Discrete-time risk sensitive portfolio optimization with proportional transaction costs

Marcin Pitera\(^1\) | Łukasz Stettner\(^2\)

\(^1\)Institute of Mathematics, Jagiellonian University, Krakow, Poland
\(^2\)Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

Abstract

In this paper we consider a discrete-time risk sensitive portfolio optimization over a long time horizon with proportional transaction costs. We show that within the log-return i.i.d. framework the solution to a suitable Bellman equation exists under minimal assumptions and can be used to characterize the optimal strategies for both risk-averse and risk-seeking cases. Moreover, using numerical examples, we show how a Bellman equation analysis can be used to construct or refine optimal trading strategies in the presence of transaction costs.

KEYWORDS

Bellman equation, long time horizon, portfolio optimization, risk sensitive portfolio, risk sensitive criterion, risk sensitive control, transaction costs

1 INTRODUCTION

Quantitative portfolio management is an important part of mathematical finance. Stimulated by the seminal work Markowitz (1952), this field has been consistently evolving during the last 70 years for both discrete and continuous time settings, see Kolm et al. (2014); Chandra (2017); Prigent (2007) and references therein for an overview. Among the considered portfolio optimization frameworks, the risk sensitive portfolio optimisation is one of the most recognized ones, see Davis and Lleo (2014); Bielecki and Pliska (1999); Fleming and Sheu (2000). Given a wealth process \(W_t\) and the risk-averse parameter \(\gamma \neq 0\), the long-run version of the risk sensitive criterion is defined as

\[
\liminf_{t\to\infty} \frac{1}{t} \frac{1}{\gamma} \ln \mathbb{E}[W_t^\gamma] \quad \text{or equivalently as} \quad \liminf_{t\to\infty} \frac{1}{t} \mu(t)(\ln W_t),
\]
where $\mu^\gamma(\cdot) := 1/\gamma \cdot \ln \mathbb{E}[\exp(\gamma(\cdot))]$ is the entropic utility. The optimality criterion presented in (1) measures the long-run normalized entropy of the log-wealth and could be seen as a non-linear extension of the Kelly criterion, see MacLean et al. (2011); Campbell and Viceira (2002a); Davis and Lleo (2021). It should be noted that the extension (1) is in fact unique within the class of cash-additive and strongly time-consistent certainty equivalents which explains why the usage of entropy is so common in multiple stochastic control applications, see Kupper and Schachermayer (2009); Cherny and Maslov (2004) for details. In fact, the risk sensitive criterion appears naturally in many portfolio investment problems and is linked to various optimality frameworks. For completeness, let us provide some examples. First, by considering the second order Taylor’s expansion of the entropic utility, around $\gamma = 0$, we get

$$\mu^\gamma(\ln W_t) = \mathbb{E}[\ln W_t] + \frac{\gamma}{2} \text{Var}[\ln W_t] + O(\gamma^2, t), \quad t > 0,$$

(2)

which shows that, for $\gamma < 0$, the risk sensitive framework is related to the mean-variance Markowitz portfolio optimization that allows time-consistent utility treatment, see Bielecki and Pliska (2003). Second, (1) is directly linked to so-called equivalent safe rate, which reflects the minimal hypothetical safe rate that would encourage the investor to invest in the risky portfolio, see Guasoni et al. (2019). Third, for $\gamma < 0$, the risk-sensitive criterion is dual to the downside risk, which is a common investment criterion in the long-run portfolio optimization, see Nagai (2012) or Pham (2015) for details. Fourth, for $\gamma > 0$, the maximization of (1) is related to the studies of power utility asymptotics and can be considered as a dual problem to upside chance probability, see Pham (2015) and Stettner (2011). Fifth, let us remark that risk sensitive criterion is an acceptability index (also called performance measure) and has many economically desirable properties, see for example, Cherny and Madan (2009); Bielecki et al. (2016). We refer to Bielecki and Pliska (2003) for an overview of economic properties of risk sensitive criterion made in reference to portfolio management. Finally, we want to recall the the entropic utility is tightly linked to a well-established utility theory in financial economics. In particular, we get the representation $\mu^\gamma(X) := U_{\gamma}^{-1}\mathbb{E}[U_\gamma(X)]$, where $U_\gamma(x) := (\exp(\gamma x) - 1)/\gamma$, which shows the connection between entropic utility and the exponential utility. When applied to the process log-wealth, one can see that $(U_{1-\gamma} \circ \ln)(x) = (x^{1-\gamma} - 1)/(1 - \gamma)$, which illustrates that the risk sensitive criterion is linked to the constant relative risk aversion (CRRA) utility, see Ingersoll (1987).

In this paper, we decided to follow the risk aversion naming convention introduced in Whittle (1990) in which the risk-averse ($\gamma < 0$), risk-neutral ($\gamma = 0$), and risk-seeking ($\gamma > 0$) case is motivated by the variance term penalty sign as presented in (2). This is motivated by the fact that that entropic utility exhibits different convexity behavior in those three environments that allows transparent presentation of our results for the full domain of risk aversion parameter. That saying, it should be emphasized that in the utility theory and portfolio optimization, most focus is set on the case when $\gamma \in (-\infty, 0) \cup (0, 1)$ and slightly different naming convention in reference to portfolio optimization is often used, see for example, Davis and Lleo (2021). Namely, the case $\gamma \to 0$ yields so-called log-optimal portfolio (Kelly portfolio), $\gamma \in (0, 1)$ corresponds to so-called overbetting (riskier than Kelly) portfolios, the case $\gamma > 1$ reflects the risk-seeking behavior and is often excluded for the analysis, while $\gamma < 0$, as in our case, is linked to a risk averse portfolio. Within this convention $\gamma = 1$ is often referred to as the risk-neutral case.

The main aim of this paper is to show that under the i.i.d. property imposed on asset’s log-returns one can solve a suitable (long-run) risk-sensitive Bellman equation under proportional transaction costs and lack of short selling. This contributes to a general theory of portfolio
management under transaction costs. In particular, the discrete time risk-sensitive portfolio optimization with proportional transaction costs was studied in a number of papers, see for example, Stettner (2005) where the discounted problem was studied, Bobryk and Stettner (1999) and Quek and Atkinson (2017), where the finite time horizon was considered, or Atkinson and Storey (2010), where risk-sensitive proportional transaction cost framework was studied under stronger assumptions. For a more general discussions about portfolio optimisation in the presence of transaction costs we refer to Davis and Norman (1990); Liu and Loewenstein (2002); Korn and Laue (2002); Muthuraman and Kumar (2006); Czichowsky and Schachermayer (2016); Czichowsky et al. (2018); Guo et al. (2021) and references therein.

We emphasize that the set of additional assumptions imposed on log-returns in this paper is minimal, that is, we only require that asset’s log-returns have finite mean and entropy. While this might be counter-intuitive at first sight, as one typically impose strong ergodic assumption on the process in order to get the existence of risk-sensitive Bellman equation solution, the i.i.d. property proves to be a plausible alternative. For an overview of the key results, we refer to Theorem 4.2, Theorem 4.3, and Theorem 5.3.

The results of this paper are presented in a self-contained entropy-based way to streamline the economic context; we hope this makes the paper more transparent and accessible to the generic mathematical finance community. That said, the results presented here are in fact linked to an extensive literature on the risk sensitive stochastic control and optimization, and are expanding this framework in reference to portfolio management, see Bielecki and Pliska (1999); Pitera and Stettner (2016). That is why in some cases we decided to present alternative formulations of the Bellman equations, to link them more directly with the (controlled) Multiplicative Poisson Equation framework.

Our work is also linked to a variety of problems studied for Markov decision processes (see e.g., Bäuerle & Jaśkiewicz (2018)) and recently studied continuous time risk sensitive problems with regime switching over finite time horizon, see Bo et al. (2019); Hata (2018); Das et al. (2018). In particular, we want to mention that the proof techniques presented in this paper are based on some novel ideas applied to vanishing-discount and span-contraction approaches, cf. Cavazos-Cadena and Hernández-Hernández (2017); Shen et al. (2013). For instance, we were able to weaken the typical assumption imposed on the negative value of the log-process, by replacing Schwarz’s inequality based approximation with a tail-based argument in one of the key steps of local contraction property proof, see Proposition 5.2. Also, by incorporating Arzelà-Ascoli theorem into the vanishing discount approach, we were able to show the existence of a regular Bellman solution.

Apart from theoretical results, we present two numerical examples. They might be interesting to a reader who is not familiar with the risk sensitive stochastic control but wants to better understand why the study of Bellman equation could improve trading performance even in a very simplistic case. In particular, by using simple approximation schemes, one can directly recover no-action strategies that are important aspect of portfolio management in the presence of transaction costs, see Czichowsky and Schachermayer (2016). This shows why the development of efficient risk sensitive approximation algorithms in the dynamic context can help to develop or benchmark trading strategies; see Fei et al. (2021); Basu et al. (2008); Borkar (2010); Arapostathis et al. (2021) where practical aspects linked to risk sensitive policy iteration algorithms are studied and Li and Hoi (2014) for an overview of online portfolio selection procedures. The optimal strategy existential results presented in this paper further motivate the adequacy of application of the aforementioned advanced numerical methods for portfolio optimization in the long-run setting under transaction costs. Note that long-run dynamics analysis often leads to efficient trading strategies,
which have consistent risk propagation. In particular, the long-run strategies are invariant to portfolio maturity, a quantity which is often unknown and which pre-setting could have a substantial impact on the trading strategy, see Campbell and Viceira (2002b) for details.

This paper is organized as follows. In Section 2, we provide the general setup, state the assumptions, and formulate suitable Bellman equations. Next, in Section 3 we focus on the discounted version of the problem, that paves the ground for the usage of the vanishing discount approach. In Section 4 we follow the vanishing discount approach in order to show the key results of this paper. Then, in Section 5 we switch to the span-contraction approach in order to strengthen the results presented in Theorem 4.3 and show how to utilize the local contraction property in the i.i.d. setting. Finally, in Section 6 we present numerical examples.

