MOD-2 HECKE ALGEBRAS OF LEVEL 3 AND 5

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Dedicated to the memory of Joël Bellaïche.

Abstract. We use deformation theory to study the big Hecke algebra acting on mod-2 modular forms of prime level $N$ and all weights, especially its local component at the trivial representation. For $N = 3, 5$, we prove that the maximal reduced quotient of this big Hecke algebra is isomorphic to the maximal reduced quotient of the corresponding universal deformation ring. Then we completely determine the structure of this big Hecke algebra. We also describe a natural grading on mod-$p$ Hecke algebras.

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1. Introduction

We study the local component at the trivial representation of big mod-2 Hecke algebras acting on all mod-2 modular forms of a fixed level $N$, focusing on $N = 3$ and $N = 5$. To describe the history of the study of mod-$p$ Hecke algebras and our contributions, we introduce some notation, informally at first.

For an integer $N$ and a prime $p$ not dividing $N$, write $M(N, F_p)$ for the space of modular forms modulo $p$ of level $\Gamma_0(N)$ and any weight in the sense of Serre and Swinnerton-Dyer [SerC, SD]. Let $A(N, F_p)$ be the shallow Hecke algebra acting on $M(N, F_p)$: this is the closed subalgebra of $\text{End}_{\mathbb{F}_p} (M(N, F_p))$ topologically generated by the Hecke operators $T_n$ for $n$ not dividing $Np$. Then $A(N, F_p)$ is a semilocal noetherian ring, which splits as a product of its completions at maximal ideals, corresponding, up to $\text{Gal}(\overline{F}_p/F_p)$-conjugacy, to semisimple Galois representations $\bar{\rho}: G_{\mathbb{Q}} \to \text{GL}_2(\overline{F}_p)$ arising from eigenforms appearing in $M(N, \overline{F}_p)$. 

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Passing to a large enough finite extension $\mathbb{F}/\mathbb{F}_p$ to resolve the Galois conjugacy, let $A(N, \mathbb{F})_{\bar{\rho}}$ denote the local component of $A(N, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}$ corresponding to the Galois representation $\bar{\rho}$.

The main object of study here is the complete noetherian local $\mathbb{F}$-algebra $A(N, \mathbb{F})_{\bar{\rho}}$ in the special case that $p = 2$, $\bar{\rho} = 1 \oplus 1$, and $N$ is a prime eventually specializing to 3 or 5.

1.1. Historical context. The structure of $A(N, \mathbb{F})_{\bar{\rho}}$ was first studied in the late 70s by Jochnowitz, who proved that $A(N, \mathbb{F})_{\bar{\rho}}$ is an infinite-dimensional $\mathbb{F}$-vector space [JSt]. In the late 90s, Khare observed that deformation theory implies that $A(N, \mathbb{F})_{\bar{\rho}}$ is noetherian, so that Jochnowitz’s result may be reinterpreted in terms of Krull dimension: $\dim A(N, \mathbb{F})_{\bar{\rho}} \geq 1$ [Kh]. There were no further developments until Nicolas and Serre revitalized the field of study of mod-$p$ Hecke algebras in 2012; we survey the subsequent progress.

2012 Nicolas and Serre use the Hecke recurrence in characteristic 2 (see proof of Theorem 10.1 for an example thereof) to show that $A(1, \mathbb{F}_2)$ is a regular local $\mathbb{F}_2$-algebra of dimension 2. More precisely, they prove that $A(1, \mathbb{F}_2) \cong \mathbb{F}_2[T_3, T_5]$.

2015 Bellaïche and Khare use very different methods — careful comparison with the characteristic-zero Hecke algebra, known to be big by the Gouvêa-Mazur infinite fern ([GM, Theorem 1]; see also [Em, Corollary 2.28]) — to show that for $N = 1$ and $p \geq 5$ the Krull dimension of $A(N, \mathbb{F})_{\bar{\rho}}$ is always at least 2, and that $A(N, \mathbb{F})_{\bar{\rho}} \cong \mathbb{F}[x, y]$ whenever $\bar{\rho}$ is deformation-theoretically unobstructed [BK].

2017 Deo (first-named author here) generalizes the Bellaïche-Khare result [BK] to all $N \geq 1$, still under the assumption $p \geq 5$ [De].

2018 Medvedovsky (second-named author here) determines the structure of $A(1, \mathbb{F}_3)$, developing the nilpotence method (Theorem 2.1 or see [MeD, MeN]), which gives the bound $\dim A(N, \mathbb{F})_{\bar{\rho}} \geq 2$ so long as the genus of $X_0(Np)$ is zero (forthcoming) by comparing the weight filtration on $M(N, \mathbb{F})$ with the nilpotence filtration. This method builds on ideas of Bellaïche [BK, appendix]; like the Nicolas-Serre method it relies on the Hecke recurrence.

2012 Monsky studies $A(N, \mathbb{F}_p)_{\bar{\rho}}$ and its subquotients for various small $N$ and $p$ by considering recurrences and power series in characteristic $p$ in the spirit of Nicolas-Serre; see https://mathoverflow.net/users/6214/paul-monsky for many conjectures. In particular, he determines the structure of $A(3, \mathbb{F}_2)$ and $A(5, \mathbb{F}_2)$: see [Mo3, Mo5, MoG]. We recover these structure theorems here (see Theorem B) using completely different methods.

All this progress has left unresolved most of the cases for $p = 2, 3$, where the best Krull dimension bound for the mod-$p$ Hecke algebra is still the Jochnowitz-Khare bound $\dim A(N, \mathbb{F})_{\bar{\rho}} \geq 1$, as well as the many cases where $\bar{\rho}$ is obstructed and the precise structure of $A(N, \mathbb{F})_{\bar{\rho}}$ is not known.

The aim of the present paper is to explore how much information one can get about the structure of $A(N, \mathbb{F}_2)_{1 \oplus 1}$ for $N$ prime by using deformation theory of Chenevier pseudorepresentations combined with the nilpotence method. In particular, we prove that the maximal reduced quotients of $A(3, \mathbb{F}_2)$ and $A(5, \mathbb{F}_2)$ are isomorphic to those of suitable deformation rings, whose structure we determine explicitly. We also determine the structure of $A(3, \mathbb{F}_2)$ and $A(5, \mathbb{F}_2)$ completely, recovering the structure results of Monsky [MoG] obtained by entirely different methods. In our proof, we use a purely local deformation condition that we call level-$N$ shape to restrict our attention to deformations that look like those coming from modular forms of level $N$ (vs. a power of $N$). The level-$N$ shape condition may be interpreted as a coarser version of Wake and Wang-Erickson’s Steinberg condition if it were extended to $p = 2$ and beyond $k = 2$ [WWE]. Finally, we identify a natural grading on a mod-$p$ Hecke algebra by the 2-Frattini quotient of the Galois group, compatible with the grading on the universal constant-determinant deformation ring described, for $p = 2$ and $\bar{\rho} = 1 \oplus 1$, by Bellaïche.

1.2. Main results. We prove a number of structure theorems, of the Hecke algebras $A(3, \mathbb{F}_2)$ and $A(5, \mathbb{F}_2)$, and of various related rings, including deformation rings, which we now introduce informally. For precise definitions, see section 2.
Let \( N \) be an odd integer and \( G_{Q,2N} \) the Galois group of the maximal extension of \( \mathbb{Q} \) unramified outside \( 2N \). We consider lifts of the trivial representation \( 1 \otimes 1 : G_{Q,2N} \rightarrow \text{GL}_2(\mathbb{F}_2) \) (viewed as a Chenevier pseudorepresentation, subsection 2.9) to local pro-2 \( \mathbb{F}_2 \)-algebras with residue field \( \mathbb{F}_2 \) subject to two conditions: the lifts must be odd (here: trace of complex conjugation is 0) and their determinant must be constant (here: determinant is 1). Such lifts are parametrized by a noetherian universal deformation ring \( \hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1} \).

The 2-adic Galois representations attached to classical modular forms of level \( N \) glue together to form the modular pseudorepresentation of \( G_{Q,2N} \) taking values in \( A(N,\mathbb{F}_2)_{1\oplus 1} \), so that the universal property of \( \hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1} \) gives us a unique local \( \mathbb{Z}_2 \)-algebra morphism, a surjection,

\[
(1.2.1) \quad \hat{\phi} : \hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1} \twoheadrightarrow A(N,\mathbb{F}_2)_{1\oplus 1}
\]

inducing the modular pseudorepresentation from the universal one (subsection 2.11). One does not expect this map to be an isomorphism for general \( N \), and certainly not for \( N \) prime (see subsection 1.4). Our first result is that for \( N = 3, 5 \), this map is in fact an isomorphism on reduced quotients. Note that for \( N = 3, 5 \) the trivial mod-2 representation is the only (semisimple) modular one, so that \( A(N,\mathbb{F}_2) = A(N,\mathbb{F}_2)_{1\oplus 1} \) (subsection 9.2).

**Theorem A** (see Corollary 11.2 and Theorem 11.3). For \( N = 3, 5 \), the surjection \( \hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1} \twoheadrightarrow A(N,\mathbb{F}_2) \) induces an isomorphism

\[
\hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1}^{\text{red}} \cong A(N,\mathbb{F}_2)^{\text{red}}.
\]

Moreover, there are explicitly describable Hecke operators \( X, Y, Z \in A(N,\mathbb{F}_2) \) with images \( \bar{X}, \bar{Y}, \bar{Z} \) in \( A(N,\mathbb{F}_2)^{\text{red}} \), respectively, and power series \( f \) and \( g \) in two variables over \( \mathbb{F}_2 \) so that the map

\[
\frac{\mathbb{F}_2[x, y, z]}{(z - f(x, y))(x - g(y, z))} \twoheadrightarrow A(N,\mathbb{F}_2)^{\text{red}}
\]

defined by \( x \mapsto \bar{X}, y \mapsto \bar{Y}, \) and \( z \mapsto \bar{Z} \) is an isomorphism.

**Theorem A** gives us, in completely explicit terms, the structure of reduced deformation rings \( \hat{\mathcal{R}}(3,\mathbb{F}_2)_{1\oplus 1}^{\text{red}} \) and \( \hat{\mathcal{R}}(5,\mathbb{F}_2)_{1\oplus 1}^{\text{red}} \), not previously known. We know of no other way of determining the structure of these rings. Moreover, **Theorem A** together with the fact that the tangent dimension of \( A(N,\mathbb{F}_2) \) is 4 (see Propositions 9.5 and 9.6) immediately implies that \( A(N,\mathbb{F}_2) \) is not reduced, the first explicit examples of nonreduced mod-\( p \) Hecke algebras for \( \Gamma_0(N) \).

We also give the following refinement of **Theorem A** on the Hecke algebra side.

**Theorem B** (see Theorem 13.1). For \( N = 3 \) and 5, there exist Hecke operators \( X, Y, Z, W \) in \( A(N,\mathbb{F}_2) \) such that the map

\[
\frac{\mathbb{F}_2[x, y, z, w]}{(xz, xw, (z + w)^2)} \twoheadrightarrow A(N,\mathbb{F}_2)
\]

sending \( x \mapsto X, y \mapsto Y, z \mapsto Z, \) and \( w \mapsto W \) is an isomorphism.

In **Theorem B** we have the first explicit structure of a mod-\( p \) Hecke algebra that is not a regular local ring. In particular, it is clear that \( A(3,\mathbb{F}_2) \) and \( A(5,\mathbb{F}_2) \) are not Gorenstein.

Additionally, we determine the structure of \( A(N,\mathbb{F}_2)^{\text{new}} \), the Hecke algebra acting on the space of mod-2 newforms in the sense of [DM], and the structure of \( A(N,\mathbb{F}_2)^{\text{pf}} \), the partially full Hecke algebra topologically generated by the action of Hecke operator \( U_N \) as well as the \( T_m \) for \( m \mid 2N \). See **Theorem 12.1** and **Corollary 12.5**.

On the deformation side, although we do not prove a structure theorem for \( \hat{\mathcal{R}}(N,\mathbb{F}_2)_{1\oplus 1} \) or its level-\( N \) quotient described in subsection 1.4, we do prove an \( R = \mathbb{T} \) theorem for \( A(N,\mathbb{F}_2)^{\text{pf}} \). See section 14 for the relevant definitions and **Theorem 14.1** for the exact statement.

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(i) Examples of nonreduced Hecke algebras for \( \Gamma_1(N) \) were previously found by the first-named author in [De], but the nilpotent elements came from diamond operators.
1.2.1. The grading on mod-$p$ Hecke algebras. The full statement of our Hecke algebra structure result (Theorem 13.1) describes a natural $(\mathbb{Z}/8\mathbb{Z})^*$-grading on $A(N,F_2)_{1\otimes 1}$, one that generalizes to mod-$p$ Hecke algebras and to local components at dihedral $\rho$: that is, $\hat{\rho}$ for which $\hat{\rho} \simeq \tilde{\rho} \otimes \omega_p^{(p-1)/2}$, where $\omega_p$ is the mod-$p$ cyclotomic character. Let $\Pi_{p,N}$ be the 2-Frattini quotient of $G_{Q,Np}$.

**Theorem C** (see Theorem 3.4). The Hecke algebra $A(N,F)$ has a $\Pi_p$-grading $A(N,F) = \bigoplus_{\ell \in \Pi_p} A(N,F)^\ell$, natural in the sense that for $m \mid Np$ we have $T_m \in A(N,F)^m$.

If $p = 2$ or if $\hat{\rho} : G_{Q,Np} \to GL_2(\mathbb{F})$ satisfies $\hat{\rho} \simeq \hat{\rho} \otimes \omega_p^{(p-1)/2}$, then the same is true for $A(N,F)_{\hat{\rho}}$.

In the statement of Theorem C we’ve implicitly identified $\Pi_p$, the 2-Frattini quotient of $G_{Q,p}$, with $\mathbb{Z}_p^* / (\mathbb{Z}_p^*)^2$.

**Theorem 3.4** itself says more: there is a corresponding $\Pi_p$-grading on the space $K(N,F)$ of forms killed by $U_p$ making $K(N,F)$ into a graded $A(N,F)$-module, and the grading is compatible with the modular pseudorepresentation map $G_{Q,Np} \to A(N,F)$, in the sense that the fibers of $G_{Q,Np} \to \Pi_p$ map to the corresponding graded components of $A(N,F)$. These statements restrict to $A(N,F)_{\hat{\rho}}$ and the $\hat{\rho}$-generalized eigenform subspace $K(N,F)_{\hat{\rho}}$ of $K(N,F)$ under the assumptions on $\hat{\rho}$ as above.

The grading in Theorem C had been previously discovered in two special cases: $(p,N) = (2,1)$ [NS1] and $(p,N) = (3,1)$ [Mc3]. Theorem C was inspired by a question of Serre, as well as a partial answer from Bellaïche, which we take the opportunity to present below.

**Theorem** (Bellaïche; see Theorem 6.3). For any odd $N$, the universal deformation ring $\hat{R}(N,F_2)_{1\otimes 1}$ has a natural $\Pi_{2N}$-grading compatible with the universal pseudorepresentation map $G_{Q,2N} \to \hat{R}(N,F_2)_{1\otimes 1}$.

1.3. Overview of proofs. We give a rough outline of the proofs of Theorems A and B. The proof of Theorem A has a spiral nature that may of be independent interest — we move several times between the deformation side and the Hecke side to achieve our results. The proof of Theorem C is more straightforward; see section 3.

For the proof of Theorem A, we first construct the universal deformation ring $\hat{R}(N,F_2)_{1\otimes 1}$ subject to our local level-$N$-shape condition. For more on this condition, see subsection 1.4 below or section 4. We use a computation of Chenevier to prove a result about the dimension of the tangent space of this restricted deformation ring $\hat{R}(N,F_2)_{1\otimes 1}$ (Corollary 7.4). Using true representations, we get a bound on the dimension of the tangent space of quotients of $\hat{R}(N,F_2)_{1\otimes 1}$ by prime ideals (Proposition 7.8). Then we use this information to prove results about quotients of the Hecke algebra. In particular, we use the nilpotence method to determine the structure of $A(N,F_2)^{v_{new}}$, the Hecke algebra acting on the space of forms killed by $U_N + 1$, which we call the very new modular forms because the newforms are those killed by $U_N^2 - 1 = (U_N + 1)^2$ (Corollary 10.3). Because $A(N,F_2)^{v_{new}}$ has dimension 2, we are able to conclude that the minimal prime ideals of $\hat{R}(N,F_2)_{1\otimes 1}$ are preimages of minimal primes of $A(N,F_2)$, so that the two rings have the same reduced quotient (Proposition 11.1). Separately, we determine the structure of $A(N,F_2)^{red}$ (Theorem 11.3), completing the proof of Theorem A; in particular, the two integral-domain quotients of $A(N,F_2)^{red}$, visible in the statement of Theorem A, are the old Hecke algebra $A(1,F_2)$ and very new Hecke algebra $A(N,F_2)^{v_{new}}$.

The proof of Theorem B is quite involved in its own way. We first determine the structure of the partially full Hecke algebra (Theorem 12.1). Then we use the partially full Hecke algebra, the two integral domain quotients found in the proof of Theorem A, and the precise description of the cotangent space $A(N,F_2)$ to find the structure of $A(N,F_2)$ (Theorem 13.1).

1.4. The level-$N$-shape deformation condition. A key tool in the proof of Theorem A is our purely local level-$N$-shape deformation condition, which may be defined for any prime $p$ and prime level $N$. The condition is simple to describe: for a pseudorepresentation lifting $\hat{\rho} : G_{Q,Np} \to GL_2(\mathbb{F})$ we ask that the restriction to the decomposition group at $N$ contains the inertia subgroup in the kernel (section 4). This captures the property of a representation with Artin conductor dividing $N$, such as those contributing to $A(N,F)_{\hat{\rho}}$, rather than those whose Artin conductor is a power of $N$, which may appear in $A(N^2,F)_{\hat{\rho}}$. 
In our setting, the level-$N$-shape deformation condition defines the universal level-$N$-shape deformation ring $R(N, \mathbb{F}_2)_{1@1}$, a nontrivial quotient of $R(N, \mathbb{F}_2)_{1@1}$ by a nilpotent ideal (Proposition 7.7). The universality induces the surjection

$$\varphi : R(N, \mathbb{F}_2)_{1@1} \twoheadrightarrow A(N, \mathbb{F}_2)_{1@1}$$

factoring $\varphi$ from (1.2.1). It is this map rather than $\varphi$ that we use in the proof of Theorem A.

We now compare our level-$N$-shape condition to both the unramified-or-$\pm$-Steinberg at-$N$ condition appearing in the work of Wake and Wang-Erickson [WWE] and the ordinary-at-$N$ condition from the work of Calegari-Specter [CS, source file (!) on arxiv]. For this it is helpful to recall that a constant-determinant pseudorepresentation lifting $\tilde{\rho} : G_{\mathbb{Q}, N_p} \to \text{GL}_2(\mathbb{F})$ to a pro-$p$ local $\mathbb{F}$-algebra $B$ with residue field $\mathbb{F}$ is described by a function $t : G_{\mathbb{Q}, N_p} \to B$ lifting $\text{tr} \tilde{\rho}$. In particular, $t$ is central: $t(gh) = t(hg)$ for all $g, h \in G_{\mathbb{Q}, N_p}$. For more on $t$, see subsection 2.9. Also let $D_N \subseteq G_{\mathbb{Q}, N_p}$ be a decomposition subgroup at $N$, $\text{Frob}_N \in D_N$ any Frobenius element, and $I_N \subseteq D_N$ the inertia-at-$N$ subgroup of $D_N$. The pseudorepresentation $t$ satisfies the level-$N$-shape condition if

$$t(d_i) = t(d) \quad \text{for } i \in I_N, \, d \in D_N.$$  

The Wake–Wang-Erickson condition is stated for residually multiplicity-free $p$-adic pseudorepresentations with $p \geq 5$ in fixed weight 2, and expressed in terms of generalized matrix algebras (GMAs), introduced earlier by Bellaïche and Chenevier [BC, Chapter 1]. It is possible to obtain an equivalent formulation of the Wake–Wang-Erickson GMA relation as a pseudorepresentation statement involving all the elements of $G_{\mathbb{Q}, N_p}$ by using the notion of the kernel of a pseudorepresentation (of an algebra; see [Be1, §2.1.2]). It is also straightforward to generalize their GMA relation to weight $k$. With these adjustments, we might expect that Wake and Wang-Erickson would call a weight-$k$ pseudorepresentation $t \in G_{\mathbb{Q}, N_p} \to B$ unramified-or-$\pm$-Steinberg at $N$ if the following conditions hold for all $g \in G_{\mathbb{Q}, N_p}$ and $i \in I_N$:

$$t(g \text{Frob}_N i) - t(g \text{Frob}_N) = \mp N^{\frac{1}{2}}(t(g i) - t(g)) \quad \text{and}$$

$$t(g i \text{Frob}_N) - t(g \text{Frob}_N) = \mp N^{\frac{1}{2}}(t(g i) - t(g)).$$

Compare to [WWE, Definitions 3.4.1 and 3.8.1]. If $t$ is an unramified-or-$\pm$Steinberg at $N$ pseudorepresentation of weight $k$, then [WWE, Lemma 3.4.4] implies that $t(i) = 2$ for all $i \in I_N$. In particular, taking $g = 1$ in (1.4.2) above recovers our local level-$N$-shape condition. Conversely, the level-$N$ shape condition implies (1.4.2) and (1.4.3) for $g$ in the local Galois group $D_N$, but not more generally for $g$ in $G_{\mathbb{Q}, N_p}$. This captures a key distinction between ours and the Wake–Wang-Erickson condition: our condition (1.4.1) is entirely local, whereas the Wake–Wang-Erickson condition [WWE, Definition 3.8.1] is global — as reflected, necessarily, in our translation-cum-generalization in (1.4.2)–(1.4.3). Let us briefly dwell on this confusing point: Definition 3.8.1 in [WWE], which relies on the purely local Definition 3.4.1 of loc. cit., requires elements in the group algebra of a local Galois group to map to zero in a “Cayley-Hamilton” (close to “faithful”) GMA carrying a representation of a global Galois group. It is therefore fundamentally global in nature.

The Calegari-Specter ordinary-at-$N$ condition from [CS] is similar to Wake and Wang-Erickson’s. Calegari and Specter enhance the deformation ring with a new variable $U$ meant to capture the behavior of $U_N$. Their condition holds if $U$, in addition to satisfying the characteristic polynomial of $\text{Frob}_N$ (see (5.1.2)), satisfies the following: for every $g \in G_{\mathbb{Q}, N_p}$ and every $i \in I_N$,

$$t(g i \text{Frob}_N) - t(g \text{Frob}_N) = t(g i) - t(g)|U = 0.$$ 

As one can see, the difference between (1.4.3) and (1.4.4) is the substitution of $U$ for $\pm N^{k/2-1}$, as one expects from the action of $U_N$ on newforms. Depending on the context, one may also expect to add a second condition here analogous to (1.4.2). We use an identity analogous to (1.4.4) for $g = e$ (see (5.2.1)) to describe the precise structure of $A(N, \mathbb{F}_2)_{1@1}$ for $N = 3, 5$ (Theorem 13.1). And we use (1.4.4) and the related condition on the other side to obtain an $R = T$ theorem for $A(N, \mathbb{F}_2)^{df}$ (Theorem 14.1).

1.5. Comparison with Monsky’s results. Finally we briefly describe the recent results of Paul Monsky, which inspired and catalyzed both [DM] and the present work, and relate them to ours. In [Mo3], motivated by recent work of Nicolas and Serre on the mod-2 level-1 Hecke algebra [NS1, NS2], Monsky was able to
determined that the Hecke algebra acting on a certain subquotient of the space $M(3, \mathbb{F}_2)$ of mod-2 level-3 modular forms, which he argued ought to be viewed as the newforms in this setting, is isomorphic to $\mathbb{F}_2[T_7, T_{13}, \varepsilon]/(\varepsilon^2)$. In [Mo5] he proved similar results in level 5. Monsky’s structure results were obtained by explicit computations with mod-2 power series and Nicolas–Serre–style Hecke recurrences; though beautiful and satisfying, they did not appear to be amenable to generalization. Our curiosity piqued, we sought to reinterpret Monsky’s results in a more conceptual way. This required several steps. First, in [DM] we defined a space $M(N, \mathbb{F}_p)^{\text{new}}$ of newforms mod $p$ for any prime-to-$p$ level $N$, and proved that Monsky’s level-3 Hecke algebra from [Mo3] is isomorphic to the Hecke algebra acting on $M(3, \mathbb{F}_2)^{\text{new}}$. Next, we used deformation theory, commutative algebra, and a dash of the nilpotence method to arrive at Theorem A, a structure theorem for $A(N, \mathbb{F}_2)^{\text{red}}$. We also conjectured the statement of Theorem B. After mutually beneficial discussions with Monsky, both he and we were able to sharpen our results using our independent methods: Monsky’s in [MoG] and ours as Theorem B here.

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# 2. Definitions and notation

In this section we review definitions, with references where appropriate, and introduce notation used in the rest of the article. The expert reader should check in with our notation for the universal pseudodeformation ring in subsection 2.10, glance at subsections 2.12 and 2.13, and otherwise skip to section 3.

## 2.1. Preliminaries.

### 2.1.1. Finite fields.**

We work with a prime $p$, and will write $\mathbb{F}$ for a finite extension of $\mathbb{F}_p$.

### 2.1.2. Rings.**

All rings are assumed to be commutative with identity. For a ring $B$, let $B^{\text{red}}$ be its maximal reduced quotient, the quotient of $B$ by its nilradical, the intersection of its prime ideals.

If $B$ is a local pro-$p$ ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$, its (reduced) tangent space is the $\mathbb{F}$-vector space $\operatorname{Tan} B := \operatorname{Hom}(\mathfrak{m}/(\mathfrak{m}^2 + p), \mathbb{F})$. If $B$ is noetherian, then its local topology agrees with its profinite topology [dSL, Proposition 2.4], so it’s better known as a complete local noetherian ring with finite residue field of characteristic $p$. Moreover, in this case $\operatorname{Tan} B$ is finite, and its dimension $d$ as an $\mathbb{F}$-vector space is the same as the dimension of the dual (reduced) cotangent space $\mathfrak{m}/(\mathfrak{m}^2 + p)$, so that $B$ is a quotient of $W(\mathbb{F})[x_1, \ldots, x_d]$, where $W(\mathbb{F}) = \mathbb{Z}_{(p)}$ is the ring of Witt vectors of $\mathbb{F} = \mathbb{F}_p$. Finally, $\dim \operatorname{Tan} B + 1 \geq \dim B$, where $\dim B$ is the Krull dimension of $B$.

\(^{(ii)}\)\url{https://services.math.duke.edu/~pierce/AROOO_2020.shtml}
2.2. Galois groups. For any number field $K$, we write $G_K = \text{Gal}(\overline{\mathbb{Q}}/K)$ for the absolute Galois group of $K$. If $N$ is any integer, write $\mathbb{Q}(N)$ for the maximal extension of $\mathbb{Q}$ unramified outside the primes dividing $N$ and $\infty$. Let $G_{\mathbb{Q},N} = \text{Gal}(\overline{\mathbb{Q}}(N)/\mathbb{Q})$.

