Exit problem as the generalized solution of Dirichlet problem

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Abstract

This paper investigates sufficient conditions for a Feynman-Kac functional up to an exit time to be the generalized viscosity solution of a Dirichlet problem. The key ingredient is to find out continuity of exit operator under Skorokhod topology, which reveals the intrinsic connection of overfitting Dirichlet boundary with fine topology. As an application, we establish the sub and supersolutions for a class of non-stationary HJB (Hamilton-Jacobi-Bellman) equations with fractional Laplacian operator via Feynman-Kac functionals associated to α-stable processes, which enables us to verify its solvability together with comparison principle and Perron’s method.

Keywords Stochastic control problem, HJB equation, Dirichlet boundary, Generalized viscosity solution, α-stable process, Fractional Laplacian operator, Fine topology.

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1 Introduction

Consider Feynman-Kac functional of the form
\[ v(x) := \mathbb{E} \left[ \int_0^\zeta e^{-\lambda s} \ell(X(s)) ds + e^{-\lambda \zeta} g(X(\zeta)) \middle| X(0) = x \right] \] (1)
and Dirichlet PDE given by
\[ -\mathcal{L} u(x) + \lambda u(x) - \ell(x) = 0 \text{ on } O, \quad \text{with } u = g \text{ on } O^c. \] (2)

In the above, \( X \) is Càdlàg Feller with generator \( \mathcal{L} \), denoted by \( X \sim \mathcal{L} \); \( O \) is a connected bounded open set in \( \mathbb{R}^d \); and \( \zeta \) is the exit time from \( \bar{O} \), denoted by \( \zeta = \tau_{\bar{O}}(X) \). Our main interest is to answer if \( v \) solves PDE in an appropriate sense:

(Q) When is the function \( v \) of (1) the generalized viscosity solution of (2)?

To illustrate the motivation of the above question, let us start with a non-stationary nonlinear Dirichlet problem as motivation. Consider
\[
\begin{cases}
-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0, & \text{on } Q_T := (0, T) \times O; \\
u = 0, & \text{on } \mathcal{P}Q_T := (0, T] \times \mathbb{R}^d \setminus Q_T.
\end{cases}
\] (3)

In the above, \( \alpha \in (0, 2] \) is the index of the fractional Laplacian operator. Such form of equations naturally arises in many applications. If \( \gamma = 1 \), then (3) becomes an HJB equation with
\[ -|\nabla_x u| = \inf_{b \in H^1_1} (b \cdot \nabla_x u), \]
see [10], also its important roles to stochastic control problem [14, 22, 25, 26]; If \( \gamma > 1 \), then (3) becomes deterministic KPZ equation, see [1]. It can also be regarded as HJB equation by seeing that
\[ -|\nabla_x u|^\gamma = \inf_{b \in \mathbb{R}^d} (-b \cdot \nabla_x u + L(b)) \]
with \( L(b) = \sup_{p \in \mathbb{R}^d} (p \cdot b - H(p)) \) being Legendre transform of the function \( H(p) = |p|^\gamma \), see Section 3.3 of [12].

For the solvability of the nonlinear PDEs, like (3), in the sense of generalized solution, most existing literatures studies analytical tools, such as the comparison principle (CP) and Perron’s method (PM), see [2] and the references therein. Such approach usually establishes unique solvability under the assumption that there exists a supersolution and a subsolution, which is not trivial to be verified and was proposed as an open question for the general case by Example 4.6 of [10].

Regarding the solvability of PDE (3), the existence of a sub and supersolution can be verified as follows. Since \( u = 0 \) is a trivially supersolution, it’s sufficient to find out a subsolution. Moreover, due to the non-positivity of \(-|\nabla u|^\gamma\), subsolution of (3) exists as long as
\[ -\partial_t u + (-\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T; \quad u = 0 \text{ on } \mathcal{P}Q_T \] (4)
is solvable. By a change of variables, one can convert the solvability of the (strong) solution of non-stationary problem (4), and further to (3), into the solvability of generalized solution of stationary problem (2) associated to some \( d + 1 \)-dimensional diffusion. The details are discussed in Section 4.

Likewise, the study of an elliptic or parabolic PDE and its interplay with the corresponding stochastic representation have a wide range of applications and many successful connections to other disciplines.
outside of mathematics. For instance in mathematical finance, the general approach to the derivative pricing is either given by the solution of a Cauchy problem or martingale approach, see [19]. The most well-known and practical one in this direction is Feynman-Kac formula, see Chapter 8 of [20]. However, a rigorous equivalence between PDE and its stochastic representation is not a trivial matter in many situations, see [16].

Different from the aforementioned references, a Feynman-Kac functional of the form (1) has its integral up to an exit time instead of a fixed time, and at least formally, such a problem usually leads to Dirichlet PDE. Nevertheless, a general verification that connects the PDE and the stochastic representation is even more difficult, due to the subtle boundary behavior of diffusions. For instance, even if the generator is simply given by Laplacian operator $L = \Delta$, it is non-trivial to show that the Feynman-Kac functional $v$ of (1) solves (2), see its proof is in Section 4.4 and 4.7 of [7]. If $L$ is given as a second order differential operator, then $X \sim L$ has almost surely continuous sample path, and the relation between Feynman-Kac functional and Dirichlet problem is discussed in [4, 13, 14, 15, 18], and the references therein. However, if $L$ is a non-local operator, the discontinuity of the diffusion $X \sim L$ gives extra difficulty in studying the boundary behavior, and there are relatively few references available for such an extension, see [3, 9].

In this paper, we work with a nonlocal operator $L$. This often corresponds to Levy jump diffusions, which become popular in the recent development in financial modeling, see [8, 11, 21]. To the best of our knowledge, the question (Q) on the verification of Feynman-Kac functional as a generalized viscosity solution of the Dirichlet PDE, has not been studied for jump diffusion in the extant literature. A few papers closely related to the question (Q), such as [9] for one dimensional non-stationary problem and [3] for multi-dimensional stationary problem, provide the following partial answer: $v$ of (1) is the (strong and thus generalized) viscosity solution of (2), if\(^1\)

- $O^c$ is the fine closure of $\bar{O}^c$ w.r.t. $L$.

However, this condition does not hold in general. For instance, if one consider the $\mathbb{R}^{d+1}$-valued process $(t, X_t) \sim \partial_t u - (-\Delta)^{\alpha/2} u$ associated to (4), the fine closure of $(\bar{Q}_T)^c$ is a proper subset of $Q_T^c$ since every point in $\{(t, x) \in \partial Q_T : t = 0\}$ is an irregular point for $(\bar{Q}_T)^c$.

In this paper, our main result (see Theorem 11 for more details) shows that $v$ of (1) is the generalized viscosity solution of (2), if

- $X$ exits from $O$ at a point of the fine closure of $\bar{O}^c$ almost surely

plus some technical conditions. Note that, according to Theorem 3.4.2 of [7] on Balayage theory, $X$ exits from $\bar{O}$ at a point of the fine closure of $\bar{O}^c$ almost surely. Example 2 provides an explicit counter-example for the statement: $v$ of (1) is the generalized viscosity solution of (2), if $X$ exits from $\bar{O}$ at a point of the fine closure of $\bar{O}^c$ almost surely.

Before closing this section, we illustrate the following simple scenario, which will be resorted to throughout the paper.

**Example 1.** Consider a Dirichlet problem on one dimensional domain given by the following setup:

\[ O = (0, 1), \ell \equiv 1, L u = \frac{1}{2} \epsilon^2 u'' + u', \lambda = 1, g \equiv 0. \]

Then (2) becomes a second order ODE

\[ -u' - \frac{1}{2} \epsilon^2 u'' + u - 1 = 0 \text{ on } (0, 1), \text{ and } u(x) = 0 \text{ for } x \geq 1 \text{ and } x \leq 0. \]

\(^1\)We refer the related description for fine topology to Chapter 3 of [7].
It is standard that, if $\epsilon > 0$, then $v_{\epsilon}$ of (1) associated to the process $X \sim L^c$ is a viscosity solution of (2), see Appendix A.4 for the explicit solution. However, if $\epsilon = 0$, then one can apply definition directly to the explicit formula $v_0(x) = -e^{-1+x} + 1$ to verify that $v_0$ is not a viscosity solution, but a generalized viscosity solution of (2). In particular, $v_0$ loses its boundary at $x = 0$, i.e. it does not satisfy the boundary condition $u(0) = 0$, while it satisfies $u(1) = 0$. \hfill \Box

Let us use our main result to explain the behavior of Example 1 above. Indeed, if $\epsilon = 0$, among the (Euclidean) boundary points of the set $\bar{O}^c = \{0, 1\}$, only $\{1\}$ is regular to $\bar{O}^c$. Hence, fine closure of $\bar{O}^c$ is $(-\infty, 0) \cup [1, \infty)$, which is a proper subset of $O^c$. and $v_0$ fails to be a (strong) viscosity solution. However, $X \sim L^0$ exits from the open set $O$ only at regular point $\{1\}$. Therefore, it is a generalized viscosity solution.

Next in Section 2, we present the precise setup and define (strong) viscosity solution and generalized viscosity solution. To avoid unnecessary confusion, we emphasize that “strong” v.s. “generalized” are in contrast for the classification of viscosity solution by its boundary behavior, see Definition 1 and Definition 2. Another possible classification is “classical” v.s. “viscosity” or “weak” solution by its smoothness. In this paper, we only focus on the former classification. Section 3 provides the analysis leading to the sufficient conditions for the existence of the generalized viscosity solution for a class of Dirichlet problems, which is our main objective. Applying this result, we are able to prove the solvability of a class of equations with fractional Laplacian operators in Section 4. At the end, we include a brief summary and some appendices.

