We introduce a novel computational framework for digital geometry processing, based upon the derivation of a nonlinear operator associated to the total variation functional. Such an operator admits a generalized notion of spectral decomposition, yielding a convenient multiscale representation akin to Laplacian-based methods, while at the same time avoiding undesirable over-smoothing effects typical of such techniques. Our approach entails accurate, detail-preserving decomposition and manipulation of 3D shape geometry while taking an especially intuitive form: non-local semantic details are well separated into different bands, which can then be filtered and re-synthesized with a straightforward linear step. Our computational framework is flexible, can be applied to a variety of signals, and is easily adapted to different geometry representations, including triangle meshes and point clouds. We showcase our method through multiple applications in graphics, ranging from surface and signal denoising to enhancement, detail transfer, and cubic stylization.

Additional Key Words and Phrases: total variation, spectral geometry

1 INTRODUCTION

In the last decades, the computer graphics community has witnessed a thriving trend of successful research in computational spectral geometry. Spectral approaches have met with considerable traction due to their generality, compact representation, invariance to data transformations, and their natural interpretation relating to classical Fourier analysis. Despite these benefits, the vast majority of current spectral methods rely upon the construction of smooth basis functions (obtained, for instance, as the eigenfunctions of the Laplace operator), leading to significant detail loss whenever the signal to be represented is not smooth. When this signal represents geometric information (e.g., the \( x, y, z \) coordinates of mesh vertices), this inevitably leads to a “lossy” geometric encoding, typically manifest as over-smoothing, vertex collapse, or spurious oscillations (so called Gibbs phenomenon) not appearing in the original geometry.

Using the TV functional, we provide a spectral decomposition (analysis) algorithm for functions \( f \) on a manifold which, similarly to Fourier decompositions, carries a canonical multiscale ordering, providing a sharp separation of the source signal into different bands; as a direct consequence to the sparsity of the gradient, frequency bands in the spectral domain assume a semantic connotation, and can be filtered and processed following classical signal processing paradigms (see Figure 1 for an example). Importantly, since \( \text{TV}(f) \)
remains finite for a wide variety of discontinuous $f$ with an appropriate adaptation of the definition (1) to be discussed in Eq. (4)), discontinuities are preserved by the transform, and is therefore well-suited for representing jumps and sharp high-frequency variations. The inverse transform (synthesis) is realized by a single linear step.

Our approach shares some common traits with Laplacian mesh processing, for which it provides a nonlinear alternative; in particular, we show that both settings can be derived as specific instances of a more general model. We demonstrate the versatility of our framework by applying it with success to several applications in geometry processing and graphics, relating its multiple incarnations to existing ad-hoc approaches, and illustrating its performance across different settings and shape representations.

1.1 Contriubtion

With this paper we introduce a new framework for geometry processing that weaves discontinuity-preserving regularization into the fabric of a spectral approach. Our key contributions are:

- We introduce a general spectral framework to analyze and process non-smooth signals defined on surfaces, as well as the surfaces themselves, which fully preserves geometric detail;
- We adopt a fast, unconditionally stable algorithm to solve the resulting nonlinear problem on non-Euclidean domains, with limited dependence on a few parameters;
- We analyze the evolution of surfaces along the flow induced by the operator, and relate it to existing geometric flows;
- We explore a wide range of possible applications in shape analysis and modelling, where we compare to state-of-the-art methods in the respective domains, demonstrating production quality results in many cases.

2 RELATED WORK

From a general perspective, ours is a filtering approach that incorporates an edge-preserving diffusion process within a spectral decomposition framework, and draws inspiration from recent progress in nonlinear image processing. In the following, we cover the works that more closely relate with these themes.

2.1 Variational mesh processing

Our methodology falls within the class of energy minimization methods for geometry processing, and carries a natural dual interpretation as a diffusion-based approach.

Diffusion-based methods. Following the work of Taubin [1995], classical editing tasks such as surface smoothing, denoising and enhancement have been traditionally phrased as variants of the diffusion PDE $\partial_t u = \Delta_M u$. Here, $u$ is a signal defined on the surface $M$, and $\Delta_M$ is its Laplace-Beltrami operator. For example, feature-aware denoising driven by local curvature has been addressed by modifying the metric on $M$ (and hence its Laplacian), leading to a rich production of anisotropic techniques [Bajaj and Xu 2003; Clarenz et al. 2000; Desbrun et al. 2000; Tasdizen et al. 2002; Fumero et al. 2019]. Higher-order filtering with volume preservation to prevent surface shrinking was proposed in [Desbrun et al. 1999]; the same work clarified the well known duality relating the diffusion to mean curvature flow, and showed how the latter notion leads to a more robust formulation for denoising applications. This was later modified in [Kazhdan et al. 2012] to resolve instabilities of the flow.

Our approach also involves a diffusion process, and comes with an associated geometric flow. We identify two key differences over prior work: (1) in our case, the operator involved is nonlinear, and (2) diffusion takes place within the level sets of the signal $u$ rather than across the boundary of the level sets. As a consequence, discontinuities are not smeared throughout the diffusion process, while contrast is reduced. In the inset figure, the source signal (left) is processed via the classical "horizontal" isotropic diffusion (middle) and with our "vertical" diffusion (right); in the latter case, the width remains constant while the amplitude decreases. When the signal encodes surface geometry, this implies that sharp features are preserved.

Energy minimization methods. Other approaches are based on formulating editing operations as the minimization of an energy. Although mathematically any variational problem can be rephrased as a diffusion problem, choosing the former enables the implementation of a richer class of algorithms. Such methods lift ideas from image processing to the surface domain. For example, in [He and Schaefer 2013] mesh denoising is done by minimizing the $L_0$ gradient norm of the vertex positions, thus suppressing low-amplitude details, as done in [Xu et al. 2011] for images; extensions to meshes of bilateral [Tomasi and Manduchi 1998] and mean shift [Comaniciu and Meer 2002] filtering were proposed in [Fleishman et al. 2003; Solomon et al. 2014]; a variant of [Zhang et al. 2014] was applied to the normal vector field in [Wang et al. 2015]; shock filters [Osher and Rudin 1990] were applied to surface geometry for feature sharpening and enhancement in [Prada and Kazhdan 2015]. Gradient-domain mesh processing [Chuang et al. 2016] is also a recent direction inspired by success in the image domain, allowing explicit prescription of target gradient fields. The above testifies to a wealthy literature of nonlinear edge preserving strategies for mesh filtering. The main drawback of these methods lies in their lack of generality. Here we provide a general framework to perform multiscale analysis and synthesis of arbitrary signals defined on geometric domains (meshes and point clouds alike); while denoising, fairing and enhancement can be easily realized within this framework, we are not limited to these applications and provide a more flexible setting for geometry processing.

TV-based methods. TV and the strictly related Mumford-Shah functional have been considered before for different tasks in geometry processing, e.g. surface reconstruction [Liu et al. 2017]. Zhang et al. [2015] consider an extension of the ROF model [Rudin et al. 1992] for denoising surfaces, which they solve with an augmented Lagrangian method followed by a geometry reconstruction step [Sun et al. 2007; Zhong et al. 2018] follow a similar approach, with the inclusion of an additional regularizer defined as the Laplacian energy of the normal coordinates, which helps dampen the staircase effect that is observed in [Zhang et al. 2015]. These methods are related to ours, as they involve the minimization of the TV of the surface normal field; we also consider this kind of energy, but in addition, we study the geometric flow associated to it. Further, our approach is not restricted to normal fields, but allows processing...
arbitrary surface signals. Tong and Tai [2016] consider the Mumford-Shah functional as a measure of sharp geometric variation for a given signal. Their focus is on the detection of features lines (i.e. the boundaries between neighboring segments) for scalar functions such as mean curvature and color texture, rather than geometry processing. The MS functional has also been considered in [Bonneel et al. 2018] in its Ambrosio-Tortorelli approximation. The authors extend the range of possible applications of [Tong and Tai 2016] by operating on the surface normal field, showing results in inpainting, denoising and segmentation tasks. In addition to lacking an analysis of the TV normal flow, none of these methods provide a spectral framework based on the TV functional for geometry processing.

