$p$-adic measures associated with zeta values and $p$-adic log multiple gamma functions

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Abstract

We study a relation between two refinements of the rank one abelian Gross-Stark conjecture: For a suitable abelian extension $H/F$ of number fields, a Gross-Stark unit is defined as a $p$-unit of $H$ satisfying some properties. Let $\tau \in \text{Gal}(H/F)$. Yoshida and the author constructed the symbol $Y_p(\tau)$ by using $p$-adic log multiple gamma functions, and conjectured that the log $p$ of a Gross-Stark unit can be expressed by $Y_p(\tau)$. Dasgupta constructed the symbol $u_T(\tau)$ by using the $p$-adic multiplicative integration, and conjectured that a Gross-Stark unit can be expressed by $u_T(\tau)$. In this paper, we give an explicit relation between $Y_p(\tau)$ and $u_T(\tau)$.

1 Introduction

Let $F$ be a totally real field, $K$ a CM-field which is abelian over $F$, $S$ a finite set of places of $F$. We assume that

- $S$ contains all infinite places of $F$, all places of $F$ lying above a rational prime $p$, and all ramified places in $K/F$.
- Let $p$ be the prime ideal corresponding to the $p$-adic topology on $F$. (Hence $p \in S$.) Then $p$ splits completely in $K/F$.

For $\tau \in \text{Gal}(K/F)$, we consider the partial zeta function

$$\zeta_S(s, \tau) := \sum_{(\frac{K/F}{a})=\tau, (a,S)=1} Na^{-s}.$$ 

Here $a$ runs over all integral ideals of $F$, relatively prime to any finite places in $S$, whose image under the Artin symbol $(\frac{K/F}{a})$ is equal to $\tau$. The series converges for Re$(s) > 1$, has a meromorphic continuation to the whole $s$-plane, and is analytic at $s = 0$. Moreover, under our assumption, we see that

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There exists the $p$-adic interpolation function $\zeta_{p,S}(s, \tau)$ of $\zeta_S(s, \tau)$.

- $\text{ord}_{s=0} \zeta_S(s, \tau), \text{ord}_{s=0} \zeta_{p,S}(s, \tau) \geq 1$.

- There exist a natural number $W$, a $p$-unit $u$ of $K$, which satisfy
  \[ \log |u\tau|_\mathfrak{P} = -W\zeta_S'(0, \tau) \quad (\tau \in \text{Gal}(K/F)). \]

Here $\mathfrak{P}$ denotes the prime ideal corresponding to the $p$-adic topology on $K$, $|x|_\mathfrak{P} := N\mathfrak{P}^{-\text{ord}_p x}$.

Gross conjectured the following $p$-adic analogue of the rank 1 abelian Stark conjecture:

**Conjecture 1** (Gr, Conjecture 3.13). Let $u$ be a $p$-unit characterized by (1) up to roots of unity. Then we have

\[ \log_p N_{K_{\mathfrak{P}}/\mathbb{Q}_p}(u\tau) = -W\zeta_{p,S}(0, \tau). \]

Dasgupta-Darmon-Pollack [DDP] proved a large part of Conjecture 1. Yoshida and the author, and independently Dasgupta formulated refinements of Conjecture 1: Let $f$ be an integral ideal of a totally real field $F$ satisfying $p \nmid f$, $H_f$ the narrow ray class field modulo $f$, $H$ the maximal subfield of $H_f$ where $p$ splits completely. Yoshida and the author [KY1] essentially constructed the invariant $Y_p(\tau)$ (Definition 6) for $\tau \in \text{Gal}(H/F)$ by using $p$-adic log multiple gamma functions. Then [KY1, Conjecture A] states that $\log_p u\tau$ (without $N_{K_{\mathfrak{P}}/\mathbb{Q}_p}$) can be expressed by $Y_p(\tau)$. On the other hand, Dasgupta constructed the invariant $u_T(b, D_f)$ (Definition 12-(iv)) by using the multiplicative integration for $p$-adic measures associated with Shintani’s multiple zeta functions. Then [Da, Conjecture 3.21] states that a modified version of $u\tau$ can be expressed by $u_T(b, D_f)$. In [Ka3, Remark 2], the author announced the following relation between these refinements.

**Theorem** (Theorem 1). Let $\eta$ be a “good” prime ideal in the sense of Definition 11. We put $T := \{\eta\}$. Then we have

\[ \log_p u_\eta(b, D_f) = -Y_p((H/F)_b) + N\eta Y_p((H/F)_{b\eta}). \]

In particular, we see that two refinements are consistent (roughly speaking, [Da, Conjecture 3.21] is a further refinement of [KY1, Conjecture A'] by $\ker \log_p$). The aim of this paper is to prove this theorem.

Let us explain the outline of this paper. In §2, we introduce Shintani’s technique of cone decompositions. We obtain a suitable fundamental domain of $F \otimes \mathbb{R}_+/E_{t,+}$, where $F \otimes \mathbb{R}_+$ denotes the totally positive part of $F \otimes \mathbb{R}$, $E_{t,+}$ is a subgroup of the group of all totally positive units. We need such fundamental domains in order to construct both of the invariants $Y_p$, $u_T$. In §3, we recall the definition and some properties of $Y_p$, which is essentially defined in [KY1] and slightly modified in [Ka3]. The classical or $p$-adic log multiple gamma function is defined as the derivative values at $a = 0$ of the classical or $p$-adic Barnes’ multiple zeta function, respectively. Then the invariant $Y_p(\tau, \iota)$ is defined in Definition 6 as a finite sum of the “difference” of $p$-adic log multiple gamma functions and classical log multiple gamma functions. Conjecture 2 predicts exact values.
of $Y_p(\tau, \iota)$. In §4, we also recall some results in [Da]. Dasgupta introduced $p$-adic measures $\nu_T(b, D_I)$ associated with special values of Shintani’s multiple zeta functions, and defined $u_T(b, D_I)$ as the multiplicative integration $\int_0^1 x \, d\nu_T(b, D_I, x)$ with certain correction terms. Dasgupta formulated a conjecture (Conjecture 3) on properties of $u_T(b, D_I)$. In §5, we state and prove the main result (Theorem 1) which gives an explicit relation between $Y_p(\tau, \iota)$ and $\log_p(u_\eta(b, D_f))$. Then we will see that Conjectures 2, 3 are consistent in the sense of Corollary 1. The key observation is Lemma 3: Dasgupta’s $p$-adic measure $\nu_\eta(b, D_f)$ is originally associated with Shintani’s multiple zeta functions. By this Lemma, we can relate $\nu_\eta(b, D_f)$ to Barnes’ multiple zeta functions and $p$-adic analogues as in Lemma 4.

