Triviality of the Aharonov–Bohm interaction in a spatially confining vacuum

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This paper explores long-range interactions between magnetically-charged excitations of the vacuum of the dual Landau–Ginzburg theory (DLGT) and the dual Abrikosov vortices present in the same vacuum. We show that, in the London limit of DLGT, the corresponding Aharonov–Bohm-type interactions possess such a coupling that the interactions reduce to a trivial factor of $e^{2\pi i \times \text{integer}}$. The same analysis is done in the SU($N_c$)-inspired $[U(1)]^{N_c-1}$-invariant DLGT, as well as in DLGT extended by a Chern–Simons term. It is furthermore explicitly shown that the Chern–Simons term leads to the appearance of knotted dual Abrikosov vortices.

I. INTRODUCTION

It has long been known that quark confinement in QCD can be modeled by means of a dual-superconductor scenario [1, 2]. This scenario suggests that the Yang–Mills vacuum can resemble that of a dual superconductor, which consists of the condensate of a magnetically charged Higgs field. The resulting dual Abelian Higgs model is a four-dimensional relativistic generalization of the Landau–Ginzburg theory of dual superconductivity. Dedicated lattice simulations support this scenario of confinement with a very high accuracy [3].

It turns out that not only the dual Abelian Higgs model but also the dual Landau–Ginzburg theory (DLGT) can be relevant to the description of the Yang–Mills vacuum. The reason is that, upon the deconfinement phase transition, large spatially-oriented Wilson loops still exhibit an area-law behavior (see Ref. [4] for the lattice results on the corresponding spatial string tension $\sigma_s$). Analytically, spatial confinement can with a good accuracy be described in terms of soft stochastic chromo-magnetic Yang–Mills fields [5], which (unlike soft chromo-electric fields) survive the deconfinement phase transition [6]. Moreover, for every temperature-dependent quantity, there exists the so-called temperature of dimensional
reduction such that, above that temperature, the contribution to the quantity at issue produced by all Matsubara frequencies $\omega_k = 2\pi Tk$ with $k \neq 0$ is negligible compared to the contribution of $\omega_0$. It should be, of course, borne in mind that, although the contributions of nonzero modes amount to at most few per cent of the static-mode contribution, these contributions are always present. For this reason, the dimensional reduction is not a phase transition with a definite critical temperature that can be determined from the thermodynamic equations. At the formal level, one can only say that the dimensional reduction of the Euclidean Yang–Mills action corresponds to the substitution

$$S_{YM} = \frac{1}{4g_{YM}^2} \int d^3x \int_0^{1/T} dx_4 (F^a_{\mu\nu})^2 \rightarrow \frac{1}{4g_T^2 T} \int d^3x (F^a_{\mu\nu})^2,$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$ is the Yang–Mills field-strength tensor. Thus, the zero-temperature Yang–Mills coupling $g_{YM}$ goes over to the temperature-dependent dimensionful coupling $g_T = g_{YM} \sqrt{T}$. The latter defines the parametric temperature dependence of all the dimensionful nonperturbative quantities upon their dimensional reduction. In particular, the spatial string tension scales with temperature as $\sigma_s \propto g_T^4$, ensuring spatial confinement in the dimensionally-reduced Yang–Mills theory. As such, this theory can be modeled by means of DLGT.

The aim of the present paper is to address topological effects that might occur in DLGT. These effects are related to the long-range interactions between the excitations of the dual-Higgs vacuum, which are described by Wilson loops, and the dual (i.e. carrying electric fluxes) Abrikosov vortices. The latter are present in the vacuum as the topologically stable solutions to the classical equations of motion. A priori one can expect the Wilson loops and Abrikosov vortices to interact only by means of massive dual vector bosons. We show that, in addition, a long-range Aharonov–Bohm-type interaction is present, which appears in the form of a Gauss’ linking number between the contour of a Wilson loop and an Abrikosov vortex. However, in the so-called London limit, which corresponds to an extreme type-II dual superconductor, the coupling of the Aharonov–Bohm-type interaction is shown to be such that the interaction trivializes, producing only an inessential factor of $e^{2\pi i \times \text{integer}}$.

The paper is organized as follows. In the next Section, we perform a path-integral duality transformation of the Wilson loop, and explicitly find the said Aharonov–Bohm-type interaction. In Section III, we generalize these results to the case of an SU($N_c$)-inspired
In Section IV, we additionally consider the effects produced in DLGT by the Chern–Simons (CS) term. First, we briefly show that, in the absence of the dual Higgs field, the CS term leads to a self-linkage of the contour of the Wilson loop. Then we perform the duality transformation of the Wilson loop in the full theory, which includes the dual Higgs field. In particular, at sufficiently large values of the Θ-parameter entering the CS term, we obtain an analytic expression for the Wilson loop. Furthermore, in the same large-Θ limit, we explicitly find knotted dual Abrikosov vortices, whose self-linkage is provided by the CS term. In Section IV, the summary of the results obtained is presented. In Appendices A and B, we provide some technical details of the calculations performed.

II. WILSON LOOP IN THE DUAL LANDAU–GINZBURG THEORY

Dual Abelian Higgs model is described by the following Euclidean action:

\[ S_{\text{DAHM}} = \int d^4 x \left\{ \frac{1}{4} F_{\mu \nu}^2 [B] + |D_\mu \varphi|^2 + \lambda (|\varphi|^2 - \eta_{4d}^2)^2 \right\}. \]

Here \( F_{\mu \nu} [B] = \partial_\mu B_\nu - \partial_\nu B_\mu \) is the strength tensor of the dual gauge field \( B_\mu \), and \( D_\mu = \partial_\mu + ig_m B_\mu \) is the covariant derivative, with \( g_m \) being the dimensionless magnetic coupling related to the electric coupling \( e \) via the Dirac quantization condition \( g_m e = 2 \pi \times (\text{integer}) \).

