ON THE TAUTOLOGICAL RINGS OF STACKS OF TWISTED CURVES

HSIAN-HUA TSENG

ABSTRACT. We introduce the tautological rings of moduli stacks of twisted curves and establish some basic properties.

0. INTRODUCTION

We work over an algebraically closed field of characteristic 0.

0.1. Background. Balanced twisted curves\(^1\) are nodal curves with specific types of stack structures possibly at nodes and isolated smooth points. Étale locally at a smooth stacky point, it looks like \([\mathbb{A}^1/\mu_r]\) for some \(r\), where \(\mu_r\) acts on \(\mathbb{A}^1\) by multiplications. Étale locally at a stacky node, it looks like \([(xy = 0)/\mu_s]\) for some \(s\), where \(\mu_s\) acts\(^2\) via \(\zeta \cdot (x, y) = (\zeta \cdot x, \zeta^{-1} \cdot y)\).

Twisted curves arise naturally in [4], [1] as a solution to the problem of compactifying spaces of maps from nodal curves to a proper Deligne-Mumford stack. More precisely, to compactify a space of maps from smooth curves of a proper Deligne-Mumford stack, one must add maps from nodal twisted curves, not just maps from nodal curves.

Later on, compact moduli spaces of certain maps from balanced\(^3\) twisted curves to a smooth proper Deligne-Mumford stack were used to construct Gromov-Witten theory of Deligne-Mumford stack, see [3].

Let 

\[ \mathcal{M}_{g,n}^{tw} \]

be the stack of prestable balanced twisted curves of genus \(g\) with \(n\) marked gerbes. The existence of \(\mathcal{M}_{g,n}^{tw}\) and some basic properties have been known to the experts. It is known that \(\mathcal{M}_{g,n}^{tw}\) is a smooth Artin stack locally of finite type. A construction of \(\mathcal{M}_{g,n}^{tw}\) using logarithmic geometry, and proofs of some of its basic properties, can be found in [10].

A variant of \(\mathcal{M}_{g,n}^{tw}\) was introduced in [8, Section 2] for the purpose of studying relations between higher genus Gromov-Witten theory and genus 0 Gromov-Witten theory of symmetric products. This variant was also used in [5] to study genus 0 Gromov-Witten theory of root gerbes. In this note, this variant is denoted by \(\mathcal{M}_{g,l,m,a}^{tw, triv}\) see (1.2) below.
0.2. This paper. In the study of Gromov-Witten theory of Deligne-Mumford stacks, the stack $\mathcal{M}_{g,n}^{tw}$ is mostly used as a placeholder. On the other hand, $\mathcal{M}_{g,n}^{tw}$ is a natural generalization of both the stack $\mathcal{M}_{g,n}$ of stable curves and the stack $\mathcal{M}_{g,n}$ of pretable curves. $\mathcal{M}_{g,n}^{tw}$ has rich geometry and deserves to be studied further. Motivated by this, this note considers one geometric aspect of $\mathcal{M}_{g,n}^{tw}$, namely the tautological ring of $\mathcal{M}_{g,n}^{tw}$. The approach taken here closely follows the work [6].

Section 1 presents a definition of the tautological rings of $\mathcal{M}_{g,n}^{tw}$ as a subring of the Chow rings of $\mathcal{M}_{g,n}^{tw}$. In this construction, the aforementioned variant $\mathcal{M}_{g,l,m,a}^{tw,triv}$ introduced in [8, Section 2] plays an important role. A key reason for working with $\mathcal{M}_{g,l,m,a}^{tw,triv}$ is that their universal curves have modular descriptions, given by the morphisms of forgetting a non-stacky marked point. This allows the tautological rings of $\mathcal{M}_{g,l,m,a}^{tw,triv}$ to be defined using natural gluing and forgetful maps, see Definition 1.2 for more details. The tautological rings of $\mathcal{M}_{g,l,m,a}^{tw,triv}$ are then defined using tautological rings of $\mathcal{M}_{g,l,m,a}^{tw,triv}$ via restrictions, see Definition 1.5 for a precise description.

A natural class of elements in the tautological rings, namely decorated strata classes, are also introduced, see Definition 1.7.

Section 2 presents an additive spanning set of the tautological rings. Namely, it is shown that the decorated strata classes span the tautological rings as $\mathbb{Q}$-vector spaces. See Theorem 2.1 for the precise result.

