High-rank minors for high-rank tensors

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Abstract

Let $d \geq 2$ be a positive integer. We show that for a class of notions $R$ of rank for order-$d$ tensors, which includes in particular the tensor rank, the slice rank and the partition rank, there exist functions $F_{d,R}$ and $G_{d,R}$ such that if an order-$d$ tensor has $R$-rank at least $G_{d,R}(l)$ then we can restrict its entries to a product of sets $X_1 \times \cdots \times X_d$ such that the restriction has $R$-rank at least $l$ and the sets $X_1, \ldots, X_d$ each have size at most $F_{d,R}(l)$. Furthermore, our proof methods allow us to show that under a very natural condition we can require the sets $X_1, \ldots, X_d$ to be pairwise disjoint.

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1 Introduction

The last few years have seen a sequence of successes in using notions of ranks for higher-dimensional tensors to solve combinatorial problems. A central idea from the breakthrough solution to the cap-set problem by Ellenberg and Gijswijt [2], which was based on a technique of Croot, Lev, and Pach [1], was reformulated by Tao [11] in terms of the notion of slice rank for tensors, leading to what is now known as the slice rank polynomial method. The slice rank was further studied by Sawin and Tao [10], and bounds shown there on the slice rank involving orderings on the coordinates were later used by Sauermann [9] to prove under suitable conditions the existence of solutions with pairwise distinct variables to systems of equations in subsets of $\mathbb{F}_p^n$ that are not exponentially sparse. Another fruitful generalisation of the idea underlying the slice rank has been the partition rank, which was defined by Naslund [8] in order to prove a polynomial upper bound on the size of subsets of $\mathbb{F}_p^n$ not containing any $k$-right corners (with $p$ a prime integer and $r \geq 1$ a positive integer) and very recently used again by Naslund [7] to prove exponential lower bounds on the chromatic number of $\mathbb{R}^n$ with multiple forbidden distances.

In this paper we will focus on high-rank minors of tensors: it is a standard fact from linear algebra that if $A$ is a matrix of rank $k$ then $A$ has a $k \times k$ minor with rank $k$, and we will study here the extent to which this statement can be generalised to notions of rank for higher-order tensors, in particular to the tensor rank, to the slice rank and to the partition rank. The results that we obtain in this direction as well as the methods that we use in their proofs will also allow us to prove that under a very natural assumption we can find a minor such that the coordinates take values in pairwise disjoint sets. As we explain in a few paragraphs, the formulation of this result also arises naturally as an analogue of the
standard inequality that every oriented graph has a bipartition such that at least a quarter of the edges go from the first part to the second.

We now define the relevant notions of higher-dimensional ranks for tensors and state our main theorems.

**Definition 1.1.** Let \( d \geq 2 \) be an integer and let \( \mathbb{F} \) be a field. An order-\( d \) tensor over \( \mathbb{F} \) is a function \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) for some finite subsets \( Q_1, \ldots, Q_d \) of \( \mathbb{N} \).

Throughout this paper we shall use the following notation. We write \( \mathbb{F} \) for an arbitrary field. If \( d \geq 2 \) is a positive integer, then \( Q_1, \ldots, Q_d \) will always stand for finite subsets of \( \mathbb{N} \), even if this is not explicitly indicated. Given an order-\( d \) tensor \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) and subsets \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \), we shall write \( T(X_1 \times \cdots \times X_d) \) for the restriction \( T' : X_1 \times \cdots \times X_d \to \mathbb{F} \) of \( T \). For each positive integer \( n \) we write \([n]\) for the set \( \{1, 2, \ldots, n\} \). Given \( x \in Q_1 \times \cdots \times Q_d \), and \( I \subset [d] \), we write \( x(I) \) for the restriction \( (x_\alpha : \alpha \in I) \) of \( x \) to its coordinates in \( I \). If \( d, s \geq 1 \) are positive integers, \( \mathbb{F} \) a field, \( T_1, \ldots, T_s \) are order-\( d \) tensors over \( \mathbb{F} \), and \( a \in \mathbb{F}^s \), then we write \( a.T \) for the linear combination \( \sum_{i=1}^s a_i T_i \).

Given a bipartition \( \{I, J\} \) of \([d]\) and points \( y : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F} \) and \( z : \prod_{\alpha \in J} Q_\alpha \to \mathbb{F} \), we write \( T(y, z) \) for the value \( T(x) \) where the element \( x \in \prod_{\alpha=1}^d Q_\alpha \) is defined by \( x_\alpha = y_\alpha \) for each \( \alpha \in I \) and \( x_\alpha = z_\alpha \) for each \( \alpha \in J \). If \( T : \prod_{\alpha=1}^d Q_\alpha \to \mathbb{F} \) is an order-\( d \) tensor, \( I \subset [d] \) and \( y \in \prod_{\alpha \in I} Q_\alpha \), then we write \( T_y : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F} \) for the order-\( |I| \) tensor defined by \( T_y(z) = T(y, z) \), with \( T(y, z) \) defined as in the previous sentence. Given arbitrary subsets \( I, J \) of \([d]\), and tensors \( T_1 : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F} \) and \( T_2 : \prod_{\alpha \in J} Q_\alpha \to \mathbb{F} \), we write \( T_1.T_2 : \prod_{\alpha \in I \Delta J} Q_\alpha \to \mathbb{F} \) for the tensor defined by

\[
(T_1.T_2)(y, z) = \sum_{x \in \prod_{\alpha \in I \cup J} Q_\alpha} T_1(y, x)T_2(z, x)
\]

for each \( y \in \prod_{\alpha \in I \cup J} Q_\alpha \) and each \( z \in \prod_{\alpha \in J \setminus I} Q_\alpha \).

**Definition 1.2.** Let \( d \geq 2 \) be an integer, and let \( T \) be an order-\( d \) tensor. We say that \( T \) has tensor rank at most \( 1 \) if there exist functions \( a_\alpha : Q_\alpha \to \mathbb{F} \) for each \( \alpha \in [d] \) such that

\[
T(x_1, \ldots, x_d) = a_1(x_1) \cdots a_d(x_d)
\]

for every \((x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d \).

We say that \( T \) has slice rank at most \( 1 \) if there exist \( \alpha \in [d] \) and functions \( a : Q_\alpha \to \mathbb{F} \) and \( b : \prod_{\alpha' \in [d], \alpha' \neq \alpha} Q_{\alpha'} \to \mathbb{F} \) such that we can write

\[
T(x_1, \ldots, x_d) = a(x_\alpha)b(x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_d)
\]

for every \((x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d \).

We say that \( T \) has partition rank at most \( 1 \) if there exist a bipartition \( \{I, J\} \) of \([d]\) with \( I, J \) both non-empty and functions \( a : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F} \) and \( b : \prod_{\alpha \in J} Q_\alpha \to \mathbb{F} \) such that we can write

\[
T(x_1, \ldots, x_d) = a(x(I))b(x(J))
\]
for every \((x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d\).

We say that the tensor rank (resp. slice rank, resp. partition rank) of \(T\) is the smallest nonnegative integer \(k\) such that there exist tensors \(T_1, \ldots, T_k\) each of tensor rank at most 1 (resp. slice rank at most 1, resp. partition rank at most 1) and such that \(T = T_1 + \cdots + T_k\). We denote by \(\text{tr } T\) the tensor rank of \(T\), by \(\text{sr } T\) the slice rank of \(T\), and by \(\text{pr } T\) the partition rank of \(T\).

Whenever \(d \geq 2\) is a positive integer and \(T\) is an order-\(d\) tensor, we always have \(\text{pr } T \leq \text{sr } T \leq \text{tr } T\): this follows from the fact that every order-\(d\) tensor with tensor rank at most 1 also has slice rank at most 1 and every order-\(d\) tensor with slice rank at most 1 also has partition rank at most 1. For \(d = 2\) the three notions coincide and are the same as the usual notion of rank for matrices. For \(d = 3\) the slice and partition ranks are the same, but are smaller than the tensor rank in general. For \(d \geq 4\) the three notions are pairwise distinct in general. In Section 2 we will show that the fact that every matrix of rank \(k\) has a \(k \times k\) minor with rank \(k\) generalises in the best way one could hope for to the tensor rank for all \(d \geq 2\): every order-\(d\) tensor \(T\) with tensor rank \(k\) has a \(k \times k \times \cdots \times k\) (\(d\) times) minor with tensor rank \(k\). In Section 3 we will however give an example which shows that this becomes false for the order-3 slice rank. As we will show in Section 4 it will nonetheless be true that if an order-\(3\) tensor is such that all its minors with size at most \(48\ell^3\) have slice rank at most \(l\) then the whole tensor has slice rank at most \(51\ell^3\). In Section 11 we will show that such an asymptotic minors property holds for the slice and partition rank for all \(d \geq 2\) as well as for a more general class of notions of rank which we will now define before stating this asymptotic result.

**Definition 1.3.** Let \(d \geq 2\) be an integer, and let \(R\) be a non-empty family of partitions of \([d]\). We say that an order-\(d\) tensor \(T\) has \(R\)-rank at most 1 if there exist a partition \(P \in R\) and for each \(I \in P\) a function \(a_I : \prod_{\alpha \in I} Q_\alpha \to \mathbb{F}\) such that we can write

\[
T(x_1, \ldots, x_d) = \prod_{I \in P} a_I(x(I))
\]

for every \((x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d\). We say that the \(R\)-rank of \(T\) is the smallest nonnegative integer \(k\) such that there exist order-\(d\) tensors \(T_1, \ldots, T_k\) with \(R\)-rank at most 1 such that \(T = T_1 + \cdots + T_k\).

We will denote by \(Rrk T\) the \(R\)-rank of \(T\). We can check that for every \(d \geq 2\), the \(R\)-rank specialised to the tensor rank, to the slice rank, and to the partition rank, by taking respectively

\[
R = \{\{1\}, \{2\}, \ldots, \{d\}\}
\]
\[
R = \{\{1\}, \{1\}^c\}, \{\{2\}, \{2\}^c\}, \ldots, \{\{d\}, \{d\}^c\}\}
\]
\[
R = \{\{I, J\} : \{I, J\} \text{ a bipartition of } [d] \text{ with } I, J \neq \emptyset\}
\]

We are now in a position to state our first main theorem.
Theorem 1.4. Let \( d \geq 2 \) be an integer, and let \( R \) be a non-empty family of partitions on \([d]\). There exist functions \( F_{d,R} : \mathbb{N} \to \mathbb{N} \) and \( G_{d,R} : \mathbb{N} \to \mathbb{N} \) such that if \( T \) is an order-\( d \) tensor with \( \text{Rrk} T \geq G_{d,R}(l) \) then there exist \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \) each with size at most \( F_{d,R}(l) \) such that \( \text{Rrk} T(X_1 \times \cdots \times X_d) \geq l \).

Another independent starting point is the following standard statement.

Proposition 1.5. Let \( G \) be an oriented graph with vertex set \( V \). There exists an ordered bipartition \((X,Y)\) of \( V \) such that the number of edges \((u,v)\) of \( G \) is at least a quarter of the total number of edges of \( G \).

This statement can be seen to be equivalent to the following: given a matrix \( A : [n] \times [n] \to \mathbb{F} \) there exist disjoint subsets \( X, Y \) of \([n]\) such that the restriction \( A(X \times Y) \) has at least a quarter as many support elements as \( A \) has outside the diagonal. A first step will be to obtain an analogue of this statement for ranks of matrices: this will be the aim of Section 11. We will then generalise this analogue in Section 11 to higher-order tensors. We note that Proposition 1.5 and its generalisation to uniform hypergraphs will themselves be involved in the proof of the general higher-order tensor case.

Let \( E \) be the set of points \((x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d \) that do not have pairwise distinct coordinates. The following definition will be central to our second main result.

Definition 1.6. Let \( d \geq 2 \) be an integer, let \( R \) be a non-empty family of partitions on \([d]\). For \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) an order-\( d \) tensor we define the essential \( R \)-rank

\[
e_{R} \text{rk} T = \min_V R \text{rk}(T + V)
\]

where the minimum is taken over all order-\( d \) tensors \( V : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) with support contained inside \( E \), and the disjoint \( R \)-rank

\[
d_{R} \text{rk} T = \max_{X_1, \ldots, X_d} R \text{rk}(T(X_1 \times \cdots \times X_d))
\]

where the maximum is taken over all \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \) with \( X_1, \ldots, X_d \) pairwise disjoint.

In the \( d = 2 \) case we will write \( \text{erk} \) and \( \text{drk} \) for respectively the associated essential rank and the disjoint rank corresponding to the usual notion of rank for matrices. For general \( d \geq 2 \), in the special cases \( R = \text{tr}, \text{sr}, \text{pr} \), we will respectively write \( \text{etr} \), \( \text{esr} \), \( \text{epr} \) for the associated essential \( R \)-rank and \( \text{dtr}, \text{dsr}, \text{dpr} \) for the associated disjoint \( R \)-rank.

It seems worthwhile to compare the essential \( R \)-rank with the disjoint \( R \)-rank, as it is straightforward to show that a tensor has essential \( R \)-rank equal to 0 if and only if it has disjoint \( R \)-rank equal to 0: the corresponding tensors are the tensors supported inside \( E \). Moreover, we can show that the disjoint \( R \)-rank is at most the essential \( R \)-rank.

Lemma 1.7. Let \( d \geq 2 \) be an integer, and let \( R \) be a non-empty family of partitions of \([d]\). For every order \( d \) tensor \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) we have

\[
d_{R} \text{rk} T \leq e_{R} \text{rk} T.
\]
Proof. Let $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ be pairwise disjoint sets and let $V : Q_1 \times \cdots \times Q_d \to \mathbb{F}$ be an order-$d$ tensor supported inside $E$. Since the support of $V$ is contained in $E$, which has empty intersection with $X_1 \times \cdots \times X_d$, we have

$$T(X_1 \times \cdots \times X_d) = (T - V)(X_1 \times \cdots \times X_d).$$

Moreover, taking a restriction of a tensor cannot increase its $R$-rank, so

$$R_{rk}(T - V)(X_1 \times \cdots \times X_d) \leq R_{rk}(T - V).$$

Therefore

$$R_{rk}T(X_1 \times \cdots \times X_d) \leq R_{rk}(T - V),$$

so taking the maximum over the $d$-tuples $(X_1, \ldots, X_d)$ of pairwise disjoint sets and the minimum over the tensors $V$ supported inside $E$ we obtain the desired inequality. \qed

Our second main result is a weak converse to this last inequality.

**Theorem 1.8.** Let $d \geq 2$ be an integer, and let $R$ be a non-empty family of partitions on $[d]$. There exists a function $G_{d,R} : \mathbb{N} \to \mathbb{N}$ such that if $T$ is an order-$d$ tensor such that $eR_{rk}T \geq G_{d,R}(l)$ then we have $dR_{rk}T \geq l$.

When $R$ is a family of partitions that corresponds to one of the notions of rank that we have already defined, it will be convenient to use the notation for the notion of rank instead of for the corresponding family of partitions when writing a pair $(d, R)$, and similarly for the indices of $F_{d,R}$, $G_{d,R}$, $G'_{d,R}$. For instance when $R$ corresponds to the partition rank for order-$d$ tensors we will write the pair $(d, R)$ as $(d, \text{pr})$, and the functions $F_{d,R}$, $G_{d,R}$, $G'_{d,R}$ as $F_{d,\text{pr}}$, $G_{d,\text{pr}}$, $G'_{d,\text{pr}}$ respectively.

The methods involved in our proofs of Theorem 1.4 and of Theorem 1.8 are similar in several ways: those that we will use to prove the latter can be viewed as a moderate complication of those that we will use to prove the former.

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**2 Optimal bounds for tensor rank minors using spanning techniques**

In this section we prove Theorem 1.4 in the case of the tensor rank. Given a vector space $V$ over a field $\mathbb{F}$, and a subset $U$ of $V$, we write $\langle U \rangle$ for the linear subspace of $V$ spanned by $U$. The steps of the proof will be modelled after those of the following standard proof of minors for matrices. Let $A : Q_1 \times Q_2 \to \mathbb{F}$ be a matrix.
1. We characterise \( \text{rk} \, A \) as the smallest \( k \geq 0 \) such that there exist vectors \( L_1, \ldots, L_k \in \mathbb{F}^Q \) such that \( A_x \in \langle L_1, \ldots, L_k \rangle \) for each \( x \in Q_1 \).

2. We extract a basis \((A_x : x \in X)\) of the linear subspace \( \langle A_x : x \in Q_1 \rangle \) spanned by all rows of \( A \), with a set \( X \) of size \( \dim(\langle A_x : x \in Q_1 \rangle) = \text{rk} \, A \).

3. It follows from the two previous steps that \( \text{rk} \, A(X \times Q_2) = \text{rk} \, A \). Iterating again on the second coordinate we are done.

As an analogue of step 1 we begin by expressing the tensor rank of an order-\( d \) tensor in terms of a notion of rank for families of order-(\( d - 1 \)) tensors.

**Definition 2.1.** Let \( V \) be a finite-dimensional vector space over \( \mathbb{F} \) and let \( S, F \subset V \). The \textit{spanning rank} \( \text{sprk}_S \, F \) of \( F \) with respect to \( S \) is the smallest nonnegative integer \( k \) such that there exist vectors \( v_1, \ldots, v_k \in S \) satisfying \( F \subset \langle v_1, \ldots, v_k \rangle \).

When \( S = V \), we have \( \text{sprk}_S \, F = \dim(F) \) for every \( F \subset V \). For any fixed \( F \subset V \), \( S \mapsto \text{sprk}_S \, F \) is decreasing for inclusion, so in particular we always have \( \text{sprk}_S \, F \geq \dim(F) \).

**Lemma 2.2.** Let \( d \geq 3 \) be a positive integer, and let \( S_{d-1} \) be the family of order-(\( d - 1 \)) tensors \( \prod_{a=2}^{d} Q_a \to \mathbb{F} \) with order-(\( d - 1 \)) tensor rank equal to 1. Then every order-\( d \) tensor \( T : \prod_{a=1}^{d} Q_a \to \mathbb{F} \) satisfies

\[
\text{tr} \, T = \text{sprk}_{S_{d-1}} \{ T_{x_1} : x_1 \in Q_1 \}.
\]

**Proof.** Let \( k \) be an nonnegative integer. If \( \text{tr} \, T = k \) then there exist functions \( a_{i,\alpha} : Q_\alpha \to \mathbb{F} \) for each \( i \in [k] \) and each \( \alpha \in [d] \) such that

\[
T(x_1, \ldots, x_d) = \sum_{i=1}^{k} \prod_{a=1}^{d} a_{i,\alpha}(x_\alpha)
\]

for every \((x_1, \ldots, x_d) \in \prod_{a=1}^{d} Q_\alpha\). Then the order-(\( d - 1 \)) tensors \( T^i = \prod_{a=2}^{d} a_{i,\alpha} \) for each \( i \in [k] \) each have order-(\( d - 1 \)) tensor rank at most 1 and \( T_{x_1} \in \langle T^1, \ldots, T^k \rangle \) for each \( x_1 \in Q_1 \). Conversely if \( \text{sprk}_{S_{d-1}} \, T = k \) then there exist order-(\( d - 1 \)) tensors \( T^1, \ldots, T^k : \prod_{a=2}^{d} Q_\alpha \to \mathbb{F} \) with tensor rank at most 1 such that for each \( x_1 \in Q_1 \) there exist \( a_1(x_1), \ldots, a_k(x_1) \in \mathbb{F} \) satisfying \( T_{x_1} = \sum_{i=1}^{k} a_i(x_1) T^i \), so we can write

\[
T(x_1, \ldots, x_d) = \sum_{i=1}^{k} a_i(x_1) T^i(x_2, \ldots, x_d)
\]

for every \((x_1, \ldots, x_d) \in \prod_{a=1}^{d} Q_\alpha\), which shows that \( T \) has tensor rank at most \( k \). \( \square \)

Our next lemma is an adaptation of the linear-algebra fact underlying step 2: the claim that a finite family of vectors of a vector space has a subfamily with the same rank as that of the original family and with the same size as its rank.
Lemma 2.3. Let $k \geq 1$ be a positive integer, let $V$ be a finite-dimensional vector space over $\mathbb{F}$, and let $S$ and $F$ be families of elements of $V$. Assume that $\sprk_S F \geq k$. Then there exists a subfamily $F'$ of $F$ with size at most $k$ such that $\sprk_S F' \geq k$.

Proof. We distinguish two cases depending on the dimension of the linear subspace $\langle F \rangle$. If $\dim \langle F \rangle \geq k$ then we take $F'$ to be a linearly independent family of $k$ elements of $F$; because $\dim \langle F' \rangle = k$, in particular $\sprk F' \geq k$. If on the other hand $\dim \langle F \rangle \leq k$ then we take $F'$ to be a maximal linearly independent family of elements of $F$; the family $F'$ has size at most $k$, and since $\langle F \rangle = \langle F' \rangle$, a family of elements of $S$ spans all elements of $F$ if and only if it spans all elements of $F'$, so $\sprk F' = \sprk F$.

We are now ready to deduce our minors result for the tensor rank.

Proposition 2.4. Let $d \geq 3$, $k \geq 1$ be positive integers and let $T : \prod_{\alpha=1}^{d} Q_\alpha \to \mathbb{F}$ be an order-$d$ tensor. Assume that $\tr T \geq k$. Then there exist sets $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ with size at most $k$ such that $\tr T(X_1 \times \cdots \times X_d) \geq k$.

Proof. By Lemma 2.2 we have $\sprk_{S_d-1}\{T_{x_1} : x_1 \in Q_1\} = \tr T$. Since $\tr T \geq k$, by Lemma 2.3 there exists a subset $X_1$ of $Q_1$ with size at most $k$ such that $\sprk_{S_d-1}\{T_{x_1} : x_1 \in X_1\} \geq k$, so applying Lemma 2.2 again we have

$$\tr T(X_1 \times \prod_{j=2}^{d} Q_j) \geq k.$$ 

Iterating this argument $d-1$ more times, which we can, since the roles of the $d$ coordinates are the same in the definition of the tensor rank, we obtain the desired sets $X_2, \ldots, X_d$. 

The remainder of this section is devoted to proving a generalisation of Proposition 2.4 to linear subspaces spanned by a fixed number of tensors. We begin by formulating such a generalisation for matrices.

Proposition 2.5. Let $s, k \geq 1$ be positive integers, and let $A_1, \ldots, A_s : Q_1 \times Q_2 \to \mathbb{F}$ be matrices. Then there exists a subset $X \subset Q_1$ with size at most $sk$ such that

$$\rk(a.A)(X \times Q_2) \geq \min(\rk(a.A), k)$$

for every $a \in \mathbb{F}^s$. Iterating a second time, there also exists a subset $Y \subset Q_2$ with size at most $sk$ such that

$$\rk(a.A)(X \times Y) \geq \min(\rk(a.A), k)$$

for every $a \in \mathbb{F}^s$.

Proof. We remove rows one by one until there are only at most $sk$ remaining rows, with the inductive step being as follows. Assume that there are still $m > sk$ remaining rows and let $Q'_1$ be the set of remaining rows. Let $\Lambda = \{a \in \mathbb{F}^s : \rk(a.A)(Q'_1 \times Q_2) \leq k\}$. We can take a family $(a^1, \ldots, a^r)$ of elements of $\mathbb{F}^s$ such that $r \leq s$ and $\Lambda \subset \langle a^1, \ldots, a^r \rangle$. For
each $j \in [r]$ the linear subspace $U_j$ of $\mathbb{F}^{Q_i}$ such that $\sum_{x \in Q_i^j} b_x (a^j A)_x = 0$ has dimension at least $m - k$, so the intersection $\bigcap_{1 \leq j \leq r} U_j$ has dimension at least $m - sk > 0$. Taking a non-zero element $b$ of this intersection and then taking $x \in Q'_1$ such that $b_x \neq 0$, removing the $x$th row does not change the rank of $(a.A)(Q'_1 \times Q_2)$ for any $a \in \Lambda$. Moreover for all $a \in \mathbb{F}^s \setminus \Lambda$ we have $\text{rk}(a.A)(Q'_1 \times Q_2) \geq k + 1$ so the rank of $(a.A)(Q'_1 \times Q_2)$ is still at least $k$ after removing the $x$th row from $Q'_1$.

We now apply Proposition 2.5 to obtain a multidimensional version of Lemma 2.3.

**Lemma 2.6.** Let $s \geq 1$ be a positive integer, let $V$ be a finite-dimensional vector space over $\mathbb{F}$, let $S$ be a family of elements of $V$, let $Q$ be a finite subset of $\mathbb{N}$ and for each $j \in [s]$ and $x \in Q$ let $T_{j,x}$ be an element of $V$. Then there exists a subset $X$ of $Q$ with size at most $sk$ such that

$$\text{sprk}((a.T)_x)_{x \in X} \geq \min(k, \text{sprk}((a.T)_x)_{x \in Q})$$

for every $a \in \mathbb{F}^s$.

**Proof.** We fix an arbitrary choice of basis $B = (b_1, \ldots, b_{\dim V})$ of $V$. For each $j \in [s]$, let $A_j : Q \times [\dim V] \to \mathbb{F}$ be the matrix such that for each $x \in Q$, the row $(A_j)_x$ is the family of coefficients of $T_{j,x}$ written in the basis $B$. By Proposition 2.5 applied to the matrices $A_1, \ldots, A_s$ there exists a set $X$ of size at most $sk$ such that

$$\text{rk}((a.T)_x)_{x \in X} \geq \min(k, \text{rk}((a.T)_x)_{x \in Q})$$

for every $a \in \mathbb{F}^s$. We now fix $a \in \mathbb{F}^s$. If $\text{rk}((a.T)_x)_{x \in Q} \geq k$ then $\text{rk}((a.T)_x)_{x \in X} \geq k$, so since $\text{sprk}((a.T)_x)_{x \in X} \geq \text{rk}((a.T)_x)_{x \in X}$ we conclude that

$$\text{sprk}((a.T)_x)_{x \in X} \geq k.$$ 

If on the other hand $\text{rk}((a.T)_x)_{x \in Q} \leq k$ then $\langle (a.T)_x \rangle_{x \in Q} = \langle (a.T)_x \rangle_{x \in X}$, so since the spanning ranks of two families with the same linear spans are equal we conclude that

$$\text{sprk}((a.T)_x)_{x \in X} = \text{sprk}((a.T)_x)_{x \in Q}. \quad \Box$$

**Proposition 2.7.** Let $d, s, k \geq 1$ be positive integers, and $T_1, \ldots, T_s : Q_1 \times \cdots \times Q_d \to \mathbb{F}$ be order-$d$ tensors. Then there exist subsets $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ all with size at most $sk$ such that

$$\text{tr}(a.T)(X_1 \times \cdots \times X_d) \geq \min(k, \text{tr} a.T)$$

for every $a \in \mathbb{F}^s$.

**Proof.** By Lemma 2.6 there exists $X_1$ with size at most $sk$ such that

$$\text{sprk}((a.T)_{x_1})_{x_1 \in X_1} \geq \min(k, \text{sprk}((a.T)_{x_1})_{x_1 \in Q_1})$$

for every $a \in \mathbb{F}^s$, so applying Lemma 2.2 to both sides we obtain

$$\text{tr}(a.T)(X_1 \times \prod_{\alpha=2}^d Q_{\alpha}) \geq \min(k, \text{tr} a.T)$$

for every $a \in \mathbb{F}^s$. Iterating this argument $d - 1$ more times we get the desired other sets $X_2, \ldots, X_d$. \quad \Box
For any fixed positive integers \( k, s \) the bound \( sk \) can be seen to be optimal by taking \( Q_1 = \cdots = Q_d = [sk] \) and taking \( T_1, \ldots, T_s \) to have pairwise disjoint supports each of size exactly \( k \) and all contained in the diagonal \( \{(x_1, \ldots, x_d) : x_1 = \cdots = x_d\} \).

