A Study of Brane Solutions in
$D$-dimensional Coupled Gravity
System

Bihn Zhou* and Chuan-Jie Zhu†
Institute of Theoretical Physics, Chinese Academy of
Sciences, P. O. Box 2735, Beijing 100080, P. R. China

September 9, 2017

Abstract

In this paper, we use only the equation of motion for an interacting system of gravity, dilaton and antisymmetric tensor to study the soliton solutions by making use of a Poincaré invariant ansatz. We show that the system of equations are completely integrable and the solution is unique with appropriate boundary conditions. Some new class of solutions are also given explicitly.

1 Introduction

Superstring theory \cite{1,2} was the leading candidate for a theory unifying all matter and forces (including gravity, in particular). Unfortunately, there are five such consistent theories, namely, $SO(32)$ type I, type IIA, type IIB, $E_8 \otimes E_8$ heterotic and $Spin(32)/Z_2$ heterotic theories. This richness is an embarrassment for pure theorists. One hope is that all these five superstring theories are just different solutions of an underlying theory. In the last few years this hope turns out to be true. The underlying theory is the so called

*E-mail: zhoub@itp.ac.cn
†E-mail: zhucj@itp.ac.cn
M theory \[3, 4, 5\]. One formulation of M theory is given in terms of Matrix theory \[6\]. Nevertheless this formulation is not background independent. It is difficult to use it to discuss non-perturbative problems and still we should rely on the study of BPS states. A thorough study of BPS states in any theory is a must for an understanding of non-perturbative phenomena.

In superstring theory and M theory, there is a plethora of BPS states. These BPS states have the special property of preserving some supersymmetry. In a low energy limit they are special solutions of the low energy supergravity theory with Poincaré invariance. However there exists also some other \(p\)-branes which don’t preserve any supersymmetry and they are also no longer Poincaré invariant. In finding these (soliton) solutions one either resort to supersymmetry or make some simple plausible assumptions. There is no definite reasoning that the so obtained solution is unique. The purpose of this paper is to fill this gap.

It is also interesting to study soliton solutions for its own sake without using any supersymmetric argument. Quite recently there are some interests \[7, 8, 9, 10\] in studying branes in type 0 string theories which have no fermions and no supersymmetry in 10 dimensions \[2\]. In these string theories, we can’t use supersymmetric arguments. So it is important to push other symmetric arguments to their limits.

In this paper we will study the coupled system of gravity \((g_{MN})\), dilaton \((\phi)\) and anti-symmetric \((A_{M_1M_2...M_{n-1}})\) tensor in any dimensions. After making a Poincaré invariant ansatz for the metric and either electric or magnetic ansatz for the anti-symmetric tensor, we derive all the equations of motion in some details. A system of five ordinary differential equations are obtained for four unknown functions (of one variable). After making some changes of the unknown functions we solved three equations explicitly. We show that the remaining two equations (for one unknown functions) are mutually compatible and a unique solution can always obtained with appropriate boundary conditions. To solve the last equation in the most generic case is surely beyond our ability because this last equation is a Riccati equation which is known to be not solvable by algebraic means. Instead we will discuss some degenerate cases and recover many known solution in the literature. Some new solutions are also found and their physical property will be discussed elsewhere \[16\].

For previous studies of soliton and brane solution in supergravity and string theories, see for example the reviews \[11, 12, 13, 14, 15\].

The extension of the result of this paper to black branes will appear in another publication \[17\]. We remark that this paper contains some unnece-
sary computational details from some experts’ point of view. Although some formulas are not new (but scattered in various places), we still think that they are necessary for an understanding of our logic.

2 The Equations of Motion and the Ansatz

Our starting point is the following action for the coupled system of gravity $g_{MN}$, dilaton $\phi$ and anti-symmetric tensor $A_{M_1\cdots M_{n-1}}$:

$$I = \int d^Dx \sqrt{-g} \left( R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} \frac{e^{a\phi}}{n!} F^2 \right),$$

where $a$ is a constant and $F$ is the field strength: $F = dA$. An overall proportional constant for the action $I$ is irrelevant for what follows.

