LD-stability for Goldie rings and their subrings of invariants

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Abstract

In this article we investigate further a notion of noncommutative transcendence degree, the lower transcendence degree, introduced by J. J. Zhang in 1998, with important connections to the classical Gelfand-Kirillov transcendence degree, noncommutative projective algebraic geometry and many open problems in ring theory. We compute the value of this invariant for many different algebras, showing that they are in fact LD-stable, which reduces the question of finding this invariant to the computation of the Gelfand-Kirillov dimension. We show that the lower transcendence degree behaves well with respect to taking invariants of the rings under consideration by actions of finite groups, and that it is Morita invariant, demonstrating that it has good theoretical properties lacking in other notions of noncommutative transcendence degree. Finally, many applications appear through the text.

Keywords: Lower transcendence degree, Gelfand-Kirillov transcendence degree, Gelfand-Kirillov dimension, division algebras

2020 Mathematics Subject Classification: 16P90, 16P50, 16W50, 16W22, 16U20

1 Introduction

Throughout the paper all rings will be algebras over an arbitrary base field k of zero characteristic, unless specified otherwise. For a prime Goldie ring A we denote by Q(A) its total quotient ring.

Transcendence degree is a very important and useful concept in the study of commutative rings and algebras. A classical result establishes the equality of the transcendence degree and the Krull dimension for a finitely generated algebra over a field. In noncommutative setting different analogs of transcendence degree were considered in [15, 6, 31, 32, 33, 34, 35, 36, 12, 10, 4]. The classical Gelfand-Kirillov transcendence degree was introduced by Gelfand and Kirillov in 1966 [15] and successfully studied in [6, 23, 11, 12], for instance. In particular, it was used in [15] to show that the skew fields of fractions of the
Weyl Algebras $A_n(k)$ (the Weyl fields) are non-isomorphic when the positive integer $n$ varies.

Let $A$ be a $k$-algebra. Recall that the Gelfand-Kirillov dimension of $A$, $GK A$, is defined as:

$$
sup V \limsup_{n \to \infty} \frac{\log(\dim V^n)}{\log n},$$

where $V$ ranges over all finite dimensional subspaces of $A$ with unity, and $V^0 = k, V^i = V.V^{i-1}, i > 0$. Then the Gelfand-Kirillov transcendence degree of $A$, $GKtdg A$, is defined as:

$$
sup V \inf_b \limsup_{n \to \infty} \frac{\log(\dim (k + bV)^n)}{\log n},$$

where $V$ ranges over all finite dimensional subspaces of $A$ with unity and $b$ over all regular elements of the algebra.

The Gelfand-Kirillov transcendence degree is in general a difficult invariant to compute, given that we have to consider all regular $b$. Zhang ([41], [42]) developed a general technique that permitted to compute the Gelfand-Kirillov transcendence degree for many important classes of algebras, such as certain quantum groups, semiprime Goldie PI-algebras, and certain Artin-Schelter regular algebras. His approach also recovered the results of [18], [6] and [23]. It is also true that the Gelfand-Kirillov transcendence degree coincides with the usual transcendence degree for any commutative field [41, Proposition 2.2]. However, in the noncommutative setting the Gelfand-Kirillov transcendence degree has some undesirable aspects. For instance ([4, Introduction]), the following very natural questions analogous to standard facts for commutative fields, in general are unknown ([41], [42]):

**Question 1.** Let $D \subset Q$ be division algebras.

1. Is it true that $GKtdg D \leq GKtdg Q$?

2. Assume $[Q : D] < \infty$, that is, $Q$ has finite dimension over $D$ (as right space). Is it true that $GKtdg D = GKtdg Q$?

Zhang [42] introduced a new concept of transcendence degree, the lower transcendence degree (henceforth denoted $LD$), for which the questions above have positive answers. This invariant is connected with noncommutative geometry ([42, Section 9]), many interesting conjectures in ring theory ([42, Section 9], [4]) and has many desirable properties. In particular, there are no known examples of division algebras for which the lower transcendence degree differs from $GKtdg$ — which is not the case of all other invariants introduced in the literature.

Following [42] we say that algebra $A$ is $LD$-stable if $LD A = GK A$. The $LD$-stability facilitates the computation of $GKtdg$ for $Q(A)$, where $A$ is a prime Goldie algebra. Namely, for an $LD$-stable prime Goldie algebra we have

$$LD A = GKtdg A = GKtdg Q(A) = LD Q(A) = GK A = GKtdg A_S,$$
for any denominator set $S$ of regular elements in $A$.

In this paper we explore applications of the $LD$ transcendence degree and establish the following result.

**Main Theorem 1.** *The following algebras are $LD$-stable:*

1. The rings of differential operators $D(L)$ and $D(A)$, where $A$ is an affine domain and $L = Q(A)$;
2. The quantum group $U_q(g)^+$ for a finite dimensional semisimple Lie algebra $g$ over an algebraically closed field and for $0 \neq q \in k$ which is not root of unity;
3. $U_q(sl(N))$;
4. All generalized Weyl algebras (under a mild condition) $D(a, \sigma)$ with a Noetherian commutative domain $D$;
5. Finite $W$-algebras of type $A$;
6. All symplectic reflection algebras and its spherical subalgebras.
7. All enveloping algebras of finite dimensional Lie superalgebras.

