The Kazhdan-Lusztig conjecture for finite W-algebras

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ABSTRACT

We study the representation theory of finite W-algebras. After introducing parabolic subalgebras to describe the structure of W-algebras, we define the Verma modules and give a conjecture for the Kac determinant. This allows us to find the completely degenerate representations of the finite W-algebras. To extract the irreducible representations we analyse the structure of singular and subsingular vectors, and find that for W-algebras, in general the maximal submodule of a Verma module is not generated by singular vectors only. Surprisingly, the role of the (sub)singular vectors can be encapsulated in terms of a ‘dual’ analogue of the Kazhdan-Lusztig theorem for simple Lie algebras. These involve dual relative Kazhdan-Lusztig polynomials. We support our conjectures with some examples, and briefly discuss applications and the generalisation to infinite W-algebras.

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1 Introduction

Since the introduction of W-algebras in conformal field theories by Zamolodchikov, there has been a tremendous effort to somehow classify a reasonable set of such algebras (see [1] for a review and further references). In that context it has been a particularly fruitful observation that a large class of W-algebras can be obtained from affine Kac-Moody algebras by hamiltonian reduction. Much of the power of this method resides in the fact that the inequivalent reductions are in one to one correspondence with a simple Lie algebraic structure: the inequivalent embeddings of sl₂ into g [2]. This also allows for a complete construction of the quantum W-algebras by a BRST procedure [3, 4].

On the other hand, relatively little is known about the representation theory of W-algebras, which in principle determines e.g. correlation functions and critical exponents of models with W-symmetry. The BRST method can be applied to study representations, by computing the cohomology of representations of the underlying Kac-Moody algebra [5]. However, this is a complicated problem, for which the general solution is still lacking.

In this paper we report on some progress in a direct approach: the analysis of the Verma modules of W-algebras. For simplicity we have first restricted to finite W-algebras. These algebras have recently been introduced as hamiltonian reductions of semisimple Lie algebras [6, 7]. They correspond to the zero mode algebra (which closes on the vacuum) of the corresponding infinite W-algebra. In this sense the representation theories of both types of algebras are also intimately linked.

Our main result is an explicit character formula for irreducible completely degenerate representations of the finite W-algebras, in terms of characters of Verma modules. The result is a generalisation of the Kazhdan-Lusztig (KL) theorems for regular integral highest weight representations of simple Lie algebras. In that case, the structure of Verma modules of g is governed by the Weyl group: the Weyl orbit of the highest weight predicts the location of singular vectors, and the Bruhat order of the Weyl group gives the embedding pattern [8]. The character formula involves an alternating sum over the orbit, with coefficients given by the KL polynomials [9].

In the case of finite W-algebras, remarkably all that happens is that the Weyl group is replaced by a poset and the KL polynomials are replaced by dual relative KL polynomials. The poset corresponds to the coset of the Weyl group of g over the Weyl group of gₛ, a minimal regular subalgebra that contains the embedded sl₂ principally. These posets also describe the structure of generalised Verma modules [10] of g, based on the parabolic subalgebra associated to gₛ. Each poset gives rise to two sets of relative KL polynomials [11], {P(x)} and \{\tilde{P}(x)\}, which are roughly speaking each others inverse. The ‘standard’ set \{P(x)\} gives the coefficients in the character formula in the case of generalised Verma modules [12]. The ‘dual’ set \{\tilde{P}(x)\}, which to our knowledge had not found any such interpretation before, describes the coefficients in the case of Verma modules of the finite W-algebras.

The outline of this letter is as follows. In section 2 the parabolic subalgebra gₛ is employed to describe the Cartan subalgebra (CSA) and roots of W-algebras. These notions are used in section 3 to define the Verma modules. We give the Kac determinant that describes the location of singular vectors. We define the completely degenerate representations, and formulate our generalisation of the KL conjectures. In section 4 two examples illustrate the results. We conclude with some comments on the case of infinite W-algebras, and mention applications.
2 Finite W-Algebras

Let g be an arbitrary simple Lie algebra and let δ be some grading element which decomposes g = g− ⊕ g0 ⊕ g+ in negative, zero and positive eigenspaces. The W-algebra is obtained from g by imposing constraints on generators of g+.

\[ g_+ - \chi(g_+) = 0, \]  

(2.1)

where \( \chi \) is some one-dimensional representation of g+. We will restrict to the case where δ is derived from some sl2-embedding since in that case the resulting W-algebra has a corresponding infinite dimensional analogue.

