Upper Tail Estimates with Combinatorial Proofs

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Abstract

We study generalisations of a simple, combinatorial proof of a Chernoff bound similar to the one by Impagliazzo and Kabanets (RANDOM, 2010).

In particular, we prove a randomized version of the hitting property of expander random walks and use it to obtain an optimal expander random walk concentration bound settling a question asked by Impagliazzo and Kabanets.

Next, we obtain an upper tail bound for polynomials with input variables in [0, 1] which are not necessarily independent, but obey a certain condition inspired by Impagliazzo and Kabanets. The resulting bound is applied by Holenstein and Sinha (FOCS, 2012) in the proof of a lower bound for the number of calls in a black-box construction of a pseudorandom generator from a one-way function.

We also show that the same technique yields the upper tail bound for the number of copies of a fixed graph in an Erdős–Rényi random graph, matching the one given by Janson, Oleszkiewicz, and Rucinski (Israel J. Math, 2002).

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1 Introduction

Motivation and previous work

Concentration bounds are inequalities that estimate the probability of a random variable assuming a value that is far from its expectation. They have a multitude of applications all across the mathematics and theoretical computer science. See, e.g., textbooks [26, 25, 4, 10] for uses in complexity theory and randomised algorithms.

A typical setting is when this variable is a function \( f(x) \) of \( n \) simpler random variables \( x = (x_1, \ldots, x_n) \) that possess a certain degree of independence and we try to bound said probability with a function decaying exponentially with \( n \) (or, maybe, \( n^\epsilon \) for some \( \epsilon > 0 \)).

The canonical examples are Chernoff-Hoeffding bounds [7, 13] for the sum of \( n \) independent random variables in [0, 1] and Azuma’s inequality [5] for martingales.

The standard technique to prove Chernoff bounds is due to Bernstein [6]. The idea is to bound \( E[e^{tf(x)}] \) for some appropriately chosen \( t \), and then to apply Markov’s inequality.

Recently, Impagliazzo and Kabanets [16] gave a different, combinatorial proof of Chernoff bound, arguing that its simplicity and nature provide additional insight into understanding concentration. What is more, their proof is constructive in a certain sense (see [16] for details).

The proof given by Impagliazzo and Kabanets is related to previous published results: in [28], Schmidt, Siegel and Srinivasan give a Chernoff bound which is applicable in case the random variables \( x = (x_1, \ldots, x_n) \) are only \( m \)-wise independent for some large enough \( m \). It
turns out that the expressions which appear in their computations have close counterparts in the proof in [16], but they still bound $E[e^{f(x)}]$, and it seems to us that the approach in [16] makes the concepts clearer and the calculations shorter.

Another work related to [16] is due to Janson, Oleszkiewicz and Ruciński [17], who give an upper tail bound (i.e., a one-sided concentration bound) for the number of subgraphs in an Erdős-Rényi random graph $G_{n,p}$. The proof given in [17] bears much relationship to the proof given in [16]. We elaborate on that in Section 3.2.

Finally, there is a connection to an argument used by Rao to prove a concentration bound for parallel repetition of two-prover games [27]. As we will see, one of the ideas in the proof given in [16] is to consider a subset of the variables $(x_1, \ldots, x_n)$. Rao also does this, with a somewhat different purpose.

Our contributions

In this paper we modify the proof of Impagliazzo and Kabanets and introduce a more general sufficient condition for concentration which we term growth boundedness (Section 3). Then, we show some applications of our framework.

First, we prove a randomized version of the hitting property of expander random walks (Theorem 4.1) and use it to obtain an optimal (up to a constant factor in the exponent) expander random walk concentration bound settling a question asked in [16] (Theorem 4.2).\footnote{Of course the bound itself is not new. Impagliazzo and Kabanets asked if such a concentration bound can be obtained from the hitting property, i.e., using the technique from [16].}

We also show that our method is quite robust: with a little more effort one can improve the constant factor to the optimal one in case of large number of steps and small deviation (Theorem 4.3).

Second, we prove an upper tail bound for polynomials with input random variables in $[0, 1]$ (Theorem 5.2). Contrary to the previous work we are aware of, we do not assume that those variables are independent, but rather that they obey a condition similar to growth boundedness.

This bound is used in a proof of a lower bound for the complexity of a black-box construction of a pseudorandom generator from a one-way function [14]. Although [14] was published earlier, the proof of the bound is not contained there, but deferred to this paper instead. We outline how the bound was used in [14] in Section 5.1.

Notation

Throughout the paper we focus on the bounds of the form $\Pr[f(x) \geq \mu(1+\epsilon)]$. We call such bounds “(multiplicative) upper tail bounds”.

Typically, we consider a probability distribution $P_x$ over some vector of random variables $x = (x_1, \ldots, x_n)$. We denote a random choice from $P_x$ as $x \leftarrow P_x$. We try to explicitly indicate randomness whenever taking probability or expectation, i.e., we write $\Pr_{x \leftarrow P_x} [...]$ and so on. For a finite set $A$, let $a \leftarrow A$ be a shorthand for a uniform random choice of an element from $A$.

