Impurity in the Tomonaga-Luttinger model: a Functional Integral Approach

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1 Introduction

The Tomonaga-Luttinger model [1,2,3] of one-dimensional (1D) strongly correlated electrons gives a striking example of non-Fermi-liquid behaviour [4,5,6]. Since the seminal paper by Haldane [6], where the notion of the Luttinger liquid (LL) has been coined and fundamentals of a modern bosonization technique have been formulated, this model and its various modifications remain at the focus of interest in research in strongly correlated systems. Without diminishing the role of standard theoretical methods, such as diagrammatic techniques or renormalization group approach, one can say that bosonization methods make the most powerful theoretical tool for strongly correlated 1D systems. All these methods, and in particularly the ‘canonical’ operator bosonization, where the creation and annihilation operators of electrons are explicitly represented in terms of Bose operators and a 4-fermionic Hamiltonian is eventually diagonalized in the bosonic representation, are described by Dmitrii Maslov in lecture notes published in this volume.

The standard operator bosonization is one of the most elegant methods developed in theoretical physics. However, by its very formulation it seems both limited to and specific for one-dimensional physics. A subject of this seminar is to demonstrate the existence and usefulness of an alternative way to bosonize 1D interacting electrons, called the ‘functional bosonization’. It is based on the Hubbard-Stratonovich decoupling of the four-fermion interaction – a typical way to “bosonize” a fermionic system in higher-dimensional problems. This method was elaborated in different ways in a set of papers [7,8,9,10,11]. In this seminar, we will describe such a functional method in the form similar to that developed earlier [10,11] for the treatment of the pure LL as well as a single-impurity problem in the Luttinger model. However, we will employ here the Keldysh technique (see for reviews [12]) rather than the Matsubara one used in [10,11].

The essence of the method is to eliminate a mixed fermion-boson term in the action (resulted from the Hubbard-Stratonovich decoupling) by a gauge transformation. Such a procedure is exact for the pure 1D Luttinger model and gives a convenient starting point for including a single backscattering impurity.

The problem of a single impurity in the LL has been actively investigated by many authors [13,14,15,16,17,18,19,20]. One of the main results of these considerations [13,14,18,19,20] was the suppression at low temperatures of the local density of states (TDoS) at the impurity site and at a distance from the impurity [11] and the related suppression of the conductance [13,14] and the X ray edge singularity [18,19,20]. Another prominent result was the dependence of the Friedel oscillations [15,16,9] on the distance from the impurity.

In this seminar, we will briefly outline only the most important results for the one-impurity problem in the Tomonaga-Luttinger model, while giving a slightly more detailed description of the functional bosonization in the Keldysh technique that was not previously published. For simplicity, we will only address a single-mode Tomonaga-Luttinger model, with one species of right- and left-moving electrons, thus omitting spin indices and considering eventually the
simplest linearized model of a single-valley parabolic electron band. We will also skip over a physical introduction, referring the reader to lecture notes by Dmitrii Maslov published in this volume.

2 Functional integral representation

Within the limitations outlined above, the most generic Hamiltonian of interacting 1D electrons in the presence of an external scattering potential \( v(x) \) can be written as

\[
\hat{H} = \int \! dx \hat{\psi}^\dagger \left[ -\frac{\partial^2}{2m} - \varepsilon_F + v(x) \right] \hat{\psi}(x) + \frac{1}{2} \int \! dx \! dx' \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') V_0(x - x') \hat{\psi}(x') \hat{\psi}(x),
\]

where \( V_0(x - x') \) is a bare electron-electron interaction. The observable quantities to be calculated with this Hamiltonian are the tunneling density of states (TDoS) and the current as a linear response to an applied field characterised by the vector potential \( A(x,t) \). We will give the results for the TDoS at the end of this presentation, but will mainly described techniques for calculating the current \( j(x,t) \). It can be written with the help of the Kubo formula in terms of the current-current correlation function thermally averaged (\( \langle \ldots \rangle_0 \)) in the equilibrium Gibbs ensemble:

\[
j(x,t) = i \int_{-\infty}^{t} \! dt' \int \! dx' \left\langle \hat{j}(x,t), \hat{j}(x',t') \right\rangle_0 A(x',t') - \frac{ne^2}{m} A(x,t),
\]

where the current operator is defined in the standard way (in the units \( \hbar = 1 \) here and elsewhere):

\[
\hat{j}(x,t) = -\frac{ie}{2m} \hat{\psi}^\dagger(x,t) \partial_x \hat{\psi}(x,t) + \text{h.c.}
\]