The equations of motion can be easily derived from the above action (1):

$$R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + S_{MN},$$

$$\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} e^{a\phi} F^{M_1 M_2 \cdots M_n} \right) = 0,$$

$$\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} g^{MN} \partial_N \phi \right) = \frac{a}{2n!} e^{a\phi} F^2,$$

where

$$S_{MN} = \frac{1}{2(n-1)!} e^{a\phi} \left( F_{M M_2 \cdots M_n} F_N^{M_2 \cdots M_n} - \frac{n-1}{n(D-2)} F^2 g_{MN} \right).$$

Of course it is impossible to solve the above system equations in their full generality. To get some meaningful solution we will make some assumptions by using symmetric arguments.

Our ansatz for a $p$-dimensional brane is as follows:

$$ds^2 = e^{2A(r)} (-dt^2 + \sum_{\alpha=1}^p (dx^\alpha)^2) + e^{2B(r)} dr^2 + e^{2C(r)} d\Omega_{\tilde{d}+1},$$

where $d\Omega_{\tilde{d}+1}$ is the square of the line element on the unit $\tilde{d}+1$ sphere which can be written as follows ($D = p + \tilde{d} + 3$):

$$d\Omega_{\tilde{d}+1} = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + (\sin \theta_1 \cdots \sin \theta_{\tilde{d}})^2 d\theta_{\tilde{d}+1}^2.$$
The brane is extended in the directions \((t, x^\alpha)\). The ansatz (8) is Poincaré invariant in these directions. For our later use in [17] we will give the formulas for the more general ansatz:

\[
ds^2 = -e^{2A(r)}dt^2 + \sum_{\alpha=1}^{p} e^{2A_\alpha(r)}(dx^\alpha)^2 + e^{2B(r)}dr^2 + e^{2C(r)}d\Omega_{\tilde{d}+1}.
\] (8)

We also note here that there is still some freedom to choose the parametrization \(r\). This freedom can be fixed by making either the choice \(C(r) = \ln r\) or \(C(r) = B(r) + \ln r\).

For the anti-symmetric tensor \(A\), we have 2 different choices. The first choice is the electric case where we take the following form for \(A\) \((x^0 = t)\):

\[
A = \pm e^{A(r)} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p.
\] (9)

This fixes \(p = n - 2\). The second choice is the magnetic case where we take the following form for the dual potential \(\tilde{A}\):

\[
\tilde{A} = \pm e^{A(r)} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p,
\] (10)

and \(p = D - n - 2\). We note that the relation between the antisymmetric tensor field strength \(F\) and its dual field strength \(\tilde{F}\) \((\equiv d\tilde{A})\) is:

\[
F_{M_1 \cdots M_n} = \frac{1}{\sqrt{-g}} e^{-a \phi} \frac{1}{(D - n)!} \epsilon^{M_1 \cdots M_n N_1 \cdots N_{D-n}} \tilde{F}_{N_1 \cdots N_{D-n}}.
\] (11)

By using this relation the ansatz (10) transform to an ansatz for \(F\):

\[
F = \pm \Lambda' \exp(\Lambda - a \phi - A - \sum_{\alpha=1}^{p} A_\alpha - B + (\tilde{d} + 1)C) \omega_{\tilde{d}+1},
\] (12)

where \(\omega_{\tilde{d}+1}\) is the volume form of the sphere \(S^{\tilde{d}+1}\) with unit radius. As it is well-known from duality, the equation of motion (3) becomes the Bianchi identity for \(\tilde{F}\) which is satisfied automatically. Nevertheless the Bianchi identity for \(F\) becomes the equation of motion for \(\tilde{F}\) which is given as follows:

\[
\frac{1}{\sqrt{-g}} \partial_{N_1} \left( \sqrt{-g} e^{-a \phi} \tilde{F}^{N_1 N_2 \cdots N_{D-n}} \right) = 0.
\] (13)

From now on we will discuss the electric case only.
We will compute the Riemann curvature by using the vilebein formalism. The coordinate index is denoted as \((M) = (t, \alpha, r, i)\) and tangent index is denoted as \((A) = (\bar{t}, \bar{\alpha}, \bar{r}, \bar{i})\), i.e. a bar over coordinate index. The moving frame is:

\[
e^{\bar{t}} = e^{A(r)} \, dt = -e_t, \tag{14}
\]

\[
e^{\bar{\alpha}} = e^{A_\alpha(r)} \, dx^\alpha = e_{\bar{\alpha}}, \tag{15}
\]

\[
e^{\bar{r}} = e^{B(r)} \, dr = e_{\bar{r}}, \tag{16}
\]

\[
e^{\bar{i}} = e^{C(r)} \sin \theta_1 \cdots \sin \theta_{i-1} d\theta_i = e_{\bar{i}}. \tag{17}
\]

