Real ideals of compact operators of complex factors

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Abstract  Real ideals of compact operators for (complex) factors are investigated. A description (up to isomorphisms) of real two-sided ideals of relatively compact operators of a complex W*-factors is given. A relative weak (RW)$_r$ convergence in a real Hilbert space is introduced. The classical Hilbert characterization of compactness of operators is extended to the compact operators in semifinite real W*-algebras.

Keywords  W*-algebras · Real factors · Ideals · Compact operators

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1 Introduction

In the present paper we investigate the real ideals of compact operators for complex factors and give a description (up to isomorphisms) of real two-sided ideal of relatively compact operators of the complex W*-factors. A concept of relative weak (RW)_r convergence in a real Hilbert space is introduced. The classical Hilbert characterization of compactness of operators is extended to the compact operators in semifinite real W*-algebras.

2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex separable Hilbert space $H$. A weakly closed $*$-subalgebra $\mathfrak{A}$ containing the identity operator $I$ in $B(H)$ is called a W*-algebra. A real $*$-subalgebra $R \subset B(H)$ is called a real W*-algebra if it is closed in the weak operator topology, $I \in R$ and $R \cap iR = \{0\}$. A real W*-algebra $R$ is called a real factor if its center $Z(R)$ consists of the elements $\{\lambda I, \lambda \in \mathbb{R}\}$, where $\mathbb{R}$ is the field of all real numbers. We say that a real W*-algebra $R$ is of the type $I_{fin}, I_{\infty}, II_1, II_\infty$, or $III_\lambda, (0 \leq \lambda \leq 1)$ if the enveloping W*-algebra $\mathfrak{A}(R)$ has the corresponding type in the ordinary classification of W*-algebras. A linear mapping $\alpha$ of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an $*$-automorphism if $\alpha(xy) = \alpha(x)\alpha(y)$; it is called an involutive $*$-antiautomorphism if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If $\alpha$ is an involutive $*$-antiautomorphism of a W*-algebra $M$, we denote by $(M, \alpha)$ the real W*-algebra generated by $\alpha$, i.e. $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$. Conversely, every real W*-algebra $R$ is of the form $(M, \alpha)$, where $M$ is the complex envelope of $R$ and $\alpha$ is an involutive $*$-antiautomorphism of $M$ (see [1,2,5,9]). Therefore we shall identify from now on the real von Neumann algebra $R$ with the pair $(M, \alpha)$.

A trace on a (complex or real) W*-algebra $N$ is a linear function $\tau$ on the set $N^+$ of positive elements of $N$ with values in $[0, +\infty]$, satisfying $\tau(u^*ux) = \tau(x)$, for an arbitrary unitary $u$ and for any $x$ in $N$.

The trace $\tau$ is said to be finite, if $\tau(I) < +\infty$; semifinite, if given any $x \in N^+$ there is a nonzero $y \in N^+$, $y \leq x$ with $\tau(y) < +\infty$.

3 Real and complex ideals of W*-algebras

Definition 1 Let $M$ be a W*-algebra. A real subspace $I$ of $M$ is called a real ideal of $M$ if $I \cdot M \subset I_c$, where $I_c$ is the smallest complex subspace of $M$, containing $I$.

It is easy to see that the subspace $I_c$ is equal to $I + iI$, therefore a real subspace $I$ is a real ideal if and only if $I \cdot M \subset I + iI$.

Since each complex subspace of $M$ is a real subspace, any complex ideal is automatically a real ideal of $M$. Let $I$ be a real ideal of $M$. If there exists a real W*-subalgebra $R$ of $M$ with $R + iR = M$, such that $I \subset R$, then $I$ is called a pure real ideal of $M$. In this case, it is obvious that we have $I \cdot R \subset I$. Note that, the reverse is not true, i.e. from $I \cdot R \subset I$ it does not follow $I \subset R$. But a complex subspace $J = I + iI$ always is a complex ideal of $M$. On the other hand if $I \subset R$ is a real subspace of $M$ and $I + iI$ is a complex ideal, then $I$ is a pure real ideal, i.e. we obtain $I \cdot R \subset I$. 
Let, now $I$ and $Q$ be pure real ideals of $M$. In general, the set $I + iQ$ is not a (complex) subspace. More precisely the set $I + iQ$ is a complex subspace if and only if $I = Q$. Therefore we consider the smallest complex subspace $J$ of $M$, containing $I$ and $Q$. Obviously $J$ is equal to $(I + Q) + i(I + Q)$. Thus, if $I$ and $Q$ are real ideals, then $J = (I + Q) + i(I + Q)$ is a complex ideal.

