Exact solutions for $U(1)$ globally invariant membranes

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\[ \begin{align*}
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\end{align*} \]

Abstract

The exact solvability problem of the nonlinear equations describing the $U(1)$ invariant membranes is studied and the general solution for the static membrane in $D = 2N + 1$-dimensional Minkowski space-time, including M-theory case $D = 11$, is obtained. The $D=5$ time-dependent elliptic cosine solution describing a family of contracting tori is also found, together with the solution corresponding to a spinning torus characterized by the presence of the critical rotation frequency $\Omega_{\text{max}} = \frac{T^{1/3}}{\sqrt{\pi}}$ expressed via the membrane tension $T$. 

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1 Introduction

A fundamental role of p-branes in M-theory [1] attracts much interest to studying the membrane \((p = 2)\) nonlinear equations. However, not so much is known about their exact classical solutions (see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10]) and a membrane realization of the string program [11]. Recently Hoppe in [10] observed that adding the \(U(1)\) global symmetry to the problem transforms the original 3-dim membrane equations into a 2-dim string-like nonlinear problem still capturing characteristic nonlinearities of the membrane dynamics.

Here the solvability of Hoppe's non-linear equations [10] and their Hamiltonian formulation are investigated [2]. We prove the exact solvability of the non-linear problem for the \(U(1)\) static membrane in \(D = 2N + 1\)-dimensional Minkowski space, including the M-theory case \(D = 11\), and present the general solution. Then the time-dependent solutions for \(D = 5\) are analyzed and two physically interesting solutions are found. The first solution is given by the family of the Jacobi elliptic cosine functions parametrized by the discrete integer number \(n\) and describing a contracting tori. The second solution describes a spinning torus characterized by the upper boundary \(\Omega_{\text{max}} = T_1^{1/3}/\sqrt{\pi}\) for the rotation frequency \(\Omega\), where \(T\) is the membrane tension.

2 The membrane dynamics

The action for a p-brane without boundaries is given by the integral in the world-volume parameters \(\xi^\alpha (\alpha = 0, \ldots, p)\)

\[
S = \int \sqrt{|G|} d^{p+1}\xi,
\]

where \(G\) is determinant of the induced metric \(G_{\alpha\beta} := \partial_\alpha x_\mu \partial_\beta x^\mu\). After splitting of the embedding \(x^\mu = (x^0, x^i) = (t, \vec{x})\) and internal coordinates \(\xi^\alpha = (\tau, \sigma^r)\), the Euler-Lagrange equations and \(p + 1\) primary constraints generated by \(S\) take the form

\[
\mathcal{P}^\mu - \partial_\tau \sqrt{|G|} G^{\rho\alpha} \partial_\alpha x^\mu, \quad \mathcal{P}^\mu = \sqrt{|G|} G^{\tau\beta} \partial_\beta x^\mu,
\]

\(1\)

\(^1\)We thank Jens Hoppe for bringing to our attention this interesting problem.

\(^2\)Here the D-dimensional Minkowski space has the signature \(\eta_{\mu\nu} = (+, -\ldots, -)\).

\(^3\) We have chosen the coordinates \(x^\mu\) to be dimensionless (similarly \(\xi^\alpha\)) by rescaling \(x^\mu \rightarrow x^\mu/\sqrt{\alpha'}\), where \(\alpha'\) parameterizes the p-brane tension \(T \sim 1/\alpha'^{p+1}\).
\( T_r := P^\mu \partial_\mu x_\mu \approx 0, \quad \tilde{U} := \langle P^\mu P_\mu - | \det G_{rs} | \rangle \approx 0, \)  \hspace{1cm} (2)

where \( P^\mu \) is the energy-momentum density. Next we follow [7] and use the orthogonal gauge

\[
\tau = x^0, \quad G_{\tau r} = - (\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \hspace{1cm} (3)
\]

g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha \beta} = \begin{pmatrix} 1 - \dot{\vec{x}}^2 & 0 \\ 0 & -g_{rs} \end{pmatrix}
\]

to simplify the metric \( G_{\alpha \beta} \). The solution of the constraint \( \tilde{U} \) (2) takes the form

\[
P_0 = \sqrt{\vec{P}^2 + g}, \quad g = \det g_{rs} \hspace{1cm} (4)
\]

and becomes the Hamiltonian density \( \mathcal{H}_0 \) of the p-brane, because \( \dot{P}_0 = 0 \) in view of Eqn. (1). Respectively the evolution of \( \vec{P} \) is described as follows

\[
\vec{P} = \partial_r \vec{x} = \sqrt{\frac{g}{1 - \dot{\vec{x}}^2}} \dot{\vec{x}}, \quad \dot{\vec{P}} = \partial_r \left( \frac{g}{P_0} g^{rs} \partial_s \vec{x} \right) \hspace{1cm} (5)
\]

and Eqns. (5) yield the second order PDE for \( \ddot{x} \)

\[
\ddot{x} = \frac{1}{P_0} \partial_r \left( \frac{g}{P_0} g^{rs} \partial_s \vec{x} \right). \hspace{1cm} (6)
\]

