Renormalization of the tensor current in lattice QCD and the $J/\psi$ tensor decay constant

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Lattice QCD calculations of form factors for rare Standard Model processes such as $B \to K \ell^+ \ell^-$ use tensor currents that require renormalization. These renormalization factors, $Z_T$, have typically been calculated within perturbation theory and the estimated uncertainties from missing higher order terms are significant. Here we study tensor current renormalization using lattice implementations of momentum-subtraction schemes. Such schemes are potentially more accurate but have systematic errors from nonperturbative artifacts. To determine and remove these condensate contributions we calculate the ground-state charmonium tensor decay constant, $f_{J/\psi}^T$, which is also of interest in beyond the Standard Model studies. We obtain $f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV}) = 0.3927(27) \text{ GeV}$, with ratio to the vector decay constant of 0.9569(52), significantly below 1. We also give $Z_T$ factors, converted to the $\overline{\text{MS}}$ scheme, corrected for condensate contamination. This contamination reaches 1.5% at a renormalization scale of 2 GeV (in the preferred Regularisation Invariant Symmetric Momentum subtraction scheme) and so must be removed for accurate results.

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I. INTRODUCTION

Rare Standard Model processes, for example those that first appear at one-loop order through so-called “penguin” diagrams, are of great interest in searches for new physics. The very low rate for the process in the Standard Model means that beyond the Standard Model searches have small backgrounds. The signal rate will also be small, however, so it is important to have firm theoretical understanding of the Standard Model contribution. This starts with the effective weak Hamiltonian, $\mathcal{H}_{\text{eff}}$, after integrating out the weak bosons. $\mathcal{H}_{\text{eff}}$ contains flavor-changing neutral-current operators that can induce, for example, rare $b \to s$ processes [1]. Sandwiched between hadronic states these operators yield matrix elements that can be converted into form factors for differential decay rates for comparison to experiment. The best way to calculate the matrix elements is by using lattice QCD. The matrix elements required are those of operators in a continuum scheme for QCD, however, ideally in the same scheme in which the Wilson coefficients for the operators in $\mathcal{H}_{\text{eff}}$ were determined (the $\overline{\text{MS}}$ scheme). This means that the lattice operators must be matched accurately to the continuum scheme. For such $b \to s$ processes tensor operators in $\mathcal{H}_{\text{eff}}$, e.g., $\bar{s}_L \sigma^{\mu\nu} b_R$, cause a particular problem for lattice to continuum renormalization, because they cannot be connected to conserved currents. We show how to solve that problem here.

An example of a rare $b \to s$ process being studied experimentally is $B \to K \ell^+ \ell^-$. A first unquenched lattice QCD calculation of this decay was performed in [2] by members of the HPQCD Collaboration and another in [3] by the Fermilab lattice and MILC collaborations. The former used highly improved staggered quark (HISQ) [4] light and strange quarks and NRQCD $b$ quarks and the latter used asqtad light and strange quarks and Fermilab $b$ quarks. In the HPQCD calculation the tensor current was renormalized using one-loop lattice QCD perturbation theory for the NRQCD-HISQ current. A 4% systematic uncertainty on the tensor form factor was then taken to account for missing higher order terms in $\alpha_s$. The Fermilab/
MILC calculation also used one-loop lattice QCD perturbation theory for the Fermilab clover-asqtad current renormalization. The $O(a_s^2)$ uncertainty on the tensor form factor was taken as 2%.

The HPQCD Collaboration has recently performed a series of $b$ physics calculations using the HISQ formalism for all quarks, working upwards in mass from that of the $c$ quark and mapping out the dependence on the heavy-quark mass [5–8]. The success of this methodology indicates the possibility of improvement on previous $B \to K$ calculations for which it would be important also to reduce the uncertainty arising from the tensor current renormalization.

Here we use a partially nonperturbative procedure for the renormalization using momentum-subtraction schemes implemented on the lattice as an intermediate scheme [9]. This produces tensor current renormalization factors with better accuracy than those used in the calculations mentioned above because the perturbative part of the calculation, the matching from momentum subtraction to the $\overline{\text{MS}}$ scheme, can be done through $a_s^2$ in the continuum. Renormalization factors calculated on the lattice in momentum-subtraction schemes suffer from nonperturbative artifacts in general. Because these survive the continuum limit they need to be removed or otherwise accounted for. The artifacts are suppressed by powers of the renormalization scale $\mu$ and can therefore be studied by performing calculations at multiple $\mu$ values, as we did for the quark mass renormalization in [10]. We show here how to remove such systematic effects in the tensor renormalization factor by calculating a simple matrix element of the tensor operator that we can determine accurately in the continuum limit. For this purpose we use the $J/\psi$ tensor decay constant $f_{J/\psi}^T$.

The vector $J/\psi$ decay constant $f_{J/\psi}^V$, calculated from the vector charmonium correlator, is related to the leptonic decay rate of the $J/\psi$ meson for a recent very accurate determination of this decay constant see [11]. In contrast there is no simple decay rate that can be related to the $J/\psi$ tensor decay constant. Two-flavor lattice QCD and QCD sum rules calculations of $f_{J/\psi}^T$ and the ratio $f_{J/\psi}^T/f_{J/\psi}^V$ were presented in [12], and we will compare to those results here. $f_{J/\psi}^T$ is required for the calculation of bounds on beyond the Standard Model charged lepton flavor violating $J/\psi$ decay rates [13] and a similar calculation for other vector mesons would extend this. The $B_c^0$ tensor decay constant appears in parametrizations of its Standard Model decay rates $B_c^0 \to e^+e^−$ [14]. Calculation of this decay constant is underway using the tensor current renormalization factors we have determined here.

In the next section we discuss the definition of the tensor current renormalization factor in the Regularisation Invariant Symmetric Momentum subtraction (RI-SMOM) and RI′-MOM momentum-subtraction schemes. In Sec. III we give details of our lattice calculation of the tensor renormalization factor. This is followed by our lattice calculation of the $J/\psi$ tensor decay constant in Sec. IV.

Our results for $f_{J/\psi}^T$ are discussed in Sec. V followed by discussion of our $Z_T$ results in Sec. VI. Finally, we give our conclusions in Sec. VII.