Let $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ denote the $p$-adic cyclotomic character, normalized so that $\chi_p(\text{Frob}_l) = \ell$ for prime $\ell \neq p$. Let $\omega_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ be the mod-$p$ cyclotomic character. Both factor through $G_{\mathbb{Q},N,p}$ for any integer $N$.

If $G$ is any quotient of $G_{\mathbb{Q}}$, we write $c$ for any complex conjugation in $G$. For any prime $\ell$, let $D_{\mathbb{Q}_\ell} \subset G_{\mathbb{Q}}$ be a decomposition group at $\ell$ and $I_{\mathbb{Q}_\ell}$ be the inertia subgroup of $D_{\mathbb{Q}_\ell}$. Then write $D_{I}(G)$, or simply $D_I$, if $G$ is clear, for the image of $D_{\mathbb{Q}_\ell}$ in $G$. Note that both $D_{\mathbb{Q}_\ell}$ and $D_I$ are only well defined up to conjugacy. Once $D_I(G)$ is fixed, we write $I_{\mathbb{Q}_\ell}(G)$, or simply $I_{\mathbb{Q}_\ell}$, for the inertia subgroup of $D_I(G)$: it is the image of $I_{\mathbb{Q}_\ell}$ in $G$, a closed normal subgroup of $D_I(G)$ with procyclic abelian quotient. We write $\text{Frob}_l$ for any Frobenius element of $D_I(G)$. If $I_{\mathbb{Q}_\ell}(G)$ is trivial, then $\text{Frob}_l$ is well defined up to conjugacy; otherwise, it is an arbitrary element in an $I_{\mathbb{Q}_\ell}(G)$-coset of $D_I(G)$, with the whole setup only defined up to conjugacy.

If $G$ is any quotient of $G_{\mathbb{Q}}$, or more generally any profinite group, write $G^{\text{pro-}2}$ for the maximal continuous pro-$2$ quotient of $G$. Also write $G^2$ for the closed subgroup of $G$ generated by the squares of elements of $G$. This is a closed normal subgroup containing the commutators $[G,G]$ (see, for example, [Ch, footnote before Lemma 5.3]), and the quotient $G/G^2 = G^{\text{pro-}2}/(G^{\text{pro-}2})^2$ is the $2$-Frattini quotient of $G$, its maximal continuous elementary $2$-group quotient. The basic theorem of Frattini theory for $p = 2$ is that generators of $G/G^2$ lift to generators of $G^{\text{pro-}2}$.

2.3. The space of modular forms of level $N$ and all weights. Fix a level $N$. For an even weight $k \geq 0$, let $M_k(N,\mathbb{Z})$ be the space of modular forms of level $\Gamma_0(N)$ and weight $k$ whose Fourier expansion at the cusp at infinity has rational integer coefficients. We view $M_k(N,\mathbb{Z})$ as a subspace of $\mathbb{Z}[q]_k$ using the $q$-expansion principle. For any ring $B$, let $M_k(N,B) := M_k(N,\mathbb{Z}) \otimes_{\mathbb{Z}} B \subset B[q]_k$, the space of modular forms of level $\Gamma_0(N)$ and weight $k$ defined over $B$. Note that for $B \subseteq \mathbb{C}$, this definition coincides with the usual notion of modular forms with Fourier coefficients in $B$ viewed as $q$-expansions. Also define $M_{\leq k}(N,B) := \sum_{k' \leq k} M_{k'}(N,B)$ and $M(N,B) := \bigoplus_{k \geq 0} M_k(N,B) \subset B[q]_k$: this is the space of all modular forms of level $\Gamma_0(N)$ defined over $B$. For a form $f \in M(N,B)$, write $a_n(f)$ for the $n$th Fourier coefficient of $f$, so that $f = \sum_{n \geq 0} a_n(f)q^n$.

From now on we assume that $p$ does not divide $N$. For $B = \mathbb{F}_p$, we have defined $M(N,\mathbb{F}_p) \subset \mathbb{F}_p[q]$, the space of all mod-$p$ modular forms of level $\Gamma_0(N)$. This is the space of mod-$p$ modular forms as studied by Serre [SerC] and Swinnerton-Dyer [SD] in level 1.

2.4. The weight filtration on mod-$p$ modular forms. In characteristic zero, spaces of $q$-expansions of modular forms are graded by their weight: if $B$ is a subring of $\mathbb{C}$, then

$$M(N,B) = \bigoplus_{k \geq 0} M_k(N,B) \quad \text{[Mi, Lemma 2.1.1].}$$

In characteristic $p$, this is not the case, and the weight grading is replaced by the weight filtration. For $p \geq 5$, let $E_{p-1}$ be the level-$1$ weight-$(p-1)$ Eisenstein series with $q$-expansion in $1+p\mathbb{Z}_{(p)}[q]$. Multiplication by $E_{p-1}$ induces Hecke-equivariant embeddings $M_k(N,\mathbb{F}_p) \hookrightarrow M_k+1(N,\mathbb{F}_p)$ for each even $k \geq 0$. In fact this is the only kind of weight ambiguity: if we define, for $i \in 2\mathbb{Z}/(p-1)\mathbb{Z}$,

\begin{equation}
M(N,\mathbb{F}_p)^i := \bigcup_{k \equiv i \text{ mod } p-1} M(N,\mathbb{F}_p), \quad \text{then } M(N,\mathbb{F}_p) = \bigoplus_{i \in 2\mathbb{Z}/(p-1)\mathbb{Z}} M(N,\mathbb{F}_p)^i.
\end{equation}

See [KHi, Theorem 2.2] or [SD, Theorem 2(iv)] for $N = 1$. To resolve the weight ambiguity we define for $f \in M(N,\mathbb{F}_p)^i$ its weight filtration

$$w(f) := \min\{k : f \in M_k(N,\mathbb{F}_p)\}.$$ 

An important property of the weight filtration for our purposes is its compatibility with powers: for any $n \geq 0$, we have $w(f^n) = nw(f)$ [JCo, proof of Fact 1.7].

For $p = 2,3$, the story is a little more subtle. We may still define the naïve weight filtration in the same way, i.e. $w(f) := \min\{k : f \in M_k(N,\mathbb{F}_p)\}$, but with this definition properties the compatibility with powers need
not be satisfied. For example, the form \( f_3 \) in \( M(3, \mathbb{F}_2) \) defined in Lemma 9.1 has \( w(f_3) = 4 \) but \( w(f_3^2) = 6 \). More dramatically, the form \( f_5 \) in \( M(5, \mathbb{F}_2) \) loc. cit. has \( w(f_5) = w(f_5^2) = 4 \). The difficulty arises because the Hasse invariant, a geometric mod-\( p \) modular form that controls the weight filtration, does not always lift to a characteristic-zero \( \Gamma_0(N) \) form in weight \( p - 1 \) for \( p = 2, 3 \). The Hasse invariant has \( q \)-expansion 1, and simple zeros at the supersingular points of \( X_0(N)_{\mathbb{F}_p} \) and nowhere else; \( E_{p-1} \) is a lift for \( p \geq 5 \). A form in \( M_k(N, \mathbb{F}_p) \) with a zero of minimal order \( s \) at each supersingular point is divisible by \( s \) copies of Hasse, and so comes from a form of lower filtration \( k - s(p - 1) \). See [Ca, §1.6–1.8] for more details.

However, the Hasse invariant does always lift to a form of weight \( p - 1 \) for \( \Gamma_1(N) \) for \( N > 1 \). For \( p = 3 \), see [KPb, §2.1] for \( N > 2 \); for \( N = 2 \), take the \( \Gamma_0(N) \) weight-2 Eisenstein series \( E_{2,N} \). For \( p = 2 \), see user Electric Penguin’s answer to MathOverflow question 228497 or [Mei, Appendix B]. Therefore we may resolve all our difficulties with the weight filtration by replacing \( w(f) \) with the weight filtration coming from \( \Gamma_1(N) \). For any \( f \in M_k(N, \mathbb{F}) \) set

\[
w_1(f) := \min \{ k' : f \text{ is the reduction of a } q \text{-expansion of a form in } M_{k'}(\Gamma_1(N), \mathbb{Z}_p) \}.
\]

Then \( w_1(f) \) is an integer satisfying \( 0 \leq w_1(f) \leq k \) and \( w_1(f) \equiv k \pmod{p - 1} \). Because \( w_1(f) \) is defined geometrically — \( \frac{k - w_1(f)}{p - 1} \) is the minimal order of a zero of \( f \) at any supersingular point of \( X_0(N)_{\mathbb{F}_p} \) — this definition resolves the problems with the naïve filtration \( w(f) \). In particular,

\[
\begin{align*}
(2.4.2) \quad w_1(f^n) &= nw(f) \quad \text{for all } n \geq 0; \\
(2.4.3) \quad w_1(f) &\leq w(f) \leq w_1(f) + 3 \quad \text{and } w_1(f) = w(f) \text{ if } p \geq 5.
\end{align*}
\]

Here (2.4.3) holds because for \( p = 2, 3 \) the weight-4 and level-1 Eisenstein form \( E_4 \), normalized so \( a_0(E_4) = 1 \), has mod-\( p \) \( q \)-expansion 1; by the \( q \)-expansion principle it is the fourth power (for \( p = 2 \)) or the square (for \( p = 3 \)) of the Hasse invariant.

In our bad examples from Lemma 9.1, \( f_3 \) and \( f_5 \) are reductions of semicuspidal\(^{(iii)} \) forms appearing with nontrivial quadratic nebentype in weight 3 and weight 2, respectively, so that their squares appear in their “true” \( \Gamma_0(N) \) weight while they themselves do not. In other words, for \( N = 3 \) we have \( w_1(f_3) = 3 \) and \( w_1(f_3^2) = w(f_3^2) = 6; \) for \( N = 5 \), \( w_1(f_5) = 2 \) and \( w_1(f_5^2) = w(f_5^2) = 4 \).

2.5. Hecke operators on mod-\( p \) modular forms. The spaces \( M_k(N, \mathbb{F}) \) carry actions of Hecke operators inherited from the action on \( M_k(N, \mathbb{Z}) \). More precisely, for prime \( \ell \nmid Np \), the action of the Hecke operator \( T_\ell \) is defined on the \( q \)-expansion of a form \( f \in M_k(N, \mathbb{F}) \) by

\[
a_m(T_\ell f) = a_{me}(f) + \ell^{k-1}a_{m/\ell}(f),
\]

where \( a_{m/\ell}(f) \) is understood to be 0 if \( \ell \nmid n \). We extend this to prime powers \( \ell^r \) via the recurrence

\[
T_\ell = T_\ell T_{\ell^{r-1}} - \ell^{k-1}T_{\ell^{r-1}} \quad \text{for } r \geq 2,
\]

and to general \( n \) with \( \gcd(n, Np) = 1 \) multiplicatively via \( T_{nm'} = T_n T_{m'} \) provided \( \gcd(n, n') = 1 \). Since \( \ell^{k-1} \) is well defined in \( \mathbb{F} \) for \( k \) in a \( (p - 1) \)-congruence class, the action of \( T_n \) for \( n \nmid Np \) on \( M_k(N, \mathbb{F}) \) extends to an action on all of \( M(N, \mathbb{F}) \) by (2.4.1).

Also inherited from characteristic-zero is the action on \( M_k(N, \mathbb{F}) \) and \( M(N, \mathbb{F}) \) of the Atkin-Lehner operators \( U_n \) for any \( n \mid N \), defined on \( q \)-expansions by \( a_m(U_n f) = a_{mn}(f) \).

Finally, the reduction of the operator \( T_p \) on \( M(N, \mathbb{Z}) \) coincides modulo \( p \) with the Atkin-Lehner operator \( U_p \), at least for \( k \geq 2 \). From now on we use \( U_p \) in place of \( T_p \) (including for \( k = 0 \)) for the at-\( p \) Hecke action on \( M(N, \mathbb{F}) \). The kernel \( K(N, \mathbb{F}) \) of \( U_p \)

\[
(2.5.1) \quad K(N, \mathbb{F}) = \{ f \in M(N, \mathbb{F}) : a_0(f) = 0 \text{ if } p \mid n \}
\]

is a key subspace of \( M(N, \mathbb{F}_p) \) in the sequel. Write \( K_k(N, \mathbb{F}) \) for \( M_k(N, \mathbb{F}) \cap K(N, \mathbb{F}) \).

All of the Hecke operators — \( T_n \) for \( (n, Np) = 1 \), \( U_n \) for \( n \mid N \), and \( U_p \) — commute.

\(^{(iii)}\)A semicuspidal form vanishes at infinity but is nonzero at at least one other cusp, so its \( q \)-expansion at \( \infty \) “looks” cuspidal without it being a cuspform.
The Artin conductor of prime $p$.

Since the actions are compatible with restriction maps, we set the conductor of $p$.

An important operator on $\mathcal{M}(N, \mathbb{F})$ is the derivation $\theta = d/d\theta$, constructed for $p > 5$ in [SD] and for $p \geq 2$ in [KRe]. The operator $\theta$ takes $f = \sum a_n \theta^n$ to $\theta(f) = \sum a_n \theta^n$. One may verify the following facts (see [KRe, II Theorem(2.1); Corollary(6)]) for (2.6.3) and (2.6.4) below):

\[
\theta^{-1} = 1 - V_p U_p \text{ is a projector onto } K(N, \mathbb{F});
\]

\[
\text{im } \theta = K(N, \mathbb{F});
\]

\[
\ker \theta = \text{im } V_p = \text{im}(p \text{th power map});
\]

\[
w_1(\theta f) \leq w_1(f) + p + 1, \text{ with equality if and only if } p \nmid w_1(f).
\]

The Hecke algebra on mod-$p$ modular forms. We denote by $\mathcal{A}_{\leq k}(N, \mathbb{F}_p)$ the $\mathbb{F}_p$-subalgebra of $\text{End}_{\mathbb{F}_p}(\mathcal{M}_{\leq k}(N, \mathbb{F}_p))$ generated by the action of the Hecke operators $T_n$ with $\text{gcd}(n, Np) = 1$ as in subsection 2.5. Since the actions are compatible with restriction maps, we set

\[
\mathcal{A}(N, \mathbb{F}_p) := \lim_{\rightarrow \mathbb{A}} \mathcal{A}_{\leq k}(N, \mathbb{F}_p) :
\]

this is the (shallow) Hecke algebra acting on the space of modular forms of level $\Gamma_0(N)$ modulo $p$. Equivalently, considering $\mathcal{M}(N, \mathbb{F}_p)$ with the discrete topology, and $\text{End}_{\mathbb{F}_p}(\mathcal{M}(N, \mathbb{F}_p))$ with the induced compact-open topology, the shallow Hecke algebra $\mathcal{A}(N, \mathbb{F}_p)$ is the closed subalgebra of $\text{End}_{\mathbb{F}_p}(\mathcal{M}(N, \mathbb{F}_p))$ generated by the $T_n$ with $\text{gcd}(n, Np) = 1$ [MeD, Proposition 2.4].

From this setup it follows that $\mathcal{A}(N, \mathbb{F}_p)$ is a pro-$p$ semilocal ring, so that it factors as a product of its localization at its maximal ideals. Moreover, the maximal ideals of $\mathcal{A}(N, \mathbb{F}_p)$ are in one-to-one correspondence with certain Galois representations, which we now describe.

To every normalized Hecke eigenform $f$ in $\mathcal{M}_{\leq k}(N, \mathbb{Q}_p)$ a construction of Eichler-Shimura and Deligne attaches a continuous Galois representation $\rho_f : G_Q \to \text{GL}_2(\mathbb{Z}_p)$ with the properties that $\rho_f$ is unramified at primes $\ell \nmid Np$ with $\text{tr } \rho_f(\text{Frob}_\ell) = a_\ell(f)$ and that $\det \rho_f = \chi^{-1}$. Reducing any $\mathcal{G}_Q$-invariant $\mathbb{Z}_p$-lattice of $\rho_f$ modulo the maximal ideal and semisimplifying gives us $\overline{\rho}_f : \mathcal{G}_Q \to \text{GL}_2(\bar{\mathbb{F}}_p)$. Note that $\overline{\rho}_f$ is independent of the chosen lattice. Galois representations of the form $\rho_f$ or $\overline{\rho}_f$ will be called $\Gamma_0(N)$-modular.$^{(iv)}$ The maximal ideals of $\mathcal{A}(N, \mathbb{F}_p)$, then, are in correspondence with $\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$-orbits of Hecke eigenforms in $\mathcal{M}(N, \mathbb{F}_p)$, which, by the Deligne-Serre lifting lemma correspond to $\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$-orbits of $\Gamma_0(N)$-modular representations $\overline{\rho} : \mathcal{G}_Q \to \text{GL}_2(\bar{\mathbb{F}}_p)$. Passing to an extension $\bar{\mathbb{F}}_p/\mathbb{F}_p$ that contains all the Hecke eigenvalues of all the mod-$p$ level-$\Gamma_0(N)$ Hecke eigenforms, we resolve the $\text{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$-conjugacy. Write $\mathcal{A}(N, \mathbb{F})$ for $\mathcal{A}(N, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F}$, and let $\mathcal{A}(N, \mathbb{F})_{\overline{\rho}}$ be the localization of $\mathcal{A}(N, \mathbb{F})$ at the maximal ideal corresponding to $\overline{\rho}$. Then $\mathcal{A}(N, \mathbb{F})_{\overline{\rho}}$ is a profinite local ring with residue field $\mathbb{F}$, and we have a decomposition of the Hecke algebra

\[
\mathcal{A}(N, \mathbb{F}) = \prod_{\overline{\rho}} \mathcal{A}(N, \mathbb{F})_{\overline{\rho}}
\]

and a corresponding decomposition of the ring of modular forms into $\overline{\rho}$-eigencomponents

\[
\mathcal{M}(N, \mathbb{F}) = \bigoplus_{\overline{\rho}} \mathcal{M}(N, \mathbb{F})_{\overline{\rho}}
\]

refining the decomposition in (2.4.1), with each $\mathcal{A}(N, \mathbb{F})_{\overline{\rho}}$ acting faithfully on $\mathcal{M}(N, \mathbb{F})_{\overline{\rho}}$. (See [BeE, I.5.1] for this kind of statement for the finite-dimensional quotients/subs and take limits.) Since Hecke operators $T_n$ are multiplicative and the prime power Hecke operators $T_{\ell^r}$ satisfy an order-2 linear recurrence in $r$ with

$^{(iv)}$ We will not use this below, but Serre reciprocity (formerly Serre’s conjecture) implies that a representation $\overline{\rho} : \mathcal{G}_{\mathbb{Q}, Np} \to \text{GL}_2(\bar{\mathbb{F}}_p)$ is $\Gamma_0(N)$-modular if and only if its determinant is an odd power of $\omega_p$ and it has prime-to-$p$ Artin conductor dividing $N$. See subsection 4.1 for more on the Artin conductor for prime $N$. 


coefficients $T_\ell$ and $\det \bar{\rho}(\text{Frob}_\ell) \in \mathbb{F}$, the local Hecke algebra $A(N, \mathbb{F})_{\bar{\rho}}$ is topologically generated as an $\mathbb{F}$-algebra by the operators $T_\ell$ for $\ell$ prime not dividing $Np$.

Write $m(N, \mathbb{F})_{\bar{\rho}}$ for the maximal ideal of $A(N, \mathbb{F})_{\bar{\rho}}$. The modified Hecke operators $T'_\ell := T_\ell - \text{tr} \bar{\rho}(\text{Frob}_\ell)$ for prime $\ell$ not dividing $Np$ are all in $m(N, \mathbb{F})_{\bar{\rho}}$; indeed, they topologically generate it.

2.8. The partially full mod-$p$ Hecke algebra. Finally, we define the partially full Hecke algebra $A(N, \mathbb{F}_p)^{pf}$, the closed subalgebra of $\text{End}_{\mathbb{F}_p}(M(N, \mathbb{F}_p))$ topologically generated by both all the Hecke operators $T_n$ with $(n, Np) = 1$ and by all the Hecke operators $U_\ell$ for $\ell | N$. As in subsection 2.7 we can alternatively define $A(N, \mathbb{F}_p)^{pf}$ as an inverse limit of finite-level partial Hecke algebras.

One can check that $A(N, \mathbb{F}_p)$ acts faithfully on $(K(N, \mathbb{F}_p), \text{End}_{\mathbb{F}_p}((K(N, \mathbb{F}_p))^{pf}))$ defined in (2.5.1), and the pairing

$$A(N, \mathbb{F}_p)^{pf} \times (K(N, \mathbb{F}_p))^{pf} \to \mathbb{F}_p$$

(2.8.1)

$$(T, f) \mapsto a_1(Tf)$$

is (continuously) perfect, inducing $A(N, \mathbb{F}_p)^{pf}$-module duality isomorphisms

$$K(N, \mathbb{F}_p)^{pf} \cong \text{Hom}_{\text{cont}}(A(N, \mathbb{F}_p)^{pf}, \mathbb{F}_p)$$

(2.8.2)

and $A(N, \mathbb{F}_p)^{pf} \cong \text{Hom}(K(N, \mathbb{F}_p), \mathbb{F}_p)$.

Here the continuity is with respect to the profinite (equivalently, the local) topology on $A(N, \mathbb{F}_p)^{pf}$ and the discrete topology on $K(N, \mathbb{F}_p)$.

In particular, for any closed ideal $J$ of $A(N, \mathbb{F}_p)^{pf}$, the duality in (2.8.2) restricts to a duality between between the $J$-torsion in $K(N, \mathbb{F}_p)$ and the quotient of $A(N, \mathbb{F}_p)^{pf}$ by $J$:

$$K(N, \mathbb{F}_p)[J] \text{ and } A(N, \mathbb{F}_p)^{pf}/J \text{ are in (continuous) duality as } A(N, \mathbb{F}_p)^{pf}\text{-modules.}$$

If $\mathbb{F}/\mathbb{F}_p$ is large enough to contain all Hecke eigenvalues of all mod-$p$ eigenforms of level $\Gamma_0(N)$, and $\bar{\rho} : G_{\mathbb{Q}, Np} \to \text{GL}_2(\mathbb{F})$ is a $\Gamma_0(N)$-modular representation, then we can define $A(N, \mathbb{F})_{\bar{\rho}}^{pf}$ as the quotient of $A(N, \mathbb{F})^{pf}$ acting faithfully on $(M(N, \mathbb{F})_{\bar{\rho}})$. Like the shallow Hecke algebra $A(N, \mathbb{F})$, the partially full $A(N, \mathbb{F})^{pf}$ will also break up into a product of the $\bar{\rho}$-components

$$A(N, \mathbb{F})^{pf} = \prod_{\bar{\rho} : \Gamma_0(N)\text{-modular}} A(N, \mathbb{F})_{\bar{\rho}}^{pf},$$

with $A(N, \mathbb{F})_{\bar{\rho}}^{pf}$ the quotient of $A(N, \mathbb{F})^{pf}$ acting faithfully on $(M(N, \mathbb{F})_{\bar{\rho}})$, and, as in (2.8.3), in duality with $K(N, \mathbb{F})_{\bar{\rho}} := K(N, \mathbb{F}) \cap (M(N, \mathbb{F})_{\bar{\rho}})$. However, $A(N, \mathbb{F})_{\bar{\rho}}^{pf}$ will not be local in general: its maximal ideals are in bijection with systems of eigenvalues $\{\alpha \cup \ell \text{ prime dividing } N\}$ of the Hecke operators $U_\ell$ with $\ell$ prime dividing $N$ appearing in mod-$p$ modular forms in $M(N, \mathbb{F})_{\bar{\rho}}$.

The natural inclusion map $A(N, \mathbb{F}_p) \hookrightarrow A(N, \mathbb{F}_p)^{pf}$ sending $T_\ell$ to $T_\ell$ for $\ell$ prime not dividing $Np$ is finite (see Corollary 5.2 below for the case of prime $N$ or [De, proof of Theorem 3, p. 23] in general) and induces finite inclusions $A(N, \mathbb{F})_\bar{\rho} \hookrightarrow A(N, \mathbb{F})_{\bar{\rho}}^{pf}$ for $\Gamma_0(N)$-modular $\mathbb{F}$-valued $\bar{\rho}$.

2.9. Pseudorepresentations. We recall the definition of a (dimension-2) pseudorepresentation: for a (topological) group $G$ and a (topological) commutative ring $B$, a (continuous) pseudorepresentation of $G$ on $B$ (of dimension 2) is a pair of (continuous) functions $(t, d) : G \to B$ satisfying the following properties:

(1) $t(1) = 2$;

(2) $d : G \to B^\times$ is a group homomorphism;

(3) $t$ is central: for all $g, h \in G$, we have $t(gh) = t(hg)$;

(4) trace-determinant identity: for all $g, h \in G$,

$$d(g)t(g^{-1}h) + t(gh) = t(g)t(h).$$

Pseudorepresentations, introduced in the form above by Chenevier in [Ch] (where they are called determinants), generalize the earlier notion introduced by Wiles [Wi] and Taylor [Tay] and further studied by Rouquier [Ro] and Nyssen [Ny] (now called pseudocharacters following [Ro]), to arbitrary characteristic. The
idea is that \((t, d)\) generalizes the data of pairs \((\tau \rho, \det \rho)\) for true representations \(\rho : G \to \GL_2(B)\): that is, if \(\rho : G \to \GL_2(B)\) is a representation, then \((\tau \rho, \det \rho)\) is a pseudorepresentation of \(G\) on \(B\). The converse is not true over for arbitrary rings \(B\), but if \(B\) is an algebraically closed field then every pseudorepresentation on \(B\) comes from a true representation [Ch, Theorem 2.12].

If \(2 \in B^\times\) and \((t, d)\) is a pseudorepresentation of \(G\) on \(B\), then \(d\) is determined by \(t\) (the trace-determinant identity for \(h = g\) gives \(d(g) = \frac{t(g)^2 - t(g^2)}{2}\)); and \(t\) is a pseudocharacter in the earlier notion of Taylor et al.

The kernel of a pseudorepresentation \((t, d) : G \to B\) of a group \(G\) on a ring \(B\) is

\[
\ker(t, d) := \{ g \in G : d(g) = 1, t(gh) = t(h) \text{ for all } h \in G \} \subset G.
\]

One can check that \(\ker(t, d)\) is a (closed) normal subgroup of \(G\). Both \(t\) and \(d\) factor through the quotient \(G/\ker(t, d)\), descending to a pseudorepresentation \((t, d) : G/\ker(t, d) \to B\) with trivial kernel. If \(\rho : G \to \GL_2(B)\) is a representation, then the kernel of \(\rho\) is contained in the kernel of the associated pseudorepresentation: \(\ker \rho \subseteq \ker(\tau \rho, \det \rho)\). An easy calculation implies that the reverse containment also holds if \(B\) is a field and \(\rho\) is absolutely irreducible.

We will work with two-dimensional pseudorepresentations of profinite groups on profinite rings, and from now on will tacitly assume that all relevant maps are continuous.

For a topological \(\Z_p\)-algebra \(B\), we say that a pseudorepresentation \((t, d)\) of \(G_\Q\) on \(B\) is unramified at some prime \(\ell\) if \(\ell_\ell(G_\Q)\) is contained in \(\ker(t, d)\). Moreover, we say that \((t, d)\) is \(\Gamma_0(N)\)-modular if \((t, d)\) is the base change of \((\tau \rho, \det \rho)\) for some \(\Gamma_0(N)\)-modular representation \(\rho\).

