Shear hydrodynamics, momentum relaxation, and the KSS bound

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Abstract

In this paper we investigate the behavior of the shear hydrodynamic response functions in a simple holographic model exhibiting momentum relaxation. We compute several stress tensor response functions in the transverse channel, and from there derive the ratio of shear viscosity to entropy density, $\eta/s$, using two different methods. The two values differ from each other, one which satisfies the KSS bound $\eta/s \geq \frac{1}{4\pi}$ and one which does not, and we discuss the causes and implications of this result.
I. Introduction

Over the past two decades, holography has shed new light and offered new perspectives on hydrodynamics, interpreting decades of work on black hole thermodynamics in terms of the behavior of the dual field theory placed at a finite temperature. One of the more stimulating results of holographic hydrodynamics has been the empirical universality of the ratio of the shear viscosity to entropy density, $\eta/s$ \[1, 2\]. Starting with the pioneering work of Kovtun, Son, and Starinets (KSS) \[3\], the ratio has been conjectured to be bounded from below, with the initial lower bound given by

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \frac{\hbar}{k_B}.$$ \hspace{1cm} (1)

While various counterexamples to the bound as written have been found \[4–19\], the notion of a fundamental bound on physically relevant quantities, e.g. hydrodynamic quantities, remains very enticing, and various attempts have been made to either generalize the bound to incorporate the known counterexamples \[20\] or otherwise extract general lessons regarding the nature of strongly coupled dynamics \[21–24\].

Physically, one way to view the ratio is to note that, up to a factor of the temperature, it governs the diffusion of the shear momentum: $T D_\perp = \eta/s$. Indeed, as is well known (see e.g. \[25–28\]), in the hydrodynamic regime, the shear momentum density response function is given by (taking the momentum $\vec{k} = k\hat{x}$ in the $x$ direction)

$$G_{TT_{xy},TT_{xy}}^R(\omega, k) = -\eta k^2 \frac{k^2}{-i\omega + \frac{\eta k^2}{s} + \frac{\eta k^2}{s}}.$$ \hspace{1cm} (2)

By utilizing the momentum conservation Ward identity, $\eta$ can also be extracted from the shear momentum current correlators by a Kubo formula:

$$\eta = -\lim_{\omega \to 0} \frac{\Im G_{TT_{xy},TT_{xy}}^R(\omega, k = 0)}{\omega}. \hspace{1cm} (3)$$

Thus, the shear viscosity can be alternatively viewed as a measure of the low frequency spectral weight of the shear momentum current density.

A common theme of recent work in holographic transport has been the incorporation of translational symmetry breaking. Practically, momentum relaxation naturally arises in physical systems, for example via lattices or quenched disorder, and there has been a lot of recent progress incorporating momentum relaxation holographically \[29–56\]. The primary
theoretical motivation for including momentum relaxation is to resolve subtleties regarding dc transport. For example, since the heat current necessarily overlaps with the conserved momentum operator at finite temperature, translation invariance prevents the heat current from relaxing and leads to an infinite dc thermal conductivity. Holography provides a particularly useful setting in which to explore effects of momentum relaxation as the strongly interacting nature of holography precludes long-lived quasiparticles which can otherwise dominate transport. The paucity of tractable calculations involving both strong coupling and momentum relaxation as well as the possible relevance of such settings in many condensed matter contexts has made holography an attractive alternative approach.

All of the holographic studies of $\eta/s$ alluded to above were in translationally invariant settings, and therefore the shear viscosity apparently does not suffer from subtleties related to momentum conservation. Indeed, it would seem we have the opposite problem in that an unambiguous definition for $\eta$ requires translational symmetry. The key ingredient in connecting the two approaches to $\eta$ is the use of the structure enforced by hydrodynamics and the resulting Ward identities. Thus, upon breaking translation invariance, we may naturally expect that the two prescriptions for calculating $\eta$ will no longer agree. Indeed, without translational symmetry, momentum is no longer a good quantum number and it’s not clear how to apply the methods above to define $\eta$. One is lead to ask which, if any, approach physically captures the notion of shear viscosity in the presence of momentum relaxation.

Of the two interpretations discussed above, the second is well defined. Namely the spectral weight of a particular component of the stress tensor makes perfect sense (provided one has a stress tensor), even if it is no longer a component of a conserved current. This perspective and the role that $\eta/s$ plays in entropy production was recently discussed in detail in [12].

On the other hand, the first interpretation is more subtle. Once translation invariance is broken, momentum no longer difffuses and so a corresponding ‘diffusion constant’ is not well defined. One expects, and we will see explicitly below, that a non-zero momentum relaxation rate $\Gamma$ will shift the hydrodynamic pole into the lower half plane. However, for a sufficiently weak momentum relaxation rate $\Gamma \ll T$, an approximate diffusion constant can be defined for length scales $\ell \ll \Gamma^{-1}$, as long as the length scale is large enough such that hydrodynamics can be trusted. Practically, one can simply search for the pole in the shear momentum density correlators and extract a diffusion constant from the momentum dependence of the
dispersion relation. However, the relation, if any, of this quantity to the $\eta$ obtained by the Kubo formula is not obvious \cite{28}. The interplay between shear momentum transport and momentum relaxation, particularly in the presence of spontaneous translational symmetry breaking, has also recently been discussed in \cite{57}.

In this brief note, we will investigate the behavior of the shear hydrodynamic response functions in a simple holographic model incorporating momentum relaxation. Our primary objective is to compare the holographic results to simple hydrodynamic arguments and also compare the notions of the shear viscosity discussed above. In particular, our key result is that the value of $\eta/s$ differs depending on whether it is obtained via the Kubo formula or via the pseudo-diffusive pole in the shear momentum density two-point function. This is not entirely surprising as the equivalence of these two approaches in traditional hydrodynamics relies on momentum conservation, which we are manifestly breaking. Explicitly, we can see that the Kubo formula calculates the spectral weight strictly at $k = 0$, and for any small but fixed momentum relaxation rate the momentum will have decayed at these length scales. Interestingly, the value obtained via the pseudo-diffusive pole satisfies the KSS bound (1), while the Kubo formula yields a value of $\eta/s < \frac{1}{4\pi}$ as shown in \cite{28}.

The rest of the paper is organized as follows. In Sec. II we study the effect of momentum relaxation in a simple hydrodynamic toy model, obtaining predictions for the retarded shear correlation functions $G_{\tau\tau}(\omega, k)$ and $G_{x\tau x\tau}(\omega, k)$. Throughout the paper we focus on $2 + 1$ dimensional systems for concreteness. In Sec. III we introduce the holographic model which will be our focus for the remainder of the paper and review its known thermodynamic and transport properties. In the remaining two sections, Sec. IV A and Sec. IV B, we evaluate the shear correlators holographically, working perturbatively in the momentum relaxation rate, and these results are compared to the predictions by our hydrodynamic toy model. Various technical details of the calculations are relegated to appendices.