2.2 Spectral geometry processing

As shown in Figure 1, our multiscale approach provides a sharp separation of an input signal into “feature bands”. We achieve this by means of a generalized spectral decomposition. Spectral geometry processing was introduced by Taubin [1995] and further developed in [Lévy 2006; Rong et al. 2008; Sorkine 2005; Zhang et al. 2010] for tasks of shape reconstruction, modelling, and deformation transfer among others. These works generalize Fourier analysis to surfaces by observing that the Laplacian eigenvectors form an orthogonal set of smooth basis functions with a canonical ordering (low to high frequencies). This basis is optimal for representing smooth functions compactly [Aflalo et al. 2014], but it gives rise to artifacts in the presence of sharp variations, thus reducing the benefits of working with a spectral representation; in the inset, we show the typical blurring effect (middle) and Gibbs oscillations (right) due to the jumps in the source signal (left), not handled well by the smooth Laplacian eigenbasis.

Variants of the Laplace operator add flexibility and feature awareness, including anisotropic [Andreux et al. 2015] and extrinsic [Liu et al. 2017] variants, see [Wang and Solomon 2019] for a recent survey. Sparsity has been analyzed in [Neumann et al. 2014], where Laplacian eigenfunctions are modified to have limited support by penalizing their $L^1$ norm. We differ in that we consider signals with sparse gradients instead. All these operators are linear, and their associated energies promote smoothness. Our operator is nonlinear, hence more powerful, and its eigenfunctions are well suited to represent jump discontinuities in the signal. This yields a cleaner separation of different features into different bands, in contrast to Laplacian-based representations, which spread individual features across the entire spectrum.

2.3 Nonlinear image processing

Finally, similar to prior research, our approach is also based on ideas originating from image processing. In this field, nonlinear variational methods have replaced linear methods due to their higher precision in approximating natural phenomena. To date, a major role has been played by the total variation (TV) energy and its minimization, originally exploited by Rudin et al. [1992] with the ROF model for image denoising. Other uses of the TV functional span image deblurring, inpainting, interpolation, image decomposition, super-resolution and stereovision, to name but a few; for a general introduction to the topic see [Chambolle et al. 2010]. Recently, a spectral theory related to the gradient flow of this functional (c.f. Bellettini et al. [2002]) on eigenfunctions, was developed by Gilboa [2013; 2014] and generalized in [Burger et al. 2016; Gilboa et al. 2015] with applications to image processing, allowing for the analysis of images at different scales by exploiting the edge-preserving property of the total variation. Here we propose to lift this latter framework from the Euclidean domain to surfaces, put it in relation to existing geometry processing models, and demonstrate several possible applications in the area of 3D graphics.

3 MATHEMATICAL PRELIMINARIES

3.1 Surfaces

We model shapes as compact and connected Riemannian manifolds $M$ embedded in $\mathbb{R}^3$, possibly with boundary $\partial M$, and with tangent bundle $\mathcal{T} M = \bigcup_{p \in M} \mathcal{T}_p M$. The manifold is equipped with a metric, i.e., an inner product defined locally at each point, $\langle \cdot, \cdot \rangle_p : \mathcal{T}_p M \times \mathcal{T}_p M \to \mathbb{R}$, making it possible to compute lengths and integrals on $M$. The intrinsic gradient $\nabla_M$, defined in terms of the metric, generalizes the notion of gradient to manifolds; similarly, the intrinsic divergence $\text{div}_M$ can be defined as the negative adjoint of the gradient, assuming Neumann boundary conditions:

$$\int_M \text{div}_M f \, dx = -\int_M \langle \nabla_M f \rangle_p \, dx , \quad (2)$$

where $f : M \to \mathbb{R}$ is a scalar function and $V : M \to \mathcal{T} M$ is a vector field tangent to the surface. In the following, to simplify the notation we will drop the subscript $M$ whenever clear from the context. We further denote by $L^p(M) = \{ f : M \to \mathbb{R} : \int_M |f(x)|^p \, dx < \infty, p \geq 1 \}$ the function space of $p$-integrable functions on $M$.

3.2 Total Variation

The total variation of a differentiable function $f$ on $M$ is defined as

$$TV(f) = \int_M \|\nabla_M f\| \, dx , \quad (3)$$

where $\| \cdot \|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_p$. This functional quantifies the amount of oscillations in a given signal. To account for non-differentiability of $f$, we adopt a weaker definition of TV in terms of continuously differentiable test vector fields $U : M \to \mathcal{T} M$ [Ben-Artzi and LeFloch 2006]:

$$TV(f) = \sup_U \left\{ \int_M f(x) \text{div}_M U(x) \, dx : \|U(x)\| \leq 1, \forall x \right\} . \quad (4)$$

Alternatively, one can define an anisotropic variant of the standard TV by considering $U(x)$ to be bounded in the $L^\infty$ norm. We refer to functions $f$ of bounded variation, i.e. $TV(f) < +\infty$, to be in $BV(M)$.

The TV functional is related to fundamental geometric notions such as the curvature of level sets of a function, through the celebrated co-area formula [Fleming and Rishel 1960]. One particular case of this relation concerns the total variation of indicator functions of closed sets; in this case, the TV corresponds to the perimeter of the set (see [Chambolle et al. 2010, Sec. 1.3] for the Euclidean setting), which generalizes to manifolds in a straight forward way:
Theorem 1. Let \( E \subset M \) be a measurable closed subset of \( M \), with smooth boundary \( \partial E \), and let \( \chi_E \) be its indicator function. Then:

\[
TV(\chi_E) = \text{length}(\partial E) .
\]

Proof: See Appendix A.1.

As we show next, this particular class of indicator functions includes the set of generalized eigenfunctions of the operator associated to the TV functional.

4 SPECTRAL TV DECOMPOSITION

In this work we introduce a new invertible transform for signals defined on surfaces, with the following key properties:
- The forward transform is non-linear;
- The inverse transform is linear;
- Any signal \( f \in BV(M) \) on the surface can be decomposed and reconstructed up to a desired accuracy;
- Discontinuities in \( f \) are preserved by the transform.

The following considerations closely follow the ideas of [Gilboa 2013, 2014] in the Euclidean setting, but extend them to functions on manifolds to enable discontinuity-preserving geometry processing.

4.1 TV flow

Consider the following energy for surface signals \( u, u_0 \in BV(M) \):

\[
E(u) = \frac{1}{2} \int_M (u(x) - u_0(x))^2 \, dx + tTV(u), \quad t > 0
\]

with the associated variational problem:

\[
\min_{u \in BV(M)} E(u) .
\]

The above seeks to minimize a functional over two terms: a fidelity term, which quantifies the distance from an initial function \( u_0 \), and a regularization term which measures the total variation of the sought minimizer. The regularization parameter \( t \) balances between the two terms. In image processing, where \( M = S \subset \mathbb{R}^2 \), Eq. (7) is known as the ROF model for image denoising [Rudin et al. 1992], and dissipates noise in flat regions of an image \( u_0 \) while preserving object contours.

Iteratively applying Eq. (7) to the output of a previous solution with small increases of \( t \) can be seen as a discretization of a time continuous total variation flow:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \text{div} \left( \frac{\nabla u}{\| \nabla u \|} \right) \quad \text{in} \ M, \\
(\nabla u, \overline{n}) &= 0 \quad \text{on} \ \partial M, \\
u(0, x) &= u_0,
\end{aligned}
\]

where \( \overline{n} \) is the unit vector normal to the boundary \( \partial M \). Equations (8) describe a physical process of isotropic diffusion within the level sets of the initial function \( u_0 \), without any diffusion across them (see Figure 2). In \( \mathbb{R}^n \), it is guaranteed that the flow converges in finite time to a constant solution under mild assumptions [Andreu et al. 2002, Cor. 1]; no proofs exist for manifolds, but in our experiments we always observed convergence.

