Statistically dual distributions and estimation of the parameters

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Abstract

The reconstruction of the parameter of the model by the measurement of the random variable depending on this parameter is one of the main tasks of statistics. In the paper the notion of the statistically dual distributions is introduced. The approach, based on the properties of the statistically dual distributions, to resolving of the given task is proposed.

Key Words: Uncertainty, Statistical duality, Measurement, Bayesian statistics

1 Introduction

As shown in refs. (Bityukov 2000, Bityukov 2002, Bityukov 2003), in frame of frequentist approach we can construct the probability distribution of the possible magnitudes of the Poisson distribution parameter to give the observed number of events $\hat{n}$ in Poisson stream of events. This distribution is described as Gamma-distribution $\Gamma_{1,\hat{n}+1}$ with the probability density, which looks like Poisson distribution of probabilities, namely,

\[ g_{\hat{n}}(\mu) = \frac{\mu^{\hat{n}}}{\hat{n}!}e^{-\mu}, \quad \mu > 0, \quad \hat{n} > -1, \]  

(1)
where \( \mu \) is a variable and \( \hat{n} \) is a parameter (in case of Poisson distributions \( \mu \) is a parameter and \( \hat{n} \) is a variable). It means, as shown below, that we can estimate the value and error of Poisson distribution parameter by the measurement of mean value of the random variable of this Poisson distribution and by using the correspondent Gamma-distribution.

Let us name such distributions, which allow to exchange the parameter and the variable, conserving the same formula for the distribution of probabilities, as **statistically dual distributions**.

Note, it is pure probabilistic (and, in this sense, frequentist) definition. As shown below, the properties of considered in this paper of two pairs of statistically dual distributions coincide with properties of conjugate families (which defined in frame of Bayesian Theory (for example, Bernardo, 1994)).

In the next Section we show that Poisson and Gamma-distributions are statistically dual distributions and that normal distribution with constant variance is statistically self-dual distribution. Despite little differences in formulae, we also point on the interdependency between the negative binomial and \( \beta - \) distributions, which allows to consider these distributions as statistically quasi-dual distributions. The interrelation of the statistical duality and the estimation of the distribution parameters are discussed in Section 3.

## 2 Statistically dual distributions

Let \( t(x, y) \) be a function of two variables. If the same function can be represented both as the distribution of the random variable \( x \) with parameter \( y \) and as the distribution of the random variable \( y \) with parameter \( x \), then these distributions can be named as **statistically dual distributions**.

The **statistical duality** of Poisson and Gamma-distributions follows from simple discourse.

Let us consider the Gamma-distribution with probability density

\[
g_x(\beta, \alpha) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}.
\]

(2)

At change of standard designations of Gamma-distribution \( \frac{1}{\beta}, \alpha \) and \( x \) for \( a, n + 1 \) and \( \mu \) we get the following formula for probability density of Gamma-distribution
\[ g_n(a, \mu) = \frac{a^{n+1}}{\Gamma(n+1)} e^{-a \mu} \mu^n, \quad (3) \]

where \( a \) is a scale parameter and \( n + 1 > 0 \) is a shape parameter. Suppose \( a = 1 \), then the formula of the probability density of Gamma-distribution \( \Gamma_{1,n+1} \) is (here we repeat Eq.1)

\[ g_n(\mu) = \frac{\mu^n}{n!} e^{-\mu}, \quad \mu > 0, \quad n > -1. \]

It is usually supposed that the probability of observing \( n \) events in the experiment is described by Poisson distribution with parameter \( \mu \), i.e.

\[ f(n; \mu) = \frac{\mu^n}{n!} e^{-\mu}, \quad \mu > 0, \quad n \geq 0. \quad (4) \]

One can see that the parameter and variable in Eq.1 and Eq.4 are exchanged in other respects the formulae are identical. As a result these distributions (Gamma and Poisson) are **statistically dual distributions**. These distributions (Bityukov 2002) are connected by identity (see, also, identities in refs. (Jaynes 1976, Frodesen 1979, Cousins 1995))

\[ \int_{\mu_1}^{\mu_2} g_m(\mu)d\mu + \sum_{i=n+1}^{m} f(i; \mu_2) + \int_{\mu_2}^{\mu_1} g_n(\mu)d\mu - \sum_{i=n+1}^{m} f(i; \mu_1) = 0, \quad (5) \]

for any real \( \mu_1 \geq 0, \mu_2 \geq 0 \), and integer \( m > n \geq 0 \).