II. Shear hydrodynamics and momentum relaxation

Before turning to our holographic analysis, we investigate a simple mechanism for momentum relaxation in the context of relativistic hydrodynamics. In this scenario, standard arguments due to Kadanoff and Martin \cite{58} allow for evaluation of the retarded two-point functions of interest. For modern discussions of the approach, see for example \cite{27, 28, 59}.
For simplicity, we focus on neutral, conformal fluids. The hydrodynamic equations of motion, i.e. the energy-momentum conservation equations, read

$$\partial_\mu T^\mu = 0. \quad (4)$$

These equations, when supplemented with the appropriate constitutive relations, govern the relaxation of energy and momentum fluctuations. We concentrate on the shear channel, taking the fluctuations to be plane waves with momentum along the $x$ direction. More explicitly, we turn on a small source for the transverse momentum density $P_y \equiv T^{ty}$, namely a velocity fluctuation $\delta v^y(t, x)$. To lowest order in $\omega$ and $k$, linear response and the constitutive relations express the fluctuations of the momentum density, $\delta P_y$, and momentum current, $\delta T^{xy}$, in terms of $\delta v^y$ as:

$$\delta P_y(\omega, k) = \chi_{PP} \delta v^y(\omega, k) = (\epsilon + P) \delta v^y(\omega, k), \quad \delta T^{xy}(\omega, k) = -i\eta k \delta v^y(\omega, k). \quad (5)$$

Here $\chi_{PP} = \epsilon + P = sT$ is the static susceptibility relating the source and response fluctuations, and $\epsilon, P, \eta, s$ are the energy density, pressure, shear viscosity, and entropy density respectively. Using these relations in the (Fourier transformed) momentum conservation equation $\partial_t \delta P_y + ik\delta T^{xy} = 0$ yields a diffusion equation for the fluctuation $\delta P_y$ with diffusion constant $D_\perp = \frac{\eta}{\epsilon + P} = \frac{\eta}{sT}$.

The analysis so far assumes momentum conservation. However, we are interested in the consequences of momentum relaxation. For simplicity, we assume that this perturbation does not change the form of the constitutive relation and that to lowest order its only effect is to modify the conservation equation. A simple ansatz for this effect is to simply add a constant momentum relaxation rate to the conservation equation

$$\partial_t \delta P_y + ik\delta T^{xy} = -\Gamma \delta P_y. \quad (6)$$

Using the constitutive relations (5), we can rewrite this result as

$$\partial_t \delta P_y + \left[\Gamma(\epsilon + P) + \eta k^2\right] \delta v^y = 0. \quad (7)$$

Appealing now to standard results of linear response [25, 27, 28], the momentum density retarded Green’s function is given by

$$G^R_{P_y, P_y}(\omega, k) = -(\epsilon + P) \frac{\Gamma + \frac{\eta}{\epsilon + P} k^2}{-i\omega + \Gamma + \frac{\eta}{\epsilon + P} k^2}. \quad (8)$$
As expected, the diffusive pole in the shear momentum correlator has been pushed further into the lower half of the complex plane by a constant amount set by the momentum relaxation rate $\Gamma$.

Unfortunately, there is no analogous approach to determine $G^R_{T\tau y, T\tau y}$ in the presence of momentum relaxation. When $\Gamma = 0$, momentum conservation implies corresponding Ward identities, which must hold as operator identities and allow us to determine $G^R_{T\tau y, T\tau y}$ from $G^R_{T\tau y, T\tau y}$ (up to contact terms). However, our modified conservation equation (6) is not such an operator identity, at least without more knowledge about the mechanism of momentum relaxation, and therefore we cannot use the same techniques. Furthermore, without a dynamical equation of motion for $T^{\tau y}$, which would go beyond our hydrodynamic considerations here, direct application of the Kadanoff-Martin approach is not possible.

III. Neutral linear axion model

In this section we introduce the holographic model that will be our focus for the remainder of this note. These models, known as linear axion models (among other names), were introduced in [60] as simple examples in which translational symmetry on the boundary can be broken while retaining a homogeneous metric, and they have been served as useful models to investigate the effects of momentum relaxation in recent years [10, 13, 28, 61, 62]. The homogeneity of the metric reduces the bulk equations of motion reduce to ODEs rather than PDEs, dramatically simplifying calculations.

The model consists of two massless scalars, $\phi^i$ with $i = 1, 2$, minimally coupled to gravity, with the action given by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{6}{L^2} - \frac{1}{2} \sum_i (\partial_\alpha \phi^i)(\partial^\alpha \phi^i) \right] + S_{GH} + S_{ct}.$$  \hspace{1cm} (9)

Here $\kappa^2 = 8\pi G_N$ is the gravitational coupling constant and $L$ is the AdS radius (which we will set to 1 throughout). This bulk action has to be supplemented with additional terms that depend only on the boundary data, namely the Gibbons-Hawking boundary term $S_{GH}$ and a counterterm action $S_{ct}$. These terms, which are only needed for the analysis of the on-shell action are given in Appendix A. The equations of motion following from the bulk
The action reads

\[ R_{ab} + 3g_{ab} = \frac{1}{2} \sum_i \partial_a \phi^i \partial_b \phi^i , \quad (10) \]

\[ \square \phi^i = 0 . \quad (11) \]

A simple class of solutions to the bulk equations of motion can be found with the following form

\[ ds^2 = \frac{1}{r^2} \left[ -f(r) dt^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2 \right] , \quad (12) \]

\[ \phi^i = m x^i , \quad (13) \]

\[ f(r) = 1 - \frac{m^2}{2} r^2 - \left( 1 - \frac{m^2 r_+^2}{2} \right) \left( \frac{r}{r_+} \right)^3 . \quad (14) \]

We’ll often rescale the radial coordinate to \( u = r/r_+ \), so the boundary and horizon are at \( u = 0 \) and \( u = 1 \) respectively. Since the \( \phi^i \) are massless scalar fields, the holographic dictionary tells us that they correspond to sources for marginal operators on the boundary (\( \Delta = d + 1 = 3 \)). Therefore this background describes a scenario where a conformal fluid at finite temperature is deformed by two marginal operators with sources linear in the spatial directions. These sources thus explicitly break the boundary translational symmetry, where the strength of the symmetry breaking is characterized by \( m \).

