Analysis of Optimal Thresholding Algorithms for Compressed Sensing

Yun-Bin Zhao* and Zhi-Quan Luo †

Abstract

The optimal thresholding is a new technique that has recently been developed for compressed sensing and signal approximation. The optimal k-thresholding (OT) and optimal k-thresholding pursuit (OTP) provide basic algorithmic frameworks that are fundamental to the development of practical and efficient optimal-thresholding-type algorithms for compressed sensing. In this paper, we first prove an improved bound for the guaranteed performance of these basic frameworks in terms of the kth order restricted isometry property of the sensing matrices. More importantly, we analyze the newly developed algorithms called relaxed optimal thresholding (ROTω) and relaxed optimal thresholding pursuit (ROTPω) which are derived from the tightest convex relaxation of the OT and OTP. Numerical results in [54] have demonstrated that such approaches are truly efficient to recover a wide range of signals and can remarkably outperform the existing hard-thresholding-type methods as well as the classic ℓ1-minimization in numerous situations. However, the guaranteed performance/convergence of these algorithms with ω ≥ 2 has not yet established. The main purpose of this paper is to establish the first guaranteed performance results for the ROTω and ROTPω.

Key words: Optimal k-thresholding, guaranteed performance, signal recovery, convex optimization, compressed sensing, restricted isometry property

1 Introduction

In signal processing, one is often interested in reconstructing a signal from the linear measurements acquired for the signal. When the signal is sparse or can be sparsely approximated, the compressed sensing theory claims that it is possible to recover the signal from far fewer measurements than the signal length (see, e.g., [13, 21, 25, 26, 30]). More practically, one may recover/reconstruct the most significant information of the signal (which can be interpreted as a few largest absolute coefficients of the signal on its redundant bases). This amounts to solving the following minimization problem with a sparsity constraint:

$$\min_{z} \{\|Az - y\|_2^2 : \|z\|_0 \leq k\},$$

where A is an m × n sensing/measurement matrix with m < n, y := Ax ∈ ℜm are the measurements of the target signal x ∈ ℜn, k is a prescribed integer number reflecting the interested
sparsity level, and $\|z\|_0$ is called the ‘$\ell_0$-norm’ counting the number of nonzero entries of $z \in \mathbb{R}^n$. Clearly, the model (1) is to find the $k$-term approximation of the target signal, which can best fit the acquired measurements. It is one of the essential models for the development of compressed sensing theory and algorithms (see, e.g., [23, 26, 30, 45]). This model also arises in other scenarios such as the subset selection problems in statistics [41, 3], low-rank matrix recovery [12, 11, 20, 31], and sparse optimization and optimal control [1, 53, 51, 39].

The thresholding method is one of the approaches that can be used to possibly solve the problem (1). This approach was first introduced by Donoho and Johnstone [23] for signal denoising problems (see also Donoho [22]). Other relatively earlier work using this technique can also be found in general areas of signal processing, [35, 27, 47] and in specific areas of compressed sensing [33, 51, 6, 2, 7]. The thresholding algorithms can roughly be grouped into the soft thresholding and the hard thresholding depending on the thresholding operators. The soft ones are usually developed from a necessary optimality condition of certain optimization problems (see [19, 22, 24, 33, 28, 50]). The hard ones can be seen as the projected Landweber iteration [37] or can be derived from the perspective of minimizing certain surrogate functions related to the underlying sparse optimization problems (see, e.g., [5, 19, 38]). The hard thresholding methods have widely been studied in the area of compressed sensing or sparse approximation [5, 6, 7, 29, 30, 4]. The latest development and applications of these methods can be found in [8, 9, 34, 16, 52, 54, 18]. Although the sparse optimization problems like (1) arising from compressed sensing are usually NP-hard [42], it does not prohibit the fast development of various computational methods for such problems. Along with thresholding, the matching pursuits (e.g., [40, 49, 43]) and convex optimization (e.g., [17, 13, 14, 55, 56, 57, 53]) are also popular methods that have been broadly studied in this area.

A unique feature of thresholding algorithms is their simple structures that are easy to implement with a low computational cost. Compared with $\ell_1$-minimization and other state-of-art algorithms, however, the numerical performance of the iterative hard thresholding (IHT) [5, 6, 30] and the hard thresholding pursuit (HTP) [29, 30] are far from satisfactory especially when the ratio of the sparsity level and the number of measurements of the signal is relatively high. Thus some acceleration techniques were introduced to possibly improve the performance of these algorithms. This includes the use of certain stepsizes (e.g., [32, 7, 15, 4]) and the Nesterov’s acceleration technique [2, 15, 44, 34, 30]. As pointed out in [54], the major drawback of existing hard-thresholding-based algorithms is the direct use of the hard thresholding operator, denoted by $\mathcal{H}_k(\cdot)$, which retains the $k$ largest magnitudes of a vector and zeroing out the remaining entries of the vector. Performing thresholding via the $\mathcal{H}_k(\cdot)$ to generate a feasible point to the problem (1) is independent of the objective function. This may cause a dramatic increase instead of the decrease of the objective value during the course of iterations. The existing acceleration techniques might help in some situations, but none of them actually serves the purpose of overcoming the intrinsic drawback of the hard thresholding operator $\mathcal{H}_k$.

To alleviate the inherent weakness of the hard thresholding operator, Zhao [54] introduced a new thresholding technique called the optimal $k$-thresholding (OT), based on which a new class of thresholding algorithms was developed, and the empirical results demonstrate that the optimal $k$-thresholding method is truly stable and efficient for signal recovery compared with
IHT and HTP. This new development may also outperform the classic $\ell_1$-minimization method in numerous situations. The OT technique is stimulated by the following idea: *The thresholding should be made to reduce the value of the objective function instead of being independent of the objective. When the operator $H_k$ is used, it should be applied to a compressible vector instead of any vector.* The optimal $k$-thresholding selects the best $k$ components of a vector that best fits the measurements among all possible $k$ entries of the vector. Such a thresholding is directly connected to the minimization of the objective function, yielding a significant numerical advantage over the existing thresholding frameworks. The encouraging empirical results for the optimal thresholding algorithms motivate us to further investigate the properties of such algorithms from a theoretical perspective. The initial analysis has been carried out in [54] to the basic OT algorithm and the optimal $k$-thresholding pursuit (OTP) as well as their tightest convex relaxation counterparts called ROT and ROTP. It was shown in [54] that the restricted isometry condition $\delta_{2k} < 0.5349$ is sufficient to guarantee the convergence of the OT and OTP, and that $\delta_{3k} \leq 1/5$ is a sufficient condition for the guaranteed performance of the ROT and ROTP. However, the convergence of the enhanced counterparts of ROTP, called the RORT2 and ROTP3, has not yet established. The simulations have demonstrated that the ROTP2 and 3 are powerful algorithms for signal recovery, and they perform remarkably better than the ROTP in signal recovery. Thus it is worthwhile to investigate the convergence of such an advanced development of thresholding methods.

The first theoretical contribution of this paper is to prove some improved convergence results for OT and OTP in terms of $\delta_k$, the $k$th order restricted isometry property. These results are summarized in Theorems 3.5 and 3.7 in this paper. The main contribution here is to establish the first convergence results for the ROTP2 and 3 algorithms. We show such results in a more general setting, i.e., the setting of ROT$\omega$ and ROTP$\omega$ algorithms which are the optimal $k$-thresholding algorithms performing $\omega$ times of data compressions at each iteration (see Section 2 for details). These methods are designed to significantly overcome the intrinsic drawback of the hard thresholding operator. For simplicity, the analysis in this paper is carried out only in noiseless settings. This analysis can be easily extended to noisy signals and inaccurate measurements under the same condition imposed on the sensing matrices.

The paper is organized as follows. The algorithms are described in Section 2. The analysis of OT and OTP algorithms is given in Section 3. The guaranteed performance of the algorithms ROT$\omega$ and ROTP$\omega$ that compress data $\omega$ times at each iteration is carried out in Section 4. Conclusions are given in the last section.

**Notation.** All vectors are column vectors unless otherwise specified. $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, and $\{0,1\}^n$ is the set of $n$-dimensional binary vectors. We use $e$ to denote the vector of ones and $I$ the identity matrix. $\|x\|_2$, $\|x\|_1$ and $\|x\|_\infty$ denote the $\ell_2$, $\ell_1$- and $\ell_\infty$-norms of the vector $x$, respectively. $\text{supp}(x)$ denotes the support of $x$ which is the index set $\{i : x_i \neq 0\}$. Given a set $S \subseteq \{1,2,\ldots,n\}$, $|S|$ denotes the cardinality of $S$ and $\overline{S} = \{1,2,\ldots,n\}\backslash S$ is the complement of $S$ with respect to $\{1,2,\ldots,n\}$. Given $x \in \mathbb{R}^n$, the vector $x_S \in \mathbb{R}^n$ is obtained by retaining the components of $x$ supported on $S$ and setting the elements outside $S$ to be zeros. That is, for every $i = 1,\ldots,n$, $(x_S)_i = x_i$ if $i \in S$; otherwise, $(x_S)_i = 0$. For vectors $x$ and $z$, $x \odot z$ is the Hadamard product (entry-wise product) of $x$ and $z$. The Hadamard product of $q$
vectors \( w^{(1)} \otimes \cdots \otimes w^{(q)} \) is also written as \( \bigotimes_{j=1}^{q} w^{(j)} \). For vectors \( x \) and \( y \), \( x \geq y \) means \( x_i \geq y_i \) for every \( i = 1, \ldots, n \). The vector \( x \) is said to be \( k \)-sparse if \( \|x\|_0 \leq k \).

2 Optimal \( k \)-thresholding algorithms

Recall that \( \mathcal{H}_k(z) \) is the \( k \)-sparse vector obtained by performing the hard thresholding on \( z \in \mathbb{R}^n \) which retains the \( k \) largest magnitudes of \( z \) and zeroes out the remaining entries of the vector. Note that \( A^T(y - Ax) \) is the negative gradient of the function \( \|y - Ax\|_2^2 / 2 \). At the current \( k \)-sparse iterate \( x^p \), the vector \( w^p := x^p + A^T(y - Ax^p) \) is produced by the classic gradient method.

To generate the next iterate \( x^{p+1} \) that satisfies the constraint of the problem (1), a simply way is to apply \( \mathcal{H}_k \) to \( w^p \), leading to the following iterative hard thresholding (IHT) scheme [5, 6]:

\[
x^{p+1} = \mathcal{H}_k \left( x^p + A^T(y - Ax^p) \right),
\]

which provides a basis for many existing methods such as the iterative hard thresholding pursuit (HTP) in [29], compressive sampling matching pursuit (CoSaMP) in [43], subspace pursuits in [18], and the graded hard thresholding in [8, 9]. It was pointed out in [54] that using \( \mathcal{H}_k \) might dramatically increase the objective value of (1) yielding \( \|y - A\mathcal{H}_k(w^p)\|_2 > \|y - Ax^p\|_2 \), unless \( \mathcal{H}_k \) is applied to a compressible vector (which is nearly \( k \)-sparse or can be approximated by a \( k \)-sparse vector). The existing hard-thresholding algorithms directly apply \( \mathcal{H}_k \) to a generally non-compressible vector such as the vector \( w^p \) generated by the gradient method. This may cause numerical oscillation, divergence or significantly slow convergence rate of the algorithms.

To overcome such a drawback of \( \mathcal{H}_k \), at the given data \( u \), we consider the minimization problem

\[
\min_w \left\{ \|y - A(u \otimes w)\|_2^2 : \mathbf{e}^T w = k, \ w \in \{0, 1\}^n \right\},
\]

which selects the best \( k \) terms of \( u \) by minimizing the objective over all possible \( k \) terms of \( u \). This idea and the definition below were introduced in [54].

**Definition 2.1** [54] Let \( w^* \) be the solution to the problem (2). The \( k \)-sparse vector

\[
Z_k^\#(u) := u \otimes w^*
\]

is called the optimal \( k \)-thresholding of \( u \), and the operator \( Z_k^\#(\cdot) \) is called the optimal \( k \)-thresholding operator.

