Coordinate Descent for MCP/SCAD Penalized Least Squares Converges Linearly

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Abstract: Recovering sparse signals from observed data is an important topic in signal/imaging processing, statistics and machine learning. Nonconvex penalized least squares have been attracted a lot of attentions since they enjoy nice statistical properties. Computationally, coordinate descent (CD) is a workhorse for minimizing the nonconvex penalized least squares criterion due to its simplicity and scalability. In this work, we prove the linear convergence rate to CD for solving MCP/SCAD penalized least squares problems.

Keywords: Nonconvex penalized least squares problems, MCP/SCAD, Coordinate descent, KL property, Linear convergence.
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1 Introduction

Considering the sparse linear estimation problem

\[ b = Ax^* + \xi, \]  

(1.1)

where the vector \( x^* \in \mathbb{R}^p \) denotes the sparse regression coefficient or sparse signal to be recovered, the vector \( \xi \in \mathbb{R}^n \) is the random error term, and the design matrix \( A \in \mathbb{R}^{n \times p} \) with \( n \ll p \) describing the system response mechanism. Throughout, we assume the matrix \( A \) has normalized column vectors \( \{A_i\} \), i.e., \( \|A_i\|_2 = 1 \) for \( i = 1, \ldots, p \). The basis pursuit \[5\] or lasso \[14\]

\[ \min_{x \in \mathbb{R}^p} F(x) = \frac{1}{2} \|Ax - b\|_2^2 + \lambda \|x\|_1, \]  

(1.2)

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is a widely used sparse recovery model. The minimizers of Lasso (1.2) enjoy attractive statistical properties [4,10,20]. The convexity of the problem (1.2) allows designing fast and global convergent algorithms. see [15] for an overview. However, the Lasso estimator tends to produce biased estimates for large coefficients [18], and hence lacks oracle property [6,7]. Several nonconvex penalty functions have been proposed to remedy this including the MCP [17] and SCAD [6,7].

Consider the following nonconvex optimization problem

$$\min_{x \in \mathbb{R}^p} F(x) = \frac{1}{2} \|Ax - b\|_2^2 + \sum_{i=1}^{p} \rho_{\lambda,\tau}(x_i),$$

(1.3)

where $\rho_{\lambda,\tau}$ is a non-convex penalty, $\lambda > 0$ is a regularization parameter, and $\tau \geq 0$ controls the degree of concavity of penalty. The nonconvex function $\rho_{\lambda,\tau}$ satisfies the requirements that it is singular at the origin in order to achieve sparsity and its derivative vanishes for large values so as to ensure unbiasedness. For SCAD, it is defined for $\tau > 2$ via

$$\rho_{\lambda,\tau}(t) = \lambda \int_0^{[t]} \min \left(1, \frac{\max(0, \lambda \tau - |s|)}{\lambda(\tau - 1)}\right) ds$$

(1.4)

and computing the integral explicitly yields the expression in Table 1. Further, variable selection consistency and asymptotic estimation efficiency were studied in [7]. MCP was devised in the same spirit as SCAD which is defined as

$$\rho_{\lambda,\tau}(t) = \lambda \int_0^{[t]} \max \left(0, 1 - \frac{|s|}{\lambda \tau}\right) ds.$$  

(1.5)

MCP minimizes the maximum concavity $\sup_{0 < t_1 < t_2} \frac{\rho'_{\lambda,\tau}(t_1) - \rho'_{\lambda,\tau}(t_2)}{t_2 - t_1}$ to satisfy unbiasedness and feature selection constraints: $\rho'_{\lambda,\tau}(t) = 0$ for any $|t| \geq \lambda \tau$ and $\rho'_{\lambda,\tau}(0^+) = \pm \lambda$. The condition $\tau > 1$ ensures the well-posedness of the thresholding operator [17]. The gradient functions of SCAD and MCP are

$$\rho'_{\lambda,\tau}(t) = \begin{cases} 0, & |t| \geq \lambda \tau, \\ \frac{\lambda \tau - \frac{1}{2} |t|}{\tau - 1}, & \lambda < |t| < \lambda \tau, \\ \frac{\lambda |t|}{\lambda}, & |t| \leq \lambda \tau \end{cases}$$

and

$$\rho'_{\lambda,\tau}(t) = \begin{cases} \lambda - \frac{|t|}{\lambda \tau}, & |t| < \lambda \tau, \\ 0, & |t| \geq \lambda \tau \end{cases}$$

respectively. We summarize the function $\rho_{\lambda,\tau}$ corresponding to Lasso, SCAD, MCP and their thresholding functions in Table 1. We plot the Lasso, MCP,
SCAD penalties, derivative of these penalties and their thresholding functions in Figure 1.

