On integration of multidimensional version of \( n \)-wave type equation

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Abstract

We represent a version of the dressing method allowing one to construct a new class of \( n \)-wave type nonlinear partial differential equations (PDEs) whose solution space may be parametrized by arbitrary functions of several variables. There is no restrictions on the dimensionality of nonlinear PDEs. The associated solution manifold is parametrized by arbitrary functions of several variables, but this freedom is not enough to provide the full integrability of nonlinear PDE.

1 Introduction

The significance of the inverse spectral transform method (ISTM) \[1, 2, 3\] is provided by the wide applicability of (1+1)- and (2+1)-dimensional nonlinear PDEs (soliton equations) which may be completely solved by this technique such as \[2, 3\] and as well as by the dressing method \[4, 5, 6, 7, 8\] (introduced as a particular realization of ISTM). However, the higher dimensional analogies of the classical (1+1)- and (2+1)-dimensional soliton equations may not be constructed in a simple way. There are some examples of such multidimensional systems (see, for instance, \[3\]), however they solution spaces are restricted to (2+1)-dimensional manifolds. A new class of partially integrable multidimensional PDEs was derived using the dressing method with the degenerate kernel of the integral operator \[9\].

Regarding non-soliton types of multidimensional integrable systems, we notice the first order PDEs integrable by the method of characteristics (equations with wave breaking solutions) \[10\], its matrix generalization \[11\], systems of hydrodynamic type \[12, 13, 14, 15\], the linearizable (C-integrable) models \[16, 17, 18, 19, 20\], linearizable equations with constraints \[21\]. Self-dual type equations (instanton equations) \[22, 23\], nonlinear equations associated with commuting vector fields \[24, 25, 26\]. Combinations of different integration algorithms may be also used for construction multidimensional PDEs \[27, 28\].

This paper is devoted to study of the multidimensional generalizations of soliton-type equations. We propose a dressing algorithm allowing one to construct a new class of \( n \)-wave type PDEs in arbitrary dimensions together with a rich manifold of particular solutions. This manifold may be parametrized by arbitrary functions of several variables, although this freedom is not enough for the complete integrability of the derived nonlinear PDEs.

To show the principal difference between the new systems and classical (2+1)-dimensional \( n \)-wave equation, let us compare the linearized general solutions to both of them. It is known that the linear limit of the general solution to the classical (2+1)-dimensional nonlinear \( N \)-wave equation \[29, 30\]

\[\left[C^{(3)}, U_{t_2}\right] - \left[C^{(2)}, U_{t_3}\right] + C^{(2)}U_{t_1}C^{(3)} - C^{(3)}U_{t_1}C^{(2)} + \left[[C^{(3)}, U], [C^{(2)}, U]\right] = 0\] (1)
has the following Fourier representation:

\[ U(t) \sim \int e^{\sum \lambda_m C^{m} t_m} R(\lambda, \mu) e^{-\sum \mu_m C^{m} t_m} d\lambda d\mu, \quad t = \{t_1, t_2, \ldots\}, \quad (2) \]

where \( \lambda \) and \( \mu \) are the scalar complex spectral parameters, \( R \) is an arbitrary function of two spectral parameters, \( C^{(m)} \) are the constant diagonal matrices. Since there are two independent spectral parameters in the exponents of eq. (2) (\( \lambda \) and \( \mu \)), the function \( U(t) \) inherits \( N(N-1) \) arbitrary scalar functions of two independent variables (say \( t_1 \) and \( t_2 \)), which coincides with the number of scalar equations in off-diagonal matrix equation (1). These arbitrary functions mean the full integrability of PDE (1). For instance, the initial value problem may be formulated for eq. (1).

Let us generalize this feature and consider the following version of eq. (2):

\[ U \sim \int e^{\sum_{i=1}^{K} \lambda_i C^{(im)} t_m} R(\lambda, \mu) e^{-\sum_{i=1}^{K} \mu_i C^{(im)} t_m} d\lambda_1 \ldots d\lambda_K d\mu_1 \ldots d\mu_K. \quad (3) \]

Here we introduce the vector complex spectral parameters: \( \lambda = \{\lambda_1, \ldots, \lambda_K\} \) and \( \mu = \{\mu_1, \ldots, \mu_K\} \), \( C^{(im)} \) are constant diagonal matrices and \( K \) is some integer. If \( C^{(im)} \) are independent, then formula (3) yields \( N(N-1) \) arbitrary functions of \( 2K \) variables \( t_i, i = 1, \ldots, K \) and, consequently, the associated nonlinear \((2K + 1)\)-dimensional version of the off-diagonal equation (1) would be completely integrable. Such equations would be of interest from the applicability standpoint because the \( N \)-wave equation is reducible from any physical system by the multiple scale expansion.

The multidimensional nonlinear PDE derived in this paper have the following discrepancies with the above ideal multidimensional model.

1. There are \( K - 1 \) independent parameters among \( K \) scalar parameters \( \lambda_i, i = 1, \ldots, K \). Consequently, eq. (3) yields arbitrary functions of \( 2(K - 1) \) variables \( t_i \).

2. There are some linear relations among the diagonal matrices \( C^{(im)} \), so that the number of arbitrary functions of \( 2(K - 1) \) independent variables is less then \( N(N - 1) \).

3. Unlike classical equation (1), the derived nonlinear PDE has the diagonal part.

The structure of this paper is following. In Sec. 2 we represent the general form of nonlinear PDE derived in this paper. Richness of the associated solution space is discussed in Sec. 3. Important relations among the number of independent variables in the nonlinear PDE, matrix dimensionality and dimensionality of the solution space are obtained in Sec. 4. In Sec. 5 we discuss the available manifold of explicite solutions for the nonlinear PDEs (in particular, soliton-type solutions) and construct some particular solutions. Results are discussed in Sec. 6. A dressing algorithm used for the derivation of the discussed nonlinear PDEs is given in Appendix A, Sec. 7. Derivation of the classical completely integrable \((2+1)\)-dimensional \( n \)-wave equation using our algorithm is given in Appendix B (Sec. 8).
2 A new type of solvable quasilinear matrix first order PDE in arbitrary dimensions

Hereafter, we denote the integration over the space of $K$-dimensional complex spectral parameter $\mu = \{\mu_1, \ldots, \mu_K\}$ by *. T.e., for any two functions $f(\mu)$ and $g(\mu)$ we have

$$f(\mu) * g(\mu) = \int f(\mu)g(\mu)d\Omega(\mu),$$

where $\Omega(\mu)$ is some measure in the $K$-dimensional space of the parameter $\mu$. We also introduce the unit $\mathcal{I}(\lambda, \mu)$ and the inverse $f^{-1}(\lambda, \mu)$ operators in a usual way:

$$f(\lambda, \nu) * \mathcal{I}(\nu, \mu) = \mathcal{I}(\lambda, \nu) * f(\nu, \mu) = f(\lambda, \mu),$$

$$f(\lambda, \nu) * f^{-1}(\nu, \mu) = f^{-1}(\lambda, \nu) * f(\nu, \mu) = \mathcal{I}(\lambda, \mu).$$

If $d\Omega(\mu) = d\mu d\bar{\mu}$ and the integration is over the whole space of the complex vector parameter $\mu$, then $\mathcal{I}(\lambda, \mu) = \delta(\lambda - \mu) = \prod_{i=1}^{K} \delta(\lambda_i - \mu_i)$. We use the double index to mark the independent variables $t_{m_1m_2}$ of the nonlinear PDE and denote the whole set of these variables by $t$. A general form of new solvable PDE is represented in the following section, Sec 2.1. A reduced form of this equation with lower matrix dimensionality is given in Sec 2.2. Details of derivation of all formulas of this section are given in Appendix A, Sec 7.

2.1 General form of matrix nonlinear PDE and its available solution space

The general form of $N \times N$ matrix nonlinear PDE with $D^2$-independent variables derived in this paper (see Appendix A, Sec 7) reads

$$\sum_{m_1,m_2=1}^{D} L^{(m_1)} (V_{t_{m_1m_2}} + V\xi(\eta^{(m_1m_2)})^T V - V\eta^{(m_1m_2)}\xi^T V) R^{(m_2)} = 0,$$

where $V$ is the matrix $N \times N$ field, $\xi$ and $\eta^{(m_1m_2)}$ are $N \times 1$ constant matrices, $L^{(m_1m_2)}$ and $R^{(m_1m_2)}$ are constant $N \times N$ matrices (not diagonal in general). This PDE is not diagonal in general, it possesses a manifold of solutions $V(t)$ given in the form

$$V(t) = -2r^T(\lambda) * (\Psi_0(\lambda, \mu; t) * R(\mu, \nu) + \mathcal{I}(\lambda, \nu))^{-1} * R^{-1}(\nu, \bar{\nu}) * r(\bar{\nu}),$$

$$\Psi_0(\lambda, \mu; t) = \varepsilon(\lambda; t) C(\lambda, \mu) \varepsilon^{-1}(\mu; t),$$

where $C(\lambda, \mu)$ is an arbitrary $N \times N$ matrix function of two spectral parameters; matrix functions of spectral parameters $R(\lambda, \mu)$ and $r(\lambda)$ are given as:

$$R(\lambda, \mu) = \mathcal{I}(\lambda, \mu) + r(\lambda)\xi^T r^T(\mu),$$

$$r(\lambda) = \sum_{j=1}^{K} g^{(j)}(\lambda) a^{(j)},$$

where $a^{(j)}$, $j = 1, \ldots, K$, are constant $N \times N$ matrices, det $a^{(j)} \neq 0$; $g^{(j)}(\lambda)$ are such diagonal matrix functions of spectral parameter that

$$|g^{(j)}| = \prod |g^{(i)}| g^{(j)}| \leq \infty, \quad i, j = 1, \ldots, K.$$
The $t$-dependence is introduced in eq. (7) through the function $\Psi_0$ by the exponent $\varepsilon(\nu; t)$

$$\varepsilon(\nu; t) = \exp \left( \sum_{m_1, m_2 = 1}^{D} T^{(m_1 m_2)}(\nu) t_{m_1 m_2} \right)$$

with

$$T^{(m_1 m_2)}(\lambda) = \sum_{j=1}^{K} \bar{g}^{(j)}(\lambda) \hat{a}^{(j; m_1)} \bar{a}^{(j; m_2)},$$

$$\bar{g}^{(1)}(\lambda) = \frac{2}{\sum_{\gamma=1}^{N} a^{(1)}_{\alpha \gamma} \xi_{\gamma 1} + \sum_{i=2}^{K} \sum_{\gamma=1}^{N} \bar{g}^{(i)}(\lambda) a^{(i)}_{\alpha \gamma} \xi_{\gamma 1}}.$$  

$$\bar{g}^{(j)}(\lambda) = \frac{2 \bar{g}^{(i)}(\lambda)}{\sum_{\gamma=1}^{N} a^{(1)}_{\alpha \gamma} \xi_{\gamma 1} + \sum_{i=2}^{K} \sum_{\gamma=1}^{N} \bar{g}^{(i)}(\lambda) a^{(i)}_{\alpha \gamma} \xi_{\gamma 1}}, \quad j = 2, \ldots, K.$$  

Here $\xi_{\gamma 1}$ are arbitrary constants, $\bar{g}^{(i)}(\lambda) = \frac{g^{(i)}(\lambda)}{g^{(1)}(\lambda)}$, $i = 2, \ldots, K$. Thus only $(K - 1)$ independent scalar spectral parameters may be introduced by eqs. (14). The constant $N \times N$ matrices $a^{(j)}$ are related with $\eta^{(m_1 m_2)}$ and arbitrary diagonal constant matrices $\hat{a}^{(i; m_1)}$ and $\bar{a}^{(i; m_2)}$, $m_1 = 2, \ldots, D$, through the equations

$$\hat{a}^{(i; m_1)} \bar{a}^{(i; m_2)} = (a^{(i)} \eta^{(m_1 m_2)})_{i1}, \quad \alpha = 1, \ldots, N, \quad m_1, m_2 = 1, \ldots, D, \quad i = 1, \ldots, K$$

with

$$\hat{a}^{(i; 11)} = 1, \quad i = 1, \ldots, K, \quad \beta = 1, \ldots, N.$$  

Eq. (15) must be solvable with respect to $a^{(i)}$. The matrices $L^{(m_1)}$, $R^{(m_2)}$ are defined in terms of $a^{(j)}$, $\hat{a}^{(j; m_1)}$, $\bar{a}^{(j; m_2)}$ through the systems:

$$\sum_{m_1 = 1}^{D} L^{(m_1)}(\hat{a}^{(i)})^T \bar{a}^{(j; m_1)} = 0, \quad i, j = 1, \ldots, K,$$

$$\sum_{m_2 = 1}^{D} \hat{a}^{(j; m_2)} \bar{a}^{(i)} r^{(m_2)} = 0, \quad i, j = 1, \ldots, K.$$  

where

$$\hat{a}^{(i)} = a^{(i)} \Gamma, \quad \Gamma = 1 - \frac{r^T * r \xi \xi^T}{1 + Q}, \quad Q = \xi r^T(\lambda) * r(\lambda) \xi.$$  

**Remark 1:** In accordance with eqs. (14), functions $\bar{g}^{(k)}(\lambda)$ depend on $(K - 1)$ arbitrary functions $\bar{g}^{(i)}(\lambda)$ ($i = 2, \ldots, K$) of the vector spectral parameter $\lambda$. Consequently, the diagonal functions $T^{(m)}(\lambda)$ depend on $(K - 1)$ independent functions $\bar{g}^{(i)}(\lambda)$. If $K = 1$, then there is no $\lambda$ in the function $\bar{g}^{(1)}$, i.e., $\bar{g}^{(1)}(\lambda) = \frac{2}{\sum_{\gamma=1}^{N} a^{(1)}_{\alpha \gamma} \xi_{\gamma 1}} = \text{const}$, and consequently the dependence on the spectral parameter disappears from $T^{(m)}$.

**Remark 2:** Using eq. (19) we may write eq. (7) in the equivalent form (see Appendix A, Sec. 7.3 for details)

$$V = -2 \Gamma r^T(\lambda) * R(\lambda, \mu) * (\Psi_0(\mu, \bar{\mu}; t) * R(\bar{\mu}, \nu) + \mathcal{I}(\mu, \nu))^{-1} * r(\nu) \Gamma^T.$$  

(20)
2.1.1 On availability of the nonlinear part in eq.(6)

Constructing the nonlinear PDE (6) one has to use such relations (17) and (18) that the nonlinear part does not completely disappear from eq.(6). We derive conditions removing the nonlinear part and then compare them with relations (17) and (18).

For this purpose we write the matrix field $V$ (7) as

$$V = -2r^T * R^{-1} * V_0 * R^{-1} * r,$$  (21)

$$V_0 = (\Psi_0 + R^{-1})^{-1}.$$  (22)

Then the nonlinear part of eq.(6) reads:

$$\sum_{m_1,m_2=1}^{D} L^{(m_1)} (V_\xi (\eta^{(m)})^TV - V\eta^{(m)}\xi^TV) R^{(m_2)} =$$

$$4 \sum_{m_1,m_2=1}^{D} L^{(m_1)} r^T * R^{-1} * V_0 * R^{-1} * r (\xi (\eta^{(m_1m_2)})^T - \eta^{(m_1m_2)} \xi^T) * r^T * R^{-1} * V_0 * R^{-1} * r R^{(m_2)}.$$ (23)

Now we transform this nonlinear part using eqs.(9), (10), (15) and write the result in components:

$$4 \sum_{m_1,m_2=1}^{D} \sum_{\gamma,\tilde{\gamma},\delta,\tilde{\delta}=1}^{N} \sum_{i,j,k=1}^{K} (L^{(m_1)})^{(\tilde{a}(i))}_\alpha (g(i) * V_0 * R^{-1} * r \xi)_\gamma (\tilde{a}^{(j;m_1)}\delta)_\delta (g(j) * R^{-1} * V_0 * g(k))_{\tilde{\delta}} (\tilde{a}^{(k)} \xi^T R^{(m_2)})_{\tilde{\beta}} -$$

$$\sum_{m_1=m_2=1}^{D} (L^{(m_1)})^{(\tilde{a}(i))}_\alpha (g(i) * V_0 * R^{-1} * g(j))_{\gamma\delta} (\tilde{a}^{(j;m_1)}\delta)_\delta (\tilde{a}^{(k)} \xi^T R^{(m_2)})_{\tilde{\beta}} (\tilde{a}^{(k)} \xi^T R^{(m_2)})_{\tilde{\beta}})$$

We see that the nonlinear part disappears if either

$$\sum_{m_1=1}^{D} (L^{(m_1)})^{(\tilde{a}(i))}_\alpha (\tilde{a}^{(j;m_1)}\delta) = 0,$$  (25)

$$\alpha, \gamma, \delta = 1, \ldots, N, \ i, j = 1, \ldots, K,$$

or

$$\sum_{m_2=1}^{D} (\tilde{a}^{(j;m_2)}(\tilde{a}^{(k)} \xi^T R^{(m_2)}))_{\gamma\beta} = 0,$$  (26)

$$\delta, \gamma, \beta = 1, \ldots, N, \ j, k = 1, \ldots, K.$$

In order to have a nonlinear part in PDE (6), conditions (25) and (26) must not be equivalent to relations (17) and (18) respectively.

