Growth and irreducibility in path-incompressible trees*

George Barmpalias and Xiaoyan Zhang

State Key Lab of Computer Science, Inst. of Software, Chinese Acad. of Sciences, Beijing, China

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Abstract. We study randomness-preserving transformations of path-incompressible trees, namely trees of finite randomness deficiency. We characterize their branching density, and show: (a) sparse perfect path-incompressible trees can be effectively densified, almost surely; (b) there exists a path-incompressible tree with infinitely many paths which does not compute any perfect path-incompressible tree with computable oracle-use.

1 Introduction

Algorithmic randomness appears in different forms: infinite bit-sequences (reals), arrays, trees, and structures. It is often essential to effectively transform one form into another, without sacrificing algorithmic complexity: transforming a real which is random with respect to a Bernoulli distribution into a uniformly random real goes back to von Neumann [18], and general cases, including non-computable distributions, have been explored [6].

We study the hardness of transformations of closed sets of random points (or trees with incompressible paths) with respect to dimensionality features: branching and accumulation points. We use $\sigma, \tau, \rho, \eta$ for bit-strings, $x, y, z$ for reals, and let $K(\sigma)$ be the prefix-free Kolmogorov complexity of $\sigma$: the length of the shortest self-delimiting program generating $\sigma$.

Definition 1.1. The (randomness) deficiency of $\sigma$ is $|\sigma| - K(\sigma)$. The deficiency of a set of strings is the supremum of the deficiencies of its members. The deficiency of $x$ is the deficiency of the set of its prefixes.

A real is random if it has finite deficiency; this is equivalent to the notion of Martin-Löf [14]. A set of strings is a tree if it is downward-closed with respect to the prefix relation $\preceq$. A path through tree $T$ is a real with all its prefixes in $T$. A tree $T$ is

- perfect if each $\sigma \in T$ has at least two $\preceq$-incomparable strict extensions in $T$
- pruned if each $\sigma \in T$ has at least one proper extension in $T$

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• *proper* if it has infinitely many paths; *positive* if the measure of its paths is positive
• *effectively proper* there is an unbounded computable lower bound on |\( T \cap 2^n \)|
• *path-incompressible* if it is pruned and has finite deficiency, as a set of strings
• *path-random* if it is pruned and all of its paths are random\(^1\)

where \( 2^n \) denotes the set of \( n \)-bit strings. Path-random trees are not always path-incompressible: it is possible that \( T \) is path-random and \( T \) has infinite deficiency. On the other hand by [12, 4] every perfect path-random tree computes a path-incompressible tree.

Our purpose is to determine when randomness can be effectively manipulated with respect to topological or density characteristics (length distribution, accumulation points etc.)

To this end, we ask: within the path-incompressible trees

(a) can every tree compute a proper tree?
(b) can every perfect tree compute a positive tree?
(c) can every proper tree compute a perfect tree?
(d) can every sparse perfect tree be effectively transformed into a denser tree?

These questions are tree-analogues of problems of randomness extraction such as effectively increasing the Hausdorff dimension of reals [15], except that the increase is now on the structural density of the tree, without loss in the algorithmic complexity of its paths.

Bienvenu and Porter [7, Theorem 5.3, Theorem 7.7] showed that incomplete randoms do not compute effectively-proper path-incompressible trees. Hirschfeldt et al. [12] showed that sufficiently random reals do not compute perfect path-random trees. A complete answer to (a) was given in [4]: incomplete randoms do not compute perfect path-random trees.

A negative answer to question (b) was given in [4]: there exists a perfect path-incompressible tree which cannot computably enumerate any positive path-random tree. Much of this work was motivated by the study of compactness in fragments of second-order arithmetic.

**Outline of our results**

Our main contribution is a partial negative answer to question (c):

**Theorem 1.2.** There is a path-incompressible effectively proper tree that does not compute any path-random perfect tree with a computable upper bound on the oracle-use.

More generally, we show that for any function \( f \) there is a path-incompressible effectively proper tree \( T \leq_T \emptyset' \oplus f \) which does not compute any path-random perfect tree within oracle-use \( f \). The difficulty of constructing a tree with these properties is due to (i) it has to be

\(^1\)all infinite trees we consider are assumed to be pruned, hence representing closed sets of \( 2^{\omega} \).
effectively proper; (ii) it has to be path-incompressible, rather than merely path-random; (iii) there is no restriction on the density of branching of the trees we want to avoid computing.

If some of these conditions are relaxed, question (c) admits a simple answer:

**Proposition 1.3.**

(a) There exists a low path-incompressible tree with two paths which computes a perfect path-random tree.

(b) There exists an effectively proper path-random tree which does not compute any perfect path-random tree.

There are precise limits on the branching-density of path-incompressible trees. Given increasing $\ell = (\ell_n)$, we say that a tree $T$ is $\ell$-perfect if each node of length $\ell_n$ in $T$ has at least two extensions of length $\ell_{n+1}$.

**Theorem 1.4.** If $\ell = (\ell_n)$ is computable and increasing the following are equivalent:

(i) $\exists$ an $\ell$-perfect path-random tree

(ii) $\exists$ an $\ell$-perfect path-incompressible tree

(iii) $\sum n 2^{-(\ell_{n+1}-\ell_n)} < \infty$.

We conclude with a positive answer to question (d) above: can a sparse perfect path-incompressible tree be effectively transformed into a denser path-incompressible tree?

**Theorem 1.5 (Informal).** Any sparse computably-perfect path-incompressible tree can be effectively transformed into an $n^2$-perfect path-incompressible tree, almost surely.

The formal statement is given in §4, but two aspects are clear: the probabilistic success of the transformation, and the bound on the density of the branching achieved.

**Our methodology.** The challenge in the study of reducibilities between path-incompressible trees is the lack of a simple representation of the associated maps between trees. Our approach is based on families of hitting-sets that intersect, and missing-sets that avoid the inverse images of the maps. The intuition comes from the Fell (hit-or-miss) topology on the space of closed sets, and the theory of measure in this space [16], where probability of a closed set of reals is characterized by the measure of its hitting-sets [1, 2].

## 2 Background and notation

We lay out additional notation and known facts that we need about the Cantor space.

Strings are ordered first by length and then lexicographically. Let

- $2^n$ denote the set of $n$-bit strings and $z \upharpoonright n$ denote the $n$-bit prefix of $z$
• $2^\omega$ denote the set of *reals* and $2^{<\omega}$ the set of binary strings.

The full binary tree represents the Cantor space, with topology generated by the sets
\[ [\sigma] := \{ z \in 2^\omega : \sigma < z \} \quad \text{and} \quad [V] = \bigcup_{\sigma \in V} [\sigma] \quad \text{for } V \subseteq 2^{<\omega}. \]

We often identify $\sigma$ with $[\sigma]$ and $V$ with $[V]$. The uniform measure on $2^\omega$ is given by
\[ \mu(\sigma) := \mu([\sigma]) = 2^{-|\sigma|} \quad \text{and} \quad \mu(V) := \mu([V]). \]

Let $\mu_\sigma(V)$ be the measure of $[V]$ relative to $[\sigma]$:
\[ \mu_\sigma(V) := \mu([V] \cap [\sigma]) / \mu(\sigma) = 2^{|\sigma|} : \mu([V] \cap [\sigma]). \]

Let $\ast$ denote the concatenation of strings and
\[ \sigma \ast U := \{ \sigma \ast \tau : \tau \in U \} \quad \text{and} \quad V \ast U := \{ \sigma \ast \tau : \sigma \in V \land \tau \in U \}. \]

### 2.1 Background on randomness of reals

A Martin-Löf test is a uniformly c.e. sequence of prefix-free sets $V_i \subseteq 2^{<\omega}$ such that $\mu(V_i) < 2^{-i}$. Following Martin-Löf [14], we say that $x$ is random if $x \not\in \cap_{i} [V_i]$ for all such tests $(V_i)$.