### 2.10. Pseudodeformations of mod-\(p\) Galois representations.

Fix a representation

\[
\tilde{\rho} : G_{\Q, Np} \to \GL_2(\F),
\]

which we assume to be semisimple and odd: that is, \(\tr \tilde{\rho}(c) = 0\). Let \(\mathcal{C}\) be the category of profinite local \(\F\)-algebras with residue field \(\F\). We will call the objects of \(\mathcal{C}\) \((\F\text{-}) coefficient algebras.

Let \(\hat{D}_{\tilde{\rho}}\) be the functor from \(\mathcal{C}\) to the category of sets sending a coefficient algebra \((B, m)\) to the set of pseudorepresentations \((t, d)\) of \(G_{\Q, Np}\) on \(B\) that reduce to the pseudorepresentation \((\tau \tilde{\rho}, \det \tilde{\rho})\) modulo \(m\) subject to the additional conditions that \(t(c) = 0\) (oddness) and that \(d = \det \tilde{\rho}\) (constant determinant). A pseudorepresentation \((t, d)\) lifting \((\tau \tilde{\rho}, \det \tilde{\rho})\) with constant determinant is obviously determined by \(t\), so by abuse of notation we call such a \(t\) a pseudodeformation of \(\tilde{\rho}\).

The functor \(\hat{D}_{\tilde{\rho}}\) is representable, represented by noetherian universal deformation ring \((\hat{R}(N, \F)_{\tilde{\rho}}, \hat{n}(N, \F)_{\tilde{\rho}})\) in \(\mathcal{C}\) equipped with a universal pseudodeformation \(\hat{\tau}^{\text{univ}} := \hat{\tau}^{\text{univ}}_{\tilde{\rho}, N} : G_{\Q, Np} \to \hat{R}(N, \F)_{\tilde{\rho}}\) of \(\tilde{\rho}\), in the sense that, for a coefficient algebra \(B\), any pseudodeformation \(t : G_{\Q, Np} \to B\) of \(\tilde{\rho}\) comes from a unique morphism \(\hat{R}(N, \F)_{\tilde{\rho}} \to B\) in \(\mathcal{C}\) of coefficient algebras:

\[
G_{\Q, Np} \xrightarrow{\hat{\tau}^{\text{univ}}} \hat{R}(N, \F)_{\tilde{\rho}} \xrightarrow{t} B.
\]

(2.10.1)

See [Ch, Propositions 3.3, 3.7] for details. Note that \(\hat{\tau}^{\text{univ}} = \tr \tilde{\rho} + \hat{\beta}^{\text{univ}}\), where \(\hat{\beta}^{\text{univ}}\) maps \(G_{\Q, Np}\) to \(\hat{n}(N, \F)_{\tilde{\rho}}\).

Following [BK, section A.2], we define, for \(\ell\) prime not dividing \(Np\), elements \(\hat{t}_\ell := \hat{\tau}^{\text{univ}}(\Frob_{\ell})\) in \(\hat{R}(N, \F)_{\tilde{\rho}}\) and \(\hat{t}_\ell' := \hat{\beta}^{\text{univ}}(\Frob_{\ell})\) in \(\hat{n}(N, \F)_{\tilde{\rho}}\).

By universality, the \(\hat{t}_\ell'\) topologically generate \(\hat{R}(N, \F)_{\tilde{\rho}}\) as an \(\F\)-algebra; hence they also generate \(\hat{n}(N, \F)_{\tilde{\rho}}\) as an ideal of \(\hat{R}(N, \F)_{\tilde{\rho}}\). If \(B\) is the trace algebra of \(t\), that is, if \(B\) is topologically generated as an \(\F\)-algebra by \(t(G_{\Q, Np})\) (equivalently, by the \(t(\Frob_{\ell})\) for \(\ell \nmid Np\)), then the unique map \(\hat{R}(N, \F)_{\tilde{\rho}} \to B\), guaranteed by universality as in (2.10.1), is surjective.
Using \( \hat{\beta} \text{univ} \) we also obtain an isomorphism of \( \mathbb{F} \)-vector spaces (standard in deformation theory; see, for example, [Go, Lemma 2.6]): if \( \mathbb{F}[\varepsilon] \) are the dual numbers, with \( \varepsilon^2 = 0 \), then
\[
(2.11.2) \quad \text{Tr} \mathcal{R}(N, \mathbb{F})_{\tilde{\rho}} = \text{Hom}(\hat{\mathfrak{n}}(N, \mathbb{F})_{\tilde{\rho}} / \hat{\mathfrak{n}}(N, \mathbb{F})^2_{\tilde{\rho}}, \mathbb{F}) \cong \bar{D}_{\rho}(\mathbb{F}[\varepsilon]).
\]
This isomorphism identifies a linear functional \( h : \hat{\mathfrak{n}}(N, \mathbb{F})_{\tilde{\rho}} \to \mathbb{F} \) factoring through \( \hat{\mathfrak{n}}(N, \mathbb{F})^2_{\tilde{\rho}} \) with the pseudodeformation \( g \mapsto \text{tr} \tilde{\rho}(g) + \varepsilon h(\hat{\beta} \text{univ}(g)) \) of \( \tilde{\rho} \).

If \( \tilde{\rho} : G_{\mathbb{Q}, Np} \to \text{GL}_2(\mathbb{F}) \) is a semisimple odd representation that factors through \( G_{\mathbb{Q}, M_p} \) for some divisor \( M \) of \( N \), then the surjective map \( G_{\mathbb{Q}, Np} \twoheadrightarrow G_{\mathbb{Q}, M_p} \) induces a natural surjection
\[
(2.11.3) \quad \tilde{\psi}_{N,M} : \mathcal{R}(N, \mathbb{F})_{\tilde{\rho}} \to \mathcal{R}(M, \mathbb{F})_{\tilde{\rho}}
\]
and its kernel is the closed ideal \( J_{N,M} \) generated by elements of the form \( \hat{\beta} \text{univ}(gi) - \hat{\beta} \text{univ}(g) \) for \( g \in G_{\mathbb{Q}, Np} \) and \( i \in I_{\ell}(G_{\mathbb{Q}, Np}) \) with \( \ell \) running over the primes that divide \( N \) but not \( M \).

### 2.11. The pseudodeformation of \( \tilde{\rho} \) carried by \( A(N, \mathbb{F})_{\tilde{\rho}} \). By gluing together all the \( \Gamma_0(N) \)-modular pseudorepresentations of \( G_{\mathbb{Q}, Np} \) that Eichler-Shimura and Deligne’s construction attaches to characteristic-zero modular eigenforms of level \( N \) and reducing modulo \( p \) (see, for example, [BeR, Step 1 of the proof of Theorem 1]) for a detailed construction for \( p = 2, N = 1 \) one obtains an odd pseudorepresentation
\[
(2.11.1) \quad \tau_{p,N \tilde{\rho}} : G_{\mathbb{Q}, Np} \to A(N, \mathbb{F})
\]
satisfying \( \tau_{p,N \tilde{\rho}}(\text{Frob}_\ell) = T_\ell \) for any prime \( \ell \nmid pN \). If we fix a \( \Gamma_0(N) \)-modular \( \tilde{\rho} : G_{\mathbb{Q}, Np} \to \text{GL}_2(\mathbb{F}) \), then by extending scalars in (2.11.1) and composing with the map \( A(N, \mathbb{F}) \to A(N, \mathbb{F})_{\tilde{\rho}} \) from the decomposition in (2.7.1) (or alternatively, by gluing together the \( \Gamma_0(N) \)-modular pseudodeformations of \( \tilde{\rho} \) and reducing mod \( p \)), one obtains a constant-determinant odd pseudodeformation of \( \tilde{\rho} \)
\[
(2.11.2) \quad \tau_{\tilde{\rho}, N \tilde{\rho}} := \tau_{p,N \tilde{\rho}} : G_{\mathbb{Q}, Np} \to A(N, \mathbb{F})_{\tilde{\rho}}
\]
again with \( \tau_{\tilde{\rho}, N \tilde{\rho}}(\text{Frob}_\ell) = T_\ell \) for \( \ell \nmid pN \).

(2.11.3) \quad \hat{\phi} : \mathcal{R}(N, \mathbb{F})_{\tilde{\rho}} \to A(N, \mathbb{F})_{\tilde{\rho}}
\]
sending \( \hat{\ell}_\ell \) to \( T_\ell \) for primes \( \ell \nmid Np \). Since \( A(N, \mathbb{F})_{\tilde{\rho}} \) is topologically generated by the \( T_\ell \), the morphism \( \hat{\phi} \) is surjective, which surjectivity tells us that \( A(N, \mathbb{F})_{\tilde{\rho}} \) is noetherian, so that its profinite topology coincides with its local topology [dSL, Proposition 2.4].

### 2.12. The nilpotence method for lower bounds on \( \dim A(N, \mathbb{F})_{\tilde{\rho}} \). We summarize the method described in [MeD, MeN] for obtaining a lower bound on the Krull dimension of a local piece of the mod-\( p \) Hecke algebra acting on a subspace of a polynomial algebra of forms. In [MeD] this method is applied for \( A = A(1, \mathbb{F}_p) \) with \( p = 2, 3, 5, 7, 13 \); more generally the method may be applied to \( A = A(N, \mathbb{F}_p)_{\tilde{\rho}} \) so long as the genus of \( X_0(Np) \) is zero.

**Theorem 2.1** (Nilpotence method [MeD, MeN]). Suppose the following conditions are satisfied.

1. \( M(N, \mathbb{F}) = \mathbb{F}[f] \) for some form \( f \in M \).
2. \( A \) is a continuous local quotient of \( A(N, \mathbb{F})_{\tilde{\rho}} \) acting faithfully on a subspace \( K \subseteq K(N, \mathbb{F}) \).
3. The maximal ideal \( \mathfrak{m} \) of \( A \) is generated by Hecke operators \( S_1, \ldots, S_d \) so that, for each \( i \),
   a. \( S_i \) is in every maximal ideal of \( A(N, \mathbb{F})_{\tilde{\rho}} \); and
   b. the sequence \( \{ S_i(f^n) \} \) satisfies an \( M \)-linear recurrence of some order \( d \), whose characteristic polynomial \( X^{d_i} + a_{i,1}X^{d_i-1} + \cdots + a_{i,d_i-1}X + a_{i,d_i} \in M[X] \) satisfies both \( \deg_f a_{i,j} \leq j \) for all \( j \)
   and \( \deg_f a_{i,d_i} = d_i \).
4. There exists a sequence of linearly independent forms \( \{ g_n \} \) in \( K \) with \( \deg_f g_n \) depending at most linearly on \( n \). (In other words, \( \deg_f g_n = O(n) \).)

Then \( \dim A \geq 2 \).

Condition (1) is crucial to the method, as it relies on the Nilpotence Growth Theorem [MeN]. If \( K = K(N, \mathbb{F}) \), then generators \( S_1, \ldots, S_d \) of \( \mathfrak{m} \) satisfying the conditions in (3) are known to exist: see [MeD, 4.3.3 and 6.3] for the case \( N = 1 \); the general case is similar.
The idea of the proof of Theorem 2.1 is as follows: by the main theorem of [MeN] and the condition on the sequence \( \{g_n\} \), the function \( h(n) \) with the property that \( m^{k(n)} \) annihilates \( \{g_0, \ldots, g_n\} \) grows slower than linearly in \( n \). By duality between \( A \) and \( K \) coming from the duality between \( A(\mathbb{N}, \mathbb{F})^{\text{old}} \) and \( K(\mathbb{N}, \mathbb{F}) \) (2.8.2), the Hilbert-Samuel function \( k \mapsto \dim \mathbb{A}/m^k \) grows faster than linearly in \( k \). Therefore the Hilbert-Samuel degree of \( A \) is strictly greater than 1, hence at least 2. But this degree is equal to the Krull dimension of \( A \) [AM, Theorem 11.14].

In practice, in condition (4) one may replace the \( f \)-degree of a form \( g \) in \( M(\mathbb{N}, \mathbb{F}) \) with its weight filtration, as described in subsection 2.4.

2.13. Oldforms and newforms mod \( p \). In this subsection, we assume \( N \) is prime. We briefly summarize the perspective of [DM].

2.13.1. The Fricke automorphism mod \( p \). The characteristic-zero Fricke involution \( w_N \) on \( M_k(\mathbb{N}, \mathbb{C}) \) sending \( f(z) \) to

\[
w_N f = f_k \left( \frac{0}{N} \right) = \frac{N^{k/2}(Nz)^{-k}}{2^{k+1}}f(wz)\]

is defined over \( \mathbb{Z}[\frac{1}{N}] \), and therefore descends to \( M_k(\mathbb{N}, \mathbb{F}_p) \). However, \( w_N \) is not in general well defined as an algebra involution on all of \( M(\mathbb{N}, \mathbb{F}_p) \): see [DM, section 3] for a details. Since this is inconvenient, we replace \( w_N \) on \( M_k(\mathbb{N}, \mathbb{F}_p) \) by \( W_N := N^{k/2}w_N \). This renormalized Fricke operator is an algebra automorphism of order dividing \( p-1 \) for \( p \) odd [DM, Proposition 3.13(3)]; for \( p = 2 \) the operator \( W_N \) coincides with \( w_N \) and is hence an algebra involution. In all cases \( W_N \) commutes with all \( T_n \) for \( n \) prime to \( Np \).

2.13.2. Old and new forms mod \( p \). We can now define the mod-\( p \) “oldforms” in level \( N \) in the following way: let \( M(\mathbb{N}, \mathbb{Q})^{\text{old}} := M(1, \mathbb{Q}) + W_N M(1, \mathbb{C}) \subset M(\mathbb{N}, \mathbb{Q}) \), as usual. Set

\[
M(\mathbb{N}, \mathbb{Z})^{\text{old}} := M(\mathbb{N}, \mathbb{Q})^{\text{old}} \cap \mathbb{Z}[\frac{1}{N}] \subset M(\mathbb{N}, \mathbb{Z});
\]

let \( M(\mathbb{N}, \mathbb{F}_p)^{\text{old}} \) be the subspace of \( M(\mathbb{N}, \mathbb{F}_p) \) obtained by reducing \( M(\mathbb{N}, \mathbb{Z})^{\text{old}} \) modulo \( p \), and finally set \( M(\mathbb{N}, \mathbb{F})^{\text{old}} := M(\mathbb{N}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{F} \) (see [DM, Section 5] for more details).

We also define the “newforms”: let \( M(\mathbb{N}, \mathbb{F})^{\text{new}} := \ker(U_{S_N}^2 - N^{-2}S_N) \), where \( S_N \) is a weight-separating operator that scales \( M_k(\mathbb{N}, \mathbb{F}) \) by \( N^k \) [DM, Section 6]. Observe that replacing \( \mathbb{F} \) by \( \mathbb{C} \) recovers the usual classical notion of newforms. Note that \( M(\mathbb{N}, \mathbb{F})^{\text{old}} \) and \( M(\mathbb{N}, \mathbb{F})^{\text{new}} \) need not be disjoint [DM, Corollary 7.1]. However, we record the following fact.

**Lemma 2.2.** The operator \( (U_{S_N}^2 - N^{-2}S_N) \) maps \( M(\mathbb{N}, \mathbb{F}) \) to \( M(\mathbb{N}, \mathbb{F})^{\text{old}} \).

**Proof.** If \( f \in M_k(\mathbb{N}, \mathbb{Z}) \), then \( f = g + h \) with \( g \in M_k(\mathbb{N}, \mathbb{Q})^{\text{new}} \) and \( h \in M_k(\mathbb{N}, \mathbb{Q})^{\text{old}} \). Since the operator \( (U_{S_N}^2 - N^{-2}S_N) \) kills newforms and preserves integrality, we see that \( (U_{S_N}^2 - N^{-2}S_N) f \in M_k(\mathbb{N}, \mathbb{Z})^{\text{old}} \). \( \square \)

2.13.3. New and old Hecke algebra quotients. We also consider \( A(\mathbb{N}, \mathbb{F})^{\text{old}} \) (respectively, \( A(\mathbb{N}, \mathbb{F})^{\text{new}} \)), the largest quotient of \( A(\mathbb{N}, \mathbb{F}) \) acting faithfully on \( M(\mathbb{N}, \mathbb{F})^{\text{old}} \) (respectively, \( M(\mathbb{N}, \mathbb{F})^{\text{new}} \)). For the shallow Hecke algebras, we have \( A(\mathbb{N}, \mathbb{F})^{\text{old}} \cong A(1, \mathbb{F}) \).

Since both \( W_N \) and \( U_N \), the operators whose actions define oldforms and newforms, commute with all the Hecke operators away from \( Np \), both the spaces \( M(\mathbb{N}, \mathbb{F})^{\text{old}} \) and \( M(\mathbb{N}, \mathbb{F})^{\text{new}} \) and the Hecke algebras \( A(\mathbb{N}, \mathbb{F})^{\text{old}} \) and \( A(\mathbb{N}, \mathbb{F})^{\text{new}} \) split into local \( \bar{\rho} \)-components.

In particular, if \( \bar{\rho} : G_{\mathbb{Q}, N_p} \to \text{GL}_2(\mathbb{F}) \) is \( \Gamma_0(\mathbb{N}) \)-modular, then \( A(\mathbb{N}, \mathbb{F})_{\bar{\rho}} \) has two quotients

\[
A(1, \mathbb{F})_{\bar{\rho}} \cong A(\mathbb{N}, \mathbb{F})_{\bar{\rho}}^{\text{old}} \quad \text{and} \quad A(\mathbb{N}, \mathbb{F})_{\bar{\rho}}^{\text{new}}.
\]

If \( \bar{\rho} \) is unramified at \( N \), then \( M(\mathbb{N}, \mathbb{F})_{\bar{\rho}}^{\text{old}} \) is nonzero, so that the left quotient \( z^{\text{old}} : A(\mathbb{N}, \mathbb{F})_{\bar{\rho}} \to A(\mathbb{N}, \mathbb{F})_{\bar{\rho}}^{\text{old}} \) is nontrivial. If \( \bar{\rho} \) further satisfies the level-raising condition at \( N \) — namely, \( \text{tr} \bar{\rho}(\text{Frob}_N) = \pm (N+1)N^{\frac{k}{2}} \),
where \( \det \rho = \omega^k \) — then \( M(N,F)^{\text{new}}_\rho \) is nonzero, so that the right quotient \( \pi^{\text{new}} : A(N,F)_\rho \to A(N,F)^{\text{new}}_\rho \) is nontrivial. See [DM, section 7] for more details.

2.13.4. New and old quotients of the partially full Hecke algebra. Finally we note that \( M(N,F)\old \) and \( M(N,F)^{\text{new}} \), as well as local pieces \( M(N,F)^{\text{old}}_\rho \) and \( M(N,F)^{\text{new}}_\rho \), are all stable by \( U_N \). Indeed, suppose \( \rho \) is \( \Gamma_0(1) \)-modular and \( f \) is in \( M(1,F)_\rho \). Since we can ignore the action of \( T_N \), or indeed of any finite set of Hecke operators, in defining the \( \rho \)-generalized eigenspace, and since \( W_N \) commutes with all Hecke operators prime to \( N \), we find that \( W_N f \in M(N,F)_\rho \). Moreover, the \( F \)-span of \( \{ f, W_N f \} \) is a \( U_N \)-stable subspace of \( M(N,F)^{\text{old}}_\rho \). And if \( \rho \) is \( \Gamma_0(N) \)-modular, then \( f \in M(N,F)_\rho \) is in \( M(N,F)^{\text{new}}_\rho \) if and only if \( f \) is in a \( U^2_N \)-eigenspace. Since \( U_N \) and \( U^2_N \) commute, this property is preserved under the action of \( U_N \), so that \( M(N,F)^{\text{new}}_\rho \), and hence \( M(N,F)^{\text{old}}_\rho \), is \( U_N \)-stable. Therefore we can define \( A(N,F)^{\text{old}}_\rho \) and \( A(N,F)^{\text{new}}_\rho \), the faithful quotients of \( A(N,F) \rho \) acting on \( M(N,F)^{\text{old}}_\rho \) and \( M(N,F)^{\text{new}}_\rho \), respectively. See section 5.

3. The grading on the mod-p Hecke algebra

In this short section we exhibit natural compatible gradings on \( K(N,F)_\rho \), \( A(N,F)_\rho \), and \( \tau^{\text{mod}}_{p,N} \) by the 2-Frattini quotient of \( \mathbb{Z}_p \mathbb{Z}_p \), which restrict to gradings on \( A(N,F)_\rho \) if \( p = 2 \) or if \( p \) is odd and \( \rho \simeq \rho \otimes \omega^{(p-1)/2} \). This result (Theorem 3.4) generalizes the \( (\mathbb{Z}/2\mathbb{Z})^\times \)-grading in the \( (p,N) = (2,1) \) setting described by Nicolas and Serre [NS1] and the \( (\mathbb{Z}/3\mathbb{Z})^\times \)-grading in the \( (p,N) = (3,1) \) setting in the forthcoming treatment of mod-3 modular forms by the second-named author [Me3]. It owes an additional debt of inspiration to Bellaïche’s universal grading result for mod-2 constant-determinant pseudodeformations of \( \rho = 1 \oplus 1 \) (Theorem 6.3).

Suppose that \( B \) is a profinite ring and \( Q \) a finite abelian group, written multiplicatively. Recall that \( B \) is \( Q \)-graded if \( B \) splits as a direct sum \( B = \bigoplus_{i \in Q} B^i \) of closed additive subgroups \( B^i \) with \( B^1 \subseteq B \) a subring and \( B^i B^j \subseteq B^{ij} \) for every \( i, j \in Q \). If \( B \) is \( Q \)-graded, then a \( B \)-module \( M \) is \( Q \)-graded if \( M = \bigoplus_{i \in Q} M^i \) with \( B^i M^j \subseteq M^{ij} \).

Now further suppose that \((t,d) : G \to B \) is a (continuous) pseudorepresentation of a profinite group \( G \) on \( B \), and that \( Q \) is equipped with a continuous quotient map \( \pi : G \to Q \). For \( i \in Q \) let \( G^i \) := \( \pi^{-1}(i) \) be the corresponding coset of \( G \). We say that \((t,d) \) is \( Q \)-graded if \( B = \bigoplus_{i \in Q} B^i \) is a \( Q \)-graded ring and for every \( i \in Q \) we have \( t(G^i) \subseteq B^i \) and \( d(G^i) \subseteq B^{i^2} \). Note that all the pseudorepresentation relations from subsection 2.9 are homogeneous with respect to such a grading.

We begin with two lemmas; the grading theorem is Theorem 3.4. These establish that \( q \)-expansions of mod-p forms can be separated into coefficients whose indices are picked out by a character of conductor \( p \). Note that Lemma 3.1 is stated in a more general form than strictly required for Theorem 3.4 (for which \( M = 1 \) in Lemma 3.1 suffices); this generality necessitates Lemma 3.2. Both are well known.

**Lemma 3.1.** If \( f = \sum_n a_n q^n \in M(N,F), \) and \( \chi \) is a quadratic Dirichlet character of modulus \( p^r M, \) where \( r \geq 0 \) is arbitrary and \( M^2 | N, \) then the following are also forms in \( M(N,F) \):

\[
(1) \ f_{\chi} := \sum_n \chi(n) a_n q^n, \quad (2) \ f_{\chi,+} := \sum_{n : \chi(n) = 1} a_n q^n, \quad (3) \ f_{\chi,-} := \sum_{n : \chi(n) = -1} a_n q^n.
\]

**Proof.** We follow Jerome Rouse’s answer to MathOverflow question #202449. Let \( \tilde{f} \in M(N,O) \) be a lift of \( f \) for some ring of integers \( O \) with residue field \( F \) in a finite extension \( L \) of \( \mathbb{Q}_p \); we first show that the analogously defined characteristic-zero forms \( \tilde{f}_\chi, \tilde{f}_{\chi,+}, \) and \( \tilde{f}_{\chi,-} \) are in \( M(Np^{2r},O) \). Indeed, since \( \chi \) has order 2 and the square of its modulus divides \( Np^{2r} \), the statement about \( \tilde{f}_\chi \) follows from [Iw, Theorem 7.4]. Let \( S \) be the squarefree product of primes dividing \( p^r M \); write \( U_S \) as usual for the operator \( \sum_n a_n q^n \to \sum a_n q^n S \) and \( V_S \) for the operator \( \sum_n a_n q^n \to \sum a_n q^n S \). By Lemma 3.2 below, \( \tilde{f}_{\chi,0} := V_S U_S f \) is in \( M(Np^{2r},O) \), so that \( \tilde{f}_{\chi,+} = \frac{1}{2}(\tilde{f} - \tilde{f}_{\chi,0} + \tilde{f}_{\chi}) \) and \( \tilde{f}_{\chi,-} = \frac{1}{2}(\tilde{f} - \tilde{f}_{\chi,0} - \tilde{f}_{\chi}) \) are both in \( M(Np^{2r},O) \). Finally, \( F_p \)-reductions of forms of level \( N \) and of level \( Np^{2r} \) coincide; indeed, Hatada [Ha, Theorem 1], generalizing earlier work of Serre for \( p \geq 3 \) [SerF], shows that every modular form of level \( Np^{2r} \) is a \( p \)-adic limit of forms of level \( N \). \( \square \)
Lemma 3.2. If \( f \) is in \( M(N, \mathbb{Z}) \) and \( \ell \) is a prime, then \( V_\ell U_\ell f \in M(\text{lcm}(N, \ell^2), \mathbb{Z}) \).

Proof. Integrality of coefficients is clearly preserved, so it suffices to establish the level. The level of \( U_\ell f \) is a priori \( N\ell \) if \( \ell \nmid N \) and \( N \) if \( \ell \mid N \). But in fact if \( \ell^2 \mid N \), then \( U_\ell f \) is of level \( N/\ell \). Indeed, if \( \ell^2 \mid N \), then any form in \( M(N, \mathbb{Q}) \) is a linear combination of \( \ell \)-new forms (which \( U_\ell \) kills), forms from level \( N/\ell \) (which \( U_\ell \) keeps at level \( N/\ell \)), and forms in the image of \( V_\ell \) coming from level \( N/\ell \) (which \( U_\ell \) sends back to level \( N/\ell \)). Finally, \( V_\ell \) raises the level from \( N\ell \) or \( N/\ell \) by a factor of \( \ell \). \( \square \)

We now return to our setting of a fixed prime \( p \) and a level \( N \) prime to \( p \). Let \( \Pi_p \) be the 2-Gr"attini quotient of \( G_{Q,p'} \); that is, \( \Pi_p = \text{Gal}(L_p/Q) \), where \( L_2 = Q(\mu_8) \) and for \( p \) odd \( L_p \) is the quadratic subfield of \( Q(\mu_p) \). We also identify \( \Pi_p \) with the 2-Gr"attini quotient of \( \text{Gal}(Q(\mu_{p^{\infty}})/Q) \simeq \mathbb{Z}_p^* \); explicitly, \( \Pi_2 = (\mathbb{Z}/8\mathbb{Z})^* \) and for \( p \) odd \( \Pi_p = F_p^* / (F_p^*)^2 \). In this way, we may think of any \( n \in \mathbb{Z} \) prime to \( p \) as having a value in \( \Pi_p \). Restriction to \( L_p \) gives a quotient map \( G_{Q,NP} \rightarrow \Pi_p \) with the property that \( \text{Frob}_t \) maps to \( \ell \) for \( \ell \nmid NP \) prime.

