Evolution of initial discontinuities in the Riemann problem for the Kaup-Boussinesq equation with positive dispersion

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We consider the space-time evolution of initial discontinuities of depth and flow velocity for an integrable version of the shallow water Boussinesq system introduced by Kaup. We focus on a specific version of this “Kaup-Boussinesq model” for which a flat water surface is modulationally stable, we speak below of “positive dispersion” model. This model also appears as an approximation to the equations governing the dynamics of polarisation waves in two-component Bose-Einstein condensates. We describe its periodic solutions and the corresponding Whitham modulation equations. The self-similar, one-phase wave structures are composed of different building blocks which are studied in detail. This makes it possible to establish a classification of all the possible wave configurations evolving from initial discontinuities. The analytic results are confirmed by numerical simulations.

PACS numbers: 05.45.Yv,47.35.Fg,47.35.Jk

The Kaup-Boussinesq water wave equation is integrable, but its generic form suffers from a dynamical instability. We study here the Riemann problem for a version of this equation which does not suffer from the same deficiency. This equation appears as an approximation of the nonlinear polarization dynamics of a two-component Bose-Einstein condensate, and in this context it is important to characterize the time evolution of simple, but experimentally relevant initial profiles.

I. INTRODUCTION

In many physical wave systems, the initial value problem treated in the long-wavelength (hydrodynamic) approximation leads to wave breaking after a finite time. As a result, the formal solution becomes multivalued, i.e. it looses its physical meaning. At the wave breaking point, the first spatial derivative of the physical variables diverges and the hydrodynamic approximation fails. This suggests that nonphysical behavior can be remedied by accounting for physical effects described by terms with higher-order derivatives in the corresponding evolution equations. For example, within the Navier-Stokes description of the dynamics of a compressible gas, the effects of viscosity are described by second-order derivative terms, and this leads, instead of wave breaking, to the formation of viscous shocks, which can often be formally described by surfaces of discontinuities in the physical variables. Formulated in this way, the theory of “shock waves” has found a number of important applications.[12]

At the same time, in many physical systems the dissipative effects may be relatively weaker than the dispersive ones, and in such cases, so-called “dispersive shock waves” (DSWs) are formed instead of viscous shocks. DSWs can be represented as modulated nonlinear oscillations whose envelope varies over characteristic distances much greater than their wavelength. In recent years such systems have attracted much attention in fluid dynamics, nonlinear optics, physics of Bose-Einstein condensates and other areas of physics (see, e.g., Refs. 3 and 4). This type of problem was studied for the first time in the context of the physics of shallow water waves whose evolution is described by the celebrated Korteweg-de Vries (KdV) equation. The equations governing the slow evolution of the envelope of the nonlinear oscillations had been derived by Whitham and later they were applied to the description of the DSW structure by Gurevich and Pitaevski. Because of the universality of the KdV equation, this approach can naturally be applied to many other physical situations. When the condition of unidirectional propagation is relaxed, shallow water waves are described by various forms of Boussinesq equations. The most convenient form for our purpose has been derived by Kaup; this is the so-called Kaup-Boussinesq (KB) equation. The KB equation is completely integrable and the well-developed methods of inverse scattering transform and finite-gap integration can be used for explicitly deriving its multi-soliton and (quasi-)periodic solution.[15]

In the applications of this theory to concrete physical problems, only the KB equations with negative dispersion has been considered so far. In this case, linear perturbations can be sought under the form of plane waves with angular frequency \( \omega \) and wavelength \( k \). In non-dimensional notations, the corresponding dispersion relation reads (\( h_0 \) is a constant depth)

\[
\omega^2 = h_0 k^2 - \frac{1}{4} k^4. \tag{1}
\]

The Whitham modulation equations were derived for this case in Ref. [12] and a complete classification of all the possible wave structures resulting from an initial discon-
The so-called cnoidal waves of the system (3) and the Whitham equations governing the modulational dynamics of a cnoidal wave. This is achieved by using the methods described, e.g., in Ref. 3 (see also Ref. 13). These techniques are based on the possibility to represent the system (3) as a compatibility condition for the linear system

\[ \psi_{xx} = A \psi, \quad \psi_t = -\frac{1}{2} B_x \psi + B \psi_x \]  

with

\[ A = h - \left( \lambda - \frac{1}{2} u \right)^2, \quad \text{and} \quad B = -\left( \lambda + \frac{1}{2} u \right), \]  

where \( \lambda \) is a free spectral parameter. Demanding that \((\psi_{xx})_t = (\psi_t)_{xx} \) for any \( \lambda \), we reproduce the KB system (3).

The second order spatial linear differential equation in (5) has two independent solutions \( \psi_+ (x, t) \) and \( \psi_- (x, t) \). Their product \( g = \psi_+ \psi_- \) satisfies the following third order equation

\[ g_{xxx} - 2 Ag_x g - 4 Ag_x = 0. \]  

Upon multiplication by \( g \), this equation can be integrated once to give

\[ \frac{1}{2} g g_{xx} - \frac{1}{4} g_x^2 - Ag^2 = P(\lambda), \]  

where the integration constant has been written as \( P(\lambda) \) since it can only depend on \( \lambda \). The time dependence of \( g(x, t) \) is determined by the equation

\[ g_t = B g_x - B_x g. \]  

We are interested in the one-phase periodic solution of the system (3). They are distinguished by the condition that \( P(\lambda) \) in (8) be a fourth degree polynomial of the form

\[ P(\lambda) = \prod_{i=1}^{4} (\lambda - \lambda_i) = \lambda^4 - s_1 \lambda^3 + s_2 \lambda^2 - s_3 \lambda + s_4. \]  

In expression (10), we chose for definiteness to order the zeroes \( \lambda_i \) according to

\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4. \]  

Then we find from Eq. (8) that \( g(x, t) \) is a first-degree polynomial in \( \lambda \), of the form

\[ g(x, t) = \lambda - \mu(x, t), \]  

where \( \mu(x, t) \) is connected with \( u(x, t) \) and \( h(x, t) \) by the relations

\[ u(x, t) = s_1 - 2 \mu(x, t), \]  

\[ h(x, t) = \frac{1}{4} s_1^2 - s_2 - 2 \mu^2(x, t) + s_1 \mu(x, t), \]  

\[ \omega^2 = h_0 k^2 + \frac{1}{4} k^4. \]  

The corresponding KB system can be written under the following non-dimensional form,

\[ h_t + (hu)_x - \frac{1}{4} u_{xxx} = 0, \]  

\[ u_t + uu_x + h_x = 0. \]  