The conformal boundary is located at \( r \to 0 \) (\( u \to 0 \)), while there is a non-degenerate horizon for \( m^2 r_+^2 < 6 \). The horizon radius determines the thermodynamic properties via

\[ 4 \pi T = \frac{1}{r_+} \left( 3 - \frac{m^2 r_+^2}{2} \right) , \quad s = \frac{1}{4G_N r_+^2} = \frac{32 \pi^3 T^2}{9 \kappa^2} . \quad (15) \]

As mentioned above, this class of backgrounds have proven useful in studies of the role of momentum relaxation in transport behavior. For small \( m \), i.e. \( m \ll T \), the momentum is almost conserved and this long-lived quantity will essentially determine the transport. This scenario falls under the name of coherent transport. However, the background specified by \((12)\)–\((14)\) exists for any \( m^2 r_+^2 < 6 \), and so by dialing the parameter \( m \), we can tune from coherent transport to incoherent transport. For a detailed analysis explicitly demonstrating the coherent-to-incoherent crossover in the thermal transport of these models, see [59].

As discussed in [34, 36, 63], the breaking of translational symmetry on the boundary is manifested in the bulk as giving a mass to the graviton. In particular, [12] emphasized how
a non-zero graviton mass generically leads to a violation of the KSS bound on the shear viscosity to entropy density ratio. For the linear axion models, the KSS ratio was shown in [12] to be given, perturbatively in $m/T$, by

$$4\pi \frac{\eta}{s} = 1 + \frac{\sqrt{3}}{16\pi} \left( 1 - \frac{3\sqrt{3}\log 3}{\pi} \right) \left( \frac{m}{T} \right)^2 + \mathcal{O} \left( \left( \frac{m}{T} \right)^4 \right). \quad (16)$$

As discussed in the introduction, while an obvious hydrodynamic interpretation of the shear viscosity breaks down in the absence of translational symmetry, it was argued that the shear viscosity, as defined by the Kubo formula, retains a fundamental interpretation as determining the low energy spectral weight and hence the rate of entropy production when subjected to a slowly varying strain $\delta g_{xy}^{(0)}$.

In the following sections, we will move away from the strict $\omega = 0$ transport behavior and study the low $\omega, k$ shear correlation functions perturbatively in $m$. The structure of these correlators is determined, at $m = 0$, by hydrodynamics, and in particular momentum conservation, for sufficiently small $\omega, k$. Therefore, we expect that there will be non-trivial interplay between the $m \to 0$ and $\omega \to 0$ limits, since for any finite $m$, the momentum is no longer conserved and therefore decays at late times.

IV. Calculating correlators

In this section we will apply standard holographic techniques to determine the correlator $G^{R}_{T^y T^y}$ in the backgrounds discussed in the previous section [28, 64, 67]. The basic strategy is to perturb our background solutions, which corresponds to turning on a small source on the boundary, solve the linearized bulk equations of motion with appropriate boundary conditions at the conformal boundary and ingoing boundary conditions at the horizon, and extract the response of the system from the near boundary behavior. Since we are interested in correlators of the shear stress tensor components $T^{ty}$ and $T^{xy}$, we will focus on the behavior of the shear metric fluctuations $\delta g_{\mu\nu} = h_{\mu\nu}$, which are coupled via the equations of motion to the scalar fluctuations $\delta \phi^2$.

Exploiting the homogeneity of our background solution, we Fourier decompose our fluctuations and take them to have the form $\delta X = X(u)e^{-i(\omega t - kx)}$, where $X \in \{ h_{ty}, h_{xy}, \delta \phi^2 \}$ and we’ve also used isotropy to set the momentum along the $x$ direction without loss of
generality. The linearized equations of motion for the shear modes then read

\begin{align*}
0 &= \frac{u^2}{r_+^2} \frac{d}{du} \left( \frac{f h''_x}{u^2} \right) + \frac{\omega}{f} (\omega h'_x + k h'_y) - m^2 h'_x + i k m \delta \phi_2, \quad (17) \\
0 &= \frac{u^2}{r_+^2} \frac{d}{du} \left( \frac{f h''_t}{u^2} \right) - k \frac{\omega}{f} (\omega h'_x + k h'_y) - \frac{m^2}{f} h'_t - \frac{i \omega m}{f} \delta \phi_2, \quad (18) \\
0 &= \frac{u^2}{r_+^2} \frac{d}{du} \left( \frac{f \delta \phi'_2}{u^2} \right) + \frac{1}{f} (\omega^2 - k^2 f) \delta \phi_2 - \frac{i m}{f} (\omega h'_t + k f h'_x), \quad (19) \\
0 &= i \omega h''_t + i k f h''_x - m f \delta \phi'_2. \quad (20)
\end{align*}

Here the primes denote \( u \) derivatives, indices are raised using the background solution from
the previous section, and for simplicity, we’ve used the gauge freedom to set \( h'_y = 0 \).

The strategy for extracting the low \( \omega, k \) behavior will be as follows. We impose the
infalling boundary conditions at the horizon by writing the radial profile of the fluctuations
as \( X(u) = f(u)^{-i \omega/4 \pi T} \tilde{X}(u) \). Plugging this form into the wave equations above we obtain a
new set of equations for \( \tilde{X} \), which we will then solve perturbatively in \( \omega, k, \) and \( m \). Once we
have these solutions, we can examine their near boundary behavior to extract the Green’s
functions of interest.

A. Solving for \( G_{T^y, T^y} \)

We first look at \( G^R_{T^y, T^y} \). This correlator is determined by the mode \( h_{ty} \), so we want
to decouple \( h_{ty} \) from \( h_{xy} \) and \( \delta \phi^2 \). This is easily accomplished by differentiating (18) and
combining with (20), which yields a third order ODE for \( h_{ty} \). If we define \( \psi_{ty} = -\frac{1}{r_+ u} h''_t \),
we can write this equation as

\begin{equation}
0 = \frac{d}{du} (f \psi'_{ty}) + \frac{r_+^2 \omega^2 - r_+^2 k^2 f - r_+^2 m^2 f - \frac{1}{u} f f'}{f} \psi_{ty}. \quad (21)
\end{equation}

In order to bring the equation to a form easier to solve we will introduce the dimensionless variables

\begin{equation}
w := \frac{\omega}{2 \pi T}, \quad q := \frac{k}{2 \pi T}, \quad M := \frac{m}{2 \pi T}, \quad R_+ := 2 \pi r_+ = 3 - \frac{M^2 R_+^2}{4} + O(M^4). \quad (22)
\end{equation}

Note that since \( R_+ \) is defined implicitly, to work perturbatively in \( M \), we will expand
\( R_+ = \frac{3}{2} - \frac{9 M^2}{16} + O(M^4) \).

