High-dimensional Change-point Detection Using Generalized Homogeneity Metrics

Shubhadeep Chakraborty\textsuperscript{1} and Xianyang Zhang\textsuperscript{2}

\textsuperscript{1}University of Washington
\textsuperscript{2}Texas A&M University

Abstract

Change-point detection has been a classical problem in statistics and econometrics. This work focuses on the problem of detecting abrupt distributional changes in the data-generating distribution of a sequence of high-dimensional observations, beyond the first two moments. This has remained a substantially less explored problem in the existing literature, especially in the high-dimensional context, compared to detecting changes in the mean or the covariance structure. We develop a nonparametric methodology to (i) detect an unknown number of change-points in an independent sequence of high-dimensional observations and (ii) test for the significance of the estimated change-point locations. Our approach essentially rests upon nonparametric tests for the homogeneity of two high-dimensional distributions. We construct a single change-point location estimator via defining a cumulative sum process in an embedded Hilbert space. As the key theoretical innovation, we rigorously derive its limiting distribution under the high dimension medium sample size (HDMSS) framework. Subsequently we combine our statistic with the idea of wild binary segmentation to recursively estimate and test for multiple change-point locations. The superior performance of our methodology compared to other existing procedures is illustrated via extensive simulation studies as well as over stock prices data observed during the period of the Great Recession in the United States.

Keywords: High Dimensionality, Multiple Change-Point Detection, Two Sample Test, Wild Binary Segmentation.

1 Introduction

Change-point detection has been a classical and well-established problem in statistics and econometrics, aiming to detect the lack of homogeneity in a sequence of time-ordered observations. It
finds abundance of applications in a wide variety of fields, for example, bioinformatics (Picard et al., 2005; Curtis et al., 2012), neuroscience (Park et al., 2015), digital speech processing (Rabiner and Schäfer, 2007), social network analysis (McCulloh, 2009), and so on. We refer the reader to Aue and Horváth (2013) and Jandhyala et al. (2013) for some recent reviews on this topic. An important problem in the detection of structural breaks in multivariate data is the detection of changes in the mean vector. The mean change problem has been extensively studied when the dimension is low and fixed. In the big data era, high-dimensional data are now frequently encountered in many scientific areas. Among recent works that tackle detection of mean change for high-dimensional data, we refer to Enikeeva and Harchaoui (2013), Jirak (2015), Cho and Fryzlewicz (2015), Wang and Samworth (2018) and Wang et al. (2019) (only to mention a few). For some recent works in detecting changes in the covariance structure in a sequence of high-dimensional observations, we mention Avanesov and Buzun (2018) and Det et al. (2020), among others.

A substantial part of the existing literature on change-point detection focuses on detecting changes in the mean or the covariance structure. In this work we aim to develop a nonparametric change-point detection procedure that is capable of detecting and localizing quite general types of changes in the data generating distribution, rather than only changes in mean or the covariance structure. To the best of our knowledge, only a few recent works deal with detecting abrupt distributional changes in a nonparametric framework for high-dimensional data. Many of the methodologies developed over the last few decades on detecting distributional changes suffer from several limitations, for example, applicability only to real-valued data or low-dimensional data, assumption that the number of true change-points is known, etc. Harchaoui and Cappé (2007) proposed a kernel-based procedure assuming a known number of change-points, which reduces its practical interest. Zou et al. (2014) proposed a nonparametric maximum likelihood approach for detecting multiple (unknown number of) change-points using BIC, but is only applicable for real-valued data. Lung-Yut-Fong et al. (2015) developed a nonparametric approach based on marginal rank statistics, which requires the number of observations to be larger than the dimension of the data. Arlot et al. (2019) proposed a kernel-based multiple change-point detection algorithm for multivariate (but fixed dimensional) or complex (non-Euclidean) data. Matteson and James (2014) proposed a procedure for estimating multiple change-point locations, namely E-Divisive, built upon an energy distance based test that applies to multivariate observations of arbitrary (but fixed) dimensions. Biau et al. (2016) rigorously derived the asymptotic distribution of the statistic proposed by Matteson and James (2014), thereby adding theoretical justifications to their methodology. However, some recent research revelations on the performance of energy distance for growing dimensions put a question on its performance when we have a sequence of high-dimensional observations. Some graph-based tests have been proposed recently by Chen and Zhang (2015) and Chu and Chen (2019) for high-dimensional data, which allow us to detect only one or two change-points. The method proposed by Chen and Zhang (2015) is more effective for location alternatives compared
to scale alternatives, and it performs poorly in estimating a single change-point location when the change does not happen near the middle of the sequence. Chu and Chen (2019) improved upon the limitations of Chen and Zhang (2015), proposing tests that are powerful for both location and scale alternatives. However, our numerical studies demonstrate that these graph-based tests perform poorly in detecting changes in the higher order moments.

Energy distance, proposed by Székely et al. (2004, 2005) and Baringhaus and Franz (2004), is a classical distance-based measure of equality of two multivariate distributions, taking the value zero if and only if the two random vectors are identically distributed. Such a complete characterization of homogeneity of distributions lends itself for reasonable use in one-sample goodness-of-fit testing and two-sample testing for equality of distributions, and has been widely studied in the literature over the last couple of years. Chakraborty and Zhang (2019) and Zhu and Shao (2019) recently showed a striking result that energy distance based on the usual Euclidean distance cannot completely characterize the homogeneity of the two high-dimensional distributions in the sense that it can only detect the equality of means and the traces of covariance matrices of the two high-dimensional random vectors. In other words, the Euclidean energy distance fails to detect in-homogeneity between two high-dimensional distributions beyond the first two moments. To overcome such a limitation, Chakraborty and Zhang (2019) proposed a new class of homogeneity metrics which inherits the desirable properties of energy distance in the low-dimensional setting. And more importantly, in the high-dimensional setup the new class of homogeneity metrics is capable of detecting the pairwise homogeneity of the low-dimensional marginal distributions, going beyond the scope of the Euclidean energy distance. The proposed class of homogeneity metrics can capture a wider range of in-homogeneity of distributions compared to the classical Euclidean energy distance in the high-dimensional framework. The core of their methodology is a new way of defining the distance between sample points (interpoint distance) in the high-dimensional Euclidean spaces.

This paper focuses on estimating an unknown number of multiple change-point locations in an independent sequence of $\mathbb{R}^p$-valued observations of sample size $n$, where $p$ can by far exceed $n$. To the best of our knowledge, we make one of the first attempts in the literature to detect and localize general type of changes in the underlying distribution beyond the first two moments. Our approach essentially rests upon distance-based nonparametric two-sample tests for homogeneity of two high-dimensional distributions. We first construct a single change-point location estimator $M_n$ based on the homogeneity metrics proposed by Chakraborty and Zhang (2019) via defining a cumulative sum process in an embedded Hilbert space. It essentially generalizes the single change-point location estimator developed by Matteson and James (2014) and Biau et al. (2016) in the high-dimensional setup, providing a unifying framework. Testing for the statistical significance of the estimated candidate change-point location necessitates determining the quantiles of the distribution of $M_n$. The key theoretical innovation of this paper is to rigorously derive the asymptotic null distribution of $M_n$ as both the dimension $p$ and the
sample size $n$ grow to infinity, with $n$ growing at a smaller rate compared to $p$. Such a setup is typically known in the literature as the high dimension medium sample size (HDMSS) framework. The intrinsic difficulty is to establish the uniform weak convergence of an underlying stochastic process under certain mild assumptions, which has been non-trivial and challenging. Because of the pivotal nature of the limiting null distribution, its quantiles can be approximated using a large number of Monte Carlo simulations. We propose an algorithm for single change-point detection based on a permutation procedure to better approximate the quantiles of the distribution of $M_n$. Subsequently, we combine the idea of Wild Binary Segmentation (WBS) proposed by Fryzlewicz (2014) to recursively estimate and test for the significance of (an unknown number of) multiple change-point locations. The superior performance of our procedure compared to the existing methodologies is illustrated over extensive simulated datasets. Further, when applied over stock prices data observed during the period of Great Recession in the United States, our method furnishes more reasonable and meaningful estimates of significant change-point locations given the historical sequence of eventualities, compared to the other existing methods. Finally, we briefly illustrate an extension of our methodology to incorporate external directed and undirected graph information. Further research along this line is well underway.

**Notations.** Denote by $\| \cdot \|$ the Euclidean norm of $\mathbb{R}^p$. Let $0_p$ be the origin of $\mathbb{R}^p$. For a set $S \subseteq [p] := \{1, 2, \ldots, p\}$ and $z = (z_1, \ldots, z_p) \in \mathbb{R}^p$, we let $\text{card}(S)$ be the cardinality of $S$ and $z_S = (z_i : i \in S)$ be a subvector of $z$ containing the components whose indices are in $S$. We use “$X \overset{d}{=} Y$” to indicate that $X$ and $Y$ are identically distributed. Let $X', X'', X'''$ be independent copies of $X$. ‘O’ and ‘o’ stand for the usual notations in mathematics : ‘is of the same order as’ and ‘is ultimately smaller than’. We use the symbol “$a \lesssim b$” to indicate that $a \leq Cb$ for some constant $C > 0$. We utilize the order in probability notations such as stochastic boundedness $O_p$ (big O in probability), convergence in probability $o_p$ (small o in probability) and equivalent order $\asymp$, which is defined as follows: for a sequence of random variables $\{Z_n\}_{n=1}^\infty$ and a sequence of real numbers $\{a_n\}_{n=1}^\infty$, $Z_n \overset{p}{\asymp} a_n$ if and only if $Z_n/a_n = O_p(1)$ and $a_n/Z_n = O_p(1)$ as $n \to \infty$. If $Z_n \overset{P}{\to} Z$ as $n \to \infty$, then we say $\text{plim}_{n \to \infty} Z_n = Z$. For a metric space $(X, \rho)$, let $\mathcal{M}(X)$ and $\mathcal{M}_1(X)$ denote the set of all finite signed Borel measures on $X$ and all probability measures on $X$, respectively. Define $\mathcal{M}_p^1(X) := \{v \in \mathcal{M}(X) : \exists x_0 \in X \text{ s.t. } \int_X \rho(x, x_0) d|v|(x) < \infty\}$. Let $1(A)$ denote the indicator function associated with a set $A$. For a compact set $T$, define $L^\infty(T) := \{f : T \to \mathbb{R} ; \|f\|_{\infty} = \sup_{t \in T} |f(t)| < \infty\}$. Denote $1_n = (1, \ldots, 1) \in \mathbb{R}^n$. Write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Finally, denote by $\lfloor a \rfloor$ the integer part of $a \in \mathbb{R}$.
2 Distance-based homogeneity tests

2.1 Generalized energy distance

Energy distance (Székely et al. (2004; 2005), Baringhaus and Franz (2004)) or the Euclidean energy distance between two random vectors $X, Y \in \mathbb{R}^p$ and $X \perp \perp Y$ with $\mathbb{E}\|X\| < \infty$ and $\mathbb{E}\|Y\| < \infty$, is defined as

$$E(X, Y) = \frac{1}{c_p} \int_{\mathbb{R}^p} \frac{|f_X(t) - f_Y(t)|^2}{\|t\|^{1+p}} \, dt,$$  \hspace{1cm} (1)

where $f_X$ and $f_Y$ are the characteristic functions of $X$ and $Y$ respectively, and $c_p = \frac{\pi^{(1+p)/2}}{\Gamma((1+p)/2)}$ is a constant with $\Gamma(\cdot)$ being the complete gamma function. Theorem 1 in Székely et al. (2005) shows that $E(X, Y) \geq 0$ and the equality holds if and only if $X \overset{d}{=} Y$. In other words, energy distance can completely characterize the homogeneity between two multivariate distributions. An equivalent expression for $E(X, Y)$ is given by

$$E(X, Y) = 2\mathbb{E}\|X - Y\| - \mathbb{E}\|X - X'\| - \mathbb{E}\|Y - Y'\|,$$  \hspace{1cm} (2)

where $(X', Y')$ is an independent copy of $(X, Y)$.

**Definition 2.1 (Generalized energy distance).** For an arbitrary metric space $(\mathcal{X}, \rho)$, the generalized energy distance between $X \sim P_X$ and $Y \sim P_Y$ where $P_X, P_Y \in M_1(\mathcal{X}) \cap M^1_p(\mathcal{X})$ is defined as

$$E_\rho(X, Y) = 2\mathbb{E}\rho(X, Y) - \mathbb{E}\rho(X, X') - \mathbb{E}\rho(Y, Y').$$  \hspace{1cm} (3)

**Definition 2.2 (Spaces of negative type).** The metric space $(\mathcal{X}, \rho)$ is said to have negative type if $\forall n \geq 2, x_1, \ldots, x_n \in \mathcal{X}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$, $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \rho(x_i, x_j) \leq 0$. Suppose $P, Q \in M_1(\mathcal{X}) \cap M^1_p(\mathcal{X})$. When $(\mathcal{X}, \rho)$ has negative type,

$$\int \rho(x_1, x_2) d(P - Q)^2(x_1, x_2) \leq 0.$$  \hspace{1cm} (4)

We say that $(\mathcal{X}, \rho)$ has strong negative type if it has negative type and the equality in (4) holds only when $P = Q$.

By Theorem 3.16 in Lyons (2013), every separable Hilbert space has strong negative type. In particular, Euclidean spaces are separable Hilbert spaces and therefore have strong negative type. If $(\mathcal{X}, \rho)$ has strong negative type, then $E_\rho(X, Y) = 0$ if and only if $X \overset{d}{=} Y$, or in other words, the completely characterization of the homogeneity of two distributions holds good in any metric spaces of strong negative type (Lyons(2013), Sejdinovic et al. (2013)). Thus the quantification of homogeneity
of distributions by the Euclidean energy distance given in (2) is just a special case when \( \rho \) is the Euclidean distance on \( \mathcal{X} = \mathbb{R}^p \). Suppose \( X_n = \{X_i\}_{i=1}^n \) and \( Y_m = \{Y_i\}_{i=1}^m \) are two independent i.i.d samples on \( X \) and \( Y \) taking values in \( (\mathcal{X}, \rho) \). A U-statistic type estimator of the generalized energy distance between \( X \) and \( Y \) is defined as

\[
\hat{E}_\rho(X_n, Y_m) = \frac{2}{nm} \sum_{i=1}^n \sum_{j=1}^m \rho(X_i, Y_j) - \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \rho(X_i, X_j)
- \frac{1}{m(m-1)} \sum_{1 \leq i \neq j \leq m} \rho(Y_i, Y_j).
\]

We refer the reader to Section A.1 in the supplementary materials of Chakraborty and Zhang (2019) for a comprehensive overview of the properties and asymptotic behavior of the U-statistic type estimator of \( E_\rho(X, Y) \) in the low-dimensional setting.

### 2.2 Generalized energy distance in high dimensions

The question of interest is how do the classical distance-based homogeneity metrics like energy distance behave in the high-dimensional framework. Consider two \( \mathbb{R}^p \)-valued random vectors \( X = (x_1, \ldots, x_p) \) and \( Y = (y_1, \ldots, y_p) \). Chakraborty and Zhang (2019) in their recent paper showed a striking result that when dimension grows high, the Euclidean energy distance between \( X \) and \( Y \) can only capture the equality of the means and the first spectral means, viz. \( \mu_X = \mu_Y \) and \( \text{tr} \Sigma_X = \text{tr} \Sigma_Y \), where \( \mu_X \) and \( \mu_Y \), and, \( \Sigma_X \) and \( \Sigma_Y \) are the mean vectors and the covariance matrices of \( X \) and \( Y \), respectively.

To illustrate, consider the case where \( X \sim N(\mu, I_p) \) with \( \mu = (1, \ldots, 1) \in \mathbb{R}^p \) and the components of \( Y \) independently follow Exponential (1) for \( 1 \leq i \leq p \). That is, \( \mu_X = \mu_Y \) and \( \text{tr} \Sigma_X = \text{tr} \Sigma_Y \) although \( X \) and \( Y \) have different distributions. The homogeneity test based on the Euclidean energy distance has trivial power in this case. Such a limitation of the classical Euclidean energy distance arises essentially due to the use of Euclidean distance. Chakraborty and Zhang (2019) proposed a new class of homogeneity metrics to overcome such a limitation of the Euclidean energy distance, which is based on a new way of defining the distance between sample points (interpoint distance) in the high-dimensional Euclidean spaces. Here we present a slightly generalized version of their distance by allowing the groups (i.e., \( S_i \)'s below) to overlap with each other.

**Definition 2.3** (Generalized Euclidean distance). Consider a collection of subsets \( \{S_i : 1 \leq i \leq g\} \) with \( S_i \subseteq [p] := \{1, 2, \ldots, p\} \) and \( \text{card}(S_i) = d_i \). Suppose \( \bigcup_{i=1}^{g} S_i = [p] \) and \( \rho_i \) is a distance of strong negative type on \( \mathbb{R}^{d_i} \) for \( 1 \leq i \leq g \). For \( z, z' \in \mathbb{R}^p \), we define the generalized Euclidean distance as

\[
\gamma(z, z') := \sqrt{\rho_1(z_{S_1}, z'_{S_1}) + \cdots + \rho_g(z_{S_g}, z'_{S_g})},
\]
which can be shown to be a valid metric on \( \mathbb{R}^p \).

