ENERGY LEVELS OF HYDROGEN-LIKE ATOMS
AND FUNDAMENTAL CONSTANTS. Part I. ⋆,†

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Abstract

The present review includes the description of theoretical methods for the investigations of the spectra of hydrogen-like systems. Various versions of the quasipotential approach and the method of the effective Dirac equation are considered. The new methods, which have been developed in the eighties, are described. These are the method for the investigation of the spectra by means of the quasipotential equation with the relativistic reduced mass and the method for a selection of the logarithmic corrections by means of the renormalization group equation. The special attention is given to the construction of a perturbation theory and the selection of graphs, whereof the contributions of different orders of $\alpha$, the fine structure constant, to the energy of the fine and hyperfine splitting in a positronium, a muonium and a hydrogen atom could be calculated.

In the second part of this article the comparison of the experimental results and the theoretical results concerning the wide range of topics is produced. They are the fine and hyperfine splitting in the hydrogenic systems, the Lamb shift and the anomalous magnetic moments of an electron and a muon. Also, the problem of the precision determination of a numerical value of the fine structure constant, connected with the above topics, is discussed.

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1 Introduction

In the last years, investigations of energy spectra of hydrogen-like systems, as a positronium, a muonium, a hydrogen atom e. t. c., are in progress. These two-particle systems can be used as a test of quantum electrodynamics (QED). The former review \cite{1} has been devoted to the consideration of this problem. The modern status of investigations in the above direction is analyzed in the present paper.

It is well-known, the Dirac equation gives the opportunity to take into account the spin-orbit interaction in hydrogenic systems and, correspondingly, to predict the fine structure of energy levels with the accuracy $O(\alpha^4)$. However, the changes in the fine structure levels, the corrections to the hyperfine splitting (HFS), the Lamb shift (splitting of the $2S_{1/2}$ and $2P_{1/2}$ levels which coincide in the Dirac theory) can be described on a basis of quantum field methods only.

In nonrelativistic quantum mechanics, the two-body problem is reduced to simpler ones, motion of the center of mass (c. m.) and motion of a particle with a reduced mass. In the relativistic case, QED, the separation of c. m. motion is impossible. Also, definition of such a notion as a potential is impossible in a usual way. The remarkable feature of quantum field methods for the description of bound states is the use of the formalism based on the two-particle Green functions. Then, the spectrum is found as the positions of its poles.

The equation for the two-fermion Green function could be written in the Schwinger form \cite{2}:  
\begin{equation}
\{(\gamma_\pi - M_1)(\gamma_\pi - M_2) - I_{12}\}G = I,
\end{equation}
where $\pi_\alpha = p_\alpha - eA_\alpha$ with $p_\alpha$ being 4-momenta of $i$ particle, $A_\alpha$ is an external field influencing $i$ particle; $e$ is an electron charge; $M_i$ is a mass operator for $i$ particle, $I_{12}$ is an interaction kernel between 1 and 2 particles; $I$ is a unit operator. Finally,  
\begin{equation}
G(x_1, x_2; x_3, x_4) = \langle 0 \left| T\{\psi_a(x_1)\psi_b(x_2)\bar{\psi}_a(x_3)\bar{\psi}_b(x_4)\} \right| 0 \rangle
\end{equation}
is the total two-particle Green function in the interaction representation, $\psi(x_i)$ are the field operators of constituent particles.

Equation (1.4) could be re-written in the Bethe-Salpeter (BS) form \cite{3}:  
\begin{equation}
G(x_1, x_2; x_3, x_4) = G_0(x_1, x_2; x_3, x_4) + G_0(x_1, x_2; x'_1, x'_2)K_{BS}(x'_1, x'_2; x_3, x_4)G(x'_3, x'_4; x_3, x_4),
\end{equation}
where $G_{a,b}$ are the Green functions of free fermions, $K_{BS}$ is the kernel for the BS equation, which is connected with the interaction operator of particles $I_{12}$ which is a sum of the two-particle irreducible Feynman diagrams. The state of the two-particle system is defined by the two-time wave function $\psi$, which is a solution of the homogeneous equation corresponding to (1.3)  
\begin{equation}
(G_0^{-1} - K_{BS})\Psi_P(x_1, x_2) = 0,
\end{equation}
\begin{equation}
\Psi_P(x_1, x_2) = \langle 0 \left| T\{\psi_a(x_1)\psi_b(x_2)\} \right| P, \nu \rangle.
\end{equation}

The ket-vector $| P, \nu \rangle$ characterizes the bound system as a whole with the 4 - momentum $P$ and a set of additional quantum numbers $\nu$.

Using a translation invariance and choosing the c. m. system $P_\mu = (E, \vec{0})$ one can obtain the wave function. It corresponds to the state of the definite value of energy $E$  
\begin{equation}
\Psi_P(x_1, x_2) = e^{-iEX_0}\phi_E(x),
\end{equation}
where $X_0$ is the time component of the c.m., $x$ are the relative coordinates.

The bound state problem can be solved in the relativistic quantum theory only approximately, by means of the perturbation methods. The primary approximation is usually chosen so that it corresponds
to the instantaneous Coulomb interaction. Then, an energy spectrum consists of the Coulomb levels found from the wave equations. The corrections to the energy spectra are obtained from the higher orders of perturbation theory:

$$\Delta E = -i\bar{\phi}_{KC}(x)(\tilde{K} + \tilde{K}G_{KC}\tilde{K} + \text{e. t. c.})\phi_{KC}(x'),$$

(1.8)

where $\tilde{K} = K_{BS} - K_{C}$, $K_{C}$ is the Coulomb part of the kernel of the BS equation, $G_{C}$ is a solution of Eq. (1.3) with the kernel $K_{C}$, $\phi_{KC}(x)$ is a solution of Eq. (1.5) with the kernels $K_{C}$. However, the state $\phi_{KC}(x)$ is time-dependent; the relation of the function $\phi_{KC}$ with a solution of the Schrödinger or Dirac equations with the Coulomb kernel is sufficiently complicated. The normalization and formulation of the boundary conditions are not clear for the wave function depending on the relative time. All the above-said influences the calculation accuracy.

The formalism based on the three-dimensional equations has been produced in relativistic bound state theory even before appearance of the covariant formalism in quantum field theory. In this connection, an important meaning did have the development of the quasipotential approach and Gross method. In these approaches quantum field equations are deprived of shortcomings of the BS equation and they are formally close to the nonrelativistic Schrödinger equation with a potential $V$. They can be considered as direct generalizations of the potential two-body theory to the relativistic case. The principal idea of these three-dimensional methods consists in a choice of a "primary" two-particle propagator which describes well physical meaning of problem. In the quasipotential approach, this choice is made by the conversion to the two-time Green function in which the parameter of a relative time $t_{a} - t_{b}$ is equated to zero. In the Gross approach, the "primary" two-particle propagator is chosen as a projector onto the mass shell of a heavy particle and an electron propagator.

## 2 Quasipotential approach in quantum field theory

The quantum field equations for a system of two particles, mentioned in the Introduction, are reduced in the three-dimensional approaches to equations like the Schrödinger ones with the quasipotential defined by the two-time Green function. In spite of the lack of a clear relativistic covariance, the quasipotential method keeps all information about properties of the scattering amplitude which could be received starting from the general principles of quantum field theory. Therefore, one can investigate both analytical properties of the scattering amplitude, its asymptotic behavior and some regularities of a potential scattering, e.g., at high energies. The renormalization procedure of the quasipotential equation is reduced to a charge and mass renormalization like the usual $S$-matrix theory.

The quasipotential method is also very efficient for determination of the relativistic and radiative corrections to energy spectra of hydrogen-like atoms. In some cases, it is convenient to define the two-particle off-shell scattering amplitude instead of the Green function:

$$G = G_{0} + G_{0}TG_{0},$$

(2.1)

which is connected with the kernel of the BS equation by the expressions:

$$T = K_{BS} + K_{BS}G_{0}T$$

(2.2)

or

$$T = K_{BS} + K_{BS}GK_{BS}.$$

(2.3)

On the mass shell ($p_{0} = Q_{0} = 0, \sqrt{p^{2} + m_{1}^{2}} = \sqrt{q^{2} + m_{2}^{2}} = E$) the amplitude $T$ is equal to the physical scattering amplitude (see Fig. 1).

The quasipotential approach is universal and symmetrical in describing both the particles. Due to this, it is used for consideration of any system of arbitrary mass particles. However, the way of equating

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2The physical sense of the quantities $p_{0}, q_{0}, \tilde{p}, \tilde{q}$ is clear from Fig. 1: $m_{1}$ and $m_{2}$ are masses of the constituent particles.
the times does not permit one to get the Dirac equation for one particle when mass of another tends to infinity. When we consider hydrogenic atoms with large nucleus charge and large mass, it would be convenient to use (as the initial approximation) the exact solution of the Dirac equation with the Coulomb potential. This procedure has been proposed in [10].

In the paper [13], the three-dimensional self-according formalism has been built, which leads to the effective (modified) Dirac equation, following to the ideas of the quasipotential approach. In contrast with the usual procedure of equating the times in the operators $\psi_a(x_1)$ and $\psi_b(x_2)$ of Eq. (1.6), the procedure of tending $x_0^2 \to \infty$ has been carried out in the above paper. After this, the following expression has been obtained for the WF in the momentum representation:

$$\Psi_p(p_0^2; \vec{p}_1, \vec{p}_2) = \frac{1}{\sqrt{2\epsilon_2}} < \vec{p}_2, \sigma_2 | \psi_a(0) | \rho, \nu >,$$  \hspace{1cm} (2.4)

in which it is easy to go over to the variables of the total and relative momenta. A similar procedure, carried out for the free Green function $G_0$ (see Eqs. (2.5), (3.4), (3.5) of the article [13]), leads us directly to the equation:

$$\eta \hat{p} + \hat{\rho} - m_1 \Psi_p(\vec{p}) = \frac{1}{(2\pi)^3} \int d\vec{q} \cdot V(\vec{p}, \vec{q}) \Psi_p(\vec{q}),$$  \hspace{1cm} (2.5)

which as $m_2 \to \infty$ goes over in the Dirac equation for the first particle in an external field. This property is a characteristic feature of the method of the effective Dirac equation (EDE).

Although the Logunov-Tavkhelidze quasipotential approach and the Gross approach (the EDE method) have different physical foundation, the formulae used to calculate the energy level corrections in composite systems are very similar.

### Quasipotential approach

**EDE Method.**

The initial propagation function for a two–fermion system:

$$\hat{G}_0(\vec{p}, \vec{q}; E) =$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} dp_0 dq_0 G_0(\vec{p}, \vec{q}; p_0, q_0; E);$$

$$G_0(\vec{p}, \vec{q}; p_0, q_0; E) =$$

$$= i(2\pi)^3 \frac{1}{p_1 - m_1} \frac{1}{p_2 - m_2} \delta^{(4)}(p - q).$$

The relative time parameter in this approach is equated to zero. In the momentum representation integration over components are usually written as $p_1 = \rho - p_2$.

In the case of the c.m.s., namely $\rho = (E, \vec{0})$, one has $p_0^2 = E - \sqrt{\rho^2 + m_2^2}$.

$$\hat{G}_0^+ = F = (2\pi)^3 \delta(\vec{p} - \vec{q})(E - \epsilon_1p - \epsilon_2p)^{-1}$$

is a free particle Green function projected onto the states of positive energies.

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3When we consider the Green function [13], it is also necessary to carry out the operation $x_0^2 \to \infty$.

4The definitions of momenta correspond to Fig.1.
The total Green function.

The total two-time Green function is connected with the off-shell scattering amplitude:

\[ \hat{G}^+ = \hat{G}_0^+ + \left[ G_0 \hat{T} G_0 \right]^+ , \]

which can be re-written using the quasipotential

\[ \hat{V} = (\hat{G}_0^+)^{-1} - (\hat{G}^+)^{-1} = (\hat{G}_0^+)^{-1} \left[ G_0 \hat{T} G_0 \right]^+ (\hat{G}^+)^{-1} = \tau \left[ 1 + \hat{G}_0^+ \tau \right]^{-1} , \]

in such a form:

\[ \hat{G}^+ = \hat{G}_0^+ + \hat{G}_0^+ \cdot \hat{V} \cdot \hat{G}^+ . \]

By means of this Green function the three-dimensional Green function \( \hat{G} \) can be defined:

\[ G(p, q; E) = i(2\pi)^2 \delta(p^2 - m^2) \hat{G}(\vec{p}, \vec{q}; E) \delta(q^2 - m^2) , \]

which satisfies the analogous equation.

The equation kernel

is the quasipotential expressed by means of the amplitude

\[ \tau = (\hat{G}_0^+)^{-1} \left[ G_0 \hat{T} G_0 \right]^+ (\hat{G}^+)^{-1} \]

as follows:

\[ \tau = \hat{V} + \hat{V} \hat{G}_0^+ \tau ; \]

is introduced by the equation:

\[ T = \hat{V}_{EDE} + \hat{V}_{EDE} \cdot g \cdot T . \]

The wave function equation.

The single-time wave function obeys to the corresponding homogeneous equation:

\[ (\hat{G}_0^+)^{-1} \Phi = \hat{V} \Phi . \]

The homogeneous equation for the wave function in a symbolic form is

\[ S^{-1} \phi = \Lambda_2 \cdot \hat{V}_{EDE} \cdot \phi . \]

The expanded form of which are
\[(\hat{p}_1 - m_1)\phi(\vec{p}, E) = (\hat{p}_2 + m_2) \int \frac{d\vec{q}}{(2\pi)^3} i\vec{V}_{\text{EDE}}(\vec{p}, \vec{q}; E) \phi(\vec{q}, E).\]

Since the WF on the muon index satisfies the Dirac equation, the above mentioned equation is re-written for the WF \(\Psi\), having the electron spin factor only:

\[(E - \epsilon_1 p - \epsilon_2 p) \Psi(\vec{p}) = \int \frac{d\vec{q}}{(2\pi)^3} (\hat{p} + m_2) \int \frac{d\vec{q}}{(2\pi)^3} i\vec{V}_{\text{EDE}}(\vec{p}, \vec{q}; E) \Psi(\vec{q}).\]

**The energy spectrum.**

After solving the obtained equations by means of perturbation theory the formulae for the corrections to energy levels of a hydrogen-like atom are [14]

\[E_n = E_n^0 + \frac{\langle n | \Delta \hat{V}^{(2)} + \hat{V}^{(4)} + ... \rangle | n \rangle}{1 + \langle n | \frac{\partial \Delta \hat{V}^{(2)}}{\partial E} | n \rangle}, \quad E_n = E_n^0 + \frac{\langle n | i\delta \hat{V}_{\text{EDE}} \rangle | n \rangle}{1 + \langle n | \frac{\partial \delta \hat{V}_{\text{EDE}}}{\partial E} | n \rangle} + \langle n | \frac{\partial \Delta \hat{V}^{(2)}}{\partial E} | n \rangle \Delta \hat{V}^{(2)} | n \rangle + ... \]

\[\Delta \hat{V}^{(2)} = \hat{V}^{(2)} - v_C, \quad v_C \text{ is the Coulomb potential}, \quad \hat{V}_{\text{EDE}} = \hat{V}_0 + \delta \hat{V}_{\text{EDE}}, \quad \hat{V}_0 \text{ is the "primary" potential.}\]

(The derivative on \(E\) is taken at the point \(E_n^0\).)

\[\eta_1 E + p_0, \vec{p} \quad \eta_1 E + q_0, \vec{q} \]
\[\eta_2 E - p_0, -\vec{p} \quad \eta_2 E - q_0, -\vec{q} \]

**Fig. 1.** The parametrization of the two-particle off-shell scattering amplitude \(T\) in c. m. s.

\[(\eta_1 = \frac{(E^2 + m_1^2 - m_2^2)}{2E^2}, \quad \eta_2 = \frac{(E^2 + m_2^2 - m_1^2)}{2E^2}).\]

It should be noted that there exists another way of constructing the quasipotential by using the physical on-mass-shell scattering amplitude but not on energy shell. However, it is extremely important to take into account the binding effects and relativistic interaction effects on precision calculations of the eigenvalues of hydrogen-like atoms, e. g., on calculations of the corrections to the Fermi energy of the HFS of the ground state with the accuracy higher than \(\alpha^5\). The efficiency of an analysis of these effects is essentially greater in the framework of the first method. The cause is that the on-shell scattering amplitude can possess singularities in the infrared region in higher orders of perturbation theory. If we are concerned with the diagrams up to the fourth order of charge, the infrared divergencies cancel one another at the end. In analysing higher orders in \(\alpha\), the singularities of the quasipotential lead to the logarithmic contributions with respect to the fine structure constant. The possibility of including these effects on the basis of the on-shell scattering amplitude is restricted. Moreover, the existence of poles
in the virtual particle propagators is a complicated factor in integrating over the relative momenta (see Section III and [15]).

When the quasipotential is built by the first method on the basis of the two-particle Green function, the total energy appears in the quasipotential directly. As a result, the normalization condition and the condition of orthogonality of the eigenfunction [16] and at the same time perturbation theory acquire a specific peculiarities.

Let the equation of the eigenvalue problem has the form:

\[
\begin{align*}
F(E) - \frac{1}{E} V(E) - V(\vec{p}, \vec{q}; E) - 1 & = 0,
\end{align*}
\]

where

\[
F(E) = \hat{G}_0^+ = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})(E - \epsilon_1 p - \epsilon_2 p)^{-1},
\]

and the quasipotential be defined by Eq. on page No. 4. Then, we have the following expression for the eigenfunction \(\Psi_n\) corresponding to the state with the eigenvalue \(E_n\):

\[
\hat{G}^+(E)W(E, E_n)\Psi_n = \frac{\Psi_n}{E - E_n},
\]

with

\[
W(E, E_n) = \frac{F^{-1}(E) - F^{-1}(E_n)}{E - E_n} - \frac{V(E) - V(E_n)}{E - E_n}.
\]

Since near the singularity \((E \simeq E_n)\) one has

\[
\hat{G}^+(E) \simeq \frac{\Psi_n \Psi^*_n}{E - E_n},
\]

the orthonormalization condition has the form:

\[
\Psi^*_m W(E_m, E_n)\Psi_n = \delta_{mn},
\]

and

\[
\Psi^*_m \Psi_n = \delta_{mn} + \Psi^*_m \frac{\partial V(E)}{\partial E} |_{E=E_n} \Psi_n, \quad (E_m = E_n).
\]

Since the energy levels of a composite system are the poles of the exact scattering amplitude and every separate term of a charge power expansion of the amplitude does not have these poles, there must exist an infinity series of the diagrams for bound states the contributions of which to the energy of a bound system have the same order on \(\alpha\). The diagrams, which this series consists of, are the reducible two-particle Feynman diagrams. However a similar binding effect takes place also for the irreducible diagrams, e. g., when calculating the Lamb shift.

The diagrams of successive Coulomb photon exchanges are shown in [17] to contribute the same order in \(\alpha\) in the infrared region when calculating the HFS of hydrogen-like systems. So, when constructing the quasipotential, the corresponding modification procedure is necessary because we must sum the infinite series of diagrams selectively. Let us introduce the Coulomb Green function \(G_C\) satisfying the equation

\[
(G_0^{-1} - K_C)G_C = I,
\]

where

\[
K_C(\vec{p}, \vec{q}) = -\frac{e(Ze)\Gamma_0}{(\vec{p} - \vec{q})^2} = v_C \Gamma_0
\]

is the Coulomb kernel \((\Gamma_0 = \gamma_{10} \gamma_{20}).\) The expression (2.13) can be rewritten in the form:

\[
G_C = G_0 + G_0 K_C G_C = G_0 + G_C K_C G_0,
\]

In view of the absence of the inverse Green function to the Green function of the system of free two fermions, it is necessary to carry out the procedure of projecting \(\hat{G}_0\) onto the states of positive energies in the quasipotential approach.

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and the total Green function can be represented similarly to (2.3):
\[ G = G_C + G_C \tilde{T} G_C, \]
\[ \tilde{T} = \tilde{K} + \tilde{K} G_C \tilde{T}, \]
\[ \tilde{T} = \tilde{K} + \tilde{K} G \tilde{K}, \]
with \( \tilde{K} = K_{BS} - K_C \).

The wave function specifying the state of the definite energy \( E \) of the two-fermion system must satisfy the quasipotential equation
\[ \left[ (\hat{G}_C^+(\vec{p}, \vec{q}; E))^{-1} - \tilde{V}(\vec{p}, \vec{q}; E) \right] \Psi_E(\vec{q}) = 0, \]
where once again an integration is implied on the repeated momenta. The corresponding quasipotential has the form:
\[ \tilde{V}(\vec{p}, \vec{q}; E) = \left[ \hat{G}_C^+(\vec{p}, \vec{q}; E) \right]^{-1} - \left[ \hat{G}_C^+(\vec{p}, \vec{q}; E) \right]^{-1} = (\hat{G}_C^+)^{-1} \left[ G_C \tilde{T} G_C \right]^+ (\hat{G}_C^+)^{-1}, \]
and \( \hat{G}_C^+ \) is the two-time Green function projected onto the positive energy states (see above).

The technique developed in [15], which uses the Coulomb Green function formalism, gives us the opportunity to take into account the successive multiplied exchange by the Coulomb photons and to write formulae for the energy level shift \( \Delta E \) with respect to the Coulomb one.