2 Shintani domains

Let $F$ be a totally real field of degree $n$, $\mathcal{O}_F$ the ring of integers of $F$, $\mathfrak{f}$ an integral ideal of $F$. We denote by $F^+_{\mathfrak{f}}$ the set of all totally positive elements in $F$ and put $\mathcal{O}_{F_{\mathfrak{f}}^+} := \mathcal{O}_F \cap F^+_{\mathfrak{f}}$, $E^+ := \mathcal{O}_F^+ \cap F^+$. We consider subgroups of $E^+_{\mathfrak{f}}$ of the following form:

$$E_{i, +} := \{ \epsilon \in E_{\mathfrak{f}}^+ \mid \epsilon \equiv 1 \mod \mathfrak{f} \}.$$ 

We identify

$$F \otimes \mathbb{R} = \mathbb{R}^n, \quad \sum_{i=1}^k a_i \otimes b_i \mapsto \left( \sum_{i=1}^k \iota(a_i)b_i \right)_{\iota \in \text{Hom}(F, \mathbb{R})},$$

where $\text{Hom}(F, \mathbb{R})$ denotes the set of all real embeddings of $F$. In particular, the totally positive part

$$F \otimes \mathbb{R}^+ := \mathbb{R}^n_+$$

has a meaning. On the right-hand side, $\mathbb{R}^+_+$ denotes the set of all positive real numbers. Let $v_1, \ldots, v_r \in \mathcal{O}_F$ be linearly independent. Then we define the cone with basis $\mathbf{v} := (v_1, \ldots, v_r)$ as

$$C(\mathbf{v}) := \{ t^t \mathbf{v} \in F \otimes \mathbb{R} \mid t \in \mathbb{R}^r_+ \}.$$ 

Here we $t^t \mathbf{v}$ denotes the inner product.

**Definition 1.** (i) We call a subset $D \subset F \otimes \mathbb{R}^+$ is a Shintani set if it can be expressed as a finite disjoint union of cones:

$$D = \bigsqcup_{i \in J} C(\mathbf{v}_j) \quad (|J| < \infty, \mathbf{v}_j \in \mathcal{O}^{(j)}_{r \mathcal{O}^+}, \ r(j) \in \mathbb{N}).$$

(ii) We consider the natural action $E^+_{\mathfrak{f}} \curvearrowright F \otimes \mathbb{R}^+$, $u(a \otimes b) := (ua) \otimes b$. We call a Shintani set $D$ a Shintani domain mod $E_{\mathfrak{f}}^+$ if it is a fundamental domain of $F \otimes \mathbb{R}^+/E^+_{\mathfrak{f}}$:

$$F \otimes \mathbb{R}^+ = \bigsqcup_{\epsilon \in E^+_{\mathfrak{f}}} \epsilon D.$$ 

When $\mathfrak{f} = (1)$, we write mod $E^+$ instead of mod $E_{(1), +}$.

Shintani [Sh, Proposition 4] showed that there always exists a Shintani domain.
3 $p$-adic log multiple gamma functions

We recall the definition and some properties of the symbol $Y_p$ defined in [KY1], [Ka3]. We denote by $\mathbb{R}_+$ the set of all positive real numbers.

**Definition 2.** Let $z \in \mathbb{R}_+$, $v \in \mathbb{R}_+^r$. Barnes’ multiple zeta function is defined as

$$\zeta(s, v, z) := \sum_{m \in \mathbb{Z}_0^r} (z + m^t v)^{-s}.$$  

This series converges for $\text{Re}(s) > r$, has a meromorphic continuation to the whole $s$-plane, and is analytic at $s = 0$. Then Barnes’ multiple gamma function is defined as

$$\Gamma(z, v) := \exp \left( \frac{\partial}{\partial s} \zeta(s, v, z) \big|_{s=0} \right).$$

Note that this definition is modified from that given by Barnes. For the proof and details, see [Yo, Chap I, §1]. Throughout this paper, we regard each number field as a subfield of $\mathbb{Q}$, and fix two embeddings $\mathbb{Q} \hookrightarrow \mathbb{C}$, $\mathbb{Q} \hookrightarrow \mathbb{C}_p$. Here $\mathbb{C}_p$ denotes the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. We denote by $\mu_{(p)}$ the group of all roots of unity of prime-to-$p$ order. Let $\text{ord}_p : \mathbb{C}_p^\times \rightarrow \mathbb{Q}$, $\theta_p : \mathbb{C}_p^\times \rightarrow \mu_{(p)}$ be unique group homomorphisms satisfying

$$|p^{-\text{ord}_p(z)}\theta_p(z)^{-1}z|_p < 1 \quad (z \in \mathbb{C}_p^\times). \quad (2)$$

**Definition 3.** Let $z \in \mathbb{Q}$, $v \in (\mathbb{Q}^\times)^r$. We assume that

$$z \in \mathbb{R}_+, \ v \in \mathbb{R}_+^r \text{ via the embedding } \mathbb{Q} \hookrightarrow \mathbb{C},$$

$$\text{ord}_p(z) < \text{ord}_p(v_1), \ldots, \text{ord}_p(v_r) \text{ via the embedding } \mathbb{Q} \hookrightarrow \mathbb{C}_p. \quad (3)$$

Then we denote by $\zeta_p(s, v, z) \ (s \in \mathbb{Z}_p - \{1, 2, \ldots, r\})$ the $p$-adic multiple zeta function characterized by

$$\zeta_p(-m, v, z) = p^{-\text{ord}_p(z)m}\theta_p(z)^{-m}\zeta(-m, v, z) \quad (m \in \mathbb{Z}_0^r). \quad (4)$$

We define the $p$-adic log multiple gamma function as

$$L\Gamma_p(z, v) := \frac{\partial}{\partial s} \zeta_p(s, v, z) \big|_{s=0}. $$

The construction of $\zeta_p(s, v, z)$ is due to Cassou-Noguès [CN1]. The author defined and studied $L\Gamma_p(z, v)$ in [Ka1]. See [Ka3] §2 for a short survey.

**Definition 4.** Let $F$ be a totally real field, $\mathfrak{f}$ an integral ideal, $D = \coprod_{j \in J} C(v_j) \ (v_j \in \mathcal{O}_{F_{\mathfrak{f}}}(j))$ a Shintani domain mod $E_\mathfrak{f}$. We denote by $\text{Hom}(F, \mathbb{R})$ (resp. $\text{Hom}(F, \mathbb{C}_p)$) the set of all embeddings of $F$ into $\mathbb{R}$ (resp. $\mathbb{C}_p$). Since we fixed embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we may identify

$$\text{Hom}(F, \mathbb{R}) = \text{Hom}(F, \mathbb{C}_p).$$
We denote by $C_f$ the narrow ideal class group modulo $f$, by $H_f$ the narrow ray class field modulo $f$. In particular, the Artin map induces

$$C_f \cong \text{Gal}(H_f/F).$$

Let $\pi : C_f \to C(1)$ be the natural projection. For each $c \in C_f$, we take an integral ideal $a_c$ satisfying

$$a_c f \in \pi(c).$$

For $c \in C_f$, $v \in \mathcal{O}_F^*$, we put

$$R(c, v) := R(c, v, a_c) := \{x \in (\mathbb{Q} \cap (0, 1])^r \mid \mathcal{O}_F \supset (x^tv)a_c f \in c\}.$$

For $c \in C_f$, $\iota \in \text{Hom}(F, \mathbb{R})$, we define

$$G(c, \iota) := G(c, \iota, D, a_c) := \sum_{j \in J} \sum_{x \in R(c, v_j)} \log \Gamma(\iota(x^tv_j), \iota(v_j)).$$

For $\iota \in \text{Hom}(F, \mathbb{R}) (= \text{Hom}(F, \mathbb{C}_p))$, we put

$$p_\iota := \{z \in \mathcal{O}_F \mid |\iota(z)|_p < 1\}.$$

Note that the prime ideal $\iota(p_\iota)$ corresponds to the $p$-adic topology on $\iota(F) \subset \mathbb{C}_p$.

