Kloosterman Sums with Primes  
and Solvability of a Congruence with Inverse Residues

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Received June 2, 2020; revised October 19, 2020; accepted November 1, 2020

To the blessed memory of Ivan Matveevich Vinogradov

Abstract—The problem of the solvability of the congruence $g(p_1) + \ldots + g(p_k) \equiv m \pmod{q}$  
in primes $p_1, \ldots, p_k \leq N$, $N \leq q^{1-\gamma}$, $\gamma > 0$, is addressed. Here $g(x) \equiv ax + bx \pmod{q}$,  
the inverse of the residue $x$, i.e., $x^{-1} \equiv 1 \pmod{q}$, $q \geq 3$, and $a, b, m, k \geq 3$ are arbitrary integers with $(ab, q) = 1$.  
The analysis of this congruence is based on new estimates of the Kloosterman sums with primes. The main result of the study is an asymptotic formula for the  
number of solutions in the case when the modulus $q$ is divisible by neither 2 nor 3.

DOI: 10.1134/S0081543821040064

1. INTRODUCTION

Let $q \geq 2$, $a$, and $b$ be integers. A complete Kloosterman sum is the exponential sum

$$S(a, b; q) = \sum_{n=1}^{q} e_q(a\overline{n} + bn).$$

(1.1)

Here $e_q(u) = e^{2\pi i u/q}$, the prime denotes summation over numbers $n$ that are coprime to the modulus, and $\overline{n}$ is the inverse of the residue $n$ modulo $q$, i.e., a solution of the congruence $n\overline{n} \equiv 1 \pmod{q}$.  
Sums of the form (1.1) in which the summation variable runs through some set different from the reduced residue system modulo $q$ are called incomplete Kloosterman sums. These include Kloosterman sums with primes

$$W_q(a, b; X) = \sum_{p \leq X} e_q(a\overline{p} + bp).$$

(1.2)

Such sums were studied by É. Fouvry and P. Michel [6], J. Bourgain [2], M. Z. Garaev [8], É. Fouvry and I. E. Shparlinski [7], P. Baker [1], and the present author [12, 13, 15] (see also [3, 11]). Estimates of these sums find application in many problems related to primes and the distribution of inverse residues for a given modulus. Of greatest interest are estimates of such sums

(a) for a maximally small length $X$ of the summation interval, and

(b) with a maximally small reducing factor.

Unfortunately, the current methods do not allow one to succeed in the two directions simultaneously;  
depending on the problem under study, one has to sacrifice either the length of the interval or the  
accuracy of the estimate.

The most precise results (in the first sense) were first proved in the case of prime modulus $q$: here a nontrivial bound has been obtained already for $X \geq q^{1/2+\varepsilon}$, and this seems to be the best

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possible result. The case of an arbitrary composite modulus $q$ is more complicated: even in the case of a “homogeneous” sum, i.e., with $b \equiv 0 \pmod{q}$, which is easier to analyze, nontrivial estimates were available until recently only for $X \geq q^{3/4+\varepsilon}$ (see [7]). In the recent papers [12, 18], this bound has been reduced to $X \geq q^{7/10+\varepsilon}$, and then even to $X \geq q^{1/2+\varepsilon}$, but at the expense of loss in the accuracy of the estimate. In the “inhomogeneous” case, which corresponds to the condition $(ab, q) = 1$, nontrivial estimates for the Kloosterman sums with primes have been obtained only for $q^{3/4+\varepsilon} \leq X \leq q^{3/2}$ (see [16]).

Bounds for the homogeneous Kloosterman sums with primes play an important role in the study of congruences of the form

$$\overline{p}_1 + \overline{p}_2 + \ldots + \overline{p}_k \equiv m \pmod{q}, \quad p_1, \ldots, p_k \leq N$$

(1.3)

(see, for example, [7, 14]). In turn, bounds for the inhomogeneous sums allow one to study congruences of a more general form. The goal of the present paper is to analyze the solvability of the congruence

$$g(p_1) + \ldots + g(p_k) \equiv m \pmod{q}, \quad g(x) \equiv ax + bx \pmod{q},$$

(1.4)

in primes $p_1, \ldots, p_k \leq N$, $N \to +\infty$, depending on the parameters $a, b, m,$ and $k$.

It has long been known that there is an analogy between the solvability problems for Diophantine equations studied by the circle method and the solvability problems for congruences modulo arbitrary composite $q$. In the latter case, the role of “major” and “minor” arcs is played by small and large divisors of $q$, respectively: the former generate the leading term, while the latter are included in the remainder. The analogy does not end here: in the leading term of the asymptotics obtained by the circle method, a factor arises (called the singular series) which is representable as an Euler product over all primes; in the same way, a factor $\zeta_k(a, b, m; q) = \zeta_k(q)$ representable as a product over all prime divisors of $q$ arises in the formula for the number $I_k(a, b, m; q, N) = I_k(N)$ of solutions of the congruence (1.4).

Accordingly, to find asymptotics for $I_k(N)$, one needs bounds for sums of the form $W_r(a, b; N)$ with “large” $r$, $r \mid q$, and approximate expressions for them with “small” $r$, $r \mid q$. For $r \gg N^{2/3}$, the desired bound follows from the above-mentioned result of [16] (see Lemma 5 below); for small $r$, $r \ll (\ln N)^A$ ($A > 1$ is a constant), the sought expression follows from the Siegel–Walfisz theorem. In the present paper we estimate the sum $W_r(a, b; N)$ for “intermediate” values of $r$, $r \mid q$ (see Theorem 3). A conditional version of this statement, which is based on the generalized Riemann hypothesis, is given by Theorem 4. In this case, an appeal to the Riemann hypothesis also makes sense because it allows one to reduce the number $k$ of variables for which the asymptotics can be obtained from seven to three. The following theorem is the first main result of the paper.

**Theorem 1.** Let $0 < \varepsilon < 0.01$ be an arbitrarily small constant, $k \geq 3$ an arbitrary fixed integer, and $q \geq q_0(\varepsilon, k)$. Let also $(ab, q) = 1$,

$$c_k = \frac{2(k + 33)}{3k + 64} \quad \text{for} \quad 3 \leq k \leq 15, \quad \text{and} \quad c_k = \frac{3k + 50}{4(k + 12)} \quad \text{for} \quad k \geq 16,$$

1The aforesaid should be understood as follows. Even in the case of an incomplete Kloosterman sum in which the variable runs through a whole interval of length $N$, a nontrivial power-saving bound for this sum can be obtained (from A. Weil’s classical results) only for $X \geq q^{1/2+\varepsilon}$. For sums of this type, in 1993–1995 A. A. Karatsuba developed a fundamentally new method that allows one to estimate these sums nontrivially even for $X \geq q^\varepsilon$, where $\varepsilon > 0$ is an arbitrarily small fixed number (and even for smaller $X$); however, this method yields only a logarithmic saving.

2Without going into detail, we only note that this is associated with the “multiplicative” property of the quantity $g(n) \equiv a\tau$ in the exponent; in general, the function $g(n) \equiv a\tau + bn$ considered in the present paper does not have such a property.
and, in addition, $q^{c_k + \varepsilon} \leq N \leq q$. Then, the number $I_k(N) = I_k(a, b, m; q, N)$ of solutions of the congruence (1.4) in primes $p_j \leq N$, $(p_j, q) = 1$, satisfies the equality

$$I_k(N) = \frac{\pi^k(N)}{q} (\zeta_k(q) + O(\Delta_k)).$$

Here, for arbitrary fixed $k, a, b$, and $m$, the quantity $\zeta_k(q) = \zeta_k(a, b, m; q)$ is a nonnegative multiplicative function of $q$. Moreover,

1. $\Delta_k = (\ln \ln N)^B(\ln N)^{-A}$ for any $k \geq 7$, with

$$A = \frac{1}{2} + \frac{25}{2}(k - 7) \quad \text{and} \quad B = 2^k - 1;$$

2. if the generalized Riemann hypothesis holds, then $\Delta_k = q^{-\varepsilon}$ for any $k \geq 3$.

The main difficulty is to find conditions under which the formula of Theorem 1 is asymptotic, or, which is the same, to study the structure of the “singular series” $\zeta_k(q)$ as a function of the parameters $k, a, b$, and $m$. The following theorem holds.

**Theorem 2.** Let $(q, 6) = 1$, and let $k \geq 3$ be an arbitrary fixed integer. Then, for any triple $(a; b; m)$ such that $1 \leq a, b, m \leq q$ and $(ab, q) = 1$, the following inequalities hold:

$$\zeta_k(q; a, b, m) \geq \begin{cases} C_1 \exp \left( - \frac{C_2 \sqrt{\ln q}}{\ln \ln q} \right) & \text{for } k = 3, \\ C_3 (\ln \ln q)^{-6} & \text{for } k = 4, \\ 10^{-5} & \text{for } k \geq 5, \end{cases}$$

where $C_j$, $j = 1, 2, 3$, are absolute constants.

The case of $(q, 6) > 1$ reduces in view of multiplicativity to the cases of $q = 3^n$ and $q = 2^n$, which require separate consideration. For certain reasons, the author’s paper [17] devoted to the analysis of the behavior of $\zeta_k(3^n)$ as a function of the parameters $k, a, b$, and $m$ turned out to be published earlier. Note only that in each of these cases, for any $k \geq 3$ and $n \geq 1$, there exist “exceptional” triples $(a; b; m)$ for which $\zeta_k(p^n) = 0$, $p = 2, 3$. In [17], a complete description of all “exceptional” triples for $p = 3$ is obtained and it is shown that $\zeta_k(3^n) > 0.02$ for any “nonexceptional” triple. The case of $p = 2$ seems to be more difficult, and the problem of describing all “exceptional” triples remains open.

The paper is organized as follows. In Section 2, we present the proofs of two estimates for the Kloosterman sum with primes: an unconditional estimate (Theorem 3) and a conditional one, based on the generalized Riemann hypothesis (Theorem 4). Both proofs follow the standard scheme (see, for example, [7, Lemma 2.1]) but use a result from sieve theory (Lemma 2) that leads to a logarithmic saving in the final estimate. The latter, however, is not principal for the original problem. In Section 3, we combine the estimates obtained with the results of [16] and give a proof of Theorem 1. Section 4 is devoted to studying the structure of the “singular series” $\zeta_k(q)$. A remarkable fact is that an important role in this question is played by exact expressions for complete Kloosterman sums $S(a, b, q)$ with prime power modulus $q = p^n$, where $p$ is a prime and $n \geq 2$ (Lemma 7). Such expressions were found by H. Salié as early as 1931; however, the author is not aware of any problem in which the explicit form of these expressions is of considerable importance. Here we also reveal a relationship between the expression for the singular series and the Ramanujan sums, which helps us to exactly calculate and found lower bounds for $\zeta_k(q)$. In Section 5, we prove Theorem 2. Finally, in Section 6, we address the question of estimating a sum over the divisors of $q$, which arises when we derive the formula of Theorem 1. Here we show, in particular, that without some new ideas
the method of this paper does not allow one to derive an unconditional formula for the number of solutions of congruences in the case when the number of unknowns is less than seven.

In the paper, we use the following notation. For integer $a$ and $b$, by $(a, b)$ we mean their greatest common divisor. In all cases, we denote by $(a; b; c)$ an ordered triple of numbers and by $(a, b, c)$ the greatest common divisor of the integers $a$, $b$, and $c$. As usual, $\varphi(n)$, $\Lambda(n)$, $\mu(n)$, and $\tau(n)$ are the Euler, von Mangoldt, Möbius, and the divisor functions, respectively, which are standard in number theory. Denote the number of different prime divisors of $n$ (without counting multiplicities) by $\omega(n)$.

2. ESTIMATES OF KLOOSTERMAN SUMS

Instead of the above sum $W_q(a, b; X)$, it is more convenient to consider the quantities

$$T_q(X) = T_q(a, b; X) = \sum_{n \leq X} \Lambda(n)e_q(a\overline{n} + bn),$$

which are also called Kloosterman sums with primes. Via the Abel transformation, an estimate for $W_q(X)$ implies an estimate for $T_q(X)$ and vice versa. The main goal of the present section is to prove the following two theorems and their corollaries.

**Theorem 3.** Let $q \geq q_0$, $a$, and $b$ be integers such that $(ab, q) = 1$. Then the following estimate holds for any $X \geq q(\tau(q))^{4}(\ln q)^{12}$:

$$T_q(X) \ll X\Delta,$$

where

$$\Delta = \frac{(\ln X)^{5/2}}{\sqrt{q}} \tau(q) + \frac{q^{1/6}}{X} (\ln X)^2 (\tau(q))^{2/3}.$$

**Corollary 1.** If the hypotheses of Theorem 3 are satisfied, then $W_q(a, b; X) \ll \pi(X)\Delta$.

**Theorem 4.** Let $q$, $a$, and $b$ be integers such that $(ab, q) = 1$, and let $X \geq q(\ln q)^4$. If the generalized Riemann hypothesis is true, then

$$T_q(X) \ll X\Delta_1,$$

where

$$\Delta_1 = \frac{\sqrt{q}}{\varphi(q)} \tau(q) + \sqrt{\frac{q}{X}} (\ln X)^2.$$

**Corollary 2.** If the hypotheses of Theorem 4 are satisfied, then $W_q(a, b; X) \ll \pi(X)\Delta_1$.

We need a number of auxiliary statements.

**Lemma 1.** Let $q \geq 2$, $a$, and $b$ be integers. Then the following estimate holds:

$$|S(a, b; q)| \leq \tau(q)\sqrt{q}(a, q)^{1/2}.$$

**Corollary 3.** If the hypotheses of Lemma 1 are satisfied, then for any $N \leq q$ we have the inequality

$$\left| \sum_{n=1}^{N} e_q(a\overline{n} + bn) \right| \leq \tau(q)\sqrt{q}(a, q)^{1/2}(\ln q + 1).$$

For the proof of Lemma 1, see [25, Appendix V, example 11] (for prime $q$) and [5] (for composite $q$). Corollary 3 follows from the lemma in a standard way.

**Lemma 2.** Let $X > 2$ and $a$ be an even number such that $2 \leq a \leq X$. Then the number of primes $p \leq X$ such that $p + a$ is also a prime does not exceed

$$\frac{cX}{(\ln X)^2} \prod_{p|a} \left(1 + \frac{1}{p} \right),$$

where $c > 0$ is an absolute constant.