The Kubo formula can be rewritten in Keldysh techniques by defining the contour \( C_K \) which runs from \( -\infty \) to the observation time \( t \) along the upper bank of the cut along the real time axis in the complex time plane and then returns to \( -\infty \) along the lower cut:

\[
j(x,t) = i \int_{C_K} \! dt' \int \! dx' \left\langle T_K \hat{j}(x,t)\hat{j}(x',t') \right\rangle_0 A(x',t') - \frac{ne^2}{m} A(x,t).
\]

Here we have introduced the chronological operator \( T_K \) time-ordering the operators in a descending order along the contour \( C_K \) (with all the times on the lower cut considered as later times as compared to those on the upper cut). Although the current operators are bilinear in Fermi-operators, for the future usage we assume in the standard way that the Fermi-operators anticommute under \( T_K \),
assuming additionally that at equal times $\hat{\psi}^\dagger(t)$ is taken at an infinitesimally later moment then $\hat{\psi}(t)$. Such an agreement is consistent with the definition of the current (Eq.3). In what follows, we will choose a more general contour (the Keldysh contour), running from minus to plus infinity above the cut and returning below the cut. In the operator language, such an extension of the contour corresponds to the insertion of extra evolution operators: having been not coupled to the observables (the current operator in our case), such evolution operators above and below the cut simply cancel each other.

Now, any expression written as the chronological operator average can be straightforwardly represented as a functional integral over the fields defined on the contour of the time ordering. We introduce the action,

$$S[\bar{\psi}, \psi] = \int_{C_K} dt \left\{ i\bar{\psi} \partial_t \psi - H[\bar{\psi}, \psi] \right\}, \quad (5)$$

with the last term being the the normal ordered Hamiltonian with $\hat{\psi}^\dagger \rightarrow \bar{\psi}$ and $\hat{\psi} \rightarrow \psi$, where $\bar{\psi}$ and $\psi$ are the anticommuting Fermi fields. Then, the the linear response current of Eq.(4) takes the form

$$\tilde{j}(x,t) = i \int_{C_K} dt' \int dx' \left\langle \tilde{j}(x,t)\tilde{j}(x',t') \right\rangle A(x',t') - \frac{ne^2}{m} A(x,t), \quad (6)$$

where the brackets stand for the functional integral

$$\langle \ldots \rangle = \int D\bar{\psi} D\psi \langle \ldots \rangle e^{iS[\bar{\psi}, \psi]}, \quad (7)$$

and the current field is defined by

$$\tilde{j}(x,t) = \frac{e}{2m_v} \bar{\psi}(x,t) \partial_x \psi(x,t) + c.c.$$

3 Effective action for the Tomonaga-Luttinger Model

For free electrons, the plane waves basis is natural. Assuming that the scattering by impurities and electron-electron interaction involve the energy scales much smaller than the Fermi energy, the plane wave basis presents a natural starting point. Therefore, we may assume that the main contribution to the functional integral above comes from the fields representing separately right-moving and left-moving electrons

$$\psi(x,t) \approx \psi_R(x,t) e^{ipFx} + \psi_L(x,t) e^{-ipFx} \quad (8)$$

where $\psi_{R,L}$ are smooth on the $p_F^{-1}$ scale. Such a separation is the essence of the Tomonaga-Luttinger model, and corresponds to the linearization of the
initially parabolic electron band. The right- and left-moving electrons can be transformed one to another due to backscattering processes. We shall neglect such processes due to the electron-electron interaction, thus keeping only the small momentum transfer part of interaction, $V(q \ll 2p_F)$. This part of interaction is non-trivial by itself as it breaks the Fermi Liquid theory in 1D. On the other hand we will keep only backscattering in the impurity potential $v(x)$ since small-momentum elastic scattering does not result in any qualitative change in the Luttinger Liquid behavior.

