The multiplicative structure on continuous polynomial valuations.

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Abstract

A canonical structure of a commutative associative filtered algebra with unit is introduced on the space of polynomial smooth valuations, and its properties are studied. Induced structure on the subalgebra of translation invariant smooth valuations has especially nice properties (it has the structure of the Frobenius algebra). We also present some applications.

0 Introduction.

The purpose of this paper is to introduce a canonical structure of a commutative associative filtered algebra with unit on the space of polynomial smooth valuations and study its properties. The induced structure on the subalgebra of translation invariant smooth valuations has especially nice properties (it is the structure of the Frobenius algebra). We will also present some applications. Part of the results of this paper was announced in [3].

Let us describe our main results in more detail. Let us recall first of all some notation and definitions. Let $V$ be a real vector space of finite dimension $n$. Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of $V$. If we fix on $V$ a Euclidean metric then we can define the Hausdorff metric $d_H$ on $\mathcal{K}(V)$ as follows:

$$d_H(A, B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\},$$
where \((U)_{\epsilon}\) denotes the \(\epsilon\)-neighborhood of a set \(U\). However the topology on \(\mathcal{K}(V)\) does not depend on the choice of a Euclidean metric on \(V\). Moreover, equipped with this topology, \(\mathcal{K}(V)\) becomes a locally compact topological space (Blaschke’s selection theorem).

**0.1 Definition.**

a) A function \(\phi : \mathcal{K}(V) \rightarrow \mathbb{C}\) is called a valuation if for any \(K_1, K_2 \in \mathcal{K}(V)\) such that their union is also convex one has

\[
\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).
\]

b) A valuation \(\phi\) is called continuous if it is continuous with respect to the Hausdorff metric on \(\mathcal{K}(V)\).

For the classical theory of valuations we refer to the surveys [17] and [16]. Let us remind the definition of a polynomial valuation introduced by Khovanskii and Pukhlikov [10], [11].

**0.2 Definition.**

A valuation \(\phi\) is called polynomial of degree at most \(d\) if for every \(K \in \mathcal{K}(V)\) the function \(x \mapsto \phi(K + x)\) is a polynomial on \(V\) of degree at most \(d\).

Note that valuations polynomial of degree 0 are called translation invariant valuations. Polynomial valuations have many nice combinatorial-algebraic properties ([10], [11]).

**0.3 Example.**

1. The Euler characteristic \(\chi\) is a continuous translation invariant valuation (remind that \(\chi(K) = 1\) for any convex compact set \(K\)).

2. Let \(\mu\) be a measure on \(V\) with a polynomial density with respect to a Lebesgue measure. Fix \(A \in \mathcal{K}(V)\). Then

\[
\phi(K) := \mu(K + A)
\]

is a continuous polynomial valuation (here \(K + A := \{k + a | k \in K, a \in A\}\)).

**0.4 Remark.**

In [3] we have announced a result that polynomial continuous valuations are dense in the space of all continuous valuations. However during the preparation of this paper we found a gap in our original argument. So we do not know if this fact is true.

Let us remind a basic definition from representation theory. Let \(\rho\) be a continuous representation of a Lie group \(G\) in a Fréchet space \(F\). A vector
ξ ∈ F is called G-smooth if the map \( g \mapsto \rho(g)\xi \) is an infinitely differentiable map from G to F. It is well known (e.g. [20], Section 1.6) that the subset \( F^{sm} \) of smooth vectors is a G-invariant linear subspace dense in F. Moreover it has a natural topology of a Fréchet space (which is stronger than that induced from F), and the representation of G in \( F^{sm} \) is continuous. Moreover all vectors in \( F^{sm} \) are G-smooth.

We will denote by \( GL(V) \) the group of all linear transformations of V, and by \( Aff(V) \) the group of all affine transformations of V.

We will especially be interested in polynomial valuations which are \( GL(V) \)-smooth. The space of \( GL(V) \)-smooth valuations polynomial of degree at most \( d \) will be denoted by \( PVal^{sm}_d(V) \). This is a Fréchet space. Let \( PVal^{sm}(V) \) denote the inductive limit of the Fréchet spaces \( PVal^{sm}_d(V) \) (with the topology of the inductive limit).

In Section 1 we define a canonical structure of commutative associative algebra with unit on \( PVal^{sm}(V) \) where the unit is the Euler characteristic. Let us give the idea of this construction. Let us denote by \( \mathcal{G}'(V) \) the linear space of valuations on \( \mathcal{K}(V) \) which are finite linear combinations of valuations from Example 0.3 (2). It turns out that \( \mathcal{G}'(V) \cap PVal^{sm}(V) \) is dense in \( PVal^{sm}(V) \) (see the proof of Lemma 1.1). Let \( W \) be another linear real vector space. Let us define the exterior product \( \phi \otimes \psi \in \mathcal{G}'(V \times W) \) of two valuations \( \phi \in \mathcal{G}'(V) \), \( \psi \in \mathcal{G}'(W) \). Let \( \phi(K) = \sum_i \mu_i(K + A_i) \), \( \psi(L) = \sum_j \nu_j(L + A_j) \).

Define

\[
(\phi \otimes \psi)(M) := \sum_{i,j} (\mu_i \otimes \nu_j)(M + (A_i \times B_j)),
\]

where \( \mu_i \otimes \nu_j \) denotes the usual product measure.

Now let us define a product on \( \mathcal{G}'(V) \). Let \( \Delta : V \hookrightarrow V \times V \) denote the diagonal imbedding. For \( \phi, \psi \in \mathcal{G}'(V) \) let

\[
\phi \cdot \psi := \Delta^*(\phi \otimes \psi),
\]

where \( \Delta^* \) denotes the restriction of a valuation on \( V \times V \) to the diagonal. By Proposition 1.4 this product is associative and commutative, the Euler characteristic is the identity element in this algebra, and this product extends (uniquely) to a continuous product on \( PVal^{sm}(V) \) (Proposition 1.10).

We have the following natural filtration on \( PVal^{sm}(V) \):

\[
\gamma_i = \{ \phi \in PVal^{sm}(V) | \phi(K) = 0 \text{ for all } K \text{ s. t. } \dim K < i \}.
\]
Then \( PVal^{sm}(V) = \gamma_0 \supset \gamma_1 \supset \cdots \supset \gamma_n \supset \gamma_{n+1} = 0 \). However this filtration is not compatible with the multiplicative structure. Nevertheless there exists another filtration \( W_i \) which can be characterized as follows.

**0.5 Theorem.** There exists a unique filtration \( W_i \) on \( PVal^{sm}(V) \) such that

1. \( \{W_i\} \) is compatible with the multiplicative structure, i.e. \( W_i \cdot W_j \subset W_{i+j} \);
2. \( \gamma_{i+1} \subset W_i \subset \gamma_i \) for all \( i \);
3. \( W_0 = \gamma_0, W_1 = \gamma_1 \);
4. \( W_i \) is a closed subspace of \( PVal^{sm}(V) \);
5. \( W_i \) is \( Aff(V) \)-invariant.

This theorem is a reformulation of Theorem 3.8 of Section 3. The explicit construction of this filtration is given in Section 3. One of the ways to describe this filtration explicitly is as follows (see Proposition 3.4):

\[ \phi \in W_i \text{ if and only if } \lim_{r \to +0} r^{-i+1} \phi(rK + x) = 0 \text{ for all } K \in K(V), x \in V. \]

We will denote the space of all translation invariant continuous valuations on \( V \) by \( Val(V) \). Remind that a valuation \( \phi \) is called \( i \)-homogeneous if \( \phi(\lambda K) = \lambda^i \phi(K) \) for all \( \lambda \geq 0, K \in K(V) \). We will denote by \( Val_i(V) \) the subspace of \( i \)-homogeneous valuations. By a result of McMullen [14] one has a decomposition:

\[ Val(V) = \bigoplus_{i=0}^{n} Val_i(V). \]  

Also one has

\[ Val_0(V) = \mathbb{C} \cdot \chi, \quad Val_n(V) = \mathbb{C} \cdot vol \]  

where the first equality is trivial, and the second one is due to Hadwiger [9]. We have also a further decomposition of these spaces with respect to parity. Namely we say that a valuation \( \phi \) is even if \( \phi(-K) = \phi(K), \forall K \in K(V) \). Similarly \( \phi \) is called odd if \( \phi(-K) = -\phi(K), \forall K \in K(V) \). The subspace of even translation invariant valuations will be denoted by \( Val^0(V) \), and the subspace of odd translation invariant valuations will be denoted by \( Val^1(V) \). Similarly \( Val_i^0(V) \) and \( Val_i^1(V) \) will denote their subspaces of \( i \)-homogeneous valuations. We obviously have

\[ Val_i(V) = Val_i^0(V) \oplus Val_i^1(V). \]  

It turns out that one can easily describe the associated graded algebra \( gr_W(PVal^{sm}(V)) \) in terms of the algebra of translation invariant smooth valuations. In Section 3 we prove the following result (Theorem 3.9).
0.6 Theorem. There exists a canonical isomorphism of graded algebras

\[ \text{gr}_W(P\text{Val}^{sm}(V)) \simeq \text{Val}^{sm}(V) \otimes \mathbb{C}[V] \]

where the \( i \)th graded term in the right hand side is equal to \( \text{Val}^{sm}_i(V) \otimes \mathbb{C}[V] \) where \( \mathbb{C}[V] \) denotes the algebra of polynomial functions on \( V \).

In the proof of this theorem we construct this isomorphism explicitly. Now let us discuss in more detail the case of translation invariant valuations. One of the main results on translation invariant valuations we will use in this paper is as follows (this was proved in [2]).

0.7 Theorem (Irreducibility Theorem). The natural representations of the group \( GL(V) \) in \( \text{Val}^0_i(V) \) and \( \text{Val}^1_i(V) \) are irreducible.