\section{Problem setup and definitions}

\subsection{Problem setup}

Let $\mathbb{D}^d$ be the space of C\"adl\"ag functions on $[0, \infty)$ with Skorokhod metric, and consider a progressively measurable process $X$ with respect to a fixed filtered probability space $(\Omega = \mathbb{D}^d, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t, t > 0\})$ satisfying the usual conditions. Throughout the paper, we assume the following without further mentioning.

\begin{itemize}
  \item[(H)] 1. $X$ is C\"adl\"ag Feller process with its infinitesimal generator $\mathcal{L}$, denoted by $X \sim \mathcal{L}$;
  2. The domain of $\mathcal{L}$ contains $C^2_0(\mathbb{R}^d; \mathbb{R})$;
  3. $O$ is a connected bounded open set in $\mathbb{R}^d$;
  4. $g, \ell \in C^{0,1}_0(\mathbb{R}^d, \mathbb{R})$ and $\lambda > 0$.
\end{itemize}

In the above, $C^2_0$ is the collection of second order differentiable functions vanishing at infinity, and $C^{0,1}_0$ is the collection of the Lipschitz functions vanishing at infinity. Given a Borel set $B$ in $\mathbb{R}^d$ and a sample path $\omega \in \mathbb{D}^d$, define the exit time as $\tau_B(\omega)$ and the exit point as $\Pi_B(\omega)$:

$$
\tau_B(\omega) = \inf\{t > 0, \omega_t \notin B\}, \quad \Pi_B(\omega) = \omega_{\tau_B(\omega)}, \quad \forall \omega \in \mathbb{D}^d.
$$

(5)

For notational convenience, denote $\zeta := \tau_{\bar{O}}$ and $\Pi := \Pi_{\bar{O}}$. We are interested in the functional $F : \mathbb{D}^d \mapsto \mathbb{R}$ given by

$$
F(\omega) = \int_{0}^{\zeta(\omega)} e^{-\lambda s} \ell(\omega_s) ds + e^{-\lambda \zeta(\omega)} g \circ \Pi(\omega), \quad \forall \omega \in \mathbb{D}^d.
$$

For a given $x \in \mathbb{R}^d$, we use $\mathbb{P}^x$ to denote the probability measure on $\mathbb{D}^d$ induced by $X$ with initial state $x$, i.e. $\mathbb{P}^x(B) = \mathbb{P}^x(\omega \in B) = \mathbb{P}(X^x \in B)$ for all Borel set $B$ of $\mathbb{D}^d$, and $\mathbb{E}^x$ to denote the
expectation operator with respect to $\mathbb{P}^x$, and the following are equivalent

$$
\mathbb{E}^x[F] = \mathbb{E}^x[F(\omega)] = \mathbb{E}[F(X^x)] = \mathbb{E}[F(X)|X_0 = x],
$$

without further explanations. We can now rewrite $v$ of (1) as

$$
v(x) := \mathbb{E}^x[F] = \mathbb{E}\left[\int_0^\zeta e^{-\lambda s}\ell(X(s))ds + e^{-\lambda \zeta}g(X(\zeta))\big|X(0) = x\right].
$$

Our main goal is to investigate if $v$ is the generalized viscosity solution of (2). For this purpose, we give the precise definition of the generalized viscosity solution below.

### 2.2 Definitions

For simplicity, if we denote

$$
G(\phi, x) = -\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x),
$$

then (2) becomes

$$
G(u, x) = 0, \text{ on } O \text{ and } u = g \text{ on } O^c.
$$

Note that the Dirichlet boundary data $g$ is given to the entire $O^c$. The reason is that, if the generator $\mathcal{L}$ is non-local, $X_\zeta$ may fall in anywhere in $O^c$. For $(u, x) \in C_0^\infty(\mathbb{R}^d) \times \mathbb{R}^d$, the value $G(u, x)$ is well defined$^3$. To generalize the definition of (7) for a possibly non-smooth function with domain $O$, we use the following test functions in place of $u$.$^4$

1. For a given $u \in USC(\bar{O})$ and $x \in \bar{O}$, we define the space of supertest functions as

$$
J^+(u, x) = \{ \phi \in C_0^\infty(\mathbb{R}^d), \text{ s.t. } \phi \geq (uI_\bar{O} + gI_{O^c})^* \text{ and } \phi(x) = u(x) \}.
$$

2. For a given $u \in LSC(\bar{O})$ and $x \in \bar{O}$, the space of subtetst functions is given by,

$$
J^-(u, x) = \{ \phi \in C_0^\infty(\mathbb{R}^d), \text{ s.t. } \phi \leq (uI_\bar{O} + gI_{O^c})_* \text{ and } \phi(x) = u(x) \}.
$$

We say a function $u \in USC(\bar{O})$ satisfies the viscosity subsolution property at some $x \in \bar{O}$, if the following inequality holds for all $\phi \in J^+(u, x)$,

$$
G(\phi, x) \leq 0.
$$

(8)

Similarly, a function $u \in LSC(\bar{O})$ satisfies the viscosity supersolution property at some $x \in \bar{O}$, if the following inequality holds for all $\phi \in J^-(u, x)$,

$$
G(\phi, x) \geq 0.
$$

(9)

In the following we define the (strong) viscosity solution of (2). Note that it does not require the viscosity property on any point $x \in \partial O$. Indeed, the concept of the viscosity solution property at $x \in \partial O$ will be needed only in the definition of the generalized viscosity solution introduced later.

**Definition 1.** 1. $u \in USC(\bar{O})$ is a viscosity subsolution of (2), if (a) $u$ satisfies the viscosity subsolution property on each $x \in O$ and (b) $u(x) \leq g(x)$ on each $x \in \partial O$.

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$^2$The superscript $x$ in $X^x$ indicates the initial state, and is omitted in the rest of the paper if no ambiguity arises.

$^3$\(C_0^\infty(\mathbb{R}^d)\) is the space of functions with derivatives of all orders vanishing at infinity.

$^4$ $f \in USC(\bar{O})$ means $f$ is upper semicontinuous in $\bar{O}$, and $f \in LSC(\bar{O})$ means $-f \in USC(\bar{O})$. Moreover, $f^*$ and $f_*$ are USC and LSC envelopes of $f$, respectively. $I_A(\cdot)$ is the indicator function of the set $A$. 

5
2. \( u \in LSC(\bar{O}) \) is a viscosity supersolution of (2), if (a) \( u \) satisfies the viscosity supersolution property on each \( x \in O \) and (b) \( u(x) \geq g(x) \) on each \( x \in \partial O \).

3. \( u \in C(\bar{O}) \) is a viscosity solution of (2), if it is the viscosity subsolution and supersolution simultaneously.

Recall that in Example 1 above, the stochastic representation \( v_\epsilon \) of (1) is the viscosity solution of equation (2) if \( \epsilon \) is strictly positive. It is not anymore for \( \epsilon = 0 \) due to the loss of the boundary value \( v_0(0) > 0 \). However, if we have a more careful check of Example 1, although \( v_0 \) violates the boundary condition at \( x = 0 \), it satisfies viscosity solution property at \( x = 0 \) according to (8) - (9). In fact, at \( x = 0 \),

1. the space of supertest functions satisfies
   \[
   J^+(v_0, 0) \subset \{ \phi \in C^\infty_c(\mathbb{R}) : \phi(0) = v_0(0) = 1 - e^{-1}, \phi'(0) \geq v_0'(0+) = -e^{-1} \}.
   \]
   Therefore, \( v_0 \) satisfies subsolution property at \( x = 0 \);

2. the space of subtest functions \( J^-(v_0, 0) \) is an empty set, because \( v_0(0) > 0 \) and \( (v_0 I_{\bar{O}})_* (0) = 0 \), and it automatically implies its supersolution property at \( x = 0 \).

The above observation suggests the following definition of the generalized viscosity solution, which relaxes the boundary condition imposed in Definition 1.

**Definition 2.** A function \( u \in C(\bar{O}) \) is said to be a generalized viscosity solution of (2), if \( u \) satisfies the viscosity solution property at each \( x \in \bar{O} \setminus \Gamma_{out} \), where

\[
\Gamma_{out} = \{ x \in \partial O : u = g \}.
\]

In the above, \( \Gamma_{out} \) stands for a subset of \( \partial O \) where the candidate solution \( u \) meet the boundary value \( g \). Generalized viscosity solution does not require \( u = g \) at all points of \( \partial O \), and requires instead the viscosity solution property at those points losing its boundary value. For convenience, we set the boundary points losing its boundary value as \( \Gamma_{in} \), i.e. \( \Gamma_{in} = \partial O \setminus \Gamma_{out} \). By definition, the generalized viscosity solution is (strong) viscosity solution if and only if \( \Gamma_{out} = \partial O \). In the previous discussion of Example 1, although \( v_0 \) is not the viscosity solution due to the loss of the boundary on \( \Gamma_{in} = \{ 0 \} \), it is indeed a generalized viscosity solution to (2).