Now we make the substitution (8) neglecting higher order derivatives of smooth functions and discarding integrals over fast oscillating terms. After some straightforward manipulations we come to the action for the Tomonaga-Luttinger model:

$$S_{TL} = S_0 + S_{int}. \quad (9)$$

The first term describes free electrons in the presence of the external scattering potential (which can be also changing in time):

$$S_0 = \int dxd\bar{t} \Psi^\dagger(x,\bar{t}) \begin{pmatrix} i\partial_R & v(x,\bar{t}) \\ \bar{v}(x,\bar{t}) & i\partial_L \end{pmatrix} \Psi(x,\bar{t}). \quad (10)$$

We have assumed (here and below) that all the time integrations are performed along the Keldysh contour, and introduced the following notations:

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \Psi^\dagger = \begin{pmatrix} \bar{\psi}_R & \bar{\psi}_L \end{pmatrix}, \quad \partial_{R/L} \equiv \partial_t \pm v_F \partial_x. \quad (11)$$

The second term in Eq. (9) gives the interaction part of the action

$$S_{int} = -\frac{1}{2} \int dx\,d' x' dt \, n(x,\bar{t}) V_0(x-x') n(x',\bar{t}), \quad \left( n \equiv \Psi^\dagger \Psi \right). \quad (12)$$

## 4 The bosonized action for free electrons

Before dealing with interacting electrons, we convert the action (10) for free electrons in the presence of the impurity potential $v$ into the action in terms of bosonic fields.

Expanding Eq. (10) in powers of $v$, we obtain the partition function by integrating over the fermion fields with the help of the Wick theorem:

$$Z_0 = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \int d^n z \, d^n z' \det g_L(z_i, z'_j) \det g_R(z'_i, z_j) \prod_{k=1}^{n} v(z_k) \bar{v}(z_k'). \quad (13)$$

We have introduced the notation $z = (x, \bar{t})$ and defined the Green functions of left- and right-moving electrons in the standard way (suppressing indices $L, R$):

$$g = -i \langle \psi(x) \bar{\psi}(z') \rangle = \begin{pmatrix} g^{++} \\ g^{+\bar{+}} & g^< \\ g^{<\bar{+}} & g^- \end{pmatrix}. \quad (14)$$
The matrix structure is defined by the position of the time arguments of $\psi$ and $\bar{\psi}$: in $g^{++}$ and $g^{--}$ both of them are, respectively, on the upper or lower branches of the contour, and in $g^<(g^>)$ the first (second) argument is on the upper branch while the second (first) is on the lower. We should remind that the functional average (7), invoked in the definition (14), automatically arranges for the time ordering along the Keldysh contour. The Green function obeys the equation

$$i\partial_{t}g_{\eta}(z,z') = \delta(x-x')\delta(t,t'),$$

(15)

Here $\delta(t,t') \equiv \delta(t-t')$ when both arguments are on the same branch of the contour and $\delta(t,t') = 0$ otherwise; $\partial_{t}$ is defined in Eq. (11). Explicitly

$$g^{R/L}_{\eta}(z,z') = -\frac{T}{2vF} \frac{1}{\sinh \pi T\left[(t-t' - i0) \mp \frac{x-x'}{vF}\right]}$$

(16)

while $g^<$ is different by the sign of $i0$. The components of $g$ obey the usual relation $g^{++} + g^{--} = g^<$ + $g^>$. In what follows, we will perform the standard Keldysh rotation, reducing all the appropriate matrices to the triangular form with the Keldysh component ($g^K = g^< + g^>$) in the upper right corner, and the retarded and advanced components ($g^\tau = (g^\eta)^* = g^{++} - g^<$) on the main diagonal (see, e.g., ref. [12]). We shall assume that the time arguments belonging to the upper and lower branch of the contour have, respectively, positive and negative infinitesimal shift into the complex plane. Note finally that in the equilibrium case presented here the (Fourier transform of the) Keldysh component is related to the (Fourier transforms of the) retarded and advanced ones via the Fermi distribution function $f_\varepsilon(T)$ as follows:

$$g^K(\varepsilon) = (1 - 2f_\varepsilon(T))(g^\tau(\varepsilon) - g^a(\varepsilon)).$$

(17)

The same relations are valid for other Green’s functions to be considered so that, wherever this does not involve an ambiguity, we will give only the retarded/advanced components in the explicit form implying that the expression like Eq. (17) is valid for the Keldysh component.