II. $Z_T$ IN THE RI-SMOM AND RI’-MOM SCHEMES

Momentum-subtraction schemes provide useful intermediate schemes in matching lattice QCD to the continuum $\overline{\text{MS}}$ scheme because they provide a way to implement the same scheme both on the lattice and in the continuum [9]. Then the continuum limit of the lattice results will be in the continuum momentum-subtraction scheme (and independent of the lattice action used) and can be matched to the $\overline{\text{MS}}$ in continuum QCD.

In both of the momentum-subtraction schemes that we consider here the wave function renormalization $Z_q$ is defined in terms of the inverse of the momentum space quark propagator $S(p)$ according to [9,15–17]

$$Z_q = -\frac{1}{12p^2} \text{Tr}[S^{-1}(p)p] .$$

As the propagator is gauge dependent it is necessary to work in a fixed gauge. Landau gauge is used throughout. Working in a fixed gauge raises the possibility of effects from Gribov copies. Here we do not address this issue and assume that such effects are negligible following general expectations and the findings of [18], which saw no observable effects at a precision below 1%.

The tensor current renormalization is defined in terms of $Z_q$ and the tensor vertex function $G_T$:

$$G_T(p_1, p_2) = \int d^4x d^4y_1 d^4y_2 e^{i\Lambda x} e^{-ip_1\cdot x} e^{ip_2\cdot y_2} \langle T^\mu(x) \rangle \times \bar{\psi}(y_1)\psi(y_2) .$$

Here $T^\mu(x)$ is the tensor current $\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$. We take the bilinears in the renormalization procedure to be nondiagonal in flavor. The renormalization of flavor singlet and nonsinglet tensor bilinears are the same on the lattice through at least two-loop level and we may therefore safely use the $Z_T$ calculated here for any flavor structure of the tensor current [19].

The wave function renormalization may be calculated using either the incoming ($p_1$) or outgoing ($p_2$) quark propagators. In the RI-SMOM scheme [17] the momenta appearing in Eq. (2) satisfy the symmetric conditions $p_1 - p_2 = q$ and $p_1^2 = p_2^2 = q^2 = \mu^2$.

The amputated tensor vertex function $\Lambda_T$ is calculated by dividing $G_T$ on either side by the quark propagators: $\Lambda_T = S^{-1}(p_2)G_T S^{-1}(p_1)$. The tensor current renormalization factor, $Z_T$, that converts the lattice current into one in the momentum-subtraction scheme may then be defined as

$$\frac{Z_T}{Z_q} = \frac{1}{144} \text{Tr}(\Lambda_T^{\mu\nu}\Lambda_T^{\nu\mu}) .$$

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Renormalization factors taking the lattice to the RI-SMOM scheme, $Z_{T}^{\overline{\text{SMOM}}}$, can be converted to the more conventional choice of the $\overline{\text{MS}}$ scheme through a calculation in continuum perturbative QCD of the SMOM-to-$\overline{\text{MS}}$ matching. For the tensor renormalization this has now been performed to three-loop order [20,21]. The RI-SMOM to $\overline{\text{MS}}$ matching factor is

$$Z_{T}^{\overline{\text{MS}}/\overline{\text{SMOM}}} (\mu, n_f) = 1 - 0.21517295 \frac{\alpha_{\overline{\text{MS}}} (\mu)}{4 \pi}$$

$$- (43.3895 - 4.103279 n_f) \left( \frac{\alpha_{\overline{\text{MS}}} (\mu)}{4 \pi} \right)^2$$

$$- (1950.76(11) - 309.8285(28) n_f) + 7.063585(58) n_f^2 \left( \frac{\alpha_{\overline{\text{MS}}} (\mu)}{4 \pi} \right)^3. \quad (4)$$

Evaluating this expression for $n_f = 4$ gives:

$$Z_{T}^{\overline{\text{MS}}/\overline{\text{SMOM}}} (\mu) = 1 - 0.0171292 \alpha_{\overline{\text{MS}}} (\mu) - 0.1707952 \alpha^2_{\overline{\text{MS}}} (\mu)$$

$$- 0.415470(55) \alpha^3_{\overline{\text{MS}}} (\mu). \quad (5)$$

We also compare to results in the RI'-MOM scheme which has a simpler kinematic setup than the RI-SMOM scheme. No momentum is inserted at the vertex and therefore there is no quark momentum, i.e., $p_1 = p_2, \ q = 0$. RI'-MOM uses the same definitions of $Z_q$ and $Z_T$ in Eqs. (1) and (3). The RI'-MOM to $\overline{\text{MS}}$ conversion is also known through $\mathcal{O}(\alpha^3)$ for the tensor current renormalization factor [22]. For $n_f = 4$ the expression is

$$Z_{T}^{\overline{\text{MS}}/\overline{\text{MOM}}} (\mu) = 1 - 0.1976305 \alpha^2_{\overline{\text{MS}}} (\mu) - 0.4768793 \alpha^3_{\overline{\text{MS}}} (\mu). \quad (6)$$

This is very similar to the RI-SMOM to $\overline{\text{MS}}$ matching in Eq. (5) although with no $\mathcal{O}(\alpha^3)$ term in Landau gauge. The situation is then very different from the case for the mass renormalization factor where the RI-SMOM matching is considerably more convergent than the corresponding RI'-MOM matching [17,20,22–25].

We tabulate the values of $Z_{T}^{\overline{\text{MS}}/\overline{\text{SMOM}}}$ and $Z_{T}^{\overline{\text{MS}}/\overline{\text{MOM}}}$ in columns 2 and 3 of Table I for different $\mu$ values. We also give the values required to run the tensor renormalization factors in the $\overline{\text{MS}}$ scheme to a reference scale of 2 GeV, denoted $R^T (2 \text{ GeV} , \mu)$. These numbers are calculated using the three-loop tensor anomalous dimension [26].

The work of [28] compares RI'-MOM and RI-SMOM renormalization for various currents. In the discussion of the tensor current presented there, uncertainties associated with missing terms in the matching to the $\overline{\text{MS}}$ scheme were added to the renormalization factors. As [28] predates the results of [21] a larger uncertainty was included on the RI-SMOM tensor renormalization result of [28] than on the RI'-MOM result. As both $\overline{\text{MS}}$ conversion factors are now known to the same order in perturbation theory this issue has been removed for the comparison between the scheme. In Sec. IV we address the issue of remaining uncertainty from unknown higher order terms in the conversion factors through our fits.

