Unbiased variance reduction in randomized experiments

Reza Hosseini, Amir Najmi
Google Inc.
Address: 1600 Amphitheatre Parkway, Mountain View, CA, 94043, USA
email: reza1317@gmail.com, amir@google.com

Abstract
This paper develops a flexible method for decreasing the variance of estimators for complex experiment effect metrics (e.g. ratio metrics) while retaining asymptotic unbiasedness. This method uses the auxiliary information about the experiment units to decrease the variance. The method can incorporate almost any arbitrary predictive model (e.g. linear regression, regularization, neural networks) to adjust the estimators. The adjustment involves some free parameters which can be optimized to achieve the smallest variance reduction given the predictive model performance. Also we approximate the achievable reduction in variance in fairly general settings mathematically. Finally, we use simulations to show the method works.

1 Introduction
Online experiments play a pivotal role in decision making for many technology companies and are widely used across the industry. From a business perspective, decision makers like to have the fastest results, with the smallest sample of impacted users. Therefore recently there has been some effort across the industry to develop experiment effect estimation methods which reduce the variance e.g. Deng et al. (2013).

Most variance reduction methods come with the price of introducing bias (which persists even asymptotically) or making restrictive modeling assumptions. One exception is discussed in Chapter 7 of Imbens and Rubin (2015) among other references: If one defines the experiment effect to be the simple expected difference between treatment (denoted by 1) and control (denoted by zero):

$$\tau = E\{Y(1)\} - E\{Y(0)\},$$

and $X$ are some auxiliary variables which are independent of the assignment procedure (assignment of units to control and treatment), then the estimated parameter for $\tau_0$ from the linear regression equation

$$Y = \alpha + \tau_0 W + X\beta + \varepsilon,$$

where $W = 1$ if the unit is in treatment and zero otherwise, is an asymptotically unbiased estimator of $\tau$ with known asymptotic variance. The proof of this result depends heavily on the squared loss function properties as well as the simplicity of the experiment effect considered as apposed to more complex metrics e.g. the ratio metric: $\tau = E\{Y(1)/Y(0)\}$. However in most cases in the technology industry, the metrics of interest are ratios rather than simple differences. (One reason for that being it is easier to think about growth as a percent increase.) Another limitation of this approach is estimating linear regression coefficients could be difficult when dealing with a large number of weak predictors which would have been possible say using regularization methods (e.g. see Hastie et al. (2009)). Our focus in this work is to develop methods which can remove these limitations.

A notable work regarding the simple difference (Equation 1) is the seminal work of Freedman (2008a,b) which evaluates regression adjustments using Neyman’s nonparametric model (Neyman (1923)). Freedman argues that regression adjustment may introduce bias (although asymptotically unbiased as discussed above); the adjustment may or may not improve precision.
As mentioned, here we are mainly concerned with more complex metrics (beyond simple difference) and our approach is not a regression adjustment, but rather uses a predictions model to adjust the estimators. It is worth noting that, using extensions of linear regression to other settings e.g. generalized linear models would even sacrifice the asymptotic unbiasedness and that is one of the reasons we do not use regression adjustments for the general settings.

A related paper to this work is Deng et al. (2013) suggesting to use a control variable (most of the time pre-treatment data on the same variable) which is uncorrelated with the assignment mechanism to decrease the variance while maintaining unbiasedness for the simple difference metric $\tau = \mathbb{E}\{Y(1)\} - \mathbb{E}\{Y(0)\}$. We know that an unbiased estimate for this effect is equal to $\Delta = \bar{Y}_t - \bar{Y}_c$, where $t$ denotes the treatment arm and $c$ denotes the control arm. Then assume we are given a control (univariate) variable $X$ for which $\mathbb{E}\{X_t\} = \mathbb{E}\{X_c\}$. Then $\Delta_{cv} = (\bar{Y}_t - \theta \bar{X}_t) - (\bar{Y}_c - \theta \bar{X}_c)$ is an unbiased estimator of $\tau$. Moreover if $X$ and $Y$ have non-zero correlation, then $\text{var}\{Y - \theta X\}$ is minimized when $\theta = \text{cov}(Y, X)/\text{var}\{X\}$ in which case the variance shrinks by a multiplicative factor of $(1 - \rho^2)$, where $\rho$ is the correlation between $Y$ and $X$. Of course, here we are not dealing with one variable $Y$ only and rather $Y_t$ and $Y_c$, while we can only pick one $\theta$. Therefore Deng et al. (2013) suggest to pick a common $\theta$ by using all the data (which still works well in practice). They also discuss application of this method to some other more complex metrics for example CTR (click through rate) using the delta method in Appendix B of the paper. Our method is more general in several ways. First we replace the variable $X$ with a predictive model which can take multiple inputs. More importantly, it allows for the experiment effect form to be the much more general case of $\tau_\theta = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\})$ for arbitrary differentiable function $g$ (with domain being the support of the variables involved), in particular for $g = \log$ this will capture the ratio metric since

$$\tau_{\log} = \log(\mathbb{E}\{Y(1)\}) - \log(\mathbb{E}\{Y(0)\}) = \log \left( \mathbb{E}\{Y(1)\}/\mathbb{E}\{Y(0)\} \right).$$

In this case, we also provide an exact equation for picking the parameter $\theta$. Moreover we also discuss further generalizations of the method to more complex metrics for example sum ratio and ratio of mean ratios involving more parameters and derive numerical equations for their optimization.

The paper is organized as follows. Section 2 discusses the various ways the experiment effect can be defined and the connection between various definitions. Section 3 develops the mathematical theory for our method. Section 4 performs a simulation study to confirm that the method retains unbiasedness while decreasing the variance (as much as expected by the theory). Finally Section 5 summarizes the results and discussed further potential applications and extensions.

2 Defining experiment effect

The experiment effect, denoted here by $\tau$ can be defined in many different ways depending on the application. Clearly the estimation method and the confidence interval calculation also depend on this choice and some definitions may have some advantages over others from a theoretical point of view. The most common definition is that of the simple difference:

$$\tau = \mathbb{E}\{Y(1)\} - \mathbb{E}\{Y(0)\},$$

can be estimated using OLS and (asymptotically unbiased) estimators with smaller variance can be obtained by incorporating predictor variables $X$ which are independent of the assignment process $W$ into the regression as discussed in Chapter 7 of Imbens and Rubin (2015).
difference has been widely considered in the literature from the seminal work of [Neyman (1923)] to more recently in [Freedman (2008a,b) and Miratrix et al. (2013)].

Here we introduce some general classes of experiment effects and discuss their interpretation and relationship. We denote the auxiliary information available to us by $X$ and assume $X$ is independent of the assignment mechanism. $X$ is a collection of predictive variables which is used in a prediction model to predict the responses of interest which appear in the experiment effect definition of interest.

One variation we can consider is by taking the expectation of $X$ outside:

$$\tau_X = \mathbb{E}_X \{ \mathbb{E}\{Y(1)|X\}\} - \mathbb{E}_X \{ \mathbb{E}\{Y(0)|X\}\},$$

which is equal to $\tau = \mathbb{E}\{Y(1)\} - \mathbb{E}\{Y(0)\}$ by the Law of Iterated Expectation.

