Gabber’s rigidity theorem for stable framed linear presheaves.

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September 13, 2018

Abstract

We extend the results of the joint work of the author with Alexey Ananievsky [AnDr18] for the case of smooth affine henselian pairs over a field which means the rigidity theorem in sense of Gabber [Gab92] for a stable homotopy invariant linear framed presheaf (of abelian groups). Precisely we prove that for such a presheaf $F$ on $Sm_k$, and a smooth affine henselian pair $(U, Z)$ over the base filed $k$, there is an isomorphism

$$F(U)/\Lambda_l \cdot F(U) \simeq F(Z)/\Lambda_l \cdot F(Z),$$

where $\Lambda_l = nh$, if $l = 2n$, $\Lambda_l = nh + 1$, if $l = 2n + 1$, for any $l \in k^\times$. As a consequence this implies the rigidity with respect to smooth henselian pairs for a $\Lambda_l$-torsion $\text{SH}(k)$-representable cohomology theory on $Sm_k$.

1 Introduction

One of the most obvious differences between the (motivic) algebraic geometry and differential geometry (and topology) is that in the algebraic one there are many non-isomorphic points. Actually any algebraic extension $K/k$ of the base filed $k$ defines a point. Moreover any non isomorphic henselian local rings over $k$ defines different points (in the sense of Grothendieck topology); in fact the set of points of the big Nisnevich site on $Sm_k$ is precisely the set of henselian local rings. The rigidity theorems allows to identify some classes of points with respect to some classes of functors on the category of schemes (or smooth schemes).

The first rigidity theorems ware proven in works by Suslin [Sus83, Main theorem] and [Sus84], by Gabber [Gab92], by Gillet Thomason [GT84, Theorem A]; later results of such type were obtained in works by Suslin and Voevodsky [SV96, Theorem 4.4], by Panin and Yagunov [PY02], by Yagunov [Ya04], by Röndigs and Østvaer [RO08],
by Hornbostel and Yagunov [HY07], by Morel [Mor11 Theorem 5.14], by Bachmann [Ba16, Corollary 40].

For the detailed review on this question we refer the reader to the introduction of [AnDr18]. Let’s note here that some of these theorems ‘identify’ in the mentioned above sense Spec $K_1$ and Spec $K_2$ for the extension of algebraically closed fields $K_1/K_2$, and some of them identify the pair of rings $A$ and $A/I$, where $(A, I)$ is a henselian pair, see def. 5.

In the case of henselian pairs most of the theorems from the list above concern the case of local henselian pairs. The main probable reasons why the case of local henselian pairs were visited much more often the the case of non-local are the following: 1) As was mentioned above such schemes plays the role of disks in topology (differential geometry) and in the precise term these are the points in the Nisnevich topology. So such theorems can be applied to compare Nisnevich sheaves with the constant sheaves on $Sm_k$, like as was done in the remarkable work by Suslin and Voevodsky [SV96]. 2) The second reason is that the rigidity for a local henselian ring of a mixed characteristic allows to transfer the computational result for fields cross the characteristic, like as was done in another (and much earlier) remarkable work by Suslin [Sus84]; and the 'local' rigidity property is enough for such a task (problem).

In the same time it is non less natural to consider the $\mathbb{A}^1$-equivalences of elements in the functors of $W$-points for all (affine) schemes $W$ over $k$, and to compare a scheme $W$ with its infinitesimal etale neighbourhood under some closed inclusion $W \hookrightarrow Y$. Using the analogy with differential geometry such neighbourhood plays the role of the tube neighbourhood of the subvariety. Probably the most known theorem of such type[5] is the theorem by Gabber [Gab92 Theorem 2] which states that for any henselian pair $(A, I)$ of $\mathbb{Z}[1/n]$-algebras there is an isomorphism

$$K_i(A, \mathbb{Z}/n\mathbb{Z}) \simeq K_i(A/I, \mathbb{Z}/n\mathbb{Z}), \forall i \geq 0.$$ 

