Stock loans with liquidation
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Abstract

We derive a “semi-analytic” solution for a stock loan in which the lender forces liquidation when the loan-to-collateral ratio drops beneath a certain threshold. We use this to study the sensitivity of the contract to model parameters.

1 Introduction

We study a stock loan contract in which the lender can force the client to liquidate. It was originally pointed out in [16] that stock loans are essentially American call options with negative interest rates. The negative rate appears since the contract is effectively discounted by \( r - \gamma \), where \( \gamma \), the loan interest rate, is generally larger than the riskless rate of return \( r \).

Concretely, a stock loan is a loan of size \( q \) obtained from a financial firm (lender) by posting shares of an asset valued at \( s \) as collateral. Such loans are usually nonrecourse in that if the stock price drops, the borrower (client) may simply forfeit ownership of the shares in lieu of repaying the loan. On the other hand, if the stock price rises, the client can regain their shares by repaying the loan (along with the accrued interest).

The liquidation clause is useful from the perspective of the lender as it reduces the amount of risk the lender is exposed to, simultaneously reducing the rational premium charged for the loan. We find that such loans are riskless in the absence of jumps, in which case neither lender nor client benefits from entering into such a loan (simultaneously motivating the need for a jump-diffusion model; in particular, we have chosen the HEM for its analytical tractability and ample freedom in shaping return distributions).

[16] studies a perpetual contract (i.e., one that does not expire) with dividends paid out to the lender. In this original model, the lender cannot take action once the contract is initiated. [17] studies stock loans under regime-switching. Optimal strategies for both perpetual and finite-maturity contracts subject to various dividend distribution schemes are

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studied in [6]. A model in which the asset is driven by a hyper-exponential jump-diffusion is considered in [4] for both perpetual and finite-maturity contracts. Some other works on stock loans are [10, 11, 9, 14, 7, 13, 15, 5, 8, 12].

2 Hyper-exponential model

Consider a stochastic process \( (X_t^x)_{t \geq 0} \) following a hyper-exponential model (HEM)

\[
X_t^x := x + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i
\]

where \((N_t)_{t \geq 0}\) is a (right-continuous) Poisson process with rate \( \lambda \), \((W_t)_{t \geq 0}\) is a standard Brownian motion, and \((Y_i)_{i=1}^{\infty}\) is a sequence of i.i.d. hyper-exponential random variables with p.d.f.

\[
x \mapsto m \sum_{i=1}^{m} p_i \eta_i e^{-\eta_i x 1_{\{x \geq 0\}}} + n \sum_{j=1}^{n} q_j \theta_j e^{\theta_j x 1_{\{x < 0\}}}.
\]

We make the following assumption throughout:

**Assumption.** \( \lambda \geq 0, p_i > 0 \) for all \( i \), \( q_j > 0 \) for all \( j \), \( 1 < \eta_1 < \cdots < \eta_m, 0 < \theta_1 < \cdots < \theta_n, m \) and \( n \) are nonnegative integers not both equal to zero, and \( \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j = 1 \) (subject to the convention \( \sum_{i=1}^{0} \cdot = 0 \)).

The Lévy exponent of the process is

\[
G(x) := \sigma^2 x^2 / 2 + \mu x + \lambda \left( \sum_{i=1}^{m} \frac{p_i \eta_i}{\eta_i - x} + \sum_{j=1}^{n} \frac{q_j \theta_j}{\theta_j + x} - 1 \right) \text{ for } x \in (-\theta_1, \eta_1).
\]

In [2, Lemma 2.1 and Remark 2.3], it is shown that for any \( \alpha \geq M(G) \) where

\[
M(G) := \inf \{ G(x) : x \in (-\theta_1, \eta_1) \} \leq 0,
\]

\( x \mapsto G(x) - \alpha \) has \( m + n + 2 \) real roots \( \beta_1, \ldots, \beta_{m+1}, -\gamma_1, \ldots, -\gamma_{n+1} \), and \(-\gamma_{n+1} \alpha\) satisfying (omitting the subscript \( \alpha \))

\[
-\infty < -\gamma_{n+1} < -\theta_n < -\gamma_n < \cdots < -\gamma_2 < -\theta_1 < -\gamma_1 \leq \beta_1 < \eta_1 < \beta_2 < \cdots < \beta_m < \eta_m < \beta_{m+1} < \infty.
\]

