Supersolid behavior in one-dimensional self-trapped Bose–Einstein condensate

Mithilesh K Parit1, Gargi Tyagi2, Dheerendra Singh1 and Prasanta K Panigrahi1,∗

1 Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata, Mohanpur-741246, West Bengal, India
2 Institute of Applied Sciences, Amity University, Sector-125, Noida-201303, Uttar Pradesh, India
E-mail: mithilesh.parit@gmail.com, gargityagi150@gmail.com, ds17ip010@iiserkol.ac.in and ppprasanta@iiserkol.ac.in

Received 19 October 2020, revised 21 March 2021
Accepted for publication 12 April 2021
Published 13 May 2021

Abstract
Supersolid is an exotic state of matter, showing crystalline order with a superfluid background, observed recently in dipolar Bose–Einstein condensate in a trap. Here, we present exact Bloch wave function of the self-trapped supersolid phase, in the presence of mean-field and beyond mean-field interaction. Our general solutions of the amended nonlinear Schrödinger equation are obtained through Möbius transform, connecting a wide class of supersolid solutions to the ubiquitous cnoidal waves. The solutions yield the supersolid phase in the self-trapped quantum matter, where an array of quantum droplets exist, accompanied by a constant condensate. For the supersolid phase, the chemical potential for one class of solutions is the same as that of self-trapped quantum droplets, and is lower for the general non-perturbative solution. Due to the destabilizing effects of fluctuations on long range order in one dimension, the realization of the supersolid phase may be possible in a finite system.

Keywords: supersolid, Bose–Einstein condensate, condensed matter physics

(Some figures may appear in colour only in the online journal)

1. Introduction
Recently, the transient supersolid phase has been identified in dipolar Bose–Einstein condensates (BECs) by Böttcher et al [1] and Tanzi et al [2]. This has been numerically well supported by Roccuzzo and Ancilotto [3]. In both the experiments, BEC from the trap is released to expand, with the interference of matter waves producing a crystalline structure, immersed in a superfluid BEC. As is well known, in one dimension (1D), spontaneous breaking of global symmetry is not allowed due to the severity of quantum fluctuations (QFs). One body density matrix shows power law decay, and it ceases to exist in the infinite volume limit. However, as noted by Popov [4, 5] and Sonin [6], the algebraic decay of the order parameter can still allow superfluidity in 1D. Furthermore, this is strictly true in the absence of long-range interaction in lower dimensions [7, 8]. Therefore, apart from the dipolar BEC, it is of great interest to investigate the existence of the supersolid phase in other quantum systems.

The supersolid phase of matter was conjectured quite some time ago [9, 10], with liquid helium being extensively investigated for its possible realization [11, 12]. Lack of clear experimental evidence from liquid helium led to alternate avenues such as ultracold atoms for its possible realization. Dipolar BEC, with competitive repulsive short-range and attractive long-range dipolar interactions, has led to unambiguous observation of this novel phase of matter [13]. The properties of the dipolar supersolid are found to be well described by the extended Gross–Pitaevskii equation, describing the mean-field (MF) dynamics with the presence of an additional quartic term [1]. Chomaz et al identified the supersolid properties in the dipolar quantum gases of 166Er and 164Dy [14]. Guo et al [15] observed two-low energy Goldstone modes, revealing the phase coherence and confirming that the array of quantum
droplets are in the supersolid phase. As mentioned earlier, in the case of the other two experiments, the BEC from the trap was released for the matter-waves to interfere and form a transient supersolid phase. Hence, it is worth investigating the possibility of realizing the supersolid phase without a trap in an ultracold scenario. This possibility arises when repulsive MF interaction is balanced by other attractive interactions, e.g. beyond mean-field (BMF) correction of Lee–Huang–Yang (LHY) type.

Recently, Petrov [16] has identified a parametric domain in a two-component BEC, where self-trapped quantum droplets form [17]. They have been observed in the free-floating condition [18–20] in dipolar Bose gas and magnetic quantum liquid [21], making them an ideal ground to explore the possibility of the self-trapped quantum liquid in the absence of a trap. Balance of the MF energy and BMF QFs [22] leads to formation of these self-trapped quantum liquid droplets. Chomaz et al. [23], demonstrated that LHY stabilization is a general feature of strongly dipolar gases, and also examined the role of QFs determining the system properties, particularly its collective mode and expansion dynamics. Various aspects of the quantum droplets have been explored: Bose–Bose droplets in-dimensional crossover regime [24] and Bose–Fermi droplets of attractive degenerate bosons and spin polarized fermions [25] have been investigated, some of which have found experimental verification. Bright soliton to droplet transitions have been demonstrated in a mixture of two states of $^{87}$K [26]. The case of 1D is distinct from those of two and three dimensions [16, 27], as MF repulsion is required to balance the quantum pressure, unlike that of attraction in higher dimensions.