Proposition 2.7 suggests the following conjecture, which would if true strengthen Theorem 1.4 in two ways: in its statement the lower bound on the \( R \)-rank of a restriction is the same as the \( R \)-rank of the original tensor in the regime where the latter is small, and the lower bounds apply to restrictions of linear combinations of several tensors. This conjecture however seems far out of reach of the methods of the present paper.

**Conjecture 2.8.** Let \( d \geq 2, s \geq 1 \) be positive integers and let \( R \) be a non-empty family of partitions on \([d]\). Then there exists a function \( F_{d,R,s,\text{same}} \) such that whenever \( T_1, \ldots, T_s \) are order-\( d \) tensors there exist \( X_1, \ldots, X_d \) of size at most \( F_{d,R,s,\text{same}}(l) \) such that

\[
Rrk(a.T)(X_1 \times \cdots \times X_d) \geq \min(l, Rrk a.T)
\]

for every \( a \in \mathbb{F}^s \).

Conjecture 2.8 would be false if we furthermore required \( F_{d,R,s,\text{same}}(l) = l \), as the next section will show for \( d = 3, s = 1 \), and \( R = \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}\}.

## 3 A counterexample to a strong bound for order-3 slice-rank minors

We thank Timothy Gowers for discovering Proposition 3.1 and constructing the example in this section, which we include here with his permission.

**Proposition 3.1.** Let \( \mathbb{F} \) be a field. Then there exists an order-3 tensor \( T : [11] \times [4] \times [15] \to \mathbb{F} \) such that \( sr T = 4 \) but whenever \( X \subset [11], Y \subset [4], Z \subset [15] \) are all of size 4, \( sr(T(X \times Y \times Z)) \leq 3 \).

For \( n_1, n_2, n_3 \) positive integers, we say that a subset \( U \subset [n_1] \times [n_2] \times [n_3] \) is an antichain if whenever \((x', y', z'), (x'', y'', z'') \in U \) are such that \( x' \leq x'', y' \leq y'', z' \leq z'' \), necessarily \((x', y', z') = (x'', y'', z'') \). In particular a set of the type

\[
\{ (x, y, z) \in [n_1] \times [n_2] \times [n_3] : x + y + z = k \}
\]

for some integer \( k \) is an antichain.

If \( U \) is a subset of \([n_1] \times [n_2] \times [n_3] \), we say that the slice covering number \( sc U \) is the smallest nonnegative integer \( k \) such that \( U \) can be covered by \( k \) slices, i.e., such that there exist nonnegative integers \( r, s, t \) with \( r + s + t = k \), and \( a_1, \ldots, a_r \in [n_1], b_1, \ldots, b_s \in [n_2], c_1, \ldots, c_t \in [n_3] \) satisfying

\[
U \subset ( \bigcup_{1 \leq i \leq r} \{ x = a_i \} ) \cup ( \bigcup_{1 \leq j \leq s} \{ y = b_j \} ) \cup ( \bigcup_{1 \leq k \leq t} \{ z = c_k \} ).
\]
The following definition will also be convenient for us. For $V$ a subset of $[n_1] \times [n_2]$, we say that the three-point line covering number $\text{lc}_3 V$ of $V$ is the smallest nonnegative integer $k$ such that $V$ can be covered by $k$ lines, with each line of one of the three types $\{x = a\}$ or $\{y = b\}$ or $\{x + y = c\}$.

It is a special case of a result of Sawin and Tao (Proposition 4 in [10]) that the slice rank $sr T$ of an order-3 tensor $T$ with support $U$ contained in an antichain is equal to the smallest number $sc U$ of (order-2) slices that suffice to cover its support $U$.

Let $S : [11] \times [4] \to \mathbb{F}$ be a matrix with support exactly equal to a subset $V$ of $[11] \times [4]$, and $T(x, y, z) = S(x, y)1_{z=x+y}$. The tensor $T' : [11] \times [4] \times [15] \to \mathbb{F}$ defined by $T' : (x, y, z) \mapsto T(x, y, 16 - z)$ has support $U'$ contained in the antichain $\{x + y + z = 16\}$, so it satisfies $sr T' = sc U'$. Since $sr T = sr T'$ and $sc U' = sc U$ we obtain $sr T = sc T$.

Let $U$ be the support of $T$. For any positive integer $k$, the intersections of the slices $\{x = a\}$, $\{y = b\}$, $\{z = c\}$ inside $[11] \times [4] \times [15]$ with the set $\{x + y + z = k\}$ are respectively the subsets $\{x = a\}$, $\{y = b\}$, $\{x + y = k - c\}$ of $[11] \times [4]$. Hence the following claim.

**Claim 1.** We have $sc U = \text{lc}_3 V$. More generally, for any $X \subset [11]$, $Y \subset [4]$, $Z \subset [15]$, $$sc(U \cap (X \times Y \times Z)) = \text{lc}_3(V \cap \mu(X, Y, Z))$$ where $\mu(X, Y, Z) = \{(x, y) \in X \times Y : x + y \in Z\}$.

We choose a set $$V = \{(2, 1), (6, 1), (11, 1), (1, 2), (11, 3), (1, 4), (6, 4), (10, 4)\} \subset [11] \times [4]$$ which we draw below as a matrix with the $x$ coordinates increasing from 1 to 11 from left to right, and the $y$ coordinates increasing from 1 to 4 from top to bottom).

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The description provided by Claim 1 together with the following two claims provides a proof of Proposition 3.1.

**Claim 2.** $\text{lc}_3 V = 4$.

**Proof.** The upper bound $sc_3 V \leq 4$ follows from taking the four lines $y = 1, y = 2, y = 3, y = 4$. We now prove the lower bound. Every line (of the type $x = a$, $y = b$, or $x + y = c$) contains at most three points. To cover all eight points with three lines, at least two of the lines must have three points. The only such lines are the lines $y = 1$ and $y = 4$. This leaves us with the task of covering the two remaining points $(1, 2)$ and $(11, 3)$ with one line, which cannot be done.

**Claim 3.** For all $X \subset [11]$, $Y \subset [4]$, $Z \subset [15]$ of size 4, $\text{lc}_3(V \cap \mu(X, Y, Z)) \leq 3$. 

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Proof. The number of values of \( x + y \) among the eight points of \( V \) is six: these values are 3, 5, 7, 10, 12 and 14. It follows that in any minor of size 4 we must remove all the points from at least two of the lines of the type \( x + y = c \).

If we remove the line \( x + y = 3 \), then we can cover the remaining points with the lines \( x = 11, \ y = 1, \ y = 4 \). Similarly if we remove the line \( x + y = 14 \), then we can cover the remaining points with the lines \( x = 1, \ y = 1, \ y = 4 \).

If we remove the line \( x + y = 7 \) (i.e. the point \( (6, 1) \)) then we can cover the remaining points with the lines \( x = 11, \ y = 4, \ x + y = 3 \). Similarly if we remove the line \( x + y = 10 \) (i.e. the point \( (6, 4) \)), then we can cover the remaining points with the lines \( x = 1, \ y = 1, \ x + y = 14 \).

The only remaining possibility is to remove both the lines \( x + y = 5 \) and \( x + y = 12 \), i.e. the points \( (1, 4) \) and \( (11, 1) \). We can then cover the remaining points with the lines \( x = 6, \ x + y = 3, \ x + y = 14 \).

4 Partition rank minors in the finite fields case using projections and analytic rank

Throughout this section and only in this section we assume that the field \( \mathbb{F} \) is finite and prove Theorem 1.4 for the partition rank in this special case. Our main result will be the following.

Proposition 4.1. Let \( d \geq 2 \) be a positive integer and let \( \mathbb{F} \) be a finite field. Then there exist \( C(d, |\mathbb{F}|), L(d, |\mathbb{F}|) \) and functions \( F_{d,\text{pr}}^{[\mathbb{F}]} : \mathbb{N} \to \mathbb{N}, \ G_{d,\text{pr}}^{[\mathbb{F}]} : \mathbb{N} \to \mathbb{N} \) satisfying

\[
F_{d,\text{pr}}^{[\mathbb{F}]}(l) \leq |\mathbb{F}|^{dl} \quad \text{and} \quad G_{d,\text{pr}}^{[\mathbb{F}]}(l) \leq l^{C(d,|\mathbb{F}|)}
\]

for all \( l \geq L(d, |\mathbb{F}|) \) and such that the following property holds. If \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) is an order-\( d \) tensor with \( \text{pr}\ T \geq G_{d,\text{pr}}^{[\mathbb{F}]}(l) \) then there exist \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \) each with size at most \( F_{d,\text{pr}}^{[\mathbb{F}]}(l) \) and such that \( \text{pr}\ T(X_1 \times \cdots \times X_d) \geq l \).

We will rely on a connection between the partition rank and the analytic rank which we begin by explaining.

Definition 4.2. Let \( T : \prod_{\alpha=1}^d Q_{\alpha} \to \mathbb{F} \) be an order-\( d \) tensor. The multilinear form \( m(T) : \prod_{\alpha=1}^d \mathbb{F}^{Q_{\alpha}} \to \mathbb{F} \) associated with \( T \) is defined by the formula

\[
m(T)(u_1, \ldots, u_d) = \sum_{x_1 \in Q_1, \ldots, x_d \in Q_d} T(x_1, \ldots, x_d)(u_1)_{x_1} \cdots (u_d)_{x_d}
\]

for each \( u_1 \in \mathbb{F}^{Q_1}, \ldots, u_d \in \mathbb{F}^{Q_d} \).

For each \( u \in \mathbb{F}^{Q_1} \) let \( u.T : \prod_{\alpha=2}^d Q_{\alpha} \to \mathbb{F} \) be the order-\( (d-1) \) tensor defined by

\[
(u.T)(x_2, \ldots, x_d) = \sum_{x_1 \in Q_1} u(x_1)T(x_1, \ldots, x_d)
\]
for every \( x_2 \in Q_2, \ldots, x_d \in Q_d \). It is straightforward to check that the \((d - 1)\)-linear form \( m(u.T) : \prod_{\alpha=2}^d F^{Q_\alpha} \to F \) satisfies
\[
m(u.T)(u_2, \ldots, u_d) = m(T)(u_1, u_2, \ldots, u_d)
\]
for every \( u_2 \in F^{Q_2}, \ldots, u_d \in F^{Q_d} \).

**Definition 4.3.** Let \( d \geq 2 \) be a positive integer, and let \( T : \prod_{\alpha=1}^d Q_\alpha \to F \) be an order-\( d \) tensor. The bias of \( T \) is defined by
\[
\text{bias}(T) = \mathbb{E}_{u_1 \in F^{Q_1}, \ldots, u_d \in F^{Q_d}} \chi(m(T)(u_1, \ldots, u_d))
\]
for any arbitrary non-trivial character \( \chi \) of \( F \).

Indeed the following interpretation shows that the right-hand side of (1) is independent of the non-trivial character \( \chi \), and furthermore that \( \text{bias}(T) \) is always a positive real number. We can write
\[
\text{bias}(T) = \mathbb{E}_{u_1 \in F^{Q_1}, \ldots, u_d \in F^{Q_d}} \chi(m(T)(u_1, \ldots, u_d))
\]
where for each \( u_1 \in \mathbb{F}^{Q_1}, \ldots, u_{d-1} \in \mathbb{F}^{Q_d} \), the linear form \( m_{(u_1, \ldots, u_{d-1})}(T) : \mathbb{F}^{Q_d} \to \mathbb{F} \) is defined by
\[
m_{(u_1, \ldots, u_{d-1})}(T)(u_d) = m(T)(u_1, \ldots, u_d)
\]
for each \( u_d \in \mathbb{F}^{Q_d} \). The main property of the bias that we will use is that we can write
\[
\text{bias}(T) = \mathbb{E}_{u_1 \in \mathbb{F}^{Q_1}} \text{bias}(u_1.T).
\]

If \( d = 2 \), then it follows from (2) that \( \text{bias} T = \mathbb{F}^{-rkT} \). Proving qualitatively that for a fixed integer \( d \geq 2 \) and a fixed finite field \( \mathbb{F} \), the bias of an order-\( d \) tensor \( T \) over \( \mathbb{F} \) tends to 0 as the partition rank of \( T \) tends to infinity and then quantifying this asymptotic relationship has been the topic of a significant line of research. The **analytic rank** of a tensor \( T \), introduced in [3], is defined to be the quantity
\[
ar T = -\log|\mathbb{F}| \text{bias} T.
\]
It is known since work of Lovett ([5], Theorem 1.7) that
\[
ar T \leq pr T
\]
and to our knowledge at the time of writing the best bounds
\[
pr T \leq A_{d,F}(ar T)
\]
in the converse direction have been obtained in works of Janzer and Milićević: Janzer showed ([4], Theorem 1.10) that for all \( r \geq 1 \) we can take \( A_{d,F}(r) = (c \log |\mathbb{F}|)^c(d)(r)^c(d) \)
for \( c \) an absolute constant and \( c'(d) = 4^d \) and Milićević showed (6, Theorem 3) that we can take \( A_{d,F}(r) = 2^O(d^2) \left(r^2 + 1\right)^{2^O(d^2)}\) for all \( r \geq 0 \).

The averaging identity (2) together with inequalities (3) and (4) between the partition and analytic ranks provides a route for our inductive argument: indeed starting from (2) we can rewrite

\[
|F|^{-\text{ar}T} = \mathbb{E}_{u \in F_Q} |F|^{-\text{ar}(u,T)}
\]

and hence obtain

\[
|F|^{-A_{d,F}^{-1}(\text{pr}T)} \geq \mathbb{E}_{u \in F_Q} |F|^{-\text{pr}(u,T)}. \tag{5}
\]

We now begin the proof of our minors result. Our first step will be to use inequality (5) to show that, in a sense that we are about to make precise, if a tensor has all its projections over the first coordinate approximately spanned by a family of order-(\( d - 1 \)) tensors that has bounded size, then the tensor has bounded partition rank.

**Proposition 4.4.** Let \( d \geq 3, q, l \geq 1 \) be positive integers. If \( T : \prod_{\alpha=1}^{d} Q_{\alpha} \to F \) is an order-\( d \) tensor over a finite field \( F \) and \( u_1, \ldots, u_l \in F^{Q_{1}} \) are such that for every \( u \in F^{Q_{1}} \) there exists \( v \in F^l \setminus \{0\} \) satisfying

\[
\text{pr}(u.T - \sum_{h=1}^{l} v_h u_h.T) \leq q,
\]

then

\[
\text{pr}T \leq A_{d,F}(l + q).
\]

**Proof.** There are only at most \( |F|^l \) linear combinations of \( u_1, \ldots, u_l \), so by the assumption and the pigeonhole principle there is some \( v \in F^l \) such that for a proportion at least \( |F|^{-l} \) of the \( u \in F^{Q_{1}} \),

\[
\text{pr}(u.T - \sum_{h=1}^{l} v_h u_h.T) \leq q.
\]

The change of variables \( u \mapsto u - \sum_{h=1}^{l} v_h u_h \) therefore shows that a proportion at least \( |F|^{-l} \) of the \( u \in F^{Q_{1}} \) is such that \( \text{pr}(u.T) \leq q \). Hence,

\[
\mathbb{E}_{u \in F^{Q_{1}}} |F|^{-\text{pr}(u.T)} \geq |F|^{-l}|F|^{-q},
\]

so by the inequality (5) we have

\[
|F|^{-A_{d,F}^{-1}(\text{pr}T)} \geq |F|^{-l}|F|^{-q},
\]

from which it follows that \( \text{pr}T \leq A_{d,F}(l + q) \).

We can without much effort deduce from Proposition 4.4 that a tensor with high partition rank has a large separated family of projections over the first coordinate.
Lemma 4.6. Let \( q, l \geq 1 \) be positive integers. Let \( T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F} \) be an order-\( d \) tensor over a finite field \( \mathbb{F} \) such that \( \text{pr} \, T \geq A_{d, \mathbb{F}}(l+q+1) \). Then there exist \( u_1, \ldots, u_l \in \mathbb{F}^{Q_1} \) such that \( \text{pr}(\sum_{h=1}^{l} v_h u_h.T) \geq q \) for every \( v \in \mathbb{F}^{l} \setminus \{0\} \).

\[ \text{Proof.} \] Using the contrapositive of Proposition 4.4 we can construct the \( u_h \) by induction on \( h = 1, \ldots, l \) as follows: we find \( u_1 \in \mathbb{F}^{Q_1} \) such that \( \text{pr} u_1.T \geq q \) and more generally at the \( h \)th step we find \( u_h \in \mathbb{F}^{Q_1} \) such that

\[ \text{pr}(u_h - \sum_{h'}^{h-1} v_{h'} u_{h'}).T \geq q \]

for every \( v \in \mathbb{F}^{h-1} \setminus \{0\} \). \( \square \)

In our next lemma we show that, conversely, if a tensor has a large separated set of projections then it has high partition rank. Unlike the proof of Proposition 4.4 the proof of the following lemma does not resort to the connection between partition rank and analytic rank (and hence works in an arbitrary field).

Corollary 4.5. Let \( q, l \geq 1 \) be positive integers. Let \( T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F} \) be an order-\( d \) tensor. Suppose that there exist \( u_1, \ldots, u_l \in \mathbb{F}^{Q_1} \) such that

\[ \text{pr}(\sum_{h=1}^{l} v_h u_h.T) \geq q \]

for every \( v \in \mathbb{F}^{l} \setminus \{0\} \). Then \( \text{pr} \, T \geq l \).

\[ \text{Proof.} \] Assume for a contradiction that \( \text{pr} \, T \leq l-1 \). Then there exist nonnegative integers \( r, s \) with \( r + s \leq l-1 \), for each \( i \in [r] \) two functions \( a_i : Q_1 \to \mathbb{F} \), \( b_i : \prod_{\alpha=2}^{d} Q_{\alpha} \to \mathbb{F} \), for each \( i \in [s] \) a bipartition \( \{ I_i, J_i \} \) of \( [d] \) with \( I_i, J_i \neq \emptyset \), \( 1 \in I_i \) and \( I_i \neq \{1\} \), and for each \( i \in [s] \) two functions \( c_i : \prod_{I_i} Q_i \to \mathbb{F} \), \( d_i : \prod_{I_i} Q_i \to \mathbb{F} \) such that

\[ T(x_1, \ldots, x_d) = \sum_{i=1}^{r} a_i(x_1)b_i(x_2, \ldots, x_d) + \sum_{i=1}^{s} c_i(x(I_i))d_i(x(J_i)) \]

for every \( x_1 \in Q_1, \ldots, x_d \in Q_d \). Let \( u_1, \ldots, u_l \) be arbitrary fixed elements of \( \mathbb{F}^{Q_1} \). For each \( h \in [l] \),

\[ u_h.T = \sum_{i=1}^{r} (u_h.a_i)b_i(x_2, \ldots, x_d) + \sum_{i=1}^{s} (u_h.c_i)(x(I_i \setminus \{1\}))d_i(x(J_i)) \]

where

\[ u_h.a_i = \sum_{x_1 \in Q_1} u_h(x_1)a_i(x_1) \in \mathbb{F} \]
and
\[(u_h, c_i) : x(I \setminus \{1\}) \mapsto \sum_{x_1 \in Q_1} u_h(x_1)c_i(x(I)).\]

Because \(l \geq r + 1\), there exists \(v \in \mathbb{F}^l \setminus \{0\}\) such that \(\sum_{h=1}^l v_h(u_h, a_i) = 0\) for all \(i \in [r]\). Hence
\[
\sum_{h=1}^l v_h(u_h, T) = \sum_{i=1}^s \left( \sum_{h=1}^l v_h(u_h, c_i)(x(I_i \setminus \{1\})) \right) d_i(x(J_i)).
\]

The right-hand side has partition rank at most \(s \leq l - 1\), a contradiction. \(\square\)

For \(d \geq 2\) we define two families of functions \(F^{[d]}_{d, pr} : \mathbb{N} \to \mathbb{N}\) and \(G^{[d]}_{d, pr} : \mathbb{N} \to \mathbb{N}\) inductively as follows. We set
\[
F^{[d]}_{2, pr}(l) = l \text{ and for each } d \geq 3, F^{[d]}_{d, pr}(l) = (|\mathbb{F}|^l F^{[d]}_{d-1, pr}(l))^{d-1},
\]
\[
G^{[d]}_{2, pr}(l) = l \text{ and for each } d \geq 3, G^{[d]}_{d, pr}(l) = A_{d, \mathbb{F}}(G^{[d]}_{d-1, pr}(l) + l + 1).
\]

The following pair of results, Proposition 4.7 and Theorem 4.8, will be proved by induction on \(d\). The base case is Theorem 4.8 for \(d = 2\). Then, for every \(d \geq 3\), Theorem 4.8 in order \(d - 1\) implies Proposition 4.7 in order \(d\), and Proposition 4.7 in order \(d\) implies Theorem 4.8 in order \(d\).

The statements that we have gathered so far in this section allow us to do the following: starting with an order-\(d\) tensor \(T\) with high partition rank, we find a large separated set of projections of \(T\), then apply Theorem 4.8 in order \((d - 1)\) to obtain sets \(X_2, \ldots, X_d\) with bounded size such that the set of projections is still separated when restricted to the order-(\(d - 1\)) minor \(X_2 \times \cdots \times X_d\), which then ensures that \(T(Q_1 \times X_2 \cdots \times X_d)\) has high partition rank. In order to also be able to restrict the first set of coordinates we furthermore want to be able to assume that the projections have bounded support: this step will constitute an important part of the proof of Proposition 4.7.

**Proposition 4.7.** Let \(d \geq 3\), \(q, l \geq 1\) be positive integers. Let \(T : \prod_{\alpha=1}^d Q_{\alpha} \to \mathbb{F}\) be an order-\(d\) tensor over a finite field \(\mathbb{F}\). If there exist \(u_1, \ldots, u_l \in \mathbb{F}^{Q_1}\) such that
\[
\text{pr} \sum_{h=1}^l v_h(u_h, T) \geq G^{[d]}_{d-1, pr}(q)
\]
for every \(v \in \mathbb{F}^l \setminus \{0\}\), then there exist \(X_2 \subset Q_2, \ldots, X_d \subset Q_d\) with size at most \(|\mathbb{F}|^l F^{[d]}_{d-1, pr}(q)\) and \(u'_1, \ldots, u'_l \in \mathbb{F}^{Q_1}\) all supported inside a subset \(X_1 \subset Q_1\) with size at most \((|\mathbb{F}|^l F^{[d]}_{d-1, pr}(q))^{d-1}\) such that
\[
\text{pr} \sum_{h=1}^l v_h(u'_h, T(X_2 \times \cdots \times X_d)) \geq q
\]
for every \(v \in \mathbb{F}^l \setminus \{0\}\).
Theorem 4.8. Let $d \geq 2$ be an integer. If $T : \prod_{\alpha=1}^{d} Q_{\alpha} \rightarrow F$ is an order-$d$ tensor over a finite field $F$ with $\text{pr} T \geq G_{d,\text{pr}}^{[F]}(l)$, then there exist sets $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ of size at most $F_{d,\text{pr}}^{[F]}(l)$ such that

$$\text{pr} T(X_1 \times \cdots \times X_d) \geq l.$$  

Proof of Proposition 4.7. For each $v \in \mathbb{F}^l \setminus \{0\}$ we can by Theorem 4.8 in dimension $d - 1$ find sets $X_{2,v} \subset Q_2, \ldots, X_{d,v} \subset Q_d$ each with size at most $F_{d-1,\text{pr}}^{[F]}(q)$ such that

$$\text{pr}(\sum_{h=1}^{l} v_h(u_h.T))(X_{2,v} \times \cdots \times X_{d,v}) \geq q.$$  

Let $X_{\alpha} = \bigcup_{v \in \mathbb{F}^l \setminus \{0\}} X_{\alpha,v}$ for each $\alpha = 2, \ldots, d$. We obtain

$$\text{pr}(\sum_{h=1}^{l} v_h(u_h.T))(X_2 \times \cdots \times X_d) \geq q$$

for all $v \in \mathbb{F}^l \setminus \{0\}$ and the sets $X_2, \ldots, X_d$ each have size at most $|\mathbb{F}|^{lF_{d-1,\text{pr}}^{[F]}(q)}$. The family $\{T_{x_1} : x_1 \in Q_1\}$ of slices $(x_2, \ldots, x_d) \mapsto T(x_1, x_2, \ldots, x_d)$ spans $\{u.T : u \in \mathbb{F}^{Q_1}\}$, so in particular spans all linear combinations $\sum_{h=1}^{l} v_h(u_h.T)$ with $v \in \mathbb{F}^l \setminus \{0\}$. For each $x_1 \in Q_1$ let $T'_{x_1}$ be the restriction

$$T_{x_1}(X_2 \times \cdots \times X_d).$$

The family of restrictions $\{T'_{x_1} : x_1 \in Q_1\}$ spans all linear combinations

$$\sum_{h=1}^{l} v_h(u_h.T)(X_2 \times \cdots \times X_d)$$

with $v \in \mathbb{F}^l \setminus \{0\}$. Because $X_2 \times \cdots \times X_d$ has size at most $(|\mathbb{F}|^{lF_{d-1,\text{pr}}^{[F]}(q)})^{d-1}$, there exists a set $X_1 \subset Q_1$ with size at most $(|\mathbb{F}|^{lF_{d-1,\text{pr}}^{[F]}(q)})^{d-1}$ such that

$$\langle\{T'_{x_1} : x_1 \in X_1\}\rangle = \langle\{T'_{x_1} : x_1 \in Q_1\}\rangle.$$  

For each $x_1 \in Q_1$ there exist coefficients $a_{x_1'}(x_1) \in \mathbb{F}$ for every $x_1' \in X_1$ such that

$$T'_{x_1} = \sum_{x_1' \in X_1} a_{x_1'}(x_1)T'_{x_1'}.$$  

For each $h \in [l]$ we define $u_h' \in \mathbb{F}^{Q_1}$ by

$$u_h'(x_1') = \sum_{x_1 \in Q_1} u_i(x_1) a_{x_1'}(x_1)$$

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for each $x'_1 \in X_1$ and by $u'_i(x'_1) = 0$ for each $x'_1 \in Q_1 \setminus X_1$. We can check that

\[ u'_h.T' = \sum_{x'_1 \in X_1} u'_i(x'_1)T'_{x_1} = \sum_{x_1 \in Q_1} u_i(x_1)(\sum_{x'_1 \in X_1} a_{x'_1}(x_1)T'_{x_1}) = \sum_{x_1 \in Q_1} u_i(x_1)T'_{x_1} = u_h.T' \]

for each $h \in [l]$. By linearity,

\[ \sum_{h=1}^{l} v_h u'_h.T' = \sum_{h=1}^{l} v_h u_h.T' \]

for each $v \in \mathbb{F}^l \setminus \{0\}$. Since taking restrictions cannot increase the partition rank, the result follows.\[\square\]

**Proof of Theorem 4.8.** Let $T$ be an order-$d$ tensor with $\text{pr} T \geq A_{d,\mathbb{F}}(G_{d-1,\text{pr}}(l) + l + 1)$. By Corollary 4.5 there exist $u_1, \ldots, u_l \in \mathbb{F}^{Q_1}$ such that

\[ \text{pr}((\sum_{h=1}^{l} v_h u_h).T) \geq G_{d-1,\text{pr}}(l) \]

for every $v \in \mathbb{F}^l \setminus \{0\}$. By Proposition 4.7 there exist $X_2 \subset Q_2, \ldots, X_d \subset Q_d$ with size at most $|\mathbb{F}^{l}F_{d-1,\text{pr}}(l)|$, a set $X_1$ with size at most $(|\mathbb{F}^{l}F_{d-1,\text{pr}}(l)|^{d-1} - 1)$ and $u'_1, \ldots, u'_l \in \mathbb{F}^{Q_1}$ all supported inside $X_1$ such that

\[ \text{pr}((\sum_{h=1}^{l} v_h u'_h).T(X_2 \times \cdots \times X_d)) \geq l \]

for every $v \in \mathbb{F}^l \setminus \{0\}$. By Lemma 4.4 we conclude that

\[ \text{pr} T(X_1 \times \cdots \times X_d) \geq l. \]

\[\square\]

**5 Disjoint rank in the matrix case**

We thank Lisa Sauermann for a sketch that led to Proposition 5.1 and which we use here with her permission.