Here ask such the question for the class of the (homotopy invariant) framed linear stable presheaves on the category $Sm_k$. This class contains all $\text{SH}(k)$-representable cohomology theories due to Voevodsky’s lemma, see [V01] or [GP14 Lemma 3.2, Proposition 3.8], which implies the functor form the category of (linear) framed correspondences to $\text{SH}(k)$. Since by definition such presheaves are defined on $Sm_k$, it is natural to ask a question about smooth henselian pairs over $k$. The answer is positive, which is the main result of the work.

**Theorem 1.** Suppose $Z \subset U$ is an affine smooth henselian pair over a field $k$ and $F: Sm_k \rightarrow \text{Ab}$ is an $\Lambda_l$-torsion homotopy invariant $\sigma$-stable linear framed presheaf for some $l \in \mathbb{Z}$, $l \in k^\times$. Then the inverse image homomorphism induces the isomorphism $F(U) \simeq F(Z)$, where $\Lambda_l = nh$, for $l = 2n$, $\Lambda_l = nh + 1$, for $l = 2n + 1$.

\footnote{actually the only one such a theorem that the author currently knows}
Theorem 2. 1) Let $\mathfrak{I}: Z \hookrightarrow U$ be a smooth affine henselian pair over a field $k$. Let $E \in \mathcal{SH}(k)$ and $\Lambda_l E = 0$ for some $l \in \mathbb{Z}$, $l \in k^\times$. Then for $p, q \in \mathbb{Z}$ the inverse image homomorphism $i^*: E^{p,q}(U) \rightarrow E^{p,q}(Z)$ is an isomorphism.

2) Let $k$ be a perfect field, and $\mathfrak{I}: Z \hookrightarrow U$ be a smooth affine henselian pair over $k$. Let $E \in \mathcal{SH}(k)$ and $\phi E = 0$ for some $\phi \in GW(k) \simeq [S, S]_{\mathcal{SH}(k)}$ such that rank $E$ is invertible in $k$. Then for $p, q \in \mathbb{Z}$ the inverse image homomorphism $i^*: E^{p,q}(U) \rightarrow E^{p,q}(Z)$ is an isomorphism.

This generalises the result of the joint work of the author with Alexey Ananievsky [AnDr18], where such statements were obtained for the case of local henselian pairs.

Like as in [AnDr18] we deduce the theorems above from theorem 3 about the pair of sections of a relative curve over $U$ with fine compactification. Originally this theorem is proven in [AnDr18, theorem 6.1] for the case of a local henselian scheme and its proof is not trivial, but the proof for an affine henselian pair is almost the same just with replacing of the local scheme by the affine pair, and with few additional comments. So we don’t repeat this proof here.

In the same time the deduction of the rigidity theorem in the Gabber’s form the theorem about the sections of relative curve in the case of local schemes is done by the standard reasoning in proof of rigidity theorems. The deduction in the case of non-local henselian pairs is more complicated and it is the content of the present work.

1.1 Relative case

Let us note that the present reasoning works over a local base scheme $S$ for affine smooth henselian pairs $(U, Z)$, $Z \subset U$, over $S$ such that there is a closed inclusion of $U$ into a projective $S$-variety $\overline{U}$, and the complement $\overline{U} \setminus U$ is of the pure relative dimension $\dim_S U - 1$.

In the same times the question on the rigidity property with respect to the pair $(S, x)$, where $S$ is local henselian scheme of a mixed characteristic and $x$ is a closed point is not of a such type; actually it is not even a smooth pair. So in the present work there is no ‘cross characteristic’ effect, like as in works by Suslin [Sus84] and Gabber [Gab92].

