CONVERGENCE OF FUNCTIONS AND THEIR MOREAU-YOSIDA ENVELOPES ON HADAMARD SPACES

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Abstract. A well known result of H. Attouch states that the Mosco convergence of a sequence of proper convex lower semicontinuous functions defined on a Hilbert space is equivalent to the pointwise convergence of the associated Moreau-Yosida envelopes. In the present paper we generalize this result to Hadamard spaces. More precisely, while it has already been known that the Mosco convergence of a sequence of convex lower semicontinuous functions on a Hadamard space implies the pointwise convergence of the corresponding Moreau-Yosida envelopes, the converse implication was an open question. We now fill this gap.

1. Introduction

Proximal mappings and Moreau-Yosida envelopes of convex functions play a central role in convex analysis. In particular, they appeared in various minimization algorithms which have recently found application in image processing and machine learning. For overviews, see for instance [7,10,17].

In the present paper we are concerned with the relation between Moreau-Yosida envelopes and the Mosco convergence. Specifically, a well known result of H. Attouch says that the Mosco convergence of a sequence of convex lower semicontinuous functions on a Hilbert space is completely characterized by the pointwise convergence of their Moreau-Yosida envelopes [1, Theorem 1.2]. Note that this result was later on extended into a certain class of Banach spaces [2, Theorem 3.26].

We briefly recall the result in a Hilbert space $H$. The domain $\text{dom } f$ of a function $f : H \to (-\infty, +\infty]$ is given by $\text{dom } f := \{ x \in H : f(x) < +\infty \}$ and $f$ is proper if $\text{dom } f \neq \emptyset$. A function $f : H \to (-\infty, +\infty]$ is called lower semicontinuous (lsc) if the level sets $\{ x \in H : f(x) \leq \alpha \}$ are closed for all $\alpha \in \mathbb{R}$. A function $f : H \to (-\infty, +\infty]$ is convex if for all $x,y \in \text{dom } f$ and all $t \in [0,1]$ the relation

$$ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) $$

holds true. For a proper convex lsc function $f : H \to (-\infty, +\infty]$, the proximal mapping $J_f : H \to H$ is defined by

$$ J_f(x) = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2}d(x,y)^2 \right\}, $$

where $d(x,y) := \| x - y \|$. For fixed $f$, we will just write $J_f$ instead of $J_f$. Indeed, the minimizer of the right-hand side exists and is unique. For $\lambda > 0$, the Moreau-Yosida envelope of $f$ is given by

$$ f_\lambda(x) := \min_{y \in H} \left\{ f(y) + \frac{1}{2\lambda}d(x,y)^2 \right\}. $$

The following definition goes back to U. Mosco [16]. A sequence $\{ f_n \}_n$ of functions $f_n : H \to (-\infty, +\infty]$ Mosco-converges to $f : H \to (-\infty, +\infty]$, abbreviated $f_n \overset{\text{M}}{\rightarrow} f$, if, for each $x \in H$, the following two conditions are fulfilled:

i) $f(x) \leq \liminf_{n \to \infty} f_n(x_n)$ whenever $x_n \overset{w}{\rightarrow} x$,

ii) there is a sequence $\{ y_n \}_n$ such that $y_n \rightarrow x$ and $f_n(y_n) \rightarrow f(x)$,

where $x_n \overset{w}{\rightarrow} x$ stands for weak convergence. A weaker type of convergence is given by $\Gamma$-convergence, see, e.g. [9,11]. Here we just have to replace the first statement in the above definition by
i) \( f(x) \leq \lim \inf_{n \to \infty} f_n(x_n) \), whenever \( x_n \to x \).

\( \Gamma \)-convergence is a suitable notion in the study of minimization problems because minimizers of the limit function \( f \) are related to approximate minimizers of the functions \( f_n \), see [9, Theorem 1.21], or, in general topological spaces [2, Theorem 1.10]. The notion of \( \tau \)-epi convergence [2, Definition 1.9], is just another name for \( \Gamma \)-convergence by [9, Proposition 1.18].

