Extrapolated sequential constraint method for variational inequality over the intersection of fixed-point sets

Mootta Prangprakhon · Nimit Nimana

Received: 25 June 2020 / Accepted: 3 January 2021 / Published online: 24 March 2021 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC part of Springer Nature 2021

Abstract
This paper deals with the solving of variational inequality problem where the constrained set is given as the intersection of a number of fixed-point sets. To this end, we present an extrapolated sequential constraint method. At each iteration, the proposed method is updated based on the ideas of a hybrid conjugate gradient method used to accelerate the well-known hybrid steepest descent method, and an extrapolated cyclic cutter method for solving a common fixed-point problem. We prove strong convergence of the method under some suitable assumptions of step-size sequences. We finally show the numerical efficiency of the proposed method compared to some existing methods.

Keywords Conjugate gradient direction · Cutter · Fixed point · Hybrid steepest descent method · Variational inequality

1 Introduction
In this paper, we consider the following variational inequality problem:

Problem 1 Let $T_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, m$, be cutters with $\bigcap_{i=1}^{m} \text{Fix} T_i \neq \emptyset$, and let $F : \mathcal{H} \to \mathcal{H}$ be $\eta$-strongly monotone and $\kappa$-Lipschitz continuous. Then, our objective is to find a point $\bar{u} \in \bigcap_{i=1}^{m} \text{Fix} T_i$ such that

$$\langle F(\bar{u}), z - \bar{u} \rangle \geq 0 \text{ for all } z \in \bigcap_{i=1}^{m} \text{Fix} T_i.$$
Attentively, Problem 1 has a bilevel structure, namely, its outer level given by the variational inequality governed by the operator $F$, while the constrained set is the inner level problem, which is the common fixed-point problem of cutter operators. We emphasize here that Problem 1 allows not only a generalization of the constrained set, but also various applications for modelling real-world problems like network location problems [19, 20, 22, 24] and machine learning [23], to name but a few.

For simplicity, we denote by $\text{VIP}(F,C)$ a variational inequality problem corresponding to an operator $F$ and a nonempty closed convex set $C$. In the literature, the simplest iterative algorithm for solving $\text{VIP}(F,C)$ is the well-known projected gradient method (PGM) [17]. The method essentially has the form:

\begin{align}
\left\{ \begin{array}{l}
    x^1 \in C \text{ is arbitrarily chosen}, \\
    x^{n+1} = \text{proj}_C(x^n - \mu F(x^n)),
\end{array} \right. 
\end{align}

for every $n \in \mathbb{N}$, where $\text{proj}_C : \mathcal{H} \to C$ is the metric projection onto $C$, $F : \mathcal{H} \to C$ is $\eta$-strongly monotone and $\kappa$-Lipschitz continuous over $C$ and $\mu \in (0, 2\eta/\kappa^2)$. It was proved that the sequence $\{x^n\}_{n=1}^{\infty}$ generated by (1) converges strongly to the unique solution of $\text{VIP}(F,C)$ in [17]. As PGM requires the use of the metric projection $\text{proj}_C$, it is perfectly suitable for the case when $C$ is simple enough in the sense that $\text{proj}_C$ has a closed-form expression. However, in many practical situations, the structure of $C$ can be highly intricate and, in consequence, $\text{proj}_C$ is difficult to evaluate. To overcome the above limitation, Yamada [34] proposed the celebrated hybrid steepest descent method (HSDM) which essentially replaces the use of $\text{proj}_C$ in (1) with an appropriate nonexpansive operator $T$. By interpreting $C$ as the fixed-point set of $T$, the method is defined by the following:

\begin{align}
\left\{ \begin{array}{l}
    u^1 \in \mathcal{H} \text{ is arbitrarily chosen}, \\
    u^{n+1} = Tu^n - \mu \beta_n F(Tu^n),
\end{array} \right. 
\end{align}

for every $n \in \mathbb{N}$, where $F : \mathcal{H} \to \mathcal{H}$ is $\eta$-strongly monotone and $\kappa$-Lipschitz continuous over $\mathcal{H}$, and $\mu \in (0, 2\eta/\kappa^2)$. Moreover, it is well-known that, under some certain conditions on $\{\beta_n\}_{n=1}^{\infty} \subset (0, 1)$, the sequence $\{u^n\}_{n=1}^{\infty}$ generated by (2) converges strongly to the unique solution of $\text{VIP}(F, \text{Fix}T)$, where $\text{Fix}T := \{x \in \mathcal{H} : Tx = x\}$ assuming that $T$ is nonexpansive. It is worth noticing that the form (2) is equivalent to the form:

\begin{align}
\left\{ \begin{array}{l}
    x^1 \in \mathcal{H} \text{ is arbitrarily chosen}, \\
    x^{n+1} = T(x^n - \mu \beta_n F(x^n)),
\end{array} \right. 
\end{align}

whenever we set $u^{n+1} = x^n - \mu \beta_n F(x^n)$ and $x^n = Tu^n$. Of course, the sequence $\{x^n\}_{n=1}^{\infty}$ also converges strongly to the unique solution of $\text{VIP}(F, \text{Fix}T)$.

Furthermore, let us note that, in the context of (2), if $F := \nabla f$ where $f : \mathcal{H} \to \mathbb{R}$ is a convex, continuously Fréchet differentiable functional, HSDM thus solves $\text{VIP}(\nabla f, \text{Fix}T)$, which is nothing else than the convex minimization problem over the fixed-point set of a nonexpansive operator. On the other hand, it is well-known that the conjugate gradient method (CGM) [12, 14, 16, 30] and the three-term conjugate gradient method (TCGM) [36–38] have great efficacy in decreasing the function $f$ value rapidly. According to these underline motivations, several modifications among HSDM, CGM, and TCGM are proposed in order to accelerate HSDM, namely, the hybrid conjugate gradient method (HCGM) [25], the hybrid three-term conjugate
gradient method (HTCGM) [18, 29], and the accelerated hybrid conjugate gradient method (AHCGM) [21]. As a matter of fact, HCGM and HTCGM are relatively similar in some basic structures and some additional conditions needed to ensure their convergences. In addressing such procedures, their common form is as follows:

\[
\begin{align*}
\begin{cases}
    x^1 \in \mathcal{H} & \text{is arbitrarily chosen}, \\
    d^1 &= -\nabla f(x^1), \\
    x^{n+1} &= T(x^n + \mu \beta_n d^n),
\end{cases}
\end{align*}
\]

for every \( n \in \mathbb{N} \), where \( \mu \in (0, 2\eta/\kappa^2) \), \( \{\beta_n\}_{n=1}^{\infty} \subset (0, 1] \) is a step size and \( \{d^n\}_{n=1}^{\infty} \in \mathcal{H} \) is a search direction. However, it is worth mentioning that the search directions of these methods are slightly different, that is, the search direction of HCGM is defined by

\[
d^n = -\nabla f(x^n) + \varphi^{(1)}_n d^{n-1},
\]

meanwhile the search direction of HTCGM is defined by

\[
d^n = -\nabla f(x^n) + \varphi^{(1)}_n d^{n-1} - \varphi^{(2)}_n w^n,
\]

for every \( n \in \mathbb{N} \), where \( \{\varphi^{(i)}_n\}_{n=1}^{\infty} \subset [0, \infty) (i = 1, 2) \) and \( \{w^n\}_{n=1}^{\infty} \in \mathcal{H} \) is arbitrarily chosen. Then, it was proved in [25] and [18] that, under some certain assumptions on \( \{\beta_n\}_{n=1}^{\infty} \subset (0, 1] \), each sequence generated by HCGM and HTCGM converges strongly to the unique solution of VIP(\( \nabla f \), Fix\( T \)) whenever \( \lim_{n \to \infty} \varphi^{(i)}_n = 0 (i = 1, 2) \), and the sequences \( \{\nabla f(x^n)\}_{n=1}^{\infty} \) and \( \{w^n\}_{n=1}^{\infty} \) are bounded.