In the context of shallow water wave physics, \( h \) is the local height of the water layer and \( u \) is a local mean flow velocity. Eq. (2) represents the dispersion relation of linear waves propagating along a uniform background characterized by the physical variables \( h_0 = \text{const} \) and \( u_0 = 0 \). It does not suffer from the instability of Eq. (1). The positive dispersion KB system (3) may be obtained in the case of capillary waves propagating on top of a thin liquid layer (see, e.g., Ref. 3). Besides this physical realization, the system (3) appears as an approximation to the Landau-Lifshitz equation for propagation of magnetization waves in easy-plane magnets and to the Gross-Pitaevskii equations for propagation of polarization waves in two-component Bose-Einstein condensates.\(^{15}\)

Motivated by these applications of the KB system (3), we consider in the present paper the so-called Riemann problem. This corresponds to the study of the time evolution of initial discontinuous profiles of the form

\[ h(x, t = 0) = h_L, \quad \text{and} \quad u(x, t = 0) = u_L \quad \text{for} \quad x < 0, \]  

\[ h(x, t = 0) = h_R, \quad \text{and} \quad u(x, t = 0) = u_R \quad \text{for} \quad x > 0. \]  

As we shall see, the resulting wave structures differ considerably from those found in Ref. 12 for the negative dispersion case. In the case of Eqs. (3) studied in the present work, the classification of the possible wave structures follows closely the scheme found for the nonlinear Schrödinger equation in Refs. 17 and 18. We shall obtain simple analytic formulae for the main parameters of the wave structures and confirm their accuracy by comparison with numerical solutions of the KB system (3).

II. PERIODIC WAVES AND WHITHAM MODULATION EQUATIONS

In this section we derive the periodic wave solutions (the so-called cnoidal waves) of the system (3) and the
which follow from a comparison of the coefficients of the different powers of \( \lambda \) on both sides of Eq. (8). The spectral parameter \( \lambda \) is arbitrary and on substitution of \( \lambda = \mu \) into Eq. (8) we obtain an equation for \( \mu \),

\[
\mu_x = 2\sqrt{-P(\mu)},
\]

while a similar substitution into Eq. (9) gives

\[
\mu_t = - (\mu + \frac{1}{2} u) \mu_x = - \frac{1}{2} s_1 \mu_x.
\]

Hence, \( \mu(x, t) \) as well as \( u(x, t) \) and \( h(x, t) \) depend only on the phase

\[
\theta = x - \frac{1}{2} s_1 t,
\]

so that

\[
V = \frac{1}{2} s_1 \left( 1 + \frac{4}{2} \sum_{i=1}^{4} \lambda_i \right)
\]

is the phase velocity of the nonlinear wave, and \( \mu(\theta) \) is determined by the equation

\[
\mu_\theta = 2\sqrt{-P(\mu)}.
\]

It follows from Eq. (15) that the variable \( \mu \) must be real. For the fourth degree polynomial (10) the real solution of (16) corresponds to oscillations of \( \mu \) in one of two possible intervals,

\[
\lambda_1 \leq \mu \leq \lambda_2 \quad \text{or} \quad \lambda_3 \leq \mu \leq \lambda_4,
\]

within which \( P(\mu) \) assumes negative values. It is well known that the solution of Eq. (16) with boundaries (17) can be expressed in terms of elliptic functions (see, e.g., Refs. 20 and 21). Without going into details, we shall list here the results which are the most relevant to our study.

- For the case

\[
\lambda_1 \leq \mu \leq \lambda_2
\]

the cnoidal wave solution of Eq. (17) with the initial condition \( \mu(0) = \lambda_1 \) is given by

\[
\mu(\theta) = \lambda_2 - \frac{(\lambda_2 - \lambda_1) \text{cn}^2 (W, m)}{1 + \frac{(\lambda_3 - \lambda_2) \text{sn}^2 (W, m)}}
\]

where \( W = \sqrt{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)} \theta \) and

\[
m = \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}
\]

is the modulus of the Jacobi elliptic functions \( \text{sn} \) and \( \text{cn} \). Substitution of (19) into (13) gives the corresponding expressions for \( u(\theta) \) and \( h(\theta) \) for a one-phase periodic nonlinear wave. Its wavelength is given by

\[
L = \int_{\lambda_1}^{\lambda_2} \frac{d\mu}{\sqrt{-P(\mu)}} = \frac{2K(m)}{\sqrt{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}},
\]

where \( K(m) \) being the complete elliptic integral of the first kind.

The soliton solution corresponds to the limit \( \lambda_3 \to \lambda_2 \) \((m \to 1)\). We obtain

\[
\mu(\theta) = \lambda_2 - \frac{\lambda_2 - \lambda_1}{\cosh^2 W + \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_2} \sinh^2 W}.
\]

This is a dark soliton solution for the variable \( \mu \). In the limit \( \lambda_2 \to \lambda_1 \) we get a small-amplitude harmonic wave

\[
\mu = \lambda_2 - \frac{1}{2} (\lambda_2 - \lambda_1) \cos[k(x - V_t)],
\]

where \( k = 2\sqrt{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \).

If \( \lambda_4 = \lambda_3 \) but \( \lambda_1 \neq \lambda_2 \), then we have again \( m = 0 \) and (22) reduces to a nonlinear trigonometric wave, but we shall not present its explicit form here (cf., e.g., Refs. 20 and 21).

- In a similar way, for the case

\[
\lambda_3 \leq \mu \leq \lambda_4
\]

the cnoidal wave solutions are of the form \( \mu(0) = \lambda_4 \)

\[
\mu(\theta) = \lambda_3 + \frac{(\lambda_4 - \lambda_3) \text{cn}^2 (W, m)}{1 + \frac{\lambda_3 - \lambda_4}{\lambda_4 - \lambda_3} \text{sn}^2 (W, m)},
\]

In the soliton limit \( \lambda_3 \to \lambda_2 \) \((m \to 1)\) we obtain

\[
\mu(\theta) = \lambda_2 + \frac{\lambda_4 - \lambda_2}{\cosh^2 W + \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_2} \sinh^2 W}.
\]

This is a bright (for \( \mu \)-variable) soliton over a constant background. In the limit \( \lambda_4 \to \lambda_3 \) we get a small-amplitude harmonic wave

\[
\mu = \lambda_3 + \frac{1}{2} (\lambda_4 - \lambda_3) \cos[k(x - V_t)],
\]

where \( k = 2\sqrt{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_1)} \).