Moreover we compute the lower transcendence degree of all above algebras. Our second main result compares the lower transcendence degree of related algebras.

**Main Theorem 2.** 1. *Let $A$ be an algebra and $G$ a finite group of auto-morphisms of $A$. If $A$ is a prime Goldie ring such that $A^G$ is also a prime Goldie ring (with some mild assumptions), then $LD A = LD A^G$. In particular, if $A$ is $LD$-stable then so is $A^G$.  

2. Let $R$ and $S$ be prime Goldie rings with a prime context between them. Then $LD R = LD S$. In particular, if $R$ is Morita equivalent to $S$ and $R$ is $LD$-stable, then $S$ is also $LD$-stable.*

Here is an outline of the paper.  

In Section 2 we recall the definition of lower transcendence degree and the main facts about this invariant that are going to be used through this paper.

In Section 3 we discuss the lower transcendence degree for rings of differential operators and prove Main Theorem 1 (1) in Proposition 1 and Theorem 2. In Corollary 1 we compute explicitly the Gelfand-Kirillov transcendence degree of the quotient division ring of the ring of differential operators on any affine commutative domain, generalizing the classical result of Gelfand and Kirillov for the Weyl fields. We also give a simple proof of a result on the embedding of quotient division rings of rings of differential operators (Theorem 3).
In Section 4 we discuss how the lower transcendence degree behaves with respect to the invariants under the action of finite groups and prove Main Theorem 2 (1) in Proposition 2, Corollary 2 and Theorem 4.

In Section 5 we discuss Galois algebras and generalized Weyl algebras. We make an assumption (†) on the embedding into a skew group ring, which is the case in all essentially all known examples. We give a simple proof of the lower bound for the Gelfand-Kirillov dimension of Galois algebras (Proposition 5) and prove Main Theorem 1 (5) in Proposition 6. Then we discuss generalized Weyl algebras and prove Main Theorem 1 (4) in Proposition 8. Some examples are given in Corollary 4. Finally, we discuss LD-stability of Ore extensions of the polynomial algebra.

Section 6 is devoted to the case of quantum groups and Lie superalgebras. We prove Main Theorem 1 (2, 3, 7) in Theorems 5, 7 and Proposition 10.

In Section 7 we consider the prime context between two rings, of which contexts for Morita equivalence is an example (Proposition 12), and prove Main Theorem 1 (2) in Theorem 9.

In Section 8, this result is used in the study of symplectic reflection algebras. First we show that the spherical subalgebra is LD-stable (Theorem 12), and then we exhibit a prime context between the symplectic reflection algebra and its spherical subalgebra (Proposition 13), thus showing that the full symplectic reflection algebra is also LD-stable (Theorem 14). This proves Main Theorem 1 (6).

Conventions. All relevant algebra invariants, such as the Gelfand-Kirillov dimension and transcendence degree, and the lower transcendence degree, are over the fixed basis field $k$. The same remark applies to the tensor product of algebras.

2 Lower transcendence degree

We recall the notion of lower transcendence degree and its main properties. We refer to [42] for motivations for this quite subtle invariant.

Let $A$ be an algebra and $V$ its subframe, that is a finite dimensional subspace of $A$ containing the identity. If for any such $V$ there exists a finite dimensional non-zero subspace $W \subset A$ such that $\dim VW = \dim W$, then we define $\text{LDA}$ as 0. Otherwise, there exists a subframe $V$ such that for every finite dimensional non-zero $W$:

$$\dim VW \geq \dim W + 1.$$ 

For a real number $d > 0$ we say that $V$ satisfies $VDI(A)_d$, the volume difference inequality (see [42]), if there exists $c > 0$ such that for every finite dimensional non-zero $W$:

$$\dim VW \geq \dim W + c(\dim W)^{(d-1)/d}.$$ 

If instead, we have:

$$\dim VW \geq \dim W + c(\dim W)^{(d-1)/d}.$$
\[ \dim V W \geq \dim W + c \dim W, \]

then we say that \( V \) satisfies \( V DI(A)_\infty \).

**Definition 1.** The lower transcendence degree of an algebra \( A \) is either 0 or:

\[ \text{LD } A = \sup_V \sup \{ d \mid V DI(A)_d \text{ holds for } V \}, \]

where \( V \) runs through all subframes of \( A \).

We shall use the expression LD-degree for the value of the lower transcendence degree. The following theorem summarizes some of its most important properties.

**Theorem 1.** [Theorem 0.3, Proposition 0.4, Proposition 2.1, Proposition 3.1] We have:

1. If \( Q \subset D \) are division algebras such that \([D : Q]\) is finite (dimension as a right space), then \( \text{LD } Q = \text{LD } D \).

2. If \( A \) is any algebra and \( S \) any denominator set of regular elements, \( \text{LD } A = \text{LD } A_S = \text{LD } S A \). In particular, if \( A \) is a prime Goldie, then \( \text{LD } A = \text{LD } Q(A) \).

3. For any algebra \( A \), \( \text{LD } A \leq \text{GKtd} A \leq \text{GK } A \); equality holds in case \( A \) is PI and prime. In particular, for a commutative field it coincides with the usual transcendence degree.