### 4 Ideals of compact operators

Let $(M, \alpha)$ be a real factor and let $\tau$ be an $\alpha$-invariant semifinite trace on $M$. A subspace $K \subset H$ is called $\tau$-finite (or finite relative to $\tau$), if $\tau(P_K) < +\infty$, where $P_K$ is the canonical projection of $H$ on $K$ with $P_K \in M$.

Now, let $K$ be a subset of $H$. A subset $K$ is called $\tau$-compact (or compact relative to $\tau$), if $K$ is approximated in the norm $\| \cdot \|_H$ by a bounded sequence of $\tau$-finite subspaces.

A real operator $A$ on $H$ (i.e. $A \in (M, \alpha)$) is called real compact relative to $\tau$ if it is an operator mapping bounded sets into relatively compact sets. We denote by $I$ (respectively, by $J$) the set of all relatively compact operators of $(M, \alpha)$ (respectively, of $M$). Let us recall the following result.

**Theorem 1** ((3,10,11)) Let $M$ be a semifinite factor and let $\alpha$ be an involutive $\ast$-antiautomorphism of $M$. Then $I$ (respectively, $J$) is a unique (nonzero) uniformly closed two-sided ideal of $(M, \alpha)$ (respectively, of $M$), and $I + iI = J$.

Now, let us recall [4] the notion of the crossed product of a $W^\ast$-algebra $M$ by a locally compact topological group $G$ by its $\ast$-automorphism. Let $\gamma : G \to Aut(M)$ be a group homomorphism such that the map $g \to \gamma_g$ is strongly continuous. Let $L_2(G, H)$ be the Hilbert space of all $H$-valued square integrable functions on $G$. We consider a $\ast$-algebra $N \subset B(L_2(G, H))$, generated by operators of the form: $\pi_\gamma(a)$ and $u(g)$, where $a \in M$, $g \in G$, and

$$(\pi_\gamma(a)\xi) = \gamma_h^{-1}(a)\xi(h), \quad (u(g)\xi)(h) = \xi(g^{-1}h),$$

$\xi = \xi(h) \in L_2(G, H), \quad g, h \in G$. The algebra $N$ is called the crossed product of $M$ by $G$, and denoted by $W^\ast(M, G)$ or $M \rtimes_\gamma G$. Moreover, there exists a canonical embedding $\pi_\gamma : M \to \pi_\gamma(M) \subset N$. Each element $x \in N$ has the form: $x = \sum_{g \in G} \pi_\gamma(x(g))u(g)$, where $x(\cdot)$ is an $M$-valued function on $G$. If $\theta$ is a $\ast$-automorphism of $M$, then for the action $\{\theta^n\}$ of the group $\mathbb{Z}$ on $M$ we denote by $W^\ast(\theta, M)$ or $M \rtimes_\theta \mathbb{Z}$ the crossed product of $M$ by $\theta$. Similarly one can define the notion of crossed product for real $W^\ast$-algebras (see [1,12–14]).

Let $M$ be a factor of type $\text{III}_\lambda (\lambda \neq 1)$ and let $\alpha$ be an involutive $\ast$-antiautomorphism of $M$. Then by [12] (see also [11]), either

- there exist a factor $N$ of type $\text{II}_\infty$ and an $\alpha$-invariant automorphism $\theta$ of $N$ such that $(M, \alpha)$ is isomorphic to the (real) crossed product $(N, \alpha) \rtimes_\theta \mathbb{Z}$, or
- there exist a factor $N$ of type $\text{II}_\infty$ and an antiautomorphism $\sigma$ of $N$ such that $(M, \alpha)$ is isomorphic to $((N \oplus N^{op}) \rtimes_\sigma \mathbb{Z}, \beta)$, where $N^{op}$ is the opposite $W^\ast$-algebra for $N$ and $\beta(x, y) = (y, x)$, for all $x, y \in N$. 