0.3. Outlook. Naturally, the next goal in this direction is to understand the genus 0 situation, thus extending the work [7] to twisted curves. More precisely, it is expected that the Chow ring and tautological ring of $\mathcal{M}_{0,n}^{tw}$ coincide, and relations in the tautological ring of $\mathcal{M}_{0,n}^{tw}$ are all obtained from WDVV relations. This will be discussed elsewhere.

A long-term, more ambitious goal, is to understand relations in the tautological rings of $\mathcal{M}_{g,n}^{tw}$.

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when \((g, n) = (1, 0)\)). Hence intersection products can be defined, making these Chow groups into \(\mathbb{Q}\)-algebras.

The stack \(\mathcal{M}^{tw}_{g, n}\) is not connected. To specify component(s) of \(\mathcal{M}^{tw}_{g, n}\) to work with, it is convenient to rephrase the setup by introducing labels for marked points. Let \(I\) be a set of \(n\) elements. Let \(\mathcal{M}^{tw}_{g, I}\) be the stack of prestable balanced twisted curves of genus \(g\) and marked gerbes labelled by \(I\). Clearly there is an isomorphism \(\mathcal{M}^{tw}_{g, I} \cong \mathcal{M}^{tw}_{g, n}\) given by choosing an identification \(I \rightarrow \{1, 2, ..., n\}\). For a function \(m : I \rightarrow \mathbb{Z}_{>0}\), let \(\mathcal{M}^{tw}_{g, I, m} \subset \mathcal{M}^{tw}_{g, I}\) be the locus parametrizing twisted curves whose stack structures at marked points are given by the function \(m\). The stacks \(\mathcal{M}^{tw, triv}_{g, I, m}\) and \(\mathcal{M}^{tw, triv}_{g, I, m}\) are defined by inverse images with respect to (1.1). The stacks \(\mathcal{M}^{tw, triv}_{g, I, m}\) and \(\mathcal{M}^{tw, triv}_{g, I, m}\) are smooth Artin stacks locally of finite type. It can be seen that \(\mathcal{M}^{tw, triv}_{g, I, m}\) and \(\mathcal{M}^{tw, triv}_{g, I, m}\) have good filtrations by finite type substacks, in the sense of [6, Definition A.2], given by loci of twisted curves with bounds on the numbers of nodes.

We are interested in the Chow rings

\[
\text{CH}^*(\mathcal{M}^{tw}_{g, I, m}), \quad \text{CH}^*(\mathcal{M}^{tw, triv}_{g, I, m}).
\]

Since (1.1) is a fiber product of cyclic gerbes, with \(\mathbb{Q}\)-coefficients there is an isomorphism

\[
\text{CH}^*(\mathcal{M}^{tw}_{g, I, m}) \cong \text{CH}^*(\mathcal{M}^{tw, triv}_{g, I, m})
\]

induced by pulling back via (1.1). In what follows, we work with \(\mathcal{M}^{tw, triv}_{g, I, m}\).

1.2. Tautological rings. In the definition of tautological rings for moduli spaces of stable curves, the modular interpretation of their universal curves plays an important role. For moduli stacks of prestable curves, such a moduli interpretation is not valid. This issue is overcome in [6]. We follow their recipe to treat the case of twisted curves, as follows.

Let \(\mathcal{A}\) be a commutative semigroup with unit \(0 \in \mathcal{A}\) such that

1. \(\mathcal{A}\) has indecomposable zero: for \(x, y \in \mathcal{A}\), if \(x + y = 0\), then we must have \(x = y = 0\);
2. \(\mathcal{A}\) has finite decomposition: if \(a \in \mathcal{A}\), then the set \(\{(a_1, a_2) | a_1 + a_2 = a\} \subset \mathcal{A} \times \mathcal{A}\) is finite.

Example 1.1. The example of \(\mathcal{A}\) that matters to us the most is the following: \(\mathcal{A}_0 = \{0, 1\}\) with \(0 + 0 = 0, 0 + 1 = 1 + 0 = 1 + 1 = 1\).

