A NOTE ON POINT-FINITE COVERINGS BY BALLS

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Abstract. We provide an elementary proof of a result by V.P. Fonf and C. Zanco on point-finite coverings of separable Hilbert spaces. Indeed, by using a variation of the famous argument introduced by J. Lindenstrauss and R.R. Phelps [9] to prove that the unit ball of a reflexive infinite-dimensional Banach space has uncountably many extreme points, we prove the following result.

Let $X$ be an infinite-dimensional Hilbert space satisfying $\text{dens}(X) < 2^{\aleph_0}$, then $X$ does not admit point-finite coverings by open or closed balls, each of positive radius.

In the second part of the paper, we follow the argument introduced by V.P. Fonf, M. Levin, and C. Zanco in [5] to prove that the previous result holds also in infinite-dimensional Banach spaces that are both uniformly rotund and uniformly smooth.

1. Introduction

A family of subsets of a real normed space $X$ is called a covering if the union of all its members coincides with $X$. A covering of $X$ is point-finite if each point of $X$ is contained in at most finitely many members of the covering.

The problem concerning existence of point-finite coverings of infinite-dimensional normed spaces by balls was considered for the first time in the paper [8] in which V. Klee asked the following question.

Problem 1.1 ([8, Question 2.6]). Let $\Gamma$ be a cardinal such that $|\Gamma| \geq \aleph_0$, does $\ell_1(\Gamma)$ (respectively $\ell_p(\Gamma)$ for $1 < p < \infty$) admit a locally finite (respectively point-finite) covering by closed balls or open balls, each of positive radius?

The question above was motivated by the results, contained in the paper itself, implying existence of a covering of $\ell_1(\Gamma)$ by pairwise disjoint closed balls of radius 1, whenever $\Gamma$ is a suitable uncountable set. In [6], V.P. Fonf and C. Zanco generalized Corson’s theorem (see Theorem 2.2 below) by proving that if a Banach space $X$ contains an infinite-dimensional closed subspace non-containing $c_0$ then $X$ does not admit any locally finite covering by bounded...
closed convex bodies. This completely solved the problem concerning locally finite coverings by balls of \( \ell_1(\Gamma) \).

More recently, V.P. Fonf and C. Zanco [7] proved that the infinite-dimensional separable Hilbert space does not admit point-finite coverings by closed balls of positive radius. Then V.P. Fonf, M. Levin and C. Zanco [5] extended the result above to separable spaces that are both uniformly smooth and uniformly rotund. However, Klee’s problem about point-finite coverings by balls of \( \ell_p(\Gamma) \) spaces \((1 < p < \infty)\) remained open in the non-separable case.

The proof of the result by V.P. Fonf and C. Zanco, contained in [7], is based on the following ingredients:

(i) [7, Proposition 2.1], a result excluding existence of certain point-finite families of slices of the unit ball in separable Banach space;

(ii) [7, Theorem 3.1], a characterization of separable isomorphically polyhedral Banach spaces via existence of point-finite countable coverings by slices of the unit sphere;

(iii) the fact that the intersection among two distinct spheres in any Hilbert space lies in some hyperplane. Indeed, this is a 3-dimensional characterization of inner product spaces [10, (15.17)];

(iv) the fact that no infinite-dimensional dual (and in particular reflexive) Banach space is polyhedral [10].

The aim of the present paper is to provide a direct and quite elementary proof of the main result contained in [7] and to present an improvement of the result contained in [5], concerning point-finite coverings by balls of Banach spaces that are both uniformly smooth and uniformly rotund. Let us start by describing the result contained in Section 2. Our Proposition 2.1 is a restatement of [7, Proposition 2.1], the elementary alternative proof presented in our paper is an immediate application of the uniform boundedness principle and it works also in the non-separable case. Theorem 2.5 excludes existence of certain point-finite families of open or closed slices of the unit ball in reflexive Banach spaces, and it is a variation of the famous argument introduced by J. Lindenstrauss and R.R. Phelps [9] to prove that the unit ball of a reflexive infinite-dimensional Banach space has uncountably many extreme points. Theorem 2.5, combined with (iii), allows us to obtain the following slight improvement of [7, Corollary 3.3].

