Inversion formulas for the linearized impedance tomography problem

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Abstract

We consider the linearized electrical impedance tomography problem in two dimensions on the unit disk. By a linearization around constant coefficients and using a trigonometric basis, we calculate the linearized Dirichlet-to-Neumann operator in terms of moments of the conduction coefficient of the problem. By expanding this coefficient into angular trigonometric functions and Legendre-Müntz polynomials in radial coordinates, we can find a lower-triangular representation of the parameter to data mapping. As a consequence, we find an explicit solution formula for the corresponding inverse problem. Furthermore, we also consider the problem with boundary data given only on parts of the boundary while setting homogeneous Dirichlet values on the rest. We show that the conduction coefficient is uniquely determined from incomplete data of the linearized Dirichlet-to-Neumann operator with an explicit solution formula provided.

1 Introduction

A classical parameter identification problem is to reconstruct certain parameter function in a second-order elliptic equation from multiple measurements of the boundary values and boundary fluxes of the solution.

Specifically, we consider in this article two types of elliptic equations on the unit ball in $\mathbb{R}^2$. Define the differential operators

$$\mathcal{L}_1 u = -\nabla.(\gamma \nabla) u \quad \mathcal{L}_2 u = -\Delta + cu,$$

where $\nabla.$ denotes divergence, $\nabla$ is the gradient, and $\gamma > 0$ and $c$ are sufficiently regular functions.

We may associate to each operator the solutions to the Dirichlet problem on the unit ball

$$\mathcal{L}_1 u = 0 \quad u = f \quad \text{on } \partial \Omega,$$

or in the second case,

$$\mathcal{L}_2 u = 0 \quad u = f \quad \text{on } \partial \Omega,$$

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where \( \Omega = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1 \} \) and \( f \) is a sufficiently regular function.

Under well-known conditions on \( \gamma \) or \( c \) and \( f \), these problems have a solution in \( H^1(\Omega) \). For instance, for \( f \in H^{1/2}(\partial \Omega) \) and \( c_1 \leq \gamma \leq c_0 \) almost everywhere, a solution in \( H^1 \) to (2) exists. Also, if \( f \in H^{1/2}(\partial \Omega) \) and, e.g., \( 0 \leq c \leq c_1 \), then a solution to (2) exists; see, e.g., [5].

For fixed parameter functions \( \gamma \) (respectively \( c \)), we may consider the Dirichlet-to-Neumann mapping,

\[
\Lambda_{1, \gamma} : f \mapsto \gamma \frac{\partial}{\partial n} u_f, \quad \Lambda_{2, c} : f \mapsto \frac{\partial}{\partial n} u_f,
\]

where \( u_f \) is the solution to (1) for \( \Lambda_{1, \gamma} \) and (2) for \( \Lambda_{2, c} \), respectively. Under mild conditions, these mappings are continuous from \( H^{1/2}(\partial \Omega) \) to \( H^{-1/2}(\partial \Omega) \).

The classical inverse problem of electrical impedance tomography, originating in the famous paper by Calderón [4], asks to reconstruct the parameter \( c \) from knowledge of the mapping \( \Lambda_{1, \gamma} \), i.e., from all pairs of \((f, \gamma \frac{\partial}{\partial n} u_f)\) of Dirichlet and Neumann values. The similar problem has been stated also for the case of \( L_2 \) (i.e., the Schrödinger equation), where \( c \) is sought to be found from \( \Lambda_{2, c} \).

Both are well-studied and classical inverse problems for partial differential equations; the central difficulty lies in the fact that only the boundary is accessible for measurements, while \( \gamma \) is sought to be reconstructed in the interior. Concerning the unique identifiability of \( \gamma \) (or \( c \)) from the Dirichlet-to-Neumann map, several landmark papers were published, for instance, by Sylvester and Uhlmann [12, 11], Nachman [9], and Astala and Päivärinta [1]. An overview of results and related problems can be found in the classical book by Isakov [5] as well as in the review articles [2, 13].

In this article we consider only the linearized versions of these problems, namely, to find \( \gamma \) or \( c \) when the Dirichlet-to-Neumann maps are linearized (with respect to \( \gamma, c \)) around a constant. It will be shown, amongst others, that one can find explicit reconstruction formulas in these cases.

## 2 Problem setup

Considering the problems (1) and (2), it is well-known that certain differences of Dirichlet-to-Neumann maps can be expressed as energy integrals: for \( f, g \in H^{1/2}(\partial \Omega) \), it holds that [5, Eq. (5.0.3)]

\[
\langle (\Lambda_{1, \gamma+1} - \Lambda_{1, 1}) f, g \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} \gamma(x) \nabla u_{f, \gamma+1}(x) \cdot \nabla u_{g, 1}(x) dx, \quad (3)
\]

\[
\langle (\Lambda_{2, c} - \Lambda_{2, 0}) f, g \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} c(x) u_{f, c}(x) u_{g, 0}(x) dx, \quad (4)
\]

where \( u_{f, \gamma+1} \) is the solution to (1) with coefficient \( \gamma + 1 \) and Dirichlet data \( f \), and \( u_{g, 1} \) is the solution to (1) with coefficient 1 (i.e., the Laplace equation) and Dirichlet data \( g \). Similarly \( u_{f, c}, u_{g, 0} \) are the solutions to (2) with coefficient \( c \) and Dirichlet values \( f \) and 0 (Laplace equation) and Dirichlet values \( g \), respectively.

Note that the right hand side in (3–4) depends in a nonlinear way on the parameter through the functions \( u_{f, \gamma+1} \) and \( u_{f, c} \). Thus, in a next step, we consider a linearization with respect to \( \gamma \) or \( c \) of the right-hand side around the
constant $\gamma = 1$ and $c = 0$. We thus view only small/moderate perturbation of $\gamma$, respectively $c$, around a constant conductivity to be of interest. This yields, the linearized impedance tomography problem with the following operators.

$$\langle \Lambda_1', f, g \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} \gamma(x) \nabla u_{f,1} \cdot \nabla u_{g,1} dx,$$

(5)

$$\langle \Lambda_2', f, g \rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega} c(x) u_{f,0} u_{g,0} dx.$$

(6)

It is well-known that under mild conditions, these operators correspond to the linearization of the associated parameter-to-data mappings. In fact, e.g., for $\gamma, c \in L^\infty(\Omega)$, one can verify that this is indeed the output of the Fréchet-derivative of these mappings.

**Definition 1.** The linearized impedance tomography problem for (1) is the problem to find the function $\gamma$ from the values $\{ \langle \Lambda_1', f, g \rangle \mid f, g \in H^{1/2}(\partial \Omega) \}$. The linearized tomography problem for (2) is the problem to find the function $c$ from the values $\{ \langle \Lambda_2', f, g \rangle \mid f, g \in H^{1/2}(\partial \Omega) \}$.

Of course, it is enough, to know $\langle \Lambda_1', f, g \rangle$ for all $f, g$ out of a basis of $H^{1/2}(\partial \Omega)$. In the next section, we find a formula for these operators in the trigonometric basis.

