String cosmology from Poisson-Lie T-dual sigma models on supermanifolds

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ABSTRACT: We generalize the formulation of Poisson-Lie T-dual sigma models on manifolds to supermanifolds. In this respect, we formulate 1+1 dimensional string cosmological models on the Lie supergroup $\mathbb{C}^3$ and its dual $(\mathbb{A}_{1,1} + 2\mathbb{A})^0_{1,0,0}$, which are coupled to two fermionic fields. Then, we solve the equations of motion of the models and show that there is a essential singularity for the metric of the original model and its dual.

KEYWORDS: Sigma Models, String Duality
1. Introduction

Two-dimensional sigma models with supermanifolds as target space have recently received considerable attention, because of their relations to superstring models. The first attempt in this direction dates back to about 3 decades ago [1], where the flat space GS superstring action was reproduced as a WZW type sigma model on the coset superspace $\mathbb{D} = 10$ Poincare supergroup $\text{SO}(9,1)$. Then, this work is extended to the curved background [2] and shown that type IIB superstring on $\text{AdS}_5 \otimes S^5$ can be constructed from sigma model on the coset superspace $\text{SU}(2,2|4)/\text{SO}(4,1) \times \text{SO}(5)$. After then, superstring theory on $\text{AdS}_3 \otimes S^3$ is related to WZW model on $\text{PSU}(1,1|2)$ [3] and also superstring theory on $\text{AdS}_2 \otimes S^2$ is related to sigma model on supercoset $\text{PSU}(1,1|2)/U(1) \times U(1)$ [4]. There are also other works in this direction, see for instance [5].

On the other hand, T-duality is the most important symmetries of string theory [6]. Furthermore, Poisson-Lie T-duality, a generalization of T-duality, does not require existence of isometry in the original target manifold (as in usual T-duality) [7]. So, the studies of Poisson-Lie T-duality in sigma models on supermanifolds and duality in superstring theories on AdS backgrounds are interesting problems. In the previous paper [8], we extended Poisson-Lie symmetry to sigma models on supermanifolds and also constructed Poisson-Lie T-dual sigma models on Lie supergroups. In this paper, we formulate Poisson-Lie T-dual sigma models on supermanifolds as an extension of the work [9]. Then, using this formalism we construct 1+1 dimensional string cosmological models as one of the first examples of string cosmological models that have a Poisson-Lie symmetry.
The paper is organized as follows. In section two, we generalize the formulation of Poisson-Lie T-dual sigma models on manifolds to supermanifolds. In section three, as an example of Poisson-Lie T-dual sigma models on supermanifolds, we construct these models by using Lie supergroup $C^3$ and its dual $(A_{1,1} + 2A_0^0)^{1,0,0}$ and choosing orbit superspace as one-dimensional space with coordinate $\{y^\alpha\} \equiv \{t\}$ as time. In this way, we obtain 1+1 dimensional string cosmological models which are coupled to two fermionic fields. In this respect we give the one-loop beta functions equations for a general sigma model on supermanifold and write string effective action on supermanifold in the beginning of this section. Then, we solve the one-loop beta functions equations for the original model and its dual and show that there are some solutions which have singular points. Then, by writing of the Kretschmann scalar invariant on supermanifolds we show that some of these singularities are essential for the model and its dual. For self containing of the paper we write some mathematical properties of matrices and tensors on supermanifold that we need in this paper, as appendix.

2. Super Poisson-Lie T-dual sigma models on supermanifolds

In the previous work, we extended Poisson-Lie symmetry to sigma models on supermanifolds and also constructed Poisson-Lie T-dual sigma models on Lie supergroups. In this section as a continuation of that work we formulate Poisson-Lie T-dual sigma models on supermanifolds. Consider a two-dimensional sigma model for the $\mathbb{R}^2$, such that $\dim\mathbb{R}^2 = \dim\mathbb{R}^2$. The $\mathbb{R}^2$ field variables $X^M = (x^\mu, y^\alpha)$ where $x^\mu (\mu = 1, \cdots, \dim G)$ are coordinates of Lie supergroup $G$ that act freely from right on supermanifolds $M$. The $\{y^\alpha\}$ are coordinates of the orbit $O = \mathbb{R}^2$. In this respect, one can construct a sigma model on $M$ with super Poisson-Lie symmetry similar to for ordinary Poisson-Lie symmetry.

Consider a linear idempotent map $K(y)$: $T^*_yO \oplus T^*_yO \oplus \mathcal{D} \longrightarrow T^*_yO \oplus T^*_yO \oplus \mathcal{D}$ where $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$. Let $(\mathcal{G}, \tilde{\mathcal{G}})$ be a Lie superbialgebra. There exists a unique Lie superalgebra structures with the following commutation relations on the vector space $\mathcal{G} \oplus \tilde{\mathcal{G}}$ such that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are Lie superalgebras and the natural scalar product on $\mathcal{G} \oplus \tilde{\mathcal{G}}$ is invariant.

\[ [x, y]_D = [x, y], \quad [x, \xi]_D = -(-1)^{|x||\xi|} ad^*_x \xi + ad^*_\xi x, \quad [\xi, \eta]_D = [\xi, \eta]_{\mathcal{G}} \quad \forall x, y, \xi, \eta \in \mathcal{G}; \quad \xi, \eta \in \tilde{\mathcal{G}}, \]

where

\[ < ad_x y, \xi > = -(-1)^{|x||y|} < y, ad^*_x \xi >, \quad < ad_x \eta, x > = -(-1)^{|\xi||\eta|} < \eta, ad^*_x x >. \]

The Lie superalgebra $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$ is invariant. Here we generalize the results of Ref. to the supermanifolds and in this direction we use the DeWitt notations for supermanifolds.

