Scattering in periodic waveguide: integral representation and spectrum decomposition

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Abstract

Scattering problems in periodic waveguides are interesting but challenging topics in mathematics, both theoretically and numerically. Due to the existence of eigenvalue, the unique solvability of such problems is not always guaranteed. To obtain a unique solution that is physically meaningful, the limiting absorption principle (LAP) is a commonly used method. LAP assumes that the limit of a family of solutions of scattering problems with absorbing material converges, as the absorbing parameter tends to 0, and the limit is the unique solution that is physically meaningful. It has been
proven that LAP holds for periodic waveguides in [Hoa11], and we call the solution obtained from LAP the LAP solution. However, the formulations of LAP solutions from these theoretical results are still very complicated, thus to application of these formulations to numerical simulations is still very challenging. In this paper, with the help of the Floquet-Bloch theory, we set up a simplified formulation for the LAP solution of scattering problems with an integral representation, when a "small enough" subset is excluded from the choice of wave numbers. It is expected that an efficient numerical method could be established based on the simplified formulation. Another important topic for scattering problems in periodic waveguides is the spectrum decomposition on periodic operators. With the newly established formulation, we have successfully established the spectrum decomposition of the operator. The operator is explicitly described by its eigenvalues and generalized eigenfunctions, which are closely related to the Bloch wave solutions. The solution is then decomposed into the generalized eigenfunctions. This gives a better understanding of the structures of the solutions.

Key words: scattering problems with periodic waveguide, limiting absorption principle, residue theorem, spectrum decomposition, generalized eigenfunction

1 Introduction

In this paper, we consider scattering problems in periodic waveguides. This topic is of great interest both in mathematics and in related technologies, e.g., nano-technology. In recent years, several mathematicians have been working on this topic and many interesting results and numerical methods have been established. Different from scattering problems by Dirichlet rough surfaces (see [Kir93, Kir94, CWR96, CE10]), the unique solvability no longer holds due to the existence of eigenvalues. Thus a proper radiation condition is always necessary to guarantee the uniqueness and the Limiting Absorption Principle (LAP) is a well-known technic. A number of numerical methods have also been established based on LAP (see [HJF06, FJ09, SZ09, FHZ09, ESZ09]). The key question of these methods is to prove that LAP holds. It is proved in [Hoa11] by the study of resolvent that LAP holds for elliptic equations with periodic coefficients, and we also refer to [FJ15] with the help of eigenvalue decomposition of elliptic operator and [KL18] by the study of singular perturbation of Fredholm operators. However, there are still some difficulties corresponds to this topic. First, although with the assumption that LAP holds, the numerical method is very complicated due to the approximation process. To established a numerical method without the approximation, we want to seek for an analytic representation for LAP solutions. Second, the structure of solution is still unclear at present, makes the study of direct and inverse scattering problems still difficult.

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The first part of this paper is devoted to establish a simplified integral representation for LAP solutions. This work is inspired by the application of the Floquet-Bloch transform in scattering by periodic surfaces (see [17A]) and periodic waveguides (see [11H, 15FJ]). The Floquet-Bloch transform builds up a “bridge” between the original problem defined in a periodic domain and a family of cell problems defined in one bounded period cell. We refer to [93K] for details of the Floquet theory. Let \( u(x) \) be the solution of the original problem that only depends on the location \( x \), the transformed field \( w(z, x) \) depends on both \( x \) and the parameter \( z \). Then \( u \) could be represented by the counter integral of \( w(z, x) \) with respect to \( z \) on a closed curve. The advantage of the Floquet-Bloch transform based method could be concluded as two points. First, the cell problems are always bounded and classical, so both the theoretical analysis and the numerical simulation for solutions \( w(z, \cdot) \) are relatively easier. Second, the original solution \( u \) could be represented by the integral of solutions of cell problems \( w(z, x) \) with respect to \( z \), which is also difficult to be carried out. Although the cell problems come originally from the Floquet-Bloch transform, and only exists with certain conditions and for certain \( z \)‘s, we can finally withdraw the transform process and consider the two problems independently. With the analytic Fredholm theory, we can prove that the solutions \( w(z, x) \) is extended to a meromorphic function in \( z \) in the whole complex plane except for \( \{0\} \). From the analysis of asymptotic behaviour of the distributions of the poles, we can finally set up an integral representation for LAP solutions. The result could also be extended to half-guide problems. This clear integral presentation also provides a simplified formulation for potential numerical methods.

The second part considers the spectrum decomposition of the periodic operator that maps the values of LAP solutions between edges of periodic cells. From a similar analysis as in [16H16], the operator is compact. In this same paper, the authors conjectured and applied the spectrum decomposition, but they did not prove the result at that time. In the second part of this paper, we finally prove the result with the help of the integral representation of the solution \( u \) by \( w(z, \cdot) \) obtained in the first part. However, due to the distribution of poles of \( w(z, \cdot) \), there are always infinite number of poles that lie in the interior of the integral curve, and the origin is the only accumulation point of the poles. Thus the residue theorem could not be applied simply to the integral. We first extend the residue theorem to the case when there are infinite number of residues lying in the interior of a curve, then consider the properties of the residues from the integral representation, where a very nice property of periodic structure is adopted. Finally, we have proved that the \( H^1 \) space defined on one periodic cell could be decomposed by generalized eigenfunctions of the periodic operators. This result shows that the periodic operator has the Jordan normal form and also describes structure of LAP solutions. We have to mention that similar results have been proved in [13H15S], the method is quite different from ours.

The rest of this paper is organised as follows. We give a brief Introduction of the scattering problem in the second section, and introduce the spectrum of the solution operator in Section 3. We formulate variational formulations for the quasi-periodic problems in the fourth section. In Section 5, we introduce the Floquet-Bloch transform and apply it to the case when the scattering problems are uniquely solvable. Then we consider a much more complicated, but more important case, i.e., when the scattering problems are not uniquely solvable in the sixth section, and set up the integral formulation for this problem. Then the result is also extended to the integral formulation of half-guide problems. In Section 7, we study the spectrum decomposition of the periodic operator, and also decompose LAP solutions by residues of quasi-periodic solutions.

2 Direct Scattering Problem

Let \( \Omega = \mathbb{R} \times \Sigma \subset \mathbb{R}^{d+1} (d = 1, 2) \) be a closed waveguide, where \( \Sigma \subset \mathbb{R}^d \) is a connected, bounded open domain (see Figure 1). The waveguide is filled up with periodic material, and the material is described by a refractive index \( q \) which is a real-valued, \( 1 \)-periodic function in \( x_1 \)-direction, i.e.,

\[
q(x_1 + 1, \vec{x}) = q(x), \quad x \in \Omega,
\]

where \( \vec{x} = (x_2, \ldots, x_d, 1) \) is a \( d \)-dimensional vector and \( x = (x_1, \vec{x}) \in \mathbb{R}^{d+1} \). We also assume that \( q \geq c_0 > 0 \) in \( \Omega \), where \( c_0 \) is a fixed positive number.
The scattering problem is modelled by the following equations:

\[ \Delta u + k^2 q u = f \quad \text{in } \Omega; \]  
\[ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \]  

where \( \nu \) is the normal outward vector, \( f \) is a function in \( L^2(\Omega) \). In this paper, we also assume that \( f \) is compactly supported.

\textbf{Remark 1.} The method in this paper may be extended to more generalized cases, such as more complicated periodic geometric structures or elliptic partial differential equations with periodic coefficients. Here we only use the settings as an example to show how the method works.

It is well known that when the wave number \( k \) is positive, the problem (1)-(2) may not be uniquely solvable in \( H^1(\Omega) \) even for homogeneous media, due to the existence of eigenvalues. Thus a proper radiation condition is required to guarantee the well-posedness of this problem. A well known way is to adopt the \textit{Limiting Absorption Principle (LAP)}, i.e., to consider the limit of a family of solutions with absorbing media when the absorption parameter tends to zero. Given any \( \varepsilon > 0 \), consider the following damped Helmholtz equation with absorption:

\[ \Delta u_\varepsilon + (k^2 + i\varepsilon) q u_\varepsilon = f \quad \text{in } \Omega; \]  
\[ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial \Omega. \]  

From Lax-Milgram theorem, the problem is uniquely solvable in \( H^1(\Omega) \). Moreover, the solution \( u_\varepsilon \) decays exponentially when \( x_1 \to \pm \infty \) (see [JLF06]). Then we consider the existence and behaviour of the limit of \( u_\varepsilon \) when \( \varepsilon \to 0^+ \).

We are especially interested in the case that the problem is not uniquely solvable, when both theoretical analysis and numerical simulation are much more complicated. For simplicity, we introduce the following operator:

\[ A u := -\frac{1}{q} (\Delta u) \]

defined in the domain

\[ D(A, \Omega) := \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \}. \]

Let the spectrum of \( A \) be denoted by \( \sigma(A) \). Then the problem (1)-(2) is uniquely solvable in \( H^1(\Omega) \) if and only if \( k^2 \notin \sigma(A) \). Thus the spectrum of \( A \) plays an important role in the well-posedness of the problem (1)-(2). From the Floquet-Bloch theory, the spectrum of \( A \) is closely related to the so-called Bloch wave solutions, which will be introduced in the following section.

\section*{3 Spectrum properties of \( A \)}

In this section, we introduce the Bloch wave solutions and the spectrum properties of operator \( A \). The mathematical basis mainly comes from the Floquet-Bloch theory in [Kuc93], and for more details we refer to [JLF06, ESZ09, FJ15].

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3.1 Bloch wave solutions

As $q$ is 1-periodic with respect to $x_1$, it is more convenient to consider problems defined in one periodic cell. Thus we defined the following domains (see Figure 1):

$$\Omega_0 := (-1/2, 1/2] \times \Sigma, \quad \text{and} \quad \Omega_j := \Omega_0 + \left( \frac{j}{\omega} \right) \text{ for any } j \in \mathbb{Z}.$$  

Then $\Omega = \bigcup_{j \in \mathbb{Z}} \Omega_j$. Define $\Gamma_j = \{-1/2 + j\} \times \Sigma$ for any $j \in \mathbb{Z}$, thus the left and right boundaries of the cell $\Omega_j$ are $\Gamma_j$ and $\Gamma_{j+1}$. Let $\partial \Omega_j := \partial \Omega \cap \Omega_j^c$. For simplicity, we assume that

$$\text{supp}(f) \subset \Omega_0.$$  

For a complex number $z \in \mathbb{C}$, define the quasi-periodic boundary condition as:

$$u|_{\Gamma_{j+1}} = z \ u|_{\Gamma_j}, \quad \frac{\partial u}{\partial x_1}|_{\Gamma_{j+1}} = z \frac{\partial u}{\partial x_1}|_{\Gamma_j}, \quad \forall j \in \mathbb{Z}. \quad (5)$$

We define the subspace of $H^1(\Omega_0)$ by:

$$H^1_{\text{qu}}(\Omega_0) := \{ \varphi \in H^1(\Omega_0) : \varphi \text{ satisfies } (5) \text{ for } j = 0 \}. \quad (6)$$

Note that, when $z = 0$, the boundary condition in (5) implies that $u|_{\Gamma_j} = \frac{\partial u}{\partial x_1}|_{\Gamma_j} = 0$ for any $j \in \mathbb{Z}$. Especially, the functions that satisfy (5) with $z = 1$ are periodic, and the subspace of periodic functions in $H^1(\Omega_0)$ is denoted by $H^1_{\text{per}}(\Omega_0)$. We can also define the operator in the quasi-periodic domain, i.e.,

$$A_z u = \frac{1}{q} (\Delta u) \text{ with its domain } D_z(A, \Omega_0) := D(A, \Omega_0) \cap H^1_{\text{qu}}(\Omega_0), \quad (7)$$

where $D(A, \Omega_0)$ is defined in the same way as $D(A, \Omega)$, where $\Omega$ is replaced by $\Omega_0$. Let $\sigma(A_z)$ be the spectrum of $A_z$. A classical result from the Floquet-Bloch theory also shows that (see [Kuc93]):

$$\sigma(A) = \bigcup_{|z| = 1} \sigma(A_z).$$

When the equation $(k^2 I - A_z) u = 0$ has a nontrivial solution in $D_z(A, \Omega_0)$, $z$ is called a Floquet multiplier, and the corresponding non-trivial solution is called a Bloch wave solution. Let $\mathbb{F}$ be the collection of all the Floquet multipliers and $\mathbb{UF} = \mathbb{F} \cap S^1$ ($S^1$ is the unit circle in $\mathbb{C}$) be the set of all unit Floquet multipliers. We conclude some properties of the Floquet multipliers from [Kuc93, JLF06, EHZZ09, FJ15, Kuc16]:

- $z \in \mathbb{F}$ if and only if $z^{-1} \in \mathbb{F}$, thus $z \in \mathbb{UF}$ if and only if $\overline{z} = z^{-1} \in \mathbb{UF}$.
- From the Hologram unique continuity, $0 \notin \mathbb{F}$.
- The number of elements in $\mathbb{UF}$ is finite.
- $\mathbb{F}$ is a discrete set and the only accumulation points of $\mathbb{F}$ are 0 and $\infty$.

3.2 The dispersion diagram

From the property (4), to study the spectrum property of $A$, it is practical to consider the spectrum of the operator $A_z$ for the cell problem when $|z| = 1$. For simplicity, we replace $z$ by $\alpha = -i \log(z)$ where $\alpha \in (-\pi, \pi]$. We also replace $A_z$ by $A_\alpha$ in this section. Then (4) becomes

$$u|_{\Gamma_{j+1}} = \exp(i\alpha) \ u|_{\Gamma_j}, \quad \frac{\partial u}{\partial x_1}|_{\Gamma_{j+1}} = \exp(i\alpha) \frac{\partial u}{\partial x_1}|_{\Gamma_j}, \quad \forall j \in \mathbb{Z}. \quad (8)$$
Denote the spectrum of $A_\alpha$ by $\sigma(A_\alpha)$. From the property of $A_\alpha$, $\sigma(A_\alpha)$ is a discrete subset of $(0, \infty)$. By rearranging the order of the points in $\sigma(A_\alpha)$ properly, we could obtain a family of analytic functions $\{\mu_n(\alpha) : n \in \mathbb{N}\}$ such that

$$\sigma(A_\alpha) = \bigcup_{n \in \mathbb{N}} \{\mu_n(\alpha)\}.$$  

Thus $\sigma(A) = \bigcup_{n \in \mathbb{N}} \cup_{\alpha \in (-\pi, \pi]} \{\mu_n(\alpha)\}$. Moreover, $\lim_{n \to \infty} \mu_n(\alpha) = \infty$ for any $\alpha \in (-\pi, \pi]$. For any $\mu_n(\alpha)$, there is also a corresponding family of eigenfunctions $\psi_n(\cdot, \alpha) \in D_{\alpha}(A, \Omega_0)$ such that

$$A_\alpha \varphi_n(\cdot, \alpha) = \mu_n(\alpha) \psi_n(\cdot, \alpha).$$  

Moreover, $\varphi_n(\cdot, \alpha)$ also depends analytically on $\alpha \in (-\pi, \pi]$. Both $\mu_n(\alpha)$ and $\varphi_n(\cdot, \alpha)$ are extended into analytic functions in $\alpha$ in a small enough neighbourhood of $(-\pi, \pi] \times \{0\}$.

**Remark 2.** From Section 3.3 in [FJ15], there is an important argument for the function $\mu_n$. As $\mu_n$ is an analytic function, it is either constant or $\mu_n'$ vanishes at finite number of points. However, the function $\mu_n$ takes constant value if and only if $A$ has a non empty point spectrum (the point spectrum of $A$ is denoted by $\sigma_p(A)$).

