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Anton Yu. Alekseev
Andreas Recknagel
Volker Schomerus

Vienna, Preprint ESI 389 (1996)
October 3, 1996

Supported by Federal Ministry of Science and Research, Austria
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Generalization of the
Knizhnik-Zamolodchikov-Equations

ANTON YU. ALEKSEEV, ANDREAS RECKNAGEL, VOLKER SCHOMERUS

1 Institut für Theoretische Physik,
ETH - Hönggerberg, CH-8093 Zürich, Switzerland
2 II. Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, D-22761 Hamburg, Germany
3 Institute of Theoretical Physics, Uppsala University,
Box 803, S-75108 Uppsala, Sweden

September 15, 1996

Abstract

In this letter we introduce a generalization of the Knizhnik-Zamolodchikov equations from affine Lie algebras to a wide class of conformal field theories (not necessarily rational). The new equations describe correlations functions of primary fields and of a finite number of their descendents. Our proposal is based on Nahm’s concept of small spaces which provide adequate substitutes for the lowest energy subspaces in modules of affine Lie algebras. We explain how to construct the first order differential equations and investigate properties of the associated connections, thereby preparing the grounds for an analysis of quantum symmetries. The general considerations are illustrated in examples of Virasoro minimal models.

e-mail: alekseev@teorfys.uu.se, anderl@itp.phys.ethz.ch,
vschomer@x4u2.desy.de
Correlation functions of chiral primary fields in the Wess-Zumino-Novikov-Witten (WZNW) model satisfy a system of first order differential equations known as Knizhnik-Zamolodchikov (KZ) equations [1]. During the last decade, they have led to a number of deep insights. First of all, the KZ-equations admit a reformulation as horizontality condition for a flat connection on a certain vector bundle of finite rank; the monodromy theory of this connection beautifully explains the (quasi) quantum group structure in the WZNW-model [2]. Another great achievement was to find explicit integral representations for the solutions of KZ-equations [3]. These formulas have revealed a link to the Bethe-Ansatz for Gaudin-models [4]. In [5], it was shown how to generalize the KZ-equations to WZNW models on higher genus Riemann surfaces; there, the equations include differentiations with respect to the moduli of the surface. Let us finally mention that the KZ-equations allow for discretizations (difference equations) which preserve much of the structure of their differential counterparts, see [6] and subsequent works.

It is not yet clear how much of this beautiful theory can be carried over to the generalized KZ-equation that we are about to discuss; in this letter, the generalized system of first order differential equations will be introduced from the flat connection point of view, therefore we are in principle in a position to discuss monodromy theory straight away. We will, however, only state some first elements here and give a detailed treatment elsewhere [7]. The main focus of the present paper is the formulation of the generalized KZ-equations itself, which is obtained by restricting the Friedan-Shenker connection [8] to a certain finite-dimensional subbundle of the (infinite-dimensional) bundle of conformal correlation functions.

In order to define those finite rank subbundles, we first need a more precise description of our framework. We consider a chiral $W$-algebra $W$ with finitely many bosonic generating fields

$$W^s(z) = \sum_{n=-\infty}^{\infty} W^s_n z^{-n-h_s}$$

(1)

of positive integer conformal dimension (spin) $h_s = s$; here, $z \in \mathbb{C}$ is the left movers’ coordinate and $W^s_n$, $n \in \mathbb{Z}$, are the Laurent modes of $W^s(z)$. We identify $W^2(z)$ (or one of the fields of dimension two, should there be several) with the energy momentum tensor $T(z)$, whose Laurent modes $L_n$
satisfy the Virasoro algebra

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0} \quad (2) \]

with central charge \( c \), whereas all the other generators \( W^s(z) \) are Virasoro primary, i.e.

\[ [L_n, W^s_m] = (n(h_s - 1) - m) W^s_{n+m} \quad (3) \]

holds for all \( n, m \in \mathbb{Z} \). \( \mathcal{W} \) is then linearly generated by these fields, their derivatives, and (derivatives of) normal-ordered products – see e.g. [9] for more details on \( W \)-algebras. For our purposes, it will also be sufficient to regard \( \mathcal{W} \) as the universal enveloping algebra of the Laurent modes of the \( W^s(z) \).