For a natural number $n$, let $[n] := \{1, \ldots, n\}$. As usual, by $\binom{n}{k}$ we denote $\prod_{i=0}^{k-1} \frac{(n-i)}{i!}$ for $n \in \mathbb{R}$ and $k \in \mathbb{N}$. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, we also identify $\binom{n}{k}$ with the set of subsets of $[n]$ of size $k$.

In particular, $(i_1, \ldots, i_m) \leftarrow [n]^m$ denotes uniform choice of $m$ elements from $[n]$ with repetition and $M \leftarrow \binom{n}{m}$ uniform choice of a subset of $[n]$ of size $m$. 
2 A Simple Proof of a Chernoff Bound

We start by presenting a short proof of a Chernoff bound in, arguably, the most basic setting.

**Theorem 2.1.** Let \( x = (x_1, \ldots, x_n) \) be i.i.d. over \( \{0, 1\}^n \) with \( \Pr[x_i = 1] = 1/2 \) and \( \epsilon \in [0, 1/2] \). Then,

\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \frac{n}{2} (1 + \epsilon) \right] \leq \exp \left( -\frac{\epsilon^2 n}{6} \right).
\]

**Proof.** Let \( m := \lceil \frac{n \epsilon}{3} \rceil \). We have

\[
E_{x \sim P_x} \left[ \left( \sum_{i=1}^{n} x_i \right)^m \right] = n^m \prod_{j=1}^{m} \Pr_{x \sim P_x} \left[ \forall j \in [m] : x_{ij} = 1 \right]
\]

\[
= n^m \prod_{j=1}^{m} \left( \Pr_{x \sim P_x} \left[ x_{ij} = 1 \mid \forall k : x_{ik} = 1 \right] \right)
\]

\[
\leq n^m \left( \frac{\epsilon}{3} \cdot 1 + \left( \frac{1 - \epsilon}{3} \cdot \frac{1}{2} \right)^m \right) = \left( \frac{n}{2} \right)^m \left( 1 + \frac{\epsilon}{3} \right)^m.
\]

Using Markov’s inequality and \( 1 + \frac{\epsilon}{3} \leq \exp \left( -\frac{\epsilon^2}{6} \right) \) for \( \epsilon \in [0, 1/2] \),

\[
\Pr \left[ \left( \sum_{i=1}^{n} x_i \right)^m \geq \left( \frac{n}{2} \right)^m (1 + \epsilon)^m \right] \leq \left( \frac{1 + \frac{\epsilon}{3}}{1 + \epsilon} \right)^m \leq \exp \left( -\frac{\epsilon^2 n}{6} \right).
\]

The above is the simplest proof of the most basic Chernoff bound we know of, and we believe that it is worthwhile to state it explicitly. It can be obtained by adapting the proof given in [16] for the given setting, although a direct adaptation yields a slightly different (and probably a bit longer) argument. Alternatively, it can be seen as an instantiation of the proof given in [17] in case one is interested in counting the number of copies of \( K_2 \) (i.e., the number of edges) in a random graph \( G_{n,p} \), after rather many simplifications that can be done for this very special case. Finally, it is a straightforward instantiation of our later proof given in Section 3.

3 Growth Boundedness

In this section we present the definition of growth-boundedness and prove that it implies concentration. In Section 3.1 we introduce growth boundedness without repetition: a variation of our concept that we use to prove the expander random walk bound.

**Definition 3.1.** Let \( \delta \geq 0 \) and \( m \in [n] \). A distribution \( P_x \) over \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n_0 \) with \( \mu := E_{x \sim P_x} [x_i] \) is \((\delta, m)\)-growth bounded if

\[
E_{x \sim P_x} \left[ \left( \sum_{i=1}^{n} x_i \right)^m \right] \leq (\mu m)^m (1 + \delta)^m.
\]

Equivalently, \( P_x \) is \((\delta, m)\)-growth bounded if and only if

\[
E_{(x_1, \ldots, x_m) \sim [n]^m} \left[ \prod_{j=1}^{m} x_j \right] \leq \mu^m (1 + \delta)^m.
\]
If random variables are over \( \{0, 1\} \), this condition reduces to

\[
\Pr_{x \sim P_x} \left[ \forall j \in [m] : x_{i_j} = 1 \right] \leq \mu^n (1 + \delta)^m.
\]

We now state our main theorem:

**Theorem 3.2.** Let \( P_x \) be a distribution over \( \mathbb{R}^n_{\geq 0} \), \( \mu := \mathbb{E}_{i \sim [n]} [x_i] \), \( \mu > 0 \), \( \epsilon \geq 0 \). If \( P_x \) is \((\delta, m)\)-growth bounded, then

\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] \leq \left( \frac{1 + \delta}{1 + \epsilon} \right)^m.
\]

**Proof.** By Markov’s inequality and growth boundedness of \( P_x \),

\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] = \Pr_{x \sim P_x} \left[ (\sum_{i=1}^{n} x_i)^m \geq (\mu n)^m (1 + \epsilon)^m \right] \leq \left( \frac{1 + \delta}{1 + \epsilon} \right)^m.
\]

There is an interesting connection between this proof (inspired by [17]) and the one used in [16], for details see Section 3.2.

