CFT approach to the $q$-Painlevé VI equation

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Iorgov, Lisovyy and Teschner established a connection between isomonodromic deformation of linear differential equations and Liouville conformal field theory at $c = 1$. In this article, we present a $q$ analogue of their construction. We show that the general solution of the $q$-Painlevé VI equation is a ratio of four tau functions, each of which is given by a combinatorial series arising in the AGT correspondence. We also propose conjectural bilinear equations for the tau functions.

Keywords: isomonodromy deformation; $q$ Painlevé equation; conformal field theory.

1. Introduction

Theory of isomonodromic deformation and Painlevé equations has long been a subject of intensive study, with many applications both in mathematics and in physics. Recent discovery of the isomonodromy/CFT correspondence [1–3] opened up a new perspective to this domain. In a nutshell, it states that the Painlevé tau functions are Fourier transform of conformal blocks at $c = 1$. The AGT relation allows one to write a combinatorial formula for the latter, thereby giving an explicit series representation for generic Painlevé tau functions.

The initial conjectures of [1–3] concern the sixth Painlevé (PVI) equation, and at present three different proofs are known. In the first approach [4] one constructs solutions to the linear problem out of conformal blocks with the insertion of a degenerate primary field. The second approach [5] also makes use of CFT ideas, but leads directly to bilinear differential equations for tau functions. The third approach [6] is based on the Riemann–Hilbert problem. It is shown that Fredholm determinant representation for tau functions reproduces the combinatorial series expansion, without recourse to conformal blocks or
the AGT relation. Painlevé equations of other types are also studied by confluence [2], in the framework of irregular conformal blocks [7, 8], through the AGT correspondence [9], and using Fredholm determinants [10].

It is natural to ask whether a similar picture holds for the $q$ difference analogue of Painlevé equations. This has been studied in [11] for the qPIII($D_8$) equation. The aim of the present article is to give a $q$ analogue of the construction of [4].

Our method is based on the work [12], where the five-dimensional AGT correspondence was studied in the light of the quantum toroidal $\mathfrak{gl}_1$ algebra (also known as the Ding–Iohara–Miki algebra). Having the central charge $c = 1$ corresponds to choosing $t = q$ where $t, q$ are the parameters of the algebra. This is a sort of classical limit, in the sense that the quantum algebra reduces to the enveloping algebra of a Lie algebra, see Section 2.1.

We use the trivalent vertex operators of [12] to introduce a $q$ analogue of primary fields and conformal blocks. As in the continuous case, a key property is the braiding relation in the presence of the degenerate field. Following the scheme of [4], we then write the solution $Y(x)$ of the Riemann problem as Fourier transform of appropriate conformal block functions. For the sake of concreteness, we restrict our discussion to the setting relevant to the qPVI equation. Also we do not discuss the issue of convergence of these series.

The above construction allows us to relate the unknown functions $y, z$ of qPVI to tau functions. More specifically, we express $y, z$ in terms of four tau functions which arise from the expansions at $x = 0$ or $x = \infty$; see (3.14), (3.15) below. On the other hand, in view of the work [13, 14] we expect that there are four more tau functions which naturally enter the picture; these are the ones related to the singular points other than $x = 0, \infty$. However, for the lack of the notion of fusion, it is not clear to us how to related them to $y, z$. The absence of fusion is the main obstacle in the $q$ analogue of isomonodromy theory. We present conjectural bilinear relations satisfied by these eight tau functions, see (3.16)–(3.23). Assuming the conjecture we give the final expression for $y, z$ in (3.24).

The plan of the article is as follows.

In Section 2, we give a brief account of the results of [12], restricting to the special case $t = q$. We introduce a $q$ analogue of primary fields, conformal blocks, degenerate (2, 1) fields and their braiding. In Section 3, we apply them to the $q$ analogue of the Riemann problem, focusing attention to the setting of the qPVI equation. We derive a formula expressing the unknowns $y, z$ in terms of four tau functions. We then propose conjectural bilinear difference equations for eight tau functions.

In appendix, we give a direct combinatorial proof of the braiding relation used in the text.

**Notation.** Throughout the article, we fix $q \in \mathbb{C}^\times$ such that $|q| < 1$. We set $[u] = (1 - q^u)/(1 - q)$, $(a; q)_N = \prod_{j=0}^{N-1} (1 - aq^j), \ (a_1, \ldots, a_k; q)_{\infty} = \prod_{j=1}^{k} (a_j; q)_{\infty}$ and $(a; q, q)_{\infty} = \prod_{j,k=0}^{\infty} (1 - aq^{j+k})$. We use the $q$ Gamma function, $q$ Barnes function and the theta function

\[
\Gamma_q(u) = \frac{(q; q)_{\infty}}{(q^u; q)_{\infty}}(1 - q)^{-1-u}, \quad G_q(u) = \frac{(q^u; q, q)_{\infty}}{(q; q, q)_{\infty}}(q; q)_{\infty}^{-1}(1 - q)^{-(u-1)(u-2)/2},
\]

\[
\vartheta(u) = q^{u(u-1)/2}\Theta_q(q^u), \quad \Theta_q(x) = (x, q/x, q; q)_{\infty},
\]

which satisfy $\Gamma_q(1) = G_q(1) = 1$ and

\[
\Gamma_q(u + 1) = [u] \Gamma_q(u), \quad G_q(u + 1) = \Gamma_q(u) G_q(u),
\]

\[
\vartheta(u + 1) = -\vartheta(u) = \vartheta(-u).
\]
A partition is a finite sequence of positive integers \( \lambda = (\lambda_1, \ldots, \lambda_l) \) such that \( \lambda_1 \geq \cdots \geq \lambda_l > 0 \). We set \( \ell(\lambda) = l \). The conjugate partition \( \lambda' = (\lambda'_1, \ldots, \lambda'_r) \) is defined by \( \lambda'_j = \# \{ i \mid \lambda_i \geq j, i' = \lambda_i \} \). We denote by \( \Lambda \) the set of all partitions. We regard a partition \( \lambda \) also as the subset \( \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i, i \geq 1 \} \) of \( \mathbb{Z}^2 \), and denote its cardinality by \( |\lambda| \). For \( \square = (i, j) \in \mathbb{Z}^2_0 \) we set \( a_j(\square) = \lambda_i - j \) and \( \ell_1(\square) = \lambda'_j - i \).

In the last formulas we set \( \lambda_i = 0 \) if \( i > \ell(\lambda) \) (resp. \( \lambda'_j = 0 \) if \( j > \ell(\lambda') \)).

2. \( q \) Analogue of conformal blocks

In this section, we collect background materials from [12]. We shall restrict ourselves to the special case \( t = q \), where formulas of [12] simplify considerably. We then introduce a \( q \) analogue of primary fields, conformal blocks and the braiding relation for the degenerate fields (analogue of \((2, 1)\) operators).

2.1 Toroidal Lie algebra

Let \( Z, D \) be non-commuting variables satisfying \( ZD = qDZ \). The ring of Laurent polynomials \( \mathbb{C}(Z^\pm 1, D^\pm 1) \) in \( Z, D \) is a Lie algebra, with the Lie bracket given by the commutator. We consider its two-dimensional central extension, which we denote by \( \mathcal{L} \). As a vector space \( \mathcal{L} \) has \( \mathbb{C} \)-basis \( Z^l D^j ((k, l) \in \mathbb{Z}^2 \setminus \{(0, 0)\}) \) and central elements \( c_1, c_2 \). The Lie bracket is defined by

\[
[Z^{k_1} D^{l_1}, Z^{k_2} D^{l_2}] = (q^{-l_1 k_2} - q^{-l_2 k_1}) Z^{k_1+k_2} D^{l_1+l_2} + \delta_{k_1+k_2,0} \delta_{l_1+l_2,0} q^{-l_2 l_1} (k_1 c_1 + l_1 c_2).
\]

We set

\[
a_r = Z^r, \quad x_n^+ = DZ^n, \quad x_n^- = Z^n D^{-1},
\]

\[
b_r = D^{-r}, \quad y_n^+ = ZD^{-n}, \quad y_n^- = D^{-n} Z^{-1},
\]

and introduce the generating series \( x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n} \), \( y^\pm(z) = \sum_{n \in \mathbb{Z}} y_n^\pm z^{-n} \). We have two Heisenberg subalgebras of \( \mathcal{L} \), \( a = \bigoplus_{r \neq 0} \mathbb{C} a_r \oplus \mathbb{C} c_1 \) and \( b = \bigoplus_{r \neq 0} \mathbb{C} b_r \oplus \mathbb{C} c_2 \); \( [a_r, a_l] = r \delta_{r+s,0} c_1, [b_r, b_s] = -r \delta_{r+s,0} c_2 \).

We say that an \( \mathcal{L} \)-module has level \((l_1, l_2)\) if \( c_1 \) acts as \( l_1 \cdot id \) and \( c_2 \) acts as \( -l_2 \cdot id \).

The Lie algebra \( \mathcal{L} \) has automorphisms \( S, T \) given by

\[
S : Z \mapsto D, \quad D \mapsto Z^{-1}, \quad c_1 \mapsto c_2, \quad c_2 \mapsto -c_1,
\]

\[
T : Z \mapsto Z, \quad D \mapsto DZ, \quad c_1 \mapsto c_1, \quad c_2 \mapsto c_1 + c_2.
\]

They satisfy \( S^4 = (ST)^6 = id \), so that the group \( SL(2, \mathbb{Z}) \) acts on \( \mathcal{L} \) by automorphisms. We have \( S(b_r) = a_r, S(y^\pm(z)) = x^\pm(z) \).

2.2 Fock representations

The most basic representations of \( \mathcal{L} \) are the Fock representations of levels \((1, 0)\) and \((0, 1)\). Following [12] we denote them by \( \mathcal{F}_{a}^{(1,0)} \) and \( \mathcal{F}_{a}^{(0,1)} \) (\( a \in \mathbb{C}^* \)), respectively.

The Fock representation \( \mathcal{F}_{a}^{(1,0)} \) is irreducible under the Heisenberg subalgebra \( a \). As a vector space we have \( \mathcal{F}_{a}^{(1,0)} = \mathbb{C}[a_{-1}, a_{-2}, \ldots]|\text{vac}\rangle \), where \( |\text{vac}\rangle \) is a cyclic vector such that \( a_r |\text{vac}\rangle = 0 \) (\( r > 0 \)). The
action of the generators $x_n^\pm$ is given by vertex operators

$$x^\pm(z) \mapsto \frac{1}{1-q^2} u^1 \exp \left( \pm \sum_{r \geq 1} \frac{1-q^{-r}}{r} a_{-r} z^r \right) \exp \left( \pm \sum_{r \geq 1} \frac{1-q^{-r}}{r} a_r z^{-r} \right).$$

The Fock representation $\mathcal{F}^{(0,1)}_a$ is the pullback of $\mathcal{F}^{(1,0)}_a$ by the automorphism $S$. It is irreducible with respect to the Heisenberg subalgebra $\mathfrak{h}$, and $y^\pm(z)$ act as vertex operators. On the other hand, the action of $a$ on $\mathcal{F}^{(0,1)}_a$ is commutative. The joint eigenvectors $|\lambda\rangle$ of $a$ are labelled by all partitions and constitute a basis of $\mathcal{F}^{(0,1)}_a$. In particular, for the vector $|\emptyset\rangle$ we have

$$a_r |\emptyset\rangle = -\frac{u^r}{1-q^r} |\emptyset\rangle, \quad x^- (z) |\emptyset\rangle = 0, \quad x^+ (z) |\emptyset\rangle = \delta(u/z)(1),$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. Formulas for the action of $a_r$ and $x^\pm(z)$ on a general $|\lambda\rangle$ can also be written explicitly. Since we do not use them in this article, we refer the reader to [12, 15].

tensor products of $N$ copies of $\mathcal{F}^{(1,0)}_a$ are closely related to representations of the $W_N$ algebra (see e.g. [16, 17]). We briefly explain this point in the case $N=2$.