Assume that $p_\iota | f$. For $c \in C_f$, $\iota \in \text{Hom}(F, \mathbb{R})$, we define

$$G_p(c, \iota) := G_p(c, \iota, D, a_c) := \sum_{j \in J} \sum_{x \in R(c, v_j)} L\Gamma_p(\iota(x^tv_j), \iota(v_j)).$$

Note that $(\iota(x^tv_j), \iota(v_j))$ satisfies the assumption [3] whenever $p_\iota \nmid f$, $x \in R(c, v_j)$.

The following map $[\ ]_p$ is well-defined by [KY1, Lemma 5.1].

**Definition 5.** We denote by $\mathbb{Q}\log_p \mathbb{Q}^\times$ (resp. $\mathbb{Q}\log \mathbb{Q}^\times$) the $\mathbb{Q}$-subspace of $\mathbb{C}_p$ (resp. $\mathbb{C}$) generated by $\log_p b$ (resp. $\pi, \log b$) with $b \in \mathbb{Q}^\times$. We define a $\mathbb{Q}$-linear map $[\ ]_p$ by

$$[\ ]_p : \mathbb{Q}\log \mathbb{Q}^\times \to \mathbb{Q}\log_p \mathbb{Q}^\times, \quad a \log b \mapsto a \log_p b, \quad a \pi \mapsto 0 \quad (a, b \in \mathbb{Q}, \ b \neq 0).$$

**Lemma 1.** Let $H$ be an intermediate field of $H_f/F$, $q$ a prime ideal of $F$, relatively prime to $f$, splitting completely in $H/F$. Then we have

$$\sum_{c \in C_{fq}, \ \text{Art}(\tau)_{|H} = \tau} G(c, \iota) \in \mathbb{Q}\log \mathbb{Q}^\times \quad (\tau \in \text{Gal}(H/F)).$$

Here $c$ runs over all ideal classes whose images under the composite map $C_{fq} \to C_f \to \text{Gal}(H_f/F) \to \text{Gal}(H/F)$ is equal to $\tau$. 

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Proof. We put \( W(c, \iota) := W(\iota(c)) \) in [KY1 (4.3)], \( V(c, \iota) := V(\iota(c)) \) in [KY1 (1.6)], and \( X(c, \iota) := G(c, \iota) + W(c, \iota) + V(c, \iota) \). Here we consider the ideal class group \( C_{\iota(f)} \) of \( \iota(F) \) modulo \( \iota(f) \). Then \( \iota(c) \) denotes the image of \( c \in C_{\iota} \) in \( C_{\iota(f)} \) under the natural map. By the definition [KY1 (4.3)] and [KY2 Appendix I, Theorem], we have \( W(c, \iota), V(c, \iota) \in \log \mathbb{Q}^\times \). Moreover [KY1 Lemma 5.5] states that

\[
\sum_{c \in C_{\iota}} \chi_q(c) X(c, \iota) \in \mathbb{Q} \log \mathbb{Q}^\times \quad (\chi \in \hat{C}_t, \ \chi([q]) = 1).
\]

Here \( \chi_q \), \([q]\) denote the composite map \( C_{\iota q} \rightarrow C_{\iota} \xrightarrow{\chi_q} \mathbb{C}^\times \), the ideal class \( \in C_{\iota} \) of \( q \), respectively. Therefore, when \( H \) is the fixed subfield under \( (H_\iota/F)_d \), it follows from the orthogonality of characters. The general case follows from this case immediately. \( \square \)

**Definition 6.** Let \( H \) be an intermediate field of \( H_\iota/F \). Assume that \( p, \notdiv f \) and that \( p, \) splits completely in \( H/F \). Then we define

\[
Y_p(\tau, \iota) := \sum_{c \in C_{\iota p}, \ \Art(\overline{\tau})|H = \tau} G_p(c, \iota) - \sum_{c \in C_{\iota p}, \ \Art(\overline{\tau})|H = \tau} G(c, \iota)_p \quad (\tau \in \Gal(H/F)).
\]

When \( \iota = \id \), we drop the symbol \( \iota \): \( Y_p(\tau) := Y_p(\tau, \id) \).

By [KY1 Proposition 5.6] (and the orthogonality of characters), we see that \( Y_p(\tau, \iota) \) depends only on \( H, f, \tau, \iota, \) not on \( D, a, \)'s. We formulated a conjecture [KY1 Conjecture A'], which is equivalent to the following Conjecture 2 by [Ka3 Proposition 6-(ii)].

**Conjecture 2.** Let \( H_\iota/H/F \) be as above: we assume that

\[
p \notdiv f, \ \text{ splits completely in } H/F.
\]

We take a lift \( \iota : H \to \mathbb{C}_p \) of \( \iota : F \to \mathbb{C}_p \) and put \( p_{H,\iota} := \{ z \in \mathcal{O}_H \mid |\iota(z)|_p < 1 \} \). Let \( \alpha_{H,\iota} \) be a generator of the principal ideal \( p_{H,\iota}^{h_{H,\iota}} \), where \( h_{H,\iota} \) denotes the class number. Then we have

\[
Y_p(\tau, \iota) = -\frac{1}{h_{H,\iota}} \sum_{c \in C_{\iota}} \zeta(0, c^{-1}) \log_p \left( \frac{\alpha_{H,\iota}^{\tau \Art(c)}}{\tau \Art(c)} \right).
\]

**Remark 1.** Roughly speaking, the above conjecture states a relation between the ratios \( p \)-adic multiple gamma functions : multiple gamma functions \) and Stark units associated with the finite place \( p, \). We also studied a relation between the same ratios and Stark units associated with real places in [Ka3]. We found a more significant relation between the ratios \( p \)-adic gamma function : gamma function \) and cyclotomic units in [Ka2].

We rewrite the definition of \( Y_p \) for later use.

