Instability of three dimensional conformally dressed black hole

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The three dimensional black hole solution of Einstein equations with negative cosmological constant coupled to a conformal scalar field is proved to be unstable against linear circularly symmetric perturbations.

From the discovery of a black hole in three dimensions [1], considerable work has been devoted to seek other black hole solutions including matter fields coupled to gravity in those dimensions [2–5]. An interesting case is the black-hole spacetime obtained by coupling conformally a massless scalar field [6], that is, a solution of the field equations which arises of extremizing the action

\[ I = \int d^3x \sqrt{-g} \left[ R + \frac{2l^{-2}}{2\kappa} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{16} R \Psi^2 \right], \]

(1)

where \(-l^{-2}\) is the cosmological constant, \(\kappa\) is the gravitational constant, and \(\Psi\) is a massless conformal scalar field.

The black-hole solution and scalar field are given by

\[ ds^2 = -\left[\frac{(2r + r_+)^2(r - r_+)}{4rl^2}\right] dt^2 + \left[\frac{(2r + r_+)^2(r - r_+)}{4rl^2}\right]^{-1} dr^2 + r^2 d\theta^2, \]

(2)

and

\[ \Psi(t, r) = \sqrt{\frac{8r_+}{\kappa(2r + r_+)}}, \]

(3)

respectively.

This metric is circularly symmetric, static and asymptotically anti-de Sitter. The scalar field is regular everywhere. The thermodynamics and geometric properties of this black hole can be found in [6].

The action (1) without cosmological constant in four dimensions (also including electromagnetism), was previously considered by Bocharova, Bronnikov and Melnikov [7] and Bekenstein [8,9]. The uncharged BBMB black hole solution looks like the extreme Reissner-Nordström metric and the scalar field is unbounded at horizon. This solution was shown to be unstable under monopole perturbations in [10].

The aim of this paper is to show the instability of three dimensional conformally dressed black hole against linear circularly symmetric perturbations.

We consider the perturbed circularly symmetric metric

\[ ds^2 = -e^{2U(t, r)} F(t, r) dt^2 + F^{-1}(t, r) dr^2 + r^2 d\theta^2, \]

(4)

where

\[ F(t, r) = F_0(r) + f(t, r), \quad \text{with} \quad F_0(r) = \frac{(2r + r_+)^2(r - 2r_+)}{4rl^2}, \]

(5)

and

\[ \Psi(t, r) = \Psi_0(r) + \psi(t, r), \quad \text{with} \quad \Psi_0(r) = \sqrt{\frac{8r_+}{(2r + r_+)}}, \]

(6)

where the constant \(\kappa\) has been absorbed by a redefinition of scalar field.

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Linearizing the Einstein equations with respect to $f$, $U$ and $\psi$ we obtain the following equations

$$0 = \frac{r_+}{r(2r + r_+)^2} f + \frac{1}{(2r + r_+)^2} \left[ \frac{r_+ F_0}{r(2r + r_+)^2} U' + A_\psi \psi \right] + \frac{r_+^2 (2r + r_+)^2}{8f^2 r^3 \Psi_0} \psi' - \frac{r_+^2 \Psi_0}{2(2r^2 - r_+ r - r_+^2)} \dot{\psi},$$

(7)

$$0 = \frac{r_+}{r(2r + r_+)^2} f + \frac{1}{(2r + r_+)^2} \left[ \frac{F_0}{r} U' + B_\psi \psi \right] + \frac{\Psi_0 (2r + r_+)^2 (12r^2 - 10r_+ r + r_+^2)}{8f^2 r^3} \psi' + \frac{(2r + r_+ \Psi_0)}{8r} \psi'',$$

(8)

$$0 = -\frac{r_+ (r + r_+)}{r^2 (2r + r_+)^2} f + \frac{1}{2F_0^2} \frac{3(8r^3 + r_+^3)}{8f^2 r^2} U' + F_0 U'' + C_\psi \psi,$$

(9)

$$0 = -\frac{(r + r_+)}{r^2 (2r + r_+)^2} \frac{f}{8f^2 r^2} + \frac{\psi_0 (2r + r_+)^2 (2r^2 - 4r_+ r - r_+^2)}{8f^2 r^3} \psi,$$

(10)

with

$$A_\psi = \frac{2r + r_+}{8f^2 r^3} \left[ \frac{(4r - r_+ r_+ - 8r^3 + r_+^3)}{8r} \right],$$

$$B_\psi = \frac{2r + r_+}{8f^2 r^3} \left[ \frac{(14r^2 - 4r_+ r - r_+^2) r_+}{8r^3} - \frac{8r^3 + r_+^3}{8r} \right],$$

$$C_\psi = \frac{2r + r_+}{4f^2 r^3} \left[ \frac{(4r^2 + r_+ r + r_+^2) r_+}{(2r + r_+ \Psi_0)} - \frac{4r^3 - r_+^3}{8r} \right].$$