For the proof, see, for example, [20, Ch. II, §4].
Lemma 3. Let $1 < N < N_1 \leq 2N$ and $1 \leq \delta \leq N$. Then

$$\sum_{N < n_2 < n_1 \leq N_1 \atop n_1 - n_2 \equiv 0 \pmod{\delta}} \Lambda(n_1)\Lambda(n_2) \ll N \sum_{1 \leq m \leq N/\delta} \sum_{d | m\delta} (\mu(d))^2 \cdot \frac{1}{d}.$$ 

Proof. Denote by $W_j$, $j = 1, 2, 3, 4$, the contributions to the sum under consideration of the pairs $(n_1; n_2)$ with, respectively,

1. $n_1 = p_1$ and $n_2 = p_2$,
2. $n_1 = p_1^\ell$ and $n_2 = p_2$,
3. $n_1 = p_1$ and $n_2 = p_2^2$,
4. $n_1 = p_1^\ell$ and $n_2 = p_2^r$.

Here $p_1, p_2$ are primes and $\ell, r$ are integers such that $2 \leq \ell, r \leq H$ where $H = \lceil \log_2 N_1 \rceil + 1$.

If $p_1 - p_2 = m\delta$, then $1 < m\delta \leq N_1 - N \leq N$, and, hence, $1 \leq m \leq N\delta^{-1}$. Therefore, by Lemma 2,

$$W_1 \leq (\ln N_1)^2 \sum_{1 \leq m \leq N/\delta} \sum_{N < p_2 < N_1 \atop p_2 \equiv 0 \pmod{\delta}} 1 \ll (\ln N)^2 \sum_{1 \leq m \leq N/\delta} \frac{N}{(\ln N)^2} \prod_{p | m\delta} \left(1 + \frac{1}{p}\right).$$

Next, to estimate $W_2$, we fix an integer $\ell \geq 2$. Then

$$\sum_{N < p_2 < p_1^\ell \leq N_1 \atop p_1^\ell - p_2 \equiv 0 \pmod{\delta}} (\ln p_1)(\ln p_2) \ll \ln N \sum_{N^{1/\ell} < p_1 \leq N^{1/\ell}} \ln p_1 \sum_{1 \leq m \leq N/\delta} \frac{1}{d} \ll \frac{N}{\delta} \ln N \sum_{N^{1/\ell} < p_1 \leq N^{1/\ell}} \ln p_1 \ll \frac{N^{1+1/\ell}}{\delta}.$$ 

The summation over $2 \leq \ell \leq H$ yields

$$W_2 \ll \frac{N^{3/2}}{\delta} \ln N.$$ 

The same estimate holds for $W_3$. Finally, for fixed $\ell, r \geq 2$, we have

$$\sum_{N < p_2 < p_1^\ell \leq N_1 \atop p_1^\ell - p_2^r \equiv 0 \pmod{\delta}} (\ln p_1)(\ln p_2) \ll \sum_{p_1 \leq N^{1/\ell}} \ln p_1 \sum_{p_2 \leq N^{1/\ell}} \ln p_2 \ll N^{-1/\ell + 1/r}.$$ 

Now, summing over $2 \leq \ell, r \leq H$, we find $W_4 \ll N$. Thus,

$$\sum_{j=1}^{4} W_j \ll N \sum_{1 \leq m \leq N/\delta} \prod_{p | m\delta} \left(1 + \frac{1}{p}\right) + \frac{N^{3/2}}{\delta} \ln N + N \ll N \sum_{1 \leq m \leq N/\delta} \sum_{d | m\delta} (\mu(d))^2 \cdot \frac{1}{d} + \frac{N^{3/2}}{\delta} \ln N.$$ 

It remains to notice that the first term is bounded from below by

$$N \sum_{1 \leq m \leq N/\delta} \sum_{d | \delta} (\mu(d))^2 \cdot \frac{1}{d} \geq \frac{N^2}{2\delta}$$ 

and therefore exceeds the second term in order of magnitude. The lemma is proved. \(\blacksquare\)
Lemma 4. Let $1 < M < M_1 \leq 2M$ and $1 < N < N_1 \leq 2N$ with $MN \leq X$, and let $|a_m| \leq \ln m$ and $|b_m| \leq \tau(m)$ for $M < m \leq M_1$. Then for the sums

$$U(M, N) = \sum'_{M < m \leq M_1} a_m \sum'_{N < n \leq N_1 \atop m n \leq X} e_q(a m n + b m n),$$

$$W(M, N) = \sum'_{M < m \leq M_1} b_m \sum'_{N < n \leq N_1 \atop m n \leq X} \Lambda(n) e_q(a m n + b m n)$$

we have the estimates

$$U(M, N) \ll \left( M \sqrt{N} + \frac{M N}{\sqrt{q}} \right) \tau(q) + N \sqrt{M} \sqrt{q} \tau(q) \sqrt{\ln X} \ln X, \quad (2.1)$$

$$W(M, N) \ll \left( M \sqrt{N} + \frac{M N}{\sqrt{q}} \right) \tau(q) + N \sqrt{M} \sqrt{q} \tau(q) (\ln X)^2. \quad (2.2)$$

Proof. Introducing general notation for the sums,

$$V(M, N) = \sum'_{M < m \leq M_1} \alpha(m) \sum'_{N < n \leq N_1 \atop m n \leq X} \beta(n) e_q(a m n + b m n),$$

and applying the Cauchy inequality, we obtain

$$|V(M, N)|^2 \leq \|\alpha\|_2^2 \sum'_{M < m \leq M_1} \left| \sum'_{N < n \leq N_1 \atop m n \leq X} \beta(n) e_q(a m n + b m n) \right|^2,$$

where

$$\|\alpha\|_2 = \left( \sum'_{M < m \leq M_1} |\alpha(m)|^2 \right)^{1/2}.$$

Next,

$$|V(M, N)|^2 \leq \|\alpha\|_2^2 \sum'_{M < m \leq M_1} \sum'_{N < n_1, n_2 \leq N_1 \atop m n_1, m n_2 \leq X} \beta(n_1) \beta(n_2) e_q(a(n_1 - n_2) m + b(n_1 - n_2) m)$$

$$\leq \|\alpha\|_2^2 \left\{ \sum'_{M < m \leq M_1} \sum'_{N < n \leq N_1 \atop m n \leq X} |\beta(n)|^2 + 2 \sum'_{N < n_2 < n_1 \leq N_1} |\beta(n_1) \beta(n_2)| \sum'_{M < m \leq M_2} e_q(a(n_1 - n_2) m + b(n_1 - n_2) m) \right\},$$

where $M_2 = \min\{M_1, X n_1^{-1}\}$. Let us split the interval $(M, M_2]$ into segments of length $q$ each (except possibly for the last one) and apply the estimates of Lemma 1 and Corollary 1. Then, using the equality $(\pi_1 - \pi_2, q) = (n_1 - n_2, q)$, we have

$$|V(M, N)|^2 \leq M |||\alpha\|_2^2 \|\beta\|_2^2 + 2 |||\alpha\|_2^2 ([M q^{-1}] \sqrt{q} \tau(q) + \sqrt{q} \tau(q) (\ln q + 1)) S,$$

where $S = \sum_{M < m \leq M_2} |\beta(n)|^2$.
where
\[
S = \sum_{N < n_2 < n_1 \leq N_1} |\beta(n_1)\beta(n_2)| (n_1 - n_2, q)^{1/2} \leq \sum_{d | q, \delta \leq N} \sum_{N < n_2 < n_1 \leq N_1} |\beta(n_1)\beta(n_2)|.
\]

In the case of the sum \( U(M, N) \), we obviously have
\[
\|\alpha\|^2 \ll M(\ln X)^2, \quad \|\beta\|^2 \ll N, \quad S \ll \sum_{d | q, \delta \leq N} \sqrt{\delta} N^2 \ll N^2 \tau(q),
\]
so that
\[
|U(M, N)|^2 \ll M^2 N(\ln X)^2 + M(\ln X)^2 \left(\frac{M}{\sqrt{q}} + \sqrt{q} \ln q\right) N^2 (\tau(q))^2
\ll M^2 N(\ln X)^2 + \frac{(MN)^2}{\sqrt{q}} (\ln X)^2 (\tau(q))^2 + MN^2 \sqrt{q}(\ln X)^3 (\tau(q))^2,
\]
which implies the desired estimate. In the second case, we have the inequalities
\[
\|\alpha\|^2 \ll M(\ln X)^3, \quad \|\beta\|^2 \ll N \ln X.
\]

Next, by Lemma 3, we find
\[
S \ll N \sum_{d | q, \delta \leq N, \delta \ll N} \sum_{1 \leq m \leq N/\delta} \frac{(\mu(d))^2}{d} \ll N \Sigma,
\]
where
\[
\Sigma = \sum_{d | q, \delta \leq N} \sum_{1 \leq m \leq N/\delta} \frac{(\mu(d))^2}{d} = \sum_{d \leq N} \frac{(\mu(d))^2}{d} \sum_{\delta | q, \delta \leq N} \frac{1}{\delta} \sum_{1 \leq m \leq N/\delta, m \delta \equiv 0 (\mod d)} 1.
\]

The inner sum does not exceed \( N(d, \delta)(d\delta)^{-1} \). Therefore,
\[
\sum_{d \leq N} \frac{(\mu(d))^2}{d} \sum_{d | q, \delta \leq N} \sqrt{\delta} N \frac{d}{d\delta} (d, \delta) = N \sum_{d \leq N} \frac{(\mu(d))^2}{d^2} \sum_{\delta | q, \delta \leq N} \frac{1}{\sqrt{\delta}} \leq N \sum_{\delta | q, \delta \leq N} \frac{1}{\sqrt{\delta}} \sum_{d = 1}^{\delta \ll \infty} \frac{(\mu(d))^2}{d^2} (d, \delta).
\]

Obviously, the sum over \( d \) coincides with
\[
\prod_p \left(1 + \frac{(p, \delta)}{p^2}\right) = \prod_{p | \delta} \left(1 + \frac{1}{p^2}\right) \prod_{p | \delta} \left(1 + \frac{1}{p}\right) \ll \prod_{p | \delta} \left(1 + \frac{1}{p}\right) \ll \sum_{d | \delta} \frac{(\mu(d))^2}{d}.
\]

Thus,
\[
\Sigma \ll N \sum_{d | q, \delta \leq N} \frac{1}{\sqrt{\delta}} \sum_{\delta \ll \infty} \frac{(\mu(d))^2}{d} \ll N \sum_{d | q, \delta \leq N} \frac{(\mu(d))^2}{d} \sum_{c | q \delta - 1} \frac{1}{\sqrt{cd}} \ll N \sum_{d | q, \delta \leq N} \frac{(\mu(d))^2}{d^{5/2}} \tau(q \delta - 1) \ll N \tau(q).
\]

Therefore, \( S \ll N^2 \tau(q) \), and so
\[
|W(M, N)|^2 \ll M^2 N(\ln X)^4 + M(\ln X)^3 \left(\frac{M}{\sqrt{q}} + \sqrt{q} \ln q\right) N^2 (\tau(q))^2
\ll M^2 N(\ln X)^4 + \frac{(MN)^2}{\sqrt{q}} (\ln X)^2 (\tau(q))^2 + MN^2 \sqrt{q}(\ln X)^3 (\tau(q))^2.
\]

This implies the assertion of the lemma. \( \Box \)
Lemma 5. Let $0 < \varepsilon < 0.1$ be an arbitrarily small fixed number and $q \geq q_0(\varepsilon)$ with $(ab, q) = 1$. Then, for any $X$ satisfying the inequalities $q^{3/4+\varepsilon} \leq X \leq (q/2)^{3/2}$, the sum $W_q(a, b; X)$ is estimated as $W_q(X) \ll Xq^\Delta$ with

$$
\Delta = (q^{3/4}X^{-1})^{1/7} + (q^{2/3}X^{-1})^{3/35} \ll \begin{cases} 
(q^{3/4}X^{-1})^{1/7} & \text{for } q^{3/4+\varepsilon} \leq X \leq q^{7/8}, \\
(q^{2/3}X^{-1})^{3/35} & \text{for } q^{7/8} \leq X \leq \left(\frac{q}{2}\right)^{3/2}.
\end{cases}
$$

Lemma 6. If $(ab, q) = 1$, then the number of solutions of the congruence $g(x) \equiv g(y) \pmod{q}$ with the conditions $1 \leq x, y \leq q$ and $(xy, q) = 1$ does not exceed $2\omega(q)+1\tau(q)q$.

For the proofs of Lemmas 5 and 6, see [16].

Proof of Theorem 3. Fixing a number $V$ such that $1 < V < \sqrt{X}$ and applying R. Vaughan’s identity in the form presented in [24, Ch. II, § 6, Theorem 1], we obtain

$$
T_q(X) = S_1 - S_2 - S_3 - S_4 + O(V)
$$

with

$$
S_1 = \sum_{m \leq V} \mu(m) \sum_{n \leq X/m} (\ln n) e_q(a\overline{mn} + bmn), \\
S_2 = \sum_{m \leq V} a_m \sum_{n \leq X/m} e_q(a\overline{mn} + bmn), \\
S_3 = \sum_{V < m \leq V^2} a_m \sum_{n \leq X/m} e_q(a\overline{mn} + bmn), \\
S_4 = \sum_{V < m \leq X/V} b_m \sum_{V < n \leq X/m} \Lambda(n) e_q(a\overline{mn} + bmn),
$$

where

$$
a_m = \sum_{r\ell = m, r, \ell \leq V} \mu(r)\Lambda(\ell), \\
b_m = \sum_{d|m, d \leq V} \mu(d).
$$

By Abel’s summation formula and Lemma 1, we estimate the inner sum $S_1(m)$ in the expression for $S_1$ as

$$
S_1(m) = \sum_{n \leq X/m} (\ln n) e_q(a\overline{mn} + bmn) = C\left(\frac{X}{m}\right) \ln \frac{X}{m} - \int_1^{X/m} C(u) \frac{du}{u},
$$

where

$$
C(u) = \sum_{n \leq u} e_q(a\overline{mn} + bmn) = \left[\frac{u}{q}\right] S(a, b; q) + \sum_{1 \leq n \leq u/q} e_q(a\overline{mn} + bmn)
$$

$$
\ll \left[\frac{u}{q}\right] \sqrt{q} \tau(q) + \sqrt{q} \tau(q) \ln q \ll \left(\frac{u}{\sqrt{q}} + \sqrt{q} \ln q\right) \tau(q).
$$

Hence,

$$
S_1(m) \ll \frac{X}{m} \frac{\tau(q)}{\sqrt{q}} \ln X + \sqrt{q} \tau(q) \ln X)^2, \quad \text{so} \quad S_1 \ll \frac{X}{\sqrt{q}} \frac{\tau(q)}{\sqrt{q}} (\ln X)^2 + V \sqrt{q} \tau(q) (\ln X)^2.
$$

The same bound also holds for the sum $S_2$. 