From Eq. (2.16) it is clear that the total two-particle Green function (projected onto the states of positive energies) has the inverse one
\[ (\hat{G}_C^+)^{-1} = (\hat{G}_C^+)^{-1} - (\hat{G}_C^+)^{-1} \hat{T}_C^+ (\hat{G}_C^+)^{-1}, \]
and the quasipotential can be presented in the form:
\[ \tilde{V}(\vec{p}, \vec{q}; E) = (\hat{G}_C^+)^{-1} \hat{T}_C^+ (\hat{G}_C^+)^{-1} = \tilde{r}_C - \tilde{r}_C \hat{G}_C^+ \tilde{r}_C + ..., \]
\[ \tilde{r}_C = (\hat{G}_C^+)^{-1} \hat{T}_C^+ (\hat{G}_C^+)^{-1} \]

After the mentioned projecting of the Coulomb Green function, one can find
\[ \hat{G}_C^+ = F + F(v_C \hat{G}_C)^+ = F + (\hat{G}_C v_C)^+ F. \]

Here, the expression has been used:
\[ \hat{G}_0 = (\Lambda^{++} F - \Lambda^{--} F') \Gamma_0, \]
where \( F \) is defined by Eq. (2.7) and
\[ F' = (2\pi)^3 \delta(\vec{p} - \vec{q})(E + \epsilon_{1p} + \epsilon_{2p})^{-1}. \]
\[ \Lambda^{++} = \Lambda^+_1(\vec{p}) \Lambda^+_2(-\vec{p}), \ \Lambda^{--} = \Lambda^-_1(\vec{p}) \Lambda^-_2(-\vec{p}), \ \Lambda^\pm(\vec{p}) \]
are projecting operators.

The following formulae are used for some transformation of Eq. (2.24):
\[ \left[ Q G_C \right]^+ = \left[ Q \Sigma \Gamma_0 \right]^+ \hat{G}_C^+, \]
\[ \left[ G_C Q \right]^+ = \hat{G}_C^+ \left[ \Sigma \Gamma_0 Q \right]^+, \]
where \( \Sigma = (1 + \Lambda^{--} F' v_C)^{-1}, \ \Sigma' = (1 + v_C \Lambda^{--} F')^{-1}, \) and \( Q \) has an arbitrary matrix structure. As a result, we get the closed equation for the function \( \hat{G}_C^+ \):
\[ \hat{G}_C^+ = F + F K_{\Sigma} \hat{G}_C^+ = F + \hat{G}_C^+ K_{\Sigma} F \]
with the kernel $K_{\Sigma} = u_1^* u_2^* \Sigma u_1 u_2$, $\Sigma = v_C \Sigma = \Sigma' v_C$. In this case, the inverse Green function $(\hat{G}_C^+)^{-1}$ is
\[
(\hat{G}_C^+)^{-1} = F^{-1} - K_{\Sigma}
\] (2.30)
and the kernel $K_{\Sigma}$ includes the projections of a Coulomb interaction onto the negative energy states
\[
K_{\Sigma} = K_C^+ + \delta K_{\Sigma} = K_C^+ - u_1^* u_2^* v_C \Lambda^- F' v_C u_1 u_2.
\] (2.31)

Let us note that $F'$, in contrast with $F$, has no singularity when $\vec{p}, \vec{q} \to 0$, $E \to m_1 + m_2$. In the approximation of the upper components ($u_i = \left(\psi_i^0\right)$, $u_i$ is a normalized Pauli spinor.) the kernel $K_C^+$ coincides with the Coulomb potential and the term, including the projecting operator $\Lambda^-$, is equal to zero. Thus, the principal part of the kernel $K_{\Sigma}$ is equal to $K_C^+$. The function $(\hat{G}_C^+)^{-1}$ coincides with the inverse Green function of the non-relativistic Schrödinger equation with the Coulomb potential.

In many cases, the quasipotential is conveniently expressed by the amplitude $\tilde{T}_0^+ = \left[G_0 \tilde{T} G_0\right]^+$. This can be simply carried out by the helpful formulae (2.27, 2.28). Since
\[
\tilde{T}_C^+ = \tilde{T}_0^+ + (\tilde{T}_0 K_C \hat{G}_C)^+ + (\hat{G}_C K_C \tilde{T}_0)^+ + (\hat{G}_C K_C \tilde{T}_0 K_C \hat{G}_C)^+,
\] (2.32)
we have after transformations (2.27, 2.28):
\[
\tilde{T}_C^+ = \tilde{T}_0^+ + \hat{G}_C (\Sigma' v_C \tilde{T}_0)^+ + (\tilde{T}_0 v_C \Gamma_0 \Sigma_0)^+ \hat{G}_C^+ + \hat{G}_C (\Sigma' v_C \tilde{T}_0 K_C \Sigma_0)^+ \hat{G}_C.
\] (2.33)
The expression (2.33) is re-written in a more convenient form:
\[
\tilde{T}_C^+ = \left(\hat{G}_C u_1^* u_2^* \Sigma \cdot I + I \cdot u_1^* u_2^*\right) \tilde{T}_0 (\Gamma_0 u_1 u_2 \cdot I + \Gamma_0 \cdot I \Sigma u_1 u_2 \hat{G}_C^+),
\] (2.34)
or, with taking into account
\[
\lambda^+ + \lambda^- = I, \quad \lambda^- = \Lambda^+_1 (\bar{p}) \Lambda^+_2 (-\bar{p}) + \Lambda^-_1 (\bar{p}) \Lambda^-_2 (-\bar{p}) + \Lambda^- -\Sigma,
\] (2.35)
in the form:
\[
\tilde{T}_C^+ = \hat{G}_C \left[F^{-1} \tilde{T}_0^+ F^{-1} + F^{-1} (\tilde{T}_0 \Gamma_0 \lambda^- \Sigma \Gamma_0)^+ + (\Sigma \lambda^- \tilde{T}_0)^+ F^{-1} + (\Sigma \lambda^- \tilde{T}_0 \Gamma_0 \lambda^- \Sigma \Gamma_0)^+\right] \hat{G}_C.
\] (2.36)

Therefore, the amplitude $\tilde{T}_C$ can be presented as the sum $\tilde{T}_C = \tilde{T}_0 + \rho$, where $\tilde{T}_0$ is the scattering amplitude including the $\tilde{T}$-matrix. The rest of the terms corresponds to interactions of higher order with subtraction of iterations.

The graphical interpretation of the amplitude $\tilde{T}_0$ is clear. It is a series of all irreducible diagrams with the exception of the one-Coulomb exchange diagram and the reducible diagram with Coulomb exchanges in intermediate states only. The corresponding quasipotential can be divided in two parts:

\[
\tilde{V}(\bar{p}, \bar{q}; E) = \tilde{V}_0^+ + \tilde{V}_\rho + \text{higher orders of perturbation expansion},
\] (2.37)
where
\[
\tilde{V}_0^+ = \tilde{V}_0 + \tilde{V}_\rho,
\] (2.38)
\[
\tilde{V}_\rho = \rho - \rho \tilde{G}_C^+ \rho - \rho \tilde{G}_C^+ \tilde{T}_0.
\] (2.39)
with
\[
\rho = F^{-1} \Delta R F^{-1},
\] (2.40)
\[
\Delta R = F (\Sigma \lambda^- \tilde{T}_0)^+ + (\tilde{T}_0 \Gamma_0 \lambda^- \Sigma \Gamma_0)^+ F + F (\Sigma \lambda^- \tilde{T}_0 \Gamma_0 \lambda^- \Sigma \Gamma_0)^+ F.
\] (2.41)
After substitution of the amplitude $\tilde{T}$ from (2.18) ($\tilde{T} \simeq \tilde{K} G_C \tilde{K}$) into (2.38) we have:
\[
\tilde{V} \tilde{T}_0 = F^{-1} \left[\tilde{K}_0^+ + (\tilde{K} G_C \tilde{K})_0^+ - \tilde{K}_0^+ F^{-1} \tilde{G}_C^+ F^{-1} \tilde{K}_0^+ + \text{ e. t. c.}\right] F^{-1},
\] (2.42)
in which the notation is used:

\[ \tilde{K}_0^+ = \left[ G_0\tilde{K}G_0 \right]^+ \]  \hspace{1cm} (2.43)

\[ (\tilde{K}G_C\tilde{K})_0^+ = \left[ G_0\tilde{K}G_C\tilde{K}G_0 \right]^+ . \]  \hspace{1cm} (2.44)

For calculations of the energy level shift up to the accuracy \( \alpha^6 \log \alpha \), it is sufficient to take into account

\[ (\tilde{K}G_C\tilde{K})_0^+ = (\tilde{K}G_0\tilde{K})_0^+ + (\tilde{K}_0K_C\tilde{K}_0)^+ + (\tilde{K}_0\Gamma_0\nu vCG_C\Gamma_0\tilde{K}_0)^+. \]  \hspace{1cm} (2.45)

Using the representation of a unit operator through projecting operators we can find out

\[
(\tilde{K}_0\Gamma_0\nu vCG_C\Gamma_0\tilde{K}_0)^+ = \tilde{K}_0^+(v_C\hat{G}_Cv_C)^+\tilde{K}_0^+ + (\tilde{K}_0\Gamma_0\nu vCG_C)^+\tilde{K}_0^+ + (\tilde{K}_0\Gamma_0\nu vCG_C\Gamma_0\nu vCG_C)^+\tilde{K}_0^+.
\]  \hspace{1cm} (2.46)

The first term of the expression derived contributes mostly in calculating the energy spectra.

Starting from the definition of the kernel \( K_C \) (2.31), we get

\[
(\tilde{\Sigma}\lambda^+\hat{G}_Cv_C)^+ = K_\Sigma(\hat{G}_Cv_C)^+ = \tilde{K}_C^+K_\Sigma.
\]  \hspace{1cm} (2.49)

Since the equality \( \lambda^+\hat{G}_C = -\lambda^--\Gamma_0^+(\Gamma_0^0 + v_C\hat{G}_C) \) holds, the sum of two last terms of the expression (2.48) can be transformed in

\[
(\tilde{\Sigma}\lambda^-\hat{G}_Cv_C)^+ + (\tilde{\Sigma}\lambda^-\Gamma_0^+\nu vCG_Cv_C)^+ = -\tilde{\Sigma}\lambda^-\Gamma_0^+\nu vCG_Cv_C)^+ = K_\Sigma - K_C^+.
\]  \hspace{1cm} (2.50)

Therefore,

\[
(v_C\hat{G}_Cv_C)^+ = K_\Sigma\tilde{K}_C^+K_\Sigma + K_\Sigma - K_C^+.
\]  \hspace{1cm} (2.51)

The iteration term in the quasipotential expression (2.42) is written in the following way using (2.22):

\[
\tilde{K}_0^+F^{-1}\hat{G}_C^+F^{-1}\tilde{K}_0^+ = \tilde{K}_0^+(F^{-1} + K_\Sigma + K_\Sigma\tilde{K}_C^+K_\Sigma)\tilde{K}_0^+.
\]  \hspace{1cm} (2.52)

Substituting the last two formulae in the quasipotential expression we can be convinced that the projection of the block \( v_C\hat{G}_Cv_C \) onto the positive-energy states drops out from the quasipotential expression (2.42). In this case

\[
\check{V}_0 = F^{-1}\left\{ K_0^+ + (K_0G_0\tilde{K})_0^+ - \tilde{K}_0^+ + (K_0\nu CG_C\tilde{K}_0)^+ - \tilde{K}_0^+K_C\tilde{K}_0^+ + K_0^+(v_C\hat{G}_Cv_C\Gamma_0\nu vCG_C\tilde{K}_0)^+ + (K_0\nu vCG_C^+\Gamma_0\nu vCG_C)^+\tilde{K}_0^+ + (K_0\Gamma_0\nu vCG_C\Gamma_0\nu vCG_C)^+\tilde{K}_0^+ \right\} F^{-1}.
\]  \hspace{1cm} (2.53)

The information about the corrections to the Coulomb levels can be obtained by constructing perturbation theory on the basis of the above presented ideas and Eq. (2.19) in the form:

\[
\left( F^{-1}(E_C) + \Delta E - K_C^+ - \check{V}(E) \right) \Psi_E = 0,
\]  \hspace{1cm} (2.54)

where

\[
\check{V} = \check{V} + K_\Sigma - K_C^+.
\]  \hspace{1cm} (2.55)
\( \Delta E = E - E_C \) is the correction to the ground state of the non-relativistic Schrödinger equation with the Coulomb potential.

If \( E' \) is an eigenvalue of the wave function of the equation with the kernel \( K_C^+ \),

\[
(\hat{G}_C^+(E'))^{-1} \Psi_{E'} = (F^{-1}(E') - K_C^+) \Psi_{E'} = 0,
\]  
(2.56)

the initial equation (2.54) gets the following form:

\[
\left( (\hat{G}_C^+(E'))^{-1} + \Delta E' - \hat{V}(E) \right) \Psi_E = 0,
\]  
(2.57)

with \( \Delta E' = E - E' \) and \( \Delta E = \Delta E' + \Delta E_C = \Delta E' + E' - E_C \). Let us assume that the eigenfunction \( \Psi_{E'} \) and the eigenvalues are known. In accordance with the methods of perturbation theory in the first order we have \( (\Psi_E = \Psi_{E'} + \Psi_I) \)

\[
\left( (\hat{G}_C^+(E'))^{-1}\Psi_I + \Delta E' \Psi_{E'} - \hat{V}(E') \Psi_{E'} \right) = 0.
\]  
(2.58)

Having multiplied both of parts of this equality by \( \Psi_{E'}^* \), on the left side and using the normalization condition we come to:

\[
\Delta E'_I = \langle \Psi_{E'} | \hat{V}(E') | \Psi_{E'} \rangle \quad (2.59)
\]

\[
\Psi_I = \left( G_C^+(E') - \frac{\Psi_{E'} \Psi_{E'}}{E - E'} \right) \hat{V}(E') \Psi_{E'}, \quad E \to E'.
\]  
(2.60)

Analogously, for the correction to the energy level in the second order we have

\[
\Delta E''_I = \langle \Psi_{E'} | \hat{V}(E') \left(1 + G_C^+(E') \hat{V}(E') \right) | \Psi_{E'} \rangle >, 
\]  
(2.61)

where

\[
G_C^+(E') = G_C^+(E') - \frac{\Psi_{E'} \Psi_{E'}}{E - E'}
\]  
(2.62)

The term including the energy derivative disappears from the final result because \( \Psi_{E'} \) is the eigenfunctions of the equation with the kernel \( K_C^+ \), which does not depend on the energy.

To determine \( \Psi_{E'} \) from Eq. (2.56) let us introduce the auxiliary function \( \Phi_{E'} \) defined by the equality

\[
\Phi_{E'}(\vec{q}) = \frac{(\epsilon_1 + \eta_1 E)(\epsilon_2 + \eta_2 E)}{2\mu(E + \epsilon_1 + \epsilon_2)} \Phi_{E'}(\vec{q}) = \\
= \frac{(2E'\epsilon_1 + E' - m_1^2)(2E'\epsilon_2 + E' - m_2^2)}{8\mu E'(E' + \epsilon_1 + \epsilon_2)} \Phi_{E'}(\vec{q}),
\]  
(2.63)

which satisfies the non-relativistic Schrödinger-like equation

\[
(g_C^{-1} - \Delta \epsilon - \delta K_C)\Phi_{E'} = 0.
\]  
(2.64)

Here

\[
g_C^{-1} = W_C - \frac{\vec{p}^2}{2\mu} - v_C, \quad W_C = E_C - m_1 - m_2,
\]  
(2.65)

\[
\Delta \epsilon = W_C + \frac{(m_1 + m_2)^2 - E'^2}{8\mu E'^2} \simeq -\Delta E_C
\]  
(2.66)

and

\[
\delta K_C = K_C^+(\vec{p}, \vec{q}) \frac{(\epsilon_1 + \eta_1 E)(\epsilon_2 + \eta_2 E)}{2\mu(E + \epsilon_1 + \epsilon_2)} - v_C \simeq K_C^+ \frac{W_C - \frac{\vec{p}^2}{2\mu}}{E' - \epsilon_1 - \epsilon_2} - v_C.
\]  
(2.67)

The Coulomb WF describing the 1S state is:

\[
\phi_C(\vec{p}) = \frac{8\pi Z\alpha\mu}{(\vec{p}^2 + Z^2\alpha^2\mu^2)^2} | \phi_C(r = 0) \ |
\]  
(2.68)
\[ | \phi_C(r = 0) |^2 = \frac{(\alpha \mu)^3}{\pi}. \] (2.69)

It satisfies the equation
\[ g_C^{-1} \phi_C = 0. \] (2.70)

The correction to the Coulomb energy level \( \Delta E_C \) and the function \( \Phi_E \) are found (as above \( \Delta E' \)) by means of a quantum-mechanical perturbation theory
\[ \Delta E_C = \langle \phi_C | \delta K_C(1 + g'_C \delta K_C) | \phi_C \rangle, \] (2.71)
\[ g'_C = g_C - \frac{\phi_C \delta \phi_C^*}{E - E_C}, \text{ when } E \to E_C. \] (2.72)

In the second order of perturbation theory the eigenfunction of Eq. (2.56) has the following form:
\[ \Psi_{E'}(\vec{p}) = \frac{(\epsilon_{1p} + m_1)(\epsilon_{2p} + m_2) - \vec{p}^2}{4m_1 m_2} \left[ \phi_C(\vec{p}) + g'_C(\vec{p}, \vec{k}) \delta K_C(\vec{k}, \vec{q}) \phi_C(\vec{q}) \right]. \] (2.73)
(Integration is implied over 3- vectors \( \vec{k} \) and \( \vec{q} \).)

The final expression for the total shift of energy levels with respect to the Coulomb level is the sum of the correction \( \Delta E_C \) and the corrections from the quasipotentials \( \delta K_S = K_S - K_C^2 \) and \( \hat{V}(E) \). In \ref{18-20} the following expressions for these corrections are presented:
\[ \Delta E = \Delta E_{KK} + \Delta E_{KV} + \Delta E_{VV}, \] (2.74)
\[ \Delta E_{KK} = \langle \phi_C | \delta K_C(1 + g'_C \delta K_C) | \phi_C \rangle + \langle \Psi_{E'} | \delta K_S(1 + G_C(E') \delta K_S) | \Psi_{E'} \rangle, \] (2.75)
\[ \Delta E_{KV} = \langle \Psi_{E'} | \hat{V}(E') G_C(E') \delta K_S + \delta K_S G_C(E') \hat{V}(E') | \Psi_{E'} \rangle, \] (2.76)
\[ \Delta E_{VV} = \langle \Psi_{E'} | \hat{V}(E') (1 + G_C(E') \hat{V}(E')) | \Psi_{E'} \rangle. \] (2.77)

The problem of construction of the kernel for the quasipotential equation was under consideration in \ref{18-20}. The kernel \( \tilde{K} \) can be expanded in the series of perturbation theory:
\[ \tilde{K} = K_T + K^{(2)}, \] (2.78)
where the index \( T \) denotes the transverse photon in the Coulomb gauge; \( K^{(2)} \) is built from the diagrams of the second order in the fine structure constant.

Let us point out that when the interaction kernel for the two-fermion system including a particle and an antiparticle is constructed, it is necessary to take into account the annihilation interaction channel \ref{21-24}.

To investigate the energy spectra of a two-particle relativistic bound system, the method has been proposed which is based on the use of the local quasipotential equation with the relativistic reduced mass in the c. m. s. \ref{22-24}. By means of the "rationalization" of the Logunov – Tavkhelidze equation \ref{25} the following equation has been obtained:
\[ \frac{b^2(E)}{2 \mu_R} - \frac{\vec{p}^2}{2 \mu_R} \Psi_E(\vec{p}) = I(E, \vec{p}) \int \frac{d\vec{q}^2}{(2\pi)^3} V(\vec{p}, \vec{q}; E) \Psi_E(\vec{q}), \] (2.79)
where
\[ I(E, \vec{p}) = \frac{(E + \epsilon_{1p} + \epsilon_{2p})(E^2 - (\epsilon_{1p} - \epsilon_{2p})^2)}{8E_1E_2E}. \] (2.80)

In accordance with the definitions of Fig. 1:
\[ E_1 = \eta_1 E = \frac{E^2 - m_1^2 + m_2^2}{2E}, \] (2.81)
\[ E_2 = \eta_2 E = \frac{E^2 - m_1^2 + m_2^2}{2E}. \] (2.82)
$E_1 + E_2 = E$ is the energy of a bound state;

$$b^2(E) = \frac{1}{4E^2} [E^2 - (m_1 + m_2)^2] [E^2 - (m_1 - m_2)^2]$$  \hspace{1cm} (2.83)

is the squared relative momentum on the energy shell. Finally,

$$\mu_R = \frac{E_1 E_2}{E} = \frac{E_1 E_2}{E_1 + E_2} = \frac{E^4 - (m_1^2 - m_2^2)^2}{4E^3}$$  \hspace{1cm} (2.84)

is the relativistic reduced mass defined in accordance with the relativistic expression of the coordinate of the center of mass. Let us mention other definitions of the relativistic reduced mass like $\tilde{\mu}_R = \frac{[E^2 - (m_1 - m_2)^2]}{4E^2}$, Ref. [23], $\mu_R' = \frac{m_1 m_2}{E}$, Ref. [20]. The quantity $m' = \sqrt{m_1 m_2}$, the effective mass, which has been defined in Ref. [27], is closely connected to the notion of the relativistic reduced mass. This definition gave the authors of [27] the opportunity to reduce the relativistic two-body problem to the case of the particle motion with the mass $m'$ in the quasipotential field. In the non-relativistic limit $E_{1,2} \rightarrow m_{1,2}$ the relativistic reduced mass $\mu_R$ becomes the well-known reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

As has been pointed out above, the quasipotential can be considered from the Lippmann-Schwinger-like equation with the scattering amplitude with the relative energies of particles $p_0 = q_0 = 0$ being equal to zero, $p_0^0 = E_1$, $p_0^2 = E_2$. On the energy shell $E = \epsilon_1 + \epsilon_2 = \epsilon_1 q + \epsilon_2 q$, $\epsilon_2 q = \sqrt{p^2 + m_1^2}$, $\epsilon_1 q = \sqrt{q^2 + m_2^2}$ we have $\tilde{I}(E, \tilde{p}) = 1$, and in this case the equation (2.79) can be solved exactly for the Coulomb interaction.