Randomness can equivalently be defined in terms of incompressibility, via Kolmogorov complexity. A prefix-free machine is a Turing machine whose domain is a prefix-free set of strings. The *prefix-free Kolmogorov complexity* of $\sigma$ with respect to prefix-free machine $M$, denoted by $K_M(\sigma)$, is the length of the shortest input $\rho$ such that $M(\rho)$ converges and outputs $\sigma$. There is an optimal prefix-free machine $U$, such that $\sigma \mapsto K_U(\sigma)$ is minimal up to a constant, with respect to all prefix-free machines. We fix an optimal prefix-free machine and let $K(\sigma)$ denote the corresponding prefix-free complexity of $\sigma$. Given $c \geq 0$, we say that $\sigma$ is *c-incompressible* if $K(\sigma) \geq |\sigma| - c$. Schnorr showed that a real is Martin-Löf random if and only if there is a finite upper bound on the randomness deficiency of its initial segments.

We use the *Kraft-Chaitin-Levin theorem* (see [10, §3.6]), which says that if

- $(\ell_\sigma)$ is uniformly approximable from above
- $S \subseteq 2^{<\omega}$ is c.e. and $\sum_{\sigma \in S} 2^{-\ell_\sigma} < 1$

there exists a prefix-free machine $M$ such that $\forall \sigma \in S$, $K_M(\sigma) \leq \ell_\sigma$. We use this theorem to prove the following useful Lemma.

**Lemma 2.1.** Let $(V_n)$ be an uniformly c.e. sequence of sets of strings. If
\[ \sum_{\sigma \in V_n} 2^{-|\sigma|} \leq 2^{-2n}, \]
then there is a constant $c$ such that $K(\sigma) \leq |\sigma| - n + c$ for all $\sigma \in V_n$. 

Proof. We use the Kraft-Chaitin-Levin theorem by setting \( S = \bigcup_n V_n \) and \( l_\sigma = |\sigma| - n \) for \( \sigma \in V_n \). The total weight of requests is
\[
\sum_n \sum_{\sigma \in V_n} 2^{-(|\sigma| - n)} = \sum_n 2^n \sum_{\sigma \in V_n} 2^{-|\sigma|} \leq \sum_n 2^n 2^{-2n} \leq 1,
\]
so by the Kraft-Chaitin-Levin theorem there is a prefix-free machine \( M \) such that \( K_M(\sigma) \leq |\sigma| - n \) for all \( \sigma \in V_n \). Then since \( K \) is minimal, there is a constant \( c \) such that \( K(\sigma) \leq K_M(\sigma) + c \) for all \( \sigma \). Then \( K(\sigma) \leq |\sigma| - n + c \) for all \( \sigma \in V_n \). \( \square \)

Given a prefix-free \( V \subseteq 2^{\omega} \) and \( k \), define \( V^k \) inductively by \( V^1 = V \) and \( V^{k+1} = V^k * V \). We use the following Lemma.

Lemma 2.2. Given prefix-free c.e. \( V \subseteq 2^{\omega} \), \( k > 0 \) and computable increasing \((t_n)\):

(i) if \( V \) contains a prefix of each nonrandom, so does \( V^k \). [13]
(ii) if \( \exists c \forall n : K(x | t_n) > t_n - c \), then \( x \) is random. In fact, there exists \( d \) such that
\[
\forall c (\forall n : K(x | t_n) > t_n - c) \Rightarrow \forall n : K(x | t_n) > n - c - d
\]
for all \( x \), where \( d \) only depends on \((t_i)\). [17]

2.2 Background on trees and randomness

All trees in this paper, unless stated to be finite, are assumed to be infinite and pruned.

A string \( \sigma \) branches in \( T \) if it has at least two incomparable extensions in \( T \). A tree \( T \) is skeletal if there is \( z \in [T] \) such that \( \sigma \in T \) branches in \( T \) if and only if \( \sigma \prec z \). This \( z \) is called the main branch of the skeletal tree. Note that skeletal trees are proper but are neither positive nor perfect.

The \( k \)-prefix of a (finite or infinite) tree \( T \) is \( T | k = T \cap 2^{\leq k} \). Given finite tree \( F \) and (finite or infinite) tree \( T \), write \( F < T \) if \( F = T | k \) for some \( k \). In this paper, all finite trees are also assumed to be “pruned”, in the sense that each of them is a prefix of an infinite pruned tree. For such a finite tree \( F \), the height of \( F \) is the unique \( k \) such that \( F = T | k \) for some \( T \).

The standard topology on the space of all trees \( C \) is generated by the basic open sets \( \ll F \gg = \{ T \in C : F < T \} \) where \( F \) is a finite tree. In the case of \( 2^{\omega} \), this coincides with the hit-or-miss and the Vietoris topologies [16, Appendix B]. We note that \( C \) is homeomorphic to a closed subset of the Cantor space. We only use the fact that \( C \) is compact [16].

The \( \sigma \)-tail of a tree \( T \) is the subtree of \( T \) consisting of the nodes that prefix or extend \( \sigma \).

The following is from [4, Corollary 1.12] and, independently, [12]:

Lemma 2.3. If \( T \) is path-random, every positive computable tree \( W \) contains a tail of \( T \). In particular, every path-random tree has a path-incompressible tail.
3 Perfect versus non-perfect path-incompressible trees

We first prove Proposition 1.3 from §1:

Proposition (Proposition 1.3).

(a) There exists a low path-incompressible tree with two paths which computes a perfect path-random tree.

(b) There exists an effectively proper path-random tree which does not compute any perfect path-random tree.

Proof. For (a), let \( z \) be a low \( \mathsf{PA} \) real. By \([5]\) there exist random \( x, y \) with \( z \equiv_T x \oplus y \). Since every \( \mathsf{PA} \) real truth-table computes a perfect path-incompressible tree, the pruned tree consisting of paths \( x, y \) has the required properties.

For (b), let \( z \) be random relative to \( \emptyset' \) so the pruned tree \( T_z \) determined by

\[
[T_z] = \{ \tau 1 * z : \tau 0 < z \}
\]

is effectively proper and path-random. By the \( \emptyset' \)-randomness of \( z \) and \([4]\), \( z \) does not compute any perfect path-random tree. Since \( T_z \equiv_T z \), the same is true of \( T_z \). \( \square \)

By \([12, 4]\) every perfect path-random tree computes a path-incompressible tree. So Theorem 1.2 is a direct consequence of the following Theorem by letting \( f \) be a function that dominates all computable functions.

Theorem 3.1. For any \( f \), there is a path-incompressible proper tree \( T \) computable in \( \emptyset' \oplus f \), such that \( T \) does not compute any path-incompressible perfect tree within oracle-use \( f \).

Toward the proof of Theorem 3.1, we set up some notations of objects that we shall use in this section. To do this, we introduce the following Lemma, which is Lemma 2.5 in \([3]\) by setting \( m_i = 1 \) and \( L_i = l_i \). With this Lemma we are able to get a tree with fast branching speed inside any non-empty \( \Pi^0_1 \) class.

Lemma 3.2. Let \( P \) be a \( \Pi^0_1 \) class and \( (l_i) \) be an increasing computable sequence of positive integers. If \( \sum 2^{l_i} - l_{i+1} < \mu(P) \), then there is a \( \Pi^0_1 \) class \( Q \subset P \) such that each string \( \sigma \) of length \( l_i \) with \( \llbracket \sigma \rrbracket \cap Q \neq \emptyset \) has at least 2 extensions \( \tau \) of length \( l_{i+1} \) such that \( \llbracket \tau \rrbracket \cap Q \neq \emptyset \).

We fix \( P \) to be the set of reals with deficiency no more than \( d \), where \( d \) is large enough so \( P \) is non-empty. We also fix \( l_i = \sum_{j \in \mathbb{Z}} (2j + c) \) where \( c \) is large enough so the condition of Lemma 3.2 is satisfied. We then fix \( Q \) as guaranteed by Lemma 3.2.

Given a skeletal tree \( T \) with main branch \( z \), we say that \( T \) is \( (l_i) \)-branching if \( |\sigma| = l_i \) and \( \sigma < z \) implies that \( \sigma \) branches in \( T \cap 2^{\leq l_{i+1}} \). Let \( \mathcal{T} \) be the class of all \( (l_i) \)-branching skeletal trees, which is a \( \Pi^0_1 \) class.
Let $\mathcal{T}(Q) = \{ T \in \mathcal{T} : \text{ } [T] \subset Q \}$ and for a finite tree $F$, let $\mathcal{T}(Q, F) = \{ T \in \mathcal{T}(Q) : F < T \}$. Since $Q$ is a $\Pi^0_1$ class of reals, $\mathcal{T}(Q)$ and $\mathcal{T}(Q, F)$ are $\Pi^0_1$ classes of trees. Lemma 3.2 guarantees that $\mathcal{T}(Q)$ is non-empty.