**Definition 7.** Let \( R \) be a subset of \( F_+ \). We assume that \( R \) can be expressed in the following form:

\[
R = \prod_{i=1}^{k} \{ (x_i + m)^{\tau} v_i \mid m \in \mathbb{Z}_{\geq 0}^r \} \quad (x_i \in \mathbb{Q}_+^r, \ v_i \in F_+^r).
\]
(i) We define
\[ \zeta_\iota(s, R) := \sum_{z \in R} \iota(z)^{-s} := \sum_{i=1}^{k} \zeta(s, \iota(v_i), \iota(x_i^t v_i)), \]
\[ L\Gamma_\iota(R) := \left. \frac{\partial}{\partial s} \zeta_\iota(s, R) \right|_{s=0} = \sum_{i=1}^{k} \log \Gamma(\iota(x_i^t v_i), \iota(v_i)). \]

(ii) Additionally we assume that each \((\iota(x_i^t v_i), \iota(v_i))\) satisfies (3). Then there exists the p-adic interpolation function
\[ \zeta_{\iota, p}(s, R) := \sum_{i=1}^{k} \zeta_p(s, \iota(v_i), \iota(x_i^t v_i)) \]
of \(\zeta(s, R)\). We define
\[ L\Gamma_{\iota, p}(R) := \left. \frac{\partial}{\partial s} \zeta_{\iota, p}(s, R) \right|_{s=0} = \sum_{i=1}^{k} L\Gamma_p(\iota(x_i^t v_i), \iota(v_i)). \]

When \(\iota = \text{id}\), we drop the symbol \(\iota\).

It follows that, for any Shintani domain \(D \mod E_+\) and for any integral ideals \(a_c\) satisfying \(a_c f \in F(c)\), we have
\[ Y_p(\tau, \iota) = \sum_{c \in \mathcal{C}_F, \text{Art}(\tau)|\mu = \tau} L\Gamma_{\iota, p}(R_c) - \left[ \sum_{c \in \mathcal{C}_F, \text{Art}(\tau)|\mu = \tau} L\Gamma_{\iota}(R_c) \right]_p, \tag{5} \]
where we put \(R_c := \{ z \in D \mid \mathcal{O}_F \supset z a_c f p_c \in c \}\). We will use the following properties of the classical or p-adic multiple gamma functions in the proof of Theorem 1.

Proposition 1. (i) Let \(R\) be as in Definition 7-(i), \(\alpha \in F_+\). Then we have
\[ L\Gamma_{\iota}(R) - L\Gamma_{\iota}(\alpha R) = \zeta_\iota(0, R) \log \iota(\alpha). \]

(ii) Let \(R\) be as in Definition 7-(ii), \(\alpha \in F_+\). Then we have
\[ L\Gamma_{\iota, p}(R) - L\Gamma_{\iota, p}(\alpha R) = \zeta_{\iota, p}(0, R) \log_p \iota(\alpha). \]

Proof. The assertions follow from \(\zeta_\iota(s, \alpha R) = \iota(\alpha)^{-s} \zeta_\iota(s, R)\) immediately. \(\square\)

We also recall Shintani’s multiple zeta functions in [Sh, (1.1)] which we need in subsequent sections.

Definition 8. (i) Let \(A = (a_{ij})\) be an \((l \times r)\)-matrix with \(a_{ij} \in \mathbb{R}_+, \ x \in \mathbb{R}^r_+, \ \chi = (\chi_1, \ldots, \chi_r) \in (\mathbb{C}^*)^r\) with \(|\chi_i| \leq 1\). Then Shintani’s multiple zeta function is defined as
\[ \zeta(s, A, x, \chi) := \sum_{(m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geq 0}} \left( \prod_{j=1}^{r} \chi_j^{m_j} \right) \left( \prod_{i=1}^{l} \left( \sum_{j=1}^{r} a_{ij} (m_j + x_j) \right) \right)^{-s}. \]

This series converges for \(\text{Re}(s) > \frac{r}{l}\), has a meromorphic continuation to the whole \(s\)-plane, is analytic at \(s = 0\).
(ii) Let \( x, \chi \) be as in (i). For \( v = (v_1, \ldots, v_r) \in F_+^r \), we consider two kinds of Shintani’s multiple zeta functions:

(a) Shintani’s multiple zeta function with \( l = 1 \):

\[
\zeta(s, v, x, \chi) = \sum_{(m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r} \left( \prod_{j=1}^r \chi_j^{m_j} \right) \left( \sum_{j=1}^r v_j (m_j + x_j) \right)^{-s}.
\]

Here we consider \( v_i \in F_+ \overset{\text{id}}{\rightarrow} \mathbb{R}_+ \).

(b) Let \( A \) be the \((n \times r)\)-matrix whose raw vectors are \( \iota(v_i) \) \( (\iota \in \text{Hom}(F, \mathbb{R}), \ n := [F : \mathbb{Q}] \) ). Then we put

\[
\zeta_N(s, v, x, \chi) := \zeta(s, A, x, \chi) = \sum_{(m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r} \left( \prod_{j=1}^r \chi_j^{m_j} \right) N \left( \sum_{j=1}^r v_j (m_j + x_j) \right)^{-s}.
\]

(iii) Let \( R \) be as in Definition 7-(i). We define

\[
\zeta_N(s, R) := \sum_{z \in R} Nz^{-s} := \sum_{i=1}^k \zeta_N(s, v_i, x_i, (1, \ldots, 1)).
\]

4 \( p \)-adic measures associated with zeta values

We consider the following two kinds of integration.

**Definition 9.** Let \( K \) be a finite extension of \( \mathbb{Q}_p \), \( O \) the ring of integers of \( K \), \( P \) the maximal ideal of \( O \).

(i) We say \( \nu \) is a \( p \)-adic measure on \( O \) if for each open compact subset \( U \subset O \), it takes the value \( \nu(U) \in K \) satisfying

(a) \( \nu(U \bigcup U’) = \nu(U) + \nu(U’) \) for disjoint open compact subsets \( U, U’ \).

(b) \( |\nu(U)|_p \)’s are bounded.

We say a \( p \)-adic measure \( \nu \) is a \( \mathbb{Z} \)-valued measure if \( \nu(U) \in \mathbb{Z} \).

(ii) Let \( \nu \) be a \( p \)-adic measure, \( f : O \to O \) a continuous map. We define

\[
\int_O f(x) d\nu(x) := \lim_{\pi \to O/P^m} \sum_{a \in O/P^m} \nu(f^{-1}(a + P^m)) f(a) \in \lim_{\pi \to O/P^m} O/P^m = O.
\]

(iii) Let \( \nu \) be a \( \mathbb{Z} \)-valued measure, \( f : (O - P^e) \to (O - P^e) \) a continuous map \( (e \in \mathbb{N}) \). We define

\[
\int_{O - P^e} f(x) d\nu(x) := \lim_{\pi \to (O - P^e) / (1 + P^m)} \prod_{\pi \in (O - P^e) / (1 + P^m)} f(a)^\nu(f^{-1}(a + P^m)) \in O.
\]
We recall the setting in [Da]. Let $F$ be a totally real field of degree $n$, $\mathfrak{f}$ an integral ideal of $F$, $p := \mathfrak{p}_\mathfrak{f}$ the prime ideal corresponding to the $p$-adic topology on $F$ induced by $\text{id}: F \hookrightarrow \mathbb{C}_p$. We assume that $p \not| \mathfrak{f}$.