Let's consider the first case. Let \((M, \alpha)\) be isomorphic to the (real) crossed product \((N, \alpha) \times_\theta \mathbb{Z}\). It is known that \((N, \alpha) \times_\theta \mathbb{Z} + i \cdot (N, \alpha) \times_\theta \mathbb{Z} = N \times_\theta \mathbb{Z}\) (see [12–14]). Let \(E : M \to N\) be the unique \(\alpha\)-invariant faithful normal conditional expectation (see [15,16]). Let us state an auxiliary lemma whose proof immediately follows from the linearity and \(\alpha\)-invariance of \(E\).

**Lemma 1** If \(S\) is an ideal in \((M, \alpha)\) and \(S_c = S + iS\), then

\[ E^{-1}(S) + iE^{-1}(S) = E^{-1}(S_c), \quad E^{-1}(S)^+ \subset E^{-1}(S_c)^+, \]

where \(E^{-1}(A) = \{x : E(x) \in A\}\).

Recall that a cone \(K \subset A^+\) is called hereditary if \(x \in A^+, y \in K\) and \(x \leq y\) implies \(x \in K\); a subalgebra \(B \subset A\) is called hereditary if the cone \(B^+\) is hereditary. It is easy to see that any two-sided ideal is hereditary.

**Lemma 2** If the cone \(E^{-1}(S_c)^+\) is hereditary, then the cone \(E^{-1}(S)^+\) is also hereditary.

**Proof** If \(x \in (M, \alpha)^+ \subset M^+, y \in E^{-1}(S)^+ \subset E^{-1}(S_c)^+\) and \(x \leq y\), then \(x \in E^{-1}(S_c)^+, \) since \(E^{-1}(S_c)^+\) is hereditary. Therefore \(\alpha(x) = x^*\) implies \(x \in E^{-1}(S)^+\). □

**Proposition 1** Let \(M\) be a semifinite factor. If \(S \subset (M, \alpha)\) is a two-sided ideal, then the linear span of \(E^{-1}(S)^+\) denoted as \(\text{span}(E^{-1}(S)^+\) is a hereditary *-subalgebra of \((M, \alpha)\) and a two-sided module over \((M, \alpha)\). Moreover, if \(S\) is a norm-closed, then \(\text{span}(E^{-1}(S)^+)\) is also norm-closed.

**Proof** By Lemma 1 and Proposition 3.3 [6] the cone \(E^{-1}(S_c)^+\) is hereditary and \(\text{span}(E^{-1}(S_c)^+)\) is a hereditary *-subalgebra of \(M\), where \(S_c = S + iS\). By Lemma 2 the cone \(E^{-1}(S)^+\) is also hereditary. Using the hereditarity of \(\text{span}(E^{-1}(S_c)^+)\) one can easily check the hereditarity of \(\text{span}(E^{-1}(S)^+\).

If \(S\) is norm-closed, then \(S^+\) is also norm-closed. The continuity of \(E\) implies that \(E^{-1}(S)^+\) is closed. Therefore \(\text{span}(E^{-1}(S)^+)\) is also closed.

Let \(x \in E^{-1}(S)^+\) and \(y \in (M, \alpha)^+\). From \(x \in E^{-1}(S_c)^+\) and \(y \in M\), by Proposition 3.3 [6] we obtain that \(yx \in \text{span}(E^{-1}(S_c)^+)\), since \(\text{span}(E^{-1}(S_c)^+)\) is a two-sided module over \(M\). On the other hand \(yx \in (M, \alpha)^+\) and \(\{a \in \text{span}(E^{-1}(S_c)^+) : \alpha(a) = a^*\} = \text{span}(E^{-1}(S)^+)\). Hence \(yx \in \text{span}(E^{-1}(S)^+)\).

Since the element \(x\) is arbitrary from \(E^{-1}(S)^+\) by linearity we obtain \(yx \in \text{span}(E^{-1}(S)^+)\) for any \(y \in \text{span}(E^{-1}(S)^+)\). Therefore \(\text{span}(E^{-1}(S)^+)\) is a left-sided module over \((M, \alpha)\). Similarly one can show that it is a right-sided \((M, \alpha)\)-module. □

Now, we put \(\mathcal{I} = \text{span}\{x \in (M, \alpha)^+ : E(x) \in I\}\), where \(I\) is the unique (nonzero) uniformly closed two-sided ideal of the semifinite real factor \((N, \alpha)\) (see Theorem 1).

