_i \xi_i} = \frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi_j} (2\pi)^3 \delta^3(p_{\chi_j}^* + p_{\xi_j}^*)$$

$$\sigma_{CM}[\chi_j(p_{\chi_j})\xi_j(p_{\xi_j}) \rightarrow \chi_i \xi_i]$$

(5.39)

\[\text{OS}\]
$$\chi_j^\epsilon \xi \to \chi_i \xi^c$$  For the initial states $\chi_j^\epsilon \xi$, we obtain

$$\sigma'_{\text{CM}}(\chi_j^\epsilon \xi \to \chi_i \xi^c) = \frac{1}{2\pi} \sum_{\hat{\alpha}, \hat{\beta}} Q_{\Phi_{\hat{\beta}} \to \chi_i \xi^c} Q_{\Phi_{\hat{\alpha}} \to \chi_j^\epsilon \xi} \frac{(E^2 - m_{\chi_j}^2)(E^2 - m_{\chi_i}^2)}{E^2 - p_{\Phi_{\hat{\beta}}}^2}.$$ (5.41)

where

$$Q_{\Phi_{\hat{\alpha}} \to \chi_i \xi^c} := h_i h_j (C_{\Phi}^{-1})_{i\hat{\alpha}} (C_{\Phi})_{\hat{\alpha}2}.\quad(5.42)$$

Hence,

$$\sum_{\alpha, \beta} \Gamma_{\chi_j^\epsilon \xi \to \Phi_{\hat{\alpha}} \to \chi_i \xi^c} := \frac{1}{4} \sum_{\alpha, \beta} \int d\Pi_{\chi^c} \int d\Pi_{\xi} (2\pi)^3 \delta^3(p_{\chi^c} + p_{\xi}) \sigma'_{\text{CM}}(\chi_j^\epsilon \xi \to \chi_i \xi^c)\bigg|_{\text{OS}}\quad(5.43)$$

$$= \frac{i}{2\pi^2} \sum_{\alpha, \beta} Q_{\Phi_{\hat{\beta}} \to \chi_i \xi^c} Q_{\Phi_{\hat{\alpha}} \to \chi_j^\epsilon \xi} m_{\Phi}^2 \left(1 - \frac{m_{\chi_j}^2}{m_{\Phi}^2}\right)\left(1 - \frac{m_{\chi_i}^2}{m_{\Phi}^2}\right)\quad(5.44)$$

$$\psi_j \psi_j^c \to \chi_i \xi^c$$  When the initial states are $\psi_j \psi_j^c$, we find

$$\sigma'_{\text{CM}}(\psi_j \psi_j^c \to \chi_i \xi^c) = \frac{1}{2\pi} \sum_{\hat{\alpha}, \hat{\beta}} Q_{\Phi_{\hat{\beta}} \to \chi_i \xi^c} Q_{\Phi_{\hat{\alpha}} \to \chi_j^\epsilon \xi} \frac{(E^2 + m_{\chi_j}^2)(E^2 - m_{\chi_i}^2)}{E^2 - p_{\Phi_{\hat{\beta}}}^2}.$$ (5.45)

where

$$Q_{\Phi_{\hat{\alpha}} \to \chi_i \xi^c} := h_i f_j (C_{\Phi}^{-1})_{i\hat{\alpha}} + h_i f_j (C_{\Phi}^{-1})_{i\hat{\alpha}} (C_{\Phi})_{\hat{\alpha}2}.\quad(5.46)$$

Hence,

$$\sum_{\alpha, \beta} \Gamma_{\psi_j \psi_j^c \to \Phi_{\hat{\alpha}} \to \chi_i \xi^c} := \frac{1}{4} \sum_{\alpha, \beta} \int d\Pi_{\psi_j} \int d\Pi_{\psi_j^c} (2\pi)^3 \delta^3(p_{\psi_j} + p_{\psi_j^c}) \sigma'_{\text{CM}}(\psi_j \psi_j^c \to \chi_i \xi^c)\bigg|_{\text{OS}}\quad(5.47)$$

$$= \frac{i}{2\pi^2} \sum_{\alpha, \beta} Q_{\Phi_{\hat{\beta}} \to \chi_i \xi^c} Q_{\Phi_{\hat{\alpha}} \to \chi_j^\epsilon \xi} m_{\Phi}^2 \left(1 - \frac{m_{\chi_j}^2}{m_{\Phi}^2}\right)\quad(5.48)$$
\[ \chi_j \xi^c \rightarrow \chi_j' \xi \]

In a similar way, we can also obtain

\[
\sum_{j, \tilde{a}} \Gamma_{\chi_j \xi^c \rightarrow \tilde{a} \rightarrow \chi_j' \xi}^{\text{scat}} := \frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi^c} (2\pi)^3 \delta^3(p_{\chi_j} + p_{\xi^c}) \sigma'_{\text{CM}}(\chi_j \xi^c \rightarrow \chi_j' \xi) \bigg|_{\text{OS}} (5.49)
\]

\[
= \frac{i}{2^{8} \pi^{2}} \sum_{j, \tilde{a}, \beta} \frac{Q_{\tilde{a} \rightarrow \chi_j' \xi} \cdot Q_{\tilde{a} \rightarrow \chi_j' \xi}}{p_{\tilde{a}} - p_{\tilde{a}}} m_{\Phi}^2 \left( 1 - \frac{m_{\chi_j}}{m_{\Phi}} \right) \left( 1 - \frac{m_{\chi_j'}}{m_{\Phi}} \right), (5.50)
\]

where

\[
Q_{\tilde{a} \rightarrow \chi_j' \xi} := h_{j}^* h_{j} (C_{\phi}^{-1})_{2\tilde{a}} (C_{\phi})_{\tilde{a}1}. (5.51)
\]

\[ \chi_j' \xi \rightarrow \chi_j' \xi \]

For the initial states \( \chi_j' \xi \), we have

\[
\sum_{j, \tilde{a}} \Gamma_{\chi_j' \xi \rightarrow \tilde{a} \rightarrow \chi_j' \xi}^{\text{scat}} := \frac{1}{4} \sum_j \int d\Pi_{\chi_j'} \int d\Pi_{\xi} (2\pi)^3 \delta^3(p_{\chi_j'} + p_{\xi}) \sigma'_{\text{CM}}[\chi_j' \xi \rightarrow \chi_j' \xi] \bigg|_{\text{OS}} (5.52)
\]

\[
= \frac{i}{2^{8} \pi^{2}} \sum_{j, \tilde{a}, \beta} \frac{Q_{\tilde{a} \rightarrow \chi_j' \xi} \cdot Q_{\tilde{a} \rightarrow \chi_j' \xi}}{p_{\tilde{a}} - p_{\tilde{a}}} m_{\Phi}^2 \left( 1 - \frac{m_{\chi_j}}{m_{\Phi}} \right) \left( 1 - \frac{m_{\chi_j'}}{m_{\Phi}} \right), (5.53)
\]

where

\[
Q_{\tilde{a} \rightarrow \chi_j' \xi} := h_{j}^* h_{j} (C_{\phi}^{-1})_{2\tilde{a}} (C_{\phi})_{\tilde{a}2}. (5.54)
\]

\[ \psi_j \psi_j' \rightarrow \chi_j' \xi \]

For the initial states \( \psi_j \psi_j' \), we obtain

\[
\sum_{j, \tilde{a}} \Gamma_{\psi_j \psi_j' \rightarrow \tilde{a} \rightarrow \chi_j' \xi}^{\text{scat}} := \frac{1}{4} \sum_j \int d\Pi_{\psi_j} \int d\Pi_{\psi_j'} (2\pi)^3 \delta^3(p_{\psi_j} + p_{\psi_j'}) \sigma'_{\text{CM}}[\psi_j \psi_j' \rightarrow \chi_j' \xi] \bigg|_{\text{OS}} (5.55)
\]

\[
= \frac{i}{2^{8} \pi^{2}} \sum_{j, \tilde{a}, \beta} \frac{Q_{\tilde{a} \rightarrow \chi_j' \xi} \cdot Q_{\tilde{a} \rightarrow \chi_j' \xi}}{p_{\tilde{a}} - p_{\tilde{a}}} m_{\Phi}^2 \left( 1 - \frac{m_{\chi_j}}{m_{\Phi}} \right) \left( 1 - \frac{m_{\chi_j'}}{m_{\Phi}} \right), (5.56)
\]

where

\[
Q_{\tilde{a} \rightarrow \chi_j' \xi} := h_{j}^* f_{j} (C_{\phi}^{-1})_{2\tilde{a}} (C_{\phi})_{\tilde{a}1} + h_{j}^* f_{j} (C_{\phi}^{-1})_{2\tilde{a}} (C_{\phi})_{\tilde{a}2}. (5.57)
\]
Moreover, we can also find

\[
\sum_{j,\alpha} \Gamma_{\text{scatt}}^{\chi_j\xi^* \to \psi_i\psi_i^*} = \frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi^*} (2\pi)^3 \delta^3(p_{\chi_j} \mp p_{\xi^*}) \sigma'_\text{CM}(\chi_j\xi^* \to \psi_i\psi_i^*) \bigg|_{\text{OS}} (5.58)
\]

\[
= \frac{i}{28\pi^2} \sum_{j,\alpha,\beta} \frac{\alpha_j\xi^* \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha}}{p_{\phi\beta} - p_{\phi\alpha}^2} m_{\phi}^2 \left( 1 - \frac{m_{\chi_j}^2}{m_{\phi}^2} \right), \quad (5.59)
\]

where

\[
Q_{\alpha\phi\alpha \to \psi_i\psi_i^*}^{\chi_j\xi^*} := f_i h_j (C_\Phi^{-1})_{1\alpha}(C_\Phi)_{\hat{a}1} + f_i^* h_j (C_\Phi^{-1})_{2\alpha}(C_\Phi)_{\hat{a}1} \quad (5.60)
\]