A striking difference between \( Z_k^\# \) and \( \mathcal{H}_k \) lies in that performing \( Z_k^\# \) is directly connected to the reduction of the objective value of the problem (1), while the operator \( \mathcal{H}_k \) does not involve such a mechanism to reduce the objective. By optimality, \( Z_k^\#(u) \) is the best \( k \)-thresholding of \( u \) in the sense that the objective at \( Z_k^\#(u) \) is smaller than or equal to the objective value at any other \( k \) terms of \( u \). In particular,

\[
\|y - AZ_k^\#(u)\|_2 \leq \|y - A\mathcal{H}_k(u)\|_2.
\]

To solve the problem (1), from the current iterate \( x^p \), using the operator \( Z_k^\# \) leads to the following iterative scheme:

\[
x^{p+1} = Z_k^\# \left( x^p + A^T(y - Ax^p) \right),
\]

which minimizes the negative gradient of the function \( \|y - Ax\|_2^2 / 2 \).
which is referred to as the optimal k-thresholding (OT) algorithm. Combining the OT with a pursuit step is called the OTP algorithm. The pursuit step is to solve the problem (4) below, which is a least squares problem over a restricted support set. By the definition of \( Z^\#_k \), the two algorithms can be explicitly described as follows.

**OT and OTP Algorithms:** Input \((A, y, k)\) and an initial point \(x^0 \in \mathbb{R}^n\). Perform the steps below until a stopping criterion is satisfied:

- **S1** At \(x^p\), set \(u^p := x^p + A^T(y - Ax^p)\). Solve the problem

\[
\min_w \{ \|y - A(u^p \otimes w)\|_2^2 : e^T w = k, w \in \{0,1\}^n \}.
\] (3)

Let \(w^*\) be the solution to this problem.

- **S2** Generate the next point \(x^{p+1}\) as follows:
  
  (a) (For OT) \(x^{p+1} = u^p \otimes w^*\).
  
  (b) (For OTP) Set \(S^{p+1} := \text{supp}(u^p \otimes w^*)\) and let \(x^{p+1}\) be the solution to

\[
\min_x \{ \|y - Ax\|_2^2 : \text{supp}(x) \subseteq S^{p+1} \}.
\] (4)

These two algorithms share the same step S1. The only difference lies in the second step. In OT, the optimal k-thresholding of \(u^p\) is directly set to be the next iterate \(x^{p+1}\), while the OTP use the pursuit step (4) to chase a point that might be better than \(u^p \otimes w^*\). The stopping criterion can be a prescribed number of iterations or other criteria such as \(\|y - Ax^{p+1}\|_2 - \|y - Ax^p\|_2 / \|y - Ax^p\|_2 \leq \varepsilon\), where \(\varepsilon\) is a given tolerance. The OT and OTP provide a basis from which the practical and efficient algorithms can be developed. Note that the binary optimization problem (3) is usually NP-hard [16, 10]. It is natural to consider the convex relaxation of this binary optimization problem, leading to the following relaxed optional k-thresholding (ROT) and the relaxed optimal k-thresholding pursuit (ROTP) methods.

**ROT and ROTP Algorithms:** Input \((A, y, k)\) and an initial point \(x^0\). Perform the steps below until a stopping criterion is satisfied:

- **S1** At \(x^p\), set \(w^p := x^p + A^T(y - Ax^p)\). Solve the convex optimization problem

\[
\min_w \{ \|y - A(u^p \otimes w)\|_2^2 : e^T w = k, 0 \leq w \leq e \}.
\] (5)

Let \(w^p\) be the solution to this problem.

- **S2** Generate \(x^{p+1}\) as follows:
  
  (a) (for ROT) \(x^{p+1} = \mathcal{H}_k(w^p \otimes w^p)\).
  
  (b) (for ROTP) Let \(x^\# = \mathcal{H}_k(w^p \otimes w^p)\), and let \(x^{p+1}\) be the solution to

\[
\min_x \{ \|y - Ax\|_2^2 : \text{supp}(x) \subseteq \text{supp}(x^\#) \}.
\]
The problem (5) is a convex quadratic optimization problem that can be solved efficiently by interior-point algorithms. As pointed out in [54], although the solution \( w^p \) of (5) may not be exactly \( k \)-sparse, but it is more compressible than the original data \( u^p \). Because of this reason, the problem (5) is referred to as a ‘data compressing problem’ which produces a compressible vector \( u^p \otimes w^p \), on which the operator \( H_k \) is used to generate the next iterate. The more compressible the vector \( u^p \otimes w^p \) becomes, the more successfully the drawback of \( H_k \) will be overcome. This motivates one to consider the next more general algorithms than ROT and ROTP. Such algorithms adopts data compression more than once at each iteration, these algorithms are still referred to as the relaxed optimal \( k \)-thresholding (pursuit) algorithms (termed \( \text{ROT}\omega \) and \( \text{ROTP} \omega \), respectively).

**\( \text{ROT}\omega \) and \( \text{ROTP} \omega \) Algorithm:** Input \((A, y, k)\). Give an integer number \( \omega \) and an initial point \( x^0 \). Repeat the following steps until a certain stopping criterion is satisfied:

**S1.** At \( x^p \), let \( u^p := x^p + A^T(y - Ax^p) \). Set \( \vartheta \leftarrow u^p \). Perform the following loops to generate the vector \( w^{(j)}, j = 1, \ldots, \omega \):

\[
\min_w \left\{ \|y - A(\vartheta \otimes w)\|_2^2 : e^T w = k, \ 0 \leq w \leq e \right\}
\]

(6)

to obtain a solution \( w^{(j)} \) and set \( \vartheta \leftarrow \vartheta \otimes w^{(j)} \).

**end**

**S2.** Let \( x^\# = H_k(u^p \otimes w^{(1)} \otimes \cdots \otimes w^{(\omega)}) \). Generate \( x^{p+1} \) as follows:

(a) (for \( \text{ROT}\omega \)) \( x^{p+1} = x^\# \).

(b) (for \( \text{ROTP} \omega \)) \( x^{p+1} \) is the solution to the problem

\[
\min_x \left\{ \|y - Ax\|_2^2 : \text{supp}(x) \subseteq \text{supp}(x^\#) \right\}.
\]

In step S1, we perform \( \omega \) times of data compression by solving the problem (6) starting from \( u^p \). Specifically, after \( j \)th compression, the \((j + 1)\)th one is to solve the following the convex quadratic optimization problem:

\[
\min_w \left\{ \|y - A \left[ \left( u^p \otimes w^{(1)} \otimes \cdots \otimes w^{(j)} \right) \otimes w \right] \|_2^2 : e^T w = k, \ 0 \leq w \leq e \right\},
\]

to which the optimal solution is denoted by \( w^{(j+1)} \). The numerical results in [54] have demonstrated that the algorithms with \( \omega = 2, 3 \) are powerful for signal recovery, compared with existing hard thresholding algorithms. In the remainder of this paper, we provide a theoretical analysis for the convergence (guaranteed performance) of the algorithms described in this section.

### 3 Theoretical performance of OT and OTP

In this section, we prove the guaranteed performance of two basic frameworks of optimal thresholding methods in terms of the \( k \)th order restricted isometry property (RIP) of a sensing matrix. Let us first recall the RIP.
**Definition 3.1** [13] Given an \(m \times n\) matrix \(A\) with \(m < n\), the \(q\)th order restricted isometry constant \(\delta_q\) of \(A\) is the smallest number \(\delta \geq 0\) such that

\[
(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2
\]

holds for all \(q\)-sparse vector \(x \in \mathbb{R}^n\).

It is evident that \(\delta_q = \max_{S \subseteq \{1, 2, \ldots, n\}, |S| \leq q} \|A_S^T A_S - I\|_2\). The following properties will be frequently used in our later analysis.

**Lemma 3.2** [13] [43] [29] (i) Let \(u, v\) be \(s\)-sparse and \(t\)-sparse vectors, respectively. If \(\text{supp}(u) \cap \text{supp}(v) = \emptyset\), then

\[
|u^T A^T A v| \leq \delta_{s+t}\|u\|_2\|v\|_2.
\]

(ii) Let \(v \in \mathbb{R}^n\) be a vector and \(S \subset \{1, 2, \ldots, n\}\) be an index set. If \(|S \cup \text{supp}(v)| \leq t\), one has

\[
\|[(I - A^T A)v]_S\|_2 \leq \delta_t\|v\|_2.
\]

We will show that recovering \(k\)-sparse signals via OT and OTP, the RIP bound \(\delta_k < \gamma^*\) or \(\delta_{k+1} \leq \gamma^*\) is very relevant, where \(\gamma^*\) is a certain number smaller than 1. We establish the convergence results for OT and OTP in terms of \(\delta_k\) by distinguishing two cases: even number \(k\) and odd number \(k\).

### 3.1 The RIP bound when \(k\) is an even number

In this section, we assume that \(k\) is an even number and denote by \(\varrho = k/2\). The following property is of independent interest.

**Lemma 3.3** Let \(z\) be a \((2k)\)-sparse vector. If \(k\) is an even number, then

\[
\|Az\|_2^2 \geq (1 - 3\delta_k)\|z\|_2^2.
\]

**Proof.** The \((2k)\)-sparse vector \(z\) can be partitioned into four \(\varrho\)-sparse vectors with disjoint supports: \(z = u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)}\), where every \(u^{(i)}\) is a \(\varrho\)-sparse vector and \(\text{supp}(u^{(i)}) \cap \text{supp}(u^{(j)}) = \emptyset\) for \(i \neq j\). Clearly,

\[
\|z\|_2^2 = \sum_{i=1}^{4} \|u^{(i)}\|_2^2. \quad (7)
\]

Since \(u^{(1)} + u^{(2)}\) and \(u^{(3)} + u^{(4)}\) are \(k\)-sparse, by the definition of the constant \(\delta_k\), we have

\[
\|A(u^{(1)} + u^{(2)})\|_2^2 \geq (1 - \delta_k)\|u^{(1)} + u^{(2)}\|_2^2 = (1 - \delta_k)(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2), \quad (8)
\]

\[
\|A(u^{(3)} + u^{(4)})\|_2^2 \geq (1 - \delta_k)\|u^{(3)} + u^{(4)}\|_2^2 = (1 - \delta_k)(\|u^{(3)}\|_2^2 + \|u^{(4)}\|_2^2). \quad (9)
\]

Note that for every \(i \in \{1, 2\}\) and \(j \in \{3, 4\}\), \(\text{supp}(u^{(i)}) \cap \text{supp}(u^{(j)}) = \emptyset\) for \(i \neq j\) and \(|\text{supp}(u^{(i)}) \cup \text{supp}(u^{(j)})| \leq 2\varrho = k\). It follows from Lemma 3.2 that

\[
|(u^{(i)})^T A^T A u^{(j)}| \leq \delta_k\|u^{(i)}\|_2\|u^{(j)}\|_2 \leq \frac{\delta_k}{2}(\|u^{(i)}\|_2^2 + \|u^{(j)}\|_2^2). \quad (10)
\]
Thus it follows from (7)-(10) that
\[
\|Az\|_2^2 = \|A(u^{(1)} + u^{(2)}) + A(u^{(3)} + u^{(4)})\|_2^2 \\
= \|A(u^{(1)} + u^{(2)})\|_2^2 + \|A(u^{(3)} + u^{(4)})\|_2^2 + 2(u^{(1)} + u^{(2)})^T A^T A(u^{(3)} + u^{(4)}) \\
\geq (1 - \delta_k) \sum_{i=1}^{4} \|u^{(i)}\|_2^2 + 2(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2) + (\|u^{(1)}\|_2^2 + \|u^{(4)}\|_2^2) \\
\quad + (\|u^{(2)}\|_2^2 + \|u^{(3)}\|_2^2) + (\|u^{(2)}\|_2^2 + \|u^{(4)}\|_2^2) \\
\geq (1 - \delta_k) \sum_{i=1}^{4} \|u^{(i)}\|_2^2 - 2\delta_k \sum_{i=1}^{4} \|u^{(i)}\|_2^2 \\
\quad = (1 - 3\delta_k) \|z\|_2^2,
\]
where the first inequality follows from (8) and (9), and the second inequality follows from (10), and the final equality follows from (7). \qed

We now prove the next technical result.