The representation theory of \( W \)-algebras, i.e. the classification of irreducible highest weight representations, can in principle be studied in a straightforward manner, although complete results are as yet available only for affine Lie algebras, for the Virasoro algebra and for some extensions and reductions thereof. On the other hand, to define and analyse the fusion of such representations is more difficult. For a large class of theories, namely the so-called quasi-rational CFTs, see below, Nahm has succeeded in giving a careful definition and in reducing the computation of fusion rules to a problem in finite-dimensional linear algebra [10]. There is hope that by these methods one can also find solutions of the associated Moore-Seiberg polynomial equations, but at present there are no simple algorithms which would allow to decipher information on braiding within this purely representation theoretic approach. We will see, on the other hand, that the relevant data are encoded in the flat connection of the generalized KZ-equation.

The cornerstone of the procedure in [10] is the discovery that the irreducible representations of a quasi-rational CFT contain finite-dimensional subspaces with special properties, and it is those subspaces that our formulation of generalized KZ-equations is based upon. Given a \( W \)-algebra \( \mathcal{W} \), one may introduce the subspace (or sub-Lie algebra) \( \mathcal{W}_{-} \) spanned by all the modes \( W_n \) with \( n \leq -h(\mathcal{W}) \), where \( h(\mathcal{W}) \) is the dimension of the field \( W(z) \). Each irreducible highest weight module \( \mathcal{V} \) of \( \mathcal{W} \) contains a subspace \( \mathcal{V}^a := \mathcal{W}_{-} \mathcal{V} \), and \( \mathcal{V} \) is called quasi-rational if \( n_{\mathcal{V}} := \dim(\mathcal{V}/\mathcal{V}^a) < \infty \). In that case, we can choose a subspace \( \mathcal{V}^a \) of \( \mathcal{V} \), called small space, such that \( \dim \mathcal{V} = n_{\mathcal{V}} \) and \( \mathcal{V} + \mathcal{V}^a \) is dense in \( \mathcal{V} \). Note that while the integer \( n_{\mathcal{V}} \) is an
invariant of the representation $\mathcal{V}$ of $\mathcal{W}$, there is some freedom of choice in selecting a small space. Often, however, there are natural choices; in particular, one can arrange $V$ such that it has a basis of $L_0$-eigenvectors. In the following, all $\mathcal{W}$-modules that occur are assumed to be quasi-rational modules, with small spaces carrying an $L_0$-grading.

A CFT with finitely generated (bosonic) $\mathcal{W}$-algebra is called quasi-rational if all the irreducible modules involved are quasi-rational. This class of CFTs seems to include all rational CFTs [11] but in addition important non-rational cases. Among those, the superconformal CFTs associated to Calabi-Yau targets in string theory might be the most interesting examples.\footnote{Note that the restriction to bosonic $\mathcal{W}$-algebras has merely been imposed for simplicity and is not essential; in [12] fermionic sectors are discussed as well.}

We gain some first insight into the construction of small spaces by testing it in a standard example: the affine Lie-algebras $\hat{\mathcal{G}}_k$. The corresponding $\mathcal{W}$-algebra is generated by currents $J^a(z), \ a = 1, \ldots, \dim \mathcal{G}$, which are primary fields of dimension $h = 1$; the Virasoro modes are given by the Sugawara construction ($h^\vee$ denotes the dual Coxeter number of $\mathcal{G}$)

$$L_m = \frac{1}{k + h^\vee} \sum_{n \geq 0} : J^a_{m-n} J^a_n : .$$

Irreducible highest weight representations $\mathcal{V}_\lambda$ of $\hat{\mathcal{G}}_k$ are induced from highest weight representations $\lambda$ of the zero mode algebra, which is isomorphic to the Lie algebra $\mathcal{G}$ itself. Since $\mathcal{W}_{--}$ is generated by the modes $J^a_n, \ n \leq -1$, $\mathcal{V}_\lambda$ contains all of $\mathcal{V}_\lambda$ except for the states of lowest energy. Consequently, a natural choice for the small space $\mathcal{V}_\lambda$ of an irreducible highest weight representation of affine Lie algebras is to take this lowest energy subspace. As we will see later, the small spaces in general contain descendant states of higher energy as well; on the other hand, one may always use ‘singular vector relations’ as in eq. (4) in order to find an explicit basis.

