ENDPOINT REGULARITY OF 2D MUMFORD-SHAH MINIMIZERS

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Abstract. We prove an $\varepsilon$-regularity theorem at the endpoint of connected arcs for 2-dimensional Mumford-Shah minimizers. In particular we show that, if in a given ball $B_r(x)$ the jump set of a given Mumford-Shah minimizer is sufficiently close, in the Hausdorff distance, to a radius of $B_r(x)$, then in a smaller ball the jump set is a connected arc which terminates at some interior point $y_0$ and it is $C^{1,\alpha}$ up to $y_0$.

1. Introduction

In this paper we study the regularity properties of the jump set of local minimizers of the Mumford-Shah energy on an open set $\Omega \subset \mathbb{R}^2$, which for $v \in SBV(\Omega)$ is given by

$$MS(v) := \int_\Omega |\nabla v|^2 dx + \mathcal{H}^1(S_v).$$

(1.1)

We say that $u : \Omega \to \mathbb{R}$ is a minimizer if $u \in SBV(\Omega)$, $MS(u) < +\infty$ and

$$MS(u) \leq MS(w) \quad \text{whenever} \{w \neq u\} \subset \subset \Omega.$$

For the notation and all the results concerning $SBV$ functions we refer to the book [5].

The Mumford-Shah functional has been proposed by Mumford and Shah in their seminal paper [22] as a variational model for image reconstruction. Since then, it has been widely studied in the literature, from the theoretical side but also from the numerical and applied ones (see [14, 16, 12, 8, 10] and also the many references in [5, Section 4.6]). Starting with the pioneering work [16], the existence of minimizers has been proved in several frameworks and with different methods, see for instance [11, 12, 21]. The most general and successful approach is that of De Giorgi and Ambrosio through the space of special functions of bounded variation that works in any dimension (see [15, 1, 5]).

The regularity theory has seen several contributions, both in two and several space dimensions, see [16, 12, 6, 2, 4, 3, 19, 18, 5]. The most important regularity problem is the famous Mumford-Shah conjecture, which states that (in 2 dimensions) the closure of the jump set $\overline{S}_u$ can be described as the union of a locally finite collection of injective $C^1$ arcs $\{\gamma_i\}$ that can meet only at the endpoints, in which case they have to form triple junctions. More
precisely, given any point \( y \in \overline{S_u \setminus \partial \Omega} \) we only have one of the following three possibilities:

(a) \( y \) belongs to the interior of some \( \gamma_i \) and thus \( S_u \), in a neighborhood of \( y \), is a single smooth arc; in this case \( y \) is called a regular point.

(b) \( y \) is a common endpoint of three (and only three) distinct arcs which form (at \( y \)) three equal angles of 120 degrees; in this case \( y \) is called a triple junction.

(c) \( y \) is the endpoint of one (and only one) arc \( \gamma_j \), i.e. it is a “loose end”; in this case \( y \) is called a crack-tip.

On the other hand, for any minimizer \( u \) it is known since the pioneering work of David \([12]\) that:

(A) If \( S_u \) is sufficiently close, in a ball \( B_r(x_0) \) and in the Hausdorff distance, to a diameter of \( B_r(x_0) \), then in the ball \( B_{r/2}(x_0) \) it is a \( C^{1,\kappa} \) arc.

(B) If \( S_u \) is close to a “spider” centered at \( x_0 \), i.e. three radii of \( B_r(x_0) \) meeting at \( x_0 \) at equal angles, then in the ball \( B_{r/2}(x_0) \) it consists of three \( C^{1,\kappa} \) closed arcs meeting at equal angles at some point \( y_0 \in B_{r/4}(x_0) \).

Up to now no similar result is known in the case where \( S_u \) is close to a single radius of \( B_r(x_0) \), namely the model case of (c) above. The best result available so far is still due to David (see \([13]\), Theorem 69.29):

(C) if \( S_u \cap B_r(x_0) \) is sufficiently close to a single radius in the Hausdorff distance, then \( S_u \cap B_{r/4}(x_0) \) consists of a single connected arc which joins some point \( y_0 \in B_{r/4}(x_0) \) with \( \partial B_{r/2}(x_0) \) and which is smooth in \( B_{r/2}(x_0) \setminus \{y_0\} \).

However, David’s result does not guarantee that such arc is \( C^1 \) up to the loose end \( y_0 \): in particular it leaves the possibility that the arc spirals infinitely many times around it. In this note we exclude the latter possibility and we prove an \( \varepsilon \)-regularity result analogous to (A) and (B) in the remaining case of crack-tips.

