Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero-Moser model

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Abstract

We consider solutions of the matrix KP hierarchy that are elliptic functions of the first hierarchical time $t_1 = x$. It is known that poles $x_i$ and matrix residues at the poles $\rho^{\alpha\beta}_i = a^{\alpha}_i b^{\beta}_i$ of such solutions as functions of the time $t_2$ move as particles of spin generalization of the elliptic Calogero-Moser model (elliptic Gibbons-Hermsen model). In this paper we establish the correspondence with the spin elliptic Calogero-Moser model for the whole matrix KP hierarchy. Namely, we show that the dynamics of poles and matrix residues of the solutions with respect to the $k$-th hierarchical time of the matrix KP hierarchy is Hamiltonian with the Hamiltonian $H_k$ obtained via an expansion of the spectral curve near the marked points. The Hamiltonians are identified with the Hamiltonians of the elliptic spin Calogero-Moser system with coordinates $x_i$ and spin degrees of freedom $a^{\alpha}_i$, $b^{\beta}_i$.

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1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is an archetypal infinite hierarchy of compatible nonlinear differential equations with infinitely many independent (time) variables \( t = \{ t_1, t_2, t_3, \ldots \} \). In the Lax-Sato formalism, the main object is the Lax operator which is a pseudo-differential operator of the form

\[
L = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \ldots
\]

(1.1)

The coefficient functions \( u_i \) are dependent variables. Among all solutions to equations of the hierarchy, of special interest are solutions which have a finite number of poles in the variable \( x \) in a fundamental domain of the complex plane. Most general solutions of this type are those for which the coefficient functions \( u_i \) are elliptic (double-periodic in the complex plane) functions of \( x \) with poles depending on the times \( t \).

The study of singular solutions to nonlinear integrable equations and dynamics of their poles was initiated in the pioneering papers [1, 2, 3, 4]. Now it is a well known subject in the theory of integrable systems. The remarkable result is that the poles of solutions to the KP equation as functions of the time \( t_2 \) move as particles of the integrable Calogero-Moser many-body system [5, 6, 7, 8] which is known to be integrable, i.e. having a large number of integrals of motion in involution. Elliptic, trigonometric and rational solutions correspond respectively to the Calogero-Moser systems with elliptic, trigonometric and rational potentials.

In the work [9], Shiota has shown that in the case of rational solutions the correspondence between the KP equation and the rational Calogero-Moser system can be extended to the whole KP hierarchy. Namely, the evolution of poles with respect to the higher time \( t_m \) was considered and it was shown that it is described by the higher Hamiltonian
flow of the rational Calogero-Moser system with the Hamiltonian $H_m = \text{tr} L^m$, where $L$ is the Lax matrix depending on the coordinates and momenta in a special way. Recently this remarkable correspondence was generalized to trigonometric and elliptic solutions to the KP hierarchy (see respectively [10, 11] and [12]). However, in the elliptic case this correspondence is no longer formulated in terms of traces of the Lax matrix (which in this case depends on a spectral parameter). Instead, the Hamiltonian $H_m$ which governs the dynamics of poles with respect to $t_m$ is shown to be obtained by expansion of the Calogero-Moser spectral curve near a distinguished marked point. It was also shown in [12] that for trigonometric and rational degenerations of elliptic solutions this construction gives the results which agree with the previously obtained ones for trigonometric and rational solutions.

There exists a matrix generalization of the KP hierarchy (matrix KP hierarchy). In matrix KP hierarchy, the coefficient functions $u_i$ in the Lax operator (1.1) are $n \times n$ matrices. Like the KP hierarchy, it is an infinite set of compatible nonlinear differential equations with infinitely many independent variables $t$ and matrix dependent variables. It is a subhierarchy of a more general multi-component ($n$-component) KP hierarchy [13, 14, 15, 16], which has an extended set of independent variables $\{t_1, t_2, \ldots, t_n\}$, $t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}$, $\alpha = 1, \ldots, n$. The matrix KP hierarchy is obtained by the restriction $t_{\alpha,m} = t_m$ for all $\alpha, m$.

The elliptic, trigonometric and rational solutions to the matrix KP equation were investigated in [17]. In the matrix case, the data of singular solutions include not only the positions of poles $x_i$ but also some “internal degrees of freedom” which are matrix residues at the poles (they were fixed in the scalar case). In the work [17] it was shown that the dynamics of the data of such solutions with respect to the time $t_2$ is isomorphic to the dynamics of a spin generalization of the Calogero-Moser system which is also known as the Gibbons-Hermsen model [18]. It is a system of $N$ particles with coordinates $x_i$ and with internal degrees of freedom represented by $n$-dimensional column vectors $a_i, b_i$ with components $a_i^\alpha, b_i^\alpha, \alpha = 1, \ldots, n$. The rank 1 matrices $\rho_i = a_i b_i^T$, where $b_i^T$ is the row vector obtained from the vector $b_i$ by transposition, represent matrix residues at the poles $x_i$. The particles pairwise interact with each other. The Hamiltonian of the elliptic model is

$$H = \sum_{i=1}^{N} p_i^2 - \sum_{i \neq k} (b_i^T a_k)(b_k^T a_i) \wp(x_i - x_k), \quad (1.2)$$

where $\wp(x)$ is the Weierstrass $\wp$-function which is the elliptic function with the only second order pole at $x = 0$ in the fundamental domain. The non-vanishing Poisson brackets are $\{x_i, p_k\} = \delta_{ik}$, $\{a_i^\alpha, b_k^\beta\} = \delta_{\alpha\beta}\delta_{ik}$. The model is known to be integrable and possessing the Lax representation with the Lax matrix $L(\lambda)$ depending on a spectral parameter $\lambda$ lying on an elliptic curve.

The extension of the isomorphism between rational and trigonometric solutions of the matrix KP equation and the Gibbons-Hermsen system to the whole hierarchy was recently made in [19] for rational solutions and in [20] for trigonometric ones. In this paper we generalize these results to elliptic solutions of the matrix KP hierarchy.

Our main result is that the dynamics of poles $x_i$ and vectors $a_i, b_i$ which parametrize matrix residues at the poles with respect to all higher times $t_m$ of the matrix KP hierarchy is Hamiltonian, with the corresponding Hamiltonians being higher Hamiltonians of the
spin elliptic Calogero-Moser model. We find them in terms of expansion of the spectral curve
\[
\det_{N \times N} \left( (z + \zeta(\lambda)) I - L(\lambda) \right) = 0
\] (1.3)
(\zeta(\lambda) is the Weierstrass \(\zeta\)-function) near some distinguished marked points at infinity. The spectral curve is a covering of the elliptic curve, where the spectral parameter \(\lambda\) lives. We show that above the point \(\lambda = 0\), there are \(n\) points at infinity \(P_\alpha = (\infty, 0)\), where \(z = \infty\), so that there are \(n\) distinguished sheets of the covering (neighborhoods of the points \(P_\alpha\)). In a neighborhood of the point \(\lambda = 0\) different branches of the function \(z(\lambda)\) such that \(z(\lambda) \to \infty\) as \(\lambda \to 0\) are defined by the equation of the spectral curve. Let us denote them by \(z_\alpha(\lambda)\) and let \(\lambda_\alpha(z)\) be inverse functions. Our main result is that
the sum over all branches \(\sum_{\alpha=1}^n \lambda_\alpha(z)\) is the generating function for the Hamiltonians \(H_m\):
\[
\sum_{\alpha=1}^n \lambda_\alpha(z) = -Nz^{-1} - \sum_{m \geq 1} z^{-m-1} H_m
\] (1.4)
or \(H_m = -\sum_{\alpha=1}^n \text{res}_\infty (z^m \lambda_\alpha(z))\). We also show that the degeneration of this construction to the rational and trigonometric cases allows one to reproduce the results of the papers [19] and [20].

The organization of the paper is as follows. In section 2 we remind the reader the main facts about the multi-component and matrix KP hierarchies. We recall both Lax-Sato approach based on Lax equations and the bilinear (Hirota) approach based on the bilinear relation for the tau-function. In section 3 we introduce elliptic solutions and discuss the corresponding double-Bloch solutions for the wave function. Section 4 contains derivation of the dynamics of poles and residues with respect to the time \(t_2\). Following [17], we derive the equations of motion together with their Lax representation. In section 5 we discuss properties of the spectral curve and define the distinguished branches of the function \(z(\lambda)\) around the point \(\lambda = 0\). Sections 6 and 7 are devoted to derivation of the Hamiltonian dynamics of respectively poles and residues in the higher times \(t_m\). In section 8 we find explicitly the first two Hamiltonians using the expansion [14] and identify them with the Hamiltonians of the spin generalization of the Calogero-Moser system. Finally, in section 9 we consider the rational and trigonometric degenerations of our construction and show that the results of the previous works are reproduced by the new approach.

## 2 The matrix KP hierarchy

Here we briefly review the main facts about the multi-component and matrix KP hierarchies following [15, 16]. We start from the more general multi-component KP hierarchy. The independent variables are \(n\) infinite sets of continuous “times”
\[
t = \{t_1, t_2, \ldots, t_n\}, \quad t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \quad \alpha = 1, \ldots, n.
\]
It is convenient to introduce also the variable \(x\) such that
\[
\partial_x = \sum_{\alpha=1}^n \partial_{t_{\alpha,1}}.
\] (2.1)
The hierarchy is an infinite set of evolution equations in the times $t$ for matrix functions of the variable $x$.

