Chiral Extrapolation: An Analogy with Effective Field Theory

Gerald V. Dunne
Department of Physics, University of Connecticut, Storrs CT 06269, USA

Anthony W. Thomas and Stewart V. Wright
Special Research Centre for the Subatomic Structure of Matter, and Department of Physics and Mathematical Physics, University of Adelaide, Adelaide SA 5005, Australia

We draw an analogy between the chiral extrapolation of lattice QCD calculations from large to small quark masses and the interpolation between the large mass (weak field) and small mass (strong field) limits of the Euler–Heisenberg QED effective action. In the latter case, where the exact answer is known, a simple extrapolation of a form analogous to those proposed for the QCD applications is shown to be surprisingly accurate over the entire parameter range.

I. INTRODUCTION

The challenge to find an accurate and reliable method of chiral extrapolation for hadronic properties calculated in lattice QCD at large quark mass is a matter of considerable current importance. While computer limitations mean that lattice simulations at physical quark masses are many years away, recent progress in chiral extrapolation suggests that it may well be possible to obtain accurate hadronic properties based on the calculations which will be possible with the next generation of supercomputers, available within just a few years, in the 10 Tera-flops range.

Fundamental to this scheme is the development of extrapolation methods which incorporate the model independent constraints of chiral symmetry [1,2], notably the leading non-analytic (LNA) behaviour of chiral perturbation theory [3,4], as well as the heavy quark limit [5]. Although these extrapolations are designed to match the leading behaviour in the extreme limits of small and large quark mass, there has been little guidance as to their reliability in the intermediate mass region. It is very unclear what precision to expect from such a simple extrapolation into the intermediate mass region, because the large mass expansion is presumably asymptotic, and the small mass limit has a log divergence plus finite corrections with a small radius of convergence. Here we attack this question from a novel direction by considering a remarkably close analogy between this problem and a well-known, exactly soluble system in effective field theory — the Euler–Heisenberg effective action [6–8]. The Euler–Heisenberg system exhibits many of the features found in the QCD calculations: at small electron mass (equivalently, strong external field) there is a logarithmic branch point, while at large mass (equivalently, weak external field) one has an asymptotic series expansion in inverse powers of mass. In this Letter, we show that a simple two-parameter interpolation formula (of the form used in the context of chiral extrapolation), which builds in the correct leading behaviour in both the small and large mass limits, yields an excellent approximation to the exact Euler–Heisenberg answer over the entire range of mass. We discuss possible consequences of this observation for the chiral extrapolation of lattice data.

Effective field theory (EFT) plays an important role in modern theoretical physics [9–11]. In pioneering work in the 1930’s, Heisenberg and Euler [6,7], and Weisskopf [8], studied the quantum corrections to classical electrodynamics associated with vacuum polarization effects. Renormalization properties and a more formal “proper-time” version were later studied by Schwinger [8]. In modern language, they computed the low energy effective action for the electromagnetic field, in leading order in the derivative expansion, by integrating out the electron degrees of freedom in the presence of a constant background electromagnetic field. This one-loop effective action can be expressed as [12]

$$S = -i \ln \det (i\partial - m),$$

where $\partial = \gamma^\nu (\partial^\nu + ieA^\nu)$, and $A^\nu$ is the fixed classical gauge potential with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. As shown in [6,7,8], this effective action can be computed in a simple closed form when the background field strength $F_{\mu\nu}$ is constant. For simplicity, we consider the case when the background is a constant magnetic field of strength $B$ (and we choose $eB$ to be positive). Then the exact, renormalized, one-fermion-loop effective action has the following integral representation:

$$S = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty ds \frac{ds}{s^2} \left( \coth s - \frac{1}{3s} \right) e^{-m^2 s/(eB)}. \quad (2)$$
The $\frac{1}{e}$ term is a subtraction of the zero field ($B = 0$) effective action, while the $\frac{2}{e}$ subtraction corresponds to a logarithmically divergent charge renormalization.