(1) Here we write some mathematical properties of matrices and tensors on supermanifold that we need in this paper, as appendix.
such that with translation of this graph to the point \( g \in G \) we have
\[
g^{-1}R_{\pm}(y)g = \text{Span}\{X_A \pm E_{AB}^\pm(g,y)\bar{X}^B\},
\]  
(2.2)
where \( \{X_A\} = \{\overrightarrow{\partial}_\alpha = \overrightarrow{\partial}_{g^\alpha}, X_i\} \) and \( \{\bar{X}^A\} = \{\overrightarrow{\partial}^{\alpha}_g, \bar{X}^i\} \) are the basis for the superspaces \( T_yO \oplus \mathcal{G} \) and \( T^*_yO \oplus \bar{\mathcal{G}} \), respectively. The matrix \( E_{AB}^\pm(g,y) \) is a \( d \)-dimensional matrix as follows:
\[
E_{AB}^\pm(g,y) = \left(\begin{array}{c}
E_{ij}^\pm(g,y) \\
\Phi_{ij}^\pm(g,y)
\end{array}\right),
\]  
(2.3)
where the minus sign (-) stands for supertranspose, i.e., \( E_{ij}^{\pm\ast} = -(1)^{ij}E_{ji}^{\pm}, \Phi_{ij}^{\pm\ast} = -(1)^{ij}\Phi_{ji}^{\pm} \). Now, similar to \([8]\) we can write vector superspaces \( R_{\pm}(y) \) as follows:
\[
g^{-1}R_{\pm}(y)g = \text{Span}\{g^{-1}X_A g \pm E_{AB}^\pm(e,y)g^{-1}\bar{X}^B g\},
\]  
(2.4)
such that
\[
E_{AB}^\pm(e,y) = \left(\begin{array}{c}
E_{ij}^\pm(e,y) \\
\Phi_{ij}^\pm(e,y)
\end{array}\right),
\]  
(2.5)
where \( e \) is the unit element of \( G \) such that \( E_{ij}^\pm(e,y), \Phi_{ij}^\pm(e,y) \) and \( E_{AB}^\pm(y) \) are subsuper-matrices with elements as functions of \( y \) with \( F_{ij}^{\pm\ast} = -(1)^{ij}F_{ji}^{\pm\ast} \). Now using the following relations:
\[
g^{-1}X_A g = A(g)_A B_B X = -(1)^B A(g)_A B_B X_B,
\]  
(2.6)
\[
g^{-1}\bar{X}^B g = B(g)^{BC} C_X + D(g)^B C \bar{X}^C = -(1)^C B(g)^{BC} C_X + D(g)^B C \bar{X}^C,
\]  
(2.7)
one can obtain an expression for the background matrix \( E_{AB}^\mp \)
\[
E_{AB}^\pm(g,y) = A(g)_A C_C \left(A(g) + E_{\mp}(e,y)B(g)\right)^{-1}D E_{B}^{\pm}(e,y),
\]  
(2.8)
where
\[
A(g)_A B = \left(\begin{array}{c}
(a(g)_{i,}^j & 0 \\
(1)^{\alpha\delta}\beta & 0
\end{array}\right), B(g)^{BC} = \left(\begin{array}{c}
b(g)_{ik}^j & 0 \\
0 & 0
\end{array}\right), D(g)^B C = \left(\begin{array}{c}
d(g)_{jk}^l & 0 \\
0 & \beta\delta
\end{array}\right).
\]  
(2.9)
Now in the same way as \([7, 8]\) and using the following equation of motion on the Drinfel’d superdouble \( D:\)
\[
< \overrightarrow{\partial}_{\pm} u^{-1} + w + v , R_{\pm}(y) >= 0, \quad l \in D,
\]  
(2.10)
where \( w = w_\alpha \overrightarrow{\partial}_g^\alpha \) and \( v^{(L,l)} = v^{(L,l)}_\alpha \overrightarrow{\partial}_g^\alpha \) are left invariant one-forms and left invariant vector fields with left derivatives on the supercoset \( M/Q \), respectively; one can obtain the following action
\[
S = \frac{1}{2} \int d\xi^+ \wedge d\xi^- \left[R^{(l)}_+ A E_{AB}^+(g,y) R^{(l)}_B - \frac{1}{4} R^{(2)}\varphi\right]
\]  
\[
= \frac{1}{2} \int d\xi^+ \wedge d\xi^- \left[R^{(l)}_+ i E_{ij}^+(g,y) R^{(l)}_j + R^{(l)}_+ i \Phi_{ij}^+(g,y) \partial y^\alpha \right]
\]  
\[
+ \partial_+ y^\alpha \Phi_i^+(g,y) R^{(l)}_i + \partial_+ y^\alpha \Phi_j^+(g,y) \partial y^\beta - \frac{1}{4} R^{(2)}\varphi(g,y),
\]  
(2.11)
where $R^{(2)}$ is the curvature of the world-sheet, $\varphi(g, y)$ is the dilaton field and

$$R^{(l)A}_{\pm} = \{R^{(l)i}_{\pm}, \partial_{\pm} y^a\}, \quad R^{(l)i}_{\pm} = (\partial_{\pm} g g^{-1})^i = \partial_{\pm} x^\mu \mu R^{(l)i},$$

(2.12)

are right invariant one-forms with left derivatives. Furthermore, using Eqs. (2.3), (2.3) and (2.9) we have

$$i E^\pm_j (g, y) = i \left( (E^{\pm}_0)^{-1} \pm \Pi \right)^{-1} j,$$

$$\Pi^{AB}(g) = B_{AC}(g) c A^{-1} B(g), \quad \Pi^{ij}(g) = b^{ik}(g) k a^{-1j}(g),$$

$$i \Phi^\pm_\alpha = i E^\pm_j (g, y) \left( (E^{\pm}_0)^{-1} \right)^k k F^\pm_\alpha,$$

$$a \Phi_\beta = a F_\beta - a F^\pm_\alpha \Pi^{kl} F^m_{\alpha l} \left( (E^{\pm}_0)^{-1} \right)^n n F^\pm_\beta,$$

$$\varphi(g, y) = \varphi^0(y^\alpha) + \ln \text{sdet}(i E^+_j (g, y)) - \ln \text{sdet}(i E^+_0 (e, y)).$$

Note that the last equation is a quantum effect and it is a generalization of bosonic case \cite{13}. In that equation, $\varphi^0$ is the scalar field and function of the variable $y^\alpha$ only. In the same way, one can obtain the dual sigma model (as \cite{13}, \cite{14} and \cite{15} for the bosonic case)

$$\tilde{S} = \frac{1}{2} \int d\xi^+ \land d\xi^- \left[ \tilde{R}^{(l)A}_{+} \tilde{E}^{+AB}(\tilde{g}, \tilde{y}) \tilde{R}^{(l)}_{+} - \frac{1}{4} R^{(2)} \tilde{\varphi}(\tilde{g}, \tilde{y}) \right]$$

$$= \frac{1}{2} \int d\xi^+ \land d\xi^- \left[ \tilde{R}^{(l)i}_{+} \tilde{E}^{+ij}(\tilde{g}, \tilde{y}) \tilde{R}^{(l)}_{+} + \tilde{R}^{(l)i}_{+} i \tilde{\Phi}^+ (\tilde{g}, \tilde{y}) \partial_- y^a \right.$$

$$\left. + \partial_+ y^a \tilde{\Phi}^+ i (\tilde{g}, \tilde{y}) \tilde{R}^{(l)}_{+} + \partial_+ y^a \tilde{\Phi}_\beta (\tilde{g}, \tilde{y}) \partial_- y^\beta - \frac{1}{4} R^{(2)} \tilde{\varphi}(\tilde{g}, \tilde{y}) \right],$$

