The band structure of the whole spectrum of an N-body cold system containing atoms with arbitrary integer spin and dominated by singlet pairing force

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Abstract
The spectra of N-boson systems with arbitrary nonzero spin $j$ have been studied. Firstly, only the singlet pairing interaction is considered, a set of exact eigenstates together with the eigenenergies are analytically obtained. The completeness of this set is proved. The analytical expression allows us to see clearly the spin textures of various states different in N and/or $j$, and to find out the similarity and relationship lying among them. Secondly, the effect of other interactions is evaluated via exact numerical calculations on the systems with a smaller $N$. Some features and notable phenomena that might emerge in high-$j$ systems, say, the ground band might have extremely high level density, have been discussed.

1. Introduction

The formation of pairs of particles is a widely established phenomenon in many-body systems. The most famous is the coupling of two electrons to form a Cooper pair in BCS theory [1, 2]. For the nucleons in a nucleus, their spins might two-by-two coupled to form the $s$- or $d$-bosons as described in the interacting boson model in nuclear theory [3]. For Bose–Einstein condensates (BEC), the coupling of two bosons to form a correlated pair with opposite momenta has been predicted by Bogoliubov 70 years ago [4], and has been confirmed by experiments recently [5].

Due to the application of optical trapping, the spin degrees of freedom are liberated, and the study of the spin textures in BEC becomes a hot topic. For the spin-1 bosons, the Hamiltonian contains two parameters $g_0$ and $g_2$ corresponding to the strengths of interaction of the $\lambda = 0$ and 2 channels. Disregarding the particle number $N$ and how $g_0$ and $g_2$ would be, it has been found theoretically as early as 1998 that the total spin $S$ and the seniority $N_1$ are both conserved and they are equal to each other (ranged from $N, N - 2, 0$ or 1). Moreover, every eigenstate is composed of a fully polarized core containing $N_1$ parallel spins together with $(N - N_1)/2$ singlet pairs (01-pair) without exception. However, if $g_0 < g_2$, a lower state in the spectrum will contain more [01]-pairs, and vice versa [6–10]. This picture can be proved via the fact that, for $j = 1$, the multiplicity of $S$ is either one (if $N - S$ is even) or zero (otherwise).

For spin-$j$ systems with $j = 2$, disregarding how the strengths $\{g_\lambda\}$ $(\lambda = 0, 2, and 4)$ would be, the seniority and the total spin remain to be good quantum numbers. Thus, the eigenstates are also composed of a core of unpaired particles together with the [01]-pairs. However, the spins in the core is not necessary to be fully polarized but can be coupled to various $S$ based on the $SO(5) \supset SO(3)$ branching rule [10–12].

For $j \geq 3$ systems, seniority is in general no more conserved (except some special cases). Related studies on these systems are scarce [13–17], the spin textures of the eigenstates remain to be clarified. It turns out that the spin degrees of freedom would increase greatly with $j$. When $N = 1000$ for an example, the number of spin states with magnetization $M = 0$ is 501 if $j = 1$. However, if $j = 3$, this number would be $7.735 \times 10^{11}$. Due to the
great increase of the dimension of the spin space, rich physics might be involved. Thus, the study of high-$f$ systems is worthy.

This paper is dedicated to the study of high-$f$ systems ($f \geq 3$). Due to the popularity of the correlated pairs, either spatial correlation or spin-spin correlation, in various many-body systems, instead of a general study, the object of this paper is concentrated to the correlated pairs in the spin space. In particular, we aim at the $|0\rangle$-pairs which arise from the singlet pairing interaction and exist extensively in $f = 1$ and 2 systems. Therefore, firstly, we consider the case that the Hamiltonian contains only the singlet pairing interaction. This ideal case has been considered previously by Isacker and Heinze [11]. Based on group algebra they obtain the entire spectrum. However, they aim essentially at $f = 2$ systems, the structures of the eigenstates with $f \geq 3$ and the inherent physical picture remain to be clarified. In this paper, instead of working with the second quantization representation and using group theory, the many-body Schrödinger equation has been solved directly and analytically by using a specially designed techniques. In this way the physical picture is rather clear, and the entire spectrum together with all the eigenstates disregarding $N$ and $f$ have been obtained. Secondly, for being closer to realistic cases, we consider that the Hamiltonian has deviated from the above ideal case in various ways. In the deviated Hamiltonian with $N = 8$ and $f = 3$ different choices of the strengths have been chosen. A numerical approach is adopted for the diagonalization of the Hamiltonian to obtain the eigenstates. An Example with $f = 4$ has also been given. We believe that the knowledge from medium-body systems would help us to understand better the many-body physics from elementary point of view.

