Certain Class of Analytic Functions with respect to Symmetric Points Defined by Q-Calculus

K. R. Karthikeyan 1, G. Murugusundaramoorthy 2, S. D. Purohit 3, and D. L. Suthar 4

1Department of Applied Mathematics and Science, National University of Science & Technology (By Merger of Caledonian College of Engineering and Oman Medical College), Muscat, Oman
2Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology Deemed to be University, Vellore, Tamilnadu, India
3Department of HEAS (Mathematics), Rajasthan Technical University, Kota, India
4Department of Mathematics, Wollo University, P. O. Box: 1145, Amhara Region, Ethiopia

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

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In this study, we familiarise a novel class of Janowski-type star-like functions of complex order with regard to \((j, k)\)-symmetric points based on quantum calculus by subordinating with pedal-shaped regions. We found integral representation theorem and conditions for starlikeness. Furthermore, with regard to \((j, k)\)-symmetric points, we successfully obtained the coefficient bounds for functions in the newly specified class. We also quantified few applications as special cases which are new (or known).

\[ f\left(e^{2\pi i k} \xi \right) = e^{2\pi i k} f\left(\xi \right). \]  \( F_k \) represents the family including all \( k \)-fold symmetric functions.

The concept of \( k \)-symmetrical function was protracted to so-called \((j, k)\)-symmetrical function by Liczberski and Połubiński in [2]. To be specific, a function \( f(\xi) \) is reported for being \((j, k)\)-symmetrical if

\[ f(\varepsilon \xi) = e^j f(\xi), \quad (\xi \in U), \]  where \( k \geq 2 \) is a fixed integer, \( j = 0, 1, 2, \ldots, k - 1 \) and \( \varepsilon = \exp(2\pi i/k) \). The family of \((j, k)\)-symmetrical functions will indeed be indicated by \( F_k^j \). We believe that \( F_k^2, F_k^0 \) and \( F_k^-j \) are quite well groups of odd functions, even functions, and \( k \)-symmetrical functions. Consider the subsequent equivalence demarcate \( f_{j,k}(\xi) \) as well

\[ f_{j,k}(\xi) = \frac{1}{k} \sum_{\varepsilon=0}^{k-1} f(\varepsilon^j \xi), \quad (f \in \mathcal{A}; k = 1, 2, \ldots; j = 0, 1, 2, \ldots, (k - 1)). \]
For $f \in \mathcal{A}$ assumed by (1) and $0 < q < 1$, the Jackson’s $q$-derivative operator or $q$-difference operator for $f \in \mathcal{A}$ is specified under (see [12–14])

$$
\mathcal{D}_q f (\xi) = \begin{cases} 
q f (0), & \text{if } \xi = 0, \\
q f (\xi) - f (q \xi) / (1 - q), & \text{if } \xi \neq 0.
\end{cases}
$$

(10)

From (10), if $f$ is assumed as in (1), we can effortlessly see that

$$
\mathcal{D}_q f (\xi) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \xi^{q^{-1}},
$$

(11)

for $\xi \neq 0$, provided the $q$-integer number $[n]_q$ is represented by

$$
[n]_q = \frac{1 - q^n}{1 - q},
$$

(12)

and take into consideration

$$
\lim_{q \to 1^{-1}} \mathcal{D}_q f (\xi) = f (0).
$$

(13)

The $q$-Jackson integral is defined by (see [6])

$$
I_q [f (\xi)] = \int_{0}^{\xi} f (t) d_q t = \xi (1 - q) \sum_{k=0}^{\infty} q^k f (\xi q^k).
$$

(14)

If the $q$-series converges, further witness that

$$
\mathcal{D}_q I_q f (\xi) = f (\xi),
$$

$$
I_q \mathcal{D}_q f (\xi) = f (\xi) - f (0),
$$

(15)

where the second equality grasps if $f$ is continuous at $\xi = 0$.

Let the classes of star-like functions of order $\eta$ ($0 \leq \eta < 1$) and convex functions of order $\eta$ ($0 \leq \eta < 1$) are symbolised by $\mathcal{S}^* (\eta)$ and $\mathcal{C} (\eta)$, respectively. In $\mathcal{A}$, we categorize the collection $\mathcal{P}$ of functions $p (\xi) \in \mathcal{A}$ with $p (0) = 1$ and $\mathcal{R} p (\xi) > 0$. The functions in the $\mathcal{P}$ class are not univalent.

