SPONTANEOUS GENERATION OF MAGNETIC FIELD IN THREE DIMENSIONAL QED AT FINITE TEMPERATURE

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Abstract

We investigate the effects of thermal fluctuations on the spontaneous magnetic condensate in three dimensional QED coupled with P-odd Dirac fermions. Our results show that the phenomenon of the spontaneous generation of the constant background magnetic field survives to the thermal corrections even at infinite temperature. We also study the thermal corrections to the fermionic condensate in presence of the magnetic field.
Recently [1] it has been shown that, within the one-loop approximation, the unique theory displaying a nontrivial ground state turns out to be the three-dimensional $U(1)$ gauge theory in interaction with Dirac fermions with a negative P-odd mass term. Indeed, in that theory there is a spontaneous generation of a non-zero constant magnetic field. So that the model could be relevant towards the effective description of planar systems in condensed matter physics.

More interestingly, it is well known that Dirac fermions with a Yukawa coupling with a scalar field develop zero mode solutions near a domain wall [2]. It turns out that the zero modes behave like massless fermions in the two-dimensional space of the wall. In this case the formation of a uniform magnetic condensate and the dynamical generation of a constant P-odd fermion mass are energetically favorable. Thus the domain wall becomes ferromagnetic. Recently it has been argued that ferromagnetic domain walls could be relevant for the formation of primordial magnetic field [3]. In view of this, it is important to ascertain if the spontaneous generation of magnetic field survives to the thermal fluctuations.

The aim of the present paper is to study the thermal corrections in the three-dimensional $U(1)$ gauge theory in interaction with fermionic fields with negative P-odd mass term. Indeed, as we said before, this theory displays a nontrivial ground state already at the one-loop level.

Let us consider the Hamiltonian in the temporal gauge $A_0(x) = 0$:

$$H = \int d^2 x \left\{ \frac{1}{2} \vec{E}^2(x) + \frac{1}{2} \vec{B}^2(x) + \psi^\dagger(x)[-i\vec{\alpha} \cdot \vec{\nabla} + \beta m] \psi(x) + e \psi^\dagger(x)\vec{\alpha} \cdot \vec{A}(x) \psi(x) \right\},$$

(1)

where we follow the Bjorken and Drell notation and we use the two-dimensional realization of the Dirac algebra:

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2,$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g^{\mu\nu} = diag(1, -1, -1).$$

Previous studies [1, 4] showed that if $m$ is negative, then the perturbative ground state is unstable toward the spontaneous formation of a constant background magnetic field. The magnetic instability is present even at the
one-loop level. To see this, we use the Furry’s representation. If we split the electromagnetic field into the fluctuation $\eta(x)$ over the background $\bar{A}(x)$:

$$A_k(x) = \bar{A}_k(x) + \eta_k(x)$$  \tag{3}

with

$$\bar{A}_k(x) = \delta_{k2} x_1 B,$$  \tag{4}

then in the one-loop approximation the Hamiltonian reads ($V$ is the spatial volume):

$$H_0 = H_\eta + H_D + V \frac{B^2}{2} = \int d^2 x \left\{ \frac{1}{2} \nabla^2(x) + \frac{1}{2} [\varepsilon_{ij} \partial_i \eta_j(x)]^2 \right\} +$$

$$+ \int d^2 x \left\{ \psi^\dagger(x) [\alpha_k (-i \partial_k - e \bar{A}_k) + \beta m] \psi(x) \right\} + V \frac{B^2}{2}$$  \tag{5}

The fermionic Hamiltonian can be diagonalized by expanding the Dirac field into the eigenstates of Dirac equation:

$$[\alpha_k (-i \partial_k - e \bar{A}_k) + \beta m] \psi(x) = E \psi(x).$$  \tag{6}

In the Furry’s representation we expand the fermionic operator $\psi(x)$ in terms of the positive and negative solutions $\psi^{(+)}_{np}$ and $\psi^{(-)}_{np}$ of Eq. (6). In the case of negative mass term $m = -|m|$ the positive solutions have eigenvalues $E_n = +\sqrt{(2 neB + m^2)}$, $n \geq 1$, while the negative ones have eigenvalues $-E_n = -\sqrt{(2 neB + m^2)}$, $n \geq 0$. Therefore we write

$$\psi(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dp \, \psi^{(+)}_{np}(x) \, a_{np} + \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dp \, \psi^{(-)}_{np}(x) \, b_{np}^\dagger.$$  \tag{7}

Thus we get

$$H_D = \int_{-\infty}^{+\infty} dp \left[ \sum_{n=1}^{\infty} E_n a_{np}^\dagger a_{np} + \sum_{n=0}^{\infty} E_n b_{np}^\dagger b_{np} \right] - \frac{eB}{2\pi} V \sum_{n=0}^{\infty} E_n.$$  \tag{8}

Observing that $\frac{eB}{2\pi} V$ is the degeneracy of the Landau levels, we see that the last term in Eq. (8) is the well known Dirac sea energy.