By Proposition 1, \(\mathcal{I}\) is hereditary.

**Lemma 3** The following is valid

\[ \mathcal{I}^+ = \{x \in (M, \alpha)^+ : x \leq y, \quad \text{for some } y \in I^+\}. \]

**Proof** Let \(x \in \mathcal{I}^+\) and \(J = I + iI, \mathcal{J}^+ = \{x \in M^+ : x \leq z, \text{ for some } z \in J^+\}. \)

Since \(x \in \mathcal{J}^+\) by Proposition 3.7 d) [6], \(x \leq z, \text{ for some } z \in J^+\). Let \(z = y + it, \)
y, t \in (N, \alpha). Then y \geq 0 (because z \geq 0) and z - x = (y - x) + it \geq 0, hence y - x \geq 0, i.e., x \leq y. Since I^+ \subset J^+, y \in I^+.

Conversely, if x \in (M, \alpha)^+ and x \leq y, for some y \in I^+, then again by Proposition 3.7 d) [6] we have x \in J^+. From \alpha(x) = x^* we have x \in I^+. \square

From Lemma 3, in particular, it follows, that a projection from (M, \alpha) is finite if and only if it majorized by some finite projection of (N, \alpha).

Let I_1 be the norm closure of I. If we apply Proposition 1, Lemma 3 and the scheme of the proofs of Propositions 4.1, 4.5 and Theorem 4.3 [6], then we can prove the following real analogue of Halpern-Kaftal’s theorem.

**Theorem 2** Let M be a factor of type III_\lambda (\lambda \neq 1) and let \alpha be an involutive \ast-antiautomorphism of M. If the real factor (M, \alpha) is isomorphic to the (real) crossed product (N, \alpha) \times_{\theta} \mathbb{Z}, then I_1 is a unique (up to an inner automorphism) smallest hereditary real C*-subalgebra of (M, \alpha), containing the ideal I, and it is a two-sided module over (N, \alpha).

Let us consider the second case. Let (M, \alpha) be isomorphic to ((N \oplus N^{op}) \times_{\theta} \mathbb{Z}, \beta), where N is a \Pi_\infty-factor, N^{op} is the opposite W*-algebra for N and \beta(x, y) = (y, x), for all x, y \in N.

Recall that, a factor N is generated by the fixed point algebra of the one parameter group \{\sigma_t^\psi : t \in \mathbb{R}\} of modular automorphisms, associated with some \alpha-invariant faithful normal semifinite weight \psi. More precisely, the W*-subalgebra M_\psi = \{x \in M : \sigma_t^\psi(x) = x, t \in \mathbb{R}\} contains a central projection p such that N is isomorphic to factor pM_\psi. In this case the real W*-algebra (M_\psi, \alpha) is isomorphic to (N \oplus N^{op}, \beta) (for more details see [1,12–14]).

Let E : (M, \alpha) \rightarrow (N \oplus N^{op}, \beta) be a faithful normal conditional expectation (see [15,16]) and let J be the unique (nonzero) uniformly closed two-sided ideal of the semifinite (complex) factor N (see Theorem 1). Similarly to the first case, we denote by I_2 the norm closure of span\{x \in (M, \alpha) : E(x) \in (J \oplus J^{op}, \beta)\}.

Applying the same reasonings, as in the first case, and the scheme of proofs of Propositions 4.1, 4.5 and Theorem 4.3 [6], we obtain one more real analogue of Halpern-Kaftal’s theorem.

**Theorem 3** Let M be a factor of type III_\lambda (\lambda \neq 1) and let \alpha be an involutive \ast-antiautomorphism of M. If the real factor (M, \alpha) is isomorphic to ((N \oplus N^{op}) \times_{\theta} \mathbb{Z}, \beta), then I_2 is the unique (up to inner automorphism) smallest hereditary real C*-subalgebra of (M, \alpha), containing the ideal I and is a two-sided module over (N, \alpha). Here I is the unique (nonzero) uniformly closed two-sided ideal of semifinite real factor (N, \alpha).