Here we introduce a generalization of the simple experiment effect defined above. For any differentiable increasing function $g$, we consider

$$\tau_g = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\}),$$

$$\tau_{g,X} = \mathbb{E}_X \{ g(\mathbb{E}\{Y(1)|X\}) - g(\mathbb{E}\{Y(0)|X\}) \}.$$  

Note that in general $\tau_{g,X} \neq \tau_g$. In most technology applications, as far as we know, metrics such $\tau_{g,X}$ which depend on auxiliary variables are not considered. However this is a common choice in statistical literature, observational studies or clinical studies. For example in Chapter 8 of [Imbens and Rubin (2015)], the authors mention that in general we can consider $\tau$ to be a function of $Y(1), Y(0), X, W$, i.e. $\tau = \tau(Y(1), Y(0), X, W)$ rather than $\tau = \tau(Y(1), Y(0))$. The reason many authors consider this is the simplifications they get by assuming models. In fact $\tau$ then could become a parameter of a well-studied model and we give an example of that here for generalized linear models.

Let’s assume that the dynamics of the system and the experiment effect could be described by a generalized linear model (GLM) with a given link function $g$:

$$g(\mathbb{E}\{Y|x, w\}) = \alpha + (x - \mu_X)\beta_w + \lambda w;$$

where $\mu_X = \mathbb{E}\{X\}$ and $\beta_w$ might depend on $w$. Then we have

$$\lambda = \tau_{g,X}.$$  

This is because:

$$\tau_{g,X} = \mathbb{E}_X \{ g(\mathbb{E}\{Y(1)|X\}) - g(\mathbb{E}\{Y(0)|X\}) \}$$

$$= \mathbb{E}_X \{ \alpha + (X - \mu_X)\beta_1 + \lambda \times 1 - (\alpha + (X - \mu_X)\beta_0 + \lambda \times 0) \}$$

$$= \mathbb{E}_X \{ (X - \mu_X)(\beta_1 - \beta_0) + \lambda \}$$

$$= \mathbb{E}_X \{ (X - \mu_X)\} (\beta_1 - \beta_0) + \lambda = \lambda.$$  

This implies that if we choose $g$ to be the link function of the GLM, then the GLM coefficient $\lambda$ is equal to the experiment effect $\tau_{g,X}$ and not necessarily $\tau_g$. Therefore one cannot extend the regression adjustment approach (for which the estimated regression coefficient remains an asymptotically unbiased estimator of $\tau$) to the general case if the purpose is to estimate $\tau_g$. Note the arbitrary dependence of $\tau_{g,X}$ on $X$ is often an undesirable property for decision makers. Unlike
regression adjustment, our adjustment method keeps the estimator asymptotically unbiased as shown in the next section.

In the above we discussed the simple difference metric \( \tau \) and its generalization to \( \tau_g = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\}) \). However we can also consider ratio effects e.g.:

\[
\tau' = \mathbb{E}\{Y(1)\}/\mathbb{E}\{Y(0)\}.
\]

Note that we have
\[
g(\tau') = \tau_g, \text{ for } g = \log,
\]
which means we can infer about the ratio metric using an appropriate \( g \).

All the experiment effects above are functions of the mean of treatment and control variables. While considering the arbitrary \( g \) transformation expends the class of estimators significantly, there are other metrics which are not covered and we discuss them here.

One popular metric considered in online experiments is “sum ratio”. Consider an experiment run on two equally sized arms (with same number of potential users) with \( t \) denoting the treatment and \( c \) denoting the control, then the sum ratio is estimated by

\[
\hat{\tau} = \sum_{i \in t} Y_i / \sum_{i \in c} Y_i = (n_t/n_c)\bar{Y}_t/\bar{Y}_c,
\]

where \( n_t \) and \( n_c \) are the number of users with impressions on each arm and \( \bar{Y}_t, \bar{Y}_c \) are the sample averages on each arm. Usually \( Y_i \) is a non-negative variable which can attain zero such as watch time or money spent. Note that not all the potential users appear in the sum as some might not have any impressions of the feature. Of course \( \hat{\tau} \) is an estimator and one needs to define a parameter which \( \hat{\tau} \) is estimating. In order to find a reasonable population parameter \( \tau \) is estimating note that \( \hat{\tau} \) can be written as

\[
\hat{\tau} = \left( (n_t/N)/(n_c/N) \right)\bar{Y}_t/\bar{Y}_c = (\bar{p}_t/\bar{p}_c)\bar{Y}_t/\bar{Y}_c,
\]

where is the (usually unobserved) number of potential users on each arm and \( \bar{p}_t, \bar{p}_c \) are the probability that a user has impression on treatment and control arms respectively.

Defining \( I(W) \) to be an indicator variable to denote if the user on the arm \( W \) has an impression of the feature \( I(W) = 1 \) or not \( I(W) = 0 \), we can define

\[
\tau = \frac{\mathbb{E}\{I(1)\}\mathbb{E}\{Y(1)|I(1) = 1\}}{\mathbb{E}\{I(0)\}\mathbb{E}\{Y(0)|I(0) = 1\}}
\]

and we have \( \hat{\tau} \to \tau \) as desired (by Delta Theorem, see [DasGupta] (2008)). Note that with \( g \) can also consider

\[
\tau_g = g(\mathbb{E}\{I(1)\}\mathbb{E}\{Y(1)|I(1) = 1\}) - g(\mathbb{E}\{I(0)\}\mathbb{E}\{Y(0)|I(0) = 1\}),
\]

which for \( g = \log \) is also equal to

\[
\tau_g = g(\mathbb{E}\{I(1)\}) - g(\mathbb{E}\{I(0)\}) + g(\mathbb{E}\{Y(1)|I(1) = 1\}) - g(\mathbb{E}\{Y(0)|I(0) = 1\})).
\]

Another popular metric considered in online experiment is constructed by comparing the ratio of two random variables on the treatment arm to the control arm. More formally consider \( Y_i \) and \( Z_i \) are two response variables and consider the ratio of ratios:

\[
\hat{\tau} = (\sum_{i \in t} Y_i / \sum_{i \in t} Z_i) / (\sum_{i \in c} Y_i / \sum_{i \in c} Z_i) = (\bar{Y}_t/\bar{Z}_t)/(\bar{Y}_c/\bar{Z}_c),
\]
which is an estimator for
\[
\tau = \left( \frac{\mathbb{E}\{Y(1)\}}{\mathbb{E}\{Z(1)\}} \right) \left/ \left( \frac{\mathbb{E}\{Y(0)\}}{\mathbb{E}\{Z(0)\}} \right) \right.
\]
We could also consider the transformed version for \( g = \log \)
\[
\tau_g = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\}) - g(\mathbb{E}\{Z(1)\}) - g(\mathbb{E}\{Z(0)\}),
\]
which has two components, each being a difference in one response. Our method developed here applies also to these more complex cases (sum ratio and ratio of mean ratios). In the discussion section we give an example of a metric for which our method does not apply directly.

