Matrix mechanics of the relativistic point particle and string in Clifford space

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Abstract

We resolve the space-time canonical variables of the relativistic point particle into inner products of Weyl spinors with components in a Clifford algebra and find that these spinors themselves form a canonical system with generalized Poisson brackets. For \(N\) particles, the inner products of their Clifford coordinates and momenta form two \(N \times N\) Hermitian matrices \(X\) and \(P\) which transform under a \(U(N)\) symmetry in the generating algebra. This is used as a starting point for defining matrix mechanics for a point particle in Clifford space. Next we consider the string. The Lorentz metric induces a metric and a scalar on the world sheet which we represent by a Jackiw-Teitelboim term in the action. The string is described by a polymomenta canonical system and we find the wave solutions to the classical equations of motion for a flat world sheet. Finally, we show that the \(SL(2,\mathbb{C})\) charge and space-time momentum of the quantized string satisfy the Poincare algebra.

1 Introduction

It is well known that a null vector can be resolved into a product of two Weyl spinors

\[
x^{AB} = c^A \cdot c^B, \quad x^\mu x_\mu = 0,
\]

where \(x^{AB}\) and \(x^\mu\) are related through the equivalence between real four-vectors and second-rank hermitian spinors

\[
V^\mu = \frac{1}{2} \sigma^\mu_{AB} V^{AB}, \quad V^{AB} = \sigma^A_{\mu} V^\mu B, \quad \sigma^\mu_{AB}, \quad \sigma^A_{\mu} V^\mu B,
\]

and \(\sigma_\mu\) are the four hermitian matrices which extend the Pauli matrices \([I]\). To resolve non-null vectors, we need something like

\[
x^{AB} = c^A \cdot c^B,
\]

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where $\bullet$ is a product which belongs to some non-commutative algebra. This problem can be compared to the somewhat similar problem of resolving the Lorentz metric $\eta_{\mu\nu}$ into vectors. The well known solution is $\eta_{\mu\nu} = \frac{1}{2}\{\gamma_\mu, \gamma_\nu\}$ where the Dirac matrices $\gamma_\mu$ generate the Clifford algebra $Cl(1, 3, \mathbb{R})$. The components of any real symmetric $4 \times 4$ matrix of signature $(1, 3)$ can therefore be expressed as the inner products (anti-commutators) of vectors (real linear combinations of $\gamma$ matrices) belonging to $Cl(1, 3, \mathbb{R})$. Real Clifford algebras are associated with real quadratic forms, but there is no similar relationship between hermitian sesquilinear forms and complex Clifford algebras $Cl(\mathbb{C})$. Instead we must use even-dimensional real Clifford algebras written in complex form [2, 3, 4, 5]. Consider a future directed time-like vector $x^\mu$. A unitary transformation followed by a non-uniform scaling can reduce $x^A$ to a diagonal matrix with ones in the diagonal and can be effected by a suitable linear transformation of $c^A$ so that (1) becomes

$$c_i \bullet c_j^* = \delta_{ij}.$$

This can be compared to the algebra of creation and annihilation operators for two fermions

$$\{a_i, a_j^\dagger\} = \delta_{ij} \cdot 1, \quad \{a_i, a_j\} = 0, \quad i, j = 1, 2. \quad (2)$$

Defining $e_i = i(a_i + a_i^\dagger)$, $e_{2+i} = a_i - a_i^\dagger$, $i = 1, 2$, the anti-commutation relations (2) become

$$\{e_i, e_j\} = -2\delta_{ij}, \quad i, j = 1, \ldots, 4$$

which generate the Clifford algebra $Cl(0, 4, \mathbb{R})$. This suggests that a solution to (1) would be to use spinors with values in the split Clifford algebra $Cl(4, 4, \mathbb{R})$ and to let $\bullet$ be the inner product (anti-commutator) of this algebra. This expectation is borne out by the following proposition

Let $V_C$ be a $2n$-dimensional complex linear space with complex conjugation $*$ and $H$ an $n \times n$ Hermitian matrix of arbitrary signature. Then the components of $H$ can be expressed as

$$H_{ij} = c_i \bullet c_j^*, \quad c_i \bullet c_j = 0, \quad i, j = 1, \ldots, n$$

where $c_i$ belong to $V_C$ and $\bullet$ is the inner product

$$a \bullet b \equiv \frac{1}{2}\{a, b\}$$

of the Clifford algebra $Cl(2n, 2n, \mathbb{R})$ on the $4n$ dimensional real linear space $V_R$ which corresponds to $V_C$.

Proof. Let $e_i, f_i, i = 1, \ldots, n$ be a basis for $V_C$ and $g_i = i(e_i + e_i^\dagger)$, $g_{n+i} = e_i - e_i^\dagger$, $h_i = i(f_i + f_i^\dagger)$, $h_{n+i} = f_i - f_i^\dagger$, $i = 1, \ldots, n$ a basis for $V_R$. Let $g_i$ and $h_i$ generate the Clifford algebra $Cl(2n, 2n, \mathbb{R})$ on $V_R$ through

$$g_i \bullet g_j = 2\delta_{ij}, \quad h_i \bullet h_j = -2\delta_{ij}, \quad g_i \bullet h_j = 0, \quad i, j = 1, \ldots, 2n.$$
Then the basis $e_i, f_i$ for $V_C$ satisfies
\[ e_i \cdot e_j^* = -\delta_{ij}, \quad f_i \cdot f_j^* = \delta_{ij}, \quad e_i \cdot e_j = f_i \cdot f_j = 0, \quad i, j = 1, \ldots, n. \]
We can create any $n \times n$ diagonal matrix of plus or minus ones by setting $c_i$ equal to either $f_i$ or $e_i$. A zero in the $k$-th entry of the diagonal can be created by $c_k = e_k + f_k$. A non-uniform scaling followed by a unitary transformation can transform this diagonal matrix into any desired $n \times n$ hermitian matrix with the same signature and can be effected by a suitable complex linear transformation of the $c_i$'s.

We shall resolve both the coordinates and momenta of the point particle into Clifford spinors
\[ x^{A\bar{B}} = c^A \cdot c^*\bar{B}, \quad p_{AB} = d_A^* \cdot d_B, \quad (3) \]
but we also need the Clifford algebra to be large enough that the inner products $c^A \cdot d_B^*$ are algebraically independent of $x$ and $p$. This can be accomplished, for example by enlarging $Cl(4,4,\mathbb{R})$ to $Cl(8,8,\mathbb{R})$ and then generating $x$ and $p$ by each their own $Cl(4,4,\mathbb{R})$ subalgebras. This makes $c \cdot d^*$ vanish. The second step is to choose two Clifford elements $h_i$ whose inner products with both $c$ and $d^*$ vanish, and to make the substitution
\[ c^A \rightarrow c^A + A_i^A h_i, \quad d_A^* \rightarrow d_A^* + B_iA h_i^*. \]
This will change $x$ and $p$ only by additive matrices that will not constrain them, and the two matrices $A$ and $B$ can be adjusted to produce any desired value of $c \cdot d^*$. Apart from this requirement, the dimension of the single-particle Clifford algebra is not of importance in our discussion.