Consider the tensor product of two Fock representations and its decomposition with respect to the Heisenberg subalgebra $a$,

$$\mathcal{F}^{(1,0)}_{a_1} \otimes \mathcal{F}^{(1,0)}_{a_2} = \mathcal{H} \otimes \Omega_{a_1 a_2},$$

where $\mathcal{H}$ is the Fock space of $a$ and $\Omega_{a_1 a_2}$ is the multiplicity space. The operators $x^\pm(z)$ act on $\mathcal{F}^{(1,0)}_{a_1} \otimes \mathcal{F}^{(1,0)}_{a_2}$ as a sum of two vertex operators. We can factor it into a product of two commuting operators,

$$(1-q^{-1}) x^\pm(z) \mapsto g^\pm(z) \cdot (u_1 u_2)^{\pm 1/2} T(z; (u_1 u_2)^{\pm 1/2}).$$

The first factor represents the action of $a$ on $\mathcal{H},$

$$g^\pm(z) = \exp \left( \pm \sum_{r \neq 0} \frac{1-q^{-r}}{2r} (a_r^{(1)} + a_r^{(2)}) z^{-r} \right),$$

where $a_r^{(1)} = a_r \otimes id, a_r^{(2)} = id \otimes a_r$. The second factor acts on $\Omega_{a_1 a_2},$

$$T(z; u) = u \Lambda^+ (z) + u^{-1} \Lambda^- (z),$$

$$\Lambda^\pm (z) = \exp \left( \pm \sum_{r \neq 0} \frac{1-q^{-r}}{2r} (a_r^{(1)} - a_r^{(2)}) z^{-r} \right).$$

and gives a free field realization of the deformed Virasoro algebra [18]. The space $\Omega_{a_1 a_2}$ is a $q$ analogue of the Virasoro Verma module with central charge $c = 1$ and highest weight $\Delta_0 = \theta^2$, where $q^{2\theta} = u_1 u_2$.  



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2.3 Trivalent vertex operators

For $N \in \mathbb{Z}$, we denote by $\mathcal{F}_{\mathcal{L}}^{(1,N)}$ the pullback of $\mathcal{F}_{\mathcal{L}}^{(1,0)}$ by the automorphism $T^N$. The following intertwiners of $\mathcal{L}$-modules are called trivalent vertex operators.

\[
\Phi = \Phi \left[ \begin{array}{c} (1, N + 1), wx \\ (0, 1), -w; (1, N), x \end{array} \right] : \mathcal{F}_{-w}^{(0,1)} \otimes \mathcal{F}_{x}^{(1,N)} \rightarrow \mathcal{F}_{wx}^{(1,N+1)},
\]

\[
\Phi^\ast = \Phi^\ast \left[ \begin{array}{c} (1, N), x; (0, 1), -w \\ (1, N + 1), wx \end{array} \right] : \mathcal{F}_{wx}^{(1,N+1)} \rightarrow \mathcal{F}_{x}^{(1,N)} \otimes \mathcal{F}_{-w}^{(0,1)}.
\]

Such non-trivial intertwiners exist and are unique up to a scalar multiple [12]. We introduce their components $\hat{\Phi}_\lambda(w), \hat{\Phi}_\lambda^\ast(w)$ relative to the basis $\{|\lambda\rangle\}$ of $\mathcal{F}_{-w}^{(0,1)}$ as follows.

\[
\Phi (|\lambda\rangle \otimes \alpha) = \hat{\lambda}(\lambda, x, w, N) \cdot \hat{\Phi}_\lambda(w)(\alpha) \quad (\alpha \in \mathcal{F}_{x}^{(1,N)}),
\]

\[
\Phi^\ast(\alpha) = \sum_{\lambda \in \Lambda} \hat{\lambda}^\ast(\lambda, x, w, N) \cdot \hat{\Phi}_\lambda^\ast(w)(\alpha) \otimes |\lambda\rangle \quad (\alpha \in \mathcal{F}_{wx}^{(1,N+1)}).
\]

The numerical factors in front are

\[
\hat{i}(\lambda, x, w, N) = \frac{q^{n(\lambda')}}{c_\lambda} f_\lambda^{-N-1} \left( \frac{x}{w} \right)^{|\lambda|}, \quad \hat{i}^\ast(\lambda, x, w, N) = \frac{q^{n(\lambda')}}{c_\lambda} f_\lambda^{N} \left( \frac{q w^N}{x} \right)^{|\lambda|},
\]

where

\[
n(\lambda) = \sum_{\square \in \Lambda} \ell_\lambda(\square), \quad n(\lambda') = \sum_{\square \in \Lambda} a_\lambda(\square),
\]

\[
f_\lambda = (-1)^{|\lambda|} q^{c_\lambda(\square)}, \quad c_\lambda = \prod_{\square \in \Lambda} (1 - q^{\ell_\lambda(\square) + a_\lambda(\square) + 1}).
\]

We normalize $\hat{\Phi}(w), \hat{\Phi}^\ast(w)$ so that $\hat{\Phi}_\lambda^\ast(w)|\text{vac}\rangle = |\text{vac}\rangle + \cdots, \hat{\Phi}_\lambda^\ast(w)|\text{vac}\rangle = |\text{vac}\rangle + \cdots$.

**Proposition 2.1** [12] The operators $\hat{\Phi}_\lambda(w), \hat{\Phi}_\lambda^\ast(w)$ are given by

\[
\hat{\Phi}_\lambda^\ast(w) =: \hat{\Phi}_\lambda^\pm(w) \eta_\lambda^\pm(w), \quad \hat{\Phi}_\lambda^\ast(w) =: \hat{\Phi}_\lambda^\pm(w) \eta_\lambda^\pm(w),
\]

\[
\eta_\lambda^\pm(w) =: \prod_{j=1}^{l(\lambda)} \prod_{i=1}^{\lambda_j} \eta^\pm(q^{j-i}),
\]

where

\[
\hat{\Phi}_\lambda^\pm(w) =: \exp \left( \mp \sum_{n \neq 0} \frac{1}{n} \frac{q^n}{1 - q^n} a_n w^{-n} \right),
\]

\[
\eta^\pm(w) =: \exp \left( \mp \sum_{n \neq 0} \frac{1 - q^n}{n} a_n w^{-n} \right).
\]
Of particular importance is their normal ordering rule. For a pair of partitions \((\lambda, \mu) \in \Lambda^2\), we introduce the Nekrasov factor by

\[
N_{\lambda, \mu}(w) = \prod_{\square \in \lambda} (1 - q^{-\ell_\lambda(\square) - a_{\mu}(\square) - 1}w) \prod_{\square \in \mu} (1 - q^{\ell_\mu(\square) + a_{\mu}(\square) + 1}w).
\]

**Proposition 2.2** [12] Let \(X, Y\) be either \(\hat{\Phi}\) or \(\hat{\Phi}^*\). Then the following normal ordering rule holds.

\[
X_\lambda(z)Y_\mu(w) =: X_\lambda(z)Y_\mu(w) : \left\{ \begin{array}{ll}
(\hat{G}_q(qw/z)N_{\mu, \lambda}(w/z))^{-1} & \text{for } X = Y, \\
\hat{G}_q(qw/z)N_{\mu, \lambda}(w/z) & \text{for } X \neq Y,
\end{array} \right.
\]

where \(\hat{G}_q(z) = (z;q,q)_\infty\).

It is useful to note the relations

\[
N_{\lambda, \mu}(w) = N_{\mu, \lambda}(w^{-1})w^{||\lambda|| + ||\mu||}f_{\lambda}f_\mu, \tag{2.2}
\]

\[
\left(\frac{q^{n(\lambda')}}{c_\lambda}\right)^2 = \frac{f_\lambda q^{-||\lambda||}}{N_{\lambda, \lambda}(1)}. \tag{2.3}
\]

### 2.4 Chiral primary field

In what follows we set

\[
\mathcal{V}_{\theta,w} := \mathcal{F}^{(0,1)}_{-q^{-\theta}w} \otimes \mathcal{F}^{(0,1)}_{-q^\theta w}.
\]

We fix its basis \(|\lambda\rangle = |\lambda_+\rangle \otimes |\lambda_-\rangle\) labelled by a pair \(\lambda = (\lambda_+, \lambda_-)\) of partitions.

Consider the compositions of trivalent vertex operators

\[
\Psi_1 : \mathcal{F}^{(1,0)}_{q^{-2\theta}w} \rightarrow \mathcal{F}^{(1,-1)}_{q^{-\theta_1-\theta_2}/x} \otimes \mathcal{F}^{(0,1)}_{-q^\theta w} \rightarrow \mathcal{F}^{(1,-2)}_{1/(xw^2)} \otimes \mathcal{F}^{(0,1)}_{-q^{-\theta_3-\theta_2}w} \otimes \mathcal{F}^{(0,1)}_{-q^\theta w},
\]

\[
\Psi_2 : \mathcal{F}^{(0,1)}_{-q^{-\theta_1}w} \otimes \mathcal{F}^{(0,1)}_{-q^\theta w} \otimes \mathcal{F}^{(1,-2)}_{1/(xw^2)} \rightarrow \mathcal{F}^{(0,1)}_{-q^{-\theta_1}w} \otimes \mathcal{F}^{(1,-1)}_{q^\theta w} \rightarrow \mathcal{F}^{(1,0)}_{1/x}.
\]

By composing them we obtain an intertwiner of \(\mathcal{L}\)-modules

\[
\mathcal{V}_{\theta_1,w} \otimes \mathcal{F}^{(1,0)}_{q^{-\theta_2}/x} \longrightarrow \mathcal{V}_{\theta_1,w} \otimes \mathcal{F}^{(1,-2)}_{1/(xw^2)} \otimes \mathcal{V}_{\theta_3,q^{-\theta_2}w} \longrightarrow \mathcal{F}^{(1,0)}_{1/x} \otimes \mathcal{V}_{\theta_3,q^{-\theta_2}w}. \tag{2.4}
\]

Taking further the vacuum-to-vacuum matrix coefficient of (2.4), we obtain a map

\[
V \left( \begin{array}{c}
\theta_2 \\
\theta_3
\end{array} ; w, x \right) : \mathcal{V}_{\theta_1,w} \longrightarrow \mathcal{V}_{\theta_3,q^{-\theta_2}w}. \tag{2.5}
\]
The matrix coefficients of (2.5) are given by
\[
\left< \mu \left| V \left( \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_1 \end{array} \right; w, x \right) | \lambda \right>
= C \frac{q^{\mu(\lambda_+^\prime)}}{c_{\lambda_+}} \frac{q^{\mu(\lambda_+^-)}}{c_{\lambda_-}} \frac{q^{\mu(\mu_+^\prime)}}{c_{\mu_+}} \frac{f_{\mu_+}}{f_{\mu_-}} \frac{q^{2\theta_1|\lambda_-|+(2\theta_3+\theta_2)+|\mu_-|}}{f_{\mu_-}^2} q^{\lambda_-} \langle \text{vac} | \Theta_{\lambda_+}(q^{-\theta_1} w) \Phi_{\lambda_-}(q^{\theta_1} w) \Phi^*_{\mu_+}(q^{-\theta_3} w) \Phi^*_{\mu_-}(q^{\theta_3-\theta_2} w) | \text{vac} \rangle,
\]
where we set $|\lambda| = |\lambda_+| + |\lambda_-|$ for $\lambda = (\lambda_+, \lambda_-) \in \Lambda^2$, and $C$ is a scalar factor independent of $\lambda, \mu \in \Lambda^2$.

We choose the normalization
\[
\left< V \left( \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_1 \end{array} ; w, x \right) \right> = \mathcal{N} \left( \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_1 \end{array} \right) q^{2\theta_1\theta_2^2} x^{\theta_1-\theta_2^2-\theta_1^2},
\]
\[
\mathcal{N} \left( \begin{array}{c} \theta_2 \\ \theta_3 \\ \theta_1 \end{array} \right) = \prod_{\epsilon, \epsilon^\prime = \pm} G_q(1+\epsilon \theta_3 - \theta_2 - \epsilon^\prime \theta_1) G_q(1+2\theta_3) G_q(1-2\theta_1),
\]
(2.7)

Here and after, $\langle \cdots \rangle$ stands for the matrix coefficient $\langle (\emptyset, \emptyset) | \cdots | (\emptyset, \emptyset) \rangle$.

We regard (2.5) as a $q$ analogue of the chiral primary field of conformal dimension $\Delta_{\theta_2} = \theta_2^2$.

### 2.5 Conformal block function

The $(m+2)$ point conformal block function is the expectation value of a product of chiral primary fields (see Fig. 1),
\[
\mathcal{F} \left( \theta_m \sigma_{m-1} \sigma_{m-2} \cdots \sigma_1 \theta_0 ; x_m, \ldots, x_1 \right) = \left< V \left( \begin{array}{c} \theta_m \\ \sigma_{m-1} \\ \sigma_{m-2} \end{array} ; w_m, x_m \right) V \left( \begin{array}{c} \theta_{m-1} \\ \sigma_{m-1} \end{array} ; w_{m-1}, x_{m-1} \right) \cdots V \left( \begin{array}{c} \theta_1 \\ \sigma_1 \end{array} ; w_1, x_1 \right) \right>
\]
where $w_p = q^{-\sigma_{p-1} - \cdots - \sigma_1} w_1$. The right-hand side is actually independent of $w_1$.

