Localized optical vortex solitons in pair plasmas

Abstract: The dynamics of short intense electromagnetic pulses propagating in a relativistic pair plasma is governed by a nonlinear Schrödinger equation with a new type of focusing-defocusing saturable nonlinearity. In this context, we provide an existence theory for ring-profiled optical vortex solitons. We prove the existence of both saddle point and minimum type solutions. Via a constrained minimization approach, we prove the existence of solutions where the photon number may be prescribed, and we get the nonexistence of small-photon-number solutions. We also use the constrained minimization to compute the soliton’s profile as a function of the photon number and other relevant parameters.

Keywords: Optical vortices, pair plasmas, nonlinear Schrödinger equation, variational methods, mountain-pass theorem

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1 Introduction

Optical vortices and vortex solitons permeate through numerous branches of fundamental and applied physics. They appear as topological defects in Bose–Einstein condensates and as phase singularities in wave propagation [1, 7, 10, 11, 14, 20]. Singular optics, focusing on wave singularities, emerged from the study of optical vortices. Such a discipline has found many applications, which include areas such as quantum information processing, wireless communications, particle interactions, and cosmology [2, 17–19, 22].

The theoretical description of a complex-valued light wave propagating in a nonlinear medium and governed by a nonlinear Schrödinger equation brings about many interesting nonlinear problems [6, 16, 21]. It is the nonlinearity appearing in such equations that help describe the properties of the structures that an electromagnetic active medium can support. Of interest is the study of the nonlinear propagation of an electromagnetic wave in pair plasmas considered by Mahajan, Shatashvili, and Berezhiani in [12], and then also in [4, 5]. The equation describing the evolution of the vector potential of an electromagnetic pulse propagating in an arbitrary pair plasma with temperature asymmetry is

$$2i\omega_0 \frac{\partial A}{\partial t} + \frac{2-c}{\omega_0^2} \frac{\partial^2 A}{\partial \xi^2} + \nabla^2_{\perp} A + f(|A|^2)A = 0,$$

(1.1)

where $A$ is the amplitude of the circularly polarized electromagnetic pulse, $\omega_0$ is the mean frequency, $\nabla^2_{\perp} = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the diffraction operator, and $\xi$ is the “co-moving” coordinate. The new focusing-defocusing nonlinearity, not derived or reported prior to the work of [12] for any known physical system, is of the form

$$f(|A|^2) = \frac{\epsilon^2}{2} \frac{\kappa |A|^2}{(1 + \kappa |A|^2)^2}.$$

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where \( I = |A|^2 \) is the intensity of the field. This type of nonlinearity belongs to the class of saturable or saturating nonlinearities, but differs from the typical and previously reported saturable nonlinearities by vanishing as the field intensity goes to infinity (i.e., in the ultrarelativistic limit \( |A| \gg 1 \) it tends to be 0) [4, 5, 7, 21]. In normalized form, other typical types of nonlinearities, appearing in studies of electromagnetic wave propagation are: \( f(I) = I \) (pure Kerr nonlinearity), \( f(I) = 1 - a I^2 \) (cubic-quintic model), and \( f(I) = I(1 + a I)^{-1} \) (saturable nonlinearity), where \( a > 0 \) is a parameter describing the nonlinearity saturation [7]. We write the saturable type nonlinearities in the form

\[
f(I) = I(1 + a I)^{-p}, \quad p = 0, 1, \ldots.
\]

In the context of nonlinear concentric couplers [18], existence of ring-profiled optical vortex solitons for the pure Kerr nonlinearity \((p = 0)\) with an external potential were given by Yang and Zhang [24] over a bounded domain, and extended to an unbounded domain by Greco [8]. In a saturable nonlinearity [21] \((p = 1, a > 0)\), solutions were established by Medina [13] over a bounded domain, and over an unbounded domain (for the single vortex) by Watanabe [23]. In this paper, we prove the existence of ring-profiled optical vortex solitons of equation (1.1) with parameters \( a > 0 \) and \( p = 2 \) over a bounded domain.

Spatially localized solutions of equation (1.1), which do not change their intensity profile during propagation, are described under the radially symmetric "m-vortex" ansatz \( A = A(r) \exp(i \lambda t + i m \theta) \), where \( \lambda \in \mathbb{R} \) is the nonlinear frequency shift and \( m \in \mathbb{Z} \) the vortex number (also referred to as the topological charge). Under this ansatz, in normalized form as in [3–5], equation (1.1) reduces to the nonlinear boundary value problem

\[
\begin{cases}
A_r + \frac{1}{r} A_r - \left( \frac{m^2}{r^2} + \lambda \right) A + \frac{A^3}{(1 + a A^2)^p} = 0, \\
A(0) = 0 = A(R),
\end{cases}
\]

(1.2)

where \( R > 0 \) can be seen as the distance from the vortex core. Solutions to (1.2) are then critical points of the functional

\[
E_\lambda(A) = \frac{1}{2} \int_0^R \left[ r A_r^2 + \left( \frac{m^2}{r^2} + \lambda \right) r A^2 - a^{-2} r \ln(1 + a A^2) + \frac{r a^{-1} A^2}{1 + a A^2} \right] \, dr.
\]

(1.3)

Our main results, governing solutions to (1.2), are the following.

