Subclass of \( p \)-valent Function with Negative Coefficients Applying Generalized Al-Oboudi Differential Operator

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Abstract

In this paper we introduce a new subclass \( \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) of \( p \)-valent functions with negative coefficient defined by Hadamard product associated with a generalized differential operator. Radii of close-to-convexity, starlikeness and convexity of the class \( \mathcal{R}^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) are obtained. Also, distortion theorem, growth theorem and coefficient inequalities are established.

1 Introduction and Definitions

Let \( \mathcal{G} \) be class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{w=2}^{\infty} l_w z^w
\]

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which are holomorphic in the open unit disk $\triangle = \{ z \in \mathbb{C} : |z| < 1 \}$.

For $f(z)$ belongs to $G$, Opoola [6] (see also [11, 12]) has introduced the following differential operator:

$$D_{\phi, \gamma}^{0, \zeta} f(z) = f(z)$$
$$D_{\phi, \gamma}^{1, \zeta} f(z) = (1 + (\phi - \gamma - 1)\zeta)f(z) - z(\gamma - \phi)\zeta + z\zeta f'(z) = D_{\phi, \zeta}^{0} f(z), \quad (1.2)$$
$$D_{\phi, \zeta}^{2, \gamma} f(z) = D_{\phi, \zeta}^{0} (D_{\phi, \zeta}^{1, \gamma} f(z)),$$
$$D_{\phi, \zeta}^{h, \gamma} f(z) = D_{\phi, \zeta}^{0} (D_{h}^{h+1, \gamma} f(z)), \quad (1.3)$$

if $f(z)$ is given by (1.1), then by (1.2) and (1.3), we see that

$$D_{\phi, \zeta}^{h, \gamma} f(z) = z + \sum_{w=2}^{\infty} (1 + (w + \phi - \gamma - 1)\zeta)h l_w z^w, \quad (1.4)$$

where $0 \leq \phi \leq \gamma, \zeta \geq 0$ and $h \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots \}$.

Let $T$ denote the subclass of $G$ consisting of the form

$$f(z) = z - \sum_{w=2}^{\infty} l_w z^w, \quad (1.5)$$

where $l_w \geq 0$ and $w \in \mathbb{N}$. This class has introduced and studied by Silverman [9].

The Hadamard product of two power series

$$f(z) = z - \sum_{w=2}^{\infty} l_w z^w, \quad g(z) = z - \sum_{w=2}^{\infty} j_w z^w$$

and it is defined in $T$ as follows:

$$(f \ast g) = f(z) \ast g(z) = f(z) = z - \sum_{w=2}^{\infty} l_w j_w z^w.$$ 

Let $G_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{w=1}^{\infty} l_{p+w} z^{p+w} \quad (1.6)$$

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that are holomorphic and \( p \)-valent in \( |z| < 1 \).

Also let \( T_p \) denote the subclass of \( G_p \) consisting of functions that can be expressed as

\[
f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}.
\]

(1.7)

The Hadamard product of two power series

\[
f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}, \quad g(z) = z^p - \sum_{w=1}^{\infty} j_{p+w} z^{p+w}
\]

and it is defined in \( T_p \) as follows:

\[
(f * g) = f(z) * g(z) = f(z) = z - \sum_{w=1}^{\infty} l_{p+w} j_{p+w} z^{p+w}.
\]

From the above differential operator, the convolution of two power series \( f(z) \) and \( g(z) \) is given by

\[
D^{h, \gamma}_{\phi, \xi, p}(f * g)(z) = p^h z^p - \sum_{w=1}^{\infty} (1 + (p + w + \phi - \gamma - 1) \xi)^h l_{p+w} j_{p+w} z^{p+w}, \quad (1.8)
\]

where \( p \in \mathbb{N} = \{1, 2, 3, \cdots\} \). Motivated by [2], [10], [7], we define a new subclass \( R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) of the class \( T_p \).

