The Triton; Low-momentum Interactions and Off-shell Effects

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Abstract

A microscopic theory of nuclei based on a 'free' scattering NN-potential is meaningful only if this potential fits on-shell scattering data. This is a necessary but not sufficient condition for the theory to be successful. It has been demonstrated repeatedly in the past that 2-body off-shell adjustments or many-body forces are necessary. It has been shown however, using Effective Field Theory (EFT) as well as formal scattering theory, that off-shell and many-body effects can not be separated. This 'equivalence theorem' allows us to concentrate on the off-shell effects. On-shell equivalent potentials can be constructed using meson-theory (Paris, Bonn etc) but in this report separable potentials are calculated by inverse scattering from NN-scattering and Deuteron data without any external parameters. Earlier calculations showed these S-state potentials in agreement with Bonn-B results in Brueckner nuclear matter calculations. They are here also used to compute the Triton binding energy and the n-D scattering length. The results are found to lie on the Phillips line defined in early calculations but like these miss the experimental point on this line and overbind the Triton. The point is reached by modifying the off-shell properties accomplished by adding a short-range repulsion without affecting fits to the experimental low-energy phase-shifts, i.e. the low energy free two-body interaction. The off-shell induced correlations result in a repulsive component in the Triton effective interactions. In nuclear matter the same effect is referred to as the dispersion correction, which is a main contributor to nuclear saturation. In finite nucleus Brueckner-Hartree-Fock calculations these same correlations give an important contribution to the selfconsistent field referred to as a rearrangement term, without which the finite nucleus would collapse. The main purpose of the present work is to illustrate that NN-correlations are as important in the Triton as they are in nuclear matter or other finite nuclei.

1 Introduction

The Triton binding energy and the related n-D scattering length has been the focus of numerous investigations dating back more than 30 years. A problem that plagued the early studies and never resolved at the time was the relation between two-body off-shell effects and many-body forces. The interest in the Triton problem and the nuclear many-body problem in general has been revitalised by the Effective Field Theory (EFT) approach to the study of nuclear interactions initiated by Weinberg and implemented by van Kolck and others. A guiding principle in these studies is that data from low energy nuclear experiments should not depend on details of the high energy components of the underlying QCD theory of the interactions, and that these can simply be included by 'contact'-interactions. A consistent derivation of nucleon interactions involves expansions ('power counting') amounting to a momentum cut-off usually denoted by $\Lambda$. One important conclusion is that contrary to earlier 'dogmas' off-shell properties of the illusory two-nucleon potential is NOT an observable entity, an observation in line with work on S-matrix theory by Haag more than 50 years ago. The message of this result is that two-nucleon off-shell properties are indistinguishable from many-body forces in a many-body system. Their relative contributions ('strengths') are indistinguishable and not subject to observation referred to as 'the equivalence theorem'. If regarded separately they are solely theoretical objects that depend on the choice of the underlying QCD Lagrangian field. Polyzou and Glöckle reached the same conclusions using formal scattering theory. One should of course also be aware of the fact that the potential itself is not an observable either.

Experimental information on the two-body N-N system consists of scattering data and the bound state, the Deuteron. The N-N scattering data analysed in terms of phase-shifts provides information on the on-shell T-matrix. The Deuteron bound state provides an additional off-shell information although this is incomplete because of the unobservable D-state probability related to the tensor interaction. As stated above other off-shell information is not observable and thus leaves undefined the off-shell part of the two-body potential if constructed solely from experimental two-body data. The various meson-theoretical NN-potentials on the market all reproduce on-shell data such as phaseshifts fairly well but differ in their off-shell predictions.

One main purpose of finding 'the' N-N potential has been for use in the many body theories of nuclear matter and finite nuclei e.g the triton. Some important results of these efforts can be summarised in the 'Coester'- and 'Phillips'- lines, that refer to binding-energies and reflect the differences in off-shell properties of the potentials

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2 relating to Brueckner theory of nuclear matter and the Triton respectively
that generate these lines. Potentials that differ in off-shell propagation differ in two-body correlations that in
turn affect the in-medium interaction. This is shown by eqs (7) - (10) for nuclear matter in Section 2.

The Triton calculations are exact, being solutions of the Faddeev equations. If this is done with some
specific on-shell NN-interaction, then the difference between the experimental and calculated Triton binding
energy (8.48 MeV) would determine the 3-body contribution (or off-shell correction), but unique only to that
specific NN-interaction. This is the message from EFT.