2.2 Reduced version of general equation (6)

The matrix dimensionality $N$ of eq.(6) is defined by the parameter $K$ and increases as $K^4$, which follows from the analysis of eqs.(15-18) and will be derived in Sec.4. However, equation
admits the following reduced version with lower matrix dimensionality and $D^2$ independent variables:

$$
\sum_{m_1, m_2=1}^{D} \hat{L}^{(m_1)} \left( U_{m_1m_2} + U p p^T \hat{a}^{(1;m_1)} \hat{a}^{(1;m_2)} s^{-1} U - U s^{-1} \hat{a}^{(1;m_1)} p p^T U \right) \hat{R}^{(m_2)} = 0, \quad (27)
$$

where $p = (1\ldots1)^T$ is $N \times 1$ matrix, $\hat{L}^{(m_1)}$, $\hat{R}^{(m_2)}$ and $s$ are arbitrary constant diagonal matrices. The diagonal matrices $\hat{a}^{(1;m_1)}$ and $\hat{a}^{(1;m_2)}$ are related with $\hat{L}^{(m_1)}$ and $\hat{R}^{(m_2)}$ by the following linear algebraic equations

$$
\sum_{m_1=1}^{D} \hat{L}^{(m_1)} \hat{a}^{(1;m_1)} = 0, \quad (28)
$$

$$
\sum_{m_1=1}^{D} \hat{a}^{(1;m_1)} \hat{R}^{(m_1)} = 0, \quad i = 0, 1, \ldots, K - 1. \quad (29)
$$

where we assume, for any diagonal matrix $b$,

$$
b^{(1;m_1)}_{N+i} = b^{(1;m_1)}_{i}, \quad b^{(1;m_1)}_{N-i} = b^{(1;m_1)}_{i}, \quad i = 0, 1, \ldots, K - 1. \quad (30)
$$

Eqs. (28) and (29) may be conventionally taken as definitions of $\hat{a}^{(1;m_1)}$ and $\hat{a}^{(1;m_1)}$ in terms of $\hat{L}^{(m_1)}$ and $\hat{R}^{(m_1)}$.

Eq. (27) possesses a manifold of solutions $U(t)$ given in the form

$$
U(t) = \frac{s \tilde{V}(t)s}{1 + Q}, \quad Q = p^T s^2 p, \quad (31)
$$

where

$$
\tilde{V}(t) = -2 \sum_{k=1}^{K} (\Pi^T)^{k-1} g^{(k)}(\lambda) * R(\lambda, \mu) * \left( \Psi_0(\mu, \tilde{\mu}; t) * R(\tilde{\mu}, \nu) + \mathcal{I}(\mu, \nu) \right)^{-1} \sum_{i=1}^{K} g^{(i)}(\nu) \Pi^{i-1}. \quad (32)
$$

Here

$$
R(\lambda, \mu) = \mathcal{I}(\lambda, \mu) + \sum_{i,j=1}^{K} g^{(i)}(\lambda) \Pi^{i-1} s p p^T s (\Pi^T)^{j-1} g^{(j)}(\mu), \quad (33)
$$

$$
\Pi = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
\end{pmatrix}, \quad (34)
$$

$\Psi_0$ is defined by eqs. (31,12), where $T^{(m_1m_2)}$ has the form

$$
T^{(m_1m_2)}(\lambda) = \sum_{j=1}^{K} \tilde{g}^{(j)}(\lambda) \Pi^{j-1} \hat{a}^{(1;m_1)} \hat{a}^{(1;m_2)} (\Pi^T)^{j-1}, \quad (35)
$$
with

\[ g^{(1)}(\lambda) = 2 \left( s + \sum_{i=2}^{K} \hat{g}^{(i)}(\lambda) \Pi^{i-1} s \right)^{-1}, \]  

\[ g^{(j)}(\lambda) = 2 \hat{g}^{(j)}(\lambda) \left( s + \sum_{i=2}^{K} \hat{g}^{(i)}(\lambda) \Pi^{i-1} s \right)^{-1}, \quad j = 2, \ldots, K, \]

and \( \hat{g}^{(i)}(\lambda) = \frac{\hat{g}^{(i)}(\lambda)}{g^{(1)}(\lambda)} \), \( i = 2, \ldots, K \), \( \hat{g}^{(1)} = 1 \). Therewith condition (11) must hold. Again, we may introduce only \((K - 1)\) independent spectral parameters using functions \( \hat{g}^{(j)} \).

Remark: Without loss of generality, we take \( \hat{L}^{(1)} = \hat{R}^{(1)} = 1, \alpha = 1, \ldots, N \), in eqs. (28,29). To provide the unique solvability of these equations, we assume hat

\[ D \geq D^{(\text{min})} = 2K \]  

and write solutions \( \hat{a}^{(1;m_1)} \) and \( \hat{a}^{(1;m_2)} \), \( m_1, m_2 = 2, \ldots, D \), in the following Cramer’s forms:

\[ \hat{a}^{(1;m_1)}_{\beta} = -\frac{\Delta^{(m_1)}_{\beta}}{\Delta_{\beta}}, \]  

\[ \hat{a}^{(1;m_2)}_{\alpha} = -\frac{\Delta^{(m_1)}_{\alpha}}{\Delta_{\alpha}}, \quad m_1, m_2 = 2, \ldots, D, \]  

where

\[ \Delta_{\beta} = \begin{vmatrix} \hat{L}^{(2)}_{\beta-K+1} & \hat{L}^{(3)}_{\beta-K+1} & \cdots & \hat{L}^{(D)}_{\beta-K+1} \\ \hat{L}^{(2)}_{\beta-K+2} & \hat{L}^{(3)}_{\beta-K+2} & \cdots & \hat{L}^{(D)}_{\beta-K+2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{L}^{(2)}_{\beta+1} & \hat{L}^{(3)}_{\beta+1} & \cdots & \hat{L}^{(D)}_{\beta+1} \\ \hat{L}^{(2)}_{\beta+K-1} & \hat{L}^{(3)}_{\beta+K-1} & \cdots & \hat{L}^{(D)}_{\beta+K-1} \end{vmatrix}, \quad \Delta_{\alpha} = \begin{vmatrix} \hat{R}^{(2)}_{\alpha-K+1} & \hat{R}^{(3)}_{\alpha-K+1} & \cdots & \hat{R}^{(D)}_{\alpha-K+1} \\ \hat{R}^{(2)}_{\alpha-K+2} & \hat{R}^{(3)}_{\alpha-K+2} & \cdots & \hat{R}^{(D)}_{\alpha-K+2} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{R}^{(2)}_{\alpha+K-1} & \hat{R}^{(3)}_{\alpha+K-1} & \cdots & \hat{R}^{(D)}_{\alpha+K-1} \end{vmatrix}. \]  

Here \( \Delta^{(m_2)}_{\beta} \) is obtained from \( \Delta_{\beta} \) by replacing the \( m_2 \)th column with the column of units. Analogously, \( \Delta^{(m_1)}_{\alpha} \) is obtained from \( \Delta_{\alpha} \) by replacing the \( m_1 \)th column with the column of units.

### 2.2.1 On availability of the nonlinear part in eq. (27)

To reveal conditions removing the nonlinear part, we write eq. (27) in components taking into account relations (28,29):

\[ \sum_{m_1, m_2=1}^{D} \hat{L}^{(m_1)}_{\alpha} \left( U_{\alpha\beta} \right)_{m_1 m_2} + \sum_{\substack{\gamma, \delta=1 \atop \gamma \neq \delta}}^{N} \hat{a}^{(1;m_1 m_2)}_{\delta} s_{\delta} \left( U_{\alpha\gamma} U_{\delta\beta} - U_{\alpha\delta} U_{\gamma\beta} \right) \hat{R}^{(m_2)}_{\beta} = 0, \]  

\( \alpha, \beta = 1, \ldots, N, \)
where \( Z(\alpha, \beta, K) = \{ \alpha \pm i, \beta \pm i, i = 1, \ldots, K - 1 \} \). We see, that the nonlinear part disappears from eq. (41) if \( N < 2K \). If \( N = 2K \), then the nonlinear part exists only if \( \alpha = \beta \), i.e. only the diagonal elements \( U_{\alpha \alpha} \) satisfy the nonlinear PDE. In general, if \( N > 2K \), then the nonlinear part exists in equations having \( \alpha = \sigma \pm i, \beta = \sigma \pm j \) with \( i, j = 0, 1, \ldots, N - 2K \). Thus, if \( 2K \leq N < 4K - 1 \) we have a ”partial nonlinearity”, i.e some elements of the matrix \( U \) satisfy the linear equations. Note that, in many cases, this is a ”hidden linearity”, i.e. all nonlinear terms (which are quadratic by construction) involve elements of the matrix \( U \) satisfying the linear PDEs. This is shown more explicitly in the example of Sec. 2.5. The system becomes fully nonlinear if \( 2(N - 2K) + 1 \geq N \), i.e. \( N \geq 4K - 1 \).

### 2.3 Hermitian reductions \( V = V^+ \) and \( U = U^+ \).

Eqs. (6) and (27) admit the Hermitian reduction

\[
V = V^+, \quad U = U^+ \quad (42)
\]

with

\[
R^{(m_1)} = (L^{(m_1)})^T, \quad \text{Im} \ L^{(m_1)} = 0, \quad t_{m_1m_2} = t_{m_2m_1}, \quad (43)
\]

\[
\text{Re} \ \xi = 0, \quad \text{Re} \ s = 0. \quad (44)
\]

\[
\text{Im} \ \eta^{(m_1m_2)} = 0, \quad \text{Im} \ \hat{a}^{(1,1m_1)} = 0, \quad (45)
\]

\[
\hat{a}^{(i;m_1)} = \hat{a}^{(i;1m_1)}, \quad m_1 = 1, \ldots, D. \quad (46)
\]

\[
C^+(\mu, \lambda) = C(\lambda, \mu), \quad (47)
\]

\[
\hat{g}^{(k)}(\lambda) = g^{(k)}(\lambda). \quad (48)
\]

This reduction reduces the number of independent variables in the nonlinear PDEs from \( D^2 \) to \( D(D + 1)/2 \). Conditions (44) assume

\[
\xi = i \xi_0, \quad s = is_0, \quad (49)
\]

where \( \xi_0 \) is a real \( N \times 1 \) matrix and \( s_0 \) is a real diagonal matrix. Condition (45) means reality of \( \eta^{(m_1m_2)} \) and \( \hat{a}^{(i;1m_1)} \). The nonlinear PDEs (6) and (27) under the Hermitian reduction read, respectively:

\[
\sum_{m_1,m_2=1}^D L^{(m_1)}(V_{m_1m_2} + iV \xi_0(\eta^{(m_1m_2)})^T V - iV \eta^{(m_1m_2)} \xi_0^T V)(L^{(m_2)})^T = 0, \quad (50)
\]

\[
\sum_{m_1,m_2=1}^D \hat{L}^{(m_1)}(U_{m_1m_2} + iUpp^T \hat{a}^{(1,1m_1)} \hat{a}^{(1,1m_2)} s_0^{-1} U - iUs_0^{-1} \hat{a}^{(1,1m_1)} \hat{a}^{(1,1m_2)} pp^T U) \hat{L}^{(m_2)} = 0. \quad (51)
\]

### 2.4 Simplest nontrivial nonlinear PDE (51): \( K = 1, \ N = 3 \)

If \( K = 1 \), then, owing to eq. (37), \( D^{(\min)} = 2 \). Setting \( D = D^{(\min)} \) we obtain \( D(D + 1)/2 = 3 \) independent variables in the nonlinear PDE. In accordance with eqs. (35) and (36), there is no spectral parameter in \( T^{(m_1m_2)} \) and, consequently, there is no arbitrary functions of independent
variables in the solution space. If \( N = 2 \) then the nonlinearity appears only in the diagonal part. The system becomes completely nonlinear if \( N = 3 \). In this case, eq. (51) reads:

\[
\sum_{m_1, m_2 = 1}^{2} \hat{L}^{(m_1)}{\alpha} \left( (U_{\alpha\beta})_{t_{m_1} m_2} - i \sum_{\delta, \gamma, \beta = 1}^{3} \hat{a}^{(1:1 m_2)}_{\delta} \hat{a}^{(1: m_1)}_{\delta} \frac{(s_0)_{\delta}}{(s_0)_{\delta}} (U_{\alpha\gamma} U_{\delta\beta} - U_{\alpha\delta} U_{\gamma\beta}) \right) \hat{L}^{(m_2)} = 0, (52)
\]

\( \alpha, \beta = 1, 2, 3, \)

which is a system of 6 independent scalar equations for three real fields \( U_{ii}, i = 1, 2, 3, \) and three complex fields \( U_{12}, U_{13} \) and \( U_{23} \).

### 2.5 PDE (51) with arbitrary functions of two independent variables in the solution space: \( K = 2, N = 6 \)

Setting \( K = 2 \) we obtain \( D^{(\text{min})} = 4 \) owing to eq. (37). Let \( D = D^{(\text{min})} \), the number of independent variables in the nonlinear PDE is \( D(D + 1)/2 = 10 \). In accordance with eqs. (35) and (36), there is one spectral parameter in \( T^{(m_1 m_2)} \), and consequently arbitrary functions of two independent variables may be introduced in solution space. It is noted in Remark 2 of Sec 2.2 that the system becomes completely nonlinear if the matrix dimensionality \( N \geq 4K - 1 = 7 \). If \( N \leq 3 \) then the system is completely linear. Consider the intermediate case \( 3 < N < 7 \). Then the matrix equation (51) can be separated into two families of scalar PDEs. First family consists of the linear PDEs for the set of fields \( U^{(\text{lin})} \). The second family consists of the non-linear PDEs for another set of fields \( U^{(\text{nl})} \). If the matrix dimensionality \( N = 4 \), then only the diagonal part of eq. (51) is nonlinear, i.e. \( \{U_{\alpha\alpha}, \alpha = 1, \ldots, N\} \in U^{(\text{lin})}, \{U_{\alpha\beta}, \alpha \neq \beta, \alpha, \beta = 1, \ldots, N\} \in U^{(\text{nl})} \). If \( N = 5 \), then the nonlinearity appears in the non-diagonal part, however this nonlinearity is trivial because each quadratic term of this nonlinearity involves fields from the set \( U^{(\text{lin})} \) and consequently the system of PDEs is essentially linear (the "hidden linearity"). The first nontrivial case corresponds to \( N = 6 \). The nonlinear system consists of 21 independent scalar equations for the fields \( U_{\alpha\beta}, \beta \geq \alpha \). Three equations are linear PDEs and 18 are nonlinear PDEs, i.e

\[
\{U_{14}, U_{25}, U_{36}\} \in U^{(\text{lin})},
\{U_{\alpha\alpha}, \alpha = 1, \ldots, 6, U_{12}, U_{13}, U_{15}, U_{16}, U_{23}, U_{24}, U_{26}, U_{34}, U_{35}, U_{45}, U_{46}, U_{56}\} \in U^{(\text{nl})}, (53)
\]

where \( U_{\alpha\alpha}, \alpha = 1, \ldots, 6, \) are the real fields and others are the complex ones. The nonlinear system (51) can be written as

\[
\sum_{m_1, m_2 = 1}^{4} \hat{L}^{(m_1)}{\alpha} \left( (U_{\alpha\beta})_{t_{m_1} m_2} - i \sum_{\gamma, \delta, K = 1}^{6} \hat{a}^{(1:1 m_2)}_{\delta} \hat{a}^{(1: m_1)}_{\delta} \frac{(s_0)_{\delta}}{(s_0)_{\delta}} (U_{\alpha\gamma} U_{\delta\beta} - U_{\alpha\delta} U_{\gamma\beta}) \right) \hat{L}^{(m_2)} = 0, (54)
\]

\( \alpha, \beta = 1, \ldots, 6, \beta \neq \alpha + 3, \)

\[
\sum_{m_1, m_2 = 1}^{4} \hat{L}^{(m_1)}{\alpha} (U_{\alpha(\alpha+3)})_{t_{m_1} m_2} \hat{L}^{(m_2)}{\alpha+3}, \alpha = 1, 2, 3. (55)
\]

If \( U_{\alpha(\alpha+3)} = 0, \alpha = 1, 2, 3, \) then eqs. (55) become identities and one has a system of 18 scalar equations (55).
3 Richness of the solution space

Now we discuss the richness of the solution space to nonlinear PDEs (6) and (27). Since the \(t\)-dependence is introduced through the diagonal matrix \(\varepsilon(\lambda, t)\) given in eq.(12) with \(T^{(m_1m_2)}\) defined by eq.(13) (or by eq.(35)), the richness of the solution space is defined by the function \(\Psi_0\) (8):

\[
(\Psi_0(\lambda, \mu; t))_{\gamma\delta} = C_{\gamma\delta}(\lambda, \mu) \exp \left[ \sum_{m_2=m_1}^D \sum_{m_1=1}^D t_{m_1m_2} \left( T_{\gamma}^{(m_1m_2)}(\lambda) - T_{\delta}^{(m_1m_2)}(\mu) \right) \right]. \tag{56}
\]

Since the number of the arbitrarily introduced variables \(t_{m_1m_2}\) coincides with the number of the linearly independent combinations of the spectral parameters in the exponent of expression (56), we have to define the number of such combinations.

