Cosmology of the type IIB superstring

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The continuous and discrete symmetries of a dimensionally reduced type IIB superstring action are employed to generate four-dimensional cosmological solutions with non-trivial Neveu-Schwarz/Neveu-Schwarz and Ramond-Ramond form-fields from the dilaton-moduli-vacuum solutions.

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Superstring theory is the most promising candidate for a unified quantum theory of the fundamental interactions including gravity [4]. The realization that the five different string theories are related non-perturbatively by duality symmetries has led to renewed interest in the type IIB theory [5,6]. The Ramond–Ramond (RR) sector of this theory is particularly relevant in view of the recent advances that have been made in understanding the relationship between D-branes of open string theory and RR gauge fields [6]. The main emphasis to date has been on black hole and membrane solutions [6]. However, cosmological models might provide one of the few observational tests of the theory and the early universe represents a natural environment in which concepts such as string duality may be quantitatively studied, at least within the context of the low energy supergravity action.

In this paper we will present four-dimensional (4–D) cosmological solutions of the type IIB theory with time-dependent Neveu–Schwarz/Neveu–Schwarz (NS–NS) and RR fields by exploiting the SL(2,R) symmetries of the theory [6,10]. Solutions with a single NS–NS or RR form-field can be directly integrated to yield simple analytic expressions [6]. Solutions with more than one form-field have also been found where the system reduces to an integrable Toda model [6,10,11]. Here we include the interactions between the NS–NS and RR form-fields on the external space, which in general leads to a more complicated system, but one which is still integrable due to the symmetries between the fields. Our results generalise solutions previously obtained by a single SL(2,R) transformation acting on the NS–NS sector solutions [12,13].

The low energy limit of the type IIB superstring is ten-dimensional (10–D) $N = 2$ chiral supergravity [14]. The NS–NS fields of the bosonic sector of this theory are the dilaton, $\Phi$, the 10–D metric, $g_{MN}$, and the antisymmetric two-form potential, $B_{1}^{MN}$. The RR sector contains a scalar axion field, $\chi$, a two-form potential, $B_{2}^{MN}$, and a four-form potential $D_{MNPQ}$. The equation of motion for the latter field cannot be derived from a covariant action [13] but we can consistently set it to zero. The effective action for the remaining fields is given by

$$ S = \int d^{10}x \sqrt{-g_{10}} \left\{ e^{-\Phi} \left[ R_{10} + (\nabla \Phi)^2 - \frac{1}{12} \left( H^{(1)} \right)^2 \right] - \frac{1}{2} (\nabla \chi)^2 - \frac{1}{12} \left( H^{(1)} \chi + H^{(2)} \right)^2 \right\}, $$

where $R_{10}$ is the Ricci curvature scalar, $g_{10} \equiv \det |g_{MN}|$ and $H^{(i)}_{MNP} \equiv \partial_{[M} B^{(i)}_{NP]}$ are the field strengths of the two-form potentials $B^{(i)}_{MN}$. To investigate 4–D cosmological solutions we compactify the 10–D spacetime on an isotropic six-torus

$$ ds_{10}^{2} = g_{\mu \nu}(x) dx^{\mu} dx^{\nu} + e^{y(x)} / \sqrt{\sigma} \delta_{ab} dX^{a} dX^{b}, $$

where $y(x)$ describes the volume of the internal space and is the only modulus field considered. We neglect moduli fields arising from the compactification of the form-fields on the internal dimensions [12,13]. The symmetries of the reduced theory become manifest in the conformally related 4–D Einstein frame

$$ \tilde{g}_{\mu \nu} = e^{\varphi} g_{\mu \nu}, $$

where $\varphi \equiv \Phi - \sqrt{3} y$ is the 4–D dilaton. In four dimensions, the three-form field strengths are dual to the gradients of pseudo-scalar axion fields, $\sigma_{i}$ [20]:

$$ \tilde{H}^{(1)}_{\mu \nu \lambda} = \tilde{z}^{\mu \nu \lambda \kappa} e^{2 \varphi} \left( \tilde{\nabla}_{\kappa} \sigma_{1} - \chi \tilde{\nabla}_{\kappa} \sigma_{2} \right), $$

$$ \tilde{H}^{(2)}_{\mu \nu \lambda} = \tilde{z}^{\mu \nu \lambda \kappa} e^{\varphi} \sqrt{3} \chi \tilde{\nabla}_{\kappa} \sigma_{2} - \chi e^{2 \varphi} \left( \tilde{\nabla}_{\kappa} \sigma_{1} - \chi \tilde{\nabla}_{\kappa} \sigma_{2} \right). $$

