In this paper, we discuss how to construct the bilinear identities for the wave functions of the $(\gamma_n, \sigma_k)$-KP hierarchy and its Hirota’s bilinear forms. First, based on the corresponding squared eigenfunction symmetry of the KP hierarchy, we prove that the wave functions of the $(\gamma_n, \sigma_k)$-KP hierarchy are equal to the bilinear identities given in Sec.3 by introducing $N$ auxiliary parameters $z_i$, $i = 1, 2, \ldots, N$. Next, we derived the bilinear equations for the tau-function of the $(\gamma_n, \sigma_k)$-KP hierarchy. Then, we obtain the bilinear equations for the tau-function of the mixed type of KP equation with self-consistent sources (KPECS), which includes both the first and the second type of KPECS as special cases by setting $n = 2$ and $k = 3$. Finally, using the relation between the Hirota bilinear derivatives and the usual partial derivatives, we show the procedure of translating the Hirota’s bilinear equations into the mixed type of KPECS.

**Keywords:** $(\gamma_n, \sigma_k)$-KP hierarchy; bilinear identity; $\tau$-function; Hirota’s bilinear form.

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1. **Introduction**

Sato theory has important applications in the theory of integrable systems. It reveals the infinite dimensional Grassmannian structure of space of tau-functions, where the tau-function are solutions for the Hirota’s bilinear form of KP hierarchy. The KP hierarchy can be expressed in terms of pseudo-differential operator and has the bilinear identities [3, 4].
Soliton equations with self-consistent sources (SESCS) are important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves. The nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high frequency wave with the ion acoustic wave in a two component homogeneous plasma. The KP equation with self-consistent sources describes the interaction of a long wave with a short wave packet propagating on the x-y plane at some angle to each other.

As an infinite dimensional integrable system, it has been generalized to large sets of integrable hierarchies by introducing new flows [7,16]. In [8], Liu and his collaborators construct an extended KP hierarchy by introducing a new vector field \( \partial_\tau \). This new extended KP hierarchy can be reduced to the \( k \)-constrained KP hierarchy, the Gelfand-Dickey hierarchy with self-consistent sources, the first type of KP equation with self-consistent sources(KPESCS) and the second type of KPESCS.

In [18], Yao and her collaborators propose a new \( (\gamma_n, \sigma_k) \)-KP hierarchy with two new time series \( \gamma_n \) and \( \sigma_k \). This new \( (\gamma_n, \sigma_k) \)-KP hierarchy can be regarded as a generalization of the extended KP hierarchy, which consists of a \( \gamma_n \)-flow, a \( \sigma_k \)-flow as well as a mixed \( \gamma_n \)- and \( \sigma_k \)-evolution equations of the eigenfunctions [8]. The \( (\gamma_n, \sigma_k) \)-KP hierarchy contains the mixed type of KP equation with self-consistent sources (KPESCS), which can also be reduced to both the first type and the second type of KPESCS as special cases. Also, the constrained flows of the \( (\gamma_n, \sigma_k) \)-KP hierarchy can be regarded as a generalization of the Gelfand-Dickey hierarchy (GDH), which contains the first, the second as well as the mixed type of GDH with self-consistent sources.

The KP hierarchy can be expressed in bilinear form using Hirota’s bilinear operators [6]. In this formalism, solutions to the KP equation can be obtained without knowing its Lax pair. Researchers have paid much attention on the subject of bilinear identities because of its importance in Sato theory. By using the bilinear identities of soliton hierarchies [2–4, 9, 15], we can derive the Hirota bilinear forms for all the equations in the hierarchies. Recently, Lin and his collaborators give the bilinear identities for the wave functions of the KP hierarchy with a squared eigenfunction symmetry in [11]. Considering the squared eigenfunction symmetry as an auxiliary flow, they also give the bilinear identities for the extended KP hierarchy. They obtain the generating functions of the Hirota bilinear forms for the extended KP hierarchy by constructing the \( \tau \)-function for the extended KP hierarchy.

This paper is organized as follows. In Section 2, we briefly recall the KP hierarchy and the \( (\gamma_n, \sigma_k) \)-KP hierarchy. In Section 3, the bilinear identities of the \( (\gamma_n, \sigma_k) \)-KP hierarchy are constructed. In Section 4, the \( \tau \)-function of the \( (\gamma_n, \sigma_k) \)-KP hierarchy is introduced. The generation functions for the Hirota bilinear form of the \( (\gamma_n, \sigma_k) \)-KP hierarchy are obtained. In Section 5, we show the procedure of translating the Hirota bilinear forms into nonlinear partial differential equations. Conclusions are given in the last section.

2. The KP hierarchy and \( (\gamma_n, \sigma_k) \)-KP hierarchy

Let

\[ L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots \]

be a pseudo-differential operator whose coefficients are considered as generators of a differential algebra \( \mathcal{A} \) [4].
The well-known KP hierarchy
\[ L_t^n = [B_n, L_n], \quad n \in \mathbb{N} \]  
(2.1)
can be constructed from the compatibility condition of the following linear systems \[3, 4\]
\[ L_\psi = \lambda \psi, \quad (2.2a) \]
\[ \frac{\partial \psi}{\partial t_n} = B_n \psi, \quad B_n = (L^n)_+, \quad n \in \mathbb{N}, \]  
(2.2b)
where \( \{t_n\} \) are the time variables with \( t_1 = x \) and \( B_n \) stands for the differential part of \( L^n \). The compatibility of \( t_n \)-flow and \( t_m \)-flow of the KP hierarchy (2.1) leads to the following zero-curvature equations
\[ (B_m)_{t_n} - (B_n)_{t_m} = [B_n, B_m], \quad m, n \in \mathbb{N}. \]  
(2.3)

Supposing that \( W = 1 + \omega_1 t^{-1} + \omega_2 t^{-2} + \cdots \) is a dressing operator satisfying
\[ \frac{\partial}{\partial t_n} W = -(W \partial W^{-1})_+ W, \quad n \in \mathbb{N}, \]  
(2.4)
then the operator \( L \) defined by
\[ L = W \partial W^{-1} \]  
(2.5)
is a solution to the KP hierarchy (2.1).