We stress that Eq. (3) is an exact, non-perturbative result. However, it can of course be expanded in two obvious limits. In the large mass limit, $m^2 \gg eB$ (which is equivalently the weak field limit), it is straightforward to develop an (asymptotic) expansion of this integral:

$$S = -\frac{e^2 B^2}{8\pi^2} \left[ -\frac{1}{45} \left( \frac{eB}{m^2} \right)^2 + \frac{4}{315} \left( \frac{eB}{m^2} \right)^4 - \frac{8}{315} \left( \frac{eB}{m^2} \right)^6 + \ldots \right]$$

(3)

Here the $B_{2n}$ are the Bernoulli numbers. The large mass expansion, Eq. (3), of the effective action has two very different expansions in the two limits of large and small electron mass. The transition expansion in powers of the coupling results in Ref. [18]:

by charge renormalization [8]). We note that, as a consequence of charge conjugation (Furry’s theorem), only even powers of $\frac{n}{m}$ appear in the perturbative expansion of Eq. (3). It is interesting to note that the series expansion of Eq. (3) is divergent, because the Bernoulli numbers grow factorially as $B_{2n} \sim (2n+1)!$ for large $n$, consistent with very general results for perturbation theory [13]. It is in fact an asymptotic series, and the proper-time integral representation in Eq. (2) is just the straightforward Borel sum of this asymptotic series [17].

The large mass limit may equivalently be characterized by the relevant length scales: the electron Compton wavelength $\lambda_e = \frac{1}{m}$, and the cyclotron radius (“magnetic length”) $\lambda_B = \frac{1}{eB}$. In terms of these length scales, the large mass limit corresponds to the situation where the electron Compton wavelength is much smaller than the cyclotron radius: $\lambda_e \ll \lambda_B$.

Since the Euler–Heisenberg system is exactly soluble, we can also use the exact integral representation to study the small mass, or strong field, limit where $m^2 \ll eB$. In terms of the length scales, in this limit the electron Compton wavelength is much greater than the cyclotron radius: $\lambda_e \gg \lambda_B$. Then, from Eq. (3), one finds (using results in Ref. [13]):

$$S = -\frac{e^2 B^2}{8\pi^2} \left\{ \frac{1}{3} + \frac{m^2}{eB} + \frac{1}{2} \left( \frac{m^2}{eB} \right)^2 \right\} \log \frac{m^2}{eB} + \left[ \frac{1}{3} - \frac{1}{3} \log 2 - 4\zeta'(1) \right] + \left[ \log \pi - 1 \right] \frac{m^2}{eB}$$

$$+ \left[ \frac{1}{3} + \frac{\gamma}{2} - \frac{1}{2} \log 2 \right] \left( \frac{m^2}{eB} \right)^2 - 4 \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k(k+1)} \left( \frac{m^2}{2eB} \right)^{k+1}$$

$$= -\frac{e^2 B^2}{8\pi^2} \left\{ \frac{1}{3} \log \frac{m^2}{eB} + 0.763969 + O \left( \frac{m^2}{eB} \log \frac{m^2}{eB} \right) \right\}.$$

(4)

Note that the coefficient, $-\frac{e^2 n^2}{24\pi^2}$, of the leading term, the log $\frac{m^2}{eB}$ term, is fixed by the (one-loop) QED beta function [19]. In (3), $\gamma$ is Euler’s constant, and $\zeta(s)$ is the Riemann zeta function [13]. Note that $\zeta'(-1) \approx -0.165421$.

It is instructive to contrast this small mass expansion, Eq. (4), with the large mass expansion, Eq. (3). In the small mass limit, analogous to the chiral limit in QCD, we see the appearance of logarithmic terms, analogous to the “chiral logs” of QCD. In addition, note that both even and odd powers of $\frac{m^2}{eB}$ appear in the small mass expansion, Eq. (4). On the other hand, in the large mass expansion, Eq. (3), there are no non-analytic log terms, and only even powers of $\frac{m^2}{eB}$ appear. So, we see that the one-loop Euler–Heisenberg effective action, which is given by the exact integral representation, has two very different expansions in the two limits of large and small electron mass. The transition between these two extreme regions is governed by whether the electron Compton wavelength, $\lambda_e$, is larger or smaller than the cyclotron radius, $\lambda_B$. In Fig. 1 we plot the exact Euler–Heisenberg effective action, Eq. (3), with an overall factor of $-\frac{e^2 B^2}{8\pi^2}$ removed, as a function of $\frac{m^2}{eB}$, and compare it to the leading large mass term $-\frac{1}{16} \left( \frac{eB}{m} \right)^2$ from Eq. (3), and to the leading small mass terms $\frac{1}{3} \log \frac{m^2}{eB} + 0.763969$ from Eq. (4). From this figure it is clear that these leading terms accurately capture the extreme behaviours of the exact result, but do not interpolate in the intermediate region where the scales are comparable.
FIG. 1. Comparison between the exact action (solid curve) for the Euler-Heisenberg model and the leading terms in the expansions about the weak (dashed curve) and strong field (dash-dot curve) limits. Note that $m^2$ is measured in units of $eB$.