This assumption holds for several different cases, see [SW02, SS02] for 1D cases and see [BDE03, KF09] for 2D cases. We do not go into details, but just simply let $\sigma_p(A) = \emptyset$. Thus for any $n \in \mathbb{N}$, $\mu_n$ is not a constant function.

For any fixed $n \in \mathbb{N}$, the graph $\{(\alpha, \mu_n(\alpha)) : \alpha \in (-\pi, \pi]\}$ is called a dispersion curve, and the collection of all the dispersion curves composes the dispersion diagram. Following [ESZ09, EHZ09], we first show the dispersion diagrams for two different examples in the waveguide $\Omega = \mathbb{R} \times (0, 1) \subset \mathbb{R}^2$:

1. **Example 1.** $q = 1$ is a constant function in $\Omega$, and its dispersion diagram is shown in Figure 2 (left). The relationship between the quasi-periodic parameter $\alpha$ and the eigenvalue could be obtained analytically:

$$\mu_{jm}(\alpha) = j^2 \pi^2 + (\alpha + 2\pi m)^2, \quad j \in \mathbb{N}, m \in \mathbb{Z}.$$  

2. **Example 2.** $q = 9$ in a disk with center $(0, 0)$ and radius 0.3 and $q = 1$ outside the disk. In this case, the functions $\mu_n$ could no longer be obtained analytically. We compute the eigenvalues with the help of the finite element method and the dispersion diagram is shown in Figure 2 (right). The calculations are carried out by the finite element method based on regular triangle meshes in the periodic cell $\Omega_0$ with the largest mesh size 0.005.

![Figure 2: Dispersion diagram. Left: Case I; Right: Case II.](image)

From left picture of Figure 2, the set $\sigma(A)$ is $(0, \infty)$. However, from the right picture of Figure 2 there are several ”stop bands” (in red) such that no eigenvalues lie in them. The bands that eigenvalues lie in are called ”pass bands”.

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Although we only show examples in two-dimensional periodic waveguides, it is easily extended to the three dimensional cases. The value $k^2$ lies either in a stop band or in a pass band. When $k^2$ lies in a pass band, from (7), there is at least one UF three dimensional cases. The value $k$ of $\psi_k$ is propagating Floquet mode. This function is also a non-trivial solution to the original problem (1) for $f = 0$, and it is called a "propagating Floquet mode". When $k^2$ lies in a pass band, there is no propagating Floquet mode and the scattering problem (1)-(2) has a unique solution in $H^1(\Omega)$. The case when $k^2$ lies in a pass band is particularly interesting and challenging. To study this case, we have to discuss more about the propagating Floquet modes.

3.3 Propagating Floquet modes

Following the notations in [FJ15], let $\{ (\alpha, \mu_n(\alpha)) : \alpha \in (-\pi, \pi) \}$ be the $n$-th dispersion curve, then the corresponding eigenfunctions $\psi_n(\cdot, \alpha)$ are propagating Floquet modes.

For any fixed $k^2 \in \sigma(A)$, there is at least one $\alpha \in (-\pi, \pi)$ such that $k^2 \in \sigma(A)$. Thus the set

$$ P := \{ \alpha \in (-\pi, \pi) : \exists n \in \mathbb{N}, \text{s.t.}, \mu_n(\alpha) = k^2 \} $$

is not empty. From the definition of $\mathbb{UF}$, it has the representation as:

$$ \mathbb{UF} = \{ \exp(i\alpha) : \alpha \in P \}. $$

The points in $P$ could be divided into the following three classes:

- When $\mu'_n(\alpha) > 0$, then $\psi_n(\cdot, \alpha)$ is propagating from the left to the right;
- when $\mu'_n(\alpha) < 0$, then $\psi_n(\cdot, \alpha)$ is propagating from the right to the left;
- when $\mu'_n(\alpha) = 0$.

Based on the above classification, we define the following three sets:

$$ P_+ := \{ \alpha \in (-\pi, \pi] : \exists n \in \mathbb{N}, \text{s.t.}, \mu_n(\alpha) = k^2 \text{ and } \pm \mu'_n(\alpha) > 0 \}; $$

$$ P_0 := \{ \alpha \in (-\pi, \pi] : \exists n \in \mathbb{N}, \text{s.t.}, \mu_n(\alpha) = k^2 \text{ and } \mu'_n(\alpha) = 0 \}. $$

Then $P = P_+ \cup P_- \cup P_0$.

**Remark 3.** It is possible that there are two (or more) different $m, n \in \mathbb{N}$ such that $\mu_m(\alpha) = \mu_n(\alpha) = k^2$, i.e., two (or more) dispersion curves may have an intersection at $(\alpha, k^2)$. In this case, $\alpha$ is treated as two (or more) different elements.

As $P$ is symmetric, we claim that the sets $P_\pm$ are symmetric also.

**Lemma 4** (Theorem 4, [FJ15]). $\alpha \in P_+$ if and only if $-\alpha \in P_-$.

As the limiting absorption principle fails when the set $P_0$ is not empty, we will assume that the following assumption holds.

**Assumption 5.** Assume that in this paper, $P_0 = \emptyset$.

The assumption is reasonable as the set $\{ k > 0 : P_0 \neq \emptyset \}$ is “small enough”. Actually, the set is countable with at most one accumulation point at $\infty$ (see Theorem 5, [FJ15]).

In our later proof of the limiting absorption principle, we also have to avoid the cases when $P_+ \cap P_- \neq \emptyset$ is not empty. Luckily, with the similar method used in the proof of Theorem 5 in [FJ15], we can also prove that this set is discrete in the following lemma.

**Lemma 6.** The set $\{ k \in \mathbb{R}_+ : P_+ \cap P_- \neq \emptyset \}$ is countable, and it has at most one accumulation point at $\infty$. 

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Proof. Assume that the set has a bounded accumulation point $k_0 \in \mathbb{R}_+$, i.e., there is a sequence $k_n$ such that

$$P_+(k_n) \cap P_-(k_n) \neq \emptyset, \quad \lim_{n \to \infty} k_n = k_0.$$ 

Thus the sequence has a strictly monotones subsequence which also converges to $k_0$. Without loss of generality, we assume that the subsequence is monotonically decreasing and is still denoted by $k_n$, i.e.,

$$k_n^2 < \cdots < k_n^2 < k_{n-1}^2 < \cdots < k_1^2.$$

This implies that for any $n \in \mathbb{N}$, there is a pair $(i_n, j_n) \in \mathbb{N} \times \mathbb{N}$ such that

$$\exists \alpha_n \in (-\pi, \pi], \text{s.t., } \mu_{i_n}(\alpha_n) = \mu_{j_n}(\alpha_n) = k_n^2$$

that satisfies

$$\mu_{i_n}^\prime(\alpha_n) > 0, \quad \mu_{j_n}^\prime(\alpha_n) < 0.$$ 

As $\lim_{n \to \infty} \mu_n(\alpha) = \infty$ for any $\alpha \in (-\pi, \pi]$, there should be a subsequences of pairs $(i_n, j_n)$ such that $i_n = i_0$ and $j_n = j_0$ where $i_0$ and $j_0$ are two constant positive integers. Still denote the subsequence of $k_n^2$ by $k_n^2$, then for any $n \in \mathbb{N}$, there is an $\alpha_n \in (-\pi, \pi]$ such that

$$\mu_{i_n}(\alpha_n) = \mu_{j_n}(\alpha_n) = k_n^2, \quad \mu_{i_n}^\prime(\alpha_n) > 0, \quad \mu_{j_n}^\prime(\alpha_n) < 0.$$ 

Define the function

$$\mu(\alpha) := \mu_{i_0}(\alpha) - \mu_{j_0}(\alpha),$$

then there is a sequence $\alpha_n \in (-\pi, \pi]$ such that

$$\mu(\alpha_n) = 0, \quad \forall n \in \mathbb{N}.$$ 

As $\mu_{i_0}$ and $\mu_{j_0}$ are both analytic functions, $\mu$ is analytic as well. Thus either $\mu$ is a constant function equals to 0, or $\alpha_n = \alpha_0$ except for a finite number of $n$’s.

For the first case, $\mu^\prime(\alpha) = 0$ for any $\alpha_n$, which contradicts with

$$\mu^\prime(\alpha_n) = \mu_{i_0}^\prime(\alpha_n) - \mu_{j_0}^\prime(\alpha_n) > 0.$$ 

For the second case, suppose there is an $N >> 1$ such that $\alpha_n = \alpha_0$ for any $n \geq N$, then $\mu_{i_0}(\alpha_n) = \mu_{j_0}(\alpha_n) = k_n^2$ implies that $k_n^2 = k_0^2$ for any $n \geq N$. This contradicts with the monotone decreasing property. Thus $k_0^2$ could not be an accumulation point, the proof is finished.

From the lemma above, the set $\{k > 0 : P_+ \cap P_- \neq \emptyset\}$ is so ”small” that it is reasonable that we ignore the case that $k$ belongs to it. Thus we make the following assumption.

**Assumption 7.** In this paper, we choose $k$ such that $P_+ \cap P_- = \emptyset$.

With Assumption 7 and 8 when $\alpha \in (-\pi, \pi]$ is an element in $P$, the propagating mode corresponds to $\alpha$ either travels to the left or to the right. This implies that the propagating modes that travels to the left and right are ”separated”. However, Assumption 7 is not a necessary condition to guarantee that LAP holds. The only reason that we make this assumption is to make sure our “simplified representation” for the LAP solution works. At the end of this paper, we also make a comment on the case that Assumption 7 is removed.

4 Variational form of quasi-periodic solutions

From the previous sections, the Bloch wave solutions are crucial for the study of the scattering problems. It is more convenient to study the problems with the help of the variational formulation. In this section,
we consider variational formulation for the following quasi-periodic problems for any $z \in \mathbb{C} \setminus \{0\}$ with right hand side $f_z \in L^2(\Omega_0)$ and depends analytically on $z$:

$$\Delta u_z + k^2 q u_z = f_z \quad \text{in } \Omega_0;$$

$$\frac{\partial u_z}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0. \quad (9)$$

We seek for the solution $u_z$ of (9)-(10) in the domain $H^1_\text{per}(\Omega_0)$. First, we transform the problem into one fixed function space $H^1_\text{per}(\Omega_0)$. Define the operator

$$\langle \zeta_z u_z \rangle (x_1, \tilde{x}) := z^{x_1} u_z(x_1, \tilde{x}),$$

and let $v_z = \zeta_z^{-1} u_z = z^{-x_1} u_z(x_1, \tilde{x})$. Then $v_z \in H^1_\text{per}(\Omega_0)$ is a periodic function that satisfies

$$\Delta v_z + 2 \log(z) \frac{\partial v_z}{\partial x_1} + (k^2 q + \log^2(z))v_z = \zeta_z^{-1} f_z \quad \text{in } \Omega_0;$$

$$\frac{\partial v_z}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0. \quad (10)$$

As $\log(z)$ is a multi-valued function, we require that $z$ lies in the branch cutting off along the negative real axis (denoted by $\mathbb{R}_{\leq} := (-\infty, 0] \times \{0\} \subset \mathbb{C}$) such that $\log(z)$ is a single valued analytic function in this branch. More explicitly, let the branch be defined as $\mathbb{C}_z := \{z \in \mathbb{C} \setminus \{0\} : -\pi < \arg(z) \leq \pi\}$, where $\arg(z)$ is the argument of the complex number $z$. From classical calculation and the Green’s formula, we obtain the variational formulation to the periodic problem, i.e., to find $v_z \in H^1_\text{per}(\Omega_0)$ that satisfies

$$\int_{\Omega_0} \left[ \nabla v_z \cdot \nabla \varphi + \log(z) \left( v_z \frac{\partial \varphi}{\partial x_1} - \frac{\partial v_z}{\partial x_1} \varphi \right) - (k^2 q + \log^2(z))v_z \varphi \right] \, dx = -\int_{\Omega_0} \left( \zeta_z^{-1} f_z \right) \varphi \, dx \quad (11)$$

for any $\varphi \in H^1_\text{per}(\Omega_0)$. The left hand side is a sesquilinear form defined in $H^1_\text{per}(\Omega_0) \times H^1_\text{per}(\Omega_0)$. From Riesz’s lemma, there is a bounded linear operator $\mathcal{K}_z \in \mathcal{L}(H^1_\text{per}(\Omega_0))$ such that

$$\langle \mathcal{K}_z v_z, \varphi \rangle = -\int_{\Omega_0} \left[ \log(z) \left( v_z \frac{\partial \varphi}{\partial x_1} - \frac{\partial v_z}{\partial x_1} \varphi \right) - (k^2 q + 1 + \log^2(z))v_z \varphi \right] \, dx,$$

where $\langle \cdot, \cdot \rangle$ is the inner product defined in $H^1_\text{per}(\Omega_0)$. Moreover, there is a $\tilde{f}_z \in H^1_\text{per}(\Omega_0)$ such that

$$-\int_{\Omega_0} f_z \varphi \, dx = \langle \tilde{f}_z, \varphi \rangle, \quad \text{for all } \varphi \in H^1_\text{per}(\Omega_0). \quad (12)$$

Thus the variational form (11) is equivalent to

$$(I - \mathcal{K}_z)v_z = \zeta_z^{-1} \tilde{f}_z.$$ 

This implies that if $I - \mathcal{K}_z$ is invertible, then

$$u_z = (I - B_z)^{-1} \tilde{f}_z, \quad \text{where } B_z = \zeta_z \mathcal{K}_z \zeta_z^{-1}.\quad$$

Now we focus on the invertibility of the operator of $I - \mathcal{K}_z$. First, $\mathcal{K}_z$ is compact and depends analytically on $z \in \mathbb{C} \setminus \mathbb{R}_{\leq}$. Especially, when $z \in S^1$, the operator $\mathcal{K}_z$ is self-adjoint. Thus the left hand side in (11) defines an analytic family of Fredholm operators $I - \mathcal{K}_z$. First, we recall the following Analytic Fredholm Theory.

**Theorem 8** (Theorem VI.14, [RS80]). Let $D$ be an open connected subset in $\mathbb{C}$, $X$ be a Hilbert space, and $\mathcal{T} : D \rightarrow \mathcal{L}(X)$ be an operator valued analytic function such that $\mathcal{T}(z)$ is compact for each $z \in D$. Then either

- $(I - \mathcal{T}(z))^{-1}$ does not exist for any $z \in D$, or
(I - T(z))^{-1} exists for all \( z \in D \setminus S \), where \( S \) is a discrete subset of \( D \). In this case, \((I - T(z))^{-1}\) is meromorphic in \( D \) and analytic in \( D \setminus S \). The residues at the poles are finite rank operators, and if \( z \in S \), then \( T(z)\varphi = \varphi \) has a nonzero solution in \( X \).

The existence and regularity of the inverse of \( I - K_z \) with respect to \( z \in \mathbb{C}_x \) is concluded in the following theorem.

**Theorem 9.** For any fixed \( k \in \mathbb{C} \) with \( \text{Re}(k) > 0 \) and \( \text{Im}(k) \geq 0 \), \( I - K_z \) is invertible for \( z \in \mathbb{C}_x \setminus \mathbb{F} \). The inverse operator depends analytically on \( z \) in \( \mathbb{C}_x \) and meromorphically on \( z \) in \( \mathbb{C}_x \).

**Proof.** From the fact that \( \mathbb{F} \) is a finite set, \((I - K_z)^{-1}\) exists for all \( z \in \mathbb{S}^1 \) except for the finite set. From the analytic Fredholm theory, there is a discrete set \( S \subset \mathbb{C}_x \) depends on \( k \) such that \( I - K_z \) is invertible. The analytic dependence of \( z \) comes from the perturbation theory.

