Physical States, Factorization and Nonlinear Structures in Two Dimensional Quantum Gravity

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ABSTRACT

The nonlinear structures in 2D quantum gravity coupled to the \((q + 1, q)\) minimal model are studied in the Liouville theory to clarify the factorization and the physical states. It is confirmed that the dressed primary states outside the minimal table are identified with the gravitational descendants. Using the discrete states of ghost number zero and one we construct the currents and investigate the Ward identities which are identified with the \(W\) and the Virasoro constraints. As nontrivial examples we derive the \(L_0, L_1\) and \(W^{(3)}_{-1}\) equations exactly. \(L_n\) and \(W^{(k)}_n\) equations are also discussed. We then explicitly show the decoupling of the edge states \(O_j (j = 0 \mod q)\). We consider the interaction theory perturbed by the cosmological constant \(O_1\) and the screening charge \(S^+ = O_{2q+1}\). The formalism can be easily generalized to potential models other than the screening charge.

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1 Introduction

Since the discovery of the double scaling limit in the matrix models \[1, 2, 3\], many efforts have been made to understand 2D quantum gravity. In the matrix model approach it was shown that the amplitudes obey the nonlinear equations called the W and the Virasoro constraints \[4, 5\]. We have tried to investigate the nonlinear structures in the Liouville theory approach \[6–15\] in order to clarify the nature of physical states and factorization properties of amplitudes. There are a few works \[12, 13\], but clear arguments have not been found yet.

In the Liouville theory it has been found that there are an infinite number of the BRST invariant states. Although the discrete states are discussed in detail \[14, 15\], the physical role of them are not sufficiently understood. Here we discuss the role of the discrete states with ghost number zero and one according to the classification by Bouwknegt, McCarthy and Pilch (BMP). We also consider the dressed primary states both inside and outside the minimal table. As shown by Kitazawa \[11\] the dressed primary states outside the minimal table no longer decouple in the combined matter-Liouville theory. Furthermore he claimed these fields to be the gravitational descendants. In this paper, using the discrete states of the ghost number zero and one, we construct symmetry currents.\footnote{The importance of the BRST invariant discrete states of ghost number zero is emphasized in the two dimensional string theory \[16, 17, 18\]. Also in ref. \[12, 13\] they play an important role.} Substituting the currents into the correlation functions of the dressed primary fields we set up the Ward identities and identify them with the W and the Virasoro constraints.

The paper is organized as follows. In Sect.2 we summarize the various BRST invariant states of the \((q + 1, q)\) minimal theory coupled to gravity and define the symmetry currents. The correlation functions are defined along the argument of Goulian and Li \[10\]. We consider the interaction theory perturbed by the cosmological constant \(O_1\) and the screening charge \(S^+ = O_{2q+1}\), where another screening charge \(S^-\) is not used. In Sect.3 we discuss the factorization properties of amplitudes. The factorization in the Liouville theory is rather similar with that of the string theory \[8, 9\]. However there is a crucial difference in the pole structures of the propagators, which leads to the nonlinear structures in the Liouville theory approach. In Sect.4 we discuss the Ward identities corresponding to the Virasoro constraints. As nontrivial examples we derive \(L_0\) and \(L_1\) equations exactly. We also discuss the \(L_n\) equations. The Ward identities corresponding to the W constraints are discussed in Sect.5, where \(W^{(3)}_{-1}\) equation is derived exactly. \(W^{(k)}_n\) equations are briefly discussed. Then we explicitly show that the edge states \(O_j (j = 0 \mod q)\) decouple from the expressions. Sect.6 is devoted to discussion. Here we argue the generalization of the formalism. If we consider the interaction theory perturbed by the scaling operator \(O_{p+q}\) instead of the screening charge \(O_{2q+1}\), we will get the gravity theory coupled to \((p, q)\) conformal matter. We also comment on the unitarity problem in the two dimensional quantum gravity.
2 BRST invariant states and correlation functions

In this section we summarize the Liouville theory approach to 2D quantum gravity coupled to the minimal conformal matter with central charge \( c_M = 1 - \frac{6}{q(q+1)} \) and establish our notations and conventions. The action for the Liouville-matter part is

\[
S_0 = \frac{1}{8\pi} \int \sqrt{g} \left( \partial_\alpha \phi \partial_\beta \phi + 2iQ_L \hat{R} \phi \right) + \frac{1}{8\pi} \int \sqrt{g} \left( \partial_\alpha \varphi \partial_\beta \varphi + 2iQ_M \hat{R} \varphi \right),
\]

where the scalar fields \( \phi \) and \( \varphi \) are the Liouville and the matter fields respectively. The background charges are

\[
iQ_L = \frac{2q + 1}{\sqrt{2q(q+1)}}, \quad Q_M = \frac{1}{\sqrt{2q(q+1)}},
\]

where we use the convention that \( Q_L \) is purely imaginary. In terms of modes

\[
i\partial \phi(z) = \sum_{n \in \mathbb{Z}} \alpha_n^L z^{-n-1}, \quad i\partial \varphi(z) = \sum_{n \in \mathbb{Z}} \alpha_n^M z^{-n-1},
\]

the Virasoro algebras are given by

\[
L_{n,M} = \frac{1}{2} \sum_{m \in \mathbb{Z}} :\alpha_n^M \alpha_{n-m}^L: - (n+1)Q_{L,M} \alpha_n^L, \alpha_n^M,
\]

where \([\alpha_n^L, \alpha_m^L] = [\alpha_n^M, \alpha_m^M] = n\delta_{n+m,0}\).

