The Perturbation Bound for the T-Drazin Inverse of Tensor and its Application

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Abstract. In this paper, let \( A \) and \( B \) be \( n \times n \times p \) complex tensors and \( B = A + E \). Denote the T-Drazin inverse of \( A \) by \( A^D \). We give a perturbation bound for \( \|B^D - A^D\| / \|A^D\| \) under condition \((W)\). Considering the solution of singular tensor equation \( A^* x = b \), \((b \in \mathbb{R}^n)\) at the same time. The optimal perturbation of T-Drazin inverse of tensors and the solution of a system of tensor equations have been given.

1. Introduction

The Drazin inverse plays an important role in many applications [1, 7, 20, 21, 25, 35]. There have been some papers on Drazin inverse of the perturbation bounds of matrix [27–31, 33, 34, 37]. Furthermore, we consider the perturbation of the Drazin inverse under the T-product of tensor. There are three monographs on the tensor [5, 19, 32]. Tensors are hyper dimensional matrices, which are the extensions of matrices. We study the generalized inverses of tensor based on Einstein product, in order to overcome high-dimension of tensor [10, 15, 22, 24]. In addition, the T-product of tensor [9, 11, 12, 14, 26] is another product which has been proven to be a useful tool in many applications [2, 9, 11, 12, 14, 16, 23, 38]. Recently, Ji and Wei [10] presented the Drazin inverse of an even-order tensor with the Einstein product. Che and Wei [3, 4, 32, 36] present the randomized algorithms for the tensor decomposition and the tensor equations.

The T-Jordan canonical form of the T-Drazin of third-order tensor inverse and the generalized tensor function are given by Miao, Qi and Wei in [17, 18], but its perturbation has not been developed yet. The perturbation of T-Drazin inverse and its application are introduced in this paper.

In this paper, let \( \mathbb{C}^{n \times n \times p} \) and \( \mathbb{R}^{n \times n \times p} \) be two sets of the \( n \times n \times p \) tensors over the complex field \( \mathbb{C} \) and the real field \( \mathbb{R} \), respectively. Let \( A \in \mathbb{C}^{n \times n \times p} \), and \( \rho_T(A) \) denote the T-spectral radius of \( A \). For positive integers \( k \) and \( n \), \([k] = [1, \cdots, n]\). We call \( O \) as a zero tensor in case of all the entries of the tensor are zero.

Now, a concept is proposed for multiplying third order tensors [9, 11, 12], based on viewing a tensor as a stake of frontal slices. Suppose \( A \in \mathbb{R}^{n \times n \times p} \) and \( B \in \mathbb{R}^{n \times n \times p} \) are third order tensors, denote their frontal...
faces as $A^{(k)} \in \mathbb{R}^{m \times n}$ and $B^{(k)} \in \mathbb{R}^{m \times n}$, respectively ($k = 1, 2, \ldots, p$). $A \in \mathbb{C}^{m \times n}$ is called as F-square tensor, if every frontal face of $A$ is square. The operation of “bcirc” was introduced in [9, 11, 12],

$$
\text{bcirc}(A) := \begin{pmatrix}
A^{(1)} & A^{(0)} & A^{(p-1)} & \cdots & A^{(2)} \\
A^{(2)} & A^{(1)} & A^{(p-1)} & \cdots & A^{(1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A^{(p)} & A^{(p-1)} & A^{(1)} & \cdots & A^{(1)} \\
\end{pmatrix},
$$

and $\text{fold}(\text{unf old}(A)) := A$. We define the corresponding inverse operation $\text{bcirc}^{-1} : \mathbb{R}^{m \times n \times p} \rightarrow \mathbb{R}^{m \times n \times p}$ such that $\text{bcirc}^{-1}(\text{bcirc}(A)) = A$.

**Definition 1.1.** [9, 11, 12](T-product) Let $A \in \mathbb{R}^{m \times n \times p}$ and $B \in \mathbb{R}^{n \times p}$ be two real tensors. Then the T-product $A \ast B$ is an $m \times n \times p$ real tensor defined by

$$
A \ast B := \text{fold}(\text{bcirc}(A) \ast \text{unf old}(B)).
$$

**Definition 1.2.** [9, 11, 12](Transpose and conjugate transpose) If $A$ is a third order tensor of size $m \times n \times p$, then the transpose $A^T$ is obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through $n$. The conjugate transpose $A^H$ is obtained by conjugate transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through $n$.

**Definition 1.3.** [9, 11, 12](Identity tensor) The $n \times n \times p$ identity tensor $I_{nmp}$ is the tensor whose first frontal slice is the $n \times n$ identity matrix, and whose other frontal slices are all zeros. It is easy to check that

$$
A \ast I_{nmp} = I_{nmp} \ast A = A \text{ for } A \in \mathbb{R}^{m \times n \times p}.
$$

For a frontal square $A$ of size $n \times n \times p$, it has inverse tensor $B \in \mathbb{R}^{n \times m \times p}(= A^{-1})$, provided that

$$
A \ast B = I_{nmp} \text{ and } B \ast A = I_{nmp}.
$$

**Definition 1.4.** [17, 18] Let $A \in \mathbb{R}^{m \times n \times p}$, then

1. The T-range space of $A$, $\mathcal{R}(A) := \text{Ran} \left( (F_p \otimes I_n) \ast \text{bcirc}(A) (F_p^H \otimes I_n) \right)$, “Ran” means the range space,
2. The T-null space of $A$, $\mathcal{N}(A) := \text{Null} \left( (F_p \otimes I_n) \ast \text{bcirc}(A) (F_p^H \otimes I_n) \right)$, “Null” represents the null space,
3. The tensor norm $\|A\| := \|\text{bcirc}(A)\|$, where $F_n$ is the discrete Fourier matrix of size $n \times n$, which is defined as [2].

$$
F_{n \times n} = \frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & w^3 & \cdots & w^{n-1} \\
1 & w^2 & w^4 & w^6 & \cdots & w^{2(n-1)} \\
1 & w^3 & w^6 & w^9 & \cdots & w^{3(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{n-1} & w^{2(n-1)} & w^{3(n-1)} & \cdots & w^{(n-1)(n-1)} \\
\end{pmatrix},
$$

where $w = e^{-2\pi i / n}$ is the primitive $n$-th root of unity in which $i = \sqrt{-1}$. $F_p^H$ is the conjugate transpose of $F_p$.

**Lemma 1.5.** [12] Suppose $A \in \mathbb{C}^{m \times n \times p}$ and $B \in \mathbb{C}^{n \times m \times p}$, then

$$
\text{bcirc}(A \ast B) = \text{bcirc}(A) \ast \text{bcirc}(B).
$$

**Remark 1.6.** Let $A, B, C \in \mathbb{C}^{m \times n \times p}$ be F-square tensors. Then $\|A \ast B \ast C\| \leq \|A\|\|B\|\|C\|$.
Proof. Since Lemma 1.5, we obtain

\[ \text{brc}(A \star B \star C) = \text{brc}(A) \text{brc}(B) \text{brc}(C). \]  

Take norm on both sides of (1) at the same time, then

\[ \| \text{brc}(A \star B \star C) \| = \| \text{brc}(A) \text{brc}(B) \text{brc}(C) \| \leq \| \text{brc}(A) \| \| \text{brc}(B) \| \| \text{brc}(C) \|. \]

According to (3) of Definition 1.4, we have

\[ \| A \star B \star C \| \leq \| A \| \| B \| \| C \|. \]

Definition 1.7. [17] (T-index) Let \( A \in \mathbb{C}^{n \times n \times p} \) be a complex tensor. The T-index of \( A \) is defined as

\[ \text{Ind}_T(A) = \text{Ind}(\text{brc}(A)). \]

Definition 1.8. [17] (T-Drazin inverse) Let \( A, X \in \mathbb{C}^{n \times n \times p} \), satisfying the following three equations

\[ \begin{align*}
A \star X &= X \star A, \\
X \star A \star X &= X, \\
A^k \star X \star A &= A^k,
\end{align*} \]

where \( \text{Ind}_T(A) = k \), then \( X \) is called by T-Drazin inverse of \( A \), which is denoted as \( A^D \).

Definition 1.9. [17] (Nilpotent tensor) Let \( A \in \mathbb{C}^{n \times n \times p} \) be nilpotent, if there exists a positive integer \( s \in \mathbb{Z} \) such that \( A^s = 0 \). If \( s \) is the smallest positive integer satisfying the equation \( A^s = 0 \), then \( s \) is called the nilpotent index of \( A \).