**Theorem 1.1.** There exist universal constants \( \varepsilon, \kappa > 0 \) with the following property. Assume that \( u \) is a local minimizer of the Mumford-Shah functional in \( B_r(x_0) \) and that \( \text{dist}_{\mu'}(S_u, \sigma) \leq \varepsilon r \) where \( \sigma \) is the horizontal radius \([x_0, x_0 + (r, 0)]\). Then there is a point \( y_0 \in B_{r/6}(x_0) \) and a \( C^{1,\kappa} \) function \( \psi : [0, r/4] \to [0, r/8] \) such that

\[
S_u \cap B_{r/4}(y_0) = \{ y_0 + (t, \psi(t)) : t \in [0, r/4] \} \cap B_{r/4}(y_0). \tag{1.2}
\]

Moreover the latter result can be combined with (A) into a single \( \varepsilon \)-regularity statement which assumes only that \( S_u \cap B_r(x_0) \) is contained in the \( \varepsilon r \)-neighborhood of a line passing through \( x_0 \), cf. with Remark 8.1.

Theorem 1.1 will be proved combining (a more precise version of) David’s statement (C) with the following weaker version of Theorem 1.1, which for
simplicity we state at scale $r = 1$ (a corresponding version for general $r > 0$

**Theorem 1.2.** There are universal constants $\varepsilon_0, \kappa, C > 0$ with the following

for some smooth function $\alpha : [0, 1] \to \mathbb{R}$ with $\sup_r r|\alpha'(r)| \leq \varepsilon_0$. Then

$$\sup_{r \in [0, 1/2]} \left( r^{1-\kappa}|\alpha'(r)| + r^{2-\kappa}|\alpha''(r)| \right) \leq C.$$  

Observe that (1.4) is easily seen to give a $C^{1,\kappa}$ estimate for the curve

From (1.4) we easily check that $|\tau'(r)| \leq Cr^{\kappa-1}$. Integrating the latter

We also observe that many of the conclusions of this paper can actually be

drawn as long as $u$ is merely a critical point of the Mumford-Shah func-

tional: this is for instance the case for Theorem 1.2. However, to keep our

Furthermore, as a consequence of Theorem 1.2, we can strengthen the

**Proposition 1.3.** The Mumford-Shah conjecture holds true for a local min-

if and only if $\nabla u \in L^{4,\infty}_{\text{loc}}(\Omega)$, i.e. if for all $\Omega' \subset \subset \Omega$ there is

$K(\Omega') > 0$ such that

$$|\{x \in \Omega' : |\nabla u(x)| > \lambda\}| \leq K\lambda^{-4}.$$  

The rest of the paper is organized as follows. In Section 2 we recall

some preliminary results, introduce a suitable system of PDEs (which is the

translation of the Euler-Lagrange conditions for (1.1) in a suitable system of

coordinates) and state the main estimate behind Theorem 1.2, which we call

**nonlinear three annuli property**, cf. Theorem 2.4. This property in inspired

by similar estimates which appear in the fundamental work of Simon on the

uniqueness of tangent cones for harmonic maps and minimal su

rfaces, [23], and in a paper of the first author with Colding and Minicozzi, see [9]. In

Section 3 we introduce a suitable linearization and study the spectrum of

a corresponding linear operator. This analysis is then used in Section 4 to

prove a linear version of Theorem 2.4, i.e. Theorem 4.3. After collecting, in

Section 5, some standard estimates for the Neumann problem, Theorem 4.3
is used in Section 6 to prove Theorem 2.4. In the last sections we then establish Theorem 1.2 and Theorem 1.1.

2. Rescalings, reparametrization and the nonlinear three annuli property

Before starting our considerations, we must introduce the model “tangent function” of a local minimizer at a loose end, which in polar coordinates is given by

\[ \text{Rad}(\theta, r) := \sqrt{\frac{2r}{\pi}} \cos \left( \frac{\theta}{2} \right) \]  

and whose singular set \( S_{\text{Rad}} \) is the open half line \( \{(t, 0) : t \in \mathbb{R}^+\} \) (in cartesian coordinates). It was conjectured by De Giorgi that \( \text{Rad} \) is a global minimizer in \( \mathbb{R}^2 \), i.e. that its restriction to any bounded open set is a minimizer in the sense introduced above. This conjecture was proved in a remarkable book by Bonnet and David, see [7].

2.1. Rescalings. From now on till the very last section, \( u \) will always denote a minimizer of the Mumford-Shah energy in \( B_1 \) satisfying the assumptions of Theorem 1.2. Keeping the notation introduced there, for \( \rho > 0 \) set

\[ u^\rho(\theta, r) := \rho^{-1/2} u(\theta + \alpha(\rho r), \rho r), \]  

\[ \alpha^\rho(r) := \alpha(\rho r). \]

Lemma 2.1. For every \( \delta > 0 \) and for every \( k \in \mathbb{N} \) there is \( \varepsilon_1 > 0 \) such that if \( u \) and \( \alpha \) are as in Theorem 1.2 with \( \varepsilon_0 \leq \varepsilon_1 \), then

\[ \|u^\rho - \text{Rad}\|_{C^k(B_2 \setminus B_1/2)} + \|\alpha^\rho\|_{C^k([1/2, 2])} \leq \delta \quad \forall \rho \leq \frac{1}{4}. \]  