**Lemma 3.4** Let \(x\) and \(z\) be two \(k\)-sparse vectors, and let \(\hat{w} \in \{0,1\}^n\) be a \(k\)-sparse binary vector such that \(\text{supp}(x) \subseteq \text{supp}(\hat{w})\). If \(k\) is an even number, then
\[
\|[(I - A^T A)(x - z)] \otimes \hat{w}\|_2 \leq \sqrt{5}\delta_k \|x - z\|_2.
\]

**Proof.** Let \(x, z, \hat{w}\) satisfy the conditions of the Lemma. We now partition the \(k\)-sparse vector \(\hat{w}\) into two binary vectors \(w'\) and \(w''\), i.e., \(\hat{w} = w' + w''\), where both \(w'\) and \(w''\) are \(\tilde{\eta}\)-sparse binary vectors with disjoint supports. Note that for any vector \(u \in \mathbb{R}^n\), we have
\[
\|u \otimes \hat{w}\|_2^2 = \|u \otimes w'\|_2^2 + \|u \otimes w''\|_2^2.
\]
Let \(v^{(1)} = (x - z) \otimes \hat{w}\) which is a \(k\)-sparse vector. Note that \((x - z) \otimes (e - \hat{w})\) is also a \(k\)-sparse vector and thus can be decomposed into \((x - z) \otimes (e - \hat{w}) = v^{(2)} + v^{(3)}\), where \(v^{(2)}\), \(v^{(3)}\) are \(\tilde{\eta}\)-sparse vectors with disjoint supports. Then \(x - z = v^{(1)} + v^{(2)} + v^{(3)}\). That is, \(x - z\) is decomposed into three vectors with disjoint supports. Since \(|\text{supp}(v^{(1)}) \cup \text{supp}(\hat{w})| \leq k\), by Lemma 3.2 we have
\[
\|[(I - A^T A)v^{(1)}] \otimes \hat{w}\|_2 = \|[(I - A^T A)v^{(1)}]_\text{supp(\hat{w})}\|_2 \leq \delta_k \|v^{(1)}\|_2.
\]
Also, we note that
\[
\|[(I - A^T A)(v^{(2)} + v^{(3)})] \otimes \hat{w}\|_2 \leq 2 \left(\|[(I - A^T A)v^{(2)}] \otimes \hat{w}\|_2^2 + \|[(I - A^T A)v^{(3)}] \otimes \hat{w}\|_2^2\right) \\
\quad = 2(\|[(I - A^T A)v^{(2)}] \otimes w'\|_2^2 + \|[(I - A^T A)v^{(2)}] \otimes w''\|_2^2) \\
\quad + \|[(I - A^T A)v^{(3)}] \otimes w'\|_2^2 + \|[(I - A^T A)v^{(3)}] \otimes w''\|_2^2) \\
\quad \leq 4\delta_k^2 (\|v^{(2)}\|_2^2 + \|v^{(3)}\|_2^2),
\]

8
where the first inequality follows from $\|a + b\|_2^2 \leq 2(\|a\|_2^2 + \|b\|_2^2)$, the equality follows from $[12]$, and the last inequality follows from Lemma $[3.2]$ due to the fact $|\text{supp}(v(i)) \cup \text{supp}(w')| \leq k$ and $|\text{supp}(v(i)) \cup \text{supp}(w'')| \leq k$ for $i \in \{2, 3\}$. Then using (13) and (14), we have

\[
\|[(I - A^T A)(x - z)] \otimes \hat{w}\|_2 \leq \|[I - A^T A]v^{(1)}\|_2 \otimes \hat{w}\|_2 + \|[I - A^T A](v^{(2)} + v^{(3)})\|_2 \otimes \hat{w}\|_2 \\
\leq \delta_k \|v^{(1)}\|_2 + 2\delta_k \sqrt{\|v^{(2)}\|_2^2 + \|v^{(3)}\|_2^2} \\
\leq \sqrt{5}\delta_k \sqrt{\|v^{(1)}\|_2^2 + \|v^{(2)}\|_2^2 + \|v^{(3)}\|_2^2} \\
= \sqrt{5}\delta_k \|z - x\|_2,
\]

where the third inequality follows from the fact $a + 2\sqrt{b} \leq \sqrt{5(a^2 + b)}$ for any numbers $a \geq 0$ and $b \geq 0$. The final equality above follows from $\|z - x\|_2 = \|v^{(1)}\|_2 + \|v^{(2)}\|_2 + \|v^{(3)}\|_2$. □

The convergence of OT and OTP in terms of $\delta_k$ is summarized as follows.

**Theorem 3.5** Let $y := Ax$ be the measurements of the $k$-sparse signal $x$, where $k$ is an even number. Let $\{x^p\}$ be the sequence generated by OT or OTP algorithm. If the RIP constant $\delta_k$ of the matrix $A$ satisfies

\[
5\delta_k^3 + 5\delta_k^2 + 3\delta_k < 1, \tag{15}
\]

then

\[
\|x^{p+1} - x\|_2 \leq \delta_k \sqrt{\frac{5(1 + \delta_k)}{1 - 3\delta_k}} \|x^p - x\|_2 = \rho \|x^p - x\|_2,
\]

where $\rho := \delta_k \sqrt{\frac{5(1 + \delta_k)}{1 - 3\delta_k}} < 1$. The condition (15) is satisfied if $\delta_k \leq 91/400 = 0.2275$. In particular, when $\delta_k \leq 9/40$, the condition (15) is guaranteed.

**Proof.** Since $x$ and $x^{p+1}$ are $k$-sparse vectors, by Lemma $[3.3]$ we immediately have

\[
\|A(x - x^{k+1})\|_2 \geq \sqrt{1 - 3\delta_k}\|x - x^{p+1}\|_2. \tag{16}
\]

Let $\hat{w} \in W^k := \{w : e^T w = k, w \in \{0, 1\}^n\}$ be a $k$-sparse binary vector such that $\text{supp}(x) \subseteq \text{supp}(\hat{w})$. Then $x = x \otimes \hat{w}$, i.e.,

\[
x \otimes (e - \hat{w}) = 0. \tag{17}
\]

Recall that $u^p = x^p + A^T(y - Ax^p)$. Since the vector $(x - u^p) \otimes \hat{w}$ is a $k$-sparse vector, we have

\[
\|A[(x - u^p) \otimes \hat{w}]\|_2 \leq \sqrt{1 + \delta_k}\|(x - u^p) \otimes \hat{w}\|_2. \tag{18}
\]

Note that $y = Ax$. Thus $x - u^p = (I - A^T A)(x - x^p)$. Since $x^p$ is $k$-sparse, applying Lemma $[3.4]$ to the $k$-sparse vector $x$ and $x^p$, one has

\[
\|[x - u^p] \otimes \hat{w}\|_2 = \|[I - A^T A](x - x^p)\|_2 \leq \sqrt{5}\delta_k\|x^p - x\|_2. \tag{19}
\]

By (17), (18) and (19), we have

\[
\|y - A(u^p \otimes \hat{w})\|_2 = \|A[x - u^p \otimes \hat{w}]\|_2 \\
= \|A[x \otimes (e - \hat{w}) + (x - u^p) \otimes \hat{w}]\|_2 \\
= \|A[(x - u^p) \otimes \hat{w}]\|_2 \\
\leq \sqrt{1 + \delta_k}\|(x - u^p) \otimes \hat{w}\|_2 \\
\leq \sqrt{1 + \delta_k} \left(\sqrt{5}\delta_k\right)\|x - x^p\|_2. \tag{20}
\]

9
Recall that \( w^* \) is the minimizer of the problem (3) with \( u^p \). For OT, \( x^{p+1} = u^p \otimes w^* \), and thus

\[
\|y - Ax^{p+1}\|_2 = \|y - A(u^p \otimes w^*)\|_2.
\]

For OTP, the iterate \( x^{p+1} \) is obtained by performing the following pursuit step:

\[
\min\{\|y - Az\|_2^2 : \text{supp}(z) \subseteq \text{supp}(u^p \otimes w^*)\},
\]

which implies that \( \|y - Ax^{p+1}\|_2 \leq \|y - A(u^p \otimes w^*)\|_2 \). Therefore, by optimality, the sequence \( \{x^p\} \) generated by OT or OTP satisfies

\[
\|y - Ax^{p+1}\|_2 \leq \|y - A(u^p \otimes w^*)\|_2 \leq \|y - A(u^p \otimes w)\|_2 \quad \text{for any } w \in W^k.
\]

In particular, since \( \hat{w} \in W^k \), we have

\[
\|y - Ax^{p+1}\|_2 \leq \|y - A(u^p \otimes \hat{w})\|_2
\]

Merging (16), (20) and (22) leads to

\[
\|x - x^{p+1}\|_2 \leq \frac{1}{\sqrt{1 - 3\delta_k}} \|y - A(u^p \otimes \hat{w})\|_2 \leq \delta_k \sqrt{\frac{5(1 + \delta_k)}{1 - 3\delta_k}} \|x - x^p\|_2.
\]

Clearly, \( \rho := \delta_k \sqrt{\frac{5(1 + \delta_k)}{1 - 3\delta_k}} < 1 \) is equivalent to that \( 5\delta_k^3 + 5\delta_k^2 + 3\delta_k < 1 \). To ensure this inequality, it is sufficient to require that \( \delta_k < \gamma^* \), where \( \gamma^* \) is the real root of the univariate equation \( 5\gamma^3 + 5\gamma^2 + 3\gamma = 1 \). It is easy to verify that \( \gamma^* > 91/400 = 0.2275 \). Thus if \( \delta_k \leq 91/400 = 0.2275 \), then \( \delta_k < \gamma^* \) is guaranteed. In particular, if \( \delta_k \leq 9/40 \) (which is the middle point between 1/4 and 1/5), then \( \delta_k < \gamma^* \) and thus the convergence of the sequence \( \{x^p\} \) is guaranteed. \( \square \)

### 3.2 The RIP bound when \( k \) is an odd number

We now consider the case when the sparsity level \( k \) is an odd number, i.e., \( k = 2q + 1 \). The following lemma is similar to Lemma 3.3.