Concerning the question (Q), we can divide it into the following two subquestions:

1. Is \( v \) of (1) a generalized viscosity solution of (2)?

2. If yes, where does \( v \) meet its boundary? i.e. what is \( \Gamma_{out} \)?

**3 The existence of generalized viscosity solution**

**3.1 Sufficient conditions - I**

As we will see in this part, Example 2 below indicates that \( v \) of (1) may not be a generalized solution of (2) just by postulating (H). Lemma 4 and Lemma 5 point out a sufficient condition: \( v \) of (1) is a generalized solution of (2), if

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5The generalized viscosity solution has been studied in the literature with slightly different form as of Definition 16, which is indeed equivalent to our Definition 2, see Appendix A.1 for details.
• $\zeta : \mathbb{D}^d \mapsto \mathbb{R}$ is continuous almost surely.

Before discussing this sufficient condition, let us briefly recall the regularity and the related fine topology, and we refer Section 3.4 of [7] for the details.

**Definition 3.** A point $x$ is said to be regular (w.r.t. $\mathcal{L}$) for the set $B$ if and only if $\mathbb{P}^x(\tau_{B^c} = 0) = 1$.

The set of all regular points for $B$ will be denoted by $B^r$ and $B^* = B \cup B^r$ is called fine closure of $B$. Obviously, the regularity of a point $x$ for a set $B$ is associated to the underlying diffusion $X \sim \mathcal{L}$. In particular, since the diffusion has right continuous path and $O$ is an open set, any point $x \in \bar{O}^c$ is regular to $\bar{O}^c$ and any point $x \in O$ is not regular to $\bar{O}^c$. Therefore, the set of all regular points to $\bar{O}^c$, denoted by $\bar{O}^c,^r$, is between the two sets $O^c$ and $\bar{O}^c$. Hence, we have $\bar{O}^c \subset \bar{O}^c,^r = \bar{O}^c,^* \subset O^c = \bar{O}^c,^* = \bar{O}^c,^*$.

It is standard, by using Ito’s formula on test functions, an interior point $x$ of the domain $\bar{O}$ satisfies viscosity solution property given that $v$ of (1) is continuous. Next, Lemma 4 shows that the same statement holds as long as $x$ is an interior point of $\bar{O}$ in fine topology, i.e. $x$ is not regular to $\bar{O}^c$.

**Lemma 4.** If $v \in C(\bar{O})$, then $v$ of (1) is a generalized viscosity solution of (2), with

$$\Gamma_{out} \supset \bar{O}^c,^* \cap \partial O.$$ 

**Proof.** Recall (6) that,

$$G(\phi, x) = -\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x).$$

1. In this part, we show $v$’s interior viscosity solution property. First, fix an arbitrary $x \in O$, and show $v$ satisfies the viscosity supersolution property, i.e.

$$G(\phi, x) \geq 0, \text{ for every } \phi \in J^-(v, x). \quad (10)$$

To the contrary, assume $G(\phi, x) < 0$ for some $\phi \in J^-(v, x)$. By the continuity of $x \mapsto G(\phi, x)$, there exists $\delta > 0$ that

$$\sup_{|y-x|<\delta} G(\phi, y) < -\epsilon/2. \quad (11)$$

Since $X$ of (19) is a Càdlàg process and $x \in O$, the first exit time satisfies $\mathbb{P}^x(\zeta > 0) = 1$. By the strong Markov property of the process $X$, we rewrite the function $v$ as, for any stopping time $h \in (0, \zeta]$

$$v(x) = \mathbb{E}^x\left[e^{-\lambda h}v(X_h) + \int_0^h e^{-\lambda s}\ell(X_s)ds\right],$$

which in turn implies that, with the fact of $\phi \in J^-(v, x)$

$$\phi(x) \geq \mathbb{E}^x\left[e^{-\lambda h}\phi(X_h) + \int_0^h e^{-\lambda s}\ell(X_s)ds\right].$$

Moreover, Dynkin’s formula on $\phi$ gives

$$\mathbb{E}^x[e^{-\lambda h}\phi(X_h)] = \phi(x) + \mathbb{E}^x\left[\int_0^h e^{-\lambda s}(\mathcal{L}\phi(X_s) - \lambda\phi(X_s))ds\right].$$

By adding up the above two formulae together, it yields that

$$\mathbb{E}^x\left[\int_0^h e^{-\lambda s}G(\phi, X_s)ds\right] \geq 0.$$
Then, we take \( h = \inf\{t > 0 : X(t) \notin B_{\delta}(x)\} \land \zeta \) in the above and note that \( h > 0 \) almost surely in \( \mathbb{P}^x \). This leads to a contradiction to (11) and implies the supersolution property at \( x \). The interior subsolution property can be similarly obtained.

2. Next we show its generalized boundary condition. We consider an arbitrary fixed \( x \in \partial O \). By Blumenthal 0-1 law, there are only two cases: either \( \mathbb{P}^x(\zeta = 0) = 1 \) or \( \mathbb{P}^x(\zeta > 0) = 1 \). In the former case, if \( \mathbb{P}^x(\zeta = 0) = 1 \), then \( v(x) = g(x) \) by its very definition (1) and hence \( \Gamma_{out} \supset \bar{O}^{c,*} \cap \partial O \) holds. In the latter case, if \( \mathbb{P}^x(\zeta > 0) = 1 \), then we shall examine its viscosity solution property.

For the viscosity supersolution property, assume (11) holds for some \( \phi \in J^-(v,x) \) and \( \delta > 0 \). Since \( \mathbb{P}^x(\zeta > 0) = 1 \), we can follow exactly the same procedure in the above proof of interior viscosity solution property to find a contradiction, which justifies the supersolution property. The subsolution property can be obtained in the similar way as the supersolution property above. \( \square \)

The following are two remarks about the results in Lemma 4.

- From the definition of generalized solution, \( \Gamma_{out} \) can be treated as part of solution. Therefore, in terms of characterization of unknown \( \Gamma_{out} \), it seems not satisfactory to have “\( \supset \)” sign instead of “\( = \)” sign as its conclusion. However, it is indeed full characterization by noting that left hand side \( \Gamma_{out} \) depends on the boundary value \( g \) via \( v \), while the right hand side \( \bar{O}^{c,*} \cap \partial O \) is invariant of \( g \). More precisely, if we temporarily sacrifice our notational simplicity by denoting \( \Gamma_{out}[g] \) for \( \Gamma_{out} \), then we could actually have

\[
\bigcap_{g \in C^{0,1}(\mathbb{R}^d, \mathbb{R})} \Gamma_{out}[g] = \bar{O}^{c,*} \cap \partial O
\]

under some mild conditions, see Appendix A.2. However, we will keep the notion \( \Gamma_{out} \) in the rest of the paper for simplicity.

- In Lemma 4, the continuity of \( v \) up to the boundary is a condition, which may not be true in general. Next, we will show an example for \( v \) of (1) being discontinuous even in the interior of the domain.

**Example 2.** Consider a problem on two dimensional domain of

\[
O = (-1,1) \times (0,1), \quad \mathcal{L}u(x) = \partial_{x_1}u(x) + 2x_1\partial_{x_2}u(x), \quad \lambda = 1 \quad \text{and} \quad \ell = 1, g \equiv 0.
\]

Then, PDE (2) becomes

\[
-\partial_{x_1}u(x) - 2x_1\partial_{x_2}u(x) + u(x) - 1 = 0, \quad \text{on} \ O, \quad \text{and} \ u(x) = 0 \ \text{on} \ O^c.
\]

In fact, the process \( X \sim \mathcal{L} \) with initial \( x = (x_1, x_2)^T \) has the following deterministic parametric representation,

\[
X_1(t) = x_1 + t, \quad X_2(t) = x_2 - x_1^2 + X_1^2(t).
\]

Therefore, the lifetime \( \zeta \) is also a deterministic number depending on its initial state \( x \), which will be denoted by \( \zeta^x \). Then, the explicit calculation leads to

\[
\zeta^x = -x_1 + \sqrt{1 - x_2 + x_1^2}, \quad \forall x \in O_1 := \{x_2 \geq x_1^2\} \cap \bar{O},
\]

\[
\zeta^x = 1 - x_1, \quad \forall x \in O_2 := \{x_2 < x_1^2, x_1 > 0\} \cap \bar{O},
\]

and

\[
\zeta^x = -x_1 - \sqrt{-x_2 + x_1^2}, \quad \forall x \in O_3 := \{x_2 < x_1^2, x_1 < 0\} \cap \bar{O}.
\]

One can observe that the mapping \( x \mapsto \zeta^x \) is discontinuous at every point on the curve \( \partial O_1 \cap \partial O_3 \), and so is \( v \) of (1), which can be rewritten as

\[
v(x) = \int_0^{\zeta^x} e^{-s}ds = 1 - e^{-\zeta^x}.
\]
Example 2 together with Lemma 4 lead us to investigate the sufficient condition for the continuity of the function $v$.

**Lemma 5.** Let $x \in \bar{O}$. If $\zeta : \mathbb{D}^d \rightarrow \mathbb{R}$ and $\Pi : \mathbb{D}^d \rightarrow \mathbb{R}^d$ are continuous in Skorokhod topology almost surely in $\mathbb{P}^x$, then $v$ of (1) is continuous at $x$ relative to $\bar{O}$, i.e. $\lim_{\bar{O} \ni y \rightarrow x} v(y) = v(x)$.