2 | PROBLEM FORMULATION

Let \((\Omega, F, (F_t)_{t \in \mathbb{N}}, \mathbb{P})\) be a discrete-time filtered probability space. Let \(d \in \mathbb{N}\) denote the number of available risky assets and let \(S(t) := (S_1(t), \ldots, S_d(t))\) denote the positive vector price process, where \(S_j(t)\) denotes the price of the \(j\)th risky asset at time \(t \in \mathbb{N}\). For a given trading strategy, we use \(N(t) = (N_1(t), \ldots, N_d(t))\) to denote the portfolio asset volume vector at time \(t\) after the portfolio is rebalanced, that is, \(N_i(t)\) denotes how much asset \(S_i\) we hold in our portfolio at time \(t\) after the rebalancing is executed. Also, we use

\[
W(t-) := \langle N(t-), S(t) \rangle \quad \text{and} \quad W(t) := \langle N(t), S(t) \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the standard scalar product, to denote portfolio wealth process at time \(t\) before and after the rebalancing, respectively. Throughout the paper we assume absence of short selling and follow the proportional transaction cost framework. This is partly encoded in the self-financing condition that is given by

\[
W(t) = W(t-) - d((N(t) - N(t-1)) \cdot S(t)),
\]

where \((\cdot)\) denotes vector point-wise product, function \(d : \mathbb{R}^d \to \mathbb{R}_+\) is the proportional transaction cost penalty given by

\[
d(x) := \langle c, [x]^+ \rangle + \langle h, [x]^\circ \rangle, \quad x \in \mathbb{R}^d,
\]

for a fixed cost rates \(c, h \in \mathbb{R}^d\) such that \(0 < c_j, h_j < 1, \ j = 1, 2, \ldots, d\), and \([x]^\pm\) denotes (component-wise) positive/negative part of \(x\). To ease the notation, we introduce portfolio loading factors (portions of the capital invested in the assets, weights) vectors given by

\[
\pi(t) := \frac{N(t) \cdot S(t)}{W(t)} \quad \text{and} \quad \pi(t-) := \frac{N(t-1) \cdot S(t)}{W(t-)}.
\]

Note that due to the absence of short selling, for any \(t \in \mathbb{N}\) and \(\omega \in \Omega\), we have \(\pi(t) (\omega), \pi(t-) (\omega) \in S\), where \(S := \{x \in \mathbb{R}^d : x_j \geq 0; \langle x, 1 \rangle = 1\}\). Before we introduce the objective function, let us present a short technical lemma which shows that capital decay \(W(t)/W(t-)\) can be expressed as a function of factor loadings.
Lemma 2.1. There is \( \mathbf{s} \in (0,1) \) and continuous function \( s : S^2 \mapsto [\mathbf{s}, 1] \) such that

\[
\frac{W(t)}{W(t-)} = s(\pi(t-), \pi(t)), \quad t \in \mathbb{N}.
\]

Proof. Let \( F : \mathbb{R}_+ \times S^2 \to \mathbb{R} \) be a function given by \( F(w, x, y) := w + d(wy - x) \) and let \( \bar{e} := \min_{j=1, \ldots, d} h_j \). Noting that \( F \) is continuous, strictly increasing in \( w \), \( \bar{e} \leq F(0, x, y) \leq \max_{j=1, \ldots, d} h_j < 1 \), and \( F(1, x, y) > 1 \), we know that there exists function \( \bar{s} : S \to [\bar{s}, 1] \) such that \( F(\bar{s}(\pi(t-), \pi(t)), \pi(t-), \pi(t)) = 1 \). On the other hand, using self-financing condition (4) we get

\[
F\left( \frac{W(t)}{W(t-)}, \pi(t-), \pi(t) \right) = \frac{W(t)}{W(t-)} + d\left( \frac{W(t)}{W(t-)} \pi(t) - \pi(t-) \right) = 1.
\]

Since \( F \) is strictly increasing with respect to \( w \) we know that \( s(\pi(t-), \pi(t)) = \frac{W(t)}{W(t-)} \). It remains to show that \( s \) is continuous. Let \( S \ni \pi_n, \pi'_n \) be such that \( \pi_n \to \pi \) and \( \pi'_n \to \pi' \), as \( n \to \infty \). Recalling that \( F \) is continuous, strictly increasing in its first argument and satisfies \( F(s(\pi_n, \pi'_n), \pi_n, \pi'_n) = 1 \) as well as \( F(s(\pi, \pi'), \pi, \pi') = 1 \), we conclude that for any subsequence \( (n_k)_{k \in \mathbb{N}} \) such that \( s(\pi_{n_k}, \pi'_{n_k}) \to \bar{s} \), for some \( \bar{s} \in [\mathbf{s}, 1] \), we get \( s = s(\pi, \pi') \). Since the same limit \( s(\pi, \pi') \) is achieved for any subsequence \( (n_k) \), we get continuity of \( s \). \( \square \)

From Lemma 2.1 we see that the trading strategy could be represented via the loading factors (5). For any given \( S \)-valued (adapted) strategy \( \pi \) we use \( W_\pi \) to denote the corresponding wealth process.

The main goal of this paper is to find strategy \( \pi \) that maximizes long run risk sensitive objective function, applied to log-wealth process. Namely, we fix a risk-sensitivity parameter \( \gamma \in \mathbb{R} \setminus \{0\} \) and consider the objective function given by

\begin{align}
J(\pi) := \liminf_{n \to \infty} & \frac{1}{n} \ln \mathbb{E}[W_\pi(n-)^\gamma] \\
= \liminf_{n \to \infty} & \frac{\mu^\gamma(\ln W_\pi(n-))}{n} \\
= \liminf_{n \to \infty} & \frac{1}{n} \mu^\gamma \left( \sum_{t=0}^{n-1} \ln \frac{W_\pi((t+1)-)}{W_\pi(t-)} \right),
\end{align}

where \( \mu^\gamma(X) := \frac{1}{\gamma} \ln \mathbb{E}[e^{\gamma X}] \) is the entropic utility function; for consistency, we also use limit notation \( \mu^0(X) := \mathbb{E}[X] \). Note that (6) is measuring time averaged entropy of portfolio’s log-return; see Bielecki and Pliska (2003) for the economical context.

Since we are interested in optimizing portfolio’s log-growth, throughout this paper we assume that the assets log-return vector \( r(t) = (r_i(t))_{i=1}^d \), where \( r_i(t) := \ln \frac{S_i(t)}{S_i(t-1)} \), is an i.i.d. vector satisfying conditions

\[ |\mu^\gamma(r_i(t))| < \infty \quad \text{and} \quad |\mathbb{E}[r_i(t)]| < \infty \quad \text{for} \quad i = 1, 2, \ldots, d, \quad (A.1) \]

which means that log-returns are integrable and have finite entropy for the prefixed risk sensitive parameter \( \gamma \in \mathbb{R} \setminus \{0\} \); here we use robust convention \( \infty - \infty = -\infty \) for \( \mathbb{E}(X) = \mathbb{E}(X_+) - \mathbb{E}(X_-) \).
Remark 2.2. From assumption \((A.1)\), using monotonicity of entropic risk with respect to risk-sensitivity parameter, we get that for any \(\delta\) between \(\gamma\) and 0, we have \(|\mu^\delta(r_i(t))| < \infty\). Also, note that assumption \((A.1)\) could be rephrased using non-entropy notation as \(\mathbb{E}[e^{\gamma r_i(t)}] < \infty\) and \(\mathbb{E}[|r_i(t)|] < \infty\), which could be linked to log-returns moment generating function finiteness.

For transparency, we also introduce an asset relative shift process \(w(t) = (w_i(t))_{i=1}^d\) given by

\[
 w(t) := e^{r(t)} = \left(\frac{S_1(t)}{S_1(t-1)}, \ldots, \frac{S_d(t)}{S_d(t-1)}\right).
\]

Let us now show how to re-express inner part of \(J(\pi)\) as a \(\pi\)-controlled process. Let \(G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) be given by \(G(x, y) := x \cdot y / \langle x, y \rangle\). Noting that

\[
 \pi(t-) = \frac{\pi(t-1) \cdot w(t)}{\langle \pi(t-1), w(t) \rangle} = G(\pi(t-1), w(t)),
\]

and rewriting the objective criterion \((6)\) as

\[
 J(\pi) = \lim\inf_{n \rightarrow \infty} \frac{1}{n} \mu^\gamma \left( \sum_{i=0}^{n-1} \ln \frac{W_{\pi}(t+1-) W_{\pi}(t)}{W_{\pi}(t-)} \right)
 = \lim\inf_{n \rightarrow \infty} \frac{1}{n} \mu^\gamma \left( \sum_{i=0}^{n-1} \ln \frac{\langle N_{\pi}(t), S(t+1) \rangle}{W_{\pi}(t)} + \ln(s(\pi(t-), \pi(t))) \right)
 = \lim\inf_{n \rightarrow \infty} \frac{1}{n} \mu^\gamma \left( \sum_{i=0}^{n-1} \ln \langle \pi(t), w(t+1) \rangle + \ln(s(G(\pi(t-1), w(t)), \pi(t))) \right),
\]

we essentially get direct restatement of the controlled log-wealth process using control process \(\pi\) and independent shifts \(w\).

In order to solve \((7)\), we introduce the associated Bellman equation. Ideally, given \(\gamma \in \mathbb{R}_+ := \mathbb{R} \setminus \{0\}\), we are looking for a function \(v : S \rightarrow \mathbb{R}\) and a constant \(\lambda \in \mathbb{R}\) which satisfy equation

\[
 \lambda + v(\pi) = \sup_{\pi' \in S} \left[ \mu^\gamma (\ln(\pi', w(1)) + \ln s(\pi, \pi') + v(G(\pi', w(1)))) \right],
\]

for \(\pi \in S\). Note that, with a slight abuse of notation, in \((8)\) we use \(\pi, \pi' \in S\) in reference to a deterministic pre-rebalancing and post-rebalancing weights rather than the whole rebalancing strategy; this convention is often used in the paper when optimality equations are considered.

For technical reasons, instead of considering \((8)\) directly, in this paper we consider its slightly modified version given by

\[
 \lambda + v(\pi, \gamma) = \gamma \sup_{\pi' \in S} \left[ \mu^\gamma (\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1} v(G(\pi', w(1), \gamma))) \right],
\]
where \( v : S \times \mathbb{R}_+ \to \mathbb{R} \) and \( \lambda \in \mathbb{R} \). This is mainly done to provide a more direct link between entropy-based formulation and classical formulation and for concise proofs. In particular, we use \( v(\pi, \gamma) \) instead of \( v(\pi) \) to emphasize the dependency between risk-parameter choice and Bellman’s equation solution, and to embed (8) into vanishing discount framework. Note that if \( v(\cdot, \gamma) \) and \( \lambda \) solves (9), then \( \gamma^{-1}v(\cdot, \gamma) \) and \( \gamma^{-1}\lambda \) solves (8). Also, Bellman’s equation (9) could be directly restated in a more classical form.

Namely, for the case \( \gamma < 0 \) we can rephrase (9) as

\[
e^v(\pi, \gamma) = \inf_{\pi' \in S} e^\gamma \ln s(\pi, \pi') \mathbb{E} \left[ e^{\gamma (\ln (\langle \pi', w(1) \rangle) - \lambda) + v(G(\pi', w(1)), \gamma)} \right],
\]

while for the case \( \gamma > 0 \) we get

\[
e^v(\pi, \gamma) = \sup_{\pi' \in S} e^\gamma \ln s(\pi, \pi') \mathbb{E} \left[ e^{\gamma (\ln (\langle \pi', w(1) \rangle) - \lambda) + v(G(\pi', w(1)), \gamma)} \right].
\]

For transparency, if not stated otherwise, we use \( v \) in reference to equation (9).

Let us now introduce a lemma that will be later helpful for establishing existence of solutions to Bellman’s equations (8) and (9) in conjunction with the vanishing discount approach. For brevity, we introduce supplementary notation

\[
z_\gamma(\pi', \pi) := \mu^\gamma \left( \ln \langle \pi, w(1) \rangle + \ln s(\pi', \pi) \right),
\]

\[
z^- := -\left| \min_i \mu^\gamma(r_i(1)) \right| - d \max_{|\gamma|} |r_i(1)| - \frac{\ln d}{|\gamma|} + \ln \bar{s},
\]

\[
z^+ := \left| \max_i \mu^\gamma(r_i(1)) \right| + d \max_{|\gamma|} |r_i(1)| + \frac{\ln d}{|\gamma|}.
\]

Note that \( z^- \) and \( z^+ \) are finite due to (A.1).