**Theorem 1.1.** Let \( p \geq 2, \alpha > 0, \) and \( R > 0 \). A necessary condition for the existence of a nontrivial localized “finite energy” solution pair \((A, \lambda)\) to (1.2) is

\[
0 < \lambda < \frac{(p - 1)^{p-1}}{a p^p} - \frac{r_0^2 + m^2}{R^2},
\]

where \( r_0 \approx 2.404825 \) is the first positive zero of the Bessel function \( J_0 \). Such solutions satisfy the exponential decay estimate

\[
0 < A(r) \leq C_\lambda \exp(-\sqrt{\lambda} r)
\]

for all \( r > R_1 \), where \( R_1 \in (0, R) \) is sufficiently large, and \( C_\lambda \) is a positive constant depending on \( \lambda \) only.

**Theorem 1.2.** Let \( p = 2 \). For every \( \lambda \) such that

\[
0 < \lambda < \frac{1}{4a} - \frac{r_0^2 + m^2}{R^2},
\]

where \( r_0 \approx 2.404825 \), and \( R > 0 \) sufficiently large, there exist at least two different types of positive solutions, i.e., \( A(r) > 0 \) for all \( r \in (0, R) \), to (1.2), one appearing as a saddle point and another as a minima of \( E_\lambda \). There also exist infinitely many pairs of solutions (not necessarily positive) to (1.2).

Typically, a saddle point corresponds to an unstable solution while a minima represents a ground state solution, which is a stable solution. An important parameter characterization of spatial optical vortex solitons is the integral

\[
N(A) = 2\pi \int_0^R r A^2 \, dr.
\]

(1.4)
Depending on the context, \( N(A) \) has different interpretations \([3, 12, 21]\). We refer to \( N(A) \) as the photon number \([12]\) and prove the following theorem.

**Theorem 1.3.** Let \( p = 2, |m| \geq 1, \) and \( a > 0 \). Subject to the prescribed photon number \( N(A) = N_0 > 0 \), there exists a “finite energy” solution pair \( (A, \lambda) \) such that \( A(r) > 0 \) for \( r \in (0, R) \), to the \( m \)-vortex equation (1.2). However, there exists no localized nontrivial “small-photon-number” solution, i.e., if \( N_0 \leq 2|m|\sqrt{a} \), then \( \lambda < 0 \) and \( A \) is not localized or \( \lambda > 0 \) and \( A \equiv 0 \).

In Sections 2–4, we prove Theorems 1.1–1.3. We use a constrained minimization to proof Theorem 1.3, and in Section 5 take advantage of this approach to numerically compute the soliton profiles and study its behavior with respect to a prescribed photon number and other relevant soliton parameters.

## 2 Proof of Theorem 1.1

To insure that the action functional \( E_A \) defined by equation (1.3) stays finite, we define the “energy” functional \( \mathcal{E} : H \to \mathbb{R} \) such that

\[
\mathcal{E}(A) = \frac{1}{2} \left\{ \int_0^R \left( rA_r^2 + \frac{A^2}{r} + \alpha^2 r \ln(1 + aA^2) + \frac{ra^{-1}A^2}{1 + aA^2} \right) \, dr \right\}.
\]  

(2.1)

with \( H \) being an appropriate function space described in Section 3. We are interested in nontrivial finite energy solutions (i.e., \( \mathcal{E}(A) < \infty \)).

**Lemma 2.1.** Let \( p \geq 2, a > 0, \) and \( R > 0 \). If \( (A, \lambda) \) is a finite energy solution pair of (1.2) and

\[
\lambda \geq \frac{(p - 1)r^{p-1}}{ap^p} - \frac{r_0^2 + m^2}{R^2},
\]

where \( r_0 \approx 2.404825 \) is the first positive zero of the Bessel function \( J_0 \), then \( A \equiv 0 \).

**Proof.** Suppose \( A \neq 0 \) and \( \lim_{r \to -0} |rA_r|A_r \neq 0 \). Then there is an \( \epsilon > 0 \) and \( r_0 \in (0, R) \) such that \( rA_r|A_r| \geq \epsilon \) for all \( r \in (0, r_0) \). It then follows that

\[
0 = \int_0^{r_0} \left( rA_r^2 \right) \, dr \leq \left( \int_0^{r_0} rA_r \, dr \right)^{1/2} \left( \int_0^{r_0} rA_r^2 \, dr \right)^{1/2},
\]

which contradicts \( \mathcal{E}(A) < \infty \). Hence, \( \lim_{r \to -0} |rA_r|A_r = 0 \) and there is a sequence \( \{r_j\}_{j=1}^{\infty} \subset \mathbb{R} \) such that \( r_j \to 0 \) as \( j \to \infty \) and \( \liminf_{j \to \infty} |r_jA_r|A_r| = 0 \). Multiplying (1.2) by \( A \) and integrating by parts, we get

\[
0 = \int_0^R \left\{ rA_r^2 + \left( \frac{m^2}{r^2} + \lambda - \frac{A^2}{(1 + aA^2)^p} \right) rA^2 \right\} \, dr.
\]