We consider this model in the so-called London limit of \( \sqrt{\lambda} \gg g_m \), that is, the extreme type-II dual superconductor. Due to the factor \( e^{-\lambda \int d^4 x (|\varphi|^2 - \eta_{4d}^2)^2} \) in the partition function, the dominant contribution to the functional integral is produced by configurations of the dual Higgs field with \( |\varphi| = \eta_{4d} \). That is, variations of the radial part of the dual-Higgs field do not matter in the London limit, which is equivalent to the fact that the condensate of this field is fully developed everywhere except of infinitely thin cores of the dual strings. Rather, it is the phase of the dual Higgs field which matters, so that \( \varphi(x) = \eta_{4d} e^{i \theta(x)} \), and the kinetic term of the dual Higgs field takes the form \( |D_\mu \varphi|^2 = \eta^2_{4d} \cdot (\partial_\mu \theta + g_m B_\mu)^2 \). Accordingly, in the London limit of interest, the action of the dual Abelian Higgs model reads

\[ S_{4d} = \int d^4 x \left\{ \frac{1}{4} F_{\mu \nu}^2 [B] + \eta^2_{4d} (\partial_\mu \theta + g_m B_\mu)^2 \right\}. \]

Notice that this action can be used to calculate the tension of a Nambu–Goto string interconnecting two static electric charges, as well as the correlation length of the two-point function of \( F_{\mu \nu} \)'s (cf. Ref. [8]). Matching these two quantities with their phenomenological
QCD-counterparts, one can readily find $\eta_{3d} \sim \sqrt{\sigma}$ and $g_m \sim \frac{1}{\alpha \sqrt{\sigma}}$, where $\sigma$ is the string tension entering the static quark-antiquark potential, and $a$ is the correlation length of the two-point correlation function of gluonic field strengths.

As was mentioned in Introduction, upon the deconfinement phase transition in QCD, the chromo-electric part of the gluon condensate vanishes (in accordance with deconfinement), while the chromo-magnetic part survives, providing an area law for large spatial Wilson loops (cf. Refs. [4–6]). The corresponding spatially confining vacuum can be modelled by means of the dual Landau–Ginzburg theory. The action of this theory,

$$S_{3d} = \int d^3x \left\{ \frac{1}{4} F_{\mu\nu} [b] + \eta_{3d}^2 (\partial_\mu \theta + \kappa b_\mu)^2 \right\},$$

(3)

follows from the action (2) upon the substitution $\int d^4x \to \beta \int d^3x$, where $\beta \equiv 1/T$ [cf. the same substitution in the Yang–Mills action (1)]. Matching the fields and parameters of the action $S_{3d}$ with those of the action $S_{4d}$, we obtain the following relations:

$$b_\mu = \sqrt{\beta} B_\mu, \quad \eta_{3d} = \sqrt{\beta} \eta_{4d}, \quad \kappa = g_m \sqrt{T}.$$  

(4)

Notice that, in terms of the phenomenological QCD parameters $\sigma$ and $a$ (cf. the previous paragraph), one gets the estimates $\eta_{3d} \sim \sqrt{\sigma \beta}$, $\kappa \sim \frac{\sqrt{T/\sigma}}{a}$.

We consider now the central object of our study, that is, the Wilson loop associated with an excitation of the dual-Higgs vacuum. In the initial dual Abelian Higgs model, it has the form $\langle W(C) \rangle_{\text{DAHM}} = \langle \exp \left( ig_m N \oint_C dx_\mu B_\mu \right) \rangle$, where the integer $N$ characterizes the magnetic charge $g_m N$ of an excitation that propagates along the contour $C$. The counterpart of this expression in the dual Landau–Ginzburg theory reads

$$\langle W(C) \rangle = \left\langle \exp \left( i\kappa N \oint_C dx_\mu b_\mu \right) \right\rangle,$$  

(5)

where we have used the above relations (4). We notice that, in the purely Maxwell theory corresponding to $\eta_{3d} = 0$ in Eq. (3), the Wilson loop has the form

$$\langle W(C) \rangle = \exp \left( -\frac{(\kappa N)^2}{2} \oint_C dx_\mu \oint_C dy_\mu D_0(x - y) \right),$$

(6)

where $D_0(x) = 1/(4\pi|x|)$ is the Coulomb propagator.

We calculate now the Wilson loop $\langle W(C) \rangle$ with the average $\langle \cdots \rangle$ corresponding to the full action (3), where $\eta_{3d} \neq 0$. To this end, we find it convenient to introduce, instead of the field $b_\mu$, a rescaled field $v_\mu = b_\mu/(\kappa N)$, and denote

$$\nu = 1/(\kappa N)^2, \quad \mu = \kappa^2 N.$$  

(7)
In terms of these notations, the Wilson loop (8) can be written as

$$\langle W(C) \rangle = \int \mathcal{D} v_\mu \mathcal{D} \tilde{\theta} \mathcal{D} \bar{\theta} e^{-\int_x \left[ \frac{1}{2!} G^2_{\mu\nu} + \eta^2 (\partial_\mu \theta + \mu v_\mu)^2 - \frac{i}{\hbar} v_\mu j_\mu \right]},$$  \hspace{1cm} (8)$$

where \( j_\mu(x; C) = \hat{f}_C \, dx_\mu(\tau) \delta(x - x(\tau)) \) is a conserved current, \( \eta \equiv \eta_{3d} \), and from now on we use the short-hand notations \( \int_x \equiv \int d^3 x \) and \( \int_p \equiv \int \frac{d^3 p}{(2\pi)^3} \). The full phase \( \theta \) of the dual Higgs field can be represented as a sum \( \theta = \tilde{\theta} + \overline{\theta} \), with \( \tilde{\theta} \) experiencing jumps by \( 2\pi \) when going around dual Abrikosov vortices, while \( \overline{\theta} \) being a Gaussian fluctuation around \( \tilde{\theta} \). The said jumps of \( \tilde{\theta} \) lead to the noncommutativity of two derivatives acting on this field (cf. Ref. [2]):

$$\langle \partial_\mu \partial_\nu - \partial_\nu \partial_\mu \rangle \tilde{\theta} = 2\pi \varepsilon_{\mu\nu\lambda} J_\lambda,$$

where \( J_\lambda \) is a current of the dual Abrikosov vortex.