The constraints \( \tilde{T}_r \) take the form \( T_r := \vec{P} \partial_r \vec{x} = 0 \) and are satisfied on the surface of the gauge conditions (3). The gauge (3) has a residual symmetry

\[
\tilde{\tau} = \tau, \quad \tilde{\sigma}^r = f^r(\sigma^s) \hspace{1cm} (7)
\]

which allows (see [12]) to simplify Eqns. (5) by choosing the additional gauge condition \( \mathcal{H}_0(= P_0) = \text{const.} = C > 0 \) resulting in

\[
\dot{\vec{x}} = \frac{1}{C \vec{P}}, \quad \dot{\vec{P}} = \frac{1}{C} \partial_r (gg^{rs} \partial_s \vec{x}).
\]

Then the equations of motion admit the canonical formulation

\[
\dot{\vec{x}} = \{ H, \vec{x} \}, \quad \dot{\vec{P}} = \{ H, \vec{P} \}, \quad \{ \mathcal{P}_i(\sigma), x_j(\tilde{\sigma}) \} = \delta_{ij} \delta^{(2)}(\sigma^r - \tilde{\sigma}^r)
\]

with the Hamiltonian given by

\[
H = \frac{1}{2C} \int d^p \sigma (\vec{P}^2 + g). \hspace{1cm} (8)
\]

Next we focus our analysis on the case of membranes \( (p = 2) \) with additional \( U(1) \) symmetry.
2.1 $U(1)$ invariant membranes

The 2-dim surface $\Sigma$ of a $U(1)$ invariant membrane has the group $U(1)$ as its isometry with the Killing vector $\frac{\partial}{\partial \sigma^2}$. Thus, the metric tensor $g_{rs}$ on $\Sigma$ is independent of $\sigma^2$, i.e. $\partial_{\sigma^2} g_{rs} = 0$. Our further analysis will be narrowed to the case of $U(1)$ invariant membranes without boundaries. The $U(1)$ membranes with boundaries are treated similarly taking into account additional boundary terms. The $U(1)$ membranes may be compact or non-compact. For the latter there are no conditions on the world-volume parameter $\sigma^1$.

The $U(1)$ membrane vector $\vec{x}$ may be chosen in the form

$$\vec{x}^T = \left( m_1 \cos \sigma^2, m_1 \sin \sigma^2, \ldots, m_N \cos \sigma^2, m_N \sin \sigma^2 \right), \quad m_a = m_a(\tau, \sigma^1). \quad (9)$$

It belongs to the $D = 2N + 1$ dimensional Minkowski space and generates the metric $g_{rs} = \partial_r \vec{x} \partial_s \vec{x}$ independent on $\sigma^2$. The geometric restrictions associated with the representation (9) are clarified from the next observation. Because any $2N$-dim vector $\vec{x}$ is fixed by $N$ pairs of its polar coordinates, the space vector $\vec{x}$ of any membrane may be presented in the form

$$\vec{x}^T(\tau, \sigma^1, \sigma^2) = \left( m_1 \cos \theta^1, m_1 \sin \theta^1, \ldots, m_N \cos \theta^N, m_N \sin \theta^N \right)$$

(where $T$ means the transposition) with $m_a = m_a(\tau, \sigma^r)$, $\theta^a = \theta^a(\tau, \sigma^r)$ parametrized by its world volume coordinates $(\tau, \sigma^1, \sigma^2)$. The ansatz (9) is obtained from the above representation by equaling all the polar angles $\theta_a$ to $\sigma^2$ and the manifestation of the radial coordinate $m_a = m_a(\tau, \sigma^1)$ independence on the parameter $\sigma^2$. It means geometrically that (9) describes a 2-dim surface swept by global rotations of a subgroup $O(2) \in SO(2N)$, parametrized by the angle $\sigma^2$, of a plane closed $\vec{m}$-curve given by its radial coordinates $m_a = m_a(\sigma^1)$ at any fixed moment $\tau$. Some of these $O(2)$ rotation subgroups will create their own $U(1)$ invariant closed 2dim surfaces. At least, it concerns of the $O(2)$ rotations with their axices lying in the plane of the $\vec{m}$-curve. It is easy to see that the ansatz (9), e.g. in $D = 2N + 1 = 5$

$$\vec{x}^T = \left( m_1 \cos \sigma^2, m_1 \sin \sigma^2, m_2 \cos \sigma^2, m_2 \sin \sigma^2 \right), \quad (10)$$

is created by the following one-parametric subgroup of $SO(4)$ rotations

$$S = \begin{pmatrix} \cos \sigma^2 & -\sin \sigma^2 & 0 & 0 \\ \sin \sigma^2 & \cos \sigma^2 & 0 & 0 \\ 0 & 0 & \cos \sigma^2 & -\sin \sigma^2 \\ 0 & 0 & \sin \sigma^2 & \cos \sigma^2 \end{pmatrix}, \quad S \in SO(4) \quad (11)$$
applied to the vector \( \vec{x}_0^T = (m_1, 0, m_2, 0) \) (lying in the \( x_1x_3 \) plane) simultaneously in the \( x_1x_2 \) and \( x_3x_4 \) planes. The similar block structure of the rotation matrix preserves for higher \( N \). Because of the arbitrariness of the \( m \)-curve other global symmetries of the membrane surface, except of the rotational \( U(1) \) symmetry, are not assumed.

