Deciphering the long-distance penguin contribution to $\bar{B}_{d,s} \to \gamma \gamma$ decays

Qin Qin$^a$, Yue-Long Shen$^b$, Chao Wang$^c$ and Yu-Ming Wang$^d$

$^a$ School of Physics, Huazhong University of Science and Technology, Luoyu Road 1037, Wuhan Hubei 430074, P.R. China
$^b$ College of Information Science and Engineering, Ocean University of China, Songling Road 238, Qingdao, Shandong 266100, P.R. China
$^c$ Department of Mathematics and Physics, Huaxiun Institute of Technology, Meicheng East Road 1, Huaian, Jiangsu 223200, P.R. China
$^d$ School of Physics, Nankai University, Weijin Road 94, Tianjin 300071, P.R. China

(Dated: July 7, 2022)

We compute for the first time the long-distance penguin contribution to the double radiative $B$-meson decays due to the purely hadronic operators acting with the electromagnetic current in the background soft-gluon field from first field-theoretical principles by introducing a novel subleading $B$-meson distribution amplitude. The numerically dominant penguin amplitude arises from the soft-gluon radiation off the light up-quark loop rather than the counterpart charm-loop effect on account of the peculiar analytical behaviour of the short-distance hard-collinear function. Importantly, the long-distance up-quark penguin contribution brings about the substantial cancellation of the known factorizable power correction possessing the same multiplication CKM parameters, thus enabling $\bar{B}_{d,s} \to \gamma \gamma$ to become new benchmark probes of physics beyond the Standard Model.

INTRODUCTION

It is widely accepted that the exclusive radiative penguin bottom-meson decays play a central role in exploring the quark-flavour dynamics of the Standard Model (SM) and in probing the nonstandard electroweak interactions at the LHCb and Belle II experiments. In particular, the double radiative $\bar{B}_{d,s} \to \gamma \gamma$ decays with non-hadronic final states offer a remarkably clean environment to address the intricate strong interaction mechanism of the heavy-hadron system with the perturbative factorization technique, in comparison with the radiative decays $\bar{B} \to V \gamma$. Phenomenologically the direct CP asymmetries of the double radiative $B$-meson decays with the linearly polarized photon states will be also highly beneficial for determining the CKM phase angle $\gamma$ [1]. Applying the QCD factorization approach the leading-power contributions to the exclusive $\bar{B}_{d,s} \to \gamma \gamma$ decay amplitudes in the heavy quark expansion have been demonstrated to be factorized into the short-distance Wilson coefficients due to the hard and hard-collinear fluctuations as well as the leading-twist bottom-meson distribution amplitude [2]. In addition, a variety of the subleading-power corrections to the radiative penguin $\bar{B}_{d,s} \to \gamma \gamma$ decay amplitudes (including the higher-twist off light-cone correction and the non-leading Fock-state effect) were investigated at tree level in the strong coupling constant with the diagrammatic factorization approach [3].

However, the persistent problem of evaluating the long-distance penguin contribution to the double radiative bottom-meson decay amplitudes in the presence of soft gluon emission remains unresolved at present. For decades the non-local subleading power correction arising from the soft gluon radiation off the charm-loop diagrams constitutes the longstanding obstacle to improve theory computations for the angular observables of $B \to K(\ast)\ell\ell$ at large hadronic recoil [5][11] (see [12][14] for more discussions on such non-local contribution at low hadronic recoil). Achieving the robust predictions of the long-distance charm-loop effect in the rare $B \to K(\ast)\ell\ell$ decays will be evidently indispensable for disentangling the genuine New Physics (NP) effect from the SM background contribution and for advancing our understanding towards the nature of the observed flavour anomalies (see for instance [15][20]). To this end, constructing the systematic theory formalism to tackle the long-distance penguin contribution to $\bar{B}_{d,s} \to \gamma \gamma$ will further shed new light on the model-independent calculation of the analogous QCD corrections to the exclusive flavour-changing neutral current (FCNC) decays $B \to K(\ast)\ell\ell$. More generally, the newly proposed framework to cope with the non-local power correction to the double radiative $\bar{B}_{d,s}$-meson decays will be in the meanwhile of paramount importance to perform the precision calculation of the radiative and electroweak penguin decays of heavy-flavour baryons [21][30].

