Quantum Hall Ground States, Binary Invariants, and Regular Graphs

Hamed Pakatchi
Massachusetts Institute of Technology
(Dated: July 6, 2018)

Extracting meaningful physical information out of a many-body wavefunction is often impractical. The polynomial nature of fractional quantum Hall (FQH) wavefunctions, however, provides a rare opportunity for a study by virtue of ground states alone. In this article, we investigate the general properties of FQH ground state polynomials. It turns out that the data carried by an FQH ground state can be essentially that of a (small) directed graph/matrix. We establish a correspondence between FQH ground states, binary invariants and regular graphs and briefly introduce all the necessary concepts. Utilizing methods from invariant theory and graph theory, we will then take a fresh look on physical properties of interest, e.g. squeezing properties, clustering properties, etc. Our methodology allows us to ‘unify’ almost all of the previously constructed FQH ground states in the literature as special cases of a graph-based class of model FQH ground states, which we call accordion model FQH states.

I. INTRODUCTION

Ever since the work of Laughlin [1], the construction of model wavefunctions for fluid states, with all particles in the lowest Landau level (LLL), has been a focal point in the study of the Fractional Quantum Hall Effect (see e.g. Ref. [2] for a review on FQHE). The Laughlin states have a filling fraction \( \nu = 1/m \), with \( m = \text{even (odd) for bosonic (fermionic)} \) electrons. In search of representations portraying more general filling fractions, the hierarchy approach [3, 4] and composite fermion approach [5] were introduced. Both approaches, however, were essentially extensions of the Laughlin’s model, not necessarily describing distinct phases of matter. That notwithstanding, Haldane’s hierarchy [3] caused an important paradigm shift. Now known as “Haldane’s sphere”, the study of the quantum Hall states in spherical geometry was initiated. The benefits of spherical geometry are three-fold:

1. Absence of boundary: No edge effect is present.
2. Compactness: Finite system size.
3. Zero genus: No topological degeneracy.

In other words, the (Riemann) sphere is the natural candidate for studying FQH ground states.

Consider a system of a thermodynamically large number \( N \) of spinless bosonic/fermionic electrons, living on a Riemann sphere \( \mathbb{C}_\infty \) of radius \( R \). A large constant magnetic field is applied normal to the sphere resulting in \( N_\phi \) flux quanta penetrating through. We further assume that the ground state of the FQH system lives in the LLL entirely. Locally (i.e. over \( \mathbb{C}_\infty - \{ \infty \} \cong \mathbb{C} \), any wavefunction in LLL is of the form:

\[
\Psi(z_1, \cdots, z_N) = \frac{P(z_1, \cdots, z_N)}{\prod_{i=1}^{N} \left[ 1 + |z_i|^2/(4R^2) \right]^{1+N_\phi/2}}
\]

with \( z_i = x_i + iy_i \) is the complex coordinate of the \( i \)th particle. If the particles are bosonic (resp. fermionic), the function \( P \) is a symmetric (resp. anti-symmetric) polynomial of local degree (i.e. highest power of \( z_1 \) in \( P \)) being at most \( N_\phi \). At the same time, for any anti-symmetric polynomial \( P_a \) in variables \( z_1, \cdots, z_N \), there exists a symmetric polynomial \( P_s \) such that \( P_a = \prod_{i<j}(z_i-z_j)P_s \). Therefore, without loss of generality, we will focus entirely on the symmetric/bosonic cases.

The concern of the current paper is the study of (model) FQH ground states, purely based on their mathematical form and general conditions they need to satisfy. A quantum Hall ground state wavefunction will be an incompressible fluid, translationally invariant (i.e. uniform) over the Riemann sphere. For \( N \) bosons and \( N_\phi \) flux quanta, these conditions can be paraphrased into:

(i) The wavefunction \( \Psi \) is the ground state of a gapped bosonic system.

(ii) \( P \) is a symmetric polynomial in \( N \) variables and of local degree \( N_\phi \).

(iii) \( P \) satisfies the following PDEs:

\[
L^+ P := \left[ \sum_{i=1}^{N} \partial_i \right] P = 0
\]

\[
L^- P := \left[ (z_1 + \cdots + z_N)N_\phi - \sum_{i=1}^{N} z_i^2 \partial_i \right] P = 0
\]

The conditions \( L^+ P = 0 \) and \( L^- P = 0 \) are respectively called highest weight and lowest weight conditions.

Henceforth we will refrain from any discussion on the gap and study polynomials satisfying the other two conditions. We call a polynomial \( P \) with properties (ii) and (iii), an \((N,N_\phi) \) FQH-like polynomial. We will review this definition in §III.

FQH-like polynomials are already concealing a wealth of information inside them. Partly, the purpose of current article is to explore them as thoroughly as we can in
full generality. In §III A, we will show that the concept of “FQH-like polynomial” is completely equivalent to the concept of the so-called “binary invariants” [thm. III.2]. Briefly, a binary $N$-form is a homogeneous polynomial in two formal variables $X,Y$:

$$
\beta_N(X,Y;\{a_r\}) = \sum_{r=0}^{N} \binom{N}{r} a_r X^{N-r} Y^r
$$

where $\{a_r\} := \{a_0, a_1, \cdots, a_N\}$ are the coefficients of $\beta_N$. Given a element $g$ in $\text{SL}_2(\mathbb{C})$, matrix multiplication transforms $(X,Y)^t \mapsto (X_g, Y_g)^t := g \cdot (X,Y)^t$. Define $g \cdot a_r$ such that $\beta_N(X_g, Y_g;\{a_r\}) = \beta_N(X,Y;\{g \cdot a_r\})$; this is called the induced action of $\text{SL}_2(\mathbb{C})$ over the space of coefficients of $\beta_N$. A binary invariant of order $N$ and degree $\delta$ is a homogeneous polynomial $Q(a_0, a_1, \cdots, a_N)$ of degree $\delta$ such that

$$
Q(g \cdot a_0, g \cdot a_1, \cdots, g \cdot a_N) = Q(a_0, a_1, \cdots, a_N)
$$

for all $g \in \text{SL}_2(\mathbb{C})$. Binary invariants have been studied since late nineteenth century, by mathematicians like Clebsch, Gordan, Cayley, Hermite, Sylvester, Petersen and Hilbert. An plethora of technology is already developed in this theory (see, e.g., [6]). For example, “Clebsch-Gordan coefficients” were first found in the study of binary invariants. We will utilize two of those tools in particular: “Cayley’s theorem” (in §III B) and “Hermite’s reciprocity theorem” (in §VII D).

The correspondence between FQH-like polynomial and binary invariants bears fruit to another correspondence, this time with regular graphs. The discussion is done in §III B. In short, a (labeled) graph $G$ of order $N$, with vertexes $\{z_1, \cdots, z_N\}$, is $\delta$-regular if the number of edges incident to each vertex $z_i$ is exactly $\delta$ (multiple edges between two distinct vertexes are allowed; no edge from a vertex to itself is allowed). In a graph $G$ of order $N$, associate a polynomial factor $(z_i - z_j)^{w_{ij}}$ to an edge with multiplicity $w_{ij}$ between $z_i$ and $z_j$ (with $w_{ij} = 0$ understood as no edge). Then the symmetric polynomial

$$
P_G(z_1, \cdots, z_N) = \mathcal{S} \left[ \prod_{i<j} (z_i - z_j)^{w_{ij}} \right]
$$

is known as symmetrized graph monomial (SGM) of $G$ [7, 8]. The connection between SGMs and FQH-like polynomials is via (a paraphrasing of) Cayley’s theorem [thm. III.3]:

1. Any $(N, \delta)$ FQH-like polynomial is a $\mathbb{C}$-linear combination of SGMs of $\delta$-regular graphs of order $N$.
2. If $G$ is any $\delta$-regular of order $N$, then either $P_G$ identically vanishes, or it is a $(N, \delta)$ FQH-like polynomial.

In other words, instead of constructing polynomials satisfying the HW and LW conditions, we can build regular graphs. Obviously, this grants the superficial benefit of being able to draw a FQH ground state. But, more importantly, graph theoretic properties of these graph translate into physical properties naturally.

In our approach to FQH ground states, the strategy would be to create “sensible” regular graphs and then take their SGM. However, a lone FQH-like polynomial does not bear any significance physically. One needs a “thermodynamic” sequence of “sensible” regular graphs, $(\cdots, G_{n-1}, G_n, G_{n+1}, \cdots)$. The size of a FQH system of filling fraction $\nu = v/d$ is determined by number of bosons $N$ and number of flux quanta $\delta$, such that

$$
\delta = \nu^{-1}N - S
$$

with $S$ being known as the shift. We will limit our attention to the case where $N_n = nv$ and $\delta_n = (n-1)d$, where $n$ is allowed to be any integer $\geq 2$ (indicating the size). The two numbers $v$ and $d$ will be called vertex and degree augmentation constants, respectively. The constructed sequence of graphs $(G_2, G_n, \cdots, G_n, \cdots)$ will be such that $G_n$ is regular of order $N_n$ and degree $\delta_n$. The full construction of “thermodynamic” sequence, called aggregation, will be presented in §IV.

Let us give a preview of what aggregation looks like. Let $J_n^+$ be the strictly upper-triangular $n \times n$ matrix with all entries above the main diagonal equal to one. The initial data of aggregation is, roughly speaking, a $v \times v$ matrix $F$, called a $(v,d)$-face matrix, satisfying the CRLE postulates:

1. The matrix $W_2 = J_2^+ \otimes F + (J_2^+ \otimes F)^t$ is the adjacency matrix of a connected graph.
2. The sum of each row, as well as each column, of $F$ is equal to the constant $d$.
3. The diagonal entries of $F$ are all non-zero.
4. The product $vd$ is even.

Given a $(v,d)$-face matrix $F$, the $N_n \times N_n$ matrix

$$
W_n = J_n^+ \otimes F + (J_n^+ \otimes F)^t
$$

is taken to be the adjacency matrix of $\delta_n$-regular graph $G_n$; i.e. the entry $(W_n)_{ij}$ is the multiplicity of the edge between $z_i$ and $z_j$. The sequence $(G_2, G_3, \cdots, G_n, \cdots)$ will be called the aggregation sequence. The graphs $G_n$, obtained via this procedure, are called an $(n,v,d)$-accordion graphs. One naturally has $G_n \subset G_{n+1}$; one should think of $G_{n+1}$ as the thermodynamic extension of $G_n$ in going from size $n$ to $n+1$.

If the polynomial $SGM(G_n)$ were to be the ground state of a local FQH Hamiltonian $H_n$, then the locality of $H_n$ should be reflected somehow in the graph $G_n$. This is indeed the case if $G_n$ is an element of an aggregation sequence. The detailed discussion and illustration of this is presented in §IV and §VII A. A few highlights of this discussion are:

- The vertex augmentation constant $v$ is the size of the clusters of the model FQH states.
• There are exactly $n$ clusters in $SGM(G_n)$, which are pairwise disjoint too.
• The subgraph restricted to any pair of clusters is a copy of $G_2$.

We interpret these properties as “any pair of clusters correlate in exactly the same way as any other pair”.

Section V is devoted to examples. In V A we revisit many of the classic model FQH ground states in the literature and show that they are all special cases of accordian model FQH ground states. These special cases include: Laughlin states [1], $\nu = 1$ Moore-Read state [9], $\nu = v/2$ $Z_v$-parafermionic states [10], Gaffnian [11], Haffnian [12] and some of the Jack polynomials [13, 14]. In §V B, we introduce a class of examples which are so-called weighted Cayley graphs. A weighted Cayley graph is a triple $(G, S, \mu)$ with $G$ a finite group, $S \subset G$ (a generator set) such that if $s \in G$ then $s^{-1} \in S$, and $\mu : S \rightarrow \mathbb{Z}_+$ (multiplicity) such that $\mu(s) = \mu(s^{-1})$. The weighted Cayley graph Cay$(G, S, \mu)$ is a graph with vertex set $G$, and an edge of multiplicity $\mu(s)$ between $g, h \in G$ if and only if $h = gs$. We show that representative graphs of parafermionic states are all weighted Cayley graphs of cyclic groups, while Gaffnian’s is a weighted Cayley graph of dihedral group [hms. V.1, V.2].

Section §VI details properties of FQH-like polynomials of the form $P_n = SGM(G_n)$, or FQH-like sequences $\Pi = SGM(G_2, G_3, \ldots, G_n, \ldots)$ with $G_n$ an $(n, v, d)$ accordian graph. We have gathered a few of the graph theoretical properties of accordian graphs in a theorem [thm. VI.1]. In §VI B we explore properties related to root partitions of $P_n$. The subsection §VI C is then about the clustering properties of $\Pi$. The clustering and root partition properties of FQH ground states are persistent, important and longstanding topics of research of the field.

Among the proposed bosonic trial FQH ground states in the literature, the $\nu = 1$ Pfaffian aka Moore-Read state [9] was perhaps the first model that was drastically different from Laughlin’s. This model was later on generalized to the so-called bosonic $(\nu = v/2)$ $Z_v$-parafermionic aka Read-Rezayi states [10]; with $Z_2$-parafermionic being the Pfaffian. Let us denote by $P_n^{vRR}(z_1, \ldots, z_n)$ the $Z_v$-parafermionic state over $nv$ bosonic electrons. A fascinating property of these states is that upon bringing $v$ particles to a common point $Z$, i.e. $z_{(n-1)v+1} = \cdots = z_{nv} = Z$, one finds

$$P_n^{vRR}(z_1, z_2, \cdots, z_{(n-1)v}, Z, Z, \cdots, Z) = \prod_{i=1}^v (Z - z_i)^2 P_{n-1}^{vRR}(z_1, z_2, \cdots, z_{(n-1)v})$$

This factorization property is now known as the $(v, 2)$-clustering property of $Z_v$-parafermionic states. Moreover, the $(v, 2)$-clustering property uniquely characterizes the $Z_v$-parafermionic states.

Consider a (thermodynamic) sequence of FQH-like polynomials ($\cdots P_{n-1}, P_n, P_{n+1}, \cdots$) describing a filling fraction $\nu$, where $P_n$ is $N = nv$ bosons (with $n \gg 1$), of local degree $N_\phi = \nu^{-1}N - d$, with $v, d$ two positive integers. One says this sequence satisfies a $(v, d)$-clustering property, if

$$P_n(z_1, z_2, \cdots, z_{(n-1)v}, Z, Z, \cdots, Z) = \prod_{i=1}^v (Z - z_i)^d P_{n-1}(z_1, z_2, \cdots, z_{(n-1)v})$$

Ever since the discovery of parafermionic states, and their clustering property, one of the main goals of the FQH scientific community has been to interpret and (at least partially) resolve the following question:

“How do we find and classify all FQH-like sequences that satisfy a clustering property?”

Parallel to physicists’ pursuit of the solution to aforementioned question, Feigin et. al. [15] discovered a connection between translationally invariant symmetric polynomials and Jack polynomials. Motivated by this mathematical work, Bernevig and Haldane [13, 14] generalized the parafermionic states to (specialized) Jack polynomials $J^\alpha(n,v,d)$ with $v + 1$ and $d$ relatively prime, $\alpha = -(v+1)/(d-1)$ and

$$J^\alpha(n,v,d) = (0^v d^v \cdots ((n-2)d^v)((n-1)d^v))$$

The notation $m^n$ signifies that $m$th orbital of LLL is occupied by $n$ bosons. These Jacks are all FQH-like polynomials. In Ref. [13], the authors conjecture that the sequence $(J^\alpha(n,v,d))_{n \geq 2}$ has a $(v, d)$-clustering property. This conjecture is now proved [16] using methods from conformal field theory (see also the related works [17], [18], [19]). A generalization of this conjecture is also proved in [20] using representation theory. However, unlike Read-Rezayi states, these model FQH ground states are not in general uniquely characterized by their clustering property. In fact, such unique characterization is quite rare.

Tantamount to clustering, the study of the structure of root partitions has always been another window into the internal structure of FQH ground states. A free bosonic state with $N$ particles, over the sphere with $N_\phi$ flux quanta, living in LLL, is of the form

$$\tilde{m}_\lambda(z_1, \cdots, z_N) = \sum_{\sigma \in \mathfrak{S}_N} z_{\sigma(1)}^{\lambda_1} z_{\sigma(2)}^{\lambda_2} \cdots z_{\sigma(N)}^{\lambda_N}$$

where, w.l.o.g. $N_\phi \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. In $\tilde{m}_\lambda$, for each $1 \leq i \leq N$ there is a boson in the LLL orbital $\lambda_i$. The sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ is called a partition of $M := N N_\phi/2$. Every $(N, N_\phi)$ FQH-like polynomial can be expressed uniquely as a linear superposition of free bosonic states. In this superposition, if $\tilde{m}_\lambda$ appears with a non-zero coefficient, one calls $\lambda$ a root partition of $P$. Given two root partitions $\lambda, \mu$, one says $\lambda$ dominates $\mu$ if $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for all $k$ (dominance is equivalent to squeezing [21]). If $P$ possesses a root
partition \( \Lambda \) which dominates all other root partitions, we will call \( \Lambda \) predominant. On another note, if \( \Lambda \) is such that there are at most \( v \) particles in any consecutive \( d \) orbitals, we say \( \lambda \) satisfies a \((v, d)\) generalized exclusion principle (GEP).

Emanated from the concepts of root partition, predominance and exclusion principle, another equivocal question regarding FQH-like polynomials is risen: “What conditions are required of a FQH-like polynomial in order for it to possess a predominant root partition \( \Lambda \) that satisfies a GEP?”

This line of thinking was initiated by Haldane and Rezayi who showed that \( \Lambda(N, 1, 2m) \) is a predominant root partition of Laughlin \( 2m \)-states over \( N \) particles. Clearly \( \Lambda(n, v, d) \) satisfies a \((v, d)\) GEP for any \( n \). For many examples in the literature, the existence of a naturally “special” root partition \( \Lambda \) with a GEP has always been known. Those root partitions are actually often similar, if not identical, to \( \Lambda(n, v, d) \). Predominance, however, was not under the spotlight until the discovery of Jack polynomial model FQH states. By definition, the (specialized) Jack polynomials \( J_{\alpha(n,v,d)} \) have \( \Lambda(n, v, d) \) as a predominant root partition. Hence, in particular, parafermionic states and Gaffnian [11] have this property too. The reversal of the above question is also of great importance for classification ambitions: “Given a partition \( \Lambda \) satisfying a generalized exclusion, does there exists a FQH-like polynomial \( P \) such that \( \Lambda \) is a predominant root partition of \( P \)? If \( \Lambda \) is chosen appropriately so that such \( P \) exists, to what degree does \( \Lambda \) characterize \( P \)”

The latter question is closely related to the pattern of zeros approach [22].

It is imperative for us to investigate these properties in our model. We will also develop (or borrow) some general tools to attack problems related to these properties. In §VI B we relate the concept of root partitions of the FQH-polynomial \( P_G = SGM(G) \) for any graph, to the orientations of \( G \) [thm. VI.2] (also see [23]). We then move on to prove that if \( G \) is a \((n, v, d)\) accordion graph, then \( \Lambda(n, v, d) \) is a root partition of \( P_G \) [thm. VI.3]. This in particular means that \( P_G \) does not identically vanish. We further conjecture that \( \Lambda(n, v, d) \) is predominant. In §VI C we review the concept of fusion. In an arbitrary fusion, \( a \) particles are brought to a common point \( Z_a \), then \( b \) particles to \( Z_b \), etc. Let \( P = SGM(G) \) for some graph \( G \), and \( P_{\text{fused}} \) be resulting polynomial after the fusion. We present a general formula for \( P_{\text{fused}} \) [thm. VI.5]. The formula translated the problem of finding the fused polynomial of \( P_{\text{fused}} \) into a search for certain vertex colorings of \( G \). Using this formula, we then move on to prove that sequences \( \Pi = SGM(G_2, G_3, \cdots, G_n) \) of \((n, v, d)\) accordion graphs obtained by aggregation from a face matrix \( F \), satisfy a \((v, d)\)-clustering property [cor. VI.6]. Finally, in §VI D, using Hermite’s reciprocity theorem [thm. VI.9] and ideas from Ref. [15], we will give a new proof to the statement: “parafermionic states, Gaffnian and Haffnian are all uniquely characterized by their clustering property”.

The proofs for all statements are gather in Appendix E. Also, all of the mathematical definitions used in the paper, specially for graph theory, are collected in Appendix A.

## II. FQH-LIKE POLYNOMIALS

The central mathematical entity in this paper is an “FQH-like polynomial”. Over a Riemann sphere of radius \( R \), with \( N_\delta \) flux quanta and \( N \) bosonic electron, a state living exclusively in the lowest Landau level (LLL) is of the form

\[
\Psi(z_1, \cdots, z_N) = \frac{P(z_1, \cdots, z_N)}{\prod_{i=1}^{N}[1+|z_i|^2/(4R^2)]^{1+N_\delta/2}}
\]

where \( P \) is a symmetric polynomial. If \( \Psi \) is a ground state, then \( P \) has to satisfy extra conditions, which leads to the notion of FQH-like polynomials. In this section we give the definition of FQH-like polynomial and review the concept of root partitions.

### A. Definition

A polynomial \( P \), with complex coefficients, is called a \((N, \delta)\) FQH-like polynomial if:

(i) \( P \) is a symmetric polynomial in \( N \) variables.

(ii) The local degree of \( P \) is \( \delta \). (The local degree of symmetric polynomial \( P(z_1, \cdots, z_N) \), is the highest power of \( z_1 \) that appears in \( P \)).

(iii) \( P \) satisfies the following set of PDEs:

\[
L^+ P := \sum_{i=1}^{N} \partial_i P = 0 \tag{2a}
\]

\[
L^2 P := \frac{N\delta}{2} - \sum_{i=1}^{N} z_i \partial_i P = 0 \tag{2b}
\]

\[
L^- P := \sum_{i=1}^{N} z_i^2 \partial_i P = 0 \tag{2c}
\]

where \( Z = z_1 + \cdots + z_N \).

The conditions \( L^+ P = 0 \) and \( L^- P = 0 \) are respectively called the highest weight (HW) and lowest weight ( LW) conditions. Note that \( 2L^2 = [L^+, L^-] \), making the condition \( L^2 P = 0 \) an automatic consequence of HW+LW conditions. In fact, one can check that \( L^+, L^- \) endow the space of polynomials with an angular momentum structure (i.e. they make the space of polynomials an infinite dimensional representation of \( \mathfrak{sl}_2 \)). Since the operator \( \sum_i z_i \partial_i \) is the Euler operator, \( L^2 P = 0 \) requires \( P \) to be homogeneous of total degree \( M = N\delta/2 \). We may also refer to \( N \) and \( M \) as number of particles and total angular momentum respectively.
B. Root Partitions

Although we will not need the concept of root partitions until § VI B, it is natural to define them alongside FQH-like polynomials. Let \( \lambda = (\lambda_1\lambda_2\cdots\lambda_N) \) be a partition (DEF.1) of \( M = N\delta/2 \), such \( \ell(\lambda) \leq N \) (DEF.2) and \( L(\lambda) \leq \delta \) (DEF.3). Denote the set of all such partitions as \( P_{N,\delta} \). Given \( \lambda \in P_{N,\delta} \), the symmetric polynomial

\[
\tilde{m}_{\lambda}(z_1, \cdots, z_N) = \mathcal{S} \left[ z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_N^{\lambda_N} \right] \tag{3}
\]

is a free bosonic state with \( N \) bosonic electrons, over a sphere with \( \delta \) flux quanta. Here \( \mathcal{S} \) is the symmetrization operator (DEF.34). In literature of symmetric polynomials, \( \tilde{m}_{\lambda} \) is called the ‘augmented’ monomial symmetric function. We often write a partition \( \lambda \) in the alternative form \( \lambda = (0^{a_0}1^{a_1} \cdots \delta^{a_\delta}) \), where \( \nu_r \) is the multiplicity (DEF.4) of \( r \) in \( \lambda \) (also see (DEF.5)). Defining \( u_\lambda = \prod_{\nu_r=0}^{\delta} \nu_r! \), the polynomial \( m_{\lambda} := u_\lambda^{-1} \tilde{m}_{\lambda} \) is the traditional monomial symmetric functions. The set of all \( m_{\lambda} \) is a \( Z \)-basis for the space of homogeneous symmetric polynomials, with coefficients in \( Z \), over \( N \) variables, having degree \( M \) and local degree \( \leq \delta \). Therefore any FQH-like polynomial \( P \) can be written uniquely as a superposition

\[
P = \sum_{\lambda} c_{\lambda} m_{\lambda} = \sum_{\lambda} \tilde{c}_{\lambda} \tilde{m}_{\lambda} \tag{4}
\]

where the sum is done over elements of \( P_{N,\delta} \) and \( c_{\lambda} \in C \) and \( \tilde{c}_{\lambda} = c_{\lambda} u_\lambda^{-1} \). A partition \( \lambda \in P_{N,\delta} \) is called a root partition of \( P \) if \( c_{\lambda} \neq 0 \). Any \((N, \delta)\)-FQH-like polynomial \( P \), by definition, has a root partition \( \lambda \) with \( L(\lambda) = \delta \).

