Interpolation of data by smooth non-negative functions

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Introduction

Continuing from [18], we prove a finiteness principle for interpolation of data by nonnegative $C^m$ functions. Our result raises the hope that one can start to understand constrained interpolation problems in which e.g. the interpolating function $F$ is required to be nonnegative.

Let us recall some notation used in [18].

We fix positive integers $m$, $n$. We write $C^m(R^n)$ to denote the Banach space of all real valued locally $C^m$ functions $F$ on $R^n$, for which the norm

$$\|F\|_{C^m(R^n)} := \sup_{x \in R^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|$$

is finite.

We will also work with the function space $C^{m-1,1}(R^n)$. A given continuous function $F : R^n \to R$ belongs to $C^{m-1,1}(R^n)$ if and only if its distribution derivatives $\partial^\beta F$ belong to $L^\infty(R^n)$ for $|\beta| \leq m$. We may take the norm on $C^{m-1,1}(R^n)$ to be

$$\|F\|_{C^{m-1,1}(R^n)} = \max_{|\beta| \leq m} \esssup_{x \in R^n} |\partial^\beta F(x)|.$$

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Expressions $c(m, n)$, $C(m, n)$, $k(m, n)$, etc. denote constants depending only on $m$, $n$; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by $C(m, n, D)$, $k(D)$, etc.

If $X$ is any finite set, then $\#(X)$ denotes the number of elements in $X$.

We are now ready to state our main theorem.

**Theorem 1** For large enough $k^# = k(m, n)$ and $C^# = C(m, n)$ the following hold.

(A) **$C^m$ FLAVOR** Let $f : E \to [0, \infty)$ with $E \subset \mathbb{R}^n$ finite. Suppose that for each $S \subset E$ with $\#(S) \leq k^#$, there exists $F^S \in C^m(\mathbb{R}^n)$ with norm $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, such that $F^S = f$ on $S$ and $F^S \geq 0$ on $\mathbb{R}^n$.

Then there exists $F \in C^m(\mathbb{R}^n)$ with norm $\|F\|_{C^m(\mathbb{R}^n)} \leq C^#$, such that $F = f$ on $E$ and $F \geq 0$ on $\mathbb{R}^n$.

(B) **$C^{m-1,1}$ FLAVOR** Let $f : E \to [0, \infty)$ with $E \subset \mathbb{R}^n$ arbitrary. Suppose that for each $S \subset E$ with $\#(S) \leq k^#$, there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with norm $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$, such that $F^S = f$ on $S$ and $F^S \geq 0$ on $\mathbb{R}^n$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with norm $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^#$, such that $F = f$ on $E$ and $F \geq 0$ on $\mathbb{R}^n$.

Our interest in Theorem 1 arises in part from its possible connection to the interpolation algorithm of Fefferman-Klartag [15, 16]. Given a function $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, the goal of [15, 16] is to compute a function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on $E$, with $\|F\|_{C^m(\mathbb{R}^n)}$ as small as possible up to a factor $C(m, n)$. Roughly speaking, the algorithm in [15, 16] computes such an $F$ using $O(N \log N)$ computer operations, where $N = \#(E)$. The algorithm is based on an easier version [10] of Theorem 1. Our present result differs from the easier version in that we have added the hypothesis $F^S \geq 0$ and the conclusion $F \geq 0$. Accordingly, Theorem 1 raises the hope that we can start to understand constrained interpolation problems, in which e.g. the interpolant $F$ is required to be nonnegative everywhere on $\mathbb{R}^n$.

For results related to Theorem 1, we refer the reader to our paper [18] and references therein.

In the following sections, we will set up the notation; then we will recall a main theorem in [18] and use it to prove Theorem 1.

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney’s seminal work [33], and including
fundamental contributions by G. Glaeser [19], Y. Brudnyi and P. Shvartsman
[4,6–9,23–31], J. Wells [32], E. Le Gruyer [21], and E. Bierstone, P. Milman,
and W. Pawłucki [1–3], as well as our own papers [10–17]. See e.g. [14] for
the history of the problem, as well as Zobin [34,35] for a related problem.

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1 Notation and Preliminaries

1.1 Background Notation

Fix $m, n \geq 1$. We will work with cubes in $\mathbb{R}^n$; all our cubes have sides parallel
to the coordinate axes. If $Q$ is a cube, then $\delta_Q$ denotes the sidelength of $Q$.
For real numbers $A > 0$, $AQ$ denotes the cube whose center is that of $Q$, and
whose sidelength is $A\delta_Q$.

A dyadic cube is a cube of the form $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$, where each
$I_v$ has the form $[2^k \cdot i_v, 2^k \cdot (i_v + 1))$ for integers $i_1, \ldots, i_n, k$. Each dyadic
cube $Q$ is contained in one and only one dyadic cube with sidelength $2\delta_Q$;
that cube is denoted by $Q^+$.

We write $B_n(x, r)$ to denote the open ball in $\mathbb{R}^n$ with center $x$ and radius
$r$, with respect to the Euclidean metric.

We write $\mathcal{P}$ to denote the vector space of all real-valued polynomials of
degree at most $(m - 1)$ on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $F$ is a real-valued $C^{m-1}$ function
on a neighborhood of $x$, then $J_x(F)$ (the “jet” of $F$ at $x$) denotes the $(m - 1)^{st}$
order Taylor polynomial of $F$ at $x$, i.e.,

$$J_x(F)(y) = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y - x)^\alpha.$$  

Thus, $J_x(F) \in \mathcal{P}$. 

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For each \( x \in \mathbb{R}^n \), there is a natural multiplication \( \odot \) on \( \mathcal{P} \) ("multiplication of jets at \( x \)) defined by setting

\[
P \odot_x Q = J_x (PQ) \quad \text{for} \quad P, Q \in \mathcal{P}.
\]

If \( F \) is a real-valued function on a cube \( Q \), then we write \( F \in C^m (Q) \) to denote that \( F \) and its derivatives up to \( m \)-th order extend continuously to the closure of \( Q \). For \( F \in C^m (Q) \), we define

\[
\| F \|_{C^m (Q)} = \sup_{x \in Q} \max_{|\alpha| \leq m} | \partial^\alpha F (x) |.
\]

The function space \( C^{m-1,1} (Q) \) and the norm \( \| \cdot \|_{C^{m-1,1} (Q)} \) are defined analogously.

If \( F \in C^m (Q) \) and \( x \) belongs to the boundary of \( Q \), then we still write \( J_x (F) \) to denote the \( (m-1) \)st degree Taylor polynomial of \( F \) at \( x \), even though \( F \) isn’t defined on a full neighborhood of \( x \in \mathbb{R}^n \).

Let \( S \subset \mathbb{R}^n \) be non-empty and finite. A Whitney field on \( S \) is a family of polynomials

\[
\bar{P} = (P^y)_{y \in S} \quad \text{(each} \quad P^y \in \mathcal{P},
\]

parametrized by the points of \( S \).

We write \( \mathcal{W} \mathcal{H} (S) \) to denote the vector space of all Whitney fields on \( S \).

For \( \bar{P} = (P^y)_{y \in S} \in \mathcal{W} \mathcal{H} (S) \), we define the seminorm

\[
\| \bar{P} \|_{C^m (S)} = \max_{x,y \in S, x \neq y, |\alpha| \leq m} \frac{| \partial^\alpha (P^x - P^y) (x) |}{|x - y|^{m-|\alpha|}}.
\]

(If \( S \) consists of a single point, then \( \| \bar{P} \|_{C^m (S)} = 0 \).)