With the new dimensionless variables, the ODE and emblackening factor reads

\begin{equation}
0 = \psi''_{ty} + \frac{f'}{f} \psi'_{ty} + \frac{R_+^2 (w^2 - q^2 f - M^2 f) - \frac{1}{w} f f'}{f^2} \psi_{ty}, \quad (23)
\end{equation}

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\[ f(u) = 1 - \frac{1}{2} M^2 R_+^2 u^2 - \left( 1 - \frac{M^2 R_+^2}{2} \right) u^3. \]  

As described above, in order to deal with the singular point of the ODE we scale out the singular power law behavior at the horizon \( u \to 1 \), writing \( \psi_{ty}(u) = f(u)^{-iw/2} \tilde{\psi}_{ty}(u) \) to obtain the following ODE for \( \tilde{\psi}_{ty} \)

\[ 0 = \tilde{\psi}_{ty}'' + (1 - iw) \frac{f'}{f} \tilde{\psi}_{ty}' + \left[ -\frac{1}{u} \frac{f'}{f} - i \frac{w}{2} f'' + \frac{w^2}{f^2} \left( R_+^2 - \frac{f'^2}{4} - \frac{R_+^2 (q^2 + M^2)}{f} \right) \right] \tilde{\psi}_{ty}. \]  

We are now ready to solve this equation perturbatively. To do so, we simply expand

\[ \tilde{\psi}_{ty} = \tilde{\psi}_{ty}(0) + iw \tilde{\psi}_{ty}(1) + q^2 \tilde{\psi}_{ty}(2) + iw M^2 \tilde{\psi}_{ty}(4) + q^2 M^2 \tilde{\psi}_{ty}(5) + \cdots. \]  

Plugging this into (25), we can solve order by order in \( w, q, \) and \( M \). Note that the factors of \( R_+ \), including those in the emblackening factor, depend on \( M \), so for consistency one needs to expand it appropriately as well.

Once we have the solution, we want to expand near the boundary \( u = 0 \) to extract the Green’s function. By expanding (21) near \( u = 0 \), or using the known asymptotics of \( h_{\mu\nu} \) and tracing through the definition of \( \psi_{ty} \), we see that near the boundary \( \psi_{ty} \) must take the schematic form \( \psi_{ty} \sim A + Bu \). More explicitly, if we write the near boundary metric perturbations as

\[ h_t^y = h_t^{y(0)} + uh_t^{y(1)} + u^2 h_t^{y(2)} + u^3 h_t^{y(3)} + \cdots, \]  

we can use the linearized equations of motion to find \( h_t^{y(1)} = 0 \) and \( h_t^{y(2)} = r_+^2 \frac{k^2 + m^2}{2} h_t^{y(0)} \), which upon plugging into our definition of \( \psi_{ty} \) tells us that \( A = 2h_t^{y(2)} \) and \( B = 3h_t^{y(3)} \). Since \( h_t^{y(3)} \) is the first variation of the on-shell action with respect to the source \( h_t^{y(0)} \) (see Appendix A), the definition of the retarded Green’s function via linear response indicates

\[ G_{T^t, T^y}^R(\omega, k) = \frac{3}{2r_+^2 \kappa^2} \frac{h_t^{y(3)}}{h_t^{y(0)}} = \frac{k^2 + m^2}{2r_+ \kappa^2} \frac{B}{A}. \]

Here the factor of \( r_+^3 \) comes from expanding near the boundary in \( u \) instead of \( r \).

To spare the reader, we’ve collected the details of obtaining the solution in Appendix B and we’ll only quote the final result, which reads

\[ C^{-1} \psi_{ty}(u) = u + \frac{3}{4} (1 - u) \left( q^2 + M^2 \right) \]

\[ -\frac{1}{12} iw \left\{ 18 + u \left[ 6 \sqrt{3} \tan^{-1} \left( \frac{1 + 2u}{\sqrt{3}} \right) + 9 \log \left( \frac{3}{1 + u + u^2} \right) - 18 - 2\sqrt{3} \right] \right\} \]  

\[ \text{(26)} \]

\[ \text{(27)} \]

\[ \text{(28)} \]
Here $C$ is an integration constant that sets the scale of the source we’ve turned on. Expanding this near the boundary we find $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} = C \left[ -\frac{3}{2} i w + \frac{3}{4} (k^2 + M^2) + \cdots \right],$$

(30)

$$\mathcal{B} = C \left[ 1 + \frac{i w}{12} \left( 18 + \sqrt{3\pi} - 9 \log 3 \right) - \frac{3}{4} (q^2 + M^2) \right].$$

(31)

From the expression for $\mathcal{A}$, we can easily find the location of the poles in $G_{T^\mu T^\nu}$:

$$w = -\frac{1}{2} (q^2 + M^2),$$

(32)

or, reinstating the factors of $T$ from (22) we get that

$$\omega = -i \left[ \frac{k^2}{4\pi T} + \frac{m^2}{4\pi T} \right].$$

(33)

Here we see that, as expected, the diffusive pole gets pushed into the lower half plane by the momentum relaxation.

All in all, we can write the shear momentum correlator as

$$G_{T^\mu T^\nu}(\omega, k) = \frac{1}{2 \kappa^2} \frac{16\pi^2 T^2}{9} \frac{k^2 + m^2}{-i\omega + \frac{k^2 + m^2}{4\pi T}}.$$ 

(34)

As an aside, we note that with this expression for $G_{T^\mu T^\nu}$, we can readily obtain the the optical thermal conductivity via the Kubo formula

$$\kappa_T(\omega) = \frac{1}{T} \frac{G_{T^\mu T^\nu}(\omega, k = 0) - G_{T^\mu T^\nu}(0, k = 0)}{i\omega} = \frac{32}{9\kappa^2} \frac{\pi^3 T^2}{-i\omega + \frac{m^2}{4\pi T}} = \frac{s}{9\kappa^2} \frac{\pi^3 T^2}{-i\omega + \frac{m^2}{4\pi T}}.$$ 

(35)

For the last equality, we’ve used our expression for the entropy density (15). Here we see explicitly that the momentum relaxation rate resolves the divergence in the DC thermal conductivity, leading to a finite Drude peak with width $\mathcal{O}(m^2/T)$, as discussed extensively in [59].