In the case of an interaction of two spinor particles with masses $m_1$, $m_2$ and charges (-e) and (Ze) the main contribution to the binding energy of particles is shown in [22] to arise from the modified Coulomb potential

$$V_C^{mod}(\tilde{p}, \tilde{q}; E) = -\frac{Ze^2}{(\tilde{p} - \tilde{q})^2} (1 + \frac{b^2(E)}{E_1 E_2}).$$  \hspace{1cm} (2.85)

Quantization of energy levels is defined by the equality analogous to the equation obtained in [28]:

$$\frac{b^2 E^2}{(b^2 + E_1 E_2)^2} = \frac{Z^2 \alpha^2}{n^2}$$  \hspace{1cm} (2.86)

$(n = 1, 2, \ldots)$ is the principal quantum number). The above equality leads to the variant of the relativistic Balmer formula [28]:

$$E_n^2 = m_1^2 + m_2^2 + 2m_1 m_2 \left(1 + \frac{(Z \alpha)^2}{n^2}\right)^{-1/2}.$$  \hspace{1cm} (2.87)

The above formula can be re-written as an expansion of the binding energy $B$ in powers $\alpha^2$:

$$B = E - m_1 - m_2 = \frac{\mu Z^2 \alpha^2}{2 n^2} + \frac{\mu Z^4 \alpha^4}{8 n^4} \left(3 - \frac{\mu^2}{m_1 m_2}\right), \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$  \hspace{1cm} (2.88)

The relativistic Balmer formula takes into account the recoil effects but does not describe the fine and hyperfine structure corresponding to the spin-orbit and spin-spin interactions. These corrections have been considered in [24] using the formalism of the local quasipotential equation with the relativistic reduced mass. In the first order of perturbation theory we have

$$\Delta E_I = \langle \Psi_C | \Delta \hat{V}_\gamma + \hat{V}_{2\gamma} + \text{e. t. c.} | \Psi_C \rangle,$$  \hspace{1cm} (2.89)

where $\Delta \hat{V}_\gamma = \hat{V}_\gamma - V_C^{mod}$ is determined by the difference of the one-photon exchange quasipotential and the modified Coulomb potential (2.85). In turn,

$$V_{2\gamma} = T_{2\gamma}^+(\tilde{p}, \tilde{q}, p_0 = 0, q_0 = 0) - \int \frac{d\tilde{k}}{(2\pi)^3} \hat{V}_\gamma(\tilde{p}, -\tilde{k}; E) \hat{V}_\gamma(\tilde{k}, \tilde{q}; E)$$  \hspace{1cm} (2.90)
In accordance with the perturbation theory methods the corrections of the second order are determined by the formula:

$$\Delta E_{II} = \langle \Psi'_C | \Delta \hat{V}_1 | \psi'_C \rangle \langle \psi'_C | \frac{\partial \Delta \hat{V}_1}{\partial E} | \Psi_C \rangle + \sum_{n=2}^{\infty} \frac{\langle \Psi'_C | \Delta \hat{V}_n | \psi'_C \rangle \langle \psi'_C | \frac{\partial \Delta \hat{V}_n}{\partial E} | \Psi_C \rangle}{E'_n - E'_n}$$

(2.91)

with using the Pauli-like eigenfunctions that are the solutions of the local quasipotential equation with the modified Coulomb potential:

$$\Psi'_C(\vec{p}) = \frac{8\pi Z\alpha \mu_{eff}(\vec{p})^2 \delta(\vec{p})}{(\vec{p}^2 + Z^2\alpha^2 \mu_{eff}^2)^{1/2}} | \Psi_C(0) \rangle \left[ 1 - \frac{1}{2} (Z\alpha)^2 \frac{1}{m_1 m_2} \right] \chi_1 \chi_2,$$

(2.92)

$$| \Psi_C(0) \rangle = \sqrt{\frac{Z^3\alpha^3 \mu_{eff}^3}{\pi}}$$

(2.93)

($\chi_{1,2}$ are Pauli two-component spinors),

$$\mu_{eff} = \frac{b_1^2(E_1 + E_2)}{E_n} = \mu_R + \frac{b_2^2(E)}{E_n} = \frac{m_1 m_2}{E_n} (1 + \frac{(Z\alpha)^2}{n^2})^{-1/2},$$

(2.94)

The corrections of the order $(Z\alpha)^2 E_F$ and $m_1 m_2 (Z\alpha)^2 E_F$ to the hyperfine structure of a muonium, obtained in this approach, are shown in Section IV of the present review.

3 Calculation techniques for finding the energy spectra in the different orders in $\alpha$

As one can see from the above-said, a calculation of the fine and hyperfine splitting of the energy levels is reduced to finding out the matrix elements of the quasipotential $V$. In the nonrelativistic case, the WF of the system lightly bound can be approximated by the Dirac $\delta$-function. The use of the Coulomb 1S- state WF gives the opportunity of considering the matrix elements when the non-zero momenta $\vec{p}, \vec{q} \neq 0$. However, the relativistic corrections are taken into account more accurately by means of the WF in the form of (2.73) when describing bound states.

In perturbation theory developed in Section II the quasipotential and the interaction kernel include the Coulomb Green function essentially. The methods of using this function are different in the problems of fine structure and hyperfine structure of the hydrogen-like atoms. In the first case, the main interest is the low-frequency region of virtual momenta where an interaction is nonrelativistic. Here, to include binding effects in the virtual states of the interaction kernel it is important to consider the block of the Coulomb exchanges as a whole, e. g., by means of the expression for the Green function of the nonrelativistic Schrödinger equation with the Coulomb potential.

In analysing the HFS of the energy levels, one should have a possibility to consider contributions of the one- or two-transverse-photon exchanges and arbitrary number of the Coulomb exchanges. It is sufficient to apply the usual expansion of the Coulomb Green function restricting ourselves to the needed number of expansion terms.

Up to the accuracy $O(\alpha^3)$, it is sufficient to use the approximate WF

$$\phi_C^{approx}(\vec{p}) = (2\pi)^3 \delta(\vec{p}) \mid \phi_C(\vec{r} = 0) \rangle, \quad E \simeq m_1 + m_2,$$

(3.1)

when the calculations of the energy of the 1S-level, based on the quasipotential $V$ constructed from the diagrams of the order $\alpha^2$ and higher.

$^6$One can restrict oneself to the diagrams of the three-photon exchanges in calculating the HFS of the ground state in a positronium and in a muonium up to the accuracy $\alpha^6 \log \alpha$. 

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The above-said statement is based on the following fact. The value of squared modulus of the WF, which is in the matrix element, has the order $\alpha^3$ in the coordinate space when $r = 0$,

$$|\phi_C(r = 0)|^2 = \frac{(\alpha \mu)^3}{\pi \alpha^3} \sim O(\alpha^3).$$

Using the well-known representation of the $\delta$ – function:

$$\delta(x) = \lim_{\alpha \to 0} \frac{1}{\pi} \frac{\alpha}{\alpha^2 + x^2},$$

we can find out

$$\frac{\pi \delta(x)}{2x^2} = \lim_{\alpha \to 0} \frac{\alpha}{(\alpha^2 + x^2)^2}$$

Thus,

$$\lim_{\alpha \to 0} \phi_C(\vec{p}) = \lim_{\alpha \to 0} \frac{8\pi \alpha \mu |\phi_C(0)|}{(\vec{p}^2 + \alpha^2 \mu^2)^2} = 8\pi |\phi_C(0)| \frac{\pi \delta(\vec{p})}{2p^2}.$$

Using the formula $\delta(\vec{p}) = \frac{\delta(p)}{2\pi p}$, which is valid in the case of the spherical symmetry, we get finally

$$\lim_{\alpha \to 0} \phi_C(\vec{p}) = (2\pi)^3 |\phi_C(0)| \delta(\vec{p}).$$

It is clear from the form of the Coulomb WF (2.68) that the main contribution in the splitting of the energy levels gives the momentum region $\vec{p}^2 \sim Z^2 \alpha^2 \mu^2$. As a result, the integrand expansion over $p/m$ is equivalent to the integral expansion as a whole over $\alpha$ provided that the integral is finite.

This property proved to be highly useful in calculations of the terms up to $\alpha^5$ to the HFS of the ground state in a positronium from the diagrams [30]:

In the calculations of the matrix elements the upper ("big") components of bispinors survived and it became possible to equate $E^2 = m^2$, $\vec{p}, \vec{q} = 0$ in the interaction amplitude corresponding to these diagrams. In the calculations of the higher orders, the situation changes because of the singular behavior of the integrand expression in the small momentum region.

Taking the one-transverse-exchange diagram as an example let us consider the extraction of the contributions of the order $\alpha^2 \log \alpha$ to the Fermi energy of the hyperfine splitting $E_F = \frac{2}{\gamma} \frac{a^4 \mu^2}{m_1 m_2} <\hat{\sigma}_1 \hat{\sigma}_2 >$.

The expression of this correction has the form:

$$\Delta E_{1T}^{hfs} = \langle \Phi'_C | F^{-1} \left[G_0 \bar{K_T} G_0\right]^+ F^{-1} | \Phi'_C \rangle,$$

$$K_T = -\frac{4\pi \alpha \Gamma_{12}(\vec{k})}{(k_0^2 - \vec{k}^2 + i\epsilon)},$$

is a kernel corresponding to the one-transverse-photon exchange diagram,

$$\Gamma_{12}(\vec{k}) = \gamma_1 \gamma_2 - \frac{(\gamma_1 \vec{k})(\gamma_2 \vec{k})}{\vec{k}^2},$$

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and \( \Phi_c \), the WF in the second order of perturbation theory, can be substituted for \( \phi_C(\vec{p}) \) defined by Eq. (2.68) in the calculations up to precision we need, with \( \phi_C(\vec{p}) \) being the exact solution of the non-relativistic Schrödinger equation with the Coulomb potential for the 1S-state.

The analytical expression of the quantity \( \Delta E_{1T}^{hfs} \) is given by the equation:

\[
\Delta E_{1T}^{hfs} = -\frac{4\alpha^2\mu^2}{(2\pi)^6} \left| \phi_C(0) \right|^2 \int \frac{d\vec{p}d\vec{q}dk_0}{(p^2 + \alpha^2\mu^2)^2(q^2 + \alpha^2\mu^2)^2} F^{-1}(p)F^{-1}(q) \times \\
\times \int dk_0dk'_0 [S_1(p_1)S_2(p_2)K_T(k_0,k'_0;\vec{p},\vec{q})S_1(q_1)S_2(q_2)]^+ \delta(p_0 - k_0)\delta(k'_0 - q_0). \tag{3.10}
\]

Using the Fourier representation of the \( \delta \)-function and the residues theory, we find:

\[
\Delta E_{1T}^{hfs} = \frac{4\alpha^2\mu^2}{(2\pi)^3} \left| \phi_C(0) \right|^2 \int \frac{d\vec{p}d\vec{q}}{|\vec{p} - \vec{q}|} \left[ \frac{u_1^*(\vec{p})u_2^*(\vec{q})(\vec{p}\cdot\vec{q})^2}{(p^2 + \alpha^2\mu^2)^2(q^2 + \alpha^2\mu^2)^2} \right] \times \\
\times \int dte^{-i(p-q)t} \left\{ \vartheta(t)\vartheta(-t)(e^{i\epsilon_1t} + e^{i\epsilon_2t} - E - i\epsilon) + \vartheta(-t)e^{i\epsilon_1t}(e^{i\epsilon_1t} + e^{i\epsilon_2t} - E - i\epsilon) \right\} \tag{3.11}
\]

after integration over the variables \( p_0, q_0, k_0, k'_0 \).

Upon extracting the spin-spin interaction from the spin-structure (see the nominator of Eq. (3.11) and using a symmetry of spin-structure expressions with respect to the substitutions \( p_i \leftrightarrow q_j \), e. g.,

\[
\int d\vec{p}d\vec{q}(\vec{p}\cdot\vec{q})^2 = \frac{1}{3}(\overline{\sigma_1}\overline{\sigma_2}) \int d\vec{p}d\vec{q}(\vec{p}\cdot\vec{q})^2, \tag{3.12}
\]

the integral expression of \( \Delta E_{1T}^{hfs} \) can be written (the case of unequal masses \( m_1 \neq m_2 \)):

\[
\Delta E_{1T}^{hfs} = \frac{\alpha^2\mu^2}{3\pi^3} |\phi_C(0)|^2 \left| \overline{\sigma_1}\overline{\sigma_2} > \int d\vec{p}d\vec{q} \frac{1}{|\vec{p} - \vec{q}|^2 + \epsilon_1p + \epsilon_2q - E - i\epsilon} \right. \\
\times \left. \frac{1}{\epsilon_1p + \epsilon_2q} \Xi \left[ \frac{M_{1q}^+}{\epsilon_1p + \epsilon_2q} + \frac{M_{2q}^+}{\epsilon_1p + \epsilon_1q} \right] + \\
+ \frac{q^2(p^2 - q^2)^2}{|\vec{p} - \vec{q}|^2} \left[ \frac{M_{1q}^+}{\epsilon_2p + \epsilon_2q} \right] + \frac{(\vec{p} - \vec{q})^2}{\epsilon_1p + \epsilon_1q} \left[ \frac{M_{2q}^+M_{1q}^+}{M_{1q}^+M_{2q}^+} - \frac{M_{1q}^+M_{2q}^+}{M_{1q}^+M_{2q}^+} \right] - \\
\left. \frac{(\vec{p} - \vec{q})^2}{(\vec{p} - \vec{q})^2} \right] \right\} \tag{3.13}
\]

The following notation has been used above:

\[
\Xi = \frac{1}{\sqrt{\epsilon_1p\epsilon_2\epsilon_1q\epsilon_2qM_{1q}^+M_{2q}^+M_{1q}^+M_{2q}^+}} \tag{14.14}
\]

\( M_{ip}^+ = \epsilon_ip + m_i, \quad M_{iq}^+ = \epsilon_iq + m_i. \)

As a rule, it is possible to estimate the order in \( \alpha \) of each of integrals to the final value of the hyperfine shift before an integration. In Ref. [13] it has been shown that the logarithmic correction of the order \( \alpha^2\log\alpha \) to the Fermi energy appears from the integral

\[
I_{st}(\log\alpha) = \frac{1}{8\pi^2} \int \frac{dp}{\epsilon_1p\epsilon_2(p^2 + \alpha^2\mu^2)^2} \int \frac{dq}{(q^2 + \alpha^2\mu^2)(p^2 - q^2)^2} = \\
\int_0^\infty \frac{p \cdot dp}{\epsilon_1p\epsilon_2(p^2 + \alpha^2\mu^2)^2} \int_0^\infty \frac{q \cdot dq}{(q^2 + \alpha^2\mu^2)(p^2 - q^2)^2} \log \frac{p + q}{p - q} = \frac{\pi^2}{2m_1m_2} \log\alpha - 1 + O(\alpha), \tag{15.15}
\]

which is agreed to be called the "standard integral".
Additional powers of $p$ or $q$ in the nominator and the additional powers of the factors $(\vec{p}^2 + \alpha^2 \mu^2)$ or $(\vec{q}^2 + \alpha^2 \mu^2)$ in the denominator of the integrand expression lead to the contributions of the order $\alpha^6$ or $\alpha^5$, respectively.

The terms in the curly brackets, containing the multiplication $\vec{p}^2 \cdot \vec{q}^2$, lead to the "standard" integral in (3.13). Moreover, the difference of the factor $|\vec{p} - \vec{q}| + \epsilon_{1p} + \epsilon_{2q} - E$ from $|\vec{p} - \vec{q}|$ in the denominator of (3.13) proved to be essential. The main contribution of this term has the order $\alpha^4$, but the next order terms in the denominator expansion result in the "standard" integral.

The other diagrams also give the contributions of the order $\sim \alpha^6 \log \alpha$ to the HFS in a muonium. It is calculated analogously. The results of calculations, Ref. [31], are shown in Table I.

Table I. The contributions to the HFS in a muonium from the diagrams of Fig. 2

| No. | Contribution $K_i$ to $\Delta E_{Mu}^{HFS}$ | No. | Contribution $K_i$ to $\Delta E_{Mu}^{HFS}$ |
|-----|------------------------------------------|-----|------------------------------------------|
| a   | $1/4$                                    | e   | $3M$                                    |
| b   | $M + 2$                                  | f   | $-2(M + 2)$                             |
| c   | $9/2$                                    | g   | $5/4$                                   |
| d   | $-(M + 2)$                               | h   | $-M$                                    |
|     | Total                                    |     | 2                                        |

In the method based on the amplitude $T$, when the constituents are on the mass shell $p_1^0 = \epsilon_{1p}, p_2^0 = \epsilon_{2p}, q_1^0 = \epsilon_{1q}, q_2^0 = \epsilon_{2q}$, the problem of correct allowance for the retarding effects appears even at the stage of calculation of contributions from the one-transverse-photon exchange diagrams. Depending on the way of representation $\omega^2$ in the denominator of a photon propagator

$$D_{\mu} = -\frac{4\pi}{\omega^2 - k^2}(\delta_{\mu} - \frac{k_{\mu}}{k^2}),$$

(3.16)

the contribution of the order $\sim \alpha^6 \log \alpha$ is different (see Table II).

Table II. The contributions to the HFS in a two-fermion system from the one-photon exchange diagram ($\omega^2$ is zero component of the photon 4-momentum).

| $\omega^2$ | $\Delta E_{T}^{HFS}(\alpha^6 \log \alpha), m_1 \neq m_2$ | $\Delta E_{T}^{HFS}(\alpha^6 \log \alpha), m_1 = m_2$ |
|------------|-------------------------------------------------|-------------------------------------------------|
| 0          | $E_F \frac{\mu^2 \alpha^2}{m_1 m_2} M \log \alpha^{-1}$ | $\frac{1}{2} E_F \alpha^2 \log \alpha^{-1}$     |
| $[\epsilon_{1p} - \epsilon_{1q}]^2$ | $E_F \frac{\mu^2 \alpha^2}{m_1 m_2} (M - 2m_m) \log \alpha^{-1}$ | 0                                               |
| $[\epsilon_{2p} - \epsilon_{2q}]^2$ | $E_F \frac{\mu^2 \alpha^2}{m_1 m_2} (M - 2m_m) \log \alpha^{-1}$ | 0                                               |
| $[(\epsilon_{1p} - \epsilon_{1q})(\epsilon_{2q} - \epsilon_{2p})]$ | $E_F \frac{\mu^2 \alpha^2}{m_1 m_2} (M + 2) \log \alpha^{-1}$ | $E_F \alpha^2 \log \alpha^{-1}$                 |

Comparing the result shown in Table II with the result obtained by the first method, the two-time Green function method,

$$\Delta E_{T}^{hfs}(\alpha^6 \log \alpha) = E_F \frac{\mu^2 \alpha^2}{m_1 m_2} (m_1 + m_2 + 2) \log \alpha^{-1},$$

(3.17)

we are convinced that it is preferable to use the symmetric form of $\omega^2$ in the quasipotential continued analytically out of the energy shell ($|\vec{p}|^2 \neq |\vec{q}|^2$).
The diagrams contributing to the order of $\sim \alpha^6 \log \alpha$ to the HFS in a muonium.

In the higher order diagrams the situation is more complicated when the above-mentioned method of passing to the mass shell is used. Firstly, the way of symmetrization, that is what method should be used when going away from the energy shell, is not clear. Secondly, the problem of existence of additional singularities complicates essentially the calculations. A loop momentum integration, for instance, should be considered in the sense of the main value. The use of various expansions of integrand and the change of integration order are problematic in a situation like that.

Thus, the most correct allowance for retard ing effects is the use of the first method for building the quasipotential. However, in this case the problem of existence of anomalously large contributions $\sim \alpha^5 \log \alpha$ to the HFS of the ground state in a two-fermion system appears already at the stage of the one-photon exchange diagram. In the second method, this trouble does not occur. This problem is not a specific feature of the quasipotential approach but it is general for the relativistic bound state theory \[4\].

The diagrams contributing to the order $\sim \alpha^5 \log \alpha$ to the hyperfine splitting of the ground state in a positronium in the direct channel are shown in Fig. 3. The corresponding quasipotential is presented in Refs. \[15, 32\].
These contributions have already been mentioned to be caused by the infrared behavior of matrix elements of the quasipotential. The existence of the iteration terms for each of the reducible diagrams improves its behavior in the infrared region. This fact permits one to avoid summing the ladder diagrams in any selected order of $\alpha$. The parameter $\alpha \mu$ is found out to behave as the regularization factor of the infrared singularities when the calculations with the exact Coulomb WF (2.65) are carried out in the first variant of the quasipotential approach. The cancellation of these anomalous terms \cite{15} is displayed in the Table III.

Fig. 3. The diagrams considered in analysing the anomalous contributions of the order $\sim \alpha^5 \log \alpha$. 
Table III. The cancellation of the anomalous contributions of the order \( \alpha^5 \log \alpha \) to the HFS in a positronium.

| No. | \( \Delta E(\alpha^5 \log \alpha) \) | No. | \( \Delta E(\alpha^5 \log \alpha) \) |
|-----|-----------------------------------|-----|-----------------------------------|
| a   | 0                                 | e   | 0                                 |
| b   | \( \frac{2\alpha}{\pi} E_F \log \alpha \) | f   | \( -\frac{2\alpha}{\pi} E_F \log \alpha \) |
| c   | \( -\frac{\alpha}{2\pi} E_F \log \alpha \) | g   | 0                                 |
| d   | \( \frac{\alpha}{2\pi} E_F \log \alpha \) | h   | 0                                 |
|     | Total |     | 0                                 |

The new corrections to the hyperfine splitting of energy levels in a muonium have been obtained by Eides et al. \[33\]–\[35\]. In these articles, in particular, the corrections of the orders \( \sim \alpha (Z \alpha \frac{m}{m_e}) \) and \( \sim Z^2 \alpha (Z \alpha \frac{m}{m_e}) \) to the Fermi energy have been calculated by means of the method of the effective Dirac equation (EDE)\[7\]. Let us consider the selection of these contributions from the diagrams with radiative photons. The remarkable feature of these articles is the use of the Fried-Yennie gauge \[39\]–\[41\] for the photon propagator

\[
D_{\mu \nu} = \frac{1}{q^2 + i\epsilon} (g_{\mu \nu} + 2q_{\mu}q_{\nu}/q^2 + i\epsilon).
\] (3.18)

The infrared singularities are softened in this gauge. Any diagram with radiative corrections has a softer behavior near mass shell than the corresponding "skeleton" diagram. An attractive property of the Fried-Yennie gauge is the possibility of carrying out the renormalization procedure on mass shell without introduction of the unphysical photon mass \( \lambda \). This feature makes it easy to estimate integrals appearing in the problems of energy levels of the hydrogen-like atoms.