We only need to prove that \( S = \mathbb{F} \). Suppose there is a \( z \in S \setminus \mathbb{F} \), then with \( f = 0 \), there is a nontrivial solution \( \varphi \in H^1_{\text{per}}(\Omega_0) \) such that \((I - K_z)\varphi = 0\). Then \( \zeta \varphi \in H^1_{\text{per}}(\Omega_0) \) is a nontrivial solution to (19)-(20), thus \( z \in \mathbb{F} \). It contradicts with the assumption that \( z \in S \setminus \mathbb{F} \). Suppose there is a \( z \in \mathbb{F} \setminus S \), then there is a \( \psi \in H^1_{\text{per}}(\Omega_0) \) that satisfies (19)-(20). Then \( \zeta^{-1}\psi \in H^1_{\text{per}}(\Omega_0) \) is a solution to (11) with vanishing right hand side. This implies \( \zeta^{-1}\psi \) satisfies \((I - K_z)(\zeta^{-1}\psi) = 0\), so \( z \in S \), it contradicts with \( z \in \mathbb{F} \setminus S \). Thus \( S = \mathbb{F} \). The proof is finished.

**Remark 10.** As was shown in the second section, \( 0 \notin \mathbb{F} \), thus \( k^2I - A_0 \) is invertible. However, as \( z = 0 \) is the branch point of the logarithm function, the operator \( K_z \) (or equivalently \( B_z \)) is not well defined at this point. Actually, it is the only accumulation point of \( \mathbb{F} \) except for the infinity.

Given any \( f_2 \in L^2(\Omega_0) \), the solution \( u_z = (I - B_z)^{-1}f_2 \). As the definition \( B_z := \zeta_zK_z\zeta^{-1}_z \) and \( I - B_z = \zeta_z(I - K_z)\zeta^{-1}_z \), the singularity of \((I - B_z)^{-1} = \zeta_z(I - K_z)^{-1}\zeta^{-1}_z \) only comes from \((I - K_z)^{-1}\). On the other hand, as the problem \( k^2I - A_z \) has a unique solution when \( z \notin \mathbb{C} \setminus \mathbb{F} \), \( u_z \) is a single-valued function. This implies that the value of \( u_z \) does not depend on the branch where \( z \) lies in. Thus it could be extended to an analytic function in \( \mathbb{C} \setminus (\mathbb{F} \cup \{0\}) \). Then the dependence of the solution \( u_z \) of the problem (19)-(20) on \( z \) could be concluded in the following theorem.

**Theorem 11.** For any \( k > 0 \) and \( f_2 \in L^2(\Omega_0) \) depends analytically on \( z \in \mathbb{C} \), (19)-(20) is uniquely solvable in \( H^1_{\text{per}}(\Omega_0) \). Moreover, \( u_z \) depends analytically on \( z \) in \( \mathbb{C} \setminus (\mathbb{F} \cup \{0\}) \) and meromorphically on \( z \) in \( \mathbb{C} \setminus \{0\} \).

At the end of this section, we discuss the distribution of the set \( \mathbb{F} \). When \( k^2 \in \sigma(A) \), the set \( \mathbb{F} \) is not empty. We define three subsets of \( \mathbb{F} \) from the definition of \( P_\pm \) and \( P_0 \) by:

\[
S^0_\pm := \{ z = \exp(i\alpha) : \alpha \in P_\pm \}, \quad S^0_0 := \{ z = \exp(i\alpha) : \alpha \in P_0 \}.
\]

Then \( \mathbb{F} = S^0_\pm \cup S^0_\pm \cup S^0_0 \). From the definitions of \( P_\pm \), when \( z \in S^0_\pm \), the corresponding propagating Floquet mode is propagating to the right; when \( z \in S^0_\pm \), the corresponding propagating Floquet mode is propagating to the left. See Figure 3 for the unit Floquet multipliers in both \( \alpha \)- and \( z \)-space. The eigenfunction corresponding to \( z \in S^0_\pm \) (resp. \( z \in S^0_\pm \)) is propagating from the left to the right (resp. from the right to the left).

We can also define the subsets of \( \mathbb{F} \setminus \mathbb{F} \) as follows:

\[
RS := \{ z \in \mathbb{F} : |z| < 1 \}; \quad LS := \{ z \in \mathbb{F} : |z| > 1 \}.
\]

The Bloch wave solution corresponds to \( z \in RS \) is evanescent while the one corresponds to \( z \in LS \) is anti-evanescent. Moreover, set

\[
S_+ := S^0_\pm \cup RS, \quad S_- := S^0_\pm \cup LS.
\]

**Remark 12.** We conclude the properties of the sets \( RS, LS \) and \( S^0_\pm \) from the properties of \( \mathbb{F} \) as follows:
From the first property of $\mathbb{F}$ and Lemma 1, the sets $S^0_+$ and $S^0_-$, $RS$ and $LS$ are symmetric, i.e.,

$$z \in S^0_+ \iff z^{-1} = \tau \in S^0_-; \quad z \in RS \iff z^{-1} \in LS. \quad (13)$$

This also implies that

$$z \in S_+ \iff z^{-1} \in S_-.$$  

- When Assumption 5 is satisfied, $S^0_0 = \emptyset$, thus $S^0_+ \cup S^0_- = UF$.
- When Assumption 7 is satisfied, $S^0_+ \cap S^0_- = \emptyset$.
- From $(13)$, if $\pm 1 \in S^0_0$ (or $\pm 1 \in S^0_0$), then $\pm 1 \in S^0_0 \cap S^0_0$. If Assumption 7 is satisfied, $S^0_0 \cap S^0_0 = \emptyset$. Then $\pm 1 \not\in UF \setminus S^0_0$. If Assumption 7 is also satisfied, $\pm 1 \not\in \mathbb{F}$.

Lemma 13. Let $k > 0$. There is a $\tau > 0$ such that $RS \in B(0, \exp(-\tau))$ and $LS \in C \setminus B(0, \exp(\tau))$.

Proof. When $k^2 \not\in \sigma(A)$, the result is obtained simply from the perturbation theory and Heine-Borel theorem. If $k^2 \in \sigma(A)$, $UF \neq \emptyset$ is a finite set. Let $UF = \{z_1, \ldots, z_N\}$. As the set $\mathbb{F}$ is discrete, for any $n = 1, \ldots, N$, there is a $\delta_n > 0$ such that $B(z_n, \delta_n) \cap \mathbb{F} = \{z_n\}$. From the perturbation theory, for any $z \in S^1 \setminus UF$, there is a $\delta_z > 0$ such that $B(z, \delta_z) \cap \mathbb{F} = \emptyset$. As $S^1 \subset \bigcup_{n=1}^N B(z_n, \delta_n) \cup \bigcup_{z \in S^1 \setminus UF} B(z, \delta_z)$, from Heine-Borel theorem, there is finite number of $z$'s on $S^1 \setminus UF$, denoted by $z_{n+1}, \ldots, z_M$ such that $S^1 \subset \bigcup_{n=1}^N B(z_n, \delta_n) \cup \bigcup_{z_{n+1}}^M B(z, \delta_z)$. Let $\tau > 0$ be a small enough number such that

$$B(0, \exp(\tau)) \setminus B(0, \exp(-\tau)) \subset \bigcup_{n=1}^N B(z_n, \delta_n) \cup \bigcup_{z_{n+1}}^M B(z, \delta_z),$$

then $(RS \cup LS) \cap T_\tau = \emptyset$. Thus $RS \in B(0, \exp(-\tau))$ and $LS \in C \setminus B(0, \exp(\tau))$. The proof is finished.

5 Floquet-Bloch transform and its application

5.1 Floquet-Bloch transform

The Floquet-Bloch transform is a very important tool in the analysis of scattering problems in PDEs in periodic structures, see [KL18a, KL18b, FJ15]. The most popular form of the Floquet-Bloch transform appears in papers is based on the Fourier series (see [Lec17, KL18a, KL18b, FJ15]). However, in the book [Kuc93], the transform is defined based on the Laurent series and we follow this definition in this paper. First, we recall the definition of the Laurent series:
Definition 14. Suppose \( \{x_n\}_{n=-\infty}^{\infty} \) is a series in \( \ell^\infty \). Then the Laurent series is defined as

\[
X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}.
\] (14)

Define the Region of Convergence (ROC) as the domain in \( \mathbb{C} \) such that the series (14) converges and is an analytic function, the coefficients are obtained by

\[
x_n = \frac{1}{2\pi i} \oint_C X(z)z^{n-1} \, dz,
\] (15)

where \( C \) is a counterclockwise closed path encircling the origin lies in ROC.

Then we extend the Laurent series to the periodic domain, i.e., for a function \( u \) defined in \( \Omega = \cup_{n \in \mathbb{Z}} \Omega_n \), define the series

\[
(F\varphi)(z, x) := \sum_{n=-\infty}^{\infty} \varphi(x_1 + n, \tilde{x})z^{-n}, \quad x \in \Omega_0, \, z \in \mathbb{C}.
\] (16)

When \( \varphi \) decays exponentially with the rate \( \gamma \), i.e., there is a \( \gamma > 0 \) and \( C > 0 \) such that \( u \) satisfies

\[
|\varphi(x_1, \tilde{x})| \leq C \exp(-\gamma|x_1|), \quad \forall \tilde{x} \in \Sigma,
\] (17)

the ROC is an annulus

\[
T_\gamma = \{ z \in \mathbb{C} : \exp(-\gamma) < |z| < \exp(\gamma) \}.
\]

Moreover, the function \((F\varphi)(z, x)\) depends analytically on \( z \in T_\gamma \). It is also easy to check that the transformed function \((F\varphi)(z, \cdot)\) is quasi-periodic (i.e., it satisfies (5)). We conclude the mapping properties of the operator \( F \) in Sobolev spaces when \( z \in S^1 \) the following proposition.

Proposition 15. The operator \( F \) has the following properties when \( z \) lies on the unit circle \( S^1 \) (see [Lec17][Kuc16]):

- \( F \) is an isomorphism between \( H^s(\Omega) \) and \( L^2(S^1; H^s(\Omega_0)) \) (where \( s \in \mathbb{R} \), where

\[
L^2(S^1; H^s(\Omega_0)) := \{ \varphi \in D'(S^1 \times \Omega_0) : \left[ \int_{S^1} \|\varphi(z, \cdot)\|_{H^s(\Omega_0)}^2 \, dz \right]^{1/2} < \infty \}.
\]

- \( F\varphi \) depends analytically on \( z \in T_\gamma \), if and only if \( \varphi \) decays exponentially at the infinity with the rate \( \gamma \).

Remark 16. Suppose the function \( \varphi(z, x) \) satisfies \( \varphi(z, \cdot) \in S(X) \) for any fixed \( z \) (where \( S(X) \) is a Sobolev space defined in the domain \( X \)). Then \( \varphi \) depends analytically on \( z \) in an open domain \( U \subset \mathbb{C} \) if for any fixed \( z_0 \in U \), the expansion

\[
\varphi(z, x) = \sum_{\ell=0}^{\infty} (z - z_0)^\ell \varphi_\ell(x), \quad \varphi_\ell \in S(X)
\]

holds uniformly in \( B(z_0, \delta) \) for a small enough \( \delta > 0 \).

When the variable \( z \) is fixed on \( S^1 \), we replace \( z \) by \( \exp(i\alpha) \) with \( \alpha \in (-\pi, \pi] \), the \( F \) has the form of the Floquet-Bloch transform defined in [Lec17][KL13][KLi16][FJ15]. Let \((J\varphi)(\alpha, x)\) be the transform defined by

\[
(J\varphi)(\alpha, x) := \sum_{j \in \mathbb{Z}} \varphi(x_1 + n, \tilde{x}) \exp(-i\alpha n), \quad x \in \Omega_0, \, \alpha \in (-\pi, \pi].
\]

Then its inverse transform has the following representation (see [Lec17][Kuc16])

\[
(J^{-1}\psi)(x_1 + n, \tilde{x}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(\alpha, x) \exp(i\alpha n) \, d\alpha, \quad x \in \Omega_0, \, n \in \mathbb{Z}.
\]
Note that if we replace \( \exp(i\alpha) \) by \( z \), then
\[
(F^{-1}\psi)(x_1 + n, \bar{\sigma}) = \frac{1}{2\pi i} \int_{\gamma} \psi(z, x) z^n (-iz^{-1}) \, dz
= \frac{1}{2\pi i} \int_{\gamma} \psi(z, x) z^{n-1} \, dz.
\]

With this result, we can obtain the following result when \( \psi(z, x) \) depends analytically on \( z \) in a neighbourhood of \( S^1 \).

**Theorem 17.** Given \( \psi(z, x) := (F\varphi)(z, x) \) for some \( \varphi \in H^s(\Omega) \) and satisfies (17) for some \( \gamma > 0 \), the inverse operator \( F \) is given by:
\[
(F^{-1}\psi)(x_1 + n, \bar{\sigma}) = \frac{1}{2\pi i} \int_{\gamma} \psi(z, x) z^{n-1} \, dz,
\]
where \( \gamma \) is a counter-clockwise closed path encircling the origin lies in the \( \text{ROC} \) \( T_\gamma \). Moreover, when \( \gamma \) is the unit circle \( S^1 \), the inverse transform has exactly the same form with \( F^{-1} \).

**Remark 18.** Note that the form of the inverse Floquet-Bloch transform \( F \) has the same form of (15).

### 5.2 Application of the Floquet-Bloch transform: when \( k^2 \notin \sigma(A) \)

In this section, we apply the Floquet-Bloch transform \( F \) to the scattering problem (11)-(2), when \( k^2 \notin \sigma(A) \). The following cases are what we are particularly interested in:

- \( k^2 \notin \sigma(A) \), i.e., \( k^2 \) lies in a stop band; or
- \( k^2 \) is replaced by \( k^2_\varepsilon := k^2 + i\varepsilon \) for some fixed \( \varepsilon > 0 \).

When either of the two conditions is satisfied, the problem (11)-(2) is uniquely solvable (see [15]).

As \( f \) is compactly supported and \( k^2 \notin \sigma(A) \), the problem (11)-(2) has a unique solution \( u \in H^1(\Omega) \). Moreover, \( u \) decays exponentially at the infinity, i.e., \( u \) satisfies (17) for some \( C > 0 \) and \( \gamma > 0 \). We define the Floquet-Bloch transform \( w(z, \cdot) := (Fu)(z, \cdot) \), then the transformed field is well-defined and depends analytically on \( z \) in \( T_\gamma \). It is also easy to check that for all \( z \in T_\gamma \), and for any fixed \( z \in T_\gamma \), \( w(z, \cdot) \in H^1_2(\Omega_0) \) satisfies
\[
\Delta w(z, \cdot) + k^2 q w(z, \cdot) = f \quad \text{in } \Omega_0;
\]
\[
\frac{\partial w(z, \cdot)}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0.
\]

Note that the source term comes from \( F(z, x) = f(x) \) for any \( z \in T_\gamma \).

From the result of the inverse Floquet-Bloch transform, we could represent the solution of the original problem as:
\[
u(x_1 + n, \bar{\sigma}) = (F^{-1}u)(x_1 + n, \bar{\sigma}) = \frac{1}{2\pi i} \int_{\gamma} w(z, x) z^{n-1} \, dz
= \frac{1}{2\pi i} \int_{\gamma} w(z, x) z^{n-1} \, dz,
\]
where \( \gamma \) is a curve encircling 0 and lies in \( T_\gamma \).