Using formula (2.6), applying the normal ordering rule (2.1) and further using (2.2), (2.3) for simplification, we obtain the following explicit expression.
\[
\mathcal{F} \left( \theta_m \sigma_{m-1} \sigma_{m-2} \cdots \sigma_1 \theta_0 ; x_m, \ldots, x_1 \right) = \prod_{p=1}^{m} \mathcal{N}(\theta_p \sigma_p \sigma_{p-1}) q^{2\theta_p \sigma_p^2} \cdot \prod_{p=1}^{m} x_p^{\sigma_p^2-\sigma_{p-1}^2} \cdot \sum_{\lambda^{(1)}, \ldots, \lambda^{(m-1)}} m-1 \prod_{p=1}^{m} \frac{q^{2\theta_p x_p^{(p)}} |\lambda^{(p)}\rangle}{\lambda^{(p+1)} x_p^{(p+1)}} \cdot \prod_{p=1}^{m} \frac{\prod_{\epsilon, \epsilon^\prime = \pm} N_{\lambda^{(p)}}^{(p)}(q^{\epsilon \sigma_p \theta_p - \epsilon^\prime \sigma_{p-1}})}{\prod_{p=1}^{m-1} \prod_{\epsilon, \epsilon^\prime = \pm} N_{\lambda^{(p)}}^{(p)}(q^{\epsilon \sigma_p - \epsilon^\prime \sigma_p})}.
\]

We have set $\sigma_0 = \theta_0, \sigma_m = \theta_{m+1}, \lambda^{(0)} = \lambda^{(m)} = (\emptyset, \emptyset)$, and the sum is taken over all $\lambda^{(p)} = (\lambda^{(p)}_+, \lambda^{(p)}_-) \in \Lambda^2, p = 1, \ldots, m - 1$. 
Formula (2.8) is the \( q \) version of the AGT formula for conformal blocks [19] specialized to the case \( t = q \). The derivation given above from the quantum algebra viewpoint is due to [12].

2.6 Degenerate field and braiding

The normalization factor (2.7) has a zero when \( \theta_2 = 1/2 \) and \( \theta_1 - \theta_3 = \pm 1/2 \). In this case we redefine the primary field by

\[
N' \left( \frac{1}{2}, \theta \pm \frac{1}{2} \right) = \lim_{\varepsilon \to 0} \frac{1}{G_q(\varepsilon)} N \left( \frac{1}{2}, \theta \pm \frac{1}{2} \right),
\]

\[
V' \left( \frac{1}{2}, \theta \pm \frac{1}{2}; w, x \right) = \lim_{\varepsilon \to 0} \frac{1}{G_q(\varepsilon)} V \left( \frac{1}{2} - \varepsilon, \theta \pm \frac{1}{2}; w, x \right).
\]

We consider the conformal block function when one of the primary fields is \( V' (\cdots) \). We keep using the same letter \( F (\cdots) \) to denote them. By choosing \( m = 2 \) and specializing parameters in (2.8), the four point conformal blocks can be evaluated in terms of Heine’s basic hypergeometric series

\[
F \left( \frac{1}{2}, \theta \pm \frac{1}{2}; x_1, x_2 \right) = \Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)^{-1} q_{\theta_1} q_{\theta_2}^2 \prod_{\mu, \mu' = \pm} G_q \left( \frac{1}{2} + \mu \theta_{\infty} - \theta_1 + \theta_0, \frac{1}{2} - \mu \theta_{\infty} - \theta_1 + \theta_0; q x_1, x_2 \right).
\]

as follows.

\[
F \left( \frac{1}{2}, \theta \pm \frac{1}{2}; x_1, x_2 \right) = N q_{\theta_2}^2 \left( \frac{q^{2\theta_1} x_2}{x_1} \right)^{e' \theta_{\infty} + 1/4}
\times F \left( \frac{1}{2} + e' \theta_{\infty} - \theta_1 + \theta_0, \frac{1}{2} + e' \theta_{\infty} - \theta_1 + \theta_0; q^{2\theta_1} x_2, 1 + 2e' \theta_{\infty} \right),
\]

\[
F \left( \theta_{\infty}, \theta_0 + \frac{e}{2}; x_2, x_1 \right) = N q_{\theta_2}^2 \left( \frac{q x_1}{x_2} \right)^{e' \theta_0 + 1/4}
\times F \left( \frac{1}{2} + \theta_{\infty} - \theta_1 + e' \theta_0, \frac{1}{2} - \theta_{\infty} - \theta_1 + e' \theta_0; q x_1, x_2 \right).
\]

Here we have set

\[
N = q^{2\theta_1 - \theta_2 x_1^{-1/4}} x_2^{-\theta_2 - \theta_2 x_1^{-1/4}} \prod_{\mu, \mu' = \pm} G_q \left( \frac{1}{2} + \mu \theta_{\infty} - \theta_1 + \mu' \theta_0 \right) G_q(1 + 2\theta_{\infty}) G_q(1 - 2\theta_0).
\]

The following braiding relation holds.
Theorem 2.3  We have

\[ V' \left( \frac{1}{\theta_0 + \frac{\epsilon}{2}}; x_1 \right) V \left( \frac{\theta_1}{\theta_0 + \frac{\epsilon}{2}}; x_2 \right) = \sum_{\epsilon = \pm} V \left( \frac{\theta_1}{\theta_0 + \frac{\epsilon}{2}}; x_2 \right) V' \left( \frac{1}{\theta_0 + \frac{\epsilon}{2}}; x_1 \right) B_{\epsilon, \epsilon'} \left[ \frac{\theta_1}{\theta_0} \frac{x_2}{x_1}, \frac{x_2}{x_1} \right], \tag{2.9} \]

where the braiding matrix is given by

\[ B_{\epsilon, \epsilon'} \left[ \frac{\theta_1}{\theta_0} \frac{x}{q} \right] = -\frac{\vartheta \left( \frac{1}{2} + \epsilon' \theta_0 + \theta_1 - \epsilon \theta_0 \right)}{\vartheta \left( 2 \theta_0 \right)} \frac{\vartheta \left( \frac{1}{2} + \epsilon' \theta_0 + \theta_1 + \epsilon \theta_0 + u \right)}{\vartheta \left( 2 \theta_1 + u \right)}. \tag{2.10} \]

with \( x = qx \).

For the vacuum matrix element, the relation (2.9) is a consequence of the known connection formula for basic hypergeometric functions. At this writing it is not clear to us whether the general case can be deduced from this and the intertwining relation. In Appendix, we give a direct combinatorial proof of the braiding relation (2.9).

The braiding matrix (2.10) has the following properties.

\[ B_{\epsilon, \epsilon'} \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right] = B_{\epsilon, \epsilon'} \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right], \quad (2.11) \]

\[ B \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right] = B \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right] q^{-2\theta_1} x^{-1}, \quad (2.12) \]

\[ \det B \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right] = \frac{\vartheta \left( 2 \theta_0 \right)}{\vartheta \left( 2 \theta_1 \right)} \frac{\vartheta \left( u \right)}{\vartheta \left( u + 2 \theta_1 \right)}, \quad (2.13) \]

\[ B \left[ \frac{\theta_1}{\theta_0} \frac{1}{q} \right] \text{ is 1-periodic in } \theta_0, \theta_1, \theta_\infty. \]

3. q-Painlevé VI equation

3.1 Riemann problem

This article deals with a non-linear non-autonomous q-difference equation called q-Painlevé VI (qPVI) equation. It is derived from deformation theory of linear q-difference equations as a discrete analogue of isomonodromic deformation [20].

A remark is in order concerning the name of the equation. Discrete Painlevé equations are classified by rational surface theory and sometimes called by the name of the surface [21]. From that viewpoint, qPVI is an equation corresponding to the surface of type \( A_3^{(1)} \). Besides, this surface has symmetry by the Weyl group \( D_5^{(1)} \). Accordingly qPVI is also referred to as the q-Painlevé equation of type \( A_3^{(1)} \), or the q-Painlevé equation with \( D_5^{(1)} \)-symmetry.

Regarding discrete Painlevé equations, pioneering researches by Ramani, Grammaticos and their coworkers are well-known. The qPVI equation, in the special case when an extra symmetry condition is fulfilled, has been studied earlier by them under the name of discrete Painlevé III equation [22].
In this section we review the content of [20] concerning the generalized Riemann problem associated with the qPVI equation.

Let \( \theta_0, \theta_1, \theta_\infty \in \mathbb{C} \). We set

\[
R_1 = \max(1, |q^{-2\theta_0}|, |tq^{-2\theta_0}|, |tq^{-2\theta_1-2\theta_\infty}|), \\
R_2 = \min(|q^{-1}|, |q^{-2\theta_1-1}|, |tq^{-2\theta_1-1}|, |tq^{-2\theta_1-2\theta_\infty-1}|),
\]

and assume that \( R_1 < R_2 \). Fix also \( \sigma \in \mathbb{C} \) and \( s \in \mathbb{C}^\infty \).

Suppose we are given \( 2 \times 2 \) matrices \( Y^\infty(x,t), Y^0(x,t), Y^0(x,t) \) with the following properties.

(i) These matrices are holomorphic in the domains

\[
Y^\infty(x,t) : R_1 < |x|, \quad Y^0(x,t) : R_1 < |x| < R_2, \quad Y^0(x,t) : |x| < R_2. \tag{3.1}
\]

(ii) They are related to each other by

\[
Y^\infty(x,t) = Y^0(x,t)B_1(x), \quad Y^0(x,t) = Y^0(x,t)B_2(x), \tag{3.2}
\]

\[
B_1(x) = \mathcal{B}\left[ \frac{\theta_1}{\theta_\infty + \frac{1}{2}}, \frac{1}{\sigma} q^{-2\theta_1} x^{-1} \right], \quad B_2(x) = \mathcal{B}\left[ \frac{\theta_1}{\sigma + \frac{1}{2}}, \frac{1}{\theta_0} q^{-2\theta_1-2\theta_\infty} \right] (0 1 s), \tag{3.3}
\]

(iii) At \( x \to \infty \) or \( x \to 0 \) they have the behaviour

\[
Y^\infty(x,t) = (1 + Y_1(t)x^{-1} + O(x^{-2})) \begin{pmatrix} x^{-\theta_\infty} & 0 \\ 0 & x^{\theta_\infty} \end{pmatrix} (x \to \infty), \tag{3.4}
\]

\[
Y^0(x,t) = G(t)(1 + O(x)) \begin{pmatrix} x^{\theta_0} & 0 \\ 0 & x^{-\theta_0} \end{pmatrix} x^{-\theta_0-\theta_1} (x \to 0). \tag{3.5}
\]

It follows that \( Y(x,t) = Y^\infty(x,t) \) is a solution of linear \( q \)-difference equations of the form

\[
Y(qx,t) = A(x,t)Y(x,t), \quad A(x,t) = \frac{A_2x^2 + A_1(t)x + A_0(t)}{(x - q^{-1})(x - tq^{-2\theta_1-1})}, \tag{3.6}
\]

\[
Y(x,qt) = B(x,t)Y(x,t), \quad B(x,t) = \frac{xI + B_0(t)}{x - tq^{-2\theta_1-2\theta_\infty}}. \tag{3.7}
\]

We have

\[
A_2 = \text{diag}(q^{-\theta_\infty}, q^{\theta_\infty}), \quad A_0(t) = tq^{-3\theta_1-\theta_0-2} G(t) \cdot \text{diag}(q^{\theta_0}, q^{-\theta_0}) \cdot G(t)^{-1},
\]

\[
\det A(x,t) = \frac{(x - q^{-2\theta_1-1})(x - tq^{-2\theta_1-2\theta_\infty-1})}{(x - q^{-1})(x - tq^{-2\theta_1-1})}.
\]

Though the argument is quite standard, we outline the derivation for completeness.
We take the determinant of the relations (3.2), (3.3) and rewrite them as

\[
\frac{(1/x, t q^{-2 \theta_1}/x; q)_\infty}{(q^{-2 \theta_1}/x, t q^{-2 \theta_1 - 2 \theta_1}/x; q)_\infty} \det Y^\infty(x, t) \\
= -\frac{\partial (2 \theta_\infty)}{\partial (2 \sigma)} \frac{(q^{2 \theta_1 + 1} x, t q^{-2 \theta_1}/x; q)_\infty}{(q x, q^{2 \theta_1 + 1} x/t; q)_\infty} q^{2 \theta_1^2 + \theta_1} x^{2 \theta_1} \det Y^0(x, t) \\
= -s \frac{\partial (2 \theta_\infty)}{\partial (2 \theta_0)} \frac{(q^{2 \theta_1 + 1} x, t q^{2 \theta_1 - 2 \theta_1}/x; q)_\infty}{(q x, q^{2 \theta_1 + 1} x/t; q)_\infty} q^{2 \theta_1^2 + \theta_1 + 2 \theta_1^2 + \theta_1} \left( \frac{q^{2 \theta_1} x}{t} \right)^{2 \theta_1} x^{2 \theta_1} \det Y^0(x, t).
\]

Both sides of this expression are single valued and holomorphic on \( \mathbb{C}^\times \). In view of the behaviour (3.4), (3.5) we conclude that

\[
\det Y^\infty(x, t) = \frac{(q^{-2 \theta_1}/x, t q^{-2 \theta_1 - 2 \theta_1}/x; q)_\infty}{(1/x, t q^{-2 \theta_1}/x; q)_\infty}.
\]

In particular, the matrices \( Y^\infty(x, t) \), \( Y^0(x, t) \) and \( Y^0(x, t) \) are generically invertible. Rewriting (3.2), (3.3) and using the periodicity of the braiding matrices, we obtain

\[
A(x, t) := Y^\infty(q x, t) Y^\infty(x, t)^{-1} = Y^0(q x, t) Y^0(x, t)^{-1} = Y^0(q x, t) Y^0(x, t)^{-1}.
\]

A similar argument shows that \( A(x, t) \) is a rational function of \( x \) with the only poles at \( x = q^{-1}, q^{-2 \theta_1 - 1} \).