We also need an elementary fact about convex sets.

**Helly’s Theorem** Let \( K_1, \ldots, K_N \subset \mathbb{R}^D \) be convex. Suppose that \( K_{i_1} \cap \cdots \cap K_{i_{D+1}} \) is nonempty for any \( i_1, \ldots, i_{D+1} \in \{1, \ldots, N\} \). Then \( K_1 \cap \cdots \cap K_N \) is nonempty.

See [22].
1.2 Shape Fields

Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, $M \in (0, \infty)$, let $\Gamma(x, M) \subseteq \mathcal{P}$ be a (possibly empty) convex set. We say that $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ is a shape field if for all $x \in E$ and $0 < M' \leq M < \infty$, we have

$$\Gamma(x, M') \subseteq \Gamma(x, M).$$

Let $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ be a shape field and let $C_w, \delta_{\text{max}}$ be positive real numbers. We say that $\vec{\Gamma}$ is $(C_w, \delta_{\text{max}})$-convex if the following condition holds:

Let $0 < \delta \leq \delta_{\text{max}}$, $x \in E$, $M \in (0, \infty)$, $P_1, P_2, Q_1, Q_2 \in \mathcal{P}$. Assume that

1. $P_1, P_2 \in \Gamma(x, M)$;
2. $|\partial^\beta (P_1 - P_2)(x)| \leq M\delta^{m-|\beta|}$ for $|\beta| \leq m - 1$;
3. $|\partial^\beta Q_i(x)| \leq \delta^{-|\beta|}$ for $|\beta| \leq m - 1$ for $i = 1, 2$;
4. $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Then

5. $P := Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 \in \Gamma(x, C_w M)$.

1.3 Finiteness Principle for Shape Fields

We recall a main result proven in [18].

**Theorem 2** For a large enough $k^\#$ determined by $m$, $n$, the following holds. Let $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$ be a $(C_w, \delta_{\text{max}})$-convex shape field and let $Q_0 \subset \mathbb{R}^n$ be a cube of sidelength $\delta_{Q_0} \leq \delta_{\text{max}}$. Also, let $x_0 \in E \cap 5Q_0$ and $M_0 > 0$ be given. Assume that for each $S \subset E$ with $\#(S) \leq k^\#$ there exists a Whitney field $\vec{P}^S = (P_x^z)_{z \in S}$ such that

$$\|\vec{P}^S\|_{\tilde{C}^m(S)} \leq M_0,$$

and

$$P^z \in \Gamma_0(z, M_0) \quad \text{for all } z \in S.$$

Then there exist $P^0 \in \Gamma_0(x_0, M_0)$ and $F \in C^m(Q_0)$ such that the following hold, with a constant $C_*$ determined by $C_w$, $m$, $n$:
\begin{itemize}
  \item $J_z(F) \in \Gamma_0(z, C_\ast M_0)$ for all $z \in E \cap Q_0$.
  \item $|\partial^\beta (F - P^0) (x)| \leq C_\ast M_0 \delta^{m - |\beta|}_{Q_0}$ for all $x \in Q_0, |\beta| \leq m$.
  \item In particular, $|\partial^\beta F (x)| \leq C_\ast M_0$ for all $x \in Q_0, |\beta| = m$.
\end{itemize}

2 \quad C^m \text{ Interpolation by Nonnegative Functions}

In this section, $c$, $C$, $C'$, etc. denote constants determined by $m$ and $n$. These symbols may denote different constants in different occurrences. For $x \in \mathbb{R}^n$ and $M > 0$, define

\begin{itemize}
  \item (1) $\Gamma_\ast (x, M) = \left\{ P \in \mathcal{P} : \text{There exists } F \in C^m (\mathbb{R}^n) \text{ with } \|F\|_{C^m(\mathbb{R}^n)} \leq M, \right.$
  \hfill $F \geq 0 \text{ on } \mathbb{R}^n, J_x (F) = P. \left\}$
\end{itemize}

It is not immediately clear how to compute $\Gamma_\ast$; we will return to this issue in a later section. Let $E \subset \mathbb{R}^n$ be finite, and let $f : E \rightarrow [0, \infty)$. Define $\vec{\Gamma}_f = (\Gamma_f (x, M))_{x \in E, M > 0}$, where

\begin{itemize}
  \item (2) $\Gamma_f (x, M) = \{ P \in \Gamma_\ast (x, M) : P (x) = f (x) \}$.
\end{itemize}

\textbf{Lemma 1} $\vec{\Gamma}_f$ is a $(C, 1)$-convex shape field.

\textbf{Proof.} It is clear that $\vec{\Gamma}_f$ is a shape field, i.e., each $\Gamma_f (x, M)$ is convex, and $M' \leq M$ implies $\Gamma_f (x, M') \subseteq \Gamma_f (x, M)$. To establish $(C, 1)$-convexity, suppose we are given the following:

\begin{itemize}
  \item (3) $0 < \delta \leq 1, x \in E, M > 0$;
  \item (4) $P_1, P_2 \in \Gamma_f (x, M)$ satisfying
  \begin{itemize}
    \item (5) $|\partial^\beta (P_1 - P_2) (x)| \leq M \delta^{m - |\beta|}$ for $|\beta| \leq m - 1$;
  \end{itemize}
  \item (6) $Q_1, Q_2 \in \mathcal{P}$ satisfying
  \begin{itemize}
    \item (7) $|\partial^\beta Q_i (x)| \leq \delta^{-|\beta|}$ for $|\beta| \leq m - 1$, $i = 1, 2$, and
  \item (8) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.
  \end{itemize}
\end{itemize}

Set
(9) \( P = Q_1 \circ x Q_1 \circ x P_1 + Q_2 \circ x Q_2 \circ x P_2. \)

We must prove that

(10) \( P \in \Gamma_f (x, CM). \)

Thanks to (4), we have

(11) \( P_1 (x) = f(x) \) and \( P_2 (x) = f(x), \)

and there exist functions \( F_1, F_2 \in C^m (\mathbb{R}^n) \) such that

(12) \( \| F_i \|_{C^m (\mathbb{R}^n)} \leq M \) \( (i = 1, 2), \)

(13) \( F_i \geq 0 \) on \( \mathbb{R}^n \) \( (i = 1, 2), \) and

(14) \( J_x (F_i) = P_i \) \( (i = 1, 2). \)

We fix \( F_1, F_2 \) as above. By (8), we have \( |Q_i(x)| \geq \frac{1}{\sqrt{2}} \) for \( i = 1 \) or for \( i = 2. \) By possibly interchanging \( Q_1 \) and \( Q_2, \) and then possibly changing \( Q_1 \) to \( -Q_1, \) we may suppose that

(15) \( Q_1 (x) \geq \frac{1}{\sqrt{2}}. \)

For small enough \( c_0, \) (7) and (15) yield

(16) \( Q_1 (y) \geq \frac{1}{10} \) for \( |y - x| \leq c_0 \delta. \)

Fix \( c_0 \) as in (16). We introduce a \( C^m \) cutoff function \( \chi \) on \( \mathbb{R}^n \) with the following properties.