**Definition 10 ([Da Definitions 3.8, 3.9])**. Let $\eta$ be a prime ideal of $F$.

(i) We say $\eta$ is good for a cone $C(v_1, \ldots, v_r)$ if $v_i \in \mathcal{O}_F - \eta$ and if $N\eta$ is a rational prime (i.e., the residue degree = 1).

(ii) We say $\eta$ is good for a Shintani set $D$ if it can be expressed as a finite disjoint union of cones for which $\eta$ is good.

**Definition 11 ([Da Definitions 3.13, 3.16, Conjecture 3.21])**. We take an element $\pi \in \mathcal{O}_{F, +}$, a prime ideal $\eta$, a Shintani domain $D \mathfrak{f} \text{ mod } E_{\mathfrak{f}+}$ satisfying the following conditions.

(i) Let $e$ be the order of $p$ in $C_\mathfrak{f}$. We fix a generator $\pi \in \mathfrak{p}^e$ satisfying $\pi \equiv 1 \mod \mathfrak{f}$.

(ii) $N\eta \geq n + 2$ and $(N\eta, \mathfrak{f}) = 1$.

(iii) The residue degree of $\eta = 1$ and the ramification degree of $\eta \leq N\eta - 2$.

(iv) $\eta$ is "simultaneously" good for $D_\mathfrak{f}, \pi^{-1}D_\mathfrak{f}$ in the following sense: There exist vectors $v_j \in (\mathcal{O}_{F, +} - \eta)^r(j)$, units $\epsilon_j \in E_{\mathfrak{f}+}$ ($j \in J'$, $|J'| < \infty$) satisfying

$$D_\mathfrak{f} = \bigsqcup_{j \in J'} C(v_j), \quad \pi^{-1}D_\mathfrak{f} = \bigsqcup_{j \in J'} \epsilon_j C(v_j).$$

**Remark 2.** Dasgupta [Da] took a suitable set $T$ of prime ideals instead of one prime ideal $\eta$. In this article, we assume that $|T| = 1$ for simplicity.

We denote by $F_p, \mathcal{O}_{F_p}$ the completion of $F$ at $p$, the ring of integers of $F_p$ respectively.

**Definition 12 ([Da Definitions 3.13, 3.17])**. Let $\pi, \eta, D_\mathfrak{f}$ be as in Definition 11, $b$ a fractional ideal of $F$ relatively prime to $\mathfrak{f}pN\eta$. We put

$$F^\times_\mathfrak{f} := \{z \in F^\times \mid z \equiv 1 \mod \mathfrak{f}\}.$$

(i) For an open compact subset $U \subset \mathcal{O}_{F_p}$, a Shintani set $D$, we put

$$\nu(b, D, U) := \zeta_N(0, F^\times_\mathfrak{f} \cap b^{-1} \cap D \cap U),$$

$$\nu_\eta(b, D, U) := \nu(b, D, U) - N\eta \nu(b^{-1}, D, U).$$

Here $\zeta_N(s, R)$ is defined in Definition 8. By [Da Proposition 3.12] we see that

- When $\eta$ is good for $D$, we have $\nu_\eta(b, D, U) \in \mathbb{Z}[N\eta^{-1}]$.
- When $\eta$ is good for $D$ and $N\eta \geq n + 2$, we have $\nu_\eta(b, D, U) \in \mathbb{Z}$. 

9
Assume that $\eta$ is good for $D$ and that $\eta \nmid p$. We define a $p$-adic measure $\nu_\eta(b, D)$ on $\mathcal{O}_F$ by

$$\nu_\eta(b, D)(U) := \nu_\eta(b, D, U).$$

Under the assumption of Definition 11, $\nu_\eta(b, D)$ is a $\mathbb{Z}$-valued measure.

For $\tau \in \text{Gal}(H_f/F)$, we put

$$\zeta_f(s, \tau) := \sum_{a \subset O_F, (a/H_f) = \tau, (a, f) = 1} N a^{-s},$$
$$\zeta_{f, \eta}(s, \tau) := \zeta_f(s, \tau) - N\eta^{1-s}\zeta_f(s, \tau(b/H_f)).$$

Here $H_f$ denotes the narrow ray class field modulo $f$.

We define $\epsilon_\eta(b, \mathcal{D}_f, \pi) := \prod_{c \in \mathcal{E}_{f,+}} \epsilon_{\nu_\eta(b, c\mathcal{D}_f \cap \pi^{-1}\mathcal{D}_f, \mathcal{O}_F)} \in E_{f,+}$,

$$u_\eta(b, \mathcal{D}_f) := \epsilon_\eta(b, \mathcal{D}_f, \pi) \pi^{\zeta_{f, \eta}(0, (H_f/F))} \int_{\mathcal{O}} x d\nu_\eta(b, \mathcal{D}_f, x) \in F_p^\times,$$

where $\mathcal{O} := \mathcal{O}_F - \pi \mathcal{O}_F$. The product in the first line is actually a finite product since $\nu_\eta(b, c\mathcal{D}_f \cap \pi^{-1}\mathcal{D}_f, \mathcal{O}_F) = 0$ for all but finite $c \in \mathcal{E}_{f,+}$.

Conjecture 3 (Da, Conjecture 3.21). Let $\pi, \eta, \mathcal{D}_f$ be as in Definition 11, $H$ the fixed subfield of $H_f$ under $(H_f/F)$.

(i) Let $\tau \in \text{Gal}(H/F)$. For a fractional ideal $b$ relatively prime to $fpN\eta$ satisfying $(H_f/F_b) = \tau$, we put

$$u_\eta(\tau) := u_\eta(b, \mathcal{D}_f).$$

Then $u_\eta(\tau)$ depends only on $\mathcal{D}_f, \eta$, not on the choices of $\mathcal{D}_f, b$.

(ii) For any $\tau \in \text{Gal}(H/F)$, $u_\eta(\tau)$ is a $p$-unit of $H$ satisfying $u_\eta(\tau) \equiv 1 \mod \eta$.

(iii) For any $\tau, \tau' \in \text{Gal}(H/F)$, we have $u_\eta(\tau\tau') = u_\eta(\tau)^{\tau'}$.  

5 The main results

We keep the notation in the previous sections: Let $F$ be a totally real field of degree $n$, $H_f$ the narrow ray class field modulo $f$. We assume that the prime ideal $p$ corresponding to the $p$-adic topology on $F$ does not divide $f$. Let $H$ be the fixed subfield of $H_f$ under $(H_f/F)$. For $\tau \in \text{Gal}(H/F)$, let $Y_p(\tau) := Y_p(\tau, \text{id})$ be as in Definition 6. For a fractional ideal $b$ relatively prime to $fpN\eta$, let $u_\eta(b, \mathcal{D}_f)$ be as in Definition 12-(iv).
Theorem 1. We have
\[
\log_p(u_\eta(b, D)) = -Y_p((H/F_b)) + N\eta Y_p((H/F_{b!})).
\]

Corollary 1. Conjecture\textsuperscript{2} implies
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{\sigma \in \text{Gal}(H/F)} \alpha_H^{(0,\sigma^{-1})(H/F)} \mod \ker \log_p,
\]
where \(p_H, h_H, \alpha_H\) are the prime ideal of \(H\) corresponding to the \(p\)-adic topology on \(H\), the class number of \(H\), a generator of \(H\).