With $U$, let $f, g$ be analytic. The function $f$ is said to be subordinate to $g$ in $U$ if the Schwarz function $\omega (\xi)$ exists in $U$ such that $|\omega (\xi)| < |\xi|$ and $f (\xi) = g (\omega (\xi))$, as shown through $f \prec g$. Whenever $g$ is univalent in $U$, consequently the subordination is identical to $f (0) = g (0)$ and $f (U) \subset g (U)$.

Using the concept of subordination for holomorphic functions, Ma and Minda [15] proposed the classes:

$$
\mathcal{S}^* (\psi) = \left\{ f \in \mathcal{A} : \frac{\xi f (\xi)}{f (\xi)} \prec \psi \right\},
$$

$$
\mathcal{C} (\psi) = \left\{ f \in \mathcal{A} : \left( 1 + \frac{\xi f (\xi)}{f (\xi)} \right) \prec \psi \right\},
$$

(16)

where $\psi \in \mathcal{W}^*$ with $\psi (0) > 0$ maps $U$ onto a region star-like with respect to 1 and symmetric with respect to the real axis. By making a choice $\psi$ to map unit disc on to some specific regions such as cardioid, parabolas, lemniscate of Bernoulli, and booth lemniscate in the right-half of the complex plane,

It is evident that $f_{j,k} (\xi)$ is a linear operator from $U$ into $U$. If $n$ is an integer, then the subsequent assumptions result directly from (4):

$$
f_{j,k} (\epsilon^j \xi) = \epsilon^{nj} f_{j,k} (\xi),
$$

$$
f'_{j,k} (\epsilon^j \xi) = \epsilon^{nj-1} f'_{j,k} (\xi) = \frac{1}{k} \sum_{p=0}^{k-1} \epsilon^{nj-p} f' (\xi),
$$

$$
f''_{j,k} (\epsilon^j \xi) = \epsilon^{nj-2} f''_{j,k} (\xi) = \frac{1}{k} \sum_{p=0}^{k-1} \epsilon^{nj-2p} f'' (\xi).
$$

(5)

Let the function $f \in \mathcal{A}$ provided by (1) and $g \in \mathcal{A}$ of the form

$$
g (\xi) = \xi + \sum_{n=2}^{\infty} Y_n \xi^n,
$$

the Hadamard product (or convolution) of these two functions is indicated by

$$
\mathcal{H} (\xi) = (f * g) (\xi) = \xi + \sum_{n=2}^{\infty} a_n Y_n \xi^n, \quad \xi \in U.
$$

(6)

Using Hadamard product, various authors studied the univalent function theory in dual with the theory of special functions, see [3–5] and references provided therein. Throughout this whole article, we will assume that $k \in \mathbb{N}$, $\epsilon = \exp (2 \pi i / k)$, and

$$
\mathcal{H}_{j,k} (\xi) = \frac{1}{k} \sum_{p=0}^{k-1} \epsilon^{nj} (f \ast g) (\epsilon^j \xi) = \xi + \cdots,
$$

(7)

where

$$
f, g \in \mathcal{A}: k = 1, 2, \ldots; j = 0, 1, 2, \ldots, (k - 1).
$$

(8)

From (7), we thus have

$$
\mathcal{H}_{j,k} (\xi) = \sum_{n=1}^{\infty} a_k Y_n \Lambda_{n,j} \xi^n, \quad (a_1 = Y_1 = 1),
$$

$$
\Lambda_{n,j} = \frac{1}{k} \sum_{p=0}^{k-1} \epsilon^{(n-j)p}.
$$

(9)

The investigation of $q$-calculus ($q$ stands for quantum) fascinated and inspired many scholars due its use in various areas of the quantitative sciences. Jackson [6, 7] was among the key contributors of all the scientists who introduced and developed the $q$-calculus theory. Just like $q$-calculus was used in other mathematical sciences, the formulations of this idea are commonly used to examine the existence of various structures of function theory. Though it is the first article in which a link was established between certain geometric nature of the analytic function and the $q$-derivative operator and the usage of $q$-calculus in function theory, a solid and comprehensive foundation is given in [8] by Srivastava. After this development, many researchers introduced and studied some useful operators in $q$-analog with the applications of convolution concepts. For example, Kanas and Raducanu [9] established the $q$-differential operator and then examined the behavior of this operator in function theory. For more applications of this operator, see [10, 11].
refer [16]. The class of Janowski star-like functions and an open disc centred with center 1.

