Probabilistic Analysis of Rule 2

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Abstract

Li and Wu proposed Rule 2, a localized approximation algorithm that attempts to find a small connected dominating set in a graph. Here we study the asymptotic performance of Rule 2 on random unit disk graphs formed from $n$ random points in an $\ell_n \times \ell_n$ square region of the plane. If $\ell_n = O(\sqrt{n/\log n})$, Rule 2 produces a dominating set whose expected size is $O(n/(\log \log n)^{3/2})$.

Keywords and phrases: coverage process, dominating set, localized algorithm, performance analysis, probabilistic analysis, Rule $k$, unit disk graph.
1 Introduction

Suppose random points $V_1, V_2, \ldots, V_n$ are selected from a connected region $Q$ in $\mathbb{R}^2$. For each $i$, let $D_1(V_i)$ be the unit disk centered at $V_i$. There is a large literature on coverage processes\[18\] that enables one to answer questions such as whether or not the random disks are likely to cover all of $Q$, i.e. whether $Q \subseteq \bigcup_{i=1}^{n} D_1(V_i)$. A variant question asks whether there is small subset of the disks whose union already covers $Q$: given $k < n$, are there indices $i_1 < i_2 < \ldots < i_k$ such that $Q \subseteq \bigcup_{j=1}^{k} D_1(V_{i_j})$. For this variant, there are several interesting ways to modify the meaning of “coverage.” For example: is there a small subset of the disks whose union is connected and contains all $n$ points $V_1, V_2, \ldots, V_n$ (but not necessarily all of $Q$)? These questions are a bit vague, but specific examples arise naturally in connection with probabilistic models for wireless networks. In particular, they are central to the probabilistic analysis of Rule 2 in this paper.

Rule 2 is a well known algorithm that was proposed by Wu and Li \[31\] as a means of increasing the efficiency of routing in ad hoc wireless networks. To describe the algorithm and a probabilistic model, we need some graph theoretic terminology. A unit disk graph has for its vertex set $V$ a finite set of points in $\mathbb{R}^2$. Given the vertex set $V$, the edge set $E$ is determined as follows: an undirected edge $e \in E$ connects vertices $u, v \in V$ (and in this case we say that $u$ and $v$ are adjacent) iff $d(u, v)$, the Euclidean distance between them, is less than one. Unit disk graphs have been used by many authors as mathematical models for the interconnections between nodes in a wireless network, and random unit disk graphs have been used as probabilistic models for these networks \[8\], \[12\], \[15\], \[16\], \[17\], \[23\], \[24\]. A dominating set in any graph $G = (V, E)$ is a subset $C \subseteq V$ such that every vertex $v \in V$ either is in the set $C$, or is adjacent to a vertex in $C$. We say $C$ is a connected dominating set if $C$ is a dominating set and the subgraph induced by $C$ is connected. Of course it is not possible for $G$ to have a connected dominating set if $G$ itself is not connected. We use the acronym “CDS” for a dominating set $C$ such that the subgraph induced by $C$ has the same number of components that $G$ has. This paper deals with a random unit disk graph model, $G_n$, which is connected with asymptotic probability one. Thus any CDS for $G_n$ will also be connected with high probability. We assume that each vertex has a unique identifier taken from a totally ordered set. For convenience, when $|V| = n$, we will use the numbers 1, 2, \ldots, $n$ as IDs, and will number the vertices accordingly. If $v_i$ is any vertex (with ID $i$), define the neighborhood $N(v_i)$ to be the set consisting of $v_i$ and any vertices in $V$ that are adjacent to $v_i$. The CDS constructed by the Rule 2 algorithm is denoted $C(V)$, and its cardinality is $|C(V)| = |\mathcal{C}(V)|$. The elements of $\mathcal{C}(V)$ are called “gateway nodes”. $C(V)$ consists of all vertices $v_i \in V$ that are not excluded under the following version of Rule 2:

**Rule 2:** Vertex $v_i$ is excluded from $\mathcal{C}(V)$ iff $N(v_i)$ contains at least one set of two vertices $v_{i_1}, v_{i_2}$ such that
\begin{itemize}
  \item $i_1 > i_2 > i$ and
  \item $\mathcal{N}(v_i) \subseteq \mathcal{N}(v_{i_1}) \cup \mathcal{N}(v_{i_2})$ and
  \item $v_{i_1}$ is adjacent to $v_{i_2}$.
\end{itemize}

Wu and Li showed that this algorithm produces a CDS. They also conjectured, based on simulation data, that it is effective in the sense that it selects a CDS that is small relative to $n$ "in the average case". In this paper we treat the analysis of Rule 2 mathematically by considering its performance when it is applied to a random unit disk graph $G_n$. Specifically, let $\ell_1 \leq \ell_2 \leq \ldots$ be a sequence of real numbers such that $\ell_n = O(\sqrt{n/\log n})$ as $n \to \infty$, but $\ell_n \geq \log n$ for all $n$. Let $Q_n$ be an $\ell_n \times \ell_n$ square region in $\mathbb{R}^2$. Select $n$ points $V_1, V_2, \ldots, V_n$ independently and uniformly randomly from an $Q_n$, and use these $n$ points as the vertex set for a unit disk graph $G_n$. We prove asymptotic estimates for the expected size of the Rule 2 dominating set. The proof involves some interesting problems in elementary geometry and geometric probability.