Acknowledgment

The author is grateful to A. Ananievsky for many helpful discussions on the problem and related questions, and his help in finding of the mistakes in the early ideas of a proof. Also the author thanks I. Panin and F. Binda for the consultations on the question on the generality of the Picard rigidity statements and applications of the proper base change theorem.
2 Preliminaries

We start with recalling of the definition of framed correspondences (see [V01] or [GP14, Definition 2.1])

Definition 1. Let $S$ be a noetherian scheme of a finite dimension. Let $X, Y$ be smooth schemes over $S$. An explicit framed correspondence of a level $n$ over $S$ is a set $(Z, V, \phi, g)$ where $Z \subset \mathbb{A}^n$ is a closed subscheme, $e: V \to \mathbb{A}^n_X$ is an etale morphism such that $e^{-1}(Z) \cong Z$, $\phi = (\phi_i)$, $0 < i \leq n$, $\phi_i$ are regular function on $V$ such that $\bigcap \{\phi_i = 0\} = Z$, and $g: V \to Y$ is a morphism of $S$-schemes.

Denote by $Fr^n_S(X, Y)$ the set of classes of explicit framed correspondences up to the equivalence relation with respect to , see the references above.

Definition 2. For an invertible function $\lambda \in \mathcal{O}^\times(S)$, let $(\lambda) \in Fr^1_S(pt, pt)$ denotes the framed correspondence given by $(0, \mathbb{A}^1_S, \lambda x, p)$, where $p: \mathbb{A}^1_S \to pt$ is the canonical projection.

Definition 3. Define an element $\Lambda_l \in ZFr_1(pt, pt)$ by the formula $\Lambda_l = nh$, for $l = 2n$, $\Lambda_l = nh + 1$, for $l = 2n + 1$.

Remark 1. For any $l \in \mathbb{Z}_{\geq 0}$, $\Lambda_l = [(Z(x^l), \mathbb{A}^1, x^n, p)] \in ZFr_1(pt, pt)$, where $Z(x^l)$ is the vanishing locus of $x^l$.

Denote by $ZFr_*(X, Y)$ the an abelian group generated by the classes of all framed correspondences between $X$ and $Y$ and relations $[Z_1, V - Z_2, \phi|_{Z_1}, g|_{Z_1}] + [Z_2, V - Z_1, \phi|_{Z_2}, g|_{Z_2}] = [Z, V - Z, \phi|_Z, g|_Z]$. A linear framed presheave over $S$ is an additive presheave on the category of linear framed correspondences with objects being smooth schemes and morphisms given by $ZFr_*(X, Y)$.

Denote by $ZF(X, Y)$ the an abelian group generated by the classes of all framed correspondences between $X$ and $Y$ and relations $[Z_1, V - Z_2, \phi|_{Z_1}, g|_{Z_1}] + [Z_2, V - Z_1, \phi|_{Z_2}, g|_{Z_2}] = [Z, V - Z, \phi|_Z, g|_Z]$ and $[\Phi \circ \sigma] = [\Phi]$. See [GP14] for details.

Now let us recall the definition of normal framed relative curves [AnDr18, Definition 2.6]:

Definition 4. Let $S$ be a scheme and $C$ be a scheme over $S$ of relative dimension $d$. A level $m$ normal framing of $C$ consists of the following data:

1. an open immersion $j: W \to \mathbb{A}^{d+m}_S$;
2. a closed immersion $i: C \to W$;
3. an étale neighborhood $(p: \tilde{W} \to W, r: C \to \tilde{W})$ of $C$ in $W$;
4. a collection of regular functions $\psi = (\psi_1, \psi_2, \ldots, \psi_m)$ on $\tilde{W}$ such that $r(C) = Z(\psi)$ where $Z(\psi)$ stands for the common zero locus of $\psi_i$-s;

5. a regular morphism $\rho: \tilde{W} \to C$ such that $\rho \circ r = id_C$.