### III. LATTICE CALCULATION OF $Z_{T}^{\overline{\text{SMOM}}}$ AND $Z_{T}^{\overline{\text{MOM}}}$

We use the highly improved staggered quark action for both valence and sea quarks. The use of staggered quarks with momentum-subtraction schemes requires some consideration as explained in [29]. As discussed there, we take physical momenta to lie in the reduced Brillouin zone $-\pi/2 \leq p_\mu \leq \pi/2$ and use momentum-space staggered quark fields at momenta $p^+ + \pi A$ where $A$ is a hypercubic vector of 1s and 0s. This multiplicity in momentum-space fields for a given physical momentum contains the staggered quark taste information. For each of these momenta we numerically solve the Dirac equation with a "momentum" source: $\hat{S} = e^{ip\cdot x}$ where $\hat{S}$ is the local tensor local tensor $[(\gamma_\mu \gamma_5 \otimes \xi_\mu \xi_5) \text{ in spin-taste notation}]$ vertex function is

$$\langle \chi(p_1^+ + \pi A) \left( \sum_x \tilde{\chi}(x)(-1)^{(x_0 + x_1)} \chi(x)e^{i(p_1^+ - p_2^+)^x}) \right. \right. \times \tilde{\chi}(p_2^+ + \pi B) \rangle \left. \left. = \frac{1}{n_{c\bar{c}} \times x_{c\bar{c}}} \sum S(p_1^+ + \pi A, x) e^{i(p_1^+ - p_2^+)^x} \right. \right. \times \left( -1 \right)^{x_0 + x_1 + x_2 + x_3 - x_0 - x_1 - x_2 - x_3} S^T(p_2^+ + \pi B, x), \quad (7)$$
making use of the $\gamma_5$ Hermiticity of $S$ in the last line. The elements of $\mathcal{B}$ are permuted compared to those of $B$ via $\mathcal{B} = B +_2 (1, 1, 1, 1)$ where $+_2$ denotes addition modulo 2.

We use the following kinematic setup, which obeys the symmetric conditions of the RI-SMOM scheme:

$$
ap_1' = \frac{2\pi}{L_x} \left( x + \theta \frac{1}{2}, 0, x + \theta \frac{1}{2}, 0 \right),$$
$$
ap_2' = \frac{2\pi}{L_x} \left( x + \theta \frac{1}{2}, -x - \theta \frac{1}{2}, 0, 0 \right),$$
$$
aq' = \frac{2\pi}{L_x} \left( 0, x + \theta \frac{1}{2}, x + \theta \frac{1}{2}, 0 \right).$$

$x$ is an integer and $\theta$ is the momentum twist applied with phased boundary conditions that we use to access arbitrary momenta [30]. For the single momentum in the RI'-MOM scheme we use $nap_1'$.

Our calculations are done on HISQ $n_f = 2 + 1 + 1$ gluon field ensembles generated by the MILC Collaboration [31,32], the details of which are given in Table II. On each ensemble we use 20 configurations except for ultrafine where only six configurations with stringent gauge fixing were available. We have checked, using other sets, that this small number of configurations is sufficient to achieve high precision given our use of momentum sources. In order to compensate for a potential underestimation of the uncertainty from the low statistics, however, we double the uncertainty on the $Z_T$ values on set 8.

Table II gives two values for the lattice spacing, reflecting the different approach to the physical quark mass limit that we take in the two parts of our calculation. Both approaches arrive at the same physical point, so this is simply a convenient choice away from the physical point. We label the two lattice spacing values $a$ and $\tilde{a}$. $a$ is determined from a calculation of $w_0/a$ [33] on each ensemble and varies as the sea quark masses are changed at fixed bare gauge coupling, $\beta$. $\tilde{a}$ is the value of the lattice spacing for physical sea quark masses at a given value of $\beta$ [11,27]. The latter definition is used for the calculation of $Z_T$ while the former is used to compute the $J/\psi$ tensor decay constant.

We use different definitions of the lattice spacing to reduce the effects of sea quark mass mistuning in the calculation. If we instead used a single definition of the lattice spacing we would have a steeper approach to the tuned sea quark mass point either in the renormalization factors or in the hadronic matrix elements. Hadronic matrix elements are sensitive to low energy scales and it is convenient to keep the value of $w_0$ fixed as the sea quark masses are varied, leading to values of $w_0/a$ that are dependent on the sea quark masses. As discussed in Appendix A of [27] the variation of hadronic quantities with the sea quark masses is similar to that of $w_0$ and so they do not vary much if $w_0$ is held fixed. Sea quark mass dependence in the hadronic quantity in lattice units is canceled by the variation of $w_0/a$. However, ultraviolet quantities such as renormalization factors have very weak sea quark mass dependence. Using $w_0/a$ values that vary with the sea masses therefore introduces unwanted dependance and so we choose to use $w_0/\tilde{a}$ defined in the physical sea quark mass limit. The sea quark mass dependence of RI-SMOM renormalization factors was studied in [10] using $w_0/\tilde{a}$ and indeed found to be tiny. We will see from the plots of our results in the next section that our strategy of using $a$ and $\tilde{a}$ does indeed lead to very little difference between results for physical and unphysical sea quark masses for the decay constant.

