BLOWUP RESULTS FOR THE FRACTIONAL SCHröDINGER EQUATION WITHOUT GAUGE INVARIANCE

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Abstract. This paper is concerned with the nonexistence of global solutions to the fractional Schrödinger equations with order \( \alpha \) and nongauge power type nonlinearity \(|u|^p\) for any space dimensions, where \( \alpha \in (0, 2] \) is assumed to be any fractional numbers. A modified test function is employed to overcome some difficulties caused by the fractional operator and to establish blowup results. Some restrictions with respect to \( \alpha, p \) and initial data in the previous literature are removed.

In this paper we consider the Cauchy problem of the following fractional nonlinear equation with non-gauge invariant type nonlinearity:

\[
\begin{aligned}
&i\partial_t u \pm (-\Delta)_{\frac{\alpha}{2}} u = \lambda |u|^p, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
&u(0,x) = u_0(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\]

with \( \lambda \in \mathbb{C}\setminus\{0\} \) and \( p > 1 \). Here operator \((-\Delta)_{\frac{\alpha}{2}}\) is realized as a Fourier multiplier with symbol \(|\xi|^\alpha\). \((-\Delta)_{\frac{\alpha}{2}} = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}\) under Fourier transform. The Cauchy problem such as (1) arises in various physical environments. Equation (1) is usually called the fractional Schrödinger equation which was used to describe particles in Lévy stochastic process, see\([3, 13, 24, 23, 1]\). It is also regarded as a degeneration of the semirelativistic equation which was used as a dynamical model of Boson star. For more related subjects, one can refer to \([2, 4, 10, 18, 19]\) and reference therein.

It is not difficult to see that in general spacial dimension \( n > 0 \), for \( s > n/2 \), (1) admits the unique time-local solution in \( H^s \) framework, where \( H^s = (1 - \Delta)^{-s/2}L^2(\mathbb{R}^n) \) is the usual inhomogeneous Sobolev space\([2, 9]\). In fact, local solution

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may be constructed by a standard contraction argument with \( H^s \hookrightarrow L^\infty \). On the other hand, we remark that (1) is scaling invariant under the transformation 
\[
    u_\rho(t, x) = \rho^{\alpha/(p-1)} u(\rho^a t, \rho x)
\]
with \( \rho > 0 \). The associated invariant space is \( H^{n/2 - \alpha/(p-1)} \). Generally, in \( H^s \) scaling subcritical case with \( s > n/2 - \alpha/(p-1) \), the global existence of solutions is expected. However, since (1) no longer keeps the charge and the energy conservation, the construction of the global solutions becomes a tricky problem.

The purpose of this paper is to show the finite time blowup of solutions to the fractional equation (1). In this direction, we remark that Ikeda-Wakasugi\cite{16} and Inui\cite{14, 17, 15} proved \( L^2 \) solution of Schrödinger equation (\( \alpha = 2 \)) blows up in finite time for any \( p \leq 1 + 4/n \). Both of their proof rely on the blowup alternative and the nonexistence of global weak solutions. Fujikawa-Ozawa \cite{11} studied the nonexistence of global weak solutions to (1) with \( \alpha = 1 \), for \( p \leq 2 \) in one dimension by transforming it into a wave equation \( \partial_t^2 v \pm \Delta v = -|\lambda|^2 \partial_t(|u|^p) \). However, this method cannot be applied to the higher dimensional case, because of the difficulty from the time derivative nonlinearity \( \partial_t^i |u|^p \). Until recently, they in \cite{8} treated (1) with \( \alpha = 1 \) again in \( n \) dimensions and established global nonexistence for the case of \( p < 1 + 1/n \), and we notice that their results are required
\[
    -\mathcal{S} \left( \int_{\mathbb{R}^n} u_0(x/R)^{-n-1} dx \right) \geq CR^{n-1/(p-1)}
\]
for some large \( R > 0 \). This is not natural enough. In the present paper, we shall remove this restriction and shall see that a similar situation occurs in (1) for any \( \alpha \in (0, 2] \) and \( p \leq 1 + \alpha/n \). In addition, we shall present the upper bound of lifespan of solutions for some special initial data.

Before stating our main results, we define weak solutions for (1) and introduce some notations used freely in the rest content.

**Definition 1.1.** Let \( T > 0 \) and \( u_0 \in L^2(\mathbb{R}^n) \). A function \( u \) is called to be a weak solution to (1) on \([0, T]\), if \( u \) belongs to \( L^1_{loc}(0, T; L^2(\mathbb{R}^n)) \cap L^1_{loc}(0, T; L^p(\mathbb{R}^n)) \) and the following identity
\[
    \int_0^T (u, i\partial_t \varphi + (-\Delta)\varphi) dt = i(u_0, \varphi(0)) + \lambda \int_0^T (|u|^p, \varphi) dt
\]
holds for any test function \( \varphi \in C([0, T]; H^\alpha(\mathbb{R}^n)) \cap C^1([0, T]; L^2(\mathbb{R}^n)) \) satisfying \( \varphi(T, x) = 0 \) for all \( x \in \mathbb{R}^n \), where \( \langle \cdot, \cdot \rangle \) is usual \( L^2 \) inner product defined by \( \langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx \), the double-sign corresponds to the sign of (1). We say \( u \) is a global weak solution to (1) if \( u \) is a weak solution to (1) on \([0, T]\) for any \( T > 0 \).

Throughout the paper, we denote
\[
    \mathcal{H}(t) = -\mathfrak{S} \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} u(t, x) dx \right), \quad \mathfrak{S}(t) = -\mathfrak{S} \int_{\mathbb{R}^n} u(t, x) dx
\]
and write \( f \lesssim g \) when \( f \leq C g \), \( f \asymp g \) when \( g \lesssim f \lesssim g \), where \( C \) denotes a suitable positive constant and may have different value from line to line. Our main result reads as follows.

