BINARY FORMS AND THE HYPERELLIPTIC SUPERSTRING ANSATZ

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ABSTRACT. We give a hyperelliptic formulation of the Ansatz of D’Hoker and Phong. We give an explicit family of binary invariants, one for each genus, that satisfies this hyperelliptic Ansatz. We also prove that this is the unique family of weight eight binary forms over the theta group on the hyperelliptic locus that satisfies this Ansatz. Furthermore, we prove that this solution may also be obtained by applying Thomae’s map to multivalued Siegel modular forms of Grushevsky and making certain choices of roots.

1. Introduction

We formulate the Ansatz of D’Hoker and Phong for the ring of binary invariants, which can be viewed as a ring of modular forms on the moduli space of hyperelliptic Riemann surfaces. We prove the existence and the uniqueness of the sequence of binary invariants $H_g$ satisfying the Ansatz. Finally, we relate our work to a sequence of multivalued Siegel modular forms constructed by Grushevsky. When Thomae’s formula is applied to Grushevsky’s multivalued Siegel modular forms, each $H_g$ may be extracted as a certain branch.

We first review the formulation of the Ansatz of D’Hoker and Phong on the Siegel upper half space $\mathcal{H}_g$, where the description of the Witt map $\Psi_{i,j}$ is simpler. The Ansatz has three parts. For each genus $g$, we seek Siegel modular forms of weight eight for the theta group, $\Xi^{(g)}[0] \in [\Gamma_g(1,2), 8]$, such that: i) For all $g_1, g_2 \in \mathbb{N}$,

$$
\Xi^{(g_1+g_2)}[0] \left( \begin{array}{cc} \Omega_1 & 0 \\ 0 & \Omega_2 \end{array} \right) = \Xi^{(g_1)}[0](\Omega_1) \Xi^{(g_2)}[0](\Omega_2),
$$

whenever $\Omega_i \in \mathcal{H}_{g_i}$ are the period matrices of compact Riemann surfaces. We can rephrase this condition in terms of the Witt map, $\Psi_{i,j}^{*} : [\Gamma_{i+j}(1,2), 8] \rightarrow [\Gamma_i(1,2), 8] \otimes [\Gamma_j(1,2), 8]$, by saying $\Psi_{g_1, g_2}^{*} \Xi^{(g_1+g_2)}[0] = \Xi^{(g_1)}[0] \otimes \Xi^{(g_2)}[0]$ on the Jacobian locus. ii) The trace of $\Xi^{(g)}[0]$ to level
one, \( \text{tr} \left( \Xi^{(g_1+g_2)}[0] \right) \in [\Gamma_8, 8] \), vanishes on all \( \Omega \in \mathcal{H}_g \) that are period matrices of compact Riemann surfaces. \( \text{iii) The family of solutions to conditions } i \text{ and } ii \text{ is uniquely determined by the genus one solution } \Xi^{(1)}[0] = \theta_4^4 \eta^{12}. \) This formulation of the Ansatz differs only slightly from the original by D’Hoker and Phong and its evolution may be traced in [4], [5], [6], [7], [11], [10], [23] and [20]. The solutions for \( g \leq 2, 3, 4 \) and 5 may be found in, for example, [5], [11], [10] and [20], respectively. Uniqueness is known for \( g \leq 4. \) It appears likely that the solution is also unique in \( g = 5 \) and that for \( g \geq 6 \) no solutions exist.

These mathematical questions owe their origin to the physics literature. We thank R. Salvati Manni for introducing us to these ideas. The chiral superstring measure \( d\nu[e] \) for a fixed theta characteristic \( e \) should take the form \( d\nu[e] = f[e]^{(g)} d\mu \), where \( d\mu \) is the Mumford measure and \( f[e]^{(g)} \) is a weight eight Teichmüller modular form on the moduli space of curves with a fixed theta characteristic \( e \). Condition \( i \) says that the measure should be the product measure on reducible curves. Condition \( ii \) says that the traced level one measure, whose integral over moduli space gives the cosmological constant, vanishes pointwise. These conditions are only required for period matrices of compact Riemann surfaces because the original interest is in Teichmüller modular forms on the moduli space of curves. For \( g \leq 3 \), period matrices are dense in \( \mathcal{H}_g \) but for \( g \geq 4 \) there is no a priori reason to expect that a solution \( f[e]^{(g)} \) on Teichmüller space will analytically extend to all of \( \mathcal{H}_g \). Thus it is remarkable that in \( g = 4 \) and 5 the solutions \( \Xi^{(g)}[e] \) exist as Siegel modular forms at all; whereas the nonexistence of the \( \Xi^{(g)}[e] \) for \( g \geq 6 \) would come as no surprise. The general existence of the Teichmüller forms \( f[e]^{(g)} \) remains open and has not even received a strict mathematical formulation—a task best reserved for those who make significant progress. Still, the above considerations have shown what the \( f[e]^{(g)} \) should be in \( g \leq 5 \) and the existence of these \( \Xi^{(g)}[0] \) is a remarkable vindication of the Ansatz of D’Hoker and Phong. For an entry into the physics literature see [17]. For Teichmüller modular forms, see [12][13].

Another probe into the existence of the hypothetical \( f[e]^{(g)} \) would be to restrict them to hyperelliptic curves, a special case that is always easier to study. If such a family exists on the moduli space of curves then it should also exist on the moduli space of hyperelliptic curves, although the uniqueness property might be lost. This idea is not new. In [18], A. Morozov studies the restriction of the \( \Xi^{(g)}[e] \) to the hyperelliptic locus and recommends the general application of Thomae’s formula to Grushevsky’s multivalued Siegel modular form—accomplished here
in section 4. We give further vindication of the Ansatz of D’Hoker and Phong by formulating it for the moduli space of hyperelliptic curves and by proving that this formulation of the Ansatz is uniquely solvable. The form of the Witt map is more complicated in the hyperelliptic case but it can be found in Tsuyumine’s work [24]. The discussion of these broad topics ends with this Introduction but one can hope that having an explicit hyperelliptic approximation to a chiral superstring measure for every genus will be of use.

The vector space of binary invariants of weight \( w \) in \( r \) variables, \( S_w(r) \), consists of those polynomials \( f \in \mathbb{C}[a_1, \ldots, a_r] \) satisfying

\[
\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \text{ with } \gamma(z) = \frac{Az + B}{Cz + D},

f(a_1, \ldots, a_r) = f(\gamma(a_1), \ldots, \gamma(a_r)) \prod_{i=1}^r (Ca_i + D)^w.
\]

From the matrix \( \begin{pmatrix} \sqrt{\chi} & 0 \\ 0 & 1/\sqrt{\chi} \end{pmatrix} \) we see that each nontrivial \( f \in S_w(r) \) is homogeneous of total degree \( wr/2 \) and from the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) that each \( f \in S_w(r) \) is a polynomial in the \( a_i - a_j \). We remark that any product of the \( a_i - a_j \) where all of the \( a_i \) occur exactly \( w \) times is an element of \( S_w(r) \). For example, if we set \( \Delta_T = \prod_{i,j \in T; i > j} (a_i - a_j) \) for \( T \subseteq \{1, 2, \ldots, r\} \), then \( \Delta_{\{1, 2, \ldots, r\}} \) is an element of \( S_{r-1}(r) \). The graded ring \( S(r) = \bigoplus_{w=0}^{\infty} S_w(r) \) is integrally closed and finitely generated over \( \mathbb{C} \). We define the star map \( * : \mathbb{C}[a_1, \ldots, a_r] \to \mathbb{C}[a_1, \ldots, a_{r-1}] \) by letting \( f^* \) be the coefficient of the highest power of \( a_r \) in \( f \); this makes \( * \) a multiplicative map. Furthermore, \( * \) is injective on \( S_w(r) \).

In the context of binary invariants \( S(2g+2) \), the theta group corresponds to a certain subgroup of the symmetric group \( S_{2g+2} \). We also call the subgroup of permutations, \( S_U \), which stabilizes the partition of \( \{1, \ldots, 2g+2\} \) into even and odd elements, the theta group although any conjugate group would serve equally well. The subspace of \( S_w(2g+2) \) fixed elementwise by \( S_U \) is written \( S_w(2g+2)(S_U) \). In the following section we will define a certain subspace \( B^k_g \subseteq S_{\frac{k}{2}}(2g+2)(S_U) \); suffice it to say here that \( B^k_g \) is the largest subspace for which applications of the Witt map \( W_{g_1,g_2} : B^k_{g_1+g_2} \to S_{\frac{k}{2}}(2g_1+2) \otimes S_{\frac{k}{2}}(2g_2+2) \) are defined. Following [24], define the Witt map for \( f \in B^k_{g_1+g_2} \) by

\[
((* \otimes *) W_{g_1,g_2}(f))(a_1, \ldots, a_{2g_1+1}, \alpha_1, \ldots, \alpha_{2g_2+1}) = \\
\text{Coeff}(t^{\frac{k}{2}}(2g_1+g_1+g_2), f(a_1, \ldots, a_{2g_1+1}, \alpha_1 + t, \ldots, \alpha_{2g_2+1} + t)).
\]
We can now state a hyperelliptic Ansatz, modeled after that of D’Hoker and Phong.

**Hyperelliptic Ansatz**

We wish to find a sequence of binary invariants $H_g \in B^8_g$ such that

- i) $\forall g_1, g_2 \in \mathbb{N} : g = g_1 + g_2, W_{g_1,g_2}(H_g) = H_{g_1} \otimes H_{g_2}$.
- ii) The symmetrization of $H_g$ vanishes: $\sum_{\sigma \in S_{2g+2}} \sigma(H_g) = 0$.
- iii) Any solution $H_g$ to i) and ii) is uniquely determined by the genus one solution $H_1 = \Delta_{1,2,3,4}(a_1 - a_3)(a_2 - a_4)$.

A solution of i) and ii) for the original Ansatz is taken to a solution of i) and ii) for the hyperelliptic Ansatz by Igusa’s $\rho$-map. The relevant commutative diagrams may be found in the final section. In order to present the solution to this hyperelliptic Ansatz, we need the following definitions.

**Definition 1.** For a finite sequence of natural numbers $e = (e_1, \ldots, e_r)$, define $\psi' = \prod_{i=1}^{r-1} (a_{e_i} - a_{e_{i+1}})$ and $\psi_e = (a_{e_r} - a_{e_1})\psi'_e$.