III. BINARY INVARIANTS, REGULAR GRAPHS AND CAYLEY’S THEOREM

In III A we will introduce binary invariants and show that “FQH-like polynomial” and “binary invariant” are equivalent concepts. In III B, utilizing Cayley’s theorem and symmetrized graph monomials, two tools developed for studying binary invariants in nineteenth century, we connect FQH-like polynomials to the theory of regular graphs. After the connection to graph theory is established, in the upcoming sections, we will pursue the graph theoretic viewpoint of FQH ground states.

Convention: For our purposes here it is more suitable to understand Riemann sphere as the complex projective line \( \mathbb{P}^1 \) (DEF.39), rather than compactification of \( C \) with a point at infinity. The way a FQH ground state is usually dealt with is a polynomial function \( P(z_1, \cdots, z_N) \) where \( z_i \) is the complex coordinates of \( i \)th particle. However, our bosonic electrons do not live on \( C \), rather Riemann sphere \( \mathbb{P}^1 \) is their host. If we denote the projective coordinates of the \( i \)th particle by \([x_i : y_i] \in \mathbb{P}^1 \), then we should understand the complex coordinate \( z_i \) as \( z_i = x_i/y_i \).

A. Binary Invariants

1. Binary forms

Let \([x_i : y_i] \) be the projective coordinates of our bosonic electrons. Construct the homogeneous polynomial

\[
\beta_N(X, Y) := \prod_{i=1}^{N}(Xy_i - Yx_i) = \prod_{i=1}^{N} \det \begin{pmatrix} X & x_i \\ Y & y_i \end{pmatrix} \tag{5}
\]

which is known as a binary \( N \)-form. By construction, \([x_i : y_i] \) are the \( N \) projective roots (DEF.40) of this binary form. But one can just as easily rewrite this binary form, upon expansion, as

\[
\beta_N(X, Y) = \sum_{r=0}^{N} \binom{N}{r} a_r X^{N-r} Y^r \tag{6}
\]

with \((a_0, \cdots, a_N) \in C^{N+1} \). Since multiplying a polynomial by a constant does not change its roots, one should actually work with \([a_0 : a_1 : \cdots : a_N] \in \mathbb{P}^N \), a point in \( N \) dimensional complex projective space (DEF.38). We call \([a_0 : a_1 : \cdots : a_N] \in \mathbb{P}^N \) the (projective) coefficients of this binary form. The key observation is that, via the above technique, we have managed to uniquely parametrize the set of \( N \) points on the Riemann sphere (the projective roots of \( \beta_N \)) by points of \( \mathbb{P}^N \) (the projective coefficients of \( \beta_N \) and vice versa. As we will see, the root-coefficient duality leads to a duality between symmetric polynomials \( P \) in complex roots \( z_1, \cdots, z_N \), and homogeneous polynomial \( P^s \) in coefficients \( a_0, \cdots, a_N \).

2. Binary duals

Given a binary form \( \beta_N \) the coefficients \( a_r \) can be obtained as a function of complex roots \( \{z_i\} := z_1, \cdots, z_N \), where as usual \( z_i = x_i/y_i \). If we denote by \( e_r(\{z_i\}) \) the \( r \)th elementary symmetric polynomial in \( N \) variables (DEF.36), then

**Proposition III.1.** When the projective roots are away from infinity \( \infty :=[1, 0] \), one can take \( a_0 = 1 \) and with that choice, for \( 1 \leq r \leq N \), one finds \([N!/r!(N-r)!] a_r = (-1)^r e_r \).  

Given a symmetric polynomial \( P \) in \( \{z_i\} \), we are looking for a homogeneous polynomial \( P^s \) in coefficients \( a_0, \cdots, a_N \) such that

\[
P(z_1, \cdots, z_N) = P^s(a_0(\{z_i\}), \cdots, a_N(\{z_i\}))
\]

The polynomial \( P^s \) will be called the binary dual of \( P \). We do this as follows: Using fundamental theorem of symmetric polynomials (DEF.37) and due to the relation in prop. III.1 between \( a_r \) and \( e_r \) for \( r > 0 \), one can find a unique \( P^s \in C[a_1, \cdots, a_N] \) such that

\[
P^s(\{z_i\}) = P_1(a_1(\{z_i\}), a_2(\{z_i\}), \cdots, a_N(\{z_i\}))
\]
The dual $P^\delta$ is now the homogenization of $P'$, i.e.
\[ P'(a_0, a_1, \ldots, a_N) = a_0^\delta P'\left(\frac{a_1}{a_0}, \ldots, \frac{a_N}{a_0}\right) \]
where $\delta$ is the local degree of $P$, aka the degree of $P'$. The reverse process $P^\delta \to P$ is obtained by evaluation $a_r \mapsto a_r(\{z_i\})$. This gives a bijection between symmetric polynomials in complex roots of $\beta_N$ and homogeneous polynomials in the (projective) coefficients of $\beta_N$. We now introduce binary invariants.

3. Binary Invariants

The group $SL_2(\mathbb{C})$ has a natural action on $\mathbb{C}^2$ via multiplication:
\[ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{C}) \Rightarrow \left(\begin{array}{c} X \\ Y \end{array}\right) \overset{g}{\mapsto} \left(\begin{array}{c} AX + BY \\ CX + DY \end{array}\right) \]
Upon the above action, a binary $N$ form $\beta_N(X,Y)$ also transforms. Let $\beta_N(X,Y,\{a_r\})$ be the binary $N$-form with coefficients $\{a_r\}$. Define $g * a_r$ such that
\[ \beta_N(AX + BY; CX + DY; \{g * a_r\}) = \beta_N(X,Y; \{a_r\}) \]
In other words, through binary forms, one finds an induced action of $SL_2(\mathbb{C})$ over the coefficient space $\mathbb{P}^N$, i.e.
\[ [a_0 : \cdots : a_N] \mapsto [g * a_0 : \cdots : g * a_N]. \]
Now consider a homogeneous polynomial $Q(a_0, a_1, \ldots, a_N)$ of degree $\delta$. We say $Q$ is a binary invariant of order $N$ and degree $\delta$, or simply an $(N, \delta)$ binary invariant, if for all $g \in SL_2(\mathbb{C})$ one has $Q(a_0, a_1, \cdots, a_N) = Q(g * a_0, \cdots, g * a_N)$. The next theorem will now unify the notion of a FQH-like polynomial and a binary invariant.

**Theorem III.2.** $P$ is an $(N, \delta)$ FQH-like polynomial if and only if $P^\delta$, the binary dual of $P$, is an $(N, \delta)$ binary invariant.

### 1. Symmetrized Graph Monomials

Symmetrized graph monomials (SGM) are a tool invented in late nineteenth century by Sylvester and Petersen [7, 8] for their study of binary invariants. In fact, it is arguable that SGMs are the birthplace of graph theory altogether. Let $G = (V, E, w)$ be a graph of order $|V| = N$ (def.9). Label the vertexes of $G$ by variables $z_1, \ldots, z_N$ (non-repeating). For each edge $z_i z_j \in E$ assign a factor $(z_i - z_j)^{w_{ij}}$ and multiply all of them. In other words
\[ \tilde{P}_G(z_1, \cdots, z_N) := \prod_{i < j} (z_i - z_j)^{w_{ij}} \] (7)
where $W = (w_{ij})$ is the adjacency matrix (def.11) of $G$. The polynomial $\tilde{P}_G$ is called a graph monomial of $G$. If one also symmetrizes the above polynomial, the result is called symmetrized graph monomial (SGM) of $G$, denoted by both $P_G$ and $SGM(G)$, i.e.
\[ SGM(G) \equiv P_G \equiv \mathcal{J}[\tilde{P}_G] \] (8)
The graph monomial implicitly depends on a certain vertex ordering of $G$ (def.10). A different ordering yields a different graph monomial. However, the SGM is label independent and in fact a graph invariant (i.e. if $G$ and $H$ are isomorphic (def.13), their SGMs are the same).

### 2. Cayley’s Theorem

In our study, the class of regular graphs play the central role. In a graph $G = (V, E)$ one defines the degree of a vertex $x \in V$ as $\delta(x) =$ number of edges incident to $x$. A graph $G$ is called $\delta$-regular or regular of degree $\delta$ if the degree of all of its vertexes is $\delta$. Henceforth an $(N, \delta)$ regular graph will mean a $\delta$-regular graph of order $N$.

**Theorem III.3** (Cayley).

1. If $G$ is an $(N, \delta)$ regular graph, then $P_G^\delta(a_0, \cdots, a_N)$ is either an $(N, \delta)$ binary invariant or it is identically zero (hence $P_G$, if non-vanishing, is of local degree $\delta$).

2. Conversely, if $Q$ is an $(N, \delta)$ binary invariant, then there exist $(N, \delta)$ regular graphs $G_1, \cdots, G_r$ and complex numbers $p_1, \cdots, p_r$ such that
\[ Q = p_1 P_{G_1}^\delta + \cdots + p_r P_{G_r}^\delta \] (9)

Consequently, any $(N, \delta)$ FQH-like polynomial $P$ is of the form $P = p_1 P_{G_1} + \cdots + p_r P_{G_r}$ for $(N, \delta)$ regular graphs $G_1, \cdots, G_r$. Naturally, it is enough to study those FQH-like polynomials which are of the form $SGM(G)$ for a single graph. It is very much possible to have two non-isomorphic regular graphs $G, H$ with $P_G = P_H$. The representation is in fact many-to-one. That being said, this
many-to-one nature poses no threat to our study. What we have seen in this section summarizes as: \((N, \delta)\) FQH-like polynomials, are the same as \((N, \delta)\) binary invariant, are intimately related to \((N, \delta)\) regular graphs.

**IV. AGGREGATION CONSTRUCTION**

Motivated by the findings of the previous section, our strategy now is to construct a class of “sensible” regular graphs which represent model FQH ground states. Once a “sensible” graph \(G\) is identified, \(SGM(G)\) will be the model ground state. In vague terms, what we require out of a “sensible” regular graph can be summarized as follows:

1. The model graph \(G\) should have a well-defined notion of a thermodynamic limit.
2. Ultimately, the symmetric \(P_G\) is supposed to be (potentially) the ground state of an effective Hamiltonian \(H\). But any such \(H\) is a local (i.e. all the interactions involve only a few other particle). We demand the graph representative \(G\) to reflect this locality manifestly.
3. The model state \(P_G\) should be a refinement of Laughlin state (although \(P_G\) is by no means obtained as a hierarchical state).

The above ideas, which are admittedly formulated in completely vague terms, will serve as compass toward our eventual construction. Before we go into the fine details of our construction, in this introductory part, we will schematically demonstrate the meaning of the words “thermodynamic limit”, “locality” and “refinement”.

The notion of a “thermodynamic limit” is related to the size of the quantum Hall system. This size is just two natural numbers: The number of bosonic electrons \(N\) and the size of the lowest Landau level \(\delta\). Since, by definition, filling fraction \(\nu\) is a thermodynamic invariant of the system, we also fix \(\nu\). This gives the constraint

\[
\delta = \nu^{-1} N - S
\]

(10)

We are particularly interested in system sizes of the form \(N_n = n\nu\) and \(\delta_n = (n - 1)d\) with \(v, d\) fixed integers (this happens for \(\nu = v/d\) and \(S = d\)). Here \(n\) is a free integer, which can grow indefinitely. For convenience, we allow \(n \geq 2\). The incorporation of “thermodynamic limit” will be done with construction of infinite sequences of regular graphs

\[
\Gamma = (G_2, G_3, \ldots, G_n, \ldots)
\]

such that \(G_n\) is a \((N_n, \delta_n)\) regular graph. We design a machinery, called aggregation, to rigorously produce these sequences. Aggregation processes a finite (small) amount of initial data, which is independent of the system size, and generates the full infinite sequence.

For illustration of “locality”, it is best to rely on a familiar example. Let \(\nu > 1\) and consider the \(\nu = v/2\) bosonic \(\mathbb{Z}_v\) parafermionic state, aka Read-Rezayi state [10] (with the \(v = 2\) case being Pfaffian or Moore-Read state [9]) over \(nv\) particles. Label the variables \(\{z_n^{(i)}\}\) with elements of two cyclic groups: \(i \in \mathbb{Z}_n\) and \(s \in \mathbb{Z}_v\), so that addition has a clear meaning. In what follows \(F^{(vRR)}\) is a \(k \times k\) matrix, and for each pair of \((i, j)\) \(\in \mathbb{Z}_n \times \mathbb{Z}_n\), the expression \(P^{(vRR)}_2(i, j)\) is a polynomial. We define them as:

\[
(F^{(vRR)})_{st} = \delta_{s,t} + \delta_{s,t+1}
\]

(11a)

\[
P^{(vRR)}_2(i, j) = \prod_{s,t \in \mathbb{Z}_n} (z_s^{(i)} - z_t^{(j)}) (F^{(vRR)})_{st}
\]

(11b)

Let us call the set \(\{z_0^{(i)}, \ldots, z_{n-1}^{(i)}\}\) the \(i\)th cluster. With these definitions, the \(\mathbb{Z}_v\)-parafermionic state becomes:

\[
P^{(vRR)}_n \left( \{z_s^{(i)}\} \right) = \mathcal{F} \left\{ \prod_{0 \leq i < j < n} P^{(vRR)}_2(i, j) \right\}
\]

(11c)

The functional form of what appears inside of symmetrization is rather special: No matter what \(n\) is, and no matter which pair of clusters \(i < j\) is chosen, the variables of the two clusters relate via a factor \(P^{(vRR)}_2\).

In §VI A we will explain that this feature translates to: any pair of clusters correlate in exactly the same fashion as any other pair of clusters do. Physically, this local correlation in the ground state is caused by the effective local Hamiltonian. Aggregation will replicate and heavily depend on this locality feature.

Let \(P\) be a \((nv, (n - 1)d)\) FQH-like polynomial over \(N = nv\) variables. Divide these variables into \(n\) sets \(B_1, \ldots, B_n\) (mutually disjoint) each of size \(v\); e.g. \(B_a = \{z_{av+1}, z_{av+2}, \ldots, z_{av+n} \}\) with \(0 \leq a \leq n - 1\). Fuse the \(v\) variables in block \(B_a\) into a point \(z_a\). Let \(P(Z_0, \ldots, Z_{n-1})\) be the polynomial obtained from \(P\) by this fusing. We say \(P\) is a “refinement of Laughlin” if

\[
P(Z_0, \ldots, Z_{n-1}) = C \prod_{0 \leq i < j \leq n-1} (Z_i - Z_j)^{vd}
\]

for some constant \(C \in \mathbb{C}\). This obviously requires \(vd\) to be even. We will see in §VI C that all graphs \(G\) obtained in aggregation construction are such that \(SGM(G)\) is a refinement of Laughlin.

**A. Bosonic Laughlin states**

The bosonic Laughlin states [1] with filling fraction \(\nu = 1/2m\) are the simplest and, to this day, the most important FQH ground states. As a first full example, we will present their aggregation sequence. The graph representation of Laughlin state with filling fraction \(\nu = 1/2\) is shown in Fig. (1) for \(N = 8\). A simple graph of order \(n\) in
The sum of each row, as well as each column, of $W_n^F$ is equal to the constant $d$.

This is known as postulate for (R)egularity. The matrix

$$W_n^F = J_n^+ \otimes F + (J_n^+ \otimes F)^t$$

is now an adjacency matrix for a regular graph $G_n^F$ of order $N_n = nv$ and degree $\delta_n = (n - 1)d$. The desired sequence, known as the aggregation sequence, is therefore obtained:

$$\Gamma^F = (G_2^F, G_3^F, \ldots, G_n^F, \ldots)$$

One of the advantages of this POV is that if one has access to explicit polynomial form of a model FQH state $P$, finding the representative graph of $P$ is almost effortless using this POV; all it takes is to identify the $F$-matrix. For example, for Read-Rezai states one immediately sees from eq. (11) that $F_{v, d}^{v, d}$ is the $F$-matrix (see Figs. (4) and (6)).

Postulate (R) alone is not enough to prevent “bad” examples from happening. In Appx. B we explain why one needs to also enforce postulates for (C)onnectiveness, (E)veness and (L)oopiness, in order to satisfactorily tame the model. In adjacency matrix POV, the said postulates reads as:

(C) $J_n^+ \otimes F + (J_n^+ \otimes F)^t$ is the adjacency matrix of a connected graph (DEF.16).

B. Aggregation and Accordion Graphs

There are three different points of view (POV) toward the aggregation process: Adjacency Matrix POV (A–POV), Ceramic POV (C–POV) and Digraph POV (D–POV). Each angle has advantages and disadvantages:

- A–POV is the easiest to work with, but it is neither canonical nor insightful.
- C–POV gives a clear and intuitive meaning to locality, but its construction is not canonical and tedious since it is inductive.
- D–POV is completely canonical but it is abstract.

We will show in Appx. C that these POVs are essentially the same. Aggregation proves to be a powerful construction. Many of the classic FQH states in the literature can be reproduced as special cases of aggregation (see §VI). Also all graphs obtained by aggregation, which we named accordion graphs, lead to model FQH states with nice properties (see §VI).

1. Adjacency Matrix POV

The most straightforward way of building our desired graphs is by means of their adjacency matrix. Let us fix two positive integers $v, d$ called respectively the vertex degree augmentation constant. For the moment let $F$ be any $v \times v$ such that

(R) The sum of each row, as well as each column, of $F$ is equal to the constant $d$.

This is known as postulate for (R)egularity. The matrix

$$W_n^F = J_n^+ \otimes F + (J_n^+ \otimes F)^t$$

is now an adjacency matrix for a regular graph $G_n^F$ of order $N_n = nv$ and degree $\delta_n = (n - 1)d$. The desired sequence, known as the aggregation sequence, is therefore obtained:

$$\Gamma^F = (G_2^F, G_3^F, \ldots, G_n^F, \ldots)$$

One of the advantages of this POV is that if one has access to explicit polynomial form of a model FQH state $P$, finding the representative graph of $P$ is almost effortless using this POV; all it takes is to identify the $F$-matrix. For example, for Read-Rezai states one immediately sees from eq. (11) that $F_{v, d}^{v, d}$ is the $F$-matrix (see Figs. (4) and (6)).

Postulate (R) alone is not enough to prevent “bad” examples from happening. In Appx. B we explain why one needs to also enforce postulates for (C)onnectiveness, (E)veness and (L)oopiness, in order to satisfactorily tame the model. In adjacency matrix POV, the said postulates reads as:

(C) $J_n^+ \otimes F + (J_n^+ \otimes F)^t$ is the adjacency matrix of a connected graph (DEF.16).
(L) The diagonal elements of $F$ are all nonzero.

(E) The product $vd$ is even.

If $F$ satisfies all of CRLE postulates we say $F$ is a face matrix. Roughly speaking, if $P$ is the SGM of $G_n^2$, then postulate (E) saves $P$ from identically vanishing, while postulate (L) makes sure that $P_{z_1 \ldots z_n} \neq 0$ but $P_{z_1 \ldots z_n z_{n+1}} = 0$. The postulates (RLE) are absolutely crucial to the theory. In contrast, postulate (C) is convenient but not, strictly speaking, necessary. One can weaken the connectivity condition, but we do not believe that would add much more depth to the theory (see Appx. B).

2. Ceramic POV

In this POV we will inductively and concretely design our graph with thermodynamic limit, locality and transitive tournaments as our ruler and compass. As a summary, the process has three ingredients:

1. A bipartite (DEF.24) $d$-regular graph $G_2$ of order $2v$ which is called the shard. We may also say $G_2$ is a $(v,d)$-shard. We require $vd = even$ due to (E).

2. A “good” drawing of the shard, called a perfect display $X$ of $G_2$. Perfect displays are the analog of face matrices in A–POV.

3. A strict method of gluing many copies of $X$ together, called transitive gluing. Transitive gluing is done over a predetermined pattern called the complete schablone:

$$\mathcal{K} = (K_2, K_3, \ldots, K_n, \ldots)$$

the infinite sequence of (simple) complete graphs. Transitive gluing is the counterpart to matrix tensor product operation in A–POV.

The idea of the construction is to make copies of our perfect display $X$ and patch them together with transitive gluing.

**Perfect Display:** Let $G_2 = (V, E)$ be a $(v, d)$-shard with partition $(A, B)$. Define a height function $h : V \to \{0, 1/v, \ldots, (v-1)/v\}$ such that $h^{-1}(p)$ has exactly one element in $A$ and one element in $B$ for all $p \in \mathbb{Z}_v$. Associated to $h$, we will assign Cartesian coordinates to the vertexes of $G_2$:

$$\text{Cartesian coordinate}(p) = \begin{cases} (-1, 0, h(p)) & p \in A \\ (+1, 0, h(p)) & p \in B \end{cases}$$

This gives a drawing of $G_2$ embedded in $xz$-plane. We call this drawing, the $h$-display of $G_2$ denoted by a symbol $X_h$. A display $X$ is said to be perfect if vertexes of the same height are adjacent.

**Proposition IV.1.** Every regular bipartite multigraph admits a perfect display.

Suppose $h$ is such that $X_h$ is perfect. Each vertex then has some coordinates $(\epsilon, 0, z)$ with $\epsilon = \pm 1$. Define a total order on the vertexes as

$$\left(\epsilon, 0, z\right) < \left(\epsilon', 0, z'\right)$$

with respect to this ordering the adjacency matrix of $G_2$ is of the form $J_2^+ \otimes F_h + (J_2^- \otimes F_h)^\dagger$ for some matrix $F_h$.

One easily shows that $F_h$ is a $(v, d)$ face matrix ($X_h$ being a perfect display ensures $F_h$ satisfies postulate (L)). The connection between $h$ and $F_h$ is how A–POV and C–POV relate to one another.

**Transitive Gluing:** The procedure of building $G_n$ out of copies of $G_2$ is now as follows: Let $X$ be a perfect display of $G_2$.

1. Take a cube, draw the complete graph $K_n$ on the top face of the cube. Drill the cube vertically over each vertex, call it a junction. Also cut the cube along the edges vertically, and call them rails. Label the junction by $0, 1, \ldots, n-1$. Let us call this setup the stencil. (see Fig. 3)

2. Choose a rail $ij$ in $K_n$ assuming $i < j$. Make a copy of $X$ and slide it along the rail $ij$ in the stencil. The $A$ partition needs to be slid into $i$th junction and the $B$ partition into $j$th junction. (see Fig. 3). Also vertexes with lower height should enter first. Repeat the same process for all rails.

3. When all copies are slid in, at any point in the cube there is either no vertexes, or there are $n - 1$ overlapping vertexes. Identify overlapping vertexes into one.

The end result is called the $n$th ceramic due to it being the result of gluing many shards together. The above process is called the transitive gluing. If one regards the display as a dipole/arc $A \rightarrow B$, going from negative to positive, then, from the top view, the ceramic will look like a transitive tournament.

As is apparent from this construction (also see Fig. (4)), “locally” the graph $G_n$ always looks like $G_2$ (for a more careful treatment of locality see §VI A). The transitive gluing is a stingy process in which once the display is known, everything is already decided. This is consistent with our philosophy of thermodynamic limit. Finally, the role of the complete schablone should clarify what is meant by “building graphs on the foundation of complete graphs” (which in turn will become the reason why $SGM(G_n)$ is a refinement of the Laughlin $vd$-state over $n$ particles; see §VI C).
We might also call \( \text{SGM} \) CRLE digraph will be called an \((\text{def.}31)\)-accordion model \( \Phi \) (def.\(31\)). Given a (multi)digraph \( D \) let the notation \( \overline{D} \) stand for the underlying undirected graph of \( D \) (forgetful functor). Using all of these canonical notions, one can simply deduce the aggregation sequence to be:

\[
TT \otimes \Phi := (TT_2 \otimes \Phi, TT_3 \otimes \Phi, \ldots, TT_n \otimes \Phi, \ldots) \quad (13)
\]

In other words, one can show that the extra data, like vertex ordering or drawing, etc., used in the other POVs are redundant. The only datum one needs is the CRLE digraph \( \Phi \). In fact, even \( \Phi \) has some redundancy: Let \(-\Phi \) be the digraph in which one reverse the direction of every arc in \( \Phi \). Then \( \overline{TT} \otimes (-\Phi) = TT \otimes \Phi \). These redundancies are the subject of Appx. C.