2. The complete spectra of high-$f$ systems under singlet pairing interaction

We assume that the temperature is so low that all the spatial degrees of freedom are frozen, and all particles fall into the same spatial state $\phi$ which is most favorable to binding. Due to the freezing, the spin–orbit coupling can be neglected. Then the high-$f$ system is governed by the spin dependent Hamiltonian

$$H_{\text{spin}} = \sum_{i<j} V_{ij}, \quad V_{ij} = \sum_{\lambda} g_{ij} \Lambda_{\lambda}^{ij},$$

where $g_{ij}$ is the weighted strength of the $\lambda$-channel ($\lambda = 0, 2, \cdots, 2f$), and the factor $\int \omega^{\lambda}(r)dr$ has been included. $\Lambda_{\lambda}^{ij}$ is the projector to the $\lambda$-channel. Let the Hamiltonian be dominated by singlet pairing interaction, namely, $g_0$ is much more negative than the other strengths. In this case the effect of the latter is smaller, thus they can be approximately given as being equal to each other. Since the spin textures will not be changed when the set $\{g_{ij}\}$ are shifted as a whole and/or when a new unit is adopted, the case with $g_{0} < g_{1} = \cdots = g_{2f}$ is equivalent to the case with $g_{0} = -1$ and $g_{1} = \cdots = g_{2f} = 0$. This simplified Hamiltonian is denoted as $H_{0|f}$. We are first going to find out all the eigenenergies and eigenstates of $H_{0|f}$ analytically. Then, when $H_{\text{spin}}$ deviates from $H_{0|f}$, the effect of the deviation is evaluated.

Let $\{\Phi_{N,S,l}\}$ be a set of normalized, symmetrized, and orthogonal spin states of a subsystem with $N_1$ particles. $S$ is the total spin and $l$ is just an index to specify further the states. Let $\chi$ denote the spin state of a particle, and $(\chi \chi)_0$ denote a singlet pair ($|0\rangle$-pair). Let $N = N_1 + 2f$ and $\mathcal{P}_{N}$ denote a summation over the $N!$ permutations of particles. Then, for the product state $\mathcal{P}_{N} \Phi_{N,S,l}(\chi \chi)_0$, we divide the $N!$ permutations into four parts as

$$\mathcal{P}_{N} = \mathcal{P}_{N,a} + \mathcal{P}_{N,b} + \mathcal{P}_{N,c} + \mathcal{P}_{N,d},$$

where $\mathcal{P}_{N,a}$ includes those permutations that the particles $i$ and $j$ are both in $\Phi_{N,S,l}(i|f = a)$, $i$ in $\Phi_{N,S,l}$ and $j$ in $(\chi \chi)_0$ (if $x = b$), $j$ in $\Phi_{N,S,l}$ and $i$ in $(\chi \chi)_0$ (if $x = c$), and both $i$ and $j$ are in $(\chi \chi)_0$ (if $x = d$). For the case $x = b$, formally we can extract the $i$-th particle from $\Phi_{N,S,l}$ by using the fractional parentage coefficients $\beta_{\chi'}$ as $\Phi_{N,S,l} = \sum_{\chi'} \beta_{\chi'} (\chi(i)\phi_{\chi'}).$ The details of $\beta_{\chi'}$ and $\phi_{\chi'}$ are irrelevant in the follows. Thus we have

$$\mathcal{P}_{N,a} \Phi_{N,S,l}(\chi \chi)_0 = \mathcal{P}^{(i|j)}_{N-2} N_1 ! \sum_{\chi'} \beta_{\chi'} (\chi(i)\phi_{\chi'})(\chi(j)\chi)_0 (\chi(\chi)_0)^{-1},$$

where $\mathcal{P}^{(i|j)}_{N-2}$ denotes a summation over all $(N-2)!$ permutations (particles $i$ and $j$ are excluded). Making use of the following specific $9\cdot j$ symbol

$$\begin{cases}
S' & S' \\
S & 0 \\
0 & S
\end{cases} = \frac{(-1)^{f+S'+S}}{(2f+1)(2S+1)},$$

