Restrictions on wave equations for passive media

Sverre Holm
Department of Informatics, University of Oslo, P. O. Box 1080, N–0316 Oslo, Norway

Martin Blomhoff Holm
Department of Economics, BI Norwegian Business School, N–0442 Oslo, Norway

(Dated: March 30, 2018)

Most derivations of acoustic wave equations involve ensuring that causality is satisfied. Here we explore the consequences of also requiring that the medium should be passive. This is a stricter criterion than causality for a linear system and implies that there are restrictions on the relaxation modulus and its first few derivatives. The viscous and relaxation models of acoustics satisfy passivity and have restrictions on not only a few, but all derivatives of the relaxation modulus. This is the important class of completely monotone systems. It is the only class where the medium is modeled as a system of springs and dampers with positive parameters. It is shown here that the attenuation as a function of frequency for such media has to increase slower than a linear function. Likewise the phase velocity has to increase monotonically. This gives criteria on which one may judge whether a proposed wave equation is passive or not, as illustrated by comparing two different versions of the viscous wave equation.

I. INTRODUCTION

In 1981 Weaver and Pao [1] showed that a wave equation’s asymptotic value for attenuation has to increase slower than a linear function of frequency. They claimed that this result followed from causality, passivity and linearity. This result seems to have gone unnoticed among researchers in acoustics. The usual criterion to apply to a wave equation’s solution is causality, assume that linearity holds, and to overlook the passivity requirement. A recent example [2] shows that the asymptote of the attenuation may increase with any power. This seems to contradict Weaver and Pao’s result, but since Ref. [2] only takes causality into account, the criterion is weaker than in Ref. [1]

One reason why Weaver and Pao’s result and the more recent work on this by Hanyga [3–5] may have been overlooked is that the results are quite abstract and mathematical. Further, some of the work above also assumes that the material can be modeled with a spring damper model without much argument for why this is so. This paper is an attempt to justify this assumption and present the result in a more accessible way.

To illustrate the point, we use the viscous wave equation. It usually comes in two versions, with either of the loss terms below. A natural question to ask is if they are completely equivalent or if one or the other has properties which the other one does not possess.

$$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \left\{ \begin{array}{l}
\frac{\varepsilon}{\gamma} \frac{\partial^3 u}{\partial t^3} = 0 \\
\frac{\gamma}{\varepsilon} \nabla^2 u = 0
\end{array} \right. \quad (1)
$$

Here $u$ can be particle displacement or pressure, $c_0$ is the speed of sound as frequency approaches zero frequency, and $\tau$ is the ratio of a viscosity and an elastic modulus. The upper version is often seen in the literature on nonlinear acoustics [6], while the lower version has a history that goes back to Stokes in 1845. The reference is p. 302 of Ref. [7].

Note that the attenuation is the same in both cases if we assume low losses or low frequencies. This can be seen from the fact that in that case one can convert one equation to the other by assuming first that the term in the curly brackets can be neglected so that $\nabla^2 u \approx \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2}$, and then replacing the Laplacian with the temporal derivative or vice versa in the loss term.

The claim in the present paper is that the set of causal wave equations is larger than the set of passive, linear materials with wave equations. Furthermore, the set of possible wave equations is even larger. This is illustrated in Fig. 1.

We start by analyzing three different forms of the constitutive equation. Then it is shown that a network of springs and dampers, where each branch has positive elasticities and viscosities, guarantees that a system is passive. This result implies that the relaxation modulus possesses a property called complete monotonicity. A consequence of this property is that the complex wavenumber is a complete Bernstein function. Such functions have specific properties that imply that the asymptotic properties of the attenuation and dispersion can be found. This is applied to the two viscous wave equations of (1) in order to judge which one that corresponds to a phys-
II. CONSTITUTIVE EQUATIONS

In acoustics, one form of a linear constitutive relation between pressure variations, \( p \), around an equilibrium pressure, \( p_0 \), and relative density variations, \( \rho / \rho_0 \), is:

\[
p(t) = h(t) \frac{p(t)}{p_0}.
\]  

(2)

In linear viscoelasticity, where much of the theory of this paper is taken from, the impulse response, \( h(t) \), is hard to measure due to the impracticality of impulsive sources, so the step response, \( G(t) \), where \( h(t) = dG(t)/dt \), is used instead. Also pressure is replaced by strain, \( \sigma(t) = -\rho(t) \), and density by stress, \( \varepsilon(t) = -\rho(t)/\rho_0 \). The 1-D constitutive equation of a linear system, from Ch. 2 of Ref. 8, is then:

\[
\sigma(t) = G(t) \frac{\partial \varepsilon(t)}{\partial t}.
\]  

(3)

The kernel \( G(t) \) is called the relaxation modulus. A linear viscoelastic material can be described in three different ways and they will be related to each other here.