To calculate the partition function (13) we use the Cauchy identity [21]:

$$\det \frac{1}{\sinh(z_i - z_j)} = (-1)^{n+1} \prod_{i < j} \sinh(z_i - z_j) \sinh(z'_i - z'_j) \prod_{i,j} \sinh(z_i - z'_j).$$

(18)

It reduces Eq. (13) for the partition function to the following one:

$$Z_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{T}{2vF}\right)^{2n} \int d^n z d^n z' \left[ \prod_{k=1}^{n} \bar{v}(z_k) v(z'_k) \right] \prod_{i < j} s(z_i - z_j) s(z'_i - z'_j)$$

(19)
with
\[ s(z - z') \equiv \sinh \pi T \left( t - t' - \frac{x - x'}{v_F} \right) \sinh \pi T \left( t - t' + \frac{x - x'}{v_F} \right) \] (20)

Introducing the bosonic Green function
\[ i G_0(z - z') = -\ln s(z - z') \] (21)

whose retarded and advanced components are Fourier transform of
\[ G_{r/a}^{\pi s}(q, \omega) = \frac{4\pi v_F}{\omega^2 \pm v_F^2 q^2}, \quad \omega_{\pm} \equiv \omega \pm i0, \] (22)

one can write
\[ Z_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{T}{2v_F} \right)^{2n} \prod_{k=1}^{n} dz_k dz'_k \epsilon^k \epsilon'^k \]
\[ \times \exp \left[ -i \sum_{i<j} \left[ G_0(z_i - z_j) + G_0(z'_i - z'_j) \right] + i \sum_{i,j} G_0(z_i - z'_j) \right]. \] (23)

This is a partition function of the Coulomb gas with the logarithmic interaction and it can be represented as the functional integral over the bosonic field \( \varphi(x,t) \):
\[ Z_0 = \int D\varphi \exp \left\{ i S_0[\varphi] + \alpha \int dz [\epsilon \varphi - \text{c.c.}] \right\}. \] (24)

Here the free bosonic action \( S_0[\varphi] \) is defined in terms of the Green’s function \( G_0 \) of Eq. (21) as follows
\[ S_0[\varphi] = \frac{1}{2} \int dz dz' \varphi(z) G_0^{-1}(z - z') \varphi(z'), \] (25)

The constant \( \alpha \) in Eq. (24) absorbs an ill-defined value of \( G_0(x,t) \) at \( x = 0, t = 0 \). We use the ultra-violet cutoff which corresponds to the scale of order \( \epsilon_F \):
\[ \alpha = \frac{T}{2v_F} e^{G_0(0)} \approx \frac{\epsilon_F}{2\pi v_F}. \]

Thus we have cast the original free fermion problem into that of interacting bosons, represented by the partition function (24). The interaction between bosons, i.e. the second term in the exponent in Eq. (24), comes from the backscattering impurity term in the original fermionic problem. The Gaussian action for noninteracting bosons, Eq. (25), can be explicitly written via \( x \) and \( t \) in the standard form
\[ S_0[\varphi] = -\frac{1}{8\pi v_F} \int dt dx \left[ (\partial_t \varphi)^2 - v_F^2 (\partial_x \varphi)^2 \right], \] (26)

although Eq. (26) is no less convenient, especially for the generalization for the interaction. The main advantage of the bosonization, either in the standard or functional form, is that including the quadratic electron-electron interaction does not substantially change the free action.
5 Gauging out the interaction

The first step in including the interaction term (12) into the bosonization scheme is to perform the Hubbard-Stratonovich transformation which can be symbolically written as

\[
\exp \{ S_{\text{int}}(\bar{\psi}, \psi) \} = \int D\phi \exp \left\{ -\frac{i}{2} \phi V^{-1}_0 \phi + \phi \Psi \Psi^\dagger \right\}.
\]  (27)