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Definition 2. If we let \(C = 1, D = 1\), \(\theta = 0\), \(b = 1\) and \(g(\xi) = \xi + \sum_{n=2}^\infty \xi^n\), then \(\mathcal{H}_j^b(0; 0; \psi; g; 1, 1) \equiv \mathcal{H}_j^{(1,1)}(\psi)[19]\) and \(\mathcal{H}_j^b(1; 0; \psi; g; 1, 1) \equiv \mathcal{H}_j^{(1,1)}(\psi)[19]\)

(2) Fixing \(\theta = 0\), \(b = 1\), \(g(\xi) = \xi + \sum_{n=2}^\infty \xi^n\) and \(\psi(\xi) = 1 + \xi(1 - \xi)\), then \(\mathcal{H}_j^b(0; 0; \psi; g; C, D)\) reduces to the class \(\mathcal{H}_j^{(1,1)}(C, D)\) ([18], Definition 5)

For completeness, we will now define \(q\)-analogue of the as follows.

Definition 3. For \(-(\pi/2) < \theta < (\pi/2)\), \(b \in C\), \(\xi \neq 0\) be defined as in (7). We say that \(f \in \mathcal{J}_j^b(\theta; \psi; g; C, D)\) if \(\mathcal{H}_j^b(\xi) = (f \ast g)(\xi)\) holds the subordination condition:

\[
1 + \frac{(1 + i \tan \theta)}{b} \left[ \frac{8q^2 \mathcal{H}_j^b(\xi) + \xi \mathcal{H}_j^b(\xi)}{(1 - \theta) \mathcal{H}_j^b(\xi) + 8 \xi \mathcal{H}_j^b(\xi) - 1} \right] < \frac{(C + 1)\psi(\xi) - (C - 1)}{(D + 1)\psi(\xi) - (D - 1)}
\]

where \(\psi \in \mathcal{P}\) and \(\psi\) is defined as in (21).

By letting \(\psi(\xi) = 1 + \xi(1 - q\xi), q \in (0, 1)\), in \(\mathcal{H}_j^b(\theta; \psi; g; C, D)\), we have

\[
1 + \frac{(1 + i \tan \theta)}{b} \left[ \frac{8q^2 \mathcal{H}_j^b(\xi) + \xi \mathcal{H}_j^b(\xi)}{(1 - \theta) \mathcal{H}_j^b(\xi) + 8 \xi \mathcal{H}_j^b(\xi) - 1} \right] = \frac{(C + 1)w(\xi) + 2 + (C - 1)qw(\xi)}{(D + 1)w(\xi) + 2 + (D - 1)qw(\xi)^2}
\]

where \(q \in (0, 1)\), \(w(\xi)\) is analytic in \(U\), and \(w(0) = 0, |w(\xi)| < 1\).

Remark 2. The impact of Janowski functions on a particular conic region was initiated by Noor and Malik [23] and was subsequently studied by various authors (see [11, 24, 25] and references provided therein).

2. Inclusion Relationships and Integral Representations of the Classes \(\mathcal{H}_j^b(\theta; \psi; g; C, D)\) and \(\mathcal{J}_j^b(\theta; \psi; g; C, D)\)

Let us begin with the following.

Theorem 1. Let \(F_{j,b}(\xi; \psi; g; C, D) = (1 - \theta) \mathcal{H}_j^b(\xi) + 8 \xi \mathcal{H}_j^b(\xi)\). If \(f \in \mathcal{H}_j^b(\theta; \psi; g; C, D)\), then

\[
\Re \left[ 1 + \frac{k(1 + i \tan \theta)}{b} \left( \frac{\xi F_{j,b}'(\xi; \psi; g; C, D)}{F_{j,b}(\xi; \psi; g; C, D) - k} \right) \right] > 0.
\]

Proof. From the definition of \(\mathcal{H}_j^b(\theta; \psi; b; C, D)\) and (18), we have
Replacing \( \xi \) by \( e^\xi \) in (25), then for all \( \nu = 0, 1, 2, \ldots, k - 1 \), we have

\[
\Re \left[ \frac{1 + (1 + i \tan \theta)}{b} \left( \frac{\partial^2 \mathcal{H}^0 \left( \frac{\xi}{\nu} \right) + \xi \mathcal{H}^0 \left( \frac{\xi}{\nu} \right)}{(1 - \vartheta) H_{j,k}(\xi) + \vartheta \xi H_{j,k}(\xi) - 1} \right) \right] > 0.
\]

(25)