Proof. The statement follows easily from the blow-up technique of Bonnet, see [6], and the higher differentiability theory of [4]. □

Corollary 2.2. For every \( \delta > 0 \) and for every \( k \in \mathbb{N} \) there is \( \varepsilon_1 > 0 \) with the following property. If \( u \) and \( \alpha \) satisfy the assumptions of Theorem 1.2 with \( \varepsilon_0 \leq \varepsilon_1 \), then

\[ \sup_{[0,2\pi] \times [0,1/2]} r^{i-1/2} |\partial_\theta^j \partial_r^i (u(\theta + \alpha(r), r) - \text{Rad}(\theta, r))| \leq \delta \quad \forall i + j \leq k, \]  

\[ \sup_{[0,1]} r^i |\alpha^{(i)}(r)| \leq \delta \quad \forall i \leq k. \]

Proof. Observe first that

\[ (\alpha^\rho)^{(i)} (r) = \rho^i \alpha^{(i)}(\rho r). \]

Taking the supremum in \( r \in [1/2, 2] \) in the latter identity, we easily infer

\[ \rho^i \|\alpha^{(i)}\|_{C^0([\rho/2, 2\rho])} = \|(\alpha^\rho)^{(i)}\|_{C^0([1/2, 2])}, \]

and hence conclude (2.6) from Lemma 2.1.
Next, from (2.2) and the $1/2$-homogeneity of Rad we conclude
\[ u(\theta + \alpha(r), r) - \text{Rad}(\theta, r) = \rho^{1/2} \left( u^\rho \left( \theta, \frac{r}{\rho} \right) - \text{Rad} \left( \theta, \frac{r}{\rho} \right) \right). \]

Differentiating the latter identity $j$ times in $\theta$ and $i$ times in $r$, we conclude
\[ \partial^j_i \partial^j_\rho (u(\theta + \alpha(r), r) - \text{Rad}(\theta, r)) = \rho^{1/2-i} \partial^i_\rho \partial^j_\theta (u^\rho - \text{Rad}) \left( \theta, \frac{r}{\rho} \right). \]
Substitute $r = \rho$ and take the supremum in $\theta$ to achieve (2.5) again from Lemma 2.1. \hfill \Box

2.2. Reparametrization. We next introduce the functions
\[ \dot{\vartheta}(t) := \alpha(e^{-t}) = \alpha^{e^{-t}}(1), \quad (2.7) \]
\[ \vartheta(t) := e^{-t} \left( \cos \dot{\vartheta}(t), \sin \dot{\vartheta}(t) \right), \quad (2.8) \]
\[ f(\phi, t) := e^{t/2} u(\phi + \vartheta(t), e^{-t}) = e^{t/2} u(\phi + \alpha(e^{-t}), e^{-t}) = u^{e^{-t}}(\phi, 1), \quad (2.9) \]
\[ \text{rad}(\phi) := \text{Rad}(\phi, 1). \quad (2.10) \]

In the next lemma we derive a system of partial differential equations for the functions $f$ and $\dot{\vartheta}$, exploiting the Euler-Lagrange conditions satisfied by $u$ and $S_u$. We also rewrite the estimates of Corollary 2.2 in terms of the new functions.

**Lemma 2.3.** If $u$ satisfies the assumptions of Theorem 1.2 and $\vartheta, f$ are given by (2.8) and (2.9), then
\[
\begin{cases}
  f_t = \frac{f}{4} + f_{\vartheta \phi} + f_{tt} + (\dot{\vartheta} f_{\phi} + \dot{\vartheta}^2 f_{\phi \phi} - 2 \dot{\vartheta} f_{t \phi} - \ddot{\vartheta} f_{\phi}) \\
  f_{\phi}(0, t) = \frac{\dot{\vartheta}(t)}{1 + \dot{\vartheta}^2(t)} \left( -\frac{1}{2} f(0, t) + f_t(0, t) \right) \\
  f_{\phi}(2\pi, t) = \frac{\dot{\vartheta}(t)}{1 + \dot{\vartheta}^2(t)} \left( -\frac{1}{2} f(2\pi, t) + f_t(2\pi, t) \right) \\
  \frac{\dot{\vartheta} + \dot{\varphi}^3 - \ddot{\varphi}}{(1 + \dot{\vartheta}^2)^{3/2}} = \left[ \left( \frac{f}{2} + \dot{\vartheta} f_{\phi} - f_t \right)^2 + f_{\phi}^2 \right]^{2\pi}_0.
\end{cases} \quad (2.11)
\]

Moreover, for every fixed $\delta > 0$ and $k \in \mathbb{N}$, the following estimates hold provided $\varepsilon_0$ in Theorem 1.2 is sufficiently small:
\[ \| \dot{\vartheta}^{(i)} \|_{C^0([\ln 2, \infty])} \leq \delta \quad \text{for all } i \leq k, \quad (2.12) \]
\[ \| \dot{\varphi}^i \dot{\varphi}^j (f - \text{rad}) \|_{C^0([0, 2\pi] \times [\ln 2, \infty])} \leq \delta \quad \text{for all } i + j \leq k. \quad (2.13) \]
Proof. Let us first introduce the unit tangent and normal vector fields to \( S_u \) denoted by \( \tau(t) \) and \( \nu(t) \):

\[
\tau(t) := \frac{\dot{\varphi}(t)}{|\dot{\varphi}(t)|}, \quad \nu(t) := \tau^\perp(t)
\]

where, given \( v = (v_1, v_2) \in \mathbb{R}^2 \), \( v^\perp = (-v_2, v_1) \).