Definition 1.10. [17] (T-core-nilpotent decomposition) Let \( A \in \mathbb{C}^{n \times n \times p} \) be a complex tensor, \( N_A \) is T-nilpotent-part of \( A \), and \( C_A \) is T-core-part of \( A \), satisfying

\[ N_A = A - C_A = (I - A^D) \star A, \]

then \( A = C_A + N_A \) is called T-core-nilpotent decomposition of \( A \).

The construction of T-core-nilpotent decomposition of a tensor is introduced in [17]. Suppose \( A \in \mathbb{C}^{n \times n \times p} \), \( P \) is an invertible tensor, \( J \in \mathbb{C}^{n \times n \times p} \) is an F-bidiagonal tensor, and \( \text{Ind}_T(A) = k \), then the T-Jordan decomposition of \( A \) is \( A = P^{-1} \star J \star P \), and

\[ \text{brc}(J) = (F_p \otimes I_n) \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_p \end{pmatrix} (F_p^H \otimes I_n), \]

where \( J_i \) can be block partitioned as

\[ J_i = \begin{pmatrix} C_i & O \\ O & N_i \end{pmatrix} = \begin{pmatrix} C_i & O \\ O & N_i \end{pmatrix} + \begin{pmatrix} O & 0 \\ 0 & O \end{pmatrix} = J_i^C + J_i^N, \]

and \( C_i \) is a nonsingular matrix, \( N_i \) is nilpotent with \( \max_{i < k} \text{Ind}(N_i) = k \), then

\[ \text{brc}(J) = \text{brc}(J^C) + \text{brc}(J^N), \]
that is
\[ A = P^{-1} \ast J \ast P = P^{-1} \ast (J^C + J^N) \ast P = C_A + N_A, \]
which is the construction of T-core-nilpotent decomposition of \( A \).

**Theorem 1.11.** [17] Let \( A \in \mathbb{C}^{n \times n \times p} \), then there is an invertible tensor \( P \in \mathbb{C}^{n \times n \times p} \) and F-bidiagonal tensor \( J \in \mathbb{C}^{n \times n \times p} \), and the T-Jordan canonical form is,
\[ A = P^{-1} \ast J \ast P, \]
where the diagonal elements of \( J_i (i = 1, 2, \cdots, p) \) are the T-eigenvalues of \( A \). The decomposition of matrix \( \text{bcirc}(J) \) is given, as follows
\[ \text{bcirc}(J) = (F_p \otimes I_n) \begin{pmatrix} J_1 & I_2 & \cdots & I_p \\ J_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_p \\ I_1 & \cdots & \cdots & I_p \end{pmatrix} (F_p^H \otimes I_n), \]
where \( J_i \) can be partitioned as \( J_i = \begin{pmatrix} J_1^i & O & \cdots & O \\ O & J_2^i & \cdots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \cdots & \cdots & J_p^i \end{pmatrix} \) is the core of the matrix \( J_i \), and \( J_i^0 \) is nilpotent, \((i = 1, 2, \cdots, p)\).

Further, the T-Drazin inverse is denoted as
\[ A^D = P^{-1} \ast J^D \ast P. \]
The decomposition of \( \text{bcirc}(J^D) \) is
\[ \text{bcirc}(J^D) = (F_p \otimes I_n) \begin{pmatrix} J_1^D & I_2^D & \cdots & I_p^D \\ J_2^D & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_p^D \\ I_1^D & \cdots & \cdots & I_p^D \end{pmatrix} (F_p^H \otimes I_n), \]
where \( J_i^D = \begin{pmatrix} (J_i^1)^{-1} & O \\ O & O \end{pmatrix} \) is the Drazin inverse of the matrix \( J_i \). \((i = 1, 2, \cdots, p)\).

**Remark 1.12.** From the T-Jordan canonical form, we know that for any complex tensor \( \mathcal{A} \in \mathbb{C}^{n \times n \times p} \) with \( \text{Ind}_T(\mathcal{A}) = k \) and \( \text{rank}_T(\mathcal{A}^k) = r \), there exists nonsingular tensor \( P \in \mathbb{C}^{n \times n \times p} \) such that
\[ \mathcal{A} = P^{-1} \ast \mathcal{J} \ast P = P^{-1} \ast \begin{pmatrix} \mathcal{J}_1 & O \\ O & \mathcal{J}_4 \end{pmatrix} \ast P, \]
and
\[ \mathcal{A}^D = P^{-1} \ast \mathcal{J}^D \ast P = P^{-1} \ast \begin{pmatrix} \mathcal{J}_1^{-1} & O \\ O & O \end{pmatrix} \ast P, \]
where \( \mathcal{J}_1 \) is the core part of tensor \( \mathcal{J} \), and \( \mathcal{J}_4^0 \) is nilpotent.

**Theorem 1.13.** [10, 17, 18](T-linear system) Let \( \mathcal{A} \in \mathbb{C}^{n \times n \times p} \) be an F-square invertible tensor with \( \text{Ind}_T(\mathcal{A}) = k \). If the T-linear tensor system
\[ \mathcal{A} \ast x = b, \quad x \in \mathcal{R}(\mathcal{A}^D), \]
where \( x, b \in \mathbb{C}^{n \times 1 \times p} \), has an unique solution, then it is given by
\[ x = \mathcal{A}^D \ast b. \quad (5) \]
Theorem 1.14. If \(\mathcal{N} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix} \in \mathbb{C}^{2n \times 2n \times p},\) where \(\mathcal{A}\) and \(\mathcal{C}\) are \(F\)-square tensors, \(\text{Ind}_T(\mathcal{A}) = k, \text{Ind}_T(\mathcal{C}) = l\), then

\[
\mathcal{N}^{(D)} = \begin{pmatrix} \mathcal{A}^{(D)} & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^{(D)} \end{pmatrix} \in \mathbb{C}^{2n \times 2n \times p},
\]

where

\[
\mathcal{X} = \sum_{\substack{s = 0 \ \text{to} \ k-1}} (\mathcal{A}^{(D)})^{s+1} \ast \mathcal{B} \ast \mathcal{C}^{(D)} + (I - \mathcal{A} \ast \mathcal{A}^{(D)}) \ast \sum_{\substack{s = 0 \ \text{to} \ k-1}} \mathcal{A}^{s} \ast \mathcal{B} \ast (\mathcal{C}^{(D)})^{s+2} - \mathcal{A}^{(D)} \ast \mathcal{B} \ast \mathcal{C}^{(D)}.
\]

Proof. There are some decompositions of matrixes \(\text{bcirc}(\mathcal{A}), \text{bcirc}(\mathcal{X}), \text{bcirc}(\mathcal{C}), \text{bcirc}(\mathcal{B})\), such that

\[
\text{bcirc}(\mathcal{A}) = (F_p \otimes I_n), \quad \text{bcirc}(\mathcal{B}) = (F_p \otimes I_n), \quad \text{bcirc}(\mathcal{C}) = (F_p \otimes I_n), \quad \text{bcirc}(\mathcal{D}) = (F_p \otimes I_n),
\]

and

\[
\text{bcirc}(\mathcal{X}) = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{pmatrix} (F_p \otimes I_n),
\]

where

\[
T_i = (A_{i}^{(D)})^2 \left( \sum_{s=0}^{n} (A_{i}^{(D)} B_i C_i) (I - CC^{(D)}) + (I - AA^{(D)}) \left( \sum_{s=0}^{n} A_i B_s (C^{(D)})_s \right) (C^{(D)})_i^2 - A_{i}^{(D)} B_i C_i^D \right).
\]

\[
i = 1, 2, \cdots, p.
\]

Expand the term \(\mathcal{A} \ast \mathcal{X}\) as follows. Since Lemma 1.5, we obtain

\[
\text{bcirc}(\mathcal{A} \ast \mathcal{X}) = \text{bcirc}(\mathcal{A}) \text{bcirc}(\mathcal{X})
\]

\[
= (F_p \otimes I_n) \begin{pmatrix} A_1 T_1 \\ A_2 T_2 \\ \vdots \\ A_p T_p \end{pmatrix} (F_p^H \otimes I_n),
\]
where

\[
A_iT_i = \sum_{s=0}^{l-1} (A_i D)^{s+1} B_i C_i^s - \sum_{s=0}^{l-1} (A_i D)^{s+1} B_i C_i^s + 1 C_i^D
\]

\[
- \sum_{s=0}^{k-1} A_i^{s+1} B_i (C_i D)^{s+2} - \sum_{s=0}^{k-1} A_i D A_i^{s+2} B_i (C_i D)^{s+2} - A_i A_i D B_i C_i
\]

\[
= \left(A_i D B_i + \sum_{s=0}^{l-2} (A_i D)^{s+2} B_i C_i^{s+1}\right) - \left(A_i D B_i C_i D + \sum_{s=0}^{l-2} (A_i D)^{s+2} B_i C_i^{s+2} C_i^D\right)
\]

\[
+ \left(\sum_{s=1}^{k-1} (A_i D B_i (C_i D)^{s+1} + A_i D B_i (C_i D)^{s+1}) - \sum_{s=1}^{k-1} (A_i D A_i^{s+1} B_i (C_i D)^{s+1} + A_i D B_i (C_i D)^{s+1}) - A_i A_i D B_i C_i
\]

\[
= A_i D B_i + \sum_{s=0}^{l-2} (A_i D)^{s+2} B_i C_i^{s+1} - A_i D B_i C_i D - \sum_{s=0}^{l-2} (A_i D)^{s+2} B_i (C_i D)^{s+2} C_i^D
\]

\[
+ \sum_{s=1}^{k-1} A_i D B_i (C_i D)^{s+1} - \sum_{s=1}^{k-1} A_i D A_i^{s+1} B_i (C_i D)^{s+1} - A_i A_i D B_i C_i. (i = 1, 2 \cdots p)
\]

Now we expand the term \(X \ast C\) as follows.