### 3 Linearized Dirichlet-to-Neumann maps in the trigonometric basis

Specifically, we now consider $\Lambda_1', \Lambda_2'$, when these operator are applied to trigonometric functions. That is, we consider the family of functions (living on the boundary of the unit disc)

$$\{ \cos(n\phi) \mid n \in \mathbb{N}_0, n \geq 0 \} \cup \{ \sin(n\phi) \mid n \in \mathbb{N}, n \geq 1 \}, \quad \phi \in [0, 2\pi].$$

These functions consist a basis of the space $H^{1/2}(\partial \Omega)$. We note that for the impedance tomography problem, we do not have to include $\cos(n\phi)$ for $n = 0$, i.e., the constant function, because it is in the nullspace of the Dirichlet-to-Neumann map and hence does not provide any information.

If $f = \cos(n\phi)$ or $f = \sin(n\phi)$, then the corresponding solutions to (1) and (2) for the parameter $\gamma = 1$ or $c = 0$ are given in polar coordinates as

$$u_f(r, \phi) = r^n \cos(n\phi) \quad \text{or} \quad u_f(r, \phi) = r^n \sin(n\phi),$$

(7)

respectively, where $r = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$.

**Definition 2.** Define the coefficients of the linearized Dirichlet-to-Neumann operators $\Lambda_1', \Lambda_2'$ in the trigonometric basis as follows:

- $K^{ij}_{cc} := \langle \Lambda_1' \cos(i\phi), \cos(j\phi) \rangle \quad i, j \in \mathbb{N}$,
- $K^{ij}_{ss} := \langle \Lambda_1' \sin(i\phi), \sin(j\phi) \rangle \quad i, j \in \mathbb{N}$,
- $K^{ij}_{sc} := \langle \Lambda_1' \sin(i\phi), \cos(j\phi) \rangle \quad i, j \in \mathbb{N}$,
- $K^{ij}_{cs} := \langle \Lambda_1' \cos(i\phi), \sin(j\phi) \rangle \quad i, j \in \mathbb{N}$.
and
\[ J_{i,j}^{cs} := \langle \Lambda' \cos(i\phi), \sin(j\phi) \rangle \quad i, j \in \mathbb{N}_0, \]
\[ J_{i,j}^{ss} := \langle \Lambda' \sin(i\phi), \sin(j\phi) \rangle \quad i, j \in \mathbb{N}, \]
\[ J_{i,j}^{sc} := \langle \Lambda' \sin(i\phi), \cos(j\phi) \rangle \quad i \in \mathbb{N}, j \in \mathbb{N}_0, \]
\[ J_{i,j}^{cc} := \langle \Lambda' \cos(i\phi), \sin(j\phi) \rangle \quad j \in \mathbb{N}, i \in \mathbb{N}_0. \]

Similarly, we may represent \( \gamma \) and \( c \) in polar coordinates as a Fourier series with respect to the angle coordinates:
\[ \gamma(r, \phi) = a_0 + \sum_{n=1}^{\infty} a_n(r) \cos(n\phi) + b_n(r) \sin(n\phi), \quad (8) \]
\[ c(r, \phi) = a_0 + \sum_{n=1}^{\infty} a_n(r) \cos(n\phi) + b_n(\phi) \sin(n\phi). \quad (9) \]

In order for \( \gamma, c \) to be \( L^2(\Omega) \) functions, it is necessary and sufficient that
\[ \sum_{n=0}^{\infty} \int a_n(r)^2 r dr + \sum_{n=1}^{\infty} \int b_n(r)^2 r dr < \infty. \quad (10) \]

We can now express the parameter-to-data operator in the trigonometric basis.

**Proposition 1.** Let \( \gamma \) be given by (8). Then
\[ K_{i,j}^{cc} = K_{i,j}^{ss} = ij\pi \zeta_{i,j} \int_0^1 r^{i+j-1} a_{|i-j|}(r) dr \quad i, j \geq 1, \]
\[ K_{i,j}^{sc} = K_{j,i}^{cs} = ij\pi \text{sign}(j-i) \int_0^1 r^{i+j-1} b_{|i-j|}(r) dr \quad i, j \geq 1, \]
where
\[ \zeta_{i,j} = \begin{cases} 1 & |i-j| \geq 1, \\ 2 & |i-j| = 0, \end{cases} \]
and \( \text{sign} \) is the sign function with \( \text{sign}(0) = 0 \).

**Proof.** It can be verified that for \( u = r^i \cos(i\phi) \) and \( v = r^j \cos(j\phi) \) and for \( u = r^i \sin(i\phi) \) and \( v = r^j \sin(j\phi) \) given in polar coordinates, we have
\[ \nabla u \cdot \nabla v = ij r^{i+j-2} \cos((i-j)\phi). \]

From
\[ \int_0^{2\pi} \cos((i-j)\phi) \cos(k\phi) d\phi = \begin{cases} \pi & k = \pm |i-j| \neq 0, \\ 2\pi & k = \pm |i-j| = 0, \\ 0 & \text{else}, \end{cases} \]
and an integration in polar coordinates gives the identity for \( K^{cc} \) and \( K^{ss} \).
(Note that the terms \( \sin(k\phi) \) in \( \gamma \) do not contribute because of the identity \( \int_0^{2\pi} \cos((i-j)\phi) \sin(k\phi) d\phi = 0 \). For \( u = r^i \cos(i\phi) \) and \( v = r^j \sin(j\phi) \), we have
\[ \nabla u \cdot \nabla v = ij r^{i+j-2} \sin((j-i)\phi), \]
Proof. The proof is based on the following integral that follow from trigonometric identities and orthogonality: for $i, j, k \in \mathbb{N}$ and $i, j, k \geq 0$,

\[
\int_{0}^{2\pi} \cos(k\phi) \cos(i\phi) \cos(j\phi) d\phi = \frac{\pi}{2} (\delta_{0,k+i+j} + \delta_{0,k-i+j} + \delta_{0,k+i-j} + \delta_{0,k-i-j}),
\]

\[
\int_{0}^{2\pi} \cos(k\phi) \sin(i\phi) \sin(j\phi) d\phi = \frac{\pi}{2} (\delta_{0,k+i-j} + \delta_{0,k-i+j} - \delta_{0,k+i+j} - \delta_{0,k-i-j}),
\]

and

\[
\int_{0}^{2\pi} \sin(k\phi) \cos(i\phi) \cos(j\phi) d\phi = 0, \quad \int_{0}^{2\pi} \sin(k\phi) \sin(i\phi) \sin(j\phi) d\phi = 0,
\]

where $\delta_{i,j}$ denotes the Kronecker delta. Thus, denote the zero-extension of the coefficients $a_{k}$ to negative indices $k$ by $\hat{a}_{k}$, we have

\[
J_{i,j}^{c_{k}} = \frac{\pi}{2} \sum_{k=0}^{\infty} \int_{0}^{1} r^{i+j+1} a_{k}(r) dr \left( \delta_{0,k+i+j} + \delta_{0,k-i+j} + \delta_{0,k+i-j} + \delta_{0,k-i-j} \right)
\]

\[
= \frac{\pi}{2} \int_{0}^{1} r^{i+j+1} \hat{a}_{-i-j}(r) + \hat{a}_{-i-j}(r) + \hat{a}_{i+j}(r) dr + \hat{a}_{i+j}(r) dr
\]

\[
= \frac{\pi}{2} \int_{0}^{1} r^{i+j+1} a_{i+j}(r) dr + \frac{\pi}{2} \int_{0}^{1} r^{i+j+1} a_{i-j}(r) dr |i-j| \geq 1 \land i, j \geq 1,
\]

\[
\frac{\pi}{2} \int_{0}^{1} r^{i+j+1} a_{i-j}(r) dr |i-j| = 0 \land (i,j) \neq (0,0),
\]

\[
3 \frac{\pi}{2} \int_{0}^{1} r^{i+j+1} a_{i-j}(r) dr |i-j| = 0 \land i = j = 0.
\]
Proposition 3. Besides this, we have the following nontrivial conditions.