The following theorem was proved by H. Attouch, see [1, Theorem 1.2] and [2, Theorem 3.26].

**Theorem 1.1.** Let \( H \) be a Hilbert space and let \( f_n : H \to (-\infty, +\infty) \), \( n \in \mathbb{N} \), be proper convex lower semicontinuous functions. Then the following statements are equivalent:

i) \( \{f_n\}_n \) converges to a function \( f : X \to (-\infty, +\infty] \) in the sense of Mosco, \( f_n \overset{M}{\to} f \).

ii) The sequence of Moreau-Yosida envelopes \( \{f_{n,\lambda}\}_n \) of \( \{f_n\}_n \) converge pointwise to the Moreau-Yosida envelope \( f_{\lambda} \) of \( f \) for all \( \lambda > 0 \).

The aim of the present paper is to generalize Theorem 1.1 to Hadamard spaces. Note that both \( \Gamma \)- and Mosco convergences have already been used in this framework. In [14], J. Jost studied harmonic mappings with metric space targets and as a tool he introduced \( \Gamma \)-convergence on Hadamard spaces. He also defined the Mosco convergence by saying that a sequence of convex lsc functions on a Hadamard space Mosco converges if their Moreau-Yosida envelopes converge pointwise [14]. In [15], K. Kuwae and T. Shioya studied both \( \Gamma \)- and Mosco convergence in Hadamard space in depth and obtained numerous generalizations. They already gave the standard definition of the Mosco convergence used in this paper (relying on the notion of weak convergence) and right after their Definition 5.7 in [15] they note “Jost’s definition of Mosco convergence... seems unfitting in view of Mosco’s original definition.”

By our main result it follows that both definitions are equivalent.

In [15, Proposition 5.12], the authors prove that the Mosco convergence of nonnegative convex lsc functions on a Hadamard space implies the pointwise convergence of their Moreau-Yosida envelopes. This result was later proved in [5, Theorem 4.1] without the nonnegativity assumption. The inverse implication was left open; see [4, Question 5.2.5]. In the present note we answer this question in the positive. As a corollary of our main result we obtain that the Mosco convergence of convex closed sets is equivalent to the Frolík-Wijsman convergence.

In [5, 15] the Mosco convergence of functions on Hadamard spaces was studied in connection with gradient flows. In particular, it was shown in [5] that the Mosco convergence of convex lsc functions on a Hadamard space implies the pointwise convergence of the associated gradient flow semigroups. Interestingly, apart from applications of Hadamard space gradient flows into harmonic mappings theory, see e.g., [14], [19, Section 8], there have been also other motivations. Most remarkably, gradient flows of a convex function on a Hadamard space appear as an important tool in Kähler geometry in connection with Donaldson’s conjecture on Calabi flows [8, 18]. It has also similarly inspired new developments in Riemannian geometry [13]. Finally, in [6], a gradient flow of a convex continuous function was used to construct a Lipschitz retraction in a Hadamard space. For discrete-time gradient flows of convex functions in Hadamard spaces and their applications, see [3, 4].

## 2. Preliminaries on Hadamard spaces

First we collect the preliminaries on Hadamard spaces required for our proof. For the details, we refer to [4]. A complete metric space \((\mathcal{H}, d)\) is called a **Hadamard space** if it is geodesic and the following condition holds true

\[
(3) \quad d(x, v)^2 + d(y, w)^2 \leq d(x, w)^2 + d(y, v)^2 + d(x, y)^2 + d(v, w)^2,
\]

for all \( x, y, v, w \in \mathcal{H} \). Recall that a metric space \((X, d)\) is **geodesic** if every two points \( x, y \in X \) are connected by a geodesic, that is, there exists a curve \( \gamma_{x,y} : [0, 1] \to X \) such that

\[
d(\gamma_{x,y}(t_1), \gamma_{x,y}(t_2)) = |t_1 - t_2|d(\gamma_{x,y}(0), \gamma_{x,y}(1)), \quad \text{for all } t_1, t_2 \in [0, 1],
\]