Before the proof, we recall that $\zeta$ and $\Pi$ were defined right after (5) as the exit time and exit point from the set $\bar{O}$. Let’s denote by $d_o$ for the Skorokhod metric in $\mathbb{D}^d$. By the definition of continuity, exit time $\zeta$ is continuous at some $\omega \in \mathbb{D}^d$, if
\[
\lim_n \zeta(\omega_n) = \zeta(\omega), \text{ as } \lim_n d_o(\omega_n, \omega) = 0.
\]
We denote as $C_\phi$ the continuity set of an arbitrarily given function $\phi : \mathbb{D}^d \rightarrow R$
\[
C_\phi = \{ \omega \in \mathbb{D}^d : \phi \text{ is continuous at } \omega \}.
\]
If there is a given probability $Q$ on a $\sigma$-algebra of $\mathbb{D}^d$, then $\phi$ is said to be continuous almost surely in $Q$, if $Q(C_\phi) = 1$. From Example 2, we see that $C_{\zeta}$ can be a proper subset of $\mathbb{D}^d$, and thus $\zeta$ may not be continuous everywhere. Lemma 5 implies that, for the continuity of $v$ at some point $x$, it suffices that the sets $C_\zeta$ and $C_\Pi$ are big enough so that $\zeta$ and $\Pi$ are continuous almost surely in $\mathbb{P}^x$, i.e. $\mathbb{P}^x(C_\zeta \cap C_\Pi) = 1$.

**Proof.** (of Lemma 5)
If $\bar{O} \ni y \rightarrow x$, then $\mathbb{P}^y$ converges to $\mathbb{P}^x$ weakly. By continuous mapping theorem (Theorem 2.7 of [6]) together with uniform boundedness of $F$, the lemma holds if $F$ is continuous almost surely in $\mathbb{P}^x$, i.e. $\mathbb{P}^x(C_F) = 1$ for the continuity set $C_F$ of $F : \mathbb{D}^d \rightarrow \mathbb{R}$. Since $\mathbb{P}^x(C_\zeta \cap C_\Pi) = 1$ is given, it suffices to show that $C_\zeta \cap C_\Pi \subseteq C_F$, i.e. “if $\zeta$ and $\Pi$ are continuous at $\omega$, then so is $F$”.

Let’s rewrite $F$ by $F = F_1 + F_2$, where
\[
F_1(\omega) = \int_0^{\zeta(\omega)} e^{-\lambda s} \ell(\omega_s) ds, \quad F_2(\omega) = e^{-\lambda \zeta(\omega)} g \circ \Pi(\omega).
\]
It is obvious that $F_2$ is continuous at a certain $\omega$ if $\zeta$ and $\Pi$ are continuous at the same $\omega$. For $F_1$, consider an arbitrary sequence $\omega_n \rightarrow \omega$ in Skorokhod metric, $|F_1(\omega_n) - F_1(\omega)|$ can be approximated by
\[
|F_1(\omega_n) - F_1(\omega)| \leq K_1 \int_0^{\zeta(\omega) \wedge \zeta(\omega_n)} e^{-\lambda s} \min \{|\omega_n(s) - \omega(s)|, K_2\} ds + K_2 |\zeta(\omega_n) - \zeta(\omega)|. \tag{12}
\]
In the above, $K_1$ and $K_2$ are two constants given by $K_1 = \max_{x \neq y} \frac{\ell(x) - \ell(y)}{x - y}$, $K_2 = 2 \max_{x \in \mathbb{R}^d} (|\ell(x)| + |g(x)|)$, of which the existence is implied by the conditions of (H). We observe that
\begin{itemize}
  \item $\omega_n \rightarrow \omega$ in Skorokhod metric implies that $\omega_n(t) \rightarrow \omega(t)$ holds for all $t \in C_\omega$, where $C_\omega$ is the continuity set of the function $\omega : [0, \infty) \rightarrow \mathbb{R}^d$, see Page 124 of [6]. Due to the fact that there are countably many discontinuities of $\omega$, we know
  \[
  \lim_{n \rightarrow \infty} |\omega_n(t) - \omega(t)| = 0
  \]
  holds for $t$ almost everywhere in Lebesgue measure.
  \item $\omega_n \rightarrow \omega$ in Skorokhod metric implies that $\zeta(\omega_n) \rightarrow \zeta(\omega)$ due to the continuity of $\zeta$.
\end{itemize}
Hence, each term of the right hand side of (12) goes to zero and uniformly bounded. Therefore, the limit of $|F_1(\omega_n) - F_1(\omega)|$ is also zero and $F$ is continuous at $\omega$. \hfill \blacksquare
3.2 Sufficient conditions - II

Lemma 4 and Lemma 5 lead us to investigate the continuity of \( \zeta \) and II, which is not always the case, as illustrated by Example 2. The main result of this section in Proposition 8 indicates that, with some technical conditions

- If starting from \( x \), \( X \) exits from \( O \) and \( \bar{O} \) at the same time, then \( \zeta \) and II are continuous a.s. in \( \mathbb{P}^x \), and thus \( v \) of (1) is a generalized solution of (2).

Note that this condition is violated in Example 2: for \( x \in \bar{O}_1 \cap \bar{O}_3, x_2 = x^2_1 \) and \( x_1 < 0 \). Thus \( \zeta \) (the exit time of \( \bar{O} \)) is \(-x_1 + \sqrt{1-x_2+x^2_1}\), while \( \hat{\zeta} \) (the exit time of \( O \) as defined below) is \(-x_1 + \sqrt{-x_2+x^2_1}\).

To proceed, we introduce the following notions. For a path \( \omega \in \mathbb{D}^d \), denote \( \omega^- \) as a Càdlàg version of \( \omega \), defined by

\[
\omega^-_0 = \omega_0, \quad \text{and} \quad \omega^-_t = \lim_{s \downarrow t} \omega_s, \quad \text{for} \quad t > 0,
\]

and the associated exit time operator:

\[
\tau^-_B(\omega) = \inf\{t > 0, \omega^-_t \notin B\}. \tag{13}
\]

If \( \omega \) is continuous, then \( \omega = \omega^- \) and \( \tau_B(\omega) = \tau^-_B(\omega) \). However, we shall not expect an equality or even an inequality between \( \tau_B \) and \( \tau^-_B \) in general, as demonstrated in the following example.

Example 3. Let \( B = (0,3) \) and a Càdlàg path \( \omega_t = |t-1| + I_{[0,1]}(t) \).

\[
\tau_B(\omega) = 1 < \tau^-_B(\omega) = 4.
\]

On the other hand, for another Càdlàg path \( \omega_t = 1 - tI_{[0,1]}(t) \),

\[
\tau_B(\omega) = \infty > \tau^-_B(\omega) = 1. \quad \square
\]

To discuss the continuity of the lifetime \( \zeta \), define

\[
\hat{\zeta}(\omega) = \tau_O(\omega), \quad \zeta^-(\omega) = \tau^-_O(\omega). \tag{14}
\]

By definition, the following inequality holds immediately,

\[
\max\{\hat{\zeta}(\omega), \zeta^-(\omega)\} \leq \zeta(\omega), \quad \forall \omega \in \mathbb{D}^d. \tag{15}
\]

Example 3 shows that neither \( \hat{\zeta} \geq \zeta^- \) nor \( \hat{\zeta} \leq \zeta^- \) is true. Interestingly, with the Càdlàg Feller process \( X \), the inequality \( \zeta^- \geq \hat{\zeta} \) holds almost surely.

Proposition 6. For any \( x \in \bar{O} \), the following identities hold:

\[
\mathbb{P}^x\{\omega^-(\zeta^-) \in \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\} = 0, \quad \mathbb{P}^x\{\hat{\zeta} \leq \zeta^- \leq \zeta\} = 1.
\]

Proof. Recall that, we defined the exit time as \( \tau_B(\omega) \) and the exit point as \( \Pi_B(\omega) \) in (5) as well as \( \tau^-_B(\omega) \) in (13) by

\[
\tau_B(\omega) = \inf\{t > 0, \omega_t \notin B\}, \quad \Pi_B(\omega) = \omega_{\tau_B(\omega)}, \quad \tau^-_B(\omega) = \inf\{t > 0, \omega^-_t \notin B\}, \quad \forall \omega \in \mathbb{D}^d.
\]

We set \( \zeta^+_1(\omega) = \zeta^-(\omega) \) if \( \omega^-(\zeta^-) \in \partial O \), otherwise infinity. Then, \( \zeta^+_1 \) is \( \mathcal{F}_{\zeta^-} \) stopping time, and hence a predictable stopping time.
If ω is discontinuous at ζ−, then ζ− is a totally inaccessible stopping time due to the jump by Meyer’s theorem; see Theorem III.4 of [23]. According to Theorem III.3 of [23], the set of predictable stopping times has no overlap with the set of totally inaccessible stopping times almost surely. Hence, we have $\mathbb{P}^x\{\omega^-(\zeta^-) \neq \omega(\zeta^-); \zeta_1(\omega) < \infty\} = 0$, which is equivalent to $\mathbb{P}^x\{\omega^-(\zeta^-) \in \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\} = 0$.

Since ω is continuous at ζ− almost surely whenever $\omega^-(\zeta^-) \in \partial O$, we have $\omega(\zeta^-) = \omega^-(\zeta^-) \in \partial O$ almost surely whenever $\omega^-(\zeta^-) \in \partial O$, i.e. $\hat{\zeta} \leq \zeta^-$ almost surely whenever $\omega^-(\zeta^-) \in \partial O$ by the definition. So we can write

$$\mathbb{P}^x\{\hat{\zeta} \leq \zeta^- | \omega^-(\zeta^-) \in \partial O\} = 1.$$ 

On the other hand, if $\omega^-(\zeta^-) \notin \partial O$, it must be $\omega^-(\zeta^-) \in O$. In this case, there exists $t_n \downarrow \zeta^-$ such that $\omega^-(t_n) \in O^c$. By the right continuity of ω, we get $\omega(\zeta^-) = \lim_{n \to \infty} \omega^-(t_n) \in O^c$ due to the closedness of $O^c$. Hence, $\hat{\zeta} \leq \zeta^-$ whenever $\omega^-(\zeta^-) \in O$. So we can write

$$\mathbb{P}^x\{\hat{\zeta} \leq \zeta^- | \omega^-(\zeta^-) \in O\} = 1.$$ 

With the fact $\mathbb{P}^x\{\omega^-(\zeta^-) \in \partial O\} \cup \{\omega^-(\zeta^-) \in O\} = 1$, we conclude $\mathbb{P}^x\{\hat{\zeta} \leq \zeta^-\} = 1$. The other inequality $\zeta^- \leq \hat{\zeta}$ is true by the definition.