To give sense to Eq. (8) for nondifferentiable \( u \) and points at which \( \nabla u(x) = 0 \), one needs to turn to the subdifferential, which for a convex functional \( J \) (TV in our case) over a function space \( X \) is formally defined as the set:

\[
\partial J(u) = \left\{ p \in X^* \mid J(v) - J(u) \geq \langle p, v - u \rangle, \ \forall v \in X \right\} ,
\]

with \( X^* \) being the dual space of \( X \). For smooth \( u \) with non-zero gradient one obtains:

\[
\partial TV(u) = \left\{ \text{div} \left( \frac{\nabla u}{\| \nabla u \|} \right) \right\} .
\]

We refer the reader to [Bellettini et al. 2002] for more details on the TV flow in a Euclidean setting.

4.2 Nonlinear eigenvalue problem

A generalized eigenfunction of the operator \( \partial J \) is a function \( u \) with \( \| u \| = 1 \) satisfying the following relation for some number \( \lambda \in \mathbb{R} \) (its corresponding generalized eigenvalue):

\[
\partial J(u) \ni \lambda u ,
\]

where \( J = TV \) in our case. This relation generalizes the classical characterization of eigenpairs of a linear operator.

Remark. For the Dirichlet functional \( D(u) = \frac{1}{2} \int_M \| \nabla u \|^2 \), one obtains \( \partial D(u) = (-\lambda u) \), such that the analogy to Eq. (8) becomes the standard diffusion equation, and Eq. (10) reduces to the standard eigenvalue problem \( \Delta u = \lambda u \) (up to a sign).

4.3 The TV flow on eigenfunctions

Let us recall the ideas from [Gilboa 2013; Gilboa et al. 2015] to motivate a spectral decomposition using the TV flow. For eigenfunctions \( u_0 : M \rightarrow \mathbb{R} \) with eigenvalue \( \lambda \), we get a simple analytic solution to the flow of Eq. (8):

\[
u(t, x) = \begin{cases} (1 - t\lambda)u_0(x) & \text{for } t \leq \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases}
\]

At each point, the eigenfunction decreases linearly at increasing \( t \), until it goes to zero. In other words, TV eigenfunctions preserve their shape (up to rescaling) under the action of the flow; Figure 2 illustrates one such eigenfunction.

The first derivative of \( u(t, x) \) with respect to time is piecewise-constant:

\[
\partial_t u(t, x) = \begin{cases} -\lambda u_0(x) & \text{for } t \leq \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases}
\]

Considering also the second derivative (in a distributional sense), we get a delta function placed exactly at the point in time where the solution \( u(t, x) \) vanishes (see Figure 3):

\[
\partial_{tt} u(t, x) = \begin{cases} +\infty & \text{for } t = \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases}
\]
The behavior of the second derivative \( \partial_{tt} u(t) \) for an eigenfunction \( u_0 \) with eigenvalue \( \lambda = \frac{1}{3} \), first and second derivatives (resp. middle and right).

**Fig. 3.** Analytic solution (left) to the 1D flow for a TV eigenfunction \( u_0 \) with eigenvalue \( \lambda = \frac{1}{3} \), first and second derivatives (resp. middle and right).

The behavior of the second derivative \( \partial_{tt} u(t) \) for an eigenfunction \( u_0 \) resembles the notion of spectrum in Fourier analysis, since sinusoidal functions (eigenfunctions of the Laplace operator) correspond to deltas in the frequency domain. This motivates the definition:

**Definition 1.** We refer to the distribution over time \( \phi_t = \partial_{tt} u(t) \) as the spectral representation of \( u_0 \), and to each individual distribution at the time scale \( t \) as a spectral component.

As we have seen, for a TV eigenfunction with eigenvalue \( \lambda \), the spectral representation has a single component: a delta at \( t = \frac{1}{\lambda} \). However, not every spectral component \( \phi_t \) is guaranteed to identify a TV eigenfunction. Further, at any given time scale \( t \), the spectral component \( \phi_t : M \to \mathbb{R} \) still has a spatial extent over \( M \). To show the entire spectral activity across different time scales, we can integrate each \( \phi_t \) and simply plot the function:

\[
s(t) = \int_M |\phi_t(x)| dx.
\]  (11)

Looking at the behavior of the function \( s(t) \) can be rather useful in practice, since it helps in defining the filtering operations required by the applications; examples of \( s(t) \) are shown in Figure 1.

The spectral representation can be translated back to the primal domain by the linear operation of integrating over time (i.e., across all spectral components \( \phi_t \)) and adding the mean of \( u_0 \):

\[
u_0 = \int_0^\infty t \phi_t dt + \text{mean}(u_0).
\]  (12)

We refer to [Gilboa 2013, 2014] for more details (with the slight difference that we decided to define the spectral decomposition without the factor \( t \) and include it in the reconstruction instead).

4.4 The spectral representation induced by the TV flow

In light of the previous discussion that TV eigenfunctions are the fundamental atoms of the spectral representation of Def. 1 (in the sense that they yield peaks in the spectral domain), one might expect to represent arbitrary signals as linear combinations of such atoms. Although there seems to be a vast number of TV eigenfunctions to allow such a representation\(^1\), computing a sparse representation turns out to be difficult since, e.g., a Rayleigh principle to compute orthogonal eigenfunctions does not hold [Benning and Burger 2013].

Interestingly, Def. 1 allows to show that the gradient flow with respect to a regularization with suitable properties does yield such a decomposition with the \( \phi_t \) representing differences of eigenfunctions, see [Burger et al. 2016]. While the TV in Euclidean spaces of dimension larger than 1 does not satisfy the required properties, the spectral representation \( \phi_t \) has been shown to still yield a meaningful, edge-preserving, data-dependent multi-scale representation in the field of image processing [Gilboa 2014; Gilboa et al. 2015], which is why we utilize it on manifolds in a similar fashion.

4.5 Computing the TV spectral decomposition

To compute the spectral representation \( \phi_t \) of a given function \( f \), we use an implicit Euler discretization of the TV flow: We compute a new iterate by solving Eq. (7) using the previous iterate in the data fidelity term starting from \( u_0 \equiv f \). The procedure is summarized in Algorithm 1.

**Algorithm 1:** Spectral TV decomposition

**input:** signal \( u_0 \), step length \( \alpha \), no. of time steps \( N 

**output:** spectral representation \( \phi_t \)

1. **begin**
2. for \( t = 0 : N \) do
3. \( u(t) \leftarrow \min_{\lambda} TV(u) + \frac{1}{2\alpha} \|u - u_0\|^2 \)
4. output \( \phi_t \leftarrow \partial_{tt} u(t) \)
5. increase \( \alpha \)
6. update \( u_0 \leftarrow u(t) \)
7. **end**

The algorithm simply evolves the input signal by \( N \) discrete steps along the TV flow; each iteration moves a step forward, with diffusion time equal to \( \alpha \). As changes happen quickly for small \( t \) and tend to become slower for larger \( t \) we iteratively increase the step size \( \alpha \) of the evolution. Subsequently, the spectral representation \( \phi_t \) is constructed incrementally using finite differences, such that the integral of Eq. (12) becomes a simple weighted sum over the \( \phi_t \). We give selection strategies for \( \alpha \) and \( N \) in Appendix B.4.

**Remark.** This yields a multiscale representation for the input, since each diffusion time \( t \) captures a feature at a different scale –
namely, the feature that vanishes at \( t \). Here, time scales play the same role as a wavelength in classical Fourier analysis.

See Figures 4 and 5 for examples of multiscale TV decomposition of color signals on two different surfaces.

Algorithmic considerations. Algorithm 1 is unconditionally stable, and line 3 corresponds to taking an implicit Euler step along the discretized TV flow. This property allows us to use arbitrary values for the step size \( \alpha \) (therefore, the resolution of the decomposition) without compromising convergence. In practice, in our experiments we use a variant of Algorithm 1 and study the evolution of the flow backwards in time by following a nonlinear inverse scale approach, that starts from a constant solution and converges to the initial function \( u_0 \). The spectral decompositions of both approaches are very similar in practice and have been proven to be identical under certain assumptions, see [Burger et al. 2016]. The inverse scale variant comes with additional numerical robustness. The complete algorithm, along with a detailed discussion on the two flows and on the role of \( \alpha \) are given in Appendices B.1 and B.2.