The physical states are identified with nontrivial cohomology classes of the BRST operator

\[
Q_{BRST} = \oint \frac{dz}{2\pi i} c(z) \left( T^L(z) + T^M(z) + \frac{1}{2} T^G(z) \right).
\]

In terms of modes

\[
Q_{BRST} = c_0 L_0 - b_0 M + \hat{d},
\]

\[
L_0 = L_0^L + L_0^M + L_0^G, \quad M = \sum_{n \neq 0} n :c_{-n} c_n:,
\]

\[
\hat{d} = \sum_{n \neq 0} c_{-n}(L_n^L + L_n^M) - \frac{1}{2} \sum_{n,m \neq 0, n+m \neq 0} (m-n) :c_{-m} c_{-n} b_{n+m}:.
\]

By introducing the lightcone-like combinations of the modes \( \alpha_n^\pm = \frac{1}{\sqrt{2}}(\alpha_n^M \pm i\alpha_n^L) \), \( n \neq 0 \) and the generalized momentum variables

\[
P^\pm(n) = \frac{1}{\sqrt{2}} \left[ (\alpha_0^M - (n+1)Q_M) \pm i(\alpha_0^L - (n+1)Q_L) \right],
\]

3
the operator $\hat{d}$ is rewritten as

$$\hat{d} = \sum_{n \neq 0} c_{-n}(\alpha^-_n P^+(n) + \alpha^+_n P^-(n))
+ \sum_{n, m \neq 0} c_{-n}(\alpha^-_{m-n} \alpha^+_{n+m} + \frac{1}{2}(m - n)c_{-n}b_{n+m}) \quad (2.10)$$

BMP [15] classify the nontrivial cohomology of the BRST charge. If $P^+(r) \neq 0$ or $P^-(s) \neq 0$ for all $r, s \in \mathbb{Z}$, $r, s \neq 0$ and $P^+(0)P^-(0) = 0$, then there exist the dressed primary states with momentums $i\alpha^L_0 = \alpha_j$ and $\alpha^M_0 = \beta_j$ parametrized by

$$\alpha_j = (2q + 1 - j)Q_M \quad \beta_j = (1 + j)Q_M \quad (2.11)$$

The corresponding fields are given by

$$O_j = \int d^2zV_j(z, \bar{z}) = \int d^2ze^{\alpha_j\phi(z, \bar{z})}e^{i\beta_j\varphi(z, \bar{z})} \quad (2.12)$$

Here we use the unusual convention for $\beta_j$. The reason why we adopt it will be mentioned after correlation functions are defined. We identify these fields with the gravitational primaries and their descendants as

$$O_{nq+k} = \sigma_n(O_k) \quad (k = 1, \cdots, q - 1; \ n \in \mathbb{Z}_{\geq 0}) \quad (2.13)$$

where $n = 0$ states are gravitational primaries. We now exclude the edge states of $k = 0$. This identification was first proposed by Kitazawa [11], who showed that the dressed primary states outside the minimal table fail to decouple in the combined matter-Liouville theory. In the following section, by deriving the non-linear structures directly in the Liouville theory approach, we will confirm this identification and show that the edge states indeed decouple.

Besides the dressed primary states there exist the nontrivial BRST invariant states called the discrete states at the momentums

$$i\alpha^L_0 = \alpha_{-(r+s)q-r} \quad \alpha^M_0 = \beta_{(r-s)q+r} \quad (2.14)$$

In this paper we only consider the discrete states of $r, s < 0$. Then there exist the states of ghost number zero, $B_{r,s}(z)$. For examples we get

$$B_{-1,-1}(z) = 1 \quad (2.15)$$

$$B_{-1,-2}(z) = [bc - \sqrt{\frac{q+1}{2q}}(\partial\phi + i\partial\varphi)]e^{\alpha_{q+1}\phi(z)}e^{i\beta_{q-1}\varphi(z)} \quad (2.16)$$

$$B_{-2,-1}(z) = [bc - \sqrt{\frac{q}{2(q+1)}}(\partial\phi - i\partial\varphi)]e^{\alpha_{q+2}\phi(z)}e^{i\beta_{q-2}\varphi(z)} \quad (2.17)$$

Note that $\beta_{r,s} = \beta_{(r-s)q+r}$, where $\beta_{r,s} = \frac{1}{\sqrt{2}}(1 + r)\beta_+ + \frac{1}{\sqrt{2}}(1 + s)\beta_-$, $\beta_+ = \sqrt{\frac{q+1}{q}}$ and $\beta_- = -\sqrt{\frac{q}{q+1}}$.
There also exist the primary states $R_{r,s}$ with the momentums (2.14). For instance

$$ R_{-1,-1}(z) = \partial \phi(z) + (2q + 1)i\partial \varphi(z) , $$ (2.18)

$$ R_{-1,-2}(z) = \frac{1}{q} \left[ \frac{q - 3}{4} \partial^2 \phi + \frac{q + 3}{4} i\partial^2 \varphi + \frac{q^2 - 2q - 1}{8q(q + 1)} \left( (\varphi)^2 - (\partial \varphi)^2 \right) \right] e^{\alpha_{3q+1}(z)} e^{i\beta_{q-1}(z)} , $$ (2.19)

$$ R_{-2,-1}(z) = \frac{1}{q + 1} \left[ \frac{q + 4}{4} \partial^2 \phi + \frac{q}{4} i\partial^2 \varphi - \frac{q^2 + 4q + 2}{8q(q + 1)} \left( (\varphi)^2 - (\partial \varphi)^2 \right) \right] e^{\alpha_{3q+2}(z)} e^{i\beta_{q-2}(z)} , $$ (2.20)

where we remove $c$ ghost in the definition of $R_{r,s}$.

Combining $R_{r,s}$ and $\bar{B}_{r,s}$ we can construct the symmetry currents

$$ W_{r,s}(z, \bar{z}) = R_{r,s}(z) \bar{B}_{r,s}(z) , \quad r, s \in \mathbb{Z}_- , $$ (2.21)

which satisfy

$$ \partial \bar{z} W_{r,s}(z, \bar{z}) = \{ \bar{Q}_{BRST}, [\bar{b}_{-1}, W_{r,s}(z, \bar{z})] \} . $$ (2.22)

Main assertion of this paper is that the Ward identities of the currents

$$ \int d^2z \partial \bar{z} \ll W_{-k,-n-k}(z, \bar{z}) \prod_{j \in S} O_j \gg_g = 0 , \quad (k = 1, \ldots q - 1; n \in \mathbb{Z}_\geq 2) $$ (2.23)

are just the $W_{q}^{(k+1)}$ constraints. The equations for $k = 1$ is the Virasoro constraints and others are the $W$ constraints.