According to the numerical hierarchy between the bottom and charm quark masses, we will apply the favored power counting scheme $m_b \gg m_c \sim \mathcal{O}(\sqrt{\Lambda m_b}) \gg \Lambda$ as advocated in [12][31][34] in establishing the perturbative factorization formulae for the penguin contractions of the effective four-quark operators accompanied by the soft gluon emission, instead of the alternative counting scheme $m_b \sim m_c \gg \Lambda$ implemented in the exclusive two-body $B$-meson decays [35][38]. Subsequently, we will report on a novel observation on the very hadronic matrix element responsible for the soft gluon radiation off the penguin diagrams. Integrating out the very hadronic QCD fluctuations embedded in this hadronic...
quantity will give rise to the generalized three-particle $B$-meson distribution amplitudes in heavy quark effective theory (HQET) defined by the non-local matrix element $\langle 0 | \bar{q} \gamma_\mu (\tau_1 n) G_{\mu \nu} (\tau_2 n) \Gamma_i h_i (0) | B_q \rangle$ (in analogy to the subleading shape function $g_{1\gamma}(\omega, \omega_1, \mu)$ discussed in [37]) rather than the conventional light-cone distribution amplitudes as previously introduced in [38, 39]. Employing the asymptotic behaviour of the generalized $B$-meson distribution amplitudes at small quark and gluon momenta and the model-independent theory constraints on these non-perturbative functions, we will proceed to demonstrate that the soft-collinear convolution integrals entering the factorized expressions of the long-distance penguin contributions converge for both the massless-quark and massive-quark loop induced pieces, by contrast with the corresponding mechanism in the FCNC decay processes $B \to K^{(*)} \ell \ell$. Phenomenological implications of the newly computed power correction to the double radiative $\bar{B}_q \to \gamma \gamma$ decay amplitudes as previously introduced in [38, 39].

The asymptotic behaviour of the generalized $B$-meson distribution amplitudes at small quark and gluon momenta and the model-independent theory constraints on these non-perturbative functions, we will proceed to demonstrate that the soft-collinear convolution integrals entering the factorized expressions of the long-distance penguin contributions converge for both the massless-quark and massive-quark loop induced pieces, by contrast with the corresponding mechanism in the FCNC decay processes $B \to K^{(*)} \ell \ell$. Phenomenological implications of the newly computed power correction to the double radiative $\bar{B}_q \to \gamma \gamma$ decay amplitudes as previously introduced in [38, 39].

**GENERAL ANALYSIS**

The effective weak Hamiltonian of the double radiative $b \to q \gamma \gamma$ transitions has been shown to be identical to the one for $b \to q \gamma$ decays [40]

$$\mathcal{H}_{\text{eff}} = \frac{4 G_F}{\sqrt{2}} \sum_{p=u,c} V_{pb} V^*_{pq} \left[ C_1(\nu) F^{(p)}_1(\nu) + C_2(\nu) F^{(p)}_2(\nu) \right] + \text{h.c.},$$

(1)

by employing the classical equations of motion [41]. We will further adopt the effective operator basis $F^{(p)}_{i,L}$ as advocated in [42] ensuring the disappearance of Dirac traces involving an odd number of $\gamma_5$ in the subsequent effective theory computations.

Up to the lowest order in the electromagnetic interaction one can conventionally cast the exclusive radiative decay amplitude for $\bar{B}_q \to \gamma \gamma$ in the following form [3]

$$\mathcal{A}(\bar{B}_q \to \gamma \gamma) = \frac{4 G_F}{\sqrt{2}} \frac{\alpha_m}{4\pi} \epsilon^{\alpha}(p) \epsilon^{\beta}(q) \times \sum_{p=u,c} V_{pb} V^*_{pq} \sum_{i=1}^8 C_i T^{(p)}_{i,\alpha \beta},$$

(2)

where the yielding hadronic tensors $T^{(p)}_{i,\alpha \beta}$ can be decomposed into the helicity form factors

$$T^{(p)}_{i,\alpha \beta} = i m_{B_q}^3 \left[ (g^{\perp}_{\alpha \beta} - i \tilde{\xi}_{\alpha \beta}^\perp) F^{(p)}_{i,L} - (g^\perp_{\alpha \beta} + i \tilde{\xi}_{\alpha \beta}^\perp) F^{(p)}_{i,R} \right],$$