The scalar equation yields

$$-\frac{1}{F_0} \dot{\psi} + \frac{1}{r} \left[ r F_0 \psi' \right]' + \frac{3}{4} \frac{1}{r} \frac{1}{r} \left[ r f \Psi_0 \right]' + F_0 \Psi_0 U' = 0.$$

(11)

We will now put the above equation in term of $\psi$ and its derivatives only. We observe that Eq. (11) is trivially integrable respect to time. The arbitrary function of $r$ that arises from the integration is set equal zero by imposing that $f$ vanishes if $\psi$ also does. This yields the following expression for $f$

$$f = \frac{(2r + r_+)^2 \Psi_0}{64f^2 r^2 (r + r_+)} \left[ (2r + r_+)(2r^2 - 4r_+ r - r_+^2) \psi + 8f^2 r^2 F_0 \psi' \right].$$

(12)

From the difference of (11) and (8) we obtain

$$U'' = -\frac{\Psi_0}{16(r + r_+)} \left[ 3 \psi + 6(2r + r_+) \psi' + (2r + r_+)^2 (\psi'' + \frac{1}{F_0} \dot{\psi}) \right].$$

(13)

Replacing (13) and (12) in (11) the equations system is decoupled and the equation for scalar field perturbation is

$$-\frac{1}{F_0} \dot{\psi} + F_0 \psi' + \frac{(2r + r_+)(6r^3 + 4r_+ r^2 - 5r_+ r + r_+^2)}{4f^2 r^2 (r + r_+)} \psi'$$

$$+ \frac{3r^4 + 4r_+ r^3 - 3r_+^2 r^2 - r_+^4}{4f^2 r^3 (r + r_+)} \psi = 0.$$ 

(14)

We proceed now to look for solution of form

$$\psi(t, r) = \chi(r) \exp(\sigma t), \quad \sigma \text{ real},$$

(15)
where $\chi(r)$ has the character of a physical perturbation, i.e., $\chi$ be a continuous bounded function. Hence, for $\sigma > 0$ the system is unstable (also being necessary to study the metric perturbations). Changing the radial coordinate by $x \equiv 2(r - r_+)/r_+$, (14) becomes a differential equation for $\chi$

$$\chi'' + P(x)\chi' + (Q(x) - \alpha^2 R(x))\chi = 0,$$

with

$$P(x) = \frac{(3x + 4)(x^2 + 6x + 6)}{x(x + 2)(x + 3)(x + 4)},$$

$$Q(x) = \frac{3x^4 + 32x^3 + 108x^2 + 144x + 48}{4x(x + 2)^2(x + 3)^2(x + 4)},$$

$$R(x) = \left[\frac{(x + 2)}{x(x + 3)^2}\right]^2, \quad \alpha = \frac{2l^2}{r_+} \sigma.$$

The above equation can be put into a Sturm-Liouville form,

$$(p(x)\chi')' + p(x)(Q(x) - \alpha^2 R(x))\chi = 0,$$

where the integrating factor is

$$p(x) = \exp\left\{\int Pdx\right\} = \frac{x(x + 3)^5}{(x + 2)(x + 4)^2}.$$

Finally, we can turn (20) into a Schrödinger-type equation, with the change of variables proposed by Liouville:

$$\eta = (p^2 R)^{1/4} \chi, \quad z = \int R^{1/2}(x)dx.$$

Thus, the scalar perturbation equation takes the familiar aspect of a Schrödinger equation with a one-dimensional potential

$$\left\{-\frac{d^2}{dz^2} + V(z)\right\} \eta = -\alpha^2 \eta,$$

where the potential $V(z)$ is

$$V(z) = (p^2 R)^{-1/4} \frac{d^2}{dz^2}(p^2 R)^{1/4} - \frac{Q}{R}.$$

Replacing the expressions for $p, R$ and $Q$ in (22) and (24), the relations between $\eta$ and $\chi$, and $z$ and $x$ are obtained

$$\eta(z) = \frac{(x + 3)^{3/2}}{x + 4} \chi(x),$$

$$z = \frac{1}{3(x + 3)} - \frac{2}{9} \log \left|\frac{x}{x + 3}\right|,$$

and the potential is

$$V(z) = \frac{x(x + 3)^2}{(x + 2)^4(x + 4)^2}(5x^3 + 36x^2 + 96x + 96).$$

We will now study the properties of the differential equation (23) in the half-line $x \geq 0$, that is, $r \geq r_+$.

First, we note from (25) that the Liouville transformation does not add singularities in the definition of $\eta$. Moreover, the change of variable (26) is a one-to-one map between the half-line $x \geq 0$ and $z \geq 0$, where spatial infinity is mapped to $z = 0$ and the horizon ($x = 0$) to $z = \infty$.