**Definition 1.1.** For \( 0 \leq \psi < 1 \), \( \varrho \geq 0 \) and \( 0 \leq \phi \leq \gamma \), \( 0 \leq \beta \leq \frac{1}{2} \), \( \zeta \geq 0 \), we let \( R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) be subclass of the class \( T_p \) consisting of functions of the form (1.7) and satisfying the analytic criterion

\[
\Re \left\{ \frac{z D^{h, \gamma}_{\phi, \xi, p}(f * g)'(z)}{D^{h, \gamma}_{\phi, \xi, p}(f * g)(z)} + \beta \frac{z^2 D^{h, \gamma}_{\phi, \xi, p}(f * g)''(z)}{D^{h, \gamma}_{\phi, \xi, p}(f * g)(z)} - \psi \right\} \geq 0 \quad \frac{z D^{h, \gamma}_{\phi, \xi, p}(f * g)'(z)}{D^{h, \gamma}_{\phi, \xi, p}(f * g)(z)} + \beta \frac{z^2 D^{h, \gamma}_{\phi, \xi, p}(f * g)''(z)}{D^{h, \gamma}_{\phi, \xi, p}(f * g)(z)} - 1.
\]

(1.9)

The main purpose of this paper is to investigate some geometric properties of the class \( R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) such as the coefficient bounds, growth and radii
of starlikeness, distortion properties, convexity and close to convexity for the class \( R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \). [5], [9], [1], [4], [8], [3], study the univalent functions for different classes.

2 Coefficient Inequalities

In the following theorem we obtain necessary and sufficient condition for a function \( f(z) \) to be in the class \( R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \). We have the following lemma useful for this work.

**Lemma 2.1.** [8] Let \( \psi \geq 0 \) and \( \nu \) be any complex number. Then \( \Re(\nu) \geq \psi \) if and only if

\[
|\nu - (1 + \psi)| < |\nu + (1 - \psi)|.
\]

**Lemma 2.2.** [8] Let \( \varrho \geq 0, 0 \leq \psi \) and \( \theta \in \mathbb{R} \). Then

\[
\Re(\nu) > \varrho|\nu - 1| + \psi
\]

if and only if

\[
\Re\left(\nu(1 + \varrho e^{i\theta}) - \varrho e^{i\theta}\right) > \psi,
\]

where \( \nu \) is a complex number.

**Theorem 2.3.** Let \( f(z) \in T_p \) be given by (1.7). Then \( f(z) \in R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \) if and only if

\[
\sum_{w=1}^{\infty} \{(p + k)(1 + (p + w - 1)\beta)(1 + \psi)((\varrho + \psi))\left[1 + (p + w + \phi - \gamma - 1)\zeta\right]h_{p+w,j_{p+w}} - \right. \leq p^h[p - \psi]. \quad (2.1)
\]

The result is sharp for the function

\[
f(z) = z - p^h(p - \psi) + p^h(p - 1)\varrho + \beta p^{h+1}(p - 1)z^h(1 - \varrho) \frac{((p + k)(1 + (p + w - 1)\beta)(1 + \psi) - (\varrho + \psi))}{\left[1 + (p + w + \phi - \gamma - 1)\zeta\right]h_{p+w}}.
\]

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Proof. If \( f(z) \in \mathcal{R}^*(p, g, \psi, q, \beta, \phi, \gamma, \zeta) \) and \( |z| = 1 \), then by Definition 1.1,
\[
\Re \left\{ \frac{z D_{\phi, \zeta, p}^{h, \gamma}((f \ast g)'(z))}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} + \frac{z^2 D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)''(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} - \psi \right\} \geq 0 \frac{z D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)'(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} + \beta \frac{z^2 D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)''(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} - 1.
\]

Using Lemma 2.2 it is sufficient to show that
\[
\Re \left\{ \left( \frac{z D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)'(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} + \beta \frac{z^2 D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)''(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} \right) (1 + \omega e^{i\theta}) - \omega e^{i\theta} \right\} \geq \psi,
\]

\[
\Re \left[ \frac{[z D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)'(z)](1 + \omega e^{i\theta}) + [\beta z^2 D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)''(z)](1 + \omega e^{i\theta}) - \omega e^{i\theta} D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)}{D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)} \right] > \psi.
\]

(2.2)

For convenience, let
\[
\mathcal{A}(z) = [z D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)'(z)](1 + \omega e^{i\theta}) + [\beta z^2 D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)''(z)](1 + \omega e^{i\theta}) - \omega e^{i\theta} D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z)
\]

and
\[
\mathcal{B}(z) = D_{\phi, \zeta, p}^{h, \gamma}(f \ast g)(z).
\]