The cigar-shaped self-trapped quantum droplets have been modeled by the amended nonlinear Schrödinger equation (NLSE), which has a repulsive MF nonlinearity like regular BEC and a quadratic nonlinearity arising from BMF [16, 17]. It has an exact flat-top large droplet solution and may have multiple droplet solutions indicated from variational approach [28].

Here, we have identified exact Bloch wave solutions of the amended NLSE, showing clearly the supersolid phase in an array of quantum droplets, necessarily accompanied by a non-vanishing condensate density. We employ Möbius transform [29, 30] to connect a wide class of desired solutions with the cnoidal functions, representing nonlinear periodic density waves, which manifest in diverse physical systems. Our procedure yields a unified picture of the self-trapped supersolid phase in different parameter domains, with periodic modulation of quantum droplets on the top of constant condensate background, as expected.

The effects of the QFs have not been accounted for in our analysis. It can destabilize this solution in the infinite volume limit. The observed supersolid phase in dipolar BEC gets naturally confined to a finite volume and shows three peaks immersed in a superfluid [1]. A similar finite size system in the quantum droplet scenario may yield the proposed supersolid phase. Furthermore, the supersolid phase may also manifest in a ring resonator geometry [31].

The paper is organized as follows. In section 2, the theory of quantum droplets is briefly described, leading to the amended NLSE, governing its dynamics in 1D. We briefly outline the parameter domain, where modulational instability (MI) can occur. This regime is conducive to the generation of extended structures like soliton trains [32, 33]. In section 3, we present supersolid solutions; a Bloch wave, necessarily possessing a constant background. Their properties in various configurations are highlighted. Finally, we conclude with the summary of results and future directions for investigation.

## 2. Dynamics of quantum droplets in one dimension

Quantum droplets have been shown to occur in the binary BECs, with repulsive intra- ($g_{\uparrow\uparrow}, g_{\downarrow\downarrow}$) and attractive inter-component interaction ($g_{\uparrow\downarrow}$), in the vicinity of the MF collapse instability point. Specifically, they appear in the regime, $0 < \delta g = g_{\uparrow\downarrow} + \sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}} \ll \sqrt{g_{\uparrow\uparrow}} g_{\downarrow\downarrow}$, with attractive inter-component interaction $g_{\uparrow\downarrow} > 0$ and repulsive intra-component average interaction $g = \sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}} > 0$. The energy density of such a homogeneous mixture has been obtained [17],

\[
\mathcal{E} = \frac{g_{\uparrow\uparrow}}{2}(n_\uparrow - g_{\downarrow\downarrow} n_\downarrow)^2 + \frac{\sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}}{2}(g_{\uparrow\uparrow} n_\uparrow + g_{\downarrow\downarrow} n_\downarrow)(g_{\uparrow\uparrow} n_\uparrow + g_{\downarrow\downarrow} n_\downarrow)^2 - \frac{2\sqrt{M}}{3\pi\hbar}(g_{\uparrow\uparrow} n_\uparrow + g_{\downarrow\downarrow} n_\downarrow)^{3/2},
\]

where $n_\uparrow$ and $n_\downarrow$ are densities of the two components, related by $n = n_\uparrow = n_\downarrow \sqrt{g_{\uparrow\downarrow}/g_{\uparrow\uparrow}}$. The first two terms in equation (1) are the MF contribution and the last term represents the LHY-BMF correction. With the assumption, $g = g_{\uparrow\uparrow} \sim g_{\downarrow\downarrow}$, we have $n = n_\uparrow = n_\downarrow$ and $0 < \delta g \ll g_{\uparrow\uparrow} \sim g_{\downarrow\downarrow}$. The energy density reduces to:

\[
\mathcal{E} = \frac{\delta g n^2}{2} - \frac{2\sqrt{M}}{3\pi\hbar}(gn)^{3/2},
\]

having the equilibrium density $n_0 = 8M\gamma^3/(9\hbar^2\pi^2\delta^2)$ and chemical potential $\mu_0 = -\delta gn_0/2$, where $g$ and $\delta g$ are the interaction strengths with unit equal to energy per density. The validity of $\mathcal{E}$ has been justified by diffusion Monte Carlo simulation [17].

