Infinite bound states and $1/n$ energy spectrum induced by a Coulomb-like potential of type III in a flat band system

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In this work, we investigate the bound states in a one-dimensional spin-1 flat band system with a Coulomb-like potential of type III, which has a unique non-vanishing matrix element in basis [1]. It is found that, for such a kind of potential, there exists infinite bound states. Near the threshold of continuous spectrum, the bound state energy is consistent with the ordinary hydrogen-like atom energy level formula with Rydberg correction. In addition, the flat band has significant effects on the bound states. For example, there are infinite bound states which are generated from the flat band. Furthermore, when the potential is weak, the bound state energy is proportional to the Coulomb-like potential strength. When the bound state energies are very near the flat band, they are inversely proportional to the natural number $n$ (e.g., $E_n \propto 1/n, n = 1, 2, 3, ...$). Further we find that the energy spectrum can be well described by quasi-classical approximation (WKB method). Finally, we give a critical potential strength $\alpha_c$ at which the bound state energy reaches the threshold of continuous spectrum. After crossing the threshold, the bound states in the continuum (BIC) may exist in such a flat band system.

I. INTRODUCTION

A lot of novel physical phenomena, for example, existences of localized flat band states [1,2], the ferromagnetism transition [3,4], localization [5,6], super-Klein tunneling [7,8], quantum Hall effects [9,10], Zitterbewegung [11,12], preformed pairs [13,14], strange metal [15,16], superconductivity/superfluidity [17,18], tunneling [8–11], quantum Hall effects [12], Zitterbewegung [13], preformed pairs [14], strange metal [15,16], super-Klein tunneling [7,8], quantum Hall effects [9,10], Zitterbewegung [11,12], in two-dimensional flat band systems, a strong Coulomb-like potential can result in a wave function collapse near the the origin. For one-dimensional case, an arbitrarily weak Coulomb-like potential also causes the wave function collapse [22].

For a potential of type II, which has a unique non-vanishing matrix element in basis [22–24], a short-ranged potential, e.g., square well potential, can cause an infinite number of bound states, even a hydrogen atom-like energy spectrum. In addition, such a kind of Coulomb-like potential can result in a $1/n$ energy spectrum [25].

In this work, we investigate the bound states in a one-dimensional spin-1 Dirac-type Hamiltonian with a Coulomb-like potential of type III, which have only one non-vanishing matrix element in basis [1]. It is found that, depending on the sign of potential strength over bound state energy, i.e., $\alpha/E$, there exist two different effective potentials. When $\alpha/E < 0$, the effective potential has a lowest point in coordinate space. The bound states can exist in the whole space. In addition, there are infinite bound states which are generated from a continuous energy spectrum. Near the threshold of continuous energy spectrum, the bound state energy is reduced to the ordinary hydrogen atom energy spectrum. When $\alpha/E > 0$, the effective potential has no lowest point. Similarly as that in Coulomb-like potential of type II [25], there are also infinite bound states which are generated from the flat band. Near the the flat band, the energy is inversely proportional to the natural number, i.e., $E \propto 1/n$. Differently from the ordinary one-dimensional bound state energy which is a parabolic function of potential strength, the bound state energy is linearly dependent on the potential strength as the strength goes to zero. For a given quantum number $n$, the bound state energy grows up with the increasing of potential strength $\alpha$. We give a critical potential strength $\alpha_c$ at which the bound state energy reaches the threshold of continuous spectrum. After crossing the threshold, the bound states may still exist in the continuous spectrum, which indicates that the bound states in a continuum (BIC) may exist in such the flat band system.