The most important abstract feature of small spaces realized in [10] is their behaviour under the ‘fusion product action’ of the $\mathcal{W}$-algebra. Here, we can avoid all subtleties connected with a precise definition of this action, since only part of this structure is needed for our purposes: We consider tensor products of $N$ carrier spaces $\mathcal{V}_i$ of quasi-rational representations.
\( V_N := \mathcal{V}_1 \otimes \ldots \otimes \mathcal{V}_N \) admits \( N \) pairwise commuting actions of the \( W \)-algebra \( \mathcal{W} \), and we write \( W_n^{(i)} \) for a mode \( W_n \) acting on the \( i \)th tensor factor of \( V_N \). Now let some tuple \( z \) of complex parameters \( z_i, \ i = 1, \ldots, N \), with \( z_i \neq z_j \) for \( i \neq j \), and define the operators

\[
M_i(W_{n-h}) := W_n^{(i)} - (-1)^n \sum_{\substack{j \neq i \\ k \geq 1}} \binom{k - 1 - n}{k - 1} (z_i - z_j)^{n-k} W_k^{(j)}
\]

for all \( n \leq 0 \) and all fields \( W(z) \in \mathcal{W} \), where \( h = h_W \) is the dimension of \( W(z) \). Denote the universal enveloping algebra of all the operators \( M_i(W_m)(z) \) by \( \mathcal{M}(z) \). This algebra could be regarded as \( \mathcal{W} \)-acting on \( V_N \) through the ‘fusion product representation’; then the elements of \( \mathcal{M}(z) \) would be obtained from contour integration of field \( W(z) \) with meromorphic 1-forms which are regular on \( \mathbb{C} \setminus z \) and vanish at infinity.

Using the basic commutation relations (2, 3), one derives the formula

\[
\partial_j M_i(W_{n-h}) + [L_{-1}^{(j)}, M_i(W_{n-h})] = -(n-1)\delta_{i,j} M_i(W_{n-1-h})
\]

and since \( \partial_i \) and \([L_{-1}^{(i)}, \cdot] \) act as derivations on the family of algebras \( \mathcal{M}(z) \), we conclude that

\[
\partial_i M + [L_{-1}^{(i)}, M] \in \mathcal{M}(z)
\]

for all elements \( M \in \mathcal{M}(z) \). Relation (6) is referred to as covariance law of the family \( \mathcal{M}(z) \).

Equipped with the algebras \( \mathcal{M}(z) \), we may reformulate the main theorem on small spaces given in [10], which can be proven by a relatively simple recursive argument:

Let \( \Phi : V_N \to S \) be a linear map into some vector space \( S \). Suppose that

1. \( \Phi \) annihilates all elements \( M \in \mathcal{M}(z) \), i.e. that \( \Phi M v = 0 \) for all \( M \in \mathcal{M}(z) \) and all \( v \in V_N \);
2. \( \Phi \) vanishes on the \( N \)-fold tensor product \( V_N \equiv V_1 \otimes \ldots \otimes V_N \subset V_N \) of small spaces.

Then \( \Phi \) vanishes on the full tensor product \( V_N \), i.e. \( \Phi \equiv 0 \).

This means that maps vanishing on \( \mathcal{M}(z) \) are completely determined as soon as we know their action on the tensor product of small spaces. We are interested in the following consequence:
Proposition 1 Let $V^*_N$ denote the dual of $V_N$ and define $E_N(z) \subset V^*_N$ to be the subspace of all linear forms on $V_N$ which annihilate elements $M \in \mathcal{M}(z)$. Then there exists an injection of $E_N(z)$ into the tensor product $V_N$ of small spaces. If this injection fails to be surjective, loosely speaking if the intersection $V_N \cap \mathcal{M}(z) V_N$ is non-empty, we say that $V_N$ contains spurious states [10].