We have

$$(\chi(i)\phi_{\chi'}) (\chi(j)\chi)_0 = \frac{1}{2f+1} (\chi(i)\chi(j)_0 (\chi\phi_{\chi'})_S + Z).$$
In $Z$ the particles $i$ and $j$ are not coupled to zero. When $H_{\text{spin}} = H(0)$, $Z$ has no contribution to energy and therefore can be neglected. After the neglect, 
\[
\Phi_{N-s,i}(\chi \chi)^{l}_{\delta} = \frac{2N_{s}J}{2f_{+} + 1} \Phi_{N-2}^{(e)}[\Phi_{N,s,i}(\chi(i)\chi(j))u(\chi(0))^{l}_{\delta-1}],
\]
and 
\[
H(0) \Phi_{N-s,i}(\chi \chi)^{l}_{\delta} = \frac{1}{2} \sum_{i \neq j} V_{6} \Phi_{N-s,i}(\chi \chi)^{l}_{\delta} = -\frac{N_{s}J}{2f_{+} + 1} \Phi_{N}[\Phi_{N,s,i}(\chi \chi)^{l}_{\delta}].
\]

The above formula remains unchanged when $\Phi_{N,s} \rightarrow \Phi_{N,s'}$.

Similarly, we have 
\[
H(0) \Phi_{N-s,i}(\chi \chi)^{l}_{\delta} = \frac{(2J_{s} - 2f_{-} + 2f_{+} - 1)J}{2f_{+} + 1} \Phi_{N}[\Phi_{N,s,i}(\chi \chi)^{l}_{\delta}].
\]

Let the set $\{ \Phi_{N,s,i} \}$ include all those states of the $N_{s}$-body subsystem full in seniority (i.e., all the $N_{s}$ particles are unpaired). Accordingly, $\Phi_{N,s,i}$ has no contribution on energy when $H_{\text{spin}} = H(0)$.

Let $\Phi_{N}$ be the operator for normalization and symmetrization, and let 
\[
\Psi_{f,\delta}^{(N)} \equiv \Phi_{N}[\Phi_{N,s,i}(\chi \chi)^{l}_{\delta}].
\]

From the above formulae we arrive at 
\[
H(0) \Psi_{f,\delta}^{(N)} = \frac{(2N_{s} - 2f_{-} + 2f_{+} - 1)J}{2f_{+} + 1} \Psi_{f,\delta}^{(N)} \equiv E_{f}^{(N)} \Psi_{f,\delta}^{(N)}
\]
which is just the Schrödinger equation. Thus, a series of eigenergies and eigenstates of $H(0)$ have been analytically obtained. The completeness of this set is given below. Each state is a product of a core (filled with $N_{s}$ unpaired particles) together with $\{0\}$-pairs. In particular, when $N_{s}$ is fixed, $E_{f}^{(N)}$ depends only on $f$, thus a group of states with the same $f$ but different in $S$ and $L$ are degenerate. When $N = 2K$ or $2K + 1$, $J$ is ranged from 0 to $K$.

Accordingly, the whole spectrum is divided into $K + 1$ bands specified by $J$, the width of each band is zero. Obviously, when $J$ is larger, more $\{0\}$-pairs are contained, therefore the energy $E_{f}^{(N)}$ is lower. The lowest band has $J = K$ and contains only one state, namely, the ground state (g.s.) $\Psi_{g}^{(N)} = \Phi_{N}[\chi \chi(0)]^{0}_{0}$ or $\Phi_{N}[\chi \chi(0)]^{0}_{N}$. The g.s. energy is $E_{g}^{(N)} = -2(K + 1)$, where $(+)$ is for odd (even $N$).