Since \( u : B_1 \subset \mathbb{R}^2 \to \mathbb{R} \) is a local minimizer of the MS energy, it is well-known that \( u \) satisfies

\[
\begin{cases}
\Delta u = 0 & \text{on } B_1 \\
\partial_\nu u = 0 & \text{on } S_u \\
k = -|\nabla u^+|^2 + |\nabla u^-|^2 & \text{on } S_u,
\end{cases}
\]

where \( k \) is the curvature of \( S_u \) given by

\[
k = \frac{1}{|\dot{\varphi}(t)|} \dot{\tau}(t) \cdot \nu(t),
\]

and \( (\nabla u)^\pm \) denotes the right and left (with respect to the vector field \( \nu \)) traces of \( \nabla u \) on \( S_u \).

We compute

\[
u(\theta, r) = r^{1/2} f(\theta - \vartheta(-\ln r), -\ln r),
\]

and

\[
u_r = r^{-1/2} \left( \frac{f}{2} - f_t + \dot{\varphi} f_{\phi} \right), \quad \nu_\theta = r^{1/2} f_{\phi}.
\]

We recall the formula for the Laplacian in polar coordinates:

\[
\Delta u = 0 \iff r^{-2} u_{\theta\theta} + r^{-1} (ru_r)_r = 0.
\]

By means of (2.16) we get

\[r^{-2} u_{\theta\theta} = r^{-3/2} f_{\phi\phi},\]

and

\[
r^{-1} (ru_r)_r = r^{-1} \left( r^{1/2} \left( \frac{f}{2} - f_t + \dot{\varphi} f_{\phi} \right) \right)_r
\]

\[= r^{-3/2} \left( \frac{f}{4} - f_t - \frac{\dot{\varphi} f_{\phi}}{2} \right) + r^{-1/2} \left( -r^{-1} \frac{f_t}{2} + r^{-1} \frac{\dot{\varphi} f_{\phi}}{2} \right)
\]

\[+ r^{-1/2} \left( r^{-1} f_{tt} - 2 \dot{\varphi} f_{t\phi} - r^{-1} \frac{\ddot{\varphi} f_{\phi}}{2} + r^{-1} \frac{\dot{\varphi}^2 f_{\phi\phi}}{2} \right)
\]

\[= r^{-3/2} \left( \frac{f}{4} - f_t + \dot{\varphi} f_{\phi} + f_{tt} - 2 \dot{\varphi} f_{t\phi} - \ddot{\varphi} f_{\phi} + \dot{\varphi}^2 f_{\phi\phi} \right).
\]

In conclusion, we get

\[
f_t = \frac{f}{4} + f_{\phi\phi} + f_{tt} + (\dot{\varphi} f_{\phi} + \dot{\varphi}^2 f_{\phi\phi} - 2 \dot{\varphi} f_{t\phi} - \ddot{\varphi} f_{\phi}).
\]

Next, we rewrite the Neumann condition in the new coordinates. Take into account that