4.6 Properties of TV eigenfunctions

We conclude by giving a more complete characterization of the TV eigenfunctions on surfaces. On the image domain, Bellettini et al. [2002] proved that any indicator function \( \chi_S : \mathbb{R}^2 \rightarrow \{0, 1\} \) of a convex closed subset \( S \subset \mathbb{R}^2 \) is a TV eigenfunction with eigenvalue \( \lambda = \frac{\text{Per}(S)}{|S|} \) if it satisfies the isoperimetric inequality:

\[
\max_{q \neq \partial S} k(q) \leq \frac{\text{Per}(S)}{|S|},
\]

requiring the maximal curvature \( k \) of \( \partial S \) to be less than or equal to the perimeter-area ratio of \( S \), intuitively excluding oblong regions.

Such results have later been generalized to \( \mathbb{R}^n \) as well as to unions of such convex sets, where the convex sets need to be sufficiently far apart for the TV spectral decomposition to yield a single \( \delta \)-peak for each of the subsets, see [Alter et al. 2005]. We conjecture (and consistently confirmed in all our experiments) that this result can be generalized to surfaces by replacing convex subsets with geodesically convex subsets of surface \( M \), making the examples in the inset figure eigenfunctions of the TV on a manifold. A patch \( C \subset M \) is geodesically convex if, given any two points \( p, q \in C \), there exists a unique minimum-length geodesic connecting them which lies entirely in \( C \). Therefore in certain practical cases (e.g., the cow’s nostrils in Figure 5) one actually gets TV eigenfunctions in the spectral decomposition.

5 IMPLEMENTATION

We implemented the framework in Matlab 2019a, and ran all the experiments on an Intel i7-4558U CPU with 16 GB RAM. A discussion on runtime performance is provided in Appendix C.

5.1 Discretization

We discretize shapes as manifold triangle meshes \((V, E, F)\), sampled at vertices \( v_i, i = \{1, \ldots |V|\} \), with the constraint that each pair of triangles \((t_i, t_j) \in F\) shares at most one edge \( e \in E \). Functions \( f : M \rightarrow \mathbb{R} \) are approximated at vertices with piecewise-linear basis elements.
tangent vector fields \( U : M \rightarrow TM \) are constant within each triangle.

The gradient operator \( G \in \mathbb{R}^{3|F| \times |V|} \) takes a function value per vertex \( v_j \) and returns a 3D vector per triangle \( t_{ijk} \in F \):

\[
Gf(t_{ijk}) = \begin{pmatrix} v_{i}^T \\ v_{j}^T \\ v_{k}^T \end{pmatrix} \begin{pmatrix} \| \epsilon_{ij} \| \| \epsilon_{ij} \| \| \epsilon_{ik} \| \| \epsilon_{ik} \| \end{pmatrix} \begin{pmatrix} f(v_i) - f(v_j) \\ f(v_j) - f(v_k) \end{pmatrix}
\]

where the notation is as per the inset figure. The divergence operator \( D \in \mathbb{R}^{|V| \times |F|} \) takes a 3D vector per triangle and returns a value per vertex. As the adjoint of the gradient, it is discretized as:

\[
D = -A^{-1} G^T \tag{13}
\]

where \( A \) is a diagonal matrix of local area elements at each vertex (shaded area in the inset) and \( T \) is a diagonal matrix of triangle areas. These discretizations follow the standard formulations as described, e.g., in [Botsch et al. 2010].

To define these operators on point clouds, we base our approach on the construction of a graph \((V, E)\), built locally at each point in a k-neighborhood. The edges of the graph are weighted by a Gaussian kernel \( w_{ij} = e^{-\frac{\|x_i - x_j\|^2}{\sigma}} \), for an appropriate \( \sigma \) depending on the point set density. Functions \( f : V \rightarrow \mathbb{R}(E) \) are defined on the nodes of the graph while vector fields \( F : E \rightarrow \mathbb{R}(V) \) are defined on the edges. The gradient and divergence operators are defined as:

\[
G_{ij} = w_{ij}(f_i - f_j) \tag{14}
\]

\[
D(F_i) = \sum_{j \in \{i,j\} \in E} w_{ij}(F_j - F_{ji}) \tag{15}
\]

In order to deal with low-quality meshes, we adopted the approach of [Sharp et al. 2019]; a more detailed discussion with an experimental comparison is given in Appendix B.4.

5.2 Minimization of Eq.7

The energy to minimize in line 3 of Alg. 1 is convex, but non-differentiable. To efficiently solve it, we discretize the problem and use the primal-dual hybrid gradient algorithm [Chambolle and Pock 2011], which is well-suited for problems of this form. The optimization procedure for an input signal \( u \) is summarized in Algorithm 2: the proximal operator \( \text{prox}_{f^*} \) is the projection of each component

**Algorithm 2: Primal-Dual Hybrid Gradient (PDHG)**

input : \( u, q, \sigma, \tau > 0, \theta \in [0,1] \)

output : minimizer \( u^* \) of Eq. (7)

begin

while not converged do

\[
\begin{align*}
\bar{u}_{k+1} &= \text{prox}_{f^*}(u_k - \tau Dq_k) \\
\bar{q}_{k+1} &= \bar{u}_{k+1} + \theta(u_{k+1} - u_k) \\
q_{k+1} &= \text{prox}_{f^*}(q_k + \sigma G\bar{u}_{k+1})
\end{align*}
\]

end

return \( u^* = u_{k+1} \)

end

of its input onto the unit \( L^2 \) ball:

\[
\text{prox}_{f^*}(q_k) = \begin{cases} q_k : \frac{\|q_k\|}{\|q_k\|} \leq 1 \\ 0 \text{ otherwise } \end{cases}
\]

5.3 TV Flow

We now study the evolution of the surface \( M \) along the TV flow. A direct way to do this is to evolve the \((x, y, z)\) vertex coordinates of \( M \); the same approach is also used to define, e.g., the mean curvature flow (MCF) and its variants [Desbrun et al. 1999; Kazhdan et al. 2012; Taubin 1995]. This approach, however, gives rise to two main issues.

First, diffusing the vertex coordinates naturally affects the metric of \( M \) at each diffusion step; this, in turn, modifies the functional \( TV(f) = \int_M \|\nabla f\| \), since the integration domain \( M \) and the attached intrinsic operators now vary along the flow. Recomputing the metric-dependent operators at each iteration of the algorithm is a possible solution, but can be highly inefficient.

Secondly, one must take care of the choice of the TV regularization for \((x, y, z)\). In fact, for \( f \) being the coordinate function on the manifold, \( \nabla f(x) \) becomes a tensor for which a suitable norm \( \|\nabla f(x)\| \) in the integral of the total variation needs to be defined. Depending on such a choice one can encourage different types of collaborative gradient sparsities [Duran et al. 2016]. For example, by choosing a fully anisotropic TV (i.e., summing the absolute value of all entries in \( \nabla f(x) \)), the transform will be orientation-dependent: the diffused vertex coordinates will tend to align to the global reference frame; see Figure 8 for examples. While this latter point can be exploited for certain stylization tasks (as we show in Section 7), it might be an undesirable effect in many other applications.

6 Geometric TV Flow

This paper we advocate the use of normal fields for encoding the geometry. This choice has three main benefits: (1) the domain upon which the operators are defined remains fixed throughout the flow; (2) using normals endows us with rotation invariance; and (3) we get interesting connections with other existing flows. Normal fields in connection with the TV functional have been considered before, e.g. in [Zhang et al. 2015; Zhong et al. 2018]. However, these approaches...
do not consider or analyze the geometric flow as we do here, and do
not provide a natural and interpretable multiscale decomposition.

