Two-Color QCD and Aharonov-Bohm Fluxes

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We investigate the effects of several Abelian Aharonov-Bohm fluxes $\phi$ on the Euclidean Dirac spectrum of light quarks in QCD with two colors. A quantitative change in the quark return probability is caused by the fluxes, resulting into a change of the spectral correlations. These changes are controlled by a universal function of $\sigma_L \phi^2$ where $\sigma_L$ is the pertinent Ohmic conductance. The quark return probability is sensitive to Abelian flux-disorder but not to $Z_2$ flux-disorder in the ergodic and diffusive regime, and may be used as a probe for the nature of the confining fields in the QCD vacuum.

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1. In a finite Euclidean volume $V$ light quarks exhibit properties analogous to that of electrons in small metallic grains \cite{weinberg}. The finite volume allows for a discrimination of length scales with the emergence of a diffusive regime characterized by the diffusion constant $D \approx 0.22 \text{ fm}^2$. Although QCD confines, the light quarks diffuse in four dimensions as a result of the spontaneous breaking of chiral symmetry \cite{weinberg}.

In two-color and two-flavor QCD the quarks are in the pseudoreal representation of the flavor group \cite{weinberg}, and the spontaneous breaking of chiral symmetry is accompanied by the occurrence of five Goldstone modes: 3 pions and a baryon and an antibaryon. The five Goldstone modes decay weakly, with a mass that satisfies the Gell-Mann–Oakes–Renner (GOR) relation \cite{weinberg}. The latter mass vanishes in the chiral limit.

The nature of the Goldstone modes and the characteristics of the light quark spectrum are intimately related in the general context of chiral disorder \cite{weinberg}. In particular, two-color QCD is characterized by diffusion (diagonal, pionic) and cooperon (interference, baryonic) contributions in the semi-classical analysis. The cooperons correspond to weak-localization (coherent back-scattering) contributions to the quark return probability \cite{weinberg}. They are soft because the baryons in two-color QCD are diquarks and soft. QCD with three colors admit only diffusions as the diquarks are expected to be heavy. As well known in disordered metals \cite{weinberg}, the weak-localization contribution is altered by the presence of an external parameter that breaks time-reversal symmetry. An example is an Aharonov-Bohm flux.

In this letter we consider the effects of several Abelian Aharonov-Bohm fluxes on the light quarks in a finite Euclidean volume with two-color QCD. In many ways, our analysis in two-color QCD will parallel recent analyses of electrons in disordered metals \cite{weinberg}. We discuss these effects on the light quark return probability, and analyze their importance on the spectral correlations. The fluxes cause a periodic interpolation in the spectral statistics between orthogonal and unitary ensembles. We show that the quark return probability is sensitive to Abelian flux-disorder but not to $Z_2$ disorder, and discuss the relevance of this result for the nature of the QCD vacuum.

2. In a finite Euclidean volume $V = L^4$ pierced by Abelian (essentially electromagnetic) fluxes $\phi_\mu = (\phi_1, \phi_2, \phi_3, \phi_4)$, the light quarks in the background gluon field $A$ satisfy the following Dirac equation

\begin{equation}
\imath \nabla[A] q_k = \lambda_k [A, \phi] q_k .
\end{equation}

subject to the boundary condition

\begin{equation}
q_k(x + L_\mu) = -e^{2\pi \phi_\mu} q_k(x).
\end{equation}

The $\phi$’s are given in units of an electromagnetic flux quantum $2h/e$ \cite{weinberg} set to 1 for convenience. Through \cite{weinberg} the quark spectrum \cite{weinberg} depends explicitly on $\phi$. We note that the Abelian fluxes through \cite{weinberg} are simple holonomies along each of the four directions of the four-volume: $e^{2\pi x_\mu \phi/L}$. The eigenvalue problem \cite{weinberg} is equivalent to

\begin{equation}
\left( \imath \nabla [A] + \frac{2\pi}{L} \gamma \cdot \phi \right) q_k = \lambda_k [A, \phi] q_k .
\end{equation}

with anti-periodic boundary conditions.