Let the wave functions and the adjoint wave functions be
\[ \psi(t, \lambda) = We^{\eta(t, \lambda)}, \quad (2.6a) \]
\[ \psi^*(t, \lambda) = (W^*)^{-1} e^{-\eta(t, \lambda)}, \quad \eta(t, \lambda) = \sum_{i \geq 1} t_i \lambda^i, \]  
(2.6b)
where \( W^* \) is the formal adjoint of \( W \) defined by \( (\sum_i a_i \partial^i)^* := \sum_i (-\partial)^i a_i \), we find that the wave function (2.6a) satisfies the KP hierarchy (2.2) while the adjoint wave function satisfies
\[ L^* \psi^* = \lambda \psi^*, \quad (2.7a) \]
\[ \frac{\partial \psi^*}{\partial t_n} = -B_n^* \psi^*, \quad B_n^* = [(L^*)^n]_+, \quad n \in \mathbb{N}. \]  
(2.7b)

Similarly, we can get the following hierarchy
\[ L^*_t = [L^*, B_n^*], \quad n \in \mathbb{N} \]  
(2.8)
from the linear systems (2.7a). If the operator \( W^* \) is a solution of
\[ \frac{\partial}{\partial t_n} [(W^*)^{-1}] = [(W^*)^{-1} \partial W^*]_+ (W^*)^{-1}, \quad n \in \mathbb{N}, \]  
(2.9)
then the adjoint operator \( L^* = -(W^*)^{-1} \partial W^* \) is a solution to the hierarchy (2.8).

For any fixed \( k \in \mathbb{N} \), by defining a new variable \( \zeta_k \) whose vector field is given by
\[ \frac{\partial \zeta_k}{\partial t} = \frac{\partial}{\partial t} - \sum_{i=1}^N \sum_{s \geq 0} \zeta_i^{s-1} \partial_i, \]
Liu and his collaborators introduce a new extended KP hierarchy [8]
\[ L_{\alpha} = [B_n, L], \quad (n \in \mathbb{N}, n \neq k), \quad \] (2.10a)

\[ L_{\alpha} = \left[ B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \] (2.10b)

\[ q_{i, t_n} = B_n(q_i), \] (2.10c)

\[ r_{i, t_n} = -B_k^* (r_i), \] (2.10d)

\[ q_{i, \tau_k} = B_k(q_i), \] (2.10e)

\[ r_{i, \tau_k} = -B_k^* (r_i), \quad i = 1, \ldots, N. \] (2.10f)

The compatibility of \( t_n \)-flow and \( \tau_k \)-flow of (2.10) gives rise to the following zero-curvature equations

\[ B_{n, \tau_k} \left( B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right) + \left[ B_n, B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] = 0. \]

For any fixed \( n, k \in \mathbb{N} \), Yao and her collaborators propose the \((\gamma_n, \sigma_k)\)-KP hierarchy with two generalized time series \( \gamma_n \) and \( \alpha_k \) in [18]

\[ L_{t_n} = [B_s, L], \quad (n \in \mathbb{N}, s \neq n, s \neq k), \] (2.11a)

\[ L_{\gamma_n} = \left[ B_n + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \quad (n \neq k), \] (2.11b)

\[ L_{\sigma_k} = \left[ B_k + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \] (2.11c)

\[ q_{i, t_n} = B_s(q_i), \] (2.11d)

\[ r_{i, t_n} = -B_s^* (r_i), \] (2.11e)

\[ \alpha_n(q_i, \sigma_k - B_k(q_i)) - \beta_k(q_i, \gamma_n - B_n(q_i)) = 0, \] (2.11f)

\[ \alpha_n(r_{i, \sigma_k} + B_k^* (r_i)) - \beta_k(r_{i, \gamma_n} + B_n^* (r_i)) = 0, \quad i = 1, \ldots, N, \] (2.11g)

where \( \alpha_n \) and \( \beta_k \) are constants, \( q_i \) and \( r_i \) (\( i = 1, 2, \ldots, N \)) are generalized eigenfunctions and adjoint eigenfunctions. It’s easy to see that the KP hierarchy can be derived from (2.11) by setting \( \alpha_n = 0 \) and \( \beta_k = 0 \). The commutativity of (2.11b) and (2.11c) under (2.11f) and (2.11g) gives rise to the following zero-curvature equations

\[ B_{n, \sigma_k} - B_{k, \gamma_n} + [B_n, B_k] + \beta_k \left[ B_n \sum_{i=1}^{N} q_i \partial^{-1} r_i \right] + \alpha_n \left[ \sum_{i=1}^{N} q_i \partial^{-1} r_i, B_k \right] = 0. \] (2.12)

Supposing that the operator \( W \) in (2.5) satisfies the following evolution equations

\[ \partial_s W = -(W \partial^s W^{-1})_s W, \quad (s \neq n, s \neq k) \] (2.13a)

\[ W_{\gamma_n} = -(W \partial^n W^{-1})_n W + \alpha_n \sum_{i=1}^{N} q_i \partial^{-1} r_i W, \quad (n \neq k) \] (2.13b)

\[ W_{\alpha_k} = -(W \partial^k W^{-1})_k W + \beta_k \sum_{i=1}^{N} q_i \partial^{-1} r_i W, \] (2.13c)
we can prove that the operator $L$ defined by (2.5) satisfies (2.11a) (2.11b) and (2.11c) (see [18] for the proof).

When we take $n = 2$, $k = 3$ and set $\gamma = \sigma = t$, $u_1 = u$, the mixed KPESCS

$$4u_t - 3\partial^{-1} u_{yy} - 12u_{tx} - u_{xxx} - 3\alpha_2 \sum_{i=1}^{N} (q_i r_i)_y + 4\beta_3 \sum_{i=1}^{N} (q_i r_i)_x,$$

$$+ 3\alpha_2 \sum_{i=1}^{N} (q_i r_i, w_i) = 0,$$  \hspace{1cm} (2.14a)

$$\alpha_2 \left( q_{ij} - q_{i,x} - 3u q_{i,x} - \frac{3}{2} q_{ij} \partial^{-1} u_y - \frac{3}{2} q_{ijy} - \frac{3}{2} \alpha_2 q_i \sum_{j=1}^{N} q_j r_j \right),$$

$$- \beta_3 \left( q_{ij} - q_{i,x} - 2u q_i \right) = 0,$$  \hspace{1cm} (2.14b)

$$\alpha_2 \left( r_{ij} - r_{i,x} - 3u r_{i,x} + \frac{3}{2} r_{ij} \partial^{-1} u_y - \frac{3}{2} r_{ijy} + \frac{3}{2} \alpha_2 r_i \sum_{j=1}^{N} q_j r_j \right),$$

$$- \beta_3 \left( r_{ij} + r_{i,x} + 2u r_i \right) = 0, \quad i, j = 1, 2, \ldots, N$$  \hspace{1cm} (2.14c)

can be obtained from (2.12), (2.11f) and (2.11g).