Let us define the correlation functions of the Liouville theory. We consider the interaction theory,

$$ S = S_0 + \mu O_1 - t O_{2q+1} , $$ (2.24)

where $O_1$ is the cosmological constant and $O_{2q+1}$ is nothing but the screening charge $S^+ = e^{i\sqrt{2}\beta_+ \varphi}$, $\beta_+ = \sqrt{(q + 1)/q}$. Note that we do not use another screening charge $S^- = e^{i\sqrt{2}\beta_- \varphi}$, $\beta_- = -\sqrt{q/(q + 1)}$ because it is not included in the definition of the scaling operators (2.13).

The matter sector has the symmetry under the constant shift $\varphi \to \varphi + 2\pi/qM$ because then the action only shifts by $2\pi i\chi$, where $\chi$ is the Euler number, such that $e^{-S}$ is invariant. Therefore we restrict the range of the zero mode integral of $\varphi$ within $0 \leq \varphi_0 \leq 2\pi/qM$. On the other hand, since $Q_L$ is purely imaginary, there is no such a symmetry for the Liouville sector so that we do not restrict the range of zero mode of $\phi$. After integrating over the zero modes of the Liouville and the matter fields the correlation functions are expressed as

$$ \ll \prod_{j \in S} O_j \gg_g = \kappa^{-\chi\mu^i} \frac{\Gamma(-s)}{\alpha_1} \frac{2\pi}{Q_M} \frac{t^n}{n!} \ll \prod_{j \in S} O_j \left( O_1 \right)^s (O_{2q+1})^n \gg_g , $$ (2.25)
where $g$ is genus, $\chi = 2 - 2g$ and
\[ s = \frac{1}{\alpha_1}[i Q L \chi - \sum_{j \in S} \alpha_j] = \frac{1}{2q} [(2q + 1) \chi - \sum_{j \in S} (2q + 1 - j)] , \tag{2.26} \]
\[ n = \frac{1}{\beta_{2q+1}}[Q_M \chi - \sum_{j \in S} \beta_j - \beta_1 s] = \frac{1}{2q} [-\chi + \sum_{j \in S} (1 - j)] . \tag{2.27} \]

We introduce the string coupling constant $\kappa$. The $\Gamma$-function comes from the zero mode integral of $\phi$. The $\varphi_0$ integral gives the Kronecker delta multiplied by $2\pi/Q_M$ which guarantees the momentum neutrality of matter sector. The expression connects between the correlators in the interaction picture $\ll \cdots \gg_g$ and ones in the free picture $< \cdots >_g$. If $s$ and $n$ are integers, the correlation functions can be calculated. However $s$ and $n$ are not integers in general. According to the argument of Goulian and Li [10] we define the correlators by analytic continuations in $s$ and $n$. Then $n!$ is defined by $\Gamma(n+1)$.

In the free picture correlators the Liouville field satisfies the equation of motion $\bar{\partial} \partial \phi = 0$. Then we get the following Ward identity for the Liouville sector,
\[
\bar{\partial} \ll \partial \phi(z) \prod_{j \in S} V_j(z_j, \bar{z}_j) \gg^{(L)}_g \\
= -\pi \alpha_1 s \mu \mu' \alpha_1 \Gamma(-s) \prod_{j \in S} V_j(z_j, \bar{z}_j)(O_1)^{s-1} \gg^{(L)}_g \\
-\pi \sum_{k \in S} \alpha_k \delta^2(z - z_k) \mu' \alpha_1 \Gamma(-s) \prod_{j \in S} V_j(z_j, \bar{z}_j)(O_1)^s \gg^{(L)}_g \tag{2.28} \\
= \pi \alpha_1 \mu \ll V_1(z, \bar{z}) \prod_{j \in S} V_j(z_j, \bar{z}_j) \gg^{(L)}_g \\
-\pi \sum_{k \in S} \alpha_k \delta^2(z - z_k) \ll \prod_{j \in S} V_j(z_j, \bar{z}_j) \gg^{(L)}_g ,
\]

where we use the operator product
\[
\partial \phi(z) e^{\alpha \phi(w, \bar{w})} = -\frac{\alpha}{z - w} e^{\alpha \phi(w, \bar{w})} \tag{2.29}
\]
in the free picture and $\bar{\partial} z = \pi \delta^2(z - w)$. In the second equality the relation $-s \Gamma(-s) = \Gamma(1 - s)$ is used. This expression indicates that in the correlator of the interaction picture the Liouville field satisfies the equation of motion
\[
\bar{\partial} \partial \phi(z, \bar{z}) = \pi \alpha_1 \mu V_1(z, \bar{z}) . \tag{2.30}
\]

For the matter sector also the same argument succeeds.

Finally we comment on the convention of the matter momentum $\beta_j$. If we adopt the convention $\beta_j = (1 - j)Q_M$ and the theory is perturbed by $S^-$, we meet the divergences when the correlation functions include the fields of $j = n(q + 1)$, $n \in \mathbb{Z}_+$. So we need the regularization. On the other hand, in our convention, such a divergence do not appear and then we can explicitly show the decoupling of the edge states $O_j$ $(j = 0 \mod q)$.
3 Factorization properties of amplitudes

The structures of factorization in 2D quantum gravity are rather similar to the string theory [8, 9]. However, in a fundamental point, they are different. The difference leads to the nonlinear structures of 2D quantum gravity. In the following we develop the argument comparing the amplitudes of the string theory and 2D quantum gravity.