In the Fried-Yennie gauge the renormalization constant for the WF, \( Z_2 \), is infrared finite and the renormalized self-energy operator has a soft behavior on mass shell

\[
\Sigma_{FY}^{(R)}(p) = (\hat{p} - m)^2 (\frac{-3\alpha\hat{p}}{4\pi m^2})(1 + O(\rho)).
\] (3.19)

\[
\rho = \frac{m^2 - p^2}{m^2} \ll 1.
\]

This is a distinctive feature of the mentioned gauge from, e. g., the Feynman gauge

\[
\Sigma_{FY}^{(R)}(p) = (\hat{p} - m)\frac{\alpha}{\pi} \left[ \log \frac{\lambda}{m} - \log \rho + 1 \right],
\] (3.20)

\[
\frac{\lambda}{m} \ll \rho \ll 1.
\]

As for the vertex function, the term corresponding to the fermion anomalous magnetic moment:

\[
-\frac{\alpha}{2\pi} \sigma_{\mu \nu} \frac{k_{\nu}}{2m}.
\] (3.21)

has a most hard behavior. However, redefining the renormalized vertex operator by means of

\[
A_{\mu}(p_1, p_2) = \gamma_{\mu} A(0, 0) - \frac{\alpha}{2\pi} \sigma_{\mu \nu} \frac{k_{\nu}}{2m} + \Lambda_{\mu}^{(R)}(p_1, p_2)
\] (3.22)

\[7\] Recently, these authors have calculated the corrections of the order \( \alpha^2 (Z \alpha) E_F \). See Section IV and \[36\]–\[38\] for the details.
one gets the expression:

$$\Lambda^{(R)}_{\mu, FY} = -\rho \frac{3\alpha}{4\pi} \frac{3}{m^2} (\hat{p} - m) \hat{p}, \quad \rho \ll 1,$$

which is valid when the transferred momentum is zero and $\rho \ll 1$. This expression agrees with the self-energy operator asymptotics owing to the Ward identity. The contributions of the term according to the anomalous magnetic moment is analyzed separately.

Let us trace the selection of graphs for the calculation of the corrections of the order $\propto \alpha(Z\alpha) \frac{m_e}{m_\mu} E_F$. Five diagrams:

exhaust the contributions to the EDE kernel, connected with the mass operator.

Here,

$${\text{=}}$$

$${\text{S}}_0$$

$${\text{=}}$$

$$\Lambda_2 \cdot \text{S}$$

with $S_0$ being the free particle propagator; $g = \Lambda_2 \cdot S$ being the projector onto the muon mass shell multiplied by an electron propagator.
The simplest diagrams with the vertex correction:

also give contributions of the order \( \sim \alpha (Z\alpha \frac{m_e}{m_c}) \).

Moreover, there are diagrams with spanned many emitted photons:

and the diagrams of the second order of perturbation theory:
In analysing the graphs, entering into the EDE kernel, with the aim of finding the corrections of the order $\sim \alpha (Z\alpha) \frac{m_e}{m_\mu}$ to the Fermi energy, it has been determined that the contributions of this order come from the diagrams of the gauge-invariant set only (see Fig. 4). Moreover, it turned out to be possible to restrict oneself by the approximated WF in the matrix element. In other words, the matrix elements are to be calculated with taking into account the upper ("big") components of electron and muon spinors, neglecting momenta of wave functions inside the diagrams. These conditions were named the "standard conditions" by the authors of [33]-[35].

Fig. 4. The complete gauge-invariant set of the diagrams for the calculation of the recoil corrections of the order $\sim \alpha (Z\alpha) E_F$ and $\sim Z^2 \alpha (Z\alpha) E_F$ to the Fermi energy of the HFS in a muonium.

---

8The diagrams, where the radiative photon spans more than two exchange photons, don’t contribute to the terms of the order $\sim \alpha (Z\alpha) E_F$ in the FY gauge.
The complete gauge-invariant set, presented by Fig. 4, leads to the infrared and ultraviolet finite matrix element in a sum. Therefore, further calculations can be done using any convenient gauge both for the exchange photons and for the radiative photons. It has been deduced by direct calculations that the anomalous magnetic moment does not lead to the corrections of the order needed. It is considered for the exchange photons and for the radiative photons. It has been deduced by direct calculations that the correction diagrams turned out, Ref. [34], to be canceled by the vertex correction diagram in the order under consideration after building the EDE kernel and perturbation theory for finding the energy levels.

As a result, the following expression is obtained for the contribution from the diagrams of Fig. 4a:

\[
\delta E_\Sigma = \alpha (Z \alpha) \frac{m_e}{\pi^2} E_F \frac{3i}{8\pi^2 \mu^2} \int_0^1 dx \int_0^1 dy \int \frac{d^4k}{k^4} \left( \frac{1}{k^2 + \mu^{-1}k_0 + i\epsilon} + \frac{1}{k^2 - \mu^{-1}k_0 + i\epsilon} \right) \times \frac{1}{-k^2 + 2k_0 + a_1^2(x,y) - i\epsilon} \left[ h_1(x,y) \cdot k_0 - h_2(x,y) \cdot (k_0^2 - \frac{2}{3}k^2) \right] \equiv \delta E_{\Sigma 1} + \delta E_{\Sigma 2},
\]

where

\[
h_1(x,y) = \frac{1 + x}{y}, \quad h_2(x,y) = \frac{1 - x}{y} \left[ 1 - \frac{2(1 + x)}{x^2 + \lambda^2 y} \right],
\]

\[
a_1^2(x,y) = \frac{x^2 + \lambda^2}{(1 - x) y}.
\]

\( \lambda \) is a non-dimensional infrared mass of a radiative photon in the electron mass units.

The main contribution of the order 1/\( \mu \) to the integral comes from the residue in the muon pole, what corresponds to the muon motion on the mass shell. The leading infrared singularity, which is proportional to \( \lambda^{-1/2} \), is also connected with this residue; the other terms are logarithmic divergent only. Therefore, it is convenient to separate the calculation of the on-shell contributions and the rest of them.

\[
\delta E_\Sigma (m. s.) = \frac{1}{2\mu} (-2I_\lambda + \frac{11\pi^2}{6}) + (I_\lambda + \frac{23\pi^2}{24}),
\]

where

\[
I_\lambda = \frac{4\pi}{3} \int_0^1 dx \left( \frac{x}{x^2 + \lambda^2} \right)^{3/2} (1 - x)^{1/2} \sim \frac{1}{\lambda^{-1/2}}
\]

is the infrared divergent integral which is subtracted after summing the singular contributions of the diagrams Fig. 4a, 4b, 4c.

By means of a number of mathematical contrivances (see [34, 35]) like dividing the integration region into two parts, small and large momenta, after the integration over the angular variables, the subtraction of the pole contribution in the integrand and use of various identities, the expressions for the contributions from all the diagrams of the gauge-invariant set have been obtained, Fig. 4:

\[
\delta E_\Sigma = \alpha (Z \alpha) E_F \left[ \log 2 - \frac{13}{4} \right] + \frac{\alpha (Z \alpha) m_e}{\pi^2} m_\mu E_F \left[ \frac{15}{4} \log \frac{m_\mu}{m_e} + 6\zeta(3) + 3\pi^2 \log 2 + \frac{\pi^2}{2} + \frac{17}{8} \right].
\]
Another method, which gives the opportunity to calculate the logarithmic contributions of the orders \( \alpha^3 \frac{m_e}{m_\mu} \log^3 \frac{m_e}{m_\mu} \) to the HFS in a muonium, is the renormalization group method, Ref. \[12\].

The contribution of the radiative-recoil corrections is equal to \( \Delta E = -E_F R_\mu \), where the quantity \( R_\mu \) is calculated in the lowest approximation from the diagrams of two-photon exchange. In Ref. \[14\], the diagrams with the radiative insertions into an electron line, photon line and to an electron vertex have been considered. The following logarithmic contributions are known from the above paper:

\[
R_\mu^{(2)} = -\frac{3\alpha}{\pi} \frac{m_e}{m_\mu} \log \frac{m_e}{m_\mu} + \left( \frac{\alpha}{\pi} \right)^2 \frac{m_e}{m_\mu} \left[ 2 \log^2 \frac{m_e}{m_\mu} + \frac{31}{12} \log \frac{m_e}{m_\mu} \right],
\]

(3.29)

which arise owing to momentum integration in the region \( m_e^2 < k^2 < m_\mu^2 \), that is in the asymptotic region for the contribution of the electron vacuum polarization to the photon propagator. The muon loop does not give contribution under momentum integration in the region \( k^2 < m_\mu^2 \). The estimation of the term of the higher order of perturbation theory is given in Ref. \[42\].

If we consider some physical quantity \( R \), calculated by perturbation theory, the following condition is to be fulfilled:

\[
\frac{\partial R}{\partial \tau} = 0,
\]

(3.30)

where the variable \( \tau = -\beta_0 \log \frac{\mu}{\Lambda} \) characterizes the used renormalization scheme (RS), \( \mu \) is an arbitrary parameter of mass dimension, \( \Lambda \) is the scale parameter \[15\], \( \beta_0 = 2/3 \) is the first coefficient in the renormalization group equation with the running coupling constant \( g \),

\[
\mu \frac{\partial g}{\partial \mu} = \beta(g) = \beta_0 g^2 + \beta_1 g^3 + ...
\]

(3.31)

Accordingly, for the quantity \( R \) in the second order of perturbation theory which is written in the form:

\[
R^{(2)} = r_0 g(1 + r_1 g),
\]

(3.32)

we have:

\[
\frac{\partial R^{(2)}}{\partial \tau} = O(g^3).
\]

(3.33)

It follows from here that

\[
\frac{\partial r_0}{\partial \tau} = 0, \quad \frac{\partial r_1}{\partial \tau} = 1.
\]

(3.34)

Thus, we can see that \( r_0 \) does not depend on the selection of the RS and \( r_1 = \tau + \rho_1 \), where the constant \( \rho_1 \) can be calculated provided that \( r_1 \) is known for some of the RS.

The dependence of the next (third) term of perturbation theory on \( \tau \) is to be arranged so as to compensate the dependence \( R^{(2)} \) on \( \tau \) up to the order \( g^4 \),

\[
\frac{\partial (R^{(2)} + \Omega^{(2)} g^3)}{\partial \tau} = O(g^4).
\]

(3.35)

Consequently,

\[
\frac{\partial \Omega^{(2)}}{\partial \tau} = r_0(2r_1 + \frac{\beta_1}{\beta_0}).
\]

(3.36)

After integration using Eqs. (3.34) we get

\[
\Omega^{(2)} = r_0 r_1 (r_1 + \frac{\beta_1}{\beta_0}) + \text{const}.
\]

(3.37)
Then, to define an arbitrary integration constant it is necessary to set the "optimal" RS in which the quantity $R^{(2)}$ is the closest to $R$. In Ref. [42] it was given by the condition

$$\Omega^{(2)}(r_1, \tau) \big|_{\tau = \tau_{\text{opt}}} = 0, \quad \tau_{\text{opt}} = -\beta_0 \log \left( \frac{\mu_{\text{opt}}}{\Lambda} \right).$$

(3.38)

If we use the RS on the mass shell ($r_1 = K_1$) as the initial RS, then we have

$$\Omega^{(2)}(K_1) = r_0 K_1 (K_1 + \frac{\beta_1}{\beta_0}) - r_0 r_1^{opt}(r_1^{opt} + \frac{\beta_1}{\beta_0}), \quad r_1^{opt} = r_1(\tau_{\text{opt}}).$$

(3.39)

We obtain from (3.29) for a physical quantity $\Delta E$, the energy of the HFS in a muonium, the following expression:

$$r_0 = -3 \frac{m_e}{m_\mu} \log \frac{m_e}{m_\mu}, \quad K_1 = -\frac{2}{3} \log \frac{m_e}{m_\mu} - \frac{31}{36}.$$

(4.40)

Also, from (3.39) we have

$$\Omega^{(2)} = -\frac{4}{3} \frac{m_e}{m_\mu} \log^3 \frac{m_e}{m_\mu} + \frac{35}{18} \frac{m_e}{m_\mu} \log \frac{m_e}{m_\mu} + A \frac{m_e}{m_\mu} \log \frac{m_e}{m_\mu},$$

(3.41)

where

$$A = -\beta_0 \log \frac{m_\mu}{\mu_{\text{opt}}} \left( \frac{35}{36} - \beta_0 \log \frac{m_\mu}{\mu_{\text{opt}}} \right).$$

(3.42)

As a result, we have

$$\Delta E = E_F (\frac{\alpha}{\pi})^3 \frac{m_e}{m_\mu} \left[ \frac{4}{3} \log \frac{m_e}{m_\mu} + \frac{35}{18} \log \frac{m_e}{m_\mu} \right] \approx -0.04 \text{ kHz.}$$

(3.43)

The first term in (3.43) is equal to the result obtained in [33] by a direct calculation. The value of $A$ in (3.41) depends on the selection of the "optimal" scheme. Its contribution to the HFS can be estimated only as $0 < -\Delta E < 1 \text{ kHz.}$

The remarkable method of calculation of the logarithmic in $\alpha$ corrections to the spectra of the QED systems has been proposed in [47]. The logarithmic in $\alpha$ corrections are pointed out in these papers to appear from the logarithmic divergent integral, with the contribution being given by the momentum region

$$\mu \alpha \leq q \leq \mu$$

(3.44)

($\mu$ is the reduced mass, $\mu = \frac{mM}{m+mM}$). The lower limit is the characteristic momentum for QED bound states (see the discussion of the WF (2.68), the upper value corresponds to the limit of applicability of the nonrelativistic approximation.

A shift of the level with the quantum numbers $n, l$ has been calculated in [47] as the matrix element of the operator $\hat{V}(q)$:

$$\hat{V}^{(N)}(q) = [A + B(\bar{\sigma}_1 \bar{\sigma}_2)] \frac{\pi \alpha^N}{mM} \log \frac{\mu}{q}$$

(3.45)

with the WF’s which are similar to the ones presented by Eq. (3.1). A calculation of the operator $\hat{V}(q)$ is reduced in this approach to a calculation of the on-shell scattering amplitude from the diagrams of the order $\alpha^N$, if we are interested in the logarithmic corrections only.

The authors of the paper [47] have used the usual quantum-mechanical perturbation theory with the substitutions

$$\sum_i \left| \frac{i > < i}{E_i - E} \right|$$

(3.46)

in the intermediate states of an interaction operator.

---

10The coefficient of the term $\log^2 \frac{m_e}{m_\mu}$ has been overestimated in [46].
For instance, for the diagrams of the order $O(\alpha^3)$ with a pure Coulomb interaction the operator $\hat{V}$ is:

$$\hat{V} = -\int \frac{d\vec{k}}{(2\pi)^3} \int \frac{d\vec{k}'}{(2\pi)^3} \frac{4\pi \alpha^3}{k^2 k'^2 (\vec{k} - \vec{k}'^2)^2} \Lambda^+_1(\vec{k}) \Lambda^+_2(\vec{k}') \Lambda^+_2(-\vec{k}) \Lambda^+_1(-\vec{k}')$$

(3.47)

With the results from other diagrams of the Coulomb photons exchange this leads to the level shift

$$\delta E_C(n, l) = \frac{\mu^5}{m^2 M^2} \alpha^6 \log \frac{1}{\alpha^2} \frac{\delta_l n}{n^3}.$$  

(3.48)

The theoretical values of the decay width of $\omega - Ps$ and $p - Ps$ and of the energy levels of fine and hyperfine structure have been obtained by I. B. Khriplovich et al. Their results are discussed in the next section. They are compared with Fell’s result, Ref. [48] calculated on the basis of the relativistic two-particle equations.

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ENERGY LEVELS OF HYDROGEN-LIKE ATOMS AND FUNDAMENTAL CONSTANTS. Part II. ⋆,†

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Abstract

The present review includes the description of theoretical methods for the investigations of the spectra of hydrogen-like systems. Various versions of the quasipotential approach and the method of the effective Dirac equation are considered. The new methods, which have been developed in the eighties, are described. These are the method for the investigation of the spectra by means of the quasipotential equation with the relativistic reduced mass and the method for a selection of the logarithmic corrections by means of the renormalization group equation. The special attention is given to the construction of a perturbation theory and the selection of graphs, whereof the contributions of different orders of $\alpha$, the fine structure constant, to the energy of the fine and hyperfine splitting in a positronium, a muonium and a hydrogen atom could be calculated.

In the second part of this article the comparison of the experimental results and the theoretical results concerning the wide range of topics is produced. They are the fine and hyperfine splitting in the hydrogenic systems, the Lamb shift and the anomalous magnetic moments of an electron and a muon. Also, the problem of the precision determination of a numerical value of the fine structure constant, connected with the above topics, is discussed.

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4 Comparison of theoretical and experimental results

4.1 Decay rate of a positronium

Quantum-electrodynamic systems, consisting of a particle and an anti-particle, have specific features. Apart from a scattering channel an annihilation channel appears in this case. A positronium atom, which is a specimen of these systems, has no stability. The life time of a positronium (or the decay rate) is the subject of precise experimental and theoretical investigations. The charge parity of a positronium, \( C = (-1)^{L+S} \) (\( L \) is the eigenvalue of an angular momentum operator, \( S \) is the eigenvalue of a total spin operator for the system under consideration), is a motion constant. Consequently, all its states are separated into the charge-even states (\( S = 1 \)) and the charge-odd states (\( S = -1 \)). The positronium total spin is also conserved and the energy levels are classified as singlet levels (\( S = 0 \), a parapositronium) and triplet levels (\( S = 1 \), an orthopositronium). The \( S \)-state (\( L = 0 \)) parapositronium has a positive parity and the \( S \)-state orthopositronium has a negative parity. As a consequence of conservation of a charge parity in electromagnetic interactions a parapositronium disintegrates into the even number of photons and an orthopositronium into the odd ones.

At present, essential disagreement exists between the theoretical and experimental values for the decay rate of an orthopositronium. The theoretical predictions are \[ \Gamma_{\text{theor}}^{(o-Ps)} = \frac{\alpha^6 m c^2}{\hbar} \frac{2(\pi^2 - 9)}{9\pi} \left[ 1 - A_3\frac{\alpha}{\pi} - \frac{1}{3} \alpha^2 \log \alpha^{-1} + B_3\left(\frac{\alpha}{\pi}\right)^2 + \ldots \right] = \Gamma_0 + \frac{m\alpha^7}{\pi^2} \left\{ \left( -1.984(2) \right) \right\} + \frac{m\alpha^8}{\pi} \log \alpha^{-1} \left[ -\frac{4}{9} \zeta(2) + \frac{2}{3} + \frac{m\alpha^8}{\pi^3} \right] + \ldots \] (4.1)

where

\[ A_3^{[51]} = -10.266 \pm 0.011, \]
\[ A_3^{[52]} = -10.282 \pm 0.003. \] (4.2)

The last experimental measurements are \[ \Gamma_{\text{exp}}^{[53]}(o-Ps) = 7.0514(14) \text{ \(\mu s\)}^{-1}, \]
\[ \Gamma_{\text{exp}}^{[54]}(o-Ps) = 7.0482(16) \text{ \(\mu s\)}^{-1}. \] (4.4)

The result of Ref. [53] has 9.4 standard deviation from the predicted theoretical decay rate and the result of Ref. [54] has 6.2 standard deviation. The coefficient \( B_3 = 1 \) in \( O(\alpha^8) \) term can contribute \( 3.5 \times 10^{-5} \text{\(\mu s\)}^{-1} \) (or 5 ppm of \( \Gamma_3 \)) only. To remove the above disagreement the coefficient \( B_3 \) should be equal to about \( \simeq 250 \pm 40 \). It is very unlikely, indeed, but this opportunity has been pointed out in Ref. [54] and cannot be rejected a priori. The calculation of the \( B_3 \) coefficient is very desirable now.

For the first time, the main contribution in the orthopositronium decay rate has been calculated in Ref. [49]:

\[ \Gamma_0(o-Ps) = -2Im(\Delta E_{3\gamma}) = \frac{2}{9\pi}(\pi^2 - 9)m\alpha^6 = 7.211 17 \text{ \(\mu s\)}^{-1}. \] (4.6)

The corrections of the \( O(\alpha) \) order to this quantity have been calculated in a numerical way at first, but later some of them have been found analytically, Refs. [47],[49]-[51], by using the

\[^1\]See Table IV for the previous experimental results.

\[^2\]Some estimations of the corrections of this order have been done in Refs. [47],[49].
Feynman gauge. The corrections arising from the diagrams with self-energy and vertex insertions have been calculated by Adkins\cite{59,60}.

\[
\Gamma_{OV} = \Gamma_0 \alpha \pi \left\{ D + \frac{3}{4(\pi^2 - 9)} \left[ -26 - \frac{115}{3} \log 2 + \frac{91}{18} \zeta(2) + \frac{443}{54} \zeta(3) + \frac{3419}{108} \zeta(2) \log 2 - R \right]\right\} = \Gamma_0 \alpha \pi \left[ D + 0.732 986 380 \right],
\]

(4.10)

where

\[
R = \int_0^1 dx \frac{\log(1 - x)}{2 - x} \left[ \zeta(2) - Li_2(1 - 2x) \right] = -1.743 033 833 7(3),
\]

(4.11)

and

\[
D = \frac{1}{2 - w - \gamma_E + \log(4\pi)}
\]

(4.12)

is the standard expression of a dimensional regularization ($2\omega$ is the space dimension.). The above results is co-ordinated with Stroscio's result\cite{58} when

\[
\Gamma_0 \alpha \pi \left[ -D - 4 - 2\log(\lambda^2/m^2) \right]
\]

(4.13)

is added to the last one. This is necessary to do because of different regularization procedures used in\cite{58} and\cite{59,60}, respectively.