In particular, if $\alpha_2 = \beta_3 = 0$ (resp., $\alpha_2 = 0$, $\beta_3 = 1$ or $\alpha_2 = 1$, $\beta_3 = 0$ or $\alpha_2 = 1$, $\beta_3 = 1$), the nonlinear equations (2.14) will be reduced to the KP equation [3, 4] (resp., the first type [12, 13, 17], or the second type [5, 8, 12], or the mixed type of KP equation with self-consistent sources [18]). The KP equation with self-consistent sources has important applications in physics [10, 13].

3. Bilinear Identities for the $(\gamma_n, \sigma_n)$-KP hierarchy

We introduce $\partial_{\gamma_i}$-flows $(i = 1, 2, \cdots, N)$ as

$$\partial_{\gamma_i} L = [q_i \partial^{-1} r_i, L], \quad i = 1, 2, \cdots, N,$$  \hspace{1cm} (3.1)

where $q_i$ and $r_i$ are the eigenfunctions and their adjoint ones, respectively to construct the bilinear identities for (2.11). According to the results given in [1], the relation between the operator $W$ and the auxiliary parameters $z_i$ $(i = 1, 2, \cdots, N)$ satisfies

$$W_{z_i} = q_i \partial^{-1} r_i W, \quad i = 1, 2, \cdots, N.$$  \hspace{1cm} (3.2)

Let $\xi (t, \lambda) = \sum_{i \neq n, k} t_i \lambda^i + \gamma_n \lambda^n + \sigma_k \lambda^k$, the action of pseudo-differential operator on $\xi (t, \lambda)$ is defined by

$$\partial^m \xi (t, \lambda) = \lambda^m,$$

$$\partial^m e^{\xi (t, \lambda)} = \lambda^m e^{\xi (t, \lambda)}$$

for any integer $m$.

Denoting $z = (z_1, z_2, \ldots, z_N)$, $t = (t_1, \ldots, t_{n-1}, \gamma_n, t_{n+1}, \ldots, t_{k-1}, \sigma_k, t_{k+1}, \cdots)$ or $t = (t_1, \ldots, t_{n-1}, \gamma_n, t_{n+1}, \ldots, t_{k-1}, \sigma_k, t_{k+1}, \cdots)$, the wave function and the adjoint wave function with auxiliary parameters $z_i$ $(i = 1, 2, \cdots, N)$ can be defined as

$$\omega (z, t, \lambda) = W e^{\xi (t, \lambda)},$$  \hspace{1cm} (3.3a)

$$\omega^* (z, t, \lambda) = (W^*)^{-1} e^{-\xi (t, \lambda)}.$$  \hspace{1cm} (3.3b)

Before giving the bilinear identities for (2.11), let’s recall a useful lemma [3]:

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Lemma 1. Let $P$ and $Q$ be two pseudo-differential operators. $Q^*$ is the formal adjoint of $Q$, then

$$\text{Res}_\lambda P \cdot Q^* = \text{Res}_\lambda P \left( e^{\xi(t,\lambda)} \right) \cdot Q \left( e^{-\xi(t,\lambda)} \right),$$

(3.4)

where $\text{Res}_\lambda \left( \sum a_i \partial^i \right) = a_{-1}$ and $\text{Res}_\lambda \left( \sum a_i \lambda^i \right) = a_{-1}$.

Now we have the following theorems:

Theorem 1. The $(\gamma_n, \sigma_k)$-KP hierarchy (2.11) is equivalent to the following bilinear identities with $N$ auxiliary variables $z_i$, $i = 1, 2, 3, \ldots, N$,

$$\text{Res}_\lambda \omega (T,t,\lambda) \omega^* (T',t',\lambda) = 0,$$

(3.5a)

$$\text{Res}_\lambda \omega_i (T,t,\lambda) \omega^* (T',t',\lambda) = q_i (T, t) r_i (T', t'),$$

(3.5b)

$$\text{Res}_\lambda \omega (T,t,\lambda) \left[ \partial^{-1} q_i (T,t') \omega^* (T',t',\lambda) \right] = -q_i (T, t),$$

(3.5c)

$$\text{Res}_\lambda \left[ \partial^{-1} r_i (T,t) \omega (T,t,\lambda) \right] \omega^* (T',t',\lambda) = r_i (T', t'), i = 1, 2, \ldots, N,$$

(3.5d)

where

$$t = (t_1, \ldots, t_{n-1}, \gamma_n, t_{n+1}, \ldots, t_{k-1}, \sigma_k, t_{k+1}, \ldots),$$

$$t = (t', t_{n-1}', \gamma_n', t_{n+1}', \ldots, t_{k-1}', \sigma_k', t_{k+1}', \ldots),$$

$$T = (z_1 - \alpha_n \gamma_n - \beta_k \sigma_k, z_2 - \alpha_n \gamma_n - \beta_k \sigma_k, \ldots, z_N - \alpha_n \gamma_n - \beta_k \sigma_k),$$

$$T' = (z_1 - \alpha_n' \gamma_n' - \beta_k' \sigma_k', z_2 - \alpha_n' \gamma_n' - \beta_k' \sigma_k', \ldots, z_N - \alpha_n' \gamma_n' - \beta_k' \sigma_k'),$$

and

$$f (T', t') = \sum g_1 \cdot g_2 \cdot f (T, t),$$

$$g_1 = (t_1 - t_{n-1})^{h_1} \cdots (t_{n-1} - t_n)^{l_{n-1}} (t_{n+1} - t_k)^{l_{n+1}} \cdots (t_{k-1} - t_{k+1})^{l_{k-1}} (t'_{k+1} - t_{k+1})^{l_{k+1}} \cdots,$$

$$g_2 = \frac{d_1^{l_1} \cdots d_{n-1}^{l_{n-1}} d_n^{l_n} \cdots d_{k-1}^{l_{k-1}} d_{k+1}^{l_{k+1}} \cdots}{(n_1-1)! \cdots (n_{n-1}! \cdots (n_k-1)! \cdots (n_{k+1}-1)! \cdots)},$$

The action of $\partial^{-1}$ on the (adjoint) wave function is taken as pseudo-differential operator acting on the exponential part of the function, e.g., $\partial^{-1} (r \omega) = \left( \partial^{-1} W \right) \left( e^{\xi(t,\lambda)} \right)$.