4. Let \( A, B \) be prime Goldie algebras. If \( B \subset A \) then \( \text{LD } B \leq \text{LD } A \). If \( A \) is a finitely generated right \( B \)-module and \( B \) is Artinian, then \( \text{LD } A = \text{LD } B \).

### 3 LD stability for rings of differential operators

In this section we will discuss rings of differential operators and and embeddings of their quotient division rings.

Let us recall the notion of differential operators on commutative algebras.

**Definition 2.** Let \( A \) be a commutative \( k \)-algebra. Define inductively \( D(A)_0 = A \) and \( D(A)_n = \{ d \in \text{End}_k A \mid [d, a] \in D(A)_{n-1}, \forall a \in A \} \). This way we obtain a natural structure of filtered associative \( k \)-algebra. If \( A \) is affine and regular, \( D(A) \) coincides with the subring of \( \text{End}_k A \) generated by \( A \) and the module \( \text{Der}_k A \) of \( k \)-derivations. In the later case, it is well known that the ring is a simple Noetherian domain ([24, Chapter 15]).

The following simple lemma will be used frequently.

**Lemma 1.** If \( A \) and \( B \) are two prime Goldie rings, \( A \subset B \), \( \text{LD } A = \text{LD } B \) and \( B \) is LD-stable, then so is \( A \).
Proof. By Theorem 1(4), \( LD A \leq GK A \leq GK B = LD B = LD A \). \( \square \)

Let \( B \) be an affine commutative domain over \( k \) (not necessarily regular) with the field of fractions \( L \) and \( n = \text{trdeg } B \).

**Proposition 1.** \( D(L) \) is LD-stable with the LD-degree \( 2n \).

Proof. By [24, 15.2.5, 15.3.10], \( D(L) \) is an Ore domain with the \( GK \) dimension \( 2n \). It contains a copy of the Weyl Algebra \( A_n(k) \). Since the Weyl Algebra is LD-stable ([24 Theorem 0.5]), we have

\[
2n = GK A_n(k) = LD A_n(k) \leq LD D(L) \leq GK D(L) = 2n.
\]

\( \square \)

**Theorem 2.** If \( B \) is an affine commutative domain over \( k \) then \( D(B) \) is LD-stable and the LD-degree is \( 2n \).

Proof. We can realize \( D(B) \) as a subset of \( D(L) \) in the following way ([24 15.5.5(iii)]): \( D(B) = \{ d \in D(L) | d(B) \subset B \} \).

The ring \( D(L) \) is a non-commutative domain with finite Gelfand-Kirillov dimension. Since \( D(B) \) is a subring of \( D(L) \), the same holds for it. Hence, \( D(B) \) does not contain a subalgebra isomorphic to the free associative algebra in two variables. It follows by a result of Jategaonkar ([22, Prop. 4.13]), that \( D(B) \) is an Ore domain. By [27 Proposition 1.8], \( Q(D(B)) = Q(D(L)) \), hence \( LD D(B) = LD D(L) \). Since \( D(L) \) is LD-stable by Proposition 1 then \( D(B) \) is LD-stable by Lemma 1. \( \square \)

**Corollary 1.** \( GKtdg Q(D(B)) = 2n \).

This is a very broad generalization of the result of Gelfand and Kirillov ([18]) that \( GKtdg Q(A_n(k)) = 2n \).

Now we are going to discuss the question of embedding of quotient division rings of algebras of differential operators. Recall the following fact obtained, by different reasonings, by Joseph ([21]), Resco ([31]), and Bavula ([3]):

- Let \( X \) and \( Y \) be two affine irreducible smooth varieties over an algebraically closed field. Denote \( D(X) := D(O(X)) \) and similarly \( D(Y) \). If \( \dim X > \dim Y \), then there is no k-algebra embedding of \( Q(D(X)) \) into \( Q(D(Y)) \).
Applying Theorem 2 and Theorem 1 we immediately obtain the following generalization of this fact.

**Theorem 3.** Let $A$ and $B$ be affine integral domains with transcendence degrees $n$ and $m$, respectively. If $n > m$ then there is no $k$-algebra embedding of $Q(D(A))$ into $Q(D(B))$.

### 4 LD stability for rings of invariants

In this section we discuss the lower transcendence degree of the subalgebra of invariants under the action of a finite group. We start with the following observation.

**Proposition 2.** Let $A$ be an Ore domain and $G$ a finite group of automorphisms of $A$. Then $LD A^G = LD A$.

**Proof.** Set $Q := Q(A)$. Then $Q(A^G) = Q^G$ (II), and we have $LD A = LD Q$ and $LD A^G = LD Q^G$ by Theorem 1. Also, $[Q : Q^G] \leq |G|$ by the Noncommutative Artin’s Lemma ([25, Lemma 2.18]), so we get $LD(Q) = LD(Q^G)$ by Theorem 1 and the result follows.

Applying Proposition 2 and Lemma 1 we obtain

**Corollary 2.** Let $A$ be an Ore domain and $G$ a finite group of automorphisms of $A$. If $A$ is $LD$-stable then so is $A^G$.

A similar result holds in more general situations. We recall the following fact

**Proposition 3.** [25, Theorem 1.15, Corollary 5.9] If $R$ is a semisimple Artinian algebra and $G$ a finite group of automorphisms of $R$, then $R^G$ is also semisimple Artinian. Moreover, $R$ is a finitely generated $R^G$ left and right module.

