Filtering in discrete systems with multiplicative perturbations and unknown input

K S Kim and V I Smagin*
National Research Tomsk State University, Tomsk, Russia

*E-mail: vsm@mail.tsu.ru

Abstract. The problem of filter synthesis for discrete systems with multiplicative perturbations and an unknown input is considered. The problem is solved on the basis of the separation principle using optimal recurrent filtering, the least squares method and smoothing procedures.

1. Introduction
Models of discrete systems with additive and multiplicative perturbations are a class of systems used in various problems of information processing in economic systems, energy systems, flight systems, communication systems, as well as in processing the results of a physical experiment with materials.

The estimation problem for discrete systems with multiplicative perturbations was considered in [1–3]. Similar estimation problems for continuous systems with multiplicative perturbations were studied in [4]. For models of systems with indefinite parameters, such problems were considered in [5]. Filtering problems for discrete systems with an unknown input and additive Gaussian perturbations were studied in [6–13]. In [6] to solve the filtering problem with unknown input, the compensation method was used, in [7–8] the least-squares method (LSM) was used to calculate estimates of the unknown input, in [9–13] nonparametric smoothing algorithms for systems with additive perturbations were used to improve the quality of estimation.

In [14] the problem of extrapolation in discrete systems with additive and multiplicative perturbations under incomplete information conditions is studied. In this article, the problem of filtering in discrete systems with multiplicative and additive perturbations with unknown input is considered. The problem is solved based on the separation principle using LSM algorithms and smoothing nonparametric procedures. The proposed method allowed to increase the accuracy of filtering.

2. Problem statement
Let the discrete systems with multiplicative perturbations and unknown input be described by a difference equation:

\[ x(k+1) = Ax(k) + \sum_{j=1}^{m} A_j x(k) \theta_j(k) + f(k) + q(k), \quad x(0) = x_0, \]  

(1)

where \( x(k) \in \mathbb{R}^n \) denote the state of the system; \( f(k) \) is unknown input; \( x_0 \) is random vector (assumed dispersion matrix \( N_0 = \mathbb{E}\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\} \) and expected value \( \bar{x}_0 = \mathbb{E}(x_0) \));
A. \( A_s \) \((s = 1...m)\) are matrices of corresponding dimensions; \( q(k) \) and \( \theta_s(k) \) are the Gaussian random sequence with characteristics:

\[
E\{q(k)\} = 0, \quad E\{q(k)q^T(j)\} = Q(\delta_{kj}), \quad E\{\theta_s(k)\} = 0, \quad E\{\theta_s(k)\theta_s^T(j)\} = \Theta_s(\delta_{kj}).
\]

Here \( E\{\cdot\} \) denotes the mathematical expectation of operator, \( ^T \) denotes matrix transposition and \( \delta_{kj} \) denotes Kronecker delta.

The observation channel is described by the formula:

\[
y(k) = Sx(k) + v(k),
\]

where \( v(k) \) is the Gaussian random sequence with characteristics: \( E\{v(k)\} = 0, \quad E\{v(k)v^T(j)\} = V(\delta_{kj}). \) It is assumed that the sequences \( q(k), \theta_s(k), v(k) \) are independent, the pair of matrices \( S, A \) is detectable, matrices \( Q, \Theta_s \) \((s = 1...m)\) are non-negative defined, \( V > 0. \)

It is required to find the corresponding optimal estimate of the state vector \( \hat{x}(k+1) \) from the observations (2) received at time \( k \) for the following criterion given in the interval \( k \in [0, T] \)

\[
J[0; T] = M \left\{ \sum_{k=0}^{T} e^T(k)e(k) \right\},
\]

where \( e(k) = x(k) - \hat{x}(k) \) is error vector.

3. Optimal filter synthesis

To solve the filtering problem, we will use the separation principle. This means that we first construct estimates of the vector \( \hat{x}(k) \) under the assumption that the vector \( \hat{f}(k) \) is known, then construct estimates of the vector \( \hat{f}(k) \) under the assumption that the estimate of the state vector is known.