Using (5) in (26), we get

\[
\Re \left[ 1 + \frac{(1 + i \tan \theta)}{b} \left( \frac{e^{2\xi} \partial^2 \mathcal{H}^0 \left( \frac{e^\xi}{\nu} \right) + e^{\xi} \mathcal{H}^0 \left( \frac{e^\xi}{\nu} \right)}{(1 - \vartheta) e^{\xi} H_{j,k}(\xi) + \vartheta e^{\xi} H_{j,k}(\xi) - 1} \right) \right] > 0, \quad (\xi \in \mathbb{U}).
\]

(26)

Suppose \( \nu = 0, 1, 2, \ldots, k - 1 \) in (27), respectively, and summing them, we arrive at

\[
\Re \left[ \frac{1 + k(1 + i \tan \theta)}{b} \left( \frac{\xi F_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi) - 1} \right) \right] > 0, \quad (\xi \in \mathbb{U}).
\]

(29)

Hence the proof.

Now, by using the following two equivalent forms (see (14), page 3)) of product rule of the \( q \)-difference operator,

\[
\mathcal{D}_q [f(\xi)g(\xi)] = g(\xi)\mathcal{D}_q [f(\xi)] + f(q\xi)\mathcal{D}_q [g(\xi)]
\]

or equivalently,

\[
\Re \left[ \frac{1 + k(1 + i \tan \theta)}{b} \left( \frac{\xi F_{j,k}(\vartheta; \xi)}{F_{j,k}(\vartheta; \xi) - 1} \right) \right] > 0, \quad (\xi \in \mathbb{U}).
\]

(30)

we can establish the following result by retracing the steps as in Theorem 1.

Theorem 2. Let \( F_{j,k}(\vartheta; \xi) = (1 - \vartheta) H_{j,k}(\xi) + \vartheta \xi H_{j,k}(\xi) \), where \( H_{j,k}(\xi) \neq 0 \). If \( f \in \mathcal{H}_q^0 (\vartheta; \theta; \psi; g; C, D) \), then

\[
\Re \left[ \frac{1 + k(1 + i \tan \theta)}{b} \left( \frac{\xi \mathcal{D}_q [F_{j,k}(\vartheta; \xi)]}{F_{j,k}(\vartheta; \xi) - 1} \right) \right] > 0.
\]

(31)

Theorem 3. Let \( f \in \mathcal{H}_q^0 (\vartheta; \theta; \psi; g; C, D) \), then

\[
\mathcal{H}_{j,k}(\xi) = \exp \left\{ \frac{1}{k} \sum_{n=0}^{k-1} e^{\xi} \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)\nu((D + 1)\psi(w(t)) - (D - 1))} dt \right\}, \quad \text{if} \ \vartheta = 0,
\]

\[
\mathcal{H}_{j,k}(\xi) = \int_0^1 \exp \left\{ \frac{1}{k} \sum_{n=0}^{k-1} e^{\xi} \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)\nu((D + 1)\psi(w(t)) - (D - 1))} dt \right\} d\xi, \quad \text{if} \ \vartheta = 1,
\]

where \( \mathcal{H}_{j,k}(\xi) \) is given by (7), \( w(\xi) \) is analytic in \( \mathbb{U} \), and \( w(0) = 0, |w(\xi)| < 1 \).

Proof. Let \( f \in \mathcal{H}_q^0 (\vartheta; \theta; \psi; g; C, D) \). In view of (20), we have

\[
\frac{\partial^2 \mathcal{H}^0(\xi) + \xi \mathcal{H}^0(\xi)}{(1 - \vartheta) H_{j,k}(\xi) + \vartheta \xi H_{j,k}(\xi) - 1} = \frac{b(C - D)[\psi(w(\xi)) - 1]}{(1 + i \tan \theta)((D + 1)\psi(w(\xi)) - (D - 1))},
\]

(33)
where \( w(\xi) \) is analytic in \( U \) and \( w(0) = 0, \ |w(\xi)| < 1 \). Substituting \( \xi \) by \( e^\zeta \xi \) in equality (33) and ensuing the steps as in Theorem 1, we get

\[
\frac{\xi F'_{j,k}(\theta; \xi)}{F_{j,k}(\theta; \xi)} - 1 = \frac{1}{k} \sum_{i=0}^{k-1} \frac{b(C - D)[\psi(w(e^i\xi)) - 1]}{(1 + i \tan \theta)[(D + 1)\psi(w(e^i\xi)) - (D - 1)]}.
\]

(34)

From this equality, we get

\[
\frac{\xi F'_{j,k}(\theta; \xi)}{F_{j,k}(\theta; \xi)} - 1 = \frac{1}{k} \sum_{i=0}^{k-1} \frac{b(C - D)[\psi(w(e^i\xi)) - 1]}{(1 + i \tan \theta)\xi[(D + 1)\psi(w(e^i\xi)) - (D - 1)]}.
\]