We obtain more convenient bounds as a corollary:

**Corollary 3.3.** Let \( \epsilon \geq 0 \) and \( P_x \) be an \((\frac{\epsilon}{3}, m)\)-growth bounded distribution over \( \mathbb{R}^n_{\geq 0} \) with \( \mu := \mathbb{E}_{i \sim [n]} [x_i] \), \( \mu > 0 \).

1. If \( \epsilon \leq \frac{1}{2} \):
\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] \leq \exp\left( - \frac{\epsilon m}{2} \right).
\]

2. If \( \epsilon \geq \frac{1}{2} \):
\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] \leq \left( \frac{4}{5} \right)^m.
\]

3. If \( \epsilon \geq 3 \):
\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] \leq 2^{-m}.
\]

**Proof.** (1) follows because \( \frac{1 + \epsilon/3}{1 + \epsilon} \leq \exp\left( - \frac{\epsilon}{2} \right) \) for \( \epsilon \in [0, \frac{1}{2}] \), (2) since \( \frac{1 + \epsilon/3}{1 + \epsilon} \leq \frac{4}{5} \) for \( \epsilon \geq \frac{1}{2} \) and (3) due to \( \frac{1 + \epsilon/3}{1 + \epsilon} \leq \frac{1}{2} \) for \( \epsilon \geq 3 \).

For example, suppose that \( x_1, \ldots, x_n \) are independent over \( \{0, 1\}^n \), \( \Pr[x_i = 1] = \mu > 0 \), and \( \epsilon \in [0, \frac{1}{2}] \).

Using that for each \( M \) with \( |M| \leq \frac{\mu n}{3} \) we have

\[
\Pr_{x \sim P_x} [x_i = 1 \mid \forall j \in M : x_j = 1] = \left( \frac{|M|}{n} + \left( 1 - \frac{|M|}{n} \right) \mu \right) \leq \frac{|M|}{n} + \mu \leq \mu \left( 1 + \epsilon \right) ,
\]

we can conclude that \( P_x \) is \((\frac{\epsilon}{3}, \lfloor \frac{\mu n}{3} \rfloor)\)-growth bounded and

\[
\Pr_{x \sim P_x} \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right] \leq \exp(-\epsilon^2 \mu n / 6).
\]
3.1 Growth boundedness without repetition

If one looks at the process in the growth boundedness definition as choosing a uniform $m$-tuple of indices $(i_1, \ldots, i_m)$ (with repetition), it is possible to make a similar argument for choosing a uniform set of indices of size $m$ instead. In particular, we find it convenient in the proof of the expander random walk bound.

Definition 3.4. Let $\delta \geq -1$ and $m \in [n]$. We say that a distribution $P_x$ over $\{0,1\}^n$ with

$$\Pr_{x \sim P_x}[\forall i \in M : x_i = 1] \leq \mu > 0$$

is $(\delta, m)$-growth bounded without repetition if

$$\Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\forall i \in M : x_i = 1] \leq \mu^m(1 + \delta)^m.$$

Theorem 3.5. Let $P_x$ be a distribution over $\{0,1\}^n$, $\mu := \Pr_{i \sim [n]}[x_i = 1]$, $\mu > 0$, $\epsilon \geq 0$, $c \in [0,1]$. If $P_x$ is $(\delta, c\epsilon \mu)$-growth bounded without repetition then

$$\Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\sum_{i=1}^{n} x_i \geq \mu n(1 + \epsilon)] \leq \left(\frac{1 + \delta}{1 + (1 - c)\epsilon}\right)^m,$$

where $m := c\epsilon \mu$.

Proof. Set $q := \Pr[\sum_{i=1}^{n} x_i \geq \mu n(1 + \epsilon)]$ and compute:

$$\mu^m(1 + \delta)^m \geq \Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\forall i \in M : x_i = 1]$$

$$\geq q \Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\forall i \in M : x_i = 1 | \sum_{i=1}^{n} x_i \geq \mu n(1 + \epsilon)]$$

$$\geq q \prod_{i=0}^{m-1} \frac{\mu n(1 + \epsilon) - i}{n - i}$$

$$\geq q \mu^m(1 + (1 - c)\epsilon)^m.$$

Corollary 3.6. Let $\epsilon \in [0, \frac{3}{4}]$ and $P_x$ be a distribution over $\{0,1\}^n$ that is $(\xi, m)$-growth bounded without repetition for some $m \leq \frac{c\epsilon \mu n}{6}$ with $\mu := \Pr_{i \sim [n]}[x_i = 1]$, $\mu > 0$. Then,

$$\Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\sum_{i=1}^{n} x_i \geq \mu n(1 + \epsilon)] \leq \exp\left(-\frac{cm}{3}\right).$$

Proof. Apply Theorem 3.5 and note that $\frac{1 + \epsilon/3}{1 + 5\epsilon/6} \leq \exp\left(-\frac{\epsilon}{3}\right)$ for $\epsilon \in [0, \frac{3}{4}]$.

3.2 Connection of [16] and [17]

Recall the proof of Theorem 3.2. In the context of [16] and [17] we find it instructive to give an alternative proof, restricted to distributions over $\{0,1\}^n$ (essentially the same as the proof of Theorem 3.5).