For the initial states \(\chi_j\xi\), we obtain

\[
\sum_{j,\alpha} \Gamma_{\text{scatt}}^{\chi_j\xi \to \psi_i\psi_i^*} = \frac{1}{4} \sum_j \int d\Pi_{\chi_j} \int d\Pi_{\xi} (2\pi)^3 \delta^3(p_{\chi_j} \mp p_{\xi}) \sigma'_\text{CM}(\chi_j\xi \to \psi_i\psi_i^*) \bigg|_{\text{OS}} (5.61)
\]

\[
= \frac{i}{28\pi^2} \sum_{j,\alpha,\beta} \frac{\alpha_j\xi \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha}}{p_{\phi\beta} - p_{\phi\alpha}^2} m_{\phi}^2 \left( 1 - \frac{m_{\chi_j}^2}{m_{\phi}^2} \right), \quad (5.62)
\]

where

\[
Q_{\phi\alpha \to \psi_i\psi_i^*}^{\chi_j\xi} := f_i h_j (C_\Phi^{-1})_{1\alpha}(C_\Phi)_{\hat{a}2} + f_i^* h_j (C_\Phi^{-1})_{2\alpha}(C_\Phi)_{\hat{a}2}. \quad (5.63)
\]

When the initial states are \(\psi_j\psi_j^*\), we have

\[
\sum_{j,\alpha} \Gamma_{\text{scatt}}^{\psi_j\psi_j^* \to \psi_i\psi_i^*} = \frac{1}{4} \sum_j \int d\Pi_{\psi_j} \int d\Pi_{\psi_j^*} (2\pi)^3 \delta^3(p_{\psi_j} \mp p_{\psi_j^*}) \sigma'_\text{CM}(\psi_j\psi_j^* \to \psi_i\psi_i^*) \bigg|_{\text{OS}} (5.64)
\]

\[
= \frac{i}{28\pi^2} \sum_{j,\alpha,\beta} \frac{\alpha_j\psi_j \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha} \alpha_{\phi\alpha}}{p_{\phi\beta} - p_{\phi\alpha}^2} m_{\phi}^2, \quad (5.65)
\]

where

\[
Q_{\phi\alpha \to \psi_i\psi_i^*}^{\psi_j\psi_j^*} := f_i h_j (C_\Phi^{-1})_{1\alpha}(C_\Phi)_{\hat{a}1} + f_i h_j (C_\Phi^{-1})_{1\alpha}(C_\Phi)_{\hat{a}2} + f_i^* h_j (C_\Phi^{-1})_{2\alpha}(C_\Phi)_{\hat{a}1} + f_i^* h_j (C_\Phi^{-1})_{2\alpha}(C_\Phi)_{\hat{a}2} \quad (5.66)
\]

### 5.3 From the time-dependent on-shell contributions to scattering

In practical applications, time-dependent expressions of decay widths are frequently used. To express the transition rates we have obtained as an integral over time, we can apply the Fourier transform to
equation 5.25 which is an integral over the total energy. To discard the off-shell divergent contribution, we rewrite
\[
\frac{(E^2 + m_{\chi_i}^2)(E^2 - m_{\chi_j}^2)}{E^2(E^2 - p_{\Phi_a}^2)} \rightarrow \frac{(m_{2_{\Phi_a}} + m_{\alpha}^2)(m_{\alpha}^2 - m_{\chi_i}^2)}{m_{2_{\Phi_a}}(E^2 - p_{\Phi_a}^2)},
\]
(5.67)
which is the on-shell prescription in this case. The Fourier transform of its energy-dependent part is written as
\[
F_{\alpha}(t - t_0) := \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iE(t-t_0)} \frac{1}{E^2 - p_{\Phi_a}^2} = -\frac{i}{2p_{\Phi_a}} e^{-ip_{\Phi_a}|t-t_0|}.
\]
(5.68)
Since \(E = m_{\chi_i}^2\) is an off-shell contribution which is well outside the resonance peak unless the relevant couplings are large, we may neglect its effect and rewrite
\[
\int_{m_{\chi_i}}^{\infty} dE \frac{1}{(E^2 - p_{\Phi_a}^2)^2} = \int_{0}^{\infty} dE \frac{1}{(E^2 - p_{\Phi_a}^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} dE \frac{1}{(E^2 - p_{\Phi_a}^2)(E^2 - p_{\Phi_a}^2)}
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dt' e^{-iE(t'-t_0)} F_{\beta}(t'-t_0) \int_{-\infty}^{\infty} dt e^{iE(t-t_0)} F_{\beta}(t-t_0) = \frac{\pi}{4p_{\Phi_{\beta}}^{*} p_{\Phi_{\alpha}}} \int_{-\infty}^{\infty} dt e^{i(p_{\Phi_{\beta}}^{*}-p_{\Phi_{\alpha}})(t-t_0)}
\]
\[
= \frac{\pi}{2p_{\Phi_{\beta}}^{*} p_{\Phi_{\alpha}}} \int_{t_0}^{\infty} dt e^{i(p_{\Phi_{\beta}}^{*}-p_{\Phi_{\alpha}})(t-t_0)} = \frac{i\pi}{2p_{\Phi_{\beta}}^{*} p_{\Phi_{\alpha}}(p_{\Phi_{\beta}}^{*}-p_{\Phi_{\alpha}})}.
\]
(5.69)
We can therefore write the transition rate in the form of the integration over time:
\[
\sum_{j,\alpha} \sum_{j,\beta} \Gamma_{\chi_j \chi_i \rightarrow \Phi_{\alpha} \rightarrow \chi_j \chi_i} := \frac{1}{2\pi^2} \sum_{j,\alpha,\beta} \sum_{j,\alpha,\beta} Q_{\Phi_{\alpha} \rightarrow \chi_j \chi_i}^{j,\alpha} Q_{\Phi_{\beta} \rightarrow \chi_j \chi_i}^{j,\beta} \frac{m_{\Phi}}{m_{\Phi}} \left(1 - m_{\chi_i}^4/m_{\chi_j}^4\right) \left(1 - m_{\chi_i}^4/m_{\chi_j}^4\right)
\]
\[
\int_{t_0}^{\infty} dt e^{i(p_{\Phi_{\beta}}^{*}-p_{\Phi_{\alpha}})(t-t_0)}
\]
(5.70)
\[
= \frac{i}{2\pi^2} \sum_{j,\alpha,\beta} \sum_{j,\alpha,\beta} \frac{Q_{\Phi_{\beta} \rightarrow \chi_j \chi_i}^{j,\beta}}{p_{\Phi_{\beta}}^{*}-p_{\Phi_{\alpha}}} \frac{m_{\Phi}}{m_{\Phi}} \left(1 - m_{\chi_i}^4/m_{\chi_j}^4\right) \left(1 - m_{\chi_i}^4/m_{\chi_j}^4\right),
\]
(5.71)
where
\[
Q_{\Phi_{\alpha} \rightarrow \chi_j \chi_i}^{j,\alpha} := h_{ij}(C_{\Phi_{\alpha}^{-1}}(C_{\Phi_{\beta}})_{\alpha 2}
\]
(5.72)
as defined before. Equation 5.71 is identical to equation 5.40, which is an almost trivial result since they are simply related to each other by the Fourier transform as long as the on-shell prescriptions are mutually consistent. The expression given by equation 5.70 is useful when we need to track the time evolution of the decay, as conventionally done in experiments or in theoretical analyses using the time-dependent solution of the effective Schrödinger equation.

If the decay widths of two quasiparticles are well-separated as in the neutral kaon system, we can always choose an intermediate time \(t = t_1 > t_0\) after which practically only the long-lived meson survives. Integrating equation 5.70 over \(t_1 < t < \infty\), it is possible to obtain an expression associated only with the long-lived one. We will investigate a kaon-like system in such a way in section 7.2.
Now we determine such an instant \( t = t_1 \), assuming \( \hat{\Gamma}_{\beta} \ll \hat{\Gamma}_{\phi} \). The time integral for \( t > t_1 \) is written as

\[
\int_{t_1}^{\infty} dt \, e^{i(p_{\phi,\beta}^2 - p_{\phi,\alpha})/(t-t_0)} = e^{i(p_{\phi,\beta}^2 - p_{\phi,\alpha})/(t_1-t_0)} \int_{t_1}^{\infty} dt \, e^{-i(m_{\phi,\beta}^2 - m_{\phi,\alpha})/(t-t_0)} e^{-\frac{i}{2}(\Gamma_{\phi,\beta} + \Gamma_{\phi,\alpha})(t_1-t_0)}. \tag{5.73}
\]

In order to be able to neglect the contribution of \( \hat{\Gamma}_{\beta} \) after \( t = t_1 > t_0 \) up to the working precision, we have to choose \( t_1 \) such that

\[
e^{-\hat{\Gamma}_{\beta}(t_1-t_0)} \lesssim O(f^2/4\pi) \ll e^{-\hat{\Gamma}_{\phi}(t_1-t_0)}, \tag{5.74}
\]

\( i.e., \)

\[
-\hbar \log \left[ O(f^2/4\pi) \right] \lesssim t_1 - t_0 \ll -\hbar \log \left[ O(f^2/4\pi) \right] \frac{1}{\hat{\Gamma}_{\phi}}, \tag{5.75}
\]

where \( \hbar \) is written for clarity. We can always find \( t_1 \) which satisfies this condition as long as \( \hat{\Gamma}_{\phi} \) and \( \hat{\Gamma}_{\beta} \) are well-separated. For convenience, we explicitly define \( t_1 \) as

\[
t_1 := -\frac{2\hbar}{\hat{\Gamma}_{\beta}} \log \left[ \frac{1}{4\pi} \sum |f_i|^2 + |h_i|^2 \right], \tag{5.76}
\]

and write the transition rate for \( t > t_1 \) as

\[
\sum_{j,\alpha} \chi_j^\alpha \rightarrow \phi_{\alpha} = \frac{1}{2\pi^2} \sum_{j,\alpha,\beta} Q_{\phi,\beta \rightarrow \phi,\alpha} \chi^\alpha \chi^\beta \frac{m_{\phi}^2}{m_{\phi}^4} \left( 1 - \frac{m_{\chi}^4}{m_{\phi}^4} \right) \left( 1 - \frac{m_{\chi}^4}{m_{\phi}^4} \right)
\]

\[
\int_{t_1}^{\infty} dt \, e^{i(p_{\phi,\beta}^2 - p_{\phi,\alpha})/(t-t_0)}, \tag{5.77}
\]

which is useful in a system like \( K^0 \rightarrow \tilde{K}^0 \) where \( \Gamma_{K^0} \gg \Gamma_{K^0} \).