We now present the main theorem of this section.

**Theorem 4.** Let $A$ be a prime Goldie ring, $G$ a finite group of automorphisms of $A$. Suppose that $G$ has the nondegenerate trace on $A$, or that the ring $A$ has no nilpotent elements. Suppose also that $B = A^G$ is a prime Goldie ring (see [25, Theorem 3.17]). Then $LD A = LD B$, and if $A$ is $LD$-stable then so is $B$. 

Proof. Set $Q = Q(A)$ and $P = Q(B)$. Then $Q^G = P$ by [25, Theorem 5.3]. Moreover, both $Q$ and $P$ are simple Artinian by Goldie’s Theorem, and hence they are prime Goldie rings. By Proposition 3 and Theorem 1(4), we have $LD P = LD Q$. Hence, $LD A = LD B$ by Theorem 1. Finally, the last claim follows from Lemma 1.

5 LD stability for Galois algebras and related algebras

In this section we apply the results of the previous two to discuss the lower transcendence degree for Galois algebras, introduced by V. Futorny and S. Ovsienko ([13], [14], [10]); and related algebras. The setup is as follows. Let $k$ be algebraically closed. Consider a pair $\Gamma \subset U$ of algebras, where $\Gamma$ is a commutative domain, finitely generated over $k$, and $U$ is an associative algebra finitely generated over $\Gamma$.

Let $K := Q(\Gamma)$, and $L$ a finite Galois extension of this field with the Galois group $G$. Let $\mathfrak{M} \subset Aut_k L$ be a monoid of automorphisms with the following separating property: if $m, m' \in \mathfrak{M}$ have the same restriction to $K$, then they coincide. Finally, we assume that $G$ acts on $\mathfrak{M}$ by conjugations.

Definition 3 ([13]). If there is an embedding of $U$ into the invariant skew monoid ring $K := (K * \mathfrak{M})^G$ such that $K U = U K = K$, then we call $U$ a Galois algebra over $\Gamma$.

We have

Proposition 4. [13, Proposition 4.2] Let $S = \Gamma - \{0\}$. Then $S$ is a left and right denominator set for $U$, and localization (on both sides) by $S$ gives us an isomorphism $U_S \cong K$.

In all known cases of Galois algebras, $\mathfrak{M}$ is an ordered semigroup, and in many cases it is $\mathbb{Z}^n, n \geq 1$ (see [10] and references therein). Note that $\mathbb{Z}^n$ is an ordered group by [30, 13.1.6].

Henceforth we make the following assumption:

(†) $\mathfrak{M}$ is isomorphic to $\mathbb{Z}^n$ for adequate $n$.

Under this condition we have:

Corollary 3. $K$ and $U$ are Ore domains.

Proof. Since $\mathbb{Z}^n$ is an ordered group, we have that $K * \mathbb{Z}^n$ is an Ore domain, and hence so is its invariant subring $K$ ([30]). By Theorem 1 the same holds for $U$. \qed
As our first application of the lower transcendence degree to Galois algebras, we recall the following result \[15, \text{Theorem 6.1}]:

- Under certain technical assumption, \( \text{GK}_U \geq \text{GK}_\Gamma + \text{Growth}(\mathfrak{M}) \).

Since we assume the condition (†), we can show this result in a straightforward way.

**Proposition 5.** Let \( U \) be a Galois Algebra in \( K \) with \( \mathfrak{M} \cong \mathbb{Z}^n \). Then

\[
\text{GK}_U \geq \text{GK}_\Gamma + n.
\]

**Proof.** To begin with, it is clear that \( \text{Growth}(\mathbb{Z}^n) \) is \( n \). Also, Corollary \[8\] and Theorem \[1\] imply that \( \text{LD}_U = \text{LD}_K \). It is also true that \( \text{LD}_K = \text{LD}_K \ast \mathfrak{M} \) by Proposition \[2\]. By \[22\], \( \text{GK}_\Gamma = \text{trdeg}_K \), which equals, as we saw, to \( \text{LD}_K \). Applying \[42\], Corollary 5.5, 2 we obtain

\[
\text{LD}_K \ast \mathfrak{M} \geq \text{trdeg}_K + n = \text{GK}_\Gamma + n.
\]

Hence, \( \text{GK}_U \geq \text{LD}_U \geq \text{GK}_\Gamma + n \).

In some cases we can establish the \( \text{LD} \)-stability and compute the \( \text{GK} \) dimension.

**Proposition 6.** If, in the notation of the proposition above, \( \text{GK}_U = \text{GK}_\Gamma + n \), then the Galois algebra is \( \text{LD} \)-stable. In particular, all finite \( W \)-algebras of type \( A \) are \( \text{LD} \)-stable.

**Proof.** The first claim is clear, and the second is \[12\], Theorem 3.3.

Next we consider the generalized Weyl algebras, introduced by V. Bavula \[2\].