3.1 Dimensionality of solution space to eq.(6)

Using expressions (13) for \(T^{(m)}\) we write the argument in the exponent of eq.(56) as

\[
\sum_{m_1, m_2=1}^D t_{m_1m_2} \left( T_{\gamma}^{(m_1m_2)}(\lambda) - T_{\delta}^{(m_1m_2)}(\mu) \right) = \tag{57}
\]

\[
\sum_{m_1, m_2=1}^D \sum_{j=1}^{K} \left( g^{(j)}(\lambda) a^{(j;m_11)}_{\gamma} \tilde{a}^{(j;1m_2)}_{\gamma} t_{m_1m_2} - g^{(j)}(\mu) a^{(j;m_11)}_{\delta} \tilde{a}^{(j;1m_2)}_{\delta} t_{m_1m_2} \right). \]

Remember that only \(K-1\) functions out of the list \(g^{(i)}, i = 1, \ldots, K\), are independent functions of spectral parameters. Since the coefficients ahead of the functions \(g^{(j)}(\lambda)\) and \(g^{(j)}(\mu), j = 1, \ldots, K\), are independent combinations of \(t_{m_1m_2}\) (in general) for each pair of \(\gamma\) and \(\delta, \gamma \neq \delta\), these \(2(K-1)\) functions may introduce up to \(2(K-1)\) independent spectral parameters in general, unless there are additional relations among \(\tilde{a}^{(j;m_11)}_{\gamma}\) with different values of \(\gamma\), like those which appear due to the reduction in Sec 2.2 where \(T^{(m_1m_2)}\) are defined by eq.(35). If \(\gamma = \delta\), then expression (57) involves only \(K-1\) independent functions \(\tilde{g}^{(j)}(\lambda) - \tilde{g}^{(j)}(\mu), j = 2, \ldots, K\) which may introduce up to \(K-1\) independent spectral parameters. Thus, the expression (56) introduces \(N(N-1)\) arbitrary functions of \(2(K-1)\) variables and \(N\) arbitrary functions of \((K-1)\) variables, provided that all components of \(C\) are independent.

3.2 Dimensionality of solution space to eq.(27)

In the case of eq.(27), we shall use expression (35) for \(T^{(m_1m_2)}\) so that the exponent in eq. (56) reads

\[
\sum_{m_2=m_1}^D \sum_{m_1=1}^D t_{m_1m_2} \left( T_{\gamma}^{(m_1m_2)}(\lambda) - T_{\delta}^{(m_1m_2)}(\mu) \right) = \tag{58}
\]

\[
\sum_{m_2=m_1}^D \sum_{m_1=1}^D \sum_{j=1}^{K} \left( g^{(j)}(\lambda) \left( \Pi^{j-1} a_{(1;m_11)}^{(1;m_21)} (\Pi^T)^{j-1} \right)_{\gamma} t_{m_1m_2} - g^{(j)}(\mu) \left( \Pi^{j-1} a_{(1;m_11)}^{(1;m_21)} (\Pi^T)^{j-1} \right)_{\delta} t_{m_1m_2} \right). \]
\[
\sum_{m_2=m_1}^{D} \sum_{m_1=1}^{D} \sum_{j=1}^{K} \left( \tilde{g}^{(j)}(\lambda) \hat{a}^{(1;m_1)}_{\gamma+j-1} \hat{a}^{(1;m_2)}_{\gamma+j-1} t_{m_1 m_2} - \tilde{g}^{(j)}(\mu) \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2} \right).
\]

In this case we may have \(2(K - 1)\) arbitrary spectral parameters only for those \(\gamma\) and \(\delta\) which satisfy the following condition:

\[
|\gamma - \delta| \geq K - 2. \tag{59}
\]

In fact, suppose that \(\gamma - \delta = \Delta > 0\), then we write eq.\((58)\) as

\[
\sum_{m_2=m_1}^{D} \sum_{m_1=1}^{D} \sum_{j=1}^{K} \left( \tilde{g}^{(j)}(\lambda) \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2} - \tilde{g}^{(j)}(\mu) \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2} \right) = \tag{60}
\]

\[
\sum_{m_2=m_1}^{D} \sum_{m_1=1}^{D} \sum_{j=1}^{D} \sum_{j=1}^{K} \left( \tilde{g}^{(j)}(\lambda) \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2} - \tilde{g}^{(j)}(\mu) \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2} \right).
\]

Let us introduce the following independent variables in eq.\((60)\):

\[
y_{1j} = \sum_{m_2=m_1}^{D} \sum_{m_1=1}^{D} \sum_{j=1}^{K} \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2}, \quad j = 1, \ldots, K, \tag{61}
\]

\[
y_{2j} = \sum_{m_2=m_1}^{D} \sum_{m_1=1}^{D} \sum_{j=1}^{K} \hat{a}^{(1;m_1)}_{\delta+j-1} \hat{a}^{(1;m_2)}_{\delta+j-1} t_{m_1 m_2}, \quad j = 1, \ldots, \Delta.
\]

All in all one has \(K + \Delta\) variables \(y_{ij}\). However, there are only \(K - 1\) independent functions \(\tilde{g}^{(i)}_\alpha\) (for fixed \(\alpha\)) of spectral parameters. Consequently, we may introduce \(i\) only \(K - 1\) independent spectral parameters if \(\Delta = 0\), \(ii\) \(K + \Delta\) independent spectral parameters if \(0 < \Delta < K - 1\), and \(iii\) \(2(K - 1)\) independent parameters if \(\Delta \geq K - 1\).

All in all, the \(N \times N\) function \(\Psi_0\) introduces \(N\) functions of \(K - 1\) variables; the number of functions of \(K + \Delta\) variables equals \(2(N - \Delta)\), \(1 \leq \Delta \leq K - 3\); the number of functions of \(2(K - 1)\) variables equals \(N^2 - N - 2 \sum_{\Delta=1}^{K-3}(N - \Delta) = (N - K + 2)(N - K + 3)\).

### 3.3 Number of arbitrary functions in the solution space under the Hermision reduction

The Hermitian reduction reduces the number of arbitrary functions in the solution space according to the number of independent elements in the matrix \(U\). Owing to the relation \((17)\), the number of arbitrary complex functions of \(2(K - 1)\) variables reduces two times in solution spaces of both eqs. \((50)\) and \((51)\) in comparison with solution spaces of eqs. \((6)\) and \((27)\) respectively. Thus, regarding eq.\((50)\), this number is \(N(N - 1)/2\) instead of \(N(N - 1)\). The number of arbitrary functions of \((K - 1)\) variables in eq.\((50)\) remains \(N\), but these functions becomes real (rather than complex). Regarding eq.\((51)\), the number of functions of \(K - 1\) variables is \(N\) and all of them are real. The number of other functions is reduced two times.
4 Relations among the number of independent variables, matrix dimensionality and dimensionality of solution space for eqs. (6) and (27)

Here we collect results regarding the relations among such important parameters of eqs. (6) and (27) as the number of independent variables in the nonlinear PDE, its matrix dimensionality and the dimensionality of the solution space. We show that all parameters may be expressed in terms of the parameter $K$ (the number of independent diagonal matrix functions $g^{(i)}(\lambda)$ in the definition of the function $r(\lambda)$, see eq. (10)). We establish the following relations.

Relations among parameters in eq. (6). It follows from eq. (57) that the solution space admits up to $N(N-1)$ arbitrary functions of $2(K-1)$ variables and up to $N$ arbitrary functions of $K-1$ variables. In addition, the matrices $L^{(m_1)}$ (or $R^{(m_1)}$), $m_1 = 1, \ldots, D$ are considered as solutions to the system of $K^2$ matrix equations (17) (or (18)). This is possible if the number of equations does not exceed the number of matrices $L^{(m_1)}$ (or $R^{(m_1)}$), $m_1 = 1, \ldots, D$, i.e

$$D \geq D^{(\text{min})} = K^2 + 1,$$

which relates parameters $D$ and $K$. Next, the system of diagonal equations (15) for any fixed $i$, $i = 1, \ldots, K$, consists of $D^2 N$ scalar equations for the $N^2$ elements of the matrix $a^{(i)}$. It may be solved for these elements if

$$N \geq N^{(\text{min})} = (D^{(\text{min})})^2 = (K^2 + 1)^2,$$

which relates parameters $N$ and $K$. Thus, the number of independent variables in the nonlinear PDE is $D^2 \geq (K^2 + 1)^2$, while the matrix dimensionality $N \geq D^2$.

Relations among parameters in eq. (27). From eqs. (58-60), it follows that the solution space admits arbitrary functions of $2(K-1)$ variables, similar to the solution space to eq. (6). However their number is $(N-K+2)(N-K+3)$ which is less then in eq. (6), while the number of arbitrary functions of $K-1$ variables is respectively bigger in comparison with this number in eq. (6) and equals $N + 2(N-1) = 3N - 1$ the same. Next, the system (28) (or the system (29)) for any fixed $\alpha$ consists of $2K - 1$ equations. It may be solved for $D$ parameters $\hat{a}^{(1:m_1)}$ (or $\hat{a}^{(1:1:m_1)}$), $m_1 = 1, \ldots, D$, if

$$D \geq D^{(\text{min})} = 2K,$$

Consequently, the number of independent variables in the nonlinear PDE is $D^2 \geq 4K^2$. The matrix dimensionality of the nonlinear PDE $N$ is also related with the parameter $K$, see the paragraph after eq. (11). Namely, the "partial nonlinearity" or "hidden linearity" appears if $N$ satisfies the following condition:

$$2K \leq N < 2D^{(\text{min})} - 1 = 4K - 1.$$

The PDE (27) becomes completely nonlinear if

$$N \geq 4K - 1.$$
5  Explicit form of particular solutions

To construct the particular solutions to eq. (6) using eq. (7) or to eq. (27) using eqs. (31,32), one has to invert the operator $\Psi_0 * R + I$. In other words, one has to solve the equation

$$\mathcal{I}(\lambda, \nu) = \theta(\lambda, \mu) * (\Psi_0(\mu, \tilde{\mu}) * R(\tilde{\mu}, \nu) + \mathcal{I}(\mu, \nu))$$

(67)

for the function $\theta$. This can be done explicitly in the particular case of the degenerate kernel $\Psi_0(\lambda, \mu)$, which is done in the next subsection.

5.1 Degenerate kernel $C(\lambda, \mu)$

Eq. (67) may be solved in the particular case of the degenerate kernel $C(\nu, \mu)$:

$$C(\nu, \mu) = \sum_{i=1}^{N_0} u^{(i)}(\nu) v^{(i)}(\mu),$$

(68)

where $N_0$ is some integer. Then we may write

$$\Psi_0(\nu, \mu; t) * R(\mu, \lambda) = \sum_{i=1}^{N_0} \phi^{(i)}(\nu; t) \psi^{(i)}(\lambda; t),$$

(69)

$$\phi^{(i)}(\nu; t) = \varepsilon(\nu; t) u^{(i)}(\nu), \quad \psi^{(i)}(\lambda) = \left( v^{(i)}(\mu; t) \varepsilon^{-1}(\mu; t) \right) * R(\mu, \lambda).$$

Substituting eq. (69) into eq. (67) we obtain:

$$\mathcal{I}(\lambda, \nu) = \theta(\lambda, \nu; t) + \sum_{i=1}^{N_0} \theta^{(i)}(\lambda, t) \psi^{(i)}(\nu; t),$$

(70)

where

$$\theta^{(i)}(\lambda, t) = \theta(\lambda, \nu; t) * \phi^{(i)}(\nu; t).$$

(71)

To derive the system of equations for $\theta^{(i)}$, let us apply $*\phi^{(k)}(\lambda; t)$ to the eq. (70). We obtain

$$\phi^{(k)}(\lambda, t) = \theta^{(k)}(\lambda, t) + \sum_{i=1}^{N_0} \theta^{(i)}(\lambda, t) S^{(ik)},$$

(72)

where

$$S^{(ik)}(t) = \psi^{(i)}(\lambda; t) * \phi^{(k)}(\lambda; t) = v^{(i)}(\mu) * v^{(k)}(\mu) + v^{(i)}(\mu) * \left( \varepsilon^{-1}(\mu; t) r(\mu) \right) \xi T r^T(\lambda) \varepsilon(\lambda; t) * u^{(k)}(\lambda).$$

(73)

Eqs. (72) represent the linear non-homogeneous system of $N_0$ matrix $N \times N$ equations for the matrices $\theta^{(k)}$, $k = 1, \ldots N_0$. The solvability of this equations must be provided by the matrices $S^{(ik)}$. In the simplest case $N_0 = 1$, eq. (72) may be easily solved:

$$\theta^{(1)}(\lambda, t) = \phi^{(1)}(\lambda, t) \left( I_N + S^{(11)}(t) \right)^{-1}.$$

(74)
After functions $\theta^{(k)}$ have been found, eq.(70) yields us expression for $\theta$:

$$\theta(\lambda, \nu, t) = \mathcal{I}(\lambda, \nu) - \sum_{i=1}^{N_0} \theta^{(i)}(\lambda, t) \psi^{(i)}(\nu; t).$$

(75)

Now we may write the expression for $V(t)$ (the solution to eq.(6)) using the formula (20) as follows:

$$V(t) = -2r^T(\lambda) * r(\lambda) \Gamma + 2r^T(\lambda) * \sum_{i=1}^{N_0} \theta^{(i)}(\lambda) \left(v^{(i)}(\mu)\varepsilon^{-1}(\mu)\right) * r(\mu),$$

(76)

where $r$ is given by eq.(10). Eq.(32) yields the following formula for $\tilde{V}(t)$:

$$\tilde{V}(t) = -2 \sum_{k=1}^{K} (\Pi^{k-1})^T g^{(k)}(\lambda) * \left(\mathcal{I}(\lambda, \nu) + r(\lambda) \xi^T r^T(\nu)\right) * \sum_{j=1}^{K} g^{(j)}(\nu) \Pi^{j-1} +$$

$$+ 2 \sum_{k=1}^{K} (\Pi^{k-1})^T g^{(k)}(\lambda) * \left(\mathcal{I}(\lambda, \mu) + r(\lambda) \xi^T r^T(\mu)\right) *$$

$$\sum_{i=1}^{N_0} \theta^{(i)}(\mu) \left(v^{(i)}(\mu)\varepsilon^{-1}(\mu)\right) * \left(\mathcal{I}(\nu, \bar{\nu}) + r(\nu) \xi^T r^T(\bar{\nu})\right) * \sum_{j=1}^{K} g^{(j)}(\bar{\nu}) \Pi^{j-1},$$

where

$$r(\lambda) \xi = \sum_{k=1}^{K} g^{(k)}(\lambda) \Pi^{k-1}.$$

(78)

Hereafter we set

$$d\Omega(\lambda) = \prod_{i=1}^{K} d\lambda_i.$$

(79)

We have to satisfy conditions (11) by the proper choice of the functions $g^{(k)}(\lambda)$. A possible choice is the following.

$$g^{(i)}_a(\lambda) = c^{(i)}_a \lambda_i \exp \frac{1}{2} \left( - \sum_{j=1}^{K} \left( w_j^2 (Re \lambda_j)^2 + w_j^2 (Im \lambda_j)^2 \right) \right).$$

(80)

Here $w_j$, $j = 1, 2$, and $c^{(i)}$, $i = 1, \ldots, K$, are the real parameters. Then we obtain

$$\varpi^{(ii)}_a = g^{(i)}_a * g^{(i)}_a = \frac{1}{2} (w_1^2 - w_2^2) \frac{\pi^K (c^{(i)}_a)^2}{2(w_1 w_2)^{K+2}}, \quad \varpi^{(ij)} = g^{(i)}_a * g^{(j)}_a = 0, \ i \neq j.$$
which must be used in $T^{(m_1 m_2)}(\lambda)$. In fact, for the system (6), we substitute $g^{(i)}$ given by eqs.(80) into eqs.(14) and obtain:

$$
\tilde{g}^{(1)}_{\alpha}(\lambda) = \frac{2}{\sum_{\gamma=1}^{N} a_{\alpha\gamma} \xi_{\gamma} + \sum_{i=2}^{K} \sum_{\gamma=1}^{N} (c_{\alpha}^{(i)} \lambda_{i})/(\lambda_{1} c_{\alpha}^{(i)}) a_{\alpha\gamma} \xi_{\gamma}},
$$

(82)

$$
\tilde{g}^{(j)}_{\alpha}(\lambda) = \frac{2(c_{\alpha}^{(j)} \lambda_{j})/(\lambda_{1} c_{\alpha}^{(i)})}{s_{\alpha} + \sum_{i=2}^{K} (c_{\alpha}^{(i)} \lambda_{i})/(\lambda_{1} c_{\alpha}^{(i)}) s_{\alpha+i-1}}, \quad j = 2, \ldots, K,
$$

(83)

which must be used in $T^{(m_1 m_2)}$ defined by eq.(13). Similarly, for the system (27), substituting eqs.(80) into eqs.(36) we obtain:

$$
\tilde{g}^{(1)}_{\alpha}(\lambda) = \frac{2 c_{\alpha}^{(1)}}{s_{\alpha} + \sum_{i=2}^{K} (c_{\alpha}^{(i)} \lambda_{i})/(\lambda_{1} c_{\alpha}^{(i)}) s_{\alpha+i-1}},
$$

(84)

$$
\tilde{g}^{(j)}_{\alpha}(\lambda) = \frac{2(c_{\alpha}^{(j)} \lambda_{j})/(\lambda_{1} c_{\alpha}^{(i)})}{s_{\alpha} + \sum_{i=2}^{K} (c_{\alpha}^{(i)} \lambda_{i})/(\lambda_{1} c_{\alpha}^{(i)}) s_{\alpha+i-1}}, \quad j = 2, \ldots, K,
$$

(85)

which must be used in $T^{(m_1 m_2)}$ defined by eq.(35).