A. Linear differential equation model

The first description is a linear differential equation between stress and strain with constant coefficients:

\[
\left[ 1 + \sum_{k=1}^{p} t_k \frac{\partial}{\partial t^k} \right] \sigma(t) = \left[ E_e + \sum_{k=1}^{q} b_k \frac{\partial}{\partial t^k} \right] \varepsilon(t).
\]  

(4)

Ch. 2.1 of Ref. 8 and many other sources state that the differential equation description is equivalent to the superposition model of (3).

B. The causal fading memory model

The second description is the convolution model of (3) where there are restrictions on the kernel, \( G(t) \), in order for it to represent a fading memory. Then changes in the past have less effect now than more recent changes. This is the hereditary model of Boltzmann 10 11. To ensure causality, the kernel of (3) also has to be zero for negative time.

The fading memory concept is not always well defined. It certainly implies that the kernel has to be non-negative and that it is monotonically decreasing. It may also imply conditions on the derivatives of higher orders. In order to reach a strict definition of fading, the concept of passivity needs to be introduced.

1. The passive fading memory model

There are at least two tests for passivity in the literature. The first is the requirement that the dissipation is non-negative at all time 12

\[
D(t) = \int_0^t \sigma(\tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau \geq 0.
\]  

(5)

To ensure that no energy is produced by the system prior to application of the input, it is necessary that the system is causal 13. Therefore the set of passive systems is a true subset of the causal system group in Fig. 1, and the lower limit in the integral can be 0 rather than \(-\infty\). This also implies that there is no need to test for causality with e.g. the Kramers-Kronig relation 13 when passivity is satisfied.

The second measure of passivity is the more general Clausius-Duhem inequality which is an expression of the second law of thermodynamics or non-negativity of the rate of entropy production 15

\[
\rho \psi'(t) + D(t) \geq 0,
\]  

(6)

where \( \psi(t) \) with \( \psi(0) = 0 \), is the isothermal free energy which equals the total strain energy stored in the springs.

In Ref. 12 a fading memory model which had a positive monotonously falling relaxation modulus with no requirements on the higher order derivatives, was found to give a non-negative dissipation according to (5). Here we will adopt the stricter result of Ref. 16 where it is shown that the memory kernel also has to be convex for it to satisfy the requirement of a non-negative rate of entropy production.

Thus the passive fading memory condition adopted here is:

\[
(-1)^n \frac{d^n G(t)}{dt^n} \geq 0, \quad t > 0, \quad n = 0, 1, 2
\]  

(7)

i.e. the kernel, \( G(t) \), is non-negative, non-increasing, and convex.

C. Spring-damper model and complete monotonicity

The third description is the spring-damper model which we introduce with two examples.

1. Kelvin-Voigt model

The left-hand model in Fig. 2 is the Kelvin-Voigt model of linear viscoelasticity. The model is expressed by a first order differential operator on the right-hand side of (4), i.e. \( p = 0, q = 1, a_1 = 0, \) and \( b_1 = \eta \) which is the viscosity. The relaxation modulus, the stress response to a unit step in strain, for the Kelvin-Voigt model is:

\[
G(t) = E_e + \eta \delta(t)
\]  

(8)

and is plotted in the upper part of Fig. 3. Observe that it is causal and fading in the sense of (7).
The Kelvin-Voigt model leads to the viscous wave equation. It is common in acoustics instead to derive the viscous wave equation from the Navier-Stokes equation combined with conservation of mass and an equation of state which only describes elasticity [17]. In that case, the material property of viscosity is mixed with the conservation of linear momentum, making it harder to distinguish empirical material properties from fundamental conservation laws. Conservation of momentum and energy (equivalent to conservation of mass in the non-relativistic case and thus the continuity equation) express fundamental physical principles in the form of invariance to spatial and temporal translations as stated in Noether’s theorem, see e.g. Ch. II of Ref. [18]. Whether all the material properties are combined in the constitutive equation as in the Kelvin-Voigt model or not, the end result is the same so these are just alternative ways of deriving Stokes’ viscous wave equation.

2. Zener model

In the Zener model, the right-hand model in Fig. 2, a first order differential term is added on the left-hand side of (4) so \( p = q = 1, a_1 = \tau_\sigma, b_1 = \eta \). The time constants in terms of the physical components are:

\[
\tau_\sigma = \eta/E_1 \leq \tau_\epsilon = \eta/E', \quad \frac{1}{E'} = \frac{1}{E_\epsilon} + \frac{1}{E_\eta}.
\] (9)

The Zener model’s relaxation modulus is:

\[
G(t) = E_\epsilon + E_\eta \left( \frac{\tau_\epsilon}{\tau_\sigma} - 1 \right) e^{-t/\tau_\sigma}.
\] (10)

It is plotted in the lower plot of Fig. 3 and it is causal and convex. The Zener model is in fact the equation of state in the relaxation model in acoustics although that is not how it is commonly presented. An example is Ref. [17] which postulates an equation of state between pressure and density variations:

\[
\tau \left( \frac{\partial \rho}{\partial t} - c_\sigma^2 \frac{\partial \rho}{\partial t} \right) + (p - c_\sigma^2 \rho) = 0,
\] (11)

where \( c_0 \) and \( c_m \) are the asymptotic values for phase velocity for low and high frequency.