Fix an \(\mathcal{A}\). Let \(a \in \mathcal{A}\). We consider the stack

\[
\mathcal{M}^{tw, triv}_{g, I, m, a}
\]

of twisted curves with labellings of irreducible components by elements of \(\mathcal{A}\). This stack is defined in [8, Section 2], where it is denoted by \(\mathcal{M}_{g, I, m, a}\). Roughly speaking, an object of \(\mathcal{M}^{tw, triv}_{g, I, m, a}\) is a twisted curve \((\mathcal{C}, \{p_i\}_{i \in I}) \in \mathcal{M}^{tw, triv}_{g, I, m}\) together with a map \(C_v \rightarrow a_{C_v}\) from the set of irreducible components \(\{C_v\}\) of the normalization of \(\mathcal{C}\) to \(\mathcal{A}\) such that \(a_{C_v}\) sum up to \(a\). The following stability
condition is required: for each irreducible component \( C_v \), either \( a_{C_v} \neq 0 \), or \( C_v \) together with its special points is a stable twisted curve.

There is a natural morphism

\[
\mathcal{M}^{tw,triv}_{g,I,m,a} \to \mathcal{M}^{tw,triv}_{g,I,m}
\]

that forgets the \( \mathcal{A} \)-valuation.

It follows from [8, Proposition 2.0.2] and properties of \( \mathcal{M}^{tw,triv}_{g,I,m,a} \) that \( \mathcal{M}^{tw,triv}_{g,I,m,a} \) is a smooth Artin stack locally of finite type, with good filtrations by finite type substacks.

Consider the stack \( \mathcal{M}^{tw,triv}_{g,I,m,a} \) where \( m' : I \cup \{ \bullet \} \to \mathbb{Z}_{>0} \) is given by \( m'|_I = m \) and \( m'(\bullet) = 1 \) (i.e. the marked point \( \bullet \) has trivial stack structure). Then there is a morphism

(1.3)

\[
\pi_\bullet : \mathcal{M}^{tw,triv}_{g,I,\{ \bullet \},m',a} \to \mathcal{M}^{tw,triv}_{g,I,m,a};
\]

given by forgetting the marked point \( \bullet \) and contracting resulting unstable components with \( \mathcal{A} \)-valuations \( 0 \). It follows from [8, Proposition 2.1.1] that (1.3) is the universal curve.

We now recall the natural gluing maps with target \( \mathcal{M}^{tw,triv}_{g,I,m,a} \). Let

(1.4)

\[
\Gamma = (V,H,E,L,g : V \to \mathbb{Z}_{\geq 0}, v : H \to V, \iota : H \to H)
\]

be a prestable graph of genus \( g \) with \( n \) markings. The required properties of \( \Gamma \) are

1. \( V \) is the set of vertices, \( g : V \to \mathbb{Z}_{\geq 0} \) is the genus assignment;
2. \( H \) is the set of half-edges, \( v : H \to V \) is the vertex assignment, and \( \iota : H \to H \) is an involution;
3. the set of 2-cycles of \( \iota \) in \( H \) is by definition the set of edges \( E \) (self-edges at vertices are permitted);
4. the set of fixed points of \( \iota \) is by definition the set of legs \( L \) (which correspond to the \( n \) markings);
5. the pair \((V,E)\) defines a connected graph \( \Gamma \) satisfying the genus condition

\[
\sum_{v \in V} g(v) + h^1(\Gamma) = g.
\]

For a prestable graph \( \Gamma \), we fix an identification \( L \simeq I \) between the set of legs and marking labels. In addition, we need to choose a map

(1.5)

\[
m_\Gamma : H \to \mathbb{Z}_{\geq 0}
\]

so that \( m_\Gamma|_L = m \) and

\[
m_\Gamma(\iota(h)) = m_\Gamma(h)
\]

for \( h \in H \), which ensures that stack structures at nodes are balanced. We also need to choose an \( \mathcal{A} \)-valuation \( a : V(\Gamma) \to \mathcal{A} \) satisfying \( \sum_{v \in V(\Gamma)} a(v) = a \).

For a vertex \( v \in V(\Gamma) \), let \( H(v) \subset H \) be the set of half-edges at \( v \) and \( L(v) \subset H(v) \) the set of legs at \( v \). \( I(v) \subset I \) is defined by the identification \( L \simeq I \).