We define the RI-SMOM and RI'-MOM schemes at zero valence quark mass to remove mass-dependent nonperturbative contributions. In order to obtain values at zero valence mass we calculate $Z_T$ at three different quark masses and extrapolate to 0 using a polynomial fit in $am_{val}$:

### Table II

| Set | Label | $\beta$ | $w_0/a$ | $w_0/\tilde{a}$ | $L_s$ | $L_t$ | $am^{sea}_{\psi}$ | $am^{sea}_{\psi}$ | $am^{sea}_{\psi}$ | $am^{sea}_{\psi}$ |
|-----|-------|---------|---------|-----------------|------|------|-----------------|-----------------|-----------------|-----------------|
| 1*  | Very coarse (vc) | 5.80    | 1.1272(7) | 1.1265(31)   | 24   | 48   | 0.0064          | 0.064           | 0.828           | 0.873           |
| 2   |       | 6.00    | 1.3826(11) | 1.4055(33)  | 24   | 64   | 0.0102          | 0.0509          | 0.635           | 0.664           |
| 3*  | Coarse (c) | 6.00    | 1.4029(9)  | 1.4055(33)  | 32   | 64   | 0.00507         | 0.0507          | 0.628           | 0.650           |
| 4   |       | 6.00    | 1.4116(9)  | 1.4055(33)  | 48   | 64   | 0.001907        | 0.05252         | 0.6382          | 0.643           |
| 5*  | Fine (f) | 6.30    | 1.9330(20) | 1.9484(33)  | 48   | 96   | 0.00363         | 0.0363          | 0.430           | 0.439           |
| 6   |       | 6.30    | 1.9518(7)  | 1.9484(33)  | 64   | 96   | 0.00120         | 0.0363          | 0.432           | 0.433           |
| 7*  | Superfine (sf) | 6.72   | 2.8960(60) | 3.0130(56)  | 48   | 144  | 0.0048          | 0.024           | 0.286           | 0.274           |
| 8*  | Ultrafine (uf) | 7.00   | 3.892(12)  | 3.972(19)   | 64   | 192  | 0.00316         | 0.0158          | 0.188           | 0.194           |
The three valence masses that we use are \( am_{\text{sea}} \), \( 2am_{\text{sea}} \), \( 3am_{\text{sea}} \). This is the same procedure as was used in [10,35]. Figure 1 shows an example of the mass dependence of \( Z_T \) for both the lattice-to-SMOM matching factor, \( Z_{\text{SMOM}}^T \), and the lattice-to-MOM factor, \( Z_{\text{MOM}}^T \). The mass dependence reflects nonperturbative artifacts (condensates) appearing in \( Z_T \) with mass-dependent coefficients. We see that the dependence is very small for the SMOM case and less so, but still relatively benign, in the MOM case.

We collect our \( Z_{\text{SMOM}}^T \) results, extrapolated to zero valence mass, for various values of \( \mu \) in Table III. The correlation matrix for these different \( \mu \) values on each ensemble is also given. Our \( Z_{\text{MOM}}^T \) results are similarly collected in Table IV.

### IV. \( J/\psi \) Tensor Decay Constant

The \( J/\psi \) tensor decay constant, \( f_{J/\psi}^T \), is defined in an analogous way to the \( J/\psi \) vector decay constant \( f_{J/\psi}^V \). \( f_{J/\psi}^T \) parametrizes the vacuum to meson matrix element of a tensor current in the following way:

\[
(0|\bar{\psi}\sigma_{\alpha\beta}\psi|J/\psi) = if_{J/\psi}^T(\mu)(\epsilon_{\alpha p \beta} - \epsilon_{\beta p \alpha}).
\]  

\( \epsilon \) is the polarization vector of the \( J/\psi \), \( p \) is the \( J/\psi \) 4-momentum and \( \mu \) is the renormalization scale for the tensor decay constant. Note that the tensor decay constant is \( \mu \)-dependent, reflecting the anomalous dimension of the continuum tensor current and unlike the vector decay constant. It is also scheme dependent and we will give results in the \( \overline{\text{MS}} \) scheme.

If one of the indices of the tensor current is in the time direction, we can extract \( f_{J/\psi}^T \) from the 2-point tensor-tensor correlation function projected onto zero spatial momentum. We construct this as

\[
C_T(t) = \frac{1}{4} \sum_x \langle (-1)^{\eta_T(x)} \text{Tr}(S(x,0)S^\dagger(x,0)) \rangle.
\]

Here \( \eta_T(x) \) is a position-dependent phase remnant of \( \sigma_{\alpha\beta} \) resulting from the use of staggered quarks. This is the same

![Graph showing valence mass dependence of tensor current renormalization factors in the RI-SMOM (upper) and RI'-MOM (lower) schemes. The values shown are for \( \mu = 2 \) GeV on set 7. Both show reasonably mild dependence on the valence mass but the dependence is smaller for RI-SMOM.](image)

**Table III.** \( Z_{\text{SMOM}}^T \) values on the ensembles in Table II at different \( \mu \) values along with the correlation matrices for these different \( \mu \) values on a given set. \( Z_{\text{SMOM}}^T \) converts the lattice tensor current into the SMOM scheme.

| Set       | \( \mu = 2 \) GeV | \( \mu = 3 \) GeV | \( \mu = 4 \) GeV | correlation                                      |
|-----------|------------------|------------------|------------------|-------------------------------------------------|
| Very coarse | 1.07293(18)     | \ldots           | \ldots           | \begin{pmatrix} 1 & 0.824 \\ 0.824 & 1 \end{pmatrix} |
| Coarse    | 1.10035(28)     | 1.036117(92)     | \ldots           | \begin{pmatrix} 1 & 0.560 & 0.375 \\ 0.560 & 1 & 0.861 \\ 0.375 & 0.861 & 1 \end{pmatrix} |
| Fine      | 1.13250(14)     | 1.064991(56)     | 1.030967(30)     | \begin{pmatrix} 1 & 0.828 & 0.866 \\ 0.828 & 1 & 0.896 \\ 0.866 & 0.896 & 1 \end{pmatrix} |
| Superfine | 1.16641(40)     | 1.09808(12)      | 1.061844(57)     | \begin{pmatrix} 1 & 0.925 \\ 0.925 & 1 \end{pmatrix} |
| Ultrafine | 1.1791(17)      | 1.11629(64)      | \ldots           | \begin{pmatrix} 1 & 0.824 \\ 0.824 & 1 \end{pmatrix} |
phase as that appearing in Eq. (7), since we use the same local tensor current. We take $\beta$ to be in the temporal direction and average $\alpha$ over spatial directions.

We compute the correlation function of Eq. (11) on the full set of ensembles with parameters summarized in Table II. The valence $c$ quark masses are chosen to be close to those giving the experimental value of the $J/\psi$ mass [11]. We will allow for mistuning of the valence $c$ quark mass in our fits to extrapolate to the continuum limit. The decay constant is determined from the ground-state amplitude and energy according to

$$f(E) = e^{-Et} + e^{-E(L_t - t)}.$$  \hspace{1cm} (12)

The temporal oscillation term appears because of our use of staggered quarks. We perform the fit using standard Bayesian fitting techniques [36] with broad priors on the parameters, as in [11].