In the Lax-Sato formalism, the main object is the Lax operator which is a pseudo-differential operator of the form

$$\mathcal{L} = \partial_x + u_1 \partial_x^{-1} + u_2 \partial_x^{-2} + \ldots$$

(2.2)

where the coefficients $u_i = u_i(x, t)$ are $n \times n$ matrices. The coefficient functions $u_k$ depend on $x$ and also on all the times:

$$u_k(x, t) = u_k(x + t_{1,1}, x + t_{2,1}, \ldots, x + t_{n,1}; t_{1,2}, \ldots, t_{n,2}; \ldots).$$

Besides, there are other matrix pseudo-differential operators $\mathcal{R}_1, \ldots, \mathcal{R}_n$ of the form

$$\mathcal{R}_\alpha = E_\alpha + u_{\alpha,1} \partial_x^{-1} + u_{\alpha,2} \partial_x^{-2} + \ldots,$$

(2.3)

where $E_\alpha$ is the $n \times n$ matrix with the $(\alpha, \alpha)$ element equal to 1 and all other components equal to 0, and $u_{\alpha,i}$ are also $n \times n$ matrices. The operators $\mathcal{L}, \mathcal{R}_1, \ldots, \mathcal{R}_n$ satisfy the conditions

$$\mathcal{L} \mathcal{R}_\alpha = \mathcal{R}_\alpha \mathcal{L}, \quad \mathcal{R}_\alpha \mathcal{R}_\beta = \delta_{\alpha\beta} \mathcal{R}_\alpha, \quad \sum_{\alpha=1}^n \mathcal{R}_\alpha = I,$$

(2.4)

where $I$ is the unity matrix. The Lax equations of the hierarchy which define evolution in the times read

$$\partial_{t_{\alpha,k}} \mathcal{L} = [B_{\alpha,k}, \mathcal{L}], \quad \partial_{t_{\alpha,k}} \mathcal{R}_\beta = [B_{\alpha,k}, \mathcal{R}_\beta], \quad B_{\alpha,k} = \left(\mathcal{L}_k \mathcal{R}_\alpha\right)_+, \quad k = 1, 2, 3, \ldots,$$

(2.5)

where $(\ldots)_+$ means the differential part of a pseudo-differential operator, i.e. the sum of all terms with $\partial_x^k$, where $k \geq 0$.

Let us introduce the matrix pseudo-differential “wave operator” $\mathcal{W}$ with matrix elements

$$\mathcal{W}_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k \geq 1} \xi_{\alpha\beta}(x, t) \partial_x^{-k},$$

(2.6)

where $\xi_{\alpha\beta}(x, t)$ are the some matrix functions. The operators $\mathcal{L}$ and $\mathcal{R}_\alpha$ are obtained from the “bare” operators $I \partial_x$ and $E_\alpha$ by ‘dressing” by means of the wave operator:

$$\mathcal{L} = \mathcal{W} \partial_x \mathcal{W}^{-1}, \quad \mathcal{R}_\alpha = \mathcal{W} E_\alpha \mathcal{W}^{-1}.$$  

(2.7)

Clearly, there is an ambiguity in the definition of the dressing operator: it can be multiplied from the right by any pseudo-differential operator with constant coefficients commuting with $E_\alpha$ for any $\alpha$.

A very important role in the theory is played by the wave function $\Psi$ and its adjoint $\Psi^\dagger$ (hereafter, $^\dagger$ does not mean Hermitian conjugation). The wave function is defined as a result of action of the wave operator to the exponential function:

$$\Psi(x, t; z) = \mathcal{W} \exp\left(xzI + \sum_{\alpha=1}^n E_\alpha \xi(t_\alpha, z)\right),$$

(2.8)

where we use the standard notation

$$\xi(t_\alpha, z) = \sum_{k \geq 1} t_{\alpha,k} z^k.$$
By definition, the operators $\partial_x^{-k}$ with negative powers act to the exponential function as $\partial_x^{-k} e^{xz} = z^{-k} e^{xz}$. The wave function depends on the spectral parameter $z$ which does not enter the auxiliary linear problems explicitly. The adjoint wave function is introduced by the formula

$$
\Psi^\dagger(x, t; z) = \exp(-xzI - \sum_{\alpha=1}^{n} E_\alpha \xi(t_\alpha; z)) \mathcal{W}^{-1}.
$$

(2.9)

Here we use the convention that the operators $\partial_x$ which enter $\mathcal{W}^{-1}$ act to the left rather than to the right, the left action being defined as $f \partial_x \equiv -\partial_x f$. Clearly, the expansion of the wave function as $z \to \infty$ is as follows:

$$
\Psi_{\alpha\beta}(x, t; z) = e^{xz+\xi(t_\beta, z)} \left( \delta_{\alpha\beta} + \xi_{(1)}^{1} z^{-1} + \xi_{(2)}^{2} z^{-2} + \ldots \right).
$$

(2.10)

As is proved in [16], the wave function satisfies the linear equations

$$
\partial_{\alpha,m} \Psi(x, t; z) = B_{\alpha,m} \Psi(x, t; z),
$$

(2.11)

where $B_{\alpha,m}$ is the differential operator (2.5), i.e. $B_{\alpha,m} = (\mathcal{W} E_\alpha \partial_x \mathcal{W}^{-1})_\alpha$. And the adjoint wave function satisfies the transposed equations

$$
-\partial_{\alpha,m} \Psi^\dagger(x, t; z) = \Psi^\dagger(x, t; z) B_{\alpha,m}.
$$

(2.12)

Again, the operator $B_{\alpha,m}$ here acts to the left. In particular, it follows from (2.11), (2.12) at $m = 1$ that

$$
\sum_{\alpha=1}^{n} \partial_{\alpha,1} \Psi(x, t; z) = \partial_x \Psi(x, t; z), \quad \sum_{\alpha=1}^{n} \partial_{\alpha,1} \Psi^\dagger(x, t; z) = \partial_x \Psi^\dagger(x, t; z),
$$

(2.13)

so the vector field $\partial_x$ can be naturally identified with the vector field $\sum_{\alpha} \partial_{\alpha,1}$.

Another approach to the multi-component KP hierarchy is provided by the bilinear formalism. In the bilinear formalism, the dependent variables are the tau-function $\tau(x, t)$ and tau-functions $\tau_{\alpha\beta}(x, t)$ such that $\tau_{\alpha\alpha}(x, t) = \tau(x, t)$ for any $\alpha$. The $n$-component KP hierarchy is the infinite set of bilinear equations for the tau-functions which are encoded in the basic bilinear relation

$$
\sum_{\nu=1}^{n} \epsilon_{\alpha\nu} \epsilon_{\beta\nu} \int_{C_\infty} dz \ z^{\delta_{\alpha\nu} + \delta_{\beta\nu} - 2} e^{\xi(t_\nu - t_\nu'; z)} \tau_{\alpha\nu}(x, t - [z^{-1}]_\nu) \tau_{\beta\nu}(x, t' + [z^{-1}]_\nu) = 0
$$

(2.14)

valid for any $t, t'$. Here $\epsilon_{\alpha\beta}$ is a sign factor: $\epsilon_{\alpha\beta} = 1$ if $\alpha \leq \beta$, $\epsilon_{\alpha\beta} = -1$ if $\alpha > \beta$. In (2.14) we use the following standard notation:

$$
(t \pm [z^{-1}]_\gamma)_\alpha = t_{\alpha,k} \pm \delta_{\alpha\gamma} z^{-k}.
$$

The integration contour $C_\infty$ is a big circle around $\infty$.

The tau-functions are universal dependent variables of the hierarchy. All other objects including the coefficient functions $u_i$ of the Lax operator and the wave functions can be expressed in terms of them. In particular, for the wave function and its adjoint we have:

$$
\Psi_{\alpha\beta}(x, t; z) = \epsilon_{\alpha\beta} \frac{\tau_{\alpha\beta}(x, t - [z^{-1}]_\beta)}{\tau(x, t)} z^{\delta_{\alpha\beta} - 1} e^{\xi(t_\beta, z)},
$$

(2.15)

$$
\Psi_{\alpha\beta}^\dagger(x, t; z) = \epsilon_{\beta\alpha} \frac{\tau_{\alpha\beta}(x, t + [z^{-1}]_\alpha)}{\tau(x, t)} z^{\delta_{\alpha\beta} - 1} e^{-\xi(t_\alpha, z)}
$$
Note that the bilinear relation (2.14) can be written in the form

\[ \oint_{C_\infty} dz \, \Psi(x; t; z) \Psi^\dagger(x; t'; z) = 0. \]  

(2.16)

The coefficient \( \xi_{\alpha\beta}^{(1)}(x, t) \) plays an important role in what follows. Equations (2.15) imply that this coefficient is expressed through the tau-functions as

\[ \xi_{\alpha\beta}^{(1)}(x, t) = \begin{cases} \\
 \frac{\epsilon_{\alpha\beta}}{\tau(x, t)}, & \alpha \neq \beta, \\
 -\partial_{t_\beta} \tau(x, t), & \alpha = \beta.
\end{cases} \]  

(2.17)

Let us point out a useful corollary of the bilinear relation (2.14). Differentiating it with respect to \( t_{\kappa,m} \) and putting \( t' = t \) after this, we obtain:

\[ \frac{1}{2\pi i} \oint_{C_\infty} dz \, z^m \Psi_{\alpha\kappa}(x; t; z) \Psi^\dagger_{\kappa\beta}(x; t; z) = -\partial_{t_{\kappa,m}} \xi_{\alpha\beta}^{(1)}(x, t) \]  

(2.18)

or, equivalently,

\[ \text{res}_{\infty} \left( z^m \Psi_{\alpha\kappa} \Psi^\dagger_{\kappa\beta} \right) = -\partial_{t_{\kappa,m}} \xi_{\alpha\beta}^{(1)}. \]  

(2.19)

The integrand in (2.18) should be regarded as a Laurent series in \( z \) and the residue at infinity is defined according to the convention \( \text{res}_{\infty} (z^{-k}) = \delta_{k1} \).