That is equation (2) is equivalent to
\[
\Re \left( \frac{\mathcal{A}(z)}{\mathcal{B}(z)} \right) \geq \psi,
\]
applying Lemma 2.1
\[
\left| \frac{\mathcal{A}(z)}{\mathcal{B}(z)} - (1 + \psi) \right| \leq \left| \frac{\mathcal{A}(z)}{\mathcal{B}(z)} + (1 - \psi) \right|
\]
\[
\Rightarrow \left| \frac{\mathcal{A}(z) - (1 + \psi)\mathcal{B}(z)}{\mathcal{B}(z)} \right| < \left| \frac{\mathcal{A}(z) + (1 - \psi)\mathcal{B}(z)}{\mathcal{B}(z)} \right|
\]
\[ |A(z) + (1 - \psi)B(z)| - |A(z) - (1 + \psi)B(z)|. \]

Now,

\[
|A(z) + (1 - \psi)B(z)| = \left| p^{h+1}z^p - \sum_{w=1}^{\infty} [1 + (p + w + \phi - \gamma - 1)\zeta]^h (p + w) + p^{h+1}(p - 1)z^p \right|
\]

\[
(p + w)l_{p+w}j_{p+w}z^{p+w} \right| (1 + ge^{i\theta}) + \left| p^{h+1}(p - 1)z^p - \sum_{w=1}^{\infty} [1 + (p + w + \phi - \gamma - 1)\zeta]^h (p + w) - (1 + \psi) \sum_{w=1}^{\infty} [1 + (p + w + \phi - \gamma - 1)\zeta]^h l_{p+w}j_{p+w}z^{p+w} \right|
\]

\[
= \left| p^{h+1}z^p + p^{h+1}z^p ge^{i\theta} - \sum_{w=1}^{\infty} [1 + (p + w + \phi - \gamma - 1)\zeta]^h (p + w) + p^{h+1}(p - 1)z^p \beta(1 + ge^{i\theta}) - \sum_{w=1}^{\infty} [1 + (p + w + \phi - \gamma - 1)\zeta]^h ge^{i\theta} l_{p+w}j_{p+w}z^{p+w} + p^{h+1}(p - 1)z^p - \psi p^{h+1}z^p \right|
\]

\[
= \left| p^{h}z^p[p - (p + w + \phi - \gamma - 1)\zeta]^h l_{p+w}j_{p+w}z^{p+w} \right|
\]
Now with $|z| = 1$

\[
\geq p^h[p-\psi+1]+p^h[q[p-1]+p^{h+1}(p-1)]1+(\psi)(1+q)-\sum_{w=1}^{\infty}(p+w)[1+(p+w-1)](1+q)
- q + (1-\psi)\left[1+(p+w+\phi-\gamma-1)\zeta\right]l_{p+w,j_{p+w}} \right]. \tag{2.3}
\]

Also,

\[
|A(z) - (1 + \psi)B(z)| = \left|p^{h+1}z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \right|
\]

\[
(p+w)l_{p+w,j_{p+w}}z^{p+w+w} + \left[ p^{h+1}(p-1)z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \right]
\]

\[
(p+w)(p+w-1)l_{p+w,j_{p+w}}z^{p+w+w} + \left[ p^{h+1}(p-1)z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \right]
\]

\[
l_{p+w,j_{p+w}}z^{p+w+w} + \left[ p^{h+1}(p-1)z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \right]
\]

\[
\left[ 1 + (p+w+\phi-\gamma-1)\zeta \right]l_{p+w,j_{p+w}}z^{p+w+w} + \left[ p^{h+1}(p-1)z^p - \sum_{w=1}^{\infty}[1+(p+w+\phi-\gamma-1)\zeta]^h \right]
\]

\[
+ \left[ 1 + (p+w+\phi-\gamma-1)\zeta \right]l_{p+w,j_{p+w}}z^{p+w+w} \right]. \tag{2.3}
\]
Now with $|z| = 1$

$$\leq p^h [\psi + 1 - p] + p^h g[p - 1] + p^{h+1} (p - 1) \beta (1 + g) + \sum_{w=1}^{\infty} (p + w) [1 + \beta (p + w - 1)] (1 + g)$$

$$- q - (1 + \psi) \left[ 1 + (p + w + \phi - \gamma - 1) \zeta \right] l_{p+w} j_{p+w} \right]. \quad (2.4)$$