**Theorem 1.2.** Let \( n > 0 \) and \( \alpha \in (0, 2] \). Assume the initial data \( u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) satisfies \( \mathcal{H}(0) > 0 \). Then, for any \( 1 < p \leq 1 + \frac{2n}{\alpha} \), problem (1) does not admit global nontrival weak solutions.
Remark 1. The same conclusion can be derived when \( \mathcal{H}(0) > 0 \) is replaced by another version \( \Re \lambda > 0 \) and \( \mathcal{S}(0) > 0 \). We note that the index \( p = 1 + \frac{\alpha}{n} \) is \( L^1(\mathbb{R}^n) \) scaling critical, it also seems to be the critical value that can be processed by our method. Compared with the critical \( L^2(\mathbb{R}^n) \) scaling index \( p_c = 1 + \frac{\alpha}{n} \), there is still a gap.

Theorem 1.3. Let \( \alpha \in (0, 2] \) and \( p \leq 1 + \alpha/n \). For any \( \lambda \in \mathbb{C}\{0\} \), \( \mathcal{H}(t) \) is monotonically increasing in its lifespan. Moreover, if \( \Re \lambda > 0 \), \( \mathcal{S}(t) \) is monotonically increasing; If \( \Re \lambda < 0 \), \( \mathcal{S}(t) \) is monotonically decreasing.

Finally, we investigate the upper estimate of the lifespan of the blowup solutions for some special initial data, which can be regarded as a refinement of the results of Ikeda-Sobajima[16] and Fujiwara-Ozawa[8].

Theorem 1.4. Let \( \alpha \in (0, 2] \) and \( p \leq 1 + \alpha/n \). Let \( u_0 = \epsilon f(x) \in C_0(\mathbb{R}^n) \) satisfy

\[-\Im \left( \frac{1}{\lambda} f(x) \right) \geq 0 \text{ for some positive constant } \epsilon_0, \text{ when } |x| \geq \epsilon_0, \text{ it holds}\]

\[-\Im \left( \frac{1}{\lambda} f(x) \right) \geq |x|^{-\zeta}, \zeta < \frac{\alpha}{p-1}.\] (3)

Then, for every \( \epsilon \in (0, \epsilon_0] \), the lifespan \( T_\epsilon \) of the solution \( u(x,t) \) to the problem (1) has the following upper bound:

\[T_\epsilon \leq \epsilon^\frac{1}{\alpha}, \zeta = \frac{\alpha}{n} - \frac{1}{p-1}.\] (4)

Corollary 1. Let \( \alpha \in (0, 2] \) and \( p \leq 1 + \alpha/n \). Let \( u_0 = \eta f(x) \in C_0(\mathbb{R}^n) \) satisfy

\[-\Im \left( \frac{1}{\lambda} f(x) \right) \geq 0 \text{ and}\]

\[-\Im \left( \frac{1}{\lambda} f(x) \right) \geq |x|^{-\zeta}|x| \leq \eta_0, \zeta < n,\] (5)

for some positive constant \( \eta_0 \). Then for every \( \eta \in [\eta_0, +\infty) \), the lifespan \( T_\eta \) of the solution \( u(x,t) \) to the problem (1) has the following upper bound:

\[T_\eta \leq \eta^\zeta, \zeta = \frac{\alpha}{n} - \frac{1}{p-1}.\] (6)

The proof of Theorems 1.2, 1.3 is based on the construction of contradiction. The key to the proof is to choose a modified test function. In fact, Fujiwara’s test function \( \psi(x) \) in [8] is not available for any \( \alpha \in (0, 2] \). Thanks to the nonlocality property of operators, it does not ensure the pointwise control \( |(-\Delta)^{\alpha/2}\psi(x)| \lesssim \psi(x) \). For this, we take the truncation of \( \psi(x) \), which can be verified to be a positive \( C^2 \)-function decaying polynomially and holding pointwise control, to overcome this difficulty and to show the essence of the solutions for (1) as far as possible. In addition, we also need a time decay function \( \chi(t) \) as an aid. Moreover, with the help of this modified test function, we also prove the monotonicity results for \( \mathcal{H}(t) \) and \( \mathcal{S}(t) \).

It is worth noticing that for the semilinear heat equation with \( p \leq 1 + \alpha/n \), the similar study has been done in literature[22, 21], where \( p = 1 + \alpha/n \) is known as Fujita exponent. Blowup phenomena for the solutions has been shown by Fujita[7], Hayakawa[12] and Sugitani[25]. The sharp upper and lower estimates for
lifespan of solutions was established in Lee-Ni[22]. However, their techniques seem to be difficult to apply to (1) because they used the positivity of heat kernel and the maximum principle. In addition, similar blowup results for damped wave equation can also be found in [6, 5], where the inequality
\[ (-\Delta)^{\alpha/2}(\omega^1)(x) \lesssim \omega^{-1}(x) \left((-\Delta)^{\alpha/2}\omega\right)(x) \]
for all \( \alpha \in (0, 2] \) and \( l \geq 1 \), for any \( \omega \) from the Schwartz class, plays an essential role. Despite of this difficulty, we could prove the blowup phenomena and lifespan estimates by using only the positivity of the nonlinear term.

The paper is organized as follows. In Section 2, we introduce the modified test function and prove the pointwise control property. Then the Section 3 is devoted to prove the main results. In particular, we give the detail proof regarding the monotonicity of Solutions and indicate the relation between the lifespan of solutions and initial data.