(35)

Upon integration, we get

\[
\log \left( \frac{F_{j,k}(\theta; \xi)}{\xi} \right) = \frac{1}{k} \sum_{i=0}^{k-1} \int_0^\xi \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)t[(D + 1)\psi(w(t)) - (D - 1)]} \, dt.
\]

(36)

or equivalently,

\[
(1 - \theta)\mathcal{F}_{j,k}(\xi) + \theta \xi \mathcal{F}_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{i=0}^{k-1} \int_0^\xi \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)t[(D + 1)\psi(w(t)) - (D - 1)]} \, dt \right\}. \tag{37}
\]

This concludes the proof of Theorem 3.

\[ \square \]

**Theorem 4.** Let \( f \in \mathcal{H}_y(\theta; \psi; g; C, D) \), then we have

\[
\mathcal{H}_{j,k}(\xi) = \xi \exp \left\{ \frac{\ln q}{(q - 1)k} \sum_{i=0}^{k-1} \int_0^\xi \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)t[(D + 1)\psi(w(t)) - (D - 1)]} \, dt \right\}, \quad \text{(if } \theta = 0),
\]

\[
\mathcal{H}_{j,k}(\xi) = \int_0^\xi \exp \left\{ \frac{\ln q}{(q - 1)k} \times \sum_{i=0}^{k-1} \int_0^\xi \frac{b(C - D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)t[(D + 1)\psi(w(t)) - (D - 1)]} \, dt \right\} \, d\zeta, \quad \text{(if } \theta = 1). \tag{38}
\]

where \( \mathcal{H}_{j,k}(\xi) \) defined by equality (7), \( w(\xi) \) is analytic in \( U \), and \( w(0) = 0, \ |w(\xi)| < 1 \).

Proof. Let \( f \in \mathcal{H}_y(\theta; \psi; g; C, D) \). In sight of Theorems 2 and 3, we have

\[
\frac{\xi F'_{j,k}(\theta; \xi)}{F_{j,k}(\theta; \xi)} - 1 = \frac{1}{k} \sum_{i=0}^{k-1} \frac{b(C - D)[\psi(w(e^i\xi)) - 1]}{(1 + i \tan \theta)\xi[(D + 1)\psi(w(e^i\xi)) - (D - 1)]}.
\]

(39)

where \( w(\xi) \) is analytic in \( U \) and \( w(0) = 0, \ |w(\xi)| < 1 \). For \( f \in \mathcal{H}(U) \) and \( 0 < q < 1 \), we obtain (see [10])

\[
I_q \frac{\zeta F(\xi)}{f(\xi)} = \frac{q - 1}{\ln q} \log f(\xi), \tag{40}
\]
where $I_{q}f$ is the Jackson $q$-integral, defined as in (14). Integrating the above equality, we get
\begin{equation}
\frac{q-1}{\ln q} \log \left( \frac{F_{j,k}(\theta; \xi)}{\xi} \right) = \frac{1}{k} \sum_{\nu=0}^{k-1} \int_{0}^{\xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)[(D+1)\psi(w(t)) - (D-1)]} \, dq, \tag{41}
\end{equation}
or equivalently,
\begin{equation}
(1-\theta)\mathcal{H}_{j,k}(\xi) + \theta \mathcal{H}_{j,k}'(\xi) = \xi \exp \left\{ \frac{\ln q}{(q-1)k} \sum_{\nu=0}^{k-1} \int_{0}^{\xi} \frac{b(C-D)[\psi(w(t)) - 1]}{(1 + i \tan \theta)[(D+1)\psi(w(t)) - (D-1)]} \, dt \right\}. \tag{42}
\end{equation}

This concludes the proof of Theorem 3.

By fixing $C = 1, D = -1, \theta = 0, b = 1$, and $g(\xi) = \xi + \sum_{n=2}^{\infty} \xi^n$ in Theorem 3, we state the subsequent result.

**Corollary 1** (see [19], Theorems 3 and 4). Let $f_{j,k}(\xi) \neq 0$ be assumed as in (4).

(i) If $f \in \mathcal{O}^{(j,k)}(\phi)$, then
\begin{equation}
f_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_{0}^{\xi} \frac{\psi(w(t)) - 1}{t} \, dt \right\}. \tag{43}
\end{equation}

(ii) If $f \in \mathcal{O}^{(j,k)}(\phi)$, then
\begin{equation}
f_{j,k}(\xi) = \int_{0}^{\xi} \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_{0}^{\xi} \frac{\psi(w(t)) - 1}{t} \, dt \right\} \, dx_{i} \xi_{j}, \tag{44}
\end{equation}

where $w(\xi)$ is analytic in $U$ and $w(0) = 0, |w(\xi)| < 1$.