The work is organized as follows. In Sec.II, the three energy bands, free particle wave functions are given. Next, we solve the bound state problem for a Coulomb-like potential of type III in Sec.III. At the end, a summary is given in Sec.IV.
II. THE MODEL HAMILTONIAN WITH A FLAT BAND

In this work, we consider a spin-1 Dirac-type Hamiltonian in one dimension, i.e.,
\[ H = H_0 + V_p(x) \]
\[ H_0 = -i v_F h S_x \partial_x + m S_z, \]
where \( V_p(x) \) is potential energy, \( H_0 \) is the free-particle Hamiltonian, \( v_F > 0 \) is Fermi velocity, and \( m > 0 \) is energy gap parameter. \( S_x \) and \( S_z \) are spin operators for spin-1 particles, i.e.,
\[ S_x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \]
in usual basis |i⟩ with \( i = 1, 2, 3 \). In the whole manuscript, we use the units of \( v_F = h = 1 \). The above Hamiltonian can be realized in photonic systems [34, 35].

When \( V_p(x) = 0 \), the free particle Hamiltonian \( H_0 \) has three eigenstates and the eigenenergies, i.e.,
\[ \langle x|−, k⟩ = ψ_{−, k}(x) = \frac{1}{2\sqrt{k^2 + m^2}} \begin{pmatrix} √k^2 + m^2 − m \\ 0 \\ 0 \end{pmatrix} e^{ikx}, \]
\[ E_{−, k} = −√k^2 + m^2; \]
\[ \langle x|0, k⟩ = ψ_{0, k}(x) = \frac{1}{2\sqrt{(k^2 + m^2)}} \begin{pmatrix} −k \\ √2m \\ k \end{pmatrix} e^{ikx}, \]
\[ E_{0, k} = 0; \]
\[ \langle x|+, k⟩ = ψ_{+, k}(x) = \frac{1}{2\sqrt{k^2 + m^2}} \begin{pmatrix} √k^2 + m^2 + m \\ √2k \\ k \end{pmatrix} e^{ikx}, \]
\[ E_{+, k} = √k^2 + m^2. \]

It is found that a flat band with zero energy (\( E_{0, k} = 0 \)) appears in between upper and lower bands (see Fig.1). The possible bound states may exist in the gaps among the three bands, i.e., \( 0 < E < m \) and \( −m < E < 0 \) (the regions A and B in the Fig.1). Some localized flat band states can be obtained by superpositions of the above wave functions \( ψ_{0, k}(x) \) [32]. The localized flat band wave functions show a logarithmic singularity near their center positions.

III. BOUND STATES IN A COULOMB-LIKE POTENTIAL OF TYPE III

In the following manuscript, we assume the potential energy \( V_p \) has following form in usual basis |i⟩ = 1, 2, 3, namely,
\[ V_p(x) = V_{11}(x) \otimes |1⟩⟨1| \]
\[ = \begin{bmatrix} V_{11}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

In the whole manuscript, we would refer such a kind of potential as potential of type III. Such a spin-dependent potential is a bit similar to the magnetic impurity potential in Kondo model, which may be realized in flat band materials of solid physics. The similar bound state problems with potential of type I and II have been investigated by Zhang and Zhu [28, 32].

Now the Schrödinger equation with three component wave functions can be written as
\[ −i∂_x ψ_1(x)/√2 = [E − m − V_{11}]ψ_1(x), \]
\[ −i∂_x [ψ_1(x) + ψ_3(x)]/√2 = Eψ_2(x), \]
\[ −i∂_x ψ_3(x)/√2 = [E + m]ψ_3(x). \]

Eliminating \( ψ_2 \) and \( ψ_3 \), we get an equation for \( ψ_1 \)
\[ −∂^2_x[E − V/2]ψ_1(x)] = E(E − m − V)ψ_1(x). \]

Further we introduce a new wave function \( ψ(x) ≡ \frac{E−V/2}{E+m}ψ_1(x) \), then the equation for \( ψ(x) \) is
\[ ∂^2_x ψ(x) + \frac{E(E − m − V)(E + m)}{E − V/2} ψ(x) = 0. \]

Let’s write it into a form of an effective Schrödinger equation (a second-order differential equation), i.e.,
\[ −∂^2_x ψ(x) + ˜V ψ(x) = ˜E ψ(x), \]
where effective total energy \( ˜E \) and effective potential \( ˜V \)
are
\[ ˜E = E^2 − m^2 < 0, \text{ for ordinary bound states}, \]
\[ ˜V = \frac{V_{11}(m + E)^2}{2(E − V_{11}/2)} \]
In the determining the effective total energy, we assume that \( V_{11}(x) \to 0 \) as \( x \to \pm \infty \).