2. Preliminaries. In this section, we collect some preliminary knowledge needed in our proofs and present some results concerning the fractional derivatives and its scaling, which will be used hereafter.

Definition 2.1. [20] Let \( \alpha \in (0, 2) \). Let \( X \) be a suitable set of functions defined on \( \mathbb{R}^n \). Then, the fractional Laplacian \( (-\Delta)^{\alpha/2} \) in \( \mathbb{R}^n \) is a nonlocal operator given by
\[ (-\Delta)^{\alpha/2} : v \in X \rightarrow (-\Delta)^{\alpha/2}v := C_{n,\alpha}P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+\alpha}}dy \]
as long as the right-hand side exists, where \( P.V. \) stands for Cauchy’s principal value, \( C_{n,\alpha} = \) is a normalization constant.

Lemma 2.2. Let \( \langle x \rangle = \sqrt{1 + (|x| - 1)^2} \) and \( 0 < \alpha \leq 2 \). Let \( \psi(x) \) be a continuous piecewise function defined by
\[ \psi(x) = \begin{cases} 1 & |x| \leq 1 \\ \langle x \rangle^{-n-\alpha} & |x| \geq 1. \end{cases} \] (8)

Then for any \( x \in \mathbb{R}^n \), \( \psi \in C^2(\mathbb{R}^n) \) and there exists a positive constant \( A_{n,\alpha} \) depending only on \( n \) and \( \alpha \) such that \( (-\Delta)^{\alpha/2}\psi \) admits the following pointwise estimate
\[ |(-\Delta)^{\alpha/2}\psi(x)| \leq A_{n,\alpha}\psi(x). \] (9)

Proof. For convenience, we denote \( r := |x| \) and consider the radial derivatives of \( \psi(x) \)
\[ \nabla \psi(x) = \frac{x}{r}\psi'(r) = \begin{cases} 0, & r \leq 1, \\ -(n + \alpha)(r - 1)^2 \langle x \rangle^{-n-\alpha-2}, & r \geq 1, \end{cases} \] (10)
and
\[ \Delta \psi(x) = \frac{n-1}{r}\psi'(r) + \psi''(r) = \begin{cases} 0, & r \leq 1, \\ -(n + \alpha)(r - 1)^2 \langle x \rangle^{-n-\alpha-2} + (n + \alpha)(n + \alpha + 2)(r - 1)^2 \langle x \rangle^{-n-\alpha-4} + (n + \alpha)\langle x \rangle^{-n-\alpha-2}, & r \geq 1. \end{cases} \] (11)
It is easily to check that \( \psi(x) \in C^2(\mathbb{R}^n) \) and \( |\Delta \psi| \leq A_{n,\alpha}\psi(x) \). So, in the following, we only treat \( 0 < \alpha < 2 \). In addition, from the above computation, we have \( ||\psi, \partial_r^2 \psi||_{L^\infty} < \infty \), which allows us to remove the principle value of integral at
the origin. Indeed, according to Definition 2.1 of fractional Laplacian, a standard change of variables leads to

\[
(-\Delta)^{\frac{\alpha}{2}} \psi(x) = -\frac{C_{n,\alpha}}{2} P.v. \int_{\mathbb{R}^n} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy \\
= -\frac{C_{n,\alpha}}{2} \lim_{\varepsilon \to 0^+} \int_{|y| \leq 1} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy \\
- \frac{C_{n,\alpha}}{2} \int_{|y| \geq 1} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy.
\]

(12)

Applying a second order Taylor expansion for \(\psi\), we have

\[
\psi(x+y) + \psi(x-y) - 2\psi(x) \lesssim \|\partial_x^2 \psi\|_{L^\infty} \frac{|y|^{n+\alpha}}{|y|^{n+\alpha-2}}.
\]

(13)

Thanks to the above estimate and \(\alpha \in (0, 2)\), we may remove the principal value of the integral at the origin to conclude

\[
(-\Delta)^{\frac{\alpha}{2}} \psi(x) = -\frac{C_{n,\alpha}}{2} \int_{\mathbb{R}^n} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy.
\]

(14)

In order to prove the desired results, here we distinguish two cases: \(|x| \leq 2\) and \(|x| \geq 2\). For \(|x| \leq 2\), we divide integral domain into the following two parts:

\[
\Gamma_1 = \{(x, y) : |y| \leq 1\}, \quad \Gamma_2 = \{(x, y) : |y| \geq 1\}.
\]

On \(\Gamma_1\), it holds

\[
\int_{\Gamma_1} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy \lesssim \|\partial_x^2 \psi\|_{L^\infty} \int_{\Gamma_1} \frac{1}{|y|^{n+\alpha-2}} dy \lesssim 1,
\]

(15)

and on \(\Gamma_2\), it has

\[
\int_{\Gamma_2} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy \lesssim \|\psi\|_{L^\infty} \int_{\Gamma_2} \frac{1}{|y|^{n+\alpha}} dy \lesssim 1.
\]

(16)

For the domain \(|x| \geq 2\), we also divide the domain into three parts:

\[
\Gamma_3 = \{(x, y) : |y| \geq 2|x|\}, \quad \Gamma_4 = \left\{(x, y) : \frac{1}{2}|x| \leq |y| \leq 2|x|\right\}, \quad \Gamma_5 = \left\{(x, y) : |y| \leq \frac{1}{2}|x|\right\}.
\]

On \(\Gamma_3\) we notice that \(|x \pm y| \geq |y| - |x| \geq |x| \geq 2\). With the help of the monotonicity of \(\psi\), we have \(\psi(x \pm y) \leq \psi(x) = \langle x \rangle^{-n-\alpha}\). Therefore it admits

\[
\int_{\Gamma_3} \frac{\psi(x+y) + \psi(x-y) - 2\psi(x)}{|y|^{n+\alpha}} dy \lesssim 4\psi(x) \int_{|y| \geq 2|x|} \frac{1}{|y|^{n+\alpha}} dy \\
\lesssim \langle x \rangle^{-n-\alpha} \int_{|y| \geq 2|x|} \frac{1}{|y|^{1+\alpha}} dy |
\]

\[
\lesssim \langle x \rangle^{-n-\alpha} |x|^{-\alpha} \\
\lesssim \langle x \rangle^{-n-2\alpha},
\]

(17)

where we used the fact \(\langle x \rangle \sim |x| - 1 \sim |x|\) for \(|x| \geq 2\).