and \( \gamma_{x,y}(0) = x \) and \( \gamma_{x,y}(1) = y \). Inequality (3) expresses the fact that Hadamard spaces have nonpositive curvature and it also implies that geodesics are uniquely determined by their endpoints.
The definition of proper and lsc functions carries over from the Hilbert space setting. A function $f : \mathcal{H} \to (-\infty, +\infty]$ is called convex if for all $x, y \in \mathcal{H}$ the function $f \circ \gamma_{x,y}$ is convex, i.e., if
\begin{equation}
 f(\gamma_{x,y}(t)) \leq tf(\gamma_{x,y}(0)) + (1-t)f(\gamma_{x,y}(1)),
\end{equation}
all $t \in [0,1]$. For a proper convex lsc function $f : \mathcal{H} \to (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping $J_{\lambda} : \mathcal{H} \to \mathcal{H}$ and the Moreau-Yosida envelope $f_{\lambda} : \mathcal{H} \to (-\infty, +\infty]$ are defined as in (1) and (2), respectively, where the Hilbert space distance has to be replaced by the Hadamard space metric. The proximal mapping is nonexpansive. The Moreau-Yosida envelope is convex and continuous. Note that in [4, p. 42] it was incorrectly claimed that the Moreau-Yosida envelope is not necessarily lsc. We now correct this statement.

**Lemma 2.1.** For any $\lambda > 0$, the Moreau-Yosida envelope $f_{\lambda}$ of a proper convex lsc function $f$ is continuous.

**Proof.** From the definition of the proximal mapping, we have
\[ f_{\lambda}(x) - f_{\lambda}(y) = f(J_{\lambda}x) - f(J_{\lambda}y) + \frac{1}{2\lambda} \left( d(x, J_{\lambda}x)^2 - d(y, J_{\lambda}y)^2 \right), \]
and hence
\begin{equation}
 |f_{\lambda}(x) - f_{\lambda}(y)| \leq |f(J_{\lambda}x) - f(J_{\lambda}y)| + \frac{1}{2\lambda} \left| d(x, J_{\lambda}x)^2 - d(y, J_{\lambda}y)^2 \right|. \tag{5}
\end{equation}
Again, by the definition of the proximal mapping, we obtain
\[ f(J_{\lambda}x) - f(J_{\lambda}y) \leq \frac{1}{2\lambda} \left( d(x, J_{\lambda}y)^2 - d(x, J_{\lambda}x)^2 \right), \]
and by the symmetry in $x$ and $y$ also
\[ f(J_{\lambda}y) - f(J_{\lambda}x) \leq \frac{1}{2\lambda} \left( d(y, J_{\lambda}x)^2 - d(y, J_{\lambda}y)^2 \right). \]
Together with the nonexpansiveness of the proximal mapping we see that the left hand side of (5) goes to 0 provided $x$ goes to $y$. 

The Moreau-Yosida envelope of a proper convex lsc function possesses the semigroup property
\[ (f_{\lambda})_{\mu} = f_{\lambda+\mu}, \quad \lambda, \mu > 0. \]
Further, for all $x \in \mathcal{H}$, we have
\begin{equation}
 \lim_{\lambda \to +0} f_{\lambda}(x) = f(x), \tag{6}
\end{equation}
see [2, Theorem 2.46].

We will also need the following two auxiliary lemmas.

**Lemma 2.2.** Let $(\mathcal{H}, d)$ be a Hadamard space and let $f : \mathcal{H} \to (-\infty, +\infty]$ be a convex lsc function. Then for all $x, y \in \mathcal{H}$ and $\lambda > 0$, we have
\begin{equation}
 f(J_{\lambda}x) + \frac{1}{2\lambda} d(x, J_{\lambda}x)^2 + \frac{1}{2\lambda} d(J_{\lambda}x, y)^2 \leq f(y) + \frac{1}{2\lambda} d(x, y)^2. \tag{7}
\end{equation}
The proof can be found in [4, Lemma 2.2.23]. The next lemma is analogous to the condition in [2, (3.83)].