The Triton probes essentially the $^1S_0$ and the $^3S_1-^3D_1$ interactions in the two-body sector. Previous works
have shown that higher angular momentum states only contribute a few tenths of an $MeV$. To be able to
single out the $S$-states is very fortunate. Firstly, because the OPE-Potential is well established for these states.
Secondly, because it can be argued that these states, in particular the $^1S_0$ state, for low momenta can be well
approximated by a separable potential in the literature referred to as the Unitary Pole Approximation or simply
UPA. $^5$ This is a consequence of the large scattering length in this state with a pole of the T-matrix near zero
momentum, and the fact that the T-matrix (and therefore also the potential) is separable for momenta in the
vicinity of the pole. With the on-shell T-matrix defined by the scattering data the off-shell is then also defined.

This may seem contradictory as the OPEP is local. It was however shown by Harms (see ref. [3]) that contrary to expectation
the OPEP is well approximated by a one-term (rank-1) separable potential for momenta $k \leq 2 fm^{-1}$.

By increasing the rank of the separable potential off-shell properties of the T-matrix calculated from this
potential can be adjusted at will. This was the theme of some earlier work in which separable potentials were
used in Brueckner nuclear matter calculations$^{[11]}$ where Deuteron data from Bonn-A,B,C potentials and Arndt
phase-shifts for all channels with $J \leq 5$ were used as input. In that initial nuclear matter work, briefly reported
below in Sect. 3, only the lowest possible rank that was needed to fit the data was used. Still, it was found and
shown below that the results for the $S$-states agreed completely with the Bonn results.

In this paper we present results of Triton-calculations with separable potentials. They may serve as a guide-
line for the more serious calculations that will eventually 'solve' the nuclear many-body problem starting from
QCD and/or EFT-methods rather than from the phenomenological approach used here.

The method of inverse scattering with separable potentials is briefly reviewed in Sect. 2. Sect. 3 shows
some results of earlier Brueckner calculations for nuclear matter that are relevant for the present work. Sect. 4
presents results of Triton and $n-D$ calculations, while Sect. 5 presents a summary and a discussion of results.

2 Separable Interaction from inverse scattering

The methods used here to calculate a separable interaction by inverse scattering were used in several previous
papers relating to Brueckner and Green's function calculations of nuclear and neutron matter as well as the
Unitary problem. $^{[11] [12] [13] [14] [15] [16]}$ The input in the calculations are phase-shifts and Deuteron data. A
potential is derived for each two-body state from these data. A rank-1 separable potential is sufficient for the
on-shell fit if all phases have the same sign. If the phase-shift changes sign such as in the
$^1S_0$ state, for low momenta the $^1S_0$ potential can be adjusted at will. This was the theme of some earlier work in which separable potentials were
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some results of earlier Brueckner calculations for nuclear matter that are relevant for the present work. Sect. 4
presents results of Triton and $n-D$ calculations, while Sect. 5 presents a summary and a discussion of results.
With the rank-1 separable interaction given by eq. (1), the solution of this equation is simply

$$K_G(k, p, P, \omega) = \frac{V(k, p)}{D_G(P, \omega)}$$

(5)

where the potential is here assumed to be attractive and where

$$D_G(P, \omega) = 1 + \frac{1}{(2\pi)^3} \int_0^\Lambda v^2(k)G(k, P, \omega)k^2dk$$

(6)

and $K_G$ is separable in functions of $k, p$ and a function of $(P, \omega)$.

Three separate kernels and the associated matrices will be considered here:

1) $G = (\omega - k^2 + \imath \eta)^{-1}$

Then

$$K_G \equiv T(k, p, \omega); \text{ the scattering-matrix}$$

with

$$T(k, k, k^2) = e^{\delta(k)} \sin \delta(k)/k$$

2) $G = P(\omega - k^2)^{-1}$

where $P$ implies the principal value.

Then

$$K_G \equiv R(k, p, \omega); \text{ the reactance-matrix}$$

with

$$R(k, k, k^2) = \tan \delta(k)/k$$

In the two cases above the 'effective' interactions are independent of $P$.

3) $G = Q(k, P)(\omega - e(k, P))^{-1}$

where $Q$ is the Pauli blocking operator and the single particle energy $e(k, P)$ includes a self-consistent mean field (self-energy).

Then

$$K_G \equiv K(k, p, P, \omega); \text{ the Brueckner reaction-matrix, in the literature sometimes denoted by } G.$$  

With the potential derived from inverse scattering it follows that the diagonal on-shell elements of the reactance matrices ($R(k, k, k^2)$) are by definition independent of $\Lambda$ (for $k < \Lambda$ of course). Effective in-medium interactions, e.g. the Brueckner $K$- (or $G$-) matrices differ from these $R$ due to the Pauli-operator and the propagator self-energy in the definition of $K$, bringing it off-shell.