By using the defining equation of \(\omega^{AB}\):

\[
d e^A + \omega^{AB} \wedge e_B = 0, \tag{18}
\]

we obtain the following expression for \(\omega^{AB}\):

\[
\omega^{\bar{t}\bar{t}} = A' e^{A-B} \, dt = A' e^{-B} e^{\bar{t}}, \tag{19}
\]

\[
\omega^{\bar{\alpha}\bar{t}} = A'_\alpha e^{A-\bar{B}} \, dy_\alpha = A'_\alpha e^{B} e^{\bar{t}}, \tag{20}
\]

\[
\omega^{\bar{t}\bar{r}} = C' e^{C-B} \sin \theta_1 \cdots \sin \theta_{i-1} d\theta_i = C' e^{B} e^{\bar{r}}, \tag{21}
\]

\[
\omega^{\bar{i}\bar{t}} = \cos \theta_i \sin \theta_{i+1} \cdots \sin \theta_{i-1} d\theta_i, \quad i > j. \tag{22}
\]

The other components are zero or can be obtained from the above by using the anti-symmetric property of \(\omega^{AB}\): \(\omega^{AB} = -\omega^{BA}\).

From \(\omega^{AB}\) we can compute \(R^{AB}\):

\[
R^{AB} = d\omega^{AB} + \omega^{AC} \wedge \eta_{CD} \omega^{DB}. \tag{23}
\]

The results are:

\[
R^{\bar{t}\bar{\alpha}} = -A' A'_\alpha e^{-2B} e^{\bar{t}} \wedge e^{\bar{\alpha}}, \tag{24}
\]

\[
R^{\bar{t}\bar{r}} = -(A' e^{A-B})' e^{-A-B} e^{\bar{t}} \wedge e^{\bar{r}}, \tag{25}
\]

\[
R^{\bar{t}\bar{\bar{r}}} = -A' C' e^{-2B} e^{\bar{t}} \wedge e^{\bar{r}}, \tag{26}
\]

\[
R^{\bar{\alpha}\bar{\bar{t}}} = -A'_\alpha A'_\beta e^{-2B} e^{\bar{\alpha}} \wedge e^{\bar{\beta}}, \tag{27}
\]

\[
R^{\bar{\alpha}\bar{r}} = -(A'_\alpha e^{A-\bar{B}})' e^{-A-B} e^{\bar{\alpha}} \wedge e^{\bar{r}}, \tag{28}
\]

\[
R^{\bar{\alpha}\bar{\bar{i}}} = -A'_\alpha C' e^{-2B} e^{\bar{\alpha}} \wedge e^{\bar{i}}, \tag{29}
\]

\[
R^{\bar{i}\bar{r}} = -(C' e^{C-B})' e^{-C-B} e^{\bar{i}} \wedge e^{\bar{r}}, \tag{30}
\]

\[
R^{\bar{i}\bar{\bar{i}}} = (e^{-2C} - (C')^2 e^{-2B}) e^{\bar{i}} \wedge e^{\bar{i}}. \tag{31}
\]
We note that all $R^{AB}$'s are of the following special form:

$$R^{AB} = f^{AB} e^A \wedge e^B,$$

where there is no summation over $A$ and $B$. $f^{AB}$ can be chosen to be symmetric and is 0 when $A = B$. By definition

$$R^{AB} = \frac{1}{2} R^{AB}_{MN} \, dx^M \wedge dx^N,$$

we have

$$R^{AB}_{MN} = f^{AB}(e^A_M e^B_N - e^A_N e^B_M),$$

and

$$R^A_M = R^{AB}_{MN} e^N_B = \sum_B f^{AB} e^A_M \equiv f^A e^A_M,$$

where we have defined $f^A$ as

$$f^A = \sum_B f^{AB},$$

which can be computed to give the following results:

$$f^t = -e^{-2B} \left( A'' + A'(A' + \sum_\alpha A'_\alpha - B' + (\tilde{d} + 1)C') \right),$$

$$f^\bar{\alpha} = -e^{-2B} \left( A'_\bar{\alpha} + A'(\bar{A}' + \sum_\beta A'_\beta - B' + (\tilde{d} + 1)C') \right),$$