Having reviewed these pertinent aspects of the Euler–Heisenberg effective action, we now turn to what appears at first glance to be a completely different problem: the calculation of hadron properties as a function of quark mass, or through the Gell-Mann–Oakes–Renner relation ($m^2 \propto m_q$), pion mass. Chiral perturbation theory permits a rigorous expansion of hadron properties about the chiral limit, where $m_\pi \to 0$. For example, for the nucleon charge radius one finds \[ \langle r^2 \rangle_E = c_1 \pm \chi_N \log \frac{m_\pi}{\mu} + c_2 m_\pi^2 + \ldots \] (5)

where $\pm$ refers to the proton or neutron respectively. (Here $\mu$ just sets the scale against which the pion mass is measured. It is arbitrary in the sense that a change in $\mu$ is equivalent to a change in the constant term, $c_1$.) Note that the charge radius diverges logarithmically in the chiral limit, with a model independent coefficient \[ \chi_N = -\frac{(1 + 5g_\Lambda^2)}{4\pi f_\pi^2}. \] (6)

On the other hand, in the large $m_\pi$ limit, heavy quark effective theory suggests that the charge radius should decrease as \[ \langle r^2 \rangle_E = \frac{\bar{c}}{m_\pi^2} + \ldots \] (7)

plus higher inverse powers of $m_\pi^2$. (In the heavy quark limit one has essentially a Coulombic problem and the charge radius is proportional to the Bohr radius which goes as $1/m_q$ and hence $1/m_\pi$.)

As discussed at length in Ref. [3], current lattice data for charge radii are confined to pion masses greater than 600 MeV. The corresponding pion Compton wavelength, $\lambda_\pi$, is then smaller than the calculated charge radius, which we may take as an indication of the size, $R$, of the source of the pion field. The lattice data shows only a very slow variation of $\langle r^2 \rangle_E$ in the mass range where the lattice calculations have been made, with no indication of a chiral log. Yet, in order to compare with the physical charge radii one must extrapolate these lattice results to the chiral regime where $\lambda_\pi \gg R$ and the chiral log is important. This is the challenge of chiral extrapolation.

We wish to draw an analogy between the Euler–Heisenberg system discussed above and this system. In this analogy, the pion Compton wavelength, $\lambda_\pi$, plays the role of the electron Compton wavelength, $\lambda_e$, and the source size, $R$, plays the role of the magnetic cyclotron radius, $\lambda_B$, (equivalently, the mass scale $\mu^2$ plays the role of the magnetic field strength $eB$). The chiral perturbation theory expansion of Eq. (5), where $\lambda_\pi \gg R$, is analogous to the leading terms
in the small mass expansion of Eq. (4), where $\lambda_e \gg \lambda_B$. The heavy quark effective theory result presented in Eq. (6), where $\lambda_\pi \ll R$, is similarly analogous to the leading term in the large mass expansion in Eq. (3) where $\lambda_e \ll \lambda_B$.

In the QCD context, following earlier studies of magnetic moments [2], where it was found that a simple Padé approximant was able to describe the mass dependence arising in a particular chiral quark model, Hackett-Jones et al. [3] extrapolated the lattice data from $m_\pi^2 > 0.4 \text{GeV}^2$ to $m_\pi^2 = 0.02 \text{GeV}^2$ (the physical point) using an interpolating formula which was chosen as the simplest two-parameter form consistent with the constraints imposed by the extreme behaviours in the large and small pion mass limits, Eq. (7) and Eq. (5) respectively. (Recall that $\chi_N$ is model independent, and note that the data could constrain no more than two parameters.) In the light of later experience [1], we choose to use a slightly modified argument in the chiral log:

$$\langle r^2 \rangle_E = \frac{c_1 \pm \chi_N \log \frac{m^2}{\mu^2 + m^2}}{1 + \bar{c}_2 m^2}.$$  

(8)

Here, rather than being arbitrary, $\mu$ assumes physical significance as the scale above which the chiral log is suppressed — of course, Eq. (8) preserves the correct behaviour in the chiral limit. From experience with moments of structure functions, magnetic moments and hadron masses, this scale is expected to be $\mu \sim 500 \text{ MeV}$. As the lattice data is not yet able to constrain $\mu$, we simply fix it to 500 MeV and adjust only $c_1$ and $\bar{c}_2$. Figure 3 shows the resulting fit to the proton charge radius and the corresponding extrapolation to the physical pion mass. As discussed in [3], this chiral extrapolation fit is closer to the physical value than a naive linear fit through the lattice data. However, in the absence of lattice data at lower quark masses, it is difficult to be more precise about the quality of the fit.

