Thermodynamics of five-dimensional static three-charge STU black holes with squashed horizons

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Abstract

We present a new expression for the five-dimensional static Kaluza-Klein black hole solution with squashed $S^3$ horizons and three different charge parameters. This black hole solution belongs to $D = 5\, N = 2$ supergravity theory, its spacetime is locally asymptotically flat and has a spatial infinity $R \times S^1 \hookrightarrow S^2$. The form of the solution is extraordinary simple and permits us very conveniently to calculate its conserved charges by using the counterterm method. It is further shown that our thermodynamical quantities perfectly obey both the differential and the integral first laws of black hole thermodynamics if the length of the compact extra-dimension can be viewed as a thermodynamical variable.

Keywords: $U(1)^3$ supergravity, squashed black hole, thermodynamics

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1. Introduction

Five-dimensional black holes [1] with squashed horizons are of special interest in Kaluza-Klein theory where the fifth dimension is assumed to be compactified into a circle. A simplest example for this in the five-dimensional Einstein-Maxwell theory is the static charged, squashed Kaluza-Klein black hole solution found by Ishihara and Matsuno [2] via applying the so-called squashing transformation to the five-dimensional Reissner-Nordström black hole solution. This black hole has horizon-topology of a $S^3$ sphere that is deformed by the squashing function. At the infinity $(r \rightarrow r_{\infty})$, the spacetime

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approaches to the direct product of a flat time dimension and the asymptotic structure of the self-dual Taub-NUT instanton with NUT charge $r_\infty/4$. In other words, the black hole spacetime is asymptotically a twisted $S^1$ fibre bundle over $M_{1,3}$, which can be interpreted as a static charged Reissner-Nordström black hole sitting on the self-dual Taub-NUT instanton $[3]$.

Sooner after the work done in Ref. [2], the squashing procedure was then successfully applied to generate new black hole solutions with squashed horizons, due to its simplicity of the method. Subsequently, a large class of new Kaluza-Klein black hole solutions in five dimensions had been constructed so far in vacuum Einstein gravity [4], Einstein-Maxwell-dilaton gravity [5, 6], $D = 5$ minimal Einstein-Maxwell-Chern-Simons supergravity [7] and $U(1)^3$ supergravity [8]. Later, several black hole solutions with squashed horizons in a five-dimensional Gödel universe had also been presented in Refs. [9, 10, 11].

In recent years, there are also a lot of attention being paid to various aspects of squashed Kaluza-Klein black holes. For example, thermodynamical properties [12, 13, 14, 15, 16, 17], Hawking radiation [18, 19, 20, 21, 22, 23], perturbation stability [24, 25, 26], quasinormal modes [27, 28], gravitational lens [29, 30, 31], geodesic motion [32] and Kerr/CFT correspondence [17] have been investigated for this kind of black holes in recent years. In particular, Cai et al. [12] first adopted the boundary counter-term method [33, 34] to investigate thermodynamics of the static charged, squashed Kaluza-Klein black holes found in Ref. [2] and showed that it is the counter-term mass, which is equal to the Abott-Deser mass, rather than the Komar mass, that obeys the differential first law when the radius of the compact extra-dimension is considered as a constant. The counter-term method was then frequently applied to calculate the boundary stress-energy tensor and the conserved charges of a large class of Kaluza-Klein black holes [4, 5, 12, 13, 15, 35] with squashed horizons. Moreover, thermodynamics of the five-dimensional squashed Reissner-Nordström black hole was investigated in details in Ref. [14] where the counter-term mass is in agreement with the mass calculated by the background subtraction method if the Kaluza-Klein monopole background is considered as a natural reference spacetime for the squashed Kaluza-Klein black hole.

Our aim of this Letter is mainly concerned with thermodynamics of the three-charge squashed Kaluza-Klein black hole solution to $D = 5, N = 2$ ungauged supergravity theory. A rotating solution in this theory was previously presented in Ref. [8] where the author obtained the solution by directly applying the squashing transformation to the three-charge Cvetič-Youm black
hole \cite{36} with two equal rotation parameters. Some thermodynamical quantities were given in \cite{8}, however they can not consistently fulfil both the differential and the integral first laws of squashed black hole thermodynamics. The reason for this is simply because the ADM mass computed there does not obey the standard first law of thermodynamics for the squashed Kaluza-Klein black holes \cite{13,14}. What is more, the contribution of three dipole charges should but had not been taken into consideration in Ref. \cite{8}, when the rotation is included. Therefore, it is clear that thermodynamical properties of this solution had not been correctly studied ever before in the previous literature, unlike the minimal case \cite{17}, and it deserves a deeper investigation of its thermodynamics, which consists of the main subject of our work.