The probability $p(t, \phi)$ for a light quark to start at $x(0)$ in $V$ and return back to the same position $x(t)$ after a proper time duration $t$, is

\begin{equation}
p(t, \phi) = \frac{V^2}{N} \left| \langle x(0)| e^{i \imath \nabla[A]+im} |x(0)\rangle^2 \right| .
\end{equation}

The averaging in \cite{weinberg} is over all gluon configurations using the unquenched two-color QCD measure. The normalization in \cite{weinberg} is per state, where $N$ is the total number of quark states in the four-volume $V$. Equation \cite{weinberg} may be written in terms of the standard Euclidean propagators for the quark field,
\[ p(t, \phi) = \frac{V^2}{N} \lim_{y \to x} \int \frac{d\lambda_1 d\lambda_2}{(2\pi)^2} e^{-i(\lambda_1 - \lambda_2) |t|} \times \left( \text{Tr} \left( S(x, y; z_1, \phi) S^\dagger(x, y; z_2, \phi) \right) \right)_A \] 

(5)

generalizing to \( \phi \neq 0 \) the results in \( \Box \). Here \( z_{1,2} = m - i\lambda_{1,2} \), and

\[ S(x, y; z, \phi) = \langle x | \frac{1}{\sqrt{|A| + i\phi}} | y \rangle. \]  

(6)

subject to the boundary condition

\[ S(x + L, y; z, \phi) = -e^{i2\pi / m} S(x, y; z, \phi). \]  

(7)

Setting \( \lambda_{1,2} = \Lambda \pm \lambda / 2 \) and neglecting the effects of \( \Lambda \) in the averaging in \( \Box \) (zero virtuality), the correlation function in \( \Box \) relates (in general) to an analytically continued pseudoscalar correlation function, as the eigenvalues \( q_k \) and \( \gamma_k q_k \) are pair-degenerate (chiral) \( \Box \).

For two-color QCD there is an extra symmetry \( \Box \) that makes the pseudoscalar correlation function degenerate with certain diquark correlation functions in the flux-free case. Indeed, for \( \phi = 0 \) and \( K = -T^2 CK \) the eigenvalues \( q_k \) and \( K q_k \) are also pair-degenerate. Here \( C \) is the charge-conjugation matrix, \( T^2 \) the color matrix and \( K \) the (right-left) complex-conjugation. For instance, for two flavors the three pions are degenerate with the spin-zero baryon and antibaryon.

In general, we may rewrite (\( \Box \)) in the form

\[ p(t, \phi) = \frac{EV^2}{2\pi N} \lim_{y \to x} \int \frac{d\lambda}{2\pi} e^{-i\lambda |t|} C_G(x, y; z, \phi) \]  

(8)

with \( z = m - i\lambda / 2 \) and \( E = \int d\Lambda \). For \( \phi = 0 \), the above symmetries allow us to write (two flavors)

\[ C_G(x, y; z; 0) = \]  

(9)

\[ + \frac{1}{2} \left( \text{Tr} \left( S(x, y; z, 0) i\gamma_5 \tau^2 S(y, x; z, 0) i\gamma_5 \tau^2 \right) \right)_A \]

\[ + \frac{1}{2} \left( \text{Tr} \left( S(x, y; z, 0) K i\gamma_5 \tau^2 S(y, x; z, 0) K i\gamma_5 \tau^2 \right) \right)_A \]

The flavor matrix \( \tau^2 \) was introduced to retain the connected parts in the correlator version \( \Box \). The first contribution is pionic (diffusion), while the second one is baryonic (cooperon). Both are bosonic and identical. The decomposition \( \Box \) is commensurate with the semi-classical description (long paths) where the time-reversed trajectories are retained \( \Box \). For \( \phi \neq 0 \), the first symmetry (chiral) is retained while the second one is upset. Indeed, now the contributions in \( \Box \) are no longer the same as the fluxes add in the cooperon, but cancel in the diffusion. For long paths, we have

\[ C_G(x, y; m, \phi) \approx \frac{1}{2V} \sum_Q e^{iQ(y-x)} \frac{\sum_1}{F^2 Q^2 + m_G^2} + \frac{1}{2V} \sum_Q e^{iQ(y-x)} \frac{\sum_2}{F^2 Q^2 + m_G^2} \]  

(10)

with \( Q_\mu = n_\mu 2\pi / L \) and \( \tilde{Q}_\mu = (n_\mu + 2\phi_\mu) 2\pi / L \). Here, \( \Sigma = |\langle q \rangle| \) and \( F \) is the weak decay constant for the Goldstone modes of mass \( m_G \), each of which are flux independent to leading order.