Using (3.4), (3.5) once again we find that \( A(x, t) \) has the form stated above. The derivation of (3.7) is quite similar.

Introduce \( y = y(t), z = z(t) \) and \( w = w(t) \) by

\[
A(x, t)_{++} = \frac{q^{\theta_\infty} w(x - y)}{(x - q^{-1})(x - tq^{-2 \theta_1 - 1})}, \\
A(x, t)_{+-} = \frac{y - tq^{-2 \theta_1 - 2 \theta_1 - 1}}{q z(y - q^{-1})},
\]

where we use \( \pm \) to indicate components of a \( 2 \times 2 \) matrix. Notation being as above, the compatibility

\[
A(x, qt) B(x, t) = B(q x, t) A(x, t)
\]

of (3.6) and (3.7) leads to the \( q \)PVI equation. Here and after we write \( \bar{f}(t) = f(q t), f_\infty(t) = f(q^{-1} t) \).

**Proposition 3.1** [20] The functions \( y, z \) solve the \( q \)PVI system

\[
\frac{y y}{a_3 a_4} = \frac{(z - t b_1)(z - t b_2)}{(z - b_3)(z - b_4)}, \quad \frac{z z}{b_3 b_4} = \frac{(y - ta_1)(y - ta_2)}{(y - a_3)(y - a_4)},
\]

with the parameters

\[
a_1 = q^{-2 \theta_1 - 1}, \quad a_2 = q^{-2 \theta_1 - 2 \theta_1 - 1}, \quad a_3 = q^{-1}, \quad a_4 = q^{-2 \theta_1 - 1}, \\
b_1 = q^{- \theta_0 - \theta_1}, \quad b_2 = q^{\theta_0 - \theta_1}, \quad b_3 = q^{\theta_1}, \quad b_4 = q^{- \theta_1}.
\]
Remark 3.2 From the compatibility condition (3.8), we obtain the expression of the matrix $B(x, t)$ in terms of $y$ and $z$, etc. In particular, the $(+, -)$ element of the matrix $B_0(q^{-1}t)$ is written as

$$B_0(q^{-1}t)_{+} = \frac{q^{1+\theta_0}y}{1-q^{1-\theta_0}z}.$$  

(3.9)

3.2 CFT construction

We are now in a position to construct solutions of the generalized Riemann problem in terms of conformal block functions. Following the method of [4], we consider the following sums of 5 point conformal blocks,

$$Y_{\epsilon, \epsilon'}^\infty(x, t) = \frac{1}{k_\infty(\epsilon)} \sum_{n \in \mathbb{Z}} s^n \mathcal{F}\left(\frac{1}{2} \epsilon, \frac{1}{2} \epsilon^\prime, \theta_t; q^{2\theta_0+2\theta_1}x, q^{2\theta_1}t\right),$$

(3.10)

$$Y_{\epsilon, \epsilon'}^0(x, t) = \frac{x^{-\theta_1}k_0(\epsilon)}{k_0(\epsilon)} \sum_{n \in \mathbb{Z}} s^n \mathcal{F}\left(\theta_1, \frac{1}{2} \epsilon, \frac{1}{2} \epsilon^\prime, \theta_t; q^{2\theta_1}x, q^{2\theta_0+2\theta_1}t\right),$$

(3.11)

$$Y_{\epsilon, \epsilon'}^0(x, t) = \frac{x^{-\theta_0-\theta_1}k_0(\epsilon)}{k_0(\epsilon)} \sum_{n \in \mathbb{Z}} s^n \mathcal{F}\left(\theta_1, \frac{1}{2} \epsilon, \theta_t, \theta_0 + \frac{1}{2} \epsilon^\prime, \theta_0; q^{2\theta_1}x, q^{2\theta_0+2\theta_1}t\right),$$

(3.12)

where

$$k_\infty(\epsilon) = q^{(\theta_0-\epsilon/2)^2-2\epsilon\theta_0}(\theta_0+\theta_1) \mathcal{N}\left(\frac{1}{2} \epsilon, \theta_\infty\right) \hat{t},$$

$$k_0(\epsilon) = k_\infty(\epsilon) q^{\theta_0^2+\theta_1^2}/2, \quad k_0(\epsilon) = k_0(\epsilon) q^{\theta_0^2+\theta_1^2/2} q^{2\theta_0} t^{-\theta_1},$$

and

$$\hat{t} = \sum_{n \in \mathbb{Z}} s^n \mathcal{F}\left(\theta_1, \theta_t, \theta_0; q^{2\theta_1}x, q^{2\theta_0}t\right).$$

We assume that these series converge in the domains (3.1). The asymptotic behaviour (3.4), (3.5) are simple consequences of the expansion (2.8) of conformal blocks. The crucial point is the validity of the connection formulas (3.2), (3.3) due to the periodicity property (2.13) of the braiding matrix. Therefore, formulas (3.10), (3.11), (3.12) solve the generalized Riemann problem.

Define the tau function by

$$\tau\left[\begin{array}{ccc} \theta_1 & \theta_t \\ \theta_0 & s, \sigma, t \end{array}\right] = q^{-2(\theta_0+\theta_1)\theta_\infty+2\theta_0\theta_1^2} G_q(1+2\theta_\infty)G_q(1-2\theta_0) \cdot \hat{t}.$$
Explicitly it has the AGT series representation
\[
\tau \left[ \begin{array}{c} \theta_1 \\ \theta_\infty \\ \theta_0 \\ \theta_t \\ \theta_\sigma \\ s, \sigma, t \end{array} \right] = \sum_{n \in \mathbb{Z}} \delta^n t^{(\sigma+n)^2-\theta_\sigma^2-\theta_0^2} C \left[ \begin{array}{c} \theta_1 \\ \theta_\infty \\ \theta_0 \\ \theta_t \\ \sigma + n \end{array} \right] Z \left[ \begin{array}{c} \theta_1 \\ \theta_\infty \\ \theta_0 \\ \theta_t \\ \sigma + n, t \end{array} \right],
\]
(3.13)
with the definition
\[
C \left[ \begin{array}{c} \theta_t \\ \theta_\infty \\ \theta_0 \end{array} | \sigma \right] = \frac{\prod_{\epsilon, \epsilon' = \pm} G_{q}(1 + \epsilon \theta_\infty - \theta_t + \epsilon' \sigma) G_{q}(1 + \epsilon \sigma - \theta_t + \epsilon' \theta_0)}{G_{q}(1 + 2 \sigma) G_{q}(1 - 2 \sigma)},
\]
\[
Z \left[ \begin{array}{c} \theta_1 \\ \theta_\infty \\ \theta_0 \end{array} | \sigma, t \right] = \sum_{\lambda = (\lambda_+, \lambda_-) \in \Lambda^2} \prod_{\epsilon, \epsilon' = \pm} N_{\theta, \lambda_+ \lambda_-}(q^{\epsilon \theta_\infty - \theta_t - \epsilon' \sigma}) N_{\lambda_+ \lambda_-}(q^{\epsilon \sigma - \theta_t - \epsilon' \theta_0}) \prod_{\epsilon, \epsilon' = \pm} N_{\lambda_+, \lambda_-}(q^{\epsilon \sigma - \epsilon' \theta_0}).
\]

Comparing the asymptotic expansion at \( x = 0 \) and \( x = \infty \), we can express the \((+, -)\) element of the matrix \( A(x, t) \) and \( B(x, t) \) in terms of tau functions.

**Theorem 3.3** The following expressions hold for \( y, z \) and \( w \) in terms of tau functions:
\[
y = q^{-2\theta_t - 11} \cdot \frac{\tau_3 \tau_4}{\tau_1 \tau_2},
\]
(3.14)
\[
z = \frac{\tau_1 \tau_2 - \tau_1 \tau_3}{q^{\theta_\infty} \tau_1 \tau_2 - q^{1-\theta_\infty} \tau_1 \tau_2},
\]
(3.15)
\[
w = q^{-1} \left( 1 - q^{1-\theta_\infty} \right) \frac{\Gamma_q(2\theta_\infty)}{\Gamma_q(2 - 2\theta_\infty)} \cdot \frac{\tau_2}{\tau_1}.
\]

Here we put
\[
\tau_1 = \tau \left[ \begin{array}{c} \theta_1 \\ \theta_\infty \\ \theta_0 \end{array} | s, \sigma, t \right],
\]
\[
\tau_2 = \tau \left[ \begin{array}{c} \theta_1 \\ \theta_\infty - 1 \end{array} \right] \left[ \begin{array}{c} \theta_0 \end{array} | s, \sigma, t \right],
\]
\[
\tau_3 = \tau \left[ \begin{array}{c} \theta_1 \\ \theta_\infty - \frac{1}{2} \end{array} \right] \left[ \begin{array}{c} \theta_0 + \frac{1}{2} \end{array} | s, \sigma + \frac{1}{2}, t \right],
\]
\[
\tau_4 = \tau \left[ \begin{array}{c} \theta_1 \\ \theta_\infty - \frac{1}{2} \end{array} \right] \left[ \begin{array}{c} \theta_0 - \frac{1}{2} \end{array} | s, \sigma - \frac{1}{2}, t \right].
\]

**Proof.** From the explicit expression of the 5-point conformal block function (2.8), we have
\[
\mathcal{F} \left( \begin{array}{c} \theta_3 \\ \theta_2 \\ \theta_1 \\ \theta_4 \end{array} \right) = N \left( \begin{array}{c} \theta_1 \\ \theta_4 \end{array} \right) q^{2\theta_1 \theta_4} \cdot x_1^2 - q_1^2 - \theta_0^2 \left( \mathcal{F} \left( \begin{array}{c} \theta_3 \\ \theta_2 \\ \theta_1 \\ \theta_4 \end{array} \right) + O(x_1) \right) = N \left( \begin{array}{c} \theta_3 \\ \theta_4 \end{array} \right) q^{2\theta_3 \theta_4} \cdot x_3^2 - \theta_3^2 - \theta_4^2 \left( \mathcal{F} \left( \begin{array}{c} \theta_2 \\ \theta_1 \\ \theta_4 \end{array} \right) + O \left( \frac{1}{x_3} \right) \right).
\]
By using this expression, we calculate the asymptotic behaviour of $Y^{\infty}(x,t)$, $Y^0(x,t)$ and get

$$G(t)_{+e'} = q^{e^0} \frac{\Gamma_q(2\varepsilon \theta_\infty)}{\Gamma_q(1 + 2e' \theta_0)} \left[ \begin{array}{c|c} \theta_1 - \varepsilon & \theta_1 + e' \sigma + \frac{1}{2}t \\
\hline 
\theta_0 - \frac{1}{2} & \theta_0 \end{array} \right], \quad Y_1(t)_{++} = \frac{\Gamma_q(2\theta_\infty)}{\Gamma_q(2 - 2\theta_\infty)} \cdot \frac{\tau_2}{\tau_1},$$

where $C = \theta_0^2 - (\theta_1 + \theta_2)^2 - \theta_\infty^2 + e'(2\theta_1 + 2\theta_2 + 1) + \varepsilon \theta_\infty$ (see (3.4)–(3.5) for $G(t)$ and $Y_1(t)$).