Let $L \equiv K - J$. While $J$ denotes the number of $\{0\}$-pairs, $L$ is related to the seniority ($L = N_{s}/2$ or $(N_{s} - 1)/2$ if $N$ is even or odd). It denotes also the bands in the order of uprising energy, say, $L = 0 (K)$ is for the bottom (top) band. The excitation energy 
\[
E_{L}^{(N)} = E_{g}^{(N)} - E_{f}^{(N)} = \frac{12(L + \delta) \pm 1}{2f_{+} + 1}L,
\]
where $+(-)$ is for odd (even $N$). As examples, the spectra of $E_{L}^{(N)}$ with $\delta = 3$ and $N = 5 \rightarrow 9$ are shown in figure 1. The spectra with $\delta = 1 \rightarrow 5$ and $N \geq 10$ (even) are shown in figure 2.

It turns out that $E_{L}^{(N)}$ does not depend on $N$, but $L$ and $(-)^{N}$, this is clearly shown in figure 1. Say, the two spectra with $N = 7$ and 9 are similar, except that the latter contains one more band at the top. We found 
\[
E_{L+1}^{(N)} - E_{L}^{(N)} = \frac{4L + 2\delta + \mu}{2f_{+} + 1},
\]
where $\mu = 2 - (-)^{N}$. Thus, the separation between the higher bands is larger. Recall that the excitation $L \rightarrow L + 1$ is realized by breaking a $\{0\}$-pair. Thus, the solidity of the $\{0\}$-pair does not depend on how many other $\{0\}$-pairs are surrounding, but depends on how many unpaired particles are surrounding, i.e., on the seniority of the state. This is shown in figures 1 and 2. The solidity depends also on $\delta$ as shown in (12). The $\{0\}$-pair with a larger $\delta$ would be easier to be broken as shown in figure 2, where the level would be lower when $\delta$ increases.

It is clear from the Schrödinger equation (10) that, if some $\{0\}$-pairs are added into (removed from) an eigenstate so that $\Psi_{f,\delta}^{(N)} \rightarrow \Phi_{f}^{(N)}, \Phi_{f}^{(N)}$, where $N_{s} = 2f_{-} + 2J_{s} + 2J \equiv N_{s}$. Then, these two states will both belong to the $L = N_{s}/2$ (if $N$ is even) band and have exactly the same core $\Phi_{N,s,i}$. They are different only in the number of $\{0\}$-pairs. Consequently, these states different in $N$ are related to each other. This relationship could be called brotherhood.

Note that, $M$, the $Z$-component of $S$, is a good quantum number. We assume $M = 0$ (the cases with $M \neq 0$ are similar when an external field is absent). When $N$ is fixed, the total number of spin states with $M = 0$ denoted as $N_{M}$ is known (say, for $\delta = 3$, $N_{M} = 8, 18, 32, \cdots$ when $N = 3, 4, 5, \cdots$). From the brotherhood, we know that, when a $\{0\}$-pair is added to every spin state of a $(N - 2)$-body system, all the spin states of the $N$-body system can be recovered except those in the top band. Therefore, the number of states contained in the top band is
In general, for the \( L \) band,
\[
\mathcal{N}_{L(1\ldots K \ldots N)} = N^L_N - N_{N-2}^L.
\]

(13)

In general, for the \( L \) band,
\[
\mathcal{N}_{L(1\ldots K \ldots N)} = N^L_2L - N_{2L-2}^L.
\]

(14)

For the bottom band with \( L = 0 \), \( \mathcal{N}_{0,1\ldots K \ldots N} = 1 \).

Summing up the above numbers, we found
\[
\sum_{L=0}^{K} \mathcal{N}_{L(1\ldots K \ldots N)} = N^L_N.
\]

(15)
3. Evaluation of the effect caused by a deviation of the Hamiltonian from singlet pairing

Let the Hamiltonian be denoted as $H \equiv \left\{ \Psi^{(N)} \right\}$ complete. The numbers $\Psi^{(N)}$ of various states are given in figures 1 and 2. It increases rapidly with $N$ (i.e., with $L$) as shown in figure 1, and with $\bar{f}$ as shown in figure 2.