**Corollary 2.** Let $f_{j,k}(\xi) \neq 0$ be assumed as in (4). If $f \in \mathcal{O}^{(j,k)}(\phi)$, then
\begin{equation}
f_{j,k}(\xi) = \xi \exp \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \int_{0}^{\xi} \frac{1 + Cw(t)}{1 + Dw(t)} \, dt \right\}, \tag{45}
\end{equation}
where $w(\xi)$ is analytic in $U$ and $w(0) = 0, |w(\xi)| < 1$.

### 3. Coefficient Inequalities for $\mathcal{K}_{s}^{\theta}(\theta; \psi; g; C, D)$ and $\mathcal{K}_{s}^{\theta}(\theta; \psi; g; C, D)$

The coefficient estimate $|a_n|$ of the defined function classes is determined in this section.

**Theorem 5.** Let $|\psi(\xi)| < |D - 1/(D+1)|, \forall \xi \in U$ and $Y_n$'s be real. If $f \in \mathcal{K}_{s}^{\theta}(\theta; \psi; g; C, D)$, then for $n \geq 2$,
\begin{equation}
|a_n| \leq \frac{1}{Y_n} \prod_{m=1}^{n-1} \frac{|(1-\theta) + \theta m|_{\Lambda_{m,j}(C-D)}L_1b - 2(1 + i \tan \theta) [\theta(m-1 - \Lambda_{m,j}) + m - (1-\theta)\Lambda_{m,j}]D|}{2 \sec \theta \theta(m+1)(m - \Lambda_{m+1,j}) + (m+1) - (1-\theta)\Lambda_{m+1,j}}. \tag{46}
\end{equation}

**Proof.** By the definition of $\mathcal{K}_{s}^{\theta}(\theta; \psi; g; C, D)$, we have
\begin{equation}
1 + \frac{(1 + i \tan \theta)}{b} \left[ \frac{\partial \mathcal{K}_{s}^{\theta}(\xi) + \mathcal{K}_{s}^{\theta}(\xi)}{(1-\theta)\mathcal{K}_{s}^{\theta}(\xi) + \mathcal{K}_{s}^{\theta}(\xi)} \right] = p(\xi). \tag{47}
\end{equation}
where $p(\xi) \in \mathbb{P}$ and satisfies the condition $p(\xi) < ((1+C)\psi(\xi) - (C-1))((D+1)\psi(\xi) - (D-1))$.

Equivalently, (47) can be rewritten as
\begin{equation}
(1 + i \tan \theta) \left[ \sum_{n=2}^{\infty} \left[ \theta n(m - 1 - \Lambda_{n,j}) + n - (1-\theta)\Lambda_{n,j} \right] Y_n a_n \right] = b \left[ \sum_{n=1}^{\infty} [(1-\theta) + \theta n] \Lambda_{n,j} Y_n a_n \right]. \tag{48}
\end{equation}

On equating the coefficient of $\xi^n$, we get
\begin{equation}
(1 + i \tan \theta) \left[ \theta(n - 1 - \Lambda_{n,j}) + n - (1-\theta)\Lambda_{n,j} \right] Y_n a_n \nonumber
\end{equation}
\begin{equation}
= b \left[ p_{n-1} \Lambda_{1,j} + p_{n-2} (1 + 2) \Lambda_{2,j} Y_2 a_2 + \ldots + p_1 \right] \cdot (1 + (n-2)\theta) Y_{n-1} Y_{n-1}. \tag{49}
\end{equation}

From Lemma 6 of [26], we have $|p_n| \leq |L_1|(C-D)/2$. On computation, we have
\begin{equation}
|a_n| \leq \frac{|b| |(C-D)|L_1}{2 \sec \theta \theta(n-1 - \Lambda_{n,j}) + n - (1-\theta)\Lambda_{n,j}} Y_n \sum_{m=1}^{n-1} [(1-\theta) + \theta m] \Lambda_{m,j} Y_m |a_m|. \tag{50}
\end{equation}
Taking \( n = 2 \), in (50), we get

\[
|a_2| \leq \frac{|b| (C - D) [L_1]}{2 \sec \theta [28(1 - \Lambda_{2,j}) + 2 - (1 - \theta)\Lambda_{2,j}]} Y_2
\]  