By Lemma 1.5, then

\[
bcirc(X \ast C) = bcirc(X)bcirc(C)
\]

\[
= (F_p \otimes I_n) \begin{pmatrix} T_1 C_1 & & \\ T_2 C_2 & & \\ & & \ddots \\ T_p C_p & & \\ \end{pmatrix} (F_p^H \otimes I_n),
\]

where

\[
T_i C_i = \sum_{s=0}^{l-1} (A_i D)^{s+2} B_i C_i^{s+1} - \sum_{s=0}^{l-1} (A_i D)^{s+2} B_i C_i^{s+2} C_i^D
\]

\[
+ \sum_{s=0}^{k-1} A_i D B_i (C_i D)^{s+1} - \sum_{s=0}^{k-1} A_i D A_i^{s+1} B_i (C_i D)^{s+1} - A_i D B_i C_i D
\]

\[
= \left(\sum_{s=0}^{l-2} (A_i D)^{s+2} B_i C_i^{s+1} + (A_i D)^{s+1} B_i C_i^D\right) - \left(\sum_{s=0}^{l-2} (A_i D)^{s+2} B_i C_i^{s+2} C_i^D + (A_i D)^{s+1} B_i C_i^D\right)
\]

\[
+ \left(B_i C_i^D + \sum_{s=1}^{k-1} A_i D B_i (C_i D)^{s+1}\right) - \left(A_i D A_i B_i C_i^D + \sum_{s=1}^{k-1} A_i D A_i^{s+1} B_i (C_i D)^{s+1}\right)
\]

\[- A_i D B_i C_i^D C_i. (i = 1, 2 \cdots p)
\]

According to \(bcirc(\mathcal{A}), bcirc(\mathcal{B}), bcirc(\mathcal{C}), bcirc(\mathcal{A}^D)\) and \(bcirc(\mathcal{C}^D)\), we obtain

\[
bcirc(\mathcal{A}^D \ast \mathcal{B}) = bcirc(\mathcal{A}^D)bcirc(\mathcal{B})
\]

\[
= (F_p \otimes I_n) \begin{pmatrix} A_1 D B_1 & & \\ A_2 D B_2 & & \\ & & \ddots \\ A_p D B_p & & \\ \end{pmatrix} (F_p^H \otimes I_n),
\]
\[ bcirc(\mathcal{B} \ast \mathcal{C}^D) = bcirc(\mathcal{B})bcirc(\mathcal{C}^D) \]
\[ = (F_p \otimes I_n) \begin{pmatrix} B_1C_1^D & B_2C_1^D & \cdots & B_pC_p^D \\ F_p \otimes I_n & \end{pmatrix}, \]
then
\[ \mathcal{A}^D \ast \mathcal{B} - \mathcal{B} \ast \mathcal{C}^D = (F_p \otimes I_n) \begin{pmatrix} A_1^DB_1 - B_1C_1^D & A_2^DB_2 - B_2C_2^D & \cdots & A_p^DB_p - B_pC_p^D \\ F_p \otimes I_n & \end{pmatrix}, \]
and
\[ \mathcal{A} \ast \mathcal{X} - \mathcal{X} \ast \mathcal{C} = (F_p \otimes I_n) \begin{pmatrix} A_1T_1 - T_1C_1 & A_2T_2 - T_2C_2 & \cdots & A_pT_p - T_pC_p \\ F_p \otimes I_n & \end{pmatrix}. \]
It is easy to see that \( \mathcal{A} \ast \mathcal{X} - \mathcal{X} \ast \mathcal{C} = \mathcal{A}^D \ast \mathcal{B} - \mathcal{B} \ast \mathcal{C}^D \), or \( \mathcal{A} \ast \mathcal{X} + \mathcal{B} \ast \mathcal{C}^D = \mathcal{A}^D \ast \mathcal{B} + \mathcal{X} \ast \mathcal{C} \).
From this it follows that
\[ \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix} \ast \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} \ast \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}, \]
so that (2) of Definition 1.8 is satisfied. To show that (3) of Definition 1.8 holds, note that
\[ \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} \ast \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix} = \begin{pmatrix} \mathcal{A}^D \ast \mathcal{A} \ast \mathcal{X} + \mathcal{X} \ast \mathcal{C}^D + \mathcal{A}^D \ast \mathcal{B} \ast \mathcal{C}^D \end{pmatrix}. \]
Thus, it is only necessary to show that \( \mathcal{A}^D \ast \mathcal{A} \ast \mathcal{X} + \mathcal{X} \ast \mathcal{C}^D + \mathcal{A}^D \ast \mathcal{B} \ast \mathcal{C}^D = \mathcal{X} \).

Finally, we will show that
\[ \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{n+2} = \begin{pmatrix} \mathcal{A}^D & \mathcal{X} \\ \mathcal{O} & \mathcal{C}^D \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^{n+1}. \]
First notice that for any \( m > 0 \),
\[ \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}^m = \begin{pmatrix} \mathcal{A}^m & \mathcal{S}_m \\ \mathcal{O} & \mathcal{C}^m \end{pmatrix}, \]
where
\[ \mathcal{S}_m = \sum_{s=0}^{m-1} \mathcal{A}^{m-1-s} \ast \mathcal{B} \ast \mathcal{C}^s, \]
(6)
it is seen that the decomposition of matrix \( bcirc(\mathcal{S}_m) \) is
\[ bcirc(\mathcal{S}_m) = (F_p \otimes I_n) \begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 & \cdots & \mathcal{S}_p \\ F_p \otimes I_n & \end{pmatrix}. \]
and
\[ S_i = \sum_{s=0}^{m-1} A_i^{m-1-s} B_i C_i^s, \quad (i = 1, 2, \cdots, p) \]

Since \( n + 2 > k \) and \( n + 2 > l \), then
\[
\begin{pmatrix} A & B \\ O & C \end{pmatrix}^{n+2} = \begin{pmatrix} A^{l+1} & X + S_{(n+2)} C^D \\ O & C^{n+1} \end{pmatrix}.
\]

Therefore, it is necessary to show that \( A^{n+2} X = \sum_{s=0}^{l-1} A^{n+2-s} B C^s \). Observe first since \( l + k < n + 1 \), by Definition 1.8, it is the case that
\[
A^i * (A^D)^j = A^{i-1} \quad \text{for } i = 1, 2, \cdots, l - 1.
\]

Thus
\[
A^{n+2} X = \sum_{s=0}^{l-1} A^{n+2-s} B C^s
\]

the decomposition of matrix \( bcirc(A^{n+2} X) \) is
\[
bcirc(A^{n+2} X) = (F_p \otimes I_n) \begin{pmatrix} A^{n+2}_1 T_1 & \cdots & A^{n+2}_p T_1 \\ \vdots & \ddots & \vdots \\ A^{n+2}_1 T_p & \cdots & A^{n+2}_p T_p \end{pmatrix} (F_p \otimes I_n),
\]

and
\[
U_i = \sum_{s=0}^{l-1} A^{n+2-s}_i B_i C_i^s - \sum_{s=0}^{l-1} A^{n+2-s}_i B_i C_i^{s+1} C^D - A^{n+2}_i B C^D, \quad (i = 1, 2, \cdots, p)
\]

