BPS Wilson loops and quiver varieties

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Received 4 May 2020, revised 8 July 2020
Accepted for publication 14 July 2020
Published 21 August 2020

Abstract

Three dimensional supersymmetric field theories have large moduli spaces of circular Wilson loops preserving a fixed set of supercharges. We simplify previous constructions of such Wilson loops and amend and clarify their classification. For a generic quiver gauge theory we identify the moduli space as a quotient of $\mathbb{C}^m$ for some $m$ by an appropriate symmetry group. These spaces are quiver varieties associated to a cover of the original quiver or a subquiver thereof. This moduli space is generically singular and at the singularities there are large degeneracies of operators which seem different, but whose expectation values and correlation functions with all other gauge invariant operators are identical. The formulation presented here, where the Wilson loops are on $S^3$ or squashed $S^3_b$ also allows to directly implement a localization procedure on these observables, which previously required an indirect cohomological equivalence argument.

Keywords: supersymmetry, Wilson loops, 3D theories, moduli spaces

(Some figures may appear in colour only in the online journal)

1. Introduction and conclusion

Three dimensional conformal field theories have an intricate spectrum of line operators. The simplest Wilson loops mirror the $1/2$ BPS Wilson loop of $\mathcal{N} \geq 2$ supersymmetric Yang–Mills theory in 4D [1–4]. Another construction was required to express the $1/2$ BPS Wilson loop in ABJM theory [5, 6]. A few years ago it was realized that the possibilities of constructing Wilson loops are much larger, starting with $\mathcal{N} = 4$ quiver gauge theories, where there is a finite degeneracy of $1/2$ BPS operators [7, 8] to all theories with $2 \leq \mathcal{N} \leq 6$ with large moduli spaces

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of Wilson loops preserving four real supercharges [9–12]. Beyond the Wilson loops there are further operators dubbed vortex loops [13–15].

In chapter 2 of a recent collaborative paper [16], Nagaoka et al presented a new formalism for constructing the family of 1/6 BPS Wilson loops in ABJM theory and identified the moduli space as two copies of the conifold. Here we adapt that formalism to arbitrary theories with \( \mathcal{N} \geq 2 \) supersymmetry and implement it for the circular Wilson loop on the 3-sphere, possibly squashed \( S^3 \). In the process several new classes of operators which have not been identified previously are presented.

In addition to uncovering these new BPS operators, this constructive approach elucidates rather opaque details of previous constructions in terms of the mathematics of quivers. To summarize the results, the Wilson loops classification goes in two steps:

(a) One chooses a quiver diagram, which is related to the quiver of the gauge theory, but not necessarily identical to it. The allowed choice corresponds to and generalizes some discrete possibilities that arise in the solutions to the equations in the previous constructions.

(b) Given the quiver, one chooses a representation thereof, assigning numbers to the nodes and linear maps to the arrows. The numbers correspond to the multiplicity of the gauge field in the Wilson loop and the linear maps encapsulate couplings of the Wilson loop to the matter fields. A residual gauge symmetry (which was missed in most previous classifications) introduces a quotient on the linear space of maps, giving spaces known as quiver varieties.

Rather than giving a detailed comparison to the previous works, the Wilson loops are constructed in the following from the ground up. The very basics of the mathematics of quivers, their representations and varieties are presented to make the paper self contained.

The construction of the Wilson loops entails certain degeneracies. Some of them are residual gauge transformations leading to the quotients. Beyond that, Wilson loops at fixed points of this action and many nearby singular orbits are actually identical as quantum operators. The connections and thus the holonomies are, say, upper triangular and since Wilson loops are traced, they do not depend on anything above the diagonal. Identifying these constructions leads to conical moduli spaces, like the aforementioned singular conifold.

In the next section we apply the techniques of chapter 2 of [16] to arbitrary theories with \( \mathcal{N} \geq 2 \) in three dimensions on \( S^3 \). We start with several examples, include the theory with one vector multiplet and several fundamental and/or anti-fundamental fields. From there we go to theories with multiple vector multiplets.

The case of the squashed sphere and theories with fields of non-canonical dimensions are studied in appendix . Some of this analysis has already been done in [12], but we use a different formalism and generalize their constructions. We note that using the language of off-shell \( \mathcal{N} = 2 \) supersymmetry allows to perform supersymmetric localization immediately without resorting to a homological-equivalence argument [17–19].

An important ingredient in the 1/2 BPS Wilson loop of ABJM theory and in most of the other operators is the coupling to Fermi fields. In the original papers this coupling had a rather subtle path dependence, which was reinterpreted in [16] as a constant shift in the bosonic connection, simplifying the expressions. The analysis here elucidates the origin of this shift as arising from the symmetry algebra and related to the curvature of \( S^3 \) (and the background vector field on the squashed sphere). This is the last term in (A.9), and it was already noticed in chapter 2 of [16], in the context of the circular Wilson loop in \( \mathbb{R}^3 \). These shifts are also crucial in the construction of the quiver representing the Wilson loops and different shifts are encoded in different (graded) quiver diagrams.
The moduli spaces are studied in section 3, again starting with several simple examples. The role of the shifts in modifying the original quiver are presented and the subsequent map between Wilson loop data and that of quiver representations is then explained.

Most of the discussion in this paper is classical, except where we point out how localization can be applied. It is an interesting question to verify to what extent the statements made here are subject to quantum corrections, as with very little supersymmetry one would expect them to arise.

A natural avenue to address that is by viewing Wilson loops as defect CFTs. This leaves many questions on the anomalous dimensions of insertions into the Wilson loop and especially the relation between the moduli spaces found here and the Zamolodchikov metric of the defect CFTs.

2. BPS Wilson loops and quiver representations

2.1. Wilson loop from vector multiplet

Any 3D theory with $\mathcal{N} \geq 2$ supersymmetry and a vector multiplet $(A_\mu, \lambda, \bar{\lambda}, \sigma, D)$ has a BPS Wilson loop of the form [1]

$$W = \text{Tr} \mathcal{P} \exp \oint (iA_\mu \dot{x}^\mu + \sigma |\dot{x}|) \, d\tau.$$  \hspace{1cm} (2.1)

We consider the Euclidean theory on $S^3$ of radius $R$, where the path is a great circle in the direction of the dreibein $e^1 = R \dot{\varphi}$. One can just as well take a circle in flat $\mathbb{R}^3$, or as done in the appendix, the squashed sphere. Using the variations in (A.4), it is easy to show that the Wilson loop is invariant under supersymmetry as long as the independent parameters $\epsilon$ and $\bar{\epsilon}$ satisfy

$$(\gamma_1 - 1)\epsilon = (\gamma_1 + 1)\bar{\epsilon} = 0. \hspace{1cm} (2.2)$$

On $S^3$ this restricts the chirality of the supercharges to $\epsilon^1$ and $\bar{\epsilon}^2$, while on $S^3_{\emptyset}$ it restricts the loop to a particular circle at $\vartheta = 0$. Denoting the corresponding supercharges $Q$ and $\bar{Q}$, these two annihilate the Wilson loop. In the following we use the two linear combinations of them $Q_{\pm} = Q \pm \bar{Q}$.