Amplitudes in the string theory can be decomposed into vertex operators and propagators

\[ D = \int_{|z| \leq 1} \frac{d^2 z}{|z|^2} z^{L_0} \bar{z}^{\bar{L}_0} = 2\pi \left( \frac{1}{H} - \lim_{\tau \to \infty} \frac{1}{H} e^{-\tau H} \right), \]  

where \( H = L_0 + \bar{L}_0 \). The last term stands for the boundary of moduli space pinching 2D surface. The intermediate states are expanded by the normalizable off-shell eigenstates of the Hamiltonian \( H \). The propagator \( 1/H \) describes the propagation of the off-shell modes.

In the Liouville theory the intermediate states are also expanded by the normalizable eigenstates of \( H \),

\[ H |p, \beta_k, N > = (p^2 - Q_L^2 + 2(\Delta_k + N - 1)) |p, \beta_k, N > , \]  

where \( p \) is real. \( \Delta_k \) is the conformal dimension of matter sector,

\[ \Delta_k = \frac{k^2 - 1}{4q(q + 1)}. \]  

The integer \( N \) stands for the oscillation level of the states. The zero level states are defined by

\[ |p, \beta_k > = e^{i(p + Q_L)\phi(0)} e^{i\beta_k \varphi(0)} |0 >_{L,M} \otimes \bar{c}_1 c_1 |0 >_G . \]  

We take the following normalization,

\[ \ll p', \beta_{k'}, N'|p, \beta_k, N \gg_{g=0} = \kappa^{-2} 2\pi \delta(p + p') \frac{2\pi}{Q_M} \delta_{k+k',0} \delta_{N,N'}. \]  

The zero mode integral of the Liouville field now produces the \( \delta \)-function.

Let us consider a channel of factorization of correlators (2.25) divided into sets \( F_1 \) and \( F_2 \) composed of the operators in \( S \). By inserting the complete set we get the expression

\[ \ll \kappa^2 Q_M \frac{2\pi}{2\pi} \sum_{N=0}^{\infty} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ll F_1 | - p, \beta_{-k}, N \gg \]

\[ \times \frac{2\pi}{p^2 + E_{k,N}} \left( 1 - \lim_{\tau \to \infty} e^{-\tau(p^2 + E_{k,N})} \right) \ll p, \beta_k, N | F_2 \gg , \]  

(3.6)
where $E_{k,N} = -Q^2_k + 2(\Delta_k + N - 1)$. The cosmological constants and the screening charges are properly factorized. Note that, since $E_{k,N}$ is always positive, the pole of the Liouville momentum $p$ lays on the imaginary axis. Therefore the $p$ integral for $1/H$ part can be deformed to the complex plane and picks up only the on-shell ($H = 0$) poles on the imaginary axis. This indicates that in the string theory the intermediate states are off-shell so that the boundary term vanishes in the limit $\tau \to \infty$, while in the 2D quantum gravity the intermediate states becomes on-shell and so we can not always ignore the boundary term. In fact it plays an important role when the correlators include the currents $W_{r,s}$.

4 Virasoro equations

The Virasoro constraints derived from the matrix model are described symbolically as

$$L_n = \frac{1}{2\lambda} \sum_{-k-l=nq} kl x_k x_l + \sum_{-k+m=nq} k x_k \partial_m + \frac{\lambda}{2} \sum_{k+l=nq} \partial_k \partial_l , \quad (4.1)$$

where $x_{mq}$ and $\partial_{mq}$, $m \in \mathbb{Z}_{\geq 0}$ are discarded. The aim of this section is to derive the Virasoro constraints as the Ward identities of the currents $W_{r,s}$.

4.1 $L_0$ equation

Let us discuss the Ward identity for the current $W_{-1,-1}$. This is rather trivial. As is easily understood from the equation (2.28) it is equivalent to the momentum neutrality conditions which are, by definition (2.25), expressed as

$$t \ll O_{2q+1} \prod_{j \in S} O_j \gg_g = n \ll \prod_{j \in S} O_j \gg_g , \quad (4.2)$$

$$-\mu \ll O_1 \prod_{j \in S} O_j \gg_g = s \ll \prod_{j \in S} O_j \gg_g , \quad (4.3)$$

where $s, n$ are given by (2.26) and (2.27). Combining these equations we get

$$(2q + 1) t \ll O_{2q+1} \prod_{j \in S} O_j \gg_g + x \ll O_1 \prod_{j \in S} O_j \gg_g + \left( \sum_{j \in S} j \right) \ll \prod_{j \in S} O_j \gg_g = 0 , \quad (4.4)$$

\footnote{From eq.(4.2) we can derive the normalization independent ratio

$$\frac{\ll O_{2q+1} O_k O_k \gg_s^2}{\ll O_{2q+1} O_{2q+1} \gg_0} = \frac{k^2}{q + 1} ,$$

which agree with the result derived in ref. [1].}
where \( x = -\mu \). This is nothing but the \( L_0 \) equation

\[
L_0 = \sum_k kx_k \partial_k
\] (4.5)

with \( x_1 = x \), \( x_{2q+1} = t \) and other \( x_j \)'s = 0.

### 4.2 \( L_1 \) equation

The first nontrivial example is the Ward identity for the current \( W_{-1,-2} \). The operator product expansion (OPE) between the current and the scaling operator is given by

\[
W_{-1,-2}(z, \bar{z}) O_k(w, \bar{w}) = \frac{1}{z-w} \frac{k^2(q+k)}{4q^3} \sqrt{\frac{q+1}{q}} O_{q+k}(w, \bar{w}) ,
\] (4.6)

where we use the notation

\[
O_k(z, \bar{z}) = \bar{c}(\bar{z})c(z)V_k(z, \bar{z}) .
\] (4.7)

such that \( O_k = \int d^2 z b_{-1} \bar{b}_{-1} \cdot O_k(z, \bar{z}) \). The derivative \( \partial_{\bar{z}} \) picks up the OPE singularity and so we get

\[
0 = \int d^2 z \partial_{\bar{z}} W_{-1,-2}(z, \bar{z}) \prod_{j \in S} O_j \gg_g
\]