For \( i \in \Pi_p \) and \( F/F_p \), let
\[
K(N,F)^i := \{ f \in K(N,F) : a_n(f) \neq 0 \Rightarrow n \in \Pi_p \} \subset K(N,F).
\]
The next lemma shows that Hecke operators act compatibly with this \( \Pi_p \)-indexing:

Lemma 3.3 (cf. [NS1, (6) and ff.]). For all \( m \) prime to \( p \) and \( i \in \Pi_p \), the Hecke operator at \( m \) maps \( K(N,F)^i \) into \( K(N,F)^{mi} \). This includes \( T_{mf} \) if \( m \nmid NP \) and \( U_m \) if \( m \mid N \).

Proof. Since the Hecke operators are multiplicative at relatively prime indices, it suffices to show this for prime-power-index Hecke operators. First let \( \ell \nmid NP \) be a prime. For a form \( f \) coming from weight \( k \) the Fourier coefficients of \( T_\ell f \) are well known:
\[
a_n(T_\ell f) = a_{n/\ell}(f) + \ell^{-1}a_{n/\ell}(f),
\]
where \( a_{n/\ell}(f) = 0 \) if \( \ell \nmid n \). The claim for \( T_\ell \) follows since \( \Pi_p \) is an elementary 2-group, so that \( \ell \equiv \ell^{-1} \) in \( \Pi_p \).

For \( r \geq 2 \) the claim for \( T_{\ell^r} \) follows by induction, since \( T_{\ell^r} = T_{\ell}T_{\ell^{r-1}} - \ell^{-1}T_{\ell^{r-2}} \). On the other hand, if \( \ell \) is a prime with \( \ell \mid N \), then \( a_n(U_{\ell^r} f) = a_n(U_{\ell^{r-1}} f) \) so that the claim for \( U_{\ell^r} \) follows. \( \square \)

Let \( \hat{\rho} \) be a \( \Gamma_0(N) \)-modular representation defined over \( \mathbb{F} \), and set \( K(N,F)^i_{\hat{\rho}} := K(N,F)^i \cap K(N,F)_{\hat{\rho}}. \)

Theorem 3.4. Suppose that \( p = 2 \) or \( p \) is odd and \( \hat{\rho} \cong \hat{\rho} \otimes \omega^{\frac{p-1}{2}}. \)

1. The space of forms has a natural \( \Pi_p \)-grading: \( K(N,F)_{\hat{\rho}} = \bigoplus_{i \in \Pi_p} K(N,F)^i_{\hat{\rho}}. \)

2. The Hecke algebra \( A(\mathbb{N,F})_{\hat{\rho}} \) has a natural \( \Pi_p \)-grading \( A(\mathbb{N,F})_{\hat{\rho}} = \bigoplus_{i \in \Pi_p} A(N,F)^i_{\hat{\rho}} \), where for \( m \nmid NP \) we have \( T_m \in A(N,F)^m_{\hat{\rho}} \). The decomposition from (1) endows \( K(N,F)_{\hat{\rho}} \) with the structure of a \( \Pi_p \)-graded \( A(N,F)_{\hat{\rho}} \)-module.

3. The decomposition from (2) and the quotient map \( G_{Q,NP} \rightarrow \Pi_p \) gives a \( \Pi_p \)-grading to the pseudorepresentation \( \tau_{\hat{\rho}}^{\text{mod}} \).

Moreover, (1)–(3) hold for \( A(N,F)_{\hat{\rho}} \times A(N,F)_{\hat{\rho} \otimes \omega^{(p-1)/2}} \) acting on \( K(N,F)_{\hat{\rho}} \oplus K(N,F)_{\hat{\rho} \otimes \omega^{(p-1)/2}} \) and for \( A(N,F)_{\hat{\rho}} \) acting on \( K(N,F)_{\hat{\rho}} \).

Proof. (1) Fix a quadratic Dirichlet character \( \chi \) on \( \Pi_p \), so that \( \chi = \omega^{\frac{p-1}{2}} \) if \( p \) is odd. For \( \varepsilon = \pm 1 \), let \( K(N,F)^{\chi,\varepsilon} = \{ g \in K(N,F)_{\hat{\rho}} : a_n(g) \neq 0 \text{ only if } \chi(n) = \varepsilon \} \). For \( f \in K(N,F)_{\hat{\rho}} \) we have, by the assumption that \( \hat{\rho} \cong \hat{\rho} \otimes \chi \) and in the notation of Lemma 3.1, \( f_{\chi,\varepsilon} \in K(N,F)^{\chi,\varepsilon}_{\hat{\rho}} \), so that \( K(N,F)_{\hat{\rho}} = K(N,F)^{\chi,\varepsilon}_{\hat{\rho}} \oplus K(N,F)^{\chi,-\varepsilon}_{\hat{\rho}} \), completing the proof for \( p \) odd. For \( p = 2 \), decompose each \( K(N,F)^{\chi,\varepsilon}_{\hat{\rho}} \) further into a direct sum of two pieces corresponding to the values of a second quadratic Dirichlet character in \( \Pi_p \).

(2) Follows formally from (1) and Lemma 3.3 as follows. Let \( B = A(N,F)_{\hat{\rho}} \). For \( i \in \Pi_p \), let
\[
B^i := \{ T \in A(N,F)_{\hat{\rho}} \mid \text{for all } j \in \Pi_p, TK(N,F)^j_{\hat{\rho}} \subseteq K(N,F)^j_{\hat{\rho}} \}. 
\]
By considering finite weight and taking limits we see that each $B^i$ is a closed $F$-submodule of $B$. Moreover, 1 in $B^1$ and $B^1B^j \subseteq B^j$: by considering $q$-expansions and using (1), it’s clear that the sum of the $B^i$ inside $B$ is direct, so that $B':= \bigoplus_{i \in \Pi_p} B^i$ is a $\Pi_p$-graded $F$-algebra. Finally by Lemma 3.3 for each $m \nmid Np$ we have $T_m \in B^m$. These operators generate $B$, so that $B = B'$.

(3) For every $i \in \Pi_p$, the coset $G^i_{Q,Np}$ is closed in $G_{Q,Np}$. By the Chebotarev density theorem, the Frobp-conjugacy classes for those primes $\ell \nmid Np$ with $\ell \equiv i \bmod \Pi_p$ are dense in $G^i_{Q,Np}$. Since $\tau^\text{mod}_p(\text{Frob}_\ell) = T_\ell$ is in $A(N,F)^i_F$ by (2), and $\tau^\text{mod}_p$ is continuous with each $A(N,F)^i_F$ closed, the claim follows.

The proofs of the second and third statement are analogous. □

Corollary 3.5. Under the conditions of Theorem 3.4 we additionally have a $\Pi_p$-grading on $A(N,F)_{\bar{\rho}}^{\text{pf}}$, and, if $N$ is prime, on $A(N,F)_{\bar{\rho}}^{\text{new}}$. These gradings are compatible with their structures as $A(N,F)_{\bar{\rho}}$-algebras and their action on $A(N,F)_{\bar{\rho}}$.

Proof. For $A(N,F)_{\bar{\rho}}^{\text{pf}}$, mimic the argument in Theorem 3.4(2), noting that Lemma 3.3 covers all Hecke operators topologically generating $A(N,F)_{\bar{\rho}}^{\text{pf}}$. For $A(N,F)_{\bar{\rho}}^{\text{new}}$, first refine Theorem 3.4(1) to give a grading on $K(N,F)_{\bar{\rho}}^{\text{new}} := K(N,F)_{\bar{\rho}} \cap M(N,F)^{\text{new}}$, on which $A(N,F)_{\bar{\rho}}^{\text{new}}$ acts faithfully. Namely, if $f \in K(N,F)_{\bar{\rho}}^{\text{new}}$ decomposes as $f = \sum_{i \in \Pi_p} f_i$ with $f_i \in K(N,F)^i_F$, then in fact each $f_i$ is in $K(N,F)_{\bar{\rho}}^{\text{new}}$. Indeed, by Lemma 3.3, the operator $U_{N}^{-1} - N^{-2}S_{N}$ is 1-graded and maps each $f_i$ to $K(N,F)^i_F$, so if it annihilates $f$, then it must annihilate each $f_i$. The grading on $A(N,F)_{\bar{\rho}}^{\text{new}}$ then follows as in Theorem 3.4(2). □

4. The level-$N$ shape deformation condition

Recall that we are assuming that $N$ is a prime different from $p$. In level 1, we expect $\mathcal{R}(1,F)_{\bar{\rho}}$ and $A(1,F)_{\bar{\rho}}$ to be isomorphic reasonably often (details in [BK]; in fact we do not know of any counterexamples). But already in prime level this is an unreasonable expectation: on the modular forms side, the prime-to-$\mathfrak{p}$ Artin conductor of a characteristic-zero $\Gamma_1(N)$-modular representation divides $N$ [CH]. But on the deformation side, the ramification at $N$ is unrestricted. In other words, by design and definition, $\mathcal{R}(N,F)_{\bar{\rho}} = \mathcal{R}(N^j,F)_{\bar{\rho}}$ for any $j \geq 1$, whereas a priori one expects $A(N^j,F)_{\bar{\rho}}$ to surject onto $A(N,F)_{\bar{\rho}}$ with nontrivial kernel. (v)

In short, to compare a Hecke algebra of prime level to a deformation ring, we will have to impose a deformation condition. We write $G = G_{Q,Np}$, $D_N = D_N(G)$ and $I_N = I_N(G)$ for brevity.

4.1. Pseudorepresentations of level-$N$ shape. Let $B$ be a $\mathbb{Z}_p$-algebra, and $(t,d) : G \to B$ a pseudorepresentation with $d$ a power of $\chi_p$. We will say that $(t,d)$ has level-$N$ shape if

\begin{equation}
\text{for every } d \in D_N \text{ and } i \in I_N, \text{ we have } t(di) = t(d). \tag{4.1.1}
\end{equation}

Equivalently, $(t,d)$ has level-$N$ shape if the kernel of $(t,d)|_{D_N}$ contains $I_N$ (see (2.9.2)).

The level-$N$ shape condition is meant to capture the notion of a representation having Artin conductor dividing $N$ for pseudorepresentations. Recall that Artin conductor of a Galois representation is defined by measuring dimensions of subspaces of invariants by filtrations of inertia groups. It is not clear how to extend this notion to general Galois pseudorepresentations, as there is no underlying space on which the Galois group is acting. But in fact, we will show that a representation has prime-to-$p$ Artin conductor dividing $N$ if and only if its associated pseudorepresentation has level-$N$ shape.

Let $K$ be a field that is also a $\mathbb{Z}_p$-algebra, and $\rho : G \to \text{GL}_K(V)$ a two-dimensional representation with det $\rho$ a power of $\chi_p$. Recall that $\rho$ has prime-to-$p$ Artin conductor $N$ (respectively, 1) if the inertial invariants $V^{1_N}$ form a one-dimensional (respectively, two-dimensional) subspace of $V$.

Proposition 4.1. Let $K$, $(\rho,V)$ be as above; let $(t,d) = (\text{tr } \rho, \det \rho)$. Then the following are equivalent.

\begin{enumerate}
\item $\rho$ has prime-to-$p$ Artin conductor dividing $N$.
\end{enumerate}

(v) In fact, one knows that $A(N^j,F)_{\bar{\rho}}$ stabilizes for $j \gg 0$. For example, if $\bar{\rho}$ is modular of level 1, then it follows from [CR, Proposition 2] that $A(N^j,F)_{\bar{\rho}} = A(N^2,F)_{\bar{\rho}}$ for every $j \geq 2$. 


(2) $\rho|_{I_N}$ is unipotent.
(3) $(t,d)|_{I_N} = (2,1)$.
(4) $(t,d)|_{D_N}$ splits over $\mathcal{R}$ as a sum of two unramified characters.
(5) $t$ has level-$N$ shape.

Proof. We show (1) $\iff$ (2) $\implies$ (4) $\implies$ (3) $\implies$ (2) and (4) $\implies$ (5) $\implies$ (3). If $\rho$ has Artin conductor dividing $N$, then $\rho|_{I_N}$ is reducible: if it is not trivial, then $\rho|_{I_N}$ has a one-dimensional invariant line $L \subset V$ and $I_N$ acts through a character on the quotient $V/L$; since $\det \rho$ is unramified at $N$, this character is trivial. In either case, $\rho|_{I_N}$ is unipotent. The converse also holds, so (1) $\iff$ (2). If $\rho|_{I_N}$ is unipotent, then since $I_N$ is normal in $D_N$ with abelian quotient, $\rho|_{D_N}$ is upper-triangularizable, possibly after a quadratic extension. Indeed, the normality guarantees that $D_N$ preserves the 1-eigenspace of $I_N$, on which it acts through its abelian quotient $D_N/I_N$, so that there’s a common eigenvector; extending $K$ may be necessary if $I_N$ acts trivially. So (2) implies (4). The implication (4) $\implies$ (3) is clear. If $(t,d)|_{I_N} = (2,1)$, then by the Brauer-Neubert theorem the semisimplification of $\rho|_{I_N}$ is trivial, so that $\rho|_{I_N}$ is unipotent, so that (3) $\implies$ (2). If $(t,d)|_{D_N}$ is a sum of unramified characters, then the semisimplification of $\rho|_{D_N}$ contains $I_N$ in its kernel. Since semisimplifying $\rho$ does not change its pseudorepresentation, we conclude that $I_N$ is contained in $\ker (t,d)|_{D_N}$, so that (4) implies (5). Finally, if $I_N$ is in the kernel of $(t,d)|_{D_N}$, then $t(i - 1) = t(1) = 2$ for all $i \in I_N$. Therefore (5) implies (3). □

4.2. Level-$N$ shape as a deformation condition. Suppose $\bar{\rho} : G_{Q,N_p} \to \text{GL}_2(\mathbb{F})$ is a semisimple representation with prime-to-$p$ Artin conductor dividing $N$. Let $D_{\bar{\rho}}$ be the functor from $C$ to sets sending a local $\mathbb{F}$-algebra $B$ in $C$ to the set of odd, constant-determinant pseudodeformations of $\bar{\rho}$ having level-$N$ shape.

Then $D_{\bar{\rho}}$ is representable by a complete noetherian $\mathbb{F}$-algebra $(\mathcal{R}(N, \mathbb{F})_{\bar{\rho}}, \mathfrak{n}(N, \mathbb{F})_{\bar{\rho}})$, the quotient of $\mathcal{R}(N, \mathbb{F})_{\bar{\rho}}$ by the closed ideal $\hat{J}_N$ generated by the set
\[
\{ \hat{\tau}^{\text{univ}}(di) - \hat{\tau}^{\text{univ}}(d) : d \in D_N(G_{Q,N_p}), i \in I_N(G_{Q,N_p}) \}.
\]
This gives us a universal pseudodeformation of $\bar{\rho}$
\[(4.2.1) \quad \tau^{\text{univ}} : G_{Q,N_p} \to \mathcal{R}(N, \mathbb{F})_{\bar{\rho}}\]
factoring through $\mathcal{R}(N, \mathbb{F})_{\bar{\rho}}$. If $\ell \nmid Np$ is prime, then set $t_\ell := \tau^{\text{univ}}(\text{Frob}_\ell)$ and $t'_\ell := t_\ell - \text{tr} \hat{\rho}(\text{Frob}_\ell)$.

As in equation (2.10.2), we can identify the tangent space of this modified universal deformation ring with deformations to the dual numbers:
\[(4.2.2) \quad \text{Tan} \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} = \text{Hom} \left( \mathfrak{n}(N, \mathbb{F})_{\bar{\rho}}/\mathfrak{n}(N, \mathbb{F})_{\bar{\rho}}^2, \mathbb{F} \right) \cong D_{\bar{\rho}}(\mathbb{F}[\varepsilon]).\]

For $\Gamma_0(1)$-modular $\rho$, write $\mathcal{R}(1, \mathbb{F})_{\bar{\rho}}$ for $\mathcal{R}(1, \mathbb{F})_{\bar{\hat{\rho}}}$, $\varphi$ for $\tilde{\varphi}$ from (2.11.3), and $\tau^{\text{univ}}$ for $\hat{\tau}^{\text{univ}}$ from (2.10.1).

Note that the map $\tilde{\psi}_{N,1} : \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} \to \mathcal{R}(1, \mathbb{F})_{\bar{\rho}}$ described in (2.10.3) factors through $\mathcal{R}(N, \mathbb{F})_{\bar{\rho}}$:
\[(4.2.3) \quad \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} \xrightarrow{\tilde{\psi}_{N,1}} \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} \xrightarrow{\psi_{N,1}} \mathcal{R}(1, \mathbb{F})_{\bar{\rho}},\]

because $\hat{J}_N$, described above, is visibly contained in $\hat{J}_{N,1} = \ker \tilde{\psi}_{N,1}$, described after (2.10.3).

4.3. Level-$N$ shape and $\Gamma_0(N)$-modular pseudorepresentations. In this subsection we establish that all $\Gamma_0(N)$-modular pseudorepresentations have level-$N$ shape and record consequences for the relationship between the Hecke algebra and the level-$N$ deformation ring. Recall our notation in this section: $N$ is prime, $G = G_{Q,N_p}$, $D_N = D_N(G)$ and $I_N = I_N(G)$.

Theorem 4.2 (Atkin-Lehner, Carayol). Let $f$ be in $S_k(N, \mathbb{Q}_p)_{\text{new}}$. Then $a_N(f) = \pm N^{(k-2)/2}$. Moreover, if $\rho : G \to \text{GL}_2(\mathbb{Q}_p)$ is the attached $p$-adic Galois representation, then $\rho|_{D_N} \sim \begin{pmatrix} Xp & \psi \psi \\ 0 & \psi \end{pmatrix}$, where $\psi$ is the unramified character sending $\text{Frob}_N$ to $a_N(f)$, and the extension $\ast$ is ramified at $N$. 
Proof. The first statement is due to Atkin and Lehner [AL, Theorem 3]. The second statement is implied by local-global compatibility established by Carayol [CH]; see also [We, Section 3] and [EPW, Lemma 2.6.1] for a statement in this context. Note that $\psi = \varepsilon \chi_p^{(k-2)/2}$ with $\varepsilon^2 = 1$.

\begin{corollary} \label{corollary4.3} 
\begin{enumerate}[label=(\arabic*)]
\item Any $\Gamma_0(N)$-modular pseudorepresentation has level-$N$ shape.
\item If $\bar{\rho} : G \to \GL_2(\mathbb{F})$ is $\Gamma_0(N)$-modular, then the pseudorepresentation $\tau_{\bar{\rho}}^{\text{mod}} : G \to A(N, \mathbb{F})_{\bar{\rho}}$ constructed in (2.11.2) has level-$N$ shape.
\item The surjection $\varphi : \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} \to A(N, \mathbb{F})_{\bar{\rho}}$ from (2.11.3) factors through $\mathcal{R}(N, \mathbb{F})_{\bar{\rho}}$, inducing a continuous surjective map $\varphi : \mathcal{R}(N, \mathbb{F})_{\bar{\rho}} \to A(N, \mathbb{F})_{\bar{\rho}}$ sending $t_\ell$ to $T_\ell$.
\end{enumerate} \end{corollary}

Proof. Any $\Gamma_0(N)$-modular pseudorepresentation comes from a $p$-adic representation $\rho_f$ attached to a $\Gamma_0(N)$-modular eigenform $f$; by Proposition 4.1, it suffices to show that $\rho_f$ satisfies any of the equivalent conditions listed there. If $f$ is a cuspidal newform, then Theorem 4.2 implies that $\rho_f$ visibly satisfies condition (4). Otherwise $f$ comes from a level-1 form, so that $\rho_f$ is unramified at $N$, and satisfies, for example, condition (3). Parts (2) and (3) follow from (1) since $\tau_{\bar{\rho}}^{\text{mod}}$ is obtained by gluing characteristic-zero $\Gamma_0(N)$-modular pseudorepresentations and then reducing modulo $p$. \qed

5. $U_N$ and the partially full Hecke algebra

We continue the notation of section 4; recall that $N \neq p$ is prime. Here we track several consequences of Theorem 4.2 and Corollary 4.3, in particular, the connection between the Atkin-Lehner operator $U_N$ and the trace of Frobenius-at-$N$ elements in the partially full Hecke algebra.

5.1. The polynomial satisfied by $U_N$. Although $\text{Frob}_N$ is not well-defined, even up to conjugacy, as an element of $G_{\mathbb{Q}, Np}$, it does determine a coset of $I_N(G_{\mathbb{Q}, Np})$ inside $D_N(G_{\mathbb{Q}, Np})$. Therefore, the level-$N$ shape of $\tau_{\bar{\rho}}^{\text{mod}}$ guarantees that

\begin{equation} \label{eq:fn}
F_N := \tau_{\bar{\rho}}^{\text{mod}}(\text{Frob}_N) \text{ is a well-defined element of } A(N, \mathbb{F})_{\bar{\rho}}.
\end{equation}

Now fix a $\Gamma_0(N)$-modular $\bar{\rho}$. In the case that $\bar{\rho}$ is unramified at $N$, the surjection $\pi_{\text{old}} : A(N, \mathbb{F})_{\bar{\rho}} \to A(1, \mathbb{F})_{\bar{\rho}}$ from (2.13.1) maps $F_N$ to $T_N$ and $a_N(\bar{\rho})$ to $\text{tr} \rho_f(\text{Frob}_N)$.

Now let $\bar{\rho}$ be an arbitrary $\Gamma_0(N)$-modular representation appearing in weight $k_{\bar{\rho}}$ and let

\begin{equation} \label{eq:pn}
P_N(X) := X^2 - \tau_{\bar{\rho}}^{\text{mod}}(\text{Frob}_N)X + \det(\text{Frob}_N) = X^2 - F_NX + N^{k_{\bar{\rho}} - 1} \in A(N, \mathbb{F})_{\bar{\rho}}[X],
\end{equation}

be the characteristic polynomial of any $\text{Frob}_N$ under $\tau_{\bar{\rho}}^{\text{mod}}$.

\begin{proposition} \label{proposition5.1} If $\bar{\rho}$ is $\Gamma_0(N)$-modular, then $P_N(U_N) = 0$ in $A(N, \mathbb{F})_{\bar{\rho}}^{pf}$.
\end{proposition}

Proof. Since $A(N, \mathbb{F})_{\bar{\rho}}^{pf}$ acts faithfully on $M(N, \mathbb{F})_{\bar{\rho}}$, and since the action of all the Hecke operators comes from their action on the characteristic-zero space $M(N, \mathbb{Q}_p)$, which has a basis of eigenforms, it suffices to show the following: for any $k \geq 0$ even and any normalized Hecke eigenform $f \in M_k(N, \mathbb{Q}_p)$, its $U_N$-eigenvalue $a_N(f)$ is annihilated by $P_{N,f}(X) := X^2 - \text{tr} \rho_f(\text{Frob}_N)X + \det \rho_f(\text{Frob}_N)$. Here $\rho_f : G_{\mathbb{Q}, Np} \to \GL_2(\mathbb{Q}_p)$ is the $p$-adic Galois representation attached to $f$.

If $f$ is a newform, then Theorem 4.2 explicitly shows that $a_N(f)$ is an eigenvalue of $\rho_f$ evaluated at any $\text{Frob}_N$ (it suffices to consider the semisimplification of the matrix loc. cit.). Thus $a_N(f)$ is a root of $P_{N,f}(X)$. On the other hand, if $f$ is an $N$-stabilization of a level-1 eigenform $g$, then $\rho_f = \rho_g$, and $a_N(f)$ is an eigenvalue of the matrix $\left( \begin{array}{cc} a_N(g) & 1 \\ -N^{k-1} & 0 \end{array} \right)$, which gives the action of $U_N$ on the two-dimensional Hecke-stable subspace with basis $g$ and $g(Nz)$. Since $a_N(g) = \text{tr} \rho_g(\text{Frob}_N)$, the characteristic polynomial of this matrix coincides with $P_{N,f}(X)$, and the claim follows. \qed
Corollary 5.2. If \( \bar{\rho} \) is \( \Gamma_0(N) \)-modular, then the map sending \( X \) to \( U_N \) gives a surjection
\[
A(N, \mathbb{F})_{\bar{\rho}}[X]/\mathcal{P}_N(X) \twoheadrightarrow A(N, \mathbb{F})_{\bar{\rho}}^{\text{pf}}
\]
compatible with the natural inclusion \( A(N, \mathbb{F})_{\bar{\rho}} \hookrightarrow A(N, \mathbb{F})_{\bar{\rho}}^{\text{pf}} \).

Proof. Follows from Proposition 5.1. See also [De, section 8]. \( \square \)

5.2. A relation between \( U_N \) and the image of \( \tau_{\bar{\rho}}^{\text{mod}} \). Write \( \tau \) for \( \tau_{\bar{\rho}}^{\text{mod}} \).

Proposition 5.3. With \( i \) any element of \( I_N(G_{\mathbb{Q}, \mathbb{Q}_p}) \), \( \text{any Frob}_N \in D_N(G_{\mathbb{Q}, \mathbb{Q}_p}) \), and \( g \in G_{\mathbb{Q}, \mathbb{Q}_p} \), we have
\[
U_N(\tau(g) - \tau(\bar{g})) = \tau(g \text{Frob}_N) - \tau(\bar{g} \text{Frob}_N).
\]
In particular, for \( g = c \) any complex conjugation,
\[
(5.2.1) \quad U_N \tau(ci) = \tau(ci \text{Frob}_N) - \tau(c \text{Frob}_N).
\]

Proof. As in the proof of Proposition 5.1 it suffices to show that for every Hecke eigenform \( f \in M(N, \mathbb{Q}_p) \), we have \( a_N(f)(\text{tr} \rho_f(gi) - \text{tr} \rho_f(\bar{g})) = \text{tr} \rho_f(gi \text{Frob}_N) - \text{tr} \rho_f(\bar{g} \text{Frob}_N) \), where \( \rho_f : G_{\mathbb{Q}, \mathbb{Q}_p} \rightarrow \text{GL}_2(\mathbb{Q}_p) \) is the \( p \)-adic Galois representation attached to \( f \). If \( f \) is old, then \( i \) is in the kernel of \( \rho_f \), so that on the left hand side \( \text{tr} \rho_f(gi) = \text{tr} \rho_f(\bar{g}) \) and on the right hand side \( \text{tr} \rho_f(gi \text{Frob}_N) = \text{tr} \rho_f(\bar{g} \text{Frob}_N) \); both sides reduce to 0. So it suffices to consider \( f \) new. In this case, Theorem 4.2 implies that there is a basis for \( \rho_f \) so that
\[
\rho_f(\text{Frob}_N) = \begin{pmatrix} a_N(f) & * \\ 0 & a_N(f) \end{pmatrix} \quad \text{and} \quad \rho_f(i) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.
\]
The main statement then follows from part (2) of Lemma 5.4 below by setting \( T = \rho_f(\text{Frob}_N) \), \( P = \rho_f(i) \), \( M = \rho_f(g) \). For the second statement, take \( g = c \) and note that \( \text{tr} \rho_f(c) = 0 \). \( \square \)

Lemma 5.4. Let \( B \) be a ring, \( M = \begin{pmatrix} a_M & b_M \\ c_M & d_M \end{pmatrix} \in M_2(B) \), upper-triangular \( T = \begin{pmatrix} a_T & b_T \\ 0 & d_T \end{pmatrix} \in M_2(B) \), and upper-triangular unipotent \( P = \begin{pmatrix} 1 & b_P \\ 0 & 1 \end{pmatrix} \in M_2(B) \). Then
\[
\begin{align*}
\text{(1)} & \quad \text{tr} MP - \text{tr} M = c_M b_P \\
\text{(2)} & \quad \text{tr} TMP - \text{tr} TM = d_T c_M b_P = d_T(\text{tr} MP - \text{tr} M)
\end{align*}
\]
Part (1) is an easy computation; (2) follows from (1) by taking \( TM \) for \( M \);

Remark 5.5. By taking \( MT \) for \( M \) in Lemma 5.4, we obtain the similar \( \text{tr} MTP - \text{tr} MT = a_T(\text{tr} MP - \text{tr} M) \), which leads to \( NU_N(\tau(gi) - \tau(\bar{g})) = \tau(g \text{Frob}_N i) - \tau(\bar{g} \text{Frob}_N) \). This and (5.2.1) are the two Calegari–Specter–style conditions that we use in Theorem 14.1. \( \triangle \)

6. The trivial \( \bar{\rho} \) mod 2

For the rest of this article, we specialize to \( p = 2 \) and \( \bar{\rho} = 1 \oplus 1 \), so that we can take \( \mathbb{F} = \mathbb{F}_2 \) and suppress \( \mathbb{F}_2 \) from notation. Recall that \( N \) is an odd prime and set \( G := G_{\mathbb{Q}, 2N} \); we’re viewing \( 1 \oplus 1 \) as a representation of \( G \). Also let \( D_N := D_N(G) \) and \( I_N := I_N(G) \). Write \( \tau := \tau_{\bar{\rho}}^{\text{mod}} \).