So, by construction the ansatz (9) describes one of the representatives of the family of the \( U(1) \) invariant surfaces created by various rotations of a closed curve in the \( 2N \) dimensional Euclidean space. Each of the members of the family has a \( U(1) \) symmetry as its inherent global symmetry.

The membrane world-volume metric \( G_{\alpha\beta} \) corresponding to (9) is

\[
G_{\alpha\beta} = \text{diag}(1 - \dot{m}^2, -m'^2, -m^2), \quad m := (m_1, \ldots, m_N).
\]

The canonical momentum \( \pi := (\pi_1, \ldots, \pi_N) \) conjugate to \( m \) and defined as

\[
\pi_a = \mathcal{P}_0 \dot{m}_a, \quad \mathcal{P}_0 = \sqrt{\frac{m^2m'^2}{1 - \dot{m}^2}},
\]

(12)

where \( m' \) and \( \dot{m} \) are partial derivatives with respect to \( \sigma \) and \( t \), after using (5) and the relations: \( \ddot{x}^2 = m^2, \quad \dot{x}^2 = \dot{m}^2, \quad \dot{x}'^2 = m'^2, \quad g = m^2m'^2 \).

Then the Hamiltonian density (11) and the constraints becomes

\[
\mathcal{H}_0 = \mathcal{P}_0 = \sqrt{\pi^2 + m^2m'^2}, \quad \dot{\mathcal{P}}_0 = 0,
\]

\[
T := T_1 = \pi m' = 0.
\]

(13)

The corresponding Hamiltonian equations of motion are transformed in Eqns.

\[
\dot{m} = \{H_0, m\} = \frac{1}{\mathcal{P}_0} \pi, \quad \dot{\pi} = \{H_0, \pi\} = \left( \frac{m^2m'}{\mathcal{P}_0} \right)',
\]

(14)

where the canonical Poisson bracket and the Hamiltonian are defined by

\[
\{\pi_a(\sigma^1), m_b(\tilde{\sigma}^1)\} = \delta_{ab} \delta(\sigma^1 - \tilde{\sigma}^1), \quad H_0 = \int d\sigma^1 \sqrt{\pi^2 + m^2m'^2}.
\]

The Hamiltonian Eqns. (14) yield the following system of PDE’s

\[
\ddot{m} = \frac{1}{\mathcal{P}_0} \left( \frac{m^2m'}{\mathcal{P}_0} \right)' - \left( \frac{1}{\mathcal{P}_0} m' \right)^2 m.
\]

(15)
which coincides with equations (5) after the substitution of (9).

The gauge condition \( P_0 = C \) transforms the constraint \( U \) to the following

\[
C^2 \ddot{m}^2 + m^2 \dot{m}'^2 - C^2 = 0,
\]

and the membrane dynamics is now completely determined by the simplified equations and constraints

\[
\begin{align*}
C^2 \ddot{m} &= (m^2 \dot{m}')' - m' \dot{m}, & \quad (16) \\
C^2 \ddot{m}^2 + m^2 \dot{m}'^2 - C^2 &= 0, & \quad \ddot{m}m' = 0. & \quad (17)
\end{align*}
\]

Eqns. (16) and (17) form the system of dynamical equations of the \( U(1) \)-invariant membrane (9) and their study is the main goal of this paper.

To this end we note that Eqns. (16) are invariant under the scaling transformations: \( \tau \rightarrow \bar{\tau} = \frac{1}{ab} \tau, \quad \sigma \rightarrow \bar{\sigma} = \frac{1}{b} \sigma, \quad m \rightarrow \bar{m} = a m \), so that \( \bar{m}(\bar{\tau}, \bar{\sigma}) = a m(\tau, \sigma) \). After performing the rescaling and fixing \( a \) and \( b \) by the condition \( a^4 b^2 C^2 = 1 \), the constraints (17) transform into

\[
C^2 \ddot{m}^2 + m^2 \dot{m}'^2 - 1 = 0, \quad \ddot{m}m' = 0.
\]

Therefore making an additional change of the time variable into \( \tau^* = \tau/C \) we find that the Eqns. (16,17) transform into

\[
\ddot{m} = (m^2 \dot{m}')' - m' \dot{m}, & \quad (19) \\
\ddot{m}^2 + m^2 \dot{m}'^2 - 1 &= 0, & \quad \ddot{m}m' = 0. & \quad (20)
\]