**Lemma 3.6** Let \( z \) be any given \((2k)\)-sparse vector, where \( k = 2q + 1 \) is an odd integer number. Then

\[
\|Az\|_2^2 \geq (1 - \delta_{k+1} - 2\delta_k)\|z\|_2^2.
\]

**Proof.** When \( k = 2q + 1 \), the \((2k)\)-sparse vector \( z \) can be partitioned into the following four sparse vectors with disjoint supports: \( z = u^{(1)} + u^{(2)} + u^{(3)} + u^{(4)} \), where \( u^{(1)} \) and \( u^{(2)} \) are \( q \)-sparse, \( u^{(3)} \) and \( u^{(4)} \) are \((q + 1)\)-sparse and \( \text{supp}(u^{(i)}) \cap \text{supp}(u^{(j)}) = \emptyset \) for \( i \neq j \). Clearly, \( \|z\|_2^2 = \sum_{i=1}^{4} \|u^{(i)}\|_2^2 \). The two inequalities below follows immediately from Definition 3.1

\[
\|A(u^{(1)} + u^{(2)})\|_2^2 \geq (1 - \delta_{2q})\|u^{(1)} + u^{(2)}\|_2^2 = (1 - \delta_{k-1})(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2),
\]

\[
\|A(u^{(3)} + u^{(4)})\|_2^2 \geq (1 - \delta_{2(q+1)})\|u^{(3)} + u^{(4)}\|_2^2 = (1 - \delta_{k+1})(\|u^{(3)}\|_2^2 + \|u^{(4)}\|_2^2).
\]

For every \( i \in \{1, 2\} \) and \( j \in \{3, 4\} \), one has \( |\text{supp}(u^{(i)}) \cup \text{supp}(u^{(j)})| \leq q + (q + 1) = k \), by Lemma 3.2, we have

\[
\|(u^{(i)})^T A^T u^{(j)}\|_2 \leq \delta_k \|u^{(i)}\|_2 \|u^{(j)}\|_2 \leq (\delta_k/2)(\|u^{(i)}\|_2^2 + \|u^{(j)}\|_2^2).
\]

\[\text{(25)}\]
By (23)-(25) and a similar proof to (11), we have
\[
\|Az\|_2^2 = \|A(u^{(1)} + u^{(2)})\|_2^2 + \|A(u^{(3)} + u^{(4)})\|_2^2 + 2(u^{(1)} + u^{(2)})^T A^T A(u^{(3)} + u^{(4)})
\]
\[
\geq (1 - \delta_{k-1})(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2) + (1 - \delta_{k+1})(\|u^{(3)}\|_2^2 + \|u^{(4)}\|_2^2) - \delta_k(\|u^{(1)}\|_2^2 + \|u^{(3)}\|_2^2)
\]
\[
+ (\|u^{(1)}\|_2^2 + \|u^{(4)}\|_2^2) + (\|u^{(2)}\|_2^2 + \|u^{(3)}\|_2^2) + (\|u^{(2)}\|_2^2 + \|u^{(4)}\|_2^2)
\]
\[
= (1 - \delta_{k+1}) \frac{4}{\delta_1^2} \|u^{(1)}\|_2^2 + (\delta_{k+1} - \delta_{k-1})(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2) - 2\delta_k \frac{4}{\delta_1^2} \|u^{(1)}\|_2^2
\]
\[
\geq (1 - \delta_{k+1} - 2\delta_k) \|z\|_2^2,
\]
where the last inequality follows from the fact \(\delta_{k-1} \leq \delta_{k+1}\). □

We now prove the convergence of OT and OTP algorithms when \(k\) is odd number.

**Theorem 3.7** Let \(y := Ax\) be the measurements of the \(k\)-sparse signal \(x\), where \(k\) is an odd number. If the RIP constants \(\delta_k\) and \(\delta_{k+1}\) of \(A\) satisfy that
\[
5\delta_k^2 \delta_{k+1}^2 + 5\delta_k^2 + 2\delta_k + \delta_{k+1} < 1,
\]
then the sequence \(\{x^p\}\), generated by the algorithm OT or OTP, converges to the signal \(x\) with error
\[
\|x^{p+1} - x\|_2 \leq \rho\|x^p - x\|_2,
\]
where
\[
\rho := \delta_{k+1} \sqrt{\frac{5(1 + \delta_k)}{1 - \delta_{k+1} - 2\delta_k}} < 1,
\]
which is ensured under (26). The condition (26) is guaranteed if \(\delta_{k+1} \leq 91/400 = 0.2275\). In particular, it is guaranteed if \(\delta_{k+1} \leq 9/40\).

**Proof.** Since \(x - x^{k+1}\) is \((2k)\)-sparse, by setting \(z = x - x^{p+1}\) in Lemma 3.6, we immediately obtain the following relation:
\[
\|A(x - x^{k+1})\|_2 \geq \sqrt{1 - \delta_{k+1} - 2\delta_k}\|x - x^{p+1}\|_2.
\]
Similar to the proof of Theorem 3.5, we still denote by \(\hat{w} \in \mathcal{W}^k := \{w : e^T w = k, w \in \{0, 1\}^n\}\) the \(k\)-sparse binary vector such that \(\text{supp}(x) \subseteq \text{supp}(\hat{w})\) and thus \(x \otimes (e - \hat{w}) = 0\). Since \((x - u^p) \otimes \hat{w}\) is \(k\)-sparse, one has
\[
\|A[(x - u^p) \otimes \hat{w}]\|_2 \leq \sqrt{1 + \delta_k}\|(x - u^p) \otimes \hat{w}\|_2.
\]
The vector \(\hat{w}\) can be partitioned as \(\hat{w} = w' + w''\), where \(w'\) is a \(\rho\)-sparse binary vector and \(w''\) is a \((\rho + 1)\)-sparse binary vector and the supports of \(w'\) and \(w''\) are disjoint. Partition the \((2k)\)-sparse vector \(x - x^p\) into three vectors \(\eta^{(1)}, \eta^{(2)}\) and \(\eta^{(3)}\) with disjoint supports such that
\[
\eta^{(1)} = (x - x^p) \otimes \hat{w}, \quad \eta^{(2)} + \eta^{(3)} = (x - x^p) \otimes (e - \hat{w}),
\]
where \( \eta^{(2)} \) and \( \eta^{(3)} \) are \( \rho \)-sparse and \((\rho + 1)\)-sparse vectors, respectively. Note that

\[
|\text{supp}(\eta^{(1)}) \cup \text{supp}(w')| \leq k, \quad |\text{supp}(\eta^{(2)}) \cup \text{supp}(w')| \leq 2\rho = k - 1, \\
|\text{supp}(\eta^{(3)}) \cup \text{supp}(w')| \leq 2\rho + 1 = k, \quad |\text{supp}(\eta^{(1)}) \cup \text{supp}(w'')| \leq k, \\
|\text{supp}(\eta^{(2)}) \cup \text{supp}(w'')| \leq 2\rho + 1 = k, \quad |\text{supp}(\eta^{(3)}) \cup \text{supp}(w'')| \leq 2\rho + 2 = k + 1.
\]

It follows from Lemma 3.2 that

\[
||[(I - A^T A)(\eta^{(2)} + \eta^{(3)})] \otimes w'||_2^2 \leq 2(||[(I - A^T A)\eta^{(2)}] \otimes w'||_2^2 + ||[(I - A^T A)\eta^{(3)}] \otimes w'||_2^2) \\
\leq 2(\delta_{k-1}^2 \| \eta^{(2)} \|_2^2 + \delta_k^2 \| \eta^{(3)} \|_2^2). \quad (29)
\]

Similarly,

\[
||[(I - A^T A)(\eta^{(2)} + \eta^{(3)})] \otimes w''||_2^2 \leq 2(\delta_{k}^2 \| \eta^{(2)} \|_2^2 + \delta_{k+1}^2 \| \eta^{(3)} \|_2^2). \quad (30)
\]

Note that \( x - x^p = \eta^{(1)} + \eta^{(2)} + \eta^{(3)} \). Then by (29) and (30), we have

\[
\|(x - u^p) \otimes \hat{w}\|_2 \\
= ||[(I - A^T A)(x - x^p)] \otimes \hat{w}\|_2 \\
= ||[(I - A^T A)(\eta^{(1)} + \eta^{(2)} + \eta^{(3)})] \otimes \hat{w}\|_2 \\
\leq ||[(I - A^T A)\eta^{(1)}] \otimes \hat{w}\|_2 + ||[(I - A^T A)(\eta^{(2)} + \eta^{(3)})] \otimes \hat{w}\|_2 \\
\leq \delta_k || \eta^{(1)} ||_2 + \left( ||[(I - A^T A)(\eta^{(2)} + \eta^{(3)})] \otimes w'||_2^2 + ||[(I - A^T A)(\eta^{(2)} + \eta^{(3)})] \otimes w''||_2^2 \right)^{1/2} \\
\leq \delta_k || \eta^{(1)} ||_2 + \left( 2(\delta_{k-1}^2 || \eta^{(2)} ||_2^2 + \delta_{k}^2 || \eta^{(3)} ||_2^2) + 2(\delta_{k}^2 || \eta^{(2)} ||_2^2 + \delta_{k+1}^2 || \eta^{(3)} ||_2^2) \right)^{1/2} \\
= \delta_k || \eta^{(1)} ||_2 + \left( 2(\delta_{k-1}^2 + \delta_{k}^2) || \eta^{(2)} ||_2^2 + 2(\delta_{k}^2 + \delta_{k+1}^2) || \eta^{(3)} ||_2^2 \right)^{1/2} \\
\leq \delta_{k+1} \left( || \eta^{(1)} ||_2 + 2 \sqrt{|| \eta^{(2)} ||_2^2 + || \eta^{(3)} ||_2^2} \right) \quad (\text{since } \delta_{k-1} \leq \delta_k \leq \delta_{k+1}) \\
\leq \delta_{k+1} \sqrt{5(|| \eta^{(1)} ||_2^2 + || \eta^{(2)} ||_2^2 + || \eta^{(3)} ||_2^2)} \\
= \sqrt{5} \delta_{k+1} || x - x^p ||_2, \quad (31)
\]

where the last inequality follows from \( a + 2\sqrt{b} \leq \sqrt{5(a^2 + b)} \) for any \( a, b \geq 0 \). Combining (28) and (31) yields

\[
\| y - A(u^p \otimes \hat{w}) \|_2 \leq \sqrt{1 + \delta_k} \| (x - u^p) \otimes \hat{w} \|_2 \leq \delta_{k+1} \sqrt{5(1 + \delta_k)} \| x - x^p \|_2. \quad (32)
\]

Note that \( w^* \) is a minimizer of the problem (3) with \( u^p = x^p + A^T(y - Ax^p) \). As we have shown in the proof of Theorem 3.5, the sequences \( \{x^p\} \) and \( \{u^p\} \) generated by OT and OTP algorithms satisfy the inequality (21), which implies that \( \| y - Ax^{p+1} \|_2 \leq \| y - A(u^p \otimes \hat{w}) \|_2 \). Combining this relation with (27) and (32) yields

\[
\| x - x^{p+1} \|_2 \leq \rho \| x - x^p \|_2,
\]

where

\[
\rho := \delta_{k+1} \sqrt{\frac{5(1 + \delta_k)}{1 - \delta_{k+1} - 2\delta_k}}.
\]
The constant $\rho < 1$ is ensured under the condition (26). Since $\delta_k \leq \delta_{k+1}$, we see that

$$5\delta_k \delta_{k+1}^2 + 5\delta_{k+1}^2 + 2\delta_k + \delta_{k+1} \leq 5\delta_{k+1}^3 + 5\delta_{k+1}^2 + 3\delta_{k+1}.$$  

Thus the condition (26) is guaranteed if $5\delta_{k+1}^3 + 5\delta_{k+1}^2 + 3\delta_{k+1} < 1$. Note that the real root $\gamma^*$ of the univariate equation $5\gamma^3 + 5\gamma^2 + 3\gamma = 1$ is approximately equal to $91/400$ and $\gamma^* > 91/400$. Therefore the condition $\delta_{k+1} \leq 91/400$ (in particular, $\delta_{k+1} \leq 9/40$) implies that $\delta_{k+1} < \gamma^*$, and thus $5\delta_{k+1}^3 + 5\delta_{k+1}^2 + 3\delta_{k+1} < 1$. □

**Remark.** To our knowledge, the best known RIP bound for the convergence of IHT and HTP is $\delta_{3k} \leq 1/\sqrt{3} \approx 0.5773$ (see [29, 30]). By adopting suitable stepsizes, the IHT and HTP may converge under the condition $\delta_{2k} < 1/3$ (see [2, 30]). It was shown in [54] that the RIP bound for the convergence of the framework of OT and OTP is $\delta_{2k} < 0.5349$. In this section, we have shown that OT and OTP are convergent under a nearly optimal RIP bound in terms of $\delta_k$ or $\delta_{k+1}$.