### 5.4 From the optical theorem

Now we discuss a method to calculate some alternative transition rates from the left-hand side of the unitarity condition 5.15. It is explicitly written as

\[
\int d\Pi_{\chi_i} \int d\Pi_{\xi_s} (2\pi)^3 \delta^3(p - p_{\chi_i} - p_{\xi_s}) \sum_{r,s} \langle \chi_i, i, \xi_{s,r} | (\mathcal{T}^\dagger - \mathcal{T}) | \chi_i, r, \xi_{s} \rangle. \tag{5.78}
\]

Note that this corresponds to applying the phase space integration 5.9 of the final states to the diagram of figure 9 without the unitarity cut and taking its imaginary part. Since

\[
\langle \chi_i, r, \xi_s | (\mathcal{T}^\dagger - \mathcal{T}) | \chi_i, r, \xi_s \rangle = (2\pi)^4 \delta^4(0) \, 2\text{Im}[\mathcal{M}(\chi_i, i, \xi_s \rightarrow \chi_i, r, \xi_s)], \tag{5.79}
\]

the scattering amplitude is written as

\[
i\mathcal{M}(\chi_i, i, \xi_s \rightarrow \chi_i, r, \xi_s) = \overline{u}_{\chi_i}(p_{\chi_i}) v_{\xi_s}(p_{\xi_s})(-ih_i)[i\Delta_{11}(p^2)(-ih_i)\overline{v}_{\xi_s}(p_{\xi_s})u_{\chi_i}(p_{\chi_i})]
\]

...
where we have defined a coefficient

$$Q_{\chi,\xi}^{\delta} := |h_i|^2 (C_{\Phi}^{-1})_{1\delta} (C_{\Phi})_{\delta1}. \quad (5.81)$$

Using \(\sum_{r,s} u_{r,\chi}^* v_{r,\xi}^* w_{s,\chi} = 4(p_\xi \cdot p_\chi)\), we can write in the CM frame

$$\sum_{r,s} \text{Im}[\mathcal{M}(\chi_i, r\xi_s^c \rightarrow \chi_i, r\xi_s^c)] = -4(p_\xi \cdot p_\chi) \sum_{\alpha} \text{Im}\left[\frac{Q_{\chi,\xi}^\alpha}{p^2 - m_{\Phi_\alpha}^2}\right]$$

$$= 2(E^2 - m_{\chi_i}^2) \sum_{\alpha} \text{Im}[Q_{\chi,\xi}^\alpha] \left(E^2 - m_{\Phi_\alpha}^2\right) + \text{Re}[Q_{\chi,\xi}^\alpha] m_{\Phi_\alpha} \Gamma_{\Phi_\alpha}. \quad (5.82)$$

As in the previous derivation, we need an on-shell prescription to remove the divergent off-shell contribution. In equation 5.82, the term \(E^2 - m_{\Phi_\alpha}^2\) is the off-shell part, and thus we should neglect it. Applying the narrow-width approximation of equation 5.27 to the remaining part, we obtain

$$\sum_{\alpha} \Gamma_{\Phi_\alpha \rightarrow \chi^\xi}^\text{opt} := \int d\Pi_{\chi_i} \int d\Pi_{\xi_s^c} (2\pi)^3 \delta^3(p_{\chi_i} + p_{\xi_s^c}) \sum_{r,s} \text{Im}[\mathcal{M}_{\text{OS}}(\chi_i, r\xi_s^c \rightarrow \chi_i, r\xi_s^c)], \quad (5.83)$$

$$= \frac{m_{\Phi}}{16\pi} \left(1 - \frac{m_{\Phi_i}^2}{m_{\Phi}^2}\right) \sum_{\alpha} \text{Re}[Q_{\chi,\xi}^\alpha]. \quad (5.84)$$

Note that the second equality is correct up to the leading order in perturbation. Similar calculations can be done for the other final states as well. For \(\Phi_\alpha \rightarrow \chi^\xi\), we obtain

$$\sum_{\alpha} \Gamma_{\Phi_\alpha \rightarrow \chi^\xi}^\text{opt} := \int d\Pi_{\chi_i} \int d\Pi_{\xi_s^c} (2\pi)^3 \delta^3(p_{\chi_i} + p_{\xi_s^c}) \sum_{r,s} \text{Im}[\mathcal{M}_{\text{OS}}(\chi_i, r\xi_s^c \rightarrow \chi_i, r\xi_s^c)]$$

$$= \frac{m_{\Phi}}{16\pi} \left(1 - \frac{m_{\Phi_i}^2}{m_{\Phi}^2}\right) \sum_{\alpha} \text{Re}[Q_{\chi,\xi}^\alpha], \quad (5.85)$$

where

$$Q_{\chi,\xi}^{\delta} := |h_i|^2 (C_{\Phi}^{-1})_{2\delta} (C_{\Phi})_{\delta2}. \quad (5.87)$$

In addition, for \(\Phi_\alpha \rightarrow \psi_i\psi_i^c\), we have

$$\sum_{\alpha} \Gamma_{\Phi_\alpha \rightarrow \psi_i\psi_i^c}^\text{opt} := \int d\Pi_{\psi_i} \int d\Pi_{\psi_i^c} (2\pi)^3 \delta^3(p_{\psi_i} + p_{\psi_i^c}) \sum_{r,s} \text{Im}[\mathcal{M}_{\text{OS}}(\psi_i, r\psi_i^c_s \rightarrow \psi_i, r\psi_i^c_s)]$$

$$= \frac{m_{\Phi}}{16\pi} \sum_{\alpha} \text{Re}[Q_{\psi,\psi}^\alpha], \quad (5.88)$$

where

$$Q_{\psi,\psi}^{\alpha} := |f_i|^2 (C_{\Phi}^{-1})_{1\alpha} (C_{\Phi})_{\alpha1} + |f_i|^2 (C_{\Phi}^{-1})_{2\alpha} (C_{\Phi})_{\alpha2}. \quad (5.90)$$

Similarly to the previous case, we cannot remove \(\sum_{\alpha}^\text{opt}\) to define a physical transition rate associated with each quasiparticle, \(\Gamma_{\Phi_\alpha \rightarrow \chi^\xi}\), since it is a complex number in general.
5.5 Comparison of transition rates

We have obtained two apparently different candidates to which the partial decay width of $\Phi_\beta$ is going to be related: $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$. The expression $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ is identical to $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$, which is trivial since they are related to each other by the Fourier transform. Even though $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$ are calculated from the right- and left-hand sides of the unitarity condition 5.15, respectively, it is not guaranteed that they are identical since the on-shell prescriptions are differently imposed in their derivations. In other words, the contributions of the intermediate on-shell mesons were considered in different ways to obtain those two transition rates.

To be explicit, let us look into each derivation with more details. In the case of $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$, the on-shell prescription is imposed on equation 5.26 for both $E^2 = m_\Phi^2$ and $E^2 = m_{\Phi_\beta}^2$, and they come from $\mathcal{M}^*$ and $\mathcal{M}$, respectively. We have obtained a quantity related to a single flavor of quasiparticles, $\Phi_\beta$, only after summing over $\bar{\beta}$. In other words, $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ involves the interference of different flavors of quasiparticles. On the other hand, $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$ is calculated from the imaginary part of $\mathcal{M}$, and no interference is involved there. In deriving $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$, only the contribution from a single flavor, $\Phi_\beta$, has been considered. Hence, it is unclear whether $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$ are identical or not.

In spite of the apparent difference, we claim that they are actually identical:

$$\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}} = \sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}. \quad (5.91)$$

Moreover, we also claim that they are consistent with the total decay widths obtained from the self-energy in the following sense:

$$\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y} = \sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}} = \sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}. \quad (5.92)$$

These claims can be verified by explicit numerical evaluations.

In order to verify equation 5.91, let us consider their fractional difference defined by

$$\text{FD}_Y := \left| \frac{\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}} - \sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}}{\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}} \right|. \quad (5.93)$$

When $\text{FD}_{\chi_1 \xi c} \lesssim O(f^2/4\pi)$, we can say that the difference of $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$ is beyond the leading order in perturbation. Let us consider the case of $Y = \chi_1 \xi c$, which will be discussed in section 7 as a numerical example that imitates the semileptonic decay of $B^0$. We choose the scale of a parameter $f$ to be in the range $-6 \leq \log_{10} |f| \leq -4$ and set the values of $f_i$ as $f_1 = f$ and $\log_{10} |f_2| = \log_{10} |f_1| + 0.0036$. All the other parameters are chosen to be identical with those in table 1. The result is shown in figure 10. Note that $\text{FD}_{\chi_1 \xi c}$ is close to the line of $f^2/4\pi$, i.e., the dotted line in the plot. In other words, the difference of $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{opt}}$ is of the next-leading order in perturbation since $\text{FD}_{\chi_1 \xi c} \sim O(f^2/4\pi)$. Hence, $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow \chi_1 \xi c}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow \chi_1 \xi c}$ are identical up to the leading order.

The verification of equation 5.92 is similar. Since the difference between $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta}^{\text{scat}}$ and $\sum_{\bar{\alpha}} \Gamma_{\Phi_\beta \rightarrow Y}^{\text{scat}}$ was simply zero in all the cases numerically checked, we do not present any figures about their comparison. Note that equations 5.91 and 5.92 have been checked and confirmed for many other numerical examples of various decay products.
Figure 10. Fractional difference of $\sum_\alpha \Gamma^{\text{scat}}_{\hat{\Phi}_\alpha \rightarrow \chi_1 \xi}$ and $\sum_\alpha \Gamma^{\text{opt}}_{\hat{\Phi}_\alpha \rightarrow \chi_1 \xi^c}$. The difference is approximately of the next-leading order in perturbation, which implies $\sum_\alpha \Gamma^{\text{scat}}_{\hat{\Phi}_\alpha \rightarrow \chi_1 \xi}$ and $\sum_\alpha \Gamma^{\text{opt}}_{\hat{\Phi}_\alpha \rightarrow \chi_1 \xi^c}$ are identical up to the leading order.