To calculate the Wilson loop (8), we perform its duality transformation. To this end, it is first convenient to introduce two auxiliary fields as follows:

$$e^{-\frac{i}{\hbar} \int_x F^2_{\mu\nu}} = \int \mathcal{D} G_\mu e^{\int_x \left[ -\frac{i}{\hbar} C^2_{\mu\nu} + i \varepsilon_{\mu\nu\lambda} \partial_\nu G_\lambda \right]}, \quad e^{-\eta^2 \int_x (\partial_\mu \theta + \mu v_\mu)^2} = \int \mathcal{D} C_\mu e^{\int_x \left[ -\frac{\eta^2}{4!} C^2_{\mu\nu} + \mu C_\mu (\partial_\mu \theta + \mu v_\mu) \right]}.$$  \hspace{1cm} (9)$$

The subsequent integration over \( \tilde{\theta} \) leads to the constraint \( \partial_\mu C_\mu = 0 \), which can be resolved by representing \( C_\mu \) as \( C_\mu = \varepsilon_{\mu\nu\lambda} \partial_\nu \varphi_\lambda \). Accordingly, \( C^2_\mu = \frac{1}{2} \Phi^2_\mu \), where \( \Phi_\mu \equiv \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu \), and \( i \int_x C_\mu \partial_\mu \tilde{\theta} = 2\pi i \int_x \varphi_\mu J_\mu \), where at the last step we have used Eq. (9). Thus, the Wilson loop (8) takes the form

$$\langle W(C) \rangle = \int \mathcal{D} J_\mu \mathcal{D} \varphi_\mu \mathcal{D} G_\mu \mathcal{D} v_\mu e^{\int_x \left[ -\frac{i}{\hbar} G^2_{\mu\nu} + \varepsilon_{\mu\nu\lambda} \partial_\nu G_\lambda (G_\lambda + \mu \varphi_\lambda) + 2\pi i \varphi_\mu \tilde{J}_\mu + \frac{1}{\hbar} v_\mu j_\mu \right]}.$$  \hspace{1cm} (10)$$

Note that, throughout this paper, we work at the entirely classical level. For this reason, the Jacobian corresponding to the change of integration variables \( \tilde{\theta} \rightarrow J_\mu \) is omitted, and the measure \( \mathcal{D} J_\mu \) in the functional integral has only a statistical (rather than a field-theoretical) meaning of counting vortices in their given configuration.

Next, noticing that the \( v_\mu \)-field enters Eq. (10) as just a Lagrange multiplier, and integrating over this field, we obtain a functional \( \delta \)-function \( \delta \left[ \varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) + \frac{1}{\hbar} \mu j_\mu \right] \). The subsequent \( G_\mu \)-integration amounts to substituting \( G_\mu \), which stems from this \( \delta \)-function, into \( e^{-\frac{i}{\hbar} \int_x G^2_\mu} \). Such a \( G_\mu \) reads \( G_\mu = -\nu \varphi_\mu - \frac{1}{\hbar} \varepsilon_{\mu\nu\lambda} \partial_\nu D_0^{xy} j_y^x \), where we have introduced short-hand notations \( D_0^{xy} \equiv 1/(4\pi|x - y|) \), \( j_y^x \equiv j_\lambda(y; C) \), and used the conservation of \( j_\mu \). Accordingly, the Wilson loop takes the form

$$\langle W(C) \rangle = \int \mathcal{D} J_\mu \mathcal{D} \varphi_\mu e^{\int_x \left[ -\frac{i}{\hbar} \Phi^2_\mu + 2\pi i \varphi_\mu \tilde{J}_\mu - \frac{i}{\hbar} \left( \varphi_\mu + \frac{1}{\hbar} \varepsilon_{\mu\nu\lambda} \int_y \partial_\mu D_0^{xy} j_y^x \right)^2 \right]}.$$  \hspace{1cm} (11)$$
or, equivalently,

or equivalently,

\[
\langle W(C) \rangle = e^{-\frac{1}{2\nu}\int_{x,y} j_{\mu}^x j_{\mu}^y D_{0}^{xy}} \int D J_\mu D \phi_\mu e^{\int_x \left( -\frac{\Phi^2}{4\eta^2} - \frac{\eta^2}{2} \phi^2 + i\phi_\mu K_\mu \right),}
\]

where

\[
K^x_\mu \equiv 2\pi J^x_\mu + i\mu \varepsilon_{\mu\nu\lambda} \int_y \partial_x^\nu D_{0}^{xy} j^y_\lambda.
\]

To perform the remaining \(\phi_\mu\)-integration, we introduce a rescaled field \(\chi_\mu \equiv \phi_\mu / (\eta \sqrt{2})\) and denote

\[
m \equiv \mu \eta \sqrt{2}.
\]

That yields

\[
\langle W(C) \rangle = e^{-\frac{1}{2\nu}\int_{x,y} j_{\mu}^x j_{\mu}^y D_{0}^{xy}} \int D J_\mu e^{\int_x \left[ -\frac{1}{8} \partial_\mu \chi_\nu - \frac{m^2}{2} \chi^2 + i\sqrt{2} \chi K_\mu \right] = e^{-\int_x \frac{1}{2} \frac{m^2}{2} \chi^2 - m K^2_\mu D_{m}^{xy}},}
\]

where \(D_{m}^{xy} \equiv e^{-m|x-y|/(4\pi|x-y|)}\) is the Yukawa propagator. Thus, the Wilson loop (5) becomes

\[
\langle W(C) \rangle = e^{-\frac{1}{2\nu}\int_{x,y} j_{\mu}^x j_{\mu}^y D_{0}^{xy}} \int D J_\mu e^{-\int_x \frac{1}{2} \frac{m^2}{2} \chi^2 - m K^2_\mu D_{m}^{xy}}.
\]

The expression standing in the last exponential in this formula can be simplified (see Appendix A for the details), that yields the following result:

\[
\langle W(C) \rangle = e^{-\frac{1}{2\nu}\int_{x,y} j_{\mu}^x j_{\mu}^y D_{0}^{xy}} \int D J_\mu e^{-\int_x \left[ \frac{1}{2}(\partial_\mu \chi_\nu - \partial_\nu \chi_\mu)^2 - \frac{m^2}{2} \chi^2 + i\Phi^2 \right] - \eta^2 \int_x \frac{1}{2} \frac{m^2}{2} \chi^2 - m K^2_\mu D_{m}^{xy}},
\]