\[
= \pi \frac{(2q+1)^2(3q+1)}{4q^3} \sqrt{\frac{q+1}{q}} t \ll O_{3q+1} \prod_{j \in S} O_j \gg_g
\]

\[
- \pi \frac{(q+1)}{4q^3} \sqrt{\frac{q+1}{q}} \mu \ll O_{q+1} \prod_{j \in S} O_j \gg_g
\]

\[
+ \pi \frac{1}{4q^2} \sqrt{\frac{q+1}{q}} \sum_{k \in S} k^2(q+k) \ll O_{q+k} \prod_{j \in S \setminus \{j \neq k\}} O_j \gg_g
\]

\[
+ \int d^2 z \ll \partial_{\bar{z}} W_{-1,-2}(z, \bar{z}) \prod_{j \in S} O_j \gg_g .
\] (4.8)

The first and the second correlators of r.h.s. come from the OPE with the screening charge \( O_{2q+1} \) and the cosmological constant \( O_1 \) respectively. Usually the last correlator would vanish because the divergence of the current is the BRST trivial (2.22). However, as discussed in Sect. 3, the boundary of moduli space pinching 2D surface are dangerous and we have to evaluate it carefully.

Using the expression of factorization we calculate the following quantity

\[
\kappa^2 Q M \sum_{N=0}^{\infty} \sum_{k=1}^{\infty} \frac{dp}{2\pi} < F_1 \int_{|z| \leq 1} d^2 z \partial_{\bar{z}} W_{-1,-2}(z, \bar{z})
\]

\[
\times D \mid -p, \beta_{-k}, N \rangle \langle p, \beta_k, N \mid F_2 > ,
\] (4.9)
where $D$ is the propagator. $F_1$ and $F_2$ are sets composed of the operators in $S$ and also the cosmological constants and the screening charges now. Since the BRST charge commutes with the Hamiltonian, there is no contribution from $1/H$ term in the propagator. While the boundary term is singular and there is a possibility that nonvanishing quantities remain in the limit $\tau \to \infty$. So we evaluate

$$
\lim_{\tau \to \infty} \kappa^2 \frac{Q_M}{2\pi} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} dp \int_{e^{-\tau} \leq |z| \leq 1} d^2 z < F_1 \left[ b_{-1}, W_{-1,-2}(z, \bar{z}) \right]
\times \bar{Q}_{\text{BRST}} \left( -\frac{2\pi}{H} e^{-\tau H} \right) | - p, \beta_{-k} > < p, \beta_k | F_2 > ,
$$

(4.10)

where we omit $N \neq 0$ modes because, as discussed below, these modes vanish at $\tau \to \infty$. Noting $\tilde{d} | - p, \beta_{-k} > = \tilde{b}_0 | - p, \beta_{-k} > = 0$ we get

$$
\bar{Q}_{\text{BRST}} \left( -\frac{2\pi}{H} e^{-\tau H} \right) | - p, \beta_{-k} > = -\pi \tilde{c}_0 e^{-\tau H} | - p, \beta_{-k} > .
$$

(4.11)

From this and the relation $[\bar{b}_{-1}, \bar{B}_{-1,-2}(\bar{z})] = -\bar{b}(\bar{z}) e^{i\alpha_{3q+1} \phi(z)} e^{i\beta_q - \varphi(z)}$, we obtain the expression

$$
\lim_{\tau \to \infty} \kappa^2 \frac{Q_M}{4\pi} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} dp A_{-1,-2}(i(-p + Q_L), \beta_{-k}) e^{-\tau (p^2 - Q_L^2 + 2\Delta_k - 2)}
\times \int_{e^{-\tau} \leq |z| \leq 1} d^2 z | z \rangle^2 \langle -2 - i\alpha_{3q+1}(-p + Q_L) - \Delta_{q-1} - \Delta_k + \Delta_{q-k} | < F_1 | - p - i\alpha_{3q+1}, \beta_{q-k} > < p, \beta_k | F_2 > ,
$$

(4.12)

where $\Delta_k$ is defined by (3.3) and

$$
A_{-1,-2}(\alpha, \beta) = \frac{1}{q} \left[ \frac{-q - 3}{4\sqrt{2}} \alpha + \frac{7q + 3}{4\sqrt{2}} \beta + \frac{q^2 - 2q - 1}{8\sqrt{q(q + 1)}} (\alpha^2 + \beta^2) + \frac{3q^2 + 2q + 1}{4\sqrt{q(q + 1)}} \alpha \beta \right].
$$

(4.13)

Changing the variable to $z = e^{-\tau x + i\theta}$, where $0 \leq x \leq 1$ and $0 \leq \theta \leq 2\pi$, the above expression is rewritten as

$$
\lim_{\tau \to \infty} \kappa^2 \frac{Q_M \tau}{2} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} dp \int_{0}^{1} dx A_{-1,-2}(i(-p + Q_L), \beta_{-k})
\times < F_1 | - p - i\alpha_{3q+1}, \beta_{q-k} > < p, \beta_k | F_2 > \exp \left[ -\tau \left( p^2 - Q_L^2 + 2\Delta_k - 2 + 2x(-i\alpha_{3q+1}(-p + Q_L) - 1 - \Delta_{q-1} - \Delta_k + \Delta_{q-k}) \right) \right].
$$

(4.14)

Since the exponential term is highly peaked in the limit $\tau \to \infty$, the saddle point estimation becomes exact. The saddle point of the $p$ integral is $p = -i\alpha_{3q+1} x$, so that (4.14) becomes

$$
\lim_{\tau \to \infty} \kappa^2 \frac{Q_M \tau}{2} \sqrt{\frac{2\pi}{2\tau}} \sum_{k=0}^{\infty} \int_{0}^{1} dx A_{-1,-2}(-\alpha_{3q+1} x + iQ_L, \beta_{-k}) \exp \left( -\tau Q_M^2 (qx - k)^2 \right)
\times < F_1 | i(x - 1)\alpha_{3q+1}, \beta_{q-k} > < -ix\alpha_{3q+1}, \beta_k | F_2 > .
$$

(4.15)
The $x$ integral is also evaluated at the saddle point

$$x = \frac{k}{q}. \quad (4.16)$$

To give nonvanishing contributions it is necessary that the saddle points are located within the interval $0 \leq k/q \leq 1$. Thus the sum of the integer $k$ is restricted within $0 < k \leq q$ and we get

$$\kappa^2 \frac{\pi}{8q^3} \sqrt{\frac{q+1}{q}} \sum_{k=1}^{q} k(q - k) < F_1 O_{q-k} > < O_k F_2 >. \quad (4.17)$$

Note that the edge state of $k = q$ vanishes because of the factor $k(q - k)$.