### 3.1 Hitting cost

We would like to use a measure on $C$. The set of all random reals has measure 1 in the Cantor space, but unfortunately the set of all path-incompressible trees has uniform measure 0 in $C$ [4]. Therefore we study and use the hitting cost to serve as an alternative.

**Definition 3.3.** Given a set of trees $G \subset \mathcal{T}(Q)$ and a set of finite strings $H$, we say that $H$ hits $G$ if $T \cap H \neq \emptyset$ for any $T \in G$. The hitting-cost of $G$ relative to $Q$, denoted by $c_Q(G)$, is

$$\inf \mu([H] \cap Q)$$

over all $H$ that hits $G$.

Observe that $T \cap H \neq \emptyset$ if and only if $T \cap [H] \neq \emptyset$, so if $[H_1] = [H_2]$ then $H_1$ hits $G$ if and only if $H_2$ hits $G$. We then use compactness to show that when working with a closed set of trees, we can take everything to be finite.

**Lemma 3.4.** If $G \subset \mathcal{T}(Q)$ is a closed set of trees and $H$ hits $G$, then there is a finite subset of $H$ that hits a clopen superset of $G$.

**Proof.** We first prove that there is a finite subset of $H$ that hits $G$. $G$ is a closed set in a compact space so it is also compact. Then note that

$$H \text{ hits } G \iff \forall T \in G, H \cap T \neq \emptyset$$

$$\iff G \subset \{ T : H \cap T \neq \emptyset \}$$

$$\iff G \subset \bigcup_{\sigma \in H} \{ T : \sigma \in T \}$$

and that each $\{ T : \sigma \in T \}$ is clopen. Now $\bigcup_{\sigma \in H} \{ T : \sigma \in T \}$ is an open cover of $G$, so it contains a finite cover of $G$. The indices of this finite cover is a finite subset of $H$ that hits $G$.

We then assume that $H$ itself is finite, and prove that it hits a clopen superset of $G$. Being a closed set, let $G = C - \bigcup_{i \in I} [F_i]$ where $F_i$ are finite trees. Then

$$H \text{ hits } G \iff G \subset \bigcup_{\sigma \in H} \{ T : \sigma \in T \}$$

$$\iff C - \bigcup_{i \in I} [F_i] \subset \bigcup_{\sigma \in H} \{ T : \sigma \in T \}$$

$$\iff C - \bigcup_{\sigma \in H} \{ T : \sigma \in T \} \subset \bigcup_{i \in I} [F_i]$$

and each $[F_i]$ is also clopen. Now $\bigcup_{i \in I} [F_i]$ is an open cover of $C - \bigcup_{\sigma \in H} \{ T : \sigma \in T \}$, which is clopen thus compact since $H$ is finite, so again $\bigcup_{i \in I} [F_i]$ contains a finite cover $\bigcup_{i \in I} [F_i]$. Then $H$ hits $C - \bigcup_{i \in I} [F_i]$ which is a clopen superset of $G$. □
Lemma 3.5. Let $\mathcal{G} \subset \mathcal{T}(Q)$ be a $\Pi^0_1$ class of trees. If $c_Q(\mathcal{G}) = 0$ then $\mathcal{G} = \emptyset$.

Proof. For each $n$, since $c_Q(\mathcal{G}) = 0$ there is $H_n$ with $\mu(\|H_n\| \cap Q) \leq 2^{-2n-1}$ that hits $\mathcal{G}$. By Lemma 3.4, there is a finite such $H_n$ that hits a clopen superset of $\mathcal{G}$. Since $Q$ is $\Pi^0_1$, there is some stage $s_n$ in the approximation of $Q$ where $\mu(\|H_n\| \cap Q_{s_n}) \leq 2^{-2n}$.

We can effectively check if a finite set of strings hits a clopen set of trees, and the set of all clopen supersets of $\mathcal{G}$ is c.e. since $\mathcal{G}$ is $\Pi^0_1$. Also given finite $H_n$ and $s_n$, we can effectively check if $\mu(\|H_n\| \cap Q_{s_n}) \leq 2^{-2n}$. Therefore given $n$ we can effectively search for such an $H_n$ and $s_n$. In this way we get a computable sequence of $(H_n)$ with $(s_n)$ such that $\mu(\|H_n\| \cap Q_{s_n}) \leq 2^{-2n}$ and that each $H_n$ hits $\mathcal{G}$. Both $\|H_n\|$ and $Q_{s_n}$ are clopen, we let $V_n$ be finite and prefix-free such that $\|V_n\| = \|H_n\| \cap Q_{s_n}$, then $(V_n)$ is uniformly computable. By Lemma 2.1, there is a constant $c$ such that $K(\sigma) \leq |\sigma| - n + c$ for all $\sigma \in V_n$, i.e. each string in $V_n$ has deficiency at least $n - c$.

Now suppose $\mathcal{G} \neq \emptyset$ and take any $T \in \mathcal{G}$. For each $n$, $H_n$ hits $\mathcal{G}$ so $\|H_n\| \cap [T] \neq \emptyset$. Take any $x \in \|H_n\| \cap [T]$. As $[T] \subset Q \subset Q_{s_n}$, $x$ is also in $\|H_n\| \cap Q_{s_n} = \|V_n\|$. So $x$ has a prefix in $V_n$ with deficiency at least $n - c$. But $x$ is also in $Q$, so this prefix has deficiency at most $d$. Taking $n = d + c + 1$ leads to a contradiction. Therefore $\mathcal{G} = \emptyset$. \hfill $\square$

3.2 Tree-functionals

We now clarify how does one tree compute another.

Definition 3.6. A tree functional is a Turing functional $\Phi$ that takes tree $T$ as an oracle and $n \in \omega$ as an input, and outputs a non-empty finite tree $\Phi(T, n)$, such that if $\Phi(T, n - 1)$ and $\Phi(T, n)$ both halt, then

- $\Phi(T, n - 1) \prec \Phi(T, n),$
- each $\sigma \in \Phi(T, n - 1)$ branches in $\Phi(T, n)$.

Let $\Phi(T) = \bigcup_n \Phi(T, n)$.

Now if $\Phi(T, n)$ halts for all $n$, we say that $\Phi$ is total on $T$. In this case, $\Phi(T)$ is an infinite tree. The second condition also makes it automatically perfect. By induction, the second condition also implies that $\Phi(T, n)$ has height at least $n$.

There is a computable enumeration of indices of Turing functionals, such that each index is an index of a tree functional, and that at least one index for each tree functional is enumerated. Let $\Phi_e$ be the $e^{th}$ tree functional in this enumeration.

The oracle-use of $\Phi(T, n)$ is the maximum $k$ such that $T \upharpoonright_k$ is accessed in the computation of $\Phi(T, n)$. By “$\Phi(T, n)$ halts within oracle-use $k$” we mean that $\Phi(T, n)$ halts and the oracle-use of $\Phi(T, n)$ is no more than $k$. Given a function $f : \omega \rightarrow \omega$, by “$\Phi$ is total on $T$ within oracle-use $f$” we mean that $\Phi(T, n)$ halts within oracle-use $f(n)$ for all $n$. 

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Given a string σ and a tree functional Φ, let \( \Phi^{-1}(σ) = \{T : σ ∈ Φ(T)\} \). Clearly \( \Phi^{-1}(σ) \) is \( \Sigma_1^0 \), and the next Lemma shows that it is indeed clopen inside any closed set where Φ is total within bounded oracle-use.

**Lemma 3.7.** Let \( G ⊂ T(Q) \) be a closed set of trees, \( f \) a function and Φ a tree functional such that Φ is total on all \( T ∈ G \) within oracle-use \( f \). Then \( G ∩ \Phi^{-1}(σ) = G ∩ \bigcup_i [F_i] \) where \( (F_i) \) are finitely many trees with height \( f(|σ|) \). In particular, \( G ∩ \Phi^{-1}(σ) \) is closed.