As a special case for illustration, if \( g = p \), \( d_i = 1 \) for all \( 1 \leq i \leq g \) and \( \rho_i \) is the Euclidean distance on \( \mathbb{R} \), then the metric boils down to \( \gamma(z, z') = \|z - z'\|_1^{1/2} = \left( \sum_{j=1}^{p} |z_j - z'_j| \right)^{1/2} \), where \( \|z\|_1 = \sum_{j=1}^{p} |z_j| \) is the \( l_1 \) or the absolute norm on \( \mathbb{R}^p \). Their new class of distance-based homogeneity metrics essentially replaces the Euclidean distance in the definition of energy distance by this proposed distance.

For fixed \( p \), \((\mathbb{R}^p, \gamma)\) is shown to have strong negative type and hence \( E_{\gamma}(X, Y) = 0 \) if and only if \( X \overset{d}{=} Y \). In other words, \( E_{\gamma}(X, Y) \) completely characterizes the homogeneity of the distributions of \( X \) and \( Y \) in the low-dimensional setting. Theorem 4.1 and Lemma 4.1 of Chakraborty and Zhang (2019) show that when \( p \) grows high and the dimensions of the sub-vectors remain fixed, \( E_{\gamma}(X, Y) \) can capture the pairwise homogeneity of the marginal distributions of \( X_{S_i} \) and \( Y_{S_i} \). Clearly \( X_{S_i} \overset{d}{=} Y_{S_i} \) for \( 1 \leq i \leq g \) implies \( \mu_X = \mu_Y \) and \( \text{tr} \Sigma_X = \text{tr} \Sigma_Y \), and therefore the proposed class of homogeneity metrics can capture a wider range of in-homogeneity of distributions compared to the Euclidean energy distance in the high-dimensional framework.

### 2.3 Two-sample t-test

Chakraborty and Zhang (2019) introduced a two-sample t-test for high-dimensional inference based on the generalized homogeneity metrics. Given the samples \( X_n \) and \( Y_m \), we first define

\[
\tilde{d}_{k, l} := \gamma(X_k, Y_l) - \frac{1}{n} \sum_{i=1}^{n} \gamma(X_i, Y_l) - \frac{1}{m} \sum_{j=1}^{m} \gamma(X_k, Y_j) + \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma(X_i, Y_j),
\]

\[
\tilde{a}_{k, k'} := \gamma(X_k, X_{k'}) - \frac{1}{n-2} \sum_{j=1}^{n} \gamma(X_k, X_j) - \frac{1}{n-2} \sum_{i=1}^{n} \gamma(X_i, X_{k'}) + \frac{1}{(n-1)(n-2)} \sum_{i,j=1}^{n} \gamma(X_i, X_j),
\]

\[
\tilde{b}_{l, l'} := \gamma(Y_l, Y_{l'}) - \frac{1}{m-2} \sum_{j=1}^{m} \gamma(Y_l, Y_j) - \frac{1}{m-2} \sum_{i=1}^{m} \gamma(Y_i, Y_{l'}) + \frac{1}{(m-1)(m-2)} \sum_{i,j=1}^{m} \gamma(Y_i, Y_j),
\]

where \( 1 \leq k, k' \leq n \) and \( 1 \leq l, l' \leq m \). Define the pooled variance estimator

\[
\hat{S}^2(X_n, Y_m) = \frac{4v_n \hat{D}^2(X_n) + 4v_m \hat{D}^2(Y_m) + 4(n-1)(m-1)\hat{C}(X_n, Y_m)}{v_n + v_m + (n-1)(m-1)},
\]
where \( \hat{D}^2 \) and \( \hat{C} \) are the sample distance variance and the cross distance covariance defined respectively by

\[
\hat{D}^2(X_n) = \frac{1}{n(n-3)} \sum_{1 \leq k \neq k' \leq n} \tilde{a}_{k,k'}^2, \quad \hat{D}^2(Y_m) = \frac{1}{m(m-3)} \sum_{1 \leq l \neq l' \leq m} \tilde{b}_{l,l'}^2,
\]

\[
\hat{C}(X_n, Y_m) = \frac{1}{(n-1)(m-1)} \sum_{k=1}^{n} \sum_{l=1}^{m} \tilde{d}_{k,l}^2,
\]

and \( v_a = a(a-3)/2 \) for \( a = n, m \). The two-sample t-statistic is defined as

\[
T(X_n, Y_m) = \frac{\hat{E}_{\gamma}(X_n, Y_m)}{a_{nm} \hat{S}(X_n, Y_m)} \quad \text{where} \quad a_{nm}^2 = \frac{1}{nm} + \frac{1}{2n(n-1)} + \frac{1}{2m(m-1)}.
\]

Note that the construction of the pooled variance estimator and hence the two-sample statistic requires \( n, m \geq 4 \). Under certain mild assumptions, it is shown in Theorem B.1 in Chakraborty and Zhang (2019) that under \( H_0 : X \overset{d}{=} Y \),

\[
T(X_n, Y_m) \overset{d}{\to} N(0, 1)
\]

as \( p \to \infty \) and \( n, m \to \infty \) at a slower rate than \( p \).

## 3 High-dimensional change-point detection

### 3.1 Problem statement

With the above background knowledge, we now turn to the change-point detection problem. Consider an independent sequence of \( \mathbb{R}^p \)-valued observations \( \{X_t\}_{t=1}^n \), where the dimension \( p \) is typically much higher than the sample size \( n \). We are concerned with testing the null hypothesis \( H_0 : X_t \sim F_0 \) for \( t = 1, \ldots, n \) against the alternative

\[
H_1 : \exists N_0 \in \mathbb{Z}^+, \quad 1 \leq t_1 < \cdots < t_{N_0} < n, \quad X_t \sim \begin{cases} F_0, & 1 \leq t \leq t_1, \\ F_1, & t_1 + 1 \leq t \leq t_2, \\ \vdots \\ F_{N_0}, & t_{N_0} + 1 \leq t \leq n, \end{cases}
\]

(5)

where the probability distributions \( F_0, F_1, \ldots, F_{N_0} \) differ on sets of non-zero measures.

### 3.2 Cumulative sum process in embedded spaces

The starting point of many change-point detection procedures is the so-called cumulative sum process. In this subsection, we illustrate the idea behind the construction of our proposed test statistic.
in Section 3.3 for estimation of a single change-point location, i.e., when $N_0 = 1$. The idea rests upon
the construction of a cumulative sum process in embedded spaces. We state the following well known
result regarding the equivalent characterization of spaces of negative type and would refer the reader
to Section 3 in Lyons (2013) for detailed discussions.

**Proposition 3.1.** A metric space $(X, \rho)$ has negative type if and only if there is a Hilbert space
$(H, \langle \cdot, \cdot \rangle_H)$ and an embedding $\phi : X \to H$ such that $\rho(x, x') = \|\phi(x) - \phi(x')\|_H^2$
for all $x, x' \in X$, where $\| \cdot \|_H = \langle \cdot, \cdot \rangle_H^{1/2}$ is the norm associated with $H$.

As $(\mathbb{R}^p, \gamma)$ has strong negative type, Proposition 3.1 ensures the existence of an embedding
$\phi : \mathbb{R}^p \to H$ for some Hilbert space $H$ such that $\gamma(x, x') = \|\phi(x) - \phi(x')\|_H^2$, where $x, x' \in \mathbb{R}^p$. Therefore,
we get

$$\langle \phi(x), \phi(x') \rangle_H = 2^{-1}(\langle \phi(x), \phi(x) \rangle_H + \langle \phi(x'), \phi(x') \rangle_H - \gamma(x, x')).$$

Define the cumulative sum process in the embedded space as

$$S_k := \frac{1}{\sqrt{n}} \sum_{t=1}^{k} \left( \phi(X_t) - \frac{1}{n} \sum_{t=1}^{n} \phi(X_t) \right)$$

for $1 \leq k \leq n - 1$. We present some basic properties of $S_k$ in the following lemma.

**Lemma 3.1.** The cumulative sum process $S_k$ can be expressed as

$$S_k = \frac{(n-k)k}{n^{3/2}} \left( \frac{1}{k} \sum_{t=1}^{k} \phi(X_t) - \frac{1}{n-k} \sum_{t=k+1}^{n} \phi(X_t) \right)$$

for $1 \leq k \leq n - 1$. Further, the squared norm of $S_k$ is given by

$$\|S_k\|_H^2 = \frac{k^2 (n-k)^2}{2 n^3} \left\{ \frac{2}{k(n-k)} \sum_{t=1}^{k} \sum_{t'=k+1}^{n} \gamma(X_t, X_{t'}) - \frac{1}{k^2} \sum_{t,t'=1}^{k} \gamma(X_t, X_{t'}) - \frac{1}{(n-k)^2} \sum_{t,t'=k+1}^{n} \gamma(X_t, X_{t'}) \right\}.$$

We provide a short proof of Lemma 3.1 in Section 3 of the supplementary material. If there
is a single change-point located at $\tau$, one expects $\|S_k\|_H^2$ after proper normalization to achieve its
maximum value at $\tau$. Motivated by this, a U-statistic type single change-point location estimator can
be constructed as

$$Q_n = \max_{1 \leq k \leq n-1} \frac{w_n(k)}{k} \tilde{\gamma}(X_{1:k}, X_{k+1:n}),$$

where $w_n(k)$ denotes a suitable weight function. For example, Matteson and James (2014) considered
$Q_n = \max_{1 \leq k \leq n-1} \frac{k(n-k)}{n} \tilde{\rho}(X_{1:k}, X_{k+1:n})$ for change point detection in the low dimensional case, where
\( \rho \) is the usual Euclidean distance.

If there is a single change-point in the sequence of observed data, the statistic \( Q_n \) gives a candidate change-point location that needs to be tested against a certain threshold. The key challenge in applying the test in practice lies in deriving the limiting distribution of \( Q_n \) under the null hypothesis.

We pointed out in Section 2.2 the limitations of the classical Euclidean energy distance in detecting in-homogeneity of two high-dimensional distributions. In that view, it is pretty expected that the methodology proposed by Matteson and James (2014) will fail to detect structural changes in a sequence of high-dimensional observations beyond the first two moments, which is supported by our extensive numerical studies in Section 4. In the next section, we shall propose a new statistic that improves over Matteson and James (2014)’s test by being powerful for a wider range of structural breaks in a sequence of high dimensional observations.

### 3.3 Single change-point detection and estimation

Our approach essentially rests upon the generalized homogeneity metric (Section 2.2) for two high-dimensional distributions and its connection with the cumulative sum process in the embedded space. Specifically, we propose the statistic

\[
M_n := \max_{4 \leq k \leq n-4} \frac{k(n-k)}{n^2} T_n(k)
\]

with \( T_n(k) := T(X_{1:k}, X_{k+1:n}) \) to test for a single change-point alternative, where the location of the change-point can be estimated via

\[
\hat{\tau}_0 := \arg \max_{4 \leq k \leq n-4} \frac{k(n-k)}{n^2} T_n(k).
\]

As the two-sample statistic \( T_n \) is based upon the generalized homogeneity metric, our methodology is capable of detecting more general changes in the underlying high-dimensional distribution, rather than merely detecting changes in the first two moments by the Euclidean energy distance. The statistical significance of the estimated change-point location \( \hat{\tau}_0 \) needs to be tested, which necessitates determining the null distribution of \( M_n \). The key theoretical innovation of this paper is to rigorously derive the asymptotic null distribution of \( M_n \) as \( n, p \to \infty \), with \( n \) growing at a smaller rate compared to \( p \). The intrinsic difficulty is to derive a uniform weak convergence result for the stochastic process \( T_n(k) \), as clearly a pointwise weak convergence result does not suffice. Towards that end, we begin with introducing some technical assumptions.
Assumption 3.1. There exist constants \( c \) and \( C \) such that uniformly over \( p \),

\[
0 < c \leq \inf_{1 \leq i \leq g} \mathbb{E} \rho_i(X_{S_i}, X'_{S_i}) \leq \sup_{1 \leq i \leq g} \mathbb{E} \rho_i(X_{S_i}, X'_{S_i}) \leq C < \infty.
\]

Define \( \tau^2 := \mathbb{E}[\gamma^2(X, X')] \). Under Assumption 3.1, it is easy to see that \( \tau \asymp p^{1/2} \). The following proposition (Proposition 4.1 in Chakraborty and Zhang (2019)) presents an expansion formula for the distance metric \( \gamma \) when the dimension is high, which plays a key role in our theoretical analysis.

Proposition 3.2. Under Assumption 3.1, we have

\[
\frac{\gamma(X, X')}{\tau} = 1 + \frac{1}{2} L(X, X') + R(X, X'),
\]

where \( L(X, X') = \frac{\gamma^2(X, X') - \tau^2}{\tau^2} \) is the leading term and \( R(X, X') \) is the remainder term. In addition, if \( L(X, X') \) is a \( o_p(1) \) random variable as \( p \to \infty \), then \( R(X, X') = O_p(L^2(X, X')) \).

Suppose \( X_t = (X_{t,S_i}, \ldots, X_{t,S_n}) \) for \( X_{t,S_i} \in \mathbb{R}^d \). Define \( H(X_k, X_t) = \tau^{-1} \sum_{i=1}^{g} d_{kl}(i) \) for \( 1 \leq k, l \leq n \), where \( d_{kl}(i) = \rho_i(X_{k,S_i}, X_{l,S_i}) - \mathbb{E}[\rho_i(X_{k,S_i}, X_{l,S_i})|X_{k,S_i}] - \mathbb{E}[\rho_i(X_{k,S_i}, X_{l,S_i})|X_{l,S_i}] + \mathbb{E}[\rho_i(X_{k,S_i}, X_{l,S_i})] \) is the double-centered distance between \( X_{k,S_i} \) and \( X_{l,S_i} \).

Assumption 3.2. As \( n, p \to \infty \),

\[
\frac{1}{n^2} \frac{\mathbb{E}[H^4(X, X')]}{\mathbb{E}[H^2(X, X')]^2} = o(1), \quad \frac{1}{n} \frac{\mathbb{E}[H^2(X, X') H^2(X', X'')]}{\mathbb{E}[H^2(X, X')]^2} = o(1),
\]

\[
\frac{\mathbb{E}[H(X, X'') H(X', X'') H(X, X''') H(X', X''')]}{\mathbb{E}[H^2(X, X')]^2} = o(1).
\]

Remark 3.1. We refer the reader to Section 2.2 in Zhang et al. (2018) for an illustration of Assumption 3.2.

Assumption 3.3. Suppose \( \mathbb{E}[L^2(X, X')] = O(\alpha_p^2) \) where \( \alpha_p \) is a positive real sequence such that \( \tau \alpha_p^2 = o(1) \) as \( p \to \infty \). Further assume that as \( n, p \to \infty \),

\[
\frac{n^4 \tau^4 \mathbb{E}[R^4(X, X')]}{(\mathbb{E}[H^2(X, X')]^2)} = o(1).
\]

Remark 3.2. We refer the reader to Remark 4.1 in Chakraborty and Zhang (2019) which illustrates some sufficient conditions under which \( \alpha_p = O(p^{-1/2}) \) and consequently \( \tau \alpha_p^2 = o(1) \) holds, as \( \tau \asymp p^{1/2} \). In similar lines of Remark D.1 in the supplementary materials of their paper, it can be argued that \( \mathbb{E}[R^4(X, X')] = O(p^{-4}) \). Further with a mild assumption that \( \sigma^2 := \lim_{p \to \infty} \mathbb{E}[H^2(X, X')] \), we have \( \mathbb{E}[H^2(X, X')] \asymp 1 \). Combining all the above, it is easy to verify that \( n^4 \tau^4 \mathbb{E}[R^4(X, X')]/(\mathbb{E}[H^2(X, X')]^2) = o(1) \) holds provided \( n = o(p^{1/2}) \).
The following theorem establishes a uniform weak convergence result of the stochastic process \( \{T_n([nr])\}_{r \in [0,1]} \) which plays a key role in deriving the limiting null distribution of \( M_n \) as \( n, p \to \infty \).

**Theorem 3.1.** Under Assumptions 3.2 and 3.3 as \( n, p \to \infty \),

\[
\left\{ T_n([nr]) \right\}_{r \in [0,1]} \overset{d}{\to} G_0 \quad \text{in } L^\infty ([0,1]),
\]

where \( G_0(r) := Q(0,1) - \frac{1}{r} Q(0,r) - \frac{1}{1-r} Q(r,1) \) for \( r \in (0,1) \) and zero otherwise, and \( Q \) is a centered Gaussian process with the covariance function given by

\[
\text{cov} (Q(a_1, b_1), Q(a_2, b_2)) = (b_1 \wedge b_2 - a_1 \lor a_2)^2 \mathbb{1} (b_1 \wedge b_2 > a_1 \lor a_2).
\]

In particular, \( \text{var} (Q(a,b)) = (b-a)^2 \mathbb{1} (b > a) \).