In this case, we can use the recurrent Kalman filter algorithm to construct the estimate

\[
\hat{x}(k+1) = A\hat{x}(k) + f(k) + K(k)[y(k+1) - S(A\hat{x}(k) + f(k))], \quad \hat{x}(0) = \bar{x}_0,
\]

where \( K(k) \) is transfer matrices of the filter, which we define from the minimum of a criterion (3) for \( k \in [0, T]. \)

Find the matrix \( K(k) \) that provides the minimum of criterion (3). For this purpose, we write the equation error vector \( e(k) \), subtracting from equation (1), equation (4):

\[
e(k+1) = x(k+1) - \hat{x}(k+1) = (A - K(k)SA)e(k) + (I - K(k)S)\sum_{s=1}^{m} A_s x(k)\theta_s(k) + (I - K(k)S)q(k) - K(k)v(k),
\]

where \( I \) is identity matrix.

Taking into account (5), we obtain the equation for the matrix \( N(k) = M\{e(k)e(k)^T\} \) in the form of the following difference equation.

\[
N(k+1) = (A - K(k)SA)N(k)(A - K(k)SA)^T + (I - K(k)S)\sum_{s=1}^{m} A_s N(k)A_s^T +
\]

\[
+ \sum_{s=1}^{m} A_s\hat{x}(k)\hat{x}(k)^T A_s^T + Q(k)(I - K(k)S)^T + K(k)V(k)K(k)^T, \quad N(0) = N_0.
\]

We represent criterion (3) in the form
\[ J[0; T] = \sum_{k=0}^{T} \text{tr} N(k), \]  
(7)

where \( \text{tr} \) is matrix trace. Then, substituting in (7) the formula (6) with a shift of one step, we get:

\[
J[0; T] = \text{tr} N_0 + \sum_{k=1}^{T} \text{tr} [(A - K(k-1)S)N(k-1)(A - K(k-1)S)^T + (I - K(k-1)S)[\sum_{j=1}^{m} A_j N(k-1)A_j^T + \\
+ \sum_{j=1}^{m} A_j \hat{\chi}(k-1)\hat{\chi}(k-1)^T A_j^T + Q(k-1)](I - K(k-1)S)^T + K(k-1)V(k-1)K(k-1)^T].
\]  
(8)

Using the rules of differentiating the trace function (\( \text{tr} \)) from the product of matrices [15]:

\[
\frac{\partial \text{tr} AXB}{\partial X} = A^T B^T, \quad \frac{\partial \text{tr} A^T XB^T}{\partial X} = BA,
\]  
(9)

from the equation

\[
\frac{\partial J[0; T]}{\partial K} = 0.
\]  
(10)

we get the expression for the matrix \( K(k) \):

\[
K(k) = (AN(k)A^T S^T + [\sum_{j=1}^{m} A_j N(k)A_j^T + \sum_{j=1}^{m} A_j \hat{\chi}(k)\hat{\chi}(k)^T A_j^T + Q(k)]S^T) \times \\
\times (SAN(k)A^T S^T + S[\sum_{j=1}^{m} A_j N(k)A_j^T + \sum_{j=1}^{m} A_j \hat{\chi}(k)\hat{\chi}(k)^T A_j^T + Q(k)]S^T + V(k))^{-1}.
\]  
(11)

Consider the stationary case, when further assumed that the pair of matrices \( A, B \) controllable and there exists a feedback control, depends on \( \hat{\chi}(k) \), which performs tracking some constant vector \( z \).