Replacing $\Delta_k$ and $\Delta_{q-k}$ with $\Delta_k + N$ and $\Delta_{q-k} + N$ in the expression (4.14), we can see that the oscillation modes vanish exponentially as $e^{-2N_\tau}$. Therefore we obtain

$$\int d^2z \ll \partial_2 W_{-1-2}(z, \bar{z}) \prod_{j \in S} O_j \gg_g$$

$$= \kappa^2 \frac{\pi}{8q^3} \sqrt{\frac{q+1}{q}} \sum_{k=1}^{q-1} k(q - k) \left[ \ll O_{q-k} O_k \prod_{j \in S} O_j \gg_{g-1} \right]$$

$$+ \frac{1}{2} \sum_{s=X,Y \atop g = g_1 + g_2} \ll O_{q-k} \prod_{j \in X} O_j \gg_{g_1} \ll O_k \prod_{j \in Y} O_j \gg_{g_2} \right]$$

The first term of r.h.s. is a variant of the boundary (4.17), where a handle is pinched.

Rescaling the normalization of the scaling operator as

$$O_l \rightarrow q \frac{\Gamma(1 + l - l \rho)}{l \Gamma(-l + l \rho)} O_l$$

and

$$t \rightarrow \frac{2q + 1 \Gamma(-2q - 1 + (2q + 1) \rho)}{q \Gamma(2q + 2 - (2q + 1) \rho)} t,$$

$$\mu \rightarrow \frac{1}{q} \frac{\Gamma(-1 + \rho)}{\Gamma(2 - \rho)} \mu,$$

where $\rho = (q + 1)/q$, we finally get the equation

$$0 = (2q + 1)t \ll O_{3q+1} \prod_{j \in S} O_j \gg_g + x \ll O_{q+1} \prod_{j \in S} O_j \gg_g$$

$$+ \sum_{k \in S} \ll O_{q+k} \prod_{j \in S \atop (j \neq k)} O_j \gg_g$$

$$+ \frac{\lambda}{2} \sum_{k=1}^{q-1} \ll O_{q-k} O_k \prod_{j \in S} O_j \gg_{g-1}$$

$$+ \frac{1}{2} \sum_{s=X,Y \atop g = g_1 + g_2} \ll O_{q-k} \prod_{j \in X} O_j \gg_{g_1} \ll O_k \prod_{j \in Y} O_j \gg_{g_2} \right],$$

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where we set $x = -\mu$ and $\lambda = -\kappa^2$. Note that the edge states do not appear in the expression as far as the operators in $S$ do not include the edge states. This equation is nothing but the Virasoro constraint

$$L_1 = \sum_k kx_k \partial_{q+k} + \frac{\lambda}{2} \sum_{k+l=q} \partial_k \partial_l$$

(4.23)

with $x_1 = x, x_{2q+1} = t$ and other $x_j$’s $= 0$.

### 4.3 $L_n$ equation

In this subsection we discuss the current $W_{-1,-n-1}$, which has the form

$$W_{-1,-n-1}(z, \bar{z}) = w(z, \bar{z})e^{\sigma(n+2)q+1}\phi(z,\bar{z})e^{\delta_{n-1}\psi(z,\bar{z})},$$

(4.24)

where $w(z, \bar{z})$ is the non-exponential part with conformal dimensions $(n+1, n)$. The OPE of the current with the screening charge gives the scaling operator $O_{(n+2)q+1} = \sigma_{n+2}(O_1)$. The factorization form can be evaluated as done in Sect 4.2 and we get

$$\sum_{k=1}^{nq} w(k) < F_1 \sigma_{nq-k} O_k F_2 > .$$

(4.25)

To calculate the factor $w(k)$ we need to know the explicit form of $w(z, \bar{z})$. For $n = 1$ we get $w(k) \propto k(q-k)$. In general $R_{-1,-n-1}(z)$ part of the current gives the $(n+1)$-th order polynomial of $k$ and the factor $w(k)$ will have the form

$$w(k) \propto k(k-q)(k-2q)\cdots(k-nq).$$

(4.26)

Then the edge states decouple and the expression can be rewritten as

$$\sum_{s=1}^{n} \sum_{r=1}^{q-1} w((s-1)q+r) < F_1 \sigma_{n-s}(O_{q-r}) > < \sigma_{s-1}(O_r) F_2 > .$$

(4.27)

Thus we identify the Ward identity for the current $W_{-1,-n-1}$ with $L_n$ constraint.

### 5 $W$ equations

The $W$ structures are more complicated than the Virasoro ones. Here we argue mainly the Ward identity for the currents $W_{-2,-1}$, in which the essence of $W$ structures is included.