Below we will denote by \( \text{Val}^{sm}(V) \), \( \text{Val}^{sm}_i(V) \), \( (\text{Val}^0_i(V))^{sm} \), \( (\text{Val}^1_i(V))^{sm} \) the subspaces of \( GL(V) \)-smooth vectors of the corresponding spaces. The space \( \text{Val}^{sm}(V) \) is a subalgebra of \( P\text{Val}^{sm}(V) \). Moreover the degree of homogeneity and parity are (obviously) compatible with the multiplication and define the structure of a bigraded algebra on \( \text{Val}^{sm}(V) \) (where the first grading is by \( \mathbb{Z} \) and the second grading is by \( \mathbb{Z}/2\mathbb{Z} \)). A non-trivial property of the multiplication is the following version of the Poincaré duality (remind that \( \text{Val}_n(V) \) is one dimensional) where for a Fréchet space \( X \) we denote by \( X^* \) its topological dual.

0.8 Theorem. The maps

\[
(\text{Val}^0_i(V))^{sm} \to ((\text{Val}^0_{n-i}(V))^*)^{sm} \otimes \text{Val}_n(V), \quad \text{and}

(\text{Val}^1_i(V))^{sm} \to ((\text{Val}^1_{n-i}(V))^*)^{sm} \otimes \text{Val}_n(V)
\]

induced by the multiplication \( (\text{Val}^0_i(V))^{sm} \otimes (\text{Val}^0_{n-i}(V))^{sm} \to \text{Val}_n(V) \) and \( (\text{Val}^1_i(V))^{sm} \otimes (\text{Val}^1_{n-i}(V))^{sm} \to \text{Val}_n(V) \), are isomorphisms.

This result is proved in Section 2 (Theorem 2.1). The proof is based on the Irreducibility Theorem 0.7. In Section 2 we also compute explicitly some examples of products of valuations (Proposition 2.2). Note that the isomorphisms in Theorem 0.8 commute with the natural action on the group \( GL(V) \). An attempt to understand these isomorphisms from purely representation theoretical point of view was done in [3].

If we forget the grading by parity then we obtain a version of the Poincaré duality on the graded algebra \( \text{Val}^{sm}(V) = \oplus_{i=0}^{n} \text{Val}^{sm}_i(V) \).
We also describe the algebra structure on the space of isometry invariant continuous valuation on an $n$-dimensional Euclidean space (as a vector space it is described by the classical Hadwiger characterization theorem [9]). This graded algebra is isomorphic to the graded algebra of truncated polynomials $\mathbb{C}[x]/(x^{n+1})$ (see Theorem 2.6 for more details).

Now let us discuss some applications of the above results to the spaces of valuations invariant under a group. Let $G$ be a compact subgroup of $GL(V)$. Let us denote by $Val^G(V)$ the space of $G$-invariant translation invariant continuous valuations. Again we have a decomposition

$$Val^G(V) = \bigoplus_{i=0}^{n} Val^G_i(V).$$

Let us write $h_i := \dim Val^G_i(V)$.

0.9 Theorem. Assume that $G$ is a compact subgroup of $GL(V)$ acting transitively on the projective space $\mathbb{P}(V)$.

(i) Then $Val^G(V)$ is finite dimensional.

(ii) $Val^G(V) \subset Val^{sm}(V)$.

(iii) $Val^G(V)$ is a finite dimensional graded subalgebra of $Val^{sm}(V)$ satisfying Poincaré duality:

$$Val^G_i(V) \otimes Val^G_{n-i}(V) \rightarrow Val^G_n(V) = \mathbb{C} \cdot \text{vol}$$

is a perfect pairing. In particular $h_i = h_{n-i}$.

(iv) $Val^G_1(V)$ is spanned by the intrinsic volume $V_1$, and $Val^G_{n-1}(V)$ is spanned by the intrinsic volume $V_{n-1}$ where the intrinsic volumes are taken with respect to a $G$-invariant Euclidean metric. Thus $h_1 = h_{n-1} = 1$.

(v) Assume in addition that $-Id \in G$. Then the numbers $h_i$ satisfy the Lefschetz inequalities:

$$h_i \leq h_{i+1} \text{ for } i < n/2.$$

For the definition of the intrinsic volumes we refer to [18]. Note that part (i) of this theorem was proved in [1] (Theorem 8.1), and part (ii) in [4] (see the proof of Corollary 1.1.3). Part (iii) is a direct consequence of Theorem 0.8. Note that the only proof we know of the equality $h_i = h_{n-i}$ for a general compact group $G$ is based on the existence of the multiplicative structure on the much larger (infinite dimensional) space $Val^{sm}(V)$ and a version of Poincaré duality for this larger algebra. Note also that Poincaré duality for it is based on the Irreducibility Theorem 0.7. Part (iv) will be proved in
Section 2 of this paper. Part (v) of Theorem 0.9 was proved in \[4\] and it is based on a version of the hard Lefschetz theorem for even valuations. We expect that this result (i.e. hard Lefschetz theorem) should be true also for odd valuations; in that case the condition \(-Id \in G\) in part (v) of Theorem 0.9 could be omitted. However we do not know this.

The paper is organized as follows. In Section 1 we define the multiplicative structure on polynomial valuations and study its properties. In Section 2 we study the properties of the subalgebra of translation invariant smooth valuations. In Section 3 we introduce and study filtrations on polynomial valuations. In Section 4 we have some further remarks and discuss some examples.

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1 The product on valuations.

Let us agree on a notation. In the rest of the paper we will denote by \(Pol_d(V)\) the space of homogeneous polynomials of degree \(d\). Let us denote by \(\mathcal{G}(V)\) the linear space of valuations on \(V\) which are finite linear combinations of valuations of the form \(K \mapsto \mu(i(K) + A)\) where \(i : V \hookrightarrow V'\) is a linear imbedding of \(V\) into a larger linear space \(V'\), and \(A \in K(V')\) is a fixed convex compact set, and \(\mu\) is a smooth measure on \(V'\).

1.1 Lemma. \(\mathcal{G}(V) \cap PValsm(V)\) is dense in \(PValsm(V)\).

Proof. Let \(\phi\) be a valuation of the above form, i.e. \(\phi(K) = \mu(i(K) + A)\) where \(i : V \hookrightarrow V'\), \(\mu\) is a smooth measure on \(V'\), and \(A \in K(V')\). We have to prove that for any \(d\) the space \(\mathcal{G}(V) \cap PValsm_d(V)\) is dense in \(PValsm_d(V)\). Let us prove it by induction in \(d\). For \(d = 0\) this is precisely McMullen’s conjecture which was proved in \[2\] (as an easy consequence of the Irreducibility Theorem 0.7). (Recall that McMullen’s conjecture from \[15\] says that the valuations of the form \(K \mapsto vol(K + A)\) are dense in the space \(Val(V)\).)

Let us assume that \(d > 0\). Then we have a continuous map \(PValsm_d(V) \rightarrow Val^{sm} \otimes Pol_d(V)\) which is defined as follows. For any \(\phi \in PValsm_d(V)\) we have

\[\phi(K + x) = P_d(K)(x) + \text{lower order terms}\]
where \( P_d \in (Val(V) \otimes Pol_d(V))^{sm} \). However by Lemma 1.5 below \((Val(V) \otimes Pol_d(V))^{sm} = Val^{sm}(V) \otimes Pol_d(V)\) So the map \( \phi \mapsto P_d \) is the desired map.

By the assumption of induction \( G(V) \cap PVal_d^{sm}(V) \) is dense in \( PVal_d^{sm}(V) \).

It follows from the case \( d = 0 \) that the image of \( G(V) \cap PVal_d^{sm}(V) \) in \( Val^{sm}(V) \otimes Pol_d(V) \) has a dense image. Hence the lemma follows. Q.E.D.

We need the following technical lemma.

1.2 Lemma. Let \( \phi_1, \ldots, \phi_s \in G(V) \). Then there exists a linear space \( V' \), a linear imbedding \( f : V \hookrightarrow V' \), smooth measures \( \mu_i \) on \( V' \), and \( A_1, \ldots, A_s \in K(V') \) such that \( \phi_i(K) = \mu_i(f(K) + A_i), i = 1, \ldots, s \).

Proof. By definition \( \phi_i \) has the form

\[
\phi_i(K) = \nu_i(g_i(K) + B_i)
\]

where \( g_i : V \hookrightarrow V'_i, B_i \in K(V'_i) \), and \( \nu_i \) are smooth measures on \( V'_i \). Let \( V' \) be the inductive limit of the system \( \{ V \xrightarrow{\phi_i} V'_i \}_{i=1}^{s} \). It would be convenient to have an explicit construction for it. Let \( T_i \) be a complement of \( g_i(V) \) in \( V'_i \). Then \( V' \) is isomorphic to \( V \oplus T_1 \oplus \cdots \oplus T_s \). With this decomposition \( f \) is the identity imbedding. Let us construct \( A_i, \nu_i \) say for \( i = 1 \). Note that \( V'_1 \) is isomorphic to \( V \oplus T_1 \). Let us denote also by \( B_1 \) the image of \( B_1 \) in \( V' \). Let \( S := V_2 \oplus \cdots \oplus V_s \). Let us fix a Lebesgue measure \( dvol_S \) on \( S \). Fix \( Q \in K(S) \) any set of volume 1. Set \( A_1 := B_1 \times Q \). Let \( \mu_1 := \nu_1 \otimes dvol_S \). These choices satisfy the lemma. Q.E.D.