Associated to \((\Gamma,m_\Gamma,a)\) is a gluing morphism

(1.6)

\[
\xi(\Gamma,m_\Gamma,a) : \mathcal{M}^{tw,triv}_{(\Gamma,m_\Gamma,a)} = \prod_{v \in V(\Gamma)} \mathcal{M}^{tw,triv}_{g(v),H(v),m_\Gamma|_{H(v)},a(v)} \to \mathcal{M}^{tw,triv}_{g,I,m,a};
\]
which is defined as follows. \( \xi_{(\Gamma, m\Phi, a)} \) sends a collection \((C_v, \{ p_h \}_{h \in H(v)})_{v \in V}\) to the curve \((C, \{ p_i \}_{i \in I})\) obtained by identifying the markings \(p_h, p_i(h)\) for each pair \((h, i(h))\) of half-edges forming an edge of \(\Gamma\). Here the gerbe structures at \(p_h\) and \(p_i(h)\) are identified via the homomorphism \(\mu_{m\Phi(h)} : \mu_{m\Phi(i(h))} : \zeta \mapsto \zeta^{-1}\). Since gerbes at nodes are not trivialized, over its image, \(\xi_{(\Gamma, m\Phi, a)}\) is the product of universal gerbes at nodes (which correspond to edges of \(\Gamma\)) and has degree \(\prod_{(h, i(h)) \in E} \frac{1}{m\Phi(h)}\); see [2, Section 5.2].

The following definition is a direct extension of [6, Definition 1.3], which is modelled on the treatment of the tautological rings of moduli stacks of stable curves in [9].

**Definition 1.2.** The tautological rings

\[
\{ R^*(M_{g,I,m,a}^{tw, \text{triv}}) \}_{g,I,m,a}
\]

is the smallest system of unital \(\mathbb{Q}\)-subalgebras of the Chow rings \(\{ CH^*(M_{g,I,m,a}^{tw, \text{triv}}) \}_{g,I,m,a}\) which is closed under taking pushforwards by the natural forgetful maps (1.3) and gluing maps (1.6).

Consider the semigroup \(A_0\) in Example 1.1. Let \(1 \in A_0\). Consider the subset

\[
\mathfrak{M}_{g,I,m}^{tw, \text{triv}} \subset \mathfrak{M}_{g,I,m}^{tw, \text{triv}}
\]

consisting of \(A\)-valued curves \((C, \{ p_i \}_{i \in I}, (a_{c_v})_{v})\) such that one of the values \(a_{c_v}\) is equal to 0. The following Lemma is a direct adaptation of [6, Proposition 2.6], whose proof can be easily adapted here as well.

**Lemma 1.3.** \(\mathfrak{M}_{g,I,m}^{tw, \text{triv}}\) is closed. The composition

\[
\mathcal{U}_{g,I,m}^{tw, \text{triv}} = \mathfrak{M}_{g,I,m}^{tw, \text{triv}} \setminus \mathfrak{M}_{g,I,m}^{tw, \text{triv}} \hookrightarrow \mathfrak{M}_{g,I,m}^{tw, \text{triv}} \to \mathfrak{M}_{g,I,m}^{tw, \text{triv}}
\]

of the inclusion and the natural map defines an isomorphism

\[
\mathcal{U}_{g,I,m}^{tw, \text{triv}} \cong \mathfrak{M}_{g,I,m}^{tw, \text{triv}}.
\]

The following extends [6, Corollary 2.7].

**Corollary 1.4.** The universal curve over \(M_{g,I,m}^{tw, \text{triv}}\) is given by

\[
\pi_* : M_{g,I,m+1}^{tw, \text{triv}} \setminus \pi^{-1}(\mathfrak{M}_{g,I,m}^{tw, \text{triv}}) \to M_{g,I,m}^{tw, \text{triv}} \setminus \mathfrak{M}_{g,I,m}^{tw, \text{triv}}
\]

**Definition 1.5.** The tautological ring

\[
R^*(M_{g,I,m,a}^{tw, \text{triv}}) \subset CH^*(M_{g,I,m,a}^{tw, \text{triv}})
\]

is defined to be the image of the restriction of \(R^*(M_{g,I,m,a}^{tw, \text{triv}})\) to the open substack from Lemma 1.3.

### 1.3. Stratum classes.

We define some basic classes in the tautological ring \(R^*(M_{g,I,m,a}^{tw, \text{triv}})\). Recall the universal curve \(\pi_*\) in (1.3). Let

\[
\sigma_i : M_{g,I,m,a}^{tw, \text{triv}} \to M_{g,I,\{\ast\},m',a}, \quad i \in I
\]

be the section corresponding to the marking labelled by \(i \in I\). Note that \(\sigma_i\) exists because marked gerbes are trivialized. Let \(\omega_{\pi_*}\) be the relative dualizing sheaf of \(\pi_*\).