Using the GOR relation \( F^2 m_G^2 = m \Sigma \) and the analytical continuation \( m \to m - i\lambda / 2 \), we find

\[ C_G(x, y; z, \phi) \approx + \frac{1}{2V} \sum_Q e^{iQ(x-y)} \frac{2\Sigma}{-i\lambda + 2m + DQ^2} \]  

\[ + \frac{1}{2V} \sum_Q e^{iQ(x-y)} \frac{2\Sigma}{-i\lambda + 2m + DQ^2} \]  

(11)

with the diffusion constant \( D = 2F^2 / \Sigma \). Inserting (11) into (8), and noting that \( E / \Delta = N \) and \( \rho = 1 / \Delta V \), with \( \Sigma = \pi \rho \), we conclude after a contour integration that

\[ p(t, \phi) = \frac{1}{2} e^{-2m |t|} \sum_Q \left( e^{-2Q^2 |t|} + e^{-2Q^2 |t|} \right). \]  

(12)

The cooperon contribution is periodic in the flux \( \phi \) with periodicity \( \phi = 0, \pm 1 / 2, \pm 1, \ldots \), and reflects on the fact that the soft part of the spectrum does not discriminate between bosonic or fermionic boundary conditions in the flux-free case. The cooperon contribution may be rewritten using Poisson’s resummation formula as

\[ p_C(t, \phi) = \frac{V}{2(4\pi D)^{1/2}} \sum_Q e^{-2m |t|} e^{i\lambda (y-x)} \cos \left( 4\pi \nu |\phi| \right). \]  

(13)

with integer \( \nu \). This result is in agreement with the one derived by Montambaux \( \Box \) in disordered metals in lower dimensions. The flux-accumulation in the cooperon part implies changes in the spectral correlations of the light quarks as we now show.

3. To describe the spectral correlations associated with \( \Box \) we will use semi-classical arguments for the two-point correlation function \( R(s, \phi) \) of the density of eigenvalues \( \Box \). Its spectral form factor \( K(t, \phi) \) is defined as

\[ R(s, \phi) = \int_{-\infty}^{+\infty} dt e^{is\Delta t} K(t, \phi) \]  

(14)

where \( \Delta = 1 / \rho V \) is a typical quantum spacing at zero virtuality and in the absence of a flux. For diffusive quarks in two-color QCD, \( K(t, \phi) \) relates to the return probability through \( \Box \)

\[ K(t, \phi) \approx \frac{\Delta^2 |t|}{(2\pi)^2} p(t, \phi) \]  

(15)

which is periodic with periodicity \( \phi = 0, \pm 1 / 2, \pm 1, \ldots \).

For \( \phi \ll 1 \), the flux periodicity may be neglected, and for times larger than the ergodic time \( \tau_{\text{erg}} = L^2 / D \) but smaller than the Heisenberg time \( \tau_H = 1 / \Delta \), the dominant contribution to the return probability stems from the zero modes in \( \Box \). Hence
\[ p(t, \phi) = \frac{1}{2} e^{-2m|t|} \left( 1 + e^{-\tilde{F}^2|t/t_H|} \right) \]  

(16)

with

\[ \tilde{F}^2 = 16\pi^2 \left( \frac{D}{L^2 \Delta} \right) \phi_\mu \phi_\mu = 16\pi^2 \sigma_L \phi_\mu \phi_\mu. \]

(17)

Here \( \sigma_L = D/(\Delta L^2) = 2(FL)^2/\pi \) is the dimensionless Ohmic conductance \([3]\). In this limit, both the return probability and the spectral form factor are only a function of the combination \( \sigma_L \phi^2 \). The result (16) could be qualitatively derived by noting that if \( 2 \times 2\pi \phi \) is the typical accumulated flux per winding path \([10]\) in the quark return probability, then \( l \) windings generate an accumulation \( \Phi = 4\pi l\phi \). Since the paths are diffusive with an arc length \( s = Ll \sim \sqrt{Dt} \), then on the average

\[ \langle \Phi^2 \rangle = \langle (4\pi \phi)^2 \rangle \sim (4\pi \phi)^2 \frac{2Dt}{L^2}. \]

(18)

Hence

\[ \langle e^{i\Phi} \rangle \sim e^{-\langle \Phi^2 \rangle/2} = e^{-\tilde{F}^2|t/t_H|} \]

(19)

which is the dephasing appearing in (16) between the diagonal and interference terms. An average \( 2\pi \) accumulation in \( \Phi \) results from quark trajectories with a long proper time \( t \sim 1/E_c \) where \( E_c = D/L^2 \) is the Thouless energy \([1]\).

