Microscopic realization of cross-correlated noise processes

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(Dated: June 1, 2010)

We present a microscopic theory of cross-correlated noise processes, starting from a Hamiltonian system-reservoir description. In the proposed model, the system is nonlinearly coupled to a reservoir composed of harmonic oscillators, which in turn is driven by an external fluctuating force. We show that the resultant Langevin equation derived from the composite system (system+reservoir+external modulation) contains the essential features of cross-correlated noise processes.

In this paper, we deal with a bath that is being nonlinarly driven by an external noise. This work is an attempt to analyze the mechanism of the action of additive, as well as multiplicative cross correlated noises, as well as to get an insight of the mutual interplay of these noises from a microscopic standpoint. This is in contrast with the traditional works which essentially treats the external cross correlated noises in a phenomenological manner. We point out the important fact that the non-linear driving becomes ubiquitous to explain (i) the actual microscopic origin of space-dependent dissipation and multiplicative noise, and (ii) the origin of external multiplicative cross correlated noise.

I. INTRODUCTION

A huge impetus has been envisaged during the last few decades towards research leading to an in-depth understanding of the detailed dynamics of systems that are subjected to external noise fields. Experimental and theoretical studies in the recent past have revealed the ubiquitous and constructive role of noises in a plethora of physical phenomena ranging from the self-organization and dissipative dynamics of systems, noise-induced transitions, phase transitions driven by noises, thermal ratchets (or Brownian motors) to even the stochastic resonance in zero-dimensional and spatially extended systems. Noises have their own role to play in a host of problems pertaining to physical, chemical, biological relevance as well as those related to economics phenomena [see Refs.1–4]. The common feature of an overwhelming majority of these studies is that the system is thermodynamically closed and the energy conservation is guided by the celebrated fluctuation-dissipation relation.5 However, in a number of situations the system may be thermodynamically open.5 There exists no fluctuation-dissipation relation for the thermodynamically open system2 and recently such a system has attained a wide attention.8–12 The origin of the noise in the thermodynamically open system driven by two or more random forces may be different. The barrier crossing dynamics with multiplicative and additive noises initiated strong interest in the early 1980s. Formerly, in most of the initial works, noise forces that were present simultaneously in stochastic physical processes have usually been treated as uncorrelated random variables, since it has assumed that they have different noise origins. However, there are some situations where noises in some open systems may have a common origin and hence can be correlated.13 If this happens, then the statistical properties of the noises should not be much different and can be correlated with each other. It has also been envisaged in many situations that strong external noises often modify the internal properties of a dynamical system, thus, in a way, necessitating that the internal and external noises be independent of each other. The case of simultaneous acting uncorrelated and correlated additive and multiplicative noise has been studied by various authors. The study of nonlinear dynamical systems perturbed cross-correlated noises has become an attractive subject in recent years.14–17 Correlated noise processes have made their presence felt in a wide range of studies, such as the statistical properties of a single mode laser15, bistable kinetics16, barrier crossing dynamics17,18, steady state entropy production19 and noise induced transport20. Recent time, it is now well accepted that the effect of correlation between additive and multiplicative noise is indispensable in explaining phenomena such as phase transition, transport of motor protein and the presence of cross-correlated noises changes the dynamics of the system.21

It was first pointed out by Fedchenko22 that the cross-correlated noises have a vital role to play in the realm of hydrodynamics of vortex flow from a common origin that appear in the time evolution equation of dimensionless modes of flow rates. In literature, numerous evidences fortify the effect of the interference among the additive and multiplicative noises in the context of dynamics, phase transition and many other relevant phenomena. While Fulinski and Telejko23 have studied the effect of correlation between the additive and multiplicative white noise in the kinetics of the bistable systems, Madureira et al’s work24 throws light on the role of the coupled effect of two correlated white noises (additive and multiplicative) on the escape-rate of double-well systems. In this context, the role of correlation between additive

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and multiplicative noises have been explored by Mei et al. during their study of the relaxation of a bistable system driven by cross-correlated noises. In one of the important developments in this direction, Hänggi and co-workers have put forth the influence of two white Gaussian noise sources that are correlated, with one being associated with an additive white noise and the other with a multiplicative white noise.