In this setting, the Hecke algebra and the deformation rings all have compatible gradings, which we describe in subsection 6.3. Moreover, the fact that the two Atkin-Lehner eigenvalues are glued together has several ramifications that we explore in subsections 6.4 and 6.5. We review what’s known in level one in subsection 6.2.
6.1. Galois group notation. We fix additional notation in use for the remainder of this document. Recall that the 2-Frattini quotient of $G$ is

$$
\Pi_{2N} = G/G^2 = \text{Gal}\left( \mathbb{Q}(i, \sqrt{2}, \sqrt{N})/\mathbb{Q} \right) \simeq \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \times \text{Gal}(\mathbb{Q}(\sqrt{N})/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times \times \{\pm 1\}.
$$

Let $\eta : G \to (\mathbb{Z}/8\mathbb{Z})^\times \times \{\pm 1\}$ be the quotient map, with components $\eta_2 : G \to (\mathbb{Z}/8\mathbb{Z})^\times$ and $\eta_N : G \to \{\pm 1\}$. For $i \in (\mathbb{Z}/8\mathbb{Z})^\times$, let $G_i := \eta_2^{-1}(i)$, so that $G = G_1 \cup G_3 \cup G_5 \cup G_7$. Further refine each $G_i$ as $G_i^+ \cup G_i^-$, with $G_i^+ := \eta^{-1}(\{1, \varepsilon\})$. Moreover, for $(i, \varepsilon) \in (\mathbb{Z}/8\mathbb{Z})^\times \times \{\pm 1\}$, we’ll let $g_i$ be an arbitrary element of $G_i^+$ and $g_i^-$ an arbitrary element of $G_i^-$. Finally, for a subset $S$ of $G$, write $\overline{S}$ for its image in $G/G^2$.

**Lemma 6.1.** For any prime $N$ we have

$$(a) \ c \ in \ G_1^+; \quad (b) \ \overline{T_N} = \overline{G_1}; \quad (c) \ \overline{D_N} = (\overline{T_N}, \overline{G_1})$$

**Proof.** Let $K_1 = \mathbb{Q}(\zeta_8)$. The first part follows from the fact that $c$ is 7 in $\text{Gal}(K_1/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times$ and that $\mathbb{Q}(\sqrt{N})$ is a totally real field. For the other parts, since $\text{Frob}_N$ in $\text{Gal}(K_1/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times$ is the element $N$, the decomposition group at $N$ in $\text{Gal}(K_1/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times$ is $(N)$. Since every prime above $N$ of $K_1$ totally ramifies in $K := \mathbb{Q}(\zeta_8, \sqrt{N})$, we know that $I_N$ is contained in the preimage $G_1$ of $\text{Gal}(K/K_1)$ in $G$ but not in $G_1^+$; the second part follows. By the same token, $D_N$ is contained in the preimage of $\langle N \rangle \subset (\mathbb{Z}/8\mathbb{Z})^\times = \text{Gal}(K_1/\mathbb{Q})$ in $G$, and we can choose $\text{Frob}_N$ such that its image is in $G_1^+$.

6.2. The structure of $\mathcal{R}(1)_{1 \oplus 1}$ and $A(1)$. Before continuing with level $N$, we briefly describe the situation in level 1. In this case, there is only one $\tilde{\rho}$, namely $1 \oplus 1$, so that $A(1) = A(1)_{1 \oplus 1}$ is a local ring. Recall that the Galois group $G_{\mathbb{Q},2}$ has 2-Frattini quotient $\Pi_2 = \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) = (\mathbb{Z}/8\mathbb{Z})^\times$; for $i \in \Pi_2$, let $G_{\mathbb{Q},2}^i$ be the coset mapping to $i$ under the natural 2-Frattini quotient map $G_{\mathbb{Q},2} \to (\mathbb{Z}/8\mathbb{Z})^\times$. We’ve established a $(\mathbb{Z}/8\mathbb{Z})^\times$-grading on $A(1)$ and $\tau$ (Theorem 3.4). We describe the structure of $A(1)$ and the isomorphism $\mathcal{R}(1)_{1 \oplus 1} \simeq A(1)$ following Nicolas, Serre, and Bellaïche [NS1, NS2, BeR].

Let $B = \mathbb{F}_2[x, y]$, an abstract $\mathbb{F}_2$-algebra that we endow with a $(\mathbb{Z}/8\mathbb{Z})^\times$-grading in two different ways as follows. Fix $i = 3$ or $i = 5$ in $(\mathbb{Z}/8\mathbb{Z})^\times$. Let $x$ have grading $i$ and $y$ have grading $-i$, so that $B^1 = \mathbb{F}_2[x^2, y^2]$, $B^i = xyB^1$, $B^{-i} = yB^1$, and $B^2 = xyB^3$.

**Theorem 6.2** (Nicolas, Serre, Bellaïche).

(1) For any choice of $g_i \in G_{\mathbb{Q},2}^i$ and $g_{-i} \in G_{\mathbb{Q},2}^{-i}$, the map

$$
\mathbb{F}_2[x, y] \rightarrow A(1) \quad \text{given by} \quad x \mapsto \tau(g_i), \quad y \mapsto \tau(g_{-i}),
$$

is an isomorphism of $(\mathbb{Z}/8\mathbb{Z})^\times$-graded $\mathbb{F}_2$-algebras. In particular, if $i = 3$ then $x \mapsto T_{13}$ and $y \mapsto T_5$ is such an isomorphism; if $i = 5$, then $x \mapsto T_{13}$ and $y \mapsto T_3$ is such an isomorphism.

(2) The map $\varphi : \mathcal{R}(1)_{1 \oplus 1} \rightarrow A(1)$ is an isomorphism; $\mathcal{R}(1)_{1 \oplus 1}$ and $\tau_{\text{univ}}$ are $(\mathbb{Z}/8\mathbb{Z})^\times$-graded.

(3) Both $\tau_{\text{univ}}$ and $\tau$ factor through $G_{\mathbb{Q},2}^{\text{pro-2}}$, preserving the grading.

The group $G_{\mathbb{Q},2}^{\text{pro-2}}$ has been studied by Markshaitis and Serre; it has presentation $\langle g, c : c^2 = 1 \rangle$ in the category of free pro-$2$-groups, where $g$ is any element that does not fix $\sqrt{2}$ [Ma].

**Theorem 6.2** begins to clarify why we eventually restrict to $N \equiv 3, 5$ mod 8 for our main results.

6.3. The grading on $\hat{\mathcal{R}}(N)_{1 \oplus 1}$ and $\mathcal{R}(N)_{1 \oplus 1}$. By Theorem 3.4 we know that the Hecke algebra $A(N)_{1 \oplus 1}$ and the modular pseudorepresentation $\tau_{\text{mod}} := \tau_{\text{mod}}^{1 \oplus 1}$ are graded by $(\mathbb{Z}/8\mathbb{Z})^\times$. Bellaïche has described a richer grading by all of $G/G^2$ on $\hat{\mathcal{R}}(N)_{1 \oplus 1}$ and $\tau_{\text{univ}} := \tau_{\text{univ}}^{1 \oplus 1}$ as well. The construction makes sense for any profinite group that is 2-finite in the sense of Mazur, but we restrict to groups of the form $G_{\mathbb{Q},2M}$.

**Theorem 6.3** (Bellaïche, unpublished). Let $M \geq 1$ be any odd level, and let $\Pi$ be the 2-Frattini quotient of $G_{\mathbb{Q},2M}$. Then $\hat{\mathcal{R}}(M)_{1 \oplus 1}$ has a natural $\Pi$-grading making $\tau_{\text{univ}}^{1 \oplus 1,M}$ into a graded $\Pi$-pseudorepresentation.
Bellå¡che’s grading on \( B = \hat{R}(M)_{1\oplus 1} \) takes the following shape: for \( i \in \Pi \), let \( G_{Q,2M}^i \subset G_{Q,2M} \) be the corresponding coset and set \( B^i \) to be the closed \( F_2 \)-submodule of \( B \) generated by \( \hat{r}^\text{univ}_{1\oplus 1,M}(G_{Q,2M}^i) \), along with 1 if \( i = 1 \). The trace-determinant identity (2.9.1) implies that \( B' := \bigoplus_{i \in \Pi} B^i \) is a \( \Pi \)-graded algebra. The same will be true for any \( \hat{r}^\text{univ}_{1\oplus 1,M}(G_{Q,2M}^i) \) grading by (6.4.5) \( \Pi \)-graded pseudorepresentation and \( \varphi : \hat{R}(N)_{1\oplus 1} \rightarrow A(N)_{1\oplus 1} \) is a map of \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-graded \( \mathbb{F}_2 \)-algebras.

**Corollary 6.4.** The level-\( N \)-shape universal deformation ring \( \hat{R}(N)_{1\oplus 1} \) has a natural \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-grading, so that \( \tau^\text{mod} \) is a \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-graded pseudorepresentation and \( \varphi : \hat{R}(N)_{1\oplus 1} \rightarrow A(N)_{1\oplus 1} \) is a map of \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-graded \( \mathbb{F}_2 \)-algebras.

**Proof.** By Theorem 6.3, \( \hat{R}(N)_{1\oplus 1} \) has a grading by \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \), so it suffices to show that the kernel of the quotient map \( \hat{R}(N)_{1\oplus 1} \rightarrow \hat{R}(N)_{1\oplus 1} \) is \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-graded. This kernel is the ideal \( J_N \) defined in subsection 4.2, topologically generated by elements of the form \( \hat{r}^\text{univ}(d) - \hat{r}^\text{univ}(d) \) for \( d \in D_N \) and \( i \in I_N \); to see that \( J_N \) is graded, we want \( \hat{r}^\text{univ}(d) \) graded \( \tau^\text{univ}(d) \) to be in the same \( \mathbb{Z}/8\mathbb{Z} \times \mathbb{F}_p \)-component of \( \hat{R}(N)_{1\oplus 1} \). Since \( \hat{r}^\text{univ} \) is graded, it suffices to know that the map \( D_N \rightarrow \text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) = \mathbb{Z}/8\mathbb{Z} \) factors through \( D_N/I_N \). But this is clear as \( \mathbb{Q}(\zeta_8) \) is unramified at \( N \).

6.4. The partially full Hecke algebra. We specialize section 5 to \( p = 2 \). In this setting, the polynomial satisfied by \( U_N \) over \( A(N) \) ((5.1.2) and Proposition 5.1) has the form

\[
(6.4.1) \quad \mathcal{P}_N(X) = X^2 + F_N X + 1.
\]

Here recall that \( F_N = \tau^\text{mod}(\text{Frob}_N) \). In particular, since \( a_N(1 \oplus 1) = 0 \), the polynomial \( \mathcal{P}_N(X) \) has a double root residually modulo \( \mathfrak{m}(N)_{1\oplus 1} \), so that from the description of the maximal ideals of the partially full Hecke algebra in subsection 2.8, it’s clear that

\[
(6.4.2) \quad A(N)_{1\oplus 1} \text{ is a complete local noetherian ring with residue field } F_2.
\]

The same will be true for any \( N \)-old \( \rho \) mod 2 satisfying the level-raising condition of subsection 2.13.3.

Let \( U'_N = U_N + 1 \), so that \( U'_N \in \mathfrak{m}(N)_{1\oplus 1} \). Then a simple manipulation of (6.4.1) implies that

\[
(6.4.3) \quad F_N = F_N U'_N - (U'_N)^2
\]

from which it’s immediately clear that

\[
(6.4.4) \quad F_N \in (\mathfrak{m}(N,F)^{1\oplus 1}).
\]

Finally, we record Proposition 5.3 in our setting: for \( i \in I_N \) we have

\[
(6.4.5) \quad U_N \tau^\text{mod}(ci) = \tau^\text{mod}(ci \text{Frob}_N) - \tau^\text{mod}(c \text{Frob}_N).
\]

6.5. New and very new forms and their Hecke algebras. Specializing subsection 2.13.2 to \( p = 2 \) tells us that the subspace of newforms inside \( M(N) \) is \( \ker(U_N^2 - N^{-2} S_N) = \ker(U'_N)^2 \). We similarly define \( K(N)_{\text{new}} := K(N) \cap M(N)_{\text{new}} \), so that

\[
(6.5.1) \quad M(N)_{\text{new}} = M(N)[(U'_N)^2] \text{ and } K(N)_{\text{new}} = K(N)[(U'_N)^2].
\]

From this description and the duality in (2.8.3) we deduce that

\[
(6.5.2) \quad A(N)_{\text{new}} = A(N)^{1\oplus 1}/(U'_N)^2.
\]

In particular, \( A(N)_{\text{new}} \) will have nilpotent elements. We therefore define, in an ad-hoc way for \( p = 2 \), the spaces of very new modular forms:

\[
M(N)_{\text{vnew}} := \ker U'_N \subseteq M(N) \text{ and } K(N)_{\text{vnew}} := K(N)[U'_N].
\]

With this definition, the newforms are generalized very new forms. Like the newforms, the very new forms break up into local components for various \( \tilde{p} \); we in particular consider

\[
M(N)_{1\oplus 1} := M(N)_{\text{vnew}} \cap M(N)_{1\oplus 1} \text{ and } K(N)_{1\oplus 1} := K(N)_{1\oplus 1}[U'_N].
\]
Let $A(N)^{v\text{new}}$ be the largest quotient of $A(N)$ acting faithfully on $M(N)^{v\text{new}}$; extending scalars as necessary, it too breaks up into local $\tilde{\rho}$-components, with $A(N)^{v\text{new}}$ the largest quotient of $A(N)$, or of $A(N)^{v\text{new}}$, acting faithfully on $M(N)^{v\text{new}}$. By (2.8.3) again,
\[(6.5.3) \quad A(N)^{v\text{new}} = A(N)^{\text{pf}}/(U_N') \quad \text{and} \quad A(N)^{v\text{new}}_{\text{red}} = A(N)^{\text{pf}}_{\text{red}}/(U_N').\]
In particular,
\[(6.5.4) \quad A(N)^{v\text{new}} \text{ acts faithfully on } K(N)^{v\text{new}}.\]

Remark 6.5. One can show that $K(N)^{v\text{new}}$, $A(N)^{v\text{new}}$, and the modular $A(N)^{v\text{new}}$-valued pseudorepresentation $\tau^{v\text{new}}$, as well as all their corresponding $\tilde{\rho}$ component analogues, are naturally and compatibly graded by $(\mathbb{Z}/8\mathbb{Z})^2/(N)$ in the sense of Theorem 3.4.

\[\triangle\]

7. Infinitesimal deformations of the trivial $\tilde{\rho}$ mod 2

We continue our notation from the previous section; in particular $N$ is an odd prime. We analyze first-order deformations of the pseudorepresentation associated to the representation $1 \oplus 1$ of $\mathcal{G}_{3,2N}$ to compute tangent space dimensions of $\mathcal{R}(N)_{\text{red}}$ and $\mathcal{R}(N)_{1\oplus 1}$ and their maximal reduced quotients. For $N \equiv 3$ or 5 modulo 8 we also find the generators of the cotangent space of $\mathcal{R}(N)_{1\oplus 1}$.

7.1. Tangent dimensions of $\mathcal{R}(N)_{1\oplus 1}$ and $\mathcal{R}(N)_{1\oplus 1}$.

Lemma 7.1. Let $G^2$ be the closed subgroup of $G$ generated by the squares of elements of $G$. Then
\[(1) \quad \text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} \cong \{ \text{set maps } b : G/G^2 \to \mathbb{F}_2 \mid b(1) = b(c) = 0 \};\]
\[(2) \quad \text{dim } \mathcal{R}(N)_{1\oplus 1} \cong \{ \text{set maps } b : G/G^2 \to \mathbb{F}_2 \mid b(1) = b(c) = 0, \ b(di) = b(d) \text{ for all } d \in \mathbb{D}_N, i \in \mathbb{T}_N \}.\]

Proof. See also [Ch, Lemma 5.3]. We use equations (2.10.2) and (4.2.2), and identify a pseudodeformation $t = \epsilon b : G \to \mathbb{F}_2[\epsilon]$ of the trivial mod-2 representation with the set-theoretic map $b : G \to \mathbb{F}_2$. The trace-determinant identity for elements $g$ and $gh$ of $G$ on $t$ simplifies to $b(g^2h) = b(gh)$, so that $b$ factors through $G/G^2$. Since the latter is abelian, the condition $t(gh) = t(hg)$ is automatically satisfied for all $g, h \in G$. The condition $t(1) = 2 = 0$ forces $b(1) = 0$; oddness forces $b(c) = 0$. The additional requirements on $b$ in (2) come from the level-$N$ shape condition. \[\Box\]

Corollary 7.2. \[\text{dim } \text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} = 6.\]

Proof. From (6.1.1), $G/G^2 = \text{Gal}(\mathbb{Q}(\sqrt{-1}, \sqrt{3}, \sqrt{N})/\mathbb{Q})$. Now we use Lemma 7.1 (1). \[\Box\]

The following lemma will help us with $\text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1}$ and follows immediately from Lemma 6.1.

Lemma 7.3.
\[(1) \quad \text{If } N \equiv 1 \text{ (mod 8), then } |\mathcal{D}_N| = |\mathcal{T}_N| = 2 \text{ and } c \notin \mathcal{D}_N.\]
\[(2) \quad \text{If } N \equiv 3, 5 \text{ (mod 8), then } |\mathcal{D}_N| = 4, |\mathcal{T}_N| = 2, \text{ and } c \notin \mathcal{D}_N.\]
\[(3) \quad \text{If } N \equiv 7 \text{ (mod 8), then } |\mathcal{D}_N| = 4, |\mathcal{T}_N| = 2, \text{ and } c \in \mathcal{D}_N \setminus \mathcal{T}_N.\]

Corollary 7.4. \[\text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} = \begin{cases} 5 & \text{if } N \equiv 1 \text{ modulo } 8 \\ 4 & \text{otherwise}. \end{cases}\]

In particular, suppose $N \equiv 3, 5$ modulo 8. Choose any $g_{N}^\pm \in G_{N}^\pm$, any $g_{-N}^\pm \in G_{-N}^\pm$, any $g_{-N}^- \in G_{-N}^-$, and any $g_7^- \in G_7^-$. The maximal ideal of $\mathcal{R}(N)_{1\oplus 1}$ is generated by the images under $\tau^{\text{univ}}$ of these four elements.

Proof. We use Lemma 7.1(2) and translate the conditions on $b$ via Lemma 7.3, taking cases. If $N \equiv 1$ (mod 8) then the level-$N$ conditions on $b$ translate to $b(\mathcal{D}_N \cup \{c\}) = \{0\}$. Hence we conclude that $\text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} = 5$. If $N \equiv 7$ (mod 8), then the conditions give $b(\mathcal{D}_N) = \{0\}$, so we get that $\text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} = 4$. And if $N \equiv 3, 5$ (mod 8), then we get $b(\mathcal{T}_N \cup \{c\}) = \{0\}$ and $b(g) = b(g')$ whenever $\{g, g'\} = \mathcal{D}_N \setminus \mathcal{T}_N$. In this case, we also get $\text{dim } \mathcal{T} \mathcal{R}(N)_{1\oplus 1} = 4$. For the refinement for $N \equiv 3, 5$ mod 8, use Lemma 6.1. \[\Box\]
7.2. Representation-theoretic lemmas. In this subsection, we will study the properties of a pseudodeformation of the trivial mod-2 representation taking values in a domain. These results will be used to control the dimension of the tangent space of \( \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \).

Before stating Proposition 7.6, we recall a result due to Chenevier that we will use in its proof.

**Lemma 7.5.** [Ch, Lemma 3.8] Suppose \( B \) is an coefficient algebra and \( t : G \to B \) is a pseudodeformation of the trivial mod-2 representation. Then \( t \) factors through \( G^{\text{pro-2}} \).

**Proposition 7.6.** Let \( B \) be an integral domain coefficient algebra and \( K \) its field of fractions with algebraic closure \( \overline{K} \). Suppose that \( t : G \to B \) is a pseudodeformation of the trivial mod-2 representation. Then \( t \) is the trace of a semisimple representation \( \rho : G \to \text{SL}_2(\overline{K}) \). Moreover:

1. \( \rho \) factors through \( G^{\text{pro-2}} \).
2. \( \rho|_{I_N} \) is unipotent: that is, \( \rho|_{I_N} \sim \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \) for some additive character \( \eta : I_N \to \overline{K} \).
3. \( \rho|_{D_N} \) has abelian image.
4. Either \( \rho(I_N) \) is trivial or \( \rho|_{D_N} \) is unipotent.
5. \( t \) has level-N shape: \( I_N \subseteq \ker t|_{D_N} \).
6. For all \( g \in G, d \in D_N, i \in I_N \), we have \( t(gi) - t(g) = t(dgi) - t(dg) \).

**Proof.** The existence of \( \rho \) with \( \text{tr} \rho = t \) is a theorem of Chenevier [Ch, Theorem 2.12].

1.Follows from Lemma 7.5, the semisimplicity of \( \rho \), and the Brauer-Nesbitt theorem.
2. By (1), the restriction \( \rho|_{I_N} \) factors through the pro-2 tame-inertia quotient of \( I_N \), which is isomorphic to \( \mathbb{Z}_2 \). Therefore \( \rho|_{I_N} \) is reducible, and hence an extension of a character \( \chi : \mathbb{Z}_2 \to \overline{K}^\times \) by \( \chi^{-1} \). We claim that \( \chi \) has finite order. Indeed, let \( a \) be a lift of a generator of the \( \mathbb{Z}_2 \)-quotient of \( I_N \), and \( \sigma \) a lift of Frob_{N} to \( D_N \). Then \( \sigma a \sigma^{-1} = a^N \) [Stacks, Lemma 0B5U5]. It follows that \( \rho(a) \) and \( \rho(a)^N \) are conjugates of each other, so that if \( \lambda \) is an eigenvalue of \( \rho(a) \), then so is \( \lambda^N \); in other words, \( \lambda^N = \lambda \pm 1 \). Therefore every eigenvalue of \( \rho(a) \) has finite order; since \( \rho(a) \) generates the image of \( \rho|_{I_N} \), the character \( \chi \) does too. Finally, any finite-order character from \( \mathbb{Z}_2 \) to a field of characteristic 2 must be trivial.
3. From (2), we get that \( \rho|_{I_N} \sim \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \) for some additive character \( \eta : I_N \to \overline{K} \) of order 1 or 2, depending on whether \( \rho \) is ramified at \( N \). Since \( N \) is odd, we have \( \rho(a)^N = \rho(a) \) in either case, so that \( \rho(a) \) and \( \rho(\sigma) \) commute.
4. Since \( I_N \) is normal in \( D_N \) with abelian quotient, \( \rho|_{D_N} = \begin{pmatrix} \alpha & \eta' \\ 0 & \alpha^{-1} \end{pmatrix} \) for some unramified character \( \alpha : D_N \to \overline{K}^\times \) and extension class \( \eta' \). Since \( \rho(D_N) \) is abelian, we have, for every \( i \) in \( I_N \) and \( d \) in \( D_N \), the matrix \( \rho(i) = \begin{pmatrix} 1 & \eta(i) \\ 0 & 1 \end{pmatrix} \) commuting with \( \rho(d) = \begin{pmatrix} \alpha(d) & \eta'(d) \\ 0 & \alpha^{-1}(d) \end{pmatrix} \). It follows that \( \eta'(d) + \eta(i) \alpha^{-1}(d) = \eta(i) \alpha(d) + \eta'(d) \). If \( \eta \) is nontrivial, then \( \alpha = \alpha^{-1} \), so \( \alpha = 1 \) as we are in characteristic 2. Otherwise \( \rho(I_N) = 1 \).
5. One checks that \( t(di) = t(d) \) for all \( d \in D_N, i \in I_N \) in both the possible settings from (4).
6. Immediate if \( \rho(I_N) \) is trivial; otherwise use Lemma 5.4.

In summary, the level-N shape condition is automatic for pseudodeformations of \( 1 \oplus 1 \) to a domain. In the next subsection we use this observation to show that the quotient map \( \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \to \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \) induced from the natural surjective map \( \mathcal{R}(N)_{1\oplus 1} \to \mathcal{R}(N)_{1\oplus 1} \) (subsection 4.2) is an isomorphism.

7.3. The tangent space to \( \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \). We now turn our attention to \( \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \), in particular its tangent dimension. For brevity, write \( \mathcal{R} := \mathcal{R}(N)_{1\oplus 1} \) and \( \mathcal{R} := \mathcal{R}(N)_{1\oplus 1}^{\text{red}} \), and \( n \) and \( n^{\text{red}} \), respectively, for their maximal ideals; keep the notation introduced above for \( G = G_{2,2N}, I_N := I_N(G/G^2) \) and \( D_N := D_N(G/G^2) \).
Lemma 7.1

7.3.1

Lemma 7.3.1

D

Proposition 7.6

Proposition 7.7

Proof. Write

\[ \overline{\mathcal{R}} \]

\[ \overline{\mathcal{R}}_{\text{red}} = \overline{\mathcal{R}}_{\text{red}} \]

Proof. The second part follows from the first, so let \( \hat{p} \) be a prime ideal of \( \overline{\mathcal{R}} \). Let \( t : G \to \overline{\mathcal{R}}/\hat{p} \) be the pseudorepresentation obtained by composing \( \hat{\rho}^{\text{univ}} : G \to \overline{\mathcal{R}} \) with the quotient map \( \overline{\mathcal{R}} \to \overline{\mathcal{R}}/\hat{p} \). By Proposition 7.6(5), \( t \) has level-\( N \) shape so that the quotient map \( \overline{\mathcal{R}} \to \overline{\mathcal{R}}/\hat{p} \) factors through \( \mathcal{R} \); let \( \alpha : \mathcal{R} \to \overline{\mathcal{R}}/\hat{p} \) be the corresponding map (that is, satisfying \( t = \alpha \circ \hat{\rho}^{\text{univ}} \)) and let \( p = \ker \alpha \). Then \( \mathcal{R}/p = \overline{\mathcal{R}}/\hat{p} \) since \( \mathcal{R}/\hat{p} \) is the trace algebra of \( t \).