It is easy to show that

$$|A(z) + (1 - \psi) B(z)| - |A(z) - (1 + \psi) B(z)| = 2 p^h [p - \psi] - 2 \sum_{w=1}^{\infty} (p + w) [1 + \beta (p + w - 1)]$$

$$(1 + g) - (q + \psi) \left[ 1 + (p + w + \phi - \gamma - 1) \zeta \right] l_{p+w} j_{p+w} \geq 0$$

$$- 2 \sum_{w=1}^{\infty} (p + w) [1 + \beta (p + w - 1)] (1 + g) - (q + \psi) \left[ 1 + (p + w + \phi - \gamma - 1) \zeta \right] l_{p+w} j_{p+w}$$

$$\geq - 2 p^h [p - \psi]$$

$$\sum_{w=1}^{\infty} (p + w) [1 + \beta (p + w - 1)] (1 + g) - (q + \psi) \left[ 1 + (p + w + \phi - \gamma - 1) \zeta \right] l_{p+w} j_{p+w}$$

$$\leq p^h [p - \psi].$$

Conversely, suppose the inequality (2.5) holds, we need to show that

$$\mathbb{R} \left[ \frac{[z D_{\phi, \zeta, \psi}^h (f * g)' (z)] (1 + \psi e^{i \theta}) + [\beta z^2 D_{\phi, \zeta, \psi}^h (f * g)'' (z)] (1 + \psi e^{i \theta})}{- \psi e^{i \theta} D_{\phi, \zeta, \psi}^h (f * g) (z)} \right] > \psi,$$

$$\mathbb{R} \left[ \frac{[z D_{\phi, \zeta, \psi}^h (f * g)' (z)] (1 + \psi e^{i \theta}) + [\beta z^2 D_{\phi, \zeta, \psi}^h (f * g)'' (z)] (1 + \psi e^{i \theta})}{- \psi e^{i \theta} D_{\phi, \zeta, \psi}^h (f * g) (z)} \right] > \psi.$$
Since $|e^{i\theta}| = 1$, hence $\Re(e^{i\theta}) \leq |e^{i\theta}| = 1$, letting $|z| \to 1$ yields, we let $H = [1 + (p + w + \phi - \gamma - 1)\zeta]^h$

$$p^h(p - \psi) + p^h[p - 1]g + \beta p^{h+1}(p - 1)(1 - g) - \sum_{w=1}^{\infty} Hl_{p+w} \frac{(p + w)(1 + \beta(p + w - 1))(1 + g) - (g + \psi)}{p^h - \sum_{w=1}^{\infty} Hl_{p+w}} > 0,$$

then we have

$$\sum_{w=1}^{\infty} Hl_{p+w} \frac{(p + w)(1 + \beta(p + w - 1))(1 + g) - (g + \psi)}{p^h - \sum_{w=1}^{\infty} Hl_{p+w}} \leq p^h(p - \psi) + p^h[p - 1]g + \beta p^{h+1}(p - 1)z^p(1 - g)$$

which completes the proof.

**Corollary 2.4.** Let $f(z) \in R^*(p, g, \psi, \theta, \beta, \phi, \gamma, \zeta)$. Then

$$l_{p+w} \leq \frac{p^h(p - \psi) + p^h[p - 1]g + \beta p^{h+1}(p - 1)z^p(1 - g)}{[1 + (p + w + \phi - \gamma - 1)\zeta]^h \left[ (p + w)(1 + \beta(p + w - 1))(1 + g) - (g + \psi) \right] j_{p+w}}.$$

Taking $\beta = 0$ in Theorem 2.3, we have the following corollary.

**Corollary 2.5.** Let $f(z) \in T_p$. Then $f(z) \in R^*(p, g, \psi, \theta, \beta, \phi, \gamma, \zeta)$ if and only if

$$\sum_{w=1}^{\infty} \{(p + k)(1 + \psi) - (g + \psi)\}[1 + (p + w + \phi - \gamma - 1)\zeta]^h \leq p^h[p - \psi].$$
3 Growth Theorem and Distortion Theorem