Proof. Let’s prove the following observations first

$$\left( \beta_k \frac{d}{d\lambda} - \alpha_n \frac{d}{d\sigma_k} \right)^m \partial^{m_1} \cdots \partial^{m_{n-1}} \partial^{m_{n+1}} \cdots \partial^{m_{k-1}} \partial^{m_{k+1}} \cdots \partial^{m_k} \left[ \omega^* (T, t, \lambda) \right],$$

(3.6a)

$$= P_{m_1m_1' \cdots m_{n-1}m_{n+1}' \cdots m_{k-1}m_{k+1}' \cdots m_k} \omega^* (T, t, \lambda),$$

$$\left( \beta_k \frac{d}{d\lambda} - \alpha_n \frac{d}{d\sigma_k} \right)^m \partial^{m_1} \cdots \partial^{m_{n-1}} \partial^{m_{n+1}} \cdots \partial^{m_{k-1}} \partial^{m_{k+1}} \cdots \partial^{m_k} \left[ r_i (T, t) \right],$$

(3.6b)

$$= P_{m_1m_1' \cdots m_{n-1}m_{n+1}' \cdots m_{k-1}m_{k+1}' \cdots m_k} r_i (T, t),$$

where $P_{m_1m_1' \cdots m_{n-1}m_{n+1}' \cdots m_{k-1}m_{k+1}' \cdots m_k}$ is a differential operator in $\partial$ since the actions of the partial derivatives $\partial_i$ (for $i \neq n, k$) and $\frac{d}{d\lambda}, \frac{d}{d\sigma_k}$ can all be written as the actions of differential operators.
Indeed, applying $\partial_{t_i} \frac{d}{d\gamma_n} \frac{d}{d\sigma_k}$ to $\omega^* (T, t, \lambda)$, the following expression can be constructed
\[
\partial_{t_i} [\omega^* (T, t, \lambda)] = -B^*_i [\omega^* (T, t, \lambda)], \tag{3.7a}
\]
\[
\frac{d}{d\gamma_n} [\omega^* (T, t, \lambda)] = \omega^*_{\gamma_n} (z, t, \lambda) |_{z=T} - \alpha_n \sum_{j=1}^{N} \omega^*_{\gamma_j} (T, t, \lambda), \tag{3.7b}
\]
\[
\frac{d}{d\sigma_k} [\omega^* (T, t, \lambda)] = \omega^*_{\sigma_k} (z, t, \lambda) |_{z=T} - \beta_k \sum_{j=1}^{N} \omega^*_{\sigma_j} (T, t, \lambda), \tag{3.7c}
\]
which can be reduced to
\[
\left( \beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k} \right) \omega^* (T, t, \lambda) = [\alpha_n B^*_i - \beta_k B^*_i] \omega^* (T, t, \lambda). \tag{3.8}
\]

Similarly, applying $\partial_{t_i} \frac{d}{d\gamma_n} \frac{d}{d\sigma_k}$ to $r_i (T, t)$ and taking (2.11e) (2.11g) into consideration, we have
\[
\partial_{t_i} [r_i (T, t)] = -B^*_i [r_i (T, t)], i = 1, \ldots, N, \tag{3.9a}
\]
\[
\alpha_n [r_i, \alpha_i (z, t)] |_{z=T} + B^*_i [r_i (T, t)] - \beta_k [r_i, \gamma_i (z, t)] |_{z=T} + B^*_i [r_i (T, t)] = 0, \tag{3.9b}
\]
\[
\frac{d}{d\gamma_n} [r_i (T, t)] = r_i, \gamma_i (z, t) |_{z=T} - \alpha_n \sum_{j=1}^{N} r_i, \gamma_j (T, t), \tag{3.9c}
\]
\[
\frac{d}{d\sigma_k} [r_i (T, t)] = r_i, \sigma_k (z, t) |_{z=T} - \beta_k \sum_{j=1}^{N} r_i, \sigma_j (T, t). \tag{3.9d}
\]

Substituting (3.9c) and (3.9d) into (3.9b), we obtain
\[
\left( \beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k} \right) r_i (T, t) = (\alpha_n B^*_i - \beta_k B^*_i) r_i (T, t). \tag{3.10}
\]

So the observations (3.6) can be obtained with the help of (3.7a), (3.8) and (3.9a), (3.10).

Now we prove the bilinear identities (3.5) from (2.11)(2.13)(2.2): To prove the bilinear identity (3.5a), it is sufficient to consider the following case
\[
\text{Res}_t \omega (T, t, \lambda) \left( \beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k} \right)^{m_0} \partial_{t_1}^{m_1} \cdots \partial_{t_{n-1}}^{m_{n-1}} \partial_{\gamma_{n+1}}^{m_{n+1}} \cdots \partial_{\gamma_k}^{m_{k-1}} \partial_{\gamma_{k+1}}^{m_{k+1}} \cdots \partial_{\sigma_{m}}^{m} \omega^* (T, t, \lambda) = 0
\]
for every $m_j \geq 0$.

