Quantization of Gauge Theory for Gauge Dependent Operators

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Based on a canonically derived path integral formalism, we demonstrate that the perturbative calculation of the matrix element for gauge dependent operators has crucial difference from that for gauge invariant ones. For a gauge dependent operator $\mathcal{O}(\phi)$ what appears in the Feynman diagrams is not $\mathcal{O}(\phi)$ itself, but the gauge-transformed one $\mathcal{O}(\omega \phi)$, where $\omega$ characterizes the specific gauge transformation which brings any field variable into the particular gauge which we have adopted to quantize the gauge theory using the canonical method. The study of the matrix element of gauge dependent operators also reveals that the formal path integral formalism for gauge theory is not always reliable.

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Ever since the emergence of nucleon spin problem, the matrix element of gauge dependent operators has aroused great interest \cite{footnote}. This is because the conventional understanding of gluon spin, gluon and quark orbital angular momentum all corresponds to gauge dependent operators. When studying these operators, people have naturally adopted the same calculation scheme as for gauge invariant ones, however some quite contradictory results were obtained \cite{footnote}. In this paper, by quantizing the gauge theory from the very beginning, we demonstrate that the perturbative calculation rules for gauge dependent operators have nevertheless critical difference from those for gauge invariant ones. By doing so we also reveal that the highly formal path integral formalism for gauge theory is not always reliable.

The theory we start with is the classical SU(N) or U(1) gauge theory. We adopt the well-defined canonical quantization approach to quantize it. Due to gauge freedom, we must specialize a gauge at first. Here we choose the most convenient axial gauge

$$n \cdot A = 0,$$

where $A$ is the gauge potential, $n$ is an arbitrary but fixed constant four-vector. Now we can apply the standard canonical quantization procedure and obtain the commutation relations and the quantum Hamiltonian. But instead of going further to use the operator method to derive the Feynman rules, we use the so obtained quantum Hamiltonian to derive the path-integral expression for the vacuum expectation value of an arbitrary operator $\mathcal{O}(\phi)$, where $\phi$ denotes collectively the gauge and matter fields. After the standard procedure which can be found in textbooks \cite{footnote}, we can write up to an irrelevant normalization factor

$$Z(\mathcal{O}) \equiv \langle \mathcal{O} \rangle_{\text{vac}} = \int [D\phi] e^{iS} \mathcal{O}(\phi) \delta(n \cdot A).$$

The notations are standard. This expression is rigorous and applies to both Abelian and non-Abelian gauge theory \cite{footnote}. Moreover, we have no requirement for the operator $\mathcal{O}$. It can be either gauge dependent or gauge invariant. However, Eq. (1) is non-perturbative and not yet perturbatively applicable. To derive the rules for perturbative calculation, we use the Faddeev-Popov trick by multiplying Eq. (1) with the identities

$$1 = \Delta_F(A,\chi) \int [D\omega] \delta(F(\omega A) - \chi),$$

$$1 = \int [D\chi] G[\chi],$$

where $F$ and $G$ are two arbitrary functions (or in general functionals), $[D\omega]$ is the invariant measure of the gauge group, $\omega A$ is the result of $A$ after a gauge transformation characterized by $\omega$. $\Delta_F$ is usually called the gauge fixing function, for example we can take our original axial gauge form $F(A) = n \cdot A$. Accidentally, it is apparent that by this method we can transit to an arbitrary gauge by choosing arbitrary $F$ and $G$ (this point will be explained further later).

Multiplying (1) with (2a) and (2b), we get

$$Z(\mathcal{O}) = \int [D\phi][D\omega][D\chi] e^{iS} \mathcal{O}(\delta(n \cdot A))$$

$$\times \Delta_F(A,\chi) \delta(F(\omega A) - \chi) G[\chi].$$

Now make the transformation

$$\phi \rightarrow \omega^{-1} \phi,$$

note that $[D\phi]$, $S$, and $\Delta_F(A,\chi)$ are gauge invariant, and integrate over $\chi$, we get

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\[ Z(\mathcal{O}) = \int [D\phi] e^{iS} \Delta_F(A, F(A)G(F(A)) \times \int [D\omega](\omega^{-1}\mathcal{O})\delta(n \cdot \omega^{-1} A) \] (6)