From the representation of the solution in Theorem 20, the solution \( u \) of (11)-(2) could be obtained from the contour integration of its Floquet-Bloch transformed field \( w(z, \cdot) \), which is a family of quasi-periodic solutions of cell problems (19)-(20). On the other hand, from Theorem 11 when \( z \in \mathbb{C} \setminus F \), the problem (19)-(20) is uniquely solvable in \( H^1_2(\Omega_0) \), and \( w(z, \cdot) \) depends analytically on \( z \in \mathbb{C} \setminus (F \cup \{0\}) \) and meromorphically on \( z \in \mathbb{C} \setminus \{0\} \). Thus \( w(z, \cdot) \) is extended meromorphically outside the ROC of the Floquet-Bloch transform of \( u \). Thus we could completely forget the Floquet-Bloch transform process and only focus on the relationship between the waveguide problem (11)-(2) and the cell problem (19)-(20). Moreover, as \( k^2 \notin \sigma(A) \), \( \cup F = \emptyset \). It is easy to check that \( F \cap T_\gamma = \emptyset \). From the analytic continuation, we could obtain the following result.
Theorem 19. When $k^2 \notin \sigma(A)$, the Floquet-Bloch transform $(Fu)(z,x)$ could be extended to an analytic function in $C \setminus (F \cup \{0\})$ by the solution $w(z,x)$ of (9)-(10). Moreover, the function is meromorphic in $C \setminus \{0\}$ with poles at $F$.

From the Cauchy integral theorem, we obtain the following integral representation of the solution $u$.

Theorem 20. Suppose $k^2 \notin \sigma(A)$. Suppose $w(z,\cdot)$ is the solution of (9)-(10) for $z \in C \setminus F$. Then the solution of (1)-(2) could be written as

$$u(x_1 + n, \tilde{x}) = (F^{-1}w)(x_1 + n, \tilde{x}) = \frac{1}{2\pi i} \oint_C w(z, x)z^{n-1}dz,$$

where $C \subset \mathbb{C}$ is a counter-clockwise closed path encircling all the points in $RS$ and not encircling any point in $LS$.

Remark 21. In this section, we build up a relationship between the unbounded problem (1)-(2) and the cell problem (9)-(10). Although the Floquet-Bloch transform only exists in $T_\sigma$, the curve $C$ could be chosen such that some parts of $C$ lies in the exterior of $T_\sigma$. This implies that we could consider the problems (1)-(2) and (9)-(10) separately, i.e., the problem (9)-(10) is not necessarily the Floquet-Bloch transform of (1)-(2) but an independent problem.

In this section, we have applied the Floquet-Bloch transform to the case that $k^2 \notin \sigma(A)$, and obtain the integral representation of the solution in Theorem 20. In the next section, we will focus on the more complicated case that $k^2 \in \sigma(A)$.

6 Limiting absorption principle

In this section, we consider the case that the scattering problem (1)-(2) has propagating modes, i.e., $k^2 \in \sigma(A)$. To this end, we consider the limiting absorption principle. First we have to consider the damped Helmholtz equation (3)-(4), i.e., to replace $k^2$ by $k^2 + i\varepsilon$ for any $\varepsilon > 0$. The corresponding quasi-periodic problems are formulated as:

$$\Delta w_\varepsilon(z,\cdot) + (k^2 + i\varepsilon)qw_\varepsilon(z,\cdot) = f \quad \text{in } \Omega_0;$$

$$\frac{\partial w_\varepsilon(z,\cdot)}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0. \quad (22)$$

Similar to the definition of $K_\varepsilon$ and $B_\varepsilon$, we denote the operators with $k^2 + i\varepsilon$ by $K^\varepsilon$ and $B^\varepsilon$. From [Kat95], the poles of the operator $(I - B_\varepsilon)^{-1}$ depends continuously on $\varepsilon$. First, we study the asymptotic behavior of distributions of the poles when $\varepsilon > 0$ is small enough.

6.1 Distribution of poles of the damped Helmholtz equations

From the result of the Floquet-Bloch theory (see [Kue93,FJ15]), $k^2 \in \sigma(A)$ implies that $UF \neq \emptyset$, i.e., there is at least one $z \in UF$ such that $k^2$ is an eigenvalue of $A_z$. As both of $S^+_0$ and $S^-_0$ are finite, and the sets $S^+_0$ and $S^-_0$, $RS$ and $LS$ are symmetric in the sense of (13), they could be written as

$$S^+_0 = \{z^+_1, \ldots, z^+_N\}; \quad RS = \{z^+_N+1, z^+_N+2, \ldots\};$$

$$S^-_0 = \{z^-_1, \ldots, z^-_N\}; \quad LS = \{z^-_{N+1}, z^-_{N+2}, \ldots\};$$

where $z^+_j = \bar{z}^-_j = (z^-_j)^{-1}$ for any integer $j = 1, \ldots, N$, and $z^+_j = (z^-_j)^{-1}$ for any integer $j \geq N + 1$. From Assumption 5 and $F = S^+_0 \cup S^-_0 \cup RS \cup LS$ and $S^+_0 \cap S^-_0 = \emptyset$.

Remark 22. In this section, we always require that Assumption 5 and 7 are satisfied. Actually, Assumption 7 is not always necessary for LAP. In [FJ15] and [KL18b], LAP has been proved in different ways without this assumption (maybe with other assumptions anyway). However, as we aim to obtain the “clear” integral representation, we have to make this assumption. If one considers the case that $S^+_0 \cap S^-_0 \neq \emptyset$, one may either refer to the two references, or find the answer by the end of this paper.
From the continuous dependence of poles, for any $z_j^\pm \in \mathbb{R}$ with $j \in \mathbb{N}$, there is a continuous function $Z_j^\pm(\varepsilon)$, such that $\{Z_j^+(\varepsilon), Z_j^-(\varepsilon) : j \in \mathbb{N}\}$ are exactly the set of all poles with respect to $k^2 + i\varepsilon$. Moreover, $\lim_{\varepsilon \to 0} Z_j^+(\varepsilon) = Z_j^+(0) = z_j^\pm$. For simplicity, we define the sets

$$
S_0^+(\varepsilon) = \{Z_1^+(\varepsilon), \ldots, Z_N^+(\varepsilon)\}; \quad RS(\varepsilon) = \{Z_{N+1}^+(\varepsilon), Z_{N+2}^+(\varepsilon), \ldots\};
$$

$$
S_0^-(\varepsilon) = \{Z_1^-(\varepsilon), \ldots, Z_N^-(\varepsilon)\}; \quad LS(\varepsilon) = \{Z_{N+1}^-(\varepsilon), Z_{N+2}^-(\varepsilon), \ldots\}.
$$

The following lemma shows the asymptotic behaviour of $|Z_j^\pm(\varepsilon)|$ as $\varepsilon \to 0$ where $j = 1, 2, \ldots, N$, i.e., $Z_j^\pm(\varepsilon) \in S_0^\pm(\varepsilon)$. For simplicity, let

$$
S_+(\varepsilon) := S_0^+(\varepsilon) \cup RS(\varepsilon), \quad S_-(\varepsilon) = S_0^-(\varepsilon) \cup LS(\varepsilon).
$$

**Lemma 23.** For any $j = 1, 2, \ldots, N$, when $\varepsilon > 0$ is small enough, the functions satisfy $|Z_j^+(\varepsilon)| < 1$ and $|Z_j^-(\varepsilon)| > 1$.

**Proof.** We follow the idea from the appendix in [JLF06]. First consider $Z_j^+(\varepsilon)$, then $z_j^+ = Z_j^+(0)$ and $z_j^+ = \exp(i\alpha_j)$ for an $\alpha_j \in P_+$. Then there is a function $\alpha(\varepsilon) = -i \log(Z_j^+(\varepsilon))$ which is differentiable with respect to $\varepsilon$. Thus

$$
Z_j^+(\varepsilon) = \exp(i\alpha(\varepsilon)), \quad \lim_{\varepsilon \to 0} \alpha(\varepsilon) = \alpha_j \text{ for an } \alpha_j \in P_+.
$$

From the perturbation theory, the function $\mu_n$ could be extended to an analytic function for $\alpha$ in a small enough neighbourhood of $(-\pi, \pi) \times \{0\}$, then there is an integer $n$ such that

$$
k^2 + i\varepsilon = \mu_n(\alpha(\varepsilon)), \quad \text{for small enough } \varepsilon > 0.
$$

Differentiate the two functions $Z_j^+$ and $\mu_n$, and let $\varepsilon = 0$, then

$$
(Z_j^+)'(0) = i\alpha'(0) \exp(i\alpha_j); \quad \mu_n'(\alpha_j)\alpha'(0) = i.
$$

As $\alpha_j \in P_+$, then $\mu_n'(\alpha_j) > 0$. Thus $\alpha'(0) \neq 0$ and

$$
(Z_j^+)'(0)Z_j^+(0) = -\mu_n'(\alpha_0)^{-1} < 0.
$$

This means that

$$
\left. \frac{d}{d\varepsilon} |Z_j^+(\varepsilon)|^2 \right|_{\varepsilon = 0} < 0.
$$

Thus for $\varepsilon > 0$ small enough, $|Z_j^+(\varepsilon)|^2 < |z(0)|^2 = 1$. The case for $Z_j^-$ is proved in the same way, thus is omitted here. 

From this lemma, we have found out the behaviour of the curves $Z_j^\pm(\varepsilon)$ for small enough $\varepsilon$:

- for any $z_j^+ \in S_0^+$, the points $Z_j^+(\varepsilon)$ converges to $z$ on the unit circle from the inside;

- for any $z_j^- \in S_0^-$, the points $Z_j^-(\varepsilon)$ converges to $z$ on the unit circle from the outside.

To make it clear, we show a visualization of the curves example of the curves in Figure 4. The red rectangles are points in $S_0^+$ and the red dimonds are points in $S_0^-$. When $\varepsilon > 0$, we can see that the curve $Z_j^-(\varepsilon)$ converges to $z_j^-$ from the exterior of the unit circle, and $Z_j^+(\varepsilon)$ converges to $z_j^+$ from the interior of the unit circle.

For the points in $RS(\varepsilon)$ or $LS(\varepsilon)$, we could also estimate their distributions for small enough $\varepsilon > 0$ in the following lemma.

**Lemma 24.** Suppose $RS \subset B(0, \exp(-\tau))$ for some $\tau > 0$ and $LS \subset \mathbb{C} \setminus \overline{B(0, \exp(\tau))}$. When $\varepsilon > 0$ is small enough, $Z_j^+(\varepsilon) \in B(0, \exp(-\tau_1))$ and $Z_j^-(\varepsilon) \in \mathbb{C} \setminus \overline{B(0, \exp(\tau_1))}$ for any $j \geq N + 1$, where $\tau_1$ takes any fixed value in $(0, \tau)$. 

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Proof. Suppose that there is a monotonically decreasing sequence \( \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots > 0 \) with \( \lim_{n \to \infty} \varepsilon_n = 0 \), such that for any \( n \in \mathbb{N} \), there is an integer \( j_n \geq N + 1 \) such that \( Z_{j_n}^+(\varepsilon_n) \) lies in \( \mathbb{C} \setminus B(0, \exp(-\tau_1)) \). For each \( j_n \), \( Z_{j_n}^+(\varepsilon) \) is a continuous function with respect to \( \varepsilon \). As \( Z_{j_n}^+(0) = z_j^+ \) lies in \( B(0, \exp(-\tau_1)) \), there is at least one \( \varepsilon_n^* \in (0, \varepsilon_n) \) such that \( |Z_{j_n}^+(\varepsilon_n^*)| = \exp(-\gamma_2) \), where \( \gamma_2 = (\gamma_1, \gamma) \). Then \( \lim_{n \to \infty} \varepsilon_n^* = 0 \).

As there are no poles on \( \partial B(0, \exp(-\tau_1)) \), for any \( z_0 \in \partial B(0, \exp(-\tau_1)) \), there is a \( \varepsilon_0 > 0 \) and \( c_0 > 0 \) such that \( I - B_c^z \) is invertible when \( z \in B(z_0, \varepsilon_0) \) and \( \varepsilon < (c_0, c_0) \). As \( \partial B(0, \exp(-\tau_1)) \subset \cup_{z_0 \in \partial B(0, \exp(-\tau_1))} B(z_0, \varepsilon_0) \), from Heine-Borel theorem, there are finite number of \( z_0 \)’s, i.e., \( z_1, z_2, \ldots, z_M \) such that \( \partial B(0, \exp(-\tau_1)) \subset \bigcup_{m=1}^{M} B(z_m, \varepsilon_0) \). Let \( \varepsilon^* = \min\{c_{z_1}, \ldots, c_{z_M} \} \), then \( I - B_c^z \) is invertible when \( z \in \partial B(0, \exp(-\tau_1)) \) and \( \varepsilon \in (\varepsilon^*, \varepsilon^*) \). As \( \lim_{n \to \infty} \varepsilon_n^* = 0 \), when \( n \in \mathbb{N} \) is large enough, \( \varepsilon_n^* \in (0, \varepsilon^*) \).

However, \( Z_{j_n}^+(\varepsilon_n^*) \) is a pole of \( I - B_c^z \), which contradicts with the result that \( I - B_c^z \) is invertible with \( z \in \partial B(0, \exp(-\tau_1)) \) and \( \varepsilon \in (\varepsilon^*, \varepsilon^*) \). Thus when \( \varepsilon > 0 \) is small enough, \( Z_{j_n}^+(\varepsilon) \in B(0, \exp(-\tau_1)) \). With the same technique, we can also prove that \( Z_j^+(\varepsilon) \in \mathbb{C} \setminus \overline{B(0, \exp(\tau_1))} \). The proof is finished.

From Lemma 23 and 24, for \( \varepsilon > 0 \) small enough, the sets \( S_+^0(\varepsilon) \) and \( RS(\varepsilon) \) have the following properties:

\[
S_+^0(\varepsilon) = S_+^0(\varepsilon) \cup RS(\varepsilon) \subset B(0, 1); \quad S_-(\varepsilon) = S_0^0(\varepsilon) \cup LS(\varepsilon) \subset \mathbb{C} \setminus B(0, 1).
\]

Moreover, any point in \( S_+^0(\varepsilon) \) approaches the unit circle from the interior of \( B(0, 1) \), and any point in \( S_0^0(\varepsilon) \) from the exterior of \( B(0, 1) \). When \( \varepsilon \) is small enough, \( RS(\varepsilon) \) lies in a ball with center 0 and radius smaller than one, while \( LS(\varepsilon) \) lies in the exterior of a larger ball with center 0 and radius larger than one.

### 6.2 Integral representation of the LAP solution

Now we are prepared to consider the LAP solution of (11)-(2) when \( k^2 \in \sigma(A) \). From Theorem 26, the solution \( u_\varepsilon (\forall \varepsilon > 0) \) of the damped problem (3)-(4) could be represented by the curve integral:

\[
u_\varepsilon(x_1 + n, \bar{x}) = \frac{1}{2\pi i} \oint_{\Gamma^1} w_\varepsilon(z, x) z^{n-1} dz.
\]

But when \( \varepsilon \to 0^+ \), the poles of \( w_\varepsilon(z, \cdot) \) approach \( S_1 \), thus the integral becomes irregular. From Theorem 20, we are aimed to find out a proper curve \( \mathcal{C} \) to replace the unit circle such that the function \( w_\varepsilon(z, x) \) is well-posed and converges uniformly when \( \varepsilon \to 0 \).

From the properties of the sets \( S_+^0(\varepsilon) \), \( RS(\varepsilon) \) and \( LS(\varepsilon) \), we are able to replace \( S_1 \) by a proper curve \( \mathcal{C}_0 \). The following is the method to define \( \mathcal{C}_0 \).