The other example of statistically dual distribution is the normal distribution with mean value \( a \) and constant variance \( \sigma^2 \):

\[ \varphi(x; a, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}, \quad (6) \]

where \( x \) is real variable, \( a \) and \( \sigma > 0 \) are real parameters. Here we also can exchange the parameter \( a \) and variable \( x \) saving the formulized description of the probability density. It allows to estimate the value of parameter \( a \) by measurement of mean value for \( x \). In this case we must consider new density of probability with the real variable \( a \) and real parameters \( x \) and \( \sigma > 0 \)

\[ \phi(a; x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}, \quad (7) \]
Such a way the normal distribution can be named as **statistical self-dual distribution**. The analogous identity (as Eq.5) for given distributions takes place

\[
\int_c^d \phi(a; c, \sigma)da + \int_c^d \varphi(x; d, \sigma)dx + \int_d^c \phi(a; d, \sigma)da + \int_d^c \varphi(x; c, \sigma)dx = 0
\]

or, simpler,

\[
\int_c^d \phi(a; b, \sigma)da - \int_c^d \varphi(x; b, \sigma)dx = 0
\]

for any real \( b, c \) and \( d \).

Let us present (Eadie 1971) the probability density of \( \beta \)-distribution as

\[
\beta(x; n, m) = \frac{(n + m + 1)!}{n!m!} x^n (1-x)^m,
\]

with real variable \( x \) (\( 0 < x < 1 \)) and integer non negative parameters \( n \) and \( m \), and the probability distribution of negative binomial distribution as

\[
P(k; n, p) = \frac{(n + k)!}{n!k!} p^{n+1} (1 - p)^k,
\]

where \( k \) is integer non negative variable, integer \( n \) and real \( p \) are parameters (\( 0 \leq p \leq 1 \)).

The formulae, which describe these distributions, are different, nevertheless, as is mentioned in ref. (Jaynes 1976), the negative binomial (Pascal) and \( \beta \)-distributions have interrelation in the form of identity

\[
\int_0^p \beta(x; n, m)dx - \sum_{k=0}^m P(k; n, p) = 0.
\]

Correspondingly, these distributions can be named as **statistically quasi-dual distributions**.

### 3 Statistical duality and estimation of the parameter of distribution

The identity (Eq.5) can be written in form (Bityukov 2000, Bityukov 2003)
\[
\sum_{i=\hat{n}+1}^{\infty} f(i; \mu_1) + \int_{\mu_1}^{\mu_2} \tilde{g}_\hat{n}(\mu)\,d\mu + \sum_{i=0}^{\hat{n}} f(i; \mu_2) = 1,
\]
i.e.
\[
\sum_{i=\hat{n}+1}^{\infty} \frac{\mu_1^i e^{-\mu_1}}{i!} + \int_{\mu_1}^{\mu_2} \frac{\mu^\hat{n} e^{-\mu}}{\hat{n}!}\,d\mu + \sum_{i=0}^{\hat{n}} \frac{\mu_2^i e^{-\mu_2}}{i!} = 1
\]
for any real \( \mu_1 \geq 0 \) and \( \mu_2 \geq 0 \) and non-negative integer \( \hat{n} \).

The definition of the confidence interval \((\mu_1, \mu_2)\) for Poisson distribution parameter \(\mu\) as (Bityukov 2000)
\[
P(\mu_1 \leq \mu \leq \mu_2) = P(i \leq \hat{n} | \mu_1) - P(i \leq \hat{n} | \mu_2),
\]
where \( P(i \leq \hat{n} | \mu) = \sum_{i=0}^{\hat{n}} \frac{\mu^i e^{-\mu}}{i!} \), allows to show that a Gamma-distribution \( \Gamma_{1,1+\hat{n}} \) is the probability distribution of different values of \(\mu\) parameter of Poisson distribution on condition that the observed value of the number of events is equal to \(\hat{n}\). This definition is consistent with identity Eq.12 in contrast with another frequentist definitions of confidence interval (for example, if we suppose in Eq.12 that \(\mu_1 = \mu_2\) we have conservation of probability). The right part of Eq.13 determines the frequentist sense of this definition.

Let us suppose that \(g_\hat{n}(\mu)\) is the probability density of parameter of the Poisson distribution if number of observed events is equal to \(\hat{n}\). It is a conditional probability density. As it follows from formulae (Eqs.1,12), the \(g_\hat{n}(\mu)\) is the density of Gamma-distribution by definition. Note, that the definition of conjugate families \(^1\) is based on using of probability density of parameter distribution (Casella, 2001).

