Abstract

The quantities $C_4$ and $\theta_{2,2,2}$ are as defined by Wanless, $C_4$ just the number of 4-loops of a graph. The construction of this paper provides a counterexample to a conjecture of Butera, Pernici, and the author about the monomer-dimer entropy, $\lambda$, of a regular bipartite lattice. The lattice we construct is not a lattice graph in its most common definition.

We let $G_1$ and $G_2$ be the graphs illustrated in Figure 1. Then given a graph $G$, we let $C_4 = C_4(G)$ be the number of subgraphs of $G$ isomorphic to $G_1$, and $\theta_{2,2,2}(G)$ be the number of subgraphs of $G$ isomorphic to $G_2$. This notation follows Section 2 of [1].

We propose the following definition for a lattice in the statistical mechanics setting: A connected infinite graph is a lattice if there is a finitely generated Abelian group of isomorphic maps on $G$ for which the set of equivalence classes of vertices is finite. Two vertices are in the same equivalence class if there is a translation (a member of the Abelian group) that carries one into the other. The lattice is $i$-dimensional if exactly $i$ of a set of generators of the group are infinite order.

The lattice (infinite graph) we construct in this paper has the following properties:
1) It is embedded in $\mathbb{R}^3$. Edges embed as straight line segments joining vertices.
2) It is invariant under translations by an integer in the three coordinate directions.
3) We consider a tessellation of $\mathbb{R}^3$ by unit cubes whose vertices have integer coordinates. Each vertex of the graph is in the interior of one of these cubes. Each cube contains a finite number of vertices.
4) Each edge either lies in the interior of some cube, or connects two vertices in nearest neighbor cubes.

Figure 1:
This lattice does not have the property of arising from a tessellation, and thus is not a lattice graph as most commonly understood, though I believe it fits the idea of being a lattice for many researchers.

For a lattice the quantities $C_4$ and $\theta_{2,2,2}$ may both be infinite, but the ratio $\theta_{2,2,2}/C_4$ makes sense. Taking a sequence of finite graphs that approach the lattice in the sense of Benjamini-Schramm \[2\] the ratios $\theta_{2,2,2}/C_4$ converge to a limit we take to be the value for the lattice.

Given any number $\kappa$ we find a three dimensional regular bipartite lattice for which

$$\frac{\theta_{2,2,2}}{C_4} > \kappa$$

(1)

$$C_6 = 0.$$  

(2)

Here $C_6$ is the number of 6-loops as $C_4$ is the number of 4-loops.

In [3] Friedland and I introduced an expansion for the monomer-dimer entropy of a regular lattice,

$$\lambda(p) = \lambda_B(p) + \sum_k d_k p^k$$

(3)

where $\lambda_B$ is the monomer-dimer entropy for the Bethe lattice. In [4] Butera, Pernici and I conjectured that for a regular bipartite lattice all the $d_k \geq 0$. Actually a slightly weaker conjecture was stated in [4], see [5] Section 3. In [2] Pernici proved in the bipartite case $d_2, d_3, d_4, d_5$ were all nonnegative using a formalism of Wanless, [1]. For $d_6$ he finds the expression

$$d_6 = \frac{5C_4}{d^6} + \frac{C_6}{2d^6} - \frac{\theta_{2,2,2}}{d^6}$$

(4)

where we have d-regularity and the bars indicate an average over the number of vertices, again as may be taken as a Benjamini-Schramm limit. We find a three dimensional regular bipartite lattice for each $\kappa$ such that

$$\frac{\theta_{2,2,2}}{C_4} > \kappa$$

and also

$$C_6 = 0.$$  

Thus there are bipartite regular lattices for which $d_6 < 0$. Pernici erred in stating otherwise.

However the conjecture that all the $d_i \geq 0$ should not be lightly dismissed. For hyper-cubic lattices of all dimensions it is shown in [4] that the first 20 such coefficients satisfy this condition! It seems quite possible there is a class of ‘nice’ lattices that satisfy the conjecture, possibly vertex-transitive lattices.

We start the construction of our lattice with a ‘root unit graph’, a finite graph. To this we apply a number of 2-lifts to end with a ‘full unit graph’. The full unit graphs are attached to yield the three dimensional lattice. We use 2-lifts similarly to as in Section 4 of [7], derived from [8]. The construction requires that $d \geq 5$.