5.6 Partial decay widths

Now we discuss how the partial decay widths of $\Phi$ and $\Phi^*$ can be defined from the transition rates we have derived so far. Assume that the initial state is tagged by $\chi^c_j$ or $\chi_j \xi^c$ using all possible $j$. Then, the partial decay widths can be defined from the corresponding transition rates as follows:

$$
\Gamma(\Phi \rightarrow Y) := \sum_{j, \tilde{\alpha}} \Gamma^{\text{scat}}_{\chi_j \xi^c \rightarrow \tilde{\Phi}_\tilde{\alpha} \rightarrow Y}, \quad \Gamma(\Phi^* \rightarrow Y) := \sum_{j} \Gamma^{\text{scat}}_{\chi_j \xi \rightarrow \hat{\Phi}_\alpha \rightarrow Y}.
$$

(5.94)

In the absence of particle-antiparticle mixing, i.e., in the limit of $f_i = 0$, these expressions reproduce the well-known formulas up to the leading order:

$$
\Gamma(\Phi \rightarrow \chi_i \xi^c) = \Gamma(\Phi^* \rightarrow \chi_i \xi) = \frac{|h_i|^2 m_{\Phi}}{16\pi} \left(1 - \frac{m_{\xi}^4}{m_{\Phi}^4}\right),
$$

(5.95)

$$
\Gamma(\Phi \rightarrow \chi_i \xi) = \Gamma(\Phi^* \rightarrow \chi_i \xi^c) = \Gamma(\Phi \rightarrow \psi_j \psi_j^c) = \Gamma(\Phi^* \rightarrow \psi_j^c \psi_j) = 0,
$$

(5.96)

since, in that case, we have $\tilde{\Phi}_1 = \Phi$, $\tilde{\Phi}_2 = \Phi^*$, $C_{\Phi} = I_2$, $\Gamma_{\tilde{\Phi}_1} = \Gamma(\Phi \rightarrow \chi_i \xi^c)$, and $\Gamma_{\tilde{\Phi}_2} = \Gamma(\Phi^* \rightarrow \chi_i \xi)$. Hence, the definitions of the partial decay widths given by equation 5.94 are reasonable.

Note that, in those definitions, the initial states such as $\psi_j \psi_j^c$ that couple to both $\Phi$ and $\Phi^*$ are excluded. Any scattering process whose initial states are $\psi_j \psi_j^c$ must have contributions of both $\psi_j \psi_j^c \rightarrow \Phi$ and $\psi_j^c \psi_j \rightarrow \Phi^*$, and they interfere with each other. Hence, defining the partial decay widths of $\Phi$ considering $\psi_j \psi_j^c \rightarrow \Phi$ while neglecting $\psi_j^c \psi_j \rightarrow \Phi^*$ does not make sense, since such a quantity is never directly related to any physical observables.

When we actually calculate a ratio of two partial decay widths, it does not really matter whether all possible $j$ is considered or not, and we can in fact use only one of them, e.g., $\chi^c_j \xi^c$ and $\chi_j \xi$ for tagging. This is because the factor relevant to the initial states is common in all the expressions, and they can be factored out to be canceled in the ratio. To be specific, we can define a common factor incorporating the overall multiplicative factor as well as the couplings and phase space factors of the initial states:

$$
\mathcal{N} := \frac{m_{\Phi}^2}{m_{\Phi}^2} \sum_j |h_j|^2 \left(1 - \frac{m_{\xi}^4}{m_{\Phi}^4}\right),
$$

(5.97)
to write the partial decay widths as

\[
\Gamma(\Phi \to \chi_i \xi_e) = i\mathcal{N}|h_i|^2 \left(1 - \frac{m^4_{\chi_i}}{m^4_{\Phi}}\right) \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{(C_{\Phi}^{-1})_1^{*}\beta}(C_{\Phi})_{\tilde{\beta}1}(C_{\Phi})_{\tilde{\alpha}1}}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}},
\]

(5.98)

\[
\Gamma(\Phi^* \to \chi_i^c \xi) = i\mathcal{N}|h_i|^2 \left(1 - \frac{m^4_{\chi_i}}{m^4_{\Phi}}\right) \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{(C_{\Phi}^{-1})_{2\beta}^{*}(C_{\Phi})_{\tilde{\beta}2}(C_{\Phi})_{\tilde{\alpha}2}}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}},
\]

(5.99)

\[
\Gamma(\Phi \to \chi_i \xi_e) = i\mathcal{N}|h_i|^2 \left(1 - \frac{m^4_{\chi_i}}{m^4_{\Phi}}\right) \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{(C_{\Phi}^{-1})_{1\beta}^{*}(C_{\Phi})_{\tilde{\beta}1}(C_{\Phi})_{\tilde{\alpha}1}}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}},
\]

(5.100)

\[
\Gamma(\Phi^* \to \chi_i^c \xi) = i\mathcal{N}|h_i|^2 \left(1 - \frac{m^4_{\chi_i}}{m^4_{\Phi}}\right) \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{(C_{\Phi}^{-1})_{2\beta}^{*}(C_{\Phi})_{\tilde{\beta}2}(C_{\Phi})_{\tilde{\alpha}2}}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}},
\]

(5.101)

\[
\Gamma(\Phi \to \psi_i \psi_i^c) = i\mathcal{N} \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{1}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}} \left[f_i(C^{-1}_{\Phi})_1^{*}\beta} + f_i(C^{-1}_{\Phi})_2^{*}\beta} (C_{\Phi})_{\tilde{\beta}1}\right]^* \left[f_i(C^{-1}_{\Phi})_1^{*}\alpha} + f_i(C^{-1}_{\Phi})_2^{*}\alpha} (C_{\Phi})_{\tilde{\alpha}1}\right],
\]

(5.102)

\[
\Gamma(\Phi^* \to \psi_i \psi_i^c) = i\mathcal{N} \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{1}{p^\beta_{\tilde{\Phi}} - p^\alpha_{\tilde{\Phi}}} \left[f_i(C^{-1}_{\Phi})_1^{*}\beta} + f_i(C^{-1}_{\Phi})_2^{*}\beta} (C_{\Phi})_{\tilde{\beta}2}\right]^* \left[f_i(C^{-1}_{\Phi})_1^{*}\alpha} + f_i(C^{-1}_{\Phi})_2^{*}\alpha} (C_{\Phi})_{\tilde{\alpha}2}\right].
\]

(5.103)

In these expressions, the factors with \( C_{\Phi} \) and \( p_{\tilde{\Phi}} \) originate from particle-antiparticle mixing. Even though their effect can sometimes be as large as \( O(1) \), they should be absent in the textbook formula of the decay width given by equation 5.8 since it cannot handle particle-antiparticle mixing as mentioned before.

### 5.7 CP asymmetry

The CP asymmetry in the decay of a meson is a quantity calculated from partial decay widths. For example, the CP asymmetry in the semileptonic decay \( B^0 \to D^- \ell^+ \nu_\ell \) is defined by

\[
A_{CP}^{SL} := \frac{\Gamma(B^0 \to D^+ \ell^- \nu_\ell) - \Gamma(B^0 \to D^- \ell^+ \nu_\ell)}{\Gamma(B^0 \to D^+ \ell^- \nu_\ell) + \Gamma(B^0 \to D^- \ell^+ \nu_\ell)}.
\]

(5.104)

In an experiment designed to observe this quantity, the initial state can be set to \( B^0 \) or \( B^0 \) by an interaction that only couples to either of them, i.e., the initial state can be tagged as we want.

Using the definitions of partial decay widths given by equation 5.94, we can define the CP asymmetry in the decays of \( \Phi \to Y \) and \( \Phi^* \to Y^c \) as follows:

\[
A_{CP} := \frac{\Gamma(\Phi^* \to Y^c) - \Gamma(\Phi \to Y)}{\Gamma(\Phi^* \to Y^c) + \Gamma(\Phi \to Y)}
\]

(5.105)

\[
= \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{\Gamma_{\tilde{\Phi}_{\tilde{\alpha}} \to \tilde{\Phi}_{\tilde{\beta}} \to Y}}{\Gamma_{\tilde{\Phi}_{\tilde{\beta}} \to \tilde{\Phi}_{\tilde{\alpha}} \to Y}} \end{\array}
\]

(5.106)

\[
= \sum_{\tilde{\alpha}, \tilde{\beta}} \frac{\Gamma_{\tilde{\Phi}_{\tilde{\alpha}} \to \tilde{\Phi}_{\tilde{\beta}} \to Y}}{\Gamma_{\tilde{\Phi}_{\tilde{\beta}} \to \tilde{\Phi}_{\tilde{\alpha}} \to Y}}
\]

(5.107)
for any $k$. As mentioned above, we can choose a single multiparticle state $\chi_k \xi^c$ for tagging the initial meson since it does not make any difference in this ratio. The CP asymmetry in the flavor-specific decay such as the semileptonic decay, for example, is given by

$$A_{\text{CP}} := \frac{\Gamma(\Phi^* \rightarrow \chi_i \xi^c) - \Gamma(\Phi \rightarrow \chi_i \xi^c)}{\Gamma(\Phi^* \rightarrow \chi_i \xi^c) + \Gamma(\Phi \rightarrow \chi_i \xi^c)}$$

$$= \sum_{\alpha, \beta} \frac{(C^{-1})_{i, \beta}(C_{\hat{\beta}})^*_{, \alpha}(C_{\hat{\beta}})^*_{, \alpha}}{p_{\hat{\beta}}^2 - m_{\Phi}^2} - \sum_{\alpha, \beta} \frac{(C^{-1})_{i, \beta}(C_{\hat{\beta}})^*_{, \alpha}(C_{\hat{\beta}})^*_{, \alpha}}{p_{\hat{\beta}}^2 - m_{\Phi}^2}.$$

\hspace{1cm} (5.108)

Now we briefly discuss the CP asymmetry in the direct decay. In the toy model presented in this paper, there does not exist CP violation in the direct decay, i.e. $\Gamma(\Phi \rightarrow \chi_i \xi^c) = \Gamma(\Phi^* \rightarrow \chi_i \xi^c)$. We can, of course, extend the toy model to allow CP violation in the direct decay by introducing an additional field. It is evident that the new field should not be a complex scalar since it would not induce a vertex loop correction as is the case only with $\Phi$ and $\Phi^*$. It also had better not be a real scalar field, since it would mix with $\Phi$ and $\Phi^*$ to complicate the analysis, which we do not want at all. The simplest way would be introducing a spin-one field such as a gauge boson $A^\mu$ which couples to $\chi_i \xi^c$ by an interaction $\sum_{\mu} \gamma_{\mu} A^\mu$. However, CP violation in the direct decay is beyond the scope of this paper, since it requires loop corrections to the vertices which correspond to the next-leading-order effects in perturbation. Note that, to keep consistency in perturbation, calculating decay widths up to the one-loop correction in the vertices requires calculating two-loop contributions in the self-energy. We will come back to this issue in section 8, where the derivation of decay widths beyond the leading order will be briefly discussed.

6 Comparison with the analysis in quantum mechanics

In this section, we calculate $\Gamma(\Phi^* \rightarrow \chi_i \xi^c)$ using the conventional method, and compare the result with equation 5.101. It turns out that there exists a discrepancy which could sometimes result in a big difference in the values of decay widths.