Since (6), then
\[
S_{(n+2)} C^D = \sum_{s=0}^{l-1} A^{n+2-s} B C^s C^D + \sum_{s=1}^{l-1} A^{n+2-s} B C^s C^D + \sum_{s=0}^{l-1} A^{n+2-s} B C^s C^D.
\]
By writing

$$\sum_{s=0}^{l} A^{n+1-s} \ast B \ast C^s \ast C^D = A^{n+1} \ast B \ast C^D + \sum_{s=1}^{l} A^{n+1-s} \ast B \ast C^s \ast C^D$$

$$= A^{n+1} \ast B \ast C^D + \sum_{s=0}^{l} A^{n-s} \ast B \ast C^{s+1} \ast C^D,$$

we obtain

$$S_{(n+2)} \ast C^D = A^{n+1} \ast B \ast C^D + \sum_{s=0}^{l-1} A^{n-s} \ast B \ast C^{s+1} \ast C^D + \sum_{s=l+1}^{n+1} A^{n+1-s} \ast B \ast C^{s-1}, \quad (8)$$

the decomposition of matrix $bcirc(S_{(n+2)} \ast C^D)$ as follows

$$bcirc(S_{(n+2)} \ast C^D) = (F_p \otimes I_n) \left( \begin{array}{cccc}
Q_1 & Q_2 & \cdots & Q_p \\
A_1 B_1 C^D_1 & A_2 B_2 C^D_2 & \cdots & A_p B_p C^D_p \\
R_1 & R_2 & \cdots & R_p \\
V_1 & V_2 & \cdots & V_p
\end{array} \right) (F_p^H \otimes I_n)$$

and

$$R_i = \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^{s+1} C_i^D, \quad V_i = \sum_{s=l+1}^{n+1} A_i^{n+1-s} B_i C_i^{s-1},$$

then

$$Q_i = A_i^{n+1} B_i C_i^D + R_i + V_i = A_i^{n+1} B_i C_i^D + \sum_{s=0}^{l-1} A_i^{n-s} B_i C_i^{s+1} C_i^D + \sum_{s=l+1}^{n+1} A_i^{n+1-s} B_i C_i^{s-1}. \quad (i = 1, 2, \cdots, p)$$
It is seen from (7) and (8) that

\[
\mathcal{A}^{n+2} \star X + S_{(n+2)} \star C^D = \sum_{s=0}^{n-1} \mathcal{A}^{n-s} \star B \star C^s + \sum_{s=n+1}^{n+1} \mathcal{A}^{n+1-s} \star B \star C^{s-1}
\]

\[
= \sum_{s=0}^{n} \mathcal{A}^{n-s} \star B \star C^s
\]

\[
= \sum_{s=0}^{n} \mathcal{A}^{n-s} \star B \star C^s
\]

\[
= S_{(n+1)}.
\]

The proof is completed.

**Definition 1.15.** (T-spectral radius) Let \( \mathcal{A} \in \mathbb{C}^{n \times n \times p} \) be an F-square tensor, then denote the spectral radius of \( \mathcal{A} \) as

\[
\rho_T(\mathcal{A}) = \rho(bcirc(\mathcal{A})) = \rho \left( (F_p \otimes I_n)bcirc(\mathcal{A})(F_H^p \otimes I_n) \right),
\]

where \( \rho_T(\mathcal{A}) \) is called by T-spectral radius of \( \mathcal{A} \).

**Definition 1.16.** [17] (T-eigenvalue) Let \( \mathcal{A} \in \mathbb{C}^{n \times n \times p} \) be an F-square tensor, then denote the eigenvalue of \( \mathcal{A} \) as

\[
\lambda_T(\mathcal{A}) = \lambda(bcirc(\mathcal{A})) = \lambda \left( (F_p \otimes I_n)bcirc(\mathcal{A})(F_H^p \otimes I_n) \right),
\]

where \( \lambda_T(\mathcal{A}) \) is called by T-eigenvalue of \( \mathcal{A} \).

### 2. Perturbation bounds

**Theorem 2.1.** Let \( \mathcal{F} \in \mathbb{C}^{n \times n \times p} \) be an F-square tensor, suppose \( \|\mathcal{F}\| < 1 \), then \( I + \mathcal{F} \) is nonsingular, and

\[
\| (I + \mathcal{F})^{-1} \| \leq \frac{1}{1 - \|\mathcal{F}\|}.
\]

**Proof.** Assume \( I + \mathcal{F} \) is singular, then there is a nonzero \( X \in \mathbb{C}^{n \times n \times p} \), such that

\[
(I + \mathcal{F}) \star X = O,
\]

furthermore

\[
I \star X = -\mathcal{F} \star X. \tag{9}
\]

Take norm on both sides of (9) at the same time, we have

\[
\|X\| = \|I \star X\| = \|\mathcal{F} \star X\| \leq \|\mathcal{F}\| \|X\|.
\]

According to \( \|X\| \leq \|\mathcal{F}\| \|X\| \), which implies \( \|\mathcal{F}\| \geq 1 \), and it is contradictory to \( \|\mathcal{F}\| < 1 \). Therefore, \( I + \mathcal{F} \) is nonsingular.

Since \( I + \mathcal{F} \) is invertible, we have \( (I + \mathcal{F}) \star (I + \mathcal{F})^{-1} = I \), then

\[
(I + \mathcal{F})^{-1} = I - \mathcal{F} \star (I + \mathcal{F})^{-1}. \tag{10}
\]

Take norm on both sides of (10) at the same time, we obtain

\[
\|(I + \mathcal{F})^{-1}\| = \|I - \mathcal{F} \star (I + \mathcal{F})^{-1}\|
\]

\[
\leq \|I\| + \|\mathcal{F}\| \|I + \mathcal{F}\|^{-1}\|
\]

\[
\leq 1 + \|\mathcal{F}\| \|(I + \mathcal{F})^{-1}\|.
\]
And then
\[ 1 \geq (1 - \|F\|)((I + F)^{-1}) \],
therefore
\[ \|(I + F)^{-1}\| \leq \frac{1}{1 - \|F\|}. \]

The proof is completed. \( \square \)

Let \( \mathcal{A}, \mathcal{B}, \mathcal{E} \in \mathbb{C}^{n \times n \times p} \) be F-square tensors, a condition \((W) [28]\) is given,
\[ (W), \mathcal{B} = \mathcal{A} + \mathcal{E} \text{ with } \text{Ind}_T(\mathcal{A}) = k, \mathcal{E} = \mathcal{A} * \mathcal{A}^D * \mathcal{E} * \mathcal{A} * \mathcal{A}^D, \text{ and } \|\mathcal{A}^D||\mathcal{E}|| < 1. \]

Now, we consider the perturbation of the T-Drazin inverse. First, let us give two lemmas of the perturbation bounds of \( \mathcal{B}^D - \mathcal{A}^D \).

**Lemma 2.2.** Suppose condition \((W)\) holds, let \( \mathcal{A} \in \mathbb{C}^{n \times n \times p} \) be a complex tensor, then there is an invertible tensor \( \mathcal{P} \in \mathbb{C}^{n \times n \times p} \) and F-bidiagonal tensor \( \mathcal{N} \in \mathbb{C}^{n \times n \times p} \). Further, the decomposition form of \( \mathcal{E} \) is
\[ \mathcal{E} = \mathcal{P}^{-1} \mathcal{N} \mathcal{P} = \mathcal{P}^{-1} \begin{pmatrix} N_1 & O & O \\ O & O & O \\ \vdots & \ddots & \ddots \\ O & O & N_p \end{pmatrix} \mathcal{P}, \]
where \( N_1 \) is the first block element of the tensor \( \mathcal{N} \), and the matrix \( \text{bcirc}(\mathcal{N}) \) has the following decomposition
\[ \text{bcirc}(\mathcal{N}) = (F_p \otimes I_n) \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_p \end{pmatrix} (F_p^H \otimes I_n), \]
where \( N_i = \begin{pmatrix} N^i_1 \\ O \\ O \end{pmatrix}, \) \( N^i_1 \) is the first block element of the matrix of \( N_i \). \( i = 1, 2, \ldots, p \)