**Definition 4.** Let \( D \) be a ring, \( \sigma = (\sigma_1, \ldots, \sigma_n) \) an \( n \)-tuple of commuting automorphisms of \( D \): \( \sigma_i \sigma_j = \sigma_j \sigma_i \), \( i, j = 1, \ldots, n \). Let \( a = (a_1, \ldots, a_n) \) be an \( n \)-tuple of non-zero divisors in the center of \( D \), such that \( \sigma_i(a_j) = a_j, j \neq i \). The generalized Weyl algebra \( D(a, \sigma) \) of rank \( n \) is generated over \( D \) by \( X_i^+, X_i^- \), \( i = 1, \ldots, n \) subject to the relations:

\[
X_i^+ d = \sigma_i(d)X_i^+; \quad X_i^- d = \sigma_i^{-1}(d)X_i^-, \quad d \in D, i = 1, \ldots, n,
\]

\[
X_i^- X_i^+ = a_i; \quad X_i^+ X_i^- = \sigma_i(a_i), \quad i = 1, \ldots, n,
\]

\[
[X_i^-, X_j^+] = [X_i^-, X_j^-] = [X_i^+, X_j^+] = 0, \quad i \neq j.
\]

We will only consider the case where \( D \) is a finitely generated commutative domain. In this case \( D(a, \sigma) \) is also a Noetherian domain. It is not true that all generalized Weyl algebras are Galois algebras, but under some very mild restriction on \( \sigma \), all of them are \( \{15\}, \text{Theorem 14} \).

We have:
Proposition 7. [13, Proposition 13, 15] \( D(a, \sigma) \) always embeds into \( D * \mathbb{Z}^n \), where the canonical generators of \( \mathbb{Z}^n \) act on \( D \) as \( \sigma_1, \ldots, \sigma_n \). Both rings have the same quotient ring.

Under very mild assumptions, we can show that the generalized Weyl algebras are \( LD \)-stable, and compute the lower transcendence degree. Namely:

Proposition 8. Let \( D(a, \sigma) \) be a generalized Weyl algebra of rank \( n \) such that for every finite dimensional vector space \( U \subset D \) there is another finite dimensional vector space \( V \supset U \) such that \( \sigma_i(V) = V, i = 1, \ldots, n \). Then \( D(a, \sigma) \) is \( LD \)-stable, and the value of the invariant is \( GK D + n \).

Proof. By Proposition [22, Corollary 5.5.2] and Theorem [1], we get \( LD D(a, \sigma) \geq GK D + n \). On the other hand, \( GK D(a, \sigma) = GK D + n \) by [26, Corollary 3.5]. So both invariants coincide.

Many interesting generalized Weyl algebras satisfy the condition of the Proposition 8 (cf. [26]). In particular, we have

Corollary 4. Every noetherian generalized down-up algebra, the \( q \)-Heisenberg algebra in three generators, \( O_q^2(so(k)) \) and every rank 1 quantum generalized Weyl algebras (with non-root of unity parameter) are \( LD \)-stable.

Proof. Follows from [26, Proposition 4.3, 4.5, Example 4.9, Proposition 4.11].

We finish this section with an analysis of \( LD \)-stability of Ore extensions of the polynomial algebra.

Proposition 9. Let \( A = \mathbb{C}[x][y; \alpha, \delta] \) (\( \alpha \) an automorphism, \( \delta \) an \( \alpha \)-derivation) be an iterated Ore extension such that the center of \( Q(A) \) is \( \mathbb{C} \). Then \( A \) is \( LD \)-stable if and only if the automorphism \( \alpha \) is locally algebraic, that is every \( t \in \mathbb{C}[x] \) is contained in a finite dimensional \( \alpha \)-stable subspace. Then \( LD A = 2 \).

Proof. It follows from [1] that under the hypothesis of the theorem, \( Q(A) \) is either the first Weyl field or \( Q(\mathbb{C}_q[x,y]) \), for \( q \neq 0, 1 \) and \( q \) not a root of unity. Hence, \( LD A = LD Q(A) = 2 \). Now \( GK A \) equals 2 if and only if \( \alpha \) is locally algebraic by [22, Theorem 12.3.3].

Remark 1. By the results of [1], if the center is bigger than the base field, then \( A \) is either the polynomial algebra in two indeterminates, or a quantum plane at a root of unity, or the first quantization of the Weyl algebra at a root of unity. In all these cases \( A \) is a PI-algebra, and hence already known to be \( LD \)-stable.
6 LD stability for quantum groups and enveloping algebras of Lie superalgebras

In this section we will discuss the LD-stability of certain quantum groups and enveloping algebras of Lie superalgebras. We will always assume that $q \in k$ is not root of unity. Let $\mathfrak{g}$ denote a finite dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. For $0 \neq q \in k$, we consider the quantized enveloping algebra $U_q(\mathfrak{g})$ given by the quantum Chevalley-Serre relations (we refer to [17], I.6.3, for standard definition).

Fix a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of the associated root system of $\mathfrak{g}$ (with $N$ positive roots). Let $w_0$ be the longest element of the Weyl group and $w_0 = s_{i_1} \ldots s_{i_N}$ a fixed reduced expression of $w_0$ with respect to the basis $\Delta$, where $s_i$ is the reflection associated to $\alpha_i$. Order all positive roots as follows

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \ldots, \beta_N = s_{i_1} \ldots s_{i_{N-1}} \alpha_N.$$ 

Define the elements $E_{\beta_i} = T_{i_1} \ldots T_{i_{j-1}} E_{i_j}$ (c.f. [17, I.6.7, I.6.8]). For $m = (m_1, \ldots, m_N) \in \mathbb{N}^N$ set $E_m^\beta = E_{\beta_1}^{m_1} \ldots E_{\beta_N}^{m_N}$. Then a basis of $U_q(\mathfrak{g})^+$ is given by $\{E_m^\beta, m \in \mathbb{N}^N\}$.