Off-shell effects are of particular interest in many-body problems. To illustrate the origin of this effect let us consider the Reaction matrix $K$ defined above, and estimate the effect of a change in the selfconsistent propagator i.e. the dependence on $\omega$.

One finds

$$K(k, p, P, \omega) = K(k, p, P, \omega) + \int dk'K(k, k', P, \omega) \left( \frac{Q(k', P)}{\omega - e(k', P)} - \frac{Q(k', P)}{\omega' - e(k', P)} \right) K(k', p, P, \omega')$$

(7)

Using

$$\Psi(k, p, P, \omega) = \Phi(k, p) + \int dk'\Phi(k, k') \frac{Q(k', P)}{\omega - k'^2} K(k', p, P, \omega)$$

(8)

where $\Psi$ and $\Phi$ are the in-medium two-body correlated and uncorrelated wave-functions respectively one finds

$$K(k, p, P, \omega') = K(k, p, P, \omega) + I_w(\omega - \omega')$$

(9)

where $I_w$ is a "wound"-integral defined by.

$$I_w = \int (\Psi_{k, p}(r) - \Phi_{k}(r))^2 dr$$

(10)

With $\omega - \omega' = \text{potential-energy (self-energy-insertion)}$, the last term in eq (10) is referred to as a dispersion correction.

4In Brueckner calculations the $\omega$ is a single particle energy $e(k, P)$ that includes the mean field.
Figure 1: The Arndt $^1S_0$ phaseshifts are shown by the solid line. Its continuation by a dotted line to $k = 10 \text{ fm}^{-1}$ shows phases used in some calculations as referred to in the text. Upper dotted curves show scattering length and effective range approximations. The broken curve at the bottom shows the repulsive phaseshifts defined in eq. (15) with $c = 0.1$ and $r_c = 0.8$ which were used in some of the calculations on the Triton shown below in Sect. 4.

The off-shell-effect that generates this correction is important for saturation and nuclear binding in general because the self-energy is density-dependent. It is important to note that by eq. (10) this off-shell effect is directly related to (un-observable) correlations. It is explicitly a three-body effect but not to be understood as a three-body force because it "is built up out of two-nucleon interactions". The effect that the momentum-cut-offs has on the two-body correlations was a subject discussed in ref.[13]. It was there already stressed that the relation between off-shell and correlation effects plays an important role in nuclear many-body physics and is intimately related to three-body correlations. A major purpose of this work is to show that these correlations play an important role also in the Triton-case. The word 'realistic' is used frequently to emphasize the quality of a specific potential, mostly referring to the fits to phase-shifts. In that sense our separable potentials are realistic as they fit phase-shifts exactly by construction. An added virtue is the agreement with the accepted realistic Bonn-potentials in the Nuclear Matter results shown below.

3 Nuclear Matter

Some earlier reports on nuclear matter calculations with separable potentials[11, 18] that are relevant for the present Triton calculations are summarised in this section. The scattering data that was used as input in the construction of the potentials were the Arndt phase-shifts[19] (Fig. 1 shows $^1S_0$ phases.) for all channels with $J \leq 5$. Other input were Deuteron properties as defined by either of the Bonn-A,B or C potentials. The original theme of this early work was to explore off-shell dependence of the in-medium interactions (easily achieved by increasing the rank) and its effect on nuclear properties. Considered as the first step only the lowest possible ranks needed to fit the data were used. Even so it was found that the results of our Brueckner calculations for the S-states agreed very well with those of the Bonn Brueckner results. Table I shows our contributions to the energy/particle from the $^1S_0$ and $^3S_1$ states compared with the BONN-B results[9]. The fermimomentum $k_f$ is in units of $\text{fm}^{-1}$ and the energies in $\text{MeV}$ per particle. (See ref. [11] for details.)

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5 The quote is from ref. [4]
The (almost) complete agreement between our separable and the BONN results may seem fortuitous. Although the respective potentials are (almost) phase-shift equivalent, any off-shell agreement is not guaranteed. The methods of constructing the potentials are indeed widely different and while the BONN are local the separable are non-local. The agreement for the $1S_0$ state may however also be a direct consequence of that the $1S_0$-potential *is* separable. (See also Sect. 1). In the $3S_1$ case the argument is somewhat different. Both fit the same Deuteron-data which implies a fit not only to scattering data but also to an off-shell energy, the binding energy of the Deuteron. This fit also includes the Deuteron wave functions (in the cases shown the BONN-B). At this point it should be observed that these are related to the un-observable D-state probability $P_D$. Results of Brueckner (and finite nucleus) calculations show a dependence on $P_D$. According to EFT, a correct calculation must then include counter-terms that eradicate this dependence.