(2.2)

Applying the inequalities

\[
\frac{A^2}{(1 + aA^2)^p} < \frac{(p - 1)r^{p-1}}{ap^p} \quad \text{and} \quad \int_0^R rA^2 \, dr \leq R \int_0^R \frac{A^2}{r} \, dr
\]

to equation (2.2), we arrive at

\[
0 > \int_0^R \left\{ rA_r^2 + \left( \frac{m^2}{R^2} + \lambda - \frac{(p - 1)r^{p-1}}{ap^p} \right) rA^2 \right\} \, dr.
\]

Using the Poincaré inequality

\[
\int_0^R rA^2 \, dr \leq \frac{R^2}{r_0^2} \int_0^R rA_r^2 \, dr,
\]
where \( r_0 (\approx 2.404825) \), we obtain

\[
0 > \left( \frac{r_0^2 + m^2}{R^2} + \lambda - \frac{(p - 1)p-1}{ap^p} \right) \int_0^R rA^2 \, dr.
\]

Consequently, if

\[
\frac{r_0^2 + m^2}{R^2} + \lambda - \frac{(p - 1)p-1}{ap^p} \geq 0,
\]

we get a contradiction and it must be the case that \( A \equiv 0 \).

The exponential decay estimate is standard and proven identically as in \([13]\). Then we have the following lemma.

**Lemma 2.2.** If \( \lambda > 0 \), then the solution pair \((A, \lambda)\) of (1.2) has the exponential decay estimate

\[
0 < A(r) \leq C_\lambda \exp(-\sqrt{\lambda}r)
\]

for all \( r \in (R_1, R) \), with \( R_1 \) sufficiently large, and \( C_\lambda \) is a positive constant depending on \( \lambda \) only.

Together Lemma 2.1 and Lemma 2.2 establish Theorem 1.1.

### 3 Proof of Theorem 1.2

Solutions to (1.2) are critical points of the functional \( E_\lambda : H \to \mathbb{R} \) given by equation (1.3), where \( H \) is taken to be the closure of \( C_0^\infty (0, R) \) with inner product

\[
(A, \lambda) = \int_0^R \left\{ rA\lambda + \left( \frac{m^2}{r^2} + \lambda \right) rA\lambda \right\} \, dr
\]

and norm \( \| A \|_H = (A, A) \). We use a similar function space as in \([8]\). Functions in the space \( H \) satisfy the boundary condition \( A(0) = 0 = A(R) \). This follows identically as in \([8]\) and we omit its justification for brevity. From the inequality

\[
\int_0^R rA^2 \, dr \leq R^2 \int_0^R \frac{A^2}{r} \, dr
\]

we get that \( H \) is embedded in the standard Sobolev space \( W^{1,2}_0(B_R) \), composed of radially symmetric functions vanishing on the boundary, with \( B_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R \} \). Critical points of \( E_\lambda \) satisfy the finite energy condition (2.1) and are smooth solutions of (1.2), which follows via a standard bootstrap argument.

Since we are only interested in nontrivial localized solutions (i.e., exponentially decaying) of (1.2), \( \lambda \) is taken in the range

\[
0 < \lambda < \frac{1}{4a} \frac{r_0^2 + m^2}{R^2}, \tag{3.2}
\]

with \( r_0 (\approx 2.404825) \). From the basic inequality

\[
as^2 - b \ln(1 + cs^2) \geq \frac{a}{c} \left( \frac{bc}{a} - 1 \right) - b \ln \left( \frac{bc}{a} \right),
\]

which holds for any \( a, b, c > 0 \) such that \( bc > a \) and all \( s \in \mathbb{R} \), we get the coercive bound

\[
E_\lambda(A) \geq \frac{1}{4} \| A \|_H^2 + \frac{1}{4} R^2 \left[ 1 - \frac{1}{2} \ln(4a^{-1}) \right]. \tag{3.3}
\]

**Lemma 3.1.** The functional \( E_\lambda : H \to \mathbb{R} \) is Palais–Smale, i.e., for any sequence \( \{A_k\}_{k=1}^{\infty} \subset H \) with the properties that (i) \( E_\lambda(A_k) \) is bounded, and (ii) \( E_\lambda'(A_k) \to 0 \) as \( k \to \infty \) as a sequence in the dual space of \( H \), there is a subsequence of \( \{A_k\}_{k=1}^{\infty} \) converging strongly in \( H \).
Proof. Let $\{A_k\}_{k=1}^\infty$ be a sequence in $H$ such that $\{E_k(A_k)\}_{k=1}^\infty$ is bounded and $E'_k(A_k) \to 0$ in $H$. From inequality (3.3) we get that $\{A_k\}_{k=1}^\infty$ is bounded in $H$. Hence, without loss of generality, $A_k \to A \in H$ (weakly). The compact embedding of $H$ in $L^p((0, R), r dr)$ for any $p \geq 1$, gives that $A_k \to A$ (strongly) in $L^p((0, R), r dr)$.