Next we assume \( V_{11} \) is a Coulomb-like potential, i.e.,

\[
V_{11}(x) = \frac{\alpha}{|x|},
\]

where \( \alpha \) describes the potential strength. The effective potential \( \hat{V} \) is

\[
\hat{V} = \frac{V_{11}(m + E)^2}{2E - V_{11}/2} = \frac{A}{|x - x_0|}.
\]

In the above equation, we introduce parameter \( A \equiv \frac{\alpha(m + E)^2}{2E} \) and \( x_0 \equiv \frac{m}{2E} \). It is shown that the effective potential \( \hat{V} \) is a shifted Coulomb-like potential with an effective potential strength \( A \) \[^{36} \], which depends on energy \( E \). The Eq.\((12)\) becomes

\[
\partial_z^2 \psi(x) + \left[ \hat{E} - \frac{A}{|x - x_0|} \right] \psi(x) = 0.
\]

In the following, we would solve the effective Schrödinger equation Eq.\((12)\) to get the bound state energies.

For \( x > 0 \), Eq.\((12)\) can be solved with some confluent hypergeometric functions. Its general solution is

\[
\psi(x) = (x - x_0)e^{-\sqrt{E}(x-x_0)} \{ c_1 \times \text{}_{1}F_{1}(a, b, 2\sqrt{-E}(x - x_0)) + c_2 \times U[a, b, 2\sqrt{-E}(x - x_0)] \}
\]

where \( \text{}_{1}F_{1}(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \) is confluent hypergeometric function, \( (a)_k = a \times (a+1) \times (a+2) \times \ldots \times (a+k-1) \), and \( c_1(c_2) \) are two arbitrary constants. \( a = 1 + \frac{4}{2\sqrt{-E}}, \ b = 2 \). When \( z \to +\infty \), \( \text{}_{1}F_{1}(a, b, z) \) has an asymptotic expansion \[^{37} \]

\[
\text{}_{1}F_{1}(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} z^{-a+b}[1 + O(1/|z|)],
\]

where \( \Gamma(a) \) is the Euler Gamma function. \( U[a, b, z] \) is a second linearly independent solution to the confluent hypergeometric equation (Tricomi function \[^{38} \]), whose asymptotic expansion is \[^{37} \]

\[
U[a, b, z] = (\frac{1}{z})^a[1 + O(1/|z|)]
\]

as \( z \to \infty \). When \( z \to 0 \), the two confluent hypergeometric functions behave as

\[
\text{}_{1}F_{1}(a, b, z) \simeq 1,
\]

\[
U[a, b, z] = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|logz|), \quad (b = 2).
\]

When \( a \) is real, \( b \) is an integer, and \( z < 0 \), the imaginary part of \( U[a, b, z] \) is proportional to \( \text{}_{1}F_{1}(a, b, z) \), i.e.,

\[
Im\{U[a, b, z]\} = -\pi i \frac{(-1)^b}{(b-1)!\Gamma(a-b+1)} \text{}_{1}F_{1}(a, b, z),
\]

where \( Im\{U\} \) is the imaginary part of \( U \). When \( a \) is real, \( b \) is an integer, and \( z > 0 \), \( Im\{U[a, b, z]\} \equiv 0 \). In addition, when the parameter \( a \to \infty \) and \( z > 0 \), the confluent hypergeometric function \( \text{}_{1}F_{1}(a, b, -z/a) \) would be transformed into Bessel functions \[^{37} \], i.e.,

\[
\lim_{a \to \infty} \text{}_{1}F_{1}(a, b, -z/a) = \frac{\Gamma(b)z^{1/2-b/2}J_{b-1}(2\sqrt{z})}{\Gamma(1+a-b)},
\]

\[
\lim_{a \to \infty} U[a, b, -z/a] = \frac{\Gamma(b)z^{1/2-b/2}H_{b-1}^{(1)}(2\sqrt{z})}{\Gamma(1+a-b)},
\]

where \( J_{b}(x) \) and \( H_{b}^{(1)}(x) \) are the \( \nu \)-th order Bessel function and Hankel function of first kind, respectively. The above properties of confluent hypergeometric functions would be very useful in the following discussions.