\footnote{We assume here and throughout the paper that $eB > 0$. Moreover, the functions $\psi^{(\pm)}_{np}(x)$ are normalized as in ref. [1].}
Now it is easy to find the vacuum energy density. Obviously we are interested in the difference between the vacuum energy in presence of the magnetic field and the perturbative vacuum energy. It is a straightforward exercise to find:

$$E(B) = \frac{1}{V} E(B) = \frac{B^2}{2} - \frac{eB}{2\pi} \sum_{n=0}^{\infty} E_n. \quad (9)$$

Introducing the dimensionless variable $\lambda = \frac{eB}{m^2}$, the function

$$g(\lambda) = \int_0^{\infty} \frac{dx}{\sqrt{\pi x}} \frac{d}{dx} \left[ \frac{e^{-x}}{1 - e^{-2x}} - \frac{e^{-x}}{2x} \right], \quad (10)$$

and subtracting a term independent on the magnetic field, we recast the energy density in the form [1]:

$$E(B) = \frac{B^2}{2} + \frac{(eB)^{3/2}}{2\pi} g(\lambda). \quad (11)$$

It is easy to show that, indeed, $E(B)$ displays a nontrivial negative minimum [1]. As a matter of fact, using the expansion [1]

$$g(\lambda) \lambda \to 0 \sim -\frac{1}{2\lambda^{3/2}} + \frac{\lambda^{1/2}}{12} \quad (12)$$

we see that near the origin

$$E \sim -\frac{eB}{4\pi} |m| + \frac{1}{24\pi} \frac{(eB)^2}{|m|} + \frac{B^2}{2}. \quad (13)$$

The negative linear term in Eq. (13) gives rise to the negative minimum in the vacuum energy density.

In the remainder of the paper we study the thermal corrections to the vacuum energy density. In particular we are interested in the phenomenon of the restoration by thermal corrections of the symmetry broken at zero temperature. Remarkably, it turns out that the spontaneous generation of the magnetic condensate survives the thermal corrections.

The relevant quantity at finite temperature is the free energy

$$F = -\frac{1}{\beta} \ln Z, \quad (14)$$
where $\beta = \frac{1}{T}$ and $Z$ is the partition function

\[
Z = \text{Tr}(e^{-\beta H}).
\]  
(15)

In the one-loop approximation we have

\[
F_0 = -\frac{1}{\beta} \ln Z_0, \quad Z_0 = \text{Tr} \left[ e^{-\beta H_0} \right].
\]  
(16)

In our approximation the free energy is the sum of the photonic and fermionic contributions. Only the latter depends on the background magnetic field $B$. Moreover it is a straightforward exercise to evaluate the partition function corresponding to the Hamiltonian $H_D$. Indeed, the calculation reduces to evaluating the partition function of a free relativistic fermions in presence of the magnetic field. Taking into account that $V\frac{eB}{2\pi}$ is the degeneracy of the Landau levels we obtain:

\[
F_0(B) = F_0(B) - \frac{1}{\beta} eB \sum_{n=1}^{\infty} \ln \left( 1 + e^{-\beta E_n} \right) - \frac{1}{\beta} eB \sum_{n=1}^{\infty} \ln \left( 1 + e^{-\beta |m|} \right) + \mathcal{E}(B).
\]  
(17)

The first term on the right hand of Eq. (17) arises from the negative and positive Landau levels with $n > 0$, the second one is the zero mode contribution and the last term is the zero temperature vacuum energy density.