**Definition 1.6.**

\[
\psi_i := \sigma_i^*(\omega_{\pi_*}) \in CH^1(M_{g,I,m,a}^{tw, \text{triv}}), \quad i \in I
\]

\[
\kappa_m := (\pi_*)^*(\psi_m^{m+1}) \in CH^m(M_{g,I,m,a}^{tw, \text{triv}}).
\]
**Definition 1.7 (Decorated stratum classes).** Let $\Gamma$ be a prestable graph of genus $g$ with $n$ markings. Let $m_\Gamma$ be as in (1.5) and $a : V(\Gamma) \to A$ an $A$-valuation with total value $a \in A$. A decoration on $(\Gamma, m_\Gamma, a)$ is a class on $\mathcal{M}_{tw,triv}((\Gamma, m_\Gamma, a)) = \prod_{v \in V(\Gamma)} \mathcal{M}_{tw,triv}^{g(v), H(v), m_{\Gamma|H(v)}, a(v)}$ of the form

\begin{equation}
\alpha = \prod_{v \in V} \left( \prod_{i \in H(v)} \psi_{v,i}^{\alpha_i} \prod_{j \geq 1} \kappa_{v,j}^{b_{v,j}} \right) \in \text{CH}^*(\mathcal{M}_{tw,triv}^{g(v), H(v), m_{\Gamma|H(v)}, a(v)}).
\end{equation}

Here $\psi_{v,i}, \kappa_{v,j} \in \text{CH}^*(\mathcal{M}_{g(v), H(v), m_{\Gamma|H(v)}, a(v)})$ are $\psi$ and $\kappa$ classes of the moduli space corresponding to the vertex $v \in V(\Gamma)$. The decorated stratum class $[(\Gamma, m_\Gamma, a), \alpha]$ is defined to be the pushforward

\begin{equation}
[(\Gamma, m_\Gamma, a), \alpha] := (\xi_{(\Gamma, m_\Gamma, a)})_* \alpha \in \text{CH}^*(\mathcal{M}_{g,I,m,a}).
\end{equation}

2. AN ADDITIVE SPANNING SET OF TAUTOLOGICAL RINGS

The main goal of this section is to prove the following extension of [6 Theorem 1.4] and [9 Proposition 11]:

**Theorem 2.1.** The tautological ring $R^*(\mathcal{M}_{g,I,m,a})$ is spanned by decorated stratum classes (1.9) as a $\mathbb{Q}$-vector space.

The treatment here follows those of [6] and [9].

Because decorated stratum classes are tautological, their span is a vector subspace of the tautological ring. Since the tautological rings are defined to be the smallest system of unital $\mathbb{Q}$-subalgebras which is closed under pushforwards by (1.3) and (1.6), we must prove two things:

1. The span of decorated stratum classes is a $\mathbb{Q}$-subalgebra, i.e. it is closed under intersection product.
2. The span of decorated stratum classes is closed under pushforwards by (1.3) and (1.6), hence the span must be the smallest such system.

The proof involves descriptions of pullbacks and pushforwards of decorated stratum classes under natural maps. We spell out the details in our case, following the treatment in [6, Section 3.2].

2.1. Closure under intersection product. We recall the combinatorial set-up. Let $A$ be an $A$-valued prestable graph.

**Definition 2.2.** ([6, Definition 3.5]) An $A$-structure on an $A$-valued prestable graph $\Gamma$, denoted by $\Gamma \to A$,

consists of choices of subgraphs $\{\Gamma_v\}_{v \in V(A)}$ and maps $V(\Gamma) \to V(A)$ and $H(\Gamma) \to H(\Gamma)$ satisfying