Let \( W \) be another linear real vector space. Let us define the exterior product \( \phi \boxtimes \psi \in G(V \times W) \) of two valuations \( \phi \in G(V), \psi \in G(W) \). By Lemma 1.2 we may assume that these valuations have the form \( \phi(K) = \sum_i \mu_i(f(K) + A_i), \psi(L) = \sum_j \nu_j(g(L) + B_j) \) where \( f : V \hookrightarrow V', g : W \hookrightarrow W' \) are imbeddings, \( A_i \in K(V'), B_j \in K(W') \). Define

\[
(\phi \boxtimes \psi)(M) := \sum_{i,j} (\mu_i \otimes \nu_j)((f \times g)(M) + (A_i \times B_j)),
\]

where \( \mu_i \otimes \nu_j \) denotes the usual product measure on \( V' \times W' \).

Let us denote by \( CVal(V) \) the closure of \( G(V) \) in the Fréchet space of all continuous valuations on \( V \) with the topology of uniform convergence on compact subsets on \( K(V) \).

1.3 Proposition. (i) For \( \phi \in G(V), \psi \in G(W) \) their exterior product \( \phi \boxtimes \psi \in G(V \times W) \) is well defined.
The exterior product is bilinear with respect to each argument.

(ii) Fix $\phi \in \mathcal{G}(V)$. Then the map $\mathcal{G}(W) \to \mathcal{G}(V \times W)$ given by $\psi \mapsto \phi \boxtimes \psi$ extends (uniquely) by continuity to a map $CVal(V) \to CVal(V \times W)$.

(iv) $$(\phi \boxtimes \psi) \boxtimes \eta = \phi \boxtimes (\psi \boxtimes \eta).$$

(v) Let $f : V \hookrightarrow V_1$, $g : W \hookrightarrow W_1$ be two imbeddings. Let $\phi \in \mathcal{G}(V_1)$, $\psi \in \mathcal{G}(W_1)$. Then $$(f \times g)^*(\phi \boxtimes \psi) = f^*\phi \boxtimes g^*\psi.$$  

Proof. Note that the parts (ii), (iv), and (v) are obvious. Now let us fix a valuation $\phi \in \mathcal{G}(V)$ of the form $\phi(K) = \mu(K + A)$, $A \in V'$. From the definition of the exterior product and the Fubini theorem one easily gets the following formula:

$$((\phi \boxtimes \psi)(M) = \int_{x \in V'} \psi(((f \times g)(M) + (f(A) \times \{0\}) \cap (\{x\} \times W)) d\mu(x).$$

Thus the right hand side in the above formula does not depend on the particular form of presentation of $\psi$. Now the parts (i) and (iii) follow easily. Q.E.D.

Now let us define a product on $\mathcal{G}(V)$. Let $\Delta : V \hookrightarrow V \times V$ denote the diagonal imbedding. For $\phi, \psi \in \mathcal{G}(V)$ let

$$\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi),$$

where $\Delta^*$ denotes the restriction of a valuation on $V \times V$ to the diagonal.

1.4 Proposition. Equipped with the above defined multiplication, $\mathcal{G}(V)$ becomes an associative commutative unital algebra where the unit is the Euler characteristic $\chi$.

Proof. The associativity follows from Proposition 1.3 (iv). The commutativity is obvious. Let us prove that the Euler characteristic $\chi$ is the unit. Let a valuation $\psi$ have the form $\psi(K) = \mu(K + A)$ where $A$ is a fixed set from $\mathcal{K}(V)$. Let $\Delta : V \to V \times V$ be the diagonal imbedding. Then by the definition of the product and the Fubini theorem we have

$$(\chi \cdot \psi)(K) = \int_{x \in V} \chi((\Delta K + (A \times \{0\}) \cap (\{x\} \times V)) d\mu(x) = \mu(K + A) = \psi(K).$$

Q.E.D.

Next let us prove the following technical lemma which is well known.
1.5 Lemma. Let $F$ be a Fréchet $G$-module. Let $S$ be a finite dimensional $G$-module. Then $(F \otimes S)^{sm} = F^{sm} \otimes S$.

Proof. It is well known and easy to see that for any finite dimensional $G$-module $S$ one has $S^{sm} = S$. Hence $F^{sm} \otimes S \subset (F \otimes S)^{sm}$. To prove the opposite inclusion let us fix $\xi \in (F \otimes S)^{sm}$. Let us also fix a basis $s_1, \ldots, s_k$ of $S$. Then $\xi = \sum_{i=1}^k f_i \otimes s_i$ with $f_i \in F$. We have to show that $f_i \in F^{sm}$. Let $\{s^*_i\}$ be the dual basis in $S^*$. One has the canonical map $t : F \otimes S \otimes S^* \rightarrow F$ given by $t(f \otimes s \otimes s^*) = s^*(s) \cdot f$. Note that $t(f \otimes s_i \otimes s^*_i) = f$. Moreover $t$ commutes with the action of $G$. Hence $t((F \otimes S \otimes S^*)^{sm}) \subset F^{sm}$. We also have $(F \otimes S)^{sm} \otimes S^* \subset (F \otimes S \otimes S^*)^{sm}$. Hence $t((F \otimes S)^{sm} \otimes S^*) \subset F^{sm}$. But $f_i = t(\xi \otimes s^*_i)$. Hence $f_i \in F^{sm}$. Q.E.D.

Let us also recall two well known results we are going to use, namely the Casselman-Wallach theorem and the L. Schwartz kernel theorem.

1.6 Theorem (Casselman-Wallach, [6]). Let $G$ be a real reductive Lie group. Let $(\rho, G, F)$ and $(\pi, G, G)$ be continuous representations of $G$ of moderate growth in Fréchet spaces $F$ and $G$. Assume in addition that $G$ is an admissible $G$-module of finite length, and

$$\mathcal{F}^{sm} = \mathcal{F}, \mathcal{G}^{sm} = \mathcal{G}.$$ 

Then any continuous morphism of $G$-modules $f : \mathcal{F} \rightarrow \mathcal{G}$ has closed image.

1.7 Theorem (L. Schwartz kernel theorem, [7]). Let $X_1$ and $X_2$ be compact smooth manifolds. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be smooth finite dimensional vector bundles over $X_1$ and $X_2$ respectively. Let $\mathcal{G}$ be a Fréchet space. Let

$$B : C^\infty(X_1, \mathcal{E}_1) \times C^\infty(X_2, \mathcal{E}_2) \rightarrow \mathcal{G}$$

be a continuous bilinear map. Then there exists unique continuous linear operator

$$b : C^\infty(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2) \rightarrow \mathcal{G}$$

such that $b(f_1 \otimes f_2) = B(f_1, f_2)$ for any $f_i \in C^\infty(X_i, \mathcal{E}_i), i = 1, 2$.

Let $PV al_d(V)$ denote the space of continuous valuations on $V$ which are polynomial of degree at most $d$. Let $\Omega^n_d$ denote the (finite dimensional) space of $n$-densities on $V$ with polynomial coefficients of degree at most $d$ (clearly $\Omega^n_d$ is canonically isomorphic to $(\oplus_{i=0}^d Sym^d V^*) \otimes |\wedge^n V^*|$ where $|\wedge^n V^*|$ denotes the space of Lebesgue measures on $V$).
Let us denote by $\mathbb{P}_+(V^*)$ the manifold of oriented lines passing through the origin in $V^*$. Let $L$ denote the line bundle over $\mathbb{P}_+(V^*)$ whose fiber over an oriented line $l$ consists of linear functionals on $l$.

We are going to construct a natural linear map

$$\Theta_{k,d} : \Omega^0_d \otimes C^\infty((\mathbb{P}_+(V^*))^k, L^{2k}) \rightarrow PVal_d(V)$$

which commutes with the natural action of the group $GL(V)$ on both spaces and induces an epimorphism on the subspaces of smooth vectors.

The construction is as follows. Let $\mu \in \Omega^0_d$, $A_1, \ldots, A_k \in \mathcal{K}(V)$. Then $\int_{\sum_{j=1}^k \lambda_j A_j}^k \mu$ is a polynomial in $\lambda_j \geq 0$ of degree at most $n + d$. This can be easily seen directly, but it was also proved in general for polynomial valuations by Khovanskii and Pukhlikov [10]. Also it easily follows that the coefficients of this polynomial depend continuously on $(A_1, \ldots, A_k) \in \mathcal{K}(V)^k$ with respect to the Hausdorff metric. Hence we can define a continuous map $\Theta'_{k,d} : \Omega^0_d \times \mathcal{K}(V)^k \rightarrow PVal_d(V)$ given by

$$(\Theta'_{k,d}(\mu; A_1, \ldots, A_k))(K) := \left. \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \right|_{\lambda_j=0} \int_{K+\sum_{j=1}^k \lambda_j A_j}^k \mu.$$

It is clear that $\Theta'_{k,d}$ is Minkowski additive with respect to each $A_j$. Namely, say for $j = 1, a, b \geq 0$, one has

$$\Theta'_{k,d}(\mu; aA'_1 + bA''_1, A_2, \ldots, A_k) = a\Theta'_{k,d}(\mu; A'_1, A_2, \ldots, A_k) + b\Theta'_{k,d}(\mu; A''_1, A_2, \ldots, A_k).$$

Remind that for any $A \in \mathcal{K}(V)$ one defines the supporting functional $h_A(y) := \sup_{x \in A} (y, x)$ for any $y \in V^*$. Thus $h_A \in C(\mathbb{P}_+(V^*), L)$. Moreover it is well known (and easy to see) that $A_N \rightarrow A$ in the Hausdorff metric if and only if $h_{A_N} \rightarrow h_A$ in $C(\mathbb{P}_+(V^*), L)$. Also any section $F \in C^2(\mathbb{P}_+(V^*), L)$ can be presented as a difference $F = G - H$ where $G, H \in C^2(\mathbb{P}_+(V^*), L)$ are supporting functionals of some convex compact sets and

$$\max\{||G||_2, ||H||_2\} \leq c||F||_2$$

where $c$ is a constant. (Indeed one can choose $G = F + R \cdot h_D$, $H = R \cdot h_D$ where $D$ is the unit Euclidean ball, and $R$ is a large enough constant depending on $||F||_2$.) Hence we can uniquely extend $\Theta'_{s,d}$ to a multilinear continuous map (which we will denote by the same letter):

$$\Theta'_{k,d} : \Omega^0_d \times (C^2(\mathbb{P}_+(V^*), L))^k \rightarrow PVal_d(V).$$

By the L. Schwartz kernel theorem (Theorem [1.7]) it follows that this map gives rise to a continuous linear map

$$\Theta_{k,d} : \Omega^0_d \otimes C^\infty(\mathbb{P}_+(V^*))^k, L^{2k}) \rightarrow PVal_d^{sm}(V).$$
which we wanted to construct.