In this dual formulation the equations of motion for the fields follow from an effective action [20]:

$$ S = \int d^{4}x \sqrt{-g} \left[ \tilde{R} - \frac{1}{2} \left( \tilde{\nabla} \varphi \right)^2 - \frac{1}{2} \left( \tilde{\nabla} y \right)^2 \right] - \frac{1}{2} e^{\sqrt{3} y + \varphi} \left( \tilde{\nabla} \chi \right)^2 - \frac{1}{2} e^{-\sqrt{3} y + \varphi} \left( \tilde{\nabla} \sigma_{2} \right)^2 - \frac{1}{2} e^{2 \varphi} \left( \tilde{\nabla} \sigma_{1} - \chi \tilde{\nabla} \sigma_{2} \right)^2. $$

(6)
The action in Eq. (6) is invariant under the global SL(2,R) transformation
\[ \tilde{y}_{\mu
u} = \tilde{y}_{\mu\nu}, \quad \tilde{\sigma} = (\Sigma^T)^{-1}\sigma, \quad \tilde{v} = v, \] (7)
where
\[ \frac{1}{2}\Phi = u \equiv \frac{1}{2}\varphi + \sqrt{3}y, \quad v = \sqrt{3}\varphi - \frac{1}{2}y \] (8)
\[ M = \begin{pmatrix} e^{2\varphi} & \chi e^{\varphi} \\ \chi e^{\varphi} & e^{-\chi} + \chi^2 e^{\varphi} \end{pmatrix}, \quad \sigma \equiv \begin{pmatrix} -\sigma_1 \\ \sigma_2 \end{pmatrix} \] (9)
and
\[ \Sigma = \begin{pmatrix} D & B \\ C & A \end{pmatrix}, \quad AD - BC = 1. \] (10)

The assumption that all fields are independent of the internal coordinates implies that the action (6) also exhibits a ‘T–duality’
\[ \tilde{y} = -y, \quad \tilde{\sigma} = -\sigma_1 + \chi \sigma_2 \] (11)
\[ \tilde{\chi} = \sigma_2, \quad \tilde{\sigma}_2 = \chi \] (11)
that inverts the volume of the internal space and leaves the dilaton and 4–D Einstein frame metric invariant [20].

This results in a second SL(2,R) symmetry that may be viewed as a mirror image of that given in Eq. (7) [20].

If we define
\[ w \equiv \frac{1}{2}\varphi - \sqrt{3}y, \quad x \equiv \frac{\sqrt{3}}{2}\varphi + \frac{1}{2}y \] (12)
\[ P = \begin{pmatrix} e^w & \sigma_2 e^w \\ \sigma_2 e^w & e^{-w} + \sigma_2^2 e^w \end{pmatrix}, \quad \rho \equiv \begin{pmatrix} \sigma_1 - \chi \sigma_2 \\ \chi \end{pmatrix}, \] (13)
the action is invariant under the SL(2,R) transformation
\[ \tilde{P} = \Sigma P \Sigma^T, \quad \tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}, \quad \tilde{\rho} = (\Sigma^T)^{-1} \rho, \quad \tilde{x} = x. \] (14)
The transformation (14) is formally equivalent to the sequence of transformations given by Eq. (1), followed by Eq. (3), followed again by Eq. (1).

It should be emphasized that neither of the symmetries (4) or (14) coincide with the SL(2,R) symmetry of the NS–NS sector alone, which mixes the 4–D dilaton and the NS–NS axion [21]. This symmetry has been employed previously to derive 4–D cosmological solutions with a non–trivial NS–NS form–field [22], but is broken due to the interaction of the RR fields.

We consider Friedmann–Robertson–Walker (FRW) cosmologies with the homogeneous and isotropic 4–D line element
\[ ds^2 = a^2(\eta) \left[-d\eta^2 + d\Omega^2_\kappa\right], \] (15)
where a is the scale factor in the string frame and dΩ^2_\kappa is the line element on the 3–space with constant curvature \( \kappa \). The field equations are given by
\[ \varphi'' + 2\frac{\dot{\alpha}}{\alpha}\varphi' = \frac{1}{2}e^{\sqrt{3}y + \varphi}\chi'^2 + \frac{1}{2}e^{-\sqrt{3}y - \varphi}\sigma'^2 \] (16)
\[ y'' + 2\frac{\dot{\alpha}}{\alpha}y' = \sqrt{3}e^{\sqrt{3}y + \varphi}\chi'^2 - \sqrt{3}e^{-\sqrt{3}y - \varphi}\sigma'^2 \] (17)
\[ \chi' + \left(2\frac{\dot{\alpha}}{\alpha} + \sqrt{3}y' + \varphi'\right)\chi' = -e^{-\sqrt{3}y + \varphi}\sigma_1\sigma_2 \] (18)
\[ \sigma_2'' + \left(2\frac{\dot{\alpha}}{\alpha} - \sqrt{3}y' + \varphi'\right)\sigma_2 = e^{\sqrt{3}y + \varphi}\chi'(\sigma_1 - \chi \sigma_2) \] (19)
\[ (\sigma_1 - \chi \sigma_2)'' + 2\left(\frac{\dot{\alpha}}{\alpha} + \varphi'\right)(\sigma_1 - \chi \sigma_2) = 0 \] (20)