\[
\partial_\nu u = 0 \iff \dot{\varphi}^\perp \cdot \nabla u = 0.
\]
On the other hand
\[ \dot{\vartheta}(t) = -e^{-t}(\cos \vartheta(t), \sin \vartheta(t)) + e^{-t}\dot{\vartheta}(t)(-\sin \vartheta(t), \cos \vartheta(t)) \]
\[ = -\vartheta(t) + \dot{\vartheta}(t)\dot{\vartheta}^\perp(t) \]  \hspace{1cm} (2.18)
and thus
\[ \dot{\vartheta}(t)^\perp = -e^{-t}(-\sin \vartheta(t), \cos \vartheta(t)) - e^{-t}\dot{\vartheta}(t)(\cos \vartheta(t), \sin \vartheta(t)) \]
\[ = -\vartheta^\perp(t) - \dot{\vartheta}(t)\vartheta(t) = -\frac{\partial}{\partial \vartheta} - \dot{\vartheta} r \frac{\partial}{\partial r}. \]  \hspace{1cm} (2.19)
We therefore infer from (2.16) that
\[ r^{1/2}\dot{\varphi} + r^{1/2}\ddot{\varphi} \left( \frac{f}{2} + \dot{\vartheta}f_{\varphi} - f_t \right) = 0, \]
in turn implying
\[
\begin{cases}
  f_{\varphi}(2\pi, t) = \frac{\dot{\vartheta}(t)}{1 + \dot{\vartheta}^2(t)} \left( -\frac{1}{2}f(2\pi, t) + f_t(2\pi, t) \right) \\
  f_{\varphi}(0, t) = \frac{\dot{\vartheta}(t)}{1 + \dot{\vartheta}^2(t)} \left( -\frac{1}{2}f(0, t) + f_t(0, t) \right). 
\end{cases}
\]  \hspace{1cm} (2.20)
Finally, we derive the equation satisfied by the scalar curvature \( k \). Differentiating (2.18) we get
\[ \dot{\vartheta}(t) = -\dot{\vartheta} + \ddot{\vartheta} \vartheta^\perp + \dot{\vartheta} \dot{\vartheta}^\perp = -\dot{\vartheta} + \ddot{\vartheta} \vartheta^\perp + \dot{\vartheta} \vartheta^\perp - \dot{\vartheta} \vartheta \]
\[ = -(1 + \dot{\vartheta}^2)\dot{\vartheta} + \ddot{\vartheta} \vartheta \]  \hspace{1cm} (2.21)
and thus we conclude
\[ k = \frac{1}{|\vartheta(t)|} \left( \frac{d}{dt} \frac{\dot{\vartheta}(t)}{|\vartheta(t)|} \right) \frac{\dot{\vartheta}^\perp(t)}{|\vartheta(t)|} = \frac{\ddot{\vartheta}(t) \cdot \dot{\vartheta}^\perp(t)}{|\vartheta(t)|^3} \]
\[ = \frac{(\dot{\vartheta} + \ddot{\vartheta}^3 - \ddot{\vartheta})|\vartheta(t)|^2}{(1 + \dot{\vartheta}^2)^{3/2}|\vartheta(t)|^3} = r^{-1} \frac{\dot{\vartheta} + \ddot{\vartheta}^3 - \ddot{\vartheta}}{(1 + \dot{\vartheta}^2)^{3/2}}. \]
As
\[ |\nabla u|^2 = (u_r)^2 + r^{-2}(u_{\vartheta})^2 = r^{-1} \left( \frac{f}{2} + \dot{\vartheta}f_{\varphi} - f_t \right)^2 + r^{-1} f_{\varphi}^2 \]
we get
\[ \frac{\dot{\vartheta} + \ddot{\vartheta}^3 - \ddot{\vartheta}}{(1 + \dot{\vartheta}^2)^{3/2}} = \left[ \left( \frac{f}{2} + \dot{\vartheta}f_{\varphi} - f_t \right)^2 + f_{\varphi}^2 \right]^{2\pi} \bigg|_0^{2\pi}. \]  \hspace{1cm} (2.22)
In terms of \( \vartheta \) the bound of \( \alpha \) in (2.6) reads as
\[ \sup_{t \in [\ln 2, \infty]} |\vartheta^{(i)}(t)| \leq C_i \delta \hspace{1cm} \text{for every } i \leq k. \]
Indeed, differentiating \( i \) times the identity \( \vartheta(t) = \alpha(e^{-t}) \) we get
\[ \vartheta^{(i)}(t) = \sum_{j=1}^{i} b_{i,j} e^{-jt} \alpha^{(j)}(e^{-t}), \]
with \( b_{i,j} \in \mathbb{R} \).

The decay (2.5) translates instead into

\[
\sup_{\theta} |\partial_{x_i}^2 \partial_{x_j}^2 (f - \text{rad})| \leq C_i \delta \quad \text{for every } t \in [\ln 2, \infty] \text{ and } i + k \leq k.
\]

Indeed, using the \( 1/2 \)-homogeneity of \( \text{Rad} \), we infer

\[
f(\phi, t) - \text{rad}(\phi) = e^{t/2} (u(\phi + \alpha(e^{-t}), e^{-t}) - \text{Rad}(\phi, e^{-t}))
\]

\[
= e^{t/2} g(\phi, e^{-t}).
\]

Therefore we conclude that (2.5) can be reformulated as

\[
\sup_{r \in [0, 1/2]} r^{i-1/2} \|\partial_{(x_i, r)}^2 g(\cdot, r)\|_{C^0} \leq C_i \delta.
\]

On the other hand, differentiating (2.23) yields

\[
\partial_{(x_i, r)}^2 (f(\phi, t) - \text{rad}(\phi)) = \sum_{\ell=0}^{i} b_{i, \ell} e^{t/2 - \ell t} \partial_{(x_i, r)}^2 g(\phi, e^{-t}),
\]

for some \( b_{i, \ell} \in \mathbb{R} \). Setting \( r = e^{-t} \), we then conclude (2.13). \( \square \)