**Definition 25.** Let the piecewise analytic curve \( \mathcal{C}_0 \) be defined by the boundary of the following domain:

\[
\mathcal{B}_0 := B(0, 1) \cup \left[ \bigcup_{n=1}^N B(z_j^+, \delta) \right] \setminus \left[ \bigcup_{n=1}^N B(z_j^-, \delta) \right].
\]

The parameter \( \delta > 0 \) is chosen so that the following conditions are satisfied:

- The intersection of any two balls \( B(z_j^+, \delta) \) and \( B(z_\ell^+, \delta) \) when \( j \neq \ell \) is empty.
- \( \left[ \bigcup_{n=1}^N B(z_j^+, \delta) \right] \cup \left[ \bigcup_{n=1}^N B(z_j^-, \delta) \right] \subset T^r_\varepsilon \), i.e., the balls do not contain any point in \( RS \cup LS \).

We refer to Figure 2 for a visualization of the choice of \( \mathcal{C}_0 \) for the example when \( n = 1 \), \( k^2 = 3\pi^2 \).

**Remark 26.** We are able to find the parameter \( \delta \) such that the first condition is satisfied under Assumption 2 while the second condition is satisfied from Lemma 15.

Thus from Lemma 23 and 24, there is a \( C = C(\delta) > 0 \) such that

\[
dist (\mathcal{C}_0, RS(\varepsilon) \cup LS(\varepsilon) \cup S_+^0(\varepsilon) \cup S_0^0(\varepsilon)) > C(\delta) \quad \text{uniformly for small enough } \varepsilon > 0.
\]

From the choice of the curve \( \mathcal{C}_0 \), the interior of the symmetric difference of \( \mathcal{B}_0 \) and \( B(0, 1) \) is

\[
\left[ \bigcup_{n=1}^N \left( B(z_j^+ \setminus \overline{B(0, 1)}) \right) \right] \cup \left[ \bigcup_{n=1}^N \left( B(z_j^- \cap B(0, 1)) \right) \right].
\]
As for small enough $\varepsilon > 0$, $(I - B_{\varepsilon}^z)^{-1}$ has no poles in this domain, then from Cauchy integral theorem, $u_\varepsilon$ has the equivalent formulation

$$u_\varepsilon(x_1 + n, \tilde{x}) = \frac{1}{2\pi i} \oint_{C_0} w_\varepsilon(z, x) z^{n-1} \, dz.$$ 

**Figure 4:** Left: the curve $C_0$; Right: a choice of $C$. Black curves: $Z_{\pm}^\varepsilon$. The red rectangles are points in $S_{0}^\varepsilon$ and the blue diamonds are points in $S_{0}^\varepsilon$. The arrows show the direction of the poles when $\varepsilon \to 0$.

**Theorem 27.** When $\varepsilon > 0$ is small enough, the function $w_\varepsilon(z, x)$ depends piecewise analytically on $z$ in $C_0$ and is uniformly bounded with respect to $\varepsilon$. Moreover,

$$\lim_{\varepsilon \to 0^+} w_\varepsilon(z, \cdot) = w(z, \cdot) \quad \text{in } H^1(\Omega_0)$$

uniformly for $z$.

**Proof.** From the choice of $C_0$, $I - B_z$ is invertible for any $z \in C_0$ and the $w(z, \cdot)$ is uniformly bounded in $H^1(\Omega_0)$. From the perturbation theory and the Heine-Borel theorem, $I - B_{\varepsilon}^z$ is invertible for any $z \in C_0$ when $\varepsilon$ is small enough. The inverse operator is uniformly bounded due to the perturbation theory. Moreover, as $C_0$ is piecewise analytic, the function $w(z, \cdot)$ is piecewise analytic as well. The limit of $w_\varepsilon(z, \cdot)$ is proved by the continuity with respect to $\varepsilon$. The proof is finished.

With the above result, we can easily obtain the following integral representation of the solution $u$ of (1)-(2) from the limiting absorption principle.

**Theorem 28.** Suppose Assumption \(A_\varepsilon\) and \(\Phi\) are satisfied. $C_0$ is chosen as is shown above. Given any $f \in L^2(\Omega_0)$ and compactly supported and $u_\varepsilon \in H^1(\Omega)$ is the unique solution of (3)-(4). Then

$$\lim_{\varepsilon \to 0^+} u_\varepsilon = u,$$

where the LAP solution $u$ has the integral representation

$$u(x_1 + n, \tilde{x}) = \frac{1}{2\pi i} \oint_{C_0} w(z, x) z^{n-1} \, dz.$$  

Moreover, for any $n \in \mathbb{Z}$, there is a constant $C = C(n) > 0$ such that

$$\|u\|_{H^1(\Omega_n)} \leq C \|f\|_{L^2(\Omega_0)}.$$ 

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Proof. From Lemma 27, \( w_\varepsilon(z, \cdot) \) is uniformly bounded with respect to \( \varepsilon \) and converges to \( w(z, \cdot) \). Then from the Lebesgue’s Dominated Convergence theorem, we can easily exchange the limit and integral, thus \( 24 \) is proved. From the uniform boundedness of the operator \((I - B_\varepsilon)^{-1}\), the functions \( w(z, \cdot) \) are also uniformly bounded in the sense of \( H^1(\Omega_0) \). Thus \( 25 \) is proved for any fixed \( n \in \mathbb{Z} \). The proof is finished.

We can also replace \( C_0 \) by any closed curve that lies in the neighbourhood of \( C_0 \) and enclose zero and all poles in \( S_+ \) (see Figure 4). The left curve is \( C_0 \), and the right is a choice of \( C \). Thus

\[
u(x_1 + n, \bar{x}) = \frac{1}{2\pi i} \oint_C w(z, x) z^{n-1} \, dz.
\]

(26)

Now we have already withdrawn the LAP process and formulate the LAP solution directly from solutions of cell problems \( 19-20 \). This also provide a nice and clear formulation for the numerical scheme without the approximation by the solutions of damped Helmholtz equations. However, we still need to know the dispersion diagram to decide the curve \( C_0 \) or \( C \). In the rest of this section, we study some properties of the LAP solutions.

6.3 Properties of LAP solutions

Denote the LAP solution of \( 1-2 \) with source \( f \in L^2(\Omega_0) \) by \( u(f) \). From previous analysis, it has the following representation:

\[
u(f)(x_1 + n, \bar{x}) = \frac{1}{2\pi i} \oint_C (I - B_\varepsilon)^{-1} z^{n-1} \, dz, \quad x \in \Omega_0.
\]

Let \( u_\varepsilon(f) \in H^1(\Omega) \) be the solution \( 33-41 \) with some \( \varepsilon > 0 \), and \( u(f) \) be the LAP solution. Then from Theorem 28

\[
u(f) = \lim_{\varepsilon \to 0} u_\varepsilon(f) \quad \text{in } H^1_{\text{loc}}(\Omega_0).
\]

Define the set

\[ \mathcal{U} := \left\{ u(f) \mid_{\Omega_0} : u(f) \in H^1_{\text{loc}}(\Omega) \text{ is the LAP solution of } 1-2 \text{ with source } f \in L^2(\Omega_0) \right\}. \]

To study the properties of \( \mathcal{U} \), we have to introduce the following integrals first. Define

\[
I_1(f, g) := \int_{\Gamma_1} \left[ \frac{\partial u(g)}{\partial x_1} u(f) - u(g) \frac{\partial u(f)}{\partial x_1} \right] \, ds;
\]

\[
I_0(f, g) := \int_{\Gamma_0} \left[ \frac{\partial u(g)}{\partial x_1} u(f) - u(g) \frac{\partial u(f)}{\partial x_1} \right] \, ds,
\]

where \( g \in L^2(\Omega_0) \) with its compact support in \( \Omega_0 \). We can also define the operators that depend on \( \varepsilon \):

\[
I_1^\varepsilon(f, g) := \int_{\Gamma_1} \left[ \frac{\partial u_\varepsilon(g)}{\partial x_1} u_\varepsilon(f) - u_\varepsilon(g) \frac{\partial u_\varepsilon(f)}{\partial x_1} \right] \, ds;
\]

\[
I_0^\varepsilon(f, g) := \int_{\Gamma_0} \left[ \frac{\partial u_\varepsilon(g)}{\partial x_1} u_\varepsilon(f) - u_\varepsilon(g) \frac{\partial u_\varepsilon(f)}{\partial x_1} \right] \, ds.
\]

Lemma 29. Suppose \( u(f) \) and \( u(g) \) be LAP solutions of \( 1-2 \) with source \( f, g \in L^2(\Omega_0) \), then \( I_0(f, g) = I_1(f, g) = 0 \).

Proof. We first consider integral \( I_1^\varepsilon(f, g) \). As both the solutions \( u_\varepsilon(f) \) and \( u_\varepsilon(g) \) satisfy the Helmholtz equation \( \Delta u + (k^2 + i\varepsilon) u = 0 \) in \( \Omega_1 \), use the Green’s formula,

\[
0 = \int_{\Omega_1} [u_\varepsilon(f) \Delta u_\varepsilon(g) - u_\varepsilon(g) \Delta u_\varepsilon(f)] \, dx
\]

\[
= \left( \int_{\Gamma_2} - \int_{\Gamma_1} \right) \left[ \frac{\partial u_\varepsilon(g)}{\partial x_1} u_\varepsilon(f) - u_\varepsilon(g) \frac{\partial u_\varepsilon(f)}{\partial x_1} \right] \, ds
\]

\[
= \int_{\Gamma_2} \left[ \frac{\partial u_\varepsilon(g)}{\partial x_1} u_\varepsilon(f) - u_\varepsilon(g) \frac{\partial u_\varepsilon(f)}{\partial x_1} \right] \, ds - I_1^\varepsilon(f, g).
\]
Thus $I^i_1(f,g) = \int_{\Gamma_n} \left[ \frac{\partial u_\epsilon(g)}{\partial x_1} - u_\epsilon'(g) \frac{\partial u_\epsilon}{\partial x_1} \right] ds$. Use the Green's formula repeatedly in $\Omega_2, \ldots, \Omega_{n-1}$, then

$$I^i_1(f,g) = \int_{\Gamma_n} \left[ \frac{\partial u_\epsilon(g)}{\partial x_1} - u_\epsilon'(g) \frac{\partial u_\epsilon}{\partial x_1} \right] ds$$

for any integer $n \geq 2$. With the result of Theorem 3.10, [CC09], there is a $C > 0$ independent of $n$ such that

$$|I^i_1(f,g)| \leq C \|u_\epsilon\|_{H^1(\Omega_n)} \|v_\epsilon\|_{H^1(\Omega_n)}.$$ 

From the exponential decay of both functions $u_\epsilon(f)$ and $u_\epsilon(g)$, $I^i_1(f,g) = 0$. As $u(f)$ and $u(g)$ are LAP solutions,

$$I_1(f,g) = \lim_{\epsilon \to 0} \int_{\Gamma_2} \left[ \frac{\partial u_\epsilon(g)}{\partial x_1} - u_\epsilon'(g) \frac{\partial u_\epsilon}{\partial x_1} \right] ds = 0.$$

We can prove with the same technique that $I^i_0(f,g) = I_0(f,g) = 0$. The proof is finished. \qed

Then we are prepared to prove the density of the space $U$ in the following lemma.

**Lemma 30.** The space $U$ is dense in $H^1(\Omega_0)$.

**Proof.** Suppose the set $U$ is not dense in $H^1(\Omega_0)$, i.e., $\overline{U} \neq H^1(\Omega_0)$, then $U^\perp \neq \emptyset$. There is a non-zero $\varphi \in C^\infty(\Omega_0)$ such that

$$\int_{\Omega_0} u(f) \overline{\varphi} \, dx = 0, \quad \forall f \in L^2(\Omega_0).$$

Let $u(\overline{\varphi})$ be the solution of (33)-(41) with the source $\overline{\varphi}$, then use the result in Lemma 25 and Green’s formula,

$$0 = \int_{\Omega_0} u(f) \overline{\varphi} \, dx = \int_{\Omega_0} u(f) \left[ \Delta u(\overline{\varphi}) + k^2 q u(\overline{\varphi}) \right] \, dx$$

$$= \int_{\Omega_0} u(\overline{\varphi}) \left[ \Delta u(f) + k^2 q u(f) \right] \, dx + \left( \int_{\Gamma_1} - \int_{\Gamma_0} \right) \left[ \frac{\partial u(\overline{\varphi})}{\partial \nu} u(f) - u(\overline{\varphi}) \frac{\partial u(f)}{\partial \nu} \right] \, ds$$

$$= \int_{\Omega_0} u(\overline{\varphi}) f \, dx + I_1(f,\varphi) - I_0(f,\varphi) = \int_{\Omega_0} u(\overline{\varphi}) f \, dx.$$ 

This implies that $u(\overline{\varphi}) = 0$ in $\Omega_0$. Thus $\varphi = 0$ in $L^2(\Omega_0)$. So the set $U$ is dense in $H^1(\Omega_0)$. The proof is finished. \qed

In the following theorem, we prove that $U$ is not only dense in, but also equal to $H^1(\Omega_0)$.

**Theorem 31.** The space $U$ is closed, thus $U = H^1(\Omega_0)$.

**Proof.** Suppose there is a sequence $\{u_n\}_{n=1}^\infty \subset L^2(\Omega_0)$ such that $u(f_n)$ is a Cauchy sequence in $H^1(\Omega_0)$. To prove that $U$ is closed, we have to find out a $f_0 \in L^2(\Omega_0)$ such that $\lim_{n \to \infty} u(f_n) = u(f_0)$. For any $n \in \mathbb{N}$, there is a $f_n \in H^1(\Omega_0)$ such that

$$\int_{\Omega_0} f_n \overline{\varphi} \, dx = \left\langle f_n, \varphi \right\rangle, \quad \forall \varphi \in H^1(\Omega_0),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the space $H^1(\Omega_0)$.

As $u(f)$ depends linearly on $f$, for any $m, n \in \mathbb{N}$, $u(f_m - f_n) = u(f_m) - u(f_n)$. Then $u(f_m - f_n)$ is the LAP solution of (33)-(42) with source $f_m - f_n$, i.e.,

$$\Delta u(f_m - f_n) + k^2 q u(f_m - f_n) = f_m - f_n \quad \text{in } \Omega_0; \quad \frac{\partial u(f_m - f_n)}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0.$$

Multiply the Helmholtz equation with $\overline{\varphi}$ where $\varphi$ is any function in $C^\infty(\Omega_0)$, and apply the Green’s formula, then

$$\int_{\Omega_0} (f_m - f_n) \overline{\varphi} \, dx = \int_{\Omega_0} \left[ k^2 q (f_m - f_n) \overline{\varphi} - \nabla u(f_m - f_n) \cdot \nabla (f_m - f_n) \right] \, dx.$$
Thus there is a constant $C = \max\{k^2\|q\|_\infty, 1\}$ such that
\[
\left| \langle \tilde{f}_m - \tilde{f}_n, \varphi \rangle \right| = \left| \int_{\Omega_0} (f_m - f_n) \varphi \, dx \right| \leq C \|u(f_m - f_n)\|_{H^1(\Omega_0)} \|\varphi\|_{H^1(\Omega_0)}.
\]
This implies that
\[
\|\tilde{f}_m - \tilde{f}_n\|_{H^1(\Omega_0)} \leq C \|u(f_m - f_n)\|_{H^1(\Omega_0)}.
\]
As \(\{u(f_n)\}_{n=1}^\infty\) is a Cauchy sequence, \(\{\tilde{f}_n\}_{n=1}^\infty\) is also a Cauchy sequence in \(H^1(\Omega_0)\). As the space \(H^1(\Omega_0)\) is closed, there is an \(\tilde{f}_0 \in H^1(\Omega_0)\) such that \(\tilde{f}_n \to \tilde{f}_0, \quad n \to \infty\). From the choice of \(\tilde{f}_m\) and \(\tilde{f}_n\), for any \(\varphi \in H^1(\Omega_0)\),
\[
\int_{\Omega_0} (f_m - f_n) \varphi \, dx = \langle \tilde{f}_m - \tilde{f}_n, \varphi \rangle \to 0, \quad m, n \to \infty.
\]
From the fact that \(H^1(\Omega_0)\) is a dense subspace of \(L^2(\Omega_0)\), for any \(\varphi \in L^2(\Omega_0)\), \(\int_{\Omega_0} (f_m - f_n) \varphi \, dx \to 0\) as \(m, n \to \infty\). Thus \(\{f_n\}\) is a Cauchy sequence in \(L^2(\Omega_0)\). From (25), \(u(f_n) \to u(f_0)\) in \(H^1(\Omega_0)\), thus \(\mathcal{U}\) is closed. Thus \(\mathcal{U} = H^1(\Omega_0)\). The proof is finished.