Note that $c^{*\bar{A}}$ and $d_A^*$ have the same commutation properties but transform differently under $SL(2,\mathbb{C})$. The complex conjugation symbol $*$ can therefore not be omitted, as it often is, because it specifies the commutation properties of the element in question. It is tacitly assumed that the inner product of elements of the same kind vanishes, and this will not be written out explicitly.

If the variation of a function $f$ of $c$ with respect to a variation of $c$ can be expressed on the form
\[ \delta f = \frac{1}{2} \{ \partial f / \partial c^A, \delta c^A \}, \]
we shall call $\partial f / \partial c^A$ a derivative of $f$ with respect to $c$. It is defined up to terms which anti-commute with arbitrary variations $\delta c$. Trivially, we have $\partial c^A / \partial c^B = \delta_A^B$. From (3) it follows that a differentiable function $f$ of $x^\mu$ has the derivative
\[ \frac{\partial f(x)}{\partial c^A} = \frac{\partial f(x)}{\partial x^\mu} \frac{1}{2} \sigma^\mu_{AB} c^B \epsilon^{A\bar{B}} = \frac{\partial f(x)}{\partial x^{A\bar{B}}} c^{\bar{B}}. \quad (4) \]

2 Clifford substructure of the relativistic point particle

Let the space-time coordinates and momenta of the relativistic point particle be resolved into Clifford spinors according to (3). The equations of motion are
obtained from the condition that the reparametrization invariant action

\[ I = 4\sqrt{m} \int d\tau \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^B}{d\tau} \cdot \frac{dc_A}{d\tau} \cdot \frac{dc^*_B}{d\tau}} \]  

is stationary under arbitrary variations of \( c(\tau) \). The momenta conjugate to \( c \) are

\[ d^*_A \equiv \frac{\partial L}{\partial \frac{dc_A}{d\tau}} = \sqrt{m} \left( \frac{1}{2} \frac{dc^E}{d\tau} \cdot \frac{dc^*_F}{d\tau} \cdot \frac{dc^*_E}{d\tau} \right)^{-\frac{1}{4}} \left( \frac{dc_A}{d\tau} \cdot \frac{dc^*_B}{d\tau} \right) \frac{dc^*_B}{d\tau}, \]

and, as expected, the Hamiltonian vanishes. A straightforward calculation using the four-vector identity

\[ V^A_E V^B_E = \frac{1}{2} \delta^B_A V^F_E V^F_E \]  

shows that the conjugate momenta \( d^*_A \) satisfy the constraint

\[ p^\mu p_\mu - m^2 = 0, \]

where \( p_\mu \) are the space-time momenta defined in (3). This happens to be the same constraint as would have been obtained from the space-time action \( \int \sqrt{\dot{x}^2} \, d\tau \). According to constrained dynamics, the Hamiltonian density is proportional to the constraint

\[ \mathcal{H}(p, e(\tau)) = e(\tau)(p^\mu p_\mu - m^2), \]

where \( e(\tau) \) is an einbein. This Hamiltonian can also be obtained from the Polyakov (‘metrical’) type of action

\[ I = \int d\tau \left( 3e(\tau)^{-\frac{1}{3}} \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^B}{d\tau} \cdot \frac{dc_A}{d\tau} \cdot \frac{dc^*_B}{d\tau} + m^2 e(\tau)} \right), \]

which recovers (5) when the equations of motion for the einbein \( e(\tau) \) are substituted back into the action. The momenta conjugate to \( c \) are

\[ d^*_A = e(\tau)^{-\frac{1}{3}} \left( \frac{1}{2} \frac{dc^E}{d\tau} \cdot \frac{dc^*_F}{d\tau} \cdot \frac{dc^*_E}{d\tau} \right)^{-\frac{1}{4}} \left( \frac{dc_A}{d\tau} \cdot \frac{dc^*_B}{d\tau} \right) \frac{dc^*_B}{d\tau}, \]

which determine the Hamiltonian density

\[ \mathcal{H}(c, d) = d^*_A \cdot \frac{dc^A}{d\tau} + c.c. - L, \]

where \( c.c. \) denotes the complex conjugate of the previous term and \( L \) is the Lagrangian in (9). A straightforward calculation gives

\[ d^*_A \cdot \frac{dc^A}{d\tau} + c.c. = 4e(\tau)^{-\frac{1}{3}} \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^B}{d\tau} \cdot \frac{dc_A}{d\tau} \cdot \frac{dc^*_B}{d\tau}}. \]
\[ p^{\mu}p_\mu = \frac{1}{2} d^A \cdot d^{*B} \frac{\partial H}{\partial d^*_A} = e(\tau)^{-\frac{1}{2}} \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^{*B}}{d\tau} \frac{dc^*_A}{d\tau} \cdot \frac{dc^{*B}}{d\tau}}, \]

which, when applied to (10), gives the Hamiltonian (8) of constrained dynamics. Hence the first order (Hamiltonian) form of the action (5) is

\[ I = \int d\tau \left( d^*_A \cdot \frac{dc^A}{d\tau} + \text{c.c.} - e(\tau)(p^\mu p_\mu - m^2) \right). \]

This action has a global \( SL(2,\mathbb{C}) \) and \( U(1) \) gauge symmetry with the conserved Noether charges

\[ J_{AB} = d^*_A \cdot c_B + d^*_B \cdot c_A, \quad j \equiv i(d^*_A \cdot c^A - d^*_A \cdot c^{*A}). \]

To obtain the correct space-time equations of motion, it is necessary to assume (as an initial value condition) that they vanish

\[ d^*_A \cdot c_B + d^*_B \cdot c_A = 0, \]
\[ d^*_A \cdot c^A - d^*_A \cdot c^{*A} = 0. \]

Since all skew-symmetric second rank tensors are proportional to \( \epsilon_{AB} \), (12) gives

\[ d^*_A \cdot c_B = \mu(\tau) \epsilon_{AB}, \quad \text{or} \quad d^*_A \cdot c^{*B} = \mu^{AB}_A, \]

with (13) saying that \( \mu(\tau) \) is real. For short, we shall refer to this condition as the 'Noether constraint'. The canonical equations of motion are obtained by independent variation of \( c \) and \( d \)

\[ \frac{dc^A}{d\tau} = \frac{\partial H}{\partial d^*_A}, \quad \frac{dd^*_A}{d\tau} = -\frac{\partial H}{\partial c^A} = -\frac{\partial H}{\partial x^{AE}} c^{*E}, \]

where we have used the differentiation rule (4). Taking the inner product of these equations with \( c^{*B} \) and \( d^*_B \) gives

\[ \frac{dx^{AB}}{d\tau} = 2 \frac{\partial H}{\partial d^*_B} c^{*B} \cdot d^*_B, \quad \frac{dp^{AB}}{d\tau} = -2 \frac{\partial H}{\partial x^{AB}} c^{*E} \cdot d^*_B, \]

which by use of the Noether constraint (14) become

\[ \frac{dx^{AB}}{d\tau} = 2 \frac{\partial H}{\partial d^*_B} \mu(\tau), \quad \frac{dp^{AB}}{d\tau} = -2 \frac{\partial H}{\partial x^{AB}} \mu(\tau). \]