Depending on the sign of \( \alpha/E \), there exists two kinds of effective potentials \( \hat{V} \) (see Fig.2). For usual bound states, the energy should satisfy \( 0 \leq E < m \) or \( -m < E < 0 \). When \( \alpha/E < 0 \), the effective potential \( \hat{V} \) has a lowest point at \( x = 0 \), i.e., \( \hat{V}_0 = -(m + E)^2 \) (see Fig.2). In addition, the effective total energy \( \hat{E} \) should be larger than the lowest point of potential, i.e, \( \hat{E} - \hat{V}_0 = 2E(m + E) > 0 \), then \( E > 0 \). So the bound states energy should be larger than zero for the case of \( \alpha/E < 0 \) (see Fig.3).

When \( \alpha/E > 0 \), the effective potential \( \hat{V} \) has no lowest point (see Fig.2). The effective potential \( \hat{V} \) is negative in the interval \( (-x_0, x_0) \), and positive in intervals \( (-\infty, -x_0) \) and \( (x_0, \infty) \). There are two infinitely high potential barriers near two ends \( x = \pm x_0 \) of the interval \( (-x_0, x_0) \). It is found that the bound state energy can be larger than zero for \( \alpha > 0 \) or smaller than zero for \( \alpha < 0 \) (see Figs.4 and 5).
A. $\alpha/E < 0$

When $\alpha/E < 0$ [$x_0 = \alpha/(2E) < 0$], the bound states can exist in the whole space $(-\infty, \infty)$. At the two ends ($x = \pm \infty$), the zero boundary conditions should be satisfied, i.e.,

$$\psi(\pm \infty) = 0.$$  \hspace{1cm} (19)

Considering Eqs. (14) and (15), the boundary conditions should be discarded. Then, the wave function is

$$\psi(x) = (x - x_0)e^{-\sqrt{-E(x-x_0)}}U[a, b, 2\sqrt{-E(x-x_0)}].$$  \hspace{1cm} (20)

In addition, due to the presence of parity symmetry ($x \rightarrow -x$), the wave functions can be classified by two distinct parities, i.e., odd and even parities. For odd parity states, the wave functions at the origin $x = 0$ should vanish. While for even parity states, the derivatives of the wave functions at the origin $x = 0$ are zero. Based on these boundary conditions, we get the bound state energy equations

$$\psi(x = 0) = 0, \text{ for odd parity states,}$$

$$\psi'(x = 0) = 0, \text{ for even parity states.}$$  \hspace{1cm} (21)

To be specific, for odd parity states, the bound state energy equation is

$$U[1 + \frac{\alpha(E + m)^2}{4E\sqrt{E^2 - m^2}} - 2, -\frac{\alpha\sqrt{m^2 - E^2}}{E}] = 0.$$  \hspace{1cm} (22)

For even parity states, the bound state energy equation is

$$U[-\frac{\alpha(E + m)^2}{4E\sqrt{E^2 - m^2}} - 2, -\frac{\alpha\sqrt{m^2 - E^2}}{E}] = 0.$$  \hspace{1cm} (23)

The results are reported in Fig. (3).

The bound state energy can be also given by the quasi-classical approximation [39], i.e., Wenzel-Kramers-Brillouin (WKB) method. With the quasi-classical approximation, the energy is

$$-\frac{\alpha\sqrt{E + m}}{2E}(\sqrt{\pi} + 2)\sqrt{\frac{1}{E} + \sqrt{\frac{m - E}{E}}}$$

$$\Delta = \frac{1}{4}, \text{ for odd parity states,}$$

$$\Delta = \frac{3}{4}, \text{ for even parity states.}$$  \hspace{1cm} (25)

B. $\alpha/E > 0$

When $\alpha/E > 0$ [$x_0 = \alpha/(2E) > 0$], due to the existence of the infinitely high potential barriers near $x = \pm x_0$, there exist two different cases.