Thus, summarizing all above, in the injective case, we can describe all (nonzero) uniformly closed two-sided real ideals of semifinite and pure infinite complex factors. Recall that [1,5], if M is an injective factor of type II, then there exists a unique conjugacy class of involutive \ast-antiautomorphisms in M; therefore there exists a unique (up to isomorphisms) real subfactor of M, generating M. If M is an injective factor of type III_\lambda (0 < \lambda < 1), then in M there exist exactly two conjugacy classes of involutive \ast-antiautomorphism; therefore there exist two (up to isomorphisms) real subfactors of M, generating M. Hence we obtain the following result.
Theorem 4 Let $M$ be a factor. Then the following assertions are true:

1. If $M$ is an injective factor of type $II_1$ or type $II_{\infty}$, then there exist (up to isomorphisms) two (nonzero) uniformly closed two-sided real ideals in $M$. One of them is the complex ideal $J$, the other is the pure real ideal $I$.

2. If $M$ is an injective factor of type $III_\lambda$ $(0 < \lambda < 1)$, then there exist (up to isomorphisms) three (nonzero) uniformly closed two-sided real ideals in $M$. One of them is the complex ideal $J$, the other two are the pure real ideals $I_1$ and $I_2$.

5 Relative weak convergence in semifinite real $W^*$-algebras

In this section, we study the relative weak (RW) convergence in a real Hilbert space. We first recall that the elements of the two-sided closed ideal $I$ generated by the projections which are finite relative to a real $W^*$-algebra $(M, \alpha)$ are called compact operators of $(M, \alpha)$.

Let $H_r$ be a real Hilbert space with $H_r + iH_r = H$. A sequence $\{ \xi_n \} \subset H_r$ (or $\subset H$) is called weakly converging to $\xi$, if for every projection $P$ which is finite relative to $B(H_r)$ (respectively, $B(H)$), $P\xi_n$ converges strongly to $P\xi$ ($P\xi_n \xrightarrow{S} P\xi$). This suggests the following generalization:

Definition 2 Let $(M, \alpha)$ be a real $W^*$-algebra in $B(H_r) \subset B(H) = B(H_r) + iB(H_r)$. We say that a sequence $\{ \xi_n \} \in H_r$ converges to $\xi$ weakly relative to $(M, \alpha)$ and briefly say (RW)$, converges or $\xi_n \xrightarrow{(RW)_r} \xi$, if

1. $\|\xi_n\|$ is bounded;
2. for every projection $P \in (M, \alpha)$ which is finite relative to $(M, \alpha)$, the sequence $\{ P\xi_n \}$ converges strongly to the element $P\xi$, i.e. $P\xi_n \xrightarrow{S} P\xi$.

Note that a weakly convergent sequence is necessarily bounded, but following the Example 2 of [7], it is easy to construct an example, in which a unbounded sequence satisfies the second condition of Definition 1.

Example Let $H$ and $K$ be infinite-dimensional separable real Hilbert spaces with orthonormal bases $\{ \eta_i \}$, $\{ \gamma_i \}$ respectively. We put $R = B(H) \otimes \mathbb{R}I_K$ and $\xi_n = \sum_{i=1}^{2n} \eta_i \otimes \gamma_i$. Since $\|\xi_n\|^2 = \sum_{i=n}^{2n} \|\eta_i\|^2 \|\gamma_i\|^2 = n$, it is an unbounded sequence. Let $P$ be a finite protection of $R$. Then there is a finite projection $P_0$ in $B(H)$ such that $P = P_0 \otimes I_K$. Without loss of generality we may assume that it is one-dimensional, i.e., that $P_0 = \langle \cdot, \cdot \rangle$, for an element $\xi \in H$ with $\|\xi\| = 1$. Then

$$
\|P\xi_n\|^2 = \left( \sum_{i=n}^{2n} P_0\eta_i \otimes \gamma_i, \sum_{j=n}^{2n} P_0\eta_j \otimes \gamma_j \right) \\
= \left( \sum_{i=n}^{2n} \langle \eta_i, \xi \rangle \otimes \gamma_i, \sum_{j=n}^{2n} \langle \eta_j, \xi \rangle \otimes \gamma_j \right) \\
= \sum_{i, j=n}^{2n} \left( \langle \eta_i, \xi \rangle \otimes \gamma_i, \langle \eta_j, \xi \rangle \otimes \gamma_j \right)_H \cdot (\gamma_i, \gamma_j)_K = \sum_{k=n}^{2n} |\langle \eta_k, \xi \rangle|^2.
$$
Since the series $\sum_{k=1}^{\infty} | < \eta_k, \xi > |^2$ converges, the sequence $\sum_{k=n}^{2n} | < \eta_k, \xi > |^2$ converges to 0, when $n \to \infty$. Hence $\| P \xi_n \| S \to P \xi = 0$, but the sequence $\{ \xi_n \}$ is unbounded. Moreover it is easy to show that the sequence $\{ \eta_n \otimes \gamma \}$ $(RW)_r$ converges to 0, but it does not converge strongly to 0, and the sequence $\{ \xi \otimes \gamma_n \}$ converges weakly to 0, but does not $(RW)_r$ converge to 0.