2.2. Wilson loop with matter

Let us assume that in addition to the vector multiplet the theory has $n$ fundamental chiral fields $(\phi^i, \psi^i, F^i)$ with $i = 1, \ldots, n$ and their conjugates. Here we take that the fields have canonical dimensions $(1/2, 1, 3/2)$ respectively, which is guaranteed for $\mathcal{N} > 2$. The case of $\mathcal{N} = 2$ with non canonical dimensions is presented in the appendix.

Using the SUSY transformation of [15, 17–20], summarized in the appendix, we have that the scalar fields satisfy (A.9)

$$R Q^{2}_+ \phi = i \partial_\phi \phi - A_\phi \phi + i R \sigma \phi - \frac{1}{2} \phi, \hspace{0.5cm} R Q^{2}_- \phi = i \partial_{\bar{\phi}} \phi + \bar{\phi} A_{\bar{\phi}} - i R \bar{\phi} \sigma + \frac{1}{2} \bar{\phi}. \hspace{1cm} (2.3)$$

To account for the factor of $1/2$, it is natural to shift the connection in (2.1) and for the purpose of coupling to the chiral fields we package the bosonic loop in a $2 \times 2$ block diagonal structure as
\[
W + 1 = -sTr \mathcal{P} \exp \int i \mathcal{L}_0 |\dot{x}| d\tau, \quad \mathcal{L}_0 = \left( \begin{array}{cc}
A_\mu^\alpha & i \sigma - \frac{1}{2R} \frac{1}{|\dot{x}|} \\
0 & 0
\end{array} \right).
\]

(2.4)

The extra 1 on the left-hand side of (2.4) accounts for the contribution from the trivial 1 \times 1 block. From now on we will absorb this term and the overall sign in front of the supertrace into \( W \).

We now place the chiral and anti-chiral fields into the off-diagonal entries of \( (N|1) \) odd-supermatrices, where the off-diagonal entries are Grassmann even.

\[
\mathcal{L}_{a,\bar{a}} = G_a + \bar{G}_{\bar{a}}, \quad G_a = \left( \begin{array}{cc}
0 & u_i \phi_i' \\
0 & 0
\end{array} \right), \quad \bar{G}_{\bar{a}} = \left( \begin{array}{c}
0 \\
0
\end{array} \right).
\]

(2.5)

\( u' \) and \( \bar{u} \) are arbitrary complex vectors (not necessarily complex conjugates). With this we can compactly write (2.3) as

\[
Q_+^2 \mathcal{L}_{a,\bar{a}} = i \mathcal{D}_0 \mathcal{L}_{a,\bar{a}} = \left( \begin{array}{c}
\bar{G}_{\bar{a}} \\
0
\end{array} \right) \mathcal{D}_0 G_a - \left[ \mathcal{L}_0, \mathcal{L}_{a,\bar{a}} \right].
\]

(2.6)

We use \( \mathcal{L}_{a,\bar{a}} \) to deform the Wilson loop to

\[
W_{a,\bar{a}} = sTr \mathcal{P} \exp \int i \mathcal{L}_{a,\bar{a}} |\dot{x}| d\tau, \quad \mathcal{L}_{a,\bar{a}} = \mathcal{L}_0 - i Q_+ \mathcal{L}_{a,\bar{a}} + \mathcal{L}_{a,\bar{a}}^2.
\]

(2.7)

Under supersymmetry transformations

\[
Q_+ \mathcal{L}_{a,\bar{a}} = Q_+ \mathcal{L}_0 - i Q_+^2 \mathcal{L}_{a,\bar{a}} + \left( Q_+ \mathcal{L}_{a,\bar{a}}, \mathcal{L}_{a,\bar{a}} \right) + \mathcal{D}_0 \mathcal{L}_{a,\bar{a}} + \left( \mathcal{L}_{a,\bar{a}}, \mathcal{L}_{a,\bar{a}} \right).
\]

(2.8)

\( \mathcal{D}_0 \) is the covariant derivative along the loop with the connection \( \mathcal{L}_0 \). We can replace it by \( \mathcal{D}_{a,\bar{a}} \) with the new connection

\[
Q_+ \mathcal{L}_{a,\bar{a}} = \mathcal{D}_{a,\bar{a}} \mathcal{L}_{a,\bar{a}} = \frac{1}{R} \mathcal{D}_0 \mathcal{L}_{a,\bar{a}} + i \left[ \mathcal{L}_0, \mathcal{L}_{a,\bar{a}} \right] + \left( \mathcal{L}_{a,\bar{a}}, \mathcal{L}_{a,\bar{a}} \right) + i \left[ \mathcal{L}_{a,\bar{a}}, \mathcal{L}_{a,\bar{a}} \right].
\]

(2.9)

This is a total derivative (or a field-valued supergauge transformation), so the Wilson loop \( W_{a,\bar{a}} \) is invariant under this supersymmetry transformation. Note that guaranteeing cancelation of this term upon integration is the reason for the supertrace in the definition of the Wilson loop (2.7). Also, the fact that one of the commutators is replaced with an anti-commutator is due to \( Q_+ \mathcal{L}_{a,\bar{a}} \) being Grassmann odd. The cancelation of the total derivative terms and all the signs were checked carefully in [5] in the case of ABJM theory and the argument carries over.

A quiver diagram representing this Wilson loop illustrated in figure 1(b), where the shift of the gauge connection in (2.4) is represented by the squiggly circle.
The notation and subsequent classification of loop operators are explained in section 2.4.

To check invariance under the second supercharge, we note that
\[ Q_- G_u = -Q_+ G_u, \quad Q_- G_\bar{u} = Q_+ G_\bar{u}, \quad Q_-^2 G_u = -Q_+^2 G_u, \quad Q_-^2 G_\bar{u} = -Q_+^2 G_\bar{u}, \]
which gives
\[ Q_- L_{u,\bar{u}} = \mathcal{D}_{u,\bar{u}} (G_u - G_\bar{u}) - 2Q_+ (G_u^2 - G_\bar{u}^2) - 2i \left[ G_u^2 - G_\bar{u}^2, G_u - G_\bar{u} \right]. \]

For the last two terms to vanish we need to require that both \( G_u \) and \( G_\bar{u} \) are nilpotent of index 2, which indeed is the case in (2.5). In other theories this may be a non-trivial constraint. If \( G \) or \( \bar{G} \) do not square to zero, we end up with 1/4 BPS loops. With some changes of signs in (2.7) we can construct 1/4 BPS loops invariant under \( Q_- \) instead of \( Q_+ \).