**Proposition 5.1.** Let $A : Q_1 \times Q_2 \rightarrow \mathbb{F}$ be a matrix. We have $\text{drk} A \geq (\text{erk} A)/3$.

**Proof.** Let $k = \text{drk} A$. There exist disjoint subsets $X \subset Q_1$, $Y \subset Q_2$ such that $\text{rk} A(X \times Y) = k$. By the standard result on minors of matrices we can furthermore require $X, Y$ to have size $k$. Let $z \in Q_1 \setminus (X \cup Y)$ and $w \in Q_2 \setminus (X \cup Y)$. If $z, w$ are distinct then $X \cup \{z\}$ and $Y \cup \{w\}$ are disjoint, so by definition of the disjoint rank,

\[ \text{rk} A((X \cup \{z\}) \times (Y \cup \{w\})) = \text{rk} A(X \times Y) \]

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whence
\[ A(z, w) = A(\{z\} \times Y)A(X \times Y)^{-1}A(X \times \{w\}). \]

If \( z = w \) then there exists a unique element \( d(z) \in \mathbb{F} \) such that
\[ A(z, z) + d(z) = A(\{z\} \times Y)A(X \times Y)^{-1}A(X \times \{w\}). \]

Let \( D : Q_1 \times Q_2 \to \mathbb{F} \) be the matrix defined by \( D(z, w) = 0 \) if \( z \neq w \) and \( D(z, w) = d(z) \) if \( z = w \). We get that for all \( z \in Q_1 \setminus (X \cup Y) \) and all \( w \in Q_2 \setminus (X \cup Y) \),
\[ (A + D)(z, w) = A(\{z\} \times Y)A(X \times Y)^{-1}A(X \times \{w\}) = (A + D)(\{z\} \times Y)(A + D)(X \times Y)^{-1}(A + D)(X \times \{w\}), \]
where the last equality holds since \( D \) is identically 0 on all three sets \( \{z\} \times Y, X \times Y, X \times \{w\} \).

Therefore
\[ \text{rk}(A + D)((X \cup (Q_1 \setminus (X \cup Y))) \times (Y \cup (Q_2 \setminus (X \cup Y)))) = \text{rk}(A + D)(X \times Y). \]

Simplifying both sides we get
\[ \text{rk}(A + D)((Q_1 \setminus Y) \times (Q_2 \setminus X)) = \text{rk} A(X \times Y) = k. \]

Moreover by subadditivity
\[ \text{rk}(A + D) \leq \text{rk}(A + D)((Q_1 \setminus Y) \times (Q_2 \setminus X)) + \text{rk}(A + D)(Y \times (U_2 \setminus X)) + \text{rk}(A + D)((Q_1 \setminus Y) \times X). \]

The first term of the right-hand side is at most \( k \) as we have just shown, and the second and third terms are each at most \( k \) since \( X, Y \) have size \( k \). So \( \text{rk}(A + D) \leq 3k \), whence \( \text{erk} A \leq 3 \text{drk} A \) as desired.

**Proposition 5.2.** Let \( s \geq 1 \) be a positive integer, let \( A_1, \ldots , A_s : Q_1 \times Q_2 \to \mathbb{F} \) be matrices, and let \( \Lambda \subset \mathbb{F}^s \). Assume that
\[ \text{erk} a.A \geq s(s + 1)l + l \]
for every \( a \in \Lambda \). Then there exist disjoint subsets \( X \subset B_1, Y \subset B_2 \) such that
\[ \text{rk}(a.A)(X \times Y) \geq l \]
for every \( a \in \Lambda \).

**Proof.** We proceed by induction on \( s \geq 1 \). Proposition 5.1 proves the \( s = 1 \) case. Let \( s \geq 2 \) be a positive integer and let us assume that Proposition 5.2 holds up to \( s - 1 \). If \( \Lambda \) is empty then we are done. Let us assume that this is not the case. We choose \( a \in \Lambda \). By Proposition 5.2 there exist disjoint subsets \( X^0 \subset Q_1, Y^0 \subset Q_2 \) such that \( \text{rk}(a.A)(X^0 \times Y^0) \geq sl \). By the standard result on minors of matrices we can furthermore find subsets \( X' \subset X_0, Y' \subset Y_0 \) both with size at most \( sl \) such that \( \text{rk}(a.A)(X' \times Y') \geq sl \). By subadditivity of the rank there exists a strict linear subspace \( W \) of \( \mathbb{F}^s \) such that
\[ \text{rk}(a.A)(X' \times Y') \geq l \]
for every $a \in \Lambda \setminus W$. By the assumption
\[ \text{rk}(a.A) \geq s(s + 1)l + l \]
for every $a \in \Lambda \cap W$ so (since a single row of column has rank at most 1) by subadditivity
\[ \text{rk}(a.A)((Q_1 \setminus Y') \times (Q_2 \setminus X')) \geq s(s - 1)l + l \]
for every $a \in \Lambda \cap W$. By the inductive hypothesis there exist disjoint subsets $X'' \subset Q_1 \setminus Y'$, $Y'' \subset Q_2 \setminus X'$ such that
\[ \text{rk}(a.A)(X'' \times Y'') \geq l \]
for every $a \in \Lambda \cap W$. Letting $X = X' \cup X''$ and $Y = Y' \cup Y''$ the sets $X$ and $Y$ are disjoint by construction and
\[ \text{rk}(a.A)(X \times Y) \geq l \]
for every $a \in \Lambda$ as desired.

6 Disjoint tensor rank using the flattening rank

Let $d \geq 2$ be a positive integer. For $I \subset [d]$, we write $E(I)$ for the set
\[ \{x(I) \in \prod_{\alpha \in I} Q_\alpha : x_{\alpha'} = x_{\alpha''} \text{ for some distinct } \alpha', \alpha'' \in I\}. \]
In particular, the set $E([d])$ is the set $E$ of elements that do not have pairwise distinct coordinates.

6.1 The order-3 case

Our proof will make use of the following notion of rank.

Definition 6.1. For $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ an order-3 tensor, let the 1-flattening rank of $T$, which we write as $\text{frank}_1 T$, be the rank of the matrix $A : Q_1 \times (Q_2 \times Q_3) \to \mathbb{F}$ defined by
\[ A(x, (y, z)) = T(x, y, z). \]
In other words $\text{frank}_1 T$ is defined as $\text{Rrk}_R T$ for $R = \{\{1\}, \{2, 3\}\}$. Likewise we define the essential 1-flattening rank for this same $R$ and denote it by $\text{efrank}_1 T = \text{eRrk}_R T$.

The structure of our proof will be as follows: starting with an order-3 tensor $T$ with high essential tensor rank, either the tensor $T$ has an order-2 slice with high essential rank, in which case we conclude using Proposition 5.1 or all order-2 slices of $T$ have bounded essential rank, in which case the essential tensor rank can be shown to be equivalent to the essential 1-flattening rank, and we will then show (using again our assumption that all order-2 slices of $T$ have bounded essential rank) that because $T$ has high essential 1-flattening rank, it must have high disjoint 1-flattening rank, which suffices to ensure that $T$ has high disjoint tensor rank.

We begin by proving our equivalence statement between the essential ranks.
Proposition 6.2. Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor such that $\text{erk} T_x \leq m$ for every $x \in Q_1$. If $\text{efrank}_1 T \leq l$ then $\text{etr} T \leq (m + 2)l^2$.

Proof. Let $r = \text{efrank}_1 T$. There exists a tensor $V$ supported inside $E$, functions $a_1, \ldots, a_r : Q_1 \to \mathbb{F}$ and functions $b_1, \ldots, b_r : Q_2 \times Q_3 \to \mathbb{F}$ such that we can write the decomposition

\[(T - V)(x, y, z) = \sum_{i=1}^{r} a_i(x) b_i(y, z).\] (6)

The functions $a_1, \ldots, a_r$ are linearly independent, so by Gaussian elimination there exists a subset $U \subset Q_1$ with size $r$ and functions $a_1^*, \ldots, a_r^* : Q_1 \to \mathbb{F}$ such that $a_i^* \circ a_i = \delta_{i,i'}$ for all $i, i' \in [r]$. For a fixed $i \in [r]$, applying $a_i^*$ to both sides of (6) we get $b_i = a_i^*(T - V)$, i.e.

\[b_i = \sum_{x \in U} a_i^*(x)(T - V)_x.\] (7)

For each $x \in U$, the support of $V_x$ is contained in the union of the row $\{y = x\}$, the column $\{z = x\}$ and the diagonal $\{y = z\}$, so (using that a row or column has rank at most 1) $\text{erk} V_x \leq 2$ and hence $\text{erk} (T - V)_x \leq m + 2$. Using (7) and subadditivity $\text{erk} b_i \leq r(m + 2)$, so by the decomposition $\text{etr} (T - V) \leq r^2(m + 2) \leq l^2(m + 2)$, so $\text{etr} T \leq l^2(m + 2)$. \hfill \Box

We will use the following upper bound in the proof of Proposition 6.4.

Remark 6.3. Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor. Then it follows from writing a tensor-rank decomposition of $T$ with minimal length that

\[\dim(T_{(y,z)} : (y, z) \in Q_2 \times Q_3) \leq \text{tr} T.\]

Throughout the next proposition and its proof we will write $Q_{2,3}$ for $Q_2 \times Q_3 \setminus E(\{2, 3\})$. We show that if an order-3 tensor has high essential 1-flattening rank and all its order-2 slices have bounded essential rank then it has high disjoint 1-flattening rank.

Proposition 6.4. Let $l, m \geq 1$ be positive integers, and let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor such that $\text{erk} T_x, \text{erk} T_y, \text{erk} T_z \leq m$ for every $x \in Q_1$, every $y \in Q_2$, and every $z \in Q_3$, respectively. Let $a, b : Q_{2,3} \to \mathbb{F}$ be functions such that

\[\dim(\langle (T_{y,z} + a(y, z)1_{x=y} + b(y, z)1_{x=z}) : (y, z) \in Q_{2,3} \rangle) \geq 10ml.\]

Then there exist pairwise disjoint subsets $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ such that

\[\dim(T_{(y,z)}(X) : (y, z) \in Y \times Z) \geq l.\] (8)

Proof. Let $l$ be the largest integer such that there exist $x_1, \ldots, x_l \in Q_1, y_1, \ldots, y_l \in Q_2, z_1, \ldots, z_l \in Q_3$ all pairwise distinct such that the restrictions

\[T_{(y_1,z_1)}(\{x_1, \ldots, x_l\}), \ldots, T_{(y_l,z_l)}(\{x_1, \ldots, x_l\})\]
are linearly independent. Taking $X = \{x_1, \ldots, x_l\}$, $Y = \{y_1, \ldots, y_l\}$, $Z = \{z_1, \ldots, z_l\}$ then ensures that (8) is satisfied. We define the following sets, which we will use throughout the remainder of the proof:

$$X_1 = \{x_1, \ldots, x_l\}$$

$$W_\alpha = Q_\alpha \setminus \{x_1, \ldots, x_l, y_1, \ldots, y_l, z_1, \ldots, z_l\} \text{ for each } \alpha \in \{1, 2, 3\}$$

$$U_1 = \{(y_1, z_1), \ldots, (y_l, z_l)\}$$

$$W_{2,3} = (W_2 \times W_3) \setminus E(\{2, 3\}).$$

We first show that we can define functions $a, b : W_{2,3} \to \mathbb{F}$ such that

$$\dim((T_{(y,z)} + a(y, z)1_{x=y} + b(y, z)1_{x=z})(W_1) : (y, z) \in W_{2,3}) \leq l.$$ 

In other words, we want to show that the matrix $A : Q_1 \times Q_{2,3} \to \mathbb{F}$ defined by

$$A((x, (y, z))) = T(x, y, z)$$

is such that the rank of the restriction $A(W_1 \times W_{2,3})$ is at most $l$ after some modifications of the entries of the type

$$A((u, (u, z)), (u, (y, u))).$$

We construct $a, b$ as follows. Let $x \in W_1$ and let $(y, z) \in W_{2,3}$. We distinguish two situations.

1. If $x \neq y$ and $x \neq z$ then by maximality of $l$, the $(l+1) \times (l+1)$ minor

$$A((X_1 \cup \{x\}) \times (U_1 \cup \{(y, z)\}))$$

has the same rank as the full-rank matrix $A(X_1 \times U_1)$, so we can write

$$A((x, (y, z))) = A((\{x\} \times U_1)A(X_1 \times U_1)^{-1}A(X_1 \times \{(y, z)\}).$$

2. If $x = y$ or $x = z$ then because $A((X_1 \cup \{x\}) \times (U_1 \cup \{(y, z)\}))$ is an $(l+1) \times (l+1)$ matrix and $A(X_1 \times U_1)$ has full rank $l$, there exists if $x = y$ a unique $a(y, z) \in \mathbb{F}$ such that

$$A((x, (y, z))) + a(y, z) = A((\{x\} \times U_1)A(X_1 \times U_1)^{-1}A(X_1 \times \{(y, z)\}))$$

and if $x = z$ a unique $b(y, z) \in \mathbb{F}$ such that

$$A((x, (y, z))) + b(y, z) = A((\{x\} \times U_1)A(X_1 \times U_1)^{-1}A(X_1 \times \{(y, z)\}).$$

The values $a(y, z), b(y, z)$ obtained in the second situation define functions $a, b : W_{2,3} \to \mathbb{F}$. Let $B : (X_1 \cup W_1) \times (U_1 \cup W_{2,3}) \to \mathbb{F}$ be the matrix defined for all $(x, (y, z)) \in W_1 \times W_{2,3}$ by

$$B((x, (y, z))) = A((x, (y, z))) + a(y, z)1_{x=y} + b(y, z)1_{x=z}$$
and by $B(x, (y, z)) = A(x, (y, z))$ for all other $(x, (y, z))$. By construction of $a, b$ we have

$$B(W_1 \times W_{2,3}) = B(W \times U_1)B(X_1 \times U_1)^{-1}B(X_1 \times W_{2,3})$$

whence $\text{rk } B(W_1 \times W_{2,3}) \leq l$. Defining the slices

$$S_{1,j} = \{x = j\} \setminus (Q_1 \times E(\{2, 3\}))$$
$$S_{2,j} = \{y = j\} \setminus (Q_1 \times E(\{2, 3\})) \cap \{x \in W_1\}$$
$$S_{3,j} = \{z = j\} \setminus (Q_1 \times E(\{2, 3\})) \cap \{x \in W_1\} \cap \{y \in W_2\}$$

for each $j \in Q_1 \setminus W_1$, each $j \in Q_2 \setminus W_2$, each $j \in Q_3 \setminus W_3$, respectively, the set $Q_1 \times Q_2, 3$ can be written as the disjoint union of the four sets

$$(W_1 \times W_{2,3}) \cup \bigcup_{j \in Q_1 \setminus W_1} S_{1,j} \cup \bigcup_{j \in Q_2 \setminus W_2} S_{2,j} \cup \bigcup_{j \in Q_3 \setminus W_3} S_{3,j}.$$  

By definition of the sets $W_1, W_2, W_3$ the sets $Q_1 \setminus W_1, Q_2 \setminus W_2, Q_3 \setminus W_3$ each have size at most $3l$. We can arbitrarily define $a, b$ on $\bigcup_{j \in Q_1 \setminus W_1} S_{1,j} = (Q_1 \setminus W_1) \times Q_{2,3}$ and obtain

$$\dim((T_{(y,z)} + a(y, z)1_{x=y} + b(y, z)1_{x=z})(Q_1 \setminus W_1) : (y, z) \in Q_{2,3})$$

to be at most $|Q_1 \setminus W_1| \leq 3l$. By assumption, for each $j \in Q_2 \setminus W_2$ we have $\text{erk } T_{y=j} \leq m$, so $a, b$ can be defined on $S_{j,2}$ such that

$$\dim((T_{(y,z)} + a(y, z)1_{x=y} + b(y, z)1_{x=z})(W_1) : (y, z) \in Q_{2,3}, y = j) \leq m.$$  

Similarly, for each $j \in Q_3 \setminus W_3$ we can define $a, b$ on $S_{j,3}$ such that

$$\dim((T_{(y,z)} + a(y, z)1_{x=y} + b(y, z)1_{x=z})(W_1) : (y, z) \in Q_{2,3}, y \in W_2, z = j) \leq m.$$  

The result follows by subadditivity.  

We are now ready to prove Theorem 1.8 for the tensor rank of order-3 tensors.

**Proposition 6.5.** Let $T$ be an order-3 tensor, and let $l \geq 1$ be a positive integer. If $\text{etr } T \geq 17500l^2$ then $\text{dtr } T \geq l$.

**Proof.** We distinguish two cases.

**Case 1:** There exists $x \in Q_1$ or $y \in Q_2$ or $z \in Q_3$ such that $	ext{erk } T_x$ or $	ext{erk } T_y$ or $	ext{erk } T_z$ is at least $3l + 2$. Without loss of generality we can assume that this occurs for some $x \in Q_1$. Then

$$\text{erk } T_x((Q_2 \setminus \{x\}) \times (Q_3 \setminus \{x\})) \geq 3l.$$  

By Proposition 5.1 there exist disjoint subsets $Y \subset Q_2 \setminus \{x\}$ and $Z \subset Q_3 \setminus \{x\}$ such that $\text{rk } T_x(Y \times Z) \geq l$. Letting $X = \{x\}$ we obtain $\text{tr } T(X \times Y \times Z) \geq l$.

**Case 2:** We have $\text{erk } T_x, \text{erk } T_y, \text{erk } T_z \leq 3l + 2$ for respectively all $x \in Q_1$, all $y \in Q_2$, all $z \in Q_3$. Since $k \geq (3l+4)(10(3l+2))^{2l}$, by Proposition 6.2 we have $\text{efrank } T \geq 10(3l+2)l$. By Proposition 6.4 there exist pairwise disjoint subsets $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ such that $\text{frank } T(X \times Y \times Z) \geq l$, so in particular $\text{tr } T(X \times Y \times Z) \geq l$. 

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6.2 The general case

We now generalise the proof to order-\(d\) tensors for an arbitrary fixed \(d \geq 3\). The structure of the proof will be similar to that of the \(d = 3\) case, although there will be some additional complications.

**Definition 6.6.** For \(T : \prod_{\alpha=1}^{d} Q_\alpha \to \mathbb{F}\) an order-\(d\) tensor, let the 1-flattening rank, denoted \(\text{frank}_1 T\), be the rank of the matrix \(A : Q_1 \times (\prod_{\alpha=2}^{d} Q_\alpha) \to \mathbb{F}\) defined by

\[
A(x_1, (x_2, \ldots, x_d)) = T(x_1, x_2, \ldots, x_d).
\]

In other words, \(\text{frank}_1 T\) is equal to \(R \text{rk}_T\) for \(R = \{\{1\}, \{2, \ldots, d\}\}\). Likewise we define the essential 1-flattening rank for this same \(R\) and denote it \(\text{efrank}_1 T = eR \text{rk}_T\).

As we have not managed to find a simpler argument we prove the following two propositions by recognising them as special cases of Corollary 11.8, a common generalisation of them which we will prove and then use in full in Section 11. The proof of Corollary 11.8 only uses Proposition 11.7, which in turn has a self-contained proof.

**Proposition 6.7.** Let \(d \geq 2, l \geq 1\) be positive integers, and let \(T : \prod_{\alpha=1}^{d} Q_\alpha \to \mathbb{F}\) be an order-\(d\) tensor. Assume that \(epr_T y \leq l\) for each \(I \subset [d]\) with \(|I| \in \{1, \ldots, d - 2\}\) and each \(y \in (\prod_{\alpha \in I} Q_\alpha) \setminus E(I^c)\). If \(e \text{frank}_1 T \leq l\) then \(e \text{tr} T \leq (4l^3)^{2d}\).

We shall also apply the following result to order-(\(d - 1\)) tensors in the proof of Proposition 6.10.

**Proposition 6.8.** Let \(d \geq 2, l \geq 1\) be positive integers and let \(T : \prod_{\alpha=1}^{d} Q_\alpha \to \mathbb{F}\) be an order-\(d\) tensor. Assume that \(epr_T y \leq l\) for each \(I \subset [d]\) with \(|I| \in \{0, \ldots, d - 2\}\) and each \(y \in (\prod_{\alpha \in I} Q_\alpha) \setminus E(I^c)\). Then \(e \text{tr} T \leq (4l^3)^{2d}\).

We shall use the following upper bound in the proof of Proposition 6.10.

**Remark 6.9.** Let \(d \geq 2, T : \prod_{\alpha=1}^{d} Q_\alpha \to \mathbb{F}\) be an order-\(d\) tensor. Then it follows from writing a tensor-rank decomposition of \(T\) with minimal length that

\[
\dim(T_{(x_2, \ldots, x_d)} : (x_2, \ldots, x_d) \in \prod_{\alpha=2}^{d} Q_\alpha) \leq \text{tr} T.
\]

In the following proposition and its proof we will write \(Q_{2, \ldots, d}\) for \((\prod_{\alpha=2}^{d} Q_\alpha) \setminus E(\{2, \ldots, d\})\).
Proposition 6.10. Let $d \geq 2$, $l \geq 1$, $m \geq 1$ be positive integers, and let $T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F}$ be an order-$d$ tensor such that $\text{epr} T_{y} \leq m$ for every $I \subset [d]$ with $|I| \in \{1, \ldots, d - 2\}$ and each $y \in (\prod_{\alpha \in I} Q_{\alpha}) \setminus E(I')$. Let $a_{2}, \ldots, a_{d} : Q_{2,\ldots,d} \to \mathbb{F}$ be functions such that

$$\dim(\langle T_{(x_{2}, \ldots, x_{d})} + \sum_{i=2}^{d} a_{i}(x_{2}, \ldots, x_{d})1_{x_{1}=x_{i}} \rangle : (x_{2}, \ldots, x_{d}) \in Q_{2,\ldots,d}) \geq d^{2}(4m^{3})^{2^{d-1}} l.$$ 

Then there exist pairwise disjoint $X_{1} \subset Q_{1}, \ldots, X_{d} \subset Q_{d}$ such that

$$\dim(\langle T_{(x_{2}, \ldots, x_{d})}(X_{1}) : (x_{2}, \ldots, x_{d}) \in X_{2} \times \cdots \times X_{d} \rangle \geq l. \quad (9)$$

Proof. Let $l$ be the largest integer such that there exist pairwise distinct $x_{i,j} \in Q_{i}$ for each $i \in [d], j \in [l]$ such that the restrictions

$$T_{(x_{2}, \ldots, x_{d},l)}(\{x_{1,1}, \ldots, x_{1,l}\}), \ldots, T_{(x_{2}, \ldots, x_{d},l)}(\{x_{1,1}, \ldots, x_{1,l}\})$$

are linearly independent. Taking $X_{1} = \{x_{1,l}, \ldots, x_{1,1}\}, \ldots, X_{d} = \{x_{d,l}, \ldots, x_{d,1}\}$ then ensures that (9) is satisfied. We define the following sets which we shall use throughout the remainder of the proof:

$$W_{i} = Q_{i} \setminus \{x_{i,j'}, j' \in [d], j' \in [s]\} \text{ for each } i \in [d]\)$$

$$U = \{(x_{2,1}, \ldots, x_{d,1}), \ldots, (x_{2,l}, \ldots, x_{d,l})\}$$

$$W_{2,\ldots,d} = (\prod_{\alpha=2}^{d} W_{\alpha}) \setminus E(\{2, \ldots, d\}).$$

We first show that we can define functions $a_{2}, \ldots, a_{d} : W_{2,\ldots,d} \to \mathbb{F}$ such that

$$\dim(\langle T_{(x_{2}, \ldots, x_{d})} + \sum_{i=2}^{d} a_{i}(x_{2}, \ldots, x_{d})1_{x_{1}=x_{i}} \rangle(W_{1}) : (x_{2}, \ldots, x_{d}) \in W_{2,\ldots,d}) \leq l.$$ 

In other words we want to show that the matrix $A : Q_{1} \times Q_{2,\ldots,d} \to \mathbb{F}$ defined by

$$A(x_{1}, (x_{2}, \ldots, x_{d})) = T(x_{1}, x_{2}, \ldots, x_{d})$$

is such that the rank of the restriction $A(W_{1} \times W_{2,\ldots,d})$ is at most $l$ after some modifications of the entries of the type

$$A(u, (u, x_{3}, \ldots, x_{d})), A(u, (x_{2}, u, x_{4}, \ldots, x_{d})), \ldots, A(u, (x_{2}, \ldots, x_{d-1}, u)).$$

We construct $a_{2}, \ldots, a_{d}$ as follows. Let $x_{1} \in W_{1}$ and $(x_{2}, \ldots, x_{d}) \in W_{2,\ldots,d}$. We distinguish two situations.