To evaluate the effect of the deviation, we present numerical results from the diagonalization of $H$ in the follows. Examples with $\bar{f} = 3$ and $N = 8$ are given in figure 3, in which $g_6$ remains to be most negative. Four cases are shown: (i) $g_2 = g_4 \leq g_6$, in this case a shift of the strengths as a whole is further performed so that $g_2 = g_4 = 0$ (figure 3(a)). (ii) $g_2 = g_4 = 0 \geq g_6$ (figure 3(b)). (iii) $g_2 > g_4$ (figure 3(c)). (iv) $g_2 < g_4$ (figure 3(d)). Let $P_{ij}^{0}$ be the probability of a pair of atoms coupled to zero. Since the key point is to check the conservation of seniority, in these figures $P_{ij}^{0}$ of all the 151 eigenstates with $M = 0$ are plotted. If a group of states have the same seniority, their $P_{ij}^{0}$ would be exactly the same. Otherwise, the distribution of $P_{ij}^{0}$ is diffused.

In figures 3(a) and 3(b) $g_2 = g_4 = 0$ is given, while $g_6$ is given at 3 values. In figure 3(a) we found that the black points (associated with $g_6 = 0$) are exactly distributed on five horizontal lines implying the conservation of seniority and, accordingly, the clear band structure. The upmost line containing only one point is for the g.s. with seniority zero (all the eight particles are in $[0]$-pairs), while the lowest line is for the top band with seniority being 8 (full in seniority). When $g_6 > 0$ (figure 3(a)), we found that the points previously belonging to a horizontal line diffuse strongly along horizontal direction but weakly along vertical direction. In particular, each diffused point has a partner point at the horizontal line, the spin states of this pair are very close to each other (see below). It implies that all the eigenstates of $H_{(0)}$ have not been seriously disturbed by $H_{\text{devi}}$ and therefore seniority remains nearly conserved. However, their energies may be seriously affected, the order of the states (i.e., the index $i$) may therefore be changed. E.g., when $g_6 = 0.3$, the lowest three states of the $L = 2$ band ($i = 5, 6,$ and 7) have $S = 3, 0,$ and 8. However, when $g_6 = 0.6$, the indexes of these three states become $i = 9, 6,$ and 6, respectively. Due to the diffusion along horizontal direction, the band widths will become broader, and the
crossover of bands may occur. E.g., when $g_6 = 0.6$, the top state of the $L = 2$ band has $i = 33$, and its energy level goes up deeply into the $L = 3$ band. This state has $S = 12$ and is found to be dominated by the component containing two $0\bar{1}$-pairs and four unpaired particles. Since the four spins coupled to $S = 12$ are aligned, this state has a smaller $P_0^i$ and is subjected to a stronger repulsion from $g_6$. This explains why this state is higher. Similarly, when $g_6 = 0.6$, the highest state of the $L = 3$ band has $i = 134$, it goes up deeply into the $L = 4$ band. This state has $S = 18$ and is dominated by one $0\bar{1}$-pairs and six unpaired particles. Since the six spins coupled to $S = 18$ are all aligned, this state has also a smaller $P_0^i$ and also subjected to a stronger repulsion from $g_6$. Whereas we found that the lowest state of the top band has $i = 41$ and $S = 0$. Therefore, the repulsion from $g_6$ can be reduced. Thus, although the seniority is nearly conserved, the energies are strongly affected by $H_{\text{dev}}$.

To evaluate the deviation quantitatively, let the eigenstates of $H_{0\bar{1}}$ be denoted as $\Psi_f$, while those of $H_{\text{dev}}$ be $\Psi_f$. For the case given in figure 3(a) with $g_6 = 0.3$, among the 151 eigenstates, 81 of them can find a partner so that $\langle \Phi_f | \Psi_f \rangle > 0.99$ and other 61 of them have $0.95 < \langle \Phi_f | \Psi_f \rangle < 0.99$. These data confirm that the deviation in the eigenstates as a whole is slight. In particular, the deviation in high-lying states is even much smaller. Say, $\langle \Phi_f | \Psi_f \rangle = 0.950$, $\langle \Phi_f | \Psi_{11} \rangle = 0.993$, and $\langle \Phi_f | \Psi_{150} \rangle = 1.000$. When $g_6$ increases from 0.3 to 0.6, the above value $81 \rightarrow 22$ and $61 \rightarrow 60$, it implies a larger deviation.