## 3 Mathematical development

In this section, we develop mathematical theory for reducing variance for general metrics as discussed in the introduction. Subsection 3.1 finds necessary and sufficient conditions for the adjustments to leave the estimators asymptotically unbiased. Subsection 3.1 uses asymptotic theory to approximate the achievable variance reduction for one of the adjustment methods which requires minimal assumption on the predictive model used for adjustment.

### 3.1 Retaining asymptotic unbiasedness

In this subsection, we introduce various adjustments and the conditions needed for each adjustment to avoid introducing bias asymptotically.

In the following, we present two general ideas to adjust estimators while keeping them unbiased. Parts (a) and (b) of lemma 3.1 use the fact that adding a random variable with expectation 0 to an estimator will not introduce bias. Part (c) notes that if we add a statistic to the estimator with the same expected value, we can avoid bias by weighting the two terms appropriately.

For simplicity we introduce the following notations. Consider \( Y \) to be a response of interest with two arms, control denoted by \( c \) and treatment denoted by \( t \) and let \( N_t, N_c \) be the sample size on the respective arms. Then we let

\[
\bar{Y}_t = \frac{1}{N_t} \sum_{i \in t} Y_i,
\]
\[
\bar{Y}_c = \frac{1}{N_c} \sum_{i \in c} Y_i,
\]
which are the average value of the response \( Y \) on treatment and control arms respectively. Now assume \( X_i = (X_{i1}, \ldots, X_{ip}) \) is a set of \( p \) predictive variables and consider a function \( h \) which only depends on \( X_i \) and the experiment arm:

\[
h : \mathbb{R}^p \times [0, 1] \rightarrow \mathbb{R},
\]
where 0 denotes the control and 1 denotes the treatment. For example \( h(X_i, 0) \) is the value of \( h \) for given predictors \( X_i \) and for a unit on the control arm. Then we use the following short hand notations:

\[
\bar{h}_t(w) = \frac{1}{N_t} \sum_{i \in t} h(X_i, w),
\]
\[
\bar{h}_c(w) = \frac{1}{N_c} \sum_{i \in c} h(X_i, w),
\]
\[
w = 0, 1
\]
Also consider a simpler function \( f \) which is only a function of the predictors (and not the experiment arm):

\[
f : \mathbb{R}^p \rightarrow \mathbb{R},
\]

and consider the short-hand notations:

\[
\tilde{f}_t = (1/N_t) \sum_{i \in t} f(X_i),
\]

\[
\tilde{f}_c = (1/N_c) \sum_{i \in c} f(X_i).
\]

Note that while for developing the theory in this section, we do not require \( h, f \) to have any particular properties, in order to get good adjustments in practice \( h, f \) are typically prediction functions which return the expected value of response given the predictor variables.

**Lemma 3.1 (Unbiased Adjustment Lemma):** Suppose \( \tau_g = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\}) \) be the defined experiment effect. Also suppose \( \theta \in \mathbb{R} \) and \( \gamma_1 + \gamma_2 = 1, \gamma_1, \gamma_2 \in \mathbb{R} \). Consider the estimator

\[
\hat{\tau}_g = g(\bar{Y}_t) - g(\bar{Y}_c)
\]

which is asymptotically unbiased in general (and unbiased if \( g(x) = x, \forall x \in \mathbb{R} \)).

Now assume \( h(X_i, W_i) \) and \( f(X_i) \) are imputation functions. Then we can consider these adjusted estimators:

(a) \( \hat{\tau}_{adj}(w) = \hat{\tau}_g - \theta(g(\bar{h}_t(w)) - g(\bar{h}_c(w))), w = 0, 1 \)

(b) \( \hat{\tau}_{adj} = \hat{\tau}_g - \theta(g(\tilde{f}_t) - g(\tilde{f}_c)) \)

(c) \( \hat{\tau}_{adj} = \gamma_1 \hat{\tau}_g + \gamma_2 (g(\bar{h}_c(1)) - g(\bar{h}_t(0))) \)

(d) \( \hat{\tau}_{adj} = \gamma_1 (g(\bar{Y}_t) - g(\bar{h}_c(0))) + \gamma_2 (g(\bar{h}_t(1)) - g(\bar{Y}_c)) \)

Then

(a) \( \hat{\tau}_{adj}(w) \) is asymptotically unbiased if \( \mathbb{E}\{g(\bar{h}_t(w))\} \rightarrow \mathbb{E}\{g(\bar{h}_c(w))\} \) which is also true if \( \mathbb{E}\{\bar{h}_t(w)\} \rightarrow \mathbb{E}\{\bar{h}_c(w)\} \).

(b) \( \hat{\tau}_{adj} \) is asymptotically unbiased if \( \mathbb{E}\{g(\tilde{f}_t)\} \rightarrow \mathbb{E}\{g(\tilde{f}_c)\} \) which is also true if \( \mathbb{E}\{\tilde{f}_t\} \rightarrow \mathbb{E}\{\tilde{f}_c\} \).

(c) \( \hat{\tau}_{adj} \) is asymptotically unbiased if

\[
\mathbb{E}\{g(\bar{h}_c(1)) - g(\bar{h}_t(0))\} \rightarrow \tau_g
\]

(d) \( \hat{\tau}_{adj} \) is asymptotically unbiased if \( \mathbb{E}\{g(\bar{Y}_t) - g(\bar{h}_c(0))\}, \mathbb{E}\{g(\bar{h}_t(1)) - g(\bar{Y}_c)\} \rightarrow \tau \).

**Proof**

(a), (b) First part of each of (a), (b) is obvious. Second part is because of Delta Theorem; see e.g. DasGupta (2008).

(c), (d) Straight-forward from the assumption and linearity of expectation.
3.1 Retaining asymptotic unbiasedness

Remark. Note that (c), (d) are in general much stronger conditions than (a) and (b).

Remark on generality of the method in Parts (a), (b) Parts (a) and (b) open the door for very flexible paradigm for variance reduction with a very wide class of predictive models since

$$\mathbb{E}\{\tilde{h}_t(w)\} \rightarrow \mathbb{E}\{\tilde{h}_c(w)\},$$

holds for almost any predictive model e.g. generalized linear models, regularization methods etc.

Remark on related alternatives to (a) and (b) We can consider related alternatives to (a), (b). E.g. consider

$$\hat{\tau}_{adj} = \hat{\tau}_g - \sum_{w \in \{0, 1\}} \theta_w (g(\tilde{h}_t(w)) - g(\tilde{h}_c(w))).$$

In this case we are adjusting the estimator by removing two terms which have asymptotic expectation of zero: one is the difference of the $g$-transformed average model prediction on the two arms assuming both were control; and one is the $g$-transformed averages on the two arms assuming both were treatment. Another alternative can be achieved by considering

$$\hat{\tau}_{adj} = \hat{\tau}_g - \theta [g(\gamma_1 \tilde{h}_t(0) + \gamma_2 \tilde{h}_t(1)) - g(\gamma_1 \tilde{h}_c(0) + \gamma_2 \tilde{h}_c(1))]$$

for $\gamma_1 + \gamma_2 = 1, \gamma_1, \gamma_2 \geq 0$. Now if we define $f(X_i) = \gamma_1 h(X_i, 0) + \gamma_2 h(X_i, 1)$ which is a weighted average of prediction of $h$ on control and treatment, we get

$$\hat{\tau}_{adj} = \hat{\tau}_g - \theta (g(\tilde{f}_t) - g(\tilde{f}_c)),$$

which is a special case of Part (b). We could interpret $f$ as a prediction function which tries to predict the value of the response well for both arms. This can motivate us to fit a prediction function to all data (including experiment and control) by ignoring the label as suggested by Deng et al. (2013).