In the parametrization

\[ \tau(\varphi) = \frac{1}{2m\mu(\tau)}, \]

these equations reduce to the canonical equations of motion

\[ \frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} \mathcal{H}(x, p) \equiv \frac{1}{2m}(p^\mu p_\mu - m^2), \]
for a relativistic point particle with proper time $\tau$. This proper time is not defined at points where $\mu$ vanishes. There will be just one such point and it represents a ‘turning point’ where the space-time trajectory has an endpoint and the underlying trajectory in Clifford space starts to reproduce it for the second time. From (15) and the Hamiltonian constraint (7), we obtain an explicit expression for $\mu(\tau)$

$$\frac{d}{d\tau}\mu(\tau) = \frac{d}{d\tau}(\frac{1}{2}d_E \bullet c^E) = e(\tau)m^2, \quad \mu(\tau) = \int_{\tau_0}^{\tau} dt \, m^2 e(t).$$

Hence $\mu(\tau)$ is determined by the mass of the particle and the ‘turning point’ $\tau_0$ of its motion.

The fact that the Noether constraint (14) leads to the conventional equations of motion (17) for $x$ and $p$, can be understood in terms of generalized Poisson brackets. We define the Poisson bracket in Clifford space as the ‘Clifford bracket’

$$\{N, M\}_{C.B.} = \frac{1}{2} \left( \left\{ \frac{\partial N}{\partial c_{A}}, \frac{\partial M}{\partial d_{A}^{*}} \right\} + \left\{ \frac{\partial N}{\partial c_{A}^{*}}, \frac{\partial M}{\partial d_{A}} \right\} - \left\{ \frac{\partial M}{\partial c_{A}}, \frac{\partial N}{\partial d_{A}^{*}} \right\} - \left\{ \frac{\partial M}{\partial c_{A}^{*}}, \frac{\partial N}{\partial d_{A}} \right\} \right),$$

where $\{,\}$ denotes the anti-commutator. This bracket is skew-symmetric in $N$ and $M$ and real when $N$ and $M$ are real. The equations of motion (15) can be written in terms of brackets

$$\frac{dc_{A}}{d\tau} = \{c_{A}, \mathcal{H}\}_{C.B.}, \quad \frac{dd_{A}^{*}}{d\tau} = \{d_{A}^{*}, \mathcal{H}\}_{C.B.},$$

which leads to

$$\frac{dx^{\mu}}{d\tau} = \{x^{\mu}, \mathcal{H}\}_{C.B.}, \quad \frac{dp_{\mu}}{d\tau} = \{p_{\mu}, \mathcal{H}\}_{C.B.}. \quad (18)$$

In general, these equations cannot be expressed solely in terms of $x$ and $p$. However, when the Noether constraint (14) holds, then by use of the differentiation rule (11), the Clifford bracket becomes proportional to the ordinary Poisson bracket:

$$\{N(x, p), M(x, p)\}_{C.B.} = \left( \frac{\partial N}{\partial x^{\mu}} \frac{\partial M}{\partial p_{\nu}} - \frac{\partial M}{\partial x^{\mu}} \frac{\partial N}{\partial p_{\nu}} \right) \left( \frac{1}{8} \sigma_{AB}^{\mu} \sigma_{AF}^{\nu} \{c_{B}^{*}, d_{F}^{*}\} + c.c. \right) = \mu \left( \frac{\partial N}{\partial x^{\mu}} \frac{\partial M}{\partial p_{\nu}} - \frac{\partial M}{\partial x^{\mu}} \frac{\partial N}{\partial p_{\nu}} \right) = \mu \{N(x, p), M(x, p)\}_{P.B.}. \quad (19)$$

After a reparametrization which absorbs $\mu$, (19) turns the equations of motion (18) into the usual space-time form (17) which ‘hides’ the Clifford substructure.

### 3 System of N particles with a U(N) symmetry

Assuming that the Clifford algebra for the point particle is $Cl(2n,2n,\mathbb{R})$, we can accommodate $N$ particles in $Cl(2nN,2nN,\mathbb{R})$ in such a way that all inner products between Clifford coordinates and momenta belonging to different
particles vanish. The generating algebra

\[ e_1^p \cdot e_j^q = \delta_{ij} \delta_{pq} \text{sign}(p), \quad e_1^p \cdot e_j^q = 0, \quad i, j = 1, \ldots, N, \quad p, q = 1, \ldots, 2n, \]  

(20)

where \( \text{sign}(p) \) denotes the sign of \( e_1^p \cdot e_1^q \), is preserved by the \( U(N) \) unitary transformation

\[ e_1^p \rightarrow U_{ih} e_1^{p'}, \quad U_{ih} U_{gh}^* = \delta_{ij}. \]

If we assemble the canonical variables \( c_A^i \) and \( d_A^i \), \( i = 1, \ldots, N \) of the \( N \) particles into the ket- and bra-vectors \( C_A \) and \( D_A \) respectively, then the corresponding space-time coordinates and momenta are elements of the \( N \times N \) diagonal matrices

\[ X^{AB} = C_A \cdot C^B, \quad P_{AB} = D_A \cdot D_B, \]

which trivially satisfy the commutation relations

\[ [X^\mu, X^\nu] = [P_\mu, P_\nu] = [X^\mu, P_\nu] = 0. \]

The equations of motion for this dynamical system can be derived from the sum

\[ I = \int Tr \left( \frac{d}{d\tau} C_A \cdot D_A + \text{c.c.} - \mathcal{H} \right) d\tau, \quad \mathcal{H} \equiv e(\tau)(P^\mu P_\mu - m^2 \cdot 1), \]

(21)

of the single-particle actions (11). The Noether constraint (14) becomes

\[ C_A \cdot D_B = \mu(\tau) \delta_A^B \cdot 1. \]

(22)

We observe that (21) and (22) are preserved by the global \( U(N) \) transformations

\[ C_A \rightarrow U C_A, \quad D_A \rightarrow D_A U^\dagger, \]

(23)

which produce the similarity transformations

\[ X^\mu \rightarrow UX^\mu U^\dagger, \quad P_\mu \rightarrow UP_\mu U^\dagger \]

of the Hermitian matrices \( X^\mu \) and \( P_\mu \). Such transformations create off-diagonal entries in \( X \) and \( P \) which correspond to artificial couplings between the Clifford coordinates and momenta belonging to different particles.

