Quantization of the topological $\sigma$-model and the master equation of the BV formalism

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Abstract

We quantize the topological $\sigma$-model. The quantum master equation of the Batalin-Vilkovisky formalism $\Delta_\rho \Psi = 0$ appears as a condition which eliminates the exact states from the BRST invariant states $\Psi$ defined by $Q \Psi = 0$. The phase space of the BV formalism is a supermanifold with a specific symplectic structure, called the fermionic Kähler manifold.

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There is much interest in the BV formalism\textsuperscript{[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]} Originally it was devised to study quantization of gauge theories\textsuperscript{[10, 11]} The geometry of the formalism was clarified in refs 1 and 4\texttextsuperscript{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}. Its extended viability has been shown by the applications to the non-critical string\textsuperscript{[3, 7]} and the string field theory\textsuperscript{[2, 3, 5, 6, 8]} But there we may recognize a conceptual departure from the original framework of the BV formalism. The anti-bracket does not follow directly from an action as defining the S-T identity\textsuperscript{12}. Recently, as one of such approaches, the BV formalism was discussed on a supermanifold with a specific symplectic structure, called the fermionic Kähler manifold\textsuperscript{13}. In this letter we show that this fermionic Kähler geometry naturally appears by quantizing the topological σ-model\textsuperscript{14}. Namely the quantum master equation $\Delta_r \Psi = 0$ with the symplectic structure of this geometry is obtained as a condition which eliminates the exact states of the BRST invariant states $\Psi$ defined by $Q \Psi = 0$. In other words $\Delta_r$ operator is equivalent to the superpartner of the BRST charge $Q$ in the topological σ-model.

The study in this direction has been motivated by the work by Lian and Zuckerman\textsuperscript{[7]}. They conjectured that in the topological conformal field theory the $G_R$(or $G_L$) charge could be connected with the $\Delta_r$ operator of the BV formalism. Although any concrete connection was not given, they wondered about the meaning of the quantum master equation in such a case.

The study has been also motivated by the arguments in ref. 15. They discussed the BRST cohomology of the topological σ-model in the right-moving sector, $Q_R \Psi = 0$ with $G_R \Psi = 0$ (or equivalently in the left-moving sector). However the $G_R$ charge does not become the $\Delta_r$ operator of the BV formalism by itself, although this cohomology equals the BRST cohomology defined by $(Q_L + Q_R) \Psi = 0$ with $(G_L + G_R) \Psi = 0$. (Namely the Dolbeault cohomology equals the de Rham cohomology on the Kähler manifold.) So we carefully study the latter cohomology. It is shown that the $G_L + G_R$ charge equals the $\Delta_r$ operator of the BV formalism on the fermionic Kähler manifold. On an off-shell state the charge $Q_L + Q_L$ becomes an exterior derivative $d$ of a differential form, while the charge $G_L + G_R$(or the $\Delta_r$ operator) its adjoint $d^*$. The last statement is generally expected in the topological field theory because of the $N = 2$ supersymmetric structure\textsuperscript{[14, 16, 17]}. Thus the quantum master equation of the BV formalism is a necessary and sufficient condition to pick out a unique representative for each BRST cohomology
class. The physical states are given by harmonic differential forms of the Kähler manifold.

Through our study of the BRST cohomology the meaning of the function $\rho$ in the master equation $\Delta_{\rho} \Psi = 0$ is considerably clarified. It is a quantum effect of operator ordering in defining the quantum $G$ charge and can be renormalized in physical states. We also give a substantial argument to show that the function $\rho$ itself is an element of the BRST cohomology class, i.e., a physical state.

To start with, it is convenient to recall the basic formulae of the symplectic geometry\textsuperscript{[4–6]}. We consider a 2D bosonic manifold $M$ parametrized by real coordinates $u^\alpha = (u^1, u^2, \cdots, u^{2D})$. Suppose that it has a symplectic structure given by a non-degenerate 2-form

$$\Omega = du^\beta \wedge du^\alpha \Omega_{\alpha\beta},$$

which is closed $d\Omega = 0$. In components these equations imply that $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$ and

$$\partial_\alpha \Omega_{\beta\gamma} + \partial_\beta \Omega_{\alpha\gamma} + \partial_\gamma \Omega_{\alpha\beta} = 0,$$

or equivalently $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$ and

$$\Omega^{\alpha\eta} \partial_\eta \Omega^{\beta\gamma} + \Omega^{\beta\eta} \partial_\eta \Omega^{\alpha\gamma} + \Omega^{\gamma\eta} \partial_\eta \Omega^{\alpha\beta} = 0.$$

by inverting $\Omega_{\alpha\beta}$ as

$$\Omega_{\alpha\beta} \Omega^{\beta\gamma} = \Omega^{\gamma\beta} \Omega_{\beta\alpha} = \delta_\alpha^\gamma.$$  