| penalty     | $\rho_{\lambda,\tau}(t)$                                                                 | $\mathcal{S}_{\lambda,\tau}^\rho(c)$                        |
|-------------|------------------------------------------------------------------------------------------|--------------------------------------------------------------|
| LASSO       | $\lambda |t|$                                                                                   | $\text{sgn}(c) \max\{|c| - \lambda, 0\}$                    |
| SCAD ($\tau > 2$) | $\begin{cases} 
\frac{\lambda^2(t+1)}{2} & |t| > \lambda \tau \\
\lambda |t| - \frac{1}{2}(t^2 + \lambda^2) & |t| \leq \lambda \tau \\
\lambda |t| & |t| \leq \lambda 
\end{cases}$ | $\begin{cases} 
0 & |v| \leq \lambda \\
\text{sgn}(v)(|v| - \lambda) & \lambda < |v| \leq 2\lambda \\
\text{sgn}(v)\frac{(\tau-1)|v| - \lambda \tau}{\tau-2} & 2\lambda < |v| \leq \lambda \tau \\
v & |v| > \lambda \tau 
\end{cases}$ |
| MCP ($\tau > 1$) | $\begin{cases} 
\lambda \left(|t| - \frac{t^2}{2\lambda \tau}\right) & |t| < \tau \lambda \\
\frac{\lambda^2 \tau}{2} & |t| \geq \tau \lambda 
\end{cases}$ | $\begin{cases} 
0 & |v| \leq \lambda \\
\text{sgn}(v)\frac{\tau(|v| - \lambda)}{\tau-1} & \lambda < |v| \leq \lambda \tau \\
v & |v| > \lambda \tau 
\end{cases}$ |

Figure 1: Lasso, SCAD, MCP penalties, derivative of these penalties and their thresholding functions.

The nonconvexity and nonsmoothness of the SCAD and MCP penalty poses challenge for solving (1.3). Several efforts has been made to handle this including local quadratic approximation (LQA) [6], local linear approximation (LLA) [21] and multi-stage convex relaxation [19], coordinate descent (CD) in either Jacobi [13] or Gauss-Seidel [3, 9] fashion. Among the above mentioned numerical methods, coordinate descent proposed in [3, 9] became a popular solver in statistical communities due to its simplicity and scalability. Numerical experiments in [3, 9] demonstrates fast convergence of CD for SCAD and MCP. However, the convergence analysis of CD is fall behind its excellent numerical performance. Indeed, in [3, 9] they showed any cluster point of the iterates is a stationary point (under the assumption that the iteration sequence...
has clusters) by using the idea developed in [16]. In this paper we fill this gap by showing linear convergence rate of CD for solving (1.3).

The rest of the paper are organized as follows. In section 2, we prove the linear convergence rate of CD. We give the conclusion in Section 3.

2 Convergence rate analysis of CD

2.1 Coordinate descent

In this section, we recall the CD algorithm [3,9] for (1.3) with SCAD and MCP penalties. The objective function reads

\[ F(x) = \frac{1}{2} \| Ax - b \|_2^2 + \sum_{i=1}^{p} \rho_{\lambda, \tau}(x_i). \]

Given the current iteration \( x^k \), we update \( x^{k+1} \) by

\[ x_i^{k+1} = \arg \min_t F(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, t, x_{i+1}^{k}, \ldots, x_p^{k}) \text{ for } i = 1, \ldots, p. \]

Some algebra shows that

\[ x_i^{k+1} \in \arg \min_t f_i(t) := \frac{1}{2}(t - c_i^k)^2 + \rho_{\lambda, \tau}(t) \]

where

\[ c_i^k = A_i^T \left( b - \sum_{j=1}^{i-1} x_j^{k+1} A_j - \sum_{j=i+1}^{p} x_j^{k} A_j \right). \] (2.1)

By the definition of the thresholding operator of \( \rho_{\lambda, \tau} \) in Table 1,

\[ x_i^{k+1} = S_{\rho_{\lambda, \tau}}(c_i^k), i = 1, 2, \ldots, p. \]

To sum up, we present the CD algorithm in the following algorithm

\textbf{Algorithm 1} Coordinate Descent

\begin{verbatim}
Given initial point \( x^0 \), parameters \( \lambda, \tau \)
repeat
    Update \( x^{k+1} \) by \( x_i^{k+1} = S_{\rho_{\lambda, \tau}}(c_i^k) \) for \( i = 1, \ldots, p \), with \( c_i^k \) in (2.1).
until Stop condition
\end{verbatim}
2.2 Preliminaries on nonsmooth analysis

To prove the convergence rate, we need the some tools in nonsmooth analysis including limiting subdifferential and KL property.