Let \( X \) be an infinite-dimensional Hilbert space.

(i) If the density character of \( X \) satisfies \( \text{dens}(X) < 2^{\aleph_0} \) then it does not admit point-finite coverings by open or closed balls, each of positive radius.

(ii) \( X \) does not admit point-finite coverings by open balls.
Finally, in Section 3, we observe that, following the argument introduced in [5], it is possible to extend this latter result to Banach spaces that are both uniformly smooth and uniformly rotund. The new ingredients in our proof are Lemma 3.5 that allows us to deal with open and closed balls at the same time, and an easy separable reduction argument used in Theorem 3.7. In particular, our results solve in negative Klee’s problem for point-finite coverings by open balls of $\ell_p(\Gamma)$ spaces ($1 < p < \infty$).

2. Point-finite coverings by slices and balls in Hilbert spaces

Throughout the paper, we consider only nontrivial real normed spaces. If $X$ is a normed space then $X^*$ is its dual Banach space. We denote by $B_X$, $U_X$, and $S_X$ the closed unit ball, the open unit ball, and the unit sphere of $X$, respectively. We denote by $U(x, \varepsilon)$ the open ball with radius $\varepsilon > 0$ and center $x$. We denote by $B(x, \varepsilon)$ the closed ball with radius $\varepsilon \geq 0$ and center $x$; in the case $\varepsilon = 0$, $B(x, \varepsilon)$ is the degenerate ball containing only the point $x$. In general, by a ball in $X$ we mean a closed ball of non-negative radius or an open ball of positive radius in $X$. For $x, y \in X$, $[x, y]$ denotes the closed segment in $X$ with endpoints $x$ and $y$, and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding “open” segment. A set $B \subset X$ will be called a body if it is closed, convex and has nonempty interior. A body is called rotund if its boundary does not contain nontrivial segments. Other notation is standard, and various topological notions refer to the norm topology of $X$, if not specified otherwise.

Let $\mathcal{F}$ be a family of nonempty sets in a normed space $X$. By $\bigcup \mathcal{F}$ we mean the union of all members of $\mathcal{F}$. A point $x \in X$ is a regular point for $\mathcal{F}$ if it has a neighbourhood that meets at most finitely many members of $\mathcal{F}$. Points that are not regular are called singular. Notice that the set of singular points is a closed set.

**Definition 2.1.** The family $\mathcal{F}$ is called:

(i) point-finite if each $x \in X$ is contained in at most finitely many members of $\mathcal{F}$;

(ii) locally finite if each $x \in X$ is a regular point for $\mathcal{F}$.

A minimal covering is a covering whose no proper subfamily is a covering. A standard application of Zorn’s lemma shows that every point-finite covering contains a minimal subcovering.

In the sequel, we say that $\mathcal{F}$ is a family of open or closed balls of $X$ if each element of $\mathcal{F}$ is an open ball (of positive radius) or a closed ball of non-negative radius (i.e., if not differently stated, we admit that $\mathcal{F}$ contains also degenerate balls).

Let us recall the following famous theorem by H.H. Corson [2].
Theorem 2.2. Let $\mathcal{F}$ be a covering of a reflexive infinite-dimensional Banach space by bounded convex sets. Then $\mathcal{F}$ is not locally finite.

In what follows, we shall use several times the following fact that immediately follows by [4, Lemma 2.2]. Let us recall that, if $T$ is a topological vector space, $\text{dens}(T)$ denotes its density character (i.e., the smallest cardinality of a dense subset of $T$).

