Thick braneworlds generated by a non-minimally coupled scalar field and a Gauss–Bonnet term: conditions for the localization of gravity

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Abstract

We consider warped five-dimensional thick braneworlds with four-dimensional Poincaré invariance originated from bulk scalar matter non-minimally coupled to gravity plus a Gauss–Bonnet term. The background field equations as well as the perturbed equations are investigated. A relationship between 4D and 5D Planck masses is studied in general terms. By imposing finiteness of the 4D Planck mass and regularity of the geometry, the localization properties of the tensor modes of the perturbed geometry are analysed to first order, for a wide class of solutions. In order to explore the gravity localization properties for this model, the normalizability condition for the lowest level of the tensor fluctuations is analysed. It is found that for the examined class of solutions, gravity in four dimensions is recovered if the curvature invariants are regular and the 4D Planck mass is finite. It turns out that both the addition of the Gauss–Bonnet term and the non-minimal coupling between the scalar field and gravity reduce the value of the 4D Planck mass compared to its value when the scalar field and gravity are minimally coupled and the Gauss–Bonnet term is absent. The above discussed analysis depends on the explicit form of the scalar field (through its non-minimal coupling to gravity), making necessary the construction of explicit solutions in order to obtain results in closed form, and is illustrated with some examples which constitute smooth generalizations of the so-called Randall–Sundrum braneworld model. These solutions were obtained by making use of a detailed singular perturbation theory procedure with respect to the non-minimal coupling parameter between the scalar field and gravity, a difficult task that we managed to perform in such a way that all the physically meaningful conditions for the localization of gravity are
fully satisfied. From the obtained explicit solutions, we found an interesting effect: when we consider a non-minimally coupled scalar–tensor theory, there arise solutions for which the symmetries of the background geometry are not preserved by the scalar matter energy density distribution. In particular, the value of the ‘5D cosmological constant’ of the asymptotically AdS5 spacetime (which is even with respect to the extra coordinate) gets different contributions at $-\infty$ and $+\infty$ from the asymptotic values of the self-interaction potential of the scalar field. Thus, an asymmetric energy density distribution of scalar matter gives rise to a spacetime which is completely even with respect to the fifth coordinate, in contrast to braneworld models derived from minimally coupled scalar–tensor theories, where both entities possess the same symmetry.

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1. Introduction

For over a decade, an interesting alternative to the standard Kaluza–Klein compactification paradigm has been put to test. The kind of scenarios based on this alternative are known as braneworld models [1–6] and allow for infinite extra dimensions, in contrast to the Kaluza–Klein idea, where extra dimensions are compactified to a very small size. The mentioned alternative requires the standard model (SM) fields to be trapped on a 4D hypersurface, called a 3-brane. Unlike ordinary matter, gravitons and exotic matter are allowed to propagate in the bulk of the higher dimensional manifold. Since gravity can propagate through all dimensions, the first important question concerning braneworld models is to check whether they give back standard 4D gravity on the brane.

In the thin braneworld model version [3–5], the branes are modelled by 4D delta functions, a mathematical complication that requires the fulfilment of the so-called Israel–Lanczos junction conditions, which, basically, dictate the way the brane must be embedded into the higher dimensional bulk to accommodate the matter degrees of freedom (SM particles) that are trapped on it. These junction conditions become much more complicated when adding to the setup higher curvature terms, for instance [7]. Moreover, in these models the curvature is singular at the location of the branes, a drawback from the gravitational point of view that can be healed in several ways.

On the other hand, thin brane models are just an idealization of the physical reality. Braneworlds, if they are to be considered as models for our world, have to be of finite thickness. Actually, at high enough energies, the SM particles might acquire a small (but non-negligible) momentum in the extra space. Indeed, the original braneworld idea put forth in [2] is consistent with a non-vanishing brane thickness $\sim m_{\text{EW}}^{-1}$ ($m_{\text{EW}}$ is the electro-weak energy scale)\(^4\). These more realistic alternatives to thin brane configurations are known, generically, as thick braneworlds, or, also, domain walls.

Thick branes might be generated in a variety of ways. One example is by replacing the delta functions in the action either by other distribution functions (see, for instance, [8] and [9]), or, alternatively, by self-interacting scalar fields minimally coupled to gravity [10–12]. Within the framework of thin (Randall–Sundrum) braneworlds, bulk scalar fields have been investigated, for instance, in [13–15], where the Einstein–Hilbert action for Randall–Sundrum branes has

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\(^4\) Electroweak interactions have been probed, precisely, at distances $m_{\text{EW}}^{-1}$. 

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been modified by considering self-interacting scalar fields non-minimally coupled to gravity. In [14], the resulting action has been further modified by adding a Gauss–Bonnet (GB) term, whereas cosmological applications have been considered in [15]. A possible influence of higher curvature terms in scalar-field-generated brane models has been studied in [16–26]. The thick braneworld approaches avoid solving the Israel–Lanczos junction conditions always present when studying braneworlds modelled by 4D delta functions (with or without the presence of scalar fields). Instead, a relevant differential equation must be solved either for the smooth distribution function or the scalar field of the setup. This task is, however, easier than facing the mathematical difficulties of solving the junction conditions when considering higher curvature terms in the setup.

In dimensions larger than 4 (in 5D, for instance), the usual Einstein–Hilbert action may be supplemented with higher order curvature corrections. For some special cases, these corrections lead to equations of motion with at most second-order derivatives of the metric with respect to spacetime coordinates [27]. A particular combination containing higher curvature terms which yields second-order differential equations is the well-known GB invariant:

$$R^2_{\text{GB}} = R^{A B C D} R_{A B C D} - 4 R^{A B} R_{A B} + R^2,$$

where $A, B, C, D = 0, 1, 2, 3, 5$. While in four dimensions the GB invariant is a topological term—in other words, it can be arranged into a four-divergence—which does not contribute to the classical equations of motion, in more than 4D the GB combination leads to a theory free of spin-2 ghosts due to higher derivatives and it appears in different higher dimensional contexts. For instance, in string theory, the heterotic string $\alpha$ correction is fixed to be the GB term in order to avoid a ghost to that order in the (tree-level) effective action [28, 29]. However, in a second-order derivative theory, there is a sector that might contain spin-2 ghosts that are not originated by higher derivatives. If we wish to study braneworlds free of such ghosts, we need to carefully monitor the overall sign of the norm of the graviton spectrum. Within the context of thick braneworld scenarios generated by a self-interacting minimally coupled scalar field, the GB invariant has been studied in connection with the localization properties of the various modes of the geometry in [16] (see also, for instance, [26]). Non-minimal interaction of the bulk (self-interacting) scalar field has been considered, for instance, in [30], where the perturbative stability of the configurations was explored, and also within the framework of Weyl geometry in [31–34].

It seems quite natural to further explore the influence of the non-minimal coupling of a self-interacting scalar field with gravity, within the framework of thick braneworlds with higher curvature terms (a GB invariant, for instance). In this work, we aim, precisely, at studying a 5D thick braneworld modelled by a smooth scalar domain wall non-minimally coupled to gravity with a GB term on the bulk. To be precise we shall consider specifically conformally flat geometries. We shall focus here on the investigation of the relationship among the localization properties of the tensor zero mode, finiteness of 4D Planck mass and smoothness of the geometry, for a wide class of solutions. This will allow us to extend and generalize, in particular, previous results obtained within the framework of braneworlds with 4D Poincaré symmetry generated by scalar fields minimally coupled to gravity [35] (see also [36] for a similar approach within braneworlds with de Sitter 3-branes). In order to do that, we analyse the tensor perturbations of the geometry.