Remark on training the data. Note that even though in Part (a) for $w = 0$, the model is predicting the values assuming the data is on the control arm, we are not required to use the control data only and we can use all the available data to fit a model and predict assuming all data is on the control arm.

The following corollary explicitly states the case for the ratio metrics, by considering and appropriate $g$.

Corollary 3.1: Suppose for non-negative valued response $Y$ with positive expectation, we are interested in $\tau' = \mathbb{E}\{Y(1)\}/\mathbb{E}\{Y(0)\}$. Then

$$\hat{\tau} = (\bar{Y}_t/\bar{Y}_c)$$

is an asymptotically unbiased estimate of $\tau'$. Also

(a) For either of $w = 0, 1$, we have

$$\hat{\tau}_{adj}(w) = \hat{\tau} \times (\tilde{h}_t(w)/\tilde{h}_c(w))^{\theta}$$

is asymptotically unbiased if

$$\mathbb{E}\{\tilde{h}_t(w)/\tilde{h}_c(w)\} \rightarrow 1.$$
\( \hat{\tau}_{adj} = \hat{\tau} \times (\tilde{f}_i / \tilde{f}_c)^\theta \) is asymptotically unbiased if
\[
\mathbb{E}\{ \tilde{f}_i / \tilde{f}_c \} \to 1.
\]

(c) \( \hat{\tau}_{adj} = \hat{\tau}^n \times (\tilde{h}_c(1) / \tilde{h}_i(0))^\gamma \) where \( \gamma_1 + \gamma_2 = 1 \) if \( \tilde{h}_c(1) / \tilde{h}_i(0) \to \tau \).

(d) \( \hat{\tau}_{adj} = (\tilde{Y}_i / \tilde{h}_c(1))^{\gamma_1} (\tilde{h}_i(0) / \tilde{Y}_c)^{\gamma_2} \) where \( \gamma_1 + \gamma_2 = 1 \) if \( (\tilde{Y}_i / \tilde{h}_c(1)), (\tilde{h}_i(0) / \tilde{Y}_c) \to \tau \).

**Proof** In order to infer about \( \tau' \) we can infer about
\[
\log(\tau') = \log(\mathbb{E}\{Y(1)\}) - \log(\mathbb{E}\{Y(0)\})
\]
which is the same as \( \tau_g = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\}) \) when \( g = \log \).

3.2 Reduction optimization and approximating achievable reduction

This subsection uses asymptotic theory to approximate the reduction in the estimator variance with the methods suggested above. We only work out the case for parts (a), (b) of Lemma 3.1
\[
\hat{\tau}_{adj}(w) = \hat{\tau}_g - \theta(g(\tilde{h}_i(w)) - g(\tilde{h}_c(w))), \quad w = 1, 2
\]
and
\[
\hat{\tau}_{adj} = \hat{\tau}_g - \theta(g(\tilde{f}_i) - g(\tilde{f}_c)).
\]

In this case we find an optimal \( \theta \) (as a function of \( g \) and the predictive power of \( h \)) to achieve maximum reduced variance.

We do not pursue parts (c), (d) type estimators as they require much stronger assumptions on the model to retain unbiasedness. The main result of this subsection is given in Theorem 3.2 which relies on the Delta Theorem in the multivariate case (see DasGupta (2008)). Much more general results are proved in Section 3.3. However the simpler cases in this section suffice for most applications. Also we are able to provide more detailed solutions for these simpler cases.

First in Theorem 3.1 we find the minimizer (\( \theta \)) for the variance of statistics of the form
\( T = g(Y) - \theta g(H) \). This is already a generalization of the main idea in Deng et al. (2013). However our adjusted estimators discussed in Lemma 3.1 are more complex and involves two differences added to each other i.e. of the form
\[ T = g(Y) - \theta g(H) - (g(Y^*) - \theta g(H^*)) \]
and Theorem 3.2 optimizes for \( \theta \) for this case.

**Theorem 3.1:** Consider the random variable \( T = g(Y) - \theta g(H) \) where \( H, Y \) are random variables with finite moments and \( g \) is a real-valued differentiable function and \( g'(\mu_H) \neq 0 \). Also assume \( \rho = \text{cor}(Y,H) \). Then

(a) \[
\text{argmin}_\theta \text{var}\{T\} \approx (g'(\mu_Y) / g'(\mu_H)) \text{cov}(Y,H) / \sigma_H^2
\]

(b) The minimum is then approximated by
\[
(g'(\mu_Y) \sigma_Y)^2 (1 - \rho^2).
\]

Moreover
\[
\text{var}\{g(Y) - \theta g(H)\} / \text{var}\{g(Y)\} \approx (1 - \rho^2)
\]
which does not depend on \( g \).
**Theorem 3.2:** Consider the random variables
\[ T_0 = g(Y) - g(Y^*), \]
\[ T_1 = g(Y) - \theta g(H), \]
\[ T_2 = g(Y^*) - \theta g(H^*), \]
\[ T = T_1 - T_2 = g(Y) - \theta g(H) - (g(Y^*) - \theta g(H^*)) \]
where \( Y, H, Y^*, H^* \) are random variables with finite moments and \( g \) is a real-valued differentiable function. Let \( \mu = (\mu_Y, \mu_H, \mu_{Y^*}, \mu_{H^*}) \) be the mean vector and \( (\sigma_Y, \sigma_H, \sigma_{Y^*}, \sigma_{H^*}) \) be the standard deviation vector. Also assume all the pairwise correlations are zero except for \( \text{cov}(Y, H) \) and \( \text{cov}(Y^*, H^*) \) and let \( \rho = \text{cor}(Y, H) \) and \( \rho^* = \text{cor}(Y^*, H^*) \). Then

(a) \[ \theta := \text{argmin}_\theta \text{var}\{T\} \approx \frac{g'(\mu_H)g'(\mu_Y)\text{cov}(Y, H) + g'(\mu_{H^*})g'(\mu_{Y^*})\text{cov}(Y^*, H^*)}{g'(\mu_H)^2\sigma_Y^2 + g'(\mu_{H^*})^2\sigma_{Y^*}^2}. \]

For \( g = \text{identity} \), we have
\[ \theta = \frac{\text{cov}(Y, H) + \text{cov}(Y^*, H^*)}{\sigma_Y^2 + \sigma_{Y^*}^2}. \]

For \( g = \log \), we have
\[ \theta = \frac{\text{cov}(Y, H)/\mu_H\mu_Y + \text{cov}(Y^*, H^*)/\mu_{H^*}\mu_{Y^*}}{\sigma_Y^2/\mu_H^2 + \sigma_{Y^*}^2/\mu_{H^*}^2}. \]