In the quantum version of this reduction [3, 4, 7] one imposes the constraint (2.1) using the BRST formalism. In the end, the W-generators are expressed in terms of g-currents. Although the BRST-construction of W-algebras is straightforward and algorithmic, the structure of the W-algebra emerges only after one has done the explicit calculations. Nonetheless: the W-algebra allows a triangular decomposition, i.e. a CSA and roots. To see this, we explore the connection between sl2 embeddings and parabolic subalgebras of g.

A parabolic subalgebra p of g is a subalgebra which contains the Borel subalgebra b of g: b ⊆ p ⊆ g, it is fixed by specifying a subset of the set of simple roots of g. Let \( \alpha_1, \ldots, \alpha_l \) be the simple roots of g, and S be any subset of \{1, \ldots, l\}. This subset defines a semisimple subalgebra \( g_S \) of g: it is generated by \( h_S \) (⊂ h the span of the \( h_i \) with \( i \in S \)) and the \( g_{\pm \alpha} \) with \( i \in S \). The set of roots is \( \Delta^S = \Delta \cap \Pi_{i \in S} Z \alpha_i \), and the set of positive roots is \( \Delta^S_+ = \Delta_+ \cap \Delta^S \). If we define the nilpotent subalgebras \( n_S = \Pi_{\alpha \in \Delta^S_+} g_\alpha \), then we have \( g_S = n_S^- \oplus h_S \oplus n_S \). The positive root generators of g outside \( g_S \) form a subalgebra \( u = \Pi_{\alpha \in \Delta^S_+} g_\alpha \) (with \( \Delta^S_+ = \Delta_+ \setminus \Delta^S_+ \)), and so do the negative root generators \( u^- = \Pi_{\alpha \in \Delta^S_+} g_{-\alpha} \). One now defines the subalgebras

\[ r = g_S + h, \quad p_S = r \oplus u. \]

(2.2)

The subalgebra r is reductive in g. \( p_S \) is the parabolic subalgebra of g associated with S. \( p_S \) has reductive part r and nilpotent part u. One can use these notions to define the associated decomposition

\[ g = u^- \oplus r \oplus u. \]

(2.3)

The two trivial cases are \( S = \emptyset \), where \( p_S = b \), and \( S = \{1, \ldots, l\} \), whence \( p_S = g \).

The relevance of this for the problem at hand is, that every sl2 embedding in g is principally embedded in a semisimple regular subalgebra \( g_S \) for some\(^1\) S. The branching of g into sl2 irreps can be done in two steps now. First consider the branching with respect to \( g_S \). By construction, the nilpotent subalgebras u and \( u^- \) will each decompose into irreps of \( g_S \). It is easy to find these irreps: define an equivalence relation on roots of u, such that for \( \alpha, \beta \in \Delta^S_+ \) we have \( \alpha \sim \beta \Leftrightarrow \alpha - \beta \in \Delta^S \). The equivalence classes in u correspond to the irreducible representations of \( g_S \) (and analogous for \( u^- \)). We label these irreps by \( \bar{\alpha} \) (and \( -\bar{\alpha} \) for \( u^- \)).