The set of level $m$ normal framings of $C$ is denoted $F_m(C)$. An open immersion $C' \subset C$ induces a map $F_m(C) \to F_m(C')$ given by

$$(j: W \to \mathbb{A}^{d+m}_S, i: C \to W, p: \tilde{W} \to W, \psi, \rho) \mapsto (j': W' \to \mathbb{A}^{d+m}_S, i': C' \to W', p': \tilde{W}' \to W', \psi', \rho')$$

with $W' = W - i(C - C')$, $\tilde{W}' = \tilde{W} - \rho^{-1}(C - C') - p^{-1}(i(C - C'))$ and the morphisms being the restrictions of the corresponding morphisms.

We continue with definitions of henselian pairs of Nisnevich neighbourhoods.

**Definition 5.** A pair $(A, I)$ of a ring $A$ and ideal $I$ is called as henselian pair iff for any $A$-algebra $C$ there is an isomorphism $Idem(C) \simeq Idem(C \otimes A/I)$; this equivalently means that for any etale rings homomorphism $A \to B$ and a ring homomorphism $B \to A/I$ there is a ring homomorphism $B \to A$ which makes the triangle being commutative.

**Definition 6.** A smooth affine henselian pair over a base field $k$ is a pair of $k$-schemes $Z$ and $U$ with a closed embedding $Z \hookrightarrow U$ such that $Z$ is smooth and affine, $U$ is essential smooth and it is a colimit of smooth affine schemes over $k$.

**Definition 7.** Let $Z \subset U$ be a closed embedding of schemes. A Nisnevich neighbourhood $(U', Z') \to (U, Z)$ is a closed embedding $Z' \subset Z'$ and an etale morphism $U' \to U$ such that $Z' \simeq U' \times_U Z \simeq Z$.

**Lemma 1.** Suppose $Z \subset U$ be affine smooth henselian pair. Let $A \in GL_n(U)$ and $A|_Z = Id_n \in GL(Z)$. Then there is a matrix $H \in GL(\mathbb{A}^1 \times U)$, $H|_{0 \times U} = A$, $H|_{1 \times U} = Id_n$.

**Proof.** Actually, the matrix $(1 - \lambda)A + \lambda Id_n$ is an element in $GL_n(\mathbb{A}^1 \times U)$ and satisfies the required properties.

Now we define what do we mean under the fine compactification of a relative curve over a spectrum of a commutative ring in the present text.
Definition 8. Let $S = \text{Spec } R$ be the spectrum of a ring and $C \to S$ be a flat morphism of relative dimension 1. We say that $(C \subset \overline{C}, \mathcal{O}(1))$ with $C$ being open and dense in $\overline{C}$ and $\mathcal{O}(1)$ being a very ample line bundle over $\overline{C}$ is a fine compactification of $C$ over $S$ if there exists $\zeta_\infty \in \Gamma(\overline{C}, \mathcal{O}(1))$ and $\zeta_c \in \Gamma(\overline{C}, \mathcal{O}(1))$ such that

1. $C = \overline{C} - Z(\zeta_\infty)$;
2. $Z(\zeta_\infty)$ is finite over $S$;
3. $Z(\zeta_c)$ is finite over $S$, $Z(\zeta_c) \subset C$.

Theorem 3. Let $Z \subset U$ be a smooth affine henselian pair over a base ring $k$, $C \to U$ be a flat morphism of relative dimension 1 admitting a fine compactification and $r_0, r_1: U \to C$ be morphisms of $U$-schemes such that $r_0|_Z = r_1|_Z$ and such that $C$ is smooth at $r_0(Z)$. Then for every $n \in \mathbb{N}$ such that $n \in k^\times$ the following holds.