The second line in Eq. (6) gives up to an irrelevant constant factor \[ \omega_0 \mathcal{O}, \] (7)

where \( \omega_0 \) is the specific function of \( A \) which satisfies \( n \cdot \omega_0 A = 0 \). (8)

So we obtain the aimed expression

\[ Z(\mathcal{O}) = \int [D\phi] e^{iS} (\omega_0 \mathcal{O})\Delta_F(A, F(A))G(F(A)). \] (9)

By choosing a Gaussian type of \( G \), and introduce the ghost field to treat the Faddeev-Popov determinant \( \Delta_F \), one can straightforwardly derive Feynman rules from Eq. (3).

We note very importantly that what appears in this final expression is \( \omega_0 \mathcal{O} \) instead of \( \mathcal{O} \) itself. This will give critical difference for gauge invariant and non-invariant operators.

First we look at the gauge invariant operator: \( \omega_0 \mathcal{O} \) equals \( \mathcal{O} \). Now Eq. (3) becomes

\[ Z(\mathcal{O}) = \int D\phi e^{iS} \mathcal{O} \Delta_F(A, F(A))G(F(A)). \] (10)

So we see that starting from a canonical quantization in the specific axial gauge, we finally arrive at an expression independent of the original gauge condition! For instance the gauge \( A_2 = 0 \) or \( A_3 = 0 \) lead to the same Eq. (10). This states that the Green’s function of gauge invariant operators are gauge independent.

For gauge dependent operators, however, \( \omega_0 \mathcal{O} \) does not equal \( \mathcal{O} \). So the perturbative calculation of the Green’s function for gauge dependent operators is not a straightforward generalization of that for gauge invariant ones. Instead, we encounter an additional complication of always having to apply a gauge transformation to the operator before inserting it into the Feynman diagrams.

It should be remarked that in Eq. (3) only the operator \( \mathcal{O} \) suffers such a gauge transformation, the other field variables such as those in the action remain the ordinary form. Accordingly, in the perturbative calculations, only the external vertex corresponding to \( \mathcal{O} \) should be the gauge transformed one, while the Feynman rules for the internal vertices and propagators generated from the expansion of the effective action are still the ordinary ones.

We note that in Eq. (3) \( \omega_0 \) is universally given by Eq. (3). However, the form of \( F \) and \( G \) can be arbitrary, i.e. the Feynman rules can be arbitrary but they give the same final results. One may wonder why Eq. (3) takes such an apparently peculiar form, especially why must we apply a transformation to \( \mathcal{O} \) and why \( \omega_0 \) still takes the form in Eq. (3) even after we have “transmitted to another gauge” by choosing an \( F(A) \) different from \( n \cdot A \). We explain below that this actually has deeper physical reasons behind.

Due to the extra degree of gauge freedom, we have to fix a gauge to do quantization. And such a gauge choice will naturally leave “traces” on the considered operator \( \mathcal{O} \) and also on the subsequent formalism, such as the to-be-derived quantum Hamiltonian and Feynman rules. Indeed, different gauges lead to different form of quantum Hamiltonian. Or to say, the form of quantum Hamiltonian is gauge dependent. Furthermore, if we stick to the operator method to calculate the Green’s functions, the original gauge choice will go throughout to manifest in the Feynman rules. However the gauge invariance principle commands that we must in the end show all the physical observables to be independent of the original gauge we chose in quantization. Within the canonical operator method, such a demonstration is essentially a direct proof of the equivalence of Feynman rules derived in different gauges, and will necessarily involve us in a detailed analysis of the Feynman diagrams. This was done for QED by Feynman \[ \xi \] and for Yang-Mills theory by Cheng and Tsai \[ \eta \].