We prepare some Lemmas in order prove this Theorem.

Lemma 2 (\textsuperscript{[CN2, Théorème 13]}). Let \(v \in F_+^r, z \in F_+, \xi = (\xi_1, \ldots, \xi_r)\) with \(\xi_i\) roots of unity, \(\neq 1\). For \(k \in \mathbb{Z}_{\geq 0}\), we have
\[
\zeta^N(-k, v, z, \xi) = \sum_{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r} \frac{\sum_{l=(l_1, \ldots, l_r), 1 \leq i \leq m_i} \binom{m_i}{l} N(z - l^tv)^k}{(1 - \xi)^m},
\]
\[
\zeta(-k, v, z, \xi) = \sum_{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r} \frac{\sum_{l=(l_1, \ldots, l_r), 1 \leq i \leq m_i} \binom{m_i}{l} (z - l^tv)^k}{(1 - \xi)^m}.
\]
Here we put \((1 - \xi)^m := \prod_{i=1}^r (1 - \xi_i)^{m_i}, \binom{m_i}{l} := \prod_{i=1}^r (-1)^{m_i-1} \binom{m_i-1}{l_i-1}\) with the binomial coefficient \(\binom{m_i-1}{l_i-1}\). The sum over \(m\) is actually a finite sum since we have \(\sum_l \binom{m_i}{l} N(z - l^tv)^k = 0\) if \(m_i\) is large enough.

Lemma 3. Let \(\nu_\eta(b, D, \mathcal{U})\) be as in Definition\textsuperscript{12}(i). Assume that \(\eta\) is good for \(D\). Then we have
\[
\nu_\eta(b, D, \mathcal{U}) = \zeta(0, F^\times_i \cap b^{-1} \cap D \cap \mathcal{U}) - N\eta \zeta(0, F^\times_i \cap b^{-1} \eta \cap D \cap \mathcal{U}).
\]

Proof. It is enough to show the statement when
- \(D\) is a cone \(C(v)\) with \(v = (v_1, \ldots, v_r), v_i \in \mathcal{O}_F - \eta\).
- \(\mathcal{U}\) is of the form \(a + p^m \mathcal{O}_{F_p}\) \((m \in \mathbb{N}, a \in \mathcal{O}_{F_p})\).

Put \(R := F^\times_i \cap b^{-1} \cap C(v) \cap (a + p^m \mathcal{O}_{F_p})\). By definition we have
\[
\nu_\eta(b, C(v), a + p^m \mathcal{O}_{F_p}) = \zeta(0, R) - N\eta \zeta(0, \{z \in R \mid \text{ord}_\eta z > 0\})
\]
\[
= \left[ \sum_{z \in R} Nz^{-s} - N\eta \sum_{z \in R, \text{ord}_\eta z > 0} Nz^{-s} \right]_{s=0}.
\]
Let $L$ be a positive integer satisfying $L \in \frak{p}^m b^{-1}$, $(\eta, L) = 1$. Then we have
\[
\sum_{z \in R} Nz^{-s} = \sum_{x \in R_a} \sum_{m \in \mathbb{Z}_{\geq 0}} N((x + m)^t(Lv))^{-s},
\]

where
\[
R_a := \{ x \in (\mathbb{Q} \cap (0, 1])^r \mid x^t(Lv) \in F_v^\times \cap b^{-1} \cap (a + \frak{p}^m \mathcal{O}_v) \}.
\]

Since $N\eta$ is a rational prime, the following homomorphism is a surjection.
\[
\mathbb{Z} \to \mathbb{Z}/N\eta \cong \mathcal{O}_F/\eta \cong \mathcal{O}_{F(\eta)}/\eta \mathcal{O}_{F(\eta)}:
\]

Here we denote the localization of $\mathcal{O}_F$ at $\eta$ by $\mathcal{O}_{F(\eta)}$. Hence for each $x \in R_a$, there exists an integer $n_x$ satisfying $x^t(Lv) \equiv n_x$ mod $\eta \mathcal{O}_{F(\eta)}$. Similarly we take $n_i$ satisfying $Lv_i \equiv n_i$ mod $\eta \mathcal{O}_{F(\eta)}$ and put $n_{Lv} := (n_1, \ldots, n_r)$. Then the following are equivalent:
\[
\ord_\eta((x + m)^t(Lv)) > 0 \iff n_x + m^t n_{Lv} \equiv 0 \text{ mod } N\eta.
\]

Let $\zeta$ be a primitive $N\eta$th root of unity. We put $\xi_x := \zeta^{n_x}$, $\xi_i := \zeta^{n_i}$, $\xi_{Lv} := (\xi_1, \ldots, \xi_r)$. Note that $\xi_i \neq 1$ for any $i$. Then we have
\[
\sum_{\lambda=1}^{N\eta-1} (\xi_x \xi_{Lv}^m)^\lambda = \begin{cases} -1 & (\ord_\eta((x + m)^t(Lv)) = 0), \\ N\eta - 1 & (\ord_\eta((x + m)^t(Lv)) > 0). \end{cases}
\]

Here we put $\xi_{Lv}^m := \prod_{i=1}^r \xi_i^m$. It follows that
\[
\sum_{x \in R_a} \sum_{\lambda=1}^{N\eta-1} \xi_x^\lambda \zeta_N(s, Lv, x, \xi_{Lv}^m) = \sum_{x \in R_a} \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{\lambda=1}^{N\eta-1} (\xi_x \xi_{Lv}^m)^\lambda N((x + m)^t(Lv))^{-s}
\]
\[
= - \sum_{z \in R} Nz^{-s} + N\eta \sum_{z \in \mathbb{Z}, \ord_\eta z > 0} Nz^{-s}.
\]

Similarly we obtain
\[
\sum_{x \in R_a} \sum_{\lambda=1}^{N\eta-1} \xi_x^\lambda \zeta(s, Lv, x, \xi_{Lv}^m) = - \sum_{z \in R} z^{-s} + N\eta \sum_{z \in \mathbb{Z}, \ord_\eta z > 0} z^{-s}
\]
\[
= \zeta(s, R) - N\eta \zeta(s, \{ z \in R \mid \ord_\eta z > 0 \}).
\]

By Lemma 2, we have for $x \in R_a$
\[
\zeta_N(0, Lv, x, \xi_{Lv}^m) = \zeta(0, Lv, x, \xi_{Lv}^m).
\]

Then the assertion follows from (6), (7), (8), (9). [\Box]

Dasgupta’s $p$-adic integration $\int dv_\eta(b, \mathcal{D}, x)$ is originally associated with special values of multiple zeta functions “with the norm” $\zeta_N(\cdots)$. By the above Lemma, we can rewrite it in terms of special values of multiple zeta functions “without the norm” $\zeta(\cdots)$.