![Graph showing the fit to the lattice QCD data for the square of the proton charge radius as a function of pion mass squared, using Eq. (8). The extrapolated value at the physical pion mass (indicated by the vertical dotted line) is shown by the solid dot with the large error bar, while the star indicates the experimentally observed value.](image)

**FIG. 2.** Fit to the lattice QCD data for the square of the proton charge radius as a function of pion mass squared, using Eq. (8). The extrapolated value at the physical pion mass (indicated by the vertical dotted line) is shown by the solid dot with the large error bar, while the star indicates the experimentally observed value.

In view of the close parallel between this hadronic problem and the Euler–Heisenberg system in QED, we return to the Euler–Heisenberg system, where we can be much more quantitative concerning the accuracy of an interpolating fit. We ask the following question. Suppose that we did not know the exact integral representation answer (2) for the effective action, but that we did know the leading terms in each of the extreme large and small mass limits. Would it then be possible to find a simple two-parameter interpolating formula, analogous to (8), that connected the extreme limits in a smooth manner? And if so, how accurate would such an interpolating formula be in the intermediate region?

The leading terms are determined as follows. In the large mass limit, this is the first term, $\frac{m^4}{m_\pi^4} \left( \frac{eB}{m_\pi} \right)^4$, in Eq. (3), corresponding to the first nonlinear correction to classical electrodynamics, whose coefficient comes from the one-fermion loop with four external photon lines, a straightforward perturbative calculation. In the small mass limit,
the leading term in Eq. (4) is the logarithmic term, \(-\frac{m^4}{24\pi^2} \left( \frac{2}{\pi} \right)^2 \log \frac{eB}{m^2} \), whose coefficient is fixed by the one-loop QED beta function [19]. Motivated by the interpolation formula, Eq. (8), which was used in the QCD case, we propose the following interpolating function for the effective action

\[
S_{\text{interpolating}} = -\frac{e^2 B^2}{8\pi^2} \left( \frac{d_1}{1 + \frac{1}{3} \log \left( \frac{m^2 + eB}{m^2 eB} \right)} - d_2 \frac{m^2}{eB} \right). 
\] (9)

This interpolating formula has the correct leading behaviour in both the large and small \(m\) limits. Figure 3 shows a comparison of the fit obtained with this form by adjusting the two parameters \(d_1\) and \(d_2\) (dash–dot curve) with the exact result (solid curve). Our best fit was obtained with parameter values: \(d_1 = 0.7059\), and \(d_2 = 1.5541\). Figure 3 also shows the percentage difference between the exact result and approximate expressions (dashed line). (Note that \(m^2\) is expressed in units of \(eB\).) Over the entire range of \(m^2\), the interpolating function is within 10% of the exact answer. Such precision is very surprising when we recall that the Euler-Heisenberg effective action has the problems (shared by the analogous QCD calculations) that the large mass expansion is asymptotic and the small mass expansion has a log divergence and a small radius of convergence.

![Figure 3](image.png)

**FIG. 3.** Comparison between the exact expression for the action in the Euler-Heisenberg model (solid line) and the interpolating approximation given in Eq. (9), which builds in the correct chiral and heavy quark limits (dot–dashed line). Note that the agreement is so good that it is difficult to distinguish between the two curves on this scale. The percentage difference between the two is indicated by the dashed line.

In summary, the Euler-Heisenberg system presents a problem which exhibits many of the mathematical complications of the chiral extrapolation problem in QCD, yet it is exactly soluble. By carefully respecting both the high and low mass limits of the exact solution, we showed how to construct a simple formula which reproduced the exact solution over the entire parameter range with surprisingly good accuracy. Of course, in the Euler-Heisenberg case we have the advantage of fitting the exact function over the entire mass range, while in QCD we have to extrapolate from large quark mass (where lattice data is available) to the chiral limit. Nevertheless, the fact that the mathematical structure of the two problems is identical, combined with the success achieved in the Euler-Heisenberg problem, gives us considerable confidence that a similar level of accuracy may be obtainable for QCD. It is therefore extremely encouraging that the chiral extrapolation of even the present crude lattice data at very large quark masses yields a physical proton charge radius within one standard deviation of the experimental value. Even more important, this result lends enormous impetus to the quest for new lattice data at lower quark mass which will better constrain the chiral extrapolation. It suggests that the next generation of supercomputers (available within 2–3 years) may well provide sufficient information that, in combination with these chiral extrapolation techniques, one should be able to calculate accurate hadron properties at the physical quark mass.
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