First we present definition of limiting subdifferential. Recall the definition of subdifferential at point \( x \) for convex function \( \partial f(x) := \{ z \in \mathbb{R}^n : f(x) - f(z) - \langle z, x - z \rangle \geq 0 \} \).

\[ (2.2) \]

when \( f \) is non-convex, one can extend subdifferential to limiting-subdifferential \[12\].

**Definition 2.1.** For a proper function \( f : \mathbb{R}^n \to [-\infty, +\infty] \), its **limiting subdifferential** at \( x \in \text{dom } f \) is defined by

\[ \partial_{\lim} f(x) := \{ \nu \in \mathbb{R}^n : \exists x^k \xrightarrow{f} x, \nu^k \xrightarrow{\nu} \nu \}, \]  
\[ (2.3) \]

with \( \lim_{z \to x^k} \inf \frac{f(z) - f(x^k) - (\nu^k, z - x^k)}{\|z - x^k\|} \geq 0, \forall k \), and \( x^k \xrightarrow{f} x \) denoting \( x^k \to x \) and \( f(x^k) \to f(x) \). We also write \( \text{dom } \partial_{\lim} f := \{ x \in \mathbb{R}^n : \partial f(x) \neq 0 \} \).

It is obvious that the limiting subdifferential coincides with the gradient for differentiable functions. Moreover, when \( f \) is convex, the limiting subdifferential equal to the subdifferential in convex analysis. Without loss of generality, we use the notation \( \partial f \) to denote limiting subdifferential in the rest of the paper. Finally, we will say that \( x^* \in \mathbb{R}^n \) is a stationary or critical point of \( f \) if \( 0 \in \partial f(x^*) \), which is a necessary condition for \( x^* \in \arg \min_x f(x) \).

Next, we recall the KL property, KL function and KL exponent which are basic tools used in convergence analysis for nonconvex problems. These results are adopted from \[12811\].

**Definition 2.2.** We say that a proper closed function \( f : \mathbb{R}^n \to [-\infty, +\infty] \) has the Kurdyka-Lojasiewicz (KL) property at \( \bar{x} \in \text{dom } \partial f \) if there exist a neighborhood \( \mathcal{N} \) of \( \bar{x} \), \( \nu \in (0, \infty] \) and a continuous concave function \( \psi : [0, \nu) \to \mathbb{R}_+ \) with \( \psi(0) = 0 \) such that:

i) \( \psi \) is a continuously differentiable on \((0, \nu)\) with \( \psi' \) over \((0, \nu)\);

ii) for all \( x \in \mathcal{N} \) with \( f(\bar{x}) < f(x) < f(\bar{x}) + \nu \), one has

\[ \psi' (f(x) - f(\bar{x})) \text{ dist } (0, \partial f(x)) \geq 1. \]  
\[ (2.4) \]

A proper closed function \( f \) satisfying the KL property at all points in \( \text{dom } \partial f \) is called a KL function.
Definition 2.3. For a proper closed function $f$ satisfying the KL property at $x \in \text{dom} \partial f$, if the corresponding function $\psi$ can be chosen as $\psi(s) = \overline{c}s^{1-\alpha}$ for some $\overline{c} > 0$ and $\alpha \in [0, 1)$, i.e., there exist $c, \epsilon > 0$ and $\nu \in (0, \infty]$ so that
\[
\text{dist}(0, \partial f(x)) \geq c(f(x) - f(\bar{x}))^\alpha
\] (2.5)
whenever $\|x - \bar{x}\| \leq \epsilon$ and $f(\bar{x}) < f(x) < f(\bar{x}) + \nu$, then we say that $f$ has the KL property at $\bar{x}$ with an exponent of $\alpha$. If $f$ is a KL function and has the same exponent $\alpha$ at any $\bar{x} \in \text{dom} \partial f$, then we say that $f$ is a KL function with an exponent of $\alpha$.

Proposition 2.1. The objective cost function $F$ defined in (1.3) is a KL function with an exponent of $\frac{1}{2}$.