Theorem 3.1. If $f(z) \in \mathcal{R}^*(p, g, \psi, \phi, \gamma, \zeta)$ and $j_{p+w} \geq j_2$, then

$$r^p - \frac{p^h(p - \psi)}{[(p+1)[1+\beta p](1+\varrho) - (\varphi + \psi)][1+(p+\phi-\gamma)\zeta]^{h_{j_{p+1}}}} \leq |f(z)| \leq r^p + \frac{p^h(p - \psi)}{[(p+1)[1+\beta p](1+\varrho) - (\varphi + \psi)][1+(p+\phi-\gamma)\zeta]^{h_{j_{p+1}}}}$$

and

$$pr^{p-1} - \frac{(p+1)p^h(p - \psi)}{[(p+1)[1+\beta p](1+\varrho) - (\varphi + \psi)][1+(p+\phi-\gamma)\zeta]^{h_{j_{p+1}}}} \leq |f(z)| \leq pr^{p-1} + \frac{(p+1)p^h(p - \psi)}{[(p+1)[1+\beta p](1+\varrho) - (\varphi + \psi)][1+(p+\phi-\gamma)\zeta]^{h_{j_{p+1}}}}.$$ 

The result is sharp for, $(|z| = r < 1)$

$$f(z) = z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w},$$

Proof. Since

$$|f(z)| = |z^p - \sum_{w=1}^{\infty} l_{p+w} z^{p+w}| \leq |z|^p + \sum_{w=1}^{\infty} l_{p+w} |z|^{p+w} \leq r^p + r^{p+1} \sum_{w=1}^{\infty} l_{p+w} \quad (3.1)$$

$$[\varphi + \psi][1+(p+w-1) \varphi + \psi][1+(p+w+\phi-\gamma-1)\zeta]^{h_{j_{p+w}}}$$

Using Theorem 2.3, we have

$$[\varphi + \psi][1+(p+w-1) \varphi + \psi][1+(p+w+\phi-\gamma-1)\zeta]^{h_{j_{p+w}}} \leq \sum_{w=1}^{\infty} l_{p+w} (p+w) [1+(p+w-1) \varphi + \psi][1+(p+w+\phi-\gamma-1)\zeta]^{h_{j_{p+w}}} \leq p^h(p - \psi)$$

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that is
\[ \sum_{w=1}^{\infty} l_{p+w} \leq \frac{p^h(p - \psi)}{(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1}} \]

using the above equation in 3.2 we have
\[ |f(z)| \leq r^p + \frac{p^h(p - \psi)}{(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1}} r^{p+1} \]

and
\[ |f(z)| \geq r^p - \frac{p^h(p - \psi)}{(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1}} r^{p+1} \]

The result is sharp for
\[ |f(z)| = z^p - \frac{p^h(p - \psi)}{(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1}} r^{p+1} \]

Similarly, since
\[ f'(z) = pz^{p-1} - \sum_{w=1}^{\infty} (p + w)l_{p+w} z^{p+w-1} \]

we have that
\[ |f'(z)| = |pz^{p-1} - \sum_{w=1}^{\infty} (p + w)l_{p+w} z^{p+w-1}| \leq p|z|^{p-1} + \sum_{w=1}^{\infty} (p + w)l_{p+w} |z|^{p+w-1} \]
\[ \leq pr^{p-1} + (p + 1) \sum_{w=1}^{\infty} l_{p+w} \quad (3.2) \]

\[ [(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1} \]
\[ \leq [(p + w)[1 + \beta(p + w - 1)](1 + \varrho) - (\varrho + \psi)][1 + (p + w + \phi - \gamma - 1)\zeta^h j_{p+w} \]

Using Theorem 2.3 we have
\[ [(p + 1)[1 + \beta p(1 + \varrho) - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta^h j_{p+1} \sum_{w=1}^{\infty} l_{p+w} \]
\[ \leq \sum_{w=1}^{\infty} l_{p+w} j_{p+w}(p+w)[1+\beta(p+w-1)](1+\varrho)-(\varrho+\psi)][1+(p+w+\phi-\gamma-1)\zeta^h j_{p+w} \]
\[ \leq p^h(p - \psi) \]