4 Guaranteed performance of ROTω and ROTPω

The convergence of the ROT and ROTP algorithms has been studied initially in [54]. The empirical results have demonstrated that the ROTP2 and ROTP3 outperform the $\ell_1$-minimization method in numerous situations. This indicates that compressing the data $u^p$ more than once before applying the operator $\mathcal{H}_k$ may significantly improve the efficiency of the hard-thresholding-based algorithms. Thus it is worth performing a theoretical analysis for the general class of ROTω and ROTPω algorithms (with $\omega \geq 2$) in order to further understand the behavior of the algorithms. At the current stage of development, however, the convergence of ROTω and ROTPω with $\omega \geq 2$ has not yet established. The purpose of this section is to establish the first convergence result for this class of algorithms under the RIP assumption. In particular, the convergence results of ROTP2 and 3 are obtained for the first time in this section. Our proof is non-trivial. We first show several useful technical results which are also of independent interest. Let us start with a property of the operator $\mathcal{H}_k$.

**Lemma 4.1** For any vector $z \in \mathbb{R}^n$ and any $k$-sparse vector $x \in \mathbb{R}^n$, one has

$$\|x - \mathcal{H}_k(z)\|_2 \leq \|(z - x)_{S \cap S^*}\|_2 + \|(z - x)_{S^* \setminus S}\|_2,$$

where $S = \text{supp}(x)$ and $S^* = \text{supp}(\mathcal{H}_k(z))$.

**Proof.** For any vector $z$, we note that $\mathcal{H}_k(z) = \arg \min_d \{\|z - d\|_2 : \|d\|_0 \leq k\}$, which implies that $\|z - \mathcal{H}_k(z)\|_2^2 \leq \|z - d\|_2^2$ for any $k$-sparse vector $d$. In particular, substituting the $k$-sparse vector $d = x + (z - x)_S$, where $S = \text{supp}(x)$, into the inequality above leads to

$$\|z - \mathcal{H}_k(z)\|_2^2 \leq \|z - x - (z - x)_S\|_2^2 = \|(z - x)_{S^*}\|_2^2 = \|z - x\|_2^2 - \|(z - x)_S\|_2^2 - 2(z - \mathcal{H}_k(z))^T (x - z).$$

Denote by $S^* = \text{supp}(\mathcal{H}_k(z))$. The relation above together with

$$\|z - \mathcal{H}_k(z)\|_2^2 = \|z - x\|_2^2 + \|x - \mathcal{H}_k(z)\|_2^2 - 2(x - \mathcal{H}_k(z))^T (x - z).$$
implies that
\[ \|x - \mathcal{H}_k(z)\|^2 \leq -\|(z - x)S\|^2 + 2(x - \mathcal{H}_k(z))^T(x - z) \]
\[ = -\|(z - x)S\|^2 + 2[(x - \mathcal{H}_k(z))S^*US]S^T(x - z)S^*US \]
\[ \leq -\|(z - x)S\|^2 + 2\|x - \mathcal{H}_k(z)\|_2\|(z - x)S^*US\|_2. \]
This further implies that \( \|x - \mathcal{H}_k(z)\|_2 \) is smaller than or equal to the largest real root of the quadratic equation \( \phi(\alpha) = \alpha^2 - 2\alpha\|(z - x)S^*US\|_2 + \|(z - x)S\|^2 = 0 \), to which the largest real root is given by
\[ \alpha^* = \frac{2\|(z - x)S^*US\|_2 + \sqrt{4\|(z - x)S^*US\|^2 - 4\|(z - x)S\|^2}}{2} = \|(z - x)S^*US\|_2 + \|(z - x)S^*\|_2. \]
The proof is complete. \( \square \)

The next lemma describes a property of the polytope \( \mathcal{P} = \{w : e^T w = k, 0 \leq w \leq e\} \).

**Lemma 4.2** Let \( \Lambda \subseteq \{1, \ldots, n\} \) be any given index set, and let \( w \) be any given vector in the polytope \( \mathcal{P} = \{w \in \mathbb{R}^n : e^T w = k, 0 \leq w \leq e\} \). Decompose the vector \( w_\Lambda \) as the sum of \( \tau \)-sparse vectors:
\[ w_\Lambda = w_{\Lambda_1} + \cdots + w_{\Lambda_q}, \tag{33} \]
where \( \Lambda_1 \cup \cdots \cup \Lambda_q = \Lambda \) and \( \Lambda_1 \) is the index set for the largest \( \tau \) elements in \( \{w_i : i \in \Lambda]\), and \( \Lambda_2 \) is the index set for the second largest \( \tau \) elements in \( \{w_i : i \in \Lambda]\), and so on. \( q \) is a nonnegative integer number such that \( |\Lambda| = (q - 1)\tau + \beta \) where \( 0 \leq \beta < \tau \). Then
\[ \|w_{\Lambda_1}\|_\infty + \cdots + \|w_{\Lambda_{q-1}}\|_\infty + \|w_{\Lambda_q}\|_\infty < (\tau + k)/\tau. \]

**Proof.** Let \( \Lambda \subseteq \{1, \ldots, n\} \) and \( w \in \mathcal{P} \) be given. Consider the vector \( w_\Lambda \) which is decomposed as \( [33] \). For every \( i = 1, \ldots, q - 1 \), sort the components of \( w \) supported on \( \Lambda_i \), i.e., \( \{w_j : j \in \Lambda_i\} \), into descending order, and denote such ordered components by \( \sigma_1^{(i)} \geq \sigma_2^{(i)} \geq \cdots \geq \sigma_\tau^{(i)} \), and denote the ordered components of \( w \) supported on \( \Lambda_q \) by \( \sigma_1^{(q)} \geq \sigma_2^{(q)} \geq \cdots \geq \sigma_\tau^{(q)} \). Then the components of the vector \( w \) supported on \( \Lambda \) are sorted into descending order as follows:
\[ \sigma_1^{(1)} \geq \sigma_2^{(1)} \geq \cdots \geq \sigma_\tau^{(1)} \geq \sigma_1^{(2)} \geq \sigma_2^{(2)} \geq \cdots \geq \sigma_\tau^{(2)} \geq \cdots \geq \sigma_1^{(q)} \geq \sigma_2^{(q)} \geq \cdots \geq \sigma_\tau^{(q)}. \tag{34} \]
Clearly, for every \( i = 1, \ldots, q \), \( \sigma_1^{(i)} \) is the largest entries of \( w_{\Lambda_i} \), i.e., \( \sigma_1^{(i)} = \|w_{\Lambda_i}\|_\infty \). For every \( i = 1, \ldots, q - 1 \), \( \sigma_\tau^{(i)} \) is the smallest entry of \( w \) on the support \( \Lambda_i \), and \( \sigma_\tau^{(q)} \) is the smallest component of \( w \) supported on \( \Lambda_q \). Therefore,
\[ \Phi(w, \Lambda) := \|w_{\Lambda_1}\|_\infty + \cdots + \|w_{\Lambda_{q-1}}\|_\infty + \|w_{\Lambda_q}\|_\infty = \sum_{i=1}^{q} \sigma_1^{(i)}. \tag{35} \]
It is sufficient to show that \( \Phi(w, \Lambda) < (\tau + k)/\tau. \) From \( [34] \), for each \( i \), the largest entry of \( w \) on the support \( \Lambda_{i+1} \) is smaller than or equal to the smallest entry of \( w \) on the support \( \Lambda_i \), i.e., \( \sigma_\tau^{(i)} \geq \sigma_1^{(i+1)} \) for every \( i \in \{1, \ldots, q - 1\} \). So we immediately see that
\[ \Phi(w, \Lambda) = \sum_{i=1}^{q} \sigma_1^{(i)} \leq \sigma_1^{(1)} + \sigma_\tau^{(1)} + \sigma_\tau^{(2)} + \cdots + \sigma_\tau^{(q-1)} \leq 1 + \sum_{i=1}^{q-1} \sigma_\tau^{(i)}, \tag{36} \]
where the last inequality follows from $\sigma_1^{(1)} \leq 1$ (since $0 \leq w \leq e$). Note that for every $i = 1, \ldots, q - 1$, $\sum_{i=1}^{q-1} \sigma_{\tau}^{(i)}$ is the sum of the smallest entries of the vector $w$ supported on $\Lambda_i$. We see from (31) that

$$\sum_{i=1}^{q-1} \sigma_{\tau}^{(i)} \leq \sum_{i=1}^{q-1} \sigma_{\tau-1}^{(i)} \leq \cdots \leq \sum_{i=1}^{q-1} \sigma_2^{(i)},$$

which together with (30) implies that $\Phi(w, \Lambda) \leq 1 + \sum_{i=1}^{q-1} \sigma_j^{(i)}$ for $j = 2, \ldots, \tau$. Adding up these $\tau - 1$ inequalities and the equality (35) altogether yields

$$\tau \Phi(w, \Lambda) \leq \tau - 1 + \sum_{i=1}^{\omega-1} \sigma_1^{(i)} + \sum_{i=1}^{\omega-1} \sigma_2^{(i)} + \cdots + \sum_{i=1}^{\omega-1} \sigma_{\tau}^{(i)} \leq \tau - 1 + \sum_{j \in \Lambda} w_j \leq \tau - 1 + \|w\|_1 = \tau - 1 + k,$$

where the final equality follows from $\|w\|_1 = e^T w = k$. Therefore $\Phi(w, \Lambda) \leq (\tau + k - 1)/\tau < (\tau + k)/\tau$, as desired. \qed

We now show a property of the vectors $w^{(j)}$ generated at S1 of ROT$\omega$ and ROTP$\omega$.

**Lemma 4.3** Let $x \in \mathbb{R}^n$ be the $k$-sparse vector satisfying $y = Ax$, and let $\hat{w} \in W^{(k)} = \{w : e^T w = k, w \in \{0,1\}^n\}$ be a binary vector such that $\text{supp}(x) \subseteq \text{supp}(\hat{w})$. At the iterate $x^p$ with $u^p = x^p + A^T(y - Ax^p)$, the vectors $w^{(1)}, \ldots, w^{(\omega)}$ are generated by the ROT$\omega$ or ROTP$\omega$. Then

$$\|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 \leq \|y - A(u^p \otimes \hat{w})\|_2 + \sum_{i=1}^{\omega-1} \|A[u^p - x] \otimes (\bigotimes_{j=1}^{i} w^{(j)}) \otimes (e - \hat{w})]\|_2. \tag{37}$$

**Proof.** Note that $\hat{w} \in W^{(k)}$ satisfies $\text{supp}(x) \subseteq \text{supp}(\hat{w})$. The first inequality below follows from the optimality of $w^{(\omega)}$:

$$\|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]\|_2 \leq \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)}) \otimes \hat{w}]\|_2$$

$$= \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)}) \otimes (e - (e - \hat{w}))]\|_2$$

$$= \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)})] + A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)}) \otimes (e - \hat{w})]\|_2$$

$$\leq \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)})]\|_2 + \|A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)}) \otimes (e - \hat{w})]\|_2$$

$$= \|y - A[u^p \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)})]\|_2 + \|A[(u^p - x) \otimes (\bigotimes_{j=1}^{\omega-1} w^{(j)}) \otimes (e - \hat{w})]\|_2,$$

where the final equality follows from $x \otimes (e - \hat{w}) = 0$ due to $\text{supp}(x) \subseteq \text{supp}(\hat{w})$. Similarly, by the optimality of $w^{(\omega-1)}, \ldots, w^{(2)}$, we obtain the following inequalities for every $\ell = \omega - 1, \omega - \ldots, 2$.
2, \ldots, 2:

\| y - A[u^p \otimes (\bigotimes_{j=1}^{\ell} w^{(j)})] \|_2 \leq \| y - A[u^p \otimes (\bigotimes_{j=1}^{\ell-1} w^{(j)})] \|_2 + \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{\ell-1} w^{(j)}) \otimes (e - \hat{w})] \|_2.

Merging the above inequalities altogether leads to the following relation:

\| y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})] \|_2 \leq \| y - A(u^p \otimes w^{(1)}) \|_2 + \sum_{i=1}^{\omega - 1} \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{i} w^{(j)}) \otimes (e - \hat{w})] \|_2.

By the optimality of \( w^{(1)} \), we have

\| y - A(u^p \otimes w^{(1)}) \|_2 \leq \| y - A(u^p \otimes \hat{w}) \|_2.