Then the transfer matrix \( K \) will also be constant and can be calculated by the formula:

\[
K = (ANA^T S^T + [\sum_{j=1}^{m} A_j NA_j^T + \sum_{j=1}^{m} A_j zz^T A_j^T + Q]S^T)(SANA^T S^T + \\
+ S[\sum_{j=1}^{m} A_j NA_j^T + \sum_{j=1}^{m} A_j zz^T A_j^T + Q]S^T + V)^{-1},
\]  
(12)

Where matrix \( N \) is determined from the solution of the matrix algebraic equation:

\[
N = (A - KSA)N(A - KSA)^T + (I - KS)[\sum_{j=1}^{m} A_j NA_j^T + \sum_{j=1}^{m} A_j zz^T A_j^T + Q](I - KS)^T + KVK^T.
\]  
(13)

If the solution of the equation (13) exists, \( N > 0 \), as well, due to the fact that the matrix

\[
(I - KS)[\sum_{j=1}^{m} A_j NA_j^T + \sum_{j=1}^{m} A_j zz^T A_j^T + Q](I - KS)^T + KVK^T
\]

positive definite, then the condition in the form of a matrix inequality will be satisfied:

\[
(A - KSA)N(A - KSA)^T - N < 0.
\]  
(14)

Condition (14) ensures the stability of the matrix filter dynamics \((A - KSA)\).
4. Estimation of the unknown input

As an algorithm for estimating an unknown input, we will use LMS algorithms; in this case, an estimate can be constructed based on minimizing an criterion [7, 8]:

$$I = \sum_{i=1}^{2} \left( \| y(t+1) - S\hat{x}(t) \|_W^2 + \| f(t) \|_W^2 \right),$$  \hspace{1cm} (15)

where $W, \bar{W}$ are positive defined weight matrices, $\hat{x}(t) = A\hat{x}(t) + r(t)$. The estimates of the unknown input constructed on the basis of minimization (15) will take the form:

$$\hat{f}^{(\text{LMS})}(k) = [S^T WS + \bar{W}]^{-1} S^T W \{ y(k+1) - S(A\hat{x}(k)) \}.$$  \hspace{1cm} (16)

The estimate of the unknown input, using additional nonparametric smoothing [11–13], is determined by the formula:

$$\hat{f}^{(\text{NP})}(k) = [S^T WS + \bar{W}]^{-1} S^T W \hat{\Omega},$$  \hspace{1cm} (17)

where $j$ component of the vector $\hat{\Omega}(k)$ is

$$\hat{\Omega}_j(k) = \sum_{i=1}^{k} \frac{1}{\mu_j} [y(i+1) - S\hat{\alpha}(i)] G\left(\frac{k-i+1}{\mu_j}\right).$$  \hspace{1cm} (18)

In relation (18), $G(\cdot)$ is a kernel function and $\mu_j$ is smoothing factor.

5. Numerical simulation

Estimates of the state vector were determined from the recurrence equation

$$\dot{x}(k+1) = A\dot{x}(k) + Bu(k) + \hat{f}(k) + K(k)(y(k) - S\dot{x}(k)), \quad \dot{x}(0) = \bar{x}_0,$$  \hspace{1cm} (19)

where the matrix $K(k)$ was calculated by the formula (12).

The simulation was performed for the following data:

$$A = \begin{pmatrix} 0.5 & 0.1 \\ -0.05 & 0.84 \end{pmatrix}, A_1 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & 0.01 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.01 & 0.01 \\ 0.01 & 0.03 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2.8 & 0 \\ 0 & 1.9 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 4.2 & 0.4 \\ 0.4 & 2.8 \end{pmatrix}, W = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad \bar{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A kernel function in (18) is Gaussian:

$$G(u) = \frac{\exp \left( \frac{-u^2}{2} \right)}{\sqrt{2\pi}}.$$

Table 1 shows the results of comparing the standard errors of the deviations of the state vector estimates for recurrent filtering algorithms, with an exact change in the input $f(k)$, when using an LMS and a modified LMS for estimating an unknown input by a nonparametric smoothing algorithm to estimate the unknown input. Averaging was performed over 100 implementations. The calculation of the standard errors of the estimation was performed according to the formulas:
\[
\sigma_{x,i} = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} (x_i(k) - \hat{x}_i(k))^2}, \quad \sigma_{f,i} = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} (f_i(k) - \hat{f}_i(k))^2} \quad (i = 1, 2).
\]

Table 1. The standard errors of the estimates of the state vector \(x(k) (\sigma_{x,i})\).

| The number of component | exact change input \(f(k)\) | LMS | LMS with nonparametric smoothing |
|-------------------------|-----------------------------|-----|---------------------------------|
| 1                       | 1.471                       | 2.223 | 1.702                           |
| 2                       | 1.279                       | 1.883 | 1.441                           |

Table 2 shows the standard errors of the deviations of the estimates of the unknown input.