It has been experienced in dealing with the dynamical systems that in several cases the effect of adding up noises makes the system more ordered. For example, it has been observed that in such systems that the lifetime is markedly prolonged upon the addition of two correlated Gaussian noise terms, vis-à-vis the lifetime in the presence of a single noise term. At the level of a Langevin type description of a dynamical system, the presence of correlation between noises can change the dynamics of the system. In recent years, there has been an increasing interest in studying the effects of the noises in the (nonlinear) dynamical systems.

The aim of the present work is to investigate the microscopic origin of mutual correlation of additive and multiplicative external noises. We hope our model can be used as an efficient tool to study mechanism of various above mentioned effects and phenomena in systems driven by cross-correlated noises, one additive and the other multiplicative.

II. THEORETICAL DEVELOPMENT

To start with, we consider the system to be coupled to a harmonic heat bath with characteristic frequency sets \( \{\omega_j\} \). The coupling between the system and the bath is, in general, considered to be nonlinear as well. Initially at \( t = 0 \), the bath is in thermal equilibrium at a temperature \( T \). At \( t = 0_+ \), the external fluctuating force \( \epsilon(t) \) is switched on, which modulates the harmonic heat bath. The Hamiltonian for the composite system thus can be written as

\[
H = H_S + H_B + H_{SB} + H_{int},
\]

where \( H_S = (p^2/2) + V(q) \) is the system’s Hamiltonian with \( q \) and \( p \) being the co-ordinate and momentum of the system of interest having unit mass. Here, \( V(q) \) is the external force field. The second and third terms in the right hand side of Eq.(1) refer to the harmonic heat bath and the interaction between the system and the latter,

\[
H_B + H_{SB} = \sum_{j=1}^{N} \left\{ \frac{p_j^2}{2} + \frac{1}{2} \omega_j^2 (q_j - c_j f(q))^2 \right\},
\]

with \( q_j \) and \( p_j \) being the bath variables. The quantity \( c_j \) is the coupling constant and \( f(q) \) is a smooth well behaved function of system variables. The last term in Eq.(1), \( H_{int} \), takes care of the interaction between the harmonic bath and the external fluctuations \( \epsilon(t) \)

\[
H_{int} = \sum_{j=1}^{N} \kappa_j h(q_j) \epsilon(t),
\]

with, \( \kappa_j \) being the strength of the interaction and \( h(q_j) \) is an arbitrary analytical function of bath variables and in general, nonlinear. The external driving force \( \epsilon(t) \) is considered to be a stationary delta correlated one with unit noise strength and follows the Gaussian statistics

\[
\langle \epsilon(t) \rangle = 0, \langle \epsilon(t) \epsilon(t') \rangle = 2\delta(t-t').
\]

Now, from Eq.(1) the various equations of motion for the system and bath variables are found to be

\[
\dot{q} = p, \quad \dot{p} = -V'(q) + \sum_{j=1}^{N} c_j \omega_j^2 (q_j - c_j f(q)) f'(q),
\]

\[
\dot{q}_j = p_j, \quad \dot{p}_j = -\omega_j^2 (q_j - c_j f(q)) - \kappa_j \frac{dh}{dq_j} \epsilon(t).
\]

Using the explicit form of the function \( h(q_j) \)

\[
h(q_j) = q_j + \frac{1}{2} q_j^2,
\]

Eq.(5) becomes

\[
\dot{q}_j + \left\{ \omega_j^2 + \kappa_j \epsilon(t) \right\} q_j = c_j \omega_j^2 f(q) + \kappa_j \epsilon(t).
\]

An analysis of Eq.(7) reveals the fact that the presence of the term \( \epsilon(t) \) in the equation essentially transforms an otherwise simple harmonic oscillator driven by a force [the terms on the right hand side of Eq.(7)] to a situation with an oscillator having fluctuating (or random) frequencies. While in the absence of \( \epsilon(t) \), we could have, at least in principle, sought for the analytic solution of the problem. In the present situation, needless to mention is the fact that we have to resort to some approximation method to solve Eq.(7). The standard system- reservoir model assumes that the any change of the system degrees of freedom does not affect the spatio-temporal evolution of the harmonic bath, while the reverse is not true. As a consequence, we use the perturbative solution of the bath to monitor the change in the system and thereby eliminate the bath variables from the system description. In nutshell, we resort to a standard approximation for the solution of Eq.(7) in which the harmonic baths remain unaffected by the system, so that we neither need to seek for a simultaneous solution of the system and bath variables, nor any explicit perturbative correction is needed for them.