The next ingredient we want to consider are anti-fundamental chiral multiplets \( \phi^\dagger \). The analog of (2.3) is
\[ R Q_+^2 \phi = i \partial_\mu \phi + i A_\mu \phi - i R \phi \sigma - \frac{1}{2} \phi \phi, \quad R Q_+^2 \phi = i \partial_\mu \phi - A_\mu \phi + i R \phi \phi + \frac{1}{2} \phi \phi. \]

\( \tilde{\phi} \) transforms similarly to \( \phi \), expect for the change in sign of the last term. It is then possible to construct matter Wilson loops with \( \phi \) and \( \tilde{\phi} \), by replacing 1/2R in \( L_0 \) with -1/2R.

If we want to include both fundamental \( \phi \) and anti-fundamental \( \phi \) fields, we can use a 3 × 3 structure which forms an \( \mathcal{N}/2 \) supermatrix
\[ \mathcal{L}_0 = \begin{pmatrix} \frac{1}{R} & 0 & 0 \\ 0 & A_\mu [i \sigma - iR] & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & v_{\dot{j}} \phi^\dagger_j \\ 0 & 0 & u_j \phi^\dagger_j \end{pmatrix}, \]
\[ \bar{G}_{\dot{\alpha} \alpha} = \begin{pmatrix} 0 & 0 \\ \bar{v}^i \phi^\dagger_i & 0 \\ 0 & 0 \end{pmatrix}. \]

Constructing \( \mathcal{L}_{\alpha\dot{\alpha}}, G_{\alpha\dot{\alpha}} \) out of these ingredients as before leads to an operator invariant under \( Q_+ \), but since \( G_{\alpha\dot{\alpha}} \) and \( \bar{G}_{\dot{\alpha} \alpha} \) do not square to zero, it is not invariant under \( Q_- \), so only 1/4 BPS.

Another approach is to incorporate an explicit phase \( \phi \to e^{-i\phi} \), which now transforms under \( Q_+^2 \) exactly like \( \phi \). We can then construct the 2 × 2 superconnection out of \( L_0 \) as in (2.4) and
\[ G_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & v_{\dot{j}} \phi^\dagger_j \\ 0 & 0 \end{pmatrix}, \quad \bar{G}_{\dot{\alpha} \alpha} = \begin{pmatrix} 0 & e^{-i\bar{v}^i \phi^\dagger_i j} \\ 0 & 0 \end{pmatrix}. \]

Again we find that \( G_{\alpha\dot{\alpha}} \) and \( \bar{G}_{\dot{\alpha} \alpha} \) are not nilpotent, so the resulting Wilson loop is 1/4 BPS, not 1/2. These Wilson loops are analogous to the ‘latitude’ Wilson loops of ABJM [21, 22], related to the 1/4 BPS circular Wilson loops of \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory in 4D [23].

The quiver diagram for the theory with both fundamental and antifundamental chiral fields is in figure 2(a). The Wilson loops that couple to all the fields is represented by the quiver in figure 2(b). The solid arrows represent \( G_u \) and the fact that \( G_u^2 \neq 0 \) is evident from the pair of consecutive arrows, so this is the graphical condition distinguishing Wilson loops preserving only \( Q_+ \) or also \( Q_- \).
Figure 2. The Wilson loop (2.13) in the theory with the quiver (a) is represented by the quiver diagram (b). It is 1/4 BPS because there is a solid arrow going into the squiggly node. For the shift to decrease by 1/2R along a solid arrow, n node should have shift 1/R as in (2.13). Alternatively, if we gague transform to (2.14), the solid line pointing into a squiggly circle indicates that the fields have explicit phases. (a) Quiver diagram of a gauge theory with SU(N) vector multiplet, n fundamental chirals and n antifundamental chirals. (b) Quiver diagram for the Wilson loop with one gauge field shifted by 1/2R and couplings to all the chirals and antichirals.

Figure 3. A contribution to the Wilson loop made of the superconnection in (2.15) when expanded to sixth order in the fermions, represented here as small circles. At these locations the operator alternates between open Wilson lines with proper connections (thick arcs) and integrals of the scalar bilinear (thin arcs).

2.3. Explicit expressions

To write the Wilson loops explicitly, we use the action of $Q$ and $\bar{Q}$ in (A.5). Equation (2.7) gives

$$L_{u,\bar{u}} = L_0 + \left( u_{ij} \bar{u}^j_i \phi_j - i u_i \psi^i \right).$$

Here $\psi_2$ and $\bar{\psi}_1$ are spinor eigenstates of $\sigma_1$, see (2.2). The Wilson loop is the supertrace of the holonomy of this superconnection.

It is worthwhile to pause and mull over this formal expression. The top left corner of the supermatrix is valued in the adjoint of SU(N), as in a usual Wilson loop. The other even entry, at the bottom right, essentially $\phi\phi$, is an SU(N) singlet. This part is not required for gauge invariance, but is crucial to guarantee supersymmetry. The odd entries, with $\psi^i$ and $\bar{\psi}_j$ transform in the fundamental and anti-fundamental of SU(N). Such fields serve as endpoints of open Wilson lines. So in terms of SU(N) Wilson loops, the $(N|1)$ Wilson loop is a linear combination of many Wilson lines. One is a closed loop with the modified connection in the upper left corner. The others are collections of open arcs with the fermions as start and endpoints and this modified connection between them. The gaps between the open arcs are filled by the trivial Wilson loops made of the singlet component. See the illustration in figure 3.
Figure 4. Constructing Wilson loops with adjoint chiral matter in the theory (a) requires doubling the quiver, with one shifted and one unshifted node (b) and (c). With all the matter couplings, the resulting Wilson loop (b) is 1/4 BPS. The subquiver (c) represents (2.16), with only chirals in the upper right block and only anti-chirals in the bottom left, so is 1/2 BPS. (a) Quiver diagram of a gauge theory with SU($N$) vector multiplet and adjoint chiral. (b) A cover of (a) with one unshifted and one shifted gauge field and all possible matter couplings. (c) A subquiver of (b) with matter couplings removed so the solid arrows only point out of the squiggly circle.