The matrix KP hierarchy is a subhierarchy of the multi-component KP one. It is obtained by the following restriction of the independent variables: \( t_{\alpha,m} = t_m \) for each \( \alpha \) and \( m \). The corresponding vector fields are related as \( \partial_{t_m} = \sum_{\alpha=1}^n \partial_{t_{\alpha,m}} \). As is clear from (2.10), the wave function for the matrix KP hierarchy has the expansion

\[ \Psi_{\alpha\beta}(x, t; z) = \left( \delta_{\alpha\beta} + \xi_{\alpha\beta}^{(1)}(t) z^{-1} + O(z^{-2}) \right) e^{xz + \xi(t, z)}, \]  

(2.20)

where \( \xi(t, z) = \sum_{k \geq 1} t_k z^k \). Equations (2.11) imply that the wave function of the matrix KP hierarchy and its adjoint satisfy the linear equations

\[ \partial_{t_m} \Psi(t; z) = B_m \Psi(t; z), \quad -\partial_{t_m} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_m, \quad m \geq 1, \]  

(2.21)

where \( B_m \) is the differential operator \( B_m = \left( \mathcal{W} \partial^m_x \mathcal{W}^{-1} \right)_+ \). At \( m = 1 \) we have \( \partial_{t_1} \Psi = \partial_x \Psi \), so we can identify \( \partial_x = \partial_{t_1} = \sum_{\alpha=1}^N \partial_{t_{\alpha,1}} \). This means that the evolution in the time \( t_1 \) is simply a shift of the variable \( x \): \( \xi^{(k)}(x, t_1, t_2, \ldots) = \xi^{(k)}(x+t_1, t_2, \ldots) \). At \( m = 2 \) equations (2.21) turn into the linear problems

\[ \partial_{t_2} \Psi = \partial_x^2 \Psi + 2V(x, t) \Psi, \]  

(2.22)

\[ -\partial_{t_2} \Psi^\dagger = \partial_x^2 \Psi^\dagger + 2 \Psi^\dagger V(x, t) \]  

(2.23)
which have the form of the matrix non-stationary Schrödinger equations with

\[ V(x, t) = -\partial_x \xi^{(1)}(x, t). \]  

(2.24)

Summing (2.18) over \( \kappa \), we obtain an analog of (2.18) for the matrix KP hierarchy:

\[ \frac{1}{2\pi i} \sum_{\nu=1}^{n} \oint_{C_{\infty}} dz z^{m} \Psi_{\alpha\nu}(x, t; z) \Psi_{\nu\beta}^\dagger(x, t; z) = -\partial_{t} \xi^{(1)}_{\alpha\beta}(x, t). \]  

(2.25)

Below we will use equations (2.18) and (2.25) for derivation of dynamics of poles and residues of elliptic solutions in higher times.

3 Elliptic solutions of the matrix KP hierarchy and double-Bloch functions

Our aim is to study solutions to the matrix KP hierarchy which are elliptic functions of the variable \( x \) (and, therefore, \( t \)). For the elliptic solutions, we take the tau-function in the form

\[ \tau(x, t) = C \prod_{i=1}^{N} \sigma(x - x_i(t)), \]  

(3.1)

where

\[ \sigma(x) = \sigma(x | \omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s^2}} \]  

and \( s = 2\omega m + 2\omega'm' \) with integer \( m, m' \)

is the Weierstrass \( \sigma \)-function with quasi-periods \( 2\omega, 2\omega' \) such that \( \text{Im}(\omega'/\omega) > 0 \). It is connected with the Weierstrass \( \zeta \)- and \( \wp \)-functions by the formulas \( \zeta(x) = \sigma'(x)/\sigma(x), \) \( \wp(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x) \). The monodromy properties of the function \( \sigma(x) \) are

\[ \sigma(x + 2\omega) = -e^{2\zeta(\omega)(x+\omega)}\sigma(x), \quad \sigma(x + 2\omega') = -e^{2\zeta(\omega')(x+\omega')}\sigma(x), \]  

(3.2)

where the constants \( \zeta(\omega), \zeta(\omega') \) are related by \( \zeta(\omega)\omega - \zeta(\omega')\omega = \pi i/2 \). The \( N \) zeros \( x_i \) of (3.1) are assumed to be all distinct.

We also assume that the tau-functions \( \tau_{\alpha\beta} \) at \( \alpha \neq \beta \) have the form

\[ \tau_{\alpha\beta}(x, t) = C_{\alpha\beta} \prod_{i=1}^{N} \sigma(x - x_i^{(\alpha\beta)}(t)), \]  

(3.3)

with

\[ \sum_i x_i(t) = \sum_i x_i^{(\alpha\beta)}(t) \text{ for all } \alpha, \beta. \]  

(3.4)

The consistency of this assumption is justified below.

Equation (2.17) together with the condition (3.4) implies that \( V(x, t) = -\partial_x \xi^{(1)} \) in the linear problem (2.22) is an elliptic function of \( x \). Therefore, one can find solutions
to (2.22) which are double-Bloch functions. The double-Bloch function satisfies the monodromy properties \( \Psi_{\alpha\beta}(x + 2\omega) = B_{\beta} \Psi_{\alpha\beta}(x) \), \( \Psi_{\alpha\beta}(x + 2\omega') = B'_{\beta} \Psi_{\alpha\beta}(x) \) with some Bloch multipliers \( B_{\beta} \), \( B'_{\beta} \). The Bloch multipliers of the wave function (2.15) are:

\[
B_{\beta} = \exp\left(2\omega z - 2\zeta(\omega) \sum_i (e^{-D_{\beta}(z)} - 1)x_i\right),
\]

\[
B'_{\beta} = \exp\left(2\omega' z - 2\zeta(\omega') \sum_i (e^{-D_{\beta}(z)} - 1)x_i\right),
\]

where the differential operator \( D_{\beta}(z) \) is

\[
D_{\beta}(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_{\beta,k}}.
\]

Since the right hand side of (2.25) is an elliptic function of \( x \), the Bloch multipliers of the adjoint wave function should be \( 1/B_{\alpha} \), \( 1/B'_{\alpha} \): \( \Psi_{\alpha\beta}^{\dagger}(x + 2\omega) = (B_{\alpha})^{-1} \Psi_{\alpha\beta}^{\dagger}(x) \), \( \Psi_{\alpha\beta}^{\dagger}(x + 2\omega') = (B'_{\alpha})^{-1} \Psi_{\alpha\beta}^{\dagger}(x) \).

Any non-trivial double-Bloch function (i.e. the one which is not just an exponential function) must have at least one pole in \( x \) in the fundamental domain. Let us introduce the elementary double-Bloch function \( \Phi(x, \lambda) \) having just one pole in the fundamental domain and defined as

\[
\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)} e^{-\zeta(\lambda)x}
\]

(here \( \zeta(\lambda) \) is the Weierstrass \( \zeta \)-function). The monodromy properties of the function \( \Phi \) follow from (3.2):

\[
\Phi(x + 2\omega, \lambda) = e^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)} \Phi(x, \lambda),
\]

\[
\Phi(x + 2\omega', \lambda) = e^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')} \Phi(x, \lambda).
\]

We see that it is indeed a double-Bloch function. It has a single simple pole in the fundamental domain at \( x = 0 \) with residue 1:

\[
\Phi(x, \lambda) = \frac{1}{x} - \frac{1}{2} \varphi(\lambda)x + \cdots, \quad x \to 0,
\]

It is easy to see that \( \Phi(x, \lambda) \) is an elliptic function of \( \lambda \). The expansion as \( \lambda \to 0 \) is

\[
\Phi(x, \lambda) = \left(\lambda^{-1} + \zeta(x) + \frac{1}{2} \left(\zeta^2(x) - \varphi(x)\right)\lambda + O(\lambda^2)\right) e^{-x/\lambda}.
\]

We will also need the \( x \)-derivatives \( \Phi'(x, \lambda) = \partial_x \Phi(x, \lambda) \), \( \Phi''(x, \lambda) = \partial_x^2 \Phi(x, \lambda) \).

It is clear from (2.15) that the wave functions \( \Psi, \Psi^{\dagger} \) (and thus the coefficient \( \xi^{(1)} \)), as functions of \( x \), have simple poles at \( x = x_i \). It is shown in [19] that the residues at these poles are matrices of rank 1. We parametrize the residues of \( \xi^{(1)} \) through the column vectors \( a_i = (a_i^1, a_i^2, \ldots, a_i^n)^T \), \( b_i = (b_i^1, b_i^2, \ldots, b_i^n)^T \) (\( T \) means transposition):

\[
\xi^{(1)}_{\alpha\beta} = S_{\alpha\beta} - \sum_i a_i^\alpha b_i^\beta \varphi(x - x_i),
\]

where \( S_{\alpha\beta} \) does not depend on \( x \). Therefore,

\[
V(x, t) = -\sum_i a_i^\alpha b_i^\beta \varphi(x - x_i).
\]
The components of the vectors \( \mathbf{a}_i, \mathbf{b}_i \) are going to be spin variables of the elliptic Gibbons-Hermsen model.

One can expand the wave functions using the elementary double-Bloch functions as follows:

\[
\Psi_{\alpha\beta} = e^{k_\beta x + \zeta(x,z)} \sum_i a_i^{\alpha} b_i^{\beta} \Phi(x - x_i, \lambda_\beta),
\]

\[
\tilde{\Psi}_{\alpha\beta} = e^{-k_\alpha x - \zeta(x,z)} \sum_i c_i^{\alpha} d_i^{\beta} \Phi(x - x_i, -\lambda_\alpha),
\]

where \( c_i^{\alpha}, c_i^{\alpha} \) are components of some \( x \)-independent vectors \( \mathbf{c}_i = (c_i^1, \ldots, c_i^n)^T \), \( \mathbf{c}_i^* = (c_i^1, \ldots, c_i^n)^T \). This is similar to expansion of a rational function in a linear combination of simple fractions.