Theorem 3.7. Let $P_x$ be a distribution over $\{0,1\}^n$, $\mu := \Pr_{i \sim [n]}[x_i = 1]$, $\mu > 0$, $\epsilon \geq 0$. If $P_x$ is $(\delta, m)$-growth bounded, then

$$\Pr_{x \sim P_x, M \sim \binom{\{0,1\}^n}{m}}[\sum_{i=1}^{n} x_i \geq \mu n(1 + \epsilon)] \leq \left(\frac{1 + \delta}{1 + \epsilon}\right)^m.$$
Proof. Set $q := \Pr \left[ \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right]$, and see that

$$\mu^n (1 + \delta)^m \geq \frac{\Pr_{x \sim \mu^n (1 + \epsilon)} \left[ \forall j \in [m] : x_{ij} = 1 \right]}{\Pr_{x \sim \mu^n (1 + \epsilon)} \left[ \forall j \in [m] : x_{ij} = 1 \mid \sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon) \right]} \geq q \mu^n (1 + \epsilon)^m.$$

The basic idea of the proof in [16] is to consider $\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right]$, where $M$ is a subset of $[n]$. Then, this is compared with $\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \mid \mathcal{E} \right]$, where $\mathcal{E}$ is the event that $\sum_{i=1}^{n} x_i \geq \mu n (1 + \epsilon)$. In fact, we have

$$\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right] \leq \Pr_{x,M} \left[ \forall i \in M : x_i = 1 \mid \mathcal{E} \right].$$

It is possible to show that for $m := E |M| \ll n$ we have $\Pr_M \left[ \forall i \in M : x_i = 1 \mid \mathcal{E} \right] \geq \mu^n (1 + \epsilon)^m$. To see the intuition of this, simply note that this probability roughly equals the probability of only selecting red balls when one chooses with repetition $m$ times out of $n$ balls, at least $\mu n (1 + \epsilon)$ of which are red. Thus,

$$\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right] \leq \mu^n (1 + \epsilon)^m.$$

Now note that this last argument only uses the probability over $M$, and so is independent of the distribution of $x$. Thus, for any distribution on which we can give a good upper bound on $\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right]$, the technique of [16] gives a concentration result.

The argument we use is very similar, but we pick $M$ as an $m$-tuple whose elements are picked independently with repetition. However, then we also have

$$n^m \Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right] = E \left[ (x_1 + \ldots + x_n)^m \right].$$

By Markov’s inequality,

$$\Pr[\mathcal{E}] = \Pr \left[ (x_1 + \ldots + x_n)^m \geq (\mu n (1 + \epsilon))^m \right] \leq \frac{\Pr_{x,M} \left[ \forall i \in M : x_i = 1 \right]}{\mu^n (1 + \epsilon)^m},$$

which is almost the same as (1).

The view in (1) is the one adopted by [16]. Bounding the $m$-th moment and using Markov is the view adopted in [17]. The above argument shows that these views are closely related, and one can argue that the connection is given by growth boundedness.

### 4 Random Walks on Expanders

#### Overview and our results

For an introduction to expander graphs, see [15] or [30, Chapter 4]. In short, a $\lambda$-expander is a $d$-regular undirected graph $G$ with the second largest (in terms of absolute value) eigenvalue of the transition matrix at most $\lambda$.

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2 Clearly $q = 0$ is not a problem.

3 The difference to the actual random experiment is that we do not keep each ball with probability $m/n$ but instead choose exactly $m$ times.
We consider a random walk on \( \lambda \)-expander starting in a uniform random vertex. It is a very useful fact in many applications that such a random walk behaves in certain respects very similarly to a random walk on the complete graph.

In particular, the so called hitting property [2, 20] states that the probability that an \( \ell \)-step random walk on a \( \lambda \)-expander \( G \) stays completely inside a set \( W \subseteq V := V(G) \) with \( \mu := |W|/|V| \) is at most \( (\mu + \lambda)^\ell \). A more general version [3] states that for each \( M \subseteq [\ell] \) the probability that a random walk stays inside \( W \) in all steps from \( M \) is at most \( (\mu + 2\lambda)^{|M|} \).

Our first result, which may be of independent interest, can be considered as a randomized version of the hitting property. Namely, we show that, given \( \epsilon > 0 \), for a relatively small random subset \( M \subseteq [\ell] \) of size \( m \) the probability that a random walk on a \( \lambda \)-expander stays inside \( W \) in all steps from \( M \) is at most \( (\mu(1 + \epsilon))^m \):

\begin{align*}
\textbf{Theorem 4.1.} & \text{ Let } G \text{ be a } \lambda \text{-expander with a distribution } P_r \text{ over } V^\ell \text{ representing an } (\ell - 1)\text{-step random walk } r = (v_1, \ldots, v_\ell) \text{ (with } v_1 \text{ being a uniform starting vertex) and } \lambda \subseteq V \text{ with } \mu := |W|/|V|. \text{ Let } \epsilon \geq 0 \text{ and } m \leq \min \left(1, \frac{\lambda + \mu}{\lambda - \mu}\right) \ell. \text{ Then,} \\
& P_{r \rightarrow P_r} \left[ \forall i \in M : v_i \in W \right] \leq (\mu(1 + \epsilon))^m.
\end{align*}

Another important property of random walks on expander graphs is the Chernoff bound estimating the probability that the number of times a random walk visits \( W \) is far from its expectation. The first Chernoff bound for expander random walks was given by Gillman [11] and the problem was treated further in numerous works [21, 24, 1, 12, 32, 8].