$$f^\bar{\bar{\alpha}} = -e^{-2B} \left( A'' + \sum_\alpha A''_\alpha + (\tilde{d} + 1)C'' + (A')^2 + \sum_\alpha (A'_\alpha)^2 \\
+ (\tilde{d} + 1)(C')^2 - B'(A' + \sum_\alpha A'_\alpha + (\tilde{d} + 1)C') \right),$$

$$f^{\bar{\bar{\bar{\alpha}}}} = -e^{-2B} \left( C'' + C'(A' + \sum_\alpha A'_\alpha - B' + (\tilde{d} + 1)C') \right.)$$

$$-\tilde{d} e^{-2C+2B} \right).$$

By making use of fact that the metric in $[3]$ is a diagonal metric we have

$$R_{MN} = R^A_M e_{AN} = f^A \delta^A_M \, g_{MN},$$

$$R_{MN} dx^M \otimes dx^N = - f^t e^t \otimes e^t + \sum_{\bar{\alpha}=1}^p f^{\bar{\alpha}} e^{\bar{\alpha}} \otimes e^{\bar{\alpha}}$$

$$+ f^\bar{\bar{\alpha}} e^{\bar{\bar{\alpha}}} \otimes e^{\bar{\bar{\alpha}}} + f^{\bar{\bar{\bar{\alpha}}}} d\Omega_{\tilde{d}+1},$$

6
To obtain the equations of motion from (2)–(4) we also need to know some expressions involving $F$. We have (for electric solution and $p = n - 2$)

$$A = \pm e^{A(r)} \, dx^0 \wedge dx^1 \cdots \wedge dx^{n-2}, \quad (43)$$

$$F = dA = \pm A' e^{A(r)} \, dr \wedge dx^0 \wedge dx^1 \cdots \wedge dx^{n-2}, \quad (44)$$

$$F_{MN} \equiv F_{MM_2 \cdots M_n} F^{M_2 \cdots M_n}, \quad (45)$$

$$F_{rr} = -(n - 1)! \, (A' e^{A(r)})^2 e^{-(n-1)A}, \quad (46)$$

$$F_{\mu\nu} = -\eta_{\mu\nu} \, (n - 1)! \, (A' e^{A(r)})^2 e^{-(n-2)A-2B}, \quad (47)$$

$$F_{\mu\nu} = 0, \quad \text{for the rest cases,} \quad (48)$$

and

$$F^2 = g^{MN} F_{MN} = -(n - 1)! \, (A' e^{A(r)})^2 e^{-(n-1)A-2B}, \quad (49)$$

The equation of motion for $g^{MN}$ gives the following equations:

$$f^g_{rr} = \frac{1}{2} (\partial_r \phi)^2 + \frac{1}{2 \cdot (n - 1)!} \, e^\phi \left( F_{rr} - \frac{(n - 1)}{n(D - 2)} \, F^2 g_{rr} \right), \quad (50)$$

$$f^g_{\mu\nu} = \frac{1}{2 \cdot (n - 1)!} \, e^\phi \left( F_{\mu\nu} - \frac{(n - 1)}{n(D - 2)} \, F^2 g_{\mu\nu} \right), \quad (51)$$

$$f^g_{ij} = \frac{1}{2 \cdot (n - 1)!} \, e^\phi \left( F_{ij} - \frac{(n - 1)}{n(D - 2)} \, F^2 g_{ij} \right), \quad (52)$$

for the $(rr)$, $(\mu\nu)$ and $(ij)$ components respectively. Substituting all $f^A$ and $F_{MN}$ into the above equations and setting $C = B + \ln r$, $A_\alpha = A$, we obtain the following three equations:

$$A'' + d(A')^2 + \frac{\tilde{d} + 1}{r} A' = \frac{\tilde{d}}{2(D - 2)} S^2, \quad (53)$$

$$B'' + dA' B' + \frac{\tilde{d}}{r} A' + \frac{\tilde{d}^2 + 1}{r} B' = -\frac{1}{2} \, \frac{n - 1}{D - 2} \, S^2, \quad (54)$$