We can write down the exponents of matrices $A(x,t)$ and $B(x,t)$ in these terms. In particular, the expressions

$$A_0(t)_{++} = q^{-3\varepsilon_1 - \theta_0^{-2}}(q^{\theta_0} - q^{\theta_0}) \frac{G(t)_{++}G(t)_{+-}}{\det G(t)}, \quad A_1(t)_{++} = (q^{\theta_0^{-1}} - q^{\theta_0})Y_1(t)_{++}$$

give the formula for $w$ and $y$. The expression of $z$ is obtained from

$$B_0(q^{-1}t)_{+-} = Y_1(t)_{+-} - Y_1(q^{-1}t)_{+-}$$

and the relation (3.9). \hfill \square

In [23], Mano constructed two parametric local solutions of the qPVI equation near the critical points $t = 0, \infty$. The above formula for $y$ extends Mano’s asymptotic expansion to all orders.

We expect that the formula (3.15) for $z$ can be simplified, see (3.24) below.

### 3.3 Bilinear equations

In [14], Tsuda and Masuda formulated the tau function of the qPVI equation to be a function on the weight lattice of $D_1^{(1)}$ with symmetries under the affine Weyl group. Motivated by [14], we consider the set of eight tau functions defined by the AGT series (3.13):

$$\tau_1 = \tau \left[ \begin{array}{c|c} \theta_1 & \theta_1 \\
\theta_\infty + \frac{1}{2} & \theta_0 \end{array} \right], \quad \tau_2 = \tau \left[ \begin{array}{c|c} \theta_1 & \theta_1 \\
\theta_\infty - \frac{1}{2} & \theta_0 \end{array} \right],$$

$$\tau_3 = \tau \left[ \begin{array}{c|c} \theta_1 & \theta_1 \\
\theta_\infty & \theta_0 + \frac{1}{2} \end{array} \right], \quad \tau_4 = \tau \left[ \begin{array}{c|c} \theta_1 & \theta_1 \\
\theta_\infty & \theta_0 - \frac{1}{2} \end{array} \right],$$

$$\tau_5 = \tau \left[ \begin{array}{c|c} \theta_1 + \frac{1}{2} & \theta_1 \\
\theta_\infty & \theta_0 \end{array} \right], \quad \tau_6 = \tau \left[ \begin{array}{c|c} \theta_1 + \frac{1}{2} & \theta_1 \\
\theta_\infty & \theta_0 \end{array} \right],$$

$$\tau_7 = \tau \left[ \begin{array}{c|c} \theta_1 + \frac{1}{2} & \theta_1 \\
\theta_\infty & \theta_0 \end{array} \right], \quad \tau_8 = \tau \left[ \begin{array}{c|c} \theta_1 + \frac{1}{2} & \theta_1 \\
\theta_\infty & \theta_0 \end{array} \right].$$

For convenience, we shift the parameter $\theta_\infty$ in the previous section to $\theta_\infty + 1/2$. 
On the basis of a computer run we put forward:

CONJECTURE 3.4 For the eight tau functions defined above, the following bilinear equations hold.

\begin{align*}
\tau_1 \tau_2 - q^{-2h_1} t \tau_3 \tau_4 - (1 - q^{-2h_1} t) \tau_5 \tau_6 &= 0, \\
\tau_1 \tau_2 - t \tau_3 \tau_4 - (1 - q^{-2h_1} t) \tau_5 \tau_6 &= 0, \\
\tau_1 \tau_2 - \tau_3 \tau_4 + (1 - q^{-2h_1} t) q^{2h_1} \tau_7 \tau_8 &= 0, \\
\tau_1 \tau_2 - q^{2h_1} \tau_3 \tau_4 + (1 - q^{-2h_1} t) q^{2h_1} \tau_7 \tau_8 &= 0, \\
\frac{\tau_5 \tau_6 + q^{-\theta_1 \theta_\infty + \theta_1 - 1/2} t \tau_7 \tau_8 - \tau_1 \tau_2}{\tau_5 \tau_6 + q^{-\theta_1 \theta_\infty + \theta_1 - 1/2} t \tau_7 \tau_8 - \tau_1 \tau_2} &= 0, \\
\frac{\tau_5 \tau_6 + q^{\theta_0 + 2h_1} \tau_7 \tau_8 - q^{\theta_1 \theta_\infty + \theta_1 - 1/2} \tau_1 \tau_2}{\tau_5 \tau_6 + q^{\theta_0 + 2h_1} \tau_7 \tau_8 - q^{\theta_1 \theta_\infty + \theta_1 - 1/2} \tau_1 \tau_2} &= 0.
\end{align*}

The solution \( y, z \) of qPVI are expressed as

\begin{align*}
y &= q^{-2h_1 - 1} t \cdot \frac{\tau_3 \tau_4}{\tau_1 \tau_2}, \\
z &= -q^{\theta_1 \theta_\infty - 1} t \cdot \frac{\tau_7 \tau_8}{\tau_5 \tau_6}.
\end{align*}

The bilinear equations (3.16)–(3.23) were originally proved in [13] in the special case when the tau functions are given by Casorati determinants of basic hypergeometric functions. It would be interesting if one can prove them in the general case by a proper extension of the methods of [5] or [6].

REMARK 3.5 Formula (3.15) for \( z \) is obtained from the asymptotic expansion at \( x = \infty \). The expansion at \( x = 0 \) gives an alternative formula for the same quantity. Comparing these (and shifting \( \theta_\infty \) to \( \theta_\infty + 1/2 \)), we arrive at the following bilinear relation:

\[
\tau_1 \tau_2 - \tau_1 \tau_2 = \frac{q^{1/2 + \theta_\infty} - q^{1/2 - \theta_\infty}}{q^{-\theta_0} - q^{\theta_0}} q^{-\theta_1 - 1} t (\tau_3 \tau_4 - \tau_3 \tau_4).
\]

This is consistent with the conjectured relations (3.20)–(3.23).

Acknowledgments

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Appendix A. Braiding relation

In this appendix, we give a direct proof of Theorem 2.3.

We rewrite the braiding relation (2.9) into the form of matrix elements \( \langle \alpha | \otimes | \beta | \otimes | \lambda | \otimes | \mu \rangle \) for any partitions \( \lambda, \mu, \alpha, \beta \). The matrix element

\[
\begin{pmatrix}
\alpha \otimes \\
| \beta | V' \\
\theta_{\infty} + \frac{\epsilon'}{2} ; x_1 \\
| \lambda | \otimes | \mu \\
\end{pmatrix}
\]

\( V \left( \frac{\theta_{\infty}}{2} + \sigma \right) ; x_2 \)

can be read off from the coefficient of \( t^{(|\lambda|+|\mu|-|\alpha|-|\beta|)} \) in the 6 point conformal block

\[
\mathcal{F} \left( \begin{array}{cccccc}
\theta_2 & \frac{1}{2} & \theta_1 & \theta_0 \\
\theta_3 & \frac{1}{2} + \epsilon \theta_{\infty} & \sigma & x_3, x_1, x_2, t \\
\end{array} \right). 
\]

Formula (2.8) tells that it is a sum over a pair of partitions. By using \( N_{\lambda,\mu}(1) = \delta_{\lambda,\mu}N_{\lambda,\lambda}(1) \), the sum can be reduced further to one over a single partition. We thus find that Theorem 2.3 is equivalent to the following proposition.

**Proposition A.1** For any partitions \( \lambda, \mu, \alpha \) and \( \beta \), we have

\[
q^{2\theta_{\infty}} \left( \frac{q^{2\theta_{\infty}} x_2}{x_1} \right)^{\epsilon \theta_{\infty} + 1/4} \prod_{\nu = \pm} \Gamma_q(\frac{1}{2} + \epsilon \theta_{\infty} - \theta_1 + \epsilon \sigma) X^{\epsilon \theta_{\infty} + 1/4}_{\lambda,\mu,\alpha,\beta}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2)
\]

\[
= \sum_{\epsilon = \pm} q^{2 \sigma} \left( \frac{q x_1}{x_2} \right)^{\epsilon \sigma + 1/4} \prod_{\nu = \pm} \Gamma_q(\frac{1}{2} + \epsilon \sigma - \theta_1 + \epsilon \theta_{\infty}) Y^{\epsilon \sigma}_\lambda(\theta_{\infty}, \theta_1, \sigma; x_1, x_2)
\]

\[
\times B_{\epsilon',\epsilon} \left( \frac{\theta_1}{\theta_{\infty}}, \frac{\frac{1}{2}}{\sigma} \frac{x_2}{x_1} \right) q^{\frac{1}{2} - \theta_1 / 2} \left( \frac{x_2}{x_1} \right)^{\theta_1}, \tag{A.1}
\]

where

\[
X^+_{\lambda,\mu,\alpha,\beta}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2) = N_{\alpha,\beta}(q^{2\theta_{\infty}})N_{\beta,\lambda}(q^{-\theta_{\infty} - 1/2 - \theta_1 - \sigma})N_{\beta,\mu}(q^{-\theta_{\infty} - 1/2 + \theta_1 + \sigma})
\]

\[
\times \sum_{\eta \in \Lambda} \left( \frac{1}{x_2} \right)^{\theta_1 / 2} \left( \frac{q^{2\theta_{\infty}} x_2}{x_1} \right)^{\theta_1 / 2} \left( q x_1 \right)^{\theta_1 / 2} \left( q^{2\theta_{\infty} + 1/2} \right) N_{\alpha,\eta}(q^{-1})N_{\eta,\lambda}(q^{\theta_{\infty} + 1/2 - \theta_1 - \sigma})N_{\eta,\mu}(q^{\theta_{\infty} + 1/2 + \theta_1 + \sigma})
\]

\[
Y^+_{\lambda,\mu,\alpha,\beta}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2) = X^+_{\lambda,\mu,\beta,\alpha}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2), \quad Y^+_{\lambda,\mu,\alpha,\beta}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2)
\]

\[
= N_{\lambda,\mu}(q^{2\sigma})N_{\alpha,\lambda}(q^{-\theta_{\infty} - 1/2 - \theta_1 - \sigma})N_{\beta,\lambda}(q^{-\theta_{\infty} + \theta_1 + 1/2})
\]

\[
\times \sum_{\eta \in \Lambda} \left( \frac{1}{x_1} \right)^{\theta_1 / 2} \left( \frac{q x_1}{x_2} \right)^{\theta_1 / 2} \left( \eta x_1 \right)^{\theta_1 / 2} \left( q^{2\theta_{\infty} + 1/2} \right) N_{\eta,\mu}(q^{-1})N_{\eta,\beta}(q^{\theta_{\infty} + 1/2 + \theta_1 + \sigma})N_{\lambda,\mu}(q^{\theta_{\infty} - \theta_1 + \sigma + 1/2})
\]

\[
Y^-_{\lambda,\mu,\alpha,\beta}(\theta_{\infty}, \theta_1, \sigma; x_1, x_2) = Y^+_{\mu,\lambda,\alpha,\beta}(\theta_{\infty}, \theta_1, -\sigma; x_1, x_2).
\]
In the following, we show that the identity (A.1) is reduced to the connection formula of basic hypergeometric functions. First, we prepare several lemmas.

**Lemma A.2** For any $\lambda, \mu \in \Lambda$ we have $N_{\lambda', \mu'}(u) = N_{\mu, \lambda}(u)$.

**Proof.** Since $a_{\lambda'}(j, i) = \ell_{\lambda}(i, j)$ for all $(i, j) \in \mathbb{Z}^2_{\geq 0}$, we have

$$N_{\lambda', \mu'}(u) = \prod_{(i, j) \in \lambda'} (1 - q^{-a_{\lambda'}(i, j) - \ell_{\mu}(i, j) - 1}u) \prod_{(i, j) \in \mu} (1 - q^{a_{\mu}(i, j) + \ell_{\lambda}(i, j) + 1}u).$$

Therefore, to prove the Lemma it suffices to show that the identity

$$\prod_{(i, j) \in \lambda, (i, j) \notin \mu} (1 - q^{-a_{\lambda}(i, j) - \ell_{\mu}(i, j) - 1}u) = \prod_{(i, j) \in \lambda, (i, j) \notin \mu} (1 - q^{a_{\mu}(i, j) + \ell_{\lambda}(i, j) + 1}u)$$

(A.2)

holds for all $\lambda, \mu \in \Lambda$. Note that the product consists of all entries $(i, j)$ such that $\lambda_i > \mu_j$ and $\mu_i + 1 \leq j \leq \lambda_i$. We show (A.2) by induction with respect to the length of $\lambda$.