As discussed above, nonlinear trigonometric waves also exit, here in the case where \( \lambda_1 = \lambda_3 \) but \( \lambda_3 \neq \lambda_4 \). If furthermore \( \lambda_3 \to \lambda_1 \) one reaches the limit of an algebraic soliton.\(^{20,21}\)

\[
\mu(\theta) = \lambda_1 + \frac{\lambda_4 - \lambda_1}{1 + (\lambda_4 - \lambda_1)^2 \theta^2}.
\]

- We now consider slowly modulated cnoidal waves. In this case the parameters \( \lambda_i \) \((i = 1, 2, 3, 4)\) become slowly varying functions of \( x \) and \( t \) changing weakly over a wavelength \( L \). Their evolution is governed by the Whitham modulation equation.\(^{31}\)

\[
\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, 4.
\]

The Whitham velocities \( v_i \) appearing in Eqs. (29) can be computed via the formulae (see, e.g., Ref. 31)

\[
v_i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left( 1 - \frac{L}{\partial \lambda_i \partial \lambda_i} \right) V, \quad i = 1, 2, 3, 4,
\]
where the phase velocity $V$ and the wavelength $L$ are given by Eqs. (15) and (21). A simple calculation yields the explicit expressions

$$v_1 = \frac{1}{2} \sum_{i=1}^{4} \lambda_i - \frac{(\lambda_4 - \lambda_1)(\lambda_2 - \lambda_1)}{K(m)}$$

$$v_2 = \frac{1}{2} \sum_{i=1}^{4} \lambda_i + \frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{K(m)}$$

$$v_3 = \frac{1}{2} \sum_{i=1}^{4} \lambda_i - \frac{(\lambda_4 - \lambda_3)(\lambda_3 - \lambda_2)}{K(m)}$$

$$v_4 = \frac{1}{2} \sum_{i=1}^{4} \lambda_i + \frac{(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)}{K(m)}$$

where $m$ is given by (20) and $K(m)$ and $E(m)$ are complete elliptic integrals of the first and second kind, respectively.

In the soliton limit $m \to 1$ (i.e., $\lambda_3 \to \lambda_2$) the Whitham velocities reduce to

$$v_1 = \frac{1}{2}(3\lambda_1 + \lambda_4), \quad v_2 = v_3 = \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_4),$$

$$v_4 = \frac{1}{2}(\lambda_1 + 3\lambda_4).$$

In a similar way, in the small amplitude limit $m \to 0$ (i.e., $\lambda_2 \to \lambda_1$) we obtain

$$v_1 = v_2 = 2\lambda_1 + \frac{(\lambda_4 - \lambda_3)^2}{2(\lambda_3 + \lambda_4 - 2\lambda_1)},$$

$$v_3 = \frac{1}{2}(3\lambda_3 + \lambda_4), \quad v_4 = \frac{1}{2}(\lambda_3 + 3\lambda_4),$$

and in another small amplitude limit ($m \to 0$ when $\lambda_3 \to \lambda_4$) we have

$$v_1 = \frac{1}{2}(3\lambda_1 + \lambda_2), \quad v_2 = \frac{1}{2}(\lambda_1 + 3\lambda_2),$$

$$v_3 = v_4 = 2\lambda_4 + \frac{(\lambda_2 - \lambda_1)^2}{2(\lambda_1 + \lambda_2 - 2\lambda_4)}.$$

III. KEY ELEMENTS OF SELF-SIMILAR WAVE STRUCTURES

The initial profiles (4) being infinitely sharp, do not involve any characteristic length. However, the dispersion relation (2) is characterized by the value of the shallow water wave velocity: $c_s = \omega/k|_{k=0} = \sqrt{g_0}$. Therefore the large scale features of the solution of this problem (with characteristic length scale much greater than the wavelength) can only depend on the self-similar variable $\xi = x/t$, which can be made non-dimensional with the help of the velocity $c_s$. This means that the large scale features of the wave pattern must be self-similar and should be composed of (possibly several) regions where $h$ and $v$ either smoothly depend on $\xi$, or consist of modulated periodic waves whose envelopes (and wavelength $L$) depend slowly on $\xi$.

In the framework of the hydrodynamic approximation these regions are separated by weak discontinuities where the physical variables have cusps. If the hydrodynamic approximation leads to non-monotonous dependence of velocities on the wave amplitude, then the wave structure can be more complicated; an example of such a situation was considered, e.g., in Ref. [20]. At first we shall consider smooth solutions of the KB system [3].

A. Dispersionless limit

For smooth enough wave patterns we can neglect the last dispersion term in the first equation of the system (3) and arrive at the so-called dispersionless equations

$$h_t + (hu)_x = 0, \quad u_t + uu_x + h_x = 0,$$

which coincide with the well-known shallow water equations. Introducing the Riemann invariants

$$\lambda_{\pm} = \frac{u}{2} \pm \sqrt{h},$$

the system (35) can be written in the following diagonal form

$$\frac{\partial \lambda_{\pm}}{\partial t} + v_{\pm}(\lambda_{\pm}, \lambda_{\pm}) \frac{\partial \lambda_{\pm}}{\partial x} = 0,$$

where $v_{\pm}(\lambda_{\pm}, \lambda_{\pm}) = \frac{1}{2}(3\lambda_{\pm} + \lambda_{\mp})$.

The physical variables are expressed in terms of $\lambda_{\pm}$ as

$$u = \lambda_+ + \lambda_-, \quad h = (\lambda_+ - \lambda_-)^2/4.$$  (38)

For the self-similar solutions one has $\lambda_{\pm} = \lambda_{\pm}(\xi)$ and the system (37) reduces to

$$\frac{d\lambda_{\mp}}{d\xi} \cdot (\xi - \xi) = 0, \quad \frac{d\lambda_{\pm}}{d\xi} \cdot (\xi - \xi) = 0.$$  (39)

This system admits a trivial solution for which $\lambda_+ = \text{const}$ and $\lambda_- = \text{const}$. It describes a uniform flow with constant $h$ and $u$. We shall call such a solution a “plateau”.