\section{A Geometric Lemma}

As observed in [20], a unit disk centered at a point $o$ cannot be completely covered with two unit disks having centers at points $u$ and $w$ ($u \neq o \neq w$): $(D_1(u) \cup D_1(w))^c \cap D_1(o) \neq \emptyset$. One might infer that a typical vertex $o$ is not likely to be be pruned under Rule 2 because no two points in $\mathcal{N}(o)$ will cover all the vertices in $\mathcal{N}(o)$. This reasoning suggests that Rule 2 will be ineffective. But such reasoning is not sound. Typically there are points $u$ and $w$ that cover all but a negligible fraction of the disk centered at $o$. The uncovered region is small enough so that it usually does not include any nodes. A more precise version of this statement is proved in the next section, but first we need to look carefully at the area of regions such as $(D_1(u) \cup D_1(w))^c \cap D_1(o)$. In particular, we need Lemma 1, which is the main result in this section.

To state Lemma 1 we adopt some notation. Throughout this section $b > 1$ will be a parameter and in terms of $b$ we let $L = [b^{1/3}(\log b)^2]$, $\delta = \frac{1}{\sqrt{b \log b}}$, and $\theta_b = \pi/L$. We fix $o = (x_o, y_o) \in \mathbb{R}^2$ and for any $r > 0$, let $D_r(o)$ be the closed disk centered at $o$ with radius $r$. We are going to partition the small disk $D_b(o)$ into $2L$ sectors as follows. Choose a new coordinate system centered at $o$, and for $0 \leq i < L$, let $Q_i$ be the sector consisting of those points $(x, y) = (r \cos \theta, r \sin \theta)$ whose polar coordinates satisfy $0 < r < \delta$ and $(i - \frac{1}{2})\theta_b \leq \theta < (i + \frac{1}{2})\theta_b$. Similarly let $R_i$ be the sector that is obtained by reflecting $Q_i$ about $o$, namely the points with $0 < r \leq \delta$ and $(i - \frac{1}{2})\theta_b < \theta - \pi < (i - \frac{1}{2})\theta_b$. The analysis of Rule 2 depends on a geometric lemma about these sectors. For any $i$, and any points $q_i \in Q_i, u_i \in R_i$, let $X(q_i, u_i)$ be the area of $(D_1(q_i) \cup D_1(u_i))^c \cap D_1(o)$, i.e. the area of the omitted region in $D_1(o)$ that is not covered by $(D_1(q_i) \cup D_1(u_i))$. Let $\tilde{q}_i$ and $\tilde{u}_i$ be the extreme points whose polar coordinates are respectively $(r, \theta) = (\delta, (i - \frac{1}{2})\theta_b)$ and $(r, \theta) = (\delta, (i + \frac{1}{2})\theta_b + \pi)$. We prove:
Lemma 1 There is a uniform constant $C > 0$ such that, for $0 \leq i < L$, and for all $q_i \in Q_i$, $u_i \in R_i$, we have $X(q_i, u_i) \leq X(\bar{q}_i, \bar{u}_i) \leq \frac{\epsilon}{\log b}$.

Proof. We prove four facts which together imply Lemma 1. In the first fact, we observe that omitted area $X(q, u)$ gets larger if we move one (or both) of the two points $q, u$ away from the origin along a radial line.

Fact 1 Let $q_1, q_2$ and $u_1, u_2$ be four points in $D_1(o)$ such that $q_1$ lies on the line segment $\overline{q_1 q_2}$ and $u_1$ lies on the line segment $\overline{o u_2}$. Then $X(q_2, u_2) \geq X(q_1, u_1)$.

Proof. It suffices to show that $D_1(q_2) \cap D_1(o) \subseteq D_1(q_1) \cap D_1(o)$ and that $D_1(u_2) \cap D_1(o) \subseteq D_1(u_1) \cap D_1(o)$. Suppose $p \in D_1(q_2) \cap D_1(o)$. Since $q_1$ lies on the line segment from $o$ to $q_2$, we have $d(q_1, p) \leq \max(d(o, p), d(q_2, p)) \leq 1$. Hence $p \in D_1(q_1) \cap D_1(o)$. By a similar same argument, $D_1(q_2) \cap D_1(o) \subseteq D_1(q_1) \cap D_1(o)$.

Fact 2 Let $a, b$ be the two points where the circles $\partial D_1(p), \partial D_1(q)$ intersect. Then, $\overline{a b} \perp \overline{p q}$, and the two line segments $\overline{a b}$ and $\overline{p q}$ intersect at their midpoints.

Proof. This follows immediately from the fact that $d(p, a) = d(p, b) = d(q, a) = d(q, b) = 1$.

Fact 3 Let $o_1, o_2$ be two points on the circle $x^2 + y^2 = \delta^2$. Then, $X(o_1, o_2)$ is a decreasing function of $\angle o_1 o o_2$.

Proof. For convenience, we will use polar coordinates. Without loss of generality, let $o_1$ be the point with polar coordinates $(r_{o_1}, \phi_{o_1}) = (\delta, \pi)$. Let $o_2$ be an arbitrary point on the circle with the polar coordinates $(\delta, \phi_2)$. By symmetry, we only need to consider the case when $o_2$ is in the first or second quadrant; we may, without loss of generality, assume that $0 \leq \phi_2 \leq \pi$. We will show that $X(o_1, o_2)$ is an increasing function of $\phi_2$, then the result follows from the fact that $\angle o_1 o o_2 = \pi - \phi_2$.