In general terms, the study of metric fluctuations in braneworld models is complicated because of the coupling between geometry and matter at the field equation level. On the
one hand, the perturbations problem is more tractable if the background metric has some isometries. In this case, the metric perturbations can be classified according to the symmetry of the problem. If the metric respects 4D Poincaré symmetry, then the different perturbation modes of the geometry can be classified in a gauge-invariant manner into scalar, vector and tensor modes under 4D Poincaré transformations. Furthermore, in [37, 30] it was shown in detail that each fluctuation mode evolves independently; then, if one studies the evolution of a specific mode, it is not necessary to worry about a possible coupling with the remaining perturbation modes. On the other hand, in order to recover the standard 4D gravity predicted by general relativity, the existence of a 4D massless spin-2 field localized on the brane is necessary. If there is a 4D massless spin-2 field in our model, it should be described by the propagating tensor modes of our 5D perturbation problem. Not only the gravitational perturbation equations, but also the background equations, are complicated and it is very difficult to analytically solve them. In spite of this difficulty, here we solve these equations for a non-singular geometry. We consider the effects of the non-minimal coupling of the bulk scalar field with the curvature as a small perturbation characterized by a small dimensionless parameter. Usually, in this kind of problem, there are regions—called boundary layer regions [38]—where the formal expansion with respect to the small parameter is not valid. Then, the expansion must be redefined for these regions. After having all the asymptotic expansions on the different regions, it is necessary to combine them to obtain an expansion valid on the entire domain.

The paper has been organized as follows. In section 2, we present the model setup and the corresponding field equations—for the respective metric ansatz—are given. In section 3, it is further shown that 4D gravity can be localized on this particular braneworld and the normalization condition for a massless spin-2 fluctuation mode is derived. The relationship between the 4D and 5D Planck masses is obtained in section 4, by integrating the action of the model with respect to the fifth dimension. In section 5, a class of solutions for which the geometry is singularity free is presented. Then, in section 6, we further construct some approximate analytical solutions for the scalar field after fixing the form of the warp factor, as well as the functional dependence of the non-minimal coupling of the scalar field to gravity. Three different cases are considered by appropriate choices of the free parameters. In the particular case when the GB term is switched off, an exact solution is obtained. In section 7, we present a brief summary of our results, and conclusions are given.

2. The model

Here we shall explore a thick braneworld described by the following 5D action (compare, for instance, with the actions investigated in [16] and [30]):

\[
S = \int d^5x \sqrt{-g} \left\{ \frac{L(\varphi) R}{2\kappa} - \alpha R_{\text{GB}}^2 + \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right\},
\]

where \(\alpha > 0\) (we choose the sign of the \(\alpha\) coupling to be positive, although it could have any sign). On the other hand, \(\kappa \approx 1/M^3\), where \(M\) is the 5D Planck mass, the function \(L(\varphi)\) is the coupling between the scalar field \(\varphi\) and the curvature (gravity), while \(V(\varphi)\) is the scalar field’s self-interaction potential and \(R_{\text{GB}}^2\) is the 5D GB term defined in equation (1). Einstein’s field equations that are derivable from action (2) are as follows:

\[
LR_{AB} = \kappa \tau_{AB} + \nabla_A \varphi \nabla_B L - \frac{1}{2} g_{AB} \Box L - \epsilon Q_{AB},
\]

where \(\epsilon = 2\alpha\kappa\) and \(\Box = \delta^{CD} \nabla_C \nabla_D\). The reduced energy–momentum tensor \(\tau_{AB}\) corresponding to the scalar matter content on the bulk takes the form

\[
\tau_{AB} = \delta_{AB} \varphi \varphi - \frac{1}{2} g_{AB} V(\varphi).
\]
The term $Q^{AB}$ is called the Lanczos tensor and represents the corrections of the GB term to the Einstein equations, which can be written in the form

$$Q^{AB} = \frac{1}{3} g^{AB} R_{\text{GB}}^{CD} - 2 R^{AB} + 4 R^{AC} R^{B}_{\text{GB}} + 4 R^{CD} R_{\text{GB}CD} - 2 R_{\text{GB}ACDE} R^{DE}_{\text{GB}}.$$ 

Finally, the remaining terms on the right-hand side of (3) come from the non-minimal coupling of the scalar field to gravity.

The Klein–Gordon equation determining the dynamics of the scalar field is obtained by varying action (2) with respect to $\phi$:

$$\Box \phi + \frac{1}{2\kappa} R L_{\phi} + \frac{\partial V}{\partial \phi} = 0,$$

where $L_{\phi} = \frac{dL}{d\phi}$.

As in [37] let us consider a warped metric in conformally flat coordinates,

$$ds^2 = a^2(w) [\eta_{\mu\nu} dx^\mu dx^\nu - dw^2],$$

where the variable $w$ is the extra coordinate and $\eta_{\mu\nu}$ is the 4D Minkowski metric. Here, for simplicity, we focus on the case where the scalar field depends only on the extra-coordinate $w$.

In terms of the above metric ansatz, the field equations (3) and (5) read

$$\psi'' + \frac{3\mathcal{H}\psi'}{\kappa a^2} + \frac{1}{2\kappa a^2} (\psi'' + \psi^2 L_{\psi}) - \frac{3}{2\kappa a^2} \mathcal{H}^2 (\mathcal{H}^2 + \mathcal{H}') - (\mathcal{H}' + 3\mathcal{H}^2) L = 0,$$

$$\psi'' + \frac{3\mathcal{H}\psi'}{\kappa a^2} - \frac{dV}{d\phi} a^2 = \frac{2L}{\kappa} (3\mathcal{H}^2 + 2\mathcal{H}') = 0,$$

respectively. Here the tilde denotes derivative with respect to the extra coordinate ($' = \frac{d}{dw}$), while $\mathcal{H} = \frac{a'}{a}$ and $q = L - \frac{a'}{2\kappa} \mathcal{H}^2$.

The above are not independent equations. As one can easily check, if one combines the first two equations, the last one is obtained.

3. Localization of gravity

As has been mentioned above, in linear perturbation theory there is no coupling between the different fluctuation modes, and hence, they evolve independently. In order to investigate if there is a 4D massless spin-2 field localized on the brane, it suffices to consider only the linear tensor fluctuations of the background metric (6),

$$ds^2 = \left[ a^2(w) \eta_{AB} + H_{AB} \right] dx^A dx^B,$$

where $H_{AB} = a^2(w) \left( \frac{2h_{\mu\nu}}{\sqrt{s}} \right)$.

Thus, the tensor $h_{\mu\nu}$ is a divergence-less and trace-less rank-2 tensor with respect to 4D Poincaré transformations [8, 37]. After taking this fact into account, perturbing equations (3) and (5), and performing some tedious algebra, the tensor fluctuation equation reads

$$q h_{\mu\nu}'' + (3\mathcal{H} q + q') h_{\mu\nu}' = \left[ q + \frac{(q - L)}{2\mathcal{H}} \right] \Box^n h_{\mu\nu} = 0,$$

where $\Box^n$ is the 4D (Minkowski) D’Alambertian. If we redefine the field $h_{\mu\nu}$ as $\Psi_{\mu\nu} = \sqrt{s(w)} h_{\mu\nu}$, equation (9) transforms into

$$\psi_{\mu\nu}'' - \frac{(\sqrt{s})''}{\sqrt{s}} \psi_{\mu\nu} - \frac{r}{s} \Box^n \psi_{\mu\nu} = 0,$$
where \( s(w) = a^3 q \) and \( r(w) = a^3 \left( q + \frac{(q-1)L}{2\Lambda} \right) \). In order to explore the mass spectrum of the metric fluctuations, we will assume separation of variables: \( \Psi_{\mu\nu} = \psi(w)\chi_{\mu\nu}(x) \), where the field \( \chi_{\mu\nu}(x) \) describes the 4D massive tensorial modes. Thus, equation (10) splits into two equations:

\[
\Box^9 \chi_{\mu\nu} + m^2 \chi_{\mu\nu} = 0,
\]

\[
\psi'' - \frac{\sqrt{s}}{\sqrt{s}} \psi' + m^2 r \psi = 0.
\]

The function \( \psi(w) \) is the \( w \)-dependent amplitude of the dynamical field \( \chi_{\mu\nu}(x) \) and defines the localization properties of the 5D field \( \Psi_{\mu\nu} \). The tensor zero mode of \( \chi_{\mu\nu}(x) \) can be identified with the 4D massless spin-2 field. This mode is localized on the brane if the associated zero-mode fluctuation wavefunction \( \psi_0(w) \) is normalizable; in other words, the norm for \( \psi_0(w) \) has to be finite,

\[
\langle \psi_0 | \psi_0 \rangle = \int_{-\infty}^{\infty} \frac{r}{s} \psi_0^2 \, dw < \infty.
\]

The massless eigenstate of (12) is \( \psi_0 = \sqrt{s} \); then the above normalization condition transforms into

\[
\langle \psi_0 | \psi_0 \rangle = \int_{-\infty}^{\infty} a^3 q \, dw - 4 \epsilon \int_{-\infty}^{\infty} \frac{a^2}{a} \, dw.
\]

It is interesting to ask: what is the relationship between the 4D Planck mass and the normalizability condition (13)? In the following sections, we will answer this question.