**Proof.** According to the Theorem 1.11, we have
\[ \mathcal{A} = \mathcal{P}^{-1} \mathcal{J} \mathcal{P} = \mathcal{P}^{-1} \begin{pmatrix} J_1 & O \\ O & J_4 \end{pmatrix} \mathcal{P}, \]
where \( J_1 \) is the first block inverse element of tensor \( \mathcal{J} \), and \( J_4^0 \) is nilpotent.
Further, we obtain
\[ \mathcal{A}^D = \mathcal{P}^{-1} \mathcal{J}^D \mathcal{P} = \mathcal{P}^{-1} \begin{pmatrix} J_1^{-1} & O \\ O & J_4^0 \end{pmatrix} \mathcal{P}, \]
where \( J_1^{-1} \) is the first block element of the tensor \( \mathcal{J}^D \).
Next, the decomposition of \( \mathcal{E} \) will be given.
Suppose that \( \mathcal{E} = \mathcal{P}^{-1} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \mathcal{P} \), then
\[ \mathcal{A} * \mathcal{A}^D * \mathcal{E} = \mathcal{P}^{-1} \begin{pmatrix} J_1 & O \\ O & J_4 \end{pmatrix} \mathcal{P} * \mathcal{P}^{-1} \begin{pmatrix} J_1^{-1} & O \\ O & J_4^0 \end{pmatrix} \mathcal{P} * \mathcal{P}^{-1} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \mathcal{P} = \mathcal{P}^{-1} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \mathcal{P}, \]
and
\[ \mathcal{E} * \mathcal{A} * \mathcal{A}^D = \mathcal{P}^{-1} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \mathcal{P} * \mathcal{P}^{-1} \begin{pmatrix} J_1 & O \\ O & J_4 \end{pmatrix} \mathcal{P} * \mathcal{P}^{-1} \begin{pmatrix} J_1^{-1} & O \\ O & J_4^0 \end{pmatrix} \mathcal{P} * \mathcal{P}^{-1} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} \mathcal{P}, \]
Lemma 2.3. Suppose condition (W) holds, let \( \mathcal{A} \in \mathbb{C}^{m \times n} \) be a complex tensor, \( \mathcal{B} = \mathcal{A} + \mathcal{E} \), then there is an invertible tensor \( \mathcal{P} \in \mathbb{C}^{m \times n} \) and \( \mathcal{J} \)-bidiagonal tensor \( \mathcal{M} \in \mathbb{C}^{n \times n} \), such that

\[
\mathcal{B}^D = \mathcal{P}^{-1} \ast \mathcal{M}^D \ast \mathcal{P},
\]

and the decomposition of the matrix \( \text{bcirc}(\mathcal{M}^D) \) is

\[
\text{bcirc}(\mathcal{M}^D) = (F_p \otimes I_n)
\begin{pmatrix}
M_1^D \\
M_2^D \\
\vdots \\
M_p^D
\end{pmatrix}
(F_p^H \otimes I_n),
\]

where \( M_i^D = \left( \begin{array}{c|c}
(M_i^1)^{-1} & O \\
O & O
\end{array} \right), \, (i = 1, 2, \ldots, p) \)

(2) \( \mathcal{A} \ast \mathcal{R}^D = \mathcal{B} \ast \mathcal{B}^D \).

Proof. (1) According to the Theorem 1.11, there is \( \mathcal{N} \in \mathbb{C}^{n \times n} \) and \( \mathcal{J} \in \mathbb{C}^{n \times n} \), then \( \mathcal{A} = \mathcal{P}^{-1} \ast \mathcal{J} \ast \mathcal{P}, \mathcal{E} = \mathcal{P}^{-1} \ast \mathcal{N} \ast \mathcal{P} \), suppose \( \mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{P}^{-1} \ast \mathcal{M} \ast \mathcal{P} \), where

\[
\text{bcirc}(\mathcal{M}) = \text{bcirc}(\mathcal{J} + \mathcal{N})
= (F_p \otimes I_n)
\begin{pmatrix}
(N_1 + I_i) \\
(N_2 + I_i) \\
\vdots \\
(N_p + I_i)
\end{pmatrix}
(F_p^H \otimes I_n),
\]

and \( I_i = \left( \begin{array}{c|c}
I_i & O \\
O & I_i
\end{array} \right) \), \( N_i = \left( \begin{array}{c|c}
N_i^1 & O \\
O & O
\end{array} \right) \), \( I_i^1 \) is the first block element of the matrix of \( I_i \), \( N_i^1 \) is the first block element of the matrix of \( N_i \), and \( I_i^0 \) is nilpotent, \((i = 1, 2, \ldots, p)\)

Therefore

\[
\text{bcirc}(\mathcal{M}^D) = (F_p \otimes I_n)
\begin{pmatrix}
(N_1 + I_i)^D \\
(N_2 + I_i)^D \\
\vdots \\
(N_p + I_i)^D
\end{pmatrix}
(F_p^H \otimes I_n),
\]

Moreover, it proves that \( N_i^1 + I_i^1 \) is invertible, where \( N_i + I_i = \left( \begin{array}{c|c}
N_i^1 + I_i^1 & O \\
O & O
\end{array} \right). \) (i = 1, 2, \ldots, p)

Now, by Theorem 1.11 and Lemma 2.2, we have

\[
\mathcal{R}^D \ast \mathcal{E} = \mathcal{P}^{-1} \ast \mathcal{J} \ast \mathcal{P} \ast \mathcal{P}^{-1} \ast \mathcal{N} \ast \mathcal{P}
= \mathcal{P}^{-1} \ast \mathcal{J} \ast \mathcal{N} \ast \mathcal{P},
\]

and the decomposition of \( \text{bcirc}(\mathcal{J}^D \ast \mathcal{N}) \) is

\[
\text{bcirc}(\mathcal{J}^D \ast \mathcal{N}) = \text{bcirc}(\mathcal{J}^D) \text{bcirc}(\mathcal{N}) = (F_p \otimes I_n)
\begin{pmatrix}
I_1^D N_1 \\
I_2^D N_2 \\
\vdots \\
I_p^D N_p
\end{pmatrix}
(F_p^H \otimes I_n),
\]
where \( J^{D}_i N_i = \begin{pmatrix} (J^{1}_i)^{-1} N^{1}_i & 0 \\ 0 & 0 \end{pmatrix} \), \((i = 1, 2, \cdots, p)\)

By Definition 1.15, we have

\[
\rho_T(J^{D} \ast N) = \rho(bcirc(J^{D} \ast N)) = \max_i \rho \left( (J^{1}_i)^{-1} N^{1}_i \right),
\]

that is

\[
\rho_T(A^{D} \ast E) = \rho_T(J^{D} \ast N) = \max_i \rho \left( (J^{1}_i)^{-1} N^{1}_i \right),
\]

thus

\[
\rho_T(A^{D} \ast E) \leq \|A^{D}\| \|E\| < 1.
\]

On the other hand, it will prove that \( J^{1}_i + N^{1}_i = J^{1}_i (I + (J^{1}_i)^{-1} N^{1}_i) \) is invertible. According to the inverse of \( J^{1}_i \), we will only prove that \( I + (J^{1}_i)^{-1} N^{1}_i \) is nonsingular. Now, we prove it by reduction to absurdity. Assume \( I + (J^{1}_i)^{-1} N^{1}_i \) is singular, then there is a nonzero vector \( x \in \mathbb{C}^{nx1} \), such that

\[
(I + (J^{1}_i)^{-1} N^{1}_i)x = 0,
\]

then

\[
x = -(J^{1}_i)^{-1} N^{1}_i)x.
\]

Therefore, -1 is the eigenvalue of matrix \( (J^{1}_i)^{-1} N^{1}_i \), denoted \( \lambda ((J^{1}_i)^{-1} N^{1}_i) = -1 \), it implies \( \rho ((J^{1}_i)^{-1} N^{1}_i) \geq 1 \).

According to (14), we obtain

\[
\rho_T(A^{D} \ast E) = \max_i \rho \left( (J^{1}_i)^{-1} N^{1}_i \right) \geq 1,
\]

which is contradictory to (15). Hence \( I + (J^{1}_i)^{-1} N^{1}_i \) is nonsingular.

(2) By Theorem 1.11, we have \( A = P^{-1} \ast J \ast P \) and \( A^{D} = P^{-1} \ast J^{D} \ast P \).

Similary, \( B = P^{-1} \ast M \ast P \) and \( B^{D} = P^{-1} \ast M^{D} \ast P \), then

\[
A \ast A^{D} = P^{-1} \ast J \ast P \ast P^{-1} \ast J^{D} \ast P = P^{-1} \ast J \ast J^{D} \ast P,
\]

and

\[
B \ast B^{D} = P^{-1} \ast M \ast P \ast P^{-1} \ast M^{D} \ast P = P^{-1} \ast M \ast M^{D} \ast P,
\]

By Lemma 1.5, we have

\[
bcirc(J \ast J^{D}) = bcirc(J)bcirc(J^{D})
\]

\[
= (F_p \otimes I_n) \begin{pmatrix} F_1^{D} & J_1^{D} \\ J_2^{D} & \ddots \\ \vdots \\ J_p^{D} \end{pmatrix} (F_p^H \otimes I_n),
\]

where \( I_p^{D} = \begin{pmatrix} (J^{1}_1)^{-1} N^{1}_1 & 0 \\ 0 & 0 \end{pmatrix}, \)(i = 1, 2, \cdots, p)
and
\[ bcirc(M \ast M^D) = bcirc(M)bcirc(M^D) = (F_p \otimes I_n) \begin{pmatrix} M_1M_1^D & M_2M_2^D & \cdots & M_pM_p^D \\ (I_p \otimes I_n) \end{pmatrix} \]
where \( J_i, J_i^D = \begin{pmatrix} I_i & \circ \circ \circ \circ \circ \circ \\ O & I_i \end{pmatrix} \) \( (i_i) \) is the first block element of the matrix of \( J_i \), and \( J_i^0 \) is nilpotent, and \( M_iM_i^D = \begin{pmatrix} M_i^1 & \circ \circ \circ \circ \circ \circ \\ O & M_i^1 \end{pmatrix} \) \( (i_i) \) is the first block element of the matrix of \( M_i. (i = 1, 2, \cdots, p) \)