**Lemma 2.3.** Let \( \gamma < 0 \) (resp. \( \gamma > 0 \)) and let us assume (A.1). Then, for any \( \delta \in [\gamma, 0] \) (resp. \( \delta \in [0, \gamma] \)) and \( \pi, \pi' \in S \) we have

\[
z_\delta(\pi', \pi) \leq z_\delta \leq z^+.
\]

**Proof.** First let us consider the case where \( \gamma < 0 \) and \( \delta \in [\gamma, 0] \). Using monotonicity and translation invariance of entropic utility we get

\[
z_\delta(\pi', \pi) \geq \mu^\delta(\ln \langle \pi, w(1) \rangle)) + \ln \bar{s}
\]

\[
\geq \mu^\delta(\min_i r_i(1)) + \ln \bar{s}
\]

\[
= \frac{1}{\gamma} \ln \mathbb{E} \left[ \max_i e^{\gamma r_i(1)} \right] + \ln \bar{s}
\]

\[
\geq \frac{1}{\gamma} \ln \left( \sum_{i=1}^d \mathbb{E} \left[ e^{\gamma r_i(1)} \right] \right) + \ln \bar{s}
\]

\[
\geq \min_i \frac{1}{\gamma} \ln \left( d \cdot \mathbb{E} \left[ e^{\gamma r_i(1)} \right] \right) + \ln \bar{s}
\]

\[
\geq -\left| \min_i \mu^\gamma(r_i(1)) \right| - \frac{\ln d}{|\gamma|} + \ln \bar{s}
\]

(12)
and
\[
  z_\delta(\pi', \pi) \leq \mu^\delta (\ln \langle \pi, w(1) \rangle) + \ln 1 \\
  \leq \mu^0 (\max_i r_i(1)) \\
  \leq d \max_i \mathbb{E}|r_i(1)|.
\]  \tag{13}

Now, for $\gamma > 0$ and $\delta \in [0, \gamma]$, using similar calculations, we get
\[
  z_\delta(\pi', \pi) \geq -d \max_i \mathbb{E}|r_i(1)| + \ln 3,
\]
\[
  z_\delta(\pi', \pi) \leq |\max_i \mu^\gamma (r_i(1))| + \frac{\ln d}{|\gamma|}.
\]  \tag{14}

Combining (12), (13), and (14) we conclude the proof. \hfill \square

In the following sections, we study the existence of the solution to the initial problem and its link to Bellman’s equation (8). For risk-averse case $\gamma < 0$, we will show that under general assumption (A.1) and can solve the recursive version of (8) in order to get the optimal constant and optimal strategy. Moreover, by imposing additional condition on $w$ that is related to mixing, one can show that (8) could be directly solved without relying on the recursive scheme. On the other hand, for risk-seeking case $\gamma > 0$ assumption (A.1) alone implies existence of a solution to (8).

For completeness, we will also show that selectors to the Bellman equation determine an optimal strategy in both cases. As already said, the results will be obtained using both vanishing discount approach as well as span-contraction approach.

### 3 DISCOUNTED PROBLEM

Before we apply the vanishing discount approach, let us provide a few remarks for the discounted version of (7). Consider $\alpha > 0$ and the discounted risk sensitive objective problem given by
\[
  \sup_{\pi} J_\alpha(\pi) = \sup_{\pi} \mu^\alpha \left( \sum_{t=0}^{\infty} e^{-\alpha t} \left[ \ln \langle \pi(t), w(t+1) \rangle + \ln s(\pi(t-), \pi(t)) \right] \right).
\]

The associated discounted analogue of Bellman equation (9) is given by
\[
  v_\alpha(\pi, \gamma) = \gamma \sup_{\pi' \in S} \left[ \mu^\gamma \left( \ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v_\alpha(G(\pi', w(1)), \gamma e^{-\alpha}) \right) \right].
\]  \tag{15}

Note that (15) is in fact linked to a series of equations which effectively should provide the formula for $(v_\alpha(\pi, \gamma e^{-n\alpha}))_{n \in \mathbb{N}}$. Nevertheless, for simplicity, we are looking for a stronger condition, that is, a function $v_\alpha$ that satisfies (15) for any value of risk sensitive parameter between $\gamma$ and 0. In other words, we want (15) to hold on $S \times \Gamma$, for
\[
  \Gamma := [\gamma_-, \gamma_+] \setminus \{0\},
\]
where $\gamma_- := \min\{0, \gamma\}$ and $\gamma_+ := \max\{0, \gamma\}$. With slight abuse of notation, if no ambiguity arise, we often use $\gamma$ to denote a generic choice from $\Gamma$. 


As before, Equation (15) could be rephrased in a classical way, that is, for $\gamma < 0$ we can restate (15) as

$$e^{\nu_\alpha(\pi, \gamma)} = \inf_{\pi' \in S} e^{\gamma \ln s(\pi, \pi')} \mathbb{E} \left[ e^{\gamma \ln(s(\pi', w(1)) + \nu_\alpha(G(\pi', w(1)), \gamma e^{-\alpha}))} \right]$$

while for $\gamma > 0$ we can rewrite (15) as

$$e^{\nu_\alpha(\pi, \gamma)} = \sup_{\pi' \in S} e^{\gamma \ln s(\pi, \pi')} \mathbb{E} \left[ e^{\gamma \ln(s(\pi', w(1)) + \nu_\alpha(G(\pi', w(1)), \gamma e^{-\alpha}))} \right].$$

We are now ready to present the main theorem of this section.

**Theorem 3.1.** Let $\gamma \in \mathbb{R} \setminus \{0\}$ and let us assume (A.1). Then, for each $\alpha > 0$, there exists a continuous and bounded function $\nu_\alpha : S \times \Gamma \to \mathbb{R}$, that is a solution to discounted Bellman's equation (15).

**Proof.** For brevity we only show the proof for $\gamma < 0$; the proof for $\gamma > 0$ is analogous. Fix $\alpha > 0$ and consider the Bellman operator linked to (15) that is given by

$$Tv(\pi, \gamma) := \gamma \sup_{\pi' \in S} \left[ \mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1} \nu(G(\pi', w(1)), \gamma e^{-\alpha})) \right],$$

for any $v : S \times \Gamma \to \mathbb{R}$. First, let us show that $T$ is $C$-Feller, that is, it transforms bounded and continuous functions into themselves. Using Lemma 2.3 we immediately get

$$\|Tv\|_{\sup} \leq |\gamma| \cdot [z^+ - z^-] + \|v\|_{\sup},$$

where $\| \cdot \|$ denotes the standard supremum norm, which shows that boundedness is preserved. Now, let us show that continuity is also preserved for continuous bounded functions. First, since $v$ is continuous, for any sequence $((\pi'_n, \gamma_n))_{n \in \mathbb{N}}$, where $\pi'_n \in S$ and $\gamma_n \in \Gamma$, satisfying $(\pi'_n, \gamma_n) \to (\pi', \gamma)$, we get

$$e^{\gamma_n \ln(s(\pi'_n, w(1)) + \nu(G(\pi'_n, w(1)), \gamma_n e^{-\alpha}))} \xrightarrow{a.s.} e^{\gamma \ln(s(\pi', w(1)) + \nu(G(\pi', w(1)), \gamma e^{-\alpha}))}.$$ 

Now, using similar reasoning as in Lemma 2.3 we know that

$$e^{\gamma \ln(s(\pi', w(1)) + \nu(G(\pi', w(1)), \gamma e^{-\alpha}))} \leq e^{\gamma \min_{r_1(1)} + \|v\|_{\sup}}.$$ 

Thus, noting that $e^{\gamma \min_{r_1(1)} + \|v\|_{\sup}} \in L^1$ due to (A.1), and using dominated convergence theorem, we get

$$\mathbb{E} \left[ e^{\gamma \ln(s(\pi', w(1)) + \nu(G(\pi', w(1)), \gamma e^{-\alpha}))} \right] \to \mathbb{E} \left[ e^{\gamma \ln(s(\pi', w(1)) + \nu(G(\pi', w(1)), \gamma e^{-\alpha}))} \right],$$

which in turn implies continuity of the mapping

$$(\pi, \pi', \gamma) \to \gamma \left[ \mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1} \nu(G(\pi', w(1)), \gamma e^{-\alpha})) \right].$$

(16)

Now, noting that $S$ is compact, we get continuity of $(\pi, \gamma) \to Tv(\pi, \gamma)$. This concludes the proof of the $C$-Feller property.
Now, we show that for \( v \equiv 0 \), the iterated sequence of operators satisfies the Cauchy condition. For any \( n \in \mathbb{N} \) and \( \delta \in \Gamma \), we get

\[
T^n 0(\pi(1), \delta) = \delta \sup_{\pi} \mu^\delta \left( \sum_{t=0}^{n-1} e^{-t \alpha} \left[ \ln(\pi(t), w(t)) + \ln s(\pi(t-), \pi(t)) \right] \right).
\]

For brevity, and with slight abuse of notation, let us introduce the abbreviated notation \( Z_t(\pi) := \ln(\pi(t), w(t)) + \ln s(\pi(t-), \pi(t)) \). For any \( n, k \in \mathbb{N} \) and \( \delta \in \Gamma \) (i.e., \( \gamma < \delta < 0 \)), noting that entropic risk is additive for independent random variables and doing similar calculations as in Lemma 2.3, we get

\[
T^{n+k} 0(\pi(1), \delta) = \delta \sup_{\pi} \mu^\delta \left( \sum_{t=0}^{n+k-1} e^{-t \alpha} Z_t(\pi) \right)
\]

\[
\leq \delta \sup_{\pi} \mu^\delta \left( \sum_{t=0}^{n-1} e^{-t \alpha} Z_t(\pi) + \sum_{t=n}^{n+k-1} e^{-t \alpha} \left( \min_{i} r_i(t) + \ln \tilde{s} \right) \right)
\]

\[
= T^n 0(\pi(1), \delta) + \delta \sum_{t=n}^{n+k-1} \mu^\delta \left( e^{-t \alpha} \left( \min_{i} r_i(t) + \ln \tilde{s} \right) \right)
\]

\[
\leq T^n 0(\pi(1), \delta) + \delta \sum_{t=n}^{n+k-1} e^{-t \alpha} \left( \mu^\delta \left( \min_{i} r_i(t) \right) + \ln \tilde{s} \right)
\]

\[
\leq T^n 0(\pi(1), \delta) + \gamma z^- \frac{e^{-n \alpha}}{1 - e^{-\alpha}}. \tag{17}
\]

Similarly, we get

\[
T^{n+k} 0(\pi(1), \delta) = \delta \sup_{\pi} \mu^\delta \left( \sum_{t=0}^{n+k-1} e^{-t \alpha} Z_t(\pi) \right)
\]

\[
\geq \delta \sup_{\pi} \mu^\delta \left( \sum_{t=0}^{n-1} e^{-t \alpha} Z_t(\pi) + \sum_{t=n}^{n+k-1} e^{-t \alpha} \left( \max_{i} r_i(t) + \ln 1 \right) \right)
\]

\[
= T^n 0(\pi(1), \delta) + \delta \sum_{t=n}^{n+k-1} \mu^\delta \left( e^{-t \alpha} \max_{i} r_i(t) \right)
\]

\[
\geq T^n 0(\pi(1), \delta) + \delta \sum_{t=n}^{n+k-1} e^{-t \alpha} \mu^\delta \left( \max_{i} r_i(t) \right)
\]

\[
\geq T^n 0(\pi(1), \delta) + \gamma z^+ \frac{e^{-n \alpha}}{1 - e^{-\alpha}}. \tag{18}
\]

Consequently, combining (17) and (18), for any \( n, k \in \mathbb{N} \) we get

\[
\sup_{\pi \in \mathcal{S}} \sup_{\delta \in \Gamma} \left| T^{n+k} 0(\pi, \delta) - T^n 0(\pi, \delta) \right| \leq (z^+ - z^-) \frac{|\gamma|}{1 - e^{-\alpha}} \frac{e^{-n \alpha}}{1 - e^{-\alpha}},
\]
which shows that the sequence of functions \((T^n_0)_{n\in\mathbb{N}}\) satisfies the Cauchy condition. Now, note that for any \(n \in \mathbb{N}\), function \(T^n_0\) is continuous and bounded due to Feller property. Consequently, as the space of bounded and continuous functions on \(S \times \Gamma\) is a Banach space, we know that there exists a bounded and continuous function \(v_\alpha : S \times \Gamma \to \mathbb{R}\) such that

\[
\sup_{\pi \in S} \sup_{\delta \in \Gamma} |T^n_0(\pi, \delta) - v_\alpha(\pi, \delta)| \to 0 \quad \text{as} \quad n \to \infty. \tag{19}
\]

Now, noting that \(T^{n+1}_0 \equiv T(T^n_0)\), we get that \(T^{n+1}_0(\pi, \delta) \to v_\alpha(\pi, \gamma)\) as well as \(T^{n+1}_0(\pi, \delta) \to Tv_\alpha(\pi, \gamma)\), which shows that \(v_\alpha\) is a fixed point of operator \(T\). □

Remark 3.2. Note that in the proof of Theorem 3.1 we have in fact showed a more direct formula for \(v_\alpha\) as due to (19) we know that there exists a limit of \(T^n_0\), as \(n \to \infty\), and we have

\[
v_\alpha(\pi, \gamma) = \lim_{n \to \infty} \left[ \gamma \sup_{\pi'} \mu\left( \sum_{t=0}^{n} e^{-\alpha_t} \left[ \ln \langle \pi(t), w(t) \rangle + \ln s(\pi(t-), \pi(t)) \right] \right) \right].
\]

In the end of this section let us show a supplementary result linked to \(v_\alpha\).