Other solutions of (39) are called simple waves. For such flows, one of the Riemann invariants is constant (say, $\lambda_-$) whereas the other one changes in such a way that the term between parenthesis in its equation is zero ($v_+ = \xi$ in the example considered). One has thus two possible types of self-similar simple waves:

$$\begin{cases} 
\lambda_- = C_{st} = \overline{\lambda}_- , \quad \text{with} \\
v_+(\overline{\lambda}_-, \lambda_+) = \frac{1}{2}(3\lambda_+ + \overline{\lambda}_-) = \xi = x/t, 
\end{cases}$$

or

$$\begin{cases} 
\lambda_+ = C_{st} = \overline{\lambda}_+ , \quad \text{with} \\
v_-(\lambda_-, \overline{\lambda}_+) = \frac{1}{2}(3\lambda_- + \overline{\lambda}_+) = \xi = x/t. 
\end{cases}$$  (41)
The constancy of one of the Riemann invariants means that \( h \) and \( u \) are related by a simple formula: either \( \lambda_+ = u/2 - h^{1/2} = \text{const} = \lambda_- = \bar{u}/2 - \bar{h}^{1/2} \), or \( \lambda_+ = u/2 + h^{1/2} = \text{const} = \lambda_- = \bar{u}/2 + \bar{h}^{1/2} \), where \( \bar{u} \) and \( \bar{h} \) are some values that fix the value of the constant Riemann invariant and can be chosen at convenience for solving a specific problem. Thus, for given values of \( \bar{u} \) and \( \bar{h} \), a simple wave corresponds to a configuration where the variables \( u \) and \( h \) are connected by the relations corresponding to one of the two parabolas drawn in the plane \((u, h)\) in Fig. 1. The parabolas cross at the point \( P = (\bar{u}, \bar{h}) \) which represents a uniform flow with constant values of \( u = \bar{u} \) and \( h = \bar{h} \) which is a trivial "plateau solution" of Eqs. (43).

The dispersionless system (35) requires continuity of the functions \( u(x, t) \) and \( h(x, t) \), but, as usual in hydrodynamics, admits jumps of their space derivatives, i.e., "weak discontinuities". Therefore a plateau solution \((\bar{u}, \bar{h})\) can be attached at one of its boundaries to a simple wave. We have here two possibilities. If the uniform flow corresponding to \( P = (\bar{u}, \bar{h}) \) matches with the simple wave along which \( \lambda_+ = \lambda_- = \text{const} \) (one of the solid parabolas in Fig. 1), then from Eq. (41) one gets for this flow

\[
\begin{align*}
\frac{1}{2}u - \sqrt{h} &= \frac{1}{2} \bar{u} - \sqrt{\bar{h}}, \\
\Rightarrow \lambda_+ &= \lambda_-.
\end{align*}
\]

This wave configuration represents a rarefaction wave (RW) propagating to the right. If it propagates into "vacuum", then Eq. (42) gives the full solution of the problem

\[
\begin{align*}
u(x, t) &= \frac{2}{3} \left( \frac{\xi}{t} + \frac{\xi}{2} + \sqrt{h} \right), \\
h(x, t) &= \frac{1}{3} \left( \frac{\xi}{t} - 2\lambda_+ \right) = \frac{1}{3} \left( \frac{\xi}{t} - \bar{u} - 2\sqrt{\bar{h}} \right)^2.
\end{align*}
\]

This represents a rarefaction wave propagating to the left. Again, it corresponds—in the hydrodynamic approximation—to the solution of the problem \( (41) \) with left boundaries \( h_L = \bar{h} \) and \( u_L = \bar{u} \) whereas at the right boundary \( h_R = 0 \) (the value of \( u_R \) is irrelevant in the space without fluid). This situation is depicted in Figs. (2a). The left edge of this rarefaction wave propagates to the left at velocity \( s_- = u_L - \sqrt{h_L} \) and the right edge propagates to the right into the empty space with velocity \( s_+ = u_L + \sqrt{h_L} \).

The values of the edge velocities \( s_\pm \) are given in the text.\( \square \) with left boundaries \( h_L = \bar{h} \) and \( u_L = \bar{u} \) whereas at the right boundary \( h_R = 0 \) (the value of \( u_R \) is irrelevant in the space without fluid). This situation is depicted in Figs. (2a). The left edge of this rarefaction wave propagates to the left at velocity \( s_- = u_L - \sqrt{h_L} \) and the right edge propagates to the right into the empty space with velocity \( s_+ = u_L + \sqrt{h_L} \).

In a similar way, if \( P = (\bar{u}, \bar{h}) \) matches with the simple wave along which \( \lambda_- = \lambda_+ = \text{const} \), then (40) yields

\[
\begin{align*}
u(x, t) &= \frac{2}{3} \left( \frac{\xi}{t} + \lambda_+ \right) = \frac{2}{3} \left( \frac{\xi}{t} + \frac{\bar{u}}{2} + \sqrt{\bar{h}} \right), \\
h(x, t) &= \frac{1}{3} \left( \frac{\xi}{t} - 2\lambda_- \right) = \frac{1}{3} \left( \frac{\xi}{t} - \bar{u} + 2\sqrt{\bar{h}} \right)^2.
\end{align*}
\]

This represents a rarefaction wave propagating to the left. Again, it corresponds—in the hydrodynamic approximation—to the solution of the problem \( (41) \) with left boundaries \( h_L = \bar{h} \) and \( u_L = \bar{u} \), see Figs. (2b). The edge velocities are equal to \( s_- = u_R - 2\sqrt{h_R} \) and \( s_+ = u_R + \sqrt{h_R} \).

It is clear that we can generalize these solutions to the cases where both sides of the rarefaction wave connect uniform flows with equal values of the corresponding Riemann invariants \( u_L/2 + \sqrt{h_L} = u_R/2 + \sqrt{h_R} \) or \( u_L/2 - \sqrt{h_L} = u_R/2 - \sqrt{h_R} \). In these cases the rarefaction wave connects two uniform flows and the corresponding distributions of \( h \) and \( u \) are shown in Fig. 3. The velocities of the edges of the rarefaction waves are given in both cases by the formulae \( s_- = u_L - \sqrt{h_L} \), \( s_+ = u_R + \sqrt{h_R} \). These values have simple physical interpretation: they are the sums of the local flow velocities \( (u_L \text{ or } u_R) \) and of the propagation velocities of small amplitude disturbances directed to the left for the left edge \( (-\sqrt{h_L}) \) and to the right for the right edge \( (+\sqrt{h_R}) \).

It is important to note that the system (35) only admits solutions of the type (40) and (41) for which the non-constant Riemann invariant increases with \( \xi = x/t \).