(2.14)

where

$$\tilde{E}^{+AB}(\tilde{g}, \tilde{y}) = \left( (\tilde{E}^+)^{-1}(\tilde{e}, \tilde{y}) + \tilde{\Pi}(\tilde{g}, \tilde{y}) \right)^{-1} AB,$$

(2.15)

such that

$$\tilde{E}^+(\tilde{e}, \tilde{y}) = (A + E^+(e, y) B)^{-1}(C + E^+(e, y) D),$$

(2.16)

and

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} i \delta^i_j & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} i \delta^j_i & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  

(2.17)

Using Eq. (2.3) for $\tilde{E}^{AB}(\tilde{e}, \tilde{y})$ one finds

$$\tilde{E}^0_{+ij}(\tilde{e}) = (E^0_0)^{-1} ij(e), \quad i \tilde{F}^+_\alpha = i \left( (E^0_0)^{-1} \right)^j j F^+_{\alpha},$$

(2.18)

$$a \tilde{F}^+_i = -a F^+_j \left( (E^0_0)^{-1} \right)^i j, \quad a \tilde{F}_\beta = a F_\beta - a \left( F^+(E^0_0)^{-1} F^+ \right)_\beta,$$

(2.19)
and from Eq. (2.15) we have
\[ \tilde{E}^{\pm ij}(\tilde{g}, y) = \left( E_0^\pm(e, y) + \tilde{\Pi}(\tilde{g}) \right)^{-1ij}, \]
\[ i\tilde{\Phi}_\alpha = \pm \tilde{E}^{\pm ij}(\tilde{g}, y) j F^\alpha, \]
\[ \alpha \tilde{\Phi}_\beta = \alpha F^\beta - \alpha F^\pm \tilde{E}^{\pm kl}(\tilde{g}, y) j F^\pm \beta, \]
\[ \tilde{\varphi}(\tilde{g}, y) = \varphi^0(y^\alpha) + \ln \text{sdet}(\tilde{E}^{\pm ij}(\tilde{g}, y)). \]

Note that in the above calculations, all relations are the same as [7] and [14], but one must be careful that matrices are supermatrices and in calculating their inverses, products, etc, one must use from the rules of superinverses, superproducts and etc [see, appendix A].

3. String cosmological models

Now by use of the actions (2.11) and (2.14) one can construct string cosmological models on supermanifold. For this purpose we note that those actions in general, have the following form:
\[ S = \frac{1}{2} \int d^n x \sqrt{G} e^{-\varphi} \left( R - 12 \nabla^P \nabla^P \varphi - 4 \nabla^M \nabla^M \varphi \right). \]

The one-loop beta functions relations for the above sigma models on supermanifolds have the following form [4]:
\[ \beta^{(G)}_{MN} = R_{MN} + \frac{1}{4} H_{MPQ} H^{QP}_{\ N} + 2 \nabla^M \nabla^N \varphi = 0, \]
\[ \beta^{(B)}_{NP} = (-1)^M \nabla^M (e^{-2\varphi} H_{MPN}) = 0, \]
\[ \beta^{(\varphi)} = -R - \frac{1}{12} H_{MN\ P} H^{PNM} + 4 \nabla^M \varphi \nabla^M \varphi - 4 \nabla^M \nabla^M \varphi = 0, \]

where
\[ H_{MN\ P} = \frac{\partial}{\partial x^N} B_{NP} + (-1)^{M(P+N)} \frac{\partial}{\partial x^P} B_{PM} + (-1)^{P(M+N)} \frac{\partial}{\partial x^M} B_{MN}, \]
is the torsion field with the following symmetry properties:
\[ H_{MN\ P} = (-1)^P H_{MPN} = (-1)^{M(P+N)} H_{NPM} = (-1)^{P(M+N)} H_{PMN}. \]

Indeed the above beta functions relations are equations of motion for the following effective action on supermanifold
\[ S_{\text{eff}} = \int d^{m,n} x \sqrt{G} e^{-\varphi} \left[ R + 4 \nabla^M \varphi \nabla^M \varphi + \frac{1}{12} H_{MN\ P} H^{PNM} \right], \]

where \( \sqrt{G} \) (with \( G = \text{sdet}(\Delta G_B) \)) and \( d^{m,n} x \) are measure and volume element on supermanifold, respectively, with \( m \) bosonic and \( n \) fermionic coordinates. Furthermore, these one-loop beta functions relations are Einstein field equations which have coupled to bosonic
and fermionic matters. For the bosonic case, in Ref. [16], it has shown that the effective action is invariant under Poisson-Lie T-duality; furthermore, it has obtained a functional relation between one-loop beta functions of the original and dual models and consequently showed that the conformality of the models are invariant under Poisson-Lie T-duality. In this way, similar to the consequence of the previous section, we expect that these proofs can be extended and satisfied to the case that the target space is a supermanifold.

3.1 An example

In this subsection we construct 1+1 dimensional string cosmological models that are coupled to two fermionic fields. In this respect, consider the Lie supergroup $C^3$ with the following Lie superalgebraic relation [11], [17]:

$$[X_1, X_3] = X_2,$$

where its dual $\tilde{G} = (A_{1,1} + 2A)_{1,0,0}^0$ has the following anticommutation relation [12]:

$$\{\tilde{X}^2, \tilde{X}^2\} = \tilde{X}^1,$$

and nonzero (anti)commutation relations for the Drinfel’d superdouble $D = (C^3, (A_{1,1} + 2A)_{1,0,0}^0)$ have the following form [12]

$$[X_1, X_3] = X_2, \quad \{\tilde{X}^2, \tilde{X}^2\} = \tilde{X}^1,$$

$$[X_1, \tilde{X}^2] = -X_2 - \tilde{X}^3, \quad \{X_3, \tilde{X}^2\} = -\tilde{X}^1,$$

such that the $\{X_1, \tilde{X}^1\}$ and $\{X_2, X_3, \tilde{X}^2, \tilde{X}^3\}$ are bosonic and fermionic basis, respectively. Note that as above discussion, the reason for choosing Lie supergroup $C^3$ is the fact that its adjoint representation is traceless so that in this way the conformality under duality is preserved [17].

Now, choosing parametrization for the Lie supergroups $C^3$ and $(A_{1,1} + 2A)_{1,0,0}^0$ we construct the model and its dual.