However, the rotating version of the three-charge squashed solution presented in Ref. \cite{8} is very complicated after performing the coordinate transformations from \((t, r)\) to \((\tau, \rho)\) even if one only considers the static case, so the expression for the solution is not suitable for our purpose to study its thermodynamical properties. For the sake of simplicity, in this Letter we shall focus on the nonrotating case only. Our strategy is to seek another new form for the three-charge static squashed black hole solution which is different from the one previously presented in Ref. \cite{8}. To derive the solution, we have applied a triple-repeated lift-boost-reduction procedure \cite{37,38} to the five-dimensional static squashed Schwarzschild black hole solution which has a normalized Killing time vector at spatial infinity. The derivation is essentially parallel to that of the static three-charge STU black hole solution in Ref. \cite{39}. The final expressions for the solution are much simpler than those presented in Ref. \cite{8} because three gauge potentials and two dilaton scalar fields associated with it all vanish at the infinity. For this form of the solution, one can very easily calculate its conserved mass and gravitational tension by using the counterterm method, and show that all thermodynamical quantities computed for our three-charge static Kaluza-Klein black hole with squashed \(S^3\) horizons perfectly satisfy both the differential and the integral first laws of squashed black hole thermodynamics if the length of the compact extra-dimension can viewed as a thermodynamical variable. When three charges are set to be equal, our results completely reproduce those obtained in Ref. \cite{12} after some suitable identifications of solution parameters.

The remaining part of our Letter is organized as follows. In Sec. \cite{2} we first present a new form of the static three-charge squashed black hole solution. Then, the boundary counterterm method is adopted to calculate the
conserved mass and gravitational tension which together with the entropy, horizon temperature, three charges and their corresponding electrostatic potentials completely satisfy both the differential and the integral first laws when we consider the length of the extra-dimension as a thermodynamical variable. Our Letter ends up with a summary of our work and the related future plan.

2. Static three-charge squashed black hole solution and its thermodynamics

In this section, we present a new simple form of the five-dimensional three-charge static squashed black hole solution and investigate its thermodynamics. To generate the solution, we start from the static squashed Schwarzschild solution after performing the appropriate coordinate transformations and use a thrice-repeated sequence \[37, 38\] of lifting to six dimensions, performing a Lorentz boost, and reducing again to \(D = 5\). The final expressions for the metric and three Abelian gauge potentials are concisely given by

\[
\begin{align*}
\rho^2 = & \left(h_1 h_2 h_3\right)^{1/3} \left[\frac{1 - \rho_1 / \rho}{h_1 h_2 h_3} d\tau^2 + \frac{1 + \rho_0 / \rho}{1 - \rho_1 / \rho} d\rho^2 \\
& + \rho (\rho + \rho_0) (d\theta^2 + \sin^2 \theta d\psi^2) + \frac{\rho_0 (\rho_0 + \rho_1)}{1 + \rho_0 / \rho} (d\phi + \cos \theta d\psi)^2\right], \\
A_I = & \frac{c_I s_I \rho_1}{h_I \rho} d\tau,
\end{align*}
\]

and three scalars are

\[
X_I = \frac{(h_1 h_2 h_3)^{1/3}}{h_I}, \quad h_I = 1 + s_I^2 \frac{\rho_1}{\rho},
\]

where \(c_I = \cosh \delta_I\), and \(s_I = \sinh \delta_I\), in which \(\delta_I\)'s are three charge parameters. In the solution, the coordinate \(\rho\) varies from 0 to \(\infty\), and \((\theta, \phi, \psi)\) are three Eulerian angles, taking the ranges \(0 < \theta < \pi\), \(0 < \psi < 2\pi\), \(0 < \phi < 4\pi\).