Combining the last two inequalities above yields the desired relation (37). □

We now bound the right-hand side of (37).

\textbf{Lemma 4.4} Under the conditions of Lemma 4.3, that is, the vectors \( x, \hat{w}, x^p, u^p \) and \( w^{(j)} \) \( (j = 1, \ldots, \omega) \) are the same as in Lemma 4.3. Then for every \( i = 1, \ldots, \omega - 1, \)

\( \Theta^{(i)} := \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{i} w^{(j)}) \otimes (e - \hat{w})] \|_2 \leq 2\delta_{3k} \sqrt{1 + \delta_k \|x - x^p\|_2} \), \hspace{1cm} (38)

and thus

\| y - A[u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})] \|_2 \leq (\delta_{2k} + 2(\omega - 1)\delta_{3k}) \sqrt{1 + \delta_k \|x - x^p\|_2}. \hspace{1cm} (39)

\textbf{Proof.} The first term in (37) is easy to bound in term of the RIP constants. Note that \( y = Ax \) and \( x \otimes \hat{w} = x \), we have

\[ \| y - A(u^p \otimes \hat{w}) \|_2 = \| A[(x - u^p) \otimes \hat{w}] \|_2 = \| A[(I - A^T A)(x - x^p)] \otimes \hat{w}] \|_2 \leq \sqrt{1 + \delta_k \|[I - A^T A](x - x^p)] \otimes \hat{w}] \|_2 \leq \delta_{2k} \sqrt{1 + \delta_k \|x - x^p\|_2}. \] \hspace{1cm} (40)

The first inequality above follows from that the vector \( [(I - A^T A)(x - x^p)] \otimes \hat{w} \) is \( k \)-sparse, and the last inequality follows from Lemma 3.3 since \| \text{supp}(x - x^p) \cup \text{supp}(\hat{w}) \| \leq 2k. \) To show (39), it is sufficient to show the bound (38) for \( \Theta^{(i)}, i = 1, \ldots, \omega - 1. \) Note that \( \text{supp}(e - \hat{w}) = \text{supp}(\hat{w}), \) the complement set of \( \text{supp}(\hat{w}) \) with respect to \( \{1, 2, \ldots, n\} \). \( \Theta^{(i)} \) can be written as

\[ \Theta^{(i)} = \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{i} w^{(j)})] \|_{\text{supp}(\hat{w})} = \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{i})] \|_{\text{supp}(\hat{w})} \|_2. \]

Let \( (w^{(1)})_{\text{supp}(\hat{w})} \) be decomposed into \( k \)-sparse vectors as follows:

\[ (w^{(1)})_{\text{supp}(\hat{w})} = (w^{(1)})_{T_1} + \cdots + (w^{(1)})_{T_{q-1}} + (w^{(1)})_{T_q}, \]
where $T_1$ is the index set for the largest $k$ elements in the set $\{ (w^{(1)})_i : i \in \text{supp}(\overrightarrow{w}) \}$, and $T_2$ is the index set for the second largest $k$ elements in this set, and so on. These index sets are mutually disjoint and the cardinality $|T_\ell| = k$ for all $\ell = 1, \ldots, q - 1$ and $|T_q| = \kappa' < k$, where $q$ and $\kappa'$ are integer numbers. As a result, we have

$$\text{supp}(\overrightarrow{w}) = T_1 \cup T_2 \cup \cdots \cup T_q, \quad |\text{supp}(\overrightarrow{w})| = (q-1)k + \kappa'.$$

Note that $w^{(1)} \in \{ w : e^T w = k, \ 0 \leq w \leq e \}$. Applying Lemma 4.2 to the vectors $w = w^{(1)}, \tau = k$ and $\Lambda = \text{supp}(\overrightarrow{w})$, we immediately have that

$$\sum_{i=1}^q \|(w^{(1)})_i\|_\infty < 2. \quad (41)$$

Define the vector $v^{(\ell)} := (\{(u^p - x) \otimes (\bigotimes_{j=1}^i w^{(j)})\})_{T_\ell}$, then

$$\{(u^p - x) \otimes (\bigotimes_{j=1}^i w^{(j)})\}_{\text{supp}(\overrightarrow{w})} = v^{(1)} + v^{(2)} + \cdots + v^{(q)}.$$

This means the vector $\{(u^p - x) \otimes (\bigotimes_{j=1}^i w^{(j)})\}_{\text{supp}(\overrightarrow{w})}$ is decomposed into $k$-sparse vectors $v^{(\ell)} \in \mathbb{R}^n, \ell = 1, \ldots, q$. Therefore,

$$\Theta^{(i)} = \left\| A \sum_{\ell=1}^q v^{(\ell)} \right\|_2 \leq \sum_{\ell=1}^q \|Av^{(\ell)}\|_2 \leq \sqrt{1 + \delta_k} \sum_{\ell=1}^q \|v^{(\ell)}\|_2. \quad (42)$$

We now estimate the term $\sum_{\ell=1}^q \|v^{(\ell)}\|_2$. By the structure of the algorithm, all the vectors $w^{(j)} \in \{ w : e^T w = k, \ 0 \leq w \leq e \}$ for $j = 1, \ldots, \omega$. Thus,

$$\|v^{(\ell)}\|_2 = \|(u^p - x) \otimes (\bigotimes_{j=1}^i w^{(j)})\|_{\text{supp}(\overrightarrow{w})} = \|(I - A^T A)(x - x^p) \otimes (\bigotimes_{j=1}^i w^{(j)})\|_{T_\ell} \leq \|(\bigotimes_{j=1}^i w^{(j)})_{T_\ell}\|_\infty \|(I - A^T A)(x - x^p)\|_{T_\ell}$$

$$\leq \|(w^{(1)})_{T_\ell}\|_\infty (\delta_{3k}) \|x - x^p\|_2, \quad (43)$$

where the inequalities above follows from Lemma 3.2 with $|T_\ell \cup \text{supp}(x - x^p)| \leq 3k$, and from the fact $0 \leq w^{(j)} \leq e$ for $j = 1, \ldots, i$, which implies that $(\bigotimes_{j=1}^i w^{(j)})_{T_\ell} \leq (w^{(1)})_{T_\ell}$. Thus merging (41)-(43) yields

$$\Theta^{(i)} \leq \sqrt{1 + \delta_k} \left[ \sum_{\ell=1}^q \|(w^{(1)})_{T_\ell}\|_\infty \delta_{3k} \|x - x^p\|_2 \right] \leq 2\delta_{3k} \sqrt{1 + \delta_k} \|x - x^p\|_2.$$ 

Substituting this bound and (40) into (37) leads to the desired bound (39).}

We now prove the main result of this section which claims that the ROT$\omega$ and ROTP$\omega$ can guarantee to recover the $k$-sparse signal $x$ if $\delta_{3k}$ is smaller than a certain number in $(0, 1)$.

**Theorem 4.5** Let $x$ be a $k$-sparse signal with measurements $y := Ax$. Let $\omega \geq 1$ be a given integer number.
(i) If the restricted isometry constant of $A$ satisfies $\delta_{3k} < \gamma(\omega)$, where $\gamma(\omega) \in (0, 1)$ is the unique real root of the univariate equation $(2\omega + 1)\gamma\sqrt{\frac{1 + \gamma}{1 - \gamma}} + \gamma = 1$, then the sequence $\{x^p\}$, generated by $\text{ROTP}$, converges to $x$ with error
\[
\|x^{p+1} - x\|_2 \leq \overline{\rho}\|x^p - x\|_2,
\]
where
\[
\overline{\rho} := (\delta_{2k} + 2\omega\delta_{3k})\sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k} < 1.
\]

(ii) If $\delta_{3k} < \gamma^*(\omega)$, where $\gamma^*(\omega) \in (0, 1)$ is the unique real root of the univariate equation
\[
\frac{1}{\sqrt{1 - \gamma^2}} \left( (2\omega + 1)\gamma\sqrt{\frac{1 + \gamma}{1 - \gamma}} + \gamma \right) = 1,
\]
then the sequence $\{x^p\}$, generated by $\text{ROTP}$, converges to $x$ with error
\[
\|x - x^{p+1}\|_2 \leq \rho'\|x - x^p\|_2,
\]
where
\[
\rho' := \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \left( (\delta_{2k} + 2\omega\delta_{3k})\sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k} \right) < 1.
\]

Proof. At the current iterate $x^p$, using the vector $u^p = x^p + A^T(y - Ax^p)$, both $\text{ROTP\omega}$ and $\text{ROTP\omega}$ algorithms generate the vectors $w^{(1)}, \ldots, w^{(\omega)} \in \{w : e^T w = k, 0 \leq w \leq e\}$ by solving the convex optimization problem (5). Denote by
\[
x^\# = \mathcal{H}_k(u^p \otimes (\bigotimes_{j=1} w^{(j)})), \ X = \text{supp}(x^\#).
\]
Since $x$ is a $k$-sparse vector with $S = \text{supp}(x)$ and $y = Ax$. By Lemma 4.1, we have
\[
\|x - x^\#\|_2 \leq \|(u^p \otimes (\bigotimes_{j=1} w^{(j)}) - x)\chi_{\cup S}\|_2 + \|[(u^p \otimes (\bigotimes_{j=1} w^{(j)})) - x]\chi_{\setminus S}\|_2.
\]
We now bound the right-hand side of (45). The second term of the right-hand side of (45) is easy to bound. By noting that $x \chi_{\setminus S} = 0$, we have
\[
\|[(u^p \otimes (\bigotimes_{j=1} w^{(j)})) - x]\chi_{\setminus S}\|_2 = \|[(u^p \otimes (\bigotimes_{j=1} w^{(j)})) - x]\chi_{\setminus S}\|_2
\]
\[
= \|[(I - A^T A)(x^p - x)) \otimes (\bigotimes_{j=1} w^{(j)})]\chi_{\setminus S}\|_2
\]
\[
\leq \|[(I - A^T A)(x^p - x)]\chi_{\setminus S}\|_2
\]
\[
\leq \delta_{3k}\|x^p - x\|_2,
\]
where the inequalities above follow from $0 \leq w^{(j)} \leq e$ for $j = 1, \ldots, \omega$ and from Lemma 3.2 with the fact $|\text{supp}(x^p - x) \cup (X \setminus S)| \leq 3k$. Substituting the bound above into (45) yields

$$
\|x - x^\#\|_2 \leq \|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_{X \cup S} + \delta_{3k} \|x^p - x\|_2.
$$

We now bound the first term of the right-hand side of (46). Let $\alpha \in (0,1)$ be any given number. Define

$$
\Theta^* := \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2.
$$

There are only two cases.