On \(\Gamma_4\), since \(|y| \sim |x|\) and

\[
\Gamma_4 \subseteq \{y \in \mathbb{R}^n : |x \pm y| \leq 3|x|\},
\]
it follows
\[
\int_{\Gamma_4} \frac{\psi(x + y) + \psi(x - y) - 2\psi(x)}{|y|^{n+\alpha}}\,dy \\
\lesssim |x|^{-n-\alpha} \left( \int_{|x+y| \leq 3|x|} \psi(x + y)\,dy + \int_{|x-y| \leq 3|x|} \psi(x - y)\,dy \right) \\
+ 2\psi(x) \int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} 1\,dy \\
\lesssim (x)^{-n-\alpha} \left( \int_{|x+y| \leq 3|x|} \psi(x + y)\,dy + (x)^{-n-\alpha}|x|^n \right) \\
\lesssim (x)^{-n-\alpha} (1 + |x|^{-\alpha}) \\
\lesssim (x)^{-n-\alpha}, \quad (18)
\]

here we used the following estimate
\[
\int_{|x-y| \leq 3|x|} \psi(x - y)\,dy = \int_{|x+y| \leq 3|x|} \psi(x + y)\,dy \\
= \int_{2 \leq |x+y| \leq 3|x|} \psi(x + y)\,dy + \int_{|x+y| \leq 2} \psi(x + y)\,dy \\
\lesssim \int_2\int r^{-\alpha}dr + \int_0^2 r^{-1}dr \lesssim 1.
\]

Next we treat the integral on the third domain \(\Gamma_5\). By applying the Taylor expansion for \(\psi\), we can deduce
\[
\int_{\Gamma_5} \frac{\psi(x + y) + \psi(x - y) - 2\psi(x)}{|y|^{n+\alpha}}\,dy \lesssim \int_{\Gamma_5} \max_{\xi \in [0,1]} \partial_x^2 \psi(x \pm \xi y) \frac{1}{|y|^{n+\alpha}}\,dy. \quad (19)
\]

On the other hand, we have the following estimate for \(\xi \in [0,1]\) and \(r = |x \pm \xi y|\),
\[
|\partial_x^2 \psi(x \pm \xi y)| \lesssim \begin{cases} 
0, & r \leq 1, \\
\frac{1}{r}(r)^{-n-\alpha-2} + (r - 1)^2(r)^{-n-\alpha-4} + (r)^{-n-\alpha-2}, & r \geq 1.
\end{cases}
\]
\[
\lesssim \begin{cases} 
0, & r \leq 1, \\
(r)^{-n-\alpha-2}, & r \geq 1.
\end{cases} \quad (20)
\]

Since \(|x \pm \xi y| \geq |x| - |\xi y| \geq |x| - |x|/2 \geq 1\), from the above estimate, it then yields
\[
|\partial_x^2 \psi(x \pm \xi y)| \lesssim (x \pm \xi y)^{-n-\alpha-2}.
\]

If \(|x \pm \xi y| \geq 2\), then \(\langle x \pm \xi y \rangle \sim |x \pm \xi y| \geq |x|/2 \gtrsim \langle x \rangle\), which implies
\[
\langle x \pm \xi y \rangle^{-n-\alpha-2} \lesssim \langle x \rangle^{-n-\alpha-2}.
\]

If \(|x \pm \xi y| \leq 2\), then \(|x|/2 \leq |x \pm \xi y| \leq 2\), which illustrates \(2 \leq |x| \leq 4\). Therefore, we may have \(\langle x \pm \xi y \rangle \sim |x| \sim 1\) and \(\langle x \pm \xi y \rangle^{-n-\alpha-2} \sim \langle x \rangle^{-n-\alpha-2}\), which yields
\[
|\partial_x^2 \psi(x \pm \xi y)| \lesssim (x)^{-n-\alpha-2}
\]
for all $\xi \in [0,1]$. By (19), we can verify

$$\int_{\Gamma_h} |\psi(x+y) + \psi(x-y) - 2\psi(x)| \frac{1}{|y|^{n+\alpha}} dy \lesssim \langle x \rangle^{-n-\alpha-2} \int_{\Gamma_h} \frac{1}{|y|^{n+\alpha}} dy \lesssim \langle x \rangle^{-n-\alpha-2} \int_0^{\sqrt{|x|/2}} \frac{1}{|y|^{\alpha-1}} |y|^{-\alpha+2} dy \lesssim \langle x \rangle^{-n-2\alpha}. \quad (21)$$

Combining (15)-(18) and (21), we conclude that

$$|(-\Delta)^{\frac{\alpha}{2}} \psi(x)| \lesssim \langle x \rangle^{-n-\alpha} \quad (22)$$

for all $x \in \mathbb{R}^n$, which completes our proof.