We are finally at a position to define our \textit{accordion family} \( \mathcal{F} \) as the collection of aggregation sequences:

\[
\mathcal{F} = \{ TT \otimes \Phi \mid \Phi \text{ is an CRLE digraph} \} \quad (14)
\]

with understanding that \( \Phi \) and \(-\Phi \) should not be considered distinct. A graph of the form \( TT_n \otimes \Phi \) with \((v,d)\)-CRLE digraph will be called an \((n,v,d)\)-\textit{accordion graph}. We might also call \( \text{SGM}(TT_n \otimes \Phi) \) an accordion model FQH ground state.

### V. EXAMPLES

As the title suggest, in this section we will go through some examples. In §V A, we will revisit many of the famous model FQH ground states in the literature. The aim is to reconcile these classic models with our model. At times we will also explore possible generalizations. We have already talked about Laughlin state [1]. Other examples we will encounter include: Moore-Read state [9], Read-Rezayi states [10], Gaffnian [11], Haffnian [12] and even some of the Bernevig-Haldane’s Jack polynomials [13, 14]. In §V B we will introduce two subclasses of the accordion family; namely \textit{circulant} and \textit{prism-circulant} mega-classes. Circulant and prism-circulant mega-classes consist entirely of certain weighted Cayley graphs of cyclic group and dihedral group respectively. Parafermionic states belong to circulant mega-class, and Gaffnian (together with its Jack polynomial generalization) belong to prism-circulant mega-class.

#### A. Classic Examples and Some Generalizations

Given any model FQH polynomial, one can always multiply it with a bosonic Laughlin-Jastrow factor and another model FQH polynomial is obtained. However, except for Laughlin states themselves, the aggregation process generates only FQH-like polynomials that are not divisible by a Laughlin. Therefore all examples presented in this section are relatively prime to Laughlin-Jastrow factor. We break our discussion into six groups:

1. **(T1) (Cyclic/Parafermionic).** These are the \( \nu = 1 \) Moore-Read state and \( \nu = k/2 \) Read-Rezayi states. The face matrix for this group is

\[
(F^{(v)}_1)_{st} = \delta_{st} + \delta_{s,t+1} \quad (15)
\]

with indices and summation being in \( Z_v \).

2. **(T2) (Simon et. al.)** In Ref. [11, appx. C] along with introduction of Gaffnian wavefunction, Simon et. al. also suggest a generalization of Gaffnian. Their construction with filling fraction \( \nu = v/(v+1) \) is equivalent to the face matrix

\[
F^{(v)}_2 = 2I_v + J^+_v + (J^+_v)^t \quad (16)
\]

where \( I_v \) is the \( v \times v \) identity matrix and \( J^+_v \) is as before.

3. **(T3) (Cyclic square)** If we take \( G_n \) to be the graph representative of \( \nu = 1 \) Moore-Read state, i.e. using

\[
\text{FIG. 3. In this figure the drawing of bipartite graph shown is a perfect display of the shard of } \mathbb{Z}_3\text{-parafermionic state. The gray box with holes (junctions) and cuts (rails) is the stencil for } n = 4. \text{ The figure illustrates how the sliding process is done into the stencil.}
\]
FIG. 4. The graphs of Laughlin 2-state, \( Z_2 \), \( Z_3 \) and \( Z_4 \) parafermionic states for \( n = 2, 3, 4 \). The \( n = 2 \) drawing in each case is a perfect display. In \( n = 3, 4 \) edges of same color are a copy of the shard. These drawings are what ceramic POV yields.

\[
F^{(2)} \text{ in (T1) case, then } \nu = 1/2 \text{ Haffnian [12] over } 2n \text{ particles, is nothing but }
\]

\[
P_{\text{Haff}} = \mathcal{S} \left[ P_{G_n}^2 \right]
\]

Using the same idea, if one defines, for all \( v > 1 \),

\[
(F_3^{(v)})_{st} = 2\delta_{st} + 2\delta_{s,t+1}
\]

(17)
a generalization of Haffnian is achieved. We name this the cyclic square class. The case corresponding to \( Z_v \)-parafermions is called the \( v \)th cyclic square.

The CRLE digraphs and shards of these three types are shown in Table I and Table II respectively.

(T4) (Prism(−1,1,1)) Bernevig and Haldane have already integrated Gaffnian wavefunctions into their model FQH state via Jack polynomials with parameter \( \alpha = -3/2 \) which have minimal angular momentum [13, 14]. The characterizing partition of these Jacks are \( \Lambda(n, 2, 3) \) (see eq. (28) for a definition) with \( n \geq 2 \). Consequently, their generalization of Gaffnian is the Jack polynomial \( J^{(\alpha)}_{\lambda} \) with \( \alpha = -(2k + 1)/2 \) and \( \lambda = \Lambda(n, 2k, 3) \). This generalization keeps \( d \) fixed but increases the size of clusters (exactly in the same fashion parafermionic states generalize Moore-Read state). Let \( v = 2k \)
find all regular graphs of order 6 and degree 4 which have non-vanishing symmetrized graph monomial. One obvious solution would be the graph representative of Pfaffian $P_3^\text{pf}$, i.e. Pfaffian over 6 particles (see Fig. (6)). Other than this, there are 4 distinct solutions (see Fig. (5)). One can explicitly check that the symmetrized graph monomial of none of them, matches Jack $J_{\Lambda(2,3,4)}^{(-4/3)}$. This means there exists no single regular graph representing the Jack in question. However, by Cayley’s theorem, this Jack has to be a superposition of $P_G$’s with $G$’s regular. Indeed one finds that

$$J_{\Lambda(2,3,4)}^{(-4/3)} = \frac{1}{72} P_2^{3rd \, sq.cyc.} - \frac{7}{90} P_3^{pf}. \quad (20)$$

where by $P_2^{3rd \, sq.cyc.}$ we mean the symmetrized graph monomial of $v = 3$ case of cyclic square class in Table II (which coincides with Type 2 in Fig. (5); also note that index 2 stands for $n = 2$). Such linear superposition is not surprising since the space of binary sextic invariants of degree 4 is two dimensional. One can easily check that $P_2^{3rd \, sq.cyc.}$ and $P_3^{pf}$ (more precisely, their binary duals) span that whole space of binary sextics.

### B. Weighted Cayley Graphs

As advertised before, we will now construct two large classes of model FQH states with interesting structure. Let us first define a weighted Cayley graph. Consider the following data:

- $(G, \cdot)$ be finite group.
- $S \subset G$ a subset (of generators) such that $1 \notin S$ and if $s \in S$ then $s^{-1} \in S$.
- $\mu : S \to \mathbb{Z}_+$ a function, called the multiplicity function, such that $\mu(s) = \mu(s^{-1})$.

One assigns a weighted graph $\text{Cay}(G, S, \mu) = (V, E, w)$, to the triple $(G, S, \mu)$ as follows:

- The vertex set $V$ is the underlying set of $G$.
- Each vertex $g$ is adjacent to the set $g \cdot S$.
- The weight of an edge $(g, gs)$ is $w(g, gs) = \mu(s)$.

The resulting (weighted) graph is called the weighted Cayley graph of $G$ with respect to $(S, \mu)$. It is often convenient to let $\mu$ and $w$ to also take zero value, with zero weight interpreted as no edge. Schematically, a Cayley graph encodes the abstract dependence of a group on a set of its generators. Weighted Cayley graphs are regular of degree $\delta = \sum_{s \in S} \mu(s)$. In this paper we are interested only in weighted Cayley graphs of cyclic groups

$$C_m = \langle q \mid q^m = 1 \rangle \quad (21)$$

and the dihedral groups

$$D_m := \langle q, x \mid q^m = x^2 = (xq)^2 = 1 \rangle \quad (22)$$
The goal is to construct graphs of the form $TT_n \otimes \Phi$ for some CRLE digraph $\Phi$ which are Cayley graphs of one of these groups. It turns out that the representative graphs of parafermionic states are a Cayley graphs of certain cyclic groups, while Gaffnian and Jack polynomial generalization of it are Cayley graphs of certain dihedrals.

The construction is based on special form of the so-called \textit{circular} matrices. Let $v$ be some positive integer and let $\mu = (\mu_0, \mu_1, \cdots, \mu_{v-1})$ be a $v$-dimensional vector of non-negative integers (note that, as usual, we choose our indices as elements of $\mathbb{Z}_v = C_v$). A circular matrix $\Omega$ corresponding to vector $\mu$ is

$$\Omega_{r,s}^{(\mu)} = \sum_{j \in \mathbb{Z}_v} \mu_j \delta_{r-j, s}$$

We call a circular matrix \textit{reflective} if given $r, s$ such that $r + s = -1$ in $\mathbb{Z}_v$, one has $\mu_r = \mu_s$. A reflective circular $v \times v$ matrix is characterized by $\lfloor v/2 \rfloor$ numbers. This brings us to our first theorem:

\textbf{Theorem V.1.} Let $F$ be a $v \times v$ reflective circular matrix corresponding to a vector $\mu$. Suppose further that

\begin{enumerate}
  \item[(L)] $\mu_0 = \mu_{v-1} \neq 0$.
  \item[(E)] If $v = 2m + 1$ is odd, then $\mu_m$ is even.
\end{enumerate}

Then $F$ is a face matrix. Suppose $\Phi$ is the corresponding CRLE digraph. Define $S_n = \bigcup_{\ell=0}^{\lfloor v/2 \rfloor} S_{n,\ell}$ where

$$S_{n,\ell} = \left\{ q^\ell \binom{n}{j} \mid j \in \mathbb{Z}_v \setminus \{0\} \right\} \subset C_nv$$

and the function $\mu : S_n \to \mathbb{Z}_{2^0}$ via $\mu(s) = \mu_{\ell}$ for all $s \in S_{n,\ell}$. Then $TT_n \otimes \Phi = \text{Cay}(C_nv, S_n, \mu)$.

We call the collection of all model FQH states with a face matrix satisfying the conditions of thm. V.1 the \textit{circular mega-class}. In $v = 1$ case, one thinks of $S_{n,1}$ to have the element $q$ twice, or alternatively $\mu_1 = 2k$, an even number; these are theLaughlin states. In the other extreme case, suppose $v \neq 1$ but the only non-zero elements of $\mu$ are $\mu_0 = \mu_{v-1} := d/2$ (in this case $d$ is necessarily even). If $d = 2$ we end up with the parafermion FQH states, which we call the \textit{cyclic class} (Fig. (6)). If $d = 4$, then the $v = 2$ case is the Haffnian and more generally this class coincides with \textit{cyclic square class}, which we defined before.

Aiming for the prism-circular mega-class, define the projection maps $\pi_k^+ : GL(2k) \to GL(k)$ (GL stands for general linear group) via

$$[\pi_k^+(M)]_{r,s} = M_{2r+2s} \quad [\pi_k^-(M)]_{r,s} = M_{2r+1,2s+1}$$

where again the indices are in the cyclic group. $\pi^+$ (resp. $\pi^-$) keeps the even-by-even (resp. odd-by-odd) submatrix of $M$ and discards the rest.

\textbf{Theorem V.2.} Let $v = 2k$ and let $F$ be a $v \times v$ matrix. Let $F^\pm = \pi_k^\pm(F)$. Suppose

\begin{enumerate}
  \item $F^+ = F^- := \tilde{F}$ for some $\tilde{F}$ which is a reflective circular matrix via a vector $\mu = (\mu_0, \cdots, \mu_{k-1})$ satisfying only the (L) condition of thm. V.1.
  \item For all $s \in \mathbb{Z}_v$ one has $F_{s,s-1} = \rho \neq 0$ a positive integer.
\end{enumerate}

All other entries of $F$ are taken to be zero. Then $F$ is a face matrix. Let $\Phi$ be the corresponding CRLE digraph. Define $T_n = S_n \cup R_n$, where $S_n = \bigcup_{\ell=0}^{\lfloor k/2 \rfloor} S_{n,\ell}$ with $S_{n,\ell} \subset C_nv \cup D_{nk}$ define similar to Eq. (24) and

$$R_n = \{ x, qx, \cdots, q^{n-2}x \}$$

Define $\mu : S_n \to \mathbb{Z}_{2^0}$ as $\mu(s) = \mu_{\ell}$ for all $s \in S_{n,\ell}$, and $\mu(r) = \rho$ for all $r \in R_n$. Then $TT_n \otimes \Phi = \text{Cay}(D_{nk}, T_n, \mu)$.

The collection of all model FQH states with a face matrix of the form of thm. V.2 is called the \textit{prism-circular mega-class}. Suppose $\mu = (m, 0, 0 \cdots, 0, m)$ a $k$-dimensional vector and let $\rho \neq 0$ be arbitrary. The corresponding model FQH state is then denoted by $\text{Prism}(2k, m, \rho)$ and the collection of all such states is called the \textit{prism class}. Special cases are:

- $\text{Prism}(2, 1, 1)$ aka Gaffnian.
- $\text{Prism}(2, 1, 2)$ aka Haffnian.
- $\text{Prism}(2, 1, 3)$ which, by force of habit, we call Iaffnian.
- $\text{Prism}(2k, 1, 1)$ aka Jack polynomial generalization of Gaffnian (T4).
- $\text{Prism}(2k, 1, 2)$ and $\text{Prism}(2k, 1, 3)$ respectively the prism generalizations of Haffnian (T5) and Iaffnian.

The shards and CRLE digraphs of some of these graphs are shown in Tables III and IV.

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\textbf{FIG. 6.} Alternative drawing of the graphs of parafermionic states. These graphs are known as \textit{circular} in the literature.

1. $F^+ = F^- := \tilde{F}$ for some $\tilde{F}$ which is a reflective circular matrix via a vector $\mu = (\mu_0, \cdots, \mu_{k-1})$ satisfying only the (L) condition of thm. V.1.
2. For all $s \in \mathbb{Z}_v$ one has $F_{s,s-1} = \rho \neq 0$ a positive integer.
VI. PROPERTIES

This final section is devoted to properties of model FQH ground states \( SGM(TT_n \otimes \Phi) \) with \( \Phi \) an CRLE digraph; i.e. accordion graphs. Throughout this introductory paragraph \( G \) stands for a \((n, v, d)\)-accordion graph, and \( P_G \) will be its SGM. In §VI A, some of the graph theoretic properties of accordion graphs are discussed. In §VI B we explain how one can read the root partitions of an SGM of a graph \( H \) from the orientations of \( H \). Utilizing that, we will prove that \( P_G \) is non-vanishing. In particular, \( P_G \) possesses a root partition

\[
\Lambda(n, v, d) = (0^v d^v \cdots [(n-2)d]^v[(n-1)d]^v)
\]

This is the unique partition of \( nv \) bosonic electrons with minimal angular momentum among partition satisfying a \((v, d)\)-Pauli exclusion. Moreover, we conjecture that any root partition of \( P_G \) can be squeezed into \( \Lambda_{n,v,d} \). In §VI C we investigate how the polynomial \( P_G \) would change upon fusing particles together in an arbitrary fashion. In particular, if \( P_n = SGM(TT_n \otimes \Phi) \), we will prove that

\[
P_{n+1}(\{z\}_m^v, Z, \cdots, Z) \propto \prod_{i=1}^{nv} \prod (Z - z_i)^d P_n(\{z\}_m^v)
\]

where \( \{z\}_m^v \) stands for the variables \( \{z_1, \cdots, z_{mv}\} \) and proportionality is up an integer factor. This is called the \((v, d)\)-clustering property. Finally, in §VI D, we will give a new proof for the fact that Gaffnian, Haffnian and Read-Rezayi states are uniquely characterized by their respective clustering property.

A. Graph Theoretic Properties

In this subsection we will gather all of the graph theoretic properties of graphs \( TT_n \otimes \Phi \) with \( \Phi \) an CRLE digraph in a big theorem. We will use these properties to prove many of the statements in this paper. For the remaining main body of this article, we aim to to keep the graph theoretic technicalities to a minimum. Some of these graph theoretic properties, however, have interesting physical interpretations.

Theorem VI.1. Let \( n, v \) be positive integers and \( \Phi \) an \((v, d)\)-CRLE digraph. Then \( \Phi \) is strongly connected. Moreover, the accordion graphs \( G_n = TT_n \otimes \Phi \) have the following properties:

1. \( G_n \) is \((nv, (n-1)d)\) regular.
2. Independence number of \( G_n \) is \( v \), i.e. \( \alpha(G_n) = v \).
3. Chromatic number of \( G_n \) is \( n \), i.e. \( \chi(G_n) = v \).
4. Clique number of \( G_n \) is \( n \), i.e. \( \omega(G_n) = n \).
5. The core of \( G_n \) is \( K_n \), i.e. \( G_n^\bullet = K_n \).
6. \( G_n \) is connected.
7. \( G_n \) possesses unique \( n \)-colorable \( u_{G_n} \).
8. The color classes of \( u_{G_n} \) are the only maximum independent sets of \( G_n \).
9. Color classes of \( u_{G_n} \) are dominating sets. If \( S \) is color class, and \( x \notin S \), then \( x \) connects to \( S \) with exactly \( d \) edges.
10. Let \( V_k \) be the union of \( k \) distinct color classes of \( u_{G_n} \). Then the induced subgraph associated to \( V_k \) is isomorphic to \( G_k \).

[for defs., see A 2: 17, 19, 23, 27, 29, 15, 28, 30, 12]
Let us interpret some of these properties physically, namely properties 2, 8 and 10. Let $\theta(G)$, which we call the *coalescence number*, be the largest $C$ such that $P_G|_{x_1=x_2=\cdots=x_C} \neq 0$. Quite generally $\theta(G) \leq \alpha(G)$ for an arbitrary $G$. We will later on prove the clustering property of $SGM(G)$ with $G$ an $(n,v,d)$-accordion graph [cor. VI.6]. A consequence of VI.6 is that $\theta(G) = v$ for $G$ an $(n,v,d)$-accordion graphs. In other words, by property 2, $\theta(G) = \alpha(G) = v$. Physically, the coalescence is the definition the size of a cluster of the model FQH state. In other words, this suggests that maximum independent sets of $G$ are to be thought as graph theoretical counterpart of clusters. Also demanding property 8, one finds that there are exactly $n$ clusters. These clusters are furthermore mutually disjoint. Moreover, by property 10, given any pair of clusters $S_1, S_2$, the induced subgraph of $S_1 \cup S_2$ is a copy of $TT_2 \otimes \Phi$, i.e. the shard. This last statement is nothing but local property of $(n,v,d)$ accordion graphs.

To see the connection between the graph theoretic locality, as in the last paragraph, and local Hamiltonians, we need the concept of the complement graph. Recall that $mK_n$ is a complete graph in which every edge has weight $m$. Let $G = (V,E,w)$ be a graph, such that $G$ is neither empty, nor $mK_n$ for any $m > 0$. Define $m = \max_{e \in E} w(e)$, as the maximum weight in $G$. Define the *complement* of $G$, i.e. $G^c$, as follows: $G$ and $G^c$ share the same vertex set, $G \cap G^c = \emptyset$ and $G \cup G^c = mK_n$. One then finds

$$P_G = \prod_{i<j} (z_i - z_j)^m \begin{cases} \mathcal{A} \left(1/P_G \right) & m \text{ odd} \\ \mathcal{A} \left(1/P_G \right) & m \text{ even} \end{cases}$$  \hspace{1cm} (26)

Here $\mathcal{A}$ is anti-symmetrization; also recall that $\tilde{P}_G$ was the graph monomial (not the SGM). Motivated by conformal field theory, the function $\mathcal{A}(1/\tilde{P}_G)$ (resp. $\mathcal{A}(1/\tilde{P}_G)$ for odd (resp. even) case) are called *correlations*. This immediately inspires us to think of $G^c$ as the *correlation graph*. A $k$-cluster in the correlation graph, is a set of $k$ pairwise $m$-adjacent (i.e. adjacent with weight $m$) vertexes; i.e. they are the only copies of $mK_n$ in $G^c$. The graph theoretic locality condition then translates to: *any pair of v-clusters are correlated in exactly the same fashion as any other pair*. These local correlations are how the local Hamiltonians leave their mark on the ground state.

### B. Orientations and Root Partitions

In introduction section, we discussed the importance of structure of root partitions of a model FQH-state. In this subsection we will bring the concept root partitions into graph theory. More precisely, for a graph $G$, we will related the root partitions of $P_G$ to orientations of $G$. We will then move on to discuss root partitions of accordion graph. Let us first give a few definitions:

1. **Squeezing, Dominance and Predominance**

Let $\lambda \in \mathcal{P}_{N,\delta}$ be a partition. Physically, $\lambda$, or more precisely $\tilde{m}_\lambda$, is a free bosonic state. Suppose $m_1, m_2, m_1', m_2'$ are non-negative integers (orbitals in the LLL) such that $m_1 < m_1' \leq m_2 < m_2'$ and $m_1 + m_2 = m_1' + m_2'$. We also demand the orbitals $m_1, m_2$ to be occupied by at least one boson. We simultaneously move a particle from orbital $m_1$ to orbital $m_1'$ and another from $m_2$ to $m_2'$. The condition $m_1 + m_2 = m_1' + m_2'$ guarantees that the angular momentum is conserved in this operation. The process is called a *squeezing operation* and transforms $\lambda$ to another partition $\lambda'$. We say $\mu$ is a *descendant* of $\lambda$, and write $\mu \preceq_{S} \lambda$, if $\mu$ is obtained from $\lambda$ via a series of squeezing operations. On a different note, we say $\lambda$ dominates $\mu$, denoted by $\mu \preceq \lambda$, if $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for all $k$. One can show that $\mu \preceq_{S} \lambda$ if and only if $\mu \preceq \lambda$ (see [24] 1.16); i.e. $\mu$ is a descendant of $\lambda$ iff $\lambda$ dominates $\mu$. If a symmetric polynomial $P$ has a root partition $\lambda$ which dominates all other root partitions, we say $\lambda$ is *predominant*.

2. **Orientations and Graph Monomials**

To give an intuition into why orientations are related to root partitions, consider a multi-graph $G$ and note that any graph monomial of $G$, i.e. $\tilde{P}_G$, is of the form $

\cdots (z_i - z_j)(z_k - z_i)(z_m - z_n) \cdots$

with each parenthesis representing an edge of $G$ (an edge with multiplicity $s$ has $s$ parentheses). To find the root partitions we need to expand and find the monomials of $\tilde{P}_G$. Note that each monomial will amount to a choice between the plus and minus variables in each parenthesis. This choice is the same as putting an orientation on each edge, and therefore on the whole graph $G$. This idea will be made precise in what follows.

3. **Orientation Types**

Let $G$ be a graph of order $N$. Label the vertexes of $G$ by $\{1, \cdots, N\}$. If $\omega$ is an orientation for $G$, then denote by $G_{\omega}$ the resulting digraph (DEF. 25). Also let the notation $\text{sgn}(\omega)$ be the sign of this orientation (DEF. 26). Given any $x \in V$ let $\delta^+_x$, called the *out-degree* of $x$, be the number of arcs going out of $x$ in $G_{\omega}$. Collect the out-degrees of every vertex in a sequence (called the out-degree sequence)

$$\Delta^+_\omega = (\delta^+_1, \delta^+_2, \cdots, \delta^+_N)$$

Upon reordering any sequence of non-negative integers becomes a partition. We call the partition associated to $\Delta^+_\omega$ the *orientation type* of $\omega$ and denote it by $\lambda_{\omega}$. Let $\Omega_\lambda$ be the set of all orientations of $G$ which are of type $\lambda$ and $\Pi_G$ be the set of all orientation types of $G$. 