This is in fact the Zener constitutive equation with \( p = -\sigma \) and \( \rho/\rho_0 = -\epsilon \), although it is not identified as such.

If the two time constants are very near each other, \( \tau_\sigma \lesssim \tau_\epsilon \), or the equivalent \( c_0 \lesssim c_m \), this model describes a relaxation model as briefly discussed in Ref. [19] and it is a building block in the multiple relaxation models for salt water and air.

3. The relaxation spectrum

The examples demonstrate that both relaxation moduli are causal and fading. They also show that the relaxation modulus may consist of a constant, a weighted impulse at time zero, and a weighted sum of exponentials as in Eq. (2.28) of Ref. [8]. The latter is a series with real, positive coefficients:

\[
G(t) = G_c + G_\delta(t) + G_\tau(t), \quad G_\tau(t) = \sum_{n=1}^{N} E_n \exp(-t/\tau_n),
\] (12)

where the terms in the sum represent series combination of springs and dampers in parallel. They may be in parallel with a spring \( G_c \geq 0 \) and another parallel damper, \( G_\delta \geq 0 \) as shown in Fig. 4. A realistic model is obtained if \( G_\tau(t) \) is modeled with spring constants, \( E_n \) and viscosities, \( \eta_n = \tau_n E_n \), which are non-negative. The series expansion for \( G_\tau(t) \) is a Prony series or a general Dirichlet series with positive weights.

4. Complete monotonicity

The relaxation spectrum of (12) satisfies a general pattern where the derivatives switch signs or are zero for all orders:

\[
(-1)^n \frac{d^n G(t)}{dt^n} \geq 0, \quad t > 0, \quad n = 0, 1, 2, \ldots
\] (13)
The criterion is stricter than that of (7) and is called complete monotonicity. The limit of the sum in (12) is a Laplace transform:

$$G(t) = \int_0^\infty G(s)e^{-st}ds.$$  \hspace{1cm} (14)

Any model that satisfies complete monotonicity can be expressed as a Laplace transform, and any complete monotone function has a non-negative Laplace transform (Ref. 20, definition 1.3).

An exponential solution, $e^{-t/\tau}$, such as in (12), is one of the simplest examples of a response function where the derivative switches sign according to (13). There are also fractional generalizations of the Kelvin-Voigt and Zener constitutive equations where the first-order derivative is replaced by a non-integer derivative of order $\alpha$. At first sight, these models may look very different from spring-damper models. The relaxation moduli of these models are described either by power law functions, $t^\alpha$, $\alpha \leq 0$, or by Mittag-Leffler function. But both of these functions are also completely monotone [8]. Actually, these relaxation moduli can also be described by an infinite relaxation series of the form of (12) as shown in Refs. 21–23.

III. THE SPECIAL ROLE OF SPRING DAMPER MODELS

The relationship between the three descriptions, the linear differential equation, the fading memory, and the spring-damper network is shown in Fig. 5. The solution to a linear differential equation can always be written in the form of convolution integral of the form of (3). But only a subset of these solutions satisfy the fading memory criterion. The fading memory subset further contains two cases:

1. The relaxation spectrum of a physical model of form (12) can always be written as a linear differential equation of the form of (4) and it will always produce the fading memory kernel of (3). Thus the subset of physical models exists as a subset of the two other descriptions.

2. There exist fading memory models which do not correspond to a spring damper system with positive coefficients.

A. A spring damper system with non-physical parameters

To illustrate the last point, we analyze the previously mentioned example of a model with a non-negative dissipation (12). This model has a positive monotonously falling relaxation modulus with no requirements on the higher order derivatives, and it was formed by a combination of a physical spring-damper and one with non-physical parameters:

$$G(t) = E_1 \exp(-t/\tau_1) + E_2 \exp(-t/\tau_2), \quad E_2 < 0.$$  \hspace{1cm} (15)

For illustration, we assume that the first time-constant is greater than the second one, $\tau_1 > \tau_2 > 0$. Because the time constant is the ratio of viscosity and elasticity, a non-physical negative elasticity, $E_2$, implies that the second viscosity also is non-physical and negative, $\eta_2 < 0$. In Ref. 12 it was found that the conditions for $G(0) \geq 0$ and $G'(0) \leq 0$ are that $E_1 + E_2 \geq 0$ and $E_1/\tau_1 + E_2/\tau_2 \geq 0$. The first condition implies that the spring constant of the physical spring is greater than that of the negative spring, and the second condition also implies that the total dissipation, (5), is non-negative. The interpretation is that the energy produced by the last spring and damper is more than absorbed by the first one.