Let $a_1, b_1$ be the two points where the circles $\partial D_1(o_1)$ and $\partial D_1(o)$ intersect, with $a_1$ in the second quadrant and $b_1$ in the third quadrant.

Let $o^*$ be a point on the circle $x^2 + y^2 = \delta^2$ so that $\partial D_1(o^*)$ meets with both $\partial D_1(o)$ and $\partial D_1(o_1)$ at $a_1$. Let $b^*, d^*$ be the other intersection points of $\partial D_1(o^*)$ with $\partial D_1(o)$ and $\partial D_1(o_1)$, respectively. For convenience, let’s denote $\phi_{o^*}$ by $\phi^*$. Figure 1 illustrates the position of $\partial D_1(o_1)$, $\partial D(o)$, and $\partial D_1(o^*)$ and their intersections.
As in the proof of Fact 2, we have $a_1, d^* \perp o_1, o^*$, $a_1, b^* \perp o, o^*$. Notice also that $o$ is on the line segment $a_1, d^*$. So,

$$\angle b^*a_1o = \angle oo^*o_1 = \angle o^*o_1o = \frac{\phi^*}{2}.$$  \hspace{1cm} (1)

It follows that

$$0 < \frac{\phi^*}{2} < \frac{\pi}{2}, \text{ and, } \sin \frac{\phi^*}{2} = \frac{\delta}{2}$$ \hspace{1cm} (2)

Now, for the point $o_2$ with polar coordinates $(\delta, \phi_2)$, let $a_2, b_2$ denote the two points where $\partial D_1(o_2)$ and $\partial D_1(o)$ intersect, and let $c_2, d_2$ denote the two points where $\partial D_1(o_2)$ and $\partial D_1(o_1)$ intersect. There are two cases to consider: $\phi_2 \leq \phi^*$, and $\phi_2 \geq \phi^*$

Case 1. $\phi_2 \leq \phi^*$.

$$\angle b^*a_1o = \angle oo^*o_1 = \angle o^*o_1o = \frac{\phi^*}{2}.$$  \hspace{1cm} (1)
Notice that $a_1, b_1$ partitions the circle $\partial D_1(o)$ into two arcs: the right section and the left section. When, $\phi_2 \leq \phi^*$, as illustrated in Figure 2, $a_2, b_2$ are both on the right section of the circle $\partial D_1(o)$ between $a_1, b_1$. Similarly, $c_2, d_2$ are both on the right section of the circle $\partial D_1(o_1)$ between $a_1, b_1$. Clearly,

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3,$$

where

- $B_1 = \text{area}(D_1(o_1)^c \cap D_1(o))$
- $B_2 = \text{area}(D_1(o) \cap D_1(o_2))$
- $B_3 = \text{area}(D_1(o_1) \cap D_1(o_2))$, the shaded area in Figure 2.

Notice that $B_3$ is the only area that depends on $\phi_2$. We shall now give an expression for $B_3$.

Let’s denote $\angle c_2 o_1 o_2 = y$. Since $\angle o_2 o_1 o = \frac{\phi_2}{2}$, we have

$$0 < y < \frac{\pi}{2}, \quad \text{and,} \quad \cos y = \delta \cos \frac{\phi_2}{2}$$

(3)

By symmetry, one can see that the shaded region is partitioned equally by the line $c_2, d_2$. So,

$$B_3 = 2(\frac{2y}{2\pi} - \frac{1}{2}(2\sin y)(\cos y)) = 2y - \sin 2y.$$

Here, the first term is the area of the sector $D_1(o_1)$ that extends from $c_2$ to $d_2$, and the second term is the area of the triangle($c_2, o_1, d_2$).

From the above two equations, we have

$$\frac{dX(o_1, o_2)}{d\phi_2} = \frac{dB_3}{d\phi_2} = \frac{dB_3}{dy} \cdot \frac{dy}{d\phi_2} = (1 - \cos 2y) \cdot \frac{\delta \sin \frac{\phi_2}{2}}{2\sin y} > 0.$$

Here the last inequality follows from the fact that $0 < \frac{\phi_2}{2}, y < \frac{\pi}{2}$. Thus $X(o_1, o_2)$ is an increasing function in $\phi_2$.

Case 2. $\phi_2 > \phi^*$. 

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Figure 3: The case when $\phi_2 > \phi^*$

One can see from Figure 3 that

$$X(o_1, o_2) = B_1 - (B_2 - B_3) = B_1 - B_2 + B_3$$

Where $B_1, B_2$ are defined the same as those in the case 1, but

$$B_3 = \text{area}(D_1(o_1) \cap D_1(o_2) \cap D_1(o)), \text{ the shaded area in Figure 3}$$

Again, $B_3$ is the only area that depends on $\phi_2$. We will now give an expression of $B_3$.