We introduce real fermion coordinates $\psi^\alpha = (\psi^1, \psi^2, \cdots, \psi^{2D})$ and think of a 4D manifold $\mathcal{M}$ parametrized by supercoordinates $s^i = (u^1, u^2, \cdots, u^{2D}, \psi^1, \psi^2, \cdots, \psi^{2D})$. Assume that the symplectic structure $\Omega_{\alpha\beta}$ is given by a generalized 2-form

$$\omega = ds^j \wedge ds^i \omega_{ij},$$

and $d\omega = 0$. In components these equations read

$$(-)^{ik} \partial_i \omega_{jk} + (-)^{ji} \partial_j \omega_{ki} + (-)^{kj} \partial_k \omega_{ij} = 0.$$
\[ \omega_{ij} = -(-)^{ij} \omega_{ji}. \]  

Here we used the short-hand notation for the grassmannian parity of the coordinates \( \varepsilon(s^i) = i \) in the sign factor. So long as the grassmannian parity is assigned as \( \varepsilon(\omega_{ij}) = i + j \), the 2-form given by eq. (5) is bosonic and defines the ordinary symplectic structure. However if the opposite grassmannian parity is assigned as \( \varepsilon(\omega_{ij}) = i + j + 1 \), then the 2-form (5) is fermionic. Eqs (6) and (7) become respectively

\[ (-)^{(i+1)(k+1)} \omega^{ij} \partial_l \omega^{jk} + (-)^{(j+1)(i+1)} \omega^{jl} \partial_l \omega^{ki} + (-)^{(k+1)(j+1)} \omega^{kl} \partial_l \omega^{ij} = 0, \]  

(8)

\[ \omega^{ij} = -(-)^{(i+1)(j+1)} \omega^{ji}, \]  

(9)

by inverting \( \omega_{ij} \) as

\[ \omega_{ij} \omega^{jk} = \omega^{kj} \omega_{ji} = \delta^k_i. \]

Then the anti-bracket is defined by

\[ \{A, B\} = A \leftrightarrow \partial_i \omega^{ij} \partial_j B. \]  

(10)

The right-derivative \( \leftrightarrow \partial_i \) is related with the left-one by

\[ A \leftrightarrow \partial_i = (-)^{i(\varepsilon(A)+1)} \partial_i A. \]

With this fermionic symplectic structure we define also a second order differential operator by

\[ \Delta_\rho \equiv \frac{1}{\rho}(\rho \omega^{ij} \partial_j), \]  

(11)

where \( \rho \) is a bosonic function of \( y^i \). It is related with the anti-bracket through

\[ \Delta_\rho(AB) = \Delta_\rho A \cdot B + (-)^{\varepsilon(A)} A \Delta_\rho B + (-)^{\varepsilon(A)} \{A, B\}. \]  

(12)

A crucial observation in this letter is that as a special solution to eqs (8) and (9) we have the fermionic symplectic structure

\[ \omega^{ij} = \begin{pmatrix} \omega^{uu} & \omega^{u\psi} \\ \omega^{\psi u} & \omega^{\psi\psi} \end{pmatrix} = \begin{pmatrix} 0 & \Omega^{\alpha\beta} \\ \Omega^{\beta\alpha} & \psi^\gamma \frac{\partial}{\partial u^\gamma} \Omega^{\alpha\beta} \end{pmatrix}, \]  

(13)
in which $\Omega^{\alpha\beta}$ is the one of the bosonic submanifold $M$ obeying eq. (3).