1. If $x_{1} \notin \{x_{2}, \ldots, x_{d}\}$ then by maximality of $l$, the $(l + 1) \times (l + 1)$ minor

   $$A((X_{1} \cup \{x_{1}\}) \times (U_{1} \cup \{(x_{2}, \ldots, x_{d})\}))$$

   has the same rank as the full rank matrix $A(X_{1} \times U_{1})$, so we can write

   $$A(x_{1}, (x_{2}, \ldots, x_{d})) = A(\{x_{1}\} \times U_{1})A(X_{1} \times U_{1})^{-1}A(X_{1} \times \{(x_{2}, \ldots, x_{d})\}).$$

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2. If \( x_1 = x_i \) for some \( i \in \{2, \ldots, d\} \) then because \( A((X_1 \cup \{x_1\}) \times (U_1 \cup \{(x_2, \ldots, x_d)\})) \)

is an \((l + 1) \times (l + 1)\) matrix and \( A(X_1 \times U_1) \) has full rank \( l \), there exists a unique

\( a_i(x_2, \ldots, x_d) \in \mathbb{F} \) such that

\[
A(x_1, (x_2, \ldots, x_d)) + a_i(x_2, \ldots, x_d) = A(\{x_1\} \times U_1)A(X_1 \times U_1)^{-1}A(X_1 \times \{(x_2, \ldots, x_d)\}).
\]

The values \( a_2(x_2, \ldots, x_d), \ldots, a_d(x_2, \ldots, x_d) \) obtained in the second situation define functions \( a_2, \ldots, a_d : W_{2, \ldots, d} \to \mathbb{F} \). Let \( B : (X_1 \cup W_1) \times (U_1 \cup W_{2, \ldots, d}) \to \mathbb{F} \) be the matrix

defined for all \((x_1, (x_2, \ldots, x_d)) \in W_1 \times W_{2, \ldots, d}\) by

\[
B(x_1, (x_2, \ldots, x_d)) = A(x_1, (x_2, \ldots, x_d)) + \sum_{i=2}^{d} a_i(x_2, \ldots, x_d)1_{x_1=x_i},
\]

and by \( B(x_1, (x_2, \ldots, x_d)) = A(x_1, (x_2, \ldots, x_d)) \) for all other \((x_1, (x_2, \ldots, x_d))\). By construction of \( a_2, \ldots, a_d \) we have

\[
B(W_1 \times W_{2, \ldots, d}) = B(W \times U_1)B(X_1 \times U_1)^{-1}B(X_1 \times W_{2, \ldots, d})
\]

and therefore \( \text{rk} B(W_1 \times W_{2, \ldots, d}) \leq l \). For each \( \alpha \in [d] \), we define the slices

\[
S_{\alpha,j} = \{x_\alpha = j\} \setminus (Q_1 \times E(\{2, \ldots, d\})) \cap \{x_1 \in W_1\} \cap \cdots \cap \{x_{\alpha-1} \in W_{\alpha-1}\}
\]

for each \( j \in Q_\alpha \setminus W_\alpha \). Then the set \( Q_1 \times Q_{2, \ldots, d} \) can be written as the disjoint union

\[
(W_1 \times W_{2, \ldots, d}) \cup \bigcup_{1 \leq \alpha \leq d, j \in Q_\alpha \setminus W_\alpha} S_{\alpha,j}.
\]

By definition all the sets \( Q_\alpha \setminus W_\alpha \) have size at most \( dl \). We can arbitrarily define \( a_2, \ldots, a_d \)

on \( \bigcup_{j \in Q_1 \setminus W_1} S_{1,j} = (Q_1 \setminus W_1) \times Q_{2, \ldots, d} \) and obtain

\[
\dim\langle(T_{x_2, \ldots, x_d} + \sum_{i=2}^{d} a_i(x_2, \ldots, x_d)1_{x_1=x_i})(Q_1 \setminus W_1) : (x_2, \ldots, x_d) \in Q_{2, \ldots, d}\rangle
\]

to be at most \( |Q_1 \setminus W_1| \leq dl \). For each \( \alpha \in \{2, \ldots, d\} \) and each \( j \in Q_\alpha \setminus W_\alpha \), by Proposition 6.7 we have \( \text{etr} T_j \leq (4m^3)^{2d-1} \), so using Remark 6.9, \( a_2, \ldots, a_d \) can be defined on \( S_{\alpha,j} \) such that

\[
\dim\langle(T_{x_2, \ldots, x_d} + \sum_{i=2}^{d} a_i(x_2, \ldots, x_d)1_{x_1=x_i})(W_1) : (x_2, \ldots, x_d) \in Q_{2, \ldots, d},
\]

\[
x_2 \in W_2, \ldots, x_{\alpha-1} \in W_{\alpha-1}, x_\alpha = j \rangle \leq (4m^3)^{2d-1}.
\]

The result follows by subadditivity. \qed
Let $G'_{d, \text{tr}}(l)$ be defined by $G'_{2, \text{tr}}(l) = l$ and for each $d \geq 3$,

$$G'_{d, \text{tr}}(l) = (2.10^6)^{2d} G'_{d-1, \text{tr}}(l)^{(3.2^{d-1})(3.2^d)}.$$  

**Proposition 6.11.** Let $T$ be an order-$d$ tensor, and let $l \geq d^2$ be a positive integer. If $\text{etr} T \geq G'_{d, \text{tr}}(l)$, then $d \text{tr} T \geq l$.

**Proof.** We prove the result by induction on $d$. The result holds for $d = 2$. Let $d \geq 3$.

We distinguish two cases.

**Case 1:** There exists $I \subset [d]$ with $|I| \in \{1, \ldots, d-2\}$ and $y \in \prod_{\alpha \in I} Q_\alpha$ with the $y_\alpha$, $i \in I$ pairwise distinct and such that $\text{epr} T_y \geq G'_{d-1, \text{tr}}(l) + d^2$. Let $d' = |I|$. Without loss of generality we can assume that $I = \{1, \ldots, d'\}$. Hence

$$\text{epr} T_y(\prod_{\alpha = d'+1}^d (Q_\alpha \setminus \{y_1, \ldots, y_{d'}\})) \geq G'_{d-1, \text{tr}}(l).$$

By the inductive hypothesis applied to the previous tensor there exist pairwise disjoint sets $X_\alpha \subset Q_\alpha \setminus \{y_1, \ldots, y_{d'}\}$ for each $\alpha \in \{d'+1, \ldots, d\}$ such that

$$\text{tr} T_y(\prod_{\alpha = d'+1}^d X_\alpha) \geq l.$$  

Letting $X_\alpha = y_\alpha$ for each $\alpha \in \{1, \ldots, d'\}$, the sets $X_1, \ldots, X_d$ are pairwise disjoint and

$$\text{tr} T(\prod_{\alpha = 1}^d X_\alpha) \geq l.$$  

**Case 2:** For all $I \subset [d]$ with $|I| \in \{1, \ldots, d-2\}$ and $y \in \prod_{\alpha \in I} Q_\alpha$ with the $y_\alpha$, $\alpha \in I$ pairwise distinct we have $\text{epr} T_y \leq G'_{d-1, \text{tr}}(l) + d^2$. Since

$$\text{etr} T \geq G'_{d, \text{tr}}(l) = (2.10^6)^{2d} G'_{d-1, \text{tr}}(l)^{(3.2^{d-1})(3.2^d)},$$

applying Proposition 6.7 we get $\text{efrank}_1 T \geq 3000F_{d-1}(l)^{3.2^{d-1}}$. By Proposition 6.10 there exist $X_1, \ldots, X_d$ pairwise disjoint such that

$$\text{frank}_1 T(\prod_{\alpha = 1}^d X_\alpha) \geq l$$

and hence

$$\text{tr} T(\prod_{\alpha = 1}^d X_\alpha) \geq l.$$  

$\square$


7 Proofs for the slice rank of order-3 tensors

7.1 Proof for order-3 slice rank minors

We here prove Theorem 1.4 in the case of the slice rank of order-3 tensors. Our proof can be summarised as follows: given an order-3 tensor $T$ with high slice rank we distinguish two cases: either there exists a large separated set of slices $T_x$, in which case we can find sets $Y, Z$ such that this is still the case after we restrict these slices to $Y \times Z$, which suffices to guarantee a high rank minor, or there does not exist such a large separated set, in which case we construct a projected tensor for which the slice rank and tensor rank are equivalent in the sense that they are large simultaneously, and conclude using our minors result in the tensor rank case.

We begin by proving a lemma showing that having a large separated set of slices guarantees a high slice rank.

**Lemma 7.1.** Let $T : Q_1 \times Q_2 \times Q_3 \rightarrow \mathbb{F}$ be an order-3 tensor, and $l \geq 1$ be an integer. If there exist $x_1, \ldots, x_l \in Q_1$ such that

$$
\text{rk}(\sum_{i=1}^{l} a_i T_{x_i}) \geq l
$$

for every $(a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\}$, then $\text{sr} T \geq l$.

**Proof.** We show the contrapositive. Assume that $\text{sr} T \leq l-1$. Then there exist nonnegative integers $r, s, t$ with $r + s + t = l$ and functions $a_i : Q_1 \rightarrow \mathbb{F}$, $b_i : Q_2 \times Q_3 \rightarrow \mathbb{F}$, $i \in [r]$, $c_j : Q_2 \rightarrow \mathbb{F}$, $d_j : Q_1 \times Q_3 \rightarrow \mathbb{F}$, $j \in [s]$, $e_k : Q_3 \rightarrow \mathbb{F}$, $f_k : Q_1 \times Q_2 \rightarrow \mathbb{F}$, $k \in [t]$ such that

$$
T(x, y, z) = \sum_{i=1}^{r} a_i(x)b_i(y, z) + \sum_{j=1}^{s} c_j(y)d_j(x, z) + \sum_{k=1}^{t} e_k(z)f_k(x, y)
$$

Let $x_1, \ldots, x_l \in Q_1$. Then we can write

$$
T_{x_h}(y, z) = \sum_{i=1}^{r} a_i(x_h)b_i + \sum_{j=1}^{s} c_j(y)d_j(x_h, z) + \sum_{k=1}^{t} e_k(z)f_k(x_h, y)
$$

for each $h \in [l]$. Because $l \geq r + 1$ there exists a function $a : Q_1 \rightarrow \mathbb{F}$ supported inside $\{x_1, \ldots, x_l\}$ such that $a \neq 0$ but $\sum_{h=1}^{l} a(x_h)a_i(x_h) = 0$ for each $i \in [r]$, so we can write

$$
\sum_{h=1}^{l} a(x_h)T_{x_h}(y, z) = \sum_{j=1}^{s} c_j(y)(\sum_{h=1}^{l} a(x_h)d_j(x_h, z)) + \sum_{k=1}^{t} e_k(z)(\sum_{h=1}^{l} a(x_h)f_k(x_h, y)).
$$

The right-hand side has rank at most $s + t \leq l - 1$, a contradiction. \qed

We next show a partial converse to the inequality $\text{sr} T \geq \text{tr} T$ which holds in the situation where all slices of $T$ of all three kinds have bounded rank.
Proposition 7.2. Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor. Let $m \geq 1$ be an integer. Assume that for all $x \in Q_1, y \in Q_2, z \in Q_3$ we have $\text{rk}T_x, \text{rk} T_y, \text{rk} T_z \leq m$. Then $\text{tr} T \leq m(sr T)^2$.

Proof. There exists a decomposition

$$
\sum_{i=1}^{r} a_i(x)b_i(y, z) + \sum_{j=1}^{s} c_j(y)d_j(x, z) + \sum_{k=1}^{t} e_k(z)f_k(x, y) \tag{10}
$$

of $T$, with $r, s, t$ nonnegative integers such that $r + s + t = sr T$. The family $\{a_1, \ldots, a_r\}$ is necessarily linearly independent. By Gaussian elimination there exists a set $X \subset Q_1$ of size at most $r$ and for each $i \in [r]$ a function $a_i^* : Q_1 \to \mathbb{F}$ supported inside $X$ such that $a_i^*a_{i'} = \delta_{i, i'}$ for all $i, i' \in [r]$. For a given $i \in [r]$, applying $a_i^*$ to both sides of (10) we get

$$
b_i(y, z) = (a_i^* T)(y, z) - \sum_{j=1}^{s} c_j(y)(a_i^* d_j)(z) - \sum_{k=1}^{t} e_k(z)(a_i^* f_k)(y).
$$

Because $\text{rk} T_x \leq m$ for each $x \in Q_1$ and $a_i^*$ is supported in a set of size at most $r$ we get by subadditivity of the rank that $\text{rk}(a_i^* T) \leq mr$ and hence $\text{rk} b_i \leq mr + s + t$. Similarly $\text{rk} d_j \leq ms + r + t$ for every $j \in [s]$ and $\text{rk} f_k \leq mt + r + s$ for every $k \in [t]$. By subadditivity

$$
\text{tr} T \leq r(mr + s + t) + s(ms + r + t) + t(mt + r + s) \leq m(sr T)^2. \quad \Box
$$

We are now ready to prove our minors result.

Proposition 7.3. Let $T : Q_1 \times Q_2 \times Q_3 \to \mathbb{F}$ be an order-3 tensor, and let $l \geq 1$ be a positive integer. If $sr T \geq 51l^3$ then there exist $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ with size at most $48l^3$ such that $sr T(X \times Y \times Z) \geq l$.

Proof. Assume $sr T \geq 51l^3$. We distinguish two cases.

Case 1: For at least one of the three coordinates $x, y, z$, which without loss of generality we can take to be the first coordinate $x$, there exist $x_1, \ldots, x_l \in Q_1$ such that

$$
\text{rk}(\sum_{i=1}^{l} a_i T_{x_i}) \geq l
$$

for every $(a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\}$. By multidimensional order-2 minors (Proposition 2.5) there are sets $Y \subset Q_2, Z \subset Q_3$ with size at most $l^2$ such that

$$
\text{rk}(\sum_{i=1}^{l} a_i T_{x_i})(Y \times Z) \geq l
$$

for every $(a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\}$. By Lemma 7.1 taking $X = \{x_1, \ldots, x_l\}$ we get $sr T(X \times Y \times Z) \geq l$. 

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Case 2: We are not in Case 1. Then we construct a decomposition
\[ T = S^1 + S^2 + S^3 + U \]
with $S^1, S^2, S^3, U$ order-3 tensors $Q_1 \times Q_2 \times Q_3 \rightarrow \mathbb{F}$ as follows. Because we are not in Case 1, there exist $r \leq l, x_1, \ldots, x_r \in Q_1$ such that for every $x \in Q_1$ there exist coefficients $a_1(x), \ldots, a_r(x) \in \mathbb{F}$ satisfying
\[ \text{rk}(T_x - \sum_{i=1}^r a_i(x)T_{x_i}) \leq l. \]
The tensor $S^1$ defined by $S^1(x, y, z) = \sum_{i=1}^r a_i(x)T_{x_i}(y, z)$ hence satisfies $\text{rk}(T_x - S^1_x) \leq l$ for every $x \in Q_1$. Similarly we can find $s, t \leq l, y_1, \ldots, y_s \in Q_2, z_1, \ldots, z_t \in Q_3$ and functions $c_1, \ldots, c_s : Q_2 \rightarrow \mathbb{F}, e_1, \ldots, e_t : Q_3 \rightarrow \mathbb{F}$ such that
\[ \text{rk}(T_y - \sum_{j=1}^s c_j(y)T_{y_j}) \leq l \]
for every $y \in Q_2$ and
\[ \text{rk}(T_z - \sum_{k=1}^t e_k(z)T_{z_k}) \leq l \]
for every $z \in Q_3$. We define the tensors $S^2$ and $S^3$ by
\[ S^2(x, y, z) = \sum_{j=1}^s c_j(y)T_{y_j}(x, z) \text{ and } S^3(x, y, z) = \sum_{k=1}^t e_k(z)T_{z_k}(x, y). \]
Let $U = T - (S^1 + S^2 + S^3)$. For each $x \in Q_1$, $\text{rk}(T_x - S^1_x) \leq l$, and $\text{rk} S^2_x, \text{rk} S^3_x \leq l$, so by subadditivity, $\text{rk} U_x \leq 3l$. Similarly for each $y \in Q_2$, $\text{rk} U_y \leq 3l$ and for each $z \in Q_3$, $\text{rk} U_z \leq 3l$. Since $tr U \geq (3l)(4l)^2 + 3l$ and $\text{sr} S^1, \text{sr} S^2, \text{sr} S^3 \leq l$, by subadditivity $\text{sr} U \geq (3l)(4l)^2$. Since $tr U \geq sr U$, we have $tr U \geq (3l)(4l)^2$. By Proposition 2.4 in the order-3 case we can find $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ each with size at most $(3l)(4l)^2$ such that $tr U(X \times Y \times Z) \geq (3l)(4l)^2$. As taking minors cannot increase the rank, it is still the case that $\text{rk} U(X \times Y \times Z)_x, \text{rk} U(X \times Y \times Z)_y, \text{rk} U(X \times Y \times Z)_z \leq 3l$ for all $x \in X, y \in Y, z \in Z$. By Proposition 1.2 $\text{sr} U(X \times Y \times Z) \geq 4l$, and hence (since $\text{sr}(S^1 + S^2 + S^3)(X \times Y \times Z) \leq 3l$) we conclude that $\text{sr} T(X \times Y \times Z) \geq l$. \qed

### 7.2 Proof for order-3 disjoint slice rank

The previous proof can be adapted to a proof for order-3 disjoint slice rank, using the same Lemma [1.1] and Proposition [1.2] as the previous proof did.

**Proposition 7.4.** Let $T : Q_1 \times Q_2 \times Q_3 \rightarrow \mathbb{F}$ be an order-3 tensor. If $\text{esr} T \geq 17500((l^2 + l + 5)(4l)^2) + 3l$ then there exist $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ pairwise disjoint such that $\text{sr} T(X \times Y \times Z) \geq l$. 

Proof. Assume that \( \text{esr} T \geq 17500((l^2 + l + 5)(4l)^3 + 3l \). We distinguish two cases.

Case 1: For at least one of the three coordinates \( x, y, z \) which without loss of generality we can take to be the first coordinate \( x \), there exist \( x_1, \ldots, x_l \in Q_1 \) such that

\[
\text{erk}(\sum_{i=1}^{l} a_i T_{x_i}) \geq (l^2 + l + 3)l
\]

for every \( (a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\} \). Using that a single row or column has rank at most 1, by subadditivity

\[
\text{erk}(\sum_{i=1}^{l} a_i T_{x_i})((Q_2 \setminus \{x_1, \ldots, x_l\}) \times (Q_3 \setminus \{x_1, \ldots, x_l\})) \geq (l^2 + l + 1)l
\]

so by Proposition 5.2 there exist disjoint subsets \( Y \subset Q_2 \setminus \{x_1, \ldots, x_l\} \), \( Z \subset Q_3 \setminus \{x_1, \ldots, x_l\} \) such that

\[
\text{rk}(\sum_{i=1}^{l} a_i T_{x_i})(Y \times Z) \geq l
\]

for every \( (a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\} \). Taking \( X = \{x_1, \ldots, x_l\} \), the sets \( X, Y, Z \) are pairwise disjoint and by Lemma 7.1 we get \( \text{sr} T(X \times Y \times Z) \geq l \).

Case 2: We are not in Case 1. Then we construct a decomposition

\( T = S^1 + S^2 + S^3 + U \)

as follows. Because we are not in Case 1, there exists \( r \leq l \) and \( x_1, \ldots, x_r \in Q_1 \) such that for every \( x \in Q_1 \) there exist coefficients \( a_1(x), \ldots, a_r(x) \) such that

\[
\text{erk}(T_x - \sum_{i=1}^{r} a_i(x) T_{x_i}) \leq (l^2 + l + 3)l.
\]

The tensor \( S^1 \) defined by \( S^1(x, y, z) = \sum_{i=1}^{r} a_i(x) T_{x_i}(y, z) \) therefore satisfies \( \text{erk}(T_x - S^1_x) \leq (l^2 + l + 3)l \) for every \( x \in Q_1 \). Similarly we can find \( s, t \leq l \), \( y_1, \ldots, y_s \in Q_2 \), \( z_1, \ldots, z_t \in Q_3 \) and functions \( c_1, \ldots, c_s : \mathbb{F}^{n_2} \to \mathbb{F} \), \( e_1, \ldots, e_t : \mathbb{F}^{n_3} \to \mathbb{F} \) such that

\[
\text{erk}(T_y - \sum_{j=1}^{s} c_j(y) T_{y_j}) \leq (l^2 + l + 3)l
\]

for every \( y \in Q_2 \) and

\[
\text{erk}(T_z - \sum_{k=1}^{t} e_k(z) T_{z_k}) \leq (l^2 + l + 3)l
\]

for every \( z \in Q_3 \). We define the tensors \( S^2 \) and \( S^3 \) by

\[
S^2(x, y, z) = \sum_{j=1}^{s} c_j(y) T_{y_j}(x, z) \quad \text{and} \quad S^3(x, y, z) = \sum_{k=1}^{t} e_k(z) T_{z_k}(x, y).
\]

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Let \( U = T - (S^1 + S^2 + S^3) \). For each \( x \in Q_1 \), \( \text{erk}(T_x - S^1_x) \leq (l^2 + l + 3)l \) and \( \text{rk} S^2_x, \text{rk} S^3_x \leq l \), so by subadditivity, \( \text{erk} U_x \leq (l^2 + l + 5)l \). Similarly for each \( y \in Q_2 \), \( \text{erk} U_y \leq (l^2 + l + 5)l \) and for each \( z \in Q_3 \), \( \text{erk} U_z \leq (l^2 + l + 5)l \). Since

\[
\text{esr} T \geq 17500((l^2 + l + 5)l(4l)^2)^3 + 3l
\]

and \( \text{srs} S^1, \text{srs} S^2, \text{srs} S^3 \leq l \), by subadditivity and then using that the essential slice rank is at most the essential tensor rank we obtain

\[
\text{etr} U \geq \text{esr} U \geq 17500((l^2 + l + 5)l(4l)^2)^3.
\]

By Proposition \ref{prop:tensor-deduction} we can find \( X \subset Q_1 \), \( Y \subset Q_2 \), \( Z \subset Q_3 \) pairwise disjoint such that \( \text{tr} U(X \times Y \times Z) \geq (l^2 + l + 5)l(4l)^2 \). As taking minors cannot decrease the essential rank, we still have \( \text{rk} U(X \times Y \times Z)_x, \text{rk} U(X \times Y \times Z)_y, \text{rk} U(X \times Y \times Z)_z \leq (l^2 + l + 5)l \) for all \( x \in X \), \( y \in Y \), \( z \in Z \). By Proposition \ref{prop:tensor-deduction} \( \text{srs} U(X \times Y \times Z) \geq 4l \), and hence (since \( \text{srs}(S^1 + S^2 + S^3)(X \times Y \times Z) \leq \text{srs}(S^1 + S^2 + S^3) \leq 3l \)) we conclude \( \text{srs} U(X \times Y \times Z) \geq l \). □

8 Deducing minors and disjoint rank for several tensors from the one-tensor case

Throughout this section, we fix \( d \geq 2 \) a positive integer and \( R \) a non-empty family of partitions of \([d]\). We assume that Theorem \ref{thm:tensor-induction} and Theorem \ref{thm:tensor-induction} hold for this choice of pair \((d, R)\) and deduce from them generalisations to several tensors for this same choice of pair \((d, R)\). In Section \ref{sec:multidimensional} we will use these multidimensional generalisations as an important part of the inductive argument that proves Theorem \ref{thm:tensor-induction} and Theorem \ref{thm:tensor-induction}.

8.1 Multidimensional minors

Let \( d \geq 2 \) be an integer and let \( R \) be a non-empty family of partitions of \([d]\). For every positive integer \( s \geq 1 \), we define the functions \( F_{d,R,s} : \mathbb{N} \to \mathbb{N} \) and \( G_{d,R,s} : \mathbb{N} \to \mathbb{N} \) by

\[
F_{d,R,1}(l) = F_{d,R}(l) \quad \text{and for each } s \geq 2, \quad F_{d,R,s}(l) = F_{d,R}(sl) + F_{d,R,s-1}(l)
\]

\[
G_{d,R,1}(l) = G_{d,R}(l) \quad \text{and for each } s \geq 2, \quad G_{d,R,s}(l) = G_{d,R}(sl).
\]

Proposition 8.1. Let \( d, s \geq 1 \) be positive integers and let \( R \) be a non-empty family of partitions of \([d]\). Then whenever \( \Lambda \) is a subset of \( \mathbb{F}^* \setminus \{0\} \), if \( T_1, \ldots, T_s : \prod_{\alpha=1}^d Q_\alpha \to \mathbb{F} \) are order-\(d\) tensors such that

\[
\text{Rrk}(\sum_{i=1}^s a_i T_i) \geq G_{d,R,s}(l)
\]

for every \((a_1, \ldots, a_s) \in \Lambda\), then there exist \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \), each with size at most \( F_{d,R,s}(l) \), such that

\[
\text{Rrk}(\sum_{i=1}^s a_i T_i)(\prod_{\alpha=1}^d X_\alpha) \geq l
\]

for every \((a_1, \ldots, a_s) \in \Lambda\).
Proof. We proceed by induction on $s$. If $s = 1$ then the result holds by the one-tensor case. Let $s \geq 2$ and assume that the result holds for $s - 1$. If $\Lambda$ is not empty then we are done. If $\Lambda$ is empty then let $a^0$ be a fixed arbitrary element of $\Lambda$, by the one-tensor case there exist $X'_1 \subset Q_1, \ldots, X'_d \subset Q_d$ all with size at most $F_{d,R}(sl)$ such that

$$\text{Rrk}(a^0.T)(X'_1 \times \cdots \times X'_d) \geq sl.$$  

By subadditivity of the $R$-rank, there exists a strict linear subspace $W$ of $\mathbb{F}^s$ such that

$$\text{Rrk}(a.T)(X'_1 \times \cdots \times X'_d) \geq l$$

for every $a \in \Lambda \setminus W$. Applying Proposition 8.1 for $s - 1$ and $\Lambda \cap W$, there exist $X''_1 \subset Q_1, \ldots, X''_d \subset Q_d$ all with size at most $F_{d,R,s-1}(l)$ such that

$$\text{Rrk}(a.T)(X''_1 \times \cdots \times X''_d) \geq l$$

for every $a \in \Lambda \cap W$. The sets $X_1, \ldots, X_d$ defined by $X_1 = X'_1 \cup X''_1, \ldots, X_d = X'_d \cup X''_d$ all have size at most $F_{d,R}(sl) + F_{d,R,s-1}(l)$, and furthermore

$$\text{Rrk}(a.T)(X_1 \times \cdots \times X_d) \geq l$$

for every $a \in \Lambda$. \hfill $\square$

8.2 Multidimensional disjoint rank

Is it the case that for every $l \geq 1$ there exists $g(l)$ such that if $T_1$ and $T_2$ are two order-3 tensors such that for all $(a, b) \in \mathbb{F}^2 \setminus \{0\}$, $\text{etr}(a_1 T_1 + a_2 T_2) \geq g(l)$ then there exist $X, Y, Z$ pairwise disjoint such that for all $(a, b) \in \mathbb{F}^2 \setminus \{0\}$, $\text{tr}(a_1 T_1 + a_2 T_2)(X \times Y \times Z) \geq l$? Without an additional assumption this is false, as the following counterexample shows.

Example 8.2. Let $k \geq 1$ be a positive integer, $b_1 : Q_3 \times Q_3 \to \mathbb{F}$, $b_2 : Q_1 \times Q_3 \to \mathbb{F}$, $b_2$ be bilinear forms with $\text{erk} b_1, \text{erk} b_2 \geq k + 3$ and let $T_1, T_2$ be the order-3 tensors defined by $T_1(x, y, z) = 1_{x=1} b_1(y, z)$, $T_2(x, y, z) = 1_{y=1} b_2(x, z)$. For any $(a_1, a_2) \in \mathbb{F}^2 \setminus \{0\}$, assuming without loss of generality that $a_1 \neq 0$ we have

$$\text{etr}(a_1 T_1 + a_2 T_2) \geq \text{erk}(a_1 T_1 + a_2 T_2)_{x=1}((Q_2 \setminus \{1\}) \times (Q_3 \setminus \{1\}))$$

$$\geq \text{erk}(a_1 T_1 + a_2 T_2)_{x=1} - 2$$

$$\geq \text{erk}(b_1) - 3$$

$$\geq k.$$  

where the third inequality comes from the fact that the order-2 slice $(T_2)_{x=1}$ has support contained inside the row $y = 1$. On the other hand whenever $X \subset Q_1, Y \subset Q_2, Z \subset Q_3$ are pairwise disjoint the element $1$ cannot be contained in both of the sets $X$ and $Y$. If $1$ is outside $X$, then $T_1(X \times Y \times Z) = 0$ and similarly if $1$ is outside $Y$, then $T_2(X \times Y \times Z) = 0$, so there exists some $(a_1, a_2) \in \mathbb{F}^2 \setminus \{0\}$ such that $(a_1 T_1 + a_2 T_2)(X \times Y \times Z) = 0$. 