1. The bound states only exist in the interval $(-x_0, x_0)$

For such a case, outside the interval $(-x_0, x_0)$, the wave function vanishes. At the two ends of the interval, the zero boundary conditions should be satisfied, i.e.,

$$\psi(\pm x_0) = 0.$$  \hspace{1cm} (28)
Taking Eq. (16) into account, \( U[a, b, 2\sqrt{-E(x-x_0)}] \) should be discarded. So the wave function is

\[
\psi(x) = (x-x_0)e^{-\sqrt{-E(x-x_0)}}F_1[a, b, 2\sqrt{-E(x-x_0)}], \quad |x| \leq x_0, \\
\psi(x) = 0, \quad |x| > x_0.
\]

(29)

Similarly, the wave functions can be classified by parities. For odd parity states, the bound state energy equation is

\[
\int_{-\infty}^{x_0} (1 + \frac{\alpha(E+m)^2}{4E\sqrt{m^2-E^2}})^2 \frac{\alpha\sqrt{m^2-E^2}}{E} = 0.
\]

(30)

For even parity states, the bound state energy equation is

\[
[-4E(2E + \alpha\sqrt{m^2-E^2})] \\
\times F_1[1 + \frac{\alpha(E+m)^2}{4E\sqrt{m^2-E^2}}^2, \frac{\alpha\sqrt{m^2-E^2}}{E} + \alpha[4E\sqrt{m^2-E^2} + \alpha(m+E)^2] \\
\times F_1[2 + \frac{\alpha(E+m)^2}{4E\sqrt{m^2-E^2}}^3, \frac{\alpha\sqrt{m^2-E^2}}{E}] = 0.
\]

(31)

The results are reported in Fig.(4).

For \( E < 0 \), with quasi-classical approximation method, the energy is given by

\[
\frac{\alpha(m+E)^{3/2}\pi}{4E\sqrt{m-E}} = n\pi,
\]

(32)

where \( n = 1, 2, 3, \ldots \). When \( \alpha \to -\infty \), the energy \( E \to -m \). It is found that the odd and even parity states have approximately same energies for a given \( n \) (see Fig.4).

This is because when \( E < 0 \), the effective total energy \( \tilde{E} = E^2 - m^2 \) is smaller than the potential energy at the origin, i.e., \( \tilde{E} - V_0 = E^2 - m^2 + (m+E)^2 = 2E(m+E) < 0 \). The origin \( x = 0 \) belongs to the classically forbidden region, then the values of wave functions near the origin would be very small. Consequently, the two boundary conditions for odd and even parity bound states, i.e., \( \psi(0) = 0 \) and \( \psi'(0) = 0 \), are basically equivalent, and then the bound state energies are doubly degenerate approximately for a given \( n \).

When \( E > 0 \), with quasi-classical approximation method, the eigen-energy is given by

\[
\frac{\alpha(m+E)^{3/2}\pi}{2E}\sqrt{\frac{2E}{m+E}} + \frac{\sqrt{m+E}}{\sqrt{m-E}}\arcsin\left(\frac{\sqrt{m-E}}{\sqrt{m+E}}\right) = (n+\Delta)\pi,
\]

(33)

where \( n = 1, 2, 3, \ldots \), and

\[
\Delta = +\frac{1}{4}, \quad \text{for odd parity states},
\]

\[
\Delta = -\frac{1}{4}, \quad \text{for even parity states}.
\]

(34)

When \( E/m \ll 1 \), the energy can be approximated by

\[
E_B = E_n = E \approx \frac{ma}{4(n+\Delta)}.
\]

(35)

where \( n = 1, 2, 3, \ldots \), \( \Delta \) takes same values as Eq.(34).