**Proof.** Take any \( T ∈ G ∩ \Phi^{-1}(σ) \). Since \( σ ∈ Φ(T) \) and \( Φ(T, |σ|) \) contains a string with length \(|σ|\) and is a prefix of \( Φ(T) \), we have \( σ ∈ Φ(T, |σ|) \). The oracle-use of \( Φ(T, |σ|) \) is no more than \( f(|σ|) \), so \( σ ∈ Φ(T') \) for any \( T' > T ↾ f(|σ|) \), i.e. \( [T ↾ f(|σ|)] ⊂ Φ^{-1}(σ) \). By taking the union over all \( T \) in \( G ∩ \Phi^{-1}(σ) \), we get

\[
G ∩ \Phi^{-1}(σ) ⊂ \bigcup_{T ∈ G ∩ \Phi^{-1}(σ)} [T ↾ f(|σ|)] ⊂ Φ^{-1}(σ).
\]

Now \( T ↾ f(|σ|) \) is a finite tree and there are only finitely many possibilities for such a tree, so the middle term is \( ∪_i [F_i] \) where \( (F_i) \) are finitely many trees with height \( f(|σ|) \). By intersecting \( G \) with all sets in the above inclusion, we get \( G ∩ \Phi^{-1}(σ) = G ∩ ∪_i [F_i] \), as desired. □

**Lemma 3.8.** Let \( F \) be a finite tree with height \( h \), \( f \) a function and Φ a tree functional such that Φ is total on all \( T ∈ T(Q, F) \) within oracle-use \( f \). Let \( n \) be such that \( l_n ≥ h \) and that \( l_n ≥ f(|σ|) \). If \( H' ∈ 2^{l_{n+1}} \) hits \( T(Q, F) ∩ Φ^{-1}(σ) \), then there is \( H ∈ 2^h \) with

\[
μ([H] − [H']) ∩ Q ≤ 2^{h−l_{n+1}}
\]

that also hits \( T(Q, F) ∩ Φ^{-1}(σ) \).

**Proof.** Let \( H = \{τ ∈ 2^h : μ([τ] − [H']) ∩ Q ≤ 2^{−l_{n+1}}\} \) so \( H ∈ 2^h \) and

\[
μ([H] − [H']) ∩ Q ≤ |H| ∗ 2^{−l_{n+1}} ≤ 2^{h−l_{n+1}}.
\]

To prove that \( H \) hits \( T(Q, F) ∩ Φ^{-1}(σ) \), suppose otherwise there is \( T ∈ T(Q, F) ∩ Φ^{-1}(σ) \) such that \( T ∩ H = ∅ \). Then

\[
τ ∈ T \text{ and } |τ| = l_n \implies τ ∉ H \text{ and } |τ| = l_n
\]

\[
⇒ μ([|τ|] − [H']) ∩ Q > 2^{−l_{n+1}}
\]

\[
⇒ \text{there is } x_1, x_2 ∈ ([|τ|] − [H']) ∩ Q \text{ with } x_1 ↾ l_{n+1} ≠ x_2 ↾ l_{n+1}.
\]

Take \( σ_1 = x_1 ↾ l_{n+1} \) and \( σ_2 = x_2 ↾ l_{n+1} \), then \( σ_1 ≠ σ_2 \) and we shall claim some properties on them. Firstly, since \( [σ_1] ∩ [σ_2] ≠ ∅ \) and \( |τ| = l_n < l_{n+1} = |σ_1| \), we have \( τ < σ_1 \). Also, as \( [σ_1] − [H'] ≠ ∅ \), with the fact that \( |σ_1| = l_{n+1} \) and \( H' ∈ 2^{l_{n+1}} \) we have \( σ_1 ∉ H' \), so \( [σ_1] ∩ [H'] = ∅ \). Finally, together with \( [σ_1] ∩ Q ≠ ∅ \) and \( [σ_1] ∩ (Q − [H']) ≠ ∅ \). The same claims hold for \( σ_2 \).
We then build an \((l_i)\)-branching skeletal tree \(T'\). We start from \(T' = T \upharpoonright l_n\) and let \(z\) be the main branch of \(T\). For each \(\tau \in T\) and \(|\tau| = l_n\), we take \(\sigma_1\) and \(\sigma_2\) extending \(\tau\) as in the previous argument. We first use Lemma 3.2 to extend \(\sigma_1\) to a real in \(Q - \|H'\|\), and put all initial segments of the real in \(T'\). Additionally, if \(\tau = z \upharpoonright l_n\), we also use Lemma 3.2 to extend \(\sigma_2\) to an \((l_i)_{i>\omega+1}\)-branching skeletal tree whose paths are in \(Q - \|H'\|\). We add the strings in this skeletal tree into \(T'\).

We can verify that \(T'\) is indeed \((l_i)\)-branching. Additionally \(T' > T \upharpoonright l_n > F\) and \([T'] \subseteq Q\), so \(T' \in T(Q, F)\). Since \([T \upharpoonright l_n] \subseteq \Phi^{-1}(\sigma)\) we also have \(T' \in \Phi^{-1}(\sigma)\). But \([H'] \cap [T'] = \emptyset\), contradicting the fact that \(H'\) hits \(T(Q, F) \cap \Phi^{-1}(\sigma)\).

\[\square\]

3.3 Envelopes

In this subsection, we fix a finite tree \(F\) with height \(h\), a function \(f\) and a tree functional \(\Phi\) such that \(\Phi\) is total on all \(T \in T(Q, F)\) within oracle-use \(f\).

This subsection is devoted to establish the fact that if the hitting cost of \(T(Q, F) \cap \Phi^{-1}(\sigma)\) is small, then \(c_Q(T(Q, F)) = 0\). We split this task into 3 steps. Firstly, if the hitting cost of \(T(Q, F) \cap \Phi^{-1}(\sigma)\) is small, then a well-behaved finite family (which we shall call a finite envelope) hits them. Then we use compactness argument to extend this family to an infinite one. Finally we use this infinite family to generate sets that hit \(T(Q, F)\), showing that \(c_Q(T(Q, F)) = 0\).

We let \(l_\tau\) be the first \(l_n\) in \((l_i)\) such that \(l_n \geq h\), \(l_n \geq f(|\sigma|)\) and that \(l_n > l_\tau\) where \(\tau\) is any proper prefix of \(\sigma\). We note that \(l_\tau\) is only determined by \(|\sigma|\), not \(\sigma\) itself. Also the last requirement implies that \(n \geq |\sigma|\).

Definition 3.9. An \(r\)-envelope is a family \((H_\sigma)\) indexed by \(\sigma \in 2^{<\omega}\) such that for all \(\sigma\),

1. \(\mu([H_\sigma] \cap Q) \leq 2^{-|\sigma|} + \sum_{l \leq l_\tau} 2^{l-l_\tau+1}\),
2. \(H_\sigma\) hits \(T(Q, F) \cap \Phi^{-1}(\sigma)\),
3. \(H_\tau \subset 2^{<l_\tau}\),
4. \(\mu((H_\sigma) - [H_0] - [H_\tau]) \cap Q) \leq \sum_{l \leq l_\tau} 2^{l-l_\tau+1}\).

A finite \(r\)-envelope is defined similarly, except that it is indexed by \(\sigma \in 2^{<n}\) for some \(n\), which is referred to as the length of the envelope.

Lemma 3.10. If there is a constant \(r\) such that \(c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) \leq 2^{-|\sigma|}\) for all \(\sigma\), then there are finite \(r\)-envelopes of any length \(n\).

Proof. For each \(|\sigma| = n\), there is \(H'_\sigma\) with \(\mu([H'_\sigma] \cap Q) \leq 2^{-|\sigma|}\) that hits \(T(Q, F) \cap \Phi^{-1}(\sigma)\). By Lemma 3.7, \(T(Q, F) \cap \Phi^{-1}(\sigma)\) is closed, so by Lemma 3.4 we can choose this \(H'_\sigma\) to be finite. Let \(L\) be in \((l_i)\) and longer than any string in this finite \(H'_\sigma\). Then \(\tau \in 2^L : \|\tau\| \in [H'_\sigma]\)
has the same measure as \( H'_\sigma \) and also hits \( T(Q, F) \cap \Phi^{-1}(\sigma) \). By repeatedly applying Lemma 3.8, there is \( H_\sigma \subset 2^L \) with

\[
\mu([H_\sigma] \cap Q) \leq \mu([H'_\sigma] \cap Q) + \sum_{l_0 \leq l \leq L} 2^{l-l_{i+1}}
\]

that also hits \( T(Q, F) \cap \Phi^{-1}(\sigma) \). Thus the family \( H_\sigma \) satisfies conditions 1, 2 and 3.