\(^1\)S is in general not uniquely determined by the embedding. Also for some g this correspondence is not complete.
These labels correspond to the eigenvalues with respect to the centre of $r$, denoted by $\bar{h}_S$, such that $h = h_S \oplus \bar{h}_S$ and

$$r = g_S \oplus \bar{h}_S.$$  

For each label $\bar{\alpha}$ there is a label for the associated irrep of $g_S$, say $\lambda$, and we have the decomposition

$$g = \bigoplus_{\lambda, \bar{\alpha} > 0} u_{\lambda, -\bar{\alpha}} \oplus (g_S \oplus \bar{h}_S) \oplus \bigoplus_{\lambda, \bar{\alpha} > 0} u_{\lambda, \bar{\alpha}}$$

In the second step, irreps of $g_S$ will break up into $\text{sl}_2$ irreps, each of which is associated with a $W$-generator. The generators divide into three sets denoted by $W^-$, $W^0$, and $W^+$ according to the decomposition (2.5).

The generators in $W^0$ form a rank-$g$-dimensional (maximal) abelian subalgebra (since the $\text{sl}_2$ is principally embedded in each simple factor of $g_S$, and $\bar{h}_S$ centralises $g_S$). Strictly speaking that does not make it into a CSA since not all generators in $W^0$ diagonalise the algebra. We find that there are three types of generators: 1) $U_1$ like generators, which are the generators of the CSA of $g$ that survived the reduction (corresponding to $\bar{h}_S$). These generators obviously diagonalise the algebra (the eigenvalues are the charges $\bar{\alpha}$); 2) central generators, since these generators commute with all other generators of the $W$-algebra (there is one such generator for each casimir of the largest simple factor in $g_S$) these generators only give rise to trivial roots; and 3) non-semisimple generators, the algebra cannot be diagonalised with respect to the adjoint action of these generators. So there are also no roots associated with these generators.

The $W$-generators in $W^+$ ($W^-$) are labelled by their charges with respect to the $U_1$-like generators (so $\pm \bar{\alpha}$). We denote the generators by $X^\mu_j, \pm \bar{\alpha}$ where $j$ is the $\text{sl}_2$ spin, $\bar{\alpha}$ the charge, and $\mu$ keeps track of degeneracies. We have the triangular decomposition familiar from Lie algebra theory:

$$W = W^- \oplus W^0 \oplus W^+.$$  

We stress that in contrast to ordinary Lie algebra theory, the rootspaces are now in general higher dimensional, due to the existence of the non-semisimple generators in $W^0$.

3 Highest weight representations and Verma modules

Given the triangular decomposition (2.6) we can define highest weight representations. By definition a highest weight representation $V$ contains a vector $v$ (highest weight vector) with the following properties: 1) the CSA acts diagonally on $v$; 2) the positive root generators annihilate $v$; 3) $V$ is generated by the action of the negative root generators.

Every highest weight representation is a quotient of a universal highest weight representation, the Verma module. These modules, denoted $M(\lambda)$, are freely generated by the negative root generators from the highest weight vector (for the moment, $\lambda$ somehow specifies the eigenvalue of the CSA on the highest weight vector). Naturally, one is interested in the irreducible highest weight representations, $L(\lambda)$, which can be obtained from the Verma modules by quotienting with respect to their maximal submodule. Therefore, one has to investigate the reducibility of the Verma modules.

Let us briefly summarise the results for a simple Lie algebra [8]. A Verma module is reducible
iff it contains a singular vector, i.e. a vector that is annihilated by the positive root generators. The weight of a singular vector is related to the highest weight \(\lambda\) by a shifted Weyl reflection

\[ w \cdot \lambda = w(\lambda + \rho) - \rho, \quad (3.1) \]

and vice versa: for all such weights in the Verma module there is at most one singular vector. Apart from the singular vectors there may also be subsingular vectors. Such vectors become singular only after one mods out the submodule generated by the singular vectors. Subsingular vectors can occur only at the same weights as the singular vectors, and therefore, the Jordan-Hölder series of a Verma module \(M(\lambda)\) (this is the decomposition of \(M(\lambda)\) in irreducible components \(L(\mu)\)) is given by a (finite) set of weights \(\mu\) of the form (3.1) which may occur with a multiplicity due to possible subsingular vectors. This results in a character formula for the irreducible representations in terms of characters of Verma modules. The coefficients in this formula are given by the KL polynomials, which have a geometrical interpretation as the dimensions of certain intersection cohomology groups associated to Schubert varieties. Restricting to regular integral weights, there is precisely one anti-dominant weight -say \(\lambda\) in the Weyl orbit. Define \(L_w = L(w \cdot \lambda)\) and \(M_w = M(w \cdot \lambda)\) for all \(w \in W\), the Weyl group of \(\mathfrak{g}\). Then we quote the following result [9]