This implies that any function in \(H^1(\Omega_0)\) is an LAP solution of (1)-(2) with some \(f \in L^2(\Omega_0)\). Define the sets
\[
\mathcal{U}^+ := \left\{ u|_{\Gamma_1} : u \in \mathcal{U} \right\}; \quad \mathcal{U}^- := \left\{ u|_{\Gamma_\infty} : u \in \mathcal{U} \right\},
\]
then the following corollary is a direct result from Theorem 31 and the trace theorem.

**Corollary 32.** \(\mathcal{U}^+ = H^{1/2}(\Gamma_1), \quad \mathcal{U}^- = H^{1/2}(\Gamma_\infty)\).

With these results, we consider the following scattering problems in half-waveguides.

**6.4 Half-guide problem and radiation condition**

In this part, we extend integral representation (26) to solutions of scattering problems in a half guide \(\Omega_+ := \bigcup_{n=1}^\infty \Omega_n\) (see Figure 5), i.e., to find the LAP solution \(u_+ \in H^1_{loc}(\Omega_+)\) such that it satisfies
\[
\Delta u_+ + k^2 u_+ = 0 \quad \text{in} \quad \Omega_+; \quad u_+ = \varphi \quad \text{on} \quad \Gamma_1.
\]
(27)

Let \(u_+(\varepsilon)\) be the one associated with \(k^2 + i\varepsilon\) for any \(\varepsilon > 0\), then from the Lax-Milgram theorem again, there is a unique solution \(u_+(\varepsilon) \in H^1(\Omega_+)\) that decays exponentially at the infinity. The solution \(u_+\) which we are looking for, is the limit of \(u_+(\varepsilon)\) as \(\varepsilon \to 0\). In the following, we also let \(\Omega_- := \bigcup_{n=-\infty}^0 \Omega_n\) for simplicity. Then \(\Omega = \Omega_- \cup \Omega_+\).

**Figure 5:** Periodic waveguide.

The uniqueness of the LAP solution is always guaranteed, when the solution exists.

**Theorem 33.** There is at most one LAP solution for any fixed \(\varphi \in H^{1/2}(\Gamma)\).
Proof. Suppose for a $\varphi \in H^{1/2}(\Gamma)$, there are two different LAP solutions $u_1$ and $u_2$. Let $u_0 := u_1 - u_2$, then it is a LAP solution that satisfies

$$\Delta u_0 + k^2 qu_0 = 0 \quad \text{in } \Omega_+; \quad u_0 = 0 \quad \text{on } \Gamma_1.$$ 

As $u_0$ is an LAP solution, it is the limit of $u_0^\varepsilon$ satisfies

$$\Delta u_0^\varepsilon + (k^2 + i\varepsilon)qu_0^\varepsilon = 0 \quad \text{in } \Omega_+; \quad u_0^\varepsilon = 0 \quad \text{on } \Gamma_1.$$ 

From the unique solvability of damped Helmholtz equations, $u_0^\varepsilon = 0$ as long as $\varepsilon > 0$. Thus $u_0 = 0$, which contradicts with $u_1 \neq u_2$. The proof is finished.

As $U^+ = H^{1/2}(\Gamma_1)$, for any $\varphi \in H^{1/2}(\Gamma_1)$, there is an $f \in L^2(\Omega_0)$ such that the LAP solution $u(f)$ of (1)-(2) satisfies $u(f)|\Gamma_1 = \varphi$. Thus for any $\varphi \in H^{1/2}(\Gamma_1)$, there is a unique LAP solution $u_+ \in H^1_{\text{loc}}(\Omega_+)$ such that it satisfies the equation (27). Moreover, the unique LAP solution to the half-guide problem (27) could be represented by the form (26). Although the choice of $f$ is multiple, the solution is still unique in $\Omega_+$. On the other hand, the LAP solution $u(f)$ is an extension of the LAP solution $u_+$ of (27) to the LAP solution of (1)-(2). As the choice of $f$ is multiple, there are multiple ways of extensions.

Remark 34. In this paper, we only show the integral representation of LAP solutions of the half guide problem (27). We do not discuss the continuous dependence of the LAP solution on the boundary value $\varphi$, but it could be proved with the method introduced in Section 5, [KL18b], with the result in Theorem 28.

Remark 35. From the analysis of half-guide problems, we can also treat more complicated cases in the similar way, for example, the conjunction of two or more different periodic half-guides, or locally perturbed periodic wave-guides.

With the help of half-guide problem, we can conclude the radiation condition as the end of this section.

Definition 36 (Radiation condition). A solution of (27) $u \in H^1_{\text{loc}}(\Gamma_1)$ satisfies the radiation condition if there is an $f \in L^2(\Omega_0)$ such that

$$u(x_1 + n, \tilde{x}) = \frac{1}{2\pi i} \oint_{C} (I - B_z)^{-1} z^{-n-1} \tilde{f} \, dz,$$

where $C$ is chosen by the criteria in Definition 25.

From the uniqueness and existence of LAP solutions, there is a unique solution of (27) that satisfies the radiation solution. This result is easily extended to full-guide problem (1)-(2).

7 Spectrum Decomposition of periodic operators

To have a better understanding of the LAP solutions, we introduce the following operators:

$$\mathcal{R}^+: u|_{\Gamma_1} \to u|_{\Gamma_2}; \quad \mathcal{R}^-: u|_{\Gamma_{-1}} \to u|_{\Gamma_{-2}}.$$ 

As $\Gamma_1 = \{1/2\} \times \Sigma$, let

$$X := H^1(\{1/2\} \times \Sigma) = H^1(\Gamma_1),$$

then with the help of a shift operator,

$$\mathcal{R}^\pm : X \to X$$

are two bounded operators. Furthermore,

$$u|_{\Gamma_{j+1}} = \mathcal{R}^+ u|_{\Gamma_j}, \quad j \geq 1; \quad u|_{\Gamma_{j-1}} = \mathcal{R}^- u|_{\Gamma_j}, \quad j \leq 0.$$
As the analysis of \( R^+ \) and \( R^- \) are exactly the same, we only consider the operator \( R^+ \). In this section, we consider the spectrum decomposition of the operator \( R^+ \). It has been conjectured and already been used in numerical simulations in [JLF06] that the generalized eigenfunctions of the boundary operator \( R^+ \) form a complete set in \( L^2(\Gamma_1) \). Now we are able to prove that the conjecture is true.

We recall two properties of the boundary operators defined in [JLF06]. In that paper, the first result was shown for the operator with \( k^2 \) be replaced by \( k^2 + i\varepsilon \), but it is easily extended to \( \varepsilon = 0 \) from the limiting absorption principle. These results are easily extended from boundary operators and domain operators.

**Theorem 37.** The operator \( R^\pm \) has the following properties:

- \( R^\pm \) is bounded from \( H^{1/2}(\Gamma_j) \) to \( H^{1/2+\delta}(\Gamma_{j+1}) \) with some \( \delta > 0 \) and \( j \geq 1 \), thus it is a compact operator form \( X \) to \( X \).
- The spectral radius \( \rho(R^\pm) \leq 1 \).

In this section, we consider the spectrum decomposition of the operators \( R^+ \) for \( d = 1 \), i.e., in two dimensional space. Thus \( \Sigma \) is an interval, for simplicity, let \( \Sigma = [0, 1] \).

### 7.1 Generalized residue theorem and application

From the representation (28), as \( w(z, \cdot) \) is meromorphic with respect to \( z \) in the interior of \( \mathcal{C} \) except for \( \{0\} \), it is conjectured that the residue theorem is available for the curve integral. However, as the number of poles that lie in the interior of \( \mathcal{C} \) is infinite and \( \{0\} \) is the only accumulation point, classic residue theorem is not available. In this section, we prove that the residue theorem could be extended to infinite number of poles when certain conditions are satisfied. We call the extension as the Generalized Residue Theorem.

**Theorem 38** (Generalized Residue Theorem). Suppose \( D \) is a simply connected open domain in \( \mathbb{C} \). Suppose \( f(z) \) is a meromorphic function in \( D \setminus \{z_0\} \), where \( z_0 \) is a point lies in \( D \). Moreover, \( z_0 \) is not a pole of \( f \) but it is the only accumulation point of poles \( \{z_j : j \in \mathbb{N}\} \). If there is a series \( \{r_n\} \) satisfies (see Figure 7)

\[
    r_1 > r_2 > \cdots > r_n > \cdots > 0, \quad \lim_{n \to \infty} r_n = 0
\]

such that \( f(z) \) does not have poles when \( |z - z_0| = r_n \) and

\[
    \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z) \, dz \to 0, \quad n \to \infty,
\]

then the following integral formula holds:

\[
    \frac{1}{2\pi i} \oint_{D_0} f(z) \, dz = \sum_{j \in \mathbb{N} : z_j \in D_0} \operatorname{Res}(f(z), z = z_j), \tag{28}
\]

where \( D_0 \subset D \) is an open subdomain of \( D \) that satisfies \( \overline{D_0} \subset D \).

**Proof.** When \( z_0 \notin D_0 \), the result is obtained directly from classic residue theorem, thus we only consider the case that \( z_0 \in D_0 \).

As \( z_0 \) is the only accumulation point of \( \{z_j : j \in \mathbb{N}\} \), the number of poles in

\[
    D_n := \{z \in D_0 : |z - z_0| > r_n\}
\]

is finite. As \( \lim_{n \to \infty} r_n = 0 \),

\[
    \bigcup_{n \in \mathbb{N}} D_n = D_0 \setminus \{z_0\}.
\]

As \( z_0 \) is the only accumulation point of poles,

\[
    \bigcup_{n \in \mathbb{N}} [D_n \cap \{z_j : j \in \mathbb{N}\}] = \{j \in \mathbb{N} : z_j \in D_0\}.
\]

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Apply the classic residue theorem to the domain $D_n$ with boundary $\partial D_0$ and $\partial B(z_0, r_n)$, then
\[
\frac{1}{2\pi i} \oint_{\partial D_0} f(z) \, dz - \frac{1}{2\pi i} \oint_{|z-z_0|=r_n} f(z) \, dz = \sum_{j \in \mathbb{N} : z_j \in D_n} \text{Res}(f(z), z = z_j).
\]
Thus
\[
\left| \sum_{j \in \mathbb{N} : z_j \in D_n} \text{Res}(f(z), z = z_j) \right| = \frac{1}{2\pi i} \oint_{\partial D_0} f(z) \, dz - \frac{1}{2\pi i} \oint_{|z-z_0|=r_n} f(z) \, dz
\]
As the integral $\frac{1}{2\pi i} \oint_{|z-z_0|=r_n} f(z) \, dz \to 0$ as $n \to \infty$, the series $\sum_{j \in \mathbb{N} : z_j \in D_n} \text{Res}(f(z), z = z_j)$ is uniformly bounded as $n \to \infty$. Thus
\[
\left| \frac{1}{2\pi i} \oint_{\partial D_0} f(z) \, dz - \sum_{j \in \mathbb{N} : z_j \in D_n} \text{Res}(f(z), z = z_j) \right| = \frac{1}{2\pi i} \oint_{|z-z_0|=r_n} f(z) \, dz \to 0
\]
as $n \to \infty$, the equation (28) is proved. The proof is finished.

Then we apply the newly established generalized residue theorem to the integral (26) on the closed curve $C$. Recall the integral representation of $u$ in (26), i.e.,
\[
u(x_1 + n, x_2) = \frac{1}{2\pi i} \oint_C w(z, x) z^{n-1} \, dz.
\]
First, we need to know the distribution of poles. From previous sections, the set $S_+$ lies in the interior of $C$. As $z_j^+ \to 0$ as $j \to \infty$, 0 is the only accumulation point of the poles. From Theorem 38, we only need to find a strictly decreasing series $\{r_\ell\}_{\ell=1}^\infty$ that satisfies $r_\ell \to 0$, such that the integral
\[
I(r_\ell) := \oint_{|z|=r_\ell} w(z, x) z^{n-1} \, dz \to 0, \quad \ell \to \infty.
\]
Before the estimation of $I(r_\ell)$, we need a classical Minkowski integral inequality.

**Lemma 39** (Theorem 202, [HLPSS]). Suppose $(S_1, \mu_1)$ and $(S_2, \mu_2)$ are two measure spaces and $F : S_1 \times S_2 \to \mathbb{R}$ is measurable. Then the following inequality holds for any $p \geq 1$
\[
\left[ \int_{S_1} \left( \int_{S_2} |F(y, z)|^p \, d \mu_2(z) \right)^{1/p} \, d \mu_1(y) \right]^{1/p} \leq \int_{S_1} \left( \int_{S_2} |F(y, z)|^p \, d \mu_2(z) \right)^{1/p} \, d \mu_1(y).
\]

**Lemma 40.** There exists a series $\{r_\ell\}_{\ell \in \mathbb{N}}$ satisfies $r_1 > r_2 > \cdots > r_\ell > \cdots > 0$ and $\lim_{n \to \infty} r_n = 0$ such that the integral $I(r_\ell) \to 0$ as $\ell \to \infty$.

**Proof.** Let $w(z, \cdot)$ be the solution of the quasi-periodic problem $[19]$, then $v_z(x) := \zeta_z^{-1} w(z, x) \in H^1_{\text{per}}(\Omega_0)$ satisfies
\[
\Delta v_z + 2 \log(z) \frac{\partial v_z}{\partial x_1} + \log^2(z) v_z = g_z := z^{-x_1} f - k^2 q v_z \quad \text{in } \Omega_0; \quad \frac{\partial v_z}{\partial x_2} = 0 \quad \text{on } \partial \Omega_0.
\]
Let $v_z$ and $g_z$ be written as the expansions of the eigenfunctions, i.e.,
\[
v_z(x_1, x_2) = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{N}} v_{j,m} z^j \exp(2i\pi j x_1) \cos(\pi mx_2);
g_z(x_1, x_2) = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{N}} g_{j,m} z^j \exp(2i\pi j x_1) \cos(\pi mx_2).
\]
Figure 6: Example of the choice of $r_\ell$. For fixed $\ell \in \mathbb{N}$, the blue solid curve is $C_1^\ell$, the red dotted curve is $C_2^\ell$.