The amended Gross–Pitaevskii equation, with cubic MF and quadratic BMF nonlinearities, has the form [17]

\[
\frac{\hbar}{i} \partial_t \psi = -\frac{\hbar^2}{2M} \partial_x^2 \psi + \delta g |\psi|^2 \psi - \frac{\sqrt{2M}}{\pi\hbar} g^{3/2} |\psi|^2 \psi.
\]

It has a constant solution: $\psi_{\text{const}} = \sqrt{\frac{2M}{\pi\hbar \delta g}} \exp(i k x - i\frac{\mu}{\hbar} t)$, with

\[
\sqrt{\mathcal{F}} \pm = \frac{\sqrt{2M\gamma^3/2}}{2\pi\hbar\delta g} \mp \frac{M\gamma^3}{\pi^2\hbar^2\delta^2 g} + \frac{\mu}{\delta g}.
\]

amplitude of the wave and $\mu = (\mu - \delta g n_0)$. In the study of MI, the of the plane wave solution, $\sqrt{\mathcal{F}}$, is perturbed by small
perturbation \(a(x,t) \ll \sqrt{P}\):
\[
a(x,t) = \Lambda \cos(kP x - \Omega t) + i \eta \sin(kP x - \Omega t),
\]
where \(kP\) is the wave number and \(\Omega\) is the frequency of the perturbation. MI in the amended NLSE has been recently investigated in [34, 35], manifesting at wave number \(kP\), where \(\Omega\) becomes complex:
\[
\Omega = \pm \sqrt{\frac{1}{2M} k^2 P^2 - 4M \left( \frac{\sqrt{2M}}{\pi \hbar} g^{3/2} \sqrt{P} - \delta gP \right)},
\]
MI occurs for \(P < \left(\frac{\sqrt{2M}}{\pi \hbar} g^{3/2} \sqrt{P} - \delta gP\right)\) and
\[
|kP| < 2 \left( \frac{\sqrt{2M}}{\pi \hbar} g^{3/2} \sqrt{P} - \delta gP \right) \equiv k_0,
\]
and is maximum at \(kP = k_0/\sqrt{2}\), and \(k_0\) is maximum at \(\mu = -\frac{GM}{\pi \hbar^2 g^2}\). The regime in the parameter domain, where MI occurs, is conducive for inhomogeneous solutions. In the following section, we investigate the possible inhomogeneous solutions for the amended NLSE, explicitly showing the presence of a supersolid phase.