Now, we can establish almost sure continuity of $\zeta$ and $\Pi$ if it starts from some point $x$ in $O$ or $\partial O \cap \bar{O}^{\circ,*}$.

**Proposition 7.** If $x \notin \partial O \setminus \bar{O}^{\circ,*}$ and $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$, then both $\zeta$ and $\Pi$ are almost surely continuous in $\mathbb{P}^x$.

**Proof.** If $x \notin \bar{O}$, then any ω with its initial state $x$ makes $\zeta(\omega) \equiv 0$ and $\Pi(\omega) \equiv x$ being constant mappings, and $\zeta$ and $\Pi$ are both continuous at $\omega$ with its initial $x \notin \bar{O}$.

If $x \in O$. A slight modification of the proof of Theorem 3.1 and Proposition 2.4 of [3] implies that, the mappings $\zeta : \mathbb{D}^d \to \mathbb{R}$ and $\Pi : \mathbb{D}^d \to \mathbb{R}^d$ are both continuous at any

$$\omega \in \Gamma := \{\omega : \omega(0) \in O\} \cap \Gamma_1 \setminus \Gamma_2$$

in Skorokhod topology, where two sets $\Gamma_1$ and $\Gamma_2$ are given by $\Gamma_1 = \{\omega : \zeta^- = \hat{\zeta} = \zeta\}$ and $\Gamma_2 = \{\omega : \omega^-(\zeta^-) \notin \partial O, \omega^-(\zeta^-) \neq \omega(\zeta^-)\}$. Moreover, Proposition 6 and the condition $\mathbb{P}^x(\zeta = \hat{\zeta}) = 1$ implies that $\mathbb{P}^x(\Gamma_1) = 1$ and $\mathbb{P}^x(\Gamma_2) = 0$. Therefore, $\mathbb{P}^x(\Gamma) = 1$ and we conclude almost sure continuity of $\zeta$ and $\Pi$ for this case.

Finally, if $x \in \partial O \cap \bar{O}^{\circ,*}$. Again using exactly the same approach of Theorem 3.1 and Proposition 2.4 of [3], we know that the mappings $\zeta : \mathbb{D}^d \to \mathbb{R}$ and $\Pi : \mathbb{D}^d \to \mathbb{R}^d$ are both continuous at any

$$\omega \in \Gamma := \{\omega : \omega(0) \in \partial O, \hat{\zeta} = \zeta = 0\}$$

in Skorokhod topology. Since $x \in \partial O \cap \bar{O}^{\circ,*}$ means $x$ is regular to $\bar{O}^c$, we have $\mathbb{P}^x(\Gamma) = 1$ and we conclude almost sure continuity of $\zeta$ and $\Pi$ for this case.

Following observation tells us that, if $x \in \partial O \setminus \bar{O}^{\circ,*}$, $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$ does not guarantee the almost sure continuity of $\zeta$ and $\Pi$ in $\mathbb{P}^x$, and additional conditions are needed in this case.

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6It is possible that $\mathbb{P}^x\{\omega^-(\zeta^-) \in \partial O\} = 0$. To avoid ambiguity, we set $\mathbb{P}(A|B) = 1$ whenever $\mathbb{P}(B) = 0$. 
Example 4. Consider $O = (0, 1)$ and $X(t) = t$. In other words, $\mathbb{P}^x\{\omega_0\} = 1$ for $\omega_0(t) = t$. Then, it satisfies both conditions $0 \in \partial O \setminus \bar{O}^{c,*}$ and $\mathbb{P}^x(\hat{\zeta} = \zeta = 1) = 1$. In particular, we recall definition of $\zeta$ of (14) that $\hat{\zeta} = 1$ instead of $\hat{\zeta} = 0$, since it is defined by hitting time $\inf\{t > 0 : \ldots\}$ instead of entrance time $\inf\{t \geq 0 : \ldots\}$. However, the sequence of process

$$\omega_n = \begin{cases} n^{-1} - 2t, & t \in [0, n^{-1}), \\ t, & t \geq n^{-1}, \end{cases}$$

satisfies $\lim_{n \to \infty} d^x(\omega_n, \omega_0) = 0$, while $\lim_{n \to \infty} \zeta(\omega_n) \to 0 \neq \zeta(\omega_0)$. $\square$

Proposition 8. Suppose $x \in \partial O \setminus \bar{O}^{c,*} \text{ and } \mathbb{P}^x(\hat{\zeta} = \zeta) = 1$. If in addition,

(A) There exists an open bounded set $O_1 \supset \bar{O} \setminus \bar{O}^{c,*}$ s.t. $\mathbb{P}^x\{\zeta_1 = \zeta\} = 1$, where $\zeta_1 = \tau_{O_1}$.

Then, both $\zeta$ and $\Pi$ are almost surely continuous in $\mathbb{P}^x$.

Proof. We first observe that

$$\mathbb{P}^x\{\zeta = \hat{\zeta} \leq \zeta_1 \leq 1 = \zeta\} = 1,$$

where $\zeta = \hat{\zeta}$ and $\zeta_1 = \zeta$ hold by assumption. For two inequalities, $\hat{\zeta} \leq \zeta_1$ holds due to the fact $O_1 \supset O$, and $\zeta_1 \leq 1$ holds by Proposition 6. Hence, this implies $\mathbb{P}^x\{\zeta = 1 = \zeta_1 = \zeta\} = 1$.

Moreover, if we define $\Pi_1 = \Pi_{O_1}$, then we have $\mathbb{P}^x\{\Pi_1 = \Pi\} = 1$, since

$$\Pi_1(\omega) = \omega(\zeta_1) = \omega(\zeta) = \Pi(\omega), \forall \omega \in \{\zeta_1 = \zeta\}.$$ 

Therefore, $x$ is an interior point of $O_1$ satisfying $\mathbb{P}^x\{\hat{\zeta}_1 = \zeta_1\} = 1$. Finally, we can apply Proposition 7 to obtain almost sure continuity of $\zeta_1$ and $\Pi_1$, which gives almost sure continuity of $\zeta$ and $\Pi$. $\square$

3.3 Sufficient conditions - III

Next, we seek for the connection between the condition $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$ used in Proposition 7 and 8 with the regularity structure. Using strong Markov property, we will show that it is indeed equivalent to the following statement:

- $X$ exits from $O$ at a point of the fine closure of $\bar{O}^{c}$ almost surely.

As a preparation, define the shift operator $\theta_t : \mathbb{D}^d \mapsto \mathbb{D}^d$ as

$$\theta_t \omega(s) = \omega(t + s), \forall s \geq 0.$$ 

This implies

$$(X_s \circ \theta_t)(\omega) = X_s(\theta_t \omega) = \theta_t \omega(s) = \omega(t + s) = X(t + s, \omega) = X_{t+s}(\omega).$$

(16)

Proposition 9. If $h \in [0, \hat{\zeta}(\omega)]$, then $\zeta \circ \theta_h(\omega) = \zeta(\omega) - h$ for all $\omega \in \mathbb{D}^d$.

Proof. By the definition of $\theta$, we have

$$\zeta \circ \theta_h(\omega) = \inf\{t > 0, \omega(t + h) \notin \bar{O}\} = \inf\{t' > h, \omega(t') \notin \bar{O}\} - h.$$ 

Observe that $\omega(t) \in O$ for all $t \in [0, h)$ due to $h \leq \hat{\zeta}$. Therefore,
1. if $\omega(h) \in \tilde{O}$, then we can write $\inf \{ t' > h, \omega(t') \notin \tilde{O} \} = \inf \{ t' > 0, \omega(t') \notin \tilde{O} \}$ by its definition of infimum;

2. if $\omega(h) \notin \tilde{O}$, then we have $\inf \{ t' > 0, \omega(t') \notin \tilde{O} \} = h$. But, we also have $\inf \{ t' > h, \omega(t') \notin \tilde{O} \} = h$ by the right continuity of $\omega$.

Hence, we always have

$$\inf \{ t' > h, \omega(t') \notin \tilde{O} \} = \inf \{ t' > 0, \omega(t') \notin \tilde{O} \} = \zeta(\omega).$$

The next lemma shows that, if the exit point $X(\hat{\zeta})$ falls in the regular point of $\tilde{O}^c$, then $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$.

**Lemma 10.** Let $x \in \tilde{O}$. If $\mathbb{P}^x(X(\hat{\zeta}) \in \tilde{O}^{c,*}) = 1$, then $\mathbb{P}^x(\hat{\zeta} = \zeta) = 1$.