Our input signal is the normal vector field \( \vec{n}_0 \) over \( \mathcal{M} \). The field \( \vec{n}_0 \) is identi
dified by the image of the Gauss map \( \Gamma : \mathcal{M} \to S^2 \), mapping points on the surface to points on the unit 2-sphere (see inset). The corresponding time continuous flow is:

\[
\frac{d\vec{n}}{dt} = \text{div}_S \left( \frac{d\Gamma}{\text{vol} S^2} \right) \quad \text{in } S^2,
\]

where \( d\Gamma \) is the differential of the Gauss map and \( \text{div}_S \) is defined with the metric of \( \mathcal{M} \), but applied to tangent fields on \( S^2 \). By dif
fusing normals according to Eq. (18), the point distribution over \( S^2 \) changes to form well-separated clusters. Intuitively, the flow tends to align the directions of normal vectors to favor piecewise-constant changes to form well-separated clusters. Intuitively, the flow tends to align the directions of normal vectors to favor piecewise-constant changes to form well-separated clusters. Intuitively, the flow tends to align the directions of normal vectors to favor piecewise-constant changes to form well-separated clusters.

Interpreting this approach from a differential geometric perspec	view provides us with a useful characterization of the normal TV flow in terms of surface curvature. In particular, the TV functional for the normal field \( \vec{n} \) reads:

\[
TV(\vec{n}) = \int_{\mathcal{M}} ||d\Gamma(x)||dx = \int_{\mathcal{M}} \sqrt{k_1^2 + k_2^2} \, dx,
\]

where \( k_1 \) and \( k_2 \) are the principal curvatures at each point. The last equality follows from the fact that \( k_1, k_2 \) are the eigenvalues of \( d\Gamma \).

As we show below, the final expression in Eq. (19) resembles the functional related to other known flows in the literature. We also see from the same expression that the TV energy of the normal field does not depend on the orientation of the normals in \( \mathbb{R}^3 \); therefore, the resulting flow is invariant to global rotations of \( \mathcal{M} \).

\[E_A(u) = \int_{\mathcal{M}} dx,
\]

where \( u \) is now the \( (x, y, z) \) embedding in \( \mathbb{R}^3 \) of the surface \( \mathcal{M} \), and we assume \( \mathcal{M} \) has no boundary. The relation to our flow emerges by approximating the membrane energy with the Dirichlet energy

\[E_D(u) = \int_{\mathcal{M}} \text{div}(u) \cdot \text{grad}(u) \, dx,
\]

\[\text{div}(u) \cdot \text{grad}(u) = \frac{1}{2} \text{div}(u^2) = \frac{1}{2} \text{div}(|u|^2) - |u|^2 + u^T \text{grad}(|u|^2).
\]

Implementation. By working with normal fields \( \vec{n} \) we are moving to manifold-valued functions (the manifold being the unit sphere). In our tests we treat each \( \vec{n}(x) \) as an element of \( \mathbb{R}^3 \), and add a normalization step projecting \( \vec{n}(x) \) back onto \( S^2 \) at each iteration.

In the discrete setting, the normal vector field \( \vec{n} \in \mathbb{R}^{3 \times |F|} \) is constant within each triangle. Its gradient (needed in Algorithm 2, line 5) is understood as the channel-wise jump discontinuity over each edge in each triangle, and the associated discrete operator can be assembled as a \( |E| \times |F| \) matrix; see Appendix B.3 for details.

Recovering vertex positions. To recover a new embedding from the filtered normals \( \vec{n} \), we follow [Prada and Kazhdan 2015]:

\[
\min_{v \in |V|^{3 \times |F|}} \| Gv - w \|_F^2 + \epsilon \| v - v_0 \|_F^2,
\]

where \( v_0 \) are the old vertex coordinates, and \( w = Gv_0 - (Gv_0 \cdot n) \odot n \) is the orthogonal component of \( Gv_0 \) with respect to \( n \) (\( \langle \cdot, \cdot \rangle \), and \( \odot \) operate row-wise). Eq. (20) seeks the set of vertices whose gradient matches the vertex field \( w \), while at the same time staying close to the initial vertices. In our tests we set \( \epsilon = 10^{-4} \) to allow large deviations from the initial shape. Solving Eq. (20) boils down to solving a screened Poisson equation, giving the sparse linear system:

\[
v = (\epsilon A - L)^{-1} (\epsilon A v_0 - D w).
\]

Here, \( L = -DG \) corresponds to the standard linear FEM discretization of the Laplace-Beltrami operator.

6.2 Related curvature flows
We compare our flow to other geometric flows used in graphics.

Mean curvature flow. The normal TV flow is closely related to the MCF, which describes the evolution of a surface under the mini
mization of the area functional (membrane energy):

\[
E_A(u) = \int_{\mathcal{M}} dx,
\]

where \( u \) is now the \( (x, y, z) \) embedding in \( \mathbb{R}^3 \) of the surface \( \mathcal{M} \), and we assume \( \mathcal{M} \) has no boundary. The relation to our flow emerges by approximating the membrane energy with the Dirichlet energy

\[E_D(u) = \int_{\mathcal{M}} \text{div}(u) \cdot \text{grad}(u) \, dx,
\]

\[\text{div}(u) \cdot \text{grad}(u) = \frac{1}{2} \text{div}(u^2) = \frac{1}{2} \text{div}(|u|^2) - |u|^2 + u^T \text{grad}(|u|^2).
\]
Perhaps more interestingly, we observe that our feature-preserving flow in 3D is given in Figure 11. Sharp geometric features get smoothed in a "vertical" fashion, preserving the overall structure. For example, the chin and pony tail maintain their shape throughout, while the braid and nose slowly disappear without affecting the neighboring geometry.

$$E_A(u) \propto D(u) = \frac{1}{2} \int_M \| \nabla u \|^2, \text{ whose gradient flow is the PDE:}$$

$$\frac{\partial u}{\partial t} = \Delta u. \quad (23)$$

Compared to Eq. (8), this PDE uses the linear operator $\Delta = \text{div}\nabla$ instead of the nonlinear operator $\text{div}\nabla u$. Denoting with $H = \frac{k_1+k_2}{2}$ the mean curvature, and observing that $\Delta u = 2\bar{H}$, the above PDE corresponds to surface motion in the normal direction, with speed proportional to curvature $H$. Critical points of the functional are minimal surfaces, which for genus-0 convex shapes correspond to spheres.

In comparison, surface evolution along the normal TV flow collapses the surface to a flat shape, with the direction of motion perpendicular to the level sets of the normal vector field $\vec{n}$. By looking at the effect of the flow on $\vec{n}$, the key difference is that the MCF tends to move points on $\mathbb{S}^2$ so as to keep them equally spaced, while the normal TV flow generates well-separated point clusters.

We illustrate the comparison in 2D in Figure 10; an example of our feature-preserving flow in 3D is given in Figure 11.

**Willmore flow.** Perhaps more interestingly, we observe that our functional $TV(\vec{n})$ resembles the Willmore (or thin plate) energy [Bobenko and Schröder 2005], a higher-order version of the area functional, defined as:

$$E_W(u) = \int_M H^2 \, dx = \int_M (k_1^2 + k_2^2) \, dx - 2\pi \chi(M). \quad (24)$$

where again $u$ encodes the coordinate functions in $\mathbb{R}^3$. The last term is the constant Euler characteristic of $M$, and thus it does not affect the flow. Compared to Eq. (19), we see that the integrand only changes by a square root operation. In this sense, we interpret the normal TV flow as a sparsity-promoting variant (in the normal domain) of the Willmore flow.

The linearization of Eq. (24) leads to a fourth-order bi-Laplacian diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta^2 u. \quad (25)$$

Our problem also involves a fourth-order PDE with respect to the coordinates; however, it is hard to solve directly due to the nonlinearity of our operator. By phrasing the PDE in terms of the normals instead of the coordinates, we decouple the problem: first we integrate the TV flow in the space of normals, which corresponds to the second-order PDE of Eq. (18), and then we integrate again to recover the vertex coordinates via the screened Poisson equation.