This observation is one of the main discoveries in this paper.
Lemma 4. Let \( \nu_\eta(b, D, \mathcal{U}) \), \( O = O_{F_p} - \pi O_{F_p} \) be as in Definition \[12\]. Assume that \( \eta \) is good for \( D \). Then we have for \( k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_p \)

\[
\int_O x^k d\nu_\eta(b, D, x) = \zeta(-k, F_i^x \cap b^{-1} \cap D \cap O) - N\eta \zeta(-k, F_i^x \cap b^{-1} \eta \cap D \cap O),
\]

\[
\int_O (x)^{-s} d\nu_\eta(b, D, x) = \zeta_\eta(s, F_i^x \cap b^{-1} \cap D \cap O) - N\eta \zeta_\eta(s, F_i^x \cap b^{-1} \eta \cap D \cap O). \tag{10}
\]

Here we put \( \langle x \rangle := p_{\text{ord}_p r_p(x)}^{-1} x \) by using \( \text{ord}_p, \theta_p \) in \[2\].

Proof. It is enough to show the statement when \( D = C(v) \) with \( v = (v_1, \ldots, v_r) \), \( v_i \in \mathcal{O}_F - \eta \). By definition we can write

\[
\int_O x^k d\nu_\eta(b, C(v), x) = \lim_{a \to 0} \sum_{m \in \mathbb{N}_r} a^k \nu_\eta(b, C(v), a(1 + p^m \mathcal{O}_{F_p})).
\]

By Lemmas \[2\], \[3\] we have

\[
a^k \nu_\eta(b, C(v), a(1 + p^m \mathcal{O}_{F_p})) = - \sum_{m \in \mathbb{N}_r} \sum_{\lambda = 1}^{N\eta - 1} \sum_{x \in R_a} \sum_{t \in \mathbb{N}} \frac{\xi_x^\lambda \sum_{1 \leq l \leq m_i} \left\{ m \atop l \right\} a^k}{(1 - \xi_L \xi_v^l)^m},
\]

where \( L, R_a, \xi_x, \xi_L \) are as in the proof of Lemma \[3\]. On the other hand, by Lemma \[2\] again, we obtain

\[
\zeta(-k, F_i^x \cap b^{-1} \cap C(v) \cap O) - N\eta \zeta(-k, F_i^x \cap b^{-1} \eta \cap C(v) \cap O)
\]

\[
= - \sum_{\pi \in \mathcal{O}_F^{1+p^m \mathcal{O}_{F_p}}} \sum_{x \in R_a} \sum_{m \in \mathbb{N}_r} \sum_{\lambda = 1}^{N\eta - 1} \xi_x^\lambda \sum_{1 \leq l \leq m_i} \left\{ m \atop l \right\} ((x - l)^t (Lv))^k \frac{1 - \xi_L \xi_v^l)^m}{1 - \xi_L \xi_v^l)^m}.
\]

By definition, we see that \( L \in p^m, x^t (Lv) \equiv \alpha \mod p^m \mathcal{O}_{F_p} \) for \( x \in R_a \). It follows that

\[
a^k \equiv ((x - l)^t (Lv))^k \mod p^m \mathcal{O}_{F_p} \quad (x \in R_a),
\]

Hence the first assertion is clear. The second assertion follows from the \( p \)-adic interpolation property \[1\]. \( \Box \)

Proof of Theorem \[1\]. For a fractional ideal \( b \), a Shintani set \( D \), an open compact subset \( \mathcal{U} \subset \mathcal{O}_{F_p} \), and for \( * = 0, p \), we put

\[
\Gamma_*(b, D, \mathcal{U}) := \Gamma_*(F_i^x \cap b^{-1} \cap D \cap \mathcal{U}),
\]

\[
\Gamma_{\eta,*}(b, D, \mathcal{U}) := \Gamma_*(b, D, \mathcal{U}) - N\eta \Gamma_*(b \eta^{-1}, D, \mathcal{U})
\]

whenever each function is well-defined. It suffices to show the following three equalities:

\[
\log_p(\varepsilon_\eta(b, D_1, \pi) \pi^{Q_\eta(0, (\frac{H}{b}))}) = [\Gamma_{\eta,*}(b, D_1, O)]_p,
\]

\[
\log_p \left( \int_O x d\nu_\eta(b, D_1, x) \right) = -\Gamma_{\eta,*}(b, D_1, O), \tag{12}
\]

\[
\Gamma_p(b, D_1, O) - [\Gamma(b, D_1, O)]_p = Y_p(\frac{H}{b}). \tag{13}
\]
Let \( \mathbf{v}_j, \epsilon_j \) \( (j \in J') \) be as in Definition 11-(iv). Since \( D_j, \pi^{-1}D_j \) are fundamental domains of \( F \otimes \mathbb{R}_+/E_{\epsilon}, \) we see that

\[
\epsilon D_j \cap \pi^{-1}D_j = \prod_{j \in J'} \epsilon_j C(\mathbf{v}_j) \quad (\epsilon \in E_{\epsilon})
\]

Namely we have

\[
\epsilon_{\eta}(b, D_j, \pi) = \prod_{j \in J'} \epsilon_j^{\nu_{\eta}(b, \epsilon_j C(\mathbf{v}_j), O_{F_p})}.
\]

By Lemma 3 we can write

\[
\nu_{\eta}(b, \epsilon_j C(\mathbf{v}_j), O_{F_p}) = \zeta(0, F_j^x \cap b^{-1} \eta \cap \epsilon_j C(\mathbf{v}_j) \cap O_{F_p}) - N \eta \zeta(0, F_j^x \cap b^{-1} \eta \cap \epsilon_j C(\mathbf{v}_j) \cap O_{F_p}).
\]

Therefore by Proposition 1(i) we obtain

\[
\log_p(\epsilon_{\eta}(b, D_j, \pi)) = \left[ \sum_{j \in J'} L \Gamma_{\eta}(b, C(\mathbf{v}_j), O_{F_p}) - \sum_{j \in J'} L \Gamma_{\eta}(b, \epsilon_j C(\mathbf{v}_j), O_{F_p}) \right]_p
\]

\[
= [L \Gamma_{\eta}(b, D_j, O_{F_p}) - L \Gamma_{\eta}(b, \pi^{-1}D_j, O_{F_p})]_p.
\]

We easily see that

\[
\zeta_{J, \eta}(0, \left( \frac{H_1/F}{b} \right)) = \nu_{\eta}(b, D_j, O_{F_p}),
\]

\[
\pi(F_j^x \cap b^{-1} \cap \pi^{-1}D_j \cap O_{F_p}) = F_j^x \cap \pi b^{-1} \cap D_j \cap \pi O_{F_p}.
\]

Hence, by Proposition 1(i) again, we get

\[
\log_p(\pi_{\eta}(0, \left( \frac{H_1/F}{b} \right))) = [L \Gamma_{\eta}(b, \pi^{-1}D_j, O_{F_p}) - L \Gamma_{\eta}(\pi^{-1}b, D_j, \pi O_{F_p})]_p.
\]

Since \( F_j^x \cap b^{-1} \cap D_j \cap O \prod(F_j^x \cap \pi b^{-1} \cap D_j \cap \pi O_{F_p}) = F_j^x \cap \pi b^{-1} \cap D_j \cap \pi O_{F_p}, \) we have

\[
L \Gamma_{\eta}(b, D_j, O) = L \Gamma_{\eta}(b, D_j, O_{F_p}) - L \Gamma_{\eta}(\pi^{-1}b, D_j, \pi O_{F_p}).
\]

Then the assertion (11) follows from (14), (15), (16).