We show first that $\angle c_2 o a_1 = \angle a_2 o c_2$ by showing that $\phi_{c_2} - \phi_{a_1} = \phi_{a_2} - \phi_{c_2}$. Then, it follows that $B_3$ is split in half by the line segment $c_2, d_2$.

\begin{align*}
\phi_{a_1} &= \phi^* + \left(\frac{\pi}{2} - \angle b^*a_1o\right) = \phi^* + \left(\frac{\pi}{2} - \frac{\phi^*}{2}\right) = \frac{\pi}{2} + \frac{\phi^*}{2} \quad (4)
\end{align*}

To find $\phi_{a_2}$, observe that, as in the proof of Fact 2, we have $a_2, b_2 \perp \overline{o_1, o_2}$. So, $\sin \angle b_2 a_2 o = \frac{\phi^*}{2}$. Comparing with (2), we see that $\sin \angle b_2 a_2 o = \sin \frac{\phi^*}{2}$. This implies that $\angle b_2 a_2 o = \frac{\phi^*}{2}$. Thus,

\begin{align*}
\phi_{a_2} &= \phi_2 + \left(\frac{\pi}{2} - \angle b_2 a_2 o\right) = \phi_2 + \left(\frac{\pi}{2} - \frac{\phi^*}{2}\right) \quad (5)
\end{align*}

Now, for $c_2$, using the fact that $c_2, o \perp \overline{o_1, o_2}$,

\begin{align*}
\phi_{c_2} &= \pi - \left(\frac{\pi}{2} - \angle c_2 a_1 o\right) = \pi - \left(\frac{\pi}{2} - \frac{\phi_2}{2}\right) = \frac{\pi}{2} + \frac{\phi_2}{2} \quad (6)
\end{align*}

It follows that $\phi_{c_2} - \phi_{a_1} = \phi_{a_2} - \phi_{c_2} = \frac{\phi_2}{2} - \frac{\phi^*}{2}$. Now, using that the circle $\partial D_1(o_1)$ in the polar system is

$$r = \sqrt{1 - \delta^2 \sin^2 \phi - \delta \cos \phi}$$
and that

\[ \phi_{d_2} = - (\pi - \phi_{c_2}) = - \left( \frac{\pi}{2} - \frac{\phi_2}{2} \right) \]  

we get

\[
B_3 = 2 \left( \int_{-\frac{x}{2} - \frac{x}{2}}^{\frac{x}{2}} \sqrt{1 - \delta^2 \sin^2 \phi - \delta \cos \phi} \, r \, dr \, d\phi + \frac{\phi_2 - \phi_1}{2\pi} \cdot \pi \right)
\]

Thus,

\[
\frac{dX(q_1, q_2)}{d\phi_2} = \frac{dB_3}{d\phi_2} = - \frac{1}{2} \left[ 1 - \delta^2 \sin^2 \left( -\frac{x}{2} + \frac{x}{2} \right) + \delta^2 \cos^2 \left( -\frac{x}{2} + \frac{x}{2} \right) \right]
\]

\[
-2\delta \cos \left( -\frac{x}{2} + \frac{x}{2} \right) \sqrt{1 - \delta^2 \sin^2 \left( -\frac{x}{2} + \frac{x}{2} \right)} + \frac{1}{2}
\]

\[
= \frac{1}{2} \left[ \delta^2 \cos^2 \frac{x}{2} - \delta^2 \sin^2 \frac{x}{2} + 2\delta \sin \frac{x}{2} \sqrt{1 - \delta^2 \cos^2 \frac{x}{2}} \right]
\]

\[
= \frac{1}{2} \left[ -\left( \sin \frac{x}{2} - \sqrt{1 - \delta^2 \cos^2 \frac{x}{2}} \right)^2 + 1 \right]
\]

\[
\geq 0
\]

The last inequality follows because \( 0 \leq \delta \sin \frac{x}{2} \leq 1 \), \( 0 \leq \sqrt{1 - \delta^2 \cos^2 \frac{x}{2}} \leq 1 \), and thus \( \left( \sin \frac{x}{2} - \sqrt{1 - \delta^2 \cos^2 \frac{x}{2}} \right)^2 < 1 \).

□

**Fact 4** Uniformly for all \( i \), we have \( X(q_i, \hat{u}_i) = O \left( \frac{1}{\pi \log b} \right) \).

**Proof.** Without loss of generality, let \( i = 0 \) and \( v = (0, 0) \). To simplify notation, define \( x_b = \delta \cos \left( -\frac{1}{2} \theta_0 \right) \), \( y_b = \delta \sin \left( -\frac{1}{2} \theta_0 \right) \). Let \( (\xi, \eta) \) be the point in the first quadrant where the circles \( x^2 + y^2 = 1 \) and \( (x - x_b)^2 + (y - y_b)^2 = 1 \) meet. Then

\[
X(q_0, \hat{u}_0) \leq 4 \int_0^\xi \sqrt{1 - x^2} - (y_b + \sqrt{1 - (x - x_b)^2}) \, dx
\]

\[
= -4y_b \xi + 4 \int_0^\xi \frac{-2x_b x + x_b^2}{\sqrt{1 - x^2} + \sqrt{1 - (x - x_b)^2}} \, dx
\]

Hence we have

\[
X(q_0, \hat{u}_0) = O(\xi y_b) + O(x_b \xi^2) + O(x_b^2 \xi).
\]

Note that \( x_b^2 + y_b^2 = \delta^2 = \frac{1}{b \log^3 b} \), that \( \xi^2 + \eta^2 = 1 \), that \( (\xi - x_b)^2 + (\eta - y_b)^2 = 1 \), that \( x_b = \delta(1 + O(\theta_0^2)) \), and that \( y_b = -\frac{\theta_0}{2} (1 + O(\theta_0^2)) \). Combining these equations, we get \( \xi = O(\delta) \). Putting this estimate back into (8), we get

\[
X(q_0, \hat{u}_0) = O \left( \frac{1}{b \log^3 b} \right).
\]
In the analysis of Rule 2 it is necessary to consider vertices in $\mathcal{G}_n$ which are close to the boundary of the square $Q_n$. For this reason we define, for $o \in \mathbb{R}^2_+$, the “truncated unit disk” $\hat{D}_i(o) := D_1(o) \cap \mathbb{R}^2_+$ and we note that $\hat{D}_i(o) \subseteq D_1(o)$, and $\hat{D}_i(o) = D_1(o)$ iff $x_o, y_o \geq 1$. Then for $L$ and $\delta$ as defined above, we have the following corollary to Lemma III.