Define $\mathcal{E}_r = \{(e_1, \ldots, e_r) \in \mathbb{N}^r : \forall i, e_i \equiv i \mod 2 \text{ and } \{e_1, \ldots, e_r\} = \{1, \ldots, r\}\}$. That is, $\mathcal{E}_r$ consists of all permutations of $\{1, \ldots, r\}$ that alternate odd and even, beginning with odd. For $g \in \mathbb{N}$, let

$$H_g = \frac{1}{2^g g + 1} \left( \Delta_{1,2,\ldots,2g+2} \right)^2 \sum_{e \in \mathcal{E}_{2g+2}} \psi_e^{-1}.$$  

**Theorem 2.** The sequence $H_g \in B^8_g$ satisfies all three conditions of the hyperelliptic Ansatz.

2. The Theorem

This section requires the following additional notation.

- $B_g = \{1, 2, \ldots, 2g + 2\}$, $U_g = \{1, 3, 5, \ldots, 2g + 1\}$, $U'_g = B_g \setminus U_g$.
- $\hat{\mathcal{E}}_r = \{e \in \mathcal{E}_r : e_r = r\}$.
- For $e \in \mathcal{E}_r$, define $e^* \in \mathcal{E}_{r-1}$ to be the sequence obtained by deleting the last term in $e$. Note that $e \mapsto e^*$ gives a natural isomorphism $\hat{\mathcal{E}}_r \rightarrow \mathcal{E}_{r-1}$.
- Define $SS = \{\sigma : \sigma \text{ a permutation of } B_g : \sigma(U_g) = U_g \text{ or } U'_g\}$, $SS = \{\sigma : \sigma \text{ a permutation of } B_g : \sigma(U_g) = U_g\}$.
- For $f(a_1, \ldots, a_r)$ a polynomial and $\sigma$ a permutation, define $\sigma(f) = f(a_{\sigma(1)}, \ldots, a_{\sigma(r)})$.

For $T \subseteq B_g$, we let $T' = B_g \setminus T$ denote the complement of $T$ in $B_g$. When $|T| = g + 1$, we note that $\Delta_T \Delta_{T'} \subseteq S_g(2g + 2)$. In fact, the ring $S(2g + 2) = \bigoplus_{g=0}^{\infty} S_g(2g + 2)$ is the integral closure of the ring generated by the $\Delta_T \Delta_{T'}$ over all $T$ with $B_g = T \bigcup T'$ and $|T| = |T'| = g$.
$g + 1$, compare Igusa [16], page 845, supplement I. For many purposes, this characterization of $S^{(g)}(2g + 2)$ obviates the need to treat this ring abstractly.

**Lemma 3.** A nontrivial $f \in S_w(r)$ has degree $w$ in each $a_i$. The star map $* : S_w(r) \to \mathbb{C}[a_i - a_j; 1 \leq i < j \leq r - 1]$ injects.

**Proof.** (Tsuyumine [24]) Consider $(\begin{smallmatrix} 0 & 1/\epsilon \\ -\epsilon & a_r \end{smallmatrix}) \in \text{SL}_2(\mathbb{C})$ for $\epsilon \neq 1$. We have $f(a_1, \ldots, a_r) = (a_r(1 - \epsilon))^w f \left( \frac{1/\epsilon}{a_r - \epsilon a_1}, \ldots, \frac{1}{a_r - \epsilon a_{r-1}}, \frac{1}{a_r(1 - \epsilon)} \right) \prod_{i=1}^{r-1} (a_r - \epsilon a_i)^w.$

We let $\epsilon \to 1$ on both sides on this equation; the limit of the left hand side is the nontrivial polynomial $f$. The limit of the right hand side does not exist if $\deg_{a_i} f > w$ and is zero if $\deg_{a_i} f < w$. Thus $\deg_{a_i} f = w$ and the same holds for each variable $a_i$. The injectivity of the star map follows from taking the limit:

$$f(a_1, \ldots, a_r) = \left( \prod_{i=1}^{r-1} (a_r - a_i)^w \right)^w \left( \frac{1}{a_r - a_1}, \ldots, \frac{1}{a_r - a_{r-1}} \right).$$

To show that the polynomial $f^*$ lies in $\mathbb{C}[a_i - a_j; 1 \leq i, j \leq r]$, it suffices to check its invariance under translations: $f^*(a_1 + \lambda, \ldots, a_{r-1} + \lambda, t) = \lim_{t \to \infty} t^{-w} f(a_1, \ldots, a_{r-1}, t - \lambda) = f^*(a_1, \ldots, a_{r-1}).$ ∎

**Proposition 4.** For any $r \in \mathbb{N}$, $\Delta_{\{1,2,\ldots,r\}} = \Delta_{\{1,2,\ldots,r-1\}}$. For $T \subseteq B_g$ with $|T| = g + 1$, $(\Delta_T \Delta_{T'})^* = \begin{cases} \Delta_{T \setminus \{2g+2\}} \Delta_T, & \text{if } 2g + 2 \in T, \\ \Delta_T \Delta_{T \setminus \{2g+2\}}, & \text{if } 2g + 2 \notin T. \end{cases}$

For $e \in \mathcal{E}_r$, we have $\ast \psi_e = -\psi_{e^*}$.

**Proof.** The proof is straightforward. ∎

For $r = r_1 + r_2$ and $j \in \mathbb{N}$, we follow Tsuyumine by defining a map

$$T_{r_1,r_2}^{(j)} : \mathbb{C}[a_1, \ldots, a_r] \to \mathbb{C}[a_1, \ldots, a_{r_1}] \otimes \mathbb{C}[\alpha_1, \ldots, \alpha_{r_2}]$$

and a valuation

$$\nu_{r_1,r_2} : \mathbb{C}[a_1, \ldots, a_r] \to \mathbb{Z}$$

$$f \mapsto \deg_{a_i} f(a_1, \ldots, a_{r_1}, \alpha_1 + t, \ldots, \alpha_{r_2} + t).$$
Definition 5. Define a valuation subring $S(2g + 2)_0$ by

$$S_w(2g + 2)_0 = \{ f \in S_w(2g + 2) : \\
\forall g_1, g_2 \in \mathbb{N} : g_1 + g_2 = g, \nu_{2g_1+1,2g_2+1}(f) \leq \frac{w}{g} (2g_1g_2 + g_1 + g_2) \}. $$

Lemma 6. Let $g_1, g_2 \in \mathbb{N}$ with $g = g_1 + g_2$. For $T \subseteq B_g$ define $\pi_1 T = \{ x : x \in T \text{ and } 1 \leq x \leq 2g_1 + 1 \} \subseteq B_g$ and $\pi_2 T = \{ x - (2g_1 + 1) : x \in T \text{ and } 2g_1 + 2 \leq x \leq 2g + 2 \} \subseteq B_g$. For $T \subseteq B_g$ with $|T| = g + 1$, we have $\Delta_T \Delta_{T'} \in S_g(2g + 2)_0$ and $T_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(\Delta_T \Delta_{T'}) = \{(\Delta_{\pi_1 T} \Delta_{(\pi_1 T')})^* \otimes (\Delta_{\pi_2 T} \Delta_{\pi_2 T'})^*\}$.

Proof. Let $m = |\pi_1 T|$ and $n = |\pi_1 T'|$. Then $m + n = 2g_1 + 1$ and $\nu_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(\Delta_T \Delta_{T'}) = m(g+1-m)+n(g+1-n) = (2g_1+1)(g+1)-(m^2+n^2)$. In this case, $m, n \in \mathbb{Z}_{\geq 0}$ and $m + n$ is odd, so the minimum of $m^2 + n^2$ occurs at $\{m,n\} = \{g_1 + 1, g_1\}$. Therefore, $\nu_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(\Delta_T \Delta_{T'}) \leq (2g_1+1)(g+1)-((g_1+1)^2+g_2^2) = 2g_1g_2 + g_1 + g_2$ and $\Delta_T \Delta_{T'} \in S_g(2g + 2)_0$.

To find the coefficient of $t^{2g_1g_2+g_1+g_2}$ in the cases of equality we may assume $|\pi_1 T'| = g_1 + 1$ and $|\pi_1 T| = g_1$; the other case follows by swapping $T$ and $T'$. We have

$$\Delta_T(a_1, \ldots, a_{2g_1+1}, a_1 + t, \ldots, a_{2g_2+1} + t) = \prod_{i,j \in \pi_1 T : i > j} (a_i - a_j) \prod_{i,j \in \pi_2 T : i > j} (a_i - a_j) \prod_{i \in \pi_2 T, j \in \pi_1 T} (a_i + t - a_j)$$

and similarly for $\Delta_{T'}$ so that

$$T_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(\Delta_T \Delta_{T'})(a_1, \ldots, a_{2g_1+1}, a_1, \ldots, a_{2g_2+1}) = \Delta_{\pi_1 T}(a_1, \ldots, a_{2g_1+1}) \Delta_{\pi_2 T}(a_1, \ldots, a_{2g_2+1})$$

$$\Delta_{\pi_1 T'}(a_1, \ldots, a_{2g_1+1}) \Delta_{\pi_2 T'}(a_1, \ldots, a_{2g_2+1}) = \Delta_{\pi_1 T}(a_1, \ldots, a_{2g_1+1}) \Delta_{(\pi_1 T') \setminus \{2g_1+2\}}(a_1, \ldots, a_{2g_1+1})$$

$$\Delta_{\pi_2 T'}(2g_2+2)(a_1, \ldots, a_{2g_2+1}) \Delta_{\pi_2 T'}(a_1, \ldots, a_{2g_2+1}).$$

Thus, $T_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(\Delta_T \Delta_{T'}) = (\Delta_{\pi_1 T} \Delta_{(\pi_1 T')})^* \otimes (\Delta_{\pi_2 T'} \Delta_{\pi_2 T'})^*$ upon comparison with Proposition 4.

Corollary 7. The map

$$T_{2g_1+1,2g_2+1} : S^{(g)}(2g + 2)_0 \to \bigoplus_{j=0}^{\infty} S_{2g+2j}((2g_1+2)^j \otimes S_{2g_2+2j}(2g_2+2)^j)$$
defined by
\[ T_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}(S_g(2g+2))_0 \to S_{g_1j}(2g_1+2)^* \otimes S_{g_2j}(2g_2+2)^* \]
is a homomorphism of graded rings.

**Proof.** We need to check that the codomain is as stated. The previous Lemma shows this for the ring generated by the \( \Delta_T \Delta_{T'} \); thus it holds for any subring of the integral closure where \( T_{2g_1+1,2g_2+1} \) is multiplicative. We know that \( T_{2g_1+1,2g_2+1} \) is multiplicative on \( S(g)(2g+2)_0 \) by the valuation condition defining \( S(g)(2g+2)_0 \).