Proposition VI.2. Let $G$ be a (multi-)graph of order $N$ equipped with predetermined labels $\{1, \ldots, N\}$. Then
\[ P_G = \sum_{\lambda \in T_G} c_\lambda \overline{m}_\lambda, \quad c_\lambda = \sum_{\omega \in \Omega_\lambda} \text{sgn}(\omega) \quad (27) \]
in other words, $\lambda$ is root partitions of $P_G$ iff $\lambda$ is an orientation type of $G$ and $\overline{c}_\lambda \neq 0$ (such $\lambda$ is called a non-vanishing orientation type of $G$).

Returning to accordion graphs $TT_n \otimes \Phi$ with $\Phi$ a $(v, d)$-CRLE digraph, there is a particular partition which is of utmost importance to us:
\[ \Lambda(n, v, d) = (0^vd^v \cdots (n-2)d^v|(n-1)d^n) \quad (28) \]
In the language of Feigin et al. [15] $\Lambda(n, v, d)$ is ‘the’ $(v, d, n v)$-admissible partition which minimizes the degree (total angular momentum) $|\lambda| = M = \sum m_{n_m}$. A partition $\lambda$ of length at most $N$ is called $(v, d, N)$-admissible if
\[ \lambda_i - \lambda_{i+v} \geq d, \quad (1 \leq i \leq N - v) \quad (29) \]
If $n_k$ is the number of bosons in orbital $m$, then the above condition is equivalent to requiring that $\sum_{j=1}^d n_{m+j-1} \leq v$ for all $m \geq 0$. That is to say, the total occupation of any consecutive $d$-orbitals is less than $v$. This is known as $(v, d)$-generalized exclusion.

Theorem VI.3. Let $\Phi$ be a $(v, d)$-CRLE digraph and $G_n = TT_n \otimes \Phi$. Then $\Lambda(n, v, d)$ is a non-vanishing orientation type of $G_n$. In particular $SGM(G_n)$ does not identically vanish and $\overline{c}_{\Lambda(n,v,d)} = n!$.

The relation between partition $\Lambda(n, v, d)$ and graph $G_n$ is more than the above theorem. Among all orientation types of $G$, one calls $\lambda$ a maximal orientation type if there exists no orientation type $\mu$ with $\lambda < \mu$ ($< \lambda$ being dominance order). One can then prove, with relative ease, that

Proposition VI.4. If $G$ is an $(n, v, d)$ accordion graph, then $\Lambda(n, v, d)$ is a maximal orientation type of $G$.

Unfortunately prop. VI.4 alone is not enough to show $\Lambda(n, v, d)$ is predominant. The dominance relation is known to be not always being a partial order, and not a total order, there might be multiple maximal elements which are mutually incomparable. In other words, $\Lambda(n, v, d)$ is predominant if and only if it is the only non-vanishing maximal orientation type of $G$. We have no proof for the uniqueness of $\Lambda(n, v, d)$. Nonetheless, based on all the numerical tests we have performed, we believe that $\Lambda(n, v, d)$ is indeed predominant.

Conjecture. Let $G$ be a $(n, v, d)$-accordion graph. Then $\Lambda(n, v, d)$ is predominant among all non-vanishing orientation types of $G$.

Note that $u_{\Lambda(n,v,d)} = (v!)^n$. If the above conjecture is true, then
\[ \frac{1}{n!(v!)^n} P_G = m_{\Lambda(n,v,d)} + \sum_{\mu < \Lambda(n,v,d)} c'_\mu m_\mu \quad (30) \]
where $c'_\mu = c_\mu/n!(v!)^n$. This is completely analogues to how (specialized) Jack polynomials relate to monomial symmetric polynomials (in both cases $c'_\mu \in \mathbb{Q}$). To emphasize the non-triviality of the conjecture, note that the altered statement: “$\Lambda(n, v, d)$ is predominant among all orientation types of $G$”, i.e. upon removing the ‘non-vanishing’ condition, is false. A counterexample for this presented in Appx. D. The non-vanishing condition, although quintessential for validity of the conjecture, is an extremely difficult to keep track of by means of combinatorics and graph theory.

C. Clustering Properties

Other than the structure of root partitions, another important (possible) property of FQH ground is $(v, d)$-clustering property. This property is related to fusion. In this section we will appropriately define an arbitrary fusion, together with some graph theoretic concepts, which allow for treating the fusion problem graph theoretically. In particular, we will define the $(v, d)$-clustering property and show that any element of the accordion family satisfies such property.

1. $(v, d)$-Clustering Property

Consider a sequence of FQH-like polynomials
\[ \Pi^{(v,d)} := (P_2, P_3, \cdots, P_n, \cdots) \quad (31) \]
where $P_n$ is over $nv$ variables and of local degree $(n-1)d$. Given any $k$, let $\{z\}_{k,v} := \{z_1, z_2, \cdots, z_{kv}\}$. We use the notation $P_{n+1,v}(Z, \{z\}_{n,v})$ for the polynomial obtained from $P_{n+1}$ by equating $z_{n+1} = z_{n+2} = \cdots = z_{(n+1)v} = Z$. We say $\Pi^{(v,d)}$ satisfies a $(v,d)$-clustering property or $\Pi^{(v,d)}$ is a $(v, d)$-clustering sequence if (for $n > 1$)
\[ P_{n+1,v}(Z, \{z\}_{n,v}) = C_{n,v} \prod_{i=1}^{nv}(Z - z_i)^d P_n(\{z\}_{n,v}) \quad (32) \]
with $C_{n,v} \in \mathbb{C}$ a constant. We sometimes also use the convention $P_1(\{z\}_{1,v}) := v!$.

The operation of bringing $v$ particles to a common point $Z$ is called a $v$-fusion. One thinks of the result of this identification as creating a $v$-composite at point $Z$. The $(v, d)$-clustering property is therefore an statement about the creation of a single $v$-composite. However, $k$-fusion has an obvious generalization in which multiple composites are created at different points. We will provide a general framework in this section for dealing with fusion problems, of general kind, for FQH-like polynomials of the form $SGM(G)$.
2. $\kappa$-fusion

Let $P$ be a FQH-like polynomial over $N$ particles. Consider a partition $\kappa = (1^{v_1}2^{v_2}\cdots g^{v_g})$ of $N$; as usual $v_r$ is the multiplicity of $r$ in $\kappa$. Also define

$$Z^{(r)} = \{Z^{(r)}_{v_1}, Z^{(r)}_{v_2}, \cdots, Z^{(r)}_{v_r}\}$$

We define $\kappa$-fusion as the operation of creating $\nu_r$ distinct $r$-composites at (distinct) positions $Z^{(r)}$ in $P$. A 1-composite is understood as an untouched variable. We will use the notation $P|_{\kappa}(Z^{(1)}, \cdots, Z^{(g)})$ for the operation.

Remark. Consider a toy FQH-like polynomial in $N$ variables $P(z_1, z_2, z_3, z_4, z_5, \cdots)$.

1. In the creation of 2-composite at $Z^{(2)}$ and a 3-composite at $Z^{(3)}$, it is implicitly assumed $Z^{(2)} \neq Z^{(3)}$. If $Z^{(2)} = Z^{(3)}$ the process is understood as the creation of a single 5-composite.

2. The identifications are done on disjoint subsets of variables; e.g., $\{z_1, z_2\} \rightarrow Z^{(2)}$ and $\{z_3, z_4, z_5\} \rightarrow Z^{(3)}$. Since $P$ is symmetric, it does not matter how we choose these subsets. The resulting polynomial $P_{\kappa}^{(1,2,3)}(Z^{(2)}, Z^{(3)}, \cdots)$ is not in general symmetric under $Z^{(2)} \leftrightarrow Z^{(3)}$.

3. Now consider the creation of two distinct 2-composites at positions $Z^{(2)}_1$ and $Z^{(2)}_2$. The fused polynomial $P_{\kappa}^{(1,2,3)}(Z^{(2)}_1, Z^{(2)}_2, z_3, z_4, \cdots)$ now would be symmetric under $Z^{(2)}_1 \leftrightarrow Z^{(2)}_2$. In particular, if the sets $\{z_1, z_2\}$ and $\{z_3, z_4\}$ are chosen for fusing, the two identification schemes

$$\begin{cases} z_1 = z_2 = Z^{(2)}_1 \\ z_3 = z_4 = Z^{(2)}_2 \end{cases}$$

lead to the same polynomial. In general, the result of a $\kappa$-fusion will be symmetric in variables $Z^{(r)}$ for any $r$.

For the rest of this subsection, we will solely focus on $\kappa$-fusions of SGMs. Introducing the concepts “$\kappa$-coloring”, “compression” and “chromatic SGM”, we will make a connect the algebraic concept of fusion to proper colorings of a graph. The bridge itself will be thm. VI.5.

3. $\kappa$-Coloring

A proper $s$-coloring $\gamma$ of a graph $G$ is an assignment of $s$ colors to the vertexes so that no two adjacent vertexes have the same color. The set of all vertexes of color $k$ is denoted by $\{k\}$ and is called a color class. Suppose $y_k$ denote the cardinality of color class $\{k\}$ and make the sequence $(y_1 y_2 \cdots y_s)$. Upon reordering this sequence, one finds a partition $\kappa$ of $N = |G|$ (the order of the graph). We call $\kappa$ the pattern of $\gamma$. A coloring $\gamma$ is called a $\kappa$-coloring if $\gamma$ is proper and its pattern is $\kappa$. We denote the set of all $\kappa$-colorings by $C_\kappa$. Two colorings $\gamma, \gamma'$ are equivalent, and write $\gamma \sim \gamma'$, if one is obtained from the other by a permutations of colors. We define $C_\kappa^* = C_\kappa/\sim$.

Finally, we define the $r$th tier of a coloring as

$$\tau^{(r)}(\gamma) = \{\text{Colors } k \text{ such that } y_k = r\}$$

We will also say tier number of color $k$ is $r$ if $k \in \tau^{(r)}(\gamma)$. All of these concepts are illustrated in Fig. (7).

4. Compression

We define a chromatic graph as graph $G$ that has a built-in coloring $\gamma^*$. We call $\gamma^*$ the intrinsic coloring of the chromatic graph. We do not require the coloring of a chromatic graph to be proper (and usually it is not). Starting from a graph $G$, and a proper coloring $\gamma$ of $G$ we will now construct a chromatic graph $G \downarrow \gamma$ called the compression of $G$ according to $\gamma$. This is done as follows:

- The vertexes of $G \downarrow \gamma$ are the colors of $\gamma$.
- The intrinsic coloring $\gamma^*$ assigns to each color, its tier number with respect to $\gamma$.
- Given any two colors of $\gamma$, say $k, k'$,

$$\# \text{ of edges between } k \text{ and } k' \text{ in } G \downarrow \gamma = \sum_{x \in [k]} \sum_{x' \in [k']} \# \text{ of edges between } x \text{ and } x' \text{ in } G$$

Figure (8) is illustrate this construction. Note that if $\gamma \sim \gamma'$ are two equivalent colorings of $G$ then $G \downarrow \gamma = G \downarrow \gamma'$. So if $\bar{\gamma}$ is the equivalence class of $\gamma$, we may also write $G \downarrow \bar{\gamma}$ without any ambiguity. Also note that the color
and \( \gamma \) a \( \kappa \)-coloring of \( G \). Let \( \tilde{P}_{G;\gamma}(Z^1; Z^2; \ldots; Z^g) \) be the graph monomial, where vertexes of color \( r \) (in coloring \( \gamma^r \)) are labeled by the elements of \( Z^{(r)} \). Then the chromatic SGM of \( G \downarrow \gamma \) is

\[
P^\chi_{G;\gamma}(Z^{(1)}; Z^{(2)}; \ldots; Z^{(g)}) = \mathcal{S}^{(1)} \cdot \mathcal{S}^{(2)} \cdot \ldots \cdot \mathcal{S}^{(g)} \left[ \tilde{P}_{G;\gamma}(Z^{(1)}; Z^{(2)}; \ldots; Z^{(g)}) \right]
\]

where \( \mathcal{S}^{(r)} \) is the symmetrization operator only on the variables in \( Z^{(r)} \) (this corresponds to symmetric group \( S_{\nu^r} \)). If \( r \) is such that \( \nu_r = 0 \), then \( \mathcal{S}^{(r)} \) is by convention the identity.

6. Fusion-Coloring Theorem

For the following theorem we will use the notion of normalized symmetrization (DEF.35). Denote by \( \tilde{P}_G \) and \( \tilde{P}_{G;\gamma} \) for the normalized SGM and normalized chromatic SGM. In other words, in the definition of SGM and chromatic SGM, we replace every symmetrization operator \( \mathcal{S}_G \) with a normalized symmetrization.

**Theorem VI.5.** Let \( G \) be any loopless multigraph of order \( N \). For a partition \( \kappa = (1^{\nu_1} \cdot 2^{\nu_2} \cdot \ldots \cdot g^{\nu_g}) \) of \( N \), the \( \kappa \)-fusion of \( P_G = \text{SGM}(G) \) can be calculated via

\[
\tilde{P}_{G|\kappa}(Z^{(1)}; \ldots; Z^{(g)}) = C_\kappa^{-1} \sum_{\eta \in S_\kappa^*} \tilde{P}_{G;\eta}^{\kappa}(Z^{(1)}; \ldots; Z^{(g)})
\]

where \( C_\kappa = N! / \prod_{r=1}^g \nu_r!(r!)^{\nu_r} \) (this is the number of set-partitions of shape \( \kappa \)).

By thm. VI.5, if one knows all \( \kappa \)-colorings of a graph \( G \), then \( \kappa \)-fusion of \( P_G \) can be easily calculated. More generally, if \( P \) is any FQH-like polynomial, by Cayley’s theorem, one can find regular graphs \( G_1, \ldots, G_s \) such that \( P \) is a superposition of \( \text{SGM}(G_i) \). Then the fusion problem of \( P \) becomes the coloring problem of graphs \( G_s \). Unfortunately, finding the colorings of a graph is a difficult problem in most instances.

Nonetheless, under special circumstances, finding \( \kappa \)-colorings of a graph is manageable, sometimes even trivial. For example, suppose \( G_n = TT_n \otimes \Phi \) with \( \Phi \) a \((v,d)\)-CRLE digraph. For finding the \((v,d)\)-clustering property, we are interested in \( v \)-fusions (i.e. \( \kappa = (1^{(\nu-1)}v) \)) of \( \text{SGM}(G_n) \). The special properties of \( G_n \), that we proved in thm. VI.1, results in the following corollary.

**Corollary VI.6.** Let \( \Phi \) be a \((v,d)\)-CRLE digraph. Then \( \text{SGM}(TT \otimes \Phi) \) has the \((v,d)\)-clustering property.

Note that, as a result, the \((v^n)\)-fusion of \( P_G \) is

\[
P_{G|v^n}(Z_1, \ldots, Z_n) = n! v^n \prod_{i<j} (Z_i - Z_j)^{vd}
\]

In this sense, all accordion model FQH states are refinements of a bosonic Laughlin state.
D. Clustering uniquely characterizes Gaffnian, Haffnian & Read-Rezayi states: new proof

In §VI C we mentioned that the \((v, 2)\) clustering property of \(Z_v\) parafermionic states fully characterizes them. We have also seen that the two cases

\[
\begin{align*}
(C1) & \text{ Pfaffian } (v = 2, d = 2), \text{ Gaffnian } (v = 2, d = 3), \\
(C2) & \text{ Read-Rezayi states } (v > 1 \text{ arbitrary}, d = 2).
\end{align*}
\]

lie in the accordion family, and therefore have their own respective clustering property. In this section, we will give a new proof that the clustering property in these cases fully characterizes them. Throughout this paper we have not used binary invariants directly, although they are the foundation of our theory. Our main tool in this subsection, is based on binary invariants and material in Ref. [15].

Let \(\Pi^{(v, d)}\) be a FQH-like sequence as before. We need two definitions to proceed:

1. Define \(W_{n,v,d}\) as the set of all partitions \(\lambda = (\lambda_1, \lambda_2, \cdots)\) of \(n(n-1)vd/2\) that are \((v-1, 2, nv)\)-admissible and have \(\lambda_1 = (n-1)d\).
2. Define \(B_{n,m}\) as the space of all binary invariants of order \(n\) and degree \(m\).

**Theorem VI.7.** Suppose \(v,d \geq 2\) is such that

1. \(W_{n,v,d} = \emptyset\) for all \(n \geq 2\).
2. \(\dim B_{2v,d} = 1\).

If \(\Pi^{(v,d)}\) is a sequence satisfying the \((v,d)\)-clustering property, then \(\Pi^{(v,d)}\) is the only FQH-like sequence which does so.

As a summary, the proof of this theorem is by induction. Assuming the hypothesis of the theorem, one takes two \((v,d)\)-clustering sequences \(\Pi^{(v,d)}, \Pi'^{(v,d)}\). The goal is, given the hypothesis, to prove that

1. The base of induction; i.e. \(P_2 = q_2 P'_2\) for some \(q_2 \in C\). Only the condition \(\dim B_{2v,d} = 1\) is required here.
2. Step of induction; i.e. if \(\lambda = (P_k = q_k P'_k)\) for some \(q_k \in C\) then \(P_{k+1} = q_{k-1} P'_{k+1}\). For non-trivial reasons, \(W_{k,v,d} = \emptyset\), validates this part.

Let us now focus on the special cases of \((C1) v = 2\) and \(d = 2, 3, 4\) and \((C2) d = 2\) and \(v > 1\) arbitrary. For both of these cases checking \(W_{n,v,d} = \emptyset\) is straightforward.

**Proposition VI.8.** If \((v,d)\) is either \((v,2)\) with \(v\) arbitrary, or \((2,d)\) with \(d = 2, 3, 4\), then \(W_{n,v,d} = \emptyset\) for all \(n \geq 2\).

Therefore, it remains to show that

\[
\begin{align*}
(C1) \dim B_{4,2} = \dim B_{4,3} = \dim B_{4,4} = 1. \\
(C2) \dim B_{2v,2} = 1.
\end{align*}
\]

The algebra of binary quartic invariants (order 4) are known to be freely generated by an invariant \(i\) of degree 2 (Pfaffian) and an invariant \(j\) of degree 3 (Gaffnian) (see [6], example 7.17). Thus the subspaces of binary quartic invariants with degrees 2, 3 and 4 (Haffnian; \(B^2\)) are one-dimensional. To show that \(\dim B_{2v,2} = 1\) let us introduce another tool of the theory of binary invariants:

**Theorem VI.9 (Hermite’s Reciprocity Theorem).** For any pair \(n, m\), we have \(\dim B_{n,m} = \dim B_{m,n}\).

So for case \((C2)\) it is enough for us to show that \(\dim(2, 2v) = 1\). But the space of binary quadratic invariants (see [6], example 7.15) is generated freely by an invariant of degree 2, namely the discriminant (aka Laughlin). So obviously \(\dim(2, 2v) = 1\) for all \(v\). Hence we have proved: **Gaffnian, Haffnian and Read-Rezayi states are all uniquely characterized by their respective clustering property.**

**ACKNOWLEDGMENTS**

I would like to thank my research supervisor Xiaogang Wen and my dear friend and colleague Michael DeMarco for their invaluable input and all of the fruitful discussions.

**Appendix A: Mathematical Definitions**

1. **Partitions**

   1. **(Partition)** A partition \(\lambda\) of a number \(M\) is a finite sequence of non-negative integers \(\lambda = (\lambda_1, \lambda_2, \cdots)\) such that \(M = \sum_i \lambda_i\) and \(\lambda_i \geq \lambda_{i+1}\) for all \(i \geq 1\). One calls \(\lambda_i\) the \(i\)th part of \(\lambda\). One also defines \(|\lambda| := M\).

   2. **(Length):** Given a partition \(\lambda\), the length of \(\lambda = (\lambda_1, \lambda_2, \cdots)\), denoted by \(|\lambda|\) is the smallest \(\ell\) such that \(\lambda_{\ell + 1} = 0\).

   3. **(Largest Part)** Given a partition \(\lambda = (\lambda_1, \lambda_2, \cdots)\) we will denote by \(L(\lambda) := \lambda_1\).

   4. **(Multiplicity)** Given a partition \(\lambda\) we will denote by let \(S(r) = \{\lambda_i\ | \ \lambda_i = r\}\), i.e. the set of all parts in \(\lambda\) that are equal to \(r\). Then \(\nu_r(\lambda) = \left|S(r)\right|\) is called the multiplicity of \(r\) in \(\lambda\). Note that one can, without any ambiguity, also write \(\lambda = (0^{\nu_0}1^{\nu_1} \cdots L^{\nu_r})\), where \(L = L(\lambda)\). One reads \(r^{\nu_r}\) as \(\lambda\) has \(r\) parts equal to \(r\).

   5. **(Lowest Landau Level)** Partitions are used in part to describe the free bosonic states in LLL. One then works with a partition of the form \(\lambda = (\lambda_1, \cdots, \lambda_N)\) of \(M\) (total angular momentum aka degree) with \(\ell(\lambda) \leq N\) (\(\ell\) the number of variables/particles), and \(L(\lambda) \leq \delta\) (where \(\delta\) is the local
degree aka the size of LLL). In the free bosonic state \( \tilde{m}_\lambda \), for each \( 1 \leq i \leq N \), there is a boson in orbital \( \lambda_i \). Writing this in multiplicity picture \( \lambda = (0^r 1^\nu_1 \cdots L^\nu_r) \), one says orbital \( r \) is occupied by \( \nu_r \) bosons. Physicists often store this information in the so called occupation pattern \((\nu_0, \nu_1, \ldots, \nu_3)\). Throughout, we have avoided this notation and stayed faithful to \( \lambda = (0^r 1^\nu_1 \cdots L^\nu_r) \) instead.

2. Graph Theory

In this paper, “graph” stands for (finite) loopless multiple/weighted undirected graph.

6. (Simple Graph) An undirected simple (finite) graph \( G = (V, E) \) consists of a (finite) set \( V \) called the vertex set and a set \( E \), called the edge set. An edge \( e \in E \) is a two element subset of \( V \). If \( \{x, y\} \in E \) one says there is an edge between \( x \) and \( y \), or \( x \) is adjacent to \( y \), and denotes it by \( xy \). An edge of the form \( xx \) is called a loop. A graph is called loopless if it does not have any loops.

7. (Multi/Weighted Graph) A multi-graph \( G = (V, E) \) is defined similar to a simple graph with only difference being the the possibility of multiple parallel edges between two vertexes \( x, y \); i.e. \( E \) can have repeated elements. Equivalently one can think of a multigraph as a triple \( G = (V, E, w) \), where \( (V, E) \) is a simple graph and \( w : E \to \mathbb{N}_+ \) is a function called the weight. In this context we call \( G \) a weighted graph. The definitions are equivalent for all intents and purposes of this paper. However, sometimes one point of view makes makes life simpler than the other.

8. (Digraph) A directed graph or a digraph is a pair of set \( D = (V, A) \) with \( V \) being the vertex set same as before. The set \( A \) called the arc set, is a set of ordered pairs of \( V \). One denotes an element of \( A \) by \( x \to y \) and calls it an arc. A directed multigraph is defined similarly, but now there is the possibility of a repeated element \( x \to y \) in \( A \) and also appearance of both \( x \to y \) and \( y \to x \) if there are two distinct edges between \( x, y \). The underlying graph of a digraph \( D \) is denote by \( \overline{D} \).

9. (Order) Given a graph \( G = (V, E) \) (weighted or not does not matter) the cardinality of the vertex set \( |V| \) is called the order of the graph.

10. (Vertex Ordering) Suppose \( G \) is of order \( N \), then one can put an ordering on \( V \) and label the vertexes by distinct elements of the set \( N := \{1, \ldots, N\} \).

11. (Adjacency Matrix) Given a weighted graph \( G = (V, E, w) \) of order \( N \) equipped with some vertex ordering, one defines the adjacency matrix of a weighted graph \( G \) as the \( N \times N \) matrix given by

\[
W = (w(ij))
\]

which stores all of the weights. The adjacency matrix is symmetric. If \( G \) is loopless, then the diagonal of \( W \) are zero. We often use the notation \( w_{ij} \) instead of the \( w(ij) \). For a weighted digraph \( D = (V, A, w) \), the definition of digraph \( D \) is the same as the undirected case, except we store the weights of the arcs. As such the adjacency matrix of a digraph is no longer symmetric.