$$dA'' + (\tilde{d} + 1) B'' + d(A')^2 + \frac{\tilde{d} + 1}{r} B'$$

$$-dA' B' + \frac{1}{2} (\phi')^2 = \frac{1}{2} \, \frac{\tilde{d}}{D - 2} \, S^2, \quad (55)$$
where \( S = \Lambda' e^{\frac{1}{2}a\phi + \Lambda - \tilde{d}A} \) and \( d = p + 1 = n - 1 \). The equation of motion for \( \phi \) is
\[
\phi'' + \left( dA' + \tilde{d}B' + \frac{\tilde{d}+1}{r} \right) \phi = -\frac{a}{2} S^2,
\] (56)
and the equation derived from the equation of motion for \( F \) is
\[
(\Lambda'(r) e^{\Lambda(r) + a\phi(r) - dA(r) + \tilde{d}B(r)} r^{\tilde{d}+1})' = 0.
\] (57)

These five equations, eqs. (53)-(55), (56) and (57), consist of the complete system of equations of motion for four unknown functions: \( A(r), B(r), \phi(r) \) and \( \Lambda(r) \). We will discuss these equations in the next two sections.

### 3 The Solvability of the Equations

In this section we will try to solve the system of the above five equations. First it is easy to integrate eq. (57) to get
\[
\Lambda'(r) e^{\Lambda(r) + a\phi(r) - dA(r) + \tilde{d}B(r)} r^{\tilde{d}+1} = C_0,
\] (58)
where \( C_0 \) is a constant of integration. If we know the other three functions \( A(r), B(r) \) and \( \phi(r) \), this equation can be easily integrated to give \( \Lambda(r) \):
\[
e^{\Lambda(r)} = C_0 \int r^\tilde{d} e^{-a\phi(r) + dA(r) - \tilde{d}B(r)} dr.
\] (59)

By using eq. (58), \( S \) can be written as
\[
S(r) = C_0 e^{-a\phi(r) - \tilde{d}B(r)} r^{-\tilde{d}+1}.
\] (60)

Now we make a change of functions from \( A(r), B(r) \) and \( \phi(r) \) to \( \xi(r), \eta(r) \) and \( Y(r) \):
\[
\xi(r) = dA(r) + \tilde{d}B(r), \quad (61)
\eta(r) = \phi(r) + a(A(r) - B(r)), \quad (62)
Y(r) = A(r) - B(r), \quad (63)
\]
or

\[ A = \frac{\xi + \tilde{d}Y}{d + \tilde{d}}, \quad (64) \]
\[ B = \frac{\xi - dY}{d + \tilde{d}}, \quad (65) \]
\[ \phi = \eta - aY. \quad (66) \]

The equations are then changed to

\[ \xi'' + (\xi')^2 + \frac{2\tilde{d} + 1}{r} \xi' = 0, \quad (67) \]
\[ \eta'' + \left( \xi' + \frac{\tilde{d} + 1}{r} \right) \eta' - \frac{a}{r} \xi' = 0, \quad (68) \]
\[ Y'' - \frac{\Delta}{2} (Y')^2 + \left( \frac{\tilde{d} - d}{D - 2} \xi' + \frac{\tilde{d} + 1}{r} + a \eta' \right) Y' \]
\[ - \frac{1}{2} (\eta')^2 - \xi'' + \frac{1}{D - 2} (\xi')^2 = 0, \quad (69) \]
\[ Y'' + \left( \xi' + \frac{\tilde{d} + 1}{r} \right) Y' - \frac{1}{r} \xi' = \frac{1}{2} S^2, \quad (70) \]

where

\[ \Delta = \frac{2\tilde{d} \Delta}{D - 2} + a^2. \quad (71) \]

The general solutions for \( \xi \) and \( \eta \) can be obtained easily from eqs. (67) and (68) and we have

\[ \xi = \ln \left| C_1 + C_2 r^{-2\tilde{d}} \right|, \quad (72) \]
\[ \eta' = \frac{2C_2 a + C_3 r^{\tilde{d}}}{r(C_2 + C_1 r^{2\tilde{d}})}, \quad (73) \]

where \( C_1, C_2 \) and \( C_3 \) are constants of integration. To make sense of the above expressions, \( C_1 \) and \( C_2 \) can’t be zero simultaneously. For \( C_1 \neq 0 \) we can always choose \( C_1 = 1 \) by a rescaling of \( t \) and \( x^\alpha \).