3. Supersolid phase

We consider propagating Bloch wave-type solutions [36]:
\[
\psi = \psi_0 \left( \frac{x - vt}{\xi} \right) \exp \left( ikx - i\frac{\mu}{\hbar} t \right),
\]
where \(\xi\) is the healing length. The equation (3) now modifies to
\[
-\frac{\hbar^2}{2M} \partial_x^2 \psi_0 + \delta g \psi_0^3 - \frac{\sqrt{2M}}{\pi \hbar} g^{3/2} \psi_0^2 - \bar{\mu} \psi_0 = 0.
\]
In the above equation, the imaginary parts are canceled due to the constraint \(v = \frac{\hbar k}{m}\), and the kinetic energy contribution \(\hbar k^2/2M\) has been included in the modified chemical potential, \(\bar{\mu} = (\mu - \hbar k^2/2M)\). Henceforth, we take \(\psi_0\) to be a real periodic function without loss of generality, taking advantage of the global \(U(1)\) symmetry of the system. For finding general solutions, appropriate Möbius transformations are employed, to connect the solution space to the ubiquitous cnoidal waves [37], satisfying: \(f'' = \pm af' = \pm \lambda f^3 = 0\) [29]. The general solution has the form
\[
\psi_0(x,t) = A + B f^\xi \left( \frac{x - vt}{\lambda \xi} \right) \frac{\left( \frac{\xi^2}{\lambda \xi} \right)}{1 + D f^\xi \left( \frac{x - vt}{\lambda \xi} \right)},
\]
Balancing of nonlinearity with dispersion leads to \(\delta = 1\) and \(\bar{\delta} = 2\) as possible choices. Here, we consider \(\delta = 1\) as it has richer structure, which allows for diverse boundary conditions to be satisfied. We now start with the periodic solution, \(\psi_0(x,t) = A + B \sin \left( \frac{x - vt}{\lambda \xi}, m \right)\), where \(m (0 \leq m \leq 1)\) is the modulus parameter which characterizes the Jacobi elliptic function, \(\sin (X, m)\) where \(X = \frac{\xi^2}{\lambda \xi}\). Its period, \(X_{\text{period}} = 4\bar{K}(m)\), which is equal to period of Jacobi elliptical function, \(\sin (X, m)\), where \(K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}\), is a complete elliptic integral of the first kind. It enables interpolation between sinusoidal and hyperbolic functions. As an example, \(\sin (X, m)_{m=0} = \sin (x)\) and \(\sin (x, m)_{m=1} = \tanh (x)\). We found that the obtained supersolid solution is necessarily accompanied by a positive background \(A = \frac{\sqrt{2M}}{\pi \hbar} g^{3/2}\), having one-third of the value of the uniform background. It is worth emphasizing that the supersolid solutions do not smoothly merge with the constant condensate in the limit \(m \to 0\), when \(B = 0\). Its amplitude, \(B = \pm \sqrt{\frac{2m}{\pi \hbar^2 g^3}} A\), never exceeds that of the background, for \(0 < m < 1\). The healing length is obtained as:
\[
\xi = \frac{3\hbar^2 \pi}{2M} \sqrt{\frac{(m + 1)\delta g}{g^3}}.
\]
It is evident that the existence of the Bloch-type solutions with a superfluid background crucially depends upon the cubic nonlinearity, BMF correction and dispersion. For a moving supersolid, \(\mu = -\hbar^2 k^2/2M = -\frac{\lambda M}{\pi \hbar^2 g^2}\), revealing that the chemical potential is bounded below, \(\mu_{\text{min}} = \mu_{0} = \frac{4M}{\pi \hbar^2 g^2} < 0\), identical to that of the self-trapped droplet [17]. Here, \(\psi_{\text{min}}/\psi_{\text{max}} = A \pm B = A \left( 1 \pm \sqrt{\frac{2m}{\pi \hbar^2 g^3}} \right) > 0\), showing clearly that the quantum droplets are immersed in a residual condensate [1, 2, 14].

The density of the residual superfluid \(n_{\text{res}} = A^2 \left( 1 - \sqrt{\frac{2m}{\pi \hbar^2 g^3}} \right)^2\). This diffused matter-wave density rules out the scenario of one atom per site, thereby overcoming the Penrose and Onsager criterion [38, 39, 43]. The relation between \(\xi\) and \(\lambda_0\) is given by
\[
\xi = \frac{3}{4\pi} \sqrt{\frac{m + 1}{2}} \lambda_0,
\]
where \(\lambda_0 = 2\pi/k_0\) (see equation (7)) is the wavelength of the small perturbation on the plane wave solution. This shows that the healing length of the supersolid phase is smaller than the wavelength of the small perturbation, \(\xi < \lambda_0\), as expected from physical considerations.

The density, \(n(x)\) of the supersolid phase is shown in figure 1, as a function of scaled position variable \(x/\xi\), in the co-moving frame. Its unit is inverse of the length. It is evident that inter-droplet spacing (peak-to-peak distance) and number of quantum droplets are inter-related. With increasing repulsive intra-component interaction \(g\), the number of droplets increases and their size decreases for a fixed value of \(x/\xi\), which is depicted in figures 1(a) and (b). For plotting, we use \(g' = \left( \frac{M}{\pi \hbar^2} \right)^{1/3} g\), with the unit of \((E^2 L)^{1/3}\), where \(E\) is the energy and \(L\) is the length.

The second exact solution of the inverse sinusoidal form:
\[
\psi_0 = \frac{A}{1 + D \sin \left( \frac{\xi^2}{\lambda \xi} \right)},
\]
with the amplitudes \(A = \frac{3\pi \hbar}{\sqrt{2M g^{3/2}}}\) and \(D = \sqrt{1 + \frac{9\pi \hbar^2}{16 M g^2} \frac{\bar{\mu}}{\bar{\delta}}\mu_{\text{min}}}\).