Following the standard approach presented in the literature [3, 4], we introduce time-dependent states $|\Phi(t)\rangle$ and $|\Phi^*(t)\rangle$, which satisfy the effective Schrödinger equation

$$i \frac{d}{dt} \begin{pmatrix} |\Phi(t)\rangle \\ |\Phi^*(t)\rangle \end{pmatrix} = H_{\text{eff}} \begin{pmatrix} |\Phi(t)\rangle \\ |\Phi^*(t)\rangle \end{pmatrix}. \hspace{1cm} (6.1)$$

Here, the effective Hamiltonian $H_{\text{eff}}$ is written in terms of the self-energy up to the one-loop order as

$$H_{\text{eff}} := m_{\Phi} \left[ 1 - \frac{1}{2} \Sigma'(m_{\Phi}^2) \right], \hspace{1cm} (6.2)$$

and it is diagonalized as

$$C_{\Phi}H_{\text{eff}}C_{\Phi}^{-1} = m_{\Phi} \left[ 1 - \frac{1}{2} (C\Sigma' C^{-1})(m_{\Phi}^2) \right] = m_{\Phi} \left[ 1 - \frac{1}{2} \Sigma'(m_{\Phi}^2) \right] = \begin{pmatrix} p_{\Phi_1} & 0 \\ 0 & p_{\Phi_2} \end{pmatrix}, \hspace{1cm} (6.3)$$
where \( p_{\hat{\phi}_\alpha} = m_{\hat{\phi}_\alpha} - i\Gamma_{\hat{\phi}_\alpha}/2 \). Hence, defining the time-dependent mass eigenstates by

\[
\begin{pmatrix}
|\hat{\Phi}_1(t)\rangle \\
|\hat{\Phi}_2(t)\rangle
\end{pmatrix} := C\Phi
\begin{pmatrix}
|\Phi(t)\rangle \\
|\Phi^*(t)\rangle
\end{pmatrix},
\]  
(6.4)

and solving the decoupled Schrödinger equation for each mass eigenstate, we obtain

\[
|\hat{\Phi}_\alpha(t)\rangle = e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}|\hat{\Phi}_\alpha\rangle.
\]  
(6.5)

Here, \( |\hat{\Phi}_\alpha\rangle := |\hat{\Phi}_\alpha(t_0)\rangle \) is the mass eigenstate that would be expressed as a fixed linear combination of basis states, and it evolves in time as a plane wave with a damping factor. Setting the initial states as

\[
|\Phi(t_0)\rangle = |\Phi\rangle, \quad |\Phi^*(t_0)\rangle = |\Phi^*\rangle,
\]  
(6.6)

i.e.,

\[
|\Phi_1\rangle = (C\Phi)_{11}|\Phi\rangle + (C\Phi)_{12}|\Phi^*\rangle, \quad |\Phi_2\rangle = (C\Phi)_{21}|\Phi\rangle + (C\Phi)_{22}|\Phi^*\rangle,
\]  
(6.7)

we can write

\[
\begin{pmatrix}
|\Phi(t)\rangle \\
|\Phi^*(t)\rangle
\end{pmatrix} = C\Phi^{-1}
\begin{pmatrix}
0 & e^{-ip_{\hat{\phi}_2}(t-t_0)} \\
e^{-ip_{\hat{\phi}_1}(t-t_0)} & 0
\end{pmatrix} C\Phi
\begin{pmatrix}
|\Phi\rangle \\
|\Phi^*\rangle
\end{pmatrix},
\]  
(6.8)

and thus

\[
|\Phi(t)\rangle = \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}[(C\Phi)^{-1}_{1\alpha}(C\phi)_{\alpha 1}|\Phi\rangle + (C\Phi)^{-1}_{1\alpha}(C\phi)_{\alpha 2}|\Phi^*\rangle],
\]  
(6.10)

\[
|\Phi^*(t)\rangle = \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}[(C\Phi)^{-1}_{2\alpha}(C\phi)_{\alpha 1}|\Phi\rangle + (C\Phi)^{-1}_{2\alpha}(C\phi)_{\alpha 2}|\Phi^*\rangle].
\]  
(6.11)

The standard interpretation of \(|\Phi(t)\rangle\) is that it is the state at time \( t \) which evolved from \(|\Phi\rangle\) at \( t = t_0 \). Using

\[
\langle \chi_i\xi^c|S|\Phi\rangle = -ih_1, \quad \langle \chi_i\xi^c|S|\Phi^*\rangle = 0, \quad \langle \chi_i^c\xi|S|\Phi\rangle = 0, \quad \langle \chi_i^c\xi|S|\Phi^*\rangle = -ih_i^*,
\]  
(6.12)

we obtain

\[
\langle \chi_i\xi^c|S|\Phi(t)\rangle = -i \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}Q^{\Phi(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i\xi^c},
\]  
(6.13)

\[
\langle \chi_i^c\xi|S|\Phi(t)\rangle = -i \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}Q^{\Phi(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i^c\xi},
\]  
(6.14)

\[
\langle \chi_i\xi^c|S|\Phi^*(t)\rangle = -i \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}Q^{\Phi^*(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i\xi^c},
\]  
(6.15)

\[
\langle \chi_i^c\xi|S|\Phi^*(t)\rangle = -i \sum_\alpha e^{-ip_{\hat{\phi}_\alpha}(t-t_0)}Q^{\Phi^*(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i^c\xi},
\]  
(6.16)

where

\[
Q^{\Phi(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i\xi^c} := h_1(C\Phi^{-1})_{1\alpha}(C\phi)_{\alpha 1}, \quad Q^{\Phi(t)}_{\hat{\phi}_\alpha\rightarrow \chi_i^c\xi} := h_i^*(C\Phi^{-1})_{1\alpha}(C\phi)_{\alpha 2},
\]  
(6.17)
As a specific example, let us consider the decay $\Phi^* \to \chi_i \xi^c$ whose initial state $|\Phi^*\rangle$ is tagged by $|\chi_i \xi^c\rangle$ at $t = t_0$. Incorporating all the information on the interaction $\chi^c_i \to \Phi^*$ as well as the phase space factor of $|\chi_i \xi^c\rangle$ into a time-independent factor $\mathcal{N}_{\Phi^* \to \chi_i \xi^c}$, we can write the decay width as

$$
\Gamma_{\Phi^* \to \chi_i \xi^c}^{\text{QM}} \defeq \int_{t_0}^{\infty} dt \left| \mathcal{N}_{\Phi^* \to \chi_i \xi^c} (\chi_i \xi^c | \Phi^*(t)) \right|^2 = \left| \mathcal{N}_{\Phi^* \to \chi_i \xi^c} \right|^2 \sum_{\alpha, \beta} \left[ Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t) \right]^* Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t) \int_{t_0}^{\infty} dt e^{i(p_{\Phi \to \chi_i \xi^c}^\beta - p_{\Phi \to \chi_i \xi^c}^\alpha)(t-t_0)} = i \left| \mathcal{N}_{\Phi^* \to \chi_i \xi^c} \right|^2 \sum_{\alpha, \beta} \left[ Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t) \right]^* Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t) \left( \frac{m^4}{m^4_{\Phi \to \chi_i \xi^c}} \right)
$$

(6.19)

Now we compare this expression with equation 5.70 obtained in the quantum field theory. First the overall factor $\mathcal{N}_{\Phi^* \to \chi_i \xi^c}$ can be identified as

$$
\left| \mathcal{N}_{\Phi^* \to \chi_i \xi^c} \right|^2 \defeq \frac{1}{2\pi^2} \sum_j |h_j|^2 m_{\Phi}^2 \left( 1 - \frac{m_{\chi_i}^2}{m_{\Phi}^2} \right) \left( 1 - \frac{m_{\xi^c}^2}{m_{\Phi}^2} \right)
$$

(6.20)

in accordance with its definition. Note that there exists a difference between equations 5.70 and 6.19, which originates from the difference in $Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t)$ and $Q_{\Phi \to \chi_i \xi^c}^{\Phi^*}(t)$. The indices $\beta$ and $\gamma$ in $(C_{\Phi \to \chi_i \xi^c})_{\beta \gamma}$ are in the opposite order. The indices $\beta$ and $\gamma$ in the current calculation are of the initial and final states of basis mesons respectively, while the reverse is true in the expression from the quantum field theory.

In fact, this derivation using the effective Hamiltonian and Schrödinger equation is flawed. Equation 6.1 is correct only when $|\Phi(t)\rangle = |\Phi\rangle$ and $|\Phi^*(t)\rangle = |\Phi^*\rangle$ since each index of the self-energy corresponds to $\Phi$ or $\Phi^*$, not any other linear combination of them, while the expressions given by equations 6.10 and 6.11 imply that $|\Phi(t)\rangle \neq |\Phi\rangle$ and $|\Phi^*(t)\rangle \neq |\Phi^*\rangle$ for $t > t_0$. Hence, the Schrödinger equation 6.1 cannot be the correct equation of motion in the presence of particle-antiparticle mixing. Moreover, any approach to find a solution solving an equation of motion cannot be valid, since such a method is supposed to give a time evolution of equation 6.5 for each decoupled degree of freedom. That solution is correct only when there exists a mass eigenstate, which is a specific linear combination of $|\Phi\rangle$ and $|\Phi^*\rangle$, such that it evolves like a plane wave while keeping its identity. In the derivation based on the quantum field theory, however, there does not exist such a fixed linear combination corresponding to each quasiparticle, as we have seen in section 4.

The discrepancy in the ordering of indices could sometimes lead to a big difference in the values of the decay widths, although it does not cause a difference in their values in many cases. As an example, let us consider $\Phi \to \psi_i \psi_i^c$, all the contributions to which can be grouped into $\Phi \to \cdots \to \Phi \to \psi_i \psi_i^c$ and $\Phi \to \cdots \to \Phi^* \to \psi_i \psi_i^c$. In the derivation using the quantum field theory, they correspond to $(C_{\Phi \to \chi_i \xi^c})_{\alpha \beta}$ and $(C_{\Phi \to \chi_i \xi^c})_{\alpha \beta}$, respectively, and the associated transition rate and its coefficient are given by equations 5.59 and 5.60. On the other hand, in the conventional derivations based on quantum mechanics, they correspond to $(C_{\Phi \to \chi_i \xi^c})_{\alpha \beta}$ and $(C_{\Phi \to \chi_i \xi^c})_{\alpha \beta}$, respectively. Since there exists an interference between those two contributions, the different ordering of the indices can sometimes give a significantly different result. We will see such an example in section 7.
7 Examples

In this section, we discuss two numerical examples in which the toy model imitates real meson systems. The first one is the $B^0 \bar{B^0}$ system, where we can see not only that the model indeed closely mimic the neutral meson system but also that the CP asymmetry calculated from the formula derived in this paper is consistent with its theoretical and experimental values currently available. This will support the validity of the formalism developed in this paper. The second example is the $K^0 - \bar{K^0}$ system, through which the nature of quasiparticles can be more clearly understood. In this system, the decay widths of two quasiparticles are well-separated, and it is expected that only one of them, $K^0_L$, would survive after some time ($t > t_1$). Even though this seems to imply that we can single out a quasiparticle in some cases, we will see that this is not totally true. It turns out that, while the interference term between $K^0_S$ and $K^0_L$ is comparable to the contribution of $K^0_L$ itself, the interference decays as fast as the contribution of $K^0_S$, and thus we must lose a part of $K^0_L$ after $t = t_1$. In other words, we can never observe each quasiparticle as it is. In both examples, we will indeed see that the contribution of each quasiparticle to the transition rate is sometimes complex, and thus it cannot be interpreted as its decay width.