12. (Subgraph) A subgraph \( H \) of simple graph \( G = (V, E) \) is a graph \( H = (V', E') \) such that \( V' \subset V \) and \( E' \subset E \). One says \( H \subset G \) is an induced subgraph if \( H \) contains all of the edges of \( G \) which start and end at a vertex of \( H \). The subgraph and induced subgraph definition for multigraph are defined similarly.

13. (Iso/Automorphism) An isomorphism \( f : G \to H \) between two simple graphs \( G = (V_G, E_G) \) and \( G = (V_H, E_H) \) is a bijection \( f_V : V_G \to V_H \) such that \( xy \in E_G \) if and only if \( f(x)f(y) \in E_H \). An isomorphism between weighted graphs furthermore requires that \( w_H(f(x)f(y)) = w_G(xy) \). An automorphism of graph \( G \) is an isomorphism \( G \to G \).

14. (Homomorphism) A homomorphism between two multigraphs \( G = (V, E) \) and \( G' = (V', E') \) is a pair of maps: \( f_V : V(G) \to V(G') \) and \( f_E : E(G) \to E(G') \) such that if \( e \in E(G) \) maps to \( f_E(e) \in E(H) \), and if \( \{x, y\} \) and \( \{x', y'\} \) are respectively the endpoints of \( e \) and \( f_E(e) \), then \( f_V \) sends \( \{x, y\} \to \{x', y'\} \) bijectively.

15. (Core) A graph \( Y \) is called a core any homomorphism from \( Y \to Y \) is an automorphism (this automatically means \( Y \) is a simple graph). The core of a graph \( G \) is a subgraph \( G^* \) such that \( G^* \) is a core and there exists a homomorphism (a retraction) \( \rho : G \to G^* \). Note that the core of a multigraph \( G \) is the same as the core of its underlying simple graph \( G_s \). It can be shown that: Every graph has a core, which is an induced subgraph and is unique up to isomorphism (see [25, lem. 6.2.2]). All complete graphs are cores.

16. (Connectedness) A graph \( G = (V, E) \) is called connected, if given any two vertexes \( x, y \in V \) there exists a finite sequence of vertexes \( x_0, x_1, \cdots, x_k \) with \( x_0 = x \) and \( x_k = y \) such that for each \( 0 \leq i \leq k - 1 \) one has \( x_i x_{i+1} \in E \) (this sequence is called a path). A digraph \( D \) is called connected if its underlying undirected graph \( \overline{D} \) is connected.

17. (Strongly Connected) A digraph \( G = (D, E) \) is called strongly connected, if given any two vertexes \( x, y \in V \) there exists a finite sequence of vertexes \( x_0, x_1, \cdots, x_k \) with \( x_0 = x \) and \( x_k = y \) such that
for each $0 \leq i \leq k - 1$ one has $x_i \to x_{i+1} \in A$ (one says $y$ is reachable from $x$). To contrast further between a connected and a strongly connected digraph, people sometimes call a connected digraph, a weakly connected digraph.

18. (Degree of a Vertex) In a loopless multi-graph $G = (V, E)$ one defines the degree of a vertex $x \in V$ as $\delta(x) =$ number of edges having one endpoint at $x$.

19. (Regular Graph) A (weighted) graph $G$ is called $\delta$-regular or regular of degree $\delta$ if the degree of all of its vertexes is exactly $\delta$. The sum of each row (or column) of the adjacency matrix of a $\delta$-regular graph is $\delta$.

20. (Out-degree and In-degree) In a multiple digraph $D = (V, A)$ (possibly with loops), given any vertex $x \in V$, the out-degree (resp. in-degree) of $x$, $\delta^+(x)$ (resp. $\delta^-(x)$), is the number of arcs in $A$ which go out of (resp. come into) $x$. A loop counts as an arc which goes both in and out of $x$.

21. (Regular Digraph) A multiple graph $D = (V, A)$ is called $d$-regular if for every vertex $x \in V$ one has $\delta^+(x) = \delta^-(x) = d$.

22. (Fully Looped) A multiple digraph $D = (V, A)$ is called fully looped if for each $x \in V$ there is at least one loop $x \to x \in A$.

23. (Independent Sets and Independence Number) Let $G = (V, E)$ be a (loopless multiple) graph. A subset $S \subseteq V$ is called an independent set if no two vertexes in $S$ are adjacent to one another. An independent set $S$ is called maximum, if there exists no independent set $S'$ with $|S'| > |S|$. The size of a maximum independent set is called the independence number of $G$ and is denoted by $\alpha(G)$.

24. (Bipartite Graph) A multi-graph $G = (V, E)$ is called bipartite if there are two independent sets $A, B$ such that $V = A \cup B$ and $A \cap B = \emptyset$. One calls $(A, B)$ the partition of $G$. If $G$ is a regular bipartite graph, then $|A| = |B|$. If $G$ is bipartite $d$-regular, then the number of edges is equal to $|E| = |A||d| = |B||d|$.

25. (Orientation) Let $G = (V, E)$ be a multi-graph. An orientation on $G$ is a function $\omega : E \to V$ such that $\omega(ij)$ is either $i$ or $j$ for any edge $ij \in E$. One interprets $\omega(ij) = i$ as the initial point of an arc $i \to j$. The resulting digraph is denoted by $G_\omega$.

26. (Sign of an Orientation) Let $G$ be a graph and $\omega_0$ an orientation on $G$. Given any other orientation $\omega$ on $G$ define $S(\omega, \omega_0)$ as the set of all edges of $G$ over which the orientations $\omega_0$ and $\omega$ disagree. We define the sign of $\omega$, with respect to $\omega_0$ as

$$\text{sgn}_{\omega_0}(\omega) = (-1)^{|S(\omega, \omega_0)|}$$

If the (loopless) graph $G = (V, E)$ is ordered, then there is a natural orientation $\omega_0$ on $G$ induced by that ordering: Since any edge is of the form $ij$, one defines $\omega_0(ij) = \min(i, j)$. In that case, when we say sign of an orientation $\omega$, we always mean with respect to $\omega_0$ and simply write $\text{sgn}(\omega)$.

27. (Proper Vertex Coloring and Chromatic Number) Let $G = (V, E)$ be a graph. A $s$-coloring is an assignment of $s$ colors to the vertexes of $V$. More precisely, any onto function $\gamma : V \to \{1, 2, \ldots, s\}$. A proper $s$-coloring is a $s$-coloring such that no two vertexes of the same color are adjacent; if $xy \in E$, then $\gamma(x) \neq \gamma(y)$. Given a proper $s$-coloring, the fibers $\gamma^{-1}(i)$, with $i \in \{1, 2, \ldots, s\}$ are called the color classes of $\gamma$. The color class of color $k$ is denoted by $[k]$. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum integer $s$, such that there exists a proper $s$-coloring for $G$.

28. (Uniquely Colorable) Let $G = (V, E)$ be a graph and $\gamma : V \to \{1, 2, \ldots, s\}$ a proper $s$-coloring of $G$. Given any permutation $\sigma \in S_n$, the function $\sigma \circ \gamma$ is also a proper coloring. The color classes of $\gamma$ and $\sigma \circ \gamma$ are exactly the same (just named differently). One says two proper $s$-colorings $\gamma, \gamma'$ are equivalent, and writes $\gamma \sim \gamma'$, if there exists $\sigma \in S_n$ such that $\gamma' = \sigma \circ \gamma$. A graph $G$ is called uniquely $n$-colorable, if $\chi(G) = n$ and, up to equivalence, $G$ has exactly one proper $n$-coloring.

29. (Clique and Clique Number) Let $G = (V, E)$ be a graph. A subset $C \subseteq V$ is called an $n$-clique if $|C| = n$ and the induced subgraph of $C$ is the complete graph $K_n$. The clique number of the graph $G$, denoted by $\omega(G)$, is the maximum integer $n$ such that an $n$-clique exists.

30. (Dominating Set) Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is called a dominating set, if for any vertex $x \notin S$, there exists a vertex $y \in S$ such that $xy \in E$.

31. (Tensor Product) Let $D = (V, A, w)$ and $D' = (V', \ A', w')$ be two weighted digraphs with loops (loops do not have to happen, but they can happen). It is convenient for us to the weight functions also take the value zero (which will mean no arc). The tensor product $D \otimes D'$ is defined as follows:

- The vertex set of $D \otimes D'$ is the Cartesian product $V \times V'$.
- If $x \to y$ with weight $w$ in $D$ and $x' \to y'$ with weight $w'$ in $D'$ then $(x, y') \to (x', y')$ with weight $ww'$ in $D \otimes D'$.

Similar to graph homomorphism, one can define a digraph homomorphism $f$ between multiple digraphs $D, D'$ with loops as a pair $f_V : V \to V'$ and
Connectivity actually has no influence. See [24, 2.4] for a proof. This is known as the {Fundamental Theorem of Symmetric Polynomials.} Note that since the local degree of all variables, \( e_k \), \( 1 \leq k \leq N \), is even.

37. (Fundamental Theorem of Symmetric Polynomials) Let \( P \) be any symmetric polynomial in \( N \) variables. Let \( z \) symbolically stand for \( z_1, \ldots, z_N \). Then there exists a unique polynomial \( Q \in \mathbb{C}[X_1, \ldots, X_N] \), such that
\[
P(z) = Q(e_1(z), e_2(z), \ldots, e_N(z))
\]
See [24, 2.4] for a proof. This is known as the fundamental theorem of symmetric polynomials. Note that since the local degree of all \( e_k \) is one, then the local degree of \( P \) is equal to degree of \( Q \).

38. (Complex Projective \( n \)-Space) Consider the space \( \mathbb{C}^{n+1} - \{0\} \) and define the equivalence relation \( (x_1, \ldots, x_{n+1}) \sim (cx_1, \ldots, cx_{n+1}) \) for any \( 0 \neq c \in \mathbb{C} \). The resulting quotient space \( \mathbb{C}^{n+1} - \{0\} / \sim \) is denoted by \( \mathbb{P}^n \), and is called the complex projective \( n \)-space. A point on \( \mathbb{P}^n \) is denoted by \([x_1 : \cdots : x_{n+1}]\), with the understanding that \([x_1 : \cdots : x_{n+1}] = [cx_1 : \cdots : cx_{n+1}] \) for any \( 0 \neq c \in \mathbb{C} \). \( \mathbb{P}^n \) is an \( n \)-dimensional complex manifold.

39. (Complex Projective Line) The space \( \mathbb{P}^1 \) is called the complex projective line and is isomorphic to Riemann sphere as a Riemann surface. Let \( U_0 = \{[x_0 : x_1] \in \mathbb{P}^1 | x_1 \neq 0 \} \) and \( U_\infty = \{[x_0 : x_1] \in \mathbb{P}^1 | x_0 \neq 0 \} \). Then \( U_0, U_\infty \) are the complex charts of \( \mathbb{P}^1 \). One has \( U_0 \approx U_\infty \approx \mathbb{C} \) and the transition map is given by \( z \mapsto 1/z \).

40. (Projective Roots) Consider a homogeneous polynomial \( f(X, Y) \). Note that if \( f(x, y) = 0 \), then for all \( c \neq 0 \) one also has \( f(cx, cy) = 0 \). One calls \([x : y] \in \mathbb{P}^1 \) a projective root of \( f \). There is a homogeneous version for fundamental theorem of algebra: A homogeneous polynomial \( f(X, Y) \) of degree \( n \) has exactly \( n \) projective roots (counting the multiplicity). Hence, one can always write
\[
f(X, Y) = A \prod_{i=1}^n (Xy_i - Yx_i)
\]
where \([x_i : y_i] \) are projective roots of \( f(X, Y) \) and \( A \) a some constant \( \mathbb{C} \)-number.

Appendix B: The CRLE Postulates

In this appendix we will explain why we have demanded the CRLE postulates. It is indeed possible to create examples which violate one or several of the postulates. Those examples, however, will potentially have highly unphysical features. In short, the CRLE postulates have been put in place to discard all of unphysical cases. Let us recall the postulates in terms of their implications for the \( v \times v \) face matrix \( F \):

\( \text{(C)} \) \( F \) is the adjacency matrix of a connected digraph.

\( \text{(R)} \) The sum of each row, as well as each column, of \( F \) is equal to the constant \( d \).

\( \text{(L)} \) The diagonal elements of \( F \) are all nonzero.

\( \text{(E)} \) The product \( vd \) is even.

If we were to order according to how much influence they have over the theory, it would be regularity (R), fully loopedness (L), evenness (E) and connectivity (C).

\( \text{(C)} \) Postulate: Connectivity actually has no influence on the theory at all. However, if \( \Phi = \Phi_1 \cup \cdots \cup \Phi_c \) have connected components, then one should demand each \( \Phi_i \) to be a CRLE digraph. In other words, (E) postulate should be enforced on each connected component separately. Let us list all of the changes this weaker set of postulates would have had:

1. (In thm. VI.1) The digraph \( \Phi \) is no longer strongly connected. However, each connected component \( \Phi_i \) would be strongly connected.
2. (In thm. VI.1) The graph \( G = T T_n \otimes \Phi \) is no longer connected. Instead, \( G \) will now have \( c \) connected components which are exactly \( G_i = T T_n \otimes \Phi_i \).

3. (In thm. VI.1) The graph \( G \) is no longer uniquely \( n \)-colorable. However, each connected component \( G_i \) is uniquely \( n \)-colorable. There properties 8 and 9 of thm. VI.1 should also be changed accordingly.

4. (In thm. VI.1) We need to replace property 10 too. If \( u_i \) be the unique \( n \)-coloring of the connected component \( G_i \). Let \( A^k_i := \{a_1^i, \ldots , a_k^i\} \) be set of \( k \)-color classes of \( u_i \) in \( G_i \). Let \( H(A^k_i) \) be the induced subgraph of \( A^k_i \). Then the graph \( \bigcup_i H(A^k_i) \) is isomorphic to \( T T_k \otimes \Phi \).

5. (In thm. VI.3) The partition \( \Lambda(n, v, d) \) is still non-vanishing, but \( c_{\Lambda(n,v,d)} = (n!)^c \) instead.

6. (In cor. VI.6) The aggregation sequence \( T T \otimes \Phi \) is still a \( (v, d) \)-clustering sequence. However, if \( P_n := SGM(T T_n \otimes \Phi) \), then

\[
P_{n+1} \vert_{Z, \{z\}, n,v} = n^c v! \prod_{i=1}^{n} (Z - z_i)^d P_n(\{z\}, n,v)
\]

i.e. the constant factor of clustering is \( n^c v! \) instead of \( nv! \).

As is apparent, nothing of any significance happens by weakening the connectivity condition. It is true that by removing connectivity one gets “new” examples not in CRLE. However, we have guessed that for any disconnected RLE \( \Phi \), there exists a CRLE digraph \( \Phi' \), such that \( SGM(T T_n \otimes \Phi) \) and \( SGM(T T_n \otimes \Phi') \) only differ by numerical factors. For example, let \( \{z, w\}_n = \{z_1, w_1, z_2, w_2, \ldots , w_n, w_n\} \) and \( \mathcal{S} \) be the symmetrization in \( \{z, w\}_n \). Then the polynomial

\[
P_n(\{z, w\}_n) = \mathcal{S} \left[ \prod_{i < j} (z_i - z_j)^2 (w_i - w_j)^2 \right]
\]

which is the SGM of the \( 2K_n \Pi 2K_n \) (disjoint union of two copies of the graph representative Laughlin 2-states), and if \( P'_n \) is the Pfaffian on \( 2n \) particles, then

\[
P_n(\{z, w\}_n) = \frac{P'_n}{n!}
\]

Similarly disjoint union of \( v \) copies yields the same polynomial (up to a numerical factor) as \( Z_v \)-parafermionic state. Admittedly these are special (well-known) cases. Nonetheless, we have yet to come up with a disconnected digraph \( \Phi \) that gives us a polynomial which is impossible to build with one (or linear superposition, in the sense of Cayley’s theorem, of several) connected \( \Phi \).

(R) POSTULATE: The regularity condition speaks for itself. Cayley’s theorem is telling us to look for regular graphs, so demanding postulate (R) is only natural.

(L) POSTULATE: Out of all of the postulates, considering the forceful impact it leaves on the theory, the fully loopedness postulate is perhaps the most non-trivial one to demand. Consider a thermodynamic sequence of FQH-like polynomials

\[
\Pi = (P_2, P_3, \cdots , P_n, \cdots)
\]

the cluster size of a symmetric polynomial \( P \) is defined as the smallest number \( \theta \) such that \( P(\theta) \), the \( r \)-fusion of \( P \), does not vanish, but \( P(\theta+1) = 0 \). Let us list the cluster size of \( \Pi \) in another sequence

\[
\Theta = (\theta_2, \theta_3, \cdots , \theta_n, \cdots)
\]

For any physically sound sequence of FQH-like polynomials, there should exists some \( n > 2 \) such that for all \( m > 1 \) one has \( \theta_m = \theta_{m+1} \). One says \( \Theta \) should eventually stabilize. This has to happen because in the thermodynamic limit, the cluster size should be finite and independent of size.

The job of (L) postulate is make certain the sequence of cluster sizes eventually stabilizes. To see why, let us consider an example violating the (L) condition, but satisfying (CRE) postulates. Consider the face matrix

\[
F_{\text{counter}} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix}
\]

If \( \Phi \) is the corresponding digraph, then \( T T_2 \otimes \Phi \) is exactly the shard of the 3rd cyclic square type (see Table II and section V A, T3). Using the adjacency matrix POV, from the face matrix \( F_{\text{counter}} \) in eq. (B1), one defines

\[
W_n = J_n^+ \otimes F_{\text{counter}} + (J_n^+ \otimes F_{\text{counter}})^t
\]

Let \( G_n \) be the graph with adjacency matrix \( W_n \). The aggregation sequence \( \Gamma = (G_2, G_3, G_4, G_5, \cdots) \) is shown in Fig. (9). Let \( P_n = SGM(G_n) \). The cluster size \( \theta_n \) of a \( P_n \) is nothing but the coalescence number of \( G_n \), i.e. \( \theta_n = \theta(G_n) \). In general one has \( \theta(G) \leq \alpha(G) \) being the independence number of \( G \). In our example, for \( n > 3 \), \( \alpha(G_n) = n \) and moreover there is exactly one independent set of size \( n \) in \( G_n \). This is illustrated in Fig. (9). By thm. VI.5, if we fuse \( n \) variables in polynomial \( P_n \) for \( n > 3 \), we find

\[
P_n \vert_{Z, z_1, \cdots , z_{2n}} = \prod_{i=1}^{2n} (Z - z_i)^{2(n-1)} Q(z_1, \cdots , z_{2n})
\]

with \( Q \) a polynomial in the remaining variables which does not identically vanish (in fact, up to a numerical factor, \( Q \) is the SGM of \( 2K_n \Pi 2K_n \)). This immediate shows that, for \( n > 3 \), in our example \( \theta(G_n) = \alpha(G_n) = n \). Consequently, the cluster size sequence \( \Theta \) never stabilizes. Tracking back the source of this unwelcome feature, \( \Theta \) could not eventually stabilize, partly due to independence number growing linearly with the system size \( n \). If
we demand (L) postulate, then the independence number \( \alpha \) is always fixed and equal to the vertex augmentation constant \( v \), no matter what system size \( n \) is chosen. Actually, one does not even need neither of CRE postulates for this to happen (see proof of thm. VI.1).

\[(R)+(L) \text{ POSTULATES:} \] The fact that \( \Theta \) eventually stabilizes is necessary, but still not really enough to make the aggregation sequences perfectly fit our intuition of what “cluster” is. Let II be a sequence of FQH-like polynomials. In the thermodynamic limit, not only the cluster size of \( P_n \) should stabilize to some number \( v \), but one also needs different clusters of size \( v \) to be disjoint.

But when do a graph \( G = \overline{TT} \otimes \Phi \) with \( \Phi \) a \((v,d)\)-CRL (but no need for E) digraph has the property that “independent sets of size \( v(G) = v \) are pairwise disjoint”? If one looks at the proof of thm. VI.1, this requires \( \Phi \) to be strongly connected. But, in the same proof, it was shown that any CRL digraph is strongly connected. In other words, the postulates \((R)+(L)\) are automatically giving us what we desired. This disjointness of maximum independent sets, is solely responsible for the sequence \( SGM(\overline{TT} \otimes \Phi) \) with \( \Phi \) a \((v,d)\)-CRL digraph, to have a \((v,d)\)-clustering property (see proof of cor. VI.6).

\[(E) \text{ POSTULATE.} \] The evenness postulate is used exactly once in the whole paper. Its use is in thm. VI.3 to show that \( \Lambda(n,v,d) \) is a non-vanishing orientation type for a \((n,v,d)\)-accordion graph \( G \). Without the condition \( vd = \text{even} \), it is not guaranteed that SGM of \( G \) is non-vanishing.

**Appendix C: Relation Between Aggregation POVs**

The three points of views (POV), that we introduced in this paper, seemingly use different data. In this section we will discuss how all of these POVs are indeed building the same graph. Let us fix the numbers \( v \) and \( d \) throughout. We will neither care about the (E) postulate, nor the (C) postulate here.

**Initial Data of AM-POV & C-POV**

Before the construction of the sequence begins, each POV feeds on a set of information.

- Adjacency POV’s initial data is a \( v \times v \) matrix \( F \) such that the sum of each row/column is \( d \), the diagonal elements are nonzero. \( F \) was called the face matrix.
- Ceramic POV’s initial data is a pair \((G_2,X)\) with \( G_2 \) a \( d \)-regular multigraph of order \( 2v \), and \( X \) a perfect display of \( G \).

We will first show that the initial data in these two POVs are completely equivalent.

(Adjacency Matrix \( \implies \text{Ceramic} \)) Consider a face matrix \( F \). Let

\[
W_2 = \begin{pmatrix} 0 & F \\ F^t & 0 \end{pmatrix}
\]

This is bipartite regular graph \( G_2 = (V,E) \) of order \( 2v \), degree \( d \), with partitions \( A,B \). If we still by the labels in which \( W_2 \) is the adjacency matrix of \( G \), then \( A = \{1,2,\cdots,v\} \) and \( B = \{v+1,v+2,\cdots,2v\} \). Define the height function \( h : V \to \mathbb{Z}_v/v \) as follows:

\[
h(x) = \begin{cases} 
\frac{(x-1)}{v} & x \leq v \\
\frac{(x-v-1)}{v} & x \geq v
\end{cases}
\]

Denote the display corresponding to this height function, \( X_F \). Note that \( X_F \) is a perfect display, since diagonal elements of \( F \) are non-zero. We call \( X_F \) the corresponding perfect display for face matrix \( F \).

(Ceramic \( \implies \text{Adjacency Matrix} \)) Conversely let \( G_2 \) be a bipartite \( d \)-regular graph and \( X \) a perfect display for it (by prop. IV.1 each bipartite regular graph has a perfect display). Define function \( \nu : X \to \{1,2,\cdots,2v\} \) as follows

\[
\nu(\epsilon,0,h) = \begin{cases} 
 hv + 1 & \epsilon = -1 \\
 hv + v + 1 & \epsilon = +1
\end{cases}
\]
Then \( \nu \) puts a total order on the vertexes of \( G_2 \). The adjacency matrix of \( G_2 \), with respect to \( \nu \), is of the form of eq. (C1). From \( W_2 \) one then reads the face matrix. We denote this by \( F_X \), the face matrix corresponding to perfect display \( X \). One easily checks that \( X_{F_X} = X \) and \( F_{X_F} = F \).

### Aggregation Data in AM-POV & C-POV

The aggregation construction itself uses some data in each POV:

- The adjacency matrix POV uses the data \( J_n^+ \) together with the tensor product operation \( \otimes \).
- Ceramic POV uses cloning and transitive gluing. Transitive gluing itself relies on a total order over the set \( \{1, 2, \cdots, n\} \), which was taken as the natural one.

As such, the end result of the graph constructed by aggregation in both adjacency matrix POV and ceramic POV is completely labeled by a pair \( (a, r) \) with \( 0 \leq a \leq n - 1 \) and \( 0 \leq r \leq v - 1 \). Here is why:

- The adjacency matrix POV gives the produces a matrix at the end of the form \( W_n = J_n^+ \otimes F + \) transpose. Such matrix already labels the vertexes by \( (a, r) \).
- In ceramic construction, there are \( n \) junctions in the stencil. There is also a height function for the perfect display which is always respected. So every vertex, at the end, can be determined by \( (j, h) \), where \( a \) is the junction it belongs to and \( h \) is its height.