**Fact 2.3.** Let $T$ be a topological space and let $\mathcal{B}$ be a point-finite family of subsets of $T$. Let us denote $\mathcal{B}' := \{ B \in \mathcal{B}; \text{int } B \neq \emptyset \}$, then $|\mathcal{B}'| \leq \text{dens}(T)$.

The following proposition is a restatement of [7, Proposition 2.1]. The elementary alternative proof presented here below is an immediate consequence of the uniform boundedness principle and it works also in the non-separable case.

**Proposition 2.4.** Let $X$ be a Banach space. Let $D \subset X^*$ be an unbounded set. For each $f \in D$, define $S_f := \{ x \in X; f(x) \geq 1 \}$. Then there exist $x \in S_X$ and an infinite set $N \subset D$ such that $x \in \text{int } S_f$, whenever $f \in N$.

**Proof.** Suppose on the contrary that, for every $x \in S_X$, the set

$$N_x := \{ f \in D; x \in \text{int } S_f \}$$

is finite. Fix $x \in S_X$ and observe that, since $N_x$ is finite, the set $x(D) \subset \mathbb{R}$ is upper-bounded. By the Banach-Steinhaus uniform boundedness principle, we get a contradiction. \[\square\]

The following theorem is the core of the results of this section and it is a variation of [9, Theorem 1.1], in which J. Lindenstrauss and R.R. Phelps proved that the unit ball of a reflexive infinite-dimensional Banach space has uncountably many extreme points.

**Theorem 2.5.** Let $X$ be an infinite-dimensional reflexive Banach space and $\{ f_n \} \subset X^* \setminus U_{X^*}$. For each $n \in \mathbb{N}$, let $S_n$ be one of the following two sets

$$\{ x \in X; f_n(x) \geq 1 \}, \quad \{ x \in X; f_n(x) > 1 \}.$$

Let us denote $S = \{ S_n \}_{n \in \mathbb{N}}$ and suppose that $S_X \subset \bigcup S$. Then $S$ is not point-finite.

**Proof.** Suppose on the contrary that, for every $x \in S_X$, the set

$$N_x := \{ n \in \mathbb{N}; x \in S_n \}$$

is finite. By Proposition 2.4, we can assume that $\{ f_n \}$ is bounded in $X^*$. For every $n \in \mathbb{N}$, let $U_n = f_n^{-1}((-\infty, 1))$ and put $U = \bigcap_n U_n$. Then $U$ is a convex set and $0 \in \text{int } U$ (since $\{ f_n \}$ is bounded in $X^*$). Moreover, $S_X \cap U = \emptyset$ and hence $U \subset U_X$. 

We claim that $U$ is open. To see this, let $x \in U \setminus \{0\}$ and suppose on the contrary that sup$_n f_n(x) = 1$. Then, since $f_n(x) < 1$ for each $n \in \mathbb{N}$ and since $\|x\| < 1$, $N_x/\|x\|$ is an infinite set. This contradiction proves our claim.

Now, for every $n \in \mathbb{N}$, put $F_n = \{x \in U; f_n(x) = p_U(x)\}$ (where $p_U$ denotes the Minkowski gauge of the set $U$) and observe that $F_n$ is closed convex and hence $w$-closed.

Fix $x \in \partial U$ and observe that, for each $n \in \mathbb{N}$, $f_n(x) \leq 1$; since $x \notin U$, there exists $n \in \mathbb{N}$ such that $f_n(x) = 1$. Hence $U = \bigcup F_n$. Since $U$ is $w$-compact, by the Baire category theorem, we can suppose without any loss of generality that $F_1$ has nonempty interior in $(U, w)$. So, there exist $x_0 \in F_1 \cap U$ and $W$, a neighbourhood of the origin in the $w$-topology, such that $(x_0 + W) \cap U \subset F_1$. Since $X$ is infinite-dimensional, there exists $y_0 \in [x_0 + (W \cap \ker f_1)] \cap \partial U \subset F_1$. Then

$$1 = p_U(y_0) = f_1(y_0) = f_1(x_0) = p_U(x_0).$$

A contradiction, since $x_0 \notin U$. □

The following observation is an easy consequence of the fact that the intersection among two distinct spheres in any Hilbert space lies in some hyperplane (see $[\Pi] (15.17)$).