The above equations are in agreement with [10], where they were derived by a change of the variable \( \sigma' \). Moreover, it was observed in [10] that for \( D = 5 \) (i.e. for \( N = 2 \)) the constraints (20) imply Eqns. (19) provided that \( \dot{m}' \) and \( \ddot{m} \) are independent [4]. In the reduced phase space with \( P_0 = C = 1 \) the Hamiltonian \( H \) and the constraints take the form

\[
H = \int d\sigma^1 \mathcal{H} = \int d\sigma^1 (\pi^2 + m^2 \dot{m}'^2), \\
U = \pi^2 + m^2 \dot{m}'^2 - 1 = 0, \quad T = \pi \dot{m}' = 0
\]

with the Hamiltonian density \( \mathcal{H}|_U = P_0 = 1 \) on the constraint surface.

\[\text{We observe that the static membrane characterized by Eqns. } \ddot{m} = 0, \text{ has to be treated on a different footing.}\]


2.2 General static solution

The static membranes play a great role in M/string theory and it is important to understand the role of the global $U(1)$ symmetry there. The obtained answer is surprising: the dynamics of static $U(1)$-invariant membrane occurs to be exactly solvable in any odd dimension $D = 2N + 1$, including $D = 11$.

To prove this we put $\dot{m} = 0$ in Eqns. (16-17) and obtain the equations

\[
\begin{align*}
(m^2 m')' - m^2 m &= 0, \\
m^2 m^2 - 1 &= 0.
\end{align*}
\]

(23) (24)

Using (24) we express $m'^2 = 1/m^2$ and transform Eqns. (23) to the form

\[
(m^2 (m^2 m')' - m = 0,
\]

(25)

where the derivative $\partial_{\sigma^1}$ appears only in the combination $m^2 \partial_{\sigma^1}$. This observation proposes a natural change of the variable $\sigma^1$ by a new one $\xi(\sigma^1)$ defined by the differential relation

\[
d\xi = \frac{1}{m^2(\sigma^1)}d\sigma^1, \quad q(\xi) = m(\sigma^1).
\]

(26)

Using the variable $\xi$ one can present Eqns. (23) and (24) in the form

\[
\partial_\xi q - q = 0, \quad (\partial_\xi q)^2 = q^2
\]

(27)

of the $N$ linear differential equations with a simple nonlinear constraint.

The $q$-system (27) is equivalent to the original $m$-system (23-24) and the solution of one gives a solution of the other. The general solution of the $q$-system may be presented in the form

\[
q = C_1 \cosh \xi + C_2 \sinh \xi, \quad C_1^2 = C_2^2 = c^2,
\]

(28)

where $C_1, C_2$ are the integration constants. It proves the claimed exact solvability of the static $U(1)$-membrane equations.

The next step is to express the general solution (28) in the $m$-representation. To this end we need to integrate Eq. (26) to obtain $\sigma^1$ as a function of $\xi$. The subsequent substitution of this solution $\xi(\sigma^1)$ to (28) will create the desired general solution in the $m$-representation. Let us realize this program.

The integration of (26) gives $\sigma^1$ as an explicit function of $\xi$

\[
\sinh 2\xi + \delta \cosh 2\xi = z(\sigma^1), \quad z := \frac{2(\sigma^1 + \sigma_0^1)}{c^2}, \quad \delta := \frac{1}{c^2}(C_1 C_2),
\]

(29)
where $\sigma^1_0$ and $\delta$ are the integration constants, and $\delta$ $(0 \leq \delta \leq 1)$ coincides with the cosine of the angle between $C_1$ and $C_2$. On the other hand taking into account the relations following from (26-28)

$$\frac{(m^1)}{4} = \mathbf{q} \partial_{\xi} \mathbf{q} = c^2 (\sinh 2\xi + \delta \cosh 2\xi)$$

(30)

and combining (30) with (29) we find $m^2$ as the explicit function of $\sigma^1$

$$m^2 = c^2 \sqrt{z^2 + 4\gamma/c^4},$$

(31)

where $\gamma$ is the integration constant. The substitution of $m^2(\sigma^1)$ (31) in Eqn. (26) and its integration in $\sigma^1$ gives the desired explicit presentation of $\xi(\sigma^1)$

$$2\xi(\sigma^1) = \arcsinh(\mu z) + \tilde{\gamma}, \quad \mu := c^2/(2\sqrt{\gamma}) = 1 - \delta^2, \quad \tilde{\gamma} = -\arctanh \delta$$

(32)

The substitution of $\xi(\sigma^1)$ (32) into the general solution (28) yields the desired general solution in the $m$-representation.