The above wave structures correspond to the conditions (a) \( \lambda_+ < \lambda_+^R \), \( \lambda_- = \lambda_-^R \) or (b) \( \lambda_+ = \lambda_+^R \), \( \lambda_- < \lambda_-^R \), as
This wave pattern can be represented approximately as a modulated nonlinear periodic wave whose parameters \( \lambda_i \) \((i = 1, 2, 3, 4, \text{ cf. Sec. III})\) change slowly along the wave structure. In this case, the two dispersionless Riemann invariants \( \lambda_k \) are replaced in the DSW region by four Riemann invariants \( \lambda_i \), cf. Fig. 4(c,d). In this region, the evolution of the DSW is determined by the Whitham equations \((49)\). In our case, when we consider of self-similar solution, all Riemann invariants depend only on \( \xi = x/t \), and the Whitham equations reduce to

\[
\frac{d\lambda_i}{d\xi} [v_i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) - \xi] = 0, \quad i = 1, 2, 3, 4. \tag{44}
\]

One of the two factors in this equation must vanish, that is, for each \( i \), either the Riemann invariant \( \lambda_i \) is constant, or \( v_i = \xi \). For the Whitham velocities \((31)\) there exist solutions for which, in the DSW region, only one of the Riemann invariants vary, whereas the others remain constant. These solutions correspond qualitatively to the same patterns as the ones depicted in Figs. 4(c,d). However, naturally, the dependence of the Riemann invariants on \( \xi \) resulting from \((44)\) differs from the one obtained from \((39)\) (different equations and different variables). As a result, the velocities of the edges of the DSW do not coincide with the velocities of the nonphysical solutions of the dispersionless equations. It nonetheless remains true that Figs. 4(c,d) schematically represent the structure of the Riemann invariants for the solutions of the Whitham equations \((44)\) in Fig. 4(c) \( \lambda_2 \) varies and \( (\lambda_1, \lambda_3, \lambda_4) \) remain constant, whereas in Fig. 4(d) \( \lambda_3 \) varies and \( (\lambda_1, \lambda_2, \lambda_4) \) remain constant. We thus arrive at the following description of these solutions:

- In the case of Fig. 4(c) where \( \lambda_+^L = \lambda_+^R, \lambda_+^L > \lambda_+^R \) we have

\[
\lambda_1 = \lambda_+^R, \quad \lambda_3 = \lambda_+^L, \quad \lambda_4 = \lambda_+^L = \lambda_+^R, \tag{45}
\]

and \( \lambda_2 \) depends on \( \xi \) according to the equation

\[
v_2(\lambda_+^R, \lambda_2, \lambda_+^L, \lambda_+^L) = \xi. \tag{46}
\]

The resulting wave pattern is obtained by substitution of these values of \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) into \((19)\) and \((13)\). The left, small amplitude, edge of the DSW propagates with the velocity

\[
s_- = 2\lambda_+^R + \frac{(\lambda_+^L - \lambda_+^R)^2}{2(\lambda_+^L + \lambda_+^R - 2\lambda_+^R)}. \tag{47}
\]

The right edge corresponds to the soliton limit, propagating with the velocity

\[
s_+ = \frac{1}{2}(\lambda_+^R + 2\lambda_+^L + \lambda_+^L). \tag{48}
\]

- In a similar way, in the case of Fig. 4(d) where \( \lambda_+^L > \lambda_+^R \) we have

\[
\lambda_1 = \lambda_+^L, \quad \lambda_2 = \lambda_+^R, \quad \lambda_4 = \lambda_+^L. \tag{49}
\]
and the dependence of $\lambda_3$ on $\xi$ is determined by the implicit equation

$$v_3(\lambda_-^R, \lambda_-^L, \lambda_3, \lambda_+^L) = \xi. \quad (50)$$

Substitution of the values of $\lambda_i$ resulting from (49) and (50) into (25) and then (13) yields the oscillatory DSW structure for the physical variables $u$ and $h$. The left edge of the DSW corresponds to the soliton limit and this soliton moves with the velocity

$$s_- = \frac{1}{2}(\lambda_-^R + 2\lambda_-^L + \lambda_+^L). \quad (51)$$

Its right edge corresponds to the small amplitude limit propagating with the velocity

$$s_+ = 2\lambda_+^L + \frac{2(\lambda_-^R - \lambda_-^L)^2}{2(\lambda_-^R + \lambda_-^L - 2\lambda_+^L)}. \quad (52)$$

### IV. CLASSIFICATION OF SOLUTIONS OF THE RIEMANN PROBLEM

For a given choice of initial conditions (4), the solution of the Riemann problem consists of combinations of the key elements listed in the preceding section: plateaus, rarefaction waves and dispersive shocks. It is important to notice that if a RW or a DSW matches with a plateau at both its left and right edges, then these plateaus share one of their (dispersionless) Riemann invariants. For example, in Figs. 4(a) and (d) we have $\lambda_-^L = \lambda_-^R$ and in Figs. 4(b,c) we have $\lambda_-^L = \lambda_-^R$.

Also, in the case of a DSW, despite the fact that the dynamics inside the shock region is described by four Riemann invariants, two of them coincide with the dispersionless invariants of one of the plateaus at the edges of the shock. Hence one may say that the value of one of the dispersionless Riemann invariants ($\lambda_-$, say, as in the case of Fig. 4(d)) is “transferred” through the DSW, although, if it were computed using formula (36), one would find that it strongly oscillates inside the DSW region. The equality $\lambda_-^L = \lambda_-^R$ (for the case of 4(d)) connects the parameters of the flow at both sides of the dispersive shock, and in this sense it plays a role similar to that of the Rankine-Hugoniot condition in the theory of viscous shocks (see Refs. 23 and 24).

Due to this property of the dispersionless Riemann invariants, the points corresponding to the edges of the DSW (or of the RW) must lie on one of the parabolas along which the value of the dispersionless Riemann invariant remains constant. Hence, the dispersionless parabolas of Fig. 1 are useful tools for the classification of all possible solutions. One should keep in mind that the parabolic arcs symbolize the different types of solutions: a physically acceptable single-valued RW (Fig. 4(a,b)) or a formal multi-valued solution (Fig. 4(c,d)) which should be replaced by a DSW correctly treated within the Whitham approach. After these preliminary remarks we can proceed to the classification of the wave structures.

![Figure 5. Regions in the $(u, h)$ plane corresponding to different types of flow. The left boundary corresponds to point $L$ of coordinates $(u_L, h_L)$. The two parabolas are defined by the equations $h = (\frac{1}{2}u - \lambda_+^L)^2$ and $h = (\frac{1}{2}u - \lambda_-^L)^2$. The type of flow depends on the region (A,B, ..., or F) in which lies the right boundary point $R$ of coordinates $(u_R, h_R)$.](image)

The left and right boundaries of the whole wave structure connect with undisturbed plateau regions whose parameters coincides with the initial conditions (4); for instance, in any situation one should always have at the left boundary: $u(x/t \leq s_-^L) = u_L$ and $h(x/t \leq s_-^L) = h_L$. Consequently, the left and right edges propagate into plateau regions represented by the two points $(u_L, h_L)$ and $(u_R, h_R)$ in the $(u, h)$ plane. We represent in Fig. 5 the two parabolas corresponding to the constant dispersionless invariants $\lambda_+^L = u_L/2 + \sqrt{h_L}$ including their branches extending beyond the tangent points with the $u$-axis (which were represented as dashes lines in Fig. 1). These parabolas cut the physical half-plane $h > 0$ into six domains labeled by the symbols $A, B, ..., F$. Depending on the domain in which the point $R$ with coordinates $(u_R, h_R)$ lies, one has one of the following possible orderings of the left and right Riemann invariants:

- A: $\lambda_-^L < \lambda_-^R < \lambda_+^R < \lambda_+^L$
- B: $\lambda_-^L < \lambda_-^R < \lambda_-^L < \lambda_+^L$
- C: $\lambda_-^L < \lambda_-^R < \lambda_-^L < \lambda_+^L$
- D: $\lambda_-^L < \lambda_-^R < \lambda_-^L < \lambda_+^L$
- E: $\lambda_-^L < \lambda_-^R < \lambda_-^L < \lambda_+^L$
- F: $\lambda_-^L < \lambda_-^R < \lambda_-^L < \lambda_+^L$

(53)

These six situations correspond to the six possible wave structures resulting from the initial discontinuous profiles (4). We shall now describe their main properties and parameters. Note that, as expected, the typology below does not depend on the absolute values of $u_R$ and $u_L$, but only on their relative positions.

(A) In this case, the two rarefaction waves represented in Fig. 2(a) and (b) are combined into a single wave structure where they are separated by an empty region (in which $h(x, t) = 0$). The velocities of the edges of the RWs are given by the formulae

$$s_-^L = u_L - \sqrt{h_L}, \quad s_-^R = u_R - \sqrt{h_R}, \quad s_+^L = u_L + 2\sqrt{h_L}, \quad s_+^R = u_R + \sqrt{h_R}. \quad (54)$$
The corresponding wave structure is displayed in Fig. 6(A). As expected, the dispersionless approximation gives a very accurate description of the solution.

In the hydrodynamic context this situation corresponds to launching two fluids in opposite directions with velocities so large that the rarefaction waves are not able to fill the empty regions between them.

(B) Here the parabolas \( u/2 + \sqrt{h} = u_L/2 + \sqrt{h_L} \) and \( u/2 + \sqrt{h} = u_R/2 - \sqrt{h_R} \) cross at the point \( P = (u_P, h_P) \) with the Riemann invariants \( \lambda_P^L = u_P/2 \pm \sqrt{h_P} \), and their equality yields the values of the physical variables

\[
\begin{align*}
    u_P &= \lambda_P^L + \lambda_P^R = \frac{1}{2}(u_L + u_R) + \sqrt{h_L} - \sqrt{h_R}, \\
    h_P &= \frac{1}{4}(\lambda_P^L - \lambda_P^R)^2 = \frac{1}{4} \left( \frac{1}{2}(u_L - u_R) + \sqrt{h_L} + \sqrt{h_R} \right)^2.
\end{align*}
\] (55)

In this case, one has two rarefaction waves separated by a plateau region which is represented by point \( P \) in the \((u, h)\) plane of Fig. 6(B). The velocities of the edges can be easily found from the self-similar solutions [40] and [41]:

\[
\begin{align*}
    s_+^L &= v_-(\lambda_+^L, \lambda_+^L) = \frac{1}{2}(3\lambda_+^L + \lambda_+^R) = u_L - \sqrt{h_L}, \\
    s_+^R &= v_+(\lambda_+^R, \lambda_+^R) = \frac{1}{2}(3\lambda_+^R + \lambda_+^L), \\
    s_-^R &= v_+(\lambda_-^R, \lambda_-^R) = \frac{1}{2}(\lambda_-^R + 3\lambda_-^L), \\
    s_-^L &= v_-(\lambda_-^L, \lambda_-^L) = \frac{1}{2}(\lambda_-^L + 3\lambda_-^R). \quad (56)
\end{align*}
\]

The corresponding wave structure is displayed in Fig. 6(B). As in case (A), the dispersionless approximation gives a very accurate description of the solution.

Here the hydrodynamic interpretation is that the two fluids are moving away from each other with velocities lower than in the previous case (A), and the rarefaction waves are now able to provide enough flux of fluid to create a plateau in the region which separates them. This plateau has a fixed value of the height \( h \) and the flow velocity \( u \).

(C) In this case the initial profile evolves to form a DSW on the left, a RW on the right and a plateau in
between. The Riemann invariants in the plateau region are \( \lambda_P^R = \lambda_P^L \) and \( \lambda_P^R = \lambda_P^L \). In the DSW region, the Riemann invariants behave as schematically represented in Fig. 4(c). The edges of the DSW propagate with velocities

\[
\begin{align*}
    s^L_\pm &= v_2(\lambda^R_\pm, \lambda^L_\pm, \lambda^L_+), \\
    s^L_+ &= v_2(\lambda^R_\pm, \lambda^L_\pm, \lambda^L_+) = \frac{1}{2} \left( \lambda^R - 2\lambda^L + \lambda^L_+ \right),
\end{align*}
\]

and the velocities of the edges of the RW are equal to

\[
\begin{align*}
    s^R_\pm &= v_+(\lambda^R_\pm, \lambda^L_\pm) = \frac{1}{2} (\lambda^R + 3\lambda^L_+), \\
    s^R_+ &= v_+(\lambda^R_\pm, \lambda^L_\pm) = \frac{1}{2} (\lambda^R + 3\lambda^L_+).
\end{align*}
\]

This situation could be interpreted as if one fluid was colliding with the other flowing away with such velocity that a plateau with increased density is formed between them. The corresponding wave structure is displayed in Fig. 7(C). The right RW and the plateau region are correctly described by the dispersionless approximation, as can be check on the upper plot of this figure where the Riemann invariants in the plateau region are equal again to \( \lambda^R = \lambda^L = \lambda^L_+ \). The velocities of the RW's edges are equal to

\[
\begin{align*}
    s^L_\pm &= v_-(\lambda^R_\pm, \lambda^L_+) = \frac{1}{2} (3\lambda^L_+ + \lambda^L_+), \\
    s^L_+ &= v_-(\lambda^R_\pm, \lambda^L_+) = \frac{1}{2} (3\lambda^L_+ + \lambda^L_+).
\end{align*}
\]

The behavior of the two dispersionless Riemann invariants in the region of the rarefaction wave corresponds to the case illustrated in Fig. 4(b). In the DSW region there are four Riemann invariants which behave as schematically represented in Fig. 4(d), and the edges of the DSW propagate with velocities

\[
\begin{align*}
    s^R_- &= v_3(\lambda^R_-, \lambda^R_+, \lambda^L_+, \lambda^L_+) = \frac{1}{2} (\lambda^R + 2\lambda^R_+ + \lambda^L_+), \\
    s^R_+ &= v_3(\lambda^R_-, \lambda^R_+, \lambda^L_+, \lambda^L_+) = 2\lambda^L_+ + \frac{(\lambda^R_+ - \lambda^R_-)^2}{2(\lambda^L_+ + \lambda^L_+ - 2\lambda^L_+)}. 
\end{align*}
\]

This situation is similar to the preceding one upon exchanging the roles of the left and right fluids; we thus do not illustrate it by a figure.