Next, we split the intervals $(V, V^2]$ and $(1, XV^{-1}]$ of variation of the variables $m$ and $n$ in the sum $S_3$ into intervals of the form $M < m \leq M_1$ and $N < n \leq N_1$ ($M_1 \leq 2M$ and $N_1 \leq 2N$) by the points $M = 2^sV$ and $N = XV^{-1}2^{-t}$, $s, t = 1, 2, \ldots$. Thus we obtain a partition of $S_3$ into sums of
the form $U(M, N)$ from Lemma 4. Applying inequality (2.1) and denoting the summation over all relevant values of $M$ and $N$ by a double prime, we obtain

$$S_3 \ll \frac{X}{\sqrt{V}} \ln X + \frac{X}{\sqrt{q}} (\ln X)^2 \tau(q) + \frac{X}{\sqrt{q}} \sqrt{q} (\ln X)^{3/2}.$$

Similarly, splitting $(V, XV^{-1})$ into intervals whose right endpoint does not exceed twice the left one and applying estimate (2.2), we obtain

$$S_4 \ll \frac{X}{\sqrt{V}} \ln X + \frac{X}{\sqrt{q}} (\ln X)^2 \tau(q) + \frac{X}{\sqrt{q}} \sqrt{q} (\ln X)^{3/2}.$$

Adding the estimates for $S_j$, $1 \leq j \leq 4$, and noticing that $V \sqrt{q} \tau(q) (\ln X)^2 \leq V \sqrt{X} (\ln X)^2$, we find

$$T_q(X) \ll V \sqrt{X} (\ln X)^2 + \frac{X}{\sqrt{q}} (\ln X)^3 \tau(q) + \frac{X}{\sqrt{q}} \sqrt{q} (\ln X)^{5/2} \tau(q).$$

Now, we specify $V$ in such a way that the first and last terms are of the same order of magnitude:

$$V \sqrt{X} (\ln X)^2 = \frac{X}{\sqrt{q}} \sqrt{q} (\ln X)^2 \tau(q), \quad \text{i.e.,} \quad V = X^{1/3} q^{1/6} (\tau(q))^{2/3}.$$

The condition $V \leq \sqrt{X}$ is equivalent to the inequality $X \geq q(\tau(q))^4$, which obviously holds for all $X$ under consideration. Thus, we obtain

$$T_q(X) \ll X^{5/6} q^{1/6} (\ln X)^2 (\tau(q))^{2/3} + \frac{X}{\sqrt{q}} (\ln X)^3 \tau(q)$$

$$\ll X \left\{ \left( \frac{q}{X} \right)^{1/6} (\tau(q))^{2/3} (\ln X)^2 + \frac{(\ln X)^{5/2}}{\sqrt{q}} \tau(q) \right\}.$$  

Theorem 3 is proved. \hfill \Box

**Proof of Corollary 1.** By Abel’s summation formula,

$$W_q(a, b; X) = \sum_{n \leq X}^{\tau} \Lambda(n) \frac{a(n)}{\ln n} + O(\sqrt{X}) = \frac{T_q(X)}{\ln X} + \int_1^X \frac{T_q(u)}{u \ln u^2} du + O(\sqrt{X}).$$

Setting $Y = q(\tau(q))^4 (\ln q)^{12}$, we estimate the sum $T_q(u)$ trivially for $2 \leq u \leq Y$ and with the use of Theorem 3 for $Y < u \leq X$. This yields

$$W_q(a, b; X) \ll \pi(X) \Delta + \frac{Y}{(\ln Y)^2} + \int_Y^X \left( u^{-1/6} q^{1/6} (\tau(q))^{2/3} + \frac{\tau(q)}{\sqrt{q}} \sqrt{\ln u} \right) du$$

$$\ll \pi(X) \Delta + \frac{Y}{(\ln Y)^2} + \frac{X \Delta}{(\ln X)^2} \ll \pi(X) \Delta. \quad \Box$$
Proof of Theorem 4. Let us represent $T_q(X)$ as

$$T_q(X) = \sum_{\nu=1}^{q} \sum_{\nu \equiv \mu \pmod{q}}^{\nu} \Lambda(n) e_q(a\nu + bn) = \sum_{\nu=1}^{q} e_q(a\nu + b\nu) \sum_{\nu \equiv \mu \pmod{q}}^{\nu} \Lambda(n).$$

The inner sum can be written as follows:

$$\sum_{\nu \leq X} \Lambda(n) \sum_{\chi \pmod{q}} \chi(n) \mu = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \mu \sum_{\nu \leq X} \chi(n) \lambda(n) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \chi(n) \mu \psi(X; \chi),$$

where

$$\psi(X; \chi) = \sum_{n \leq X} \chi(n) \lambda(n).$$

Thus,

$$T_q(X) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \psi(X; \chi) \sum_{\nu=1}^{q} \chi(n) e_q(a\nu + b\nu) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \psi(X; \chi) S(a, b, q; \chi),$$

where

$$S(a, b, q; \chi) = \sum_{\nu=1}^{q} \chi(n) e_q(a\nu + b\nu).$$

Separating the contribution of the principal character $\chi_0$, we obtain

$$T_q(X) = \frac{\psi(X; \chi_0)}{\varphi(q)} S(a, b; q) + R, \quad \text{where} \quad |R| \leq \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \psi(X; \chi) \cdot |S(a, b, q; \chi)|.$$

It is easy to see that

$$\psi(X; \chi_0) = \psi(X) + O((\ln X)^2) \ll X.$$

Moreover, it follows from the generalized Riemann hypothesis that $\psi(X; \chi) \ll \sqrt{X}(\ln X)^2$ for any character $\chi \neq \chi_0$ (see, for example, [4, Ch. 20]). Thus,

$$T_q(X) \ll \frac{X}{\varphi(q)} \sqrt{q} \tau(q) + \frac{\sqrt{X}}{\varphi(q)} (\ln X)^2 S, \quad \text{where} \quad S = \sum_{\chi \pmod{q}} |S(a, b, q; \chi)|.$$

Applying the Cauchy inequality, we find

$$S^2 \leq \varphi(q) \sum_{\chi \pmod{q}} \left| \sum_{\nu=1}^{q} \chi(n) e_q(a\nu + b\nu) \right|^2 = \varphi(q) \sum_{\chi \pmod{q}} \sum_{\mu, \nu=1}^{q} \chi(\nu) \lambda(\mu) e_q(a(\nu - \mu) + b(\nu - \mu))$$

$$= \varphi(q) \sum_{\mu, \nu=1}^{q} e_q(a(\nu - \mu) + b(\nu - \mu)) \sum_{\chi \pmod{q}} \chi(\nu) \lambda(\mu) = \varphi^3(q).$$

Finally, we obtain

$$T_q(X) \ll \frac{X}{\varphi(q)} \sqrt{q} \tau(q) + \sqrt{X} \sqrt{\varphi(q)} (\ln X)^2 \ll X \Delta_1. \quad \Box$$

The proof of Corollary 2 is similar to that of Corollary 1 with the only difference that as $Y$ we should take $q(\ln q)^4$.
3. PROOF OF THEOREM 1

First of all, we have

$$I_k(N) = \frac{1}{q} \sum_{c=1}^{q} e_q(-cm)W_q^k(ac, bc; N).$$

If \((c, q) = \delta\), then \(q = \delta r\) and \(c = \delta f\) for some \(f\) and \(r\) such that \((r, f) = 1\). Hence,

$$I_k(N) = \frac{1}{q} \sum_{r \mid q} \sum_{f=1}^{r} e_r(-mf)W_{br}^k(a\delta f, b\delta f; N).$$  (3.1)

To prove the unconditional result, we set \(L = \ln N\), \(F = L^C\), \(C = 60\), and \(G = 2N^{2/3}\) and split the sum in (3.1) into three parts \(I_k^{(1)}, I_k^{(2)}, \text{ and } I_k^{(3)}\) corresponding to the intervals \(1 \leq r \leq F\), \(F < r \leq G\), and \(G < r \leq q\) (some of the sums may be empty). The first of these sums gives the leading term of the asymptotics for \(I_k(N)\), and the other sums enter the remainder term.

**Calculation of \(I_k^{(1)}\).** First, notice that

$$W_{br}(a\delta f, b\delta f; N) = \sum_{p \leq N \atop \text{prime}} e_p(f(a\bar{p} + bp)) = W_r(a, bf; N) + \theta_1\omega(q), \quad |\theta_1| \leq 1. \quad (3.2)$$

Next,

$$W_r(a, bf; N) = \sum_{h=1}^{r} e_r(f(a\bar{h} + bh)) \sum_{p \leq N \atop p \equiv h \text{ (mod } r)} 1 = \sum_{h=1}^{r} e_r(f(a\bar{h} + bh))\pi(N; r, h).$$

By the Siegel–Walfisz theorem for \(1 < r \leq F\), we have

$$\pi(N; r, h) = \frac{\pi(N)}{\varphi(r)} + O\left(\pi(N)e^{-c\sqrt{\varphi(r)}}\right), \quad c > 0.$$\n
Thus,

$$W_{br}(a\delta f, b\delta f; N) = \frac{\pi(N)}{\varphi(r)} S(a, bf; r) + O(r\pi(N)e^{-c\sqrt{\varphi(r)}}).$$

Therefore,

$$W_{br}^k(a\delta f, b\delta f; N) = \frac{\pi^k(N)}{\varphi^k(r)} S^k(a, bf; r) + O\left(\frac{\pi^k(N)}{\varphi^k(r)} |S(a, bf; r)|^{k-1}e^{-c\sqrt{\varphi(r)}}\right) + O(\pi^k(N)r^{k-1}e^{-ck\sqrt{\varphi(r)}})$$

$$= \frac{\pi^k(N)}{\varphi^k(r)} S^k(a, bf; r) + O(\pi^k(N)e^{-c_1\sqrt{\varphi(r)}}), \quad c_1 = \frac{2}{3}c.$$

Returning to the sum \(I_k^{(1)}\), we obtain

$$I_k^{(1)} = \frac{1}{q} \sum_{r \mid q \atop 1 \leq r \leq F} e_r(-mf)\left\{\frac{\pi^k(N)}{\varphi^k(r)} S^k(a, bf; r) + O(\pi^k(N)e^{-c_1\sqrt{\varphi(r)}})\right\}$$

$$= \frac{\pi^k(N)}{q} \sum_{r \mid q \atop 1 \leq r \leq F} A_k(r) + O\left(\frac{\pi^k(N)}{q} e^{-c_2\sqrt{\varphi(r)}}\right),$$

$$A_k(r) = \frac{\pi^k(N)}{q} \sum_{m \leq F} e_m(-mf)\left\{S^k(a, bf; r) + O(\pi^k(N)e^{-c_1\sqrt{\varphi(r)}})\right\}.$$
where
\[ A_k(r) = A_k(a, b, m; r) = \frac{1}{\varphi^k(r)} \sum_{\substack{f=1 \\ (f,r)=1}}^r e_r(-mf) S^k(af, bf; r), \quad c_2 = \frac{1}{3} c. \]

Now, let us show that \( A_k(r) \) is a multiplicative function of \( r \) (for fixed \( a, b, m, \) and \( k \)). Indeed, let \( r = r_1 r_2 \) with \( (r_1, r_2) = 1 \) and \( r_1, r_2 > 1 \). If \( y \) and \( z \) run through the reduced residue systems modulo \( r_1 \) and \( r_2 \), then \( x = yr_2 + zr_1 \) runs through the reduced residue system modulo \( r \) and, in this case,
\[ x \equiv \overline{y} r_2^2 r_2 + \overline{z} r_1^2 r_1 \pmod{r}, \]
where \( \overline{y} \equiv r_2 \overline{r}_2 \equiv 1 \pmod{r_1} \) and \( \overline{z} \equiv r_1 \overline{r}_1 \equiv 1 \pmod{r_2} \). Thus,
\[
S(u, v; r) = \sum_{\substack{z=1 \\ (z,r)=1}}^r \exp\left(\frac{2\pi i}{r} (u\overline{x} + vx)\right) = \sum_{\substack{y=1 \\ (y,r_1)=1}}^{r_1} \sum_{\substack{z=1 \\ (z,r_2)=1}}^{r_2} \exp\left(\frac{2\pi i}{r_1 r_2}(u\overline{y} r_2^2 r_2 + u\overline{z} r_1^2 r_1 + vy r_2 + vz r_1)\right)
\]
\[ = \sum_{\substack{y=1 \\ (y,r_1)=1}}^{r_1} \exp\left(\frac{2\pi i}{r_1}(u\overline{y} r_2^2 + vy)\right) \sum_{\substack{z=1 \\ (z,r_2)=1}}^{r_2} \exp\left(\frac{2\pi i}{r_2}(u\overline{z} r_1^2 + vz)\right). \]

Replacing \( y \) and \( z \) by \( y \overline{r}_2 \) and \( z \overline{r}_1 \), respectively, we obtain
\[
S(u, v; r) = \sum_{\substack{y=1 \\ (y,r_1)=1}}^{r_1} \exp\left(\frac{2\pi i}{r_1}(u\overline{y} + vy)\right) \sum_{\substack{z=1 \\ (z,r_2)=1}}^{r_2} \exp\left(\frac{2\pi i}{r_2}(u\overline{z} + vz)\right) = S(u \overline{r}_2, v \overline{r}_2; r_1) S(u \overline{r}_1, v \overline{r}_1; r_2). \]