(3)

thanks to the QED Ward-Takahashi identities and the transversality of the on-shell photons. Here we have introduced the shorthand notations for brevity

$$g^{\perp}_{\alpha \beta} = g_{\alpha \beta} - \frac{n_\alpha n_\beta}{2} - \frac{\bar{n}_\alpha \bar{n}_\beta}{2}, \quad \tilde{\xi}_{\alpha \beta}^\perp = \frac{1}{2} \varepsilon_{\alpha \beta \rho \tau} n^\rho n^\tau, \quad (4)$$

by defining two light-cone vectors $n_\mu$ and $\bar{n}_\mu$ which satisfy the kinematic constraints $p_\mu = m_{B_q} n_\mu / 2$ and $q_\mu = m_{B_q} n_\mu / 2$. It is interesting to note that only the left-handed form factors $F^{(p)}_{i,L}$ will survive at leading order in the heavy quark expansion on account of the helicity conservation of the QCD interaction at high energy. Explicitly, the resulting factorization formula for $F^{(p)}_{i,L}$ at leading power can then be written as

$$\sum_{i=1}^8 C_i F^{(p)}_{i,L} = - \frac{Q_q \bar{f}_{B_q}}{m_{B_q}} \frac{m_{\bar{b}}(\nu)}{m_{B_q}} V^{(p)}_{\tau,\text{eff}}(m_b, \mu, \nu) \times K^{-1}(m_b, \mu) \int_0^\infty \frac{d\omega}{\omega} \phi_B^+(\omega, \mu) J(m_b, \omega, \mu),$$

(5)

where the desired expressions for the effective hard function $V^{(p)}_{\tau,\text{eff}}$, the perturbative matching coefficient $K$ and the hard-collinear function $J$ at the one-loop accuracy can be found in [3, 43, 44].

**QCD FACTORIZATION FOR THE LONG-DISTANCE PENGUIN CONTRIBUTION**

We are now in a position to explore factorization properties of the long-distance penguin contribution to the double radiative $\bar{B}_q \to \gamma \gamma$ decay amplitude by inspecting the partonic diagram displayed in Figure 1. Integrating out the hard-collinear quark loop one can readily derive the flavour-changing scattering amplitude of $g(\ell) + b(\nu) \to \gamma q \gamma$ governed by the effective weak Hamiltonian $\mathcal{H}_{\text{eff}}$ by discarding the subleading-power terms in $\Lambda/m_b$ and by invoking the on-shell con-
where the perturbative penguin function is given by
\[ \mathcal{M}(g + b \to q + \gamma) = i \frac{4 G_F g_{em} g_s}{\sqrt{2} 4 \pi^2} \sum_{p = u, c} V_{pb} V^*_{pq} \left\{ \right. \\
\left. \left( C_2 - \frac{C_4}{2N_c} \right) Q_p \left[ F(z_p) - 1 \right] + 6 C_6 \sum_{q'} Q_{q'} \left[ F(z_{q'}) - 1 \right] \\
+ \left[ \left( C_3 - \frac{C_4}{2N_c} \right) + 16 \left( C_5 - \frac{C_6}{2N_c} \right) \right] Q_q \left[ F(z_q) - 1 \right] \right\} \times \left[ \bar{q}(\bar{q}) \gamma_\beta P_L G_{\mu\alpha} \tilde{F}^{\mu\beta} b(v) \right] \frac{p^\alpha}{(p - \bar{t})^2}, \tag{6} \]
where the perturbative penguin function is given by
\[ F(x) = 4 x \arctan^2 \left( \frac{1}{4 x - 1} \right), \tag{7} \]
and we have further employed the conventions
\[ z_p = \frac{m_B^2 - i 0^+}{(p - \bar{t})^2}, \quad \tilde{F}^{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{8} \]

Apparently, the hard-scattering kernel of the partonic amplitude displayed in [6] depends on the unique component \( n \cdot \ell \) of the soft-gluon momentum at leading order in the heavy quark expansion. Moreover, it is straightforward to verify that the bottom-quark loop diagram with an insertion of the QCD penguin operator \( P_6 \) can only bring about the subleading-power effect, by virtue of the asymptotic behaviour of \( F(x) - 1 \sim \mathcal{O}(1/x) \) at large \( x \), as it should be. It remains necessary to remark that the long-distance penguin contraction mechanism does not give rise to the non-trivial strong phase for the radiative \( B_q \to \gamma \gamma \) amplitude in consequence of the space-like four-momentum \( (p - \bar{t}) \), by contrast with the counterpart contribution to the radiative leptonic \( B_q \to \gamma \ell \nu \) decay.