(17) \( 0 \leq \chi \leq 1 \) on \( \mathbb{R}^n; \chi = 0 \) outside \( B_n (x, c_0 \delta); \chi = 1 \) in a neighborhood of \( x; \)

(18) \( |\partial^\beta \chi| \leq C \delta^{-|\beta|} \) on \( \mathbb{R}^n, \) for \( |\beta| \leq m. \)

We then define \( \tilde{\theta}_1 = \chi \cdot Q_1 + (1 - \chi) \) and \( \tilde{\theta}_2 = \chi \cdot Q_2. \)

These functions satisfy the following: \( \tilde{\theta}_i \in C^m (\mathbb{R}^n) \) and \( |\partial^\beta \tilde{\theta}_i| \leq C \delta^{-|\beta|} \)
on \( \mathbb{R}^n \) for \( |\beta| \leq m, \) \( i = 1, 2; \) \( \tilde{\theta}_1 \geq \frac{1}{10} \) on \( \mathbb{R}^n; \) \( J_x (\tilde{\theta}_1) = Q_1 \) for \( i = 1, 2; \) outside \( B_n (x, c_0 \delta) \) we have \( \tilde{\theta}_1 = 1 \) and \( \tilde{\theta}_2 = 0. \) Setting \( \theta_i = \tilde{\theta}_i \cdot \left( \tilde{\theta}_1^2 + \tilde{\theta}_2^2 \right)^{-1/2} \)
for \( i = 1, 2, \) we find that
\[ \theta_i \in C^m(\mathbb{R}^n) \text{ and } |\partial^\beta \theta_i| \leq C\delta^{-|\beta|} \text{ on } \mathbb{R}^n \text{ for } |\beta| \leq m, \ i = 1, 2; \]

\[ \theta_1^2 + \theta_2^2 = 1 \text{ on } \mathbb{R}^n; \]

\[ J_x (\theta_i) = Q_i \text{ for } i = 1, 2 \text{ (here we use (8)); and} \]

\[ \text{outside } \mathcal{B}_n(x, c_0\delta) \text{ we have } \theta_1 = 1 \text{ and } \theta_2 = 0. \]

Now set

\[ F = \theta_1^2 F_1 + \theta_2^2 F_2 = F_1 + \theta_2^2 (F_2 - F_1) \text{ (see (20)).} \]

Clearly \( F \in C^m(\mathbb{R}^n) \). By (14), we have \( J_x (F_2 - F_1) = P_2 - P_1 \); hence (5) yields the estimate

\[ |\partial^\beta (F_2 - F_1)(x)| \leq CM\delta^{m-|\beta|} \text{ for } |\beta| \leq m - 1. \]

Together with (12), this tells us that

\[ |\partial^\beta (F_2 - F_1)| \leq CM\delta^{m-|\beta|} \text{ on } \mathcal{B}_n(x, c_0\delta) \text{ for } |\beta| \leq m. \]

Recalling (19), we deduce that

\[ |\partial^\beta (\theta_2^2 \cdot (F_2 - F_1))| \leq CM\delta^{m-|\beta|} \text{ on } \mathcal{B}_n(x, c_0\delta) \text{ for } |\beta| \leq m. \]

Together with (12) and (23), this implies that

\[ |\partial^\beta F| \leq CM \text{ on } \mathcal{B}_n(x, c_0\delta), \]

since \( 0 < \delta \leq 1 \) (see (3)). On the other hand, outside \( \mathcal{B}_n(x, c_0\delta) \) we have \( F = F_1 \) by (22), (23); hence \( |\partial^\beta F| \leq CM \) outside \( \mathcal{B}_n(x, c_0\delta) \) for \( |\beta| \leq m \), by (12). Thus, \( |\partial^\beta F| \leq CM \) on all of \( \mathbb{R}^n \) for \( |\beta| \leq m \), i.e.,

\[ \|F\|_{C^m(\mathbb{R}^n)} \leq CM. \]

Also, from (13) and (23) we have

\[ F \geq 0 \text{ on } \mathbb{R}^n; \]

and (9), (14), (21), (23) imply that

\[ \text{(24) } \|F\|_{C^m(\mathbb{R}^n)} \leq CM. \]
(26) \( J_x (F) = Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 = P. \)

Since \( F \in C^m (\mathbb{R}^n) \) satisfies (24), (25), (26), we have

(27) \( P \in \Gamma_r (x, CM). \)

Moreover,

(28) \( P (x) = (Q_1 (x))^2 f (x) + (Q_2 (x))^2 f (x) = f (x), \)

thanks to (8), (9), (11).

From (27), (28) we conclude that \( P \in \Gamma_f (x, CM) \), completing the proof of Lemma 1. \qed

**Lemma 2** Let \( (P^x)_{x \in E} \) be a Whitney field on the finite set \( E \), and let \( M > 0 \). Suppose that

(29) \( P^x \in \Gamma_r (x, M) \) for each \( x \in E \),

and that

(30) \( |\partial^\beta (P^x - P^{x'}) (x)| \leq M |x - x'|^{m-|\beta|} \) for \( x, x' \in E \) and \( |\beta| \leq m - 1. \)

Then there exists \( F \in C^m (\mathbb{R}^n) \) such that

(31) \( \| F \|_{C^m (\mathbb{R}^n)} \leq CM, \)

(32) \( F \geq 0 \) on \( \mathbb{R}^n \), and

(33) \( J_x (F) = P^x \) for all \( x \in E. \)

**Proof.** We modify slightly Whitney’s proof [33] of the Whitney extension theorem. We say that a dyadic cube \( Q \subset \mathbb{R}^n \) is “OK” if \( \# (E \cap 5Q) \leq 1 \) and \( \delta_Q \leq 1 \). Then every small enough \( Q \) is OK (because \( E \) is finite), and no \( Q \) of sidelength \( \delta_Q > 1 \) is OK. Also, let \( Q, Q' \) be dyadic cubes with \( 5Q \subset 5Q' \). If \( Q' \) is OK, then also \( Q \) is OK. We define a Calderón-Zygmund (or CZ) cube to be an OK cube \( Q \) such that no \( Q' \) that strictly contains \( Q \) is OK. The above remarks imply that the CZ cubes form a partition of \( \mathbb{R}^n \); that the sidelengths of the CZ cubes are bounded above by 1 and below by some positive number; and that the following condition holds.
“Good Geometry”: If \(Q, Q' \in CZ\) and \(\overline{Q \cap Q'} \neq \emptyset\), then \(\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q\).

We classify CZ cubes into three types as follows.

**Type 1** if \(E \cap 5Q \neq \emptyset\)

**Type 2** if \(E \cap 5Q = \emptyset\) and \(\delta_Q < 1\).

**Type 3** if \(E \cap 5Q = \emptyset\) and \(\delta_Q = 1\).

Let \(Q \in CZ\) be of Type 1. Since \(Q\) is OK, we have \(#(E \cap 5Q) \leq 1\). Hence \(E \cap 5Q\) is a singleton, \(E \cap 5Q = \{x_Q\}\). Since \(P^{x_Q} \in \Gamma_\ast (x_Q, M)\), there exists \(F_Q \in C^m(\mathbb{R}^n)\) such that

\[
\|F_Q\|_{C^m(\mathbb{R}^n)} \leq M, \quad F_Q \geq 0 \text{ on } \mathbb{R}^n, \quad J_{x_Q}(F_Q) = P^{x_Q}.
\]

We fix \(F_Q\) as in (35).