1. If $2 \in k^\times$ then
   $$\langle \sigma_C^m \rangle \circ r_1 - \langle \sigma_C^m \rangle \circ r_0 = H \circ i_1 - H \circ i_0 + n \Theta \circ (h \boxtimes \text{id}_U)$$
   for some $m \in \mathbb{N}$, $H \in ZF_S^n(\mathbb{A}^1 \times U, C)$ and $a \in ZF_{m-1}^U(U, C)$.
2. If $2 = 0$ in $k$ then
   $$\langle \sigma_C^m \rangle \circ r_1 - \langle \sigma_C^m \rangle \circ r_0 = H \circ i_1 - H \circ i_0 + n \Theta \circ (\sigma \boxtimes \text{id}_U)$$
   for some $m \in \mathbb{N}$, $H \in ZF_m^U(\mathbb{A}^1 \times U, C)$ and $a \in ZF_{m-1}^U(U, C)$.

Here $i_0, i_1: U \to \mathbb{A}^1 \times U$ are the closed immersions given by $\{0\} \times U$ and $\{1\} \times U$ respectively.

Proof. The proof is the same as for [AnDr18, theorem 6.1]. One point that we need to note is the following: by the reasoning of [AnDr18, theorem 6.1] we get equality of the framed correspondences $\tilde{r}_0 = r_0 \circ \langle A_0 \rangle$ and $\tilde{r}_1 = r_1 \circ \langle A_1 \rangle$, where $\langle A_i \rangle \in Fr_n(pt_U, pt_U)$ are framed correspondences of the level $n$ over the base $U$ given by matrix $A_i \in GL(U)$ and such that $A_0|_Z = A_1|_Z$. So applying lemma 1 we get the claim.

Corollary 1. Under the notation of theorem 3 suppose that $F$ is framed linear $\sigma$-stable presheave over $U$, and suppose that one of the following condition holds:

1. $2 \in k^\times$;
2. $2 = 0$ in $k$.

Then $r_1^* = r_2^*: F(U) \to F(C)$.
3 The main theorem

**Theorem 4.** Suppose \( Z \subset U \) is an affine smooth henselian pair over a filed \( k \) and \( F: \text{Sm}_k \to \text{Ab} \) is an \( \Lambda_1 \)-torsion homotopy invariant \( \sigma \)-stable linear framed presheaf for some \( l \in \mathbb{Z}, l \in k^* \). Then the inverse image homomorphism induce the isomorphism \( F(U) \cong F(Z) \).

**Lemma 2.** Let \( Z \subset U \) be a smooth affine henselian pair; let \( \tilde{U} \) be a smooth affine scheme, and \( i: Z \to \tilde{U} \) be a closed embedding such that \( (\tilde{U})^h_Z = U \). Then there an etale morphism \( e: U' \to \tilde{U} \) and a closed embedding \( i': Z \to U' \), \( i = e \circ i' \) and a smooth retraction \( r: U \to Z \) such that \( U = (U')^h_Z \).

**Proof.** Consider a closed embedding \( U' \to \mathbb{A}^N_k \). Consider the etale morphism of affine varieties \( t: N_{Z/U} \to \mathbb{A}^N_k \). Define \( U'' = t^{-1}(U') \subset N_{Z/U} \). Then since \( t \) is etale, \( (U'')^h_Z = (U')^h_Z = U \). On other side the canonical projection \( N_{Z/U} \to Z \) induces the desired retraction \( U = (U'')^h_Z \to Z \). \( \square \)

New we start some construction, which summarized in lemma, and used in the further part of the proof. Consider an arbitrary smooth affine henselian pair \( Z \subset U \). By definition \( U = (U_1)_Z^h \) for some smooth affine scheme \( U_1 \) with a closed embedding \( Z \subset U_1 \). By lemma we can assume in addition that there is a retraction \( r: \tilde{U}_1 \to Z \).

Let \( T_Z \) be a vector bundle on \( Z \) such that \( T_Z \) is trivial; let \( \tilde{N} \) be a vector bundle on \( Z \) such that \( N_{Z/U} \oplus \tilde{N} \) is trivial. Let \( T = r^*(T_Z) \) and \( \tilde{N} = r^*(N_{Z/U}) \). Then \( \tilde{T}|_Z = T, \tilde{N}|_Z = \tilde{N} \).