(b) The minimum is then approximated by
\[ g'(\mu_Y)^2\sigma_Y^2 + g'(\mu_{Y^*})^2\sigma_{Y^*}^2 - \delta, \]
where
\[ \delta = \frac{(g'(\mu_H)g'(\mu_Y)\text{cov}(Y, H) + g'(\mu_{H^*})g'(\mu_{Y^*})\text{cov}(Y^*, H^*))^2}{g'(\mu_H)^2\sigma_Y^2 + g'(\mu_{H^*})^2\sigma_{Y^*}^2}. \]

Therefore \( \text{var}\{T_0\} - \text{var}\{T\} \rightarrow \delta. \)

For \( g = \text{identity} \), we have
\[ \delta = \frac{(\text{cov}(Y, H) + \text{cov}(Y^*, H^*))^2}{\sigma_Y^2 + \sigma_{Y^*}^2}. \]

For \( g = \log \), we have
\[ \delta = \frac{(\text{cov}(Y, H)/\mu_H\mu_Y + \text{cov}(Y^*, H^*)/\mu_{H^*}\mu_{Y^*})^2}{\sigma_Y^2/\mu_H^2 + \sigma_{Y^*}^2/\mu_{H^*}^2}. \]
Remark. The assumptions is Part (b) are only used to approximate the decrease in the variance and are not needed for the method to work. Even if these assumptions do not hold, we can expect a decrease in variance. Moreover, in most practical cases, these assumption approximately hold. For example in most useful interpolation models we expect even stronger assumptions to hold e.g. we expect $\mu_H = \mu_L$ meaning that the model predictions on the experiment and control arm are the same on average, which must be true for most models considering the experiment units are chosen at random and independent from their $x_i$ which is used by the interpolation function $f$. 

(c) $\min\{\theta_1, \theta_2\} \leq \theta \leq \max\{\theta_1, \theta_2\}$, where $\theta_i$ is the argmin for minimizing the variance for $T_i$, $i = 1, 2$.

(d) If we further assume $g'(\mu_H)\sigma_H = g'(\mu_{H^*})\sigma_H$ and $\rho g'(\mu_Y)\sigma_Y = \rho g'(\mu_{Y^*})\sigma_{Y^*}$, we have

$$\var\{T\} = (1 - \rho^2)\var\{Y\} + (1 - (\rho^*)^2)\var\{Y^*\} \leq (1 - \min\{\rho^2, \rho^{*2}\})\var\{T_0\}$$

\textbf{Proof} See Appendix.

Now let’s apply this theorem to our problem.

\textbf{Corollary 3.3 (Variance Reduction):} Suppose $\tau_e = g(\mathbb{E}\{Y(1)\}) - g(\mathbb{E}\{Y(0)\})$ be the defined experiment effect. Also consider the consistent (raw) estimator

$$\hat{\tau}_e = g(\bar{Y}_1) - g(\bar{Y}_0).$$

Assume $f(X_i)$ is an interpolation function which does not depend on the experiment arm, e.g. $f(X_i) = h(X_i, 0)$ and consider the adjusted estimator

$$\hat{\tau}_{adj} = \hat{\tau}_e - \theta(g(\bar{f}_1) - g(\bar{f}_0)).$$

We denote $f(X_1)$ by $H_i$ for simplicity. Also we denote $(Y_i, f(X_i))$ when $i \in t$ (chosen at random) by $(Y_t, H_t)$ and $(Y_c, H_c)$ for the control arm. Let

$$\rho_c = \text{cor}(Y_c, H_c) \quad \rho_t = \text{cor}(Y_t, H_t)$$

Then

(a) the optimal $\theta$ which minimizes $\hat{\tau}_{adj}$ is between $\theta_1$ and $\theta_2$ which minimize the variance of $g(\bar{Y}_c) - \theta g(\bar{f}_c)$ and $g(\bar{Y}_t) - \theta g(\bar{f}_t)$ respectively:

$$\theta_1 = \text{cov}(Y_c, H_c)/\var\{H_c\}, \quad \text{and} \quad \theta_2 = \text{cov}(Y_t, H_t)/\var\{H_t\}$$

(b) Moreover if we assume

$$g'(\mu_H)\sigma_{H_c} = g'(\mu_{H^*})\sigma_{H_c} \quad \text{and} \quad \rho_c g'(\mu_Y)\sigma_Y = \rho_i g'(\mu_{Y^*})\sigma_{Y^*}$$

$$\var\{\hat{\tau}_{adj}\}/\var\{\hat{\tau}\} \leq (1 - \min\{\rho^2_c, \rho^2_t\}).$$

\textbf{Proof} See Appendix.
The reduction in the confidence interval length, can then be approximated by

$$\frac{\text{raw CI length} - \text{adj CI length}}{\text{raw CI length}} \approx 1 - \sqrt{1 - \rho^2}$$

(2)

Figure 1 depicts this relationship. For example with a correlation of 0.6 we can expect 20% reduction and with a correlation of 0.8 we can expect 40% reduction and with 0.9 correlation we get close to 50%. To get an idea for cost savings, lets compare this to the reduction achieved by increasing the sample size from $n$ to $kn$ for $k > 1$. In this case the reduction is equal to

$$\frac{\text{raw CI length} - \text{adj CI length}}{\text{raw CI length}} = \frac{\sigma / \sqrt{n} - \sigma / \sqrt{kn}}{\sigma / \sqrt{n}} = 1 - 1 / \sqrt{k}$$

This means for a 20% reduction we need to multiply sample size by 1.5 (as compared with 0.6 correlation with adjustment) and to get 40% reduction need to almost triple the sample size.

![Fig. 1: Left: The reduction in CI (confidence intervals) length as a function of cross-validated correlation between predictions and observed. Middle: The reduction in CI length as a function of increase in sample size. Right: The practical sample size gain compared to the cross-validated correlation.](image)

**3.3 Adjusting other complex metrics**

In this subsection we discuss using these methods for other complex metrics discussed at the end of Section 2.

First consider the sum ratio:

$$\hat{\tau} = \sum_{i \in t} Y_i / \sum_{i \in c} Y_i = (n_t / n_c) \tilde{Y}_t / \tilde{Y}_c,$$

and note that since we are assuming not all potential users appear in the sum, $n_t$ and $n_c$ can be thought of random variables themselves (if they are fixed variables in the experiment by design then we can simply revert back to the mean ratio case). Considering the second term in the above is simply the estimator for the mean ratio, we can replace the second term by its adjusted version by multiplying the estimator by $[\tilde{Y}_t / \tilde{Y}_c]^\theta$ which leaves the estimator asymptotically unbiased. Therefore in summary we can use the exact same adjustment as the mean ratio for the sum case. In the appendix we have performed simulations for this case.