If $M$ is a Kähler manifold, it is endowed with a complex structure which is covariantly constant:

$$D_{\gamma}J_{\beta}^{\alpha} = 0,$$

and $J_{\beta}^{\alpha} J_{\gamma}^{\beta} = -\delta_{\gamma}^{\alpha}$. We assume the metric $g_{\alpha\beta}$ to be of type (1, 1), i.e.,

$$g_{\alpha\beta} = g_{\gamma\delta} J_{\alpha}^{\gamma} J_{\beta}^{\delta}.$$

The symplectic structure $\Omega_{\alpha\beta}$ is given by

$$\Omega_{\alpha\beta} = g_{\alpha\gamma} J_{\beta}^{\gamma}.$$

The closure property (2) follows from eq. (14). We also obtain the self-duality condition of the Riemann tensor

$$R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\eta\sigma} J_{\eta}^{\gamma} J_{\sigma}^{\delta},$$

taking the covariant derivative of eq. (14). According to the previous discussion we may generalize the Kähler manifold $M$ to a supermanifold $\mathcal{M}$ with the fermionic symplectic structure given by eq. (13). In this case the complex structure $J_{\alpha\beta}$ may be also extended to $J_{i\ j}$ as

$$J_{i\ j} = \begin{pmatrix} J_{\alpha\beta}^{\alpha} & 0 \\ 0 & J_{\beta\alpha}^{\beta} \end{pmatrix}.$$

It is natural to define the metric of the supermanifold $\mathcal{M}$ by

$$\gamma_{ij} = -\omega_{ik} J_{j}^{k}.$$

Then the metric is fermionic, $\varepsilon(\gamma_{ij}) = i + j + 1$ and of type (1, 1). It is given by

$$\gamma_{ab} = \partial_{a} \partial_{b} K,$$

in which $K$ is a fermionic Kähler potential. Here we have used the complexified coordinates $u^{a} = (u^{a}, u^{\bar{a}})$, $a, \ a = 1, 2, \cdots, D$. This supermanifold $\mathcal{M}$ is a fermionic version of the Kähler manifold $M$, which has been discussed in refs 13. It is called the fermionic Kähler manifold.
The topological \(\sigma\)-model\(^{14}\) is constructed on the (bosonic) Kähler manifold discussed just above. In the real field representation the action is given by

\[
S = \int d^2x \left[ -\frac{1}{4} H^{\mu \alpha} H_{\mu \alpha} + H^{\mu \alpha}_\alpha \partial_\mu u^\alpha 
- i \rho^{\mu \alpha}_\alpha J^{\alpha}_\beta D_\mu \psi^\beta - \frac{1}{8} R^{\gamma \delta}_{\alpha \beta} \rho^\mu_{\gamma \delta} \rho^\mu_{\rho \delta} \psi^\alpha \psi^\beta \right]
\]

Here we have spin 0 fermions \(\psi^\alpha\), spin 1 fermions \(\rho^\mu_{\alpha \mu}\) and spin 1 bosons \(H^{\alpha}_\mu\) in addition to spin 0 bosons which are coordinates of the Kähler manifold \(M\). The spin 1 fields are constrained by the self-duality conditions

\[
\rho^{\mu \alpha}_\alpha = \varepsilon^{\mu \nu} J^{\alpha}_\beta \rho^{\nu \beta}, \quad H^{\mu \alpha}_\alpha = \varepsilon^{\mu \nu} J^{\alpha}_\beta H^{\nu \beta}.
\]

Here \(\varepsilon^{\mu \nu}\) is the anti-symmetric tensor in two dimensions. By the replacement \(\psi^\alpha = J^{\alpha}_\beta \chi^\beta\) and the use of eq. (15) the action (16) goes back to the original form in ref. 14. It is invariant by the BRST transformations

\[
\delta u^\alpha = i J^{\alpha}_\beta \psi^\beta,
\delta \psi^\alpha = i J^{\alpha}_\beta,\gamma (J^{\beta}_\eta \psi^\eta) (J^{\gamma}_\delta \psi^\delta),
\delta \rho_{\mu \alpha} = H^{\mu \alpha} - i \rho_{\mu \beta} \Gamma^{\beta}_{\alpha \gamma} (J^{\gamma}_\delta \psi^\delta),
\delta H^{\alpha}_\mu = -i \Gamma^{\alpha}_{\beta \gamma} (J^{\beta}_\eta \psi^\eta) H^{\gamma}_\mu - \frac{1}{2} R^{\gamma \delta}_{\alpha \beta} \rho^\mu_{\beta \gamma} \psi^\gamma \psi^\delta,
\]