We can proceed to evaluate the five-point amplitude of \( g(\ell) + b(v) + \bar{q}(k) \to \gamma(p) + \gamma(q) \) with the diagrammatic factorization technique by integrating out the anti-hard-collinear quark propagator in Figure 1 subsequently
\[ \left\langle \gamma(p) \gamma(q) \bar{q}(\bar{q}) \gamma_\beta P_L G_{\mu\alpha} \tilde{F}^{\mu\beta} b(g(\ell)b(v)\bar{q}(k)) \right\rangle \]
\[ = \frac{i g_{em} e_q}{(q - k)^2} \epsilon^{\mu\beta\lambda\tau} p_\lambda \epsilon^*_\tau(p) \epsilon^*_\beta(q) \]
\[ \times \left[ \bar{q}(k) \gamma_\rho \bar{q}(\bar{q}) \gamma_\beta P_L G_{\mu\alpha}(\ell) b(v) \right] + \mathcal{O}(\alpha_s), \tag{9} \]
where the yielding short-distance matching coefficient depends on the component \( n \cdot k \) (rather than \( \bar{n} \cdot k \)) of the soft-quark momentum in the leading-power approximation. As a consequence, it becomes evident to introduce the subleading \( B \)-meson distribution amplitude defined by the HQET matrix element of the three-body non-local operator \( \bar{q}_6(\tau_n) G_{\mu\nu}(\tau_\bar{n}) \gamma_\lambda h_1(0) \) for the sake of describing the soft QCD dynamics encoded in the long-distance quark contribution to \( B_q \to \gamma \gamma \). Constructing the general parametrization of the emitted effective matrix element with the covariant tensor formalism [10] (in analogy to the Lorentz decomposition for the corresponding light-cone matrix element [39]) allows us to derive the soft-collinear factorization formula of the soft-gluon radiative correction to the left-handed helicity form factor
\[ \sum_{i = 1}^8 C_i F_{i,L}^{(p), \text{soft}} a_q = -\frac{Q_2 f_{B_q}}{m_{B_q}} \int_0^\infty d \omega_1 \int_0^\infty d \omega_2 \int_0^\infty \frac{d \omega_3}{\omega_3} \left\{ \left[ C_2 - \frac{C_4}{2N_c} \right] Q_p \left[ F(z_p) - 1 \right] + 6 C_6 Q_q \left[ F(z_q) - 1 \right] \right\} \]
\[ \times \Phi_G(\omega_1, \omega_2, \mu) + \mathcal{O}(\alpha_s), \tag{10} \]
where the electric-charge relation for the light-flavour quarks \( Q_u + Q_d + Q_s = 0 \) and the vanishing penguin function \( F(0) \) have been applied to simplify the obtained perturbative kernel. It is customary to define two light-cone variables \( \omega_1 = n \cdot k \) and \( \omega_2 = \bar{n} \cdot \ell \) such that the resulting hard-collinear function develops a peculiar dependence on \( \omega_2 \) via the dimensionless quantity \( z_p = -m_B^2/(m_B \omega_2) \) (apart from an overall factor 1/\( \omega_2 \)). Including the higher-order QCD corrections to the partonic diagrams in Figure 1 will generate the non-trivial hard functions (instead of “1” at tree level) from matching the effective four-quark operators \( P_i^{(p)} \) onto SCET and simultaneously result in the interesting impacts on the (anti)-hard-collinear matching coefficients appeared in [10]. However, the fundamental property that the determined short-distance matching functions merely depend on the two dimensional variables \( \omega_1 \) and \( \omega_2 \) remains valid beyond the leading-order accuracy, thus justifying the appearance of the soft matrix element \( \langle 0 \vert \bar{q}_6(\tau_n) G_{\mu\nu}(\tau_\bar{n}) \gamma_\lambda h_1(0) \vert B_q \rangle \) (see [17] for further discussions in a different context). Schematically, the factorized expression for the long-distance penguin correction to \( B_q \to \gamma \gamma \) can be cast in the form \( \mathcal{H}_i \mathcal{J} \star \mathcal{J} \star \Phi_G \), which resembles the very pattern for the \( Q_i'' \to Q_\tau \) contribution to \( B \to X_s \gamma \) [37]. We also mention in passing that the newly computed long-distance soft gluon radiative correction appears to preserve the large-recoil symmetry of the two transversality amplitudes and does not affect the right-handed form factors \( F_{i,R}^{(p)} \) in agreement with the earlier observation on the “resolved photon” contribution to \( B \to \gamma \ell \nu \) at twist-two [48] (see also [39, 50] for an estimate of the soft contribution with the dispersion approach).