In Table 1 we summarize the three flows. A qualitative comparison is shown in Figure 12, where we use stable variants of the existing flows that prevent singularities.

| Flow | Functional | PDE |
|------|------------|-----|
| MCF | $E_{\text{MCF}}(u) = \int_M dx$ | $\frac{\partial u}{\partial t} = \Delta u$ |
| Willmore | $E_{W}(u) = \int_M (k_1^2 + k_2^2) \, dx$ | $\frac{\partial u}{\partial t} = \Delta^2 u$ |
| TV  | $E_{TV}(\vec{n}) = \int_M \sqrt{k_1^2 + k_2^2} \, dx$ | $\frac{\partial u}{\partial t} = \text{div}\left(\frac{\vec{n}}{\|\vec{n}\|}\right)$ |

In particular, the $p$-Dirichlet energy with $p = 1$ corresponds to the TV functional, while for $p = 2$ we retrieve the standard Dirichlet energy. By studying the solution for the evolution along the $p$-Laplacian flow of a unit ball, we observe convergence to a cube for $p = 1$ (TV), to a sphere for $p = 2$ (Dirichlet energy) and to an octahedron for $p \approx 2$, which is a consistent behavior with the definition of $p$-norms (see Figure 13). From the example we see clearly that the TV flow tends to piecewise-flat solutions, and further observe that other $p$-Laplacian flows carry other properties that may be worth investigating in the future. We refer to [Bühler and Hein...].
The TV spectral framework lends itself well to a variety of applications for graphics; for each application, we compare with other approaches from the state of the art to better position our method in terms of achievable quality. While these results are intended to demonstrate the flexibility of the framework, they are by no means exhaustive and could serve as a basis for follow-up work.

### 7.1 Detail removal

Perhaps the most straightforward application is the direct filtering of signals and geometric features living on the given surface. One such example is shown in Figure 1, where we apply a band-pass \( p = 1 \) or \( p = 2 \) to an example of detail transfer. In Figure 15 we compare to five other top performing methods from the literature, specifically tailored to edge-preserving filtering, while we show a comparison with Laplacian-based filtering in Figure 17.

7.2 Detail transfer

A remarkable application of the spectral TV framework is to transfer sharp geometric details between shapes. Consider two surfaces \( M, N \) and a bijection \( \pi : M \rightarrow N \) between them. Further, let \( \phi_t \) and \( \psi_t \) be the TV spectral representations of the normal fields on \( M \) and \( N \) respectively. The idea is to transfer some components of \( \phi_t \) to \( \psi_t \) via \( \pi \), and then compute the inverse transform of \( \psi_t \) to obtain a new version of shape \( N \) with additional semantic details.

For this to work, recall that \( \phi_t : M \rightarrow \mathbb{R}^3 \) is a vector field on \( M \) for any given \( t \in \mathbb{R}_+ \), and similarly for \( \psi_t : N \rightarrow \mathbb{R}^3 \). Therefore, we can transfer details from \( M \) to \( N \) by evaluating the integral:

\[
\hat{u}(x) = \int t(\hat{\psi}_t(x) + I^b_u(\phi_t) \circ \pi^{-1}(x))dt 
\]

where \( [a, b] \) is the range of time scales left untouched by the filter; the normal field \( \bar{n} \) is decomposed by applying Algorithm 2 to each component of \( \bar{n} \) separately and coupling the channels by projecting on the unit sphere at each step. The above filter relies upon one main property of the TV decomposition: The separation of geometric details into different spectral bands, ensured by the underlying assumption that most interesting shapes have a sparse gradient. This simple filter already provides high-quality selective suppression of semantic detail while preserving sharp geometric features. In Figure 14 we compare to five other top performing methods from the literature, specifically tailored to edge-preserving filtering, while we show a comparison with Laplacian-based filtering in Figure 15.

### 7.3 Stylization

We further exploit some basic properties of the TV flow for artistic rendition of 3D shapes. In particular, we show how with a proper choice of the TV functional, we get the desired effect of mimicking the stylistic features typical of voxel art. A similar effect of "cubic stylization" was achieved in [Huang et al. 2014] for constructing...
Nonlinear Spectral Geometry Processing via the TV Transform

Fig. 14. Top row: Example of high-frequency detail removal, implemented as a low-pass filter on the spectral TV representation of the source shape (leftmost). We get a “rejuvenation” effect on the face model (second column), and compare our solution with state-of-the-art approaches, specifically tailored for edge-preserving mesh filtering (third to fifth column). Bottom row: To better highlight the differences, we plot the mean curvature (clamped to $[-30, 30]$ and color-coded, growing from blue to red) on the filtered surfaces. Our method is the most effective at removing wrinkles and obtains sharper feature lines compared to other approaches, which either tend to over-smooth or retain spurious vertex noise. A quantitative comparison is reported in Table 2.

Fig. 15. Comparison between low-pass filtering of a source shape (left) using 300 Laplacian eigenfunctions (middle), and with our spectral TV representation using just 11 spectral components out of 25 (right). The full shape has approximately 70K vertices. Laplacian-based smoothing correctly removes details, but also has a smoothing effect on the rest of the shape. In contrast, our approach correctly preserves the underlying structure.

Fig. 16. Example of detail transfer. The wrinkles on the source shape $X$ (left) are captured with a high-pass filter in the spectral TV decomposition of $X$, added to the spectral representation of the target (middle), and finally resynthesized (right). The TV decomposition allows to inject tiny geometric details into the target, without affecting the other bands (e.g., nose, mouth and ears are not affected by the transfer). Note that source and target shape do not have the same mesh connectivity, nor the same number of vertices.

7.4 Denoising and enhancement

Finally, we demonstrate the application of the spectral TV framework to more classical tasks of geometry processing, namely surface denoising and enhancement. For these tests, we use both triangle meshes and point clouds. Once the spectral decomposition is computed, filtering and reconstruction can be performed in real time, as shown in the accompanying video.

Fairing. When dealing with point clouds, denoising and fairing tasks are notoriously challenging due to the difficulty of distinguishing feature points from outlier noise. This confusion often requires pre-processing steps to localize sharp features and filter out the rest. In our setting, the edge-aware property of the TV functional...
We formulate denoising as a simple low-pass filter in the spectral domain. Our method also supports localized operations by means of masks defined well separated in the spectrum; differently, the Laplacian-based filtering approach of [Chuang et al. 2016] amplifies high-frequency components in the spectral representation of a signal. Together with the jump-preserving property, one gets the formality of frequency bands that tend to cluster together features with similar semantics. For example, the hair strands on a head model can be captured sharply. The general approach to mesh filtering that we described also retracts the steps of classical pipelines adopted in the linear case (transform – filter – resynthesize), making our nonlinear framework more accessible and of potentially broader impact.

8 DISCUSSION AND CONCLUSIONS

We presented a new spectral processing framework for surface signals and 3D geometry. The framework is based on ideas developed in the last decade within the nonlinear image processing community, but whose extension and application to geometric data have been lagging behind to date. The main contribution of this paper is the extension of denoising of point clouds, where we compare with the recent bilateral filtering approach of [Digne and de Franchis 2017]. On the top-right of each point cloud, we plot its normal field on the unit sphere. Points are color-coded according to their cosine similarity (light to dark red, with values reported in the figure) with the ground truth normals (in solid cyan).

**Enhancement.** While denoising can be seen as a suppression of high-frequency components in the spectral representation of \( \hat{\mathbf{n}} \), one can similarly obtain feature enhancement by amplifying those components; see Figure 20 for an example. Here we compare to the gradient-based mesh processing approach of [Chuang et al. 2016], which implements feature sharpening and smoothing by scaling the gradient of coordinate functions and solving for new coordinates via a Poisson equation. Conceptually, our spectral TV representation allows for more accurate control on specific features, since these are well separated in the spectrum; differently, the Laplacian-based approach adopted in [Chuang et al. 2016] encodes geometric features by spreading them out across the entire spectrum.

**Localized operations.** We conclude by mentioning that our framework also supports localized operations by means of masks defined over regions of interest, which could be useful in view of an interactive application for mesh editing (see the supplementary video). Let \( R \subset M \) be a region of \( M \), and let \( I \) be an arbitrary spectral filter. Then, for a signal \( u \) with spectral components \( \hat{u}_l \), localized spectral filtering can be carried out simply by evaluating:

\[
\tilde{u}(x) = \rho(x) \int_M tI(\hat{u}_l(x))dt + (1 - \rho(x))u(x),
\]

where \( \rho : M \rightarrow [0, 1] \) is an indicator function for \( R \).