The agreements in Table I are consistent with the agreements with the BONN half-shell reactance matrices for these states as shown in ref. [11] (See figs 4 and 6 in this reference.) For the $SD$ and $DD$ matrices these agreements were however not good and correspondingly there was a 0.55$MeV$ difference in the $3D_1$ contribution to the binding. The sources of these agreements and discrepancies were explained in more detail in ref.[11]. Less good agreements were also found in states for which, unlike the $S$-states their corresponding $T$-matrix does not have a pole close to zero energy. Noticeable was in particular the disagreement in the $3P_1$ state with our binding $1.48MeV$ more repulsive than the BONN. This was consistent with more repulsion in off-shell reactance matrix elements shown in Fig. 5 of ref.[11] A rank one potential was used for this state as this was sufficient for fit of the phase-shifts in this case. With an increase to rank-2 the off-shell repulsion was corrected and simultaneously the binding from this state. [18] It should also be observed that for many of the high angular momentum states with small phase-shifts the *phase-shift approximation* [20] is good, making potentials for these states less needed.

The effect of varying high-energy phase-shifts (beyond those determined experimentally ) was investigated already in ref. [11]. Although the potentials would change with such variations, on-shell properties defined by the known phase-shifts would of course remain the same. In contrast, half-shell Reactance matrices did as expected change, but less so than the potentials, especially for the low momenta relevant for nuclear structure calculations. The high-energy phases of choice were therefore the straight line extension of the Arndt phases shown in Fig. 1. Larger changes simulating increased short ranged repulsions are used below (Sect. 4.) in the Triton calculations affecting also nuclear matter results.

The effect of cutoffs $\Lambda$, with phaseshifts equal to zero for $k > \Lambda$ was investigated in some later reports. [12, 13]. Some results from these that are relevant for comparison with the Triton calculations are shown in Fig. 2 (From ref. [13] ) It shows results of Brueckner calculations of potential energy contributions in nuclear matter at saturation density. In each of the three sets of curves there is a lower and an upper curve. In the propagator $G$ that defines the Brueckner $K$-matrix the mean field (self-energy) is included in the upper curve, but not in the lower. This off-shell effect generated by the mean field is seen to be repulsive but starts to decrease when the cutoff $\Lambda$ decreases below $\Lambda \sim 3.0fm^{-1}$ and approaches zero as $\Lambda \rightarrow k_F = 1.35fm^{-1}$. For $\Lambda < k_f$ the range of momenta that contribute to the binding will be too small and hence the sharp decrease in binding. The off-shell effect is seen to be much smaller for the singlet than it is for the triplet $S$ state. Eq. (9) above shows an important relation between the two-body correlation and these off-shell effects. From this relation one would conclude that the correlations are much larger (the wound-integral much larger) in case of the triplet than in the singlet. Tensor-correlations are absent in the $1S_0$-state. In this state the correlation is mainly due to the short ranged repulsion and is smaller. Figs (5-8) in ref.[13] shows a difference between the

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**TABLE I**

| $k_f$ | BONN-B | SEPARABLE | BONN-B | SEPARABLE |
|------|--------|-----------|--------|-----------|
| 1.35 | 16.66 | 16.57     | 21.34  | 21.33     |
| 1.60 | 22.62 | 22.76     | 26.59  | 26.27     |
| 1.90 | 28.72 | 29.84     | 31.36  | 31.45     |

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6For a detailed discussion see ref. [11]
7The reason for showing Reactance rather than e.g. Brueckner Reaction matrix half-shell is that the latter as shown in Sect. 2 depend on 4 rather than 2 variables.
correlated wave-functions for the two states in accordance with the above. Further evidence is the near absence of correlations shown in ref. [13] Figs (9,10) when $\Lambda = 2 fm^{-1}$.

The effect of the self-energy insertion in the Brueckner propagators can be interpreted in two different ways. i) As a change in the interaction (correlation) between two nucleons i and j due to the presence of a third k or ii) as the reduced interaction of nucleon k with i because i is correlated with (partly excited by) nucleon j reducing occupation $n_i$. The first interpretation invokes a three-body force because it explicitly pictures the modification of the interaction between two nucleons due to the presence of a third. It is however built up out of two-nucleon interactions and is not intrinsically a three-body force, and is in nuclear many-body theory (mostly) referred to as a three-body term. (See also end of Sect. 2) It is a consequence of two-body correlations that are (nearly) independent of the medium. There is some medium-dependence here but already included by the $Q$-operator and the mean-field itself. A three-body force on the other hand would be due to a change of these correlations due to a presence of a third in the medium, e.g. by a polarisation of the mesonic fields.