Let \(Q \in CZ\) be of Type 2. Then \(\delta_Q \leq 1\) but \(Q^+\) is not OK; hence \(#(E \cap 5Q^+) \geq 2\). We pick \(x_Q \in E \cap 5Q^+\). Since \(P^{x_Q} \in \Gamma_\ast (x_Q, M)\), there exists \(F_Q \in C^m(\mathbb{R}^n)\) satisfying (35). We fix such an \(F_Q\).

Let \(Q \in CZ\) be of Type 3. Then we set \(F_Q = 0\). In place of (35), we have the trivial results

\[
\|F_Q\|_{C^m(\mathbb{R}^n)} = 0 \text{ and } F_Q \geq 0 \text{ on } \mathbb{R}^n.
\]

Thus, we have defined \(F_Q\) for all \(Q \in CZ\), and we have defined \(x_Q \in E \cap 5Q^+\) for all \(Q\) of Type 1 or Type 2. Note that

\[
J_x(F_Q) = P^x \text{ for all } x \in E \cap 5Q.
\]

Indeed, if \(Q\) is of Type 1, then (37) follows from (35) since \(E \cap 5Q = \{x_Q\}\). If \(Q\) is of Type 2 or Type 3, then (37) holds vacuously since \(E \cap 5Q = \emptyset\). Now suppose \(Q, Q' \in CZ\) and \(\overline{Q \cap Q'} \neq \emptyset\). We will show that

\[
\left|\partial^\beta (F_Q - F_{Q'})\right| \leq CM\delta_Q^{m-|\beta|} \text{ on } \overline{Q \cap Q'} \text{ for } |\beta| \leq m.
\]

To see this, suppose first that \(Q\) or \(Q'\) is of Type 3. Then \(\delta_Q\) or \(\delta_{Q'}\) is equal to 1, hence \(\delta_Q \geq \frac{1}{2}\) by (34). Consequently, (38) asserts simply that
\[(39) \left| \partial^\beta (F_Q - F_Q') \right| \leq CM \text{ on } \frac{65}{64}Q \cap \frac{65}{64}Q' \text{ for } |\beta| \leq m,\]

and (39) follows at once from (35), (36). Thus, (38) holds if \( Q \) or \( Q' \) is of Type 3. Suppose that neither \( Q \) nor \( Q' \) is of Type 3. Then \( x_Q \in E \cap 5Q^+ \), \( x_{Q'} \in E \cap 5(Q')^+ \), \( \frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset \), \( \frac{1}{2} \delta_Q \leq \delta_{Q'} \leq 2 \delta_Q \). Consequently,

\[(40) |x_Q - x_{Q'}| \leq C \delta_Q, \text{ and}\]

\[(41) |x - x_Q|, |x - x_{Q'}| \leq C \delta_Q \text{ for all } x \in \frac{65}{64}Q \cap \frac{65}{64}Q'.\]

Applying (35) to \( Q \) and to \( Q' \), we find that

\[(42) \left| \partial^\beta (F_Q - P^{x_Q}) (x) \right| \leq CM |x - x_Q|^{m-|\beta|} \leq CM \delta_Q^{m-|\beta|}, \text{ and}\]

\[(43) \left| \partial^\beta (F_Q' - P^{x_{Q'}}) (x) \right| \leq CM |x - x_{Q'}|^{m-|\beta|} \leq CM \delta_Q^{m-|\beta|},\]

for \( x \in \frac{65}{64}Q \cap \frac{65}{64}Q', |\beta| \leq m.\)

Also, (30), (40), (41) imply that

\[(44) \left| \partial^\beta (P^{x_Q} - P^{x_{Q'}}) (x) \right| \leq CM \delta_Q^{m-|\beta|} \text{ for } x \in \frac{65}{64}Q \cap \frac{65}{64}Q', |\beta| \leq m.\]

(Recall, \( P^{x_Q} - P^{x_{Q'}} \) is a polynomial of degree at most \( m - 1 \).)

Estimates (42), (43), (44) together imply (38) in case neither \( Q \) nor \( Q' \) is of Type 3. Thus, (38) holds in all cases.

Next, as in Whitney \[33\], we introduce a partition of unity

\[(45) 1 = \sum_{Q \in CZ} \theta_Q \text{ on } \mathbb{R}^n,\]

where each \( \theta_Q \in C^m(\mathbb{R}^n), \) and

\[(46) \text{support } \theta_Q \subset \frac{65}{64}Q, |\partial^\beta \theta_Q| \leq C \delta_Q^{-|\beta|} \text{ for } |\beta| \leq m, \theta_Q \geq 0 \text{ on } \mathbb{R}^n.\]

We define

\[(47) F = \sum_{Q \in CZ} \theta_Q F_Q \text{ on } \mathbb{R}^n.\]

Thus, \( F \in C^m_{\text{loc}}(\mathbb{R}^n) \) since \( CZ \) is a locally finite partition of \( \mathbb{R}^n \), and \( F \geq 0 \) on \( \mathbb{R}^n \) since \( \theta_Q \geq 0 \) and \( F_Q \geq 0 \) for each \( Q \). Let \( \hat{x} \in \mathbb{R}^n \), and let \( \hat{Q} \) be the one and only CZ cube containing \( \hat{x} \). Then for \( |\beta| \leq m, \) we have
\( \partial^\beta F(\hat{x}) = \partial^\beta F_Q(\hat{x}) + \sum_{Q \in CZ} \partial^\beta (\theta_Q \cdot (F_Q - F_{\hat{Q}})) (\hat{x}) \).

A given \( Q \in CZ \) enters into the sum in (48) only if \( \hat{x} \in \frac{\delta_Q}{64} Q \); there are at most \( C \) such cubes \( Q \), thanks to (34). Moreover, for each \( Q \in CZ \) with \( \hat{x} \in \frac{\delta_Q}{64} Q \), we learn from (38) and (46) that
\[
|\partial^\beta (\theta_Q \cdot (F_Q - F_{\hat{Q}})) (\hat{x})| \leq CM\delta_Q^{m-|\beta|} \leq CM \text{ for } |\beta| \leq m, \text{ since } \delta_Q \leq 1.
\]

Since also \( |\partial^\beta F_Q(\hat{x})| \leq CM \) for \( |\beta| \leq m \) by (35), (36), it now follows from (48) that \( |\partial^\beta F(\hat{x})| \leq CM \) for all \( |\beta| \leq m \). Here, \( \hat{x} \in \mathbb{R}^n \) is arbitrary. Thus, \( F \in C^m(\mathbb{R}^n) \) and \( \|F\|_{C^m(\mathbb{R}^n)} \leq CM \).

Next, let \( x \in E \). For any \( Q \in CZ \) such that \( x \in \frac{\delta_Q}{64} Q \), we have \( J_x(F_Q) = P^x \), by (37). Since support \( \theta_Q \subset \frac{\delta_Q}{64} Q \) for each \( Q \in CZ \), it follows that \( J_x(\theta_Q F_Q) = J_x(\theta_Q) \odot_x P^x \) for each \( Q \in CZ \), and consequently,
\[
J_x(F) = \sum_{Q \in CZ} J_x(\theta_Q F_Q) = \left[ \sum_{Q \in CZ} J_x(\theta_Q) \right] \odot_x P^x = P^x, \text{ by (45).}
\]

Thus, \( F \in C^m(\mathbb{R}^n) \), \( \|F\|_{C^m(\mathbb{R}^n)} \leq CM \), \( F \geq 0 \) on \( \mathbb{R}^n \), and \( J_x(F) = P^x \) for each \( x \in E \).