Recall the Levendorskii-Soibelman relations:

$$E_{\beta_i} E_{\beta_j} - q^{(\beta_i, \beta_j)} E_{\beta_j} E_{\beta_i} = \sum_{m \in \mathbb{N}^N} z_m E_m^\beta,$$

where $1 \leq i < j \leq N$, $z_m \in \mathbb{Q}(q^{\pm 1})$ and it is zero unless $m_r = 0$ for $r \leq i, r \geq j$ ([17] Proposition 1.6.10).

In order to proceed, we will need the following important notions.

Definition 5. A total ordering $\preceq$ on the monoid $\mathbb{N}^m$ is called linear admissible if:

- $x \preceq y$ implies $x + z \preceq y + z, \forall x, y, z \in \mathbb{N}^m$
- $(0, \ldots, 0)$ is the smallest element.

Definition 6. A multi-filtration of an algebra $A$ is a family $\mathcal{F} = \{F_x(A) | x \in \mathbb{N}^m\}$ of subspaces such that:

- $F_x(A) \subset F_y(A)$ if $x \preceq y$;
- $F_x(A)F_y(A) \subset F_{x+y}(A)$;
- $\bigcup_{x \in \mathbb{N}^m} F_x(A) = A$;
- $1 \in F_0(A),$

where $\preceq$ is a linear admissible total ordering on $\mathbb{N}^m$. 

11
Given the basis described above for $U_q(\mathfrak{g})^+$, we can order $\mathbb{N}^N$ lexicographically and get an admissible ordering $\preceq$. Also, define a multi-filtration $F_m = \{E^p_\beta | p \preceq m\}$, $m \in \mathbb{N}^N$. This multi-filtration is finite dimensional and the associated graded algebra is the quantum affine space with generators $E_\beta_i, i = 1, \ldots, N$ and relations

$$E_\beta_i E_\beta_j = q^{(\beta_i, \beta_j)} E_\beta_j E_\beta_i,$$

following the Levendorskii-Soibelman relations (cf. [17 I.6.11]).

The quantum affine spaces are LD-stable ([42, Corollary 6.3(1)]), and since the multi-filtration is finite dimensional, one can use [39, Theorem 2.8] to conclude that $GK U_q(\mathfrak{g})^+ = GK \text{gr} U_q(\mathfrak{g})^+$ is the same as the Gelfand-Kirillov dimension of the quantum affine space, which is known to be $N$ [17 II.9.6-9].

Applying [42, Theorem 4.3] we get

**Theorem 5.** $U_q(\mathfrak{g})^+$ is LD-stable, and $LD U_q(\mathfrak{g})^+$ equals the number of positive roots of $\mathfrak{g}$.

Now we consider $U_q(\mathfrak{sl}_N)$ and the extended quantum group $U_q^{\text{ext}}(\mathfrak{sl}_N)$, cf. [11 7.1]. For the rest of this section we assume that the base field is $\mathbb{C}$.

**Theorem 6.** ([11 Theorem 7.1]) $Q(U_q^{\text{ext}}(\mathfrak{sl}_N)) \cong Q(\mathbb{K}_q[x, y] \otimes N^{-1} \otimes \mathbb{K}_q^\otimes (N-1)(N-2)/2)$, where $\mathbb{K}_q[x, y]$ is the quantum plane, and $\mathbb{K} = \mathbb{C}(Z_1, \ldots, Z_{N-1})$, a purely transcendental extension.

This is the quantum analogue of the Gelfand-Kirillov Conjecture for $\mathfrak{sl}_N$ (cf. [17 I.2.11, II.10.4]). Since the quantum plane has $GK$ dimension 2 over its base field and since it is LD stable, we can immediately compute $LD U_q^{\text{ext}}(\mathfrak{sl}_N)$ as (*)

$$GK \mathbb{K}_q[x, y] \otimes N^{-1} \otimes \mathbb{K}_q^\otimes (N-1)(N-2)/2 = GK \mathbb{C} \mathbb{K} + 2N - 2 + N^2 - 3N + 2 = N^2 - 1,$$

using [22 Proposition 3.11, 3.12].

On the other hand, by [29], $GK U_q(\mathfrak{sl}_N) = N^2 - 1$. Since $U_q(\mathfrak{sl}_N)$ embeds into the extended quantum group ([11 7.1]), $LD U_q(\mathfrak{sl}_N) \leq N^2 - 1$ by (*). Hence, by Theorem 1 we get

**Theorem 7.** $U_q(\mathfrak{sl}_N)$ is LD-stable and $LD U_q(\mathfrak{sl}_N)$ equals $N^2 - 1$.

We now briefly discuss the LD-stability of envelopping algebras of finite dimensional Lie superalgebras.

**Proposition 10.** Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra, $\mathfrak{g}_0$ the even part, $\mathfrak{g}_1$ the odd. Then $U(\mathfrak{g})$ is LD-stable, and the value of this invariant is $\dim \mathfrak{g}_0$. 