We will study this map $\Theta_{k,d}$. Note that it depends on $k$ and $d$ which will be fixed from now on. Let us denote by $\Theta_d$ the sum of the maps $\bigoplus_{k=0}^{n-1} \Theta_{k,d}$. Thus $\Theta_d : \Omega_n^d \otimes \bigoplus_{k=0}^{n-1} C^\infty(\mathbb{P}_+(V^*)^k, L^{2k}) \longrightarrow PVal_d^{sm}(V)$.

1.8 Lemma. The map $\Theta_d$ has a dense image.

1.9 Corollary. The map $\Theta_d$ is an epimorphism.

This corollary follows immediately from Lemma 1.8 and the Casselman-Wallach theorem (Theorem 1.6). So let us prove Lemma 1.8.

Proof of Lemma 1.8. We will prove it by induction in $d$. For $d = 0$ this is just McMullen’s conjecture proved in [2]. Assume that the statement of Lemma 1.8 holds for $d - 1$. It is sufficient to show that the map $\text{Sym}^d(V^*) \otimes \Lambda^n V^* \otimes \bigoplus_{k=0}^{n-1} C^\infty(\mathbb{P}_+(V^*)^k, L^{2k}) \rightarrow (PVal_d(V)/PVal_{d-1}(V))^{sm}$ is onto. However the last space is equal to $(\text{Sym}^d(V^*) \otimes Val(V))^{sm} = \text{Sym}^d(V^*) \otimes Val^{sm}(V)$ (using Lemma 1.5). Thus we have reduced the claim again to the case $d = 0$. Q.E.D.

1.10 Proposition. The multiplication

$$(PVal_i(V))^{sm} \otimes (PVal_j(V))^{sm} \longrightarrow (PVal_{i+j}(V))^{sm}$$

is continuous.

Proof. Since this map commutes with the action of $GL(V)$ it is sufficient to prove that the map

$$(PVal_i(V))^{sm} \otimes (PVal_j(V))^{sm} \longrightarrow PVal_{i+j}(V)$$

is continuous. In fact we will prove a bit more. We will prove that the exterior product of valuations is a continuous bilinear map

$$(PVal_i(V))^{sm} \times (PVal_j(W))^{sm} \longrightarrow PVal_{i+j}(V \times W).$$

For the simplicity of notation we will assume that $W = V$.

Let us fix a large natural number $N \geq 2$. Let $\phi \in (PVal_i(V))^{sm}$, $\psi \in (PVal_j(V))^{sm}$ be such that $\phi = \sum p \omega_p \otimes l_p$, $\psi = \sum q \omega'_q \otimes l'_q$ where $\omega_p \in \Omega^n_i$, $\omega'_q \in \Omega^n_j$, $l_p, l'_q \in \bigoplus_{k=0}^{n-1} C^\infty(\mathbb{P}_+(V^*)^k, L^{2k})$ and

$$\max\{\sum_p ||\omega_p||, \sum_q ||\omega'_q||, \sum_p ||l_p||_{2N}, \sum_q ||l'_q||_{2N}\} \leq 1$$
where \( ||\cdot|| \) is a norm on \( \Omega_n^p \), and \( ||\cdot||_N \) is a norm on \( \bigoplus_{k=0}^{n-1} C^{2N}(\mathbb{P}_+(V^*)^k, L^2) \).

Then it is easy to see that for a constant \( C \) (depending on \( n \) and \( k \) only) such that one can write

\[
l_p = \sum_{s=1}^{\infty} \gamma_{s,p,1} \otimes \cdots \otimes \gamma_{s,p,k}, \quad l'_q = \sum_{s=1}^{\infty} \gamma'_{s,q,1} \otimes \cdots \otimes \gamma'_{s,q,k}
\]

where \( \gamma_{s,p,i}, \gamma'_{s,q,i} \in C^\infty(\mathbb{P}_+(V^*), L) \) and

\[
\max \left\{ \sum_{s} \sum_{k=0}^{n-1} \prod_{t=1}^{k} ||\gamma_{s,p,t}||_N, \sum_{s} \sum_{k=0}^{n-1} \prod_{t=1}^{k} ||\gamma'_{s,q,t}||_N \right\} < C
\]

where \( C \) is a constant which might be different from the previous \( C \). Every \( \gamma \in C^\infty(\mathbb{P}_+(V^*), L) \) can be written as \( \gamma = \gamma_1 - \gamma_2 \) such that \( \gamma_1, \gamma_2 \) are supporting functionals of convex sets and \( \max \{||\gamma_1||_N, ||\gamma_2||_N\} \leq C||\gamma||_N \) (using the same argument as in the construction of the maps \( \Theta_{k,d} \)). Hence we may assume that all \( \gamma_{s,p,t}, \gamma'_{s,q,t} \) are supporting functionals of convex sets \( A_{s,p,t}, A'_{s,q,t} \) respectively. Let \( \Omega_{p,q} := \omega_p \wedge \omega'_q \); it is a top-degree form on \( V \times V \).

Then by the definition of the exterior product of valuations

\[
(\phi \boxtimes \psi)(K) = \sum_{s=1}^{\infty} \sum_{k=0}^{n-1} \sum_{p,q} \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \frac{\partial^{k'}}{\partial \mu_1 \cdots \partial \mu_{k'}} \left| \int_{0}^{1} \right| K + \sum_{m=1}^{k} \lambda_m (A_{s,p,m} \times 0) + \sum_{m'=1}^{k'} \mu_{m'} (0 \times A'_{s,q,m'}) \cdot \Omega_{p,q}.
\]

Note that

\[
\frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \frac{\partial^{k'}}{\partial \mu_1 \cdots \partial \mu_{k'}} \left| \int_{0}^{1} \right| K + \sum_{m=1}^{k} \lambda_m (A_{s,p,m} \times 0) + \sum_{m'=1}^{k'} \mu_{m'} (0 \times A'_{s,q,m'}) \cdot \Omega_{p,q} = \prod_{m} ||\gamma_{s,p,m}||_0 \prod_{m'} ||\gamma_{s,q,m'}||_0 \cdot \prod_{m} ||\gamma'_{s,p,m}||_0 \prod_{m'} ||\gamma'_{s,q,m'}||_0 \cdot \Omega_{p,q}.
\]

The last integral is a polynomial in \( \lambda_m, \mu_{m'} \) of degree at most \( i + j + 2n \). Hence in order to get an estimate on its derivatives it is sufficient to estimate the polynomial itself for all \( \lambda_m, \mu_{m'} \) of absolute value at most 1. If the set \( K \) is
uniformly bounded then the whole Minkowski sum of sets under the integral is uniformly bounded. Hence all the integrals are uniformly bounded, and we get

\[ \left| (\phi \boxtimes \psi)(K) \right| \leq C \sum_{s=1}^{\infty} \sum_{k,k'=0}^{n-1} \left( \prod_{m} \| \gamma_{s,p,m} \| \prod_{m'} \| \gamma_{s,q,m'} \| \right). \]

The last sum is estimated by a constant. Q.E.D.

2 Translation invariant valuations.

In this section we discuss the subalgebra \( (Val(V))^{sm} \) of smooth translation invariant valuations. It has a structure of a Frobenius algebra, namely it satisfies a version of the Poincaré duality with respect to the natural grading.

Remind that by a result of Hadwiger [9] the space \( Val_n(V) \) is one dimensional and it is spanned by Lebesgue measure. First observe that the multiplication \( (Val^0_n(V))^{sm} \otimes (Val^1_{n-i}(V))^{sm} \to Val_n(V) \) is trivial since the product of even and odd valuations must be odd, and all valuations of the maximal degree of homogeneity are even. The main result of this section is

2.1 Theorem. (i) The map \( (Val^0_n(V))^{sm} \otimes (Val^0_{n-i}(V))^{sm} \to Val_n(V) \) is a perfect pairing. More precisely the induced map

\[ (Val^0_i(V))^{sm} \to (Val^0_{n-i}(V)^*)^{sm} \otimes Val_n(V) \]

is an isomorphism.

(ii) For \( 1 \leq i \leq n-1 \), the map \( (Val^1_i(V))^{sm} \otimes (Val^1_{n-i}(V))^{sm} \to Val_n(V) \) is a perfect pairing in the above sense.

Before we prove the theorem we will need a proposition which is of independent interest.

2.2 Proposition. Let \( V \) be an \( n \)-dimensional Euclidean space. Let

\[ \phi(K) = V(K[i], A_1, \ldots, A_{n-i}), \psi(K) = V(K[n-i], B_1, \ldots, B_i) \]

where \( A_p, B_q \in K(V) \) are fixed. Then

\[ (\phi \cdot \psi)(K) = \binom{n}{i}^{-1} V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) vol(K). \]
First we have the following simple identity.