Since the star map is injective, the Witt map \( W_{g_1,g_2} \) is well-defined by the following:

**Definition 8.** Let \( g_1, g_2 \in \mathbb{N} \) with \( g_1 + g_2 = g \). The graded ring homomorphism \( W_{g_1,g_2} : S(g)(2g+2)_0 \to \oplus_{j=0}^\infty S_{g_1j}(2g_1+2) \otimes S_{g_2j}(2g_2+2) \) is defined on \( S_g(2g+2)_0 \) by \((\ast \otimes \ast) \circ W_{g_1,g_2} = T_{2g_1+1,2g_2+1}^{(2g_1g_2+g_1+g_2)}\).

Intuitively, the \( T \) map pulls apart a hyperelliptic surface and the star map opens up a hyperelliptic surface at a branch point; so the Witt map pulls apart a hyperelliptic surface into two pieces and then closes up the individual pieces.

**Proposition 9.** Given \( g = g_1 + g_2 \), with \( g_1, g_2 \in \mathbb{N} \), we have
1. \( \nu_{2g_1+1,2g_2+1} \Delta_{B_g} = (2g_1+1)(2g_2+1) \) and \( T_{2g_1+1,2g_2+1}^{(4g_1g_2+2g_1+2g_2+1)} \Delta_{B_g} = \Delta_{(1,\ldots,2g_1+1)} \otimes \Delta_{(1,\ldots,2g_2+1)} \)
2. \( \nu_{2g_1+1,2g_2+1} \Delta_{U_g} = (g_1+1)g_2 \) and \( \nu_{2g_1+1,2g_2+1} \Delta_{U_2} = g_1(g_2+1) \)
3. If \( e \in E_{2g+2} \) such that \( \{e_1, \ldots, e_{2g_1+1}\} = \{1, \ldots, 2g_1+1\} \), define \( e_L = (e_1, \ldots, e_{2g_1+1}) \) and \( e_R = (e_{2g_1+2} - (2g_1+1), \ldots, e_{2g+2} - (2g_1+1)) \). Then \( T_{2g_1+1,2g_2+1}^{(2)} \psi_e = -\psi_{e_L} \otimes \psi_{e_R} \).

**Proof.** The proof is straightforward.

**Proposition 10.** Let \( g = g_1 + g_2 \), with \( g_1, g_2 \in \mathbb{N} \). Denote \( \delta = j(2g_1g_2 + g_1 + g_2) \) and let \( h \in S_{g_1j}(2g+2)_0 \). Suppose \( h = f_1 f_2 \) with both \( f_1, f_2 \) polynomials. Suppose \( \nu_{2g_1+1,2g_2+1} f_1 = \delta_1 \). Then \( T_{2g_1+1,2g_2+1}^{(\delta)} h = T_{2g_1+1,2g_2+1}^{(\delta_1)} f_1 \cdot T_{2g_1+1,2g_2+1}^{(\delta - \delta_1)} f_2 \)

**Proof.** The fact that \( h \) satisfies the valuation property implies that \( \nu_{2g_1+1,2g_2+1} f_2 \leq \delta - \delta_1 \). The result then follows easily.

**Definition 11.** For subgroups \( G \) of the symmetric group \( S_{2g+2} \), define
\[ S_w(2g+2)_0(G) = \{ f \in S_w(2g+2)_0 : \forall \sigma \in G, \sigma(f) = f \} \]
For even \( k \), define \( B^k_g = S_{k g}(2g+2)_0(S^2) \).
This space of binary invariants, $B_g^k$, is analogous to $[\Gamma_g(1,2),k]$, the Siegel modular forms of degree $g$ and weight $k$ for the theta group. It remains to define the concept of a cusp form on $B_g$. For $1 \leq m,n \leq r$, define $\Phi_{mn}$ on a polynomial $f \in \mathbb{C}[a_1, \ldots, a_r]$ by $\Phi_{mn}f = f$ with $a_m = 0 = a_n$. For $m \neq n$, define $\Phi_{mn} : S_{gj}(2g+2) \to \mathbb{C}[a_i; 1 \leq i \leq 2g+2, i \neq m,n]$ by $\Phi_{mn} = \Phi_{mn}f/\prod_{\ell \neq m,n}a_\ell^{-1}$ and reindexing the variables if necessary.

**Lemma 12.** Let $m, n, j \in \mathbb{N}$. We have $\Phi_{mn} : S_{gj}(2g+2) \to S_{(g-1)j}(2g)$.

**Proof.** Consider $\Delta_T \Delta_{T'} \in S_g(2g+2)$. If $\{m, n\} \subseteq T$ or $\{m, n\} \subseteq T'$ then $\Phi_{mn}(\Delta_T \Delta_{T'}) = 0$. Otherwise we may relabel so that $m \in T$ and $n \in T'$ and then $\Phi_{mn}(\Delta_T \Delta_{T'}) = \pm \Delta_{T\setminus\{m\}} \Delta_{T'\setminus\{n\}}$, which is indeed in $S_{g-1}(2(g-1)+2)$ after potential reindexing. Because the $\Phi_{mn}$ map is multiplicative, the codomain is shown to be as stated by taking integral closure. \( \square \)

**Definition 13.** An element $f \in S^{(g)}(2g+2)$ is called a cusp form if $\Phi_{mn}(f) = 0$ for all distinct $m, n \in B_g$.

**Theorem 14.** (Main Theorem) For each $g \in \mathbb{N}$, define

\[
H_g = \frac{1}{2g} \sum_{\ell \in \mathbb{E}_{2g+2}} \frac{-1}{\psi_\ell}.
\]

Then the following conditions hold:

- $H_1 = \Delta_{B_1} \Delta_{\psi_1'}$.
- $H_g \in B_g^8$.
- For all $g_1, g_2 \in \mathbb{N}$ with $g_1 + g_2 = g$, we have that
  \[ W_{g_1, g_2} H_g = H_{g_1} \otimes H_{g_2} \]
- $\sum_{\sigma \in S_{2g+2}} \sigma(H_g) = 0$.
- $\Phi_{ij} H_g = 0$ for all $i \neq j$.

**Proof.** It is straightforward to check that $H_1 = \Delta_{B_1} \Delta_{\psi_1}$. The last two conditions are also easily checked: Consider the polynomial $G = \frac{1}{2g} \sum_{\ell \in \mathbb{E}_g} \frac{-1}{\psi_\ell}$, so that $H_g = \Delta_{B_g} G$. For any two $i \neq j$, $\Phi_{ij} H_g = \Phi_{ij} \Delta_{B_g} \Phi_{ij}(G) = 0 \cdot \Phi_{ij}(G) = 0$. Next, we prove that $\sum_{\sigma \in S_{2g+2}} \sigma(H_g)$ is trivial. We have

\[
\sum_{\sigma \in S_{2g+2}} \sigma(H_g) = \frac{1}{2g} \frac{1}{\Delta_{B_g}} \sum_{\sigma \in S_{2g+2}} \sum_{\ell \in \mathbb{E}_g} \frac{-1}{\sigma(\psi_\ell)}
\]

because $\Delta_{B_g}^2$ is invariant under all $\sigma \in S_{2g+2}$. Define a polynomial by $G = \frac{1}{2g} \Delta_{B_g} \sum_{\sigma \in S_{2g+2}} \sum_{\ell \in \mathbb{E}_g} \frac{-1}{\sigma(\psi_\ell)}$. Since $\Delta_{B_g}$ is alternating and
since $\Delta_{B_g} \tilde{G}$ is invariant under $S_{2g+2}$, then $\tilde{G}$ must be alternating. This implies that $\tilde{G}$ is a multiple of $\Delta_{B_g}$. But $\deg \tilde{G} < \deg \Delta_{B_g}$ forces $\tilde{G} = 0$.

Next, we show that $H_g \in B_g^8$. First, it is clear from the construction that $H_g \in S_w(2g + 2)$ where $w = 4g$ because it is a sum whose terms are products of the form $(a_i - a_j)$ where $i, j$ are of opposite parity such that in the product each $a_i$ appears exactly $w$ times. Second, $H_g$ is invariant under $SS$ because

$$\sum_{e \in E_{2g+2}} \frac{1}{\psi_e} = \sum_{\sigma \in SS} \frac{1}{\sigma(\psi_{e_0})}$$

for any particular $e_0 \in SS$, and because applying any $\tau \in SS$ we have $\tau(\psi_{e_0}) = \psi_{e_1}$ for some $e_1 \in E_{2g+2}$. Third, the valuation property will be evident when we find the image of the Witt map.

From the definition of the Witt map, we need to prove that

$$\text{Coeff}(t^{8g_1g_2+4g_1+4g_2}, H_g(a_1, \ldots, a_{2g_1+1}, \alpha_1 + t, \ldots, \alpha_{2g_2+1} + t)),$$

is equal to $(\ast H_{g_1})(a_1, \ldots, a_{2g_1+1}) \cdot (\ast H_{g_2})(a_1, \ldots, \alpha_{2g_2+1})$. Note from Proposition $9$ that the maximal power of $t$ in the expansion of the factor $\Delta_{B_g}^2(a_1, \ldots, a_{2g_1+1}, \alpha_1 + t, \ldots, \alpha_{2g_2+1} + t)$ is $t^{2(2g_1+1)(2g_2+1)}$ and that its coefficient is $\Delta_{\{1, \ldots, 2g_1+1\}}(a_1, \ldots, a_{2g_1+1}) \Delta_{\{1, \ldots, 2g_2+1\}}(a_1, \ldots, \alpha_{2g_2+1})$. We claim that $\nu_{2g_1+1,2g_2+1}(\psi_e) \geq 2$. For simplicity, let $C = \{1, \ldots, a_{2g_1+1}\}$ and $D = B_g \setminus C$. Then $\nu_{2g_1+1,2g_2+1}(\psi_e) = \deg_t(\psi_e(a_1, \ldots, a_{2g_1+1}, \alpha_1 + t, \ldots, \alpha_{2g_2+1} + t))$ is the number of transitions between the sets $C$ and $D$ in the sequence $e_1, \ldots, e_r, e_1$. This number is clearly at least 2 and is exactly 2 if and only if all the numbers in $C$ are together and all the numbers in $D$ are together (where we have to view the sequence with wrap-around); call the set of such $e$ the set $\mathcal{F}$. In particular, we just proved that $H_g$ satisfies the valuation condition $\nu_{2g_1+1,2g_2+1}(H_g) \leq 2(2g_1+1)(2g_2+1) - 2 = 4(2g_1g_2 + g_1 + g_2)$ so that $H_g \in S_{4g(2g+2)}(SS) = B_g^8$.