3.1.1 Model

For the Lie supergroups $C^3$ we choose the following parametrization:

$$g = e^{x_1}e^{\psi_2}e^{\chi_3},$$

Note that in those work it has shown that for preserving of conformal invariant, the trace of the adjoint representation of the structure constants related to the Lie group $G$ and its dual $\tilde{G}$ must be zero.

4Of course similar to the bosonic case [14] these proofs are very lengthy and we leave those calculations to the another work.

5Note that these (anti)commutation relations have been written in the nonstandard basis. If someone wishes to write these relations in the standard basis, it suffices to multiply the structure constants of the anticommutators by $i = \sqrt{-1}$.

6Note that the structure of supergroup is fully defined by specifying a coproduct and antipode on the space of functions on it. Our local parametrizations (3.11) and (3.44) together with the superalgebra relations (3.8) and (3.9) give however an equivalent information if we take the respective underlying bosonic subgroups contractible.
where the $x$ is bosonic parameter and $\psi, \chi$ are fermionic ones. Now using Eqs. (2.6), (2.7) and (2.13) we have

$$\Pi^{ij}(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.12)$$

Finally, choosing the orbit $O = \mathcal{M}/G$ as a one-dimensional space with time coordinate $\{y^\alpha\} \equiv \{t\}$, using the Eqs. (2.11), (2.13) and assuming $\alpha F_\beta = f(t)$ and $\beta F_\beta^\pm = 0$ we obtain the following action for the original model:

$$S = \frac{1}{2} \int d\xi^+ \wedge d\xi^- \left[ \partial_+ t \; f(t) \; \partial_- t + \partial_+ x \; \frac{1}{a(t)} \; \partial_- x - \partial_+ \psi \; \frac{1}{e(t)} \; \partial_- \chi \\
+ \partial_+ \chi \; \frac{1}{e(t)} \; \partial_- \psi - \partial_+ \chi \; \frac{x}{e^2(t)} \; \partial_- \chi - \frac{1}{4} R(2) \varphi^0(t) \right], \quad (3.13)$$

where we have chosen the constant matrix $(E_0^+)^{-1}$ as follows:

$$(E_0^+)^{-1}{}^{ij} = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & 0 & e(t) \\ 0 & -e(t) & 0 \end{pmatrix}. \quad (3.14)$$

Such that for this model we have

$$G_{AB} = \begin{pmatrix} f(t) & 0 & 0 & 0 \\ 0 & \frac{1}{a(t)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{e(t)} \\ 0 & 0 & -\frac{1}{e(t)} & 0 \end{pmatrix}, \quad B_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x}{e^2(t)} \\ 0 & 0 & \frac{x}{e^2(t)} & 0 \end{pmatrix}. \quad (3.15)$$

Now one can construct the beta functions equations (3.2)-(3.4) for the action (3.13) with assuming $\varphi = \varphi^0(t) = 0$. For this action using (3.5) one can obtain the nonzero components of $H_{MNP}$ as follows:

$$H_{033} = x \frac{d}{dt} \left( \frac{1}{e^2(t)} \right), \quad H_{133} = \left( \frac{1}{e^2(t)} \right), \quad (3.16)$$

so for this example $H_{MN} = H_{MPQ}H^{QP}_{\;\;\;\;N} = 0$ and $H^2 = H_{MNP}H^{P\;\;NM} = 0$; hence, the beta function relations are rewritten as follows:

$$\beta^{(G)}_{MN} = R_{MN} = 0, \quad (3.17)$$

$$\beta^{(B)}_{NP} = (-1)^M \sqrt{\varphi} {\sqrt{\varphi}}^M H_{MNP} = 0, \quad (3.18)$$

$$\beta^{(\varphi)} = R = 0, \quad (3.19)$$

where the nonzero components of $R_{MN}$ are $R_{00}, R_{11}$ and $R_{23}$. Note that in this way we have a Ricci flat supermanifold. After some calculations the relations (3.17) have the following
forms

\[ R_{00} = \frac{1}{2} \left[ \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) - \frac{\dot{\dot{f}}}{2af} - \frac{\dot{a}^2}{2a^2} - 2 \frac{d}{dt} \left( \frac{\dot{e}}{e} \right) + \frac{\dot{e}^2}{e^2} \right] = 0, \quad (3.20) \]

\[ R_{11} = \frac{1}{2} \left[ \frac{d}{dt} \left( \frac{\dot{a}}{a^2f} \right) + \frac{\dot{\dot{f}}}{2a^2f^2} + \frac{\dot{a}^2}{2a^3f} + \frac{\dot{e}a}{a^2ef} \right] = 0, \quad (3.21) \]

\[ R_{23} = \frac{1}{2} \left[ \frac{d}{dt} \left( \frac{\dot{e}}{e^2f} \right) + \frac{\dot{\dot{f}}}{2e^2f^2} - \frac{\dot{e}a}{2e^2af} + \frac{\dot{e}^2}{2e^2f} \right] = 0, \quad (3.22) \]

where dot stands for time derivative. Furthermore, the relation (3.19) leads to

\[ R = \frac{1}{f} \left[ \frac{d}{dt} \left( \frac{\dot{a}}{a} - \frac{2\dot{e}}{e} \right) + \frac{\dot{\dot{f}}}{2ae} + \frac{\dot{a} \dot{e}}{2ae} - \frac{\dot{e}^2}{2a^2} \right] = 0, \quad (3.23) \]

and only the nonzero component of \( \beta^{(B)}_{NP} \) is

\[ \beta^{(B)}_{33} = \frac{-2x}{f e^2} \left[ \frac{d}{dt} \left( \frac{\dot{e}}{e} \right) - \frac{\dot{\dot{f}}}{2e^2f^2} - \frac{\dot{e}a}{2e^2af} + \frac{\dot{e}^2}{2e^2} \right] = \frac{-4x}{e(t)} R_{23} = 0. \quad (3.24) \]

Now, by combination of Eqs. (3.21), (3.22) and (3.23) we obtain the following constraints:

\[ \frac{d}{dt} \ln(e(t)) = 0, \quad (3.25) \]

\[ \frac{d}{dt} \ln(a(t)) = 3 \frac{d}{dt} \ln(e(t)). \quad (3.26) \]

After substituting the constraint (3.25) into the Eq. (3.20) we obtain the following equation:

\[ \frac{d^2}{dt^2} \ln(a(t)) - \frac{1}{2} \frac{d}{dt} \ln(a(t)) \frac{d}{dt} \ln(f(t)) - \frac{1}{4} \left( \frac{d}{dt} \ln(a(t)) \right)^2 = 0. \quad (3.27) \]