Hence, \( A \ast A^D = B \ast B^D \). The proof is completed. □

**Theorem 2.4.** Let \( A, B, E \in \mathbb{C}^{n \times n} \) be F-square tensors, \( A^D \) is T-Drazin inverse of \( A \), if \( E = A \ast A^D \ast E = E \ast A \ast A^D \), \( \text{Ind}_T(A) = k, B = A + E \) and \( \|A^D \ast E\| < 1 \), then
1. \( B^D - A^D = -B^D \ast E \ast A^D \ast (B - A) \ast A^D \),
2. \( B^D = (I + A^D \ast E)^{-1} \ast A^D = A^D \ast (I + E \ast A^D)^{-1} \),
3. \( \|A^D - E\| \leq \frac{\|A^D - E\|}{1 - \|A^D - E\|}. \)

**Proof.** (1) According to Lemma 2.3, we have \( A \ast A^D = B \ast B^D \), then
\[
B^D - A^D = -B^D \ast E \ast A^D + B^D - A^D + B^D \ast (B - A) \ast A^D
\]
\[
= -B^D \ast E \ast A^D + B^D \ast B^D \ast (B - A) \ast A^D
\]
\[
= -B^D \ast E \ast A^D + B^D - A^D \ast A^D + B^D \ast A^D
\]
\[
= -B^D \ast E \ast A^D,
\]
that is
\[
B^D - A^D = -B^D \ast E \ast A^D. \tag{16}
\]
Similarly,
\[
B^D - A^D = -A^D \ast E \ast B^D + B^D \ast A^D \ast (B - A) \ast A^D
\]
\[
= -A^D \ast E \ast B^D + B^D - A^D \ast A^D + A^D \ast B^D
\]
\[
= -A^D \ast E \ast B^D + B^D - A^D \ast B^D + A^D \ast A^D
\]
\[
= -A^D \ast E \ast B^D,
\]
that is
\[
B^D - A^D = -A^D \ast E \ast B^D. \tag{17}
\]
(2) By (16), we have
\[
B^D = (I + E \ast A^D) = A^D.
\]
Since \( \rho_T(E \ast A^D) = \rho_T(A^D \ast E) \), then \( \rho_T(E \ast A^D) = \rho_T(A^D \ast E) \leq \|A^D \ast E\| < 1 \), therefore \( I + E \ast A^D \) is nonsingular, then
\[
B^D = A^D \ast (I + E \ast A^D)^{-1}. \tag{18}
\]
By (17), we obtain

$$(I + A^D \ast E) \ast B^D = A^D.$$ 

Since $\|A^D \ast E\| < 1$, therefore $I + A^D \ast E$ is nonsingular, then

$$B^D = (I + A^D \ast E)^{-1} \ast A^D. \quad (19)$$

(3) By Theorem 2.1, and take norm on both sides of (19) at the same time, then

$$\|B^D\| = \|(I + A^D \ast E)^{-1} \ast A^D\|$$

$$\leq \|I + A^D \ast E\| \cdot \|A^D\|$$

$$\leq \frac{\|A^D\|}{1 - \|A^D \ast E\|}.$$ 

Therefore

$$\|B^D\| \leq \frac{\|A^D\|}{1 - \|A^D \ast E\|}. \quad (20)$$

Take norm on both sides of (17) at the same time, then

$$\|B^D - A^D\| = \| - A^D \ast E \ast B^D\|$$

$$\leq \|A^D \ast E\| \cdot \|B^D\|.$$ 

Divide $\|A^D\|$ on both sides at the same time, we obtain

$$\frac{\|B^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D \ast E\| \cdot \|B^D\|}{\|A^D\|},$$

$$\frac{\|B^D - A^D\|}{\|A^D\|} \leq \left( \frac{\|A^D \ast E\|}{\|A^D\|} \right) \cdot \left( \frac{\|A^D \ast E\|}{\|A^D\|} \right).$$

Therefore

$$\frac{\|B^D - A^D\|}{\|A^D\|} \leq \left( \frac{\|A^D \ast E\|}{\|A^D\|} \right) \cdot \left( \frac{\|A^D \ast E\|}{\|A^D\|} \right). \quad (21)$$

The proof is completed.

**Corollary 2.5.** Suppose condition (W) holds, let $\mathcal{A}, \mathcal{B}, \mathcal{E} \in \mathbb{C}^{n \times n \times p}$ be F-square tensors, then

$$\frac{\|\mathcal{A}\|}{1 + \|\mathcal{A}\| \cdot \|\mathcal{E}\|} \leq \|\mathcal{B}\| \leq \frac{\|\mathcal{A}\|}{1 - \|\mathcal{A}\| \cdot \|\mathcal{E}\|}.$$ 

**Proof.** According to Theorem 2.4, we have $\mathcal{B}^D = \mathcal{A}^D \ast (I + \mathcal{E} \ast \mathcal{A}^D)^{-1}$, then

$$\mathcal{A}^D = \mathcal{B}^D \ast (I + \mathcal{E} \ast \mathcal{A}^D). \quad (22)$$

Taking norm on both sides of (22) at the same time, we obtain

$$\|\mathcal{A}^D\| = \|\mathcal{B}^D \ast (I + \mathcal{E} \ast \mathcal{A}^D)\| \leq \|\mathcal{B}^D\| \cdot \|I + \mathcal{E} \ast \mathcal{A}^D\|.$$
Hence
\[ \|B^D\| \geq \frac{\|A^D\|}{\|I + E \ast A^D\|}. \] (23)

According to \( \|I + E \ast A^D\| \leq \|I\| + \|E \ast A^D\| \leq 1 + \|E\||A^D\| \), then
\[ \frac{1}{1 + \|E\||A^D\|} \leq \frac{1}{\|I + E \ast A^D\|}. \]

Multiply \( \|A^D\| \) on both sides at the same time, we obtain
\[ \frac{\|A^D\|}{1 + \|E\||A^D\|} \leq \frac{\|A^D\|}{\|I + E \ast A^D\|} \leq \frac{\|A^D\|}{\|B^D\|}. \]

By (23), then
\[ \frac{\|A^D\|}{1 + \|E\||A^D\|} \leq \frac{\|A^D\|}{\|I + E \ast A^D\|} \leq \frac{\|A^D\|}{\|B^D\|}. \]

On the other hand, by (20), it shows that
\[ \|B^D\| \leq \|A^D\||I + A^D \ast E|^{-1}| \leq \frac{\|A^D\|}{1 - \|A^D\||E|}. \]

Therefore
\[ \frac{\|A^D\|}{1 + \|A^D\||E|} \leq \frac{\|B^D\|}{1 - \|A^D\||E|}. \]

The proof is completed.

**Theorem 2.6.** Let \( A, B \in C^{n \times n \times p} \) be F-square tensors, if \( \|E\||A^D\| < 1 \), and \( K_D(A) = \|A\||A^D\| \), then
\[ \frac{\|B^D - A^D\|}{\|A^D\|} \leq \frac{K_D(A)||E||/\|A\|}{1 - K_D(A)||E||/\|A\|}. \]

**Proof.** From (21), we have
\[ \frac{\|B^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D \ast E\|}{1 - \|A^D \ast E\|} \leq \frac{\|A^D\||E|}{1 - \|A^D\||E|} = \frac{\|A\||A^D\||E|/\|A\|}{1 - \|A\||A^D\||E|/\|A\|} = \frac{K_D(A)||E||/\|A\|}{1 - K_D(A)||E||/\|A\|}. \]

where \( K_D(A) = \|A\||A^D\| \).

The proof is completed.

**Remark 2.7.** If \( \text{Ind}_{-1}(A) = 1 \), then condition \((W)\) is reduced to \( B = A + E, E = A \ast A \ast E \ast A \ast A \ast A \), and \( \|A\||E| < 1 \). Thus under these assumes, we can get a perturbation bound for the group inverse of the tensor.

**Remark 2.8.** If \( \text{Ind}_{-1}(A) = 0 \), i.e., \( A \) is nonsingular, then condition \((W)\) is reduced to \( B = A + E, \) and \( \|A^{-1}\||E| < 1 \). We also obtain a perturbation bound on the common tensor inverse.
3. Applications

In this section, we consider the T-linear system. Let $B \in \mathbb{C}^{n \times n \times p}$ be an F-square tensor, and $y, b, c, f \in \mathbb{C}^{n \times 1 \times p}$ are tensors.

$$B \ast y = c, \quad y \in \mathbb{R}(B^D),$$

where $B = A + E, c = b + f \in \mathbb{R}(B^D)$.