Consider a sub-sequence $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+s}$ in $\lambda$ such that

$$\lambda_n \leq \mu_n, \quad \lambda_i > \mu_j \quad (i = n + 1, \ldots, n + s), \quad \lambda_{n+s+1} \leq \mu_{n+s+1}.$$

Put $\check{\lambda} = (\lambda_{n+1}, \ldots, \lambda_{n+s})$ and $\check{\mu} = (\mu_{n+1}, \ldots, \mu_{n+s})$. Then we have for $i = 1, \ldots, s$ and $\mu_i + 1 \leq j \leq \check{\lambda}_i$

$$\ell_{\mu}(n + i, j) = \ell_{\check{\mu}}(i, j), \quad \ell_{\check{\lambda}}(n + i, j) = \ell_{\check{\lambda}}(i, j),$$

because $\lambda_n \leq \mu_n$ and $\lambda_{n+s+1} \leq \mu_{n+s+1}$. We further divide a partition $\check{\lambda}$ into sub-sequences $\check{\lambda}_{m+1}, \ldots, \check{\lambda}_{m+r}$ such that

$$\check{\lambda}_{m+1} \leq \check{\mu}_m, \quad \check{\lambda}_i > \check{\mu}_{i-1} \quad (i = m + 2, \ldots, m + r), \quad \check{\lambda}_{m+r+1} \leq \check{\mu}_{m+r}.$$

Put $\check{\lambda} = (\check{\lambda}_{m+1}, \ldots, \check{\lambda}_{m+r})$ and $\check{\mu} = (\check{\mu}_{m+1}, \ldots, \check{\mu}_{m+r})$. Then we have again for $i = 1, \ldots, r$ and $\check{\mu}_i + 1 \leq j \leq \check{\lambda}_i$

$$\ell_{\check{\mu}}(m + i, j) = \ell_{\check{\mu}}(i, j), \quad \ell_{\check{\lambda}}(m + i, j) = \ell_{\check{\lambda}}(i, j),$$

because $\check{\lambda}_{m+1} \leq \check{\mu}_m$ and $\check{\lambda}_{m+r+1} \leq \check{\mu}_{m+r}$. Replacing $\lambda, \mu$ by $\check{\lambda}, \check{\mu}$ we may assume without loss of generality that

$$\lambda_1 > \mu_1, \quad \lambda_i > \mu_{i-1} \quad i = 2, \ldots, \ell(\lambda).$$

(A.3)

If $\ell(\lambda) = 1$, it is easy to check (A.2).

Suppose (A.2) is true for $\ell(\lambda) = \ell - 1$. Let $\check{\lambda}$ satisfy $\ell(\check{\lambda}) = \ell$. Assuming (A.3), we choose the number $k$ such that $\mu_{\ell-(k+1)} > \lambda_\ell \geq \mu_{\ell-k}$.
Writing \( \lambda = (v, \lambda_\ell) \), we compare (A.2) for \( \lambda \) and for \( \nu \). In the left hand side of (A.2), the factors for \( i < \ell \) remain the same as for \( \nu \). The product of factors for \( i = \ell \) is expressed as

\[
\prod_{j=0}^{\lambda_{\ell}-\mu_{\ell}-1} \left( 1 - q^{-j}u \right) / \prod_{j=1}^{k} (1 - q^{-\lambda_{\ell}+\mu_{\ell}+j}u).
\]

(A.4)

In the right hand side of (A.2), the factors for \( j > \lambda_\ell \) remain the same as for \( \nu \). The product of factors for \( i = \ell \) is equal to

\[
\lambda_{\ell}-\mu_{\ell}-1 \prod_{j=0}^{\lambda_{\ell}-\mu_{\ell}-1} (1 - q^{-j}u).
\]

The products of factors for \( i = \ell - k, \ldots, \ell - 1 \) and \( j \leq \lambda_\ell \) are equal to

\[
\lambda_{\ell}-\mu_{\ell} \prod_{m=1}^{\lambda_{\ell}-\mu_{\ell}} (1 - q^{\ell-i-m}u)
\]

before adding the \( \ell \)-th row, and after they become

\[
\lambda_{\ell}-\mu_{\ell} \prod_{m=0}^{\lambda_{\ell}-\mu_{\ell}-1} (1 - q^{\ell-i-m}u).
\]

Namely, we lose the factor \( \prod_{i=\ell}^{\ell-k} (1 - q^{-\lambda_{\ell}+\mu_{\ell}+i}u) \) and gain the factor \( \prod_{m=1}^{k} (1 - q^m u) \). Therefore, the outcome can be written as (A.4).

We introduce the following operations on partitions \( \lambda \in \Lambda \):

\[
\tilde{\lambda} = (\lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1),
\]

(A.5)

\[
r_n(\lambda) = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+2}, \ldots) \quad (n \in \mathbb{Z}_{\geq 0}).
\]

(A.6)

Here we identify a partition \( \lambda \) with a sequence \((\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0, 0, \ldots)\).

**Lemma A.3** Let \( \lambda, \eta \) be partitions. Then \( N_{\eta,\lambda}(q^{-1}) \neq 0 \) if and only if \( \eta = r_n(\lambda) \) for some \( n \geq 0 \).

**Proof.** It is easy to show the ‘if’ part.

To show the ‘only if’ part, assume that \( \eta \) satisfies \( N_{\eta,\lambda}(q^{-1}) \neq 0 \). Then we have

\[
\ell_\eta(i,j) + a_\eta(i,j) + 2 \neq 0 \quad \forall (i,j) \in \eta,
\]

(A.7)

\[
\ell_\lambda(i,j) + a_\lambda(i,j) \neq 0 \quad \forall (i,j) \in \lambda.
\]

(A.8)
Using them we show

(i) If \( \ell(\lambda) < i \leq \ell(\eta) \) then \( \eta_i = 1 \).

(ii) If \( i \leq \min(\ell(\lambda), \ell(\eta)) \) then \( \eta_i \leq \lambda_i + 1 \).

(iii) If \( i \leq \min(\ell(\lambda), \ell(\eta)) \) and \( \eta_i \leq \lambda_i \), then \( i < \ell(\lambda) \) and \( \eta_i \leq \lambda_{i+1} \).

(iv) If \( i \leq \min(\ell(\lambda) - 1, \ell(\eta)) \), then \( \eta_i \geq \lambda_{i+1} \).

To see (i), suppose there exists such an \( i \) that \( \ell(\lambda) < i \leq \ell(\eta) \) and \( \eta_i \geq 2 \). Then \( (n'_2, 2) \in \eta_i, \ell_i(n'_2, 2) = 0 \) and \( a_\lambda(n'_2, 2) = -2 \), in contradiction with (A.7). Since \( \eta_i > 0 \) for \( i \leq \ell(\eta) \), we must have \( \eta_i = 1 \).

To see (ii), suppose there exists such an \( i \) \( \leq \min(\ell(\lambda), \ell(\eta)) \) that \( \eta_i > \lambda_i + 1 \). Take the largest such \( i \) and let \( j = \lambda_i + 2 \). Then \( (i, j) \in \eta_i, \ell_i(i, j) = 0 \) and \( a_\lambda(i, j) = -2 \), which contradicts (A.7).

Under the assumption of (iii), we have \((i, \eta_i) \in \lambda_i \) and \( a_\lambda(i, \eta_i) = 0 \). Further if \( i = \ell(\lambda) \), or else \( i < \ell(\lambda) \) and \( \lambda_{i+1} < \eta_i \), then we have \( \ell_\lambda(i, \eta_i) = 0 \), which contradicts (A.8).

To show (iv), first assume \( i \leq \ell(\lambda) - 2 \) and \( \eta_i < \lambda_{i+1} \), \( \eta_i = \lambda_{i+1} \) hold. Then \((i, \eta_i + 1) \in \lambda_i \), \( a_\lambda(i, \eta_i + 1) = -1 \), and \( \ell_\lambda(i, \eta_i + 1) = 1 \), which contradicts (A.8). Similarly if \( i = \ell(\lambda) - 1 \) and \( \eta_{\ell(\lambda) - 1} < \lambda_{\ell(\lambda)} \), then the same conclusion takes place.

Now we prove the Lemma. If \( \eta_i = \lambda_i + 1 \) for all \( i \leq \ell(\lambda) \), then together with (i) we have \( \eta = r_\lambda(\lambda) \). Otherwise, from (ii), (iii) and (iv) there exists an \( n \) such that \( \eta_i = \lambda_i + 1 \) \( (1 \leq i \leq n) \) and \( \eta_i = \lambda_{i+1} \) for all \( n < i \leq \ell(\lambda) - 1 \). Moreover if \( \ell(\eta) \geq \ell(\lambda) \) then \( \eta_{\ell(\lambda)} \leq \eta_{\ell(\lambda) - 1} = \lambda_{\ell(\lambda)} \) which contradicts (iii). Therefore \( \ell(\eta) = \ell(\lambda) - 1 \) and \( \eta = r_n(\lambda) \).

**Lemma A.4** Let \( \lambda, \mu \in \Lambda \) and \( n \in \mathbb{Z}_{\geq 0} \). Using the notation (A.5) and (A.6), we set

\[
\ell = \ell(\lambda), \quad k = \ell(\mu), \quad \eta = r_n(\lambda), \quad \gamma = (r_n(\mu'))', \quad \tilde{\eta} = \begin{cases} \tilde{\eta} & (n \leq \ell - 1), \\ (\lambda, 1^{n-\ell+1}) & (n \geq \ell). \end{cases}
\]

Then we have

\[
\frac{N_{\eta, \lambda}(q^{-1})}{N_{\eta, \gamma}(1)} = \frac{N_{\eta, \lambda}(q^{-1})}{N_{\eta, \gamma}(1)} (1 - q^{\tilde{\eta} - |\lambda|}), \quad (A.9)
\]

\[
N_{\mu, \lambda}(u) = N_{\mu, \lambda}(q^{-1} u) \prod_{j=1}^{\mu_\lambda} \frac{1 - q^{j-1} u^{\ell - i + \mu(i, 1)}}{1 - q^{j-1} u^{\ell - i + \mu(i, 1) + 1}}, \quad (A.10)
\]

\[
N_{\mu, \lambda}(u) = N_{\mu, \lambda}(q^{-1} u) \prod_{j=1}^{\mu_\lambda} \frac{1 - q^{j-1} u^{\ell - i + \mu(i, 1) + 1}}{1 - q^{j-1} u^{\ell - i + \mu(i, 1) + 2}}, \quad (A.11)
\]

\[
\frac{N_{\mu, \lambda}(u)}{N_{\mu, \eta}(qu)} = \frac{N_{\mu, \lambda}(q^{-1} u)}{N_{\mu, \eta}(u)} (1 - u), \quad (A.12)
\]

\[
\frac{N_{\mu, \lambda}(u)}{N_{\mu, \eta}(qu)} = \frac{N_{\mu, \lambda}(q^{-1} u)}{N_{\mu, \eta}(q^2 u)} \frac{1 - q^{\gamma - |\lambda| + 4 - \tilde{\eta}}}{1 - qu}, \quad (A.13)
\]
By the definition we have
\[ a_\lambda(i,j) = a_\lambda(i,j + 1) \quad (i = 1, \ldots, \ell, j \in \mathbb{Z}_{>0}), \]
\[ \ell_\lambda(i,j) = \ell_\lambda(i,j + 1) \quad ((i,j) \in \tilde{\lambda}), \]
\[ a_\eta(i,j) = a_\eta(i,j + 1) \quad (i = 1, \ldots, \min(\ell(\tilde{\eta}), \ell), j \in \mathbb{Z}_{>0}), \]
\[ \ell_\eta(i,j) = \begin{cases} \ell_\eta(i,j + 1) & ((i,j) \in \tilde{\eta}, n \leq \ell - 1 \text{ or } n \geq \ell \text{ and } j \geq 2), \\
  n + 1 - i & (i > 0, j = 1, n \geq \ell). \end{cases} \]

Using these relations we show below (A.9)–(A.14).