2.3 Claim.

\[ V(K[i], A_1, \ldots, A_{n-i}) = (n(n-1) \cdots (i+1))^{-1} \left. \frac{\partial^{n-i}}{\partial \lambda_1 \cdots \partial \lambda_{n-i}} \right|_0 \text{vol}(K + \sum_{j=1}^{n-i} \lambda_j A_j). \]

**Proof** of Proposition 2.2. Using Claim 2.3 we have

\[ (\phi \cdot \psi)(K) = (n(n-1) \cdots (i+1))^{-1} (n(n-1) \cdots (n-i+1))^{-1}. \]

\[ \left. \frac{\partial^{n-i}}{\partial \lambda_1 \cdots \partial \lambda_{n-i}} \right|_0 \left. \frac{\partial^i}{\partial \mu_1 \cdots \partial \mu_i} \right|_0 \text{vol}_{2n}(\Delta(K) + \sum_{j=1}^{n-i} \lambda_j (A_j \times 0) + \sum_{l=1}^i \mu_l (0 \times B_l)) \]

where \( \Delta : V \hookrightarrow V \times V \) is the diagonal imbedding. Again by Claim 2.3

\[ (2n \cdots (n+1)) \cdot V_{2n}(\Delta(K)[n]; A_1 \times 0, \ldots, A_{n-i} \times 0; 0 \times B_1, \ldots, 0 \times B_i). \]

Hence we obtain

\[ (\phi \cdot \psi)(K) = \binom{2n}{n} \binom{n}{i}^{-1} V_{2n}(\Delta(K)[n]; A_1 \times 0, \ldots, A_{n-i} \times 0; 0 \times B_1, \ldots, 0 \times B_i). \]

We will need a lemma.

2.4 Lemma. Let \( X = Y \oplus Z \) be an orthogonal decomposition of a Euclidean space \( X \). Let \( \dim X = N, \dim Y = n \). Let \( M \in K(Y), A_1, \ldots, A_{N-n} \in K(X) \).

Then

\[ V_N(M[n]; A_1, \ldots, A_{N-n}) = \binom{N}{n}^{-1} \text{vol}_n(M) V_{N-n}(Pr_Z A_1, \ldots, Pr_Z A_{N-n}) \]

where \( Pr_Z \) denotes the orthogonal projection onto \( Z \).

Let us postpone the proof of this lemma and continue proving Proposition 2.2. In our situation \( X = V \times V, Y = \Delta(V) = \{(x, x)\}, Z = \{(x, -x)\} \). Note that

\[ Pr_Z((x, 0)) = \frac{1}{2}(x, -x), \quad Pr_Z((0, x)) = \frac{1}{2}(-x, x). \]
Let us denote by $\Delta'$ the imbedding $V \hookrightarrow V \times V$ given by $\Delta'(x) = (x, -x)$.

Using Lemma 2.4 we get

$$(\phi \cdot \psi)(K) = (\binom{n}{i}^{-1} \nu_n(\Delta(K)))\nu_n\left(\frac{\Delta'A_1}{2}, \ldots, \frac{\Delta'A_{n-i}}{2}, \frac{-\Delta'B_1}{2}, \ldots, \frac{-\Delta'B_i}{2}\right)$$

$$= \left(\binom{n}{i}^{-1} V(A_1, \ldots, A_{n-i}, -B_1, \ldots, -B_i) \nu(K).$$

Q.E.D.

**Proof** of Lemma 2.4. We may assume that $A_1 = \cdots = A_{N-n} = A$. We have

$$V_N(M[n], A[N-n]) = \frac{(N-n)!}{N!} \frac{d^n}{d\varepsilon^n}|_0 \nu_N(A + \varepsilon M).$$

We have

$$\nu_N(A + \varepsilon M) = \int_{z \in Z} \nu_N((A + \varepsilon M) \cap (z + Y)) dz$$

$$= \int_{z \in Z} \nu_N((A \cap (z + Y)) + \varepsilon M) dz$$

$$= \int_{z \in Pr_Z A} (\varepsilon^n \nu_n M + O(\varepsilon^{n-1})) dz$$

$$= \varepsilon^n \nu_{N-n}(Pr_Z A) \nu_n M + O(\varepsilon^{n-1}).$$

This proves Lemma 2.4. Q.E.D.

**Proof** of Theorem 2.1. The kernel of the map

$$(Val^0_i(V))^{sm} \rightarrow (Val^0_{n-i}(V)^*)^{sm} \otimes Val_n(V)$$

is $GL(V)$-invariant subspace (and similarly in the odd case). Hence by the Irreducibility Theorem 0.7 it must be either trivial or coincide with the whole space. We will show that the multiplication in non-trivial. Assume that it is proved. Then by the Irreducibility Theorem 0.7 the image the above map is a dense subspace, and hence it is equal to the whole space by the Casselman-Wallach theorem. Thus it remains to check in both cases that the product is non-zero.

First let us check it in the even case. By Proposition 2.2 the product of the intrinsic volumes $V_i \cdot V_{n-i} \neq 0.$
Now let us consider the odd case. We want to show that the multiplication

\((Val^1(V))^{sm} \otimes (Val^{1-i}(V))^{sm} \rightarrow Val^{n}(V)\)

is non-zero. We have

\((Val^\varepsilon(V))^{sm} \otimes (Val^\delta(V))^{sm} \rightarrow (Val^{\varepsilon+\delta}(V))^{sm}\)

where the addition of upper indexes is understood modulo 2.

2.5 Lemma. The multiplication

\((Val^\varepsilon(V))^{sm} \otimes (Val^\varepsilon(V))^{sm} \rightarrow (Val^0_2(V))^{sm}\)

is non-zero for \(\varepsilon \in \mathbb{Z}/2\mathbb{Z}\) and hence has a dense image.

First let us finish proving Theorem 2.1 assuming this lemma. Consider the multiplication maps (we will omit for brevity the upper sign \(sm\)):

\(Val^1(V) \otimes (Val^0(V))^{\otimes(i-1)} \rightarrow Val^1(V)\);

\(Val^1 \otimes (Val^0)^{\otimes(n-1-i)} \rightarrow Val^{n-i}\).

It is sufficient to showing that the composition

\((Val^1(V) \otimes (Val^0(V))^{\otimes(i-1)}) \otimes (Val^1(V) \otimes (Val^0(V))^{\otimes(n-1-i)}) \rightarrow Val^{n}(V)\)

is non-zero. This is equivalent to show that

\(Val^1(V) \otimes Val^1(V) \otimes (Val^0(V))^{\otimes(n-2)} \rightarrow Val^{n}(V)\)

is non-zero. Note that the image \((Val^0(V))^{\otimes(n-2)} \rightarrow Val^{0}_{n-2}(V)\) is non-zero since the power of the intrinsic volume \((V_1)^{n-2}\) does not vanish. Hence this image is a dense subspace in \(Val^0_{n-2}(V)\). Hence it is sufficient to show that the map

\(Val^1(V) \otimes Val^1(V) \otimes Val^0_{n-2}(V) \rightarrow Val^{n}(V)\)

is non-zero. By Lemma 2.5 the map \(Val^1(V) \otimes Val^1(V) \rightarrow Val^0_2(V)\) has a dense image. As we have previously proved the map \(Val^0_2(V) \otimes Val^0_{n-2}(V) \rightarrow Val^{n}(V)\) is non-zero. This immediately implies Theorem 2.1. Q.E.D.

Proof of Lemma 2.5. First let reduce the statement to the case \(n = 2\). Let us fix a 2-dimensional subspace \(W \subset V\). It is easy to see that the restriction map \(Val(V) \rightarrow Val(W)\) is non-zero and its image is \(GL(W)\)-invariant. Hence by the Irreducibility Theorem the image of this restriction is
a dense subspace. Moreover the multiplication commutes with the restriction by Proposition 1.3(v). Hence it is sufficient to show that

\[ \text{Val}_1^1(W) \otimes \text{Val}_1^1(W) \longrightarrow \text{Val}_2(W) \]

is non-zero. By Proposition 2.2 we have for \( A, B \in K(W) \):

\[ V(\cdot, A) \cdot V(\cdot, B) = \kappa V(A, -B) \cdot \text{vol}(\cdot), \]

where \( \kappa \) is a non-zero normalizing constant.

Now let us choose valuations \( \phi(\cdot) := V(\cdot, A) - V(\cdot, -A), \psi(\cdot) := V(\cdot, B) - V(\cdot, -B) \) with \( A \) and \( B \) to be chosen later. Then we get

\[ (\phi \cdot \psi)(K) = 2\kappa(V(A, B) - V(A, -B)) \cdot \text{vol}(K). \]

If we choose \( A \) and \( B \) so that \( V(A, B) \neq V(A, -B) \) then the above product does not vanish. Q.E.D.

For any subgroup \( G \) of the group of linear transformations of the space \( V \) let us denote by \( \text{Val}_G(V) \) the space of translation invariant continuous valuations invariant with respect to \( G \). Let now \( V \) be a Euclidean space of dimension \( n \). Also \( \text{Val}_G^i(V) \) denote the subspace of \( \text{Val}_G(V) \) of \( i \)-homogeneous valuations. Then it immediately follows from McMullen's theorem [14] that \( \text{Val}_G(V) = \bigoplus_{i=0}^{n} \text{Val}_G^i(V) \). Let \( O(n) \) denote the full orthogonal group, and \( SO(n) \) denote the special orthogonal group. Let \( V_i \) denote the \( i \)th intrinsic volume (see e.g. [18]). It was proved by Hadwiger that for any \( i \) \( \text{Val}_i^{O(n)} = \text{Val}_i^{SO(n)} = C \cdot V_i \). Now we describe the algebra structure on the space \( \text{Val}^{O(n)}(V) \).

**2.6 Theorem.** There exists an isomorphism of graded algebras \( \mathbb{C}[x]/(x^{n+1}) \simeq \text{Val}^{O(n)}(V) \) given by \( x \mapsto V_1 \).

**Proof.** For any \( i \) the valuation \((V_i)^i \in \text{Val}_i^{O(n)} \). Hence it must be proportional to \( V_i \). By Proposition 2.2 the constant of proportionality does not vanish. This implies the result. Q.E.D.