Figure 5. Quiver diagrams for more general Wilson loops in the theory with adjoint matter (a). With $p$ copies of the shifted gauge field and $q$ copies unshifted, on the diagonal of $\mathcal{L}_0$, allowing for all matter couplings (b) gives 1/4 BPS Wilson loops, or restricting to only half the coupling (c) gives 1/2 BPS loops.

Before attempting to describe the most general Wilson loop arising in this way, as is done in the next section, here is another example for a field theory with one vector multiplet and several adjoint chirals. We view the matter field as if it is in the bi-fundamental of two copies of the gauge group, so define the doubled $2 \times 2$ structure (or $(N|N)$ superconnection)

$$
\mathcal{L}_{u,\bar{u}} = \begin{pmatrix}
A_{\mu} \frac{\xi^u}{|x|} - i \sigma + u \bar{\mu} \bar{\phi}^i \bar{\phi}_j & \frac{1}{2R} - i u \bar{\psi}_j^l \\
-i \bar{u} \bar{\psi}^l_i & A_{\mu} \frac{\xi^u}{|x|} - i \sigma - u \bar{\mu} \bar{\phi}_j \phi^i
\end{pmatrix}.
$$

(2.16)

Note that the two diagonal blocks have the same gauge field and $\sigma$, but a different combination of the scalars and a different constant shift (which cannot be gauged away by a single valued gauge transformation). The Wilson loop made out of this connection is in a representation of SU($N|N$) and its quiver is in figure 4(c).

2.4. General theories and quiver representations

Following the examples above, we can construct a very general Wilson loop as follows.

The first step is the construction of the block-diagonal superconnection $\mathcal{L}_0$ with any number of copies of shifted or unshifted gauge fields (with the appropriate $\sigma$). We use gauge transformations as in (2.14) to make all the shifts 0 or 1/2$R$. The vector multiplets are nodes of the original quiver and any vector field appearing unshifted in $\mathcal{L}_0$ is represented again as a circle, with its degeneracy written inside. A shifted vector field is indicated by a wiggly circle, again decorated by its degeneracy. Not all nodes of the original quiver have to be represented in the new quiver, but some may be doubled, as in the example in figure 4. The Wilson loop in the same theory with more copies of the gauge fields ($p$ shifted and $q$ unshifted) is represented by the quivers in figure 5.
Given $L_0$ we construct the off-diagonal matrices $G$ and $\overline{G}$ with the matter fields with entries only connecting shifted and unshifted entries in $L_0$. The shifts endow all the matrices with $\mathbb{Z}_2$ gradings. We can split $L_0$ into a shifted and unshifted blocks in which case $G$ and $\overline{G}$ are in the two complementary blocks. The resulting superconnection is then clearly a supermatrix.

In the quiver representing the Wilson loop, the non-zero entries in $G$ are represented by solid arrows, and of course mirror the chiral multiplets in the original quiver. $\overline{G}$, which includes the anti-chiral fields, is represented by dashed arrows. These matrices incorporate couplings similar to $u_i$ above, furnishing the Wilson loops with continuous parameters. They are rectangular matrices, i.e., linear maps between spaces associated with the nodes, forming a quiver representation.

We thus find a construction involving a discrete choice of quiver, which is a sub-quiver of a double cover of the original quiver, and continuous parameters. In section 3, we study the continuous parameters in greater detail and distinguish the moduli spaces of $1/2$ BPS and $1/4$ BPS Wilson loops.

As a further example, consider ABJ(M) theory, as illustrated in figure 6. The gauge theory has two nodes, and to construct Wilson loops we need to grade it. There are two possibilities, indicated in the right two diagrams there (class I and II in the classification of [9–12]). The middle diagram represents Wilson loops where the gauge field of the left node is shifted by $1/2\mathbb{R}$ and the second node is unshifted. The first appears $q$ times in the diagonal of $L_0$ and the second $p$ times. The right most diagram has the shift on the other gauge fields. Note that we can view the union of the two diagrams on the right as a double cover of the original quiver, where each node has a shifted and unshifted copy and we retain only the solid arrows out of the shifted (squiggly) nodes. Since the cover is disconnected, we get two options, unlike the case when the quiver has odd cycles as in figure 5.

In the graphical representation of the Wilson loops as quivers, the arrows represent the scalar couplings. As mentioned around (2.11), while $Q_+$ is preserved with coupling to any of the matter fields, for the loop to be invariant under $Q_-$ we have to impose the nilpotency condition $G^2 = \overline{G}$. $G$ are the solid arrows and $G^2$ are two consecutive solid arrows. Thus the graphical condition for $1/2$ BPS loops is that all nodes have either all outgoing solid arrows (the squiggly circles) or all ingoing (unsquiggly), and likewise for dashed arrows.

More precisely, the dashed arrows need to point in the opposite direction, otherwise also $(G + \overline{G})^2 = 0$, rendering the last term in (2.7) trivial. Formally this operator is $1/2$ BPS (it was called class III and class IV in [9–12]), but $L$ in such cases is (block) upper/lower triangular with the same diagonal as $L_0$. As explained in the next section, the Wilson loops constructed from such a connection are identical as quantum operators to those made from $L_0$.

The two quiver diagrams on the right of figure 6 are the possible Wilson loops preserving $Q_+$ and $Q_-$(which makes them $1/6$ BPS with respect to the $\mathcal{N} = 6$ of ABJ(M)). If we include the other couplings, giving a total of $8pq$ parameters, we have $1/12$ Wilson loops represented by the quiver diagram in figure 7. Note that the solid arrows going into the squiggly circle violate the grading rule in (2.13), so we are actually in the setting of (2.14), where we included...
explicit angle dependent phases and these Wilson loops are generalizations of the ‘latitude’ Wilson loops of [21, 22].

2.5. More cases

Let us discuss three extra ingredients: squashing the sphere, theories with non-canonical dimensions and the theory on $\mathbb{R}^3$.

The construction for the squashed sphere $S^3_b$, presented in the appendix, follows the exact same structure and the only modification is changing $1/2R$ to $(1 + b^2)/4bR$. The classification of 1/2 BPS loops, based on $\mathbb{Z}_2$ graded quivers (with squiggly and unsquiggly nodes) which are a cover and/or subquiver of the original gauge theory quiver remains the same, as we allow only the two shifts 0 and $(1 + b^2)/4bR$.

The classification of 1/4 BPS loops is different, since now a double shift to $(1 + b^2)/2bR$, cannot be gauged away, as it is not an integer multiple of $1/R$. This excludes the latitude Wilson loops as in (2.14), but the construction in (2.13) is still valid. If $b + 1/b$ is rational, then after several shifts we could make a gauge transformation back to 0, but otherwise we need to rely on a $\mathbb{Z}$ grading of the quiver, which exists if it is a tree. If the quiver has loops, we need in principle to take an infinite cover of it, to get a $\mathbb{Z}$ graded quiver and then base the Wilson loop on a finite subquiver of it. As usual there is also the doubling of arrows representing the anti-chiral fields.