\begin{align*}
J_{i,j}^{s\cdot s} &= \frac{\pi}{2} \sum_{k=0}^{\infty} \int_0^1 r^{i+j+1} a_k(r) dr \left( \delta_{0,k+i-j} + \delta_{0,k-i+j} - \delta_{0,k+i+j} - \delta_{0,k-i-j} \right) \\
&= \frac{\pi}{2} \int_0^1 r^{i+j+1} \left( \hat{a}_{j-i}(r) + \hat{a}_{i-j}(r) - \hat{a}_{-j-i}(r) - \hat{a}_{j+i}(r) \right) dr \\
&= -\frac{\pi}{2} \int_0^1 r^{i+j+1} a_{j+i}(r) dr + \frac{\pi}{2} \int_0^1 r^{i+j+1} a_{i-j}(r) dr, \\
\text{Moreover, for } i &\geq 1 \text{ and } j \geq 0, \\
J_{i,j}^{s\cdot c} &= \frac{\pi}{2} \sum_{k=0}^{\infty} \int_0^1 r^{i+j+1} b_k(r) dr \left( \delta_{0,j+i-k} + \delta_{0,j-i+k} - \delta_{0,j+i+k} - \delta_{0,j-i-k} \right) \\
&= \frac{\pi}{2} \int_0^1 r^{i+j+1} \left( \hat{b}_{j+i}(r) + \hat{b}_{j-i}(r) - \hat{b}_{-j-i} - \hat{b}_{j+i} \right) dr \\
&= \frac{\pi}{2} \int_0^1 r^{i+j+1} b_{j+i}(r) dr + \frac{\pi}{2} \int_0^1 r^{i+j+1} b_{i-j}(r) dr.
\end{align*}

As a consequence, we can characterize what “algebraic” condition the linearized Dirichlet-to-Neumann map in the trigonometric basis has to satisfy. We have some trivial conditions that arise from the symmetry of the Dirichlet-to-Neumann map, namely,

\begin{equation}
K^{cc}, K^{ss} \text{ are symmetric, } \quad K^{cs} = K^{scT}, \\
J^{cc}, J^{ss} \text{ are symmetric, and } J^{cs} = J^{scT}.
\end{equation}

Besides this, we have the following nontrivial conditions.

**Proposition 3.** Let \( K^{cc}, K^{ss}, K^{cs}, K^{sc} \) and \( J^{cc}, J^{ss}, J^{cs}, J^{sc} \) be as above. Then, besides of (11) we have that

\begin{equation}
K^{cs} = -K^{csT} \quad K^{sc} = -K^{scT},
\end{equation}

i.e., they are antisymmetric.

In the Schrödinger case, we have that

\( J^{sc} - J^{cs} \) is antisymmetric,

and

\( (J^{ss} - J^{cc})_{i,j \geq 0} \) and \( (J^{sc} + J^{cs})_{i,j \geq 0} \) are Hankel matrices.

Note that Hankel matrices have entries \( A_{i,j} \) that only depend on \( i + j \); in the formulas above, the entries in the Hankel matrices involve the terms \( \int_0^1 r^{i+j+1} a_{i+j}(r) dr \) and \( \int_0^1 r^{i+j+1} b_{j+i}(r) dr \), respectively.

By rearranging the entries in the Dirichlet-to-Neumann map, the identification problem can be rephrased differently: Define

\begin{align*}
d_{i,k}^c &= \frac{1}{i(i+k)\pi} \left( \frac{K^{cc}}{i_{i+k}^{i+k}} \right), \\
d_{i,k}^s &= \frac{1}{i(i+k)\pi} \left( \frac{K^{cs}}{i_{i+k}^{i+k}} \right), \quad i = 1, \ldots, \quad k = 0, \ldots
\end{align*}

(13)
then, in order to find \( \gamma \) in the form

\[
\gamma(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(r) \cos(n\phi) + b_n(r) \sin(n\phi),
\]

we have to solve the moment problems

\[
\int_0^1 r^{2i+k-1} a_k(r) dr = d^{c,k}_{i,i} \quad i = 1, \ldots, k = 0, \ldots
\]

\[
\int_0^1 r^{2i+k-1} b_k(r) dr = d^{s,k}_{i,i} \quad i = 1, \ldots, k = 0, \ldots
\]

Similarly, in the Schrödinger case, we define

\[
d^{c,0}_{i,i} := \begin{cases} 
\frac{1}{\pi} J_{0,0}^{cc} & i = 0, \\
\frac{1}{\pi} (J^{cc} + J^{ss})_{i,i} & i \geq 1,
\end{cases}
\]

\[
d^{c,k}_{i,i} := \begin{cases} 
\frac{1}{\pi} (J^{cc} - J^{ss})_{k,0} & i = 0, \\
\frac{1}{\pi} (J^{cc} + J^{ss})_{i,i+k} & i \geq 1, k \geq 1,
\end{cases}
\]

\[
d^{s,k}_{i,i} := \begin{cases} 
\frac{1}{\pi} (J^{cs}_{0,k} + J^{sc}_{k,0}) & i = 0, \\
\frac{1}{\pi} (J^{cs} - J^{sc})_{i,i+k} & i \geq 1, k \geq 1.
\end{cases}
\]

Then, to find \( c \) in the form

\[
c(r, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(r) \cos(n\phi) + b_n(r) \sin(n\phi),
\]

we have to solve the moment problems

\[
\int_0^1 r^{2i+k+1} a_k(r) dr = d^{c,k}_{i,i} \quad i = 0, \ldots, k = 0, \ldots
\]

\[
\int_0^1 r^{2i+k+1} b_k(r) dr = d^{s,k}_{i,i} \quad i = 0, \ldots, k = 0, \ldots
\]

where, by \((10)\), the coefficients \( a_k, b_k \) are sought such that \( \sqrt{r} a_k(r), \sqrt{r} b_k(r) \) are in \( L^2([0,1]) \).

Thus, up to a shift in the index \( i \), both parameter identification problems lead to the same moment problems. In the next section, we study in detail their solution by Müntz-Legendre polynomials.