**Proof.** By Proposition 9, we can write

$$\zeta = \hat{\zeta} + \zeta \circ \theta_\zeta$$

for the time-shift operator $\theta$. Therefore, we have

$$\mathbb{P}^x(\zeta = \hat{\zeta}) = \mathbb{P}^x(\zeta \circ \theta_\zeta = 0) = \mathbb{E}^x[\mathbb{P}^x(\hat{\zeta} = \zeta) = 1].$$

To summarize, the next theorem provides a sufficient condition for the stochastic representation $\mathbb{E}[F]$ in (1) to be a generalized viscosity solution of (2).

**Theorem 11.** Suppose $\mathbb{P}^x(X(\hat{\zeta}) \in \tilde{O}^{c,*}) = 1$ holds for all $x \in \tilde{O}$, and condition (A) of Proposition 8 holds for all $x \in \partial O \setminus \tilde{O}^{c,*}$. Then, $v$ of (1) is a generalized viscosity solution of (2) with

$$\Gamma_{out} \supset \tilde{O}^{c,*} \cap \partial O.$$ 

In particular, if $\tilde{O}^{c,*} = O^c$, then $\Gamma_{out} = \partial O$ and $v$ of (1) is a viscosity solution of (2).

**Proof.** The first statement is a consequence of Lemma 4, Lemma 10 and Proposition 7 and Proposition 8. For the second statement, if $\tilde{O}^{c,*} = O^c$, then we have $\partial O \supset \Gamma_{out} \supset \tilde{O}^{c,*} \cap \partial O \supset \partial O$, which implies all sets are equal indeed.

Returning back to Example 1, if $\epsilon = 0$, condition (A) is trivially satisfied for $x = 0 \in \partial O \setminus \tilde{O}^{c,*}$ with $O_1 = (-1,1)$. Similarly, the next immediate consequence of Theorem 11 prepares us for the non-stationary problems in the next section.

**Corollary 12.** Let $O$ be some cylinder set of the form $O = (a, b) \times A$. If we assume that

1. $\tilde{A}^{c,*} = A^c$ w.r.t. $X_{-1} := (X_2, \ldots, X_d)$; and
2. $X_1$ is a subordinate process,

then $v$ of (1) is a generalized viscosity solution of (2) with $\Gamma_{out} \supset \{ a \} \times A$.

**Proof.** Since $X$ is Feller, both $X_1$ and $X_{-1}$ are Feller. Notice that $\tilde{O}^{c,*} = \tilde{O} \setminus (\{ a \} \times A)$. By taking $O_1 = (a - 1, b) \times O$, it is an easy consequence of Theorem 11.
4 Applications to non-stationary problems

In this section, we apply the above results to solve two non-stationary equations involved with fractional Laplacian operators, one being linear and the other non-linear. Recall that a fractional Laplacian operator can be written by, for \( \alpha \in (0, 2) \)

\[-(\Delta)^{\alpha/2}\phi(x) = C_d \int_{\mathbb{R}^d \setminus \{0\}} \frac{[\phi(x + y) - \phi(x) - y \cdot D\phi(x)I_{B_1}(y)]}{|y|^{d+\alpha}} dy\]

with some normalization constant \( C_d \). If \( \phi \) is given by a function on \( (t, x) \in \mathbb{R} \times \mathbb{R}^d \), then \( (\Delta)^{\alpha/2}\phi \) is understood as \( (\Delta_x)^{\alpha/2}\phi(t, x) = (\Delta)^{\alpha/2}\phi(t, \cdot)(x) \).

Given a state space \( O \), its non-stationary (parabolic) domain \( Q_T \) and its non-stationary boundary \( \mathcal{P}Q_T \) is defined by

\[Q_T := (0, T) \times O, \quad \mathcal{P}Q_T := (0, T) \times \mathbb{R}^d \setminus Q_T.\]

Let \( b \) be a vector field \( \mathbb{R}^d \to \mathbb{R}^d \) known as a drift, \( \sigma \) be a constant in \( \mathbb{R}^d \times \mathbb{R}^d \) known as a volatility.

The first equation to be studied is a linear equation of the form

\[
\partial_t u + b \cdot \nabla_x u - |\sigma|^\alpha (\Delta)^{\alpha/2} u + \ell = 0 \text{ on } Q_T, \quad u = 0 \text{ on } \mathcal{P}Q_T. \quad (17)
\]

The second equation is the non-linear equation

\[-\partial_t u - |\nabla_x u|^\gamma + (\Delta_x)^{\alpha/2} u + 1 = 0 \text{ on } Q_T, \quad u = 0 \text{ on } \mathcal{P}Q_T. \quad (18)\]

For the viscosity solution of non-stationary problem (17) and (18), it can be similarly defined as in Definition 2 for the stationary problem, in the \( \mathbb{R}^{d+1} \) domain \( Q_T \), with additional requirement on boundary condition on \( \mathcal{P}Q_T \).

Consider a general form of non-stationary problem

\[G(u, t, x) = 0, \text{ on } Q_T, \quad u = 0 \text{ on } \mathcal{P}Q_T.\]

1. For a given \( u \in USC(\bar{Q}_T) \) and \( (t, x) \in \bar{Q}_T \), we define the space of supertest functions as

\[J^+(u, t, x) = \{ \phi \in C_0^\infty(\mathbb{R}^{d+1}) \text{, s.t. } \phi \geq (uI_{\bar{Q}_T})^* \text{ and } \phi(t, x) = u(t, x) \}.\]

2. For a given \( u \in LSC(\bar{Q}_T) \) and \( (t, x) \in \bar{Q}_T \), the space of subtest functions is given by,

\[J^-(u, t, x) = \{ \phi \in C_0^\infty(\mathbb{R}^{d+1}) \text{, s.t. } \phi \leq (uI_{\bar{Q}_T})_* \text{ and } \phi(t, x) = u(t, x) \}.\]

The function \( u \in USC(\bar{Q}_T) \) satisfies the viscosity subsolution property at some \( x \in \bar{Q}_T \), if \( G(\phi, t, x) \leq 0 \), for \( \forall \phi \in J^+(u, t, x) \). Similarly, a function \( u \in LSC(\bar{Q}_T) \) satisfies the viscosity supersolution property at some \( x \in \bar{Q}_T \), if \( G(\phi, t, x) \geq 0 \), \( \forall \phi \in J^-(u, t, x) \).

**Definition 13.** A function \( u \in C(Q_T) \) is said to be the viscosity solution (of \( (17) \) or \( (18) \)), if (i) \( u \) satisfies the viscosity property at each \( (t, x) \in Q_T \); (ii) \( u \equiv 0 \) on \( \mathcal{P}Q_T \cap \partial Q_T \).

4.1 Linear equation

As an example, for Lipschitz continuous \( b \), there exists a unique strong solution for

\[dX(t) = b(X_t)dt + \sigma dJ_t, \quad (19)\]
where $J$ is the driving noise process given by an isotropic $\alpha$-stable process for some $\alpha \in (0, 2)$ with its generating triplet (see notions of Lévy process in [17] or [5])

$$A = 0, \nu(dy) = \frac{1}{|y|^{d+\alpha}} dy, \ b = 0.$$ 

Particularly, the process $X$ of (19) is a Feller process and has a Càdlàg version with its generator $L$ with its domain $D(L) \supset C^2(\mathbb{R}^d)$. In particular, if $\phi \in C^2(\mathbb{R}^d)$ is given, then $L$ is consistent to the following integro-differential operator,

$$L\phi(x) = b(x) \cdot \nabla \phi(x) - |\sigma|^\alpha (-\Delta)^{\alpha/2} \phi(x). \quad (20)$$

With obvious extension of $L\phi(x)$ to partial operator $L_x\phi(t, x)$ by $L_xu(t, x) = L\phi(t, \cdot)(x)$, PDE (17) becomes

$$\partial_t u + L_xu + \ell = 0 \text{ on } Q_T, \text{ and } u = 0 \text{ on } \mathcal{P}Q_T.$$ 

Next, we will present a formal derivation of the above non-stationary PDE into a stationary PDE and its associated random process. If (17) has a smooth solution $u$ in $Q_T$, then the change of variable of

$$y = (t, x) \in \mathbb{R}^{d+1}, \ w(y) = e^{\lambda t}u(t, x) \quad (21)$$

with arbitrarily given constant $\lambda > 0$ implies that $w$ satisfies following stationary equation with the domain in $\mathbb{R}^{d+1}$,

$$-L_1w(y) + \lambda w(y) - \ell_1(y) = 0 \text{ on } Q_T, \text{ and } w(y) = 0 \text{ on } \mathcal{P}Q_T \cap \partial Q_T, \quad (22)$$

where

$$L_1w(y) = (\partial_t u + L_xu)(t, x), \ \ell_1(y) = e^{\lambda t}(y_1, y_{-1})$$

and $y_{-1} = [y_2, \ldots, y_{d+1}]^T$ is a $d$-dimensional column vector with elements of the vector $y$ except the first scalar $y_1$. In particular, $L_1$ is the generator of $\mathbb{R}^{d+1}$-valued Markov process $s \mapsto Y_s = (t+s, X_{t+s})$ for $X$ of (19), which follows the following dynamics

$$dY_s = b_1(Y_s)dt + \sigma dJ_t, \ Y_0 := y = (t, x), \quad (23)$$

where $b_1(y) = \begin{bmatrix} 1 \\ b(t, x) \end{bmatrix}$ and $\sigma_1 = \begin{bmatrix} 0_{1 \times d} \\ I_d \end{bmatrix}$, with $d \times d$ identity matrix $I_d$ and $d$-dimensional zero row vector $0_{1 \times d}$.