In the case of theories with $\mathcal{N} = 2$ supersymmetry, the dimensions of the chiral fields are not protected by supersymmetry and may vary under renormalization group flow. To account for this, one can assign them non-canonical dimensions from the onset, where the scalars in the chiral multiplet have dimension $\Delta$ instead of 1/2. In this case the shift should be $\Delta/R$. If all the chirals have the same dimension, the discussion remains as in the squashed $S^3_b$ case, but if they are different, then the shifts are not all multiples of a basic shift.

To get 1/2 BPS loops we need the same types of quivers as before, with each node either having all ingoing solid arrows or all outgoing solid arrows. Generically, if all dimensions are different, only one arrow is allowed per pair of nodes and if more exist, we need to take a cover of the quiver to account for that. A simple illustration is the theory with one vector and $n$ fundamental chirals. Instead of the single flavor node with $v^i$ in the vector space $\mathbb{C}^n$, we have $n$ flavor nodes, each with a different shift and a single $v^i$ coupling. In the next section the moduli spaces of the loop operators are presented. In the case of a single shift, it is $(\mathbb{C}^n)^2 // \mathbb{C}^{*,1,1}$, while with different shifts it turns out to be only $\mathbb{C}^n$.

The 1/4 BPS loops are based on the same infinite cover of the quiver without the requirement of nodes being all ingoing or outgoing. Again one cannot generically perform gauge transformations to get loops like in (2.14), but rather use constructions as in (2.13).

Note that in all of these cases, both with $S^3_b$ and generic $\Delta$, we can still usually assign a grading to the cover of the quiver based on the distance of a node from a fixed one. This integer grading does not necessarily match the shifts, but it still guarantees that $L$ is a supermatrix. This
is violated if we can form odd loops, when $b + 1/b$ is rational or when three dimensions align as in $\Delta_1 + \Delta_2 = \Delta_3$.

Projecting from $S^3$ to $\mathbb{R}^3$, the Wilson loop can become a circle or an infinite straight line. For the circle, one replaces $R$ in the shifts with the radius of the circle as in [16], so the discussion remains identical to $S^3$. In the case of the straight line, taking $R \to \infty$ leads all the shifts to vanish.

In this case there is no need to grade the quiver or take its cover (just including the dashed arrows). There are no restrictions on constructing $1/4$ BPS loops, and the supersymmetry is doubled if $G^2 = \overline{G}^2 = 0$, as usual. As already pointed out in [12], a Wilson line for a triangular quiver need not satisfy the $\mathbb{Z}_2$ grading and $L$ is not a supermatrix.

In theories with $\mathcal{N} \geq 4$ there are Wilson loops preserving more than the minimal set of supercharges. In the present formulation it is hard to identify the points of enhanced supersymmetry on the moduli space, as it relates to other details of the theory like the superpotential. In the case of ABJM an argument based on enhanced SU(3) symmetry was presented in [16]. Also the fermionic latitudes of ABJM preserve 2 complex supercharges, while the construction here guarantees only one.

A last point we have not touched upon yet are the representation of the Wilson loops and in all the expressions above the supertrace is assumed to be in the fundamental representation of the large matrix. One could perform the trace, of course also in higher dimensional representations. We have also not discussed when the different constructions are really different or are reducible. This question should be answered by whether the quiver representation is irreducible, as discussed in one example around (3.2) below.

### 2.6. Localization

Since the constructions presented above relies on $\mathcal{N} = 2$ off-shell supersymmetry on the sphere, for all the $1/2$ BPS Wilson loops we can immediately apply supersymmetric localization in this formalism. The bosonic Wilson loop was already localized in [17] and the generalization to the loops coupled to the matter fields requires no further work, since at the localization locus the matter fields (as well as the gauge field) vanish. We are left with the field $\sigma$, fixed to a constant, so the operators are identical to the usual bosonic loops, or combination thereof in the extra even blocks of the superconnection. In the case of the theory with one vector and $n$ fundamental chirals, the expectation value of the Wilson loop is then given by the matrix model

$$
\langle W_{a,b} \rangle = \left\langle -s \text{Tr} e^{ \frac{2 \pi \sigma + \gamma i}{i} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}} \right\rangle_{\text{M.M.}} = \langle \text{Tr} e^{2 \pi \sigma} \rangle_{\text{M.M.}} + 1.
$$

M.M. represents matrix model calculations, the usual result of the localization procedure.

Note that we have made no reference to the action of the field theory, and the entire discussion goes through as long as it is supersymmetric. We can have Yang–Mills and chiral actions, that do not effect the matrix model. The Chern–Simons action, Fayet–Iliopoulos and mass terms do appear in the matrix model action, as usual [17–19].

The cases that are $1/4$ BPS require new techniques to perform localization with only $Q_+$. For the latitude of ABJM theory, a matrix model was proposed in [22]. It is possible to come up with generalizations of this proposal for all theories, but we do not pursue that here.
3. Moduli spaces

In the example of a vector with \( n \) fundamental chirals and an \((N|1)\) superconnection, the Wilson loop \( W_{u,\bar{u}}(2.7) \) is defined in terms of two complex \( n \)-component vectors \( u_i \) and \( \bar{u}_i \), but it is invariant under the (constant) gauge transformation

\[
L_{u,\bar{u}} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & x \end{pmatrix} L_{u,\bar{u}} = L_{xu,\bar{u}/x}.
\] (3.1)

We find the equivalence relation \((u, \bar{u}) \sim (xu, \bar{u}/x)\), giving the moduli space \((\mathbb{C}^n)^2/\mathbb{C}^*_{-1,1}\), where the subscripts represent the weights of this action on each copy of \( \mathbb{C}^n \).

To get a nice space out of this quotient requires either further identifications or some resolution. For example consider \( \bar{u} = 0 \), where \( \bar{G} = 0 \), so \( L_{u,\bar{u}} \) is upper triangular and the diagonal pieces are the same as \( L_0 \). Any product of such a superconnection is still triangular so the Wilson loop operator that we get from supertracing such products is identical to the original Wilson loop without matter. Their expectation value and correlators with any other operators are identical.

Based on this, we should identify the \( \bar{u} = 0 \) subspace as well as that of \( u = 0 \) with the origin (the Wilson loop with \( L_0 \)). An alternative description of this space is that of complex \( n \)-dimensional matrices of rank one. For \( n = 2 \) we have the four coordinates \( P = u_1\bar{u}'_1, R = u_1\bar{u}'_2, S = u_2\bar{u}'_1, T = u_2\bar{u}'_2 \) satisfying \( PT - RS = 0 \) inside \( \mathbb{C}^4 \), which is the equation for the singular conifold.