**Lemma 3.3.** Let \(\gamma \in \mathbb{R} \setminus \{0\}\), \(\alpha \in (0, 1)\), and let \(v_\alpha\) solve (15). Then, for any \(\pi, \bar{\pi} \in S\) and \(\delta \in \Gamma\) we have

\[
\inf_{\pi'} \delta \ln \left( \frac{s(\pi, \pi')}{s(\bar{\pi}, \pi')} \right) \leq v_\alpha(\pi, \delta) - v_\alpha(\bar{\pi}, \delta) \leq \sup_{\pi'} \delta \ln \left( \frac{s(\pi, \pi')}{s(\bar{\pi}, \pi')} \right). \tag{20}
\]

**Proof.** The proof of (20) follows directly from (15). Indeed, using (15) it is easy to note that for any \(\pi \in S\), we get

\[
v_\alpha(\pi, \delta) \leq \delta \sup_{\pi' \in S} \left[ \mu(\ln(\pi', w(1)) + \delta^{-1}v_\alpha(\pi', w(1), \delta e^{-\alpha})) \right] + \sup_{\pi' \in S} \delta \ln s(\pi, \pi'),
\]

\[
v_\alpha(\pi, \delta) \geq \delta \sup_{\pi' \in S} \left[ \mu(\ln(\pi', w(1)) + \delta^{-1}v_\alpha(\pi', w(1), \delta e^{-\alpha})) \right] + \inf_{\pi' \in S} \delta \ln s(\pi, \pi').
\]

from which (20) follows. □

## 4 VANISHING DISCOUNT APPROACH

Fix \(\bar{\pi} \in S\) and for any \(\alpha \in (0, 1)\) and \(n \in \mathbb{N}\) define

\[
\bar{v}_\alpha(\pi, \gamma) := v_\alpha(\pi, \gamma) - v_\alpha(\bar{\pi}, \gamma), \tag{21}
\]

\[
\lambda^{(n)}_\alpha := v_\alpha(\bar{\pi}, e^{-\alpha n}) - v_\alpha(\bar{\pi}, e^{-\alpha(n+1)}). \tag{22}
\]

\[
v^{(n)}_\alpha(\pi, \gamma) := \bar{v}_\alpha(\pi, \gamma e^{-\alpha n}). \tag{23}
\]
where \( v_\alpha \) is a solution to the discounted Bellman equation (15). First, let us show that sequence introduced in (22) is uniformly bounded.

**Lemma 4.1.** Let \( \gamma \in \mathbb{R} \setminus \{0\} \) and let us assume (A.1). Then,

\[
\sup_{\alpha \in (0,1)} \sup_{n \in \mathbb{N}} |\lambda^{(n)}_\alpha| < \infty.
\]

**Proof.** We only show the proof for \( \gamma < 0 \); the proof for \( \gamma > 0 \) is analogous. Let us fix \( n \in \mathbb{N} \) and \( \alpha \in (0,1) \). Using Lemma 3.3, for any \( n \in \mathbb{N} \) and \( \pi' \in S \), we get

\[
|v_\alpha(G(\pi', w(1)), \gamma e^{-\alpha(n+1)}) - v_\alpha(\hat{\pi}, \gamma e^{-\alpha(n+1)})| \leq |\gamma| e^{-\alpha n} \cdot |\ln \tilde{s}|.
\]

(24)

Consequently, recalling that \( v_\alpha \) is a solution to the discounted Bellman equation, rewriting \( v_\alpha(\hat{\pi}, \gamma e^{-\alpha(n+1)}) \) using (15), applying (24), and then Lemma 2.3, we get

\[
\lambda^{(n)}_\alpha = v_\alpha(\hat{\pi}, \gamma e^{-\alpha n}) - v_\alpha(\hat{\pi}, \gamma e^{-\alpha(n+1)})
\]

\[
\leq \gamma e^{-\alpha n} \sup_{\pi' \in S} \left[ \mu^e \gamma^{-\alpha n} (\ln \langle \pi', w(1) \rangle + \ln s(\hat{\pi}, \pi') - (\gamma e^{-\alpha n})^{-1} (\gamma e^{-\alpha n}) \cdot |\ln \tilde{s}|) \right]
\]

\[
\leq \gamma e^{-\alpha n} \left( \sup_{\pi' \in S} z^e \gamma^{-\alpha n}(\hat{\pi}, \pi') - |\ln \tilde{s}| \right)
\]

\[
\leq \gamma (z^- - |\ln \tilde{s}|).
\]

Similarly, we get \( \lambda^{(n)}_\alpha \geq \gamma (|\ln \tilde{s}| + z^+) \). Noting that both upper and lower bound is independent of \( \alpha \) and \( n \), we conclude the proof. \( \square \)

Now, we present two main results of this section, which shows that under (A.1) one could find a sequence of functions solving the iterated Bellman equation. These functions could be used to find optimal strategy and related optimal value for the problem (6). While Theorem 4.2 is in fact true under both risk-averse (\( \gamma < 0 \)) and risk-seeking (\( \gamma > 0 \)) case, in the latter case we can show that iteration is not required, that is, we can directly solve (9); this is stated in Theorem 4.3.

**Theorem 4.2.** Let \( \gamma < 0 \) and let us assume (A.1). Then, there exists a sequence of constants \( \lambda^{(n)} \), \( n \in \mathbb{N} \), and a sequence of continuous bounded functions \( v^n(\cdot, \gamma) : S \to \mathbb{R} \), \( n \in \mathbb{N} \), such that the recursive Bellman equation

\[
v^{(n)}(\pi, \gamma) + \lambda^{(n)} = \gamma \sup_{\pi' \in S} \left[ \mu^e (\ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v^{(n+1)}(G(\pi', w(1)), \gamma)) \right]
\]

is satisfied for any \( n \in \mathbb{N} \). Moreover, the constant \( \Lambda := \lim \inf_{n \to \infty} 1/|\gamma n| \cdot \sum_{i=1}^{n} \lambda^{(i)} \) is the optimal value for the problem (6), that is, we get \( \Lambda = \sup_{\pi} J(\pi) \), and the optimal (iterated) strategy is defined by the selectors to the recursive Bellman equation.

**Proof.** Fix \( \gamma < 0 \). First, observe that the family of functions \( \{ \pi \to \bar{v}_\alpha(\pi, \delta) \} \), indexed by \( \alpha \in (0,1) \) and \( \delta \in \Gamma \) is both uniformly bounded and equicontinuous. Indeed, both uniform boundedness
and equicontinuity follows directly from Lemma $3.3$ since for any $\pi, \tilde{\pi} \in S$ we get

$$\sup_{\alpha \in (0,1)} \sup_{\delta \in \Gamma} |\bar{v}_\alpha(\pi, \delta)| \leq |\gamma| \cdot |\ln s|,$$

$$\sup_{\alpha \in (0,1)} \sup_{\delta \in \Gamma} |\bar{v}_\alpha(\pi, \delta) - \bar{v}_\alpha(\tilde{\pi}, \delta)| < |\gamma| \sup_{\pi' \in S} \left| \frac{s(\pi, \pi')}{s(\tilde{\pi}, \pi')} \right|.$$ 

Thus, by the Arzela-Ascoli theorem, we know that there exists a decreasing sequence $(\alpha_i)_{i \in \mathbb{N}}$, such that $\alpha_i \in (0,1), \alpha_i \searrow 0$ as $i \to \infty$, and for any $n \in \mathbb{N}$, we have

$$v^{(n)}(\pi, \gamma) \to v(\pi, \gamma), \quad \pi - \text{uniformly},$$

for some continuous and bounded function $v^{(n)}(\cdot, \gamma) : S \to \mathbb{R}$. Second, from Lemma $4.1$, we know that the sequence $(\alpha_i)_{i \in \mathbb{N}}$, can be chosen in such a way that for any $n \in \mathbb{N}$ we also have

$$\lambda^{(n)}(\pi, \gamma) \to \lambda^{(n)}, \quad i \to \infty,$$

where $(\lambda^{(n)})_{n \in \mathbb{N}}$ is some sequence of real numbers. Third, by combining Bellman equation (15) with (21) and (22), for any $n \in \mathbb{N}$, we get

$$v^{(n)}(\pi, \gamma) + \lambda^{(n)}(\pi, \gamma) = \gamma e^{-n\alpha_i} \sup_{\pi' \in S} \left[ \mu^{\gamma e^{-n\alpha_i}} (\ln(\pi', w(1)) + \ln s(\pi, \pi')) + (\mu^{\gamma e^{-n\alpha_i}})^{-1} v^{(n+1)}(G(\pi', w(1)), \gamma) \right].$$

(27)

Now, noting that the limit values are also bounded, for each $n \in \mathbb{N}$, we can take the limit in (27), as $i \to \infty$, and get

$$v^{(n)}(\pi, \gamma) + \lambda^{(n)} = \gamma \sup_{\pi' \in S} \left[ \mu^{\gamma} (\ln(\pi', w(1)) + \ln s(\pi, \pi')) + \gamma^{-1} v^{(n+1)}(G(\pi', w(1)), \gamma) \right],$$

(28)

which concludes the first part of the proof. Now, iterating the sequence (28), starting from $n = 1$, and using tower property of entropic utility, for any $n \in \mathbb{N}$, we get

$$\sum_{i=1}^{n} \lambda^{(n)} = \gamma \sup_{\pi} \left[ \mu^{\gamma} \left( \sum_{i=1}^{n-1} Z_i(\pi) + \gamma^{-1} \left[ v^{(n+1)}(G(\pi(n), w(n)), \gamma) - v^{(0)}(\pi(0), \gamma) \right] \right) \right],$$

where $Z_i(\pi) := \ln(\pi(t), w(t)) + \ln s(\pi(t-), \pi(t))$. Dividing both sides by $\frac{1}{\gamma^n}$, noting that the sequence of functions $(v^{(n)})$ is uniformly bounded by $\pm |\gamma| \cdot |s|$, and taking the limes inferior of both sides, we get

$$\Lambda = \sup_{\pi} \liminf_{n \to \infty} \frac{1}{n} \mu^{\gamma} \left( \sum_{i=1}^{n-1} Z_i(\pi) \right) = \sup_{\pi} J(\pi),$$

which concludes the proof. Also, note that for any admissible strategy $\pi$ we get $\Lambda \geq J(\pi)$, while for the strategy $\tilde{\pi}$ determined by the iterated sequence $(v^{(n)})$ we get $\Lambda = J(\tilde{\pi})$. \hfill \Box
Now, let us show that for $\gamma > 0$, the result in Theorem 4.2 could be strengthened. Namely, while for $\gamma < 0$ we had to consider recursively defined sequence of Bellman equations to recover the optimal value, and so on, for $\gamma > 0$, we can directly solve (9).