Next, differentiating (16) at \( s = 0, \) we obtain

\[
- \int_{O} \log_p x \, d\eta_{x}(b, D_j, x) = L \eta_{b}(b, D_j, O).
\]

By definition, we have \( \log_p(\int_{O} x \, d\eta_{x}(b, D_j, x)) = \int_{O} \log_p x \, d\eta_{x}(b, D_j, x). \) Hence the assertion (12) is clear.

Finally we prove (13). Let \( D \) be a Shintani domain mod \( E_+. \) For each \( c \in C_{fp}, \) we take an integral ideal \( a_c \) satisfying \( a_c \mathfrak{f} \in \pi(c), \) and put \( R_c := \{ z \in D \mid O_F \supseteq z a_c \mathfrak{fp} \in c \}. \) By (5) we can write

\[
Y_p\left( \left( \frac{H_1/F}{b} \right) \right) = \sum_{c \in C_{fp}, \text{ Art}(\mathfrak{f})|_{\mu} = (\frac{H_1/F}{b})} L \Gamma_p(R_c) - \left[ \sum_{c \in C_{fp}, \text{ Art}(\mathfrak{f})|_{\mu} = (\frac{H_1/F}{b})} L \Gamma(R_c) \right]_p.
\]
Since $H$ is the fixed subfield under $(\frac{H/F}{p})$, we may replace

$$\sum_{c \in C_p, \, \overline{\pi}=(\frac{H/F}{p})} \cdots = \sum_{k=0}^{e-1} \sum_{c \in C_p, \, \overline{\pi}=[bp^{-k}]} \cdots,$$

where $\overline{\pi}$ denotes the image under $C_p \to C_f$, $[a]$ denotes the ideal class in $C_f$ of a fractional ideal $a$. On the other hand, we can write for $* = \emptyset, p$

$$L\Gamma_*(b, D_f, O) = \sum_{k=0}^{e-1} L\Gamma_*(b, D_f, p^kO_{k_p}^\times).$$

Therefore it suffices to show that we have for each $k$

$$(\sum_{c \in C_p, \, \overline{\pi}=[bp^{-k}]} L\Gamma_p(R_c)) - L\Gamma_p(b, D_f, p^kO_{k_p}^\times) = [(\sum_{c \in C_p, \, \overline{\pi}=[bp^{-k}]} L\Gamma(R_c)) - L\Gamma(b, D_f, p^kO_{k_p}^\times)]_p. \quad (17)$$

We fix $k$. Whenever $\overline{\pi} = [bp^{-k}]$, $\pi(c) \in C_{(1)}$ is constant, so we may put $a_c$ to be a fixed integral ideal $a_0$. Then we have

$$\prod_{c \in C_p, \, \overline{\pi}=[bp^{-k}]} R_c = \{z \in (a_0f_p)^{-1} \cap D \mid (za_0f_p, f_p) = 1, \, [za_0f_p] = [bp^{-k}] \text{ in } C_f\}.$$

Let $\alpha_0 \in F_+^*$ be a generator of the principal ideal $(a_0f_p)(bp^{-k})^{-1}$. Then the following are equivalent:

$$[za_0f_p] = [bp^{-k}] \Leftrightarrow [(z\alpha_0)] = [(1)] \Leftrightarrow \exists \epsilon \in E_+ \text{ such that } z\alpha_0 \equiv 1 \mod f.$$

Hence, taking a representative set $E_0$ of $E_+/E_{f,+}$, we can write

$$\{z \in (a_0f_p)^{-1} \cap D \mid (za_0f_p, f_p) = 1, \, [za_0f_p] = [bp^{-k}] \text{ in } C_f\} = \prod_{\epsilon \in E_0} (\epsilon\alpha_0)^{-1}(F_f^\times \cap b^{-1} \cap \epsilon\alpha_0 D \cap p^kO_{k_p}^\times).$$

Namely we have for $* = \emptyset, p$

$$\sum_{c \in C_p, \, \overline{\pi}=[bp^{-k}]} L\Gamma_*(R_c) = \sum_{\epsilon \in E_0} L\Gamma_*(((\epsilon\alpha_0)^{-1}(F_f^\times \cap b^{-1} \cap \epsilon\alpha_0 D \cap p^kO_{k_p}^\times))). \quad (18)$$

On the other hand, $D_f' := \bigcup_{\epsilon \in E_0} \epsilon\alpha_0 D$ becomes another Shintani domain mod $E_{f,+}$, and we can write for $* = \emptyset, p$

$$L\Gamma_*(b, D_f', p^kO_{k_p}^\times) = \sum_{\epsilon \in E_0} L\Gamma_*(F_f^\times \cap b^{-1} \cap \epsilon\alpha_0 D \cap p^kO_{k_p}^\times). \quad (19)$$
Then the assertion (17), replacing $\mathcal{D}_f$ with $\mathcal{D}'_f$, follows from (18), (19) and Proposition 1.

We conclude the proof of (13) by showing that

$$L\Gamma_p(b, \mathcal{D}_f, p^k \mathcal{O}_{F_p}^\times) - L\Gamma_p(b, \mathcal{D}'_f, p^k \mathcal{O}_{F_p}^\times) = [L\Gamma(b, \mathcal{D}_f, p^k \mathcal{O}_{F_p}^\times) - L\Gamma(b, \mathcal{D}'_f, p^k \mathcal{O}_{F_p}^\times)]_p.$$ 

Note that the independence on the choice of $\mathcal{D}_f$ is also discussed in [Da, §5.2] under certain conditions. Similarly to [Yo, Chap. III, Lemma 3.13], we see that there exist cones $C(v_j)$ and units $u_j \in E_{i,+}$ ($j \in J''$) which satisfy

$$\mathcal{D}_f = \prod_{j \in J''} C(v_j), \quad \mathcal{D}'_f = \prod_{j \in J''} u_j C(v_j).$$

Therefore it suffices to show that

$$L\Gamma_p(b, C(v_j), p^k \mathcal{O}_{F_p}^\times) - L\Gamma_p(b, u_j C(v_j), p^k \mathcal{O}_{F_p}^\times)$$

$$= [L\Gamma(b, C(v_j), p^k \mathcal{O}_{F_p}^\times) - L\Gamma(b, u_j C(v_j), p^k \mathcal{O}_{F_p}^\times)]_p.$$ 

It follows from Proposition 1 since $F_i^\times \cap b^{-1} \cap u_j C(v_j) \cap p^k \mathcal{O}_{F_p}^\times = u_j(F_i^\times \cap b^{-1} \cap C(v_j) \cap p^k \mathcal{O}_{F_p}^\times)$.

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