Since the OPE of $W_{-2,-1}(z, \bar{z})$ with $O_k(0,0)$ is regular, we need to evaluate the OPE

$$W_{-2,-1}(z, \bar{z})O_k(0,0) \int d^2V_i(w, \bar{w}) = \frac{1}{z}C(k,l)O_{k+l-q}(0,0).$$

(5.1)
After contracting all operators the coefficient $C(k, l)$ is calculated as

$$C(k, l) = A_{-2,-1}(\alpha_k, \beta_k)I_1 + A_{-2,-1}(\alpha_l, \beta_l)I_{-1} + a_{-2,-1}(k, l)I_0,$$

(5.2)

where

$$A_{-2,-1}(\alpha, \beta) = \frac{1}{q+1} \left[ \frac{q+4}{4\sqrt{2}} \alpha - \frac{7q+4}{4\sqrt{2}} \beta - \frac{q^2+4q+2}{8\sqrt{q(q+1)}} (\alpha^2+\beta^2) + \frac{3q^2+4q+2}{4\sqrt{q(q+1)}} \alpha \beta \right]$$

(5.3)

and

$$a_{-2,-1}(k, l) = A_{-2,-1}(\alpha_k + \alpha_l, \beta_k + \beta_l) - A_{-2,-1}(\alpha_k, \beta_k) - A_{-2,-1}(\alpha_l, \beta_l).$$

(5.4)

The integrals $I_n$ ($n = 0, \pm 1$) are defined by

$$I_n = \int d^2 y |y|^{2(k+l-2q)/q} |1-y|^{-2l/q} (1-y)^n,$$

(5.5)

where we introduce the variable $y = w/z$. They are calculated as (see ref. [19])

$$I_0 = \pi D(k, l; k+l-q), \quad I_1 = \frac{q-l}{k} I_0, \quad I_{-1} = \frac{q-k}{l} I_0,$$

(5.6)

where $D$ function is defined by

$$D(a, b; c) = \frac{\Gamma(1+a-ap\rho)\Gamma(1+b-bp\rho)\Gamma(-c+cp)}{\Gamma(-a+ap\rho)\Gamma(-b+bp\rho)\Gamma(1+c-cp)}.$$ 

(5.7)

Thus we get

$$C(k, l) = \frac{\pi}{2\sqrt{q(q+1)}} (k+l-q) D(k, l; k+l-q).$$

(5.8)

Note that, if $k + l - q = nq$, $n \in \mathbb{Z}_+$, the coefficient $C(k, l)$ vanishes because $D(a, b; c)$ vanishes at $c = nq$, $n \in \mathbb{Z}_+$.

We also need to calculate the following boundary

$$\lim_{\tau \to \infty} \kappa^2 Q M \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{dp}{2\pi} < F_1 \left\{ \int_{e^{-\tau} \leq |z| \leq 1} d^2 z \bar{\partial} W_{-2,-1}(z) \int_{|w| \leq |z|} d^2 w V_l(w) 
+ \int_{e^{-\tau} \leq |z| \leq 1} d^2 z V_l(z) \int_{|w| \leq |z|} d^2 w \bar{\partial} W_{-2,-1}(w) \right\} \times \frac{-2\pi}{H} e^{-\tau H} - p, \beta_{-k} > < p, \beta_k | F_2 >.$$ 

(5.9)

As done in the previous section the integrals of $p$ and $z$ are evaluated by using the saddle point method,

$$\kappa^2 \frac{\pi}{2(l-q)} \sum_{k=1}^{l-q} \left\{ A_{-2,-1}(\alpha_{-k}, \beta_{-k}) \tilde{I}_0 + A_{-2,-1}(\alpha_l, \beta_l) \tilde{I}_2 + a_{-2,-1}(-k, l) \tilde{I}_1 \right\} 
\times < F_1 O_{l-k-q} > < O_k F_2 >.$$ 

(5.10)
Note that the sum of \( k \) is restricted within the interval \( 0 < k \leq l-q \), where \( l-q > 0 \). The integrals \( I_n \) \((n = 0, 1, 2)\), which comes from the \( w \) integrals, are defined by

\[
I_n = \int d^2y |y|^{2(l-k-q)/2} |1 - y|^{2(q-l)/2} \left( \frac{1}{1 - y} \right)^n ,
\]

where the region \( |y| \leq 1 \), \( y = w/z \) is given by the first integral of \( w \) in (5.9) and the region \( |y| \geq 1 \), \( y = z/w \) is given by the second. They are calculated as

\[
\tilde{I}_0 = -\pi \frac{(l - q)^2}{q^2} \frac{1}{D(k, l - k - q; l)} , \quad \tilde{I}_1 = \frac{k}{l - q} \tilde{I}_0 , \quad \tilde{I}_2 = \frac{k(k + q)}{l(l - q)} \tilde{I}_0 .
\]

Thus we get the expression

\[
-\frac{k^2 \pi^2}{4q^2 \sqrt{q(q + 1)}} \sum_{k=1}^{l-q-1} k(l-k-q)D^{-1}(k, l-k-q; l) < F_1 O_{l-k-q} >> O_k F_2 > .
\]

Here also, due to the existence of the inverse of \( D \) function and the factor \( k(l-k-q) \), the edge states indeed decouple.

Combining the boundary contributions (5.1) and (5.13) and rescaling the fields by using relations (4.19-21) we get the following Ward identity

\[
2(2q + 1)^2 t^2 \ll O_{q+2} \prod_{j \in S} O_j \gg_g + 2(2q + 1) xt \ll O_{q+2} \prod_{j \in S} O_j \gg_g \\
+ 2(2q + 1)t \sum_{k \in S} k \ll O_{q+k+1} \prod_{j \in S \setminus \{j \neq k\}} O_j \gg_g + 2x \sum_{k \in S} k \ll O_{k+1} \prod_{j \in S \setminus \{j \neq k\}} O_j \gg_g \\
+ 2 \sum_{j \neq k, l \in S} kl \ll O_{k+l-q} \prod_{j \in S \setminus \{j \neq k\}} O_j \gg_g \\
+ (2q + 1) \lambda t \sum_{k=2}^{q-1} \left[ \ll O_{q+1-k} O_2 \prod_{j \in S} O_j \gg_{g-1} + \frac{1}{2} \sum_{S-X \cup Y} \ll O_{q+1-k} \prod_{j \in X} O_j \gg_{g_1} \ll O_k \prod_{j \in Y} O_j \gg_{g_2} \right] \\
+ \lambda \sum_{l \in S} \sum_{k=1}^{l-q-1} l \left[ \ll O_{l-k-q} O_k \prod_{j \in S \setminus \{j \neq l\}} O_j \gg_{g-1} + \frac{1}{2} \sum_{S-X \cup Y} \ll O_{l-k-q} \prod_{j \in X \setminus \{j \neq l\}} O_j \gg_{g_1} \ll O_k \prod_{j \in Y \setminus \{j \neq l\}} O_j \gg_{g_2} \right] = 0 ,
\]