The proof of this theorem is non-trivial, requiring the finite dimensional weak convergence and stochastic equicontinuity of the process \( \{T_n([nr])\}_{r \in [0,1]} \) to be established (see Theorem 10.2 in Pollard et al., 1990). Because of its extremely long and technical nature, we relegate it to the supplementary materials.

**Remark 3.3.** Theorem B.1 in the supplementary materials of Chakraborty and Zhang (2019) essentially proves that for fixed \( r \in (0,1) \), \( T_n([nr]) \overset{d}{\to} N(0,1) \) as \( n, p \to \infty \), under the same Assumptions 3.2 and 3.3. Note that in Theorem 3.1, for fixed \( r \in (0,1) \), \( G_0(r) \) has a Gaussian distribution with zero mean. From the covariance structure of the gaussian process \( Q \) given in Theorem 3.1, it is not hard to verify that \( \text{var} (G_0(r)) = 1 \). This illustrates that the uniform weak convergence result established in Theorem 3.1 in fact generalizes the pointwise weak convergence result proven in Theorem B.1 in Chakraborty and Zhang (2019).

As a consequence of Theorem 3.1, we derive the limiting null distribution of \( M_n \), which serves as the main theoretical innovation of the paper.

**Theorem 3.2.** Under Assumptions 3.2 and 3.3 as \( n, p \to \infty \),

\[
M_n \overset{d}{\to} \sup_{r \in (0,1)} r (1 - r) G_0(r).
\]

Theorem 3.2 follows from Theorem 3.1 and the continuous mapping theorem. It is to be noted that the limiting null distribution is pivotal in nature and distribution free. Consequently the quantiles of the limiting distribution can be approximated via a large number of Monte Carlo simulations.

**Remark 3.4.** Table 1 below provides the simulated quantiles of the limiting distribution of \( M_n \) (based on 2000 Monte Carlo replications) with \( \{X_t\}_{t=1}^n \) generated from the \( N(0, I_p) \) distribution with \( n = 500 \) and \( p = 1000 \). \( Q_\alpha \) denotes the 100\((1-\alpha)\)th quantile of this distribution.
Table 1: Simulated quantiles of the limiting distribution of $M_n$.

| 100(1 – $\alpha$)% | 90% | 95% | 99% |
|---------------------|-----|-----|-----|
| $Q_{\alpha}$        | 0.566 | 0.642 | 0.810 |

With the limiting null distribution of $M_n$ being rigorously established, we now present in Algorithm 1 the pseudocode of the procedure to test for $H_0$ against the single change-point alternative. We use a permutation procedure to approximate the quantiles of the distribution of $M_n$, aiming to achieve more accurate results.

Algorithm 1 Single change-point detection

1: Input : $\mathbb{R}^p$-valued observations $\{X_1, \ldots, X_n\}$; level of significance $\alpha \in (0, 1)$; number of permutation replicates $B$.
2: Compute the value of the test statistic $M_n$ and the candidate change-point location $\hat{\tau}_0$.
3: for $j = 1, 2, \ldots, B$ do
4: Generate a random permutation of the observations $\{X_1, \ldots, X_n\}$.
5: Compute the value of the test statistic for the permuted data, call it $M^j_n$.
6: end for
7: Compute $\tilde{Q}_\alpha$, the 100(1 – $\alpha$)\textsuperscript{th} percentile of $\{M^1_n, \ldots, M^B_n\}$.
8: if $M_n > \tilde{Q}_\alpha$ then
9: Reject $H_0$ at level $\alpha$.
10: Return $\hat{\tau}_0$ as the estimated change-point.
11: end if

3.4 Recursive estimation of multiple change-point locations

In practice, both the number and locations of change-points are unknown and need to be estimated. We need a ‘greedy’ procedure to sequentially detect multiple change-point locations, with each stage relying on the previously detected change-points, which are never re-visited. We combine our proposed test statistic $M_n$ with the WBS procedure proposed by Fryzlewicz (2014) to recursively estimate and test for significant multiple change-point locations.

The main idea is quite simple. In the beginning, instead of computing the statistic $M_n$ over the entire sample $\{X_1, \ldots, X_n\}$, we randomly draw (hence the term ‘wild’) $M$ sub-samples $\{X_{s_m}, \ldots, X_{e_m}\}$, $1 \leq m \leq M$, where $s_m, e_m$ are integers satisfying $1 \leq s_m \leq n - 7$ and $s_m + 7 \leq e_m \leq n$. We compute the statistic

$$T_{s_m, e_m}(b) = \frac{(e_m - b)(b - s_m + 1)}{(e_m - s_m + 1)^2}T(X_{s_m:b}; X_{(b+1):e_m})$$

for each sub-sample with $b$ ranging over $\{s_m + 3, \ldots, e_m - 4\}$. We require $s_m + 7 \leq e_m$ to ensure there are $e_m - s_m + 1 \geq 8$ observations in the sub-sample $\{X_{s_m}, \ldots, X_{e_m}\}$. We choose the largest maximizer over
all the sub-samples to be the first change-point candidate to be tested against a certain threshold. We
determine that threshold using a permutation procedure with $B$ replicates. If the candidate change-
point location turns out to be statistically significant, the same procedure is then repeated to the left
and right of it. The recursive search quits a bisected sub-interval if either it does not contain at least 8
observations, or, if no further significant change-point locations are detected within that sub-interval.
We illustrate in Algorithm 2 the pseudocode of the WBS procedure for detecting significant multiple
change-point locations within a generic interval $(s, e)$.

**Algorithm 2** WBS procedure for multiple change-point detection

```
1: function WBS(s, e)
2:    if $e - s < 7$ then
3:        STOP;
4:    else
5:        $M_{s,e} := \{1 \leq m \leq M : s \leq s_m \leq e_m - 7 \leq e - 7\}$.
6:        $(m_0, b_0) := \arg\max_{m \in M_{s,e}} T_{s_m,e_m}(b)$. 
7:        for $j = 1, 2, \ldots, B$ do
8:            Generate a random permutation of the observations $\{X_s, \ldots, X_e\}$.
9:            Compute $T^j := \max_{m \in M_{s,e}, b = s_m + 3, \ldots, e_m - 4} T_{s_m,e_m}(b)$ based on the permuted observations.
10:       end for
11:       Compute $\zeta_\alpha$, the $100(1 - \alpha)^{th}$ percentile of $\{T^1, \ldots, T^B\}$.
12:       if $T_{s_{m_0},e_{m_0}}(b_0) > \zeta_\alpha$ then
13:           Add $b_0$ to the set of estimated change-points.
14:           WBS($s, b_0$)
15:           WBS($b_0 + 1, e$)
16:       end if
17:    end if
18: end function
```

4 Numerical studies

4.1 Simulation studies

In this subsection, we examine the finite sample performance of our proposed methodology for single
and multiple change-point detection via extensive simulation studies. Throughout the discussions, we
consider the distance $\gamma(z, z') = \|z - z\|_1^{1/2}$ for the proposed procedure. We first consider the following
examples of the single change-point alternative.

**Example 4.1** (No structural break).
1. \( X_t \overset{i.i.d.}{\sim} N(0, I_p) \) for \( 1 \leq t \leq n \).

2. \( X_t \overset{i.i.d.}{\sim} N(0, \Sigma) \) for \( 1 \leq t \leq n \), where \( \Sigma = (\sigma_{ij})_{i,j=1}^p \) with \( \sigma_{ij} = 0.7^{i-j} \).

3. For each \( i = 1, \ldots, p \), \( \{X_{1,i}, \ldots, X_{n,i}\} \) is generated independently from the ARCH(2) model \( X_{t,i} = \sigma_{t,i} \epsilon_{t,i} \), with \( \sigma_{t,i}^2 = \alpha_0 + \alpha_1 X_{t-1,i}^2 + \alpha_2 X_{t-2,i}^2 \), where \( \epsilon_{t,i} \overset{i.i.d.}{\sim} N(0, 1) \) for \( 1 \leq t \leq n \). We consider \( \alpha_0 = 10^{-6}, \alpha_1 = 0.008 \) and \( \alpha_2 = 0.001 \).

4. For each \( i = 1, \ldots, p \), \( \{X_{1,i}, \ldots, X_{n,i}\} \) is generated independently from the GARCH(1,1) model \( X_{t,i} = \sigma_{t,i} \epsilon_{t,i} \), with \( \sigma_{t,i}^2 = \alpha_0 + \alpha_1 X_{t-1,i}^2 + \beta_1 \sigma_{t-1,i}^2 \), where \( \epsilon_{t,i} \overset{i.i.d.}{\sim} N(0, 1) \) for \( 1 \leq t \leq n \). We consider \( \alpha_0 = 10^{-6}, \alpha_1 = 0.001 \) and \( \beta_1 = 0.001 \).

**Example 4.2** (Single change-point in mean).

1. \( X_t \overset{i.i.d.}{\sim} N(0, I_p) \) for \( 1 \leq t \leq \lfloor n/2 \rfloor \) and \( X_t \overset{i.i.d.}{\sim} N(\mu, I_p) \) for \( \lceil n/2 \rceil + 1 \leq t \leq n \), where \( \mu = (0.6, \ldots, 0.6) \in \mathbb{R}^p \).

2. \( X_t \overset{i.i.d.}{\sim} N(0, \Sigma) \) for \( 1 \leq t \leq \lfloor n/2 \rfloor \) and \( X_t \overset{i.i.d.}{\sim} N(\mu, \Sigma) \) for \( \lceil n/2 \rceil + 1 \leq t \leq n \), where \( \Sigma = (\sigma_{ij})_{i,j=1}^p \) with \( \sigma_{ij} = 0.7^{i-j} \), and \( \mu = (0.6, \ldots, 0.6) \in \mathbb{R}^p \).

**Example 4.3** (Single change-point in higher order moments).

1. \( X_t \overset{i.i.d.}{\sim} N(\mu, I_p) \) with \( \mu = (1, \ldots, 1) \in \mathbb{R}^p \) for \( 1 \leq t \leq \lfloor n/2 \rfloor \) and \( X_t \overset{i.i.d.}{\sim} \text{Exponential}(1) \) for \( i = 1, \ldots, p \) and \( \lceil n/2 \rceil + 1 \leq t \leq n \).

2. \( X_t = (X_{t,1}, \ldots, X_{t,p}) \overset{i.i.d.}{\sim} \text{Poisson}(1) - 1 \) for \( 1 \leq t \leq \lfloor n/2 \rfloor \) and \( X_t = (X_{t,1}, \ldots, X_{t,\lfloor \beta p \rfloor}, X_{t,\lceil \beta p \rceil + 1}, \ldots, X_{t,p}) \overset{i.i.d.}{\sim} \text{Rademacher}(0.5) \) for \( \lceil n/2 \rceil + 1 \leq t \leq n \).

3. \( X_t = R^{1/2} Z_{1t} \) for \( 1 \leq t \leq \lfloor n/2 \rfloor \) and \( X_t = R^{1/2} Z_{2t} \) for \( \lceil n/2 \rceil + 1 \leq t \leq n \), where \( R = (r_{ij})_{i,j=1}^p \) with \( r_{ii} = 1 \) for \( i = 1, \ldots, p \), \( r_{ij} = 0.25 \) if \( 1 \leq |i - j| \leq 2 \) and \( r_{ij} = 0 \) otherwise, \( Z_{1t} \overset{i.i.d.}{\sim} N(0, I_p) \) and \( Z_{2t} = (Z_{2t,1}, \ldots, Z_{2t,p}) \) - 1.

In Example 4.3 there is a distributional change along the sequence of observations, although the mean vectors and the covariance matrices of the underlying distributions remain the same. Or in other words, the change is in the higher order moments of the underlying distributions.

We try \( n = 100, p = 100, 200 \) and \( \beta = 1/2 \). We implement Algorithm 1 with \( B = 199 \) permutation replicates and a significance level of \( \alpha = 0.05 \). We cluster the observations based on the estimated significant change-point location and compute the Adjusted Rand Index (ARI) (Morey and Agresti,
The ARI is a positive value between 0 and 1. The ARI value is 0 when there is no change-point estimated when there does exist one (or more), or there is no change-point but the method estimates one (or more) change-point locations. The ARI value is 1 when the estimation is perfect. Higher the value of ARI, more accurate is the estimation of the change-point location. We consider 100 simulations of each of the above examples, for each of which we compute the ARI value and report it in the table below. We compare our test with

- the E-Divisive procedure proposed by Matteson and James (2014) with $R = 199$ random permutations (using the ‘ecp’ R package); (denote by MJ)
- the test based on the graph-based original scan statistic proposed by Chen and Zhang (2015) with 199 permutations (using the ‘gSeg’ R package); (denote by CZ)
- the max-type edge-count test proposed by Chu and Chen (2019) with 199 permutations (using the ‘gSeg’ R package); (denote by CC) and
- the INSPECT procedure proposed by Wang and Samworth (2018) (using the ‘InspectChange-point’ R package) (denote by WS).

Table 2: Comparison of average ARI values for different methods over 100 simulations.

|       | n  | p  | Our test | MJ | CC | CZ | WS |
|-------|----|----|----------|----|----|----|----|
| Ex 4.1| 1  | 100| 0.98     | 0.97| 0.97| 0.97| 0.00|
|       | 1  | 200| 0.97     | 0.98| 0.96| 0.96| 0.00|
|       | 2  | 100| 0.93     | 0.97| 0.91| 0.92| 0.00|
|       | 2  | 200| 0.97     | 0.97| 0.95| 0.98| 0.00|
|       | 3  | 100| 0.96     | 0.94| 0.98| 0.91| 0.00|
|       | 3  | 200| 0.97     | 0.95| 0.96| 0.96| 0.00|
|       | 4  | 100| 0.95     | 0.95| 0.99| 0.93| 0.00|
|       | 4  | 200| 0.97     | 0.96| 0.92| 0.92| 0.00|
| Ex 4.2| 1  | 100| 1.00     | 1.00| 0.999| 0.999| 1.00|
|       | 1  | 200| 1.00     | 1.00| 0.999| 0.999| 1.00|
|       | 2  | 100| 0.984    | 0.986| 0.867| 0.946| 0.981|
|       | 2  | 200| 0.996    | 0.996| 0.978| 0.983| 0.993|
| Ex 4.3| 1  | 100| 0.993    | 0.014| 0.004| 0.027| 0.390|
|       | 1  | 200| 1.00     | 0.030| 0.007| 0.037| 0.414|
|       | 2  | 100| 0.999    | 0.034| 0.001| 0.059| 0.468|
|       | 2  | 200| 1.00     | 0.032| 0.001| 0.055| 0.502|
|       | 3  | 100| 0.978    | 0.024| 0.021| 0.065| 0.402|
|       | 3  | 200| 0.992    | 0.029| 0.006| 0.040| 0.363|

Although the methodology proposed by Wang and Samworth (2018) is aimed at detecting a mean shift for high dimensional data, we compare our method with theirs to illustrate that our method can capture inhomogeneities among a sequence of high-dimensional observations beyond the first moment.
The results from Table 2 indicate that almost all the methods perform nearly equally good when there is no true change-point or when there is a mean shift. When there is no true change-point, the procedure proposed by Wang and Samworth (2018) still detects one, leading to a zero ARI value. Although our methodology is developed for an i.i.d. sequence of observations, the results for Examples 4.1.3-4.1.4 show that our method does reasonably well even when there is relatively weak conditional heteroskedasticity and temporal dependence among the sequence of observed data.

Most interestingly, when there is a change in distribution among the sequence of the high-dimensional observations beyond the first two moments, our method performs way better than any of the other competitors in terms of accurately estimating the single change-point location. That it clearly beats the E-Divisive procedure, is quite expected as the Euclidean energy distance fails to capture any in-homogeneity between two high-dimensional distributions beyond the first two moments. Our results indicate that although the methods proposed by Chen and Zhang (2015) and Chu and Chen (2019) are effective for location and scale alternatives, they perform poorly in detecting changes in higher order moments and are outperformed by our method.

The following examples illustrate the performance of Algorithm 2 in the case of a two change-points alternative.

**Example 4.4** (Two change-points in mean).

1. \( X_t \overset{i.i.d.}{\sim} N(0, I_p) \) for \( 1 \leq t \leq \lfloor n/3 \rfloor \) and \( 2 \lfloor n/3 \rfloor + 1 \leq t \leq n \), and \( X_t \overset{i.i.d.}{\sim} N(\mu, I_p) \) for \( \lfloor n/3 \rfloor + 1 \leq t \leq 2 \lfloor n/3 \rfloor \), where \( \mu = (0.6, \ldots, 0.6) \in \mathbb{R}^p \).