Quite similar to what was done for initial data, one can check that the labeled graphs \( G_n \) obtained by aggregation in adjacency POV with face \( F \) is the same labeled graph obtained from ceramic POV with perfect display \( X_F \) (and vice versa).

### Fully Labeled Digraph POV

Let us start with a face matrix \( F \). This matrix can be understood as the adjacency matrix of a fully looped \( d \)-regular digraph \( \Phi F \) which is already ordered. Conversely if \( \Phi \) is \( (v, d) \)-RL digraph and one puts an order on the vertexes, then its adjacency matrix is a face matrix. So we define the initial data of fully labeled digraph POV as an ordered \( (v, d) \)-RL digraph.

The tensor product operation of ordered digraphs also translates exactly into tensor product of the adjacency matrices. At the same time \( J_n^+ \) is the adjacency matrix of transitive tournament \( TT_n \) labeled according to the natural total order in \( \{1, 2, \cdots, n\} \). Therefore the aggregation data of fully labeled digraph POV is a labeled

transitive tournament (which is take to be according to the natural order), a labeled \((v, d)\)-RL digraph \( \Phi \) together with the tensor product operation and forgetful functor.

### Redundancies

The whole point now is that the operation \( TT_n \otimes \Phi \) is completely independent of labeling. No matter what labeling is chosen for \( TT_n \) or for \( \Phi \) the end result is the same digraph. This leads to three different redundancies:

1. **The ordering on \( \Phi \) is irrelevant.** There are exactly \( n! \) such orderings, one for each element of the symmetric group \( \mathfrak{S}_n \). Let \( \sigma \in \mathfrak{S}_n \).

   - (C-POV:) Let \( h \) be the height function for perfect display \( X \) and the perfect display corresponding to \( h_\sigma = \sigma \circ h \) be denoted by \( X_\sigma \). Then this redundancy is simply stating the obvious fact that \( G_n(X) = G_n(X_\sigma) \) in ceramic POV.

   - (AM-POV) Let \( M_\sigma \) be the permutation matrix of \( \sigma \). The message of this redundancy is the adjacency matrices

     \[
     W_\sigma^\sigma = J_n^+ \otimes (M_\sigma F M_\sigma^t) + (J_n^+)^t \otimes (M_\sigma F^t M_\sigma^t)
     \]

     all describe the same graph. But it is a know fact that two adjacency matrices describe the same graph if and only if they are permutations of one another.

2. **The ordering on \( TT_n \) is irrelevant.** There are exactly \( n \) \( n! \) such orderings, one for each element of the symmetric group \( \mathfrak{S}_n \). Let \( \sigma \in \mathfrak{S}_n \).

   - (C-POV) A different ordering on \( TT_n \) is equivalent to a different ordering of junctions in the transitive gluing. Unsurprisingly, the ceramic construction does not depend on which total order is chosen.

   - (AM-POV) Of course, the adjacency matrices

     \[
     W_\sigma^\sigma = (M_\sigma J_n^+ M_\sigma^t) \otimes F + (M_\sigma (J_n^+)^t M_\sigma^t) \otimes F^t
     \]

     also describe the same graph too.

3. **Define \( -\Phi \) as the digraph obtained from \( \Phi \) by reversing the direction of every arc.** Then \( TT_n \otimes \Phi = TT_n \otimes -\Phi \). This is non-trivial and we prove it here. In AM-POV \( \Phi \mapsto -\Phi \) is equivalent to \( F \mapsto F^t \). In C-POV, under the effect \( f \) this operation, the vertex \( \epsilon \) at \((\epsilon, 0, h)\) with \( \epsilon = \pm1 \), maps to \((-\epsilon, 0, h)\). In other words, this operation flips the display around the z-axis. This redundancy is best seen in AM-POV. Let \( \sigma \in \mathfrak{S}_n \) be the permutation which sends \( j \) to \( n - j \) for all \( j \in \{1, \cdots, n\} \). Then \((J_n^+)^t = M_\sigma J_n^+ M_\sigma^t \).

Therefore the adjacency matrix

\[
J_n^+ \otimes F^t + (J_n^+)^t \otimes F = (M_\sigma J_n^+ M_\sigma^t) \otimes F + (M_\sigma (J_n^+)^t M_\sigma^t) \otimes F
\]
and adjacency matrix \( J_n^+ \otimes F + (J_n^+)^t \otimes F^t \) describe the same graph.

With that it should be clear how these three points of view related to one another.

Appendix D: Counterexample

This appendix is devoted to an example of \((n, v, d)\)-accordion graph \(G\), which has an orientation type \(\lambda\) such that \(\Lambda(n, v, d)\) and \(\lambda\) are incomparable with dominance partial order. Consider the face matrix

\[
F = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}
\]

Define \(J_n = J_n^+ + J_n^-\). Note that since \(F\) is already symmetric \((J_n^+ \otimes F) = (J_n^+)^t \otimes F\). Therefore define \(G\) as the graph which has

\[
W_4 = J_4 \otimes F
\]
as its adjacency matrix. We label the vertexes according to the tensor product notation by \((a, r)\) where \(a = 1, 2, 3, 4\) and \(r = 1\). With this notation, an orientation of \(G\) which has \(\Lambda(4, 2, 4) = (12, 12, 8, 8, 4, 4, 0, 0)\) as its orientation type is shown in Fig. (10). An orientation with type \(\lambda = (12, 11, 10, 9, 3, 2, 1, 0)\) is shown in Fig. (11). One can check easily that \(\Lambda(4, 2, 4)\) and \(\lambda\) are incomparable with dominance partial order. That being said, one can also check that \(\lambda\) is a vanishing orientation type of \(G\).

Appendix E: Proofs

Proposition III.1

The elementary symmetric polynomials in \(N\) variables \(z_1, \cdots, z_N\) can be found from their generating function:

\[
E(t) := \prod_{i=1}^{N} (1 + z_i t) = \sum_{r=0}^{N} e_r(z_1, \cdots, z_n) t^n
\]

If the projective roots are away from infinity, i.e. \([1 : 0]\), with the definition \(z_i = x_i/y_i\), the binary form \(\beta_N(X, Y)\),

\[
\beta_N(X, Y) = \prod_{j=1}^{N} y_j \prod_{i=1}^{N} (X - Y z_i)
\]

\[
= \prod_{j=1}^{N} y_j \sum_{r=0}^{N} (-1)^r e_r(z_1, \cdots, z_N) X^{N-r} Y^r
\]

Since all roots are away from infinity, then \(\prod_{j=1}^{N} y_j \neq 0\). By comparing to the definition of coefficients of binary form \(a_0 = \prod_{j=1}^{N} y_j \neq 0\). Since the coefficients should be treated as points \(\mathbb{P}^N\), one can then take \(a_0 = 1\). With that choice, then

\[
\binom{N}{r} a_r = (-1)^r e_r(z_1, \cdots, z_N)
\]

Theorem III.2

The proof of this theorem is mostly calculation. We will shorten the calculations and leave the details to the reader.

The linear Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\) is the three (complex) dimensional vector space with the Pauli matrices \(\sigma^+, \sigma^-, \frac{1}{2} \sigma^z\) as its basis. These three are the infinitesimal generators of \(\text{SL}_2(\mathbb{C})\). Take elements in \(\text{SL}_2(\mathbb{C})\) infinitesimally close to identity:

\[
\begin{align*}
(X, Y) \xrightarrow{1 + \epsilon \sigma^+} (X + \epsilon Y, Y) \\
(X, Y) \xrightarrow{1 + \frac{\epsilon^2}{2} \sigma^z} ((1 + \frac{\epsilon}{2}) X, (1 - \frac{\epsilon}{2}) Y) \\
(X, Y) \xrightarrow{1 + \epsilon \sigma^-} (X, Y + \epsilon X)
\end{align*}
\]
Remember that to find the transformations \( g \ast a_r \), we need to satisfy \( \beta_N(g(X,Y); \{ g \ast a_r \}) = \beta_N(X,Y; \{ a_r \}) \), or better yet, \( \beta_N(g^{-1}(X,Y); \{ a_r \}) = \beta_N(X,Y; \{ g \ast a_r \}) \).

Now

\[
\begin{align*}
\beta_N(X,Y) & \left( 1 + \epsilon \sigma^+ \right) \sum_{k=0}^{N} (\epsilon)^k [a_k - ek_{a_{k-1}}]X^{N-k}Y^k \\
\beta_N(X,Y) & \left( 1 + \epsilon \sigma^+ \right) \sum_{k=0}^{N} (\epsilon)^k [a_k - \epsilon(N-k)ak_{a_{k+1}}]X^{N-k}Y^k \\
\beta_N(X,Y) & \left( 1 + \epsilon \sigma^- \right) \sum_{k=0}^{N} (\epsilon)^k [a_k - \epsilon(N-k)ak_{a_{k+1}}]X^{N-k}Y^k
\end{align*}
\]

This means that via the induces action of \( \text{SL}_2(\mathbb{C}) \) on the space of coefficients \( \mathbb{P}^N \), one has

\[
\begin{align*}
\left( 1 + \epsilon \sigma^+ \right) & \sum_{k=0}^{N} (\epsilon)^k [a_k - ek_{a_{k-1}}] \\
\left( 1 + \epsilon \sigma^+ \right) & \sum_{k=0}^{N} (\epsilon)^k [a_k - \epsilon(N-k)ak_{a_{k+1}}] \\
\left( 1 + \epsilon \sigma^- \right) & \sum_{k=0}^{N} (\epsilon)^k [a_k - \epsilon(N-k)ak_{a_{k+1}}]
\end{align*}
\]

Note that, although this seemingly depends on \( a_{N+1} \) and \( a_{-1} \), these two auxiliary entities always are multiplied by a zero. Define the following operators

\[
\Delta = \sum_{k=0}^{N} \frac{ak}{\partial \partial ak}, \quad \ell^+ = \sum_{k=1}^{N} ka_k - 1 \frac{\partial}{\partial \partial ak}, \\
\ell_+ = \frac{N}{2} \Delta - \sum_{k=1}^{N} ka_k \frac{\partial}{\partial \partial ak}, \quad \ell^- = \sum_{k=0}^{N-1} (N-k)ak_{a_{k+1}} \frac{\partial}{\partial \partial ak}
\]

Operator \( \Delta \) is just the Euler operator, measuring the degree of a homogeneous polynomial \( Q \). One easily checks that the following are equivalent:

1. \( Q \) is a binary invariant.
2. \( Q \) is invariant under induced action of \( 1 + \epsilon \sigma^+ \) and \( 1 + \epsilon \sigma^+ \) (since \( \text{SL}_2(\mathbb{C}) \) is the exponential of \( \mathfrak{s} \mathfrak{l}_2(\mathbb{C}) \)).
3. \( \ell^+ Q = \ell^+ Q = \ell^- Q = 0 \).

We want to find \( \ell^\pm \), \( \ell^\pm \) in terms of roots. To do so, define the operator \( D_m : \mathbb{C}(z_1, \ldots, z_N) \rightarrow \mathbb{C}(z_1, \ldots, z_N) \) (where \( \mathbb{C}(z_1, \ldots, z_N) \) is the ring of formal power series) as

\[
D_m = -\sum_{j=0}^{N-1} z_j^m \frac{\partial}{\partial z_j}
\]

for all \( m \in \mathbb{Z} \). One can check that \( [D_m, D_n] = (m - n)D_{m+n} \). This is a differential representation of Wick algebra on \( \mathbb{C}(z_1, \ldots, z_N) \).

**Claim.** Let \( e_k(z_1, \ldots, z_N) \) be the kth elementary symmetric polynomial. Then for \( 1 \leq k \leq N \), one has

\[
\begin{align*}
D_1 e_k & = - (N+1-k)e_{k-1} \\
D_0 e_k & = -ke_k \\
D_{k+1} e_k & = -e_1 e_k + (k+1)e_{k+1}
\end{align*}
\]

with the convention \( e_0 = 1 \).

**Proof.** First of all note that \( D_0 \) is the just negative of the Euler operator. Since \( e_k \) is homogeneous of order \( k \), the identity \( D_0 e_k = -ke_k \) is trivial. We therefore need to show the \( \pm 1 \) case.

Define \( p_k(z_1, \ldots, z_N) = \sum_{j=1}^{N} z_j \in \mathbb{C}(z_1, \ldots, z_N) \). Clearly, \( D_m p_k = -k p_{m+k} \) for all \( m \). In particular if \( k \geq 1 \) then \( p_k \) is a symmetric polynomial, and \( D_{k+1} p_k \) is also a symmetric polynomial (with \( p_0 = N \)). Newton’s identities (see [24, 2.11]) state that

\[
k e_k = \sum_{r=1}^{N} (-1)^{r-1} e_{k-r} p_r
\]

We therefore prove our assertions by induction on \( k \). For the base of induction note that \( p_1 = e_1 \). We leave the details of the calculation to the reader. \( \square \)

Now using prop. III.1, for the case \( a_0 \neq 0 \) but other than that an arbitrary constant, \( (\epsilon)^k a_k = (-1)^k a_k e_k \), for \( 1 \leq k \leq N \) one finds

\[
\begin{align*}
D_1 a_k & = ka_k \\
D_0 a_k & = -ka_k \\
D_{k+1} a_k & = -e_1 e_k - (N-k)ak_{a_{k+1}}
\end{align*}
\]

Note that \( e_1 = z_1 + \cdots + z_N = Z = -Na_1/a_0 \). Combining these together, we find that in the coefficient space,

\[
\begin{align*}
D_1 & = \sum_{k=1}^{N} ka_k \frac{\partial}{\partial \partial ak}, \quad D_0 = -\sum_{k=1}^{N} ka_k \frac{\partial}{\partial \partial ak} \\
D_{k+1} & = -Z \sum_{k=1}^{N} ak \frac{\partial}{\partial \partial ak} - \sum_{k=1}^{N}(N-k)ak_{a_{k+1}} \frac{\partial}{\partial \partial ak}
\end{align*}
\]

Both sums in \( D_{k+1} \) start from \( k = 1 \). However, if we were to add their \( k = 0 \) terms, it would become \( C := (-Za_0 - Na_1)\partial/\partial a_0 \). But \( Z = -Na_1/a_0 \), so \( C = 0 \). We will therefore lift the sum index in \( D_{k+1} \) to \( k = 0 \). Now suppose we restrict the action of all of these operator to symmetric polynomials \( P \) in \( z_1, \ldots, z_N \) with local degree \( \delta \) (which means \( \Delta P^\delta = \delta P^\delta \)). Then

\[
\begin{align*}
\ell^+ & = D_{-1} = -\sum_{j=1}^{N} \frac{\partial}{\partial z_j} = -L^+ \\
\ell^+ & = \frac{N\delta}{2} + D_0 = \frac{N\delta}{2} - \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j} = L^+ \\
\ell^- & = -Z\delta - D_{-1} = -Z\delta + \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j} = -L^-
\end{align*}
\]

This shows that \( P \) is a \( (N, \delta) \) FQH-like polynomial if and only if \( P^\delta \) is a binary invariant of order \( N \) and degree \( \delta \).

**Discussion:** Let \( g \in \text{SL}_2(\mathbb{C}) \). The action of \( g \) on coefficients is translated to an action on the projective root. In fact, under \( g \)

\[
(x_i, y_i) \rightarrow g \cdot (x_i, y_i)
\]
Exactly like \((X,Y)^t\) transforms. This shows given \(P\), the binary dual \(P^\circ\) is a binary invariant if and only if by uniformly transforming all of the projective roots by any \(g \in \text{SL}_2(C)\), the polynomial \(P\) (or more precisely the section corresponding to \(P\)) stays invariant. At the same time, thinking of Riemann sphere as the complex projective line, since the automorphism group of \(C^2\) are \(\text{GL}_2(C)\), the automorphism group of \(\mathbb{P}^1\) is simply \(\text{PGL}_2(C)\). The action of this automorphism is by matrix multiplication. Moreover \(\text{PGL}_2(C) = \text{SL}_2(C)/\pm 1\). In other words, binary invariance is basically synonymous with invariance under the natural action of the automorphism group of the Riemann sphere. The special \(\text{SL}_2(C)\) transformations of interest are:

1. \(\exp(t \sigma^+)=\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\)
   which is a translation by amount \(t\) for points in \(\mathbb{P}^1 - \{\infty\}\) (while fixing the infinity).
2. \(\exp(t \sigma^-)=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\)
   which is a translation by amount \(t\) for points in \(\mathbb{P}^1 - \{0\}\) (while fixing zero).
3. \(\exp((ln r + i\theta)\sigma^\pm)\)
   which is a scaling by amount \(r\) followed by a rotation by amount \(\theta\) in \(\mathbb{P}^1 - \{\infty\}\) (since the transition map between the two charts of \(\mathbb{P}^1\) is \(z \mapsto 1/z\), the transformation in \(\mathbb{P}^1 - \{0\}\) is automatic).

Note that, since the two translations generate the whole automorphism group, \(P\) is a FQH-like polynomial, if and only if it is translation invariant over the Riemann sphere.

**Theorem III.3 (Cayley)**

See Elliott [26, §72-74 & §88-89].

**Proposition IV.1**

We will prove here that any bipartite (multiple) graph has a perfect matching. Then using that perfect matching one constructs a perfect display. But first, we need to define a few concepts:

Given a graph \(G = (V,E)\), a matching \(M\) in \(G\) is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. A matching \(M\) is maximum if there is no other matching \(M'\) with \(|M'| > |M|\). A matching \(M\) is perfect if every vertex of \(G\) appears as the end point of some (and therefore only one) edge in \(M\). Any perfect matching is maximum. If \(X \subset V\), the neighborhood of \(X\), denoted by \(N(X)\) is the set of all vertexes \(x \in V - X\) that are adjacent to some vertex in \(X\). We say a set \(S \subset E\) covers a set \(X \subset V\) if every \(x \in X\) is the end point of some \(e \in S\). With all of those definitions, we can now state the famous **Hall’s marriage theorem**

**Theorem (Hall’s Marriage Theorem)**. Let \(G\) be a simple bipartite graph with bipartition \((A,B)\). Then there is a matching \(M\) which covers \(A\) if and only if \(|N(X)| \geq |X|\) for every \(X \subset A\).

For three different proves of this, see [27, thm. 2.1.2].

As a corollary of Hall’s marriage theorem we have:

**Corollary.** Let \(G\) be a \(d\)-regular bipartite multi-graph. Then \(G\) has a perfect matching.

**Proof.** We will actually prove that if \(G\) is the \(d\)-regular bipartite multigraph, then the underlying simple graph of \(G\), which we denote by \(G_s\), has a perfect matching. By underlying simple graph we mean when all parallel edges are treated as a single edge.

Let the partitions of \(G\) be \(A, B\). Take any subset \(X \subset A\). The number of edges with one end in \(X\) is \(\epsilon_X = |X|d\) due to \(d\)-regularity. Let \(N(X) \subset B\) be the neighborhood of \(X\). The number of edges with one end in \(N(X)\) is \(\epsilon_{N(X)} = |N(X)|d\). But any edge incident to \(X\) is also incident to \(N(X)\) by definition of neighborhood. So \(\epsilon_X \leq \epsilon_{N(X)}\), or \(|X| \leq |N(X)|\). Note that the neighborhood of \(X\) in \(G\) is the same as neighborhood of \(X\) in \(G_s\). So now by Hall’s marriage theorem we find that \(G_s\) has a perfect matching. But one can also treat \(G_s\) as a subgraph of \(G\). So \(G\) has a perfect matching.

Now we prove that any bipartite \(d\)-regular multi-graph \(G\) of order \(2v\) admits a perfect display. Fix some perfect matching \(M\) of \(G\). Label the vertexes in \(A\) and \(B\) as \((-1,j)\) and \((+1,j')\) respectively (with \(j, j' \in \mathbb{Z}_v\)) such that each edge in \(M\) is of the form \((-j)(+j)\): i.e. it connects \((-j)\) to \((+j)\).

**Theorem V.1**

The proof of this theorem is completely analogous to thm. V.2 (This theorem is just the (ee) case of thm. V.2).

**Theorem V.2**

We will use the notation used in thm. VI.1 for this proof. Although this theorem was shown first in the paper, thm. VI.1 uses nothing discussed in thm. V.2.

**Elements of Dihedral Group:** \(D_m = \langle q, x \mid q^m = x^2 = (xq)^2 = 1 \rangle\). All elements of \(D_m\) can be written as \(q^ix^k\) with \(0 \leq i < m\) and \(k = 0, 1\). One has \(q^i x q^j = q^{i+j} x_r\).

**Face Matrix** : Let us recall the face matrix \(F\) since we will use it often: Here \(v = 2k\) and \(F\) is a \(v \times v\) matrix. Let \(F^\pm = \pi_k^\pm(F)\). Suppose

1. \(F^+ = F^- := \tilde{F}\).
2. For all $s \in \mathbb{Z}_n$ one has $F_{s,s-1} = \rho \neq 0$.

where $\tilde{F}$ is a $k \times k$ matrix given by $\tilde{F}_{rs} = \sum_{j=0}^{k-1} \mu_j \delta_{s-r,j}$ such that if $r + s = -1 \pmod{k}$ then $\mu_r = \mu_s$. Also $\mu_0 \neq 0$. With these conditions clearly $F$ is a face matrix.

Let $(a, r)$ be the labeling we used in proof of thm. VI.1. We label the vertexes of $G_n = \overline{T T}_n \otimes \overline{\Phi}$ by the elements of $D_{nk}$ as follows:

\[
\begin{cases}
(a, 2i) & \mapsto q^{ni+a} \\
(a, 2i + 1) & \mapsto q^{ni+a}q^{n-1}x
\end{cases}
\]

where $0 \leq i < k - 1$. Suppose $0 \leq a < b < n$. Fix some $0 \leq \ell \leq \lfloor k/2 \rfloor$. Define $\ell'$ such that $\ell + \ell' = -1 \pmod{k}$.

According to the data in the face matrix

- $(ee)$ $q^{ni+a}$ connects to $q^{n(i+\ell)+b}$ and $q^{n(i+\ell')+b}$ with $F_{i,i+\ell} = F_{i,i+\ell'} = \mu_\ell$ edges.
- $(oo)$ $q^{ni+a}q^{n-1}x$ connects to $q^{n(i+\ell)+b}q^{n-1}x$ and $q^{n(i+\ell')}$ with $F_{i,i+\ell} = F_{i,i+\ell'} = \mu_\ell$ edges.

These edges (all of which are of multiplicity $\mu_\ell$) are all of the form $(g, gs)$ with $g \in D_{2k}$ and $s \in S_{n,\ell}$ with $S_{n,\ell} = \left\{ q^{\pm(n+\ell+i)} \mid j \in \mathbb{Z}_k \setminus \{0\} \right\}$.

And any pair of elements of the form $(g, gs)$ with $g \in D_{2k}$ and $s \in S_{n,\ell}$ has already been accounted for in even-by-even (ee) and odd-by-odd (oo) cases. This brings us to even-by-odd and odd-by-odd cases.

- $(eo)$ $(a, 2i)$ is connected to $(b, 2i - 1)$, i.e. $q^{ni+a}$ is connected to $q^{n(i-1)+b}q^{n-1}x$ with $\mu_\ell$ edges.
- $(oe)$ $(a, 2i + 1)$ is connected to $(b, 2i)$, i.e. $q^{ni+a}q^{n-1}x$ is connected to $q^{ni+b}$ with $\mu_\ell$ edges.

These edges (all of which are of multiplicity $\rho$) are all of the form $(g, gs)$ with $g \in D_{2k}$ and $s \in R_{n,\ell}$ with $R_{n} = \{ x, qx, \cdots , q^{n-2}x \}$.