Further, if \( f \) runs through the reduced residue system modulo \( r = r_1 r_2 \), then \( f \) can be represented as \( f = sr_2 + tr_1 \) with \( s \) and \( t \) running through the reduced residue systems modulo \( r_1 \) and \( r_2 \), respectively. Thus we obtain
\[
S(af, bf; r) = S(af \overline{r}_2, bf \overline{r}_2; r_1) S(af \overline{r}_1, bf \overline{r}_1; r_2) = S(as, bs; r_1) S(at, bt; r_2)
\]
and
\[ e_r(-mf) = e_{r_1}(-ms) e_{r_2}(-mt). \]

Hence we conclude that
\[
A_k(r) = \frac{1}{\varphi^k(r)} \sum_{\substack{f=1 \\ (f,r)=1}}^r e_r(-mf) S^k(af, bf; r)
\]
\[ = \frac{1}{\varphi^k(r_1)} \sum_{\substack{s=1 \\ (s,r_1)=1}}^{r_1} e_{r_1}(-ms) S^k(as, bs; r_1) \cdot \frac{1}{\varphi^k(r_2)} \sum_{\substack{t=1 \\ (t,r_2)=1}}^{r_2} e_{r_2}(-mt) S^k(at, bt; r_2)
\]
\[ = A_k(r_1) A_k(r_2), \]
as required. If \( (u, v, r) = 1 \), then by Lemma 1
\[ |A_k(r)| < \frac{\varphi(r)^{k/2}(\tau(r))^k}{\varphi^k(r)} \ll_r r^{-k/2+1+\varepsilon}, \tag{3.3} \]
Thus, we obtain

\[
\left| \sum_{F < r \leq q, r \mid q} A_k(r) \right| \ll_\varepsilon \sum_{r > F} r^{-k/2+1+\varepsilon} \ll_\varepsilon F^{-k/2+2+\varepsilon}
\]

for \( k \geq 2 \). Hence,

\[
I_k^{(1)} = \frac{\pi^k(N)}{q} \chi_k(q) + O_\varepsilon \left( \frac{\pi^k(N)}{q} \left( F^{-k/2+2+\varepsilon} + e^{-\varepsilon/2} \right) \right)
\]

\[
= \frac{\pi^k(N)}{q} \chi_k(q) + O_\varepsilon \left( \frac{\pi^k(N)}{q} L^{-C(k/2-2-\varepsilon)} \right), \tag{3.4}
\]

where

\[
\chi_k(q) = \chi_k(a, b, m; q) = \sum_{r \mid q} A_k(q).
\]

Obviously, \( \chi_k(q) \) is a multiplicative function for fixed \( a, b, m, \) and \( k \).

**Estimation of** \( I_k^{(2)} \) **and** \( I_k^{(3)} \). **Denote by** \( E_j, j = 2, 3 \), the intervals \( F < r \leq G \) and \( G < r \leq q \). **Then, according to (3.2), we have**

\[
|I_k^{(1)}| \leq \frac{1}{q} \sum_{r \mid q} \sum_{r \in E_j (f, r) = 1} |W_r(af, bf; N) + \theta_1 \omega(q)|^k \leq 2^{k-1} \sum_{r \mid q} \sum_{r \in E_j (f, r) = 1} |W_r(af, bf; N)|^k + \omega^k(q)
\]

\[
\leq 2^{k-1} \sum_{r \mid q, r \in E_j} W_r^{k-2} \sum_{r \in E_j} |W_r(af, bf; N)|^2 + \left( \frac{2\omega(q)}{q} \right)^k \sum_{r \mid q} \varphi(r)
\]

\[
= 2^{k-1} \sum_{r \mid q, r \in E_j} W_r^{k-2} r J_r(N) + (2\omega(q))^k,
\]

where \( W_r = \max_{(f, r) = 1} |W_r(af, bf; N)| \) and \( J_r(N) \) stands for the number of solutions of the congruence

\[
g(p_1) \equiv g(p_2) \pmod{r}
\]

in primes \( 1 < p_1, p_2 \leq N \). **Splitting the variation ranges of** \( p_1 \) **and** \( p_2 \) **into intervals of length** \( r \) **and applying Lemma 6, we find**

\[
r J_r(N) \leq (\lfloor N/r \rfloor + 1)^2 \cdot 2^\omega(r) r^2 \tau(r) \ll (N^2 + r^2) (\tau(r))^2.
\]

If \( j = 2 \), then \( r \leq G = 2N^{2/3} \) **for any** \( r \in E_j \), **so that** \( N \geq r^{3/2} \) **and** \( N^2 + r^2 \ll N^2 \). **Using Corollary 3, we obtain**

\[
W_r \ll \pi(N) \left\{ \left( \frac{r}{N} \right)^{1/6} L^2(\tau(r))^{2/3} + L^{5/2} \frac{\tau(r)}{\sqrt{r}} \right\} \ll \pi(N) \left\{ N^{-1/18} L^2(\tau(r))^{2/3} + L^{5/2} \frac{\tau(r)}{\sqrt{r}} \right\}.
\]

Thus,

\[
I_k^{(2)} \ll \frac{1}{q} \sum_{F < r \leq G, r \mid q} \pi^{k-2}(N) \left\{ \left( \frac{(\tau(r))^2 L^6}{N^{1/6}} \right)^{(k-2)/3} + L^{5(k-2)/2} \frac{(\tau(r))^{k-2}}{\tau(k-2)/4} \right\} N^2(\tau(r))^2
\]

\[
\ll \frac{\pi^k(N)}{q} \sum_{F < r \leq G, r \mid q} \left\{ L^2 \left( \frac{(\tau(r))^2 L^6}{N^{1/6}} \right)^{(k-2)/3} + L^{5k/2-3} \frac{(\tau(r))^{k-2}}{\tau(k-2)/4} \right\} (\tau(r))^2
\]
where

Now, the required assertion follows from equality (3.4) and estimates (3.5) and (3.6).

hence,

By the condition $k \geq 7$, the first term is not greater than $N^{-(k-2)/20} \leq N^{-1/4}$ in order of magnitude. Since $(k-2)/4 \geq 5/4 > 1$, the series in $r$ converges. Using Mardzhanishvili’s inequality [19], we can easily conclude that this sum is bounded in order of magnitude by

\[
\frac{(\ln F)^{2k-1}}{F(k-6)/4} \ll \frac{(\ln L)^{2k-1}}{L^{C(k-6)/4}}.
\]

Thus we obtain

\[
I_k^{(2)} \ll \frac{\pi^k(N)}{q} \left\{ N^{-1/4} + L^{5k/2-3} \frac{(\ln L)^{2k-1}}{L^{C(k-6)/4}} \right\} \ll \frac{\pi^k(N)}{q} \frac{(\ln L)^{2k-1}}{L^A},
\]

where

\[
A = \frac{C}{4} (k - 6) - \frac{5k}{2} + 3 = \frac{25}{2} (k - 7) + \frac{1}{2}.
\]

Applying the same arguments in the case of $j = 3$, but with the estimate of Corollary 1 replaced by that of Lemma 5, for any fixed $\varepsilon$ we obtain

\[
W_r \ll (N^{6/7}q^{3/28} + N^{32/35}r^{2/35})r^{\varepsilon/(4k)} \ll \pi(N)\left(N^{-1/7}q^{3/28} + N^{-3/35}q^{2/35}\right)q^{\varepsilon/(3k)},
\]

\[
rI_r(N) \ll (N^2 + r^2)(\tau(q))^2 \ll q^2(\tau(q))^2 \ll \pi^2(N) N^{-2}q^{2+\varepsilon/6};
\]

hence,

\[
I_k^{(3)} \ll \frac{\pi^k(N)}{q} \sum_{G < r \leq q, r \mid q} \left( N^{-(k-2)/7}q^{3(k-2)/28} + N^{-3(k-2)/35}q^{2(k-2)/35} \right) N^{-2}q^2q^{\varepsilon/2-2\varepsilon/(3k)}
\]

\[
= \frac{\pi^k(N)}{q} q^{\varepsilon/2} \left( N^{-(k-2)/7-2}q^{3(k-2)/28+2} + N^{-3(k-2)/35-2}q^{2(k-2)/35+2} \right)
\]

\[
\ll \frac{\pi^k(N)}{q} q^{\varepsilon/2} \left\{ \left( \frac{q^{\alpha_k}}{N} \right)^{(k+12)/7} + \left( \frac{q^{\beta_k}}{N} \right)^{(3k+64)/35} \right\},
\]

where

\[
\alpha_k = \frac{3k + 50}{4(k + 12)} \quad \text{and} \quad \beta_k = \frac{2(k + 33)}{3k + 64}.
\]

Since $\max\{\alpha_k, \beta_k\} = c_k$, for $N \geq q^{c_k+\varepsilon}$ we have

\[
I_k^{(3)} \ll \frac{\pi^k(N)}{q} q^{\varepsilon/2}(q^{-(\varepsilon(k+12)/7} + q^{-(3k+64)/35})
\]

\[
\ll \frac{\pi^k(N)}{q} \left( q^{-\varepsilon(2k+17)/14} + q^{-\varepsilon(6k+93)/70} \right) \ll \frac{\pi^k(N)}{q} q^{-\varepsilon}.
\]

Now, the required assertion follows from equality (3.4) and estimates (3.5) and (3.6).
The conditional result is proved by similar arguments; however, we have to set \( F = N^{1/3} \) and \( G = 2N^{2/3} \) to this end. If the generalized Riemann hypothesis is true, then

\[
\pi(x; r, c) = \frac{\pi(x)}{\varphi(r)} + O(\sqrt{x} \ln x)
\]

for any \( r \) and \( c \) with \( (r, c) = 1 \) (see [4, Ch. 20]; the implicit constant is absolute). Proceeding to the calculation of \( I_k^{(1)} \) and using Corollary 3, we find

\[
W_r(a_f, b_f; N) = \frac{\pi(N)}{\varphi(r)} S(a_f, b_f; r) + O(\varphi(r)\sqrt{N}L),
\]

\[
W_k^g(a_f, b_f; N, r) = \frac{\pi^k(N)}{\varphi^k(r)} S^k(a_f, b_f; r) + O\left(\frac{\pi^{k-1}(N)}{\varphi^{k-1}(r)}(\sqrt{\tau(r)})^{k-1} \varphi(r)\sqrt{N}\right)
\]

+ \( O(\varphi(r)N^{k/2}L^k) \),

so that

\[
I_k^{(1)} = \frac{\pi^k(N)}{q} \sum_{1 \leq r \leq F, r|q} A_k(r) + O\left(\frac{N^{k/2}}{q} \sum_{1 \leq r \leq F, r|q} \varphi^{k+1}(r)\right) + O\left(\frac{\pi^{k-1}(N)}{q} \sqrt{N} \sum_{1 \leq r \leq F, r|q} \left(\frac{\sqrt{\tau(r)}}{\varphi^{k-3}(r)}\right)^{k-1}\right).
\]

The second term in (3.7) does not exceed

\[
\frac{N^{k/2}L^k}{q} F^{k+1} \tau(q) \ll \frac{N^k}{q} F \frac{N^{k/2}}{L^2} \tau(q) \ll \frac{N^{k-2}/6}{q} F L^{2k} \tau(q) \ll \frac{\pi(N)}{q} q^{-\varepsilon}
\]

in order of magnitude. Next, since \( r/\varphi(r) \ll \ln r \ll \ln L \), the last term in (3.7) can be estimated as

\[
\frac{\pi^k(N)}{q} \sum_{1 \leq r \leq F, r|q} \left(\frac{r}{\varphi(r)}\right)^{k-3} \left(\frac{\tau(r)}{\varphi^{k-3}(r)}\right)^{k-1} \frac{r^{(k-1)/2}}{r^{k-3}} \ll \frac{\pi^k(N)}{q} \frac{L^2}{\sqrt{N}} (\ln L)^{k-3} \tau(q) \sum_{1 \leq r \leq F, r|q} r^{-(k-5)/2}
\]

\[
\ll \frac{\pi^k(N)}{q} \frac{L^2}{\sqrt{N}} (\ln L)^{k-3} \tau(q) \left(1 + F^{-(k-5)/2}\right)
\]

\[
\ll \frac{\pi^k(N)}{q} L^2 (\ln L)^{k-3} \tau(q) \left(N^{-1/2} + N^{-(k-2)/6}\right) \ll \frac{\pi^k(N)}{q} q^{-\varepsilon}.
\]

To estimate \( I_k^{(2)} \), we apply Corollary 2:

\[
I_k^{(2)} \ll \frac{1}{q} \sum_{F < r \leq G, r|q} \pi^{k-2}(N) \left\{ \left(\frac{\sqrt{\tau(r)}}{\varphi(r)}\right)^{k-2} + \left(\frac{\pi\tau(r)}{N}\right)^{k-2} \right\} (\sqrt{\tau(r)}) \ll \frac{\pi^k(N)}{q} \frac{L^2}{\sqrt{N}} (\ln L)^{k-2} \sum_{F < r \leq G, r|q} \left(\frac{\tau(r)}{\sqrt{r}}\right)^{k-2} \ll \frac{\pi^k(N)}{q} \frac{L^2}{\sqrt{N}} (\ln L)^{k-2} \sum_{F < r \leq G, r|q} \left(\frac{\tau(r)}{\sqrt{r}}\right)^{k-2} + (GN^{-1})^{(k-2)/2} L^{2k-2}
\]

\[
\ll \frac{\pi^k(N)}{q} \frac{L^2}{\sqrt{N}} (\ln L)^{k-2} \sum_{F < r \leq G, r|q} \left(\frac{\tau(r)}{\sqrt{r}}\right)^{k-2} + N^{-(k-2)/6} L^{2k-2}
\]

\[
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\]
Denote by \(V\) give a “probabilistic” interpretation of the first term of the formula for \(I_k^{(3)}\). Thus, we obtain

\[
\ll \frac{\pi^k(N)}{q} L^2 \left( F^{-(k-2)/2} (\ln L)^{k-2} (\tau(q))^{k+1} + N^{-(k-2)/6} (\tau(q))^2 L^{2(k-2)} \right) 
\ll \frac{\pi^k(N)}{q} N^{-(k-2)/6} (\tau(q))^{k+1} L^{2k-2} \ll \frac{\pi^k(N)}{q} - q^{-\varepsilon}.
\] (3.8)

Finally, note that the above unconditional inequality (3.6) for \(I_k^{(3)}\) remains valid in this case as well. Now, the required assertion follows from (3.6)–(3.8).