In the solution above, we see that the “diffusion” constant, that is the coefficient of $k^2$ in the pole (33), is unchanged from the $\text{AdS}_4$ value, $\frac{\rho}{sT} = \frac{1}{4\pi T}$. To see how the momentum relaxation affects this “diffusion”, one needs to continue to higher orders, including terms of

1 Here we denote that thermal conductivity by $\kappa_T$ in order to differentiate it from the gravitational coupling $\kappa^2$. 
order $M^2w$ and $M^2q^2$. It is in fact possible to solve this equation non-perturbatively in $M$, to lowest order in $w$ and $q$. That is, taking $\tilde{\psi}_{ty} = \tilde{\psi}_{ty}^{(0)}(u; M) + w\tilde{\psi}_{ty}^{(1)}(u; M) + q^2\tilde{\psi}_{ty}^{(2)}(u; M)$, and substituting into (25), one can solve for $\tilde{\psi}_{ty}^{(0)}(u; M)$, $\tilde{\psi}_{ty}^{(1)}(u; M)$, and $\tilde{\psi}_{ty}^{(2)}(u; M)$ with generic $M$. As the expressions are rather unwieldy, we won’t present this full solution, though we present the necessary equations in Appendix B. The “hydrodynamic” pole is obtained by finding the zero in $A$, and working to the next non-trivial order in $M^2$, one finds the following for $A$

\begin{equation}
C^{-1}A = -\frac{3}{2} \left[ 1 - \frac{1}{24} \left( 9\log 3 - 9 - 3\sqrt{3}\pi \right) M^2 \right] iw + \frac{3}{4} q^2 + \frac{3}{4} \left( 1 - \frac{3}{8} M^2 \right) M^2 + \cdots ,
\end{equation}

which gives a pole at

\begin{equation}
iw = \frac{q^2}{2} \left[ 1 + \frac{1}{24} \left( 9 + \sqrt{3}\pi - 9\log 3 \right) M^2 \right] + \frac{M^2}{2} + O(M^4).
\end{equation}

Therefore, upon restoring factors of $T$, we see that the “diffusion” constant is given by

\begin{equation}
D_\perp = \frac{1}{4\pi T} \left[ 1 + \frac{1}{24} \left( 9 + \sqrt{3}\pi - 9\log 3 \right) M^2 \right] + \cdots .
\end{equation}

Comparing this with the result found in [12], reproduced in (16), we see that the ratio $4\pi\eta/s$ obtained from the “diffusion” constant of the pole in $G_{Tty,Tty}^R$ does not agree with the value found via the Kubo formula. However, it is curious to note that the value (38) satisfies the KSS bound (1).

B. Calculating $G_{Txy,Txy}$

We now turn to the correlator $G_{Txy,Txy}^R$. Unlike the $T^{ty}$ case, here it won’t be possible to fully decouple the metric and scalar fluctuations for $k \neq 0$. We’ll start by considering the simpler $k = 0$ case, which in any case will allow us to rederive the results for the shear viscosity in [12].

1. $k = 0$

At $k = 0$, the $h_x^y$ equation of motion completely decouples $h_x^y$ from the other modes, as can be easily read off from (17). With a little rearranging, the $k = 0$ equation reads

\begin{equation}
0 = \frac{d}{du} \left( \frac{fh_x^y u^2}{u^2} \right) + \frac{r_x^2 (\omega^2 - m^2 f)}{u^2 f} h_x^y .
\end{equation}
Again, a quick check of the on-shell action (see Appendix A) confirms that the Green's function can be read off of the solution (satisfying infalling boundary conditions at the horizon) to this equation via

\[ G^{R}_{T^{x}T^{y}} = \frac{3}{2r_{+}^{3}\kappa^{2}} \frac{h_{x}^{y(3)}}{h_{y}^{y(0)}} = \frac{3(m^{2} - \omega^{2})}{2r_{+}^{2}\kappa^{2}} \frac{h_{x}^{y(3)}}{h_{y}^{y(2)}}. \]  

Here we’ve used the equations of motion to relate the subleading behavior \( h_{x}^{y(2)} \) to the leading behavior \( h_{x}^{y(0)} \), simply as a technical convenience. With the two point function in hand, we can compute the shear viscosity and modulus via:

\[ \eta = -\lim_{\omega \to 0} \frac{\text{Im} \ G^{R}_{T^{x}T^{y}}(\omega, k = 0)}{\omega}, \quad G = \lim_{\omega \to 0} \text{Re} \ G_{T^{x}T^{y}}(\omega, k = 0). \]  

To explicitly determine the two point function, we solve (39) perturbatively as in the previous section. The details are relegated to the appendix, but to orient ourself we note that the object we’ll need is the ratio \( R = \frac{h_{x}^{y(2)}}{h_{y}^{y(0)}} \), which we will expand for small \( \omega \) as

\[ R = a + ib + \epsilon w + ifw + gw^{2} + ihw^{2} + \mathcal{O}(w^{3}). \]  

where we \( a, \ldots, h \) are real, \( m \)-dependent constants. Note in particular that we are taking the small \( \omega \) limit first. Time reversal symmetry implies that \( \text{Im} \ G_{T^{x}T^{y}}(\omega) \) is odd in \( \omega \), and so we must take \( b = 0 \) (as can be explicitly checked). Therefore we can expand the two point function as

\[ G^{R}_{T^{x}T^{y}}(\omega, k = 0) = \frac{3m^{2}}{2\kappa^{2}a} \left[ 1 - \frac{\epsilon + if}{a} \frac{\omega}{2\pi T} + \mathcal{O}(\omega^{2}) \right]. \]  

Using this result and (41), we can write the shear viscosity and modulus as

\[ \eta = \frac{3m^{2}}{4\pi T\kappa^{2}a} \frac{f}{a^{2}}, \quad G = \frac{3m^{2}}{2\kappa a}. \]  

As we’ll see shortly, \( a \sim \mathcal{O}(1) \) while \( f \sim m^{-2} \) for small \( m \), and so the \( m \to 0 \) limit reproduces the expected results for the planar AdS black brane.

Now let’s solve (39) for \( h_{y}^{y}(u) \) exactly as we did for \( \psi_{ty} \) in the previous section. Namely, we can take the ansatz \( h_{x}^{y}(u) = f(u)^{-\frac{i\pi}{2}} F(u) \), to impose infalling boundary conditions, and expand the remaining factor in \( w \) and \( x \)

\[ F(u) = F_{0}(u) + wF_{1}(u) + \sum_{n=1}^{N} M^{2n} H_{n}(u) + \sum_{n=1}^{N} w M^{2n} J_{n}(u) + \ldots. \]  

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The explicit solutions for $F_0(u), F_1(u)$, etc. can be found in Appendix C. Expanding these near $u = 0$ we can find $h_y^{(3)}$ and $h_y^{(2)}$ and thus $G_{T^{xy}, T^{xy}}^R$.

Using the solutions in Appendix C we find the small $x$ behavior

$$a = -\frac{3}{2} + \frac{1}{4}(\sqrt{3} - 3 \log 3)M^2 + O(M^3),$$  

$$f = \frac{9}{4x^2} - \frac{1}{8}(3 + 3\sqrt{3}) + O(x^2).$$

Putting together the results, we obtain:

$$4\pi \frac{\eta}{s} = 1 + \left(\frac{\pi}{3\sqrt{3}} - \log 3\right)\left(\frac{3m}{4\pi T}\right)^2 + O(x^3),$$

which agrees with the shear viscosity obtained in [12], and

$$G = -\frac{4\pi T}{3}m^2 - \frac{1 + \pi\sqrt{3} - 3 \log 3}{8\pi T}m^4 + O(m^5).$$

Here we see that the shear viscosity as computed by the Kubo formula disagrees with the value obtained from $G_{T^{xy}, T^{xy}}^R$ and that the shear modulus is non-trivial in these backgrounds. This result on the shear modulus has been previously emphasized in these models in [13, 61].