Take the expansion into the equation, we finally get the equation for the coefficients

$$[-4\pi^2j^2 - \pi^2m^2 + 2i \log(z)\pi j + \log^2(z)] v_{z}^{j,m} = g_{z}^{j,m}.$$  

Let $z = re^{i\theta}$ for a small enough $r > 0$ and $\theta \in [-\pi, \pi]$ (it could be replaced by any interval with length $2\pi$), then $|z| = r$ and $\log(z) = \log(r) + i\theta$.

$$c_{j,m,z} v_{z}^{j,m} = g_{z}^{j,m},$$

where the coefficient is defined by

$$c_{j,m,z} := (-4\pi^2j^2 - \pi^2m^2 - 2\theta\pi j + \log^2(r) - \theta^2) + 2i (\pi j + \theta) \log(r).$$

1) First we consider the case that $|\theta| \geq \varepsilon > 0$ for some small enough $\varepsilon > 0$. Then for any $j \in \mathbb{Z}$ and $m \in \mathbb{N}$, the coefficient

$$|c_{j,m,z}| \geq |\text{Im}(c_{j,m,z})| = 2|\log(r)| |\pi j + \theta| \geq 2\varepsilon |\log(r)|.$$

Thus $2\varepsilon |\log(r)||v_z^{j,m}| \leq |g_z^{j,m}|$, implies that

$$2\varepsilon |\log(r)||v_z^{j,m}| \leq \|g\|_{L^2(\Omega_0)} \leq \|z^{-x_0} f\|_{L^2(\Omega_0)} + k^2 \|q\|_{\infty} \|v_z\|_{L^2(\Omega_0)}.$$  

In this case, when $\varepsilon$ depends on $r$ and $2\varepsilon |\log(r)| \to \infty$ as $r \to 0^+$,

$$\|v_z\|_{L^2(\Omega_0)} \leq \frac{\|z^{-x_0} f\|_{L^2(\Omega_0)}}{2\varepsilon |\log(r)| - k^2 \|q\|_{\infty}} \leq \frac{\|z^{-x_0} f\|_{L^\infty(\Omega_0)} \|f\|_{L^2(\Omega_0)}}{2\varepsilon |\log(r)| - k^2 \|q\|_{\infty}} \leq \frac{r^{-1/2} \|f\|_{L^2(\Omega_0)}}{2\varepsilon |\log(r)| - k^2 \|q\|_{\infty}}.$$  

Thus

$$\|w(z, \cdot)\|_{L^2(\Omega_0)} = \|z^{x_0} v_z\|_{L^2(\Omega_0)} \leq \|z^{x_0} f\|_{L^\infty(\Omega_0)} \|v_z\|_{L^2(\Omega_0)} \leq \frac{\|f\|_{L^2(\Omega_0)}}{r(2\varepsilon |\log(r)| - k^2 \|q\|_{\infty})}. \tag{29}$$

2) Second we consider the case that $\theta \in (-\varepsilon, \varepsilon)$. When $j \neq 0$, the coefficient

$$|c_{j,m,z}| \geq |\text{Im}(c_{j,m,z})| = 2|\log(r)| |\pi j + \theta| \geq \pi |\log(r)|.$$  

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When $j = 0$, then

$$|c_{j,m,z}| \geq |\text{Re}(c_{j,m,z})| = |\log^2(r) - \pi^2 m^2 - \theta^2|.$$  

We choose $r_\ell = \exp\left(-\pi \sqrt{\ell^2 + (\ell + 1)^2}\right)$, then

$$|\log^2(r_\ell) - \pi^2 m^2| \geq |\log^2(r_\ell) - \pi^2 \ell^2| = |\log^2(r_\ell) - \pi^2 (\ell + 1)^2| = \frac{(2\ell + 1)\pi^2}{2}.$$  

Thus when $|\theta|$ is small enough,

$$|c_{j,m,z}| \geq |\log^2(r_\ell) - \pi^2 m^2 - \theta^2| \geq \pi^2 \ell.$$  

With the same technique, we can prove that when $\ell$ is large enough,

$$\|w(z,\cdot)\|_{L^2(\Omega_\ell)} = \|z^{\frac{n}{2}} v_z\|_{L^2(\Omega_\ell)} \leq \frac{1}{r(\pi^2 \ell - k^2 \|q\|_\infty)} \|f\|_{L^2(\Omega_\ell)}.$$

(30)

Now we are prepared to consider the integral $I(r_\ell)$. Let $\varepsilon_\ell := (\ell^2 + (\ell + 1)^2)^{-1/4}$, then $r_\ell \to 0$, $\varepsilon_\ell \to 0$, $|\log(r_\ell)| \varepsilon_\ell \to \infty$, $\ell \to \infty$.

Define $C_1^\ell = \{r_\ell \exp(i\theta) : |\theta| \leq \varepsilon_\ell\}$ and $C_2^\ell = \{r_\ell \exp(i\theta) : \varepsilon_\ell < \theta < 2\pi - \varepsilon_\ell\}$, then $C_1^\ell \cup C_2^\ell = \partial B(0, r_\ell)$. We can write the integral $I(r_\ell)$ as two parts (see Figure [6]):

\[ \int_{\partial B(0, r_\ell)} w(z, x) z^{-n-1} \, dz \]
\[ = \int_{C_1^\ell} w(z, x) z^{-n-1} \, dz + \int_{C_2^\ell} w(z, x) z^{-n-1} \, dz \]
\[ = i \int_{-\varepsilon_\ell}^{\varepsilon_\ell} w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta} \, d\theta + i \int_{\varepsilon_\ell}^{2\pi - \varepsilon_\ell} w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta} \, d\theta . \]

From (30), using Lemma [39] the $L^2(\Omega_\ell)$-norm of the first term is bounded by

\[ \left\| i \int_{-\varepsilon_\ell}^{\varepsilon_\ell} w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta} \, d\theta \right\|_{L^2(\Omega_\ell)} \leq \left( \int_{\Omega_\ell} \left| \int_{-\varepsilon_\ell}^{\varepsilon_\ell} w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta} \, d\theta \right|^2 \, dx \right)^{1/2} \]
\[ \leq \int_{-\varepsilon_\ell}^{\varepsilon_\ell} \left( \int_{\Omega_\ell} |w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta}|^2 \, dx \right)^{1/2} \, d\theta \]
\[ \leq \frac{2 \pi r_\ell^{n-1} \varepsilon_\ell}{\pi^2 \ell - k^2 \|q\|_\infty} \|f\|_{L^2(\Omega_\ell)} \to 0 \]

as $\ell \to \infty$, with any fixed integer $n \geq 1$.

For the second term, we use (29) and Lemma [39] again,

\[ \left\| i \int_{\varepsilon_\ell}^{2\pi - \varepsilon_\ell} w(r_\ell e^{i\theta}, x) r_\ell^{n} e^{i\theta} \, d\theta \right\|_{L^2(\Omega_\ell)} \leq \frac{2 \pi r_\ell^{n-1}}{2 |\log(r_\ell)\varepsilon_\ell| - k^2 \|q\|_\infty} \|f\|_{L^2(\Omega_\ell)} \to 0 \]

as $\ell \to \infty$. Thus

$$\int_{\partial B(0, r_\ell)} w(z, x) z^{-n-1} \, dz \to 0$$

in $L^2(\Omega_\ell)$, $\ell \to \infty$.

The proof is finished. \[ \square \]

With the above results, we have finally arrived at the following theorem.
Theorem 41. Suppose \( S_+ = \{ z^+_1, \ldots, z^+_J, \ldots \} \). Then for \( n \geq 1 \),
\[
    u(x_1 + n, x_2) = \sum_{j=1}^{\infty} \text{Res} \left( w(z, x) z^{n-1}, z = z^+_j \right).
\]  
(31)

Proof. The proof of \( \text{(31)} \) comes directly from Theorem \( \text{I.8} \) and Lemma \( \text{II} \).

We can also extend the result in Theorem \( \text{I.11} \) to the case when \( n \leq -1 \). We can not apply the generalized residue theorem directly, as in this case, Lemma \( \text{II} \) no longer holds. To deal with this problem, we consider the generalized residue theorem in the exterior of \( C \), with the only accumulation point of poles at the infinity. The main idea is to find a series \( \{ r_\ell \}_{\ell=1}^{\infty} \) satisfies
\[
    r_1 < r_2 < \cdots < r_\ell < \cdots < \infty, \quad r_\ell \to \infty,
\]  
(32)
such that no poles lie on \( |z| = r_\ell \) and \( I(r_\ell) \to 0 \) as \( \ell \to \infty \). Then we can prove that
\[
    \frac{1}{2\pi i} \oint_{C_0} f(z) \, dz = \sum_{j \in \mathbb{N}} \text{Res}(f(z), z = z_j)
\]
where \( D_0 \) is the subset of \( \mathbb{C} \) in the exterior of \( C \). Apply the result in Theorem \( \text{II.11} \) again, we can finally obtain the formulation
\[
    u(x_1 + n, x_2) = \sum_{j=1}^{\infty} \text{Res} \left( w(z, x) z^{n-1}, z = z^+_j \right).
\]
(33)

From Theorem \( \text{II.11} \) the solution is decomposed into the combination of residues of the function \( w(z, x) \) at the poles distribute in the interior (exterior) of the curve \( C \). Especially, the function \( u \) in \( \Omega_1 \) has the representation
\[
    u(x_1 + 1, x_2) = \sum_{j=1}^{\infty} \text{Res} \left( w(z, x), z = z^+_j \right), \quad x \in \Omega_0.
\]

In the next section, we discuss the structure of the residue \( \text{Res} \left( w(z, x), z = z^+_j \right) \), for any \( j \in \mathbb{N} \).

7.2 Eigenvalue decomposition of the operator \( R^+ \)

From the last section, the solution \( u \) has been written as the composition of residues at poles \( z^+_j \) for \( j \in \mathbb{N} \). To study the eigenvalue problems of \( R^+ \), we fix one residue \( \text{Res}(w(z, x), z^+_j) \) where \( w(z, \cdot) \in H^{1}_{\Omega_0} \) is the solution of \( \text{(19)-(20)} \), and study the property of this function. For simplicity, define the operator
\[
    \mathcal{L}_j : \mathcal{P}(C, L^2(\Omega_0)) \to H^1(\Omega_0)
\]
\[
    p(z) f(x) \mapsto \frac{1}{2\pi i} \oint_{|z-z^+_j| = \delta_j} (I - B_z)^{-1} p(z) \tilde{f}(x) \, dz
\]
where \( p \) is a polynomial, \( \tilde{f} \in H^1(\Omega_0) \) is obtained from \( \text{(12)} \) and the space \( \mathcal{P}(C, L^2(\Omega_0)) \) is defined by
\[
    \mathcal{P}(C, L^2(\Omega_0)) := \{ p(z) f(x) : p(z) \text{ is a polynomial}, \ f \in L^2(\Omega_0) \}.
\]

As any function in this space depends analytically on \( z \), we can defined the norm in this space by
\[
    \| p(z) f(x) \|_{\mathcal{P}(C, L^2(\Omega_0))} = \| p \|_{L^\infty(C)} \| f \|_{L^2(\Omega_0)}.
\]

Then \( \mathcal{L}_j \) is bounded from \( \mathcal{P}(C, L^2(\Omega_0)) \) to \( X \). Then we define the following space
\[
    \mathcal{E}_j^+ := \{ \mathcal{L}_j(p(z)f(x)) : p \text{ is a polynomial}, \ f \in L^2(\Omega_0) \} \subset X,
\]
thus
\[
    \mathcal{E}_j^+ = \{ \text{Res}(p(z)w(z, x), z = z^+_j) : w(z, x) \in H^{1}_1(\Omega_0) \text{ is the solution of } \text{(19)-(20)} \}.
\]

From Theorem \( \text{I.8} \) \( \mathcal{E}_j^+ \) is a finite dimensional space. Let the trace operator \( \Upsilon : H^1(\Omega_0) \mapsto H^{1/2}(\Gamma_1) \), and \( \Upsilon \mathcal{E}_j^+ \) be the space of all the functions in \( \mathcal{E}_j^+ \) that restricted on the boundary \( \Gamma_1 \).
Remark 42. Although the operator $\mathcal{L}_j$, which comes from generalized residue theorem directly, is defined for $f$ only depends on $x$, we can simply extend the definition of the operator in the space for functions with the form of $p(z)f(x)$, where $f \in L^2(\Omega_0)$ and $p(z)$ is a polynomial.

From Corollary \textbf{32} $U^+ = H^{1/2}(\Gamma_1) = X$. As $U^+ = \left\{ \sum_{j=1}^{\infty} \mathcal{Y} \mathcal{L}_j f : f \in L^2(\Omega_0) \right\}$,

$$X = H^1(\Gamma_1) = \text{span}(\mathcal{Y} \mathcal{E}^+_j : j \in \mathbb{N}).$$

Then we study the property of the space $\mathcal{E}^+_j$. As $w(z, x)|_{\Gamma_{j+1}} = zw(z, x)|_{\Gamma_j}$, we have

$$\mathcal{R}^+(\mathcal{Y} \mathcal{L}_j (p(z)f)) = \mathcal{Y} \left( \frac{1}{2\pi i} \oint_{|z-z_j^+| = \delta_j} (I - B_z)^{-1} z^{p(z)} \tilde{f}(x) \, dz \right) = \mathcal{Y} \mathcal{L}_j (zp(z)f).$$

Denote the $m$-th power of $\mathcal{R}^+$ by $\mathcal{R}^{+,m}$, then it satisfies

$$\mathcal{R}^{+,m} : u|_{\Gamma_j} \mapsto u|_{\Gamma_{j+m}}$$

with the representation

$$\mathcal{R}^{+,m}(\mathcal{Y} \mathcal{L}_j p(z)f) = \mathcal{Y} \left( \frac{1}{2\pi i} \oint_{|z-z_j^+| = \delta_j} (I - B_z)^{-1} z^{m(p(z)} \tilde{f} \, dz \right) = \mathcal{Y} \mathcal{L}_j (z^m p(z)f).$$

From the analysis above, we can conclude the following result.

Lemma 43. Let $\tilde{p}(z)$ be a polynomial of $z$ with arbitrary positive degree. Then

$$\tilde{p}(\mathcal{R}^+) (\mathcal{Y} \mathcal{L}_j p(z)f) = \mathcal{Y} \mathcal{L}_j (\tilde{p}(z)p(z)f).$$

With this lemma, we are prepared to study the structure of the set $\mathcal{E}^+_j$.

Theorem 44. For any $j \in \mathbb{N}$, $z_j^+ \in S_+$ is an eigenvalue of $\mathcal{R}^+$. Moreover, any non-zero element in $\mathcal{Y} \mathcal{E}^+_j$ is a generalized eigenfunction of $\mathcal{R}^+$ associated with the eigenvalue $z_j^+$, and $\mathcal{Y} \mathcal{E}^+_j$ is an invariant subspace of the operator $\mathcal{R}^+$.

Proof. As $z_j^+ \in F$ for any $j \in \mathbb{N}$, it is an eigenvalue of $\mathcal{R}^+$ as there is a Bloch wave solution corresponds to $z_j^+$.

As $(I - B_z)^{-1}$ is a family of meromorphic operators with respect to $z$ and $z_j^+$ is a pole of it, in a small punctured neighbourhood of $z_j^+$ denoted by $\hat{B}(z_j^+, \delta_j)$, it has the Laurent series expansion:

$$(I - B_z)^{-1} = \sum_{\ell = -M}^{\infty} (z - z_j^+)^\ell B_\ell,$$  \hspace{1cm} (34)

where $M > 0$ is a positive integer and $B_\ell$ is bounded and the series converges $\hat{B}(z_j^+, \delta_j)$ where $\delta_j > 0$ is sufficiently small. From Lemma \textbf{28}

$$\left( \mathcal{R}^+ - z_j^+ I \right)^M (\mathcal{Y} \mathcal{L}_j p(z)f) = \mathcal{Y} \mathcal{L}_j ((z - z_j^+)^M p(z)f)$$

$$= \mathcal{Y} \left( \frac{1}{2\pi i} \oint_{|z-z_j^+| = \delta_j} (I - B_z)^{-1} (z - z_j^+)^M \tilde{f} \, dz \right)$$

$$= \mathcal{Y} \left( \frac{1}{2\pi i} \oint_{|z-z_j^+| = \delta_j} \sum_{\ell = -M}^{\infty} (z - z_j^+)^\ell B_\ell \right) (z - z_j^+)^M p(z) \tilde{f} \, dz$$

$$= \mathcal{Y} \left( \frac{1}{2\pi i} \oint_{|z-z_j^+| = \delta_j} \sum_{\ell = 0}^{\infty} (z - z_j^+)^\ell \left( B_{\ell - M} \tilde{f} \right) p(z) \, dz \right).$$

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Thus for the integrand \((I - B_z)^{-1}(z - z_j^+)M^p(z)\tilde{f}, z_j^+\) is a removable singularity. From the residue theorem, the integral equals to 0. Thus
\[
(R^+ - z_j^+ I)^M(\mathcal{L}_j^p(z)f) = 0,
\]
implies that \(\mathcal{L}_j^p(p(z)f)\) is a generalized eigenfunction of \(R^+\) associated with the eigenvalue \(z_j^+\).