Some further conclusions regarding off-shell (three-body) effects can be deduced from Fig. 2 as follows. For each value of $\Lambda$ in Fig. 2 a new potential is calculated for each state. Consider first the upper lines in each of the three sets. Although the potentials change, the figure shows that all binding energies are essentially constant for $\Lambda > \sim 3 fm^{-1}$. Consider the Brueckner $K$-matrix from which the curves are calculated to be the effective two-body interaction that includes the three-body term discussed above. Invoking EFT, discussed above, a proper renormalisation of two- and three-body forces should leave the calculated energy independent of $\Lambda$. This implies that contributions from 3-body forces are constant for $\Lambda > \sim 3 fm^{-1}$, because the two-body alone is constant there but also that the 3-body forces would have to become increasingly repulsive for $\Lambda < \sim 3 fm^{-1}$. But a change in 2-body off-shell repulsion can also have the same effect. It can be done by adding a contact force. This will be done in the 3-body calculations below.

The difference between the two curves, the lower and the upper in each of the three sets is a consequence of the relation given by eq. (2) the dispersion-correction. It contributes a repulsive medium(density)-dependent component to the effective interaction. In Brueckner calculations this is a major contribution to the saturation of nuclear matter. It likewise contributes an important component (often referred to as a “rearrangement” potential) to the selfconsistent Brueckner Hartree-Fock potential in finite nucleus calculations. Without it the finite nucleus collapses. It justifies (in part) the density dependent part of the Skyrme interactions.
The Brueckner theory does of course not give an exact solution of the many body problem while the Faddeev equation does give an exact solution of the 3-body problem for a given set of potentials. It therefore seems of interest to find the results of using our potentials in Triton and n-D calculations.

4 Three-body Calculations

The three-body problem in nuclear physics has been a subject of interest for a long time. $^3H$ and $^3He$ calculations showed a dependence on two-body off-shell properties. Three-body forces were introduced (e.g. ref.[22] to seek improved fits to experimental data.[23] The three-body system also provides a good testing ground for EFT methods.(e.g. refs[24][25][26])

The formalism necessary for calculating the Triton energy $E_T$ and the $n-D$ scattering length $^2a$ using the Faddeev equations [27] is well documented. Calculations are substantially simplified using separable rather than local potentials with numerous reported results. Separable parametrizations of the Paris potential has for example also been used in three-nucleon calculations. [28] Only the simplest version of the Faddeev equation(s) for a spin-independent rank-1 separable interaction acting only in the two-body $S$-state is shown here:

$$\chi(q) = \frac{2}{D(E_T - \frac{1}{2}q^2)} \int \frac{v(|k + \frac{1}{2}q|)v(|q + \frac{1}{2}k|)}{q^2 + q \cdot k + k^2 - E_T} \chi(q)dk$$

with (cf eq. [6])

$$D(s) = 1 + \frac{1}{(2\pi)^2} \int_0^{\Lambda} v^2(k)(s - k^2)^{-1}k^2dk$$

The notations are the standard: $E_T$ is the Triton energy, $k$ is the relative momentum of two particles and $q$ is the momentum of the third. This equation is conveniently solved for $E_T$ using the Malfliet-Tjon method [29].

The equations relevant for the present calculations with spin-dependent and higher rank potentials are, for example, found in refs [30][31]. The paper by Dabrowski et al.[32] is also exceptionally helpful with detailed presentation of the formalism in particular regarding $^2a$. Fig. 3 shows our results for $E_T$ vs $^2a$ for our separable potentials for the different values of $\Lambda$ shown in Figs 4 and 5. Early work with separable potentials were found to lie along a "Phillips" line [33]. Harms[34] fits these data by (see ref.[3]):

$$^2a = 0.75(E_T + 8.5) + 0.75fm$$

where $E_T$ is in MeV. Our results are seen to agree with these early results that are also supported by the EFT method.[24] Fig. 4 shows the Triton energy as a function of cutoff. Figs 4 and 5 show that both $E_T$ and $^2a$ are fairly constant for $\Lambda > 5fm^{-1}$ with maxima at $\sim 4fm^{-1}$ followed by a minimum at $\sim 2fm^{-1}$(not shown in Fig. 5). The decrease in energy for small $\Lambda$ is similar to that for nuclear matter as shown in Fig. 2 at approximately the same $\Lambda$ but the Triton results differ from those of nuclear matter in the sharp increase at $\Lambda \sim 4fm^{-1}$.