To discuss this general solution we note that because of the global invariance of Eqns. (23-24) of the $m$-system under the scaling transformations $\tilde{\sigma}^1 = b\sigma^1$, $\tilde{m} = \sqrt{b}m$ one can put $\mu = 1$ in (32) without loss of generality. This choice of $\mu$ is equivalent to $\delta = \tilde{\gamma} = (C_1 C_2) = 0$ and results in

$$2\xi(\sigma^1) = \arcsinh z, \quad m(\sigma^1)^2 = c^2 \sqrt{z^2 + 1}, \quad z := 2(\sigma^1 + \sigma^1_0)/c^2$$

(33)

producing a simplified representation of the general solution of Eqns. (23-24)

$$m(\sigma^1) = C_1 \sqrt{\cosh(2\xi) + 1 \over 2} + C_2 \sqrt{\cosh(2\xi) - 1 \over 2} =$$

$$C_1 \sqrt{\sqrt{1 + z^2 + 1 \over 2}} + C_2 \sqrt{\sqrt{1 + z^2 - 1 \over 2}}.$$

(34)

If the parameter $\sigma^1$ was bounded, e.g. $\sigma^1 \in [0, 2\pi]$ - the case discussed in more details in Section 3 - then the solution (34) would be bounded too. It corresponds to a membrane with the boundary, since the periodicity condition $m(0) = m(2\pi)$ cannot be satisfied. On the other hand, if we let $\sigma^1 \in [0, \infty)$ then the general solution describes a $U(1)$-invariant membrane without a boundary.

$^5$Since equations (16) and (17) can be derived without referring to the Moser theorem 

[12], the assumptions of the theorem (in particular the compactness of the membrane) can be omitted allowing one to consider noncompact membrane solutions.
The solution (34) shows that the angle \( \theta(\sigma^1) \) between \( \mathbf{m} \) and \( \mathbf{m}' \) at any point \( \sigma^1 \) is given by the relation
\[
\mathbf{m} \mathbf{m}' = \cos \theta = (1 + z^{-2})^{-1/2},
\]
and goes to zero when \( \sigma^1 \) increases. The asymptotic behavior of \( \mathbf{m}(\sigma^1) \) when \( \sigma^1 \to \infty \) is described by the vector function \( \mathbf{m}_\infty(\sigma^1) \) is
\[
\mathbf{m}(\sigma^1) \to \mathbf{m}_\infty = C\sqrt{2\sigma^1}, \quad C := (C_1 + C_2)/\sqrt{2}, \quad C^2 = c^2. \quad (35)
\]
The comparison of the product \( \mathbf{m}_\infty^2 \mathbf{m}'_\infty^2 = c^4 \) with the constraint (24) hints that \( \mathbf{m}_\infty \) could be a special, exact solution \( \mathbf{m}^{(pl)}(\sigma^1) \) of Eqns. (23-24) provided that \( C \) satisfies the additional constraint \( C^2 = c^2 = 1 \) that implies
\[
\mathbf{m}^{(pl)}(\sigma^1) = \pm \mathbf{n}\sqrt{2\sigma^1}, \quad (36)
\]
where \( \mathbf{n} \) is a unit constant vector with \( N \) components. The substitution of \( \mathbf{m}^{(pl)}(\sigma^1) \) into Eqns. (23) proves that \( \mathbf{m}^{(pl)}(\sigma^1) \) actually is a special solution of (23) characterized by the additional collinearity constraint
\[
\mathbf{m}^{(pl)}(\sigma^1) \mathbf{m}'^{(pl)}(\sigma^1) = 1. \quad (37)
\]
The constraint (37) means that the vectors \( \mathbf{m}^{(pl)} \) and \( \mathbf{m}'^{(pl)} \) are parallel, but their lengths \( \rho^{(pl)}(\sigma^1) = \sqrt{2\sigma^1} \) and \( \rho'^{(pl)}(\sigma^1) = \frac{1}{\sqrt{2\sigma^1}} \) are inversely proportional. So, the gradient \( \rho^{(pl)} \) is equal to the infinity at the end point \( \sigma^1 = 0 \) of the interval \([0, \infty)\), and respectively to zero at \( \sigma^1 = \infty \).

For the first nontrivial case \( N = 2 \), corresponding to \( D = 5 \), the constraints (28) for the 2-vectors \( C_1 \) and \( C_2 \) of the \( \mathbf{m} \)-plane are explicitly solved by introducing two arbitrary constants \( C \) and \( D \)
\[
C_1 = (C; D), \quad C_2 = (D, -C), \quad C, D \in \mathbb{R}.
\]
In this case the solution (34) yields a curve in the \( \mathbf{q} \)-plane parametrized by \( \xi \)
\[
q_1^2 - q_2^2 = (C^2 - D^2) + 2CD \sinh(2\xi) \quad (38)
\]
which is obtained from the rectangular hyperbola:
\[
\tilde{q}_1 = c \cosh \xi, \quad \tilde{q}_2 = c \sinh \xi \quad (39)
\]
\[
\tilde{q}_1^2 - \tilde{q}_2^2 = c^2, \quad c^2 = C^2 + D^2
\]
rotated in the $q$-plane: $q_1 = \tilde{q}_1 \cos \alpha - \tilde{q}_2 \sin \alpha$, $q_2 = \tilde{q}_1 \cos \alpha + \tilde{q}_1 \sin \alpha$ with \( \cos 2\alpha = (C^2 - D^2)c^{-2} \), \( \sin 2\alpha = -2CDc^{-2} \). Respectively, the equation of the rotated hyperbola parametrized by $\sigma^1$ has the form