**Corollary 2** There is a uniform constant $C > 0$ such that, for all $o \in \mathbb{R}^2_+$ such that $D_0(o) \subseteq \mathbb{R}^2_+$, for $0 \leq i < L$, and for all $q_i \in Q_i$, $u_i \in R_i$, we have $\hat{X}(q_i, u_i) \leq X(\tilde{q}_i, \tilde{u}_i) \leq \frac{C}{b \log b}$, where $\hat{X}(q, u)$ is the area of $(D_1(q) \cap D_1(u))^c \cap \hat{D}_1(o)$.

**Proof.** Clearly $\hat{X}(q_i, u_i) \leq X(q_i, u_i)$ since $\hat{D}_1(o) \subseteq D_1(o)$. So the result follows from Lemma III (since $\tilde{q}_i, \tilde{u}_i \in D_0(o) \subseteq \mathbb{R}^2_+$).

\[\square\]

### 3 Local Coverage by Two Discs

Recall that under Rule 2 a vertex $v_i$ is excluded from $\mathcal{C}(V)$ if there are two adjacent vertices, $v_{i_1}, v_{i_2} \in \mathcal{N}(v_i)$, with higher IDs than $v_i$ which also ‘cover’ $v_i$, i.e. $\mathcal{N}(v_i) \subseteq \mathcal{N}(v_{i_1}) \cup \mathcal{N}(v_{i_2})$. In the analysis of Rule 2 we will distinguish vertices in $\mathcal{N}(v_i)$ with higher ID than $v_i$ by coloring them blue; all other vertices in $\mathcal{N}(v_i)$ are colored white. With this in mind, we consider in this section a two-colored random unit disk graph and prove a local coverage result.

Let $w$ and $b$ be positive integers such that $w < b(\log b)^2$ and, as before, let $L = \lfloor b^{1/3}(\log b)^{3/2} \rfloor$ and $\delta = \frac{1}{b \log b}$. Fix $o \in \mathbb{R}^2_+$ such that $D_0(o) \subseteq \mathbb{R}^2_+$ and select $w+b$ points independently and uniform randomly from the truncated disk $\hat{D}_1(o)$. Color the first $w$ points white, and the remaining $b$ points blue. Form a random (improperly colored) unit disk graph $\hat{\mathcal{H}}_{w,b}$ by putting an edge between two of the $w+b$ colored points iff the distance between them is one or less. Our goal in this section is to prove that, with high probability, $\hat{\mathcal{H}}_{w,b}$ contains a dominating set consisting of two blue vertices that are adjacent to each other.

For $0 \leq i < L$, let $Q_i, R_i$ denote the sectors of $D_0(o)$ as defined in the previous section and let $N(Q_i), N(R_i)$ respectively be the number of blue vertices of $\hat{\mathcal{H}}_{w,b}$ that lie in $Q_i$ and $R_i$. Let $\tau_b = \sum_{i=0}^{L-1} I_i$ where, in this section only, the $I_i = 1$ if and only if $N(R_i) = N(Q_i) = 1$ (and otherwise $I_i = 0$.) We note that the distribution of $\tau_b$ depends on the position of $o$ and we indicate this dependence by using the notation $Pr_o(\tau_b \in \cdot)$. Provided $o$ is not too close to the boundary of $\mathbb{R}^2_+$, we can obtain uniform bounds on the tail of the distribution of $\tau_b$.

**Lemma 3** $Pr_o \left( \tau_b < \frac{b^{1/3}}{16 \log b} \right) = O \left( \frac{\log^{2} b}{b^2} \right)$ uniformly for all $o \in \mathbb{R}^2_+$ such that $D_0(o) \subseteq \mathbb{R}^2_+$.

**Proof.** Let $|\hat{D}_1(o)|$ denote the area of $\hat{D}_1(o)$, let $\hat{\lambda} = \frac{\hat{\lambda}(o)}{|D_1(o)|}$, and define

$$\hat{p} = \frac{\text{Area}(Q_i)}{|\hat{D}_1(o)|} = \frac{\pi \delta^2 / 2L}{|\hat{D}_1(o)|} = \frac{\hat{\lambda}}{2b \log b} \left( 1 + O \left( \frac{1}{b^{1/3} \log^2 b} \right) \right).$$

(10)

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The expected value of $I_i$ depends on $o$:

$$E_o(I_i) = b(b - 1)\hat{p}^2(1 - 2\hat{p})^{b - 2} = \frac{\hat{\lambda}^2}{4 \log^5 b} \left( 1 + O\left(\frac{1}{\log^4 b}\right)\right). \quad (11)$$

Hence

$$E_o(\tau_o) = LE_o(I_i) = \frac{b^{1/3}\hat{\lambda}^2}{4 \log^5 b} \left( 1 + O\left(\frac{1}{\log^4 b}\right)\right). \quad (12)$$

We likewise have, for $i \neq j$,

$$E_o(I_i, I_j) = b(b - 1)(b - 2)(b - 3)\hat{p}^4(1 - 4\hat{p})^{b - 4} = \frac{\hat{\lambda}^4(a)}{16 \log^{16} b} \left( 1 + O\left(\frac{1}{\log^4 b}\right)\right). \quad (13)$$