Next, let us review some sequential methods used for solving the common fixed-point problem (in short, CFPP). Namely, let \( T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, m \), be nonlinear operators, the problem is to find \( x^* \in \bigcap_{i=1}^{m} \text{Fix} T_i \), provided that the intersection is nonempty. A classical sequential method for solving CFPP was developed from an iterative method introduced by Kaczmarz [26] who firstly aimed to solve a linear system in \( \mathbb{R}^n \). The method was referred to the cyclic projection method (CPM) or Kaczmarz method (KM) which has the form:

\[
\begin{align*}
\begin{cases}
    x^1 \in \mathcal{H} & \text{is arbitrarily chosen}, \\
    x^{n+1} &= \text{proj}_{C_m} \cdots \text{proj}_{C_1} x^n,
\end{cases}
\end{align*}
\]

where proj\( _{C_i} \) are the metric projections onto the linear equations \( C_i \subset \mathcal{H}, i = 1, 2, \ldots, m \). After that, the general case when \( C_i \subset \mathcal{H}, i = 1, 2, \ldots, m \), are nonempty closed and convex subsets was considered by Bregman [4]. It was proved that the sequence generated by (7) converges weakly to a solution of CFPP. As the interest in the aforementioned results continuously increases, it is well-known that, under some additional hypotheses, the convergence of CPM is true for a wider class of operators such as nonexpansive operators or cutter operators [5, 6, 8, 27, 31]. In particular, the latter is a key tool of a method called the cyclic cutter method (CCM) whose weak convergence was proved by Bauschke and Combettes [2]. In order to accelerate the convergence of CCM, Cegielski and Censor [9] proposed the so-called extrapolated cyclic cutter method (ECCM) which essentially requires the use of an
appropriate step-size function $\sigma : \mathcal{H} \to (0, \infty)$ to speed up numerically the convergence behavior. Indeed, let $T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, m$, be cutters with $\bigcap_{i=1}^{m} \text{Fix}T_i \neq \emptyset$, define $T := T_mT_{m-1}\cdots T_1$, $S_0 := I_d$, and $S_i := T_iT_{i-1}\cdots T_1$, then they defined the step-size function $\sigma$ as

$$
\sigma(x) := \begin{cases} 
\sum_{i=1}^{m}(Tx - S_{i-1}x, S_i x - S_{i-1}x) & \text{for } x \notin \bigcap_{i=1}^{m} \text{Fix}T_i, \\
\frac{1}{\|Tx - x\|^2} & \text{otherwise}.
\end{cases}
$$

Moreover, it was shown that ECCM converges weakly whenever the cutter operators $T_i, i = 1, 2, \ldots, m$, satisfy the demi-closedness principle. Along the line of [9], Cegielski and Nimana [10] indicated that there are some practical situations in which the value of the extrapolation function $\sigma$ can be enormously large, which consequently may produce some uncertainties in numerical experiments. In order to avoid these situations, they proposed an algorithm called the modified extrapolated cyclic subgradient projection method (MECSPM). The main idea of this method is to map each iterate obtaining from ECCM via the last subgradient projection. If the constrained sets are nonempty closed convex sets, the modification is nothing else than the projecting a sequence generated by ECCM into the last constraint set. To conclude, the aforementioned methods used for solving variational inequality problem and common fixed-point problem are concisely summarized in Table 1. Another direction of solving VIP($F, C$) is the generalization of HSDM where the subset $C$ can be represented as the intersection of infinitely many fixed-point sets of some certain quasi-nonexpansive operators; see some interesting works of Cegielski and Zalas [11], Aoyama and Kohsaka [1], Cegielski [7], and Gibali et al. [15].

The main contribution of this paper is an iterative algorithm called the extrapolated sequential constraint method with conjugate gradient direction (ESCoM-CGD) used for solving the variational inequality problem over the intersection of the fixed-point sets. To construct the algorithm, we utilize some ideas of the aforementioned methods, namely, HCGM [25] and MECSPM [10]. Under the context of cutter operators and some certain conditions, we establish strong convergence of the proposed algorithm. In order to demonstrate the effectiveness and the performance of the algorithm, we present numerical results and numerical comparisons of the algorithm with some existing methods such as HCGM and HTCGM.

The remainder of this paper is organized as follows. In Section 2, we collect some useful definitions and results needed in the paper. In Section 3, we introduce ESCoM-CGD used for solving Problem 1 and subsequently analyze its convergence result. In Section 4, we derive an important situation of the considered problem by means of the subgradient projection. In Section 5, the efficacy of ESCoM-CGD is illustrated by some numerical results. Finally, we give some concluding remarks in Section 6.

2 Preliminaries

Throughout the paper, $\mathcal{H}$ is always a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and with the norm $\| \cdot \|$. For a sequence $(x^n)_{n=1}^{\infty}$, the expressions $x^n \rightharpoonup x$ and $x^n \to x$
Table 1  Summary of the corresponding iterative methods used for solving Problem 1

| Reference                      | Problem       | Method     | Constrained operator |
|-------------------------------|---------------|------------|----------------------|
| Goldstein [17]                | VIP($F, C$)   | PGM        | Metric projection    |
| Yamada [34]                   | VIP($F, FixT$)| HSDM       | Nonexpansive         |
| Iiduka and Yamada [25]        | VIP($\nabla f, FixT$) | HCGM       | Nonexpansive         |
| Iiduka [18]                   | VIP($\nabla f, FixT$) | HTCGM      | Nonexpansive         |
| Bregman [4]                   | CFPP          | CPM        | Metric projection    |
| Bauschke and Combettes [2]    | CFPP          | CCM        | Cutter               |
| Cegielski and Censor [9]      | CFPP          | ECCM       | Cutter               |
| Cegielski and Nimana [10]     | CFPP          | MECSPM     | Subgradient projection|
| This work                     | VIP($F, \bigcap_{i=1}^m FixT_i$) | ESCoM-CGD  | Cutter               |

denote $\{x^n\}_{n=1}^\infty$ converges to $x$ weakly and converges to $x$ in norm, respectively. $Id$ represents the identity operator on $\mathcal{H}$.

An operator $F : \mathcal{H} \to \mathcal{H}$ is said to be $\eta$-strongly monotone if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2$, for all $x, y \in \mathcal{H}$, and is said to be $\kappa$-Lipschitz continuous if there exists a constant $\kappa > 0$ such that $\|Fx - Fy\| \leq \kappa \|x - y\|$, for all $x, y \in \mathcal{H}$.

The following lemma found in [34, Lemma 3.1(b)] will be useful in the sequel.

**Lemma 1** Suppose that $F : \mathcal{H} \to \mathcal{H}$ is $\eta$-strongly monotone and $\kappa$-Lipschitz continuous. For any $\mu \in (0, 2\eta/\kappa^2)$ and $\beta \in (0, 1]$, define the operator $T^{\beta} : \mathcal{H} \to \mathcal{H}$ by $T^{\beta} := Id - \mu \beta F$. Then

$$\|T^{\beta} x - T^{\beta} y\| \leq (1 - \beta \tau) \|x - y\|,$$

for all $x, y \in \mathcal{H}$, where $\tau := 1 - \sqrt{1 + \mu^2 \kappa^2 - 2 \mu \eta} \in (0, 1]$.

**Remark 1** It is worth to notice that the well-definedness of the parameter $\tau \in (0, 1]$ is guaranteed by the assumption of $F$. Indeed, the monotonicity of $F$ and the Cauchy-Schwarz inequality yield that $\eta \|x - y\|^2 \leq \langle F(x) - F(y), x - y \rangle \leq \|F(x) - F(y)\| \|x - y\|$, and hence $\eta \|x - y\| \leq \|F(x) - F(y)\|$. Due to the Lipschitz continuity of $F$, we obtain $\|F(x) - F(y)\| \leq \kappa \|x - y\|$, which implies that $0 < \eta \leq \kappa$.

Thus, we have $0 < \frac{2\eta}{\kappa^2}$. Setting $\mu \in (0, \frac{2\eta}{\kappa^2})$, we obtain

$$0 \leq (1 - \mu \kappa)^2 \leq 1 + \mu^2 \kappa^2 - 2 \mu \eta < 1.$$

Therefore

$$0 < 1 - \sqrt{1 + \mu^2 \kappa^2 - 2 \mu \eta} \leq 1,$$

which means that $\tau \in (0, 1]$. 

\[ Springer \]
Below, some concepts of quasi-nonexpansivity of operators are presented for the sake of further use. More details can be found in [6, Section 2.1.3].

An operator \( T : \mathcal{H} \to \mathcal{H} \) with \( \text{Fix} T \neq \emptyset \) is said to be quasi-nonexpansive if \( \| Tx - z \| \leq \| x - z \| \), for all \( x \in \mathcal{H} \) and for all \( z \in \text{Fix} T \), is said to be \( \rho \)-strongly quasi-nonexpansive, where \( \rho \geq 0 \), if \( \| Tx - z \|^2 \leq \| x - z \|^2 - \rho \| Tx - x \|^2 \), for all \( x \in \mathcal{H} \) and for all \( z \in \text{Fix} T \), and, is said to be a cutter if \( \langle x - Tx, z - Tx \rangle \leq 0 \), for all \( x \in \mathcal{H} \) and for all \( z \in \text{Fix} T \).

The following facts can be found in [7, Fact 2.1 and 2.3].

**Fact 1** If \( T : \mathcal{H} \to \mathcal{H} \) is quasi-nonexpansive, then \( \text{Fix} T \) is closed and convex.

**Fact 2** Let \( T : \mathcal{H} \to \mathcal{H} \) be an operator. The following conditions are equivalent:

1. \( T \) is a cutter.
2. \( \langle Tx - x, z - x \rangle \geq \| Tx - x \|^2 \) for every \( x \in \mathcal{H} \) and \( z \in \text{Fix} T \).
3. \( T \) is 1-strongly quasi-nonexpansive.

We recall a notion of the demi-closedness principle in the following definition.