From $E'_k(A_k) \to 0$ in $H$ we get the existence of a sequence $\{\epsilon_k\}_{k=1}^\infty$ such that $\epsilon_k \geq 0$ for all $k \in \mathbb{N}$, $\epsilon_k \to 0$ as $k \to \infty$, and

$$\left| \int_0^R \left( rA_k \tilde{A}_r + \left( \frac{m^2}{r^2} + \lambda \right) rA_k \tilde{A} - \frac{rA_k^3 \tilde{A}}{(1 + aA_k^2)^2} \right) dr \right| \leq \epsilon_k \|A\|_H$$

for every $\tilde{A} \in H$. Letting $k \to \infty$ in inequality (3.4) then gives

$$\int_0^R \left( rA_k \tilde{A}_r + \left( \frac{m^2}{r^2} + \lambda \right) rA_k \tilde{A} - \frac{rA_k^3 \tilde{A}}{(1 + aA_k^2)^2} \right) dr = 0. \quad (3.5)$$

Let $\tilde{A} = A_k - A$ in inequality (3.4) and equation (3.5). Negating the resulting equation (3.5) and inserting the result into the resulting inequality (3.4), we get

$$\left| \int_0^R \left( \frac{rA_k^3}{(1 + aA_k^2)^2} - \frac{A^3(A_k - A)}{(1 + aA_k^2)^2} \right) dr \right| \leq \epsilon_k \|A_k - A\|_H.$$ 

Consequently,

$$\|A_k - A\|_H^2 \leq \int_0^R \left( \frac{A^3}{(1 + aA_k^2)^2} - \frac{A^3(A_k - A)}{(1 + aA_k^2)^2} \right) dr + \epsilon_k \|A_k - A\|_H,$$

and this may be rewritten as

$$\|A_k - A\|_H^2 \leq 2 \int_0^R \left( \frac{A^3}{(1 + aA_k^2)^2} |A_k - A| \right) dr + \epsilon_k \|A_k - A\|_H \quad (3.6)$$

For all $q \geq \frac{3}{2}$, we have

$$\int_0^R |A_k^q (A_k - A)| r dr \leq \left( \int_0^R |A_k|^{3q/2} r dr \right)^{2/3} \left( \int_0^R |A_k - A|^3 r dr \right)^{1/3} \to 0 \quad (3.7)$$

and

$$\int_0^R |A^q (A - A)| r dr \leq \left( \int_0^R |A|^{3q/2} r dr \right)^{2/3} \left( \int_0^R |A - A|^3 r dr \right)^{1/3} \to 0 \quad (3.8)$$

as $k \to \infty$. Using inequalities (3.7) and (3.8) in inequality (3.6), we get $A_k \to A$ (strongly) in $H$.

The functional $E_A$ is even, coercive, and satisfies the Palais–Smale condition. By the Ljusternik–Schnirelman theorem due to Rabinowitz [9], $E_A$ has infinitely many pairs of critical points. Then Ekeland’s variational principle [9, 15] gives the existence of $\tilde{A} \in H$ such that

$$E'_A(\tilde{A}) = 0 \quad \text{and} \quad E_A(\tilde{A}) = \inf_{A \in H} E_A(A).$$

Thus $\tilde{A}$ is a minima of $E_A$ over $H$. It remains to prove that $\tilde{A}$ is nontrivial and positive. Let $h : \mathbb{R} \to \mathbb{R}$ be given by

$$h(s) = \frac{1}{2} As^2 - a^{-2} \ln(1 + as^2) + \frac{a^{-1}s^2}{1 + as^2}, \quad (3.9)$$
For each $\lambda$ satisfying inequality (3.2) there is a $k > 0$ such that $h(k) < 0$. With this choice of $k$ and by setting $R > 2$ for convenience, consider the test function

$$A_0(r) = \begin{cases} kr, & 0 \leq r \leq 1, \\ k, & 1 \leq r \leq R - 1, \\ k(R - r), & R - 1 \leq r \leq R. \end{cases} \quad (3.10)$$

It follows that

$$E_\lambda(A_0) \leq k^2 \left(R + m^2 \left(\ln(R - 1) + R^2 \ln \left(\frac{R}{R - 1}\right)\right) + \frac{1}{4}M(2R - 1) + \frac{1}{4}h(k)(R^2 - 2R)\right) < 0$$

for $R$ sufficiently large and $M = \max_{s \in [0, k]} h(s)$. Therefore, $E_\lambda(\tilde{A}) < 0$ and $\tilde{A} \neq 0$.