Next consider the ratio of ratios metrics. The estimator in that case in the log scale is given by

$$\hat{\tau}_g = g(\tilde{Y}_t) - g(\tilde{Z}_t) - (g(\tilde{Y}_c) - g(\tilde{Z}_c))$$

$$= g(\tilde{Y}_t) - g(\tilde{Y}_c) - (g(\tilde{Z}_t) - g(\tilde{Z}_c)).$$

For general $g$ the above expression could be considered as a $g$-transformed difference of difference which has many applications. For example when we compare the experiment and treatment arms difference with a baseline.
Now we can adjust each of the two components on the right side individually without introducing bias as

\[ \hat{\tau}_{adj} = g(\bar{Y}_t) - g(\bar{Y}_c) - \theta_Y (g(\bar{f}_Y, t) - g(\bar{f}_Y, c)) - \left( g(\bar{Z}_t) - g(\bar{Z}_c) - \theta_Z (g(\bar{f}_Z, t) - g(\bar{f}_Z, c)) \right). \]

This implies an adjusted estimator for \( \tau \) is

\[ \left( \bar{Y}_t / \bar{Y}_c \right) \left( \bar{Z}_c / \bar{Z}_t \right)^{\theta_Y} \left( \bar{f}_Y(t) / \bar{f}_Y(c) \right)^{\theta_Y} \left( \bar{f}_Z(t) / \bar{f}_Z(c) \right)^{\theta_Z}. \]

Note that we do not need to require \( \theta_Y = \theta_Z \) of course and could pick the individual \( \theta \) which minimizes each term separately. However that would not necessarily give us the global argmin. Indeed it is possible to find the optimal values for this problem and much more general class of metrics (using linear algebra) and we publish those results in future manuscripts.

4 Simulations

This section performs a simulation study to test the variance reduction method in Part (a) of Lemma 3.1 with \( g = \log \) which is equivalent to the Mean Ratio metric in which case the raw estimator is equal to

\[ \hat{\tau} = \frac{\sum_{i \in t} Y_i(1)/N_t}{\sum_{i \in c} Y_i(0)/N_c}, \]

and the adjusted is equal to

\[ \hat{\tau}_{adj} = \hat{\tau} \times \left( \frac{\sum_{i \in t} h(X_i, 0)/N_t}{\sum_{i \in c} h(X_i, 0)/N_c} \right)^{\theta}. \]

We examine the existence of bias in the new method and the amount of variance reduction which is compared to the theoretical reduction approximated in Equation Corollary 3.3. More simulation studies are performed for various other scenarios and the results are included in the appendix. We generate a large population of users (1,000,000) to play the role of the whole (super) population and use smaller sample sizes to test the methods.

In this simulation, we assume a set of users are exposed to different versions of the same feature. Each user may have a different number of impressions (denoted by \( \text{Imp} \) in the following) to the feature. In each impression the user has a chance to spend certain amount denoted by \( A \)) of time/money on the feature. Also the user have a chance to either have an explicit interaction (denoted by \( \text{Interact} \)) with the feature or not each time.

We first simulate a 100,000 population of users with random user attributes (country and gender) and then for those users, we simulate impression counts using these model assumptions:

\[ \text{Imp}(u) \sim \text{ZTP}(\mu_1), \]

\[ \mu_1 = \exp(X(u)\beta_1 + \theta_1 W(u) + \varepsilon_1(u)), \]

where \( W(u) = 1 \) if \( u \) is in the treatment arm and zero otherwise; \( \text{ZTP} \) stands for Zero-Truncated Poisson. The reason we use \( \text{ZTP} \) instead of Poisson is to avoid having user imbalance in the predictors (country and gender here) the two arms due to experiment which would then result in bias.

We use country and gender as our predictors here. Figure 2 depicts the number of users for each slice of the predictors (country/gender) and for each arm. At the population level we observe approximate balance in how the users in each arm are distributed across country and gender which
means the appearance of users in the data from various slices of the prediction variables is not impacted by the experiment as desired.

Next conditional on the impression, we simulate interactions and amounts. For amount

\[ A(u) | \text{Imp}(u) \sim \exp(\mu_2) \]

\[ \mu_2 = \exp(X(u)\beta_2 + \theta_2 W(u) + \varepsilon_2(u)) \]

and for the interaction:

\[ \text{Interact}(u) | \text{Imp}(u) \sim \text{Bernoulli}(\mu_3) \]

\[ \mu_3 = \logit^{-1}(X(u)\beta_3 + \theta_3 W(u) + \varepsilon_3(u)) \]

where \( \logit^{-1}(x) = \exp(x)/(1 + \exp(x)) \). Also \( \varepsilon_1(u), \varepsilon_2(u), \varepsilon_3(u) \) model the user random effect and assumed to be multivariate normal.

![Image](image.png)

**Fig. 2:** Check for population imbalance across predictors

We consider the three response variables: impression, amount and interaction and the predictors: country and gender. To test our method, we consider the Mean Ratio Metric for each response variable:

\[ \tau = \frac{\mathbb{E}\{Y(1)\}}{\mathbb{E}\{Y(0)\}} \]

defined over the (super) population of all units eligible and \( t, c \) denote treatment and control respectively. Then assuming we have data from a random experiment, we can estimate \( \tau \) by the following unbiased estimator which we call the raw method:

\[ \text{raw estimate} = \frac{\left( \sum_{i \in t} y_i / n_t \right)}{\left( \sum_{i \in c} y_i / n_c \right)} \]

where \( n_t, n_c \) denote the sample size on the treatment and the control arm respectively.

Then to calculate adjusted estimators we use linear regression to fit a predictive model for each response. We fit two regression models for each response:

(i) using control data only: the data is fitted only to the control arm data
Simulations

(ii) using all data (including treatment data): the data is fitted to both arms, using the experiment label as a predictor – however when predicting, we assume all units are on control arm.

(iii) using all data but dropping the experiment label as a predictor

and we use the multiplicative adjustment from Corollary 5.1 Part (a):

\[
a = \left( \frac{\sum_{i \in \mathcal{I}} h(x_i; 0)/n_t}{\sum_{i \in \mathcal{I}} h(x_i; 0)/n_c} \right) / \left( \frac{\sum_{i \in \mathcal{C}} h(x_i; 0)/n_c}{\sum_{i \in \mathcal{C}} h(x_i; 0)/n_c} \right),
\]

where \( h \) denotes the model prediction function and \( x_i \) denotes the value of the predictors for user \( i \). Also note that the 0 in the second component of \( h(x_i; 0) \) prescribe to the model to assume the user has been on the control arm. This is automatically the case for (i), but for two if we use the experiment arm as a categorical variable in the model, when performing the interpolation we need to make sure to predict the response for all users assuming they were on the control arm.

The results from these two models are given Tables 1, 2 and 3 respectively. The cross-validated correlation for each response is given in the column \( cv_{cor} \), the optimal \( \theta \) and the theoretical reduction in the variance are also given. There is little difference between the model performance in this example and we expect both adjustment methods produce similar reduction.

| model_formula       | cv_cor | theta | sd_ratio |
|---------------------|--------|-------|----------|
| 1 imp_count: gender+country | 0.67   | 0.97  | 0.74     |
| 2 obs_interact: gender+country  | 0.75   | 1.00  | 0.66     |
| 3 obs_amount: gender+country    | 0.72   | 1.00  | 0.69     |

Tab. 1: Prediction accuracy for various responses when using control data only.

| model_formula       | cv_cor | theta | sd_ratio |
|---------------------|--------|-------|----------|
| 1 imp_count: gender+country+expt_id | 0.69   | 0.99  | 0.72     |
| 2 obs_interact: gender+country+expt_id  | 0.74   | 1.00  | 0.67     |
| 3 obs_amount: gender+country+expt_id    | 0.66   | 1.00  | 0.75     |

Tab. 2: Prediction accuracy for various responses when using all the data in building the model and the experiment label. However we assume all data is on control arm when predicting.

| model_formula       | cv_cor | theta | sd_ratio |
|---------------------|--------|-------|----------|
| 1 imp_count: gender+country | 0.65   | 1.00  | 0.76     |
| 2 obs_interact: gender+country | 0.69   | 1.00  | 0.72     |
| 3 obs_amount: gender+country    | 0.62   | 1.00  | 0.78     |

Tab. 3: Prediction accuracy for various responses when using all the data in building the model but not using the experiment label.