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This kind of counterexample does not show up if we consider matrices instead of order-3 tensors, because for matrices it is not possible to fit a matrix of arbitrarily large essential rank into a single row or column, whereas it is possible to fit an order-3 tensor of arbitrarily large essential tensor rank into a single order-2 slice. However, the statement becomes true if we require in addition that the high essential rank assumption on the linear combinations \(a.T\) holds even after removing sufficiently many order-\((d-1)\) slices of each of the \(d\) kinds.

Let \(d \geq 2\) be an integer and let \(R\) be a non-empty family of partitions of \([d]\). For every positive integer \(s \geq 1\), we define the functions \(H_{d,R,s} : \mathbb{N} \to \mathbb{N}\) and \(G'_{d,R,s} : \mathbb{N} \to \mathbb{N}\) by

\[
H_{d,R,1}(l) = 0 \quad \text{and} \quad \text{for each } s \geq 2, \quad H_{d,R,s}(l) = F_{d,R}(sl) + H_{d,R,s-1}(l) \\
G'_{d,R,1}(l) = G'R(l) \quad \text{and} \quad \text{for each } s \geq 2, \quad G'_{d,R,s}(l) = G'R(G_{d,R}(sl)).
\]

For each \(d \geq 2\), we can deduce multidimensional order-\(d\) disjoint \(R\)-rank from one-dimensional order-\(d\) disjoint \(R\)-rank and one-dimensional order-\(d\) \(R\)-rank minors.

**Proposition 8.3.** Let \(d \geq 2\), \(s \geq 1\) be positive integers, let \(R\) be a non-empty family of partitions of \([d]\), let \(F\) be a field and let \(\Lambda\) be a subset of \(\mathbb{F}^s \setminus \{0\}\). Let \(T_1, \ldots, T_s : \prod_{\alpha=1}^d Q_{\alpha} \to \mathbb{F}\) be order-\(d\) tensors and suppose that for all \(Y_1 \subset Q_1, \ldots, Y_d \subset Q_d\) all with size at most \(H_{d,R,s}(l)\),

\[eRrk(a.T)(\prod_{\alpha=1}^d (Q_{\alpha} \setminus Y_{\alpha})) \geq G'_{d,s,R}(l)\]

for every \((a_1, \ldots, a_s) \in \Lambda\). Then there exist \(X_1 \subset Q_1, \ldots, X_d \subset Q_d\) pairwise disjoint such that

\[Rrk(a.T)(\prod_{\alpha=1}^d X_{\alpha}) \geq l\]

for every \((a_1, \ldots, a_s) \in \Lambda\).

**Proof.** We proceed by induction on \(s\). If \(s = 1\) then the result holds by the one-tensor case. Let \(s \geq 2\) and assume that the result holds for \(s - 1\). If \(\Lambda\) is empty then the result holds. We assume that this is not the case and choose an arbitrary \(a \in \Lambda\). By the one-tensor case there exist \(X_1^0, \ldots, X_d^0\) pairwise disjoint such that

\[Rrk(a^0.T)(\prod_{\alpha=1}^d X_{\alpha}^0) \geq G_{d,R}(sl)\]

By the one-tensor case of order-\(d\) \(R\)-rank minors we can find subsets \(X_{\alpha}'\) of \(X_{\alpha}^0\) for each \(\alpha = 1, \ldots, d\) with size at most \(F_{d,R}(sl)\) such that

\[Rrk(a^0.T)(\prod_{\alpha=1}^d X_{\alpha}') \geq sl\]
By subadditivity of the $R$-rank, there exists a strict linear subspace $W$ of $\mathbb{F}^s$ such that

$$Rrk(a.T)(\prod_{\alpha=1}^d X'_\alpha) \geq l$$

for every $a \in \Lambda \setminus W$. Taking $Y_\alpha = \bigcup_{1 \leq \alpha \leq d} X'_\alpha$ for each $\alpha = 1, \ldots, d$ and applying the current proposition for $s - 1$ there exist subsets $X''_\alpha \subset Q_\alpha \setminus X'_\alpha$ with the $X''_\alpha$ all pairwise disjoint and such that

$$Rrk(a.T)(\prod_{\alpha=1}^d X''_\alpha) \geq l$$

for every $a \in \Lambda \cap W$. Letting $X_\alpha = X'_\alpha \cup X''_\alpha$ for each $\alpha = 1, \ldots, d$ the sets $X_\alpha$ are pairwise disjoint, and

$$Rrk(a.T)(\prod_{\alpha=1}^d X_\alpha) \geq l$$

for every $a \in \Lambda$ as desired. \hfill $\square$

**Remark 8.4.** We note that Example 8.2 occurs neither for the slice rank nor for the partition rank: this is because for these notions of rank for order-$d$ tensors, every tensor supported inside an order-$(d-1)$ slice has rank at most 1. For these notions of rank we can replace $(G'_{d,R,s}, H_{d,R,s})$ by $(G'_{d,R,s} + dH_{d,R,s}, 0)$ in the statement of Proposition 8.3. Throughout the remainder of the present paper, all applications of Proposition 8.3 will involve only the partition rank and we will assume this replacement whenever we use Proposition 8.3.

9 Extending the proof for order-3 slice rank minors to order-4 partition rank minors

In this section we adapt the proof of Proposition 7.3 to a proof for order-4 partition rank minors. Compared with the proof of Proposition 7.3 the main novelty in the proofs will be how to navigate between the different notions of rank, but many of the central ideas will be the same as for the proof of Proposition 7.3. The following notions of rank will be relevant to the argument. We will write

$$pr T$$
for the order-4 partition rank of $T$

$$pr_{(2,2)} T$$
for $Rrk T$ with $R = \{\{1\}, \{2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}\}$

$$(1 \times sr) T$$
for $Rrk T$ with $R = \{\{1\}, \{2\}, \{3, 4\}\}, \{\{1\}, \{3\}, \{2, 4\}\}, \{\{1\}, \{4\}, \{2, 3\}\}\}.$

We will refer to the second notion as the $(2,2)$-partition rank and to the third notion as the 1-enhanced slice rank.

The overall structure of the proof will be as follows: starting from any of the first two previously listed notions of $R$-rank, either we can identify a simple lower-dimensional
structure that guarantees high \( R \)-rank, in which case we can conclude by Proposition 8.1 applied to slices of strictly smaller order, or we cannot, in which case the task at hand reduces to proving minors for the notion of rank listed immediately thereafter. If we reduce twice and end up considering the 1-enhanced slice rank then we will able to conclude in rather short order due to the particular form of \( R \) in this case and the fact that we already know how to prove Theorem 1.4 for the slice rank of order-3 tensors, but in the light of the proof of the case of general \( R \) discussed in Section 11 it is conceptually worthwhile to point out that had we not had this simple concluding argument, then another possibility would have been to try to iterate our proof techniques one more time to reduce to the order-4 tensor rank, which would have led to some additional complications in the proof similar to those of Section 11.

We begin by deducing Theorem 1.4 for the order-4 partition rank from Proposition 8.1 for the order-3 slice rank and from Theorem 1.4 for the order-4 \((2,2)\)-partition rank. The proof will distinguish two cases depending on the existence of a large set of order-3 slices that is sufficiently separated for the notion of order-3 slice rank. The first lemma that we will use is an analogue of Lemma 7.1.

**Lemma 9.1.** Let \( T : Q_1 \times Q_2 \times Q_3 \times Q_4 \to \mathbb{F} \) be an order-4 tensor, and let \( l \geq 1 \) be an integer. Suppose that there exist \( x_1, \ldots, x_l \in Q_1 \) such that
\[
\text{sr}(\sum_{i=1}^{l} a_iT_{x_i}) \geq l
\]
for every \( a \in \mathbb{F}^l \setminus \{0\} \). Then \( pr T \geq l \).

**Proof.** Assume for a contradiction that \( pr T \leq l - 1 \). Then there exist nonnegative integers \( r_1, r_2, r_3, r_4, r' \) with \( r_1 + r_2 + r_3 + r_4 + r' = pr T \) such that we can write a decomposition
\[
T(x, y, z, w) = \sum_{i=1}^{r_1} a_{1,i}(x)b_{1,i}(y, z, w) + \sum_{i=1}^{r_2} a_{2,i}(y)b_{2,i}(x, z, w) + \sum_{i=1}^{r_3} a_{3,i}(z)b_{3,i}(x, y, w) + \sum_{i=1}^{r_4} a_{4,i}(w)b_{4,i}(x, y, z) + U(x, y, z, w)
\]
for some order-4 tensor \( U \) with \( pr_{(2,2)} U = r' \). Since \( r_1 \leq l - 1 \) there exists a function \( a : Q_1 \to \mathbb{F} \) supported inside \( \{x_1, \ldots, x_l\} \) such that \( a.a_{1,i} = 0 \) for each \( i \in [r_1] \). Applying \( a \) to both sides of the decomposition (11) we obtain \( \text{sr} a.T \leq l - 1 \), a contradiction. \( \square \)

Our next statement is an analogue of Proposition 7.2. It provides us with an equivalence between the partition rank and the \((2,2)\)-partition rank of an order-4 tensor under the assumption that all order-3 slices of the tensor have bounded slice rank.

**Proposition 9.2.** Let \( k, m \geq 1 \) be two positive integers, let \( T : Q_1 \times Q_2 \times Q_3 \times Q_4 \to \mathbb{F} \) be an order-4 tensor, and suppose that \( \text{sr} T_x \leq m, \text{sr} T_y \leq m, \text{sr} T_z \leq m, \text{sr} T_w \leq m \) for each \( x \in Q_1, y \in Q_2, z \in Q_3, w \in Q_4 \). If \( pr T \leq l \), then \( pr_{(2,2)} T \leq ml^2 \).
Proof. Starting from the decomposition (11) we can find a subset \( X \subset Q_1 \) with size at most \( r_1 \) and functions \( a_{i,j}^* : Q_1 \to \mathbb{F} \) such that \( a_{i,i}, a_i = \delta_{i,i} \) for all \( i, i' \in [r] \). For a fixed \( i \in [r] \), applying \( a^*_i \) to both sides of (11) yields the bound \( \text{sr} b_{1,i} \leq m r_1 + r_2 + r_3 + r_4 + r' \). We obtain similar bounds for the quantities \( \text{sr} b_{2,i}, \text{sr} b_{3,i}, \text{sr} b_{4,i} \) and conclude the desired inequality by subadditivity of the \( (2,2) \)-partition rank.

We are now ready to show that to obtain order-4 partition rank minors it suffices to show order-4 \( (2,2) \)-partition rank minors together with multidimensional order-3 slice rank minors.

Proposition 9.3. If Theorem 1.4 holds for \((d, R) = (4, \text{pr}_{(2,2)})\) and Proposition 8.1 holds for \((d, R) = (3, \text{sr})\) then Theorem 1.4 holds for \((d, R) = (4, \text{pr})\) with

\[
F_{4,\text{pr}}(l) = \max(F_{3,\text{sr},l}(l), F_{4,\text{pr}_{(2,2)}}(\{(G_{3,\text{sr},l}(l) + 4l)(5l)^2\})
\]

\[
G_{4,\text{pr}}(l) = G_{4,\text{pr}_{(2,2)}}(\{(G_{3,\text{sr},l}(l) + 4l)(5l)^2\}) + 4l.
\]

Proof. Let \( T \) be an order-4 tensor with \( \text{pr} T \geq G_{4,\text{pr}_{(2,2)}}(\{(G_{3,\text{sr},l}(l) + 4l)(5l)^2\}) + 4l \). We distinguish two cases.

Case 1: For one of the possible four coordinates, which without loss of generality we can assume to be the first coordinate, there exist \( x_1, \ldots, x_l \in Q_1 \) such that

\[
\text{pr} \left( \sum_{i=1}^l b_i T_{x_i} \right) \geq G_{3,\text{sr},l}(l)
\]

for every \( b \in \mathbb{F}^l \setminus \{0\} \). Then by Proposition 8.1 for \((3, \text{sr})\) there exist \( Y \subset Q_2, Z \subset Q_3, W \subset Q_4 \) each with size at most \( F_{3,\text{sr},l}(l) \) such that

\[
\text{pr} \left( \sum_{i=1}^l b_i T_{x_i} \right)(Y \times Z \times W) \geq l
\]

for every \( b \in \mathbb{F}^l \setminus \{0\} \) and hence \( T(X \times Y \times Z \times W) \geq l \).

Case 2: If we are not in Case 1, then, proceeding as in the proof of Proposition 7.3 we can write

\[
T = S^1 + S^2 + S^3 + S^4 + U
\]

with

\[
S^1(x, y, z, w) = \sum_{i=1}^{r_1} a_{i,1}(x) T_{x_i}(y, z, w)
\]

for some \( r_1 \leq l \) and \( x_1, \ldots, x_l \in Q_1, a_1, \ldots, a_l : Q_1 \to \mathbb{F} \), with \( S^2, S^3, S^4 \) order-4 tensors with similar expressions (with the singled out coordinate being \( y, z, w \) respectively) and with \( U \) an order-4 tensor such that every order-3 slice of \( U \) (of any of the four types) has slice rank at most \( G_{3,\text{sr},l}(l) + 4l \). In particular \( S^1, \ldots, S^4 \) have partition rank at most \( l \), so by

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subadditivity $\pr U \geq k-4l$, so $\pr_{(2,2)} U \geq k-4l$. Since $k = G_{4, \pr_{(2,2)}} ((G_{3, sr, l}(l) + 4l)(5l) + 4l)$, by Theorem 1.4 for $(d, R)$ there exist $X, Y, Z, W$ with size at most $F_{4, \pr_{(2,2)}} ((G_{3, sr, l}(l) + 4l)(5l) + 4l)$ such that

$$\pr_{(2,2)} U(X \times Y \times Z \times W) \geq (G_{3, sr, l}(l) + 4l)(5l)^2.$$  

Moreover every order-3 slice of $U$ (of any of the four types) has slice rank at most $G_{3, sr, l}(l) + 4l$ (so in particular, this is still true for the order-3 slices of the restriction $U(X \times Y \times Z \times W)$). By Proposition 9.2 we have

$$\pr U(X \times Y \times Z \times W) \geq 5l$$

and hence

$$\pr T(X \times Y \times Z \times W) \geq l$$

by subadditivity. \qed

We now begin the second phase of the arguments of this section, where we deduce Theorem 1.4 for $(d, R) = (4, \pr_{(2,2)})$ from Proposition 8.1 for $(d, R) = (2, \rk)$ and from Theorem 1.4 for $(d, R) = (4, 1 \times sr)$. Before the deduction we will again prove a separation lemma and an equivalence. Their bounds will be slightly worse than those of the respective corresponding statements that we have encountered so far and we will highlight the relevant computations.

**Lemma 9.4.** Let $T : Q_1 \times Q_2 \times Q_3 \times Q_4 \rightarrow F$ be an order-4 tensor, and let $l \geq 1$ be an integer. If there exist $(x_1, y_1), \ldots, (x_l, y_l) \in Q_1 \times Q_2$ such that

$$\rk(\sum_{i=1}^{l} a_i T(x_i, y_i)) \geq (l-1) + 1$$

for every $(a_1, \ldots, a_l) \in F^l \setminus \{0\}$ then $\pr_{(2,2)} T \geq l$.

**Proof.** We prove the contrapositive. Assume $\pr_{(2,2)} T \leq l-1$. Then there exist nonnegative integers $r, s, t \geq 0$ with $r + s + t = \pr_{(2,2)} T$ such that we can write a decomposition

$$T(x, y, z, w) = \sum_{i=1}^{r} a_i(x, y)b_i(z, w) + \sum_{j=1}^{s} c_j(x, z)d_j(y, w) + \sum_{k=1}^{t} e_k(x, z)f_k(y, w).$$

For each $h \in [l]$ we can write

$$T_{x_h, y_h}(z, w) = \sum_{i=1}^{r} a_{i,x_h, y_h}b_i(z, w) + \sum_{j=1}^{s} c_{j,x_h}(z)d_j(y, w) + \sum_{k=1}^{t} e_{k,x_h}(z)f_k(y, w).$$

Because $l \geq r + 1$ there exists a function $a : Q_1 \rightarrow F$ supported inside $\{(x_1, y_1), \ldots, (x_l, y_l)\}$ such that $a \neq 0$ but $\sum_{h=1}^{l} a(x_h, y_h)a_i(x_h, y_h) = 0$ for each $i \in [r]$. Hence,

$$\sum_{h=1}^{l} a(x_h, y_h)T_{x_h, y_h}(z, w) = \sum_{j=1}^{s} \sum_{h=1}^{l} c_{j,x_h}(z)d_j(y, w) + \sum_{k=1}^{t} \sum_{h=1}^{l} e_{k,x_h}(z)f_k(y, w).$$

The right-hand side has rank at most $sl + tl \leq (l - 1)l$, a contradiction. \qed

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We next prove our equivalence statement.

**Proposition 9.5.** Let \( T : Q_1 \times Q_2 \times Q_3 \times Q_4 \to \mathbb{F} \) be an order-4 tensor, and let \( m \geq 1 \) be an integer. Assume that \( \text{rk} T_{(z,w)}, \text{rk} T_{(y,w)}, \text{rk} T_{(y,z)} \leq m \) for respectively all \((z,w) \in Q_3 \times Q_4, \) all \((y,w) \in Q_2 \times Q_4, \) and all \((y,z) \in Q_2 \times Q_3 \). If \( \text{pr}_{(2,2)} T \leq l \), then \((1 \times r)T \leq (m+l)^2 \).

**Proof.** There exists a decomposition

\[
\sum_{i=1}^{r} a_i(x,y)b_i(z,w) + \sum_{j=1}^{s} c_j(x,z)d_j(y,w) + \sum_{k=1}^{t} e_k(x,z)f_k(y,w)
\]

of \( T \) with \( r+s+t = \text{pr}_{(2,2)} T \). The family \( \{b_1, \ldots, b_r\} \) is linearly independent. By Gaussian elimination there exist functions \( b^*_i : Q_3 \times Q_4 \to \mathbb{F} \) all supported inside a set \( U \subset Q_3 \times Q_4 \) of size \( r \) such that \( b^*_i b_i = \delta_{i,i'} \) for all \( i, i' \in [r] \). For a given \( i \in [r] \), applying \( b^*_i \) to \( T \) we get

\[
a_i = b^*_i T - \sum_{j=1}^{s} b^*_i (c_j d_j) - \sum_{k=1}^{t} b^*_i (e_k f_k).
\]

Because for each \((z,w) \in U\), the order-2 slice \( T_{(z,w)} \) has rank at most \( m \) and \( b^*_i \) is supported inside a set of size at most \( r \), by subadditivity of the rank we have \( \text{rk} b^*_i T \leq mr \). For each \( j \in [s] \),

\[
(b^*_i (c_j d_j))(z,w) = \sum_{h=1}^{l} b^*_i(z_h,w_h)c_j(x,z_h)d_j(y,w_h)
\]

so \( \text{rk}(b^*_i (c_j d_j)) \leq l \). Similarly for each \( k \in [t] \), \( \text{rk} b^*_i (e_k f_k) \leq l \). Therefore by subadditivity \( \text{rk} a_i \leq mr + ls + lt \). Similarly \( \text{rk} d_j \leq lr + ms + lt \) and \( \text{rk} f_k \leq lr + ls + mt \), so

\[
(1 \times sr)T \leq r(mr + ls + lt) + s(lr + ms + lt) + t(lr + ls + mt) \leq (m+l)^2.
\]

We are again ready to write the step leading to Theorem \[A\] for order-4 \((2,2)\)-partition rank minors.

**Proposition 9.6.** If Theorem \[A\] holds for \((d,R) = (4,1 \times sr)\) then Theorem \[A\] holds for \((d,R) = (4, \text{pr}_{(2,2)})\) with

\[
F_{4,\text{pr}_{(2,2)}}(l) = F_{4,1 \times sr}(112l^4)
\]

\[
G_{4,\text{pr}_{(2,2)}}(l) = G_{4,1 \times sr}(112l^4) + 3l.
\]

**Proof.** Let \( T \) be an order-4 tensor such that \( \text{pr}_{(2,2)} T \geq G_{4,1 \times sr}(112l^4) + 3l \).

**Case 1:** For at least one of the three types of slices \( T_{(z,w)}, T_{(y,w)}, T_{(y,z)} \), which without loss of generality we can assume to be the direction with \( z, w \) as fixed coordinates there exist \((z_1, w_1), \ldots, (z_l, w_l) \in Q_3 \times Q_4 \) such that

\[
\text{rk}(\sum_{i=1}^{l} a_i T_{(z_i, w_i)}) \geq l^2
\]
for every $a \in \mathbb{F}^l \setminus \{0\}$. Then by Proposition 2.5 there exist $X \subset Q_1$, $Y \subset Q_2$ with size at most $l^2$ such that

$$\text{rk}(\sum_{i=1}^l a_i T(z_i, w_i))(X \times Y) \geq l^2$$

for every $a \in \mathbb{F}^l \setminus \{0\}$. Letting $Z = \{z_i : 1 \leq i \leq l\}$, $W = \{w_i : 1 \leq i \leq l\}$, by Lemma 9.3 we get $\Pr_{(2,2)} T(X \times Y \times Z \times W) \geq l$.

**Case 2:** We are not in Case 1. Then there exist $(z_i, w_i), i = 1, \ldots, r$ with $r \leq l$ such that for all $(z, w) \in Q_3 \times Q_4$, there exist $A_1(z, w), \ldots, A_r(z, w)$ such that

$$\text{rk}(T(z, w) - \sum_{i=1}^r A_i(z, w) T(z_i, w_i)) \leq l^2.$$

We write $S_{34}^{34}(x, y, z, w) = \sum_{i=1}^r A_i(z, w) T(z_i, w_i)(x, y)$. Similarly we define two other tensors $S_{24}^{24}$ and $S_{23}^{23}$ (constructed using linear combinations of slices of the types $T(y, w)$ and $T(y, z)$ respectively), and write

$$T = S_{34}^{34} + S_{24}^{24} + S_{23}^{23} + U.$$ 

The tensors $S_{34}^{34}, S_{24}^{24}, S_{23}^{23}$ have $(2,2)$-partition rank at most $l$, so $\Pr_{(2,2)} U \geq k - 3l$ and hence $(1 \times \text{sr}) U \geq k - 3l$. Since $k \geq G_{4,1 \times \text{sr}}(112l^4) + 3l$, by $1 \times \text{sr}$ minors there exist $X \subset Q_1, Y \subset Q_2, Z \subset Q_3, W \subset Q_4$ of size at most $F_{4,1 \times \text{sr}}(112l^4)$ such that

$$(1 \times \text{sr}) U(X \times Y \times Z \times W) \geq 112l^4 \geq ((l^2 + 2l) + 4l)(4l)^2.$$ 

The order-2 slices of $U$ with pairs of fixed coordinates $(z, w), (y, w)$, and $(y, z)$ all have rank at most $l^2 + 2l$, so this is still the case for the corresponding order-2 slices of $U(X \times Y \times Z \times W)$. By Proposition 9.5

$$\Pr_{(2,2)} U(X \times Y \times Z \times W) \geq 4l$$

and hence

$$\Pr_{(2,2)} T(X \times Y \times Z \times W) \geq l. \quad \square$$

Since the proof of Proposition 9.3 had as its second case a reduction to the notion of $(2 - 2)$-partition rank, where all parts of all partitions of the corresponding family $R$ have size at most 2, we can ask whether the proof of Proposition 9.6 could not have reduced to a family where these parts have size at most 1, in other words reduced to the tensor rank. What stands in the way of doing so is that given $l$ a fixed nonnegative integer and $S_{12}^{12}, S_{34}^{34}$ order-4 tensors of the type

$$S_{12}^{12}(x, y, z, w) = \sum_{i=1}^r a_i^{12}(x, y) T_{x_i, x_i}(z, w)$$
\[ S^{34}(x, y, z, w) = \sum_{i=1}^{s} a^{34}_i(z, w) T_{zi,w}(x, y) \]

with \( r, s \leq l \) such that for each \((x, y) \in Q_1 \times Q_2\), \( \text{rk}(T - S^{12})(x, y) \leq l \) and for each \((z, w) \in Q_3 \times Q_4\), \( \text{rk}(T - S^{34})(z, w) \leq l \), it is not necessarily true that the ranks of the slices of the type \((T - S^{12} - S^{34})(x, y)\) and \((T - S^{12} - S^{34})(z, w)\) are bounded: to conclude this we would also want to know that the functions \( a^{12}_i \) and \( a^{34}_i \) have bounded rank. This difficulty led us to to the structure of the proof in Section 10.

We finish by proving Theorem \ref{thm:1} for the 1-enhanced slice rank using its product structure.

**Proposition 9.7.** Theorem \ref{thm:1} holds for \((d, R) = (4, 1 \times sr)\), with

\[
F_{4,1 \times sr}(l) = F_{3, sr}(l)
\]
\[
G_{4,1 \times sr}(l) = lG_{3, sr}(l).
\]

**Proof.** Let \( T \) be an order-4 tensor with \((1 \times sr)T \geq k\). Let \( A : Q_1 \times (Q_2 \times Q_3 \times Q_4) \rightarrow \mathbb{F} \) be the matrix defined by \( A(x, (y, z, w)) = T(x, y, z, w) \). We distinguish two cases depending on the value of \( \text{frank}_1 T = \text{rk} A \).

**Case 1:** \( \text{frank}_1 T \geq l \). Then by the standard minors result on matrices there exist subsets \( X \subseteq Q_1 \) and \( U \subseteq Q_2 \times Q_3 \times Q_4 \), both with size \( l \), such that \( \text{rk} A(X_1 \times U) = l \). Then the canonical projections \( Y, Z, W \) of \( U \) on the second, third and fourth coordinate axes respectively have size at most \( l \), and we have \( \text{frank}_1 T(X \times Y \times Z \times W) \geq l \) whence \((1 \times sr)T(X \times Y \times Z \times W) \geq l \).

**Case 2:** \( \text{frank}_1 T \leq l \). Letting \( l' = \text{frank}_1 T \) there exist \( x_1, \ldots, x_{l'} \in Q_1 \) and functions \( a_1, \ldots, a_{l'} : Q_1 \rightarrow \mathbb{F} \) such that

\[
T(x, y, z, w) = \sum_{i=1}^{l'} a_i(x) T_{xi}.
\]

By this decomposition we have \((1 \times sr)T \leq \sum_{i=1}^{l'} \text{sr} T_{xi} \), so there exists \( i \in [l'] \) such that \( \text{sr} T_{xi} \geq G_{3, sr}(l) \). By Proposition \ref{prop:minor_bound} there exist \( Y \subseteq Q_2, Z \subseteq Q_3, W \subseteq Q_4 \) with size at most \( F_{3, sr}(l) \) such that \( \text{sr} T_{xi}(Y \times Z \times W) \geq l \). Letting \( X = \{x_i\} \) we obtain \((1 \times sr)T(X \times Y \times Z \times W) \geq l \).

\[ \square \]

10 **Additional difficulties for order-4 tripartition rank minors**

For \( T \) an order-4 tensor, let the **tripartition rank** \( \text{trp} T \) of \( T \) be the value of \( \text{Rrk} T \) for \( R \) the set of partitions of \{1, 2, 3, 4\} into three (non-empty) parts.