### 4. Planck masses

Consider a spacetime of the following form:

\[
dx^2 = a^2(w) [\tilde{g}_{\mu\nu}(x) \, dx^\mu \, dx^\nu - dw^2],
\]

where \( \tilde{g}_{\mu\nu}(x) \) is an arbitrary 4D metric. Then, the relationship between 4D and 5D Planck masses can be obtained if we perform a dimensional reduction by integrating (2) with respect to the \( w \) coordinate. Since the coupling function \( L(\phi) \) depends only on the extra coordinate \( w \), the 5D theory (2) is reduced to a 4D Einstein–Hilbert effective action plus the corrections that come from the scalar matter and higher curvature terms of the bulk,

\[
S_4 \simeq M_{Pl}^2 \int d^4x \sqrt{|\tilde{g}_4|} \tilde{R}_4 + \cdots,
\]

where the subscript 4 labels quantities computed with respect to the 4D metric \( \tilde{g}_{\mu\nu}(x) \).

A way to find the Einstein–Hilbert part of the 4D effective action consists in considering spacetime (15) as a conformal transformation of the metric \( \tilde{g}_{AB} \) as follows [39]:

\[
\tilde{g}_{AB} \rightarrow g_{AB} = a^2(w) \tilde{g}_{AB} = a^2(w) \begin{pmatrix} 0 & 1 \\ \frac{\tilde{g}_{\mu\nu}(x)}{a} & -1 \end{pmatrix},
\]

and rewriting action (2) in terms of the quantities defined with respect to \( g_{AB} \). Since we need only to find the Einstein–Hilbert part of the 4D effective action, it is only necessary to apply the above conformal transformation to the terms \( \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \) and \( a \tilde{R}_{GB}^2 \) in the 5D action. Therefore, the following expressions display the terms we need:

\[
R = a^{-2}(\tilde{R} - 8\tilde{\Box} \phi - 12(\tilde{\nabla} \phi)^2),
\]

\[
\tilde{R} = a^2(\tilde{R} + 8\tilde{\Box} \phi - 12(\tilde{\nabla} \phi)^2),
\]

\[
\tilde{R}_{GB} = a\tilde{R}_{GB},
\]

\[
\tilde{g}_{\mu\nu} = a^2 \tilde{g}_{\mu\nu}.
\]

\[
S_4 = M_{Pl}^2 \int d^4x \sqrt{|g_4|} R_4 + \cdots,
\]

where the subscript 4 labels quantities computed with respect to the 4D metric \( g_{\mu\nu}(x) \).
let us assume that the curvature invariants and warp factor should take care of the behaviour of the curvature invariants at spatial infinity. In consequence, and consider the class of solutions where asymptotically
\[ R_{AB} = \tilde{C}_A - \frac{1}{34} \left[ -8 \tilde{R}_{AB} \tilde{R}^{AB} + \frac{5}{2} \tilde{R}^2 - 12 \tilde{R} (\tilde{\nabla} \tilde{\nabla}) \tilde{R} + 48 \tilde{R} A \tilde{B} \tilde{A} \tilde{B} \tilde{R} + 48 \tilde{R} A \tilde{B} \tilde{A} \tilde{B} \tilde{R} + 72 (\tilde{\nabla}) \tilde{R} (\tilde{\nabla} \tilde{\nabla}) \tilde{R} - 72 (\tilde{\nabla} \tilde{\nabla}) (\tilde{\nabla} \tilde{\nabla}) \tilde{R} + 144 (\tilde{\nabla} \tilde{\nabla}) (\tilde{\nabla} \tilde{\nabla}) \tilde{R} \right]. \]
where \( \tilde{R} = \ln a \). The quantities with a tilde are defined with respect to the metric \( \tilde{g}_{AB} \) and \( \tilde{C}^2 = \tilde{C}^{ABCD} \tilde{C}_{ABCD} \), where \( \tilde{C}_{ABCD} \) is the Weyl tensor computed with the metric \( \tilde{g}_{AB} \).

After separating the 4D and 5D contributions of the above quantities, substituting them into (2) and integrating with respect to the \( w \) coordinate, we obtain the following expression for \( M_{Pl} \):
\[
M_{Pl}^2 \simeq M^3 \int_{-\infty}^{\infty} a^3(w) \left[ L(\psi) + \frac{4}{a^2} (\dot{H}^2 + 2H') \right] d\omega = M^3 \int_{-\infty}^{\infty} a^3(w) q d\omega + 8 M^3 \varepsilon [a']^\infty_{-\infty}. \tag{18}
\]
As one can see, the Planck mass \( M_{Pl} \) is closely related to \( \langle \psi_0 | \psi_0 \rangle \), but an extra term \( \int_{-\infty}^{\infty} \frac{a^2}{\omega^2} d\omega \) arises in (14). A clearer relation between \( M_{Pl} \) and \( \langle \psi_0 | \psi_0 \rangle \) can be obtained if we analyse the smoothness conditions for the curvature invariants.

5. Smoothness of geometry

A realistic thick braneworld model should not have singularities in the geometry. Since we are considering thick braneworlds, the corresponding warp factors, and hence the geometric invariants of the theory, must be smooth at the position of the branes. However, for some kinds of solutions, naked singularities can develop at the boundaries of the manifold [35, 36] and we should take care of the behaviour of the curvature invariants at spatial infinity. In consequence, let us assume that the curvature invariants and warp factor \( a \) are regular in the whole spacetime and consider the class of solutions where asymptotically
\[
a(w \to \infty) \simeq \frac{1}{w^\gamma}, \tag{19}
\]
with \( \gamma \) being a positive constant. For metric (6), the curvature invariants are
\[
\begin{align*}
R &= \frac{4}{a^2} (2H' + 3H^2), \\
R^{AB} R_{AB} &= \frac{4}{a^4} (5H^4 + 9H^2 + 6H' H^2), \\
R^{ABCD} R_{ABCD} &= \frac{4}{a^4} (4H^2 + 6H^2).
\end{align*}
\]
By calculating the asymptotic behaviour of the previous expressions at \( w \to \infty \), we obtain
\[
R \simeq w^{2(\gamma - 1)}, \quad R^{AB} R_{AB} \simeq R^{ABCD} R_{ABCD} \simeq w^{4(\gamma - 1)}.
\]
Thus, by imposing smoothness at infinity, the values of the constant \( \gamma \) are restricted to the interval \( 0 < \gamma \leq 1 \). Under this restriction for \( \gamma \), we obtain
\[
M_{Pl}^2 \simeq M^3 \int_{w_{\infty}}^{\infty} a^3 \omega d\omega + \cdots \quad \text{and} \quad \langle \psi_0 | \psi_0 \rangle \simeq \int_{w_{\infty}}^{\infty} a^3 \omega d\omega + \cdots, \tag{20}
\]
where \( \cdots \) denote finite terms and the interval \( (w_{\infty}, \infty) \) formally represents the range where approximation (19) is valid. Then, for this class of solutions where the geometry is regular, a finite 4D Planck mass implies the localization of gravity on the brane.
Figure 1. Behaviour of the different curvature invariants with respect to the extra space ($a_0 = b = 1$). The thick line represents the behaviour of $R^2$, the dashed line corresponds to $R^{ab}R_{ab}$ as a function of $x$, while the thin line represents the behaviour of the Kretschmann scalar $R^{abcd}R_{abcd}$.