To solve Eq.(7) we assume

\[
q_j(t) = q_j^0(t) + \kappa_j q_j^1(t),
\]
with $|\kappa_j| < 1$ ($\kappa_j$ is small, and of the same order for all $j$) and both $q_j^0(t)$ and $q_j^1(t)$ [representing a small perturbation around $q_j^0(t)$] satisfy the equations

$$\dot{q}_j^0(t) + \omega_j^2 q_j^0(t) = c_j \omega_j^2 f_q(t),$$

and

$$\dot{q}_j^1(t) + \omega_j^2 q_j^1(t) = -q_j^0(t)\epsilon(t) - \epsilon(t),$$

respectively. Eqs. (9)-(10) have been written on the physical ground that at $t = 0$ the heat bath is in thermal equilibrium, which in turn is modulated by external fluctuations $\epsilon(t)$, at $t = 0$. Under such condition, $q_j(0) = q_j^0(0)$ and $p_j(0) = p_j^0(0)$. This also implies that $q_j^1(0) = p_j^1(0) = 0$. Making use of the formal solution(s) of Eq. (9), namely,

$$q_j^0(t) = q_j^0(0) \cos \omega_j t + \frac{p_j^0(0)}{\omega_j} \sin \omega_j t$$

$$+ c_j \omega_j \int_0^t dt' \sin \omega_j (t - t') f(t'),$$

we obtain the following expression for the bath variable $q_j(t)$

$$q_j(t) = c_j f(q) + \{q_j^0(0) - c_j f(q(0))\} \cos \omega_j t + \frac{p_j^0(0)}{\omega_j} \sin \omega_j t$$

$$- c_j \int_0^t dt' \cos \omega_j (t - t') f'(q(t')) q_j(t')$$

$$- \frac{\kappa_j}{\omega_j} \int_0^t dt' \sin \omega_j (t - t') \epsilon(t') \left[1 + q_j^0(t')\right].$$

Now inserting Eq. (12) into Eq. (11), we obtain the dynamical equation for the system variable as

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) - f'(q) \int_0^t dt' \gamma(t - t') f'(q(t')) p(t')$$

$$f'(q) \sum_{j=1}^N c_j \kappa_j \omega_j \int_0^t dt' \sin \omega_j (t - t') \epsilon(t') q_j^0(t')$$

$$+ f'(q(t)) \xi(t) + f'(q(t)) \pi(t)$$

(13)

where the random force $\xi(t)$ and the memory kernel $\gamma(t)$ are given by

$$\xi(t) = \sum_{j=1}^N c_j \omega_j^2 \left[\{q_j^0(0) - c_j f(q(0))\} \cos \omega_j tight.$$

$$+ \frac{p_j^0(0)}{\omega_j} \sin \omega_j t],$$

$$\gamma(t) = \sum_{j=1}^N c_j^2 \omega_j^2 \cos \omega_j t.$$ (14)

In Eq. (13), $\pi(t)$ is the fluctuating force generated due to the linear part of the coupling function $h(q_j)$ and is given by

$$\pi(t) = -\int_0^t dt' \varphi(t - t') \epsilon(t'),$$ (16)

where

$$\varphi(t) = \sum_{j=1}^N c_j \omega_j \kappa_j \sin(\omega_j t).$$ (17)

The form of Eq. (13) reflects that the system dynamics is effectively modulated by three fluctuating forces, $\xi(t)$, the internal thermal noise and the other two fluctuating forces due to driving of the bath by an external random force $\epsilon(t)$. If the external noise-bath coupling is linear, i.e., when $h(q_j) = q_j$, the fourth term disappears and we obtain our earlier results. $9,29,30$ It is important to note that all the effective fluctuating forces are multiplicative in nature due to the presence of the system variable function $f'(q)$. The statistical properties of $\xi(t)$ can be derived by using suitable canonical thermal distribution of bath coordinates and momenta at $t = 0$. We assume that the bath variables $\{q_j(0), p_j(0)\}$ are distributed according to the Gaussian form with the probability distribution function, $W\{q_j(0), p_j(0)\} = (1/Z) \exp\left[-\left[H_B + H_{SB}\right]/kB T\right]$ where $Z$ is the partition function. Then, the statistical properties of the fluctuating force $\xi(t)$ become

$$\langle \xi(t) \rangle = 0, \langle \xi(t) \xi(t') \rangle = 2\sigma(t - t') k_B T.$$ (15)