The healing length \(\xi = \sqrt{\frac{2m}{\pi \hbar^2 g^3}}\) and \(\mu_{\text{min}} = \frac{4M}{\pi \hbar^2 g^2} \frac{\bar{\delta}}{\bar{\mu}}\). Hence,
Figure 1. The density of the supersolid phase as a function of scaled position is depicted for different values of intra-component interaction (g) and modulus parameter, with \( \delta g = 0.1 \) (black: \( m = 0.2 \) and red: \( m = 0.5 \)), for both the panels. The value of \( g' = 10 \) for panel (a) and \( g' = 12 \) for panel (b). Panel (c) shows the pictorial representation of several quantum droplets in a finite domain.

The minimum chemical potential is again identical to that of the self-trapped droplet [17].

We now proceed to study the general periodic solution of the amended NLSE,

\[
\psi_0 = \frac{A + B \sn \left( \frac{x - vt}{\xi}, m \right)}{1 + D \sn \left( \frac{x - vt}{\xi}, m \right)},
\]

(15)

The parameter \( D \) is the coefficient of the function \( \sn(x, m) \) in the denominator and can be physically pictured as a local modulation to the solution \( A + B \sn \left( \frac{x - vt}{\xi}, m \right) \). For explicitness, we concentrate on the sinusoidal solutions \((m = 0)\) having the lowest wavelength. A straightforward but tedious calculation yields

\[
\begin{align*}
A &= \frac{3g_1 - g_2 X_2}{3g_1 X_2 - g_2}, \\
B &= \frac{-X_2}{2X_2 X_1}, \\
D &= \frac{-g_1}{\sqrt{g_2^2 - 4g_1^2}}, \\
1 &= \frac{4g_2^2}{g_1^2}, \\
\end{align*}
\]

and \( X_2 = \frac{g_2\pm\sqrt{g_2^2-4g_1^2}}{g_1}, \) with \( g_1 = \delta g \) and \( g_2 = \frac{\sqrt{2\pi\hbar^2}}{\delta g} g^{3/2} \). The chemical potential \(- \frac{g_1^2}{g_2} < \mu < -\frac{g_1^2}{2g_2} \) and \( \mu_{\text{min}} \) is less than that of the value of the previous solutions for the self-trapped supersolids (equations (11) and (14)) or quantum droplets [17]. As mentioned earlier, for a pure 1D system the celebrated Mermin–Wagner [40], Hohenberg [41], and Coleman [42] theorem forbids a uniform condensate phase for spontaneously broken global symmetry. Therefore, these solutions may occur in a finite system, as depicted in figure 1.

It may also appear as a transient phase. It is worth mentioning that, for the case of cigar-shaped BEC, the Gross–Pitaevskii equation does not yield a constant background with periodic modulation of the Bloch form considered here. The presence of both MF and BMF energies are crucial for the existence of these solutions here. The soliton trains observed in the attractive BEC are composed of bright solitons in the absence of a superfluid background [44]. For the case of NLSE with a phase-locked source, periodic solutions with constant background are permissible [29]. In the case of NLSE, cnoidal wave solutions with a nontrivial phase have been shown to yield a supersolid phase with the desired Goldstone modes [45]. In optical fibers, analogous periodic solutions in the temporal domain have been experimentally observed as frequency combs [46]. Recently, NLSE with cubic-quadratic nonlinearity and phase-locked with a source has been investigated, wherein the parity even and kink type solutions have been identified [47]. The cubic-quadratic equation with a phase-locked source has also been investigated through the fractional linear transformations, for attractive Kerr nonlinearity and repulsive BMF correction [48]. The behavior of solitons under nonlinearity management has been explored [49]. In the following section, the dispersion relation for supersolid is investigated, distinctly showing that an increase in the repulsive inter-component interaction decreases the energy.
Figure 2. The energy per atom of the supersolid as a function of $k$ is depicted for different values of repulsive intra-component interaction and modulus parameter, with $m = 0.5$. It shows that increasing repulsive interaction, while keeping the competitive intra- and inter-component interaction constant, decreases the total energy per atom. The value of the intra-component interaction is $g' = 10$ for panel (b). Increasing the value of the modulus parameter leads to broader dispersion, due to increasing periodicity of the solution.