Using the fusion product representation of $W$ on $V_N$, one would obtain the following immediate application of this proposition: Consider a tensor product of two modules $V_{\alpha} \otimes V_{\beta}$ and assume that the fusion product action of $W$ on $V_{\alpha} \otimes V_{\beta}$ can be decomposed into a sum of irreducible representations, i.e. $V_{\alpha} \otimes V_{\beta} \cong \bigoplus_\gamma \bigoplus_{i=1}^{N_{\gamma}} V_{\gamma}$ as $W$-modules; then the small space dimensions $n_{\alpha}$ and the fusion rules $N_{\alpha \beta}^{\gamma}$ satisfy the inequality [10]

$$n_{\alpha} n_{\beta} \geq \sum_{\gamma} N_{\alpha \beta}^{\gamma} n_{\gamma}.$$ 

In this situation, the inequality is strict if and only if $V_{\alpha} \otimes V_{\beta}$ contains spurious states.

Having collected these statements, we are prepared to begin our construction of the generalized KZ-equations. We allow the coordinates $z_i$ to sweep out an open subset $U$ of the configuration space. A map $F : U \to V^*_N$ is called holomorphic if the functions $h_{F(z)}^{_{|\nu}}$ are holomorphic for all $\nu \in V_N$. The trivial infinite rank vector bundle $U \times V^*_N$ can be equipped with the following flat Friedan-Shenker (FS) connection (defined on holomorphic sections $F(z)$),

$$\nabla^FS_N = \sum_{i=1}^{N} dz_i \nabla^FS_i \quad \text{with}$$

$$\nabla^FS_i F(z) = \partial_i F(z) - F(z) L^{(i)}_{-1},$$

where the last term is understood as composition of linear maps on $V_N$. Flatness of the connection $\nabla^FS_N$ holds because of the commutation relation $[L^{(i)}_{-1}, L^{(j)}_{-1}] = 0$ and because $L^{(i)}_{-1}$ does not depend on the $z_j$.

The crucial observation for our purposes is that $\nabla^FS_N$ restricts to a (flat) connection on a certain finite rank subbundle. Let us define the space of generalized conformal blocks as spanned by all holomorphic $F(z)$ which
vanish on $\mathcal{M}(z)$, i.e. $F(z) \in E_N(z)$. For such $F(z)$, we evaluate $\nabla_i^{FS} F(z)$ on $\mathcal{M}(z)$ and find

$$
\langle \nabla_i^{FS} F(z) | M(z)v \rangle = (\partial_i F(z)| M(z)v) - L_{-1}^{(i)} \langle F(z)| M(z)v \rangle
= -\langle F(z)| \partial_i M(z)v \rangle - \langle F(z)| [L_{-1}^{(i)}, M(z)]v \rangle .
$$

In the last step we have dropped two terms which are zero due to our assumption on $F(z)$. The covariance law (6) for the family of algebras $\mathcal{M}(z)$ finally gives

$$
\langle F(z)| \mathcal{M}(z)v \rangle = 0 \Rightarrow \langle \nabla_i^{FS} F(z) | M(z)v \rangle = 0
$$

for all $M(z) \in \mathcal{M}(z)$ and $v \in V_N$. When combined with Proposition 1 above, the result may be expressed as follows:

**Proposition 2** (Generalized KZ-connection) Suppose that the irreducible highest weight modules $V_i$ are quasi-rational. Then the Friedan-Shenker connection (7) descends to a flat connection

$$
\nabla_N = \sum_i dz_i \nabla_i \quad \text{with} \quad (8)
$$

$$
\nabla_i \equiv \partial_i - A_i = \nabla_i^{FS} \mid_\mathcal{E}
$$

on the finite rank vector bundle $\mathcal{E}$ of generalized conformal blocks. The fibres $E_N(z)$ of $\mathcal{E}$ can be embedded into the tensor product $V_N$ of small spaces $V_i \subset V_N$.

The generalized KZ-equation is then simply the horizontality condition

$$
\nabla_N F(z) = \sum_i dz_i (\partial_i - A_i) F(z) = 0 . \quad (9)
$$

At least if there are no spurious states, it is relatively easy to obtain explicit formulas for the connection. We recall this for the case of affine Lie algebras first and then give some new examples.