2. $k \neq 0$

Now we relax the condition $k = 0$. By taking appropriate combinations of (17) and (20), we obtain the following equation for $\psi_{xy} = \frac{L}{h_y^{(2)} x}$

$$0 = \partial_u (f \partial_u \psi_{xy}) - R_+^2 \left( M^2 + q^2 - \frac{u^2}{f} + \frac{f'}{R_+^2} u \right) \psi_{xy} - \frac{M^2 R_+^2 f'}{u} \varphi,$$

where $\varphi = h_y^{(2)} - i\frac{\varphi}{M}_2$. We see that the scalar fluctuation doesn’t fully decouple from the $h_y^{(2)}$ fluctuation, and so we also need to solve for $\varphi$, using its equation of motion

$$0 = u^2 \partial_u \left( \frac{f \partial_u \varphi}{u^2} \right) - R_+^2 \left( M^2 + q^2 - \frac{u^2}{f} \right) \varphi.$$

To extract the Green’s function from $\psi_{xy}$, we proceed as we did for the $t y$ fluctuations and note that its near boundary expansion is given by

$$\psi_{xy} = 2h_y^{(2)} + 3h_y^{(3)} u + \cdots,$$

and so using $2h_y^{(2)} = r_+^2(\omega^2 - m^2)h_y^{(0)}$, we have

$$G_{T^{xy}, T^{xy}}^R = \frac{3}{2} \frac{h_y^{(2)} h_y^{(3)}}{h_y^{(0)} \omega^2 - m^2 B} = \frac{2r_+^2 \kappa^2}{A},$$
where $B = \psi'_{xy}(0) = 3h^y_{x}$ and $A = \psi_{xy}(0) = 2h^y_{x}$.

To impose the infalling boundary conditions, we write $\psi_{xy}(u) = f^{-i\mathcal{w}(u)}\tilde{\psi}_{xy}(u)$ and $\varphi(u) = f^{-i\mathcal{w}}\tilde{\varphi}(u)$ and expand as:

\[
\tilde{\psi}_{xy}(u) = \tilde{\psi}_{xy}^{(0)}(u) + w\tilde{\psi}_{xy}^{(1)}(u) + q^2\tilde{\psi}_{xy}^{(2)}(u) + M^2\tilde{\psi}_{xy}^{(3)}(u) + \cdots ,
\]

\[
\tilde{\varphi}(u) = \tilde{\varphi}^{(0)}(u) + w\tilde{\varphi}^{(1)}(u) + q^2\tilde{\varphi}^{(2)}(u) + M^2\tilde{\varphi}^{(3)}(u) + \cdots .
\]

The equations governing $\tilde{\psi}_{xy}^{(0)}$, $\tilde{\psi}_{xy}^{(1)}$, and $\tilde{\psi}_{xy}^{(2)}$ are the same as the corresponding ones for $\tilde{\psi}_{ty}^{(i)}$ (as they must by symmetry), and one can easily find $\tilde{\varphi}^{(0)} = C_1$ (see Appendix C). The only ODE which is changed is the one for $\tilde{\psi}_{xy}^{(3)}$:

\[
0 = \left[(1-u^3)\partial^2_{uu} - 3u(u\partial_u - 1)\right]\tilde{\psi}_{xy}^{(3)} - \frac{9}{4} \left(\tilde{\psi}_{xy}^{(0)} + 3u\tilde{\varphi}^{(0)}\right).
\]

The solution to the above equation, requiring that non-analytic terms vanish and that $\tilde{\psi}_{xy}^{(3)}(u = 1) = 0$, is:

\[
\tilde{\psi}_{xy}(u) = \frac{3}{4}(C_0 - 3C_1),
\]

where the constants $C_0$ and $C_1$ are determined by $\tilde{\psi}_{xy}^{(0)} = C_0u$ and $\tilde{\varphi}^{(0)} = C_1$.

Combining this with the expressions for $\tilde{\psi}_{xy}^{(i)} = \tilde{\psi}_{ty}^{(i)}$ (for $i < 3$) from Appendix C, we can readily expand $\psi_{xy}$ near the boundary to find an expression of the form $\psi_{xy}(u) = A + Bu + \cdots$, where

\[
A = -\frac{3}{2} iwC_0 + \frac{3}{4} q^2 C_0 + \frac{3}{4} (C_0 - 3C_1) M^2,
\]

\[
B = C_0 + \frac{iw}{4} C_0 \left[6 + \frac{\pi\sqrt{3}}{3} - 3\log 3\right] - \frac{3}{4} C_0 q^2 - \frac{3}{4} (C_0 - 3C_1) M^2.
\]

The values of $C_0$ and $C_1$ can be fixed in terms of the source $h^y_{x}(0)$ by recalling the definitions of $\varphi$ and $A$

\[
\varphi(u = 0) = h^y_{x}(0) = C_1,
\]

\[
\psi_{xy}(u = 0) = A = r^2_+ (m^2 - \omega^2) h^y_{x}(0) = r^2_+ (m^2 - \omega^2) C_1.
\]

Using our expression for $A$, (58), we obtain

\[
\frac{C_1}{C_0} = \frac{-2iw + q^2 + M^2}{3(2M^2 - w^2)}.
\]
Here we’ve used $4\pi Tr_+ \sim 3$, which is valid since to the order at which we are working. With this ratio in hand, we can readily calculate the Green’s function

$$G_{T^{xy},T^{xy}}(\omega, k) = \frac{4\pi T}{3} \frac{2m^2 - \omega^2}{-i\omega + \frac{k^2 + m^2}{4\pi T}}.$$  \hspace{1cm} (63)

We note that this result does not reproduce the values for $\eta/s$ we found directly at $k = 0$, as we have not worked to high enough order. Furthermore the $m \to 0$ limit and the $\omega \to 0$ limits don’t commute and to obtain a sensible $\eta/s$, one must first send $m \to 0$ and then calculate $\eta$, which of course yields $\eta/s = 1/4\pi$. It would be insightful to continue the calculation of $G_{T^{xy},T^{xy}}^{R}$ to higher order in $M$, but we were unable to do so.

Acknowledgments

We are grateful to Luca Delacrétaz and Sean Hartnoll for insightful discussions. TC was funded for part of the project by the Stanford Physics Department.