---

**Fig. 17.** Replacing the inscription from a source (top left) to a target model (top right), using Laplacian coating transfer [Sorkine et al. 2004] (bottom left), and our detail transfer approach (bottom right). The smoothing applied to the target causes the former approach to fail at preserving edges, while the transfer of high frequencies in the TV spectrum preserves them. The target model was obtained by applying our cubic stylization method of Section 7.3 to the source model before superimposing the inscriptions.

**Fig. 18.** The oriented point cloud on the top left (obtained by additive Gaussian noise on the vertex coordinates) is denoised by applying a low-pass filter to the spectral TV representation of its normal field (bottom left). On the top-right of each point cloud, we plot its normal field on the unit sphere. Points are color-coded according to their cosine similarity (light to dark red, with values reported in the figure) with the ground truth normals (in solid cyan).
Fig. 19. Example of cubic stylization. The scene on the left is evolved along the anisotropic normal TV flow (i.e., each dimension of the normal vectors is treated as a separate scalar function, and these are regularized jointly via the sum of anisotropic TV penalties), which forces the normals to align with the global coordinate axes; the final result is shown on the right. Each shape was processed individually, and then placed in the scene.

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Fig. 20. Top row: Spectral TV enhancement of a lion-shaped vase. Assigning a larger weight to the high-frequency spectral components has an effect of pronouncing the details of the 3D model, while preserving the sharp features. Bottom row: Comparison with [Chuang et al. 2016]. Their method comes in two variants. On the left, their direct sharpening results in a noisy mesh. On the right, feature enhancement is alternated with smoothing 3 times, at the cost of detail loss (e.g. around the whiskers).

...the adoption of learning-based techniques to predict task-specific filters, as done in [Moeller et al. 2015] for image denoising. More generally, the integration within geometric deep learning pipelines is a promising direction that we are eager to pursue. We also anticipate that the study of the more general p-Laplacian operator on surfaces might lead to a richer class of algorithms for geometry processing. One promising idea is to define the operator adaptively, selecting a value for p depending, e.g., on curvature. Finally, a more in-depth analysis of the normal TV flow as a robust alternative for mesh fairing is essential, due to its feature preservation properties and its connection to other geometric flows.
A MATHMATICAL DERIVATIONS

A.1 Proof of Theorem 1

For functions defined on Euclidean spaces (as opposed to surfaces, which we prove here), we refer to [Fleming and Rishel 1960].

THEOREM. Let \( E \subset M \) be a measurable closed subset of surface \( M \), with smooth boundary \( \partial E \), and let \( \chi_E \) be its indicator function. Then:

\[
TV(\chi_E) = \text{length}(\partial E).
\]

PROOF. We split the proof by first proving \( TV(\chi_E) \leq \text{length}(\partial E) \) and then \( TV(\chi_E) \geq \text{length}(\partial E) \). Let \( V \) be the tangent vector field that attains the supremum in the weak definition of \( TV \) (see Eq. (4)), and let \( n \) be the unit normal vector to \( \partial E \). We have:

\[
TV(\chi_E) = \int_M \chi_E \div V \, dx = \left| \int_M \chi_E \div V \, dx \right| = \left| \int_E \div V \, dx \right|
\]

We now apply the divergence theorem and Cauchy-Schwarz inequality:

\[
\left| \int_{\partial E} (V, n) \, d\ell \right| \leq \int_{\partial E} |(V, n)| \, d\ell \leq \int_{\partial E} 1 \, d\ell = \text{length}(\partial E).
\]

For the other direction, we pick an arbitrary vector field \( U, s.t. \ |U| \leq 1 \) and \( U = n \) on \( \partial E \). This vector field is guaranteed to exist since \( \partial E \) is smooth. Then, applying divergence theorem:

\[
\int_M \chi_E \div V \, dx = \int_E \div U \, dx = \int_E (U, n) \, d\ell = \int_{\partial E} 1 \, d\ell = \text{length}(\partial E).
\]
Finally, from Eq. (4) we get that:

$$\text{length}(\partial E) = \int_M \chi_E \text{div} \phi d\text{vol} \leq \sup_{|\phi| \leq 1} \int_M \chi_E \text{div} \phi d\text{vol} = TV(\chi_E).$$

Therefore, it follows that:

$$TV(\chi_E) = \text{length}(\partial E).$$

\[\square\]

**B.1 Inverse scale space**

The inverse flow starts from the projection of the input signal \(u_0\) into the kernel of the TV (yielding the mean of \(u_0\)) and converges in an inverse fashion to \(u_0\) according to the PDE:

\[
\begin{align*}
\partial_t p(t) &= u_0 - u(t) \\
p(t) &\in \partial TV(u(t)) \\
p(0) &= 0
\end{align*}
\]  

(Eq. 33)

The inverse flow shares many properties with the forward TV flow and has finite time extinction (in finite dimensions after inverting time \(t \to 1/t\) to account for the inverse nature of the flow; see [Burger et al. 2016, Proposition 5]). The full algorithm is summarized below. Other than reaching stationarity in an inverse fashion, the main difference with the forward flow is that the solution has a piecewise-constant behavior in time, as opposed to the linear behavior for the forward case, as seen in Section 4.3. This implies that it is sufficient to take a single derivative of the primal variable \(u(t)\) to compute a spectral component \(\phi_t\) (line 7 of Algorithm 3).

**Algorithm 3:** Inverse scale decomposition

**Input:** signal \(u_0\), max diffusion time \(\alpha\), no. of time steps \(N\)

**Output:** spectral representation \(\phi_t\)

1. begin
2. \(v(0) \leftarrow 0\)
3. \(u(0) \leftarrow \text{mean}(u_0)\)
4. for \(t = 0 : N\) do
5. \(u(t+1) \leftarrow \text{arg min}_u \text{TV}(u) + \frac{1}{2\Delta t} \|u - (u_0 + v(t))\|^2_2\)
6. \(v(t+1) \leftarrow v(t) - (u_0 - u(t+1))\)
7. output \(\phi_t(t) \leftarrow u(t+1) - u(t)\)
8. end
9. end

**B.2 Forward vs inverse flow and the role of \(\alpha\)**

We show the discretization of the forward and inverse flows and the computation of the related time scales, to provide a better understanding of the inverse relation w.r.t. time between the two and to give better insight on the role of the regularization parameter \(\alpha\). The inverse flow is given by Eq. (33). Its discretization yields:

\[
\frac{p(t^{k+1}) - p(t^k)}{\Delta t} = u_0 - u(t^{k+1})
\]

which – due to \(p(t) \in \partial TV(u(t))\) – is the optimality condition to:

\[
\min_u \frac{1}{2} \|u - u_0 - \frac{1}{\Delta t} p(t^k)\|^2 + TV(u).
\]

On the other hand, the update formula in our Algorithm 3 solves:

\[
u_{\text{new}} \leftarrow \arg \min_u \frac{1}{2\Delta t} \|u - (u_0 + v(t))\|^2_2 + TV(u).
\]

Therefore, we identify \(\alpha\) with \(1/\Delta t\), and \(v(t)\) with \(\frac{1}{\Delta t} p(t^k)\). For the latter it is important to look at the update equation for \(p(t^{k+1})\):

\[p(t^{k+1}) = p(t^k) - \Delta t \frac{1}{\Delta t} p(t^k)\]

Dividing this equation by \(\Delta t^{k+1}\) yields:

\[
\frac{1}{\Delta t^{k+1}} p(t^{k+1}) = \frac{1}{\Delta t^{k+1}} p(t^k) - \frac{\Delta t^k}{\Delta t^{k+1}} (u_0 - u(t^{k+1})).
\]

Now inserting our relation of \(v\) and \(p\) we obtain:

\[v(t^{k+1}) = \frac{\Delta t^k}{\Delta t^{k+1}} v(t^k) - \frac{\Delta t^k}{\Delta t^{k+1}} (u_0 - u(t^{k+1})).
\]

This is the update formula in line 6 of Algorithm 3 in the case of variable step sizes. Finally the discretization in Eq. (33) means that:

\[t^{k+1} = t^k + \Delta t^k = \sum_{i=0}^k \Delta t^k.
\]

Thus, considering quantities on a continuous time scale we interpret:

\[u(t^{k+1}) = u(t^0 + \sum_{i=0}^k \Delta t^k).
\]

Interestingly, the inverse scheme is extremely forgiving in terms of truncation errors given by the finite number of iterations in Alg. 2, and does not accumulate them as proven by Yin and Osher in [2013]. Therefore, one could also consider to trade computational efficiency against error by relying on the latter property of the inverse flow.