**Definition 1** An operator \( T : \mathcal{H} \to \mathcal{H} \) is said to satisfy the demi-closedness (DC) principle if \( T - \text{Id} \) is demi-closed at 0, that is, for any sequence \( \{x^n\}_{n=1}^{\infty} \subset \mathcal{H} \), if \( x^n \rightharpoonup y \in \mathcal{H} \) and \( \| (T - \text{Id})x^n \| \to 0 \), then \( Ty = y \).

Furthermore, we recall that an operator \( T : \mathcal{H} \to \mathcal{H} \) is said to be nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \), for all \( x, y \in \mathcal{H} \). It is worth mentioning that if \( T : \mathcal{H} \to \mathcal{H} \) is a nonexpansive operator with \( \text{Fix} T \neq \emptyset \), then the operator \( T \) satisfies the DC principle (see [35, Lemma 2]).

For an operator \( T : \mathcal{H} \to \mathcal{H} \) and a real number \( \lambda \in [0, 2] \), the operator \( T_{\lambda} := (1 - \lambda)\text{Id} + \lambda T \) is called a relaxation of \( T \) and \( \lambda \) is called a relaxation parameter. Actually, in many situations, the relaxation parameter which is greater than 2 may yield a superiority of algorithmic convergence property. So, we are now in a position to recall a generalized relaxation of an operator. The generalized relaxation of an operator \( T : \mathcal{H} \to \mathcal{H} \) is defined by \( T_{\sigma,\lambda} : \mathcal{H} \to \mathcal{H} \) where \( \sigma : \mathcal{H} \to (0, \infty) \) is a step-size function. If \( \sigma (x) \geq 1 \) for all \( x \in \mathcal{H} \), then the operator \( T_{\sigma,\lambda} \) is called an extrapolation of \( T \). In the case that \( \sigma (x) = 1 \), for all \( x \in \mathcal{H} \), the generalized relaxation of \( T \) is reduced to the relaxation of \( T \), that is \( T_{\sigma,\lambda} = T_{\lambda} \). We denote here that \( T_{\sigma} := T_{\sigma,1} \). For any \( x \in \mathcal{H} \), it can be noted that

\[
T_{\sigma,\lambda}x = x + \lambda \sigma (x) (Tx - x),
\]
i.e., \( T_{\sigma,\lambda}x = x + \lambda (T_{\sigma}x - x) \), and

\[
\text{Fix} T_{\sigma,\lambda} = \text{Fix} T_{\sigma} = \text{Fix} T,
\]
for any \( \lambda \neq 0 \).

The following lemma plays an important role in proving our convergence result. The proof can be found in [6, Section 4.10].
Lemma 2 Let \( T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, m, \) be cutters with \( \bigcap_{i=1}^{m} \text{Fix}T_i \neq \emptyset, \) and denote \( T := T_m T_{m-1} \cdots T_1. \) Let \( \sigma : \mathcal{H} \to (0, \infty) \) be defined by (8), then the following properties hold:

(i) For any \( x \notin \text{Fix}T, \) we have
\[
\sigma(x) \geq \frac{1}{2} \sum_{i=1}^{m} \frac{\|S_i x - S_{i-1} x\|^2}{\|T x - x\|^2} \geq \frac{1}{2m},
\]
where \( S_0 = \text{Id} \) and \( S_i = T_i T_{i-1} \cdots T_1. \)

(ii) The operator \( T_\sigma \) is a cutter.

3 Algorithms and convergence results

In this section, we start with introducing a new iterative algorithm for solving Problem 1 and subsequently study its convergence result. For the sake of convenience, we denote the following notations: the compositions \( T := T_m T_{m-1} \cdots T_1, \) \( S_0 := \text{Id}, \) and \( S_i := T_i T_{i-1} \cdots T_1, i = 1, 2, \ldots, m, \) where \( T_i : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, m, \) are cutters with \( \bigcap_{i=1}^{m} \text{Fix}T_i \neq \emptyset. \)

The iterative method for solving Problem 1 is presented as follows.

Algorithm 1 ESCoM-CGD.

**Initialization:** Given \( \mu \in (0, 2\eta/\kappa^2), \{\beta_n\}_{n=1}^{\infty} \subset (0, 1], \) \( \{\varphi_n\}_{n=1}^{\infty} \subset [0, \infty) \) and a positive sequence \( \{\lambda_n\}_{n=1}^{\infty}. \) Choose \( x^1 \in \mathcal{H} \) arbitrarily and set \( d^1 = -F(x^1). \)

**Iterative Steps:** For a current iterate \( x^n \in \mathcal{H} \) and direction \( d^n \in \mathcal{H} (n \in \mathbb{N}), \) calculate as follows:

**Step 1.** Compute \( y^n \) and the step size as
\[
y^n := x^n + \mu \beta_n d^n
\]
and
\[
\sigma(y^n) := \begin{cases} \frac{\sum_{i=1}^{m} (Ty^n - S_{i-1} y^n, S_i y^n - S_{i-1} y^n)}{\|Ty^n - y^n\|^2}, & \text{for } y^n \notin \bigcap_{i=1}^{m} \text{Fix}T_i, \\ 1, & \text{otherwise.} \end{cases}
\]

**Step 2.** Compute the next iterate and the search direction as
\[
\begin{align*}
x^{n+1} & := T_m (y^n + \lambda_n \sigma(y^n)(Ty^n - y^n)), \\
d^{n+1} & := -F(x^{n+1}) + \varphi_{n+1} d^n. \tag{9}
\end{align*}
\]

Update \( n := n + 1 \) and return to **Step 1.**
Remark 2  

(i) In the case of $m = 1$, $\lambda_n \equiv 1$, and $\sigma(y^n) \equiv 1$, Algorithm 1 becomes HCGM considered in [25]. Furthermore, if $\varphi_n \equiv 0$, Algorithm 1 is the same as HSMD investigated by Yamada [34].

(ii) If $F \equiv 0$, Algorithm 1 forms a generalization of MECSPM [10] in the sense of the operators $T_i$, $i = 1, \ldots, m$, are assumed to be subgradient projections. Moreover, if the operator $T_m$ in (9) is omitted from the method, Algorithm 1 coincides with ECCM [9].

(iii) Note that Algorithm 1 is not feasible in the sense that the generated sequence $\{x^n\}_{n=1}^{\infty}$ need not belong to the constrained set. Moreover, the step size $\sigma(y^n)$ may have large values for some $n \in \mathbb{N}$. These situations may yield the instabilities of the method. To avoid this situation, let us observe that if the operator $T_m$ is the metric projection onto a nonempty closed convex and bounded set $C_m$, and the initial point $x^1$ is chosen from $C_m$, then the iterate $x^n \in C_m$ ($n \in \mathbb{N}$), which subsequently yields the boundedness of $\{x^n\}_{n=1}^{\infty}$. In this case, even if we can not gain the feasibility of the method, it is very worth to note that the presence of $T_m$ in (9) ensures us that the generated sequence $\{x^n\}_{n=1}^{\infty} \subset C_m$, which may yield the numerical stabilities of the method; see [10, Section 4] further discussion and some numerical illustrations.

It is worth noting that the existence and uniqueness of the solution to Problem 1 are guaranteed by the above conditions according to [13, Theorem 2.3.3]. In order to analyze the main convergence theorem, we present a series of preliminary convergence results which indicate some important properties of the sequences generated by Algorithm 1. To begin with, the boundedness of the sequences is investigated in the following lemma.

Lemma 3  Let the sequences $\{x^n\}_{n=1}^{\infty}$, $\{y^n\}_{n=1}^{\infty}$ and $\{d^n\}_{n=1}^{\infty}$ be given by Algorithm 1. Suppose that $\lim_{n \to \infty} \varphi_n = 0$. If $\{x^n\}_{n=1}^{\infty}$ is bounded, then the sequences $\{F(x^n)\}_{n=1}^{\infty}$, $\{d^n\}_{n=1}^{\infty}$, and $\{y^n\}_{n=1}^{\infty}$ are bounded.