Note that for any \(p(z)f(x), \mathcal{L}_j^p(z(z)f) \in \mathcal{L}_j^p_{z_j^+}.\) From Lemma\([33]\) \(R^+ [\mathcal{L}_j^p(p(z)f)] = \mathcal{L}_j^p(z(p(z))f) \in \mathcal{L}_j^p_{z_j^+}\). Thus \(\mathcal{L}_j^p_{z_j^+}\) is an invariant subspace in \(X\) of the operator \(R^+.\) The proof is finished.

This implies that \(\mathcal{L}_j^p_{z_j^+} \subseteq \mathcal{N}\left((R^+ - z_j^+ I)^M\right).\) From the fact that the generalized eigenfunctions are linearly independent, we have the following corollary.

**Corollary 45.** The subspaces \(\{\mathcal{L}_j^p_{z_j^+}\}_{j \in \mathbb{N}}\) are linearly independent and
\[
X = \bigoplus_{j \in \mathbb{N}} \mathcal{L}_j^p_{z_j^+}.
\]

As \(R^+\) is a compact operator and \(z_j^+\) is an eigenvalue of it, let the Riesz number for \((R^+ - z_j^+ I)\) be \(\nu_+ (j)\), which implies
\[
\mathcal{N}((R^+ - z_j^+ I)0) \subseteq \mathcal{N}((R^+ - z_j^+ I)^1) \subseteq \cdots \subseteq \mathcal{N}((R^+ - z_j^+ I)^{\nu_+ (j)}) = \mathcal{N}((R^+ - z_j^+ I)^{\nu_+ (j)+1});
\]
\[
X = (R^+ - z_j^+ I)^0 (X) \supset (R^+ - z_j^+ I)^1 (X) \supset \cdots \supset (R^+ - z_j^+ I)^{\nu_+ (j)} (X) = (R^+ - z_j^+ I)^{\nu_+ (j)+1} (X).
\]
Thus for any fixed \(j\),
\[
\mathcal{L}_j^p_{z_j^+} \subseteq \mathcal{N}\left((R^+ - z_j^+ I)^M\right) \subset \mathcal{N}\left((R^+ - z_j^+ I)^{\nu_+ (j)}\right).
\]

Then we have the following decomposition.

**Corollary 46.** The space \(X\) has the following form
\[
X = \bigoplus_{j \in \mathbb{N}} \mathcal{N}\left((R^+ - z_j^+ I)^{\nu_+ (j)}\right)
\]
and \(R^+\) has the Jordan normal form in the space \(\bigoplus_{j \in \mathbb{N}} \mathcal{N}\left((R^+ - z_j^+ I)^{\nu_+ (j)}\right)\).

Now we have proved that the generalized eigenfunctions of \(R^+\) form a complete set in \(X\). We can also conclude the same results for \(R^-\). This result implies that any LAP solution is composed by finite number of propagating modes (elements in \(E_j^{\pm}\) for \(j = 1, 2, \ldots, N\)), and infintiy number of evanescent modes (elements in \(E_j^{0}\) for \(j = N + 1, N + 2, \ldots\)).

### 7.3 Decomposition of LAP solutions

With the help of the spectrum decomposition of \(R^+\), we would like to discuss the decomposition of the LAP solution with the help of the spectrum decomposition. As
\[
u_j (x_1 + n, x_2) = \sum_{j=1}^{\infty} L_j(z^{-1}f), \]
the LAP solution is decomposed into infinite number of generalized eigenfunctions corresponds to \(z_j^+\).

Recall \(S_+ = S_+^0 \cup RS, S_+^0 = \{z_1^+, \ldots, z_N^+, z_{N+1}^+\}\) and \(RS = \{z_{N+2}^+, \ldots\}\), let
\[
u_j (x_1 + n, x_2) = L_j(z^{-1}f).
\]

As any \(z_j^+\) is an isolated singularity, for any \(j \in \mathbb{N}\), there is a \(\delta_j > 0\) small enough such that \(B(z_j^+, \delta_j) \cap S_+ = \{z_j^+\}\). Thus
\[
u_j (x_1 + n, x_2) = \frac{1}{2\pi i} \oint_{|z - z_j^+| = \delta_j} (I - B_z)^{-1} z^{-1} f dz.
\]

As $(I - B_z)^{-1}$ exists and is uniformly bounded for $|z - z_j^+| = \delta_j$, there is a $C > 0$ such that 

$$
\|(I - B_z)^{-1} f\|_{H^1(\Omega_0)} \leq C \|f\|_{L^2(\Omega_0)}.
$$

Then for any $n \geq 1$, from Lemma 39

$$
\|u_j(x_1 + n, x_2)\|_{L^2(\Omega_0)} = \left\| \frac{1}{2\pi i} \oint_{|z-z_j^+|=\delta_j} (I - B_z)^{-1} z^{n-1} \tilde{f} \, dz \right\|_{L^2(\Omega_0)}
$$

$$
= \left( \int_{\Omega_0} \left| \frac{1}{2\pi i} \oint_{|z-z_j^+|=\delta_j} (I - B_z)^{-1} z^{n-1} \tilde{f} \, dz \right|^2 \, dx \right)^{1/2}
$$

$$
\leq \frac{1}{2\pi} \oint_{|z-z_j^+|=\delta_j} \left( \int_{\Omega_0} \left| (I - B_z)^{-1} z^{n-1} \tilde{f} \right|^2 \, dx \right)^{1/2} \, dz
$$

$$
\leq C \|f\|_{L^2(\Omega_0)} \oint_{|z-z_j^+|=\delta_j} |z|^{n-1} \, dz
$$

$$
= C \|f\|_{L^2(\Omega_0)} (\delta_j + |z_j^+|)^{n-1}.
$$

The result could be easily extended to the $H^1$-norm and $n \leq -1$. Thus we conclude the result in the following theorem.

**Theorem 47.** For any $j \in \mathbb{N}$, given a small enough $\delta > 0$, there is a constant $C = C(\delta) > 0$ such that for any $n \geq 1$,

$$
\|u_j\|_{H^1(\Omega_n)} \leq C (\delta + |z_j^+|)^{n-1}; \quad \|u_j\|_{H^1(\Omega_{n-\alpha})} \leq C (\delta + |z_j^-|)^{-n+1}.
$$

Thus for any $z_j^+ \in RS$, $u_j$ decays exponentially when $|x_1| \to \infty$.

From the theorem above, we know that the modes $u_j$ for $j \geq N + 1$ are evanescent for $|z_j^+| < 1$ uniformly. Then we only need to discuss the modes with respect to $z_j^+ \in S_0^0$. To study the structure of $L_j f$, we have to consider the eigenvalue decomposition of the operator $A$. Suppose for $z_j^+ \in S_0^0$, let $(\mu_j(z), \phi_j(z, \cdot))$ be the eigenvalues and eigenfunctions such that

$$
A \psi_j(\cdot, z) = \mu_j \psi_j(\cdot, z),
$$

and both of them depends analytically on $z$ in a small enough neighbourhood of $z$ in a small neighbourhood of $S_1$. (Note that these are $(\mu_j(\alpha)), \phi_j(\alpha, \cdot)$ defined in Section 3, where $z = \exp(\alpha i)$. Thus the solution to the problem $A w(z, \cdot) + k^2 w(z, \cdot) = q^{-1} f$ has the representation

$$
w(z, x) = \sum_{j=1}^{\infty} \frac{\langle q^{-1} f, \psi_j(\cdot, z) \rangle}{\mu_j(z) - k^2} \psi_j(x, z).
$$

From Assumption 6 as $\mu_j(z) \neq 0$ for fixed $k > 0$ and $\mu_j(z) = k$, there are simple poles of $w(z, \cdot)$ on the unique circle. This implies that $M = 1$ in the expansion 61. Thus $L_j f$ is the eigenfunction corresponding to the eigenvalue $\mu_j^+$. We can also define the operator $L_j^-$ in the similar way, i.e., for a $z_j^- \in S_0^0$,

$$
L_j^- f := \frac{1}{2\pi i} \oint_{|z-z_j^-|=\delta_j} w(z, x) \, dz.
$$

The arguments for $L_j^-$ are easily extended to $L_j^+$, thus we can determine the structure of $\{L_j f : f \in L^2(\Omega_0)\}$ and $\{L_j^- f : f \in L^2(\Omega_0)\}$ in the following theorem.

**Theorem 48.** For any $z_j^+ \in S_0^0$, $L_j f$ is a Bloch wave solution corresponds to $z_j^+$; for any $z_j^- \in S_0^0$, $L_j^- f$ is a Bloch wave solution corresponds to $z_j^-$. 

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With this result, we obtain the final result that the LAP solution consists of finite number of Bloch wave solutions corresponds to $z^* \in S^0_+$ and infinite number of exponentially decay solutions where the decaying rate are closely related to $|z^*|$ with $z^* \in RS$.

With the method introduced in [FJ15], we can also consider the case that Assumption 7 does not hold. Suppose $S^0_+ \cap S^0_- \neq \emptyset$. We still use the same notation $u, u_z, w, w_z$, then

$$u_x(x_1 + n, x_2) = \frac{1}{2\pi i} \oint_{S^0_+} w_z(z, x)z^{n-1} \, dz.$$  

We consider the limit $\varepsilon \to 0^+$. Suppose $z_0$ is the only element in the set $S^0_+ \cap S^0_-$, suppose there are two continuous functions $Z^\pm_0(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0^+} Z^\pm_0(\varepsilon) = z_0, \quad |Z^+_0(\varepsilon)| < 1, \quad |Z^-_0(\varepsilon)| > 1$$  

for small enough $\varepsilon > 0$.

Figure 7: Example of the domains $B_+(z_0, \delta)$ and $B_-(z_0, \delta)$ for $z_0 \in S^0_+ \cap S^0_-$.  

Let $B_+(z_0, \delta) := B(0, 1) \cap B(z_0, \delta)$ and $B_-(z_0, \delta) := B(0, 1) \setminus B(z_0, \delta)$ (see Figure 7), then we define

$$u^0_x(x_1 + n, x_2) = \frac{1}{2\pi i} \oint_{\partial B_+(z_0, \delta)} w_z(z, x)z^{n-1} \, dz,$$

and

$$u'^x_x(x_1 + n, x_2) := u_x(x_1 + n, x_2) - u^0_x(x_1 + n, x_2)$$

$$= \frac{1}{2\pi i} \oint_{\partial B_-(z_0, \delta)} (I - B^\varepsilon_z)^{-1} z^{n-1} \tilde{f} \, dz.$$  

Similar to Definition 25, we can also define the new domain $B_\delta$ by

$$B_\delta := B(0, 1) \setminus \overline{B(z_0, \delta)} \cup \bigcup_{z \in S^0_+ \setminus \{z_0\}} B(z, \delta) \setminus \bigcup_{z \in S^0_- \setminus \{z_0\}} \overline{B(z, \delta)},$$

where $\delta > 0$ satisfies all the criteria in Definition 25. Let $C_0 = \partial B_\delta$, then we can easily prove that

$$u'^x_x(x_1 + n, x_2) = \frac{1}{2\pi i} \oint_{C_0} (I - B^\varepsilon_z)^{-1} z^{n-1} \tilde{f} \, dz.$$
and 
\[ u_n^\varepsilon(x_1 + n, x_2) \to u^\varepsilon(x_1 + n, x_2) := \frac{1}{2\pi i} \oint_{\mathcal{C}_0} (I - B_z^{-1})z^{n-1}\tilde{f} \, dz. \]

Thus we only need to consider the limit of \( u_0^\varepsilon \) as \( \varepsilon \to 0 \). For any fixed \( \varepsilon > 0 \), as \( Z_0^+(\varepsilon) \) is the only pole in the domain \( B_+(z_0, \delta) \) for \( (I - B_z)^{-1} \), \( u_0^\varepsilon(x_1 + 1, x_2) \in \mathcal{N}(I - B_z) \) thus it is the eigenfunction of \( A_{z_0^+(\varepsilon)} \) with the eigenvalue \( k^2 + i\varepsilon \). From the continuous dependence of eigenvalues and eigenfunctions on \( \varepsilon \),
\[ u_0^\varepsilon(x_1 + 1, x_2) \to u^0(x_1 + 1, x_2), \quad \varepsilon \to 0, \]
where \( u^0 \) is a non-trivial solution of (19)-(20) with \( z = z_0 \) and \( f = 0 \). Thus we can extend \( u^0 \) quasi-periodically to a Bloch wave solution in any \( \Omega_0 \).

This result is easily extended to the case when there are multiple numb er of poles in \( S_0^+ \cap S_0^- \), thus we can conclude the result as follows.

**Theorem 49.** Suppose \( S_0^+ \cap S_0^- = \{z_0^1, \ldots, z_0^M\} \) and the pole corresponds to \( z_0^j \) with \( \varepsilon > 0 \) are denoted by \( Z_{0,j}^{\varepsilon, \pm} \) where \( j = 1, 2, \ldots, N(\ell) \), \( N(\ell) \) is a positive integer depens on \( \ell \). Moreover, \( |Z_{0,j}^{\varepsilon, +}| < 1 \) and \( |Z_{0,j}^{\varepsilon, -}| > 1 \) for \( \varepsilon > 0 \). Then the problem (11)-(12) has a unique LAP solution with the representation
\[
\begin{align*}
  \tilde{u}_{0,j}^{+}(x_1 + n, x_2) & = \sum_{f=1}^{M} \sum_{j=1}^{N(\ell)} \tilde{u}_{0,j}^{f, +}(x_1 + n, x_2) + \frac{1}{2\pi i} \oint_{\mathcal{C}_0} (I - B_z)^{-1}z^{n-1}\tilde{f} \, dz; \\
  \tilde{u}_{0,j}^{-}(x_1 - n, x_2) & = \sum_{f=1}^{M} \sum_{j=1}^{N(\ell)} \tilde{u}_{0,j}^{f, -}(x_1 - n, x_2) + \frac{1}{2\pi i} \oint_{\mathcal{C}_0} (I - B_z)^{-1}z^{n-1}\tilde{f} \, dz.
\end{align*}
\]

Here \( \tilde{u}_{0,j}^{f, \pm} = \lim_{\varepsilon \to 0} u_0^\varepsilon \) and \( \tilde{u}_{0,j}^{f, \pm} \in \mathcal{N}(I - B_{z_0^j}^{z_0^j}) \) is an eigenfunction. \( \mathcal{C}_0 \) is the boundary of
\[
B(0, 1) \setminus \bigcup_{f=1}^{M} B(z_0^j, \delta) \cup \left[ \bigcup_{z \in S_0^+ \setminus \{z_0\}} B(z, \delta) \right] \setminus \left[ \bigcup_{z \in S_0^- \setminus \{z_0\}} B(z, \delta) \right],
\]
\( \tilde{\mathcal{C}}_0 \) is the boundary of
\[
B(0, 1) \cup \bigcup_{f=1}^{M} B(z_0^j, \delta) \cup \left[ \bigcup_{z \in S_0^+ \setminus \{z_0\}} B(z, \delta) \right] \setminus \left[ \bigcup_{z \in S_0^- \setminus \{z_0\}} B(z, \delta) \right],
\]

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