A Remark on the Kelvin Transform for a Quasilinear Equation

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The p-harmonic equation $\text{div}(|\nabla u|^{p-2}\nabla u) = 0$ has certain applications to quasiregular mappings [B-I], to non-linear potential theory, where the corresponding variational integral defines the $p$-Capacity, and in Physics it describes certain non-Newtonian fluids. The equation is invariant under Euclidean motions and the aim of this note is to study the behaviour under reflections in spheres.

The Euler–Lagrange equation for the variational integral

$$\int |\nabla u(x)|^p dx, \quad (1 < p < \infty)$$

is the p-harmonic equation

$$\text{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad (1 < p < \infty).$$

The solutions are called p-harmonic functions. Usually Eqn (2) is interpreted in the weak sense: a function $u$ in the Sobolev space $W^{1,p}_{\text{loc}}(G)$, $G$ being a domain in the Euclidean n-dimensional space $\mathbb{R}^n$, is p-harmonic in $G$, if

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = 0$$

whenever $\eta \in C_0^\infty(G)$. It is known that the weak solutions are (equivalent to functions) of class $C^{1,\alpha}_{\text{loc}}(G)$. This means that the gradient $\nabla u$ is locally Holder continuous with the exponent $\alpha = \alpha(n, p) > 0$, cf. [D,E,Lew,T,U]. (In the plane the best Holder exponent is known [I-M].)
The integral \( \int |\nabla u|^p \) is conformally invariant when \( p = n \) (“the border-line case”). This important observation was made by Ch. Loewner in 1959, cf. [Loe]. In the n-dimensional space this means that the n-harmonic functions are invariant under Mobius transformations. See [B, Chapter 3] and [A] for Mobius transformations in \( \mathbb{R}^n \). Since all p-harmonic functions are preserved under Euclidean motions and homothetic transformations, the crucial point is easily reduced to the reflection (or inversion)

\[
x^* = \frac{x}{|x|^2}
\]

in the unit sphere and the proof is a matter of a simple calculation. If \( u \) is n-harmonic in the domain \( G \) in \( \mathbb{R}^n \), so is

\[
v = v(x) = u\left(\frac{x}{|x|^2}\right)
\]

in the reflected domain \( G^* \). If necessary, define \( 0^* = \infty \).

If \( p \neq n \), p-harmonicity is not, in general, preserved under reflections in spheres: the Mobius invariance brakes down.

For \( p = 2 \) Eqn (2) reduces to the Laplace equation \( \Delta u = 0 \). Even though harmonic functions in higher dimensions are not preserved under reflections in spheres, this deficiency is fortunately compensated by the celebrated Kelvin transform\[^1\]: if \( u \) is harmonic in \( G \), then

\[
v = v(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)
\]

is harmonic in the reflected domain \( G^* \). See [H, Section 1.9]. A recent application is given in [Leu].

Note that formulae (4) and (5) can be written as

\[
v = v(x) = |x|^{\frac{p-n}{p-1}} u\left(\frac{x}{|x|^2}\right)
\]

valid for \( p = 2 \) and \( p = n \). One might ask whether there is, for general \( p \), some counterpart to the Kelvin transform. Although there is intuitive support in favour of a positive conjecture, the answer is plainly ”no”. To some extent

\[^1\]W. Thomson (Lord Kelvin), Journal de mathématiques pures et appliquées 12 (1847), p. 256.
this answer is unexpected, to say the least. This constitutes a serious obstacle for the development of some parts of the theory for p-harmonic functions.

**THEOREM** For a given $p$, $1 < p < \infty$, there is no radial function $\rho \not\equiv 0$ such that

$$v = v(x) = \rho(|x|) u\left(\frac{|x|}{|x|^2}\right)$$

(7)

is p-harmonic in the reflected domain $G^*$, whenever $u$ is p-harmonic in $G, G \subset \mathbb{R}^n$, except in the cases $p = 2$ and $p = n$.

**Proof.** Let us first point out that a direct calculation of $\text{div}(|\nabla v|^{p-2} \nabla v)$ does not lead to anything comprehensible. A more "experimental approach" yields the non-existence of $\rho$: transform sufficiently many p-harmonic functions according to (7) to get a contradiction.

Transforming the p-harmonic function

$$u(x) = \begin{cases} |x|^{(p-n)/(p-1)}, & \text{if } p \neq n, \\ \ln(|x|), & \text{if } p = n, \end{cases}$$

(8)

we immediately get the necessary condition

$$\rho(|x|) = |x|^{(p-n)/(p-1)},$$

(9)

since, as it is easily seen, the only radial p-harmonic functions are essentially those in (7), variants like $A|x - a|^{(p-n)/(p-1)} + B$ being included. Thus we are back at (6), indeed.

A routine calculation shows that the functions

$$u = u(x_1, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_j^2)^{(p-j)/2(p-1)},$$

$j = 1, 2, \ldots, n$ are p-harmonic, when $x_1^2 + x_2^2 + \cdots + x_j^2 \neq 0$. But transforming them according to (6), we shall obtain p-harmonic functions only for $p = n$, $p = 2$, and $j = n$. Indeed, it is here sufficient, although not virtually simpler, to transform only the first function $u(x) = x_1$ in order to arrive at a contradiction with the necessary choice (9). Now the following technical lemma concludes our proof.

**LEMMA** The function $v = v(x) = |x|^\alpha x_1$ is p-harmonic only in the following cases: (i) $\alpha = 0$, (ii) $\alpha = -2$ and $p = n$, (iii) $\alpha = -n$ and $p = 2$, and (iv) $\alpha = -1$ and $p = 3 - n$. 

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Proof. A lengthy routine calculation gives the expression
\[
\text{div}(|\nabla v|^{p-2}\nabla v) = |\nabla v|^{p-4} \alpha |x|^{3\alpha-4} x_1 \{[\alpha + n + (p - 2)(2\alpha + 3)]|x|^2 \\
+ (\alpha + 2)[\alpha(\alpha + n) + (p - 2)(\alpha^2 + \alpha - 1)] x_1^2\},
\]
where
\[
|\nabla v|^2 = |x|^{2(\alpha - 1)} \{\alpha(\alpha + 2)x_1^2 + |x|^2\}.
\]
This yields the desired result. (Case (iv) does not occur for \(1 < p < \infty\).)

Epilogue. Of course one might try to replace the reflection by some kind of a distorted inversion like
\[
x^* = \frac{x}{|x|^\beta}
\]
where \(\beta = \beta(n, p) > 0\). Indeed, the disappointing news is that no "Kelvin transform" even of the type
\[
v = v(x) = \rho(|x|) u\left(\frac{x}{|x|^\beta}\right)
\]
exists for all \(p\)-harmonic functions \(u\) if \(p \neq n\) and \(p \neq 2\). The proof is similar to the above one.

This leaves us with little or no hope to map an unbounded domain onto a bounded one in the \(p\)-harmonic setting. However, sometimes one can proceed as in [L1], using \(p\)-superharmonic functions. In the complex plane there is, fortunately, a rich structure partly compensating for this lack [L2].

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