Recently, calculations of these corrections have been completed, Ref.\cite{61}, in the Fried – Yennie gauge

\[
\Gamma_{SE} = \frac{ma^7}{\pi^2} \left[ -\frac{13}{54} \zeta(3) + \frac{461}{108} \zeta(2) \log 2 - \frac{251}{72} \zeta(2) - \frac{29}{6} \log 2 - \frac{9}{2} \right] = \frac{ma^7}{\pi^2}(-0.007 132 904) = \Gamma_0 \alpha \pi (-0.036 911 113),
\]

(4.14)

\[
\Gamma_{OV} = \frac{ma^7}{\pi^2} \left[ -\frac{88}{54} \zeta(3) - \frac{299}{216} \zeta(2) \log 2 + \frac{49}{18} \zeta(2) + \frac{13}{6} \log 2 - 2 - \frac{1}{6} R \right] = \frac{ma^7}{\pi^2} \left( 0.732 986 380 \right) = \Gamma_0 \alpha \pi (3.793 033 599).
\]

(4.15)

The contributions from the remained diagrams, with a radiative insertion in a vertex of an internal photon; with two vertices spanned by radiative photon; the diagram taking into account binding effects
and the annihilation diagram (see Fig. 1 in [31]), have been calculated numerically. As a sum the $O(\alpha)$ corrections are jointed to give

$$\frac{ma^7}{\pi^2} [-1.987 84(11)] = \Gamma_0 \frac{\alpha}{\pi} [-10.286 6(6)].$$

(4.17)

Then

$$\Gamma_{3}^{\text{theor}}(o - Ps) = 7.038 \, 236(10) \mu s^{-1}.$$  

(4.18)

The above result is the most precise theoretical result at present.

To solve the existing disagreement between theory and experiment, the 5-photon mode of $o - Ps$ decay and the 4-photon mode of $p - Ps$ decay have been under consideration in [62, 63]. The following theoretical evaluations were obtained:

$$\Gamma_{2}^{\text{theor}}(o - Ps) = 0.177(\frac{\alpha}{\pi})^2 \simeq 0.96 \cdot 10^{-6},$$

(4.19)

$$\Gamma_{2}^{\text{theor}}(p - Ps) = 0.274(\frac{\alpha}{\pi})^2 \simeq 1.48 \cdot 10^{-6},$$

(4.20)

and

$$\Gamma_{3}^{\text{theor}}(o - Ps) = 0.0189(11)\alpha^2 \Gamma_0,$$

(4.21)

$$\Gamma_{3}^{\text{theor}}(p - Ps) = 0.01389(6)ma^7.$$  

(4.22)

They are in agreement with one another and with the results of the previous papers [64].

$$\Gamma_{4}^{\text{theor}}(p - Ps) = 0.01352 \, ma^7 = 11.57 \cdot 10^{-3} s^{-1}.$$  

(4.23)

In the connection of the present situation concerning the decay rate, investigations of alternative decay modes for this system (e. g., $o - Ps \rightarrow \gamma + a$, $a$ is an axion, a pseudo-scalar particle with mass $m_a < 2m_e$) are of present interest, Refs. [66]-[71]. In the paper [69], the following experimental limits of the branching of the decay width have been obtained:

$$Br = \frac{\Gamma(o - Ps \rightarrow \gamma + a)}{\Gamma(o - Ps \rightarrow 3\gamma)} < 5 \cdot 10^{-6} - 1 \cdot 10^{-6} \quad (30 \, \text{ppm}),$$

(4.24)

provided that $m_a$ is in the range 100 – 900 keV. In the case of the axion mass less than 100 keV (This is implied by Samuel’s hypothesis [70]. According to it $m_a < 5.7 \, keV, g_{ae+e^-} \sim 2 \cdot 10^{-8}$) the limits of $Br$ are the following [71]:

$$Br = 7.6 \cdot 10^{-6}, \quad \text{if} \quad m_a \sim 100 \, keV,$$

(4.25)

$$Br = 6.4 \cdot 10^{-5}, \quad \text{if} \quad m_a < 30 \, keV.$$  

(4.26)

These limits are about 2 orders of magnitude less than the value which is necessary to remove the disagreement.

13The uncalculated yet $O(\alpha^8)$ corrections are not taken into account here.

14As a consequence of conservation of an angular momentum and isotropic properties of the coordinate space, an orthopositronium has to decay into the odd number of photons and a parapositronium into the even ones, see above.

15The result of Ref. [65] is not correct, four times less than the above cited results. The explanation of this was given in [65].

16The proposed values don’t cause the contradictions in comparing theoretical and experimental results of the anomalous magnetic moment (AMM) of an electron.
Finally, a decay $o - Ps \rightarrow nothing$ (that is into weak-interacting non-detected particles) has been investigated in Ref. [73]. The obtained result

$$\frac{\Gamma(o - Ps \rightarrow nothing)}{\Gamma(o - Ps \rightarrow 3\gamma)} < 5.8 \cdot 10^{-4} \quad (350 \text{ ppm})$$

(4.27)

excludes an opportunity that this decay mode is the cause of disagreement between theory and experiment.

The decay of $o - Ps$ into two photons which breaks the CP- invariance, as has been mentioned in [74, 75], was experimentally rejected in [76].

It should be marked that the contribution of a weak interaction has been studied in [78]. However, because of the factor $m_e^2/M_W^2$ it cannot influence the final results. In the cited articles the weak decay modes have been estimated as

$$\frac{\Gamma(p - Ps \rightarrow 3\gamma)}{\Gamma(p - Ps \rightarrow 2\gamma)} \simeq \frac{\Gamma(o - Ps \rightarrow 4\gamma)}{\Gamma(o - Ps \rightarrow 3\gamma)} \simeq \alpha (G_F m_e^2 g_V)^2 \simeq 10^{-27},$$

(4.29)

where $G_F$ is the Fermi constant for a weak interaction,

$$g_V = 1 - 4\sin^2\Theta_W \simeq 0.08,$$

(4.30)

$\Theta_W$ is the Weinberg angle. The present experimental limits are [79, 81, 82]

$$\frac{\Gamma(p - Ps \rightarrow 3\gamma)}{\Gamma(p - Ps \rightarrow 2\gamma)} \leq 2.8 \cdot 10^{-6},$$

(4.31)

$$\frac{\Gamma(o - Ps \rightarrow 4\gamma)}{\Gamma(o - Ps \rightarrow 3\gamma)} \leq 8 \cdot 10^{-6}.$$  

(4.32)

In Table IV all experimental results for the $o - Ps$ decay rate, known to us, are presented.

Table IV.

| Year | Reference | $\Gamma_3(o - Ps), \mu s^{-1}$ | Error, ppm | Technique |
|------|-----------|-------------------------------|------------|-----------|
| 1968 | [84]      | 7.262(15)                     | 2070       | gas       |
| 1973 | [85]      | 7.262(15)                     | 2070       | gas       |
| 1973 | [86]      | 7.275(15)                     | 2060       | gas       |
| 1976 | [87]      | 7.104(6)                      | 840        | powder $SiO_2$ |
| 1976 | [88]      | 7.09(2)                       | 2820       | vacuum    |
| 1978 | [89]      | 7.056(7)                      | 990        | gas       |
| 1978 | [90]      | 7.045(6)                      | 850        | gas       |
| 1978 | [91]      | 7.050(13)                     | 1840       | vacuum    |
| 1982 | [92]      | 7.122(12)                     | 1680       | vacuum    |
| 1987 | [93]      | 7.051(5)                      | 710        | gas       |
| 1987 | [53]      | 7.031(7)                      | 1000       | vacuum    |
| 1989 | [54]      | 7.0516(13)                    | 180        | gas       |
| 1990 | [55]      | 7.0514(14)                    | 200        | gas       |
| 1990 | [56]      | 7.0482(16)                    | 230        | vacuum    |

Regarding the results for the decay rate of a parapositronium, the situation was highly favorable until

\[\text{Like that Glashow [72] spoke out the hypothesis of the decay into invisible "mirror" particles.}\]

\[\text{The physics ground of these speculations is possible existence of an unisotropic vector field with non-zero vacuum expectation [77], with which an electron and a positron could interact}\]

\[\mathcal{L} = g\bar{\psi}O_{\alpha\beta}\psi A^\alpha \Omega^\beta,\]

(4.28)

\[\mathcal{L}\] is the interaction Lagrangian.

\[\text{The results of the papers [81, 82] and [83] could be accounted as rough estimations.}\]
the last time. The theoretical value, found out as early as in the fifties, Refs. [96, 97], is

\[ \Gamma_{\text{theor}}^2(p - Ps) = -2Im(\Delta E_{2\gamma}) = \frac{1}{2} \alpha^5 \frac{mc^2}{\hbar} \left[ 1 - \frac{\alpha}{\pi} \left( 5 - \frac{\pi^2}{4} \right) \right] = 7.9852 \text{ ns}^{-1}, \quad (4.33) \]

The above value, confirmed in [98, 99], coincides with the direct experimental result with good accuracy

\[ \Gamma_{\text{exp}}^2(p - Ps) = 7.994 \pm 0.011 \text{ ns}^{-1}. \quad (4.34) \]

The experimental values of the parapositronium decay rate are shown in Table V.

**Table V.**

| Year | Reference | \( \Gamma_2(p - Ps), \text{ ns}^{-1} \) | Error, % | Technique |
|------|-----------|--------------------------------|---------|-----------|
| 1952 | [100]     | 7.63(1.02)                     | 13      | gas       |
| 1954 | [101]     | 9.45(1.41)                     | 15      | gas       |
| 1970 | [102]     | 7.99(11)                       | 1.38    | gas       |
| 1982 | [103]     | 7.994(11)                      | 0.14    | gas       |

In Refs. [51, 99], it has been pointed out that it is necessary to add the logarithmic corrections to \( \alpha \) to Harris and Brown result. In the paper [47] these corrections to \( \Gamma_3(o - Ps) \) and \( \Gamma_2(p - Ps) \) have been re-calculated, with the result of the decay rate of a parapositronium differing from the one found out before, Refs. [51, 99]:

\[ \Gamma_{\text{47}}^2(p - Ps, \alpha \log \alpha) = \frac{m \alpha^5}{2} \cdot \frac{2 \alpha^2 \log \alpha}{\alpha}, \quad (4.35) \]

\[ \Gamma_{\text{51, 99}}^2(p - Ps, \alpha^2 \log \alpha) = \frac{m \alpha^5}{2} \cdot \frac{2 \alpha^2 \log \alpha}{3 \alpha}. \quad (4.36) \]

Finally, we would like to mention a quite unexpected result, presented in Remiddi’s (and collaborators) talk [103]. The calculations carried out by them lead to the additional contribution

\[ \Gamma_{\text{103}}^2(p - Ps, \alpha \log \alpha) = \frac{m \alpha^5}{2} \left( \frac{\alpha}{\pi} \right)^2 \log \alpha, \quad (4.37) \]

which is explained by them to appear as a result of taking into account the dependence of the interaction kernel on the relative momenta (see, e. g., Fig. 1).

The above-mentioned leads to necessity to continue calculations of the decay rates of an orthopositronium as well as a parapositronium using more accurate relativistic methods, e. g., the quasipotential approach.

### 4.2 Hyperfine splitting

#### 4.2.1 Positronium

Comparison of theoretical and experimental results of the hyperfine splitting of the ground state of a positronium and a muonium was considered for a long time as correction of our understanding.

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\(^{20}\)The branching of the decay rates of a para- and an orthopositronium, \( \Gamma_2(p - Ps) \), had been measuring in the experiments of 1952 and 1954. The presented results are recalculated by means of the first direct experimental value, Ref. [84]. \( \Gamma_3(o - Ps) = 7.262(15) \text{ \mu s}^{-1} \).

\(^{21}\)It also deserves an attention The new approach to the positronium lifetime calculation, proposed by A. A. Pivovarov, Ref. [104], also deserves some attention.
the bound state problem. At first, the quantity of this splitting was estimated in a positronium as $9.4 \cdot 10^{-4}$ $\pm$ $1.4 \cdot 10^{-4}$ $eV$ in 1951, Ref. [105]. The last measured result is, Ref. [106]:

$$\Delta E_{hfs}^{exp}(Ps) = 203.389.10 \pm 0.74 \, MHz \quad (3.6 \, ppm).$$

In Table VI all published values of precise experimental measurements of the hyperfine splitting of the ground state of a positronium are presented.

Table VI.

| Year | Reference | $\Delta E, \, GHz$ | Error, ppm |
|------|-----------|-------------------|------------|
| 1952 | [107]     | 203.2(3)          | 1 500      |
| 1954 | [108]     | 203.350(50)       | 250        |
| 1955 | [109]     | 203.380(40)       | 200        |
| 1957 | [110]     | 203.330(40)       | 200        |
| 1970 | [102]     | 203.403(12)       | 60         |
| 1972 | [111]     | 203.396(5)        | 24         |
| 1975 | [112]     | 203.387 0(16)     | 8          |
| 1975 | [113]     | 203.384 9(12)     | 6          |
| 1977 | [114]     | 203.3847(16)      | 8          |
| 1978 | [115]     | 203.387 5(16)     | 8          |
| 1983 | [116]     | 203.389 10(74)    | 3.6        |

All the conducted experiments are based on the techniques using an observation of Zeeman transitions in $Ps$ and the further substitution of the results into the well-known Breit-Rabi equation. $\Delta E$, the energy of the HFS, is deduced from it.

At present, the theoretical result determined [4, 31, 51], [116]–[120] is

$$\Delta E_{hfs}^{theor}(Ps) = m\alpha^4 \left[ \frac{7}{12} - \frac{\alpha}{\pi} \left( \frac{8}{9} + \log \frac{2}{2} \right) + \frac{5}{24} \alpha^2 \log \alpha - 1 + O(\alpha^2) \right] \approx 203.400.3 \, MHz. \quad (4.39)$$

The coefficient 1 of the term of an order $\sim \alpha^6$ can contribute $\approx 18.7 \, MHz$ to the energy of the HFS. The estimated uncertainty is almost 50 ppm, an order of magnitude greater than the experimental one. After calculations of the corrections of an order $O(\alpha^2)$ and $O(\alpha^3 \log \alpha)$ to the Fermi energy the theoretical error would decrease to 1 ppm. The work in this direction has been started since the seventies [44], [121]–[130].

The first contributions of this order have been calculated from the diagrams of a one-photon annihilation, Ref. [122],

$$\Delta E_{hfs}^{1}\gamma = \frac{\alpha^4 R_{\infty}}{\pi^2} \left( \frac{3}{4} \zeta(3) - \frac{1}{3} \zeta(2) log 2 - \frac{1}{6} \zeta(2) - 4 log 2 + \frac{3}{2} - i \pi \left( \frac{4}{3} \zeta(2) - 2 \right) \right). \quad (4.40)$$

The contribution of a three-photon annihilation [123], which has also been calculated analytically, has recently been corrected, Ref. [124],

$$\Delta E_{hfs}^{2}\gamma = \frac{\alpha^4 R_{\infty}}{\pi^2} \left( 1 + \frac{35}{9} \pi^2 + \frac{41}{4} + \pi^2 log 2 - \frac{85}{4} \zeta(3) - i \pi (5 - \frac{\pi^2}{4}) \right). \quad (4.41)$$

This expression contributes numerically the small value $Re(\Delta E_{3}\gamma) = -0.966 \, MHz$. It has been confirmed by the authors of [123] in Ref. [124].

The analytical expression of the contribution from the two-photon annihilation diagram is

$$\Delta E_{hfs}^{2}\gamma = -\frac{\alpha^4 R_{\infty}}{2\pi^2} \left[ 1 + \frac{35}{9} \pi^2 + \left( \frac{41}{4} + \pi^2 log 2 - \frac{85}{4} \zeta(3) - i \pi (5 - \frac{\pi^2}{4}) \right) \right].$$

22The only contribution coming from electron-positron loops is essential in the case of positronium because of the smallness of $m_e$.
At present, it gives the greatest contribution $Re(\Delta E_{2\gamma}) = -13.13 \text{ MHz}$ numerically as compared to other corrections of this order.

It is to be mentioned that in calculating corrections of the order $\sim \alpha^6$ different authors used different approximate methods and, consequently, the comparison of their results is rather difficult. So, the authors of Ref. [128] investigated the case of the static interaction kernel, which is the fourth component of the vector potential independent of the relative times (the method of an effective potential). They found the result up to the third order of perturbation theory

$$\Delta E_{III} = \frac{1}{12} \alpha^4 R_\infty \left[ -\frac{1}{2}(1 + \log \frac{1}{2} \alpha) + \frac{2}{\pi} G + \frac{1}{\pi} (1 - 4F) \right] = \alpha^4 R_\infty \left[ \frac{1}{24} \log \alpha^{-1} + 0.031 \right] = (1.9 + 0.3) \text{ MHz},$$

(4.43)

where

$$G = \int_0^1 \frac{\tanh p}{p} dp = 0.91596\ldots$$

(4.44)

is the Catalan constant;

$$F = \int_0^1 \frac{(\tanh p)^2}{p} dp = 0.3897\ldots$$

(4.45)

In [44, 131], the contributions of various diagrams have been calculated by numerical methods. The importance of finding the contributions $\sim \alpha^6$ from the diagrams uncalculated until now was pointed out (see Fig. 7 in Ref. [14]).

For the completeness let us mention the recent calculation, Ref. [131] of the contributions from a weak interaction to the HFS of the ground level of positronium. The obtained result, $\Delta E_{\text{weak}} = 4.76 \times 10^{-14} eV = 1.15 \times 10^{-5} \text{ MHz}$, is far from the present experimental precision.

### 4.2.2 Muonium

In the previous reviews [35, 44], [135]–[137] the following theoretical result for the HFS of the ground state in a muonium was given:

$$\Delta E_{\text{HFS}}^{\text{theor}}(Mu) = E_F (1 + a_\mu) \left[ 1 + a_e + \frac{3}{2} (Z \alpha)^2 + \epsilon + \frac{\delta_\mu}{1 + a_\mu} \right],$$

(4.46)

with

$$\epsilon = \alpha (Z \alpha) \left( \log 2 - \frac{5}{2} \right) - \frac{8 \alpha (Z \alpha)^2}{3 \pi} \log (Z \alpha) \left[ \log (Z \alpha) - 4 + \frac{281}{480} \right] +$$

23The first calculations have been done some years ago [132, 134].
\[ + \frac{\alpha(Z\alpha)^2}{\pi} \cdot (15.38(29)) + \frac{\alpha^2(Z\alpha)}{\pi} D, \]  
\[ \delta_\mu = - \frac{3(Z\alpha)}{\pi} \cdot \frac{m_e m_\mu}{m_\mu^2 - m_e^2} \log \frac{m_\mu}{m_e} + (Z\alpha)^2 \frac{m_e m_\mu}{(m_e + m_\mu)^2} [-2\log(Z\alpha) - 
- 8\log 2 + \frac{31}{18}], \]  

(4.47)  

(4.48)

The classical works\[138\]–\[140]\ were devoted to calculations of the non-recoil corrections without taking into account of finiteness of mass of a heavy particle. In Refs.\[144\],\[130\] the result for the corrections of the order \( \alpha(Z\alpha)E_F \) has been confirmed and the contribution of the order \( \alpha(Z\alpha)^2E_F \) has been obtained numerically. The leading recoil corrections have been calculated by many authors\[25\],\[119\],\[120\],\[135\],\[141\]–\[143\]. In Refs.\[34\],\[35\] the contributions of the diagrams with radiative insertions in electron and muon lines, which depend on \( \frac{m_e}{m_\mu} \), the mass ratio, have been calculated analytically\[34\].

\[ \delta_\mu(m_e/m_\mu; \text{electron line}) = \frac{\alpha(Z\alpha)}{\pi^2} \frac{m_e}{m_\mu} \left[ \frac{15}{4} \log \frac{m_\mu}{m_e} + 6\zeta(3) + 3\pi^2\log 2 + \frac{\pi^2}{2} + \frac{17}{8} \right], \]  

\[ \delta_\mu(m_e/m_\mu; \text{muon line}) = \frac{Z^2\alpha(Z\alpha)}{\pi^2} \frac{m_e}{m_\mu} \left[ \frac{9}{2} \zeta(3) - 3\pi^2\log 2 + \frac{39}{8} \right]. \]  

(4.49)  

(4.50)

In Ref.\[144\], the above-mentioned result has been confirmed for the contributions of an electron line. The calculations were carried out in the Fried–Yennie gauge for radiative photons.