In order to have a physically consistent model, it is not enough to satisfy the above conditions. Since some terms in relations (14), (19) and in the definition of $L(\phi)$ itself have contributions with negative sign, it is additionally required to check whether the following conditions are satisfied:

$$L(\phi) > 0, \quad \langle \psi | \psi \rangle > 0, \quad M_{\text{Pl}}^2 > 0,$$

where a positive value of the coupling $L(\phi)$ is required in order to have a positive definite quadratic Hamiltonian for the metric tensor modes [30], while the positive norm of the graviton fluctuation modes is required in order to have a theory free of spin-2 ghosts.

6. Some particular solutions

Let us consider the case where the coupling between the scalar field and gravity takes the form [13] (see also [30])

$$L = 1 - \frac{\xi}{2} \phi^2.$$

If the parameter $\xi = 0$, the model reduces just to the one explored in [16] (the bulk scalar field is minimally coupled to the curvature). Hence, the parameter $\xi$ switches between models with minimal and non-minimal couplings, respectively. On the other hand, we consider a regular metric that interpolates between two asymptotically AdS$_5$ [16] spacetimes. This geometry is described by the following warp factor:

$$a = \frac{a_0}{\sqrt{1 + (bw)^2}},$$

where $\frac{1}{2}$ characterizes the width of the thick brane and the parameter $a_0$ is related to the radius of the asymptotic AdS$_5$ space. For this geometry, all of the quadratic curvature invariants are regular and asymptotically constant, as shown in figure 1.

The scalar field $\phi$ and the self-interaction potential $V(\phi)$ can be determined using the first two equations of the system (7), since only two equations are independent. Let us replace the
variable $w$ by the dimensionless variable $x = bw$. In terms of this new variable, the second equation in (7) can be rewritten as

$$\xi \varphi'' + \frac{2\xi x}{1 + x^2}\varphi' + \varphi^2 (\xi - \kappa) - \frac{3\xi}{2} \frac{\varphi^2}{(1 + x^2)^2} = - \frac{3}{(1 + x^2)^2} + \frac{4\epsilon b^2}{a_0^3} \frac{3x^2}{(1 + x^2)^3}. \quad (24)$$

In general when $\xi \neq 0$, it is difficult to solve the above equation. A way to (approximately) solve it is to assume the parameter $\xi$ as a small perturbation, i.e. to apply a perturbative analysis. It will be easier and more transparent to split the analysis into three separated cases corresponding to different field configurations (theories):

(I) a GB term plus a scalar field minimally coupled to gravity ($\xi = 0$) [16];

(II) a scalar field non-minimally coupled to gravity without the GB term ($\epsilon = 0$);

(III) the general case where $\epsilon \neq 0$ and the small parameter $\xi \neq 0$.

**Case (I): Minimally coupled theory with the GB term.** If one sets $\xi = 0$ and considers the case where $a_0 = 2\sqrt{\epsilon b}$, the background solution is [37]

$$\varphi(x) = \pm \varphi_0 \frac{x}{\sqrt{1 + x^2}} + \varphi_1^0, \quad (25)$$

$$V(\varphi) = V_0 \left[ 3 \left( \frac{\varphi - \varphi_1^0}{\varphi_0} \right)^4 - 6 \left( \frac{\varphi - \varphi_1^0}{\varphi_0} \right)^2 + 1 \right], \quad (26)$$

where $\varphi_0 = \sqrt{\frac{2}{\epsilon}}$ and $V_0 = \frac{4}{\epsilon \pi^2}$. The $\pm$ sign describes two possible solutions of $\varphi$ with the constants $\varphi_1^0$. In this case, the 4D Planck mass does not depend explicitly on the parameters of the scalar field; moreover, definition (19) tells us that the addition of a GB term to a model with a scalar field minimally coupled to gravity in the action slightly reduces the value of the 4D Planck mass:

$$M_{Pl}^2 \sim \frac{4 M_\ast^3 a_0^3}{3} b, \quad (27)$$

comapred to its value when there is no GB term in the model

$$M_{Pl}^2 \sim \frac{2 M_\ast^3 a_0^3}{b}. \quad (28)$$

The above relation implies that the zero tensor mode is normalizable and all conditions (21) are trivially satisfied.

As can be seen from (26), the self-interaction potential of the scalar field $V(\varphi)$ interpolates between two identical constant values.

In cases (II) and (III), we will investigate solutions of (24) when the non-minimal coupling is small compared to the GB and Einstein–Hilbert contributions to action (2), characterized by the parameters $\epsilon$ and $\kappa$, respectively. Thus, it is convenient to make the quantities in (24) dimensionless since one wishes to study the field configurations independently of the choice of the units of measure. Let us consider the following redefinitions: $\varphi = \sqrt{\epsilon} \varphi$ and $\epsilon = \xi \ll 1$.

Under these new dimensionless variables, the field equation (24) takes the following form:

$$\epsilon \phi'' + \frac{2\epsilon x}{1 + x^2}\phi' + \phi^2 (\epsilon - 1) - \frac{3}{2} \frac{\phi^2}{(1 + x^2)^2} = - \frac{3}{(1 + x^2)^2} + \frac{4\epsilon b^2}{a_0^3} \frac{3x^2}{(1 + x^2)^3}. \quad (29)$$

In these two cases, we will assume that the solution can be formally expanded in powers of $\epsilon$:

$$\phi = \phi_0(x) + \epsilon \phi_1(x) + o(\epsilon \phi_1(x)). \quad (30)$$

If the first-order term $\epsilon \phi_1(x)$ in (30) is uniformly 'small' with respect to the zeroth-order $\phi_0(x)$ when $\epsilon \rightarrow 0$ over some region in the domain of variation $D$ of $\phi$, in other words, if $\epsilon \phi_1(x) = o(\phi_0(x))$ when $\epsilon \rightarrow 0$ over some region $\in D$, the expansion is called asymptotic expansion up to first order of the field on this region.
By substituting this expansion into (29), one can find approximate solutions of the field equation. However, a subtlety arises when we try to apply this naive procedure to (29) since in this case the term containing the second derivative is multiplied by the small perturbation parameter, namely, if one sets \( \varepsilon = 0 \) in (29), the differential equation that arises is no longer of second order but of first order. Therefore, it is not possible to generate all the approximate solutions to the field equation (29) with two arbitrary boundary conditions for \( \phi \) at \( x = \pm \infty \), because the differential equations for \( \phi_0 \) and \( \phi_1 \) will be of first order. Thus, the study of the boundary value problem poses a dilemma: there is only one unknown arbitrary constant but there are two fixed boundary conditions to be satisfied. This leads to what is generally known as a singular perturbation or boundary layer problem [38, 40–43].

In this kind of problem, in general, it is not possible to find a single uniformly valid asymptotic expansion for the field, in agreement with two arbitrary boundary conditions. What one can do is to propose two different asymptotic expansions for regions that contain the boundaries \( x = -\infty \) and \( x = \infty \), such that each one of them is uniformly valid in the corresponding region. Although, as we will see, in our case it is not enough to have two asymptotic expansions, we shall require another asymptotic expansion in the neighbourhood of the origin \( (x = 0) \).

In addition to the mentioned complications, in this kind of problem there is usually a region of rapid variation of the field and/or of its first derivatives, known as the boundary layer. In this region, the term that contains the second derivative in the field equation is no longer negligible; thus, expansion (30) and the corresponding solutions are not valid in this region.