We see that the parameter $\lambda_1$ appears only through the ratios $\lambda_j/\lambda_1$ in eqs.(82), (83) and, consequently, in $T^{(m_1 m_2)}(\lambda)$. Thus, there are $K-1$ complex parameters $\tilde{\lambda}_i = \lambda_i/\lambda_1$, $i = 2, \ldots, K$ in the function $\varepsilon(\lambda; t)$. This is the maximal number of independent spectral parameters which can be introduced in eq.(14) (or (36)).

### 5.2 Multisoliton solutions to eqs.(6) and (27)

In order to obtain the multisoliton solution, we have to introduce $\delta$-functions in $v^{(i)}$ and $u^{(i)}$:

$$
u^{(i)}(\lambda) = \delta(\lambda - p_i), \quad v^{(i)}(\mu) = v_0^{(i)} \delta(\mu - q_i),
$$

(86)

where we use notation

$$
\delta(\lambda - p_i) = \prod_{j=1}^{D} \delta(\lambda_j - p_{ji}), \quad \delta(\mu - q_i) = \prod_{j=1}^{D} \delta(\mu_j - q_{ji}).
$$

(87)

Here $v_0^{(i)}$ are constant matrices and $p_i = \{p_{1i}, \ldots, p_{Di}\}$ and $q_i = \{q_{1i}, \ldots, q_{Di}\}$ are some vector complex parameters, $p_i \neq q_i$, $i, j = 1, \ldots, N_0$. Eqs.(72) read as:

$$
\varepsilon(p_k; t) \delta(\lambda - p_k) = \theta^{(k)}(\lambda; t) + \sum_{i=1}^{N_0} \theta^{(i)}(\lambda; t) S^{(ik)},
$$

(88)

where $S^{(ik)}$ are given by eq.(73) after substitution (84):

$$
S^{(ik)}(t) = v_0^{(i)} \ast \left( \varepsilon^{-1}(q_i; t) r(q_i) \right) \xi \xi^T \left( r^T(p_k) \varepsilon(p_k; t) \right),
$$

(89)

and we take into account that $v^{(i)} \ast u^{(j)} = 0$, $i, j = 1, \ldots, N_0$. We may collect the $t$-dependence in the RHS of equation (88) multiplying it by $\varepsilon^{-1}(p_k; t)$ from the right and introducing the function

$$
\hat{\theta}^{(i)}(\lambda; t) = \theta^{(i)}(\lambda; t) v_0^{(i)} \varepsilon^{-1}(q_i).
$$

(90)
Then eq. (86) yields
\[ \delta (\lambda - p_k) = \sum_{i=1}^{N_0} \hat{\theta}^{(i)}(\lambda, t) \hat{S}^{(ik)}, \] (89)
where
\[ \hat{S}^{(ik)} = \varepsilon (q_i, t) (v_0^{(i)})^{-1} \varepsilon^{-1} (p_i, t) \delta_{ik} + r(q_i) \xi^T r^T (p_k). \] (90)

### 5.2.1 Non-singularity conditions

In general, functions \( \hat{\theta}^{(i)} \) have singularities in the \( t \)-space which lead to the singularities in the solutions \( V(t) \) and \( U(t) \) to the nonlinear PDEs. To derive the non-singularity conditions we rewrite the system (89) in the following row matrix form:
\[ [\delta (\lambda - p_1) \ldots \delta (\lambda - p_{N_0})] = [\hat{\theta}^{(1)} \ldots \hat{\theta}^{(N_0)}]^T S, \] (91)
where the matrix \( S \) has the following block structure:
\[ S = \begin{pmatrix} S^{(11)} & S^{(12)} & \cdots & S^{(1N_0)} \\ S^{(21)} & S^{(22)} & \cdots & S^{(2N_0)} \\ \vdots & \vdots & \ddots & \vdots \\ S^{(N_01)} & S^{(N_02)} & \cdots & S^{(N_0N_0)} \end{pmatrix}. \] (92)

In other words, the elements of \( S \) read
\[ S_{N(i-1)+\alpha, N(j-1)+\beta} = \hat{S}^{(ij)}_{\alpha, \beta}. \] (93)

Solution \( \theta^{(i)}(\lambda, t) \) has no singularities if
\[ |\det S| \neq 0 \ \forall t. \] (94)

Using eq. (90) we may write
\[ S_{ij} = \varepsilon_i \hat{v}_{ij}^{-1} \hat{\varepsilon}_j + \hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)}, \] (95)
where the matrices \( \varepsilon, \hat{v} \) and \( \hat{r}^{(i)}, i = 1, 2, \) are defined by their elements as (similar to eq. (93))
\[ \hat{v}_{N(i-1)+\alpha, N(j-1)+\beta} = (v_0^{(i)})_{\alpha, \beta} \delta_{ik}, \] (96)
\[ \varepsilon_{N(i-1)+\alpha} = (q_i)_{\alpha}, \ \hat{\varepsilon}_{N(k-1)+\beta} = (\varepsilon^{-1}_i (p_k), \]
\[ \hat{r}_{N(i-1)+\alpha}^{(1)} = (r(q_i))_{\alpha}, \ \hat{r}_{N(k-1)+\beta}^{(2)} = (\xi^T r^T (p_k)). \]

Let us derive the constraints for the elements of \( v_0^{(i)} \) and \( \xi \) which provide condition (94). The later has the following explicite form:
\[
\left| \prod_{k,l=1}^{N_0} \frac{\varepsilon_k \hat{\varepsilon}_l}{\det \hat{v}} \left( 1 + \sum_{i,j=1}^{N_0} \frac{\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{v}_{ij}}{\varepsilon_i \hat{\varepsilon}_j} \right) \right| \equiv \left| \prod_{k,l=1}^{N_0} \frac{\varepsilon_k \hat{\varepsilon}_l}{\det \hat{v}} \left( 1 + \sum_{i,j=1}^{N_0} \frac{\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{v}_{ij}}{\varepsilon_i \hat{\varepsilon}_j} \right) \right| \neq 0. \] (97)
The first factor $\prod_{k,l=1}^{N_0N} \frac{\varepsilon_k \tilde{\varepsilon}_l}{\det \hat{v}}$ is always positive if $\det \hat{v} \neq 0$, which will be assumed hereafter. Thus, condition (98) reduces to
\[
\left| \left( 1 + \sum_{i,j=1}^{N_0} \frac{\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij}}{\varepsilon_i \varepsilon_j} \right) \right| \neq 0. \tag{98}
\]

The analysis of condition (98) depends on whether combinations $\varepsilon_i \varepsilon_j$ depend on $t$. We consider the real matrix $v_0$. Then condition (98) may be simply satisfied if the arguments of the matrix exponent $\varepsilon$ are either real or imaginary. First, we assume that these arguments are real and consider two cases.

1. Let combinations $\varepsilon_i \tilde{\varepsilon}_j$ depend on $t$ for all $i$ and $j$. Condition (98) is satisfied if coefficients ahead of all exponents are positive, i.e.
\[
\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij} > 0, \quad \forall i, j. \tag{99}
\]

2. Let $\varepsilon_i \tilde{\varepsilon}_i = 1$ and combinations $\varepsilon_i \tilde{\varepsilon}_j$ depend on $t$ if $i \neq j$. Then condition (98) reads
\[
\left| \left( 1 + \sum_{i=1}^{N_0} \hat{r}_{i}^{(1)} \hat{r}_{i}^{(2)} \hat{\nu}_{ii} + \sum_{i,j=1}^{N_0} \frac{\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij}}{\varepsilon_i \varepsilon_j} \right) \right| \neq 0. \tag{100}
\]
which holds if
\[
\text{sign} \left( \hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij} \right) = \text{sign} \left( 1 + \sum_{i=1}^{N_0} \hat{r}_{i}^{(1)} \hat{r}_{i}^{(2)} \hat{\nu}_{ii} \right), \quad i \neq j. \tag{101}
\]
Second, we assume that the arguments of the exponent $\varepsilon$ are imaginary and consider two cases analogous to pp.1.2.

1. Let combinations $\varepsilon_i \tilde{\varepsilon}_j$ depend on $t$ for all $i$ and $j$. Then the oscillating terms in condition (98) have amplitudes $|\hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij}|$. Consequently, the condition (98) holds if
\[
1 - \sum_{i,j=1}^{N_0} \hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij} > 0, \tag{102}
\]

2. Let $\varepsilon_i \tilde{\varepsilon}_i = 1$ and combinations $\varepsilon_i \tilde{\varepsilon}_j$ depend on $t$ if $i \neq j$. Then condition (98) reads as (100), which holds if
\[
\text{abs} \left( 1 + \sum_{i,j=1}^{N_0} \hat{r}_{i}^{(1)} \hat{r}_{i}^{(2)} \hat{\nu}_{ii} \right) - \sum_{i,j=1}^{N_0} \text{abs} \left( \hat{r}_{i}^{(1)} \hat{r}_{j}^{(2)} \hat{\nu}_{ij} \right) > 0. \tag{103}
\]

The analysis of condition (98) in general case may be done similarly but it is more cumbersome so that we do not represent it here.

Let us write the general solution $\theta^{(i)}(\lambda, t)$ to eq. (89) as a linear combination of the delta-functions:
\[
\hat{\theta}^{(i)}(\lambda, t) = \sum_{j=1}^{N_0} w^{(ji)}(t) \delta(\lambda - p_j), \tag{104}
\]
Then the functions \( V \) and \( \tilde{V} \) given by the formulas (76) and (77) respectively read:

\[
V(t) = -2 \sum_{k=1}^{K} (a^{(k)})^T \varphi^{(k)} \Gamma^T + \sum_{i,j=1}^{N_0} 2r^T(p_j)w^{(ji)}r(q_i),
\]

\[
\tilde{V}(t) = C_0 + \sum_{n,i=1}^{N_0} C_{1n} w^{(ni)}(t) \tilde{C}_{1i},
\]

where the constant matrices \( C_0 \) and \( C_{ij} \) are given by the following expressions:

\[
C_0 = -2 \sum_{k=1}^{K} (\Pi^T)^{k-1} \varphi^{(k)} \Pi^{k-1} - \sum_{j=1}^{K} (\Pi^T)^{k-1} \varphi^{(k)} \Pi^{k-1} s p p^T s^T (\Pi^T)^{j-1} \varphi^{(j)} \Pi^{j-1}
\]

\[
C_{1n} = \sum_{k=1}^{K} (\Pi^T)^{k-1} \left( \varphi^{(k)} g^{(n)}(p_n) + \varphi^{(k)} \Pi^{k-1} s p p^T s^T (\Pi^T)^{l-1} g^{(l)}(p_n) \right),
\]

\[
\tilde{C}_{1n} = \sum_{k=1}^{K} \left( \varphi^{(k)} g^{(n)}(q_n) + \sum_{l=1}^{K} g^{(l)}(q_n) (\Pi)^{l-1} s p p^T s^T (\Pi^T)^{k-1} \varphi^{(k)} \right) (\Pi)^{k-1}.
\]

**One-soliton solution.** Let \( N_0 = 1, v_0^{(1)} = v_0 \). Then eq. (74) yields

\[
\hat{\theta}^{(1)}(\lambda, t) = \delta(\lambda - p_1) \hat{\theta}_0^{(1)}(t), \quad \hat{\theta}_0^{(1)}(t) = \varepsilon(p_1, t) \left( 1 + v_0 \varepsilon^{-1}(q_1, t) r(q_1) \xi^T r(p) \varepsilon(p_1, t) \right)^{-1},
\]

so that \( w^{(1)} = \hat{\theta}_0^{(1)} v_0 \varepsilon^{-1}(q_1) \). Substituting this \( w^{(1)} \) into eq. (106) we obtain

\[
V(t) = -2 \sum_{k=1}^{K} (a^{(k)})^T \varphi^{(k)} \Gamma^T + 2r^T(p_1) \varepsilon(p_1) (1 + v_0 \varepsilon^{-1}(q_1) r(q_1) \xi^T r(p_1) \varepsilon(p_1))^{-1} v_0 \varepsilon^{-1}(q_1) r(q_1) \xi^T r(p_1) \varepsilon(p_1) - 2 \sum_{k=1}^{K} (a^{(k)})^T \varphi^{(k)} \Gamma^T + 2r^T(p_1) \hat{\theta}_0(t) r(q_1),
\]

where \( \varepsilon \) is given by eq. (12) with \( T^{(m_1 m_2)} \) from eq. (13) and \( \tilde{g}^{(i)} \) from eq. (2). Then

\[
\hat{\theta}_0(t) = \left( \varepsilon(q_1) v_0^{-1} \varepsilon^{-1}(p_1) + r(q_1) \xi^T r(p_1) \right)^{-1},
\]

Now we turn to eq. (107), which reads

\[
\tilde{V}(t) = C_0 + 2C_{11} \hat{\theta}_0(t) \tilde{C}_{11},
\]
where the constant matrix $C_0$ is given by eq.(108), while expressions (109) for $C_{11}$ and (110) for $\tilde{C}_{11}$ read:

\[ C_{11} = \sum_{j=1}^{K} \sum_{k=1}^{K} (\Pi^T)^{k-1} \left( g^{(k)}(p_1) + \xi^{(k)} \Pi^{k-1} s p p^T s^T \sum_{l=1}^{K} (\Pi^T)^{l-1} g^{(l)}(p_1) \right), \]

\[ \tilde{C}_{11} = \sum_{k=1}^{K} \left( g^{(k)}(q_1) + \sum_{l=1}^{K} g^{(l)}(q_1) (\Pi^T)^{l-1} s p p^T s^T (\Pi^T)^{k-1} \xi^{(k)}) (\Pi)^{-k}. \]

In eq.(114), $\varepsilon$ is given by eq.(12) with $T^{l_{m_1}m_2}$ from eq.(85) and $\tilde{g}^{(i)}$ from eq.(83).

Non-singularity condition (98) for solutions $V$ and $\tilde{V}$ reads in the case $N_0 = 1$ as

\[ 1 + \sum_{\alpha,\beta=1}^{N} \frac{(r(q_1)\xi_\alpha(r(p_1)\xi_\beta(v_0))\beta_\alpha}{\varepsilon_\alpha(q_1,t)\varepsilon_\beta^{-1}(p_1,t)} \neq 0 \ \forall t. \] (116)

We may simply satisfy this condition if the arguments of the matrix exponent $\varepsilon$ is either real or imaginary, rewriting eqs.(99-103) for the case $N_0 = 1$ as follows.