We want to determine a formula for $\nabla_i F(z)$ when $F(z)$ vanishes on $\mathcal{M}(z)$. For these $F(z)$, $\nabla_i F(z)$ vanishes on $\mathcal{M}(z)$ too and hence it suffices to evaluate the form $\nabla_i F(z)$ on states in the tensor product of small spaces, i.e. to compute

$$
\langle \nabla_i F(z) | v \rangle \quad \text{for all} \quad v \in V_N .
$$
In the case of affine Lie algebras, we can exploit the fact that all the operators $J^{a(i)}_n$ with $n \geq 1$ vanish on $V_N$. With the help of eq. (4) we find that

$$
\langle \nabla_i F(z), v \rangle
= \partial_i \langle F(z)|v \rangle - \frac{1}{k + \hbar^2} \sum_a \langle F(z)|J^{a(i)}_{i-1} J^{a(i)}_0 v \rangle
= \partial_i \langle F(z)|v \rangle - \frac{1}{k + \hbar^2} \sum_{a,j \neq i} \frac{1}{z_i - z_j} \langle F(z)|J^{a(j)}_0 J^{a(i)}_0 v \rangle .
$$

When the operators $J^{a(i)}_0$ are replaced by their representation matrices $t^a_i$ on the highest weight space $V_i$, our short computation produces the usual KZ-connection. Even though it is rather standard and easy to follow, we would like to describe the individual steps once more to distinguish particular properties of the example from generic features of the calculation.

We start the evaluation of $\nabla_i$ by inserting its definition. This produces an operator $L^{(i)}_{-1}$ acting on the vector $v$ in the tensor product $V_N$ of small (or highest weight) spaces. The result is no longer in $V_N$ and can be expressed as a sum of terms where generators of $W_{\lambda}$ act on $V_N$, or – to be more specific – where $J^{a(i)}_{-1}$ act on $J^{a(i)}_0 v \in V_N$. From the definition of our operators $M_i(W_n)$ and the requirement that $\langle F(z)|$ vanishes on $M(z)$ we obtain

$$
\langle F(z)|W^{(i)}_{n-h} = \sum_{\substack{k \geq 1 \atop k \neq 1}} (-1)^n \binom{k - 1 - n}{k - 1} (z_i - z_j)^{n-k} \langle F(z)|W^{(j)}_{k-h} .
$$

which is used to reshuffle the generators of $W_{\lambda}$ appearing on the lhs. In our case, where $h = 1$, it serves to replace operators $J^{a(i)}_{-1}$ by an infinite sum of $J^{a(j)}_m$ with $m \geq 0$. Since all $J^{a(j)}_m$ with $m > 0$ vanish on the lowest energy subspace, only the first terms $J^{a(j)}_0$ act non-trivially on $V_N$ and contribute to the connection.

Most of this analysis applies to more general examples of $W$-algebras, too. Only the discussion of the infinite sum in the end is based on some exceptional features of the affine Lie algebras and their representation theory. In general, a larger (but finite) number of terms in the sum may contribute because $h$ may be larger than 1 and because $W^{(j)}_n v, v \in V_N$, need not
vanish for $n > 0$. Moreover, the action of operators $W_n^{(j)}$ can result in vectors outside of $V_N$ so that iterated reshuffling with the help of eq. (10) is necessary. This means that an explicit formula for the generalized KZ-connection will have more terms than in the WZNW case – the strategy for its derivation, however, remains unchanged.

The degenerate models of the Virasoro algebra already display most aspects of the generic scenario. Their $W$-algebra is generated by the Virasoro field $T(z)$ with central charge $c = 1 - 6(p-q)^2/pq$, and their highest weight modules are induced from vectors $|h\rangle$ with

$$L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = 0$$

for all $n > 0$. For $h = h(r,s) = (pr - qs)^2 - (p - q)^2)/4pq$ with $p, q$ real and $r, s$ integer, those modules are quasi-rational and the dimension of the small space $V = V_{p,q}^{r,s}$ can be computed to be

$$n \equiv n_{r,s}^{p,q} := \text{dim}(V_{r,s}^{p,q}) = \begin{cases} \min\{rs, (p-s)(q-r)\} & \text{if } p, q \text{ integer} \\ rs & \text{else} \end{cases}.$$  