And any pair of elements of the form $(g, gs)$ with $g \in D_{2k}$ and $s \in R_{n}$ has already been accounted for. So with $T_n$ and $\mu$ as defined in the statement of the theorem, $\overline{T T}_n \otimes \overline{\Phi} = \text{Cay}(D_{nk}, T_n, \mu)$.

\[\text{Theorem VI.1}\]

Let us first address why $\Phi$ is strongly connected.

\[\text{Claim. Any connected finite regular digraph is strongly connected.}\]

\[\text{Proof. The proof uses the concept of a “cut” and two lemmas related to the concept. Let } D = (V, A) \text{ be digraph. A cut } C = (S, T^+, T^-), \text{ is a triple of sets: A subset of the vertex set } S \subset V, \text{ and two so-called cut-s sets } T^+ \subset A. T^+ \text{ is the set of arcs going from } S \to V - S \text{ while } T^- \text{ is the set of arc going from } V - S \to S. \text{ By abuse of language, we will refer to } S \text{ as the cut. Also, in } D, \text{ we say } y \text{ is reachable from } x \text{ if there exists a directed path from } x \to y. \]

\[\text{Lemma. Let } D = (V, A) \text{ be a connected digraph. Suppose } y \in V \text{ is not reachable from } x \in V \text{ and } x \neq y. \text{ Then there exists a cut } S \text{ such that}\]

1. $x \in S$ and $y \in V - S$.
2. The cut-set $T^+$ is empty.
3. The cut-set $T^-$ is non-empty.

\[\text{Proof. Define } S' \subset V \text{ as the set of all vertexes that are reachable from } x. \text{ Then let } S = \{ x \} \cup S'. \text{ Clearly } y \notin S \text{ and } T^+ = \emptyset. \text{ Moreover, if } D \text{ is connected, } T^- \text{ cannot be empty.}\]

\[\text{Lemma. Suppose } D = (V, A) \text{ is a digraph such that the in-degree of each vertex is equal to its out-degree. Then for any cut } S, \text{ the cut-sets are of equal size, i.e. } |T^+| = |T^-|.\]

\[\text{Proof. For each } x \in S \text{ let } d_{\text{ext}}^+(x) \text{ be the number arcs going from } x \text{ to } S - V, \text{ and } d_{\text{ext}}^-(x) \text{ the number of arcs coming from } S - V \text{ into } x. \text{ Let } d(x) = d^+(x) = d^-(x) \text{ (since the out-degree and in-degree are equal). Define }\]

\[d_{\text{int}}^+(x) = d(x) - d_{\text{ext}}^+(x)\]

Here, $d_{\text{int}}^+(x)$ (resp. $d_{\text{int}}^-(x)$) is the number of arcs of the form $S \to S$ that start at $x$ (resp. end at $x$). Note that $\sum_{x \in S} d_{\text{ext}}^+(x) = |T^+|$. Define

\[\Delta = \sum_{x \in S} d_{\text{int}}^+(x) - d_{\text{int}}^-(x) = |T^+| - |T^-|\]

Consider an internal arc, i.e. $x \to y$ with both $x, y \in S$. Note that this arc is contributing one to $d_{\text{int}}^+(x)$ and one to $d_{\text{int}}^-(y)$. As such, $\Delta = 0$, resulting in $|T^+| = |T^-|$.\]

Now let $D$ be a connected $d$-regular digraph. Suppose, however, that $D$ is not strongly connected, i.e. there exists $x, y \in D$ with no directed path from $x$ to $y$. Then by lemma there exists a cut with $T^+ = \emptyset$ but $T^- \neq \emptyset$. But, the second lemma we must have $|T^+| = |T^-|$. This is a contradiction, showing that $D$ is strongly connected.

Instead of proving the properties quoted in the statement for $G_n$, we will prove a collection of facts, numbered as $(\#1 n)$ about accordion graphs. Combining these facts together gives the theorem. We will start by assuming $\Phi$ is just fully looped and nothing else. As we move further into the discussion, we impose more conditions on $\Phi$ and discuss the implications of those new conditions. In particular, in this whole theorem we will never need the (E) postulate.

As mentioned, we start by assuming $\Phi$ is fully looped digraph of some order $v$. Let $G_n = \overline{T T}_n \otimes \overline{\Phi}$. Label the vertexes of $TT_n$ with elements of $\mathbb{Z}_n$ such that $j \to j$
iff $i < j$. Also label the vertexes of $\Phi$ by elements of $\mathbb{Z}_n$ arbitrarily. Let $F$ be the adjacency matrix of $\Phi$ in this ordering. Then the vertexes of $G_n$ are labeled by $(a, r) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Define

$$A_a = \{(a, r) \mid r \in \mathbb{Z}_n\}, \quad R_r = \{(a, r) \mid a \in \mathbb{Z}_n\}$$

We call $A_a$ the $a$th axis and $R_r$ the $r$th rung. By definition of tensor product of digraphs, we know the following facts:

(#1) The order of $G_n$ is $nv$.

(#2) Since $TT_n$ has no loops, no two vertexes belonging to the same axis are adjacent. In other words, each axis is an independent set of size $v$. Moreover, obviously $A_a \cap A_b = \emptyset$ if $a \neq b$.

(#3) The diagonal entry, $F_{r,r}$ is counting the number loops over vertex $r$ in $\Phi$. Since $\Phi$ is fully looped, $F_{r,r} \neq 0$. Moreover, due to tensor product definition, given some $r \in \mathbb{Z}_n$ and $a \neq b \in \mathbb{Z}_n$ the vertexes $(a, r)$ and $(b, r)$ of $G_n$ are adjacent by $F_{r,r} \neq 0$ parallel edges (or an edge with weight $F_{r,r}$). In other words, each rung is a clique of size $n$.

(#4) Take any two vertexes $(a, r)$ and $(b, s)$, with $a < b$, then the edge between them has a weight $F_{r,s}$ (with the understanding that a zero weight means no edge). This independence from the numbers $a, b$ shows that if $T_k$ is the union of any $2 \leq k \leq n$ distinct axes, then the induced subgraph of $T_k$ is isomorphic to $G_k$.

These facts are basically the unraveling of the definition of tensor product. We now turn our attention away from tensor product.

Claim (#5). The independence number of $G_n$ is $v$; i.e. $\alpha(G_n) = v$.

Proof. Let $S$ be any independent set of $G_n$ and $r \in \mathbb{Z}_n$. Then since $R_r$ is a clique, the intersection $S \cap R_r$ is either empty or a singleton. Since there are only $v$ rungs, $|S| \leq v$ necessarily. Since $A_0$ is an independent set of size $v$, the independence number is $v$.

Claim (#6). The clique number of $G_n$ is $n$; i.e. $\omega(G_n) = n$.

Proof. Let $C$ be any clique. Since each axis is an independent set, $C \cap A_0$ is either empty or a singleton for each $a \in \mathbb{Z}_n$. The rest is similar to (#5).

Claim (#7). The chromatic number of $G_n$ is $n$; i.e. $\chi(G_n) = n$.

Proof. We do know that $G_n$ is $n$ colorable by painting each axis with a different color, so $\chi(G_n) \leq n$. Suppose there exists a proper $(n - 1)$-coloring $c$ with color classes $[1], [2], \ldots, [n - 1]$. Since $R_0$ has $n$ elements, by pigeonhole principle, there exists a color class of $c$ containing two or more vertexes of $R_0$. But all vertexes in $R_0$ are adjacent, and therefore, in a proper coloring cannot have the same color. This is a contradiction, which proves $\chi(G_n) = n$.

Claim (#8). The core of $G_n$ is $K_n$; i.e. $G_n^* = K_n$.

Proof. First of all it is very well-known fact that complete graphs are cores (and easy to show). We also know that $K_n \subseteq G_n$, since each rung is a clique. At the same time tensor product of digraphs already comes with a natural projection homomorphism of digraphs $\pi : TT_n \otimes \Phi \rightarrow TT_n$. Upon applying the forgetful functor, this descends to a homomorphism of undirected graphs $\overline{\pi} : \overline{TT_n} \otimes \Phi \rightarrow \overline{TT_n} = K_n$. Therefore $K_n$ is the core of $G_n$.

Claim (#9). Let $\Phi$ be a fully looped digraph. Then the following are equivalent

(a) $\Phi$ has $c$ connected components.
(b) $H_2 = \overline{TT_n} \otimes \Phi$ has $c$ connected components.
(c) $H_n = \overline{TT_n} \otimes \Phi$ has $c$ connected components for all $n \geq 2$.

Proof. First of all, (c) $\rightarrow$ (b) is trivial. For the converse (b) $\rightarrow$ (c), take two vertexes $(a, r)$ and $(b, s)$ of $H_n$ with $a < b$. If $(0, r)$ and $(1, s)$ are connected in $H_2$, then $(a, r)$ and $(a + 1, s)$ are connected by a path $\gamma$ in the induced subgraph of $A_0 \cup A_{a+1}^1$ in $H_n$. By augmenting $\gamma$ path, if necessary, with $(a + 1, s) - (a + 2, s) - \cdots - (b, s)$ (these vertexes all belong to $R_s$, so such path exists), one finds a path from $(a, r)$ to $(b, s)$. On the other hand, suppose $(0, r)$ and $(1, s)$ are not connected in $H_2$. Then $(0, s)$ and $(1, r)$ are not connected in $H_2$ either. Now suppose $(a, r)$ and $(b, s)$ are connected in $H_n$ by some path

$$(a, r) = (a_1, r_1) - (a_2, r_2) - \cdots - (a_n, r_n) = (b, s)$$

By going into the induced subgraph of $A_b \cup A_{a_n-1}$, due to existence of the edge $(a_{n-1}, r_{n-1}) - (b, s)$, we conclude that $(0, s)$, $(1, s)$, $(0, r_{n-1})$ and $(1, r_{n-1})$ all belong to the same connected component of $H_2$. Repeating the same process for each edge, inductively we will find that $(0, r)$ and $(1, s)$ will belong to the same connected component. This is a contradiction, which will result in the proof of (b) $\rightarrow$ (c).

Before we start, recall that connectedness properties of a digraph $D$ are defined as the connectedness properties of $\overline{D}$.

We first prove that $\Phi$ is connected if and only if $H_2$ is connected. Let the labeling of $\Phi$ be same way we have done before.

Suppose $H_2$ is connected. Take vertexes $i, j$ in $\Phi$. One can find a path from $(0, i)$ to $(1, j)$ in $H_2$. Say this path
is a finite sequence of the form \((\epsilon_k, m_k)\) with \(\epsilon_k = 0\) if \(k = \text{odd}\) and \(\epsilon_k = 1\) for \(k = \text{odd}\). But then simply consider the sequence \((m_k)\) as a sequence of vertexes of \(\Phi\). This is a path connecting \(i\) to \(j\) in \(\overline{\Phi}\).

Conversely suppose \(\Phi\) is connected and take two vertexes \((0, i)\) and \((0, j)\) in \(H_2\). Let \(i = i_1 - i_2 - \cdots - i_{m+1} = j\) be a path connecting \(i\) to \(j\) in \(\overline{\Phi}\). We now make a path between \((0, i)\) and \((0, j)\) in \(H_2\). Define

\[
\gamma_k = \begin{cases} 
(0, i_k) - (1, i_{k+1}) - (0, i_{k+1}) & \text{if } i_k \rightarrow i_{k+1} \\
(0, i_k) - (1, i_{k+1}) - (0, i_{k+1}) & \text{if } i_k \leftarrow i_{k+1}
\end{cases}
\]

where by “if \(i_k \rightarrow i_{k+1}\)” we mean if the direction of the edge \(i_k - i_{k+1}\) in \(\overline{\Phi}\) is in fact \(i_k \rightarrow i_{k+1}\) in \(\Phi\). Clearly \(\gamma_k\) is a path from \((0, i_k)\) to \((0, i_{k+1})\) in \(H_2\). This brings us to the purpose of strong connectivity. Re-notice that since \(\Phi\) is strongly connected, for each \(a, s\) there exists an edge between \((a, s)\) and \((a, r)\) for some \(r\) is a singleton for every \(a\) in \(\Phi\). Then \(\gamma_k = (0, i_k) - (1, i_{k+1}) - (0, i_{k+1})\) is a path connecting \(i\) to \(j\) in \(\overline{\Phi}\). Finding paths in the other cases, e.g. for \((0, i)\) and \((1, j)\), is now immediate from this construction.

Finally suppose \(\Phi = \Phi^1 \cup \cdots \cup \Phi^c\) with each \(\Phi^i\) being a connected component of \(\Phi\). Then \(H_2^i = \overline{T} T_2 \otimes \overline{\Phi^i}\) is connected, and by construction of tensor product, it is obvious that \(H_2 = H_2^1 \cup \cdots \cup H_2^c\) with \(H_2^i\) being connected components of \(H_2\). The converse is also trivial.

This brings us to the purpose of strong connectivity. Recall that since strongly connectivity of \(\Phi\) is a consequence of (R) and (C) together, we are technically using the (R) condition too. But, for the next few claims, it suffices to assume \(\Phi\) is strongly connected, but not necessarily regular.

**Claim (**#10**). If \(\Phi\) is fully looped and strongly connected, then the only maximum independent sets of \(G_n\) are the axes of \(G_n\).**

**Proof.** The maximum independent sets of \(G_n\) are of size \(a(G_n) = v\). Let \(S\) be such set, then by necessity \(S \cap R_r\) is a singleton for every \(r\). Therefore

\[
S = \{(a_0, 0), (a_1, 1), \ldots, (a_{v-1}, v-1)\}
\]

for some \(a_0, \ldots, a_{v-1} \in \mathbb{Z}_n\). Let \(a = \min\{a_0, \ldots, a_{v-1}\}\) and define \(s \in \mathbb{Z}_n\) such that \((a, s) \in S\) (i.e. \(a = a_s\)). Suppose there exists an arc \(s \rightarrow r\) in \(\Phi\). Then for all \(b > a\) there exists an edge between \((a, s)\) and \((b, r)\). By definition of independent set, \((b, r) \notin S\) for all \(b > a\). Then by minimality of \(a\), and since \(S \cap R_r\) is a singleton, one must necessarily have \(a_r = a\) too, i.e. \((a, r) \in S\). Now note that since \(\Phi\) is strongly connected, for each \(r \in \Phi\), with \(r \neq s\), there is a directed path \(s \rightarrow r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_m = r\). A simple induction argument then shows that \(a_r = a\) for all \(r \neq s\). Therefore \(S = A_a\).

**Claim (**#11**). If \(\Phi\) is fully looped and strongly connected, then \(G_n\) is uniquely \(n\)-colorable. If \(\mathcal{C}_n\) is this unique \(n\)-coloring, the color classes of \(\mathcal{C}_n\) are exactly the axes of \(G_n\).**

**Proof.** We have already proved that \(\chi(G_n) = n\). Since \(G_n\) has \(nv\) and \(a(G_n) = v\), if \(c\) is any proper \(n\)-coloring, each color class of \(c\) is necessarily of size \(v\). Since the axes of \(G_n\) are the only possible independent sets of size \(v\), \(G_n\) is uniquely colorable.

As mentioned in the main body, it is possible to discard the (C) condition by changing the (R,E) condition to: \(\Phi\) is \(d\)-regular and if \(\Phi^i\) is a connected component of \(\Phi\), then \(|\Phi^i| = \text{even}\). If one insists on doing this, then all the results we have shown in this paper should be applied to connected components first. Nothing of any meaningful significance is gained by doing so.

Finally, we will also add the (R) condition. Suppose \(\Phi\) is \(d\)-regular digraph (no other condition is necessary). Then one immediately finds that

\[
\text{(\#12) } G_n \text{ is a regular graph of degree } (n-1)d.
\]

Finally, we take the combination of CRL gives us:

**Claim (**#13**). If \(\Phi\) is fully looped, connected and \(d\)-\regular, then each maximum independent set \(S\) of \(G_n\) is dominating. Furthermore, if \(x \notin S\), then \(x\) connects to \(S\) with exactly \(d\) edges.**

**Proof.** Fix some axis \(A_n\) and some vertex \((b, r)\) with \(a \neq b\). Let \(H_{ab}\) denote the induced subgraph of \(A_n \cup A_b\). We know that \(H_{ab}\) is bipartite, with partitions being \(A_n\) and \(A_b\), and regular of degree \(d\). So \((b, r)\) connects to \(A_n\) with exactly \(d\)-edges. So in particular, \(A_n\) is dominating.

This concludes the theorem.

---

**Proposition VI.2**

Consider \(G = (V, E)\) be a loopless multi-graph with \(V = \{1, 2, \ldots, N\}\) already ordered. Let \(I = \{(i, j) \in V^2 \mid i < j\}\) and also let \(W = \{(w_{ij})\}\) be the adjacency matrix of \(G\). Then the graph monomial of \(G\) with these conventions is

\[
\overline{P}_G(z_1, \ldots, z_N) = \prod_{(i,j) \in I} (z_i - z_j)^{w_{ij}}
\]

This is a multiplication of \(|E|\) linear factors of the form \((z_i - z_j)\). More precisely, given any edge \(e = ij \in E\) (there are actually \(w_{ij}\) copies of \(ij \in E\) since \(G\) is multiple), choosing either \(z_i\) or \(-z_j\) is equivalent to putting the direction on \(e\). Doing the same for all edges, i.e. making \(|E|\) choices, will give an orientation \(\omega\) in \(G\). Let \(\omega_0\) be the orientation induced by natural ordering of the labels, i.e. \(i \rightarrow j\) if \(i < j\) and \(ij \in E\). We associate the orientation \(\omega_0\) to when out of every linear factor \(z_i - z_j\) we choose \(z_i\) (i.e. the variable with smaller index). Define \(S(\omega, \omega_0) \subseteq E\) as the set of all edges \(e\) such that \(\omega(e) \neq \omega_0(e)\). Define \(\text{sgn}(\omega) = (-1)^{|S(\omega, \omega_0)|}\). Then note that if \(\omega\) involves the choice of \(-z_j\) out of \(s\) linear forms, then \(|S(\omega, \omega_0)| = s\). Given an orientation \(\omega\), let \(G_\omega\) be the corresponding digraph. Make the sequence

\[
\Delta^+(\omega) = (\delta_1^+, \delta_2^+, \ldots, \delta_N^+)
\]
where $\delta_i$ is the number of arcs going out $i$ in $G$. Note that $\delta_i$ is exactly the number of times variable $z_i$ has been chosen overall. Make the convention

$$z^{\Delta^+(\omega)} := \frac{\delta_1}{z_1} \cdot \frac{\delta_2}{z_2} \cdots \frac{\delta_N}{z_N}$$

Combining all of this, we find that

$$P_G(z_1, \ldots, z_N) = \sum_{\omega} \text{sgn}(\omega) z^{\Delta^+(\omega)}$$

where the sum is done over all orientations of $G$. Given any orientation $\omega$, the sequence $\Delta^+(\omega)$ can be reordered into a partition of $|E|$. We call this partition, $\lambda(\omega)$, the orientation type of $\omega$. Let $\Omega_\lambda$ be the set of all orientations of type $\lambda$, and let $T_G$ be the set of all orientation types of $G$. Clearly if $\omega \in \Omega_\lambda$, then by definition of orientation type,

$$\mathcal{F}(z^{\Delta^+(\omega)}) = m_\lambda(z_1, \ldots, z_N)$$

with $m_\lambda$ the free bosonic state corresponding to $\lambda$. Therefore, we immediately obtain

$$P_G = \sum_{\lambda \in T_G} c_\lambda m_\lambda, \quad c_\lambda = \sum_{\omega \in \Omega_\lambda} \text{sgn}(\omega)$$



**Theorem VI.3**

Let $\tilde{\omega}$ be an orientation of $G_n = T T_n \otimes \Phi = (V_n, E_n)$ that makes it into the digraph $T T_n \otimes \Phi$. In other words, this construction one can already read this orientation $\omega$ before applying the forgetful functor. We want to first show that $\tilde{\omega}$ is of type $\Lambda(n, v, d)$. Using the notation we used in the proof of thm. VI.1, if there is an edge between $(a, r)$ and $(b, s)$ with $a < b$, then $(a, r) \rightarrow (b, s)$ according to $\omega$. As a result, $\delta^+(0, r) = \delta_0 = (n-1)d$ for all $r \in \mathbb{Z}_r$. In other words, any edge with one end in $A_0$ will go out of $A_0$. Furthermore, since by deleting the zeroth axis the resulting graph is $G_{n-1}$, by induction on $n$ one finds that the type of $\tilde{\omega}$ is

$$\left( \delta_n \cdots \delta_0 \delta_{n-1} \cdots \delta_{n-1} \cdots \delta_2 \cdots \delta_2 \right)$$

which is exactly $\Lambda(n, v, d)$.

Let us find all orientations of type $\Lambda(n, v, d)$. Suppose $\omega$ is of type $\Lambda(n, v, d)$. Let us define $\omega_n = \omega$ for further convenience. Note that the set $X_\theta(\omega) \subset V$ which consists of all vertexes of out-degree $\delta_n = (n-1)d$ is an independent set of size $v$. At the same time, the set $X_{n-1}(\omega) \subset V_n$ of all vertexes of $V_n$ with out-degree zero (in $G_n$ with orientation $\omega_n$) is an independent set of size $v$ too. In other words, $\omega$ decomposes $V_n$ as

$$V_n = \bigcup_{a \in \mathbb{Z}_n} X_a(\omega)$$

with each $X_a$ a color class of $uG_n$. Note that $X_0(\omega) = A_0$. For any $\omega \in \Lambda(n, v, d)$, let $\sigma(a)$ be such that $X_a(\omega) = A_{\sigma(a)}$. This $\sigma$ is a permutation of elements of $Z_n$. Let $H_{a,b}$ be the induced subgraph of $A_a \cup A_b$ in $G_n$. If $a < b$, then in $\omega$ all edges in subgraph $H_{a,b}$ will go out of $A_{\sigma(a)}$ into $A_{\sigma(b)}$. Therefore there are exactly $n!$ such orientations.

Let $\omega \in \Lambda(n, v, d)$. By what we just showed, over each $H_{a,b}$, either $\omega$ and $\tilde{\omega}$ agree on orientation of every edge of $H_{a,b}$, or they disagree on every last one of them. Let $S(\tilde{\omega}, \omega)$ be the number of edges of $G_n$ over which $\omega, \tilde{\omega}$ disagree. By what we just showed, and since $H_{a,b} \simeq G_2$ has $vd$ edges, $|S(\tilde{\omega}, \omega)|$ is a multiple of $vd$. But $vd = \text{even}$, so $\omega$ and $\tilde{\omega}$ have the same sign. (This is exactly why postulate (E) is put in place). Since by construction $\text{sgn}(\tilde{\omega}) = 1$, we find that

$$c_{\Lambda(n, v, d)} = \sum_{\omega \in \Lambda(n, v, d)} \text{sgn}(\omega) = n! \neq 0$$

In particular, $P_G$ does not identically vanish.

**Proposition VI.4**

Given a partition $\lambda$, define $S_k(\lambda) = \sum_{i=1}^k \lambda_i$, i.e. the $k$th partial sum. As usual, we prove by induction. First off, $\Lambda(2, v, d)$ is trivially a maximal in $G_2$. So we need to take care of induction step.

Suppose now that $\Lambda(n-1, v, d)$ is a maximal orientation type of $G_{n-1}$, but there exists an orientation $\omega$ of $G_n$ with a type $\lambda \neq \Lambda(n, v, d)$ that dominates $\Lambda(n, v, d)$; i.e. assume $S_k(\lambda) \geq S_k(\Lambda(n, v, d))$ for all $k$. Then, in particular, for $k = 1$ we have $\lambda_1 \geq \delta_n = (n-1)d$. The only way this can happen is if $\lambda_1 = \delta_n$. Similarly one shows that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \delta_n$. Define $\lambda = (\lambda_{n+1}, \lambda_{n+2}, \cdots)$, i.e. remove the first $v$ parts of $\lambda$. For $k > v$, the inequality $S_k(\lambda) \geq S_k(\Lambda(n, v, d))$ implies $\Lambda(n-1, v, d) \leq \lambda'$. At the same time, the set of all vertexes that have out-degree $\delta_n$ in $\omega$ is an independent set of size $v$. Therefore this set is also a color class of the unique coloring. Denote this set by $X$. The induced subgraph of $V - X$ is isomorphic to $G_{n-1}$. Let $\omega'$ be the orientation on $G_{n-1}$ induced by $\omega$. The type of $\omega'$ is exactly $\lambda'$. Note that if $\lambda \neq \Lambda(n, v, d)$, then $\lambda' \neq \Lambda(n-1, v, d)$. Therefore we have found an orientation type in $G_{n-1}$ which dominates $\Lambda(n, v, d)$. This contradicts maximality of $\Lambda(n-1, v, d)$ in $G_{n-1}$. 

\qed
**Theorem VI.5**

**Set-Partitions:** For this proof we will need the concept of a set-partition and quite a few terminologies related to it. Let \( X \) be a finite set. A set-partition \( \pi \) of \( X \) is the determined by the following data:

1. \( \pi \) is a set of subsets of \( X \); i.e. if \( U \in \pi \), then \( B \supseteq U \).