For the canonical formulation it is convenient to write the action (16) in terms of the independent fields alone. In representation \(\varepsilon^{0}_1 = -\varepsilon^{1}_0 = 1\) and \(\varepsilon^{0}_0 = \varepsilon^{1}_1 = 0\) the constraints (17) read

\[
\rho^{1}_1 = \rho^{0}_0 J^{\beta}_\alpha, \quad H^{1}_1 = H^{0}_0 J^{\beta}_\alpha.
\]

By using these equations we eliminate the fields \(\rho^{1}_1\) and \(H^{1}_1\) from the action. It becomes

\[
S = \int d^2x \left[ -\frac{1}{2} H^{\alpha}_0 H^{\alpha}_0 + H^{0}_0 \partial_0 u^\alpha + H^{0}_0 J^{\alpha}_\beta \partial_1 u^\beta 
- i \rho^{0}_0 J^{\alpha}_\beta D_0 \psi^\beta + i \rho^{0}_0 D_1 \psi^\alpha - \frac{1}{4} R^{\gamma \delta}_{\alpha \beta} \rho^\gamma_{\beta \gamma} \rho^\delta_{\rho \delta} \psi^\alpha \psi^\beta \right]
\]

by eq. (15). We carry out the canonical formulation at the euclidean time \(x_0 = 0\). The standard Noether procedure gives the Hamiltonian and the BRST charge

\[
H = \int dx_1 T_{00}
\]
\[
= \int dx_1 \left[ \frac{1}{2} H_0^\alpha H_0^\alpha - H_0^\alpha J_\beta^\alpha \partial_1 u^\beta \\
- i \rho_{0\alpha} D_1 \psi^\alpha + \frac{1}{4} R_{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \rho_0 \gamma \rho_0 \delta \psi^\alpha \psi^\beta \right]
\]

\[
Q = i \int dx_1 \left[ (H_0^\alpha - i \rho_{0\sigma} J_\eta^\sigma \Gamma^\gamma_{\alpha\beta} \psi^\beta)(J_\gamma^\alpha \psi^\gamma) + i \rho_{0\sigma} J_\eta^\sigma J_\eta^\alpha,\beta (J_\gamma^\alpha \psi^\gamma)(J_\delta^\beta \psi^\delta) \right].
\] (19)

For these quantities we find that

\[
T_{00} = \delta G_{00},
\]

with

\[
G_{00} = \frac{1}{2} g^{\alpha\beta} \rho_{0\alpha} [H_0^\beta - 2 \Omega_\beta^\gamma \partial_1 u^\gamma],
\] (20)

which is the hallmark of the topological field theory.

We calculate the canonical conjugate momenta for \(u^\alpha\) and \(\psi^\alpha\):

\[
\pi_\alpha^u = H_0^\alpha - i \rho_{0\sigma} J_\gamma^\sigma \Gamma^\gamma_{\alpha\beta} \psi^\beta,
\]

\[
\pi_\alpha^\psi = i \rho_{0\beta} J_\alpha^\beta,
\]

and set up the Poisson brackets as usual:

\[
\{ \pi_\alpha^u(0, x_1), u^\beta(0, y_1) \}_{PB} = \delta_\alpha^\beta \delta^{(1)}(x_1 - y_1),
\]

\[
\{ \pi_\alpha^\psi(0, x_1), \psi^\beta(0, y_1) \}_{PB} = \delta_\alpha^\beta \delta^{(1)}(x_1 - y_1).
\]

As a consistency check it is shown that the Hamiltonian \(H\) and the \(Q\) charge are generators of the Euler-Lagrange equations and the BRST transformations (18) respectively. Consequently it follows that

\[
\{ Q, G_{00} \}_{PB} = T_{00},
\]

\[
\{ Q, Q \}_{PB} = 0
\]

Then we write the \(Q\) charge and the space integral of \(G_{00}\), called the \(G\) charge, in terms of the canonical variables alone:

\[
Q = i \int dx_1 [\pi_\alpha^u (J_\gamma^\alpha \psi^\gamma) + \pi_\eta J_\eta^\alpha,\beta (J_\gamma^\alpha \psi^\gamma)(J_\delta^\beta \psi^\delta) ],
\] (21)

\[
G = -\frac{i}{4} \int dx_1 [\Omega^\alpha^\beta \pi_\alpha^u \pi_\beta^u - \Omega^\alpha^\beta \psi^\gamma \pi_\alpha^\psi \pi_\beta^\psi - 4 \pi_\alpha^\psi \partial_1 u^\alpha ].
\] (22)
The last formula has been derived by using formula

\[ \Omega^\alpha [\beta \Gamma_{\alpha\delta}^\gamma] = \Omega^{\beta\gamma}_{\cdot\cdot}. \]