2.3. The nonlinear three annuli property. Given any couple \((f, \vartheta)\) solution of system (2.11) we decompose \( f \) along its components parallel and \((L^2)\) orthogonal to \( \text{rad} \):

\[
f(\phi, t) = (1 + \gamma(t)) \text{rad}(\phi) + v(\phi, t),
\]

with

\[
\int v(\phi, t) \text{rad}(\phi) \, d\phi = 0.
\]

Given this decomposition we denote by \( \mathcal{L}(v, \vartheta, a, b) \) a functional depending on an arbitrary solution \((f, \vartheta)\) of system (2.11) and on any open interval \([a, b]\) such that the corresponding solution is defined on \([0, 2\pi] \times [a, b]\) (strictly speaking the functional depends on \( f \) but since it neglects the projection of \( f \) parallel to \( \text{rad} \) we prefer to use the variable \( v \)).

**Theorem 2.4.** There exist \( \eta, C, \delta, T > 0 \) and \( k \in \mathbb{N} \setminus \{0, 1, 2\} \) universal constants and a functional \( \mathcal{L}(v, \vartheta, a, b) \) with the following properties. First of all,

\[
C^{-1} \int_{a}^{b} \left( \|v(\cdot, t)\|_{W^{2,2}}^2 + \vartheta^2(t) \right) \, dt \leq \mathcal{L}(v, \vartheta, a, b)
\]

\[
\leq C \int_{a}^{b} \left( \|v(\cdot, t)\|_{W^{2,2}}^2 + \vartheta^2(t) \right) \, dt,
\]

(2.24)

for all \((f, \vartheta, a, b)\). Secondly, if \((f, \vartheta)\) is defined on \([0, 2\pi] \times [0, 3T]\) with

\[
\|f - \text{rad}\|_{C^k} + \|\vartheta\|_{C^k} \leq \delta,
\]

onto \( \mathbb{R} \).
\[ V(t) = \sum_{i=1}^{n} \lambda_i e_i, \]

where \( \lambda_i \) are the eigenvalues of the matrix \( A \) and \( e_i \) are the corresponding eigenvectors. The function \( V(t) \) represents the potential energy of the system.

3. Linear problem and spectral analysis

As already mentioned, the proof of Theorem 2.4 will be achieved through a suitable linearization in Section 6, which contains the corresponding computations leading to the relevant system. The latter is introduced here and consists of two unknown functions \( w : [0, 2\pi] \times [a, b] \to \mathbb{R} \) and \( \lambda : [a, b] \to \mathbb{R} \):

\[
\begin{cases}
    w_t = \frac{w}{4} + w_{\phi\phi} + w_{tt} - (\ddot{\lambda} - \dot{\lambda}) \text{rad}_\phi \\
    w_\phi(0, \cdot) = -w_\phi(2\pi, \cdot) = -\frac{\dot{\lambda}}{\sqrt{2\pi}} \\
    \dot{\lambda} - \ddot{\lambda} = \frac{1}{\sqrt{2\pi}} \left[ (w(0, \cdot) + w(2\pi, \cdot)) - 2(w_t(0, t) + w_t(2\pi, t)) \right].
\end{cases}
\]

3.1. Odd-even splitting. Given a solution \( (w, \lambda) \) of (3.1) we introduce its odd and even parts (w.r.t. \( x = \pi \)):

\[
\begin{align*}
    w^o(\phi, t) &:= \frac{1}{2} (w(\phi, t) - w(2\pi - \phi, t)) \\
    w^e(\phi, t) &:= \frac{1}{2} (w(\phi, t) + w(2\pi - \phi, t)).
\end{align*}
\]

By linearity of the equations we conclude easily that

\[
\begin{cases}
    w^o_t = \frac{w^o}{4} + w^o_{\phi\phi} + w^o_{tt} \\
    w^o_\phi(2\pi, \cdot) = w^o(0, \cdot) = 0,
\end{cases}
\]

and thus \( (w^e, \lambda) \) is also a solution of (3.1). Therefore in what follows we assume, with a slight abuse of notations, that \( w \) satisfies also the condition

\[
w(\phi, t) = w(2\pi - \phi, t).
\]
Summarizing:

\[
\begin{align*}
  w_t &= \frac{w}{4} + w_{\phi\phi} + w_{tt} - (\ddot{\lambda} - \dot{\lambda})\text{rad}_{\phi} \\
  w_{\phi}(0, \cdot) &= -\frac{\dot{\lambda}}{\sqrt{2\pi}} \\
  w(\phi, \cdot) &= w(2\pi - \phi, \cdot) \\
  \dddot{\lambda} - \dot{\lambda} &= \sqrt{\frac{2}{\pi}} [w(0, \cdot) - 2w_t(0, \cdot)]
\end{align*}
\]

We next define \( z = w - \lambda \text{rad}_{\phi} \). The first equation then becomes

\[
z_t = \frac{z}{4} + z_{\phi\phi} + z_{tt}.
\]