To observe that $\tilde{A}(r) > 0$ for all $r > 0$, replace the nonlinearity

$$j(s) = \frac{\alpha^2s^2}{1 + as^2} - \alpha^2 \ln(1 + as^2)$$

in $E_\lambda$ by

$$j_+(s) = \begin{cases} \frac{\alpha^2s^2}{1 + as^2} - \alpha^2 \ln(1 + as^2), & s \geq 0, \\ 0, & s < 0. \end{cases}$$

Let $h_+(s) = \frac{1}{2} \tilde{\lambda}s^2 - j_+(s)$ and note that $h_+(s) = h(s)$ for all $s \geq 0$. It follows that there is a $k > 0$ such that $h_+(k) < 0$ and, consequently, with $E_+^\lambda : H \to \mathbb{R}$ given by

$$E_+^\lambda(A) = \frac{1}{2} ||A||_H^2 - \frac{1}{2} \int_0^R rj_+(A) \, dr,$$

satisfies $E_+(A_0) < 0$ for $R$ sufficiently large. Moreover, similarly to before, we get that $E_+$ is a $C^1(H; \mathbb{R})$ functional, coercive, and satisfies the Palais–Smale condition. Therefore, there exists a nontrivial $\tilde{A}$ in $H$ such that $(E_+^\lambda)'(\tilde{A}) = 0$,

$$E_+^\lambda(\tilde{A}) = \inf_{A \in H} E_+^\lambda(A) < 0,$$

and is also a smooth solution to

$$A_{rrr} + \frac{1}{r} A_r = \left(\frac{m^2}{r^2} + \lambda\right)A - \frac{1}{2} j_+(A),$$

with $A(0) = 0 = A(R)$. Suppose there is an $\tilde{r} \in (0, R)$ such that $\tilde{A}(\tilde{r}) < 0$. Then there would also be an $r_0 \in (0, R)$ such that $\tilde{A}(r_0) < 0$, $\tilde{A}_r(r_0) = 0$, and $\tilde{A}_{rr}(r_0) \geq 0$. Thus,

$$0 \leq \tilde{A}_{rr}(r_0) = \left(\frac{m^2}{r_0^2} + \lambda\right)\tilde{A}(r_0) - \frac{1}{2} j_+(\tilde{A}(r_0)) < 0,$$

a contradiction. Therefore, $\tilde{A}(r) > 0$ for all $r \in (0, R)$, and $j_+(\tilde{A}) = j'\tilde{A}(\tilde{A})$. Consequently, there is at least one strictly positive solution to (1.2) which minimizes $E_\lambda$.

It remains to show the existence of a saddle point solution. Using $A(0) = 0$, we have

$$A^2(r) = 2 \int_0^r A(s)A_s(s) \, ds \leq 2 \left( \int_0^r sA_s^2(s) \, ds \right)^{1/2} \left( \int_0^r \frac{A_s^2(s)}{s} \, ds \right)^{1/2} \leq C||A||_H^2,$$

where $C > 0$ independent of $A$. Consequently, $||A||_{L^\infty} \leq \sqrt{C}||A||_H$ with $|| \cdot ||_{L^\infty} : H \to \mathbb{R}$ being the infinity norm on $L^\infty[0, R]$. From this embedding we can conclude that $0 \in H$ is a local minima of $E_\lambda$.

**Lemma 3.2.** There are constants $\delta > 0$ and $C_0 > 0$ such that for every $A \in H$ with $0 < ||A||_H < \delta$ we have $E_\lambda(A) > 0$ and $E_\lambda(A) \geq C_0$ for $||A||_H = \delta$.

**Proof.** Let $h : \mathbb{R} \to \mathbb{R}$ be as defined by equation (3.9). For any $\epsilon > 0$, there is a $s_0 > 0$ so that

$$h(s) \geq \frac{1}{2}(\lambda - \epsilon)s^2$$

for all $|s| < s_0$. 


Let $\delta = s_0/\sqrt{C} > 0$, where $C > 0$ is the constant from the embedding of $H$ in $L^\infty[0, R]$, and let $A \in H$ be such that $0 < \|A\|_H < \delta$. It then follows that

$$|A(r)| \leq \|A\|_\infty \leq \sqrt{C}\|A\|_H < \sqrt{C}\delta = s_0$$

and

$$E_A(A) = \int_0^R \left\{ rA_r^2 + \frac{m^2}{r} A^2 + \frac{1}{2} rA^2 + rh(A) \right\} dr \geq \frac{1}{4}\|A\|_H^2 + \frac{1}{4}(\lambda - \epsilon) \int_0^R rA^2 \, dr.$$

Choose $\epsilon = \lambda$ to get $E_A(A) \geq \frac{1}{4}\delta^2 = C_0 > 0$, and the claim follows.

\[ \square \]

**Lemma 3.3.** Let $\delta > 0$. There exists an element $A_0$ in $H$ such that $\|A_0\|_H^2 > \delta$ and $E_A(A_0) < 0$.