Now let us prove part (iv) of Theorem 0.9. Namely let us prove that if \( G \) is a compact subgroup of \( GL(V) \) acting transitively on the projective space \( \mathbb{P}_+(V) \) then \( \text{Val}_G^1(V) = C \cdot V_1 \) and \( \text{Val}_G^{n-1} = C \cdot V_{n-1} \). Here we assume that we consider the intrinsic volumes \( V_i \) with respect to a Euclidean metric on \( V \) invariant under \( G \) (which is unique up to a proportionality). It was
proved by McMullen [15] that every translation invariant continuous \((n-1)\)-homogeneous valuation \(\phi\) has the form
\[
\phi(K) = \int_{S^{n-1}} f(x) dS_{n-1}(K, x)
\]
where \(dS_{n-1}(K, \cdot)\) denotes the \((n-1)\)-area measure of \(K\) (see [18]), and \(f\) is a continuous function on the sphere \(S^{n-1}\). Moreover \(f\) can be chosen to be orthogonal (with respect to the Lebesgue measure on \(S^{n-1}\)) to any linear functional on the sphere, and under this restriction \(f\) is defined uniquely by \(\phi\). Hence in our situation \(f\) can be chosen \(G\)-invariant. If \(G\) acts transitively on the projective space then it acts transitively on the sphere. Hence \(f\) must be constant. Hence \(\phi \in C \cdot V_{n-1}\).

Now let us consider the case of \(Val^G(V)\). The result follows from the previous case and the Poincaré duality. A more elementary way to see it is as follows. Let us construct a canonical imbedding \(Val_1(V) \hookrightarrow C^{-\infty}(P_+(V), L^* \otimes |\omega|)\) where the target space is the space of generalized sections of the line bundle \(L^* \otimes |\omega|\) over \(P_+(V)\), where \(L\) was defined in Section 1, and \(|\omega|\) is the line bundle of densities over \(P_+(V)\). This imbedding (in fact in a more general situation) was first considered by Goodey and Weil [8]; now we essentially reproduce their argument. It is well known that any valuation \(\phi \in Val_1(V)\) is Minkowski additive, i.e. \(\phi(\lambda A + \mu B) = \lambda \phi(A) + \mu \phi(B)\) for all \(A, B \in K(V)\) and \(\lambda, \mu \geq 0\). Note that the supporting functional of any convex compact set is a continuous section of the line bundle \(L\) over \(P_+(V)\) defined in Section 1. However any \(C^\infty\)-section \(F\) of \(L\) can presented as a difference of two smooth supporting functionals of convex compact sets, \(F = G - H\) such that \(\max\{\|G\|_2, \|H\|_2\} \leq \|F\|_2\). Hence using the Minkowski additivity of \(\phi\) we can extend it uniquely to a continuous linear functional on \(C^\infty(P_+(V), L)\). Clearly we get a continuous imbedding \(Val_1(V) \hookrightarrow C^{-\infty}(P_+(V), L^* \otimes |\omega|)\).

However if the group \(G\) acts transitively on the sphere then the space of \(G\)-invariant (generalized) sections is at most one dimensional. It follows that \(Val^G_1(V) = C \cdot V_1\). Q.E.D.

### 3 Filtrations on polynomial valuations.

Let us define the following filtration on \(PVal^{sm}(V)\):
\[
\gamma_i = \{\phi \in PVal^{sm}(V) | \phi(K) = 0 \forall K \text{ such that } \dim K < i\}.
\]
Then $PVal_{sm}(V) = \gamma_0 \supset \gamma_1 \supset \cdots \supset \gamma_n \supset \gamma_{n+1} = \{0\}$. Note that

\[
\gamma_i \cap Val_{sm}(V) = (Val_{i-1}^{sm}(V))^{sm} \oplus Val_{i}^{sm}(V) \oplus \cdots \oplus Val_{n}^{sm}(V).
\]

Let us introduce another filtration on $(PVal)^{sm}$. Set

\[
W_i(PVal_{d}^{sm}) := \sum_{k=0}^{n-i} \text{Im}(\Theta_{k,d})
\]

where the maps $\Theta_{k,d}$ were defined in Section 1. We have

\[
PVal_{d}^{sm} = W_0(PVal_{d}^{sm}) \supset W_1(PVal_{d}^{sm}) \supset \cdots \supset W_n(PVal_{d}^{sm}).
\]

Moreover $W_n(PVal_{d}^{sm})$ coincides with densities on $V$ polynomial of degree at most $d$.

3.1 Lemma. For any $d$

\[
W_i(PVal_{d}^{sm}) \cap Val^{sm}(V) = Val_{i}^{sm}(V) \oplus Val_{i+1}^{sm}(V) \oplus \cdots \oplus Val_{n}^{sm}(V).
\]

Proof. Obviously we have an inclusion

\[
Val_{i}^{sm}(V) \oplus Val_{i+1}^{sm}(V) \oplus \cdots \oplus Val_{n}^{sm}(V) \subset W_i(PVal_{d}^{sm}) \cap Val^{sm}(V).
\]

Since $W_i(PVal_{d}^{sm}) \subset \gamma_i$ (by Proposition 3.7(i) below) we have

\[
W_i(PVal_{d}^{sm}) \cap Val^{sm}(V) \subset ((Val_{i}^{sm}(V) \oplus Val_{i+1}^{sm}(V) \oplus \cdots \oplus Val_{n}^{sm}(V)) \oplus Val_{i-1}^{1}(V).
\]

It is sufficient to show that $W_i(PVal_{d}^{sm}) \cap Val_{i-1}^{1} = 0$. Assume that $\phi \in W_i(PVal_{d}^{sm}) \cap Val_{i-1}^{1}(V), \phi \neq 0$. By the Poincaré duality there exists $\psi \in (Val_{n-1+i}^{sm}(V))^{sm}$ such that $\phi \cdot \psi \neq 0$. We may also assume that $\psi(K) = vol(K[n-i+1],A[i-1])$ where $A \in K(V)$ is fixed. By Proposition 1.3(iii) for fixed $\psi$ the map $CVal(V) \rightarrow CVal(V)$ given by $\xi \mapsto \xi \cdot \psi$ is a continuous map (with the topology of uniform convergence on compact subsets of $K(V)$). By using the construction of multiplication one easily checks the following claim.

3.2 Claim.

$\psi \cdot W_i(PVal_{d}^{sm}) = 0$.

Hence by continuity it follows that $\psi \cdot W_i(PVal_{d}^{sm}) = 0$. We get a contradiction. Q.E.D.
3.3 Proposition. $W_i(PVal_d^{sm})$ is a closed subspace of $PVal_d^{sm}$.

Proof. The statement follows from the Casselman-Wallach theorem (Theorem 1.6) and the fact that the target of the map $\Theta_{k,d}$ is an admissible $GL(V)$-module of finite length. Q.E.D.

3.4 Proposition. Let $\phi \in PVal_d^{sm}$. If $\phi \in W_i(PVal_d^{sm})$ for some $d' \geq d$ then

$$\lim_{r \to 0^+} \frac{1}{r^{i-1}} \phi(rK + x) = 0 \forall x \in X, \forall K \in K(V).$$

Conversely if the condition $(\star)$ holds then $\phi \in W_i(PVal_d^{sm})$.

Proof. Let us assume that $\phi \in PVal_d^{sm}$. Then

$$\int_{rK + x_0 + \sum_{j=1}^{n-i} \lambda_j A_j} F(x) dx = F(x_0) \text{vol}(rK + \sum_{j=1}^{n-i} \lambda_j A_j) + \text{higher order terms}.$$ 

The condition $(\star)$ follows.

Let us prove the other direction. Let $\phi \in PVal_d^{sm}$ satisfy $(\star)$. We will prove the result by induction in $d$. For $d = 0$ the statement is clear. Let us assume now that the statement is true for valuations of degree of polynomiality less than $d$. We have

$$\phi(K + x) = P_d(K)(x) + \text{lower order terms}.$$ 

Thus $P_d \in Val^{sm} \otimes Pol_d$. From the condition $(\star)$ we get:

$$0 = \lim_{r \to 0^+} \frac{1}{r^{i-1}} \phi(rK + x) = \lim_{r \to 0^+} \frac{1}{r^{i-1}} P_d(rK)(x) + \cdots.$$ 

Hence $P_d \in (Val_i^{sm} \oplus \cdots \oplus Val_n^{sm}) \otimes Pol_d$. It follows from the Casselman-Wallach theorem that there exists $\psi \in W_i(PVal_d^{sm})$ such that $\psi(K + x) = P_d(K)(x) + \cdots$. Applying the assumption of induction to the valuation $\phi - \psi$ we prove the statement. Q.E.D.

We get immediately the following corollary.

3.5 Corollary. (i) For $d' > d$ $W_i(PVal_d^{sm}) \cap PVal_d^{sm} = W_i(PVal_d^{sm})$.

(ii) Let $f : U \to V$ be an injective imbedding of linear spaces. Then $f^*(W_i(PVal_d^{sm}(V))) \subset W_i(PVal_d^{sm}(U))$.

Using part (i) of this corollary let us define the filtration $W_i$ on all smooth polynomial valuations $PVal^{sm}$ so that $W_i \cap PVal_d^{sm} = W_i(PVal_d^{sm})$. 

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3.6 Theorem.

\[ W_i(PVal_{d_1}^{sm}) \otimes W_i(PVal_{d_2}^{sm}) \subset W_{i+i_2}(PVal_{d_1+d_2}^{sm}). \]

**Proof.** It is clear from the definitions that

\[ W_i(PVal_{d_1}^{sm}(V)) \otimes W_i(PVal_{d_2}^{sm}(W)) \subset W_{i+i_2}(PVal_{d_1+d_2}(V \times W)). \]

Now the statement follows from Corollary 3.5(ii). Q.E.D.