   An element \( B \in \pi \) is called a block of \( \pi \).

2. If \( B, B' \) are two blocks, then \( B \cap B' = \emptyset \).

3. If \( \pi = \{B_1, \ldots, B_i\} \), then \( \bigcup_{j=1}^i B_j = X \).

without loss of generality, suppose the blocks are labeled such that \( |B_1| \geq |B_2| \geq \cdots \geq |B_t| \). If \( \lambda_i = |B_i| \), then \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \) is a partition of \( |X| \). One says \( \pi \) is of shape \( \lambda \). We denote the set of all set-partitions of shape \( \lambda \) by \( \mathcal{P}_\lambda \). With \( \lambda_1 = b \), in the multiplicity picture, one writes \( \lambda = (1^{m_1}2^{m_2} \cdots b^{m_b}) \). Then the number of set-partitions of shape \( \lambda \) is

\[
C_{\lambda} := |\mathcal{P}_\lambda| = \frac{N!}{\prod_{r=1}^b m_r! (r!)^{m_r}}
\]

**Tiers of a Set-Partition:** For purposes of fusion, it is convenient to do one further refinement. Let \( X \) be a set and \( \pi \) a set-partition of \( X \). Let \( f : \pi \to \mathbb{N} \) be the cardinality function, i.e. \( B \mapsto |B| \). We call the \( f^{-1}(r) \subset \pi \) the \( r \)th tier of \( \pi \), denoted by \( \tau(r) \). Given a block \( B \in \pi \), we say the tier number of \( B \) is \( r \) if \( B \in \tau(r) \).

**Internally Ordered Set-Partitions:** We call the collection of a set-partition \( \pi \), together with bijections \( L^{(r)} : \tau^{(r)}(\pi) \to \{1, 2, \ldots, \nu_r\} \) and internally ordered set-partition and we denote it by \( \mathcal{P}_\lambda \). If \( \nu_r = 0 \), the datum is \( L^{(r)} \) is not necessary. The maps \( L^{(r)} \) put a total order on the tiers of \( \pi \). We will never write these \( L^{(r)} \) maps explicitly, however, for an internally ordered \( \mathcal{P}_\lambda \), when we write

\[
\tau^{(r)} = \{B_1^{(r)}, \ldots, B_{\nu_r}^{(r)}\}
\]

the index \( j \) of \( B_j^{(r)} \) should be understood as the total order put on \( \tau^{(r)} \). We denote the set of internally ordered set-partitions of shape \( \lambda \) by \( \mathcal{P}_\lambda \). There are \( \epsilon_\lambda := |\mathcal{P}_\lambda| = \frac{N!}{\prod_{r=1}^b (r!)^{m_r}} \) internally ordered set-partitions of shape \( \lambda \).

**Conventions:** We reserve a few symbols before we start the proof. Throughout \( \kappa = (1^{\nu_1}2^{\nu_2} \cdots g^{\nu_g}) = (\kappa_1, \kappa_2, \ldots, \kappa_s) \); i.e. these symbols are reserved:

- \( \kappa \) is a partition of the number \( N \), the number of variables/particles/vertexes.
- \( \kappa_i \) is ith part of \( \kappa \).
- \( s \) is the length of \( \kappa \) is \( s \); meaning \( \kappa_s \neq 0 \).
- \( g = \kappa_1 \); i.e. \( g \) is the largest part of \( \kappa \).
- Given any number \( 1 \leq r \leq g \), the multiplicity of \( r \) in \( \kappa \) is \( \nu_r \geq 0 \). Therefore \( s = \sum_{r=1}^g \nu_r \) and \( N = \sum_{r=1}^g r \nu_r \).

**k-fusion:** The first real step in our journey down the road of this proof, is to translate the definition \( \kappa \)-fusion into algebraic language. Let \( S = \{z_1, \ldots, z_N\} \) denote our set of variables. Let \( \mathfrak{p} = \mathcal{P}_\kappa(S) \). Then the \( r \)th tier is

\[
\tau^{(r)} = \{B_1^{(r)}, B_2^{(r)}, \ldots, B_{\nu_r}^{(r)}\}
\]

Also recall that for \( 1 \leq r \leq g \) we defined

\[
Z^{(r)} = \{Z_1^{(r)}, Z_2^{(r)}, \ldots, Z_{\nu_r}^{(r)}\}
\]

Associated to an internally ordered set-partition \( \mathfrak{p} \), we define a \( \mathbb{C} \)-algebra homomorphism \( \phi_\mathfrak{p} : \mathbb{C}[S] \to \mathbb{C}[Z^{(1)}, \ldots, Z^{(g)}] \) by sending variables in block \( B_j^{(r)} \) to the variable \( Z_j^{(r)} \) (for all \( r \) and \( j \)). This \( \phi_\kappa \) fuses the variables of a polynomial \( f \in \mathbb{C}[S] \) according to the ordered set-partition \( \mathfrak{p} \).

**Operator \( D_\kappa \):** Given any permutation of \( N \), i.e. \( \sigma \in \mathfrak{S}_N \), by abuse of notation, let us denote the isomorphism \( \sigma : \mathbb{C}[S] \to \mathbb{C}[S] \) via \( f(z_1, \ldots, z_N) \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(N)}) \). Given any \( \mathbb{C} \)-algebra homomorphism \( \psi : \mathbb{C}[S] \to R \), with \( R \) some \( \mathbb{C} \)-algebra, we define \( \sigma^* \psi = \psi \circ \sigma \) (pullback).

Now define \( M_\kappa = \{\phi_\mathfrak{p} \mid \mathfrak{p} \in \mathcal{P}_\kappa \} \). Note that if \( \phi \in M_\kappa \), then \( \sigma^* \phi \in M_\kappa \) and moreover \( \sigma^* : M_\kappa \to M_\kappa \) is a bijection. Using all of that, now define the \( \mathbb{C} \)-linear map \( D_\kappa : \mathbb{C}[S] \to \mathbb{C}[Z^{(1)}, \ldots, Z^{(g)}] \) as

\[
D_\kappa = e_\kappa^{-1} \sum_{\phi \in M_\kappa} \phi
\]

i.e. as the “average” of \( M_\kappa \). Note that for any \( \sigma \in \mathfrak{S}_N \), one has \( \sigma^* D_\kappa = D_\kappa \). Let us make a two remarks about \( D_\kappa \):

1. If \( P \) is a symmetric polynomial in variables \( S \), then for any two \( \phi, \phi' \in M_\kappa \) one has \( \phi(P) = \phi'(P) = P|_\kappa \).

2. Due to \( \sigma^* : M_\kappa \to M_\kappa \) being a bijection, one finds that \( \sigma^* D_\kappa = D_\kappa \) for any \( \sigma \in \mathfrak{S}_N \).

Now combining the two, and recalling that the normalized symmetrization was \( \mathcal{F} = N^{-1} \sum_{\sigma} \sigma^* D_\kappa = D_\kappa \), we find

\[
D_\kappa \circ \mathcal{F} = N^{-1} \sum_{\sigma} \sigma^* D_\kappa = D_\kappa
\]

So in particular, if \( \check{P} \in \mathbb{C}[S] \), then

\[
\check{P} = \mathcal{F}[\check{P}] \implies \check{P}|_\kappa = D_\kappa[\check{P}]
\]

This is specially nice, since we do not have to deal with the pesky symmetrization operator anymore. At the same time, this brings out right outside of the door of graph theory and SGM construction. All we need to do is to understand what \( \phi_\mathfrak{p} \) does to a graph monomial \( P_G \).
Effect of $\phi_{\pi}$ on graph monomials: Let $G = (V, E)$ be a multigraph of order $N$, which we have already labeled with $S = \{z_1, z_2, \ldots, z_N\}$. Let $\tilde{P}_G$ be the graph monomial with respect to this order. Let $\pi$ be an internally ordered partition as before. Note that if some block $B_j^{(r)}$ contains a pair of adjacent vertexes, then $\phi_{\pi} \tilde{P}_G = 0$. However, if there are no such blocks in $\pi$, then each block is an independent set. Therefore $\pi$ induces a $\kappa$-coloring on $G$. This induced $\kappa$-coloring, $\gamma_{\pi}$, explicitly defined as follows: Define the color set as

$$
\text{Col} = \{(r, j) \mid 1 \leq r \leq g, 1 \leq j \leq \nu_r\}
$$

with the understanding that if $\nu_r = 0$ for some $r$, then there is no element of the form $(r, \bullet)$ in Col. This set is obviously of size $N$. We define the coloring $\gamma_{\pi}$ as a mapping $\gamma_{\pi} : V \rightarrow \text{Col}$. Take vertex $z$ of $G$, if it belongs to the block $B_j^{(r)}$ of $\pi$, then $\gamma_{\pi}(z) = (r, j)$.

The $\kappa$-coloring $\gamma_{\pi}$, in turn, leads to the construction of a colored multi-graph $H_{\pi}$ which is already labeled with elements $Z(1) \cup \cdots \cup Z(g)$. The construction of $H_{\pi}$ is as follows:

- The vertex set is simply the set-partition $\pi$ (we forget the order of $\pi$). In other words, a vertex of $H_{\pi}$ is a block of $\pi$.
- The vertex $B_j^{(r)}$ is labeled by $Z_j^{(r)}$.
- The coloring $\gamma_{\pi}$ is found by assigning to each block of $\pi$, its tier number.
- Finally for edges:

$$
\# \text{edges between blocks } B \text{ and } B' \text{ in } H_{\pi} = \sum_{z \in B} \sum_{z' \in B'} \text{ edges between vertexes } z \text{ and } z' \text{ in } G
$$

Once we forget the extra labels $Z_j^{(r)}$, which the construction will automatically do later on, $H_{\pi}$ becomes the compression $G \downarrow \gamma_{\pi}$. Now, by construction of $H_{\pi}$, one finds that

$$
(\phi_{\pi} \tilde{P}_G)(Z(1), \ldots, Z(g)) = \tilde{P}_{H_{\pi}}(Z(1), \ldots, Z(g))
$$

where the graph monomial of $H_{\pi}$ is calculated with respect to the labels already on it.

A Step in the Reverse Direction: Again let $G = (V, E)$ be a multigraph of order $N$, with predetermined labels in $S$. Take a $\kappa$-coloring $\gamma : V \rightarrow \{1, 2, \ldots, s\}$ b. For $1 \leq k \leq s$, let $[k] = \gamma^{-1}(k)$ be the color class for color $k$. Then the set

$$
\pi_{\gamma} = \{[k] \mid 1 \leq k \leq s\}
$$

is clearly an element of $p_{\kappa}$. We can turn $\pi_{\gamma}$ into an internally ordered set-partition in $\prod_{r=1}^g (\nu_r)!$ different ways. If $\pi_{\gamma} = \pi_{\gamma'}$, then some $\kappa$-coloring $\gamma$ together with an internal order, then we say $\pi$ is a “good” element of $p_{\kappa}$. Define

$$
M_{\kappa}^{\text{good}} = \{\phi_{\pi} \mid \pi \text{ is a good element of } p_{\kappa}\}
$$

We have already seen that if $\phi_{\pi} \notin M_{\kappa}^{\text{good}}$, then $\phi_{\pi}(\tilde{P}_G) = 0$. So it enough to work with $M_{\kappa}^{\text{good}}$ only. Let us understand $M_{\kappa}^{\text{good}}$ a bit better:

1. Recall that two $s$-colorings $\gamma_1, \gamma_2$ said to be equivalent, $\gamma \sim \gamma'$ if $\gamma = \gamma \circ \sigma$ for some $\sigma \in S_\kappa$. If that happens, then the sets $\pi_{\gamma}$ and $\pi_{\gamma'}$ are exactly the same. Therefore if $\gamma \in C_{\kappa}$ we denote its equivalence class by $\gamma \in C_{\kappa}^\pi \sim \gamma'$; then the construction of $M_{\kappa}^{\text{good}}$ really depends on $\pi$ and not $\gamma$. In fact if $\eta, \eta'$ are not equivalent, then they will lead to distinct set-partitions $\pi_{\eta}$ and $\pi_{\eta'}$

2. Now let $\eta \in C_{\kappa}^\pi$, which by abuse of language, we still call a “$\kappa$-coloring” and $\pi_{\eta}$ the corresponding set-partition. Construct $\pi_{\eta} = \pi_{\eta}^{\text{good}}$ with putting some internal order on $\pi_{\eta}$. For any $1 \leq r \leq g$ and $\sigma(r) \in \eta_{\nu_r}$ and any $C$-algebra homomorphism $\psi : R \rightarrow \mathbb{C}(Z^{(r)})$, define $\sigma(r)^{\psi} = \sigma(r)^{\psi}(\text{pushforward})$. Now given any $\pi : \pi_{\eta}$, which is $\pi_{\eta}$ together with some internal ordering, there exists $\sigma(r) \in \eta_{\nu_r}$ for each $1 \leq r \leq g$, such that

$$
\phi_{\pi} = \sigma^{(1)} \otimes \sigma^{(2)} \otimes \cdots \otimes \sigma^{(g)} \phi_{\pi_{\eta}}
$$

This is pretty much the definition of internal ordering. In the above, we are using the isomorphism $\mathbb{C}(Z(1), \ldots, Z(g)) \simeq \mathbb{C}(Z(1)) \otimes \cdots \otimes \mathbb{C}(Z(g))$.

Finishing Touch: Note that $c_{\kappa} = c_{\kappa} \prod_{r=1}^g \nu_r!$

By what we have shown so far,

$$
\tilde{P}_G|_{\kappa} = \frac{c_{\kappa}}{\prod_{r=1}^g \nu_r!} \sum_{\phi \in M_{\kappa}^{\text{good}}} \phi \tilde{P}_G(Z(1), \ldots, Z(g)) = \frac{c_{\kappa}}{\prod_{r=1}^g \nu_r!} \sum_{\eta \in C_{\kappa}^\pi} \prod_{r=1}^g \frac{1}{\nu_r} \sum_{\sigma(r) \in \eta_{\nu_r}} \bigotimes_{r=1}^g \sigma^{(r)}(\tilde{P}_{H_{\eta_{\nu_r}}}^{\eta_{\nu_r}}(Z(1), \ldots, Z(g)))
$$

$$
= \frac{c_{\kappa}}{\prod_{r=1}^g \nu_r!} \sum_{\eta \in C_{\kappa}^\pi} \tilde{P}_{\eta_{\nu_r}}^{\eta_{\nu_r}}(Z(1), \ldots, Z(g))
$$

Where the normalized symmetrization operator $\mathcal{S}^{(r)}$ only acts on the variables $Z^{(r)}$. Now note that, due this symmetrizations, we no longer need to be careful about how to label the vertexes of the colored graph $H_{\eta_{\nu_r}}$ which have the same color. In other words, the vertexes of rth color (rth tier) can be labeled by $Z^{(r)}$ in any fashion one desires. This allows us to get rid of labeling of $H_{\eta_{\nu_r}}$ completely. This forgetful process turns the data in $H_{\eta_{\nu_r}}$ simply to the compression $G \downarrow \eta$ (by $G \downarrow \eta$ we mean $G \downarrow \gamma$ for some $\gamma \in \eta$). Now by definition of chromatic SGM, which we was given in the main body of the paper, one concludes that:

$$
\tilde{P}_G|_{\kappa}(Z(1), \ldots, Z(g)) = \frac{c_{\kappa}}{\prod_{r=1}^g \nu_r!} \sum_{\eta \in C_{\kappa}^\pi} \tilde{P}_{\eta_{\nu_r}}^{\eta_{\nu_r}}(Z(1), \ldots, Z(g))
$$
Corollary VI.6

Let $G_n = \overline{TT_n} \otimes \Phi = (V_n, E_n)$ with $\Phi$ a $(v, d)$-CRLE digraph. Also let $P_n$ denote the SGM of $G_n$. To calculate $P_n|_v$ we need to find proper colorings of type $(v^1(n-1)v^v)$. This is a $s = (n - 1)v + 1$ coloring problem. Up to equivalence of colorings, finding a $P_n|_v$ on a digraph $G_n$ is $G_n$-equivalent to finding an independent set of size $v$, which we denote by $F_n$ (recall that $F_n$ is an equivalence class of $\gamma$). There are exactly $n$ such independent sets, which are color classes of unique coloring $u_{G_n}$.

Now the induced subgraph of $V_n - S_\tau$ in $G_n$ is $G_n-1$. Moreover, we know that $S_\tau$ is a dominating set of $G_n$ and every vertex in $x \in G_n-1$ connects to $S_\tau$ with $d$ edges. So upon compression $G_n \downarrow \gamma$ one ends up with a vertex set $V' = V_{n-1} \cup \{x\}$, an edge set $E' = E_{n-1} \cup E_x$, and a coloring $\gamma'$, where

- $x$ is the so called “fat” vertex; This is the vertex $S_\tau$ compresses to.
- $(V_{n-1}, E_{n-1})$ is the graph $G_n-1$, which compression does not touch.
- For every vertex $x \in G_n-1$, the edge subset $E_x$ contains $d$ parallel edges connecting $x$ to $X$. Those are the only edges in $E_x$.
- The coloring $\gamma'$ colors $x$ red and all other vertexes blue.

The normalized chromatic SGM of $G_n \downarrow \gamma$ is now:

$$\hat{P}_{G_n \downarrow \gamma}(z; z_1, \ldots, z_{(n-1)v}) =$$

$$\mathcal{T} \left[ \prod_{i=1}^{(n-1)v} (Z - z_i)^d \hat{P}_{G_{n-1}}(z_1, \ldots, z_{(n-1)v}) \right] =$$

$$\prod_{i=1}^{(n-1)v} (Z - z_i)^d \hat{P}_{G_{n-1}}(z_1, \ldots, z_{(n-1)v})$$

where the symmetrization is over the usual variables $z_1, \ldots, z_{(n-1)v}$. Now that $c_n^{-1} = v!/(v - (n-1)v)/((n-1)v)!$. Since this is completely independent from $\gamma$, and since there are $n$ such colorings (equivalence classes), we conclude that (in the unnormalized version)

$$P_n|_v(Z; z_1, \ldots, z_{(n-1)v}) =$$

$$nv! \prod_{i=1}^{(n-1)v} (Z - z_i)^d P_{n-1}(z_1, \ldots, z_{(n-1)v})$$

Meaning the aggregation sequence $SGM(\overline{TT} \otimes \Phi)$ satisfies a $(v, d)$-clustering property. □

Claim. $R$ is either zero or of local degree $\delta$.

Proof. Assume $R \neq 0$, i.e. $P \neq Q$. By going into binary dual picture, $P^i = P^i - Q^i \neq 0$. Since both $P^i, Q^i$ are homogeneous of degree $\delta$, and $R^i \neq 0$, the degree of $R^i$ is also $\delta$. Therefore, the local degree of $R$ is also $\delta$.

Let us give a few definitions before we proceed. Let the notation $J^{(\alpha)}$ stand for a (specialized) Jack polynomial with partition $\lambda$ and parameter $\alpha$ [28]. For a reference on Jack polynomials see [29].

Claim. Let $f = J^{(\alpha)}$ be over $N$ variables, of degree $M$ and local degree $\delta$. If $\lambda = (\lambda_1, \lambda_2, \ldots)$, under these conditions, $\lambda$ is a partition of $M$, with $\ell(\lambda) \leq N$ and $\lambda_1 = \delta$.

Proof. A Jack $J^{(\alpha)}$ is homogeneous, symmetric and has $\lambda$ as the predominant root partition (see [29]). So if $f$ is over $N$ variables and of degree $M$, then $\lambda$ is a partition of $M$ of length at most $N$. Furthermore, by predominance, the local degree of $f$ is the local degree of $\mu \lambda$. Therefore if local degree of $f$ is $\delta$, then $\lambda_1 = \delta$.

Let $F_N(k)$ denote the space of all symmetric polynomials $f$ in $N$ variables such that $f|_{k+1} = 0$. Also suppose $k, r$ are such that $k + 1$ and $r - 1$ are relatively prime and let $\alpha(k, r) = - (k + 1)/(r - 1)$. For those $k, r$, define the space $I^N(k, r)$ as the $C$-span of specialized Jack polynomials $J^{(\alpha(k, r))}$ over $N$ variables in which $\lambda$ is a $(k, r, N)$-admissible partition. The main tool we now need is Theorem 4.2 of [15]. This theorem states that $F_N(k) = I^N(k, 2)$.

Claim. Let $P, Q$ be as before. If $P|_v = Q|_v$ and $W_{n,v,d} = 0$, then $P = Q$.

Proof. Suppose not! Then $R = P - Q \neq 0$ but one has $R|_v = 0$. In other words, $R \in F_N(v - 1) = I^N(v - 1, 2)$. So $R$ is a $C$-linear combination of Jacks in $I(v - 1, 2)$, say

$$R = c_1 J^{(\alpha_1)} + \cdots + c_p J^{(\alpha_p)}$$

let $L = \{\lambda_1, \ldots, \lambda_p\}$. Since deg $R = M$ and $R$ is homogeneous, all elements of $L$ are partitions of $M$ which are $(v - 1, 2, N)$-admissible. Moreover, since local degree of $R$ is $\delta$, the largest part of all $\lambda \in L$ are $\leq \delta$ and there exist at least one $\rho = (\rho_1, \rho_2, \ldots)$ in $L$ with $\rho_1 = \delta$. By definition of $W_{n,v,d}$, we conclude that $\rho \in W_{n,v,d}$. But by hypothesis of the theorem, $W_{n,v,d} = 0$ for all $n \geq 2$. This is a contradiction, proving $P = Q$.

With $i = 1, 2$, let $\Pi^{(i)} = (P_2^{(i)}, P_3^{(i)}, \ldots, P_n^{(i)}, \ldots)$ be two sequences of FQH-like polynomials such that

1. $P_n^{(i)}$ is over $nv$ variables and of local degree $(n-1)d$.
2. $\Pi^{(i)}$ has the $(v, d)$-clustering property.

By what we have proved, if $k \geq 2$ and $q \in C$ is such that $P_k^{(i)} = q_k P_k^{(i)}$, then there is $q_k \in C$ for all $k$ such that $P_k^{(i)} = q_k P_k^{(i)}$. At the same time, if $\dim B_{v,d} = 1$,
then one necessarily has $P_2^{(1)} = q_2 P_2^{(2)}$ for some $q_2 \in \mathbb{C}$. So up to constant factors, the sequences $\Pi^{(1)}$ and $\Pi^{(2)}$ are the same.

**Proposition VI.8**

**Case** $(v, 2)$: A partition $\lambda$ in $W_{n,v,2}$ is characterized by the following conditions:

1. $\lambda$ satisfies the $(v - 1, 2)$-generalized Pauli exclusion.
2. $\tilde{m}_\lambda$ lives in a Landau level of size $\delta_n = 2(n - 1)$.
3. $\tilde{m}_\lambda$ has angular momentum $M_n = n(n - 1)v$.
4. $\lambda$ has length at most $nv$ (or $\tilde{m}_\lambda$ is for $nv$ particles).

Let us find the highest angular momentum a partition satisfying (1) and (2) can have. This is achieved by putting $v - 1$ particles in the last orbital, then skip an orbital, then another $v - 1$ bosons in orbital $2(n-2)$, etc. This partition (which is actually $\lambda(v - 1, 2, n)$) has angular momentum $M_{\text{max}} = n(n - 1)(v - 1)$. Since $M_{\text{max}} < M_n$, conditions (1), (2) and (3) cannot be satisfied simultaneously. Therefore $W_{n,v,2} = \emptyset$ for all $n \geq 2$.

**Theorem VI.9 (Hermite’s Reciprocity Theorem)**

See Elliot [26, §131] or Kung-Rota [30, thm. 4.3].

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[28] Depending on the source, the parameter $\alpha$ might be defined differently. In Ref. [15], $J^{(\alpha)}_n$ in $n$ variables is an eigenvector of

$$H_\alpha = \sum_{i=1}^n (z_i \partial_i)^2 + \alpha \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j)$$

while, for example, in the definition [13] (and most physics sources involving Jacks) $J^{(\alpha)}_n$ is defined as an eigenvector of $H_{1/\alpha}$. The two notions are obviously equiv-
alent, but result in a discrepancy of notation in the literature. We are using the definition in Ref. [13].

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