7.1 $B^0 - \bar{B^0}$

Here, we choose the model parameters such that it imitates the $B^0 - \bar{B^0}$ system. The input parameters of the toy model are presented in Table 1, and the corresponding output parameters are given in Table 2. The mixing matrix $C_\Phi$ is found to be

\[ C_\Phi = \begin{pmatrix} 0.519268 + i0.480066 & -0.707034 \\ 0.519268 + i0.480066 & 0.707034 \end{pmatrix} = \begin{pmatrix} p & q \\ p & -q \end{pmatrix}, \]

which is a non-unitary matrix:

\[ C_\Phi^2 = \begin{pmatrix} 2|p|^2 & 0 \\ 0 & 2|q|^2 \end{pmatrix} = \begin{pmatrix} 1.00021 & 0 \\ 0 & 0.999794 \end{pmatrix}. \]

The components of $C_\Phi$ are chosen to satisfy $|p|^2 + |q|^2 = 1$. Note that the CP asymmetry defined by equation 5.107 indeed gives a value that is consistent with the observed values of the CP asymmetry in the semileptonic decay $B^0 \rightarrow D^- \mu^+ \nu_\mu X$ or $B^0 \rightarrow D^+ \mu^- \bar{\nu}_\mu X$, where $X$ denotes any additional decay products. The predicted values in the Standard Model (SM) are also provided for comparison. The deviation from the unitarity of $C_\Phi$ is of the order of the CP asymmetry, $O(10^{-4})$, which is much larger than the typical perturbative correction, $O(f^2/4\pi)$, with $O(10^{-12})$.

The transition rates calculated from the input parameters are presented in Table 3. The values of $\Gamma_{\Phi_{\alpha} \rightarrow 0}$ are sometimes complex, and thus $\Sigma_{\Phi_{\alpha}}$ cannot be arbitrarily removed from $\Sigma_{\Phi_{\alpha}} \Gamma_{\Phi_{\alpha} \rightarrow 0}$ to define a partial decay width of $\Phi_{\alpha}$. Also note $\Gamma_{\Phi_{\alpha} \rightarrow \psi_1 \psi_1'} = \Gamma_{\Phi_{\alpha} \rightarrow \psi_1 \psi_1}$. Nevertheless, we still have the identities: $\sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow 0} = \sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow 0}$ and $\sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow 0}$ = $\sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow 0}$. We can also see that the ratio of decay widths obtained by the method of this paper is sometimes consistent with the value from the conventional approach, e.g., $\Gamma(\Phi \rightarrow \chi_2 \xi') / \Gamma(\Phi \rightarrow \chi_1 \xi')$, while it is sometimes different, e.g., $\Gamma(\Phi \rightarrow \psi_1 \psi_1') / \Gamma(\Phi \rightarrow \psi_1 \psi_1')$.

Using various input parameters, we can see more clearly the difference between the results from quantum mechanics and the quantum field theory. Let us first define the ratios of decay widths as follows:

\[ R_{\Phi_{\alpha} \rightarrow 0}^{\text{QFT}} := \frac{\sum_{j \in \alpha} \Gamma_{\chi_j \xi \rightarrow \Phi_{\alpha} \rightarrow \psi_2 \psi_2'}}{\sum_{j \in \alpha} \Gamma_{\chi_j \xi \rightarrow \Phi_{\alpha} \rightarrow \psi_1 \psi_1'}}, \quad R_{\Phi_{\alpha} \rightarrow 0}^{\text{tot}} := \frac{\sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow \psi_2 \psi_2'}}{\sum_{\alpha} \Gamma_{\Phi_{\alpha} \rightarrow \psi_1 \psi_1'}}, \quad R_{\Phi_{\alpha} \rightarrow 0}^{\text{QM}} := \frac{\Gamma_{\Phi \rightarrow \psi_2 \psi_2'}}{\Gamma_{\Phi \rightarrow \psi_1 \psi_1'}}. \]
Table 1. Model parameters for $\Phi \rightarrow \chi \xi^c$ to mimic the semileptonic decay $B^0 \rightarrow D^- \mu^+ \nu_{\mu} X$ or $B^0 \rightarrow D^+ \mu^- \nu_{\mu} X$. Note that $\mu_1$ is an argument of a logarithmic function, and $\mu_2$ therefore corresponds to a contribution of around $O(10 f^2/4 \pi)$ to the self-energy. The masses $m_{\chi_i}$ are arbitrarily chosen.

| Output parameter for $B^0, B^0$ | Value | SM prediction |Observed value |
|---|---|---|---|
| $m_\Phi$ | 5279.63 MeV [5] | | |
| $m_{\chi_1}$ | 2900 MeV | | |
| $m_{\chi_2}$ | 4500 MeV | | |
| $f_1$ | $10^{-6} \, e^{0.1190}$ | | |
| $f_2$ | $10^{-7} \, e^{-0.3124}$ | | |
| $h_1$ | $10^{-5} \, e^{0.032}$ | | |
| $\mu_1$ | $m_\Phi$ | | |
| $\mu_2$ | $10^3 \cdot m_\Phi$ | | |

Table 2. Output parameters corresponding to the input parameters given in table 1. The observed values available in the references are in fact $(1/2) \sum \Gamma_{\chi_i} = \Gamma_{B^0} = (1.520 \pm 0.004) \, \text{ps}^{-1}$, $\Delta m_{\chi_i}/\Gamma_{\chi_i} \approx \Delta m_{B^0}/\Gamma_{B^0} = 0.770 \pm 0.004$, and $\Delta \Gamma_{\chi_i}/\Gamma_{\chi_i} \approx \Delta \Gamma_{B^0}/\Gamma_{B^0} = -(0.2 \pm 1.0) \cdot 10^{-2}$. In addition, we have used $\text{Re}[\epsilon]/(1 + |\epsilon|^2) = \text{Re}[\epsilon_B]/(1 + |\epsilon_B|^2) \approx (|q/p|^2 - 1)/4$.

Note that $R_{\chi_1,\chi_2}^{\text{tot}}$ is an inclusive quantity for $\Phi$, $\Phi^*$, and their interference, while the others are only for $\Phi$ since it is tagged as such. Varying the phase of $f_2$ such that $-\pi/2 < \text{Arg}[f_2/f_1] \leq \pi/2$, we obtain the variation of ratios as shown in figure 11a, where the differences are obviously presented. The variation of the CP asymmetry, $A_{\text{CP}}$, in $\Phi^* \rightarrow \chi_1 \xi^c$ and $\Phi \rightarrow \chi_2 \xi^c$ is also shown in figure 11b. Note that $A_{\text{CP}} = 0$ when $\text{Arg}[f_2/f_1] = n\pi/2$ where $n$ is an integer.

7.2 $K^0, \bar{K}^0$

Now we consider an example in which the toy model imitates the $K^0, \bar{K}^0$ system. In $K^0, \bar{K}^0$ mixing, the total decay widths of quasiparticles are well-separated, and the analysis in terms of the integration over time discussed in section 5.3 is therefore useful. The model parameters for $\tilde{\Phi}_L \rightarrow \chi_1 \xi^c$ (\tilde{\Phi}_L := \tilde{\Phi}_2) to mimic $K^0_L \rightarrow \pi^+ \ell^+ \bar{\nu}_\ell$ are given in tables 4 and 5. The mixing matrix is given by

\[
C_\Phi = \begin{pmatrix} 0.705943 & 0.708265 + i0.00214715 \\ 0.705943 & -0.708265 - i0.00214715 \end{pmatrix} = \begin{pmatrix} p & q \\ p & -q \end{pmatrix},
\]

which is a non-unitary matrix

\[
C_\Phi^\dagger C_\Phi = \begin{pmatrix} 2|p|^2 & 0 \\ 0 & 2|q|^2 \end{pmatrix} = \begin{pmatrix} 0.996712 & 0 \\ 0 & 1.003329 \end{pmatrix},
\]
Table 3. Transition rates corresponding to the input parameters given in table 1.

| Output parameter for $B^0_d - B^0_d$ | Value               |
|---------------------------------------|---------------------|
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $3.37378 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $7.61146 \cdot 10^{-13}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $1.09955 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $5.19857 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.70083 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.70083 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.10070 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.13581 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.00035 \cdot 10^{-9}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.59928 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $1.17276 \cdot 10^{-14} + 33.25827 \cdot 10^{-17}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.10070 \cdot 10^{-10} - 33.25827 \cdot 10^{-17}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.59928 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $1.05035 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $1.05035 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.10070 \cdot 10^{-10}$ MeV |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $0.519533$ |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $1.01390$ |
| $\sum_{\alpha,\beta} \Gamma_{\Phi \rightarrow \chi_1 \xi}^{\text{pcast}}$ | $2.90553$ |