Throughout this section we prove Theorem \ref{thm:1} for \((d, R) = (4, \text{trp})\). The structure of the proof will be as follows. Let \( T \) be an order-4 tensor with large tripartition rank. Rather
than the two cases involved in the proofs of the previous section, the proof will distinguish between three cases (the second of our steps below is a preparation for the last two cases).

1. If the tensor $T$ has a large separated set of order-2 slices, without loss of generality of the type $T(x_i, y_i)$, then we can restrict them to a product $Z \times W$ with $Z, W$ of bounded size such that the restrictions are still separated, and containing the $(x_i, y_i)$ in a box $X \times Y$ with bounded size ensures that the tripartition rank of $T(X \times Y \times Z \times W)$ is large.

2. If the tensor $T$ does not have such a large separated set, then we find a bounded number of $(x_i, y_i)$ and of functions $A_i : Q_1 \times Q_2 \to \mathbb{F}$ such that we can approximate each slice $T_{x,y}$ by
   \[
   \sum_i A_i(x, y)T(x_i, y_i)(z, w)
   \]
   and similarly for the five other kinds of order-2 slices. Using a sequence of scales to define the approximation we can ensure that the functions $A_i$ can to be thought of as independent coordinates.

3. If at least one of the functions $A_i$ has high rank, or similarly for their analogues for one of the five other kinds of order-2 slices, then taking minors $X \times Y$ and $Z \times W$ such that $A_i(X \times Y)$ has high rank and the slices $T_{x,y}(Z \times W)$ are still separated with thresholds in a sequence of scales ensures that the tripartition rank of $T(X \times Y \times Z \times W)$ is large.

4. If all the functions $A_i$ have bounded rank, and similarly for their analogues for all five other kinds of order-2 slices, then we can decompose $T = S + U$ with $S$ a tensor with bounded tripartition rank and $U$ a tensor such that all slices $U_{(x,y)}$ with $(x, y) \in Q_1 \times Q_2$ have bounded rank. Using an equivalence between the tripartition rank and the (order-4) tensor rank we are then able to conclude by applying order-4 tensor rank minors.

We begin as usual with the relevant separation result. Because the tripartition rank is at least as big as the $(2, 2)$-partition rank, the following lemma follows from Lemma 9.4.

**Lemma 10.1.** Let $T$ be an order-4 tensor, and suppose that there exist $(x_1, y_1), \ldots, (x_l, y_l) \in Q_1 \times Q_2$ such that
   \[
   \operatorname{rk}(\sum_{i=1}^{l} a_i T(x_i, y_i)) \geq l(l - 1) + 1
   \]
   for every $a \in \mathbb{F}^l \setminus \{0\}$. Then $\operatorname{trp} T \geq l$.

We next want to obtain another condition which guarantees that a tensor has high tripartition rank and will allow us to conclude in the second of the three cases of the proof. To prepare for the proof of this condition we first prove a lemma that states that the order-2 slices of a tensor with bounded tripartition rank can be well-approximated by a linear combination of a bounded number of order-2 tensors.

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Lemma 10.2. Let \( l \geq 1 \) be a positive integer and let \( T \) be an order-4 tensor with \( \text{trp} \, T \leq l \). Then there exist functions \( A_1, \ldots, A_l : Q_1 \times Q_2 \rightarrow \mathbb{F} \), each with rank at most 1, and functions \( B_1, \ldots, B_l : Q_3 \times Q_4 \rightarrow \mathbb{F} \) such that

\[
\text{rk}(T_{x,y}) - \sum_{i=1}^{l} A_i(x,y)B_i \leq l
\]

for every \((x, y) \in Q_1 \times Q_2\).

Proof. There exists a decomposition

\[
T(x, y, z, w) = \sum_{i=1}^{r} a_i(x) b_i(y) c_i(z, w) + \sum_{i=1}^{r'} F_i(x, y, z, w)
\]

for some nonnegative integers \( r, r' \) with \( r + r' = \text{trp} \, T \leq l \), where for each \( i \in [r'] \), the function \( F_i \) is of one of the five types

\[
\begin{align*}
& a(x)b(z)c(y, w), \\
& a(x)b(w)c(y, z), \\
& a(y)b(z)c(x, w), \\
& a(y)b(w)c(x, z), \\
& a(z)b(w)c(x, y).
\end{align*}
\]

We take \( A_i = a_i \) and \( B_i = c_i d_i \) for each \( i \in [r] \). For any given \((x, y) \in Q_1 \times Q_2\) and each \( i \in [r'] \) the rank of \((F_i)_{(x,y)}\) is at most 1, so the desired inequality follows by subadditivity. \( \square \)

We can now deduce our second condition that guarantees that an order-4 tensor has high tripartition rank.

Proposition 10.3. Let \( M, m, l', l \geq 1 \) be four integers, let \( T : Q_1 \times Q_2 \times Q_3 \times Q_4 \rightarrow \mathbb{F} \) be an order-4 tensor, and let functions \( A_1, \ldots, A_{l'} : Q_1 \times Q_2 \rightarrow \mathbb{F} \), functions \( B_1, \ldots, B_{l'} : Q_3 \times Q_4 \rightarrow \mathbb{F} \) be such that the three following conditions hold.

(i) For all \( a \in \mathbb{F}^{l'} \setminus \{0\} \), \( \text{rk}(\sum_{i=1}^{l'} a_i B_i) \geq M \).

(ii) There exists \( j \in [l'] \) such that \( \text{rk} \, A_j \geq l \).

(iii) For all \( y \in Q_1 \times Q_2 \), \( \text{rk}(T_{(x,y)} - \sum_{i=1}^{l'} A_i(x,y)B_i) \leq m \).

If \( M \geq (m + l)l' \), then \( \text{trp} \, T \geq l \).

Proof. Assume for contradiction that \( \text{trp} \, T < l \). By Lemma 10.2 there exist functions \( C_1, \ldots, C_{l-1} : Q_1 \times Q_2 \rightarrow \mathbb{F} \) and functions \( D_1, \ldots, D_{l-1} : Q_3 \times Q_4 \rightarrow \mathbb{F} \) such that \( \text{rk} \, C_i = 1 \) for each \( i \in [l - 1] \) and such that

\[
\text{rk}(T_{(x,y)} - \sum_{i=1}^{l-1} C_i(x,y)D_i) \leq l - 1
\]

for every \((x, y) \in Q_1 \times Q_2\). Then \( \text{rk} \, C_i \leq 1 \) for each \( i \in [l - 1] \), but by assumption \( \text{rk} \, A_j \geq l \), so by subadditivity of the rank, \( A_j \) does not belong to the linear span of

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$C_1, \ldots, C_{l-1}$. It follows that there exists a function $u : Q_1 \times Q_2 \to F$ supported in a subset $U$ of $Q_1 \times Q_2$ with size at most $l$ such that $u.C_i = 0$ for each $i \in [l-1]$ but $u.A_j \neq 0$. We can write $u.T = \sum_{(x,y) \in U} u(x,y)T(x,y)$. On the one hand, applying (12), subadditivity, and the fact that $u.C_i = 0$ for each $i \in [l-1]$ we obtain

$$\text{rk}(u.T) \leq (l-1)|U| \leq (l-1)l. \quad (13)$$

On the other hand, by assumption (iii) and subadditivity of the rank, we have

$$\text{rk}(u.T - \sum_{i=1}^{\nu}(u.A_i)B_i) \leq m|U| \leq ml.$$ 

Since $u.A_j \neq 0$, by applying assumption (i) to $(a_1, \ldots, a_\nu) = (u.A_1, \ldots, u.A_\nu)$ we have $\text{rk} \sum_{i=1}^{\nu}(u.A_i)B_i \geq M$, and therefore by subadditivity

$$\text{rk}u.T \geq M - ml.$$ 

From this inequality and (13) we obtain $(l-1)l \geq M - ml$ and therefore $M \leq (m+l-1)l$, a contradiction. \hfill \square

In order to conclude in the third case of our argument we prove an equivalence between the tripartition rank and the tensor rank for order-4 tensors.

**Proposition 10.4.** Let $T$ be an order-4 tensor such that $\text{rk}T_{(x,y)} \leq m$, $\text{rk}T_{(x,z)} \leq m$, $\text{rk}T_{(x,w)} \leq m$, $\text{rk}T_{(y,z)} \leq m$, $\text{rk}T_{(y,w)} \leq m$, $\text{rk}T_{(z,w)} \leq m$ for respectively all $(x, y) \in Q_1 \times Q_2$, all $(x, z) \in Q_1 \times Q_3$, all $(x, w) \in Q_1 \times Q_4$, all $(y, z) \in Q_2 \times Q_3$, all $(y, w) \in Q_2 \times Q_4$, all $(z, w) \in Q_3 \times Q_4$. If $\text{trp} T \leq l$, then $\text{tr} T \leq (m+l)l^2$.

**Proof.** We start with a decomposition

$$T(x, y, z, w) = \sum_{i=1}^{r_{12}} A_{12,i}(x, y)B_{12,i}(z, w) + \sum_{i=1}^{r_{13}} A_{13,i}(x, z)B_{12,i}(y, w) + \sum_{i=1}^{r_{14}} A_{14,i}(x, w)B_{12,i}(y, z) + \sum_{i=1}^{r_{23}} A_{23,i}(y, z)B_{12,i}(x, w) + \sum_{i=1}^{r_{24}} A_{24,i}(y, w)B_{12,i}(x, z) + \sum_{i=1}^{r_{34}} A_{34,i}(z, w)B_{12,i}(x, y)$$

where the functions $A_{12,i}, A_{13,i}, A_{14,i}, A_{23,i}, A_{24,i}, A_{34,i}$ all have rank 1, and $r_{12} + r_{13} + r_{14} + r_{23} + r_{24} + r_{34} = \text{trp} T$.

The functions $A_{12,i}, i \in [r_{12}]$ are linearly independent, so we can find functions $A^*_{12,i}, i \in [r_{12}]$ all supported inside a subset $U \subset Q_1 \times Q_2$ such that $A^*_{12,i}A_{12,j} = \delta_{i,j}$ for every $i, i' \in [r_{12}]$. Applying $A^*_i$ to the decomposition we obtain that for each $i \in [r]$, $\text{rk} B_{12,i} \leq r_{12}m + (r_{13} + r_{14} + r_{23} + r_{24} + r_{34})l$, and similar bounds for the ranks of the functions $B_{13,i}, B_{14,i}, B_{23,i}, B_{24,i}, B_{34,i}$. The result follows by subadditivity. \hfill \square

We are now ready to deduce Theorem 1.4 for the tripartition rank of order-4 tensors.
Proposition 10.5. Theorem 1.4 holds for \((d, R) = (4, \text{trp})\), and for each \(l \geq 2\) we may take
\[
F_{4, \text{trp}}(l) = G_{4, \text{trp}}(l) = 300l^{3l+9}.
\]

Proof. Let \(l \geq 2\) and let \(T\) be an order-4 tensor with \(\text{trp} T \geq 300l^{3l+6}\). We distinguish three cases, as follows.

Case 1: For at least one of the six types of order-2 slices \(T_{(x,y)}, T_{(x,z)}, T_{(x,w)}, T_{(y,z)}, T_{(y,w)}, T_{(z,w)}\), which without loss of generality we can assume to be of the first type, there exist \((x_1, y_1), \ldots, (x_l, y_l) \in Q_1 \times Q_2\) such that
\[
\text{rk}(\sum_{i=1}^{l} a_i T_{(x_i, y_i)}) \geq l^2
\]
for every \(a \in \mathbb{F}^l \setminus \{0\}\). By Proposition 2.5 we can find \(Z \subset Q_3, W \subset Q_4\) with size at most \(l^3\) such that
\[
\text{rk}(\sum_{i=1}^{l} a_i T_{(x_i, y_i)})(Z \times W) \geq l^2
\]
for every \(a \in \mathbb{F}^l \setminus \{0\}\). Letting \(X = \{x_i : 1 \leq i \leq l\}\) and \(Y = \{y_i : 1 \leq i \leq l\}\), the sets \(X, Y\) have size at most \(l\) and by Lemma 10.1 we get
\[
\text{trp} T(X \times Y \times Z \times W) \geq l.
\]

Let us now assume that we are not in Case 1. We here prepare Cases 2 and 3. For each of the six types of order-2 slices we can define a process as follows: we will here describe it for the first and second coordinates taken to be the fixed coordinates. If there exists \((x, y)\) such that
\[
\text{rk} T_{(x,y)} > l^2(l^l - 1)/(l - 1)
\]
then we let \((x_1, y_1) = (x, y)\), otherwise we stop the process. Thereafter, if there exists \((x, y)\) such that for all \(a_1 \neq 0,\)
\[
\text{rk}(T_{(x,y)} - a_1 T_{(x_1, y_1)}) > l^2(l^{l-1} - 1)/(l - 1)
\]
then we let \((x_2, y_2) = (x, y)\), and otherwise we stop. More generally, at the \(i\)th step of the process, if there exists \((x, y)\) such that for all \((a_1, \ldots, a_{i-1}) \neq 0,\)
\[
\text{rk}(T_{(x,y)} - (a_1 T_{(x_1, y_1)} + \cdots + a_{i-1} T_{(x_{i-1}, y_{i-1})})) > l^2(l^{l-i+1} - 1)/(l - 1)
\]
then we let \((x_i, y_i) = (x, y)\), and otherwise we stop.

The process must necessarily stop after at most \(l - 1\) iterations: if it did not, then we would have \((x_i, y_i), i = 1, \ldots, l\) satisfying \((14)\) for all \((a_1, \ldots, a_l) \in \mathbb{F}^l \setminus \{0\}\); in other words we would be in Case 1, which we have assumed that we are not. Let \(l' \leq l - 1\) be
the number of iterations after which the process stops. The family \( \{ T(x_i,y_i) : 1 \leq i \leq l' \} \) satisfies the following two properties: the inequality

\[
\text{rk}(\sum_{i=1}^{l} a_i T(x_i,y_i)) \geq l^2 (l^{l'-1} - 1)/(l-1)
\]

holds for every \( a \in \mathbb{F}^{l'} \setminus \{0\} \), and for each \((x,y) \in Q_1 \times Q_2\) there exists a (unique) element \((A_1(x,y), \ldots, A_{l'}(x,y)) \in \mathbb{F}^{l'}\) such that

\[
\text{rk}(T(x,y)) - \sum_{i=1}^{l'} A_i(x,y) T(x_i,y_i) \leq l^2 (l^{l'-1} - 1)/(l-1).
\]

We now distinguish two further cases (Cases 2 and 3 below) depending on whether for at least one of the six types of order-2 slices, there exists \( j \in [l'] \) such that \( A_j \) has high rank.

**Case 2:** For at least one of the six types of order-2 slices \( T(x,y), T(x,z), T(x,w), T(y,z), T(y,w), T(z,w) \), which without loss of generality we can assume to be the first type, there exists \( j \in \{1, \ldots, l'\} \) such that \( \text{rk} A_j \geq l \). By the standard result on minors of matrices there exist \( X, Y \) with size at most \( l \) such that \( \text{rk} A_j(X \times Y) \geq l \). Moreover, by (15) and the standard result on minors of matrices there exist \( Z, W \) with size at most \( l' \) such that

\[
\text{pr}(\sum_{i=1}^{l'} a_i T(x_i,y_i))(Z \times W) \geq l^2 (l^{l'-1} - 1)/(l-1)
\]

for every \((a_1, \ldots, a_{l'}) \in \mathbb{F}^{l'} \setminus \{0\}\). Let \( T' = T(X \times Y \times Z \times W) \). Applying Proposition 10.3 to the tensor \( T(X \times Y \times Z \times W) \), the functions \( A_i(X \times Y) \) for each \( i \in [l'] \), the functions \( B_i(Z \times W) \) for each \( i \in [l'] \), and the parameters \( m = l^2 (l^{l'-1} - 1)/(l-1) \) and \( M = l(m+l) = l^2 (l^{l'-1} - 1)/(l-1) \), we obtain that \( \text{trp} T(X \times Y \times Z \times W) \geq l \).

**Case 3:** We are not in Case 1 or Case 2. We define a decomposition

\[
T = S^{12} + S^{13} + S^{14} + S^{23} + S^{24} + S^{34} + U
\]

as follows. We define the tensor \( S^{12} \) by

\[
S^{12}(x,y,z,w) = \sum_{i=1}^{l_{12}} A_{i,12}(x,y) T(x_i,y_i)(z,w)
\]

where \( l_{12} \) is the value of \( l' \) obtained in the process above for the slices of the type \( T(x,y) \), and we define the tensors \( S^{13}, S^{14}, S^{23}, S^{24}, S^{34} \) similarly for the slices of the type \( T(x,z), T(x,w), T(y,z), T(y,w), T(z,w) \) respectively.

For each \( i \in \{1, \ldots, l_{12}\} \) we have \( \text{rk} A_i \leq l \), so by (15), \( \text{trp} S^{12} \leq l l^2 (l^{l'-1} - 1)/(l-1) \leq l^{l'+3} \). We similarly show that this upper bound holds for the tripartition ranks of the tensors \( S^{12}, S^{13}, S^{14}, S^{23}, S^{24}, S^{34} \).
Let \((x, y) \in Q_1 \times Q_2\) be fixed. Since \(l^2(l^l - 1)/(l - 1) \leq l^{l+2}\) we have \(rk(T - S_{12})_{x,y} \leq l^{l+2}\); moreover, \(rk S_{13}^{(x,y)}, rk S_{14}^{(x,y)}, rk S_{23}^{(x,y)}, rk S_{24}^{(x,y)}, rk S_{34}^{(x,y)}\) are at most \(trp S_{13}, trp S_{14}, trp S_{23}, trp S_{24}, trp S_{34}\) respectively, so are each at most \(l^{l+3}\), so by subadditivity \(rk U_{x,y} \leq 6l^{l+3}\). We similarly show that this upper bound holds for the ranks of the five other types of order-2 slices of \(U\).

Since \(trp T \geq 300l^{3d+9}\), by subadditivity \(trp U \geq 300l^{3d+9} - 6l^{l+3} \geq 294l^{3d+9}\), so \(tr U \geq k - 294l^{3d+9}\). Applying Proposition \([2.4]\) to \(U\) we obtain that for \(k\) large enough we can find \(X, Y, Z, W\) with size at most \(294l^{3d+9}\) such that

\[
tr U(X \times Y \times Z \times W) \geq (7l^{l+3})^2(6l^{l+3}).
\]

The slices \(U(X \times Y \times Z \times W)_{(x,y)}\) all have rank at most \(6l^{l+3}\), so applying Proposition \([10.4]\) we obtain that

\[
trp U(X \times Y \times Z \times W) \geq 7l^{l+3},
\]

and hence that

\[
trp T(X \times Y \times Z \times W) \geq l. \quad \square
\]

11 The inductive proof in the general case

11.1 Definitions

In this section we prove Theorem \([1.4]\) in full. For \(d \geq 2\) an integer and \(R\) a non-empty family of partitions of \([d]\), we shall say that a set \(C \subset [d]\) is a largest part for \(R\) if there exists \(P \in R\) with \(C \subseteq P\) and if moreover \(|C| = \max_{P \in R, I \in P} |I|\). We shall write \(R_{tr} = \{\{1\}, \ldots, \{d\}\}\) for the family of partitions corresponding to the tensor rank.

Our proof strategy is to proceed by induction, with the base case of the induction being the case \(R = R_{tr}\), for which we already know by Proposition \([2.4]\) that Theorem \([1.4]\) holds. Otherwise, we choose a largest part \(C \subset [d]\) for \(R\) that will remain fixed throughout the inductive step. The following notions of rank will be relevant to the proof:

\[
R_+ = \{P \in R : C \subseteq P\}
\]

\[
R_- = \{P \in R : C \not\subseteq P\}
\]

\[
R_{\text{comp}} = \{P \setminus \{C\} : P \in R_+\}
\]

\[
R_{\text{new}} = \{P \cup \{I, J\} : P \in R_{\text{comp}}, \{I, J\} \text{ a bipartition of } C\}
\]

\[
R' = R_- \cup R_{\text{new}}.
\]

We emphasize that although \(R_+, R_-, R', R_{\text{new}}\) are notions of rank for order-\(d\) tensors \(\prod_{\alpha=1}^d Q_\alpha \to \mathbb{F}\), \(R_{\text{comp}}\) is a notion of rank for tensors \(\prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F}\). We will refer to \(R'\) as the down-shadow of \(R\) (with respect to \(C\)). Informally, the set \(R'\) is the set \(R\), modified such that whenever \(C\) appears in a partition \(P\) of \(R\), the partition \(P\) is replaced by all partitions which are identical to \(P\) in all ways except that the part \(C\) is split into two non-empty parts. Starting from a non-empty family \(R\) of partitions on \([d]\), choosing
a largest part $C$ for $R$, building a down-shadow $R^1$ of $R$ with respect to $C$, then iterating by again choosing a largest part $C^1$ for $R^1$, building a down-shadow $R^2$ of $R^1$ with respect to $C^1$, and so forth we end up at the family $R_{tr}$ corresponding to the tensor rank after at most $2^d$ iterations, because every iteration forbids one additional subset of $[d]$ (the selected largest part) from belonging to any later family $R^i$, and the subsets of $[d]$ are forbidden by decreasing size.

We now summarise the structure of the inductive step, which can largely be viewed as a generalisation of the structure of the proof in Section 10. Let $R \neq R_{tr}$ be a non-empty family of partitions on $[d]$, and let $T$ be an order-$d$ tensor with large $R$-rank.

1. If the tensor $T$ has a large separated set of slices $T_{y_1}, \ldots, T_{y_l}$ with $y_1, \ldots, y_l \in \prod_{\alpha \in C^c} Q_{\alpha}$, then by applying Proposition 8.1 for $(|C|, pr)$ we can restrict this set of slices to a product $\prod_{\alpha \in C} X_{\alpha}$ such that the restrictions of the slices to this product are still separated and the sets $X_{\alpha}, \alpha \in C$ have bounded size. Containing $\{y_1, \ldots, y_l\}$ in a box $\prod_{\alpha \in C} X_{\alpha}$ with bounded size ensures that the $R$-rank of $T(\prod_{\alpha=1}^d X_{\alpha})$ is large.

2. If the tensor $T$ does not have such a large separated set, then we can find a bounded number of $y_i \in \prod_{\alpha \in C^c} Q_{\alpha}$ and functions $A_i : \prod_{\alpha \in C^c} Q_{\alpha} \to F$ such that for each $y \in \prod_{\alpha \in C^c} X_{\alpha}$ we can approximate $T_y$ by

$$\sum_i A_i(y)T_{y_i}$$

Using a sequence of scales to define the approximation we can ensure that the functions $A_i$ can be thought of as independent coordinates.

3. If at least one of the functions $A_i$ has high $R_{comp}$-rank, then by applying Theorem 1.4 for $(d-|C|, R_{comp})$ and Proposition 8.1 for $(|C|, pr)$ we can find $\prod_{\alpha \in C^c} X_{\alpha}$ and $\prod_{\alpha \in C} X_{\alpha}$, respectively, such that $A_i(\prod_{\alpha \in C^c} X_{\alpha})$ has high $R_{comp}$-rank and the slices $T_{y_i}(\prod_{\alpha \in C} X_{\alpha})$ are still separated in a sequence of scales. This ensures that the $R$-rank of $T(\prod_{\alpha=1}^d X_{\alpha})$ is large.

4. If all the functions $A_i$ have bounded $R_{comp}$-rank, then we have a decomposition $T = S + U$ where $S$ is a tensor with bounded $R$-rank and $U$ is a tensor such that every slice $U_y$ with $y \in \prod_{\alpha \in C^c} X_{\alpha}$ has bounded partition rank. We can then conclude using an equivalence statement between the $R$-rank and the $R'$-rank together with Theorem 1.2 for $R'$-rank.

Whenever $T : \prod_{\alpha=1}^d Q_{\alpha} \to F$ is an order-$d$ tensor with $Rrk T = k$, there exist nonnegative integers $r, r'$ with $r + r' = k$, functions $A_i : \prod_{\alpha \in C^c} Q_{\alpha} \to F$, $B_i : \prod_{\alpha \in C} Q_{\alpha} \to F$ with $R_{comp}rk A_i = 1$ for each $i \in [r]$, and for each $i \in [r']$ a function $F_i : \prod_{\alpha=1}^d Q_{\alpha} \to F$ with $Rrk F_i = 1$ such that for all $(x_1, \ldots, x_d) \in \prod_{\alpha=1}^d Q_{\alpha}$ we can write

$$T(x_1, \ldots, x_d) = \sum_{i=1}^r A_i(x(C))B_i(x(C)) + \sum_{i=1}^{r'} F_i(x). \quad (18)$$

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Furthermore the family \( \{A_1, \ldots, A_r\} \) is linearly independent: if this were not the case then without loss of generality \( A_r \) would be a linear combination of \( A_1, \ldots, A_{r-1} \) and we would be able to rewrite

\[
\sum_{i=1}^{r} A_i(x(C))B_i(x(C)) = \sum_{i=1}^{r-1} A_i(x(C))B'_i(x(C))
\]

for some functions \( B'_i : \prod_{\alpha \in C} Q_\alpha \rightarrow \mathbb{F} \) and hence obtain an \( R \)-rank decomposition of \( T \) with length strictly smaller than \( k \).

### 11.2 The general case for \( R \)-rank minors

In the case of a general non-empty family \( R \) of partitions, the formulation of the separation statement that we shall use will be the following.

**Lemma 11.1.** Let \( d \geq 2 \) be a positive integer, let \( R \neq R_{tr} \) be a non-empty family of partitions of \([d]\), and let \( C \) be a largest part for \( R \). Let \( l \geq 1 \) be a positive integer. If \( T \) is an order-\( d \) tensor and \( y_1, \ldots, y_l \in \prod_{\alpha \in C} Q_\alpha \) are such that

\[
\pr(\sum_{i=1}^{l} a_i T_{y_i}) \geq l(l - 1) + 1
\]

for every \( a \in \mathbb{F}^{l} \setminus \{0\} \), then \( Rrk T \geq l \).

**Proof.** Assume that \( Rrk T \leq l - 1 \). We consider an \( R \)-rank decomposition of \( T \) as in (18). Because the family \( \{A_1, \ldots, A_r\} \) is linearly independent and \( l \geq r+1 \), there exists a function \( u : \prod_{\alpha \in C} Q_\alpha \rightarrow \mathbb{F} \) supported inside \( \{y_1, \ldots, y_l\} \) such that \( u \neq 0 \) but \( u.A_i = 0 \) for each \( i \in [r] \). Applying \( u \) to the decomposition (18) we find that

\[
u.T = \sum_{i=1}^{r'} u.F_i(x).
\]

Let \( i \in [r'] \). There exists \( P \in R_- \) such that we can write

\[
F_i(x) = \prod_{I \in P} a_I(x(I))
\]

for some functions \( a_I : \prod_{\alpha \in I} Q_\alpha \rightarrow \mathbb{F} \). Because \( C \) is a largest part for \( R \) and \( Rrk F_i = 1 \), \(|C|\) is an upper bound on the sizes of all sets \( I \in P \); since \( C \notin P \), there exist distinct \( I_1, I_2 \in P \) such that \( C \cap I_1 \) and \( C \cap I_2 \) are both non-empty, so we have \( pr(F_i)_y \leq 1 \) for each \( y \in \prod_{\alpha \in C} Q_\alpha \). Hence \( pr u.F_i \leq l \) for each \( i \in [r'] \), and therefore \( pr u.T \leq l r' \leq l(l - 1) \). \( \square \)

To prepare for the proof of Lemma 11.3, we write the following approximation of the slices of a tensor with bounded \( R \)-rank.
Lemma 11.2. Let \( d \geq 2 \) be a positive integer, let \( R \neq R_{1d} \) be a non-empty family of partitions of \([d]\), and let \( C \) be a largest part for \( R \). Let \( l \geq 1 \) be a positive integer. If \( T \) is an order-\( d \) tensor with \( R \text{rk} T \leq l \) then there exist functions \( A_1, \ldots, A_l : \prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F} \) with \( R_{\text{comp}} \)-rank at most 1 and functions \( B_1, \ldots, B_l : \prod_{\alpha \in C} Q_\alpha \to \mathbb{F} \) such that

\[
\text{pr}(T_y - \sum_{i=1}^{l} A_i(y)B_i) \leq l
\]

for every \( y \in \prod_{\alpha \in C^c} Q_\alpha \).