where \( \lambda = -k^2 \) and \( x = -\mu \). The edge states are removed. The first term is given by choosing the two screening charges as \( O_k \) and \( O_l \) in eq.(5.1). The second
corresponds to choosing the cosmological constant and the screening charge, and so on. This is nothing but the $W^{(3)}_{-1}$ constraint described as

$$W^{(3)}_{-1} = \sum_{-l-k+m=-q} lkx_lx_k\partial_m + \lambda \sum_{-l+k+m=-q} lx_l\partial_k\partial_m$$

with $x_1 = x$, $x_{2q+1} = t$ and other $x_j$'s = 0.

In $W^{(3)}_n$ algebra there exists the three derivative term

$$\lambda^2 \sum_{l+k+m=nq} \partial_l\partial_k\partial_m$$

This term can be calculated as a variant of the boundary (5.9) by replacing $V_l$ with $\kappa^2(QM/2\pi)(1/h_l)V_{-l} < O_l F_3 >$, where $1/h_l = \int dp (p^2 + E_{l,0})^{-1} = \pi/lQ_M$. For $n = -1$ this term vanishes.

In general cases it is necessary to calculate the following operator product

$$W_{-k,-n-k}(z, \bar{z})O_l(0,0) \int V_{l_2} \cdots \int V_{l_k} \propto \frac{1}{z^{O_{nq+l_1+\cdots+l_k}(0,0)}}.$$

This OPE corresponds to the single derivative term of $W^{(k+1)}_n$ constraint

$$W^{(k+1)}_n = \sum_{-l_1-\cdots-l_k+m=nq} l_1 \cdots l_kx_{l_1} \cdots x_{l_k}\partial_m + \cdots$$

If we take $l_1 = \cdots = l_k = 2q+1$, this produces the operator $O_{(n+2k)q+k} = \sigma_{n+2k}(O_k)$. We also need to calculate the the boundary pinching 2D surface for the two derivative term. The terms with more derivatives are calculated as variants of the two derivative term.

Finally we comment on the OPE algebra of the currents $W_{-k,-n-k}$. We identify the Ward identities of the currents $W_{-k,-n-k}$ with the $W^{(k+1)}_n$ constraints which form the $W_q$ algebra. Contrary to this the conservation of the momentums indicates that the operator algebra of the currents forms rather the $W_\infty$ algebra than the $W_q$ algebra. Here we conjecture that in the correlation functions the $W_\infty$ algebra reduces to the $W_q$ algebra and the currents of $k \geq q$ becomes redundant. The similar argument appears in the matrix model approach [20].

6 Discussion

Until now we considered the interaction theory perturbed by the cosmological constant $O_1$ and the screening charge $O_{2q+1}$. The formalism is easily generalized to the arbitrary potential model,

$$S = S_0 - \sum_j x_jO_j.$$
If we take $x_1 = -\mu$, $x_{p+q} = t$ and other $x_j$'s = 0, we will get the $(p, q)$ conformal theory coupled to gravity. Replacing the screening charge $O_{2q+1}$ with the operator $O_{p+q}$, the definition of the correlation function changes into

$$\ll \prod_{j \in S} O_j \gg_g = \kappa^{-\chi} \mu^s \frac{\Gamma(-s)}{\alpha_1} \frac{2\pi}{Q_M \Gamma(s)} \frac{t^n}{n!} < \prod_{j \in S} O_j (O_1)^s (O_{p+q})^n >_g , \quad (6.2)$$

where

$$s = \frac{1}{p + q - 1} [(p + q)\chi - \sum_{j \in S} (p + q - j)] , \quad (6.3)$$

$$n = \frac{1}{p + q - 1} [-\chi + \sum_{j \in S} (1 - j)] . \quad (6.4)$$

The Ward identities of this model are easily derived by using the results (4.6), (4.17), (5.1) and (5.13), where we have only to lay the roles that the screening charge did on the operator $O_{p+q}$.

It may be straightforward to generalize the formalism to the supersymmetric model. The physical states of the superconformal model coupled to gravity are studied in ref. [21, 22].

The realization of the nonlinear structures is due to the pole structure of the propagator. As a result only the states satisfying the $H = 0$ condition survive in the intermediate line. The $H = 0$ condition is the defining-equation of the quantum gravity, so it seems to be natural that the $H = 0$ condition is preserved in the intermediate line of the quantum gravity.

The unitarity is also an important issue in the quantum gravity. In the Liouville theory approach, from the hermiticity of the Virasoro algebra $L_n = L_{-n}$, we can see that the Liouville and the matter fields have the positive and the negative metric respectively. Together with the $b$ and $c$ ghosts they form the BRST quartet, so the no-ghost theorem goes well as in the string theory.

As a nontrivial model there is the 2D quantum dilaton gravity [24, 25]. This model has the feature that, if the matter's degree of freedom is greater than the critical value determined by the theory, there appears the region where two negative metric fields exist. Naively this region is dangerous in the unitarity and so, in order that the theory is well-defined, it seems that the matter's degree of freedom should be restricted. Furthermore it is shown that, in this case, the curvature singularity disappears by the quantum effects. It may give the another interpretation of the Hawking radiation.

5Note that the $H = 0$ condition is realized in the correlators, but not for the states in general. For instance $HV_{i}(1)|_{\text{phys}} \neq 0$. 

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