Since $t^{8g_1g_2+4g_1+4g_2} = t^{2(2g_1+1)(2g_2+1)} / t^2$, if $\nu_{2g_1+1,2g_2+1}(\psi_e) > 2$ for an $e$, then $\text{Coeff}(t^{8g_1g_2+4g_1+4g_2}, \Delta_{B_g}^2 / \psi_e(a_1, \ldots, a_{2g_1+1}, \alpha_1 + t, \ldots, \alpha_{2g_2+1} + t)) = 0$. Thus we have that

$$T_{(8g_1g_2+4g_1+4g_2)} H_g = T_{(2g_1+1,2g_2+1)}(\Delta_{B_g}^2) \cdot \frac{2^{-g}}{g+1} \sum_{e \in \mathcal{F}} \frac{-1}{T_{2g_1+1,2g_2+1}^{(2)} \psi_e},$$

Since $\psi_e$ is unchanged when $e$ is cyclically rotated, we may rotate $e$ so that the set $C$ comes first and then the set $D$. To this end, let

$$\mathcal{F} = \{ e \in \mathcal{E}_{2g+2} : \{e_1, \ldots, e_{2g_1+1}\} = \{1, \ldots, a_{2g_1+1}\} \}.$$
factor of $g + 1$:

$$T_{2g_1+1,2g_2+1}(8g_1g_2+4g_1+4g_2)H_g = T_{2g_1+1,2g_2+1}(8g_1g_2+4g_1+4g_2+2) \frac{1}{2g} \left( \frac{\Delta B_g^2}{2g} \right) \sum_{e \in \tilde{F}} \frac{-1}{2g_{1+2g_2+1} \psi_e}.$$ 

Then by Proposition 9

$$T_{2g_1+1,2g_2+1}H_g = \Delta^2_{\{1,\ldots,2g_1+1\}}(a_1,\ldots,a_{2g_1+1}) \Delta^2_{\{1,\ldots,2g_2+1\}}(\alpha_1,\ldots,\alpha_{2g_2+1})$$

$$\frac{1}{2g_{1+2g_2}} \sum_{e_L \in \tilde{E}_{2g_1+1}} \sum_{e_R \in \tilde{E}_{2g_2+1}} -1 \cdot -1 \psi'_{e_L}(a_1,\ldots,a_{2g_1+1}) \psi'_{e_R}(\alpha_1,\ldots,\alpha_{2g_2+1}).$$

because we can view each $e \in \tilde{F}$ as the concatenation of two pieces $e_L$ and $e_R$; that is given an $e \in \tilde{F}$, we have corresponding $e_L = (e_1,\ldots,e_{2g_1+1})$ and $e_R = (e_{2g_1+2} - (2g_1 + 1),\ldots,e_{2g_2+2} - (2g_1 + 1))$. 

On the other hand,

$$*H_{g_1} = \frac{1}{2g_1} \left( *\Delta B_{g_1} \right)^2 \frac{1}{2g_{1+2g_2}} \sum_{e \in \tilde{E}_{2g_1+2}} -1 \psi_e^*$$

$$= \frac{1}{2g_1} \left( \Delta_{\{1,\ldots,2g_1+1\}} \right)^2 \sum_{e \in \tilde{E}_{2g_1+2}} -1 \psi_e^*$$

$$= \frac{1}{2g_1} \left( \Delta_{\{1,\ldots,2g_1+1\}} \right)^2 \sum_{e \in \tilde{E}_{2g_1+2}} -1 \psi_e^*$$

and similarly for $*H_{g_2}$. 

Now it is easy to see that $T_{2g_1+1,2g_2+1}H_g = *H_{g_1} \otimes *H_{g_2}$. 

We thank R. Salvati Manni for bringing the following consequence to our attention: As A. Morozov points out in [15], the fact that $\Delta B_g$ divides $H_g$ implies that, for variables $x$ and $y$ and $P_T(x) = \prod_{i \in T}(x-a_i)$, 

$$\sum_{\sigma \in S_{2g+2}} \sigma ((P_U(x)P_{U^*}(y) - P_U(y)P_{U^*}(x))H_g) = 0.$$ 

The reason for this is that the complete symmetrization must be divisible by $(x - y)^2 \Delta^2_{B_g}$. Along with $\sum \sigma(H_g) = 0$, this identity is equivalent to the non-renormalization of the 2 and 3-point functions.

3. Uniqueness

We now prove some propositions aimed at proving the uniqueness of the family $H_g$. 

Proposition 15. For any \( r \in \mathbb{N} \), let \( f \in S_w(r) \). Then
\[
(T_{r-3,3}^{(3w)})(a_1, \ldots, a_{r-3}, \alpha_1, \alpha_2, \alpha_3) = \text{Coeff}(t^{3w}, f(a_1, \ldots, a_{r-3}, t, t, t)).
\]
Furthermore, \( f(a_1, \ldots, a_{r-3}, u, u, u) = \prod_{i=1}^{r-3} (u - a_i)^w \cdot (T_{r-3,3}^{(3w)})(\frac{1}{u - a_1}, \ldots, \frac{1}{u - a_{r-3}}, \alpha_1, \alpha_2, \alpha_3). \)

Proof. Since each variable in \( f \) occurs to degree \( w \), it is clear that
\[
(2) \quad \text{Coeff}(t^{3w}, f(a_1, \ldots, a_{r-3}, t + \alpha_1, t + \alpha_2, t + \alpha_3) = \text{Coeff}(t^{3w}, f(a_1, \ldots, a_{r-3}, t, t, t))
\]
and that this expression is really independent of \( \alpha_1, \alpha_2, \alpha_3 \). Therefore
\[
(T_{r-3,3}^{(3w)})(a_1, \ldots, a_{r-3}, \alpha_1, \alpha_2, \alpha_3) = \text{Coeff}(t^{3w}, f(a_1, \ldots, a_{r-3}, t, t, t)).
\]
Fix \( 0 < \epsilon < 1 \). Let \( u \) be a variable. Let \( \gamma(z) = \frac{1}{u-\epsilon^2} \). Then \( f \in S_w(r) \) implies that
\[
f(a_1, \ldots, a_{r-3}, u, u, u) = \prod_{i=1}^{r-3} (u - \epsilon a_i)^w \cdot (u - \epsilon u)^{3w} f\left(\frac{1}{u - \epsilon a_1}, \ldots, \frac{1}{u - \epsilon a_{r-3}}, \frac{1}{u - (1-\epsilon)u}, \frac{1}{u - (1-\epsilon)u}\right).
\]
Let us expand \( f(a_1, \ldots, a_{r-3}, t, t, t) \) in powers of \( t \) as
\[
(3) \quad f(a_1, \ldots, a_{r-3}, t, t, t) = (T_{r-3,3}^{(3w)})(a_1, \ldots, a_{r-3}, \alpha_1, \alpha_2, \alpha_3) t^{3w} + G(a_1, \ldots, a_{r-3}, t),
\]
where \( \deg_f G < 3w \). Then
\[
f(a_1, \ldots, a_{r-3}, u, u, u)
\]
\[
= \prod_{i=1}^{r-3} (u - \epsilon a_i)^w \cdot ((T_{r-3,3}^{(3w)})(\frac{1}{u - \epsilon a_1}, \ldots, \frac{1}{u - \epsilon a_{r-3}}, \alpha_1, \alpha_2, \alpha_3)(1/\epsilon)^{3w}
\]
\[
+ (1 - \epsilon)^{3w} G(\frac{1/\epsilon}{u - \epsilon a_1}, \ldots, \frac{1/\epsilon}{u - \epsilon a_{r-3}}, (1/\epsilon) u / (1-\epsilon)u))
\]
\[
= \prod_{i=1}^{r-3} (u - \epsilon a_i)^w \cdot ((T_{r-3,3}^{(3w)})(\frac{1/\epsilon}{u - \epsilon a_1}, \ldots, \frac{1/\epsilon}{u - \epsilon a_{r-3}}, \alpha_1, \alpha_2, \alpha_3)(1/\epsilon)^{3w}
\]
\[
+ (1 - \epsilon)^{3w}(\text{terms where } (1 - \epsilon)^{\beta} \text{ occurs with } \beta > -3w))
\]
Taking the limit as \( \epsilon \to 1 \) gives the desired result. \( \square \)

Proposition 16. Let \( f \in S_w(2g + 2)(SS) \). If \( f(a_1, \ldots, a_{2g+2}) = 0 \) whenever \( a_i = a_j = a_k \) with distinct \( i, j, k \) not all of the same parity, then either \( f = 0 \) or \( \deg f \geq g(g + 1) \).
Proof. Assume \( f \neq 0 \). For each integer \( 0 \leq j \leq g+1 \), define a polynomial \( h_j \) by

\[
h_j(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}) = f(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}, b_{g+1-j}).
\]

Note \( \deg f \geq \deg h_j \) for each \( j \). Note that \( h_{g+1} = f \), so \( h_{g+1} \neq 0 \). Then let \( m \) be the minimum such that \( h_m \neq 0 \). Since \( f \) is invariant under \( SS \), then \( h_m \) is invariant under swapping within the \( x_i \) or within the \( y_i \), and \( h_m \) is invariant under swapping within the \( b_i \). Note \( h_m = 0 \) whenever \( b_i = b_j \) with \( i \neq j \), Thus

\[
h_m = \prod_{0 \leq i < j \leq g+1-m} (b_i - b_j) \cdot k(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}),
\]

for some polynomial \( k \). Then \( k \) would be alternating under swapping within the \( b_i \) which implies that \( k \) is a multiple of each \( (b_i - b_j) \). Thus

\[
h_m = \prod_{0 \leq i < j \leq g+1-m} (b_i - b_j)^2 \cdot k_2(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}),
\]

for some polynomial \( k_2 \). Now, also \( h_m = 0 \) whenever any \( b_i = x_j \) or \( b_i = y_j \). Thus

\[
h_m = \prod_{0 \leq i < j \leq g+1-m} (b_i - b_j)^2 \cdot \prod_{i,j} (b_i - x_j)(b_i - y_j) \cdot k_3(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}),
\]

for some polynomial \( k_3 \). Then the homogeneous degree is

\[
\deg h_m \geq (g+1-m)(g-m) + 2(g+1-m)m = (g+1-m)(g+m).
\]

If \( m = 0 \), then this says \( \deg h_0 \geq (g+1)g \) and \( \deg f \geq (g+1)g \) follows. If \( m > 0 \), then \( h_{m-1} = 0 \). This says that \( h_m = 0 \) if \( x_m = y_m \). Thus \( h_m \) is a multiple of \( (x_m - y_m) \), and so \( h_m \) is a multiple \( (x_i - y_j) \) for all \( i, j \). Thus

\[
h_m = \prod_{0 \leq i < j \leq g+1-m} (b_i - b_j)^2 \cdot \prod_{i,j} (b_i - x_j)(b_i - y_j) \cdot \prod_{i,j} (x_i - y_j)(b_i - y_j) \cdot k_4(x_1, y_1, \ldots, x_j, y_j, b_1, \ldots, b_{g+1-j}),
\]

for some polynomial \( k_4 \). Then

\[
\deg h_m \geq (g+1-m)(g+m) + m^2 = (g+1)g + m.
\]

Then \( \deg f \geq (g+1)g + m \) and the proposition is proved. \( \square \)
Corollary 17. For any $g \in \mathbb{N}$ with $g \geq 2$, let $f \in S_{g-1}(2g+2)(SS)$. If $T_{2g-1,3}^{(3g-3)} f = 0$, then $f = 0$.