\(-\gamma_1 = \beta_1\) is a possibility, in which case there are only \( m + n + 1 \) distinct roots.
3 Stock loans with liquidation

We consider a stock loan contract similar to [16]. At the initial time, the client borrows an amount $q$ from the lender using one share of the stock with initial price $e^x$ as collateral. The stock follows (in log space) the process $X$ defined in §2 with $\mu := r - \delta - \sigma^2/2 - \lambda \zeta$, where $r$ is the risk-free rate, $\delta$ is the dividend rate, $\sigma$ is the volatility, and $\zeta := \mathbb{E} e^{Y_t} - 1$. The loan is continuously compounded at the rate $\gamma$. At any (stopping) time $t$, the client can choose to redeem the stock and pay back $qe^{\gamma t}$. If the loan-to-collateral ratio $qe^{\gamma t} - X_t$ exceeds the level $d$, the lender liquidates the loan, forcing the client to pay back $qe^{\gamma t}$. During the collateral period, stock dividends are collected by the lender.

Remark. Transfer-of-title stock loans were shut down by the SEC and IRS between 2007-2012 and reclassified as fully taxable sales at inception (see also [1]). It stands to reason that the case of $\delta > 0$ implies a transfer-of-title (since the lender receives the dividends), and hence is subject to tax considerations. The case of $\delta = 0$ corresponds to dividends being immediately reinvested in the stock and returned to the client upon redemption, and is thus arguably the more relevant case for the current era. Other dividend distribution schemes are visited in [6].

If we assume that the client is able to pick a time to redeem the stock from $\mathcal{T}$, the set of $[0, +\infty)$ stopping times, the value of this contract is

$$
\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ e^{-r(\tau \wedge \pi)} \left( e^{X_{\tau \wedge \pi}} - qe^{\gamma(\tau \wedge \pi)} \right)^+ \right] \text{ where } \pi := \inf \left\{ t \geq 0 : qe^{\gamma t} - X_t \geq d \right\}.
$$

It is understood that the expression in the expectation is zero whenever $\tau \wedge \pi = +\infty$. However, due to the presence of the stopping time $\pi$, it is not a simple matter to show that the above is a free boundary problem with respect to $x$. This makes the above problem difficult if we are seeking analytical solutions (see, e.g., [16, Proposition 3.1]). Instead, we consider the simpler

$$
v(x) := \sup_{u \in (0,d)} \mathbb{E} \left[ e^{-r(\tau \wedge \pi)} \left( e^{X_{\tau \wedge \pi}} - qe^{\gamma(\tau \wedge \pi)} \right)^+ \right] \text{ where } \pi := \inf \left\{ t \geq 0 : qe^{\gamma t} - X_t \geq d \right\} \text{ and } \tau := \inf \left\{ t \geq 0 : qe^{\gamma t} - X_t \leq u \right\},
$$

in which the client picks instead a level $u$ to stop at based on the loan-to-collateral ratio.

Assumption. $0 < d \leq 1, \delta \geq 0, \text{ and } \gamma \geq r$. 

The following is a trivial consequence of the definition of $v$.

Lemma 1. $(e^x - q)^+ \leq v(x) \leq e^x \text{ everywhere and } v(x) = (e^x - q) \text{ for } x \leq \ln(q/d)$. 

The following is a trivial, but interesting aside: it establishes that the client has no reason to take out a stock loan in the absence of (downward) jumps in the collateral value (equivalently, the lender is exposed to no risk).