The motion of a classical point particle can be described by a set of integral curves in the phase space \( (x^\mu, p_\mu) \). From the foregoing it follows that the coordinates of these integral curves are eigenvalues of \( X^\mu \) and \( P_\mu \).

4 Matrix Mechanics

The unitary system obtained in the foregoing has the structure of a finite dimensional form of matrix mechanics and suggests the quantization

\[ c \rightarrow C, \quad d^* \rightarrow D, \quad c \cdot d^* \rightarrow C \cdot D, \quad c \cdot c^* \rightarrow C \cdot C, \quad d \cdot d^* \rightarrow D \cdot D, \]

(24)
\[ \{ c^A, M(x, p) \}_{C.B.} \to \frac{1}{i\hbar} [X^{AB}, M(X, P)] D^B, \]

\[ \{ d_A^*, M(x, p) \}_{C.B.} \to \frac{1}{i\hbar} \zeta^B [P_{A\bar{B}}, M(X, P)], \quad (25) \]

By use of the Noether constraint (22), it follows that

\[ \{ N(x, p), M(x, p) \}_{C.B.} \to \frac{\mu}{i\hbar} [N(X, P), M(X, P)], \]

and consequently the well-known rule \( \{ , \}_{P.B.} \to \frac{1}{i\hbar} [\cdot, \cdot] \). To show that this is a valid procedure, we shall derive matrix mechanics from a variational principle. If we simply used the sum of the single-particle actions, the Noether constraint would become too weak. Instead, we must require that all real linear combinations of the single-particle actions are stationary and obey the constraints. This corresponds to the action

\[ I = \int d\tau \sum_{i=1}^N \phi_i L_i, \quad L_i = d_i^A \cdot \frac{dc_i^A}{d\tau} + \text{c.c.} - \mathcal{H}(p_i, e(\tau)), \quad (26) \]

where the coefficients \( \phi_i \) are arbitrary real constants, and \( L_i \) are the single-particle Lagrangians. When \( \Phi \) denotes the \( N \times N \) diagonal matrix with \( \phi_i \) along its diagonal and \( P \) is diagonal, the action (26) can be written as

\[ I = \int dt Tr \left( \Phi \left( \frac{d}{dt} C^A \cdot D_A + \text{h.c.} - \mathcal{H} \right) \right), \quad \mathcal{H} \equiv e(\tau)(P^\mu P_\mu - m^2 \cdot 1). \quad (27) \]

This action is preserved by the unitary transformation (23) with \( \Phi \) transforming according to

\[ \Phi \to U\Phi U^\dagger, \quad \Phi^\dagger = \Phi. \]

The diagonal matrices \( \Phi \) and \( P_\mu \) trivially satisfy the unitarily invariant conditions

\[ \frac{d\Phi}{d\tau} = [\Phi, P_\mu] = [P_\mu, P_\nu] = 0. \quad (28) \]

Conversely, these conditions ensure that the action (27) can be gauged back into (26). The conserved \( SL(2,\mathbb{C}) \) and \( U(N) \) Noether charges corresponding to the action (27) are

\[ J_{AB} = Tr \left( \Phi (\tilde{C}_A \cdot \tilde{D}_B + \tilde{C}_B \cdot \tilde{D}_A) \right), \quad j = i(\Phi \tilde{C}^A \cdot \tilde{D}_A - \text{h.c.}). \]

Requiring that they vanish for all values of \( \Phi \) gives (22).

For Hamiltonians which are polynomial functions of \( X \) and \( P \), the derivatives of \( Tr(\mathcal{H}) \) with respect to \( \tilde{C}^A \) and \( \tilde{D}_A \) are well defined and can be written on the form

\[ \frac{\partial Tr(\mathcal{H})}{\partial \tilde{C}^A} = \tilde{E}, \quad \frac{\partial Tr(\mathcal{H})}{\partial X^{AE}} = \frac{\partial}{\partial \tilde{D}_A} \tilde{D}_E, \quad \frac{\partial Tr(\mathcal{H})}{\partial P_{AE}} = \tilde{E}, \quad \frac{\partial Tr(\mathcal{H})}{\partial P_{AE}} = \tilde{D}_E, \]
where $\partial T r(\mathcal{H})/\partial X$ and $\partial T r(\mathcal{H})/\partial P$ are matrix functions of $X$ and $P$. The equations of motion are obtained by requiring the action $\mathcal{S}$ to be stationary for all $\Phi$ which satisfy (28). By independent variation of the action $\mathcal{S}$ with respect to $C$ and $D$, we obtain

$$\frac{d}{d\tau} \begin{cases} C \mathcal{A} = \frac{\partial T r(\mathcal{H})}{\partial P}, & \frac{d}{d\tau} \mathcal{B} = \frac{\partial T r(\mathcal{H})}{\partial X} \mathcal{A}, \\ D \mathcal{A} = -\mathcal{B} \frac{\partial T r(\mathcal{H})}{\partial X} \mathcal{A}, & \frac{d}{d\tau} \mathcal{B} = \frac{\partial T r(\mathcal{H})}{\partial P}. \end{cases}$$

Note that while $\Phi$ does not affect the equations of motion, it may be necessary to keep it to ensure convergence in the infinite-dimensional case.

Taking the inner product on both sides of these equations with $\mathcal{C}$ and $\mathcal{D}$ and applying the Noether constraint (22) and the reparametrization (16), we obtain

$$\frac{dX}{d\tau} = \frac{\partial T r(\mathcal{H})}{\partial P}, \quad \frac{dP}{d\tau} = -\frac{\partial T r(\mathcal{H})}{\partial X}, \quad \mathcal{H} = \frac{1}{2m}(P\mu P_\mu - m^2 \mathbf{1}).$$

The dynamical system so obtained describes a general class of unitarily invariant systems which includes, but is not limited to, systems of independent particles. Systems of independent particles are obtained by adding the commutation relations

$$[X^\mu, X^\nu] = [X^\mu, P_\nu] = 0, \quad [P^\mu, P_\nu] = 0, \quad [P^\mu, X_\nu] = 0,$$

which are also preserved by the equations of motion. For $\hbar \neq 0$, the couplings between different particles can no longer be gauged away and the $N$ integral curves of classical dynamics are replaced with an infinite and irreducible system of coupled paths. The commutation relations (31) allow $\partial T r(\mathcal{H})/\partial X$ and $\partial T r(\mathcal{H})/\partial P$ to be written as commutators, turning (29) into

$$\frac{dX^\mu}{d\tau} = \frac{1}{i\hbar}[X^\mu, \mathcal{H}], \quad \frac{dP_\mu}{d\tau} = \frac{1}{i\hbar}[P_\mu, \mathcal{H}].$$

These equations of motion taken together with the commutation relations (31), are formally identical to Matrix Mechanics in the Heisenberg picture and correspond to the Clifford bracket quantization (24)-(25).