4. STRUCTURE OF THE SINGULAR SERIES

Below we investigate the properties of the “singular series” \(\varkappa_k(q) = \varkappa_k(a, b, m; q)\). First of all, we give a “probabilistic” interpretation of the first term of the formula for \(I_k(N)\) obtained in Theorem 1.

Since \(\varkappa_k(q)\) is multiplicative in \(q\), it suffices to consider the case of \(q = p^n\) with prime \(p\) and \(n \geq 1\). Denote by \(V_k(q) = V_k(a, b, m; q)\) the number of solutions of the congruence

\[
g(x_1) + \ldots + g(x_k) \equiv m \pmod{q}
\] (4.1)

subject to the conditions \(1 \leq x_j \leq q\) and \((x_j, q) = 1, j = 1, 2, \ldots, k\). Clearly,

\[
V_k(q) = \frac{1}{q} \sum_{f=1}^{q} e_q(-mf) S^k(af, bf; q).
\]

Next, for \(n \geq 2\) we have

\[
A_k(p^n) = \frac{1}{\varphi^k(p^n)} \left\{ \sum_{f=1}^{p^n} e_{p^n}(-mf) S^k(af, bf; p^n) - \sum_{f=1}^{p^{n-1}} e_{p^n}(-mf p) S^k(af p, bfp; p^n) \right\}
\]

\[
= \frac{1}{\varphi^k(p^n)} \left\{ \sum_{f=1}^{p^n} e_{p^n}(-mf) S^k(af, bf; p^n) - p^k \sum_{f=1}^{p^{n-1}} e_{p^n-1}(-mf) S^k(af, bf; p^{n-1}) \right\}
\]

\[
= \frac{1}{p^{k(n-1)}(p-1)^k} \left( p^k V_k(p^n) - p^{k+n-1} V_k(p^{n-1}) \right) = \frac{p}{(p-1)^k} \left( \frac{V_k(p^n)}{p^{k-1}(n-1)} - \frac{V_k(p^{n-1})}{p^{k-1}(n-2)} \right).
\]

Similarly, we have

\[
A_k(p) = \frac{pV_k(p) - (p-1)^k}{(p-1)^k} = \frac{pV_k(p)}{(p-1)^k} - 1.
\]

Thus, we obtain

\[
\varkappa_k(p^n) = \sum_{\mu=0}^{n} A_k(p^\mu) = 1 + \left( \frac{pV_k(p)}{(p-1)^k} - 1 \right) + \sum_{\mu=2}^{n} \frac{p}{(p-1)^k} \left( \frac{V_k(p^\mu)}{p^{(k-1)(\mu-1)}} - \frac{V_k(p^{\mu-1})}{p^{(k-1)(\mu-2)}} \right)
\]

\[
= \frac{p}{(p-1)^k} \frac{V_k(p^n)}{\varphi^k(p^n)} = p^k V_k(p^n) \frac{p^k}{\varphi^k(p^n)}.
\]

Since \(V_k(q)\) is multiplicative (which follows readily from the Chinese remainder theorem), in the case of an arbitrary modulus \(q\) we have

\[
\varkappa_k(q) = \frac{q}{\varphi^k(q)} \prod_{p^n \mid q} V_k(p^n) = \frac{qV_k(q)}{\varphi^k(q)}.
\]
Thus, the first term in the expression for $I_k(N)$ takes the form
\[
\frac{\pi^k(N)}{q} qV_k(q) = \left( \frac{\pi(N)}{\varphi(q)} \right)^k V_k(q).
\] (4.2)

The ratio $\pi(N)/\varphi(q)$ represents the “density” of primes $p \leq N$ ($N < q$) in the reduced residue system. Hence, the factor $(\pi(N)/\varphi(q))^k$ in (4.2) is the “probability” that all components of the $k$-tuple $(x_1, \ldots, x_k)$ satisfying the congruence (4.1) are primes. In other words, this factor expresses the “probability” that this solution of (4.1) is also a solution of the congruence (1.4).

In order to study the further properties of $z_k(q)$, we need explicit expressions for $S(a,b;p^n)$ and $A_k(a,b,m;p^n)$ with $n \geq 2$.

**Lemma 7.** Let $p \geq 3$ be prime and $(ab,p) = 1$. If $ab$ is a quadratic nonresidue modulo $p$, then $S(a,b;p^n) = 0$ for all $n \geq 2$. Otherwise, denoting $p^n$ by $q$, for an arbitrary solution $\nu$ of the congruence $ab \equiv \nu^2 \pmod{q}$ we have
\[
S(a,b;q) = S(\nu,\nu;q) = \begin{cases} 
2\sqrt{q}\cos\frac{4\pi\nu}{q} & \text{if } n \text{ is even}, \\
2\sqrt{q}\left(\frac{\nu}{q}\right)\cos\left(\frac{4\pi\nu}{q} + \frac{\pi s}{2}\right) & \text{if } n \text{ is odd},
\end{cases}
\]
where $s = 0$ for $p \equiv 1 \pmod{4}$ and $s = 1$ for $p \equiv 3 \pmod{4}$.

Here $(\frac{a}{q})$ is the Jacobi symbol. For the proof of this lemma, see [22].

**Corollary 4.** Under the hypotheses of Lemma 7, for all $n \geq 2$ and $q = p^n$ we have the estimate
\[
|S(a,b;q)| < 2\sqrt{q}.
\]

**Remark 1.** Corollary 4 is sufficient for most number-theoretical applications of the Kloosterman sums. In this connection, it is quite remarkable that the formulas for the Kloosterman sums from Lemma 7 play an essential role in deriving the explicit formulas for $A_k(q)$ below. The relationship between the formulas for the Kloosterman sums and the expressions for $A_k(q)$ is expressed in terms of the well-known Ramanujan sums.

Recall that the Ramanujan sum $c_q(a)$ (see [21]) is defined by
\[
c_q(a) = \sum_{f=1}^{q} e^{2\pi iaf/q} = \sum_{f=1}^{q} \cos\frac{2\pi af}{q}.
\]

It is well known that
\[
c_q(a) = \frac{\varphi(q)}{\varphi(q/\delta)} \mu\left(\frac{q}{\delta}\right), \quad \delta = (q,a)
\]
(see, for example, [9, Ch. XVI, Theorem 272]). In particular, if $q = p^n$, $n \geq 2$, and $\delta = p^r$, then
\[
c_q(a) = \begin{cases} 
0 & \text{if } 0 \leq r \leq n - 2, \\
-p^{n-1} & \text{if } r = n - 1, \\
\varphi(p^n) & \text{if } r = n.
\end{cases}
\] (4.3)

In addition, we need the following quantities:
\[
uq(a) = \Re w_q(a) \quad \text{and} \quad vq(a) = \Im w_q(a), \quad \text{where} \quad w_q(a) = \sum_{f=1}^{q} \left(\frac{f}{q}\right) e^{2\pi iaf/q}.
\]
Lemma 8. If \( p \geq 3 \) is a prime, \( n \) is an odd number, and \( q = p^n \), then we have the equalities

\[
\begin{align*}
    u_q(a) &= \begin{cases} 
    \left( \frac{b}{p} \right) p^{n-1/2} & \text{if } a = bp^{n-1}, \ p \equiv 1 \pmod{4}, \\
    0 & \text{otherwise},
    \end{cases} \\
    v_q(a) &= \begin{cases} 
    \left( \frac{b}{p} \right) p^{n-1/2} & \text{if } a = bp^{n-1}, \ p \equiv 3 \pmod{4}, \\
    0 & \text{otherwise.}
    \end{cases}
\end{align*}
\]

Proof. Indeed, let \( a = bp^r \), where \((b,p) = 1\) and \( 0 \leq r \leq n - 1 \). Then, setting \( f = g + p^{n-r}h \), where \( 1 \leq g \leq p^{n-r} \) and \( 1 \leq h \leq p^r \), we obtain

\[
w_q(a) = \sum_{g=1}^{p^{n-r}} \sum_{h=1}^{p^r} \left( \frac{g}{p} \right) \exp \left( 2\pi i \frac{b}{p^{n-r}}(g + p^{n-r}h) \right) = p^r \sum_{g=1}^{p^{n-r}} \left( \frac{g}{p} \right) \exp \left( 2\pi i \frac{g}{p^{n-r}} \right).
\]

If \( n - r = 1 \), then the sum over \( g \) coincides with the Gauss sum

\[
\sum_{g=1}^{p} \left( \frac{g}{p} \right) e^{2\pi ig/p} = \begin{cases} 
\sqrt{p}, & p \equiv 1 \pmod{4}, \\
i\sqrt{p}, & p \equiv 3 \pmod{4}.
\end{cases}
\]

Otherwise, setting \( g = s + p^{n-r-1}t \), where \( 1 \leq s \leq p^{n-r-1} \) and \( 1 \leq t \leq p \), we have

\[
w_q(a) = p^r \left( \frac{b}{p} \right) \sum_{s=1}^{p^{n-r-1}} \left( \frac{s}{p} \right) \exp \left( 2\pi i \frac{s}{p^{n-r}} \right) \sum_{t=1}^{p} e^{2\pi it/p} = 0.
\]

The lemma is proved. \( \Box \)

Lemma 9. Let \( p \geq 3, n \geq 2, q = p^n \), and let \((a; b; m)\) be an arbitrary triple of integers satisfying the conditions \( 1 \leq a, b, m \leq q \), \((a,b,p) = 1\), and \((\frac{ab}{p}) = 1\). Finally, let \( \nu \) be an arbitrary solution of the congruence \( ab \equiv \nu^2 \pmod{q} \). Then the following equalities hold:

\[
A_{3}(q) = \begin{cases} 
\frac{q\sqrt{q}}{\varphi^3(q)} (c_q(m + 6\nu) + c_q(m - 6\nu) + 3c_q(m + 2\nu) + 3c_q(m - 2\nu)) & \text{if } n \text{ is even,} \\
\frac{q\sqrt{q}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) (u_q(m + 6\nu) + u_q(m - 6\nu) + 3u_q(m + 2\nu) + 3u_q(m - 2\nu)) & \text{if } n \text{ is odd and } p \equiv 1 \pmod{4}, \\
\frac{q\sqrt{q}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) (v_q(m + 6\nu) - v_q(m - 6\nu) - 3v_q(m + 2\nu) + 3v_q(m - 2\nu)) & \text{if } n \text{ is odd and } p \equiv 3 \pmod{4}.
\end{cases}
\]

Proof. By Lemma 7, for even \( n \) we have

\[S(af, bf; q) = S(\nu f, \nu f; q) = 2\sqrt{q} \cos \frac{4\pi \nu f}{q}.
\]

Hence,

\[
A_{3}(q) = \frac{1}{\varphi^3(q)} \sum_{j=1}^{q} \cos \frac{2\pi mf}{q} \left( 2\sqrt{q} \cos \frac{4\pi \nu f}{q} \right)^3 = \frac{8q\sqrt{q}}{\varphi^3(q)} \sum_{j=1}^{q} \cos \frac{2\pi mf}{q} \cos^3 \frac{4\pi \nu f}{q}.
\]
Next, using the identity $4 \cos^3 \vartheta = \cos(3\vartheta) + 3 \cos \vartheta$, we obtain

$$A_3(q) = \frac{8q \sqrt{\varphi}}{\varphi^3(q)} \sum_{f=1}^{q'} \cos \frac{2\pi mf}{q} \left( \frac{1}{4} \cos \frac{12\nu f}{q} + \frac{3}{4} \cos \frac{4\nu f}{q} \right)$$

$$= \frac{2q \sqrt{\varphi}}{\varphi^3(q)} \sum_{f=1}^{q'} \left( \cos \frac{2\pi mf}{q} \cos \frac{12\nu f}{q} + 3 \cos \frac{2\pi mf}{q} \cos \frac{4\nu f}{q} \right)$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \sum_{f=1}^{q'} \left\{ \cos \left( \frac{2\pi f}{q} (m + 6\nu) \right) + \cos \left( \frac{2\pi f}{q} (m - 6\nu) \right) + 3 \cos \left( \frac{2\pi f}{q} (m + 2\nu) \right) + 3 \cos \left( \frac{2\pi f}{q} (m - 2\nu) \right) \right\}$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \left( c_q (m + 6\nu) + c_q (m - 6\nu) + 3c_q (m + 2\nu) + 3c_q (m - 2\nu) \right).$$

In the case when $n$ is odd and $p \equiv 1 \pmod{4}$, Lemma 7 yields

$$A_3(q) = \frac{1}{\varphi^3(q)} \sum_{f=1}^{q'} \cos \frac{2\pi mf}{q} \left( 2 \left( \frac{\nu f}{q} \right) \sqrt{\varphi} \cos \frac{4\nu f}{q} \right)^3$$

$$= \frac{8q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \cos \frac{2\pi mf}{q} \cos \frac{3 \cdot 4\nu f}{q}$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \left\{ \cos \left( \frac{2\pi f}{q} (m + 6\nu) \right) + \cos \left( \frac{2\pi f}{q} (m - 6\nu) \right) + 3 \cos \left( \frac{2\pi f}{q} (m + 2\nu) \right) + 3 \cos \left( \frac{2\pi f}{q} (m - 2\nu) \right) \right\}$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) u_q (m + 6\nu) + u_q (m - 6\nu) + 3u_q (m + 2\nu) + 3u_q (m - 2\nu).$$