**Proof.** Let $A_0$ be the test function defined by equation (3.10). Then its norm on $H$ is

$$\|A_0\|^2 = k^2 \left( R + m^2 \left[ \ln(R - 1) + R^2 \ln\left( \frac{R}{R - 1} \right) - 2 \right] + \frac{1}{2} \lambda R \left( R - \frac{1}{3} \right) \right),$$

which is $O(R^2)$ for $R$ sufficiently large. Thus, for any $\delta > 0$ we can choose $R$ such that $\|A_0\|_H^2 > \delta$. Let the function $h : R \to R$ be as defined by equation (3.9) and let $M = \max_{s \in [0, k]} h(s)$. Recall that there is a $k$ such that $h(k) < 0$, and for $R$ sufficiently large we have

$$E_A(A_0) < 0.$$ 

Therefore, for $R$ sufficiently large and $k$ such that $h(k) < 0$, we have $E_A(A_0) < 0$ and $\|A_0\|_H^2 > \delta$, as claimed. \[ \square \]

Let $\Gamma = \{ g \in C([0, 1]; H) \mid g(0) = 0, g(1) = A_0 \}$. Then Lemmas 3.1–3.3 and the fact $E_A(0) = 0$ allow us to apply the mountain-pass theorem [9, 15] to get the existence of a nontrivial critical point at the level $\eta_0$ defined by

$$\eta_0 = \inf_{g \in \Gamma} \max_{t \in [0, 1]} E_A(g(t)).$$

It follows identically as prior that such a solution is also positive.

### 4 Proof of Theorem 1.3

Consider the functional

$$J(A) = \frac{1}{2} \int_0^R \left\{ rA_r^2 + \frac{m^2}{r} A^2 - rA^{-2} \ln(1 + A^2) + \frac{rA^{-1}A^2}{1 + A^2} \right\} \, dr$$

defined over the nonempty admissible class of functions

$$\mathcal{A} = \{ A(r) \text{ is absolutely continuous on } [0, R], A(0) = 0 = A(R), \mathcal{E}(A) < \infty \},$$

and the constrained minimization problem

$$J_0 = \inf_{A \in \mathcal{A}} \{ J(A) \mid N(A) = N_0 > 0 \}, \quad (4.1)$$

where $\mathcal{E}$ is as given by (2.1). Problem (1.2) may be viewed as a nonlinear eigenvalue problem. The nonlinear frequency shift $\lambda$ is undetermined and appears as an eigenvalue of the constrained minimization problem (4.1). We use similar ideas as in [24] to prove Theorem 1.3.

Note that $J$ satisfies the coercive bound

$$J(A) \geq \frac{1}{2} \int_0^R \left\{ rA_r^2 + \frac{m^2}{r} A^2 \right\} \, dr - \frac{\alpha^{-1}}{2\pi} N_0. \quad (4.2)$$

So the minimization problem (4.1) is well-defined. Since $|A|_1 \leq |A|_r$ and $J$ and $N$ are even functionals, we get $J(A) \geq J(|A|)$ and $N(A) = N(|A|)$. Let $\{ A_j \}_{j=1}^{\infty}$ be a minimizing sequence of (4.1). Such a sequence may be
viewed as consisting of nonnegative radially symmetric functions defined over the disk

\[ B_R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R \} \]

and vanishing on its boundary. From inequality (4.2) there then exists a \( C > 0 \), independent of \( j \), such that

\[ c \geq \int_0^R rA^2_j, dr + \int_0^R \frac{A^2_j}{r} dr. \tag{4.3} \]

The radially symmetric reduced norm

\[ \|A\|^2 = \int_0^R rA^2_j, dr + \int_0^R \frac{A^2_j}{r} dr \]

and inequalities (4.3) and (3.1) give the embedding of \( J \) in \( W^{1,2}_0(B_R) \). From inequality (4.3) we get that \( \|A_j\|_{\infty} \) is bounded in \( W^{1,2}_0(B_R) \). Consequently, without loss of generality, \( A_j \to A \) (weakly) to \( A \in W^{1,2}_0(B_R) \). The compact embedding \( W^{1,2}(B_R) \subseteq L^p(B_R) \), \( p \geq 1 \), then gives that \( A_j \to A \) (strongly) in \( L^p(B_R) \) for every \( p \geq 1 \). Hence, \( A \) is radially symmetric and satisfies the boundary condition \( A(R) = 0 \).

In the weak topology of \( W^{1,2}_0(B_R) \), we have that the functional

\[ J(A) = \int_0^R r \ln(1 + aA^2) \, dr \]

is continuous. From inequality (4.3) and using Fatou’s lemma, we get

\[ \int_0^R rA^2_j, dr \leq \liminf_{j \to \infty} \int_0^R rA^2_{j,}, dr, \]

\[ \int_0^R \frac{A^2_j}{r} dr \leq \liminf_{j \to \infty} \int_0^R \frac{A^2_{j,}}{r} dr, \]

and, as a result, we get the weak lower semi-continuity of the functional \( J \), i.e.,

\[ J(A) \leq \liminf_{j \to \infty} J(A_j). \]

Therefore, \( J(A) = J_0 \) and \( N(A) = \lim_{j \to \infty} N(A_j) = N_0 \). The finite-energy condition holds and \( rA^2_j, A^2_j/r, rA^2_j, \) and \( r \ln(1 + aA^2) \) are all in \( L(0, R) \).