To identify Eq. (13) as a generalized Langevin equation for nonequilibrium open system, we need to impose some conditions on the coupling co-efficients $c_j$ and $\kappa_j$, on the bath frequencies $\omega_j$ and on the number $N$ of the bath oscillators that ensure $\gamma(t)$ to be indeed dissipative. A sufficient condition for $\gamma(t)$ to be dissipative is that it is positive definite and decreases monotonically with time. These conditions are achieved if $N \rightarrow \infty$ and if $c_j \omega_j^2$ and $\omega_j$ are sufficiently smooth functions of $j$. As $N \rightarrow \infty$ one replaces the sum by an integral over $\omega$ weighted by a density of state $\rho(\omega)$. Thus, to obtain a finite result in the continuum limit, the coupling function $c_j = c(\omega)$ and $\kappa_j = \kappa(\omega)$ are chosen as

$$c(\omega) = \frac{c_0}{\omega \sqrt{\tau_c}}, \quad \text{and} \quad \kappa(\omega) = \kappa_0 \omega \sqrt{\tau_c}.$$ (18)

Consequently, $\gamma(t)$ and $\varphi(t)$ reduce to

$$\gamma(t) = \frac{c_0^2}{\tau_c} \int_0^\infty d\omega \rho(\omega) \cos(\omega t)$$

and

$$\varphi(t) = c_0 \kappa_0 \int_0^\infty d\omega \rho(\omega) \omega \sin(\omega t),$$ (19) (20)

where $c_0$ and $\kappa_0$ are constants and $\omega_c = 1/\tau_c$ is the cut-off frequency of the bath oscillators. $\tau_c$ may be regarded as the correlation time of the bath oscillators and $\rho(\omega)$ is the density of modes of the heat bath which is assumed to be Lorentzian: $\rho(\omega) = 2\tau_c/\pi(1 + \omega^2 \tau_c^2)$. With these forms of $\rho(\omega)$, $c(\omega)$ and $\kappa(\omega)$, $\gamma(t)$ and $\varphi(t)$ take the following forms:

$$\gamma(t) = \frac{c_0^2}{\tau_c} \exp(-|t|/\tau_c) = \frac{\gamma}{\tau_c} \exp(-|t|/\tau_c),$$

$$\varphi(t) = \frac{c_0 \kappa_0}{\tau_c} \exp(-|t|/\tau_c).$$ (21) (22)
For $\tau_c \to 0$, Eqs. (21,22) become $\gamma(t) = 2\gamma(t)$, where $\gamma = c_0^2$ and $\varphi(t) = 2c_0\kappa_0\delta(t)$; and consequently, one obtains the $\delta$-correlated noise processes describing Markovian dynamics.

The third term on the right hand side of Eq. (13) can be written as

$$\dot{f}[q(t)] \int_0^t dt'[e(t')] \int_0^{t'} dt'' \left\{ \sum_j c_j^2\kappa_j\omega_j^2 \sin\omega_j(t - t') \times \sin\omega_j(t' - t'') \right\} f[q(t'')].$$

where we have used the particular solution for $q_0(t)$ [see Eq. (11)]. The above expression can be rewritten, using known trigonometric identity, as

$$\frac{1}{2} f'[q(t)] \int_0^t dt'[e(t')] \int_0^{t'} dt'' f[q(t'')] \sum_j c_j^2\kappa_j\omega_j^2 \times \cos\omega_j(t - 2t' + t'') + \frac{1}{2} f'[q(t)] \int_0^t dt'[e(t')] \int_0^{t'} dt'' f[q(t'')] \sum_j c_j^2\kappa_j\omega_j^2 \times \cos\omega_j(t - t'').$$