**Theorem 3.1.** Suppose condition (W) holds, let $y, x, b, c, f \in \mathbb{C}^{n \times 1 \times p}$ and $\|A^D\|\|E\| < 1$, then

$$\frac{\|y - x\|}{\|x\|} \leq K_D(A) \left( \frac{\|E\|}{\|A\|} + \frac{\|f\|}{\|b\|} \right).$$

**Proof.** According to Theorem 1.13, we obtain $x = A^D \ast b$, and by (5), one can obtain

$$x = A^D \ast b.$$

Similarly

$$y = B^D \ast c = (A + E)^D \ast (b + f).$$

Since $B^D - A^D = -(B^D \ast E \ast A^D)$, then

$$y - x = (A + E)^D \ast (b + f) - A^D \ast b$$

$$= (A + E)^D \ast b + (A + E)^D \ast f - A^D \ast b$$

$$= ((A + E)^D - A^D) \ast b + (A + E)^D \ast f$$

$$= -B^D \ast E \ast A^D \ast b + (A + E)^D \ast f$$

$$= -((A + E)^D) \ast E \ast (A + E)^D \ast f.$$  \hspace{1cm} (24)

Hence

$$y - x = -((A + E)^D) \ast E \ast x + (A + E)^D \ast f.$$  \hspace{1cm} (24)

Due to Corollary 2.5, and take norm on both sides of (24) at the same time, then

$$\|y - x\| = \| -((A + E)^D) \ast E \ast x + (A + E)^D \ast f\|$$

$$\leq \|((A + E)^D)\|\|E\|\|x\| + \|(A + E)^D\|\|f\|$$

$$= \|(A + E)^D\| (\|E\|\|x\| + \|f\|)$$

$$= \|B^D\| (\|E\|\|x\| + \|f\|)$$

$$\leq \frac{\|A\|\|A^D\|\|E\|}{1 - \|A^D \ast E\|} \left(\|E\| + \frac{\|f\|}{\|b\|}\right)$$

$$\leq \frac{\|A\|\|A^D\|\|x\|}{1 - \|A^D \ast E\|} \left(\|E\| + \frac{\|f\|}{\|b\|}\right)$$

$$\leq \frac{K_D(A)\|x\|}{1 - K_D(A)\|E\|/\|A\|} \left(\|E\| + \|f\|/\|b\|\right).$$

The proof is completed. \hfill \Box
4. One-sided Perturbation of T-Drazin Inverse

**Lemma 4.1.** Let $A \in \mathbb{C}^{n \times n \times p}$, $E \in \mathbb{C}^{n \times n \times p}$ be complex tensors, and $E = A \ast A^D \ast E$, then there is an invertible tensor $P \in \mathbb{C}^{n \times n \times p}$ and F-bidiagonal tensor $N \in \mathbb{C}^{n \times n \times p}$. Further, the decomposition form of $E$ is

$$E = P^{-1} \ast N \ast P = P^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ O & O \end{array} \right) \ast P,$$

where $N_1$ and $N_2$ are block elements of tensor $N$.

And the matrix $bcirc(N)$ has the following decomposition

$$bcirc(N) = (F_p \otimes I_n) \left( \begin{array}{ccc} N_1 \\ N_2 \\ \vdots \\ N_p \end{array} \right) (F_p^H \otimes I_n),$$

where $N_i = \left( \begin{array}{cc} N_{1i} & N_{2i} \\ O & O \end{array} \right)$, $N_{1i}$ and $N_{2i}$ are block elements of the matrix of $N_i$. ($i = 1, 2, \cdots, p$)

**Proof.** According to the Theorem 1.11, we have

$$A = P^{-1} \ast J \ast P = P^{-1} \ast \left( \begin{array}{c} J_1 \\ O \\ J_0_4 \end{array} \right) \ast P, \quad (25)$$

where the first block element $J_1$ is inverse in tensor $J$, and $J_0^4$ is nilpotent.

Further, we obtain

$$A^D = P^{-1} \ast J^D \ast P = P^{-1} \ast \left( \begin{array}{c} J_1^{-1} \\ O \\ J_0^4 \end{array} \right) \ast P, \quad (26)$$

where the first block element $J_1^{-1}$ of the tensor $J^D$.

Next, the decomposition of $E$ will be given. Suppose $E = P^{-1} \ast N \ast P = P^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ N_3 & N_4 \end{array} \right) \ast P$, then

$$A \ast A^D \ast E = P^{-1} \ast \left( \begin{array}{c} J_1 \\ O \\ J_0_4 \end{array} \right) \ast P \ast P^{-1} \ast \left( \begin{array}{c} J_1^{-1} \\ O \\ O \end{array} \right) \ast P \ast \left( \begin{array}{cc} N_1 & N_2 \\ N_3 & N_4 \end{array} \right) \ast P = p^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ N_3 & N_4 \end{array} \right) \ast P. \quad (27)$$

By $E = A \ast A^D \ast E$ and (27), we obtain

$$E = P^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ N_3 & N_4 \end{array} \right) \ast P = P^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ O & O \end{array} \right) \ast P \quad (28)$$

Hence $E = P^{-1} \ast N \ast P = P^{-1} \ast \left( \begin{array}{cc} N_1 & N_2 \\ O & O \end{array} \right) \ast P$, and

$$bcirc(N) = (F_p \otimes I_n) \left( \begin{array}{ccc} N_1 \\ N_2 \\ \vdots \\ N_p \end{array} \right) (F_p^H \otimes I_n).$$

The proof is completed. □
Lemma 4.2. Let $A \in \mathbb{C}^{n \times n \times p}$, $E \in \mathbb{C}^{n \times n \times p}$ be complex tensors, and $E = A \ast A^D \ast E$, $\|A^D \ast E\| < 1$, $B = A + E$, such that

$$B^D = A^D - A^D \ast E \ast (I + A^D \ast E)^{-1} \ast A^D + \sum_{s=0}^{k-1} (A^D - A^D \ast E \ast (I + A^D \ast E)^{-1} \ast A^D)^{s+2} \ast E \ast (I - A \ast A^D) \ast A^D.$$ 

Proof. According to the Theorem 1.11, then there is an invertible tensor $P \in \mathbb{C}^{n \times n \times p}$ such that

$$B = A + E $$

$$= P^{-1} \ast J \ast P + P^{-1} \ast N \ast P $$

$$= P^{-1} \ast \left( \begin{array}{ll} J_1 & N_1 \\ O & J_0^D \end{array} \right) \ast P + P^{-1} \ast \left( \begin{array}{ll} N_1 & N_2 \\ O & O \end{array} \right) \ast P $$

$$= P^{-1} \ast \left( \begin{array}{ll} J_1 + N_1 & N_2 \\ O & J_0^D \end{array} \right) \ast P.$$ 

By Theorem 1.14, we have

$$B^D = P^{-1} \ast \left( \begin{array}{ll} J_1 & N_1 \ O & J_0^D \end{array} \right) \ast P $$

$$= P^{-1} \ast \left( \begin{array}{ll} (J_1 + N_1)^D \ O \ (J_0^D)^D \end{array} \right) \ast P $$

$$= P^{-1} \ast \left( \begin{array}{ll} (J_1 + N_1)^{-1} \ O \ (J_0^D)^{D} \end{array} \right) \ast P,$$

where $J_0^D$ is nilpotent and

$$X = \sum_{s=0}^{k-1} \left( (J_1 + N_1)^{-1} \right)^{s+2} \ast N_2 \ast (J_0^D)^s \ast \left( I - J_0^D \ast (J_0^D)^D \right) $$

$$+ \left( I - (J_1 + N_1) \ast (J_1 + N_1)^{-1} \right) \ast \sum_{s=0}^{k-1} (J_1 + N_1)^s \ast N_2 \ast (J_0^D)^{s+2} $$

$$- (J_1 + N_1)^D \ast N_2 \ast (J_0^D)^D $$

$$= \sum_{s=0}^{k-1} \left( (J_1 + N_1)^{-1} \right)^{s+2} \ast N_2 \ast (J_0^D)^s.$$

Therefore

$$B^D = P^{-1} \ast \left( \begin{array}{ll} (J_1 + N_1)^{-1} \ O \ \sum_{s=0}^{k-1} ((J_1 + N_1)^{-1})^{s+2} \ast N_2 \ast (J_0^D)^s \end{array} \right) \ast P $$

$$= A^D - A^D \ast E \ast (I + A^D \ast E)^{-1} \ast A^D $$

$$+ \sum_{s=0}^{k-1} (A^D - A^D \ast E \ast (I + A^D \ast E)^{-1} \ast A^D)^{s+2} \ast E \ast (I - A \ast A^D) \ast A^D.$$ 