We also need to show that the boundary condition \( A(0) = 0 \) is satisfied. Let \( \{A_j\}_{j=1}^{\infty} \) be a sequence in \( W^{1,2}(e, R) \), where \( e \in (0, R) \). For any \( e \in (0, R) \), \( \{A_j\}_{j=1}^{\infty} \) is bounded in \( W^{1,2}(e, R) \). The compact embedding \( W^{1,2}(e, R) \subseteq C[e, R] \) then gives \( A_j \to A \) uniformly on \( [e, R] \) as \( j \to \infty \). So for any pair \( r_1, r_2 \in (0, R) \) such that \( r_1 < r_2 \) and using \( C \) from (4.3), we get

\[ |A^2_j(r_2) - A^2_j(r_1)| \leq 2 \int_{r_1}^{r_2} |A_j(r)A_j, r(r)| \, dr \leq 2C^{1/2} \left( \int_{r_1}^{r_2} \frac{A^2_j}{r} \, dr \right)^{1/2}. \]

Let \( j \to \infty \) in the above inequality. From the uniform convergence of \( A_j \to A \), we get

\[ |A^2(r_2) - A^2(r_1)| \leq 2C^{1/2} \left( \int_{r_1}^{r_2} \frac{A^2}{r} \, dr \right)^{1/2}. \]

Since \( A^2/r \in L(0, R) \), we may take the limit \( r_1, r_2 \to 0 \) to get the existence of

\[ \xi_0 = \lim_{r \to 0} A^2(r) = 0, \]

and the desired result follows, i.e., \( A(0) = 0 \). Therefore, the function \( A \), obtained as the limit of the minimizing sequence \( \{A_j\}_{j=1}^{\infty} \), is a solution to the constrained minimization problem (4.1), and there is a parameter \( \lambda \) such that the pair \((A, \lambda)\) solves (1.2). Furthermore, such a solution is positive, i.e., \( A(r) > 0 \) for all \( r \in (0, R) \). To see
this, suppose there is an \( r_0 \in (0, R) \) such that \( A(r_0) = 0 \). Then \( r_0 \) would be a minimum point of \( A(r) \) and \( A'(r_0) = 0 \). Then, by the uniqueness of the initial value problem of ordinary differential equations, \( A(r) = 0 \) for all \( r \in (0, R) \), contradicting the constraint \( N(A) = N_0 > 0 \).

We now prove that if \( N_0 \) is too small and \( \lambda > 0 \), then \( A \) is identically zero. To this end, let \( (A, \lambda) \) be the solution pair obtained above. As in Section 2, we are justified to multiply (1.2) by \( rA(r) \) and integrate by parts on \( (0, R) \) to get

\[
-\int_0^R rA^2 dr - \left( \frac{m^2}{r^2} + \lambda \right) rA^2 dr = -\int_0^R \frac{ra^{-1}A^4}{(1 + AA^2)^2} dr \geq -a^{-1} \int_0^R rA^4 dr. \tag{4.4}\]

From Schwarz’s inequality and \( A(0) = 0 \), we get

\[
A^2(\rho) = \int_0^\rho 2A(r)A'(r) dr \leq 2 \left( \int_0^\rho rA^2 dr \right)^{1/2} \left( \int_0^\rho \frac{A^2(r)}{\rho} dr \right)^{1/2}. \]

Multiplying by \( \rho A^2(\rho) \), integrating from 0 to \( R \), and inserting \( N(A) = N_0 \) > 0, we obtain

\[
\int_0^R rA^4 dr \leq \frac{N_0}{\pi} \left( \int_0^R rA^2 dr \right)^{1/2} \left( \int_0^R \frac{A^2}{\rho} dr \right)^{1/2}. \tag{4.5}\]

From the basic inequality \( ab \leq \epsilon a^2 + \frac{b^2}{2\epsilon} \) for every \( a, b \in \mathbb{R} \) and \( \epsilon > 0 \), we then get

\[
\int_0^R rA^4 dr \leq \epsilon \int_0^R rA^2 dr + \frac{N_0^2}{4\pi^2\epsilon} \int_0^R \frac{A^2}{\rho} dr. \tag{4.5}\]

We insert inequality (4.5) in (4.4) and use the constraint \( N(A) = N_0 \) to arrive at

\[
(\alpha^{-1}\epsilon - 1) \int_0^R rA^2 dr + \left( \frac{N_0^2}{4\pi^2m^2} - \frac{1}{\epsilon} \right) \int_0^R \frac{A^2}{\rho} dr - \lambda N_0 \geq 0. \]

Let \( \epsilon = \frac{N_0^2}{(4\pi^2m^2)} \) to get

\[
\left( \frac{\alpha^{-1}}{4\pi^2m^2} - 1 \right) \int_0^R rA^2 dr - \lambda N_0 \geq 0. \]

Thus, if \( N_0 \leq 2\pi m|\sqrt{\alpha} \) and \( \lambda > 0 \), then \( A \equiv 0 \). Otherwise, if \( N_0 \leq 2\pi m|\sqrt{\alpha} \), then \( \lambda < 0 \) and the solution is not localized.

5 Numerical results

The constrained minimization approach of Section 4 allows us to use a finite element formalism to compute the soliton’s profile for any prescribed photon number \( N_0 \). To compare our results with those in [4, 12], we take \( \alpha = 1 \) and \( R = 40 \) in all our computations. The approach here differs from that presented in [4, 12] and is of value by allowing the photon number to be prescribed first and then computing the soliton’s profile.