As already pointed out in chapter 2 of [16], the moduli space in the case of the \( 1/6 \) BPS Wilson loops in ABJM theory is two copies of the same singular conifold. Indeed the discussion above carries over to the case of the two quiver diagrams on the right of figure 6 for \( p = q = 1 \), each giving one copy of the conifold.

Going back to the theory with \( n \) fundamentals, we can construct more complicated Wilson loops based on \( 3 \times 3 \) block matrices, coupling to more than one copy of the gauge field, represented by the quiver diagram in figure 8. Let us label the couplings \( u_i, \bar{u}', u'_i \) and \( \bar{u}' \) such that we have (note that \( G \) and \( \bar{G} \) are nilpotent)

\[
L_0 = \begin{pmatrix} A_{\mu} \frac{i^\mu}{|x|} - i\sigma + \frac{1}{2R} & 0 & 0 \\ 0 & A_{\mu} \frac{i^\mu}{|x|} - i\sigma + \frac{1}{2R} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & u_i \bar{\phi}'_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{u}' \bar{\phi}_i & \bar{u}' \bar{\phi}_i & 0 \end{pmatrix}.
\] (3.2)
Now we have the gauge freedom of $GL(2, \mathbb{C})$ acting on the $2 \times 2$ block. Giving the moduli space $(\mathbb{C}^{2n})^2/\text{GL}(2, \mathbb{C})$ (with inverse action on one of the two vector spaces). Again some care is needed to identify the singular orbits of this action and reduce to the conical space. Starting with the case of a single chiral ($n = 1$), the dimension of the quotient seems to vanish and in fact there are only singular orbits. The quotient acts on the rank-one matrix constructed out of $u_1 u_1^*, u_1 u_2^*, u_2 u_2^*$ by conjugation and can therefore diagonalize it, amounting to setting $u_1' = u_2' = 0$. This implies that the Wilson loops in this case are reducible to a block diagonal form with one $2 \times 2$ block as in (2.15) and another block with only the gauge field and no coupling to the chirals, as in (2.1). The same is true if we include more copies of the gauge field in the theory with a single chiral, but for more than one chiral ($n > 1$) these spaces are more interesting.

The analysis in the case of 1/4 BPS Wilson loops is identical, the moduli space can be constructed from the quiver describing the Wilson loop.

Taking the example of the theory with $n$ fundamental and $\bar{n}$ anti-fundamental chiral fields, the discussion after (2.12) suggests two possible constructions of 1/4 BPS loops. We can get 1/2 BPS loops based on (2.13) with say $v = \bar{u} = 0$, so coupling to only $\phi$ and $\bar{\phi}$. In this case, not only $G^2 = \mathbb{T}^2 = 0$, but also $(G + \mathbb{T})^2 = 0$, which is similar to the upper-triangular example discussed above. These Wilson loops (class III and IV of [9–12]) are identical to the analog bosonic Wilson loops. Ignoring such trivial Wilson loops, we have a moduli space which is the union of two cones meeting at a point: $(\mathbb{C}^n)^2/C_{1,-1}^* \oplus (\mathbb{C}^n)^2/C_{1,-1}^*$.

The moduli space of 1/4 BPS loops is much larger. The construction in (2.14) is similar to (2.5) but with a pair of $n + \bar{n}$ fields, so the moduli space is $(\mathbb{C}^{n+\bar{n}})^2/C_{1,-1}^*$. The construction in (2.13) is similar to (3.2), but the symmetry is only $(\mathbb{C}^n)^2$, so the moduli space is $(\mathbb{C}^n)^2/C_{1,-1}^* \oplus (\mathbb{C}^n)^2/C_{1,-1}^*$. Since this construction anyhow preserves only one supercharge, there is really no reason to restrict the blocks to only the $n$ or $\bar{n}$ fields, we could restore the full effective flavor group and with two copies of matter blocks have the moduli space $(\mathbb{C}^{2n+2\bar{n}})^2/\text{GL}(2, \mathbb{C})$.

3.1. Quiver varieties

Analyzing the full spectrum of BPS Wilson loops seems to be a daunting problem where starting from the most general $\mathcal{L}_0$, $G$, we need to understand the residual symmetry and analyze the resulting quotient.

Luckily this problem is exactly the theory of quiver varieties (see e.g. [24–27]) which is where the full power of the quiver representation of the BPS Wilson loops presented in the figures in section 2 comes to the fore. To make the connection, recall that a representation of a quiver corresponds to assigning a vector space to each node and a linear map to each edge. We identified $\mathcal{L}_0$ with the node information, since it includes the gauge fields. In particular, the vector space is $\mathbb{C}^p$ if there are $p$ blocks with the same gauge field (with identical shifts). $G$ and $\overline{G}$ represent the linear maps, as they have entries for each chiral (and anti-chiral) field between the appropriate blocks in $\mathcal{L}_0$. Note that $G$ and $\overline{G}$ have independent $n$ and $\bar{n}$ parameters. As far as representations of quivers, this implies the doubling of edges, where every arrow is augmented by another one with the opposite orientation, the dashed arrows in our quivers.

We stress that the ranks of the gauge groups play no role here. The overall size of the matrices depends on them, but we are not higgsing the vector multiplet, so have no freedom within each $N \times N$ block associated to an SU($N$) gauge field and it counts as a single copy of $\mathbb{C}$ in the quiver representation. This is different for flavor nodes (or framing in the mathematical language), where as we saw in the examples above, a single SU($n$)
flavor node introduced a copy of $\mathbb{C}^n$. Also, in this case we do not mod out by the action of $GL(n, \mathbb{C})$, as we distinguish between the different fundamental fields (say, by assigning them masses).

The spaces of the $u, \bar{u}$ parameters modulo symmetry are exactly the varieties associated to the corresponding quiver. The classification of quiver representations reproduces all the moduli spaces outlined above and provides the answer for the classification of all Wilson loop operators in any other 3D quiver gauge theory. This is the main result of this paper and provides an organizing principle to previous attempts at classifying such line operators. This also raises many questions, under current investigation:

• Quiver varieties appear in other contexts in supersymmetric field theories, most notably in the question of the Coulomb branch of 3D $\mathcal{N} = 4$ theories, as pioneered by Hanany and Witten [28]. Why the classification of line operators and of Coulomb vacua may be related is unclear.