Appendix

A. On-shell Action

In order to be self contained, we present in this appendix the terms in the on-shell action needed to compute the Green’s functions found in the text. As usual, the bulk action must be supplemented by the Gibbons-Hawking term and the appropriate counter-terms to render the action finite as the cutoff surface at $r = \epsilon$ is sent to $r = 0$ [68, 69]. For convenience we reproduce these terms here:

$$S_{GH} + S_{ct} = -\frac{1}{2\kappa^2} \int d^3 x \sqrt{-\gamma} \left[-2K + R[\gamma] + \frac{4}{L^2} - \frac{1}{2} \sum_i \gamma^{ab} \partial_a \phi^i \partial_b \phi^i \right].$$ \hspace{1cm} (A1)

Here $\gamma$ is the induced metric on the conformal boundary, $K$ is the trace of the extrinsic curvature of the boundary, and $R[\gamma]$ is the Ricci scalar of the induced metric.

To find the Green’s functions, we evaluate the full action, both the bulk and boundary terms, on solutions to the equations of motion to quadratic order in the fluctuations. As is typically the case, the full action actually evaluates to a boundary term; in particular, the bulk terms can actually be rewritten as a total derivative up to the equations of motion. To
evaluate, we use the near boundary expansions,

\[
\begin{align*}
    h^y_t(r) &= h^{y(0)}_t + rh^{y(1)}_t + r^2h^{y(2)}_t + r^3h^{y(3)}_t + \cdots, \\
    h^y_x(r) &= h^{y(0)}_x + rh^{y(1)}_x + r^2h^{y(2)}_x + r^3h^{y(3)}_x + \cdots, \\
    \delta\phi_2(r) &= \delta\phi^{(0)}_2 + r\delta\phi^{(1)}_2 + r^2\delta\phi^{(2)}_2 + r^3\delta\phi^{(3)}_2 + \cdots,
\end{align*}
\]

(A2)  
(A3)  
(A4)

and note that the equations of motion automatically only the coefficients of \( r^0 \) and \( r^3 \) are independent, e.g. \( h^{y(1)}_t \) and \( h^{y(2)}_t \) are fixed in terms of \( h^{y(0)}_t \) while \( h^{y(3)}_t \) is not. However, this independence is short lived as we also have to impose the appropriate boundary conditions at the horizon, which imposes a relation between the two coefficients. Nevertheless, the dictionary tells us that the leading terms correspond to sources and we’ll see shortly that the \( \mathcal{O}(r^3) \) terms encode the responses. Since we are only interested in the stress tensor correlators, we set \( \delta\phi^{(0)}_2 = 0 \).

Carrying through the exercise of plugging these expansions into the action, we find (up to terms quadratic in the sources, which correspond to contact terms we ignore)

\[
S_{\text{on-shell}} = \frac{3}{4\kappa^2} \int \frac{\text{d}\omega \text{d}k}{(2\pi)^2} \left[ -h^{y(0)}_t(-\omega,-k)h^{y(3)}_t(\omega,k) + h^{y(0)}_x(-\omega,k)h^{y(3)}_x(\omega,k) \right].
\]

(A5)

Here we see explicitly that the response, obtained by the derivative of the action with respect to the source, is directly given by \( h^{y(3)}_t \) and \( h^{y(3)}_x \), and so upon dividing by the source, we obtain the expressions given for the Green’s functions in the text

\[
G^{R}_{T_tT_t}(\omega,k) = \frac{3}{2\kappa^2} \frac{h^{y(3)}_t}{h^{y(0)}_t}, \quad \quad \quad G^{R}_{T_xT_x}(\omega,k) = \frac{3}{2\kappa^2} \frac{h^{y(3)}_x}{h^{y(0)}_x}.
\]

(A6)

While the expansions given above, namely those of the precise metric and scalar fluctuations, are physically transparent, we saw in the text that decoupling the equations of motion was aided by defining combinations of the bare fields, e.g. \( \psi_{ty} \) and \( \psi_{xy} \). It is a simple matter, discussed in the text, to use the definitions of these gauge invariant combinations to relate their near boundary expansions to the near boundary expansions of \( h^y_t \) and \( h^y_x \).

B. \( G_{T_tT_t} \) fluctuation equations

Plugging the perturbative expansion (26) into (25), one obtains the following equations

\[
0 = \left[ (1 - u^3)\partial_u^2 - 3u^2\partial_u + 3u \right] \tilde{\psi}^{(0)}_{ty},
\]

(B1)
Each of these can be readily solved. Requiring the solutions to be regular at the horizon and \( \tilde{\psi}^{(i)}_{ty}(1) = 0 \) for \( i > 0 \), we find the following solutions:

\[
\tilde{\psi}^{(0)}_{ty}(u) = Cu, \\
\tilde{\psi}^{(1)}_{ty}(u) = -\frac{C}{12} \left\{ 18 + u \left[ 9 \log \left( \frac{3}{1 + u + u^2} \right) + 6 \sqrt{3} \tan^{-1} \left( \frac{1 + 2u}{\sqrt{3}} \right) - 18 - 2 \sqrt{3} \pi \right] \right\}, \\
\tilde{\psi}^{(2)}_{ty}(u) = \tilde{\psi}^{(3)}_{ty}(u) = \frac{3C}{4} (1 - u).
\]

These are all the terms needed for the solution given in (29).

To obtain the order \( M^2 \) contribution to the dispersion relation, we need to solve the higher order terms in the expansion (26). These terms are governed by the following equations:

\[
0 = \left[ (1 - u^3) \partial_u^2 - 3u^2 \partial_u + 3u \right] \tilde{\psi}^{(4)}_{ty} + \frac{9}{8} \left[ u(2 - 3u) \partial_u + (1 - 3u) \right] \tilde{\psi}^{(0)}_{ty}, \\
+ \frac{9}{8} u \left[ u(1 - u) \partial_u^2 + (2 - 3u) \partial_u + 3 \right] \tilde{\psi}^{(1)}_{ty} + 3u \left( u \partial_u + 1 \right) \tilde{\psi}^{(3)}_{ty}, \\
0 = \left[ (1 - u^3) \partial_u^2 - 3u^2 \partial_u + 3u \right] \tilde{\psi}^{(5)}_{ty} + \frac{9}{16} \left( 4 \tilde{\psi}^{(3)}_{ty} - 3 \tilde{\psi}^{(0)}_{ty} \right), \\
+ \frac{9}{8} u \left[ u(1 - u) \partial_u^2 + (2 - 3u) \partial_u + 3 \right] \tilde{\psi}^{(2)}_{ty}.
\]

Using the solutions, (B5)–(B7), for \( \tilde{\psi}^{(i)}_{ty} \) for \( i < 4 \), we can solve these, again imposing regularity at the horizon and \( \tilde{\psi}^{(4,5)}_{ty} = 0 \), to find:

\[
C^{-1} \tilde{\psi}^{(4)}_{ty}(u) = \frac{\sqrt{3}}{8} (1 + u) \left[ \pi - 3 \tan^{-1} \left( \frac{1 + 2u}{\sqrt{3}} \right) \right] + \frac{9}{16(1 + u + u^2)} \left[ 1 + u^2 - 2u^3 - (1 - u^3) \log \left( \frac{3}{1 + u + u^2} \right) \right], \\
C^{-1} \tilde{\psi}^{(5)}_{ty}(u) = -\frac{3 \sqrt{3}}{16} u \left[ \pi - 3 \tan^{-1} \left( \frac{1 + 2u}{\sqrt{3}} \right) \right].
\]

Series expanding these solutions near the boundary leads to the order \( wM^2 \) and \( q^2M^2 \) terms in (36).
1. Non-perturbative hydrodynamic terms

As mentioned in the text, the dispersion relation can actually be solved for exactly in \( M \). The solution itself is very lengthy and not worth producing directly here, so we will simply present the equations needed to obtain the solution and leave it to the reader to plug the equations below into Mathematica.