**Lemma 2.** If $n = 0$ or $\lambda = 0$, then $v(x) = (e^x - q)^+$ everywhere.

**Proof.** Note that in this case, $qe^{\gamma \pi(\omega) - X^x_{\tau(\omega)}} = d$ for $\mathbb{P}$-almost all $\omega$ such that $\pi(\omega) < \infty$. Therefore, for any stopping time $\tau$, $qe^{\gamma [\tau(\omega) \wedge \pi(\omega)] - X^x_{\tau(\omega) \wedge \pi(\omega)}} \leq d \leq 1$ and hence $qe^{\gamma [\tau(\omega) \wedge \pi(\omega)]} \leq e^{X^x_{\tau(\omega) \wedge \pi(\omega)}}$ for $\mathbb{P}$-almost all $\omega$ such that $\tau(\omega) \wedge \pi(\omega) < \infty$. It follows that

$$v(x) \leq \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ e^{-r(\tau \wedge \pi)} \left( e^{X^x_{\tau \wedge \pi}} - q e^{\gamma (\tau \wedge \pi)} \right) \right] \leq \sup_{t \geq 0} \left\{ e^{x - \delta t} - q e^{(\gamma - r)t} \right\} = e^x - q \text{ for } x \geq \ln q. \quad \blacksquare$$

Returning to our objective, define a drift-adjusted process $(\tilde{X}_t^x)_{t \geq 0}$ by $\tilde{X}_t^x := X_t^x - \gamma t$ with Lévy exponent $\tilde{G}(x) := G(x) - \gamma x$. Letting $f(x) := (e^x - q)^+$, we can write

$$v(x) = \sup_{u \in (0, d)} \mathbb{E} \left[ e^{(\gamma - r)\tilde{\tau}} f(\tilde{X}^x_{\tilde{\tau}}) \right] \text{ where } \tilde{\tau} := \inf \left\{ t \geq 0 : \tilde{X}^x_t \notin (h, H) \right\},$$

$$h := \ln(q/d), \text{ and } H := \ln(q/u). \quad (2)$$

For fixed values of $h$ and $H$, we may compute the expectation

$$\mathbb{E} \left[ e^{(\gamma - r)\tilde{\tau}} f(\tilde{X}^x_{\tilde{\tau}}) \right]$$

using Proposition 5 of the appendix (see, in particular, expression (7)). Before we can do so, we must ensure the finitude of the expectation.

**Assumption.** Either (i) $\delta > 0$ or (ii) $\delta = 0$ and $\frac{dG}{dx}(1) < 0$.

Subject to the above, [4, Theorem 3.1] holds, repeated below for convenience.

**Proposition 3.** $\mathbb{E}[\sup_{t \geq 0} e^{(\gamma - r)t} f(\tilde{X}^x_t)] < \infty$.

Some of the computations required to apply Proposition 5 to (3) are summarized below.

**Lemma 4.** Let $q > 0, \ln q \leq h < H$, and $f(x) := (e^x - q)^+$. Then,

$$f^u_0 = e^H - q, \quad f^u_i = \frac{e^H}{\eta_i - 1} - \frac{q}{\eta_i} \quad \text{for } 1 \leq i \leq m;$$

$$f^d_0 = e^h - q, \quad f^d_i = \frac{e^{-h} q^{1+\theta_i}}{\theta_j (1+\theta_j)} + \frac{e^h - q}{\theta_j} \quad \text{for } 1 \leq j \leq n;$$

where $f^u_i$ and $f^d_j$ are defined in Proposition 5.
4 Numerical results

4.1 Algorithm

Using Proposition 5 of the appendix, a direct computation shows that the expectation in (2) is continuous as a function of \( u \). This inspires the algorithm below for approximating \( v \) at \( x \). We write \( \tilde{\tau}(u) \) to stress the dependence of \( \tilde{\tau} \) (defined in (2)) on \( u \). Below, \( f \) is given by

\[
 f(x) := (e^x - q)^+. 
\]

**STOCK-LOAN(\( x; N \))**

1. \( V := f(x) \)
2. if \( x > \ln(q/d) \)
3. for \( j = 1, 2, \ldots, N \)
4. \( u_j := jd/(N+1) \)
5. \( V := \max\{\mathbb{E}[e^{(\gamma-r)\tilde{\tau}(u_j)} f(\tilde{X}_{\tilde{\tau}(u_j)})], V\} \)
6. return \( V \)

The integer \( N \geq 1 \) controls the accuracy of the algorithm. The expectation is computed using Proposition 5 and Lemma 4. There are various modifications one can make to speed up this algorithm (e.g., using a non-uniform grid \( \{u_j\} \)), though we do not visit them here.