Figure 11. Results obtained by varying the relative phase of $f_1$ and $f_2$. And its components satisfy $|p|^2 + |q|^2 = 1$. Defining the partial decay width of $\Phi_L$ as

$$\Gamma(\Phi_L \rightarrow Y) := \sum_{X,\tilde{\alpha}} \Gamma_{X \rightarrow \tilde{\Phi}_\tilde{\alpha}}^{\tilde{\alpha}} = \sum_{j,\tilde{\alpha}} \Gamma_{\chi_j \xi \rightarrow \tilde{\Phi}_\tilde{\alpha}}^{\tilde{\alpha}} = \sum_{j,\tilde{\alpha}} \Gamma_{\chi_j \xi \rightarrow \tilde{\Phi}_\tilde{\alpha}}^{\tilde{\alpha}} + \sum_{j,\tilde{\alpha}} \Gamma_{\chi_j \xi \rightarrow \tilde{\Phi}_\tilde{\alpha}}^{\tilde{\alpha}} + \sum_{j,\tilde{\alpha}} \Gamma_{\chi_j \xi \rightarrow \tilde{\Phi}_\tilde{\alpha}}^{\tilde{\alpha}}, \hspace{1cm} (7.6)$$
we write the asymmetry in the decay of $\Phi_L$ as

$$A_L := \frac{\Gamma(\Phi_L \to \chi_i \xi^c) - \Gamma(\Phi_L \to \chi_i \xi^c)}{\Gamma(\Phi_L \to \chi_i \xi^c) + \Gamma(\Phi_L \to \chi_i \xi^c)} \quad (7.7)$$

The transition rates corresponding to the input parameters are given in table 6. Above all, note the difference between $\sum_\alpha \Gamma^{t>0}_{X \to \Phi_\alpha \to Y}$ and $\sum_\alpha \Gamma^{t=0}_{X \to \Phi_\alpha \to Y}$. To the latter, only $\Phi_L$ contributes, since $\Phi_S$ has already decayed completely up to the working precision at $t = t_1$. The interference between $\Phi_S$ and $\Phi_L$ for $0 < t < t_1$ is, however, comparable to the contribution only of $\Phi_L$, and thus there exists a difference between $\sum_\alpha \Gamma^{t>0}_{X \to \Phi_\alpha \to Y}$ and $\sum_\alpha \Gamma^{t=0}_{X \to \Phi_\alpha \to Y}$. It is easier to understand this statement from the time integral given by equation 5.73: among $\sum_\alpha \Gamma^{t>0}_{X \to \Phi_\alpha \to Y}$ and $\sum_\alpha \Gamma^{t=0}_{X \to \Phi_\alpha \to Y}$. The quasiparticle $\Phi_\alpha$ is a degree of freedom which is dynamically generated, and it is not possible to define a decay width $\Gamma(\Phi_\alpha \to Y)$ such that it truly considers everything related to $\Phi_\alpha$, and its neglected contribution to $\sum_\alpha \Gamma^{t>0}_{X \to \Phi_\alpha \to Y}$ is comparable to $\sum_\alpha \Gamma^{t=0}_{X \to \Phi_\alpha \to Y}$. Note also that $\Gamma^{t>0}_{X \to \Phi_\alpha \to Y}$ is sometimes a complex number, as is shown in table 6, and it cannot be interpreted as a decay width.

The big difference between $\sum_\alpha \Gamma^{t>0}_{X \to \Phi_\alpha \to \psi_1 \psi_1^c}$ and $\sum_\alpha \Gamma^{t=0}_{X \to \Phi_\alpha \to \psi_1 \psi_1^c}$ is particularly noteworthy. This happens because the transition $\Phi_L \to \psi_1 \psi_1^c$, which defines the latter, gets hugely suppressed although the final states $\psi_1 \psi_1^c$ almost evenly couple to $\Phi$ and $\Phi^*$. Note that, in the real $K^0 \bar{K}^0$ system, $K^0_L \to \pi^0 \pi^0, \pi^+ \pi^-$ get suppression while $K^0_S \to \pi^0 \pi^0, \pi^+ \pi^-$ are the dominant decay channels of $K^0_S$. Such an effect is a general characteristic of particle-antiparticle or flavor mixing when the coupling is almost universal: $|f_1| \sim |f_2| > 10 |\beta_3|$ is assumed for this example. In the perspective of CP violation, this occurs because $\Phi_L$ is almost CP-odd while $\psi_1 \psi_1^c$ is CP-even, which is the conventional explanation. In the perspective of dynamics, there exists a close cancellation among the various self-energy contributions to $\Phi_L$ when the couplings are almost universal, and thus $\Phi_L \to \psi_1 \psi_1^c$ gets suppressed so that $\Phi_L$ lives longer than $\Phi_S$.

### Table 4

| Input parameter for $K^0 \bar{K}^0$ | Value          |
|-----------------------------------|----------------|
| $m_\phi$                          | 497.611 MeV [5]|
| $m_{\chi_1}$                     | 100 MeV        |
| $m_{\chi_2}$                     | 300 MeV        |
| $f_1$                             | $10^{-5.366 \pm 0.001058}$ |
| $f_2$                             | $10^{-6.366 \pm 0.001058}$ |
| $h_1$                             | $10^{-5.79}$    |
| $\mu_1$                          | $m_\phi$       |
| $\mu_2$                          | $10^{258} m_\phi$|

Table 4. Model parameters for $\Phi_L \to \chi_i \xi^c$ to mimic $K^0_L \to \pi^- \ell^+ \nu_\ell$. The masses $m_{\chi_i}$ are arbitrarily chosen.

## 8 Beyond the leading order in perturbation

In this paper, we have calculated the decay widths of mesons only up to the leading order in perturbation. In some cases, however, we must go beyond the leading order. The CP violation in the direct decay of mesons, for example, requires consideration of the loop corrections to the vertices, and the
Table 5. Output parameters corresponding to the input parameters given in table 1. The observed values of lifetime directly available in the references are $h/\Gamma_{S} = (0.8954 \pm 0.0004) \cdot 10^{-10}$ s and $h/\Gamma_{L} = (5.116 \pm 0.021) \cdot 10^{-8}$ s.

| Output parameter for $K^0\rightarrow K^0$ | Value | Observed value |
|----------------------------------------|-------|----------------|
| $p_{\Phi_S} = p_{\Phi_1}$              | $497.611 - i3.6762 \cdot 10^{-12}$ MeV | - |
| $p_{\Phi_L} = p_{\Phi_2}$              | $497.611 - 16.44910 \cdot 10^{-15}$ MeV | - |
| $\Gamma_{S} = \Gamma_{\Phi_1}$         | $7.35240 \cdot 10^{-12}$ MeV | $(7.351 \pm 0.003) \cdot 10^{-12}$ MeV [6] |
| $\Gamma_{L} = \Gamma_{\Phi_2}$         | $1.28984 \cdot 10^{-12}$ MeV | $(1.287 \pm 0.005) \cdot 10^{-14}$ MeV [6] |
| $\sum_{j,\alpha} \Gamma_{j} = 2 \cdot \Gamma_{S}$ | $7.36529 \cdot 10^{-12}$ MeV | - |
| $\Delta m_{\Phi} = m_{\Phi_1} - m_{\Phi_2}$ | $3.38370 \cdot 10^{-12}$ MeV | $(3.384 \pm 0.006) \cdot 10^{-12}$ MeV [6] |
| $|\epsilon| = 2.23699 \cdot 10^{-3}$ | | $(2.228 \pm 0.011) \cdot 10^{-3}$ [5] |
| $A_L = 3.28785 \cdot 10^{-3}$ | | $(3.32 \pm 0.06) \cdot 10^{-3}$ [5] |

Table 6. Transition rates corresponding to the input parameters given in table 4. The values of $\Gamma_{j,\Phi_\alpha}$ are complex in general, and thus each mass eigenstate cannot be separately considered. Note the difference between $\sum_{j,\alpha} \Gamma_{j} = 2 \cdot \Gamma_{S}$ and $\sum_{j,\alpha} \Gamma_{j,\Phi_\alpha} = 2 \cdot \Gamma_{S}$. In addition, $t_1 = 8.47360 \cdot 10^{12}$ MeV$^{-1} = 5.57743 \cdot 10^{-9}$ s, whose definition is given by equation 5.76.

| Output parameter | Value | Calculation |
|------------------|-------|-------------|
| $\sum_{j,\alpha} \Gamma_{j,\Phi_1}$ | $2 \cdot \Gamma_{S}$ | $3.20630 \cdot 10^{-15} + i1.13720 \cdot 10^{-17}$ MeV |
| $\sum_{j,\alpha} \Gamma_{j,\Phi_2}$ | $2 \cdot \Gamma_{S}$ | $6.77910 \cdot 10^{-17} - i1.13720 \cdot 10^{-17}$ MeV |
| $\sum_{j,\alpha} \Gamma_{j,\Phi_3}$ | $2 \cdot \Gamma_{S}$ | $4.20503 \cdot 10^{-17}$ MeV |
| $\sum_{j,\alpha} \Gamma_{j,\Phi_4}$ | $2 \cdot \Gamma_{S}$ | $4.23277 \cdot 10^{-17}$ MeV |
| $\Gamma_{\Phi,\Phi_1 \rightarrow \Phi_2 \rightarrow \Phi_1}$ | $2 \cdot \Gamma_{S}$ | $7.69482 \cdot 10^{-19}$ MeV |
| $\Gamma_{\Phi,\Phi_2 \rightarrow \Phi_1 \rightarrow \Phi_2}$ | $2 \cdot \Gamma_{S}$ | $0.869311$ |
| $\Gamma_{\Phi,\Phi_3 \rightarrow \Phi_2 \rightarrow \Phi_3}$ | $2 \cdot \Gamma_{S}$ | $0.976160$ |
| $\Gamma_{\Phi,\Phi_4 \rightarrow \Phi_2 \rightarrow \Phi_4}$ | $2 \cdot \Gamma_{S}$ | $0.998262$ |

Calculation of the decay widths should be done up to the next-leading order in perturbation. Here, we briefly discuss what kind of changes would be needed to go beyond the leading order.
Above all, the next-leading order calculation of the decay widths requires consideration of the followings:

1. Two-loop corrections in the self-energy.
2. One-loop corrections in the vertices.
3. \( R_{\Phi_a} \neq 1 \).
4. Distinction between \( \Sigma(p^2_{\Phi_1}) \) and \( \Sigma(p^2_{\Phi_2}) \).
5. Distinction between phase space factors from two different quasiparticles.

Note that we have been using \( \Sigma(p^2_{\Phi_1}) \) which is correct up the leading order. In addition, both \( p^2_{\Phi_1} = m^2_{\Phi_1} - i m_{\Phi_1} \Gamma_{\Phi_1} \) and \( p^2_{\Phi_2} = m^2_{\Phi_2} - i \Gamma_{\Phi_2}/2 \) have been used so far, but they are no longer mutually consistent beyond \( \mathcal{O}(f^2/4\pi) \).