Nogga et al.[35] show $V_{low-k}$ results of $E_T$ as a function of $\Lambda$ for AV18 and CD Bonn potentials. Their curves also show minima although at a slightly smaller value of $\Lambda$, $1.6fm^{-1}$ compared to ours at $2fm^{-1}$. They do not show the increase at around $4fm^{-1}$. More significant is that separable potentials in general show appreciably over-binding of the Triton as opposed to local potentials that show under-binding. This has of course been known for a long time, but this difference has never been well understood. As shown above in Table I, there is almost perfect agreement for nuclear matter in the $S$-states so why is there a large difference in Triton energy where $S$ states dominate. The answer must be simply that the two cases, nuclear matter and the Triton explore different parts of the interactions. One can however conclude that the dependence on cutoffs $\Lambda$ for the Triton, reminiscent of that for nuclear matter is (in our case) understood as a reflection of the change in off-shell properties of the potentials as a function of $\Lambda$.

Fig. 5 shows the $^2a$ scattering length as a function of cutoff. It shows a $\Lambda$ dependence very much like in Fig. 4 for the energy. This is consistent with the Phillips line in Fig. 8.

Our results show an over-binding of the Triton. It is of interest to find if and what modifications of the interaction can be made to decrease the binding and if this would or would not mean a shift off the Phillips line. It seems apparent that the strength of the in-medium interaction would have to be decreased but can this be done without changing the low momentum on-shell interaction that is fixed experimentally? In this equation the form-factors $v(k)$ in the numerator are only needed for the low momenta $k$ of nucleons in the Triton. They
Figure 3: The solid line is the Phillips line defined by eq. [13]. The crosses are results with the separable potentials from inverse scattering using the Arndt phase-shifts and with the Bonn-B Deuteron parameters while the squares are with Bonn-C. The dot at the top of the Phillips line is the experimental point. See also Fig. 6.
Figure 4: Shown is the Triton energy as a function of cutoff. The crosses joined by broken lines are with Bonn-B Deuteron parameters while the squares are with Bonn-C.
Figure 5: Shown is the $n - D$ scattering length $^2a$ as a function of cutoff. The crosses joined by broken lines are with Bonn-B Deuteron parameters while the squares are with Bonn-C.
are fixed by the input data. The only possible freedom of change is in the denominator $D_C$; that depends on the medium and involves an integration over high momenta. The form-factors $v(k)$ at these high momenta relate to the un-known short ranged part of the potential that therefore leaves room for a parametrisation. One can argue that the potential defined by eq. (2) is a functional of $\delta(k)$ i.e. a function of the phases at all momenta. As a consequence it is not possible to independently vary high and low components of the form-factor $v(k)$ by a similar change in phases. Rather than the potential it is the high and low momentum components of the in-medium interaction that should be discussed but it was found in ref. [11] that high energy phases only weakly affect low-energy parts of the reactance-matrix which is a fair approximation of the effective interaction. The potential on the other hand was in fact shown to be affected for all momenta.

Our discussion above together with eq. (14) implies that a short-ranged repulsive potential increases two-body correlations and induces a repulsive term in the effective in-medium interaction. And it is easily incorporated in the potential without destroying the fit to low-energy data.

Incorporation of a short-ranged repulsive potential is done by using a rank-2 separable potential as follows (also used in earlier works, e.g.[32, 36]):

$$V(k, p) = h(k)h(p) - g(k)g(p)$$

(14)

with a repulsive and an attractive part respectively. The $h(k)$ form-factor was determined by an inverse scattering method using repulsive phase-shifts of the following form

$$\delta_s(k) = kr_c/(1 + c * k^2)$$

(15)

where $r_c$ would be related to the range of the assumed repulsion. The constant $c = 0.1$ was chosen to limit $\delta(10 f m^{-1}) < \pi/2$, $10 f m^{-1}$ being the cutoff used in these particular calculations. With $h(k)$ given, the form-factor $g(k)$ was determined by inverse scattering to reproduce the experimental phase-shifts known only for $k \leq 4.35 f m^{-1}$ but extended to $10 f m^{-1}$ by our parametrisation of the repulsive part as shown in Fig. 11. The inverse scattering method used in ref. [11] for the $^1S_0$-state was that of Bolsterli and McKenzie[37] while with the present rank-2 potential the method of Fuda was used instead. [38] In the previous work the Fuda method was only used for the $^3S_1$ potential.