It is easy to see that the following assertions are true.

**Lemma 4**

$\xi_n \xrightarrow{RW} \xi \iff \xi_n \xrightarrow{(RW)_r} \xi$  \hspace{1cm} (1)

Here $\xi_n \xrightarrow{RW} \xi$ means that $\| \xi_n \|$ is bounded and $P \xi_n \xrightarrow{S} P \xi$, for every projection $P \in M$ (see [7, Definition 1]).

**Proof** The proof of implication “$\Rightarrow$” is obvious. Let’s prove the implication “$\Leftarrow$”. Assume, that there is some projection $p \in M$ with $\| p \xi_n \| \not\to 0$. Since the projections $p$ and $\alpha(p)$ are equivalent we have $\| \alpha(p) \xi_n \| \not\to 0$ (see [1]). It is easy to see that $a = p + \alpha(p) \in (M, \alpha)$ and $a \geq p$ (because $\alpha(p) \geq 0$), therefore $\| a \xi_n \| \not\to 0$. Then exists a spectral projection $e \in (M, \alpha)$ of an element $a$ with $\| e \xi_n \| \not\to 0$. It is a contradiction with the assumption. \hfill \Box

The following theorem is a generalization of Hibert’s characterization of the compact operators.

**Theorem 5** (Theorem 7, [7]) An element $A$ is compact in $M$ iff it maps $(RW)$ converging sequences into strongly converging ones.

We have the following real analogue of the above characterization.

**Theorem 6** An element $A$ is compact in $(M, \alpha)$ iff it maps $(RW)_r$ converging sequences into strongly converging ones.

**Proof** Firstly, let us prove the sufficiency. Let $I$ be the two-sided closed (pure real) ideal generated by the projections which are finite relative to a real W*-algebra $(M, \alpha)$ and put $J = I + iI$. As it was noticed above $J$ is the two-sided closed (complex) ideal, generated by the projections which are finite relative to a W*-algebra $M$. If we put

$$I_1 = \{ A \in (M, \alpha) : A \xi_n \xrightarrow{S} 0, \text{ for any sequence } \{ \xi_n \} \subset H_r \text{ with } \xi_n \xrightarrow{(RW)_r} 0 \},$$

then by (1) for

$$J_1 = \{ A \in M : A \xi_n \xrightarrow{S} 0, \text{ for any sequence } \{ x_n \} \subset H \text{ with } \xi_n \xrightarrow{(RW)} 0 \},$$

we obtain $I_1 \subset J_1 = J = I + iI$. Here the equality $J_1 = J$ is valid by Theorem 5.

Since $I_1 \subset (M, \alpha)$, we obtain $I_1 \subset I$. Therefore $A \in I$, i.e. $A$ is compact relatively to $(M, \alpha)$.
Now we shall prove the necessity. It suffices to show that \( I \subset I_1 \). Suppose that \( K \in I \) and \( \xi_n^{(RW)} \xrightarrow{S} \xi \). Without loss of generality we can assume that \( \xi = \theta \). We repeat step by step the scheme of proof of Theorem 1.3 [8], to obtain the real analogue of the generalized Rellich condition for \( K \), i.e., for every \( \lambda > 0 \) there is a projection \( P \) of \( (M, \alpha) \) such that \( \| KP \| \leq \lambda \) and \( \| \xi \| \) is bounded by definition. Since \( \lambda \) is arbitrary, we have \( \xi \in I_1 \). Hence \( K \in I_1 \), and therefore \( I_1 \subset I \).  

Theorem 6 shows that Hilbert’s characterization of the compact operators remains valid in semifinite real \( W^* \)-algebras.

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