$$m_1^2 - m_2^2 = C^2 - D^2 + 4CD(\sigma^1 + \sigma_0^1)/c^2. \quad (40)$$

On the other hand the asymptotic solution (36) for $D = 5$

$$m^{(pl)}(\sigma^1) = \pm n\sqrt{2}\sigma^1, \quad n = (\cos \psi_0, \sin \psi_0), \quad (41)$$

defines a membrane whose surface is a plane going through the origin of the 4-dimensional Euclidean $\vec{x}$-space (10) $\vec{N}\vec{x} = 0$, $\vec{N} = (-\sin \psi_0, -\sin \psi_0, \cos \psi_0, \cos \psi_0, \cos \psi_0)$, \( \vec{x}_{\sigma} = \pm \sqrt{2}\sigma^1(\cos \psi_0 \cos \sigma^2, \cos \psi_0 \sin \sigma^2, \sin \psi_0 \cos \sigma^2, \sin \psi_0 \sin \sigma^2) \) is the world-volume vector of the $U(1)$ invariant membrane embedded in the five dimensional Minkowski space-time. Thus, we established that equations (23), (24) are exactly solvable and their solutions (34), (36) correspond to non-compact membranes having the above described fixed shapes.

### 2.3 The D=5 static solution in polar coordinates

Here we study time dependent membrane solutions in D=5 using the polar representation of the 2-dimensional vector $m$ proposed in [10]. To this end we analyse how the polar representation works in the discussed static case.

For $D = 5$, the two-dimensional vector $m$ and the constraint (24) are

$$m = \rho(\sigma^1) \left( \cos \psi(\sigma^1), \sin \psi(\sigma^1) \right), \quad (44)$$

$$\rho^2(\rho'{}^2 + \psi'\rho^2) = 1, \quad (45)$$

hence the two components of a vector equation (23) take the form

$$f_1 \rho \cos \psi + f_2 \rho^2 \sin \psi = 0, \quad f_1 \rho \sin \psi - f_2 \rho^2 \cos \psi = 0,$$

$$f_1 = 2\rho^2 \psi' - \rho'{}^2 - \rho\rho'', \quad f_2 = 4\rho' \psi + \rho \psi''$$
which generically imply that

\[ f_1 = f_2 = 0. \]  

(46)

Because of the presence of the \( \psi' \) and \( \psi'' \) in \( f_2 \), we first consider the case when \( \psi \) is a linear function \( \psi = k\sigma^1 + \psi_0 \).

For \( k = 0 \) the solution is \( \rho = \pm \sqrt{2\sigma^1} \) and describes the non-compact membrane (11). The curve \( m_2 = F(m_1) \) in the plane \( x_1x_3 \) is a straight line and its rotation (11) yields a two parameter family of planes in \( \mathbb{R}^4 \) given by

\[ A(x_1 - x_3 \tan \psi_0) + B(x_2 - x_4 \tan \psi_0) = 0, \quad A, B \in \mathbb{R}. \]

For the case of \( k \neq 0 \) we find that the solution of (45) is

\[ \rho(\sigma^1) = \pm \sqrt{\sin(2k\sigma^1 + c)} / k, \]

where \( k \) and \( c \) are such that \( \frac{1}{k} \sin(2k\sigma^1 + c) \geq 0 \). However, it turns out that this solution does not satisfy (46) or equivalently Eqns. (16), because

\[ \ddot{m} - (m^2m')' + m^2m = 4k\rho(\sigma^1)(\sin(3k\sigma^1 + c), -\cos(3k\sigma^1 + c)) \neq 0. \]

It shows that the ansatz \( \psi = k\sigma^1 + \psi_0, k \neq 0 \), is a solution of the constraint (15) but not of the static-membrane equations.

Next, assuming that \( \rho \neq 0 \) and that \( \psi \) is not linear in \( \sigma^1 \), we find from Eqn. (16) (i.e. from \( f_2 = 0 \)) that \( \rho^2\psi' = 1 \) which substituted to (45) and (16) (i.e. to \( f_1 = 0 \)) gives

\[ \rho = \pm \sqrt{2/b} \sqrt{(a + b\sigma^1)^2 + 1}, \quad \psi = \pm \arctan(a + b\sigma^1) / 2 + \phi, \quad b > 0. \]  

(47)

Note that the parameters \( a \) and \( b \) could be argued from the scaling invariance of the equations: \( \sigma^1 \rightarrow a + b\sigma^1, \quad m \rightarrow \sqrt{b}m \).