The $J/\psi$ tensor decay constant is then calculated from the ground-state amplitude and energy according to

$$f_{J/\psi}^T = Z_T \sqrt{\frac{2A^T_i}{E^t_i}}.$$  \hspace{1cm} (13)

Here the ground state energy, $E^t_i$, is the mass of the $J/\psi$ as we implement Eq. (10) for a $J/\psi$ at rest.

As we have used the local tensor current with taste $\xi_\alpha \xi_\tau$, $E^t_i$ is the mass of the $J/\psi$ of that taste. Because of taste splitting effects this is expected to differ from the local $J/\psi$ with taste $\xi_\tau$. The values of the local $J/\psi$ mass on the ensembles used here were given in [11] and we collect the values for the taste $\xi_\tau \xi_\tau$ in Table V. As taste-breaking effects are a discretization effect we should see the difference between the two masses $[\Delta(M_{J/\psi})]$ decrease as the lattice spacing is decreased. This is shown in Fig. 2. Note that even on the coarsest ensemble the difference is only 6 MeV, about 0.2% of the $J/\psi$ mass. A fit to the mass difference of the form

$$\Delta(M_{J/\psi})(a) = c_1 a s (1/a) (am_c)^2 + c_2 (am_c)^4$$  \hspace{1cm} (14)

is included in the figure. This is the expected form for taste effects as the HISQ action is improved to remove tree-level $(am_c)^2$ errors [4]. The fit works well, with a $\chi^2$/dof of 0.4.

The values of $af_{J/\psi}^T/Z_T$ extracted from our 2-point correlator fits on the ensembles in Table II are given in Table V.

An important goal of this analysis is to investigate the size of systematic effects arising from nonperturbative contamination of $Z_T$ and show how to remove them. Doing this requires analysis of a physical quantity sensitive to the tensor current renormalization, for which we use the $J/\psi$ tensor decay constant in the $\overline{MS}$ scheme at a reference scale of 2 GeV. This is obtained by taking the product of

| Set           | $\mu = 2$ GeV | $\mu = 3$ GeV | $\mu = 4$ GeV | correlation |
|---------------|---------------|---------------|---------------|-------------|
| Very coarse   | 1.08435(42)   | ...           | ...           | (1 0.637)   |
| Coarse        | 1.10970(58)   | 1.04631(16)   | ...           | (1 0.384 0.393) |
| Fine          | 1.13949(47)   | 1.06979(13)   | 1.037388(39)  | (1 0.384 1 0.609) |
| Superfine     | 1.17449(71)   | 1.10045(25)   | 1.063735(93)  | (1 0.103 0.155) |
| Ultrafine     | 1.1845(29)    | 1.1181(14)    | ...           | (1 0.234 0.234) |

TABLE IV. RI-MOM equivalents ($Z_T^{\overline{MOM}}$) of the RI-SMOM values in Table III.
FIG. 2. The taste splitting of the $J/\psi$ masses of tastes $\xi_p\xi_p$ and $\xi_p$ as a function of $(am_c)^2$. Clearly this difference decreases with the lattice spacing. The fit shown has the form of Eq. (14).

several quantities: the unrenormalized $J/\psi$ tensor decay constant $a f_{J/\psi}^T(Z_T)$ from Table V; the perturbative renormalization factor that converts this to a momentum-subtraction scheme at scale $\mu$ from Table III or Table IV (although for convenience here we use SMOM notation); the perturbative matching from the momentum-subtraction scheme to $\overline{\text{MS}}$

\[
 f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV}, \mu, a) = R^T(2 \text{ GeV}, \mu) Z_{T}^{\overline{\text{MS}}/\text{SMOM}}(\mu) 
 \times Z_{T}^{\text{SMOM}}(\mu, a) (a f_{J/\psi}^T/Z_T)/a. 
\]

Note that the first three factors above, combined, constitute $Z_{T}^{\overline{\text{MS}}}(2 \text{ GeV}, a)$ i.e., the renormalization factor that takes the decay constant from the lattice scheme to the $\overline{\text{MS}}$ scheme at a renormalization scale of 2 GeV, up to discretization effects and nonperturbative artifacts present in $Z_{T}^{\text{SMOM}}$.

We fit the results from Eq. (15) as a function of lattice spacing and $\mu$ values in order to obtain a physical value for $f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV})$ in the continuum limit. The fit form used is

This is designed to capture the lattice spacing and $\mu$ dependence of $Z_T$ as well as the discretization and quark mass effects in $a f_{J/\psi}^T/Z_T$. We take $\mu_{\text{ref}}$ to be 2 GeV and include results from $\mu$ values of 2, 3 and 4 GeV and multiple values of $a$.

The first square brackets of Eq. (16) allow for discretization effects in the raw lattice values for $a f_{J/\psi}^T$ through an even polynomial in powers of the $c$ quark mass in lattice units, $am_c$, as appropriate for a charmonium quantity. The next terms in that bracket then account for mistuning of the sea quark masses away from their physical values and mistuning of the valence $c$ quark mass, respectively. This part of the fit is the same form as that used for the $J/\psi$ vector decay constant in [11].