Proof. Follows from Corollary 5.2 of [8]. \qed

Last, we recall the main conditions to prove the convergence of general algorithms for nonconvex problems. Let $H : \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi-continuous function and $\{x^k\}_{k=0}^\infty$ a sequence generated by some optimization method. Assume the following conditions are satisfied:

- (H1) The sequence $\{H(x^k)\}_{k=0}^\infty$ is monotonically decreasing thus converging. In particular for any finite starting point $x^0 \in \mathbb{R}^p$, there exists some positive constant $\theta$, such that the sequence $\{x^k\}_{0}^\infty$ satisfies
  \[
  H(x^k) - H(x^{k+1}) \geq \theta\|x^k - x^{k+1}\|_2^2;
  \] (2.6)

- (H2) For each $k \in \mathbb{N}$, there exists some $d^{k+1} \in \partial H(x^{k+1})$, such that
  \[
  \|d^{k+1}\| \leq C\|x^{k+1} - x^k\|_2,
  \] (2.7)
  where $C > 0$;

- (H3) There exists a subsequence $\{x^{k_i}\}_{i=0}^\infty$ of $\{x^k\}_{k=0}^\infty$, s.t.,
  \[
  x^{k_i} \to x^* \text{ and } H(x^{k_i}) \to H(x^*);
  \] (2.8)

Proposition 2.2. Let $H : \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi-continuous function. Consider a sequence $\{x^k\}_{k \in \mathbb{N}}$ that satisfies (H1) – (H3). If $H$ has the KL property at some cluster point $x^* \in \mathbb{R}^p$ specified in (H3), then the sequence $\{x^k\}_{k \in \mathbb{N}}$ convergences to $\bar{x} = x^*$ as $k$ goes to infinity, and $\bar{x}$ is a critical point of $H$. Moreover the sequence $\{x^k\}_{k \in \mathbb{N}}$ has a finite length, i.e.
\[
\sum_k \|x^k - x^{k+1}\| < \infty.
\]

Proof. Follows from Lemma 2.6 of [2]. \qed
2.3 Linear convergence rate

**Theorem 2.1.** Let $\{x^k\}_{k=0}^\infty$ be the sequence generated by CD **Algorithm 1** for objective cost function $F$ defined in (1.3) with SCAD or MCP penalty. If the sequences admits a accumulation point $x^*$, then

- (a) (H1) holds, i.e.,
  \[ F(x^k) - F(x^{k+1}) \geq \theta \|x^k - x^{k+1}\|_2^2; \]

- (b) (H2) holds, i.e., for each $k \in \mathbb{N}$, there exists some $d^{k+1} \in \partial F(x^{k+1})$, such that
  \[ \|d^{k+1}\| \leq C \|x^{k+1} - x^k\|_2; \]

- (c) Let $x^k \rightarrow x^*$, then we have
  \[ F(x^k) \rightarrow F(x^*); \]

- (d) $\{x^k\}_{k=0}^\infty$ converges to $x^*$. The sequence $\{x^k\}_{k \in \mathbb{N}}$ has a finite length, i.e.,
  \[ \sum_{k} \|x^k - x^{k+1}\| < \infty. \]

- (e) $\{x^k\}_{k=0}^\infty$ converges to $x^*$ linearly.

**Proof.** (a) We use $\rho$ to short for $\rho_{\lambda, \tau}$. Let $\theta = (1+\min\{\rho''(|t|), 0\})/2$. For fixed $k, i$, by the definition of CD **Algorithm 1**, 

\[ x_i^{k+1} = S_{\lambda, \tau}(c_i^k) \in \arg\min_t f_i(t) = \frac{1}{2}(t - c_i^k)^2 + \rho(t). \]

We can conclude that

\[ 0 \in (x_i^{k+1} - c_i^k) + \partial \rho(x_i^{k+1}) := \partial f_i(x_i^{k+1}) \tag{2.9} \]

and

\[
\begin{align*}
F(x_0^{k+1}, ..., x_{i-1}^{k+1}, x_i^k, ..., x_p^{k+1}) - F(x_0^{k+1}, ..., x_i^{k+1}, x_i^k, x_{i+1}^{k+1}, ..., x_p^{k+1}) \\
= f_i(x_i^k) - f_i(x_i^{k+1}) \\
\geq \theta |x_i^k - x_i^{k+1}|^2,
\end{align*}
\]

where we use the strong convexity of $f_i$ in the last inequality. Then,

\[ F(x^k) - F(x^{k+1}) \geq \theta \|x^k - x^{k+1}\|_2^2. \]
follows by summarizing the above display over all coordinates. We then easily obtain
\[
\sum_{k=1}^{\infty} \|x^k - x^{k+1}\|^2 \leq F(x^0)/\theta < \infty
\]
and \(\lim_{k \to \infty} \|x^k - x^{k+1}\| = 0\).