Note that

$$\pi \geq |\hat{D}_1(o)| \geq \frac{\pi}{4}, \quad (14)$$

and therefore

$$1 \leq \hat{\lambda}(o) \leq 4. \quad (15)$$

Therefore we have uniformly for all $o \in \mathbb{R}^2_+$ such that $D_{\delta}(o) \subseteq \mathbb{R}^2_+$

$$Var(\tau_o) = O\left(\frac{b^{1/3}}{\log^4 b}\right). \quad (16)$$

Observe that

$$\Pr_o \left( \tau_o < \frac{b^{1/3}}{16 \log^5 b} \right) \leq \Pr_o \left( \tau_o \leq \frac{1}{2}E_o(\tau_o) \right) \leq \Pr_o \left( |\tau_o - E_o(\tau_o)| > \frac{1}{2}E_o(\tau_o) \right). \quad (17)$$

The lemma now follows from [16], [17] and Chebyshev’s inequality. \hfill \Box

Recall our assumptions that $w < b(\log b)^{3/2}$, that $\delta = \frac{1}{b^{1/3} \log b}$, and that $x_o, y_o \geq \delta$. With these assumptions, we have:

**Theorem 4** There is a constant $c > 0$, independent of the position of $o$, such that with probability at least $1 - \frac{c}{(\log b)^{3/2}}$, the random graph $H_{w,b}$ has a connected dominating set that consists of two blue vertices in $D_{\delta}(o)$.

**Proof.**

Let $T_o \subseteq \{0, 1, 2, 3, \ldots, L - 1\}$ be the random subset of indices such that $i \in T_o$ iff $N(Q_i) = N(R_i) = 1$. If $T_o = \emptyset$, define $Y = \min T_o$ to be the smallest of the indices in $T_o$; otherwise, if $T_o = \emptyset$, set $Y = -1$.

Define the random variable $X_o$ as follows: If $\tau_o = |T_o| = 0$ then $X_o = 0$; otherwise, if $T_o = \{i_1, i_2, \ldots, i_{\tau_o}\}$ and $i_1 < i_2 < \ldots < i_{\tau_o}$, then $X_o = 1$ iff $Q_{i_1} \cup R_{i_1}$ contains a blue connected dominating set for $H_{w,b}$.

Let $B = \{g_1, g_2, \ldots, g_b\}$ be the set of blue nodes, selected independently and uniform randomly from $D_1(o)$. Define $Z = B \cap D_{\delta}(o)$ to be set of blue points that fall near the origin $o$, and let $Z = |Z|$ be the number of these points. Then
\[
\Pr_o(X_b = 0) \leq \Pr_o\left( X_b = 0, \tau_b \neq 0, Z \leq \frac{2\lambda b^{1/3}}{(\log b)^2} \right) + \Pr_o(\tau_b = 0) + \Pr_o\left( Z > \frac{2\lambda b^{1/3}}{(\log b)^2} \right).
\]  

(18)

Note that \( Z \) has a binomial distribution: \( Z \sim Bin(b, \lambda b^2) \) where \( \lambda \) is as defined in the proof of Lemma 3. If \( \beta = \frac{2\lambda b^{1/3}}{(\log b)^2} \), then by Chernoff’s inequality,

\[
\Pr_o(Z \geq \beta) \leq \exp(-b^{1/3}/4(\log b)^2).
\]  

(19)

By Lemma 3 \( \Pr_o(\tau_b = 0) = O\left(\frac{\log b}{b^{1/3}}\right) \). Therefore

\[
\Pr_o(X_b = 0) \leq \Pr_o(X_b = 0, \tau_b \neq 0, Z \leq \beta) + O\left(\frac{\log b}{b^{1/3}}\right).
\]  

(20)

Now we decompose the first term on the right side of (20) according to the value of \( Y \).

\[
\Pr_o(X_b = 0, \tau_b \neq 0, Z \leq \beta) = \sum_{k=0}^{L-1} \Pr_o(X_b = 0) | Y = k, Z \leq \beta \Pr_o(Y = k, Z \leq \beta).
\]  

(21)

(The redundant condition \( \tau_b \neq 0 \) need not be included on the right side of (21) because it a consequence of the condition \( Y \geq 0 \).) We have

\[
\Pr_o(X_b = 0) | Y = k, Z \leq \beta = \sum_S \Pr_o(X_b = 0) | Z = S, Y = k \Pr_o(Z = S | Y = k, Z \leq \beta)
\]  

(22)

where the sum is over subsets \( S \subseteq [b] \) such that \( 2 \leq |S| \leq \beta \).

\[
Pr(X_b = 0) | Z = S, Y = k = 1 - Pr(X_b = 1) | Z = S, Y = k,
\]  

so it is enough to find a lower bound for \( Pr(X_b = 1) | Z = S, Y = k \).