To be explicit, let us consider \( \Gamma(\Phi \to \chi_i \xi^c) \) and \( \Gamma(\Phi^* \to \chi_i^c \xi) \). Introducing effective couplings \( \tilde{h}_i \), \( \tilde{h}_i^c \) that incorporate the vertex-loop corrections to \( h_i \), \( h_i^c \), we can write the partial decay widths up to the next-leading order as

\[
\Gamma(\Phi \to \chi_i \xi^c) := \sum_{j,\tilde{a}} \Gamma_{\chi_j \xi^c \to \tilde{a}_{\Phi_a} \to \chi_i \xi^c} = \frac{i}{2\pi^2} \sum_{j,\tilde{a},\tilde{b}} \frac{Q_{\chi_j \xi^c \to \tilde{a}_{\Phi_a} \to \chi_i \xi^c}}{p_{\Phi_a}^2 - p_{\Phi_0}^2} \left[ m_{\Phi_a}^2 \left( 1 - \frac{m_{\chi_j}^2}{m_{\Phi_a}^2} \right) \left( 1 - \frac{m_{\chi_i}^2}{m_{\Phi_a}^2} \right) + m_{\Phi_0}^2 \left( 1 - \frac{m_{\chi_j}^2}{m_{\Phi_0}^2} \right) \left( 1 - \frac{m_{\chi_i}^2}{m_{\Phi_0}^2} \right) \right],
\]

(8.1)

where

\[
Q_{\chi_j \xi^c \to \tilde{a}_{\Phi_a} \to \chi_i \xi^c} := \tilde{h}_j \tilde{h}_i^c R_{\Phi_a} (C^{-1}_{\Phi_a})_{\tilde{a}1} (C_{\Phi_a})_{\tilde{a}1},
\]

(8.2)

and

\[
\Gamma(\Phi^* \to \chi_i^c \xi) := \sum_{j,\tilde{a}} \Gamma_{\chi_j^c \xi \to \tilde{a}_{\Phi_a} \to \chi_i \xi} = \frac{i}{2\pi^2} \sum_{j,\tilde{a},\tilde{b}} \frac{Q_{\chi_j^c \xi \to \tilde{a}_{\Phi_a} \to \chi_i \xi}}{p_{\Phi_a}^2 - p_{\Phi_0}^2} \left[ m_{\Phi_a}^2 \left( 1 - \frac{m_{\chi_j}^4}{m_{\Phi_a}^4} \right) \left( 1 - \frac{m_{\chi_i}^4}{m_{\Phi_a}^4} \right) + m_{\Phi_0}^2 \left( 1 - \frac{m_{\chi_j}^4}{m_{\Phi_0}^4} \right) \left( 1 - \frac{m_{\chi_i}^4}{m_{\Phi_0}^4} \right) \right],
\]

(8.3)

where

\[
Q_{\chi_j^c \xi \to \tilde{a}_{\Phi_a} \to \chi_i \xi} := \tilde{h}_j^c \tilde{h}_i R_{\Phi_a} (C^{-1}_{\Phi_a})_{\tilde{a}2} (C_{\Phi_a})_{\tilde{a}2}.
\]

(8.4)

Note the difference of these expressions from equations 5.98 and 5.99 which were calculated up to the leading order. In the current calculation with a higher precision, the summations over \( j \) and \( \tilde{a}, \tilde{b} \) are no longer separable, and thus we cannot expect any cancellation that could simplify the expression of a ratio of decay widths, as in equation 5.108. The detailed discussion is beyond the scope of this paper.
9 Conclusion

In summary, we have seen that, in the presence of heavy particle-antiparticle mixing, the decoupled degree of freedom that propagates like a free particle until it decays should be interpreted as a quasiparticle, e.g., an emergent particle dynamically generated by interactions. Contrary to popular belief, it cannot be expressed as a single linear combination of the basis states, and thus the standard approach using the Schrödinger equation cannot properly investigate the physics of mixing. Hence, a method to describe the phenomenology of heavy particle-antiparticle mixing fully in the framework of the quantum field theory has been developed. Since the mixing effect can be correctly taken into account only when the particles and antiparticles appear as intermediate states of physical processes, their decay widths are calculated from the scattering mediated by on-shell quasiparticles. The results have been compared with those from the conventional approach based on quantum mechanics, and a discrepancy has been found which could sometimes make big differences in theoretical predictions. Hence, in spite of the great success of the quantum mechanical approach so far, the physics of the meson mixing and CP violation should be reanalyzed using the method developed in this paper, all the more because we are observing anomalies in meson physics.

Acknowledgement

This work was supported by the National Center for Theoretical Sciences, Hsinchu.

A Explicit calculations of the self-energy

For the loops of $\chi_i(k)\xi^c(p-k)$ and $\chi^\dagger_i(k)\xi(p-k)$, we can write

$$i\Sigma^{\chi\xi}(p^2) := -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (-i\hbar)^2 \tr\left[ \frac{i}{i k - m_{\chi_i}} \frac{i}{i \bar{k} - \bar{p}} \right] (-ih_i) = \frac{|h_i|^2}{2} \int \frac{d^d k}{(2\pi)^d} \tr[(\bar{k} - m_{\chi_i})(\bar{p} - \bar{k})]$$

$$= \frac{|h_i|^2}{2} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{4(k \cdot p)}{[(1 - x)(k^2 - m_{\chi_i}^2) + x(p^2 - (1 - x)m_{\chi_i}^2)]^2},$$

(A.1)

where $d = 4 - 2\epsilon$ is the spacetime dimension. Note that the factor 1/2 is simply the coefficient of the Taylor expansion of the exponential function $e^{i\int dx\overline{E}_\alpha}$ that appears in the two-point function. For the current theory of two fermions and one complex scalar field, there exists no symmetric combination of contractions which would cancel this factor. Manipulating the Feynman parameter and the momentum integration in the standard way, we rewrite

$$\int \frac{d^d k}{(2\pi)^d} \frac{4(k \cdot p)}{[(1 - x)(k^2 - m_{\chi_i}^2) + x(p^2 - (1 - x)m_{\chi_i}^2)]^2} = \int \frac{d^d \ell}{(2\pi)^d} \frac{4x p^2}{[\ell^2 + x(1 - x)p^2 - (1 - x)m_{\chi_i}^2]^2},$$

(A.2)

where $\ell = k - xp$. Hence,

$$\int \frac{d^d k}{(2\pi)^d} \tr[(\bar{k} - m_{\chi_i})(\bar{p} - \bar{k})] = \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{4x p^2}{[\ell^2 + x(1 - x)p^2 - (1 - x)m_{\chi_i}^2]^2}$$

$$= 4p^2 \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}\Gamma(2)} [-x(1 - x)p^2 + (1 - x)m_{\chi_i}^2]^{-(2-d)/2}$$

$$= i \frac{4p^2}{(4\pi)^2} \int_0^1 dx \left[ \frac{1}{\epsilon - \gamma + \log 4\pi - \log \{-x(1 - x)p^2 + (1 - x)m_{\chi_i}^2\} + \mathcal{O}(\epsilon)} \right]$$
\[= ip^2 \frac{1}{8\pi^2} \left[ \frac{1}{\epsilon} - \gamma + \log 4\pi + 2 + \frac{m_{\chi_1}^2}{p^2} - \frac{m_{\chi_2}^4}{p^4} \log m_{\chi_2}^2 - \left(1 - \frac{m_{\chi_1}^4}{p^4}\right) \log (p^2 - m_{\chi_1}^2) \right.\]
\[+ \left(1 - \frac{m_{\chi_1}^4}{p^4}\right)i\pi + O(\epsilon) \right], \quad (A.3)\]

where we have used \(\log(-1) = -i\pi\). For complex-valued \(p^2\), it is straightforward to apply analytic continuation to this expression, which we do not discuss since the effect of the imaginary part of \(p^2\) is negligible up to the working precision of the paper when \(p^2 \approx p_\Phi^2\). Defining

\[\log \Lambda^2 := \frac{1}{\epsilon} - \gamma + \log 4\pi + 2, \quad (A.4)\]

we finally obtain

\[\Sigma_{0,\chi,\xi}(p^2) := \frac{|h_i|^2}{16\pi^2} \left[ 1 - \frac{m_{\chi_1}^4}{p^2} \right] \left[ m_{\chi_1}^2 \left(1 + \frac{m_{\chi_2}^4}{p^2}\right)^{-1} \log \left(\frac{m_{\chi_2}^2}{\Lambda^2}\right) - \log \left(\frac{|p^2 - m_{\chi_1}^2|}{\Lambda^2}\right) + i\pi \right], \quad (A.5)\]

where \(i\Sigma_{0,\chi,\xi}(p^2) = ip^2\Sigma_{0,\chi,\xi}(p^2)\). In a similar way, we can also obtain the self-energy contribution from the loop of \(\psi_i\psi_i^c\):

\[\Sigma_{0,\psi,\psi^c}(p^2) := \frac{1}{16\pi^2} \left[ |f_i|^2 |f_i^c|^2 \right] \left[ \log \left(\frac{|p^2|}{\Lambda^2}\right) + i\pi \right], \quad (A.6)\]

and the self-energy matrix of \(\Phi_\alpha\) is written as

\[\left(\Sigma_{0}\right)_{\beta\alpha}(p^2) = \sum_i \left[ \left(\Sigma_{0,\psi,\psi^c}\right)_{\beta\alpha}(p^2) + \delta_{\beta\alpha} \Sigma_{0,\chi,\xi}(p^2) \right]. \quad (A.7)\]

References

[1] A. Capolupo, C.-R. Ji, Y. Mishchenko, and G. Vitiello, Phenomenology of flavor oscillations with non-perturbative effects from quantum field theory, Phys. Lett. B 594 (2004) 135, arXiv:hep-ph/0407166.

[2] M. Veltman, Unitarity and causality in a renormalizable field theory with unstable particles, Physica 29 (1963) 186.

[3] K. Anikeev et al., B Physics at the Tevatron: Run II and Beyond, arXiv:hep-ph/0201071.

[4] U. Nierste, Three lectures on meson mixing and CKM phenomenology, arXiv:hep-ph/0904.1869.

[5] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018).

[6] J. Beringer et al. (Particle Data Group), Phys. Rev. D 86, 010001 (2012).

[7] A. Lenz and U. Nierste, Theoretical update of \(B_s - \bar{B}_s\) mixing, JHEP 06 (2007) 072, arXiv:hep-ph/0612167.

[8] R. Aaij et al. (LHCb Collaboration), Measurement of the Semileptonic CP Asymmetry in \(B_0 - \bar{B}_0\) Mixing, Phys. Rev. Lett. 114, 041601 (2015). arXiv:hep-ph/1409.8586.