In the Clifford space description, the picture independence of quantum mechanics can be made explicit by coupling an auxiliary unitary gauge connection to the (vanishing) unitary Noether charge. It transforms according to

$$\Gamma \rightarrow UTU^\dagger - i\frac{dU}{d\tau} U^\dagger, \quad \tilde{\Gamma}(\tilde{\tau}) = \Gamma(\tau) \frac{d\tau}{d\tilde{\tau}},$$

and when the ordinary derivatives are replaced by the gauge covariant derivatives

$$\nabla_\tau \tilde{V} \equiv \left( \frac{d}{d\tau} - i\Gamma(\tau) \right) \tilde{V}, \quad \nabla_{\tilde{\tau}} \tilde{V} \equiv \left( \frac{d}{d\tilde{\tau}} - i\tilde{\Gamma}(\tilde{\tau}) \right) \tilde{V},$$
\[ \nabla_\tau X^\mu = \frac{1}{i\hbar}[X^\mu, \mathcal{H}], \quad \nabla_\tau P_\mu = \frac{1}{i\hbar}[P_\mu, \mathcal{H}] . \]

The Heisenberg picture corresponds to the gauge \( \Gamma = 0 \). In the local gauge \( \Gamma(\tau) = -\frac{i}{\hbar}H \), the commutators on the left and right hand side of (32) cancel out and \( X \) and \( P \) become stationary. It therefore corresponds to the Schrödinger picture.

### 5 The state vector

In the foregoing we have seen that when the point particle is described relative to Clifford space, the classical and the quantum particle become objects in the same formal system. This makes it possible to compare directly the classical and the quantum measurement principles.

Let us first consider the classical system \( h = 0 \). In section 4 we found that in a unitary frame where \( X \) is diagonal, the paths in Clifford space constitute a family of integral curves consisting of eigenvalues of \( X \) and \( P \). When, for example, the space-time position \( x \) of the particle is being measured at some time \( \tau \), a good measurement would therefore be expected to return an eigenvalue \( x_i(\tau) \) of \( X(\tau) \). The corresponding Clifford coordinate \( c_i(\tau) \) (which for short we shall also call an eigenvalue) can be expressed as a unitarily invariant expectation value \( E \) in terms of a state vector \( |s> \):

\[ c_i(\tau) = E(C^A) \equiv <s| C^A \).

To see this, we expand \( C^A(\tau) \) in terms of \( c_i(\tau) \):

\[ C^A(\tau) = \sum_r |x_r(\tau)\rangle c^A_r(\tau), \quad c^A_r(\tau) \bullet c^*_s(\tau) = \delta_{rs} x^A_r(\tau), \]

where \( |x_r(\tau)\rangle \) denotes the eigenvectors of \( X^\mu(\tau) \) with eigenvalues \( x_i^\mu(\tau) \). It follows that the expectation value \( E(C^A) \) returns the correct value \( c_i \) of a measurement when the state vector \( |s> \) is set equal to the eigenvector \( |x_i> \).

Conversely, if the expectation value coincides with an eigenvalue, we would expect a good measurement to return this value. For the purpose of predicting the outcome of future measurements, the state vector must be subject to a time evolution. In classical dynamics, we expect that after a measurement has been performed, the expectation value must stay on the integral curve corresponding to this measurement. In the unitary gauge where the paths are integral curves, \( X(\tau) \) is diagonal and \( \Gamma = 0 \). In this frame the eigenvectors \( |x_i> \) can be chosen to be constants of motion and the state vector must therefore also be a constant of motion. This leads to the gauge invariant time evolution

\[ \nabla_\tau |s\rangle = \left( \frac{d}{d\tau} - i\Gamma \right) |s\rangle = 0. \quad (33) \]
For the classical system, these measurement principles merely represent a
different way of formulating the traditional initial value problem. Remarkably,
however, they also apply to the non-classical system, regardless of the fact that
the way they were derived is no longer valid. In the quantum system where X
and P do not commute, the assumption that measurements must be expressed
through a single state vector imposes restrictions on which type of measurements
can be performed. The time evolution \( \langle C \rangle \) also holds true, as follows from the
fact that the state vector is known to be a constant of motion in the Heisenberg
picture \( \Gamma = 0 \). The difference between the classical and the quantum systems
becomes clear when we expand the expectation value \( E(C) \) in terms of the
eigenvalues \( c_i \):

\[
E(C^A(\tau)) \equiv <s | C^A(\tau) = <s | x_i(\tau) > c_i(\tau).
\]

In the classical system, in the gauge \( \Gamma = 0 \), both \(<s | \) and \(|x_i > \) are constants
of motion and hence the expectation value \( E(C(\tau)) \) is equal to one of the eigen-
values \( c_i(\tau) \). The outcome of a measurement is therefore predictable. This is
not surprising since it was used to derive the time evolution of the state vector.
The role of the state vector in the classical system is simply to select an integral
curve. In the non-classical system, in the gauge \( \Gamma = 0 \), the state vector is also
stationary, but the eigenvectors \(|x_i > \) undergo a unitary time evolution. After
a measurement has been performed, the expectation value therefore develops
into a complex linear combination of different eigenvalues \( c_i(\tau) \). Accordingly,
the outcome of a measurement is no longer predictable, but instead occurs with
statistical frequencies given by the Born rule.

In the non-relativistic limit, the proper time \( \tau \) is equal to the expectation
value of \( X^0 \) which represents the ‘physical’ time \( t \equiv <s | X^0 | s > : \)

\[
\frac{dt}{d\tau} = <s | \bar{\nabla}_\tau X^0 | s > = \frac{1}{m} <s | P^0 | s > \approx 1,
\]

where we have used the time evolution of the state vector and the equations
of motion for \( X^0 \). Restricting the equations of motion \( \mu = 1, 2, 3, \) to the
Hamiltonian density \( H \) effectively reduces to the non-relativistic Hamiltonian
density

\[
\tilde{H} = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2).
\]

Taken together with the corresponding commutation relations, this system is
identical to that of non-relativistic Matrix Mechanics. The Schrödinger picture
corresponds to the non-relativistic gauge condition \( \Gamma(t) = -\frac{i}{\hbar} H \) which turns the
time evolution \( \langle C \rangle \) of the state vector into the matrix form of the Schrödinger
equation.