This follows directly from the fact that homogeneous massless scalar fields have the same equation of state as a maximally stiff fluid [23]. This remains true when the form–fields are included, despite their couplings to the
dilaton and moduli $\tilde{u}$, since the 4–D Einstein frame metric is invariant under the SL(2,R) transformations in Eqs. (10) and (14). Thus, the Einstein frame scale factor always has the simple evolution described by Eq. (26) and this is singular as $\eta \to 0$ (or $k^{1/2} \eta \to \pm \pi/2$ for $k > 0$).

The general FRW solution containing a single excited RR form field can be generated by applying the SL(2,R) transformation (7). The trajectories in (27–29) for different choices of the $\psi$ field. We obtain

\[
\alpha^2 = \frac{\alpha_{n}}{2} \left[ (\tau/\tau)_{*} ((1+\sqrt{3}\cos \chi) + (\tau/\tau)_{*} (1-\sqrt{3}\cos \chi)) \right] ^{1/n}
\]

\[
e^\psi = \frac{e^{\phi}}{21/m} \left[ (\tau/\tau)_{*} (n \sqrt{3} \sin \xi_{1} + (\tau/\tau)_{*} (n \sqrt{3} \sin \xi_{2})) \right] ^{1/m}
\]

\[
\psi = \psi_{*} + K^{-1} \left[ (\tau/\tau)_{*} (\sqrt{3} \cos \xi_{1} - (\tau/\tau)_{*} (\sqrt{3} \cos \xi_{2})) \right] ^{1/m}
\]

where $K = \pm e^{(\phi_{n}/n)+(\psi_{n}/m)}$ and the field $\psi$ represents the field $\chi$ and $\sigma_{2}$ depending upon which of these fields is excited. These solutions interpolate between two asymptotic regimes where the form–fields vanish and the trajectories in $(\varphi, y)$ space become straight lines. If the asymptotic trajectory comes in at an initial angle $\xi_{1}$ to the $\varphi$ axis, it leaves at an angle $\xi_{2}$. The values of the parameters $n, m, \xi_{1}$ and $\xi_{2}$ for different choices of form–field are given in Table I. Note that for each form–field there is a characteristic angle $\theta$ such that $1/n = \cos \theta$, $1/m = \sin \theta$ and $\xi_{2} = 2\theta - \xi_{1}$.

The general solution with non-trivial $\chi$ and constant $\sigma_{2}$ (i.e., vanishing three–form field strengths $H^{(2)}$) is obtained by applying the SL(2,R) transformation. The transformed fields $\tilde{u}$ and $\tilde{\chi}$ have the form

\[
e^{\tilde{u}} = \left[ 2C(D + C\chi) \right] \cosh(u + \Delta)
\]

\[
\tilde{\chi} = \chi_{*} \pm \frac{1}{2C(D + C\chi)} \tanh(u + \Delta)
\]

where $e^{\Delta} \equiv [(D + C\chi)/C]$. The introduction of a non–constant $\tilde{\chi}$ field places a lower bound on $u$, and hence the 10–D dilaton field, $\Phi = 2u$. A typical solution with $\chi' \neq 0$ is shown in Fig. 1. The RR field interpolates between two asymptotic vacuum solutions where $\chi' \to 0$. Trajectories that come in from infinity $(u \to \infty)$ at

| $\psi$ | $1/n$ | $1/m$ | $\xi_{1}$ | $\xi_{2}$ |
|------|------|------|------|------|
| $\chi$ | $1/2$ | $\sqrt{3}/2$ | $\xi_{*}$ | $(2\pi/3) - \xi_{*}$ |
| $\sigma_{2}$ | $1/2$ | $-\sqrt{3}/2$ | $\xi_{*}$ | $-(2\pi/3) - \xi_{*}$ |
| $\sigma_{1}$ | $1$ | $0$ | $\xi_{*}$ | $-\xi_{*}$ |