Table 2. The standard errors of the estimates \(f(k) (\sigma_{f,i})\).

| The number of component | LMS | LMS with nonparametric smoothing |
|-------------------------|-----|---------------------------------|
| 1                       | 2.655 | 0.959                           |
| 2                       | 2.516 | 0.658                           |

Tables 1 and 2 show that the use of LSM for estimates of the unknown input with additional smoothing using an algorithm and nonparametric smoothing reduces the mean-square errors of state vector filtering estimates and the mean-square errors of estimates of the unknown input vector.

6. Conclusion
A solution to the problem of filter synthesis for a linear discrete model with additive and multiplicative perturbations with an unknown input is obtained. It is shown that the use of smoothing algorithms for evaluating an unknown input can improve the accuracy of the estimation.

Acknowledgments
This work was supported by the RFBR according to the research project № 19-31-90080.

References
[1] Mohanty N C and Soong T T 1977 Linear filtering in multiplicative noise Information and Control 34 141–7
[2] Stoica A-M and Yaesh I 2008 Kalman–type filtering for discrete–time stochastic systems with state–dependent noise. Proc. Mathematical Theory of Network and Systems-MTN 28 July–1 August Blacksburg VA pp 1–6
[3] Wu Y, Zhang Q and Shen Z 2016 Kalman filtering with multiplicative and additive noises Proc. 12th World Congress on Intelligent Control and Automation (WCICA) pp 483–7
[4] Germani A, Manes C and Palumbo P 2002 Linear filtering for bilinear stochastic differential systems with unknown inputs. IEEE Transactions on Automatic Control 47 1726–30
[5] Yang F, Wang Z and Hung Y S 2002 Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noises. IEEE Trans. on Automatic Control 47 1179–83
[6] Smagin V I 2015 State estimation for nonstationary discrete systems with unknown input using compensations Russian Physics Journal 58 1010–7
[7] Janczak D and Grishin Yu 2006 State estimation of linear system with unknown input and uncertain observation using dynamic programming Control and Cybernetics 4 851–62
[8] Gillijns S and Moor B 2007 Unbiased minimum-variance input and state estimation for linear discrete-time systems Automatica 43 111–6
[9] Hsien C-S 2010 On the optimality of two-stage Kalman filter for systems with unknown input. *Asian Journal of Control* **12** 510–523

[10] Witczak M 2014 Fault diagnosis and fault-tolerant control strategies for non-linear systems Chapter 2 Unknown input observers and filters *Lecture Notes in Electrical Engineering* Springer International Publishing Switzerland pp 19–56

[11] Smagin V I, Koshkin G M and Udod V A 2015 State Estimation for Linear Discrete-Time Systems with Unknown Input Using Nonparametric Technique. *ACSR-Advances in Computer Science Research* **18** 675–7

[12] Smagin V I and Koshkin G M 2015 Kalman filtering and control algorithms for systems with unknown disturbances and parameters using nonparametric technique *Proceedings 20th International Conference on Methods and Models in Automation and Robotics* (MMAR 2015) Miedzyzdroje Poland pp 247–51

[13] Koshkin G M and Smagin V I 2016 Kalman filtering and forecasting algorithms with use of nonparametric functional estimators. *Springer Proceedings in Mathematical Statistics* ed Ricardo Cao, et al **175** 75–84

[14] Kim K S and Smagin V I 2019 Extrapolation in discrete systems with multiplicative perturbations at incomplete information. *Tomsk State University Journal of Control and Computer Science* **47** 49–56

[15] Athans M 1968 The matrix minimum principle. *Informat. and Contr* **11** 592–606