(51)

On substituting \( n = 2 \) in (46), we can see the hypothesis is true for \( n = 2 \). Now, taking \( n = 3 \) in (50), we get

\[
|a_3| \leq \frac{|b| (C - D) [L_1]}{2 \sec \theta [38(2 - \Lambda_{3,j}) + 3 - (1 - \theta)\Lambda_{3,j}]} Y_3 \left[ 1 + (1 + \theta)\Lambda_{2,j} Y_2 |a_2| \right]
\]

\[
\leq \frac{|b| (C - D) [L_1]}{2 \sec \theta [38(2 - \Lambda_{3,j}) + 3 - (1 - \theta)\Lambda_{3,j}]} Y_3 \left[ 1 + \frac{1}{2} \sec \theta \left[ 20(1 - \Lambda_{2,j}) + 2 - (1 - \theta)\Lambda_{2,j} \right] |b| (C - D) [L_1] \right].
\]

(52)

If we let \( n = 3 \), in (46), we have

\[
|a_3| \leq \frac{1}{Y_3} \left[ \frac{|b| (C - D) [L_1]}{2 \sec \theta [28(1 - \Lambda_{3,j}) + 2 - (1 - \theta)\Lambda_{2,j}]} \times \frac{(C - D) [L_1] b (1 + \theta)\Lambda_{2,j} - 2 (1 + \theta)\Lambda_{2,j} D}{2 \sec \theta [38(2 - \Lambda_{3,j}) + 3 - (1 - \theta)\Lambda_{3,j}]} \right]
\]

\[
\leq \frac{1}{Y_3} \left[ \frac{|b| (C - D) [L_1]}{2 \sec \theta [38(2 - \Lambda_{3,j}) + 3 - (1 - \theta)\Lambda_{3,j}]} \times \frac{|b| (C - D) [L_1] (1 + \theta)\Lambda_{2,j} + 2 \sec \theta [28(1 - \Lambda_{2,j}) + 2 - (1 - \theta)\Lambda_{2,j}]}{2 \sec \theta [28(1 - \Lambda_{2,j}) + 2 - (1 - \theta)\Lambda_{2,j}]} \right]
\]

\[
\leq \frac{1}{Y_3} \left[ \frac{|b| (C - D) [L_1]}{2 \sec \theta [38(2 - \Lambda_{3,j}) + 3 - (1 - \theta)\Lambda_{3,j}]} \times \frac{1 + (1 + \theta)\Lambda_{2,j} |b| (C - D) [L_1]}{2 \sec \theta [28(1 - \Lambda_{2,j}) + 2 - (1 - \theta)\Lambda_{2,j}]} \right].
\]

(53)

Hence, the hypothesis of the theorem is true for \( n = 3 \). Now, let us suppose (46) is valid for \( n = 2, 3, \ldots, r \). On using triangle inequality in (46), we get

\[
|a_r| \leq \frac{1}{Y_r} \prod_{m=1}^{r-1} \left[ (1 - \theta) + \delta m \Lambda_{m,j} |b| (C - D) [L_1] + 2 \sec \theta \left[ \delta m (m - 1 - \Lambda_{m,j}) + m - (1 - \theta)\Lambda_{m+1,j} \right] \right]
\]

(54)

By induction hypothesis, we have

\[
\frac{|b| (C - D) [L_1]}{2 \sec \theta [\delta (r - 1 - \Lambda_{r,j}) + r - (1 - \theta)\Lambda_{r,j}]} Y_r \left[ \sum_{m=1}^{r-1} [(1 - \theta) + \delta m \Lambda_{m,j} Y_m |a_m|] \right]
\]

\[
\leq \frac{1}{Y_r} \prod_{m=1}^{r-1} \left[ (1 - \theta) + \delta m \Lambda_{m,j} |b| (C - D) [L_1] + 2 \sec \theta \left[ \delta m (m - 1 - \Lambda_{m,j}) + m - (1 - \theta)\Lambda_{m+1,j} \right] \right]
\]

(55)

From the above inequality, we have
\[
\prod_{n=1}^{r} \left\{ \frac{[(1 - \theta) + 8m]\Lambda_{m,j}[b](C - D)[L_i] + 2 \sec \theta [8m(m - 1 - \Lambda_{m,j}) + m - (1 - \theta)\Lambda_{m,j}]}{2Y_{r+1} \sec \theta [8(m + 1)(m - \Lambda_{m+1,j}) + (m + 1) - (1 - \theta)\Lambda_{m+1,j}]} \right\} \geq \frac{|b|(C - D)[L_i]}{2 \sec \theta [8(r + 1)(r - \Lambda_{r+1,j}) + (r + 1) - (1 - \theta)\Lambda_{r+1,j}]} Y_{r+1} \left[ \sum_{n=1}^{r} [(1 - \theta) + 8m]\Lambda_{m,j}Y_{n}\{a_{m}\} \right],
\]

which implies that inequality (46) is true for \( n = r + 1 \). Hence the proof of the theorem. \( \square \)