FIG. 1. Trajectories in $(\varphi, y)$ field–space for the dilaton–moduli–vacuum solution (solid line) with $\xi_{*} = \pi/9$. The dashed, dot–dashed and dotted lines represent the three single form–field solution with $\psi = \chi, \sigma_{2}$ and $\sigma_{1}$, respectively, obtained by the appropriate SL(2,R) transformation of the dilaton–moduli–vacuum solution.
fields \( \varphi \) and \( y \) is shown in Fig. [1]. The presence of a non-vanishing \( \sigma'_1 \) enforces a lower bound on the value of the 4–D dilaton, \( \varphi \geq \varphi_* \). Trajectories that come from infinity \( (\varphi \to \infty) \) at an angle \( \xi_1 = \xi_* \), where \(-\pi/2 \leq \xi_* \leq \pi/2 \), are reflected in the line \( \varphi = \varphi_* \) back out at an angle \( \xi_2 = -\xi_* \). No trajectories can reach \( \varphi \to -\infty \) unless \( \sigma'_1 = 0 \).

We now consider solutions where two of the form–fields are non-vanishing but the third is zero. The field equations (13) and (22) imply that the only consistent solution of this type arises when \( H^{(1)} = 0 \). From Eq. (4), \( \sigma'_1 = \chi \sigma'_2 \) and this allows \( \sigma_1 \) to be eliminated. Eqs. (18) and (19) may be integrated directly to yield \( \tilde{a}^2 e^{\varphi + \sqrt{3} y} \chi' = L \) and \( \tilde{a}^2 e^{\varphi - \sqrt{3} y} \sigma'_2 = J \), where \( J \) and \( L \) are arbitrary constants. Defining a new time parameter \( T \equiv \int^0 dq'/\tilde{a}^2 \propto \ln |\tau| \) and new variables \( q_1 \equiv \varphi \pm (y/\sqrt{3}) \) implies that the field equations for the dilaton and moduli may be expressed as

\[
\begin{align*}
\ddot{q}_+ &= J^2 e^{q_+ - 2q_-} \\
\ddot{q}_- &= L^2 e^{q_- - 2q_+}
\end{align*}
\]  
(33)

and the Friedmann constraint (21) gives

\[
\frac{1}{8} (\dot{q}_+ + \dot{q}_-)^2 + \frac{3}{8} (\dot{q}_+ - \dot{q}_-)^2 + V = \frac{3a_4^2}{2},
\]  
(35)

where a dot denotes \( d/dT \) and the potential

\[
V = \frac{1}{2} \left( J^2 e^{q_+ - 2q_-} + L^2 e^{q_- - 2q_+} \right)
\]  
(36)

Equations (33–35) correspond to those of the SU(3) Toda system (21). This has recently been studied in similar models by a number of authors (8,9,11). The general solution is of the form (11)

\[
\begin{align*}
e^{q_-} &= \sum_{i=1}^3 A_i e^{-\lambda_i T} , \\
e^{q_+} &= \sum_{i=1}^3 B_i e^{\lambda_i T} ,
\end{align*}
\]  
(37)

where \( \sum_i \lambda_i = 0 \), so that \( \lambda_{\text{min}} < 0 \) and \( \lambda_{\text{max}} > 0 \). This gives the asymptotic solution for \( \varphi \) and \( y \) as \( T \to -\infty \):

\[
\begin{align*}
e^{\varphi} &\sim e^{-(\lambda_{\text{max}} - \lambda_{\text{min}}) T/2} , \\
e^{y} &\sim e^{\sqrt{3}(\lambda_{\text{max}} + \lambda_{\text{min}}) T/2} ,
\end{align*}
\]  
(38)

while as \( T \to +\infty \) we have

\[
\begin{align*}
e^{\varphi} &\sim e^{(\lambda_{\text{max}} - \lambda_{\text{min}}) T/2} , \\
e^{y} &\sim e^{\sqrt{3}(\lambda_{\text{max}} + \lambda_{\text{min}}) T/2} .
\end{align*}
\]  
(39)

As in the single form-field solutions discussed above, the asymptotic solutions correspond to straight lines in the \((\varphi,y)\) plane. We see that trajectories that come from infinity \( (\varphi \to \infty) \) at an angle \( \xi_* \) are reflected back out at an angle \(-\xi_* \). This is exactly the qualitative behaviour of the NS–NS dilaton–moduli–axion solution. However, the range of allowed asymptotic trajectories is more restricted than in the pure NS–NS case. The potential in the constraint Eq. (43) is bounded from above and we therefore require that \(|y| \leq \varphi/\sqrt{3}\) asymptotically.