where \(\hat{L}(j,J) = \varepsilon_{\mu\nu\lambda} \int_{x,y} J^\mu_\nu \partial_\lambda \chi \chi \int_x \frac{1}{2} \frac{m^2}{2} \chi^2 - m K^2_\mu D_{m}^{xy}\) is the Gauss’ linking number of the contour \(C\) and a dual Abrikosov vortex. The exponential \(e^{\frac{2\pi i}{m\nu} \hat{L}(j,J)}\) in Eq. (14) formally describes a long-range Aharonov–Bohm-type interaction of the dual-Higgs excitation with the dual Abrikosov vortex. However, recalling the notations introduced in Eq. (7), we have \(\frac{1}{\mu \nu} = N\). For this reason, the obtained interaction turns out to be trivial, i.e. \(e^{\frac{2\pi i}{m\nu} \hat{L}(j,J)} = 1\). Thus, we conclude that integer-charged excitations of the dual-Higgs vacuum do not interact with the dual Abrikosov vortices by means of the long-range Aharonov–Bohm-type interaction. Rather, the interaction between the excitations of the dual-Higgs vacuum and the dual Abrikosov vortices is provided by the dual vector boson, through the factor \(e^{-2\pi i N \varepsilon_{\mu\nu\lambda} \int_{x,y} J^\mu_\nu \partial_\lambda \chi \chi \int_x \frac{1}{2} \frac{m^2}{2} \chi^2 - m K^2_\mu D_{m}^{xy}}\).

### III. Generalization to the SU\((N_c)\)-Inspired Case

In this Section, we generalize the result (14) to the SU\((N_c)\)-inspired case. The corresponding theory is invariant under the \([U(1)]^{N_c-1}\)-group, which is the maximal...
Abelian subgroup of SU($N_c$). A counterpart of Eq. (8) in this theory reads

$$\langle W_b(C) \rangle = \int \mathcal{D}v_\mu \left( \prod_a \mathcal{D}\tilde{\theta}_a \mathcal{D}\tilde{\theta}_a \right) \mathcal{D}k \delta \left( \sum_a \tilde{\theta}_a \right) e^{-\int_x \left[ \frac{1}{4N_c} F_{\mu\nu}^2 + \eta^2 \sum_a (\partial_\mu \theta_a + \mu q_a v_\mu)^2 - ik \sum_a \theta_a - \frac{i}{2} v_\mu J^0_\mu \right]} \tag{15}$$

Here $v_\mu = (v_\mu^1, \ldots, v_\mu^{N_c-1})$, the index $a = 1, \ldots, \frac{N_c(N_c-1)}{2}$ labels positive roots $q_a$’s of the SU($N_c$)-group, and the fact that this group is special imposes a constraint $\sum_a \theta_a = 0$ on the phases $\theta_a$’s of the dual Higgs fields. Similarly to Eq. (9), we have $\theta_a = \hat{\theta}_a + \tilde{\theta}_a$, where $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \hat{\theta}_a = 2\pi J^a_\mu$, with $J^a_\mu$ being a current of the dual Abrikosov vortex of the $a$-th type. The constraint $\sum_a \tilde{\theta}_a = 0$ is further imposed in Eq. (15) by means of a Lagrange multiplier $k(x)$. Next, since the current $j^b_\mu$ describes a magnetically charged excitation of the vacuum, it is directed along some of the root vectors, $q_b$, where “$b$” is a certain fixed index from the set $1, \ldots, \frac{N_c(N_c-1)}{2}$. Therefore, one can write $j^b_\mu = q_b j_\mu$. Introducing auxiliary fields $C^a_\mu$’s as

$$e^{-\eta^2 \int_x \sum_a (\partial_\mu \theta_a + \mu q_a v_\mu)^2} = \int \prod_a \mathcal{D}C^a_\mu e^{\int_x \left[ -\frac{1}{4\eta^2} (C^a_\mu)^2 + iC^a_\mu (\partial_\mu \theta_a + \mu q_a v_\mu) \right]}$$

one obtains, similarly to the 4-d case considered in Refs. [7, 11], the following result:

$$\int \left( \prod_a \mathcal{D}\tilde{\theta}_a \mathcal{D}\tilde{\theta}_a \right) \mathcal{D}k \delta \left( \sum_a \tilde{\theta}_a \right) e^{-\int_x \left[ \eta^2 \sum_a (\partial_\mu \theta_a + \mu q_a v_\mu)^2 - ik \sum_a \theta_a \right]} =$$

$$\int \left( \prod_a \mathcal{D}J^a_\mu \mathcal{D}\varphi^a_\mu \right) \delta \left( \sum_a J^a_\mu \right) e^{\int_x \left[ -\frac{1}{4\eta^2} (\varphi^a_\mu)^2 + i\mu q_a v_\mu \partial_\nu \varphi^a_\mu + 2\pi i \varphi^a_\mu J^a_\mu \right]}.$$

Here, it has been taken into account that $\sum_a q_a = 0$, owing to which the $k$-integration yields just an inessential global normalization constant. Furthermore, the constraint $\sum_a \tilde{\theta}_a = 0$ went over into $\sum_a J^a_\mu = 0$, which means that the theory actually contains $\frac{N_c(N_c-1)}{2} - 1$ types of mutually independent vortices (cf. Refs. [7, 11] for a similar constraint for the dual strings).

To further perform the integration over $v_\mu$, it is convenient to introduce the fields $u^a_\mu = q_a v_\mu$, and use the formula [11, 12] $\sum q^a q^b = \frac{N_c}{2} \delta^{ab}$. Recalling that $j^b_\mu = q_b j_\mu$, we can then represent the $v_\mu$-dependent part of the action as

$$\int_x \left[ \frac{1}{4\nu} F_{\mu\nu}^2 - i v_\mu \left( \mu \varepsilon_{\mu\nu\lambda} q_a \partial_\nu \varphi^a_\lambda + \frac{1}{\nu} \varphi^a_\mu j_\mu \right) \right] = \int_x \left[ \frac{1}{2N_c\nu} (\partial_\nu u^a_\mu - \partial_\mu u^a_\nu)^2 - i u^a_\mu K^a_\mu \right],$$

where $K^a_\mu = \mu \varepsilon_{\mu\nu\lambda} \partial_\nu \varphi^a_\lambda + \frac{1}{\nu} \delta^{ab} j_\mu$. Then the Gaussian integration over $u^a_\mu$’s readily yields the action $\frac{N_c}{4} \int_{x,y} K^a_\mu D^b_\mu K^a_b$, which can be further simplified by representing $K^a_\mu$ as
\[ K^a_\mu = \varepsilon_{\mu\nu\lambda} \partial_\nu \left( \mu \varphi^a_\lambda + \frac{1}{\nu} \delta^{ab} \varepsilon_{\alpha\beta} \int_y \partial_\alpha D_0^{xy} j_\beta^y \right). \] In this way, we obtain the following \((N_c > 2)\)-counterpart of Eq. (11):

\[ \langle W_b(C) \rangle = \int \left( \prod_a \mathcal{D}J^a_\mu \mathcal{D}\varphi^a_\mu \right) \delta \left( \sum_a J^a_\mu \right) e^{\int_x \left\{ \frac{1}{8\pi^2} (\Phi^a_\mu)^2 + 2\pi i \varepsilon^a_\mu \partial_\nu \varepsilon^a_\nu \right\}}. \]