**Observation 2.6.** Let $X$ be a Hilbert space and let $B$ be a closed (open, respectively) ball intersecting the unit sphere $S_X$. Then there exists a closed (open, respectively) slice $S$ of $B_X$ such that $S_X \cap B$ coincide with $S_X \cap S$.

We are now ready to prove the main result of this section.

**Theorem 2.7.** The following assertions hold true.

(i) Let $B$ be a covering of a separable infinite-dimensional Hilbert space by closed or open balls. Suppose that $B$ is point-finite, then $|B| = 2^{\aleph_0}$.

(ii) If we suppose that $\Gamma$ is an infinite set such that $|\Gamma| < 2^{\aleph_0}$, $\ell_2(\Gamma)$ does not admit a point-finite covering by open or closed balls, each of positive radius.

(iii) Let $B$ be a covering of an infinite-dimensional Hilbert space by open balls. Then $B$ is not point-finite.

**Proof.** Let us observe that (ii) follows easily by (i), indeed assume on the contrary that $B$ is a point-finite cover of $\ell_2(\Gamma)$ by open or closed balls, each of positive radius. Since the density character of $\ell_2(\Gamma)$ is $|\Gamma|$, by Fact $2.3$ we have $|B| < 2^{\aleph_0}$. Let us consider $Y = \ell_2 \subset \ell_2(\Gamma)$ and observe that

$$B' := \{B \cap Y; B \in B, B \cap Y \neq \emptyset\}$$

is a cover of a separable infinite-dimensional Hilbert space by open or closed balls such that $|B'| < 2^{\aleph_0}$. By (i), we get a contradiction.
Similarly, (i) implies (iii). Indeed, if $\mathcal{B}$ is a cover of an infinite-dimensional Hilbert space $X$ by open balls and we consider $Y = \ell_2 \subset X$, we have that

$$\mathcal{B'} := \{ B \cap Y; B \in \mathcal{B}, B \cap Y \neq \emptyset \}$$

is a cover of a separable infinite-dimensional Hilbert space by open balls. By Fact 2.3, $\mathcal{B'}$ is countable. By (i), $\mathcal{B'}$ (and hence $\mathcal{B}$) is not point-finite.

It remains to prove (i). Let $\mathcal{B}$ be a point-finite cover of a separable infinite-dimensional Hilbert space $X$ by open or closed balls. Since $|X| = 2^{\aleph_0}$ and $\mathcal{B}$ is point-finite, we clearly have $|\mathcal{B}| \leq 2^{\aleph_0}$. Now, suppose on the contrary that $|\mathcal{B}| < 2^{\aleph_0}$. Since the origin of $X$ is contained in finitely many members of $\mathcal{B}$, if we denote

$$\mathcal{B''} := \{ B \in \mathcal{B}; 0 \notin B \},$$

there exists $R_0 > 0$ such that, for each $r \geq R_0$, $rS_X$ is contained in $\bigcup \mathcal{B'}$.

Let us consider the set $A \subset [R_0, \infty)$ defined by

$$A := \{ r \geq R_0; \exists B \in \mathcal{B'} \text{ such that } B \subset rS_X \};$$

that is, $A$ is the set of all $r \geq R_0$ such that $rS_X$ contains a degenerate ball $B \in \mathcal{B'}$. It is clear that $|A| \leq |\mathcal{B'}| = |\mathcal{B}| < 2^{\aleph_0}$ and hence there exists $\rho \in [R_0, \infty) \setminus A$. By the separability of the space, it is clear that the family

$$\mathcal{B''} := \{ B \in \mathcal{B'}; B \cap \rho S_X \neq \emptyset \},$$

is countable (indeed, each element in $\mathcal{B''}$ has nonempty interior). Moreover, $\rho S_X$ is contained in $\bigcup \mathcal{B''}$. By Observation 2.6, there exists a countable point-finite family $\mathcal{S}$ of closed or open slices of $\rho B_X$ which covers $\rho S_X$ and such that $0 \notin \mathcal{S}$, whenever $S \in \mathcal{S}$. By Theorem 2.5, we get a contradiction. $\square$