Results of $E_T$ vs $^2S_1$ are shown by the crosses in Fig. 6. Like the earlier results shown in Fig. 3 they lie along a line although slightly off from the the Phillips-line defined by eq. (13). The three crosses are from bottom to top obtained with $r_c = 0.6, 0.7$ and $0.8 f m$ respectively with the latter being very close to the experimental point shown by the dot. This upward movement along the Phillips-line with increased short-ranged repulsion has been shown earlier. [32] Using the arguments above an increase in short ranged high-momentum repulsion results in an increased repulsion in the diagonal elements of the low-momentum effective interaction because of the increased 2-body correlations, that are due to a change in un-observable high-energy off-diagonal elements. This interpretation is substantiated by Fig. 7 showing the half-shell reactance matrix, $R(k, p, p^2)$, for $r_c = 0.4, 0.6$ and $0.8 f m$. The lowest, broken curve is with the ‘original’ phases shown by the dotted curve in Fig. 11. In agreement with comments above, $R$ is independent of $r_c$ for $k < \sim 3$ but shows significance dependence on $r_c$ for the larger momenta.

The result of this part of our investigation may be interpreted as showing the role of the three-body force. The two-body potentials in Fig. 7 are different at each point along the Phillips-line but are all phase-equivalent. They all fit the same on-shell data for $k \leq \Lambda$. They differ by yielding different in medium (non-observable) off-shell interactions. Alternatively if not modifying the 2-body off-shell one can instead add a three-body force to reproduce the Triton data. Of course, EFT indicates that the two statements are equivalent.

The rather drastic change in off-shell NN effective interactions has a likewise effect on Brueckner nuclear matter calculations. The contribution to binding in the $^1S_0$ and $^3S_1 - ^3D_1$ states at $k_F = 1.35 f m^{-1}$ was in the earlier calculations[11] $35.77 MeV/A$. With the repulsive core calculations this contribution to binding now varies between 36.22 for $r_c = 0$ to 31.79 for $r_c = .8 f m$. (These particular results are all obtained with $\Lambda = 10 f m^{-1}$.) So the trend is the same for the Triton and the Nuclear Matter calculations, but while the Triton binding can be reproduced it leaves Nuclear matter even more under-bound. Remember however that the Triton-calculation is exact while the Brueckner is not.

(There earlier result $35.77 MeV/A$ and the recent $36.22 MeV/A$ are both for $r_c = 0$. The (small) difference is because of the different inverse scattering methods as described above that may result in some off-shell difference. The two methods should give on-shell equivalent potentials.)

The short ranged repulsion represented by the formfactor $h(k)$ is believed to be related to the 2-body contact term in EFT.[5] In the renormalisation process, decreasing $\Lambda$ as in the calculations leading to the results shown

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8 A constant repulsion, a contact force, is expected to have the same effect, but the form chosen seemed more practical.
Figure 6: Similar to Fig 3 but with the $^1S_0$ phase-shifts modified as explained in the text. The deuteron parameters are those of Bonn-B. The calculated point close to the experimental (the dot) is obtained by a fit adjusting high energy phase-shifts to $r_c = 0.8 fm^{-1}$ while $c = 0.1$ (see text).
Figure 7: Solid lines show half-shell reactance matrix elements for three different values of short-range repulsive parameter $r_c$. Parameter ranges from 0.4 fm (bottom solid curve) to 0.8 fm (top solid curve) and momentum $p = 1 fm^{-1}$. The broken (lowest) curve is without any short-ranged repulsion i.e. with the Arndt phases extrapolated to $\delta(10 fm^{-1}) = 0.0$ as shown by the dotted curve in Fig. 1.

In Figs 3 to 5 the off-shell (three-body) contributions are also cut. Without the contact term the half-shell reactance matrix $R(k, p, p_2) = 0$ for $k > \Lambda$. So while the on-shell parts of $R$ are left essentially untouched by the renormalisation, the off-shell parts are decreased. That is the source of the triton energy $E_T$ in Fig. 4 not being independent of cut-off as required by a proper renormalisation. A proper renormalisation procedure has to include three-(many-) body forces or (according to ref.[6]) equivalently, maintain off-shell parts of the interaction.