**Theorem 4.3.** Let $\gamma > 0$ and let us assume (A.1). Then, there exists a constant $\lambda$ and a continuous bounded functions $v(\cdot, \gamma) : S \to \mathbb{R}$ that solves Bellman’s equation (9), that is, we get

$$v(\pi, \gamma) + \lambda = \gamma \sup_{\pi' \in S} \left[ \mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1}v(G(\pi', w(1)), \gamma)) \right].$$

Moreover, the constant $\Lambda := \lambda / \gamma$ is the optimal value for the problem (6), that is, we get $\Lambda = \sup_\pi J(\pi)$, and the optimal strategy is defined by the selector to the Bellman equation.

**Proof.** Fix $\gamma > 0$. The first part of the proof is analogous to the proof of Theorem 4.3. Applying similar reasoning, we get that there exists sequence of constants $\lambda(n)$ and bounded continuous functions $v(n)(\cdot, \gamma) : S \to \mathbb{R}$, $n \in \mathbb{N}$ such that

$$v(n)(\pi, \gamma) + \lambda(n) = \gamma \sup_{\pi' \in S} \left[ \mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1}v(n+1)(G(\pi', w(1)), \gamma)) \right], \quad (29)$$

Now, let us show that for $\gamma > 0$ the sequence $(\lambda(n))_{n \in \mathbb{N}}$ is non-decreasing. First, note that for any random variable $Z$, the mapping $\gamma \to \ln E[e^{\gamma Z}]$ is convex; this could be easily shown using Hölder inequality by considering $U = e^{(1-\theta)\gamma Z}$ and $V = e^{\theta \gamma Z}$, for $\theta \in [0,1]$, with $\frac{1}{p} + \frac{1}{q} = 1$, such that $p = \frac{1}{1-\theta}$ and $q = \frac{1}{\theta}$, and taking logarithm of both sides. Consequently, since supremum of a family of convex functions is convex, we get that for any $\alpha \in (0,1)$ and $\pi \in S$ the mapping $\gamma \to v(\alpha)(\pi, \gamma)$ is also convex. In particular, for any $n \in \mathbb{N}$, we get

$$\frac{v(\alpha)(\pi, \gamma e^{-\alpha n}) - v(\alpha)(\pi, \gamma e^{-\alpha(n+1)})}{\gamma e^{-\alpha n}(1 - e^{-\alpha})} \geq \frac{v(\alpha)(\pi, \gamma e^{-\alpha(n+1)}) - v(\alpha)(\pi, \gamma e^{-\alpha(n+2)})}{\gamma e^{-\alpha(n+1)}(1 - e^{-\alpha})}, \quad (30)$$

which implies $e^{-\alpha}\lambda(\alpha)^{(n)} \leq \lambda(\alpha)^{(n+1)}$. Now, letting $i \to \infty$ in the decreasing sequence $(\alpha_i)_{i \in \mathbb{N}}$ defined in the proof of Theorem 4.2, we get

$$\lambda(n) = \lim_{i \to \infty} e^{-\alpha_i} \lambda(\alpha_i)^{(n)} \leq \lim_{i \to \infty} \lambda(\alpha_i)^{(n+1)} = \lambda(n+1),$$

which concludes this part of the proof. Second, from (30) we get

$$e^{-\alpha}(v(\alpha)^{(n+1)}(\pi, \gamma) - v(\alpha)^{(n)}(\pi, \gamma) + \lambda(\alpha)) \leq v(\alpha)^{(n+2)}(\pi, \gamma) - v(\alpha)^{(n+1)}(\pi, \gamma) + \lambda(\alpha)^{(n+1)}.$$

Again, taking the limit $i \to \infty$, for the decreasing sequence $(\alpha_i)_{i \in \mathbb{N}}$ defined in the proof of Theorem 4.2, we get

$$v(n+1)(\pi, \gamma) - v(n)(\pi, \gamma) + \lambda(n) \leq v(n+2)(\pi, \gamma) - v(n+1)(\pi, \gamma) + \lambda(n+1).$$

Consequently, the sequence of functions $z(n)(\pi, \gamma) := v(n+1)(\pi, \gamma) - v(n)(\pi, \gamma) + \lambda(n)$ is increasing w.r.t. $n$. As the sequence $z(n)$ is equicontinuous and bounded, there exists a continuous bounded
function $z(\cdot, \gamma) : S \to \mathbb{R}$ such that $z^{(n)}(\cdot, \gamma) \to z(\cdot, \gamma)$, as $n \to \infty$. Now, since $\lambda^{(n)} \not\to \lambda$ for some $\lambda \in \mathbb{R}$, as $n \to \infty$, we get

$$[v^{(n+1)}(\pi, \gamma) - v^{(n)}(\pi, \gamma)] \to z(\pi, \gamma) - \lambda, \quad n \to \infty.$$ 

Now, note that for any $\pi \in S$ we get $z(\pi, \gamma) = \lambda$ as otherwise, the sequence $(v^{(n)}(\pi, \gamma))_{n \in \mathbb{N}}$, that could be represented by

$$v^{(n+1)}(\pi, \gamma) = \sum_{i=1}^{n} [v^{(i+1)}(\pi, \gamma) - v^{(i)}(\pi, \gamma)] + v^{(1)}(\pi, \gamma), \quad n \in \mathbb{N},$$

would be unbounded which would lead to contradiction as $|v^{(n)}(\pi, \gamma)| < |\gamma| \cdot |\ln s|$, for $n \in \mathbb{N}$. This implies

$$\lim_{n \to \infty} [v^{(n+1)}(\pi, \gamma) - v^{(n)}(\pi, \gamma)] \to 0, \quad n \to \infty.$$ 

(31)

Due to Arzela-Ascoli theorem, as the mapping $n \to v^{(n)}(\cdot, \gamma)$ is equicontinuous and uniformly bounded, we can choose a subsequence $(n_k)_{k \in \mathbb{N}}$ and continuous bounded function $v(\cdot, \gamma) : S \to \mathbb{R}$ such that

$$v^{(n_k)}(\pi, \gamma) \to v(\pi, \gamma), \quad \pi - \text{uniformly}.$$ 

Finally, as $\lambda^{(n)} \not\to \lambda$, $n \to \infty$, recalling (31), and letting $n \to \infty$ in (28) we get

$$v(\pi, \gamma) + \lambda = \gamma \sup_{\pi' \in S} [\mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + \gamma^{-1}v(G(\pi', w(1)), \gamma))],$$

which concludes the proof. \qed

5 \hspace{1cm} SPAN-CONTRACTION APPROACH

In Section 4 we have shown that for $\gamma > 0$ one can solve directly Bellman equation (9); see Theorem 4.3. On the other hand, for $\gamma < 0$, we were only able to obtain the recursive scheme as presented in Theorem 4.3. In this section, we show that the solution to (9) exists also for $\gamma < 0$ under relatively weak ergodic assumptions imposed on asset log-returns. We follow the span-contraction approach; see for example, Pitera and Stettner (2016). In the span-contraction approach the parameter $\gamma < 0$ is kept fixed in a sense that we do not need to introduce the discounting scheme. Consequently, to ease the exposition, rather than using notation from (9), we revert to the one from (8): we fix one $\gamma < 0$ and write $v(\pi)$ rather than $v(\pi, \gamma)$. As usual, we use $B(S)$ to denote the space of continuous and bounded functions $v : S \to \mathbb{R}$. For any $v \in B(S)$ we introduce supremum norm and span semi-norm notation

$$\|v\| := \sup_{\pi \in S} |v(\pi)| \quad \text{and} \quad \|v\|_{sp} := \sup_{\pi, \pi' \in S} \frac{v(\pi) - v(\pi')}{2}.$$
Note that those norms are bound by relation
\[
\inf_{d \in \mathbb{R}} \|v + d\| = \|v\|_{sp},
\]
see Pitera and Stettner (2016) or Hairer and Mattingly (2011) for details. Now, we introduce additional assumption that relates to ergodicity and plays a central role in the span-contraction approach. Namely, for any \(\delta \in (0, 1/d)\), we assume that
\[
\sup_{A \in B(S)} \sup_{\pi, \pi' \in S_\delta} \left( \mathbb{P}[G(\pi, w(1)) \in A] - \mathbb{P}[G(\pi', w(1)) \in A] \right) < 1,
\]
where \(S_\delta := \{\pi \in S : \min_i \pi_i \geq \delta\}\) identifies a set of strategies in which we allocate at least \(\delta\) proportion of capital to each asset. This assumption is related to mixing and states that whatever our initial (non-degenerated) allocation is, we expect to be in some common set with positive probability.

Remark 5.1 (Ergodicity/mixing assumption relevance). Recalling that \(G(x, y) = x \cdot y / \langle x, y \rangle\), for any \(\pi \in S\) we get \(G(\pi, w(1)) = \frac{\pi \cdot w(1)}{(\pi, w(1))}\), which shows that (A.2) is in fact related to assumptions imposed on \(w(1)\). Nevertheless, we decided to present (A.2) in its classical form, to show the connection to mixing. One can show that assumption (A.2) is satisfied by any log Levy process, even with ergodic economic factors, see (Duncan et al., 2011, Proposition 1) for details. Also, one can notice that if \(r(1)\) has full support, then the assumption (A.2) is automatically satisfied.

Now, for any \(\delta \in (0, 1/d)\) we introduce operator
\[
T_\delta v (\pi) := \sup_{\pi' \in S_\delta} \left[ \gamma \mu^\pi' \left( \ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v(G(\pi', w(1))) \right) \right], \quad v \in B(S).
\]
It is relatively easy to show that operator \(T_\delta\) is \(C\)-Feller. In particular, using similar reasoning as in Lemma 3.3, for any \(v \in B(S)\) we get
\[
\|T_\delta v\|_{sp} \leq |\gamma| \sup_{\pi, \pi', \pi' \in S} \ln \left[ \frac{s(\pi, \pi')}{s(\pi', \pi')} \right] \leq -|\gamma| \ln \delta := K,
\]
which implies boundedness of \(T_\delta v\), for any \(v \in B(S)\). Let us now show that \(T_\delta\) is a local contraction.