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that is
\[ \sum_{w=1}^{\infty} t_{p+w} \leq \frac{p^h(p - \psi)}{[(p + 1)[1 + \beta p][1 + \varrho] - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta]^{h_j_{p+1}}} \]
using the above equation in 3.2
\[ |f'(z)| \leq pp^{p-1} + \frac{(p + 1)p^h(p - \psi)}{[(p + 1)[1 + \beta p][1 + \varrho] - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta]^{h_j_{p+1}}} r^p \]
and
\[ |f'(z)| \geq pp^{p-1} - \frac{(p + 1)p^h(p - \psi)}{[(p + 1)[1 + \beta p][1 + \varrho] - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta]^{h_j_{p+1}}} r^p \]
\[ |f'(z)| \geq 1 - \frac{(p + 1)p^h(p - \psi)}{[(p + 1)[1 + \beta p][1 + \varrho] - (\varrho + \psi)][1 + (p + \phi - \gamma)\zeta]^{h_j_{p+1}}} r^p. \]
This completes the proof. \( \square \)

4 Radii of Univalent Starlikeness, Convexity and Close to Convexity

**Theorem 4.1.** If \( f(z) \in R^*(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta) \), then \( f(z) \) is starlike of order \( \sigma \) \( (0 \leq \sigma < 1) \) in the disc \( |z| < r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma) \), where

\[
\begin{align*}
r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma) &= \left\{ \begin{array}{l}
(1 - \sigma)[(p + w)[1 + \beta(p + w - 1)](1 + \varrho) - (\varrho + \psi)] \\
(1 - \sigma)[(p + w - \sigma)p^h(p - \psi) + p^h(p - 1)\varrho + \beta p^h(p - 1)\varrho^+(1 - \varrho)]
\end{array} \right\} \left( \frac{1}{p^{p+1}} \right),
\end{align*}
\]
(4.1)

where \( p \in \mathbb{N} = \{1, 2, 3, \ldots \} \).

**Proof.** It is sufficient to show that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sigma \quad (0 \leq \sigma < 1)
\]

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for $|z| < r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zp^{p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p+w-1}}{zp^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} - 1 \right|$$

$$= \left| \frac{zp^{p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p+w-1}}{zp^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} - 1 \right|$$

$$= \left| \frac{zp^{p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p+w-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}}{zp^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} \right|$$

$$= \frac{(p-1)z^{p-1} - \sum_{w=1}^{\infty} (p+w-1)l_{p+w}z^{p+w-1}}{zp^{p-1} - \sum_{w=1}^{\infty} l_{p+w}z^{p+w-1}} \leq 1 - \sigma$$  \hspace{1cm} (4.2)

Hence by Theorem 2.3, (4.2) will be true if

$$\frac{\sum_{w=1}^{\infty} (p+w-\sigma)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq 1.$$  

and hence

$$|z| \leq \left\{ \begin{array}{l}
(1-\sigma)[(p+w)[1+\beta(p+w-1)](1+\varrho) - (\varrho+\psi)] \\
[p^{h}[p-\psi] + p^{h}[p-1]g + \beta p^{h+1}(p-1)z^{p}(1-\varrho)]\end{array} \right\}^{1\over p^{w-1}}$$  \hspace{1cm} (4.3)

setting

$$|z| = r_1(p, g, \psi, \varrho, \beta, \phi, \gamma, \zeta, \sigma)$$

we get the desired result.
Theorem 4.2. If \( f(z) \in \mathcal{R}^*(p,g,\psi,\varrho,\beta,\phi,\gamma,\zeta) \), then \( f(z) \) is convex of order \( \sigma(0 \leq \sigma < 1) \) in the disc \( |z| < r_2(p,g,\psi,\varrho,\beta,\phi,\gamma,\zeta,\sigma) \), where

\[
r_2(p,g,\psi,\varrho,\beta,\phi,\gamma,\zeta,\sigma) =
\frac{1}{\rho_{p+w}}
\begin{pmatrix}
\inf_{p+w}
\left\{
\frac{(1-\sigma)(p+w)(1+\beta(p+w-1)|(1+\varrho)-(\varrho+\psi)|}{[1+(p+w+\phi-\gamma-1)\zeta]^h \rho_{p+w}}
\right\}
\end{pmatrix}
\]

(4.4)

where \( p \in \mathbb{N} = \{1, 2, 3, \cdots \} \).