Let \( Z' \) and \( \tilde{U}' \) be the total spaces of a vector bundles \( \tilde{T} \) and \( \tilde{T} \oplus \tilde{N} \). Then \( T_{Z'} \) and \( N_{Z'/U'} \) are equal to the inverse images of the vector bundles \( \tilde{T} \oplus T_Z \) and \( \tilde{N} \oplus N_{Z/U} \). Hence \( T_{Z'} \) and \( N_{Z'/U'} \) are trivial.

Furthermore, since for any etale morphism \( V \to \tilde{U} \) the schemes \( U'' = V \times_{\tilde{U}} U' \) and \( Z'' = V \times_{\tilde{U}} Z' \) are the total spaces of the inverse images of the vector bundles \( N_{Z/U} \) and \( \tilde{T} \). Thus since \( T_{Z''} \) and \( N_{Z''/U''} \) are equal to the inverse images of \( T_{Z'} \) and \( N_{Z'/U'} \), we get the following.

**Lemma 3.** For any smooth affine henselian pair \( Z \subset U \) there is a diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & \tilde{U}' \\
\downarrow{p} & & \downarrow{p} \\
Z & \xrightarrow{i} & \tilde{U}.
\end{array}
\]

such that (1) \( \tilde{U}, \tilde{U}' \) are smooth affine schemes, \( Z \to \tilde{U} \); (2) \( Z' \to \tilde{U}' \) are closed embeddings; (3) \( (\tilde{U})^h_Z = U \); (4) \( N_{Z'/U'} \), \( T_{Z'} \) are trivial vector bundles on \( Z' \); (5) all four
squares are commutative. Moreover there is a diagram such as above and such that for any etale morphism $V \to \tilde{U}$ the base change of the square \ref{lemma2} satisfies the properties (1), (2), (5).

**Lemma 4.** Let \ref{lemma2} be a diagram as in lemma\ref{lemma2} satisfying properties (1), (2), (5) and such that for any etale morphism $V \to \tilde{U}$ the base change of the square \ref{lemma2} satisfies the properties (1), (2), (5). Let $U = (\tilde{U})_2^b$, $U' = (\tilde{U}')_2^b$, be henselizations.

Suppose for some $a \in \mathbb{Z}Fr_*(pt, pt)$ for any homotopy invariant $\sigma$-stable a-periodical liner framed presheave $F$ the inverse image homomorphism induces the isomorphism $(i')^*: F(U') \to F(Z')$, then for any such a presheaf $F$ the inverse image homomorphism $i^*: F(U) \to F(Z)$ is an isomorphism too.

**Proof.** To get the claim it is enough to prove that for any Nisnevich neighbourhood $(V, Z) \to (U, Z)$, we have the equality $[i \circ r] = [j] \in \mathbb{ZF}(U, V)$, where $j: U \to V$. In the same way by assumption of the lemma applying to the presheaf $F = \mathbb{ZF}(-, \tilde{U}')$ we have $[i' \circ r''] = [j'] \in \mathbb{ZF}(U', V')$, where $j': U' \to V'$, $V' = V \times_\tilde{U} \tilde{U}'$. Now the claim follows since for any homotopy $h \in \mathbb{ZF}(U' \times \mathbb{A}^1, V')$ the formula $p_Z \circ h \circ id_{\mathbb{A}^1} \times j^U$ gives us an element in $\mathbb{ZF}(U \times \mathbb{A}^1, V)$.

**Lemma 5.** Let $Z \subset U$ be affine smooth henselian pair, $T_Z$ and $N_{Z/U}$ are trivial. Then there is a sequence $Z = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = U$ of essentially smooth closed subschemes, $\dim V_i = \dim Z + i$, $N_{V_i/V_{i+1}} \simeq O_{V_i}$.