Proof of (A.9). First, we consider the case \( n \leq \ell - 1 \). We have
\[
\frac{N_{\eta,\lambda}(q^{-1})}{N_{\tilde{\eta},\tilde{\lambda}}(q^{-1})} = \prod_{(i,j) \in \eta} \left(1 - q^{-\ell_\eta(i,j)+a_\eta(i,j)-2}\right) \prod_{(i,j) \in \tilde{\eta}} \left(1 - q^{\ell_\eta(i,j)+a_\eta(i,j)}\right) \\
\times \frac{1}{\prod_{(i,j) \in \lambda, \ell < \ell} (1 - q^{\ell_\lambda(i,j)+a_\lambda(i,j)+1})} \\
= \prod_{i=1}^{\ell} \left(1 - q^{-\ell+i+\lambda_i}\right) \prod_{i=1}^{\ell-1} \left(1 - q^{\ell-i+\eta_i-1}\right),
\]
and
\[
\frac{N_{\tilde{\eta},\tilde{\lambda}}(1)}{N_{\eta,\lambda}(1)} = \prod_{(i,j) \in \tilde{\eta}} \left(1 - q^{-\ell_\eta(i,j)+a_\eta(i,j)-1}\right) \prod_{(i,j) \in \lambda} \left(1 - q^{\ell_\eta(i,j)+a_\eta(i,j)+1}\right) \\
\times \frac{1}{\prod_{i=1}^{\ell} (1 - q^{-\ell+n+1} - \eta_i) (1 - q^{\ell-i+\eta_i-1})}.
\]

Since \( \eta = (\lambda_1 + 1, \ldots, \lambda_n + 1, \lambda_{n+2}, \ldots, \lambda_{\ell}) \) and \( |\tilde{\eta}| - |\lambda| = -\ell + n + 1 - \lambda_{n+1} \), we obtain (A.9) for \( n \leq \ell - 1 \).

Second, we consider the case \( n \geq \ell \). We have
\[
\frac{N_{\eta,\lambda}(q^{-1})}{N_{\tilde{\eta},\tilde{\lambda}}(q^{-1})} = \prod_{i=1}^{\ell} \left(1 - q^{-(n-i)(\lambda_{i-1})-2}\right) \prod_{i=1}^{\ell} \left(1 - q^{-(\ell-i)(\lambda_{i-1})-2}\right) \left(1 - q^{\ell-i+\lambda_i}\right) \\
\times \frac{1}{\prod_{i=1}^{\ell} (1 - q^{-(n+1-i)(\lambda_{i-1})-2}) \prod_{i=1}^{n+1} (1 - q^{-(n+1-i)(\lambda_{i-1})-2})} \\
= \prod_{i=1}^{\ell} \left(1 - q^{-\ell+i+\lambda_i}\right) \left(1 - q^{\ell-i+\lambda_i}\right) \left(1 - q^{-(n-i)\lambda_i}\right).
and

\[
\frac{N_{\tilde{\eta},\tilde{\eta}}(1)}{N_{\eta,\eta}(1)} = \frac{(1 - q^{-(\sigma - \ell - 1)}(1 - q^{\sigma - \ell + 1})}{\prod_{i=1}^{\ell} (1 - q^{\ell - i + 1}) (1 - q^{\ell - i + \tilde{\lambda}})}.
\]

Since \(|\tilde{\eta}| - |\lambda| = n + 1 - \ell\), we obtain (A.9) for \(n \geq \ell\).

**Proof of (A.10).** First, we consider the case \(n \leq \ell - 1\). Then by \(\ell(\tilde{\eta}) \leq \ell - 1\), we obtain

\[
\frac{N_{\mu,\eta}(u)}{N_{\mu,\tilde{\eta}}(u)} = \frac{\prod_{(i,j) \in \mu, \ell \geq \ell} (1 - q^{-\ell(i,j)+j-1}u) \prod_{i=1}^{\ell} (1 - q^{\ell - i + a_\mu(i,1)}u)}{\prod_{j=1}^{\mu} (1 - q^{-\ell(i,j)+j-2}u) \prod_{i=1}^{\ell} (1 - q^{\ell - i + a_\mu(i,1)}u)}.
\]

Second, we consider the case \(n \geq \ell\). Then, we have \(a_\eta(i+1,j) = a_\eta(i,j)\) if \(\ell < i \leq n\) and \(\ell(\tilde{\eta}) = \ell(1,1) + 1\) for \(i \in \mathbb{Z}_{\geq 0}\). Hence, we obtain

\[
\frac{N_{\mu,\eta}(u)}{N_{\mu,\tilde{\eta}}(u)} = \frac{\prod_{(i,j) \in \mu, \ell \geq \ell} (1 - q^{-\ell(i,j)+j-1}u) \prod_{i=1}^{\ell} (1 - q^{\ell - i + a_\mu(i,1)}u)}{\prod_{j=1}^{\mu} (1 - q^{-\ell(i,j)+j-2}u) \prod_{i=1}^{\ell} (1 - q^{\ell - i + a_\mu(i,1)}u)}.
\]

**Proof of (A.11).** In (A.10), choose \(n = \ell - 1\), then rename \(\ell\) by \(\ell + 1\) and \(\lambda_i\) by \(\lambda_i - 1\). We obtain (A.11).

**Proof of (A.12).** This follows from (A.11) and (A.10).

**Proof of (A.13).** Using the relation (2.2) and (A.11), we have

\[
\frac{N_{\mu,\lambda}(u)N_{\mu,\eta}(q^2u)}{N_{\mu,\tilde{\lambda}}(u)N_{\mu,\tilde{\eta}}(qu)} = q^{(|\eta| - |\lambda|) - k} \frac{N_{\lambda,\mu}(u^{-1})N_{\tilde{\lambda},\tilde{\eta}}((q^2u)^{-1})}{N_{\lambda,\tilde{\eta}}((qu)^{-1})N_{\tilde{\lambda},\eta}((qu)^{-1})}
\]

\[
= q^{(|\eta| - |\lambda|) - k} \prod_{j=1}^{k+1} \frac{1 - q^{j-1}u^{-1}}{1 - q^{-\ell(k+1,j)+j-2}u^{-1}} \prod_{j=1}^{k+1} \frac{1 - q^{-\ell(k+1,j)+j-3}u^{-1}}{1 - q^{j-2}u^{-1}}
\]

\[
\times \prod_{i=1}^{k} \frac{1 - q^{-j+1}u^{-1}}{1 - q^{j+i+a_\eta(i,1)}u^{-1}}.
\]
First, we consider the case \( n \leq \ell - 1 \). By definition, we have for \( k + 1 \leq n \)

\[
\ell_n(k + 1, j) = \begin{cases} 
\ell_\lambda(k + 1, j) - 1 & (j = 1, \ldots, \lambda_{n+1}), \\
n - (k + 1) & (j = \lambda_{n+1} + 1), \\
\ell_\lambda(k + 1, j - 1) & (j = \lambda_{n+1} + 2, \ldots, \lambda_1 + 1),
\end{cases}
\]

and for \( k + 1 \geq n + 1 \)

\[\ell_n(k + 1, j) = \ell_\lambda(k + 1, j) - 1 \quad (j = 1, \ldots, \eta_{k+1}).\]

Hence, we obtain for \( k + 1 \leq n \),

\[
\frac{N_{\mu, \lambda}(u)N_{\mu, \eta}(q^2u)}{N_{\mu, \lambda}(qu)N_{\mu, \eta}(qu)} = q^{n - |\lambda| - k} \prod_{j=1}^{\lambda_{k+1}} \frac{1 - q^{j-1}u^{-1}}{1 - q^{-\ell_\lambda(k+1)+j-2}u^{-1}} \prod_{j=1}^{\lambda_{n+1}} (1 - q^{-\ell_\lambda(k+1)+j-2}u^{-1}) \\
\times (1 - q^{k+1-n+\lambda_{n+1}-2}u^{-1}) \frac{\prod_{j=\lambda_{n+1}+2}^{\lambda_{k+1}+1} (1 - q^{-\ell_\lambda(k+1)+j-3}u^{-1})}{\prod_{j=1}^{\lambda_{k+1}+1} (1 - q^{-2}u^{-1})} \\
= q^{n - |\lambda| - k} \times \frac{1 - q^{k-n+\lambda_{n+1}-1}u^{-1}}{1 - q^{-1}u^{-1}}.
\]

We also obtain for \( k + 1 \geq n + 1 \),

\[
\frac{N_{\mu, \lambda}(u)N_{\mu, \eta}(q^2u)}{N_{\mu, \lambda}(qu)N_{\mu, \eta}(qu)} = q^{n - |\lambda| - k} \prod_{j=1}^{\lambda_{k+1}} \frac{1 - q^{j-1}u^{-1}}{1 - q^{-\ell_\lambda(k+1)+j-2}u^{-1}} \prod_{j=1}^{\lambda_{k+2}} (1 - q^{-\ell_\lambda(k+1)+j-2}u^{-1}) \\
\times \prod_{i=n+1}^{k} \frac{1 - q^{k-i+\alpha_i(i)+1}u^{-1}}{1 - q^{k-i+\alpha_i(i+1)}u^{-1}} \\
= q^{n - |\lambda| - k} \times \frac{1 - q^{k-n+\lambda_{n+1}-1}u^{-1}}{1 - q^{-1}u^{-1}}.
\]

Since \( |\eta| - |\lambda| = n - \lambda_{n+1} \), we obtain (A.13) for \( n \leq \ell - 1 \).

Second, we consider the case \( n \geq \ell \). By the definition, we have

\[\ell_\eta(i, j) = \ell_\lambda(i, j - 1) \quad ((i, j) \in \eta, j \geq 2),\]

\[\ell_\eta(i, 1) = n - i \quad (i \in \mathbb{Z}_{>0}).\]
Hence, we obtain

\[
\frac{N_{\mu,\lambda}(u)N_{\mu,\eta}(q^2u)}{N_{\mu,\lambda}(qu)N_{\mu,\eta}(qu)} = q^{\eta - |\lambda|-k} \prod_{j=1}^{k+1} \frac{1 - q^{-1}u^{-1}}{1 - q^{-\ell \delta_{j(k+1)} + j - 2}u^{-1}} \prod_{j=1}^{n+1} \frac{1 - q^{-\ell \delta_{j(k+1)} + j - 3}u^{-1}}{1 - q^{-2}u^{-1}} \cdot \frac{1 - q^{n+1-k}u^{-1}}{1 - q^{-1}u^{-1}}
\]

\[
= q^{\eta - |\lambda|-k} \prod_{i=n+1}^{k} \frac{1 - q^{-i}u^{-1}}{1 - q^{-i-1}u^{-1}}
\]

\[
= q^{\eta - |\lambda|-k} \cdot \frac{1 - q^{-n+k-1}u^{-1}}{1 - q^{-1}u^{-1}}.
\]

Since $|\eta| - |\lambda| = n$, we obtain (A.13) for $n \geq \ell$.

Proof of (A.14). By (A.11), we have

\[
N_{\mu,\lambda}(u) = N_{\gamma,\lambda}(q^{-1}u) \prod_{j=1}^{\gamma+1} \frac{1 - q^{-1}u^{-1}}{1 - q^{-\ell \delta_{j(\ell+1)} + j - 2}u^{-1}} \prod_{i=1}^{\ell} (1 - q^{\ell+\alpha_{\gamma}(i,1)+1}u).
\]

Recall that $\gamma = (s_{n}(\mu'))'$. First, we consider the case $n \leq \mu_1 - 1$. By the definition, we have

\[
\ell_{\gamma}(i,j) = \begin{cases} 
\ell_{\mu}(i,j) + 1 & ((i,j) \in \gamma, j \leq n), \\
\ell_{\mu}(i,j + 1) & ((i,j) \in \gamma, j \geq n + 1),
\end{cases}
\]

\[
a_{\gamma}(i,j) = \begin{cases} 
a_{\mu}(i,j) - 1 & (i = 1, \ldots, \mu_{n+1}', j \in \mathbb{Z}_{\geq 0}), \\
n - j & (i = \mu_{n+1}', 1, j \in \mathbb{Z}_{\geq 0}), \\
a_{\mu}(i - 1,j) & (i,j) \in \gamma, i \geq \mu_{n+1}' + 2, j \in \mathbb{Z}_{\geq 0}).
\end{cases}
\]

Hence, for $\ell + 1 \leq \mu_{n+1}'$ we have

\[
N_{\mu,\lambda}(u) = N_{\gamma,\lambda}(q^{-1}u) \prod_{j=1}^{\mu_{\ell+1}'-1} (1 - q^{-1}u^{-1}) \prod_{i=1}^{\ell} (1 - q^{\ell+\alpha_{\gamma}(i,1)}u) 
\]

\[
= N_{\gamma,\lambda}(q^{-1}u) \prod_{j=1}^{\mu_{\ell+1}'} \frac{1 - q^{-\ell \delta_{j(\ell+1)} + j - 3}u^{-1}}{1 - q^{-\ell \delta_{j(\ell+1)} + j - 2}u^{-1}} \cdot \frac{1 - q^{-\mu_{n+1}'+\ell+n-1}u}{1 - q^{\mu_{\ell+1}'-1}u} 
\]

\[
\times \prod_{i=1}^{\ell-1} (1 - q^{\ell+\alpha_{\gamma}(i,1)}u) \cdot (1 - q^{\mu_{\ell+1}'-1}u).
\]
Similarly, for $\ell + 1 = \mu'_{n+1} + 1$ we have

$$N_{\gamma, \lambda}(u) = N_{\gamma, \lambda}(q^{-1}u) \prod_{j=1}^{n} \frac{1 - q^{j-1}u}{1 - q^{-\ell_{\mu}(i,j)+j-2}u} \prod_{i=1}^{\ell} (1 - q^{-i+\mu(i,1)}u) \cdot (1 - q^{\mu(i-1)}u)$$

$$= N_{\gamma, \lambda}(q^{-1}u) \prod_{j=1}^{\mu_{\ell}} \frac{1 - q^{j-1}u}{1 - q^{-\ell_{\mu}(i,j)+j-2}u} \prod_{i=1}^{\mu'_{n+1}} (1 - q^{-i+\mu(i,1)}u) \cdot (1 - q^{\mu(i-1)}u).$$

Here, we use the inequality $\mu_{\ell+1} \leq n$.