Proof  Let $\bar{u} \in \bigcap_{i=1}^{m} \text{Fix} T_i$ be given. By the triangle inequality and the $\kappa$-Lipschitz continuity of $F$, we obtain:

$$\|F(x^n)\| \leq \|F(x^n) - F(\bar{u})\| + \|F(\bar{u})\| \leq \kappa \|x^n - \bar{u}\| + \|F(\bar{u})\|.$$ 

Since $\{x^n\}_{n=1}^{\infty}$ is bounded, we immediately get that $\{F(x^n)\}_{n=1}^{\infty}$ is also bounded. According to the boundedness of $\{F(x^n)\}_{n=1}^{\infty}$, we have $\|F(x^n)\| \leq M$ for all $n \in \mathbb{N}$. Now, since $\lim_{n \to \infty} \varphi_n = 0$, we have some integer $n_0 \in \mathbb{N}$ such that $\varphi_n < \frac{1}{2}$ for all $n \geq n_0$. It turns out from (9) that, for all $n \geq n_0$,

$$\|d^{n+1}\| \leq \frac{1}{2}(2M) + \frac{1}{2}\|d^n\| \leq \max\{2M, \|d^n\|\} \leq \cdots \leq \max\{2M, \|d^{n_0}\|\}. \quad (10)$$

This proves the boundedness of $\{d^n\}_{n=1}^{\infty}$. Since $y^n = x^n + \mu \beta_n d^n$, we consequently obtain that $\{y^n\}_{n=1}^{\infty}$ is also bounded. \[\square\]
Remark 3 We point out that if the parameter $\varphi_n \equiv 0$, one can make use of the strong monotonicity and Lipschitz continuity of $F$ to ensure the boundedness of the sequence $\{x^n\}_{n=1}^{\infty}$; see [11, Lemma 9] for the proving lines. Furthermore, we also have that the boundedness of the sequences $\{x^n\}_{n=1}^{\infty}$ and $\{F(x^n)\}_{n=1}^{\infty}$ are equivalent. However, this is not sufficient for the boundedness of the sequence $\{x^n\}_{n=1}^{\infty}$ generated by Algorithm 1 though the equivalence of the bounded hypotheses of $\{x^n\}_{n=1}^{\infty}$ and $\{F(x^n)\}_{n=1}^{\infty}$ still holds. Of course, if the operator $T_m$ is the metric projection onto a nonempty closed convex and bounded sets, then it is clearly that $\{x^n\}_{n=1}^{\infty}$ is a bounded sequence. If not, we need to make use of the boundedness of the search direction $\{d^n\}_{n=1}^{\infty}$ and the parameter $\{\varphi_n\}_{n=1}^{\infty}$. Actually, by the definition of search direction, we have for all $u \in H$ and $n \geq 2$,

$$
\|x^n\| \leq \|x^n - u\| + \|u\| \leq \frac{1}{\eta} \|F(x^n) - F(u)\| + \|u\| \\
\leq \frac{1}{\eta} \|\varphi_n d^{n-1} - d^n\| + \frac{1}{\eta} \|F(u)\| + \|u\|,
$$

which yields that the sequence $\{x^n\}_{n=1}^{\infty}$ is bounded. We know from Lemma 3 that the boundedness the search direction $\{d^n\}_{n=1}^{\infty}$ is implied by the boundedness of $\{x^n\}_{n=1}^{\infty}$. In this situation, we can obtain under the assumptions of $F$ and $\{\varphi_n\}_{n=1}^{\infty}$ that the boundedness of the sequences $\{x^n\}_{n=1}^{\infty}$, $\{F(x^n)\}_{n=1}^{\infty}$ and $\{d^n\}_{n=1}^{\infty}$ are equivalent.

Before continuing the analysis, for $n \in \mathbb{N}$ and $\bar{u} \in \bigcap_{i=1}^{m} \text{Fix} T_i$, let us denote the following terms:

$$
\xi_n := \mu^2 \beta_n^2 \|d^n\|^2 + 2\mu \beta_n \|x^n - \bar{u}\| \|d^n\| \text{ and } \alpha_n := \beta_n \tau.
$$

In particular, for $n \geq 2$, we denote

$$
\delta_n := \frac{2\mu}{\tau} \left( \varphi_n \langle y^n - \bar{u}, d^{n-1} \rangle + \langle y^n - \bar{u}, -F(\bar{u}) \rangle \right).
$$

The aforementioned notations give rise to the following lemmas which demonstrate some crucial inequalities needed in proving our main convergence result.

**Lemma 4.** Let the sequences $\{x^n\}_{n=1}^{\infty}$, $\{y^n\}_{n=1}^{\infty}$ and $\{d^n\}_{n=1}^{\infty}$ be given by Algorithm 1. Suppose that $\{\lambda_n\}_{n=1}^{\infty} \subset [\varepsilon, 2 - \varepsilon]$ for some constant $\varepsilon \in (0, 1)$. Then, for all $n \in \mathbb{N}$ and $\bar{u} \in \bigcap_{i=1}^{m} \text{Fix} T_i$, there holds:

$$
\|x^{n+1} - \bar{u}\|^2 \leq \|x^n - \bar{u}\|^2 - \frac{\lambda_n (2 - \lambda_n)}{4m} \sum_{i=1}^{m} \|S_i y^n - S_{i-1} y^n\|^2 + \xi_n.
$$
Proof According to Lemma 2(ii), it is worth noting here that $T_\sigma$ is a cutter. By utilizing the quasi-nonexpansivity of $T_m$ and the properties of $T_\sigma$ in Fact 2, for all $n \in \mathbb{N}$, we have

$$
\|x^{n+1} - \bar{u}\|^2 = \|T_m(y^n + \lambda_n \sigma(y^n)(Ty^n - y^n)) - \bar{u}\|^2
\leq \|y^n + \lambda_n \sigma(y^n)(Ty^n - y^n) - \bar{u}\|^2
= \|y^n - \bar{u}\|^2 + \lambda_n^2 \|\sigma(y^n)(Ty^n - y^n)\|^2
+ 2\lambda_n \langle y^n - \bar{u}, \sigma(y^n)(Ty^n - y^n)\rangle
= \|y^n - \bar{u}\|^2 + \lambda_n^2 \|T_\sigma y^n - y^n\|^2 + 2\lambda_n \langle y^n - \bar{u}, T_\sigma y^n - y^n\rangle
\leq \|y^n - \bar{u}\|^2 + \lambda_n^2 \|T_\sigma y^n - y^n\|^2 - 2\lambda_n \|T_\sigma y^n - y^n\|^2
= \|y^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2.
$$

(11)

By the definition of $(\gamma^n)_{n=1}^{\infty}$, we have

$$
\|x^{n+1} - \bar{u}\|^2 \leq \|x^n + \mu \beta_n d^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2
= \|x^n - \bar{u}\|^2 + \mu^2 \beta_n^2 \|d^n\|^2 + 2\mu \beta_n \langle x^n - \bar{u}, d^n\rangle
- \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2
\leq \|x^n - \bar{u}\|^2 + \mu^2 \beta_n^2 \|d^n\|^2 + 2\mu \beta_n \|x^n - \bar{u}\| \|d^n\|
- \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2
= \|x^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2 + \xi_n
= \|x^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \sigma^2(\gamma^n) \|T y^n - y^n\|^2 + \xi_n.
$$

Thanks to Lemma 2(i), we finally have

$$
\|x^{n+1} - \bar{u}\|^2 \leq \|x^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \left(\frac{1}{4} \left( \sum_{i=1}^{m} \|S_i y^n - S_{i-1} y^n\|^2 \right) \right)^2 \frac{\|T y^n - y^n\|^2}{4m}
+ \xi_n
= \|x^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \frac{\|T y^n - y^n\|^2}{4m}
+ \sum_{i=1}^{m} \|S_i y^n - S_{i-1} y^n\|^2 + \xi_n,
$$

which completes the proof. \qed

Lemma 5 Let the sequences $(x^n)_{n=1}^{\infty}$, $(y^n)_{n=1}^{\infty}$ and $(d^n)_{n=1}^{\infty}$ be given by Algorithm 1. Suppose that $(\lambda_n)_{n=1}^{\infty} \subset [\epsilon, 2 - \epsilon]$ for some constant $\epsilon \in (0, 1)$. Then, for all $n \geq 2$ and $\bar{u} \in \bigcap_{i=1}^{m} \text{Fix} T_i$, there holds:

$$
\|x^{n+1} - \bar{u}\|^2 \leq (1 - \alpha_n) \|x^n - \bar{u}\|^2 + \alpha_n \delta_n.
$$

Proof By (11), we have

$$
\|x^{n+1} - \bar{u}\|^2 \leq \|y^n - \bar{u}\|^2 - \lambda_n (2 - \lambda_n) \|T_\sigma y^n - y^n\|^2.
$$
Since \( \{\lambda_n\}_{n=1}^{\infty} \subset [\varepsilon, 2 - \varepsilon] \) for some constant \( \varepsilon \in (0, 1) \), we obtain that
\[
\|x^{n+1} - \bar{u}\|^2 \leq \|y^n - \bar{u}\|^2. \tag{12}
\]

Then, for all \( n \geq 2 \), we have
\[
\|y^n - \bar{u}\| = \|x^n + \mu \beta_n d^n - \bar{u}\|
= \|x^n + \mu \beta_n (-F(x^n) + \varphi_n d^{n-1}) - \bar{u}\|
= \|(x^n - \mu \beta_n F(x^n)) - (\bar{u} - \mu \beta_n F(\bar{u})) + \mu \beta_n (\varphi_n d^{n-1} - F(\bar{u}))\|. \tag{13}
\]