**Case 1.** $\Theta^* \leq \alpha \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2$. In this case, since $y = Ax$, we have

$$
\|y - A[u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}]\|_2 = \|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_2
$$

$$
= \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2 + \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2
$$

$$
\geq \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2 - \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2
$$

$$
= \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2 - \Theta^*
$$

$$
\geq (1 - \alpha) \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2
$$

$$
\geq (1 - \alpha) \sqrt{1 - \delta_{2k} \|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_{X \cup S}}
$$

where the last inequality follows from Definition 3.1 and the fact $|X \cup S| \leq 2k$. This, together with Lemma 4.4 implies that

$$
\|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_{X \cup S} \leq \frac{1}{(1 - \alpha) \sqrt{1 - \delta_{2k}}} \|y - A[u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}]\|_2
$$

$$
\leq \frac{\delta_{2k} + 2(\omega - 1)\delta_{3k}}{1 - \alpha} \sqrt{\frac{1 + \delta_{k}}{1 - \delta_{2k}}} \|x - x^p\|_2.
$$

**Case 2.** $\Theta^* > \alpha \|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2$. In this case, by noting that

$$
\|A[(u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)}) - x]_{X \cup S}\|_2 \geq \sqrt{1 - \delta_{2k} \|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_{X \cup S}}
$$

we obtain

$$
\|u^p \otimes \bigotimes_{j=1}^{\omega} w^{(j)} - x\|_{X \cup S} \leq \frac{\Theta^*}{\alpha \sqrt{1 - \delta_{2k}}}. \quad (48)
$$
So it is sufficient to bound the term $\Theta^{*}$. The idea is similar to the proof of Lemma 4.4. Since $x_{\cup U S} = 0$, $\Theta^{*}$ can be written as

$$\Theta^{*} = \| A[(u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}))_{\cup U S}] \|_2 = \| A[(u^p - x) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{\cup U S} \|_2.$$  

Let $q$ and $\kappa$ are integer numbers such that $|X \cup S| = (q - 1)k + \kappa$, where $0 \leq \kappa < k$. Let $(w^{(1)})_{\cup U S}$ be decomposed into $k$-sparse vectors as follows:

$$(w^{(1)})_{\cup U S} = (w^{(1)})_{S_1} + \cdots + (w^{(1)})_{S_q},$$

where $S_1$ is the index set for the largest $k$ elements in the set $\{(w^{(1)})_i : i \in X \cup S\}$, and $S_2$ is the index set for the second largest $k$ elements in $\{(w^{(1)})_i : i \in X \cup S\}$, and so on. $S_q$ is the index set for the remaining $\kappa$ element in this set. The index sets $S_\ell, \ell = 1, \ldots, q$ are mutually disjoint and $|S_\ell| = k$ for all $\ell = 1, \ldots, q - 1$ and $|S_q| = \kappa < k$. Clearly, $X \cup S = S_1 \cup S_2 \cup \cdots \cup S_q$.

Applying the Lemma 4.2 with $w = w^{(1)}, \tau = k$ and $\Lambda = X \cup S$ yields the following inequality:

$$\sum_{\ell=1}^{q} \| (w^{(1)})_{S_\ell} \|_\infty < 2. \tag{49}$$

Define the vector $z^{(\ell)} := [(u^p - x) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{S_\ell}$, then

$$[(u^p - x) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{\cup U S} = z^{(1)} + z^{(2)} + \cdots + z^{(q)}.$$  

Therefore,  

$$\Theta^{*} = \| A \sum_{\ell=1}^{q} z^{(\ell)} \|_2 \leq \sum_{\ell=1}^{q} \| A z^{(\ell)} \|_2 \leq \sqrt{1 + \delta_k} \sum_{\ell=1}^{q} \| z^{(\ell)} \|_2, \tag{50}$$

where the last inequality follows from the definition of $\delta_k$ and the fact that every $z^{(\ell)}$ is $k$-sparse.

We now estimate the term $\sum_{\ell=1}^{q} \| z^{(\ell)} \|_2$. Note that

$$\| z^{(\ell)} \|_2 = \| [(u^p - x) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{S_\ell} \|_2 = \| [(I - A^T A)(x - x^p)) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{S_\ell} \|_2$$

$$= \| [(I - A^T A)(x - x^p)) \otimes (\bigotimes_{j=1}^{\omega} w^{(j)})]_{S_\ell} \|_2$$

$$\leq \| (\bigotimes_{j=1}^{\omega} w^{(j)})_{S_\ell} \|_\infty \| [(I - A^T A)(x - x^p))]_{S_\ell} \|_2$$

$$\leq \| (w^{(1)})_{S_\ell} \|_\infty \| x - x^p \|_2, \tag{51}$$

where the last inequality follows from the fact $0 \leq w^{(j)} \leq e$ for all $j = 1, \ldots, \omega$ and from Lemma 3.2 with $|S_\ell \cup \text{supp}(x - x^p)| \leq 3k$. Thus combining (49), (50) and (51), we obtain

$$\Theta^{*} \leq \sqrt{1 + \delta_k} \sum_{\ell=1}^{q} \| (w^{(1)})_{S_\ell} \|_\infty \delta_{3k} \| x - x^p \|_2 \leq 2 \delta_{3k} \sqrt{1 + \delta_k} \| x - x^p \|_2.$$  

20
Substituting this into (48), we get
\[
\|u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x\|_{X \cup S} \leq \frac{2\delta_{3k}}{\alpha} \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} \|x - x^p\|_2.
\]

Thus combining (47) for Case 1 and (52) for Case 2 yields
\[
\|u^p \otimes (\bigotimes_{j=1}^{\omega} w^{(j)}) - x\|_{X \cup S} \leq \max \left\{ \frac{\delta_{2k} + 2(\omega - 1)\delta_{3k}}{1 - \alpha}, \frac{2\delta_{3k}}{\alpha} \right\} \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} \|x - x^p\|_2,
\]
which holds for any given number \(\alpha \in (0, 1)\). It is very easy to verify that
\[
\min_{\alpha \in (0, 1)} \max \left\{ \frac{\delta_{2k} + 2(\omega - 1)\delta_{3k}}{1 - \alpha}, \frac{2\delta_{3k}}{\alpha} \right\} = 2\omega \delta_{3k} + \delta_{2k}.
\]
This minimum value attains at
\[
\alpha = \frac{2\delta_{3k}}{2\omega \delta_{3k} + \delta_{2k}}.
\]
It follows from (46) that
\[
\|x - x^\#\|_2 \leq \left(2\omega \delta_{3k} + \delta_{2k}\right) \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k} \leq (2\omega + 1)\delta_{3k} \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k}.
\]

(i) By the structure of the ROT\(\omega\), \(x^{k+1} = x^\#\). Thus the desired result for ROT\(\omega\) follows immediately from (53). By noting that \(\delta_k \leq \delta_{2k} \leq \delta_{3k}\), the constant
\[
\tilde{\rho} := (2\omega \delta_{3k} + \delta_{2k}) \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k} \leq (2\omega + 1)\delta_{3k} \sqrt{\frac{1 + \delta_k}{1 - \delta_{2k}}} + \delta_{3k}.
\]
The right-hand side of the above inequality is smaller than 1 provided that \(\delta_{3k} < \gamma(\omega)\), where \(\gamma(\omega) \in (0, 1)\) is the positive real root of the following univariate equation of \(\gamma:\)
\[
g_\omega(\gamma) = (2\omega + 1)\gamma \sqrt{\frac{1 + \gamma}{1 - \gamma}} + \gamma - 1 = 0.
\]
It is very easy to verify that for a given integer number \(\omega \geq 1\), the above univariate equation has a unique real root \(\gamma(\omega)\) in the interval \((0, 1)\). In fact, when \(\gamma \in (0, 1)\) and \(\gamma \to 0\), we see that \(g_\omega(\gamma) < 0\), and when \(\gamma \to 1\), we see that \(g_\omega(\gamma) > 0\). Also, the function \(g_\omega(\gamma)\) is strictly increasing over \((0, 1)\). This implies that the univariate equation \(g_\omega(\gamma) = 0\) has a unique real solution in the interval \((0, 1)\).

(ii) We now establish the convergence of ROTP\(\omega\). Note that the first step (i.e., the step S1 of the algorithm) is the same as that of ROT\(\omega\). Therefore, the relation (53) remains valid to ROTP\(\omega\) which treats \(x^\#\) as an intermediate point instead of the next iterate \(x^{p+1}\). Using this intermediate point \(x^\#\), the ROTP\(\omega\) algorithm solve the following least-squares problem:
\[
\min_z \{\|y - Az\|^2 : \text{supp}(z) \subseteq \text{supp}(x^\#)\},
\]
to which the solution is set to be \(x^{k+1}\). By optimality, the vector \(x^{p+1}\) must satisfy the relation \([A^T(y - Ax^{p+1})]_{\text{supp}(x^\#)} = 0\) which, together with \(y = Ax\), implies that \([I - A^TA](x -
\[ x^{p+1}_{\text{supp}(x^#)} = (x-x^{p+1})_{\text{supp}(x^#)}. \] Since \( \text{supp}(x^{p+1}) \subseteq \text{supp}(x^#) \) which implies \( x^{p+1} - x^# \)
\[ \text{supp}(x^#) = 0, \] we then have that
\[ (x - x^{p+1})_{\text{supp}(x^#)} = (x - x^# + x^# - x^{p+1})_{\text{supp}(x^#)} = (x - x^#)_{\text{supp}(x^#)}. \]

Therefore,
\[ \|x - x^{p+1}\|_2^2 = \|(x - x^{p+1})_{\text{supp}(x^#)}\|_2^2 + \|(x - x^{p+1})_{\text{supp}(x^#)}\|_2^2 \]
\[ = \|[(I - A^T A)(x - x^{p+1})]_{\text{supp}(x^#)}\|_2^2 + \|(x - x^#)_{\text{supp}(x^#)}\|_2^2 \]
\[ \leq \delta_{2k}^2 \|x - x^{p+1}\|_2^2 + \|x - x^#\|_2^2. \]

The first term in the final inequality above follows from Lemma 3.2 since \(|\text{supp}(x - x^{p+1}) \cup \text{supp}(x^#)| \leq 2k\). Therefore,
\[ \|x - x^{p+1}\|_2 \leq \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \|x - x^#\|_2 \leq \rho' \|x - x^p\|_2. \]

where the last inequality follows from (53) and the constant \( \rho' \) is given as
\[ \rho' = \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \left( 2\omega \delta_{3k} + \delta_{2k} \right) \left( \frac{1 + \delta_{k}}{1 - \delta_{2k} + \delta_{3k}} \right) < 1, \]
which is guaranteed if \( \delta_{3k} \leq r^*(\omega) \), where \( r^*(\omega) \in (0,1) \) is the real root of the following univariate equation in variable \( \gamma \):
\[ \frac{1}{\sqrt{1 - \gamma^2}} \left[ (2\omega + 1)\gamma \sqrt{\frac{1 + \gamma}{1 - \gamma}} + \gamma \right] = 1. \]

By an analysis similar to (i), it is very easy to verify that the root \( \gamma^*(\omega) \) of the above equation in \((0,1)\) is unique. □

Given a specific \( \omega \), the values of \( r(\omega) \) and \( r^*(\omega) \) can be immediately obtained. As a result, the guaranteed performance of ROTP, ROTP2 and ROTP3 (which correspond to the cases \( \omega = 1,2,3 \) respectively) can be immediately obtained from Theorem 4.5. The convergence results for ROTP2 and ROTP3 are summarized in the corollary below, which is established for the first time.

**Corollary 4.6** Let \( x \) be a \( k \)-sparse signal with measurements \( y := Ax \).

(i) If \( \delta_{3k} \leq 1/7 \), then the sequence \( \{x^p\} \), generated by ROTP2, converges to \( x \) with error
\[ \|x - x^{p+1}\|_2 \leq \rho' \|x - x^p\|_2. \]

where
\[ \rho' = \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \left( 4\delta_{3k} + \delta_{2k} \right) \left( \frac{1 + \delta_{k}}{1 - \delta_{2k} + \delta_{3k}} \right) < 1. \] (54)
(ii) If $\delta_{3k} \leq 1/9$, then the sequence $\{x^p\}$, generated by ROTP3, converges to $x$ with error

$$\|x - x^{p+1}\|_2 \leq \rho''\|x - x^p\|_2,$$

where

$$\rho'' = \frac{1}{\sqrt{1 - (\delta_{2k})^2}} \left( (6\delta_{3k} + \delta_{2k}) \sqrt{1 + \delta_{k}} \frac{1 + \delta_{k}}{1 - \delta_{2k} + \delta_{3k}} \right) < 1.$$

The proof of the above corollary is straightforward. In fact, when $\omega = 2$, we can verify that the unique root $r^*(2)$ of the equation (44) in $(0, 1)$ is larger than $1/7$. For $\omega = 3$, the unique root $r^*(3)$ in $(0, 1)$ of the equation (44) is larger than $1/9$. The corollary follows from Theorem 4.5 immediately. The convergence results in [54] for ROT and ROTP (corresponding to $\omega = 1$) can be reobtained immediately from Theorem 4.5 as well. Similar to Corollary 4.6, the first convergence results for ROT2 and ROT3 can be obtained immediately from Theorem 4.5. Briefly, the RIP bounds $\delta_{3k} < 1/7$ and $\delta_{3k} < 1/9$ are the sufficient conditions for the convergence of ROT2 and ROT3, respectively.