6 The classical string in Clifford space

The world sheet of the relativistic string in Clifford space is described by the
coordinate functions \( c^A(\tau, \sigma) \) with values in the generating space of the infinite-
dimensional Clifford algebra obtained from (20) by letting $N \to \infty$. We follow the convention that $\mu, \nu, \ldots$ denote the space-time indices and $\alpha, \beta, \ldots$, the world sheet indices. Differentiation with respect to the world sheet parameters $\sigma^\alpha = \tau, \sigma$ will be written as $\partial_\alpha$.

It is well known that for a string which resides in space-time, the Lorentz metric $\eta_{\mu\nu}$ induces a metric on the world sheet through the tangent derivatives $\partial_\alpha x^\mu$. For a string which resides in Clifford space, we use the complex vectors

$$V^\mu_\alpha \equiv \sigma^\mu_{AB} c^A \bullet \partial_\alpha e^B,$$

which have the real part $\partial_\alpha x^\mu$. These vectors induce the Hermitian tensor

$$g_{\alpha\beta} \equiv V^\mu_\alpha V'^\nu_\beta \eta_{\mu\nu}, \quad g^*_{\alpha\beta} = g_{\beta\alpha},$$
on the Clifford worldsheet, which can be decomposed into a real symmetric tensor $h_{\alpha\beta}$ and a real scalar $\phi$:

$$g_{\alpha\beta} = h_{\alpha\beta} + i\phi \sqrt{h} \epsilon_{\alpha\beta}, \quad h_{\alpha\beta} \equiv g_{(\alpha\beta)}, \quad \phi \equiv -\frac{1}{2} i h^{-1/2} \epsilon^{\alpha\beta} g_{\alpha\beta}, \quad h \equiv |\det(h_{\alpha\beta})|.$$

The reparametrization invariant string generalization of the point particle action (9) is

$$I = \hat{\int} d\tau d\sigma \left( \sqrt{3} \sqrt{W^\mu W_\mu} - m^2 \right) \sqrt{h}, \quad W^\mu \equiv \frac{1}{2} \sigma^\mu_{AB} h^{\alpha\beta} \partial_\alpha \partial_\beta e^B \sqrt{h}, \quad (34)$$

To write this action in an explicit covariant first order form, we use De Donder-Weyl covariant canonical variables [9, 10, 11]. The polymomenta density conjugate to $c$ is

$$d^*_{A} \equiv \sqrt{h} d^*_{A} \equiv \frac{\partial L}{\partial (\partial_\alpha c^A)} = (W^\nu W_\nu)^{-\frac{1}{2}} W^\mu \sigma^\mu_{AB} h^{\alpha\beta} \partial_\beta e^B \sqrt{h},$$

where $L$ is the Lagrangian density in (34). This leads to the expressions

$$\frac{1}{2} h_{\alpha\beta} d^*_{A} \bullet d^*_{B} \gamma^{\gamma\delta} d^\gamma_{A} \bullet d^\delta_{B} = \frac{3}{2} \sqrt{h} W^\mu W_\mu, \quad d^*_{A} \bullet \partial_\alpha c^A + c.c. = 4 \sqrt{h} W^\mu W_\mu,$$

from which we obtain the De Donder-Weyl scalar Hamiltonian density

$$\mathcal{H} \equiv d^*_{A} \bullet \partial_\alpha c^A + c.c. - L = \sqrt{h} (p^\mu p_\mu + m^2), \quad p^\mu \equiv \frac{1}{2} \sigma^\mu_{AB} h_{\alpha\beta} d^*_{A} \bullet d^\beta_{B},$$

and hence the first order form

$$I_M = \int d\tau d\sigma L_M, \quad L_M = \sqrt{h} \left( d^*_{A} \bullet \partial_\alpha c^A + c.c. - (p^\mu p_\mu + m^2) \right)$$

of the Polyakov action (34). Without kinetic terms for $h_{\alpha\beta}$ and $\phi$, the equations of motion for the metric would lead to a singular metric. The simplest such
kinetic term, is a Jackiw-Teitelboim term 6, 7 which describes a world sheet with constant curvature

\[ I_{JT} = \int d\tau d\sigma L_{JT}, \quad L_{JT} = \sqrt{h} \left( \phi(R(h_{\alpha\beta}) - 2\Lambda) \right). \]

The equations of motion obtained from

\[ \partial_\alpha c^A = h_{\alpha\beta} p^{A\hat{E}} d^{\hat{E}}, \quad \partial_\alpha (d_\alpha^A) = 0, \]

are

\[ \frac{1}{\sqrt{h}} \frac{\partial L}{\partial h_{\alpha\beta}} = R(h_{\alpha\beta}) - 2\Lambda = 0, \]

\[ \frac{1}{\sqrt{h}} \frac{\partial L}{\partial \phi} = -(h^{\alpha\beta} \nabla^2 + \Lambda h^{\alpha\beta} - \nabla^\alpha \nabla^\beta) \phi + T^{\alpha\beta} = 0, \]

where \( \nabla_\alpha \) denotes the covariant derivative.

To find the wave solutions, we consider a flat world sheet (\( \Lambda = 0 \)) and the parametrization \( h_{\alpha\beta} = \eta_{\alpha\beta} \). Travelling waves with a spatial period of \( 4\pi \) are described by

\[ c^A = k^A + l^A \tau + \sum_{n \neq 0} a_n A e^{\frac{i}{2} n(\tau + \sigma)} + \sum_{n \neq 0} b_n A e^{\frac{i}{2} n(\tau - \sigma)}, \quad 0 \leq \sigma \leq \pi. \]

With the exception of \( a_n \bullet a^*_n \) and \( b_n \bullet b^*_n \), we assume that the inner products between different coefficients vanish. This leads to the space-time trajectories

\[ x^{AB} = k^A \bullet k^B + l^A \bullet l^B \tau^2 + \sum_{n \neq 0} \left( a_n^A \bullet a^*_n^B + b_n^A \bullet b^*_n^B + a_n^A \bullet a^*_n e^{in(\tau + \sigma)} + b_n^A \bullet b^*_n e^{in(\tau - \sigma)} \right), \]

with a spatial period of \( 2\pi \). When \( p^2 \equiv p_\mu p^\mu \neq 0 \), (38) can be solved with respect to \( p^{AB} \) and \( d^2_{\hat{E}} \), giving

\[ p^2 p^{AB} = \eta^{\alpha\beta} \partial_\alpha c^A \bullet \partial_\beta c^{*B} = l^A \bullet l^{*B}, \quad d_{\hat{E}} = (p^2)^{-2} l_{\hat{E}} \bullet l_A \partial_\beta c^A. \]

It follows that \( p_\mu \) is constant and, since \( c^A \) satisfies the free wave equation, the polymomenta \( d^2_A \) must satisfy their equation of motion (36). The space-time coordinates (39) satisfy \( \square x^{AB} = 2l^A \bullet l^{*B} \) and therefore the Clifford string is
not a substructure of the bosonic string. In a time interval of given length, it approaches the bosonic string in the limit \( t^A \to 0 \) with \( l^A \cdot l^B \tau \) held fixed.

