ON THE MATTILA-SJOLIN THEOREM FOR DISTANCE SETS

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Abstract. We extend a result, due to Mattila and Sjolin, which says that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then the distance set $\Delta(E) = \{|x-y| : x, y \in E\}$ contains an interval. We prove this result for distance sets $\Delta_B(E) = \{||x-y||_B : x, y \in E\}$, where $|| \cdot ||_B$ is the metric induced by the norm defined by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature. We also obtain some detailed estimates pertaining to the Radon-Nikodym derivative of the distance measure.

1. Introduction

The classical Falconer distance conjecture, originated in 1985 ([2]) says that if the Hausdorff dimension of a compact subset of $\mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the set of distances, $\Delta(E) = \{|x-y| : x, y \in E\}$ is positive. Falconer ([2]) proved the first result in this direction by showing that $\mathcal{L}^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. See also [3] and [7] for a thorough description of the problem and related ideas. The best currently known results are due to Wolff in two dimensions, and to Erdogan ([1]) in dimensions three and greater. They prove that $\mathcal{L}^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d}{2} + \frac{1}{3}$.

In another direction, an important addition to this theory is due to Mattila and Sjolin ([8]) who proved that if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$, then $\Delta(E)$ not only has positive Lebesgue measure, but also contains an interval. This is accomplished by showing that the natural measure on the distance set has a continuous density. It was previously shown by Mattila ([6]) that if the ambient dimension is two or three, then the density of the distance measure is not in general bounded if the Hausdorff dimension of the underlying set $E$ is smaller than $\frac{d+1}{2}$. In higher dimensions, this question is still open for the Euclidean metric, but has been resolved if the Euclidean metric is replaced by a metric induced by a norm defined by a suitably chosen paraboloid. See [5].

In this paper we give an alternative proof of the Mattila-Sjolin result and extend it to more general distance sets $\Delta_B(E) = \{||x-y||_B : x, y \in E\}$, where $|| \cdot ||_B$ is the
norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.

Our main result is the following.

**Theorem 1.1.** Let $E$ be a compact subset of $\mathbb{R}^d$, $d \geq 2$, with Hausdorff dimension, denoted by $s$, greater than $\frac{d+1}{2}$. Let $\mu$ be a Frostman measure on $E$. Let $\sigma$ denote the Lebesgue measure on $\partial B$. Define the distance measure $\nu$ by the relation

$$\int h(t)d\nu(t) = \int \int h(||x - y||_B)d\mu(x)d\mu(y),$$

where $|| \cdot ||_B$ is the norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.

- **i)** Then the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure.

- **ii)** We have

$$\frac{\nu((t - \epsilon, t + \epsilon))}{2\epsilon} = M(t) + R(t),$$

where

$$M(t) = \int |\hat{\mu}(\xi)|^2\hat{\sigma}(t\xi)t^{d-1}d\xi$$

is the density of $\nu$ and

$$\sup_{0 < \epsilon < \epsilon_0} |R(t)| \lesssim \epsilon_0^{s - \frac{d+1}{2}}.$$

- **iii)** Moreover, $M \in C^{s - \frac{d+1}{2}}(I)$ for any interval $I$ not containing the origin, where $[u]$ denotes the smallest integer greater than or equal to $u$. In particular, $M$ is continuous away from the origin if $s > \frac{d+1}{2}$ and therefore $\Delta_B(E)$ contains an interval in view of i).

- **iv)** Suppose that $s > k + \alpha$, where $k$ is a non-negative integer and $0 < \alpha < 1$. Then the $k$th derivative of the density function of $\nu$ is Hölder continuous of order $\alpha$.

**Remark 1.2.** Metric properties of $|| \cdot ||_B$ are not used in the proof of Theorem 1.1.

Let $\Gamma$ be a star shaped body in the sense that for every $\omega \in S^{d-1}$ there exists $1 < r_0(\omega) < 2$ such that $\{r\omega : 0 \leq r \leq r_0(\omega)\} \subset \Gamma$ and $\{r\omega : r > r_0\} \cap \Gamma = \emptyset$. Define $||x||_\Gamma = \inf\{t > 0 : x \in t\Gamma\}$ and let $\Delta_\Gamma(E) = \{||x - y||_\Gamma : x, y \in E\}$. Let $\sigma_\Gamma$ denote the Lebesgue measure on the boundary of $\Gamma$. Then if $|\sigma_\Gamma(\xi)| \lesssim |\xi|^{-\frac{d+1}{2}}$, the conclusion of Theorem 1.1 holds with the same exponents.
1.1. Sharpness of results: As we note above, Mattila’s construction ([6]) shows that if the Hausdorff dimension of $E$ is smaller than $\frac{d+1}{2}$, $d = 2, 3$, then the density of distance measure is not in general bounded in the case of the Euclidean metric. Moreover, Mattila construction can be easily extended to all metrics generated by a bounded convex body $B$ with a smooth boundary and non-vanishing Gaussian curvature.