By utilizing the inequalities (12), (13), the fact that
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle,
\]
for all \( x, y \in H \), and Lemma 1, for all \( n \geq 2 \), we have
\[
\|x^{n+1} - \bar{u}\|^2 \leq \|(x^n - \mu \beta_n F(x^n)) - (\bar{u} - \mu \beta_n F(\bar{u})) + \mu \beta_n (\varphi_n d^{n-1} - F(\bar{u}))\|^2
\leq \|(x^n - \mu \beta_n F(x^n)) - (\bar{u} - \mu \beta_n F(\bar{u}))\|^2
+ 2 \langle x^n - \mu \beta_n F(x^n) - \bar{u} + \mu \beta_n \varphi_n d^{n-1}, \mu \beta_n (\varphi_n d^{n-1} - F(\bar{u})) \rangle
\leq (1 - \beta_n \tau) \|x^n - \bar{u}\|^2
+ 2 \mu \beta_n \|x^n + \mu \beta_n (-F(x^n) + \varphi_n d^{n-1}) - \bar{u}, \varphi_n d^{n-1} - F(\bar{u})\|
= (1 - \beta_n \tau) \|x^n - \bar{u}\|^2 + 2 \mu \beta_n \langle y^n - \bar{u}, \varphi_n d^{n-1} - F(\bar{u}) \rangle
\leq (1 - \beta_n \tau) \|x^n - \bar{u}\|^2 + 2 \mu \beta_n \langle y^n - \bar{u}, d^{n-1} \rangle
+ 2 \mu \beta_n \langle y^n - \bar{u}, -F(\bar{u}) \rangle
\leq (1 - \beta_n \tau) \|x^n - \bar{u}\|^2
+ \beta_n \tau \left[ \frac{2 \mu}{\tau} \left( \varphi_n \langle y^n - \bar{u}, d^{n-1} \rangle + \langle y^n - \bar{u}, -F(\bar{u}) \rangle \right) \right]
= (1 - \alpha_n) \|x^n - \bar{u}\|^2 + \alpha_n \delta_n,
\]
which completes the proof. \( \square \)

In order to prove our main theorem, we need the following lemma which can be found in [28, Lemma 3.1].

**Lemma 6** Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of nonnegative real numbers such that there exists a subsequence \( \{a_{n_j}\}_{j=1}^{\infty} \) of \( \{a_n\}_{n=1}^{\infty} \) with \( a_{n_j} < a_{n_{j+1}} \) for all \( j \in \mathbb{N} \), and define the set of indexes \( \{v(n)\}_{n \geq n_0} \) by
\[
v(n) = \max \{ k \in [n_0, n] : a_k < a_{k+1} \}.
\]

Then, the following properties hold:

(i) \( \{v(n)\}_{n \geq n_0} \) is nondecreasing.

(ii) \( \lim_{n \to \infty} v(n) = \infty. \)

(iii) \( a_{v(n)} \leq a_{v(n)+1} \) and \( a_n \leq a_{v(n)+1} \) for all \( n \geq n_0. \)

Another important tool for proving our main result is stated in the next lemma, where its proof can be found in [33, Lemma 2.5].
Lemma 7 Let \( \{a_n\}_{n=1}^\infty \) be a sequence of nonnegative real numbers such that \( a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n \), for all \( n \in \mathbb{N} \) where the sequences \( \{\alpha_n\}_{n=1}^\infty \subset [0, 1] \) and \( \{\delta_n\}_{n=1}^\infty \subset \mathbb{R} \) satisfy \( \sum_{n=1}^\infty \alpha_n = \infty \) and \( \limsup_{n \to \infty} \delta_n \leq 0 \). Then \( \lim_{n \to \infty} a_n = 0 \).

Now, we are in a position to present our main theorem. The proof of this theorem goes along with the technique used in [32, Theorem 11].

Theorem 3 Let the sequence \( \{x^n\}_{n=1}^\infty \) be given by Algorithm 1. Suppose that \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^\infty \beta_n = \infty, \lim_{n \to \infty} \varphi_n = 0, \) and \( \{\lambda_n\}_{n=1}^\infty \subset [\varepsilon, 2-\varepsilon] \) for some constant \( \varepsilon \in (0, 1) \). If \( \{x^n\}_{n=1}^\infty \) is bounded and \( \{T_i\}_{i=1}^m \) satisfies the DC principle, then the sequence \( \{x^n\}_{n=1}^\infty \) converges strongly to \( \bar{u} \), the unique solution of Problem 1.

Proof Assume that \( \{x^n\}_{n=1}^\infty \) is bounded and \( \{T_i\}_{i=1}^m \) satisfies the DC principle. For simplicity, we denote \( a_n := \|x^n - \bar{u}\|^2 \) for all \( n \in \mathbb{N} \). Due to Lemma 3 and the assumption \( \lim_{n \to \infty} \beta_n = 0 \), we obtain
\[
\lim_{n \to \infty} \xi_n = 0.
\]
To prove the strong convergence of the theorem, we consider the following two cases of the sequence \( \{a_n\}_{n=1}^\infty \) according to its behavior.

Case 1. Suppose that there exists \( n_0 \in \mathbb{N} \) such that \( a_{n+1} \leq a_n \) for all \( n \geq n_0 \). It is clear that \( \{a_n\}_{n=1}^\infty \) is convergent. By utilizing Lemma 4 and the fact \( \lim_{n \to \infty} \xi_n = 0 \), we obtain
\[
0 \leq \limsup_{n \to \infty} \frac{\lambda_n (2 - \lambda_n)}{4m} \sum_{i=1}^m \|S_i y^n - S_{i-1} y^n\|^2 \\
\leq \limsup_{n \to \infty} (a_n - a_{n+1} + \xi_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} a_{n+1} + \lim_{n \to \infty} \xi_n = 0.
\]
Thus, we have
\[
\lim_{n \to \infty} \frac{\lambda_n (2 - \lambda_n)}{4m} \sum_{i=1}^m \|S_i y^n - S_{i-1} y^n\|^2 = 0.
\]
Since \( \lambda_n \in [\varepsilon, 2-\varepsilon] \) for an arbitrary constant \( \varepsilon \in (0, 1) \), we have \( \varepsilon^2 \leq \lambda_n (2 - \lambda_n) \), which implies that, for all \( i = 1, 2, \ldots, m \),
\[
\lim_{n \to \infty} \|S_i y^n - S_{i-1} y^n\| = 0.
\] (14)
On the other hand, since \( \{y^n\}_{n=1}^\infty \) is a bounded sequence, so is the sequence \( \{(y^n - \bar{u}, -F(\bar{u}))\}_{n=1}^\infty \). Now, let \( \{y^{nk}\}_{k=1}^\infty \) be a subsequence of \( \{y^n\}_{n=1}^\infty \) such that
\[
\limsup_{n \to \infty} \langle y^n - \bar{u}, -F(\bar{u}) \rangle = \lim_{k \to \infty} \langle y^{nk} - \bar{u}, -F(\bar{u}) \rangle.
\]
Due to the boundedness of the sequence \( \{y^n\}_{n=1}^{\infty} \), there exists a weakly cluster point \( z \in \mathcal{H} \) and a subsequence \( \{y^{n_k}\}_{k=1}^{\infty} \) of \( \{y^n\}_{n=1}^{\infty} \) such that \( y^{n_k} \rightharpoonup z \in \mathcal{H} \). According to (14), let us note that

\[
\lim_{j \to \infty} \| (T_1 - I) y^{n_{k_j}} \| = \lim_{j \to \infty} \| S_1 y^{n_{k_j}} - S_0 y^{n_{k_j}} \| = 0.
\]

Then, the DC principle of \( T_1 \) yields that \( z \in \text{Fix} T_1 \). Furthermore, we note that the facts \( y^{n_{k_j}} \rightharpoonup z \) and

\[
\lim_{j \to \infty} \| (T_1 y^{n_{k_j}} - T_1 z) - (y^{n_{k_j}} - z) \| = 0
\]

lead to \( T_1 y^{n_{k_j}} \rightharpoonup z \). Furthermore, we observe that

\[
\lim_{j \to \infty} \| (T_2 - I) T_1 y^{n_{k_j}} \| = \lim_{j \to \infty} \| S_2 y^{n_{k_j}} - S_1 y^{n_{k_j}} \| = 0.
\]

By invoking the DC principle of \( T_2 \), we then obtain \( z \in \text{Fix} T_2 \). By continuing the same argument used in the above proving lines, we obtain that \( z \in \bigcap_{i=1}^{m} \text{Fix} T_i \). As \( \bar{u} \) is the unique solution to Problem 1, we have

\[
\limsup_{n \to \infty} \langle y^n - \bar{u}, -F(\bar{u}) \rangle = \lim_{k \to \infty} \langle y^{n_k} - \bar{u}, -F(\bar{u}) \rangle = \lim_{j \to \infty} \langle y^{n_{k_j}} - \bar{u}, -F(\bar{u}) \rangle = \langle z - \bar{u}, -F(\bar{u}) \rangle \leq 0. \tag{15}
\]

Now, the assumption that \( \lim_{n \to \infty} \varphi_n = 0 \), the boundedness of the sequences \( \{y^n\}_{n=1}^{\infty} \) and \( \{d^n\}_{n=1}^{\infty} \) and the relation (15) yield that

\[
\limsup_{n \to \infty} \delta_n = \limsup_{n \to \infty} \frac{2\mu}{\tau} \left( \frac{\varphi_n}{\tau} \langle y^n - \bar{u}, d^{n-1} \rangle + \langle y^n - \bar{u}, -F(\bar{u}) \rangle \right) \leq 0. \tag{16}
\]

According to Lemma 5, we have, for all \( n \geq 2 \), that

\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n.
\]

To reach the conclusion of this case, we observe that \( \{\alpha_n\}_{n=1}^{\infty} \subset [0, 1] \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \) which are following the assumptions of \( \beta_n \) and the property of \( \tau \). Therefore, by applying this and the relation (16), Lemma 7 yields that \( \lim_{n \to \infty} \| x_n - \bar{u} \| = 0 \).