In order to find an asymptotic expansion uniformly valid in a boundary layer located at \( x_b \), characterized in size by the function \( \delta(\varepsilon) \), it is suitable to magnify this region by rescaling the variable \( x \) with the aid of the stretched or boundary layer variable:

\[
\zeta = \frac{x - x_b}{\delta(\varepsilon)}, \quad \text{with} \quad \delta(\varepsilon) = o(1) \quad \text{when} \quad \varepsilon \to 0.
\]

In terms of this variable, the field transforms into

\[
\phi(x) = \phi(x_b + \delta(\varepsilon)\zeta) \equiv \Phi(\zeta).
\]

The next step is to consider a different expansion for \( \Phi(\zeta) \) that we hope to be uniformly valid on the boundary layer region. How to choose \( \delta(\varepsilon) \) in order to satisfy this condition will be discussed later.

On the other hand, in order to obtain an approximation on the whole domain \( D \), it is necessary to know how to match two different adjacent asymptotic expansions. The principal hypothesis of this method is to assume that there is an intermediate region where two different asymptotic expansions give the same result. By following [40], the idea consists in defining on the intermediate domain a new stretched variable:

\[
\zeta_0 = \frac{x - x_b}{\delta_0(\varepsilon)}, \quad \text{with} \quad \delta_0(\varepsilon) = o(1), \quad \delta(\varepsilon) = o(\delta_0(\varepsilon)) \quad \text{when} \quad \varepsilon \to 0,
\]

and rewriting the asymptotic expansions in terms of \( \zeta_0 \) up to some order\(^7\) and then, after expanding both of them, asking for their equality in terms of an arbitrary function \( \delta_0(\varepsilon) \) under the restrictions written in (33). This matching procedure serves us to fix the constants that appear in the expansion for \( \Phi(\zeta) \) on regions which are not connected with the boundary conditions.

A description of the solution on certain regions consists of two expansions which must be combined to form a composite expansion. This is done by adding the expansions and then

\(^7\) Not necessarily the same order for both asymptotic expansions.
Subtracting the part that is common to both, yielding an approximation to the solution on the above-mentioned region. After determining all the composite expansions uniformly valid on different regions, the resulting approximation to the solution valid on the whole interval can be obtained by joining together all of them.

**Case (II): Non-minimally coupled theory without the GB term.** In this case, equation (29) can be written as

$$\varepsilon \phi \phi'' + \frac{2 \varepsilon x}{1 + x^2} \phi \phi' + \phi'^2 (\varepsilon - 1) - \frac{3}{2} \phi^2 (1 + x^2)^2 = -\frac{3}{2} \phi^2 (1 + x^2)^2,$$

and the consistency conditions (21) can be reduced to the following pair of constraints:

$$L(\phi) = 1 - \frac{5}{2} \phi^2 > 0$$

and

$$M_{pl}^2 > 0.$$  

**(IIa) Case without boundary conditions.** These last restrictions do not require any initial or boundary conditions to be fixed. Thus, to begin with, let us study the class of approximate solutions generated by (30). By substituting this expansion into (34), it is not difficult to obtain the equations for the first and second approximations, $\phi_0(x)$ and $\phi_1(x)$, respectively:

$$\phi_0'^2 - \frac{3}{2} \phi_0^2 (1 + x^2)^2 = 0,$$

and

$$2 \phi_0'^2 - \frac{2x}{1 + x^2} \phi_0 \phi_0' - \phi_0 \phi_0'' - \phi_0'^2 + \frac{3}{2} \phi_0^2 (1 + x^2)^2 = 0.$$  

The above equations give us the following solutions:

$$\phi_0^\pm = \pm \sqrt{3} \arctan(x) + A_0^\pm,$$

$$\phi_1^\pm = \pm \frac{3}{4} \arctan(x) \mp \frac{3}{2} \arctan^2(x) - \frac{3}{2} A_0^\pm \arctan^2(x) + A_1^\pm,$$

where the signs $\pm$ mean two possible solutions for each order of approximation, and $A_0^\pm$ and $A_1^\pm$ are the arbitrary constants. One can check that the above solutions generate an asymptotic expansion $\phi^\pm = \phi_0^\pm + \varepsilon \phi_1^\pm$ on the whole domain $D = (-\infty, \infty)$ of the field variable. In figure 2, it is shown that there are configurations of the scalar field with a kink-like behaviour.
Figure 3. Self-interaction potential $V(\phi(x))$ up to first order in $\epsilon$ for case (IIa) (in all cases, we have chosen $\kappa = 1$, $a_0 = 1$, $b = 1$ and $\epsilon = 0.01$). The following values of the constant parameters have been chosen: for $\phi^+ = 0$—thin line, $\phi^- = 3$—thick line, while the dashed line corresponds to the choice $A^+_0 = 3$, for $\phi^-$. Moreover, in contrast to the previous case, the self-interaction potential, in general, interpolates between two different negative constant values, as shown in figure 3; these asymptotic values can be written as follows:

$$V^+(\pm \infty) \sim \frac{3b^2}{4a_0^2\kappa} \left[ -8 + \epsilon \left( 2A_0^+ \pm \sqrt{3} \pi \right)^2 \right],$$

$$V^-(\pm \infty) \sim \frac{3b^2}{4a_0^2\kappa} \left[ -8 + \epsilon \left( 2A_0^- \mp \sqrt{3} \pi \right)^2 \right],$$

where $V^+$ and $V^-$ correspond to the solutions $\phi^+$ and $\phi^-$, respectively. This asymmetric asymptotic behaviour of the matter energy density of the system may seem surprising if one takes into account the fact that the spacetime is asymptotically AdS 5. However, the uneven character of the energy distribution of the scalar matter at $-\infty$ and $+\infty$ is compensated by its non-minimal coupling to gravity, rendering an asymptotically AdS 5 spacetime which is even with respect to the fifth coordinate. Furthermore, up to first order in $\epsilon$, the 4D Planck mass is

$$M_{Pl}^2 \sim \frac{a_0^2M^3}{b} \left\{ 2 - \epsilon \left[ \frac{3}{4\pi^2} + (A_0^\pm)^2 - 6 \right] \right\}. \quad (39)$$

As we expected from (19), the above formula tells us that the effect of the non-minimal coupling between the scalar field and gravity reduces the value of the 4D Planck mass compared to its value when there is only minimal coupling. By taking into account the requirements $L(\phi) > 0$ and $M_{Pl}^2 > 0$ mentioned above, one can show that the values of $\epsilon$ are bounded,

$$\epsilon < \frac{2}{\sqrt{3} \frac{\pi^2}{4} + |A_0^\pm|^2}. \quad (40)$$

In figure 4, the function $L(\phi) = 1 - \frac{\pi}{2} \phi^2$ for several sets of values of the parameters is shown. As one can see, the condition $L(\phi) > 0$ is obviously satisfied.

As one can see from (37) and (38), after fixing the sign of the solutions, only one free boundary condition is left. In consequence, it is not possible to generate all of the approximate solutions of (34) by using only one expansion of $\phi$. 

Figure 4. The function $L(\phi) = 1 - \frac{\pi}{2} \phi^2$ for several sets of values of the parameters is shown.
Figure 4. Behaviour of the function $L(\phi)$—up to first order in $\varepsilon$—versus the variable $x$ for case (IIa). We have chosen three different sets of values of the free constants ($\varepsilon = 0.01$ in all cases): (i) $A_0^+ = 0, \phi_0^+$—thin line; (ii) $A_0^+ = 3, \phi_0^+$ (thick line); and (iii) $A_0^- = 3, \phi_0^-$—dashed line.

For arbitrary boundary conditions, it is difficult to solve (34). Hence, here we solve the field equation just for the case where the field vanishes at $x = \pm \infty$. These boundary conditions are interesting because these are the only boundary conditions that will appear in the next relevant case where $\xi \neq 0$ and $\varepsilon \neq 0$.