## 4 Inversion Formula

We recall the definition of the Müntz-Legendre polynomials (see, e.g., \([3]\)):

**Definition 3.** For a sequence of real numbers with disjoint elements, \( \Theta := (\lambda_i)_i \), \( \lambda_i \geq -\frac{1}{2}, i = 0, \ldots \), the Müntz-Legendre polynomials are defined as

\[
L_n(x) = \sum_{k=0}^{n} c_{k,n} x^{\lambda_k} \quad c_{k,n} = \frac{\prod_{j=0}^{n-1} (\lambda_k + \lambda_j + 1)}{\prod_{j=0, j \neq k} (\lambda_k - \lambda_j)}, \quad n = 0, \ldots
\]
Here the product over an empty set of indices is by definition 1. It is well-known that the functions \((L_n)_n\) are orthogonal \[3, \text{Theorem 2.4}\] but not normalized with respect to the \(L^2([0,1])\)-inner product. These polynomials are named after C. Müntz, who proved the famous result \[3\] that the powers \((x^λ_i)\), span the space \(L^2([0,1])\) if and only if the series \(\sum_{i=0}^{∞} \frac{1}{λ_i} \) diverges.

The functions \(L_n\) are up to a normalization constant identical to the result of a Gram-Schmidt procedure applied to the monomials \(x^λ_i\). In particular, it follows that

\[
(L_n(x), \text{span}\{x^λ_i \mid i < n\})_{L^2([0,1])} = 0,
\]

and it is easy to verify (e.g., by induction) that

\[
\text{span}\{L_n(x) \mid 0 ≤ n ≤ N\} = \text{span}\{x^λ_i \mid 0 ≤ n ≤ N\}.
\]

The coefficients of the monomials \(x^λ_i\) with respect to the Müntz-Legendre-polynomials can be explicitly calculated:

**Lemma 1.** For a sequence \(Θ = (λ_i)\) as in Definition \[3\] let

\[
A_{l,n} := (L_n(x), x^λ_i)_{L^2([0,1])} \quad \text{for } l, n = 0, \ldots \tag{19}
\]

Then

\[
A_{l,n} = \frac{Π_{j=0}^{n-1}(λ_l - λ_j)}{Π_{j=0}^{n}(1 + λ_l + λ_j)},
\]

in particular \(A_{l,n} = 0\) for \(n > l\).

**Proof.** From the orthogonality it follows that \(A_{l,n} = 0\) for \(n > l\). Thus, consider the case of \(n ≤ l\). The Müntz-Legendre polynomials satisfy the recurrence \[3\]

\[
xL_n'(x) - xL_{n-1}'(x) = λ_nL_n(x) + (1 + λ_{n-1})L_{n-1}(x).
\]

Multiply this identity by \(x^λ_i\) and integrate by parts to obtain

\[
- (λ_l + 1) \int_0^1 x^λ_i L_n(x)dx + x^λ_{i+1}L_n(x)|^1_0
\]

\[
= [- (λ_l + 1) \int_0^1 x^λ_i L_{n-1}(x)dx + x^λ_{i+1}L_{n-1}(x)|^1_0]
\]

\[
= λ_nA_{l,n} + (1 + λ_{n-1})A_{l,n-1}.
\]

We have that \(L_0(1) = 1\), \[3\], thus,

\[
-(λ_l + 1)A_{l,n} + (λ_l + 1)A_{l,n-1} = λ_nA_{l,n} + (1 + λ_{n-1})A_{l,n-1},
\]

which gives the recursion

\[
(1 + λ_l + λ_n)A_{l,n} = (λ_l - λ_{n-1})A_{l,n-1}.
\]

Since \(L_0(x) = x^λ_0\), we have \(A_{l,0} = \frac{1}{λ_0 + λ_l + 1}\) and thus

\[
A_{l,n} = \frac{Π_{j=0}^{n-1}(λ_l - λ_j)}{Π_{j=0}^{n}(1 + λ_l + λ_j)}
\]
Now \((A_{l,n})\) can be viewed as an infinite-dimensional lower triangular matrix. Thus, its inverse can be calculated by back-substitution. The next lemma gives an explicit formula for the inverse (in the sense of matrix-multiplication).

**Lemma 2.** Given the infinite-dimensional matrix from Lemma 1. Then \((A_{l,n})_{l,n}\) has the following inverse

\[
R_{\alpha,\beta} = \begin{cases} 
(1 + 2\lambda_{\alpha}) \frac{\prod_{j=0}^{\alpha-1} (1 + \mu_{\beta} + \lambda_{\alpha})}{\prod_{j=0,j\neq \beta} (\lambda_{\alpha} - \lambda_{j})} & \beta \leq \alpha, \\
0 & \beta > \alpha.
\end{cases}
\]

**Proof.** Since both matrices are lower triangular, a matrix-multiplication involves only finitely many terms. For \(\beta < \alpha\),

\[
\sum_{s=\beta}^{\alpha} R_{\alpha,s} A_{s,\beta} = (1 + 2\lambda_{\alpha}) \sum_{s=\beta}^{\alpha} \frac{\prod_{j=0}^{\alpha-1} (1 + \lambda_{s} + \lambda_{j})}{\prod_{j=0,j\neq \beta} (\lambda_{s} - \lambda_{j})} \prod_{j=0}^{\alpha-1} (1 + \lambda_{s} + \lambda_{j})
\]

\[
= (1 + 2\lambda_{\alpha}) \sum_{s=\beta}^{\alpha} \frac{\prod_{j=\beta+1}^{\alpha} (1 + \lambda_{s} + \lambda_{j})}{\prod_{j=\beta,j\neq s} (\lambda_{s} - \lambda_{j})}.
\]

By substituting \(\mu_{s} = \lambda_{s+\beta}\) we have

\[
\sum_{s=\beta}^{\alpha} R_{\alpha,s} A_{s,\beta} = (1 + 2\lambda_{\alpha}) \sum_{s=0}^{\alpha-\beta} \frac{\prod_{j=0}^{\alpha-\beta-1} (1 + \mu_{s} + \mu_{j})}{\prod_{j=0,j\neq s} (\mu_{s} - \mu_{j})}.
\]

We now prove that

\[
T_{n,\kappa} := \sum_{s=0}^{n} \frac{\mu_{s}^{\kappa}}{\prod_{j=0,j\neq s} (\mu_{s} - \mu_{j})} = 0 \quad \text{for any } \kappa < n.
\]

As in [3], we make use of contour integrals. Indeed, by the residue theorem, we may express \(T_{n,\kappa}\) as

\[
2\pi i T_{n,\kappa} = \int_{\Gamma} x^{\kappa} \prod_{j=0}^{n} \frac{1}{x - \mu_{j}} \ dx,
\]

where \(\Gamma\) is a contour in the complex plane that encloses all \(\mu_{j}\). If \(\Gamma\) is a circle with large enough radius \(R\), we may estimate

\[
|2\pi iT_{n,\kappa}| \leq 2\pi R \frac{R^{\kappa}}{|R - \max_{j=0,N} ||\mu_{j}||^{n+1}|},
\]