In dimensions four and higher, all we know at the moment (see the main result in [5]) is that there exists a bounded convex body $B$ with a smooth boundary and non-vanishing curvature, such that the density of the distance measure is not in general bounded if the Hausdorff dimension of the underlying set $E$ is less than $\frac{d+1}{2}$. We do not know what happens when the Hausdorff dimension of $E$ equals $\frac{d+1}{2}$ in any dimension and for any smooth metric.

It would be very interesting if any of these results actually depended on the underlying convex body $B$ in a non-trivial way. This would mean that smoothness and non-vanishing Gaussian curvature of the level set do not tell the whole story. There is some evidence that this may be the case. See, for example, [4], where connections between problems of Falconer type and distribution of lattice points in thin annuli are explored.

If the Hausdorff dimension of $E$ is less than $\frac{d}{2}$, then the density of the distance measure, for any metric induced by a bounded convex body $B$ with a smooth boundary and non-vanishing curvature is not in general bounded by a construction due to Falconer ([2]).

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2. Proof of Theorem 1.1

2.1. Proof of items i) and ii). The proof of item i) of Theorem 1.1 is due to Falconer ([2]) and Mattila ([6]). This brings us to item ii). Recall that every compact set $E$ in $\mathbb{R}^d$, of Hausdorff dimension $s > 0$ possesses a Frostman measure (see e.g. [7], p. 112), which is a probability measure $\mu$ with the property that for every ball of radius $r^{-1}$, denoted by $B_{r^{-1}},$

$$\mu(B_{r^{-1}}) \lesssim r^{-s},$$

where here, and throughout, $X \lesssim Y$, with the controlling parameter $r$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon r^\epsilon Y$. Let
\[\nu^\varepsilon(t) = \frac{\nu((t - \varepsilon, t + \varepsilon))}{2\varepsilon} = \frac{1}{2\varepsilon} \mu \times \mu \{ (x, y) : t - \varepsilon \leq ||x - y||_B \leq t + \varepsilon \}.\]

We shall prove that \(\lim_{\varepsilon \to 0} \nu^\varepsilon(t)\) exists and is a \(C^{(s - \frac{d+1}{2})}\) function.

Let \(\rho\) be a smooth cut-off function, identically equal to 1 in the unit ball and vanishing outside the ball of radius 2. Let \(\rho^\varepsilon(x) = \varepsilon^{-d} \rho(x/\varepsilon)\). It is not hard to check that one can construct a \(\rho\) such that the difference between \(\nu^\varepsilon(t)\) and

\[
\int \int \sigma_t * \rho^\varepsilon(x - y)d\mu(x)d\mu(y),
\]

where \(\sigma_t\) is the surface measure on the set \(\{ x : ||x||_B = t \}\), is \(o(\varepsilon)\), so there is no harm in taking (2.1) as the definition of \(\nu^\varepsilon(t)\). By the Fourier inversion formula,

\[
\nu^\varepsilon(t) = \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_t(\xi) \hat{\rho}(\varepsilon \xi) d\xi
\]

\[
= \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_t(\xi) d\xi - \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_t(\xi)(1 - \hat{\rho}(\varepsilon \xi)) d\xi
\]

\[= M(t) + R^\varepsilon(t).\]

We shall prove that \(M(t)\) is a \(C^{(s - \frac{d+1}{2})}\) function and that \(\lim_{\varepsilon \to 0} R^\varepsilon(t) = 0\). We start with the latter. We shall need the following stationary phase estimate. See, for example, \([10], [9]\) or \([11]\).

**Lemma 2.1.** Let \(\sigma\) be the surface measure on a compact piece of a smooth convex surface in \(\mathbb{R}^d\), \(d \geq 2\), with everywhere non-vanishing Gaussian curvature. Then

\[|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}},\]

where here, and throughout, \(X \lesssim Y\) means that there exists \(C > 0\) such that \(X \leq CY\).

We shall also need the following well-known estimate. See, for example, \([3]\) and \([7]\).

**Lemma 2.2.** Let \(\mu\) be a Frostman measure on a compact set \(E\) of Hausdorff dimension \(s > 0\). Then

\[
\int_{2^j \leq ||\xi|| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 d\xi \lesssim 2^{j(d-s)},
\]

and, consequently,

\[
\int |\hat{\mu}(\xi)|^2 |\xi|^{-\gamma} d\xi = c \int \int |x - y|^{-d+\gamma} d\mu(x)d\mu(y) \lesssim 1
\]

if \(\gamma > d - s\).
To prove the lemma, observe that
\[ \int_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 d\xi \lesssim \int |\hat{\mu}(\xi)|^2 \psi(2^{-j} \xi) d\xi, \]
where \( \psi \) is a suitable smooth function supported in \( \{ x \in \mathbb{R}^d : 1/2 \leq |x| \leq 4 \} \) and identically equal to 1 in the unit annulus. By definition of the Fourier transform and the Fourier inversion theorem, this expression is equal to
\[ 2^d \int \int \hat{\psi}(2^j (x-y)) d\mu(x)d\mu(y) \lesssim 2^{j(d-s)} \]
since \( \hat{\psi} \) decays rapidly at infinity.