**Proposition 5.2.** Let \(\gamma < 0\) and let us assume (A.1) and (A.2). Then, for each \(\delta \in (0, 1)\), the operator \(T_\delta\) is a local contraction under \(\| \cdot \|_{sp}\), that is, there exists \(L_\delta : \mathbb{R}_+ \rightarrow (0, 1)\) such that
\[
\|T_\delta v_1 - T_\delta v_2\|_{sp} \leq L_\delta(M)\|v_1 - v_2\|_{sp},
\]
for \(v_1, v_2 \in C(S)\), such that \(\|v_1\| \leq M\) and \(\|v_2\| \leq M\).
Proof. Let us fix \( \delta \in (0, 1/d) \). For any \( \nu \in B(S) \) and \( \pi \in S \) let

\[
\tilde{\mu}_{(\pi, \nu)}(B) := \frac{\mathbb{E}[1_{B}(G(\pi, w(1))e^{\gamma \ln \langle \nu', w(1) \rangle + \gamma s(\pi, \nu')} + G(\pi, w(1)))]}{\mathbb{E}[e^{\gamma \ln \langle \nu', w(1) \rangle + \gamma s(\pi, \nu')} + G(\pi, w(1))]}, \quad B \in B(\mathbb{R}^d)
\]
denote the projection measure for \( \nu \) with rebalancing \( \pi \), and let

\[
\pi_{\nu} := \arg \max_{\pi \in S_\delta} \left[ \gamma \mu(\nu'(\pi, w(1)) + \ln s(\pi, \nu') + \gamma^{-1} \nu(G(\pi', w(1))) \right]
\]
denote the optimal rebalancing (induced by operator \( T_\delta \)) given \( \nu \in C(S) \) and initial state \( \pi \in S_\delta \). First, let us show that for any \( \nu_1, \nu_2 \in C(S_\delta) \) and \( \pi, \pi' \in S_\delta \) we get

\[
T_\delta \nu_1(\pi) - T_\delta \nu_2(\pi) - (T_\delta \nu_1(\pi') - T_\delta \nu_2(\pi')) \leq \|\nu_1 - \nu_2\|_{sp} \cdot \|\mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}\|_{var},
\]

where \( \mathbb{H}_{\nu_1, \nu_2} := \tilde{\mu}(\nu_1, \nu_2) - \tilde{\mu}(\nu_1, \nu_2) \) is a signed measure and \( \| \cdot \|_{var} \) is the total variation norm given by

\[
\|\mathbb{H}\|_{var} := \int_{\mathbb{R}^d} |\mathbb{H}|(dx),
\]

where \( |\mathbb{H}| \) is the total variation of \( \mathbb{H} \); see Pitera and Stettner (2016) for details. Using dual representation for entropic risk, and performing similar calculations as in (Pitera and Stettner, 2016, Lemma 1) we get

\[
T_\delta \nu_1(\pi) - T_\delta \nu_2(\pi) - (T_\delta \nu_1(\pi') - T_\delta \nu_2(\pi')) \leq \int_{\mathbb{R}^d} [\nu_1(x) - \nu_2(x)] \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}(dx).
\]

Now, recalling that for any \( \nu \in B(S) \) we have \( \inf_{d \in \mathbb{R}} \|\nu + d\| = \|\nu\|_{sp} \), we know there exists \( d \in \mathbb{R} \) such that

\[
\|\nu_1 - \nu_2\|_{sp} = \sup_{x \in \mathbb{R}^d} (\nu_1(x) - \nu_2(x) + d) = -\inf_{x \in \mathbb{R}^d} (\nu_1(x) - \nu_2(x) + d)
\]

Consequently, for \( A \) denoting the positive set of measure \( \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'} \), we get

\[
\int_{\mathbb{R}^d} [\nu_1(x) - \nu_2(x)] \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}(dx) = \int_{\mathbb{R}^d} [\nu_1(x) - \nu_2(x) + d] \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}(dx)
\]

\[
\leq \|\nu_1 - \nu_2\|_{sp} \left( \int_{A} \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}(dx) - \int_{A^c} \mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}(dx) \right)
\]

\[
= \|\nu_1 - \nu_2\|_{sp} \cdot \|\mathbb{H}_{\nu_1, \nu_2, \pi, \pi'}\|_{var},
\]

which concludes this step of the proof. Now, let us fix \( M \in \mathbb{R}_+ \) and show that there exists constant \( L(M) \in (0, 1) \) such that for any \( \nu_1, \nu_2 \in C(S) \) satisfying \( \|\nu_1\| \leq M \) and \( \|\nu_2\| \leq M \), and \( \pi, \pi' \in S \),
we get
\[ \|\hat{\mu}^{v_1,v_2}_{\pi_1,\pi_2}\|_{\text{var}} \leq 2L(M). \]  
(35)

Suppose that (35) is not satisfied. Recalling that
\[ \|\hat{\mu}^{v_1,v_2}_{\pi_1,\pi_2}\|_{\text{var}} = 2 \sup_{B \in B(\mathbb{R}^d)} |\tilde{\mu}(\pi_1,v_1,B) - \tilde{\mu}(\pi_1',v_2,B)|, \]
we get that there exists a sequence of sets \(B_n \in B(\mathbb{R}^d)\), functions \(v_n, v'_n \in B(S)\) satisfying \(\|v_n\| \leq M\) and \(\|v'_n\| \leq M\), and weights \(\pi_{v_n}, \pi'_{v_n} \in S_S\) such that
\[ \tilde{\mu}(\pi_{v_n},v_n)(B_n) \to 1 \quad \text{and} \quad \tilde{\mu}(\pi_{v_n'},v'_n)(B_n) \to 0, \quad \text{as } n \to \infty. \]  
(36)

Now, let \(r_+(1) := \max_i r_i(1)\) and \(r_-(1) := \min_i r_i(1)\). For any \(B \in B(\mathbb{R}^d), \pi \in S\) and \(v \in B(S)\), we get
\[ \tilde{\mu}(\pi,v)(B) = \frac{\mathbb{E}[1_B(G(\pi,v(1))e^{\gamma r_+(1) + v(G(\pi,v(1)))})]}{\mathbb{E}[e^{\gamma r_+(1) + v(G(\pi,v(1)))}]}, \]
\[ \geq \frac{\mathbb{E}[1_B(G(\pi,v(1))e^{\gamma r_+(1) + \inf_{\pi' \in S} v(\pi')})]}{\mathbb{E}[e^{\gamma r_-(1) + \sup_{\pi' \in S} v(\pi')}]}, \]
\[ = e^{-2\|v\|_{\text{sp}}} \frac{\mathbb{E}[1_B(G(\pi,v(1))e^{\gamma r_+(1)})]}{\mathbb{E}[e^{\gamma r_-(1)}]} \geq e^{-2\|v\|_{\text{sp}}} \frac{\mathbb{E}[1_{\{G(\pi,v(1)) \in B\}}e^{\gamma r_+(1)}]}{\mathbb{E}[e^{\gamma r_-(1)}]} . \]  
(37)

Following similar arguments as in (12) we get that \(\mathbb{E}[e^{\gamma r_-(1)}] < \infty\). Thus, combining (36) and (37), and recalling that \(\|v_n\| \leq M\) and \(\|v'_n\| \leq M\) for any \(n \in \mathbb{N}\), we get that
\[ \mathbb{E}[1_{\{G(\pi_{v_n},v(1)) \in B_n'\}}e^{\gamma r_+(1)}] \to 0 \quad \text{and} \quad \mathbb{E}[1_{\{G(\pi_{v_n},v(1)) \in B_n\}}e^{\gamma r_+(1)}] \to 0, \quad \text{as } n \to \infty. \]

Consequently, for \(B_n := \{G(\pi_{v_n},w(1)) \in B_n'\}\) and \(N \geq 0\), letting \(n \to \infty\), we get
\[ \mathbb{P}[B_n \cap \{\gamma r_+(1) \geq -N\}] = \mathbb{E}[1_{B_n \cap \{\gamma r_+(1) \geq -N\}}e^{\gamma r_+(1)}e^{-\gamma r_+(1)}] \leq \mathbb{E}[1_{B_n \cap \{\gamma r_+(1) \geq -N\}}e^{\gamma r_+(1)}e^N] \leq e^N \mathbb{E}[1_{B_n}e^{\gamma r_+(1)}] \to 0. \]

Now, noting that \(\mathbb{P}[B_n] \leq \mathbb{P}[B_n \cap \{\gamma r_+(1) \geq -N\}] + \mathbb{P}[\gamma r_+(1) < -N]\), letting \(n \to \infty\) and then \(N \to \infty\), we get that
\[ \mathbb{P}[\{G(\pi_{v_n},w(1)) \in B_n'\}] = \mathbb{P}[B_n] \to 0, \quad \text{as } n \to \infty. \]
Similarly, one can show that $\mathbb{P}[G(\pi_{v_n}, w(1)) \in B_n] \to 0$, as $n \to \infty$. Therefore, as $n \to \infty$, we have

$$\mathbb{P}[G(\pi_{v_n'}, w(1)) \in B_n] - \mathbb{P}[G(\pi_{v_n}, w(1)) \in B_n] \to 1.$$  \hspace{1cm} (38)

Noting that $\pi_{v_n}, \pi_{v_n'} \in S_\delta$, we get that (38) contradicts (A.2), which concludes the proof of (35). Combining (34) with (35) we conclude the proof.

Finally, we are ready to show the main result of this section; note that while in Theorem 5.3 we adjusted notation from $v(\cdot, \nu)$ to $v(\cdot)$, the presented conclusions are consistent with those presented in Theorem 4.3 (for $\gamma < 0$ instead of $\gamma > 0$).

**Theorem 5.3.** Let $\gamma < 0$ and let us assume (A.1) and (A.2). Then, there exists a constant $\lambda$ and a continuous bounded functions $v(\cdot, \nu) : S \to \mathbb{R}$ that solves (9), that is, we get

$$v(\pi) + \lambda = \gamma \sup_{\pi' \in S} [\mu'(\ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v(G(\pi', w(1))))].$$

Moreover, the constant $\Lambda := \lambda / \gamma$ is the optimal value for the problem (6), that is, we get $\Lambda = \sup_{\pi, J(\pi)}$, and the optimal strategy is defined by the selectors to the Bellman equation.

**Proof.** For any fixed $\delta \in (0, 1/d)$, combining (33) with Proposition 5.2 we get that there exists a unique (up to an additive constant) function $v_\delta \in B(S)$ and a constant $\lambda_\delta \in \mathbb{R}$ such that

$$v_\delta(\pi) + \lambda_\delta = \gamma \sup_{\pi' \in S_\delta} [\mu'(\ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v_\delta(G(\pi', w(1))))].$$  \hspace{1cm} (39)

Now, let $\bar{\delta} := (1/d, \ldots, 1/d)$ and $\tilde{v}_\delta(\pi) := v_\delta(\pi) - v_\delta(\bar{\delta})$ for $\delta \in (0, 1/d)$; note that $\tilde{v}_\delta$ also solves (39) and we get $\|\tilde{v}_\delta\| < 2K$ due to (33) and property $\bar{v}_\delta(\bar{\pi}) = 0$. Thus, the family of functions $\{\tilde{v}_\delta\}_{\delta \in (0, 1/d)}$ is uniformly bounded and equicontinuous since, by (39), we get

$$\tilde{v}_\delta(\pi) - \tilde{v}_\delta(\bar{\pi}) \leq |\gamma| \sup_{\pi' \in S_\delta} \left| \ln s(\pi, \pi') \right|.$$  \hspace{1cm} (40)

Consequently, using Arzela-Ascoli theorem, we know that there exists a function $v \in B(S)$ and a subsequence $(\delta_n)_{n \in \mathbb{N}}$, $\delta_n \searrow 0$, such that $v_{\delta_n} \to v$ (uniformly) and $\lambda_{\delta_n} \to \lambda$, as $n \to \infty$. Thus, due to uniform convergence, we get

$$v(\pi) + \lambda = \gamma \sup_{\pi' \in S_0} [\mu'(\ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v(G(\pi', w(1))))],$$  \hspace{1cm} (41)

where $S_0 := \bigcup_{\delta \in (0, 1/d)} S_\delta$. Now, since right hand side of (41) is a continuous function of $\pi'$, we also get

$$v(\pi) + \lambda = \gamma \sup_{\pi' \in S} [\mu'(\ln \langle \pi', w(1) \rangle + \ln s(\pi, \pi') + \gamma^{-1} v(G(\pi', w(1))))],$$

which concludes the proof. \qed
In this section we illustrate the implications of Theorem 4.3 and Theorem 5.3. Namely, for exemplary dynamics, we approximate the optimal strategies and maximize the objective function (6) using the associated Bellman’s equations. For brevity, we study only the risk-averse case $\gamma < 0$ and focus on a simple low-dimensional setting in which one can directly approximate optimal policies following the standard policy iteration algorithm under known dynamics, see Whittle (1990) for details. It should be noted that the policy iteration scheme could be also applied to a more realistic high-dimensional market setting in which transition kernel might be unknown, change in time, and so on. For an overview of more advanced methods based on Reinforced Learning, Q-learning, and so on, we refer to Fei et al. (2021); Basu et al. (2008); Borkar (2010); Arapostathis et al. (2021). The goal of this section is to illustrate how to numerically approximate the solution (8) and recover the optimal policy used to construct optimal trading strategy. Namely, by considering iterations of operator $T$ given by

$$Tv(\pi) = \sup_{\pi' \in S} \left[ \mu'(\ln(\pi', w(1)) + \ln s(\pi, \pi') + v(G(\pi', w(1)))) \right], \quad v \in B(S),$$

we get a series of functions $T_0, T^2, \ldots, T^n$ that should converge in the span norm, as $n \to \infty$, to the solution of the Bellman’s equation (8). By studying the difference of the consequent iterations, we have control over so-called regret that can tell us how far from the optimal policy we are, see Fei et al. (2021). Let us now present two illustrative examples that show why detailed look into Bellman’s equation could lead to a more optimal trading choice under transaction costs when confronted with plausible alternatives.