Proof. Suppose we have an $f \in S_{g-1}(2g+2)(SS)$ with $T_{2g-1,3}^{(3g-3)} f = 0$. Proposition 15 with $w = g-1$ implies that $f(a_1, \ldots, a_{2g-1}, u, u, u) = 0$. By symmetry under $SS$, this implies $f(a_1, \ldots, a_{2g+2}) = 0$ whenever three of the $a_i$ are equal with not all three indices of the same parity. By Proposition 16, we have either $f = 0$ or $\deg f \geq (g+1)g$. But if $f \neq 0$, then $f \in S_{g-1}(2g+2)$ implies that $\deg f = \frac{1}{2}(g-1)(2g+2) = g^2 - 1 < (g+1)g$, a contradiction. Hence $f = 0$. □

Proposition 18. Any cusp form $h \in B_g^k$ must be of the form

$$h = \Delta_{B_g} \Delta_{U_g} \Delta_{U_g'} f,$$

where $f \in S_{\frac{1}{2}kg - 3g-1}(2g+2)$.

Proof. Since $\Phi_{ij} h = 0$ for any $i \neq j$, then $h = 0$ whenever $a_i = a_j$. This forces $(a_i - a_j)$ to be a divisor of $h$. Thus $h = \Delta_{B_g} h_2$ for some polynomial $h_2$. Since $h$ is invariant under $SS$ and $\Delta_{B_g}$ is alternating under $SS$, then $h_2$ must be alternating under $SS$, which means that $h_2$ changes sign whenever $a_i$ and $a_j$ are swapped with $i, j$ of the same parity. This implies $h_2 = 0$ whenever $a_i = a_j$ with $i, j$ of the same parity. So $h_2$ must be a multiple of $\Delta_{U_g}$. Thus $h = \Delta_{B_g} \Delta_{U_g} \Delta_{U_g'} f$ with $f$ a polynomial. Since $h$ has weight $\frac{1}{2}kg$ and $\Delta_{B_g} \Delta_{U_g} \Delta_{U_g'}$ has weight $3g+1$, then $f$ has the asserted weight. □

Lemma 19. Let $f$ be a cusp form in $S^{(g)}(2g+2)$. For $i, j < 2g+2$, we have $\Phi_{ij}(*f) = 0$.

Proof. Let $f \in S_{g\ell}(2g+2)$. We have

$$\Phi_{ij}(*f) = \Phi_{ij} \text{Coeff}(t^{\ell \ell}, f(a_1, \ldots, a_{2g+1}, t))$$

$$= \text{Coeff}(t^{\ell \ell}, f(a_1, \ldots, a_{2g+1}, t)) \text{ with } a_i = a_j = 0$$

$$= \text{Coeff}(t^{\ell \ell}, f(a_1, \ldots, a_{2g+1}, t)) \text{ with } a_i = a_j = 0$$

$$= 0.$$ □

Proposition 20. Let $f \in B_g^k$ be a binary invariant with respect to the theta group. Suppose $W_{g-1,1} f = h_2 \otimes h_1$. Then $*\Phi_{2g+1,2g+2} f = (-1)^{k/2}(*h_2)\Phi_{2,3}(*h_1)a_1^{-k/2}$ and $*\Phi_{2g,2g+2} f = (*h_2)\Phi_{1,3}(*h_1)(-a_2)^{-k/2}$. In particular, if $W_{g-1,1} f = h_2 \otimes h_1$ where $h_1$ is a cusp form, then $f$ is a cusp form.
Proof. Just write out
\[(*h_2)(a_1, \ldots, a_{2g-1})(*h_1)(\alpha_1, \alpha_2, \alpha_3) = T_{2g-1,3}f\]
\[= \text{Coeff}(t^{\frac{k}{2}(3g-2)}, f(a_1, \ldots, a_{2g-1}, \alpha_1 + t, \alpha_2 + t, \alpha_3 + t))\]
Then
\[(*h_2) \bar{\Phi}_{2g}(*h_1) = \text{Coeff}(t^{\frac{k}{2}(3g-2)}, f(a_1, \ldots, a_{2g-1}, \alpha_1 + t, t, t))\]
\[= \text{Coeff}(t^{\frac{k}{2}(3g-2)}, f(a_1 - t, \ldots, a_{2g-1} - t, \alpha_1, 0, 0))\]
On the other hand,
\[*\Phi_{2g+1,2g+2}f = \left(\frac{f(a_1, \ldots, a_{2g}, 0, 0)}{(a_1 \cdots a_{2g})^{k/2}}\right)\]
\[= \text{Coeff}(t^{k(g-1)/2}, f(a_1, \ldots, a_{2g-1}, t + a_{2g}, 0, 0))\]
\[= \text{Coeff}(t^{k(g-1)/2}, f(a_1 - t, \ldots, a_{2g-1} - t, a_{2g}, 0, 0))\]
where we used the fact that \(*\Phi_{2g+1,2g+2}f\) is invariant under translations in the last equality using Lemmas 3 and 12. Since the highest term in \(t\) in the denominator is \((-1)^{(2g-1)k/2}a_{2g}^{k/2}t^{(2g-1)k/2}\), then
\[*\Phi_{2g+1,2g+2}f = (-1)^{(2g-1)k/2}(a_{2g})^{-k/2}.\]
\[\text{Coeff}(t^{k(g-1)/2}, f(a_1 - t, \ldots, a_{2g-1} - t, a_{2g}, 0, 0))\]
\[= (-1)^{k/2}(a_{2g})^{-k/2}.\]
\[\text{Coeff}(t^{k(3g-2)/2}, f(a_1 - t, \ldots, a_{2g-1} - t, a_{2g}, 0, 0))\]
This proves \(*\Phi_{2g+1,2g+2}f = (-1)^{k/2}(*h_2)\bar{\Phi}_{2,3}(*h_1)\alpha_1^{-k/2}\) and similarly \(*\Phi_{2g,2g+2}f = (-1)^{k/2}(*h_2)\bar{\Phi}_{1,3}(*h_1)\alpha_2^{-k/2}\).

Now suppose that \(W_{g-1,1}f = h_2 \otimes h_1\) where \(h_1\) is a cusp form. Then by Lemma 19 \(\bar{\Phi}_{2,3}(*h_1) = 0\) and so \(*\Phi_{2g+1,2g+2}f = 0\) and thus \(\Phi_{2g,2g+2}f = 0\). Similarly \(\Phi_{2g,2g+2}f = 0\). The invariance of \(f\) under \(SS\) implies \(\Phi_{i,j}f = 0\) for all \(i \neq j\) and so \(f\) is a cusp form. \(\square\)

Proposition 21. The Witt map \(W_{g-1,1}\) is injective on \(B^8_g\).

Proof. Let \(h \in B^8_g\) and suppose \(W_{g-1,1}h = 0\). Then \(T_{2g-1,3}^{(12g-8)}h = *(W_{g-1,1}h) = 0\). By Proposition 20 we deduce that \(h\) is a cusp form. By Proposition 18 we know that any cusp form \(h \in B^8_g\) must be of the form \(h = \Delta_B \Delta_{U_g} \Delta_{V_g} f\) where \(f \in S_{g-1}(2g + 2)\). From Proposition 9 we know \(\nu_{g-1,1}(\Delta_B \Delta_{U_g} \Delta_{V_g}) = (2g - 1)3 + (g)1 + (g - 1)2 = 9g - 5\).
Then Proposition [10] says that

\[ T_{2g-1,3}^{(12g-8)}h = T_{2g-1,3}^{(9g-5)}(\Delta_{B_g} \Delta_{U_g} \Delta_{U'_g}) \cdot T_{2g-1,3}^{(3g-3)}f. \]

Since \( T_{2g-1,3}^{(12g-8)}h = 0 \) and \( T_{2g-1,3}^{(9g-5)}(\Delta_{B_g} \Delta_{U_g} \Delta_{U'_g}) \neq 0 \), then \( T_{2g-1,3}^{(3g-3)}f = 0 \).

Since \( f \in S_{g-1}(2g + 2) \), then Corollary [17] implies \( f = 0 \). So \( h = 0 \), completing the proof. \( \square \)

**Theorem 22.** The family \( H_g \) as given in Theorem [14] is the unique family that satisfies the first three conditions stated in that Theorem.

**Proof.** Suppose by way of contradiction there is another family \( K_g \) that satisfies the first three conditions of Theorem [14]. Let \( g_0 \) be the smallest index such that \( K_{g_0} \neq H_{g_0} \). Then \( g_0 \geq 2 \) by the first condition. Use the third condition to check

\[
W_{g_0-1,1}(K_{g_0} - H_{g_0}) = W_{g_0-1,1}K_{g_0} - W_{g_0-1,1}H_{g_0} = K_{g_0-1} \otimes K_1 - H_{g_0-1} \otimes H_1 = 0.
\]

Since \( W_{g_0-1,1} \) is injective on \( \mathcal{B}_{g_0}^8 \) by Proposition [21], and \( K_{g_0} - H_{g_0} \in \mathcal{B}_{g_0}^8 \) by the second condition, then \( K_{g_0} - H_{g_0} = 0 \), which is a contradiction. \( \square \)

### 4. Remarks on Grushevsky’s Construction

In [10], Grushevsky gave a uniform construction of Siegel modular cusp forms that satisfied the Ansatz in genera \( g = 1, 2, 3, 4 \):

\[
\Xi^{(g)}[0] = \frac{1}{2^g} \left( \sum_{i=0}^{g} (-1)^i 2^\frac{i}{2}(i-1)G_{1,2^{i-1}}^{(g)} \right)
\]

where

\[
G_{i,r}^{(g)} = \sum_{V \subseteq \mathbb{F}_2^{2g}} \left( \prod_{\zeta \in V} \theta[\zeta] \right)^r
\]

and where the sum is over isotropic subspaces \( V \) of dimension \( i \).