The proof of Lemma 2 is complete. \( \blacksquare \)

**Theorem 3 (Finiteness Principle for Nonnegative \( C^m \) Interpolation)**

*There exist constants \( k^#, C \), depending only on \( m, n \), such that the following holds.*

Let \( E \subset \mathbb{R}^n \) be finite, and let \( f : E \to [0, \infty) \). Let \( M_0 > 0 \). Suppose that for each \( S \subset E \) with \( \#(S) \leq k^# \), there exists \( P^x = (P^x)_{x \in S} \in \text{Wh}(S) \) such that

- \( P^x \in \Gamma_f(x, M_0) \) for each \( x \in S \), and
- \( |\partial^\beta (P^x - P^y)(x)| \leq M_0|x - y|^{m-|\beta|} \) for \( x, y \in S \), \( |\beta| \leq m - 1 \).

Then there exists \( F \in C^m(\mathbb{R}^n) \) such that

- \( \|F\|_{C^m(\mathbb{R}^n)} \leq CM_0 \),
- \( F \geq 0 \) on \( \mathbb{R}^n \), and
- \( F = f \) on \( E \).
Proof. Suppose first that $E \subset \frac{1}{2}Q_0$ for a cube $Q_0$ of sidelength $\delta_{Q_0} = 1$.
Pick any $x_0 \in E$. (If $E$ is empty, our theorem holds trivially.)
Let $S \subset E$ with $\#(S) \leq k^*$.
Our present hypotheses supply the Whitney field $\overline{P^S}$ required in the hypotheses of Theorem 2.
Hence, recalling Lemma 1 and applying Theorem 2, we obtain

(49) $P^0 \in \Gamma_f(x_0, CM_0)$

and

(50) $F^0 \in C^m(Q_0)$

such that

(51) $J_x(F^0) \in \Gamma_f(x, CM_0)$ for all $x \in E \cap Q_0 = E$

and

(52) $|\partial^\beta (P^0 - F^0)| \leq CM_0$ on $Q_0$, for $|\beta| \leq m$.

From (1), (2), (49), we have $|\partial^\beta P^0(x_0)| \leq CM_0$ for $|\beta| \leq m - 1$.
Since $P^0$ is a polynomial of degree at most $m - 1$, and since $x_0 \in E \subset Q_0$ with $\delta_{Q_0} = 1$, it follows that $|\partial^\beta P^0| \leq CM_0$ on $Q_0$ for $|\beta| \leq m$.
Together with (52), this tells us that

(53) $|\partial^\beta F^0| \leq CM_0$ on $Q_0$ for $|\beta| \leq m$.

Note that $F^0$ needn’t be nonnegative.
Set $P^x = J_x(F^0)$ for $x \in E$. Then

(54) $P^x \in \Gamma_f(x, CM_0)$ for $x \in E$, and

(55) $|\partial^\beta (P^x - P^y)(x)| \leq CM_0 |x - y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m - 1$.

By Lemma 2, there exists $F \in C^m(\mathbb{R}^n)$ such that

(56) $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$,

(57) $F \geq 0$ on $\mathbb{R}^n$, and
\( (58) \) \( J_x(F) = P^x \) for each \( x \in E \).

From (54) and (2), we have \( P^x(x) = f(x) \) for each \( x \in E \); hence, (58) implies that

\( (59) \) \( F(x) = f(x) \) for each \( x \in E \).

Our results (56), (57), (59) are the conclusions of our theorem. Thus, we have proven Theorem 3 in the case in which \( E \subset \mathcal{Q}_0 \) with \( \delta_{Q_0} = 1 \).

To pass to the general case (arbitrary finite \( E \subset \mathbb{R}^n \)), we set up a partition of unity \( 1 = \sum \chi_{\nu} \) on \( \mathbb{R}^n \), where each \( \chi_{\nu} \in C^m(\mathbb{R}^n) \) and \( \chi_{\nu} \geq 0 \) on \( \mathbb{R}^n \), \( \| \chi_{\nu} \|_{C^m(\mathbb{R}^n)} \leq C \), support \( \chi_{\nu} \subset \frac{1}{2}Q_\nu \), with \( \delta_{Q_\nu} = 1 \), and with any given point of \( \mathbb{R}^n \) belonging to at most \( C \) of the \( Q_\nu \).

For each \( \nu \), we apply the known special case of our theorem to the set \( E_\nu = E \cap \frac{1}{2}Q_\nu \) and the function \( f_\nu = f|_{E_\nu} \). Thus, we obtain \( F_\nu \in C^m(\mathbb{R}^n) \), with \( \| F_\nu \|_{C^m(\mathbb{R}^n)} \leq CM_0 \), \( F_\nu \geq 0 \) on \( \mathbb{R}^n \), and \( F_\nu = f \) on \( E \cap \frac{1}{2}Q_\nu \).

Setting \( F = \sum \chi_{\nu} F_\nu \in C^m_{\text{loc}}(\mathbb{R}^n) \), we verify easily that \( F \in C^m(\mathbb{R}^n) \), \( \| F \|_{C^m(\mathbb{R}^n)} \leq CM_0 \), \( F \geq 0 \) on \( \mathbb{R}^n \), and \( F = f \) on \( E \).

This completes the proof of Theorem 3. \( \blacksquare \)

**Remark** Conversely, we make the following trivial observation: Let \( E \subset \mathbb{R}^n \) be finite, let \( f : E \to [0, \infty) \), and let \( M_0 > 0 \). Suppose \( F \in C^m(\mathbb{R}^n) \) satisfies \( \| F \|_{C^m(\mathbb{R}^n)} \leq M_0 \), \( F \geq 0 \) on \( \mathbb{R}^n \), and \( F = f \) on \( E \). Then for each \( x \in E \), we have

- \( P^x = J_x(F) \in \Gamma_f(x, M_0) \) by (1), (2); and
  \[ |\partial^\beta (P^x - P^y)(x)| \leq CM_0|x - y|^{m-|\beta|} \] for \( x, y \in E \), \( |\beta| \leq m - 1 \).

Therefore, for any \( S \subset E \), the Whitney field \( \overline{P}^S \) = \( (P^x)_{x \in S} \in W\mathcal{H}(S) \) satisfies

- \( P^x \in \Gamma_f(x, CM_0) \) for \( x \in S \), and
  \[ |\partial^\beta (P^x - P^y)(x)| \leq CM_0|x - y|^{m-|\beta|} \] for \( x, y \in S \), \( |\beta| \leq m - 1 \).

**Note that Theorem 1 (A) follows easily from Theorem 3.**
3 Computable Convex Sets

In this section, we discuss computational issues regarding the convex set

$$\Gamma_\ast(x, M) = \left\{ J_x(F) : F \in C^m(R^n), \|F\|_{C^m(R^n)} \leq M, F \geq 0 \text{ on } R^n \right\}.$$  

We write $c$, $C$, $C'$, etc., to denote constants determined by $m$ and $n$. These symbols may denote different constants in different occurrences.

We will define convex sets $\tilde{\Gamma}_\ast(x, M) \subset P$, prove that

$$\tilde{\Gamma}_\ast(x, cM) \subset \Gamma_\ast(x, M) \subset \tilde{\Gamma}_\ast(x, CM)$$

for all $x \in R^n$, $M > 0$, and explain how (in principle) one can compute $\tilde{\Gamma}_\ast(x, M)$.