4. Energy and momentum of the supersolid phase

Before proceeding with the energy-momentum computation, it is worth comparing the static amended NLSE for the quantum droplet and supersolid phase. For this purpose, we consider equation (9) in the form of a Schrödinger eigenvalue problem with zero momentum:

$$\frac{-\hbar^2}{2M} \partial_x^2 \psi_0 + \delta g \psi_0^3 - \frac{\sqrt{2M}}{\pi \hbar} g^{3/2} \psi_0^2 = \bar{\mu} \psi_0. \quad (16)$$

The ansatz solution of Petrov and Astrakharchik [17] describes a bound state in the case of the quantum droplet with $\mu_0 < \mu < 0$. In the case of the currently studied supersolid phase with $\psi_0 = A + B \text{sn}(x, m)$, the potential in the corresponding eigenvalue problem is of the form $V(x, m) = C_1 + C_2 \text{sn}(x, m) + C_3 \text{sn}^2(x, m)$. This Lamé type potential is periodic, having a band spectrum [50]. As is shown above, the corresponding minimum chemical potential coincides with that of the quantum droplet: $\mu_{\text{min}} = \mu_0 = -\frac{3\hbar^2}{2M} g^{3/2} \psi_0^2 < 0$. For our general ansatz (equation (15)), the chemical potential is still lower than $\mu_0$.

We now study the dispersion relation of the supersolid phase by computing energy and momentum per atom, showing explicitly that the nature of energy dispersion is that of the acoustic phonon type. The energy and momentum density [51, 52] can be computed for a supersolid in a box of finite length ($L$). Computed expressions for number density ($N$), energy density ($E$) and momentum density ($P(x)$) are provided in the appendix. It is observed that for a given periodicity (fixed modulus parameter $m$), stronger intracomponent repulsion reduces the energy per atom; this is depicted in figure 2(a).

As is seen in figure 2(b), varying periodicity changes the curvature of the dispersion. A comparative flattening of the energy density is observed for greater periodicity.

5. Conclusion

In conclusion, we have obtained exact supersolid solutions in 1D self-trapped quantum matter. The supersolid phase occurs in a free-floating condition, unlike the case of dipolar BEC, wherein the condensate needs to be released from a trap for its formation. The exact Bloch waves obtained here represent a phase-coherent array of quantum droplets immersed in a residual condensate background [1, 53]. As the chemical potential for the general supersolid phase is lower than the self-trapped droplet, keeping in mind the destabilizing effect of QFs on long-range order in 1D, its realization in a finite system is of deep experimental interest. The nature of fluctuations in a dipolar system [54] has been investigated earlier, which may throw light on fluctuations in the supersolid phase in the appropriate parameter domain. The possibility of supersolid behavior and phase transitions between supersolid and quantum fragments have been predicted, with two- and three-body contact interactions, which may be studied by the procedure employed here to investigate the subtle differences between these quantum systems [55]. Recently, self-bound droplets have been investigated, incorporating the beyond LHY correction [56]. The effect of the same on the supersolid behavior is worth investigating.

Acknowledgments

We thank Professor R B MacKenzie and Dr T Mishra for fruitful discussions. MK Parit acknowledges the financial...
support from REDXY Center for Artificial Intelligence, IISER Kolkata, funded by Silicon Valley Community Foundation through Grant No. 2018-191175 (5618) and Council of Scientific and Industrial Research, New Delhi, Government of India for Junior Research Fellowship. D Singh is thankful to IISER Kolkata for the financial support. G Tyagi is thankful to IISER Kolkata for hospitality and financial support from the Inter-disciplinary Cyber Physical Systems (ICPS) program of the Department of Science and Technology (DST), India through Grant No. DST/ICPS/QuEST/Theme-1/2019/6.

Data availability statement

No new data were created or analysed in this study.

Appendix. Energy and momentum of the supersolid phase

Here, we present the number, energy and momentum densities for \( \psi_0(x, t) = (A + B \sin \left( \frac{\omega x - \delta t}{h} \right) \) exp (i\( kx - \omega t \)) [51]. Explicitly, \( E = \int \mathcal{H} \, dx \), where \( \mathcal{H} \) is the Hamiltonian density,

\[
\mathcal{H} = \frac{\hbar^2}{2M} |\partial_x \psi|^2 + \frac{\delta g}{2} |\psi|^4 - \frac{2\sqrt{2M}}{3\hbar} g^{1/2} |\psi|^4, \tag{17}
\]

and the momentum, \( P = \frac{i\hbar}{2} \int (\psi^* \partial_x \psi - \psi \partial_x \psi^*) \, dx \). The corresponding dispersion relation is obtained for a finite size system of length \( L \) (from \(-L/2\) to \( L/2\)). The values of \( A \), \( B \), and \( \xi \) are provided in the text.