and this expression tends to 0 for \(\kappa < n\) as \(R \to \infty\). Thus (21) is shown. Expanding the product in the numerator on (20) gives the same terms as in (21). If follows that \(RA\) is zero below the diagonal. Above the diagonal, all entries are trivially 0 by the lower triangular structure. Thus \(RA\) is a diagonal matrix, which uniquely fixes the inverse of \(A\) up to a scaling of the rows. Since the diagonal entries of the inverse are the inverse of the diagonal entries of \(A\), the diagonal entries of \(R\) are known and as stated in the lemma. Thus \(R\) is the inverse of \(A\).
We are now ready to solve the moment problems. Note, however, that we require $a, b$ to be in a weighted $L^2$-space $\|a\|^2_{L^2([0,1], r dr)} := \int a(r)^2 r dr$. Taking this into account, we may rewrite the moment problem as having the coefficients $(a_k(r), r^{2(i-1)+k})_{L^2([0,1], r dr)}$, $i = 1, \ldots$, given (and similar for $b$). It makes sense to expand $a(r)$ and $b(r)$ into Müntz-Legendre polynomials which are orthogonal with respect to the weighted $L^2$-inner product. This can be achieved by defining the Müntz-Legendre polynomials based on the monomials $x^\lambda$, but where the coefficients $c_{k,n}$ are taken as those for the sequence $\lambda_i + \frac{1}{2}$. The corresponding Müntz-Legendre polynomials $L_n(r)$ are then orthogonal with respect to the $L^2([0,1], r dr)$-product. Using $\lambda_i = 2(i - 1)$, $i = 1, \ldots$, leads to the following definition:

**Definition 4.** For $k = 0, \ldots$ fixed, we define the polynomials

$$L_n^k(x) := \sum_{l=0}^n c_{l,n}^k x^{2l+k}, \quad n = 0, \ldots$$

with

$$
c_{l,n}^k := \frac{\Pi_{j=0}^{n-1}(2l + k + 2j + k + 2)}{\Pi_{j=0,j\neq l}^{n-1}(2(l-j))} = \frac{\Pi_{j=0}^{n-1}(l + j + k + 1)}{\Pi_{j=0,j\neq l}^{n-1}(l-j)}.
$$

As explained above, the coefficients $c_{l,n}^k$ are those that are obtained for the Müntz-Legendre polynomials based on the sequence $\Theta = (2i + k + \frac{1}{2})_{i=0,\ldots}$.

From the orthogonality of the original Müntz-Legendre polynomials it follows easily that

$$\langle L_n^k(x), L_m^k(x) \rangle_{L^2([0,1], x dx)} = 0 \quad n \neq m$$

and

$$\langle L_n^k(x), \text{span}\{x^{2l+k} \mid l = 0, \ldots n-1\} \rangle_{L^2([0,1], x dx)} = 0,$$

and the functions $(L_n^k(x))_{n=0}^N$ span the space with basis the monomials $(x^{2n+k})_{n=0}^N$.

We now arrive at the inversion formula for (15).

**Proposition 4.** Let $k \in \mathbb{N}_0$ be fixed. Then $a_k \in L^2([0,1], r dr)$ in (15) is uniquely determined by $d_i^k$ and can be found by

$$a_k(r) = \sum_{n=0}^\infty L_n^k(r) p_n$$

where

$$p_n = \sum_{l=0}^n \frac{(2(-1)^{n-l}(2n+k+1))}{l!(n-l)!} d_{l+1}^k, \quad n = 0, \ldots$$

The same holds for $b$ with $d_i^k$ in place of $d_i^k$.

**Proof.** By Müntz’ theorem and its construction, $L_n^k(r)$ is an orthogonal basis for $L^2([0,1], r dr)$, thus $a_k$ can be expanded as in (23). Fix $k$ and plug the expansion into (15) to obtain

$$\sum_{n=0}^\infty p_n \langle L_n^k(r), r^{2(i-1)+k} \rangle_{L^2([0,1], r dr)} = d_i^k \quad i = 1, \ldots.$$  

We have that the matrix entries

$$\hat{A}_{n,i} := \langle L_n^k(r), r^{2(i-1)+k} \rangle_{L^2([0,1], r dr)} \quad i = 1, \ldots$$
equal \( \tilde{A}_{n,i} = A_{n,i-1} \), where \( A \) is the matrix in (19) defined for the sequence \( \Theta = (2i + k + \frac{1}{2}) \), \( i = 0, \ldots \). Thus, by Lemma 2, we can invert (25) by

\[
p_n = \sum_{l=0}^{n} R_{n,l} d_{l+1}^{c,k}
\]

with \( R \) as in Lemma 2 for the sequence \( \Theta \), i.e., for \( l \leq n \)

\[
R_{n,l} = (2 + 4n + 2k) \prod_{j=0, j \neq l}^{n} (2(l - j))
\]

\[
= 2(2n + k + 1) \prod_{j=0}^{n} (1 + k + (l + j))
\]

\[
= 2(-1)^{n-l}(2n + k + 1) \prod_{j=0}^{n-l} (1 + k + (l + j)) \frac{l!}{(n-l)!}.
\]

Now we collect the results into a theorem:

**Theorem 1.** Let \( K^{cc}, K^{ss}, K^{sc}, K^{sc} \) contain the entries of the linearized Dirichlet-to-Neumann map in the trigonometric basis as in Definition 2 with a coefficient \( \gamma \in L^2(\Omega) \). Then \( \gamma \) can be reconstructed from the expression in polar coordinates

\[
\gamma(r, \phi) = \frac{1}{2} \sum_{n=0}^{\infty} L_n^0(r)p_{n,0} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} L_n^k(r) \cos(k\phi)p_{n,k}
\]

\[
+ \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} L_n^k(r) \sin(k\phi)q_{n,k}
\]

(26)

with

\[
p_{n,k} = \sum_{l=0}^{n} 2(-1)^{n-l}(2n + k + 1) \prod_{j=0}^{n-l} (1 + k + (l + j)) \frac{K_{l+1,l+1+k}^{cc}}{l!(n-l)!},
\]

\[
q_{n,k} = \sum_{l=0}^{n} 2(-1)^{n-l}(2n + k + 1) \prod_{j=0}^{n-l} (1 + k + (l + j)) \frac{K_{l+1,l+1+k}^{cs}}{l!(n-l)!}.
\]

and \( L_n^k(r) \) defined in Definition 4.

Of course, a similar formula holds for \( c \) in the Schrödinger case. Note that the functions \( L_n^k(r) \cos(k\phi) \), \( n, k = 0, \ldots \) and \( L_n^k(r) \sin(n\phi) \), \( n = 0, \ldots, k = 1, \ldots \) provide an orthogonal basis in \( L^2(\Omega) \), thus, we can find conditions for \( \gamma \) being in \( L^2(\Omega) \) in terms of the coefficients \( p_{n,k}, q_{n,k} \).

**Corollary 1.** \( K^{cc}, K^{ss}, K^{sc}, K^{sc} \) are the entries of the linearized Dirichlet-to-Neumann map in the trigonometric basis with \( \gamma \in L^2(\Omega) \) if and only if (11) and (12) is satisfied and

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2n + k + 1} p_{n,k}^2 + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2n + k + 1} q_{n,k}^2 < \infty
\]

with \( p_{n,k}, q_{n,k} \) given in Theorem 1.
Proof. According to [3][Theorem 2.4], we can calculate the $L^2([0,1], rdr)$-norm of $L^n_k(r)$ to $\frac{1}{\sqrt{4n^2+2k^2+2}}$, and after an normalization and Parseval’s identity, the result follows.