We extend these \( H_\sigma \) to a finite envelope of length \( n \). Inductively for all \( |\sigma| < n \) (from \( |\sigma| = n - 1 \) to \( |\sigma| = 0 \)), let \( H \) be such that \( [H] = [H_{r0}] \cup [H_{r1}] \).

We claim that \( H \) hits \( T(Q, F) \cap \Phi^{-1}(\sigma) \). To see this, observe that if \( T \in T(Q, F) \cap \Phi^{-1}(\sigma) \), then since \( \sigma \in \Phi(T) \) and \( \Phi \) is total on \( T \), at least one of \( \sigma 0 \) or \( \sigma 1 \) is in \( \Phi(T) \). So

\[
T(Q, F) \cap \Phi^{-1}(\sigma) \subset (T(Q, F) \cap \Phi^{-1}(\sigma 0)) \cup (T(Q, F) \cap \Phi^{-1}(\sigma 1)).
\]

Therefore by the fact that \( H_{r0} \) hits \( T(Q, F) \cap \Phi^{-1}(\sigma 0) \) and \( H_{r1} \) hits \( T(Q, F) \cap \Phi^{-1}(\sigma 1) \), we have that \( H_{r0} \cup H_{r1} \) hits \( T(Q, F) \cap \Phi^{-1}(\sigma) \), and so does \( H \).

Again by repeatedly applying Lemma 3.8, there is \( H_\sigma \subset 2^L \) with

\[
\mu([H_\sigma] - [H]) \cap Q) \leq \sum_{l_0 \leq l \leq L} 2^{l-l_{i+1}}
\]

that also hits \( T(Q, F) \cap \Phi^{-1}(\sigma) \). Thus conditions 2, 3 and 4 are satisfied. Condition 1 is satisfied for \( |\sigma| = n \), and for \( |\sigma| < n \) it is automatically implied by condition 4.

**Lemma 3.11.** If there are finite \( r \)-envelopes for any length \( n \), then there is an infinite \( r \)-envelope.

**Proof.** By the assumption, for each \( n \) let \( H_n = (H_{n, \sigma}) \) be a finite \( r \)-envelope of length \( n \). We define another sequence of finite \( r \)-envelopes \( H'_n \) of length \( n \) such that \( H'_{n-1} \) is the restriction of \( H'_n \) to \( \sigma \in 2^{L_n-1} \). Then \( H'_n \) is an infinite \( r \)-envelope.

To choose this sequence, we start from \( H'_0 \) being the empty envelope of length 0. In each stage \( n \), find \( H'_n \) extending \( H'_{n-1} \) such that \( H'_n \) is the restriction of \( H_m \) for infinitely many \( m \). Such an extension can always be found since there are only finitely many choices for a finite envelope of a certain length, but infinitely many restrictions of \( H_m \).

Before proceeding, we simplify the term in condition 1 of an envelope. If \( l_\sigma = l_n \) then

\[
\sum_{l_0 \leq l} 2^{l-l_{i+1}} = \sum_{i=n}^{\infty} 2^{-l_{i+1}} = \sum_{i=n}^{\infty} 2^{-2l-c-1} = \frac{2^{-2n-c+1}}{3} \leq 2^{-|\sigma|-c} \leq 2^{-|\sigma|},
\]

therefore condition 1 implies that \( \mu([H_\sigma] \cap Q) \leq 2^{1+|\sigma|} \).

**Lemma 3.12.** If there is an infinite \( r \)-envelope \( (H_\sigma) \) then \( c_Q(T(Q, F)) = 0 \).
Proof. For any real \( z \), let \( \text{hit}(z, n) = |\{ \sigma \in 2^n : z \in [H_\sigma] \}| \).

As \([H_\sigma]\) could contain more than \([H_{\sigma_0}] \cup [H_{\sigma_1}]\), \( \text{hit}(z, n) \) is generally not non-decreasing in \( n \). However, by condition 4 of the envelope, the measure of all \( z \in Q \) such that \( \text{hit}(z, n) \) is decreasing at \( n = |\sigma| \) is no more than \( 2^n \sum_{l_0 \leq l < l_0} 2^{l-l_0} \). Fix any \( m \). Let \( V_m \) be the set of all \( z \) such that \( \text{hit}(z, n) \) is decreasing at any \( n \geq m \), then

\[
\mu(V_m \cap Q) \leq \sum_{n \geq m} 2^n \sum_{l_0 \leq l < l_0} 2^{l-l_0} \leq \sum_{l \geq m} 2^{2l-\epsilon-1} \sum_{n \leq l} 2^n \leq 2^{1-m-\epsilon}.
\]

For \( z \notin V_m \), \( \text{hit}(z, n) \) is non-decreasing on \( n \geq m \), so we can set \( \text{hit}(z) = \lim_n \text{hit}(z, n) \). Let

\[
U^n_k = \{ z : \text{hit}(z, n) \geq 2^k \} \quad \text{and} \quad U_{k,m} = \bigcup_{n \geq m} U^n_k.
\]

For a particular \( n \), consider the sum

\[
\sum_{\sigma \in 2^n} \mu([H_\sigma] \cap Q) \leq 2^{1+r}
\]

and note that each set of reals contributing its measure to \( U^n_k \cap Q \) also contributes \( 2^k \) times its measure to the above sum, so \( \mu(U^n_k \cap Q) \leq 2^{1+r-k} \). Note that \( U^n_k - V_m \) is non-decreasing for \( n \geq m \), so

\[
\mu((U_{k,m} - V_m) \cap Q) \leq \lim_n \mu(U^n_k \cap Q) \leq 2^{1+r-\epsilon}.
\]

Note that \( V_m \) and \( U_{k,m} \) are open. Let \( H \) be finite, prefix-free and such that \([H] = V_m \cup U_{k,m}\), we claim that \( H \) hits \( \mathcal{T}(Q, F) \), so

\[
c_\mathcal{Q}(\mathcal{T}(Q, F)) \leq \mu((V_m \cup U_{k,m}) \cap Q) \leq 2^{1-m-\epsilon} + 2^{1+r-\epsilon},
\]

and since \( m \) and \( k \) are arbitrary we have \( c_\mathcal{Q}(\mathcal{T}(Q, F)) = 0 \). To prove the claim, suppose otherwise there is \( T \in \mathcal{T}(Q, F) \) such that \([T] \cap V_m = \emptyset \) and \([T] \cap U_{k,m} = \emptyset \). Then for any \( z \in [T] \), \( \text{hit}(z) \) is defined, so \( \text{hit}(z, n) \) is bounded.

\( T \) is a skeletal tree, let \( z_0 \) be its main branch. \( \Phi(T) \) is perfect and \( \text{hit}(z_0, n) \) is bounded, so there is \( \sigma_0 \in \Phi(T) \) with \( z \notin [H_{\sigma_0}] \). Since \( H_{\sigma_0} \) is finite, there is \( \tau < z \) such that \([\tau] \cap [H_{\sigma_0}] = \emptyset \).

There are only finitely many paths \( z_1, \cdots, z_n \) in \([T]\) not prefixed by \( \tau \). Inductively in \( k \) (from 1 to \( n \)), since \( \text{hit}(z_k, n) \) is bounded, there is \( \sigma_k \in \Phi(T) \) with \( \sigma_k > \sigma_{k-1} \) and \( z_k \notin [H_{\sigma_k}] \). Let \( \sigma = \sigma_n \). Now \([H_{\sigma_0}] \supset [H_{\sigma_1}] \supset \cdots \supset [H_{\sigma_n}] = [H_\sigma] \).