**Remark 2.** A similar result may be derived with some $L^1(\mathbb{R}^n)$ functions decaying faster than $\langle x \rangle^{-n}$. Indeed, for $1 \leq \alpha < 2$, there exists positive constants $A_{n,\alpha}$ depending only on $n$ and $\alpha$ such that $(-\Delta)^{\alpha/2}$ admits the following pointwise estimate

$$|(-\Delta)^{\alpha/2} e^{-|\cdot|^2}(x)| \leq A_{n,\alpha} \langle x \rangle^{-n-\alpha}. \quad (23)$$

**Lemma 2.3.** [6] Let $0 < \alpha \leq 2$, $x \in \mathbb{R}^n$ and $\psi$ be a smooth function satisfying $\partial_t^2 \psi \in L^\infty(\mathbb{R}^n)$. For any $R > 0$, let $\psi_R$ be a function defined by

$$\psi_R(x) := \psi(x/R). \quad (25)$$

Then, for all $x \in \mathbb{R}^n$, $(-\Delta)^{\frac{\alpha}{2}} \psi_R(x)$ admits the following scaling property

$$(-\Delta)^{\frac{\alpha}{2}} \psi_R(x) = R^{-\alpha}((-\Delta)^{\frac{\alpha}{2}} \psi)(x/R). \quad (26)$$

3. **Proof of main results.** In this section, we prove main theorems based on the definition of weak solution and the construction of contradiction. For this, we also introduce two auxiliary functions to evolve the desired results.

**Proof of Theorem 1.2.** We assume $u$ is a global weak solution to (1), then $u$ will satisfy

$$\int_0^T \left( u, i\partial_t \varphi + (-\Delta)^{\alpha/2} \varphi \right) dt = i\langle u_0, \varphi(0) \rangle + \lambda \int_0^T (|u|^p, \varphi) dt \quad (24)$$

for all test function such that $\varphi(T,\cdot) = 0$ for all $T > 1$. Let $R$ and $T$ be large parameters in $\mathbb{R}_+$. we first introduce the following auxiliary functions:

$$\psi_R(x) := \psi(x/R), \quad \chi(t) = (1 - \frac{t}{T})^\delta, \quad (25)$$

where $\delta \gg 1$ is a properly large constant. Then we define the test function

$$\varphi(t,x) := \psi_R(x)\chi(t). \quad (26)$$
From (24), it yields
\[
\int_0^T (|u|^p, \varphi(t,x))dt - \Im\left(\frac{u_0}{\lambda}(\psi_R(x))\right)
\]
\[
= - \Re\left(\frac{i\delta}{\lambda T} \int_0^T (u, \varphi(x,t)(1-t/T)^{-1})dt\right)
\]
\[
+ \Re\left(\frac{1}{\lambda} \int_0^T (u, \chi(t)(-\triangle)^{\alpha/2}(\psi_R(x)))dt\right)
\]
\[
=: J_1 + J_2.
\] (27)

In what follows we proceed the estimates for \(J_1\) and \(J_2\), respectively. Indeed, applying Holder’s inequality with \(1/p + 1/p' = 1\), we have
\[
|J_1| \lesssim \frac{1}{T} \int_0^T |(u, \varphi(x,t)(1-t/T)^{-1})|dt
\]
\[
\lesssim \frac{1}{T} \int_0^T (|u|^p, \varphi(x,t))^{1/p}(\varphi(x,t), (1-t/T)^{-p'})^{1/p'}dt
\]
\[
\lesssim \frac{1}{T} I_R^{1/p} \left( \int_0^T (\varphi(x,t), (1-t/T)^{-p'})dt \right)^{1/p'},
\] (28)

where
\[
I_R(T) := \int_0^T (|u|^p, \varphi(t,x))dt.
\]

Note that the uniform boundedness of \(\chi(t)\), by the change of variable \(\tilde{x} := x/R\), we get
\[
|J_1| \lesssim T^{1/p'-1} R^{1/p} \left( \int_{\mathbb{R}^n} \psi_R(x)dx \right)^{1/p'}
\]
\[
\lesssim T^{1/p'-1} R^{1/p} \left( \int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-\alpha} d\tilde{x} \right)^{1/p'}
\]
\[
\lesssim T^{1/p'-1} R^{1/p} R^{n/p'}.\] (29)

Similarly, with the help of Holder’s inequality again and Lemma 2.2 and Lemma 2.3, we can deduce
\[
|J_2| \lesssim \int_0^T |(u, \chi(t)(-\triangle)^{\alpha/2}(\psi_R(x)))|dt
\]
\[
\lesssim R^{-\alpha} \int_0^T (|u|, \chi(t)(-\triangle)^{\alpha/2}\psi_R(x/R))dt
\]
\[
\lesssim R^{-\alpha} I_R^{1/p} \left( \int_0^T \chi(t) dt \int_{\mathbb{R}^n} \psi_R(x)dx \right)^{1/p'}
\]
\[
\lesssim R^{n/p'-\alpha} I_R^{1/p} R^{1/p'}.\] (30)
Substituting the estimates (29)-(30) into (27), we have
\[
\int_0^T (|u|^p, \varphi(t, x)) dt - 3 \left( \frac{u_0, \psi_R(x)}{\lambda} \right) \\
\lesssim I_{R}^{1/p} (T^{1/p'} - 1) \lesssim R^{n/p'} + R^{n/p' - \alpha} T^{1/p'} \lesssim I_{R}^{1/p} R^{n/p'} T^{1/p'} (T^{-1} + R^{-\alpha}). \tag{31}
\]
In the following, we consider three cases with respect to the index \( p \):