Moreover, we can characterize which parameter $\gamma$ are identifiable from finite data, i.e., when only trigonometric function up to a certain frequency are used, and we can completely characterize the finite-data Dirichlet-to-Neumann matrices.

Corollary 2. Given matrices $K_{i,j}^{cc}$, $K_{i,j}^{ss}$, $K_{i,j}^{sc}$, $K_{i,j}^{cs}$, where $i,j = 0,\ldots,N$ and $k,l = 1,\ldots,N$ that satisfy (11) and (12). Then there exists a $\gamma \in L^2(\Omega)$ such that these matrices contain the coefficients of the linearized Dirichlet-to-Neumann map for the impedance tomography problem for the trigonometric basis \{cos($l\phi$)$\cup$ sin($k\phi$)\} for $l = 0,\ldots,N$, $k = 1,\ldots,N$. The parameter $\gamma$ can be expressed as

$$
\gamma(r, \phi) = \frac{1}{2} \sum_{n=0}^{N-1} L^0_n(r)p_{n,0} + \sum_{n=0}^{N-1} \sum_{k=1}^{N-(n+1)} L^k_n(r)\cos(k\phi)p_{n,k}
+ \sum_{n=0}^{N-1} \sum_{k=1}^{N-(n+1)} L^k_n(r)\sin(k\phi)q_{n,k},
$$

(27)

and the coefficients $q_{n,k}$, $p_{n,k}$ are uniquely specified by the matrices $K_{cc}^{cc}$, $K_{cc}^{ss}$, $K_{cc}^{sc}$, $K_{cc}^{cs}$.

Proof. We may define infinite-dimensional matrices $K_{cc}^{cc}$, $K_{cc}^{ss}$, $K_{cc}^{sc}$, $K_{cc}^{cs}$ by extending the given matrices with 0 for indices that are larger than $N$. These matrices satisfy again (11) and (12). The inversion formula (26) gives then a $\gamma$ which induces a Dirichlet-to-Neumann map with these matrices. The coefficients in (27) only involve the given matrices with indices smaller than $N$.

Remark 1. Of course, (27) is not the only parameter that solves the impedance tomography problem with finite data, but we may add higher terms as in (26) at our wish because they are in the nullspace. What components to add is, of course, subjective and may be justified by a regularization approach (or a Bayesian perspective) that chooses a $\gamma$ that fits additional needs.

Remark 2. Equation (27) also shows the typical difficulty of impedance tomography: Since the functions $L^k_n(r)$ are quite flat close to the center and similar to each other there, the resolution in the interior is typically quite bad and becomes worse as we approach the center of the disk. By expanding inclusion sets into the Müntz-Legendre basis, it is possible to provide estimates for the resolution limits of inclusions in the linearized case with finite and noisy data.

Remark 3. In the Schrödinger case, the situation is slightly different: Given the first $n \times n$ portion of the linearized Dirichlet-to-Neumann map, we can still recover the coefficients of $c$ as in (27). However, the Hankel-part provides additional information, namely the moments $\int_0^1 a_l r^{l+1} dr$ and $\int_0^1 b_l r^{l+1} dr$ for $l = n + 1,\ldots,2n$, which could be used to get additional information about $c$. Note, however, that we cannot extend the $J_{cc}^{cc}$, $J_{cc}^{ss}$, $J_{cc}^{sc}$, $J_{cc}^{cs}$ by 0 as in the proof of Corollary [2] as this would destroy the Hankel structure. Thus, this means that the first $n \times n$ part of the matrices have to satisfy certain compatibility condition with matrix entries of higher frequencies larger than $n$. 

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Remark 4. Since coefficients of the form \((27)\) are uniquely specified by the finite-data linearized Dirichlet-to-Neumann operator and the inversion formula provides an explicit inverse that is bounded (though with a terrible \(n\)-dependent bound), we may employ the inverse function theorem to conclude that also coefficients of the form \((27)\) are locally uniquely determined by the (nonlinear) Dirichlet-to-Neumann mapping in the finite-data case (that is, involving \(f, g\) only up to certain frequencies \(n\)). Here locally means that \(\gamma - 1\) with \(\gamma\) as in \((27)\) must be sufficiently small in the \(L^\infty\)-norm and the bound on the \(L^\infty\)-norm depends on \(n\).

5 Incomplete measurements

In this section we consider the impedance tomography problem with incomplete measurements. That is, we assume only parts of the boundary accessible for measurements and impose homogeneous Dirichlet condition on the rest. Results on uniqueness in this case are scarce, in particular, in the two-dimensional case. For the 3D (and nonlinear) case, results on unique identifiability can be found in [7, 6].

5.1 Incomplete measurements on the half disk

Specifically, we first consider the problem on the upper half disk where the upper half circle is accessible to measurements.

\[
\Omega_h = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y > 0\}
\]

\[
\Gamma = \partial \Omega_h \cap \{y > 0\} \quad \Gamma_0 = \partial \Omega_h \cap \{y = 0\}.
\]

We consider the problem

\[
-\nabla \cdot (\gamma \nabla) u_f = 0 \quad \text{in } \Omega
\]

\[
u_f = \begin{cases} f & \text{on } \Gamma, \\ 0 & \text{on } \Gamma_0. \end{cases}
\]

(28)

As data we consider the Neumann data again on \(\Gamma\) which introduces the incomplete Dirichlet-to-Neumann operator

\[
\Lambda_{1, \gamma}^{\text{inc}} : f|\Gamma \rightarrow \gamma \frac{\partial}{\partial n} u_f|\Gamma.
\]

Here, we assume that \(f\) is so that the Dirichlet problem has a solution in \(H^1(\Omega_h)\), which is the case when the zero-extension of \(f\) to \(\partial \Omega_h\),

\[
f^e := \begin{cases} f & \text{on } \Gamma \\ 0 & \text{on } \Gamma_0 \end{cases}
\]

(29)

is in \(H^{\frac{1}{2}}(\partial \Omega_h)\), which we assume throughout. We denote that spaces as \(H^{\frac{1}{2}}_0(\partial \Omega_h)\). Is is not difficult to verify that the incomplete Dirichlet-to-Neumann operator can be written as

\[
\langle \Lambda_{1, \gamma}^{\text{inc}} f, g \rangle_{H^{\frac{1}{2}}_0, H^{\frac{1}{2}}_0} = \langle \Lambda_{1, \gamma} f^e, g^e \rangle_{H^{\frac{1}{2}}, H^{\frac{1}{2}}},
\]
By this formula, we find for the linearized problem (around $\gamma = 1$) the following expression

$$\langle \Lambda^{inc} \gamma, f, g \rangle_{H^0_\gamma, \partial H_\gamma} = \int_{\Omega_h} \gamma(x) \nabla u_{f,1}(x) \cdot \nabla u_{g,1}(x) dx,$$

(30)

where $u_{f,1}, u_{g,1}$ are solutions to (28) with $\gamma = 1$, (i.e., the Laplace equation) with the respective Dirichlet boundary conditions.