Note that the auxiliary ‘Hubbard-Stratonovich’ (HS) bosonic field \( \phi \) here is different from the field \( \phi \) in Eqs. (24) and (26). Substituting this representation into the full action (9), we bring the partition function to the following form:

\[
Z = \int D\phi D\Psi^\dagger D\Psi e^{iS_0[\phi] + iS[\Psi, \phi]}.
\]  (28)

The action \( S[\Psi, \phi] \) for fermions interacting with the HS field is given by

\[
S[\Psi, \phi] = \int d^2z \Psi^\dagger(z) \left( i\partial_{\text{R}} - i\phi \right) \bar{\psi}(z) \left( i\partial_{\text{L}} - i\phi \right) \Psi(x, t).
\]  (29)

To cast this integral into the form identical to that of the previous section we apply the local gauge transformation,

\[
\psi_\eta(z) \equiv \psi_\eta(x, t) \rightarrow \psi_\eta(x, t) e^{i\theta_\eta(x, t)} \quad \text{with} \quad i\partial_{\eta} \theta_\eta(x, t) = \phi(x, t),
\]  (30)

which removes the bosonic field \( \phi \) from the diagonal part of the action but at a cost: the off-diagonal terms are rotated with the factors \( e^{\pm i\theta} \), and the Jacobian of the transformation \( J \) changes the quadratic in \( \phi \) part of the action.

It is shown in Appendix A that the Jacobian \( J \) of the gauge transformation can be represented as

\[
\ln J[\phi] = -\frac{i}{2} \int dz dz' \phi(z) \Pi(z, z') \phi(z').
\]  (31)

The polarization operator \( \Pi \) is given in the random phase approximation (RPA) by

\[
\Pi = \sum_{\eta=R,L} \Pi_\eta, \quad \Pi_\eta(z - z') = ig_\eta(z - z')g_\eta(z' - z),
\]  (32)

where \( g_\eta(x - x', t - t') \) is the free electron Green function given by Eq. (16). It is well known that the RPA is exact for the Luttinger Liquid [4]. Note that we give in Appendix A a very simple and straightforward proof of this.

The quadratic in \( \phi \) contribution of the Jacobian, Eq. (31), should be added to the quadratic term in Eq. (27). This results in the free bosonic action with the kernel corresponding to the screened interaction:

\[
S[\phi] = -\frac{1}{2} \int dz dz' \phi(z) V^{-1}(z - z') \phi(z'), \quad V^{-1} = V_0^{-1} + \Pi.
\]  (33)
The above expression should be understood in the operator sense: $V$ and $\Pi$ are the operators whose kernels are defined with the appropriate time and spatial dependence on the Keldysh contour.

Using the expressions for the retarded and advanced components of $g_\eta$ (with $\varepsilon_\pm = \varepsilon \pm i0$),

\begin{align}
g^{r/a}_R(q, \varepsilon) &= (\varepsilon - v_F q)^{-1}, & g^{r/a}_L(q, \varepsilon) &= (\varepsilon + v_F q)^{-1},
\end{align}

one finds the appropriate components of the polarisation operator,

\begin{align}
\Pi^{r/a}_R(q, \omega) &= -\frac{1}{2\pi} \frac{v_F q}{\omega_\pm - v_F q}, & \Pi^{r/a}_L(q, \omega) &= \frac{1}{2\pi} \frac{v_F q}{\omega_\pm + v_F q},
\end{align}

and thus the total polarisation operator as

\begin{align}
\Pi^{r/a}(q, \omega) &= -\frac{1}{\pi} \frac{v_F q^2}{(\omega_\pm)^2 - v^2_F q^2},
\end{align}

Assuming that the Fourier transform of the forward-scattering pair interaction only weakly depends on momentum, i.e. $V_0(q \ll 2p_F) \approx \text{const} \equiv \bar{V}$, and substituting Eq. (35) into the free bosonic action (33), one finds the components of the free HS bosonic propagator as follows:

\begin{align}
V^{r/a}(q, \omega) &= \frac{\omega_\pm^2 - v^2_F q^2}{\omega_\pm - v^2_F q^2} \bar{V}, & V^K(q, \omega) &= \tanh \left( \frac{\omega}{2T} \right) \left[ V^R(q, \omega) - V^A(q, \omega) \right].
\end{align}