In contrast, instead of demonstrating gauge invariance at the level of Feynman diagrams, the advantage of path-integral formalism is that we can already get rid of the original gauge condition at an intermediate stage before deriving the Feynman rules. (To be equivalent to the canonical formalism, our path-integral expression is derived by the operator method using the canonically-derived quantum Hamiltonian.) The crucial step is that we can obtain Eq. (3), in which what appears is the original gauge invariant classical action \( S \). (This can be done at least in the axial gauge, in which the quantum Hamiltonian takes a relatively simpler form, than, say, in the Coulomb gauge.) Now, the subsequent procedure in Eqs. (3)- (6) is essentially converting \( n \cdot A \) into \( n \cdot A' = n \cdot \omega A = (F(A) - \chi) \) by coordinate transformation, then integrating over \( \chi \) with a weight-functional \( G[\chi] \). The \( \Delta_F(A, \chi) \) is nothing but the Jacobian of this coordinate transformation, which due to the compactness of the gauge group can be chosen gauge invariant. This step is mathematically trivial, and by it alone we are still in the original axial gauge, but now expressed as \( n \cdot A' = 0 \). However, since \( [D^2 \phi], \Delta_F(\omega A, \chi) \), and critically the action \( S(\omega \phi) \) are gauge invariant, we can changed their “\( \phi \) back to \( \phi \). Now we are in another arbitrarily chosen gauge \( (F(A) - \chi = 0) \), and can commit an arbitrary weight \( G[\chi] \) to each \( \chi \). Therefore, the freedom to choose arbitrary \( F \) and \( G \) is a consequence of the gauge invariance of \( S \) in Eq. (3). Namely, despite the appearance of an explicit \( \delta(n \cdot A) \), the original gauge condition \( n \cdot A \) has essentially disappeared out in Eq. (3) (apart from \( O(\phi) \)), because we can convert \( n \cdot A \) into an arbitrary form by changing \( A \) to \( \omega A \) but leave the \( A \) else-
where unaltered. Therefore later on when we convert $G$ and $\Delta_F$ into an effective action and expand it to derive Feynman rules, we already know for sure that these Feynman rules are equivalent to each other, without having to study the details of Feynman diagrams. This is the convenient path-integral method of demonstration that gauge theory still reserves gauge invariance after quantization. Namely, the “trace” of the original gauge choice has disappeared out in the final Feynman rules.

However, the “trace” left on the considered operator $O$ by the original gauge condition will not disappear, if $O$ is not gauge invariant. And such a “trace” should be “visible” no matter whatever tricks we play. Indeed, in the original expression Eq. (3), the gauge condition is enforced by the delta function. And after we adopted the Faddeev-Popov trick to transit to an arbitrary gauge fixing form $G(F(A))$, this $G(F(A))$ no longer enforce the original gauge condition on $O$. However we get a gauge transformation on $O$ characterized by $\omega_q$. In fact, by the definition in Eq. (3), the role of $\omega_q$ on $O$ is nothing but bringing it to the initial axial gauge in which we start the canonical quantization.

In one word, once we start with the expression Eq. (3), which is obtained by canonical quantization in the axial gauge, the subsequent calculation will exhibit such a gauge condition anyhow, unless the considered operator $O$ is gauge invariant. And no matter what kind of gauge fixing functions $F$ and $G$ we choose to derive the Feynman rules, the final matrix element we obtain should be the same (as long as the Faddeev-Popov trick is justified) and should be regarded as the result in the original axial gauge; the freedom of choosing Feynman rules for calculation is simply because the quantized gauge theory still reserves gauge invariance, as was demonstrated by the path-integral formalism. If we want to study what the matrix element of a gauge dependent operator would be in another gauge, we have to begin the initial canonical quantization in that gauge (which might not necessarily lead to a simple form like Eq. (3)). Merely adopting the Faddeev-Popov trick to shift the form of $F$ and $G$ only changes the gauge for Feynman rules (which are equivalent in different gauges), but not the gauge for the whole matrix element.

For example, the conventional understanding of quark or electron orbital angular momentum corresponds to the gauge dependent operator

$$
\hat{L}_q = \int d^3x \psi^\dagger \left( \vec{x} \times \frac{1}{i} \vec{\partial} \right) \psi.
$$

(11)

Whether $\hat{L}_q$ gives gauge invariant matrix element in a nucleon helicity eigenstate will determine whether quark’s orbital contribution to nucleon spin is meaningful. According to our above discussion, we cannot calculate $\hat{L}_q$’s matrix element as we do for the gauge invariant operators such as quark spin. Actually we don’t really know at all the expression of $\hat{L}_q$’s matrix element in an arbitrary gauge except for the gauges in which we know how to do canonical quantization.