Case 2. Suppose that there exists a subsequence \( \{a_{n_k}\}_{k=1}^{\infty} \) of \( \{a_n\}_{n=1}^{\infty} \) such that \( a_{n_k} < a_{n_{k+1}} \) for all \( k \in \mathbb{N} \), and let \( \{\nu(n)\}_{n=1}^{\infty} \) be defined as in Lemma 6. Then, for all \( n \geq n_0 \), we have

\[
a_{\nu(n)} \leq a_{\nu(n)+1}, \tag{17}
\]

and

\[
a_n \leq a_{\nu(n)+1}. \tag{18}
\]
By utilizing Lemma 4 and the inequality (17), we have
\[ 0 \leq a_{v(n)+1} - a_{v(n)} \leq -\frac{\lambda_v(n)(2 - \lambda_v(n))}{4m} \sum_{i=1}^{m} \|S_i y^{v(n)} - S_{i-1} y^{v(n)}\|^2 + \xi_v(n), \]
and hence
\[ \frac{\lambda_v(n)(2 - \lambda_v(n))}{4m} \sum_{i=1}^{m} \|S_i y^{v(n)} - S_{i-1} y^{v(n)}\|^2 \leq \xi_v(n), \]
for all \( n \geq n_0 \). According to the fact \( \lim_{n \to \infty} \xi_v(n) = 0 \) and the fact that \( \epsilon^2 \leq \lambda_v(n)(2 - \lambda_v(n)) \), for all \( i = 1, 2, \ldots, m \), we obtain
\[ \lim_{n \to \infty} \|S_i y^{v(n)} - S_{i-1} y^{v(n)}\| = 0. \] (19)

Now, let \( \{y^{v(n_k)}\}_{k=1}^\infty \subset \{y^{v(n)}\}_{n=1}^\infty \) be a subsequence such that
\[ \limsup_{n \to \infty} \langle y^{v(n)} - \bar{u}, -F(\bar{u}) \rangle = \lim_{k \to \infty} \langle y^{v(n_k)} - \bar{u}, -F(\bar{u}) \rangle. \]

By proceeding the similar argument to those used in Case 1, the relation (19) and the DC principle of each \( T_i \) yield that, for any subsequence \( \{y^{v(n_{kj})}\}_{j=1}^\infty \) of \( \{y^{v(n_k)}\}_{k=1}^\infty \), we get that \( y^{v(n_{kj})} \rightharpoonup z \in \bigcap_{i=1}^{m} \text{Fix} T_i \). Furthermore, we have
\[ \limsup_{n \to \infty} \langle y^{v(n)} - \bar{u}, -F(\bar{u}) \rangle = \lim_{k \to \infty} \langle y^{v(n_k)} - \bar{u}, -F(\bar{u}) \rangle = \lim_{j \to \infty} \langle y^{v(n_{kj})} - \bar{u}, -F(\bar{u}) \rangle = \langle z - \bar{u}, -F(\bar{u}) \rangle \leq 0. \]
As a result, we simultaneously obtain
\[ \limsup_{n \to \infty} \delta_{v(n)} \leq 0. \] (20)

In the light of Lemma 5, we have
\[ 0 \leq a_{v(n)+1} \leq (1 - \alpha_{v(n)}) a_{v(n)} + \alpha_{v(n)} \delta_{v(n)}, \]
and hence
\[ 0 \leq a_{v(n)+1} - a_{v(n)} \leq \alpha_{v(n)} (\delta_{v(n)} - a_{v(n)}). \]
Since \( \alpha_{v(n)} > 0 \), we obtain
\[ a_{v(n)} \leq \delta_{v(n)}. \]

Thanks to (20), we have
\[ \limsup_{n \to \infty} a_{v(n)} \leq \limsup_{n \to \infty} \delta_{v(n)} \leq 0, \]
which leads to
\[ \lim_{n \to \infty} a_{v(n)} = 0. \]
By utilizing the inequality (18) together with this, we have
\[
0 \leq \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_{\nu(n)+1} = 0.
\]
Hence, we finally obtain that \( \lim_{n \to \infty} a_n = 0 \) as desired.

**Remark 4**  
(i) The step-size sequences \( \{\varphi_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) in Theorem 3 are, for instance, \( \varphi_n = \frac{1}{(n+1)a} \) with \( a > 0 \) and \( \beta_n = \frac{1}{(n+1)b} \) with \( 0 < b \leq 1 \) for all \( n \in \mathbb{N} \).

(ii) It can be noted that the DC principle assumed in Theorem 3 will be satisfying in many cases, for instance, the operators \( T_i, i = 1, \ldots, m \), are nonexpansive, or, in particular, the metric projections onto closed convex sets. Moreover, this still holds true when the operators \( T_i, i = 1, \ldots, m \), are subgradient projections of continuous convex functions which are Lipschitz continuous on bounded subsets; see further discussion in the next section.

## 4 Variational inequality problem with inequality constraints

In this section, we will consider the solving of the variational inequality problem over the finite family of continuous convex functional constraints and a simple closed convex and bounded constraint by applying the results obtained in the previous section.

Let \( C_i := \{x \in \mathcal{H} : c_i(x) \leq 0\} \) be a sublevel set of a continuous and convex function \( c_i : \mathcal{H} \to \mathbb{R}, i = 1, \ldots, m - 1 \), and \( C_m \subset \mathcal{H} \) be a simple closed convex and bounded sets. Let \( F : \mathcal{H} \to \mathcal{H} \) be \( \eta \)-strongly monotone and \( \kappa \)-Lipschitz continuous, we consider the variational inequality of finding a point \( \bar{u} \in \bigcap_{i=1}^{m} C_i \) such that
\[
\langle F(\bar{u}), z - \bar{u} \rangle \geq 0 \quad \text{for all } z \in \bigcap_{i=1}^{m} C_i.
\]

Assume that \( \bigcap_{i=1}^{m} C_i \neq \emptyset \). Let us consider, for each \( i = 1, \ldots, m - 1 \), since each \( C_i \) is the sublevel set of the function \( c_i \), we define the operator \( T_i : \mathcal{H} \to \mathcal{H} \) to be a subgradient projection relative to \( c_i \), \( P_{c_i} : \mathcal{H} \to \mathcal{H} \), namely, for every \( x \in \mathcal{H} \),
\[
P_{c_i}(x) := \begin{cases} 
    x - \frac{c_i(x)}{\|g_i(x)\|^2} g_i(x) & \text{if } c_i(x) > 0, \\
    x & \text{otherwise},
\end{cases}
\]
where \( g_i(x) \in \partial c_i(x) := \{g \in \mathcal{H} : \langle g, y - x \rangle \leq c_i(y) - c_i(x), \forall y \in \mathcal{H}\} \), is a subgradient of the function \( c_i \) at the point \( x \). Since \( c_i, i = 1, \ldots, m - 1 \), are continuous and convex, we ensure that the subdifferential sets \( \partial c_i(x), i = 1, \ldots, m - 1 \), are nonempty, for every \( x \in \mathcal{H} \); see [3, Proposition 16.17]. Note that the subgradient projection \( P_{c_i} \) is a cutter and \( \text{Fix} P_{c_i} = C_i \), for all \( i = 1, \ldots, m - 1 \); see [6, Lemma 4.2.5 and Corollary 2.4.6].
Moreover, since $C_m$ is the nonempty closed convex and bounded, we define the operator $T_m : H \to H$ to be a metric projection onto $C_m$ written by $\text{proj}_{C_m} : H \to H$, i.e., for every $x \in H$, we have

$$\| x - \text{proj}_{C_m} x \| = \inf_{y \in C_m} \| x - y \|.$$  

Note that the metric projection $\text{proj}_{C_m}$ is also a cutter and $\text{Fix}(\text{proj}_{m}) = C_m$, see [6, Theorem 2.2.21]. These mean that the operators $T_i, i = 1, \ldots, m$, are cutters and $\bigcap_{i=1}^{m} \text{Fix} T_i \neq \emptyset$.