\[ \text{ch} L_w = \sum_{y \leq w} \epsilon_y \epsilon_w P_{y,w}(1) \text{ch} M_y, \quad (3.2) \]

where the \(P_{y,w}(x)\) are the KL polynomials and \(\epsilon_w = (-1)^{l(w)}\) where \(l(w)\) is the length of \(w\).

We will propose an analogous formula for irreducible, completely degenerate representations of finite \(W\)-algebras. The Verma modules of the finite \(W\)-algebras are generated from a highest weight vector \(|\lambda^\mu\rangle\) which satisfies\(^2\)

\[ X_0^\mu |\lambda^\mu\rangle = \lambda^\mu |\lambda^\mu\rangle, \quad X_\alpha^\mu |\lambda^\mu\rangle = 0. \quad (3.3) \]

Ordering the roots with multiplicities, a basis for the Verma module is given by the states

\[ X_-^{\mu_1} X_-^{\mu_2} \cdots X_-^{\mu_n} |\lambda^\mu\rangle \quad \text{for} \quad (\bar{\alpha}_1, \mu_1) \geq (\bar{\alpha}_2, \mu_2) \geq \cdots \geq (\bar{\alpha}_n, \mu_n) \quad (3.4) \]

The anti-involution \(X_\alpha^\mu \mapsto X_-^\alpha\) allows one to define an invariant bilinear form on the Verma module. The determinant of this form is the Kac determinant, which contains information about the reducibility of the Verma module in the usual way.

**Conjecture 1** Consider the finite \(W\)-algebra associated to the \(\mathfrak{sl}_2\) embedding that is principal in \(\mathfrak{g}_S \subseteq \mathfrak{g}\). The Kac determinant at position \(\bar{\beta}\) in the Verma module \(M(\Lambda)\) is given by

\[ M_{\bar{\beta}}(\Lambda) = \prod_{k > 0} \prod_{\alpha \in \Delta_+^S} (\langle \Lambda + \rho, \alpha \rangle - \frac{k}{2} \langle \alpha, \alpha \rangle)^{P(\bar{\beta} - k\bar{\alpha})} \quad (3.5) \]

\(^2\)We ignore the spin label of the generators.
Here $P(\bar{\alpha})$ is the Kostant partition function for the restricted roots $\bar{\alpha}$ with their multiplicities (which gives the multiplicities of the W Verma module). The W-weights are parametrised by a $g$-weight $\Lambda$ in terms of invariants of the Weyl group $W_S$ of $g_S$.

Let us give some explanation. Products of factors in (3.5) corresponding to roots in the same irrep of $g_S$ are invariant\(^3\) under $W_S$. Therefore $M_{\bar{\beta}}(\Lambda)$ is a $W_S$-invariant polynomial, $M_{\bar{\beta}}(w \cdot \Lambda) = M_{\bar{\beta}}(\Lambda)$ for all $w \in W_S$. This means that it can be expressed in terms of the fundamental polynomial invariants of $W_S$. There are precisely rank $g$ independent invariants $W_S$, with degrees corresponding to the exponents of $g_S$ [13]. Our claim is that these invariants correspond one-to-one with the eigenvalues of the CSA generators of the W-algebra, in such a way that (3.5) is the Kac determinant, in completely factorised form. The degree one invariants correspond to the generators from $h_S$. Furthermore there is one central generator for each Casimir of the largest simple factor of $g_S$. The rest of the invariants is associated with the non-semisimple generators.

In principle, the explicit parametrisation of the W-weights in terms of $g$-weights $\Lambda$ can be extracted from the BRST-construction, which expresses the W-generators in terms of $g$-generators. However, then one also needs to determine the highest weight vector which in general is rather involved. A more practical way is to compute a few determinants directly to fix the parametrisation.