(IIIb) Case with imposed boundary conditions. Let us consider equation (34) under the following boundary conditions:

$$\phi(-\infty) = \phi(\infty) = 0.$$  

First of all, it is easy to show that if $\phi(x)$ is a solution of (34) under (41), then $\phi(-x)$ is a solution of the field equation too, with the same boundary conditions. This implies that the solution of the field equations under conditions (41) is an even function.

Let us solve perturbatively (34) under the boundary conditions (41). The expansion outside the boundary layers is described by solutions (37) and (38). As one can see, it is not possible to construct a single expansion which satisfies both boundary conditions at the same time. Therefore, we need to find asymptotic expansions valid on the boundaries. In general, this is a difficult task, but in our case we shall see that if one chooses any expansion $\phi^+$ or $\phi^-$ for a domain that contains the boundary $x = +\infty$ and the remaining solution for a region that contains the point $x = -\infty$, the constants that appear in the solutions can be fixed in such a way that the boundary conditions can be completely satisfied.

In what follows, we will denote with the index $\alpha$ the case where $\phi^-$ is defined on the region that contains the boundary $x = -\infty$ and $\phi^+$ on the region that contains the boundary $x = +\infty$; otherwise the subscript $\beta$ will be used. To begin with, let us discuss in detail the case $\alpha$. The other case is similar; therefore, only the final results will be presented.

By setting $A_0^+ = A_0^- = A_0 = -\frac{\sqrt{3}}{4} \pi$ and $A_1^+ = A_1^- = A_1 = \frac{\sqrt{3}}{4} \pi (\frac{\sqrt{3}}{8} - 1)$, we can write the approximate solution to (34) under (41) for regions that contain the boundaries in the form

$$\phi(x) = -\sqrt{3} \arctan(x) + A_0 - \frac{\sqrt{3}}{4} \varepsilon \left[ 2 \arctan(x) - A_0^2 \arctan(x) - \arctan^3(x) + \sqrt{3} A_0 \arctan^2(x) \right] + \varepsilon A_1, \quad \text{for all } x \in D^-,$$  

(42)
\( \phi_0^+ = \sqrt{3} \arctan(x) + A_0 + \frac{\sqrt{3}}{2} x \arctan(x) - A_0^2 \arctan(x) + \arctan^3(x) \)
\[ - \sqrt{3} A_0 \arctan^2(x) + \varepsilon A_1, \quad \text{for all } x \in D^+, \quad (43) \]
where the domains \( D^- \) and \( D^+ \) contain the points \( x = -\infty \) and \( x = +\infty \), respectively.

If one naively supposes that there are no intermediate boundary layers between the two boundary regions, the next step is to find a common region where the two expansions are valid. In our case, it is not difficult to find it because \( \phi_0^+ \) and \( \phi_0^- \) can be matched order by order at \( x = 0 \). Thus, the domains \( D^\pm \) can be taken as \( D^- = (-\infty, 0] \) and \( D^+ = [0, \infty) \), respectively.

These two domains allow us to define a general solution (valid on the entire domain) for the field. As one can see from (42) and (43), the latter solution is continuous but its derivative is not continuous at \( x = 0 \). In other words, on a neighbourhood of the origin, the first derivative of the field changes ‘rapidly’ from negative to positive values. Therefore, we have a boundary layer behaviour of the first derivative on a neighbourhood of the point \( x = 0 \).

The above analysis tells us that our initial hypothesis about the absence of intermediate boundary layers is false; then we need to find an asymptotic expansion valid on a neighbourhood of the origin. As we mentioned before, it is convenient to define a new variable
\[ \zeta = \frac{x}{\delta(\varepsilon)}, \quad \text{with} \quad \delta(\varepsilon) = o(1), \quad \text{when} \quad \varepsilon \to 0, \quad (44) \]
and to perform a new expansion for \( \Phi_0(x) = \Phi(\xi) \) over the domain that contains the origin \( x = 0 \):
\[ \Phi(\zeta) = \Phi_0(\zeta) + \varepsilon \Phi_1(\zeta). \quad (45) \]

In terms of the new variable (44), equation (34) can be rewritten as
\[ \varepsilon \Phi \frac{d^2 \Phi}{d\zeta^2} + 2 \varepsilon \delta \frac{\zeta}{1 + (\delta \zeta)^2} \frac{d\Phi}{d\zeta} + (\varepsilon - 1) \left( \frac{d\Phi}{d\zeta} \right)^2 - \frac{3}{2} \varepsilon \delta^2 \frac{\Phi^2}{(1 + (\delta \zeta)^2)^2} = 0. \quad (46) \]

By substituting (45) into the previous equation, we obtain
\[ - \left( \frac{d\Phi_0}{d\zeta} \right)^2 + \varepsilon \left[ \Phi_0 \frac{d^2 \Phi_0}{d\zeta^2} - 2 \frac{d\Phi_0}{d\zeta} \frac{d\Phi_1}{d\zeta} + \left( \frac{d\Phi_0}{d\zeta} \right)^2 \right] + \varepsilon \delta^2 \left[ 2 \varepsilon \Phi_0 \frac{d\Phi_0}{d\zeta} - \frac{3}{2} (\Phi_0)^2 \right] \]
\[ + \varepsilon^2 \left[ \Phi_0 \frac{d^2 \Phi_0}{d\zeta^2} + 2 \frac{d\Phi_0}{d\zeta} \frac{d\Phi_1}{d\zeta} + \Phi_1 \frac{d^2 \Phi_0}{d\zeta^2} - \left( \frac{d\Phi_1}{d\zeta} \right)^2 \right] + \cdots = -3 \delta^2 + \cdots. \quad (47) \]

In order to determine \( \Phi_0 \) and \( \Phi_1 \), we need to know how to choose \( \delta(\varepsilon) \). The idea is to make a well-balanced choice of the variable \( \xi \) or, equivalently, of the \( \delta(\varepsilon) \) function, such that the equations for the approximations \( \Phi_0 \) and \( \Phi_1 \) contain as much information as possible as \( \varepsilon \to 0 \). This selection is called the distinguished limit of the expansion (45) [38–42]. Let us put it in different words: if we choose \( \delta(\varepsilon) \) in a way different from the distinguished limit, all the terms that arise in the equations for \( \Phi_0 \) and \( \Phi_1 \) will be contained in the equations for these functions in the distinguished limit.

In our case we can show that \( \delta = \varepsilon \) corresponds to the distinguished limit of (45); therefore, by using (47) the equations for the zero- and first-order approximations of \( \Phi \) on the boundary layer region are
\[ \frac{d\Phi_0}{d\zeta} = 0, \quad (48) \]
By solving the above system of equations, the field $\Phi(\zeta)$ on the boundary layer domain adopts the form

$$\Phi(\zeta) = \Phi_0 + \varepsilon \left[ c_2 - \Phi_0 \ln \cosh \left( \sqrt{3} \frac{\zeta + c_1}{\Phi_0} \right) \right],$$

(50)

where $c_1$, $c_2$ and $\Phi_0$ are constants. These constants can be determined by matching the expansion of (42), up to some order, with the above solution on a negative neighbourhood of the point $x = 0$, and by matching the expansion of solution (43), up to some order, with (50) on a positive neighbourhood of the point $x = 0$, respectively. Let us next define the intermediate stretched variable

$$\zeta_0 = \frac{x}{\delta_0(\varepsilon)}, \quad \text{with} \quad \delta_0(\varepsilon) = o(1), \quad \varepsilon = o(\delta_0(\varepsilon)) \quad \text{when} \quad \varepsilon \to 0.$$  

(51)

By expanding the zeroth-order approximations of (42) and (43) and the boundary layer solution (50) in terms of $\delta_0(\varepsilon)$, we obtain

$$\Phi_0^\pm \sim \phi_0^\pm + \varepsilon(\pm \sqrt{3} c_1 + c_2 + \Phi_0 \ln 2),$$

(52)

where in the $\pm$ symbol, the $-$ sign stands for quantities defined on a negative neighbourhood of the origin; similarly, the $+$ sign denotes quantities defined on a positive neighbourhood of the origin.