First, we introduce a two-dimensional toy example based on finite state-space dynamics. Its aim is to show that even under simplistic setting, one can meaningfully increase the trading performance by solving a Bellman’s equation and easily recover optimal barrier-hit strategy with a simple numerical scheme.

Second, we introduce a three-dimensional example based on correlated Gaussian noise with drift. Note that while in this paper we follow an i.i.d. framework under which very generic assumptions (A.1) and (A.2) are sufficient to guarantee the existence to Bellman’s equation (8), similar reasoning could be applied to a more generic Markovian setting, see for example, Pitera and Stettner (2016) where the discrete-time version of the Merton’s inter-temporal capital asset pricing model (CAPM) is considered.

**Example 6.1 (Toy example).** To illustrate how to approximate Bellman equation’s solution and how the solution is linked to portfolio performance let us present a simple two-dimensional example based on finite state-space dynamics. Let $d = 2, \gamma = -0.5$, and let $S(t) = S(0) \cdot \prod_{i=1}^t e^{r(i)}$, $t \in \mathbb{N}$, where $\{r(t)\}_{t \in \mathbb{N}}$ is an i.i.d. sequence of log-returns such that

$$\mathbb{P}[r_i(t) = (\ln 1.5, \ln 0.5)] = \frac{1}{2}, \quad \mathbb{P}[r_i(t) = (\ln 0.6, \ln 1.8)] = \frac{1}{2}.$$  

(43)

Note that the sequence $\{r(t)\}_{t \in \mathbb{N}}$ satisfies assumption (A.1). Let us assume that the proportional transaction cost penalty is antisymmetric with 10%/20% costs, that is, let the penalty function be given by $d(x) := \langle c, [x]^+ \rangle + \langle h, [x]^− \rangle$, $x \in \mathbb{R}^2$, for $c = (0.1, 0.2)$ and $h = (0.2, 0.1)$. Noting that for any $(\pi_1, \pi_2) \in S$ we get $(\pi_1, \pi_2) = (\pi_1, 1 - \pi_1)$ and using notation $\bar{v}(\pi) := v(\pi, 1 - \pi)$ as well as
For the dynamics introduced in (43), we get that (44) is equal to

\[
\sup_{\pi' \in [0,1]} \left[ \ln s(\bar{\pi}, \bar{\pi}') - \frac{\ln 2}{\gamma} + \frac{1}{\gamma} \ln \left( e^{\gamma \ln(0.5 + \pi')} + e^{\gamma \ln(1.8 - 1.2 \pi')} \right) \right],
\]

so that a simple univariate iteration scheme could be applied to recover the value of \( T \bar{v} \) given \( \bar{v} \). In fact, simple numerical calculations show that the value of iterated value function (centered with constant from (32) to increase calculation robustness) stabilizes relatively fast. This is illustrated in Figure 1.

Noting that the convergence rate is fast, we decided to set \( n = 8 \) and use optimal strategy induced by \( T^{80} \) in the optimization. To illustrate the usefulness of risk-sensitive approach we study the asymptotic dynamics of the wealth function under various trading strategies for the dynamics introduced in (43). Namely, we consider four trading strategies: (1) Static buy-and-hold asset 1 strategy; (2) Static buy-and-hold asset 2 strategy; (3) Optimal proportion strategy; (4) Dynamic strategy induced by risk-sensitive framework; the optimal proportion strategy (3) have been chosen in such a way that the final portfolio value is the highest, that is, we decided to choose the trajectory-optimal fixed ratio strategy.

To compare the performance of trading strategies on a single long trajectory, we simulate a single realization of \((r(t))_{t=1}^T\), for \( T = 5000 \), with initial capital \( W(0) = 1 \). Then, we apply the trading strategies to the dataset. The log-wealth evolution as well as trading strategy profiles could be found in Figure 2.

From the plot a couple of things can be deduced. First, the risk-sensitive strategy outperforms all other strategies and shows the benefits of dynamic rebalancing. In particular, while the static strategies lead to losses, the dynamic risk-sensitive strategy produces relatively stable gains. Second, as expected, the risk-sensitive induced strategy is a barrier-trading strategy. In other word, if
In the left exhibit, we present the values of the log-wealth process $\ln W(t)$ for a single trajectory and $t = 1, \ldots, 5000$ under Example 6.1 dynamics. One can see that the risk-sensitive strategy is outperforming all others and leads to stable time-growth. The right exhibit presents the trading profile of all strategies as a function of the first weight. As expected, the risk sensitive strategy is in fact a barrier strategy, that is, no trading is made in we are inside a predefined interval (ca. $[0.38, 0.75]$) and a push-back in initiated if we fall outside of it. [Color figure can be viewed at wileyonlinelibrary.com]

for a given day $t$ the (proportion) value $\pi(t−)$ falls outside of some interval (ca. $[0.38, 0.75]$), then the trading strategy pushes it into this interval with cost-efficient approach. On the other hand, if the value is inside the interval, then it is optimal to not impose any trading. Third, we see that it is advised to study dynamic trading strategies even in the most simplistic transaction-cost setting—the performance improvement coming from applying risk-sensitive strategy to the problem looks material even when confronted with optimal proportion strategy.

Next, to fully understand the balance between the pay-off and the risk, we decided to calculate cumulative log-wealth for multiple trajectories and check their performance using assessment metrics. Namely, we consider the (time-normalized) mean, standard deviation, and entropy function $\mu^\gamma$; note the time-averaged entropy of log-wealth corresponds directly to the risk-sensitive objective function (6). For completeness, we also added values for entropy second-order Taylor expansion based on mean and variance which illustrates the link between risk-sensitive framework and mean-variance framework. Note that while mean-variance trading might lead to time-inconsistency, the risk-sensitive trading is time-consistent, see Bielecki and Pliska (2003); Bielecki et al. (2021) for details. Also, to better understand the difference between risk-sensitive and risk-neutral frameworks, we decided to include the results for strategy similar to (4) but with risk-sensitivity parameter set to $\gamma = −0.0005$, which approximates the risk-neutral setting, that is, strategy for $\gamma$ close to zero, which is near Kelly (log-growth) optimal, see Di Masi and Stettner (2006). With slight abuse of notation, we call it the risk-neutral strategy. The aggregated results are presented in Figure 3.

By looking at Figure 3 we see that the risk-sensitive trading strategy outperforms trading strategies (1)–(3). While the normalized log-mean for the risk-neutral strategy is higher compared to risk-sensitive strategy (which is in fact expected as the log-mean reflects directly the objective function in the risk-neutral setting), it also results in higher variance. In other words, in the risk-sensitive case, the smaller mean (ca. 2% decrease) is compensated by risk decrease (ca. 25% variance reduction) which seems like a plausible trade-off. This suggests that the risk-sensitive
The table presents time-normalized performance metrics for trading strategies introduced in Example 6.1. The outputs are based on a strong Monte Carlo sample of size 20,000 applied to log-wealth at time $T = 250$. For completeness, we also added the results for risk-neutral dynamic strategy.

| Strategy                  | Mean  | Std   | Mean + $\frac{7}{2}$ Variance | Entropy ($\mu^\gamma$) |
|---------------------------|-------|-------|--------------------------------|------------------------|
| Buy-and-hold asset 1      | -0.053| 0.029 | -0.106                         | -0.101                 |
| Buy-and-hold asset 2      | -0.053| 0.041 | -0.156                         | -0.146                 |
| Optimal proportion       | 0.005 | 0.009 | 0.000                          | 0.000                  |
| Risk-sensitive            | 0.049 | **0.009** | **0.044**                  | **0.044**              |
| Risk-neutral              | **0.050** | 0.012 | 0.042                          | 0.043                  |

**Example 6.2** (Gaussian noise with a drift). In this example, we focus on a three-dimensional correlated Gaussian noise with a positive drift. Let $d = 3$, and let $S(t) = S(0) \cdot \prod_{i=1}^{t} e^{r(i)}$, $t \in \mathbb{N}$, where $\{r(t)\}_{t \in \mathbb{N}}$ is an i.i.d. sequence of log-returns such that $r(t) \sim \mathcal{N}(\mu, \Sigma)$, where

$$
\mu := 0.001 \cdot (2.5, 1.5, 2.0), \quad \Sigma := 0.0008 \cdot \begin{bmatrix} 3.0 & -1.0 & -0.5 \\ -1.0 & 1.5 & 0.5 \\ -0.5 & 0.5 & 2.0 \end{bmatrix}.
$$

Next, let $\gamma = -5$ and let the penalty function be given by $d(x) = \langle c, x \rangle + \langle h, x \rangle$, $x \in \mathbb{R}^3$, for $c = 0.004 \cdot (2.0, 1.6, 1.0)$ and $h = 0.004 \cdot (1.0, 1.6, 2.0)$. The parameters are fixed in such a way that we have both positive and negative correlation in assets; note the correlation coefficients for $r_i(t)$ are (ca.) equal to $\rho_{12} = -0.48$, $\rho_{13} = -0.20$ and $\rho_{23} = 0.29$. Also, the transaction costs reflect pay-off between return (mean) and risk (variance). The assumption (A.1) is satisfied for $\{r(t)\}_{t \in \mathbb{N}}$ as the moment generating function for Gaussian random variables exists and assumption (A.2) is satisfied as the support of $\{r(t)\}_{t \in \mathbb{N}}$ is full.

First, as in Example 6.1, we approximate the solution to Bellman’s equation. The approximation was made on a discrete $\pi$-grid of step-size 0.02. By analyzing the consequent differences and the shape of the approximated value functions, we decided to set $n = 5$, that is, use approximated value of $T^50$ to determine the trading strategy. The approximation results are illustrated in Figure 4; note that for any $(\pi_1, \pi_2, \pi_3) \in S$ we get $\pi_3 = 1 - (\pi_1 + \pi_2)$, so it is sufficient to analyze a two-dimensional graphs.