To this end, let \( \mathcal{A} \) be composed of \( N \) orthonormal functions

\[
\{\psi_j\}_{j=1}^N \subset W^{1,2}_0(B_R) \]

and defined under the inner product

\[
\langle A, \tilde{A} \rangle = 2\pi \int_0^R rA\tilde{A} dr, \quad A, \tilde{A} \in \mathcal{A}, \]

whose form is suggested by equation (1.4). We use the formalism

\[
A = \sum_{j=1}^N a_j \psi_j
\]
to approximate $A \in \mathcal{A}$, where $a_1, \ldots, a_N \in \mathbb{R}$, and insert it into the infinite-dimensional constrained optimization problem (4.1) to get the finite-dimensional problem

$$J_0 = \min \left\{ F(a) = f \left( \sum_{j=1}^{N} a_j \psi_j \right) \left| \sum_{j=1}^{N} a_j^2 = N_0, \ a = (a_1, \ldots, a_N) \in \mathbb{R}^N \right. \right\}. \quad (5.1)$$

Note that $F : \mathbb{R}^N \to \mathbb{R}$ is continuous and defined over a compact set. So (5.1) is well-defined and has a solution. Via MATLAB’s Optimization Toolbox [25], we solve (5.1).

To compute the nonlinear frequency shift $\lambda$, recall that there is a Lagrange multiplier $\xi \in \mathbb{R}$ such that

$$\langle J'(A), \tilde{A} \rangle = \xi \langle N'(A), \tilde{A} \rangle$$

for all $\tilde{A} \in \mathcal{H}$, with $\mathcal{H}$ as defined in Section 4. This weak formulation of (1.2) gives $\lambda = 4\pi \xi$. Let $\tilde{A} = A$ and $N(A) = N_0 > 0$. Then

$$\lambda = -\frac{2\pi}{N_0} \int_{0}^{R} \left\{ rA_r^2 + \frac{m^2}{r} A^2 - \frac{rA^4}{(1+A^2)^2} \right\} dr.$$

We use twenty basis functions ($N = 20$) in the implementation. Figure 1 illustrates the soliton’s profile for varying photon number $N_0$ and fixed charge $m = 1$. Similarly to [4, 12], we observe the formation of localized wave structures with flat-top shapes for large $N_0$. In principle, such flat-top solitons may be created with a width as large as desired by increasing the distance $R$ from the vortex core and photon number $N_0$.

In Figure 2, for $N_0 = 100$, $N_0 = 5000$, and fixed $m = 1$, we present the soliton amplitude $A$ over a spatial domain and observe the formation of the flat-top soliton.

Table 1 shows the corresponding nonlinear frequency shift $\lambda$ and maximum soliton amplitude for each $m$. As $m$ is increased, $\lambda$ increases and approaches the critical value $\lambda_{cr} = 0.2162$. Moreover, the soliton amplitude increases and seems to be bounded above by a certain critical value $A_{cr} = 1.5$. Our numerical results are in agreement with those presented in [4, 12].

In Figure 3, we show the behavior of the soliton’s profile for fixed $N_0 = 500$ and a varying $m$. Table 2 shows that both $\lambda$ and the maximum amplitude of the soliton decrease as $m$ increases.

| $N_0$ | 50 | 100 | 500 | 1000 | 2500 | 5000 |
|-------|----|-----|-----|------|------|------|
| $\lambda$ | 0.0123 | 0.0901 | 0.1816 | 0.1960 | 0.2053 | 0.2089 |
| $A_{\text{max}}$ | 0.2024 | 0.6157 | 1.1761 | 1.3338 | 1.4571 | 1.4920 |

Table 1: Nonlinear frequency shift $\lambda$ and maximum amplitude $A_{\text{max}}$ of single charged ($m = 1$) soliton solutions for varying photon number $N_0$. 

Figure 1: Soliton’s amplitude $A$ for single charged ($m = 1$) soliton solutions and increasing photon number $N_0$. 

Figure 2: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 3: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 4: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 5: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 6: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 7: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 8: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 9: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 10: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 11: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 12: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 13: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 14: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 15: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 16: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 17: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 18: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$. 

Figure 19: Soliton’s amplitude $A$ for different $m$ and fixed $N_0 = 500$. 

Figure 20: Soliton’s amplitude $A$ for different $N_0$ and fixed $m = 1$.
Figure 2: Surface plot of soliton’s amplitude $A$ for photon numbers (a) $N_0 = 100$ and (b) $N_0 = 5000$, both with fixed topological charge $m = 1$.

Figure 3: Solitons amplitude for varying $A$, for varying topological charge $m$.

| $m$ | 1     | 2     | 3     | 4     | 5     |
|-----|-------|-------|-------|-------|-------|
| $\lambda$ | 0.1816 | 0.1599 | 0.1388 | 0.119  | 0.1009 |
| $A_{\text{max}}$ | 1.1761 | 0.9929 | 0.8594 | 0.7520 | 0.6683 |

Table 2: Nonlinear frequency shift $\lambda$ and maximum amplitude $A_{\text{max}}$ of multiple charged ($m = 1, \ldots, 5$) soliton solutions.
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