If $n$ is odd and $p \equiv 3 \pmod{4}$, then from the identity $4 \sin^3 \vartheta = -\sin(3\vartheta) + 3 \sin \vartheta$ and Lemma 7 we obtain

$$A_3(q) = -\frac{8q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{q} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \cos \frac{2\pi mf}{q} \sin \frac{3 \cdot 4\nu f}{q}$$

$$= \frac{8q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \cos \frac{2\pi mf}{q} \left\{ \frac{1}{4} \sin \frac{12\nu f}{q} - \frac{3}{4} \sin \frac{4\nu f}{q} \right\}$$

$$= \frac{2q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \left\{ \cos \left( \frac{2\pi mf}{q} \sin \frac{12\nu f}{q} - 3 \cos \frac{2\pi mf}{q} \sin \frac{4\nu f}{q} \right) \right\}$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \sum_{f=1}^{q'} \left( \frac{f}{q} \right) \left\{ \sin \left( \frac{2\pi f}{q} (m + 6\nu) \right) - \sin \left( \frac{2\pi f}{q} (m - 6\nu) \right) - 3 \sin \left( \frac{2\pi f}{q} (m + 2\nu) \right) + 3 \sin \left( \frac{2\pi f}{q} (m - 2\nu) \right) \right\}$$

$$= \frac{q \sqrt{\varphi}}{\varphi^3(q)} \left( \frac{\nu}{p} \right) \left( v_q (m + 6\nu) - v_q (m - 6\nu) - 3v_q (m + 2\nu) + 3v_q (m - 2\nu) \right).$$
Lemma 10. Under the hypotheses of Lemma 9, we have
\[ A_4(q) = \frac{q^2}{\varphi^4(q)} \left( c_q(m + 8\nu) + c_q(m - 8\nu) + 4(-1)^{\nu(p-1)/2} (c_q(m + 4\nu) + c_q(m - 4\nu)) + 6c_q(m) \right). \]

Lemma 11. Under the hypotheses of Lemma 9, we have
\[ A_5(q) = \begin{cases} \frac{q^2\sqrt{q}}{\varphi^5(q)} \left( c_q(m + 10\nu) + c_q(m - 10\nu) + 5c_q(m + 6\nu) + 5c_q(m - 6\nu) + 10c_q(m + 2\nu) + 10c_q(m - 2\nu) \right) & \text{if } n \text{ is even}, \\ \frac{q^2\sqrt{q}}{\varphi^5(q)} \left( \frac{\nu}{p} \right) (u_q(m + 10\nu) + u_q(m - 10\nu) + 5u_q(m + 6\nu) + 5u_q(m - 6\nu) + 10u_q(m + 2\nu) + 10u_q(m - 2\nu)) & \text{if } n \text{ is odd and } p \equiv 1 \pmod{4}, \\ \frac{q^2\sqrt{q}}{\varphi^5(q)} \left( \frac{\nu}{p} \right) (-v_q(m + 10\nu) + v_q(m - 10\nu) + 5v_q(m + 6\nu) - 5v_q(m - 6\nu) - 10v_q(m + 2\nu) + 10v_q(m - 2\nu)) & \text{if } n \text{ is odd and } p \equiv 3 \pmod{4}. \end{cases} \]

The proofs of Lemmas 10 and 11 are similar to that of Lemma 9 and are based on the identities
\[ 8\cos^4 \vartheta = \cos(4\vartheta) + 4\cos(2\vartheta) + 3, \quad 8\sin^4 \vartheta = \cos(4\vartheta) - 4\cos(2\vartheta) + 3 \]
and, respectively,
\[ 16\cos^5 \vartheta = \cos(5\vartheta) + 5\cos(3\vartheta) + 10\cos \vartheta, \quad 16\sin^5 \vartheta = \sin(5\vartheta) - 5\sin(3\vartheta) + 10\sin \vartheta. \]

Now, we proceed to estimating \( A_k(q) \).

Lemma 12. Let \( p \geq 3 \) be a prime. Then for all \( s \geq n \geq 2 \) and \( k \geq 5 \) we have the equalities
\[ \left| \sum_{r=n}^{s} A_k(p^r) \right| < 2 \left( \frac{2p}{p-1} \right)^{k-1} \frac{p^{-n(k/2-1)}}{1-p^{-k/2-1}}. \]

Proof. According to Lemma 7, we have
\[ |A_k(p^r)| < 2 \left( \frac{2p}{p-1} \right)^{k-1} p^{-r(k/2-1)}. \]

Summing these estimates over all \( n \leq r \leq s \), we arrive at the required assertion. \( \square \)

Lemma 13. Let \( p \geq 3 \) be a prime. Then the following inequality holds for all \( k \geq 5 \):
\[ |A_k(p)| < \left( \frac{(2\sqrt{p})^k}{(p-1)^{k-1}} \left( 1 + \frac{1}{4p} \right) \right). \]

Proof. By Lemma 1, we have \( |S(u,v;p)| < 2\sqrt{p} \) for all \( u \) and \( v \) such that \( (uv,p) = 1 \). Hence,
\[ |A_k(p)| \leq \frac{1}{\varphi^k(p)} \sum_{f=1}^{p-1} |S(af,bf;p)|^k < \frac{(2\sqrt{p})^{k-2}}{\varphi^k(p)} \sum_{f=1}^{p-1} |S(af,bf;p)|^2 \]
\[ = \frac{(2\sqrt{p})^{k-2}}{\varphi^k(p)} \left( \sum_{f=1}^{p} |S(af,bf;p)|^2 - |S(0,0;p)|^2 \right) = \frac{(2\sqrt{p})^{k-2}}{\varphi^k(p)} (pI - (p-1)^2), \]
where \( I \) is the number of solutions of the congruence
\[ g(x) \equiv g(y) \pmod{p}, \quad 1 \leq x, y \leq p - 1. \]
It is obvious that $I \leq 2(p - 1)$. Therefore,

$$|A_k(p)| \leq \frac{(2\sqrt{p})^{k-2}}{(p - 1)^k} (2p(p - 1) - (p - 1)^2) = \frac{(2\sqrt{p})^{k-2}}{(p - 1)^{k-1}} (p + 1) = \frac{(2\sqrt{p})^k}{(p - 1)^{k-1}} \left( \frac{1}{4} + \frac{1}{4p} \right).$$

The lemma is proved. □

**Lemma 14.** Let $p \geq 3$ be a prime, and let $n \geq 3$ if $p = 3$ and $n \geq 2$ if $p \geq 5$. Then the following inequalities hold for all $s \geq n$ and all $a$ and $b$ such that $(\frac{ab}{p}) = 1$:

$$\sum_{r=n}^{s} A_3(p^r) \leq \begin{cases} 3\left(\frac{p}{p-1}\right)^4 p^{-n/2} & \text{for } n \equiv 0 \pmod{2}, \\ 3\left(2 - \frac{1}{p}\right)\left(\frac{p}{p-1}\right)^4 p^{-(n+1)/2} & \text{for } n \equiv 1 \pmod{2}. \end{cases}$$

In particular,

$$\sum_{r=2}^{s} A_3(p^r) \leq \frac{3p^3}{(p - 1)^4}, \quad \sum_{r=4}^{s} A_3(p^r) \leq \frac{3p^2}{(p - 1)^4},$$

$$\sum_{r=3}^{s} A_3(p^r) \leq \frac{3p(2p - 1)}{(p - 1)^4}, \quad \sum_{r=5}^{s} A_3(p^r) \leq \frac{3(2p - 1)}{(p - 1)^4}.$$

**Proof.** First of all, we show that each of the formulas (4.4) in Lemma 9 for $A_3(q)$ contains at most one nonzero term. Indeed, suppose the contrary. Then it follows from relations (4.3) and Lemma 8 that $p^{n-1}$ divides at least two numbers among $m \pm 6v$ and $m \pm 2\nu$. But then the difference of these numbers is also divisible by $p^{n-1}$. However, this difference has the form $\pm 4\nu$, $\pm 8\nu$, or $\pm 12\nu$ and obviously cannot be divisible by $p^{n-1}$. A contradiction.

Next, let $n \leq r \leq s$. Since $|c_q(a)| \leq \varphi(q)$ and $|u_q(a)|, |v_q(a)| \leq p^{r-1/2}$ for $q = p^r$, it follows that

$$|A_3(p^r)| \leq \frac{q\sqrt{q}}{\varphi^3(q)} \cdot 3\varphi(q) = \frac{3q\sqrt{q}}{\varphi^2(q)} = \frac{3p^{3r/2}}{p^{2(r-1)}(p - 1)^2} = 3\left(\frac{p}{p-1}\right)^2 p^{-r/2}$$

for even $r$ and

$$|A_3(p^r)| \leq \frac{q\sqrt{q}}{\varphi^3(q)} \cdot 3p^{r-1/2} = \frac{3p^{3r/2}p^{r-1/2}}{p^{3(r-1)}(p - 1)^3} = 3\left(\frac{p}{p-1}\right)^3 p^{-(r+1)/2}$$

for odd $r$. Then, if $n = 2h$ is even, we have

$$\sum_{r=n}^{s} A_3(p^r) \leq \sum_{r \geq 2h \ (\text{mod } 2)} 3\left(\frac{p}{p-1}\right)^2 p^{-r/2} + \sum_{r \geq 2h+1 \ (\text{mod } 2)} 3\left(\frac{p}{p-1}\right)^3 p^{-(r+1)/2}$$

$$= 3\left(\frac{p}{p-1}\right)^2 \sum_{\ell=h}^{\infty} p^{-\ell} + 3\left(\frac{p}{p-1}\right)^3 \sum_{\ell=h}^{\infty} p^{-\ell-1}$$

$$= 3\left(\frac{p}{p-1}\right)^3 p^{-h} + 3\left(\frac{p}{p-1}\right)^3 \frac{p^{-h}}{p-1} = 3\left(\frac{p}{p-1}\right)^4 p^{-h} = 3\left(\frac{p}{p-1}\right)^4 p^{-n/2}.$$
Similarly, if \( n = 2h + 1 \), we have

\[
\left| \sum_{r=n}^{s} A_3(p^r) \right| \leq \sum_{r \geq 2h+1 \atop r \equiv 1 \pmod{2}} \frac{3}{p-1} \frac{p^{-r+1/2}}{p} + \sum_{r \geq 2h+2 \atop r \equiv 0 \pmod{2}} \frac{3}{p-1} \frac{p^{-r/2}}{p} = 3 \left( \frac{p}{p-1} \right)^3 \frac{p^{-h}}{p-1} + 3 \left( \frac{p}{p-1} \right)^2 \frac{p^{-h}}{p-1} = 3 \left( 2 - \frac{1}{p} \right) \left( \frac{p}{p-1} \right)^4 p^{-(n+1)/2}.
\]

The lemma is proved. \( \square \)

**Lemma 15.** Under the hypotheses of Lemma 14, we have

\[
\left| \sum_{r=n}^{s} A_4(p^r) \right| \leq 6 \left( \frac{p}{p-1} \right)^4 p^n.
\]

**Proof.** Applying the same arguments as above, we conclude that the formula of Lemma 10 for \( A_4(q) \) contains at most one nonzero term. This implies the estimate

\[
|A_4(q)| \leq \frac{6q^2 \varphi(q)}{\varphi^3(q)} = \frac{6q^2}{\varphi^3(q)} = \frac{6p^{2r}}{p^3(r-1)(p-1)^3} = 6 \left( \frac{p}{p-1} \right)^3 p^r.
\]

Summing over all \( n \leq r \leq s \), we arrive at the required result. \( \square \)

**Lemma 16.** Let \( p \geq 3 \) be a prime, \( n \geq 2 \), and \( n \equiv \gamma \pmod{2} \), \( \gamma = 0, 1 \). Then the following inequality holds for all \( s \geq n \) and all \( a \) and \( b \) with \( (ab, q) = 1 \):

\[
\left| \sum_{r=n}^{s} A_5(p^r) \right| \leq 10 \left( \frac{p}{p-1} \right)^4 \frac{p^2 - (-1)^\gamma(p-1)}{p^2 + p + 1} p^{-3n/2 - \gamma/2}.
\]

**Proof.** Let us first prove the estimate

\[
|A_5(p^r)| \leq 10 \left( \frac{p}{p-1} \right)^{4+\delta} p^{-3r/2 - \gamma/2}, \quad \delta = \begin{cases} 0 & \text{for } r \equiv 0 \pmod{2}, \\ 1 & \text{for } r \equiv 1 \pmod{2}. \end{cases} \tag{4.5}
\]

Indeed, let \( r \) be even. Suppose that the expression for \( A_5(p^r) \) in Lemma 11 contains two nonzero terms. Then the difference of the arguments of the corresponding Ramanujan sums \( c_q \) is divisible by \( p^r-1 \). However, all such differences have the form

\[
\pm 4\nu, \quad \pm 8\nu, \quad \pm 12\nu, \quad \pm 16\nu, \quad \pm 20\nu.
\]

Obviously, this is impossible for \( p \geq 7 \) and \( r \geq 2 \) or for \( p = 3, 5 \) and \( r \geq 3 \). Hence, in these cases we have

\[
|A_5(p^r)| \leq \frac{q^2 \sqrt{q}}{\varphi^3(q)} \cdot 10 \varphi(q) = \frac{10q^2 \sqrt{q}}{\varphi^3(q)} = 10 \left( \frac{p}{p-1} \right)^4 p^{-3r/2}. \tag{4.6}
\]

If \( p = 5 \) and \( r = 2 \), i.e., if \( q = 5^2 \), then \( 5^{r-1} = 5 \) may only divide the numbers \( m \pm 10\nu \) and the nonzero terms in the formula for \( A_5(q) \) must coincide with the sum \( c_q(m + 10\nu) + c_q(m - 10\nu) \). Since this sum is not greater than \( 2\varphi(q) < 10\varphi(q) \) in absolute value, estimate (4.6) holds in this case as well.

If \( p = 3 \) and \( r = 2 \), i.e., if \( q = 3^2 \), then \( p^{r-1} = 3 \) may only divide the differences

\[
\pm \{(m + 6\nu) - (m - 6\nu)\}, \quad \pm \{(m + 10\nu) - (m - 2\nu)\}, \quad \pm \{(m - 10\nu) - (m + 2\nu)\}.
\]

Hence, the nonzero terms in the formula of Lemma 11 must coincide with one of the sums

\[
5c_q(m + 6\nu) + 5c_q(m - 6\nu), \quad c_q(m + 10\nu) + 10c_q(m - 2\nu), \quad c_q(m - 10\nu) + 10c_q(m + 2\nu).
\]
Obviously, the first of these sums does not exceed \(10\varphi(9)\) in modulus, so inequality (4.6) holds. In the remaining two cases, we notice that the numbers \(m + 10n\) and \(m - 2n\) (respectively, \(m - 10n\) and \(m + 2n\)) cannot be divisible by 9 simultaneously: otherwise, \(\nu\) would be divisible by 3. Hence, the Ramanujan sums \(c_9(m \pm 10n)\) and \(c_9(m \mp 2n)\) cannot be equal to \(\varphi(9)\) simultaneously, and at least one of these sums must be equal to either 0 or -3. Now, we can easily verify that in all such cases the expression \(c_9(m \pm 10n) + 10c_9(m \mp 2n)\) does not exceed \(10\varphi(9) - 3 < 10\varphi(9)\) in absolute value. Hence we obtain (4.6).