This expression can finally be brought to the form similar to that of Eq. (14). Indeed, proceeding in the same way as from Eq. (11) to Eq. (14), we obtain the following final result:

\[ \langle W_b(C) \rangle = e^{-N_c \frac{\nu}{4} \int x,y j^x_\mu j^y_\mu D^{xy}_m} \int \prod_a \mathcal{D}J^a_\mu \delta \left( \sum_a J^a_\mu \right) e^{-(2\pi^2)^2 \int_{x,y} j^x_\mu j^y_\mu D^{xy}_m + 2\pi i \varepsilon_{\mu\nu\lambda} \int_{x,y} j^x_\mu j^y_\nu \partial_\lambda D^{xy}_m}, \]

where \(m = \mu \eta \sqrt{N_c} \nu\) generalizes Eq. (13) for the mass of the dual vector boson. Thus, Eq. (16) represents the sought generalization of Eq. (14) to the case of \(N_c > 2\). We notice that, while the strength of the \((j \times j)\)-interaction becomes \((N_c/2)\) times larger compared to that of Eq. (14), the coefficient at the linking number remains the same. Therefore, much as in the SU(2)-inspired case, in the general SU\((N_c)\)-inspired model considered in this Section, the Aharonov–Bohm-type interaction between the integer-charged excitations of the dual Higgs vacuum and the dual Abrikosov vortices yields only a trivial factor of \(e^{2\pi i \times (\text{integer})}\).

**IV. DUAL WILSON LOOP AND ITS INTERACTION WITH ABRIKOSOV VORTICES IN THE PRESENCE OF A CHERN–SIMONS TERM**

We extend now the analysis performed in Section II to the case where the CS term is included. This term is known to produce self-linkage of the contour of a Wilson loop [10], and we expect that it would lead to a similar effect for the dual Abrikosov vortices. To start with, we again consider the theory where the dual Higgs field is absent, that is equivalent to setting \(\eta = 0\). The Wilson loop in such a theory is given by the following extension of Eq. (8):

\[ \langle W(C) \rangle = \int \mathcal{D}v_\mu e^{-\int_x \left[ \frac{1}{4\pi^2} F_{\mu\nu}^2[v] + i\Theta \varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu v_\lambda - B v_\mu j_\mu \right]}, \]
where the dimensionality of the new parameter \( \Theta \) is \((\text{mass})^2\). Imposing the gauge-fixing condition \( \partial_\mu v_\mu = 0 \), we obtain the saddle-point equation

\[-\partial^2 v_\mu + im\varepsilon_{\mu\nu\lambda} \partial_\nu v_\lambda = ij_\mu, \quad \text{where} \quad m = 2\Theta v.\]

Seeking a solution in the form \( v_\mu = U_\mu + iV_\mu \), we get a system of equations

\[\partial^2 U_\mu + m\varepsilon_{\mu\nu\lambda} \partial_\nu V_\lambda = 0, \quad -\partial^2 V_\mu + m\varepsilon_{\mu\nu\lambda} \partial_\nu U_\lambda = j_\mu. \quad (17)\]

The first of these equations can be solved with respect to \( U_\mu \) as

\[U^x_\mu = m\varepsilon_{\mu\nu\lambda} \int_y D^x_0 \partial^y_\nu V^y_\lambda. \quad (18)\]

Differentiating the second equation (17), and applying the maximum principle, one gets \( \partial_\mu V_\mu = 0 \). Using this relation, one further obtains from Eq. (18): \( \varepsilon_{\mu\nu\lambda} \partial_\nu U_\lambda = mV_\mu \). The substitution of this formula into the second equation (17) yields for that equation a remarkably simple form

\[(-\partial^2 + m^2)V_\mu = j_\mu.\]

Therefore, one has \( V^x_\mu = \int_y (D^x_0 - D^x_m) \partial^y_\nu j^y_\lambda \).

Altogether, the resulting Wilson loop has the form

\[\langle W(C) \rangle |_{\eta=0} = \exp \left\{ \frac{1}{2\nu} \int_{x,y} \left[ -j^x_\mu D^x_0 j^y_\mu + \frac{i}{m} \varepsilon_{\mu\nu\lambda} j^x_\mu \partial^y_\nu (D^x_0 - D^x_m) \right] \right\}. \quad (19)\]

Recalling the definition of the parameter \( \nu \) from Eq. (7), we observe that the obtained Eq. (19) extends Eq. (6) to the case of \( \Theta \neq 0 \). Clearly, the \( \Theta \)-term leads to a self-linkage of the contour \( C \), as well as to a short-range self-interaction of this contour by means of the Yukawa propagator \( D^x_m \). We also notice that, when \( \Theta \to 0 \) in Eq. (19), one recovers Eq. (6). Indeed, in this limit, one has \( \frac{1}{m}(D^x_0 - D^x_m) \to \frac{1}{4\pi} \), so that

\[\frac{1}{m} \int_{x,y} j^x_\mu j^y_\lambda \partial^y_\nu (D^x_0 - D^x_m) = \frac{1}{m} \int_{x,y} j^x_\mu (D^x_0 - D^x_m) \partial^y_\nu j^y_\lambda \to \frac{1}{4\pi} \int_{x,y} j^x_\mu \partial^y_\nu j^y_\lambda = 0,\]

since \( \int_x j^x_\mu = 0 \).