4.2 Lender’s perspective

To aid our understanding, we also consider the contract from the lender’s perspective. In particular, let \( u \) be the map satisfying \( v(x) = x - u(x) \). This equation has the interpretation that the client retains ownership of the stock (initially valued at \( e^x \)) and is short the contract \( u \). Conversely, the lender is long the contract \( u \). Simple algebra along with the fact \( \mathbb{E}[e^{X_t - rt}] = e^{x-\delta t} \) reveals

\[
 u(s) = \inf_{u \in (0,d]} \mathbb{E} \left[ e^{-\delta \tau} \left( e^{X_{\tau\wedge \pi}} \left( e^{\delta(\tau\wedge \pi)} - 1 \right) + \min \left\{ e^{X_{\tau\wedge \pi}}, qe^{\delta(\tau\wedge \pi)} \right\} \right) \right] \tag{4} 
\]

From the form (4), it is easy to see that \( u \) includes dividend flows to the lender (the case in which dividends are immediately reinvested in the stock and returned to the client upon prepayment is equivalent to taking \( \delta = 0 \); we refer to [6, Section 3] for an explanation).

4.3 Jump risk

Figure 1 shows the stock loan value under varying jump arrival rates \( \lambda \). As stipulated by Lemma 2, in the absence of jumps, the loan is riskless (i.e. \( v(x) = (e^x - q)^+ \), or equivalently, \( u(x) = \min\{e^x, q\} \)). However, as \( \lambda \) increases, the lender is exposed to more risk. This
Table 1: Default parameters

| Parameter                  | Value |
|----------------------------|-------|
| Model                      | Double exponential \( (m = n = 1) \) |
| Risk-free rate             | \( r \) 0.05 |
| Dividend rate              | \( \delta \) 0.02 |
| Volatility                 | \( \sigma \) 0.15 |
| Loan interest rate         | \( \gamma \) 0.07 |
| Initial collateral value   | \( e' \) 100 |
| Loan value                 | \( q \) 80 |
| Liquidation ratio          | \( d \) 80/90 \( \approx 90\% \) |
| Jump arrival rate          | \( \lambda \) 0.5 |
| Mean up-jump scaling factor| \( \frac{1}{\eta_1} \) 1/2.3 |
| Mean down-jump scaling factor| \( \frac{1}{\theta_1} \) 1/1.8 |
| Up-jump probability        | \( p_1 \) 0.09 |
| Down-jump probability      | \( q_1 \) \( 1 - p_1 = 0.91 \) |

coincides with our intuition: as the probability of a downward jump is increased, so too is the probability that \( X^+_t \) jumps below the loan value \( q \).

The lender is exposed to the most risk at a point between \( x = \ln(q/d) \) and \( x \to \infty \). This is explained as follows:

▷ If the collateral is very close to \( \ln(q/d) \), the event that a downward jump brings \( X^+_t \) below the loan value before the lender is able to liquidate is unlikely.

▷ If the collateral is very large and the client chooses to prepay, the lender can retrieve the loan value in its entirety.

4.4 Rational values

Denote by \( c \) an up-front premium. At time zero, the client essentially exchanges the amount \( e^x - q + c \) for a stock loan [16]. It follows that \( \gamma \) and \( c \) are required to satisfy the following identity to preclude arbitrage:

\[
v(x; \gamma) = e^x - q + c \quad \text{(equivalently, } u(x; \gamma) = q - c)\]

(5)

Table 2 computes some rational values of \( \gamma \), and \( c \) satisfying (5).

Note that for \( x \leq \ln(q/d) \), the rational premium is \( c = (q - e^x)^+ \) (see Figure 1 for further intuition). In this case, neither client nor lender has any reason to enter into the contract. For \( x \geq k \) where \( k := \inf\{x > \ln(q/d): u(x) \geq q\} \), the rational premium is \( c = 0 \), as the contract is exercised immediately. It stands to reason that the only region of interest
Figure 1: Effect of varying jump arrival rates $\lambda$. 