• The spaces found here are all cones, and except for special cases like $\mathbb{C}$, are singular. There are natural resolutions of these spaces, and it would be interesting to find whether there is a way to deform the loop operators such that the moduli spaces of the deformed loop operators are the resolved spaces. It would also be interesting to understand whether more equivalences between loops exist, possibly for specific theories, leading to more intricate singular loci.

• Though we used the term moduli space, it may be more appropriate to call them parameter spaces. It would be interesting to understand whether it has a physical interpretation, such that the metric is meaningful. A natural question is then whether this structure persists at the quantum level, as studied for the finite degeneracy of loops preserving 8 supercharges in [29, 30]. Is the degeneracy lifted and/or is the metric corrected.

• Assel and Gomis studied 1/2 BPS line operators (Wilson and vortex loops) in 3D $\mathcal{N} = 4$ theories [31]. Their operators have double the amount of supersymmetry studied here, so it is not surprising that the answers are different. Still, it would be good to connect to that work by either specializing the current work to enforce more supersymmetry, or generalize their work to less supersymmetric theories.

• Related to that, it would be interesting to study the moduli spaces of vortex loops.

• It would be interesting to understand the holographic duals of these Wilson loops in theories with known holographic duals, like ABJM theory [32]. Despite some recent progress on that question [33], a full understanding of the bosonic 1/6 BPS is still lacking.

Acknowledgments

I am very grateful to all the collaborators on the recent roadmap paper [16] who stimulated this new examination of old questions. In particular the extensive exchanges with M Probst, D Trancanelli and M Trépanier were the seed of this work. I am also grateful to A Hanany, C Herzog, Y Lekili, M Meineri, D Panov and E Segal for fruitful conversations. This work is supported by an STFC Grant number ST/P000258/1.

Appendix. $\mathcal{N} = 2$ theories on the squashed sphere

This appendix repeats the construction of BPS Wilson loops presented in section 2 for theories with matter fields of non-canonical dimensions on the squashed sphere $S^3_b$, and fills in some details about the supersymmetry transformations.
The issue with non-canonical dimensions arise for theories with $\mathcal{N} \leq 2$, which can have non-trivial renormalization group flows such that the IR dimensions differ from the canonical ones. In such cases it is possible to construct UV theories with arbitrary dimensions such that the result of the localization calculation is a function of these dimensions. Using $F$-extremization [34, 35] allows then to find the correct IR dimensions and plug it into all other calculations in that theory, in this case the Wilson loops. We denote the dimensions of the chiral multiplet fields $(\phi, \psi, F)$ by $(\Delta, \Delta + 1/2, \Delta + 1)$.

For the squashed sphere we use the conventions of [15] (with the replacement $\varphi_1 \rightarrow \chi$ and $\varphi_2 \rightarrow \varphi$), which are slight modifications of those in [19] (see footnote 12 of [15]). In particular the metric on the squashed sphere $S^3_\delta$ is

$$\mathrm{ds}^2 = R^2 \left( f(\vartheta)^2 \mathrm{d}\vartheta^2 + 2b^2 \sin^2 \vartheta \, \mathrm{d}\chi^2 + b^{-2} \cos^2 \vartheta \, \mathrm{d}\varphi^2 \right); \quad f(\vartheta) = \sqrt{b^{-2} \sin^2 \vartheta + b^2 \cos^2 \vartheta}.$$

(A.1)

and dreibein

$$e^1 = Rb^{-1} \cos \vartheta \, \mathrm{d}\varphi, \quad e^2 = -Rb \sin \vartheta \, \mathrm{d}\chi, \quad e^3 = R f(\vartheta) \mathrm{d}\vartheta. \quad (A.2)$$

The spinors by default have upper indices are are lowered with $-i\sigma_2$ such that

$$\bar{\psi}\lambda = \lambda\psi, \quad \bar{\psi}\gamma^\mu\lambda = -\lambda\gamma^\mu\bar{\psi}, \quad (\gamma^\mu\bar{\psi})\lambda = -\bar{\psi}\gamma^\mu\lambda. \quad (A.3)$$

The variation of the fields in the vector multiplet are

$$\delta A_\mu = \frac{i}{2} (\gamma_\mu \lambda - \bar{\lambda} \gamma_\mu \epsilon), \quad \delta \sigma = \frac{i}{2} (\bar{\lambda} \epsilon - \lambda \epsilon),$$

$$\delta \lambda = -\frac{1}{2} \gamma^{\mu\nu} \epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu \epsilon D_\mu \sigma + \frac{2i}{3} \sigma \gamma^\mu D_\mu \epsilon,$$

$$\delta \bar{\lambda} = -\frac{1}{2} \gamma^{\mu\nu} \epsilon F_{\mu\nu} + D\epsilon - i\gamma^\mu \epsilon D_\mu \sigma - \frac{2i}{3} \sigma \gamma^\mu D_\mu \epsilon,$$

$$\delta D = -\frac{i}{2} \gamma^\mu D_\mu \lambda + \frac{i}{2} D_\mu \bar{\lambda} \gamma^\mu \epsilon + \frac{i}{2} \bar{\lambda} \epsilon \sigma \epsilon \bar{\lambda} \sigma + i \frac{1}{6} \frac{i}{2} D_\mu \epsilon \gamma^\mu \lambda + \bar{\lambda} \gamma^\mu D_\mu \epsilon. \quad (A.4)$$

For the chiral multiplet we have

$$\delta \phi = \bar{\epsilon} \psi, \quad \delta \bar{\phi} = \epsilon \bar{\psi},$$

$$\delta \psi = i\gamma^\mu \epsilon D_\mu \phi + i\epsilon \sigma \phi + \frac{2i\Delta}{3} \gamma^\mu D_\mu \epsilon \phi + \bar{\epsilon} F,$$

$$\delta \bar{\psi} = i\gamma^\mu \epsilon D_\mu \bar{\phi} + i\bar{\epsilon} \sigma \bar{\phi} + \frac{2i\Delta}{3} \bar{\epsilon} \gamma^\mu D_\mu \bar{\epsilon} + \bar{\epsilon} \bar{F},$$

$$\delta F = \epsilon (i\gamma^\mu D_\mu \psi - i\sigma \psi - i\lambda \phi) + \frac{i}{3} (2\Delta - 1) D_\mu \epsilon \gamma^\mu \psi,$$

$$\delta \bar{F} = \bar{\epsilon} (i\gamma^\mu D_\mu \bar{\psi} - i\bar{\epsilon} \sigma \bar{\psi} + i\bar{\epsilon} \bar{\lambda} \bar{\phi}) + \frac{i}{3} (2\Delta - 1) D_\mu \bar{\epsilon} \gamma^\mu \bar{\psi}. \quad (A.5)$$
For $b \neq 1$ a BPS Wilson loop can be either along the $\varphi$ direction at $\vartheta = 0$, or along $\chi$ at $\vartheta = \pi/2$. We focus on the former, but everything works for the other case as well. One also needs to turn on a background field that the spinors are charged under

$$V_\mu \, dx^\mu = -\frac{1}{2} \left( 1 - \frac{1}{b f(\vartheta)} \right) d\varphi - \frac{1}{2} \left( 1 - \frac{b}{f(\vartheta)} \right) d\chi.$$  \hfill (A.6)