Proof. It is sufficient to show that

\[
\left| \frac{zf''(z)}{f'(z)} + 1 \right| = \left| \frac{p(p-1)z^{p-1} - \sum_{w=1}^{\infty} (p+w)(p+w-1)l_{p+w}z^{p+w-1}}{p^{2p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p+w-1}} \right|
= \frac{p(p-1) - \sum_{w=1}^{\infty} (p+w)(p+w-1)l_{p+w}z^{p}}{p - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p}} \leq 1 - \sigma \tag{4.5}
\]

then

\[
\sum_{w=1}^{\infty} \frac{(p+w)(p+w-\sigma)l_{p+w}z^{p+w-1}}{1 - \sigma} \leq 1 - \sigma.
\]

Hence, by Theorem 2.3 (4.5) will be

\[
\frac{(p+w)(p+w-\sigma)l_{p+w}z^{p+w-1}}{1 - \sigma} \leq \frac{1}{\sum_{w=1}^{\infty} l_{p+w}}
\]

\[
\frac{(p+w)(p+w-\sigma)l_{p+w}z^{p+w-1}}{1 - \sigma} \leq \frac{[\rho(p-\psi) + p^h|p-1|\varrho + \beta p^{h+1}(p-1)z^p(1-\varrho)]}{l_{p+w}}
\]

and hence

\[
|z| \leq \left\{ \frac{(1-\sigma)(p+w)(1+\beta(p+w-1)|(1+\varrho)-(\varrho+\psi)|}{[1+(p+w+\phi-\gamma-1)\zeta]^h \rho_{p+w}} \right\}^{\frac{1}{p+w-1}}
\]

(4.6)
setting

\[ |z| = r_2(p, g, \psi, \rho, \phi, \gamma, \zeta, \sigma) \]

we get the desired result. \( \square \)

**Theorem 4.3.** If \( f(z) \in \mathcal{R}^*(p, g, \psi, \rho, \phi, \gamma, \zeta) \), then \( f(z) \) is close to convex of order \( \sigma \) \((0 \leq \sigma < 1)\) in the disc \( |z| < r_3(p, g, \psi, \rho, \phi, \gamma, \zeta, \sigma) \), where

\[
r_3(p, g, \psi, \rho, \phi, \gamma, \zeta, \sigma) = \inf_{p+w} \left\{ \frac{(1-\sigma)(p+w)[1+\beta(p+w-1)][1+\theta-(\varphi+\psi)]}{[1+(p+w+\phi-\gamma-1)\zeta^h j_{p+w}]} \left(\frac{1}{p+w-1}\right) \right\},
\]

(4.7)

where \( p \in \mathbb{N} = \{1, 2, 3, \ldots\} \).

**Proof.** It is sufficient to show that

\[ |f'(z) - 1| \leq 1 - \sigma \quad (0 \leq \sigma < 1) \]

for \( |z| \leq r_3(p, g, \psi, \rho, \phi, \gamma, \zeta, \sigma) \) we have

\[
|f'(z) - 1| = \left| pz^{p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}z^{p+w-1} - 1 \right| \leq p|z|^{p-1} - \sum_{w=1}^{\infty} (p+w)l_{p+w}|z|^{p+w-1} - 1 \leq 1 - \sigma. \tag{4.8}
\]

If

\[
\sum_{w=1}^{\infty} (p+w)l_{p+w}|z|^{p+w-1} \leq \frac{1}{1-\sigma},
\]

then by Theorem 2.3 (4.8) will be true if

\[
\frac{(p+w)l_{p+w}|z|^{p+w-1}}{1-\sigma} \leq \frac{[(p+w)[1+\beta(p+w-1)][1+\theta-(\varphi+\psi)][1+(p+w+\phi-\gamma-1)\zeta^h j_{p+w}]}{[p^h(p-\varphi)+p^h|p-1|\theta+\beta p^{h+1}(p-1)z^p(1-\theta)]},
\]

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and hence

\[
|z| \leq \left\{ \frac{[(p + w)[1 + \beta(p + w - 1)](1 + \varrho) - (\varrho + \psi)]}{[1 + (p + w + \phi - \gamma - 1)\zeta]\eta_{p+w}} \right\}^{\frac{1}{p+w-1}} \tag{4.9}
\]

setting

\[|z| = r_3(p, g, \psi, \varrho, \phi, \gamma, \zeta, \sigma)\]

we get the desired result. The proof is complete.

\textbf{Remark 4.4.} If we put \(p = 1\) in Theorems 2.3, 3.1 and 4.1 we obtain the corresponding result studied by Godwin and Opoola \[7\].

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