Given conjecture 1 we can now define the completely degenerate representations, which have a maximal number of vanishing factors in the Kac determinant. They are labelled by dominant regular integral weights of $g$ (up to reflections in $W_S$).

The $W_S$ invariance of the W-weights and the form (3.5) of the Kac determinant implies that the weights of the W-singular vectors are on the orbit of the coset $W_S \setminus W$. This coset inherits the Bruhat ordering of the Weyl group $W$, making it into a poset [13]. The embedding diagram of the W-singular vectors follows from this ordering on the poset.

To find the irreducible representations of finite W-algebras we need to settle the question of the subsingular vectors. In contrast to the case of simple Lie algebras, these vectors do show up in the completely degenerate representations. It follows from the Kac determinant that the weight of these subsingular vectors coincide with the weight of a singular vector. For the character formula this means that non-trivial coefficients may appear, as in the case of the general KL conjecture for simple Lie algebras. By studying some explicit examples we have found that these multiplicities are given by dual KL polynomials. This is formulated in the following:

**Conjecture 2** The KL theorem for finite W-algebras is given by

$$\text{ch} L^S_\tau = \sum_{\sigma \leq \tau \in W_S \setminus W} \epsilon_\sigma \epsilon_\tau \bar{P}^S_{\sigma, \tau}(1) \text{ch} M^S_\sigma,$$

where the $\bar{P}^S_{\sigma, \tau}(x)$ are the dual relative KL polynomials associated to the subgroup $W_S$.

Here $M^S_\tau$ and $L^S_\tau$ respectively denote a Verma module and an irreducible module for the W-algebra associated to the regular subalgebra $g_S$. They are now labelled by an element $\bar{\alpha}$. Note that $w(\alpha) \sim \alpha$ for $\alpha \in \bar{\Delta}_S^+$ and $w \in W_S$, and that if $\alpha \sim \beta$, then $\beta = w(\alpha)$ for some $w \in W_S$.\(^3\)
of the poset $W_S \setminus W$. We recall that these posets play a role in the theory of generalised Verma modules, analogous to the full Weyl group for the Verma modules [10]. Generalised Verma modules are certain quotients of Verma modules, defined when the highest weight is dominant integral with respect to some regular subalgebra $g_S$ of $g$. There is a ‘relative’ KL conjecture [12], which expresses the character of the irreducible representations in terms of characters of generalised Verma modules. The coefficients are given in terms of ‘standard’ relative KL polynomials $P^S\sigma$, which are defined for every such quotient of Coxeter groups [11]. There is a second set of relative KL polynomials $\hat{P}^S\sigma$, called ‘dual’ since they are the inverse of the standard relative KL polynomials in some sense. The relative KL polynomials are related to the standard KL polynomials [9] by

$$P^S\sigma,\tau = P_{w_Sx,w_SY}, \quad \hat{P}^S\sigma,\tau = \sum_{w \in W_S} \epsilon_w P_{wx,y}$$

(3.7)

where $x$ and $y$ are the minimal representatives of $\sigma$ and $\tau$ respectively and $w_S$ is the longest element in $W_S$. Remarkably, the dual polynomials appear in the above conjecture for the $W$-algebras.

4 Examples

We have explicitly verified the results of the preceding section for all finite $W$-algebras from $\mathfrak{sl}_2, \mathfrak{sl}_3$ and $\mathfrak{sl}_4$. We discuss two examples here. The main focus is on the structure of the $W$-algebra and the representations. For details on the construction of the algebra we refer to [7], the calculation of the relative KL-polynomials can be found in [14].