1. The total $A$-value of $\Gamma_v$ is $a(v)$, where $a : V(A) \to A$ is the $A$-valuation of $A$.
2. The map $V(\Gamma) \to V(A)$ is surjective, and the inverse image of $v \in V(A)$ are precisely the vertices of $\Gamma_v$.
3. The $A$-valuation of $\Gamma$ is given by the composition of $V(\Gamma) \to V(A)$ and $a : V(A) \to A$.
4. The map $H(\Gamma) \to H(\Gamma)$ is injective, and identifies half-edges $H(v)$ of $A$ with legs of $\Gamma_v$.
5. The maps $V(\Gamma) \to V(A)$ and $H(\Gamma) \to H(\Gamma)$ respect the incidence relation of half-edges and vertices, and the pairs of half-edges forming edges. (In particular, $E(A) \subseteq E(\Gamma)$.)
Given an $A$-structure $\Gamma \to A$ on $\Gamma$, the map $m_\Gamma : H(\Gamma) \to \mathbb{Z}_{\geq 0}$ and the injection $H(A) \to H(\Gamma)$ defines the map

$$m_A : H(A) \to \mathbb{Z}_{\geq 0}.$$ 

Then there is a natural gluing map

$$\xi_{(\Gamma \to A,m_\Gamma,a)} : \mathcal{M}^w_{(\Gamma,m_\Gamma,a)} \to \mathcal{M}^w_{(A,m_A,a)}.$$  

We have

1. If $H(A) \to H(\Gamma)$ maps $i \in H(v) \subset H(A)$ to $j \in H(w) \subset H(\Gamma)$, then $\xi^*_{(\Gamma \to A,m_\Gamma,a)} \psi_{v,i} = \psi_{w,j}$.
2. $\xi^*_{(\Gamma \to A,m_\Gamma,a)} \kappa_{v,l} = \sum_{w \in \Gamma(v)} \kappa_{w,l}$.

These can be proven by ways analogous to their counterparts for moduli stacks of prestable curves, see [6, Section 3.2]. These properties allow, in a straightforward manner, a description of the pull-back $\xi^*_{(\Gamma \to A,m_\Gamma,a)} \alpha$ of a decoration $\alpha$ on $\mathcal{M}^w_{(A,m_A,a)}$ as in (1.8).

Suppose a prestable graph $\Gamma$ admits $A$ and $B$-structures $f_A : \Gamma \to A$ and $f_B : \Gamma \to B$. As in [6, Section 3.2], the pair $(f_A, f_B)$ is called a generic $(A,B)$-structure on $\Gamma$ if each half-edge of $\Gamma$ corresponds to either a half-edge of $A$ or a half-edge of $B$. In other words, the injections $H(A) \to H(\Gamma)$ and $H(B) \to H(\Gamma)$ have disjoint images and their union is the whole $H(\Gamma)$.

With a generic $(A,B)$ structure on $\Gamma$, the maps $m_A : H(A) \to \mathbb{Z}_{\geq 0}$ and $m_B : H(B) \to \mathbb{Z}_{\geq 0}$ uniquely determine a map $m_{\Gamma} : H(\Gamma) \to \mathbb{Z}_{\geq 0}$.

**Proposition 2.3.** Let $A$ and $B$ be $\mathcal{A}$-valued prestable graphs for $\mathcal{M}^g_{l,m,a}$. Let $m_A$ and $m_B$ be as in (1.5). Then the fiber product of the gluing maps $\xi_{(A,m_A,a)} : \mathcal{M}^w_{(A,m_A,a)} \to \mathcal{M}^w_{g,l,m,a}$ and $\xi_{(B,m_B,a)} : \mathcal{M}^w_{(B,m_B,a)} \to \mathcal{M}^w_{g,l,m,a}$ is given by the disjoint union

$$\prod_{\Gamma \in G_{A,B}} \mathcal{M}^w_{(\Gamma,m_\Gamma,a)}$$

taken over the set $G_{A,B}$ of isomorphism classes of generic $(A,B)$ structures on prestable graphs $\Gamma$, with two projection maps given by $\xi_{(\Gamma \to A,m_\Gamma,a)}$ and $\xi_{(\Gamma \to B,m_\Gamma,a)}$. Moreover, the top Chern class of the excess bundle

$$E_{(\Gamma,m_\Gamma,a)} = \xi_{(\Gamma \to A,m_\Gamma,a)} N_{\xi_{(\Gamma \to B,m_\Gamma,a)}} / N_{\xi_{(\Gamma \to A,m_\Gamma,a)}}$$

is given by

$$c_{\text{top}}(E_{(\Gamma,m_\Gamma,a)}) = \prod_{e=(h,h') \in E(A) \cap E(B) \subset E(\Gamma)} (-\psi_h - \psi_{h'}) / m_\Gamma(h).$$  (2.1)

The proof of this Proposition is similar to that of [6, Proposition 3.6]. The only thing to notice is the factor $1/m_\Gamma(h)$ in (2.1), which accounts for the fact that the gerbe at nodes are not trivialized.