By matching the above expansions, the following values for the constants $\Phi_0$, $c_1$ and $c_2$ are obtained:

$$\Phi_0 = -\sqrt{3} \pi \frac{\varepsilon}{2},$$

$$c_1 = 0,$$

$$c_2 = \sqrt{3} \pi \frac{\ln 2}{2}.$$  

Note that, up to this point, the description of our approximate solution is realized by three well-defined asymptotic expansions (which form two pairs), which can be joined together to obtain a solution that is valid on the entire domain $D = \{ -\infty < x < +\infty \}$. One pair of these asymptotic expansions is uniformly valid on the region $\{ x \leq 0 \} \in D$, while the other pair is uniformly valid on the region $\{ x \geq 0 \} \in D$. Thus, we can construct two composite expansions defined on the regions $x \leq 0$ and $x \geq 0$, respectively, that will be collected to generate an approximate solution defined on the whole domain of variation $D$.

As we mentioned above, the composite expansion for the region $x \leq 0$ (or $x \geq 0$) is obtained by adding two asymptotic expansions uniformly valid on the corresponding region and expressed in terms of the same variable, and then by subtracting the part that is common to both of them. In our case, it is convenient to construct the composite expansions for the positive and negative regions ($x \leq 0$ and $x \geq 0$) separately.

In other words, the composite expansion (either for $x \leq 0$ or $x \geq 0$) can be expressed as follows:

$$\phi_c^\pm(x) = \phi_0^\pm(x) + \Phi \left( \frac{x}{\varepsilon} \right) - \Phi_{\text{com}}(x),$$

(53)

8 In order to include the first-order approximation $\phi_1^\pm(x)$ in the matching process, it is necessary to calculate the contribution of $\varepsilon^2 \Phi_2(\zeta)$ to expansion (45). However, this case is analytically more complex and, thus, will not be considered here.
where $\phi_{\text{com}}(x)$ is the common part to both asymptotic expansions. It should be noted that in our case, by virtue of (52), the common functions for the expansions on the $x \leq 0$ and $x \geq 0$ regions can be described in a single functional form,

$$\phi_{\text{com}}(x) = \sqrt{3} \left( |x| - \frac{\pi}{2} \right). \quad (54)$$

Thus, the solution is

$$\phi^{a}(x) = \phi(x) = \sqrt{3} \left\{ \arctan |x| - \left( |x| + \frac{\pi}{2} \right) + \frac{\pi}{2} \varepsilon \ln \left[ 2 \cosh \left( \frac{2x}{\pi \varepsilon} \right) \right] \right\}. \quad (55)$$

By following the same scheme described above, we can show that $\phi^{a}(x) = -\phi^{b}(x)$.

As is shown in figure 5(a), both solutions are bounded, continuous even functions, with continuous derivatives on the whole domain $D$. In figure 5(b), we have zoomed $\phi^{b}(x)$ on a region near to the origin of coordinates to appreciate more clearly the smooth behaviour of this profile. Furthermore, like in case (I), the self-interaction potential interpolates between two identical negative constant values given by (see figure 6(a))

$$V(\infty) \sim -\frac{6b^{2}}{\alpha \kappa}. \quad (56)$$
On the other hand, the 4D Planck mass is

$$M_{Pl}^2 \sim a_0^3 \frac{M^3}{b} [2 - 3\varepsilon(\pi - 2)]$$ \hspace{1cm} (56)

again, the non-minimal coupling effects diminish $M_{Pl}^2$ with respect to a simpler model where the field is minimally coupled to gravity and the GB term is absent. By using $L(\phi) > 0$ and the above obtained result (56), the restriction for the values of $\varepsilon$ is

$$\varepsilon < \frac{8}{3\pi^2}.$$  

In figure 7(a), the function $L(\phi)$ is shown for both solutions. As we wished, the condition $L(\phi) > 0$ is satisfied.

Case (III): Non-minimal coupling and the GB term. Like in the first case, the relation $a_0 = 2\sqrt{\varepsilon}b$ is imposed; then the field equation (29) is transformed into

$$\varepsilon \phi \phi'' + \frac{2\varepsilon x}{1 + x^2} \phi \phi' + \phi^2 (\varepsilon - 1) - \frac{3}{2} \frac{\varepsilon \phi^2}{(1 + x^2)^2} = - \frac{3}{(1 + x^2)^3}.$$  

(57)

In this case, we shall modify the condition $\langle \psi | \psi \rangle > 0$ of (21) by a stronger and more tractable one, where the weight function $r/s$ is not negative. Under this redefinition of (21), the admissible profiles for $\phi$ are constrained by the following condition:

$$f(x) = \frac{1}{1 + x^2} - \frac{\varepsilon \phi^2}{x} \geq 0;$$  

(58)

then, when $x \to \infty$, the field vanishes. Based on this fact, we will study approximate solutions to (57) under the boundary conditions:

$$\phi(-\infty) = \phi(\infty) = 0.$$  

(59)

Of course, this boundary value problem is weaker than (57) under the restrictions (21) and (58). Thus, after finding the solutions, it is necessary to check whether such restrictions are fulfilled.

As mentioned above, we will find solutions of (57) in terms of expansion (30). By substituting such an expansion into (57), the equations for the zeroth and first approximations become

$$\phi_0'^2 - \frac{3}{(1 + x^2)^3} = 0,$$

$$2\phi_0'\phi_1' - \phi_0'^2 - \phi_0 \phi_0'' = \frac{2x \phi_0 \phi_0'}{1 + x^2} + \frac{3}{2} \frac{\phi_0^2}{(1 + x^2)^2} = 0.$$  

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and possess the following solutions:

\[ \phi_0^\pm = \pm \sqrt{3} \frac{x}{\sqrt{1+x^2}} + B_0^\pm, \]

\[ \phi_1^\pm = \frac{1}{4\sqrt{3}} \left\{ \pm \frac{21x}{\sqrt{1+x^2}} \mp 3 [5 + (B_0^\pm)^2] \arcsinh(x) - 4\sqrt{3}B_0^\pm \ln(1+x^2) \right\} + B_1^\pm, \]

where \( B_0^\pm \) and \( B_1^\pm \) are constants. As in the previous case, for each order of perturbation we have two solutions characterized by the signs \( \pm \). Since we have well-defined boundary conditions, we must set the values of the above constants. Again, with a single solution it is not possible to satisfy both conditions simultaneously. Then, motivated by case (II), we have two possibilities of finding asymptotic expansions which are valid on the boundaries. The first one where \( \phi^- \) is valid on a region that contains \( x = -\infty \), and \( \phi^+ \) is valid on a region that contains \( x = +\infty \). By following the same notation used in case (II), we will denote this choice by an \( \alpha \) index. The second one corresponds to choosing the expansions in reverse order and will be denoted by a \( \beta \) index. Let us consider the first case in detail.