$\mathfrak{sl}_2 \to \mathfrak{sl}_2 + \mathfrak{sl}_1$

Consider the finite $W$-algebra associated with the $\mathfrak{sl}_2$ embedding that is principal in $g_S = \mathfrak{sl}_2 \subset \mathfrak{sl}_3$. The parabolic subalgebra is fixed by choosing the $\mathfrak{sl}_2$ associated with the root $\alpha_1$ (i.e. $S = \{1\}$). The W-algebra corresponding to this embedding is a polynomial deformation of $\mathfrak{sl}_2$, which was discussed in [6]. We first summarise their results. There are 4 generators $\{e, f, h, w_2\}$ with $w_2$ a central element and

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h^2 + w_2. \quad (4.1)$$

On a Verma module with highest weights $(h, w_2)$ the Kac determinant at weight $(h - 2p, w_2)$ reads

$$\prod_{k=1}^{p} (w_2 + (h + 1 - k)^2 + \frac{1}{3}(k^2 - 1))$$

(4.2)

The completely degenerate representations occur for

$$h = \frac{2}{3}(p + k) - 1, \quad w_2 = -\frac{4}{9}(p^2 - pk + k^2) - \frac{1}{3}$$

(4.3)

where $p$ is an arbitrary positive integer and $k$ is a positive integer smaller than $p$. These representations consist of reducible $p$-dimensional representations with an irreducible $k$-dimensional
To rederive these results in our framework we need the parametrisation of the W-weights \( h, w_2 \) in terms of invariants of the Weyl group \( Z_2 \) of \( g \) which is generated by \( w_{\alpha_1} \). Using the explicit expressions for the W-generators in terms of the \( sl_3 \) currents from the BRST-construction we find

\[
\begin{align*}
    h &= \frac{1}{3}(2\Lambda_1 + 4\Lambda_2 + 3), \\
    w_2 &= -\frac{4}{3} < \Lambda, \Lambda + 2\rho > -1.
\end{align*}
\]

With this parametrisation, the Kac determinant (4.2) factorises as

\[
\prod_{k=1}^{p} (\Lambda_2 + 1 - k)(\Lambda_1 + \Lambda_2 + 2 - k),
\]

which is in agreement with (3.5) since \( \Delta_+^S \) consists of the roots \( \alpha_2 \) and \( \alpha_1 + \alpha_2 \). Also, one may now parametrise the completely degenerate representations (4.3) in terms of \( \Lambda \) using (4.4), it is easy to verify that this gives rise to two sets of solutions

\[
\begin{align*}
    \begin{cases}
        \Lambda_1 = p - k - 1 \\
        \Lambda_2 = k - 1
    \end{cases},
\end{align*}
\]

\[
\begin{align*}
    \begin{cases}
        \Lambda_1 = k - p + 1 \\
        \Lambda_2 = p - 1
    \end{cases}.
\end{align*}
\]

The first set corresponds exactly to the integral regular weights in the dominant Weyl chamber, the second set consists of the weights in the \( w_{\alpha_1} \)-reflected Weyl chamber.

To arrive at a character formula we consider the Jordan-Hölder-decomposition of the Verma module \( M_{h,w_2} \). Fix the parametrisation (4.6) of the completely degenerate weights by taking \( \Lambda \) in the dominant Weyl chamber. Clearly \( w_2 \) is Weyl-invariant, but the Weyl-orbit of the weight \( h \) of the singular vector (4.4) consists of three elements in the coset \( Z_2 \setminus S_3 \) (see figure 1). Denoting the Verma modules by \( M_\sigma \) with \( \sigma = \{e, 2, 21\} \) and the irreducible representation

![Diagram](image)

Figure 1: Embedding structure of the singular vectors in the completely degenerate Verma module of W-algebras associated with the embedding \( 3 \rightarrow 2 + \frac{1}{2} \) (left) and \( 4 \rightarrow 2 + 2 \) (right). The numbers denote simple reflections, \( hw \) is the highest weight.

at the same weight by \( L_\sigma \), as in the previous section, it is easy to see that

\[
\begin{align*}
    \text{ch} M_{21} &= \text{ch} L_{21} + \text{ch} L_2 + \text{ch} L_e, \\
    \text{ch} M_2 &= \text{ch} L_2 + \text{ch} L_e, \\
    \text{ch} M_e &= \text{ch} L_e.
\end{align*}
\]