**The forward flow** (Eq. 8) is given by:

\[
\begin{align*}
\frac{\partial p}{\partial t}(t) &= -p(t), \\
p(t) &\in \partial TV(u(t)), \\
u(0) &= u_0.
\end{align*}
\]

The implicit Euler discretization of such a flow yields:

\[\frac{u(t^{k+1}) - u(t^k)}{\Delta t^{k+1}} = -p(t^{k+1}) \Rightarrow 0 = \frac{1}{\Delta t^{k+1}} (u(t^{k+1}) - u(t^k)) + p(t^{k+1})\]

which is the optimality condition to:

\[\min_u \frac{1}{2\Delta t^{k+1}} \|u - u(t^k)\|^2 + TV(u).
\]

Comparing this to our Algorithm 1, we have to identify \(\alpha\) with \(\Delta t^k\). Again, the discretization yields:

\[t^{k+1} = t^k + \Delta t^k = \sum_{i=0}^k \Delta t^k.
\]
We expose four parameters: the step size $\alpha$, the number of spectral components $N$, the interior step sizes $\sigma$, and the resolution $\tau$. We can choose arbitrarily small or large values for $\alpha$. This determines the resolution of the spectral representation; to get a good separation of features, one should increment $\alpha$ according to a non-uniform sampling of the spectral domain, with higher sampling density within the bands having stronger spectral activity. In practice, following Algorithm 3, we proceed by estimating the maximum diffusion time by choosing the smallest value for $\alpha$ that makes the fidelity term negligible in the following energy:

$$
\frac{1}{2\alpha} \int_M (u(x) - \text{mean}(u))^2 dx + TV(u).
$$

For such an $\alpha$, minimizing the energy above corresponds to minimizing the total variation alone. This ensures that no spectral activity exists beyond the diffusion interval $(0, \alpha)$. Once the maximum possible $\alpha$ is estimated, we rescale this value as $\alpha \leftarrow C\alpha$, at each iteration (with $C \in [0, 1]$, typically $C = 0.7$). Decreasing $\alpha$ by a different schedule would still ensure a perfect reconstruction of the input signal; no details are ever lost in the analysis and synthesis process. Instead, the specific choice affects the resolution of the spectral representation; to get a good separation of features, one should increment $\alpha$ according to the strategy above.

### B.3 Edge-based gradient

The gradient operator for piecewise-constant signals defined on mesh triangles can be defined, for each pair of adjacent triangles, as the jump discontinuity across the shared edge. The operator is assembled as a $|E| \times |F|$ matrix $G_e$, with the same zero pattern as the edge-to-triangle adjacency matrix; it contains $\pm 1$ as its values, s.t. each row sums up to zero. The divergence operator is defined as:

$$
D_e = -T^{-1} G_e^T A_e ,
$$

where $A_e$ is a diagonal matrix of edge lengths, i.e., the area elements for this function space, and $T$ is the matrix of triangle areas.

### B.4 Parameters and stability

We expose four parameters: the step size $\alpha$, the number of spectral components $N$ (both used in Algorithm 1), and the interior step sizes $\sigma$, $\tau$ in Algorithm 2. The only parameter which requires tuning is $\alpha$, while the others can be set automatically, accordingly.

**Step size $\alpha$**. Since Algorithms 1 and 3 are **unconditionally stable**, we can choose arbitrarily small or large values for $\alpha$. This determines the resolution of the spectral representation; to get a good separation of features, one should increment $\alpha$ according to a non-uniform sampling of the spectral domain, with higher sampling density within the bands having stronger spectral activity. In practice, following Algorithm 3, we proceed by estimating the maximum diffusion time by choosing the smallest value for $\alpha$ that makes the fidelity term negligible in the following energy:

$$
\frac{1}{2\alpha} \int_M (u(x) - \text{mean}(u))^2 dx + TV(u).
$$

For such an $\alpha$, minimizing the energy above corresponds to minimizing the total variation alone. This ensures that no spectral activity exists beyond the diffusion interval $(0, \alpha)$. Once the maximum possible $\alpha$ is estimated, we rescale this value as $\alpha \leftarrow C\alpha$, at each iteration (with $C \in [0, 1]$, typically $C = 0.7$). Decreasing $\alpha$ by a different schedule would still ensure a perfect reconstruction of the input signal; no details are ever lost in the analysis and synthesis process. Instead, the specific choice affects the resolution at which the spectral components are extracted, and depends on the specific application, as briefly discussed in the limitations paragraph of Section 8.1.

**Number of spectral components $N$**. This parameter can be determined automatically, since it is equivalent to the number of rescalings of $\alpha$ required to cover the interval $(0, \alpha)$, assuming $\alpha$ is chosen according to the strategy above.

**Interior step sizes $\sigma$, $\tau$**. To ensure stability, these parameters must be chosen so as to satisfy:

$$
\tau \sigma \leq \min \{\|e\| \in E \} \|G\|_0^2 ,
$$

where $\|G\|_0 = \sup \{ \|Gu\|_{\|u\|} : u \in L^2(M) \}$ is the operator norm, and the numerator denotes the minimum edge length on the mesh. We estimate $\|G\|_0$ with the **normest** function in Matlab and set $\sigma = \tau$.

**Mesh quality.** Like most geometry processing algorithms, our spectral decomposition is susceptible to low-quality input meshes.

---

**Example |V| |F| N | Time | Rec Time**

| Figure 4 | 6079 | 2000 | 50 | 12.9 s | - |
| Figure 6 | 2590 | 2000 | 40 | 3.9 s | - |
| Figure 14 | 23308 | 45953 | 30 | 16.59 s | 0.16 s |
| Figure 15 | 44313 | 86622 | 25 | 31.27 s | 0.46 s |
| Figure 1 | 157056 | 314112 | 30 | 180.5 s | 2.50 s |
| Figure 20 | 50002 | 100000 | 20 | 23.69 s | 0.41 s |
| Figure 9 | 50002 | 100000 | 20 | 28.5 s | 0.52 s |

Table 3. Table of computational times for spectral TV. Left to right: input, number of vertices, number of faces, number of spectral components computed, total time for computing the spectral decomposition, total time for recovering vertex coordinates, when needed (i.e., solution of Eq. (20)).

Tiny angles cause an increase in the gradient norm $\|G\|_0$, leading to a decrease in convergence speed of the diffusion process. To address this, we adopt the approach of [Sharp et al. 2019], which decouples the triangulation used to describe the domain from the one used in the decomposition algorithm. In the inset figure, we show an example where degenerate triangles (visible from the bottom of the drill, bottom row) cause distortion when we run the TV flow on the normal field (blue curve). With intrinsic triangulations (red curve), we get the expected results.

### C. RUNTIME STATISTICS

The time performance of the spectral decomposition depends on a few factors: complexity of the domain, time steps, and number of spectral components to compute. A single iteration of Alg.(2) has a convergence rate of $\frac{1}{\tau}$ w.r.t. the primal-dual gap (see [Chambolle et al. 2010]). The number of iterations to bring the gap error below a desired threshold however depends on the internal step sizes $\sigma$, $\tau$ (see Appendix B.4). To provide a good intuition for the actual performance we report in Table 3 the runtimes for some of the examples showcased in the paper. The computations were done in Matlab with an unoptimized single-threaded implementation, setting the error threshold of PDHG to $10^{-4}$ and a maximum number of iterations to 1000 and 300, for scalar and vector signals respectively.