When \( n \gg 1 \), the bound state energy is

\[
E_n \approx \frac{ma}{4n} \propto 1/n.
\]

(36)

It indicates that near the flat band, the bound state energies are proportional to potential strength \( \alpha \), and they are inversely proportional to the natural number \( n \) for large quantum number, which are similar to the case of Coulomb-like potential of type II.

For a given quantum number \( n \), the bound state energy grows up with the increasing of potential strength \( \alpha \) (see Fig.4). When \( \alpha \) reaches a critical value \( \alpha_c \), the bound state energy \( E \) would reach the threshold of upper continuous spectrum, i.e., \( E = m \). When energy approaches the threshold \( m \), the parameter \( a = 1 + \frac{E}{2\sqrt{\frac{m-E}{m}} \sqrt{m-E}} \to \infty \). Using Eq.(18), the Eq.(30) and Eq.(31) can be represented by

\[
J_1(2\alpha_c) = 0, \quad \text{for odd parity states},
\]

\[
\alpha J_2(2\alpha_c) - J_1(2\alpha_c) = 0, \quad \text{for even parity states}.
\]

(37)

It should be emphasized that Eq.(37) is exact result for \( E = m \).

Further using the asymptotic formula of Bessel functions, i.e, \( J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x - \nu\pi/2 - \pi/4) \) as \( x \to \infty \), the critical potential strength \( \alpha_c \) can be approximated by

\[
\alpha_c \approx \frac{(1/4 + n)\pi}{2}, \quad \text{for odd parity states},
\]

\[
\alpha_c \approx \frac{(-1/4 + n)\pi}{2}, \quad \text{for even parity states}.
\]

(38)
where natural number \( n = 1, 2, 3, \ldots \). After crossing these critical values, the bound states may still exist and they would form the bound states in a continuum (BIC). We would give detailed discussions on the existence of bound states in the continuum (BIC) elsewhere [42].

2. The bound states exist in the whole space \((-\infty, \infty)\)

For such a case, outside the interval \((-x_0, x_0)\), the wave function does not vanish. In addition, at the two ends of the interval \((-x_0, x_0)\), the wave function takes some finite values, i.e.,

\[
\psi(\pm x_0) \neq 0. \tag{39}
\]

Using Eqs. (14) and (17), it is found that the wave function can be represented with

\[
\psi(x) = (x - x_0)e^{-\sqrt{E(x - x_0)}}Re\{U[a, b, 2\sqrt{-E(x - x_0)}]\}, \tag{40}
\]

where \(Re\{U\}\) is the real part of \(U\).

Similarly, the wave functions can be classified by parities. For odd parity states, the bound state energy equation is

\[
Re\{U[1 + \frac{\alpha(E + m)^2}{4E\sqrt{m^2 - E^2}}, 2, \frac{-\alpha\sqrt{m^2 - E^2}}{E}]\} = 0. \tag{41}
\]

For even parity states, the bound state energy equation is

\[
Re\{U[\frac{\alpha(E + m)^2}{4E\sqrt{m^2 - E^2}}0, \frac{-\alpha\sqrt{m^2 - E^2}}{E} - 2U[\frac{\alpha(E + m)^2}{4E\sqrt{m^2 - E^2}}, 1, \frac{-\alpha\sqrt{m^2 - E^2}}{E}]\} = 0. \tag{42}
\]

The results are reported in Fig.(5).

Similarly, for \( E < 0 \), with quasi-classical approximation method, the energy is given by

\[
\frac{\alpha(m + E)^{3/2}}{4E\sqrt{m - E}} = (n - 1/2)\pi, \tag{43}
\]

where \( n = 1, 2, 3, \ldots \).