As shown by Navarro [8], the equations of motion for \( h_{\alpha \beta} \) and \( \phi \) are equivalent to the single equation

\[
\nabla_\alpha \nabla_\beta \phi = -h_{\alpha \beta} \Lambda \phi + h_{\alpha \beta} T - T_{\alpha \beta}.
\]

(40)

Since \( p_\mu \) is constant, we can set \( p_\mu p^\mu = m^2 \). This makes the trace \( T \) of the energy-momentum tensor \( \varepsilon \) vanish and \( \varepsilon_1 \) reduce to

\[
\partial_\alpha \partial_\beta \phi = -m^2 \eta_{\alpha \beta} + \eta_{\gamma \delta} d^\gamma_A \cdot d^\delta_B (\partial_\alpha \cdot \partial_\beta).
\]

(41)

The trace of the right hand side vanishes and \( \phi \) therefore satisfies the free wave equation with the general solution \( \phi = \phi_L(\tau + \sigma) + \phi_R(\tau - \sigma) \). Reinserting this solution into \( \varepsilon_1 \) gives the differential equation for \( \phi_L \)

\[
\phi''_L(\tau + \sigma) = \frac{1}{2} m^2 - \frac{1}{4} m^{-2} l_A \cdot l_B \sum_{\alpha \neq \beta} n^2 (a^\alpha_A \cdot a^\beta_B - a^\beta_A \cdot a^\alpha_B) e^{in(\tau + \sigma)},
\]

(42)

and the corresponding one for \( \phi_R \) obtained from \( \varepsilon_2 \) by \( a \to b \) and \( \sigma \to -\sigma \).

Integrating these equations yields the dilaton field

\[
\phi = k + n \sigma^a + \frac{1}{2} m^2 (\sigma^2 + \sigma^2) - \frac{1}{4} m^{-2} l_A \cdot l_B \sum_{\alpha \neq \beta} n^2 a^\alpha_A \cdot a^\beta_B (\tau + \sigma)^2
\]

\[
+ \frac{1}{2} n^2 b^A_n \cdot b^B_n (\tau - \sigma)^2 + a^A_n \cdot a^B_n e^{in(\tau + \sigma)} + b^A_n \cdot b^B_n e^{in(\tau - \sigma)}.
\]

When \( \Gamma \) is a space-like curve connecting two points on the boundaries of the world sheet, the total Clifford momentum \( d^A_{tot} \) of the string can be determined by integration of the conserved current density \( d^A_\alpha \)

\[
d^A_{tot} = \int_{\Gamma} d\sigma^\alpha \epsilon_{\alpha \beta} d^\beta_A.
\]

(43)

It is reasonable to assume that the total Clifford momentum determines the total space-time momentum of the string in the same way as for the point particle. For \( a_n = b_n = 0 \), we get \( d^A_{tot} \cdot d^B_{tot} = \pi^2 P_{AB} \) which identifies \( p_\mu \) and \( m \) as the space-time momentum and mass of the non-vibrating string of unit length.

The spinning string is described by a subset of the trajectories \( \varepsilon_3 \)

\[
c^1 = k \tau + a e^{i\frac{1}{2} (\tau + \sigma)} + b e^{i\frac{1}{2} (\tau - \sigma)},
\]

\[
c^2 = l \tau + a e^{-i\frac{1}{2} (\tau + \sigma)} + b e^{-i\frac{1}{2} (\tau - \sigma)},
\]

where \( a \cdot a^* = b \cdot b^* \), \( k \cdot k^* = l \cdot l^* \) and all other inner products vanish. This produces the space-time trajectories

\[
x = \frac{1}{2} (c^1 \cdot c^2 + c^3 \cdot c^3) = 2a \cdot a^* \cos(\tau) \cos(\sigma),
\]

\[
y = \frac{1}{2} (c^1 \cdot c^2 - c^1 \cdot c^2) = 2a \cdot a^* \sin(\tau) \cos(\sigma),
\]

\[
z = \frac{1}{2} (c^1 \cdot c^2 - c^1 \cdot c^2) = 0,
\]

\[
t = \frac{1}{2} (c^1 \cdot c^2 + c^3 \cdot c^3) = 2a \cdot a^* + k \cdot k^* \tau^2.
\]
The quantum string

If the quantized string is going to represent a physical particle, the Lorentz charge and space-time momentum of the string must satisfy the Poincaré algebra. The conserved $\mathcal{SL}(2, \mathbb{C})$ and $U(1)$ Noether current densities are

\[
\mathcal{J}_{AB} \equiv (c_A \cdot d_B^\alpha + c_B \cdot d_A^\alpha), \quad i^\alpha \equiv i (c^A \cdot d_A^\alpha - \text{c.c.}).
\]

Let $\Gamma$ be a space-like curve connecting two fixed points on the boundaries of the world sheet and let $\sigma^\alpha(u), \sigma^\alpha(u')$ and $\sigma^\alpha(u'')$ be three points on this curve. Then we can define the world sheet scalars

\[
\mathcal{j}_{AB} \equiv v^\alpha \epsilon^\beta A \mathcal{J}_{AB} = c_A \cdot d_B^\alpha + c_B \cdot d_A^\alpha, \quad d_A^\alpha \equiv v^\alpha \epsilon^\beta A d^\alpha B + \text{c.c.},
\]

and the total $\mathcal{SL}(2, \mathbb{C})$ charge

\[
\mathcal{j}_{AB}^{\text{tot}} \equiv \hat{\iota}_\Gamma d\sigma^\alpha \epsilon^\beta A \mathcal{J}_{AB} = \int du \mathcal{j}_{AB},
\]

which is path independent. The Clifford bracket is defined as

\[
\left\{ \mathcal{j}_{AB}^{\prime}, \mathcal{j}_{EF}^{\prime\prime} \right\}_{\text{C.B.}} \equiv \int du \frac{1}{2} \left( \left\{ \frac{\partial \mathcal{j}_{AB}^{\prime}}{\partial c^{\prime G}}, \frac{\partial \mathcal{j}_{EF}^{\prime\prime}}{\partial d_{\prime G}} \right\} + \left\{ \frac{\partial \mathcal{j}_{AB}^{\prime}}{\partial c^{\prime G}}, \frac{\partial \mathcal{j}_{EF}^{\prime\prime}}{\partial d_{\prime G}} \right\} - \left\{ \frac{\partial \mathcal{j}_{EF}^{\prime\prime}}{\partial c^{\prime G}}, \frac{\partial \mathcal{j}_{AB}^{\prime}}{\partial d_{\prime G}} \right\} - \left\{ \frac{\partial \mathcal{j}_{EF}^{\prime\prime}}{\partial c^{\prime G}}, \frac{\partial \mathcal{j}_{AB}^{\prime}}{\partial d_{\prime G}} \right\} \right),
\]

where unprimed variables depend on $u$, and variables with a single prime or a double prime depend on $u'$ and $u''$, respectively. By means of the functional differentiation rule $\partial f^A(u')/\partial f^B(u) = \delta^A_B (u' - u)$, the Clifford brackets reduce to

\[
\left\{ \mathcal{j}_{AB}^{\prime}, \mathcal{j}_{EF}^{\prime\prime} \right\}_{\text{C.B.}} = \left( \mathcal{j}_{AE} \epsilon_{FB} + A \leftrightarrow B \right) + E \leftrightarrow F \delta(u' - u''),
\]

\[
\left\{ \mathcal{j}_{AB}^{\prime}, \mathcal{j}_{EF}^{\prime\prime} \right\}_{\text{C.B.}} = 0. \tag{44}
\]