(4.7)
Inverting these relations we obtain
\[
\text{ch}L_{21} = \text{ch}M_{21} - \text{ch}M_2, \quad \text{ch}L_2 = \text{ch}M_2 - \text{ch}M_e, \quad \text{ch}L_e = \text{ch}M_e.
\] (4.8)

From this one reads off the multiplicities \( \tilde{P}_{\sigma,\tau} \) in eq. (3.6), and they indeed coincide with the KL-polynomials of this example (see table 1).

| \( P \) | \( e \) | 2 | 21 |
|-------|-----|----|----|
| \( e \) | 1 | 1 | 1 |
| 2 | 1 | 1 |
| 21 | 1 |

| \( \tilde{P} \) | \( e \) | 2 | 21 |
|-------|-----|----|----|
| \( e \) | 1 | 1 | 0 |
| 2 | 1 | 1 |
| 21 | 1 |

Table 1: Relative KL matrices for \( S = \{1\} \) in \( \mathfrak{sl}_3 \).

\( \mathfrak{sl}_2 \rightarrow 2 + 2 \)

Consider the W-algebra that corresponds to the \( \mathfrak{sl}_2 \) embedding that is principal in \( \mathfrak{g}_S = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \subset \mathfrak{sl}_4 \), with \( S = \{1,3\} \). It has 7 generators \( \{w_2, J^a, S^a\} \) where \( a = 0, +, - \). The generator \( w_2 \) is central and
\[
[J^a, J^b] = f^{ab}_c J^c, \quad [J^a, S^b] = f^{ab}_c S^c, \quad [S^a, S^b] = (w_2 - C_2) f^{ab}_c J^c,
\] (4.9)

with normalisation \( f^{0+}_+ = f^{0-}_- = f^{+-}_0 = 1 \). So the \( J^a \) form an \( \mathfrak{sl}_2 \) subalgebra, and the \( S^a \) are in the adjoint representation, the nonlinearity is hidden in the last commutator in (4.9) where \( C_2 = 2(J^0 J^0 + J^+ J^- + J^- J^+) \) is the Casimir of the \( \mathfrak{sl}_2 \) subalgebra, and this term does not commute with the \( S^a \).

The CSA is generated by \( \{J^0, S^0, w_2\} \), but \( S^0 \) does not diagonalise the algebra so the weight lattice is one-dimensional with multiplicities \( P(i) = i + 1 \). Using (3.5) we find that the Kac determinant \( M_{j-p}(j, s, w_2) \) at weight \( j - p \) reads
\[
\prod_{k=1}^p \left(s^2 - (w_2 - j(j + 2) - (j + 1 - k)^2)(j + 1 - k)^2\right)^{p-k+1},
\] (4.10)

where we have used the parametrisation
\[
j = \frac{1}{2}(\Lambda_1 + 2\Lambda_2 + \Lambda_3 + 2), \quad s = \frac{1}{4}(\Lambda_1 - \Lambda_3)(\Lambda_1 + \Lambda_3 + 2),
\] (4.11)

and \( w_2 = \langle \Lambda, \Lambda + 2\rho \rangle > +4 \). These weights are invariant under the Weyl subgroup \( Z_2 \times Z_2 \) generated by the reflections \( \{1,3\} \), For a dominant integral weight the orbit under the coset \( (Z_2 \times Z_2) \setminus S_4 \) is depicted in figure 1.

At first sight, this embedding pattern is puzzling: all singular vectors are descendant of the singular vector with weight \( w_2 \cdot \Lambda \). So if one divides out the submodule generated by the
singular vectors one obtains an infinite dimensional representation, with character $\text{ch}L_{2132} = \text{ch}M_{2132} - \text{ch}M_{213}$. However, as is signaled by the non-trivial multiplicity in the KL-matrix in table 2: there is a subsingular vector with weight $w_{213} \cdot \Lambda$. Adding this vector to the maximal submodule -taking into account the overlap of the submodules- results in the correct finite dimensional irreducible representation with character

$$\text{ch}L_{2132} = \text{ch}M_{2132} - \text{ch}M_{213} - \text{ch}M_2 + \text{ch}M_e.$$  