Now we are in a position to quantize the topological \( \sigma \)-model. By the replacement

\[ \pi_\alpha^u(0, y) = \frac{\delta}{\delta u^\alpha(0, y)} \equiv \partial_\alpha(0, y), \]

\[ \pi_\alpha^\psi(0, y) = \frac{\delta}{\delta \psi^\alpha(0, y)} \equiv D_\alpha(0, y). \]

The \( Q \) and \( G \) charges become quantum operators. The quantum Hamiltonian is obtained through \( H = [Q, G]_+. \) Physical states \( \Psi \) of the theory are defined by

\[ Q\Psi = 0. \quad (23) \]

They are also assumed to obey the condition\(^{[16]}\)

\[ G\Psi = 0. \quad (24) \]

It will be later shown that this extra requirement does a right thing. Let us consider an off-shell state given by

\[ \Psi(0, x) = \phi_{\alpha_1\alpha_2\cdots\alpha_N}(0, x)\psi^{\alpha_1}(0, x)\psi^{\alpha_2}(0, x)\cdots\psi^{\alpha_N}(0, x), \quad N \leq 2D. \quad (25) \]

in which \( \phi_{\alpha_1\alpha_2\cdots\alpha_N}(0, x) \) is a function of \( u^\alpha(0, x) \) with no derivative. Acting on this state the quantum \( G \) charge drops the last term of the classical expression (22) in order to fulfil the condition (24). Then the corresponding classical \( G \) charge satisfies \( \{G, G\}_PB = 0. \) Therefore we require that the quantum \( G \) charge is nilpotent as well as the \( Q \) charge. Eqs (23) and (24) respectively obtain the following normal-ordered expressions

\[ Q\Psi(0, x) = i \int dy[J^\alpha_{\beta}\psi^\beta\partial_\alpha + J^\eta_{\alpha,\beta}(J^\gamma_\alpha\psi^\gamma)(J^\delta_\beta\psi^\delta)D_\eta \Psi(0, x) = 0, \quad (26) \]

\[ G\Psi(0, x) = -i \int dy[\Omega^{\alpha\beta}D_{\alpha}\partial_\beta - \Omega^{\alpha\beta}_{\cdot\cdot}\psi^\gamma D_\alpha D_\beta]\Psi(0, x) = 0. \quad (27) \]

It is worth checking that in this operator realization the \( Q \) and \( G \) charges indeed satisfy

\[ Q^2 = 0 \quad \text{and} \quad G^2 = 0. \quad (28) \]
Remarkably eq. (27) can be put in a geometrical form of the BV formalism such that

$$G_\rho \Psi(0,x) = -\frac{i}{4} \Delta_\rho \Psi(0,x) = 0,$$

(29)

with $\Delta_\rho$ defined by eq. (11), in which $\omega^{ij}$ is the symplectic structur of the fermionic Kähler manifold given by eq. (13) and $\partial_i = (\partial_\alpha(0,y), D_\alpha(0,y))$. Here we should understand the index $i$ in the formula (11) as standing also for the space coordinate $y$. A little calculation shows that

$$G_\rho \Psi(0,x) = G\Psi(0,x) - \frac{i}{4} \delta^{(1)}(0) \{ \log \rho(0,x), \Psi(0,x) \}. $$

(30)

Here the second piece is the anti-bracket given by (10), which represents a quantum effect due to operator ordering. In front of it we have the well-known quantum singularity of the $\Delta_\rho$ operator[9] (but the one-dimensional $\delta$-singularity in our canonical formulation). Therefore the representation (29) is more general than (27). In order to ensure that $G_\rho$ also satisfies eq. (28) it is sufficient to impose the condition on $\rho$ [4]

$$G_\rho(0,x) = 0,$$

(31)

for the r.h.s. of eq. (30) is calculated as

$$G_\rho \Psi(0,x) = \frac{1}{\rho(0,x)} G[\rho(0,x)\Psi(0,x)] - \frac{1}{\rho(0,x)} [G\rho(0,x)] \cdot \Psi(0,x),$$

(32)

by eq. (12). Thus we have shown that the condition (29) becomes the quantum master equation of the BV formalism together with the nilpotency condition (31). The relevant phase space of the BV formalism is the fermionic Kähler manifold discussed in refs 13.