The general solution for the above equation has the following form:

\[ a(t) = c_1 e^{-A(t)}, \quad (3.28) \]

where

\[ A(t) = \int^t \frac{\sqrt{f(t')}dt'}{[\sqrt{f(t')}dt' + c_2]} \quad c_1, c_2 \in \mathbb{R}. \quad (3.29) \]

Furthermore one can obtain the following special class of solutions for the above equation:

(i) \( a(t) = \frac{a_0}{(t - \alpha_0)}, \quad e(t) = e_0, \quad f(t) = \frac{f_0}{(t - \alpha_0)}. \quad (3.30) \)

(ii) \( a(t) = a_0 e^{b_0 t}, \quad e(t) = e_0, \quad f(t) = f_0 e^{-b_0 t}. \quad (3.31) \)

On the other hand, by substituting the constraint (3.26) into the Eq. (3.20) we obtain the following equation:

\[ \frac{d^2}{dt^2} \ln(e(t)) - \frac{1}{2} \frac{d}{dt} \ln(f(t)) \frac{d}{dt} \ln(e(t)) + \frac{1}{4} \left( \frac{d}{dt} \ln(e(t)) \right)^2 = 0. \quad (3.32) \]
The general solution for the above equation has the following form:

\[ e(t) = c_1 e^{tA(t)}, \]

for which we obtain the following special class of solutions:

(iii) \( a(t) = a_0 e^{\frac{t}{2}c_0t}, \quad e(t) = e_0 e^{c_0t}, \quad f(t) = f_0 e^{\frac{t}{2}c_0t}, \quad (3.34) \)

(iv) \( a(t) = \frac{a_0}{(t - \beta_0)^3}, \quad e(t) = \frac{e_0}{(t - \beta_0)^2}, \quad f(t) = \frac{f_0}{(t - \beta_0)^3}, \quad (3.35) \)

(v) \( a(t) = a_0(t - \gamma_0)^3, \quad e(t) = e_0(t - \gamma_0)^2, \quad f(t) = \frac{f_0}{(t - \gamma_0)} \), \quad (3.36) \)

where \( a_0, e_0, f_0, b_0, c_0, \alpha_0, \beta_0 \) and \( \gamma_0 \) are real constants. We see that the class (3.30), (3.35) and (3.36) of solutions have singular points at \( t = \alpha_0, t = \beta_0 \) and \( t = \gamma_0 \), respectively.

To investigate the type of singular points we write the Kretschmann scalar invariant for the model as follows:

\[ K = R^{IJKL} R_{IJKL}. \quad (3.37) \]

Using the matrix representation, we rewrite this formula in the following form\(^7\):

\[
K = -(1)^{I+J+L+J+1N+1L+1J+1L+1J+1K+1J+1L+1N+1L+1K} \text{str} \left[ G^{-1} R_{MN} G^{-1} R_{IJ} \right] G^{JN} G^{IM},
\]

where \( (R_{MN})_{PQ} = R_{MNPQ} \). Now using the following form of the metric of the original model

\[
ds^2 = dx^A G_{AB} dx^B = (-1)^{AB} G_{AB} dx^A dx^B = f(t) dt^2 + \frac{1}{a(t)} dx^2 - \frac{1}{e(t)} \psi d\chi + \frac{1}{e(t)} d\chi d\psi,
\]

and after some calculations one can obtain the general form of the Kretschmann scalar invariant for the model as follows:

\[
K = \frac{1}{f^2} \left[ \left( \frac{d}{dt} \left[ \frac{a}{2} \dot{a} - \frac{1}{2} \frac{a}{2} \dot{a}^2 \right] \right)^2 + 2 \left( \frac{d}{dt} \left[ \frac{e}{2} \dot{e} - \frac{1}{2} \frac{e}{2} \dot{e}^2 \right] \right)^2 + \frac{\dot{a}^2 \dot{e}^2}{2 a^2 e^2} + \frac{3 \dot{e}^4}{4 e^4} \right].
\]

For solutions (i) and (ii) the Kretschmann scalar invariant vanishes and for the solutions (iii), (iv) and (v) we have

\[
K_{(iii)} = \frac{21 c_0^4}{4 f_0^2} e^{-c_0 t},
\]

\[
K_{(iv)} = \frac{84}{f_0^2} (t - \beta_0)^2,
\]

\[
K_{(v)} = \frac{84}{f_0^2} (t - \gamma_0)^2.
\]

We see that in the latter case the Kretschmann scalar invariant is singular for the point \( t = \gamma_0 \); therefore this singular point is essential.

---

\(^7\)Note that this matrix representation is useful for simplifying the computations.
3.1.2 Dual model

In the same way, one can construct the dual model on the Lie supergroup $(A_{1,1} + 2A)^0_{1,0,0}$ using the following parametrization:

$$\tilde{g} = e^{\tilde{x}_1}e^{\tilde{y}_2}e^{\tilde{z}_3}. \quad (3.44)$$

In this case, we find

$$\tilde{\Pi}_{ij}(\tilde{g}) = \begin{pmatrix} 0 & 0 & -\tilde{\psi} \\ 0 & 0 & 0 \\ \tilde{\psi} & 0 & 0 \end{pmatrix}, \quad (3.45)$$

and using the equations (2.14)-(2.20) the dual action is obtained as

$$\tilde{S} = \frac{1}{2} \int d\xi^+ \wedge d\xi^- \left[ \partial_+ t f(t) \partial_- \tilde{x} + \partial_+ \tilde{x} a(t) \partial_- \tilde{\psi} + \left( a(t)e(t)\tilde{\psi} - \frac{a(t)e(t)\tilde{\psi}}{2} \right) \partial_- \tilde{\psi} 
+ \partial_+ \tilde{\psi} \left( a(t)e(t)\tilde{\psi} + \frac{a(t)e(t)\tilde{\psi}}{2} \right) \partial_- \tilde{x} - \partial_+ \tilde{x} e(t) \partial_- \tilde{x} + \partial_+ \tilde{x} e(t) \partial_- \tilde{\psi} - \frac{1}{4} R^{(2)} \tilde{\phi} \right], \quad (3.46)$$

such that for this model we have

$$\tilde{G}_{AB} = \begin{pmatrix} f(t) & 0 & 0 & 0 \\ 0 & a(t) & -\frac{a(t)e(t)}{2} & 0 \\ 0 & -\frac{a(t)e(t)}{2} & 0 & e(t) \\ 0 & 0 & -e(t) & 0 \end{pmatrix}, \quad \tilde{B}_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a(t)e(t)\tilde{\psi} & 0 \\ 0 & -a(t)e(t)\tilde{\psi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.47)$$