By using Lemma 1 and observation (3.6a), we have
\[
\text{Res}_t \omega (T, t, \lambda) \left( \beta_k \frac{d}{d\gamma_n} - \alpha_n \frac{d}{d\sigma_k} \right)^{m_0} \partial_{t_1}^{m_1} \cdots \partial_{t_{n-1}}^{m_{n-1}} \partial_{\gamma_{n+1}}^{m_{n+1}} \cdots \partial_{\gamma_k}^{m_{k-1}} \partial_{\gamma_{k+1}}^{m_{k+1}} \cdots \partial_{\sigma_{m}}^{m} \omega^* (T, t, \lambda)
\]
\[
= \text{Res}_x W e^{\tilde{\xi} (t, \lambda)} P_{m_0 m_1 \cdots m_{n-1} m_{n+1} \cdots m_{k-1} m_{k+1} \cdots m_{m}} \left( W^* \right)^{-1} e^{-\tilde{\xi} (t, \lambda)}
\]
\[
= \text{Res}_x W (W)^{-1} P_{m_0 m_1 \cdots m_{n-1} m_{n+1} \cdots m_{k-1} m_{k+1} \cdots m_{m}} = 0
\]
so the bilinear identity (3.5a) holds.
Notice that \( W_c = q_i \partial^{-1} r_i W, i = 1, 2, \ldots, N \), we get
\[
\text{Res}_\lambda \omega_c (T, t, \lambda) \left( \beta_k \frac{d}{d\theta_k} - \alpha_i \frac{d}{d\theta_i} \right)^m \partial_{t_1}^m \cdots \partial_{t_{n-1}}^{m_{n-1}} \partial_{t_{n+1}}^{m_{n+1}} \cdots \partial_{t_l}^{m_l} \omega^* (T, t, \lambda)
\]
\[
= \text{Res}_\lambda \omega_c (T, t, \lambda) P_m \omega (m_1, \ldots, m_l) T_{m_1} \cdots T_{m_l} \omega^* (T, t, \lambda)
\]
\[
= \text{Res}_\lambda q_i (T, t) \partial^{-1} r_i (T, t) W e^{\xi(t, \lambda)} P_m \omega (m_1, \ldots, m_l) \omega^* (T, t, \lambda)
\]
\[
= q_i (T, t) P_m \omega (m_1, \ldots, m_l) \omega^* (T, t, \lambda)
\]
so the bilinear identity (3.5b) is proved.

Similarly, we have the following bilinear identity
\[
\text{Res}_\lambda \omega (T, t, \lambda) \omega^* (T', t', \lambda) = q_i (T, t) r_i (T', t').
\]  
(3.11)

By substituting \( \omega_c^* = -r_i \partial^{-1} q_i \omega^* \) into (3.11) and \( \omega_c = q_i \partial^{-1} r_i \omega \) into (3.5b) respectively, the bilinear identities (3.5c) and (3.5d) can be proved.

**Theorem 2.** If \( q_i (T, t), r_i (T, t) \ (i = 1, 2, \ldots, N) \),
\[
\omega (T, t, \lambda) = \left( 1 + \sum_{i \geq 1} \omega_i \lambda^{-i} \right) e^{\xi(t, \lambda)},
\]
and
\[
\omega^* (T, t, \lambda) = \left( 1 + \sum_{i \geq 1} \omega_i^* \lambda^{-i} \right) e^{-\xi(t, \lambda)}
\]
satisfy the bilinear identities (3.5), then the pseudo-differential operators \( L = W \partial W^{-1} \) and functions \( q_i \) and \( r_i \) are solutions to the \( (\gamma_3, \sigma_2) \)-KP hierarchy (2.11).

**Proof.** For any \( m \geq 1 \), denoting \( \bar{W} = 1 + \sum_{i \geq 1} \omega_i^* \partial^{-i} \) and taking (3.5a) and Lemma 1 into account, we have
\[
\text{Res}_\lambda \partial \bar{W} \omega^* \partial^m = \text{Res}_\lambda \omega (T, t, \lambda) (-\partial)^m \bar{W} \omega^* (T, t, \lambda)
\]
\[
= \text{Res}_\lambda \omega (T, t, \lambda) (-\partial)^m \omega^* (T, t, \lambda)
\]
\[
= 0,
\]
which implies that the negative part of \( (\bar{W} \omega^*)_\_ \) is 0. Noticing that the non-negative part of \( (\bar{W} \omega^*)_+ \) is 1, we have \( \bar{W} = (W^*)^{-1} \).

For any \( m > 0 \), the following computation
\[
\text{Res}_\lambda \partial \bar{W} \omega \partial^{-1} (-\partial)^m = \text{Res}_\lambda \omega (T, t, \lambda) \partial \omega (T, t, \lambda)
\]
\[
= q_i (T, t) \partial \omega (T, t, \lambda)
\]
leads to
\[
(W_\gamma W^{-1})_+ = q_i (T, t) \partial^{-1} r_i (T, t).
\]
Since the non-negative part of \( (W_\gamma W^{-1})_+ \) is 0, then (3.2) holds.

From the definition of \( W_\gamma \), we know that \( (W_\gamma + L_\gamma W)_+ = 0 \).
For $m > 0$ and $s \neq n, s \neq k$, we have the following computation

$$
\text{Res}_q(W_0 W^{-1} + L^e_\gamma) \cdot \partial^m = \text{Res}_q(W_0 W^{-1} + (W \partial^s W^{-1}) \cdot \partial^m
$$

$$
= \text{Res}_q(W_0 e^{\xi(t, \lambda)} \cdot (-\partial)^m (W^* e^{-\xi(t, \lambda)} + \text{Res}_q(W \partial^s W^{-1} - (W \partial^s W^{-1}) \cdot \partial^m
$$

$$
= \text{Res}_q(W_0 e^{\xi(t, \lambda)} \cdot (-\partial)^m (W^* e^{-\xi(t, \lambda)} + \text{Res}_q(W \partial^s e^{\xi(t, \lambda)} \cdot (-\partial)^m W^* e^{-\xi(t, \lambda)}
$$

$$
= \text{Res}_q \omega_k(T, t, \lambda) (-\partial)^m \omega^*(T, t, \lambda) = 0,
$$

which means that (2.13a) holds.

Similarly, we can get

$$
\frac{d}{d\gamma_n} [W(T, t)] = -L^n_\gamma W,
$$

(3.12a)

$$
\frac{d}{d\sigma_k} [W(T, t)] = -L^k_\gamma W,
$$

(3.12b)

i.e.,

$$
W_{\gamma_n}(z, t)|_{z=T} - \alpha_n \sum_{i=1}^{N} q_i \partial_r W = -L^n_\gamma W,
$$

$$
W_{\sigma_k}(z, t)|_{z=T} - \beta_k \sum_{i=1}^{N} q_i \partial_r W = -L^k_\gamma W.
$$

So (2.13b) and (2.13c) are proved.