The second set of square brackets in Eq. (16) allows for effects from the lattice calculation of $Z_T$ in the momentum-subtraction scheme at scale $\mu$. We expect discretization effects in this case to appear as even powers of $\bar{a} \mu/\pi$. The missing $\alpha_4^s$ term in the matching from momentum-subtraction to $\overline{\text{MS}}$ schemes is allowed for with coefficient $c_{a1}$ and a similar effect for the running, with coefficient $c_{a2}$. The terms on the final line allow for the condensate contamination of $Z_T$ coming from its nonperturbative calculation on the lattice. The condensate contamination (discussed in Sec. II) and the running from $\mu$ to 2 GeV in the $\overline{\text{MS}}$ scheme. These last two factors are given in Table I. This gives us the results that we will fit:

\[
 f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV}, \mu, a) = R^T(2 \text{ GeV}, \mu) Z_{T}^{\overline{\text{MS}}/\text{SMOM}}(\mu) 
 \times Z_{T}^{\text{SMOM}}(\mu, a) (a f_{J/\psi}^T/Z_T)/a. 
\]

We take priors on all the coefficients of the fit in Eq. (16) of $0 \pm 1$, except for three terms. We take a prior of $0 \pm 0.1$ for $h_c^{\text{sea}}$ based on [11], and $0 \pm 0.5$ for $c_{a1}$ and $0 \pm 0.4$ for $c_{a2}$ based on the lower order terms in Eqs. (5) and (6) and in [26]. We also take $0.4 \pm 0.1$ GeV for the prior for the physical value of $f_{J/\psi}^V(\overline{\text{MS}}, 2 \text{ GeV})$ based on the expectation that it should be close in value to $f_{J/\psi}^V$. We include five terms in each of the sums over discretization effects and three terms in the sum over condensate contributions.

Our results using the RI-SMOM $Z_T$ from Sec. III with the fit of Eq. (16) are shown in Fig. 3. The $\chi^2$/dof is 0.19 giving a continuum value with condensate contributions from $Z_T$ removed of

\[
 f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV}) = 0.3889(33) \text{ GeV (int.SMOM)}. 
\]
The phrase “int. SMOM” here indicates that the result uses the intermediate RI-SMOM scheme. Note that the $\chi^2$/dof increases significantly, to 2.5, if the $\mu$-dependent terms that survive the continuum limit, that is condensate terms and $\alpha_s^4$ terms, are removed from the fit.

The black hexagon in Fig. 3 shows this result [the $f_{J/\psi}^{T}(\overline{\text{MS}}, 2 \text{ GeV})$ fit parameter in Eq. (16)]. This is the physical value of the tensor decay constant, with discretization and quark-mass-mistuning effects extrapolated away and condensate contributions and $\alpha_s^4$ errors removed. Note that this value is lower than the value obtained from simply taking the continuum limit of the 2 GeV results (blue line), mainly because of condensate contamination at $\mu = 2 \text{ GeV}$. This underlines the necessity of performing the calculation at multiple values of $\mu$ in the RI-SMOM scheme before running all of the results to a reference scale, in this case 2 GeV, in order to determine and remove systematic $\mu$-dependent errors.

The difference between the black hexagon and the continuum limit of the lines for the different $\mu$ values can be thought of as a correction that needs to be applied to the $Z_T$ values that connect the lattice results and the $\overline{\text{MS}}$ value at 2 GeV [i.e., $Z_T^{\overline{\text{MS}}}(2 \text{ GeV}, \alpha)$] that combines the first three factors on the right-hand side of Eq. (15) so that they are independent of $\mu$. This will give a corrected $Z_T^{\overline{\text{MS}}}$ that can then be used in future calculations. The correction depends on the intermediate momentum-subtraction scheme used and the condensate contamination that it has as well as $\alpha_s^4$ errors in the matching to $\overline{\text{MS}}$.

We define a $\mu$-dependent subtraction, $c^{\text{SMOM}}(\mu)$, to apply to the values of $Z_T^{\overline{\text{MS}}}$ from the combination of the $c_{\text{cond}}^{(j)}$ terms in Eq. (16) along with the $c_{a1}$ and $c_{a2}$ terms. It is difficult for the fit to completely separate these different $\mu$-dependent contributions and as a result the individual coefficients are not as well determined as the total correction (because the fit parameters are correlated). The full correction is shown in Fig. 4 plotted against $\mu^2$, and significantly nonzero values are seen across the $\mu^2$ range, with the correction at the $\sim 1.5\%$ level for $\mu = 2 \text{ GeV}$. These values, and their correlation matrix, are given in Table VI. If we extract the condensate contributions to the correction separately, values with the same central values are obtained but with uncertainties that are about 40% larger at $\mu = 2 \text{ GeV}$. If the corrected $Z_T$ value is denoted $Z_T^{\overline{\text{MS}}, \epsilon}$ and the uncorrected value $Z_T^{\overline{\text{MS}}, a}$,

$$Z_T^{\overline{\text{MS}}, \epsilon}(2 \text{ GeV}, \alpha) = Z_T^{\overline{\text{MS}}, a}(2 \text{ GeV}, \alpha) - c^{\text{SMOM}}(\mu).$$

A corrected value for $Z_T^{\overline{\text{MS}}}$ is then readily derived using the results in Tables III, I and VI.

We also examine $f_{J/\psi}^{T\nu}$ using a tensor current renormalization obtained in the RI’-MOM scheme on the lattice. In this case we use the conversion to $\overline{\text{MS}}$ in Eq. (6) and calculate the RI’-MOM equivalent of Eq. (15). The results and the fit to Eq. (16) are shown in Fig. 5. We see that the final continuum result with condensate contributions and $\alpha_s^4$ errors removed agrees with that given by intermediate

| $\mu$ (GeV) | $c^{\text{SMOM}}(\mu)$ | Correlation matrix |
|------------|----------------|------------------|
| 2          | 0.0153(36)    | 1.0              | 0.9889 0.9249 |
| 3          | 0.0074(24)    | 0.9889           | 1.0 0.9708   |
| 4          | 0.0041(16)    | 0.9249           | 0.9708 1.0   |

TABLE VI. Values, with uncertainties, and correlation matrix for the correction $c^{\text{SMOM}}(\mu)$ to be applied to renormalization factors for the tensor current when using the RI-SMOM intermediate scheme.
RI-SMOM renormalization factors. The $\chi^2$/dof of this fit is 0.4 giving a final result of
\[ f^T_{J/\psi}(\overline{\text{MS}}, 2 \text{ GeV}) = 0.3847(37) \text{ GeV (int. MOM)}. \] (19)

Dropping both condensate and $\alpha_s^4$ terms from the fit increases the $\chi^2$/dof here to 8.2.

There is more difference between the 2 GeV and the 3 and 4 GeV values in the RI’-MOM case than in the RI-SMOM case. This is reflected in the larger coefficient for the $1/\mu^4$ condensate term in the fit of $-1.19(49)$. The size of the correction, $C_{\text{MOM}}(\mu)$, needed for $Z_T$ when the RI’-MOM scheme is used is shown in Fig. 6. It can be seen that the correction is larger than for the RI-SMOM case, because of larger condensate effects. It is not surprising that condensate effects are larger in the RI’-MOM scheme than in RI-SMOM since this has been shown to be true in several other renormalization factors in the past [16,35] and is also consistent with the mass dependence seen in Fig. 1.