**Theorem 6.** Let \(|\psi(\xi)| < |D - 1/(D + 1)|\) for all \( \xi \in \mathbb{U} \) and \( Y_n \) be real. If \( f \in \mathcal{O}_{\mathcal{X}}^b(\theta; \psi; g; C, D) \), then for \( n \geq 2 \),

\[
|a_n| \leq \frac{1}{n+1} \prod_{n=1}^{n} \left[ \frac{[(1 - \theta) + 8[m]q] \Lambda_{m,j}(C - D)_{q} b - 2(1 + i \tan \theta)[q8[m]q[(m - 1)q - \Lambda_{m,j}] + [m]q - (1 - \theta)\Lambda_{m,j}]D}{2 \sec \theta [8q(m + 1)q[(m + 1)q - \Lambda_{m+1,j}] + (m + 1) - (1 - \theta)\Lambda_{m+1,j}]} \right].
\]  

**Proof.** From the definition of \( \mathcal{O}_{\mathcal{X}}^b(\theta; \psi; g; C, D) \), we have

\[
(1 + i \tan \theta) \left[ \sum_{n=1}^{\infty} \left[ 8[n]q[(m - 1)q - \Lambda_{m,j}] + [m]q - (1 - \theta)\Lambda_{m,j} \right] Y_{n}\{a_{n}\} \right] = b \left[ \sum_{n=1}^{\infty} \left[ (1 - \theta) + 8[n]q\Lambda_{m,j}Y_{n}\{\xi^n\} \right] \right].
\]

Equating the coefficient of \( \xi^n \) and retracing the steps as in Theorem 5, we get

\[
|a_n| \leq \frac{|b|(C - D)[L_i]}{2 \sec \theta [8(m + 1)q[(m + 1)q - \Lambda_{m+1,j}] + (m + 1) - (1 - \theta)\Lambda_{m+1,j}]} \left[ \sum_{n=1}^{n} [(1 - \theta) + 8[m]q\Lambda_{m,j}Y_{m}[a_{m}]] \right].
\]

(59)

Now, by repeating the processes in Theorem 5, we acquire the required outcome. \( \square \)

If we let \( \psi(\xi) = 1 + \xi/(1 - q\xi) \) in Theorem 6, we have the following.

**Corollary 3.** Let \( f \in \mathcal{O}_{\mathcal{X}}^b(\theta; \psi; g; C, D) \) and \( Y_n \) be real, then for \( n \geq 2 \),

\[
|a_n| \leq \frac{1}{n+1} \prod_{n=1}^{n} \left[ \frac{[(1 - \theta) + 8[m]q\Lambda_{m,j}(C - D)(1 + q)b - 2(1 + i \tan \theta)[q8[m]q((m - 1)q - \Lambda_{m,j}) + [m]q - (1 - \theta)\Lambda_{m,j}] D(1 + q) + (1 - q)]}{2 \sec \theta [8q(m + 1)q[(m + 1)q - \Lambda_{m+1,j}] + (m + 1) - (1 - \theta)\Lambda_{m+1,j}]} \right].
\]

(60)

If we let \( \theta = 0, \ b = 1, \ g(\xi) = \xi + \sum_{n=1}^{\infty} \xi^n, \) and \( \psi(\xi) = 1 + \xi/1 - \xi \) in Theorem 5, we have the following.

**Corollary 4.** (see \[18], Theorem 2) Let \( f \in \mathcal{O}^{(j,k)}(C, D) \) and \( Y_n \) be real, then for \( n \geq 2 \),

\[
|a_n| \leq \prod_{n=1}^{n} \Lambda_{m,j}[\Lambda_{m,j} - 1 + m] / (m + 1) - \Lambda_{m+1,j}.
\]

(61)

**4. Conclusion**

Very few studies have been showed on analytic functions with regard to \((j,k)\)-symmetric points. Since we have articulated the problem differently so as to deviate from the similar studies, only few special cases could be discussed. Furthermore, by swapping the ordinary differentiation with quantum differentiation, we have tried at the discretization of some of the familiar findings.
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest.

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