In order to evaluate the infinite sum in Eq. (17) we expand the logarithm to obtain

\[
I_1(B) = -\frac{eB}{\pi \beta} \sum_{n=1}^{\infty} \ln \left( 1 + e^{-\beta E_n} \right) = -\frac{eB}{\pi \beta} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} e^{-\beta E_n k}.
\]  
(18)

Now using \[5\]

\[
\int_0^{\infty} e^{-ax^2} \frac{x^2}{x^2+1} dx = \frac{1}{2} \sqrt{\pi} e^{-\sqrt{ab}}, \quad a > 0, \quad b > 0
\]  
(19)

we perform the sum over $n$

\[
\sum_{n=1}^{\infty} e^{-\beta k E_n} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} dx e^{-\frac{x^2}{2}} \frac{\beta^2 k^2 m^2}{x^2 e^{-\frac{\beta^2 k^2 m^2}{2}}} \quad \frac{1}{e^{\frac{\beta^2 k^2 m^2}{2}} - 1}
\]  
(20)
and obtain

$$I_1(B) = -\frac{eB}{\frac{1}{2} \beta} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^\infty dx \, e^{-\frac{x^2}{4}} \frac{1}{e^{\frac{\beta^2 k^2 m^2}{x^2}} - 1}. \quad (21)$$

Subtracting the contribution at $B = 0$ we get

$$\tilde{I}_1(B) = -\frac{eB}{\frac{1}{2} \beta} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^\infty dx \, e^{-\frac{x^2}{4}} \frac{1}{e^{\frac{\beta^2 k^2 m^2}{x^2}} - 1} - \frac{x^2}{2eB\beta^2 k^2}. \quad (22)$$

Thus we have

$$\mathcal{F}_0(B) = \tilde{I}_1(B) - \frac{1}{\beta} \frac{eB}{2\pi} \ln \left(1 + e^{-\beta|m|}\right) + \mathcal{E}(B). \quad (23)$$

This last expression is amenable to a numerical evaluation. In Figure 1 we display the free energy density $\mathcal{F}_0(B)$ (in units of $|m|^3$) for four different temperatures $\tilde{T} = T|m|$ and $\alpha = \frac{|m|}{\epsilon} = 0.1$. We see that by increasing the temperature the negative minimum never disappears. In other words, the spontaneous magnetic field survives at any temperature. Similar results hold even in presence of the Chern-Simons term [6]. Interesting enough, it turns out that the asymptotic curve at infinite temperature can be evaluated in closed form. Indeed, from Eq.(18) we get for $\tilde{\beta} = \beta|m| << 1$

$$I_1(B) = \frac{eB}{2\pi \beta} \ln 2 + \frac{eB}{2\pi} \sum_{n=1}^{\infty} E_n + \mathcal{O}(\beta^2). \quad (24)$$

On the other hand, in the same approximation we have

$$-\frac{eB}{2\pi \beta} \ln \left(1 + e^{-\beta|m|}\right) = -\frac{eB}{2\pi \beta} \ln 2 + \frac{eB}{4\pi} |m| + \mathcal{O}(\beta^2). \quad (25)$$

From Equations (11),(23),(24) and (25), it follows the remarkable simple result:

$$\mathcal{F}_0(B) = -\frac{eB}{4\pi} |m| + \frac{B^2}{2} + \mathcal{O}(\beta^2). \quad (26)$$

Again we see that the negative minimum at finite temperature is due to the linear term in $|m|$. Note that the slope of the linear term coincides with the
one at zero temperature. This can be seen clearly in Fig. 2 where we display
the free energy density without the classical energy $\frac{B^2}{2}$.

Equation (26) displays a negative minimum at

$$eB^* = \frac{e^2|m|}{4\pi}.$$  \hfill (27)

The condensation energy is:

$$\mathcal{F}_0(B^*) = -\frac{1}{2}eB^*|m| = -\frac{e^2|m|^2}{32\pi^2}.$$  \hfill (28)

Note that the minimum Eq. (27) coincides with the zero-temperature mini-
mum in the "weak-coupling" region $\alpha = \frac{|m|}{e^2} >> 1$ \[1\]:

$$\frac{eB^*}{m^2} = \frac{1}{4\pi\alpha}.$$  \hfill (29)

Thus, even if the system at $T = 0$ lies in the strong coupling region $\alpha << 1$, the thermal corrections drive it in the weak-coupling region where the condensation energy is maximum. We feel that this phenomenon gives a rather convincing evidence that the higher-order radiative corrections do not modify the spontaneous generation of the magnetic condensate. So that the spontaneous generation of a background magnetic field is a genuine feature of the theory.

The magnetic condensate vanishes in the massless limit. However it is concealable that in the massless theory there is the spontaneous generation of the dynamical mass and, at the same time, of the magnetic condensate. A complete treatment of the problem can be done by using the formalism of the composite operator effective potential \[7\].

As a preliminary step it is important to investigate the fermionic condensate with the thermal corrections in the massless limit.

In the P-invariant formulation of the theory, it is known \[8\] that the fermion condensate $\langle \bar{\psi}\psi \rangle$ in presence of a constant magnetic field is extremely unstable. Indeed, the condensate disappears as soon as a heat bath is introduced. Moreover the condensate becomes nonanalytic at finite temperature.