2. \( X_t \overset{i.i.d.}{\sim} N(0, \Sigma) \) for \( 1 \leq t \leq \lfloor n/3 \rfloor \) and \( 2 \lfloor n/3 \rfloor + 1 \leq t \leq n \), and \( X_t \overset{i.i.d.}{\sim} N(\mu, \Sigma) \) for \( \lfloor n/3 \rfloor + 1 \leq t \leq 2 \lfloor n/3 \rfloor \), where \( \Sigma = (\sigma_{ij})_{i,j=1}^p \) with \( \sigma_{ij} = 0.7^{|i-j|} \) and \( \mu = (0.6, \ldots, 0.6) \in \mathbb{R}^p \).

**Example 4.5** (Two change-points in higher order moments).

1. \( X_t \overset{i.i.d.}{\sim} N(\mu, I_p) \) with \( \mu = (1, \ldots, 1) \in \mathbb{R}^p \) for \( 1 \leq t \leq \lfloor n/3 \rfloor \) and \( 2 \lfloor n/3 \rfloor + 1 \leq t \leq n \), and \( X_{t,i} \overset{i.i.d.}{\sim} \text{Exponential}(1) \) for \( i = 1, \ldots, p \) and \( \lfloor n/3 \rfloor + 1 \leq t \leq 2 \lfloor n/3 \rfloor \).

2. \( X_t = \underbrace{(X_{t,1}, \ldots, X_{t,p})}_{\sim \text{Poisson}(1)} - 1 \) for \( 1 \leq t \leq \lfloor n/3 \rfloor \) and \( 2 \lfloor n/3 \rfloor + 1 \leq t \leq n \), and \( X_{t,[1:|\beta p|]} \overset{i.i.d.}{\sim} \text{Poisson} (1) - 1 \), and \( X_{t,([|\beta p|+1):p]} \overset{i.i.d.}{\sim} \text{Rademacher} (0.5) \) for \( \lfloor n/3 \rfloor + 1 \leq t \leq 2 \lfloor n/3 \rfloor \).

The results from Table 3 indicate that almost all the methods perform nearly equally good in Example 4.4.1, and our methodology as well as the E-Divisive procedure perform considerably better than the rest in Example 4.4.2. Most interestingly, when there are two changes in distribution among the sequence of the high-dimensional observations, our method performs way better than any of the
other competitors in terms of accurately estimating the two change-point locations. Again, as the Euclidean energy distance fails to capture any in-homogeneity between two high-dimensional distributions beyond the first two moments, it is no wonder that our method clearly beats the E-Divisive procedure. And quite evidently our method performs significantly better than the graph-based methods proposed by Chen and Zhang (2015) and Chu and Chen (2019) in detecting and localizing general type of changes in the underlying distribution beyond the first two moments.

Table 3: Comparison of average ARI values for different methods over 100 simulations.

|     | n  | p  | Our test | MJ  | CC  | CZ  | WS  |
|-----|----|----|----------|-----|-----|-----|-----|
| Ex 4.4 | (1) | 100 | 100 | 0.991 | 1 | 0.975 | 0.960 | 0.916 |
|     | (1) | 100 | 200 | 0.979 | 1.000 | 0.996 | 0.959 | 0.886 |
|     | (2) | 100 | 100 | 0.962 | 0.978 | 0.747 | 0.885 | 0.643 |
|     | (2) | 100 | 200 | 0.978 | 0.994 | 0.912 | 0.935 | 0.619 |
| Ex 4.5 | (1) | 100 | 100 | 0.969 | 0.024 | 0.487 | 0.412 | 0.383 |
|     | (1) | 100 | 200 | 0.982 | 0.054 | 0.496 | 0.413 | 0.279 |
|     | (2) | 100 | 100 | 0.978 | 0.028 | 0.214 | 0.528 | 0.179 |
|     | (2) | 100 | 200 | 0.982 | 0.028 | 0.244 | 0.534 | 0.172 |

4.2 Real data illustration

We consider the daily closed stock prices of \( p = 72 \) companies under the Consumer Defensive sector, listed under the NYSE and NASDAQ stock exchanges, on the first dates of each month during the time period between January 1, 2005 and December 31, 2010. The data has been downloaded from Yahoo Finance via the R package ‘quantmod’. At each time \( t \), denote the closed stock prices of these companies by \( X_t = (X_{t,1}, \ldots, X_{t,p}) \) for \( 1 \leq t \leq 72 \). We consider the stock returns \( S_t^X = (S_{t,1}^X, \ldots, S_{t,p}^X) \) for \( 1 \leq t \leq 71 \), where \( S_{t,i}^X = \log(X_{t+1,i}/X_{t,i}) \) for \( 1 \leq i \leq p \).

According to the US National Bureau of Economic Research, the Great Recession began in the United States in December 2007. The government responded with an unprecedented $700 billion bank bailout in October 2008 and $787 billion fiscal stimulus package in February 2009 to save existing jobs, provide temporary relief programs for those most affected by the recession, invest in infrastructure, education, health and renewable energy, etc. The recession officially lasted till June 2009, thus extending over 19 months.

When a recession or an economic slowdown occurs, markets tend to become volatile, prompting investors to sell stocks. Although all industrial sectors are susceptible to economic changes, some are less sensitive or more resistant to recessions (for example, Consumer Defensive, Utility or Healthcare sectors) compared to some others (for example, Real Estate, Finance, Oil and Gas, Automobiles, etc.). The goal is to consider stock returns of companies under a sector that is known to perform
relatively well even when a recession hits the market, and see if our proposed methodology and the
other state-of-the-art methods can detect change-points in the stock returns data.

Figure 1: Time series plots of the stock returns for six companies under the Consumer Defensive sector. The solid red lines represent the change-point locations detected by our methodology. The dotted blue and gray lines represent the change-point locations detected by the procedures proposed by Chu and Chen (2019) and Chen and Zhang (2015), respectively.

When applied on the stock returns dataset (with $n = 71$ and $p = 72$) for the Consumer Defensive sector:

- our procedure with $\gamma(x, x') = \|x - x'\|_1^{1/2}$ detects two change-points, viz. September 1, 2007 and January 1, 2009, which seems quite reasonable given the historical sequence of eventualities;

- the E-Divisive procedure by Matteson and James (2014) fails to detect any change-point over that time period;

- the test based on the graph-based original scan statistic proposed by Chen and Zhang (2015) detects only one change-point, viz. March 1, 2009;
• the max-type edge-count test proposed by Chu and Chen (2019) detects two change-points, viz. May 1, 2008 and September 1, 2008; and
• the methodology proposed by Wang and Samworth (2018) detects as many as 18 change-points over the aforesaid period of time.

5 Discussions

Two important questions still remain unaddressed. First, how to perform the partitioning or grouping in Definition 2.3 optimally in practice, and second, whether it is possible to completely characterize the homogeneity between two high-dimensional random vectors. We present two examples below where external undirected or directed graph information is available to guide the partition involved in the definition of the generalized Euclidean distance. In both cases, the corresponding generalized energy distance can completely characterize the homogeneity between two high-dimensional random vectors. In this way, the external graph information can be incorporated in our change-point detection and estimation procedure to improve its efficiency.

5.1 Undirected graph parameterized by exponential family

Suppose the distribution of $X = (x_1, \ldots, x_p)$ can be represented by an exponential family of the form

$$\exp \left\{ \sum_{C \in \mathcal{I}} \theta_C \phi_C(\tilde{x}_C) - A(\theta) \right\},$$

where $\mathcal{I}$ denotes a collection of subsets of $[p]$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_p)$ and $\theta = \{\theta_C : C \in \mathcal{I}\}$. A special case is the pairwise graphical models, where we have

$$\sum_{C \in \mathcal{I}} \theta_C \phi_C(\tilde{x}_C) = \sum_{i \in [p]} \theta_i \phi_i(\tilde{x}_i) + \sum_{(i,j) \in E} \theta_{ij} \phi_{ij}(\tilde{x}_{ij}),$$

with $\tilde{x}_{ij} = (\tilde{x}_i, \tilde{x}_j)$ and $E \subseteq [p] \times [p]$ denoting the set of edges. Examples include the Gaussian graphical models and Ising models. Suppose there does not exist a vector $\{\beta_C : C \in \mathcal{I}\}$ such that $\sum_{C \in \mathcal{I}} \beta_C \phi_C(\tilde{x}_C)$ is equal to a constant for all $\tilde{x}$, which is the so-called minimal representation condition. Let $\mu_C = E_\theta[\phi_C(x_C)]$ be the mean parameter. Note that

$$\frac{\partial A(\theta)}{\partial \theta_C} = E_\theta[\phi_C(x_C)].$$
Therefore, the gradient $\nabla A$ defines a map from the canonical parameter $\theta$ to the mean parameter $\{\mu_C : C \in \mathcal{I}\}$. Under the minimal representation condition, $\nabla A$ is a one-to-one map, see e.g., Proposition 3 in Chapter 3 of Wainwright and Jordan (2008). Denote by $p_C$ the marginal distribution of $x_C$ for $C \in \mathcal{I}$. Note that the mean parameter is determined by the set of marginal distributions $\{p_C : C \in \mathcal{I}\}$ as $\mu_C = \int \phi_C(\bar{x}_C)p_C(\bar{x}_C)d\bar{x}_C$. Hence the collection of the marginal distributions $\{p_C : C \in \mathcal{I}\}$ completely determines the distribution of $X$. Suppose $Y$ can be presented by the same exponential family with possibly different parameter value. We see that

$$X \overset{d}{=} Y \text{ if and only if } x_C \overset{d}{=} y_C \text{ for all } C \in \mathcal{I}.$$ 

To incorporate the potentially useful graph information, we modify our change-point detection procedure by using the graph-guided generalized Euclidean distance given by

$$\gamma(z, z') = \sqrt{\sum_{C \in \mathcal{I}} \rho_C(z_C, z'_C)}.$$ 

We illustrate the usefulness of this strategy through a toy example below.

**Example 5.1** (Fully visible Boltzmann machine). Let $X = (X_1, \ldots, X_p)$ be a $p$-variate binary random vector where $X_i \in \{-1, 1\}$ for $1 \leq i \leq p$. Suppose the probability density function of $X$ is given by

$$f(\bar{x}; b, M) = \frac{1}{Z(b, M)} \exp \left( \frac{1}{2} \bar{x}^\top M \bar{x} + b^\top \bar{x} \right),$$

where $M$ is a symmetric $p \times p$ matrix with zero diagonal entries, $b$ is a $p \times 1$ vector and $Z(b, M)$ is a normalizing constant. This model is popularly known as a fully visible Boltzmann machine (FVBM; Nguyen and Wood (2016)). For $1 \leq t \leq \lfloor n/2 \rfloor$, we generate $X_t$ independently from a FVBM with $b = 0.1 \times \mathbf{1}_p$, $M(a, b) = 0.1$ for $|a - b| = 1$ and $M(a, b) = 0$ otherwise, for $1 \leq a, b \leq p$. For $\lfloor n/2 \rfloor + 1 \leq t \leq n$, we generate $X_t$ independently from another FVBM with $b = 0.5 \times \mathbf{1}_p$, $M(a, b) = 0.3$ for $|a - b| = 1$ and $M(a, b) = 0$ otherwise. In this case, we have $\mathcal{I} = \{\{1, 2\}, \{2, 3\}, \ldots, \{p - 1, p\}\}$. We use the R package BoltzMM to generate data from a FVBM, which works only for $p < n$. We consider $n = 50$ and $p = 25$, and implement our test with $B = 199$ permutation replicates and at a significance level $\alpha = 0.05$. Here the partitioning or the grouping for the generalized homogeneity metric in Definition 2.3 is induced by $\mathcal{I}$. Table 4 reports the average ARI values for the different tests over 100 simulations. Although $p < n$ in the considered example, our test outperforms Matteson and James (2014)’s test and does quite better than the others in terms of accurately estimating the change-point location.
Table 4: Comparison of average ARI values over 100 simulations.

|           | Our test | MJ  | CC  | CZ  | WS  |
|-----------|----------|-----|-----|-----|-----|
|           | 0.974    | 0.000 | 0.311 | 0.712 | 0.931 |

5.2 Directed graph/Bayesian networks

As the second example, we consider the Bayesian network, a versatile and widely used probabilistic framework for modeling high-dimensional distributions with structure. A Bayesian network specifies a probability distribution in terms of a directed acyclic graph (DAG), where each component of $X$ is represented by a node in the DAG. To describe the probability distribution, one specifies the conditional probabilities $P(x_i|x_{\pi(i)})$ for $1 \leq i \leq p$, where $\pi(i) \subseteq [p]$ represents the set of parents of $x_i$ in the DAG. The joint distribution of $X$ has a factorization based on these conditional probabilities

$$P(X) = \prod_{i=1}^{p} P(x_i|x_{\pi(i)}).$$

Markov Chain is a special case of the Bayesian networks. But Bayesian networks are much more versatile and are universal. They can interpolate between product measures and arbitrary distributions over the nodes as the DAG becomes denser and denser. Suppose the distributions of $X$ and $Y$ are two Bayesian networks on the same DAG. Then we have

$$X \overset{d}{=} Y \quad \text{if and only if} \quad x_{i \cup \pi(i)} \overset{d}{=} y_{i \cup \pi(i)} \text{ for all } i = 1, 2, \ldots, p.$$ 

In this case, to incorporate the DAG information, the change-point test can be constructed based on the DAG-guided distance defined as

$$\gamma(z, z') = \sqrt{\sum_{i=1}^{p} \rho_i(z_{i \cup \pi(i)}, z'_{i \cup \pi(i)})}.$$ 

We illustrate the applicability of this strategy via the following numerical example.

**Example 5.2** (Directed Chain). We consider the model: $X_i = \phi X_{i-1} + \epsilon_i$ for $2 \leq i \leq p$ and $X_1 = \epsilon_1$. In matrix notations, we can write the model as $X = (I_p - L)^{-1} \epsilon$, where $L = (L_{ij})_{i,j=1}^{p}$ is a lower triangular matrix with $L_{i,i-1} = \phi$ and $L_{i,j} = 0$ if $i - j > 1$. To generate $\{X_t\}_{t=1}^{n}$, we consider $\epsilon_t \overset{i.i.d.}{\sim} N(\mu, I_p)$ with $\mu = (1, \ldots, 1) \in \mathbb{R}^p$ for $1 \leq t \leq \lfloor n/2 \rfloor$ and $\epsilon_t \overset{i.i.d.}{\sim} \text{Exponential}(1)$ for $i = 1, \ldots, p$ and $\lfloor n/2 \rfloor + 1 \leq t \leq n$.

We consider $n = 100$, $p = 100$ and $\phi = 0.5$, and implement our test with $B = 199$ permutation
replicates and at a significance level $\alpha = 0.05$. Here the partitioning considered for constructing the generalized homogeneity metric is $\{X_1\}, \{X_1, X_2\}, \{X_2, X_3\}, \ldots, \{X_{p-1}, X_p\}$. Table 5 reports the average ARI values for the different tests over 100 simulations, which illustrates that our test beats the others in terms of accurately estimating the change-point location.

| Our test | MJ | CC | CZ | WS |
|----------|----|----|----|----|
| 0.949    | 0.050 | 0.010 | 0.070 | 0.376 |

### 6 Future directions

There are several open problems that seem intriguing for future investigations. First, extension of our methodology for a weakly dependent high-dimensional time series seems to be of obvious interest, as temporal dependence is natural to expect in many practical applications. The problem, though quite interesting, seems absolutely non-immediate and non-trivial from both methodological and theoretical perspectives because of the additional complexity brought in by the temporal dependence.

Second it would be intriguing to develop theoretical consistency results for the WBS procedure we implemented for multiple change-point detection, similar to Theorem 3.2 in Fryzlewicz (2014). The latter was proved in a much simpler setting, in the context of detecting changes in the mean in a sequence of univariate observations. This again looks technically non-trivial, and we would leave it as a potential topic for future research.

### References

Arlot, S., Celisse, A. and Harchaoui, Z. (2019). A Kernel Multiple Change-point Algorithm via Model Selection. *Journal of Machine Learning Research*, 20, 1-56.

Aue, A. and Horváth, L. (2012). Structural breaks in time series. *Journal of Time Series Analysis*, 34(1) 1-16.

Avanesov, V. and Buzun, N. (2018). Change-point detection in high-dimensional covariance structure. *Electronic Journal of Statistics*, 12(2), 3254-3294.

Baringhaus, L. and Franz, C. (2004). On a New Multivariate Two-sample Test. *Journal of Multivariate Analysis*, 88(1), 190-206.

Biau, G., Bleakley, K. and Mason, D.M. (2016). Long Signal Change-point Detection. *Electronic Journal of Statistics*, 10(2), 2097-2123.
CHAKRABORTY, S., and ZHANG, X. (2019). A New Framework for Distance and Kernel-based Metrics in High Dimensions. arXiv:1909.13469.