We are interested in a condition on the second derivative in order to have a non-negative rate of entropy, (6). By differentiating and setting the derivative at $t = 0$ of order $n$ to zero, the condition of an alternating sign of the derivative, as in (13), is satisfied up to and including order $n$ if the following condition holds:

$$\frac{E_1}{\tau_1^n} + \frac{E_2}{\tau_2^n} \geq 0.$$  \hspace{1cm} (16)

In the limit as $n \to \infty$, the second elasticity, $E_2$, has to approach zero in order to satisfy this requirement because $\tau_1$ is the greatest of the two time constants.

When $n = 2$ in (16), this is an example of a fading memory system which satisfies the requirement for non-negative rate
of entropy production. It is a system consisting of a physical branch and a non-physical branch. We may therefore dismiss many of the fading memory systems that satisfy the Clausius–Duhamel inequality if we impose the additional criterion that each subbranch should be a physical system, i.e. have positive elasticity and viscosity.

B. Complete monotone models

As shown in the previous section, the two main models of acoustics are spring-damper models. Also in linear viscoelasticity, the constitutive equations in the form of spring-damper models play a fundamental role.

Here we therefore surmise that completely monotone models, i.e. those based on springs and dampers, play a special role in fading memory systems. The hypothesis is that they are better physically motivated than models which only have positive, monotone, and convex relaxation moduli, despite that these models also satisfy the Clausius-Duhem inequality. The spring-damper models may however only be a subset of the interesting models, but a very important subset.

IV. ANALYSIS OF THE WAVE EQUATION

The fact that most of the interesting systems have a relaxation modulus, \(G(t)\), which is a completely monotone function has consequences for the asymptotic properties of the solution to the wave equation.

A. Wavenumber as a function of relaxation modulus

First we need to express the wave equation in terms of the relaxation function. The following is a less rigorous version of the wave equation derivation of Ref. 5. It builds on transforming the main equations to the frequency domain [19]. The constitutive equation [3] is then

\[
\sigma(\omega) = i\omega G(\omega)\varepsilon(\omega),
\]

(17)

The frequency domain version of the conservation of momentum is:

\[
\rho_0 \frac{\partial^2 u}{\partial t^2} = \nabla \sigma \iff \rho_0 (i\omega)^2 u(\omega) = -ik \sigma(\omega).
\]

(18)

and conservation of mass gives:

\[
\varepsilon(t) = \frac{\partial u}{\partial x} \iff \varepsilon(\omega) = -ik u(\omega).
\]

(19)

Now insert [19] in [17] to eliminate \(\varepsilon(\omega)\) and then use [18] to eliminate \(u(\omega)\). The wave number is then:

\[
k^2(\omega) = \frac{\rho_0 \omega^2}{i\omega G(\omega)}.
\]

(20)

Equation (20) is now made more general by using the Laplace transform rather than the Fourier transform, i.e. substitution of \(s = i\omega\):

\[
K^2(s) = -k^2(s) = \frac{1}{\rho_0 s G(s)}.
\]

(21)

The new variable \(K\) is related to the wavenumber by \(K = ik\). Taking the square root gives:

\[
K(s) = ik(s) = \sqrt{\frac{s}{\rho_0 s G(s)}}.
\]

(22)

This shows how the wavenumber depends on the relaxation modulus and corresponds to Eq. (15) in Ref. 5.

B. Bernstein property

In order to proceed, we will need the definition of a class of functions which are related to completely monotone ones. Bernstein functions are non-negative functions where the derivative is completely monotone:

\[
f(t) \geq 0, \quad (-1)^n t^n f(t) \leq 0, \quad t > 0, \quad n = 1, 2, \ldots
\]

(23)

Examples of such functions are \(1 - e^{-t/\tau}\) and \(t^\alpha, 0 < \alpha \leq 1\).

A subclass of Bernstein functions is the set of complete Bernstein functions. These functions are defined in Appendix A.1 and will be called CBF from now on. Likewise completely monotone functions will be denoted by CM.

If the relaxation modulus \(G(t)\) is CM, then the wavenumber \(K(s)\) is a CBF. The argument is formally given in Appendix A.2 and in Ref. 3 and follows these steps:

1. If \(G(t)\) is CM, then the Laplace transform \(G(s)\) as given by (14) is also CM, since it is a combination of a positive constant, a positive impulse, and positively weighted exponentials

2. If \(G(s)\) is CM, then \(sG(s)\) is CBF

3. A CBF applied to a second CBF produces a third CBF. Both \(sG(s)\) and the square root are CBF, so then \(\sqrt{sG(s)}\) is also CBF

4. If \(\sqrt{sG(s)}\) is CBF then \(\frac{s}{\sqrt{sG(s)}}\) is CBF

The consequence of this argument is that the wavenumber \(K(s)\) of (22) is a complete Bernstein function.