In the outer region to the horizon, $z \geq 0$, the potential $V(z)$ behaves as a positive, concave, monotonically decreasing function, which has its maximum at $z = 0$ ($V(z = 0) = 5$) and goes to zero if $z \to \infty$. Although we have a positive definite potential, it is possible to find negative eigenvalues since we only consider the half-line.
In order to study the behavior of the solution in the neighborhood of \( x = 0 \), we consider Eq. (16). This is a Fuchsian equation since its two singularities, \( x = 0 \) and \( x = \infty \), are regular singular points. Applying Frobenius method, we conclude that, around \( x = 0 \), the linearly independent solutions take the form
\[
\chi_\pm(x) = x^{\pm 2\alpha} u_\pm(x),
\]
where \( u_\pm \) are analytic functions. We observe that \( \chi_+ \) is the regular solution and vanishes at \( x = 0 \) (assuming \( \alpha \) positive). This implies that \( \eta = 0 \) at \( z = \infty \).

The behavior of the solution at \( x = \infty \) is analyzed by replacing \( x = 1/w \). Then Eq. (16) takes the asymptotic form
\[
\frac{d^2\chi}{dw^2} - \frac{1}{w} \frac{d\chi}{dw} + \frac{3}{4w^2} \chi = 0.
\]
The roots of the associated indicial equation are \( 1/2, 3/2 \). Thus \( \chi \) is asymptotically given by
\[
\chi(x) \sim x^{-\frac{1}{2}}.
\]
We note that the perturbation \( \chi \) has the same asymptotic form as the starting solution \( \Psi_0 \). Replacing (30) into (25), we observe that \( \eta(z = 0) \equiv \eta_0 \) is a constant. It is easy to see that there are no bounded solutions if \( \eta_0 = 0 \). To see that, assume \( \eta_0 = 0 \) y \( \eta'(0) > 0 \). From (23), we note that \( \eta'(z) \) is an increasing function and \( \eta \) diverges. If \( \eta'(0) < 0 \), \( \eta \) diverges to negative values. In the same way, it is straightforward to show that the bounded solutions of (23) cannot have zeros since \( \eta \) and its second derivative has the same sign.

Using the results of [11] it is possible to establish that there exist bounded solutions to (23) with the boundary conditions \( \eta(0) \neq 0 \) and \( \eta(\infty) = 0 \). Now, we turn over to the metric perturbations.

From (12) it is inferred that
\[
f(r_+) = \left( \frac{3}{2} \right)^{7/2} \frac{r_+^2}{4\ell^2} \psi(r_+),
\]
and therefore, \( f \) vanishes at horizon.

The asymptotic behavior of \( f \) is
\[
f \sim x^{3/2}(\psi + 2x\psi'), \quad (x \to \infty),
\]
and given that \( \psi \sim (x^{-1/2} + \text{const.} x^{-3/2}) \exp(\sigma t) \), then,
\[
f \sim [\text{const.} + O(x^{-1})] \exp(\sigma t).
\]
We see that spatial dependence of \( f \) acts modifying the coefficients associated to \( r^0 \) and \( r^{-1} \) of \( F_0 \), that is, only modifies the position of the horizon. It is a sensible initial perturbation.

The only remaining point is to deal the \( U \) perturbation. From (13), we obtain
\[
U' \sim \frac{\text{const.}}{r^3}, \quad \text{that is, } \ U \sim \frac{\text{const.}}{r^2}.
\]
Therefore, \( U \) modifies the mass asymptotically. Near the horizon,
\[
U' = x^{\frac{2\alpha}{9} - 1} + (\frac{4\alpha}{9} - 1)x^{\frac{4}{9}(\alpha - 9)}.
\]
Thus, \( U' = 0 \) at the horizon if \( \alpha > 9 \), or equivalently, \( \sigma > 9r_+/2\ell^2 \), which proves the instability of the solution.

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[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992); M. Bañados, C. Teitelboim, M. Henneaux and J. Zanelli, Phys. Rev. D48, 1506 (1993).
[2] K. C. K. Chan and R. B. Mann, Phys. Rev. D50, 6385 (1994); erratum, D52, 2600 (1995).
[3] K. C. K. Chan and R. B. Mann, Phys. Lett. B371,199 (1996).
[4] P. M. Sá, A. Kleber and J. P. S. Lemos, Class. Quant. Grav. 13, 125 (1996).
[5] K. C. K. Chan, Phys. Rev. D55, 3564 (1997).
[6] C. Martínez and J. Zanelli, Phys. Rev. D54, 3830 (1996).
[7] N. Bocharova, K. Bronnikov and V. Melnikov, Vestn. Mosk. Univ. Fiz. Astron. 6, 706 (1970).
[8] J. D. Bekenstein, Ann. Phys. (N.Y.) 82, 535 (1974).
[9] J. D. Bekenstein, Ann. Phys. (N.Y.) 91, 75 (1975).
[10] K. A. Bronnikov and Y. N. Kireyev, Phys. Lett. A67, 95 (1978).
[11] E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley (1969), p 445.