One can see that (3.11) is a double-Bloch function with Bloch multipliers

\[
B_\beta = e^{2\omega (k_\beta - \zeta(\lambda_\beta)) + 2\zeta(\omega, \lambda_\beta)}, \quad B'_\beta = e^{2\omega' (k_\beta - \zeta(\lambda_\beta)) + 2\zeta(\omega', \lambda_\beta)}
\]

and (3.12) has Bloch multipliers \((B_\alpha)^{-1}\) and \((B'_\alpha)^{-1}\). These Bloch multipliers should coincide with (3.5). Therefore, comparing (3.5) with (3.13), we get

\[
2\omega(k_\beta - z - \zeta(\lambda_\beta)) + 2\zeta(\omega, \lambda_\beta + (e^{-D_\beta(z)} - 1) \sum_i x_i) = 2\pi in,
\]

\[
2\omega'(k_\beta - z - \zeta(\lambda_\beta)) + 2\zeta(\omega', \lambda_\beta + (e^{-D_\beta(z)} - 1) \sum_i x_i) = 2\pi in'
\]

with some integer \( n, n' \). These equations can be regarded as a linear system. The solution is

\[
k_\beta - z - \zeta(\lambda_\beta) = 2n' \zeta(\omega) - 2n \omega',
\]

\[
\lambda_\beta + (e^{-D_\beta(z)} - 1) \sum_i x_i = 2n \omega' - 2n' \omega.
\]

Shifting \( \lambda_\beta \) by a suitable vector of the lattice spanned by \( 2\omega, 2\omega' \), we can represent the connection between the spectral parameters \( k_\beta, z, \lambda_\beta \) in the form

\[
\begin{cases}
  k_\beta = z + \zeta(\lambda_\beta), \\
  \lambda_\beta = (1 - e^{-D_\beta(z)}) \sum_i x_i.
\end{cases}
\]

(3.14)

These two equations for three spectral parameters \( k_\beta, z, \lambda_\beta \) determine the spectral curve, with the index \( \beta \) numbering different sheets of it. Another description of the same spectral curve is obtained below as the spectral curve of the spin generalization of the Calogero-Moser system (it is given by the characteristic polynomial of the Lax matrix \( L(\lambda) \) for the spin Calogero-Moser system). As we shall see below, it has the form \( R(k, \lambda) = 0 \), where \( R(k, \lambda) \) is a polynomial in \( k \) whose coefficients are elliptic functions of \( \lambda \). These coefficients are integrals of motion in involution. The spectral curve in the form \( R(k, \lambda) = 0 \) appears if one excludes \( z \) from the equations (3.14). Equivalently, one can represent the spectral curve as a relation connecting \( z \) and \( \lambda_\beta \):

\[
R(z + \zeta(\lambda_\beta), \lambda_\beta) = 0.
\]

(3.15)
The function $z(\lambda)$ defined by this equation is multivalued, $z_\beta(\lambda)$ being different branches of this function. Then the function $\lambda_\beta(z)$ is the inverse function to the $z_\beta(\lambda)$. Using the same arguments as in [12], one can see that the second equation in (3.14) can be written as

$$\lambda_\beta(z) = D_\beta(z) \sum_i x_i = \sum_{j \geq 1} \frac{z^{-j}}{j} V_j^{(\beta)}, \quad V_j^{(\beta)} = \partial_{t_\beta,j} \sum_i x_i,$$

where $\lambda_\beta(z)$ should be understood as expansion of the $\beta$-th branch of the function $\lambda(z)$ in negative powers of $z$ near $z = \infty$.

4 Dynamics of poles and residues in $t_2$

We first consider the dynamics of the poles and residues with respect to the time $t_2$. Following Krichever’s approach, we consider the linear problems (2.22), (2.23), and substitute the pole ansatz (3.11), (3.12) for the wave functions.

Consider first the equation for $\Psi$. After the substitution, we see that the expression has poles at $x = x_i$ up to the third order. Equating coefficients at the poles of different orders at $x = x_i$, we get the conditions:

- At $\frac{1}{(x-x_i)^2}$: $b'_i a^\nu_i = 1$;
- At $\frac{1}{(x-x_i)^2}$: $-\frac{1}{2} \dot{x}_i c_i^{\beta} - \sum_{j \neq i} b'_i a'_j c_j^{\beta} \Phi(x_i - x_j, \lambda_\beta) = k_\beta c_i^{\beta}$;
- At $\frac{1}{x-x_i}$: $\partial_{t_2} (a_i^\alpha c_i^{\beta}) = (k_\beta^2 - z^2 + \varphi(\lambda_\beta)) a_i^\alpha c_i^{\beta} - 2 \sum_{j \neq i} a_i^\alpha b'_i a'_j c_j^{\beta} \Phi'(x_i - x_j, \lambda_\beta) - 2 c_i^{\beta} \sum_{j \neq i} a_i^\alpha b'_i a'_j c_j^{\beta} \varphi(x_i - x_j)$,

where dot means $t_2$-derivative. Here and below summation over repeated Greek indices numbering components of vectors from 1 to $n$ is implied, unless otherwise stated. Similar calculations for the linear problem for $\Psi^\dagger$ lead to the conditions

- At $\frac{1}{(x-x_i)^2}$: $b'_i a^\nu_i = 1$ (the same as above);
- At $\frac{1}{(x-x_i)^2}$: $-\frac{1}{2} \dot{x}_i c_i^{\alpha} - \sum_{j \neq i} c_j^{\alpha} b'_j a'_i \Phi(x_i - x_j, \lambda_\alpha) = k_\alpha c_i^{\alpha}$;
- At $\frac{1}{x-x_i}$: $\partial_{t_2} (c_i^{\alpha} b_i^{\beta}) = -(k_\beta^2 - z^2 + \varphi(\lambda_\alpha)) c_i^{\alpha} b_i^{\beta} + 2 \sum_{j \neq i} c_j^{\alpha} b'_j a'_i b_i^{\beta} \Phi'(x_j - x_i, \lambda_\alpha) + 2 c_i^{\alpha} \sum_{j \neq i} b'_i a'_j b_j^{\beta} \varphi(x_i - x_j)$.

Here we have used the obvious property $\Phi(x, -\lambda) = -\Phi(-x, \lambda)$. 

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The conditions coming from the third order poles are constraints on the vectors \( \mathbf{a}_i, \mathbf{b}_i \). The other conditions can be written in the matrix form

\[
\begin{align*}
\left\{ \begin{array}{l}
(k_\beta I - L(\lambda_\beta))\mathbf{c}^\beta = 0 \\
\dot{\mathbf{c}}^\beta = M(\lambda_\beta)\mathbf{c}^\beta,
\end{array} \right. \\
\mathbf{c}^{*\alpha}(k_\alpha I - L(\lambda_\alpha)) = 0 \\
\dot{\mathbf{c}}^{*\alpha} = \mathbf{c}^{*\alpha}M^*(\lambda_\alpha)
\end{align*}
\]

(no summation over \( \alpha, \beta \)), where \( \mathbf{c}^\beta = (c_1^\beta, \ldots, c_N^\beta)^T \), \( \mathbf{c}^{*\alpha} = (c_1^{*\alpha}, \ldots, c_N^{*\alpha}) \), are \( N \)-dimensional vectors, \( I \) is the unity matrix, and \( L(\lambda), M(\lambda), M^*(\lambda) \) are \( N \times N \) matrices of the form

\[
L_{ij}(\lambda) = -\frac{1}{2} \lambda_{ij} - (1 - \delta_{ij})b^\nu_{ij}\Phi(x_i - x_j, \lambda),
\]

\[
M_{ij}(\lambda) = (k^2 - z^2 + \varphi(\lambda) - \Lambda_i)\delta_{ij} - 2(1 - \delta_{ij})b^\nu_{ij}\Phi'(x_i - x_j, \lambda),
\]

\[
M^*_{ij}(\lambda) = -(k^2 - z^2 + \varphi(\lambda) - \Lambda^*_i)\delta_{ij} + 2(1 - \delta_{ij})b^\nu_{ij}\Phi'(x_i - x_j, \lambda).
\]

Here

\[
\Lambda_i = \frac{\dot{a}_i}{a_i} + 2 \sum_{j \neq i} \frac{a_j b_j a_i}{a_i} \varphi(x_i - x_j), \quad -\Lambda_i^* = \frac{\dot{b}_i}{b_i} - 2 \sum_{j \neq i} \frac{b_j a_j b_i}{b_i} \varphi(x_i - x_j)
\]

do not depend on the index \( \alpha \) (there is summation over \( \nu \) but no summation over \( \alpha \)). In fact one can see that \( \Lambda_i = \Lambda_i^* \), so that \( M^*(\lambda) = -M(\lambda) \). Indeed, multiplying equations (4.6) by \( a_\alpha b_\alpha \) (no summation here!), summing over \( \alpha \) and summing the two equations, we get \( \Lambda_i - \Lambda_i^* = \partial_\nu (a_\alpha b_\alpha) = 0 \) by virtue of the constraint \( a_\alpha b_\alpha = 1 \).

Differentiating the first equation in (4.1) by \( t_2 \), we get the compatibility condition of equations (4.1):

\[
(\dot{L} + [L, M])\mathbf{c}^\beta = 0.
\]

One can see, taking into account equations (4.6), which we write here in the form

\[
\dot{a}_i = \Lambda_i a_i - 2 \sum_{j \neq i} \frac{a_j b_j a_i}{a_i} \varphi(x_i - x_j),
\]

\[
\dot{b}_i = -\Lambda_i b_i + 2 \sum_{j \neq i} b_j a_j b_i \varphi(x_i - x_j)
\]

(in this form they are equations of motion for the spin degrees of freedom) that the off-diagonal elements of the matrix \( \dot{L} + [L, M] \) are equal to zero. Vanishing of the diagonal elements yields equations of motion for the poles \( x_i \):

\[
\ddot{x}_i = 4 \sum_{j \neq i} b_j a_j b_i a_i \varphi'(x_i - x_j).
\]

The gauge transformation \( a_\alpha \rightarrow a_\alpha q_i, \ b_\alpha \rightarrow b_\alpha q_i^{-1} \) with \( q_i = \exp\left(\int_{t_2}^{t_3} \Lambda_i dt\right) \) eliminates the terms with \( \Lambda_i \) in (4.8), (4.9), so we can put \( \Lambda_i = 0 \). This gives the equations of motion

\[
\dot{a}_i = -2 \sum_{j \neq i} a_j b_j a_i \varphi(x_i - x_j), \quad \dot{b}_i = 2 \sum_{j \neq i} b_j a_j b_i \varphi(x_i - x_j).
\]
Together with (4.10) they are equations of motion of the elliptic Gibbons-Hermsen model. Their Lax representation is given by the matrix equation $\dot{L} = [M, L]$. It states that the time evolution of the Lax matrix is an isospectral transformation. It follows that the quantities $\text{tr} L^m(\lambda)$ are integrals of motion. In particular,

$$H_2 = \sum_{i=1}^{N} p_i^2 - \sum_{i \neq j} b_i^\mu a_j^\nu b_j^\rho a_i^\nu \varphi(x_i - x_j) = \text{tr} L^2(\lambda) + \text{const}$$  \hspace{1cm} (4.12)

is the Hamiltonian of the elliptic Gibbons-Hermsen model. Equations of motion (4.10), (4.11) are equivalent to the Hamiltonian equations

$$\dot{x}_i = \frac{\partial H_2}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H_2}{\partial x_i}, \quad \dot{a}_i^\alpha = \frac{\partial H_2}{\partial b_i^\alpha}, \quad \dot{b}_i^\alpha = -\frac{\partial H_2}{\partial a_i^\alpha}.$$ \hspace{1cm} (4.13)