Integrating both sides of these equations with respect to $u'$ and $u''$, we obtain the path-independent relations

\[
\left\{ \mathcal{j}_{AB}^{\text{tot}}, \mathcal{j}_{EF}^{\text{tot}} \right\}_{\text{C.B.}} = \left( \mathcal{j}_{AE}^{\text{tot}} \epsilon_{FB} + A \leftrightarrow B \right) + E \leftrightarrow F, \quad \left\{ \mathcal{j}_{AB}^{\text{tot}}, \mathcal{j}_{EF}^{\text{tot}} \right\}_{\text{C.B.}} = 0,
\]

which are turned into the algebra

\[
[\mathcal{J}_{AB}, \mathcal{J}_{EF}] = i\hbar \left( \mathcal{J}_{AE} \epsilon_{FB} + A \leftrightarrow B \right) + E \leftrightarrow F, \quad [\mathcal{J}_{AB}, \mathcal{J}_{EF}^{\dagger}] = 0,
\]

\[
\mathcal{J}_{AB} \equiv \int_\Gamma d\sigma^\alpha \epsilon^\beta A \mathcal{J}_{AB}^{\alpha}, \quad \mathcal{J}_{AB}^{\beta} \equiv C_A \cdot D_B + C_B \cdot D_A \tag{45}
\]

\[15\]
by the quantization $c \rightarrow \hat{c}$, $d^\alpha \rightarrow \hat{d}^\alpha$, $\{.,\}_{C.B.} \rightarrow \frac{1}{\hbar}[.]$. When this algebra is written in terms of the non-Hermitian 3-vector $N_k$

$$N_1 \equiv \frac{i}{4}(J_{22} - J_{11}), \quad N_2 \equiv -\frac{1}{4}(J_{11} + J_{22}), \quad N_3 \equiv -\frac{i}{2}J_{12}$$

and its Hermitian conjugate $N_k^\dagger$, it becomes the direct sum of two $su(2)$ algebras and can be identified with the Lorentz algebra. Since $v^\alpha$ is a space-like vector and its weight is different from 1, there exists a parametrization in which $v^\alpha = (0, 1)$ and where (44) becomes the equal-time commutation relations

$$[J_{AB}^1(\tau, \sigma), J_{EF}^1(\tau, \sigma')] = i\hbar \left((J_{AE}^1(\tau, \sigma) \epsilon_{FB} + A \leftrightarrow B) + E \leftrightarrow F \right) \delta(\sigma - \sigma'),$$

$$[J_{AB}^1(\tau, \sigma), J_{EF}^1(\tau, \sigma')] = 0.$$

The total space-time momentum of the string is obtained from (43) as $p^\text{tot}_{EF} \equiv \frac{d^*_{d_{\text{tot}}}}{\quad} \cdot \frac{d_{\text{tot}}}{\quad}$, and yields the Clifford brackets

$$\{p^\text{tot}_{EF}, J_{AB}^1\}_{C.B.} = \epsilon_{EAP_{FB}} + \epsilon_{EBP_{FA}}, \quad \{p^\text{tot}_{AB}, p^\text{tot}_{EF}\}_{C.B.} = 0,$$

with the quantum form

$$[P_{EF}, J_{AB}] = i\hbar(\epsilon_{EAP_{FB}} + \epsilon_{EBP_{FA}}), \quad [P_{AB}, P_{EF}] = 0,$$

$$P_{AB} \equiv \hat{D}_B \cdot \hat{D}_A, \quad P_{\beta} \equiv \int d\sigma^\alpha \epsilon_{\beta \alpha} \hat{D}_A.$$

When the algebra (43), (46) is written in terms of the skew-symmetric tensor $M_{\mu\nu}$ defined as $N_i \equiv \epsilon_{ijk}(N_k + N_k^\dagger)$, $M_{k0} \equiv i(N_k - N_k^\dagger)$, it becomes the Poincare algebra for $M_{\mu\nu}$ and $P_{\mu}$. The usual proof that orbital angular momentum can only take on integral values of $\hbar$ does not apply to $J_{AB}$. Clifford strings with half-integral spin could provide a more detailed picture of a fermion than is possible in a space-time description.

The unitary Noether current can be analyzed in the same way, giving the equal-time commutation relations

$$[I^1(\tau, \sigma), I^1(\tau, \sigma')] = [I^1(\tau, \sigma), J_{AB}^1(\tau, \sigma')] = 0, \quad I^\alpha \equiv \frac{\hat{C}^A}{\quad} \cdot \frac{D_A}{\quad} - h.c..$$

8 Conclusion

We have described the dynamics of the classical point particle in terms of the Clifford substructure of its canonical variables and shown that this substructure itself forms a canonical system. Compared to the space-time description, this description offers a conceptually simpler road to matrix mechanics because the Clifford algebra inherently supports the unitary symmetry of quantum states through its generating algebra. Unlike the point particle, we found that the
relativistic string in Clifford space is not a substructure of the bosonic string in space-time. We found the wave solutions for a flat world sheet and derived the equal-time commutation relations for the $SL(2, \mathbb{C})$ and unitary Noether currents.

There are good reasons to believe that a four-dimensional Minkowski space does not suffice to accommodate the particle physics of the Standard Model. The Clifford model discussed in the foregoing is limited to a four-dimensional Minkowski space because it is based on complex Weyl spinors. Since Weyl spinors are an integral part of the model, it is difficult to see how the dimension of space-time can be increased without replacing the complex numbers with a higher dimensional algebra. The complex numbers correspond to the Clifford algebra $Cl(0,1,\mathbb{R})$. Increasing the dimension, we find $Cl(0,2,\mathbb{R})$ which corresponds to the quaternions and $Cl(0,3,\mathbb{R})$ which can be deformed into the octonions. For algebraic reasons [12, 13, 14], such spinors would be expected to generate a six-dimensional and a ten-dimensional Minkowski space respectively.

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