For a path in \([T]\) prefixed by \( \tau \), since \([\tau] \cap [H_{\sigma_0}] = \emptyset \) it is not in \([H_\sigma]\). For a path \( z_k \) in \([T]\) not prefixed by \( \tau \), since \( z_k \notin [H_{\sigma_k}] \), it is not in \([H_\sigma]\). So \([T] \cap [H_\sigma] = \emptyset \). But \( H_\sigma \) hits \( \mathcal{T}(Q, F) \cap \Phi^{-1}(\sigma) \), and indeed \( \sigma \in \Phi(T) \) so \( T \in \Phi^{-1}(\sigma) \), a contradiction. \( \square \)
3.4 Proof of Theorem 3.1

Combining Lemma 3.10, Lemma 3.11, Lemma 3.12 and Lemma 3.5, we get the following.

Lemma 3.13. Let $F$ be a finite tree, $f$ a function and $\Phi$ a tree functional such that $\Phi$ is total on all $T \in T(Q, F)$ within oracle-use $f$. If there is a constant $r$ such that $c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) \leq 2^{-|\sigma|}$ for all $\sigma$, then $T(Q, F) = \emptyset$.

To prove Theorem 3.1 we need a final Lemma.

Lemma 3.14. Let $F$ be a finite tree, $f$ a function and $\Phi$ a tree functional such that $\Phi$ is total on all $T \in T(Q, F)$ within oracle-use $f$. If there is no constant $r$ such that $c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) \leq 2^{-|\sigma|}$ for each $\sigma$, then for each $m$, there is some $T \in T(Q, F)$ such that $\Phi(T)$ has deficiency at least $m$.

Proof. For each $n$, enumerate $V_n$ by the following loop, starting with $s = 0$.

1. Find $t$ and $\sigma$ with $t > s$ and $|\sigma| > 2n$, such that $[\sigma] \cap V_{n,s} = \emptyset$ and $$c_Q(T(Q_s, F) \cap \Phi_t^{-1}(\sigma)) > 2^{2n+1-|\sigma|}.$$ Enumerate the least such $\sigma$ into $V_n$.

2. Wait for a stage $r$ such that $T(Q_r, F) \cap \Phi_r^{-1}(\sigma) = \emptyset$. Set $s := r$ and go back to step 1.

Consider a single loop. From $T(Q_s, F) \cap \Phi_t^{-1}(\sigma)$ to $T(Q_r, F) \cap \Phi_t^{-1}(\sigma) = \emptyset$ only trees that have a path in $Q_s - Q_r$ are removed, so $Q_s - Q_r$ hits $T(Q_s, F) \cap \Phi_t^{-1}(\sigma)$. Therefore $$\mu(Q_s - Q_r) = \mu((Q_s - Q_r) \cap Q_s) > 2^{2n+1-|\sigma|},$$ so $2^{-|\sigma|} \leq 2^{-2n} \mu(Q_s - Q_r)$. Taking the sum over all loops, taking into account that the last loop could be incomplete (where $r$ is never found) and that $|\sigma| > 2n$, we have $$\sum_{\sigma \in V_n} 2^{-|\sigma|} \leq 2^{-2n-1} + 2^{-2n-1} = 2^{-2n}.$$ Now $V_n$ is uniformly c.e., using Lemma 2.1 there is a constant $c$ such that $K(\sigma) \leq |\sigma| - n + c$ for all $\sigma \in V_n$. Then $K(\sigma) \leq |\sigma| - m$ for all $\sigma \in V_{m+c}$, i.e. each string in $V_{m+c}$ has deficiency at least $m$.

There is $\sigma$ with $|\sigma| > 2n$ such that $c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) > 2^{2(m+c)+1-|\sigma|}$, since otherwise by taking $r$ the maximum of $2(m+c) + 1$ or the constant needed for all $|\sigma| \leq 2n$, we have $c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) \leq 2^{-|\sigma|}$ which contradicts the condition.

As $Q_s \supset Q$ we have $c_{Q_s} \geq c_{Q_r}$, so $c_Q(T(Q, F) \cap \Phi^{-1}(\sigma)) > 2^{2(m+c)+1-|\sigma|}$. Eventually such a $\sigma$ is enumerated into $V_{m+c}$, and then eventually the construction of $V_{m+c}$ gets stuck at step 2. Therefore $T(Q, F) \cap \Phi^{-1}(\sigma) \neq \emptyset$, i.e. there is $T \in T(Q, F)$ with $\sigma \in \Phi(T)$. With $\sigma \in V_{m+c}$, $\Phi(T)$ has deficiency at least $m$, as desired. $\square$

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Finally we can prove Theorem 3.1.

Proof of Theorem 3.1. We build the tree from $F_{-1} = \{ \lambda \}$ which is extendable in $\mathcal{T}(Q)$, and for each index $e$ and constant $m$, we extend the tree $F_{(e,m)-1}$ to some $F_{(e,m)}$ that is extendable in $\mathcal{T}(Q)$, satisfying

$$R_{e,m}: \text{for any tree } T > F_{(e,m)}, \text{ if } \Phi_e \text{ is total on } T \text{ within oracle-use } f, \text{ then } \Phi_e(T) \text{ has deficiency at least } m.$$ 

Finally we let $T = \bigcup_i F_i$. Then for any $e$, if $\Phi_e$ is total on $T$ within oracle-use $f$, then $\Phi_e(T)$ has infinite deficiency. Thus this $T$ does not compute any path-incompressible perfect tree within oracle-use $f$, as desired.

Now we show how to extend $F_{(e,m)-1}$ to some $F_{(e,m)}$ extendable in $\mathcal{T}(Q)$, satisfying $R_{e,m}$. For simplicity write $F = F_{(e,m)-1}$. One of the following two cases happens.

- $\Phi_e$ is not total on some $T \in \mathcal{T}(Q,F)$ within oracle-use $f$.

There is some $n$ such that $\Phi_e(T,n)$ does not halt within oracle-use $f(n)$. Let $F' = T \upharpoonright f(n)$, then $F < F' < T$ and for any $T' > F'$, $\Phi_e(T')$ does not halt within oracle-use $f(n)$. Therefore there is $n$ and $F'$ such that

1. $F' > F$ and $F'$ is extendable in $\mathcal{T}(Q)$, i.e. $\mathcal{T}(Q,F') \neq \emptyset$,

2. $\Phi_e(F',n)$ uses oracle no more than $F'$ and does not halt.

Using oracle $\emptyset'$ and $f$, we can effectively check if $n$ and $F'$ witness that this case happens. If we found such $n$ and $F'$, let $F_{(e,m)} = F'$, then for any $T > F_{(e,m)}$, $\Phi_e(T)$ is not total within oracle-use $f$.

- $\Phi_e$ is total on all $T \in \mathcal{T}(Q,F)$ within oracle-use $f$.

By Lemma 3.13, since $\mathcal{T}(Q,F)$ is non-empty by the previous steps of the construction, there is no constant $r$ such that $c_Q(\mathcal{T}(Q,F) \cap \Phi_e^{-1}((\sigma))) \leq 2^{-|\sigma|}$. Then by Lemma 3.14, there is some $T \in \mathcal{T}(Q,F)$ such that $\Phi_e(T)$ has deficiency at least $m$, i.e. it contains some $\sigma$ with deficiency at least $m$. Then again there is some $n$ such that $\sigma \in \Phi_e(T,n)$, and similarly there is $n$ and $F'$ such that

1. $F' > F$ and $F'$ is extendable in $\mathcal{T}(Q)$, i.e. $\mathcal{T}(Q,F') \neq \emptyset$,

2. $\Phi_e(F',n)$ uses oracle no more than $F'$ and has deficiency at least $m$.

Also using oracle $\emptyset'$, we can effectively check if $n$ and $F'$ witness that this case happens. If we found such $n$ and $F'$, let $F_{(e,m)} = F'$, then for any $T > F_{(e,m)}$, $\Phi_e(T)$ has deficiency at least $m$.

$\square$
4 Perfect path-incompressible trees

We examine the density of branching in path-incompressible trees, and the possibility of effectively increasing it. The density of branching can be formalized in terms of a computable increasing sequence \((\ell_n)\) as follows.