In figure 3(b) the vertical diffusion of points is also slight, thus the near conservation of seniority also holds, in particular for higher states. However, the widths of the bands become very broad. When $g_6 = -0.6$, the energy of the bottom state of every band is close to the g.s.. For example, the bottom state of the top band with $L = 4$ has $i = 1$ and $S = 24$. Thus, this state is fully polarized and it becomes the g.s.. Accordingly, the width of the top band covers all the spectrum. When both $g_6$ and $g_0$ are negative, there is a competition between alignment and pairing. When alignment exceeds pairing, as in the above case, a change of phase of the g.s. would occur. Similarly, the blue point for the bottom state of the $L = 3$ band has $i = 2$ and $S = 18$. It is composed of 6 aligned particles together with a $0\bar{1}$-pair. While the blue point for the bottom state of the $L = 2$ band has $i = 4$ and $S = 12$ containing four aligned particles and two $0\bar{1}$-pairs. Thus, when $g_6$ is sufficiently negative, all the unpaired particles are aligned in the bottom states of every band. Alternatively, the attraction from $g_6$ would be minimized in $S = 0$ states. Therefore, among the five bands, four top states have $S = 0$ (the appearance of five $S = 0$ states is prohibited because, for $N = 8$ and $J = 3$, the multiplicity of $S = 0$ state is 4).

For $J = 3$, it has been pointed out that there is a critical value $g_6 = (18g_4 - 11g_2)/7 \equiv g_{\text{crit}}$ (disregarding the value of $g_0$, at which the seniority is strictly conserved [11]. Figures 3(c) and 3(d) are for the case in the neighborhood of $g_{\text{crit}}$. In figure 3(c), $g_2 > g_{\text{crit}}$ and $g_{\text{crit}}$ becomes negative. Accordingly, due to the attraction from $g_6$, this figure is similar to figure 3(b). Whereas in figure 3(d), $g_2 < g_{\text{crit}}$ and $g_{\text{crit}}$ becomes positive. Accordingly, due to the repulsion from $g_6$, this figure is more or less similar to figure 3(a). Note that, for figures 3(a), 3(b), and when $g_6 = 0$, all the states belonging to the same band are degenerate. However, when $g_6 = g_{\text{crit}}$, the states belonging to the same band are not degenerate. The band is further divided into a few pieces. Only the states in a piece are degenerate [11].

One more example with $J = 4$ is given in figure 4. For $N = 8$, there are totally 526 $M = 0$ eigenstates. $P_0^0$ of the lowest 100 states are plotted. For higher states with $i > 100$, $P_0^0$ of them are less than 0.09. When $i > 250$, most $P_0^0$ are close to zero. Figure 4 demonstrates that, with $H_{\text{dev}}$, a larger $J$ does not further spoil the near conservation of seniority. The corresponding band structure remains clear.

4. Final remarks

For $H_{0\bar{1}}$, the Schrödinger equation of the $N$-boson system with arbitrary spin $J > 0$ has been solved analytically. The complete spectrum together with the set of eigenstates have been obtained, and the completeness of the set has been proved. It is found that the band structure based on the conservation of seniority holds also when $J \geq 3$. Comparison has been made among the spectra different in $N$ and/or $J$. Similarity and relationship among their eigenstates have been demonstrated. In particular, for $H_{\text{dev}}$, the effect of the deviation has been studied via strict calculations on $P_0^0$. The following points are mentioned.