Assuming that the arguments of the matrix exponent $\varepsilon$ are real, we consider two following cases.

1. Let $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1)$ depend on $t$ for all $\alpha$ and $\beta$. Then condition (99) reads

\[ (r(q_1)\xi_\alpha(r(p_1)\xi_\beta(v_0))\beta_\alpha > 0, \ \forall \alpha, \beta. \]

2. Let $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1) = I_N$ and $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1)$ depend on $t$ for all $\alpha \neq \beta$. Then condition (101) reads

\[ \text{sign}(r(q_1)\xi_\alpha(r(p_1)\xi_\beta(v_0))\beta_\beta) = \text{sign}(1 + \sum_{\gamma=1}^{N} (r(q_1)\xi_\gamma(r(p_1)\xi_\gamma(v_0))\gamma), \ \forall \alpha \neq \beta. \] (118)

If the matrix exponent $\varepsilon$ has the imaginary arguments we consider two other cases. 1. Let $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1)$ depend on $t$ for all $\alpha$ and $\beta$. Then condition (102) reads

\[ 1 - \sum_{\alpha,\beta=1}^{N} (r(q_1)\xi_\alpha(r(p_1)\xi_\beta(v_0))\beta_\alpha > 0, \] (119)

2. Let $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1) = I_N$ and $\varepsilon_\alpha(q_1)\varepsilon_\beta^{-1}(p_1)$ depend on $t$ for all $\alpha \neq \beta$. Then condition (103) reads

\[ \text{abs}(1 + \sum_{\alpha=1}^{N} (r(q_1)\xi_\alpha(r(p_1)\xi_\alpha(v_0))\alpha) - \sum_{\alpha,\beta=1}^{N} \text{abs}(r(q_1)\xi_\alpha(r(p_1)\xi_\beta(v_0))\beta_\alpha) > 0. \] (120)

### 5.3 Hermitian reduction

The Hermitian reduction requires relations (43-48), so that eqs.(82) and (83) for $\tilde{g}^{(i)}$ read respectively

\[ \tilde{g}^{(1)}_\alpha(\lambda) = \frac{-2i}{\sum_{\gamma=1}^{N} a^{(1)}_{\alpha\gamma}(\xi_0)_{\gamma_1} + \sum_{i=2}^{K} \sum_{\gamma=1}^{N} (c^{(i)}_{\alpha} f_{i\lambda_1} / c^{(i)}_{\alpha} f_{\lambda_1}) a^{(i)}_{\alpha\gamma}(\xi_0)_{\gamma_1}}, \]

\[ \tilde{g}^{(j)}_\alpha(\lambda) = \frac{-2i\lambda_j c^{(j)}_{\alpha} / c^{(1)}_{\alpha} f_{\lambda_1}}{\sum_{\gamma=1}^{N} a^{(1)}_{\alpha\gamma}(\xi_0)_{\gamma_1} + \sum_{i=2}^{K} \sum_{\gamma=1}^{N} (c^{(i)}_{\alpha} f_{i\lambda_1} / c^{(i)}_{\alpha} f_{\lambda_1}) a^{(i)}_{\alpha\gamma}(\xi_0)_{\gamma_1}}, \ j = 2, \ldots, K, \]
In eq.(80), we take
\[ \varepsilon = \sum_{i=2}^{K} \left( c_{\alpha}^{(i)} \lambda_{i}/c_{\alpha}^{(1)} \lambda_{1} \right) (s_{0})_{\alpha+i-1} \]

so that it reads
\[ \bar{g}_{\alpha}^{(j)}(\lambda) = \frac{-2i\lambda_{j}/c_{\alpha}^{(1)} \lambda_{1}}{(s_{0})_{\alpha} + \sum_{i=2}^{K} (c_{\alpha}^{(i)} \lambda_{i}/c_{\alpha}^{(1)} \lambda_{1}) (s_{0})_{\alpha+i-1}}, \quad j = 2, \ldots, K. \]

In this case eq.(121) and (122) tell us that we always deal with the oscillating matrix exponent \( \varepsilon \), so that cases (103) and (120) realize the nonsingular solutions to the nonlinear PDEs (50) and (51).

### 5.4 Examples of Explicit Solutions

We consider a solution to eq.(54), corresponding to \( K = 2, D = 4, N = 6 \). The number of independent variables in the nonlinear PDE is \( D(D+1)/2 = 10 \). We fix parameters \( L^{(m_{1})} \) and \( s_{0} \) as
\[ L^{(1)} = I_{N}, \quad L^{(m_{1})} = \left\{ \alpha^{m_{1}-1}, \quad m_{1} = 2, 3, 4, \quad \alpha = 1, 2, 3 \right\}, \quad s_{0} = I_{N}, \quad (123) \]

Elements of the matrices \( a^{(1;m_{1})} \equiv \hat{a}^{(1;m_{1})} \), \( m_{1} = 2, 3, 4 \), may be found using eqs. (38):
\begin{align*}
\hat{a}^{(1;12)} &= \text{diag}(-5/4, -11/6, -1, 1/2, 13/12, -5/12), \quad \hat{a}^{(1;13)} = \text{diag}(1/8, 1, -1/4, -1/9, 3/8, -1/2), \quad (124) \\
\hat{a}^{(1;14)} &= \text{diag}(1/8, -1/6, 1/12, -1/18, 1/24, -1/12). 
\end{align*}

In eq.(80), we take
\[ w_{1} = 1, \quad w_{2} = 2, \quad c_{i} = \frac{1}{2} I_{n}, \quad \Rightarrow \quad \kappa_{\alpha}^{i} = \frac{3}{25}, \quad i = 1, 2, \quad (125) \]

so that it reads
\[ g_{\alpha}^{(i)}(\lambda) = \frac{\lambda_{i}}{\pi} \exp \frac{1}{2} \left( -\sum_{j=1}^{2} \left( (Re \lambda_{j})^{2} + 4(Im \lambda_{j})^{2} \right) \right), \quad i = 1, 2. \quad (126) \]

Let
\[ p_{11} = -i, \quad p_{12} = -i h, \quad q_{11} = i, \quad q_{12} = i h, \quad (127) \]

where \( h \) is a positive constant. Then eqs. (122) read
\[ \bar{g}^{(1)}(p_{1}) = \frac{-2i}{1+h}, \quad \bar{g}^{(2)}(p_{1}) = \frac{-2ih}{1+h}. \quad (128) \]

Represent \( \varepsilon(p_{1}, t) \) in the form
\[ \varepsilon(t) \equiv \varepsilon(p_{1}, t) = \text{diag}(e^{iX_{1}}, \ldots, e^{iX_{6}}), \quad (129) \]

\[ X_{\alpha} = \frac{-2}{1+h} \sum_{m_{1}, m_{2}=1}^{4} \left( a^{(1;m_{1})}_{\alpha} a^{(1;m_{2})}_{\alpha} + h a^{(1;m_{1})}_{\alpha+1} a^{(1;m_{2})}_{\alpha+1} \right) t_{m_{1}m_{2}}. \]
Taking $\hat{C}_{11} = C_{11}^+$ we write eq. (114) as

$$V(t) = C_0 + 2C_{11}\hat{\theta}_0(t)C_{11}^+,$$

where

$$\hat{\theta}_0(t) = \left((\varepsilon(t)v_0^{-1}\varepsilon^{-1}(t) - \frac{(1 + h)^2e^{-4(1 + h^2)}}{\pi^2})E\right)^{-1}$$  

(131)

and $E$ is the $6 \times 6$ matrix of units (don’t mix it with the unit matrix!). Substituting values (127) for $p_{1i}$ and $q_{i1}$ into eqs. (108) and (115) we obtain

$$C_0 = -2 \sum_{k=1}^{K}(\Pi^k)^{-1} \varepsilon^{(k)} \Pi^{k-1} +$$

$$2 \sum_{j=1}^{K} \sum_{k=1}^{K}(\Pi^k)^{-1} \varepsilon^{(k)} \Pi^{k-1} p p^T (\Pi^j)^{-1} \varepsilon^{(j)} \Pi^{j-1} = -\frac{3}{8} I_N + \frac{9}{128} E$$

(132)

$$C_{11} = \frac{1}{\pi} \sum_{j=1}^{K} \sum_{k=1}^{K}(\Pi^k)^{-1} (p_{ik} - \varepsilon^{(k)} \Pi^{k-1} p p^T \sum_{l=1}^{K}(\Pi^l)^{-1} p_{J}) \exp \frac{1}{2} \left(-\sum_{j=1}^{K} \left((Re \lambda_j)^2 + 4(Im \lambda_j)^2\right)\right) =$$

$$\frac{ie^{-2(1+h^2)}}{\pi} \left(-I_N - h\Pi^T + \frac{3(1 + h)E}{16}\right).$$

Now we have to fix the matrix $v_0$.

**Example 1.** As a simple case let

$$v_0 = \begin{pmatrix}
v_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad \text{(133)}$$

Condition (120) requires the following expression for $v_1$:

$$v_1 = -5 + \frac{(1 - d)\pi^2}{(1 + h)^2} \exp \left(4(1 + h^2)\right), \quad d > 0,$$

(134)

where $d$ is an arbitrary positive parameter. In this case solution $U$ depends on the single variable $Z_1 = X_3 - X_2$.

Now we fix $h = 1/4$. The absolute values of all elements have oscillating behavior. They may be characterized by the double amplitude $U_{ij}^{\text{ampd}} = |U_{ij}^{\text{max}}| - |U_{ij}^{\text{min}}|$ and by the average value $U_{ij}^{\text{avr}} = (|U_{ij}^{\text{max}}| + |U_{ij}^{\text{min}}|)/2$, which are collected in Table 1 for three values of $d$: $d = 0.1$, 0.01 and 0.001. Different shapes of absolute values $|U_{ij}|$ are shown in Fig. Ia-c.

In Fig. Ia, we represent the absolute values $|U_{12}|$ (the upper curve) and $|U_{26}|$ (the lower curve) as functions of $Z_1$ for $d = 0.1$. Functions $|U_{23}|$, $|U_{24}|$ and $|U_{25}|$ have the shape of $|U_{26}|$ as well, while the absolute values of all other elements have the shape of $|U_{12}|$.

In Fig. Ib, we represent the absolute values $|U_{23}|$ (the big amplitude curve) and $|U_{22}|$ (the small amplitude curve) as functions of $Z_1$ for $d = 0.01$. The absolute values of all other elements have the shape of $|U_{23}|$. 

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Figure 1: Absolute values of some elements $U_{ij}$ for $h = 1/4$ and $d = 0.1, 0.01, 0.001$ (from the top to the bottom); $v_0$ is given in eq. (133).
Table 1: The amplitudes $U_{ampl}^{ij}$ and the average values $U_{avr}^{ij}$ for three values of $d$: $d = 0.1$, 0.01 and 0.001; $h = 1/4$ and $v_0$ is given in eq.(133).

|       | $d = 0.1$ |       | $d = 0.01$ |       | $d = 0.001$ |
|-------|-----------|-------|------------|-------|-------------|
|       | $U_{ampl}^{ij}$ | $U_{avr}^{ij}$ | $U_{ampl}^{ij}$ | $U_{avr}^{ij}$ | $U_{ampl}^{ij}$ | $U_{avr}^{ij}$ |
| $U_{11}$ | 3.638 | 48.201 | 269.660 | 440.384 | 5239.94 | 3207.18 |
| $U_{12}$ | 0.223 | 1.497 | 6.357 | 8.908 | 106.415 | 63.6566 |
| $U_{13}$ | 0.939 | 15.059 | 81.562 | 135.842 | 1609.39 | 982.867 |
| $U_{14}$ | 1.217 | 14.920 | 83.372 | 134.937 | 1608.53 | 983.301 |
| $U_{15}$ | 1.286 | 14.885 | 83.824 | 135.842 | 1605.05 | 982.861 |
| $U_{16}$ | 1.286 | 14.879 | 83.824 | 134.710 | 1609.39 | 982.867 |
| $U_{22}$ | 0.010 | 2.416 | 0.127 | 2.275 | 32.5786 | 18.8045 |
| $U_{23}$ | 0.040 | 0.279 | 1.976 | 1.990 | 32.6842 | 18.7749 |
| $U_{24}$ | 0.069 | 0.276 | 1.960 | 1.993 | 32.6606 | 18.7808 |
| $U_{25}$ | 0.079 | 0.275 | 1.976 | 1.990 | 32.6842 | 18.7749 |
| $U_{26}$ | 0.079 | 0.275 | 1.976 | 1.990 | 32.6842 | 18.7749 |
| $U_{33}$ | 0.229 | 2.387 | 24.656 | 39.587 | 491.631 | 300.228 |
| $U_{34}$ | 0.310 | 5.337 | 25.212 | 42.299 | 492.705 | 302.681 |
| $U_{35}$ | 0.332 | 5.324 | 25.354 | 42.227 | 492.975 | 302.545 |
| $U_{36}$ | 0.332 | 5.324 | 25.354 | 42.227 | 492.975 | 302.545 |
| $U_{44}$ | 0.406 | 2.311 | 25.775 | 39.039 | 493.774 | 299.168 |
| $U_{45}$ | 0.430 | 5.328 | 25.916 | 41.951 | 494.042 | 302.017 |
| $U_{46}$ | 0.430 | 5.275 | 25.916 | 41.945 | 494.042 | 302.011 |
| $U_{55}$ | 0.455 | 2.287 | 26.057 | 38.899 | 494.309 | 298.902 |
| $U_{56}$ | 0.455 | 5.269 | 26.057 | 41.881 | 494.309 | 301.883 |
| $U_{66}$ | 0.455 | 2.287 | 26.057 | 38.899 | 494.309 | 298.902 |

Finally, in Fig.1c, we represent the absolute values $|U_{23}|$ (the upper curve) and $|U_{22}|$ (the lower curve) for $d = 0.001$. The absolute values of all other elements have the shape of $|U_{23}|$.

We see from Table 1, that the maximal amplitude has $|U_{11}|$ in all three cases.

**Example 2.** As a more complicated example, we take

$$
\begin{pmatrix}
  v_1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 0 & 1 & 1 & 1 & 1 \\
  1 & 1 & 0 & 1 & 1 & 1 \\
  1 & 1 & 1 & 0 & 1 & 1 \\
  1 & 1 & 1 & 1 & 0 & 1 \\
  1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}

(135)

Condition (120) requires

$$
v_1 = -30 + \frac{(1 - d)\pi^2}{(1 + h)^2} \exp \left( 4(1 + h^2) \right), \quad d > 0.
$$

(136)

In this case solutions depend on the five variable $Z_i = X_{i+1} - X_1, \ i = 1, \ldots, 5$. Absolute values of elements $U_{11}$ and $U_{22}$ as functions of $Z_4$ and $Z_5$ with $h = 1/4$, $d = 0.001$ (and fixed $Z_1 = Z_2 = Z_3 = 0$) are depicted in Fig.2. These are lattices of lumps. Absolute values of all other elements have the form of $|U_{11}|$. 
Figure 2: Elements $U_{11}$ and $U_{22}$ for $h = 1/4$, $d = 0.001$, $v_0$ given in eq.(135) and $Z_1 = Z_2 = Z_3 = 0$.

Similar to the previous example, the absolute values $|U_{ij}|$ as functions of $Z_4$ and $Z_5$ with fixed $Z_1 = Z_2 = Z_3 = 0$ may be characterized by the double amplitudes $U_{ij}^{\text{ampl}} = |U_{ij}^{\text{max}}| - |U_{ij}^{\text{min}}|$ and by the average values $U_{ij}^{\text{avr}} = (|U_{ij}^{\text{max}}| + |U_{ij}^{\text{min}}|)/2$, which are collected in Table 2. It is

| $U_{ij}$ | $U_{ij}^{\text{ampl}}$ | $U_{ij}^{\text{avr}}$ |
|----------|----------------------|----------------------|
| $U_{11}$ | 5108.45              | 2628.74              |
| $U_{12}$ | 91.2786              | 48.1621              |
| $U_{13}$ | 1581.04              | 813.069              |
| $U_{14}$ | 1581.04              | 813.069              |
| $U_{15}$ | 1580.12              | 813.526              |
| $U_{16}$ | 1579.89              | 813.647              |
| $U_{22}$ | 1.61317              | 1.58563              |
| $U_{23}$ | 28.2704              | 14.1456              |
| $U_{24}$ | 28.2704              | 14.1399              |
| $U_{25}$ | 28.211               | 14.1696              |
| $U_{26}$ | 28.1961              | 14.177               |
| $U_{33}$ | 489.299              | 249.155              |
| $U_{34}$ | 489.299              | 252.174              |
| $U_{35}$ | 489.064              | 252.297              |
| $U_{36}$ | 489.006              | 252.326              |
| $U_{44}$ | 489.299              | 249.155              |
| $U_{45}$ | 489.064              | 252.291              |
| $U_{46}$ | 489.006              | 252.326              |
| $U_{55}$ | 488.726              | 249.441              |
| $U_{56}$ | 488.641              | 252.503              |
| $U_{66}$ | 488.55               | 249.529              |

Table 2: The amplitudes $U_{ij}^{\text{ampl}}$ and the average values $U_{ij}^{\text{avr}}$ for $d = 0.001$, $h = 1/4$ and $v_0$ from eq.(135).

remarkable, that if we put to zero any other three parameters $Z_i$, then we obtain the same shapes for the absolute values $|U_{ij}|$ as functions of two remaining parameters $Z_i$. 