12
Proof. $U(g)$ is a finitely generated and free left and right module over $U(g_0)$. By [12], Proposition 3.1, $LDU(g) = GKU(g_0) = dimg_0$. \qed

7 Prime Contexts and lower transcendence degree

In this section we will discuss how the lower transcendence degree behaves with respect to certain contexts between prime Goldie rings. In particular, we show that $LD$-degree is a Morita invariant of those rings. We begin by recalling the following well known result.

Proposition 11. Let $S M_R$ be an $S - R$ bimodule, $S = \text{End}_R(M_R)$, and $M$ be finitely generated projective on the right. Let $\text{RMP}_S^\ast = \text{hom}_R(M_R, R_R)$, which is finitely generated projective (left) $R$-module. Consider the $R - R$ bimodule map $M^* \otimes_S M \mapsto R$. If $m \in M$ is different from 0 then $M^* m \neq 0$.

Proof. By the dual basis lemma ([24, Lemma 5.2]), there is a finite collection \( \{m_i\}_{i=1}^n \subset M \), \( \{g_i\}_{i=1}^n \subset M^* \) such that, for any \( x \in M \), \( x = \sum m_i g_i(x) \). Hence, if \( m \neq 0 \) then \( g_i(m) \neq 0 \) for some \( i \). \qed

We refer to [24, 1.1.6] and [20, 3.12] for generalities on Morita contexts and recall the fundamental theorem of Morita theory.

Theorem 8. [20, 3.12, 3.14] Let $R$ and $S$ be two Morita equivalent rings. Then we have a Morita context

\[
\begin{bmatrix}
R & M^* \\
M & S
\end{bmatrix},
\]

where $M$ is a bimodule $SM_R$, $S = \text{End}(M_R)$ and $RMP_S^\ast$ is isomorphic as bimodule to both $\text{hom}(M_R, R_R)$ and $\text{hom}(SM_S, S)$. We also have an isomorphism of bimodules between $SM_R$ and $\text{hom}(M^*_{SM}, M^*)$, $\text{hom}(RMP^*, R)$; $SM, RMP^*$, $M_R$, $M^*_S$ are all finitely generated projective modules.

We now recall the notion of prime contexts introduced in [28] (cf. also [24, 3.6.])

Definition 7. [24, 3.6.5] Let $R$, $S$ be prime rings, $V$ a $R - S$ bimodule and $W$ a $S - R$ bimodule. Then the Morita context

\[
\begin{bmatrix}
R & V \\
W & S
\end{bmatrix},
\]

is called a prime context if for $v \neq 0 \in V, s \neq 0 \in S, w \neq 0 \in W, vW, Vw$ and $VSW$ are all non-zero.

Morita equivalences are a particular case of prime contexts, as the below result shows.
Proposition 12. If $R$ and $S$ are prime rings, then a Morita context, which is a Morita equivalence for them, is also a prime context.

Proof. Follows from Theorem 8 and Proposition 11.

The main result of this section is the following:

Theorem 9. Let $R$, $S$ be prime Goldie rings with a prime context between them. Then $LD R = LD S$. In particular, this is the case if $R$ and $S$ are Morita equivalent. Since $GK$ dimension is also Morita invariant ([24, 8.2.9(iii)]) then Morita equivalence rings are $LD$-stable simultaneously.

Proof. Given two prime Goldie rings $R$, $S$ and a prime context between them, $Q(R)$ is Morita equivalent to $Q(S)$ by [24, 3.6.9]. Call $Q = Q(R)$, $P = Q(S)$. By the Morita theory, $P = \text{End}(M_Q)$ for some finitely generated $Q$-module, and $Q$ is simple Artinian by Goldie’s Theorem. Hence, every module over $Q$ is completely reducible, $Q = \bigoplus_{i=1}^{n} L$ is a direct some of copies of some simple right $Q$-module $L$, which is the unique simple $Q$-module up to isomorphism. Set $D = \text{End}(L_Q)$. Then $D$ is a division ring and $Q \cong M_n(D)$. We have $LD Q = LD D$ by [12, Corollary 3.2(3)]. Moreover, $M_Q$ is a direct sum of a finite number of $L$’s, and so $P$ is a matrix ring over $D$. We conclude that $LD P = LD D = LD Q$. Hence, $LD R = LD S$ by Theorem 1.

8 Symplectic reflection algebras

In this section we are going to use the results of the previous section to study symplectic reflection algebras. The question of weather they are Morita Equivalent or not is a very subtle one ([5]). But we show that, nonetheless, they always belong to a prime context, and show that they are $LD$-stable.

Definition 8. [8] Let $V$ be a complex vector space of dimension $2n$, with a non-degenerate skew-symmetric form $\omega$. Let $\Gamma$ be a finite subgroup of $\text{SP}_{2n}(\mathbb{C})$ generated by symplectic reflections, that is, by the elements $g \in \Gamma$ such that $1 - g$ has rank two. Then $\Gamma$ is called a finite symplectic reflection group.