CHEN, H. and ZHANG, N. (2015). Graph-based Change-point Detection. Annals of Statistics, 43(1), 139-176.

CHU, L. and CHEN, H. (2019). Asymptotic Distribution-free Change-point Detection for Multivariate and Non-Euclidean Data. Annals of Statistics, 47(1), 382-414.

CURTIS, R., XIANG, J., PARIKH, A., KINNAIRD, P. and XING, E.P. (2012). Enabling Dynamic Network Analysis Through Visualization in TVNViewer. BMC Bioinformatics, 13(204).

DETTE, H., PAN, G. and YANG, Q. (2020). Estimating a Change Point in a Sequence of Very High-Dimensional Covariance Matrices. Journal of the American Statistical Association, to appear.

ENIKEEVA, F. and HARCHAOUI, Z. (2019). High-dimensional change-point detection under sparse alternatives. Annals of Statistics, 47(4) 2051-2079.

FRYZLEWICZ, P. (2014). Wild Binary Segmentation for Multiple Change-point Detection. Annals of Statistics, 42(6) 2243-2281.

HARCHAOUI, Z. and CAPPE, O. (2007). Retrospective Change-point Estimation with Kernels. IEEE Workshop on Statistical Signal Processing.

JANDHYALA, V., FOTOPOULOS, S., MACNEILL, I. and LIU, P. (2013). Inference for single and multiple change-points in time series. Journal of Time Series Analysis, 34(4) 423-446.

JIRAK, M. (2015). Uniform Change Point Tests in High Dimension. Annals of Statistics, 43(6) 2451–2483.

LUNG-YUT-FONG, A., LÉVY-LEDUC, C. and CAPPE, O. (2015). Homogeneity and Change-point Detection Tests for Multivariate Data Using Rank Statistics. Journal de la Société Française de Statistique, 156(4), 133-162.

NGUYEN, H.D. and WOOD, I.A. (2016). Asymptotic Normality of the Maximum Pseudolikelihood Estimator for Fully Visible Boltzmann Machines. IEEE Transactions on Neural Networks and Learning Systems, 27(4), 897-902.

LYONS, R. (2013). Distance Covariance in Metric Spaces. Annals of Probability, 41(5) 3284-3305.

MATTeson, D.S. and JAMES, N.A. (2014). A Nonparametric Approach for Multiple Change Point Analysis of Multivariate Data. Journal of the American Statistical Association, 109(505), 334-345.

McCulloh, I. (2009). Detecting Changes in a Dynamic Social Network. PhD thesis, Institute for Software Research, School of Computer Science, Carnegie Mellon University. CMU-ISR-09-104.

MOREY, L.C. and AGRESTI, A. (1984). The Measurement of Classification Agreement : An Adjustment to the Rand Statistic for Chance Agreement. Educational and Psychological Measurement, 44(1), 33-37.
Park, Y., Wang, H., NCappöbauer, T., Vaziri, A. and Priebe, C.E. (2015). Anomaly Detection on Whole-Brain Functional Imaging of Neuronal Activity using Graph Scan Statistics. In ACM Conference on Knowledge Discovery and Data Mining (KDD), Workshop on Outlier Definition, Detection, and Description (ODDx3).

Picard, F., Robin, S., Lavielle, M., Vaisse, C. and Daudin, J.-J. (2005). A Statistical Approach for Array CGH Data Analysis. BMC Bioinformatics, 6(27).

Pollard, D. (1990). Empirical Processes: Theory and Applications. NSF-CBMS Regional Conference Series in Probability and Statistics, 2 i-iii+v+vii-viii+1-86.

Rabiner, L.R. and Schäfer, R.W. (2007). Introduction to Digital Speech Processing. Foundations and Trends in Signal Processing, 1(1-2), 1-194.

Sejdinovic, D., Sriperumbudur, B., Gretton, A. and Fukumizu, K. (2013). Equivalence of Distance-based and RKHS-based Statistics in Hypothesis Testing. Annals of Statistics, 41(5) 2263-2291.

Székely, G. J. and Rizzo, M. L. (2004). Testing for Equal Distributions in High Dimension. InterStat, 5.

Székely, G. J. and Rizzo, M. L. (2005). Hierarchical Clustering via Joint Between-within Distances: Extending Ward’s Minimum Variance Method. Journal of Classification, 22(2) 151-183.

Wang, R., Volgushev, S. and Shao, X. (2019). Inference for Change Points in High Dimensional Data. arXiv:1905.08446.

Wainwright, M. J. and Jordan, M. I. (2008). Graphical Models, Exponential Families, and Variational Inference. Now Publishers Inc.

Wang, T. and Samworth, R.J. (2018). High Dimensional Change Point Estimation via Sparse Projection. Journal of the Royal Statistical Society, Series B, 80(1), 57-83.

Zhang, X., Yao, S. and Shao, X. (2018). Conditional Mean and Quantile Dependence Testing in High Dimension. Annals of Statistics, 46(1) 219-246.

Zhu, C. and Shao, X. (2019). Interpoint Distance based Two Sample Tests in High Dimension. arXiv:1902.07279.

Zou, C., Yin, G., Feng, L. and Wang, Z. (2014). Nonparametric Maximum Likelihood Approach to Multiple Change-point Problems. Annals of Statistics, 42(3) 970-1002.
A Sketch of the proof of Theorem 3.1

For the ease of notation, we write \( \hat{E}_{n,k} = \hat{E}_\gamma(X_{1:k}, X_{k+1:n}) \), \( \hat{D}^2_{1:k} = \hat{D}^2(X_{1:k}) \), \( \hat{D}^2_{k+1:n} = \hat{D}^2(X_{k+1:n}) \) and \( \hat{C}_{1,k,n} = \hat{C}(X_{1:k}, Y_{k+1:n}) \). From the proof of Lemma D.1 in the supplementary materials of Chakraborty and Zhang (2019), we can write under \( H_0 \),

\[
\hat{E}_{n,k} = L_{n,k} + R_{n,k},
\]

where

\[
L_{n,k} = \frac{1}{k(n-k)} \sum_{i_1=1}^{k} \sum_{i_2=k+1}^{n} H(X_{i_1}, X_{i_2}) - \frac{1}{k(k-1)} \sum_{1 \leq i_1 < i_2 \leq k} H(X_{i_1}, X_{i_2}) - \frac{1}{(n-k)(n-k-1)} \sum_{k+1 \leq i_1 < i_2 \leq n} H(X_{i_1}, X_{i_2}),
\]

\[
R_{n,k} = \frac{2\tau}{k(n-k)} \sum_{i_1=1}^{k} \sum_{i_2=k+1}^{n} R(X_{i_1}, X_{i_2}) - \frac{\tau}{k(k-1)} \sum_{1 \leq i_1 < i_2 \leq k} R(X_{i_1}, X_{i_2}) - \frac{\tau}{(n-k)(n-k-1)} \sum_{k+1 \leq i_1 \neq i_2 \leq n} R(X_{i_1}, X_{i_1}).
\]

Following the discussions in Section D in the supplementary materials of Chakraborty and Zhang (2019), the variance of \( L_{n,k} \) is given by

\[
V_{n,k} := a^2_{k,n-k} \mathbb{E}[H^2(X, X')] =: V_{n,k;1} + V_{n,k;2} + V_{n,k;3},
\]

which can be estimated by

\[
\hat{V}_{n,k} = \frac{4\hat{C}_{1,k,n}}{k(n-k)} + \frac{2\hat{D}^2_{1:k}}{k(k-1)} + \frac{2\hat{D}^2_{k+1:n}}{(n-k)(n-k-1)} =: \hat{V}_{n,k;1} + \hat{V}_{n,k;2} + \hat{V}_{n,k;3}.
\]

Define

\[
\tilde{T}_{1,n,k} = \frac{\hat{E}_{n,k}}{\sqrt{\hat{V}_{n,k}}}.
\]
For $1 \leq l < k < m - 1 \leq n - 1$, define $\tilde{S}_n(k, m) := \sum_{i_2=k+1}^{m} \sum_{i_1=k}^{i_2-1} H(X_{i_1}, X_{i_2})$ and

\[
\tilde{L}_n(k; l, m) = \frac{1}{(k-l+1)(m-k)} \sum_{i_2=k+1}^{m} \sum_{i_1=l}^{k} H(X_{i_1}, X_{i_2}) - \frac{1}{(k-l+1)(k-l)} \sum_{l \leq i_1 < i_2 \leq k} H(X_{i_1}, X_{i_2})
\]

(14)

Let $\tilde{S}_n(k, m) = 0$ and $\tilde{L}_n(k; l, m) = 0$ for $k \leq l$ or $k \geq m - 1$ or $m > n$. From (10) and (14), it is easy to see that $L_{n,k} = \tilde{L}_n(k; 1, n)$. With the definition of $\tilde{S}_n(k, m)$ as above, we can write

\[
\tilde{L}_n(k; l, m) = - \frac{1}{(k-l+1)(k-l)} \tilde{S}_n(l, k) - \frac{1}{(m-k)(m-k-1)} \tilde{S}_n(k+1, m)
+ \frac{1}{(k-l+1)(m-k)} \{ \tilde{S}_n(l, m) - \tilde{S}_n(l, k) - \tilde{S}_n(k+1, m) \}
\]

(15)

Denote $S_n(a, b) := \tilde{S}_n([na] + 1, [nb])$ for any $0 \leq a < b \leq 1$. Further let $l = [na] + 1, k = [nr]$, and $m = [nb]$ for $0 \leq a < r < b \leq 1$. Therefore, from (15) we have

\[
\tilde{L}_n(k; l, m) = - \frac{1}{(k-l+1)(k-l)} S_n(a, r) - \frac{1}{(m-k)(m-k-1)} S_n(r, b)
+ \frac{1}{(k-l+1)(m-k)} \{ S_n(a, b) - S_n(a, r) - S_n(r, b) \}
\]

(16)

For $1 \leq l < k < m - 1 \leq n - 1$, further define

\[
\tilde{V}_n(k; l, m) := a^2_{k-l+1,m-k} V_0 =: \tilde{V}_{n1}(k; l, m) + \tilde{V}_{n2}(k; l, m) + \tilde{V}_{n3}(k; l, m),
\]

(17)

where $V_0 := E H^2(X, X')$. From (11) and (17), it is easy to check that $V_{n,k} = \tilde{V}_n(k; 1, n)$. Also define

\[
\hat{V}_n(k; l, m) := \frac{4 \tilde{D}_{l,k,m}}{(k-l+1)(m-k)} + \frac{2 \tilde{D}_{l,k}}{(k-l+1)(k-l)} + \frac{2 \tilde{D}_{l+1,m}}{(m-k)(m-k-1)}
\]

(18)

Let $\tilde{V}_n(k; l, m) = 1$ and $\hat{V}_n(k; l, m) = 1$ for $k \leq l$ or $k \geq m - 1$ or $m > n$. From (12) and (18), it is not hard to see that $\hat{V}_{n,k} = \tilde{V}_n(k; 1, n)$.

**Theorem A.1.** Under Assumption $3.2$ as $n, p \to \infty$,

\[
\left\{ \frac{\sqrt{2}}{n \sqrt{V_0}} S_n(a, b) \right\}_{a,b \in [0,1]} \xrightarrow{d} Q \quad \text{in } L^\infty([0,1]^2),
\]
where $Q$ is a centered Gaussian process with the covariance function given by

$$\text{cov}(Q(a_1, b_1), Q(a_2, b_2)) = (b_1 \land b_2 - a_1 \lor a_2)^2 \mathbf{1}(b_1 \land b_2 > a_1 \lor a_2).$$

In particular, $\text{var}(Q(a, b)) = (b - a)^2 \mathbf{1}(b > a)$.

The proof of Theorem A.1 is given in Section B. From (16) and (17), we can write

$$\frac{\bar{L}_n(k; l, m)}{\sqrt{V_n(k; l, m)}} = \frac{n}{\sqrt{2}a_{k-l+1,m-k}} \times \left[ -\frac{1}{(k - l + 1)(k - l)} \frac{\sqrt{2}S_n(a, r)}{n\sqrt{V_0}} \right. \left. - \frac{1}{(m - k)(m - k - 1)} \frac{\sqrt{2}S_n(r, b)}{n\sqrt{V_0}} + \frac{1}{(k - l + 1)(m - k)} \frac{S_n(a, b) - S_n(a, r) - S_n(r, b)}{n\sqrt{V_0}} \right].$$

Combining the above with Theorem A.1, it is not hard to see that as $n, p \to \infty$,

$$\left\{ \frac{\bar{L}_n([nr]; [na] + 1, [nb])}{\sqrt{V_n([nr]; [na] + 1, [nb])}} \right\}_{a, r, b \in [0, 1]} \quad \overset{d}{\longrightarrow} \quad G \quad \text{in} \; L^\infty([0, 1]^3), \quad (19)$$

where

$$G(r; a, b) := \frac{1}{\sqrt{(r-a)(b-r)} + \frac{1}{(r-a)^2} + \frac{1}{(b-r)^2}} \times \left[ -\frac{1}{(r-a)^2} Q(a, r) - \frac{1}{(b-r)^2} Q(r, b) \right. \left. + \frac{1}{(r-a)(b-r)} \{Q(a, b) - Q(a, r) - Q(r, b)\} \right]$$

for $0 \leq a < r < b \leq 1$ and zero otherwise. As a further consequence, when $a = 0$ and $b = 1$, we have

$$\left\{ \frac{\bar{L}_n([nr]; 1, n)}{\sqrt{V_n([nr]; 1, n)}} \right\}_{r \in [0, 1]} \quad \overset{d}{\longrightarrow} \quad G_0 \quad \text{in} \; L^\infty([0, 1]), \quad (20)$$

where

$$G_0(r) := \frac{1}{\sqrt{r(1-r) + \frac{1}{r^2} + \frac{1}{(1-r)^2}}} \times \left[ -\frac{1}{r^2} Q(0, r) - \frac{1}{(1-r)^2} Q(r, 1) \right. \left. + \frac{1}{r(1-r)} \{Q(0, 1) - Q(0, r) - Q(r, 1)\} \right] \quad (21)$$

for $0 < r < 1$ and zero otherwise. The second equality in (21) follows from some straightforward calculations.
Now for $1 \leq l < k < m - 1 \leq n - 1$, define $\tilde{R}_n(k, m) := \sum_{i_2=k+1}^{m} \sum_{i_1=k}^{k-1} \tau R(X_{i_1}, X_{i_2})$ and

$$\tilde{Q}_n(k; l, m) := \frac{2\tau}{(k - l + 1)(m - k)} \sum_{i_2=k+1}^{m} \sum_{i_1=l}^{k} \tau R(X_{i_1}, X_{i_2}) - \frac{\tau}{(k - l + 1)(k - l)} \sum_{l \leq i_1 \neq i_2 \leq k} R(X_{i_1}, X_{i_2}).$$

(22)

Define $\tilde{R}_n(k, m)$ and $\tilde{Q}_n(k; l, m)$ to be zero otherwise. Comparing (10) and (22), it is easy to verify that $R_{n,k} = \tilde{Q}_n(k; 1,n)$. With the definition of $\tilde{R}_n(k, m)$ as above, clearly we have

$$\tilde{Q}_n(k; l, m) = \frac{2}{(k - l + 1)(m - k)} \left\{ \tilde{R}_n(l, m) - \tilde{R}_n(l, k) - \tilde{R}_n(k + 1, m) \right\}$$

$$- \frac{2}{(k - l + 1)(k - l)} \tilde{R}_n(l, k) - \frac{2}{(m - k)(m - k - 1)} \tilde{R}_n(k + 1, m).$$

(23)

Denote $R_n(a, b) := \tilde{R}_n([na] + 1, [nb])$ for any $0 \leq a < b \leq 1$. Therefore we have from (23)

$$\tilde{Q}_n(k; l, m) = \frac{2}{(k - l + 1)(m - k)} \left\{ R_n(a, b) - R_n(a, r) - R_n(r, b) \right\}$$

$$- \frac{2}{(k - l + 1)(k - l)} R_n(a, r) - \frac{2}{(m - k)(m - k - 1)} R_n(r, b).$$

(24)

Define $G_n(a, b) := \frac{1}{n\sqrt{V_0}} R_n(a, b)$.