**Proof.** Let $U'$ be an affine scheme and $Z \hookrightarrow U'$ be an embedding such that $U = (U')^Z_2$. Consider a closed embedding $U' \subset \mathbb{A}^N_k$. Let $t_1 \ldots t_{\dim Z}$ be the basis of $T_Z$, $t_{\dim Z+1} \ldots t_{\dim U'}$ be the basis of $N_{Z/U}$, $t_{\dim U'+1} \ldots t_{N}$ be the basis of $N_{U', \mathbb{A}^N_k}$. Let $\hat{t}_i$ denotes the dual basis of $\Omega(\mathbb{A}^N_k)|_Z$. Choose a lift $f_i \in k[\mathbb{A}^N_k]$ of $\hat{t}_i$ to a regular functions on the affine space. The desired filtration is given by $Z(f_1 \ldots f_i) \times \mathbb{A}^N_k = U = V_{n-i}$.

**Proposition 1.** Suppose $Z \subset U$ is an affine smooth henselian pair over a field $k$, $T_Z$ is trivial, $\text{codim}_U Z = 1$, $N_{Z/U}$ is trivial, and $F: Sm_k \to Ab$ is as in theorem\ref{thm1}. Then the inverse image homomorphism induces the isomorphism $F(U) \simeq F(Z)$.

**Proof.** It follows from lemma\ref{lemma2} that there are a smooth affine $U'$, a closed embedding $Z \to U'$, and a retraction $r: U' \to Z$ such that $(U')^Z_2 = U$.

Since $U'$ is affine there is an embedding $U' \to \mathbb{A}^N_k$. Since $T_{U'}$ is trivial, $N_{U'/ \mathbb{A}^N_k}$ is stable trivial. Hence for some $N_1 \in \mathbb{Z}$, $N_{U'/ \mathbb{A}^N_k}$ is trivial. Redenote now $N_1$ by $N$, but do not change $U'$, $U$, and $Z$. Then $N_{U'/ \mathbb{A}^N_k}$ is trivial. Hence there is a vector of functions $(\phi_i)_{i=n+1, \ldots, N}$, $\phi_i \in k[\mathbb{A}^N_k]$, and such that $Z(\phi) = U' \cap \hat{U}$ for some closed subscheme $\hat{U}' \in \mathbb{A}^N_k$.

Let $W$ be the closure of $U'$ in $\mathbb{P}^N_k$, and let $X = W \times U \subset \mathbb{P}^N_U$. Let $\Gamma$ and $\Delta$ denotes the graphs of morphisms $i' \circ r$ and can. Denote by $\Delta_Z$ the graph of the morphism $Z \to W$.
Consider the closed subscheme $E = U' \times_Z U \subset X \setminus \mathbb{P}^{N-1}_{U} \subset \mathbb{A}^{N-1}_U$. (Warring: we do not work with the closure of $E$ in $\mathbb{P}^{N-1}_U$.) Since $T_Z$ is trivial, and $N_{Z/U}$ is trivial, then $N_{E/X}$ is stable trivial. Let $\mathbb{A}^{N}_U \to \mathbb{A}^{M}_U$ be an embedding such that $N_{E/(X \times \mathbb{A}^{M-N})}$ is trivial. Let us redenote now the scheme $U' \times \mathbb{A}^{M-N} \subset \mathbb{A}^{M}$ by $U'$, and $M$ by $N$. Then consequently we redenote the schemes $W$ and $X$, namely $W = U'$, and $X = W \times U$. Note that here we do not change that schemes $U$ and $Z$, though now $\dim U' \neq \dim U$ and $U \neq (U')^0_2$. On the other side after such a replacement we have got that $N_{E/X}$ is trivial. In the same time $N_{U'/A}^N$ is still trivial, and so there is a choice of the functions $\phi$ for a new $U'$ as well.