Additionally, the novel subleading distribution amplitude (perhaps more appropriately called soft function) of the \( B \)-meson \( \Phi_G \) in (10) is defined in terms of the effective matrix element of the three-body non-local operator with quark and gluon fields localized on two distinct light-cone directions.
where the two soft Wilson lines $S_n$ and $\bar{S}_n$ essential to maintain gauge invariance are given by

$$S_n(x) = P \exp \left[ i g_s \int_{-\infty}^{0} dt \ n \cdot A_s(x + t n) \right],$$

$$S_n(x) = P \exp \left[ i g_s \int_{-\infty}^{0} dt \ n \cdot A_s(x + t n) \right].$$

It is apparent that the non-local HQET matrix element on the left-hand side of (11) can be described by the familiar three-particle light-cone distribution amplitude when taking the limit $\tau_1(2) \to 0$. We are therefore led to the following three important and model-independent normalization conditions at tree level

$$\int_{0}^{\infty} d\omega_1 \Phi_G(\omega_1, \omega_2, \mu) = \int_{0}^{\infty} d\omega_1 \Phi_4(\omega_1, \omega_2, \mu),$$

$$\int_{0}^{\infty} d\omega_2 \Phi_G(\omega_1, \omega_2, \mu) = \int_{0}^{\infty} d\omega_2 \Phi_5(\omega_1, \omega_2, \mu),$$

$$\int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \Phi_G(\omega_1, \omega_2, \mu) = \frac{\lambda_E^2 + \lambda_H^2}{3},$$

where the explicit definitions of the twist-four and twist-five light-cone distribution amplitudes $\Phi_4$ and $\Phi_5$ can be found in [39] and the hadronic quantities $\lambda_E^2$ and $\lambda_H^2$ can be defined by the effective matrix elements of the local chromoelectric and chromomagnetic operators [51]. Furthermore, the asymptotic behaviour of $\Phi_G$ at small quark and gluon momenta can be predicted with the dispersion technique as widely adopted in the explorations of the two-particle and three-particle $B$-meson light-cone distribution amplitudes [51]. Starting with the HQET correlation function

$$\Pi_G = i \int \! d^4 x \exp \left( -i \omega \cdot x \right) \langle 0| \left\{ \left[ (\bar{q}_s S_n)(\tau_1 n) \right] \right. \right.$$

$$\left. \left( S_n^\dagger S_n^\dagger S_n^\dagger S_n^\dagger S_n(\tau_2 \bar{n}) \right) \left. \bar{n}^\nu \gamma_\mu^\nu \gamma_5 (S_n^\dagger h_v) \right\} |0\rangle, \quad (11)$$

be of interest to investigate the non-trivial impact of the renormalization-group evolution on the generalized distribution amplitude $\Phi_G$ in the future.