To set up the problem, we expand \( \tilde{\psi}_{ty} \) as before but letting the functions depend on \( M \):

\[
\tilde{\psi}_{ty}(u) = \tilde{\psi}_{ty}^{(0)}(u; M) + iw\tilde{\psi}_{ty}^{(1)}(u; M) + q^2\tilde{\psi}_{ty}^{(2)}(u; M) + \cdots . \tag{B12}
\]

Since we aren’t working perturbatively in \( M \), we don’t need to expand \( R_+ \) in \( M \), and therefore, we can work directly in terms of the emblackening factor. Then the coupled equations governing the expansion (B12) are

\[
0 = \partial_u \left[ f \partial_u \tilde{\psi}_{ty}^{(0)} \right] + \frac{3}{2u^2} \left( 2 - M^2 R^2 u^2 - 2f \right) \tilde{\psi}_{ty}^{(0)}, \tag{B13}
\]

\[
0 = \partial_u \left[ f \partial_u \tilde{\psi}_{ty}^{(0)} \right] + \frac{3}{2u^2} \left( 2 - M^2 R^2 u^2 - 2f \right) \tilde{\psi}_{ty}^{(1)} - \partial_u f \partial_u \tilde{\psi}_{ty}^{(0)} - \frac{1}{2} \tilde{\psi}_{ty}^{(0)} \partial_u^2 f, \tag{B14}
\]

\[
0 = \partial_u \left[ f \partial_u \tilde{\psi}_{ty}^{(2)} \right] + \frac{3}{2u^2} \left( 2 - M^2 R^2 u^2 - 2f \right) \tilde{\psi}_{ty}^{(2)} - R^2 \tilde{\psi}_{ty}^{(0)}. \tag{B15}
\]

These equations can, with a bit of patience, be solved upon using the expression for the emblackening factor, imposing regularity at the horizon and \( \tilde{\psi}_{ty}^{(1)}(1) = \tilde{\psi}_{ty}^{(2)}(1) = 0 \). The solutions obtained in this fashion can be readily expanded for small \( M \) to check against the perturbative expressions given previously.

C. Solving \( \varphi(r) \)

By combining equations (39) and (45), we deduce that

\[
h_{y''} + \left[ \frac{f'}{f} - \frac{2}{u} \right] h_{y'} + \frac{R_+^2 (w^2 - q^2 f - M^2 f)}{f^2} h_y = 0 \tag{C1}
\]

Plugging in the ansatz \( h_y(u) = f(u)^{-i\pi} F(u) \):

\[
0 = F'' + \left[ (1 - iw) \frac{f'}{f} - \frac{2}{u} \right] F' + \left[ \frac{i w f'}{u f} - \frac{1}{2} \frac{w f''}{f} + \frac{w^2}{f^2} \left( R_+^2 - \frac{f'^2}{4} \right) - \frac{R_+^2 (q^2 + M^2)}{f} \right] F.
\]

We can thus solve \( F(u) \) perturbatively:

\[
F(u) = F_0(u) + wF_1(u) + q^2 G_1(u) + w^2 F_2(u) + x^2 H_1(u) + wx^2 J_1(u) + x^4 H_2(u) + \ldots \tag{C2}
\]
We obtain 4 equations:

\[ 0 = \left[ (1 - u^3) \partial_u^2 - \left( \frac{2 + u^3}{u} \right) \partial_u \right] F_0, \quad \text{(C3)} \]
\[ 0 = \left[ (1 - u^3) \partial_u^2 - \left( \frac{2 + u^3}{u} \right) \partial_u \right] F_1 + 3iu^2 \partial_u F_0, \quad \text{(C4)} \]
\[ 0 = \left[ (1 - u^3) \partial_u^2 - \left( \frac{2 + u^3}{u} \right) \partial_u \right] G_1 - \frac{9}{4} F_0, \quad \text{(C5)} \]
\[ 0 = \left[ (1 - u^3) \partial_u^2 - \left( \frac{2 + u^3}{u} \right) \partial_u \right] H_1 - \frac{u(u + 2)(1 - u)}{2(1 + u + u^2)} \partial_u F_0 - F_0. \quad \text{(C6)} \]

Each of these can be readily solved. Requiring the solutions to be regular at the horizon and \( F(1) - F_0(1) = 0 \), we find the following solutions

\[ F_0(u) = C, \quad \text{(C7)} \]
\[ F_1(u) = 0, \quad \text{(C8)} \]
\[ H_1(u) = -\frac{C}{18} \left[ 2\pi \sqrt{3} - 9 \log 12 - 6\sqrt{3} \arctan \left( \frac{1 + 2u}{\sqrt{3}} \right) + 9 \log \left( -i + \sqrt{3} - 2iu \right) \right. \]
\[ \left. + 9 \log \left( i + \sqrt{3} + 2iu \right) \right], \quad \text{(C9)} \]

together with the following equations for \( J_1(u), F_2(u) \) and \( H_2(u) \):

\[ 0 = J_1'' - \left[ \frac{3u^2}{1 - u^3} + \frac{2}{u} \right] J_1' - iC \frac{1 + u + u^2 + 6u^3}{2(1 - u^3)(1 + u + u^2)}, \quad \text{(C11)} \]
\[ 0 = F_2'' - \left[ \frac{3u^2}{1 - u^3} + \frac{2}{u} \right] F_2' - \frac{9C}{4} \frac{(1 + u)(1 + u^2)}{(1 - u)(1 + u + u^2)^2}, \quad \text{(C12)} \]
\[ 0 = H_2'' - \left[ \frac{3u^2}{1 - u^3} + \frac{2}{u} \right] H_2' - \frac{C}{18} \left[ \frac{9u^2(2u^2 + 2u - 1) - 2\sqrt{3} \pi (1 + u + u^2)^2}{(1 - u)(1 + u + u^2)^3} + 6\sqrt{3} \arctan \left( \frac{2u+1}{\sqrt{3}} \right) + 9 \log 3 - 9 \log (1 + u + u^2) \right] \right]. \quad \text{(C13)} \]

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