We see that $\dot{x}_i = 2p_i$ and the Lax matrix is expressed through the momenta as follows:

$$L_{ij}(\lambda) = -p_i \delta_{ij} - (1 - \delta_{ij}) b_{ij}^\nu a_i^\nu \Phi(x_i - x_j, \lambda).$$ \hspace{1cm} (4.14)

As we shall see, the higher time flows are also Hamiltonian with the Hamiltonians being linear combinations of spectral invariants of the Lax matrix, i.e. linear combinations of traces of its powers $\text{tr} L^j(\lambda)$. It is not difficult to see that

$$G^{\alpha \beta} = \sum_i a_i^\alpha b_i^\beta$$ \hspace{1cm} (4.15)

are integrals of motion for all time flows: $\partial_m G^{\alpha \beta} = 0$. Indeed, we have

$$\partial_m \left( \sum_i a_i^\alpha b_i^\beta \right) = \sum_i \left( b_i^\beta \partial_m h^\alpha_m - a_i^\alpha \partial_m b_i^\beta \right)$$

and this is zero because $H_m$ is a linear combination of $\text{tr} L^j(\lambda)$ and

$$\sum_i \left( b_i^\beta \text{tr} \left( \frac{\partial L}{\partial b_i^\beta} L_i^{j-1} \right) - a_i^\alpha \text{tr} \left( \frac{\partial L}{\partial a_i^\alpha} L_i^{j-1} \right) \right)$$

$$= \sum_i \sum_{l,k} \left( b_i^\beta \frac{\partial L}{\partial b_i^\beta} L_i^{j-1} - a_i^\alpha \frac{\partial L}{\partial a_i^\alpha} L_i^{j-1} \right)$$

$$= \sum_i \sum_{l,k} (\delta_{lk} - \delta_{il}) b_i^\beta a_k^\alpha \Phi(x_l - x_k) L_i^{j-1} = 0.$$

A simple lemma from linear algebra states that eigenvalues $\nu_\alpha$ of the $n \times n$ matrix $G$ (4.15) coincide with nonzero eigenvalues of the rank $n N \times N$ matrix $F$ with matrix elements

$$F_{ij} = b_i^\nu a_j^\nu$$ \hspace{1cm} (4.16)

(we assume that $n \leq N$). Indeed, consider the rectangular $N \times n$ matrix $A_{i\alpha} = a_i^\alpha$ and $B_{i\beta} = b_i^\beta$, then $G = A^T B$, $F = B A^T$ and a straightforward verification shows that traces of all powers of these matrices coincide: $\text{tr} G^m = \text{tr} F^m$ for all $m \geq 1$. This means that their nonzero eigenvalues also coincide. Note that $\text{tr} G = \sum_{\alpha=1}^{n} \nu_\alpha = N$. 

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5 The spectral curve

The first of the equations (4.1) determines a connection between the spectral parameters
$k = k_\beta, \lambda = \lambda_\beta$ which is the equation of the spectral curve:

$$R(k, \lambda) := \det \left( kI - L(\lambda) \right) = 0. \quad (5.1)$$

As we already mentioned, the spectral curve is an integral of motion. The matrix $L = L(\lambda)$ has an essential singularity at $\lambda = 0$. It can be represented in the form $L = V\tilde{L}V^{-1}$, where $V$ is the diagonal matrix $V_{ij} = \delta_{ij} e^{-\zeta(\lambda)x_i}$. Matrix elements of $\tilde{L}$ do not have any essential singularity in $\lambda$. We conclude that

$$R(k, \lambda) = \sum_{m=0}^{N} R_m(\lambda)k^m,$$

where the coefficients $R_m(\lambda)$ are elliptic functions of $\lambda$ with poles at $\lambda = 0$. They can be represented as linear combinations of the $\wp$-function and its derivatives, coefficients of this expansion being integrals of motion. Fixing their values, we obtain an algebraic curve $\Gamma$ which is an $N$-sheet covering of the initial elliptic curve $E$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega, 2\omega'$. 

In a neighborhood of the point $\lambda = 0$ the matrix $\tilde{L}(\lambda)$ can be represented as

$$\tilde{L}(\lambda) = \lambda^{-1}(I - F) + O(1),$$

where $F$ is the rank $n$ matrix (4.16) (recall that $n \leq N$). This matrix has $N-n$ vanishing eigenvalues and $n$ nonzero eigenvalues $\nu_\alpha$, $\alpha = 1,\ldots,n$. They are time-independent quantities because as we have shown above they coincide with eigenv alues of the matrix $G$ (4.15) which is an integral of motion. Therefore, we can write

$$\det \left( kI - L(\lambda) \right) = \prod_{\alpha=1}^{n} \left( k - (1 - \nu_\alpha)\lambda^{-1} - h_\alpha(\lambda) \right) \prod_{j=n+1}^{N} \left( k - \lambda^{-1} - h_j(\lambda) \right),$$

where $h_\alpha, h_j$ are regular functions of $\lambda$ near $\lambda = 0$. This means that the function $k$ has simple poles on all sheets at the points of the curve $\Gamma$ located above $\lambda = 0$. Now, recalling the connection between $k$ and $z$ given by the first equation in (3.14), we have

$$\det \left( (z + \zeta(\lambda))I - L(\lambda) \right) = \prod_{\alpha=1}^{n} \left( z + \nu_\alpha\lambda^{-1} - h_\alpha(\lambda) \right) \prod_{j=n+1}^{N} \left( z - h_j(\lambda) \right). \quad (5.2)$$

We see that $n$ sheets of the curve $\Gamma$ lying above a neighborhood of the point $\lambda = 0$ are distinguished. There are $n$ points at infinity above $\lambda = 0$: $P_1(\infty) = (\infty_1, 0), \ldots P_n(\infty) = (\infty_n, 0)$. In the vicinity of the point $P_\alpha(\infty)$ the function $\lambda = \lambda_\alpha(z)$ has the following expansion:

$$\lambda = \lambda_\alpha(z) = -\nu_\alpha z^{-1} + O(z^{-2}). \quad (5.3)$$

As it is shown in [17], the points $P_\alpha(\infty) \in \Gamma$ are the marked points, where the Baker-Akhiezer function on the spectral curve has essential singularities.
With the expansion (5.3) at hand, we can make a more detailed identification of the wave function (2.15) with the expansion (2.20) and the wave function (3.11). The expansion of the function (3.11) as

$$\Psi_{\alpha \beta} = e^{z x + \xi(t, z)} \sum_i \left( a_i^\alpha b_i^\beta \nu^{-1}_\beta + \lambda_\beta \nu^{-1}_\beta \left( a_i^\alpha d_i^\beta + a_i^\alpha t_i^\beta \zeta(x - x_i) \right) + O(\lambda^2_\beta) \right),$$

where we took into account that the identification implies the expansion

$$c_i^\beta = \nu^{-1}_\beta e^{-x_i \xi(\lambda)} \left( b_i^\beta + \lambda_\beta d_i^\beta + O(\lambda^2_\beta) \right), \quad \lambda_\beta \to 0. \quad (5.4)$$

Therefore, taking into account (5.3), we can write

$$\Psi_{\alpha \beta} = e^{z x + \xi(t, z)} \left( \sum_i a_i^\alpha b_i^\beta \right) \left( \sum_i a_i^\alpha b_i^\beta \right) \left( S_{\alpha \beta} - \sum_i a_i^\alpha b_i^\beta \zeta(x - x_i) \right) + O(z^{-2}).$$

Comparing with (2.20), we conclude that

$$\sum_i a_i^\alpha b_i^\beta = \nu_\alpha \delta_{\alpha \beta}. \quad (5.5)$$

It is easy to see that the Hamiltonian and the Lax matrix are invariant with respect to the gauge transformation

$$a_i \to W^{-1} a_i, \quad b_i^T \to b_i^T W \quad (5.6)$$

with arbitrary non-degenerate $n \times n$ matrix $W$. Therefore, after the transformation $G \to W^{-1} gW$ the matrix $G$ can always be regarded as diagonal matrix, as in (5.5), with the eigenvalues being the same as nonzero eigenvalues $\nu_\alpha$ of the $N \times N$ matrix $F$.

### 6 Dynamics of poles in the higher times

Our basic tool is equation (2.18). Substituting $\Psi, \Psi^\dagger$ in the form (3.11), (3.12) and $\xi^{(1)}$ in the form (3.9), we have:

$$\frac{1}{2\pi i} \oint_{C_{\infty}} dz \sum_{i<j} z^m a_i^\alpha c_i^\nu c_j^\mu b_j^\beta \Phi(x - x_i, \lambda_\nu) \Phi(x - x_j, -\lambda_\nu) \quad (6.1)$$

$$\quad = \sum_i \partial_{\nu, m} x_i a_i^\alpha b_i^\beta \varphi(x - x_i) + \sum_i \partial_{\nu, m} \left( a_i^\alpha b_i^\beta \right) \zeta(x - x_i)$$

(no summation over $\nu$ here!). Equating the coefficients in front of the second order poles at $x = x_i$, we get the relation

$$\partial_{\nu, m} x_i = \text{res}_\infty \left( z^m c_i^\nu c_i^\nu \right) = \text{res}_\infty \left( z^m c^\nu E_i c^\nu \right), \quad (6.2)$$

where $E_i$ is the diagonal $N \times N$ matrix with matrix elements $(E_i)_{jk} = \delta_{ij} \delta_{ik}$ (again, no summation over $\nu$). Summing over $i$, we get

$$\partial_{\nu, m} \sum_i x_i = \text{res}_\infty \left( z^m c^\nu c^\nu \right) \quad (6.3)$$
Comparing with equation (3.16), we conclude that
\[ c^\alpha c^\alpha = -\nu_\alpha z^{-2} + \sum_{m \geq 2} z^{-m-1} \partial_{t_{a,m}} \sum_i x_i = -\lambda_\alpha (z) \] (6.4)
(no summation over \( \alpha \)). Now, we note that \( E_i = -\partial_{p_i} L \) and continue (6.2) as the following chain of equalities, using (4.1), (4.2), (6.4):

\[ \partial_{t_m} x_i = \sum_{\nu} \res \left( z^m c^{*\nu} E_i c^\nu \right) = -\sum_{\nu} \res \left( z^m c^{*\nu} \partial_{p_i} L(\lambda_\nu) c^\nu \right) \]

\[ = -\sum_{\nu} \res \left( z^m \partial_{p_i} \left( c^{*\nu} L(\lambda_\nu) c^\nu \right) \right) + \sum_{\nu} \res \left( z^m (\partial_{p_i} c^{*\nu}) L(\lambda_\nu) c^\nu \right) + \sum_{\nu} \res \left( z^m c^{*\nu} L(\lambda_\nu) \partial_{p_i} c^\nu \right) \]

\[ = -\sum_{\nu} \res \left( z^m \partial_{p_i} \left( c^{*\nu} L(\lambda_\nu) c^\nu \right) \right) + \sum_{\nu} \res \left( z^m (\partial_{p_i} c^{*\nu}) L(\lambda_\nu) c^\nu \right) \]

\[ = \sum_{\nu} \res \left( z^m \lambda_\nu (z) \partial_{p_i} L(\lambda_\nu) c^\nu \right). \]