Here we introduced the renormalized velocity $v$ which defines the effective coupling constant $g$:

\begin{align}
v^2 &\equiv v_F^2 + \frac{v_F \bar{V}}{\pi}, & g &\equiv \frac{v_F}{v}.
\end{align}

Therefore, the gauge transformation (30) reduces the action in (28) to

\begin{align}
S = S[\theta] + S[\Psi^\dagger, \Psi; \theta].
\end{align}

Its fermionic part is given by

\begin{align}
S[\Psi^\dagger, \Psi; \theta] &= \int dz \Psi^\dagger(z) \begin{pmatrix} i\partial_R & ve^{-i\theta} \\ \bar{v} e^{i\theta} & i\partial_L \end{pmatrix} \Psi(z)
\end{align}

with $\theta = \theta_R - \theta_L$, while its bosonic part $S[\theta]$ is defined via the field $\phi$ by Eqs. (39), (30) and (37). It is convenient to write it explicitly as an integral over the field $\theta$, which is straightforward since $\phi$ is linearly related to the field $\phi$ as in Eq. (30). Thus we arrive at the following explicit expression for $S[\theta]$ in Eq. (38):

\begin{align}
S[\theta] = \frac{1}{2} \int dz \, dz' \theta(z) G^{-1}_B(z - z') \theta(z').
\end{align}
The Gaussian kernel of this interaction, $G_B$, can be represented as

$$G_B = G - G_0,$$

where $G_0$ is defined by Eqs. (21) and (22), while $G$ has the standard triangular matrix structure in the Keldysh space, with the Fourier transform of its retarded/advanced component given by

$$G^{r/a}(q,\omega) = \frac{4\pi v_F}{\omega^2 - v^2 q^2},$$

i.e. it differs from $G_0$ only by substituting $v$ for $v_F$ in the denominator. Finally, the expressions for fermion density and current in new variables become

$$n = \bar{\psi}_R \psi_R + \bar{\psi}_L \psi_L + \frac{1}{2\pi} \partial_x \theta, \quad j = v_F \left[ \bar{\psi}_R \psi_R - \bar{\psi}_L \psi_L \right] - \frac{1}{2\pi} \partial_t \theta$$

which can be seen by keeping source terms coupled to original fermionic fields when calculating the Jacobian.

The effective action (38) is quadratic in fermionic fields which can now be easily integrated out. Before doing so, let us stress that the representation of Eqs. (38)–(40) seems to be more convenient for some problems than the fully bosonized action. To perform the fermionic integration, we note that the fermionic part of the action, Eq. (39), differs from that for the free electrons, Eq. (10), only by the substitution $v \rightarrow v e^{-i\theta}$. Therefore, repeating the same procedure as in the previous section, we represent this part of the action with the help of the bosonic field $\phi$ so that the full action in Eq. (38) goes (in symbolical notations) to

$$S[\phi,\theta] = \frac{1}{2} \theta G_B^{-1} \theta + \frac{1}{2} \phi G_0^{-1} \phi + \alpha \left[ ve^{-i(\phi+\theta)} + c.c. \right].$$

Introducing the new bosonic field, $\Theta \equiv \theta + \phi$, and noting again the relation (41) we arrive at the standard fully bosonized action,

$$S[\Theta] = \frac{1}{2} \int dz dz' \Theta(z) G^{-1}(z-z') \Theta(z') + \alpha \int dz \left[ v(z) e^{-i\Theta(z)} + c.c. \right],$$

with Eq. (43) transforming to

$$n = n_0 + \frac{1}{2\pi} \partial_x \Theta, \quad j = -\frac{1}{2\pi} \partial_t \Theta.$$