**Theorem A.2.** Under Assumption 3.3 as $n, p \to \infty$, $\sup_{a,b \in [0,1]} |G_n(a,b)| = o_p(1)$.

The proof of Theorem A.2 is given in Section B. From (17) and (24), we have

$$\frac{\tilde{Q}_n(k; l, m)}{\sqrt{V_n(k; l, m)}} = \frac{1}{a_{k-l+1,m-k}\sqrt{V_0}} \times \left[ \frac{2}{(k - l + 1)(m - k)} \left\{ R_n(a, b) - R_n(a, r) - R_n(r, b) \right\} \right.$$

$$\left. - \frac{2}{(k - l)(k - l + 1)} R_n(a, r) + \frac{2}{(m - k)(m - k - 1)} R_n(r, b) \right].$$

Multiplying both the numerator and denominator above by $n^2$, it is not hard to see that as a consequence of Theorem A.2, we have

$$\sup_{a,r,b \in [0,1]} \left| \frac{\tilde{Q}_n([nr]; [na] + 1, [nb])}{\sqrt{V_n([nr]; [na] + 1, [nb])}} \right| = o_p(1) \quad \text{as } n, p \to \infty .$$

(25)

As a special case, putting $a = 0$ and $b = 1$, we get from (25)

$$\sup_{r \in [0,1]} \left| \frac{\tilde{Q}_n([nr]; 1, n)}{\sqrt{V_n([nr]; 1, n)}} \right| = o_p(1) \quad \text{as } n, p \to \infty .$$

(26)
Theorem A.3. Under Assumptions 3.2 and 3.3, as \( n, p \to \infty \),
\[
\sup_{a, r, b \in [0, 1]} \left| \frac{\hat{V}_n([nr] ; \lfloor na \rfloor + 1, \lfloor nb \rfloor)}{V_n([nr] ; \lfloor na \rfloor + 1, \lfloor nb \rfloor)} - 1 \right| = o_p(1).
\]

The proof of Theorem A.3 is given in Section B. As a special case, putting \( a = 0 \) and \( b = 1 \), we get from Theorem A.3 that
\[
\sup_{r \in [0, 1]} \left| \frac{\hat{V}_n([nr] ; 1, n)}{\tilde{V}_n([nr] ; 1, n)} - 1 \right| = o_p(1) \quad \text{as} \quad n, p \to \infty.
\]

With all the above, the proof of Theorem 3.1 can be completed as below.

Proof of Theorem 3.1. Combining (9) and (13) with (20) and (26) yields that
\[
\left\{ \hat{r}_n([nr]) \right\}_{r \in [0, 1]} \xrightarrow{d} G_0 \quad \text{in} \quad L^\infty ([0, 1]),
\]
as \( n, p \to \infty \). This equipped with (27) completes the proof of Theorem 3.1.

All along our derivations, we use the simple facts that for \( 0 < a \leq 1, \lfloor na \rfloor \sim n \) and
\[
\lim_{n \to \infty} \frac{\lfloor na \rfloor}{n} = \lim_{n \to \infty} \frac{na - \{na\}}{n} = a
\]
as \( n \to \infty \), since \( 0 \leq \{na\} < 1 \).

B Technical Appendix

Proof of Lemma 3.1. For the first part, simply note that some direct calculations yield
\[
S_k = \frac{1}{\sqrt{n}} \sum_{t=1}^{k} \left\{ \phi(X_t) - \frac{1}{n} \sum_{t=1}^{n} \phi(X_t) \right\} = \frac{1}{\sqrt{n}} \left\{ \sum_{t=1}^{k} \phi(X_t) - \frac{k}{n} \sum_{t=1}^{k} \phi(X_t) - \frac{k}{n} \sum_{t=k+1}^{n} \phi(X_t) \right\}
= \frac{k(n-k)}{n^{3/2}} \left\{ \frac{1}{k} \sum_{t=1}^{k} \phi(X_t) - \frac{1}{n-k} \sum_{t=k+1}^{n} \phi(X_t) \right\}.
\]

(29)
For the second part, note that following equation (6) in the main paper, the expression of $S_k$ in equation (29) and some elementary calculations, we can write

$$
\frac{n^3}{k^2(n-k)^2} \|S_k\|^2 = \frac{1}{k^2} \sum_{t,t'=1}^{k} \langle \phi(X_t), \phi(X_{t'}) \rangle_H + \frac{1}{(n-k)^2} \sum_{t,t'=k+1}^{n} \langle \phi(X_t), \phi(X_{t'}) \rangle_H
$$

$$
- \frac{2}{k(n-k)} \sum_{t=1 \atop t'=k+1}^{k} \sum_{t'=k+1}^{n} \langle \phi(X_t), \phi(X_{t'}) \rangle_H
$$

$$
= \frac{2}{k(n-k)} \sum_{t=1 \atop t'=k+1}^{n} \gamma(X_t, X_{t'}) - \frac{1}{k^2} \sum_{t,t'=1}^{k} \gamma(X_t, X_{t'}) - \frac{1}{(n-k)^2} \sum_{t,t'=k+1}^{n} \gamma(X_t, X_{t'}). \quad \diamond
$$

**Proof of Theorem A.1.** To establish the uniform weak convergence of $\sqrt{\frac{2}{n\sqrt{V_0}}} S_n(a, b)$, by Theorem 10.2 in Pollard (1990) we need to show

T1. the finite dimensional convergence, viz.

$$
\left( \sqrt{\frac{2}{n\sqrt{V_0}}} S_n(a_1, b_1), \ldots, \sqrt{\frac{2}{n\sqrt{V_0}}} S_n(a_s, b_s) \right) \overset{d}{\longrightarrow} \left( Q(a_1, b_1), \ldots, Q(a_s, b_s) \right)
$$

as $n, p \to \infty$ for fixed $0 \leq a_i < b_i \leq 1$, $1 \leq i \leq s$, and

T2. asymptotic stochastic equicontinuity of $\sqrt{\frac{2}{n\sqrt{V_0}}} S_n(a, b)$ on $[0, 1]^2$, viz. for any $x > 0$,

$$
\lim_{\delta \downarrow 0} \limsup_{n,p \to \infty} P \left( \sup_{\| (a, b) - (c, d) \| \leq \delta} \left| \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(a, b) - \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(c, d) \right| \right) = 0.
$$

To prove T1, we will consider the case $s = 2$ and the general case can be proved in a similar fashion. By Cramér-Wold theorem, it is equivalent to prove

$$
\alpha_1 \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(a_1, b_1) + \alpha_2 \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(a_2, b_2) \overset{d}{\longrightarrow} \alpha_1 Q(a_1, b_1) + \alpha_2 Q(a_2, b_2) \quad (30)
$$

for any fixed $\alpha_1, \alpha_2 \in \mathbb{R}$, as $n, p \to \infty$. As $0 \leq a_i < b_i \leq 1$, $i = 1, 2$, we consider the following three cases: i) $a_1 \leq a_2 \leq b_2 \leq b_1$, ii) $a_1 \leq a_2 \leq b_1 \leq b_2$, and iii) $a_1 \leq b_1 \leq a_2 \leq b_2$. Consider case (ii). We will prove T1 and T2 for this case and similar arguments can prove them for the other two cases.
Proof of T1. We can write

\[
\alpha_1 \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(a_1, b_1) + \alpha_2 \frac{\sqrt{2}}{n\sqrt{V_0}} S_n(a_2, b_2)
\]

\[
= \frac{\sqrt{2}}{n\sqrt{V_0}} \left\{ \alpha_1 \sum_{i=|na_1|+1}^{\lfloor nb_2 \rfloor - 1} H(X_i, X_j) + \alpha_2 \sum_{i=|na_2|+1}^{\lfloor nb_2 \rfloor - 1} H(X_i, X_j) \right\}
\]

\[
= \sum_{i=|na_1|+2}^{\lfloor nb_2 \rfloor} \bar{\xi}_{n,i},
\]

where

\[
\bar{\xi}_{n,i} := \frac{\sqrt{2}}{n\sqrt{V_0}} \left\{ \begin{align*}
\alpha_1 \xi_{1,i} & \quad \text{if } |na_1| + 2 \leq i \leq |na_2| + 1, \\
\alpha_1 \xi_{1,i} + \alpha_2 \xi_{2,i} & \quad \text{if } |na_2| + 2 \leq i \leq |nb_1|, \\
\alpha_2 \xi_{2,i} & \quad \text{if } |nb_1| + 1 \leq i \leq |nb_2|,
\end{align*} \right.
\]

with \( \xi_{1,i} = \sum_{j=|na_1|+1}^{i-1} H(X_i, X_j) \) and \( \xi_{2,i} := \sum_{j=|na_2|+1}^{i-1} H(X_i, X_j) \). Define \( F_i := \sigma(X_i, X_{i-1}, \ldots) \). By Theorem 3.2 and Corollary 3.1 in Hall and Heyde (1980), it suffices to show

P1. For each \( n \geq 1 \), \( \{ \sum_{i=2}^{\lfloor n b_2 \rfloor - |na_1|} \bar{\xi}_{n,i} | F_{i=2} \} \) is a sequence of zero mean, square integrable martingales;

P2. \( V_n := \sum_{i=1}^{\lfloor nb_2 \rfloor - |na_1|} \mathbb{P} \left[ \bar{\xi}_{n,i} \mid F_{i=1} \right] \xrightarrow{P} \alpha_1^2 (b_1 - a_1)^2 + \alpha_2^2 (b_2 - a_2)^2 + 2 \alpha_1 \alpha_2 (b_1 - a_2)^2 \), as \( n, p \to \infty \);

P3. \( \sum_{i=|na_1|+2}^{\lfloor nb_2 \rfloor} \mathbb{E} \left[ \bar{\xi}_{n,i}^4 \right] \to 0 \), as \( n, p \to \infty \).

From Theorem 3.2 in Hall and Heyde (1980), the variance of \( \alpha_1 Q(a_1, b_1) + \alpha_2 Q(a_2, b_2) \) should be \( \text{plim}_{n,p \to \infty} V_n \) as in P2. From there, it is intuitive that

\[
\text{cov} \left( Q(a_1, b_1), Q(a_2, b_2) \right) = (b_1 \wedge b_2 - a_1 \vee a_2)^2 1 \left( b_1 \wedge b_2 > a_1 \vee a_2 \right).
\]

To show P1, it is easy to see that \( \bar{\xi}_{n,|na_1|+1} \) is square integrable, \( \mathbb{E} \left( \bar{\xi}_{n,|na_1|+1} \right) = 0 \) and \( F_2 \subseteq F_I \). To prove P3, note that using the power mean inequality

\[
\left| \sum_{i=1}^{n} a_i \right|^r \leq n^{r-1} \sum_{i=1}^{n} |a_i|^r
\]
for \( a_i \in \mathbb{R}, 1 \leq i \leq n, n \geq 2 \) and \( r > 1 \), we can write

\[
\sum_{i=[na_1]+2}^{[nb_2]} E [\tilde{\xi}_{n,i}^4] = \sum_{i=[na_1]+2}^{[na_2]+1} E [\tilde{\xi}_{n,i}^4] + \sum_{i=[na_2]+2}^{[nb_1]} E [\tilde{\xi}_{n,i}^4] + \sum_{i=[nb_1]+1}^{[nb_2]} E [\tilde{\xi}_{n,i}^4]
\]

\[
= \sum_{i=[na_1]+2}^{[na_2]+1} E \left( \left( \frac{\sqrt{2}}{n V_0} \xi_{1,i} \right)^4 \right) + \sum_{i=[na_2]+2}^{[nb_1]} E \left( \left( \frac{\sqrt{2}}{n V_0} \xi_{1,i} + \frac{\sqrt{2}}{n V_0} \xi_{2,i} \right)^4 \right)
\]

\[
+ \sum_{i=[nb_1]+1}^{[nb_2]} E \left( \left( \frac{\sqrt{2}}{n V_0} \xi_{2,i} \right)^4 \right)
\]

\[
\lesssim \frac{1}{n^4 V_0^2} \left( \alpha_1^4 \sum_{i=[na_1]+2}^{[nb_1]} E [\xi_{1,i}^4] + \alpha_2^4 \sum_{i=[na_2]+2}^{[nb_2]} E [\xi_{2,i}^4] \right).
\]

(34)

We have essentially used the definitions in (32) in the above calculations. Now for the first summand in the right hand side of (34), using (32), we have

\[
\frac{1}{n^4 V_0^2} \sum_{i=[na_1]+2}^{[nb_1]} E [\xi_{1,i}^4] = \frac{1}{n^4 V_0^2} \sum_{i=[na_1]+2}^{[nb_1]} E \left[ \left( \sum_{j=[na_1]+1}^{i-1} H(X_i, X_j) \right)^4 \right]
\]

\[
= \frac{1}{n^4 V_0^2} \sum_{i=[na_1]+2}^{[nb_1]} \left\{ \sum_{j=[na_1]+1}^{i-1} E[H^4(X_i, X_j)] + 3 \sum_{[na_1]+1 \leq j_1 \neq j_2 \leq i-1} E[H^2(X_i, X_{j_1}) H^2(X_i, X_{j_2})] \right\}
\]

\[
= \frac{1}{n^4} O \left( \frac{n^2 E[H^4(X, X')] + n^3 E[H^2(X, X') H^2(X, X')]}{(E[H^2(X, X')])^2} \right).
\]

(35)

Similar expressions hold for the second summand in the right hand side of (34). With this, it is easy to see that under Assumption 3.2

\[
\sum_{i=[na_1]+2}^{[nb_2]} E [\tilde{\xi}_{n,i}^4] = o(1) \quad \text{as } n, p \to \infty ,
\]

which completes the proof of P3. To prove P2, write

\[
V_n = \sum_{i=2}^{[nb_2]-[na_1]} E \left[ \tilde{\xi}_{n,[na_1]+i}^2 | \mathcal{F}_{[na_1]+i-1} \right] = \sum_{i=[na_1]+2}^{[nb_2]} E \left[ \tilde{\xi}_{n,i}^2 | \mathcal{F}_{i-1} \right]
\]

(36)
where we have simply substituted \( l = [na_1] + i \). From (36) we have

\[
V_n = \sum_{l=\lfloor na_1 \rfloor + 1}^{\lfloor nb_1 \rfloor} \mathbb{E} \left[ \left( \frac{\sqrt{2}}{n\sqrt{V_0}} \alpha_1 \xi_{1,l} \right)^2 \right] F_{l-1} + \sum_{l=\lfloor na_2 \rfloor + 2}^{\lfloor nb_2 \rfloor} \mathbb{E} \left[ \left( \frac{\sqrt{2}}{n\sqrt{V_0}} \alpha_2 \xi_{2,l} \right)^2 \right] F_{l-1}
\]

\[
= \frac{2}{n^2 V_0} \left( \alpha_1^2 \sum_{l=\lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \mathbb{E} \left[ \xi_{1,l}^2 \right] F_{l-1} + \alpha_2^2 \sum_{l=\lfloor na_2 \rfloor + 2}^{\lfloor nb_2 \rfloor} \mathbb{E} \left[ \xi_{2,l}^2 \right] F_{l-1} \right) + 2 \alpha_1 \alpha_2 \sum_{l=\lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \mathbb{E} \left[ \xi_{1,l} \xi_{2,l} \right] F_{l-1}
\]

\[
= \alpha_1^2 V_1 n + \alpha_2^2 V_2 n + 2 \alpha_1 \alpha_2 V_3 n,
\]

where

\[
V_1 n = \frac{2}{n^2 V_0} \sum_{l=\lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \mathbb{E} \left[ \xi_{1,l}^2 \right] F_{l-1},
\]

\[
V_2 n = \frac{2}{n^2 V_0} \sum_{l=\lfloor na_2 \rfloor + 2}^{\lfloor nb_2 \rfloor} \mathbb{E} \left[ \xi_{2,l}^2 \right] F_{l-1},
\]

\[
V_3 n = \frac{2}{n^2 V_0} \sum_{l=\lfloor na_2 \rfloor + 2}^{\lfloor nb_1 \rfloor} \mathbb{E} \left[ \xi_{1,l} \xi_{2,l} \right] F_{l-1}.
\]

Using the definition of \( \xi_{1,l} \) from (32), we can write

\[
V_1 n = \frac{2}{n^2 V_0} \sum_{l=\lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \sum_{j_1,j_2=\lfloor na_1 \rfloor + 1}^{l-1} \mathbb{E} \left[ H(X_l, X_{j_1}) H(X_l, X_{j_2}) \right] F_{l-1},
\]

and therefore

\[
\mathbb{E} [V_1 n] = \frac{2}{n^2 V_0} \sum_{l=\lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \sum_{j=\lfloor na_1 \rfloor + 1}^{l-1} \mathbb{E} \left[ H^2(X_l, X_j) \right],
\]

as \( \mathbb{E} [H(X_l, X_{j_1}) H(X_l, X_{j_2})] = 0 \) for \( j_1 \neq j_2 \). Using the fact that \( V_0 = \mathbb{E} [H^2(X, X')] \), some straightforward
Following the proof of Lemma D.1 in the supplementary materials of Chakraborty and Zhang (2019), we have

calculations yield

\[
\mathbb{E} [V_{1n}] = \frac{2}{n^2 V_0} \sum_{l = \lfloor na_1 \rfloor + 1}^{\lfloor nb_1 \rfloor} \mathbb{E} [H^2(X, X')] = \frac{2}{n^2} \left(\frac{|nb_1| - |na_1|}{2}\right)
\]

\[
= \frac{1}{n^2} (|nb_1| - |na_1|) (|nb_1| - |na_1| - 1)
\]

\[
\rightarrow (b_1 - a_1)^2,
\]

as \( n \to \infty \). Define \( L_l(j_1, j_2) := \mathbb{E} \left[ H(X_l, X_{j_1}) H(X_l, X_{j_2}) \mid F_{l-1} \right] \). Then from (39) we can write

\[
V_{1n} = \frac{2}{n^2 V_0} \sum_{l = \lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \sum_{j_1, j_2 = \lfloor na_1 \rfloor + 1}^{\lfloor nb_1 \rfloor} L_l(j_1, j_2),
\]

and therefore

\[
\text{var} (V_{1n}) = \frac{4}{n^4 V_0^2} \sum_{l, l' = \lfloor na_1 \rfloor + 2}^{\lfloor nb_1 \rfloor} \sum_{j_1, j_2 = \lfloor na_1 \rfloor + 1}^{\lfloor nb_1 \rfloor} \sum_{j_1', j_2' = \lfloor na_1 \rfloor + 1}^{\lfloor nb_1 \rfloor} \text{cov} \left( L_l(j_1, j_2), L_{l'}(j_1', j_2') \right).
\]