This is illustrated phenomenologically by keeping the contact force $h(k)$ in eq. (14) intact, but cutting the attractive force $g(k)$ such that $g(k) = h(k)$ for $k > \Lambda$. Calculations were then made for a few values of $\Lambda$ and for each value of $\Lambda$ the repulsion, the parameter $r_c$ in the contact force $h^2(k)$, was adjusted to maintain the experimental value $E_T \sim -8.5 MeV$ and simultaneously also $^2a = 0.78 fm$ as shown by the point close to the experimental point in Fig. 6. The adjustments were made keeping $c$ in eq. (15) constant at $c = 0.1$ and resulted in $r_c = 0.49, 0.68$ and $0.78 fm$ for $\Lambda = 2, 3$ and $4 fm^{-1}$ respectively. Thus the smaller the cut-off the weaker also the required repulsive contact force. If instead $c$ and $r_c$ are both kept constant while cutting the attractive force, the Triton energy will increase and so will $^2a$ while the two move along the Phillips line. Fig. 8 shows the Triton energy as a function of $\Lambda$ in this case together with the three crosses when $r_c$ is adjusted to values just shown above.

Fig. 9 shows the half-shell reactance matrices for the three values of $\Lambda$. It shows results similar to those of Fig. 7. It is important to realise that had the repulsive force also been cut, one would have $R(k, p, p^2) \equiv 0$ for $k > \Lambda$.

5 Summary and Discussion

Separable NN-potentials were used in Triton calculations, with emphasis on off-shell dependence. The on-shell properties were determined from Deuteron data and scattering phase-shifts by inverse scattering. It seems reasonable to assume that it would only be necessary to consider low-energy phase-shifts as long as energies compatible with those expected in the Triton are included. This certainly imposes a necessary condition. But it is not sufficient. A theory of a many-body system requires also off-shell input. The present calculations find
Figure 8: The crosses joined by a broken line shows Triton energy as a function of \( \Lambda \) with \( c = 0.1 \) and \( r_c = 0.8 \). The three separate crosses show results after adjusting \( r_c \) as explained in the text.
The momentum-range has to be raised appreciably to adequately include the off-shell scatterings. Figs 6 to 9 illustrate this situation.

Many past results with phase-shift equivalent potentials have ascribed differences between experimental and microscopically calculated energies as being alternatively due to off-shell or three-body contributions. The 'equivalence' theorem has however changed this picture. Two-body off-shell is not observable and off-shell effects and three-body contributions cannot be uniquely separated other than in relation to a specific potential-model.

Experiments only provide on-shell data with some off-shell information from the Deuteron. Off-shell information can be obtained from theoretical input but specific only to that chosen input. With a 2-body potential so defined, the calculation of the Triton would then give information on the three-body force required together with that specific two-body force.

It was argued above that the $S$ state potentials are separable for low momenta as in UPA. This implies that the off-shell is also defined for low momenta without any additional theory other than that of the importance of a pole in the $T$-matrix.

It was illustrated above that the vanishing of the off-shell scatterings with decreasing $\Lambda$ could be corrected for by adding a 2-body short-ranged (contact) term to the attractive long-ranged part. This term has the effect of increasing correlations and will by eq. (10) add a repulsive component to the long-ranged attractive in-medium interaction, which is of importance in a low density medium.

Some 'effective' in-medium forces have a similar structure but with a density-dependent 2-body term, e.g. induced by a 3-body contact force. One example is the Skyrme-force. Another would be $V_{low-k}$ plus a three-body force. Our effective force represented by the $K-$matrix has a different in-medium (density) dependence in that both the long- and the short-ranged parts are medium-dependent as shown by eq. (5), with $V(k,p)$ given by eq. (11). Our effective force reduces to the $R-$matrix at zero density irrespective of the value of $\Lambda$. A $V_{low-k}$ would also do so but only for small enough cut-offs $\lambda$.

The model forces (functions of the cut-off) that are a result of this report are fitted to Triton binding energy and $^2\alpha$ scattering length. As mentioned above, these forces lead to under-binding of nuclear matter in Brueckner theory. This may not be significant because unlike the Triton calculation it is not exact. Of more interest would e.g. be $^4He$.

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9With the limit taken properly to $\delta$ rather than to $tan\delta$
An unanswered question is why the separable (non-local) potential that agreed with Bonn-potential results for $S$-state Brueckner calculations overbinds the Triton while, as is well-known, realistic (local) potentials in general underbind. It is however also been known for many years and shown by separate authors that separable potentials fitted to low energy phase-shifts (although not from inverse scattering) overbind. The answer may again lie in differences in off-shell properties but a closer investigation is justified. It is however also believed that a greater significance should be given to the "exact" Triton rather than to approximate nuclear matter calculations.

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