Figure 3 checks if the adjustment (in the log scale) is on average unbiased. To that end, we sample 500 users from all the data randomly multiple times and calculate the adjustment in each case. We expect to see a distribution which is symmetric around zero in the absence of bias. We observe that in all cases the adjustment is unbiased.
Fig. 3: Check for imbalance in the adjustment when $g = \log$. The top panels are using the control data only. The middle plots use all the data and use the experiment label in training. The bottom plots are using all data but do not use the experiment label.

Next we vary the sample size of users from 500 to 20,000 and for each sample size, we sample from the population 1000 times to calculate the estimators 1000 times for both raw and adjusted cases. Figure 4 depicts the mean of the estimator for the three responses on the left panels and the variance on the right panels. We only show adjustment methods based on (i) and (ii) as (iii) was very similar to (ii). It is clear that both adjustment methods leave the estimator unbiased while reducing the variance significantly regardless of the sample size. Figure 5 depicts the ratio of the standard deviation adjusted estimators to the raw estimator which is approximated to be

$$\text{ratio} = \frac{\text{sd(adj estimator)}}{\text{sd(raw estimator)}} \approx \sqrt{1 - \rho^2},$$

using Theorem 3.1 and its corollaries. For $\rho \approx 0.70$ this ratio is approximately 0.71 which is what we observe in the figure. Note that this ratio does not depend on the sample size according to the theory and the figure confirms that for this case. Other simulation results for varying correlation values (presented in the appendix) were also consistent with the theory.
Fig. 4: Mean Ratio Metric estimator’s mean and variance across sample size.
Finally, Figure 6 varies the sample size from 500 to 10000. For each sample size, we sample from the population once, and for that one sample we calculate a CI using the bucketed JackKnife method with 50 buckets. The bucketed jack-knife method is similar to regular jack knife method but takes out one bucket for each calculation, rather than one unit. We observe that the adjusted confidence intervals are within the raw confidence intervals most of the time, while all converging toward a common value.
5 Discussion

This paper developed a method to decrease the variance of a wide class of experiment effects metrics while leaving the estimator (asymptotically) unbiased.

Our goal was to keep the method very flexible in two ways. (1) The method is applicable to a wide variety of metrics. (2) The interpolation/prediction function used in the method can be quite arbitrary i.e. any machine learning algorithm could be used for that purpose. The second requirement is a useful one in the context of technology industry at the moment where many powerful complex machine learning algorithms have been developed. However their statistical properties are almost a black-box to the practitioners. Given any such predictive model, we introduced adjusted estimators which remain asymptotically unbiased. The adjusted estimator also includes extra parameters which can be optimized for a given predictive model.

We think these methods can have other applications beyond experiment analysis. For example we can use these adjustments also in the context of observational studies to estimate casual effects since the adjustments perform balancing with respect to the prescribed auxiliary variables.

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A Proofs

Proof (Proof of Theorem 3.1)
(a) Define \( f(H,Y;\theta) = g(Y) - \theta g(H) \). Also assume that the bivariate distribution of \((H,Y)\) has the mean \( \mu = (\mu_H, \mu_Y) \). Then by applying the Taylor series approximation to \( f \) (see [DasGupta (2008)]):

\[
f(H,Y) = f(\mu) + \nabla(f)(\mu).[(H,Y) - \mu] = f(\mu) + \frac{\partial f}{\partial h}(\mu)(H - \mu_H) + \frac{\partial f}{\partial y}(\mu)(Y - \mu_Y).
\]

We can approximate the variance of \( f(H,Y) \) by

\[
\text{var}\{f(H,Y;\theta)\} \approx \frac{\partial f}{\partial h}(\mu)^2 \text{var}\{H\} - 2\frac{\partial f}{\partial h}(\mu)\frac{\partial f}{\partial y}(\mu)\text{cov}(Y,H) + \frac{\partial f}{\partial y}(\mu)^2 \text{var}\{Y\}.
\]

By replacing

\[
\frac{\partial f}{\partial h}(\mu) = -\theta g'(\mu_H), \quad \frac{\partial f}{\partial y}(\mu) = g'(\mu_Y),
\]

we get:

\[
\text{var}\{f(H,Y;\theta)\} = \theta^2(g'(\mu_H)\sigma_H)^2 + (-2g'(\mu_H)g'(\mu_Y)\text{cov}(Y,H))\theta + (g'(\mu_Y)\sigma_Y)^2.
\]

The above expression is a 2nd-order convex polynomial of the form \( p(\theta) = a\theta^2 + b\theta + c \) which is minimized at \(-b/(2a)\) with minimum being \(-b^2/4a + c\). Therefore

\[
\arg\min_{\theta} f(X,H;\theta) = \frac{g'(\mu_Y)}{g'(\mu_H)} \frac{\text{cov}(Y,H)}{\sigma_H^2}.
\]

(b) The minimum can be obtained by replacing \( \theta \) from (a). For the second part, note that by Taylor series approximation we have

\[
\text{var}\{g(Y)\} = g'(\mu_Y)\text{var}\{Y\},
\]

and the result follows.

Proof (Proof of Theorem 3.2)
To prove (a) and (b) define

\[
f(Y,H,Y^*,H^*;\theta) = g(Y) - \theta g(H) - \left(g(Y^*) - \theta g(H^*)\right)
\]

Then after by Taylor series expansion, we can approximate \( f \)

\[
f(Y,H,Y^*,H^*;\theta) \approx f(\mu) + \nabla(f)(\mu).[(Y,H,Y^*,H^*) - \mu].
\]

Therefore

\[
\text{var}\{T\} \approx \sum_{U \in \{Y,H,Y^*,H^*\}} \frac{\partial f}{\partial u}(\mu)^2 \text{var}\{U\} - 2\frac{\partial f}{\partial h}(\mu)\frac{\partial f}{\partial y}(\mu)\text{cov}(Y,H) - 2\frac{\partial f}{\partial h^*}(\mu)\frac{\partial f}{\partial y^*}(\mu)\text{cov}(Y^*,H^*)
\]

\[
= a\theta^2 + b\theta + c,
\]
where,
\[ a = g'(\mu_H)^2 \sigma_H^2 + g'(\mu_{H^*})^2 \sigma_{H^*}^2, \]
\[ b = -2(g'(\mu_H)g'(\mu_Y)\text{cov}(Y,H) + g'(\mu_{H^*})g'(\mu_{Y^*})\text{cov}(Y^*,H^*)) \]
\[ c = g'(\mu_Y)^2 \sigma_Y^2 + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2. \]

This is a quadratic function of \( \theta \) for which the argmin is equal to \(-b/(2a)\) and the minimum is equal to \(-b^2/(4a) + c\) which gives (a) and (b).