Regarding \( z \) as an independent variable, we apply the same argument as in [12] to obtain
\[ \partial_{t_m} x_i = -\sum_{\nu} \res \left( z^m \partial_{p_i} \lambda_\nu (z) \right). \] (6.5)

In this way we obtain the first half of the higher Hamiltonian equations for poles
\[ \partial_{t_m} x_i = \frac{\partial H_m}{\partial p_i} \] (6.6)
with the Hamiltonian
\[ H_m = \sum_{\alpha=1}^{n} \res \left( z^m \lambda_\alpha (z) \right). \] (6.7)

The second half of the Hamiltonian equations for poles can be obtained by taking the \( t_2 \)-derivative of (6.2) and using (4.1), (4.2). In this way we obtain:
\[ \partial_{t_{\nu,m}} x_i = \res \left( z^m c^{*\nu} [E_i, M(\lambda_\nu)] c^\nu \right). \] (6.8)

A straightforward verification shows that
\[ [E_i, M(\lambda)] = 2\partial_{x_i} L(\lambda). \] (6.9)
Recalling also that \( \dot{x}_i = 2p_i \), we rewrite (6.8) as
\[ \partial_{t_{\nu,m}} p_i = \res \left( z^m c^{*\nu} \partial_{x_i} L(\lambda_\nu) c^\nu \right) \] (6.10)
(no summation over \( \nu \)). With the relation (6.10) at hand, one can repeat the chain of equalities after equation (6.4) with the change \( \partial_{p_i} \to \partial_{x_i} \) to obtain
\[ \partial_{t_m} p_i = \sum_{\nu} \res \left( z^m \partial_{x_i} \lambda_\nu (z) \right), \] (6.11)
so that
\[ \partial_t m p_i = - \frac{\partial H_m}{\partial x_i} \]  
(6.12)
with the same Hamiltonian \[ (6.7) \].

Let us make some comments on a more general case when the tau-function for elliptic solutions has a slightly more general form
\[ \tau(x, t) = e^{Q(x, t)} \prod_{i=1}^{N} \sigma(x - x_i(t)), \]  
(6.13)
where
\[ Q(x, t) = c(x + t_1)^2 + (x + t_1) \sum_{j \geq 2} a_j t_j + b(t_2, t_3, \ldots) \]  
(6.14)
with some constants \( c, a_j \) and a function \( b(t_2, t_3, \ldots) \). Repeating the arguments leading to \[ (6.5) \], one can see that now the first equation in \[ (3.14) \] will be modified as
\[ k_\beta = z - \alpha(z) + \zeta(\lambda_\beta), \quad \alpha(z) = 2cz^{-1} + \sum_{j \geq 2} \frac{a_j}{j} z^{-j}. \]  
(6.15)
Instead of \[ (6.5) \] we will have
\[ \partial_t m x_i = - \sum_{\nu} \text{res} \left( z^m \partial_{\nu} \lambda_{\nu}(z)(1 - \alpha'(z)) \right), \]  
(6.16)
so the Hamiltonian for the \( m \)-th flow will be a linear combination of \( H_m \) and \( H_j \) with \( 1 \leq j < m \).

7 Dynamics of spin variables in the higher times

The Hamiltonian dynamics of spin variables in the higher times can be derived by analysis of first order poles in \[ (6.1) \]. Equating coefficients in front of first order poles, we get the relation
\[ \partial_{t, m} (a_i^\alpha b_j^\beta) = \text{res} \left( z^m \sum_{j \neq i} a_i^\alpha c_j^\nu c_j^\nu b_j^\beta \Phi(x_i - x_j, -\lambda_{\nu}) + z^m \sum_{j \neq i} a_j^\alpha c_j^\nu c_i^\nu b_i^\beta \Phi(x_i - x_j, \lambda_{\nu}) \right) \]
which can be rewritten as
\[ a_i^\alpha \left[ \partial_{t, m} b_j^\beta + \text{res} \left( z^m c_j^\nu \sum_{j \neq i} c_j^\nu b_j^\beta \Phi(x_j - x_i, \lambda_{\nu}) \right) \right] \]
\[ + b_i^\beta \left[ \partial_{t, m} a_i^\alpha - \text{res} \left( z^m c_i^\nu \sum_{j \neq i} c_j^\nu a_j^\alpha \Phi(x_i - x_j, \lambda_{\nu}) \right) \right] = 0. \]
Now we notice that
\[ \frac{\partial L_{jk}(\lambda)}{\partial a_i^\alpha} = -\delta_{ijk}(1 - \delta_{jk})b_j^\alpha \Phi(x_j - x_i, \lambda), \]
\[ \frac{\partial L_{jk}(\lambda)}{\partial b_i^\beta} = -\delta_{ijk}(1 - \delta_{jk})a_k^\beta \Phi(x_i - x_k, \lambda), \]  
(7.1)
so the equation above can be written as

\[
a_{i}^{\alpha} \left[ \partial_{\nu,m} b_{i}^{\beta} - \text{res} \left( z^{m} c^{*\nu} \frac{\partial L(\lambda_{\nu})}{\partial a_{i}^{\alpha}} c^{\nu} \right) \right] + b_{i}^{\beta} \left[ \partial_{\nu,m} a_{i}^{\alpha} + \text{res} \left( z^{m} c^{*\nu} \frac{\partial L(\lambda_{\nu})}{\partial b_{i}^{\alpha}} c^{\nu} \right) \right] = 0. \tag{7.2}
\]

Having this equation at hand, one can repeat the chain of equalities after equation (6.4) with the changes \(\partial p_{i} \rightarrow \partial/\partial a_{i}^{\beta}, \partial p_{i} \rightarrow \partial/\partial b_{i}^{\alpha}\) to obtain

\[
a_{i}^{\alpha} P_{i}^{\beta} - b_{i}^{\beta} Q_{i}^{\alpha} = 0, \tag{7.3}
\]

where

\[
P_{i}^{\beta} = -\partial_{t} b_{i}^{\beta} + \sum_{\nu} \text{res} \left( z^{m} \frac{\partial}{\partial a_{i}^{\beta}} \lambda_{\nu}(z) \right) = -\partial_{t} b_{i}^{\beta} - \frac{\partial H_{m}}{\partial a_{i}^{\beta}}, \tag{7.4}
\]

\[
Q_{i}^{\alpha} = \partial_{t} a_{i}^{\alpha} + \sum_{\nu} \text{res} \left( z^{m} \frac{\partial}{\partial b_{i}^{\alpha}} \lambda_{\nu}(z) \right) = \partial_{t} a_{i}^{\alpha} - \frac{\partial H_{m}}{\partial b_{i}^{\alpha}}. \tag{7.5}
\]

It follows from (7.3) that

\[
\frac{Q_{i}^{\alpha}}{a_{i}^{\alpha}} = \frac{P_{i}^{\beta}}{b_{i}^{\beta}} = \Lambda_{i}^{(m)},
\]

and the equations (7.4), (7.5) acquire the form

\[
\partial_{t} a_{i}^{\alpha} = a_{i}^{\alpha} \Lambda_{i}^{(m)} + \frac{\partial H_{m}}{\partial b_{i}^{\alpha}}, \tag{7.6}
\]

\[
\partial_{t} b_{i}^{\beta} = -b_{i}^{\beta} \Lambda_{i}^{(m)} - \frac{\partial H_{m}}{\partial a_{i}^{\beta}}. \tag{7.7}
\]

The gauge transformation \(a_{i}^{\alpha} \rightarrow a_{i}^{\alpha} q_{i}^{(m)}, b_{i}^{\alpha} \rightarrow b_{i}^{\alpha} (q_{i}^{(m)})^{-1}\) with \(q_{i}^{(m)} = \exp \left( \int^{t_{m}} \Lambda_{i}^{(m)} \, dt \right)\) eliminates the terms with \(\Lambda_{i}^{(m)}\), so we can put \(\Lambda_{i}^{(m)} = 0\). We obtain the Hamiltonian equations of motion for spin variables in the higher times:

\[
\partial_{t} a_{i}^{\alpha} = \frac{\partial H_{m}}{\partial b_{i}^{\alpha}}, \quad \partial_{t} b_{i}^{\alpha} = -\frac{\partial H_{m}}{\partial a_{i}^{\alpha}}. \tag{7.8}
\]

with \(H_{m}\) given by (6.7).