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6  Conclusions

In this paper we propose an algorithm for derivation of a new class of multidimensional nonlinear PDEs together with a manifold of their particular solutions in arbitrary dimensions. The freedom of the solution space is characterized by arbitrary functions of any number of independent variables. However the nonlinear PDEs are not completely integrable because the increase of the number \( K \) of independent variables in the list of arguments of the above functions causes the increase of the dimensionality \( D^2 \) of the nonlinear PDEs: \( D^2 \sim K^2 \) in eq.(27) and \( D^2 \sim K^4 \) in eq.(6). Thus the problem of suppressing the dimensionality of the nonlinear PDEs is very important.

Although the general nonlinear PDE (6) does not assume the diagonal form of the coefficients \( L^{(m_1)} \) and \( R^{(m_2)} \), the reduction considered in Sec.7.5.2 result in the diagonal coefficients in the linear part of the nonlinear PDE (27) which is more suitable for the physical applications, in particular, in study of the multiple-scale expansions of the various physical systems.

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7  Appendix A. Derivation of the nonlinear PDEs (6) and (27)

Our algorithm deriving the nonlinear PDEs (6) and (27) is based on the following linear integral equation for the matrix function \( W(\lambda; t) \):

\[
P(\lambda) = W(\mu; t) * \Psi(\mu, \lambda; t) + W(\lambda; t) \equiv W(\mu; t) * \left( \Psi(\mu, \lambda; t) + I_1(\mu, \lambda) \right),
\]

where \( P(\mu), \Psi(\mu, \lambda; t), W(\lambda; t) \) are the \( N \times N \) matrix functions of arguments, \( \lambda, \mu, \nu \) are complex vector parameters defined above in eq.(3). Remind that we mark the independent variables of nonlinear PDEs by the double index (i.e. \( t_{m_1m_2} \)) and denote the whole set of them by \( t = (t_{m_1m_2} : m_1, m_2 = 1, \ldots, D) \). Function \( \Psi \) is the kernel of the integral operator. Here \( * \) means the integration over the space of vector spectral parameter defined in eqs.(4) and (5). We require that eq.(137) is uniquely solvable for \( W \), i.e. the operator \( * (\Psi(\mu, \lambda; t) + I_1(\mu, \lambda)) \) is invertible:

\[
W(\lambda; t) = P(\mu) * (\Psi(\mu, \lambda; t) + I(\mu, \lambda))^{-1}.
\]

Let us introduce the dependence on the additional parameters \( t_{m_1m_2} \) through the function \( \Psi \), which is defined by the following system of linear PDEs with the coefficients independent on \( t \):

\[
\Psi_{t_{m_1m_2}}(\lambda, \mu; t) = \left( B^{(m_1m_2)}(\lambda, \nu) + A(\lambda)C^{(m_1m_2)}P(\nu) \right) * \Psi(\nu, \mu; t) - \Psi(\lambda, \nu; t) * (B^{(m_1m_2)}(\nu, \mu) - A(\nu)C^{(m_1m_2)}P(\mu)), \quad m_1, m_2 = 1, \ldots, D,
\]

where \( B^{(m_1m_2)}(\lambda, \nu) \) and \( A(\lambda) \) are \( N \times N \) matrix functions of spectral parameters.

7.1  System of compatible linear equations for \( W(\lambda; t) \).

The basic result of this subsection is represented in the following theorem.
Theorem 1. Let matrices $B^{(m_1m_2)}(\lambda, \mu)$ satisfy the following external constraints:

$$\sum_{m_1=1}^{D} L^{(m_1)} P(\lambda) \ast (B^{(m_1m_2)}(\lambda, \mu) - A(\lambda)C^{(m_1m_2)} P(\mu)) = 0, \ m_2 = 1, \ldots, D, \ (140)$$

where $L^{(m_1)}$ are some $N \times N$ constant matrices. Then the matrix function $W(\lambda; t)$ obtained as the solution to the integral equation (137) with the kernel $\Psi$ defined by eq.(139) is a solution to the following system of compatible linear equations

$$E^{(m_2)}(\lambda; t) := -2W(\mu; t) \ast A(\mu) \quad \text{Eq. (138)}$$

$$\sum_{m_1=1}^{D} L^{(m_1)} \left( W_{m_1m_2}(\lambda; t) + V(t)C^{(m_1)}(\lambda; t) + W(\mu; t) \ast (B^{(m_1m_2)}(\mu, \lambda) + A(\lambda)C^{(m_1m_2)} P(\lambda)) = 0, \ m_2 = 1, \ldots, D, \ (141)$$

where

$$V(t) = -2W(\mu; t) \ast A(\mu) \quad \text{Eq. (138)}$$

does not depend on the spectral parameters.

Proof: To derive eq.(141), we differentiate eq.(137) with respect to $t_{m_1m_2}$. Then, in view of eq.(139), one gets the following integral equation:

$$\mathcal{E}^{(m_1m_2)}(\mu; t) := P(\nu) \ast (B^{(m_1m_2)}(\nu, \mu) - P(\nu)C^{(m_1m_2)} A(\mu)) = E^{(m_1m_2)}(\nu; t) \ast (\Psi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu)), \quad (143)$$

$$\mathcal{E}^{(m_1m_2)}(\lambda; t) = W_{m_1m_2}(\lambda; t) + V(t)C^{(m_1m_2)}W(\lambda; t) + W(\lambda; t) \ast (B^{(m_1m_2)}(\mu, \lambda) + A(\lambda)C^{(m_1m_2)} P(\lambda)).$$

Consider the following combination of eqs.(143): $\sum_{m_1=1}^{D} L^{(m_1)} \mathcal{E}^{(m_1m_2)}$. Then, using the external constraints (140), we result in:

$$\sum_{m_1=1}^{D} L^{(m_1)} \mathcal{E}^{(m_1m_2)} := \sum_{m_1=1}^{D} L^{(m_1)} E^{(m_1m_2)}(\nu; t) \ast (\Psi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu)) = 0. \quad (144)$$

Since operator $\ast (\Psi(\nu, \mu; t) + \mathcal{I}_1(\nu, \mu))$ is invertible, eq.(144) is equivalent to eq.(141). $\blacksquare$

We refer to the constraints (140) as the external constraints since they involve matrices $L^{(m_1)}$ which do not appear in the integral equation (137) as well as in the system of the linear PDEs (139) defining the function $\Psi$.

System (141) is an analogy of the overdetermined system of linear equations in the classical inverse spectral transform method. According to that method, the system of nonlinear PDEs for the potentials of the overdetermined linear system appears as the compatibility condition for this linear system. However, the system of nonlinear PDEs may not be obtained as the compatibility condition in our case because of the term $W(\mu; t) \ast (B^{(m_1m_2)}(\mu, \lambda) + A(\lambda)C^{(m_1m_2)} P(\lambda))$ in eq.(141). This term must be considered as a new spectral function because it involves the integration over the spectral parameter. Therefore we represent another algorithm of derivation of nonlinear PDEs in Sec.7.2.
7.2 First order nonlinear PDEs for the field $V(t)$

**Theorem 2.** In addition to the eqs.\(\text{(137,139)}\) and the external constraints \(\text{(140)}\), we impose another set of external constraints:

$$\sum_{m_1=1}^{D} \sum_{m_2=1}^{D} (B^{(m_1,m_2)}(\lambda, \nu) + A(\lambda)C^{(m_1,m_2)}P(\nu)) \ast A(\nu)R^{(m_2)} = 0,$$

where $R^{(m_2)}$ are some $N \times N$ constant matrices. Then the $N \times N$ matrix function $V(t)$ is a solution to the following system of nonlinear PDEs:

$$\sum_{m_1,m_2=1}^{D} L^{(m_1)} \left( V_{m_1,m_2} + V^{C^{(m_1,m_2)}V} \right) R^{(m_2)} = 0. \quad (146)$$

**Proof:** Applying operator $\ast(-2A)$ to eq.\(\text{(141)}\) from the right one gets the following equation

$$E^{(m_2)}(t) = E^{(m_2)}(\lambda; t) \ast A(\lambda) :=$$

$$\sum_{m_1=1}^{D} L^{(m_1)} \left( V_{m_1,m_2} + V^{C^{(m_1,m_2)}V} + U^{(m_1,m_2)} \right) = 0,$$

which introduces a new set of fields $U^{(m_1,m_2)}$

$$U^{(m_1,m_2)}(t) = -2W(\mu; t) \ast (B^{(m_1,m_2)}(\mu, \nu) + A(\mu)C^{(m_1,m_2)}P(\nu)) \ast A(\nu). \quad (148)$$

Due to the external constraints \(\text{(145)}\), we may eliminate these fields using a proper combinations of eqs.\(\text{(147)}\). Namely, combinations $\sum_{m_2=1}^{D_2} E^{(m_2)} R^{(m_2)}$ results in the system \(\text{(146)}\). □

The nonlinear PDE \(\text{(146)}\) is the basic form of the PDEs considered hereafter. First it was derived in \(\text{[31]}\). However, the acceptable structure for the constant matrix coefficients $L^{(m_1)}$, $R^{(m_2)}$ and $C^{(m_1,m_2)}$ as well as the richness of the solution space have not been investigated to the full extent therein. Note that the classical $(2+1)$-dimensional $N$-wave equation is embedded in the general eq.\(\text{(146)}\).

Below we show that the structure of the coefficients $L^{(m_1)}$, $R^{(m_2)}$ and $C^{(m_1,m_2)}$ is defined by the solution $\Psi$ to system of linear PDEs \(\text{(139)}\). In Sec. 7.3, choosing the special form for the constant matrices $C^{(m_1,m_2)}$, we represent a family of solutions to the system of linear PDEs \(\text{(139)}\) leading to the non-classical type of multidimensional PDEs \(\text{(6)}\).

7.3 Construction of the kernel $\Psi$ as the solution to the system of linear PDEs \(\text{(139)}\). Special form of the matrices $C^{(m_1,m_2)}$

In this section, we denote $m = (m_1,m_2)$ for the sake of brevity; accordingly, $\sum_{m} \equiv \sum_{m_1,m_2=1}^{D}$. In order to construct the explicite solutions to the nonlinear PDE \(\text{(146)}\), we have to find the explicite form of the function $\Psi$ solving the system of linear PDEs \(\text{(139)}\). Let $\Psi$ have the following structure:

$$\Psi(\lambda, \mu) = \chi(\lambda, \nu) \ast \left( \varepsilon(\nu; t)C(\nu, \tilde{\nu})\tilde{\varepsilon}(\tilde{\nu}, t) \right) \ast \tilde{\chi}(\tilde{\nu}, \mu), \quad (149)$$
where
\[
\varepsilon(\nu; t) = e^{\sum_m T^{(m)}(\nu) t_m}, \quad \bar{\varepsilon}(\nu; t) = e^{-\sum_m \bar{T}^{(m)}(\nu) t_m}.
\] (150)

Here $\chi$ and $\bar{\chi}$ are the $N \times N$ invertible matrix operators, $T^{(m)}$ and $\bar{T}^{(m)}$ are the diagonal $N \times N$ matrix functions of the spectral parameter. Substituting function $\Psi$ given by eq.(149) into eq.(139) we obtain
\[
\left( \chi T^{(m)} - (B^{(m)} + AC^{(m)} P) \right) * \chi = 0, \quad m = 1, 2, \ldots, N
\] (151)

Each of two terms in this equation must be identical to zero, which suggests us the following two equations relating $C^{(m)}$, $B^{(m)}$, $T^{(m)}$ and $\bar{T}^{(m)}$:
\[
\chi T^{(m)} - (B^{(m)} + AC^{(m)} P) \neq 0, \quad m = 1, 2, \ldots, N
\] (152)
\[
\bar{T}^{(m)} \chi - \bar{\chi} \neq 0, \quad m = 1, 2, \ldots, N
\] (153)

Solving eq.(152) with respect to $B^{(m)}$ we obtain:
\[
B^{(m)} = (\chi T^{(m)})^{-1} - AC^{(m)} P,
\] (154)

which defines the operator $B^{(m)}$. Substituting this expression into eq.(153) and applying the operator $*\chi$ from the right side we obtain:
\[
R(\lambda, \mu) T^{(m)}(\mu) - \bar{T}^{(m)}(\lambda) R(\lambda, \mu) = 2r(\lambda) C^{(m)} \bar{r}(\mu), \quad \forall m,
\] (155)

where we introduce notations
\[
R(\lambda, \mu) = \bar{\chi} \neq \chi, \quad r(\lambda) = \bar{\chi}(\lambda, \nu) * A(\nu), \quad \bar{r}(\mu) = P(\nu) * \chi(\nu, \mu).
\] (156)

Thus, we have to find functions $R$, $r$, $\bar{r}$, $T$, $\bar{T}$ and constant matrices $C^{(m)}$ satisfying the eqs.(155).

Remark, that there is a particular solution to the system (155) leading to the classical integrable $(2+1)$-dimensional $N$-wave equation, which is considered in the Appendix B, Sec.8.

Here we study another solution to the system (155) which results in a new multidimensional nonlinear $N$-wave type equation.

In order to resolve the eq.(155) we propose the following form of the constant matrix $C^{(m)}$:
\[
C^{(m)} = \xi^{(m)} - \eta^{(m)} \eta
\] (157)

and take the function $R(\lambda, \mu)$ in form (9). Here $\xi$ and $\eta^{(m)}$ are $N \times 1$ constant matrices, while $\xi^{(m)}$ and $\eta$ are $1 \times N$ constant matrices. Substituting eqs.(9) and (157) into eq.(155) we obtain:
\[
\mathcal{I}(\lambda, \mu)(T^{(m)}(\mu) - \bar{T}^{(m)}(\lambda)) + r(\lambda) \xi^{(m)} \bar{r}(\mu).T^{(m)}(\mu) - \bar{T}^{(m)}(\lambda) r(\lambda) \xi^{(m)} \bar{r}(\mu) = 2r(\lambda) \xi^{(m)} \bar{r}(\mu) - 2r(\lambda) \eta^{(m)} \bar{r}(\mu).
\] (158)

Eq.(158) may be splitted into three following relations:
\[
\bar{T}^{(m)}(\lambda) = T^{(m)}(\lambda),
\] (159)
\[
r(\lambda) \eta^{(m)} = \frac{1}{2} T^{(m)}(\lambda) r(\lambda) \xi,
\] (160)
\[
\xi^{(m)} \bar{r}(\mu) = \frac{1}{2} \eta^{(m)} \bar{r}(\mu) T^{(m)}(\mu).
\] (161)
Note that owing to eq. (159) we have \( \tilde{\varepsilon} = I_N \), i.e.
\[
\tilde{\varepsilon}(\lambda; t) = \varepsilon^{-1}(\lambda; t).
\]
(162)

Let us analyze eqs. (160) and (161). Eq. (160) may be resolved for \( T^m(\lambda) \):
\[
T^m_\alpha(\lambda) = 2 \sum_{\gamma=1}^N \frac{r_{\alpha\gamma}(\lambda) \eta_{\gamma1}}{\sum_{\gamma=1}^N r_{\alpha\gamma}(\lambda) \xi_{\gamma1}},
\]
(163)

In other words, we relate the \( \lambda \)-dependence of the diagonal elements of \( T^m(\lambda) \) with the \( \lambda \)-dependence of the elements of the matrix \( r(\lambda) \). Now we turn to eq. (161). Substituting eq. (163) into eq. (161) we obtain
\[
\sum_{\gamma=1}^N r_{\alpha\gamma}(\lambda) \eta_{\gamma1} = \sum_{\gamma=1}^N \tilde{r}_{\alpha\gamma}(\lambda) \xi_{\gamma1},
\]
(164)

which relates elements of \( r \) and \( \tilde{r} \). In particular, condition (164) becomes an identity if equation \( (161) \) is the transposition of eq. (160) i.e.
\[
\tilde{r}(\lambda) = r^T(\lambda), \quad \eta = \xi^T, \quad \xi^{(m)} = (\eta^{(m)})^T.
\]
(165)

Thus, we satisfy eq. (153) using the matrices \( C^{(m)} \) and \( R \) given by eq. (157) and (9) respectively, identifying \( \tilde{T}^m = T^m \) (eq. (159)), relating \( T^m(\lambda) \) with \( r(\lambda) \) by eq. (163) and using additional relations (165). Substituting eqs. (157) and (165) into eq. (146) we obtain the nonlinear PDE (6). Expression (142) for \( V \) may be written as (7) using eqs. (149) and (156). Representation (20) for \( V \) may be derived from eq. (7) as follows. First, write the explicit expression for the inverse of the operator \( R \),
\[
R^{-1}(\lambda, \mu) = \mathcal{I}(\lambda, \mu) - \frac{r(\lambda) \xi \xi^T r^T(\mu)}{1 + Q},
\]
(166)

where scalar \( Q \) is introduced in eq. (19). Next, write the combinations \( r^T * R^{-1} \) and \( R^{-1} * r \) as follows:
\[
r^T(\nu) * R^{-1}(\nu, \lambda) = \left( 1 - \frac{r^T * r \xi \xi^T}{1 + Q} \right) r^T(\lambda) = \Gamma r^T(\lambda), \quad \tag{167}
\]
\[
R^{-1}(\lambda, \nu) * r(\nu) = r(\lambda) \left( 1 - \frac{\xi \xi^T r^T * r}{1 + Q} \right) = r(\lambda) \Gamma^T, \quad \tag{168}
\]

where \( \Gamma \) is given in eq. (19). Finally, using eqs. (167) and (168), we transform eq. (7) into eq. (20).