Let $y_1 \ldots y_{n-1}$ be the basis of $N_{E/X}$. Let $y'_i$ be the image of $y_i$ in $N_{(E \times_U Z)/(X \times_U Z)}$, and let $y''_i \in I(\Delta_Z)/I(\Delta_Z)^2 = N_{\Delta_Z/A_U}$ be any lift of $y_1 \ldots y_{n-1}$. Denote $W_\infty = W \cap \mathbb{P}^{N-1}_k$ and let $n = \dim U$. Using Serre’s theorem [Ha77, theorem 5.2] we find sections $w_1 \ldots w_{n-1} \in \Gamma(\mathbb{P}^{N-1}_k, \mathcal{O}(b))$ such that $Z(w_1|_{W_\infty}, \ldots, w_{n-1}|_{W_\infty})$ is finite (over $k$). Denote $Y_\infty = Z(w_1|_{W_\infty}, \ldots, w_{n-1}|_{W_\infty})$.

Then by Serre’s theorem again for some $l \in \mathbb{Z}$ there are sections $s_1 \ldots s_{n-1} \in \Gamma(\mathbb{P}^{N-1}_U, \mathcal{O}(b))$, $s_i|_{\mathbb{P}^{N-1}_U} = w_i$, $s_i|_{Z(\mathbb{P}^{N-1}_Z)} = y''_i$. Let $\mathcal{C} = \mathcal{C}(s_1 \ldots s_{n-1}) \cap X \subset \mathbb{P}^{N}_U$, and $C = \mathcal{C} \cap \mathbb{A}^{N}_U$. Then $\mathcal{C}$ is projective variety over $U$ of the pure relative dimension one, $\Gamma \cup \Delta \subset \mathcal{C}$, $\mathcal{C}$ is smooth over $U$ at $\Delta \cup \Gamma$, and the functions $(s_1/t_\infty^d, \ldots, s_{n-1}/t_\infty^d, \phi_{n+1}, \ldots, \phi_{N})$ defines a framing on $\mathbb{A}^{N}_U$. Next let’s see that $Z(t_\infty|_{\mathcal{C}}) = \mathbb{P}^{N-1}_U \cap \mathcal{C} = W_\infty \times U$ is finite over $U$. Now we apply Serre’s theorem also one time to get $d \in \mathbb{Z}$ and $\zeta_\infty \in \Gamma(\mathbb{P}^{N}_U, \mathcal{O}(d))$ such that $\zeta_\infty$ is invertible on $Y_\infty$.

Thus we’ve got the relative framed curve $C \cap \mathbb{A}^{N}_U$ with a fine compactification $(C, \mathcal{C}, \mathcal{O}(d), t_\infty^d, \zeta_\infty)$ over $U$, and a pair of sections $\Delta$ and $\Gamma$. Now applying corollary 1 we get the claim. \qed

**Proof of the theorem.** Consequently applying lemma 3, lemma 5, proposition 1 and lemma 4 we get the claim immediately. \qed

**Corollary 2.** 1) Let $1: Z \subseteq U$ be a smooth affine henselian pair over a field $k$. Let $E \in \mathcal{SH}(k)$ and $\Lambda E = 0$ for some $l \in \mathbb{Z}$, $l \in k^\times$ (see [Jar00] and [MV99] for $\mathcal{SH}(k)$). Then for $p, q \in \mathbb{Z}$ the inverse image homomorphism $i^*: E^{p,q}(U) \to E^{p,q}(Z)$ is an isomorphism.

2) Let $k$ be a perfect field, and $1: Z \subseteq U$ be a smooth affine henselian pair over $k$. Let $E \in \mathcal{SH}(k)$ and $\phi E = 0$ for some $\phi \in GW(k) \simeq [S, S]_{SH(k)}$ such that $E$ is invertible in $k$. Then for $p, q \in \mathbb{Z}$ the inverse image homomorphism $i^*: E^{p,q}(U) \to E^{p,q}(Z)$ is an isomorphism.

**Proof.** The same as for [AnDr18 Theorem 7.10, Corollary 7.11] \qed
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