C. Consequences of the Bernstein property

If \(K(s)\) is a CBF, the same applies to \(k(s) = K(s)/i\). Every CBF can be written in the following form (exact definition in Appendix A.1):

\[
K(s) = a + bs + \beta(s),
\]

(24)
where \( a, b \geq 0, a = K(0), b = \lim_{s \to \infty} K(s)/s \) and \( \beta(s) \) is a term with sublinear growth. That means that the exponent \( y \) in a power law expression \( s^y \) has to be less than or equal to one as \( s \to \infty \).

Often this expression can be even more simplified by noting that the constant \( a = K(0) \) may be zero. From (22) that means that one needs to find \( \lim_{s \to \infty} sG(s) = \lim_{s \to \infty} G(t) \). The last expression is found from the final value theorem for the Laplace transform. In Ref. 4 it is argued that this limit always is positive, but from the expression for the relaxation modulus in (12) it can be seen that the limiting value is \( G_\infty \), the spring across the terminals in Fig. 4. The presence of this spring is what distinguishes a solid from a liquid [9], so only for those materials that can be considered to be solids one has \( \lim_{s \to \infty} sG(s) > 0 \) and thus \( a = 0 \). In contrast to Ref. 4 the following argument is however not dependent on \( a \) being 0.

The wave number can also be expressed by the attenuation, \( \alpha(\omega) \), and the phase velocity, \( c_p(\omega) \);

\[
k(\omega) = \frac{\omega}{c_p(\omega)} - i\alpha(\omega) = \omega \left( \frac{1}{c_\infty} + d(\omega) \right) - i\alpha(\omega), \quad (25)
\]

The phase velocity has been decomposed into a constant asymptotic value and a frequency dependent component \( d(\omega) \) which is the excess dispersion. Expressed with Laplace transforms this is

\[
K(s) = ik(s) = \frac{s}{c_\infty} + id(s) + \alpha(s). \quad (26)
\]

This expression is combined with (24), noting that \( b = 1/c_\infty \). This parameter may be 0 for instance for the wave equation derived from the Kelvin-Voigt equation where \( c_\infty = \infty \). Equating real and imaginary parts gives

\[
\alpha(s) = \Re \beta(s) + a, \quad d(s) = \Im \beta(s). \quad (27)
\]

D. Asymptotic properties

Equation (27) shows that both the attenuation and the excess dispersion are proportional to \( \beta(s) \), which is a term with sublinear growth. This implies that both attenuation \( \alpha(\omega) \) and excess dispersion \( d(\omega) \) also must have sublinear growth. For the attenuation this means:

\[
\lim_{\omega \to \infty} \alpha(\omega)/\omega = 0 \quad \text{or} \quad \lim_{\omega \to \infty} \alpha(\omega) \propto \omega^y, \quad y \leq 1. \quad (28)
\]

The asymptotic result for the excess dispersion compared to (25) implies that the phase velocity is a non-decreasing function of frequency:

\[
c_p(\omega) \geq 0, \quad \frac{dc_p(\omega)}{d\omega} \geq 0. \quad (29)
\]

V. VISCOUS WAVE EQUATIONS

The results of the previous section are the tools required to decide among the two viscous wave equations of (1).
Thus the attenuation increases proportionally with $\omega^2$ for low frequencies and the phase velocity decreases. For the high frequency/high loss case, the attenuation increases in proportion to $\omega^{3/2}$ and the phase velocity decreases with frequency. The properties are illustrated in Fig. 6 with the dash-dot line. These results violate the conditions of both (28) and (29) and implies that the upper version of the wave equation of (1) is not derived from a spring damper constitutive equation and is probably not passive.

\[ k^2 - \frac{\omega^2}{c_0^2} + i\omega \tau k^2 = 0 \Rightarrow k = \frac{\omega}{c_0} \left( 1 + i\omega \tau \right)^{1/2}. \quad (35) \]

For low frequencies/low losses, $\omega \tau \ll 1$, the approximate wavenumber is

\[ k \approx \frac{\omega}{c_0} \left( 1 - i\frac{\omega \tau}{2} - \frac{3(\omega \tau)^2}{8} \right). \quad (36) \]

The asymptotic value for attenuation is the same as for the previous case as expected, but the phase velocity now increases with frequency:

\[ c_{p,low}(\omega) = c_0(1 + \frac{3}{8}(\omega \tau)^2). \quad (37) \]