**Definition 4.1.** Given computable increasing \(\ell = (\ell_n)\), tree \(T, z \in [T]\) we say that

- \(T\) is \(\ell\)-perfect if for almost all \(n\), each \(\sigma \in T \cap 2^{\ell_n}\) has \(\geq 2\) extensions in \(T \cap 2^{\ell_{n+1}}\)
- \(z\) is \((\ell, T)\)-branching if for almost all \(n\), each \(z \upharpoonright \ell_n\) has \(\geq 2\) extensions in \(T \cap 2^{\ell_{n+1}}\),

where almost all \(n\) means all but finitely many \(n\).

We use the following assembly of items from [3]:

**Lemma 4.2.** Let \(\ell = (\ell_i)\) be computable and increasing, and \(P\) be a positive \(\Pi^0_1\) tree with \([P]\) consisting entirely of randoms.

(i) If \(\sum_i 2^{-(\ell_{i+1}-\ell_i)} = \infty\), every \((\ell, P)\)-branching \(z\) is incomplete; also there are arbitrarily large \(n\) and \(\sigma \in 2^{\ell_n} \cap P\) which have exactly one extension in \(2^{\ell_{n+1}} \cap P\).

(ii) If \(\sum_i 2^{-(\ell_{i+1}-\ell_i)} < \infty\), there exists an \(\ell\)-perfect tree \(T \subseteq P\) and an injection \(z \mapsto f(z) \in [T]\) such that \(z \leq_T f(z)\).

**Proof.** The first part of (i) is by [3, Lemma 2.8], taking into account that a random is difference-random iff it does not compute \(0'\). The second part of (i) is by [3, Lemma 2.4 & Corollary 2.9]. Clause (ii) is [3, Lemma 2.2].

**4.1 Density of branching in path-incompressible trees**

We characterize the density of branching that a perfect path-random tree can have:

**Theorem 4.3.** Given computable increasing \(\ell = (\ell_n)\), the following are equivalent:

(a) \(\exists\) an \(\ell\)-perfect path-random tree

(b) \(\exists\) an \(\ell\)-perfect path-incompressible tree

(c) \(\sum_n 2^{-(\ell_{n+1}-\ell_n)} < \infty\).

**Proof.** Implication (b)→(a) is trivial, while (c)→(b) follows from Lemma 4.2 (ii).

By Lemma 2.3 we get (a)→(b). For ¬(c)→¬(b), let \(\mathcal{F}_T\) be the class of trees \(T\) such that each \(\sigma \in T \cap 2^n\) has exactly two extensions in \(T \cap 2^{\ell_{n+1}}\). Let \(P\) be a \(\Pi^0_1\) pruned tree \(P\) of finite deficiency and let \(\mathcal{F}_T(P)\) be the restriction of \(\mathcal{F}_T\) to trees \(T\) with \(T \subseteq P\). Then

(i) \(Q := \{\sigma : \exists T \in \mathcal{F}_T(P), \sigma \in T\}\) is a \(\Pi^0_1\) subtree of \(P\), by compactness.
(II) each \( \sigma \in Q \cap 2^f_n \) has at least two extensions in \( Q \cap 2^f_{n+1} \).

If \( F_T(P) \neq \emptyset \) then \( Q \) is infinite; so for \( \neg (c) \Rightarrow \neg (b) \) it remains to show that \( Q \) is finite. Assuming otherwise, by \( \neg (c) \) and the second clause of Lemma 4.2(i), there exists \( n \) and \( \sigma \in Q \cap 2^f_n \) which has at most one extension in \( 2^f_{n+1} \cap Q \). But this contradicts (II) above. \( \square \)

Next, we consider the branching density along a path in a path-incompressible \( \Pi^0_1 \) tree \( P \). It is known that Turing-hard members of \( P \) have low density in \( P \), hence sparse branching.

**Definition 4.4.** Given \( Q \subseteq 2^\omega \), the *density of \( z \) in \( Q \) is given by*

\[
\rho(Q \mid z) := \liminf_n \mu(Q \mid \|z \restriction_n\|) = \liminf_n 2^n \cdot \mu(Q \cap \|z \restriction_n\|).
\]

A real is a *positive-density point* if it has positive-density in every \( \Pi^0_1 \) tree that it belongs to.

Bienvenu et al. [8] showed that a random \( z \) is a positive-density point iff \( z \not\in_T \emptyset' \). Positive-density random reals are not necessarily *density-1 reals* (density tends to 1) and the complete characterization of density-1 reals is an open problem. However a positive-density random real can have arbitrarily high density, in an appropriately chosen \( \Pi^0_1 \) tree \( P \):

**Lemma 4.5.** If \( z \) is a positive-density random real and \( \epsilon > 0 \), there exists \( \Pi^0_1 \) pruned tree \( P \) of finite deficiency such that \( z \in P \) and the \( P \)-density of \( z \) is \( > 1 - \epsilon \).

**Proof.** Since \( z \) is random, \( \exists c := \forall n K(z \restriction_n) \geq n - c \). Let \( V_0 \) be a c.e. prefix-free set such that

\[ \|V_0\| = \|(\sigma : K(\sigma) < |\sigma| - c)\| \]

and let \( P_0 \) be the \( \Pi^0_1 \) pruned tree consisting of the strings with no prefix in \( V_0 \). Then \( z \in [P_0] \) and since \( z \) is a positive-density real, there exists \( \delta > 0 \) such that the \( P_0 \)-density of \( z \) is \( > \delta \), so

\[ \forall \tau < z : \mu_\tau(V_0) < 1 - \delta. \]

Let \( k \) be such that \( (1 - \delta)^k < \epsilon \) and let \( V := (V_0)^k \), so for each \( \tau \in P_0 \), \( \mu_\tau(V) \leq (\mu_\tau(V_0))^k < \epsilon \).

By Lemma 2.2 (i), \( V \) is c.e. and contains a prefix of every non-random real. Let \( P \) be the pruned \( \Pi^0_1 \) tree such that \( |P| = 2^\omega - |V| \).

Then \( P \supseteq P_0, z \in P \), and \( P \) contains only random reals. So \( \mu_\tau(P) > 1 - (1 - \mu_\tau(P_0))^k \) and

\[ \tau < z \Rightarrow \mu_\tau(P) > 1 - (1 - \mu_\tau(P_0))^k > 1 - (1 - \delta)^k > 1 - \epsilon \]

which shows that the density of \( z \) in \( P \) is \( > 1 - \epsilon \). \( \square \)

We now show a gap theorem: an incomplete random real can be everywhere branching inside some path-incompressible \( \Pi^0_1 \) tree \( P \), but the branching density of a Turing-hard random real in such \( P \) is precisely and considerably more sparse.
Theorem 4.6. Given computable increasing \( \ell = (\ell_n) \), the following are equivalent:

(a) \( \forall \) path-incompressible \( \Pi^0_1 \) tree \( P \), \( \exists (\ell, P) \)-branching \( z \geq_T \emptyset' \)

(b) \( \exists \Pi^0_1 \) path-incompressible tree \( P \) and \( (\ell, P) \)-branching \( z \geq_T \emptyset' \)

(c) \( \sum_n 2^{-(\ell_{n+1} - \ell_n)} < \infty \).

If \( z \not\geq_T \emptyset' \) is random, \( \ell_n = n \), \( \exists \) path-incompressible \( \Pi^0_1 \) tree \( P \) such that \( z \) is \( (\ell, P) \)-branching.

Proof. By Lemma 4.2 (ii) we get (c)\( \rightarrow \)(a) while (a)\( \rightarrow \)(b) is trivial. By Lemma 4.2 (i) we get (b)\( \rightarrow \)(c). For the last clause note that if \( z \) has density \( > \frac{1}{2} \) in \( P \), it is \( (\ell, P) \)-branching for \( \ell_n = n \). Hence the last clause follows from the characterization of incomplete reals as positive-density points by Bienvenu et al. [8], and Lemma 4.5. \( \Box \)

4.2 Increasing the density of branching

We are interested in effectively transforming a perfect path-incompressible tree into one with more dense branching, without significant loss in the deficiency. To this end, we give a positive answer on certain conditions.