### 6 Tunnelling density of states near a single impurity

As an application of the formalism developed above we will calculate the density of states in the vicinity of a single impurity in the Luttinger Liquid characterised by the following local time-independent potential:

$$v(x) = \lambda v_F \delta(x)$$
where $\lambda$ is the dimensionless impurity strength. The potential (46) should be substituted into the action (44). Then one integrates out the fields with $x \neq 0$ which results in the local action in terms of $\Theta(t) \equiv \Theta(x=0,t)$:

$$S_{imp} = \frac{1}{2} \int \frac{dtdt'}{C_K} \Theta(t) G^{-1}_{imp}(t-t') \Theta(t') + 2i\alpha \lambda v_F \int \frac{dt}{C_K} \cos \Theta(t)$$

(47)

where the Fourier transform of the retarded/advanced components of the inverse Gaussian kernel, $G_{imp}(t)$, are obtained by integrating the Green’s functions of Eq. (42) over all momenta:

$$G^{r/a}_{imp}(\omega) = \int \frac{dq}{2\pi} \frac{4\pi v_F}{\omega^2 \pm v^2 q^2} = -ig \frac{2\pi}{\omega \pm i0}.$$  

(48)

We now employ the self-consistent harmonic approximation (see, e.g., [22, 23]), i.e. substitute the impurity $\cos \Theta$ term with the quadratic one:

$$i 2av_F \lambda \int dt \cos \Theta(t) \to -\frac{i}{2} \Lambda \int dt \Theta^2(t)$$

The coefficient $\Lambda$ is to be found from the condition that this substitution is optimal,

$$\frac{\partial}{\partial \Lambda} \left[ 2av_F \lambda \langle \cos \Theta \rangle - \frac{1}{2} \Lambda \langle \Theta^2 \rangle \right] = 0,$$  

(49)

where the averages are taken with the effective action symbolically represented as

$$S_{eff} = \frac{1}{2} \Theta \left( G^{-1}_{imp} + \Lambda \right) \Theta.$$  

(50)

Solving self-consistently Eq. (49) with the action (50) (which involves preserving the proper analytical structure, with $\Lambda$ being of the standard matrix structure in the Keldysh space) one finds with the logarithmic accuracy:

$$\Lambda^{r/a} = \pm \varepsilon_F \left( \frac{av_F}{\varepsilon_F \lambda} \right)^{1/2} \simeq \pm \varepsilon_F \lambda^{1/2}. $$

(51)

Including the source terms of Eq. (45) does not cause any principal difficulties but makes the transformations more cumbersome. Therefore, we do not describe such transformations in the framework of this seminar presentation. Instead, we simply present the results for the tunnelling density of states [following from the long time asymptotics of the full electron Green function calculated in the SCHA with the bosonized action (38) as a function of the distance from the impurity. We refer the reader interested in detail to our previous publication [11] (albeit for the Matsubara rather than the Keldysh functional integrals). We present the explicit expressions for the TDoS smoothed over the length scale much larger than $p_F^{-1}$ in three different regions:
The regions of different behavior of the tunneling density of states are sketched in the figure. There and in Eq. (52), $\tilde{x} \equiv gp_{p} x$, $\tilde{\varepsilon} \equiv \varepsilon / \varepsilon_{r}$, and the renormalized impurity strength $\Lambda \equiv |\Lambda^{r,a}|$ is given by Eq. (51). Equation (52a) describes TDoS in the vicinity of impurity, in full correspondence with the original results of Kane and Fisher [13] obtained for the TDoS at $x = 0$, i.e. exactly at the impurity. In addition, we have established here the TDoS dependence on the impurity strength $\Lambda \equiv \lambda$. The region of applicability of Eq. (52a) corresponds to the diagonally hatched region in Fig. 1. Equation (52c) gives the TDoS at very large distances from the impurity. As expected, it coincides with a well-known result for the TDoS in the homogenous Luttinger liquid. Its region of applicability is horizontally hatched in Fig. 1. In the intermediate region, vertically hatched in Fig. 1, the TDoS depends both on the energy and the distance from the impurity. This analytic dependence given by Eq. (52b) describes the crossover from the impurity-induced dip in the TDoS to the bulk behavior. Finally, the unhatched region for $\tilde{\varepsilon} < \alpha$ corresponds to small energies,
\[ \varepsilon < T, \] where the energy dependence saturates (by \( \varepsilon \to T \)) in all the three lines of Eq. (52).