Impagliazzo and Kabanets [16] apply their technique to obtain a bound for random walks on expander graphs, but in case of deviations smaller than \( \lambda \) they lose a factor of \( \log \left( \frac{1}{\lambda} \right) \) in the exponent. They then ask if their technique can be modified to avoid this loss.

We answer this question affirmatively: using Theorem 4.1 we immediately obtain a bound that matches the known ones and does not suffer from the additional \( \log \left( \frac{1}{\lambda} \right) \) factor while preserving the simplicity of the proof.

\begin{align*}
\textbf{Theorem 4.2.} & \text{ Let the setting be as in Theorem 4.1 with } \mu > 0. \text{ Define } P_x \text{ over } \{0, 1\}^\ell \text{ as } x_i = 1 \iff v_i \in W \text{ and let } \epsilon \in \left[0, \frac{\lambda}{\mu}\right]. \text{ Then,} \\
& P_{x \rightarrow P_x} \left[ \sum_{i=1}^\ell x_i \geq \mu \ell(1 + \epsilon) \right] \leq 2 \exp \left( -\frac{(1 - \lambda)\epsilon^2 \mu \ell}{18} \right).
\end{align*}

Furthermore, we demonstrate robustness of our method by improving the exponent to \( \frac{1 - \lambda}{1 + \lambda} \frac{\epsilon^2 \ell}{2} + o(\epsilon^2) \ell \), which is optimal for fixed \( \lambda, \mu \) and \( \epsilon \rightarrow 0_+ \) and \( \ell \rightarrow \infty \):

\begin{align*}
\textbf{Theorem 4.3.} & \text{ Let the setting be as in Theorem 4.1 with } \mu \in (0, 1). \text{ Define } P_x \text{ over } \{0, 1\}^\ell \text{ as } x_i = 1 \iff v_i \in W \text{ and let } \epsilon \in [0, 1]. \text{ Then, there exists } c_\mu \text{ that depends only on } \mu \text{ such that} \\
& P_{x \rightarrow P_x} \left[ \sum_{i=1}^\ell x_i \geq \mu \ell(1 + \epsilon) \right] \leq 2 \exp \left( -\frac{1 - \lambda}{1 + \lambda} \cdot \frac{\mu}{1 - \mu} \cdot \frac{\epsilon^2 \ell}{2} + c_\mu \epsilon^3 \ln \left( \frac{1}{\epsilon} \right) \ell \right).
\end{align*}

For a proof of Theorem 4.3 see the full paper. In the following we prove Theorems 4.1 and 4.2.
Proofs

First, we need a coupling argument: let \( m, \ell \in \mathbb{N}, m \leq \ell \) be given. We consider the distribution \( D_{m, \ell} \) defined by the following process:

- Pick uniformly \( M \leftarrow \left( \begin{array}{} m \\ \ell \end{array} \right) \) and let \( M := \{x_1, \ldots, x_m\} \) with \( x_1 < \ldots < x_m \).
- Let \( d_i := x_i \) and \( d_i := x_i - x_{i-1} \) for \( i > 1 \).

A bijection shows that (2) is due to Cauchy-Schwarz inequality (note there are at most \( \text{Claim 4.6.} \)

\[ \text{Proof.} \]

Let \( M \leftarrow \left( \begin{array}{} m \\ \ell \end{array} \right) \) and let \( \sum_{i=1}^{m} d_i \leq \ell \) and \( d_i > 0 \). We now couple \( D_{m, \ell} \) with independent random variables (see full paper for the proof):

\[ \text{Lemma 4.4.} \]

Let \( 0 < m \leq \ell \). There exists a distribution over \( (d_1, \ldots, d_m, e_1, \ldots, e_m) \) such that:

- \( e_i \leq d_i \) for \( 1 \leq i \leq m \).
- \( (d_1, \ldots, d_m) \) is distributed according to \( D_{m, \ell} \).
- \( (e_1, \ldots, e_m) \) are i.i.d. with \( e_i \) in \( \mathbb{N}_+ \) and \( \Pr[e_i = k] \leq \frac{2m}{\ell} \) for every \( k \).

\[ \text{Proof of Theorem 4.1.} \]

Pick \( M \leftarrow \left( \begin{array}{} m \\ \ell \end{array} \right) \) and let \( (d_1, \ldots, d_m) \) be as in the definition of \( D_{m, \ell} \).

\[ \text{Lemma 4.5.} \]

\[ \Pr_{\omega \leftarrow \left( \begin{array}{} m \\ \ell \end{array} \right)} \left[ \forall i \in M : v_i \in W \right] \leq \mathbb{E}_{M \leftarrow \left( \begin{array}{} m \\ \ell \end{array} \right)} \left[ \prod_{i=1}^{m} (\mu + \lambda^d) \right]. \]

\[ \text{Proof.} \]

Let \( v := \left( \frac{1}{\ell}, \ldots, \frac{1}{\ell} \right) \) be the vector of the uniform distribution on \( V \) and let \( P_W \) be a diagonal \( n \times n \) matrix with \( (P_W)_{uu} = 1 \) if \( u \in W \) and \( (P_W)_{uu} = 0 \) otherwise. Note that \( P_W^2 = P_W \).