We prove the following theorem:

**Theorem 2.** Let $\gamma \in L^2(\Omega_h)$. Then $\gamma$ is uniquely determined by the linearized incomplete Dirichlet-to-Neumann map $\Lambda^{inc}_{1,\gamma}$. In fact, it suffices to have in (30) $f(\phi) = \sin(n \phi)$ and $g(\phi) = \sin(k \phi)$, $\phi \in [0, \pi]$ for $n, k = 1, \ldots$

**Proof.** Setting $f_n = \sin(n \phi)$ and, by using polar coordinates, we obtain the associated solution $u_{f_n,1}(r, \phi) = r^n \sin(n \phi)$, $r \in (0, 1)$, $\phi \in [0, \pi]$, and similar for $g_k(\phi) = \sin(k \phi)$ with $u_{g_k,1}(r, \phi) = r^k \sin(k \phi)$, $n, k \geq 1$. By the Schwarz’ reflection principle we may extend $u$ (antisymmetrically with respect to $y = 0$) to a harmonic map on the unit disk and obtain the functions

$$u_{f_n,1}(r, \phi) = r^n \sin(n \phi), \quad u_{g_k,1}(r, \phi) = r^k \sin(k \phi), \quad r \in (0, 1), \phi \in [0, 2\pi].$$

Extending $\gamma$ symmetrically to the lower half disk by

$$\tilde{\gamma}(x, -y) = \gamma(x, y), \quad y < 0,$$

we observe that the integral in (30) over the lower half disk equals that of the upper half disk. Thus, we can express the linearized incomplete Dirichlet-to-Neumann map via the linearization of the full Dirichlet-to-Neumann map on the unit disk with $\tilde{\gamma}$. That is

$$\langle \Lambda^{inc}_{1,\tilde{\gamma}} f_n, g_k \rangle = \frac{1}{2} \langle \Lambda^{inc}_{1,\tilde{\gamma}} \tilde{f}_n, \tilde{g}_k \rangle,$$

where $\tilde{f}_n$ and $\tilde{g}_k$ are the antisymmetric extensions to the lower half, i.e., $\tilde{f}_n(\phi) = \sin(n \phi)$, $\tilde{g}_k(\phi) = \sin(k \phi)$, for $\phi \in [0, 2\pi]$. Finally, we may expand $\gamma$ on the upper half disk into a pure cosine series, i.e., using polar coordinates

$$\gamma(r, \phi) = \sum_{n=0}^{\infty} a_n(r) \cos(n \phi), \quad \phi \in [0, \pi],$$

which, by our symmetric extension immediately gives the expansion of $\tilde{\gamma}$

$$\tilde{\gamma}(r, \phi) = \sum_{n=0}^{\infty} a_n(r) \cos(n \phi), \quad \phi \in [0, 2\pi].$$

Now we may use the formula for $\Lambda^{inc}_{1,\tilde{\gamma}}$ to conclude

$$2 \langle \Lambda^{inc}_{1,\tilde{\gamma}} f_n, g_k \rangle = nk \pi \zeta_{n,k} \int_0^1 r^{n+k-1} a_{|n-k|}(r) dr.$$

Thus, by considering ($\Lambda^{inc}_{1,\gamma} f_n, g_{n+l}$), for $n = 1, \ldots, l = 0, \ldots$, and the completeness of the function families ($r^{2n+l-1}$)$_{n=1}^{\infty}$ for $l = 0, \ldots$ that follows from Müntz’ theorem, we have shown that the coefficients $a_n$ are uniquely determined, which completes the proof.  

$\square$
An inversion formula is provided by (26) using only a cosine expansion,
\[
\gamma(r, \phi) = \frac{1}{2} L_0^0(r)p_{n,0} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} L_k^0(r) \cos(k \phi)p_{n,k},
\]
and noting that with our sine boundary functions
\[
f_n(\phi) = \sin(n \phi), \quad g_k(\phi) = \sin(k \phi),
\]
we have
\[
2(\Lambda_{inc}^{1,\gamma} f_l, g_j) = K_{ss}^{l,j} \quad \text{leading to the coefficient formula}
\]
\[
p_{n,k} = \sum_{l=0}^{n} 2(-1)^{n-l}(2n + k + 1) \Pi_{l=0}^{n-1} (1 + k + (l + j)) \frac{n!}{l!(n-l)!} \frac{1}{(l+1)(l+1+k)\pi}.
\]

\[\tag{31}\]

5.2 Incomplete measurements on the disk

We now study a similar problem as before but on the full disk and where the measurements are available only on an interval on the boundary. Specifically, we assume access only to an interval on the unit circle on the upper half of the form
\[
\phi \in I := [\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha],
\]
where, \(0 < \alpha < \frac{\pi}{2}\) and \(\phi\) is the angular coordinate. The restriction to \(\alpha < \frac{\pi}{2}\) is only for convenience and could be dropped. Again considering the linearized case, we study boundary value problems with given \(f\) defined on \(I\).
\[
\Delta u_{f,1} = 0 \quad \text{in } \Omega = \{ (x, y) \in \mathbb{R}^2 | x^2 + y^2 < 2 \leq 1 \},
\]
\[
\begin{align*}
    u_{f,1} & = f \quad \text{on } I \cap \partial \Omega, \\
    u_{f,1} & = 0 \quad \text{on } \partial \Omega \setminus I.
\end{align*}
\]
\[\tag{33}\]

As above, we have to restrict ourselves to functions \(f\) for which the zero extension to \(\partial \Omega\) is in \(H^1_s\), or, equivalently, to Dirichlet values where the problem \(\tag{33}\) has a solution in \(H^1(\Omega)\).

We study the analogous linearized problem with data available only on \(I\), that is, we define the linearized incomplete Dirichlet-to-Neumann operator
\[
\langle \Lambda_{inc}^{1,\gamma} f_n, g_k \rangle := \int_{\Omega} \gamma \nabla u_{f,1} \cdot \nabla u_{g,1} \, dx \quad f^e, g^e \in H^\frac{1}{2}.
\]
\[\tag{34}\]

Here \(f^e, g^e\) denotes the zero extension to the whole boundary as before.

We proof the following theorem:

**Theorem 3.** Let \(\gamma \in L^2(\Omega)\) be such that \(\gamma = 0\) in a neighborhood of the endpoints of \(I\). Let the linearized incomplete Dirichlet-to-Neumann operator \(\Lambda_{inc}^{1,\gamma}\) be defined in \(\Omega\) with data on the interval \(I\). Then \(\gamma\) is uniquely determined by the values
\[
\left\{ \langle \Lambda_{inc}^{1,\gamma} f, g \rangle, \mid f^e, g^e \in H^\frac{1}{2} \right\}.
\]

The proof is based on conformal mappings; more specifically on the following lemma:
Lemma 3. There exists a conformal map $\psi$ from the upper half of the unit disk to the full unit disk which can be extended to a homeomorphism from and to the closures of the respective sets such that the left and right endpoints of the half disk, $(-1,0)$ and $(1,0)$ are mapped to the left and right endpoints of the interval in $\Omega$.