(E) In this case the initial profile evolves in two DSWs separated by a plateau, the parameters of which are \( \lambda_P^R = \lambda_P^L \) and \( \lambda_P^R = \lambda_P^L \). The DSW's edges propagate with velocities

\[
\begin{align*}
    s^L_- &= v_2(\lambda^R_-, \lambda^L_-, \lambda^L_+, \lambda^L_+) = \frac{1}{2} (\lambda^R + 2\lambda^L_+ + \lambda^L_+), \\
    s^L_+ &= v_2(\lambda^R_-, \lambda^L_-, \lambda^L_+, \lambda^L_+) = \frac{1}{2} (\lambda^R + 2\lambda^L_+ + \lambda^L_+), \\
    s^R_- &= v_3(\lambda^R_-, \lambda^R_+, \lambda^L_+, \lambda^L_+) = 2\lambda^L_+ + \frac{(\lambda^R_+ - \lambda^R_-)^2}{2(\lambda^L_+ + \lambda^L_+ - 2\lambda^L_+)}. 
\end{align*}
\]

Here we have a collision of two fluids with ‘moderate’ velocities: the two DSWs do not overlap, but a central plateau region of increased height is formed. This situation is represented in Fig. 7(E). Again, the theoretical approach quite accurately describes the numerical results (cf. the bottom row).

The upper part of the figure illustrates a phenomenon already present in case (C): the large nonlinear oscillations in the DSW regions are associated with locally negative values of \( h(x,t) \). This phenomenon is clearly seen in Fig. 8 which represents \( h \) as a function of \( x \) for the two configurations (C) and (E) considered in Fig. 7. Although extended regions of constant and negative values of \( h \) lead to a dynamical instability (as clearly seen from the dispersion relation (2)), nothing forbids local excursions of \( h \) below 0, and this is confirmed by the excellent agreement of the numerical and theoretical results presented in Fig. 8. Of course, in this case, the interpretation of \( h \) as being the height of a fluid surface becomes meaningless, but, as explained in the introduction, the physical model behind the nonlinear equations (E) can have an origin different from shallow water physics.

(F) In this configuration the two fluids collide with velocities so large that the central plateau observed in case (E) disappears: the DSWs overlap and, on the basis of a local excursions of \( h \) below 0, and this is confirmed by the excellent agreement of the numerical and theoretical results presented in Fig. 8. Of course, in this case, the interpretation of \( h \) as being the height of a fluid surface becomes meaningless, but, as explained in the introduction, the physical model behind the nonlinear equations (E) can have an origin different from shallow water physics.

Our numerical simulations show that this is not the case: the collision of the DSW is here associated with a numerical instability which we attribute to a physical dynamical instability of the region of the overlapping DSWs.

V. DAM BREAK AND PISTON PROBLEM

In view of the particularities noticed in the precedent section – possible negative values of \( h \) and dynamical
instabilities – it is interesting to study in more detail two model cases illustrating the specificity of the Kaup-Boussinesq system. The first one is the dam break problem which corresponds to a particular case of initial conditions \( s \). The second case is the piston problem. It does not pertain to the same class of initial conditions, but nevertheless provides an instructive insight on non-modulated cnoidal waves whose stability is questioned by the results obtained in case (F).

A. Dam break problem

This is the case where a semi-infinite constant height of water expands into empty space, which would be a model of flow after the abrupt breaking of a dam. Such a configuration is schematically described by an initial condition of type \( \text{(C)} \) with

\[
h_R = 0 \quad \text{and} \quad u_R = 0.
\]

On the basis of physical intuition, one expects that the time evolution of this initial profile will result in a rarefaction wave expanding into vacuum, as for the case illustrated in Fig. 2(a). This is not quite correct: such a situation is only reached when \( u_L \) is sufficiently negative. More specifically, the initial condition (C) pertains either to case (A) when \( u_L < -2\sqrt{h_L} \), either to case (D) when \( |u_L| < 2\sqrt{h_L} \), either to case (F) when \( u_L > 2\sqrt{h_L} \). In other words, this is only when the initial left velocity (the initial velocity of the water of the dam) is negative enough that a rarefaction wave is observed. This behavior is different from the one observed in the similar case for the nonlinear Schrödinger equation. In the present case, the most natural situation where \( u_L = 0 \) (the water in the dam is initially steady) pertains to case (D) for which the dam break leads to a DSW where the field \( h \) becomes negative. Only when \( u_L \) becomes negative enough (in practice, when it becomes lower than \(-2\sqrt{h_L}\), i.e., when one reaches the regime (A)) does the excursion of \( h(x,t) \) below zero disappear.

This point deserves a slightly more detailed discussion: for the dam break problem in case (D) one can easily check that the plateau region has a vanishing extension \((s_L^+ = \frac{1}{2} \lambda_2^L = s_R^L)\). The behavior of the Riemann invariants is depicted in Fig. 9. In this case \( \lambda_1 = \lambda_2 \) and the DSW is described by a nonlinear trigonometric wave whose large amplitude boundary (at \( s_R^L \)) corresponds to an algebraic soliton of type \( \text{(E)} \) for which the largest value of \( \mu \) is \( \lambda_0^L = \lambda_2^L \). Then, from (13), the corresponding extremal value of \( h \) is \(-\frac{1}{2} \lambda_1^L = -\frac{1}{2}(3u_L + \sqrt{h_L})\); it is always negative, and only vanishes when one leaves regime (D) to enter regime (A), i.e., when \( u_L \leq -2\sqrt{h_L} \); in this case the plateau \( P \) and the right boundary \( R \) coincide and the flow is of the type exemplified in Fig. 2(a), which corresponds to a RW expanding into empty space, as intuitively expected.

FIG. 8. \( h \) as a function of \( \xi \) for the same configurations as the ones depicted in Fig. 7(C) and (E). The pink solid line represents the results of the numerical simulations and the black solid line is the theoretical result. Note the numerous excursions of \( h(x,t) \) below zero.