To prove (c), note that if we let
\[ a_1 = g'(\mu_H)g'(\mu_Y)\text{cov}(Y,H) \]
\[ a_2 = g'(\mu_{H^*})g'(\mu_{Y^*})\text{cov}(Y^*,H^*) \]
\[ b_1 = g'(\mu_H)^2 \sigma_H^2 \]
\[ b_2 = g'(\mu_{H^*})^2 \sigma_{H^*}^2. \]

Then in (a) we showed argmin\( \theta \) \( \text{var}\{T\} \approx \frac{a_1 + a_2}{b_1 + b_2} \) and from Theorem 3.1 we have \( \theta_i = a_i/b_i, \ i = 1, 2. \)

The proof is complete by noting
\[ \min\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\} \leq \frac{a_1 + a_2}{b_1 + b_2} \leq \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\} \]

To prove (d) note that by the assumptions given:
\[ g'(\mu_H)\sigma_H = g'(\mu_{H^*})\sigma_{H^*} \text{ and } \rho g'(\mu_Y)\sigma_Y = \rho g'(\mu_{Y^*})\sigma_{Y^*}, \]

we conclude \( a_1 = a_2 \) and \( b_1 = b_2 \). This allows us to rewrite \( \delta \) as \( \delta = \frac{(a_1 + a_2)^2}{b_1 + b_2} = \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} \). By Part (b), we have
\[ \text{var}\{T\} = g'(\mu_Y)^2 \sigma_Y^2 + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 - (a_1 + a_2)^2/(b_1 + b_2) \]
\[ = g'(\mu_Y)^2 \sigma_Y^2 + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 - a_1^2/b_1 - a_2^2/b_2 \]
\[ = g'(\mu_Y)^2 \sigma_Y^2 - \frac{a_1^2}{b_1} + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 - \frac{a_2^2}{b_2} \]
\[ = g'(\mu_Y)^2 \sigma_Y^2 - \rho^2 g'(\mu_Y)^2 \sigma_Y^2 + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 - \rho^2 g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 \]
\[ = g'(\mu_Y)^2 \sigma_Y^2 (1 - \rho^2) + g'(\mu_{Y^*})^2 \sigma_{Y^*}^2 (1 - \rho^*2) = (1 - \rho^2)\text{var}\{g(Y)\} + (1 - \rho^*2)\text{var}\{g(Y^*)\} \]
\[ \leq (1 - \min\{\rho^2, \rho^{*2}\})\text{var}\{g(Y)\} + (1 - \min\{\rho^2, \rho^{*2}\})\text{var}\{g(Y^*)\} \]
\[ = (1 - \min\{\rho^2, \rho^{*2}\})(\text{var}\{g(Y)\} + \text{var}\{g(Y^*)\}) = (1 - \min\{\rho^2, \rho^{*2}\})\text{var}\{T_0\}. \]

**Proof** (Proof of Corollary 3.3)

The adjusted estimator is equal to
\[ \hat{\theta}_{adj} = \hat{\theta}_{g} - \theta \left( g(\hat{\theta}_H) - g(\hat{\theta}_c) \right) \]
\[ = g(\hat{\theta}_H) - g(\hat{\theta}_c) - \theta \left( g(\hat{\theta}_H) - g(\hat{\theta}_c) \right) \]
\[ = \left( g(\hat{\theta}_H) - \theta g(\hat{\theta}_H) \right) - \left( g(\hat{\theta}_c) - \theta g(\hat{\theta}_c) \right) \]
The form in Equation 3 was convenient for showing unbiasedness. And we use the form in Equation 4 for deriving the variance. Now we can use Theorem 3.2 by letting \( Y = \bar{Y}_t, \ Y^* = \bar{Y}_c, \ H = \bar{f}_t, \ H^* = \bar{f}_c, \) and also noting that

\[
\rho_c = \text{cor}(Y_c, H_c) = \text{cor}(\bar{Y}_c, \bar{H}_c) \quad \rho_t = \text{cor}(\bar{Y}_t, \bar{H}_t)
\]

\[\blacksquare\]

B Simulations for various scenarios

Here we perform a few more simulation studies for various scenarios.

B.1 Simulation results for Sum Ratio

This subsection presents the results for the Sum Ratio Metric for the same simulated data in Tables 1, 2 and 3 (in the main text).

Figure 7 depicts the mean and the standard deviation of the raw and adjusted estimators for Sum Ratio. Figure 8 depicts the ratio of the standard deviation of the adjusted estimators versus the raw estimator. We observe that the estimators remain unbiased while the variance is decreased, albeit the decrease in the variance is less than the decrease for the Mean Ratio case.
Fig. 7: Sum Ratio Metric estimator’s mean and variance across sample size.
B.2 Simulation with no predictive power

This subsection considers a simulation in which the auxiliary variables do not have any prediction power. It is desirable if the methods we introduced in this paper do not increase the variance significantly, which might be the case due to the extra variance introduced by the model uncertainty which generates some variance in the adjustment. Tables 4 and 5 show the predictive power of the models for this case and the expected reduction in the variance. Figure 8 depicts the mean and the standard deviation of the raw and adjusted estimators. Figure 9 depicts the ratio of the standard deviation of the adjusted estimators versus the raw estimator. Fortunately we cannot observe any increase in the variance using the adjusted methods.

| model_formula          | cv_cor | theta | sd_ratio |
|-----------------------|--------|-------|----------|
| 1 imp_count: gender+country | 0.03   | 0.70  | 1.00     |
| 2 obs_interact: gender+country | -0.03  | 0.00  | 1.00     |
| 3 obs_amount: gender+country | -0.04  | 0.00  | 1.00     |

Tab. 4: Prediction accuracy for various responses using only control data.

| model_formula          | cv_cor | theta | sd_ratio |
|-----------------------|--------|-------|----------|
| 1 imp_count: gender+country+expt_id | 0.16   | 1.00  | 0.99     |
| 2 obs_interact: gender+country+expt_id | 0.14   | 0.67  | 0.99     |
| 3 obs_amount: gender+country+expt_id | 0.19   | 0.91  | 0.98     |

Tab. 5: Prediction accuracy for various responses when using all the data in building the model and the experiment label. However we assume all data is on control arm when predicting.
Fig. 9: Mean Ratio Metric estimator’s mean and variance across sample size. This is for the scenario with no predictive power.
Fig. 10: The ratio of standard deviations of adjusted and raw estimators for Mean Ratio. This is for the scenario with no predictive power.
Fig. 11: Mean Ratio Metric estimator’s mean and variance across sample size. This is for the scenario with experiment impacting the user populations appearing in the data.
B.3 Simulation results for Ratio of Mean Ratios

We use the same simulation settings as the simulation in the main text here. The result is given in Figure 13 which show the adjusted estimator is unbiased while decreasing the variance.

Fig. 12: Comparing the CI length convergence across varying sample sizes.

Fig. 13: The ratio of mean ratios.
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