Proof. A conformal map that takes the upper half disk to the unit disk is well known: The combination of the mappings $\theta_0 = \frac{1+z}{1-z}$ and $z \rightarrow z^2$ gives the map $\theta_1(z) = (1+z)^2/(1-z)^2$ [16] Ex 2, 3, pp. 210 which takes the upper half disk to the upper half plane, while the fractional transform $\theta_2(z) := \frac{z+1}{z-1}$ maps the upper half plane to the unit disk. Combining these two maps gives

$$\theta(z) = \theta_2 \circ \theta_1 = \frac{(1+z)^2 - i(1-z)^2}{(1+z)^2 + i(1-z)^2},$$

which maps the upper half disk to the unit disk and leaves the half-disk’s endpoints at $z = \pm 1 + 0i$ invariant. Moreover, the boundary $\{ x^2 + y^2 = 1, y \geq 0 \}$ is mapped to itself, while the lower part of the boundary, $\Omega_1 \cap \{ y = 0 \}$, is mapped to the lower half of the unit circle. By Caratheodory’s theorem the mappings extends to homeomorphisms, but for this example, we can calculate explicitly that the mappings on the boundary are invertible. Finally, we observe that the mapping is smooth on the closure of the upper half disk. To achieve a mapping with the specifications about the endpoints, we consider the conformal automorphisms of the unit disc,

$$\sigma(z) = e^{i\mu} \frac{z-w}{\overline{w}z-1} \quad \mu \in [0,2\pi], |w| < 1.$$ The conditions $\theta(1) = e^{i(\frac{\pi}{2} - \alpha)}$ and $\theta(-1) = e^{i(\frac{\pi}{2} + \alpha)}$ can be satisfied by $\mu = \pi$ and $w = -i\frac{\cos(\alpha)}{1+\sin(\alpha)}$. It is easily verified that $|w| < 1$ holds such that $\sigma$ is indeed an automorphism of the unit circle. Setting $\psi = \sigma \circ \theta$ provides the desired map.

We note that the inverse of the conformal map is given by

$$\psi^{-1} = \theta_0^{-1} \circ \sqrt{\theta_2^{-1} \circ \sigma^{-1}},$$

where $\theta_0^{-1} = \frac{1+z}{1-z}, \theta_2^{-1} = i\frac{z+1}{z-1}$ and $\sqrt{ }$ is the complex square root with branch cut at the negative imaginary axis. By calculating the derivative of $\psi$, we observe that $|D\psi|$ vanishes only at the end points of the circle $(1,0)$ and $(-1,0)$. Since they are mapped to the endpoints of $I$, the inverse is smooth on $\Omega$ away from these points.

Proof of Theorem. Start with the functions $\tilde{f}^c$ and $\tilde{g}^c$ that are defined on the boundary of the upper half disk by (19) with $f = \sin(n\phi)$ and $g = \sin(k\phi)$ on the upper boundary $\Gamma$ (and extended by zero to $\Gamma_0$). Consider their images by the conformal map of Lemma 3

$$f^c = \tilde{f} \circ \psi \quad g^c = \tilde{g} \circ \psi.$$ These are continuous functions on the unit circle and they are supported in the interval $I$ by construction. Moreover, since $f^c, g^c$ are the boundary values of the harmonic functions $\tilde{u}_{f}(r, \phi) = r^n \sin(n\phi) = \text{Im}(z^n)$ and $r^k \sin(k\phi) = \text{Im}(z^k)$, the so defined functions $f^c, g^c$ correspond (by conformality) to harmonic functions

$$u_{f,1} = \text{Im}(\phi^{-1})^n \quad u_{g,1} = \text{Im}(\phi^{-1})^k.$$
We have by conformal invariance of the Dirichlet integral that
\[ \int_{\Omega} |\nabla u_{f,1}|^2 \, dx = \int_{\Omega_h} |\nabla \tilde{u}_{0,j}|^2 \, dx \]
and
\[ \int_{\Omega} |u_{f,1}|^2 \, dx = \int_{\Omega_h} |\tilde{u}_{0,j}|^2 |D\psi(x)| \, dx \leq \sup_{x \in \Omega_h} |D\psi(x)| \int_{\Omega_h} |\tilde{u}_{0,j}|^2 < \infty \]
because \(|D\psi(x)|\) is smooth. Thus, \(u_{f,1}\) (and clearly similarly \(u_{g,1}\)) are in \(H^1\), thus, \(f^\epsilon\), and \(g^\epsilon\) are in \(H^{1/2}\) and the restriction to \(I\) can be used as data for \((\Lambda_{inc}^{\epsilon,\gamma} f, g)\). We then have
\[ \langle \Lambda_{inc}^{\epsilon,\gamma} f, g \rangle = \int_{\Omega} \gamma \nabla u_{f,1} \cdot \nabla u_{g,1} \, dx = \int_{\Omega_h} \gamma(\psi(x)) \nabla \tilde{u}_{f,1} \cdot \nabla \tilde{u}_{g,1} \, dx, \]
where \(\tilde{u}_{f,1}(r, \phi) = r^n \sin(n \phi)\) and \(\tilde{u}_{g,1} = r^k \sin(k \phi)\). We verify that \(\gamma(\psi^{-1})\) is in \(L^2(\Omega_h)\):
\[ \int_{\Omega_h} \gamma(\psi(x))^2 \, dx = \int_{\Omega} \gamma(x)^2 |D\psi^{-1}(x)| \, dx \leq \sup_{x \in \text{supp}(\gamma)} |D\psi^{-1}(x)| \int_{\Omega} \gamma(x)^2 \, dx < \infty. \]
The later inequality holds because \(\psi^{-1}\) is smooth away from the interval endpoints. Thus \(\gamma \circ \psi\) is uniquely determined by Theorem 2 and, hence, so is \(\gamma\).

The inversion formula in this case is
\[ \gamma(r, \phi) = \left[ \frac{1}{2} L_n^0(r) p_{n,0} + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} L_n^k(r) \cos(k \phi) p_{n,k} \right] \circ \psi^{-1} \]
with coefficients given by (31), where \(f_n, g_n\) are the transformed sine functions as in the proof.

Also in this situation it is possible to study the finite-data case and again derive a formula as in Corollary 2. The “flatness” of the Müntz-Legendre polynomials is transformed by the conformal map to the region opposite of the data interval \(I\), which shows the expectable fact that variations in the parameter \(\gamma\) that are located opposite to the data site are hardest to reconstruct.

### 6 Final comments

We have shown that the expansion of the parameters into trigonometric/Müntz-Legendre polynomials is an interesting tool for the linearized impedance tomography problem as it allows for an explicit inversion formula and a transparent characterization of what can be identified in the finite data case.

When it comes to numerical calculations, even with an explicit inversion formula, problems may occur because of, e.g., rounding errors. Note that the formula shows the typical features of inversion in the case of ill-posed problems, namely values of opposite signs have to be added, which may lead to cancellation. Thus, it is a good idea to include a regularization also here.
the other hand, the inversion formula certainly provides a fast solution method compared to a PDE-based approach as it operates directly on the data space and no interior grid has to be used.

It would be interesting to analyze the corresponding three-dimensional case. There, the trigonometric expansion in the angular coordinate is naturally replaced by spherical harmonics. When also expanding $\gamma$ in this basis, we similarly come to integrals that involve triple combinations of spherical harmonics, which leads to quite complicated combinatorial coefficients. We do not know whether an approach for an inversion formula succeeds in this case.

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