### NUMERICAL IMPLICATIONS

We now turn to address the phenomenological implications of the soft-gluon radiative correction to the penguin contractions of the effective four-quark operators on the double radiative $B\to \gamma\gamma$ decay amplitudes. To achieve this goal, we first need to construct the acceptable non-perturbative model for the subleading distribution amplitude $\Phi_G$ fulfilling the third relation in (13) as well as the obtained asymptotic behaviour

$$\Phi_G(\omega_1, \omega_2, \mu_0) = \frac{\lambda_E^2 + \lambda_H^2}{6} \omega_1 \omega_2 \exp \left( -\frac{\omega_1 + \omega_2}{\omega_0} \right) \frac{\Gamma(\beta + 2)}{\Gamma(\alpha + 2)} U \left( \beta - \alpha, 4 - \alpha, \frac{\omega_1 + \omega_2}{\omega_0} \right),$$

at the reference scale $\mu_0 = 1.0 \text{ GeV}$, motivated from the suggested three-parameter ansatz for the twist-two distribution amplitude $\phi_B^n(\omega, \mu_0)$ [50]. The remaining shape parameters $\omega_0$, $\alpha$ and $\beta$ can be further determined by enforcing the first and second normalization relations in (13) and employing the concrete model of the twist-four and twist-five light-cone distribution amplitudes $\Phi_{4,5}(\omega_1, \omega_2, \mu_0)$ as implemented in [34]. The conventional HQET distribution amplitudes on the light-cone appearing in the established factorization formulae of the helicity form factors $F^{(\mu)}_{i,L(R)}$ [34] will also be in demand in the subsequent numerical investigations, and we will apply the same phenomenological model as presented in this reference, with the exceptions of updated intervals for the inverse moments $\lambda_{B_d} = (275 \pm 75) \text{ MeV}$ and $\lambda_{B_s} = (325 \pm 75) \text{ MeV}$ [57] (see [55] for a recent determination of $\lambda_{B_s}/\lambda_{B_d}$ with the method of QCD sum rules). The allowed intervals of additional theory input parameters entering our numerical studies are identical to the ones collected in [3].

We present the theory predictions for the long-distance penguin contribution to the double radiative $B\to \gamma\gamma$ decay form factors including the obtained uncertainties from varying the factorization scale $\mu$ in Figure 2 where the numerical predictions of the previously computed factorizable power corrections at tree level [3] are also presented for the illustration purpose. Interestingly, the newly computed soft gluon radiation from the up-quark penguin contraction appears to generate the substantial cancellation of the combined factorizable power

$$(0|(\bar{q}_s S_n)(\tau_1 n) (S_n^\dagger S_n^\dagger S_n^\dagger S_n^\dagger S_n(\tau_2 \bar{n}) \bar{n}^\nu \gamma_\mu^\nu \gamma_5 (S_n^\dagger h_v))(0)|\bar{B}_v) = 2 f_B(\mu) m_B \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \exp \left[ -i(\omega_1 \tau_1 + \omega_2 \tau_2) \right] \Phi_G(\omega_1, \omega_2, \mu),$$

where the two soft Wilson lines $S_n$ and $\bar{S}_n$ essential to maintain gauge invariance are given by

$$(0|(\bar{q}_s S_n)(\tau_1 n) (S_n^\dagger S_n^\dagger S_n^\dagger S_n^\dagger S_n(\tau_2 \bar{n}) \bar{n}^\nu \gamma_\mu^\nu \gamma_5 (S_n^\dagger h_v))(0)|\bar{B}_v) = 2 f_B(\mu) m_B \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \exp \left[ -i(\omega_1 \tau_1 + \omega_2 \tau_2) \right] \Phi_G(\omega_1, \omega_2, \mu),$$

where $\omega$ enables us to extract the desired asymptotic behaviour $\Phi_G(\omega_1, \omega_2, \mu) \sim \omega_1 \omega_2^2$ at $\omega_1, \omega_2 \to 0$ immediately, which further indicates that the convolution integrals in the factorized expression (10) converge. We restrict ourselves to the leading-order accuracy in this letter, however, it will

$$(0|(\bar{q}_s S_n)(\tau_1 n) (S_n^\dagger S_n^\dagger S_n^\dagger S_n^\dagger S_n(\tau_2 \bar{n}) \bar{n}^\nu \gamma_\mu^\nu \gamma_5 (S_n^\dagger h_v))(0)|\bar{B}_v) = 2 f_B(\mu) m_B \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \exp \left[ -i(\omega_1 \tau_1 + \omega_2 \tau_2) \right] \Phi_G(\omega_1, \omega_2, \mu),$$