Moreover,

\[
z_{\phi}(0, t) = w_{\phi}(0, t) - \lambda(t)\text{rad}_{\phi}(0) = -\frac{\dot{\lambda}(t)}{\sqrt{2\pi}} + \frac{\lambda(t)}{2\sqrt{2\pi}}.
\]

Finally, observe that

\[
z(0, t) = w(0, t) \quad z_t(0, t) = w_t(0, t).
\]

We therefore conclude

\[
\begin{align*}
  z_t &= \frac{z}{4} + z_{\phi\phi} + z_{tt} \\
  z_{\phi}(0, \cdot) &= -\frac{\dot{\lambda}}{\sqrt{2\pi}} + \frac{\lambda}{2\sqrt{2\pi}} \\
  z(\phi, \cdot) &= z(2\pi - \phi, \cdot) \\
  \dddot{\lambda} - \dot{\lambda} &= \sqrt{\frac{2}{\pi}} [z(0, \cdot) - 2z_t(0, \cdot)].
\end{align*}
\]

3.2. Reduction to Ventcel boundary condition. We rewrite the boundary conditions for \( z \) as

\[
\begin{align*}
  \sqrt{2\pi}z_{\phi}(0, t) &= - (\partial_t - \frac{1}{2})\lambda(t) \\
  -2\sqrt{\frac{2}{\pi}}(\partial_t - \frac{1}{2})z(0, t) &= (\partial_t - 1)\partial_t\lambda(t).
\end{align*}
\]

We then conclude that

\[
\sqrt{2\pi}(\partial_t - 1)\partial_t z_{\phi}(0, t) - 2\sqrt{\frac{2}{\pi}}(\partial_t - \frac{1}{2})^2z(0, t) = 0.
\]

(3.4)
On the other hand using the equation we have
\[
(\partial_t - 1)\partial_t z_\phi = z_{\phi tt} - z_\phi t = -\frac{z_\phi}{4} - z_{\phi \phi \phi}
\]
\[
(\partial_t - 1)^2 z = z_{tt} - z_t + \frac{z}{4} = -z_{\phi \phi}.
\]
Plugging the last two identities into (3.6) we conclude
\[
\left[ \frac{\pi}{2} \left( \frac{z_\phi}{4} + z_{\phi \phi \phi} \right) - z_{\phi \phi} \right](0, t) = 0.
\] (3.7)
If we introduce the new unknown \( \zeta(\phi, t) := z_\phi(\phi, t) \), we then conclude that it satisfies the following system of identities on \( 0, 2\pi \times [a, b] \)
\[
\begin{cases}
\zeta_t = \frac{\zeta}{4} + \zeta_{\phi \phi} + \zeta_{tt} \\
\zeta_{\phi}(0, \cdot) = \frac{\pi}{2} \left( \frac{\zeta(0, \cdot)}{4} + \zeta_{\phi \phi}(0, \cdot) \right) = 0 \\
\zeta(\phi, \cdot) = -\zeta(2\pi - \phi, \cdot).
\end{cases}
\] (3.8)

3.3. The Ventsel boundary condition. Consider the following space:
\[ \mathcal{O} := \{ g \in W^{1,2}([0, 2\pi]) : g(\phi) = -g(2\pi - \phi) \} . \]
For every \( g \in \mathcal{O} \) we look for solutions \( \sigma \in \mathcal{O} \) of the following equation:
\[
\begin{cases}
\frac{\sigma}{4} + \sigma_{\phi \phi} = g \\
\sigma_{\phi}(0) = \frac{\pi}{2} \left( \sigma(0) + \sigma_{\phi}(0) \right) .
\end{cases}
\] (3.9)
This is equivalent to find a solution \( \sigma \in \mathcal{O} \) of
\[
\begin{cases}
\frac{\sigma}{4} + \sigma_{\phi \phi} = g \\
\sigma_{\phi}(0) = \frac{\pi}{2} g(0) .
\end{cases}
\] (3.10)
Introducing the new unknown
\[ \tau(\phi) := \sigma(\phi) - \frac{\pi}{2} g(0) (\phi - \pi) , \]
and the new function
\[ h(\phi) := g(\phi) - \frac{\pi}{8} g(0) (\phi - \pi) \] (3.11)
we are reduced to finding a solution \( \tau \in \mathcal{O} \) of
\[
\begin{cases}
\frac{\tau}{4} + \tau_{\phi \phi} = h \\
\tau_{\phi}(0) = 0 .
\end{cases}
\] (3.12)
Observe that, if we find a solution $\xi \in W^{1,2}([0,2\pi])$ of
\[
\begin{align*}
\frac{\xi}{4} + \xi_{\phi\phi} &= h \\
\xi_{\phi}(2\pi) &= \xi_{\phi}(0) = 0,
\end{align*}
\]
(3.13)
it then suffices to set
\[
\tau(\phi) = \frac{\xi(\phi) - \xi(2\pi - \phi)}{2}.
\]
On the other hand given that the operator $A(\xi) = \xi_{\phi\phi} + \frac{\xi}{4}$ is self-adjoint on $X := \{ \xi \in W^{1,2} : \xi_{\phi}(2\pi) = \xi_{\phi}(0) = 0 \}$ with the $L^2$ scalar product, the condition of solvability of (3.13) is that $h$ is $L^2$ orthogonal to the kernel of $A$ on $X$. Such kernel is 1-dimensional and generated by $\cos \frac{\phi}{2}$. In view of (3.11) such condition is equivalent to
\[
\int_0^{2\pi} g(\phi) \cos \frac{\phi}{2} d\phi = -\pi g(0) \quad (3.14)
\]
Since
\[
\int_0^{2\pi} (\phi - \pi) \cos \frac{\phi}{2} d\phi = -8,
\]
we conclude the following