For high frequencies/high losses the approximate wavenumber differs from the previous case

\[ k \approx \frac{\omega}{c_0} (i\omega \tau)^{-1/2} = \frac{\omega \tau^{-1/2}}{c_0} \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \omega^{-1/2}. \quad (38) \]

which results in

\[ \alpha_{high}(\omega) = \frac{\sqrt{\pi}}{2c_0} \omega^{1/2} \quad c_{p,high}(\omega) = 2c_0 \sqrt{\frac{\pi}{2}} \omega^{1/2}. \quad (39) \]

Thus the attenuation and the phase velocity both increase asymptotically with $\sqrt{\omega}$ above a crossover frequency. Also the phase velocity is a non-decreasing function of frequency. These characteristics are seen in the solid line plots of Fig. 6 and are in agreement with the conditions of both (28) and (29).

It should not be surprising that the wave equation with a mixed derivative viscous term is the more physical one. It can be derived from a constitutive equation consisting of a spring and a damper in parallel, the Kelvin-Voigt model of Fig. 5 (left-hand side). The dynamic modulus, which is the Fourier transform of the impulse response of (2), is $H(\omega) = \sigma(\omega)/\varepsilon(\omega) = E + i\eta \omega$. $G(\omega)$ derives from a step response given by (17) and when inserted into (20) the result is (35) with $c_0^2 = E/p_0$, showing that this viscous wave equation indeed builds on a physical constitutive equation.

VI. DISCUSSION

The example in the previous section illustrates the usefulness of the asymptotic result. The first analysis, that of (1) with only temporal derivatives in the loss term, leads to the conclusion that there is no constitutive equation of the spring damper type for it. The second analysis of the viscous wave equation does not give so definite an answer, due to the way the implications flow in Appendix A 2. We cannot say for sure that there exists such a constitutive equation, as there is a possibility that the conditions of (28) and (29) may be satisfied in other ways also. In the case of the viscous wave equation, we have other ways of verifying that it indeed is rooted in a spring damper model, because it is straightforward to derive it from the Kelvin-Voigt model. Thus the lack of agreement with the asymptotic result tells us definitely that there is no spring-damper constitutive equation, but the opposite is more ambiguous [4].

The spring damper constitutive equations all satisfy both passivity criteria: positive dissipation and the more stringent Clausius-Duhem criterion. There is a possibility that there may be other medium models that are passive, and this is a topic which may be further investigated. But the spring damper or completely monotone class is the model behind two of the most common acoustic attenuation descriptions, the viscous loss and the relaxation loss models. It must therefore be the most important subclass of passive systems.

One kind of medium where the requirement for complete monotonicity may be too strong is a porous medium. The Biot model predicts three wave modes: one shear mode, and two compressional modes. It has recently been shown that the shear wave solution is exactly equivalent to a Zener model [23]. However, the two compressional wave modes are only approximately equivalent to spring damper models. The solution to the fast wave equation is approximately that of a spring in parallel with a Maxwell element, i.e. a Zener model, and the slow wave can be approximated to that of a Maxwell element [23]. Per mode, these solutions therefore cannot be described exactly by two or three term spring-damper models and it is an open question if they can be described by a more complex network of spring-damper models according to (12) or not. The passivity criterion in this case is also more complex to formulate as one of the features of the Biot model is that energy is converted from one compressional wave mode to the other as discussed in Ch. 3 of Ref. [26].

Fractional wave equations, where the loss term is described by non-integer derivatives, are also of interest. As mentioned in Sec. II C 4 all the basic fractional constitutive equations are completely monotonic. Fractional wave equations which have been derived from e.g. the fractional Newton-Kelvin-Voigt, or Zener constitutive equations therefore satisfy the asymptotic results of (28) and (29). But some of the fractional wave equations which have been derived by simply substituting non-integer derivatives for the integer order temporal or spatial derivatives in a conventional wave equation do not. This analysis has already been performed for some of the proposed fractional wave equations [19, 27, 28].

An alternative to the poroelastic model for sediment acous-
tics is the grain shearing model [29]. It has been shown that
the grain shearing mechanism is a non-Newtonian linearly
increasing viscosity which may be approximated by a frac-
tional derivative [30]. From that it follows that the shear wave
model builds on a fractional Newton constitutive equation,
and the compressional wave solution corresponds to a frac-
tional Kelvin-Voigt model [31]. Furthermore its variants such
as the viscous grain shearing model [32] can also be derived
from a combination of ordinary and fractional spring damper
models [33], so all of these models satisfy complete mono-
tonicity.

In light of our derivation, Weaver and Pao11 can also be
reinterpreted. Their medium model is different in that it is
not a strain stress response model, but rather a propagating
wave from one place in the material to another. Then they
require that the transfer function satisfies \( H(\omega) \geq 0 \) for pas-
sivity. That resembles more the dissipation-free criterion
than the entropy-rate criterion. They show, in a way which is dif-
ferent from ours, that the complex wave number,

\[
\mathbf{f} = \mathbf{s} \mathbf{G} \mathbf{t}
\]

is compatible with passivity.