From Figure 4 we see that the approximated value function is regular with maximal value around point $(0.37, 0.40)$. Recalling that entropy utility function has a second order Taylor expansion $\mu^\gamma(X) \approx \mathbb{E}[X] + \frac{\gamma^2}{2} \text{Var}[X]$, one would expect that this point (approximate) corresponds to an optimal allocation obtained using Markowitz portfolio optimization. This could be easily verified by solving a quadratic programming problem of the form

$$
\pi^* := \arg\min_{\pi \in \mathbb{S}} (\pi^T \mu + \frac{\gamma^2}{2} \pi^T \Sigma \pi),
$$

for which we get $\pi^* \approx (0.3705357, 0.4017857, 0.2276786)$. Let us now confront the (approximated) trading strategy with other alternative strategies as in Example 6.1. Namely, we consider five trading strategies: (1) Static buy-and-hold asset 1 strategy; (2) Static buy-and-hold asset 2 strategy; (3) Static buy-and-hold asset 3 strategy; (4) Markowitz proportion strategy; (5) Dynamic strat-
FIGURE 4 The plot illustrates approximated value function (left) and iteration convergence rates (right) for Example 6.2. The left exhibit shows the approximated values of $T^0$ as a function of the first and second weight; note that we used centering constant from (32). The right plot shows logarithm of the mean squared distance between the subsequent (centered) iterations for $n = 1, \ldots, 5$. [Color figure can be viewed at wileyonlinelibrary.com]

FIGURE 5 In the left exhibit, we present the values of the log-wealth process $\ln W(t)$ for a single trajectory and $t = 1, \ldots, 5000$ under Example 6.2 dynamics. One can see that the risk-sensitive strategy is outperforming all others. The right exhibit presents the structure of the risk-sensitive trading strategy—no trading is executed if we are nearby Markowitz-induced optimal point and a push-back strategy is applied if we fall outside of the black-point area. [Color figure can be viewed at wileyonlinelibrary.com]

egy induced by risk-sensitive framework. In strategy (4) we shift the allocation to the static state induced by (46), that is, we follow the optimal Markowitz strategy for risk aversion $\gamma$ under no transaction costs. While the full illustration of the trading strategy obtained via Bellman’s equation approximation (as presented in Figure 2) is problematic (it would require four-dimensional plot) we can analyze strategy structure by looking at no-action points as well as shift points, that is, sets of $\pi$’s such that no trading is executed if we are in the state $\pi$ and sets of all $\pi$’s that are the target state for some pre-trading initial state. The trading results for an exemplary long single trajectory as well as simplified strategy profile is presented in Figure 5.
Next, we analyze trading performance by looking into (time-averaged) performance metrics introduced in Example 6.1. The aggregated results are presented in Figure 6.

From Figure 6 we see that risk-sensitive trading strategy is outperforming all other strategies and has the highest entropy, as expected. While the variance for Markowitz strategy is slightly smaller, the Markowitz allocation leads to smaller mean—the payoff between the two is better for risk-sensitive strategy as could be seen by looking into both Entropy and Mean + $\frac{\gamma}{2}$ Variance performance criterions. To better understand the difference between risk-sensitive strategy and Markowitz strategy it is best to look into trading intensity. While the Markowitz strategy has homogeneous trading intensity (as expected, as the strategy should always push the allocation back to the fixed point), the risk-sensitive strategy shows more intense trading on rare occasions, that is, when the process falls outside of the zone presented in Figure 5. In fact, while the trading for Markowitz strategy was initiated for almost all of the considered days, the trading for risk-sensitive strategy was initiated in only 176 days (ca. 3.5% of sample size). Moreover, the aggregated trading intensity for risk-sensitive strategy, measured, for example, by cumulative product of capital decays, is much smaller. This shows that intense re-balancing could in fact negatively impact performance, especially in the transaction cost regime.

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ORCID
Marcin Pitera https://orcid.org/0000-0003-2469-8864

REFERENCES
Arapostathis, A., Biswas, A., & Pradhan, S. (2021). On the policy improvement algorithm for ergodic risk-sensitive control. Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 151(4), 1305–1330.
Atkinson, C., & Storey, E. (2010). Building an optimal portfolio in discrete time in the presence of transaction costs. Applied Mathematical Finance, 17(4), 323–357.
Basu, A., Bhattacharyya, T., & Borkar, V. S. (2008). A learning algorithm for risk-sensitive cost. Mathematics of Operations Research, 33(4), 880–898.
Bäuerle, N., & Jaśkiewicz, A. (2018). Stochastic optimal growth model with risk sensitive preferences. Journal of Economic Theory, 173, 181–200.
Bielecki, T. R., Chen, T., & Cialenco, I. (2021). Time-inconsistent markovian control problems under model uncertainty with application to the mean-variance portfolio selection. *International Journal of Theoretical and Applied Finance*, 24(01), 2150003.

Bielecki, T. R., Cialenco, I., Drapeau, S., & Karliczek, M. (2016). Dynamic assessment indices. *Stochastics*, 88(1), 1–44.

Bielecki, T. R., & Pliska, S. R. (1999). Risk-sensitive dynamic asset management. *Applied Mathematics & Optimization*, 39(3), 337–360.

Bielecki, T. R., & Pliska, S. R. (2003). Economic properties of the risk sensitive criterion for portfolio management. *Review of Accounting and Finance*, 2, 3–17.

Bo, L., Liao, H., & Yu, X. (2019). Risk sensitive portfolio optimization with default contagion and regime-switching. *SIAM Journal on Control and Optimization*, 57(1), 366–401.

Bobryk, R., & Stettner, Ł. (1999). Discrete time portfolio selection with proportional transaction costs. *Probability and Mathematical Statistics*, 19(2), 235–248.

Borkar, V. S. (2010). Learning algorithms for risk-sensitive control. In *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems–MTNS*, volume 5.

Campbell, J. Y., & Viceira, L. M. (2002a). *Strategic asset allocation: Portfolio choice for long-term investors*. Clarendon Lectures in Economic.

Campbell, J. Y., & Viceira, L. M. (2002b). *Strategic asset allocation: Portfolio choice for long-term investors*. Oxford University Press.

Cavazos-Cadena, R., & Hernández-Hernández, D. (2017). Vanishing discount approximations in controlled Markov chains with risk-sensitive average criterion. *Advances in Applied Probability*, 50(1), 204–230.

Chandra, P. (2017). *Investment analysis and portfolio management*. McGraw-Hill Education.

Cherny, A. S., & Madan, D. B. (2009). New measures for performance evaluation. *The Review of Financial Studies*, 22(7), 2571–2606.

Cherny, A. S., & Maslov, V. P. (2004). On minimization and maximization of entropy in various disciplines. *Theory of Probability & Its Applications*, 48(3), 447–464.

Christensen, S., Irle, A., & Ludwig, A. (2017). Optimal portfolio selection under vanishing fixed transaction costs. *Advances in Applied Probability*, 49(4), 1116–1143.

Czichowsky, C., Peyre, R., Schachermayer, W., & Yang, J. (2018). Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs. *Finance and Stochastics*, 22(1), 161–180.

Das, M. K., Goswami, A., & Rana, N. (2018). Risk sensitive portfolio optimization in a jump diffusion model with regimes. *SIAM Journal on Control and Optimization*, 56(2), 1550–1576.

Davis, M. H. A., & Lleo, S. (2014). *Risk-sensitive investment management* (vol. 19). World Scientific.

Davis, M. H. A., & Lleo, S. (2021). Risk-sensitive benchmarked asset management with expert forecasts. *Mathematical Finance*, 31(4), 1162–1189.

Di Masi, G. B., & Stettner, Ł. (2006). Remarks on risk neutral and risk sensitive portfolio optimization. In *From stochastic calculus to mathematical finance* (pp. 211–226). Springer.

Duncan, T., Pasik-Duncan, B., & Stettner, Ł. (2011). Growth optimal portfolio selection under proportional transaction costs with obligatory diversification. *Applied Mathematics & Optimization*, 63(1), 107–132.

Fei, Y., Yang, Z., & Wang, Z. (2021). Risk-sensitive reinforcement learning with function approximation: A debiasing approach. In *International Conference on Machine Learning* (pp. 3198–3207). PMLR.

Fleming, W. H., & Sheu, S. J. (2000). Risk-sensitive control and an optimal investment model. *Mathematical Finance*, 10(2), 197–213.

Guasoni, P., Tolomeo, A., & Wang, G. (2019). Should commodity investors follow commodities’ prices? *SIAM Journal on Financial Mathematics*, 10(2), 466–490.

Guo, S., Gu, J., & Ching, W. (2021). Adaptive online portfolio selection with transaction costs. *European Journal of Operational Research*, 295(3), 1074–1086.

Hairer, M., & Mattingly, J. C. (2011). Yet another look at Harris’ ergodic theorem for Markov chains. In *Seminar on Stochastic Analysis, Random Fields and Applications VI* (pp. 109–117). Springer.
Hata, H. (2018). Risk-sensitive portfolio optimization problem for a large trader with inside information. *Japan Journal of Industrial and Applied Mathematics*, 35(3), 1037–1063.

Ingersoll, J. E. (1987). *Theory of financial decision making* (vol. 3). Rowman & Littlefield.

Kolm, P. N., Tütüncü, R., & Fabozzi, F. J. (2014). 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research*, 234(2), 356–371.

Korn, R., & Laue, S. (2002). Portfolio optimisation with transaction costs and exponential utility. *Stochastic Processes and Related Topics. Proceedings of the 12th Winter School, Siegmundsburg, Germany*, 171–188.

Kupper, M., & Schachermayer, W. (2009). Representation results for law invariant time consistent functions. *Mathematics and Financial Economics*, 2(3), 189–210.

Li, B., & Hoi, S. (2014). Online portfolio selection: A survey. *ACM Computing Surveys (CSUR)*, 46(3), 1–36.

Liu, H., & Loewenstein, M. (2002). Optimal portfolio selection with transaction costs and finite horizons. *The Review of Financial Studies*, 15(3), 805–835.

MacLean, L. C., Thorp, E. O., & Ziemba, W. T. (2011). *The Kelly capital growth investment criterion: Theory and practice*. World Scientific.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1), 77–91.

Muthuraman, K., & Kumar, S. (2006). Multidimensional portfolio optimization with proportional transaction costs. *Mathematical Finance*, 16(2), 301–335.

Nagai, H. (2012). Downside risk minimization via a large deviations approach. *The Annals of Applied Probability*, 22(2), 608–669.

Pham, H. (2015). Long time asymptotics for optimal investment. In *Large deviations and asymptotic methods in finance* (pp. 507–528). Springer.

Pitera, M., & Stettner, Ł. (2016). Long run risk sensitive portfolio with general factors. *Mathematical Methods of Operations Research*, 83(2), 265–293.

Prigent, J.-L. (2007). *Portfolio optimization and performance analysis*. CRC Press.

Quek, G., & Atkinson, C. (2017). Portfolio selection in discrete time with transaction costs and power utility function: A perturbation analysis. *Applied Mathematical Finance*, 24(2), 77–111.

Shen, Y., Stannat, W., & Obermayer, K. (2013). Risk-sensitive Markov control processes. *SIAM Journal on Control and Optimization*, 51(5), 3652–3672.

Stettner, Ł. (2005). Discrete time risk sensitive portfolio optimization with consumption and proportional transaction costs. *Applicationes Mathematicae*, 4(32), 395–404.

Stettner, Ł. (2009). Long time growth optimal portfolio with transaction costs. In *Optimality and risk – Modern trends in mathematical finance* (pp. 237–250). Springer.

Stettner, Ł. (2011). Asymptotics of HARA utility from terminal wealth under proportional transaction costs with decision lag or execution delay and obligatory diversification. In *Advanced mathematical methods for finance* (pp. 509–536). Springer.

Whittle, P. (1990). *Risk-sensitive optimal control*. Wiley.

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