Proof. We consider an \( R \)-rank decomposition of \( T \) as in (19). As shown in the proof of Lemma 11.1, for each \( i \in [r'] \) and each \( y \in \prod_{\alpha \in C^c} Q_\alpha \) we have \( \text{pr}(F_{i})_{y} \leq 1 \). The result follows by subadditivity of the partition rank.

We now deduce the following condition, which ensures that a tensor has high \( R \)-rank and which will be used in the second of the three main cases of our proof.

Lemma 11.3. Let \( d \geq 2 \) be a positive integer, let \( R \neq R_{1d} \) be a non-empty family of partitions of \([d]\), let \( C \) be a largest part for \( R \), and let \( M, m, l' \), \( l \geq 1 \) be four positive integers such that \( M \geq (m + l)l' \). Let \( T : \prod_{\alpha = 1}^{d} Q_\alpha \to \mathbb{F} \) be an order-\( d \) tensor, and suppose that \( A_1, \ldots, A_{l'} : \prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F} \) and \( B_1, \ldots, B_{l'} : \prod_{\alpha \in C} Q_\alpha \to \mathbb{F} \) are functions such that the three following conditions hold.

(i) For all \( a \in \mathbb{F}^{l'} \setminus \{0\} \), \( \text{pr}(\sum_{i=1}^{l'} a_iB_i) \geq M \).

(ii) There exists \( j \in [l'] \) such that \( R_{\text{comp} \text{rk}}(A_j) \geq l \).

(iii) For all \( y \in \prod_{\alpha \in C^c} Q_\alpha \), \( \text{pr}(T_y - \sum_{i=1}^{l'} A_i(y)B_i) \leq m \).

Then \( \text{rk} T \geq l \).

Proof. Assume for a contradiction that \( \text{rk} T < l \). Then by Lemma 11.2 there exist functions \( C_1, \ldots, C_{l-1} : \prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F} \) and \( D_1, \ldots, D_{l-1} : \prod_{\alpha \in C} Q_\alpha \to \mathbb{F} \) such that \( R_{\text{comp} \text{rk}} C_i = 1 \) for each \( i \in [l-1] \) and such that

\[
\text{pr}(T_y - \sum_{i=1}^{l-1} C_i(y)D_i) \leq l - 1 \tag{19}
\]

for every \( y \in \prod_{\alpha \in C^c} Q_\alpha \). Let \( j \) be given by assumption (ii). Because \( R_{\text{comp}} C_i \leq 1 \) for each \( i \in [l-1] \), but \( R_{\text{comp}} A_j \geq l \), the subadditivity of the \( R_{\text{comp} \text{rk}} \)-rank implies that \( A_j \) does not belong to the linear span of \( C_1, \ldots, C_{l-1} \), so there exists a function \( u : \prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F} \) supported in a subset \( U \) of \( \prod_{\alpha \in C^c} Q_\alpha \) with size at most \( l \) such that \( u.C_i = 0 \) for each \( i \in [l-1] \) but \( u.A_j \neq 0 \). We can write \( u.T = \sum_{y \in U} u(y)T_y \). On the one hand, applying (19), subadditivity, and the fact that \( u.C_i = 0 \) for each \( i \in [l-1] \) we obtain the bound

\[
\text{pr}(u.T) \leq (l-1)|U| \leq (l-1)l. \tag{20}
\]
On the other hand by assumption (iii) and by subadditivity of the partition rank, we have

$$\text{pr}(u.T - \sum_{i=1}^{\nu}(u.A_i)B_i) \leq m|U| \leq ml.$$  

Since \(u.A_j \neq 0\), by applying assumption (i) to \((a_1, \ldots, a_\nu) = (u.A_1, \ldots, u.A_\nu)\) we have \(\text{pr}\sum_{i=1}^{\nu}(u.A_i)B_i \geq M\), so by subadditivity

$$\text{pr}(u.T) \geq M - ml.$$  

From this inequality and (20) we find that \((l - 1)l \geq M - ml\), so \(M \leq (m + l - 1)l\), a contradiction.

We next formulate our equivalence statement between the \(R\)-rank and \(R'\)-rank.

**Proposition 11.4.** Let \(d \geq 2\) be a positive integer and let \(R \neq R_1\) be a non-empty family of partitions of \([d]\). Let \(C\) be a largest part for \(R\), let \(R'\) be the down-shadow of \(R\) with respect to \(C\), and let \(l, m \geq 1\) be positive integers. Let \(T : \prod_{\alpha=1}^{d} \mathbb{Q}_\alpha \rightarrow \mathbb{F}\) be an order-\(d\) tensor such that \(RrkT \leq l\). Assume that

$$\text{pr}T_y \leq m$$

for every \(y \in \prod_{\alpha \in C} \mathbb{Q}_\alpha\). Then \(R'rkT \leq l(ml + l^2 + 1)\).

**Proof.** We consider an \(R\)-rank decomposition of \(T\) with minimal length as in (18). Because \(\{A_1, \ldots, A_r\}\) is linearly independent we can find a subset \(U \) of \(\prod_{\alpha \in C} \mathbb{Q}_\alpha\) with size \(r \leq l\) and functions \(u_i : \prod_{\alpha \in C} \mathbb{Q}_\alpha \rightarrow \mathbb{F}\) for each \(i \in [r]\) such that \(u_i.A_{i'} = 1_{i=i'}\) for all \(i, i' \in [r]\). For each \(i \in [r]\), applying \(u_i\) to (18) we obtain

$$B_i = u_i.T - \sum_{i'=1}^{r'} u_{i}.F_{i'} = \sum_{y \in U} u_i(y)T_y - \sum_{i'=1}^{r'} \sum_{y \in U} u_i(y)(F_{i'})_y.$$  

By the assumption of the lemma we have that \(\text{pr}T_y \leq l\) for each \(y \in U\), so

$$\text{pr}\sum_{y \in U} u_i(y)T_y \leq |U|m \leq lm.$$  

For each \(i' \in [r']\), because \(RrkF_{i'} = 1\) we have \(\text{pr}(F_{i'})_y \leq 1\) (as shown in the proof of Lemma 11.1), and hence

$$\text{pr}\sum_{i'=1}^{r'} \sum_{y \in U} u_i(y)(F_{i'})_y \leq r'|C| \leq l^2.$$  

So by (21), \(pr B_i \leq lm + l^2\). For each \(i \in [r]\) we have \(RrkA_iB_i \leq 1\) and furthermore \(pr B_i \leq lm + l^2\), so \(R'rkA_iB_i \leq lm + l^2\). For each \(i' \in [r']\) we have \(RrkF_{i'} \leq 1\) so \(R'rkF_{i'} \leq 1\).

Using (18) and subadditivity we obtain \(R'rkT \leq r(lm + l^2) + r' \leq l((lm + l^2) + l \leq l(ml + l^2 + 1)\) as desired.  

\[51\]
The next lemma encapsulates an approximation process that we shall use in the proof of Proposition 11.6.

**Lemma 11.5.** Let \( d' \geq 2, l \geq 1 \) be positive integers, and let \( (T_y)_{y \in Y} \) be a family of order-\( d' \) tensors indexed by some finite set \( Y \). Let \( D : [l] \to \mathbb{R}_+ \) be a decreasing function. Assume that there do not exist \( y_1, \ldots, y_l \in Y \) satisfying

\[
pr\left( \sum_{i=1}^{l} a_i T_{y_i} \right) \geq D(l)
\]

for all \( a \in \mathbb{F}^l \setminus \{0\} \). Then there exist \( l' \in \{0, \ldots, l - 1\} \) and a subset \( Y' = \{y_1, \ldots, y_{l'}\} \) of \( Y \) such that the two following conclusions hold.

1. For all \( (a_1, \ldots, a_{l'}) \in \mathbb{F}^{l'} \setminus \{0\} \) we have \( pr(\sum_{i=1}^{l'} a_i T_{y_i}) \geq D(l') \).
2. For all \( y \in Y \), there exist \( a_1(y), \ldots, a_{l'}(y) \in \mathbb{F} \) such that \( pr(T_y - \sum_{i=1}^{l'} a_i(y) T_{y_i}) \leq D(l' + 1) \).

**Proof.** We construct the set \( Y' \) inductively as follows. If we can find \( y_1 \in Y \) such that \( pr(T_{y_1}) \geq D(1) \) then we continue, and otherwise we stop. Thereafter, if we can find \( y_2 \) such that

\[
pr(a_1 T_{y_1} + a_2 T_{y_2}) \geq D(2)
\]

for every \( (a_1, a_2) \in \mathbb{F}^2 \setminus \{0\} \) then we continue, and otherwise we stop. For each \( j \in [l], \) at the \( j \)th step, provided that we have continued up to this step, if we can find \( y_j \) such that

\[
pr\left( \sum_{i=1}^{j} a_i T_{y_i} \right) \geq D(j)
\]

for all \( (a_1, \ldots, a_j) \in \mathbb{F}^j \setminus \{0\} \) then we continue, and otherwise we stop. The process stops after at most \( l - 1 \) steps: if it did continue up to and including the \( l \)th step then since \( D \) is decreasing the resulting family \( \{y_1, \ldots, y_l\} \) would contradict the assumption. Let \( l' \) be the total number of iterations completed. That (i) is satisfied follows from the fact that \( D \) is decreasing, and that (ii) is satisfied follows from the criterion which led to stopping at step \( l' + 1 \).

**Proposition 11.6.** Let \( d \geq 2 \) be a positive integer, let \( R \neq R_{str} \) be a non-empty family of partitions of \([d]\), let \( C \) be a largest part for \( R \), and let \( R' \) be the down-shadow of \( R \) with respect to \( C \). If Proposition 8.7 holds for \((|C|, pr)\), Theorem 1.4 holds for \((d - |C|, R_{\text{comp}})\) and Theorem 1.4 holds for \((d, R')\) then Theorem 1.4 holds for \((d, R)\).

**Proof.** Let \( l \) be a fixed positive integer, and let \( k \) be a large positive integer that we will fix later depending on \( l \) and let \( T \) be an order-\( d \) tensor with \( RrkT \geq k \).

We distinguish three cases.
Case 1: There exist \( y_1, \ldots, y_l \in \prod_{\alpha \in C^c} Q_{\alpha} \) such that the order-\(|C|\) slices \( T_{y_1}, \ldots, T_{y_l} \) of \( T \) satisfy

\[
\text{pr}(\sum_{i=1}^{l} a_i T_{y_i}) \geq G_{|C|, \text{pr}, l}(l^2)
\]

for all \( a \in \mathbb{F}^l \setminus \{0\} \). By Proposition 8.1 for \((|C|, \text{pr})\) we can find sets \( X_\alpha : \alpha \in C \) all with size at most \( F_{|C|, \text{pr}, l}(l^2) \) such that

\[
\text{pr}(\sum_{i=1}^{l} a_i T_{y_i})(\prod_{\alpha \in C} X_\alpha) \geq l^2
\]

for every \( a \in \mathbb{F}^l \setminus \{0\} \). For each \( \alpha \in C^c \), let the set \( X_\alpha \) be the image of the canonical projection of \( \{y_1, \ldots, y_l\} \) on to the \( \alpha \)th coordinate. Then we obtain sets \( X_\alpha, \alpha \in C^c \) all with size at most \( l \). By Lemma 11.1 we conclude that \( \text{Rrk} T(\prod_{\alpha=1}^{l} X_\alpha) \geq l \).

If we are not in Case 1 then there exist \( y_1, \ldots, y_l \in \prod_{\alpha \in C^c} Q_{\alpha} \) and for each \( y \in \prod_{\alpha \in C^c} Q_{\alpha} \) coefficients \( a_1(y), \ldots, a_l(y) \in \mathbb{F} \), such that

\[
\text{pr}(T_y - \sum_{i=1}^{l} a_i(y) T_{y_i}) \leq G_{|C|, \text{pr}, l}(l^2).
\]  

(23)

Let \( D : [l] \to \mathbb{R}_+ \) be the decreasing function defined by \( D(l) = G_{|C|, \text{pr}, l}(l^2) \) and for each \( l' \in [l - 1] \), \( D(l') = G_{|C|, \text{pr}, l}(ID(l' + 1) + l^2) \). By Lemma 11.3 applied to \( Y = \prod_{\alpha \in C^c} Q_{\alpha} \) there exist \( l' \in \{0, \ldots, l - 1\} \) and \( y_1, \ldots, y_{l'} \in Y \) such that the two following statements hold.

(i) For all \( (a_1, \ldots, a_{l'}) \in \mathbb{F}^{l'} \setminus \{0\} \) we have the inequality \( \text{pr}(\sum_{i=1}^{l'} a_i T_{y_i}) \geq D(l') \).

(ii) For each \( y \in \prod_{\alpha \in C^c} Q_{\alpha} \) there exist \( A_1(y), \ldots, A_{l'}(y) \in \mathbb{F} \) satisfying the inequality

\[
\text{pr}(T_y - \sum_{i=1}^{l'} A_i(y) T_{y_i}) \leq D(l' + 1).
\]

Case 2: There exists \( j \in [l'] \) satisfying

\[
\text{R}_{\text{comp rk}} A_j \geq G_{d - |C|, \text{comp}}(l).
\]

By Theorem 11.3 for \((d - |C|, \text{comp})\) minors, there exist \( X_\alpha, \alpha \in C^c \) each with size at most \( F_{d - |C|, \text{comp}}(l) \) such that

\[
\text{R}_{\text{comp rk}} A_j(\prod_{\alpha \in C^c} X_\alpha) \geq l.
\]  

(24)

Moreover by (i) and Proposition 8.1 for \((|C|, \text{pr})\) there exist \( X_\alpha : \alpha \in C \) each with size at most \( F_{|C|, \text{comp}, l}(lD(l' + 1) + l^2) \) such that

\[
\text{pr}(\sum_{i=1}^{l'} a_i T_{y_i})(\prod_{\alpha \in C} X_\alpha) \geq lD(l' + 1) + l^2.
\]  

(25)
for every \((a_1, \ldots, a_{l'}) \in \mathbb{F}^l \setminus \{0\}\). By \((25)\), \((24)\) and \((ii)\) and applying Lemma 11.3 to the tensor \(T(\prod_{\alpha=1}^d X_\alpha)\), the functions \(A_i(\prod_{\alpha \in C^e} X_\alpha)\) for each \(i \in [l']\), the functions \(B_i(\prod_{\alpha \in C^e} X_\alpha)\) for each \(i \in [l']\), and the parameters \(m = D(l' + 1)\) and \(M = l(m + l)\), we obtain \(\text{Rrk} T' \geq l\).

**Case 3:** We are not in Case 1 and also not in Case 2. Let 
\[ S(x) = \sum_{i=1}^{l'} A_i(x(C^e))T_{y_i}(x(C)). \]
Since \(l' \leq l\) and for each \(i \in [l']\) we have \(\text{Rrk} A_i \leq G_{d-|C|, \text{Rcomp}}(l)\), we get \(\text{Rrk} S \leq lG_{d-|C|, \text{Rcomp}}(l)\). The tensor \(U = T - S\) is such that for each \(y \in \prod_{\alpha \in C^e} Q_\alpha\) the slice \(U_y\) has partition rank at most \(D(1)\). Moreover by subadditivity \(\text{Rrk} U \geq k - lG_{d-|C|, \text{Rcomp}}(l)\).

Since every tensor with \(R'\)-rank equal to 1 also has \(R\)-rank equal to 1, we have \(R'\text{rk} U \geq R\text{rk} U \geq k - lG_{d-|C|, \text{Rcomp}}(l)\). Applying Theorem 1.4 for \((d, R')\) to \(U\) we obtain that for
\[ k \geq G_{d,R'}(4(l + lG_{d-|C|, \text{Rcomp}}(l))^2D(1)) + lG_{d-|C|, \text{Rcomp}}(l) \]
we can find \(X_1, \ldots, X_d\) with size at most
\[ F_{d,R'}(4(l + lG_{d-|C|, \text{Rcomp}}(l))^2D(1)) \]
such that
\[ R'\text{rk} U(\prod_{\alpha=1}^d X_\alpha) \geq 4(l + lG_{d-|C|, \text{Rcomp}}(l))^2D(1). \]
Because \(U(\prod_{\alpha=1}^d X_\alpha)\) is a restriction of \(U\), it is still the case that the slices \(U(\prod_{\alpha=1}^d X_\alpha)_y\) with \(y \in \prod_{\alpha \in C^e} X_\alpha\) of this tensor all have partition rank at most \(D(1)\). Applying Proposition 11.4 we obtain
\[ \text{Rrk} U(\prod_{\alpha=1}^d X_\alpha) \geq l + lG_{d-|C|, \text{Rcomp}}(l) \]
and hence
\[ \text{Rrk} T(\prod_{\alpha=1}^d X_\alpha) \geq l. \]

**11.3 The general case for disjoint \(R\)-rank**

We begin by proving our equivalence result between the essential \(R\)-rank and the essential \(R'\)-rank.

**Proposition 11.7.** Let \(d \geq 2\) be a positive integer, let \(R \neq R_{tr}\) be a non-empty family of partitions of \([d]\), let \(C\) be a largest part for \(R\), and let \(d'\) be the size of \(C\). Let \(l, m \geq 1\) be positive integers. Let \(T : \prod_{\alpha=1}^d Q_\alpha \to \mathbb{F}\) be an order-\(d\) tensor such that \(\text{eRrk} T \leq l\). Assume that
\[ \text{epr} T_y \leq m \]
for every \(y \in (\prod_{\alpha \in C^e} Q_\alpha) \setminus E(C^e)\). Then \(\text{eR'rk} T \leq l^2(m + d'(d - d')) + l^3 + l\).
Proof. Because $eRrk T \leq l$ there exists a tensor $V$ supported inside $E$ such that $Rrk(T - V) \leq l$. We consider an $R$-rank decomposition of $T - V$ with minimal length, which by [18] we can write

$$
(T - V)(x) = \sum_{i=1}^{r} A_i(x(C^c))B_i(x(C)) + \sum_{i=1}^{r'} F_i(x)
$$

(26)

where $r + r' \leq l$, with $R_{\text{comp}}\text{rk} A_i = 1$ for each $i \in [r]$ and $R_{\text{rk}} F_i = 1$ for each $i \in [r']$. The restrictions $A_i((\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c))$ are linearly independent. (If they were not, then

$$
\text{Rrk}(T - V - \sum_{i=1}^{r} A_i(E(C^c))B_i) \leq (r - 1) + r' < l
$$

would hold and since for each $i \in [r]$ the support of the product $A_i(E(C^c))B_i$ is contained in $E(C^c) \times \prod_{\alpha \in C^c} Q_\alpha \subset E$, we would have $eRrk T < l$.) Therefore, we can find a subset $U$ of $(\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c)$ with size $r \leq l$ and functions $u_i : \prod_{\alpha \in C^c} Q_\alpha \to \mathbb{F}$ for each $i \in [r]$ with supports all contained inside $U$ (so in particular, contained outside $E(C^c)$) such that $u_i.A_{i'} = 1_{i=i'}$ for every $i, i' \in [r]$. For each $i \in [r]$, applying $u_i$ to (26) we obtain

$$
B_i = u_i.(T - V) - \sum_{i'=1}^{r'} u_i.F_{i'} = \sum_{y \in \mathcal{U}} u_i(y)(T - V)_y - \sum_{i'=1}^{r'} \sum_{y \in \mathcal{U}} u_i(y)(F_{i'})_y.
$$

(27)

By our assumption, for each $y \in \mathcal{U}$ we have $epr T_y \leq m$, so there exists an order-$d'$ tensor $V'^y$ supported in

$$
E(C) = \{ x(C) : x_{\alpha'} = x_{\alpha''} \text{ for some distinct } \alpha', \alpha'' \in C \}
$$

such that $\text{pr}(T_y - V'^y) \leq m$. The slice $V_y$ of $V$ has its support contained inside the union of $E(C)$ and of

$$
\{ x(C) : \text{ there exist } \alpha' \in C \text{ and } \alpha'' \in C^c \text{ such that } x_{\alpha'} = y_{\alpha''} \}.
$$

We can write $V_y = V_{y,\text{int}} + V_{y,\text{ext}}$ where $V_{y,\text{int}}$ and $V_{y,\text{ext}}$ are the restrictions of $V_y$ to these two respective sets. The tensor $V_{y,\text{ext}}$ has support contained in the union of $d'(d - d')$ order-$(d' - 1)$ slices of $\prod_{\alpha \in C} Q_\alpha$, so $\text{pr} V_{y,\text{ext}} \leq d'(d - d')$. The tensor $V'^{y} = V'^y - V_{y,\text{int}}$ is supported inside $E(C)$ and we have by subadditivity

$$
\text{pr}((T - V)_y - V'^y) \leq \text{pr}(T_y - V'^y) + \text{pr}(V_{y,\text{int}} - V_y) = \text{pr}(T_y - V'^y) + \text{pr}(V_{y,\text{ext}}) \leq m + d'(d - d')
$$

Hence, for each $y \in \mathcal{U}$ there exists a tensor $U'^y : \prod_{\alpha \in C} Q_\alpha \to \mathbb{F}$ such that we can write

$$
(T - V)_y = V'^y + U'^y,
$$

where $\text{pr} U'^y \leq m + d'(d - d')$. For each $i \in [r]'$, using (27) we get

$$
B_i = \sum_{y \in \mathcal{U}} u_i(y)(V'^y + U'^y) - \sum_{i'=1}^{r'} \sum_{y \in \mathcal{U}} u_i(y)(F_{i'})_y.
$$

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Substituting into (26) we obtain

\[(T - V)(x) = \sum_{i=1}^{r} A_i(x^{c^c}) \left( \sum_{y \in U} u_i(y)(V^{ny} + U^y)(x(C)) - \sum_{i'=1}^{r'} \sum_{y \in U} u_i(y)F_i'(y, x(C)) \right) + \sum_{i=1}^{r'} F_i(x) = T_1 + T_2 + T_3 + T_4,\]

where

\[T_1(x) = \sum_{i=1}^{r} A_i(x^{c^c}) \sum_{y \in U} u_i(y)V^{ny}(x(C))\]
\[T_2(x) = \sum_{i=1}^{r} A_i(x^{c^c}) \sum_{y \in U} u_i(y)U^y(x(C))\]
\[T_3(x) = -\sum_{i=1}^{r} A_i(x^{c^c}) \sum_{i'=1}^{r'} \sum_{y \in U} u_i(y)F_i'(y, x(C))\]
\[T_4(x) = \sum_{i=1}^{r'} F_i(x).\]

For each \(y \in U\), the support of \(V^{ny}\) is contained in \(E(C)\), so the support of \(T_1\) is contained in \((\prod_{a \in C^c} Q_a) \times E(C)\) and hence contained in \(E\). For each \(y \in U\) and each \(i \in [r]\) we have \(R_{comp} A_i \leq 1\) and \(pr U^y \leq m + d'(d - d')\), so \(R'rk T_2 \leq r|U|(m + d'(d - d')) \leq l^2(m + d'(d - d')).\) For each \(y \in U\), each \(i \in [r]\), and each \(i' \in [r']\) we have \(R_{comp} A_i \leq 1\) and \(pr(F_i') \leq 1\) (since \(R_{rk} F_i' \leq 1\), this last inequality holds for the same reason as that why it did in the the proof of Lemma 11.1), so \(R'rk T_3 \leq rr'|U| \leq l^3\). For each \(i' \in [r']\) we have \(R_{rk} F_i' \leq 1\) so \(R'rk F_i' \leq 1\), and therefore \(R'rk T_4 \leq r' \leq l\). It follows that \(T\) coincides outside of \(E\) with a tensor \((T_2 + T_3 + T_4)\) which has \(R'\)-rank at most \(l^2(m + d'(d - d')) + l^3 + l\), so \(eR'rk T \leq l^2(m + d'(d - d')) + l^3 + l\). \(\Box\)

From Proposition 11.7 we can deduce the following corollary, from which Proposition 6.7 and Proposition 6.8 from Section 6 can be deduced.

**Corollary 11.8.** Let \(d \geq 2\) be a positive integer, and let \(R \neq R_{tr}\) be a non-empty family of partitions of \([d]\). Let \(l \geq d^2\) be a positive integer. If \(T : \prod_{a=1}^{d} Q_a \rightarrow F\) is an order-\(d\) tensor such that \(eRrk T \leq l\) and furthermore \(etr T_y \leq l\) for every \(I \subset [d]\) with \(|I| \in \{1, \ldots, d - 2\}\) and for every \(y \in (\prod_{a \in I} Q_a) \setminus E(I^r)\), then \(etr T \leq (4l^3)^{2d}\).

**Proof.** We repeatedly apply Proposition 11.7 we let \(R_1 = R\), then inductively define families \(R_2, R_3, \ldots\) of partitions of \([d]\) as follows: as long as \(R_i \neq R_{tr}\) we choose \(C_i\) a largest part for \(R_i\) and obtain the down-shadow \(R_{i+1} = R_i'\) of \(R_i\) with respect to \(C_i\). We
stop the process when \( R_i = R_{tr} \), which necessarily occurs after at most \( 2^d \) iterations, since the \( i \)th iteration rules out the set \( C_i \) from all partitions of all \( R_{i'} \) with \( i' \geq i \), and the sets \( C_i \) have decreasing size. Using the assumption \( l \geq d^2 \) it is simple to check that whenever \( m \geq l \) we have \( l^2(m + d'(d - d')) + l^3 + l \leq 4l^2m \). The claim follows. \( \square \)

For \( d \geq 2 \) an integer and \( T : Q_1 \times \cdots \times Q_d \to \mathbb{F} \) an order-\( d \) tensor let
\[
Z(T) = \{ (x_1, \ldots, x_d) \in Q_1 \times \cdots \times Q_d : T(x_1, \ldots, x_d) \neq 0 \}
\]
be the support of \( T \), and let \( eZ(T) = Z(T) \setminus E \). In the proof of Proposition 11.10 we shall use the following generalisation of Proposition 1.5 to ordered \( d \)-uniform hypergraphs.

**Lemma 11.9.** Let \( T \) be an order-\( d \) tensor such that \( |eZ(T)| \geq k \). Then there exist \( X_1, \ldots, X_d \) pairwise disjoint such that
\[
|Z(T(X_1 \times \cdots \times X_d))| \geq k/d!.
\]

**Proof.** Without loss of generality we may assume that \( Q_1 = \cdots = Q_d \). Indeed, letting \( Q = \bigcup_{1 \leq a \leq d} Q_a \) we can extend \( T \) to a tensor \( T' \) supported on \( Q^d \) such that \( Z(T') = Z(T) \) and \( Z(T') \setminus E = Z(T) \setminus E \) by setting all new entries to take the value 0. Provided that the claim holds for \( T' \), with sets \( X_1', \ldots, X_d' \) we then deduce it for \( T \) by taking \( X_1 = X_1' \cap Q_1, \ldots, X_d = X_d' \cap Q_d \), since \( Z(T(X_1 \times \cdots \times X_d)) = Z(T'(X_1' \times \cdots \times X_d')) \).