For $\ell + 1 \geq \mu'_{n+1} + 2$ we have

$$N_{\gamma, \lambda}(u) = N_{\gamma, \lambda}(q^{-1}u) \prod_{j=1}^{\mu_{\ell}} \frac{1 - q^{j-1}u}{1 - q^{-\ell_{\mu}(i,j)+j-2}u} \prod_{i=1}^{\mu'_{n+1} + n-1} (1 - q^{-i+\mu(i,1)}u) \cdot (1 - q^{\ell_{\mu(i,1)}u}).$$

Here, we use the inequality $\mu_{\ell} \leq n$. Since $|\gamma| - |\mu| = n - \mu'_{n+1}$, we obtain (A.14) for $n \leq \mu_1 - 1$.

Second, we consider the case $n \geq \mu_1$. By the definition, we have

$$\ell_{\gamma}(i,j) = \ell_{\mu}(i-1,j) \quad ((i,j) \in \gamma, i \geq 2),$$

$$a_{\gamma}(i,j) = \begin{cases} n - j & (i = 1, j \in \mathbb{Z}_{>0}), \\ a_{\mu}(i-1,j) & ((i,j) \in \gamma, i \geq 2, j \in \mathbb{Z}_{>0}). \end{cases}$$

Hence, we obtain

$$N_{\gamma, \lambda}(u) = N_{\gamma, \lambda}(q^{-1}u) \prod_{j=1}^{\mu_{\ell}} \frac{1 - q^{j-1}u}{1 - q^{-\ell_{\mu}(i,j)+j-2}u} \prod_{i=2}^{\ell} (1 - q^{-i+\mu(i-1,1)+1}u) \cdot (1 - q^{\ell_{\mu(i,1)}u}).$$

Since $|\gamma| - |\mu| = n$, we obtain (A.14) for $n \geq \mu_1$.

Lemma A.4 enables us to reduce $\lambda$ to $\tilde{\lambda}$ of $X_{\lambda,\mu,\alpha,\beta}^\varepsilon$ and $Y_{\lambda,\mu,\alpha,\beta}^\varepsilon$. Denote by $T_1$ the $q$-shift operator with respect to $x_1$: $(T_1 f)(x_1, \ldots, x_n) = f(qx_1, \ldots, x_n)$. 

\[\Box\]
LEMMA A.5 For partitions $\lambda, \mu, \alpha, \beta$, we have

$$X_{\lambda, \mu, \alpha, \beta}^\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = C(1 - q^{-\epsilon\theta_\infty-\theta_1-\sigma-1/2})q^{\epsilon\theta_\infty-\theta_1-\sigma-1/2}$$

$$\times (q^{-\ell(\lambda)-\epsilon\theta_\infty+\theta_1+\tau+1/2}T_1 - 1) \left(X_{\lambda, \mu, \alpha, \beta}^+(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2}; x_1, x_2)\right),$$

$$Y_{\lambda, \mu, \alpha, \beta}^+(\theta_\infty, \theta_1, \sigma; x_1, x_2) = C(1 - q^{\pm\theta_\infty-\theta_1-\sigma-1/2})$$

$$\times (1 - q^{-\ell(\lambda)+1+2\sigma}T_1) \left(Y_{\lambda, \mu, \alpha, \beta}^+(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2}; x_1, x_2)\right),$$

$$Y_{\lambda, \mu, \alpha, \beta}^-(\theta_\infty, \theta_1, \sigma; x_1, x_2) = C(1 - q^{-2\sigma})\frac{x_2}{x_1}$$

$$\times (1 - q^{-\ell(\lambda)}T_1) \left(Y_{\lambda, \mu, \alpha, \beta}^-(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2}; x_1, x_2)\right),$$

where

$$C = \prod_{j=1}^{\ell(\lambda)} \frac{1 - q^{i+\epsilon\theta_\infty-\theta_1-\sigma-1/2}}{1 - q^{-\ell(\lambda)+j+\theta_\infty-\theta_1-\sigma-3/2}} \prod_{i=1}^{\ell(\lambda)-1} (1 - q^{i+\epsilon\theta_\infty-\theta_1-\sigma-1/2})$$

$$\times \prod_{j=1}^{\ell(\lambda)} \frac{1 - q^{i+\epsilon\theta_\infty-\theta_1-\sigma-1/2}}{1 - q^{-\ell(\lambda)+j+\theta_\infty-\theta_1-\sigma-3/2}} \prod_{i=1}^{\ell(\lambda)-1} (1 - q^{i+\epsilon\theta_\infty-\theta_1-\sigma-1/2})$$

$$\times q^{\ell(\lambda)-|\alpha|-|\beta|}X_{\lambda, \mu, \alpha, \beta}^{-\ell(\lambda)}. $$

Proof of Proposition A.1. By induction. If all partitions $\lambda, \mu, \alpha$ and $\beta$ are equal to $\emptyset$, then the identity (A.1) is the connection formula of basic hypergeometric functions. By definition we have the symmetries

$$X_{\lambda, \mu, \alpha, \beta}^\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = X_{\lambda, \mu, \alpha, \beta}^\epsilon(\theta_\infty, \theta_1, -\sigma; x_1, x_2),$$

$$X_{\lambda, \mu, \alpha, \beta}^-\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = X_{\lambda, \mu, \alpha, \beta}^+\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2),$$

$$Y_{\lambda, \mu, \alpha, \beta}^-\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = Y_{\lambda, \mu, \alpha, \beta}^+\epsilon(\theta_\infty, \theta_1, -\sigma; x_1, x_2),$$

$$Y_{\lambda, \mu, \alpha, \beta}^-\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = Y_{\lambda, \mu, \alpha, \beta}^+\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2),$$

$$B_{+,-} \left[ \begin{array}{c} \theta_1 \\ \frac{1}{2} \sigma \\ \frac{1}{2} x_2 \\ \sigma x_1 \end{array} \right] = B_{+,-} \left[ \begin{array}{c} \theta_1 \\ \frac{1}{2} \sigma \\ \frac{1}{2} x_2 \\ \sigma x_1 \end{array} \right], \quad B_{+,-} \left[ \begin{array}{c} \theta_1 \\ \frac{1}{2} \sigma \\ \frac{1}{2} x_2 \\ \sigma x_1 \end{array} \right] = B_{+,-} \left[ \begin{array}{c} \theta_1 \\ \frac{1}{2} \sigma \\ \frac{1}{2} x_2 \\ \sigma x_1 \end{array} \right],$$

and by Lemma A.2 we have

$$X_{\lambda, \mu, \alpha, \beta}^\epsilon(\theta_\infty, \theta_1, \sigma; x_1, x_2) = Y_{\lambda', \mu', \alpha', \beta'}^\epsilon(\sigma, \theta_1, \theta_\infty, (q_{x_1})^{-1}, (q_{x_2})^{-1}).$$

Recall also the relations (2.11) and (2.12) for the braiding matrix. Because of these relations, it suffices to show that for any partitions $\lambda, \mu, \alpha$ and $\beta$, the connection formula (A.1) is equivalent to the formula (A.1) for $\tilde{\lambda}, \mu, \alpha$ and $\beta$ acting by a $q$-difference operator.
By Lemma A.5, the left-hand side of (A.1) is computed as follows:

\[
q^2 \left( \frac{q^{2\bar{t}_1}x_2}{x_1} \right)^{\theta_\infty+1/4} \prod_{\nu''=\pm} \Gamma_q \left( \frac{1}{2} + \theta_\infty - \theta_1 + \nu'' \sigma \right) \left( \frac{q^{T} q^{-\theta_\infty - \theta_1 - \sigma - 1/2}}{\Gamma_q(1 + 2\theta_\infty)} \right) X^+_{\lambda, \mu, \alpha, \beta}(\theta_\infty, \theta_1, \sigma; x_1, x_2)
\]

\[
= q^{\theta_\infty} \left( \frac{q^{2\bar{t}_1}x_2}{x_1} \right)^{\theta_\infty+1/4} \prod_{\nu''=\pm} \Gamma_q \left( \frac{1}{2} + \theta_\infty - \theta_1 + \nu'' \sigma \right) \right) \left( \frac{q^{T} q^{-\theta_\infty - \theta_1 - \sigma - 1/2}}{\Gamma_q(1 + 2\theta_\infty)} \right) \times C q^{\theta_\infty - \theta_1 - \sigma - 1/2} \left( q^{-\ell(\lambda) - \theta_\infty + \theta_1 + \sigma + 1/2} T_1 - 1 \right) \left( X^+_{\lambda, \mu, \alpha, \beta}(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2}; x_1, x_2) \right)
\]

\[
= C q^{\theta_\infty - \theta_1 - \sigma - 3/4} \frac{1 - q^{4\theta_\infty - \theta_1 - \sigma - 1/2}}{1 - q} \times \frac{\prod_{\nu''=\pm} \Gamma_q \left( \frac{1}{2} + \nu'' \theta_\infty - \theta_1 + \nu'' \sigma \right)}{\Gamma_q(1 + 2\sigma + \frac{1}{2})} Y^e_{\lambda, \mu, \alpha, \beta}(\theta_\infty, \theta_1, \sigma; x_1, x_2)
\]

Similarly, by Lemma A.5, the right-hand side of (A.1) is computed as follows:

\[
\sum_{\nu''=\pm} q^{\nu'' \theta_\infty - \theta_1 - \sigma - 1/2} \prod_{\nu''=\pm} \Gamma_q \left( \frac{1}{2} + \nu'' \theta_\infty - \theta_1 + \nu'' \sigma \right) \times B_{\nu''} \left[ \frac{1}{2} \theta_1 + \nu'' \frac{1}{2} \right] \left( \frac{x_2}{x_1} \right)^{\theta_1} \left( \frac{x_2}{x_1} \right)^{\theta_1 - (\nu'' + 1)/2} \frac{Y^e_{\lambda, \mu, \alpha, \beta}(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2})}{\Gamma_q(1 + 2\sigma + \frac{1}{2})}
\]

\[
= C q^{\theta_\infty - \theta_1 - \sigma - 3/4} \frac{1 - q^{4\theta_\infty - \theta_1 - \sigma - 1/2}}{1 - q} \times \frac{\prod_{\nu''=\pm} \Gamma_q \left( \frac{1}{2} + \nu'' \theta_\infty - \theta_1 + \nu'' \sigma \right)}{\Gamma_q(1 + 2\sigma + \frac{1}{2})} Y^e_{\lambda, \mu, \alpha, \beta}(\theta_\infty, \theta_1 + \frac{1}{2}, \sigma + \frac{1}{2})
\]

Therefore, the identity (A.1) with parameters \(\theta_\infty, \theta_1, \sigma\) for \(\lambda, \mu, \alpha, \beta\) is reduced to the identity (A.1) with parameters \(\theta_\infty, \theta_1 + 1/2, \sigma + 1/2\) for \(\bar{\lambda}, \mu, \alpha, \beta\).
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