We may then use

$$\tilde{\Gamma}_f(x, M) = \left\{ P \in \tilde{\Gamma}_\ast(x, M) : P(x) = f(x) \right\}$$

in place of $\Gamma_f(x, M)$ in the statement of Theorem 3. (The assertion in terms of $\tilde{\Gamma}_f$ follows trivially from (2) and the original assertion in terms of $\Gamma_f$.)

To achieve (2), we will define

$$\tilde{\Gamma}_\ast(x, M) = \left\{ MP(\cdot + x) : P \in \tilde{\Gamma}_0 \right\},$$

for a convex set $\tilde{\Gamma}_0$.

We will prove that

$$\Gamma_\ast(0, c) \subset \tilde{\Gamma}_0 \subset \Gamma_\ast(0, C).$$

Property (2) then follows at once from (1), (4), and (5).

Thus, our task is to define a convex set $\tilde{\Gamma}_0$ satisfying (5), and explain how (in principle) one can compute $\tilde{\Gamma}_0$.

Recall that $P$ is the vector space of $(m - 1)$-jets. We will work in the space of $m$-jets. In this section, we let $P^+$ denote the vector space of real-valued polynomials of degree at most $m$ on $R^n$, and we write $J_x^+(F)$ to denote the $m$th-degree Taylor polynomial of $F$ at $x$, i.e.,

$$J_x^+(F)(y) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} (\partial^\alpha F(x)) \cdot (y - x)^\alpha.$$  

We define
\[ \left\{ P \in \mathcal{P}^+ : |\partial^\beta P(0)| \leq 1 \text{ for } |\beta| \leq m; \ P(x) + |x|^m \geq 0 \text{ for all } x \in \mathbb{R}^n; \right. \]

and for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[ P(x) + \varepsilon |x|^m \geq 0 \text{ for } |x| \leq \delta. \]

Later, we will discuss how \( \Gamma_0^+ \) may be computed in principle.

We next establish the following result.

**Lemma 3** For small enough \( c \) and large enough \( C \), the following hold.

(A) If \( F \in C^m(\mathbb{R}^n) \), \( \|F\|_{C^m(\mathbb{R}^n)} \leq c \), \( F \geq 0 \) on \( \mathbb{R}^n \), then \( J_0^+(F) \in \Gamma_0^+ \).

(B) If \( P \in \Gamma_0^+ \), then there exists \( F \in C^m(\mathbb{R}^n) \) such that \( \|F\|_{C^m(\mathbb{R}^n)} \leq C \), \( F \geq 0 \) on \( \mathbb{R}^n \), and \( J_0^+(F) = P \).

**Proof.** (A) follows trivially from Taylor’s theorem. We prove (B).

Let \( P \in \Gamma_0^+ \) be given. We introduce cutoff functions \( \varphi, \chi \in C^m(\mathbb{R}^n) \) with the following properties.

(7) \( \|\chi\|_{C^m(\mathbb{R}^n)} \leq C \), \( \chi = 1 \) in a neighborhood of \( 0 \), \( \chi = 0 \) outside \( B_n(0, 1/2) \), and \( 0 \leq \chi \leq 1 \) on \( \mathbb{R}^n \).

(8) \( \|\varphi\|_{C^m(\mathbb{R}^n)} \leq C \), \( \varphi = 1 \) for \( 1/2 \leq |x| \leq 2 \), \( \varphi \geq 0 \) on \( \mathbb{R}^n \), and \( \varphi(x) = 0 \) unless \( 1/4 < |x| < 4 \).

For \( k \geq 0 \), let

(9) \( \varphi_k(x) = \varphi(2^k x) \) (\( x \in \mathbb{R}^n \)).

Thus,

(10) \( \|\varphi_k\|_{C^m(\mathbb{R}^n)} \leq C 2^{mk} \), \( \varphi_k \geq 0 \) on \( \mathbb{R}^n \), \( \varphi_k(x) = 1 \) for \( 2^{-1-k} \leq |x| \leq 2^{1-k} \), \( \varphi_k(x) = 0 \) unless \( 2^{-2-k} \leq |x| \leq 2^{2-k} \).

Also, for \( k \geq 0 \), we define a real number \( b_k \) as follows.

(11) \( b_k = 0 \) if \( P(x) \geq 0 \) for \( |x| \leq 2^{-k} \); \( b_k = -\min \{ P(x) : |x| \leq 2^{-k} \} \) otherwise.

Since \( P \in \Gamma_0^+ \), the \( b_k \) satisfy the following:

(12) \( 0 \leq b_k \leq 2^{-mk} \) for all \( k \geq 0 \).
(13) $b_k \cdot 2^{mk} \to 0$ as $k \to \infty$.

By definition of the $b_k$, we have also for each $k \geq 0$ that

(14) $P(x) + b_k \geq 0$ for $|x| \leq 2^{-k}$.

We define a function $\tilde{F}$ on the closed unit ball $B_n(0, 1)$ by setting

(15) $\tilde{F}(x) = P(x) + \sum_{k=0}^{\infty} b_k \varphi_k(x)$ for $x \in B_n(0, 1)$.

(The sum contains at most $C$ nonzero terms for any given $x$.)

We will check that

(16) $\tilde{F} \geq 0$ on $B_n(0, 1)$.

Indeed, $\tilde{F}(0) = P(0) \geq 0$ since each $\varphi_k(0) = 0$ and $P \in \Gamma_0^+$. For $\hat{x} \in B_n(0, 1) \setminus \{0\}$ we have $2^{-1-k} \leq |\hat{x}| \leq 2^{-k}$ for some $\hat{k} \geq 0$.

We then have $\varphi_{\hat{k}}(\hat{x}) = 1$ by (10), hence $P(\hat{x}) + b_{\hat{k}} \varphi_{\hat{k}}(\hat{x}) \geq 0$ by (14). Since also $b_k \varphi_k(\hat{x}) \geq 0$ for all $k$, it follows that

$$\tilde{F}(\hat{x}) = [P(\hat{x}) + b_{\hat{k}} \varphi_{\hat{k}}(\hat{x})] + \sum_{k \neq \hat{k}} b_k \varphi_k(x) \geq 0,$$

completing the proof of (16).

Next, we check that

(17) $\tilde{F} \in C^m \left( B_n(0, 1) \right)$, $\|\tilde{F}\|_{C^m \left( B_n(0, 1) \right)} \leq C$, $J_\partial^+ \left( \tilde{F} \right) = P$.

To see this, let

(18) $\tilde{F}_K = P + \sum_{k=0}^{K} b_k \varphi_k$ for $K \geq 0$.

Since $P \in \Gamma_0^+$, we have $|\partial^\beta P(0)| \leq 1$ for $|\beta| \leq m$, hence

(19) $\|P\|_{C^m \left( B_n(0, 1) \right)} \leq C$.