3. Point-finite coverings by balls of Banach spaces that are both uniformly rotund and uniformly smooth

The aim of this section is to show that, following the argument introduced in [5], it is possible to extend Theorem 2.7 to Banach spaces that are both uniformly rotund and uniformly smooth. The next two results coincide with [5, Proposition 2.3] and [5, Fact 2.4], respectively. Observe that, if we use Proposition 2.4 instead of [7, Proposition 2.1], both the proofs presented in [5] work also in the non-separable case and even if we consider families of open or closed balls.

**Proposition 3.1.** Let $\mathcal{B} = \{ B_n \}_{n \in \mathbb{N}}$ be a countable family of open or closed balls in a uniformly smooth Banach space $X$. Let us denote by $R_n$ the radius of $B_n$ ($n \in \mathbb{N}$) and suppose that $R_n \to \infty$. If $\mathcal{B}$ is not locally finite, then it is not point-finite.
Fact 3.2. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable collection of open or closed balls in a uniformly rotund Banach space $X$. Let us denote by $R_n$ the radius of $B_n$ ($n \in \mathbb{N}$). Let $b > 0$ and $x_0 \in X$. Suppose that, for each $n \in \mathbb{N}$, $R_n > b$ and $x_0 \notin \text{int}B_n$. If

$$F_n = \text{conv}(B_n \setminus U(x_0, b))$$

and $\text{dist}(x_0, F_n) \to 0$ then $R_n \to \infty$.

The next lemma coincides with [5, Lemma 2.5]. Observe that in their statement it is not necessary to require that the members of $\mathcal{F}$ are closed.

Lemma 3.3. Let $X$ be a reflexive Banach space. Let $x_0 \in X$, $a > b > c > 0$ and $\mathcal{F}$ a collection of convex subsets of $X$ contained in $B(x_0, a) \setminus U(x_0, c)$ such that $\mathcal{F}$ covers $B(x_0, a) \setminus U(x_0, b)$. Then $\mathcal{F}$ is not locally finite in $X$.

The next lemma coincides with [5, Lemma 2.6]. Observe that it holds also in the case $X'$ is a closed infinite-dimensional subspace. Moreover, in their statement it is not necessary to require that the members of $\mathcal{F}$ are closed. Indeed, it is sufficient in its proof to use Fact 3.2 and Lemma 3.3 instead of [5, Fact 2.4] and [5, Lemma 2.5], respectively.

Lemma 3.4. Let $X$ be both uniformly rotund and uniformly smooth. Consider a closed infinite-dimensional subspace $X' \subset X$ and let $x_0 \in X'$, $a > 0$. Assume that $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ is a countable point-finite collection of open or closed balls and $\mathcal{F} = \{F_n\}_{n=1}^{\infty}$ is a countable collection of convex sets such that: $\mathcal{F}$ covers $B(x_0, a) \cap X'$, $F_n \subset B_n \cap B(x_0, a)$ and $x_0 \notin \text{int}B_n$, whenever $n \in \mathbb{N}$. Then there is a point $y \in B(x_0, a) \cap X'$, $y \neq x_0$, that is a singular point for $\mathcal{F}$.