FIG. 9. The upper plot is a sketch of the behavior of the Riemann invariants describing the dam break problem when \( |u_L| < 2\sqrt{h_L} \) (type (D) configuration). The dispersionless Riemann invariants describing the boundary conditions at the right are \( \lambda_1^R = \lambda_2^R = 0 \), and at the left, \( \lambda_1^L = \frac{1}{2} u_L \). The two lower plots compare the results of the numerical simulations (pink solid lines) with the theoretical results (black solid lines) for the boundary condition \( h_L = 1 \) and \( u_L = 0 \).
B. Piston problem

The piston problem corresponds to the situation where a hard wall (the piston) is moving (in the case considered here, with a constant positive velocity $V$) with respect to a steady fluid. We work henceforth in the rest frame of the piston. In this frame the piston is located at $x = 0$, the fluid is incoming from the right with a constant velocity $u_R = -V$ and a fixed constant depth $h_R$. The boundary condition on the piston is $u(0, t) = 0$: the fluid in contact with the piston is at rest with respect to it. The boundary condition for the height is taken as $h(0, t) = 0$, or $h_x(0, t) = 0$. These two conditions, of Dirichlet or Neumann type, are equivalent if treated within the Whitham approach, since the corresponding profiles differ only locally near 0, over characteristic lengths of order 1.

For intermediate velocities $V$, the profile is of the type characterized by the arrangement of Riemann invariants displayed in Fig. 1(d): there is a plateau in contact with the piston, then, at its right, a DSW, and finally a plateau corresponding to the right boundary condition, characterized by $\lambda^R = \frac{1}{2} u_R \pm \sqrt{h_R}$. The plateau in contact with the piston is characterized by a height $h_L$ (unknown at this point) and a velocity $v_L = 0$, hence $\lambda_L^L = \pm \sqrt{h_L}$. The constancy of the lower Riemann invariant across the structure of Fig. 1(d) yields $\lambda_L^L = \lambda^R$ which fixes the value of $h_L = (\sqrt{h_R} - u_R/2)^2$. The velocities of the edges of the DSW are determined from (51) and (52). In particular, the boundary between the left plateau and the DSW has a velocity $s_- = \sqrt{h_R} - V/2$. This velocity vanishes when $V = 2\sqrt{h_R}$. For piston velocities $V$ larger than this threshold, the plateau in contact with the piston disappears and the structure of the flow changes: in a good approximation it is represented by a stationary, non-modulated cnoidal wave (SCW) in contact with the piston. This SCW is connected to its right with a partial DSW, itself connecting to a plateau defined by the right boundary condition. This corresponds to the arrangement of Riemann invariants displayed in Fig. 10.

The fact that the cnoidal wave located in the region $0 < \xi < s_-$ in Fig. 10 is stationary reflects in the relation $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ which fixes its phase velocity to zero. This yields

$$\lambda_3 + \lambda_4 = -\lambda_1 - \lambda_2 = -u_R = V. \quad (63)$$

Another condition is obtained by imposing that there is no flux through the piston, therefore the average flux vanishes: $\langle u \rangle = 0$ (it is evident that there is no contribution of the last term in the first Eq. 3 to the average flux, since $u_{xx} \equiv 0$ by virtue of local periodicity). The relation (63) yields $s_1 = 0$ and Eqs. (13) now read $u = -2\mu$ and $h(\mu) = -s_2 - 2\mu^2$ where $s_2 = \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2)^2 + \lambda_3 \lambda_4$. Hence we get the condition

$$s_2(\mu) + 2\langle \mu^3 \rangle = 0, \quad (64)$$

where $\langle \mu^n \rangle = L^{-1} \int \frac{1}{2} \mu^n d\mu / \sqrt{-P(\mu)}$. Condition (64) is fulfilled for $s_3 = (\lambda_1 + \lambda_2)(\lambda_3 \lambda_4 - \lambda_1 \lambda_2) = 0$. Together with (63) this yields a system for the yet unknown quantities $\lambda_3$ and $\lambda_4$. The obvious solutions are (taking into account the ordering (11)) $\lambda_3 = -\lambda_1$ and $\lambda_3 = -\lambda_2$. Then, in the SCW, the height $h$ oscillates between the two opposite values $\pm 2V \sqrt{h_R}$ and the velocity $v$ between $-V \pm 2\sqrt{h_R}$. The velocities of the edges of the DSW are $s_+ = v_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $s_- = v_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 2\sqrt{h_R} + V - h_R/(V + \sqrt{h_R})$. The wavelength of the large amplitude edge of the DSW (at $\xi = s_-$) is given by

$$L = \frac{2K(m^*)}{V}, \text{ where } m^* = \frac{4h_R}{V^2} \quad (65)$$

is the modulus (20) of the elliptic functions; $m^* < 1$ in the present case since $V > 2\sqrt{h_R}$. As is checked in Fig. 10 these predictions are in excellent agreement with the numerical simulations.

Two comments are in order here: first, the numerical simulations show that the cnoidal wave is weakly modulated, and, as suggested in Refs. 12 and 25 it should be more accurately described as a two phase solution. However, as seen in Fig. 10 the modulation is small, and the approximate description of the structure as a SCW is quite accurate. Second, and more important, we have here an example of dynamically stable non-modulated cnoidal wave with large amplitude ($h$ oscillate between $\pm 5$ in the SCW region of Fig. 10). This is quite different from the situation observed in the preceding section (region F) in Fig. 5 where an expected SCW resulting from the collision of two DSWs has proven unstable. This example shows that the wave structures in the piston problem may have properties quite different from those arising from the evolution of initial discontinuities.

VI. CONCLUSION

In this paper we have developed a full classification of the wave patterns evolving dynamically from initial
discontinuities according to the Kaup-Boussinesq equation with positive dispersion. At variance with the case of negative dispersion considered in Ref. [12], the classification used here follows closely the one for the nonlinear Schrödinger equation [13], although there are a number of technical differences caused by the possible negative value of the “height” field and also by different representations of dark and bright “soliton trains” and corresponding changes of the Whitham modulation equations. This common behavior of the positive dispersion Kaup-Boussinesq and the nonlinear Schrödinger equation is related to the common sign of dispersion in both equations and will be clarified in a forthcoming publication [15].

Our results can find applications as approximations of the dynamics of polarization waves in two-component Bose-Einstein condensates [21] and of magnetic systems with easy-plane anisotropy [25]. Work in this direction is in progress.

ACKNOWLEDGMENTS

We are grateful to M. A. Hoefer for useful discussions. AMK thanks Laboratoire de Physique Théorique et Modèles Statistiques (Université Paris-Sud, Orsay) where this work was started, for kind hospitality. This work was supported by the Russian Foundation for Basic Research (project no. 16-01-00398) and by the French ANR under grant n° ANR-15-CE30-0017 (Harlab project).

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