The spectral rigidity around zero virtuality (and also in bulk) follows from \([1,12]\)

\[ \Sigma_2(N, \phi) = \int_{-N}^{+N} ds \langle N - |s| \rangle R(s, \phi) \]

(20)

with \( N = E/\Delta \gg 1 \). In particular

\[ \Sigma_2(N, \phi) = \frac{1}{2\pi^2} \ln \left( 1 + \frac{N^2}{\alpha^2} \right) \left( 1 + \frac{N^2}{\tilde{\alpha}^2} \right) \]

(21)

with \( \alpha = 2m/\Delta \) and \( \tilde{\alpha} = \alpha + \tilde{F}^2 \). For \( \tilde{\alpha} \gg 1 \) the spectral rigidity is reduced by about a factor of 2. It is bracketed by the orthogonal (no flux) and unitary (finite flux) results.

For large fluxes, the non-zero momentum contributions have to be retained to enforce the proper \( \phi = 0, \pm 1/2, \pm 1, \ldots \) flux-periodicity. For simplicity, consider the case \( \phi_\mu = (0, 0, 0, \phi) \) with only one-flux retained. For long proper times, the zero modes along the 1, 2, 3 directions contribute only, giving

\[ p(t, \phi) = \frac{1}{2} e^{-2m|t|} \times \sum_n \left( e^{-4\pi^2n^2\sigma_L|t/t_H|} + e^{-4\pi^2(n+2\phi)^2\sigma_L|t/t_H|} \right) \]

(22)

which is the analogue of a diffusion in \( d=1 \). For \( t < \tau_{erg} = L^2/D \) the diffusive paths are short and do not accumulate enough flux. The spectral rigidity in this case is just

\[ \Sigma_2(N, \phi) = \frac{1}{2\pi^2} \sum_n \ln \left[ \left( 1 + \frac{N^2}{\alpha^2} \right) \left( 1 + \frac{N^2}{\tilde{\alpha}^2} \right) \right] \]

(23)

with \( \alpha_n = \alpha + 4\pi^2\sigma_L n^2 \) and \( \tilde{\alpha}_n = \alpha + 4\pi^2\sigma_L (n + 2\phi)^2 \).

For \( N, \sigma_L \gg \alpha \), (23) simplifies to

\[ \Sigma_2(N, \phi) = \Sigma_2(N, 0) = \frac{1}{2\pi^2} \ln \left( 1 + 4\sigma_L \cos^2(2\pi\phi) \right) \]

(24)

in agreement with a result derived by Montambaux \([3]\) in the context of disordered metals. We note that for a small flux (24) is in agreement with (21). Interestingly enough, the conductance \( \sigma_L \) of the chiral vacuum is directly accessible from the spectral rigidity through (24) providing for a direct measurement of this important quantity in disordered QCD. (24) may be assessed using current lattice QCD simulations.

4. Since the quark return probability and the spectral rigidity are sensitive to flux-variations in a finite Euclidean volume, they could be used to probe the flux-content of the two-color QCD vacuum – in particular monopole-antimonopole rich vacua which are likely to generate flux-disordered vacua. Although our analysis so far has focused on Abelian electromagnetic Aharonov-Bohm fluxes, we speculate that in the maximally projected gauge, colored magnetic monopoles and anti-monopoles generate colored and Abelian fluxes that act randomly on the light quarks in the vacuum.