Applying both sides of (3.12) to $e^{\xi(t, \lambda)}$ respectively, we can get the following equalities

$$
\frac{d}{d\gamma_n} \omega(T, t, \lambda) = B_n \omega(T, t, \lambda),
$$

$$
\frac{d}{d\sigma_k} \omega(T, t, \lambda) = B_k \omega(T, t, \lambda).
$$

Then according to (3.5c), we find

$$
\frac{d}{d\gamma_n} [q_i(T, t)] = -\text{Res}_\lambda \frac{d}{d\gamma_n} \omega(T, t, \lambda) \cdot \partial^{-1} (q_i(T', t') \omega^*(T', t', \lambda))
$$

$$
= L^n_+ (-\text{Res}_\lambda \omega(T, t, \lambda) \cdot \partial^{-1} (q_i(T', t') w^*(T', t', \lambda)))
$$

$$
= L^n_+ (q_i(T, t)),
$$

which leads to

$$
\alpha_n \sum_{i=1}^{N} q_i z_j (T, t) + q_i, \gamma (z, t)|_{z=T} - L^n_\gamma q_i (T, t) = 0.
$$

(3.13)

Similarly, we have

$$
\beta_k \sum_{i=1}^{N} q_i z_j (T, t) + q_i, \sigma (z, t)|_{z=T} - L^k_\gamma q_i (T, t) = 0.
$$

(3.14)

So equation (2.11f) can be proved by combining (3.13) and (3.14).

Equation (2.11g) can also be proved in the similar way from (3.5d).
4. Tau-Function for \((\gamma_i, \sigma_k)\)-KP hierarchy

Since the (adjoint) wave functions of \((\gamma_i, \sigma_k)\)-KP hierarchy satisfy the same bilinear equation (3.5a) just as the (adjoint) wave functions of the original KP hierarchy do (if one considers \(z_i, i = 1, 2, \ldots, N\) as auxiliary parameters), it is reasonable to assume the existence of tau-function and make the following assumptions

\[
\omega(T, t, \lambda) = \frac{\tau(z, t - [\lambda])}{\tau(z, t)} e^{\xi(t, \lambda)}, \quad (4.1a)
\]

\[
\omega^*(T, t, \lambda) = \frac{\tau(z, t + [\lambda])}{\tau(z, t)} e^{-\xi(t, \lambda)}, \quad (4.1b)
\]

where the \(\sim\) over a function \(f(z, t)\) is defined as

\[
\sim f(z, t) \equiv f(z - \alpha_n \gamma_n - \beta_k \sigma_k, \ldots, z_N - \alpha_n \gamma_n - \beta_k \sigma_k, t)
\]

with \([\lambda] = \left(\frac{1}{\lambda_{1, 2}, \ldots}\right), \ z = (z_1, z_2, \ldots, z_N), \ t = (t_1, \ldots, t_{n-1}, \gamma_n t_{n+1}, \ldots, t_{k-1}, \sigma_k, t_{k+1}, \ldots).\) For example, according to the definition (4.2), we have

\[
\sim \tau(z, t) = \tau(T, t),
\]

\[
\sim \tau(z, t - [\lambda]) = \tau(R, t - [\lambda]),
\]

where

\[
R = (R_1, R_2, \ldots, R_N), \quad R_i = z_i - \alpha_n \left(\gamma_n - \frac{1}{n\lambda_i}\right) - \beta_k \left(\sigma_k - \frac{1}{k\lambda_i}\right), \quad i = 1, 2, \ldots, N.
\]

According to [2, 11], assume

\[
q_i(z, t) = \frac{k_i(z, t)}{\tau(z, t)}, \quad r_i(z, t) = \frac{\rho_i(z, t)}{\tau(z, t)}, \quad i = 1, 2, \ldots, N.
\]

we can get the following results

\[
\partial^{-1} (r_i(T, t) \omega(T, t, \lambda)) = \frac{\rho_i(z, t - [\lambda])}{\lambda \tau(T, t)} e^{\xi(t, \lambda)}, \quad i = 1, 2, \ldots, N, \quad (4.3a)
\]

\[
\partial^{-1} (q_i(T, t) \omega^*(T, t, \lambda)) = -\frac{k_i(z, t + [\lambda])}{\lambda \tau(T, t)} e^{-\xi(t, \lambda)}, \quad i = 1, 2, \ldots, N. \quad (4.3b)
\]

Substituting (4.1) (4.3) and (4.4) into (3.5), we have

\[
\text{Res}_\lambda \tau(z, t - [\lambda]) \tau(z, t') e^{\xi(t', \lambda)} = 0, \quad (4.5a)
\]

\[
\text{Res}_\lambda \tau(z, t - [\lambda]) \tau(z, t') e^{\xi(t', \lambda)} = 0, \quad (4.5b)
\]

\[
-\text{Res}_\lambda \tau(z, t - [\lambda])(\partial_{\lambda} \log \tau(z, t)) \tau(z, t') e^{\xi(t', \lambda)} = k_i(z, t) \rho_i(z, t'), \quad (4.5c)
\]

\[
\text{Res}_\lambda \lambda^{-1} \tau(z, t - [\lambda]) k_i(z, t') e^{\xi(t', \lambda)} = k_i(z, t) \tau(z, t'), \quad (4.5d)
\]

\[
\text{Res}_\lambda \lambda^{-1} \rho_i(z, t - [\lambda]) \tau(z, t') e^{\xi(t', \lambda)} = \rho_i(z, t') \tau(z, t). \quad (4.5e)
\]
Denoting \( y = (y_1, y_2, \cdots) \) and setting \( t \) and \( t' \) as \( t+y \) and \( t-y \) respectively, we can write the equalities (4.5) as the following systems with Hirota bilinear derivatives \( \tilde{D} \) and \( D_i \)'s:

\[
\begin{align*}
\sum_{i=0}^{\infty} p_i (2y) p_{i+1} \left(-\tilde{D}\right) \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \tau(T,t) \cdot \tau(T,t) &= 0, \quad (4.6a) \\
- \sum_{i=0}^{\infty} p_i (2y) \left[ \partial_x \log \tau(T,t+y) \right] p_{i+1} \left(-\tilde{D}\right) \tau(T,t+y) \cdot \tau(T,t-y) &= \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \tau_x(T,t) \cdot \tau(T,t), \quad (4.6b) \\
+ \sum_{i=0}^{\infty} p_i (2y) p_{i+1} \left(-\tilde{D}\right) \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \tau_{z_i} (T,t) \cdot \tau(T,t) &= \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \kappa (T,t) \cdot \rho_i (T,t), \quad (4.6c) \\
\sum_{i=0}^{\infty} p_i (2y) p_i \left(-\tilde{D}\right) \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \tau(T,t) \cdot \kappa (T,t) &= \exp \left( \sum_{j=1}^{\infty} y_j D_j \right) \kappa (T,t) \cdot \tau(T,t), \quad (4.6d)
\end{align*}
\]

where \( \tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \cdots) \), \( D_i \) is the Hirota bilinear derivative defined by \( D_i f \cdot g = f_i g - f g_i \) and \( p_i (y) \) is the \( i \)-th Schur polynomial given by \( \exp \sum_{i=1}^{\infty} y_i \lambda^i = \sum_{i=0}^{\infty} p_i (y) \lambda^i \).

Let \( y = 0 \), the equation (4.6b) can be converted into the following forms by using the Hirota bilinear operator

\[
\begin{align*}
\kappa (T,t) \rho_i (T,t) + D_x \tau_x (T,t) \cdot \tau(T,t) &= \kappa (T,t) \rho_i (T,t) + D_y \tau_y (T,t) \cdot \tau(T,t) = \kappa (T,t) \rho_i (T,t) + \frac{1}{2} D_y D_x \tau (T,t) \cdot \tau(T,t). \quad (4.7)
\end{align*}
\]

By setting \( n = 2 \), \( k = 3 \) and comparing the coefficient of \( y_3 \) in equation (4.6a) and the coefficient of \( y_2, y_3 \) in equation (4.6c) (4.6d), we can obtain

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\[
\kappa_i(z,t) \rho_i(z,t) + \frac{1}{2} D_1 D_2 \tau(z,t) \cdot \tau(z,t) = 0, \tag{4.8a}
\]

\[
\begin{bmatrix}
D_1 + 3 \left( D_2 - \alpha \sum_{i=1}^{N} D_{\alpha i} \right) \\
\end{bmatrix}^2 - 4D_1 \left( D_3 - \beta \sum_{i=1}^{N} D_{\beta i} \right) \tau(z,t) \cdot \tau(z,t) = 0, \tag{4.8b}
\]

\[
\begin{bmatrix}
4 \left( D_3 - \beta \sum_{i=1}^{N} D_{\beta i} \right) + 3D_1 \left( D_2 - \alpha \sum_{i=1}^{N} D_{\alpha i} \right) - D_1^2 \right) \tau(z,t) \cdot \kappa_i(z,t) = 0, \tag{4.8c}
\]

\[
\begin{bmatrix}
4 \left( D_3 - \beta \sum_{i=1}^{N} D_{\beta i} \right) + 3D_1 \left( D_2 - \alpha \sum_{i=1}^{N} D_{\alpha i} \right) - D_1^2 \right) \rho_i(z,t) \cdot \tau(z,t) = 0, \tag{4.8d}
\]

\[
\begin{bmatrix}
D_1^2 + \left( D_2 - \alpha \sum_{i=1}^{N} D_{\alpha i} \right) \right) \tau(z,t) \cdot \kappa_i(z,t) = 0, \tag{4.8e}
\]

\[
\begin{bmatrix}
D_1^2 + \left( D_2 - \alpha \sum_{i=1}^{N} D_{\alpha i} \right) \right) \rho_i(z,t) \cdot \tau(z,t) = 0, \quad i = 1, 2, \ldots, N. \tag{4.8f}
\]

The bilinear equations (4.8) correspond to the mixed type of KP equation with self-consistent sources [5], which can be reduced to the first type or the second type of KP equation with self-consistent sources by setting \( \alpha_2 = 0 \) or \( \beta_3 = 0 \) respectively. The KP equation with self-consistent sources describes the interaction of a long wave with a short-wave packet propagating on the \( x, y \) plane at an angle to each other [13].

5. The Procedure of Getting Nonlinear Equation from Hirota’s Bilinear Equation

At the beginning of this section, we recall two identities for arbitrary functions \( \tau(t) \) and \( \kappa(t) \), which is proved in [11]

\[
\exp \left( \sum_i \delta_i D_i \right) \kappa \cdot \tau = e^{2 \cosh \left( \sum_i \delta_i D_i \right) \log \tau} \cdot \sum_i \delta_i (\kappa \cdot \tau), \tag{5.1a}
\]

\[
\cosh \left( \sum_i \delta_i D_i \right) \tau \cdot \tau = e^{2 \cosh \left( \sum_i \delta_i D_i \right) \log \tau}. \tag{5.1b}
\]

By defining \( u = \frac{\partial^2}{\tau^2} \log \tau(x \equiv t_1) \), \( q = \frac{\kappa}{\tau} \), \( r = \frac{\partial}{\tau} \), we can get [11]

\[
\frac{1}{\tau^2} \sum_{n=0}^{\infty} \frac{\left( \sum_i \delta_i D_i \right)^n}{n!} \kappa \cdot \tau = \exp \left[ 2 \sum_{n=1}^{\infty} \frac{\left( \sum_i \delta_i D_i \right)^{2n}}{(2n)!} \partial^{-2} u \right] \cdot \sum_i \delta_i q, \tag{5.2a}
\]

\[
\frac{1}{\tau^2} \sum_{n=1}^{\infty} \frac{\left( \sum_i \delta_i D_i \right)^{2n}}{(2n)!} \tau \cdot \tau = \exp \left[ 2 \sum_{n=1}^{\infty} \frac{\left( \sum_i \delta_i D_i \right)^{2n}}{(2n)!} \partial^{-2} u \right], \tag{5.2b}
\]
and similarly, we have

$$\frac{1}{\tau^2} \sum_{n=0}^{\infty} \left( \frac{\sum_i \delta_i D_i}{n!} \right)^n \rho \cdot \tau = \exp \left[ \frac{2 \sum_{n=0}^{\infty} \left( \frac{\sum_i \delta_i D_i}{2n!} \right)^{2n}}{\partial^2 u} \right] \sum_i \delta_i \partial_i r. \quad \text{(5.2c)}$$