Since the discretization effects in $f^T_{J/\psi}$ are similar to those in $f^V_{J/\psi}$ on the same set of gluon field ensembles we expect to be able to extract the ratio of the two decay constants to a higher precision than can be obtained from the individual quantities. We may also be able to see a clearer indication of the size of nonperturbative effects in the ratio.

We show the ratio of $f^T_{J/\psi}/f^V_{J/\psi}$ in Fig. 7 using $Z_T$ and $Z_V$ determined in the RI-SMOM scheme. We neglect any correlations between the raw values of the decay constants on each lattice ensemble because the statistical uncertainties are so small. We fit the values of the ratio to Eq. (16) and obtain a result for the ratio in the continuum limit with nonperturbative contamination effects removed of
\[ \frac{f^T_{J/\psi}(\overline{\text{MS}}, 2 \text{ GeV})}{f^V_{J/\psi}} = 0.9569(52) (\text{int. SMOM}). \] (20)

The fit has a $\chi^2$/dof of 0.2.

As discussed in [35] the RI-SMOM $Z_V$ contains no nonperturbative contamination because of the protection of the Ward-Takahashi identity and likewise no perturbative matching of SMOM to $\overline{\text{MS}}$ is needed. Therefore the condensate and $\alpha_s^4$ terms returned by the fit to the ratio of the tensor and vector $J/\psi$ decay constants should agree with those from the fit to just the tensor decay constant. We find that this is the case for each coefficient individually and for the $Z_T$ correction factor obtained from their combination which we show for the ratio fit in Fig. 8.

Because the RI’-MOM determination of $Z_V$ has condensate contamination (since it is not protected by a
Ward-Takahashi identity [35]) and perturbative matching is needed to reach $\overline{\text{MS}}$ we cannot perform the same analysis for that case. We give an error budget for our result for the decay constant ratio $f_{J/\psi}^T = f_{J/\psi}^V$ in Table VII. We can leverage this ratio and the vector decay constant determined in [11] to get a slightly more precise value of the tensor decay constant:

$$f_{J/\psi}^T(\overline{\text{MS}}, 2 \text{ GeV}) = 0.3927(27) \text{ GeV (int. SMOM)}.$$ (21)

The vector decay constant result of [35] includes QED effects from the nonzero electric charge of the valence charm quarks. We have not included any electromagnetic effects here. However, the QED effect on the vector decay constant was at the 0.2% level and we expect some cancellation of these effects in the decay constant ratio, so we neglect these effects here.

V. DISCUSSION: $f_{J/\psi}^T$

As discussed in Sec. I there is no experimental observable available to which we can compare our tensor current decay constant value. Theoretical results using light-cone wave functions were presented in [37] and using QCD sum rules in [12]. A lattice QCD result using twisted-mass quarks on gluon field ensembles with only $u/d$ quarks in the sea ($n_f = 2$) was also given in [12]. The RI'-MOM scheme was used to renormalize the lattice tensor and vector currents in that case, without studying or removing nonperturbative condensate contamination. We compare our results to these in Fig. 9 where the reduction in uncertainty that we have achieved here can clearly be seen.

A comparison plot of values of the decay constant ratio $f_{J/\psi}^T(2 \text{ GeV})/f_{J/\psi}^V$ is shown in Fig. 10. This ratio is expected to be below 1 [12] but we see that earlier results were not able to demonstrate this conclusively. Our value for the ratio is 8σ below 1. The value that we obtain for the ratio is just over 1σ lower than the sum rules determination of [12] and is over 2σ lower than the lattice QCD result of that work (using their σ values). In the lattice QCD calculation both the tensor and vector current were renormalized in the RI'-MOM scheme without accounting for nonperturbative contamination. Our results indicate that this could lead to a discrepancy with our results of the size seen.

VI. DISCUSSION: $Z_T$

In the discussion presented above in Sec. IV we ran all of our results, after converting to $\overline{\text{MS}}$, to a common scale of

![Figure 8](image)

**FIG. 8.** Correction $C_{\text{SMOM}}(\mu)$ for the tensor current renormalization factor, $Z_T$, determined from a fit to the ratio of the $J/\psi$ tensor and vector decay constants using intermediate renormalization factors in the RI-SMOM scheme. This agrees with the results shown in Fig. 4, as expected because the lattice RI-SMOM vector current renormalization factor has no condensate contamination [35].

![Figure 9](image)

**FIG. 9.** A comparison plot of results for the tensor $J/\psi$ decay constant in the $\overline{\text{MS}}$ scheme at a scale of 2 GeV. The top two results are from this work [using both RI-SMOM (Eq. (21)) and RI'-MOM (Eq. (19)) intermediate schemes] and then we include results from [12,37].

| Effect | Value |
|--------|-------|
| $Z_T$ | 0.12 |
| $Z_V$ | 0.14 |
| Missing $\alpha_s^4$ term | 0.06 |
| Statistics | 0.41 |
| Sea mistuning | 0.04 |
| Condensates | 0.07 |
| Total | 0.54 |

**TABLE VII.** Error budget for the ratio of the $J/\psi$ vector and tensor decay constants. “Statistics,” the dominant uncertainty, refers to statistical errors in the amplitudes needed for the decay constants. The uncertainties coming from the renormalization factors, $Z_T$ and $Z_V$, are much smaller and are dominated by the contribution from the (doubled) statistical uncertainties on the low statistics ultrafine lattices, set 8. The “Missing $\alpha_s^4$” and “Condensates” error contributions come from the terms in the fit from which the $Z_T$ correction (discussed in the text) is constructed.
2 GeV and then determined and subtracted a correction that depends on $\mu$. This correction needs to be applied to our $Z_T$ values for future use. The scale of 2 GeV allows us to compare directly to the results of \cite{12} in Sec. V. However, another scale is useful when computing form factors for semileptonic $B$ decay processes. Then differential rates are calculated as functions of products of the form factors and appropriate Wilson coefficients of the weak Hamiltonian. These Wilson coefficients are scale dependent and are typically calculated at a scale equal to the $b$ pole mass, 4.8 GeV, see for example \cite{38}. We therefore present our $Z_T$ values run to this scale. If the $b$ quark running mass of 4.2 GeV were used instead of the pole mass then the values would be approximately 1% larger.