Now, suppose that \(r \geq 3\) is odd. In the cases when \(p \geq 7\) and \(n \geq 2\) or \(p = 3, 5\) and \(n \geq 3\), the formula for \(A_3(q)\) contains at most one nonzero term, so that

\[
|A_3(p^r)| < \frac{q^2 \sqrt{q}}{\varphi^5(q)} \cdot 10p^{r-1/2} = \frac{10p_{7r/2-1/2}}{p^{5r/2}(p-1)^5} = 10\left(\frac{p}{p-1}\right)^5 p^{-3r/2-1/2}. \tag{4.7}
\]

Applying the same arguments as above, we conclude that estimate (4.7) is valid in the remaining cases as well. Thus, inequality (4.5) holds for all \(p\) and \(r\) under consideration.

Next, for even \(n = 2h\) and any \(s \geq n\), we have

\[
\left| \sum_{r=n}^s A_3(p^r) \right| < \sum_{\ell \geq h} 10\left(\frac{p}{p-1}\right)^4 p^{-3\ell/2} + \sum_{\ell \geq h} 10\left(\frac{p}{p-1}\right)^5 p^{-3(\ell+1)/2-1/2} = 10\left(\frac{p}{p-1}\right)^4 \sum_{\ell = h}^{+\infty} p^{-3\ell} + 10\left(\frac{p}{p-1}\right)^5 \sum_{\ell = h}^{+\infty} p^{-3\ell-2} = 10\left(\frac{p}{p-1}\right)^4 p^{-3h} + 10\left(\frac{p}{p-1}\right)^5 p^{-3h-2} = 10\left(\frac{p}{p-1}\right)^6 p^2 + p - 1 p^{-3n/2}.
\]

Finally, for odd \(n = 2h + 1\), we obtain

\[
\left| \sum_{r=n}^s A_3(p^r) \right| < \sum_{\ell \geq h} 10\left(\frac{p}{p-1}\right)^5 p^{-3(\ell+1)/2-1/2} + \sum_{\ell \geq h+1} 10\left(\frac{p}{p-1}\right)^4 p^{-3\ell/2} = 10\left(\frac{p}{p-1}\right)^5 \sum_{\ell = h}^{+\infty} p^{-3\ell-2} + 10\left(\frac{p}{p-1}\right)^4 \sum_{\ell = h+1}^{+\infty} p^{-3\ell} = \left(\frac{p}{p-1}\right)^5 \frac{p^{-3h+1}}{p^3 - 1} + 10\left(\frac{p}{p-1}\right)^4 \frac{p^{-3h}}{p^3 - 1} = 10\left(\frac{p}{p-1}\right)^6 \frac{p^2 + 1}{p^2 + p - 1} p^{-3h-2}.
\]

Noticing that \(3h + 2 = 3n/2 + 1/2\), we arrive at the required result. \(\Box\)

5. PROOF OF THEOREM 2

To prove Theorem 2, we need a few auxiliary lemmas.

**Lemma 17.** Let \(k \geq 3\) be fixed, and let \(p \geq 7\) be prime. Then the inequality

\[
\zeta_k(q) = \zeta_k(a, b, m; q) > c_1 = \frac{1}{23}
\]

holds for all \(q = p^n\), \(n \geq 1\), and for any triple \((a; b; m)\) such that \(1 \leq a, b, m \leq q\) and \((ab, p) = 1\).

**Proof.** Indeed, using the estimates of Lemmas 12 and 13, as well as the fact that

\[
A_k(p^n) = 0 \quad \text{for} \quad n \geq 2 \quad \text{if} \quad e = \left(\frac{c}{p}\right) = -1, \quad c \equiv ab \pmod{p}, \tag{5.1}
\]

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\]
we easily conclude that

\[
|\varphi_k(q) - 1| \leq |A_k(p)| + \sum_{\nu=2}^{n} |A_k(p^\nu)| \leq \frac{(2\sqrt{p})^k}{(p-1)^{k-1}} \left( \frac{1}{4} + \frac{1}{4p} \right) + (1 + e) \left( \frac{2p}{p-1} \right)^{k-1} \frac{p^{2-k}}{1-p^{1-k/2}}
\]

\[
\leq \frac{(2\sqrt{p})^k}{(p-1)^{k-1}} \left( \frac{1}{4} + \frac{1}{4p} + \frac{1}{p^{k/2-1-1}} \right).
\]

Denote the right-hand side of (5.2) by \( h(p; k) \). Then

\[
\frac{\partial h}{\partial p} = -\frac{(2\sqrt{p})^k}{(p-1)^{k-1}} \left\{ \left( \frac{k}{2} \left( 1 + \frac{1}{p} \right) - 1 \right) \left( \frac{1}{4} + \frac{1}{4p} + \frac{1}{p^{k/2-1-1}} \right) \\
+ \frac{p-1}{p^2} \left( \frac{1}{4} + \left( \frac{k}{2} - 1 \right) \frac{p^{k/2}}{(p^{k/2-1-1})^2} \right) \right\},
\]

\[
\frac{\partial h}{\partial k} = -\frac{(2\sqrt{p})^k}{(p-1)^{k-1}} \left\{ \left( \frac{1}{4} + \frac{1}{4p} + \frac{1}{p^{k/2-1-1}} \right) \ln \frac{p-1}{2\sqrt{p}} + \frac{1}{2} \frac{p^{k/2-1}}{(p^{k/2-1-1})^2} \ln p \right\}.
\]

It is not difficult to verify that

\[
\frac{\partial h}{\partial p} < 0 \quad \text{and} \quad \frac{\partial h}{\partial k} < 0
\]

for all \( p \geq 7 \) and \( k \geq 6 \). Hence, the function \( h(p, k) \) is decreasing in this domain in each of the variables. A direct calculation shows that \( h(7, 6) = 0.865398 \ldots \). Therefore,

\[
h(p, k) \leq h(7, 6), \quad \varphi_k(q) > 1 - h(7, 6) = 0.134602 \ldots > c_1
\]

for any \( p \geq 7 \), \( k \geq 6 \), and \( q = p^n \), \( n \geq 1 \).

Next, since \( \partial h/\partial p < 0 \) for any \( p, k \geq 3 \), the functions \( h(p, k) \), \( k = 3, 4, 5 \), are decreasing in \( p \). Therefore, the equalities \( h(11, 4) = 0.721600 \ldots \) and \( h(23, 3) = 0.955938 \ldots \) imply that

\[
\varphi_k(q) > 1 - h(11, 4) > 0.2784 \ldots > c_1 \quad \text{for any} \quad k = 4, 5 \quad \text{and} \quad p \geq 11,
\]

\[
\varphi_k(q) > 1 - h(23, 3) > 0.04406 \ldots > c_1 \quad \text{for any} \quad k = 3, 4, 5 \quad \text{and} \quad p \geq 23.
\]

Thus, it suffices to check the assertion of the lemma for \( k = 3 \) and \( 11 \leq p \leq 19 \) as well as for \( p = 7 \) and \( 3 \leq k \leq 5 \).

Since \( S(a, b; p) = S(1, c; p) \) for \( c \equiv ab \pmod{p} \), we have \( S(fa, fb; p) = S(1, cf^2; p) \) for any \( f \) with \( (f, p) = 1 \). If \( f \) runs through the reduced residue system modulo \( p \), then \( f^2c \) runs through all quadratic residues or nonresidues depending on whether \( c \) is a quadratic residue modulo \( p \) or not.

By virtue of (5.1), the inequalities of Lemmas 14 and 15 imply that

\[
|\varphi_k(q) - 1| < \frac{2}{(p-1)^k} \sum_{\nu} |S(1, \nu; p)|^k + \frac{1}{2} (1 + e) \Phi_k(p), \quad (5.3)
\]

where \( \nu \) runs through all residues modulo \( p \) such that \( \left( \frac{\nu}{p} \right) = e \) and

\[
\Phi_3(p) = \frac{3p^3}{(p-1)^2}, \quad \Phi_4(p) = \frac{6p^2}{(p-1)^2}, \quad \Phi_5(p) = \frac{10p^3}{(p-1)^6} \frac{p^2 - p + 1}{p^2 + p + 1}.
\]
A direct calculation yields the following estimates (the left column corresponds to the case \(e = 1\), and the right, to the case \(e = -1\)):

\[
\begin{align*}
|\kappa_4(7^n) - 1| &\leq 0.382716\ldots, & |\kappa_4(7^n) - 1| &\leq 0.641975\ldots, \\
|\kappa_5(7^n) - 1| &\leq 0.119817\ldots, & |\kappa_5(7^n) - 1| &\leq 0.474595\ldots, \\
|\kappa_3(11^n) - 1| &\leq 0.860875\ldots, & |\kappa_3(11^n) - 1| &\leq 0.498271\ldots, \\
|\kappa_3(13^n) - 1| &\leq 0.692316\ldots, & |\kappa_3(13^n) - 1| &\leq 0.494440\ldots, \\
|\kappa_3(17^n) - 1| &\leq 0.542971\ldots, & |\kappa_3(17^n) - 1| &\leq 0.400066\ldots, \\
|\kappa_3(19^n) - 1| &\leq 0.499344\ldots, & |\kappa_3(19^n) - 1| &\leq 0.378563\ldots.
\end{align*}
\]

Hence,

\[
\begin{align*}
\kappa_k(7^n) > 1 - 0.641975\ldots > 0.358024 > c_1 & \quad (k = 4, 5), \\
\kappa_3(11^n) > 1 - 0.860875\ldots > 0.139124 > c_1, \\
\kappa_3(13^n) > 1 - 0.692316\ldots > 0.307683 > c_1, \\
\kappa_3(17^n) > 1 - 0.542970\ldots > 0.45702 > c_1, \\
\kappa_3(19^n) > 1 - 0.499344\ldots > 0.500655 > c_1.
\end{align*}
\]

Thus, it remains to consider the case of \(k = 3\) and \(p = 7\). If \(n \geq 2\), then Lemma 14 implies the estimate

\[
|\kappa_3(7^n) - \kappa_3(7^2)| \leq \sum_{f=3}^{n} A_3(7^f) \leq \frac{3 \cdot 7 \cdot 13}{64} = \frac{91}{432}.
\]

At the same time, a direct calculation by formulas (4.4) shows that the minimum value of

\[
\kappa_3(7^2) = 1 + \frac{1}{6^3} \sum_{f=1}^{6} S^3(1, (\nu f)^2, 7) \cos \frac{2\pi mf}{7}
\]

\[
+ \frac{1}{6^3} (c_{72}(m + 6\nu) + c_{72}(m - 6\nu) + 3c_{72}(m + 2\nu) + 3c_{72}(m - 2\nu))
\]

is \(7/9\) (the minimum is taken over all possible values of \(1 \leq m, \nu \leq 7^2\) with \((\nu, 7) = 1\); this minimum is attained for \(m = 7\mu, \mu = 1, \ldots, 6\), and any \(\nu \not\equiv 0 \pmod{7}\)). Hence, for any \(n \geq 2\), we have

\[
\kappa_3(7^n) \geq \kappa_3(7^2) - \frac{91}{432} > \frac{7}{9} - \frac{91}{432} = \frac{245}{432} = 0.567129\ldots > c_1.
\]

Finally, in the case \(n = 1\), the desired estimate follows from the inequalities

\[
|\kappa_3(7) - 1| \leq \frac{2}{63} (|S(1, 1; 7)|^3 + |S(1, 2; 7)|^3 + |S(1, 4; 7)|^3) < 0.381509.
\]

The lemma is proved. □

**Lemma 18.** Let \(k \geq 3\) be fixed, \(n \geq 1\), and \(q = 5^n\). Then the inequality

\[
\kappa_k(q) = \kappa_k(a, b, m; q) > c_2 = \frac{1}{22}
\]

holds for any triple \((a; b; m)\) such that \(1 \leq a, b, m \leq q\) and \((ab, 5) = 1\).

**Proof.** First, suppose that \(k \geq 4\). Set \(e = (\frac{ab}{5})\). If \(e = -1\), then Lemma 7 implies the inequality

\[
|\kappa_k(q) - 1| \leq |A_k(5)|.
\]
If $e = 1$, then by Lemmas 12 and 15 we have

$$|\varkappa_k(q) - 1| \leq |A_k(5)| + \sum_{r=2}^n A_k(5^r) \leq |A_k(5)| + f(k),$$

where

$$f(k) = \begin{cases} 
\frac{75}{128} & \text{for } k = 4, \\
\frac{20 \cdot 2^{-k}}{1 - 5^{1 - k/2}} & \text{for } k \geq 5.
\end{cases}$$

Applying the same arguments as in the proof of the previous lemma, we conclude that

$$|A_k(5)| \leq \frac{2}{4^k} \sum_{\nu: (\frac{5}{\nu})=e} |S(1, \nu; 5)|^k = 2(|\xi|^k + |\eta|^k),$$

where

$$\xi = \frac{1}{4} S(1, 1; 5) = \frac{3 - \sqrt{5}}{8}, \quad \eta = \frac{1}{4} S(1, 4; 5) = \frac{3 + \sqrt{5}}{8}$$
in the case of $e = 1$ and

$$\xi = \frac{1}{4} S(1, 2; 5) = -\frac{\sqrt{5} + 1}{4}, \quad \eta = \frac{1}{4} S(1, 3; 5) = \frac{\sqrt{5} - 1}{4}$$
in the case of $e = -1$. Thus,

$$|\varkappa_k(q) - 1| < h(k; a, b), \quad \text{where} \quad h(k; a, b) = \begin{cases} 
2(|\xi|^k + |\eta|^k) & \text{for } e = -1, \\
2(|\xi|^k + |\eta|^k) + f(k) & \text{for } e = 1.
\end{cases}$$

If $k \geq 5$, then the function $h(k; a, b)$ is decreasing in $k$ in both cases $e = \pm 1$. Therefore,

$$|\varkappa_k(q) - 1| < \max_{a, b} h(5; a, b) < 0.813922 \ldots,$$

and so $\varkappa_k(q) > 0.186078 \ldots > c_2$. Similarly, for $k = 4$ we have

$$|\varkappa_4(q) - 1| < \max_{a, b} h(4; a, b) < 0.953125 \ldots$$

and $\varkappa_4(q) > 0.046875 \ldots > c_2$.