By Lemma 2.1 and Lemma 2.2, we have
\[ |R^s(t)| \lesssim \int_{|\xi| > \frac{1}{8}} |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \]
\[ \lesssim \sum_{j > \log_2(1/\epsilon)} \int_{2^j \leq |\xi| \leq 2^{j-1}} |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \]
\[ \lesssim \sum_{j > \log_2(1/\epsilon)} 2^{j(d-s)} 2^{-j \frac{d-1}{2}} \lesssim \epsilon^{s-\frac{d+1}{2}}, \]
and thus \( \lim_{\epsilon \to 0} R^s(t) = 0 \). (To handle \( |R^s(t)| \) over the integral when \( |\xi| < \frac{1}{8} \), we notice that \( (1 - \hat{\rho}(\epsilon \xi)) \) is 0 when \( \xi = 0 \) and, by continuity, is small in a neighborhood about 0. Then, we may dilate and re-define \( \rho \) to get that \( (1 - \hat{\rho}(\epsilon \xi)) \) is small in a neighborhood of our choosing.) This calculation establishes all the claims in part ii) of Theorem 1.1

2.2. Proof of item iii). Once again, by Lemma 2.1, we have
\[ |M(t)| \lesssim \int |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}} d\xi \]
and by the calculation identical to the one in the previous paragraph, we see that this quantity is \( \lesssim 1 \) if the Hausdorff dimension of \( E \) is greater than \( \frac{d+1}{2} \). Continuity follows by the Lebesgue dominated convergence theorem. The convergence of the integral allows us to differentiate inside the integral sign. We obtain
\[ M'(t) = \int |\hat{\mu}(\xi)|^2 \frac{d}{dt} \left\{ t^{d-1} \hat{\sigma}(t\xi) \right\} d\xi. \]
We have
\[ \frac{d}{dt} \left\{ t^{d-1} \hat{\sigma}(t\xi) \right\} = (d-1)t^{d-2} \hat{\sigma}(t\xi) + t^{d-1} \nabla \hat{\sigma}(t\xi) \cdot \xi. \]
Applying Lemma 2.1 once more, the best we can say is that
\[ \left| \frac{d}{dt} \{ t^{d-1}\tilde{\sigma}(t\xi) \} \right| \lesssim |\xi|^{-\frac{d+1}{2}+1}. \]

Repeating the argument in 2.2, we see that \( M'(t) \) exists if the Hausdorff dimension of \( E \) is greater than \( \frac{d+1}{2} + 1 \). Proceeding in the same way one establishes that
\[ \frac{d^m}{dt^m} \{ t^{d-1}\tilde{\sigma}(t\xi) \} \lesssim |\xi|^{-\frac{d+1}{2}+m} \]
and the conclusion of Theorem 1.1 follows.

2.3. **Proof of item iv).** We shall deal with the case \( k = 0 \), as the other cases follow from a similar argument. Let
\[ \lambda(t) = t^{d-1}\tilde{\sigma}(t\xi). \]

We must show that
\[ |M(u) - M(v)| \leq C|u - v|^\alpha. \]

We have
\[ M(u) - M(v) = \int |\hat{\mu}(\xi)|^2 (\lambda(u) - \lambda(v)) d\xi \]
\[ = \int |\hat{\mu}(\xi)|^2 (\lambda(u) - \lambda(v))^\alpha (\lambda(u) - \lambda(v))^{1-\alpha} d\xi. \]

Now,
\[ \lambda(u) - \lambda(v) = (u - v)\lambda'(c), \]
where \( c \in (u, v) \), by the mean-value theorem. It follows that
\[ |\lambda(u) - \lambda(v)|^\alpha \leq |u - v|^\alpha |\lambda'(c)|^\alpha. \]

On the other hand,
\[ |\lambda(u) - \lambda(v)|^{1-\alpha} \leq |\lambda(u)|^{1-\alpha} + |\lambda(v)|^{1-\alpha}. \]

We have already shown above that
\[ |\lambda(u)| \lesssim |\xi|^{-\frac{d+1}{2}} \text{ and } |\lambda'(u)| \lesssim |\xi|^{-\frac{d+1}{2}+1}. \]

It follows that
\[ |M(u) - M(v)| \lesssim |u - v|^\alpha \int |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d+1}{2}+\alpha} d\xi \lesssim |u - v|^{-\alpha}, \]
where the last step follows by Lemma 2.2, and so the item iv) follows.
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