To simplify notation, let \( \gamma = X(\tilde{q}_0, \tilde{u}_0) \), and recall that \( \gamma = O\left(\frac{1}{b^{1/3}}\right) \). In this section of the paper, define \( |D_b(o)| = \frac{\pi}{b^{2/3}(\log b)^2} \) to be the area of the disk \( D_b(o) \), and let \( |\hat{D}_1(o)| = Area(\hat{D}_1(o)) \). An important observation is that, once we have specified \( b - |S| \) the number of blue points that fall outside \( D_b(o) \), the locations in \( D_b(o) \cap \hat{D}_1(o) \) of these \( b - |S| \) points are independent of the locations of the \( |S| \) blue points in \( D_b(o) \), and are also independent of the locations of the white points. Hence

\[
\Pr_o(X_b = 1) | Z = S, Y = k \geq \left( 1 - \frac{|D_b(o)|}{|\hat{D}_1(o)|} \right)^{\gamma} \left( 1 - \frac{|S|}{|\hat{D}_1(o)|} \right)^{\gamma} \left( 1 - \frac{\gamma}{|\hat{D}_1(o)|} \right)^{\gamma} \left( 1 - \frac{C}{b^{2/3}(\log b)^2} \right)^{b - |S| + w}
\]  

(24)

\[
\geq \left( 1 - \frac{C}{b^{2/3}(\log b)^2} \right)^{b - |S| + w}
\]  

(25)
for some constant $C$ that is independent of $o$. With our assumption $w < b(\log b)^{3/2}$ we get, for all sufficiently large $b$, the lower bound

$$\Pr_o(X_b = 1|Z = S, Y = k) \geq \left(1 - \frac{C'}{b(\log b)^3}\right)^{b(\log b)^{3/2}} \geq 1 - \frac{C''}{(\log b)^{3/2}}$$

(26)

for some constants $C'$ and $C''$ which are independent of $Z, Y,$ and $o$. Hence

$$\Pr_o(X_b = 0) \leq c(\log b)^{3/2}$$

(27)

for some constant $c$ that is independent of the point $o$.

□

4 Analysis of Rule 2

Let $U$ be the number of nodes that become non-gateways when Rule 2 is applied to the random graph $G_1$: $U = \sum_i I_i$ where (in this section) the indicator variable $I_i = 1$ iff the node with ID $i$ becomes a non-gateway under Rule 2. Assume that there is a positive constant $\bar{c}$ such that, for all $n > 1$, $\log n \leq \ell_n \leq \bar{c}\sqrt{n \log n}$. Let $\xi_n = \alpha_n \ell_n^2$, where $(\alpha_n)$ is any sequence of real numbers satisfying the following three conditions:

- $\alpha_n = o(n)$ as $n \to \infty$.
- $\xi_n = \frac{\alpha_n}{\ell_n^2} \to \infty$ as $n \to \infty$.
- For all sufficiently large $n$, $\frac{16\alpha_n}{\log^{3/2} \xi_n} < \alpha_n$.

For example, if $\ell_n = \Theta(\sqrt{n / \log n})$, then the sequence $\alpha_n = \frac{32n}{(\log \log n)^t}$ satisfies the three conditions. On the other hand, if $\ell_n = \Theta((n / \log n)^t)$ for some fixed positive $t < 1/2$, then $\alpha_n = \frac{n}{\log n}$ satisfies the three conditions above. With these three assumptions, our goal is to prove

**Theorem 5** $E(U) \geq n - O(\alpha_n)$.

**Proof.** The idea of the proof is to use Theorem 4 to bound the probability that a typical vertex $V_i$ is pruned by Rule 2. In this case the blue vertices correspond to nodes in $D_1(V_i)$ with IDs higher than $i$, and the white vertices correspond to nodes in $D_1(V_i)$ with lower IDs. Let $r = \frac{1}{\log^{3/2} \xi_n}$, and let $A_i$ be the event that $D_r(V_i) \subseteq Q_n$. Then

$$\Pr(A_i) = \frac{(\ell_n - 2r)^2}{\ell_n^2} \geq 1 - \frac{4r}{\ell_n},$$

(28)

Let $\hat{D}_1(V_i) = D_1(V_i) \cap Q_n$ be the set of points in $Q_n$ whose distance from $V_i$ is one or less, and let $|\hat{D}_1(V_i)|$ be the area of $\hat{D}_1(V_i)$. Let $\rho_i^{(b)}$ denote the number
of nodes in $\hat{D}_1(V_i)$ having a label that is larger than $i$, and let $\rho_i^{(w)}$ be the number of nodes in $\hat{D}_1(V_i)$ having a label that is smaller than $i$. Then, given the location of the $i$'th vertex $V_i$, $\rho_i^{(b)}$ has a Binomial($n - i, \frac{|\hat{D}_1(V_i)|}{\ell_n^2}$) distribution. Define $\mu_b = \mu_b(i)$ to be the expected value of $\rho_i^{(b)}$ given the location of the $i$'th point:

$$\mu_b = E(\rho_i^{(b)} | V_i) = \frac{(n - i)|\hat{D}_1(V_i)|}{\ell_n^2}. \quad (29)$$

Similarly $\rho_i^{(w)}$ has a Binomial($i - 1, \frac{|\hat{D}_1(V_i)|}{\ell_n^2}$) distribution, and we define $\mu_w = \mu_w(i)$ to be the expected value:

$$\mu_w = E(\rho_i^{(w)} | V_i) = \frac{(i - 1)|\hat{D}(V_i)|}{\ell_n^2}. \quad (30)$$

If $A_i$ occurs, then by Chebyshev’s inequality,

$$\Pr(|\rho_i^{(b)} - \mu_b(i)| < \frac{\mu_b}{2} | A_i) \geq 1 - \frac{16\ell_n^2}{n - i}, \quad (31)$$

and similarly for $\rho_i^{(w)}$.