Next, applying Theorem 11, we can show that the Feynman-Kac functional associated to the random process (23) is a generalized viscosity solution of the stationary PDE (21), and with additional regularity conditions, the generalized viscosity solution of (21) coincide with the viscosity solution of (17) in the sense of Definition 13.

To proceed, define the cone $C(y, \theta)$ with the direction $y$ and aperture $\theta$ by, for $y \in \mathbb{R}^d \setminus \{0\}$ and $\theta \in (0, \pi)$,

$$C(y, \theta) = \{x \in \mathbb{R}^d : x \cdot y > |x| \cdot |y| \cdot \cos \theta\}.$$ 

Denote a truncated cone by an open Ball $B_r$ centered at 0 with radius $r > 0$ by $C_r(y, \theta)$, i.e. $C_r(y, \theta) = C(y, \theta) \cap B_r$. We say that $O$ satisfies exterior cone condition with $C_r(x)(v_x, \theta_x)$, if there exists $r(x) : \mathbb{R}^d \to \mathbb{R}^+$, $v_x : \mathbb{R} \to \mathbb{R}^d \setminus \{0\}$, and $\theta_x : \mathbb{R}^d \to (0, \pi)$, such that for each $x \in \partial O$, its associated truncated exterior cone $x + C_r(x)(v_x, \theta_x)$ belongs to the complement of $O$, i.e.

$$x + C_r(x)(v_x, \theta_x) \subset O^c.$$ 

With the Feller process $X \sim \mathcal{L}$ for the $L$ of (20) and a bounded open set $O$ satisfying exterior cone condition, Proposition 19 of Section A.3 shows that every point of $\partial O$ is regular to $\overline{O}^c$, if the one of the conditions (A1) - (A3) is satisfied.
(A1) $|\sigma| > 0$ and $\alpha \geq 1$;
(A2) $|\sigma| > 0$ and $b \equiv 0$;
(A3) $b(x) \cdot v_x > 0$ for all $x \in \partial O$.

Now, we are ready to present the main result of this section.

**Corollary 14.** Let $b$ be Lipschitz and $\sigma$ be a constant, and $O$ be a bounded open set satisfying exterior cone condition with $C_{r(x)}(v_x, \theta_x)$. In addition, we assume that $(b, \sigma)$ satisfies one of the conditions (A1) - (A3) above. Then the function $v_1$ defined by

$$v_1(t, x) = E^{t,x}[\int_t^{\zeta \wedge T} \ell(s, X_s)ds]$$

is a viscosity solution of (17), where $\zeta$ is defined as lifetime $\tau_O(X)$ for $X$ in (19).

**Proof.** Setting $w(t, x) = e^{\lambda t}v_1(t, x)$ and $r = s - t$, we have

$$w(t, x) = e^{\lambda t}E^{t,x}\left[\int_t^{\zeta \wedge T} \ell(s, X_s)ds\right] = e^{\lambda t}E^{t,x}\left[\int_0^{\zeta \wedge T - t} \ell(r + t, X_{r+t})dr\right].$$

With $Y_s = (t + s, X_{t+s})$ as a $d+1$ dimensional process, and $\zeta_1$ as the lifetime of $Y$ in the state space $\bar{Q}_T$, $Y$ follows the dynamic of (23) with initial state $Y_0 = (t, x)$, and $\zeta_1$ satisfies

$$\zeta_1 := \tau_{\bar{Q}_T}(Y) = \zeta \wedge T - t.$$

Therefore, $w$ can be represented in terms of $Y$:

$$w(t, x) = e^{\lambda t}E^{t,x}\left[\int_0^{\zeta_1} \ell(Y_r)dr\right].$$

Since $Y_1(r) = t + r$, a further substitution of $\ell_1(y) = e^{\lambda t}\ell(t, x)$ leads to

$$w(t, x) = E^{t,x}\left[\int_0^{\zeta_1} e^{-\lambda r} e^{\lambda (t+r)} \ell(Y_r)dr\right] = E^y\left[\int_0^{\zeta_1} e^{-\lambda r} \ell_1(Y_r)dr\right].$$

By Corollary 12, $w$ is a generalized viscosity solution of (22). Therefore, $v_1$ is the viscosity solution of (17) by Definition 13.

**4.2 Non-stationary nonlinear equation**

In this part, we will discuss the non-linear equation (3) mentioned in the introduction,

$$\begin{cases} 
-\partial_t u - |\nabla_x u|^\gamma + (-\Delta_x)^{\alpha/2} u + 1 = 0, & \text{on } Q_T; \\
 u = 0, & \text{on } \mathcal{P}Q_T.
\end{cases}$$

As a starting point, we recall the consequence of the comparison principle and Perron’s method, see also [2, 10]:

- (CP + PM) Suppose the comparison principle holds and Perron’s method is valid. If there exists sub and supersolution, then (3) is uniquely solvable.
From now on, we will call the above statement as (CP + PM). To concentrate on the application of Feynman-Kac functional as a generalized viscosity solution, we will not pursue the validity of (CP+PM) and take it as granted in the discussion below. The next proposition shows that, our results about the linear equation (17) above help establish the semi-solutions of (3).

**Proposition 15.** Let \( O \) be a bounded open set satisfying exterior cone condition with \( C_{r(x)}(v_x, \theta_x) \).

1. If \( \gamma \geq 1 \) and \( \alpha \in (0, 2) \), then there exist viscosity sub- and supersolutions of (3).

2. Moreover, if (CP + PM) holds for (3), then there exists unique viscosity solution.

**Proof.** First \( u = 0 \) is supersolution. On the other hand, Corollary 14 confirms that the stochastic representation \( v_1 \) of (24) with \( X \sim -(-\Delta_x)^{\alpha/2} \) is the viscosity solution for

\[
\begin{cases}
-\partial_t u + (-\Delta_x)^{\alpha/2}u + 1 = 0, & \text{on } Q_T := (0, T) \times B_1; \\
u = 0, & \text{on } \mathcal{P}Q_T := (0, T] \times \mathbb{R}^d \setminus Q_T.
\end{cases}
\]

By non-negativity of \( |\nabla_x u|^\gamma \), \( v_1 \) is also a viscosity subsolution of (3). Unique existence is a straightforward application of (CP + PM). \( \square \)

## 5 Summary

In this paper, we have shown that \( v \) of (1) is the generalized viscosity solution of (2), if \( X \) exits from \( O \) at a point of the fine closure of \( \bar{O}^c \) almost surely, see Theorem 11. To the best of our knowledge, this is the first result for the verification of Feynman-Kac functional as the generalized viscosity solution of the Dirichlet problem in the presence of jump diffusion. We also provide Example 2 where the assumptions in Theorem 11 do not hold and the Feynman-Kac functional fails to be continuous. Not to distract the readers from the main idea, we have rather strict condition (H.4) on \( g, \ell, \) and \( \lambda \). However, this condition could be appropriately relaxed as long as \( F \) in (1) is integrable.

Although the proof of Theorem 11 is mainly probabilistic, it gives an alternative constructive proof for the existence of generalized viscosity solution on Integro-Differential equation with Dirichlet boundary, which could be utilized for the solvability of nonlinear equation together with Perron’s method. In other words, Theorem 11 together with the probabilistic regularity, e.g. as in Proposition 19, yields a purely analytical result on the solvability question of Dirichlet problem. As an application, we considered an \( \mathbb{R}^{d+1} \)-valued process on a cylinder domain \( Q_T = (0, T) \times O \), see Corollary 14. If \( X_1 \) is uniform motion in time (i.e., \( dX_1(t) = dt \)) and \( X_{-1} = (X_2, \ldots, X_{d+1}) \) is an \( \mathbb{R}^d \)-valued process having each point of \( \partial O \) regular to \( \bar{O}^c \), then the corresponding Feynman-Kac functional is easily verified as the generalized viscosity solution of the stationary problem (22). Moreover, if one replace the uniform motion \( X_1 \) by a subordinate process, we can also verify the condition of Theorem 11 analogously. It is also desirable in the future to check that, the value of associated stochastic control problem (or nonlinear Feynman-Kac functional) is the solution of (3) constructed from Perron’s method.

## A Appendix

### A.1 Equivalence of definitions on the generalized solution

In this section we show that the following definition is actually equivalent to Definition 2.
Definition 16. 1. \( u \in USC(\bar{O}) \) is a generalized viscosity subsolution of (2), if (a) \( u \) satisfies the viscosity subsolution property at each \( x \in O \) and (b) \( u \) satisfies at the boundary
\[
\min \{-\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x), u(x)\} \leq 0, \forall x \in \partial O \text{ and } \forall \phi \in J^+(u, x).
\]
2. \( u \in LSC(\bar{O}) \) is a generalized viscosity supersolution of (2), if (a) \( u \) satisfies the viscosity supersolution property at each \( x \in O \) and (b) \( u \) satisfies at the boundary
\[
\max \{-\mathcal{L}\phi(x) + \lambda\phi(x) - \ell(x), u(x)\} \geq 0, \forall x \in \partial O \text{ and } \forall \phi \in J^-(u, x).
\]
3. \( u \in C(\bar{O}) \) is a viscosity solution of (2), if it is the viscosity subsolution and supersolution simultaneously.

Proposition 17. Let \( x \in \partial O \).

1. If \( v \in USC(\bar{O}) \) and \( v(x) < 0 \), then \( v \) satisfies the viscosity subsolution property at \( x \);
2. If \( v \in LSC(\bar{O}) \) and \( v(x) > 0 \), then \( v \) satisfies the viscosity supersolution property at \( x \),
and thus Definitions 2 and 16 are equivalent.