3.7 Proposition. (i) For any \( i \)

\[ \gamma_{i+1} \subset W_i \subset \gamma_i. \]

(ii) \( W_1 = \gamma_1. \)

**Proof.** Let us prove part (i). Let us prove first the inclusion \( \gamma_{i+1} \subset W_i \). Let \( \phi \in \gamma_{i+1}(PVal_{d}^{sm}) \). We will prove the statement by induction in \( d \). Assume first that \( d = 0 \). Then \( \gamma_{i+1}(Val^{sm}) = (Val_{i}^{1})^{sm} \oplus Val_{i+1}^{sm} \oplus \cdots \subset W_{i}(Val^{sm}) = Val_{i}^{sm} \oplus Val_{i+1}^{sm} \oplus \cdots. \)

Now let us assume that the statement holds for valuations polynomial of degree less than \( d \). Let us prove it for \( d \). We have

\[ \phi(K + x) = P_d(K)(x) + \text{lower order terms}. \]

Thus \( P_d \in \gamma_{i+1}(Val^{sm}) \otimes Pol_{d} \subset W_{i}(Val^{sm}) \otimes Pol_{d} \). It follows from the case \( d = 0 \) and the Casselman-Wallach theorem that there exists \( \psi \in W_{i}(PVal_{d}^{sm}) \cap \gamma_{i+1}(PVal_{d}^{sm}) \) such that

\[ \psi(K + x) = P_d(K)(x) + \text{lower order terms}. \]

Applying the induction assumption to the valuation \( \phi - \psi \) we obtain the result.

The second inclusion \( W_i \subset \gamma_i \) follows from the fact that if \( \dim K < i \) then

\[ \int_{K} \sum^{n-i}_{j=1} \lambda_j A_j \mu = O(\lambda^{n-i+1}) \text{ where } \lambda := \sqrt{\sum^{n-i}_{j=1} \lambda_j^2} \rightarrow 0. \]

Let us prove part (ii). It remains to prove the inclusion \( \gamma_1 \subset W_1 \). Assume that \( \phi \in \gamma_1 \cap PVal_{d}^{sm} \). We will prove by induction in \( d \) that \( \phi \in W_1 \). For \( d = 0 \) the statement is clear. As previously we can write

\[ \phi(K + x) = P_d(K)(x) + \text{lower order terms} \]
with $P_d \in \gamma_1(Val^{sm}) \otimes Pol_d = W_1(Val^{sm}) \otimes Pol_d$. Again the Casselman-Wallach theorem implies that there exists $\psi \in W_1(PVal_d^{sm})$ such that

$$\psi(K + x) = P_d(K)(x) + \text{lower order terms}.$$

Applying the assumption of induction to the valuation $\phi - \psi$ we prove the result. Q.E.D.

The next theorem gives an axiomatic characterization of the filtration $W_i$.

3.8 Theorem. Assume that we have another filtration $\{\tilde{W}_i\}$ on $PVal_d^{sm}$ such that

1. $\{\tilde{W}_i\}$ is compatible with the multiplicative structure, i.e. $\tilde{W}_i \cdot \tilde{W}_j \subset \tilde{W}_{i+j}$;
2. $\gamma_{i+1} \subset \tilde{W}_i \subset \gamma_i$ for all $i$.
3. $W_0 = \gamma_0$, $\tilde{W}_1 = \gamma_1$;
4. $\tilde{W}_i \cap PVal_d^{sm}$ is a closed subspace of $PVal_d^{sm}$ for all $i, d$.
5. $\tilde{W}_i$ is $Aff(V)$-invariant.

Then $\tilde{W}_i = W_i$.

Proof. One has to prove that $\tilde{W}_i \cap PVal_d^{sm} = W_i(PVal_d^{sm})$ for all $i, d$. As previously the general case reduces easily to the case $d = 0$. Let us prove the statement in this case. We may assume that $i > 1$. Remind that $\gamma_1 = \tilde{W}_1$. Also $\tilde{W}_i \cdot \tilde{W}_1 \subset \tilde{W}_{i+1}$. Hence $W_{i+1} \supset (Val_1^{0})^{sm}$. Hence $\gamma_{i+1} \supset (Val_1^{0})^{sm}$. This is a contradiction.

Let us consider the associated graded algebra $gr_W(PVal^{sm}) := \bigoplus_{i=0}^{n} W_i/W_{i+1}$. The next result gives a description of it.
3.9 Theorem. There exists a canonical isomorphism of graded algebras

\[ \text{gr}_W(P\text{Val}^{sm}(V)) \simeq \text{Val}^{sm}(V) \otimes \mathbb{C}[V] \]

where the ith graded term in the right side is equal to \( \text{Val}^{sm}_i(V) \otimes \mathbb{C}[V] \) where \( \mathbb{C}[V] \) denote the algebra of polynomial functions on \( V \).

Proof. Let us construct the isomorphism explicitly. Let us define a map

\[ Q : W_i \longrightarrow \text{Val}^{sm}_i \otimes \mathbb{C}[V] \]

by \( Q(\phi)(K)(x) = \lim_{r \to 0} r^{-i} \phi(rK + x) \). First notice that \( Q(\phi) \) is indeed a translation invariant valuation. Let us check it say for \( x = 0 \). We have

\[ \phi(r(K + a)) = r^iQ(\phi)(K + a)(0) + o(r^i). \]

On the other hand

\[ \phi(r(K + a)) = \phi(rK + ra) = r^iQ(\phi)(K)(ra) + o(r^i) = r^iQ(\phi)(K)(0) + o(r^i). \]

Hence \( Q(\phi)(K + a)(0) = Q(\phi)(K)(0) \).

Next the kernel of \( Q \) is equal to \( W_{i+1} \) by Proposition 3.4. Let us show that \( Q \) is surjective. Let us check that for any \( d \) the map

\[ Q : W_i(P\text{Val}^{sm}_d) \longrightarrow \text{Val}^{sm}_i \otimes \mathbb{C}[V]_{\leq d} \]

is surjective (where \( \mathbb{C}[V]_{\leq d} \) denote the space of polynomials of degree at most \( d \)). Both spaces are admissible Fréchet representations of \( GL(V) \) of finite length. Hence by the Casselman-Wallach theorem it is sufficient to show that \( Q \) has dense image. Let \( \phi \in W_i(P\text{Val}^{sm}_d) \) be a valuation of the form

\[ \phi(K) = \frac{\partial^{n-i}}{\partial \lambda_1 \ldots \partial \lambda_{n-i}} \left| \lambda_j = 0 \int_{K + \sum_j \lambda_j A_j} f(x) \, dx \right. \tag{4} \]

where \( A_1, \ldots, A_{n-i} \in \mathcal{K}(V) \) are fixed, and \( f \) is a polynomial of degree at most \( d \). Then it is easy to see that \( Q(\phi)(K)(x) = f(x) \frac{\partial^{n-i}}{\partial \lambda_1 \ldots \partial \lambda_{n-i}} \left| \lambda_j = 0 \right. \text{vol}(K + \sum_j \lambda_j A_j) \). Since by the (proved) McMullen conjecture the valuations of the form \( \frac{\partial^{n-i}}{\partial \lambda_1 \ldots \partial \lambda_{n-i}} \left| \lambda_j = 0 \right. \text{vol}(K + \sum_j \lambda_j A_j) \) are dense in \( \text{Val}_i \), we deduce that \( Q \) has dense image.

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It remains to prove that $Q$ is a homomorphism of algebras. Let $\phi \in W_i(PVal_d^{sm})$ be a representative of an element from $W_i/W_{i+1}$ of the form (4), and $\psi \in W_j(PVal_{d'}^{sm})$ be a representative of an element of $W_j/W_{j+1}$ of the form
\[
\frac{\partial^{n-j}}{\partial \mu_1 \cdots \partial \mu_{n-j}} \bigg|_{\mu=0} \int_{K+\sum_{i=1}^{n-j} \mu_i B_i} g(x)dx
\]
where $B_1, \ldots B_{n-j} \in K(V)$ are fixed, and $g$ is a polynomial of degree at most $d'$. Then
\[
(\phi \boxtimes \psi)(K) = \int_{K+\sum_{m} \lambda_m(A_m \times \{0\}) + \sum_{i} \mu_i(\{0\} \times B_i)} f(x)g(y)dxdy.
\]
Then clearly
\[
Q(\phi \boxtimes \psi)(K)((x,y)) = \int_{K+\sum_{m} \lambda_m(A_m \times \{0\}) + \sum_{i} \mu_i(\{0\} \times B_i)} \frac{\partial^{n-i}}{\partial \lambda_1 \cdots \partial \lambda_{n-i}} \frac{\partial^{n-j}}{\partial \mu_1 \cdots \partial \mu_{n-j}} \bigg|_{0} vol(K + \sum_{m} \lambda_m(A_m \times \{0\}) + \sum_{i} \mu_i(\{0\} \times B_i)) = (Q(\phi) \boxtimes Q(\psi))(K)((x,y)).
\]
The result follows. Q.E.D.

4 Further remarks.

In Theorem 2.6 of this paper we have described the algebra $Val_O^{(n)}(\mathbb{R}^n) = Val_{SO}^{(n)}(\mathbb{R}^n)$ of isometry invariant continuous valuations on an $n$-dimensional Euclidean space $\mathbb{R}^n$. We would like to study the space of isometry invariant continuous valuations in all dimensions simultaneously. More precisely assume that $i_n : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is the standard isometric imbedding when the last coordinate vanishes. It follows from the Hadwiger characterization theorem that for any $k \leq n$ the restriction map $i_n^* : Val_k^{(n+1)}(\mathbb{R}^{n+1}) \longrightarrow Val_k^{(n)}(\mathbb{R}^n)$ is an isomorphism. In other words for a fixed $k$ the sequence $Val_k^{(n)} \xrightarrow{i_n^*} Val_k^{(n+1)} \xrightarrow{i_{n+1}^*} \ldots$ stabilizes. Let us denote this limit vector space by $Val_k^{(\infty)}$. Consider the stable algebra of isometry invariant valuations
\[
Val^{(\infty)} := \oplus_{k=0}^{\infty} Val_k^{(\infty)}.
\]
From Theorem 2.6 we easily deduce the following statement:
4.1 Proposition. The graded algebra \( \text{Val}^{O(\infty)} = \text{Val}^{SO(\infty)} \) is isomorphic to the graded algebra of polynomials in one variable \( \mathbb{C}[x] \) with the grading given by the degree of a polynomial.