Also, (10) and (12) give

$$\|b_k \varphi_k\|_{C^m \left( B_n(0, 1) \right)} \leq C \text{ for each } k.$$

Since any given $x \in B_n(0, 1)$ belongs to at most $C$ of the supports of the $\varphi_k$, it follows that
\[
\sum_{k=0}^{K} b_k \varphi_k \leq C. \tag{20}
\]

From (18), (19), (20), we see that

\[
\tilde{F}_K \in C^m\left(\mathbb{B}_n(0,1)\right) \quad \text{and} \quad \left\| \tilde{F} \right\|_{C^m(\mathbb{B}_n(0,1))} \leq C. \tag{21}
\]

Also, (10) and (18) tell us that

\[
J_0^+\left(\tilde{F}_K\right) = P \quad \text{for each} \quad K. \tag{22}
\]

Furthermore for \(K_1 < K_2\), (18) gives \(\tilde{F}_{K_2} - \tilde{F}_{K_1} = \sum_{K_1 < k \leq K_2} b_k \varphi_k\). Let \(\epsilon > 0\). From (10) and (13) we see that

\[
\max_{K_1 < k \leq K_2} \left\| b_k \varphi_k \right\|_{C^m(\mathbb{B}_n(0,1))} < \epsilon \quad \text{if} \quad K_1 \text{ is large enough.}
\]

Since any given point lies in support \(\varphi_k\) for at most \(C\) distinct \(k\), it follows that

\[
\left\| \sum_{K_1 < k \leq K_2} b_k \varphi_k \right\|_{C^m(\mathbb{B}_n(0,1))} \leq C\epsilon \quad \text{if} \quad K_2 > K_1 \text{ and} \quad K_1 \text{ is large enough.}
\]

Thus, \((\tilde{F}_K)_{K \geq 0}\) is a Cauchy sequence in \(C^m(\mathbb{B}_n(0,1))\). Consequently, \(\tilde{F}_K \to \tilde{F}_\infty\) in \(C^m(\mathbb{B}_n(0,1))\)-norm for some \(\tilde{F}_\infty \in C^m(\mathbb{B}_n(0,1))\). From (21) and (22), we have

\[
\left\| \tilde{F}_\infty \right\|_{C^m(\mathbb{B}_n(0,1))} \leq C \quad \text{and} \quad J_0^+\left(\tilde{F}_\infty\right) = P.
\]

On the other hand, comparing (15) to (18), and recalling that any given \(x\) belongs to support \(\theta_k\) for at most \(C\) distinct \(k\), we conclude that \(\tilde{F}_K \to \tilde{F}\) pointwise as \(K \to \infty\).

Since also \(\tilde{F}_K \to \tilde{F}_\infty\) pointwise as \(K \to \infty\), we have \(\tilde{F}_\infty = \tilde{F}\).

Thus, \(\tilde{F} \in C^m\left(\mathbb{B}_n(0,1)\right), \left\| \tilde{F} \right\|_{C^m(\mathbb{B}_n(0,1))} \leq C\), and \(J_0^+\left(\tilde{F}\right) = P\), completing the proof of (17).

Finally, we recall the cutoff function \(\chi\) from (7), and define \(F = \chi \tilde{F}\) on \(\mathbb{R}^n\).
From (16), (17), and the properties (7) of \( \chi \), we conclude that
\[
F \in \mathcal{C}^m(\mathbb{R}^n), \|F\|_{\mathcal{C}^m(\mathbb{R}^n)} \leq C, \quad F \geq 0 \text{ on } \mathbb{R}^n, \quad J_0^+(F) = P.
\]
Thus, we have established (B).

The proof of Lemma 3 is complete. ■

Now let \( \pi : \mathcal{P}^+ \to \mathcal{P} \) denote the natural projection from \( m \)-jets at 0 to \( (m-1) \)-jets at 0, namely, \( \pi P = J_0(P) \) for \( P \in \mathcal{P}^+ \).

We then set \( \tilde{\Gamma}_0 = \pi \Gamma_0^+ \).

From the above lemma, we learn the following.

\((A')\) Let \( F \in \mathcal{C}^m(\mathbb{R}^n) \) with \( \|F\|_{\mathcal{C}^m(\mathbb{R}^n)} \leq c \), \( F \geq 0 \) on \( \mathbb{R}^n \). Then \( J_0(F) \in \tilde{\Gamma}_0 \).

\((B')\) Let \( P \in \tilde{\Gamma}_0 \). Then there exists \( F \in \mathcal{C}^m(\mathbb{R}^n) \) such that \( \|F\|_{\mathcal{C}^m(\mathbb{R}^n)} \leq C \), \( F \geq 0 \) on \( \mathbb{R}^n \), and \( J_0(F) = P \).

Recalling the definition (1), we conclude from \((A')\), \((B')\) that \( \Gamma_*(0,c) \subset \tilde{\Gamma}_0 \subset \Gamma_*(0,C) \).

Thus, our \( \tilde{\Gamma}_0 \) satisfies the key condition (5).

We discuss briefly how the convex set \( \tilde{\Gamma}_0 \) may be computed in principle. Recall [20] that a semialgebraic set is a subset of a vector space obtained by taking finitely many unions, intersections, and complements of sets of the form \( \{P > 0\} \) for polynomials \( P \). Any subset of a vector space \( V \) defined by \( E = \{x \in V : \Phi(x) \text{ is true}\} \), where \( \Phi \) is a formula of first-order predicate calculus (for the theory of real-closed fields) is semialgebraic; moreover, there is an algorithm that accepts \( \Phi \) as input and exhibits \( E \) as a Boolean combination of sets of the form \( \{P > 0\} \) for polynomials \( P \). For any given \( m, n \), we see, by inspection of the definitions of \( \Gamma_0^+ \) and \( \tilde{\Gamma}_0 \), that \( \Gamma_0^+ \subset \mathcal{P}^+ \) is defined by a formula of first-order predicate calculus; hence, the same holds for \( \tilde{\Gamma}_0 \subset \mathcal{P} \).

Therefore, in principle, we can compute \( \tilde{\Gamma}_0 \) as a Boolean combination of sets of the form \( \{P \in \mathcal{P} : \Pi(P) > 0\} \), where \( \Pi \) is a polynomial on \( \mathcal{P} \).

In practice, we make no claim that we know how to compute \( \tilde{\Gamma}_0 \).

It would be interesting to give a more practical method to compute a convex set satisfying (5).

### 4 \( \mathcal{C}^{m-1,1} \) Interpolation by Nonnegative Functions

In this section we will establish Theorem 1 (B) and discuss computational issues for \( \mathcal{C}^{m-1,1} \) interpolation by nonnegative functions.
We note that the derivatives $\partial^\beta F$ of $F \in C^{m-1,1}(\mathbb{R}^n)$ of order $|\beta| \leq m - 1$ are continuous. Also, Taylor's theorem holds in the form

$$\left| \partial^\beta F(y) - \sum_{|\beta|+|\gamma| \leq m-1} \frac{1}{\gamma!} [\partial^{\gamma+\beta} F(x)] \cdot (y-x)^\gamma \right| \leq C \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \cdot |y-x|^{m-|\beta|}$$

for $x, y \in \mathbb{R}^n$.

Similar remarks apply to $C^{m-1,1}(Q)$ and $C^m(Q)$ for cubes $Q \subset \mathbb{R}^n$.

Therefore, we may repeat the proofs [18] of Lemmas 1 and 2 in Section 2, to derive the following results.