Proof. 1. If \( v \in USC(\bar{O}) \) and \( v(x) < 0 \), then \( J^+(v, x) = \emptyset \) and it implies the viscosity subsolution property at \( x \). 2. The other case of \( v \in LSC(\bar{O}) \) and \( v(x) > 0 \) follows an analogous proof.

In order to check the equivalent between Definitions 2 and 16, it suffices to check it is the case for \( x \in \partial O \).

If \( v \) is a generalized viscosity solution at \( x \) according to Definition 2, then either the viscosity solution property is satisfied, or \( v(x) = 0 \). It implies that viscosity sub- and supersolution properties in Definition 16 are satisfied.

If \( v \) is a generalized viscosity solution at \( x \) according to Definition 16, there are three cases to discuss: if \( v(x) = 0 \), then it satisfies Definition 2. If \( v(x) > 0 \), then Definition 16-1 implies that \( v \) satisfies viscosity subsolution property at \( x \). On the other hand, case 2 above shows that \( v \) also satisfies viscosity supersolution property at \( x \). The case of \( v(x) < 0 \) follows similarly.

A.2 Characterization of \( \Gamma_{out} \)

From the definition (1) of \( v \) and Definition 2 of \( \Gamma_{out} \), we observe that \( \Gamma_{out} \) depends on the function \( g \) via \( v \), and we explicitly write it as \( \Gamma_{out}[g] \) for \( \Gamma_{out} \) in this section. From Lemma 4, we have
\[
\bigcap_{g \in C_b^{0,1}(\mathbb{R}^d, \mathbb{R})} \Gamma_{out}[g] \supset \bar{O}^{c,*} \cap \partial O.
\]
On the other hand, Definition (1) yields an estimate
\[
v(x) \leq 1 + \|f\|_{\infty} + \|g\|_{\infty} p(x),
\]
where \( p(x) = \mathbb{E}_x[e^{-\lambda\zeta}] \).

Take an arbitrary \( x_0 \in \partial P \setminus \bar{O}^{c,*} \) and
\[
g(x) = e^{-|x-x_0|} \frac{\|f\|_{\infty} + 1}{1 - p(x_0)}.
\]
If we assume
\[
\mathbb{P}^x(\zeta < \infty) = 1 \text{ for all } x,
\]
(25)
then \( p(x_0) \in (0, 1) \) and the above \( g \) is well defined strictly positive function in \( C^{0, 1}_0(\mathbb{R}^d) \) with
\[
g(x_0) = \|g\|_{\infty} = \frac{\|\ell\|_{\infty} + 1}{1 - p(x_0)} > v(x_0).
\]
This implies \( v \) has no way to meet \( g \) at \( x_0 \) and
\[
x_0 \notin \bigcap_{g \in C^{0, 1}_0(\mathbb{R}^d, \mathbb{R})} \Gamma_{\text{out}}[g].
\]
By arbitrariness of \( x_0 \), we can have

**Lemma 18.** Assume (25) holds. If \( v \in C(\overline{O}) \), then \( v \) of (1) is a generalized viscosity solution of (2), with
\[
\bigcap_{g \in C^{0, 1}_0(\mathbb{R}^d, \mathbb{R})} \Gamma_{\text{out}}[g] = \overline{O}^{c, \ast} \cap \partial O.
\]

### A.3 Regularity under the exterior cone condition

In this part, we will prove the regularity condition used in Corollary 14. Recall that, the diffusion \( X \) of (19) is given in the form of
\[
dX(t) = b(X_t)dt + \sigma dJ_t.
\]

**Proposition 19.** Let \( b \) be Lipschitz and \( \sigma \) be a constant, and \( O \) be a bounded open set satisfying exterior cone condition with \( C_{r(x)}(\mathbb{R}^d, \mathbb{R}) \). In addition, we assume that \( (b, \sigma) \) satisfies one of the conditions of (A1) - (A3). Then, any point in \( O^c \) is regular for the set \( \overline{O}^c \) with respect to the process (19), i.e. \( O^c = \overline{O}^{c, r} \).

**Proof.** By right continuity of the sample path, \( \overline{O}^c \subset \overline{O}^{c, r} \) and \( O \cap \overline{O}^{c, r} = \emptyset \). Therefore, we only need to verify \( \partial O \subset \overline{O}^{c, r} \) as below.

Fix \( x \in \partial O \), and let \( Y = X \cdot v_x \) be the projection of the process \( X \) of (19) on the unit vector \( v_x \) pointing the direction of the exterior cone. Then, \( Y \) has a representation of
\[
dY(t) = \hat{b}(X_t)dt + \hat{\sigma} d\hat{J}(t), \quad Y(0) = x \cdot v_x,
\]
where \( \hat{b}(x) = b(x) \cdot v_x \), \( \hat{\sigma} = |v_x| \sigma \), and \( \hat{J} \) is isotropic one dimensional \( \alpha \)-stable process with its generating triplets \( A = 0, \nu(dz) = \frac{1}{|z|^\alpha} \; dz, \; b = 0 \). To see that \( \hat{J} \) is indeed an \( \alpha \)-stable process, notice that the characteristic function of \( J_1 \) is
\[
\mathbb{E}[\exp\{iu \cdot J_1\}] = e^{-c_0|u|^\alpha}, \quad \forall u \in \mathbb{R}^d,
\]
for some normalizing constant \( c_0 \). Therefore, the characteristic function of \( \hat{J}_1 \) is
\[
\mathbb{E}[\exp\{iu \cdot \hat{J}_1\}] = \mathbb{E}[\exp\{iuv_x \cdot J_1\}] = e^{-c_0|u v_x|^\alpha} = e^{-c_0|u|^\alpha}, \quad \forall u \in \mathbb{R},
\]
and hence \( \hat{J} \) is an \( \alpha \)-stable process.

Note that, by the definition of the exterior cone condition, the regularity of \( x \) to \( \overline{O}^c \) with process \( X \) can be implied by the regularity of \( y = x \cdot v_x \) to the open line segment \((y, y + r_x)\) with respect to the process \( Y \). Moreover, due to the right continuity of the sample path, it’s equivalent to check the regularity of \( y \) w.r.t. the half line \((y, \infty)\). Equivalently, to obtain the property \( \mathbb{P}^x \{ \tau_{\overline{O}} = 0 \} = 1 \) for the \( x \in \partial O \), it’s enough to check \( \mathbb{P}^y \{ \tau_{(-\infty, y]}(Y) = 0 \} = 1 \).

1. If \( \alpha \geq 1 \), consider
\[
\hat{Y}_t = y - \sup_{x \in \overline{O}} |b(x)| t + \hat{\sigma} \hat{J}(t).
\]
Note that $\tilde{Y}_t \leq Y_t$, but $\tilde{Y}$ is Type C process and $\mathbb{P}^y(\tau_{(-\infty,y]}(\tilde{Y}) = 0) = 1$. Therefore, $\mathbb{P}^y(\tau_{(-\infty,y]}(Y) = 0}) = 1$.

2. If $b \equiv 0$, then $X$ is simply an isotropic Levy process and $\mathbb{P}^y(\tau_{(-\infty,y]}(Y) = 0} = 1$.

3. If $\alpha < 1$ and $\hat{b}(x) = b(x) \cdot \mathbf{v}_x > 0$, then define

$$h := \inf \{ t \geq 0 : \hat{b}(X_t) < \frac{1}{2} b(x) \}.$$

Due to the right continuity of $t \mapsto \hat{b}(X_t)$, we have $h > 0$ $\mathbb{P}^x$-almost surely. Consider

$$\hat{Y}_t = y + \frac{1}{2} \hat{b}(x)t + \hat{\sigma} \hat{J}(t),$$

then $Y_t \geq \hat{Y}_t$ on $(0,h)$. Moreover, by Theorem 47.5 of [24], $\hat{Y}$ is a Type B process with $\frac{1}{2} \hat{b}(x) > 0$, hence $\mathbb{P}^y(\tau_{(-\infty,y]}(\hat{Y}_t) = 0} = 1$. Therefore, we have $\mathbb{P}^y(\tau_{(-\infty,y]}(Y) = 0} = 1$.

A.4 Explicit solution of a singularly perturbed Dirichlet problem on reaction-convection-diffusion equation

For the illustration purpose on our problem, we adopt an example on a singularly perturbed reaction-convection-diffusion equation, whose solution can be explicitly computable. Consider

$$\frac{1}{2} \epsilon^2 u'' + u' - u + 1 = 0, \text{ on } O := (0,1), \quad u(0) = u(1) = 0.$$

If $\epsilon > 0$, then there exists unique $C^2(O) \cap C(\bar{O})$ solution, whose representation can be explicitly written by

$$u(x) = 1 + \frac{(1 - e^{\lambda_1})e^{\lambda_2 x} + (e^{\lambda_2} - 1)e^{\lambda_1 x}}{e^{\lambda_1} - e^{\lambda_2}}.$$

where

$$\lambda_1 = \frac{\sqrt{1 + 2 \epsilon^2} - 1}{\epsilon^2}, \quad \text{and} \quad \lambda_2 = \frac{-\sqrt{1 + 2 \epsilon^2} - 1}{\epsilon^2}.$$

The above explicit solution is obtained by the simple SageMath code implemented by CoCalc - a cloud computing platform, see

https://cocalc.com/share/3e846d30-3d24-467c-a0d8-58549b3cf8f6/2018-06-21-155129.ipynb?viewer=share

Interestingly, it can provide solutions only for $\epsilon > 0$.

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