Substituting the above expressions into eqs. (69) and (70), we obtain

\[ Y'' - \frac{\Delta}{2} (Y')^2 + Q(r)Y' = R(r) \quad (74) \]
and

\[ S^2 = \Delta \left( Y' - \frac{2C_2 + C_3 a r^\tilde{d}}{r(C_2 + C_1 r^{2\tilde{d}})} \right)^2 + \frac{K}{\Delta(C_2 + C_1 r^{2\tilde{d}})^2} r^{2\tilde{d} - 2}, \quad (75) \]

where

\[ Q(r) = \frac{\tilde{d} + 1}{r} + \frac{2C_2(\Delta - \tilde{d}) + C_3 a r^\tilde{d}}{r(C_2 + C_1 r^{2\tilde{d}})}, \quad (76) \]

\[ R(r) = \frac{2C_2^2(\Delta - \tilde{d}) + 2C_2 C_3 a r^\tilde{d} + 2C_1 C_2 \tilde{d}(2\tilde{d} + 1) r^{2\tilde{d}} + \frac{1}{2} C_3^2 r^{2\tilde{d}}}{r^2(C_2 + C_1 r^{2\tilde{d}})^2}, \quad (77) \]

where

\[ K = C_3^2(\Delta - a^2) + 8C_1 C_2 \Delta \tilde{d}(\tilde{d} + 1), \quad (78) \]

is a constant.

From eq. (75) we see that if \( K \geq 0 \), there is no restriction for the interval of \( r \). If the constant \( K < 0 \), \( r \) can only take such values that where \( S^2 \geq 0 \).

Notice that our system of equations of motion is an over determined system: four unknown functions satisfying five equations. We have solved three equations and there are two equations, eqs. (74) and (75), remaining with one unknown functions \( Y(r) \). Now we would like to show that these two equations actually give no constraints on \( Y(r) \) and effectively there is only one equation, i.e., we can solve either one of them and the other one will be satisfied automatically.

Because this proof is a rather involved algebraic calculation which we did it by using Mathematica using computer, we will only give the steps of the proof.

First we change eq. (73) by taking the logarithm of both sides. The logarithm of \( S \) can be rewritten as a function of \( Y, \xi \) and \( \eta \) by using eq. (60). Now we differentiate both sides with respect to \( r \) to eliminate \( C_0 \). Substituting \( \xi \) and \( \eta' \) with the explicit results given in eqs. (72) and (73) we obtain an equation containing only the unknow function \( Y \) and its first and second derivatives \( Y' \) and \( Y'' \).

In the second step we eliminate \( Y'' \) by using eq. (74). We found that the remaining equation involving \( Y \) and \( Y' \) is just an identity. This completes our proof that eq. (74) and eq. (75) are compatible and we can use either one to solve for \( Y \). It seems that eq. (74) is more suitable for solving \( Y \).

\[ ^1 \] Otherwise we would obtain an equation containing \( Y \) and \( Y' \) only. It could be used to solve \( Y \).
Now we start to solve equation (74). By a change of function from $Y$ to $y$:
\[ y = Y' - \frac{1}{\Delta} Q(r), \]  
(79)
eq \]  
which is known as a kind of Ricatti equation. Here
\[ \tilde{R}(r) = R(r) - \frac{1}{\Delta} Q'(r) - \frac{1}{2\Delta} Q^2(r) \]
\[ = -\frac{d^2 - 1}{2\Delta r^2} + \frac{\tilde{\Lambda}}{2\Delta (C_2 + C_1 r^{2d})^2} \]  
(81)
with
\[ \tilde{\Lambda} = K - 4C_1 C_2 d^2 \]
\[ = C_3^2 (\Delta - a^2) + 4C_1 C_2 d(2\bar{d}\Delta + 2\Delta - \bar{d}). \]  
(82)

It is well-known that the above Ricatti equation can’t be solved algebraically for generic $\tilde{R}(r)$. If we know any special solution to (80), a general construction could be used to find the general solution to eq. (74). Denoting the special solution to (80) by $y_0(r)$, the general solution to eq. (74) is given as follows:
\[ Y(r) = \int y_0(r')dr' - \frac{2}{\Delta} \ln \left| C_4 + C_5 \int e^{\Delta \int y_0(r'')dr''} dr' \right| + \frac{1}{\Delta} \int Q(r')dr', \]  
(83)
where $C_4$ and $C_5$ are two constants of integration.