where $\omega$ enables us to extract the desired asymptotic behaviour $\Phi_G(\omega_1, \omega_2, \mu) \sim \omega_1 \omega_2^2$ at $\omega_1, \omega_2 \to 0$ immediately, which further indicates that the convolution integrals in the factorized expression (10) converge. We restrict ourselves to the leading-order accuracy in this letter, however, it will

$$(0|(\bar{q}_s S_n)(\tau_1 n) (S_n^\dagger S_n^\dagger S_n^\dagger S_n^\dagger S_n(\tau_2 \bar{n}) \bar{n}^\nu \gamma_\mu^\nu \gamma_5 (S_n^\dagger h_v))(0)|\bar{B}_v) = 2 f_B(\mu) m_B \int_{0}^{\infty} d\omega_1 \int_{0}^{\infty} d\omega_2 \exp \left[ -i(\omega_1 \tau_1 + \omega_2 \tau_2) \right] \Phi_G(\omega_1, \omega_2, \mu),$$

where $\omega$ enables us to extract the desired asymptotic behaviour $\Phi_G(\omega_1, \omega_2, \mu) \sim \omega_1 \omega_2^2$ at $\omega_1, \omega_2 \to 0$ immediately, which further indicates that the convolution integrals in the factorized expression (10) converge. We restrict ourselves to the leading-order accuracy in this letter, however, it will
FIG. 2. Theory predictions for the soft-gluon radiative corrections to the left-handed helicity form factors of $\bar{B}_d \to \gamma \gamma$ with the perturbative uncertainties from varying the factorization scale in the interval $\mu \in [1.0, 2.0]$ GeV (green bands), where we further display the numerical results for the combined factorizable power corrections as previously derived in [3] (blue bands) for a comparison. The shorthand notations for the weighted helicity form factors $F_{L}^{(p), X} = \sum_i C_i F_{i,L}^{(p), X}$ have been introduced here for convenience.

corrections derived in [3]. On the contrary, the long-distance charm-quark penguin mechanism will only lead to the rather minor impact on the left-handed helicity form factor $\sum_i C_i F_{i, L}^{(c)}$ numerically as indicated by Figure 2. This intriguing observation can be attributed to the peculiar analytical behaviour of the perturbative penguin function $F(z_p)$ entering the soft-collinear factorization formula (10) such that the yielding result of $|F(z_p)|$ is approximately one order of magnitude lower than the corresponding up-quark penguin contribution $|F(0)| = 1$. Along the same vein, one can readily observe that the soft-gluon radiative corrections to the factorizable quark loops cannot generate numerically important contributions to the exclusive $\bar{B}_d \to \gamma \gamma$ helicity amplitudes due to the CKM suppression of the up-quark penguin contraction and the dynamical suppression of the charm-quark penguin contribution as discussed above. Bearing in mind the utmost importance of understanding the charming penguin contribution in unveiling the genuine NP effects embedded in the semileptonic $B \to K^{(*)}\ell\bar{\ell}$ decays (see for instance [55–60]), the achieved robust control of such long-distance penguin contribution in the double radiative $B$-meson decays with the diagrammatic factorization technique evidently makes these FCNC decay processes most suitable for probing the nonstandard four-fermion $b \to q f \bar{f}$ interaction at the high-luminosity Belle II experiment [61] and for performing the dedicated parton tomography of the composite heavy-quark hadron system in the QCD framework.

We are now ready to explore the numerical impacts of the long-distance penguin contribution on the CP-averaged branching fractions, the two polarization fractions and the CP-violating observables for $\bar{B}_d \to \gamma \gamma$ (see [3] for their explicit definitions). It turns out that such subleading power corrections can enhance the theory predictions for the mixing induced CP asymmetries $A_{CP}^{\text{mix}, \parallel}$ and $A_{CP}^{\text{mix}, \perp}$ by approximately an amount of $\mathcal{O}(30\%)$ with the default inputs, while yielding insignificant effects in the remaining observables numerically. Moreover, our numerical prediction for the ratio of the two branching fractions $\mathcal{B}(B_s \to \gamma \gamma) : \mathcal{B}(B_d \to \gamma \gamma)$ allows for extracting the high-profile hadronic quantity $\lambda_{B_s} : \lambda_{B_d}$ with the improved systematic uncertainty at the level of $(5 – 10\%)$, when confronting with the anticipated precision measurements.