Proposition 7.8. Let \( p \) be a prime ideal of \( \mathcal{R} \). Then

\[ \dim \text{Tan} \mathcal{R}/p \leq \begin{cases} 5 & \text{if } N \equiv 1 \text{ modulo } 8; \\ 2 & \text{if } N \equiv 3, 5 \text{ modulo } 8; \\ 3 & \text{if } N \equiv 7 \text{ modulo } 8. \end{cases} \]

Proof. Write \( K \) for the field of fractions of \( B := \mathcal{R}/p \). From Proposition 7.6, \( t \) is the trace of a unique semisimple representation \( \rho : G \to \text{SL}_2(\overline{\mathcal{K}}) \). If \( \rho \) is unramified at \( N \), then \( \alpha \) factors through the quotient \( \mathcal{R} \to \mathcal{R}(1) \); see (2.9.2) and (4.2.3). By Theorem 6.2,

\[ \dim \text{Tan} B \leq \dim \text{Tan} \mathcal{R}(1) = 2. \]

Otherwise, (7.3.1) above and Proposition 7.6(4),(6) tell us that \( t \) satisfies the following properties:

1. \( t(c) = t(D_N) = 0 \);
2. \( t(g) - t(g_i) = t(dg) - t(dgi) \) for \( g \in G, d, i \in D_N, i \in I_N \).

Let \( b \in T_B \). By following the maps in (7.3.1) we see that \( b \) is a set map from \( G/G^2 \cong (\mathbb{Z}/2\mathbb{Z})^3 \) to \( \mathbb{F}_2 \) subject to the same conditions as \( t \) projected to \( G/G^2 \). We use Lemma 7.3 repeatedly. If \( N \equiv 1 \text{ mod } 8 \), then \( \{c\} \cup \overline{D}_N \) has size 3, so that \( b \) has at most \( 8 - 3 = 5 \) degrees of freedom and \( \dim \text{Tan} B \leq 5 \). Otherwise, we can choose \( i \in I_N \) and \( d \in \overline{D}_N \) so that \( \{1, i, d, di\} \) are the distinct elements of \( \overline{D}_N \). The condition (2) for \( g \notin \overline{D}_N \) implies that the sum of the \( b \)-values on \( G/G^2 - \overline{D}_N \) is zero. Since \( \overline{D}_N \) has size 4 itself, that is a total of 5 independent conditions on \( b \), so that \( \dim \text{Tan} B \leq 3 \). Finally, for \( N \equiv 3, 5 \text{ mod } 8 \), the condition \( b(c) = 0 \) is an additional independent condition, so that \( \dim \text{Tan} B \leq 2 \).

In any case since \( \mathcal{R}(1) \cong \mathbb{F}_2[x, y] \) is a quotient of \( \mathcal{R} \) (Theorem 6.2), the following corollary is immediate.

Corollary 7.9. If \( N \equiv 3, 5 \text{ mod } 8 \), then \( \dim \hat{\mathcal{R}} = \dim \mathcal{R} = 2 \).

We now analyze the tangent dimension of \( \mathcal{R}_{\text{red}} \).
Proposition 7.10. \( \dim \text{Tan } \mathcal{R}^{\text{red}} \leq \begin{cases} 5 & \text{if } N \equiv 1 \mod 8, \\ 3 & \text{otherwise.} \end{cases} \) In particular, suppose \( N \equiv 3,5 \mod 8 \). Choose
\[
g^+_N \in G^+_N, \quad g^-_N \in G^-_N, \quad g^-_{-N} \in G^-_{-N}, \quad \text{and } g^-_7 \in G^-_7.
\]
Then the maximal ideal of \( \mathcal{R}^{\text{red}} \) is generated by \( \tau^{\text{red}}(g^+_N) \) and any two of \( \tau^{\text{red}}(g^-_N), \tau^{\text{red}}(g^-_{-N}), \) and \( \tau^{\text{red}}(g^-_7) \).

Proof. The proof is similar to that of Proposition 7.8 and repeatedly uses Lemma 7.3. By Proposition 7.6, the following hold modulo every prime ideal of \( \mathcal{R} \), so that they hold in \( \mathcal{R}^{\text{red}} \):
\[
\begin{align*}
(1) & \quad \tau^{\text{red}}(c) = \tau^{\text{red}}(I_N) = 0, \\
(2) & \quad \tau^{\text{red}}(d_i) = \tau^{\text{red}}(d) \text{ for all } d \in D_N \text{ and } i \in I_N, \\
(3) & \quad \tau^{\text{red}}(gi) = \tau^{\text{red}}(dgi) - \tau^{\text{red}}(dg) \text{ for all } g \in G, d \in D_N, \text{ and } i \in I_N.
\end{align*}
\]
For \( N \equiv 1 \mod 8 \), then \( \mathbf{I}_N = \mathbf{D}_N \), and the analysis and the conclusion are analogous to those of Proposition 7.8. Otherwise, \( \mathbf{I}_N \) has size 2 and index 2 in \( \mathbf{D}_N \), which in turn has has index 2 in \( G/G^2 \). For any \( b \in \mathbf{I}_{\mathcal{R}^{\text{red}}} \), the first condition tells us that \( b \) is zero on the 3-element set \( \mathbf{I}_N \cup \{c\} \). The second condition tells us that the \( b \)-values on the two elements of \( \mathbf{D}_N \) that are not in \( \mathbf{I}_N \) coincide. Now the third condition, as in the proof of Proposition 7.8, means that the sum of the \( b \)-values on \( G/G^2 - \mathbf{D}_N \) is zero. Thus we get 5 independent conditions, so that \( \dim \text{Tan } \mathcal{R}^{\text{red}} \leq 3 \). For the refinement in the case \( N \equiv 3,5 \mod 8 \), compare to Corollary 7.4. \( \square \)

8. Hecke tangent dimensions at the trivial \( \tilde{\rho} \) mod 2

Here we study \( A(N)_{1 \oplus 1} := A(N, \mathbb{F}_2)_{1 \oplus 1} \), the local component of trivial mod-2 representation of the big Hecke algebra acting on \( M(N, \mathbb{F}_2) \). Recall that \( N \) is an odd prime.

8.1. The shallow Hecke algebra and its reduced quotient. As a consequence of the surjective map \( \mathcal{R}(N)_{1 \oplus 1} \twoheadrightarrow A(N)_{1 \oplus 1} \) (Corollary 4.3(3)) as well as Corollary 7.4 and Proposition 7.10 we get the following:

Corollary 8.1.

(1) \( \dim \text{Tan } A(N)_{1 \oplus 1} \leq \begin{cases} 5 & \text{if } N \equiv 1 \mod 8, \\ 4 & \text{otherwise.} \end{cases} \)

(2) \( \dim \text{Tan } A(N)_{1 \oplus 1}^{\text{red}} \leq \begin{cases} 5 & \text{if } N \equiv 1 \mod 8, \\ 3 & \text{otherwise.} \end{cases} \)

8.2. The partially full Hecke algebra of level \( N \). From the discussion in subsection 6.4 \( A(N)_{1 \oplus 1}^{\text{pf}} \) is a complete local noetherian ring, an \( A(N)_{1 \oplus 1} \)-algebra whose maximal ideal \( \mathfrak{m}^{\text{pf}} := \mathfrak{m}(N)_{1 \oplus 1}^{\text{pf}} \) is generated by \( U'_N := U_N + 1 \) together with \( \mathfrak{m} := \mathfrak{m}(N)_{1 \oplus 1} \). Moreover with \( \tau : G \rightarrow A(N)_{1 \oplus 1} \) the modular pseudodefomation of \( 1 \oplus 1 \), for any \( \text{Frob}_N \) element in \( D_N \), the element \( F_N := \tau(\text{Frob}_N) \in A(N)_{1 \oplus 1} \) is well defined independent of the choice and satisfies \( F_N \in (\mathfrak{m}^{\text{pf}})^2 \). Finally, (6.4.4) implies that for any \( i \in I_N \) and any complex conjugation \( c \), so that
\[
U'_N \tau(c) = \tau(c) + \tau(c \text{ Frob}_N) + \tau(c \text{ Frob}_N),
\]
so that \( \tau(c) + \tau(c \text{ Frob}_N) = \tau(c \text{ Frob}_N) = \text{Or, more generally, by } U'_N, \tau(1), \tau(g^7), \text{ and } \tau(g^7_{-N}) \text{ for any } g^7 \in G^7 \text{ and any } g^7_{-N} \in G^7_{-N}. \)

Lemma 8.2. If \( N \equiv 3,5 \mod 8 \), then \( \dim \text{Tan } A(N)_{1 \oplus 1}^{\text{pf}} \leq 3 \), with the maximal ideal generated by \( U'_N, \tau(c), \text{ and } \tau(c \text{ Frob}_N) \) — or, more generally, by \( U'_N, \tau(g^7), \) and \( \tau(g^7_{-N}) \) for any \( g^7 \in G^7 \) and any \( g^7_{-N} \in G^7_{-N}. \)

Proof. Let \( i \in I_N \) be a lift of a generator of its \( \mathbb{Z}_2 \) tame-inertia quotient. It follows from the proof of Corollary 7.4 that the images of \( \tau(c), \tau(c \text{ Frob}_N), \tau(c \text{ Frob}_N) \) and \( \tau(c \text{ Frob}_N) \) are a \( \mathbb{F}_2 \)-basis for the cotangent space \( \mathfrak{m}/\mathfrak{m}^2 \). Therefore, so are
\[
\tau(c), \tau(c \text{ Frob}_N), \tau(\text{Frob}_N), \text{ and } \tau(c) + \tau(c \text{ Frob}_N) + \tau(c \text{ Frob}_N). \]
So these four elements generate \( \mathfrak{m} \) as in ideal. Therefore \( \mathfrak{m}^{\text{pf}} \) is generated by these four elements together with \( U'_N \). Since both \( \tau(\text{Frob}_N) \) and \( \tau(c) + \tau(c \text{ Frob}_N) + \tau(c \text{ Frob}_N) \) are in \( (\mathfrak{m}^{\text{pf}})^2 \), the cotangent space of the partially full Hecke algebra is spanned by the images of \( U'_N, \tau(c), \text{ and } \tau(c \text{ Frob}_N). \) The claim follows. \( \square \)
8.3. The Hecke algebra on very new forms. As above, let \( i \in I_N \) be a lift of the generator of its \( \mathbb{Z}_2 \) quotient, \( \text{Frob}_N \in D_N \) any Frobenius-at-\( N \) element, and \( c \in G \) any complex conjugation.

**Lemma 8.3.** The element \( F_N = \tau^{mod}(\text{Frob}_N) \) of \( A(N)_{1\oplus 1} \) maps to zero in \( A(N)^{vnew}_{1\oplus 1} \).

**Proof.** Since \( U'_N = 0 \) in \( A(N)^{vnew}_{1\oplus 1} \) and \( F_N = F_N U'_N - (U'_N)^2 \) \((6.4.3)\), we have \( F_N = 0 \) in \( A(N)^{vnew}_{1\oplus 1} \). \( \square \)

**Lemma 8.4.** If \( N \equiv 3, 5 \pmod{8} \), then \( \dim \text{Tan} A(N)^{vnew}_{1\oplus 1} \leq 2 \). Its maximal ideal is generated by the images of \( \tau(c \text{Frob}_N) \) or \( \tau(c \text{Frob}_N) \). More generally still, it is generated by any \( \tau(g_7) \) and \( \tau(\xi N) \).

**Proof.** Let \( m^{vnew} \) be the maximal ideal of \( A(N)^{vnew}_{1\oplus 1} \). By construction \( U'_N \) kills \( M(N)^{vnew}_{1\oplus 1} \). Therefore there is a surjective map \( A(N)^{pf}_{1\oplus 1} \rightarrow A(N)^{vnew}_{1\oplus 1} \) sending \( U'_N \) to zero and restricting the action of the other Hecke operators. The proof of Lemma 8.2 (and using the same notation) tells us that the images of \( \tau^{mod}(ci) \) and \( \tau^{mod}(c \text{Frob}_N) \) span \( m^{pf}/(m^{pf})^2, U'_N \) as an \( \mathbb{F}_2 \)-vector space. That space surjects onto \( m^{vnew}/(m^{vnew})^2 \), proving the lemma. \( \square \)

9. Mod-2 modular forms of level 3 and 5

In this section, we restrict ourselves to \( N = 3, 5 \) and study the properties of \( A(N) := A(N, \mathbb{F}_2) \) in these cases. We determine the structure of \( M(N) := M(N, \mathbb{F}_2) \) (subsection 9.1), prove that \( A(N) \) is a local ring (subsection 9.2), and compute the exact dimensions of the tangent space of \( A(N) \).

9.1. \( M(3) \) and \( M(5) \) are polynomial algebras. In level 1, Swinnerton-Dyer showed that \( M(1) = \mathbb{F}_2[\overline{\Delta}] \), where \( \overline{\Delta} = q + q^9 + q^{25} + \cdots \) is the mod-2 \( q \)-expansion of \( \Delta \), the unique normalized cuspform of level 1 and weight 12 [SD, Theorem 3]. The fact that \( M(1) \) is a polynomial algebra is a key ingredient for all the known proofs of the structure of \( A(1) \) [NS2, MeN, GG, MoV]. In levels 3 and 5, we use an analogous structure result to allow us to apply the nilpotence method [MeN] to prove that \( \dim A(N)^{vnew} \geq 2 \) (Theorem 10.1). The structure of \( M(3) \) and \( M(5) \) is certainly known to experts and we make no claim of originality.

**Lemma 9.1.** If \( N = 3, 5 \), then \( M(N) = \mathbb{F}_2[f_N] \) for some \( f_N \in M(N) \). More precisely,
\[
\begin{align*}
\tau_3 &= E_{4}^{3\text{-crit}} = q + q^2 + q^3 + q^4 + q^6 + q^9 + q^{12} + q^{16} + q^{18} + O(q^{20}) \in M_3(\mathbb{F}_2), \\
\tau_5 &= E_{4}^{5\text{-crit}} = q + q^2 + q^4 + q^5 + q^6 + q^{10} + q^{16} + q^{18} + O(q^{20}) \in M_5(\mathbb{F}_2).
\end{align*}
\]

**Remark 9.2.** For \( p \geq 5 \), the subspace of \( M(N, \mathbb{F}_p) \) coming from weights divisible by \( p - 1 \) is the coordinate algebra of \( X_0(N)_{\overline{\mathbb{F}_p}} \) with the supersingular points removed, with the weight filtration corresponding to the maximal order at the poles [SerC, Corollaire 2]; for \( p = 2, 3 \) something similar is true with the filtration adjusted. For example, \( M(7, \mathbb{F}_2) = \mathbb{F}_2[f_7, f_7^{-1}] \) for \( f_7 = \eta(q^7)\eta(q)^{-1} \) a Hauptmodul on \( X_0(7) \).

**Proof of Lemma 9.1.** In all cases, we determine \( M(N, \mathbb{Z}) \) and reduce modulo 2.

[CWase \( N = 3 \):] We claim that \( M(3, \mathbb{Z}) \) is generated by \( e_2 = E_{2,3} \), the unique Eisenstein series in weight 2; \( g_4 := E_{4}^{3\text{-crit}} \), the normalized semiscandial Eisenstein eigenform in weight 4; and \( h_6 \), which captures a congruence in weight 6; with a single relation: \( g_4^2 = e_2 h_6 \). More precisely, let
\[
\begin{align*}
e_2 := E_{2,3} = E_2 - 3 E_2(q^3) = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + O(q^6) \in M_2(3, \mathbb{Z}), \\
g_4 := E_{4}^{3\text{-crit}} = E_4 - E_4(q^3) = q + 9q^2 + 27q^3 + 73q^4 + 126q^5 + O(q^6) \in M_4(3, \mathbb{Z}),
\end{align*}
\]
and let
\[
c_6 := q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6) \in S_6(3, \mathbb{Z})
\]
be the unique normalized cuspform. Then \( e_2 g_4 \) and \( c_6 \) are congruent modulo 27, so set
\[
h_6 := \frac{e_2 g_4 - c_6}{27} = q^2 + 6q^3 + 27q^4 + 80q^5 + O(q^6) \in M_6(3, \mathbb{Z}).
\]
We claim that monomials in \( e_2, g_4, \) and \( h_6 \) give a Victor-Miller basis in each even weight \( k \geq 0 \), by which here we mean simply a set of forms \( \{ f_{k,0}, \ldots, f_{k,d_k k-1} \} \), where \( d_k = \dim M_k(3, \mathbb{C}) = 1 + \left\lfloor \frac{k}{6} \right\rfloor \), satisfying \( f_{k,i} = q^i + O(q^{i+1}) \) for each \( i, k \). For \( 0 \leq i \leq \left\lfloor \frac{k}{6} \right\rfloor \) we define the even Victor-Miller elements: let 
\[
 f_{k,2i} := e_2^{k/2-3i} h_6^i.
\]
For \( 0 \leq i \leq \left\lfloor \frac{k}{6} \right\rfloor \) we define the odd Victor-Miller elements: let 
\[
 f_{k,2i+1} := e_2^{k/2-3i} g_4^i h_6^i.
\]
To see that these \( \{ f_{k,i} \} \) define a basis of \( M_k(3, \mathbb{C}) \), observe that for even \( k \), we have 
\[
 \left\lfloor \frac{k}{6} \right\rfloor + 1 + \left\lfloor \frac{k-1}{6} \right\rfloor + 1 = 1 + \left\lfloor \frac{k}{3} \right\rfloor;
\]
the Victor-Miller shape of their \( q \)-expansions guarantees that they are also a basis for \( M_k(3, \mathbb{Z}) \).

The dimension formulas guarantee that the relation \( g_4^2 = e_2 h_6 \) is the only one. In other words,
\[
 M(3, \mathbb{Z}) = \mathbb{Z}[e_2, g_4, h_6] / (g_4^2 - e_2 h_6)
\]
as a graded algebra. Finally, modulo 2 we have \( e_2 \equiv 1 \), so that our relation becomes \( h_6 = g_4^2 \), so that we can conclude that \( M(3) = M(3, \mathbb{F}_2) \) is a polynomial algebra in \( g_4 \) over \( \mathbb{F}_2 \).

Case \( N = 5 \): Very similar to \( N = 3 \), except that forms of weight 2 and 4 already generate. Again, let 
\[
 e_2 := E_{2,5} = 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 8q^5 + O(q^6) \quad \text{and} \quad g_4 := E_4^{\text{crit}} = q + 9q^2 + 28q^3 + 73q^4 + 125q^5 + O(q^6).
\]
Let \( c_4 = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 + O(q^6) \) be the unique normalized cuspform of weight 4. Since \( c_4 \) and \( g_4 \) have a congruence modulo 13, let 
\[
 d_4 := \frac{g_4 - c_4}{13} = q^2 + 2q^3 + 5q^4 + 10q^5 + O(q^6) \quad \in M_4(5, \mathbb{Z}).
\]
Using the standard dimension result \( \dim M_k(5, \mathbb{C}) = 1 + 2 \left\lfloor \frac{k}{6} \right\rfloor \), combined with a Victor-Miller–basis construction similar to above, one can prove that, as a graded algebra,
\[
 M(5, \mathbb{Z}) = \mathbb{Z}[e_2, g_4, d_4] / (g_4^2 - e_2 d_4 - 4g_4 d_4 + 8d_4^2).
\]
Over \( \mathbb{F}_2 \) we have \( e_2 = 1 \), so that the relation reduces to \( g_4^2 = d_4 \), so that \( M(5, \mathbb{F}_2) = \mathbb{F}_2[g_4] \).

9.2. \( A(3) \) and \( A(5) \) are local rings. By the discussion in subsection 2.7, to show that \( A(3) \) and \( A(5) \) are local \( \mathbb{F}_2 \)-algebras, it suffices to prove the following lemma. Recall that \( G = \Gamma_0(2N) \).

Lemma 9.3. For \( N = 3, 5 \), if \( \bar{\rho} : G \to \text{GL}_2(\mathbb{F}_2) \) is a \( \Gamma_0(N) \)-modular representation, then \( \bar{\rho} = 1 \oplus 1 \).

Proof. It is clear that the only reducible semisimple such \( \bar{\rho} \) is \( 1 \oplus 1 \), so it remains to prove that there are no such irreducible \( \bar{\rho} \). Since \( M(N) \) is a polynomial algebra, one may use an elementary method of Serre, as sketched in [BK, footnote p. 398] in level 1. Alternatively, this fact follows from Serre reciprocity (formerly Serre’s conjecture). We give a third argument that is in between these in terms of machinery involved.

Suppose \( \bar{\rho} : G \to \text{GL}_2(\mathbb{F}_2) \Gamma_0(N) \)-modular. We first show that the image of \( \bar{\rho} \) is a finite 2-group. By a theorem of Tate, an irreducible \( \bar{\rho} \) must be ramified at \( N \) [Tat]. Since the level is prime, \( \bar{\rho}|_{I_N} \simeq (\frac{1}{0} \frac{1}{1}) \), where \( * \) is a nontrivial extension (Theorem 4.2). Thus \( * \) is an additive character \( I_N \to \mathbb{F}_2 \), which must factor through the tame inertia quotient of \( I_N \), an abelian group isomorphic to \( \prod_{p \mid 2, \text{prime}} \mathbb{Z}_p \). The only nontrivial such \( * \) factors through \( \mathbb{Z}_2 / 2\mathbb{Z}_2 = \mathbb{F}_2 \), so that \( |\bar{\rho}(I_N)| = 2 \). Following the proof of part (4) of Proposition 7.6, it follows that \( \bar{\rho}|_{D_N} \simeq (\frac{1}{0} \frac{1}{1}) \) which implies that \( \bar{\rho}(D_N) \) is an elementary abelian 2-group. Therefore, it follows that \( \bar{\rho}|_{G_{2N}} \) is unramified outside \( \{ 2, \infty \} \). From [Sc, Corollary to Theorem B] (see also [MT]), it follows that \( \bar{\rho}|_{G_{2N}} \) is reducible. From class field theory it follows that the maximal abelian extension of \( \mathbb{Q}(\sqrt{-N}) \) unramified outside \( \{ 2, \infty \} \) is a pro-2 extension of \( \mathbb{Q}(\sqrt{-N}) \). Hence, the image \( \bar{\rho}(G) \) is a 2-group, finite since \( \bar{\rho} \) is continuous, as claimed. But any 2-subgroup of \( \text{GL}_2(\mathbb{F}_2) \) is contained in a unipotent Borel. Therefore \( \bar{\rho} \) is reducible, hence trivial.

Corollary 9.4. \( A(3) \) and \( A(5) \) are pro-2 noetherian local rings with residue field \( \mathbb{F}_2 \).

9.3. Tangent spaces to \( A(3, \mathbb{F}_2) \) and \( A(5, \mathbb{F}_2) \). For \( N = 3, 5 \), we can strengthen and refine Corollary 8.1. Let \( m(N) \) be the maximal ideal of \( A(N) \), and \( m(N)_{\text{red}} \) be the maximal ideal of \( A(N)_{\text{red}} \). We state the result for \( N = 3 \) and \( N = 5 \) separately, and then combine these to get a general result.

Proposition 9.5.

1. \( \dim \text{Tan}(A(3)) = 4 \), and the maximal ideal is generated by \( T_{11}, T_5, T_{13}, \) and \( T_7 \).
2. The maximal ideal of \( A(3)_{\text{red}} \) is generated by \( T_{11} \) and any two of \( T_5, T_7, T_{13} \).
3. In (1) and (2), \( T_{q'} \) may replace \( T_q \) if \( q' \) is a prime congruent to \( q \) modulo 24.
Proposition 9.6.
(1) dim \text{Tan} A(5) = 4, and the maximal ideal is generated by \(T_{13}, T_3, T_{11} \) and \(T_7\).
(2) The maximal ideal of \(A(5)_{\text{red}}\) is generated by \(T_{13}\) and any two of \(T_3, T_7, T_{11}\).
(3) In (1), (2), \(T_q'\) may replace \(T_q\) if \(\text{Frob}_q = \text{Frob}_q'\) in \(\text{Gal}(\mathbb{Q}(i, \sqrt{2}, \sqrt{5})/\mathbb{Q})\).

Corollary 9.7. For \(N = 3, 5\), we have
(1) \(\dim \text{Tan} A(N) = 4\), with the maximal ideal generated by \(\tau(g_N^+), \tau(g_N^-), \tau(g_5^+),\) and \(\tau(g_5^-)\).
(2) \(m(N)_{\text{red}}\) is generated by \(\tau_{\text{red}}(g_N^+), \tau_{\text{red}}(g_N^-), \tau_{\text{red}}(g_5^+),\) and \(\tau_{\text{red}}(g_5^-)\).

Corollary 9.7 follows from Corollary 7.4 and Propositions 7.10, 9.5 and 9.6.

Proof of Proposition 9.5. By Corollary 9.4 we know that \(A(3) = A(3)_{\text{red}}\) is already local. Write \(m, m_{\text{red}}\) for \(m(3), m(3)_{\text{red}}\), respectively. The method of Proposition 7.8 and Proposition 7.10 will yield explicit spanning sets for \(m/m^2\) and \(m_{\text{red}}/(m_{\text{red}})^2\); we then establish their linear independence in \(m/m^2\) by exhibiting their action on forms explicitly.

Recall that \(G_{Q,2,3}/G_{Q,2,3}^2 = \text{Gal}(Q(\sqrt{-1}, \sqrt{2}, \sqrt{3})/Q) = \text{Gal}(Q(\mu_{24})/Q)\), so every element is determined by its action on the triple \(v_3 = (\sqrt{-1}, \sqrt{2}, \sqrt{3})\) and may be represented by \(\text{Frob}_q\) for \(q\) in \(\{73, 5, 7, 11, 13, 17, 19, 23\}\) (or other prime representatives of their congruence classes modulo 24).