Since \( \Xi^{(g)}[0] \) is multivalued for \( g > 4 \), it is natural to ask whether some branch is single valued on the Jacobian locus. In [11], S. Grushevsky and R. Salvati Manni showed that, if single valued, \( \text{tr}(\Xi^{(g)}[0]) \) is a multiple of \( J^{(g)} \), the difference of the theta series of the two classes of even unimodular rank 16 lattices. For \( 1 \leq g \leq 3 \), \( J^{(g)} \) is trivial whereas \( J^{(4)} \) is the Schottky form defining the Jacobian locus, see [15]. The long open problem of whether \( J^{(g)} \) vanishes on the Jacobian locus for \( g > 4 \) was resolved negatively in [11]; thus \( \Xi^{(g)}[0] \) stops solving
the Ansatz for $g > 4$. However, it is known [22] that $J^{(g)}$ always vanishes on the hyperelliptic locus and we will show that $\rho_g(\Xi^{(g)}[0])$ does have a branch that solves the hyperelliptic Ansatz. Thus, the intricate pattern discovered by Grushevsky in the construction of $\Xi^{(g)}[0]$ properly belongs to the hyperelliptic locus even though the same pattern happens to define a Siegel modular form for $g \leq 4$. We need some definitions and lemmas.

We refer to [9] and [19] for standard theory on Siegel modular forms. The action of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_g(\mathbb{R})$ on $\Omega \in \mathcal{H}_g$ is $M \cdot \Omega = (A \Omega + B)(C \Omega + D)^{-1}$. Let $\Gamma \subseteq \Gamma_g = \text{Sp}_g(\mathbb{Z})$ be a subgroup of finite index. The vector space of Siegel modular forms of degree $g$, weight $k$ and character $\chi$, written $[\Gamma, \chi, k]$ is the set of holomorphic functions $f : \mathcal{H}_g \rightarrow \mathbb{C}$, bounded at the cusps for $g = 1$, that satisfy $f|_k M = \chi(M)f$ for all $M \in \Gamma$ where $(f|_k M)(\Omega) = \det(C \Omega + D)^{-k}f(M \cdot \Omega)$. We define the graded rings: $M(k_0)(\Gamma, \chi) = \sum_{j=0}^\infty [\Gamma, \chi, jk_0]$.

The set $\{S \subseteq B_g : |S| \text{ even} \}$ is a group under the symmetric difference $\oplus$. The quotient group $\{S \subseteq B_g : |S| \text{ even} \}/\{\emptyset, B_g\}$ treats each $S$ as equivalent to its complement $S'$. In fact, we have an explicit isomorphism

$$\eta : (\{S \subseteq B_g : |S| \text{ even} \}/\{\emptyset, B_g\}, \oplus) \rightarrow (\mathbb{F}_2^{2g}, +)$$

given by $\eta_S = \sum_{i \in S} \eta_i$ and

$$\eta_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

$$\vdots$$

$$\eta_{2i-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \eta_{2i} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 0 \end{pmatrix},$$

$$\vdots$$

$$\eta_{2g-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad \eta_{2g} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

$$\eta_{2g+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \quad \eta_{2g+2} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We treat the elements $\zeta \in \mathbb{F}_2^{2g}$ as theta characteristics; i.e., we consider the action $\zeta \mapsto M \cdot \zeta$ of $\text{Sp}_g(\mathbb{F}_2)$ on $\mathbb{F}_2^{2g}$ given by

$$\theta[\zeta]|_{1/2} M \in (\text{eighth roots of unity}) \theta[M \cdot \zeta],$$

(7)
Explicitly, we have
\[
(A \ B \ C \ D) \cdot \zeta = \left(\begin{array}{cccc}
A & B \\
C & D
\end{array}\right) \zeta + \left(\begin{array}{c}
(A'C)_0 \\
(B'D)_0
\end{array}\right)
\]
where \((X)_0\) denotes the vector formed from the diagonal of \(X\) and \(\zeta\) is treated as a column vector. In general this action is affine and is linear precisely when we have \(M \in \Gamma_g(1, 2)(\mathbb{F}_2)\); this can be taken as the definition of \(\Gamma_g(1, 2)\). Frobenius found a complete set of invariants for this action, \([14]\), page 212. For \(\zeta = \left[\begin{array}{c}a \\ b\end{array}\right]\), \(\zeta_1, \zeta_2, \zeta_3 \in \mathbb{F}_2^g\), we put
\[
e_*(\zeta) = (-1)^{a \cdot b},
\]
\[
e(\zeta_1, \zeta_2, \zeta_3) = e_*(\zeta_1)e_*(\zeta_2)e_*(\zeta_3)e_*(\zeta_1 + \zeta_2 + \zeta_3).
\]

Theorem 23. Let \(\text{Sp}_g(\mathbb{F}_2)\) act on theta characteristics in \(\mathbb{F}_2^{2g}\) as in equation \([7]\). Two sequences, \((\zeta_1, \ldots, \zeta_m)\) and \((\xi_1, \ldots, \xi_m)\), are in the same \(\text{Sp}_g(\mathbb{F}_2)\)-orbit if and only if sending \(\zeta_i \mapsto \xi_i\) preserves
- all linear relations with an even number of summands,
- all \(e_*\) values and
- all \(e\) values.

Given any permutation \(\sigma\) of \(B_g\), we can induce a linear map \(\bar{\sigma} : \mathbb{F}_2^{2g} \to \mathbb{F}_2^{2g}\) by \(\eta_S \mapsto \eta_{\sigma(S)}\) for \(S \subseteq B_g\) with \(|S|\) even. This action is induced by an element \(M \in \text{Sp}_g(\mathbb{F}_2)\) if and only if \(\sigma\) preserves \(e_*\), in view of the linearity of \(\bar{\sigma}\). One can check, or see \([21]\), page 824, that \(e_*(\eta_S) = (-1)^{\frac{1}{2}g + 1 - |S \oplus U|}\), so that for \(\sigma \in S_S\) there exists an \(M \in \text{Sp}_g(\mathbb{F}_2)\) such that \(\eta_{\sigma(S)} = M \cdot \eta_S\). This \(M\) is uniquely determined because the \(\eta_{\{i,j\}}\) span \(\mathbb{F}_2^{2g}\) and we have \(M \in \Gamma_g(1, 2)(\mathbb{F}_2)\) because \(\bar{\sigma}\) is linear. We will have use for a certain character on \(\Gamma_g(1, 2)\). Define \(\kappa\) by \(\theta[0]|M = \kappa \theta[0]\). Then \(\kappa^4\) gives a real character of \(\Gamma_g(1, 2)(\mathbb{F}_2)\), or of \(\Gamma_g(1, 2)\). From Igusa \([14]\), page 182, we know that \(\kappa^4\) is given by \((A \ B \ C \ D) \mapsto (-1)^{\text{tr}(D - I_n)}\).

We now connect the traditional marking of a hyperelliptic curve with Igusa’s \(\rho\)-homomorphism. Let \(W \subseteq \mathbb{C}^{2g+2}\) be the quasiprojective variety of points with distinct coordinates. There is a morphism \(h : W \to \Gamma_g(2) \setminus \mathcal{H}_g\) that sends \(a = (a_1, \ldots, a_{2g+2}) \in W\) to the \(\Gamma_g(2)\)-class of the period matrix \(\Omega(a)\) for the traditional marking \([19]\) of a hyperelliptic curve \(y^2 = \prod_{i=1}^{2g+2} (x - a_i)\). The \(\rho_g\) map follows Thomae’s formula, given below, and for all \(f, g \in [\Gamma_g(2), k]\) with \(\rho_g(g) \neq 0\) we
have the important property, [21], page 777.

\[(8) \quad \frac{\rho_g(f)}{\rho_g(g)} = \frac{f \circ h}{g \circ h}.\]

**Lemma 24.** Let \(\sigma \in SS\) and \(M \in \Gamma_g(1, 2)\) with \(\eta_{\sigma(S)} = M \cdot \eta_S\) for all \(S \subseteq B_g\) with \(|S|\) even. For all \(a \in W\), we have \(h(a^\sigma) = M(h(a))\).

**Proof.** Let \(C\) be the Riemann surface given by the hyperelliptic curve \(y^2 = \prod_{i=1}^{2g+2}(x - a_i)\). Let \(w : C \to \text{Jac}(C) = \mathbb{C}^g/\Lambda(\Omega(a))\) be the Abel-Jacobi map, where \(\Lambda(\Omega) = \mathbb{Z}^g + \Omega\mathbb{Z}^g\). In the traditional marking of a hyperelliptic curve \(C\) we have \(w((a_i, 0)) = \frac{1}{2}(\Omega(a), I) \eta_i\), see [21], page 824, and the Lemma follows from this as we explain.