Now, in the continuum limit, and for Markovian internal dissipation, one easily obtains that

$$\sum_j c_j^2\kappa_j\omega_j^2 \cos\omega_j(t - 2t' + t'') = 2c_0^2\kappa_0\delta(t - 2t' + t'')(25)$$

and

$$\sum_j c_j^2\kappa_j\omega_j^2 \cos\omega_j(t - t'') = 2c_0^2\kappa_0\delta(t - t'').$$

Now keeping in mind that $t > t' > t''$ one can show that after integration, the first term in expression (24) will not contribute, while the second term transpires to $c_0^2\kappa_0 f[q(t)] f'[q(t)]\epsilon(t)$.

Taking into consideration all the above assumptions and assuming that the system variables evolve much more slowly in comparison to the external noise $\epsilon(t)$, in the limit $\tau_c \to 0$, Eq. (13) reduces to

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) - \gamma[f'(q)]^2p + f'(q)\xi(t) + f'(q)\pi(t) + \gamma\kappa_0 f[q(t)] f'[q(t)]\epsilon(t).$$

where the dressed noise $\pi(t)$ obeys the correlation function $\langle \pi(t)\pi(t') \rangle = 2c_0^2\kappa_0\delta(t - t')$. Now for linear system-bath coupling, i.e., for $f[q] = q$, we obtain, from Eq. (27), the Langevin equation for the Brownian particle

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) - \gamma p + \xi(t) + \pi(t) + \eta(t).$$

with $\eta(t) = \gamma\kappa_0\epsilon(t)$. The noise $\xi(t)$ is thermal noise and makes its presence felt due to the coupling of the system with the reservoir. The other two noises, $\pi(t)$ and $\eta(t)$, appear additively and multiplicatively, respectively. As the origin of these two noise processes is the same, that is the driving of the bath modes by $\epsilon(t)$, we expect a cross correlation between them. Thus, it is easy to show that

$$\langle \pi(t)\pi(t') \rangle = c_0^2\kappa_0\delta(t - t') = \langle \eta(t)\pi(t') \rangle.$$

Eq. (29) reveals the fact that the two noise processes $\pi(t)$ and $\eta(t)$ are mutually correlated with $\kappa_0$ playing the role of degree of correlation. In writing Eq. (29) we redefine the noise $\eta(t)$ by absorbing the negative sign.

From the very mode of the present development it is clear that we use the paradigm of a hierarchy of interacting Hamiltonians which underlies the reduced-dimensional nonequilibrium Langevin equations so as to highlight which interaction is the driver for space and time dependent correlations. Using an analytical treatment of the system + reservoir-bath + driving-bath hierarchy, we are able to demonstrate that the origin for the nonstationary colored noise is the nonlinearity in the driving term.

In this context we want to mention some allied formalisms made by Hernandez and his group which bears a close kinship with our present development. Very recently Popov and Hernandez used a hierarchy of interacting Hamiltonians to address the question of multiple temperature baths, and the analytical approach used is similar to that taken here. In this development they have addressed a generalized construction for the effective temperature of a tagged particle connected to an arbitrary number of time-dependent inhomogeneous reservoirs. In a number of papers, Hernandez and co-workers explored the extent to which different molecular scale perturbations can drive molecular systems to exhibit dynamic and metastable properties accessible only within nonequilibrium conditions. In these development they have also provided an illustration of the nonstationary dissipation through the justification for the use of the hierarchy of Hamiltonians analytically. In this context Hernandez and co-workers have also discussed treatments of an external driving potential which lead to a rephrasing of the frictional terms.

**III. CONCLUSION**

In conclusion, using the microscopic Hamiltonian picture, we have constructed a Langevin equation that apart from including the thermal noise, originating from the heat bath, includes two external mutually correlated noises, one appears additively and the other multiplicatively. In almost all of the traditional works, the external cross-correlated noises are treated phenomenologically. Our approach is an attempt to understand the underlying mechanism of additive and multiplicative cross correlated noises and to realize their mutual interplay from a microscopic viewpoint.
IV. ACKNOWLEDGEMENTS

We would like to thank the anonymous reviewer for critical reading of our paper and various critical suggestions. Financial support from CSIR, India [01(2257)/08/EMR-II] is thankfully acknowledged.

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