On the other hand, in the P-odd formulation of QED in (2+1)-dimensions it turns out that \[1\]
Therefore in the massless limit and in presence of a magnetic field, the fermionic condensate is a non-vanishing quantity only in the case of a negative mass term.

In our approximation it is straightforward to evaluate the fermion condensate at finite temperature. Indeed, from Eq. (7) and taking into account that

\[
\langle a_{np} a_{np} \rangle_\beta = 1 - \langle a_{np} a_{np}^\dagger \rangle_\beta = \frac{1}{e^{\beta E_n} + 1},
\]

\[
\langle b_{np} b_{np}^\dagger \rangle_\beta = 1 - \langle b_{np}^\dagger b_{np} \rangle_\beta = \frac{1}{e^{-\beta E_n} + 1},
\]

we get

\[
\langle \bar{\psi} \psi \rangle_\beta = \frac{eB}{2\pi} \frac{1}{e^{\beta m} + 1} - \frac{mcB}{2\pi} \sum_{n=1}^{\infty} \frac{1}{E_n} \tanh \left( \frac{\beta}{2} E_n \right).
\]

Equation (33) holds for both positive and negative mass. The subscripts $\beta$ in Eqs. (31) and (33) stand for the thermal average with respect to the Hamiltonian $H_0$, Eq. (5).

In the massless limit the expansion parameter is $\hat{\beta} = \beta |m|$. This means that the massless limit coincides with the high temperature limit. A straightforward calculation gives:

\[
\langle \bar{\psi} \psi \rangle_\beta = \frac{eB}{4\pi} + O(\hat{\beta}).
\]

Comparing Eq. (34) with Eq. (30) we see that the order of the limits $m \to 0$ and $\beta \to \infty$ is not commutative. Note that Eq. (34) implies that in presence of a heat bath even with an infinitesimal temperature the massless limit is symmetric:

\[
\lim_{m \to 0^+} \langle \bar{\psi} \psi \rangle_\beta = \lim_{m \to 0^-} \langle \bar{\psi} \psi \rangle_\beta = \frac{eB}{4\pi}.
\]
and it corresponds to half filled zero modes. It is remarkable that in the high temperature limit both the free energy and fermion condensate are accounted for with an effective system composed of half filled zero modes at zero temperature.

In conclusion we have investigated (2+1)-dimensional QED coupled with P-odd Dirac fermions at finite temperature. We find that the spontaneous generation of a constant background magnetic field is stable towards the thermal fluctuations. Moreover, it turns out that at high temperature the thermal fluctuations tend to increase both the condensation energy and the strength of the induced magnetic field.

In addition we found that the finite temperature fermion condensate in the massless limit is different from zero and temperature independent. However it seems that, due to the non commutativity of the massless limit and the zero temperature limit, the fermion condensate is nonanalytic. The nonanalyticity vanishes if at zero temperature one assumes the rule of the symmetric massless limit.

Let us, finally, compare the results of the present work with Ref. [6]. The authors of Ref. [6] consider the model studied in Ref. [4] at finite temperature and in the one-loop approximation. In the first paper of Ref. [1] one can found a critical comparison at zero temperature. At finite temperature our results agree with the ones of Ref. [6] as concern the survival of the spontaneous generation of the constant background field to the thermal corrections even at infinite temperature. Moreover we agree on the fact that the coefficient of the term linear in the magnetic field is accounted for by the contributions due to the zero modes. However, we find that the thermal corrections to the above mentioned coefficient are small and vanish at infinite temperature, while in Ref. [6] the coefficient grows logarithmically with the temperature. As a consequence both the induced magnetic field and the negative condensation energy grow with the temperature, so that one must supply from an external source, via the chemical potential, the energy needed in order to induce the states with a nonzero magnetic field.

For these reasons we believe that the model of Ref. [6] is artificial and without direct physical applications. On the other hand, as we have already discussed, our model is relevant for the dynamics of four dimensional fermions localized on the two-dimensional space of the domain walls.
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FIG. 1 Free energy density (in units of $|m|^3$) versus $\lambda$ for $\alpha = 0.1$. Full line $T = 0$, dashed line $\hat{T} = 1$, dotted line $\hat{T} = 5$, and dash-dotted line $\hat{T} \to \infty$. 
FIG. 2 Free energy density (in units of $|m|^3$) versus $\lambda$ for $\alpha = 0$. Temperature values as in Fig. 1.