Moreover, it proves that $N_1 + J_1$ is invertible. Let consider spectral radius of $A^D \ast E$. Since (26) and (28), then

$$A^D \ast E = P^{-1} \ast J^D \ast P \ast P^{-1} \ast N \ast P $$

$$= P^{-1} \ast J^D \ast N \ast P.$$
and the decomposition of the matrix $bcirc(J^D \ast N)$ is

$$bcirc(J^D \ast N) = bcirc(J^D) bcirc(N)$$

$$= (F_p \otimes I_n) \begin{pmatrix} J^{D_1}_1 N_1 & J^{D_2}_2 N_2 & \cdots & J^{D_p}_p N_p \end{pmatrix},$$

where $J^{D_i}_i N_i = \begin{pmatrix} (J^{D_i}_i)^{-1} N_i & (J^{D_i}_i)^{-1} N_i \end{pmatrix}$. ($i = 1, 2, \cdots, p$)

Similarly, we obtain

$$E \ast A = P^{-1} \ast N \ast P \ast P^{-1} \ast J^D \ast P$$

and the decomposition of the matrix $bcirc(N \ast J^D)$ is

$$bcirc(N \ast J^D) = bcirc(N) bcirc(J^D)$$

$$= (F_p \otimes I_n) \begin{pmatrix} N_1 J^{D_1}_1 & N_2 J^{D_2}_2 & \cdots & N_p J^{D_p}_p \end{pmatrix},$$

where $N_i J^{D_i}_i = \begin{pmatrix} (N_i J^{D_i}_i)^{-1} & (N_i J^{D_i}_i)^{-1} \end{pmatrix}$. ($i = 1, 2, \cdots, p$)

By Definition 1.15, we have

$$\rho_T(J^D \ast N) = \rho(bcirc(J^D \ast N))$$

$$= \rho((F_p \otimes I_n) bcirc(J^D \ast N)(F_p^H \otimes I_n))$$

$$= \max_i \rho((J^{D_i}_i)^{-1} N_i)$$

$$= \rho((F_p \otimes I_n) bcirc(N \ast J^D))(F_p^H \otimes I_n))$$

$$= \rho(bcirc(N \ast J^D))$$

$$= \rho_T(N \ast J^D),$$

that is

$$\rho_T(\mathcal{A}^D \ast \mathcal{E}) = \max_i \rho((J^{D_i}_i)^{-1} N_i) = \max_i \rho((N_i J^{D_i}_i)^{-1}) = \rho_T(\mathcal{E} \ast \mathcal{A}^D), \quad (29)$$

further

$$\rho_T(\mathcal{E} \ast \mathcal{A}^D) = \rho_T(\mathcal{A}^D \ast \mathcal{E}) \leq \|\mathcal{A}^D \ast \mathcal{E}\| < 1. \quad (30)$$

On the other hand, it will prove that $J_1 + N_1 = J_1 \ast (I + (J_1)^{-1} \ast N_1)$ is invertible. According to the inverse of $J_1$, we will only prove that $I + (J_1)^{-1} \ast N_1$ is nonsingular. Now, we prove it by reduction to absurdity. Assume $I + (J_1)^{-1} \ast N_1$ is singular, then there is a nonzero tensor $y \in \mathbb{C}^{n \times n \times p}$, such that

$$(I + (J_1)^{-1} \ast N_1) \ast y = O,$$
then
\[ y = -((F_1^{-1} * N_1) * y), \]
and the decomposition of \( b_{\text{circ}}((F_1^{-1} * N_1) \) is
\[
b_{\text{circ}}((F_1^{-1} * N_1) = b_{\text{circ}}((F_1^{-1}) b_{\text{circ}}(N_1)) = (F_p \otimes I_n) \begin{pmatrix} (I_1^T)^{-1} N_1^1 \\ (I_2^T)^{-1} N_2^1 \\ \vdots \\ (I_p^T)^{-1} N_p^1 \end{pmatrix} (F_p^H \otimes I_n). \]

Therefore, by Definition 1.16, then \(-1\) is the eigenvalue of tensor \((F_1^{-1} * N_1)\), denoted
\[ \lambda_T\((F_1^{-1} * N_1)\) = \lambda\left(b_{\text{circ}}((F_1^{-1} * N_1))\right) = \lambda\left((F_p \otimes I_n) b_{\text{circ}}((F_1^{-1} * N_1) (F_p^H \otimes I_n)\right) = \lambda\left((I_1^T)^{-1} N_1^1\right) = -1, \]
it implies \( \max_i \rho\left((I_1^T)^{-1} N_1^1\right) \geq 1. \)
According to (29), we obtain
\[ \rho_T(E * A^D) = \rho_T(A^D * E) = \max_i \rho\left((I_1^T)^{-1} N_1^1\right) \geq 1, \]
which is contradictory to (30).
Hence \( I + (F_1^{-1} * N_1) \) is nonsingular. The proof is completed. \( \square \)

**Theorem 4.3.** Let \( A \in \mathbb{C}^{n \times n \times p}, E \in \mathbb{C}^{n \times n \times p} \) be complex tensors, \( \|A^D * E\| < 1 \), and \( B = A + E \) with \( \text{Ind}_T(A) = k \). Suppose that \( E = A * A^D * E \), then
\[
\frac{\|B^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D * E\|}{1 - \|A^D * E\|} + \sum_{s=0}^{k-1} \frac{K_B(A)^{s+1}}{(1 - \|A^D * E\|)^s} \|E\| \|A * A^D\|,
\]
where \( K_B(A) = \|A\||A||A^D||. \)

**Proof.** Since Lemma 4.2, we have
\[
B^D - A^D = -A^D * E * (I + A^D * E)^{-1} * A^D \] 
\[+ \sum_{s=0}^{k-1} \left(A^D - A^D * E * (I + A^D * E)^{-1} * A^D\right)^{s+2} * E * (I - A * A^D) * A^D, \]
(31)

taking norm on both sides of (31) at the same time, then
\[
\|B^D - A^D\| \leq \|A^D * E * (I + A^D * E)^{-1} * A^D\| 
\[+ \sum_{s=0}^{k-1} \left\|(A^D - A^D * E * (I + A^D * E)^{-1} * A^D)^{s+2} * E * (I - A * A^D) * A^D\right||
\leq \|A^D * E\| \|I + A^D * E\|^{-1} \|A^D\| 
\[+ \sum_{s=0}^{k-1} \left(\|A^D\| + \|A^D * E\| \|I + A^D * E\|^{-1} \|A^D\|\right)^{s+2} \|E\| \|I - A * A^D\| \|A^D\|. \]
by Theorem 2.1, we have

\[
\|B^D - A^D\| \leq \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s
+ \sum_{s=0}^{k-1} \left(\|A^D\| \|A^D \ast E\| \left(1 + \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)\right)^{s+2}\frac{1}{\|A\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s,
\]

that is

\[
\|B^D - A^D\| \leq \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s,
\]

(32)

divide \|A^D\| on both sides of (32) at the same time, we obtain

\[
\frac{\|B^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D \ast E\|}{1 - \|A^D \ast E\|} \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \frac{1}{\|A\|}\|E\| \|A \ast A^D\| \|A\|^s
+ \sum_{s=0}^{k-1} \left(\|A^D\| \|A^D \ast E\| \left(1 + \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)\right)^{s+2}\frac{1}{\|A\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s,
\]

\[
= \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \frac{1}{\|A\|}\|E\| \|A \ast A^D\| \|A\|^s
+ \sum_{s=0}^{k-1} \left(\|A^D\| \|A^D \ast E\| \left(1 + \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)\right)^{s+2}\frac{1}{\|A\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s
\]

\[
= \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \frac{1}{\|A\|}\|E\| \|A \ast A^D\| \|A\|^s
+ \sum_{s=0}^{k-1} \left(\|A^D\| \|A^D \ast E\| \left(1 + \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)\right)^{s+2}\frac{1}{\|A\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s
\]

\[
= \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)^{s+2} \frac{1}{\|A\|}\|E\| \|A \ast A^D\| \|A\|^s
+ \sum_{s=0}^{k-1} \left(\|A^D\| \|A^D \ast E\| \left(1 + \|A^D \ast E\| \left(\frac{1}{1 - \|A^D \ast E\|}\right)\right)^{s+2}\frac{1}{\|A\|}\right)^{s+2} \|E\| \|A \ast A^D\| \|A\|^s
\]

where \(K_p(A)^{s+1} = (\|A\|\|A^D\|)^{s+1}\). The proof is completed. □

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