The curve \( m_2 = F(m_1) \) in the \( x_1x_3 \) plane is now given by

\[ m_1^2 - m_2^2 = 2\cos(2\phi) / b - 2(a + b\sigma^1) \sin(2\phi) / b, \]

(48)

which is recognized as the hyperbola \( m_1^2 - m_2^2 = \frac{2}{b} \) rotated in that plane through the angle \( \phi \) (cp. (14)). Therefore, this solution is equivalent to the non-compact membrane (10) that can be seen by writing

\[ a = \frac{2\sigma_0}{C^2 + D^2}, \quad b = \frac{2}{C^2 + D^2}, \quad \sin(2\phi) = \frac{-2CD}{C^2 + D^2}, \quad \cos(2\phi) = \frac{C^2 - D^2}{C^2 + D^2} \]

and next identification of \( \phi \) with \( \alpha \). The corresponding implicit equation of the membrane surface is given by (cp. (10))

\[ (x_1^2 + x_2^2 - x_3^2 - x_4^2 - 2/b \cos(2\phi))^2 = b^2 \sin(2\phi)^2 ((x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 - 4/b^2), \]

\[ x_1x_4 = x_2x_3. \]
3 Time dependent solutions

Here we consider a class of solutions corresponding to the following ansatz

\[ m = \rho(\xi) (\cos \psi(\eta), \sin \psi(\eta)), \quad \xi = a_1 \sigma + a_2 \tau, \quad \eta = a_3 \sigma + a_4 \tau. \] (49)

In that case the constraints (17) become

\[ a_1 a_2 \rho_\xi^2 + a_3 a_4 \psi_\eta^2 = 0, \quad a_2^2 \rho_\xi^2 + a_4^2 \rho^2 \psi_\eta^2 + \rho^2 (a_1^2 \rho_\xi^2 + a_3^2 \rho^2 \psi_\eta^2) = 1 \]

and one can see that if \( a_i = 0 \) for at least one of \( i \) then the solutions are either trivial or correspond to

\[
\begin{align*}
a) \quad m &= \rho(\tau) (\cos \psi(\sigma^1), \sin \psi(\sigma^1)), \\
b) \quad m &= \rho(\sigma^1) (\cos \psi(\tau), \sin \psi(\tau)), \\
c) \quad m &= \rho(\sigma^1) (\cos \psi(\sigma^1), \sin \psi(\sigma^1)), \\
d) \quad m &= \rho(\tau) (\cos \psi(\tau), \sin \psi(\tau)),
\end{align*}
\]

from which one can exclude the previously studied static case c). The purely time dependent case d) yields a degenerate world-volume metric, as \( \det g = 0 \) everywhere, and describes the closed (in \( \sigma^2 \)) \( U(1) \) invariant null string moving with the velocity of light (\( m^2 = 1 \)).

On the other hand if \( a_i \neq 0 \) for all \( i \) then the first constraint is solved by

\[ \rho = \rho_0 e^{\frac{a_1 a_2 a_3 a_4 \psi_\xi^2}{a_1 a_2 a_3 a_4}}, \quad \psi = \psi_0 \eta + \psi_1, \quad a_1 a_2 a_3 a_4 > 0 \]

in contradiction with the second constraint hence we can narrow the discussion to a) and b) that correspond to compact membranes.

Since there are two possible topologies corresponding to a \( U(1) \) invariant, compact, orientable membrane surface \( \Sigma \) without boundaries - the sphere \( S^2 \) with the genus \( g = 0 \) and the torus \( T^2 \) with \( g = 1 \) - we shall take advantage of using the Gauss-Bonnet theorem

\[ \frac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{g} K = \chi(\Sigma) = 2 - 2g, \] (50)

where \( \chi(\Sigma) \) is the Euler characteristic and \( K \) is the Gauss curvature

\[ K = \frac{1}{2} \frac{1}{m^2 m'^2} \left( \frac{1}{\sqrt{m^2 m'^2}} m'' \right), \] (51)
given by $\vec{x}$ (10). The integration range of $\sigma^1$ is $[0, \pi]$ or $[0, 2\pi]$ for $S^2$ or $T^2$ respectively. The integration over $\sigma^2 \in [0, 2\pi]$ in (50) gives the equation
\[
\int d\sigma^1 \left( \frac{1}{\sqrt{m^2 m^2'}} \right)' = 4g - 4 \tag{52}
\]
with the vanishing integrand for the cases a) and b). It results in $g = 1$ and proves that the a) and b) cases describe the surface $\Sigma$ which is a torus.