Now, in order to construct an iterative method for solving the problem (21), we recall the notations $T := T_m T_{m-1} \ldots T_1$, $S_0 := I d$, and $S_i := T_i T_{i-1} \ldots T_1, i = 1, 2, \ldots, m$. Furthermore, for every $x \in H$, we denote $u_i := S_i x$, thus $u_0 = x$ and $u_i = T_i u_{i-1}, i = 1, 2, \ldots, m$. Firstly, let us note from [9, Remark 10, the equation (37)] that the step-size function $\sigma : H \to [0, +\infty)$ which is defined in (8) can be written in the equivalent form as

$$\sigma(x) = \frac{\| Tx - x \|^2 + \sum_{i=1}^{m} \| u_i - u_{i-1} \|^2}{2\| Tx - x \|^2}.$$  

Now, for every $i = 1, \ldots, m - 1$, and $x \notin \bigcap_{i=1}^{m} \text{Fix} T_i$, we note that

$$u_i = T_i u_{i-1} = u_{i-1} - \frac{\max\{c_i(u_{i-1}), 0\}}{\| g_i(u_{i-1}) \|^2} g_i(u_{i-1}),$$

where $g_i(u_{i-1})$ is a subgradient of the function $c_i$ at the point $u_{i-1}$. For simplicity, we use throughout the convention that $\frac{\max\{c_i(u_{i-1}), 0\}}{\| g_i(u_{i-1}) \|^2} = 0$ whenever $\max\{c_i(u_{i-1}), 0\} = 0$. Subsequently, we have

$$\| u_i - u_{i-1} \|^2 = \left( \frac{\max\{c_i(u_{i-1}), 0\}}{\| g_i(u_{i-1}) \|} \right)^2.$$  

Therefore, the step-size function $\sigma : H \to [0, +\infty)$ is nothing else than

$$\sigma(x) := \begin{cases} 
\frac{\| Tx - x \|^2 + \| \text{proj}_{C_{m-1}} u_{m-1} - u_{m-1} \|^2 + \sum_{i=1}^{m-1} \left( \frac{\max\{c_i(u_{i-1}), 0\}}{\| g_i(u_{i-1}) \|^2} \right)^2}{2\| Tx - x \|^2}, & \text{for } x \notin C, \\
1, & \text{otherwise.}
\end{cases}$$  

According to the above convention and Lemma 2(i), we can ensure that the step-size function $\sigma(x)$ is well-defined and nonnegative which is bounded from below by $\frac{1}{2m}$, for every $x \in H$.

Now, we are in position to propose the method for solving the problem (21) as the following algorithm.
Remark 5 Observe that Algorithm 2 is nothing else than a particular case of ESCoM-CGD (Algorithm 1). Moreover, as we have mentioned in Remark 2 (iii), we underline here again that the initial point $x_1$ in Algorithm 2 is particularly chosen in the nonempty closed convex and bounded subset $C_m$ rather than in the whole space $H$ $(n \in \mathbb{N})$, calculate as follows:

**Step 1.** Compute $y^n$ as
$$y^n := x^n + \mu \beta_n d^n.$$  
**Step 2.** Set $u^n_0 := y^n$ and compute the estimates
$$u^n_i := u^n_{i-1} - \frac{\max \{c_i(u^n_{i-1}), 0 \}}{\|g_i(u^n_{i-1})\|^2} g_i(u^n_{i-1}), \ i = 1, \ldots, m - 1,$$
where $g_i(u^n_{i-1})$ is a subgradient of the function $c_i$ at the point $u^n_{i-1}$, and subsequently compute
$$u^n_m := \text{proj}_{C_m} u^n_{m-1}.$$  
**Step 3.** Compute a step size as
$$\sigma(y^n) := \begin{cases} \frac{\|Ty^n - y^n\|^2 + \|u^n_m - u^n_{m-1}\|^2 + \sum_{i=1}^{m-1} \left( \frac{\max \{c_i(u^n_{i-1}), 0 \}}{\|g_i(u^n_{i-1})\|^2} \right)^2}{2\|Ty^n - y^n\|^2}, & \text{for } y^n \notin C, \\ 1, & \text{otherwise.} \end{cases}$$
**Step 4.** Compute a next iterate and a search direction as
$$\begin{cases} x^{n+1} := \text{proj}_{C_m}(y^n + \lambda_n \sigma(y^n)(u^n_m - y^n)), \\ d^{n+1} := -F(x^{n+1}) + \varphi_{n+1} d^n. \end{cases}$$
Update $n := n + 1$ and return to **Step 1.**

The following corollary is a consequence of Theorem 3.

**Corollary 1** Let the sequence $\{x^n\}_{n=1}^\infty$ be given by Algorithm 2. Suppose that $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^\infty \beta_n = \infty$, $\lim_{n \to \infty} \varphi_n = 0$, and $\{\lambda_n\}_{n=1}^\infty \subseteq [\varepsilon, 2 - \varepsilon]$ for some constant $\varepsilon \in (0, 1)$. If one of the following conditions hold:

(i) The functions $c_i$, $i = 1, \ldots, m - 1$, are Lipschitz continuous relative to every bounded subset of $H$;

(ii) The functions $c_i$, $i = 1, \ldots, m - 1$, are bounded on every bounded subset of $H$;
(iii) The subdifferentials \( \partial c_i, i = 1, \ldots, m - 1 \), map every bounded subset of \( \mathcal{H} \) to a bounded set, then the sequence \( \{ x^n \}_{n=1}^{\infty} \) converges strongly to \( \bar{u} \), the unique solution of the problem (21).

**Proof** Observe that the convergence Theorem 3 is dependent on the assumptions that the sequence \( \{ x^n \}_{n=1}^{\infty} \) is bounded and the operators \( T_i, i = 1, \ldots, m \), satisfy the DC principle. If we verify that these two mentioned assumptions are true, the convergence is a consequence of Theorem 3.

Now, note that the generated sequence \( \{ x^n \}_{n=1}^{\infty} \) is bounded. On the other hand, it is noted from [6, Theorem 4.2.7] that for a continuous convex function which is satisfying (i), we have that its corresponding subgradient projection will be satisfying the DC principle. Consequently, this means that the operators \( T_i, i = 1, \ldots, m - 1 \), are satisfying the DC principle. Moreover, we know from [3, Proposition 16.20] that for a continuous convex function, the assumptions (i)–(iii) are equivalent. This gives us that these three assumptions are the sufficient conditions for the fact that operators \( T_i, i = 1, \ldots, m - 1 \), are satisfying the DC principle. Furthermore, since the metric projection \( \text{proj}_{C_m} \) is a nonexpansive operator (see [6, Theorem 2.2.21]), it follows that \( T_m \) is also satisfying the DC principle. Hence, the assumptions of Theorem 3 are satisfied, and we therefore conclude that the sequence \( \{ x^n \}_{n=1}^{\infty} \) converges strongly to the unique solution of the problem (21) as desired.

**Remark 6**

(i) It is very important to note that the continuity of \( c_i, i = 1, \ldots, m \), and the assumptions (i)–(iii) used in Corollary 1 can be dropped whenever the whole Hilbert space \( \mathcal{H} \) is finite dimensional; see [3, Corollary 8.40 and Proposition 16.20] for further details.

(ii) An example of the simple closed convex and bounded set \( C_m \) in a general Hilbert space is nothing else than a closed ball \( C_m := \{ x \in \mathcal{H} : \| x - z \| \leq r \} \), where \( z \in \mathcal{H} \) is the center, and \( r > 0 \) is the radius. In particular, if \( \mathcal{H} = \mathbb{R}^n \), the finite-dimensional Euclidean space, an additional example is a box constraint \( C_m := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \), where \( a_i, b_i \in \mathbb{R} \) with \( a_i \leq b_i, i = 1, \ldots, n \). For the closed-form formulae of these simple sets, the reader may consult [6, Subsections 4.1.6 and 4.1.7].

### 5 Numerical result

In this section, we report the convergence of ESCoM-CGD by the minimum-norm problem to a system of homogeneous linear inequalities with box constraint. Suppose that we are given a matrix \( A = [a_1 | \cdots | a_m] \top \in \mathbb{R}^{m \times k} \) of predictors \( a_i = (a_{i1}, \ldots, a_{ik}) \in \mathbb{R}^k \), for all \( i = 1, \ldots, m \). The approach of the considered problem with a box constraint is to find the vector \( x \in \mathbb{R}^k \) that solves the problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| x \|^2 \\
\text{subject to} & \quad Ax \leq 0_{\mathbb{R}^m}, \\
& \quad x \in [u, v]^k,
\end{align*}
\]

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or equivalently, in the explicit form,

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \| x \|^2 \\
\text{subject to} & \quad \langle a_i, x \rangle \leq 0, i = 1 \ldots, m, x \in [u, v]^k,
\end{align*}
\]

where \( u, v \in \mathbb{R} \) with \( u \leq v \).