In the next section we will solve some degenerate cases. In this way we got the well-known $p$-brane solution.

4 Some Degenerate Cases

4.1 Case of $\tilde{\Lambda} = 0$

In this case $\tilde{R}$ was reduced to
\[ \tilde{R}(r) = -\frac{d^2 - 1}{2\Delta r^2}. \]  
(84)
It is easy to find a special solution to eq. (80):

$$y_0 = \frac{\tilde{d} - 1}{\Delta r}. \quad (85)$$

By using eq. (83) we have

$$Y = -\frac{2}{\Delta} \ln \left| C_4 + C_5 r^{-\tilde{d}} \right| - \frac{\Delta - \tilde{d}}{\Delta \tilde{d}} \ln \left| C_1 + C_2 r^{-2\tilde{d}} \right|$$

$$+ \frac{C_3 a}{2d \Delta \sqrt{-C_1 C_2}} \ln \left| \frac{\sqrt{-C_1 C_2} + C_2 r^{-\tilde{d}}}{\sqrt{-C_1 C_2} - C_2 r^{-\tilde{d}}} \right|, \quad (86)$$

where we have redefined $C_4$ and $C_5$. From this we get

$$A(r) = \frac{\tilde{d}}{\Delta(D - 2)} \ln \left| \frac{C_1 + C_2 r^{-2\tilde{d}}}{(C_4 + C_5 r^{-\tilde{d}})^2} \right|$$

$$+ \frac{a C_3}{2d \Delta \sqrt{-C_1 C_2}} \ln \left| \frac{\sqrt{-C_1 C_2} + C_2 r^{-\tilde{d}}}{\sqrt{-C_1 C_2} - C_2 r^{-\tilde{d}}} \right|, \quad (87)$$

$$B(r) = \frac{\Delta + a^2}{2 \Delta \tilde{d}} \ln \left| C_1 + C_2 r^{-2\tilde{d}} \right| + \frac{2d}{\Delta(D - 2)} \ln \left| C_4 + C_5 r^{-\tilde{d}} \right|$$

$$- \frac{C_3 a d}{2d \Delta(D - 2) \sqrt{-C_1 C_2}} \ln \left| \frac{\sqrt{-C_1 C_2} + C_2 r^{-\tilde{d}}}{\sqrt{-C_1 C_2} - C_2 r^{-\tilde{d}}} \right|, \quad (88)$$

$$\phi(r) = \frac{C_3 d}{\Delta(D - 2) \sqrt{-C_1 C_2}} \ln \left| \frac{\sqrt{-C_1 C_2} + C_2 r^{-\tilde{d}}}{\sqrt{-C_1 C_2} - C_2 r^{-\tilde{d}}} \right|$$

$$- \frac{a}{\Delta} \ln \left| C_1 + C_2 r^{-2\tilde{d}} \right| + \frac{2a}{\Delta} \ln \left| C_4 + C_5 r^{-\tilde{d}} \right| + \ln C_6,$$ 

$$\Lambda'(r) e^\Lambda = \frac{C_0 C_6^{1-a}}{r^{d+1}(C_4 + C_5 r^{-\tilde{d}})^2}. \quad (89)$$

One can check explicitly that the other equation (75) is also satisfied if we have

$$C_0 = \pm 2 \tilde{d} \sqrt{\frac{|C_1 C_5^2 + C_2 C_4^2|}{\Delta}} C_6^{1-a}. \quad (90)$$

From the structure of the above solution it is consistent to set $C_3 = 0$. Because of $\tilde{A} = 0$, this requires $C_1 C_2 = 0$. The two cases $C_1 = 0$ or $C_2 = 0$
are actually equivalent. They are related by a \( r \to 1/r \) symmetry which is discussed in [16]. So we set \( C_2 = 0 \), and \( C_1 = C_4 = 1 \) by a conventional choice of scale. The solution simplifies to:

\[
A(r) = -\frac{2\tilde{d}}{(D-2)\Delta} \ln \left( 1 + C_5 r^{-\tilde{d}} \right),
\]

\[
B(r) = \frac{2d}{(D-2)\Delta} \ln \left( 1 + C_5 r^{-\tilde{d}} \right),
\]

\[
\phi(r) = \frac{2a}{\Delta} \ln \left( 1 + C_5 r^{-\tilde{d}} \right),
\]

\[
e^{\Lambda(r)} = \frac{2}{\sqrt{\Delta}} \frac{1}{(1 + C_5 r^{-\tilde{d}})}.
\]

This is the well-known BPS \( p \)-brane solution.