Note that for the above action the nonzero components of $\tilde{H}$ have the following forms:

$$\tilde{H}_{012} = \frac{d}{dt}(a(t)e(t))\tilde{\psi}, \quad \tilde{H}_{122} = -2a(t)e(t). \quad (3.48)$$

Also, by taking the $\varphi = \varphi^0(t) = 0$ and using the last equation in (2.20) we find

$$\tilde{\varphi} = \ln(\frac{a(t)}{e^2(t)}). \quad (3.49)$$

Now using the Eq. (3.48) we obtain that $\tilde{H}_{MN} = \tilde{H}_{MNP}\tilde{H}^{QP}N = 0$ and $\tilde{H}^2 = \tilde{H}_{MNP}\tilde{H}^{PNM} = 0$; in this way the relations (3.2)-(3.4) take the following forms for the dual model:

$$\beta(\tilde{G})_{MN} = \tilde{R}_{MN} + 2\tilde{\nabla}_M\tilde{\nabla}_N\tilde{\varphi} = 0, \quad (3.50)$$

$$\beta(\tilde{B})_{NP} = (-1)^M\tilde{\nabla}_M(e^{-2\tilde{\varphi}}\tilde{H}_{MNP}) = 0, \quad (3.51)$$

$$\beta(\tilde{\varphi}) = -\tilde{R} + 4(\tilde{\nabla}\tilde{\varphi})^2 - 4\tilde{\nabla}^2\tilde{\varphi} = 0, \quad (3.52)$$

where the nonzero components of $\tilde{R}_{MN}$ and $\tilde{\nabla}_M\tilde{\nabla}_N\tilde{\varphi}$ have the following forms, respectively, (see, appendix A)
\[ \ddot{R}_{00} = \frac{d}{dt}(\frac{\dot{e}}{e} - \frac{\dot{a}}{2a}) + \frac{\dot{a}f}{2af} - \frac{\dot{e}f}{2ef} - \frac{\dot{a}^2}{4a^2} + \frac{\dot{e}^2}{2e^2}, \quad (3.53) \]

\[ \ddot{R}_{11} = -\frac{d}{dt}(\frac{\dot{a}}{2f}) - \frac{\dot{a}f}{4f^2} + \frac{\dot{a}e}{2ef} + \frac{\dot{a}^2}{4af}, \quad (3.54) \]

\[ \ddot{R}_{12} = -\frac{1}{2} \ddot{R}_{11} \dddot{\psi}, \quad (3.55) \]

\[ \ddot{R}_{23} = -\frac{d}{dt}(\frac{\dot{e}}{2f}) - \frac{\dot{e}f}{4f^2} - \frac{\dot{a}e}{2ef} + \frac{\dot{e}^2}{ef}, \quad (3.56) \]

\[ \overleftarrow{\nabla}_0 \overleftarrow{\nabla}_0 \dddot{\phi} = \frac{d^2}{dt^2} \ln\left(\frac{a}{e^2}\right) - \frac{\dot{f}}{2f} \frac{d}{dt} \ln\left(\frac{a}{e^2}\right), \quad \overleftarrow{\nabla}_1 \overleftarrow{\nabla}_1 \dddot{\phi} = \frac{\dot{a}}{2f} \frac{d}{dt} \ln\left(\frac{a}{e^2}\right), \quad (3.57) \]

\[ \overleftarrow{\nabla}_2 \overleftarrow{\nabla}_2 \dddot{\phi} = -\frac{1}{2} (\overleftarrow{\nabla}_1 \overleftarrow{\nabla}_1 \dddot{\phi}) \dddot{\psi}, \quad \overleftarrow{\nabla}_2 \overleftarrow{\nabla}_3 \dddot{\phi} = \frac{\dot{e}}{2f} \frac{d}{dt} \ln\left(\frac{a}{e^2}\right), \quad (3.58) \]

and the $\beta(H)_{NP} = 0$ leads to

\[ (-2 \frac{\overleftarrow{\nabla}^M \dddot{\phi}}{\partial x^M}) \dddot{H}^M_{NP} + (-1)^M \overleftarrow{\nabla}^M \dddot{H}_{MNP} \]

\[ = (-2 \overrightarrow{\nabla}^M \dddot{\phi}) \dddot{\bar{H}}^M_{NP} + (-1)^{M+L+L(N+P)} \bar{G}^L^M \dddot{\bar{H}}_{MNP} \frac{\overrightarrow{\nabla}^L}{\partial x^L} \]

\[ - (-1)^{(N+P)(M+Q)} \dddot{\bar{H}}^Q_{NP} \dddot{\bar{Q}}_{ML} - (-1)^{P(N+Q)} \dddot{\bar{H}}^Q_{MNP} \dddot{\bar{Q}}_{NL} - \dddot{\bar{H}}_{NMNP} \dddot{\bar{Q}}_{PL} = 0. \quad (3.59) \]

The dilatonic contribution to the $\beta(\dddot{\phi})$ is

\[ (\overleftarrow{\nabla} \dddot{\phi})^2 - \nabla^2 \dddot{\phi} = (\overrightarrow{\nabla} \dddot{\phi})(\overleftarrow{\nabla} \dddot{\phi}) - \overrightarrow{\nabla} \overleftarrow{\nabla} \dddot{\phi} \]

\[ = (-1)^{M+L+N} \dddot{G}^MN \left[ (\dddot{\phi} \frac{\overrightarrow{\nabla} \overleftarrow{\nabla} \dddot{\phi}}{\partial x^N}) - (\dddot{\phi} \frac{\overrightarrow{\nabla} \overleftarrow{\nabla} \dddot{\phi}}{\partial x^M}) + (\dddot{\phi} \frac{\overrightarrow{\nabla} \overleftarrow{\nabla} \dddot{\phi}}{\partial x^P}) \dddot{\bar{Q}}_{NP} \right] \]

\[ = \frac{1}{f} \left[ \frac{d}{dt} \left( \frac{2\dot{e}}{e} - \frac{\dot{a}}{a} \right) + \frac{\dot{a}^2}{2a^2} + \frac{\dot{e}^2}{e^2} + \frac{\dot{a}f}{2af} - \frac{\dot{e}f}{ef} - \frac{2\dot{a}e}{ae} \right]. \quad (3.60) \]