The Root Unit Graph

This graph has black vertices:

$$c_i, \quad i = 1, \ldots, d$$

and white vertices:

$$t, b, lx, ly, lz, rx, ry, rz, f_i, \quad i = 1, \ldots, d - 5$$
The edges are:

\[(t, c_i) \quad i = 1, \ldots, d\]
\[(b, c_i) \quad i = 1, \ldots, d\]
\[(c_i, f_j) \quad i = 1, \ldots, d \quad j = 1, \ldots, d - 5\]
\[(l_x, c_1), (l_y, c_1), (l_z, c_1)\]
\[(r_x, c_i) \quad i = 2, \ldots, d\]
\[(r_y, c_i) \quad i = 2, \ldots, d\]
\[(r_z, c_i) \quad i = 2, \ldots, d\]

It has a subgraph, the ‘root central subgraph’. The root central subgraph has black vertices:

\[c_i, \quad i = 1, \ldots, d\]

and white vertices:

\[t, b\]

Its edges are:

\[(t, c_i) \quad i = 1, \ldots, d\]
\[(b, c_i) \quad i = 1, \ldots, d\]

The quantities \(\theta_{2,2,2}\) and \(C_4\) for the root central subgraph are easily computed to be

\[C_4 = \frac{d(d-1)}{2}\]  \(5\)
\[\theta_{2,2,2} = \frac{d(d-1)(d-2)}{6}\]  \(6\)

Exploitation of the root central subgraph is the key idea in this paper.

Thank Heaven for 2-lifts

The full unit graph is constructed from the root unit graph through a sequence of \(s\) 2-lifts. Each edge in the full unit graph projects in a natural sense onto an edge of the root unit graph. \(2^s\) edges of the full unit graph project onto each edge of the root unit graph (likewise \(2^s\) vertices project onto each vertex). The edges that project onto edges of the root central subgraph have all been chosen parallel, none crossed (this is also in a natural sense). Thus there are \(2^s\) disjoint subgraphs of the full unit graph isomorphic to the root central subgraph. After \(s\) 2-lifts any vertex may be assigned a binary integer of length \(s\), that we call its ‘level’. If a zero appears in the \(i^{th}\) spot then at the \(i^{th}\) 2-lift the lower vertex was chosen, if a one appears the upper vertex chosen. (One arrives at the vertex by following a sequence of \(s\) splittings.)

The sequence of 2-lifts are chosen so that for the full unit graph there are no 6-loops, and 4-loops only in the \(2^s\) subgraphs isomorphic to the root central subgraph.

Connecting the Graph

We associate a full unit graph to each of the aforementioned unit cubes that tessellate \(\mathbb{R}^3\). Corresponding to each nearest neighbor pair of cubes we create a connection. Let cubes \(a\) and \(b\)
be nearest neighbors, and suppose cube $b$ is obtained from cube $a$ by the translation $x \to x + 1$. Then we connect the full unit graph associated to $a$ to the full unit graph associated to $b$ by identifying each of the $2^4$ vertices that project onto vertex $rx$ of the graph $a$ with one of the $2^4$ vertices that project onto $lx$ of the graph $b$ in a level preserving way. Nearest neighbor cubes lined up in the other two directions are treated analogously. The lattice or graph thus constructed will have $C_6 = 0$ and $\frac{g_{2,2,2}}{C_4} = \frac{g_{2,2,2}}{4}$.

Locating the Graph

It remains to associate to each vertex of the lattice a point in $R^3$. To do this it is clearly sufficient to locate each vertex of a single full unit graph included in the construction. We let $G$ be the full unit graph associated to the unit cube $[0,1] \times [0,1] \times [0,1]$, and $g$ its root unit graph. We divide the vertices of $g$ into five disjoint sets $S_1$, $S_2$, $S_3$, $S_4$ and $S_5$ with $S_1 = \{rx\}$, $S_2 = \{ry\} \cup S_3 = \{rz\}$ and $S_4 = \{lx, ly, lz\}$. We let $S_i$ be the vertices in $G$ that project to vertices in $S_i$. A 'try' is an assignment of the vertices in $S_5$ into a set of disjoint points in $(1/3,2/3) \times (1/3,2/3) \times (1/3,2/3)$, an assignment of $S_1$ into a set of disjoint points in $(2/3,1) \times (1/3,2/3) \times (1/3,2/3)$, an assignment of $S_2$ into a set of disjoint points in $(1/3,2/3) \times (2/3,1) \times (1/3,2/3)$, and an assignment of $S_3$ into a set of disjoint points in $(1/3,2/3) \times (1/3,2/3) \times (2/3,1)$. Note that the assignments for $S_1$, $S_2$, and $S_3$ determine an assignment for $S_4$ by the way connections were determined. For example, if a vertex in $S_1$ is assigned a point $w$ in $R^3$, then the vertex of the same level that projects onto $lx$ is assigned the point $w - (1,0,0)$. A try is a 'good try' if the edges of our full unit graph map into line segments that may intersect only in a vertex. But almost all tries are good tries, we pick a good try to fix the assignment. The resulting embedding of the lattice satisfies all our conditions.

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