The contributions of the vacuum polarization diagrams have been calculated earlier, Refs.\[143\],\[146\]

\[ \delta_\mu(m_e/m_\mu; \text{v.p.}) = \left( \frac{\alpha}{\pi} \right)^2 \frac{m_e}{m_\mu} \left[ -2\log^2 \left( \frac{m_\mu}{m_e} \right) - \frac{8}{3} \log \left( \frac{m_\mu}{m_e} \right) - \frac{28}{9} - \frac{\pi^2}{3} \right]. \]  

(4.51)

The leading logarithmic corrections with respect to \( m_e/m_\mu \) have been considered in\[42\],\[46\],\[24\]

\[ \delta_\mu(m_e/m_\mu \log \frac{m_\mu}{m_e}) = - \frac{\alpha^2(Z\alpha)}{\pi^4} \cdot \frac{m_e}{m_\mu} \left[ \frac{4}{3} \log^2 \left( \frac{m_\mu}{m_e} \right) - \frac{4}{3} \log \left( \frac{m_\mu}{m_e} \right) \right]. \]  

(4.52)

Moreover, the program of calculation of the pure radiative corrections of the order \( \alpha^2(Z\alpha)E_F \) (that is of the \( D \) coefficient in\[4.47\]) will be finished soon\[24\], Refs.\[38\],\[37\]. The \( \alpha^2(Z\alpha)E_F \) corrections, induced by the diagrams with the insertions of vacuum polarization loops into external photons (see Figs. 1a–1c in the cited paper) have been calculated in\[38\]–a

\[ \Delta E(\alpha^2(Z\alpha)) = \frac{\alpha^2(Z\alpha)}{\pi} E_F \left\{ - \frac{4}{3} \log^2 \frac{1 + \sqrt{5}}{2} - \frac{20}{9} \sqrt{5} \log \frac{1 + \sqrt{5}}{2} + \frac{608}{45} \log 2 + \frac{\pi^2}{9} - \frac{38}{15} + \frac{91639}{37800} \right\} \approx -2.23 \frac{\alpha^2(Z\alpha)}{\pi} E_F \approx \left\{ \begin{array}{c} -1.2 \text{kHz for } M\mu \\ -0.34 \text{kHz for } H \end{array} \right\}; \]  

(4.53)

induced by the diagrams with the insertions of vacuum polarization loops into radiative photons (see Fig. 1 in the cited paper) have been calculated in\[38\]–b,

\[ \Delta E(\alpha^2(Z\alpha)) = \frac{\alpha^2(Z\alpha)}{\pi} E_F \left\{ - \frac{149}{270} + \frac{2}{9\pi} \int_0^1 dq \cdot D(q) \left[ \frac{3}{1 + q} \arctg\sqrt{\frac{2q}{1-q}} + \right. \right. \]

\[ + \frac{1}{1 + q} \arctg\sqrt{\frac{2q}{1-q}} \right\]. \]

24 Originally, these corrections have been found by numerical methods, Ref.\[130\].

25 The result of the paper\[12\], obtained by the technique of the preceding section, has been completed in\[40\].

26 Recently, Ref.\[38\], Kinoshta has presented the preliminary result of calculations of the last remained diagram of the order \( \alpha^2(Z\alpha) \) (with cross virtual photons). It is \( \Delta E(\alpha^2(Z\alpha)) \approx (-0.64 \pm 0.06) \frac{\alpha^2(Z\alpha)}{\pi} E_F = -0.353(33) \text{ kHz for } M\mu \), what gives the opportunity to reduce the theoretical error in\[4.53\] to 0.17 kHz.
\[ D(q) \] is the total elliptic integral; induced by the diagrams with the insertions of the light-to-light scattering sub-diagrams (see Fig. 1 in the cited paper) have been calculated in Ref. 27–c,

\[ \Delta E(\alpha^2(Z\alpha)) \simeq -0.482 13 \ldots \frac{\alpha^2(Z\alpha)}{\pi} E_F \simeq \left\{ \begin{array}{l}
-0.26 \text{ kHz for } Mu \\
-0.084 \text{ kHz for } H
\end{array} \right\}. \]  

Let us mention the first calculated recoil corrections of the second order in \( \frac{m_e}{m_\mu} \), Ref. 147, the corrections of the \( (Z\alpha)^2 E_F \) and \( \frac{m_e}{m_\mu}(Z\alpha)^2 E_F \) orders, calculated in the quasipotential approach, Ref. 22,

\[ \Delta E = E_F \left\{ 1 + (Z\alpha)^2 \left[ \frac{3}{2} + \frac{m_e m_\mu}{(m_e + m_\mu)^2} \left( \frac{19}{2} - \frac{1}{72} - \frac{2}{3\pi^2} \right) \right] \right\}; \]  

the contributions of the hadron vacuum polarization \[ \delta_p^{\text{hadrons}} = 3.7520 \pm 0.2373 \left( \frac{\alpha}{\pi} \right)^2 \frac{m_e m_\mu}{m_\mu^2} \simeq 0.250 \pm 0.016 \text{ kHz}; \]  

and the estimations of the weak interaction contributions, Refs. 132, 133:

\[ \Delta E(\text{weak int.}) = \frac{3}{4\sqrt{2\pi}}(Z\alpha)^{-1}G_F m_e m_\mu E_F \simeq 0.065 \text{ kHz}. \]  

As a result of inclusion of the above contributions, the theoretical predictions for the HFS of the ground state in a muonium is

\[ \Delta E_{\text{hfs}}^{\text{theor}}(Mu) = 4463303.0(0.2)(1.3)(0.6) \text{ kHz}. \]  

The first uncertainty arises from the experimental error in determination of \( \alpha \), Ref. 143, the second one is from the experimental error in determination of the ratio of electron mass and muon mass, Ref. 150, the third one is from the coefficient \( D \) that is not finally calculated.

Regarding the problem of experimental measuring of \( \Delta E_{\text{hfs}}^{\text{exp}}(Mu) \) the published values for the HFS of the ground state in a muonium are summed in Table VII beginning from its discovery in 1960, Ref. 151, 28.

Table VII.

| Year | Reference | \( \Delta E, kHz \) | Error, ppm |
|------|-----------|-------------------|-----------|
| 1962 | 153       | 4461300(2200)     | 493       |
| 1964 | 154       | 4463330(190)      | 43        |
| 1964 | 155       | 4463150(60)       | 13        |
| 1969 | 156       | 4463260(40)       | 9.0       |
| 1969 | 157       | 4463317(21)       | 4.7       |
| 1970 | 158       | 4463302.2(8.9)    | 2.0       |
| 1971 | 159       | 4463311(12)       | 2.7       |
| 1971 | 160       | 4463301.17(2.3)   | 0.5       |
| 1972 | 161       | 4463240(120)      | 26.9      |
| 1973 | 162       | 4463304.0(1.8)    | 0.4       |
| 1975 | 163       | 4463302.2(1.4)    | 0.3       |
| 1977 | 164       | 4463302.35(52)    | 0.12      |
| 1980 | 165       | 4463302.90(27)    | 0.06      |
| 1982 | 166       | 4463302.88(16)    | 0.036     |

27. The three-photon exchange diagrams have not taken into account there.

28. The result of Ref. 152 of 1961 has the meaning of rough estimation, \( \Delta E = 2250 - 9000 \text{ MHz} \).
We conclude that the agreement between theoretical and experimental results is excellent. On the basis of this fact one can get the value of the fine structure constant (see (4.199) by comparing the expression (4.46) and the recent experimental result of Table VII.

4.2.3 Hydrogen

The analytical result of calculations of the quantum-electrodynamics non-recoil corrections to the HFS in a hydrogen is obviously the same as in a muonium (4.46, 4.47). The numerical value for it is

$$\Delta E(QED) = 1.420.451.95(14) \, MHz.$$ \hfill (4.60)

The recoil corrections and dynamics correction caused by the nuclear structure have been calculated in the classical works [141], [167]-[169] which have been completed by the results of [166], [170]-[175].

$$\Delta E(\text{structure}) = E_F \left[ \delta_p(Zemach) + \delta_p(\text{recoil}) + \delta_p(\text{polarizability}) \right],$$ \hfill (4.61)

$$\delta_p(Zemach) = -2\mu a < r_p \geq -38.72(56) \, ppm,$$ \hfill (4.62)

$$\delta_p(\text{recoil}) \simeq 5.68 \, ppm,$$ \hfill (4.63)

$$| \delta_p(\text{polarizability}) | < 4 \, ppm,$$ \hfill (4.64)

where $<r_p>$ is the average proton radius connected with the charge distribution.

For the discussion of the contributions of proton polarizability, which have been obtained on the basis of the data of the deep-inelastic scattering of polarized electrons on a nucleon target you see Ref. [176].

Table VIII.

| Year | Reference | Isotope | $\Delta E, kHz$ | Error, ppb |
|------|-----------|---------|----------------|------------|
| 1948 | [178]     | H       | 1420410(6)     | 4224       |
| 1948 | [178]     | D       | 327384(3)      | 9164       |
| 1952 | [179]     | H       | 1420405.1(2)   | 141        |
| 1952 | [179]     | D       | 327384.24(8)   | 244        |
| 1955 | [180]     | H       | 1420405.73(5)  | 35         |
| 1955 | [180]     | D       | 327384.302(30) | 92         |
| 1956 | [181]     | H       | 1420405.80(6)  | 42         |
| 1956 | [182]     | H       | 1420405.726(30)| 21         |
| 1956 | [182]     | D       | 327384.349(5)  | 15         |
| 1960 | [183]     | H       | 1516701.396(30)| 20         |
| 1962 | [184]     | H       | 1420405.762(4) | 2.8        |
| 1962 | [184]     | H       | 1420405.7491(60)| 4.2       |
| 1962 | [184]     | T       | 1516701.4768(60)| 4.0       |
| 1963 | [185]     | H       | 1420405.751827(20)| 0.014     |
| 1963 | [186]     | H       | 1420405.751800(28)| 0.019     |
| 1964 | [187]     | H       | 1420405.751827(20)| 0.014     |
| 1965 | [188]     | H       | 1420405.751778(16)| 0.011     |
| 1965 | [189]     | H       | 1420405.751785(16)| 0.011     |
| 1966 | [190]     | H       | 1420405.751781(16)| 0.011     |
| 1966 | [191]     | H       | 1420405.7517860(46)| 0.003     |
| 1966 | [192]     | H       | 1420405.7517864(17)| 0.001     |
| 1970 | [193]     | H       | 1420405.7517667(9)| 0.0006     |

\textsuperscript{29}New experiment at the LANL, Los Alamos, is expected to improve the experimental precision by a factor 5, cited in [38].

\textsuperscript{30}It is necessary, of course, to make the corresponding substitutions, $m_\mu \rightarrow m_p$ and $a_\mu \rightarrow a_p$.

\textsuperscript{31}The following results are presented in relative units of $E_F$. 
The published experimental values of the measurings of the HFS of the ground state in a hydrogen and a deuterium are presented in Table VIII.32

The difference of values between theory and experiment can be written as

$$\frac{\Delta E_{\text{theor}} - \Delta E_{\text{exp}}}{E_F} = (-0.48 \pm 0.56 \pm \text{uncalculated terms}) \text{ ppm}. \quad (4.66)$$

The uncertainty 0.56 ppm arises because of the error in the experimental value of the fine structure constant and, mostly, the inaccuracy of the data of the proton elastic formfactor. The corrections uncalculated so far could contribute about 1 ppm, Refs. [166, 194].

4.2.4 Muonic helium atom

The muonic helium atom, which is $^4$He$^{++} \mu^- e^-$, has been experimentally discovered in 1975, Refs. [195, 196]. From the point of view at a number of electrons this system can be considered as a heavy isotope of a hydrogen having the "pseudo-nucleus" ($^4$He$^{++} \mu^-$)$^+$, which has a middle size between the intrinsic nuclear size and the intrinsic atomic one ($\sim 130 \text{ fm}$). The first measurings of the HFS, carried out in 1980, Refs. [197]-[199], led to the following results:

$$\Delta E_{\text{EFP}}^{197} = 4464.95(6) \text{ MHz} \quad (13 \text{ ppm}), \quad (4.67)$$

$$\Delta E_{\text{EFP}}^{198} = 4464.02(10) \text{ MHz} \quad (22 \text{ ppm}), \quad (4.68)$$

and

$$\Delta E_{\text{EFP}}^{199} = 4465.004(29) \text{ MHz} \quad (6.5 \text{ ppm}). \quad (4.69)$$

The last experimental result gave the opportunity to find the magnetic moment of a negative charged muon

$$\frac{\mu_{\mu^-}}{\mu_p} = 3.18328(15) \quad (47 \text{ ppm}), \quad (4.70)$$

what allowed one to check the predictions of the CPT- invariance. According to it, the magnetic moments of a particle and an anti-particle are to be equal. This quantity for a positive charged muon, which was measured more accurately in the experiments of the muonium HFS, Ref. [150],

$$\frac{\mu_{\mu^+}}{\mu_p} = 3.1833461(11) \quad (0.36 \text{ ppm}), \quad (4.71)$$

and, also, in the experiments on observations of the muon spin rotation in liquid, the Larmour precession, Ref. [200]:

$$\frac{\mu_{\mu^+}}{\mu_p} = 3.1833441(17) \quad (0.53 \text{ ppm}), \quad (4.72)$$

is in agreement with (4.70) to the precision of several tens of ppm. For comparison, the agreement between the electron and positron magnetic moments is 0.13 ppm, Ref. [201], and is 7500 ppm between the proton and anti-proton magnetic moments, Ref. [202].

Theoretical description of a muonic helium atom, which is generally connected with the theoretical methods used for muonium, can be found in Refs. [203-206]. We should like to note essential contributions of relativistic and radiative corrections to the Fermi energy in this system (M is the "pseudo-nucleus" mass),

$$E_F = \frac{16}{3} \alpha^2 R_\infty \frac{m_e}{m_\mu} \left(1 + \frac{m_e}{M}\right)^{-3} = 4516.96 \text{ MHz}. \quad (4.73)$$

The only experimental result of the HFS of the $n = 2$ state ($n$ is the principal quantum number) in a hydrogen, known to us, is $\Delta E_{\text{EFP}}^{197} (2S,H) = 177556.6(3) \text{ kHz}$, Ref. [177]. It satisfies the Breit formula, Ref. [138],

$$\Delta E(2S) = \frac{\Delta E(1S)}{(8 - 5\alpha^2)}, \quad (4.65)$$
The obtained theoretical values are

\[
\begin{align*}
\Delta E^{th}_{203} &= 4\,465.1(1.0) \text{ MHz}, \\
\Delta E^{th}_{204} &= 4\,462.6(3.0) \text{ MHz}, \\
\Delta E^{th}_{205} &= 4\,464.8(5) \text{ MHz}, \\
\Delta E^{th}_{206} &= 4\,460 \text{ MHz}.
\end{align*}
\]

The first value has been obtained by means of the variational methods; the second and third ones, on the basis of direct calculation of the Feynman diagrams by perturbation theory method; the fourth, in the framework of the Born-Oppenheimer theory.

### 4.3 Fine structure

#### 4.3.1 Positronium and muonium

For the first time the fine structure interval has been investigated in a positronium in \[209\]

\[
\Delta E(2^3S_1 - 2^3P_2) = 8628.4 \pm 2.8 \text{ MHz} \quad (1000 \text{ ppm}).
\]

In contrast with a hydrogen atom, the mentioned levels are not degenerated in the order of \(\alpha^2 R_{\infty}\).

The experiments of this kind permit checking the validity of QED for the excited states of pure leptonic systems.

The recent experiments \[210\] achieved the accuracy of 300 ppm \[34\]

\[
\begin{align*}
\Delta E(2^3S_1 - 2^3P_2) &= 8619.6(2.7)(0.9) \text{ MHz}, \\
\Delta E(2^3S_1 - 2^3P_1) &= 13\,001.3(3.9)(0.9) \text{ MHz}, \\
\Delta E(2^3S_1 - 2^3P_0) &= 18\,804.1(10.0)(1.7) \text{ MHz}.
\end{align*}
\]

The theoretical predictions for the first excited states, based on the BS equation, are, Refs. \[17, 48, 116, 117, 211\] (see also \[212, 213\]),

\[
\begin{align*}
E(1^3S_1) &= R_{\infty} \left\{ -\frac{1}{2} + \frac{49}{96} \alpha^2 + \frac{3}{2\pi} \alpha^3 \log \alpha^{-1} + \frac{\alpha^3}{\pi} \left[ -\frac{1}{15} + \frac{4}{3} \log 2 - \frac{4}{3} \log R(1,0) \right] + A_{1S} \alpha^4 \log \alpha^{-1} + B_{1S} \alpha^4 + \ldots \right\}, \\
E(2^3S_1) &= \frac{1}{8} R_{\infty} \left\{ -1 + \frac{65}{192} \alpha^2 + \frac{3}{2\pi} \alpha^3 \log \alpha^{-1} + \frac{\alpha^3}{\pi} \left[ \frac{97}{120} + \frac{1}{6} \log 2 - \frac{4}{3} \log R(2,0) \right] + A_{2S} \alpha^4 \log \alpha^{-1} + B_{2S} \alpha^4 + \ldots \right\}, \\
E(2^3P_2) &= \frac{1}{8} R_{\infty} \left\{ -1 - \frac{43}{960} \alpha^2 - \frac{\alpha^3}{\pi} \left[ \frac{1}{45} + \frac{4}{3} \log R(2,1) \right] + A_{P2} \alpha^4 \log \alpha^{-1} + B_{P2} + \ldots \right\}, \\
E(2^3P_1) &= \frac{1}{8} R_{\infty} \left\{ -1 - \frac{47}{192} \alpha^2 - \frac{\alpha^3}{\pi} \left[ \frac{5}{36} + \frac{4}{3} \log R(2,1) \right] + A_{P1} \alpha^4 \log \alpha^{-1} + B_{P1} \alpha^4 + \ldots \right\}, \\
E(2^3P_0) &= \frac{1}{8} R_{\infty} \left\{ -1 - \frac{95}{192} \alpha^2 - \frac{\alpha^3}{\pi} \left[ \frac{25}{72} + \frac{4}{3} \log R(2,1) \right] + A_{P0} \alpha^4 \log \alpha^{-1} + B_{P0} \alpha^4 + \ldots \right\},
\end{align*}
\]

\[34\]The calculation of some of the contributions can also be found in \[207, 208\], but the numerical results shown there are highly different from the experimental results.

\[34\]The first error in these expressions is statistical, the second one is systematical.
where \( \log R(n, l) \) is the Bethe logarithm\(^{214, 217}\):

\[
\begin{align*}
\log R(1, 0) & \simeq 2.9841285, \\
\log R(2, 0) & \simeq 2.8117699, \\
\log R(2, 1) & \simeq -0.0300167.
\end{align*}
\]

(4.87) (4.88) (4.89)

The coefficients \( A_{1S} \) and \( A_{2S} \) have been calculated recently, but the formulae obtained in\(^{47, 48}\) differ from each other\(^3\):

\[
\begin{align*}
\Delta E & \left( \alpha^6 \log \alpha \right) = 5 \frac{m \alpha^6 \log \alpha - \delta_{00} \delta_{l1}}{n}, \\
\Delta E & \left( \alpha^6 \log \alpha \right) = 1 \frac{m \alpha^6 \log \alpha - \delta_{00} \delta_{l1}}{n^3}.
\end{align*}
\]

(4.90) (4.91)

The coefficients \( B \) are not yet calculated. The coefficient 1 of the term \( \alpha^4 \log \alpha^{-1} R/8 \) gives the contribution \( 5.7 \text{MHz} \) and of the term \( \alpha^4 R/8 \) gives \( 1.2 \text{MHz} \).

There is a simple formula for the \( \alpha^2 R_{\infty} \) contributions to the S- levels of an electron-positron system\(^{212}\):

\[
E(\alpha^2 R_{\infty}) = \frac{m \alpha^4}{n^3} \left[ \frac{11}{64} \frac{1}{n} + \frac{7}{12} \delta_{1S} \right].
\]

(4.92)

The contributions of the order \( \alpha^3 R \) and higher arise from the allowance for the radiative corrections, a vertex function, a vacuum polarization, an electron-positron self-energy and an annihilation interaction channel. Probably, the formula for the S- states of the electron-positron system, which has been derived in Ref.\(^{212}\), is not correct, as is shown, e. g., in Refs.\(^{13, 213}\). It can be seen from the result of the last paper that these corrections are not proportional to \( 1/n^2 \):

\[
\Delta E(\alpha^3 R_{\infty}) = \frac{m \alpha^5}{8 \pi n^3} \left\{ \frac{14}{3} \left[ \frac{7}{15} + \log \frac{2}{n} + \frac{n-1}{2n} + \sum_{k=1}^{n} \frac{1}{k} \right] + \frac{14}{3} \log 2 - 6 \log \alpha - \frac{16}{3} \log R(n, 0) - 4(\frac{16}{9} + \log 2) \delta_{1S} \right\}.
\]

(4.93)

The numerical values of theoretical predictions, with taking into account the logarithmic corrections, are the following, Refs.\(^{17, 48}\):

\[
\begin{align*}
\Delta E(2^3 S_1 - 1^3 S_1) &= 1233607211.7 \text{MHz}; & 1233607221.69 \text{MHz}, \\
\Delta E(2^3 S_1 - 2^3 P_1) &= 8627.7 \text{MHz}; & 8626.21 \text{MHz}, \\
\Delta E(2^3 S_1 - 2^3 P_0) &= 13013.3 \text{MHz}; & 13011.86 \text{MHz}, \\
\Delta E(2^3 S_1 - 2^3 P_0) &= 18498.5 \text{MHz}; & 18497.10 \text{MHz}.
\end{align*}
\]

Due to the development of the experimental methods based on the Doppler-free two-photon spectroscopy, now it is possible to measure the "gross structure" interval \((1S - 2S)\) at the accuracy level of some \( \text{MHz} \). These experiments give the opportunity to find the value of the fundamental constant as the Rydberg constant most precisely, see below.

In Ref.\(^{216}\), the latest results of the measurements of the \( \Delta E(1^3 S_1 - 2^3 S_1) \) interval are presented for a positronium and a muonium\(^\dagger\):

\[
\Delta E_{\text{exp}}^P(2^3 S_1 - 1^3 S_1) = 1233607218.9 \pm 10.7 \text{MHz} = \frac{3}{8} c R_{\infty} - 83516.6 \pm 10.7 \text{MHz},
\]

(4.98)

\(^{35}\)To check the results of Fell and Khriplovich et al. the experiments at the accuracy level of \( \sim 10 \text{ppm} \) are necessary.

\(^{36}\)The first column contains the result of Khriplovich et al.; and the second one, the result of Fell.

\(^{37}\)The results of previous experiments are

\[
\Delta E_{\text{exp}}^P = \frac{3}{16} c R_{\infty} - 41.4(5) \text{GHz};
\]

(4.94)
\[ \Delta E_{\text{exp}}^{\text{Mu}}(2^3S_1 - 1^3S_1) = 2\,455\,527\,936 \pm 120 \pm 140\,MHz. \]  
(4.99)

In the case of a muonium, there is an agreement with theory, Ref. [224],

\[ \Delta E_{\text{theor}}^{\text{Mu}}(2^3S_1 - 1^3S_1) = 2\,455\,527\,959(3.6)\,MHz. \]  
(4.100)

In the case of a positronium the value is 16.6(10.7)\,MHz greater than the Fulton’s theoretical result. However, since many authors consider the formula for \( S \)-states to be incorrect, it is preferable to compare the experimental value (4.98) with the value of Ref. [213], where the fine structure has been found by using a simple potential method, Ref. [223],

\[ \Delta E_{\text{theor}}^{\text{Ps}}(2^3S_1 - 1^3S_1) = 38R_\infty - 83\,507.4\,MHz, \]  
(4.101)

\[ \Delta E_{\text{theor}}^{\text{Mu}}(2^3S_1 - 1^3S_1) = 2\,455\,528\,055\,MHz, \]  
(4.102)

or with the latest results of Fell or Khriplovich et al.