Lemma 3.5. Let $A$ be an open convex subset of an infinite-dimensional Banach space $X$. Let $A_1, \ldots, A_n$ be nonempty convex sets in $X$ such that, for each $i = 1, \ldots, n$ and $x \in \partial A_i$, there exists a hyperplane $\Gamma$ supporting $\overline{A}_i$ at $x$ such that $\Gamma \cap \overline{A}_i = \{x\}$. Define $D = A \setminus (A_1 \cup \cdots \cup A_n)$, then $D \subset \text{int}(\text{conv}(D))$.

Proof. Suppose that $x \in D$, let us prove that $x \in \text{int}(\text{conv}(D))$. If $x \in \text{int}D$ there is nothing to prove. Suppose that $x \in \partial D$, without any loss of generality, we can suppose that there exists $1 \leq m \leq n$ such that:

(i) $x \in \partial A_1 \cap \ldots \cap \partial A_m$;

(ii) $x \notin \overline{A}_k$, whenever $m < k \leq n$.

For each $i = 1, \ldots, m$, let $\Gamma_i$ be a hyperplane supporting $\overline{A}_i$ at $x$ such that $\Gamma_i \cap \overline{A}_i = \{x\}$. Since $X$ is infinite-dimensional, $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_m$ is an infinite-dimensional affine subset of $X$. Since $A$ is open, there exists $\varepsilon > 0$ such that $B(x, 2\varepsilon) \subset A \setminus \left(\bigcup_{m < k \leq n} A_k\right)$. Let $v_j \in \Gamma \cap B(x, \varepsilon)$ ($j = 1, 2$) be such that $x \in \langle v_1, v_2 \rangle$ and let $0 < \delta < \varepsilon$ be such that $B(v_j, \delta) \cap \overline{A}_i = \emptyset$ ($j = 1, 2$, $i = 1, \ldots, m$). Then clearly $B(x, \delta) \subset \text{conv}(D)$ and the proof is concluded. \qed
Using the previous lemma we obtain the following easy variation of [5, Lemma 2.7]. For the sake of completeness, we include a proof.

**Lemma 3.6.** Let $X$ be both uniformly rotund and uniformly smooth. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable point-finite family of open or closed balls in $X$. Let $Y \subset X$ be a separable infinite-dimensional closed subspace of $X$ and suppose that $B'_n = B_n \cap Y \neq \emptyset$ ($n \in \mathbb{N}$) and that $\{B'_n\}_{n \in \mathbb{N}}$ is a covering of $Y$. Put $B^\#_1 = B'_1$ and, for each $n \in \mathbb{N}$, define

$$B^\#_{n+1} = \begin{cases} \text{conv}((B'_{n+1} \setminus (B'_1 \cup \cdots \cup B'_n))) & \text{if } B'_{n+1} \text{ is a closed set in } Y; \\ \text{int}_Y(\text{conv}((B'_{n+1} \setminus (B'_1 \cup \cdots \cup B'_n)))) & \text{if } B'_{n+1} \text{ is an open set in } Y. \end{cases}$$

Then $\mathcal{B}^\# = \{B^\#_n\}_{n=1}^\infty$ is a point-finite covering of $Y$. Moreover, for every $n \in \mathbb{N}$, we have that $B^\#_n \subset B'_n \subset B_n$ and any $x_0 \in \bigcup_{n \in \mathbb{N}} \text{int}_Y B'_n$ is a regular point for $\mathcal{B}^\#$.

**Proof.** Observe that, since $X$ is uniformly rotund, for each $n \in \mathbb{N}$, one of the following conditions hold:

(i) $\overline{B'}_n Y$ is a rotund body in $Y$;

(ii) $B'_n$ is a singleton.

In any case, for each $n \in \mathbb{N}$ and $x \in \partial_Y B'_n$, there exists a hyperplane $\Gamma$ in $Y$ supporting $\overline{B'}_n Y$ at $x$ such that $\Gamma \cap \overline{B'}_n Y = \{x\}$. Applying Lemma 3.5, we have that $B'_{n+1} \setminus (B'_n \cup \cdots \cup B'_1) \subset B^\#_{n+1}$, whenever $n \in \mathbb{N}$. Hence, $\mathcal{B}^\#$ is a covering of $Y$.