We proceed now to the duality transformation of the Wilson loop in the full theory, where the dual Higgs field is present and its condensation does take place, i.e. \( \eta \neq 0 \). The corresponding extension of Eq. (8) reads

\[\langle W(C) \rangle = \int Dv_\mu D\tilde{\Theta} D\tilde{\theta} e^{-\int_x \left[ \frac{i}{4} F^2_\mu[v] + \eta^2 (\partial_\mu \tilde{\theta} + \nu v_\mu)^2 + i\Theta \varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu v_\lambda - \frac{1}{2} v_\mu j_\mu \right]}. \quad (20)\]
The transformation leading from Eq. (8) to Eq. (10) remains the same, so that the counterpart of Eq. (10) in the presence of the CS term has the form

\[ \langle W(C) \rangle = \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu \mathcal{D}G_\mu \mathcal{D}v_\mu e^{\int_x \left\{ -\frac{\xi}{2} G_\mu^2 - \frac{\kappa}{2\pi} \phi_\mu^2 + i\eta_\mu \left[ \varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) + \frac{i}{2} j_\mu \right] + 2\pi i \varphi_\mu J_\mu \right\}}. \]

Unlike the case where the CS term was absent, the field \( v_\mu \) now ceases to be a Lagrange multiplier. Nevertheless, since the \( v_\mu \)-integration is Gaussian, it can be performed exactly, and we proceed to this integration.

The corresponding saddle-point equation for \( v_\mu \) reads \( \varepsilon_{\mu\nu\lambda} \partial_\nu v_\lambda = \frac{1}{2\kappa} k_\mu \), where we have denoted \( k_\mu = \varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) - \frac{1}{2} j_\mu \). Owing to the conservation of \( k_\mu \), a solution to this saddle-point equation reads \( v_\mu^s = \frac{1}{2\kappa} \varepsilon_{\mu\nu\lambda} \partial_\nu \int_y D_0^{xy} k_\lambda^y \). Plugging this solution back into the exponent \( e^{\int_x v_\mu (k_\mu - \Theta \varepsilon_{\mu\nu\lambda} \partial_\nu v_\lambda)} \), and using the above explicit expression for \( k_\mu \), we obtain, upon some algebra, the following formula:

\[ \langle W(C) \rangle = e^{\frac{1}{2\kappa} \varepsilon_{\mu\nu\lambda} \int_{x,y,j} \varepsilon^{jxy} D_0^{xy} \times} \]

\[ \times \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu \mathcal{D}G_\mu \mathcal{D}v_\mu e^{\int_x \left\{ -\frac{\xi}{2} G_\mu^2 - \frac{\kappa}{2\pi} \phi_\mu^2 + \frac{\xi e^2 m^2}{4\pi} \phi_\mu^2 \right\} \sum \left[ \varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) + \frac{i}{2} \eta_\mu \right] + \frac{1}{2\kappa} (G_\mu + \mu \varphi_\lambda) j_\mu + 2\pi i \varphi_\mu J_\mu \right\}. \]

(21)

Here, the argument of the first exponent coincides with the term containing the Gauss’ self-linking number of the contour \( C \), which was present already in Eq. (19). In addition, the functional integral in Eq. (21) describes interactions of the dual-Higgs excitation with the dual Abrikosov vortices, as well as their self-interactions in the presence of the CS term.

In order to visualize all these interactions, let us perform the \( G_\mu \)-integration first. Representing the saddle-point expression for \( G_\mu \) in the form \( G_\mu = L_\mu + iN_\mu \), we obtain a system of two saddle-point equations:

\[ \varepsilon_{\mu\nu\lambda} \partial_\nu L_\lambda - mN_\mu + n_\mu = 0, \quad \varepsilon_{\mu\nu\lambda} \partial_\nu N_\lambda + mL_\mu = 0, \]

where we have denoted \( n_\mu = \mu \varepsilon_{\mu\nu\lambda} \partial_\nu \varphi_\lambda + \frac{1}{\kappa} j_\mu \). Owing to the conservation of \( n_\mu \), we find a solution to these equations in the form

\[ L_\mu = -\varepsilon_{\mu\nu\lambda} \int_y D_m^{xy} \partial_y n_\lambda, \quad N_\mu = m \int_y D_m^{xy} n_\mu. \]

Plugging the corresponding saddle-point expression for \( G_\mu \) back into Eq. (21), we obtain, after some algebra, the following general result:

\[ \int \mathcal{D}G_\mu e^{\int_x \left( -\frac{\xi}{2} G_\mu^2 + \frac{\kappa}{2\pi} \phi_\mu^2 + \frac{\xi e^2 m^2}{4\pi} \phi_\mu^2 \right) \sum \left[ \varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) + \frac{i}{2} \eta_\mu \right] + \frac{1}{2\kappa} (G_\mu + \mu \varphi_\lambda) j_\mu + 2\pi i \varphi_\mu J_\mu \right\}. \]
\[
\langle W(C) \rangle \to \int \mathcal{D}v_\mu e^{-\frac{i}{\hbar} \int x y \left( \frac{1}{4} F_{\mu \nu}^2 - \frac{1}{2} v_\mu J_\mu \right)} = e^{-\frac{i}{\hbar} \int x y j_\mu^T D^{\mu}_0}.
\]

Therefore, the remaining \( \varphi_\mu \)-integration in Eq. (21) should also yield Eq. (6) in this limit. The limit of \( \nu \to 0 \) can thus serve as a check for Eq. (22). The right-hand side of Eq. (22) simplifies in this limit to the form

\[
e^{-\frac{i}{\hbar} \int x y j_\mu^T D^{\mu}_0 - \mu_\varepsilon_{\mu \nu} \int x y \left( j_\mu^T D^{\mu}_0 - D^{\nu}_0 j_\nu^T \right)} \times
\]

and the Wilson loop (21) becomes

\[
\langle W(C) \rangle \to e^{-\frac{i}{\hbar} \int x y j_\mu^T D^{\mu}_0 + \frac{1}{32 \pi^2} \varepsilon_{\mu \nu \lambda} \int x y \left( j_\mu^T D^{\mu}_0 - D^{\nu}_0 j_\nu^T \right)} \times
\]

\[
\int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{i \left( -\frac{1}{m^2} \Phi_\mu^2 + 2 \pi i \varphi_\mu J_\mu \right) - \mu_\varepsilon_{\mu \nu} \int x y \varphi_\mu^T D^{\nu}_0}.
\]

The Gaussian \( \varphi_\mu \)-integration in this formula yields, upon some algebra,

\[
\langle W(C) \rangle \to e^{-\frac{i}{\hbar} \int x y j_\mu^T D^{\mu}_0} \int \mathcal{D}J_\mu e^{-\left( 2 \pi \eta \right)^2 \int x y j_\mu^T D^{\mu}_0}.
\]

Recalling the normalization of the integration measure \( \mathcal{D}J_\mu \), discussed in Appendix A, we indeed recover the expected result (23). Thus, our check of Eq. (22) was successful.