The results in this paper can be easily generalized to the cases when the signal is not exactly $k$-sparse and the measurements are inaccurate. In this cases, the measurements take the form $y = Ax + \nu$, where $x$ is the signal to recover (which is not necessarily $k$-sparse) and $\nu$ is a noise vector. Taking ROTP2 as an example, it is easily to show that under the same assumption of Corollary 4.6 i.e., $\delta_{3k} < 1/7$, the sequence $\{x^p\}$ generated by ROTP2 approximates the largest $k$ magnitudes of the signal $x$ (denoted by $x_S$) with error

$$\|x^{p+1} - x_S\|_2 \leq \rho'\|x^p - x_S\|_2 + C\|\nu\|_2$$

where $\rho'$ is given in [54] and the constant $C$ is determined only by the RIP constants $\delta_{k}, \delta_{2k}$ and $\delta_{3k}$. The details are omitted here.

5 Conclusions and future work

The newly developed optimal $k$-thresholding algorithms (OT and OTP) can exactly recover $k$-sparse signals if the restricted isometry constant of the sensing matrix satisfies $\delta_k < 0.2275$ when $k$ is even and $\delta_{k+1} < 0.2275$ when $k$ is an odd number. Such guaranteed performance conditions governing the sparse signal recovery are nearly optimal. Recall that Cai and Zhang [11] have proved that $\delta_k < 1/3$ is a sufficient condition for the guaranteed recovery of $k$-sparse signals via $\ell_1$ minimization. A clear question is whether the RIP bounds for OT and OTP established in this paper can be improved to $\delta_k < 1/3$ or $\delta_{k+1} < 1/3$? Given an integer number $\omega$ (the number of times for data compression in every iteration), it turns out that the algorithms ROT$\omega$ and ROTP$\omega$ can guarantee to recover the sparse signal if the sensing matrix satisfies the condition in Theorem 4.5. As special cases, the convergence of the ROT2 and ROTP2 can be guaranteed under the bounds $\delta_{3k} < 1/7$ and $\delta_{3k} < 1/9$, respectively. An immediate question is whether these theoretical results can be improved.
References

[1] A. Beck and Y.C. Eldar, Sparsity constrained nonlinear optimization: Optimality conditions and algorithms, *SIAM J. Optim.*, 23 (2013), pp. 1480–1509.

[2] A. Beck and M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.*, 2 (2009), pp. 183–202.

[3] D. Bertsimas, A. King and R. Mazumder, Best subset selection via a modern optimization Lens, *Ann. Statist.*, 44 (2016), pp. 813–852.

[4] J. Blanchard, J. Tanner and K. Wei, CGIHT: Conjugate gradient iterative hard thresholding for compressed sensing and matrix completion, *IEEE Trans. Signal Process.*, 63 (2015), pp. 528–537.

[5] T. Blumensath and M. Davies, Iterative hard thresholding for sparse approximation, *J. Fourier Anal. Appl.*, 14 (2008), pp. 629–654.

[6] T. Blumensath and M. Davies, Iterative hard thresholding for compressed sensing, *Appl. Comput. Harmon. Anal.*, 27 (2009), pp. 265–274.

[7] T. Blumensath and M. Davies, Normalized iterative hard thresholding: Guaranteed stability and performance, *IEEE J. Sel. Top. Signal Process.*, 4 (2010), pp. 298–309.

[8] J.-U. Bouchot, A generalized class of hard thresholding algorithms for sparse signal recovery. In: Fasshauer G., Schumaker L. (eds) Approximation Theory XIV: San Antonio 2013. Springer Proceedings in Mathematics & Statistics, 83 (2014), pp. 45–63.

[9] J.-U., Bouchot, S. Foucart and P. Hitczenki, Hard thresholding pursuit algorithms: Number of iterations, *Appl. Comput. Harmon. Anal.*, 41 (2016), pp. 412–435.

[10] C. Buchheim and E. Traversi, Quadratic combinatorial optimization using separable underestimators, *INFORMS Journal on Computing*, 30 (2018), pp. 424–637.

[11] T. Cai and A. Zhang, Sharp RIP bound for sparse signal and low-rank matrix recovery, *Appl. Comput. Harmon. Anal.*, 35 (2013), pp. 74–93.

[12] E.J. Candès and Y. Plan, Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. *IEEE Trans. Inform. Theory*, 57 (2011), pp. 2342–2359.

[13] E.J. Candès and T. Tao, Decoding by linear programming, *IEEE Trans. Inform. Theory*, 51 (2005), pp. 4203–4215.

[14] E.J. Candès, M. Wakin and S. Boyd, Enhancing sparsity by reweighted $\ell_1$ minimization, *J. Fourier Anal. Appl.*, 14 (2008), pp. 877–905.

[15] V. Cevher, On accelerated hard thresholding methods for sparse approximation, Proc. SPIE 8138, Wavelets and Sparsity XIV, 813811, 2011.
[16] W.A. Chaovalitwongse, I.P. Androulakis and P.M. Pardalos, Quadratic integer programming: Complexity and equivalent forms. In: Floudas C., Pardalos P. (eds) Encyclopedia of Optimization, Springer, Boston, MA, 2008.

[17] S.S. Chen, D.L. Donoho and M.A. Saunders, Atomic decomposition by basis pursuit, SIAM J. Sci. Comput., 20 (1998), pp. 33–61.

[18] W. Dai, and O. Milenkovic, Subspace pursuit for compressive sensing signal reconstruction, IEEE Trans. Inform. Theory, 55 (2009), pp. 2230–2249.

[19] I. Daubechies, M. Defries and C. De Mol, An iterative thresholding algorithm for linear inverse problems with a sparsity constraint, Comm. Pure Appl. Math., 57 (2004), pp. 1413–1457.

[20] M.A. Davenport and J. Romberg, An overview of low-rank matrix recovery from incomplete observations, IEEE J. Sel. Topics Signal Process., 10 (2016), No. 4, pp. 608–622.

[21] D.L. Donoho, Compressed sensing, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.

[22] D.L. Donoho, De-noising by soft-thresholding, IEEE Trans. Inform. Theory, 41 (1995), pp. 613–627.

[23] D.L. Donoho and I. Johnstone, Idea spatial adaptation via wavelet shrinkage, Biomatrika, 81 (1994), pp. 425–455.

[24] M. Elad, Why simple shrinkage is still relevant for redundant representation, IEEE Trans. Inform. Theory, 52 (2006), pp. 5559–5569.

[25] M. Elad, Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing, Springer, New York, 2010.

[26] Y.C. Eldar and G. Kutyniok, Compressed Sensing: Theory and Applications, Cambridge University Press, 2012.

[27] M. Figueiredo and R. Nowak, An EM algorithm for wavelet-based image restoration, IEEE Trans. Image Process., 12 (2003), pp. 906–916.

[28] M. Fornasier and R. Rauhut, Iterative thresholding algorithms, Appl. Comput. Harmon. Anal., 25 (2008), pp. 187-208.

[29] S. Foucart, Hard thresholding pursuit: An algorithm for compressive sensing, SIAM J. Numer. Anal., 49 (2011), pp. 2543–2563.

[30] S. Foucart and H. Rauhut, A Mathematical Introduction to Compressive Sensing, Springer, NY, 2013.

[31] S. Foucart and S. Subramanian, Iterative hard thresholding for low-rank recovery from rank-one projections, Linear Algebra Appl., 572 (2019), pp. 117–134.
[32] R. Garg and R. Khandekar, Gradient descent with sparsification: An iterative algorithm for sparse recovery with restricted isometry property, Proceeding ICML 2009, Montreal, Canada, pp. 337-344.

[33] K. Herrity, A. Gilbert and J. Tropp, Sparse approximation via iterative thresholding, in IEEE ICASSP 2006, pp. 624–627.

[34] R. Khanna, and A. Kyrillidis, IHT dies hard: Provable accelerated iterative hard thresholding, in Proceedings of the AISTATS, Lanzarote, Spain, 84 (2018), pp. 188–198.

[35] N. Kingsbury and T. Reeves, Redundant representation with complex wavelets: How to achieve sparsity, in IEEE ICIP 2003, Barcelona, pp. 45–48.

[36] A. Kyrillidis and V. Cevher, Matrix recipes for hard thresholding methods, J. Math. Imag. Vision, 48 (2014), pp. 235–265.

[37] L. Landweber, An iteration formula for Freholm integral equations of the first kind, Amer. J. Math., 73 (1951), pp. 615–624.

[38] K. Lange, MM Optimization Algorithms, SIAM, Philadelphia, 2016.

[39] H. Liu and R.F. Barber, Between hard and soft thresholding: Optimal iterative thresholding algorithms, arXiv, July 2019.

[40] S. Mallat and Z. Zhang, Matching pursuits with time-frequency dictionaries, IEEE Trans. Signal Process., 41 (1993), pp. 3397–3415.

[41] A. Miller, Subset Selection in Regression, CRC Press, Washington, 2002.

[42] B.K. Natarajan, Sparse approximate solutions to linear systems, SIAM J. Comput., 24 (1995), pp. 227-234.

[43] D. Needell and J.A. Tropp, CoSaMP: Iterative signal recovery from incomplete and inaccurate samples, Appl. Comput. Harmon. Anal. 26 (2009), pp. 301–321.

[44] Y. Nesterov, Introductory Lectures on Convex Optimization: A Basic Course, Volume 87, Springer Science and Business Media, 2013.

[45] N. Nguyen, D. Needell and T. Woolf, Linear convergence of stochastic iterative greedy algorithms with sparse constraints, IEEE Trans. Inform. Theory, 63 (2017), pp. 6869–6895.

[46] J. Shen and P. Li, A tight bound of hard thresholding, J. Machine Learning Res., 18 (2018), pp. 1–42.

[47] J. Starck, M. Nguyen, and F. Murtagh, Wavelet and curvelet for image deconvolution: A combined approach, J. Signal Process., 83 (2003), pp. 2279–2283.
[48] A. Suggala, K. Bhatia, P. Ravikumar and P. Jain, Adaptive hard thresholding for near-optimal consistent robust regression, arXiv, March 2019.

[49] J.A. Tropp and A.C. Gilbert, Signal recovery from random measurements via orthogonal matching pursuit, IEEE Trans. Inform. Theory, 53 (2007), pp. 4655–4666.

[50] S. Voronin, H.J. Woerdeman, A new iterative firm-thresholding algorithms for inverse problems with sparsity constraints, Appl. Comput. Harmonic Anal., 35 (2013), pp. 151–164.

[51] A. Wachsmuth, Iteration hard-thresholding applied to optimal control problems with $L^0(\Omega)$ control cost, SIAM J. Control Optim., 57 (2019), pp. 854–879.

[52] A. Zaki, P. Mitra, L. Rasmussen and S. Chartterjee, Estimate exchange over network is good for distributed hard thresholding pursuit, Signal Processing, 156 (2019), pp. 1–11.

[53] Y.-B. Zhao, Sparse Optimization Theory and Methods, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2018.

[54] Y.-B. Zhao, Optimal $k$-thresholding algirithms for sparse optimization problems, SIAM J. Optim. (to appear). https://arxiv.org/abs/1909.00717

[55] Y.-B. Zhao and D. Li, Reweighted $\ell_1$-minimization for sparse solutions to underdetermined linear systems, SIAM J. Optim., 22 (2012), pp. 893–912.

[56] Y.-B. Zhao and M. Kocvara, A new computational method for the sparsest solutions to systems of linear equations, SIAM J. Optim., 25 (2015), pp. 1110–1134.

[57] Y.-B. Zhao and Z.-Q. Luo, Constructing new reweighted $\ell_1$-algorithms for the sparsest points of polyhedral sets, Math. Oper. Res., 42 (2017), pp. 57–76.