At spatial infinity \((\rho \to \infty)\), we have \(h_I \to 1\) and \(X_I \to 1\), so three gauge potentials \(A_I\) tend to zero and two dilaton scalar fields \((\varphi_1, \varphi_2)\) behave asymptotically like

\[
\varphi_1 = \frac{(s_1^2 + s_2^2 - 2s_3^2) \rho_1}{\sqrt{6} \rho} + O(\rho^{-2}), \quad \varphi_2 = \frac{(s_1^2 - s_2^2) \rho_1}{\sqrt{2} \rho} + O(\rho^{-2}),
\]
while the metric (1) approaches to
\[ ds^2 = -d\tau^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\psi^2) + \rho_0(\rho_0 + \rho_1)(d\phi + \cos \theta d\psi)^2. \] \( (5) \)

Just like the uncharged solution \( (\delta I = 0) \), the asymptotic structure of our solution (1) is also a four-dimensional flat Minkowski spacetime with a compact extra-dimension. That is, its spacetime is locally asymptotically flat and has an asymptotic boundary topology \( \mathbb{R} \times S^1 \hookrightarrow S^2 \), and \( \partial_\phi \) generates the twisted \( S^1 \) fibre bundle at spatial infinity, with a constant size \( 2\pi \sqrt{\rho_0(\rho_0 + \rho_1)} \). The horizon topology is, however, obviously a squashed \( S^3 \) sphere.

In the case where all three charges are set to equal \( (s_I = s) \), we find that by setting
\[ \tilde{\rho} = \rho + s^2 \rho_1, \quad \tilde{\rho}_0 = \rho_0 - s^2 \rho_1, \quad \rho_+ = c^2 \rho_1, \quad \rho_- = s^2 \rho_1, \]
and rescaling the gauge potential by a factor \( \sqrt{3} \), the above solution exactly reproduces the static squashed Reissner-Nordström black hole solution presented in Refs. \([2, 12]\).

We have verified that our solution solves the full set of equations of motion derived from the Lagrangian of the \( D = 5, N = 2 \) supergravity theory whose action is
\[ I = \frac{1}{16\pi G} \int_M d^5 x \left\{ \sqrt{-g} \left[ R - \frac{1}{2}(\partial \varphi_1)^2 - \frac{1}{2}(\partial \varphi_2)^2 - \sum_{I=1}^3 \frac{1}{4} X_I^{-2} F_{\mu\nu}^{I} F_{\mu\nu}^{I} \right] + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{-h} d^4 x \right\}, \]
where the Chern-Simons term is included for the completeness, but it makes no contribution to the action in the nonrotating case. In the boundary Gibbons-Hawking term, \( K \) is the trace of extrinsic curvature \( K_{ij} = (n_{ij} + n_{i,i})/2 \) for the boundary \( \partial M \) with the induced metric \( h_{ij} \), \( R \) is the bulk scalar curvature, and \( F_I = dA_I \) are strengths associated to three \( U(1) \)'s gauge fields. Two dilaton scalar fields \( (\varphi_1, \varphi_2) \) are related to three scalars \( X_I \) by
\[ X_1 = e^{-\varphi_1/\sqrt{6} - \varphi_2/\sqrt{2}}, \quad X_2 = e^{-\varphi_1/\sqrt{6} + \varphi_2/\sqrt{2}}, \quad X_3 = e^{2\varphi_1/\sqrt{6}}. \] \( (7) \)

It is now the position to investigate thermodynamical properties of our solution given above. An suitable approach for this aim is to make use of
the counterterm method \cite{33, 34} to compute the conserved charges since our solution is also an asymptotically flat spacetime with boundary topology \( R \times S^1 \hookrightarrow S^2 \). In the method, a counterterm, which is a functional only of the curvature invariants of the induced metric on the boundary, is added to the boundary term at infinity, so that a regular gravitational action is obtained without any modification of the equations of motion.

We consider the following simple counterterm proposed by Mann and Stelea \cite{33}

\[
I_{ct} = \frac{1}{8\pi G} \int d^4 x \sqrt{-h} \sqrt{2R},
\]

where \( R \) is the Ricci scalar with respect to the boundary metric \( h_{ij} \). Varying the action \((6)\) with this counterterm leads to the boundary stress-energy tensor

\[
T_{ij} = \frac{1}{8\pi G} \left[ K_{ij} - Kh_{ij} - \Psi (R_{ij} - R h_{ij}) - h_{ij} h^{kl} \Psi_{;kl} + \Psi_{;ij} \right],
\]

where \( \Psi = \sqrt{2/R} \), and the covariant derivative is defined with respect to the induced metric \( h_{ij} \) on the boundary.