If we let $D_i$ be the event that both of the inequalities $|\rho_i^{(b)} - \mu_b(i)| < \frac{\mu_b}{2}$ and $|\rho_i^{(w)} - \mu_w(i)| < \frac{\mu_w}{2}$ are satisfied, then

$$\Pr(D_i | A_i) \geq 1 - \frac{16\ell_n^2}{n - i} - \frac{16\ell_n^2}{i - 1}. \quad (32)$$

Combining (32) and (29), we get

$$\Pr(D_i \cap A_i) \geq \left(1 - \frac{16\ell_n^2}{n - i} - \frac{16\ell_n^2}{i - 1}\right) \left(1 - \frac{4r}{\ell_n}ight). \quad (33)$$

Now let $\lambda_n = n - \alpha_n$, then clearly

$$E(U) \geq \sum_{\lambda_n} \Pr(I_i = 1) \geq \sum_{\lambda_n} \Pr(I_i = 1 | D_i \cap A_i) \Pr(D_i \cap A_i) \quad (34)$$

To obtain a lower bound for the right hand side of inequality (34), we prove

**Lemma 6** There is a constant $\tilde{c} > 0$ such that for all sufficiently large $n$ and all $\alpha_n \leq i < \lambda_n$, $\Pr(I_i = 1 | D_i \cap A_i) \geq 1 - \frac{i^{\tilde{c}}}{(\log \lambda_n)^{x}}$.

**Proof.** We begin by noting that given the event $D_i \cap A_i$ and $\alpha_n \leq i < \lambda_n = n - \alpha_n$, we have

$$\rho_i^{(w)} < \frac{3}{2} \mu_w(i) = \frac{3(i - 1)|\hat{D}_1(V_i)|}{2\ell_n^2} \leq \frac{3\pi n}{2\ell_n^2}. \quad (35)$$
Similarly

\[ \rho^{(b)}_i > \frac{1}{2} \mu_b(i) = \frac{(n-i)|\hat{D}_1(V_i)|}{2\ell^2_n} > \frac{\alpha_n \pi}{8\ell^2_n} = \frac{\xi_n \pi}{8} \tag{36} \]

It follows from inequalities (35) and (36) and from the conditions on the sequences \(\langle \xi_n \rangle\) and \(\langle \alpha_n \rangle\) that, given \(D_i \cap A_i\) and \(\alpha_n \leq i < \lambda_n\),

\[ \rho^{(b)}_i (\log \rho^{(b)}_i)^{3/2} \geq \rho^{(w)}_i. \tag{37} \]

Next we consider the conditional probability \(\Pr(I_i = 1|\rho^{(b)}_i, \rho^{(w)}_i, V_i, D_i \cap A_i)\) where the values of \(\rho^{(b)}_i\) and \(\rho^{(w)}_i\) and the location of \(V_i\) are consistent with the event \(D_i \cap A_i\). In this case, it follows from inequality (36) that

\[ \delta(\rho^{(b)}_i) := \left( \frac{1}{\rho^{(b)}_i} \right)^{1/3} \log(\rho^{(b)}_i) \leq \frac{1}{(\xi_n/3)^{1/3} \log(\xi_n/3)} \leq \frac{1}{(\log(\xi_n/3))^{3/2}} = r. \tag{38} \]

Since the event \(A_i\) implies \(D_r(V_i) \subseteq \mathbb{R}^2_+\), it follows from (38) that \(D_{\delta(\rho^{(b)}_i)}(V_i) \subseteq \mathbb{R}^2_+\). Finally, it follows from Theorem 4 that for some fixed positive constant \(\tilde{c}\)

\[ \Pr(I_i = 1|\rho^{(b)}_i, \rho^{(w)}_i, V_i, D_i \cap A_i) \geq 1 - \frac{c}{(\log(\rho^{(b)}_i))^{3/2}} \geq 1 - \frac{\tilde{c}}{(\log(\xi_n^{(b)}))^{3/2}} \tag{39} \]

for all sufficiently large \(n\) and all \(\alpha_n \leq i < \lambda_n\). The lemma now follows from (39).

Recall that \(\lambda_n = n - \alpha_n\), that \(\alpha_n = o(n)\), that \(\xi_n = \frac{\alpha_n}{\ell^2_n} \to \infty\) as \(n \to \infty\), and that for all sufficiently large \(n\), \(\alpha_n > \frac{16n}{(\log \xi_n)^{3/2}}\). So it follows from Lemma 6 and (33) and (34), that

\[ E(U) \geq n - 2\alpha_n + o(\alpha_n). \]

5 Discussion

In this final section, assume \(\ell_n = \Theta((\frac{n}{\log n})^t)\) for some fixed positive \(t \leq \frac{1}{2}\). For all sufficiently large \(n\), the expected size of the Rule 2 dominating set is at least \(\ell^2_n/4\) (See Theorem 5 of [17]). There is a gap between this lower bound and the \(O(\alpha_n)\) upper bound in Theorem 5. For example, when \(t = 1/2\), the lower and upper bounds for the expected size of the Rule 2 dominating set are respectively \(\Theta(n/\log n)\) and \(\Theta(n/(\log \log n)^{3/2})\). For \(t < 1/2\) the gap is even wider: the lower and upper bounds are respectively \(\Theta((n/\log n)^{-t})\) and \(\Theta(\frac{n}{\log n})\). We conjecture that, in fact, the expected size of the Rule 2 dominating set is \(\Theta(\ell^2_n)\).
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