Let \( A_G \) be the probability transition matrix of \( G \). Let us denote the spectral norm of a matrix with \( || \cdot || \). We bound the probability of a random walk staying in \( W \) on indices of \( M \) using a standard technique. In particular, we use (for the proof see [30, Claim 4.21]):

\[ \text{Claim 4.6.} \]

\[ ||P_W A_G^d P_W|| \leq \mu + (1 - \mu)\lambda^k. \]

Fix \( M \). First of all, by induction (and noting that \( vA_G = v \)):

\[ \Pr_{r \leftarrow P_r} [\forall i \in M : v_i \in W] = |vP_W \prod_{i=2}^{m} A_G^{d_i} P_W|_1. \]

Estimate:

\[ |vP_W \prod_{i=2}^{m} A_G^{d_i} P_W|_1 \leq \sqrt{\mu \lambda^k} \cdot |vP_W \prod_{i=2}^{m} A_G^{d_i} P_W| \]

\[ \leq \sqrt{\mu \lambda^k} \cdot |vP_W| \prod_{i=2}^{m} ||P_W A_G^{d_i} P_W|| \]

\[ = \mu \prod_{i=2}^{m} ||P_W A_G^{d_i} P_W|| \]

\[ \leq \prod_{i=1}^{m} (\mu + \lambda^d), \]

where (2) is due to Cauchy-Schwarz inequality (note there are at most \( \mu n \) non-zero coordinates in the final vector), (3) follows from \( ||AB|| \leq ||A|| \cdot ||B|| \), (4) from \( ||vP_W|| = \sqrt{\mu} \) and (5) from Claim 4.6.  

\[ \text{STACS 2015} \]
The hope is that \((d_1, \ldots, d_m)\) behave “almost” like i.i.d. uniform random variables. This is indeed true, and by Corollary 4.4 we have \((e_1, \ldots, e_m)\) such that \(e_i \leq d_i\) and \(e_i\) are i.i.d. with \(e_i \in \mathbb{N}_+\) and \(\text{Pr}[e_i = k] = \frac{2m}{\ell}\) for each \(k\).

Putting this fact together with Lemma 4.5:

\[
\text{Pr}_{\ell \leftarrow \binom{\ell}{m}} \left[ \forall i \in M : v_i \in W \right] \leq \mathbb{E} \left[ \prod_{i=1}^{m} (\mu + \lambda^{e_i}) \right] = \prod_{i=1}^{m} (\mu + \mathbb{E}[\lambda^{e_i}]) \leq \left( \mu + \frac{2m}{\ell} \cdot \frac{\lambda}{1-\lambda} \right)^{m} \leq \mu^{\ast}(1 + \epsilon)^{m}.
\]

An immediate corollary of Theorem 4.1 is:

\begin{itemize}
  \item \textbf{Corollary 4.7.} Let the setting be as in Theorem 4.1. Define \(P_x\) over \(\{0,1\}^\ell\) as \(x_i = 1 \iff v_i \in W\). Then, \(P_x\) is \((\epsilon, \min(\ell, \lfloor 1 - \lambda^{m \mu_{\ell} / 6} \rfloor)\)-growth bounded without repetition.
\end{itemize}

\textbf{Proof of Theorem 4.2.} Combine Corollary 4.7 with Corollary 3.6 (setting \(m = \lfloor (1 - \lambda^{\frac{n_{\ell}^{\mu_{\ell}}}}) / 6 \rfloor\)).

## 5 Polynomial Concentration

In certain applications it is desired to bound the concentration not only of the sum, but rather of a (low-degree) polynomial of some random variables.

In the case when (informally) the polynomial is such that the change in its value is bounded when the value of a single input variable is changed the Azuma’s inequality can be applied to bound concentration.

If this is not so, one can use techniques that were invented by Kim and Vu [22] and developed in a body of work that followed (in particular [31, 29]). In the special case of a multilinear low-degree polynomial \(p(v)\) and an independent distribution of input variables \(P_v\) their concentration bound can be expressed, very roughly speaking, as a function of \(\mu_0\) and \(\mu' = \max_{K \neq \emptyset} \mathbb{E}[\partial_K p(v)]\), where \(\mu_0\) is the expectation of \(p(v)\) and \(\mu' = \max_{K \neq \emptyset} \mathbb{E}[\partial_K p(v)]\).

We obtain a bound in similar spirit. It is not tight in general, but can be applied to arbitrary polynomials with positive coefficients over input random variables in \([0,1]\) and is tight in the case of \textit{elementary symmetric polynomials} \(e_k(v) := \sum_{|S|=k} \prod_{i \in S} v_i\) (see the full paper for a proof).