Our final task is to solve eqs. (26) and (29). As can be seen from eq. (32) the quantum effect of operator ordering is a multiplicative renormalization of the state. The function $\rho$ should be further constrained by

$$Q_\rho(0,x) = 0,$$

(33)

in order that both equations become

$$Q\Psi_{\rho}(0,x) = 0, \quad G\Psi_{\rho}(0,x) = 0,$$

(34)
with the renormalized state $\Psi_\rho = \rho \Psi$. Hence $\rho$ can be regarded as a physical state as well. By eq. (25) we may put the renormalized state in the form

$$
\Psi_\rho(0, x) = A_{\alpha_1 \alpha_2 \ldots \alpha_N} (J^{\alpha_1}_{\beta_1} \psi^{\beta_1}) (J^{\alpha_2}_{\beta_2} \psi^{\beta_2}) \cdots (J^{\alpha_N}_{\beta_N} \psi^{\beta_N}),
$$

with a new function $A_{\alpha_1 \alpha_2 \ldots \alpha_N}$. Here the $x$-dependence was not written explicitly. Then the first equation in (34) becomes

$$
\partial_{\alpha_0} A_{\alpha_1 \alpha_2 \ldots \alpha_N} \pm \text{cyclic permutations} = 0,
$$

while the second one

$$
D^{\alpha_1} A_{\alpha_1 \alpha_2 \ldots \alpha_N} = 0,
$$

with the covariant derivative for the Kähler manifold. Thus the respective operators $Q$ and $G$ turn into an exterior derivative $d$ and its adjoint $d^*$ of the differential $N$-form $A$. Solutions to these equations are given by harmonic differential forms of the Kähler manifold. This result proves that it was right to impose the condition (24).

In this letter we have quantized the topological $\sigma$-model by the canonical formalism. The BRST cohomology of the physical states defined by $Q \Psi = 0$ has been worked out. We have imposed the condition $G \Psi = 0$ and shown that it is equivalent to the quantum master equation of the BV formalism. The relevant phase space is the fermionic Kähler manifold discussed in refs 13. As generally expected in the topological field theory$^{[14,16,17]}$, the conditions $Q \Psi = 0$ and $G \Psi = 0$ are written by an exterior derivative $d$ of a differential form and its adjoint $d^*$. Thus the quantum master equation is a necessary and sufficient condition to eliminate exact states of the BRST cohomology. The physical states of the topological $\sigma$-model are given by harmonic differential forms of the Kähler manifold. Our discussions have considerably clarified the meaning of the function $\rho$ in the quantum master equation. Namely it takes into account a quantum effect of operator ordering in defining the quantum $G$ charge. The physical states $\Psi$ get a multiplicative normalization from this quantum effect such that $\Psi_\rho = \rho \Psi$. We have also found that the function $\rho$ itself is an element of the BRST cohomology by eqs (31) and (33).
Finally the reader might worry about the $\delta$-singularity of the $G$ operator which appeared in the calculations like eq. (30) and (35). The origin of this divergence can be traced back to the Hamiltonian through the relation $H = [Q, G]_+$. It contains the same singularity. This is the usual phenomenon in the Schrödinger representation of the quantum field theory. Also he might worry about the quantum singularity in defining a local state such as eq. (25). In this regard the situation is worse in the canonical quantization in the right(or left)-moving frame\cite{15}. In that case the coordinates of the Kähler manifold no longer commute with each other. Consequently a product of the coordinates at the same space-time point is more divergent than in our case. However because of the global topological nature of the theory there might be a circumstance in which such singularities do not come in. In mathematical words there could exist a Hilbert subspace in which both $Q$ and $G$ charges are realized merely by the usual derivatives instead of the functional ones. Note that the topological $\sigma$-model has instanton solutions. It is promising to find such a situation by a semi-classical quantization around an instanton background. It is interesting to demonstrate this conjecture.

Acknowledgments

The author is grateful to A. Van Proeyen for the discussions and reading the manuscript. He thanks the Research Council of K.U. Leuven for the financial support.

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