- There is brotherhood among the eigenstates different in $N$. Therefore, the knowledge from few-body systems (which can be obtained rigorously) can be used to evaluate the low-lying states of many-body systems.
- The case with a positive $g_6$ is noticeable. When all the strengths change their signs so that $g_0 \rightarrow +1$, all the above discussions remain valid unless that the whole spectrum is reversed. In this case, the ground band might have an extremely high level density and well protected by a very large energy gap $E_{\text{gap}} = (2N + 2J - 3)/2J + 1$. Let $N = 1000$ as an example. When $J = 2$, the number of $M = 0$ states contained in the ground band is $1.672 \times 10^6$. However, when $J = 3$, this number is $7.716 \times 10^6$. The great difference demonstrates that, for $H_{\text{dev}}$, the increase of $J$ could lead to extremely high level density in the neighborhood of the lowest level.
For a general Hamiltonian with $f = 3$, there is a zone in the parameter space lying along the line $g_6 = (18g_4 - 11g_2)/7$ in which the conservation of seniority holds roughly. In this zone the whole spectrum can be divided into bands. The states in a band have their $P_0$ close to each other, i.e., these states have the same average numbers of the $[0]$-pairs. For $f = 4$, it is believed that there would also be zones in which the band structure holds. Nonetheless, the case with $f = 4$ remains to be further clarified.

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Data availability statement

No new data were created or analysed in this study.

Appendix. Diagonalization of the Hamiltonian based on the Fock-states

For $f = 3$ systems, we introduce the Fock-states $|\alpha\rangle \equiv |N_0^\alpha, N_1^\alpha, \ldots, N_7^\alpha\rangle$, where $N_\mu^\alpha$ is the number of particles in magnetic component $\mu$ ($-3 \leq \mu \leq 3$). Obviously, $\sum_\mu N_\mu^\alpha = N$ and $\sum_\mu \mu N_\mu^\alpha = M$, the total magnetization. They are adopted as basis-states for the diagonalization of $H_{\text{spin}}$. The matrix element is

$$
\langle \alpha' | H_{\text{spin}} | \alpha \rangle = \frac{1}{2} \sum_{\mu'\nu'} \delta_{\mu'+\nu',\mu+\nu} \sum_\lambda g_{\lambda} C_{2\mu',\mu+\nu'} C_{3\mu,\mu+\nu'}
$$

$$
\times (\delta_{\mu'\nu'} \delta_{\mu\nu} N_\mu^\alpha N_\nu^\alpha N_\mu^\alpha N_\nu^\alpha \delta_{|\alpha'||\alpha|} + \delta_{\mu'\nu'} \delta_{\mu\nu} N_\mu^\alpha N_\nu^\alpha (N_\mu^\alpha - 1) N_\nu^\alpha \delta_{|\alpha'||\alpha|} + \delta_{\mu'\nu'} \delta_{\mu\nu} N_\mu^\alpha (N_\nu^\alpha - 1) N_\mu^\alpha \delta_{|\alpha'||\alpha|} + \delta_{\mu'\nu'} \delta_{\mu\nu} (N_\nu^\alpha - 1) N_\mu^\alpha \delta_{|\alpha'||\alpha|})
$$

where $|\alpha'\rangle \equiv |N_0^{\alpha'}, N_1^{\alpha'}, \ldots, N_7^{\alpha'}\rangle$, $\delta_{\mu\nu} = 1$ or 0 (if $\mu = \nu$ or $\mu \neq \nu$), $\delta_{|\alpha'||\alpha|} = 1 - \delta_{|\alpha'||\alpha|}$, $[\alpha]_\mu$ denotes the set of seven numbers $N_0^\alpha, \ldots, N_7^\alpha$, $|\alpha|_\mu$ ($\mu = \nu$) denotes the set $[\alpha]$ except that $N_\mu^\alpha$ is changed to $N_\mu^\alpha - 1$ and $N_\nu^\alpha$ is changed to $N_\nu^\alpha - 1$, $[\alpha]_\mu$ denotes the set $[\alpha]$ except that $N_\mu^\alpha$ is changed to $N_\mu^\alpha - 2$, $\delta_{|\mu|\alpha} = 1$ (if all the seven numbers
in $[\beta]$ are one-to-one identical to those in $[\alpha]$) or 0 (otherwise), the Clebsch–Gordan coefficients have been introduced. With these matrix elements, the diagonalization can be carried out numerically.

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