(a) Number density

The indefinite integral, \( N = \int |\psi|^2 \, dx \) is obtained as

\[
N(x) = (A^2 m + B^2) (x - tv) + B^2 \xi E \times \left( \text{am} \left( \frac{tv - x}{\xi} \right) \right) m \right) + 2AB\sqrt{m} \xi \times \log \left( \text{dn} \left( \frac{x - tv}{\xi} \right) \right) m \right) + \sqrt{m} \text{cn} \left( \frac{x - tv}{\xi} \right) m \right). \tag{18}
\]

(b) Energy density

\[
E(x) = \frac{E_1 + E_2 + E_3 + E_4 + E_5 \times \lambda E}{6m^2 \xi^2}. \tag{19}
\]

\[
E_1 = 2 \left( 3A^2 m^2 \xi^2 (x - tv) \right) (A^2 g_1 - A g_2 + D_1 k^2) \right) + B^2 g_1 (m + 2) \xi^2 (x - tv), \tag{20}
\]

\[
E_2 = 2 \left( B^2 m \right) \left( 9A^2 \xi^2 (2A g_1 - g_2) (x - tv) \right) + D_1 tv \left( -3k^2 \xi^2 + 2m + 1 \right) \right) + D_1 x \left( 3k^2 \xi^2 + m - 1 \right) \right) \right) \right). \tag{21}
\]

\[
E_3 = 2B^2 \xi E \times \left( \text{am} \left( \frac{x - tv}{\xi} \right) \right) m \right) \times \left( \xi^2 \left( -9Am (2A g_1 - g_2) - 2B^2 g_1 (m + 1) \right) + D_1 m \left( -3k^2 \xi^2 + m + 1 \right) \right) \right) \right) \right). \tag{22}
\]

\[
E_4 = B^2 m \xi \text{cn} \left( \frac{x - tv}{\xi} \right) m \right) \times \left( (3B^2 (4A g_1 - g_2) + 2 \left( B^2 g_1 \xi^2 + D_1 m \right) \right) \times \text{sn} \left( \frac{x - tv}{\xi} \right) m \right). \tag{23}
\]

\[
E_5 = 3B^2 \xi \text{cn} \left( \frac{x - tv}{\xi} \right) m \right) + \sqrt{m} \text{cn} \left( \frac{x - tv}{\xi} \right) m \right). \tag{24}
\]

(c) Momentum density

\[
P(x) = \frac{h \kappa}{M} \left( (A^2 m + B^2) (x - tv) \right) + B^2 \xi E \times \left( \text{am} \left( \frac{tv - x}{\xi} \right) \right) m \right) + 2AB\sqrt{m} \xi \times \log \left( \text{dn} \left( \frac{x - tv}{\xi} \right) \right) m \right) - \sqrt{m} \text{cn} \left( \frac{x - tv}{\xi} \right) m \right). \tag{26}
\]

For the supersolid confined in a box of finite length \( L \) (from \(-L/2\) to \( L/2\)), number, \( N = N(L/2) - N(-L/2) \), energy, \( E = E(L/2) - E(-L/2) \), and momentum, \( P = P(L/2) - P(-L/2) \). The dispersion relation has been plotted in the main text.

ORCID IDs

Mithilesh K Parit https://orcid.org/0000-0003-4375-7014
Gargi Tyagi https://orcid.org/0000-0002-8795-9905
Dheerendra Singh https://orcid.org/0000-0002-5185-9443
Prasanta K Panigrahi https://orcid.org/0000-0001-5812-0353

References

[1] Böttcher F, Schmidt J-N, Wenzel M, Hertkorn J, Guo M, Langen T and Pfau T 2018 Phys. Rev. X 9 011051
[2] Tanzi L, Lucioni E, Famà F, Catani J, Fioretti A, Gabbanini C, Bisset R N, Santos L and Modugno G 2019 Phys. Rev. Lett. 122 130405
[3] Rocuzzo A M and Ancilotto F 2019 Phys. Rev. A 99 041601(R)
[4] Popov V N 1983 Functional Integrals in Quantum Field Theory and Statistical Physics (Dordrecht: Reidel) and references therein