The points \(a, a^\sigma \in W\) both define \(C\) but the traditional markings, see page 3.76 of [19], will differ. Let \(\begin{pmatrix} B \\ A \end{pmatrix}\) be the standard homology basis corresponding to \(a \in W\) and \(\Omega(a)\) the period matrix computed from this basis. Similarly, let \(\begin{pmatrix} \tilde{B} \\ \tilde{A} \end{pmatrix}\) correspond to \(a^\sigma\) so that

\[
\begin{pmatrix} \tilde{B} \\ \tilde{A} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} B \\ A \end{pmatrix}
\]

for some \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Sp}_g(\mathbb{Z})\) and we have \(\Omega(a^\sigma) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \langle \Omega(a) \rangle = (\alpha \Omega(a) + \beta)(\gamma \Omega(a) + \delta)^{-1}\). The Abel-Jacobi maps \(w : \text{Div}^0(C) \to \mathbb{C}^g/\Lambda(\Omega(a))\), \(\tilde{w} : \text{Div}^0(C) \to \mathbb{C}^g/\Lambda(\Omega(a^\sigma))\), are related by \(\tilde{w} = (\Omega(a)\gamma' + \delta')^{-1}w\). Thus we have

\[
(\Omega(a)\gamma' + \delta')^{-1}w((a_{\sigma(i)}, 0) - (a_{\sigma(j)}, 0)) \equiv \tilde{w}((a_i^\sigma, 0) - (a_j^\sigma, 0)) \equiv \frac{1}{2}(\Omega(a^\sigma), I) \eta_{\{i,j\}} \mod \Lambda(\Omega(a^\sigma)),
\]

\[
\frac{1}{2}(\Omega(a), I) M' \eta_{\{i,j\}} \equiv \frac{1}{2}(\Omega(a), I) M \cdot \eta_{\{i,j\}} \equiv \frac{1}{2}(\Omega(a), I) \eta_{\{\sigma(i),\sigma(j)\}} \equiv w((a_{\sigma(i)}, 0) - (a_{\sigma(j)}, 0)) \equiv \frac{1}{2}((\Omega(a)\gamma' + \delta')\Omega(a^\sigma), \Omega(a)\gamma' + \delta') \eta_{\{i,j\}} \equiv \frac{1}{2}(\Omega(a), I) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}' \eta_{\{i,j\}} \mod \Lambda(\Omega(a)).
\]

Thus we have \(M \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mod 2\) and, along with \(\Omega(a^\sigma) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \langle \Omega(a) \rangle\), this implies \(\Gamma_g(2)\Omega(a^\sigma) = \Gamma_g(2)M(\Omega(a))\). Since \(\Gamma_g(2)\) is normal in \(\Gamma_g(1, 2)\), this is \(h(a^\sigma) = M(h(a))\). \(\Box\)

**Definition 25.** Define a subset \(S \subseteq \{1, \ldots, 2g + 2\}\) to be balanced if \(S\) contains an equal number of even numbers and odd numbers, and unbalanced otherwise.
Here is Thomae’s formula: If $T \subset B_g$ with $|T| = g + 1$, we have

\begin{equation}
\rho : \theta[\eta_U \otimes T]^4 \mapsto (-1)^{\frac{g + 1}{2}} \cdot \prod_{i < j \in T} (a_i - a_j) \prod_{i < j \in T'} (a_i - a_j).
\end{equation}

If $\zeta$ cannot be put into the form $\eta_U \otimes T$ with $|T| = g + 1$, then $\theta[\zeta] \mapsto 0$. It is simple to see that $S = U_g \otimes T$ with $|T| = g + 1$ if and only if $S$ is balanced. Therefore, when $S$ is unbalanced, then we have $\rho(\theta[\eta_S]) = 0$. By the isomorphism [8], we say that a theta characteristic is balanced if it can be written as $\eta_S$ with $S$ balanced.

We can now give the commutative diagrams that show that a solution of the Ansatz of D’Hoker and Phong goes to a solution of the hyperelliptic Ansatz under Igusa’s $\rho$-homomorphism; these commutative diagrams are deduced from those in Tsuyumine [21].

**Proposition 26.** Let $g_1, g_2, g \in \mathbb{N}$ such that $g_1 + g_2 = g$. There exists an $M_{g_1, g_2} \in \Gamma_g(1, 2)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
M^2(\Gamma_g(2)) & \rho_{g_1} \circ M_{g_1, g_2} & S^g(2g + 2)_0 \\
\Psi^*_{g_1, g_2} \downarrow & & \downarrow W_{g_1, g_2} \\
M^2(\Gamma_g(2)) \otimes M^2(\Gamma_g(2)) & \rho_{g_1} \otimes \rho_{g_2} & S^g(2g_1 + 2) \otimes S^g(2g_2 + 2) \\
\end{array}
\]

Let $\kappa^4$ be the character of $\Gamma_g(1, 2)$ given by $(\begin{array}{cc} A & B \\
C & D \end{array}) \mapsto (-1)^{A(D - T_g)}.

\rho_g : M^2(\Gamma_g(1, 2), \kappa^4) \to \mathcal{B}_g.$

**Proof.** The commutative diagram actually given in [21], page 786, and there only in the case of $(g_1, g_2) = (g - 1, 1)$, is

\[
\begin{array}{ccc}
M^2(\Gamma_g(2)) & \rho_g & S^g(2g + 2)_0 \\
\Psi^*_{g_1, g_2} \downarrow & & \downarrow T_{2g_1 + 1, 2g_2 + 1} \\
M^2(\Gamma_g(2)) \otimes M^2(\Gamma_g(2)) & (\ast \otimes \ast)(\rho_{g_1} \otimes \rho_{g_2}) & S^g(2g_1 + 2)^* \otimes S^g(2g_2 + 2)^* \\
\end{array}
\]

However, the proof of the general case is the same. In this article, we fix $\rho_g$ to be the map induced by the traditional marking of a hyperelliptic curve; therefore the commutative diagram from Tsuyumine holds for $\rho_g \circ M_{g_1, g_2}$ for some $M_{g_1, g_2} \in \text{Sp}_g(\mathbb{Z})$. Applying this commutative diagram to $\theta[0]^8$ we see that $M_{g_1, g_2} \in \Gamma_g(1, 2)$.

In order to show that $\rho_g$ sends $M^2(\Gamma_g(1, 2), \kappa^4)$ to $\mathcal{B}_g$, let $f \in [\Gamma_g(1, 2), \kappa^{2k}, k]$, and consider $\sigma \rho_g(f)$ for $\sigma \in SS$. Notice first that

\[
\sigma(\rho_g(\theta[0]^{2k})) = \sigma((\epsilon_g \Delta_U \Delta_U)^{k/2} = (\epsilon_g \Delta_U \Delta_U)^{k/2} = \rho_g(\theta[0]^{2k}).
\]
Therefore, using equation 8 and Lemma 24, we have
\[
\frac{(\sigma(\rho_\eta(f)))(a)}{(\sigma(\rho_\eta[0]2^k))(a)} = \frac{f(h(a^\sigma))}{\theta[0]2^k(h(a^\sigma))} = \frac{f(M \cdot h(a))}{\theta[0]2^k(M \cdot h(a))} = \frac{(f|_M)(a)}{(\rho_\eta[0]2^k(M \cdot h(a))}(\rho_\eta[0]2^k(h(a))) = \frac{(f|_M)(a)}{(\rho_\eta[0]2^k(h(a)))}.
\]
Thus, \(\sigma \rho_\eta(f) = \rho_\eta(f)\). □

**Lemma 27.** Let \(V\) be an isotropic subspace with all balanced elements. Then there exists a partitioning of \(\{1, \ldots, 2g + 2\}\) into balanced subsets of 2 elements each (call them \(u_1, \ldots, u_{g+1}\)), and a subspace \(H \subseteq \mathbb{F}_2^{g+1}\) with \(\dim H = \dim V\) such that
\[
V = \{\eta_{S_h} : h \in H, \text{ where } S_h = \bigcup_{i : h_i \neq 0} u_i\}\]

**Proof.** Using the isomorphism (6), we view \(V\) as a set of balanced subsets \(\{S_j\}\). Given that the symmetric difference \(S_{j_1} \oplus S_{j_2}\) is balanced by hypothesis, then the intersection \(S_{j_1} \cap S_{j_2}\) is also balanced. More generally, we can prove that the intersection of any number of these balanced subsets will be balanced. Then the Venn Diagram of intersections of all the \(S_j\) will give a partitioning of \(\{1, \ldots, 2g + 2\}\) into balanced subsets. We can make a finer partition into balanced subsets of 2 elements each such that each \(S_j\) is a union of subcollection of this partition. The result follows. □

**Lemma 28.** Fix genus \(g\) and fix an \(d \in \mathbb{N}\) with \(0 \leq d \leq g\). For any subspace \(V\) of dimension \(d\), there exists a polynomial \(Q_V \in \mathbb{Z}[a_1, \ldots, a_{2g+2}]\) in the \(a_i\) such that we have
\[
\rho(\prod_{\zeta \in V} \theta[\zeta]^8) = Q_V^{2d}.
\]
Furthermore, \(Q_V^2\) is unique.

**Proof.** Note that if \(V\) is not isotropic or if \(V\) contains any unbalanced theta characteristics, then \(Q_V = 0\) suffices. So assume \(V\) is isotropic and contains only balanced elements. Let \(H \subseteq \mathbb{F}_2^{g+1}\) and a partitioning of \(\{1, \ldots, 2g + 2\}\) into balanced subsets \(u_1, \ldots, u_{g+1}\) of 2 elements each, as in Lemma 27. Note \(|H| = 2^d\). Now, we have
\[
\rho(\prod_{\zeta \in V} \theta[\zeta]^4) = \pm \prod_{i > j} (a_i - a_j)^{r_{ij}}
\]
where \(r_{ij}\) are exponents that we will calculate. There are two cases: \(i, j\) of the same or different parity.
Case: \(i, j\) are of the same parity. Let \(i \in u_a\) and \(j \in u_b\) with \(a \neq b\). Then \(\rho(\theta[\eta_{S_h}]^4)\) contains a factor of \((a_i - a_j)\) if and only if \(i, j\) are both in or both not in \(S_h \oplus U\) which happens (because they are of the same parity) if and only if they are both in or not in \(S_h\) which is if and only if \(h_a = h_b\). Since \(H\) is a vector subspace of \(\mathbb{F}_2^{q+1}\), then the number of \(h \in H\) for which \(h_a = h_b\) is either \(|H|\) or \(\frac{1}{2}|H|\) because the set of such is the kernel of the linear map \(H \to \mathbb{F}_2\) by \(h \mapsto h_a - h_b\). So \(r_{ij} = 2^d\) or \(2^{d-1}\) in this case.

Case: \(i, j\) are of opposite same parity. Let \(i \in u_a\) and \(j \in u_b\).

Subcase: \(a = b\). Then \(i, j\) are always either both in or both not in an \(S_h\). Then \(S_h \oplus U\) will contain one of the \(i, j\) but not the other. This means \((a_i - a_j)\) will not appear in \(\rho(\theta[\eta_{S_h}]^4)\) Thus \(r_{ij} = 0\) in this subcase. Subcase: \(a \neq b\). Then \(\rho(\theta[\eta_{S_h}]^4)\) contains a factor of \((a_i - a_j)\) if and only if \(i, j\) are both in or both not in \(S_h \oplus U\) which happens if and only if exactly one of \(i, j\) is in \(S_h\) which is if and only if \(h_a \neq h_b\). Since the set of such \(h\) is the complement in \(H\) of the kernel of the linear map \(H \to \mathbb{F}_2\) by \(h \mapsto h_a - h_b\), then \(r_{ij} = |H| - |H|\) or \(|H| - \frac{1}{2}|H|\). So \(r_{ij} = 0\) or \(2^{d-1}\) in this subcase.