**Lemma 4** For $x \in \mathbb{R}^n$, $M > 0$, let

$$\Gamma'_s(x, M) = \left\{ P \in \mathcal{P} : \exists F \in C^{m-1,1}(\mathbb{R}^n) \text{ such that } \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq M, F \geq 0 \text{ on } \mathbb{R}^n, J_x(F) = P \right\}.$$ 

Let $f : E \to [0, \infty)$, where $E \subset \mathbb{R}^n$ is finite. For $x \in E$, $M > 0$, let

$$\Gamma'_f(x, M) = \{ P \in \Gamma'_s(x, M) : P(x) = f(x) \}.$$ 

Then $\tilde{\Gamma}_f := (\Gamma'_f(x, M))_{x \in E, M > 0}$ is a $(C, 1)$-convex shape field, where $C$ depends only on $m$, $n$.

**Lemma 5** Let $E$, $f$, $\Gamma'_s(x, M)$ be as in Lemma 4, and let $M > 0$, $\bar{P} = (P^x)_{x \in E} \in \text{Wh}(E)$. Suppose we have $P^x \in \Gamma'_s(x, M)$ for all $x \in E$, and $|\partial^\beta (P^x - P^y)| \leq M |x-y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m - 1$. Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that $J_x(F) = P^x$ for all $x \in E$, and $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM$, where $C$ depends only on $m$, $n$.

Similarly, by making small changes in the proof [18] of Theorem 3, we obtain the following result.

**Lemma 6** There exist $k^\#$, $C$, depending only on $m$, $n$ for which the following holds.

Let $E \subset \mathbb{R}^n$ be finite, let $f : E \to [0, \infty)$, and let $M_0 > 0$. Suppose that for each $S \subset E$ with $\#(S) \leq k^\#$ there exists $\bar{P}^S = (P^x)_{x \in S} \in \text{Wh}(S)$ such that $P^x \in \Gamma'_f(x, M_0)$ for all $x \in S$, and $|\partial^\beta (P^x - P^y)| \leq M_0 |x-y|^{m-|\beta|}$ for $x, y \in S$, $|\beta| \leq m - 1$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM_0$, $F \geq 0$ on $\mathbb{R}^n$, and $F = f$ on $E$. 

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Now we can easily deduce the following result.

**Theorem 4 (Finiteness Principle for Nonnegative \( C^{m-1,1} \)-Interpolation)**

There exists constants \( k^\#, C \), depending only on \( m, n \) for which the following holds.

Let \( f : E \to [0, \infty) \), with \( E \subset \mathbb{R}^n \) arbitrary (not necessarily finite). Let \( M_0 > 0 \). Suppose that for each \( S \subset E \) with \( \#(S) \leq k^\# \) there exists \( \vec{P} = (P^x)_{x \in S} \in \text{Wh}(S) \) such that

1. \( P^x \in \Gamma'(x, M_0) \) for all \( x \in S \),
2. \( |\partial^\beta (P^x - P^y)(x)| \leq M_0 |x - y|^{m-|\beta|} \) for \( x, y \in S \), \( |\beta| \leq m - 1 \).

Then there exists \( F \in C^{m-1,1}(\mathbb{R}^n) \) such that

1. \( \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM_0 \),
2. \( F \geq 0 \), and
3. \( F = f \) on \( E \).

**Proof.** Suppose first that \( E \subset Q \) for some cube \( Q \subset \mathbb{R}^n \). Then by Ascoli’s theorem,

\[
\left\{ F \in C^{m-1,1}(Q) : \|F\|_{C^{m-1,1}(Q)} \leq CM_0, \ F \geq 0 \ on \ Q \right\} \equiv X
\]

is compact in the \( C^{m-1}(Q) \)-norm topology.

For each finite \( E_0 \subset E \), Lemma 6 tells us that there exists \( F \in X \) such that \( F = f \) on \( E_0 \).

Consequently, there exists \( F \in X \) such that \( F = f \) on \( E \). That is,

(1) \( F \in C^{m-1,1}(Q), \ \|F\|_{C^{m-1,1}(Q)} \leq CM_0, \ F \geq 0 \ on \ Q, \ F = f \ on \ E \).

We have achieved (1), assuming that \( E \subset Q \).

Now suppose \( E \subset \mathbb{R}^n \) is arbitrary.

We introduce a partition of unity \( 1 = \sum_v \theta_v \) on \( \mathbb{R}^n \), with \( \theta_v \geq 0 \) on \( \mathbb{R}^n \), \( \theta_v \in C^m(\mathbb{R}^n), \ \|\theta_v\|_{C^m(\mathbb{R}^n)} \leq C \), support \( \theta_v \subset Q_v \) for a cube \( Q_v \subset \mathbb{R}^n \), with (say) \( \delta_{Q_v} = 1 \), and such that any given \( x \in \mathbb{R}^n \) has a neighborhood that intersects at most \( C \) of the \( Q_v \). (Here \( C \) depends only on \( m, n \).)
Applying our result (1) to $f|_{E\cap Q_{\nu}} : E \cap Q_{\nu} \to [0, \infty)$ for each $\nu$, we obtain functions $F_{\nu} \in C^{m-1,1}(Q_{\nu})$ such that $\|F_{\nu}\|_{C^{m-1,1}(Q_{\nu})} \leq CM_{0}$, $F_{\nu} \geq 0$ on $Q_{\nu}$, $F_{\nu} = f$ on $E \cap Q_{\nu}$.

(Here $C$ depends only on $m, n$.)

We define $F = \sum_{\nu} \theta_{\nu} F_{\nu}$ on $\mathbb{R}^{n}$. One checks easily that $\|F\|_{C^{m-1,1}(\mathbb{R}^{n})} \leq C'M_{0}$ with $C'$ determined by $m, n$; $F \geq 0$ on $\mathbb{R}^{n}$; and $F = f$ on $E$.

This completes the proof of Theorem 4. ■

Note that Theorem 4 easily implies Theorem 1 (B).

As in the case of nonnegative $C^{m}$-interpolation, we want to replace $\Gamma'_{*}(x, M)$ by something easier to calculate. In the $C^{m-1,1}$-setting, it is enough to make the following observation.

Define

$$\tilde{\Gamma}'_{0} = \left\{ P \in \mathcal{P} : |\partial^{\beta} P(0)| \leq 1 \text{ for } |\beta| \leq m-1 \text{ and } P(x) + |x|^{m} \geq 0 \text{ for all } x \in \mathbb{R}^{n} \right\}.$$ 

Then

$$\Gamma'_{*}(0, c) \subset \tilde{\Gamma}'_{0} \subset \tilde{\Gamma}'_{*}(0, C),$$

with $c, C$ depending only on $m, n$.

Indeed, the first inclusion in (2) is immediate from the definitions and Taylor’s theorem. To prove the second inclusion, we let $P \in \tilde{\Gamma}'_{0}$ be given, and set $F(x) = \chi(x)(P(x) + |x|^{m})$, where $\chi$ is a nonnegative $C^{m}$ function with norm at most $C_{*}$ (depending only on $m, n$), satisfying $J_{0}(\chi) = 1$ and support $\chi \subset B_{n}(0, 1)$.

We then have $F \in C^{m-1,1}(\mathbb{R}^{n})$, $\|F\|_{C^{m-1,1}(\mathbb{R}^{n})} \leq C$ (depending only on $m, n$), $F \geq 0$ on $\mathbb{R}^{n}$, $J_{0}(F) = P$. Hence, $P \in \tilde{\Gamma}'_{*}(0, C)$, completing the proof of (2).

This concludes our discussion of interpolation by nonnegative $C^{m-1,1}$ functions.

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