We consider now large values of the \( \Theta \)-parameter, namely such that

\[
\Theta \gg \kappa \mu \eta.
\]

According to Eq. (7), such large \( \Theta \)'s imply \( m \gg \kappa \eta \), that makes the action in the exponentials on the right-hand side of Eq. (22) local, and brings the Wilson loop to the form

\[
\langle W(C) \rangle \to e^{-\frac{i}{2m^2} \int x j_\mu^2 + \frac{4 \pi^2}{3 \kappa \eta^2} \varepsilon_{\mu \nu \lambda} \left( \int x y j_\mu^T D^{\mu}_0 \right) \times
\]

\[
\int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{i \left[ -\frac{1}{m} \Phi_\mu^2 + \frac{1}{2 \kappa \eta^2} \left( \partial_\mu \varphi_\mu \right)^2 - \mu_\varepsilon_{\mu \nu} \int x y \varphi_\mu^T D^{\nu}_0 \right]}.
\]

Furthermore, in the same limiting case (24), the \( \varphi_\mu \)-integration in this formula can also be performed analytically. Referring the reader for the details to Appendix B, we present here the final result of this integration:

\[
\langle W(C) \rangle \to e^{-\frac{i}{2m^2} \int x j_\mu^2 - \frac{4 \pi}{m^4} \varepsilon_{\mu \nu \lambda} \int x j_\mu \partial_\nu j_\lambda} \times
\]
\[
\times \int \mathcal{D}_\mu e^{-\eta^2 \int_{x,y} R^\mu_{\mu'} R^\mu_{\mu''} D^{xy}_{\mu'} D^{xy}_{\mu''} + \frac{i\Theta}{\Theta} \int_{x,y} \left[ R^\mu_{\mu'} R^\mu_{\mu''} D^{xy}_{\mu'} D^{xy}_{\mu''} - 4\pi J^x_\mu \left( \pi J^y_\lambda + \frac{\mu}{m} j^y_\lambda \right) \partial^\nu D^{xy}_0 \right]}. \tag{26}
\]

In this formula, \( \mathcal{M} \equiv \frac{\mu^2}{\Theta} \), and \( R^\mu_\mu \equiv 2\pi J^\mu_\mu + \frac{\mu}{m} j^\mu_\mu \). Remarkably, in the limit (24), the initial CS term for the velocity, \( i\Theta \varepsilon_{\mu\nu\lambda} v^\mu \partial^\nu v^\lambda \) from Eq. (20), leads to the appearance of its counterpart \( \frac{i\Theta}{\Theta} \varepsilon_{\mu\nu\lambda} j^\mu_\lambda \) for the current \( j^\mu_\lambda \), while the self-linkage of the contour \( C \), described by the first exponential in Eq. (21), disappears. Rather, we observe the appearance of a self-linkage of the dual Abrikosov vortices, as well as of their linkage with the contour \( C \), as described by the term \( \frac{4\pi i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J^x_\mu \left( \pi J^y_\lambda + \frac{\mu}{m} j^y_\lambda \right) \partial^\nu D^{xy}_0 \) in the Lagrangian. In particular, the part \( \frac{4\pi i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J^x_\mu \left( \pi J^y_\lambda + \frac{\mu}{m} j^y_\lambda \right) \partial^\nu D^{xy}_0 \) of this expression yields in the action the same term \( -\frac{2\pi i}{\mu^2} \hat{L}(j, J) \) as in the absence of the CS term (cf. the end of Section II). Thus, in the presence of the CS term, the Aharonov–Bohm-type interaction of the dual-Higgs excitation with the dual Abrikosov vortex gets trivial in the limit (24). Rather, the term \( \frac{4\pi i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J^x_\mu \left( \pi J^y_\lambda + \frac{\mu}{m} j^y_\lambda \right) \partial^\nu D^{xy}_0 \) means that the CS term makes dual Abrikosov vortices knotted as long as the condition
\[
\frac{\mu^2}{\Theta} \neq \frac{2\pi \text{integer}}{}
\]
is met, where the parameter \( \mu \) is defined in Eq. (7).

V. SUMMARY

The spatial confinement in the dimensionally-reduced high-temperature gluodynamics can be modelled by means of the dual Landau–Ginzburg-type theory. In this paper, we have explored interactions between an excitation of the dual-Higgs vacuum and the dual Abrikosov vortices, which are present in such a theory. For this purpose, starting with the simplest SU(2)-inspired case, we have performed a duality transformation of the corresponding Wilson loop (5). The resulting Eq. (14) contains a long-range Aharonov–Bohm-type interaction of the dual-Higgs excitation with the dual Abrikosov vortices, which is represented by the Gauss’ linking number. However, we have found the coefficient at this linking number to be \( 2\pi i \times \text{(integer)} \), which makes the said Aharonov–Bohm-type interaction trivial. In Section III, we have obtained the same trivialization for the case of the SU(\( N_c \))-inspired dual Landau–Ginzburg-type theory, and in Section IV — at the sufficiently large values of the \( \Theta \)-parameter in the theory extended by the CS term. Thus, in all these cases, massless interactions drop out altogether from the dual formulation of the Wilson loop, so that the interactions between the dual-Higgs excitation and the dual Abrikosov vortices are mediated.
entirely by the dual vector bosons. Finally, we have explicitly demonstrated a qualitatively
novel phenomenon of the appearance of knotted dual Abrikosov vortices due to the CS term.

Acknowledgments

This work was supported by the Portuguese Foundation for Science and Technology (FCT,
program Ciência-2008) and by the Center for Physics of Fundamental Interactions (CFIF)
at Instituto Superior Técnico (IST), Lisbon. The author is grateful to the whole staff of the
Department of Physics of IST for their cordial hospitality.