For the latter part we proceed as in the proof of [5, Lemma 2.7]. For $n \in \mathbb{N}$, let us denote by $R_n$ the radius of the ball $B_n$. Assume on the contrary that, for some $\tilde{n} \in \mathbb{N}$, $x_0 \in \text{int}_Y B^\#_{\tilde{n}}$ is a singular point for $\mathcal{B}^\#$. Then there exists a subsequence of the integers $\{n_i\}_{i \in \mathbb{N}}$ such that, for each $i \in \mathbb{N}$: (a) $n_i > \tilde{n}$; (b) $x_0 \notin B'_{n_i}$; (c) for every $j \geq i$, $B(x_0, 1/i)$ intersects the set $B^\#_{n_j}$. Note that $B^\#_n \subset \text{conv}(B'_n \setminus B'_0)$, whenever $i \in \mathbb{N}$. Then $B(x_0, 1/i)$ intersects $\text{conv}(B_{n_j} \setminus B_0)$, whenever $i \in \mathbb{N}$ and $j \geq i$. Let $b > 0$ be such that $B(x_0, b) \subset B_0$, then it holds

$$\text{conv}(B_{n_j} \setminus B(x_0, b)) \cup \text{conv}(B_{n_j} \setminus B_0).$$

Since, for each $i \in \mathbb{N}$, $x_0 \notin B_{n_i}$, from Fact 3.2 we get that $R_{n_i} \to \infty$. By Proposition 3.1, this contradicts the assumption that $\mathcal{B}$ is point-finite. \qed

**Theorem 3.7.** Let $X$ be an infinite-dimensional Banach space. Suppose that $X$ is both uniformly rotund and uniformly smooth. Then the following assertions hold true.

(i) If $\text{dens}(X) < 2^{\aleph_0}$ then $X$ does not admit a point-finite covering by open or closed balls, each of positive radius.

(ii) If $\mathcal{B}$ is a cover of $X$ by open balls then $\mathcal{B}$ is not point-finite.
A note on point-finite coverings by balls.

Proof. (i) Suppose on the contrary that $\mathcal{B}$ is a point-finite covering of $X$ by open or closed balls, each of positive radius. By Fact 2.3 we have $|\mathcal{B}| < 2^{\aleph_0}$.

Let $Y$ be a separable infinite-dimensional closed subspace of $X$. Let us denote

$$\mathcal{B}' = \{ B \cap Y; B \in \mathcal{B}, B \cap Y \neq \emptyset \}.$$ 

Clearly $\mathcal{B}'$ is a point-finite covering of $Y$ and passing to a subcovering we can suppose that $\mathcal{B}'$ is a minimal covering of $Y$. If we denote

$$\mathcal{C}' = \{ C \in \mathcal{B}'; \text{int}_Y C \neq \emptyset \}, \quad \mathcal{D}' = \{ D \in \mathcal{B}'; |D| = 1 \},$$