(b) Recall that \(F(x) = \frac{1}{2} \|Ax - b\|^2 + \sum_{i=1}^{p} \rho(x_i)\). Consider the limiting subdifferential at \(x^{k+1}\)
\[
\partial F(x^{k+1}) = A^T (Ax^{k+1} - b) + \begin{pmatrix}
\partial \rho(x_1^{k+1}) \\
\partial \rho(x_2^{k+1}) \\
\vdots \\
\partial \rho(x_p^{k+1})
\end{pmatrix}
\]
By (2.9), we have \(-x_i^{k+1} + c_i^k \in \partial \rho(x_i^{k+1})\), i.e.,
\[
-x_i^{k+1} + A_i^T \left( b - \sum_{j=1}^{i-1} x_j^{k+1} A_j - \sum_{j=i+1}^{p} x_j^k A_j \right) \in \partial \rho(x_i^{k+1})
\]
\[
-x_i^{k+1} + x_i^{k+1} + A_i^T \left[ \sum_{j=i+1}^{p} (x_j^{k+1} - x_j^k) A_j \right] \in \partial \rho(x_i^{k+1}) + A_i^T (Ax^{k+1} - b)
\]
\[
\sum_{j=i+1}^{p} (x_j^{k+1} - x_j^k) A_i^T A_j \in \partial \rho(x_i^{k+1}) + A_i^T (Ax^{k+1} - b).
\]
Let \(d_i^{k+1} = \sum_{j=i+1}^{p} (x_j^{k+1} - x_j^k) A_i^T A_j\). Then the above display shows
\(d^{k+1} \in \partial F(x^{k+1})\).

With \(|A_i^T A_j| \leq 1\), we can conclude that,
\[
|d_i^{k+1}|^2 = \left| \sum_{j=i+1}^{p} (x_j^{k+1} - x_j^k) A_i^T A_j \right|^2 \leq (p-i) \sum_{j=i+1}^{p} |x_j^{k+1} - x_j^k|^2 \leq (p-i) \|x^{k+1} - x^k\|^2.
\]
Then,
\(\|d^{k+1}\| \leq p\|x^{k+1} - x^k\|\).

(c) Follows from the continuity of \(F\). Moreover, the assumption that there exist a accumulation point implies \((H3)\) holds.
This can be easily verified by applying \textbf{Proposition 2.2} with KL property.

By Proposition 2.1, $F$ admits the KL property with exponent of $1/2$. Then, using the Definition 2.3 and (b) we have

$$C^2 \|x^k - x^{k+1}\|^2 \geq \text{dist}^2(0, \partial F(x^{k+1})) \geq c^2 (F(x^{k+1}) - F(x^*)).$$

Let $A_{k+1} = \sum_{i=k+1}^{\infty} \|x^i - x^{i+1}\|^2$. The above display and (a) implies,

$$C^2 (A_k - A_{k+1}) \geq \text{dist}^2(0, \partial F(x^{k+1})) \geq c^2 \theta^2 A_{k+1},$$

which leads to

$$A_{k+1} \leq \frac{C^2}{C^2 + c^2 \theta^2} A_k := \nu^2 A_k,$$

where $\nu \in (0, 1)$ obviously. From the above display and the finite length of $\{x^k\}$ in (d), we know that there exists some $\eta > 0$, such that

$$A_k \leq \nu^{2k} \eta^2$$

and

$$\|x^k - x^{k+1}\| \leq \nu^k \eta.$$

Then by triangle inequality and the convergence of $\{x_k\}$ to $x^*$ we have

$$\|x^k - x^*\| \leq \sum_{i=k}^{\infty} \|x^i - x^{i+1}\| \leq \frac{\nu^k \eta}{1 - \xi},$$

i.e, $x^k$ globally converges to $x^*$ linearly.

3 Conclusion

In this work, we prove the linear convergence rate of coordinate descent method for solving MCP/SCAD penalized least squares problems. In the proof we use the assumption that the sequences admits a accumulation point. Removing this assumption is an interesting question for further study.

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