Appendix A: Some details of the derivation of Eq. (14)

With the use of Eq. (12), and owing to the conservation of \( j_\mu \), one has

\[
-\eta^2 \int_{x,y} K_i^x K_i^y D_{m}^{xy} = -(2\pi\eta)^2 \int_{x,y} J_\mu^i J_\mu^i D_{m}^{xy} - 4\pi i \mu \eta^2 \int_{x,y} D_{m}^{xy} J_\mu^i \partial_{\nu} \bar{D}_{0}^{yz} j_\lambda^z + \\
+ (\mu \eta)^2 \int_{x,y} D_{m}^{xy} \left( \partial_{\nu} \int_{z} D_{0}^{xz} j_\lambda^z \right) \left( \partial_{\nu} \int_{u} D_{0}^{yu} j_\lambda^u \right) \tag{A1}
\]

We furthermore assume the standard normalization \( \langle \hat{1} \rangle = 1 \) of the functional average, which
implies a division by the functional integral \( \int \mathcal{D} J_\mu e^{-(2\pi\eta)^2 \int_{x,y} J_\mu^i J_\mu^i} \) corresponding to the
first term on the right-hand side of Eq. (A1). Thus, we always imply that the measure \( \mathcal{D} J_\mu \)
is normalized by a division by this integral.

The last term in Eq. (A1) can be represented, through the integration by parts, as

\[
(\mu \eta)^2 \int_{x,y} D_{m}^{xy} j_\mu^x \int_{u} D_{0}^{yu} j_\mu^u .
\]

In the second term on the right-hand side of Eq. (A1), one can use the equality

\[
\partial_{\nu} \int_{z} D_{0}^{xz} j_\lambda^z = \int_{z} D_{0}^{xz} \partial_{\nu} j_\lambda^z,
\]

which yields the same y-integration as in Eq. (A2): \( \int_{y} D_{m}^{xy} D_{0}^{yz} = \int_{y} D_{m}^{xy} D_{0}^{yz} = \frac{1}{m^2} (D_{0}^{yu} - D_{m}^{yu}) \).

In the second term on the right-hand side of Eq. (A1), one can use the equality

\[
\partial_{\nu} \int_{z} D_{0}^{xz} j_\lambda^z = \int_{z} D_{0}^{xz} \partial_{\nu} j_\lambda^z,
\]

which yields the same y-integration as in Eq. (A2): \( \int_{y} D_{m}^{xy} D_{0}^{yz} = \frac{1}{m^2} (D_{0}^{yu} - D_{m}^{yu}) \). Upon the subsequent integration by parts, we obtain for this term the following expression: \( \frac{2\pi i}{\mu \nu \epsilon \mu \lambda} \int_{x,y} J_\mu^i J_\nu^j \partial_{\lambda} D_{0}^{ij} \). Noticing also the definition of the Gauss’
linking number, \( \hat{L}(j, J) = \epsilon \mu \nu \lambda \int_{x,y} J_\mu^i j_\nu^j \partial_{\lambda} D_{0}^{ij} \), we arrive at Eq. (14).
Appendix B: Some details of the derivation of Eq. (26)

For $\Theta$'s obeying condition (24), one readily obtains the inequality

$$\frac{\mu^2}{\Theta^2 \nu} \ll \frac{1}{\eta^2}. \quad (B1)$$

Owing to this inequality, the term $\frac{\mu^2}{8\Theta \nu} (\partial_\mu \varphi_\mu)^2$ in Eq. (25) can be neglected in comparison with the absolute value of the term $-\frac{1}{8\eta^2} \Phi^2_{\mu \nu}$. The resulting Gaussian $\varphi_\mu$-integration can be performed by seeking the saddle-point function in the form $\varphi_\mu = \varphi_\mu^{(1)} + i\varphi_\mu^{(2)}$, and solving the so-emerging system of equations for $\varphi_\mu^{(1)}$ and $\varphi_\mu^{(2)}$. The result can be written as

$$\int D\varphi_\mu e^{I[\cdots]} = e^{\frac{1}{2} \int_y \left[-R_\mu \varphi_\mu^{(2)} - S_\mu \varphi_\mu^{(1)} + i \left(R_\mu \varphi_\mu^{(1)} - S_\mu \varphi_\mu^{(2)}\right)\right]}, \quad (B2)$$

where $R_\mu \equiv 2\pi J_\mu + \frac{\mu}{m} j_\mu$ and $S_\mu \equiv \frac{\mu}{m^2} \varepsilon_{\mu \nu \lambda} \partial_\nu j_\lambda$ are respectively the real and the imaginary parts of the current which couples to $\varphi_\mu$ in Eq. (25). The obtained real and imaginary parts of the saddle-point function $\varphi_\mu$ entering Eq. (B2) read

$$\varphi_\mu^{(1)} = \frac{2\mu \eta^2 \varepsilon_{\mu \nu \lambda}}{\Theta \mathcal{M}^2} \int_y \left[2\pi J_\lambda^y + \frac{\mu}{m} \left(1 - \frac{\mathcal{M}}{m}\right) j_\lambda^y\right] \partial_\nu \left(D^{xy}_{\mathcal{M}} - D^{xy}_0\right) - \frac{1}{m^2} \int_y j_\lambda^y \partial_\nu D^{xy}_0, \quad (B3)$$

and

$$\varphi_\mu^{(2)} = 2\eta^2 \int_y D^{xy}_{\mathcal{M}} \left[2\pi J_\mu^y + \frac{\mu}{m} \left(1 - \frac{\mathcal{M}}{m}\right) j_\mu^y\right], \quad (B4)$$

with the new mass parameter $\mathcal{M} \equiv \frac{\mu^2 \eta^2}{\Theta}$. Furthermore, in the limit (B1) at issue, the $O(\mathcal{M}/m)$-terms in Eqs. (B3) and (B4) should be neglected compared to 1. That yields the following saddle-point expressions for $\varphi_\mu^{(1)}$ and $\varphi_\mu^{(2)}$:

$$\varphi_\mu^{(1)} = \frac{2\Theta}{\mu^2 \varepsilon_{\mu \nu \lambda}} \int_y R_\lambda^y \partial_\nu \left(D^{xy}_{\mathcal{M}} - D^{xy}_0\right), \quad \varphi_\mu^{(2)} = 2\eta^2 \int_y R_\mu^y D^{xy}_{\mathcal{M}}. \quad (B5)$$

Substituting them into Eq. (B2), we obtain for the Wilson loop in the limit (24) expression (26).

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