We now assume that \( Q_1 = \cdots = Q_d = Q \). For each \( u \in Q \) we send \( u \) to a set \( X_{\alpha(u)} \) by choosing the \( \alpha(u) \) independently and uniformly at random. For each \( x \in Z(T) \setminus E \), the probability that \( x \in Z(T(X_1 \times \cdots \times X_d)) \) is equal to \( 1/d! \), so the expected size of \( Z(T(X_1 \times \cdots \times X_d)) \) is \( eZ(T)/d! \), and in particular for at least one of the choices of \( u \) this is the case. \( \square \)

The following proof will involve applying Proposition 8.3. However since it will be used only for the partition rank, we can assume as explained in Remark 8.4 that for any positive integers \( d \geq 2, s \geq 1 \) we have \( H_{d,pr,s} = 0 \) at the cost of increasing \( G'_{d,pr,s} \). Throughout the proof of the following proposition, the notation \( G'_{d,pr,s} \) will refer to the quantity obtained after the increase rather than before.

**Proposition 11.10.** Let \( d \geq 2 \) be a positive integer, let \( R \neq R_{tr} \) be a non-empty family of partitions of \([d]\), let \( C \) be a largest part for \( R \), and let \( R' \) be the down-shadow of \( R \) with respect to \( C \). If Proposition 8.3 holds for \(|C|, pr\), Theorem 7.8 holds for \((d - |C|, R_{comp})\), Theorem 7.4 holds for \((d - |C|, R_{comp})\), and Theorem 7.8 holds for \((d, R')\), then Theorem 7.8 holds for \((d, R)\).

**Proof.** Let \( l \) be a fixed positive integer, let \( k \) be a large positive integer that we shall fix later depending on \( l \), and let \( T \) be an order-\( d \) tensor with \( eR_{rk}T \geq k \). We distinguish three cases.

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Case 1: There exist \( y_1, \ldots, y_{dl'} \in (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) such that the order-\(|C|\) slices \( T_{y_1}, \ldots, T_{y_{dl'}} \) of \( T \) satisfy

\[
epr(\sum_{i=1}^l a_i T_{y_i}) \geq G'_{|C|, pr, d}(l^2) + d^2 l
\]

for every \( a \in \mathbb{F}^l \setminus \{0\} \). By Lemma 11.3 there exist pairwise disjoint \( X_\alpha, \alpha \in C^c \) such that \( \{y_1, \ldots, y_{dl'}\} \cap \prod_{\alpha \in C^c} X_\alpha \) contains at least \( l \) elements, which without loss of generality we can assume to be \( y_1, \ldots, y_l \). Furthermore (by requiring for each \( \alpha \in C^c \) the set \( X_\alpha \) to be the image of \( \{y_1, \ldots, y_l\} \) by the canonical projection on the \( \alpha \)th coordinate) we can require each of the \( X_\alpha, \alpha \in C^c \) to have size at most \( l \). By subadditivity

\[
epr(\sum_{i=1}^l a_i T_{y_i})(\prod_{\alpha \in C} X_\alpha) \geq G'_{|C|, pr, d}(l^2)
\]

for every \( a \in \mathbb{F}^l \setminus \{0\} \). By Proposition 8.3 for \(|C|, pr\) we can find pairwise disjoint subsets \( X_\alpha \subset Q_\alpha \setminus \cup_{\alpha' \in C^c} X_{\alpha'} \) for each \( \alpha \in C \) such that

\[
\text{pr}(\sum_{i=1}^l a_i T_{y_i})(\prod_{\alpha \in C} X_\alpha) \geq l^2
\]

for every \( a \in \mathbb{F}^l \setminus \{0\} \). By construction the sets \( X_\alpha, \alpha \in [d] \) are pairwise disjoint and by Lemma 11.1 we have \( \text{rk} T(\prod_{\alpha=1}^d X_\alpha) \geq l \).

If we are not in Case 1 then there exist \( y_1, \ldots, y_{dl'} \in (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) such that for each \( y \in (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) there exist coefficients \( a_1(y), \ldots, a_{dl'}(y) \in \mathbb{F} \) with

\[
epr(T_y - \sum_{i=1}^{dl'} a_i(y) T_{y_i}) \leq l^2. \tag{28}
\]

Let \( D : [dl'] \to \mathbb{R}_+ \) be the decreasing function defined by \( D(dl') = G'_{|C|, pr, d}(l^2) + d^2 l \) and \( D(l') = G'_{|C|, pr, dl}(lD(l'+1) + l^2) + d^2 F_{d-|C|, r_{comp}}(l) \) for each \( l' \in [dl'-1] \). By Lemma 11.5 (which also holds for the essential partition rank instead of the partition rank, as it suffices to replace the partition rank by the essential partition rank everywhere in its proof) applied to \( Y = (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) there exist \( l' \in \{0, \ldots, dl'-1\} \) and \( \{y_1, \ldots, y_{dl'}\} \in (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) such that the two following statements hold.

(i) For all \( (a_1, \ldots, a_{dl'}) \in \mathbb{F}^{dl'} \setminus \{0\} \) we have the inequality \( \text{epr}(\sum_{i=1}^{dl'} a_i T_{y_i}) \geq D(l') \).

(ii) For each \( y \in (\prod_{\alpha \in C^c} Q_\alpha) \setminus E(C^c) \) there exist \( A_1(y), \ldots, A_{dl'}(y) \in \mathbb{F} \) such that \( \text{epr}(T_y - \sum_{i=1}^{dl'} A_i(y) T_{y_i}) \leq D(l'+1) \).
Case 2: There exists $j \in [l']$ such that

$$eR_{\text{comp}}rk A_j \geq G'_{d-|C|, \text{comp}}(G_{d-|C|, \text{comp}}(l)).$$

By Theorem 1.4 for $(d - |C|, \text{comp})$, there exist pairwise disjoint $X_{\alpha, \text{pre}} \subset Q_\alpha$ for each $\alpha \in C^c$ such that

$$R_{\text{comp}}rk A_j(\prod_{\alpha \in C^c} X_{\alpha, \text{pre}}) \geq G'_{d-|C|, \text{comp}}(l).$$

By Theorem 1.8 for $(d - |C|, \text{comp})$, there exist subsets $X_\alpha \subset X_{\alpha, \text{pre}}$ for each $\alpha \in C^c$ all with size at most $F_{d-|C|, \text{comp}}(l)$ such that

$$R_{\text{comp}}rk A_j(\prod_{\alpha \in C^c} X_\alpha) \geq l. \quad (29)$$

By (i) and subadditivity

$$epr(\sum_{i=1}^{l'} a_i T_{y_i})(\prod_{\alpha \in C} (Q_\alpha \setminus \cup_{\alpha' \in C^c} X_{\alpha'})) \geq G'_{|C|, \text{pr}, \text{comp}}(lD(l' + 1) + l^2)$$

for every $(a_1, \ldots, a_{l'}) \in \mathbb{F}^{l'} \setminus \{0\}$. By Proposition 8.3 for $(|C|, \text{pr})$ there exist subsets $X_\alpha \subset Q_\alpha \setminus (\cup_{\alpha' \in C^c} X_{\alpha'})$ for each $\alpha \in C$ which are pairwise disjoint and such that

$$\text{pr}(\sum_{i=1}^{l'} a_i T_{y_i})(\prod_{\alpha \in C} X_\alpha) \geq lD(l' + 1) + l^2 \quad (30)$$

for every $(a_1, \ldots, a_{l'}) \in \mathbb{F}^{l'} \setminus \{0\}$. By (30), (29) and (ii) and applying Lemma 11.3 to the tensor $T' = T(\prod_{\alpha=1}^{d} X_\alpha)$, the functions $A'_i = A_i(\prod_{\alpha \in C^c} X_\alpha)$ for each $i \in [l']$, the functions $B'_i = B_i(\prod_{\alpha \in C^c} X_\alpha)$ for each $i \in [l']$ and the parameters $m = D(l' + 1)$ and $M = l(m + l)$, we obtain $R_{\text{rk}} T' \geq l$.

Case 3: We are not in Case 1, and also not in Case 2. Let $S$ be the tensor defined by

$$S(x) = \sum_{i=1}^{l'} A_i(x(C^c))T_{y_i}(x(C)).$$

Since for each $i \in [l']$ we have $eR_{\text{comp}}rk A_i \leq G'_{d-|C|, \text{comp}}(G_{d-|C|, \text{comp}}(l))$, we get

$$eR_{\text{rk}} S \leq dl^l G'_{d-|C|, \text{comp}}(G_{d-|C|, \text{comp}}(l)).$$

The tensor $U = T - S$ is such that for each $y \in (\prod_{\alpha \in C^c} X_\alpha) \setminus E(C^c)$ the $C$-slice $U_y$ satisfies $epr U_y \leq D(1)$. Moreover by subadditivity

$$eR_{\text{rk}} U \geq k - dl^l G'_{d-|C|, \text{comp}}(G_{d-|C|, \text{comp}}(l)).$$

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Since every tensor with $R'$-rank equal to 1 also has $R$-rank equal to 1, we have $eR'rk U \geq eRrk U \geq k - d!lG'_{d-[C],R_{\text{comp}}}(G_{d-[C],R_{\text{comp}}}(l))$. Applying Theorem 12.3 for $(d, R')$ to $U$ we obtain that for

$$k \geq G'_{d,R'} \left(4d^2(l + d!lG'_{d-[C],R_{\text{comp}}}(G_{d-[C],R_{\text{comp}}}(l)))^2 D(1) + d!lG'_{d-[C],R_{\text{comp}}}(G_{d-[C],R_{\text{comp}}}(l))\right)$$

we can find pairwise disjoint $X_1, \ldots, X_d$ such that

$$R'rk U \left(\prod_{\alpha=1}^{d} X_{\alpha}\right) \geq 4d^2(l + d!lG'_{d-[C],R_{\text{comp}}}(G_{d-[C],R_{\text{comp}}}(l)))^2 D(1).$$

Because $U\left(\prod_{\alpha=1}^{d} X_{\alpha}\right)$ is a restriction of $U$, it is still the case that the $C$-slices of this tensor all have essential partition rank at most $D(1)$. Applying Proposition 11.7 we obtain

$$Rrk U \left(\prod_{\alpha=1}^{d} X_{\alpha}\right) \geq l + d!lG'_{d-[C],R_{\text{comp}}}(G_{d-[C],R_{\text{comp}}}(l))$$

and hence

$$Rrk T \left(\prod_{\alpha=1}^{d} X_{\alpha}\right) \geq l. \quad \square$$

12 A simple minors argument in the case of rank powers

Before concluding we would like to devote a short section to discuss a technique which allows us to obtain reasonably good bounds for minors when the family of partitions can be factored in a simple way in this sense, whereas the slice rank and the partition rank cannot.

**Definition 12.1.** Let $D \geq 1$, $d_1, \ldots, d_D \geq 2$ be positive integers and let $R_1, \ldots, R_D$ be non-empty families of partitions of $[d_1], \ldots, [d_D]$, respectively. Let $R = R_1 \times \cdots \times R_D$ be the family of partitions of $\{(i, j) : 1 \leq i \leq d, 1 \leq j \leq D_d\}$ defined by $\{\{P_{1, \alpha} \cup \cdots \cup P_{d, \alpha}\} : P_{1, \alpha} \in R_{1, \alpha}, \ldots, P_{d, \alpha} \in R_{d, \alpha}\}$, where a partition $P_{i,j}$ of $[d_i]$ is identified with the corresponding partition of $\{i\} \times [d_i]$.

We shall use the following notion of flattening rank.

**Definition 12.2.** For $T : \prod_{\alpha=1}^{d_1} Q_{1,\alpha} \times \prod_{\alpha=1}^{d_2} Q_{2,\alpha} \rightarrow \mathbb{F}$ an order-$(d_1 + d_2)$ tensor, let the flattening rank of $T$, denoted by $frank T$, be the rank of the matrix $A : (\prod_{\alpha=1}^{d_1} Q_{1,\alpha}) \times (\prod_{\alpha=1}^{d_2} Q_{2,\alpha}) \rightarrow \mathbb{F}$ defined by

$$A((x_{1,1}, \ldots, x_{1,d_1}), (x_{2,1}, \ldots, x_{2,d_2})) = T(x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}).$$
We remark that if $T$ is such that $(R_1 \times R_2)\text{rk} T \leq k$ then $\text{frank} T \leq k$: this follows from checking that if $(R_1 \times R_2)\text{rk} T = 1$ then $\text{frank} T \leq 1$. The proof that we are about to start can be viewed as a generalisation of the proof of Proposition 9.7.

**Proposition 12.3.** Let $d_1, d_2 \geq 2$ be positive integers, and let $R_1, R_2$ be non-empty families of partitions of respectively $[d_1]$, $[d_2]$. If Theorem [1.4](#) holds for $(d_1, R_1)$ with bounds $F_{d_1, R_1}$ and $G_{d_1, R_1}$ and for $(d_2, R_2)$ with bounds $F_{d_2, R_2}$ and $G_{d_2, R_2}$ then it holds for $(d_1 + d_2, R_1 \times R_2)$ with bounds

$$F_{d_1 + d_2, R_1 \times R_2}(l) = \max (l, F_{d_1, R_1}(l), F_{d_2, R_2}(l))$$

$$G_{d_1 + d_2, R_1 \times R_2}(l) = lG_{d_1, R_1}(l)G_{d_2, R_2}(l).$$

**Proof.** Let $T$ be a tensor with $(R_1 \times R_2)\text{rk} T \geq lG_{d_1, R_1}(l)G_{d_2, R_2}(l)$. We distinguish two cases.

**Case 1:** We have $\text{frank} T \geq l$. Then letting $A$ be as above and using the standard result on minors of matrices, there exist subsets $X^1, X^2$ of $\prod_{\alpha=1}^{d_1} Q_{1, \alpha}$ and $\prod_{\alpha=1}^{d_2} Q_{2, \alpha}$, respectively, with size at most $l$, such that $\text{frank} A(X^1 \times X^2) \geq l$. Let $X_{1, \alpha}$ be the projection of $X^1$ on to the $(1, \alpha)$th coordinate axis for each $\alpha = 1, \ldots, d_1$, and similarly let $X_{2, \alpha}$ be the projection of $X^2$ on to the $(2, \alpha)$th coordinate axis for each $\alpha = 1, \ldots, d_2$. The tensor

$$T' = T(X_{1,1} \times \cdots \times X_{1,d_1} \times X_{2,1} \times \cdots \times X_{2,d_2})$$

does not satisfy $\text{frank} T' \geq l$ and hence

$$(R_1 \times R_2)\text{rk} T' \geq l.$$

**Case 2:** We have $\text{frank} T \leq l$. We then let $\ell' = \text{frank} T$. There exist tensors $T_{1,1}, \ldots, T_{1,\ell'} : \prod_{\alpha=1}^{d_1} Q_{1, \alpha} \to \mathbb{F}$ and tensors $T_{2,1}, \ldots, T_{2,\ell'} : \prod_{\alpha=1}^{d_2} Q_{2, \alpha} \to \mathbb{F}$ such that we can write

$$T(x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}) = \sum_{i=1}^{\ell'} T_{1,i}(x_{1,1}, \ldots, x_{1,d_1})T_{2,i}(x_{2,1}, \ldots, x_{2,d_2})$$

for all $x_{1,1} \in Q_{1,1}, \ldots, x_{1,d_1} \in Q_{1,d_1}, x_{2,1} \in Q_{2,1}, \ldots, x_{2,d_2} \in Q_{2,d_2}$. Moreover, the families of tensors $\{T_{1,1}, \ldots, T_{1,\ell'}\}$ and $\{T_{2,1}, \ldots, T_{2,\ell'}\}$ are both linearly independent (if they were not, then we would have $\text{frank} T < \ell'$). For each $i \in [\ell']$, by definition of the $(R_1 \times R_2)$-rank we have

$$(R_1 \times R_2)\text{rk}(T_{1,i}T_{2,i}) \leq R_1\text{rk} T_{1,i} R_2\text{rk} T_{2,i}$$

so by subadditivity

$$(R_1 \times R_2)\text{rk} T \leq \sum_{i=1}^{\ell'} R_1\text{rk} T_{1,i} R_2\text{rk} T_{2,i}.$$

Since $(R_1 \times R_2)\text{rk} T \geq lG_{d_1, R_1}(l)G_{d_2, R_2}(l)$ and $\ell' \leq l$, there exists $i \in [\ell']$ such that $R_1\text{rk} T_{1,i} \geq G_{d_1, R_1}(l)$ or $R_2\text{rk} T_{2,i} \geq G_{d_2, R_2}(l)$. Without loss of generality let us assume
that \( R_1 \text{rk} T_{1,i} \geq G_{d,R_1}(l) \). Because the family \( \{T_{2,1}, \ldots, T_{2,l'}\} \) is linearly independent there exists a function \( u : \prod_{\alpha=1}^{d_2} Q_{2,\alpha} \to \mathbb{F} \) supported inside a subset \( U \) of \( \prod_{\alpha=1}^{d_2} Q_{2,\alpha} \) with size at most \( l' \leq l \) such that \( u.T_{2,i'} = 1_{i'=i} \) for all \( i' \in [l'] \), so \( u.T = T_{1,i} \). For each \( \alpha \in [d_2] \) let \( X_{2,\alpha} \) be the projection of \( U \) on the \( (2, \alpha) \) th coordinate axis. The sets \( X_{2,\alpha} \) all have size at most \( l' \leq l \). By Theorem 1.4 for \( (d_1, R_1) \) there exist sets \( X_{1,\alpha}, \alpha \in [d_1] \) with size at most \( F_{d_1,R_1}(l) \) and such that

\[
R_1 \text{rk} T_{1,i}(X_{1,1} \times \cdots \times X_{1,d_1}) \geq l.
\]

Letting

\[
T' = T(X_{1,1} \times \cdots \times X_{1,d_1} \times X_{2,1} \times \cdots \times X_{2,d_2})
\]

we have \( R_1 \text{rk} u.T' \geq l \). This ensures that \( (R_1 \times R_2) \text{rk} T \geq l \): indeed if we can write

\[
T'(x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}) = \sum_{i=1}^r T_{1,i}'(x_{1,1}, \ldots, x_{1,d_1})T_{2,i}'(x_{2,1}, \ldots, x_{2,d_2})
\]

for some tensors \( T_{1,i}' : \prod_{\alpha=1}^{d_1} Q_{1,\alpha} \to \mathbb{F} \), \( T_{2,i}' : \prod_{\alpha=1}^{d_2} Q_{2,\alpha} \to \mathbb{F} \) and some positive integer \( r \) then

\[
u.T' = \sum_{i=1}^r T_{1,i}'(u.T_{1,i})
\]

and hence \( R_1 \text{rk} u.T' \leq r \).

Using induction on \( D \) by applying the previous proposition to \( R_1 \times \cdots \times R_{D-1} \) and \( R_D \) we obtain the following bounds.

**Corollary 12.4.** Let \( D \geq 1 \), \( d_1, \ldots, d_D \geq 2 \) be positive integers, let \( R_1, \ldots, R_D \) be families of non-empty partitions of respectively \( [d_1], \ldots, [d_D] \), and let \( R = R_1 \times \cdots \times R_D \). If for each \( j \in [D] \) Theorem 1.4 holds for \( (d_j, R_j) \) with bounds \( F_{d_j,R_j} \) and \( G_{d_j,R_j} \) then Theorem 1.4 holds for \( (d_1 + \cdots + d_D, R_1 \times \cdots \times R_D) \) with the bounds

\[
F_{d_1+\cdots+d_D,R_1\times\cdots\times R_D}(l) = \max(l, \max_{1 \leq j \leq D} F_{d_j,R_j}(l))
\]

\[
G_{d_1+\cdots+d_D,R_1\times\cdots\times R_D}(l) = l^{D-1} \prod_{j=1}^d G_{d_j,R_j}(l).
\]

**Corollary 12.5.** Let \( D \geq 1 \), \( d \geq 2 \) be positive integers and let \( R \) be a family of non-empty partitions of \( [d] \). If Theorem 1.4 holds with the bounds \( F_{d,R} \) and \( G_{d,R} \), then Theorem 1.4 holds with the bounds

\[
F_{Dd,R \otimes D}(l) = \max(l, F_{d,R}(l))
\]

\[
G_{Dd,R \otimes D}(l) = l^{D-1} G_{d,R}(l).
\]
13 Open problems

The results in this paper still leave open a number of related strengthenings. The bounds that we obtain for Theorem 1.4 are very probably suboptimal: the functions $F_{d,R}$ and $G_{d,R}$ that we obtain are merely those resulting from the current proof, and nothing particularly suggests that they are close to the optimal bounds. On the contrary, our guess would be that it is possible to take both $F_{d,R}(l)$ and $G_{d,R}(l)$ to be linear in $l$.

**Conjecture 13.1.** Let $d \geq 2$ be an integer and let $R$ be a non-empty family of partitions on $[d]$. Then there exist constants $A(d, R), B(d, R) > 0$ such that whenever $T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F}$ is an order-$d$ tensor with $\text{rank} T \geq l$, there exist $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ with size at most $A(d, R)l$ such that

$$\text{rank} T(X_1 \times \cdots \times X_d) \geq B(d, R)l.$$  

Aside from improving our current bounds, there are two additional statements involving minors that we would expect to be true. The first involves obtaining a minor of bounded size with the same rank as the original tensor, and the second involves obtaining a full-rank minor (that is, of rank equal to the sizes of each of the $d$ sets) of size tending to infinity with the rank of the original tensor.

**Conjecture 13.2.** Let $d \geq 2$ be an integer, and let $R$ be a non-empty family of partitions on $[d]$. Then there exists a function $F_{d,R,\text{same}} : \mathbb{N} \to \mathbb{N}$ such that whenever $T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F}$ is an order-$d$ tensor with $\text{rank} T \geq l$, there exist $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ with size at most $F_{d,R,\text{same}}(l)$ such that

$$\text{rank} T(X_1 \times \cdots \times X_d) \geq l.$$  

**Conjecture 13.3.** Let $d \geq 2$ be an integer, and let $R$ be a non-empty family of partitions on $[d]$. There exists a function $G_{d,R,\text{full}} : \mathbb{N} \to \mathbb{N}$ such that whenever $T : \prod_{\alpha=1}^{d} Q_{\alpha} \to \mathbb{F}$ is an order-$d$ tensor with $\text{rank} T \geq G_{d,R,\text{full}}(l)$, there exist $X_1 \subset Q_1, \ldots, X_d \subset Q_d$ with size at most $l$ such that

$$\text{rank} T(X_1 \times \cdots \times X_d) \geq l.$$  

It is worth noting that conjectures 13.1, 13.2 and 13.3 are still unproved for very simple cases, such as for the slice rank for order-3 tensors. As the methods in this paper involve losses rather quickly, we expect that new, more precise methods would be required to make progress on them: arguments with no losses at all in the direction where the bound is sharp will necessarily be involved in any attempt on conjectures 13.2 and 13.3. Still on the topic of strengthening Theorem 1.4 but in another direction, we can ask about the dependence of the bounds in the case of several tensors. We would expect that in Proposition 8.1 for fixed $d, R, l$ we can take the dependence of $F_{d,R,s}(l)$ to be linear in $s$ and that we can take $G_{d,R,s}(l)$ to be independent of $s$, as was shown in Proposition 2.7 to be the case for the tensor rank.
Conjecture 13.4. Let \( d \geq 2 \) be a positive integer, and let \( R \) be a non-empty family of partitions on \([d]\). For every positive integer \( l \geq 1 \), there exist quantities \( F_{d,R}(l) \), \( G_{d,R}(l) \) such that whenever \( s \geq 1 \) is a positive integer, if \( T_1, \ldots, T_s : \prod_{a=1}^d Q_a \to \mathbb{F} \) are order-\( d \) tensors such that

\[
Rrk(a.T) \geq G_{d,R}(l)
\]

for every \( a \in \mathbb{F}^s \) then there exist \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \) each with size at most \( s F_{d,R}(l) \) such that

\[
Rrk(a.T)(X_1 \times \cdots \times X_d) \geq l
\]

for every \( a \in \mathbb{F}^s \).

We note that the bound \( F_{d,R,s}(l) \leq F_{d,R}(sl) + F_{d,R,s-1}(l) \) resulting from the proof of Proposition 8.1 would give only a quadratic bound in \( s \) even if we assume that \( F_{d,R}(l) \) can indeed be taken to be linear in \( l \).

In the tensor rank case we were able to show that the optimistic bounds in the generalisation of the standard statement on minors of matrices holds indeed do hold. We can ask whether there is a simple characterisation of all non-empty families \( R \) of partitions on \([d]\) for which this is the case.

Question 13.5. Let \( d \geq 2 \) be an integer. For which families \( R \) of partitions of \([d]\) is it true that if \( T : \prod_{a=1}^d Q_a \to \mathbb{F} \) is an order-\( d \) tensor with \( RrkT = l \), then there exist \( X_1 \subset Q_1, \ldots, X_d \subset Q_d \) all with size \( l \) such that \( RrkT(X_1 \times \cdots \times X_d) = l \)?

Regarding Theorem 1.8 we can also ask for the optimal bounds on the value of the disjoint rank for a given essential rank.

Conjecture 13.6. Let \( d \geq 2 \) be an integer, and let \( R \) be a non-empty family of partitions on \([d]\). Then \( d Rrk(T) \geq (e Rrk(T))/d \) for every order-\( d \) tensor \( T \).

The example of the slice rank shows that a better bound cannot hold in general: if \( Q_1 = \cdots = Q_d = [n] \) for some large positive integer \( n \), then whenever \( X_1, \ldots, X_d \) are pairwise disjoint subsets of \([n]\) we have \( sr T(X_1 \times \cdots \times X_d) \leq \min(|X_1|, \ldots, |X_d|) \leq n/d \), but a simple counting argument in (say) \( \mathbb{F} = F_5 \) shows that there exist tensors \( T : [n]^d \to F_5 \) with essential slice rank at least \( n(1 - o(1)) \). We have briefly attempted the case \( d = 2 \) of Conjecture 13.6 but found it to be, perhaps unexpectedly, a problem of at least moderate difficulty already.

Conjecture 13.7. Let \( A : Q_1 \times Q_2 \to \mathbb{F} \) be a matrix. Then \( drk A \geq (erk A)/2 \).

One more question that seems natural to draw attention to regarding the main theorems of this paper is the extent to which we can relax the requirement that \( R \) be a family of partitions of \([d]\) to \( R \) being an arbitrary non-empty element of \( \mathcal{P}(\mathcal{P}([d])) \). Although this can lead to notions of rank that may be behave in somewhat surprising ways (for instance if \( d = 3 \) and \( R = \{(\{1,2\}, \{1,3\}), \{(1,2), \{2,3\}\}, \{(1,3), \{2,3\}\}\} \) then the tensor \( 1_{x=y=z} = 1_{x=y} 1_{x=z} \) has \( R \)-rank one), we believe that Theorems 1.3 and 1.8 are probably still true in this case, at least for tensors for which the rank is well-defined.
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