**Case 1:** \( p < 1 + \alpha/n \). Due to \( \mathcal{H}(0) > 0 \) and the fact \( \psi_R(x) \to 1 \) as \( R \to \infty \), then there exists a sufficiently large constant \( R_0 > 0 \) such that
\[
-3 \left( \frac{u_0, \psi_R(x)}{\lambda} \right) > 0 \tag{32}
\]
for all \( R > R_0 \). On the other hand, for sufficiently large \( T \), we take \( R = T^{\theta} \) with \( \theta > 0 \) satisfying \( \theta < \frac{1}{n(p-1)} \) for \( \alpha \theta \geq 1 \) and \( \theta > \frac{1}{np - n} \) for \( \alpha \theta < 1 \), where \( p' \) is the conjugate of \( p \). By (31) and (32), we have
\[
I_R \lesssim I_{R}^{1/p} R^{n/p'} T^{1/p'} (T^{-1} + T^{-\theta \alpha}) \lesssim I_{R}^{1/p} T^{(n\theta + 1)/p'} + \max \{-1, -\theta \alpha\}. \tag{33}
\]
Applying Young's inequality we conclude that
\[
\frac{I_{R}}{p'} \lesssim T^{(n\theta + 1)/p' + \max \{-1, -\theta \alpha\}}. \tag{34}
\]
It is easy to check \( p < 1 + \alpha/n \) implies \( (n\theta + 1)/p' + \max \{-1, -\theta \alpha\} < 0 \). Passing to the limit as \( T \to \infty \), we derive that \( u = 0 \), a.e.. Using (31) again, we have
\[
-3 \left( \frac{u_0, \psi_R(x)}{\lambda} \right) \leq 0 \tag{35}
\]
for all \( R \geq 1 \), which contradicts our assumption (32).

**Case 2:** \( p = 1 + \alpha/n \). In this case, the above selection of \( \theta \) will lead to \( (n\theta + 1)/p' + \max \{-1, -\theta \alpha\} = 0 \). In fact, the desirable range of \( \theta \) is an empty set except \( \theta = 1/\alpha \). Then, applying Young' inequality, we have
\[
I_R \lesssim 1, \tag{36}
\]
which follows \( I_R \leq C \) as \( T \to \infty \). By Levi's theorem on monotone convergence, we have
\[
\int_0^\infty \int_{\mathbb{R}^n} |u|^p dx dt = \lim_{T \to \infty} \int_0^T (|u|^p, \varphi(t, x)) dt = \lim_{T \to \infty} I_R \leq C, \tag{37}
\]
which implies \( u \in L^p(\mathbb{R}^+ \times \mathbb{R}^n) \). From (31), we may arrive at
\[
\int_0^T (|u|^p, \varphi(t, x)) dt - 3 \left( \frac{u_0, \psi_R(x)}{\lambda} \right) \lesssim 1. \tag{38}
\]
Instead of \( \chi(t) \), we define a new version of \( \chi(t) \) as follows
\[
\tilde{\chi}(t) = \begin{cases} 
\frac{1}{2}, & t \in [0, \frac{1}{2}], \\
\frac{1}{2} - (\frac{1}{2} - t)^2, & t \in \left[\frac{1}{2}, 1\right], \\
0, & t \in [1, \infty).
\end{cases} \tag{39}
\]
It is easy to check \( \tilde{\chi}(t) \in C^1[0, \infty) \) and \( |\partial_t \tilde{\chi}(t)| \lesssim \tilde{\chi}(t)/T \), where \( \tilde{\chi}(t) = \tilde{\chi}(t/T) \).

Then, we define the test function
\[
\tilde{\varphi}(t, x) := \psi_R(x) \tilde{\chi}(t/T).
\]
From (24), it yields
\[
\int_0^T (|u|^p, \bar{\varphi}(t,x)) dt - \Im \left( \frac{u_0, \psi_R(x)}{\lambda} \right) 
\]
\[
= - \Re \left( \frac{i}{\lambda} \int_0^T (u, \psi_R(x) \partial_t \eta_T(t)) dt \right) 
\]
\[
+ \Re \left( \frac{1}{\lambda} \int_0^T (u, \eta_T(t)(-\triangle)^{\alpha/2}(\psi_R(x))) dt \right) 
\]
\[
\lesssim T^{-1/p} I_{R,T}^{1/p} R^{n/p'} + T_{R}^{1/p} R^{n/p' - \alpha} T^{1/p'}.
\] 
(40)

where
\[
\bar{I}_{R,T} = \int_{\frac{T}{2}}^T (|u|^p, \bar{\varphi}(t,x)) dt, \quad \bar{I}_R = \int_0^T (|u|^p, \bar{\varphi}(t,x)) dt 
\]

For any fixed \( R \), we note that \( \bar{I}_{R,T} \) goes to 0 as \( T \to \infty \) since \( \int_{\infty}^{0} \int_{\mathbb{R}^n} |u|^p dx dt < \infty \), which implies
\[
T^{-1/p} \bar{I}_{R,T}^{1/p} R^{n/p'} \lesssim - \frac{1}{2} \Im \left( \frac{u_0, \psi_R(x)}{\lambda} \right) 
\]
for sufficiently large \( T \). Then we have
\[
\int_0^T (|u|^p, \bar{\varphi}(t,x)) dt - \frac{1}{2} \Im \left( \frac{u_0, \psi_R(x)}{\lambda} \right) \lesssim \bar{I}_{R}^{1/p} R^{n/p' - \alpha} T^{1/p'}.
\] 
(41)

Let \( R = T^\theta \) with \( \theta > \frac{1}{\alpha p - n} \). Passing to the limit as \( T \to \infty \), we have \( \int_{\infty}^{0} \int_{\mathbb{R}^n} |u|^p dx dt \lesssim 0 \), which implies \( u = 0 \), a.e.. By (41), we can deduce
\[
- \Im \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} u_0 dx \right) \lesssim 0
\]
(42)
which contradicts our assumption \( H(0) > 0 \).