If the loss term has a mix of spatial and temporal derivatives it
exist any physical constitutive equation for the material, while
if the loss term has a mix of spatial and temporal derivatives it
is compatible with passivity.

VIII. ACKNOWLEDGEMENT

We thank Professor Alexander Lion for helpful discussions
concerning passivity and the Clausius-Duhem inequality.

Appendix A: Mathematical properties

1. Definition and representation of a complete Bernstein
function

A function \( f: (0, \infty) \to R \) is a Bernstein function if \( f(t) \geq 0 \),
all derivatives exist, and the sign of the derivatives alternate as
in (23). A Bernstein function has the following representation

\[
f(t) = a + bt + \int_0^\infty (1 - e^{-t}) \tilde{\mu}(dr)
\]

where \( a, b \geq 0 \) and \( \tilde{\mu} \) is a measure on \((0, \infty)\) satisfying

\[
\int_0^\infty \min\{1, r\} \tilde{\mu}(dr) < \infty.
\]

A Bernstein function is said to be complete if the Lévy mea-
sure \( \tilde{\mu} \) has a completely monotone density. A complete Ber-
stein function \( f: (0, \infty) \to R \) has the following representation

(24)

\[
f(t) = a + bt + \int_0^\infty \frac{1}{1 + r} \sigma(dr)
\]

where \( a, b \geq 0 \), \( a = f(0) \), \( b = \lim_{t \to \infty} f(t)/t \) and \( \sigma \) is a mea-
sure on \((0, \infty)\) such that \( \int_0^\infty 1/(1 + r) \sigma(dr) < \infty \).

2. Proof of complete Bernstein property of wavenumber

The proof of the numbered points in Sec. IV B proceeds in
these steps [3].

1. \( G(t) \in CM \Rightarrow G(s) \in CM \). The Laplace transform \( G(s) \)
of a CM time domain function, \( G(t) \), is also CM by Theorem 1.4 of Ref. [20].

2. \( G(s) \in CM \Rightarrow sG(s) \in CBF \). If \( G(s) \in CM \), then it is has
a Stieltjes representation: \( G(s) = s^2 + \int_0^\infty \frac{1}{1 + r} \mu(dr) \)
where \( a, b \geq 0 \) are non-negative constants and \( \mu \) is a measure on \((0, \infty)\) such that \( \int_0^\infty 1/(1 + r) \mu(dr) < \infty \), see
Definition 2.1 in Ref. [20]. It follows that \( sG(s) = a + 
bs + \int_0^\infty \frac{1}{1 + r} \mu(dr) \) is a complete Bernstein function since it
is equivalent to (A2).

3. \( sG(s) \in CBF \Rightarrow \sqrt{sG(s)} \in CBF \). Corollary 7.6 in
Ref. [20] states that if \( f_1, f_2 \) are CBF, then \( f_1(f_2) \) is CBF.
Since \( s^\alpha \) with \( 0 \leq \alpha \leq 1 \) is CBF, \( \sqrt{sG(s)} \) is also CBF.

4. \( \sqrt{sG(s)} \in CBF \Rightarrow \frac{s}{\sqrt{sG(s)}} \in CBF \). Theorem 6.2 in
Ref. [20] states that \( sG(s)/s \) is a Stieltjes function if \( \sqrt{sG(s)} \) is CBF. Furthermore, the inverse of a Stieltjes
function is a CBF by Theorem 7.3 in Ref. [20]. As a re-
result, \( s/\sqrt{sG(s)} \) is CBF.
[1] R. L. Weaver and Y. H. Pao, “Dispersion relations for linear wave propagation in homogeneous and inhomogeneous media,” Journ. Math. Phys. 22, 1909–1918 (1981).

[2] Michael J Buckingham, “Frequency power-law attenuation and dispersion in marine sediments,” J. Acoust. Soc. Am. 137, 2281–2288 (2015).

[3] M. Seredyńska and Andrzej Hanyga, “Relaxation, dispersion, attenuation, and finite propagation speed in viscoelastic media,” J. Math. Phys. 51, 092901 (2010).

[4] A Hanyga, “Wave propagation in linear viscoelastic media with completely monotonic relaxation moduli,” Wave Motion 50, 909–928 (2013).

[5] Andrzej Hanyga, “Dispersion and attenuation for an acoustic wave equation consistent with viscoelasticity,” J. Comp. Acoust. 22, 1450006 (2014).

[6] M. F. Hamilton and D. T. Blackstock, Nonlinear Acoustics (Academic Press, New York and London, 1998).

[7] G. G. Stokes, “On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids,” Trans. Cambridge Philos. Soc. 8 (1845).