If we consider an Abelian flux-disordered vacuum characterized by a Gaussian distributed flux with a mean

\[ \langle \langle \phi_\mu \phi_\nu \rangle \rangle = \kappa_\mu^2 \delta_{\mu \nu} \]

(25)

then the quark return probability can be easily estimated from \([12]\) using the Poisson form \([3]\) for the cooperon part. If we split the quark return probability \( p = p_D + p_C \), then the diffusion part \( p_D \) is flux-insensitive

\[ \langle \langle p_D(t, \phi) \rangle \rangle = \frac{V}{4(\pi Dt)^2} e^{-2m|t|} \]

(26)

while the cooperon part \( p_C \) is flux-sensitive

\[ \langle \langle p_C(t, \phi) \rangle \rangle = \frac{V}{4(\pi Dt)^2} \sum_{\{l\}} e^{-2m|t| - \frac{\sigma_L}{2\pi Dt} - \frac{\sigma_L}{\pi Dt}}. \]

(27)

We note that the periodicity in \( \phi = 0, \pm 1/2, \pm 1, \ldots \) of the quark return probability implies that the latter is likely insensitive to a \( Z_2 \) flux-disordered vacuum. If these effects extend to an Abelian flux-rich vacuum, a simple way to detect them is to measure the relative ratio of the quark return probabilities for \( N_c = 2 \) and \( N_c = 3 \). A flux sensitivity implies a \( t \)-dependent ratio close to \( 1/2 \) (as opposed to 1), assuming that the vacuum structure does not change appreciably from two to three colors.

5. Finally, since the transition from an orthogonal to unitary ensemble sets in the ergodic regime with \( t > \tau_{erg} \),
the ensuing spectral statistics are amenable to an analysis in 0-dimension (matrix model). The simplest realization that is chiral and embodies the essentials of a single flux is [13]

$$\mathcal{M}(\varphi) = \left( \frac{\im \varphi}{A + \im \varphi B / \sqrt{N}} A + \im \varphi B / \sqrt{N} \right)^{\im m}$$  \hspace{1cm} (28)

where $A$ and $B$ are real $N \times N$ symmetric and antisymmetric random matrices respectively, with a fixed variance $\Sigma^2/N$. For $\varphi = 0$ and $m = 0$ the matrices are chiral orthogonal, while for $\varphi \neq 0$ they are in general chiral unitary, hence the interpolation. The $1/\sqrt{N}$ in the off-diagonal matrix elements follows from second order perturbation theory since the typical level shift induced by such terms is

$$\Delta E = \langle \sum_n \frac{|\Psi_n^*(I) B_{IJ} \Psi_0(J)|^2}{E_n - E_0} \rangle_{AB} \cdot \left( \frac{\varphi}{\sqrt{N}} \right)^2$$  \hspace{1cm} (29)

where the $\Psi_n$’s are generic eigenfunctions of (28) with eigenvalues $E_n$ for $B = 0$. In the mesoscopic (ergodic) limit, the sum is dominated by the modes where the energy denominator is of order $\Delta \sim 1/N$. If the states are delocalized then the eigenfunctions are of order $1/\sqrt{N}$, hence the interpolation. The $1/\sqrt{N}$ in the $A^B$ limit, the sum is dominated by the modes where the energy denominator is of order $\Delta \sim 1/N$. If the states are delocalized then the eigenfunctions are of order $1/\sqrt{N}$. Since the sum over $I, J$ is of order $N^2$, and the averaging over $B$ is of order $\Sigma^2/N$, the second order shift is $\Delta E \sim 1/N$ and comparable to the mesoscopic level spacing $\Delta \sim 1/N$. This result is consistent with the observation made using non-chiral matrices [14]. We recall that in $d \neq 0$ dimension, the corresponding terms are down by $1/\sqrt{N}$ as suggested by (3).

6. We have shown that in two-color QCD, Abelian Aharonov-Bohm fluxes cause changes in the quark return probability with important consequences on the spectral form factor and the spectral statistics. The change is controlled by a universal function of $n^2 \sigma_L \phi^2$, where $n$ is the number of applied fluxes. This analysis offers a simple measurement of the Ohmic conductance $\sigma_L$. When combined with the value of the chiral condensate $\Sigma$, this leads to an independent assessment of the pion weak-decay constant $F$ and a simple vindication of the GOR relation in two-color QCD.

The sensitivity of the quark return probability to Abelian fluxes for two-color QCD raises the interesting possibility of addressing the nature of the confining fields in the QCD vacuum. Indeed, the quark return probability for two colors can discriminate between a $\mathbb{Z}_2$ flux-disordered phase and an Abelian flux-disordered one, as it is blind to the former. Also, if the flux-disordering is generic in two and three colors QCD, quantitative changes in the proper time behavior of the pertinent quark return probabilities and spectral rigidities are expected. The present results can be tested by using lattice QCD simulations, or continuum models of the QCD vacuum [3,13,18].

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