Thus, the asymptotic solutions are restricted to the range \(-\pi/6 \leq \xi_* \leq \pi/6\), where \( V \leq 3a_4^2/2 \).

The general FRW solutions to the type IIB string action presented in Eq. (3) can be generated by applying an appropriate sequence of SL(2,R) transformations on the dilaton–moduli–vacuum solution presented in Eqs. (22–23). The transformations are given by Eq. (4), followed by Eq. (11), followed by Eq. (10). This includes the above Toda system as a special case, where \( H^{(1)} = 0 \). The procedure is readily extended to other situations including anisotropic and inhomogeneous metrics. One can also obtain solutions with all three form–fields non–zero [21,13] by applying the SL(2,R) transformation given in Eq. (4) to the single form–field \( \psi = \sigma_1 \) solutions presented in Eqs. (23,30). However, these solutions correspond only to the particular case where \( H^{(2)} \propto H^{(1)} \).

The general solution exhibits a sequence of bounces between asymptotic vacuum states. A typical solution is shown in Figs. 2 and 3. The time–dependence of the fields \( \chi \) and \( \sigma_2 \) induces lower bounds on the variables \( u \) and \( w \), respectively, as seen in the single form–field solutions. In the general solution this results in a lower bound on \( \varphi = u + w \).

The general type IIB solution contains a non-vanishing NS–NS form–field, but can always be obtained from a Toda system with \( H^{(1)} = 0 \) by a single SL(2,R) transformation (9). The asymptotic behaviour of \( \varphi \) and \( y \) is invariant under this transformation. This follows since \( u \to \infty \) asymptotically for all solutions in the Toda system and, from Eq. (31), we obtain \( \tilde{u} \to u \) in the general solution. We also have \( \tilde{v} = v \) and thus \( \varphi \) and \( y \) are invariant.

*An exceptional case is when \( u \to u_* \) asymptotically, where \( u_* \) is a constant. In this case \( \tilde{u} \to \text{constant} \), though not necessarily \( u_* \), but the qualitative behaviour is the same.
ant in this limit. Thus, trajectories in \((\varphi, y)\) field-space come in at an angle \(\xi_*\) and leave at an angle \(-\xi_*\), where \(-\pi/6 \leq \xi_* \leq \pi/6\).

In conclusion, therefore, we have shown that from the dimensionally reduced type IIB superstring action \(S\), the general FRW cosmological solution with non-vanishing RR and NS–NS form–fields interpolates between asymptotic dilaton–moduli–vacuum solutions, where the form–fields vanish. These early– and late–time limiting solutions correspond to straight lines in the \((\varphi, y)\) plane at an angle \(\xi_*\) to the \(\varphi\)–axis and are related via a reflection symmetry \(\xi_* \rightarrow -\xi_*\). This is strikingly similar to the pure NS–NS solution \(\mathcal{P}\). On the other hand, when the RR fields are non–vanishing, the initial and final trajectories in \((\varphi, y)\) space are restricted to the wedge \(-\pi/6 \leq \xi_* \leq \pi/6\), in contrast to the NS–NS solutions, where they are bounded by \(-\pi/2 \leq \xi_* \leq \pi/2\).

This places important restrictions on the range of cosmological solutions found. Firstly, the scale factor in the string frame, \(a\), is always bounded from below. The solutions are still singular in this frame, however, and approach a curvature singularity in the limit \(\tau \rightarrow 0\). When the RR fields vanish, the axion field \(\sigma_1\) is dual to the NS–NS three-form field strength. This field is minimally coupled in the conformally related frame with scale factor \(e^\varphi a\). The metric in this frame is non–singular when \(|\xi_*| \leq \pi/6\) \[23\]. It is interesting that this is precisely the bound placed on the allowed trajectories by the RR fields. Moreover, the spectrum of quantum fluctuations induced in the NS–NS axion field during an inflationary “pre-Big Bang” epoch \[23\] has a spectral index given by \(n_s = 4 - 2\sqrt{3}\cos\xi_*/\pi\) \[23\]. The RR fields limit the allowed range to \(4 - 2\sqrt{3} \leq n_s \leq 1\). Thus, RR fields in this model lead to the NS–NS axion possessing a spectrum with \(n_s \leq 1\) and this might have interesting consequences for the formation of large-scale structure in our universe.

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