Let \( (\ell_n) \) increasing and computable, and let \( T_\ell \) be the pruned trees such that each \( \sigma \in T \cap 2^{\leq \ell_n} \) has one or two extensions in \( T \cap 2^{\leq \ell_{n+1}} \). The uniform measure \( \nu \) on \( T_\ell \) is induced by

\[
\nu([T \upharpoonright \ell_n]) = \frac{1}{|T_{\ell_n}|} \quad \text{for } T \in T_\ell, \text{ where } T_{\ell_n} := \{ T \upharpoonright \ell_n : T \in T_\ell \}
\]

where \( T \upharpoonright \ell_n := T \cap 2^{\leq \ell_n} \) and \([T \upharpoonright \ell_n] \) denotes the set of trees in \( T_\ell \) that have \( T \upharpoonright \ell_n \) as a prefix.

Theorem 1.5 is a special case of the following, for \( m_n = n^2 \).

Theorem 4.7. Let \( \ell = (\ell_n), m = (m_n) \) be computable and increasing such that

\[
\ell_{n+1} - \ell_n \geq m_{n+1} - m_n \quad \text{and} \quad \sum_n 2^{-(m_{n+1} - m_n - n)} < \infty.
\]

There exists a truth-table map \( \Phi : T_\ell \rightarrow T_m \) such that for \( T \in T_\ell \):

- if \( T \) is path-incompressible, so is \( \Phi(T) \)
- with \( \nu \)-probability 1, \( T \) is \( \ell \)-perfect and \( \Phi(T) \) is \( m \)-perfect.

Toward the proof, we need to be specific regarding the deficiency of the trees, so consider a \( \Pi^0_1 \) pruned tree containing the \( c \)-incompressible reals:

\[
P_c = \{ \sigma \mid \forall \rho \leq \sigma, K(\rho) \geq |\rho| - c \} \quad \text{and} \quad T_\ell(P_c) := \{ T \in T_\ell : [T] \subseteq [P_c] \}.
\]

For Theorem 4.7 it suffices to define a truth-table \( \Phi : T_\ell \rightarrow T_m \) such that for \( T \in T_\ell \):

(a) if \( T \) is path-incompressible, so is \( \Phi(T) \): \( \exists d \forall c \Phi(T_\ell(P_c)) \subseteq T_m(P_{c+d}) \)
(b) with \( \nu \)-probability 1, \( T \) is \( \ell \)-perfect and \( \Phi(T) \) is \( m \)-perfect

The required map \( \Phi \) will be defined by means of a family of sets of strings.

**Definition 4.8.** Given increasing \( m = (m_i), \ell = (\ell_i) \), a \((m, \ell)\)-family \( H \) is a family \( (H_\sigma) \) of finite subsets of \( 2^{<\omega} \) indexed by the \( \sigma \in \{ 2^{m_n} : n \in \mathbb{N} \} \) such that for each \( \sigma \in 2^{m_n}, \tau \in 2^{m_{n+1}}:\n\)

\[
(\sigma < \tau \Rightarrow \|H_\tau\| \leq \|H_\sigma\|) \land H_\sigma \subseteq 2^{\ell_n} \land \mu(H_\sigma) \leq 2^{-|\sigma|}.
\]

Given an \((m, \ell)\)-family \( H \) define the \((H, m, \ell)\)-map: \( \Phi(T; n) = \{ \sigma \in 2^{m_n} : T \cap H_\sigma \neq \emptyset \} \).

If \( \sigma \mapsto H_\sigma \) is computable, \( \Phi \) defines a truth-table map from \( T_\ell \) to \( T_m \).

It remains to define a computable \((m, \ell)\)-family \( H := (H_\sigma) \) such that conditions (a), (b) above hold for the corresponding truth-table map \( \Phi \).

**Construction.** Let \( H_\lambda = \{ \lambda \} \) and inductively assume that \( H_\sigma, \sigma \in 2^{m_i}, i < n \) have been defined. For each \( \sigma \in 2^{m_{n-1}}, \rho \in 2^{m_n-m_{n-1}} \) let

\[
H_{\sigma\rho} := \bigcup_{\tau \in H_\sigma} \{ \tau' \in 2^{\ell_n} : \tau * \rho < \tau' \},
\]

so \( \mu(H_{\sigma\rho}) = \mu(H_\sigma) \cdot 2^{-|\rho|}, \mu(H_\sigma) = 2^{-|\rho|} \). Let \( \Phi \) be the \((H, m, \ell)\)-map, so

\[
\sigma \in \Phi(T) \iff H_\sigma \cap T \neq \emptyset.
\]

Let \( \Phi \) be the truth-table functional induced by \((H_\sigma)\).

**Verification.** Toward (a), consider a prefix-free machine \( M \) such that

\[
\forall n, c \forall \sigma \in 2^{m_n} : (K(\sigma) \leq |\sigma| - c \Rightarrow \forall \tau \in H_\sigma : K_M(\tau) \leq |\tau| - c).
\]

Such \( M \) exists by the Kraft-Chaitin-Levin theorem (see [10, §3.6]) since \( \mu(H_\sigma) = 2^{-|\sigma|} \), so the weight of its descriptions is bounded by 1. Let \( d \) be such that \( K \leq K_M + d \) so

\[
K(\sigma) \leq |\sigma| - c - d \Rightarrow \forall \tau \in H_\sigma : K(\tau) \leq |\tau| - c
\]

for each \( n, c \) and \( \sigma \in 2^{m_n} \). Hence for all \( \tau \):

\[
\left( \tau \in T \cap 2^{\ell_n} \land K(\tau) > |\tau| - c \right) \Rightarrow \forall \sigma \in \Phi(T) \cap 2^{m_n}, K(\sigma) > |\sigma| - c - d.
\]

Since \((m_n), (\ell_n)\) are increasing and computable, by Lemma 2.2 (ii) this proves (a).

For (b), we first show that the \( \nu \)-probability of \( T \) containing an isolated path is 0. The probability that \( \sigma \in T \cap 2^{\ell_n} \) does not branch at the next level is the probability that two independent trials with replacement pick the same extension, which is \( 2^{-(\ell_{n+1} - \ell_n)} \). Since \( |T \cap 2^{\ell_n}| \leq 2^n \), the probability that this occurs for some \( \sigma \in T \cap 2^{\ell_n} \) is \( 2^{n-(\ell_{n+1} - \ell_n)} \). By the hypothesis

\[
\sum_n 2^{-(\ell_{n+1} - \ell_n - n)} < \infty
\]
so by the first Borel-Cantelli lemma, with probability 1 the tree $T$ is $\ell$-perfect.

For (b), it remains to show that the $\nu$-probability of $\Phi(T)$ containing an isolated path is 0. If $\sigma \in \Phi(T) \cap 2^{m_n}$ does not branch at the next level, the corresponding $\tau \in H_\sigma \cap T$ gets both of its two extensions from the same $H_{\sigma, \rho}, \rho \in 2^{m_{n+1}-m_n}$. The probability of this event $E_\tau$ is $2^{-(m_{n+1}-m_n)}$ and, since $|T \cap 2^\ell| \leq 2^n$, the probability that $E_\tau$ occurs for some $\tau \in T \cap 2^\ell$ is $\leq 2^{n-(m_{n+1}-m_n)}$. By the hypothesis there exists $b$ such that

$$\sum_n 2^{-(m_{n+1}-m_n-n)} < b$$

so the probability that the above event occurs in more than $2^c$ many levels of $T$ is $\leq b \cdot 2^{-c}$. By the first Borel-Cantelli lemma, with probability 1, $\Phi(T)$ is $m$-perfect. This completes the verification of (b) and the proof of Theorem 4.7.

5 Conclusion and discussion

We studied the extent to which the branching in a path-incompressible tree can be effectively altered, without significant deficiency increase. We showed that there is a path-incompressible proper tree that does not compute any path-incompressible prefect tree with a computable upper bound on the oracle-use.

We also explored the limits of effective densification of perfect path-incompressible trees, and in this context the following question seems appropriate: given computable increasing $\ell = (\ell_n), m = (m_n)$ with $\ell_n \gg m_n \gg n^2$, is there an $\ell$-perfect path-incompressible tree which does not compute any $m$-perfect path-incompressible tree?

Our methodology relied on the use of hitting-families of open sets, for expressing maps from trees to trees. We suggest that this framework can give analogous separations between classes of trees of different Cantor-Bendixson rank. Applications are likely in the study of compactness in fragments of second-order arithmetic [12, 4, 9, 11].

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