### 3.1 Contracting tori

Let us at first consider the ansatz a). The constraints (17) are satisfied provided $\dot{\rho}^2 + \dot{\psi}'^2 \rho^4 = 1$ which implies that $\dot{\psi}'^2 = \omega^2 = \text{const.}$ and
\[
F \left( \arccos(\sqrt{\omega} \rho); 1/\sqrt{2} \right) = \pm \sqrt{2} \omega (\tau + \tau_0), \quad \tau_0 \in \mathbb{R},
\]
where $F(\phi; k)$ is the elliptic integral of the first kind. It follows that for fixed $\tau$, $m(\sigma^1, \tau)$ is bounded, so that the membrane is compact and in view of $K = 0$ corresponds to a torus. That implies that $\omega = n$ ($n \in \mathbb{Z}$) due to the periodicity condition $m(\sigma^1, \tau) = m(\sigma^1 + 2\pi, \tau)$. Thus the solution reads
\[
\rho(\tau; n) = \frac{1}{\sqrt{n}} cn \left( \sqrt{2n} (\tau + \tau_0), \frac{1}{\sqrt{2}} \right), \quad \psi(\sigma^1) = n \sigma^1 + \psi_0, \quad \tau_0, \psi_0 \in \mathbb{R}, \quad n \in \mathbb{Z},
\]
where $cn(\phi, k)$ is the Jacobi elliptic cosine function. If the initial data are such that with $\dot{\rho}(\tau_0) > 0$ then the solution describes an expanding torus which at some point reaches the maximal size $\rho_{\text{max}} = 1/\sqrt{n}$ and then shrinks to a point after a finite time $K(1/\sqrt{2})/\sqrt{2n}$ (where $K(1/\sqrt{2}) = 1.8451$ is the quarter period of the elliptic cosine function).

The dynamical equation (16) is automatically satisfied because $m'$ and $\dot{m}$ are independent in this case. The implicit presentation of time dependent surface $\Sigma(\tau)$ corresponding to the solution is given by
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{1}{n} cn \left( \sqrt{2n} (\tau + \tau_0), \frac{1}{\sqrt{2}} \right)^2, \quad x_1 x_4 = x_2 x_3.
\]

### 3.2 Spinning torus

Finally we study the torus ansatz b) characterized by $m^2 = \rho^2$ and $m'^2 = \rho'^2$ that results in $K = 0$ and the following solutions of the constraints (17)
\[
\rho(\sigma^1)^2 = - \left( \omega \sigma^1 \pm \tilde{B}/\omega \right)^2 + \omega^{-2}, \quad \psi(\tau) = \omega \tau + \psi_0, \quad \omega, \psi_0, \tilde{B} \in \mathbb{R}.
\]
Note that $\sigma^1$ is bounded since $\rho(\sigma^1)^2 > 0$ and the membrane is compact similarly to the a) case. The periodicity condition $\rho(0)^2 = \rho(2\pi)^2$ fixes the integration constant $\tilde{B}: \omega^2 \rho = \mp \tilde{B}$ representing the solution in the form

$$\rho(\sigma^1)^2 = \omega^{-2} - (\pi - \sigma^1)^2 \omega^2, \hspace{1em} \psi(\tau) = \omega \tau + \psi_0$$

(53)

describing a rotating torus with the preserved shape. Here $m'$ and $m$ stay independent hence the ansatz b) with (53) is the solution of (16). The function $\rho(\sigma^1)^2$ has a maximum at $\sigma^1 = \pi$ with $\rho(\pi) = 1/\omega$ and a minimum at $\sigma^1 = 0$ with

$$\rho(0) = \rho(2\pi) = \frac{1}{\omega} \sqrt{1 - \pi^2 \omega^4}$$

which shows that $\omega$ has the upper limit $\omega \leq \omega_{\text{max}} = 1/\sqrt{\pi}$. Restoring the physical units ([Ω] = $L^{-1}$) for $\omega$ with help of the membrane tension $T$, one can present the critical frequency in the form

$$\Omega_{\text{max}} = T^{1/3} \omega_{\text{max}} = T^{1/3} / \sqrt{\pi}$$

This time the implicit description of $\Sigma(t)$ is given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \omega^{-2} - (\pi - \arctan x_2/x_1)^2 \omega^2, \hspace{1em} x_1 x_4 = x_2 x_3.$$  

4 Summary

The $U(1)$ globally symmetric membranes realizing the idea of reduction of the 3-dim nonlinear problem to a 2-dim string-like one were studied. Such type of reductions could sharpen the role of the methods developed in the theory of 2D chiral models [13] for the membrane quantization problem. To this end an attempt to integrate the $U(1)$ membrane equations was undertaken here. We established the exact solvability of these nonlinear equations for the static membranes in the odd dimensions $D = 5, 7, 9, 11, ..., 2N + 1$ of the Minkowski space, and found their general solution. The general solution for $D = 5$ includes the infinitely extended membranes whose surface is a 2-dimensional plane. We also found the time dependent solutions for $D = 5$ that show a deep connection of the elliptic functions with the membrane dynamics (see also [14]). These functions encode peculiarities of the hidden structure of the membrane dynamics different from the string dynamics. We hope that the solutions presented here will prove useful in further study of the integrability of the membrane dynamics along the line studied in [6, 14].
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