Let \(i \in I_3\) be nontrivial, and let \(d \in \overline{I}_3 - I_3\) be the element fixing \(\sqrt{3}\). Then the correspondence is as follows.

| elt. | lift | action on \(v_3\) | sample | local at 3 | elt. | lift | action on \(v_3\) | sample | local at 3 |
|------|------|-----------------|--------|----------|------|------|-----------------|--------|----------|
| 1    | \(g_1^3\) | (0, 0, 0) | \(73\) | trivial | \(c\) | \(g_1^3\) | (1, 0, 0) | \(23\) | - |
| \(i\) | \(g_1^3\) | (0, 0, 1) | 17 | in \(I_3\) | \(ci\) | \(g_1^3\) | (1, 0, 1) | 7 | - |
| \(d\) | \(g_3^7\) | (1, 1, 0) | 11 | in \(\overline{I}_3\) | \(cd\) | \(g_3^7\) | (0, 1, 0) | 13 | - |
| \(id\) | \(g_3^7\) | (1, 1, 1) | 19 | in \(\overline{I}_3\) | \(cid\) | \(g_3^7\) | (0, 1, 1) | 5 | - |

Comparing this chart to Corollary 7.4 tells us that \(m(3)\) is generated by, for example, \(T_{11}, T_5, T_{13},\) and \(T_7\). For the generators of \(m(3)_{\text{red}}\), similarly use Proposition 7.10. It remains to see that \(T_{11}, T_5, T_{13},\) and \(T_7\) are linearly independent in \(m/m^2\). Let \(\Delta(q) = \sum_{n \geq 0} q^n\) be the mod-2 \(q\)-expansion of the Ramanujan \(\Delta\)-function, so that \(\Delta\) and \(\Delta' := \Delta(q^2)\) are both forms in \(M(3)\). It is straightforward to verify the following table, for example using SageMath [Sage].

| \(f\) | \(T_3(f)\) | \(T_7(f)\) | \(T_{11}(f)\) | \(T_{13}(f)\) |
|------|------------|------------|---------------|---------------|
| \(\Delta\) | 0 | 0 | 0 | 0 |
| \(\Delta^3\) | 0 | 0 | \(\Delta\) | 0 |
| \(\Delta^5\) | \(\Delta\) | 0 | 0 | \(\Delta\) |
| \(\Delta'\) | 0 | 0 | 0 | 0 |
| \(\Delta^2\Delta'\) | \(\Delta\) | \(\Delta'\) | 0 | 0 |

From the table, it is clear that \(\Delta^3, \Delta^5\) and \(\Delta^2\Delta'\) are three forms annihilated by \(m^2\) but not by \(m\). Now suppose \(T = aT_3 + bT_7 + cT_{11} + dT_{13}\) is in \(m^2\) for some \(a, b, c, d \in \mathbb{F}_2\). Then \(T(\Delta^3) = c\Delta\) and \(T(\Delta^5) = (a + d)\Delta\) and \(T(\Delta^2\Delta') = a\Delta + b\Delta'\) are all three equal to zero, so \(a = b = c = d = 0\). \(\square\)

Proof of Proposition 9.6. The proof for \(N = 5\) is analogous; we highlight a few details. The Frattini field \(K = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{5})\) has index 2 in \(\mathbb{Q}(\mu_{40})\). So \(\text{Frob}_q\) in \(\text{Gal}(K/\mathbb{Q})\) depends on congruences modulo 40. Table of elements in \(\text{Gal}(K/\mathbb{Q})\) with local-at-5 subgroups and action on \(v_5 = (\sqrt{-1}, \sqrt{2}, \sqrt{5})\):
The forms $\Delta^3$, $\Delta^5$, and $\Delta^2 \Delta(q^5)$ again serve as witnesses for the linear independence of $\{T_3, T_7, T_{11}, T_{13}\}$ in $m(5)/m(5)^2$. The details are left to the reader. \hfill \Box

10. Structure of $A(N, F_2)^{\text{new}}$ for $N = 3, 5$

We now focus on $A(N)^{\text{new}}$, the Hecke algebra acting faithfully on the very new forms $M(N)^{\text{new}} = \ker(U_N+1)$. In subsection 10.1 we use the nilpotence method (Theorem 2.1) to show that $\dim A(N)^{\text{new}} \geq 2$. In subsection 10.2 we deduce that $A(N)^{\text{new}}$ is a complete regular local $F_2$-algebra of dimension 2 (Corollary 10.2) and deduce that $\dim \Tan A(N)^{\text{red}} = 3$ (Corollary 10.4).

Recall that $U'_N = U_N + 1$. Let $m^{\text{new}}$, $m^{\text{pf}}$ be the maximal ideals of $A(N)^{\text{new}}$, $A(N)^{\text{pf}}$, respectively.

10.1. Lower bound on $\dim A(N)^{\text{new}}$ by the nilpotence method.

**Theorem 10.1.** For $N = 3, 5$, the Krull dimension of $A(N)^{\text{new}}$ is at least 2.

**Proof.** Recall from (6.5.2) and (6.5.3) the presentations of $A(N)^{\text{new}}$ and $A(N)^{\text{new}}$ as quotients of $A(N)^{\text{pf}}$: to wit, $A(N)^{\text{new}} = A(N)^{\text{pf}}/(U'_N)^2$ and $A(N)^{\text{new}} = A(N)^{\text{pf}}/(U'_N)$. It is thus clear that $U'_N$ is nilpotent in $A(N)^{\text{new}}$, so that $\dim A(N)^{\text{new}} = \dim A(N)^{\text{new}}$. It therefore suffices to show that $\dim A(N)^{\text{new}} \geq 2$.

We prove this with the nilpotence method, by showing that the four conditions of Theorem 2.1 are satisfied for $A := A(N)^{\text{new}}$.

**Condition (1):** By Lemma 9.1, $M(N) = F_2[f]$ for a form $f = f_N$.

**Condition (2):** Set $K := K(N)^{\text{new}}$ as in (6.5.1), by (2.8.3) in duality with $A = A(N)^{\text{pf}}/(U'_N)^2$.

**Condition (3):** The Hecke operators $U'_N$ as well as every $T_i$ for $\ell \nmid 2N$ act locally nilpotently on $M(N)$ and are therefore in $m^{\text{pf}}$. By [MeD, Proposition 6.2] for general level, generalizing [NS1, Théorème 3.1], given any prime $\ell \nmid 2N$, the sequence $\{T_i(f^n)\}$ satisfies a linear recurrence over $M(N)$ whose companion polynomial $P_{t,f} = X^{t+1} + a_1 X^t + \cdots + a_{t+1}$ has coefficients $a_j \in M(N)$ with $\deg a_j \leq j$ and $\deg a_{t+1} = \ell + 1$. Equivalently, both the $f$-degree and the total degree (as a polynomial in $X$ and $f$) of $P_{t,f}$ coincide with its $X$-degree. By a similar argument, $\{U_N(f^n)\}$ satisfies an $M(N)$-linear recurrence whose polynomial $P_{N,f}$ also has its $f$-degree and total degree coincide with its $X$-degree $N$. Therefore $\{U'_N(f^n)\} = \{U_N(f^n) + f^n\}$ will satisfy an $M(N)$-linear recurrence with characteristic polynomial $P'_{N,f} = (X - f)P_{N,f}$; the degree constraints of $P'_{N,f}$ follow from those of $P_{N,f}$.

For completeness, we include below both $P_{N,f}$ and $P_{t,f}$ for $T_i$ that generate $m^{\text{pf}}$ (Lemma 8.2) and hence its quotient $m^{\text{new}}$. For $N = 3$ we may take $\ell = 13$ and $\ell = 7$; for $N = 5$ we take $\ell = 11$ and $\ell = 7$; see the element tables in the proofs of Propositions 9.5 and 9.6.
We seek a linearly independent sequence of forms \( \{g_n\} \) in \( K(N)_{\text{new}} \) with the property that \( \deg_f g_n \) grows no faster than linearly in \( n \). By (2.4.2) we may replace \( \deg_f (g_n) \) in this estimate with its \( \Gamma_1(N) \)-weight filtration \( w_1(g_n) \).

Write \( f = f_N \) and let \( u = w_1(f_N) \); by §2.4 we know that \( u = 3 \) \((N = 3)\) or \( u = 2 \) \((N = 5)\).

For \( n \) odd, consider the \( \mathbb{F}_2 \)-vector space \( W_n \) spanned by the forms \( \theta(f), \theta(f^3), \ldots, \theta(f^n) \) inside \( M(N) \subset \mathbb{F}_2[q] \). On one hand, by (2.6.2) we have \( W_n \subseteq K(N) \). On the other hand, by (2.4.2) for each \( i \), we have

\[
\dim \theta(f^i) \leq w_1(f^i) + 3 = ui + 3.
\]

Therefore (2.4.3) implies that \( W_n \subseteq K(N)_{u+6} \). Moreover, we claim that

\[
\dim W_n = \frac{n-1}{2}.
\]

Indeed, \( f = q + O(q^2) \), so that for odd \( i \) we have \( \theta(f^i) = q^i + O(q^{i+1}) \). Thus \( \theta(f), \theta(f^3), \ldots, \theta(f^n) \) are linearly independent, and there are \( \frac{n-1}{2} \) of them.

Now consider the image of \( W_n \) under the operator \((U'_N)^2\). By Lemma 2.2, we have \((U'_N)^2 W_n \subseteq M(N)_{\text{old}} \). More precisely, since \( \theta \) commutes with \( U'_N \), we have

\[
(U'_N)^2 W_n \subseteq \theta(M_{u+6}(N)_{\text{old}}).
\]

where we write

\[
M_k(N)_{\text{old}} := M_k(1) + W_N M_k(1).
\]

We now analyze \( \dim \theta(M_k(N)_{\text{old}}) \) for any even \( k \geq 0 \). Standard dimension formulas (for example, [DS, Theorem 3.5.1]) tell us that \( \dim M_k(1) = \frac{k}{12} + O(1) \). Since \( W_N \) is an involution and the sum in (10.1.3) is direct on cuspforms [DM, Proposition 5.4], we conclude that for any even \( k \geq 0 \), we have

\[
\dim M_k(N)_{\text{old}} = \frac{k}{6} + O(1).
\]

On the other hand, by (2.6.3), (2.4.2), and (2.4.3), the kernel of \( \theta \) on \( M_k(N)_{\text{old}} \) certainly contains the squares of forms in \( M(\theta)_{\text{old}} \). Thus by (10.1.4) we obtain

\[
\dim \ker (\theta | M_k(N)_{\text{old}}) \geq \frac{k}{12} + O(1),
\]

so that combining (10.1.4) and (10.1.5) gives

\[
\dim \theta(M_k(N)_{\text{old}}) \leq \frac{k}{12} + O(1).
\]

Finally, we return to (10.1.2). By (10.1.1), the operator \((U'_N)^2\) maps a space of dimension \( \frac{n-1}{2} \) to a space whose dimension, by (10.1.6), grows no faster than \( \frac{n-1}{12} + O(1) \). Therefore,

\[
\dim \ker ((U'_N)^2 | W_n) \geq \frac{(6-u)n}{12} + O(1).
\]
Since \( u \leq 3 \), we can certainly choose \( g_n \in \ker((U'_N)^2|W_n) \subset K(N)_{\text{new}} \), at least for \( n \gg 0 \), so that the sequence \( \{g_n\} \) is linearly independent. Moreover, \( u_1(g_n) \leq u_n \), as required.

Finally, Theorem 2.1 allows us to conclude that \( \dim A(N)^{\text{new}} = \dim A(N)^{\text{new}} \geq 2 \), as desired.

\[ \square \]

10.2. Main result and corollaries.

**Corollary 10.2.** For \( N = 3, 5 \), \( A(N)^{\text{new}} \) is a complete regular local \( \mathbb{F}_2 \)-algebra of dimension 2.

The proof is immediate from Theorem 10.1 and Lemma 8.4. To give a more precise statement, we endow \( \mathbb{F}_2[y, z] \) with a grading by \( \langle \mathbb{Z}/8\mathbb{Z} \rangle / \langle \psi \rangle \simeq \{ \pm 1 \} \) by giving \( y \) and \( z \) both the grading \(-1\). Also write \( \tau^{\text{new}} \) for the pseudorepresentation \( G_{\mathbb{Q}, 2N} \rightarrow A(N)_{1_{\mathbb{Q}, 21}}^{\text{new}} \) coming from \( \tau^{\text{mod}} \).

**Corollary 10.3.** For \( N = 3, 5 \), the map \( y \mapsto \tau^{\text{new}}(g_\psi^\pm) \) and \( z \mapsto \tau^{\text{new}}(g_\psi^-) \) gives a graded isomorphism \( \mathbb{F}_2[y, z] \cong A(N)^{\text{new}} \) of \( \langle \mathbb{Z}/8\mathbb{Z} \rangle / \langle \psi \rangle \)-graded \( \mathbb{F}_2 \)-algebras. In particular:

- The map \( \mathbb{F}_2[y, z] \rightarrow A(3)^{\text{new}} \) given by \( y \mapsto T_{13} \) and \( z \mapsto T_7 \) is an isomorphism of \( \text{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \)-graded algebras.
- The map \( \mathbb{F}_2[y, z] \rightarrow A(5)^{\text{new}} \) given by \( y \mapsto T_{11} \) and \( z \mapsto T_7 \) is an isomorphism of \( \text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \)-graded algebras.

**Proof.** Corollary 10.2 and Lemma 8.4. For explicit generators, see Propositions 9.5 and 9.6.

**Corollary 10.4.** For \( N = 3, 5 \), we have \( \dim \text{Tan} A(\mathrm{N})^{\text{red}} = 3 \).

**Proof.** On one hand, by Theorem 6.2 the level-1 Hecke algebra \( A(1) \) is a dimension-2 domain, so that the quotient map \( A(N) \rightarrow A(1) \) factors through \( A(N)^{\text{red}} \). Corollary 10.2 and the same logic imply that the map \( A(N) \rightarrow A(N)^{\text{new}} \) factors through \( A(N)^{\text{red}} \) as well. On the other hand, Propositions 9.5 and 9.6 tell us that \( \dim \text{Tan} A(\mathrm{N})^{\text{red}} \) is at most 3. Therefore it suffices to prove, for example, that the kernel \( J_1 \) of the induced quotient map \( A(\mathrm{N})^{\text{red}} \rightarrow A(1) \) is nonzero. We show this by proving that \( J_1 + J_{\text{new}} \neq J_{\text{new}} \), where \( J_{\text{new}} \) is the kernel of the induced map \( A(\mathrm{N})^{\text{new}} \rightarrow A(\mathrm{N})^{\text{new}} \).

Since \( A(\mathrm{N})^{\text{new}} = A(\mathrm{N})^{\text{red}}/(U_N') \), the relation in (6.4.3) implies that \( J_{\text{new}} \), when projected to \( A(1) \), contains \( T_N = \tau^\text{mod}(\text{Frob}_N) \), one of the two generators of the maximal ideal of \( A(1) \). Therefore \( A(\mathrm{N})^{\text{red}}/(J_{\text{new}} + J_1) \) is a quotient of \( A(1)/(T_N) \), so has Krull dimension at most 1. Since \( A(\mathrm{N})^{\text{red}}/J_{\text{new}} = A(\mathrm{N})^{\text{new}} \) has Krull dimension 2, we must have \( J_{\text{new}} + J_1 \neq J_{\text{new}} \), so that \( J_1 \neq 0 \), as desired.

\[ \square \]

11. Structure of \( \mathcal{R}(\mathrm{N}, \mathbb{F}_2)^{\text{red}} \) and \( A(\mathrm{N}, \mathbb{F}_2)^{\text{red}} \) for \( N = 3, 5 \)

We are now ready to state and prove one of the main results of this article where for \( N = 3, 5 \) we describe the structure of \( A(\mathrm{N})^{\text{red}} := A(\mathrm{N}, \mathbb{F}_2)^{\text{red}} \) precisely, by proving that it is isomorphic to \( \mathcal{R}(\mathrm{N})^{\text{red}} := \mathcal{R}(\mathrm{N}, \mathbb{F}_2)^{\text{red}} \). In other words, we prove Theorem A.

**Proposition 11.1.** For \( N = 3, 5 \), \( \mathcal{R}(\mathrm{N}) \) has two minimal prime ideals, and the quotient by each of them factors through \( \varphi : \mathcal{R}(\mathrm{N}) \rightarrow A(\mathrm{N}) \).

**Proof.** By Corollary 7.9, the Krull dimension of \( \mathcal{R}(\mathrm{N}) \) is 2. On one hand, its quotient \( \mathcal{R}(1) \) is isomorphic to \( A(1) \), which is a domain of Krull dimension 2 (Theorem 6.2). Therefore the kernel \( p_{\text{old}} := J_{N, 1} \) of the map \( \psi_{N, 1} : \mathcal{R}(\mathrm{N}) \rightarrow \mathcal{R}(1) \) from (4.2.3) is a maximal prime ideal of \( \mathcal{R}(\mathrm{N}) \). Moreover, \( \psi_{N, 1} \) factors through \( A(\mathrm{N}) \), since again its target is isomorphic to \( A(1) \). On the other hand, \( A(\mathrm{N})^{\text{new}} \) is also a quotient, via \( A(\mathrm{N}) \), of \( \mathcal{R}(\mathrm{N}) \); and for \( N = 3, 5 \) we know that \( A(\mathrm{N})^{\text{new}} \) is also a domain of Krull dimension 2 (Corollary 10.2). Therefore the kernel \( p_{\text{new}} \) of the corresponding surjection \( \mathcal{R}(\mathrm{N}) \rightarrow A(\mathrm{N})^{\text{new}} \) is also a
Lemma 8.3
Proposition 11.1
any prime ideal
all the minimal primes of

By The surjections
contains
not contain all contractions of primes of

A quotient
F
is a quotient of a complete regular local F₂-algebra of Krull dimension 2, which has A(N)vnew ≃ F₂[y, z] as a quotient. In other words, R(N)/J ≃ A(N)vnew and J = pnew. Finally, by Proposition 7.8, any prime ideal p of R(N) that does not contain pold contains the set \( \tau^{univ}(D_N) \) and hence, in particular \( \tau^{univ}(Frob_N) \). Therefore, any such p contains J = pnew, as claimed.

Corollary 11.2. The surjections \( \tilde{R}(N) \rightarrow R(N) \overset{\sim}{\rightarrow} A(N) \) induce isomorphisms
\[ \tilde{R}(N)^{\text{red}} \simeq R(N)^{\text{red}} \simeq A(N)^{\text{red}}. \]

Proof. By Proposition 11.1, all the minimal primes of R(N) contain ker \( \varphi \), so that they also correspond to the minimal primes of A(N). Hence, that ker(\( \varphi \)) is nilpotent. By Proposition 7.7, the primes of \( \tilde{R}(N) \) are all contractions of primes of R(N).

Theorem 11.3. For N = 3, 5, the reduced Hecke algebra \( A(N)^{\text{red}} \) is isomorphic to \( \frac{F_2[a, b, c]}{(ab)} \), with
\[ a \in \tau^{\text{univ}}(g_3^\pm) + (m(N)^{\text{red}})^2, \quad b \in \tau^{\text{univ}}(g_7^\pm) + (m(N)^{\text{red}})^2, \quad \text{and} \quad c = \tau^{\text{univ}}(g_7^\pm). \]

(1) For N = 3, let \( f \in F_2[x, y] \) be the power series satisfying \( f(T_{11}, T_{13}) = T_7 \) in A(1), and let \( g \in F_2[y, z] \) be the power series satisfying \( g(T_{13}, T_7) = T_{11} \) in \( A(N)^{\text{vnew}} \): that is,
\[ f = xy + x^3 y + x^2 y^3 + x^5 y^5 + x^3 y^3 + x^5 y^5 + x^3 y^7 + x y^9 + x y O((x^2, y^2)^5) \]
and \( g = y z + z^2 + y z^3 + y^2 z^2 + y^3 z + y^3 z^3 + y^2 z^4 + O(y^4, (y, z)^8) \). Then the map
\[ \frac{F_2[x, y, z]}{(z - f(x, y))(x - g(y, z))} \rightarrow A(N)^{\text{red}} \quad \text{induced by} \quad x \mapsto T_{11}, y \mapsto T_{13}, z \mapsto T_7 \]
is an isomorphism.

(2) For N = 5, let \( f' \in F_2[x, y] \) be the power series satisfying \( f'(T_{13}, T_{11}) = T_7 \) in A(1), and let \( g' \in F_2[y, z] \) be the power series satisfying \( g'(T_{11}, T_7) = T_{13} \) in \( A(N)^{\text{vnew}} \). Then the map
\[ \frac{F_2[x, y, z]}{(z - f'(x, y))(x - g'(y, z))} \rightarrow A(N)^{\text{red}} \quad \text{induced by} \quad x \mapsto T_{13}, y \mapsto T_{11}, z \mapsto T_7 \]
is an isomorphism.
Proof. Following the notation of the proof of Proposition 11.1, write $p_{\text{old}}^{\text{red}}$ and $p_{\text{vnew}}^{\text{red}}$ for the images of $p_{\text{old}}$ and $p_{\text{vnew}}$ respectively, in $A(N)^{\text{red}}$. These are the two minimal primes of the reduced ring $A(N)^{\text{red}}$, so that in particular $p_{\text{old}}^{\text{red}} \cap p_{\text{vnew}}^{\text{red}} = (0)$. We know from Corollary 10.4 that $\dim \ker A(N)^{\text{red}} = 3$; moreover by putting Corollary 9.7, Theorem 6.2, and Corollary 10.3 together, we know that can choose an ordered triple of generators $(X, Y, Z)$ of $m(N)^{\text{red}}$ so that the first two generate $m(1)$ and the second and third generate $m(N)^{\text{vnew}}$: namely $X = \tau(g_{3, N}^+)$, $Y = \tau(g_{7, N}^+)$, and $Z = \tau(g_{7, N}^-)$. In particular for $N = 3$ we can choose $(T_{11}, T_{13}, T_7)$; for $N = 5$ we choose $(T_{13}, T_{11}, T_7)$. Then we can find a unique power series $f \in F_2[x, y]$ so that $Z = f(X, Y)$ in $A(1)$; more precisely, $f$ is in $m_2^N[z, x, y]$ because $Z$ maps to 0 in the cotangent space of $A(1)$. Similarly, there is a unique power series $g \in m_2^N[y, z]$ with $X = g(Y, Z)$ in $A(N)^{\text{vnew}}$.

Now consider the map $\alpha : F_2[x, y, z] \to A(N)^{\text{red}}$ given by $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$. By construction, $z - f(x, y) \in p_{\text{old}}^{\text{red}}$ and $x - g(y, z) \in p_{\text{vnew}}^{\text{red}}$. Since furthermore,

$$\frac{F_2[x, y, z]}{(z - f(x, y))} \simeq F_2[x, y] = A(1),$$

we have $\alpha^{-1}(p_{\text{old}}^{\text{red}}) = (z - f(x, y))$. Similarly, $\alpha^{-1}(p_{\text{vnew}}^{\text{red}}) = (x - g(y, z))$. By considering degrees, it is clear that the elements $z - f(x, y)$ and $x - g(y, z)$ of the UFD $F_2[x, y, z]$ have no common factors — they are nonassociative irreducibles — so that the intersection and the product of the ideals they generate agree:

$$\ker \alpha = \alpha^{-1}(p_{\text{old}}^{\text{red}} \cap p_{\text{vnew}}^{\text{red}}) = \alpha^{-1}(p_{\text{old}}^{\text{red}}) \cap \alpha^{-1}(p_{\text{vnew}}^{\text{red}}) = (z - f(x, y))(x - g(y, z)).$$

Since the images of $x - g(y, z)$, $y$, and $z - f(x, y)$ form a basis of the cotangent space of $F_2[x, y, z]$, the reduced Hecke algebra has the structure $F_2[a, b, c]/(ab)$ as claimed. The computation of $f$ and $g$ in the case $N = 3$ and $(X, Y, Z) = (T_{11}, T_{13}, T_7)$ proceeds by linear algebra by constructing a basis of $K(1)$ (respectively, $K(3)^{\text{vnew}}$) dual to the monomials in $x$ and $y$ of $A(1)$ (respectively, $y$ and $z$ of $A(3)^{\text{vnew}}$).

Remark 11.4. The obstacle to extending Theorem 11.3 from $N = 3, 5$ to all primes $N \equiv 3, 5 \pmod {8}$ is the lower bound on the Krull dimension of $A(N)^{\text{vnew}}$ from Theorem 10.1, which implies that $A(N)^{\text{vnew}} \simeq F_2[y, z]$ (Corollary 10.2). The limitation is twofold.

- **Condition (1)** of the nilpotence method (Theorem 2.1) requires that the ambient space of modular forms be a polynomial algebra over a finite field. Using Riemann-Hurwitz and the ideas described in Remark 9.2, one can show that this happens if and only if $X_0(2N)$ has genus 0. This is a technical, not a conceptual, limitation to the nilpotence method, but we do not know of any workarounds at the moment.

- **Condition (4)** of the nilpotence method requires as input an infinite sequence of forms in $K(N)^{\text{vnew}}$ whose filtration grows at most linearly. In the proof of Theorem 10.1, we essentially obtain such a sequence from dimension formulas for $M_k(1) = M_k(1)_{1 \oplus 1}$ and $M_k(N) = M_k(N)_{1 \oplus 1}$, from which we can deduce information about $\dim M_k(N)^{\text{vnew}}_{1 \oplus 1}$. But in general, standard dimension formulas are insufficient: they must be refined for various $\vec{p}$.

There are several known $\vec{p}$-dimension formula techniques: Bergdall-Pollack [BP, Proposition 6.9(a)] uses Ash-Stevens, which requires $p \geq 5$; Jochnowitz [JCo, Lemma 6.4], uses $\theta$ twists for $p \geq 5$ and the Eichler-Selberg trace formula, but is limited by dimension $< p$ in characteristic $p$; Anni-Gitza-Medvedovsky (forthcoming, currently also only for $p \geq 5$) refines Jochnowitz’s trace formula work with deeper congruences; see also [AGM]. All of these rely on propagation from low weight and none of them yet allow for $p = 2$. The low-weight piece in our context is done: since the level-raising condition for $1 \oplus 1$ modulo 2 is satisfied at every odd $N$, [DM, Theorem 2(2b)] guarantees that $K_k(N)^{\text{vnew}}_{1 \oplus 1}$ is nonempty for some weight $k$. But the propagation part for $p = 2$ awaits further development.

Despite these limitations, one would not be surprised to discover that $R(N)^{\text{red}}_{1 \oplus 1} \simeq A(N)^{\text{red}}_{1 \oplus 1} \simeq F_2[a, b, c]/(ab)$ for all primes $N \equiv 3, 5 \pmod {8}$.\[\triangle\]
12. Structure of $A(N, \mathbb{F}_2)^{pf}$ for $N = 3, 5$

We now determine the structure of $A(N)^{pf}$ for $N = 3, 5$. Let $\tau^{mod} : G_{\mathbb{Q}, 2N} \rightarrow A(N)^{pf}_{1 \oplus 1}$ be the pseudorepresentation obtained by composing $\tau^{mod}$ with the natural injection $A(N)_{1 \oplus 1} \rightarrow A(N)^{pf}_{1 \oplus 1}$. In this section we write $\tau$ for $\tau^{pf}$. Recall that $c \in G_{\mathbb{Q}, 2N}$ is a complex conjugation.

**Theorem 12.1.** Let $N = 3, 5$, and let $Y = \tau(g^\pm_{-N})$ and $Z = \tau(ci)$ for any choice of $g^\pm_{-N}$, $c$ and $i \in I_N$. Then the map

$$F_2[y, z, u] \rightarrow A(N)^{pf}_{1 \oplus 1} \text{ induced by } u \mapsto U'_N, \quad y \mapsto Y \quad z \mapsto Z$$

is an isomorphism. If $N = 3$ then $(Z, Y)$ may be $(T_{13}, T_7)$; if $N = 5$ then $(Z, Y)$ may be $(T_{11}, T_7)$.

The proof has a number of steps. We first determine the structure of $A(N)^{pf,old}$. Recall from subsection 2.13.4 that $M(N)^{old} = M(1) + W_N M(1)$ is a subspace of $M(N)$ stable under the action of both $A(N)$ and $U_N$, and $A(N)^{pf,old}$ is the largest quotient of $A(N)^{pf}$ acting faithfully on $M(N)^{old}$.