We will not discuss the physical meaning of these solutions here. For details see [16] or the reviews [11, 12, 13, 14, 15].

4.2 Special Cases of \( \tilde{\Lambda} \neq 0 \)

In this subsection we will solve some special cases which didn’t fall into the case \( \tilde{\Lambda} = 0 \).

4.2.1 \( C_1 = 0 \)

In this case eq. (80) becomes

\[
y' - \frac{\Delta}{2} y^2 = -\frac{\tilde{d}^2 - 1}{2\Delta r^2} + \frac{C_3^2 (\Delta - a^2)}{2C_2^2 \Delta} r^{2\tilde{d}-2}
\]

which has the following two special solutions:

\[
y_{\pm} = \frac{\tilde{d} - 1}{\Delta r} \pm i \frac{C_3}{C_2} \frac{\sqrt{\Delta - a^2}}{\Delta} r^{\tilde{d}-1}.
\]

Eventhough the above solutions are complex, we can still obtain a real solution. For details see [16].

4.2.2 \( C_2 = 0 \)

This case is quite similar to the case \( C_1 = 0 \) discussed in the last subsection. In fact there is a symmetry \( r \to 1/r \) as mentioned above [16].
4.2.3

So far we have discussed all the possible solutions when either $C_1$ or $C_2$ is zero. When none of them is zero, things become more complicated. The interested reader can find more special cases in [16] not discussed here.

Acknowledgments

We would like to thank Han-Ying Guo, Yi-hong Gao, Ke Wu, Ming Yu, Zhujun Zheng and Zhong-Yuan Zhu for discussions. This work is supported in part by funds from Chinese National Science Fundation and Pandeng Project.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, 2 vols., Cambridge University Press, 1987.

[2] J. Polchinski, *String Theory*, 2 vols., Cambridge University Press, 1998.

[3] C. M. Hull and P. K. Townsend, *Unity of Superstring Dualities*, Nucl. Phys. B438 (1995) 109, [hep-th/9410167].

[4] E. Witten, *String Theories in Various Dimensions*, Nucl. Phys. B443 (1995) 85, [hep-th/9503124].

[5] J. H. Schwarz, *The Power of M Theory*, Phys. Lett. B367 (1996) 97, [hep-th/9510086].

[6] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M Theory as a Matrix Model: a Conjecture*, Phys. Rev. D55 (1997) 30, [hep-th/9610043].

[7] I. R. Klebanov and A. A. Tseytlin, *D-Branes and Dual Gauge Theories in Type 0 Strings*, preprint [hep-th/9811033]: *Asymptotic Freedom and Infrared Behavior in the Type 0 String Approach to Gauge Theory*, preprint [hep-th/9812089]: *A Non-supersymmetric Large N CFT from Type 0 String Theory*, preprint [hep-th/9901101].
[8] J. A. Minahan, *Glueball Mass Spectra and Other Issues for Supergravity Duals of QCD Models*, preprint [hep-th/9811156]; *Asymptotic Freedom and Confinement from Type 0 String Theory*, preprint [hep-th/9902074].

[9] G. Ferretti and D. Martelli, *On the construction of gauge theories from non critical type 0 strings*, preprint [hep-th/9811208].

[10] O. Bergman and M. R. Gaberdiel, *Non-BPS States in Heterotic-Type IIA Duality*, preprint [hep-th/9901014].

[11] M. J. Duff, R. R. Khuri and J. X. Lu, *String Solitons*, Physics Reports 259 (1995) 213.

[12] K. S. Stelle, *BPS Branes in Supergravity*, preprint [hep-th/9803116].

[13] R. Argurio, *Brane Physics in M-Theory*, Doctoral Thesis [hep-th/9807171].

[14] B. A. Abers and Pioline, *U-Duality and M-Theory*, preprint [hep-th/9809039].

[15] R. D’Auria and P. Frè, *BPS Black Holes in Supergravity*, preprint [hep-th/9812160].

[16] B. Zhou, ITP Doctoral Thesis, 1999.

[17] B. Zhou and C. -J. Zhu, *A Study of Black Brane Solutions in D-Dimensional Coupled Gravity System*, to appear.