The projection formula yields the following description of products of decorated stratum classes:

$$[(A, m_A, a), \alpha] \cdot [(B, m_B, a), \beta] = \sum_{\Gamma \in G_{A,B}} (\xi_{(\Gamma,m_\Gamma,a)} \ast (\xi^*_{(\Gamma \to A,m_\Gamma,a)} \alpha \cdot \xi^*_{(\Gamma \to B,m_\Gamma,a)} \beta) \cdot c_{\text{top}}(E_{(\Gamma,m_\Gamma,a)}).$$  (2.2)

The above discussion shows that the right-hand side of (2.2) is contained in the span of decorated stratum classes. This shows that the span is closed under intersection product.
2.2. Closure under pushforwards.

2.2.1. Gluing maps. Pushing forward via (1.6) is straightforward to understand. Let $\Gamma_0$ be a graph, $m_{\Gamma_0} : H(\Gamma_0) \to \mathbb{Z}_{\geq 0}$, and $a_0 : V(\Gamma_0) \to \mathcal{A}$ an $\mathcal{A}$-valuation. Suppose for each $v \in V(\Gamma_0)$, we have the data of a genus $g(v)$ graph $\Gamma_v$, a map $m_{\Gamma_v} : H(\Gamma_v) \to \mathbb{Z}_{\geq 0}$, and an $\mathcal{A}$-valuation $a_v : V(\Gamma_v) \to \mathcal{A}$, satisfying the following conditions:

1. For each $v \in V(\Gamma_0)$, there is an embedding of the set of half-edges $H(\Gamma_0)|_v \subset H(\Gamma_v)$ so that $m_{\Gamma_v}$ restricts to $m_{\Gamma_0}|_{H(\Gamma_0)|_v}$.
2. For each $v \in V(\Gamma_0)$, $\sum_{v' \in V(\Gamma_v)} a_v(v') = a_0(v)$.

For each $v \in V(\Gamma_0)$, we can replace $v$ by the graph $\Gamma_v$ and replace half-edges in $H(\Gamma_0)|_v$ by their images in $H(\Gamma_v)$. This yields a new graph $\Gamma$ with $V(\Gamma) = \cup_{v \in V(\Gamma_0)}V(\Gamma_v)$ and $H(\Gamma) = \cup_{v \in V(\Gamma_0)}H(\Gamma_v)$. Define $m_{\Gamma}$ by $m_{\Gamma}|_{H(\Gamma_v)} = m_{\Gamma_v}$, and define $a : V(\Gamma) \to \mathcal{A}$ by $a|_{V(\Gamma_v)} = a_v$.

This construction describes a natural gluing map

$$
\prod_{v \in V(\Gamma_0)} \mathcal{M}_{tw, triv}^{(\Gamma, m_{\Gamma_v}, a_v)} \to \mathcal{M}_{tw, triv}^{(\Gamma, m_{\Gamma}, a)}.
$$

The pushforward of a class $\prod_{v \in V(\Gamma_0)}[(\Gamma_v, m_{\Gamma_v}, a_v), \alpha_v]$ under this gluing map is given by

$$
\frac{1}{\prod_{h \in E(\Gamma_0)} m_{\Gamma_0}(h)}[(\Gamma, m_{\Gamma}, a), \alpha]
$$

where $\alpha$ is formed by combining $\alpha_v$ using $V(\Gamma) = \cup_{v \in V(\Gamma_0)}V(\Gamma_v)$.

2.2.2. Forgetful maps. We begin with an analogue of [6, Proposition 3.11].

Let $[(\Gamma, m_{\Gamma}, a), \alpha] \in R^s(\mathcal{M}_{g, \bullet \cup \{\bullet\}, m', \mathcal{A}})$ with $\alpha = \prod_{v \in V(\Gamma)} \alpha_v$. Let $v \in V(\Gamma)$ be the vertex incident to $\bullet$ and let $\Gamma'$ be the graph obtained from $\Gamma$ by forgetting $\bullet$ and stabilizing if the vertex $v$ becomes unstable. $v$ becomes unstable exactly when $g(v) = 0$, $a(v) = 0$, and $v$ has two other half-edges. By stability, the curve component corresponding to $v$ is a genus 0 stable twisted curve with three marked points. By assumption the marked point corresponding to $\bullet$ has trivial stack structure: $m_{\Gamma'}(\bullet) = 1$. Stabilization contracts this curve component, hence the other marked points have opposite stack structure. Hence $m_{\Gamma'} : H(\Gamma') \to \mathbb{Z}_{\geq 0}$ is well-defined from $m_{\Gamma}$. $a' : V(\Gamma') \to \mathcal{A}$ is naturally defined from $a$.