**Lemma 2.3.** Let $\mathcal{H}$ be a Hadamard space and $\{f_n\}_n$ a sequence of functions $f_n : \mathcal{H} \to (-\infty, +\infty]$. Suppose that, for some $\lambda > 0$, the Moreau-Yosida envelopes fulfill $f_{n,\lambda} \to f_\lambda$ as $n \to \infty$, for some proper convex lsc function $f : \mathcal{H} \to (-\infty, +\infty]$. Then, given $x \in \mathcal{H}$, there exists $n_0 \in \mathbb{N}$ and $r > 0$ such that for all $n \geq n_0$ and all $y \in \mathcal{H}$ we have
\begin{equation}
 f_n(y) \geq -r d(x, y)^2 - r. \tag{8}
\end{equation}
Proof. As \( f \) is proper, the function \( f_\lambda \) is everywhere finite. By assumption, there is some \( n_0 \) such that for all \( n \geq n_0 \), we have \( |f_{n,\lambda}(x) - f_\lambda(x)| \leq 1 \). Thereby, from
\[
\limsup_n \left\{ f_n(z) + \frac{1}{2\lambda} d(x, z)^2 \right\}
\]
we obtain for any \( y \in \mathcal{H} \) and \( n \geq n_0 \) that
\[
\limsup_n f_n(y) \geq f_{n,\lambda}(x) - \frac{1}{2\lambda} d(x, y)^2 \geq f_\lambda(x) - 1 - \frac{1}{2\lambda} d(x, y)^2.
\]
This yields the assertion for \( r := \max \left\{ \frac{1}{2\lambda}, 1 - f_\lambda(x) \right\} \).

The definition of Mosco convergence in Hadamard spaces requires a notion of weak convergence. For a bounded sequence \( \{x_n\} \) of points \( x_n \in \mathcal{H} \), the function \( \omega: \mathcal{H} \to [0, +\infty) \) defined by
\[
\omega(x; \{x_n\}) := \limsup_{n \to \infty} d(x, x_n)^2
\]
has a unique minimizer, which is called the asymptotic center of \( \{x_n\} \), see [4, p. 58]. A sequence \( \{x_n\} \) weakly converges to a point \( x \in \mathcal{H} \) if it is bounded and \( x \) is the asymptotic center of each subsequence of \( \{x_n\} \), see [4, p. 103]. We write \( x_n \overset{w}{\to} x \). Note that \( x_n \to x \) implies \( x_n \overset{w}{\to} x \). Then the definition of Mosco convergence given in the previous section carries over to functions defined on Hadamard spaces.

3. Main Result

The implication \( i) \Rightarrow ii) \) in Theorem 1.1 has been generalized to Hadamard spaces in [5, Theorem 4.1]:

**Theorem 3.1.** Let \( \mathcal{H} \) be a Hadamard space and \( \{f_n\} \) a sequence of proper convex lsc functions \( f_n: \mathcal{H} \to (-\infty, +\infty] \) which converges to a proper function \( f: \mathcal{H} \to (-\infty, +\infty] \) in the sense of Mosco. Then, for all \( \lambda > 0 \) and all \( x \in \mathcal{H} \), we have
\[
f_{n,\lambda}(x) \to f_\lambda(x) \quad \text{as } n \to \infty.
\]

Our main result is the inverse implication.

**Theorem 3.2.** Let \( \mathcal{H} \) be a Hadamard space, \( \{f_n\} \) a sequence of proper convex lsc functions \( f_n: \mathcal{H} \to (-\infty, +\infty] \), and \( f: \mathcal{H} \to (-\infty, +\infty] \) a proper convex lsc function. Assume that for each \( \lambda > 0 \) the sequence of Moreau-Yosida envelopes \( \{f_{n,\lambda}\} \) converges pointwise to the Moreau-Yosida envelope \( f_\lambda \). Then \( f_n \overset{M}{\to} f \) as \( n \to \infty \).