In the usual discussion of gauge theory with path-integral approach, people sometimes simply ignore the gauge freedom and write down a formal expression as if all field variables were independent:

$$
Z(O) = \int [D\phi] e^{iS} O(\phi). \tag{12}
$$

Then by following the same steps as from Eqs. (4) to (6), we can get

$$
Z(O) = \int [D\phi] e^{iS} \Delta_F(A, F(A)) G(F(A)) \times \int [D\omega](\omega^{-1} O). \tag{13}
$$

Different from Eq. (3), we don’t have delta function in the second line to pick up a specific $\omega$ as a function of $A$. The $\omega$ here is purely an integration variable independent of the field variable. If $O$ is gauge invariant and hence $\omega^{-1} O = O$, the infinite integration $\int [D\omega]$ factorizes out as an irrelevant normalization factor, so we get the same perturbatively applicable expression as what we obtained by starting with strict canonical quantization,

$$
Z(O) = \int [D\phi] e^{iS} O \Delta_F(A, F(A)) G(F(A)). \tag{14}
$$

Since Eq. (14) can be derived from the same Eq. (12) for arbitrary $F$ and $G$, this formally states that $Z(O)$ is independent of $F$ and $G$ up to an irrelevant constant factor, or to say, the Feynman rules in different gauges are equivalent. This is again the same conclusion as we discovered by starting with well-defined canonical method.

However, we remark that such a formal procedure of starting with the ill-defined infinite expression Eq. (12) is not always reliable. Such unreliability simply often does not manifest for gauge invariant operator, but can be seen by studying gauge dependent operators. Generally speaking, when $O$ is gauge dependent we are unable to arrive at the perturbatively applicable expression Eq. (14) from Eq. (3). But there are some exceptions. For instance we take the product of electron field at different points:

$$
O = \psi^\dagger(x) \psi(y). \tag{15}
$$

$\omega O$ will be

$$
\omega O = e^{i\omega(x) - i\omega(y)} \psi^\dagger(x) \psi(y). \tag{16}
$$

Therefore when $x \neq y$, $\psi^\dagger(x) \psi(y)$ is gauge dependent. However, we see in Eq. (13) that the gauge transformed phase factor $e^{i\omega(x) - i\omega(y)}$ can be absorbed into the irrelevant integration over $\omega$. Therefore we can obtain the same expression for $Z(\psi^\dagger(x) \psi(y))$ as for gauge invariant operators, and would conclude that $Z(\psi^\dagger(x) \psi(y))$ can be calculated with the usual Feynman rules in an
arbitrary gauge and is gauge independent! This however contradicts the results obtained by starting with the well-defined canonical quantization. And in fact, we known that the Fermion two point function is gauge dependent. A toy model demonstration of how such error arises can be found in Ref. [9].

By some more delicate tricks, the authors in Ref. [10] adopted the above formal path-integral formalism to demonstrate that if we use the same calculation rules as for gauge invariant operators, the gauge dependent quark orbital angular momentum operator will give gauge independent matrix element for a nucleon helicity state. This however is refuted by explicit 1-loop calculations [11].

Finally, we mention that Cheng and Tsai [12] has ever pointed out that the Feynman rules derived via the Fadeev-Popov approach often exhibit singularities (for example in the Coulomb and axial gauges). Such singularities must be dealt with care, otherwise at the two-loop level of non-Abelian gauge theory the Feynman rules in the Coulomb, covariant, and axial gauges derived via the Fadeev-Popov approach will be inconsistent with each other [12].

In summary, we demonstrated that the calculation of the matrix element of gauge dependent operators is not a straightforward generalization of that for gauge invariant ones, and must be carefully derived from the very beginning of gauge theory quantization. We must distinguish the effects of initial gauge choice (in quantization) on the Feynman rules and on the studied operator or matrix element. Feynman rules in different gauge are demonstrated to be equivalent but the gauge condition on the studied operator might bring non-trivial difference. The formal path-integral formalism of starting with an ill-defined infinite expression, which often gives correct results for gauge invariant operators, is however not always reliable, as can be seen by studying the matrix element of gauge dependent operators.

Note added: We call the interested readers’ attention to a most recent paper by Hoodbhoy and Ji [13]. We got quite some inspiration from [13] in developing this paper, however our opinions towards gauge theory quantization are rather different from that in [13].

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