Of course, this minimum-norm problem can be written in the form of Problem 1 as: finding \( x^* \in \bigcap_{i=1}^{m+1} \text{Fix}(\text{proj}_{C_i}) \) such that

\[
\langle x^*, x - x^* \rangle \geq 0 \text{ for all } x \in \bigcap_{i=1}^{m+1} \text{Fix}(\text{proj}_{C_i}).
\]

where the constrained sets \( C_i := \{ x \in \mathbb{R}^k : \langle a_i, x \rangle \leq 0 \}, i = 1, \ldots, m \), are half-spaces and \( C_{m+1} := \{ x \in \mathbb{R}^k : x \in [u, v]^k \} \) is a box constraint. It is clear that this variational inequality problem satisfies all assumptions of Problem 1 by setting \( F := \text{Id} \), the identity operator, which is 1-strongly monotone and 1-Lipschitz continuous, and \( T_i := \text{proj}_{C_i}, i = 1, \ldots, m + 1 \), the metric projections onto \( C_i \) which are cutters with \( \text{Fix}(T_i) = C_i \) and satisfying the demi-closed principle. Moreover, since the box constrained set \( C_{m+1} \) is bounded, we have that the generated sequence \( \{x^n\}_{n=1}^\infty \) is a bounded sequence which means that the assumptions of Theorem 3 are satisfying. All the experiments were performed under MATLAB 9.6 (R2019a) running on a MacBook Air 13-inch, early 2015 with a 1.6GHz Intel Core i5 processor and 4GB 1600MHz DDR3 memory. All CPU times are given in seconds.

We generate the matrix \( A \) in \( \mathbb{R}^{m \times k} \) where \( m = 1000 \) and \( k = 200 \) by uniformly distributed random generating between \((-5, 5)\) and choose the box constraint with boundaries \( u = -1 \) and \( v = 1 \). The initial point is a vector whose all coordinates are normally distributed randomly chosen in \((0, 1)\). In order to justify the advantages of the proposed Algorithm 1, we thus choose the hybrid conjugate gradient method (HCGM) [25] and hybrid three-term conjugate gradient method (HTCGM) [18, Algorithm 6] as the benchmarks for the numerical comparisons. In this situation, we set the operator considered in [18, 25] by \( T := \text{proj}_{C_{m+1}} \text{proj}_{C_m} \cdots \text{proj}_{C_1} \), which is a nonexpansive operator. Since the minimum-norm solution has the unique solution, in all following numerical experiments, we terminate the experimented methods when the norm becomes small, i.e., \( \|x^n\| \leq 10^{-6} \). We use 10 samplings for different randomly chosen matrix \( A \) and the initial point when performing each combination, and the presented results are averaged. We manually select the involved parameters of each compared algorithm and show some results when it achieves the fairly best performance.

Firstly, we demonstrate the effectiveness of step-size sequence \( \varphi_n := \frac{1}{(n+1)^a} \); where \( a > 0 \), when ESCoM-CGD, HCGM, and HTCGM are applied for solving the above minimum-norm problem. We choose different values \( a = 0.005, 0.01, 0.05 \), and 0.1, and fix the corresponding parameter \( \mu = 1 \), the step-size sequence \( \beta_n = \frac{1}{(n+1)^{0.5}} \), and additionally set \( \lambda_n = 0.7 \) for ESCoM-CGD. We plot the number of iterations and computational time in seconds with respect to different choices of \( a \) in Fig. 1.
According to the plots in Fig. 1, we see that the larger the value of $a$ yields the faster convergence in the senses that it needs the smaller number of iterations and less computational time. We also see that the proposed ESCoM-CGD is really faster than other methods, where the best result is observed for $a = 0.1$. Notice that HCGM and HTCGM are very sensitive to the value of $a$, while ESCoM-CGD seems not. In fact, for $a = 0.005$, HCGM and HTCGM require more than 550 iterations, whereas for $a = 0.1$, those require approximately 50 iterations.

Next, we testify the influence of step-size sequence $\beta_n := \frac{1}{(n+1)^{b}}$; where $0 < b \leq 1$, for the tested methods. We fix $\mu = 1$, $\lambda_n = 0.7$, and $\phi_n = \frac{1}{(n+1)^{0.1}}$. We choose different values of $b$ in the interval $(0, 1]$, namely, $b = 0.01, 0.05, 0.1, \text{and } 0.5$. The number of iterations and computational time in seconds for each choice of $b$ are plotted in Fig. 2.
It can be seen from Fig. 2 that ESCoM-CGD gives the best results for all values $b$. Moreover, their number of iterations and computational time seem indifferent to the different choices of $b$. For HCGM and HTCGM, we observe the numbers of iterations as well as computational time decrease when the values $b$ grow up. For the exact results, the value $b = 0.01$ is the best choice for ESCoM-CGD; however, the value $b = 0.5$ is the best choice for both HCGM and HTCGM, which is coherent with the assertions in [18, 25].

In Fig. 3, we illustrate behavior of the methods with respect to parameter $\mu \in (0, 2)$. We fix $\varphi_n = \frac{1}{(n+1)^{0.1}}$ and $\lambda_n = 0.7$. Moreover, we fix the best choices $\beta_n = \frac{1}{(n+1)^{0.01}}$ for ESCoM-CGD and $\beta_n = \frac{1}{(n+1)^{0.5}}$ for both HCGM and HTCGM. We choose different values $\mu = 10^{-4}, 10^{-3}, 0.01, 0.1, 0.5, 1.0, 1.5, 1.9$. According to the plots, we observe that the very small value of $\mu = 10^{-4}$ yields the best
Table 2  Best choice of parameters used for performing ESCoM-CGD, HCGM [25], and HTCGM [18]

| Parameter | $\varphi_n$ | $\beta_n$ | $\mu$ | $\lambda_n$ |
|-----------|-------------|------------|-------|------------|
| ESCoM-CGD | $\frac{1}{(n+1)^{0.1}}$ | $\frac{1}{(n+1)^{0.01}}$ | $10^{-4}$ | 1.2 |
| HCGM [25] | $\frac{1}{(n+1)^{0.1}}$ | $\frac{1}{(n+1)^{0.5}}$ | $10^{-4}$ | – |
| HTCGM [18] | $\frac{1}{(n+1)^{0.1}}$ | $\frac{1}{(n+1)^{0.5}}$ | $10^{-4}$ | – |

results for all methods. As a matter of fact, even if the best result for ESCoM-CGD is obtained for very small value $\mu$, we see that the method with large value $\mu > 1$ also performs well. The overall best result is observed for HTCGM; this means that the assertion in [18] is confirmed again.

As it is well-known, the presence of an appropriate relaxation parameter $\lambda_n \in (0, 2)$ in MECSPM, or even the state-of-the-art relaxation methods, can make the methods converge faster. Now, we demonstrate the influence of the relaxation parameter $\lambda_n$ when ESCoM-CGD is performed for solving the considered problem. We fix $\mu = 10^{-4}$, $\varphi_n = \frac{1}{(n+1)^{0.1}}$, and $\beta_n = \frac{1}{(n+1)^{0.01}}$. We test a set of parameters $\lambda_n \in \{0.1, 0.2, \ldots, 1.9\}$, and plot the number of iterations and computational time with respect to different choices of $\lambda_n$ in Fig. 4.

According to the curves in Fig. 4, we see that the relaxation parameter $\lambda_n$ behaves significantly well convergence for a wide range of choices. In fact, we observe the faster convergence is obtained for some intermediate choices of $\lambda_n \in [0.8, 1.5]$, and the exactly best result is observed for $\lambda_n = 1.2$. This observation relatively conforms to the numerical experiments in [10].

Finally, to showcase the superiority of our ESCoM-CGD, we compare the methods for various sizes $(m, k)$ of randomly matrix $A$. We fix the corresponding parameters as in Table 2. To show performance of the methods, the numbers of iterations with respect to the size of $A$ are plotted in Fig. 5. Moreover, we also present computational time in seconds with respect to the sizes $(m, k)$ in Table 3.

The plots in Fig. 5 show that ESCoM-CGD gives the best convergence results for all choices of $(m, k)$. Moreover, we see that HCGM and HTCGM reach the optimal
tolerance at most the same number of iterations. Likewise, the results given in Table 3 reveal that ESCoM-CGD reaches the optimal tolerance faster than both HCGM and HTCGM. It is worth noting that when the size (20000, 5000), ESCoM-CGD requires computational time less than other two methods approximately 40 s. This underlines the essential superiority of the proposed ESCoM-CGD.

### 6 Conclusion

The object of this work was the solving of a variational inequality problem governed by a strongly monotone and Lipschitz continuous operator over the intersection of fixed-point sets of cutter operators. We associated to it the so-called extrapolated sequential constraint method with conjugate gradient direction. We proved strong convergence of the generated sequence of iterates to the unique solution to the considered problem. Our numerical experiments show that the proposed method has a better convergence behavior compared to other two methods. For future work, one may consider and analyze a variant of the proposed method by using some constrained selections, e.g., the so-called dynamic string averaging procedure, for dealing with the constrained operators.

**Acknowledgements** The authors are thankful to the Editor and two anonymous referees for comments and remarks which improved the quality and presentation of the paper. Mootta Prangprakhon was partially supported by Science Achievement Scholarship of Thailand (SAST), and Faculty of Science, Khon Kaen University. Nimit Nimana was partially supported by Thailand Science Research and Innovation under the project IRN62W0007.

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