(b) $C^1$ discontinuity at $e^x = q/d = 90$. 

50 100 150 200 250 300
Initial collateral value $s$
50 55 60 65 70 75 80 85
$U(s)$

(a)

50 100 150 200 250 300
Initial collateral value $s$
60 65 70 75 80 85
$U(s)$

(b) $C^1$ discontinuity at $e^x = q/d = 90$. 

Figure 1: Effect of varying jump arrival rates $\lambda$. 

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is \( \ln(q/d) < x < k \). Parameterizations in which \( k = \ln(q/d) \) (i.e. the free boundary is “collapsed”) are, therefore, uninteresting.

In particular, note the \( C^1 \) discontinuity occurring at \( x = \ln(q/d) \) (Figure 1), corresponding to liquidation. In the event that the free boundary is collapsed, \( u(x) = \min\{e^x, q\} \), and this discontinuity is removed.

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## A Laplace transform of first passage time to two barriers

We require an expression for the Laplace transform of a process to two flat barriers \( h \) and \( H \) with \( h < H \). Formally, let

\[
\tau := \inf \{ t \geq 0 : X_t \notin (h, H) \}.
\]

A generalization of the Laplace transform of \( \tau \) is studied in [3]. A trivial modification of the authors’ result is repeated below.

**Proposition 5.** Let \( f \) be a nonnegative measurable function such that

\[
\int_0^\infty f(y + H)e^{-\eta(y)}dy \quad \text{and} \quad \int_{-\infty}^0 f(y + h)e^{\theta(y)}dy
\]

are integrable for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Let \( \alpha \geq M(G) \) and \( x \in (h, H) \). Let \( N \) be an
\((m + n + 2) \times (m + n + 2)\) matrix given by

\[
N := \begin{bmatrix}
1 & \cdots & 1 & x^{\gamma_1} & \cdots & x^{\gamma_{n+1}} \\
1 & \cdots & 1 & x^{\gamma_1} & \cdots & x^{\gamma_{n+1}} \\
\eta_1 - \beta_1 & \cdots & \eta_1 - \beta_{m+1} & \eta_1 + \gamma_1 & \cdots & \eta_1 + \gamma_{n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\eta_m - \beta_1 & \cdots & \eta_m - \beta_{m+1} & \eta_m + \gamma_1 & \cdots & \eta_m + \gamma_{n+1} \\
\theta_1 + \beta_1 & \cdots & \theta_1 + \beta_{m+1} & \theta_1 - \gamma_1 & \cdots & \theta_1 - \gamma_{n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\theta_n + \beta_1 & \cdots & \theta_n + \beta_{m+1} & \theta_n - \gamma_1 & \cdots & \theta_n - \gamma_{n+1}
\end{bmatrix}
\tag{6}
\]

where \(\beta_1, \ldots, \beta_{m+1}\) and \(\gamma_1, \ldots, \gamma_{n+1}\) are the real roots of \(x \mapsto G(x) - \alpha\) satisfying (1) and \(\bar{x} := \bar{e}^{h-H}\). If \(N\) is nonsingular,

\[
\mathbb{E} \left[ e^{-\bar{x}^T f(X_T)} \right] = \varpi(x) N^{-1} f
\tag{7}
\]

where \(\varpi(x)\) is a row vector defined as

\[
\varpi(x) := \left( e^{\beta_1(x-H)}, \ldots, e^{\beta_{m+1}(x-H)}, e^{-\gamma_1(x-h)}, \ldots, e^{-\gamma_{n+1}(x-h)} \right),
\]

and \(f\) is a column vector such that \(f = (f^u_0, \ldots, f^u_m, f^d_0, \ldots, f^d_n)^\top\) where

\[
\begin{aligned}
f^u_0 &= f(H), &
f^d_i &= \int_0^\infty f(y + H) e^{-\eta_i y} dy & \text{ for } 1 \leq i \leq m; \\
f^d_0 &= f(h), &
f^d_j &= \int_{-\infty}^0 f(y + h) e^{\theta_j y} dy & \text{ for } 1 \leq j \leq n.
\end{aligned}
\]

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