If the boundary geometry has an isometry generated by the Killing vector \( \xi \), then \( T_{ij} \xi^j \) is divergence free, so the conserved charge associated with it is given by

\[
Q = \int_{\Sigma} d^3 S_i T_{ij} \xi^j,
\]

which represents the conserved mass \( M_{ct} \) in the case when \( \xi = \partial_\tau \), and the gravitational tension \( T \) if \( \xi = \partial_\phi \).

After some calculation, we find the needed components of the stress tensor as follows

\[
T_{\tau\tau} = \frac{\rho_0 + \rho_1 (2 + s_1^2 + s_2^2 + s_3^2)}{2\rho^2} + O(\rho^{-3}),
\]

\[
T_{\phi\phi} = \frac{2\rho_0 + \rho_1}{2\rho^2} + O(\rho^{-3}),
\]

\[
T_{\psi\psi} = \frac{\rho_0 + \rho_0 \rho_1 - \rho_1^2}{8\rho^3} + O(\rho^{-4}).
\]

Then it is straightforward to calculate the conserved mass and gravitational
tension as
\[ M_{ct} = \pi \sqrt{\rho_0(\rho_0 + \rho_1)} \left[ \rho_0 + (2 + s_1^2 + s_2^2 + s_3^2) \rho_1 \right], \quad (14) \]
\[ \mathcal{T} = \frac{\rho_0}{2} + \frac{\rho_1}{4}. \quad (15) \]

On the horizon, the entropy \( S = A/4 \) and temperature \( T = \kappa/(2\pi) \) can be easily obtained as
\[ S = 4\pi^2 c_1 c_2 c_3 \sqrt{\rho_0 \rho_1^{3/2}} (\rho_0 + \rho_1), \quad (16) \]
\[ T = \frac{1}{4\pi c_1 c_2 c_3 \sqrt{\rho_1 (\rho_0 + \rho_1)}}, \quad (17) \]
and three electrostatic potentials are given by
\[ \Phi_I = (A_I^\mu \chi^\mu) \bigg|_{\rho = \rho_1} = \frac{s_I}{c_I}. \quad (18) \]

Finally, we have three electric charges corresponding to the gauge potentials by completing the integral
\[ Q_I = \frac{1}{16\pi} \int_{S^3} X_I^{-2} * F_I = \pi c_1 s_1 \rho_1 \sqrt{\rho_0 (\rho_0 + \rho_1)}. \quad (19) \]

It is not difficult to verify that the above thermodynamical quantities completely satisfy both the different and the integral first laws of black hole thermodynamics
\[ dM_{ct} = T dS + \Phi_1 dQ_1 + \Phi_2 dQ_2 + \Phi_3 dQ_3 + 4\pi T dL, \quad (20) \]
\[ M_{ct} = \frac{3}{2} T S + \Phi_1 Q_1 + \Phi_2 Q_2 + \Phi_3 Q_3 + 2\pi T L, \quad (21) \]

where \( 2\pi L \) is the length of the extra-dimension, and \( L = \sqrt{\rho_0(\rho_0 + \rho_1)} \) can be identified with twice of the NUT charge. In the equal-charge case \((Q_i = Q/\sqrt{3})\), our results exactly recover those obtained in Ref. [12] where the radius \( 2\pi L \) of the extra-dimension was considered as a constant. Here, we view it as a thermodynamical variable for the self-consistence of the integral first law.
3. Conclusions

In this Letter, we have presented a new form for the static three-charge STU black hole solution with squashed horizons. The expressions for the metric, three gauge potentials and two scalar fields are rather simple and very convenient for us to investigate its thermodynamics. By means of the counterterm method, all thermodynamical quantities of our solution are easily calculated and have been shown to fulfil both differential and integral first laws of the squashed black hole thermodynamics when the length of the extra-dimension is considered as a thermodynamical variable.

It is interesting to extend the present work to the rotating charged case where the form of the squashed black hole solution is definitely inconvenient for studying its thermodynamics. A promising routine to resolve this difficulty is to regenerate this solution from the neutral seed of the rotating Kaluza-Klein squashed black hole solution with a Killing time vector normalized at spatial infinity.

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