The fine structure of the first excited state \((n = 2)\) in a muonium has been investigated in the papers [224]-[226], see below Subsection 4.5.2.

Let us mention the works [78, 227], in which the problem of validity of the CP-invariance in the lepton sector has been under consideration. The experimental and theoretical limits were obtained there for transitions like \(2^3S_1 \to 2^1P_1\).

### 4.3.2 Hydrogen and Deuterium

The results of measuring of the \(1S - 2S\) interval have been reported in [228]-[236]. The importance of these investigations is clear, because they give information about the Lamb shift value of the \(1S\)-level, which is impossible to find by means of the radiofrequency spectroscopy method used in the experiments to find the \(n = 2\) Lamb shift. They also gave a possibility to determine the numerical value of the Rydberg constant with the highest precision.

In Table IX the results of the measurements of the \(1S - 2S\) interval and \(1S\)-level Lamb shift are given. The energy characteristics are presented in MHz.

\[ \Delta E_{\text{exp}}^{\text{Ps}}(2^2S_{1/2} - 2^2P_{3/2}) = 1\,233\,607\,185(15)\,MHz\]  
(4.95)

\[ \Delta E_{\text{exp}}^{\text{Mu}}(2^2S_{1/2} - 2^2P_{3/2}) = 1\,233\,607\,142(10.7)\,MHz, \]  
(4.96)

for a positronium; and

\[ \frac{1}{4}\Delta E_{\text{exp}}^{\text{Mu}} = 613\,881\,924 \pm 30 \pm 35\,MHz, \]  
(4.97)

for a muonium (see for the discussion Ref. [221]).

38See Refs. [237, 238] for the review of earlier results of investigation of the interval \(2^2S_{1/2} - 2^2P_{3/2}\).

39The first successful attempt had been made, Ref. [239], as early as 1955 to measure this value \(E_{\text{exp}}^{\text{exp}}(LS, 1S) = 7.9 \pm 1.1\,GHz\). See also [240], where the value of the isotopic splitting has been discussed in a hydrogen for the first time as a result of observations the Layman lines in a hydrogen and in a deuterium.
Table IX

| Year | Reference | Isotope | $\Delta E(1S - 2S)$ | $E(LS, 1S)$ |
|------|-----------|---------|---------------------|------------|
| 1975 | 228       | $D$     | -                   | 8300(300)  |
| 1975 | 228       | $H$     | -                   | 8600(800)  |
| 1975 | 229       | $D$     | -                   | 8250(110)  |
| 1975 | 229       | $H$     | -                   | 8200(100)  |
| 1980 | 230       | $D$     | -                   | 8177(30)   |
| 1980 | 230       | $H$     | -                   | 8151(30)   |
| 1986 | 231       | $H$     | 2466 061 395.6(4.8) | 8184.8(5.4)|
| 1986 | 232       | $H$     | 2466 061 397(25)    | 8182(25)   |
| 1987 | 233       | $H$     | 2466 061 413.8(1.5) | 8173.3(1.7)|
| 1989 | 234       | $H$     | 2466 061 413.19(1.75) | 8173.9(1.9)|
| 1989 | 235       | $D$     | 2466 732 408.5(7)   | 8183.7(6)  |
| 1989 | 235       | $H$     | 2466 061 414.1(8)   | 8172.6(7)  |
| 1990 | 236       | $H$     | 2466 061 413.182(45)| 8172.804(83)|

The result of Ref. [235] of the 1S Lamb shift value has been obtained when the following value of the Rydberg constant was used:

$$R_\infty = 109 737.315 714(19) \text{ cm}^{-1}.$$  (4.103)

that is the average value of Refs. [258, 259]. There are some values of the isotopic splitting $\Delta E(H - D)$, received on the basis of this technique

\[
\begin{align*}
\Delta E_{[229]}^{[\text{exp}]}(H - D) &= 670 993(56) \text{ MHz}, \\
\Delta E_{[230]}^{[\text{exp}]}(H - D) &= 670 992.3(6.3) \text{ MHz}, \\
\Delta E_{[235]}^{[\text{exp}]}(H - D) &= 670 994.33(64) \text{ MHz}, \\
\Delta E_{[241]}^{[\text{exp}]}(H - D) &= 670 994.337(22) \text{ MHz}.
\end{align*}
\]  (4.104)

These values are to be compared with the theoretical values:

\[
\begin{align*}
\Delta E_{[234]}^{[\text{theor}]}(H - D) &= 670 994.39(12) \text{ MHz}, \\
\Delta E_{[241]}^{[\text{theor}]}(H - D) &= 670 994.414(22) \text{ MHz}.
\end{align*}
\]  (4.105)

which are in agreement with the last experimental results within an error.

The numerical values of the 1S- level Lamb shift, obtained through the theoretical calculations, Refs. [242, 248], you can find in Refs. [234, 235]

\[
\begin{align*}
E_{[234]}^{[\text{theor}]}(H, LS, 1S) &= 8 172.89(9) \text{ MHz}, \\
E_{[235]}^{[\text{theor}]}(H, LS, 1S) &= 8 173.03(9) \text{ MHz}, \\
E_{[235]}^{[\text{theor}]}(D, LS, 1S) &= 8 184.08(12) \text{ MHz}.
\end{align*}
\]  (4.110)

The proton charge radius is supposed in Ref. [235] to be equal to 0.862(12) fm, see Ref. [250]. and the deuteron charge radius used in Ref. [241] is 1.9627(38) fm, Ref. [251].

Provided that the Lamb shift value is known from theory, information about the Rydberg constant could be received by means of comparison of the calculated value of the $1S - 2S$ interval with the above-mentioned experimental one, see the next subsection.

---

40The result shown in the Table has been obtained by using $R_\infty$ of [253]. Better agreement with the theoretical result is achieved when $R_\infty$ of [253] is used, $E(LS, 1S) = 8 174.8(8.7) \text{ MHz}$.

41The previous measuring of the proton radius, Ref. [249], should not be ignored because it allows one to get a better agreement with theoretical predictions for some experiments.
4.4 Rydberg constant

The latest measurements of the Rydberg constant are the following.

Table X.

| Year | Reference | \( R, \text{ cm}^{-1} \) | Interval |
|------|-----------|--------------------------|----------|
| 1974 | [252]     | 109737.3143(10)          | 2P-3D    |
| 1978 | [253]     | 109737.31476(32)         | 2S-3P    |
| 1980 | [254]     | 109737.31513(85)         | 2S-3P, 2P-3D |
| 1981 | [255]     | 109737.31521(11)         | 2S-3P    |
| 1986 | [256]     | 109737.31492(22)         | 1S-2S    |
| 1986 | [257]     | 109737.3150(11)          | 1S-2S    |
| 1986 | [258]     | 109737.31569(7)          | 2S-3P    |
| 1986 | [259]     | 109737.31569(6)          | 2S-8D, 10D |
| 1987 | [260]     | 109737.31573(3)          | 2S-4P    |
| 1987 | [261]     | 109737.31571(7)          | 1S-2S    |
| 1989 | [262]     | 109737.315709(18)        | 2S-8D,10D, 12D |
| 1989 | [263]     | 109737.31569(8)          | 1S-2S    |
| 1989 | [264]     | 109737.31573(3)          | 1S-2S    |
| 1992 | [265]     | 109737.3156841(42)       | 1S-2S    |
| 1992 | [266]     | 109737.3156830(31)       | 2S-8S, 8D |

4.5 Lamb shift

4.5.1 Hydrogen

In Ref. [261], the results of optical measurements of the Lamb shift of the \( 1S \)-level in a hydrogen have been reported.

\[
E(\text{LS}, 1S, H) = 8172.82(11) \text{ MHz} \quad (13 \text{ ppm}). \tag{4.114}
\]

The technique is based on comparison of frequencies of the two-photon transitions between the \( 1S - 2S \) and \( 2S - 4S, 4D \) levels. It highly differs from the experimental technique of an indirect measuring of this quantity, see the preceding subsection, by the two-photon Doppler-free spectroscopy methods in \( 1S - 2S \) transitions. The first optical measurement of the Lamb shift of the \( 4S \) level in a hydrogen has also been given in Ref. [261]:

\[
E(\text{LS}, 4S, H) = 131.66(4) \text{ MHz} \quad (300 \text{ ppm}). \tag{4.115}
\]

This value could be compared with theoretical predictions of [238]:

\[
E_{\text{theor}}^{[238]} (\text{LS}, 4S, H) = 133.084(1) \text{ MHz}, \tag{4.116}
\]

\[
E_{\text{theor}}^{[238]} (\text{LS}, 4S, D) = 133.254(3) \text{ MHz}, \tag{4.117}
\]

and with the radio-frequency measurements, Refs. [262]-[264],

\[
E^{\text{exp}}_{\text{262}} (\text{LS}, 4S, D) = 133(10) \text{ MHz}, \tag{4.118}
\]

\[
E^{\text{exp}}_{\text{263}} (\text{LS}, 4S, H) = 133.18(59) \text{ MHz}, \tag{4.119}
\]

\[
E^{\text{exp}}_{\text{264}} (\text{LS}, 4S, H) = 132.53^{+0.58}_{-0.78} \text{ MHz}. \tag{4.120}
\]

The discussion concerning the \( n = 3 \) Lamb shift in a hydrogen atom can be found in Ref. [265]. Below we reproduce the results presented in this article:

\[
E^{\text{exp}}_{\text{262}} (\text{LS}, 3S, D) = 314.93(40) \text{ MHz}, \tag{4.121}
\]

16
\[ E_{exp}^{266} (LS, 3S, H) = 313.6(2.9) \text{ MHz}, \]  
\[ E_{exp}^{266} (LS, 3S, D) = 315.3(8) \text{ MHz}, \]  
\[ E_{exp}^{266} (LS, 3S, H) = 315.11(89) \text{ MHz}, \]  
\[ E_{exp}^{266} (LS, 3S, H) = 314.819(48) \text{ MHz}, \]  
\[ E_{theor}^{243} (LS, 3S, H) = 314.898(3) \text{ MHz}. \]

The precision of these experiments approaches the precision of measurements of the \( n = 2 \) Lamb shift, which were of great importance for checking the predictions of QED. In Table XI the results of all the experiments for \( n = 2 \) level are given.

Table XI.

| Year | Reference | \( \Delta E, \text{ MHz} \) | Error, ppm |
|------|-----------|----------------|-----------|
| 1953 | [268] | 1057.774(100) | 94.5      |
| 1969 | [269] | 1057.772(63)  | 59.6      |
| 1970 | [270] | 1057.90(6)    | 56.7      |
| 1975 | [271] | 1057.892(20)  | 18.9      |
| 1979 | [272] | 1057.862(20)  | 18.9      |
| 1981 | [273] | 1057.845(9)   | 8.5       |
| 1982 | [274] | 1057.8594(19) | 1.8       |
| 1983 | [275] | 1057.851(2)   | 1.9       |

The above results depend essentially on the parameter \( \tau \), the life time of the \( 2P \) state. Since the experimental data of this constant are absent, the value was found theoretically, Refs. [274, 275]. After taking into account relativistic corrections one has

\[ \gamma_{rel.} = 4\pi c \left( \frac{2}{3} \right)^8 R_H \alpha^3 (1 + \alpha^2 \log \frac{9}{8}) = \left( \frac{2}{3} \right)^8 \frac{m e^4}{\hbar^3} \frac{\alpha^3 (1 + \alpha^2 \log \frac{9}{8})}{1 + \frac{m}{\mu_p}}, \]  

(\( R_H \) is the Rydberg constant with allowance for the finite mass of a proton.).

The leading radiative corrections, the self-energy and the vacuum polarization, contribute additionally

\[ \gamma_{rad.} = 4\pi c \left( \frac{2}{3} \right)^8 R_H \alpha^3 \left[ R(2, 1) \frac{1}{8} - R(1, 0) - \log \frac{1}{\alpha^2} - \frac{1}{64} - \frac{19}{30} \right]. \]  

\( R(n, l) \) is the Bethe logarithm.

Then, we have the numbers

\[ \gamma = \frac{1}{\tau} = 6.264 \text{881 } 2(20) \times 10^8 \text{ c}^{-1}, \]  
\[ \tau = 1.596 \text{199 } 46(48) \times 10^{-9} \text{ c} \]  

(with taking into account the above-shown corrections).

In the case of a deuterium \( (n = 2) \) we know the following experimental results:

\[ E_{exp}^{268} (LS, D) = 1059.00(6) \text{ MHz}, \]  
\[ E_{exp}^{268} (LS, D) = 1059.24(3) \text{ MHz}. \]  

Let us note that further improvement of the accuracy of the \( n = 2 \) Lamb shift experimental values in a hydrogen faces the serious difficulties; namely, the natural width of the \( 2P \) state is about \( \sim 100 \text{ MHz} \).
The total theoretical formula for the Lamb shift \( n = 2 \) in a hydrogen is, Ref. 277:

\[
\Delta E_{LS} = \Delta E_{2S_{1/2}} - \Delta E_{2P_{1/2}} = \frac{\alpha(Z\alpha)^4m^4}{6\pi} \left\{ \frac{1}{8\mu} + \log(Z\alpha)^2 - 2.207909 + \right. \\
+ \pi Z\alpha \left[ \frac{427}{128} - \frac{3}{2} \log 2 \right] + (Z\alpha)^2 \left[ \frac{3}{4} \log^2(Z\alpha)^2 + (4\log 2 + \frac{55}{48} \log(Z\alpha)^2) \right] + \\
+ \left( Z\alpha \right)^2 \left[ G_{s.e.}(Z\alpha) + G_{v.p.}(Z\alpha) \right] + \alpha \left( \frac{0.323}{\pi} \right) + \\
+ \frac{(Z\alpha)^5 m^2}{6\pi M} \left\{ \frac{1}{4} \log(Z\alpha)^2 - 2.39977 + \frac{3\pi}{4} Z\alpha \left[ \frac{5}{2} + \log(2Z\alpha)^{-1} - 4, 25 \right] \right\} + \\
+ \frac{1}{12} (Z\alpha)^4 m^3 < r_p^2 > - \frac{1}{48} \left( Z\alpha \right)^4 m^3 \left[ \frac{35}{4} \log 2 - \frac{39}{5} + \frac{31}{192} \right] + \\
+ \left( -0.415 \pm 0.004 \right),
\]

(4.133)

where the self-energy and vacuum polarization contributions \((G_{s.e.} \text{ and } G_{v.p.})\) can be expanded in the Wichmann-Kroll form 242-c. 278:

\[
G_{v.p.} = -\frac{1199}{2100} + \frac{5}{128} \pi(Z\alpha) \log(Z\alpha)^2 - 0.5(Z\alpha) + \ldots;
\]

(4.134)

\[
G_{s.e.} = -24.1 + 7.5 \langle Z\alpha \rangle \log(Z\alpha)^2 + 12.3 \langle Z\alpha \rangle + 1.2.
\]

(4.135)

They give \(-24.0 \pm 1.2\), Ref. 277, as a sum in the case of a hydrogen atom 244.

The numerical value of the Lamb shift of the \( n = 2 \) level in a hydrogen, corrected into account the new calculated corrections, was given in 277

\[
E_{[277]}^{[277]}(LS, H) = 1057.855 \pm 0.011, MH z \quad \text{when } < r_p > = 0.805(11),
\]

(4.136)

\[
E_{[277]}^{[277]}(LS, H) = 1057.873 \pm 0.011 MHz, \quad \text{when } < r_p > = 0.862(12).
\]

(4.137)

For the earlier theoretical works see Refs. 117-c, 174, 282-284, where the recoil corrections of the order \((Z\alpha)^5 \frac{m^2}{M} \) have been calculated. The corrections obtained from the diagrams of radiative exchanges, which have the order \( \propto (Z\alpha)^4 \frac{m^3}{M} \), \( \propto (Z\alpha)^5 \frac{m^3}{M} \), have been calculated in the external field approximation in Ref. 247. The corrections of the order \( \propto (Z\alpha)^4 \frac{m^3}{M} \) can be found in Refs. 226, 283. The contributions arising after taking into account the finite size of a proton, have been discussed in 277, where the corrections of the order \( \propto (Z\alpha)^6 \frac{m^3}{M} \) have been found out.

The correction of the order \( \alpha^2 \langle Z\alpha \rangle^5 m \), which is the binding correction from the two-loop radiative exchange diagrams, has not yet been calculated 43. It is extremely desirable to calculate it in order to achieve the theoretical precision \( \sim 1 \text{kHz} \).

### 4.5.2 Muonium

At present, two experimental results are known for the Lamb shift \( 2S_{1/2} \rightarrow 2P_{1/2}, J = 1 \) in a muonium

\[
E_{[280]}^{[280]}(LS, Mu) = 1070^{+12}_{-15} \pm 2 \text{MHz},
\]

(4.138)

---

42 For the case of other ions \((Z \neq 1)\) see Refs. 242, 278.

43 On the basis of the new analytical method relied on division into the low and high energy part, K. Pachucki calculated, Ref. 280, the principal contribution to \( G(s.e.) \). For the coefficient \( A_{60} \) he presented \( A_{60}(LS) = -30.928901(1) \) and \( A_{60}(2S) = -31.84047(1) \), which are much more accurate than the previous calculations, Ref. 243, 244, 246, 281, and the approximation of P. J. Mohr, Ref. 242-b.

44 See Ref. 285, devoted to calculation of the corrections of this order from the diagrams with polarization insertions into external Coulomb legs and from the diagrams with the radiative insertions into an electron line and one polarization insertion into a Coulomb leg. The calculation of the remained diagrams is in progress by this group.
\[ E_{LS}^{exp} (\mu) = 1054 \pm 22 \text{ MHz}. \]  

(4.139)

The theoretical value was given by Owen, Refs. 224-226:

\[ E_{LS}^{theor} (\mu) = 1047.03 \text{ MHz}, \]  

(4.140)

as well as the fine structure interval: \(2P_{3/2} - 2P_{1/2}\)

\[ E_{FS}^{theor} (\mu) = 10921.50 \text{ MHz}. \]  

(4.141)

In contrast with a hydrogen atom, we have not come across with the problems of structure in a muonium, analogously to the calculations of the HFS in such a system.

4.5.3 Helium

The precision of experimental measuring, Refs. 288-289, of the frequencies of transitions between Rydberg states of \(^4\)He\(^+\) reached such values by the beam-foil spectroscopy methods, which made it possible to check the Lamb shift value \(2S_{1/2} - 2P_{1/2}\) in a helium within the corrections of the order \(\sim \alpha (Z \alpha)^6 mc^2\). In Table XII all the experimental results are presented concerning the measuring of this quantity, including the early ones based on the Lamb – Rutherford technique, Ref. 268.

### Table XII

| Year | Reference | \(E(\text{LS}), \text{ MHz}\) | Error, ppm |
|------|-----------|--------------------------|------------|
| 1950 | 290       | 14020(100)               | 7130       |
| 1952 | 291       | 14021(60)                | 4280       |
| 1955 | 292       | 14043(13)                | 930        |
| 1957 | 293       | 14040.2(1.8)             | 128        |
| 1971 | 294       | 14046.2(1.2)             | 85         |
| 1979 | 295       | 14040.9(2.9)             | 207        |
| 1987 | 288       | 14041.9(1.5)             | 107        |
| 1988 | 286       | 14042.2(35)              | 25         |

The first theoretical investigations of the Lamb shift in a helium atom have been carried out in Ref. 297. The modern calculations, Refs. 237, 243, 279, 289, give the different results (see also 244).

\[ E_{LS}^{theor} (\mu) = 14044.5(5.2) \text{ MHz}, \]  

(4.142)

\[ E_{LS}^{theor} (\mu) = 14045.12(55) \text{ MHz}, \]  

(4.143)

\[ E_{LS}^{theor} (\mu) = 14042.36(55) \text{ MHz}, \]  

(4.144)

\[ E_{LS}^{theor} (\mu) = 14042.26(50) \text{ MHz}. \]  

(4.145)

The latest calculations of \(G_{s.e.}(Z = 2)\) give

\[ G_{298}^{298} (Z = 2) = -22.8 \pm 2.0, \]  

(4.146)

\[ G_{299}^{299} (Z = 2) = -22.0 \pm 0.3. \]  

(4.147)

Besides, we would like to mention the recent works devoted to investigation of highly excited states of a helium atom, Refs. 299-304. The Table of the latest results for ions of other atoms was shown in Ref. 289. The investigation in a muonic helium can be found in Ref. 305.

\(^{45}\)The calculations of Ref. 288 have been fulfilled by using the new value of the nucleus radius 1.673(1) \(fm\), Ref. 296, and the obtained result is in excellent agreement with experiment.
4.6 The anomalous magnetic moment (AMM)

4.6.1 Electron

The opportunity of calculations of the AMM of an electron is guaranteed by the renormalizability of QED by means of the expansion in the perturbation series in $\frac{\alpha}{\pi}$ with the finite coefficients $a_i$.

$$\frac{g - 2}{2} = a_e(QED) = a_{II} + a_{IV}(\frac{\alpha}{\pi})^2 + a_{V}(\frac{\alpha}{\pi})^3 + a_{VI}(\frac{\alpha}{\pi})^4 + \ldots$$  (4.148)

At present, the value of AMM of an electron is known up to the eighth order. In the papers [306]-[310], the calculations of this order have been finished. The obtained result is the following:

$$a_{e,\text{VIII}}^{\text{theor}} = -1.434(138).$$  (4.149)

If we use the most precise value for $\alpha$, the fine structure constant, Ref. [312], defined by means of the quantum Hall effects,

$$\alpha^{-1} = 137.035 997 9(32) \times 0.024 \text{ ppm},$$  (4.150)

then the most precise value for the AMM of an electron is:

$$a_e^{\text{theor}} = 1 159 652 140(27.1)(5.3)(4.1) \times 10^{-12},$$  (4.151)

that is in agreement with the experimental values for an electron and a positron with the precision of 1.7 standard deviation, Ref. [149].

$$a_e^{\text{exp}} = \begin{cases} 1 159 652 188.4(4.3) \times 10^{-12}, \\ 1 159 652 187.9(4.3) \times 10^{-12}. \end{cases}$$  (4.152, 4.153)

The theoretical result includes the analytically calculated contributions of the 2nd and 4th orders:

$$a_e^{\text{III}} = 0.5,$$  (4.154)

$$a_e^{\text{IV}} = \left[ 197 \frac{\pi^2}{144} + \frac{\pi^2 \log 2}{12} + \frac{3 \zeta(3)}{4} \right] = -0.328 478 965 \quad (\text{footnote} [\text{III}]),$$  (4.155)

$$a_e^{\text{VI}} = \left( \frac{m_e}{m_\mu} \right)^2 = \left( \frac{m_e}{m_\mu} \right)^2 + O \left( \left( \frac{m_e}{m_\mu} \right)^4 \log \frac{m_e}{m_\mu} \right) = 5.198 \times 10^{-7} \quad (\text{footnote} [\text{IV}]),$$  (4.156)

and the contribution of the sixth order, Refs. [322]-[325], consisting of 72 diagrams (with not all of them being calculated analytically), which have been corrected in Ref. [310]:

$$a_{e,\text{VII}}^{\text{theor}} = 1.176 11(42) \quad (\text{footnote} [\text{V}]).$$  (4.157)

---

46 The presented result is much more accurate than the preliminary one, $a_{e,\text{VIII}} = (-0.8 \pm 2.5)$.

47 The most considerable uncertainty (27.1) arises from the uncertainty of the fine structure constant; the second one, from the numerical calculations of $a_{e,\text{VII}}^{\text{theor}}$; the third one, from the numerical calculations of $a_{e,\text{VIII}}^{\text{theor}}$.