### 8 How to obtain the first two Hamiltonians

In order to find the Hamiltonians, we need to expand the spectral curve near \(\lambda = 0\). Using the expansion (3.8), we represent the equation of the spectral curve as

\[
\det \left( zI + F\lambda^{-1} + Q + S\lambda + O(\lambda^{2}) \right) = 0, \tag{8.1}
\]

where the matrices \(Q, S\) are

\[
Q_{ij} = p_{i} \delta_{ij} + (1 - \delta_{ij}) F_{ij} \zeta(x_{i} - x_{j}), \tag{8.2}
\]

\[
S_{ij} = \frac{1}{2} (1 - \delta_{ij}) F_{ij} \left( \zeta^{2}(x_{i} - x_{j}) - \varphi(x_{i} - x_{j}) \right). \tag{8.3}
\]
We set
\[ z = -\frac{\omega}{\lambda}, \] (8.4)
then the equation (8.1) acquires the form
\[ \det \left( \omega I - F - Q\lambda - S\lambda^2 + O(\lambda^3) \right) = 0. \] (8.5)
This equation has \( n \) roots \( \omega_\alpha \) such that
\[ \omega_\alpha = \omega_\alpha(\lambda) = \nu_\alpha + \omega_1^{(\alpha)} \lambda + \omega_2^{(\alpha)} \lambda^2 + O(\lambda^3) \] (8.6)
and \( N - n \) roots which are \( O(\lambda) \). These roots are eigenvalues of the matrix \( F + Q\lambda + S\lambda^2 + O(\lambda^3) \). Expressing \( \lambda \) through \( z \) from equation (8.4) and expanding in powers of \( z^{-1} \), we have
\[ \lambda_\alpha = -\frac{\nu_\alpha}{z} + \nu_\alpha \omega_1^{(\alpha)} z^{-2} - (\nu_\alpha^2 \omega_2^{(\alpha)} + \nu_\alpha (\omega_1^{(\alpha)})^2) z^{-3} + O(z^{-4}). \] (8.7)
Then
\[ H_1 = -\sum_\alpha \nu_\alpha \omega_1^{(\alpha)}, \]
\[ H_2 = \sum_\alpha (\nu_\alpha^2 \omega_2^{(\alpha)} + \nu_\alpha (\omega_1^{(\alpha)})^2). \] (8.8)
We regard the matrix \( Q\lambda + S\lambda^2 \) as a small variation of the matrix \( F \). The idea is to find the variation of the eigenvalues (the corrections \( \omega_1^{(\alpha)} \lambda + \omega_2^{(\alpha)} \lambda^2 \) in (8.6)) using first two orders of the perturbation theory.

Let \( \psi^{(i)} \) be a basis in the \( N \)-dimensional space and \( \tilde{\psi}^{(j)} \) be the dual basis such that \( (\tilde{\psi}^{(i)} \psi^{(j)}) = 0 \) at \( i \neq j \). We take first \( n \) vectors to be
\[ \psi_i^{(\alpha)} = b_i^{\alpha}, \quad \tilde{\psi}^{(\alpha)} = a_i^{\alpha}, \]
then
\[ (\tilde{\psi}^{(\alpha)} \psi^{(\beta)}) = \sum_i a_i^{\alpha} b_i^{\beta} = \nu_\alpha \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, n. \]
These vectors are eigenvectors of the (non-perturbed) matrix \( F \) with nonzero eigenvalues:
\[ F \psi^{(\alpha)} = \nu_\alpha \psi^{(\alpha)}, \quad \tilde{\psi}^{(\alpha)} F = \nu_\alpha \tilde{\psi}^{(\alpha)}. \] (8.9)
The other \( N - n \) vectors are chosen to be orthonormal:
\[ (\tilde{\psi}^{(i)} \psi^{(j)}) = \delta_{ij}, \quad i, j = n + 1, \ldots, N. \]

In the first order of the perturbation theory we have:
\[ \omega_1^{(\alpha)} = \frac{(\tilde{\psi}^{(\alpha)} Q \psi^{(\alpha)})}{(\psi^{(\alpha)} \tilde{\psi}^{(\alpha)})}. \] (8.10)
The next coefficient, \( \omega_2^{(\alpha)} \), is obtained in the second order of the perturbation theory as
\[ \omega_2^{(\alpha)} = \frac{(\tilde{\psi}^{(\alpha)} S \psi^{(\alpha)})}{(\psi^{(\alpha)} \tilde{\psi}^{(\alpha)})} + \sum_{j \neq \alpha} \frac{(\tilde{\psi}^{(\alpha)} Q \psi^{(j)}) (\tilde{\psi}^{(j)} Q \psi^{(\alpha)})}{(\psi^{(\alpha)} \tilde{\psi}^{(\alpha)}) (\psi^{(j)} \tilde{\psi}^{(j)}) (\nu_\alpha - \nu_j)}. \] (8.11)
In the denominator of the last term $\nu_j = \nu_\beta$ at $j = \beta$, $\beta = 1, \ldots, n$ and $\nu_j = 0$ at $j = n + 1, \ldots, N$.

Using these formulas, we have:

$$\sum_\alpha \nu_\alpha \omega_1^{(\alpha)} = \sum_\alpha (\tilde{\psi}^{(\alpha)} Q \psi^{(\alpha)}) = \sum_\alpha \sum_{ij} a_i^\alpha Q_{ij} b_j^\alpha = \sum_{i,j} F_{ji} Q_{ij} = \text{tr} (FQ), \quad (8.12)$$

$$\sum_\alpha (\nu_\alpha^2 \omega_2^{(\alpha)} + \nu_\alpha (\omega_1^{(\alpha)})^2) = \sum_\alpha \frac{\nu_\alpha (\tilde{\psi}^{(\alpha)} Q \psi^{(\alpha)}) (\tilde{\psi}^{(\beta)} Q \psi^{(\alpha)})}{\nu_\alpha (\nu_\alpha - \nu_\beta)} + \sum_{j=n+1}^N \sum_\alpha (\tilde{\psi}^{(\alpha)} Q \psi^{(j)}) (\tilde{\psi}^{(j)} Q \psi^{(\alpha)}) + \sum_\alpha \nu_\alpha (\tilde{\psi}^{(\alpha)} \psi^{(\alpha)})^2 + \sum_\alpha \nu_\alpha (\tilde{\psi}^{(\alpha)} S \psi^{(\alpha)})$$

$$= \sum_{\alpha, \beta} \nu_\alpha^{-1} (\tilde{\psi}^{(\alpha)} Q \psi^{(\alpha)}) (\tilde{\psi}^{(\beta)} Q \psi^{(\alpha)}) + \sum_{j=n+1}^N \sum_\alpha (\tilde{\psi}^{(\alpha)} Q \psi^{(j)}) (\tilde{\psi}^{(j)} Q \psi^{(\alpha)}) + \sum_\alpha \nu_\alpha (\tilde{\psi}^{(\alpha)} S \psi^{(\alpha)})$$

$$= \sum_{i,j,l} (\sum_\alpha \nu_\alpha^{-1} a_i^\alpha b_l^\alpha + \sum_{r=n+1}^N \tilde{\psi}_i^{(r)} \psi_l^{(r)}) Q_{ij} F_{jk} Q_{kl} + \sum_\alpha \nu_\alpha a_i^\alpha S_{ij} b_j^\alpha.$$

But

$$\sum_\alpha \nu_\alpha^{-1} a_i^\alpha b_l^\alpha + \sum_{r=n+1}^N \tilde{\psi}_i^{(r)} \psi_l^{(r)} = \delta_{il}$$

(the completeness relation), and so finally we obtain

$$\sum_\alpha (\nu_\alpha^2 \omega_2^{(\alpha)} + \nu_\alpha (\omega_1^{(\alpha)})^2) = \text{tr} (QFQ) + \text{tr} (FSF). \quad (8.13)$$

From (8.12) we obtain

$$H_1 = -\text{tr} (FQ) = -\sum_i p_i F_{ii} - \sum_{i \neq j} F_{ij} F_{ji} \zeta(x_i - x_j) = -\sum_i p_i, \quad (8.14)$$

which is indeed the first Hamiltonian. The calculation of (8.13) is more involved. We have, after some cancellations:

$$\text{tr} (QFQ) = \sum_i p_i^2 + \sum_{k \neq i} \sum_{j \neq i} F_{ij} F_{jk} F_{ki} \zeta(x_i - x_j) \zeta(x_k - x_i),$$

$$\text{tr} (FSF) = \frac{1}{2} \sum_i \sum_{i \neq j} F_{li} F_{ij} F_{jl} (\zeta^2(x_i - x_j) - \varphi(x_i - x_j)).$$

Therefore,

$$H_2 = \sum_i p_i^2 - \sum_{i \neq j} F_{ij} F_{ji} \varphi(x_i - x_j) + \mathcal{F}, \quad (8.15)$$

where

$$\mathcal{F} = \sum_i F_{ij} F_{jk} F_{ki} \zeta(x_i - x_j) \zeta(x_k - x_i) + \frac{1}{2} \sum_i F_{ij} F_{jk} F_{ki} \left(\zeta^2(x_i - x_j) - \varphi(x_i - x_j)\right) = 0. \quad (8.16)$$

Here $\sum'$ means summation over all distinct indices $ijk$. The proof of identity (8.16) is given in Appendix A. To conclude, we have reproduced the correct Hamiltonians $H_1$ and $H_2$ within our approach.
9 Rational and trigonometric limits

In the rational limit \( \omega, \omega' \to \infty \), \( \sigma(\lambda) = \lambda, \Phi(x, \lambda) = (x^{-1} + \lambda^{-1})e^{-x/\lambda} \) and the equation of the spectral curve becomes

\[
\det\left( zI - L_{\text{rat}} + \lambda^{-1}F \right) = 0, \tag{9.1}
\]

where

\[
(L_{\text{rat}})_{ij} = -\delta_{ij}p_i - (1 - \delta_{ij})\frac{b'_i a'_j}{x_i - x_j}, \tag{9.2}
\]

is the Lax matrix of the spin generalization of the rational Calogero-Moser model. Let us rewrite the equation of the spectral curve in the form

\[
\det \left( \lambda I + \frac{1}{zI - L_{\text{rat}}} \right) = 0. \tag{9.3}
\]

Expanding the determinant, we have:

\[
\lambda^N + \sum_{j=1}^{n} D_j(z) \lambda^{N-j} = 0, \quad D_1(z) = \text{tr} \left( F \frac{1}{zI - L_{\text{rat}}} \right), \tag{9.4}
\]

where we took into account that rank of \( F \) is equal to \( n \leq N \). Let us note that the functions \( \lambda_{\nu}(z) \) are different nonzero roots of equation (9.4) and the sum of these roots is equal to \(-D_1(z)\). Therefore, we can write

\[
H_m = -\sum_{\nu} \text{res}_\infty \left( z^m \lambda_{\nu}(z) \right) = \text{res}_\infty \left( z^m \text{tr} \left( F \frac{1}{zI - L_{\text{rat}}} \right) \right) = \text{tr} (FL_{\text{rat}}^m). \tag{9.5}
\]

It is straightforward to check the commutation relation

\[
[X, L_{\text{rat}}] = F - I, \quad X = \text{diag} (x_1, \ldots, x_N). \tag{9.6}
\]

Substituting it into (9.5), we see that

\[
H_m = \text{tr} L_{\text{rat}}^m. \tag{9.7}
\]

This is the result of paper [19] obtained there by another method.