This number is just the energy level where the first singular vector in the Verma module occurs: Observe that all elements $L_n$, $n \leq -2$, are in $\mathcal{W}_-$ so that we can choose a small space $V$ spanned by the states $L_{-1}^{\nu} |h\rangle$ with $\nu = 0, 1, \ldots$; when $\nu = n$, we find relations of the form

$$L_{-1}^{n} |h\rangle + |v\rangle = 0$$

with some vector $|v\rangle \in V^n$, see e.g. [13], which shows that $\dim V \leq n$; a closer look at the structure of the normal ordered products [9] tells that indeed none of the vectors $L_{-1}^{\nu} |h\rangle$, $\nu = 0, \ldots, n - 1$, is contained in $V^n$.

Let us now restrict to the case $(r,s) = (2,1)$ for which one has

$$(L_{-1}^2 - \alpha L_{-2})|h(2,1)\rangle = 0 \quad \text{with} \quad \alpha = \frac{2(2h(2,1) + 1)}{3}. \quad (11)$$

Thus, each small space is two-dimensional and we choose the one with basis vectors $|h\rangle, L_{-1}|h\rangle$. Following the procedure outlined above for WZNW models, one may derive formulas for the generalized KZ-connection corresponding to $N$-fold tensor products of $V_{2,1}$. We state an explicit result for
\[ \nabla_i = \partial_i - A_i^N \] in the case \( N = 2 \), where we obtain the following two \( 2^2 \times 2^2 \)-matrices, evaluated in the basis \( v_1 = |h\rangle \otimes |h\rangle \), \( v_2 = |h\rangle \otimes L_{-1}|h\rangle \), \( v_3 = L_{-1}|h\rangle \otimes |h\rangle \) and \( v_4 = L_{-1}|h\rangle \otimes L_{-1}|h\rangle \):

\[
A_1^2(z) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & \alpha & 1 \\
\frac{z_1 - z_2}{(z_1 - z_2)^2} & \frac{z_1 - z_2}{(z_1 - z_2)^2} & \alpha & 0 \\
\frac{(\alpha + 2)\alpha}{(z_1 - z_2)^2} & -\alpha^2 & 0 & 0
\end{pmatrix}
\] (12)

\[
A_2^2(z) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
\frac{\alpha h}{(z_1 - z_2)^2} & 0 & \frac{z_2 - z_1}{(z_1 - z_2)^2} & 0 \\
0 & 0 & \frac{\alpha h}{(z_1 - z_2)^2} & 1 \\
\frac{(\alpha + 2)\alpha h}{(z_1 - z_2)^2} & -\alpha^2 & \frac{(z_1 - z_2)^2}{(z_2 - z_1)^2} & 0
\end{pmatrix}
\] (13)

The null vector (11), which already allowed us to determine the dimension of the small space, is again essential during the (multiple) reshuffling in the computation of \( A_i^2 \), cf. the remarks on the WZNW calculation. Above, we have tacitly assumed that \( p \) and \( q \) are chosen in such a way that no spurious states occur, which otherwise (e.g. for \( p/q = 2/5 \)) would have to be taken care of in the reshuffling procedure.

At this stage, let us recall a more general aspect of singular vectors: It is well known that by conformal covariance their existence leads to higher order differential equations on the conformal blocks. Of course, those can be replaced by systems of first order differential equations – a procedure which is, however, highly non-unique. From this point of view, the remarkable feature of the generalized KZ-equations presented here is that they provide a canonical first order system.

We would also like to remark that for coset models and certain generalizations, first order differential equations for conformal correlators are known for some time [14], [15]. It might well be that an investigation of the relation between those results and our generalized KZ-equations proves to be useful for obtaining integral representation of the correlation functions in a general case.

Turning back to the explicit formulas (12,13) given above, we detect an obvious difference to the ordinary KZ-connection: The connection matrices \( A_i^2 \) contain higher order poles. The appearance of higher singularities is somewhat disturbing since we expect the term \( \nabla_i \) in our generalized KZ-connection to determine monodromy properties of the solutions when the
‘particle’ at \(z_i\) is moved around the one at \(z_j\). Such monodromies can be read off from the simple poles of the connection if there are no poles of higher order. So how is it possible to determine the monodromy from the generalized KZ-connection?