In all cases, we get \(r_{ij} = 0, 2^{d-1}, \) or \(2^d\). Note that \(r_{ij}\) is a multiple of \(2^{d-1}\) in all cases. Thus there exists a polynomial \(Q_v\) such that \(\rho(\prod_{\zeta \in V} \theta[\zeta]^4) = \pm Q_v^{2^{d-1}}\). Squaring both sides completes the proof. \(\square\)

**Lemma 29.** Let \(r, w \in \mathbb{N}\). If \(Q\) is a polynomial with real coefficients such that \(Q^4 \in S_{4w}(r)\), then \(Q^2 \in S_{2w}(r)\).

**Proof.** Take any \(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}_2(\mathbb{R})\) and \(\gamma(z) = \frac{Az + B}{Cz + D}\). We need to show that \(Q(a_1, \ldots, a_r)^2 = Q(\gamma(a_1), \ldots, \gamma(a_r))^4 \prod_{i=1}^r (Ca_i + D)^{2w}\) for all \(a_i\). We know that

\[
Q(a_1, \ldots, a_r)^4 = Q(\gamma(a_1), \ldots, \gamma(a_r))^4 \prod_{i=1}^r (Ca_i + D)^{4w}
\]

for all \(a_i\). Since \(Q\) has real coefficients, then for real values of \(a_i\), Equation (10) is an equality of real numbers to the fourth power. Thus we can take the positive square root of both sides and obtain that \(Q(a_1, \ldots, a_r)^2 = Q(\gamma(a_1), \ldots, \gamma(a_r))^2 \prod_{i=1}^r (Ca_i + D)^{2w}\) for all real values of the \(a_i\). That is, the multivariable polynomial \(Q(a_1, \ldots, a_r)^2 - Q(\gamma(a_1), \ldots, \gamma(a_r))^2 \prod_{i=1}^r (Ca_i + D)^{2w}\) is zero for all real values of its variables and must be the zero polynomial. Hence \(Q^2 \in S_{2w}(r)\). \(\square\)

**Lemma 30.** For any subspace \(V\) of theta characteristics, and any \(\sigma \in SS\), define

\[\sigma \cdot V = \{ \eta_{\sigma(S)} : \eta_S \in V \}\].
Then
\[ \sigma(Q_V^2) = Q_{\sigma V}^2. \]

Furthermore, when \( V \) is an isotropic subspace consisting of balanced elements, then \( \sigma \cdot V \) is also such a subspace.

**Proof.** For \( \zeta = \eta_S \in V \), we have that \( \pm(a_i - a_j) \) occurs in \( \rho(\theta[\zeta]^4) \) if and only if \( i, j \in U_g \oplus S \) or \( i, j \in U_g' \oplus S \). Because \( \sigma(U_g) = U_g \) or \( \sigma(U_g) = U_g' \), then \( \pm(a_i - a_j) \) occurs in \( \rho(\theta[\zeta]^4) \) if and only if \( \sigma(i), \sigma(j) \in U_g \oplus \sigma(S) \) or \( \sigma(i), \sigma(j) \in U_g' \oplus \sigma(S) \). Thus \( \pm(a_i - a_j) \) occurs in \( \rho(\theta[\zeta]^4) \) if and only if \( \pm(a_{\sigma(i)} - a_{\sigma(j)}) \) occurs in \( \rho(\theta[\eta_{\sigma(S)}]^4) \). Thus \( \sigma(\rho(\theta[\zeta]^4)) = \pm \rho(\theta[\eta_{\sigma(S)}]^4) \).

Letting \( d = \dim V \), we have by Lemma 28 that \( \sigma(Q_V^d) = Q_{\sigma V}^d \). Since \( Q_V \) and \( Q_{\sigma V} \) have real coefficients, by an argument similar to that of Lemma 29 we have that \( \sigma(Q_V^d) = Q_{\sigma V}^d \).

Now let \( V \) be an isotropic subspace of balanced elements. By Lemma 27 there exists a partitioning of \( \{1, \ldots, 2g + 2\} \) into balanced subsets of \( 2 \) elements each (call them \( u_1, u_2, \ldots, u_{2g + 1} \)) and a subspace \( H \subseteq \mathbb{F}_2^{g+1} \) with \( \dim H = \dim V \) such that \( V = \{ \eta_{S_h} : h \in H, \text{ where } S_h = \bigcup_{i: h_i \neq 0} u_i \} \). Then
\[
\sigma \cdot V = \{ \eta_{\sigma(S_h)} : h \in H, \text{ where } \sigma(S_h) = \bigcup_{i: h_i \neq 0} \sigma(u_i) \},
\]
and so \( \sigma \cdot V \) is a subspace, and in fact a subspace of balanced elements. \( \square \)

For any subspace \( V \) of theta characteristics, we will use the notation \( Q_V \) as prescribed by Lemma 28, with the understanding that \( Q_V^2 \) is unique given \( V \). Note that \( Q_V = 0 \) unless \( V \) is isotropic and contains only balanced elements.

Fix \( g = g_1 + g_2 \) with \( g_1, g_2 \in \mathbb{N} \) for the following discussion, which parallels that of [10]. For any theta characteristic \( \zeta \), the Witt map \( W_{g_1, g_2} \) on Siegel modular forms yields
\[
W_{g_1, g_2}(\theta[\zeta])^8 = \theta[\pi_1 \zeta]^8 \theta[\pi_2 \zeta]^8
\]
where \( \pi_1 \zeta \) is the projection of \( \zeta \) onto the left \( 2g_1 \) coordinates and \( \pi_2 \zeta \) is the projection of \( \zeta \) onto the right \( 2g_2 \) coordinates. Let \( V \subseteq \mathbb{F}_2^g \) be a subspace of theta characteristics. Then
\[
W_{g_1, g_2} \left( \prod_{\zeta \in V} \theta[\zeta]^8 \right) = \prod_{\zeta_1 \in \pi_1 V} \theta[\zeta_1]^8 \prod_{\zeta_2 \in \pi_2 V} \theta[\zeta_2]^8
\]
where \( d_i = \dim \pi_i V \).

Since the eighth powers are modular forms on \( \Gamma(2) \) and the appropriate Witt maps, \( \Psi_{g_1, g_2}^* \) and \( W_{g_1, g_2} \), are equivariant with respect to the
ρ-map, we get that
\[ W_{g_1,g_2} \left( Q_V^4 \right) = Q_{\pi_1 V}^{4,2-d_1} Q_{\pi_2 V}^{4,2-d_2}. \]

Since \( Q_V^4 \in S_{8g}(2g+2) \) and \( Q_V \) has real coefficients, then by Lemma \ref{lemma29} we have that \( Q_V^2 \in S_{4g}(2g+2) \). The important point here is that \( Q_V^2 \) has the correct valuation and we can apply the Witt map \( W_{g_1,g_2} \) to it. Then we must have
\[ W_{g_1,g_2} \left( Q_V^2 \right) = Q_{\pi_1 V}^{2,1+d-d_1} Q_{\pi_2 V}^{2,1+d-d_2}. \]

Now letting \( V \) vary over subspaces of dimension \( d \), we get the following.

\[
W_{g_1,g_2} \left( \sum_{V \subseteq \mathbb{F}_2^{2g}} Q_V^2 \right) = \sum_{0 \leq d_1, d_2 \leq d} N_{d_1,d_2; d} \cdot \sum_{V_1 \subseteq \mathbb{F}_2^{2g_1}} Q_{V_1}^{2,1+d-d_1} \sum_{V_2 \subseteq \mathbb{F}_2^{2g_2}} Q_{V_2}^{2,1+d-d_2},
\]

where \( N_{d_1,d_2; d} \) is the number of \( V \subseteq \mathbb{F}_2^{2g_1} \oplus \mathbb{F}_2^{2g_2} \) of dimension \( d \) that have \( \pi_i V = V_i \) \( (i = 1, 2) \) given fixed \( V_1, V_2 \) of dimensions \( d_1, d_2 \) respectively. The formula proven in \cite{ref10} is
\[
N_{d_1,d_2; d} = \prod_{j=0}^{d_1+d_2-d-1} \frac{(2^{d_1} - 2^j)(2^{d_2} - 2^j)}{(2^{d_1+d_2-d} - 2^j)}
\]
for \( 0 \leq d_1, d_2 \leq d \leq d_1 + d_2 \) and it is 0 otherwise. Now define
\[
\tilde{G}_d^{(g)} = \sum_{V \subseteq \mathbb{F}_2^{2g}} Q_V^2.
\]

Note that by summing over all subspaces, by Lemma \ref{lemma30} we have that \( \tilde{G}_d^{(g)} \) is invariant under \( SS \). We have
\[
(11) \quad W_{g_1,g_2} \tilde{G}_d^{(g)} = \sum_{0 \leq d_1, d_2 \leq d \leq d_1+d_2} N_{d_1,d_2; d} \tilde{G}_d^{(g_1)} \tilde{G}_d^{(g_2)}
\]

Define
\[
(12) \quad K^{(g)} = \frac{1}{2^g} \left( \sum_{i=0}^{g} (-1)^i 2^{i(i+1)/2} \tilde{G}_i^{(g)} \right).
\]

**Proposition 31.** Let \( g = g_1 + g_2 \) with \( g_1, g_2 \in \mathbb{N} \). Then
\[
W_{g_1,g_2} K^{(g)} = K^{(g_1)} \otimes K^{(g_2)}
\]
Proof. Expand both sides using Equations 11 and 12 and prove that the coefficient of $\tilde{G}(g_1)\tilde{G}(g_2)$ is the same for all $n, m$. The key identity is

$$(-1)^n 2^{\frac{n(n-1)}{2}} (-1)^m 2^{\frac{m(m-1)}{2}} = \sum_{i=0}^{n+m} (-1)^i 2^{\frac{i(i-1)}{2}} N_{n,m;i}$$

which is proven in [10]. □

**Theorem 32.** The family $K^{(g)}$ satisfies the hyperelliptic Ansatz. Furthermore, $K^{(g)} = H_g$ for all $g$.

Proof. We already know that $K^{(g)} \in S_{4g}(2g+2)_{0}(SS) = B^g_0$ is a family of modular forms that satisfy the splitting property and that $K^{(1)}$ satisfies the base condition. By Theorem 22 on uniqueness, we must have that the family $K^{(g)}$ equals the family $H_g$. □

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