In [4] we have described in geometric terms the vector space \( \text{Val}^{U(m)}(\mathbb{C}^m) \) of translation invariant unitarily invariant continuous valuations on a Hermitian space \( \mathbb{C}^m \). It would be of interest to describe the algebra structure of this space (compare with Theorem 0.9). However it might be of interest as well to describe the stable algebra of translation invariant unitarily invariant continuous valuations \( \text{Val}^{U(\infty)} \) which is defined similarly to the previous case using the following lemma.

4.2 Lemma. Let \( i_n : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \) be the standard Hermitian imbedding such that the last coordinate vanishes. For any \( k \) the restriction map \( i_n^* : \text{Val}^{U(n+1)}_k(\mathbb{C}^{n+1}) \rightarrow \text{Val}^{U(n)}_k(\mathbb{C}^n) \) is an isomorphism provided \( n \geq k \).

Proof. In [2] we have shown that \( \dim \text{Val}^{U(n)}_k(\mathbb{C}^n) = 1 + \min\{[k/2], n - [k/2]\} \). Hence if \( n \geq k \) we have \( \dim \text{Val}^{U(n)}_k(\mathbb{C}^n) = \dim \text{Val}^{U(n+1)}_k(\mathbb{C}^{n+1}) = 1 + [k/2] \). Hence it suffices to prove that \( i_n^* \) is injective. Let \( \mathbb{R}Gr_k(\mathbb{C}^n) \) denote the Grassmannian of real \( k \)-dimensional subspaces of \( \mathbb{C}^n \). In [1], [2] we have used an imbedding of \( \text{Val}^0_k(\mathbb{C}^n) \) to the space \( C(\mathbb{R}Gr_k(\mathbb{C}^n)) \) of continuous functions on the Grassmannian as follows. Let \( \phi \in \text{Val}^0_k(\mathbb{C}^n) \). For any \( E \in \mathbb{R}Gr_k(\mathbb{C}^n) \) consider the restriction \( \phi|_E \in \text{Val}^k(E) \). By a result by Hadwiger [2] \( \text{Val}^k(E) \) coincides with the space of Lebesgue measures on \( E \). Hence \( \phi|_E = f(E)\text{vol}_E \) where \( \text{vol}_E \) is the Lebesgue measure induced by the Hermitian metric. Thus \( \phi \mapsto [E \mapsto f(E)] \) defines a map

\[
\text{Val}^0_k(\mathbb{C}^n) \rightarrow C(\mathbb{R}Gr_k(\mathbb{C}^n)).
\]

The non-trivial fact due to D. Klain [12], [13] is that this map is injective. Let us consider the restriction of this map to \( U(n) \)-invariant valuations. Then its image is contained in the space of \( U(n) \)-invariant continuous functions on \( \mathbb{R}Gr_k(\mathbb{C}^n) \). Hence it is enough to prove that the restriction map \( C(\mathbb{R}Gr_k(\mathbb{C}^n))^{U(n+1)} \rightarrow C(\mathbb{R}Gr_k(\mathbb{C}^n))^{U(n)} \) is injective if \( n \geq k \) where the restriction is considered under the imbedding \( \mathbb{R}Gr_k(\mathbb{C}^n) \hookrightarrow \mathbb{R}Gr_k(\mathbb{C}^{n+1}) \). It is enough to check that each \( U(n+1) \)-orbit in \( \mathbb{R}Gr_k(\mathbb{C}^{n+1}) \) intersects non-trivially with \( \mathbb{R}Gr_k(\mathbb{C}^n) \). This fact follows immediately from the explicit description of \( U(n) \)-orbits on \( \mathbb{R}Gr_k(\mathbb{C}^n) \) in terms of Kähler angles due to H. Tasaki [19]. Q.E.D.
We would like to notice that the next interesting case to classify is the algebra of translation invariant continuous valuations on the quaternionic space $\mathbb{H}^n$ invariant under the group $Sp(n)Sp(1)$.

A Appendix.

In this appendix we prove that all spaces we work with in Section 1 satisfy the assumptions of the Casselman-Wallach theorem. First we remind the notion of representation of moderate growth following [20], Ch. 11.

Let $G$ be a real reductive group. Assume that $G$ can be imbedded into the group $GL(N, \mathbb{R})$ for some $N$ as a closed subgroup invariant under the transposition. Let us fix such an imbedding $p : G \hookrightarrow GL(N, \mathbb{R})$. (In our applications $G$ will be either $GL(n, \mathbb{R})$ or a direct sum of several copies of $GL(n, \mathbb{R})$.) Let us introduce a norm $| \cdot |$ on $G$ as follows:

$$|g| := \max\{p(g), p(g^{-1})\}$$

where $|| \cdot ||$ denotes the usual operator norm in $\mathbb{R}^N$.

A.1 Definition. Let $(\pi, G, V)$ be a smooth representation of $G$ in a Fréchet space $V$ (namely $V^{sm} = V$). One says that this representation has moderate growth if for each continuous semi-norm $\lambda$ on $V$ there exists a continuous semi-norm $\nu_\lambda$ on $V$ and $d_\lambda \in \mathbb{R}$ such that

$$\lambda(\pi(g)v) \leq ||g||^{d_\lambda} \nu_\lambda(v)$$

for $g \in G, v \in V$.

The proof of the next lemma can be found in [20], Lemmas 11.5.1 and 11.5.2.

A.2 Lemma. (i) If $(\pi, G, H)$ is a continuous representation of $G$ in a Banach space $H$ then $(\pi, G, H^{sm})$ has moderate growth.

(ii) If $(\pi, G, V)$ is a representation of moderate growth. Let $W$ be a closed $G$-invariant subspace of $V$. Then $W$ and $V/W$ have moderate growth.

The next lemma is obvious.

A.3 Lemma. Let $G_1$ be a closed reductive subgroup of a reductive group $G$. Assume that the image of $G_1$ in $GL(N, \mathbb{R})$ under the map $p : G \hookrightarrow GL(N, \mathbb{R})$ is closed under the transposition. Let $(\pi, G, H)$ has moderate growth. Then the restriction of this representation to $G_1$ also has moderate growth.
In Section 1 we have discussed the space $PVal_d(V)$ of continuous valuations on $V$ polynomial of degree $d$. Equipped with the topology of uniform convergence on compact subsets of $K(V)$ it is a Fréchet space. We will need a more precise statement.

**A.4 Lemma.** The space $PVal_d(V)$ is a Banach space.

**Proof.** Let $\phi \in PVal_d(V)$. By a result due to Khovanskii and Pukhlikov [10] the function $t \mapsto \phi(tK)$ is a polynomial in $t \geq 0$ of degree at most $n + d$ for any $K \in K(V)$. Thus $\phi(tK) = \sum_{i=0}^{n+d} t^i \phi_i(K)$ where $\phi_i$ is $i$-homogeneous, namely $\phi_i(tK) = t^i \phi_i(K)$ for any $\lambda > 0$, $K \in K(V)$. Let us denote by $PVal_{d,i}(V)$ the space of $i$-homogeneous continuous valuations polynomial of degree at most $d$. By the previous discussion one has

$$PVal_d(V) = \bigoplus_{i=0}^{n+d} PVal_{d,i}(V).$$

But it is clear that $PVal_{d,i}(V)$ is a Banach space with the norm given by

$$||\phi|| = \sup\{||\phi(K)|| \mid K \in K(V), K \subset D\}$$

where $D$ is the unit ball.

**A.5 Proposition.** $PVal_d^{sm}(V)$ has moderate growth as $GL(V)$-module.

**Proof.** This immediately follows from Lemmas A.2 and A.4. Q.E.D.

**A.6 Proposition.** In the notation of Section 1, the space $\Omega_d^p \otimes C^\infty((P_+(V^*))^k, L^{\mathbb{Z}_k})$ has moderate growth as $GL(V)$-module.

This proposition is an immediate consequence of the next more general proposition.

**A.7 Proposition.** Let the group $GL(V)$ act transitively on compact smooth manifolds $X_1, \ldots, X_k$. Let $\mathcal{E}_i$ be a finite dimensional $GL(V)$-equivariant vector bundle over $X_i$, $i = 1, \ldots, k$. Then the space $C^\infty(X_1 \times \cdots \times X_k, \mathcal{E}_1 \times \cdots \times \mathcal{E}_k)$ has moderate growth as $GL(V)$-module.

**Proof.** Let us denote by $G := (GL(V))^k$. The group $G$ acts transitively on the manifold $X := X_1 \times \cdots \times X_k$, and the vector bundle $\mathcal{E} := \mathcal{E}_1 \times \cdots \times \mathcal{E}_k$ is $G$-equivariant. Consider the Banach space $H = C(X, \mathcal{E})$. Since $G$ acts transitively on $X$ the space of $G$-smooth vectors in $H$ is equal to $C^\infty(X, \mathcal{E})$. Hence by Lemma A.2 the space $C^\infty(X, \mathcal{E})$ has moderate growth as $G$-module. Consider the diagonal imbedding $G_1 := GL(V) \hookrightarrow G$. By Lemma A.3 $C^\infty(X, \mathcal{E})$ has moderate growth as $G_1$-module. Q.E.D.
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