**Lemma 12.2.** For any prime $N \equiv 3, 5 \mod 8$ we have $A(N)^{pf,old}_{1 \oplus 1} \cong F_2[U'_N, Y]$. More precisely, given any $g^\pm_{-N}$, the map $F_2[u, y] \rightarrow A(N)^{pf,old}_{1 \oplus 1}$ given by $u \mapsto U'_N$ and $y \mapsto \tau(g^\pm_{-N})$ is an isomorphism. In particular, $A(3)^{pf,old} \cong F_2[U'_3, T_{13}]$ and $A(5)^{pf,old} \cong F_2[U'_5, T_{11}]$.

**Proof.** Recall that the Hecke algebra $A(N)$ sits inside $A(N)^{pf}$ as a closed subalgebra, with the latter finite over the former (subsection 2.8 and Corollary 5.2). Moreover, the image of $A(N)_{1 \oplus 1}$ under the natural surjection $A(N)^{pf}_{1 \oplus 1} \rightarrow A(N)^{pf,old}_{1 \oplus 1}$ is $A(N)^{old}_{1 \oplus 1} \cong A(1)$. Therefore $A(N)^{pf,old}$ is finite over $A(1)$, so $\dim A(N)^{pf,old}_{1 \oplus 1} = \dim A(1) = 2$ (Theorem 6.2). We now look at generators closely: the maximal ideal of $A(1)$ is generated by the images $F_N$ and $Y$ of $Frob_N$ and $\tau(g^\pm_{-N})$, respectively. Therefore, the finiteness of $A(N)^{pf,old}_{1 \oplus 1}$ over $A(1)$ implies that the maximal ideal of $A(N)^{pf,old}_{1 \oplus 1}$ is generated by $F_N$, $Y$, and $U'_N = U_N + 1$. Since, by (6.4.3), the image of $F_N$ in the cotangent space of $A(N)^{pf,old}_{1 \oplus 1}$ is 0, the maximal ideal of $A(N)^{pf,old}_{1 \oplus 1}$ is generated by $Y$ and $U'_N$. As $\dim A(N)^{pf,old}_{1 \oplus 1} = 2$, this is a minimal generating set, and the map

$$F_2[y, u] \rightarrow A(N)^{pf,old}_{1 \oplus 1}$$

given by $y \mapsto Y = \text{image of } \tau(g^\pm_{-N})$ and $u \mapsto U'_N$ is an isomorphism. \hfill $\Box$

**Lemma 12.3.** For $N = 3, 5$ we have $\dim \text{Tan} A(N)^{pf} = 3$. The maximal ideal is generated by $U'_N$ and any choice of $\tau(g^\pm_{-N})$ and $\tau(g^\mp_{7})$. For example, the maximal ideal of $A(3)^{pf}$ is generated by $T_{13}$, $T_7$, and $U'_3$, and the maximal ideal of $A(5)^{pf}$ is generated by $T_{11}$, $T_7$, and $U'_5$.

**Proof.** On one hand, $\dim \text{Tan} A(N)^{pf} \leq 3$ (Lemma 8.2). On the other hand, Lemma 12.2 implies that $\dim A(N)^{pf} \geq 2$, so that $\dim \text{Tan} A(N)^{pf} \geq 2$ as well. If $\dim \text{Tan} A(N)^{pf} = 2$, then $A(N)^{pf}$ is a regular local ring of dimension 2 and hence isomorphic to its quotient $A(N)^{pf,old}$. But that would imply that the annihilator of $M(N)^{old}$ in $A(N)^{pf}$ is trivial, which would mean that the same is true for the annihilator $p_{old}$ of $M(N)^{old}$ in $A(N)$, which in turn would mean that $A(N) \simeq A(1)$. But that is absurd — for example, Proposition 11.1 shows that $A(N)$ has two distinct minimal primes, so that $p_{old} \neq (0)$. The precise statement about generators follows from the analysis in Lemma 8.2. \hfill $\Box$

We are now ready to prove the structure theorem for $A(N)^{pf}$ (Theorem 12.1).

**Proof of Theorem 12.1.** From Lemma 12.3 we know that $Y = \tau(g^\pm_{-N})$, $Z = \tau(ci)$ and $U'_N$ generate the maximal ideal of $A(N)^{pf}$. Let $\beta : F_2[y, z, u] \rightarrow A(N)^{pf}$ be the map sending $y \mapsto Y$, $z \mapsto Z$, and $u \mapsto U'_N$.

**Claim 1:** $\ker \beta \subseteq (zu)$: Note that $\tau^{mod}_{2, 1}(ci) = \tau^{mod}_{2, 1}(c) = 0$ for any complex conjugation $c$ and $i \in I_N$. Here $\tau^{mod}_{2, 1} : G_{\mathbb{Q}, 2} \rightarrow A(1)$ is the level-one modular pseudorepresentation. Hence, $\beta(z)$ annihilates $M(N)^{old}$. 


Consider the natural surjections $A(N)^{pf} \to A(N)^{pf,old}$ and $A(N)^{pf} \to A(N)^{vnew}$. Precomposed with $\beta$ and the isomorphism from Lemma 12.2, the former becomes the quotient map $F_2[y, z, u] \to F_2[y, u]$, with kernel $(z)$. Indeed, from the observation above, we see that $z$ is in the kernel and Lemma 12.2 then implies that the kernel is $(z)$. Similarly, precomposed with $\beta$ and the isomorphism from Corollary 10.2 the latter becomes the quotient map $F_2[y, z, u] \to F_2[y, z]$, with kernel $(u)$.

$$\begin{array}{ccc}
F_2[y, z, u] & \xrightarrow{\beta} & A(N)^{pf} \\
\kappa=(z) & \downarrow & \kappa=(u) \\
\text{ker}(z) & \xrightarrow{\beta} & A(N)^{vnew}
\end{array}$$

(12.0.1)

Since both the maps $F_2[y, z, u] \to A(N)^{pf,old}$ and $F_2[y, z, u] \to A(N)^{vnew}$ factor through $\beta$, its kernel $\text{ker}(\beta)$ is contained in $(z) \cap (u) = (zu)$.

**Claim 2:** $\text{ker} \beta \supset (zu^2)$: Since $\beta(u^2)M(N) = (U'_N)^2M(N) \subseteq M(N)^{old}$ (Lemma 2.2) and $\beta(z)$ annihilates $M(N)^{old}$, the operator $\beta(zu^2)$ annihilates $M(N)$.

**Claim 3:** $\text{ker} \beta = (zu^2)$: By Claim 1 above, any element of $\text{ker} \beta$ is of the form $zut$ for some $t \in F_2[y, z, u]$. Suppose $t \notin (u)$. Then $\beta(zt)$ does not annihilate $M(N)^{vnew}$, so that by (6.5.4) there is an $f \in K(N)^{vnew}$ with $\beta(zt)(f) \neq 0$. By Lemma 12.4 below, $f$ is in the image of $U'_N$. This means that if $U'_N(g) = f$ for some $g \in K(N)$, then $\beta(zut)(g) \neq 0$. Hence $zut \notin \text{ker} \beta$.

To complete the argument, we show that every very new form in $\text{ker} U_2$ is in the image of $U'_N$:

**Lemma 12.4.**

Let $f$ be in $K(N)^{vnew}$ for a prime $N \neq 1 \mod 8$. There exists $g \in K(N)$ with $U'_N(g) = f$.

**Proof.** Write $f = f_1 + f_3 + f_5 + f_7$, with $f_i \in K(N)$ as in Theorem 3.4. Since $U_N$ is an $N$-graded operator and $f \in K(N)^{vnew} = \text{ker}(U_N + 1)$, we have $U_N f = f$, so that $U_N f_i$ must equal $f_{N,i}$. For $i \neq 1, N \mod 8$, let $f_{i,i} := f_i + f_i$. Then $U'_N(f_{1,i}) = f$.

We deduce the structure of $A(N)^{new}$. Write $\tau$ for the composition $G_{Q,2N} \xrightarrow{\tau_{mod}} A(N) \to A(N)^{new}$.

**Corollary 12.5.** Let $N = 3, 5$, and let $Y = \tau^{new}(y_{-N}^+) and Z = \tau^{new}(ci)$ for any choice of $y_{-N}^+$, complex conjugation $c$, and $i \in I_N$. Then the following map is an isomorphism:

$$\mathbb{F}_2[y, z, u]/(u^2) \to A(N)^{new} \text{ defined by } u \mapsto U'_N, \quad y \mapsto Y, \quad z \mapsto Z.$$

If $N = 3$ then we may take $(Z, Y) = (T_{13}, T_7)$; if $N = 5$ then $(Z, Y)$ may be $(T_{11}, T_7)$.

**Proof.** Theorem 12.1 and (6.5.2).

13. Structure of $A(N, \mathbb{F}_2)$ for $N = 3, 5$

Finally, we determine the structure of $A(N)$ for $N = 3, 5$. In other words, we prove Theorem B.

**Theorem 13.1.** Let $N = 3$ or $5$. Choose any $Y = \tau_{mod}(y_{-N}^+)$, and $Z = \tau_{mod}(ci)$ for any choice of $c$ and any $i \in I_N$. Recall that $F_N = \tau_{mod}($Frob$ _N)$, Then the map

$$\mathbb{F}_2[x, y, z, w]/(xz, xw, (z + w)^2) \to A(N) \text{ induced by } x \mapsto F_N, \quad y \mapsto Y, \quad z \mapsto Z, \quad w \mapsto U_N Z$$

is an isomorphism of $(\mathbb{Z}/2\mathbb{Z})^\times$-graded algebras. Here the grading on $\mathbb{F}_2[x, y, z, w]/(xz, xw, (z + w)^2)$ is defined by $x$ having grading $N$, $y$ and $w$ having grading $-N$, and $z$ having grading 7.
Proof. Denote \( \tau^{\text{mod}} \) by \( \tau \) throughout the proof. From Corollary 9.7, \( A(N) \) is topologically generated by \( F_N, Y, Z \), and any \( Y' = \tau(g_{-N}^c) \) chosen so that \( Y = \tau(g_{-N}^c) \). First we adjust these generators slightly. Recall that (5.2.1) gives us that, for any \( i \in I_N \),
\[
U_N Z = U_N \tau(c_i) = \tau(c_i \text{ Frob}_N) - \tau(c \text{ Frob}_N);
\]
moreover, the image of the set \( \{Y, Y'\} \) in \( m(N)/m(N)^2 \) is the same as the image of the set \( \{\tau(c_i \text{ Frob}_N), \tau(c \text{ Frob}_N)\} \). Therefore we can replace \( Y' \) by \( U_N Z \) in the generating set.

Now let \( F_2[x, y, z, w] \) be an abstract power series ring endowed with the grading as described in the statement of Theorem 13.1. By the discussion above, the map \( \gamma \) defined by sending \( x \mapsto F_N, y \mapsto Y \), \( z \mapsto Z \), and \( w \mapsto U_N Z \) gives us a surjection of \((\mathbb{Z}/8\mathbb{Z})^a\)-graded \( F_2 \)-algebras \( \gamma : F_2[x, y, z, w] \rightarrow A(N) \). We now view \( A(N) \) as a subalgebra of \( A(N)^{\text{pf}} \cong F_2[y_1, z_1, u_1]/(z_1u_1^2) \) via \( \delta \), the inverse of the map \( \beta \) from Theorem 12.1, in order to understand \( \ker \gamma \):
\[
F_2[x, y, z, w] \xrightarrow{\gamma} A(N) \subset A(N)^{\text{pf}} \xrightarrow{\delta} F_2[y_1, z_1, u_1]/(z_1u_1^2).
\]
Here the \( y \) and the \( z \) of the first ring map to the image of \( y_1 \) and \( z_1 \), respectively, of the last.

We show that \((w+z)^2, xz \) and \( xw \) are all in \( \ker \gamma = \ker \delta \circ \gamma \). The first is simple: \( \delta \circ \gamma (w+z) = 0 \) and \((w+z)^2 \in \ker \gamma \). For the second and third, recall that \( F_N = F_N U_N + (U_N)^2 \) (6.4.3), so that \( \delta \circ \gamma (xw) = 0 \), as \( F_N \) is a multiple of \( U_N \) and \((U_N)^2 \) is in \( \ker \delta \). Moreover \( \delta \circ \gamma (xz) = 0 \). Therefore both \( xw \) and \( xz \) are in \( \ker \gamma \).

Finally, we claim that \( \ker \gamma = (w+z)^2, xw, xz) \). For simplicity, reparametrize by letting \( w_0 = w+z \) (note that this is not a graded parameter!) and rephrase the claim in the following way: consider the map
\[
\eta : F_2[x, y, z, w_0] \rightarrow F_2[y_1, z_1, u_1]
\]
sending \( x \mapsto w_0^2/(1+u_1), y \mapsto y_1, z \mapsto z_1, \) and \( w_0 \mapsto z_1u_1 \). Then our claim \( \ker \gamma = (w+z)^2, xw, xz) \) is equivalent to the claim \( \eta((w_0^2, w_0, xz)) = (z_1u_1^2) \). Note that we have shown that \( \eta((w_0^2, w_0, xz)) \) \( \subset (z_1u_1^2) \); our goal is to show that there is nothing else in \( \eta^{-1}((z_1u_1^2)) \). Observe that any \( f \) in \( F_2[x, y, z, w_0] \) is equivalent modulo \((w_0^2, xw, xz) \) to a power series of the form \( g = a(x, y) + w_0b(y, z) + c(y, z) \) for some \( a, b, c \in F_2[x, y] \) and \( y, c \in F_2[y, z] \). Then \( \eta(g) = a(u_1^2/(1+u_1), y_1) + z_1u_1b(y_1, z_1) + c(y_1, z_1) \). By inspection it is clear that \( \eta(g) \) is a multiple of \( z_1u_1^2 \) only if \( g = 0 \). Therefore \( \eta^{-1}((z_1u_1^2)) = (w_0^2, xw, xz) \), as claimed. Returning to our original graded parametrization, we have shown that \( \ker \gamma = ((w+z)^2, xw, xz) \). Note that this is a graded ideal, as expected.

Note that the element \( z + w \) in \( F_2[x, y, z, w_0]_{(x, z, xw, (z+w)^2)} \) is nilpotent, and the reduced quotient
\[
\left( \frac{F_2[x, y, z, w]}{(x, z, xw, (z+w)^2)} \right)_{\text{red}} \cong \frac{F_2[y, z]}{(xz)}
\]
matches the results of Theorem 11.3.

14. An \( R = \mathbb{T} \) theorem for \( A(N, F_2)^{\text{pf}}_{1@1} \)

In this section we prove an \( R = \mathbb{T} \) theorem for the partially full Hecke algebra \( A(N)^{\text{pf}} \) for \( N = 3, 5 \). To construct a deformation ring surjecting onto \( A(N)^{\text{pf}} \) we interpolate the deformation conditions of Wake–Wang–Erickson [WWE] and Calegari–Specter [CS, arxiv source file].

Let \( U \) be a formal variable, and let \( \mathcal{R}(N)^{\text{pf}}_{1@1} := \mathcal{R}(N, F_2)^{\text{pf}}_{1@1} \) be the quotient of the polynomial ring \( \mathcal{R}(N)_{1@1}[U] \) by the closed ideal \( J \) generated by the following elements:
\begin{enumerate}
  \item \( U^2 = \tau^{\text{univ}}(\text{Frob}_N)U + 1 \),
  \item \( \tau^{\text{univ}}(g \text{ Frob}_N) = \tau^{\text{univ}}(g \text{ Frob}_N) - U(\tau^{\text{univ}}(g) - \tau^{\text{univ}}(g)), \quad \text{for } g \in G_{\mathbb{Q}, Np} \) and \( i \in I_N \);
\end{enumerate}
Remark 5.5

Theorem 12.1, and Proposition 5.3, ker $A$

Note that If Proposition 15.1.

Hecke algebras for $\Gamma_0(5)$. We also include some unanswered questions.

We are now ready to state and prove the main result of this section.

Theorem 14.1. For $N = 3, 5$, the tuple $(\tau^{\text{univ}}, U_N)$ induces an isomorphism 

$$R(N)_{1@1}^{\text{pf}} \simeq A(N)^{\text{pf}}.$$  

Proof. Write $\tau^{\text{pf}} : G_{\mathbb{Q}, 2N} \to A(N)^{\text{pf}}$ for the modular pseudorepresentation $\tau^{\text{mod}} : G_{\mathbb{Q}, 2N} \to A(N)$ composed with the natural inclusion $A(N) \to A(N)^{\text{pf}}$. By (5.1.2), Proposition 5.3, and Remark 5.5, the element $U_N$ of $A(N)^{\text{pf}}$ satisfies (1)–(3), inducing the surjective morphism 

$$\phi^{\text{pf}} : R(N)_{1@1}^{\text{pf}} \to A(N)^{\text{pf}}.$$  

Conditions (1)-(3), together with the proof of Lemma 8.2, implies that the cotangent space of $R(N)_{1@1}^{\text{pf}}$ is spanned by the images of $U + 1$, $\tau^{\text{univ}}(ci)$ and $\tau^{\text{univ}}(c \text{Frob}_N)$. Note that 

$$\phi^{\text{pf}}(X + 1) = U_N, \quad \phi^{\text{pf}}\left(\tau^{\text{univ}}(ci)\right) = \tau^{\text{mod}}(ci), \quad \text{and} \quad \phi^{\text{pf}}\left(\tau^{\text{univ}}(c \text{Frob}_N)\right) = \tau^{\text{mod}}(c \text{Frob}_N).$$

From Theorem 12.1, we know that there exists a surjective map 

$$\beta : F_2[y, z, u] \to A(N)^{\text{pf}}$$ sending $y \mapsto \tau^{\text{mod}}(c \text{Frob}_N)$, $z \mapsto \tau^{\text{mod}}(ci)$, $u \mapsto U_N.$

Therefore, the surjective map 

$$\beta' : F_2[y, z, u] \to R(N)_{1@1}^{\text{pf}}$$ sending $y \mapsto \tau^{\text{univ}}(c \text{Frob}_N)$, $z \mapsto \tau^{\text{univ}}(ci)$, $u \mapsto U + 1.$

is a lift of $\beta$: that is, $\phi^{\text{pf}} \circ \beta' = \beta.$ Combining conditions (2) and (3) for $g = c$, we obtain $U \tau^{\text{univ}}(ci) = U^{-1} \tau^{\text{univ}}(ci)$; i.e., $(U + 1)^2 \tau^{\text{univ}}(ci) = 0$. Thus $zu^2 \in \ker \beta'$. On the other hand, by Theorem 12.1, $\ker \beta = (zu^2)^\ast$. Since $\phi^{\text{pf}} \circ \beta' = \beta$ and $\beta'$ is surjective, it follows that $\ker \beta = \ker \beta'$ so that $\phi^{\text{pf}}$ is injective. Since $\phi^{\text{pf}}$ is surjective by construction, our claim is proved. 

Note that, for $N = 3, 5$, the trace algebras of $\tau^{\text{mod}}$ and $\tau^{\text{univ}}$ in $A(N)^{\text{pf}}$ and $R(N)_{1@1}^{\text{pf}}$ are $A(N)$ and $R(N)_{1@1}$, respectively. So we obtain the following immediate corollary.

Corollary 14.2. For $N = 3, 5$, under the isomorphism obtained in Theorem 14.1, $A(N)$ is isomorphic to the image of $R(N)_{1@1}$ in $R(N)_{1@1}^{\text{pf}}$.

15. COMPLEMENTS AND QUESTIONS

In this last section we gather some easy-to-deduce information about Hecke algebras and deformation rings of levels closely related to $\Gamma_0(3)$ and $\Gamma_0(5)$. We also include some unanswered questions.

15.1. Hecke algebras for $\Gamma_0(9)$ and $\Gamma_0(25)$ modulo 2. The analysis we have already done allows us to determine the structure of the reduced quotients $A(9, F_2)^{\text{red}}$ and $A(25, F_2)^{\text{red}}_{1@1}$ with minimal additional work.

Proposition 15.1. If $N = 3$ or 5, then we have an isomorphism 

$$A(9, F_2)^{\text{red}}_{1@1} \simeq A(N^2, F_2)^{\text{red}}_{1@1} \simeq A(N, F_2)^{\text{red}}_{1@1}.$$  

Note that $A(9, F_2) = A(9, F_2)_{1@1}$ is a local ring, whereas $A(25, F_2)$ has two local components.
Proof. On the deformation side, the rings $\hat{R}(N^2, F_2)_{1\oplus 1}$ and $\hat{R}(N, F_2)_{1\oplus 1}$ coincide by definition. On the Hecke side, restriction to modular forms of level $N$ induces a surjective morphism $A(N^2, F_2)_{1\oplus 1} \twoheadrightarrow A(N, F_2)_{1\oplus 1}$. Combining these with the map $\hat{\varphi}$ from (2.11.3) gives us

$$\hat{R}(N, F_2)_{1\oplus 1} \xrightarrow{\hat{\varphi}} A(N^2, F_2)_{1\oplus 1} \twoheadrightarrow A(N, F_2)_{1\oplus 1}. \tag{15.1.1}$$

Taking reduced quotients, along with Corollary 11.2, completes the proof. \hfill $\Box$

Since characteristic-zero Galois representations attached to a form of level $N^2$ — for example, to the twist of a level-one eigenform by a Dirichlet character modulo $N$ — need not have level-$N$ shape, we do not expect $\hat{\varphi}$ to factor through $R(N, F_2)_{1\oplus 1}$.

15.2. Hecke algebras of for $\Gamma_1(3)$ and $\Gamma_1(5)$ modulo 2. We can achieve similar results for $\Gamma_1(3)$ and $\Gamma_1(5)$. The construction of the spaces of mod-2 modular forms and the Hecke algebra are similar to those for $\Gamma_0(N)$ described in subsections 2.3 and 2.7. For odd $N$ we define spaces $M_k(\Gamma_1(N), F_2)$ of mod-2 modular forms of level $\Gamma_1(N)$ as reductions of $q$-expansions of characteristic-zero $\Gamma_1(N)$-modular forms of weight $k$ whose $q$-expansion at the infinity cusp is integral. We also let $A_k(\Gamma_1(N), F_2)$ be the Hecke algebra generated by the action of the Hecke operators prime to $2N$ acting on $M_k(\Gamma_1(N), F_2)$. As described in subsection 2.4, we have embeddings $M_k(\Gamma_1(N), F_2) \hookrightarrow M_{k+1}(\Gamma_1(N), F_2)$ induced by multiplication by $E_{1,\chi, N}$ lifting the Hasse invariant, so that restriction defines projections $A_{k+1}(\Gamma_1(N), F_2) \twoheadrightarrow A_k(\Gamma_1(N), F_2)$, and we can form the Hecke algebra

$$A(\Gamma_1(N), F_2) := \lim_{\kappa \to \infty} A_k(\Gamma_1(N), F_2) \quad \text{acting on} \quad M(\Gamma_1(N), F_2) := \bigcup_k M_k(\Gamma_1(N), F_2).$$

Both the Hecke algebra and the space of forms break up into $\hat{\rho}$ components as in (2.7.1), (2.7.2).

Proposition 15.2. For $N = 3, 5$ restriction to forms of level $\Gamma_0(N)$ induces the isomorphism

$$A(\Gamma_1(N), F_2)_{1\oplus 1}^{\text{red}} \simeq A(N, F_2)_{1\oplus 1}^{\text{red}}.$$

For $N = 3$ we further have the isomorphism $A(\Gamma_1(N), F_2)_{1\oplus 1}^{\text{red}} \simeq A(N, F_2)_{1\oplus 1}$. \hfill $\Box$

Proof. Although the modular pseudorepresentation taking values in $A(\Gamma_1(N), F_2)_{1\oplus 1}$ does not have constant determinant, and hence does not factor through $\hat{R}(N, F_2)_{1\oplus 1}$, its image in $A(\Gamma_1(N), F_2)_{1\oplus 1}^{\text{red}}$ does $[\text{De},$ Lemma 5 and ff.], so that universality gives us a surjective morphism

$$\hat{R}(N, F_2)_{1\oplus 1} \twoheadrightarrow A(\Gamma_1(N), F_2)_{1\oplus 1}^{\text{red}}.$$

On the other hand, restriction from forms of level $\Gamma_1(N)$ to forms of level $\Gamma_0(N)$ gives a surjective map $A(\Gamma_1(N), F_2)_{1\oplus 1} \twoheadrightarrow A(N, F_2)_{1\oplus 1}$. Now take reduced quotients and apply Corollary 11.2.

The second statement comes from equality on the space of forms. Indeed, for $k$ even, we already have $M_k(3, \mathbb{Z}_2) = M_k(\Gamma_1(3), \mathbb{Z}_2)$ since the only even Dirichlet character modulo 3 is the trivial one. And for $k$ odd, we have $M_k(\Gamma_1(3), F_2) \subset M_{k+1}(3, \mathbb{F}_2)$ with the embedding induced by multiplying by the Hasse invariant lifted by $E_{1,\chi}$, where $\chi$ is the nontrivial mod-3 Dirichlet character. Thus $M(\Gamma_1(3), F_2) = M(3, F_2)$, and the claim follows. \hfill $\Box$

15.3. Remaining questions. We close with a number of questions not answered in this text.

(1) Can Theorems A and B be generalized to all primes $N \equiv 3, 5 \mod 8$? See Remark 11.4 for a discussion of the limitations of our methods.

Let $\mathcal{R}'(N, F_2)_{1\oplus 1}$ be the deformation ring with conditions (1.4.2) and (1.4.3) imposed. Recall that $\mathcal{R}(N, F_2)_{1\oplus 1}$ is the quotient of $\hat{R}(N, F_2)_{1\oplus 1}$ subject to the purely local level-$N$ shape condition (1.4.1), whereas $\mathcal{R}'(N, F_2)_{1\oplus 1}$
a priori satisfies a global condition (see also the discussion in subsection 1.4). One can show that the surjection from $R(N,F_2)_{1\oplus 1}$ to $A(N,F_2)_{1\oplus 1}$ factors through $R'(N,F_2)_{1\oplus 1}$, so that there are natural surjective maps

\[(15.3.1) \quad R(N,F_2)_{1\oplus 1} \to R'(N,F_2)_{1\oplus 1} \to A(N,F_2)_{1\oplus 1}.
\]

Furthermore, Corollary 11.2 implies that all three rings have the same reduced quotient.

(2) Is the second map in (15.3.1) an isomorphism?

(3) Does the first surjection in (15.3.1) have a nontrivial kernel? Note that by Corollary 14.2 this map is an isomorphism if and only if the structure map $R(N,F_2)_{1\oplus 1} \to R'(N,F_2)_{1\oplus 1}$ is injective.

(4) For $N = 3, 5$, is the map $\hat{R}(N,F_2)_{1\oplus 1} \to A(N^2,F_2)_{1\oplus 1}$ from (15.1.1) an isomorphism?

(5) In general, given a modular representation $\bar{\rho} : G_{\mathbb{Q},N} \to GL_2(\mathbb{F})$ and a prime $\ell \nmid Np$, can one determine the structure of the level $N\ell$ Hecke algebra $A(N\ell,F)_{\bar{\rho}}$ from the structure of the level $N$ Hecke algebra $A(N,F)_{\bar{\rho}}$?

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