**Remark 3.** From the analysis of Theorem 1.2, for \( p > 1 + \frac{\alpha}{n} \), we can also deduce if
\[
\Im \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} u(0) \langle x \rangle^{-n - \alpha} dx \right) < 0,
\]
then there is no local weak solution to the equation (1).

Indeed, for \( T \ll 1 \), let \( R = T^\theta \) with \( \theta > 0 \). By (31) and (32), we have
\[
I_R \lesssim I_{R}^{1/p} R^{n/p'} T^{1/p'} (T^{-1} + T^{-\alpha}) \lesssim I_{R}^{1/p} T^{(n\theta + 1)/p'} \min(-1, -\theta \alpha).
\] 
(44)

Applying Young’s inequality we conclude that
\[
\frac{1}{2} I_R \lesssim T^{(n\theta + 1)/p'} \min(-1, -\theta \alpha).
\] 
(45)
Taking \( \theta > \frac{1}{\alpha(p-1)} \) for \( \alpha \theta \leq 1 \) and \( \theta < \frac{1}{\alpha p - n} \) for \( \alpha \theta > 1 \), we have \( (n\theta + 1) + p' \min(-1, -\theta \alpha) > 0 \) when \( p > 1 + \alpha/n \). Let \( T \to 0 \), then \( T^{(n\theta + 1)/p'} \min(-1, -\theta \alpha) \to 0 \) and \( I_R \to 0 \). Using (31) again, we have
\[
- \Im(u_0/\lambda, \psi_R(x)) \lesssim 0
\]
(46)
for all \( 0 < R < 1 \), which contradicts to (43).
Proof of Theorem 1.3. Let
\[ \mathcal{H}_R(t) = -\Im \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} u(x,t)\psi_R(x)dx \right). \]

Obviously, \( \mathcal{H}_R(t) \to \mathcal{H}(t) \) as \( R \to \infty \). Next, we use differential inequality to analyze blow up properties for \( \mathcal{H}(t) \). A straightforward computation shows
\[
\frac{d}{dt} \mathcal{H}_R(t) = \Re \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} i\partial_t u(x,t)\psi_R(x)dx \right)
= \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx - \Re \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} u(x,t)(-\triangle)^{\alpha/2} \psi_R(x)dx \right)
\geq \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx - \frac{1}{|\lambda|} \int_{\mathbb{R}^n} |u(x,t)(-\triangle)^{\alpha/2} \psi_R(x)|dx
\geq \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx - |\lambda|^{-1} R^{-\alpha} \int_{\mathbb{R}^n} |u(x,t)|\psi_R(x)dx. \tag{47}
\]

By the Holder and Young inequalities, we have
\[
|\lambda|^{-1} R^{-\alpha} \int_{\mathbb{R}^n} |u(x,t)|\psi_R(x)dx
\leq A_{n,\alpha} \left( \int_{\mathbb{R}^n} |u(x,t)|^p \psi_R(x)dx \right)^{1/p} \left( \int_{\mathbb{R}^n} \psi_R(x)dx \right)^{1/p'}
\leq A_{n,p,\alpha} \Re(\lambda)^{-\alpha'} R^{n-\alpha'} \int_{\mathbb{R}^n} \psi(x)dx + \int_{\mathbb{R}^n} |u(x,t)|^p \psi_R(x)dx, \tag{48}
\]

Combining the above inequalities, one has
\[
\frac{d}{dt} \mathcal{H}_R(t) \geq -\Re(\lambda)^{-\alpha'} R^{n-\alpha'}, \tag{49}
\]
then, for any \( t > 0 \), when \( p < 1 + \frac{\alpha}{n-\alpha} \), we have \( n - \alpha' < 0 \) and \( R^{n-\alpha'} \to 0 \) as \( R \to \infty \), which implies
\[ \mathcal{H}(t) \geq \mathcal{H}(0) \]
and also means \( \mathcal{H}(t) \) is monotonically increasing.

Similarly, put
\[ \tilde{\mathcal{H}}_R(t) = -\Im \left( \int_{\mathbb{R}^n} u(x,t)\psi_R(x)dx \right). \]

Obviously, \( \tilde{\mathcal{H}}_R(t) \to \tilde{\mathcal{H}}(t) \) as \( R \to \infty \). Next, we use differential inequality to analyze blow up properties for \( \tilde{\mathcal{H}}(t) \). A straightforward computation shows
\[
\frac{d}{dt} \tilde{\mathcal{H}}_R(t) = \Re \left( \int_{\mathbb{R}^n} i\partial_t u(x,t)\psi_R(x)dx \right)
= \Re \lambda \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx - \Re \left( \int_{\mathbb{R}^n} u(x,t)(-\triangle)^{\alpha/2} \psi_R(x)dx \right)
\leq \Re \lambda \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx + \int_{\mathbb{R}^n} |u(x,t)(-\triangle)^{\alpha/2} \psi_R(x)|dx
\leq \Re \lambda \int_{\mathbb{R}^n} |u|^p \psi_R(x)dx + R^{-\alpha} \int_{\mathbb{R}^n} |u(x,t)|\psi_R(x)dx, \tag{50}
\]
In particular, in the second line, we used the Parseval’s formula. By the Holder and Young inequalities, we have

$$R^{-\alpha} \int_{\mathbb{R}^n} |u(x,t)| \psi_R(x) dx \leq A_{n,\alpha} R^{-\alpha} \left( \int_{\mathbb{R}^n} |u(x,t)|^p \psi_R(x) dx \right)^{1/p} \left( \int_{\mathbb{R}^n} \psi_R(x) dx \right)^{1/p'}$$

$$\leq A_{n,p,\alpha} \Re(-\lambda)^{-p'/p} R^{n-\alpha p'} - \Re \lambda \int_{\mathbb{R}^n} |u(x,t)|^p \psi_R(x) dx \quad (51)$$