48 The hypothesis of Ref. [313], that this disagreement (1.7σ) is caused by existence of a scalar electron (supersymmetric particle), but not only by the experimental error of $\alpha$ and the numerical integration inaccuracy, is not of the present interest.

49 The history of experimental results is presented in [314].

50 Recently the contribution $a_{e,\text{IV}}$ was recalculated in the Fried-Yennie gauge. The result agrees with the previous ones [310], [313], [320].

51 The term of the fourth order $a_{e,\text{IV}}(\frac{m_e}{m_\mu})^2$, caused by existence of the diagrams with the radiative insertion of $\tau$-lepton into a vertex, is $\left( \frac{m_e}{m_\mu} \right)^2$ times smaller (4.156). The term $a_{e,\text{V}}(\frac{m_\mu}{m_\tau}, \frac{m_\tau}{m_e})$ of the order $(\frac{\alpha}{\pi})^3(\frac{m_\mu}{m_\tau})^2(\frac{m_\tau}{m_e})^2$ is negligible small for the modern level of experimental accuracy.

52 The Samuel’s result [326], $a_{e,\text{V}} = 1,184(5)$, is now considered to be overestimated.
as well as the small corrected terms appearing as a result of taking into account the muon, $\tau$-lepton and hadron vacuum polarization loops and the contribution of a weak interaction

$$\Delta a_e (\text{muon}) = 2.804 \cdot 10^{-12},$$  
$$\Delta a_e (\tau - \text{lepton}) = 0.010 \cdot 10^{-12},$$  
$$\Delta a_e (\text{hadron}) = 1.6(2) \cdot 10^{-12},$$  
$$\Delta a_e (\text{weak int.}) = 0.05 \cdot 10^{-12}.$$  

The recent analytically calculated contribution of the light-to-light subdiagrams in the sixth order of the AMM, $a_e$, is of the present interest, Ref. [327]:

$$a_{e, I}^{I, \text{VI}}(\gamma \gamma) = \frac{5}{6} \zeta(5) - \frac{5}{18} \pi^2 \zeta(3) - \frac{41}{540} \pi^4 - \frac{2}{3} \pi^2 \log^2 2 + \frac{2}{3} \log^4 2 +$$  
$$+ 16 a_4 - \frac{4}{3} \zeta(3) - 24 \pi^2 \log 2 + \frac{931}{54} \pi^2 + \frac{5}{9} \simeq 0.371 \ 005 \ 292 \ 1 \ldots,$$

$$a_4 = \sum_{n=1}^{\infty} \frac{1}{2n^4} \simeq 0.517 \ 479 \ 061 \ldots,$$

what agrees with the numerical estimations

$$a_{e, I}^{I, \text{VI}}(\gamma \gamma) = 0.36(4),$$  
$$a_{e, I}^{I, \text{VI}}(\gamma \gamma) = 0.37112(8),$$  
$$a_{e, I}^{I, \text{VI}}(\gamma \gamma) = 0.370986(20),$$

but disagrees with $^{53}$

$$a_{e, I}^{I, \text{VI}}(\gamma \gamma) = 0.398(5).$$

### 4.6.2 Muon

The AMM of a lepton of mass $m_1$ can be written in the following most general form:

$$a_i = a_1 + a_2 \left( \frac{m_1}{m_2} \right) + a_2 \left( \frac{m_1}{m_3} \right) + a_3 \left( \frac{m_1}{m_2}, \frac{m_1}{m_3} \right),$$  

where $m_2$ and $m_3$ are masses of other leptons.

Like for the AMM of an electron one has

$$a_i = a_{i, I} \left( \frac{1}{\pi} \right) + a_{i, IV} \left( \frac{1}{\pi} \right)^2 + a_{i, VI} \left( \frac{1}{\pi} \right)^3 + a_{i, VII} \left( \frac{1}{\pi} \right)^4 + \ldots$$

It is clear that $a_{2, I} = a_{3, I} = a_{3, IV} = 0$ because the Feynman diagrams, which could contribute to these terms, are absent.

Let us consider the modern status of investigations of the muon AMM. The value $a_3$ is known from the calculations of the electron AMM. The value $a_{2, IV} \left( \frac{m_\mu}{m_e} \right)$ is not so small in contrast with the electron AMM because of the large quantity $m_\mu/m_e$. The result is known analytically, Ref. [333]

$$a_{2, IV} \left( \frac{m_\mu}{m_e} \right) = \frac{1}{3} \log \frac{m_\mu}{m_e} - \frac{25}{36} \pi^2 m_e - 4 \left( \frac{m_e}{m_\mu} \right)^2 \log \frac{m_\mu}{m_e} + 3 \left( \frac{m_e}{m_\mu} \right)^2 + O \left( \frac{m_e}{m_\mu} \right)^3 =$$

$$= 1.094 \ 259 \ 6 \ldots$$

From the above formula the contribution

$$a_{2, IV} \left( \frac{m_\mu}{m_e} \right) \simeq 7.807(5) \cdot 10^{-5}$$

$^{53}$Recently, this result has been superseded by the authors, Ref. [339].
can be deduced, Ref. [331], what gives the contribution to the $a^\mu$ equal to $421.2(3) \times 10^{-12}$.

The term of the sixth order $a_2,VI(m_\mu/m_e)$ which arises from 18 Feynman diagrams, containing vacuum polarization loops, is also known analytically, Refs. [321]-[334]

$$a_{2,VI}(\frac{m_\mu}{m_e};v.p.) = a_e^{VI}(v.p.) + \left(\frac{\alpha}{\pi}\right)^3 \left[\frac{1075}{216} - \frac{25}{3} \zeta(2) + 10\zeta(2) \log 2 - 3\zeta(3) + 3c_4 + \left(\frac{31}{27} + \frac{2}{3} \zeta(2) - 4\zeta(2) \log 2 + 3\zeta(3)\right) \log \frac{m_\mu}{m_e} + \frac{2}{9} \log^2 \frac{m_\mu}{m_e} + O(\frac{m_\mu}{m_e})\right] =$$

$$= 1.94404 \left(\frac{\alpha}{\pi}\right)^3,$$

where

$$c_4 = \frac{11}{648} \pi^4 - \frac{1}{27} \pi^2 \log^2 2 - \frac{1}{27} \log^4 2 - \frac{8}{9} a_4.$$  \hspace{1cm} (4.173)

The corrections of the order $O(m_e/m_\mu)$ are known from Refs. [334, 337]

$$a_{2,VI}(\frac{m_e}{m_\mu}; v.p.) = \left(\frac{m_e}{m_\mu}\right)^4 \left[-\frac{13}{18} \pi^3 - \frac{16}{3} \pi^2 \log 2 + \frac{3199}{1080} \pi^2 \right] + \left(\frac{m_e}{m_\mu}\right)^2 \left[\frac{10}{3} \log^2 \frac{m_\mu}{m_e}\right] - \frac{11}{9} \log \frac{m_\mu}{m_e} - \frac{14}{3} \pi^2 \log 2 - 2\zeta(3) + \frac{49}{12} \pi^2 + \frac{131}{54} + \left(\frac{m_e}{m_\mu}\right)^3 \left[\frac{4}{3} \pi^2 \log \frac{m_\mu}{m_e} + \frac{35}{12} \pi^3 - \frac{16}{3} \pi^2 \log 2 - \frac{5771}{1080} \pi^2 \right] + \left(\frac{m_e}{m_\mu}\right)^4 \left[-\frac{25}{9} \log^3 \frac{m_\mu}{m_e} - \frac{1369}{180} \log^2 \frac{m_\mu}{m_e} + \frac{1369}{180} \log^2 \frac{m_\mu}{m_e}\right] +$$

$$= \left(-2\zeta(3) + 4\pi^2 \log 2 - 2\zeta(3) + 4\pi^2 - \frac{7496}{675} \log \frac{m_\mu}{m_e} - \frac{8}{9} \pi^2 \log 2 + \frac{80}{3} c_4 + \frac{10}{9} \log^2 2 + \frac{411}{32} \zeta(3) + \frac{88}{48} \pi^2 \log 2 - \frac{1061}{646} \pi^2 - \frac{274511}{54000} \right) + O((\frac{m_\mu}{m_e})^5).$$  \hspace{1cm} (4.175)

As a result the numerical value is $a_{2,VI}^\mu = 1.9204550(2)$.

For the first time, the contribution of 6 diagrams containing the light-to-light subdiagrams has numerically been calculated in Refs. [329, 335, 336]. The analytical result has been obtained recently, Ref. [338]:

$$a_{2,VI}(\frac{m_\mu}{m_e}; \gamma \gamma) = \frac{2}{3} \pi^2 \log \frac{m_\mu}{m_e} + \frac{59}{270} \pi^4 - 3\zeta(3) - \frac{10}{3} \pi^2 + \frac{2}{3} + \frac{m_\mu}{m_e} \left[\frac{4}{3} \pi^2 \log \frac{m_\mu}{m_e} - \frac{196}{3} \pi^2 \log 2 + \frac{424}{9} \pi^2 \right] + \left(\frac{m_\mu}{m_e}\right)^2 \left[\frac{2}{3} \log^3 \frac{m_\mu}{m_e} + \left(\frac{1}{3} \pi^2 - \frac{20}{9}\right) \log \frac{m_\mu}{m_e}\right] - \frac{16}{135} \pi^4 + 4\zeta(3) - \frac{283}{12} \pi^2 + \frac{61}{3} \log \frac{m_\mu}{m_e} + \frac{4}{3} \zeta(3) \pi^2 - \frac{61}{270} \pi^4 + 3\zeta(3) + \frac{25}{18} \pi^2 - \frac{283}{12} \pi^2 + \frac{m_\mu}{m_e} \left[\frac{10}{9} \pi^2 \log \frac{m_\mu}{m_e} - \frac{11}{9} \pi^2 \right] +$$

$$+ \left(\frac{m_\mu}{m_e}\right)^4 \left[\frac{7}{9} \log^3 \frac{m_\mu}{m_e} + \frac{41}{18} \log^2 \frac{m_\mu}{m_e} + \left(\frac{13}{9} \pi^2 + \frac{517}{108}\right) \log \frac{m_\mu}{m_e}\right] + \frac{2}{3} \zeta(3) + \frac{191}{216} \pi^2 + \frac{13283}{2592} \right] + O((\frac{m_\mu}{m_e})^5) = 20.9479242(9).$$  \hspace{1cm} (4.176)

Thus, the latest evaluations of this order of the muon AMM give the following numbers, respectively, Refs. [329, 335, 336]:

$$a_{2,VI}(\frac{m_\mu}{m_e}; v.p.) = 1.9200(14), \quad 1.92045(5), \quad 1.920450(2);$$

$$a_{2,VI}(\frac{m_\mu}{m_e}; \gamma \gamma) = 20.9471(29), \quad 20.9469(18), \quad 20.9479242(9);$$

$$a_{2,VI}(\frac{m_\mu}{m_e}; \text{sum}) = 22.8671(33), \quad 22.8674(18), \quad 22.8683792(11).$$

The term of the eighth order to the muon AMM was calculated from 469 Feynman diagrams, each

$^{54}$The result of Ref. [336], $a_{2,VI}^\mu(\gamma \gamma) = 21.32(5)$, has been invalidated by Samuel in Refs. [335, 339], thus finishing the discussion with Kinoshita, see above.
containing electron loops of the type of vacuum polarization, or the light-to-light subdiagrams. The result of numerical calculations of the eighth order was presented in \[340\]–\[355\]:

\[
a_{\mu, \text{VIII}}(m_\mu / m_e) = 126.92(41) \quad (4.177)
\]

In Refs. \[342\]–\[344\] the asymptotic (when \(m_\mu / m_e \to 0\)) contributions to the muon AMM have been obtained in the analytical form by using the renormalization group technique. They come from the diagrams of the eighth order with one loop of electron vacuum polarization, Ref. \[344\],

\[
a_{\mu, \text{VIII}}(v.p.) = -\frac{1}{32} \log \frac{m_\mu}{m_e} + \frac{17}{48} + \frac{5}{8} \zeta(2) - \zeta(2) \log 2 + \frac{99}{128} \zeta(3) - \frac{5}{4} \zeta(5) = -0.290987\ldots \quad (4.178)
\]

and from the diagrams with two loops of electron vacuum polarization, Ref. \[342, 343\],

\[
a_{\mu, \text{VIII}}(v.p.) = \frac{1}{12} \log^2 \frac{m_\mu}{m_e} + \left[ \frac{1}{3} \zeta(3) - \frac{2}{3} \right] \log \frac{m_\mu}{m_e} + \frac{1531}{1728} + \frac{5}{12} \zeta(2) - \frac{1025}{1152} \zeta(3) = 1.452570\ldots \quad (4.179)
\]

Moreover, the result for the diagram of the tenth order (additional loop of vacuum polarization) is now known, Refs. \[344, 345\],

\[
a_{\mu, \text{X}}(v.p.) = -\frac{1}{24} \log^2 \left( \frac{m_\mu}{m_e} \right) + \left[ -\frac{5}{3} \zeta(5) + \frac{33}{32} \zeta(3) - \frac{4}{3} \zeta(2) \log 2 + \frac{5}{6} \zeta(2) + \frac{161}{288} \right] + \frac{125}{36} \zeta(5) - \frac{275}{128} \zeta(3) + \frac{25}{9} \zeta(2) \log 2 - \frac{16}{9} \zeta(2) - \frac{3409}{3456} = -1.3314\ldots \quad (4.180)
\]

The numerical estimation of the tenth order is, Ref. \[340\],

\[
a_2, \text{X}(\frac{m_\mu}{m_e}) = 570(140). \quad (4.181)
\]

The terms

\[
a_3, \text{VII}(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}) = 5.24(1) \times 10^{-4} \quad (4.182)
\]

\[
a_3, \text{VIII}(\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}) = 0.079(3) \quad (4.183)
\]

\[55\] The preliminary value has been given in \[337\], \(a_{\mu, \text{VIII}} = 140(6)\).

\[56\] When we have submitted this review to print we learned about Kinoshita’s recalculation \[341\] of his result of 1990, namely, of the contribution of the eight-order vertices containing sixth-order one-electron-loop vacuum polarization subdiagrams. Including new result, which is close to the asymptotic analytic result of Broadhurst et al. \[344\], the value \(a_{\mu, \text{VIII}}(m_\mu / m_e) = 127.55(41)\) supersedes the result \(4.177\). Thus, the improved value of \(4.184\) is \(a_\mu(QED) = 1165846984(17)(28) \times 10^{-12}\).

\[57\] When we have submitted this review to print we learned about the remarkable results of the paper \[346\] where the \((n + 1)\)-loop contributions to muonic anomaly have been obtained analytically from the 1-loop diagrams with insertion of the \(n\)-loop photon propagator containing \(n - 1\) electron loops.

\[58\] See also Ref. \[347\] where the same technique has been used for calculation of the Kallan-Simanzik \(\beta\)-function in the eighth order.
have also been calculated there. Combining the above results with the term $a_1$ known from Kinoshita obtained the pure quantum-electrodynamics contribution:

$$a_\mu(QED) = 1165846947(46)(28) \times 10^{-12}. \quad (4.184)$$

The recalculated result of Samuel, Ref. [335], is

$$a_\mu(QED) = 1165846950(28)(27) \times 10^{-12}. \quad (4.185)$$

After adding the hadron contribution, Refs. [348]-[351]:

$$a_\mu(hadron) = 7.03(19) \times 10^{-8} \quad (4.186)$$

and the weak interaction one, Refs. [352, 353]

$$a_\mu(weak\,int.) = 181(1) \times 10^{-11} \quad (4.187)$$

one gets the following value for the muon AMM, Refs. [335, 340], respectively:

$$a_\mu,\, theor^{[335]} = 116591902(77) \times 10^{-11} \quad (4.188)$$

$$a_\mu,\, theor^{[340]} = 116591920(191) \times 10^{-11}. \quad (4.189)$$

which are in good agreement with the known experimental values, Ref. [354]

$$a_\mu^{exp} = 1165937(12) \times 10^{-9},$$

$$a_\mu^{exp} = 1165911(11) \times 10^{-9}. \quad (4.190)$$

Let us point out that the experiments planned at BNL, Brookhaven, will increase twenty times the accuracy of experiment (up to $\pm0.05 \times 10^{-8}$), Ref. [355].

### 4.6.3 $\tau$ – lepton

The theoretical calculations of the $\tau$ – lepton AMM have been carried out in Refs. [356, 357]. The obtained estimations are the following:

$$a_\tau = 11773(3) \times 10^{-7}. \quad (4.191)$$

Experimentally, the quantity $a_\tau$ is investigated less completely than theoretically. The direct experiment is lack. The cause of this bad situation is a very small life-time of $\tau$- lepton, Ref. [358], $T = 2.957(32) \times 10^{-13}c$. However, recently the experimental methods to check theoretical predictions have been proposed, Refs. [359]-[362]. Moreover, some constraints, Ref. [363], have been obtained from the analysis of electroweak experimental data:

$$-8 \times 10^{-3} \leq a_\tau \leq 1 \times 10^{-2} \quad (2\sigma). \quad (4.192)$$

For the completeness let us reproduce the data on the energy levels of $\tau^+\tau^-$ atom, Ref. [364]. It can be made in $e^+e^-$ annihilation below $\tau$ pair threshold. It is clear that the energy levels are defined from the formula:

$$E_n = \frac{m_\tau c^2\alpha^2}{4n^2} = -\frac{23.7}{n^2} \text{keV}, \quad (4.193)$$

---

59 The first error is the estimation of the theoretical uncertainty, mostly from $a_{2,\, V\, I}(\frac{m_e}{m_\tau})$, the second one is from the uncertainty of $\alpha$ determined in quantum Hall effect experiments.

60 The recent result of Dubička et al. [351] is more exact than the result presented by Kinoshita, $a_\mu^{(hadron)} = 6.986 \pm 0.042 \pm 0.016 \times 10^{-8}$. The model and experimental errors were also given there.

61 The shown uncertainty is caused by the lack of information about Higgs mass.
where \( m_\tau = 1777.8 \pm 0.7 \pm 1.7 \, MeV/c^2 \), Ref. \[365\], has been used. The decay widths are
\[
\Gamma(\tau^+\tau^- \to 2\gamma) = \frac{\alpha^5 m_\tau c^2}{2n^3} = \frac{1.8 \times 10^{-2}}{n^3} \, eV,
\]
\[
\Gamma(\tau^+\tau^- \to 3\gamma) = \frac{2(\pi^2 - 9)\alpha^6 m_\tau c^2}{9n^3} = \frac{1.7 \times 10^{-5}}{n^3} \, eV.
\]
(4.194)
(4.195)

More complete consideration of \( \tau \)-physics you can find in the excellent reviews, Ref. \[366\].

4.7 Fine structure constant

The experimental values for the fine structure constant are following.

a) The fine structure constant defined from a quantum Hall effect is, Ref. \[312\],
\[
\alpha^{-1}(QHE) = 137.0359979(32) \, (0.024 \, ppm); \tag{4.196}
\]
b) on the basis of calculations of the eighth order of the AMM of an electron and its comparison with the experimental values, Refs. \[149, 310\],
\[
\alpha^{-1}(ae) = 137.03599222(51)(63)(48) \, (0.0069 \, ppm); \tag{4.197}
\]
c) on the basis of Josephson’s effects, Ref. \[367\],
\[
\alpha^{-1}(JE) = 137.0359770(77) \, (0.056 \, ppm); \tag{4.198}
\]
d) on the basis of comparison of the experimental and theoretical results of the HFS in a muonium \[137\],
\[
\alpha^{-1}(\mu-hfs) = 137.035992(22) \, (0.16 \, ppm). \tag{4.199}
\]

The results (a) and (b) agree with each other up to the error level 0.05 ppm, thus proving the validity of QED to the above accuracy. However, further enhancing of experimental accuracy of measuring of \( \alpha^{-1}(QHE) \) and on the basis of other methods is desirable.

A possible origin of the discrepancy between (4.196) and (4.197) is a possible subquark structure of an electron, Ref. \[368\], and the some unknown type of an interaction mediated by particles more massive than \( W^\pm \) and \( Z^0 \)-bosons.

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\[62\] The first uncertainty originates from the experimental uncertainty, the second one is from the uncertainty of the sixth order calculations, the third one is from the uncertainty of the eighth order calculations. The present result, Ref. \[310\], is 3 times more precise than the one in \[312\] and, moreover, than the ones obtained by the other methods.

\[63\] However, the contribution, caused by this interaction, can not be large because of the \( m/M \) term. The value \( a_{\text{theor}} \) is more sensitively to the case when there is an unknown light particle interacting with an electron, Ref. \[7\].
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