In conclusion, we have demonstrated how to develop the formalism of bosonization based on the functional integral representation of observable quantities within the Keldysh formalism. We have derived in this way the fully bosonized action for the interacting electrons in the presence of the scattering potential given by Eq. (44), and illustrated its usage on the example of the TDoS on a single impurity, Eq. (52). Let us stress finally that the intermediate representation of Eqs. (38)–(40), which still contains the part quadratic in fermion fields appears to be more convenient for some problems than the fully bosonized action.

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A Jacobian of the gauge transformation

The Jacobian of the gauge transformation (30) can be defined as

\[ J = J_R J_L \]
with

\[ J_{\eta}[\phi] = \frac{\int D\psi e^{-\int dz \bar{\psi}(\partial_n - i\phi)\psi}}{\int D\psi e^{-\int dz \bar{\psi}\partial_n\psi}} = e^{\text{Tr} \ln[1 - ig_\eta \phi]}, \] (53)

where the Green functions of non-interacting right- or left-moving electrons, obeying \( \partial_0 g_\eta = \hat{I} \), are given by Eq. (16). Note that in the matrix components \( g^< (t) \) and \( g^> (t) \) the time argument should be understood, respectively, as \( t \pm i0 \). The exponent in Eq. (53) can be represented as infinite series in the HS-field \( \phi \):

\[ \ln J_{\eta}[\phi] = -\sum_{n=1}^{\infty} \frac{i^n}{n!} \text{Tr} [g_\eta \phi]^n \] (54)

The \( n \)-th order term in \( \phi \) is proportional to the loop \( \Gamma^\eta_n \) with \( n \) external lines corresponding to \( \phi \)'s, each loop being built of the \( n \) Green functions \( g_\eta \):

\[ \text{Tr} [g_\eta \phi]^n \propto \int \left[ \prod_{i=1}^{n} dz_i \phi(z_i) \right] \Gamma_n (z_1; \ldots; z_n), \] (55)

where the \( n \)-th order vertex is given by

\[ \Gamma_n (z_1; \ldots; z_n) = \prod_{i=1}^{n} g_\eta (z_i; z_{i+1}), \quad (z_{n+1} = z_1). \]

Introducing the new variables \( \xi = e^{2\pi T \left[ \frac{\phi}{2} \right] - t} \), one represents the Green functions of Eq. (19) as

\[ g_R (\xi_1, \xi_2) \propto \frac{\sqrt{\xi_1 \xi_2}}{\xi_1 - \xi_2} \]
so that the vertex becomes
\[ \Gamma_n(\xi_1; \ldots; \xi_n) \propto \gamma_n(\xi_1; \ldots; \xi_n) \prod_{i=1}^{n} \xi_i, \quad \gamma_n = \prod_{i=1}^{n} \frac{1}{\xi_i - \xi_{i+1}}, \] (56)

with the boundary condition \( \xi_{n+1} = \xi_1 \). Since only the symmetric part of the vertex contributes to the integral (55), we may symmetrize \( \gamma_n \):
\[ \gamma_n \mapsto \frac{A_N(\xi_1; \ldots; \xi_n)}{\prod_{i<j}(\xi_i - \xi_j)}, \] (57)
where \( A_N \) is an absolutely antisymmetric polynomial of the \( N \)-th order which depends on \( n \) variables. A simple power counting gives \( N = n(n - 3)/2 \) while the possible minimal order of such a polynomial is \( N_{\text{min}} = n(n + 1)/2 \). This is not self-contradictory for \( n = 1 \) and \( n = 2 \) only. All other terms must therefore be equal to zero. The term with \( n = 1 \) is cancelled due to electroneutrality. The only non-vanishing vertex then is the one with two legs:
\[ \ln J_n[\phi] = \frac{i}{2} \int \! dz \! dz' \phi(z) \Pi_\eta(z; z') \phi(z'), \] (58)
where
\[ \Pi_\eta(z; z') = ig_\eta(z - z')g_\eta(z' - z). \] (59)

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