When \( E > 0 \), with quasi-classical approximation method, the energy is given by

\[
\frac{\alpha(m + E)\sqrt{2E}}{2E}\left[\frac{1}{\sqrt{m + E}} + \frac{1}{\sqrt{m - E}}\arcsin\left(\frac{\sqrt{m - E}}{\sqrt{m + E}}\right)\right] = (n + \Delta)\pi, \tag{44}
\]

where \( n = 1, 2, 3, \ldots \), and

\[
\Delta = -\frac{1}{4}, \text{ for odd parity states},
\]

\[
\Delta = -\frac{3}{4}, \text{ for even parity states}. \tag{45}
\]

Comparing Fig.5 with Fig.4, we see they have similar energy spectra. However, there is a \( \pi/2 \) phase difference in quasi-classical approximation wave functions between the two cases [see also Eqs. (32) and (43) or Eqs. (34) and (44)]. This is because the boundary conditions of the wave functions at \( x = \pm x_0 \) are different for two cases [see Eqs. (28) and (39)].

When energy approaches the threshold \( m \), the parameter \( \alpha \to \infty \). Using Eq. (15), Eq. (41) and Eq. (42) can be represented by

\[
\text{Im}\{H_1^{(1)}(2\alpha_c)\} = 0, \text{ for odd parity states},
\]

\[
\text{Im}\{H_0^{(1)}(2\alpha_c)\} = 0, \text{ for even parity states}, \tag{46}
\]

where \( \text{Im}\{H^{(1)}\} \) is the imaginary part of \( H^{(1)} \). It should be emphasized that Eq. (16) is exact result for \( E = m \).
Further using the asymptotic formula of Hankel functions, i.e., $H^{(3)}_n(x) \sim \sqrt{\frac{2}{\pi x}} \exp[i(x-\nu\pi/2-\pi/4)]$ as $x \to \infty$, the critical potential strength $\alpha_c$ can be approximated by

$$\alpha_c \simeq \frac{(-1/4+n)\pi}{2}, \text{ for odd parity states,}$$

$$\alpha_c \simeq \frac{(-3/4+n)\pi}{2}, \text{ for even parity states,} \quad (47)$$

where natural number $n = 1, 2, 3, \ldots$.

Finally, it should be remarked that the above two different choices of boundary conditions at $x = x_0$, i.e., Eqs. (28) and (39), correspond to two different self-adjoint extensions of Hamiltonian operator [43]. Further more, an arbitrarily linear combination of above two kinds of wave functions would form a new self-adjoint extensions of Hamiltonian operator. It is expected that the system would have a new energy spectrum which is an interpolation of Fig.4 and Fig.5. In this sense, the Hamiltonians with different boundary conditions would be different physical systems.

IV. SUMMARY

In conclusion, we investigate the bound states for a one-dimensional spin-1 Dirac model with a Coulomb-like potential of type III. We get the bound state energies by solving an effective Schrödinger equation. It is found that the bound state energies can be well described by the quasi-classical approximation method. Depending on the sign of potential strength over energy, i.e., $\alpha/E$, there exists two kinds of effective potentials. For the case of $\alpha/E < 0$, there exists an infinite number of bound states near the threshold of upper continuous spectrum. For large quantum number, the bound energy is consistent with the ordinary hydrogen atom bound state energy.

For $\alpha/E > 0$, similarly as that of Coulomb-like potential of type II, there also exists an infinite number of bound states which are generated from the flat band. When the bound state energies are very near the flat band, they are proportional to the Coulomb-like potential strength $\alpha$ and form a $1/n$ energy spectrum. We should emphasize that the existences of infinite bound states induced by potential are not limited to long ranged Coulomb potential. Even for short-ranged potential, e.g., square well potential, there are also infinite bound states. The infinite bound states may also exist in the Cornel-like potential and the quark-antiquark interactions which usually have infinite values at infinities as required by quark confinements.

The above results would provide some useful insights in the understanding of flat band properties in many-body physics. For example, the infinite bound states induced by weak potential imply that an arbitrarily small interaction would dominate the physics. Since the bound states can appear for both repulsive and attractive potentials, then one can expect that even a repulsive interaction may result in superfluid/superconductor pairing states in flat band [44]. It is expected that $1/n$ energy spectrum may be observed experimentally in near future [40, 41]. In addition, the existence of bound states in the continuous spectrum (BIC) needs further investigations.

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