Comparing the coefficient of $\left( \delta_i \right)^j$, $j \geq 0$, we can get the relations between the Hirota bilinear derivatives and the usual partial derivatives. Here are some of them

$$\begin{align*}
\frac{D_t^1 \epsilon \tau}{\tau^2} &= 2u_{1,1} + 12u^2, \quad \frac{D_t^1 \epsilon \tau}{r^2} = 2u, \quad \frac{D_t^1 \epsilon \tau}{r^3} = 2\partial^{-1} u_3 \\
\frac{D_t^2 \epsilon \tau}{\tau^2} &= r_{1,1,1} + 6u r_1, \quad \frac{D_t^2 \epsilon \tau}{r^2} = r_{1,2} + 2r \partial^{-1} u_2 \\
\frac{D_t^2 \epsilon \tau}{r^3} &= r_2, \quad \frac{D_t^2 \epsilon \tau}{r^4} = r_3, \quad \frac{D_t^2 \epsilon \tau}{r^5} = r_{1,1} + 2u r \\
\frac{D_t^2 \epsilon \tau}{r^6} &= q_{1,1,1} + 6u q_1, \quad \frac{D_t^2 \epsilon \tau}{r^7} = q_{1,2} + 2q \partial^{-1} u_2 \\
\frac{D_t^2 \epsilon \tau}{r^8} &= q_2, \quad \frac{D_t^2 \epsilon \tau}{r^9} = q_3, \quad \frac{D_t^2 \epsilon \tau}{r^{10}} = q_{1,1} + 2u q
\end{align*}$$

(5.3)

where the subscripts $i, j, \cdots$ of $u_{1,1}, \cdots, r_{1,2}, \cdots, q_{1,1}, \cdots$ denote the derivatives with respect to the variables $t_i, t_j, \cdots$.

We also have the following expressions

$$\begin{align*}
\frac{D_t^1 \epsilon \tau}{\tau^2} &= 2\partial^{-1} u_{z_i}, \quad \frac{D_t^1 \epsilon \tau}{r^2} = 2\partial^{-1} u_{z_i}, \quad \frac{D_t^1 \epsilon \tau}{r^3} = 2\partial^{-1} u_{z_i} \\
\frac{D_t^2 \epsilon \tau}{\tau^2} &= r_{z_i} + 2r \partial^{-1} u_{z_i} \\
\frac{D_t^2 \epsilon \tau}{r^3} &= q_{z_i} + 2q \partial^{-1} u_{z_i}
\end{align*}$$

(5.4)

where the subscripts $z_i$ ($i = 1, 2, 3, \cdots N$) of $u$ denote the derivatives with respect to the variables $z_i$ ($i = 1, 2, 3, \cdots, N$).

By using (5.3) and (5.4), we can write (4.8) as the following nonlinear partial differential equations

$$\begin{align*}
\partial^{-1} \partial_x u &+ r_i q_i = 0, \\
\left( u_{xxx} + 12u u_x + 4\beta_3 \sum_{k=1}^{N} u_{x_k} - 4u_t \right)_x + 3 \left( \partial_y - \alpha_2 \sum_{k=1}^{N} \partial_{x_k} \right)^2 u &= 0, \\
-4q_{i,j} + 4\beta_3 q_{i,x_k} + 3q_{i,x_k} + 6q_{i} \left( \partial_y - \alpha_2 \sum_{k=1}^{N} \partial_{x_k} \right) \partial^{-1} u - 3\alpha_2 \sum_{k=1}^{N} q_{1,x_k} + q_{1,x_k} + 6u q_{i,x} &= 0, \\
4r_{i,j} - 4\beta_3 r_{i,x_k} + 3r_{i,x_k} + 6r_{i} \left( \partial_y - \alpha_2 \sum_{k=1}^{N} \partial_{x_k} \right) \partial^{-1} u - 3\alpha_2 \sum_{k=1}^{N} r_{1,x_k} - r_{1,x_k} - 6u r_{i,x} &= 0, \\
q_{i,x} + 2u q_i - q_{i,x} + \alpha_2 \sum_{k=1}^{N} q_{i,x_k} &= 0, \\
r_{i,x} + 2u r_i - r_{i,x} - \alpha_2 \sum_{k=1}^{N} r_{i,x_k} &= 0, \quad i = 1, 2, 3, \cdots, N.
\end{align*}$$

(5.5)

Equations (2.14) can be obtained by eliminating the auxiliary variables $z_i$ in (5.5). So we can see that the Hirota bilinear equations (4.8) correspond to the mixed type of KP equation with self-consistent sources (2.14).
6. Conclusions

The bilinear identities for the \((\gamma_n, \sigma_k)\)-KP hierarchy [18] are constructed in this paper, which could be seen as the generating functions of all the Hirota’s bilinear equations for the zero-curvature forms in the \((\gamma_n, \sigma_k)\)-KP Hierarchy. Many integrable 2+1 dimensional equations with self-consistent sources are included as special cases of this hierarchy. We have shown that the Hirota’s bilinear forms (4.8) correspond to the mixed type of KPESCS, which can be reduced to the first and the second type of KPESCS.

With the help of \(N\) auxiliary flows (\(\partial_{z_i}\)–flow), we obtain the bilinear identities of the whole \((\gamma_n, \sigma_k)\)-KP hierarchy, which have many important applications. For example, taking the intimate relation between quasi-periodic solutions and bilinear identity into account, we investigate the quasi-periodic solutions for the \((\gamma_n, \sigma_k)\)-KP Hierarchy. Under proper constraints, the \((\gamma_n, \sigma_k)\)-KP hierarchy can be reduced to Gelfand-Dickey hierarchy (GDH), KdV equation, Bonssinesq equation and many other equations with self-consistent sources. So the bilinear identities of the \((\gamma_n, \sigma_k)\)-KP hierarchy can help us learn the relation among these equations’ bilinear identities. We will investigate these problems in future.

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