it is clear that $\mathcal{B}' = \mathcal{C}' \cup \mathcal{D}'$ and that $\mathcal{C}'$ is countable (since $Y$ is separable). Hence, $\bigcup \mathcal{C}'$ is a Borel subset of $Y$. By the fact that $\mathcal{B}'$ is minimal we have that $\bigcup \mathcal{D}' = Y \setminus \bigcup \mathcal{C}'$. Hence, $\bigcup \mathcal{D}'$ is a Borel subset of a Polish space such that $|\bigcup \mathcal{D}'| < 2^{\aleph_0}$. By [11, Theorem 3.2.7], $|\mathcal{D}'| = |\bigcup \mathcal{D}'| \leq \aleph_0$, and hence $\mathcal{B}'$ is countable. Let $\mathcal{B}' = \{ B'_n \}_{n \in \mathbb{N}}$ and suppose that, for each $n \in \mathbb{N}$, $B'_n = B_n \cap Y$ for some $B_n \in \mathcal{B}$. Now, we proceed as in the proof of [5, Theorem 1.5]. Consider the covering $\mathcal{B}^\#$ of $Y$ from Lemma 3.6 and let $S \subset Y$ be the set of the points that are singular for $\mathcal{B}^\#$. By Theorem 2.2, we have $S \neq \emptyset$ and, by Lemma 3.6, we have $S \subset \bigcup_n \partial Y' B'_n$. Since $S$ is closed in $Y$, by the Baire category theorem, there are $m \in \mathbb{N}$, $x_0 \in S$, and $a > 0$ such that $S \cap B(x_0, a) \subset \partial Y' B'_m$. Observe that we have two possibilities: $B'_m$ is a singleton or $\overline{B'_m}^Y$ is a rotund body in $Y$. In any case, there exists a closed hyperplane $Y'$ in $Y$ passing through $x_0$ and intersecting $\overline{B'_m}^Y$ only at $x_0$. Then, by applying Lemma 3.4 to the families $\mathcal{F} = \{ B^\# \cap B(x_0, a); B^\# \in \mathcal{B}^\# \}$ and $\{ B_n \}_{n \in \mathbb{N}}$, with respect to the subspace $Y'$, we get a contradiction.

The proof of (ii) is similar but easier. Indeed, observe that if $Y$ and $\mathcal{B}'$ are defined as above then we clearly have that $\mathcal{B}'$ is countable and then we can proceed as in the previous point.

In the non-separable case, non-existence of coverings by balls satisfying certain condition, were recently proved in the papers [3, 4]. In [4], the authors showed that if $X$ is LUR or uniformly smooth then it does not admit star-finite coverings by closed balls, each of positive radius (we recall that a family of sets is called star-finite if each of its members intersects only finitely many other members of the family). The results contained in [3] imply that if $X$ is LUR or Fréchet smooth then it does not admit tilings by closed balls. However, the following problem remains open, even in the case $X$ is a Hilbert space.

Problem 3.8. Is it possible to generalize (i) in Theorem 3.7 to the case $\text{dens}(X) \geq 2^{\aleph_0}$?
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REFERENCES

[1] D. Amir, Characterizations of inner product spaces, Operator Theory: Advances and Applications, 20, Birkhauser Verlag, Basel, 1986.
[2] H.H. Corson, Collections of convex sets which cover a Banach space, Fund. Math. 49 (1961), 143–145.
[3] C.A. De Bernardi and L. Veselý, Tilings of normed spaces, Canad. J. Math. 69 (2017), 321–337.
[4] C.A. De Bernardi, J. Somaglia and L. Veselý, Star-finite coverings of Banach spaces, arXiv:2002.04308.
[5] V.P. Fonf, M. Levin, and C. Zanco, Covering $L^p$ spaces by balls, J. Geom. Anal. 24 (2014), 1891–1897.
[6] V.P. Fonf and C. Zanco, Covering a Banach space, Proc. Amer. Math. Soc. 134 (2006), 2607–2611.
[7] V.P. Fonf and C. Zanco, Covering the unit sphere of certain Banach spaces by sequences of slices and balls, Canad. Math. Bull. 57 (2014), 42–50.
[8] V. Klee, Dispersed Chebyshev sets and coverings by balls, Math. Ann. 257 (1981), 251–260.
[9] J. Lindenstrauss and R.R. Phelps, Extreme point properties of convex bodies in reflexive Banach spaces, Israel J. Math. 6 (1968), 39–48.
[10] J. Lindenstrauss, Notes on Klee’s paper: “Polyhedral sections of convex bodies”, Israel J. Math. 4 (1966), 235–242.
[11] S.M. Srivastava, A course on Borel sets, Graduate Texts in Mathematics, 180, Springer-Verlag, New York, 1998.

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