Following the proof of Lemma D.1 in the supplementary materials of Chakraborty and Zhang (2019), we have \( \mathbb{E} L_l(j_1, j_2) = 0 \) for \( j_1 \neq j_2 \), and

\[
\mathbb{E} \left[ L_l(j_1, j_2) L_{l'}(j_1', j_2') \right] =
\begin{cases}
\mathbb{E} \left[ H^2(X_l, X_{j_1}) H^2(X_{l'}, X_{j_1}) \right] & \text{if } j_1 = j_2 = j_1' = j_2', \\
\mathbb{E} \left[ H(X_l, X_{j_1}) H(X_l, X_{j_2}) H(X_{l'}, X_{j_1}) H(X_{l'}, X_{j_2}) \right] & \text{if } j_1 = j_1' \neq j_2 = j_2' \text{ or } j_1 = j_2' \neq j_1' = j_2, \\
\mathbb{E} \left[ H^2(X_l, X_{j_1}) \right] \mathbb{E} \left[ H^2(X_{l'}, X_{j_1'}) \right] & \text{if } j_1 = j_2 \neq j_1' = j_2'.
\end{cases}
\]

where the above expression holds for \( l = l' \) as well. Therefore

\[
\text{var} (V_{1n}) = \frac{4}{n^4 V_0^2} \left\{ \sum_{l = l'}^{l-1} \left( \mathbb{E} \left[ H^2(X_l, X_{j_1}) \right] \mathbb{E} \left[ H^2(X_{l'}, X_{j_1'}) \right] \right) \right. \\
+ 2 \sum_{|na_1| + 1 \leq j_1 \neq j_2 \leq l-1} \mathbb{E} \left[ H(X_l, X_{j_1}) H(X_l, X_{j_2}) H(X_{l'}, X_{j_1}) H(X_{l'}, X_{j_2}) \right] \right\}
\]

\[
+ 2 \sum_{|na_1| + 2 \leq l < l' \leq |nb_1|} \left\{ \sum_{j_1 = |na_1| + 1}^{l-1} \mathbb{E} \left[ H^2(X_l, X_{j_1}) \right] \mathbb{E} \left[ H^2(X_{l'}, X_{j_1}) \right] \right. \\
+ 2 \sum_{|na_1| + 1 \leq j_1 \neq j_2 \leq l-1} \mathbb{E} \left[ H(X_l, X_{j_1}) H(X_l, X_{j_2}) H(X_{l'}, X_{j_1}) H(X_{l'}, X_{j_2}) \right] \right\}.
\]
This implies
\[\var(V_{1n}) = \frac{1}{V_0} O\left(n^{-1} \mathbb{E} \left[H^2(X, X') H^2(X, X'')\right] + \mathbb{E} \left[H(X, X'') H(X', X'') H(X, X')\right]\right)\]
\(\Rightarrow \var(V_{1n}) = o(1),\)

as \(n, p \to \infty,\) under Assumption 3.2. Combining (41) and (42), we get
\[\mathbb{E} \left[(V_{1n} - (b_1 - a_1)^2)^2\right] = \var(V_{1n}) + \left\{\mathbb{E}[V_{1n}] - (b_1 - a_1)^2\right\}^2 = o(1),\]

which combined with Chebyshev's inequality implies
\[V_{1n} \xrightarrow{P} (b_1 - a_1)^2 \quad \text{as} \quad n, p \to \infty.\] (43)

Likewise it can be shown that as \(n, p \to \infty,\)
\[V_{2n} \xrightarrow{P} (b_2 - a_2)^2 \quad \text{and} \quad V_{3n} \xrightarrow{P} (b_1 - a_2)^2.\] (44)

Combining (43) and (44), we get from (37)
\[V_n \xrightarrow{P} \alpha_1^2 (b_1 - a_1)^2 + \alpha_2^2 (b_2 - a_2)^2 + 2 \alpha_1 \alpha_2 (b_1 - a_2)^2.\] (45)

This completes the proof of P2, and thereby the proof of T1, i.e., the finite dimensional convergence. ♦

**Proof of T2.** Denote \(u = (a, b)\) and \(v = (c, d).\) Also define \(W_n(u) := \frac{\sqrt{2}}{n \sqrt{V_0}} S_n(u)\) for \(u \in [0, 1]^2.\) To prove the stochastic equicontinuity of \(W_n(u)\) for \(u \in [0, 1]^2,\) we need to show for any \(\epsilon > 0\)

\[\lim_{\delta \downarrow 0} \lim_{n, p \to \infty} \mathbb{E} \sup_{u, v \in [0, 1]^2, \kappa(u, v) < \delta} |W_n(u) - W_n(v)| = 0,\]

where \(([0, 1]^2, \kappa)\) is compact. By Theorem A.8 in Li and Racine (2007), it suffices to show that \(\forall u, v \in [0, 1]^2,\)

\[\mathbb{E} |W_n(u) - W_n(v)|^\alpha \lesssim \kappa^\gamma(u, v)\] (46)

for some \(\alpha > 0\) and \(\gamma > 1.\) For our purpose, we choose \(\kappa(u, v) = \|u - v\|_1^{1/2}\) for \(u, v \in [0, 1]^2.\) Note that \([0, 1]^2 \subseteq \mathbb{R}^2\) is compact (closed and bounded) with respect to the metric \(\rho(u, v) = \|u - v\|_1.\) It is easy to verify that \([0, 1]^2\) is closed and bounded (and hence compact) with respect to the metric \(\kappa(u, v) = \rho^{1/2}(u, v)\) as well.

Choosing \(\alpha = 2\) and \(\gamma = 2,\) we will prove that \(\forall u, v \in [0, 1]^2,\)

\[\mathbb{E} |W_n(u) - W_n(v)|^2 \lesssim \kappa^2(u, v),\] (47)
which will complete the proof. Towards that end, consider the case $a < c < d < b$. We will show that \[47\] holds in this case, and similar arguments will do the job for the other cases. Observe that

\[
W_n(u) - W_n(v) = \frac{\sqrt{2}}{nV_0} S_n(a, b) - \frac{\sqrt{2}}{nV_0} S_n(c, d)
\]

\[
= \frac{\sqrt{2}}{nV_0} \left[ \sum_{i=[na]+2}^{[nc]-1} \sum_{j=[na]+1}^{i-1} H(X_i, X_j) - \sum_{i=[nc]+2}^{[nc]-1} \sum_{j=[nc]+1}^{i-1} H(X_i, X_j) \right]
\]

\[
= \frac{\sqrt{2}}{nV_0} \left[ \sum_{i=[nd]+1}^{[nc]} \sum_{j=[na]+1}^{i-1} H(X_i, X_j) + \sum_{i=[nb]}^{[nc]} \sum_{j=[na]+1}^{i-1} H(X_i, X_j) \right]
\]

\[
= I + II + III + IV + V.
\]

By power mean inequality,

\[
(I + II + III + IV + V)^2 \leq I^2 + II^2 + III^2 + IV^2 + V^2.
\] (49)

Now

\[
E[I^2] = \frac{2}{n^2V_0} \sum_{i_1, i_2=[na]+2}^{[nc]-1} \sum_{j_1=[na]+1}^{i_1-1} \sum_{j_2=[na]+1}^{i_2-1} E[H(X_{i_1}, X_{j_1}) H(X_{i_2}, X_{j_2})].
\]

Clearly $E[H(X_{i_1}, X_{j_1}) H(X_{i_2}, X_{j_2})] = 0$ if the cardinality of the set $\{i_1, j_1\} \cap \{i_2, j_2\}$ is 0 or 1. Therefore we have

\[
E[I^2] = \frac{2}{n^2V_0} \sum_{i=[na]+2}^{[nc]+1} \sum_{j=[na]+1}^{i-1} E[H^2(X_i, X_j)] = \frac{2}{n^2V_0} \sum_{[na]+1 \leq j < i \leq [nc]} V_0
\]

\[
= \frac{1}{n^2} ([nc] - [na]) ([nc] - [na] - 1).
\] (50)

Note that

\[
|nc| - |na| - 1 \leq nc - na + na - |na| - 1
\]

\[
= n(c-a) + \{|na| - 1\}
\]

\[
\leq n(c-a);
\] (51)
as \( \{na\} \leq 1 \). Therefore we have from \([50]\) and \([51]\)

\[
\mathbb{E}[I^2] \lesssim c - a .
\]

Likewise it can be shown that

\[
\mathbb{E}[V^2] \lesssim b - d .
\]

Now

\[
\mathbb{E}[II^2] = \frac{2}{n^2V_0} \sum_{i_1,i_2=\lfloor nc \rfloor + 1}^{\lfloor nd \rfloor} \sum_{j_1,j_2=\lfloor na \rfloor + 1}^{\lfloor nc \rfloor} \mathbb{E}[H(X_{i_1},X_{j_1})H(X_{i_2},X_{j_2})]
\]

\[
= \frac{2}{n^2V_0} \sum_{i=\lfloor nc \rfloor + 1}^{\lfloor nd \rfloor} \sum_{j=\lfloor na \rfloor + 1}^{\lfloor nc \rfloor} \mathbb{E}[H^2(X_i,X_j)]
\]

\[
= \frac{2}{n^2} \left( \lfloor nd \rfloor - \lfloor nc \rfloor \right) \left( \lfloor nc \rfloor - \lfloor na \rfloor \right)
\]

\[
\lesssim \frac{1}{n} \left[ n(c - a) + 1 \right]
\]

\[
\lesssim c - a .
\]

Similarly it can be shown that

\[
\mathbb{E}[III^2] \lesssim c - a \quad \text{and} \quad \mathbb{E}[IV^2] \lesssim b - d .
\]

Combining \([52]\)-\([55]\) with \([48]\) and \([49]\), we get

\[
\mathbb{E} \left[ \left| W_n(u) - W_n(v) \right|^2 \right] \lesssim (c - a) + (b - d) = \|u - v\|_1 = \kappa^2(u, v) .
\]

This proves \([47]\) and thereby completes the proof of T2.

Combining the above results, we complete the proof of Theorem A.1.

\begin{proof}[Proof of Theorem A.2] Again consider the subset \([0,1]^2 \subseteq \mathbb{R}^2\) equipped with the metric \(\kappa(u, v) = \|u - v\|_1^{1/2}\) for \(u, v \in [0, 1]^2\). By Theorem 1 in Andrews (1992), we essentially need to show

A1. \([0,1]^2\) is totally bounded with respect to the metric \(\kappa\);

A2. Pointwise convergence : \(G_n(u) \xrightarrow{P} 0 \quad \forall u \in [0, 1]^2\) as \(n, p \to \infty\);

A3. Asymptotic stochastic equicontinuity : for any \(\epsilon > 0\),

\[
\lim_{\delta \downarrow 0} \lim_{n,p \to \infty} \sup_{P} \left( \sup_{u,v \in [0,1]^2} |G_n(u) - G_n(v)| \right) \leq \delta = 0 .
\]

\end{proof}
As $[0, 1]^2$ is compact with respect to the metric $\kappa$, it is therefore totally bounded. To see A2, note that for fixed $u \in [0, 1]^2$, using Chebyshev’s inequality we have for any $\epsilon > 0$

$$P(\{|G_n(u)| > \epsilon\}) \leq \frac{1}{\epsilon^2} \mathbb{E} G_n^2(u) = \frac{1}{n^2 \epsilon^2 V_0} \mathbb{E} R_n^2(a, b).$$  \hspace{1cm} (56)$$

Recalling that $R_n(a, b) = \tilde{R}_n((an) + 1, [bn])$ and the definition of $\tilde{R}_n(k, m)$, it is not hard to verify that

$$R_n^2(a, b) = \sum_{[na]+1 \leq i_1 < i_2 \leq [nb]} \sum_{[na]+1 \leq i'_1 < i'_2 \leq [nb]} \tau^2 R(X_{i_1}, X_{i_2}) R(X_{i'_1}, X_{i'_2}).$$

Therefore by Hölder’s inequality, we have

$$\mathbb{E} R_n^2(a, b) \leq \sum_{[na]+1 \leq i_1 < i_2 \leq [nb]} \sum_{[na]+1 \leq i'_1 < i'_2 \leq [nb]} \tau^2 \left( \mathbb{E} R^2(X_{i_1}, X_{i_2}) \right)^{1/2} \left( \mathbb{E} R^2(X_{i'_1}, X_{i'_2}) \right)^{1/2}$$

$$= \tau^2 \left[ \frac{1}{2} (|nb| - [na]) (|nb| - [na] - 1) \left( \mathbb{E} R^2(X, X') \right)^{1/2} \right]^2$$

$$= O\left( n^4 \tau^2 \mathbb{E} R^2(X, X') \right)$$

$$= O\left( n^4 \left( \tau^4 \mathbb{E} R^4(X, X') \right)^{1/2} \right).$$  \hspace{1cm} (57)$$

Combining (56) and (57), we get

$$P(\{|G_n(u)| > \epsilon\}) = O\left( \frac{n^2}{\epsilon^2 \mathbb{E} H^2(X, X')} \left( \tau^4 \mathbb{E} R^4(X, X') \right)^{1/2} \right) = O\left( \frac{1}{\epsilon^2} \left[ \frac{n^4 \tau^4 \mathbb{E} R^4(X, X')}{(\mathbb{E} H^2(X, X'))^2} \right]^{1/2} \right).$$  \hspace{1cm} (58)$$

Under Assumption 3.3, it is easy to see from (58) that

$$P(\{|G_n(u)| > \epsilon\}) = o(1),$$

which implies $G_n(u) \xrightarrow{P} 0$ for any fixed $u \in [0, 1]^2$ as $n, p \to \infty$. This proves A2.