7.3.1 Special representation of the function \( r(\lambda) \)

Hereafter we consider \( r(\lambda) \) as a linear combination of \( K \) arbitrary diagonal functions \( g^{(i)}(\lambda) \) \( (i = 1, \ldots, K) \) of the spectral parameter \( \lambda \), i.e. \( r \) is defined by eq. (10). Number \( K \) coincides with the dimensionality of the spectral parameter space introduced in the Introduction. The structure (10) will be important in Sec. 7.4 to satisfy the external constraints (140) and (145). Emphasize that \( N \times 1 \) matrix \( r^T * r \xi \) in the definition of \( \Gamma \) (19) must be finite, i.e.
\[
|(r^T * r \xi)_{\alpha1}| < \infty, \quad \alpha = 1, \ldots, N.
\]
(169)
This holds if conditions (11) are valid.

Let us show that the representation (10) leads to \((K - 1)\) arbitrary functions of spectral parameters in the exponent of the function \(\varepsilon(\lambda)\) (and \(\tilde{\varepsilon}(\lambda)\)), see eqs.(12). In fact, substituting eq.(10) into eq.(163) one gets:

\[
T^{(m_1m_2)}(\lambda) = \sum_{j=1}^{K} \tilde{g}^{(j)}(\lambda) \hat{a}^{(j;m_1m_2)}
\]  

(170)

where the elements of the diagonal matrices \(\tilde{g}^{(j)}\) are defined by the formulas

\[
\tilde{g}^{(j)}(\lambda) = \frac{2g^{(j)}(\lambda)}{\sum_{i=1}^{K} \sum_{\gamma=1}^{N} g^{(i)}(\lambda) a^{(i)}_{\alpha\gamma} \xi_{\gamma 1}}
\]  

(171)

and the elements of the diagonal matrices \(\hat{a}^{(j;m_1m_2)}\) are defined as

\[
\hat{a}^{(j;m_1m_2)} = \left( a^{(j)} \eta^{(m_1m_2)} \right)_{\alpha 1}.
\]  

(172)

Eq.(170) shows that all \(T^{(m_1m_2)}\) are linear combinations of \(K\) diagonal functions \(\tilde{g}^{(j)}(\lambda), j = 1,\ldots, K\). Note that eq.(171) for \(\tilde{g}^{(j)}\) can be written in terms of ratios \(\hat{g}^{(i)}_{\alpha}(\lambda) = \frac{g^{(i)}_{\alpha}(\lambda)}{g^{(1)}_{\alpha}(\lambda)}\), \(i = 1,\ldots, K\) with \(g^{(1)} = 1\), see eq.(14). Thus \(\tilde{g}^{(k)}(\lambda)\) are parametrized by \((K - 1)\) arbitrary functions \(\hat{g}^{(i)}(\lambda)\) \((i = 12,\ldots, K)\) of the vector spectral parameter \(\lambda\). Consequently, the diagonal functions \(T^{(m)}(\lambda)\) depend on \((K - 1)\) independent parameters \(\hat{g}^{(i)}\) which is noted in Remark 1 of Sec.2.1.

### 7.4 External constraints (140) and (145)

Now we have to satisfy the external constraints (140) and (145). First, using the substitution \((B^{(m)} - AC^{(m)}P) = \tilde{\chi}^{-1} T^{(m)} \tilde{\chi}\) (which follows from the eq.(153)), we transform the constraint (140) to the following form

\[
\sum_{m_1} L^{(m_1)} P \star (\tilde{\chi}^{-1} T^{(m)} \tilde{\chi}) \star \tilde{\chi} = 0.
\]  

(173)

Or, applying \(\star \tilde{\chi}^{-1}\) and using notations (156) with relations (159,165), we write eq.(173) as

\[
\sum_{m_1} L^{(m_1)} r^T \star R^{-1} T^{(m)} = 0.
\]  

(174)

In a similar way, we transform the constrain (145) using substitution \((B^{(m)} + AC^{(m)}P) = \chi \star (T^{(m)} \chi^{-1})\) (following from eq.(152)) with notations (156) and applying \(\chi^{-1} \star\). As a result we obtain

\[
\sum_{m_2} T^{(m)} R^{-1} \star r R^{(m_2)} = 0.
\]  

(175)
Now, substituting eqs. (167) and (168) into constraints (174) and (175) we obtain

\[
\sum_{m_1=1}^{D} L^{(m_1)} \Gamma r^T(\lambda) T^{(m)}(\lambda) = 0, \quad m_2 = 1, \ldots, D, \\
\sum_{m_2=1}^{D} T^{(m)}(\lambda) r(\lambda) \Gamma^T R^{(m_2)} = 0, \quad m_1 = 1, \ldots, D,
\]

where \(T^{(m)}\) is related with \(r\) by eq. (163). For convenience, we introduce the notation \(\text{diag } A\) for the diagonal matrix with the diagonal elements \(A_\alpha\), where \(A\) is \(N \times 1\) matrix. Let us substitute \(T^{(m)}\) from eq. (163) into eqs. (176, 177) and multiply the result by the non-degenerate diagonal matrix \(\text{diag}(r^* \xi)\) from the right and left sides respectively:

\[
\sum_{m_1=1}^{D} L^{(m_1)} \Gamma r^T(\lambda) \text{diag}(r(\lambda) \eta^{(m_1m_2)}) = 0, \quad m_2 = 1, \ldots, D, \tag{178}
\]

\[
\sum_{m_2=1}^{D} \text{diag}(r(\lambda) \eta^{(m_1m_2)}) r(\lambda) \Gamma^T R^{(m_2)} = 0, \quad m_2 = 1, \ldots, D, \tag{179}
\]

Eqs. (178, 179) represent the system of nonlinear equations for the elements of \(r\) involving constant (non-diagonal in general) matrices \(L^{(m_1)}, R^{(m_2)}\).

### 7.4.1 External constraints (178) and (179) with \(r(\lambda)\) given by eq. (10)

Let us substitute eq. (10) into eqs. (178) and (179) obtaining

\[
\sum_{i,j=1}^{i,j \geq i} Z^{(m_2:ij)} g^{(i)}(\lambda) g^{(j)}(\lambda) = 0, \\
\sum_{i,j=1}^{i,j \geq i} g^{(i)}(\lambda) g^{(j)}(\lambda) \tilde{Z}^{(m_2:ij)} = 0,
\]

where

\[
Z^{(m_2:ii)} = \sum_{m_1=1}^{D} L^{(m_1)} \hat{a}^{(i)}(\lambda) \hat{a}^{(i;m_1m_2)}, \tag{182}
\]

\[
Z^{(m_2:ij)} = \sum_{\text{perm}(i,j)} \sum_{m_1=1}^{D} L^{(m_1)} \hat{a}^{(i)}(\lambda) \hat{a}^{(j;m_1m_2)}, \tag{183}
\]

\[
\tilde{Z}^{(m_1:ii)} = \sum_{m_1=1}^{D} \hat{a}^{(i;m_1m_2)} (\hat{a}^{(i)})^T R^{(m_2)}, \tag{184}
\]

\[
\tilde{Z}^{(m_1:ij)} = \sum_{\text{perm}(i,j)} \sum_{m_1=1}^{D} \hat{a}^{(i;m_1m_2)} (\hat{a}^{(i)})^T R^{(m_2)}, \tag{185}
\]

and

\[
\hat{a}^{(i)} = \Gamma (a^{(i)})^T. \tag{186}
\]
Assuming the linear independence of \( g^{(i)}(\lambda, i = 1, \ldots, K) \), we conclude that eqs. (180,181) hold if

\[
Z^{(m_2;i;j)} = 0, \quad m_2 = 1, \ldots, D, \quad (187)
\]
\[
\tilde{Z}^{(m_1;i;j)} = 0, \quad m_1 = 1, \ldots, D, \quad i, j = 1, \ldots, K. \quad (188)
\]

In other words, system of equation (180,181) depending on the spectral parameter is equivalent to system (187,188) independent on the spectral parameter.

### 7.5 Derivation of nonlinear PDEs (6) and (27): reduction of the systems (187) and (188)

Hereafter we consider the case of non-degenerate matrices \( \hat{a}^{(i;m_1m_2)} \). In order to find the allowed coefficients \( L^{(m_1)} \) and \( R^{(m_2)} \) in the nonlinear PDE (6) we have to find the non-degenerate matrices \( L^{(m_1)} \) and \( R^{(m_2)} \) satisfying the systems of equations (187) and (188) respectively. However this system is too large to admit the non-degenerate matrices \( L^{(m_1)} \) and \( R^{(m_2)} \) as solutions. We reduce the number of equations in this system imposing the decomposition for the constant diagonal matrices \( \hat{a}^{(i;m_1m_2)} \) into other non-degenerate diagonal matrices \( \hat{a}^{(i;m_11)} \) and \( \hat{a}^{(i;1m_2)} \).

\[
\hat{a}^{(i;m_1m_2)} = \hat{a}^{(i;m_11)} \hat{a}^{(i;1m_2)}, \quad m_1, m_2 = 1, \ldots, D, \quad i = 1, \ldots, K, \quad (189)
\]

with

\[
\hat{a}^{(i;11)} = I_N, \quad i = 1, \ldots, K. \quad (190)
\]

In this case, putting \( i = j \) in eq. (187) and multiplying it by \( (\hat{a}^{(i;1m_2)})^{-1} \) from the right we eliminate superscript \( m_2 \), i.e. eq. (187) with \( i = j \) yields:

\[
Z^{(1;ii)} = \sum_{m_1=1}^{D} L^{(m_1)} \hat{a}^{(i;i)} \hat{a}^{(i;m_11)} = 0, \quad i = 1, \ldots, K. \quad (191)
\]

In turn, each of eqs. (187) with \( i \neq j \) might be represented in the form

\[
\sum_{m_1=1}^{D} L^{(m_1)} \left( \hat{a}^{(i)} \hat{a}^{(j;m_11)} \hat{a}^{(i;1m_2)} + \hat{a}^{(j)} \hat{a}^{(i;m_11)} \hat{a}^{(j;1m_2)} \right) = 0, \quad i, j = 1, \ldots, K, \quad i \neq j. \quad (192)
\]

Require the linear independence of matrices \( \hat{a}^{(i;1m_2)} \), \( i = 1, \ldots, K \). Then each of eqs. (192) must be separated into two equations. As a result, we obtain

\[
\sum_{m_1=1}^{D} L^{(m_1)} \hat{a}^{(i)} \hat{a}^{(j;m_11)} = 0, \quad i, j = 1, \ldots, K, \quad i \neq j. \quad (193)
\]

We combine systems (191) and (193) in the following single formula:

\[
\sum_{m_1=1}^{D} L^{(m_1)} \hat{a}^{(i)} \hat{a}^{(j;m_11)} = 0, \quad i, j = 1, \ldots, K. \quad (194)
\]
Similarly, multiplying eq. (188) with \( i = j \) by \( (\hat{a}^{(i;m_1)})^{-1} \) from the left we eliminate the superscript \( m_1 \), while each of equations (188) with \( i \neq j \) can be separated into two equations (require the linear independence of matrices \( \hat{a}^{(i;m_1)} \), \( i = 1, \ldots, K \)). Thus, the system (188) yields

\[
\sum_{m_2=1}^{D} \hat{a}^{(i;m_2)}(\hat{a}^{(j)})^T R^{(m_2)} = 0, \quad i, j = 1, \ldots, K.
\]

(195)

We obtain the general solution to the system (194) in Sec. 7.5.1. After that we consider a reduced form of this system in Sec. 7.5.2.

### 7.5.1 Derivation of general nonlinear PDE (6)

The system (194) consists of \( K^2 \) matrix \( N \times N \) equations and may be considered as a system for the constant matrices \( L^{(m_1)} \), \( m_1 = 1, \ldots, D \). Similarly the system of \( K^2 \) equations (195) must be solved for the constant matrices \( R^{(m_2)} \), \( m_2 = 1, \ldots, D \). In order to avoid additional constraints on the matrices \( \hat{a}^{(i;m_1)} \), \( \hat{a}^{(i;m_2)} \) and \( \hat{a}^{(i)} \), we need

\[
D > K^2,
\]

(196)
i.e. the minimal value \( D^{(\text{min})} \) must be \( D^{(\text{min})} = K^2 + 1 \). Then the system (194) may be considered as a system for \( D^{(\text{min})} - 1 \) matrices \( L^{(m_1)} \), \( m_1 = 1, \ldots, D^{(\text{min})} - 1 \) with arbitrary \( L^{(i)} \), \( i = D^{(\text{min})} - 1 \), \( \ldots \), \( D \), while the system (195) may be considered as a system for \( R^{(m_2)} \), \( m_2 = 1, \ldots, D^{(\text{min})} - 1 \) with arbitrary \( R^{(i)} \), \( i = D^{(\text{min})} \), \( \ldots \), \( D \).

What is the corresponding matrix dimensionality \( N \) of the nonlinear PDE (6)? The answer to this question is closely related to the resolvability of relations (189). Remember that \( \hat{a}^{(i;m_1)m_2} \) must satisfy their definitions (172), which, in view of eq. (189), takes the form (15). Let us consider the system (172) as a system for \( \hat{a}^{(i)} \). We may write it in the matrix form as follows. Let \( \tilde{D} \) be the number of different \( \hat{a}^{(i;m_1)m_2} \) for any \( i \). In general, this number is \( \tilde{D} = D^2 \).

Introduce the \( N \times \tilde{D} \) matrices \( \hat{\xi} \) and \( \hat{A}^{(i)} \)

\[
\hat{\xi} = \begin{pmatrix}
\eta^{(11)} \cdots \eta^{(1D)} \\
\eta^{(11)} \cdots \eta^{(21)} \\
\eta^{(1D)} \cdots \eta^{(21)} \\
\cdots \cdots \cdots \\
\eta^{(N1)} \cdots \eta^{(N1)} 
\end{pmatrix},
\]

(177)

\[
\hat{A}^{(i)} = \begin{pmatrix}
\hat{a}^{(i;11)} \cdots \hat{a}^{(i;1D)} \\
\hat{a}^{(i;11)} \cdots \hat{a}^{(i;21)} \\
\hat{a}^{(i;1D)} \cdots \hat{a}^{(i;21)} \\
\cdots \cdots \cdots \\
\hat{a}^{(i;N1)} \cdots \hat{a}^{(i;N1)} 
\end{pmatrix}.
\]

(178)

Elements of these matrices read:

\[
\hat{\xi}_{\beta j} = \eta^{(m_1m_2)}_{\beta j} \\
\hat{A}^{(i)}_{\beta j} = \hat{a}^{(i;m_1m_2)}_{\beta j} = \hat{a}^{(i;m_1)}_{\beta} \hat{a}^{(i;m_2)}_{j},
\]

\[
m_1, m_2 = 1, \ldots, D, \quad \beta = 1, \ldots, N, \quad j = 1, \ldots, \tilde{D}.
\]

Now we may write system (172) in the following matrix form:

\[
a^{(i)} \hat{\xi} = \hat{A}^{(i)}.
\]

(200)
The matrices $a^{(i)}$ may be uniquely found if $N = N^{(min)} = \tilde{D}$ and $\det \hat{\xi} \neq 0$:

$$a^{(i)} = \hat{A}^{(i)} \hat{\xi}^{-1}. \quad (201)$$

Herewith the elements $a^{(i;1m_2)}_\beta$ and $a^{(i;1m_1)}_\beta$ are arbitrary parameters. If $N > \tilde{D}$, then matrices $a^{(i)}$ are not unique and some elements of the matrices $a^{(i)}$ might be arbitrary as well (we assume that $\det \hat{\xi} \neq 0$). Having defined matrices $a^{(i)}$, we fix all coefficients in the systems (194) and (195). Now these systems lead to eqs. (17) and (18) respectively and can be solved for $L^{(m_1)}$ and $R^{(m_2)}$.