To state the answer we need some more notations. Denoting the \(L_0\)-highest weight of some \(\mathcal{W}\)-module \(\mathcal{V}\) by \(h\), we define a map \(d : V \to V\) by restricting \(L_0 - h\) to the small space \(V\), i.e. \(d = L_0|_V - h\). In this way, \(V\) is endowed with a \((\mathbb{Z}\text{-valued})\) energy grading. From \(d\) one obtains \(N\) maps \(d_i\) on \(\mathcal{V}_N\) so that \(d_i\) detects the energy grading in the \(i\)th factor of the tensor product \(\mathcal{V}_N\), i.e. \(d_i = d^{(i)}\). Whenever \(d\) does not vanish, the holomorphic, matrix-valued functions

\[
D_{ij}(z) := (z_i - z_j)^{d_i + d_j}
\]  

(14)

are non-trivial and can be used to prove the following result:

**Proposition 3** Let \(\nabla_N^{ij}\) be obtained from \(\nabla_N\) by a holomorphic gauge transformation with \(D_{ij}(z)\) (defined as in eq. (14)), i.e.

\[
\nabla_N^{ij} \equiv D_{ij}(z)\nabla_N d^{-1}(z) = \sum_k d_k \nabla_N^{ij}.
\]

Then the term \(\nabla_N^{ij}\) has at most simple poles in \((z_i - z_j)\). Moreover, the residue at these poles is independent of \(N\).

In other words, one can simply read off the monodromy from the residue of the generalized connections after having performed an explicitly known gauge transformation. This residue, and hence the monodromy, is not influenced by the possible presence of ‘particles’ at \(z_k\) with \(k \neq i, j\).

Let us also note that the monodromy does not depend (except from a similarity transformation) on whether we move \(z_i\) around \(z_j\) or \(z_j\) around \(z_i\). A mathematical formulation of this property involves the permutation operators \(P_{ij} : \mathcal{V}_N \to \mathcal{V}_N\) which permute the \(i\)th and \(j\)th factor in the tensor product \(\mathcal{V}_N\) of small spaces. In addition, let us define maps \(\kappa_{ij}\) which exchange the \(i\)th and \(j\)th coordinate so that

\[
\kappa_{ij}(z_i) = z_j, \quad \kappa_{ij}(dz_i) = dz_j, \quad \kappa_{ij}(\partial_i) = \partial_j
\]

etc. Our next proposition states that the generalized KZ-connection is symmetric under the simultaneous exchange of small spaces and coordinates.
Proposition 4 The connection $\nabla_N$ is symmetric in the sense that

$$P_{ij} \nabla_N P_{ij} = \kappa_{ij}(\nabla_N)$$

holds for all indices $i, j = 1, \ldots, N$, where $P_{ij}$ and $\kappa_{ij}$ denote permutation of small spaces and coordinates, respectively.

Here, we have just described the first elements of a full monodromy theory for the generalized KZ-equations. Being flat and symmetric, the connection $\nabla_N$ determines a representation of the braid group $B_N$ on the tensor product $V_N$ of small spaces. For the ordinary KZ-equations, this monodromy representation was computed explicitly by Drinfel’d [2]. He showed that it may be obtained from a certain quasi-Hopf algebra and compared the latter with the quantum universal enveloping algebras $U_q(\mathcal{G})$. Some parts of his analysis generalize to our framework and admit the reconstruction of a quasi-Hopf algebra. An explicit formula for the universal $R$-matrix, for instance, is based on Proposition 3. We plan to discuss the monodromy theory in a forthcoming paper [7]. Let us nevertheless remark that existence of the quasi-Hopf algebras is already guaranteed by abstract reconstruction theorems in [16]. The latter, however, make no predictions about the carrier spaces for representations of the quasi-Hopf algebra. Our analysis leads us to identify these carrier spaces with the small spaces of quasi-rational models and produces a canonical quasi-Hopf algebra.

It is a pleasure to thank W. Nahm for stimulating and helpful discussions. We also thank J. Fröhlich, M. Gaberdiel, K. Gawedkzi and J. Teschner for useful comments. Hospitality of the Erwin-Schrödinger-Institute, Vienna, where part of this work was done, is gratefully acknowledged.

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