Most importantly, as opposed to prior results, it does not require the input variables to be independent, but rather \textit{almost independent} in a certain sense (for simplicity we limit ourselves to multilinear polynomials and inputs in \([0,1]\), full treatment can be found in the full paper):

\begin{itemize}
  \item \textbf{Definition 5.1.} Let \(P_v\) be a distribution over \([0,1]^\ell\), \(\delta \geq 0\) and \(m \in [\ell]\). \(P_v\) is \((\delta, m)\)-almost independent if for each \(M \subseteq [\ell]\) with \(|M| \leq m\)

\[
\text{Pr}_{v \sim P_v} \left[ \forall i \in M : v_i = 1 \right] \leq (1 + \delta)^m \prod_{i \in M} \text{Pr}_{v \sim P_v} [v_i = 1] .
\]
\end{itemize}
Let us state our main theorem of this section. Let $P_v$ be a $(\delta, km)$-almost independent distribution. Let $p(v)$ be a multilinear polynomial of degree $k$ with positive coefficients. Our way to deal with dependencies in $P_v$ is to state the bound in terms of $P^*_v$, which is the distribution of independent variables with the same marginals as $P_v$ (i.e., each $v^*_i$ has the same distribution as $v_i$).

We express the concentration in terms of

$$
\mu^*_i := \max_{|K| \leq i} E_{v \sim P^*_v} [\partial_K p(v)].
$$

Note that $\mu^*_i$ is the expectation of $p(v)$ under $P^*_v$.

**Theorem 5.2.** Let the setting be as above and $\epsilon > 0$. Then,

$$
Pr_{v \sim P^*_v} \left[ p(v) \geq \mu^*_0 (1 + \epsilon) \right] \leq \left( \frac{(1 + \delta)^k \left( 1 + \sum_{i=1}^k \binom{k}{i} \mu^*_i \right)}{\mu^*_0} \right)^m.
$$

**Proof outline.** Write $p(v)$ as a sum of binary random variables (corresponding to the monomials) $x_1, \ldots, x_n$. Due to Theorem 3.2 it is enough to show that $(x_1, \ldots, x_n)$ are $(\delta', m)$-growth bounded, where $1 + \delta' = (1 + \delta) \left( 1 + \sum_{i=1}^k \binom{k}{i} \mu^*_i \right) / \mu^*_0$.

Since $P_v$ is $(\delta, km)$-almost independent, this task can be further reduced to showing that if $v$ is distributed according to $P^*_v$ instead of $P_v$, then $(x_1, \ldots, x_n)$ are $(\delta'', m)$-growth bounded, where $1 + \delta'' = (1 + \sum_{i=1}^k \binom{km}{i} \mu^*_i)$.

Fix $s < m$ and $(i_1, \ldots, i_s) \in [n]^s$ and let $M$ be the set of all indices $j$ such that $v_j$ influences at least one of $x_{i_1}, \ldots, x_{i_s}$ (note that $|M| \leq km$).

We write $p(v) = \sum_{K \subseteq M: |K| \leq k} p_K(v)$, where $p_K(v)$ consists of those monomials whose variables intersected with $M$ are exactly $K$. Observe that

$$
E_{v \sim P^*_v} \left[ p_K(v) \mid \forall i : v_i = 1 \right] \leq E_{v \sim P^*_v} \left[ \partial_K p(v) \right].
$$

To get growth boundedness for $x_1, \ldots, x_n$ we proceed by induction and bound

$$
Pr_{v \sim P^*_v, i_{s+1} = [n]} \left[ x_{i_{s+1}} = 1 \mid \forall j \in [s] : x_{i_j} = 1 \right] = \frac{1}{\mu^*_0} E_{v \sim P^*_v} \left[ p(v) \mid \forall i \in M : v_i = 1 \right]
$$

$$
\leq \frac{1}{\mu^*_0} \sum_{K \subseteq M: |K| \leq k} E_{v \sim P^*_v} \left[ \partial_K p(v) \right]
$$

$$
\leq \frac{\mu^*_0}{\mu^*_0} \left( 1 + \sum_{i=1}^k \binom{km}{i} \mu^*_i \right).
$$

Let $\mu' := \max_{i \in [k]} \mu^*_i$. Since $\sum_{i=1}^k \binom{km}{i} \leq (km)^k$, we have:

**Corollary 5.3.** Let the setting be as in Theorem 5.2. Then,

$$
Pr_{v \sim P_v} \left[ p(v) \geq \mu^*_0 (1 + \epsilon) \right] \leq \left( \frac{(1 + \delta)^k (1 + \binom{km}{i} \mu')}{\mu^*_0} \right)^m.
$$
5.1 An application in [14]

In [14] the authors prove a lower bound on the complexity of a black-box construction of a pseudorandom generator from a one-way function.

Part of their proof consists in using Theorem 5.2 to show a concentration bound for a certain polynomial. The proof of Theorem 5.2 is not included in [14], but deferred to this paper instead. Since the input variables of the polynomial are not independent, to the best of our knowledge no previous work is applicable to this case.

The following random process is considered: pick a permutation $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$ u.a.r. and consider the distribution $P_g$ over $2^n$ random variables $g := \{g_{x,y} : x, y \in \{0, 1\}^n\}$ defined as $g_{x,y} = 1$ if $f(x) = y$ and $g_{x,y} = 0$ otherwise.