Again this result is reproduced correctly by (3.6) using the KL-polynomials of table 2.

| $P$ | e  | 2   | 21  | 23  | 213 | 2132 |
|-----|----|-----|-----|-----|-----|------|
| e   | 1  | 1   | 1   | 1   | 2   | 1    |
| 2   | 1  | 1   | 1   | 1   | 1   | 1    |
| 21  | 1  | 1   | 1   | 1   |     |      |
| 23  | 1  | 1   | 1   |     |     |      |
| 213 | 1  | 1   |     |     |     |      |
| 2132| 1  |     |     |     |     |      |

Table 2: Relative KL matrices for $S = \{1, 3\}$ in $\mathfrak{sl}_4$.

5 Discussion

We have studied the representation theory of finite W-algebras. After recognizing the relevance of parabolic subalgebras to the description of the structure of W-algebras, we have defined the Verma modules and given the Kac determinant (3.5). This allowed us to find the completely degenerate representations of the finite W-algebras. To extract the irreducible representations we have analysed the structure of singular and subsingular vectors, and found that for W-algebras, in general the maximal submodule of a Verma module is not generated by singular vectors only. Surprisingly, the role of the (sub)singular vectors can be encapsulated in terms of a ‘dual’ analogue (3.6) of the KL-theorem for simple Lie algebras, which involves the dual relative KL polynomials.

Note that the correspondence between $\mathfrak{sl}_2$ embeddings and parabolic subalgebras may not be one-to-one. An ambiguity in the choice of parabolic subalgebras leads to different types of Verma modules for the same W-algebra (corresponding to a different choice of positive W-roots), and different character formulas. For finite irreps, this results in some interesting character identities.

In all the examples we studied we were able to define some limit in which the W-algebra reduces to the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$, i.e. the W-algebra can be viewed as a deformation of $\mathfrak{g}_0$. The deformation results in the truncation of $\mathfrak{g}_0$ irreps to W irreps. This is not so surprising since the W-algebras are symmetry algebras of non-abelian Toda models, i.e. $\mathfrak{g}_0$ sigmamodels with a non-trivial selfinteraction [6]. The relation to the quantum Miura transformation of [7] needs to be investigated.

For the W-algebra $4 \to 2 + 2$ there is a nice physical interpretation of the truncation. The algebra (4.9) is the dynamical symmetry algebra of a particle moving in a 3-dimensional
space with constant curvature under the influence of a Coulomb potential [15]. For negative curvature, the Casimirs $D_1, D_2$ of the algebra (4.9) are subjected to the physical constraints

$$D_1 = -J.S = 0, \quad D_2 = -S^2 - \frac{1}{2}C_2^2 + (w_2 - 2)C_2 = \nu^2,$$

(5.1)

where $\nu \equiv R/R_B$ measures the curvature [15]. The Hamiltonian $H = -w_2/2$ has a spectrum $E_j = -(j(j + 2) + \nu^2/(j + 1)^2)/2$, as follows directly from (5.1). For zero curvature, where the algebra reduces to the (linear) Runge-Lenz algebra $O(4)$, all states with $E < 0$ are bound states. The non-zero negative curvature effectively truncates this bound spectrum at the point where the energy levels start to cross. It is easily verified that is exactly as predicted by the Kac determinant (4.10).

Obviously, one would like to generalise these results to infinite W-algebras, and in fact this seems rather straightforward, since all the necessary ingredients already exist. The parametrisation of the W-weights is fixed by the analysis of the underlying finite W-algebra, and the relative KL polynomials are also defined, although they are no longer related by an inversion relation. The explicit verification will be a lot harder though, and a general proof of the conjectures is lacking so far. Work on this is in progress.

Acknowledgement. KdV would like to thank G. Watts, A. Kent and F. Malikov for discussions, and P. Bowcock for drawing our attention to [15]. PvD acknowledges discussions with K. Pilch and P. Bouwknegt.

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