Finally to prove A3, again by Theorem A.8 in Li and Racine (2007), it will suffice to show that $\forall u, v \in [0, 1]^2$

$$\mathbb{E} \left[ |G_n(u) - G_n(v)|^2 \right] \lesssim \kappa^2(u, v).$$  \hspace{1cm} (59)$$

Similar to the proof of T2 before in the proof of Theorem A.1 we will show that (59) holds in the case
\[ a < c < d < b. \] Similar arguments can prove (59) for other cases. Similar to the proof of T2, now we have

\[ G_n(u) - G_n(v) = \frac{1}{n\sqrt{V_0}} R_n(a, b) - \frac{1}{n\sqrt{V_0}} R_n(c, d) \]

\[ = \frac{\tau}{n\sqrt{V_0}} \left\{ \sum_{i=|na|+1}^{nc} \sum_{j=|na|+1}^{nc} R(X_i, X_j) - \sum_{i=|na|+1}^{nc} \sum_{j=|na|+1}^{nc} R(X_i, X_j) \right\} \]

\[ = \frac{\tau}{n\sqrt{V_0}} \left\{ \sum_{i=|na|+1}^{nc} \sum_{j=|na|+1}^{nc} R(X_i, X_j) + \sum_{i=|na|+1}^{nc} \sum_{j=|na|+1}^{nc} R(X_i, X_j) \right\} \]

\[ =: I_G + II_G + III_G + IV_G + V_G. \]

By power mean inequality,

\[ (I_G + II_G + III_G + IV_G + V_G)^2 \lesssim I_G^2 + II_G^2 + III_G^2 + IV_G^2 + V_G^2. \] (61)

Now

\[ \mathbb{E}[I_G^2] = \frac{\tau^2}{n^2 V_0} \sum_{i_1, i_2=|na|+1}^{nc} \sum_{j_1=|na|+1}^{nc} \sum_{j_2=|na|+1}^{nc} \mathbb{E} \left[ R(X_{i_1}, X_{j_1}) R(X_{i_2}, X_{j_2}) \right]. \] (62)

Again using Hölder’s inequality and similar arguments as used in deriving (57), we get from (62)

\[ \mathbb{E}[I_G^2] = \frac{\tau^2}{n^2 V_0} \left[ \sum_{i=|na|+1}^{nc} \sum_{j=|na|+1}^{nc} \mathbb{E} R^2(X_i, X_j) \right]^{1/2} \]

\[ = \frac{\tau^2}{n^2 V_0} \frac{(|nc| - |na|)^2 (|nc| - |na| - 1)^2}{4 n^2} n^2 \mathbb{E} R^2(X, X'). \] (63)

Using the fact that \(|nc| - |na| - 1 \leq n(c - a), (c - a)^2 \leq (c - a)\) and Hölder’s inequality, we get from (63)

\[ \mathbb{E}[I_G^2] \lesssim (c - a) \left( \frac{n^2 \tau^2 \mathbb{E} R^2(X, X')}{\mathbb{E} H^2(X, X')} \right) \leq (c - a) \left( \frac{n^2 \tau^2 \left( \mathbb{E} R^4(X, X') \right)^{1/2}}{\mathbb{E} H^2(X, X')} \right) \]

\[ \leq (c - a) \left( \frac{n^4 \tau^4 \mathbb{E} R^4(X, X')}{\left( \mathbb{E} H^2(X, X') \right)^2} \right)^{1/2} \].
Under Assumption 3.3, \( n^4 \tau^4 \frac{E R^4(X,X^\prime)}{E H^2(X,X^\prime)\hat{I}} = o(1) \) as \( n,p \to \infty \), and hence \( n^4 \tau^4 \frac{E R^4(X,X^\prime)}{E H^2(X,X^\prime)} \) must be a bounded sequence in \( n \) and \( p \). Therefore we have from (64)

\[
\mathbb{E}[I^3_G] \lesssim c - a.
\]

Likewise it can be shown that

\[
\mathbb{E}[II^3_G] \lesssim c - a, \quad \mathbb{E}[III^3_G] \lesssim c - a, \quad \mathbb{E}[IV^3_G] \lesssim b - d, \quad \mathbb{E}[V^3_G] \lesssim b - d.
\]

Combining (65)-(66) with (60) and (61), we get

\[
\mathbb{E}|G_n(u) - G_n(v)|^2 \lesssim (c - a) + (b - d) = \|u - v\|_1 = \kappa^2(u,v).
\]

This proves (59) and thereby completes the proof of A3 and hence the theorem. \( \diamond \)

**Proof of Theorem A.3.** Clearly it suffices to prove

\[
\sup_{a,r,b \in [0,1]} \left| \frac{\hat{V}_{n,\eta}(\lfloor nr \rfloor; \lfloor na \rfloor + 1, \lfloor nb \rfloor)}{\hat{V}_{n,\eta}(\lfloor nr \rfloor; \lfloor na \rfloor + 1, \lfloor nb \rfloor)} - 1 \right| = o_p(1)
\]

as \( n,p \to \infty \) for \( \eta = 1, 2, 3 \). We will prove it for \( \eta = 2 \) and other cases can be proved in a similar fashion. Denote \( \omega(n; a, r) := (\lfloor nr \rfloor - \lfloor na \rfloor) (\lfloor nr \rfloor - \lfloor na \rfloor - 3) \). From equations (17) and (18), we can write

\[
\frac{\hat{V}_{n,2}(\lfloor nr \rfloor; \lfloor na \rfloor + 1, \lfloor nb \rfloor)}{\hat{V}_{n,2}(\lfloor nr \rfloor; \lfloor na \rfloor + 1, \lfloor nb \rfloor)} = \frac{4}{\hat{V}_0} \hat{D}^2_{n; a + 1; b} = \frac{8}{\hat{V}_0 \omega(n; a, r)} \sum_{i=\lfloor na \rfloor + 2}^{\lfloor nr \rfloor} \sum_{j=\lfloor na \rfloor + 1}^{i-1} \hat{A}^2_{i,j},
\]

where \( A_{i,j} = \gamma(X_i, X_j) \) and \( \hat{A} \) is the U-centered version of \( A \). The last equality above in (68) follows from the definition of \( \hat{D}^2_{1:k} \) in Section 2.2.

Define \( C_n(a, r) := 8 \sum_{i=\lfloor na \rfloor + 2}^{\lfloor nr \rfloor} \sum_{j=\lfloor na \rfloor + 1}^{i-1} \hat{A}^2_{i,j}. \) Then we need to prove that

\[
\sup_{a,r \in [0,1]} \left| \frac{n^2}{\omega(n; a, r)} \left( \frac{1}{n^2 \hat{V}_0} C_n(a, r) - \frac{\omega(n; a, r)}{n^2} \right) \right| = o_p(1)
\]

as \( n,p \to \infty \). Define \( J_n(a, r) := \frac{1}{n^2 \hat{V}_0} C_n(a, r) \) and \( \tilde{J}_n(a, r) := J_n(a, r) - \frac{\omega(n; a, r)}{n^2} \). Note that if we can prove

\[
\sup_{a,r \in [0,1]} |\tilde{J}_n(a, r)| = o_p(1)
\]

as \( n,p \to \infty \), then (69) will follow by Slutsky’s theorem.

Denote \( u = (a, r) \) and \( u' = (a', r') \). Consider the subset \([0,1]^2 \subseteq \mathbb{R}^2\) equipped with the metric \( \bar{\kappa}(u, u') := \|u - u'\| \) for \( u, u' \in [0,1]^2 \). By Theorem 1 in Andrews (1992), it suffices to show
B1. \([0, 1]^2\) is totally bounded with respect to the metric \(\overline{\kappa}\);

B2. Pointwise convergence : \(\overline{J}_n(u) \xrightarrow{P} 0 \quad \forall u \in [0, 1]^2\) as \(n, p \to \infty\);

B3. Asymptotic stochastic equicontinuity : for any \(\epsilon > 0\),

\[
\lim_{\delta \downarrow 0} \limsup_{n, p \to \infty} P \left( \sup_{u, u' \in [0, 1]^2, \overline{\kappa}(u, u') \leq \delta} \left| \overline{J}_n(u) - \overline{J}_n(u') \right| \right) = 0.
\]

To argue B1, note that \([0, 1]^2 \subseteq \mathbb{R}^2\) is compact (closed and bounded) with respect to the \(l_2\) distance. It is easy to check that \([0, 1]^2\) is compact (and therefore totally bounded) with respect to the metric \(\overline{\kappa}\) as well. B2 is equivalent to showing \(\omega(n; a, r) C_n C^*_n \xrightarrow{p} 1\) as \(n, p \to \infty\). The proof of B2 will follow similar lines of Lemma D.4 in the supplementary materials of Chakraborty and Zhang (2019), which essentially proves the pointwise convergence result under Assumptions 3.2 and 3.3.

Finally to prove B3, again by Theorem A.8 in Li and Racine (2007) it will suffice to show that \(\forall u, u' \in [0, 1]^2\)

\[
\mathbb{E} \left| \overline{J}_n(u) - \overline{J}_n(u') \right|^2 \lesssim \kappa^2(u, v). \tag{71}
\]

Similar to the proof of T2 earlier in the proof of Theorem A.1, we will show that (71) holds in the case \(a < a' < r' < r\). Similar arguments can prove (71) for other cases. Note that using triangle inequality and power mean inequality, we can write

\[
\left| \overline{J}_n(u) - \overline{J}_n(u') \right|^2 \lesssim \left| J_n(u) - J_n(u') \right|^2 + \left| \frac{\omega(n; a, r)}{n^2} \right|^2 - \frac{\omega(n; a', r')}{n^2} \right|^2. \tag{72}
\]

For \(a_i, b_i \in \mathbb{R}\) with \(|a_i|, |b_i| \leq 1\) for \(1 \leq i \leq n\), the product comparison lemma (Lemma 9.7.1 in Resnick, 1999) yields

\[
\left| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right| \leq \sum_{i=1}^n |a_i - b_i|. \tag{73}
\]

This yields

\[
\left| \frac{\omega(n; a, r)}{n^2} \right|^2 - \frac{\omega(n; a', r')}{n^2} \right|^2 \leq \left( \frac{2}{n} \left( \left| \frac{nr-r'}{n^2} \right| + \left| \frac{na-n'a}{n^2} \right| \right) \right)^2 \tag{74}
\]

\[
\lesssim \frac{1}{n^2} \left( 1 + 2 \left| \frac{nr-2r'}{n^2} \right| + \left| \frac{na-n'a}{n^2} \right| \right),
\]

where we have used the product comparison lemma and power mean inequality to get the first and the second inequalities, respectively. Following (51), we can write

\[
|nr - nr'| \leq 1 + n(r - r') \quad \text{and} \quad |na' - na| \leq 1 + n(a' - a).
\]
With this and using power mean inequality once again, we have from (74)

$$\left| \frac{\omega(n; a, r)}{n^2} - \frac{\omega(n; a', r')}{n^2} \right| \lesssim (a - a')^2 + (r - r')^2 = \|u - u'\|^2 = \tilde{\kappa}^2(u, u').$$  \hfill (75)

Similar to equation (60) in the proof of Theorem A.2, we have

$$J_n(u) - J_n(u') = \frac{1}{n^2 V_0} C_n(a, r) - \frac{1}{n^2 V_0} C_n(a', r')$$

\begin{align*}
&= \frac{8}{n^2 V_0} \left\{ \sum_{i=\lfloor n a \rfloor + 2}^{\lfloor n a' \rfloor} \sum_{j=\lfloor n a \rfloor + 1}^{\lfloor n a' \rfloor} \bar{A}_{i,j}^2 - \sum_{i=\lfloor n a' \rfloor + 2}^{\lfloor n a' \rfloor + 1} \sum_{j=\lfloor n a' \rfloor + 1}^{\lfloor n a' \rfloor + 1} \bar{A}_{i,j}^2 \right\} \\
&= \frac{8}{n^2 V_0} \left\{ \sum_{i=\lfloor n a \rfloor + 2}^{\lfloor n a' \rfloor} \sum_{j=\lfloor n a \rfloor + 1}^{\lfloor n a' \rfloor} \bar{A}_{i,j}^2 + \sum_{i=\lfloor n a' \rfloor + 1}^{\lfloor n a' \rfloor + 1} \sum_{j=\lfloor n a \rfloor + 1}^{\lfloor n a' \rfloor + 1} \bar{A}_{i,j}^2 + \sum_{i=\lfloor n r \rfloor + 1}^{\lfloor n r' \rfloor} \sum_{j=\lfloor n r \rfloor + 1}^{\lfloor n r' \rfloor + 1} \bar{A}_{i,j}^2 \right\} \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{align*}  \hfill (76)

By power mean inequality,

$$(J_1 + J_2 + J_3 + J_4 + J_5)^2 \lesssim J_1^2 + J_2^2 + J_3^2 + J_4^2 + J_5^2,$$  \hfill (77)

and therefore

$$\mathbb{E} \left| J_n(u) - J_n(u') \right|^2 \lesssim \mathbb{E} \left[ J_1^2 \right] + \mathbb{E} \left[ J_2^2 \right] + \mathbb{E} \left[ J_3^2 \right] + \mathbb{E} \left[ J_4^2 \right] + \mathbb{E} \left[ J_5^2 \right].$$  \hfill (78)

Consider the term $J_1$. Clearly

$$\mathbb{E} \left[ J_1^2 \right] = \left( \mathbb{E} \left[ J_1 \right] \right)^2 + \text{var} \left( J_1 \right).$$  \hfill (79)

Lemma D.3 in the supplementary materials of Chakraborty and Zhang (2019) essentially proves that under Assumptions 3.2 and 3.3 var $ \left( J_1 \right) = o(1)$ as $n, p \to \infty$. Following (68), it is not hard to see that

$$J_1 = \frac{\omega(n; a, a')}{n^2} \frac{\tilde{V}_{n,2}(\lfloor na' \rfloor; \lfloor na \rfloor + 1, \lfloor nr \rfloor)}{\tilde{V}_{n,2}(\lfloor na' \rfloor; \lfloor na \rfloor + 1, \lfloor nr \rfloor)}.$$

Following (28) and the proof of Lemma D.2 in the supplementary materials of Chakraborty and Zhang (2019), it can be verified that under Assumption 3.3 as $n, p \to \infty$

$$\frac{1}{(a' - a)^2} \mathbb{E} \left[ J_1 \right] \to 1,$$  \hfill (80)

43
i.e., $\mathbb{E}[J_1]$ and hence $\mathbb{E}[J_1^2]$ is a bounded sequence in $n$ and $p$. Therefore we can write

$$\mathbb{E}[J_1^2] \lesssim (a - a')^4 \leq (a - a')^2.$$  \hspace{1cm} (81)

In the same way we can obtain

$$\mathbb{E}[J_2^2] \lesssim (r - r')^2.$$  \hspace{1cm} (82)

To obtain upper bounds for the terms $\mathbb{E}[J_2^2]$, $\mathbb{E}[J_3^2]$ and $\mathbb{E}[J_4^2]$, we first introduce the double centered distance $\bar{A}_{i,j} := A_{i,j} - \mathbb{E}[A_{i,j}|X_i] - \mathbb{E}[A_{i,j}|X_j] + \mathbb{E}[A_{i,j}]$ for $i \neq j$. We define $\bar{L}(X_i, X_j)$ and $\bar{R}(X_i, X_j)$ in a similar way. Following the proof of Lemma D.3 in the supplementary materials of Chakraborty and Zhang (2019), we can argue in a similar fashion that $\text{var}(J_2)$, $\text{var}(J_3)$ and $\text{var}(J_4)$ are $o(1)$ as $n, p \to \infty$. Moreover, we have $\mathbb{E}[\bar{A}_{i,j}^2] \lesssim \mathbb{E}[\bar{A}_{i,j}^2]$, $\bar{L}(X_i, X_j) = \frac{1}{2} \overline{H}(X_i, X_j)$ and

$$A_{i,j} = \frac{\tau}{2} \bar{L}(X_i, X_j) + \tau \bar{R}(X_i, X_j) = \frac{1}{2} H(X_i, X_j) + \tau R(X_i, X_j).$$  \hspace{1cm} (83)

With all these, we can write

$$\frac{1}{V_0} \mathbb{E}[\bar{A}_{i,j}^2] \lesssim \frac{1}{4} + \frac{\tau^2}{V_0} \mathbb{E}[R^2(X, X')] \leq \frac{1}{4} \left( \mathbb{E} \left[ \frac{\tau^4}{V_0^2} \bar{R}^4(X, X') \right] \right)^{1/2},$$  \hspace{1cm} (84)

where the first and the second inequalities follow from power mean inequality and Hölder’s inequality, respectively. This implies

$$\mathbb{E}[J_2] = \frac{8}{n^2 V_0} \sum_{i=[na']+1}^{[nr']-1} \sum_{j=[na]+1}^{[na']-1} \mathbb{E}[\bar{A}_{i,j}^2] \lesssim \frac{1}{n^2} \left[ (nr' - [na'])([na'] - [na]) + O \left( \left( \frac{n^4 \tau^4}{V_0^2} \mathbb{E}[\bar{R}^4(X, X')] \right)^{1/2} \right) \right],$$  \hspace{1cm} (85)

where we have used the fact that $\mathbb{E}[\bar{R}^4(X, X')] = O(\mathbb{E}[\bar{R}^4(X, X')]$. Following (51) and under Assumption 3.3, we have from (85)

$$\mathbb{E}[J_2] \lesssim (r' - a')(a' - a) + o(1),$$  \hspace{1cm} (86)

and therefore $(\mathbb{E}[J_2])^2 \lesssim (a - a')^2$, which in turn implies $\mathbb{E}[J_2^2] \lesssim (a - a')^2$ as $\text{var}(J_2) = o(1)$ (and hence is a bounded sequence in $n$ and $p$).

In similar lines, we can show that $\mathbb{E}[J_3^2] \lesssim (a' - a)^2$ and $\mathbb{E}[J_4^2] \lesssim (r - r')^2$. Combining all these, we have from (78)

$$\mathbb{E} \left| J_n(u) - J_n(u') \right|^2 \lesssim (a - a')^2 + (r - r')^2 = \|u - u'\|^2 = \tilde{\kappa}^2(u, u').$$  \hspace{1cm} (87)
Finally combining (72), (75) and (87), we get

$$E \left| \tilde{J}_n(u) - \tilde{J}_n(u') \right|^2 \lesssim \tilde{\kappa}^2(u,v),$$

which completes the proof of B3 and hence Theorem A.3.

References

Andrews, D.W.K. (1992). Generic Uniform Convergence. *Econometric Theory*, 8(2) 241-257.

Chakraborty, S., and Zhang, X. (2019). A New Framework for Distance and Kernel-based Metrics in High Dimensions. arXiv:1909.13469.

Hall, P. and Heyde, C. C. (1980). Martingale Limit Theory and Its Applications. *Academic press*.

Li, Q. and Racine, J. C. (2007). Nonparametric Econometrics: Theory and Practice. *Princeton University press*.

Pollard, D. (1990). Empirical Processes: Theory and Applications. *NSF-CBMS Regional Conference Series in Probability and Statistics*, 2 i-iii+v+vii-viii+1-86.

Resnick, S. I. (1999). *A Probability Path*. Springer.