The random variables in $g$ are not independent, but it is easy to check that they are $(1, 2^{n-1})$-almost independent. Also, the corresponding independent distribution $P^*_g$ has expectation $2^n$ for each $g_{x,y}$.

Fix $k \leq \frac{n}{100 \log n}$ [14] defines a certain multilinear polynomial $p(g)$ of degree at most $k$ such that $\mu^*_0 \leq 2^{n/15}$ and $\mu' \leq 2^{n/15}$ (we omit the details).

[14] needs to show that (for $n$ big enough):

$$\Pr_{g \sim P_g} \left[ p(g) \geq 2^{n/10} \right] \leq 2^{-2^{n/100k}}.$$

To this end, calculate using Corollary 5.3 and setting $\delta := 1$, $\epsilon := 2^{9n/100}/\mu^*_0$ and $m := 2^{n/100k}$:

$$\Pr_{g \sim P_g} \left[ p(g) \geq \mu^*_0 + 2^{9n/100} \right] \leq \left( \frac{2k \max \left( \frac{2k^{2k^{2k^{n/100}}} \mu'}{\mu^*_0^3} \right)^{2^{9n/100k}}}{\mu^*_0^3} \right)^{2k \max \left( \frac{2k^{2k^{2k^{n/100}}} \mu'}{\mu^*_0^3} \right)^{2k \max \left( \frac{2k^{2k^{2k^{n/100}}} \mu'}{\mu^*_0^3} \right)^{2k^{2k^{2k^{n/100}}} \mu'}} \right) \leq 2^{-2^{n/100k}}.$$

5.2 Other applications

We note that despite the fact that the deviation for which we applied our theorem in Section 5.1 is big relative to the expectation, one can obtain meaningful bounds also for very small deviations.

This can be seen by taking a restricted version of Theorem 5.2:

► Theorem 5.4. Let $P_v$ be a distribution of independent variables (i.e., $P_v = P^*_v$) over $[0, 1]^\ell$. Let $p(v)$ be as in Theorem 5.2 and $\epsilon \in [0, \frac{1}{2}]$. Then:

$$\Pr_{v \sim P_v} \left[ p(v) \geq \mu(1 + \epsilon) \right] \leq 2 \exp \left( - \frac{\epsilon}{6k} \left( \frac{2k^{2k^{2k^{n/100}}} \mu'}{\mu^*_0^3} \right)^{1/k} \right).$$

Proof. Note that $P_v$ are $(0, \ell)$-almost independent. Take $m := \left\lceil \frac{1}{k} \left( \frac{2k^{2k^{2k^{n/100}}} \mu'}{\mu^*_0^3} \right)^{1/k} \right\rceil$, obtain $(\xi, m)$-growth boundedness as in Corollary 5.3 and apply Corollary 3.3.1. ◀

It was pointed out to us that a generalisation of the result of Latala and Łochowski [23] might be applicable (together with [9]). However, moment bound in [23] is optimal only up to a constant in the exponent that depends on the degree and the degree is non-constant in our setting.
For example, in a representative setting when Azuma-like methods fail: consider the polynomial that counts the triangles in Erdős–Rényi random graph $G_{n, n - \frac{3}{4}}$, i.e., $p(v) = \sum_{(a, b, c) \in \mathcal{E}} v_{ab} v_{ac} v_{bc}$. We compute $\mu = \Theta(n^{3/4})$ and $\mu' = \Theta(1)$.

For $\epsilon \in [0, \frac{3}{16}]$ Theorem 5.4 gives:

$$\Pr_{v \sim P} \left[ p(v) \geq \mu(1 + n^{-\epsilon}) \right] \leq \exp(-\Omega(n^{1/4 - 4\epsilon/3})).$$

This is comparable to the bound from [22] (which was the first paper to give a good bound in this setting). Better bounds are known, in particular we revisit the triangle counting in Section 6.

For some more discussion on the tightness of Theorem 5.2, see the full paper.

6 Counting Subgraphs in Random Graphs

In the proof of the polynomial concentration bound we consider values $\mu_i^*$ which are maxima of expectations of $\partial K p(v)$ over sets $K$ of size $i$. Each such value yields a contribution of $\mu_i^* \approx km_i$ (proportional to the number of partial derivatives of this type in the subset of input variables of size $km$) and the “quality” of a concentration bound depends, roughly, on the maximum such contribution.

In principle, nothing prevents us from considering a different, possibly finer, division of partial derivatives into a constant number of classes, each with its own contribution.

In particular, it is an obvious fact that the number of occurrences of a fixed subgraph $H$ in a random Erdős–Rényi graph (for some of the work on the problem see [18, 17, 19]) can be expressed in terms of a multilinear polynomial. In this setting we may divide the partial derivatives into classes corresponding to subgraphs of $H$. Interestingly, this yields an upper tail bound proof that is basically isomorphic to the famous one of Janson, Oleszkiewicz and Ruciński [17].

Our result holds in the setting of almost-independent distributions, readily applicable, for example, to $G_{n, m}$ random graphs (of course the proof of [17] also generalises to those settings).

For details, see the full paper.

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