[8] Francesco Mainardi, “Fractional calculus and waves in linear viscoelasticity: An introduction to mathematical models,” (Imperial College Press, London, UK, 2010) pp. 1–347.

[9] Nicholas W Tschoegl, The phenomenological theory of linear viscoelastic behavior: An introduction (Springer-Verlag Berlin, 1989) reprinted in 2012.

[10] Ludwig Boltzmann, “Zur theorie der elastischen nachwirkung (On the theory of hereditary elastic effects),” Ann. Phys. Chem Bd. 7, 624–654 (1876).

[11] Hershel Markovitz, “Boltzmann and the beginnings of linear viscoelasticity,” Trans. Soc. Rheol. (1957-1977) 21, 381–398 (1977).

[12] P. Akyildiz, RS Jones, and K Walters, “On the spring-dashpot representation of linear viscoelastic behaviour,” Rheol acta 29, 482–484 (1990).

[13] Piero Triverio, Stefano Grivet-Talocia, Michel S Nakhla, Flavio G Canaveri, and Ramachandra Achar, “Stability, causality, and passivity in electrical interconnect models,” IEEE Trans. Adv. Packag. 30, 795–808 (2007).

[14] Kendall R Waters, Joel Mobley, and James G Miller, “Causality-imposed (Kramers-Kronig) relationships between attenuation and dispersion,” IEEE Trans. Ultrason. Ferroelectr., Freq. Control 52, 822–823 (2005).

[15] Alexander Lion, “On the thermodynamics of fractional damping elements,” Continuum Mech. Therm. 9, 83–96 (1997).

[16] Peter Haupt and Alexander Lion, “On finite linear viscoelasticity of incompressible isotropic materials,” Acta Mech 159, 87–124 (2002).

[17] David T Blackstock, Fundamentals of physical acoustics (John Wiley & Sons, 2000).

[18] Lev Davidovich Landau and Evgenii Mikhailovich Lifshitz, Mechanics, 3rd Edition: Vol. 1 of Course of Theoretical Physics (Elsevier, 1976).

[19] Sverre Holm and Sven Peter Nåsholm, “A causal and fractional all-frequency wave equation for lossy media,” J. Acoust. Soc. Am. 130, 2195–2202 (2011).

[20] René L Schilling, Renming Song, and Zoran Vondracek, Bernoulli functions: theory and applications (Walter de Gruyter, 2012).

[21] B Gross, “On creep and relaxation,” J Appl Phys 18, 212–221 (1947).

[22] M Caputo and F Mainardi, “Linear models of dissipation in anelastic solids,” La Rivista del Nuovo Cimento (1971-1977) 1, 161–198 (1971).

[23] Sverre Holm, “Spring-damper equivalents of the fractional, poroelastic, and poroelastic models for elastography,” Submitted for publication, preprint [arXiv:1703.09515] (2017).

[24] Sverre Holm and Ralph Sinkus, “A unifying fractional wave equation for compressional and shear waves,” J. Acoust. Soc. Am. 127, 542–548 (2010).

[25] J Geertsm and D. C. Smit, “Some aspects of elastic wave propagation in fluid-saturated porous solids,” Geophysics 26, 169–181 (1961).

[26] Nicholas P Chotiros, Acoustics of the Seabed as a Poroelastic Medium (Springer, 2017).

[27] S. Holm, S. P. Nåsholm, F. Prieur, and R. Sinkus, “Deriving fractional acoustic wave equations from mechanical and thermal constitutive equations,” Comput. Math. Appl. 66, 621–629 (2013).

[28] Sverre Holm and Sven Peter Nåsholm, “Comparison of fractional wave equations for power law attenuation in ultrasound and elastography,” Ultrasound. Med. Biol. 40, 695–703 (2014).

[29] Michael J Buckingham, “Wave propagation, stress relaxation, and grain-to-grain shearing in saturated, unconsolidated marine sediments,” J. Acoust. Soc. Am. 108, 2796–2815 (2000).

[30] Vikash Pandey and Sverre Holm, “Linking the fractional derivative and the Lomnitz creep law to non-Newtonian time-varying viscosity,” Phys. Rev. E 94, 032606 (2016).

[31] Vikash Pandey and Sverre Holm, “Connecting the grain-shearing mechanism of wave propagation in marine sediments to fractional order wave equations,” J. Acoust. Soc. Am. 140, 4225–4236 (2016).

[32] Michael J Buckingham, “On pore-fluid viscosity and the wave properties of saturated granular materials including marine sediments,” J. Acoust. Soc. Am. 122, 1486–1501 (2007).

[33] V Pandey and S Holm, “Connecting the viscous grain-shearing mechanism of wave propagation in marine sediments to fractional calculus,” in 78th EAGE Conference and Exhibition 2016 (2016).