Finally, by substituting the relations \(3.53\) and \(3.61\) into the Eqs. \(3.50\)-\(3.52\) we obtain the following equations:

\[ \frac{d}{dt} \left( \frac{\dot{a}}{a} - \frac{2\dot{e}}{e} \right) - \frac{\dot{a}f}{2af} + \frac{\dot{e}f}{ef} - \frac{\dot{a}^2}{6a^2} + \frac{\dot{e}^2}{3e^2} = 0, \quad (3.61) \]

\[ \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) - \frac{\dot{a}f}{2af} - \frac{3\dot{a}^2}{2a^2} + \frac{3\dot{a}e}{ae} = 0, \quad (3.62) \]

\[ \frac{d}{dt} \left( \frac{\dot{e}}{e} \right) - \frac{\dot{e}f}{2ef} - \frac{3\dot{a}e}{2ae} + \frac{3\dot{e}^2}{e^2} = 0, \quad (3.63) \]

\[ \frac{d}{dt} \left( \frac{\dot{a}}{a} + \frac{\dot{e}}{e} \right) - \frac{\dot{a}f}{2af} - \frac{\dot{e}f}{2ef} + \frac{3\dot{a}e}{2ae} - \frac{3\dot{a}^2}{2a^2} + \frac{3\dot{e}^2}{e^2} = 0, \quad (3.64) \]
\[ \frac{d}{dt} \left( \frac{d\tilde{a}}{dt} \right) - \frac{2\dot{\tilde{a}}}{\tilde{e}} = \frac{\dot{a}}{2af} + \frac{\dot{f}}{ef} + \frac{3\dot{a}^2}{2ae} - \frac{5\dot{a}^2}{6a^2} - \frac{17e^2}{6e^2} = 0, \quad (3.65) \]

Now, by combination of Eqs. (3.61) and (3.65) we find the following constraint:

\[ \frac{d}{dt} \ln(a(t)) = \frac{9 + \sqrt{5}}{4} \frac{d}{dt} \ln(e(t)), \quad (3.66) \]

and the result of combination Eqs. (3.62) and (3.64) is Eq. (3.63), then by substituting the constraint (3.66) into Eq. (3.63), we obtain the following equation:

\[ \frac{d^2}{dt^2} \ln(e(t)) - \frac{1}{2} \frac{d}{dt} \ln(e(t)) \frac{d}{dt} \ln(f(t)) - \frac{3(1 + \sqrt{5})}{8} \left( \frac{d}{dt} \ln(e(t)) \right)^2 = 0, \quad (3.67) \]

where the general solution for the above equation has the following form:

\[ e(t) = c_1 e^{-\frac{a}{4(1 + \sqrt{5})}A(t)}, \quad (3.68) \]

for which we obtain the following special class of solutions

\[ (i)^\pm : \quad a(t) = \tilde{a}_0 e^{\frac{9 + \sqrt{5}}{4}t}, \quad e(t) = \tilde{e}_0 e^\tilde{\alpha} t, \quad f(t) = \tilde{f}_0 e^{-\frac{3(1 + \sqrt{5})}{4}t}, \quad (3.69) \]

\[ (ii)^\pm : \quad a(t) = \tilde{a}_0 (t - \tilde{\beta}_0)^{\frac{9 + \sqrt{5}}{12}}, \quad e(t) = \tilde{e}_0 (t - \tilde{\beta}_0)^{\frac{4}{3}}, \quad f(t) = \frac{\tilde{f}_0}{(t - \tilde{\beta}_0)^{\frac{9 + \sqrt{5}}{4}}}, \quad (3.70) \]

where \( \tilde{a}_0, \tilde{e}_0, \tilde{f}_0, \tilde{\alpha}_0 \) and \( \tilde{\beta}_0 \) are real constants. Using the following form of the metric of the dual model

\[ ds^2 = f(t)dt^2 + a(t) d\tilde{x}^2 - \frac{a(t)\tilde{\psi}}{2} d\tilde{\psi} d\tilde{x} - \frac{a(t)\tilde{\psi}}{2} d\tilde{x} d\tilde{\psi} - e(t) d\psi d\chi + e(t) d\chi d\tilde{\psi}, \quad (3.71) \]

and after some calculations one can obtain the general form of the Kretschmann scalar invariant for the dual model as follows:

\[ \tilde{K} = \frac{1}{f^2} \left[ \left( \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) - \frac{\dot{a}^2}{2af} \right)^2 + 2 \left( \frac{d}{dt} \left( \frac{\dot{e}}{e} \right) - \frac{\dot{e}^2}{2ef} + \frac{\ddot{e}^2}{2e^2} \right)^2 + \frac{a^2 e^2}{2a^2 e^2} + \frac{3\dot{e}^4}{4e^4} \right]. \quad (3.72) \]

For solutions \( (i)^\pm, (ii)^+ \) and \( (ii)^- \) the Kretschmann scalar invariant is given by

\[ \tilde{K}_{(i)^\pm} = \frac{(269 + 111\sqrt{5})\tilde{a}_0^4}{8\tilde{f}_0^2} e^{\frac{a}{4(1 + \sqrt{5})}t}, \quad (3.73) \]

\[ \tilde{K}_{(ii)^+} = \frac{22.82}{\tilde{f}_0^2} (t - \tilde{\beta}_0)^{\frac{1 + \sqrt{5}}{2}}, \quad (3.74) \]

\[ \tilde{K}_{(ii)^-} = \frac{2.28}{\tilde{f}_0^2} \frac{1}{(t - \tilde{\beta}_0)^{\frac{1 + \sqrt{5}}{2}}}. \quad (3.75) \]

We see that in the latter case the Kretschmann scalar invariant is singular for the point \( t = \tilde{\beta}_0 \); therefore this singular point is essential. Note that the form and coefficients of the Kretschmann scalar invariants for the original model and its dual are the same for all solutions and as we expect the feature of essential singularity of the metric of the model and its dual are preserved under duality, because duality transformation is a canonical transformation.
4. Conclusion

In this paper, as a continuation of Ref. [8] we extended the results of Ref. [7] to the supermanifolds by using of the formulation of Poisson-Lie T-dual sigma models on supermanifolds. Then, using this formalism we constructed 1+1 dimensional string cosmological models as an example which has super Poisson-Lie symmetry. Also one can construct other models by using other Lie superbialgebras\(^8\) such as \((C^5_{\rho=0}, \tilde{G}_{\alpha,\beta,\gamma})\) and \((C^2_{\rho=-1}, \tilde{G}_{\alpha,\beta,\gamma})\) of Ref. [12]. Furthermore, in this way one can construct the 2+1 and 3+1 dimensional string cosmological models that have super Poisson-Lie symmetry [18].

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A. Some properties of matrices and tensors on supervector space and supermanifolds

In this appendix we collect a few relevant details concerning properties of matrices and tensors on supervector space which feature in the main text, appear as supertranspose, superdeterminant, supertrace, etc [8].