Let, finally, $k = 3$. Then a direct calculation using the formulas of Lemma 9 shows that the minimum values of

$$\begin{align*}
\varkappa_3(5) &= 1 + \frac{1}{4^3} \sum_{f=1}^{4} S^3(1, (\nu f)^2, 5) \cos \frac{2\pi m f}{5}, \\
\varkappa_3(5^2) &= \varkappa_3(5) + \frac{1}{4^3} \left( c_{5^2} (m + 6\nu) + c_{5^2} (m - 6\nu) + 3 c_{5^2} (m + 2\nu) + 3 c_{5^2} (m - 2\nu) \right), \\
\varkappa_3(5^3) &= \varkappa_3(5^2) + \frac{1}{4^3} \left( \frac{\nu}{5} ( u_{5^3} (m + 6\nu) + u_{5^3} (m - 6\nu) + 3 u_{5^3} (m + 2\nu) + 3 u_{5^3} (m - 2\nu) \right)
\end{align*}$$

are $35/64$, $15/32$, and $15/32$, respectively. If $n \geq 4$, then by Lemma 14 we obtain the inequality

$$|\varkappa_3(5^n) - \varkappa_3(5^4)| \leq \frac{27}{256}.$$ 

Hence,

$$\varkappa_3(5^n) \geq \frac{15}{32} - \frac{27}{256} = \frac{93}{256}.$$ 

The lemma is proved. □
Lemma 19. Let \((q, 6) = 1\). Then the inequality
\[
\varkappa_k(q) = \varkappa_k(a, b, m; q) > 10^{-5}
\]
holds for any triple \((a; b; m)\) such that \(1 \leq a, b, m \leq q\) and \((ab, q) = 1\) and for any \(k \geq 5\).

Proof. Let us represent \(q\) as a product \(q_1q_2\) with all prime divisors of \(q_2\) being greater than 11 and with \(q_1\) either equal to 1 or having all its prime divisors in the set \(\{5, 7, 11\}\). Suppose that \(q_1 = 5^\alpha \cdot 7^\beta \cdot 11^\gamma > 1\) with \(\alpha, \beta, \gamma \geq 0\); then, by Lemmas 17 and 18, we have
\[
\varkappa_k(q_1) = \varkappa_k(5^\alpha) \varkappa_k(7^\beta) \varkappa_k(11^\gamma) > \frac{1}{22} \cdot \frac{1}{23^2}.
\]
Next, let \(q_2 = p_1^{\alpha_1} \ldots p_s^{\alpha_s} \neq 1\), where \(13 \leq p_1 < \ldots < p_s\). According to (5.2), for \(\nu = 1, \ldots, s\) we have
\[
\varkappa_k(p_\nu^{\alpha_\nu}) > 1 - \frac{(2\sqrt{p_\nu})^k}{(p_\nu - 1)^{k-1}} \left( \frac{1}{4} \left( 1 + \frac{1}{p_\nu} \right) + \frac{1}{p_\nu^{k/2 - 1}} \right) = 1 - h(p_\nu, k).
\]
Above, we have shown that the function \(h(p, k)\) is decreasing in each of the variables \(p\) and \(k\) in the domain \(p \geq 13, k \geq 5\). Hence, for each such pair, we have \(0 < h(p, k) \leq h(13, 5) = 0.273667\ldots\). It is easy to verify that
\[
\ln(1 - h(p, k)) > -\frac{6}{5} h(p, k)
\]
and
\[
h(p, 5) = \frac{32p^{5/2}}{p^4} \left( \frac{p}{p - 1} \right)^4 \left\{ \frac{1}{4} \left( 1 + \frac{1}{p} \right) + \frac{1}{p^{3/2 - 1}} \right\} < \frac{13}{p^{3/2}}.
\]
Therefore,
\[
\prod_{p \geq 13} (1 - h(p, k)) \geq \prod_{p \geq 13} (1 - h(p, 5)) = \exp \left( \sum_{p \geq 13} \ln(1 - h(p, 5)) \right) > \exp \left( -\frac{6 \cdot 13}{5} \sum_{p \geq 13} \frac{1}{p^{1/2}} \right) > \frac{1}{8}.
\]
Finally, we obtain
\[
\varkappa_k(q) > \frac{1}{8 \cdot 22 \cdot 23^2} > 10^{-5}.
\]
The lemma is proved. \(\square\)

Lemma 20. There exist absolute positive constants \(C_1, C_2,\) and \(C_3\) such that the inequalities
\[
\varkappa_3(q) > C_1 \exp \left( -\frac{C_2 \sqrt{\ln q}}{\ln \ln q} \right) \quad \text{and} \quad \varkappa_4(q) > \frac{C_3}{(\ln \ln q)^6}
\]
hold for any \(q\) coprime to 6 and for any triple \(1 \leq a, b, m \leq q\) such that \((ab, q) = 1\).

Proof. Setting
\[
q_1 = \prod_{p^\alpha || q, p \leq 19} p^\alpha \quad \text{and} \quad q_2 = \prod_{p^\alpha || q, p \geq 23} p^\alpha,
\]
by Lemmas 17 and 18 we have
\[
\varkappa_k(q_1) > 22^{-1} \cdot 23^{-5}
\]
for \(k = 3, 4\). Next, one can easily verify that
\[
h(p, 3) < \frac{4.6}{\sqrt{p}} \quad \text{and} \quad h(p, 4) < \frac{6}{p}
\]
for any \( p \geq 23 \). Therefore,

\[
\varkappa_3(q_2) \geq \exp \left( \sum_{p|q} \ln \left( 1 - \frac{4.6}{\sqrt{p}} \right) \right) \gg \exp \left( -4.6 \sum_{p|q} \frac{1}{\sqrt{p}} \right),
\]

\[
\varkappa_4(q_2) \geq \exp \left( \sum_{p|q} \ln \left( 1 - \frac{6}{p} \right) \right) \gg \exp \left( -6 \sum_{p|q} \frac{1}{p} \right).
\]

Now, it remains to notice that

\[
\sum_{p|q} \frac{1}{\sqrt{p}} \ll \frac{\ln q}{\ln \ln q} \quad \text{and} \quad \sum_{p|q} \frac{1}{p} \leq \ln \ln q + O(1).
\]

The lemma is proved. \( \square \)

Now, Theorem 2 follows from Lemmas 19 and 20.

6. ADDENDUM

When we derived the unconditional estimate for \( I_k^{(2)} \) in the proof of Theorem 1, the fact that \( k \geq 7 \) played an essential role. Indeed, for such \( k \), the sum

\[
\sum_{r|q, r \leq G} \frac{(\tau(r))^k}{r^{(k-2)/4}} = \sum_{r|q, r \leq F} \frac{(\tau(r))^k}{r^{(k-2)/4}} + O \left( \frac{(\tau(q))^{k+1}}{G} \right) = \sum_{r|q, r > F} \frac{(\tau(r))^k}{r^{(k-2)/4}} + O(N^{-2/3+\varepsilon})
\]

turns out to be small and is included in the remainder term. Therefore, it is natural to ask whether this sum is also small for \( k \leq 6 \)? It is easy to show that this is the case for “almost all” \( q \). However, it turns out that in the general case this is not so even for \( k = 6 \). Moreover, the estimate

\[
\sum_{r|q, r > F} \frac{(\tau(r))^k}{r^{(k-2)/4}} = \sum_{r|q, r > F} \frac{(\tau(r))^6}{r} = O(1)
\]

fails for such \( k \). Below we will show that for any arbitrarily large fixed constant \( A > 1 \) there exists an infinite sequence of moduli \( q \) such that the sums

\[
\Sigma(q; A) = \sum_{d|q, d > (\ln q)^4} \frac{1}{d}
\]

do not satisfy the estimate \( \Sigma(q; A) = O(1) \); moreover, for these moduli, \( \Sigma(q; A) \to +\infty \) as \( q \to +\infty \).

To this end, we need a few definitions. Namely, denote by \( \Psi(x, y) \) (\( \Psi_2(x, y) \)) the number of those integers \( n \leq x \) (respectively, square-free integers \( n \leq x \)) that are not divisible by primes \( p > y \) (it is assumed that \( 2 \leq y \leq x \)). Then, for a fixed \( \varepsilon > 0 \), we have

\[
\Psi(x, y) = xy \vartheta(u) \left( 1 + O \left( \frac{\ln(1 + u)}{\ln y} \right) \right), \quad (6.1)
\]

\[
\Psi_2(x, y) = \frac{6}{\pi^2} \Psi(x, y) \left( 1 + O \left( \frac{\ln^2(1 + y)}{\ln y} \right) \right), \quad u = \frac{\ln x}{\ln y}, \quad (6.2)
\]

The first relation holds uniformly in \( e^{(\ln x)^2+\varepsilon} \leq y \leq x \), and the second, uniformly in \( x^{\varepsilon} \leq y \leq x \) (for the proof of equality (6.1), see, for example, [23, Ch. III.5, Corollary 9.3]; equality (6.2) can be derived from (6.1) by the method pointed out in [10]). Denote by \( \varrho(u) \) the Dickman function defined by the conditions

\[
\varrho(u) \equiv 1 \quad \text{for} \quad 0 < u \leq 1 \quad \text{and} \quad u \varrho'(u) + \varrho(u - 1) = 0 \quad \text{for} \quad u > 1.
\]
Now, we take an arbitrary \( A > 1 \) and suppose that \( X \geq X_0(A) > 2 \). Let \( q = \prod_{p \leq X} p \). According to the prime number theorem, we have
\[
\ln q = \sum_{p \leq X} \ln p = X(1 + O(e^{-c\sqrt{\ln X}})), \quad c > 0,
\]
so that \( X^A = (\ln q)^A(1 + o(1)) \). Next, by the Mertens formula, we have
\[
S = \sum_{d \mid q} \frac{1}{d} = \prod_{p \leq X} \left( 1 + \frac{1}{p} \right) = \prod_{p \leq X} \left( 1 - \frac{1}{p} \right)^{-1} = \frac{6}{\pi^2} e^\gamma (\ln X)(1 + O(e^{-c\sqrt{\ln X}})). \tag{6.3}
\]
Next, let
\[
S_1 = \sum_{d \mid q, d \leq X^A} \frac{1}{d}.
\]
Then, denoting by \( P^+(n) \) the greatest prime divisor of an integer \( n \geq 2 \), we have
\[
S_1 = \sum_{n \leq X^A, P^+(n) \leq X} \frac{(\mu(n))^2}{n} = \sum_{n \leq X} \frac{(\mu(n))^2}{n} + \sum_{X < n \leq X^A, P^+(n) \leq X} \frac{(\mu(n))^2}{n} = S_2 + S_3.
\]
Using the basic property of the Möbius function, one can easily show that
\[
S_2 = \frac{6}{\pi^2} \left( \ln X + \gamma + 2 \sum_p \frac{\ln p}{p^2 - 1} \right) + O\left( \frac{\ln X}{\sqrt{X}} \right).
\]
Further, by Abel’s summation formula,
\[
S_3 = \frac{1}{X^A} \left( \Psi_2(X^A, X) - \Psi_2(X, X) \right) + \int_X^{X^A} \left( \Psi_2(u, X) - \Psi_2(X, X) \right) \frac{du}{u^2}
\]
\[
= \frac{\Psi_2(X^A, X)}{X^A} + \int_X^{X^A} \Psi_2(u, X) \frac{du}{u^2} - \frac{\Psi_2(X, X)}{X}.
\]
Since
\[
\Psi_2(X, X) = \frac{6}{\pi^2} X + O(\sqrt{X}),
\]
we conclude from (6.2) that
\[
S_3 = \frac{6}{\pi^2} (1 + O(\Delta)) \int_X^{X^A} g\left( \frac{\ln u}{\ln X} \right) \frac{du}{u} + \frac{6}{\pi^2} g(A)(1 + O(\Delta)) - \frac{6}{\pi^2} + O\left( \frac{1}{\sqrt{X}} \right)
\]
\[
= \frac{6}{\pi^2} \left( \int_1^{A} g(v) dv \right) (\ln X)(1 + O(\Delta)), \quad \Delta = \frac{\ln^2(A + 1)}{\ln X}.
\]
Thus,
\[
S_1 = \frac{6}{\pi^2} \left( 1 + \int_1^{A} g(v) dv \right) (\ln X)(1 + O(\Delta)) = \frac{6}{\pi^2} \left( \int_0^{A} g(v) dv \right) (\ln X)(1 + O(\Delta)). \tag{6.4}
\]
Finally, subtracting (6.4) from (6.3), we find

$$\Sigma(q; A) = \sum_{d \mid q, d > X} \frac{1}{d} = \frac{6}{\pi^2} \left( e^\gamma - \int_0^A \varrho(v) \, dv \right) (\ln X) \left( 1 + O(\Delta) \right).$$

Using the identity

$$\int_0^{+\infty} \varrho(v) \, dv = 2\varrho(2) + 3\varrho(3) + 4\varrho(4) + 5\varrho(5) + \cdots = e^\gamma$$

(see, for example, [23, Ch. III.5, Exercise 2]), we conclude that

$$\Sigma(q; A) = c(A)(\ln X)(1 + O(\Delta)), \quad c(A) = \int_A^{+\infty} \varrho(v) \, dv > 0.$$

Therefore, $$\Sigma(q; A) \to +\infty$$ as $$X \to +\infty$$.

ACKNOWLEDGMENTS

I am grateful to the referee for carefully reading the manuscript and providing valuable comments.

FUNDING

This work is supported by the Russian Science Foundation under grant 19-11-00001.

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Translated by I. Nikitin