We now pass to the trigonometric limit. We choose the period of the trigonometric (or hyperbolic) functions to be \( \pi i/\gamma \), where \( \gamma \) is some complex constant (real for hyperbolic functions and purely imaginary for trigonometric functions). The second period tends to infinity. The Weierstrass functions in this limit become

\[
\sigma(x) = \gamma^{-1} e^{-\frac{1}{6} \gamma^2 x^2} \sinh(\gamma x), \quad \zeta(x) = \gamma \coth(\gamma x) - \frac{1}{3} \gamma^2 x. \tag{9.8}
\]

The tau-function for trigonometric solutions is [20]

\[
\tau = \prod_{i=1}^{N} \left( e^{2\gamma x} - e^{2\gamma x_i} \right), \tag{9.9}
\]

so we should consider

\[
\tau = \prod_{i=1}^{N} \sigma(x - x_i) e^{\frac{1}{6} \gamma^2 (x-x_i)^2 + \gamma x + x_i}. \tag{9.9}
\]
Similarly to the KP case \cite{12}, equation (3.14) with this choice acquires the form
\[
k_\beta = z + \gamma \coth(\gamma \lambda_\beta).
\] (9.10)

The trigonometric limit of the function $\Phi(x, \lambda)$ is
\[
\Phi(x, \lambda) = \gamma \left( \coth(\gamma x) + \coth(\gamma \lambda) \right) e^{-\gamma x \coth(\gamma \lambda)}.
\]

For further calculations it is convenient to pass to the variables
\[
w_i = e^{2\gamma x_i}, \quad (9.11)
\]
and introduce the diagonal matrix $W = \text{diag}(w_1, w_2, \ldots, w_N)$. In this notation, the equation of the spectral curve acquires the form
\[
\det \left( W^{1/2}(zI - (L_{\text{trig}} - \gamma I))W^{-1/2} + \gamma (\coth(\gamma \lambda) - 1)F \right) = 0,
\] (9.12)
where $L_{\text{trig}}$ is the Lax matrix of the spin Calogero-Moser model with matrix elements
\[
(L_{\text{trig}})_{ij} = -p_i \delta_{ij} - \frac{(1 - \delta_{ij}) \gamma F_{ij}}{\sinh(\gamma (x_i - x_j))} = -p_i \delta_{ij} - 2(1 - \delta_{ij}) \frac{\gamma w_i^{1/2} w_j^{1/2} F_{ij}}{w_i - w_j}.
\] (9.13)

Some simple transformations allow one to bring the equation of the spectral curve to the form
\[
\det \left( \omega I + 2\gamma W^{-1/2} F W^{1/2} \right) = 0, \quad \omega = e^{2\gamma \lambda} - 1.
\] (9.14)

Expanding the determinant, we have:
\[
\omega^N + \sum_{j=1}^N K_j(z) \omega^{N-j} = 0,
\] (9.15)
where we took into account that rank of $F$ is equal to $n \leq N$. In particular,
\[
K_1 = \text{tr} Y, \quad K_2 = \frac{1}{2} (\text{tr}^2 Y - \text{tr} Y^2),
\]
where $Y$ is the matrix
\[
Y = 2\gamma W^{-1/2} F W^{1/2} \frac{1}{zI - (L_{\text{trig}} - \gamma I)}.
\]

The coefficients $K_j$ are expressed through the elementary symmetric polynomials $e_j = e_j(\omega_1, \ldots, \omega_n)$ of nonzero roots $\omega_\nu = \omega_\nu(z)$ of this equation as $K_j = (-1)^j e_j(\omega_1, \ldots, \omega_n)$. Therefore,
\[
\sum_\nu \lambda_\nu(z) = \frac{1}{2\gamma} \sum_\nu \log(1 + \omega_\nu(z)) = \frac{1}{2\gamma} \log \prod_\nu (1 + \omega_\nu(z))
\]
\[
= \frac{1}{2\gamma} \log \left( \sum_{j=0}^n e_j(\omega_1, \ldots, \omega_n) \right) = \frac{1}{2\gamma} \log \left( \sum_{j=0}^n (-1)^j K_j \right).
\]
From this we conclude that
\[
\sum_{\nu} \lambda_{\nu}(z) = \frac{1}{2\gamma} \log \det \left[ I - 2\gamma W^{-1/2} FW^{1/2} \frac{1}{zI - (L_{\text{trig}} - \gamma I)} \right]. \tag{9.16}
\]

Starting from this point, one can literally repeat the corresponding calculation from [12] with the change of the rank 1 matrix $E$ to the rank $n$ matrix $F$ and using the easily proved relation
\[
[L_{\text{trig}}, W] = 2\gamma(W^{1/2}FW^{1/2} - W). \tag{9.17}
\]

The result is
\[
\sum_{\nu} \lambda_{\nu}(z) = \frac{1}{2\gamma} tr\left(\log(I - z^{-1}(L_{\text{trig}} + \gamma I)) - \log(I - z^{-1}(L_{\text{trig}} - \gamma I))\right)
\]
\[
= -\frac{1}{2\gamma} \text{tr} \sum_{m \geq 1} \frac{z^{-m}}{m} ((L_{\text{trig}} + \gamma I)^m - (L_{\text{trig}} - \gamma I)^m) \tag{9.18}
\]
and
\[
H_m = \frac{1}{2\gamma(m + 1)} \text{tr}\left((L_{\text{trig}} + \gamma I)^{m+1} - (L_{\text{trig}} - \gamma I)^{m-1}\right) \tag{9.19}
\]
which agrees with the result of paper [20].

**Appendix A: Proof of identity (8.16)**

Here we prove identity (8.16) $\mathcal{F} = 0$, where
\[
\mathcal{F} = \sum \mathcal{F}_{ij} \mathcal{F}_{jk} \mathcal{F}_{ki} \zeta(x_i - x_j) \zeta(x_k - x_i) + \frac{1}{2} \sum \mathcal{F}_{ij} \mathcal{F}_{jk} \mathcal{F}_{ki} \left(\zeta^2(x_i - x_j) - \varphi(x_i - x_j)\right).
\]

and $\sum$’ means summation over all distinct indices $ijk$. Using the behavior of the $\zeta$-function under shifts by periods
\[
\zeta(x + 2\omega) = \zeta(x) + 2\eta, \quad \zeta(x + 2\omega') = \zeta(x) + 2\eta',
\]
one can see that $\mathcal{F}$ is a double-periodic function of any of $x_i$. Consider, for example, the shift of $x_1$ by $2\omega$. The terms in $\mathcal{F}(x_1 + 2\omega) - \mathcal{F}(x_1)$ proportional to $\eta^2$ are:
\[
-(2\eta)^2 \sum_{j \neq k \neq 1} F_{ij} F_{jk} F_{ki} + \frac{1}{2} (2\eta)^2 \sum_{j \neq k \neq 1} F_{ij} F_{jk} F_{ki} + \frac{1}{2} (2\eta)^2 \sum_{i \neq k \neq 1} F_{1k} F_{ki} F_{i1} = 0.
\]

The terms proportional to $\eta$ are:
\[
-2\eta \sum_{j \neq k \neq 1} F_{ij} F_{jk} F_{ki} \zeta(x_1 - x_k) - 2\eta \sum_{j \neq k \neq 1} F_{ij} F_{jk} F_{ki} \zeta(x_1 - x_j)
\]
\[
+2\eta \sum_{j \neq k \neq 1} F_{j1} F_{1k} F_{kj} \zeta(x_j - x_k) + 2\eta \sum_{j \neq k \neq 1} F_{j1} F_{1k} F_{kj} \zeta(x_k - x_j)
\]
\[
+2\eta \sum_{j \neq k \neq 1} F_{1k} F_{1j} F_{kj} \zeta(x_1 - x_j) + 2\eta \sum_{j \neq k \neq 1} F_{1j} F_{1k} F_{kj} \zeta(x_1 - x_j) = 0.
\]
Therefore, we see that $F(x_1+2\omega) = F(x_1)$. The double-periodicity in all other arguments is established in the same way.

Next, the function $F$ as a function of $x_1$ may have poles only at the points $x_i$, $i = 2, \ldots, N$. The second order poles cancel identically in the obvious way. We find the residue at the simple pole at $x_1 = x_2$ as follows:

$$- \sum_{k \neq 1,2} F_{12} F_{2k} F_{k1} \zeta(x_1 - x_k) - \sum_{j \neq 1,2} F_{1j} F_{j2} F_{21} \zeta(x_1 - x_j)$$

$$+ \sum_{k \neq 1,2} F_{21} F_{1k} F_{k2} \zeta(x_1 - x_k) + \sum_{j \neq 1,2} F_{2j} F_{j1} F_{12} \zeta(x_1 - x_j) = 0.$$

Vanishing of the residues in all other points and for all other variables can be proved in the same way. We see that the function $F$ is a regular elliptic function and, therefore, it must be a constant. To find this constant, we set $x_j = j \varepsilon$ and tend $\varepsilon$ to 0. Thanks to the fact that $\zeta(x) = x^{-1} + O(x^3)$, $\wp(x) = x^{-2} + O(x^2)$ as $x \to 0$, we find:

$$F = \frac{1}{\varepsilon^2} \sum F_{ij} F_{jk} F_{ki} \zeta + O(\varepsilon^2)$$

Making the cyclic changes of the summation variables $(ijk) \to (jki)$ and $(ijk) \to (kij)$, we have:

$$F = \frac{1}{3\varepsilon^2} \sum F_{ij} F_{jk} F_{ki} \left( \frac{1}{(i-j)(k-i)} + \frac{1}{(j-k)(i-j)} + \frac{1}{(k-i)(j-k)} \right) + O(\varepsilon^2)$$

$$= \frac{1}{3\varepsilon^2} \sum F_{ij} F_{jk} F_{ki} \left( \frac{j-k}{(i-j)(j-k)(k-i)} \right) + O(\varepsilon^2) = O(\varepsilon^2).$$

Therefore, we conclude that $F = 0$ and the identity (8.16) is proved.

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