Proof. Observe that \( f(x) \geq f_\lambda(x) \geq f(J_\lambda x) \). For \( x \in \text{dom} f \), it holds by [4, Proposition 2.2.26] that
\[
\lim_{\lambda \to 0^+} J_\lambda x = x \quad \text{so that the lower semicontinuity of } f \text{ implies}
\]
\[
f(x) = \lim_{\lambda \to 0^+} f_\lambda(x) = \lim_{\lambda \to 0^+} f(J_\lambda x).
\]

**Step 1 (Limsup Inequality).** Let us show that, given \( x \in \mathcal{H} \), there exists a sequence \( y_n \to x \) with \( \limsup_{n \to \infty} f_n(y_n) \leq f(x) \). If \( f(x) = \infty \), then there is nothing to prove. Assume therefore \( x \in \text{dom} f \).

Together with the assumption that \( f_{n,\lambda}(x) \to f_\lambda(x) \) for all \( x \in \mathcal{H} \) as \( n \to \infty \), we obtain
\[
f(x) = \lim_{\lambda \to 0^+} f_\lambda(x) = \lim_{\lambda \to 0^+} \lim_{n \to \infty} f_{n,\lambda}(x).
\]

By the diagonalization lemma, see [2, Lemma 1.18], there exists a sequence \( \{\lambda_n\} \) with \( \lim_{n \to \infty} \lambda_n = 0 \) such that
\[
f(x) = \lim_{n \to \infty} f_{n,\lambda_n}(x)
\]
\[
= \lim_{n \to \infty} \left( f_n(J_{\lambda_n}^r x) + \frac{1}{2\lambda_n} d(x, J_{\lambda_n}^r x)^2 \right),
\]
where \( J_{\lambda_n}^r = J_{\lambda_n} \). Hence \( f(x) \geq \limsup_{n \to \infty} f_n(J_{\lambda_n}^r x) \). We put \( y_n := J_{\lambda_n}^r x \) and show that \( y_n \to x \). Indeed, inserting (8) into (12), we have
\[
f(x) \geq \limsup_{n \to \infty} \left( \frac{1}{2\lambda_n} - r \right) d(x, y_n)^2 - r
\]
and we can conclude that \( y_n \to x \).

**Step 2.** Let us show that \( J^n_\lambda x \to J_\lambda x \). From the previous step, we know that there exists a sequence \( y_n \to J_\lambda x \) with \( \limsup_{n \to \infty} f_n(y_n) \leq f(J_\lambda x) \). Then we obtain

\[ f_\lambda(x) = f(J_\lambda x) + \frac{1}{2\lambda} d(x, J_\lambda x)^2 \leq \limsup_{n \to \infty} \left( f_n(y_n) + \frac{1}{2\lambda} d(x, y_n)^2 \right) \]

and by (7) further

\[ f_\lambda(x) \geq \limsup_{n \to \infty} \left( f_{n, \lambda}(x) + \frac{1}{2\lambda} d(J^n_\lambda x, y_n)^2 \right) = f_\lambda(x) + \limsup_{n \to \infty} \frac{1}{2\lambda} d(J^n_\lambda x, y_n)^2. \]

Hence we conclude \( J^n_\lambda x \to J_\lambda x \).

**Step 3 (Liminf Inequality).** Let \( x_n \rightharpoonup x \). We have to prove \( \liminf_{n \to \infty} f_n(x_n) \geq f(x) \). By definition of the Moreau-Yosida envelope and (7) we have

\[ f_n(x_n) \geq f_n(J^n_\lambda x_n) + \frac{1}{2\lambda} d(x_n, J^n_\lambda x_n)^2 \]

\[ \geq f_n(J^n_\lambda x) + \frac{1}{2\lambda} d(x, J^n_\lambda x)^2 + \frac{1}{2\lambda} d(J^n_\lambda x_n, J^n_\lambda x)^2 + \frac{1}{2\lambda} d(x_n, J^n_\lambda x)^2 - \frac{1}{2\lambda} d(x, J^n_\lambda x)^2. \]

By the nonpositive curvature inequality in (3) we obtain

\[ f_n(x_n) \geq f_n(J^n_\lambda x) + \frac{1}{2\lambda} d(J^n_\lambda x, x_n)^2 - \frac{1}{2\lambda} d(x, x_n)^2. \]