**Proposition 2.4.**

1. If $v$ remains stable after forgetting $\bullet$, then

$$
(\pi_\bullet)_*[(\Gamma, m_{\Gamma}, a), \alpha] = (\xi_{(\Gamma', m_{\Gamma'}, a')})_*((\pi_v)_*\alpha_v \cdot \prod_{w \neq v} \alpha_w),
$$

where $\pi_v$ is the forgetful map of $\bullet$ of the vertex $v$.

2. Suppose $v$ becomes unstable after forgetting $\bullet$. If $\alpha_v \neq 1$, then $[(\Gamma, m_{\Gamma}, a), \alpha] = 0$. If $\alpha_v = 1$, then

$$
(\pi_\bullet)_*[(\Gamma, m_{\Gamma}, a), \alpha] = [(\Gamma', m_{\Gamma'}, a'), \prod_{w \neq v} \alpha_w].
$$
The proof of Proposition 2.4 is similar to that of [6, Proposition 3.11] and is omitted.

We can study pushforwards of products of $\kappa$ and $\psi$ classes under $\pi_*$, as follows. A product
\[
\prod b \kappa_b^e \cdot \prod_{i \in I} \psi_i^{l_i} \cdot \psi_i^{l_i} \in R^*(\mathcal{M}_{g,\{\bullet\},m',a})
\]
can be written as
\[
(\pi_*)^* \left( \prod b \kappa_b^e \cdot \prod_{i \in I} \psi_i^{l_i} \right) \cdot \psi_i^{l_i} + \text{boundary terms}.
\]
This uses a straightforward analogue of [6, Proposition 3.10] and known intersection formulas on $\mathcal{M}_{g,\{\bullet\},m',a}$. Applying $(\pi_*)^*$ yields
\[
\left( \prod b \kappa_b^e \cdot \prod_{i \in I} \psi_i^{l_i} \right) \cdot \kappa_{l_i-1} + (\pi_*)^*(\text{boundary terms}).
\]
The boundary terms can be handled by induction together with Proposition 2.4.

We conclude that $(\pi_*)^*$ preserves the span of decorated stratum classes.

REFERENCES

[1] D. Abramovich, A. Corti, A. Vistoli, Twisted bundles and admissible covers, Comm. Algebra 31 (2003), no. 8, 3547–3618.
[2] D. Abramovich, T. Graber, A. Vistoli, Algebraic orbifold quantum products, in: Orbifolds in mathematics and physics (Madison, WI, 2001), 1–24, Contemp. Math., 310, Amer. Math. Soc., 2002.
[3] D. Abramovich, T. Graber, A. Vistoli, Gromov-Witten theory of Deligne-Mumford stacks, Amer. J. Math., 130(5):1337–1398, 2008.
[4] D. Abramovich, A. Vistoli, Compactifying the space of stable maps, J. Amer. Math. Soc. 15 (2002), 27–75.
[5] E. Andreini, Y. Jiang, H.-H. Tseng, Gromov-Witten theory of root gerbes I: structure of genus 0 moduli spaces, J. Differential Geom. Vol. 99, no. 1 (2015), 1–45.
[6] Y. Bae, J. Schmitt, Chow rings of stacks of prestable curves I, Forum. Math. Sigma Volume 10, e28, [arXiv:2012.09887]
[7] Y. Bae, J. Schmitt, Chow rings of stacks of prestable curves II, J. Reine Angew. Math. 800, 55–106 (2023), [arXiv:2107.09192]
[8] K. Costello, Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products, Ann. Math. (2), 164(2):561–601, 2006.
[9] T. Graber, R. Pandharipande, Constructions of non-tautological classes on moduli spaces of curves, Michigan Math. J., 51(1):93–109, 2003.
[10] M. Olsson, On (log) twisted curves, Comp. Math. 143 (2007), 476–494.