On the other hand: if \(g_\hat{n}(\mu)\) is not equal to this probability density and the probability density of the Poisson parameter is the other function \(h(\mu; \hat{n})\) then there takes place another identity
\[
\sum_{i=\hat{n}+1}^{\infty} f(i; \mu_1) + \int_{\mu_1}^{\mu_2} h(\mu; \hat{n})\,d\mu + \sum_{i=0}^{\hat{n}} f(i; \mu_2) = 1.
\]

\(^1\)Given a family \(\mathcal{F}\) of pdf’s \(f(x|\theta)\) indexed by a parameter \(\theta\), then a family, \(\Pi\) of prior distributions is said to be *conjugate* for the family \(\mathcal{F}\) if the posterior distribution of \(\theta\) is in the family \(\Pi\) for all \(f \in \mathcal{F}\), all priors \(\pi(\theta) \in \Pi\) and all possible data sets \(x\).
This identity is correct for any real $\mu_1 \geq 0$ and $\mu_2 \geq 0$ too. The sums in the left part of this equation determine the boundary conditions on the confidence interval.

If we subtract Eq.14 from Eq.12 then we have

$$\int_{\mu_1}^{\mu_2} (g_\hat{n}(\mu) - h(\mu; \hat{n}))d\mu = 0. \quad (15)$$

We can choose the $\mu_1$ and $\mu_2$ by the arbitrary way. Let us make this choice so that $g_\hat{n}(\mu)$ is not equal $h(\mu; \hat{n})$ in the interval $(\mu_1, \mu_2)$ and, for example, $g_\hat{n}(\mu) > h(\mu; \hat{n})$ and $\mu_2 > \mu_1$. In this case we have

$$\int_{\mu_1}^{\mu_2} (g_\hat{n}(\mu) - h(\mu; \hat{n}))d\mu > 0 \quad (16)$$

and we have contradiction (i.e. $g_\hat{n}(\mu) = h(\mu; \hat{n})$ everywhere except, may be, a finite set of points). As a result we can mix Bayesian ($g_\hat{n}(\mu)$) and frequentist ($f(k; \mu)$) probabilities without any logical inconsistencies. The identity (Eq.12) leaves no place for any prior except uniform ($\pi(\mu) = \text{const}$). As a result, we can construct the distribution of the errors in the estimation of the parameter of Poisson distribution by single measurement of a casual variable and, correspondingly, the confidence intervals, estimate the parameter by several measurements, take into account systematics and statistical uncertainties of measurements at statistical conclusions about the quality of planned experiments (Bityukov 2000, Bityukov 2003, Bityukov 2004).

For the normal distribution the identity (Eq.8) can be written in case of the observed value of random variable $\hat{x}$ as

$$\int_{-\infty}^{\hat{x}-c} \varphi(x; \hat{x}, \sigma)dx + \int_{\hat{x}-c}^{\hat{x}+d} \phi(a; \hat{x}, \sigma)da + \int_{\hat{x}+d}^{\infty} \varphi(x; \hat{x}, \sigma)dx = 1 \quad (17)$$

for any real $c$ and $d$ or, in analogy with Eq.12 for the case of Poisson-Gamma distributions,

$$\int_{-\infty}^{\hat{x}-c} \varphi(x; \hat{x} - c, \sigma)dx + \int_{\hat{x}-c}^{\hat{x}+d} \phi(a; \hat{x}, \sigma)da + \int_{\hat{x}+d}^{\infty} \varphi(x; \hat{x} + d, \sigma)dx = 1 \quad (18)$$

Bayesian methods are supposed that $g_\hat{n}(\mu) = \frac{f(\hat{n}; \mu) \cdot \pi(\mu)}{\int f(\hat{n}; \mu) \cdot \pi(\mu)d\mu}$, where $\pi(\mu)$ is the prior probability density for $\mu$. 

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for any real \( c \geq 0 \) and \( d \geq 0 \).

These identities (18,19) also allow to say that conditional distribution (if observed value is \( \hat{x} \)) of true value of the parameter \( a \) obeys normal distribution with mean value \( \hat{x} \) and constant variance \( \sigma^2 \) (i.e. here, in contrast with previous example, \( \hat{x} \) is the unbiased estimator of the parameter \( a \)). As a result we can construct the distribution of the error and the confidence intervals of parameter \( a \), take into account systematics and statistical uncertainties and so on in accordance with standard analysis of errors (Eadie 1971).

So, the statistical duality allows to connect the estimation of the parameter with the measurement of the random variable of the distribution under study.

\section{Conclusion}

In the paper the notion of the statistically dual distributions is introduced. The relation between the measurement of casual variable and the estimation of the given distribution parameter for two pairs of statistically dual distributions is presented. The proposed approach allows to construct the distribution of measurement error of the estimator of distribution parameter by the using of statistically dual distribution.

Both considered cases of statistically dual distributions (which are introduced in frame of frequentist approach), namely Poisson distribution versus Gamma distribution and Normal distribution versus Normal distribution, also belong to conjugate families (which are defined in frame of Bayesian approach). For example (Bityukov 2003 ACAT), the distributions conjugated to Poisson distributions were built by Monte Carlo method (i.e. in frame of frequentist approach). The hypotheses testing confirms that these distributions are Gamma-distributions as expected in this case.

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