In Table VIII we give the corrected results for $Z_T$ in the MS scheme at a scale equal to the $b$ quark pole mass calculated from intermediate values of $Z_T$ in the RI-SMOM scheme at 2 and 3 GeV. We use a notation $Z_T(\mu_{\text{SMOM}}|\mu_{\text{MS}})$ where $\mu_{\text{SMOM}}$ is the scale at which the RI-SMOM calculation was performed and $\mu_{\text{MS}}$ is the final scale at which the MS result is presented. It can be seen that the addition of the correction results in $Z_T$ values that agree for different intermediate scales once run to the same final scale (this would not be true for uncorrected values). We also give the correlations between these numbers in Table IX.

\section{VII. Conclusions}

We have shown here that it is possible to renormalize lattice tensor currents to give accurate results for continuum matrix elements in the MS scheme using nonperturbative determination of intermediate renormalization factors in momentum-subtraction schemes. A key requirement is that the nonperturbative renormalization factors should be obtained at multiple values of the renormalization scale, $\mu$, so that $\mu$-dependent nonperturbative (condensate) contamination of $Z_T$ can be fitted and removed. This contamination would otherwise give a systematic error of 1.5\% using the RI-SMOM scheme and 3\% using the RI'-MOM scheme in our calculation.

In order to do this we have determined the $J/\psi$ tensor decay constant, $f_{J/\psi}$ so that we can study the continuum limit of a tensor current matrix element. Using $n_f = 2 + 1 + 1$ HISQ lattices and the local tensor current, we obtain a 0.7\%-accurate value for $f_{J/\psi}$ of [repeating Eq. (21)]

\begin{table}[h]
\centering
\caption{ZT values converting lattice results involving the tensor current to the MS scheme, run to a renormalization scale of the $b$ quark pole mass. The notation $Z_T(\mu_1|\mu_2)$ indicates that the intermediate $Z_T$ has been calculated in the RI-SMOM scheme at a scale of $\mu_1$ and then converted to the MS scheme and run to a scale of $\mu_2$. The superscript denotes that these renormalization constants have been corrected for nonperturbative artifacts and $a_s^2$ errors in $Z_T^{\text{SMOM}}$ as described in the text. The results with intermediate scales of 2 and 3 GeV then agree well with each other and either can be used.}
\begin{tabular}{lcc}
\hline
Set & $Z_T(2 \text{ GeV}|m_b)$ & $Z_T(3 \text{ GeV}|m_b)$ \\
\hline
vc & 0.9493(42) & \ldots \\
c & 0.9740(43) & 0.9707(25) \\
f & 1.0029(43) & 0.9980(25) \\
sf & 1.0342(43) & 1.0298(25) \\
uf & 1.0476(42) & 1.0456(25) \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Correlation matrix of the corrected $Z_T$ values from Table VIII. These correlations are large because the matching, running and correction terms are all correlated.}
\begin{tabular}{cccccccccc}
\hline
 & (vc, 2) & (c, 2) & (f, 2) & (sf, 2) & (uf, 2) & (c, 3) & (f, 3) & (sf, 3) & (uf, 3) \\
\hline
(vc, 2) & 1.0 & 0.99750 & 0.99854 & 0.99475 & 0.93231 & 0.98398 & 0.98611 & 0.98713 & 0.96385 \\
(c, 2) & 0.99750 & 1.0 & 0.99777 & 0.99430 & 0.93294 & 0.98314 & 0.98371 & 0.98487 & 0.96243 \\
(f, 2) & 0.99854 & 0.99777 & 1.0 & 0.99605 & 0.93562 & 0.98045 & 0.98323 & 0.98423 & 0.96263 \\
(sf, 2) & 0.99475 & 0.99430 & 0.99605 & 1.0 & 0.93197 & 0.97361 & 0.97632 & 0.98097 & 0.95627 \\
(uf, 2) & 0.93231 & 0.93294 & 0.93562 & 0.93197 & 1.0 & 0.90439 & 0.90777 & 0.90941 & 0.96855 \\
(c, 3) & 0.98398 & 0.98314 & 0.98045 & 0.97361 & 0.90439 & 1.0 & 0.99909 & 0.99807 & 0.96824 \\
(f, 3) & 0.98611 & 0.98371 & 0.98323 & 0.97632 & 0.90777 & 0.99909 & 1.0 & 0.99868 & 0.96951 \\
(sf, 3) & 0.98713 & 0.98487 & 0.98423 & 0.98097 & 0.90941 & 0.99807 & 0.99868 & 1.0 & 0.96909 \\
(uf, 3) & 0.96383 & 0.96243 & 0.96263 & 0.95627 & 0.96855 & 0.96824 & 0.96951 & 0.96909 & 1.0 \\
\hline
\end{tabular}
\end{table}
This uses our preferred intermediate RI-SMOM scheme and makes use of the determination of the ratio of tensor to vector decay constants and the fact that the vector current renormalization is protected by the Ward-Takahashi identity in this scheme [35]. We also obtain a 0.5%-accurate value for the ratio itself [repeating Eq. (20)],

\[
\frac{f_J^{T}}{f_J^{V}} = 0.9569(52) \text{ (int. SMOM).} \quad (23)
\]

This shows unequivocally that the ratio is less than 1.

Finally, in Tables VIII and IX, we give \(Z_T\) renormalization factors that can be used, for example, in a future determination (underway) of the tensor form factor for the rare flavor-changing neutral current process \(B \to K\ell^{+}\ell^{-}\) using HISQ quarks. These \(Z_T\) values take results determined with the local HISQ lattice tensor current and convert them into values in the \(\overline{\text{MS}}\) scheme at the scale of \(m_b\), to be multiplied by Wilson coefficients from the effective weak Hamiltonian determined at this scale. We have corrected these \(Z_T\) values so that they are free of the systematic error from condensate contamination of the intermediate momentum-subtraction scheme.

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