We consider the standard basis for the supervector spaces so that in writing the basis as a column matrix, we first present the bosonic base, then the fermionic one. The transformation of standard basis and its dual basis can be written as follows:

\[
e'_{i} = (-1)^{j} K_{i j} e_{j}, \quad e^{i} = K^{-st_{i}}_{j} e^{j},
\]

(A.1)

where the transformation matrix \(K\) has the following block diagonal representation [8]

\[
K = \begin{pmatrix} A & C \\ D & B \end{pmatrix},
\]

(A.2)

where \(A, B\) and \(C\) are real submatrices and \(D\) is pure imaginary submatrix\(^9\). Here we consider the matrix and tensors having a form with all upper and lower indices written in the right hand side.

The transformation properties of upper and lower right indices to the left one for general tensors are as follows:

\[
i T^{k}_{j l \ldots} = T^{ik}_{j l \ldots}, \quad j T^{ik}_{l \ldots} = (-1)^{j} T^{ik}_{j l \ldots}.
\]

(A.3)

Let \(K, L, M\) and \(N\) be the matrices whose their elements indices have different positions. Then, we define the supertranspose for these matrices as follows:

\[
K^{st \ i}_{j} = (-1)^{ij} K_{j}^{i}, \quad L^{st \ i}_{j} = (-1)^{ij} L^{j}_{i},
\]

\(^8\)Note that these Lie superbialgebras have zero supertrace for the adjoint representation of the generators so that in this way the conformality is preserved under duality transformation [16].

\(^9\)For further details, one may refer to DeWitt’s book [8], p.24.
\[ M_{ij}^t = (-1)^i M_{ji}, \quad N_{ij}^t = (-1)^j N_{ji}. \] (A.4)

For the matrix \( K \) whose elements \( iK^j \) have the left index in the lower position and the right index in the upper position, we define the supertrace as follows:
\[
\text{str} K = (-1)^i i K^i = K_i^i, \tag{A.5}
\]
when \( K \) is expressed in the block form (A.2) the supertrace become
\[
\text{str} K = \text{tr} A - \text{tr} B, \tag{A.6}
\]
where 'tr' denotes the ordinary trace.

If the submatrix \( B \) in the block form (A.2) is a nonsingular, then the superdeterminant for the matrix \( K \) is defined by
\[
\text{sdet} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = \det (A - CB^{-1}D)(detB)^{-1}, \tag{A.7}
\]
and if the submatrix \( A \) is nonsingular, then
\[
\text{sdet} \begin{pmatrix} A & C \\ D & B \end{pmatrix} = (\det(B - DA^{-1}C))^{-1} (\det A). \tag{A.8}
\]

If both \( A \) and \( B \) are nonsingular, then the inverse matrix for (A.2) has the following form:
\[
\begin{pmatrix} A & C \\ D & B \end{pmatrix}^{-1} = \begin{pmatrix} (1_m - A^{-1}CB^{-1}D)^{-1}A^{-1} & -(1_m - A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1} \\ -(1_n - B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1} & (1_n - B^{-1}DA^{-1}C)^{-1}B^{-1} \end{pmatrix}, \tag{A.9}
\]
where \( m \) and \( n \) are dimensions of submatrices \( A \) and \( B \), respectively.

If \( f \) be a differentiable function on \( \mathbb{R}^m \times \mathbb{R}^n \) (\( \mathbb{R}^m \) are subset of all real numbers with dimension \( m \) and \( \mathbb{R}^n \) are subset of all odd Grassmann variables with dimension \( n \)), then relation between the left partial differentiation and right ones is given by
\[
\frac{\nabla}{\partial x^i} f = (-1)^{(i|f|+1)} f \frac{\partial}{\partial x^i}, \tag{A.10}
\]
where \( |f| \) indicates the grading of \( f \).

If \( f \) be a scalar field, \( \vec{X} = X^i \frac{\partial}{\partial x^i} \) a contravariant vector field and \( \omega = \omega_i dx^i \) a covariant vector field, then one finds covariant derivative in explicit components form as follows:
\[
f \nabla_i = (-1)^{(i|f|)} \nabla_i f = f \frac{\partial}{\partial x^i}, \tag{A.11}
\]
\[
X^i \nabla_j = (-1)^{(i(|X|+i)} \nabla_j X^i = X^i \frac{\partial}{\partial x^j} + (-1)^{(i+1)} X^k \Gamma^i_{kj}, \tag{A.12}
\]
\[
\omega_i \nabla_j = (-1)^{(i(|\omega|+i)} \nabla_j \omega_i = \omega_i \frac{\partial}{\partial x^j} - \omega_k \Gamma^k_{ij}, \tag{A.13}
\]
where $\Gamma^i_{jk}$ are called the components of the connection $\nabla$.

If the supersymmetric matrix $AG_B$ (its inverse denotes to $A^G_B$ and $G^{AB} = (-1)^{AB}G^{BA}$) be the components of metric tensor field on a Riemannian supermanifold, then, in a coordinate basis, the components of the connection and Riemann tensor field are given by

$$
\Gamma^M_{NP} = (-1)^Q G^{MQ} \Gamma_{QN}^P = \left(\frac{1}{2}G^{MQ}\frac{\partial}{\partial x^P} + (-1)^{NP}G^{QP}\frac{\partial}{\partial x^N}\right)
$$

$$
R^I_{JKL} = -\Gamma^I_{JK} \frac{\partial}{\partial x^L} + (-1)^{KL} \Gamma^I_{JL} \frac{\partial}{\partial x^K} + (-1)^{K(J+M)} \Gamma^I_{M} \Gamma^M_{JL}
$$

\[\text{(A.14)}\]

also, for the curvature tensor field, the Ricci tensor and the curvature scalar field we have

$$
R_{IJKL} = G^{IM} R_{MJKL}. \quad \text{(A.16)}
$$

$$
R_{IJ} = (-1)^{K(I+1)} R^K_{IKJ}. \quad \text{(A.17)}
$$

$$
R = R^M_M = \text{str}(R_{MN}G^{NM}). \quad \text{(A.18)}
$$

To lower and raise indices denoting tensor field components, one can use of the tensor fields $G$ and $G^{-1}$ as follows:

$$
T^C_{A_1,\ldots,A_r} {\%^B_1,\ldots,B_S} = (-1)^{(E+C)(B_1,\ldots,B_S)} T_{A_1,\ldots,A_r,E,B_1,\ldots,B_S} G^{EC}. \quad \text{(A.19)}
$$

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