**CONCLUSIONS**

In conclusion, we have presented the first computation of the long-distance penguin contribution to the double radiative $B$-meson decay amplitudes by applying the perturbative factorization approach. Adopting the power counting scheme $m_b \gg m_c \sim \mathcal{O}(\sqrt{\Lambda} m_b) \gg \Lambda$, we demonstrated further that the novel subleading $B$-meson distribution amplitude $\Phi_Q$ defined by the three-body HQET operator with partonic fields localized on two different light-ray directions (instead of the conventional light-cone distribution amplitude) emerged naturally in the resulting factorization formula (10). Phenomenologically the soft-gluon radiative off the factorizable up-quark loop appeared to bring about more pronounced effect in comparison with the corresponding charm-quark penguin mechanism thanks to the peculiar analytical behaviour of the short-distance matching function (7). In addition, the observed destructive interference between the long-distance penguin contribution and the available factorizable power correction from [3] enabled us to determine the pivotal ratio of the two inverse moments $\lambda_{B_d,s}$ with the reduced theory uncertainty. Our analysis will be evidently beneficial for exploring the intricate charming penguin dynamics encoded in a large variety of the radiative and electroweak penguin decay processes including $B \to K^{*}\gamma$ and $B \to K^{(*)}\ell\bar{\ell}$, which are generally recognized as the flagship probes of physics beyond the SM at the LHC.
ACKNOWLEDGEMENTS

The research of Q.Q. is supported by the National Natural Science Foundation of China with Grant No. 12005068. C.W. is supported in part by the National Natural Science Foundation of China with Grant No. 12105112 and the Natural Science Foundation of Jiangsu Education Committee with Grant No. 21KJB140027. The research of Y.L.S. is supported by the National Natural Science Foundation of China with Grant No. 11735010 and 12075125, and the Natural Science Foundation of Tianjin with Grant No. 19JCYJJC6110.

*qqin@hust.edu.cn

†corresponding author: shenylmeteor@ouc.edu.cn

‡corresponding author: chaowang@nankai.edu.cn
[48] Y.-M. Wang and Y.-L. Shen, JHEP **05**, 184 (2018), arXiv:1803.06667 [hep-ph].

[49] V. M. Braun and A. Khodjamirian, Phys. Lett. B **718**, 1014 (2013), arXiv:1210.4453 [hep-ph].

[50] Y.-M. Wang, JHEP **09**, 159 (2016), arXiv:1606.03080 [hep-ph].

[51] A. G. Grozin and M. Neubert, Phys. Rev. D **55**, 272 (1997), arXiv:hep-ph/9607366.

[52] V. M. Braun, D. Y. Ivanov, and G. P. Korchemsky, Phys. Rev. D **69**, 034014 (2004), arXiv:hep-ph/0309330.

[53] A. Khodjamirian, T. Mannel, and N. Offen, Phys. Rev. D **75**, 054013 (2007), arXiv:hep-ph/0611193.

[54] C.-D. Lü, Y.-L. Shen, Y.-M. Wang, and Y.-B. Wei, JHEP **01**, 024 (2019), arXiv:1810.00819 [hep-ph].

[55] A. Khodjamirian, R. Mandal, and T. Mannel, JHEP **10**, 043 (2020), arXiv:2008.03935 [hep-ph].

[56] M. Beneke, V. M. Braun, Y. Ji, and Y.-B. Wei, JHEP **07**, 154 (2018), arXiv:1804.04962 [hep-ph].

[57] M. Beneke, C. Bobeth, and R. Szafron, Phys. Rev. Lett. **120**, 011801 (2018), arXiv:1708.09152 [hep-ph].

[58] A. Cerri et al., CERN Yellow Rep. Monogr. **7**, 867 (2019), arXiv:1812.07638 [hep-ph].

[59] M. Ciuchini, M. Fedele, E. Franco, A. Paul, L. Silvestrini, and M. Valli, (2021), arXiv:2110.10126 [hep-ph].

[60] W. Altmannshofer and F. Archilli, in 2022 Snowmass Summer Study (2022) arXiv:2206.11331 [hep-ph].

[61] W. Altmannshofer et al. (Belle-II), PTEP **2019**, 123C01 (2019), arXiv:1810.00819 [hep-ph].