The two supercharges preserving the Wilson loops are parametrized by the Killing spinors

$$\epsilon = \frac{1}{\sqrt{2}} \left( e^{i(\varphi + \chi + \vartheta)/2} \right), \quad \bar{\epsilon} = \frac{1}{\sqrt{2}} \left( -e^{-i(\varphi + \chi + \vartheta)/2} \right).$$  \hfill (A.7)

Clearly at $\vartheta = 0$ we have $\sigma_1 \epsilon = \epsilon$ and $\sigma_1 \bar{\epsilon} = -\bar{\epsilon}$. We find

$$\bar{\epsilon} \epsilon \equiv \bar{\epsilon} \epsilon_a = 1, \quad \nu^\mu = \bar{\epsilon} \gamma^\mu \epsilon = \left( \frac{b}{R} \frac{1}{Rb}, 0 \right).$$  \hfill (A.8)

We denote the corresponding supercharges $Q$ and $\bar{Q}$, such that $Q \Psi = \partial_\vartheta \Psi$, and likewise for $\bar{Q}$. With $Q_{\pm} = Q \pm \bar{Q}$ we have the double variation of the scalars in the chiral multiplet

$$Q_{\pm}^2 \phi = [\delta_\vartheta, \delta_\vartheta] \phi = i \nu^\mu (\partial_\mu + i A_\mu) \phi + i \bar{\sigma} \phi - \Delta \left( \frac{1}{R f(\vartheta)} + \nu^\mu V_\mu \right) \phi,$$

$$Q_{\pm,1}^2 \phi = [\delta_\vartheta, \delta_\vartheta] \phi = i \nu^\mu (\partial_\mu \phi - i \bar{\sigma} A_\mu) - i \bar{\sigma} \phi + \Delta \left( \frac{1}{R f(\vartheta)} + \nu^\mu V_\mu \right) \phi.$$  \hfill (A.9)

Similar expressions exist for the other fields, which is necessary to prove closure of the off-shell SUSY algebra, but not for the details of our construction.

At $\vartheta = 0$ the last term in (A.9) is $-\Delta(b + 1/b)/2R$. In the following we parametrize the deviation from $\Delta = 1/2$ and $b = 1$ using

$$\Delta' = \frac{\Delta}{2} \left( b + \frac{1}{b} \right), \quad q = \exp \pi i (2\Delta' - 1) = -\exp \pi i (b + b^{-1}).$$  \hfill (A.10)

The original bosonic loop (2.1) can then be written (up to a phase and shift) as

$$q W + 1 = -s \text{Tr } \mathcal{P} \exp \int i \mathcal{L}_0 |\dot{x}| \, d\tau, \quad \mathcal{L}_0 = \begin{pmatrix} A_\mu \dot{x}^\mu \sqrt{|x|} - i \sigma + \Delta' R \frac{b}{b} & 0 \\ 0 & 0 \end{pmatrix}. \hfill (A.11)$$

Such a structure is required in order to couple the Wilson loop to the matter fields. Instead of the usual supertrace in (2.4) (and the 1/2 BPS Wilson loop [5]), in this case we need a $q$-deformed sum (a simple rescaling gives the form $q^{1/2} W + q^{-1/2}$).

Following the steps in the main text we introduce $\mathcal{G}$ as in (2.6)

$$Q_{\pm}^2 G_{a,\bar{a}} = i \mathcal{D}_0 G_{a,\bar{a}} = \frac{i}{R} \partial_\vartheta G_{a,\bar{a}} - [\mathcal{L}_0, G_{a,\bar{a}}], \quad G_{a,\bar{a}} = \frac{1}{R^{\Delta-1/2}} \begin{pmatrix} 0 \, u_{\bar{a}} \phi^j \\ 0 \, 0 \end{pmatrix}. \hfill (A.12)$$

We needed to introduce explicit powers of the the radius $R$ into $\mathcal{G}$, to give it dimension 1/2. We are assuming that all the fundamental fields $\phi^j$ have the same dimension, otherwise we require further modifications to a larger superconnection with different shifts in $\mathcal{L}_0$ as in (2.13) and appropriate powers of $R$ in the different blocks of $\mathcal{G}$. 


We use the new $G_{u,\bar{u}}$ to define a superconnection which has dimension one and from it we get the Wilson loop

$$ W_{u,\bar{u}} = sTr \mathcal{P} \ exp \int i\mathcal{L}_{u,\bar{u}} |\dot{x}| d\tau, \quad \mathcal{L}_{u,\bar{u}} = L_0 - iQ_+ G_{u,\bar{u}} + G_{u,\bar{u}}^2. \quad (A.13) $$

As in (2.9), we find that the supersymmetry variation is a total derivative

$$ Q_+ \mathcal{L}_{u,\bar{u}} = \mathcal{D}_{u,\bar{u}} G_{u,\bar{u}}. \quad (A.14) $$

The argument for the cancelation of the boundary terms from integrating this is similar to the argument for the requirement of sTr in the main text, or the argument used in [5]. As discussed around (2.11), we can show that this Wilson loop is also invariant under $Q_-$ if $G^2_u = G^2_{\bar{u}} = 0$, where these are the chiral and anti-chiral parts of $G_{u,\bar{u}}$.

Using the anti-fundamental chiral fields also works as before, but now $\tilde{\mathcal{L}}_0$ is

$$ \tilde{\mathcal{L}}_0 = \begin{pmatrix} A_\mu \dot{x}^\mu & i\sigma - \frac{\Delta'}{R} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (A.15) $$

For generic $\Delta'$, this is not gauge equivalent to $\mathcal{L}_0$. In fact, even before the deformation we find that the analog of (A.11) is

$$ -sTr \mathcal{P} \ exp \int i\tilde{\mathcal{L}}_0 |\dot{x}| d\tau = q^{-1}W + 1. \quad (A.16) $$

This is a different linear combination and once we incorporate the matter fields the two constructions are really inequivalent. One can combine fundamental and anti-fundamentals as in (2.13), but not the ‘latitudes’ of (2.14). The construction of the generic Wilson loop in such a theory is outlined in section 2.5.

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