Let us show that \( f_n(J^n_\lambda x) \) converges as \( n \to \infty \). Consider

\[ f_{n, \lambda}(x) = f_n(J^n_\lambda x) + \frac{1}{2\lambda} d(x, J^n_\lambda x)^2, \]

\[ f_\lambda(x) = f(J_\lambda x) + \frac{1}{2\lambda} d(x, J_\lambda x)^2. \]

By assumption we have \( f_{n, \lambda}(x) \to f_\lambda(x) \), and by Step 2 also \( J^n_\lambda x \to J_\lambda x \) as \( n \to \infty \). This implies

\[ f_n(J^n_\lambda x) \to f(J_\lambda x). \]

By the definition of the weak limit of \( \{x_n\}_n \), for every subsequence \( n_k \to \infty \), we have

\[ \limsup_{k \to \infty} d(J_\lambda x, x_{n_k})^2 \geq \limsup_{k \to \infty} d(x, x_{n_k})^2. \]

By the triangle inequality we obtain \( d(J^n_\lambda x, x_{n_k}) \geq |d(J^n_\lambda x, J_\lambda x) - d(J_\lambda x, x_{n_k})| \). Using Step 2 and (15) results in

\[ \limsup_{k \to \infty} d(J^n_\lambda x, x_{n_k})^2 \geq \limsup_{k \to \infty} (d(J^n_\lambda x, J_\lambda x) - d(J_\lambda x, x_{n_k}))^2 \]

\[ = \limsup_{k \to \infty} d(J_\lambda x, x_{n_k})^2 \]

\[ \geq \limsup_{k \to \infty} d(x, x_{n_k})^2. \]

Rearranging, we get

\[ 0 \leq \limsup_{k \to \infty} d(J^n_\lambda x, x_{n_k})^2 + \liminf_{k \to \infty} (-d(x, x_{n_k})^2) \]

\[ \leq \limsup_{k \to \infty} (d(J^n_\lambda x, x_{n_k})^2 - d(x, x_{n_k})^2) \]

and, as the subsequence was arbitrary,

\[ \liminf_{n \to \infty} (d(J^n_\lambda x, x_{n_k})^2 - d(x, x_{n_k})^2) \geq 0. \]

Returning to (13), the previous equation and (14) yield

\[ \liminf_{n \to \infty} f_n(x_n) \geq f(J_\lambda x). \]

If \( x \in \text{dom} f \), then from (11) we obtain

\[ \liminf_{n \to \infty} f_n(x_n) \geq f(x). \]
For $x \notin \text{dom } f$ we can repeat the above conclusions for the finite continuous convex functions $g_n := f_{n, \mu}$ and $g = f_{\mu}$ for some fixed $\mu > 0$ instead of $f_n$ and $f$. Note that the assumptions are fulfilled by the semigroup property of the Moreau-Yosida envelopes. Finally we let $\mu \to +0$ and invoke (6). This concludes the proof. □

Recall that a sequence of convex closed sets $C_n \subset H$ converges to a convex closed set $C \subset H$ in the sense of Frolík-Wijsman if the respective distance functions converge pointwise; that is, if $d(x, C_n) \to d(x, C)$ for each $x \in H$. This concept originated in [12, 20]. On the other hand, a sequence of convex closed sets $C_n \subset H$ converges to a convex closed set $C \subset H$ in the sense of Mosco if the indicator functions $\iota_{C_n}$ converge in the sense of Mosco to the indicator function $\iota_C$. The following is a direct consequence of our main result.

**Corollary 3.3** (Frolík-Wijsman convergence). A sequence of convex closed sets $C_n \subset H$ converges to a convex closed set $C \subset H$ in the sense of Frolík-Wijsman if and only if it converges to $C$ in the sense of Mosco.

**Proof.** Observe that the Moreau-Yosida envelope of $\iota_C$ with $\lambda = \frac{1}{2}$ is precisely the distance function squared $d(\cdot, C)^2$ and apply Theorems 3.1 and 3.2. □

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