The data $(V, \omega, \Gamma)$ is called a symplectic triple. We also assume the triple to be indecomposable, that is we assume that $V$ can not be expressed as a direct sum of two non-trivial $\Gamma$-invariant subspaces. Let $W$ be a complex reflection group acting on a vector space $h$ and hence on its dual $h^\ast$. Then, for $V = h \oplus h^\ast$ define a bilinear form

$$\omega((y, f), (u, g)) = g(y) - f(u), y, u \in h, f, g \in h^\ast.$$ 

With the diagonal action of $W$ we get then an indecomposable triple, which subsumes the case of rational Cherednik algebras.
For each symplectic reflection $s \in \Gamma$ let $\omega_s$ be the form with radical $\ker 1 - s$ such that $\omega_s = \omega$ on $\text{im} 1 - s$. Choose $t \in \mathbb{C}$ a complex parameter, and calling $S$ the set of symplectic reflections in $\Gamma$, let $c : S \mapsto \mathbb{C}$ be a complex valued function in the set of symplectic reflections, invariant under conjugation in $\Gamma$.

**Definition 9.** (Symplectic reflection algebras) Consider the tensor algebra on $V, T(V)$, with the natural action of $\Gamma$ extended from that on $V$. The symplectic reflection algebra, henceforth denoted $H_{t,c}$, is the quotient of $T(V) \ast \Gamma$ by the relations

$$[x, y] = t\omega(x, y) + \sum_{s \in S} c_s \omega_s(x, y)s, x, y \in V.$$

**Remark 2.** We will assume $t \neq 0$ since otherwise the symplectic reflection algebra is PI (cf. [7]), and hence the question of LD-stability is already settled by the results in [42] and the following theorem. Nonetheless, everything that follows holds without this assumption.

We recall the following result, which shows that the symplectic reflection algebras are prime Goldie rings.

**Theorem 10.** [7, Theorem 4.4] Symplectic reflection algebras are prime Noetherian algebras.

We also have the following important notion. Let $e = 1/|\Gamma| \sum_{h \in \Gamma} h$ be an idempotent. Define the spherical subalgebra of the symplectic reflection algebra as $U_{t,c} := e H_{t,c} e$. Its unit is $e$. The same argument as in [7, Theorem 4.4] shows that it is an Ore domain.

For symplectic reflection algebras we have a finite dimensional filtration $F$ given by $F_{-1} = 0$, $F_0 = \mathbb{C} \Gamma$, $F_1 = \mathbb{C} \Gamma \oplus \mathbb{C} V, F_i = F_i^1, i \geq 2$. This filtration clearly induces a filtration (also denoted by $\mathcal{F}$) on $U_{t,c}$. We have

**Theorem 11.** [8, Theorem 1.3]

- $\text{gr}_F H_{t,c} \cong S(V) \ast \Gamma$;
- $\text{gr}_F U_{t,c} \cong S(V)^\Gamma$.

This shows, in particular, that the spherical has a finite dimensional filtration whose associated filtered algebra is a domain, and hence, using the filtered techniques from [42] we have:

**Theorem 12.** The algebra $U_{t,c}$ is LD-stable, and the value of the lower transcendence degree is $\dim V$.

**Proof.** We will use Theorem [11]. The algebra $S(V)^\Gamma$, being a commutative domain, is $LD$-stable. By [22, Theorem 4.5(a)] and [24, 8.2.9], $\text{GK} S(V)^\Gamma = \dim V$. Hence, $\text{GK} U_{t,c} = \dim V$ by [22, Proposition 6.6]. Now the statement follows from [42, Theorem 4.3(4)].
Let us now find a prime context between the symplectic reflection algebra and its spherical subalgebra. To simplify the notation we set \( H := H_{t,c} \) and \( U := U_{t,c} \). Recall the following result of Etingof and Ginzburg.

**Theorem 13.** \([8, \text{Theorem 1.5}]\) Let us consider the right \( U \)-module \( H_e \), and the left \( U \)-module \( eH \).

1. We have an isomorphism \( eH \cong \text{Hom}_U(H_e, U) \), where \( x \in eH \) goes to the map \( F_x(y) = xy \), for \( y \in H_e \).
2. In an analogue fashion, \( H_e \cong \text{Hom}_U(eH, U) \). In particular, the modules \( H_e, eH \) are reflexive.
3. The left action of \( H \) on \( H_e \) induces an isomorphism \( H \cong \text{End}_U(H_e) \).

We have

**Proposition 13.**

\[
\begin{bmatrix}
U & eH \\
He & H
\end{bmatrix},
\]

is a prime context.

**Proof.** Let \( 0 \neq x \in eH \). By Theorem 13 (1), it induces a non-zero homomorphism in \( \text{Hom}_U(H_e, U) \), given by left multiplication by \( x \). So, indeed, \( xH_e \neq 0 \). Similarly, if \( 0 \neq y \in H_e \), then \( eH_y \neq 0 \), by Theorem 13 (2). Finally, if \( 0 \neq s \in H \), then \( sHe \neq 0 \) by Theorem 13 (3), and hence \( eHsH_e \) is non-zero by the preceding reasoning. \( \square \)

Combining Theorems 9, 12 and Proposition 13 we immediately obtain

**Theorem 14.** \( H_{t,c} \) is LD-stable, and the value of the lower transcendence degree is \( \dim V \).

**Acknowledgments**

V. F. is supported in part by the CNPq (304467/2017-0) and by the Fapesp (2018/23690-6); J.S. is supported by the Fapesp (2018/18146-5); I. S. is supported by the CNPq (304313/2019-0) and by the Fapesp (2018/23690-6).
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