Like in case (II), a boundary layer behaviour of the first derivative on the neighbourhood of the origin arises. In order to overcome this problem, we repeat the same procedure applied to case (II): we perform the change of variable (44) and express (57) in terms of it:

\[ \varepsilon \Theta \frac{d^2 \Theta}{d\zeta^2} + 2\varepsilon \delta^2 \frac{\zeta}{1+(\delta \zeta)^2} \Theta \frac{d\Theta}{d\zeta} + (\varepsilon - 1) \left( \frac{d\Theta}{d\zeta} \right)^2 - \frac{3}{2} \varepsilon \delta^2 \frac{\Theta^2}{(1+(\delta \zeta)^2)^3} = - \frac{3\delta^2}{(1+(\delta \zeta)^2)^3}, \]  

(60)

where \( \Theta(\zeta) \) is the field written in terms of the \( \zeta \) variable. When one compares (47) to the above equation, the only difference appears on the right-hand side of the equations, but, after expanding (30) on the boundary layer region, we consider only the first-order term in \( \delta \). Thus, the equations for the two first approximations of the field on the boundary layer domain are the same as we obtained in the previous case and the solution takes the form

\[ \Theta(\zeta) = \Theta_0 + \varepsilon \left[ d_2 - \Theta_0 \ln \cosh \left( \sqrt{3} \frac{\zeta + d_1}{\Phi_0} \right) \right], \]

(61)

where \( \Theta_0 \) is constant and describes the zeroth-order approximation of the field, and \( d_1 \) and \( d_2 \) are the arbitrary constants. By matching the three expansions, we have

\[ \Theta_0 = -\sqrt{3}, \]

\[ d_1 = 0, \]

\[ d_2 = \sqrt{3} \ln 2. \]

Finally, the solution for the field is

\[ \phi^\alpha(x) = \phi(x) = \sqrt{3} \left\{ \frac{|x|}{\sqrt{1+x^2}} - 1 - |x| + \varepsilon \ln \left[ 2 \cosh \left( \frac{2}{\sqrt{3}} \right) \right] \right\}, \]

(62)

By applying the same procedure as in case (IIb), one can show that \( \phi^\beta(x) = -\phi^\alpha(x) \). From figure 8 we see that both profiles of the field are continuous, bounded even functions, with continuous derivatives on the whole domain. The asymptotic value of the self-interaction potential reads

\[ V(\infty) \sim -\frac{6b^2}{\alpha^2}, \]
Figure 8. The solutions for the field $\phi$ in case (III); the thin line represents $\phi^\alpha(x)$ and the thick line denotes $\phi^\beta(x)$. Both profiles tend asymptotically to zero. We chose $\epsilon = 0.01$.

Figure 9. Self-interaction potential versus $x$ for case (III) ($\kappa = 1, a_0 = 1, b = 1$ and $\epsilon = 0.01$).

and its profile is shown in figure 9; it is similar to the one illustrated in figure 6(a). Furthermore, the value of $M_{Pl}^2$ is

$$M_{Pl}^2 \sim a_0^3 M^3 \left( \frac{4}{3} - \epsilon \right).$$

(63)

By comparing the previous expression to (27), one can see the effect of the non-minimal coupling: it reduces the value of the 4D Planck mass compared to its value in a model in which the scalar field is minimally coupled to gravity and there is a GB term on the bulk. The consistency conditions (21) together with modification (58) imply that

$$\epsilon < \frac{2}{3}.$$  

(64)

As one can see from figure 10, the consistency condition (58) is fully satisfied for these constrained values of $\epsilon$. 

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In order to compare cases (II) and (III), we plot $\phi^\beta(x)$ and the self-interaction potentials for both cases in figures 11 and 12, respectively. Although there are some small differences for these quantities near the origin of the extra coordinate, similarly to the results obtained in [44], one can see that there is no significant difference between both cases.

As we have already mentioned, in all the analysed cases the consistency conditions (21) are satisfied. Since the couplings $L(\phi)$ are bounded, the corresponding zero tensor modes are localized on the brane. Therefore, in our model (2) there are field configurations that give rise to a regular and asymptotically AdS$_5$ geometry with a 4D massless spin-2 field localized on the brane. Unlike the zero tensor modes, it is much more difficult to analytically find the full massive spectrum. In [30], it was shown that in case (II), where the GB term is absent, the massive spectrum is continuous for the class of solutions (19). In other words, for case (II) we have a localized zero tensor mode and a tower of continuous massive modes without a mass gap. It is more difficult to characterize the massive spectrum for cases (I) and (II) because the
Figure 12. Comparison of the self-interaction potentials. The thin line represents the self-interaction potential for case (IIb) and the thick line represents the self-interaction potential for case (III) ($\kappa = 1$, $a_0 = 1$, $b = 1$ and $\varepsilon = 0.01$).

equation for the mass spectrum does not have the form of a Schrödinger equation, at least in terms of the coordinate $x = bw$; thus, these cases will not be considered here, but they will be treated in a later work.

7. Conclusions

We have explored a thick braneworld model where the matter scalar field is coupled non-minimally to the Einstein–Hilbert term; in addition to this, there is a Gauss–Bonnet (GB) term in the bulk. We compute the 4D Planck mass in terms of the 5D Planck mass and quantities related both to the matter and to the geometry. In contrast to theories where the matter field is minimally coupled to the Einstein–Hilbert term, in our model $M_{\text{Pl}}$ depends explicitly on the matter content of the bulk due to the non-minimal coupling of the scalar field to gravity. In addition to this, by imposing certain natural conditions on the parameters of the model (see (21)), the expression for the 4D Planck mass (see (19)) tells us that if a non-minimally coupled scalar field and/or a bulk GB term are considered, the predicted 4D Planck mass will be smaller than that resulting in a model where the scalar field is minimally coupled to gravity and the GB term is absent.

On the other hand, a relation among the smoothness of geometry, finiteness of the 4D Planck mass and localization of the tensorial modes was studied for a wide class of solutions. Our results show that if the geometry is regular, a finite 4D Planck mass implies the localization of gravity on the brane. In the general analysis described above, some assumptions were made about the regularity of the geometry and its asymptotic behaviour.

We further explored an example in which we applied this general analysis to a regular, asymptotically AdS$_5$ geometry. Since the 4D Planck mass and the normalization condition explicitly depend on the profile of the scalar field, it is important to solve the equation for $\varphi$ and then to check carefully whether conditions (21) are fully satisfied, since in such quantities there are terms with negative sign contributions. We perturbatively solved the equation for $\varphi$ by considering a small parameter defined in terms of the strength of the non-minimal coupling. In order to solve the latter equation, it was necessary to apply the singular perturbation method.
since the scalar field appeared to have different scales of variation on different regions, a situation known in the literature as the boundary layer problem. The application of such a method is a difficult task in general, since the involved differential equations are hard to solve analytically in exact or approximate form. However, for a wide class of solutions we managed to obtain perturbative analytic expressions for $\varphi$ which satisfy all the physically meaningful conditions (21).

In order to elucidate the physical effects of the inclusion of the non-minimal coupling and of the GB term, we further considered three cases in which we switched on/off the corresponding parameters. When the model has minimal coupling with a GB term, the solution can be obtained exactly and $\varphi$ has the form of a kink/anti-kink. In the second case, when the non-minimal coupling is considered and the GB term is switched off, a class of perturbative solutions is obtained with kink-like behaviour.

For this latter case, we can observe an interesting effect: there are some configurations where the self-interaction potential of the scalar field approaches different asymptotic values at $-\infty$ and $+\infty$, mimicking two distinct cosmological constants at both ends of the extra coordinate. Since the spacetime is asymptotically AdS$_5$, this means that the non-minimal coupling of the scalar field to gravity also contributes to the ‘total’ cosmological constants at $-\infty$ and $+\infty$, compensating the unevenness of the scalar energy distribution in order to asymptotically render an even AdS$_5$ spacetime. Usually, in general relativity the symmetries of the geometric background are preserved by the matter energy distribution as a consequence of self-consistency of the Einstein equations. We see that this is not the case anymore when one considers a system with non-minimal coupling between the scalar field and gravity. In particular, for a symmetric or even with respect to the extra coordinate geometry, the ‘total’ cosmological constants of the asymptotically AdS$_5$ spacetime get distinct contributions from the asymptotic values of the self-interaction potential of the scalar field at $-\infty$ and $+\infty$, and the non-minimal coupling between the scalar matter and gravity.

We further imposed asymptotically vanishing boundary conditions on the field configuration of the second case, yielding a physically meaningful solution which is even with respect to the fifth coordinate. This solution has been compared with the solution of the third case in which the GB term is also considered. We found that both solutions have a very similar behaviour.

Moreover, for these last two cases we also found that the consistency conditions (21) impose bounds on the perturbation parameter $\varepsilon$ and that they render a smaller 4D Planck mass compared to the case in which both the non-minimal coupling and the GB term are absent.

It will be interesting to further consider the non-minimal coupling of the scalar field with the GB term. The investigation of the resulting model will be the subject of forthcoming work.

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