### 7.5.2 Derivation of reduced equation (27): solution to eq. (15) with $N < D^2$

Since $N \geq \tilde{D} \sim K^4$, the matrix dimensionality $N$ is very large and increases very fast with the number $K$ of independent functions $g^{(k)}(\lambda)$. In this section we decrease the matrix dimensionality and reduce the nonlinear PDE (6) to eq. (27).

Let $N < \tilde{D}$. Now $\hat{\xi}$ and $\hat{A}^{(i)}$ are rectangular $N \times \tilde{D}$ matrices and may be represented in the following block forms:

$$\hat{\xi} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix}, \quad \hat{A}^{(i)} = \begin{pmatrix} \hat{A}_1^{(i)} \\ \hat{A}_2^{(i)} \end{pmatrix}, \quad (202)$$

where $\hat{\xi}_1$ and $\hat{A}_1^{(i)}$ are the square $N \times N$ matrices, while $\hat{\xi}_2$ and $\hat{A}_2^{(i)}$ are the rectangular $N \times (\tilde{D} - N)$ matrices. Split eq. (200) into two following equations:

$$a^{(i)} \hat{\xi}_1 = \hat{A}_1^{(i)}, \quad (203)$$
$$a^{(i)} \hat{\xi}_2 = \hat{A}_2^{(i)}, \quad i = 1, \ldots, K. \quad (204)$$

Let $\hat{\xi}_1$ be invertible. Without loss of generality we take $\hat{\xi}_1 = I_N$. Then eq. (203) defines $a^{(i)}$:

$$a^{(i)} = \hat{A}_1^{(i)}, \quad i = 1, \ldots, K. \quad (205)$$

In particular, if $N = D$, then $a^{(i)}_{aam_1} = a^{(i;1m_1)}_a$. Substituting eq. (205) into eq. (204), we obtain:

$$\hat{A}_1^{(i)} \hat{\xi}_2 = \hat{A}_2^{(i)}, \quad i = 1, \ldots, K. \quad (206)$$

Eq. (206) with $i = 1$ defines $\hat{\xi}_2$ (assume $\det \hat{A}_1^{(i)} \neq 0$):

$$\hat{\xi}_2 = (\hat{A}_1^{(i)})^{-1} \hat{A}_2^{(i)}. \quad (207)$$

It is obvious that eqs. (206) with $i > 1$ hold if

$$\hat{A}_2^{(i)} = \Pi^{(i)} \hat{A}_2^{(i)}, \quad \hat{A}_1^{(i)} = \Pi^{(i)} \hat{A}_1^{(i)}, \quad \Pi^{(1)} = I_N, \quad (208)$$

where $\Pi^{(i)}$ are constant matrices and $I_N$ is the $N \times N$ identity matrix. However, one has to remember relations (189), which mean that the elements of $\hat{A}_2^{(i)}$ are related with the elements of $\hat{A}_1^{(i)}$. Therefore not any constant matrices $\Pi^{(i)}$ can be taken in relations (208). The allowed matrices $\Pi^{(i)}$ are those that rearrange rows in the matrices. Thus, all matrices $\hat{A}^{(i)}$ consist of the same rows taken in different orders.
Having relations (208) among \( \hat{\alpha}^{(i)} \) and, consequently, the similar relations among \( \alpha^{(i)}, \hat{\alpha}^{(i)} \) and \( \hat{\alpha}^{(i;m_1 m_2)} \),

\[
a^{(i)} = \Pi^{(i)} a^{(1)}, \quad \hat{\alpha}^{(i)} = \hat{\alpha}^{(1)} (\Pi^{(i)})^T, \quad \hat{\alpha}^{(i;m_1 m_2)} = \Pi^{(i)} \hat{\alpha}^{(1;m_1 m_2)} (\Pi^{(i)})^T,
\]

we establish the equivalence between equations (194) with \( i = j \), i.e. only one of them is independent:

\[
\sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(i)} \hat{\alpha}^{(i;m_1 1)} = 0 \iff \sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} (\Pi^{(i)})^T \Pi^{(i)} \hat{\alpha}^{(1;m_1 1)} = 0 \Rightarrow
\]

\[
\sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} \hat{\alpha}^{(1;m_1 1)} = 0.
\]

Eqs. (194) with \( i \neq j \) read

\[
\sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} (\Pi^{(i)})^T \Pi^{(j)} \hat{\alpha}^{(1;m_1 1)} = 0, \quad i \neq j, \quad i, j = 1, \ldots, K.
\]

Similarly, eqs. (195) get the following form

\[
\sum_{m_2 = 1}^{D} \hat{\alpha}^{(1;m_2 1)} (\hat{\alpha}^{(1)})^T R^{(m_2)} = 0,
\]

\[
\sum_{m_2 = 1}^{D} \hat{\alpha}^{(1;m_2 1)} (\Pi^{(j)})^T \Pi^{(i)} (\hat{\alpha}^{(1)})^T R^{(m_2)} = 0, \quad i \neq j, \quad i, j = 1, \ldots, K.
\]

Note that not all equations in the systems (211) and (213) are independent. We discuss this property using a particular representation of \( \Pi^{(i)} \) in terms of the matrix \( \Pi \) which shifts the rows: \( \Pi^{(i)} = \Pi^{i - 1} \). Then eqs. (209) read

\[
a^{(i)} = \Pi^{i - 1} a^{(1)}, \quad \hat{\alpha}^{(i)} = \hat{\alpha}^{(1)} (\Pi^T)^{i - 1}, \quad \hat{\alpha}^{(i;m_1 m_2)} = \Pi^{i - 1} \hat{\alpha}^{(1;m_1 m_2)} (\Pi^T)^{i - 1}.
\]

Only those of eqs. (211) (and (213)) are independent which have different values \( i - j \) (this increment might be either positive or negative). Thus, we reduce the system (210, 211) to

\[
\sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} \hat{\alpha}^{(i;m_1 1)} = 0 \Rightarrow \sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} \Pi^{i - 1} \hat{\alpha}^{(1;m_1 1)} = 0, \quad i = 1, \ldots, K
\]

\[
\sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(i)} \hat{\alpha}^{(1;m_1 1)} = 0 \Rightarrow \sum_{m_1 = 1}^{D} L^{(m_1)} \hat{\alpha}^{(1)} (\Pi^T)^{i - 1} \hat{\alpha}^{(1;m_1 1)} = 0, \quad i = 2, \ldots, K,
\]

which is a system of \( 2K - 1 \) equations. Or, introducing matrices \( \tilde{L}^{(m_1)} = L^{(m_1)} \hat{\alpha}^{(1)} \), we may represent system (213) by the single formula

\[
\sum_{m_1 = 1}^{D} \tilde{L}^{(m_1)}_{\alpha \beta} \hat{\alpha}^{(1;m_1 1)} = 0 \Rightarrow \sum_{m_1 = 1}^{D} \tilde{L}^{(m_1)}_{\alpha (\beta \pm i)} \hat{\alpha}^{(1;m_1 1)} = 0,
\]

\[
\hat{\alpha}^{(1;m_1 1)} = \hat{\alpha}^{(1;m_1 1)} \hat{\alpha}^{(1;m_1 1)} = \hat{\alpha}^{(1;m_1 1)} = \hat{\alpha}^{(1;m_1 1)},
\]

\[
\tilde{L}^{(m_1)}_{\alpha(N+i)} = \tilde{L}^{(m_1)}_{\alpha i}, \quad \tilde{L}^{(m_1)}_{\alpha(N-i)} = \tilde{L}^{(m_1)}_{\alpha(-i)}, \quad i = 0, 1, \ldots, K - 1.
\]
Thus, the total number of equations reduces from $K^2$ in system (194) to $(2K - 1)$ in the system (216). Consequently, now $D^{(\min)} = 2K$, which provides the solvability of system (216) with respect to $\hat{L}^{(m_1)}$.

Similarly, the system (212-213) reduces to

$$
\sum_{m_2=1}^{D} \hat{a}_{(1;m_2)}^{(i;1m_2)}(\hat{a}^{(1)})^T R^{(m_2)} = 0 \quad \Rightarrow \quad \sum_{m_2=1}^{D} \hat{a}_{(1;m_2)}^{(i;1m_2)}(\Pi^{i-1}(\hat{a}^{(1)})^T R^{(m_2)} = 0, \quad i = 1, \ldots, (221)
$$

which is also a system of $2K - 1$ equations. System (217) may be written in terms of the matrices $\hat{R}^{(m_2)} = (\hat{a}^{(1)})^T R^{(m_1)}$ as

$$
\sum_{m_2=1}^{D} \hat{a}_{(1;m_2)}^{(i;1m_2)} \hat{R}_{\alpha\beta}^{(m_2)} = 0 \quad \Rightarrow \quad \sum_{m_2=1}^{D} \hat{a}_{\alpha}^{(1;m_2)} \hat{R}_{(\alpha\mp1)\beta}^{(m_2)} = 0, 
$$

Thus, the total number of equations reduces from $K^2$ in the system (195) to $(2K - 1)$ in the system (217), i.e. $D^{(\min)} = 2K$ provides solvability of system (218) with respect to $\hat{R}^{(m_2)}$.

The systems (216) and (218) allow us to introduce the diagonal matrices $\hat{L}^{(m_1)}$ and $\hat{R}^{(m_2)}$:

$$
\hat{L}_{\alpha\beta}^{(m_1)} = \delta_{\alpha\beta} \hat{L}_{\beta}^{(m_1)}, 
$$

$$
\hat{R}_{\alpha\beta}^{(m_2)} = \delta_{\alpha\beta} \hat{R}_{\beta}^{(m_2)}. 
$$

Now the systems (216) and (218) may be written as eqs. (228) and (229) respectively which may be solved for $\hat{a}^{(1;m_1)}$ and $\hat{a}^{(1;m_2)}$, $m_1, m_2 = 2, \ldots, D$, see Remark 1 in Sec. 2.2, eqs. (38-39).

Next, we turn to nonlinear equation (6) and write it in the form

$$
\sum_{m_1, m_2=1}^{D} \hat{L}_{m_1m_2}^{(m_1)} \left( \hat{V}_{m_1m_2} + \hat{V}(\hat{a}^{(1)})^T \eta^{(m_1m_2)} \eta^{(m_1m_2)} \hat{V} - \hat{V}(\hat{a}^{(1)})^T \eta^{(m_1m_2)} \eta^{(m_1m_2)} \hat{V} \right) \hat{R}_{m_2}^{(m_2)} = 0, 
$$

where

$$
\hat{V} = (\hat{a}^{(1)})^{-1} V((\hat{a}^{(1)})^T)^{-1}, 
$$

Eq. (222) may be transformed to eq. (32) using eq. (20) for $\hat{V}$, eq. (10) for $r$ and eq. (186) for $\hat{a}^{(i)}$. Taking into account relations (32), (28) and (29) we may rewrite eq. (221) in components eliminating some of the nonlinear terms:

$$
\sum_{m_1, m_2=1}^{D} \hat{L}_{m_1m_2}^{(m_1)} \left( \hat{V}_{m_1m_2} + \frac{1}{1 + Q} \sum_{\gamma=1}^{N} \sum_{\delta=1}^{N} s_{\gamma\delta}^{(1;m_1m_2)} \left( \hat{V}_{\gamma\delta} \hat{V}_{\delta\beta} - \hat{V}_{\alpha\beta} \hat{V}_{\beta\delta} \right) \right) \hat{R}_{m_2}^{(m_2)} = 0, 
$$

$\alpha, \beta = 1, \ldots, N.$
where we introduce the diagonal matrix $s$ with the diagonal elements $s_\alpha$,

$$s_\alpha = \sum_{\gamma=1}^{N} a^{(1)}_{\alpha\gamma} \xi_{\gamma 1}, \quad (224)$$

and $Z(\alpha, \beta, K) = \{\alpha \pm i, \beta \pm i, i = 1, \ldots, K - 1\}$. Assuming the invertibility of the matrix $a^{(1)}$, we can use eq. (224) as the definition of the $N \times 1$ matrix $\xi$ in terms of the arbitrary parameters $s_\alpha$:

$$\xi = (a^{(1)})^{-1} sp. \quad (225)$$

Finally, let us introduce the function $U(t)$ with elements

$$U_{\alpha\beta}(t) = s_\alpha \tilde{V}_{\alpha\beta}(t) s_\beta. \quad (226)$$

Then eq. (223) results in eq. (41). Introducing $N \times 1$ matrix $p = (1 \ldots 1)^T$ and using eqs. (189) we may write eq. (41) in the matrix form (27). In turn, expressions (14) for $\tilde{g}(\lambda; t)$ in virtue of eq. (225) yield eqs. (36)

8 Appendix B. Diagonal eq. (155) with diagonal $C^{(m)}$: classical (2+1)-dimensional $n$-wave equation

According to Sec. 7.3, eq. (155) is the principal equation in our algorithm. It defined what kind of nonlinear PDEs can be derived. Here we consider such solution to this equation that generates the classical (2+1)-dimensional $n$-wave equation.

Let eq. (155) be diagonal. This implies that matrices $R(\lambda, \mu)$, $r(\lambda)$ and $\tilde{r}(\mu)$ defined by eqs. (156) are diagonal too. Let

$$T^{(m)}(\lambda) = \lambda C^{(m)}, \quad \tilde{T}^{(m)}(\mu) = \mu C^{(m)} \quad (227)$$

(here $m = (m_1 m_2)$). Then $C^{(m)}$ may be canceled out from the eq. (155) which yields

$$R(\lambda, \mu) = \frac{r(\lambda) \tilde{r}(\mu)}{\mu - \lambda}. \quad (228)$$

Since $R$, $r$ and $\tilde{r}$ are diagonal, the conditions (140) and (145) (or (174) and (175)) are satisfied if

$$\sum_{m_1} L^{(m_1)} C^{(m_1 m_2)} = 0, \quad (229)$$

$$\sum_{m_2} C^{(m_1 m_2)} R^{(m_2)} = 0. \quad (230)$$

Because of eqs. (227), the number of arguments in arbitrary functions (parameterizing solution space) is restricted. In fact, eq. (57) now reads

$$\sum_{m_1, m_2=1}^{D} t_{m_1 m_2} (T^{(m_1 m_2)}(\lambda) - T^{(m_1 m_2)}_{\delta}(\mu)) = \sum_{m_1, m_2=1}^{D} t_{m_1 m_2} \left( C^{(m_1 m_2)}(\lambda) - C^{(m_1 m_2)}_{\delta}(\mu) \right). \quad (231)$$
We see that only two independent parameters $\lambda$ and $\mu$ appear in this expression if $\delta \neq \gamma$. If $\delta = \gamma$, then there is only one parameter $\lambda - \mu$. Thus we may introduce $(N^2 - N)$ arbitrary functions of two independent variables and $N$ arbitrary functions of one independent variable in the solution space. This means that we may provide the full solution space only to $(2+1)$-dimensional nonlinear PDE with $(N^2 - N)$ scalar fields. To obtain the classical form of this PDE we take the real diagonal matrices $C^{(m_1 m_2)}$, $L^{(m_1)}$ and $R^{(m_2)}$ with

\[
L^{(m_1)} = R^{(m_1)}, \quad C^{(m_1 m_1)} = 0, \quad C^{(m_1 m_2)} = -C^{(m_2 m_1)}, \quad t_{m_1 m_2} = -t_{m_2 m_1},
\]

\[
L^{(1)} = C^{(23)} \equiv C^{(1)} = I_N, \quad L^{(2)} = -C^{(13)} \equiv -C^{(2)}, \quad L^{(3)} = C^{(12)} \equiv C^{(3)}
\]

and introduce the new variables

\[
t^{(1)} \equiv -t^{(23)}, \quad t^{(2)} \equiv t^{(13)}, \quad t^{(3)} \equiv t^{(12)},
\]

Then eq. (146) becomes the classical $(2+1)$-dimensional nonlinear PDEs (1), coupling three variables $t_i, i = 1, 2, 3$ [29, 30].

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