Combining the above inequalities, one has

$$\frac{d}{dt} \mathcal{H}_R(t) \lesssim -\Re(\lambda)^{-p'} R^{n-\alpha p'}, \quad (52)$$

then, for any $t > 0$, when $p < 1 + \frac{\alpha}{n-\alpha}$, we have $n - \alpha p' < 0$ and $R^{n-\alpha p'} \to 0$ as $R \to \infty$, which implies

$$\mathcal{H}(t) \leq \mathcal{H}(0)$$

and also means $\mathcal{H}(t)$ is monotonically decreasing. On the contrary, we can also prove $\mathcal{H}(t)$ is monotonically increasing when $\Re \lambda > 0$ in a similar way.

**Proof of Theorem 1.4.** Here we consider the weak solution again and apply the test function $\varphi(x,t) = \chi(t) \psi_R(x)$, where $\chi(t)$ and $\psi_R(x)$ are given by (25). According to the definition 1.1, we have

$$\lambda \int_0^T \int_{\mathbb{R}^n} |u|^p \varphi dx dt + i \int_{\mathbb{R}^n} u_0 \varphi(0) dx \quad = \quad \int_0^T \int_{\mathbb{R}^n} u(-\triangle)^{\alpha/2} \varphi dx dt - i \int_0^T \int_{\mathbb{R}^n} u \varphi_1(x,t) dx dt. \quad (53)$$

Using (31),(32) and $\chi(0) = 1$, one has

$$\lambda \int_0^T \int_{\mathbb{R}^n} |u|^p \varphi dx dt \geq \Re \left( \frac{\epsilon}{\lambda} \int_{\mathbb{R}^n} f(x) \psi_R(x) dx \right) 
\lesssim R^n T (T^{-1} + R^{-\alpha} p'), \quad (54)$$

which implies

$$-\Im \left( \frac{\epsilon}{\lambda} \int_{\mathbb{R}^n} f(x) \psi_R(x) dx \right) \lesssim R^\alpha T^{1-p'} + TR^{n-\alpha p'}. \quad (55)$$

Let $R = T^{1/\alpha}$. The above estimate can be written as

$$-\Im \left( \frac{\epsilon}{\lambda} \int_{\mathbb{R}^n} f(x) \psi_R(x) dx \right) \lesssim T^{\frac{\alpha}{\alpha} + 1-p'}. \quad (56)$$

On the other hand, we treat the initial datum $u_0(x) = \epsilon f(x)$ to obtain a lower bound. Using the hypothesis $-\Im \left( \frac{1}{\lambda} f(x) \right) \gtrsim |x|^{-\epsilon}, |x| \geq \epsilon_0$ for some constant $\epsilon_0 > 0$
and passing to the following scaled variable \( y = \frac{x}{T^{1/\alpha}} \), we can deduce for \( R > 4\epsilon_0 \),

\[
-\Im \left( \frac{\epsilon}{N} \int_{R^n} f(x)\psi_R(x)dx \right) \gtrsim \epsilon \int_{|x| \geq \epsilon_0} |x|^{-\zeta} \psi_R(x)dx \\
\gtrsim \epsilon T^{\frac{n-\zeta}{\alpha}} \int_{|y| \geq \frac{\epsilon_0}{T^{1/\alpha}}} |y|^{-\zeta} \psi(y)dy \\
\gtrsim \epsilon T^{\frac{n-\zeta}{\alpha}} \int_{1 \leq |y| \geq \frac{\epsilon_0}{T^{1/\alpha}}} |y|^{-\zeta} \psi(y)dy \\
\gtrsim \epsilon T^{\frac{n-\zeta}{\alpha}} \left\{ 1, \left( \frac{\epsilon_0}{T^{1/\alpha}} \right)^{n-\zeta}, n \geq \zeta, \left( \frac{\epsilon_0}{T^{1/\alpha}} \right)^{n-\zeta}, n < \zeta, \right\}
\]

where \( \left( \frac{n-\zeta}{\alpha} \right)_+ = \max(\frac{n-\zeta}{\alpha}, 0) \). From (56) and (57), it follows

\[
\epsilon \lesssim T^\kappa,
\]

where \( \kappa = \frac{n}{\alpha} + 1 - p' - \max\{\frac{n-\zeta}{\alpha}, 0\} = -\frac{1}{p-1} + \frac{\min(\zeta, n)}{\alpha} < 0 \). Therefore, we finally deduce

\[
T \lesssim \epsilon^{\frac{1}{\kappa}},
\]

which completes the proof of the theorem.

\[\Box\]

**Proof of Corollary 1.** We choose \( R < \eta_0 \), and modify (57) as

\[
-\Im \left( \frac{\eta}{N} \int_{R^n} f(x)\psi_R(x)dx \right) \gtrsim \eta \int_{|x| \leq \eta_0} |x|^{-\zeta} \psi_R(x)dx \\
\gtrsim \eta T^{\frac{n-\zeta}{\alpha}} \int_{|y| \leq \frac{\eta_0}{T^{1/\alpha}}} |y|^{-\zeta} \psi(y)dy \\
\gtrsim \eta T^{\frac{n-\zeta}{\alpha}} \int_{1 \leq |y| \leq \frac{\eta_0}{T^{1/\alpha}}} |y|^{-\zeta} \psi(y)dy \\
\gtrsim \eta T^{\frac{n-\zeta}{\alpha}},
\]

(58)

In a similar way to the above, we can get the final conclusion, here we omit the details.

\[\Box\]

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