**Lemma 3.1.** If $g \in \mathcal{O}$ satisfies the condition
\[
\int_0^{2\pi} g(\phi) \cos \frac{\phi}{2} d\phi = -\pi g(0) \quad (3.14)
\]
then there is a solution $\sigma \in \mathcal{O} \cap W^{3,2}$ of (3.9).

In fact we are going to state a stronger version of this lemma, namely

**Definition 3.2.** Consider the space
\[
W := \{ g \in \mathcal{O} : (3.14) \text{ holds} \}.
\]

**Proposition 3.3.** For every $g \in W$ there is a unique solution $\sigma = T(g) \in W \cap W^{3,2}$ of (3.9). The bilinear form
\[
\langle \alpha, \beta \rangle = \int_0^{2\pi} (\alpha_{\phi\beta\phi} - \frac{1}{4} \alpha \beta)
\]
is a scalar product on $W$, which makes it into an Hilbert space. Moreover the induced norm equivalent to $\| \cdot \|_{W^{1,2}}$. With such structure $T$ is a self-adjoint compact non-positive operator on $W$.

**Proof.** The existence follows from Lemma 3.1 because if $\sigma \in \mathcal{O}$ is any solution of (3.9), so is
\[
\sigma(\phi) + c \cos \frac{\phi}{2}
\]
for any choice of the constant $c$. Thus setting $c$ appropriately, we find a solution in $W$. If $\sigma_1, \sigma_2 \in W$ are two solutions of (3.9), then $\xi := \sigma_1 - \sigma_2$ is a solution in $W$ of

$$\begin{cases}
\frac{\xi}{4} + \xi_{\phi\phi} = 0 \\
\xi_{\phi}(2\pi) = \xi_{\phi}(0) = 0.
\end{cases}$$

Thus

$$\xi(\phi) = c \cos \frac{\phi}{2}.$$  

On the other hand formula (3.14) for $\xi$ becomes

$$c \int_0^{2\pi} \cos^2 \frac{\phi}{2} d\phi = -\pi c$$

which implies $c = 0$. We conclude that the operator $T$ is well-defined.

We next claim the existence of a constant $\omega_0 > 0$ such that

$$\frac{1}{4} \int g^2 \leq (1 - \omega_0) \int g_{\phi}^2$$

for every $g \in W$. First observe that

$$\frac{1}{4} \int g^2 \leq \int g_{\phi}^2 \quad \forall g \in \mathcal{O}$$

and that equality happens if and only if $g = c \cos \frac{\phi}{2}$. This can be achieved because we can write each element in $g$ as a Fourier-series expansion

$$g(\phi) = \sum_{k=1}^{\infty} \alpha_k \cos \frac{k\phi}{2}.$$

So, if our claim were false, there would be a sequence of functions $g_k \in W$ with the property that

$$\frac{1}{4} \int g_k^2 \geq \left(1 - \frac{1}{k}\right) \int (g_k)_{\phi}^2.$$

Since $\int g_k = 0$, normalizing the $L^2$ norm of each $g_k$ to 1 and using the compact embedding of $W^{1,2}$ into $L^2$, up to subsequences we can assume that $g_k$ converges strongly in $L^2$ to a (nontrivial) element $g \in W$ for which equality in (3.16) is attained. However this is a contradiction because it would imply that $W$ contains $\cos \frac{\phi}{2}$. The validity of (3.15) proves easily that $\langle \cdot, \cdot \rangle$ makes $W$ into a Hilbert space.

We now check that $T$ is self-adjoint. Let $g, h \in W$ and consider $u = T(g), v = T(h)$, namely elements in $W \cap W^{3,2}$ solving

$$\begin{cases}
\frac{u}{4} + u_{\phi\phi} = g \\
u_{\phi}(0) = \frac{\pi}{2} \left(\frac{u(0)}{4} + u_{\phi\phi}(0)\right).
\end{cases}$$

(3.17)
and
\[
\begin{aligned}
\begin{cases}
\frac{v}{4} + v_{\phi\phi} = g \\
v_{\phi}(0) = \frac{\pi}{2} \left( \frac{v(0)}{4} + v_{\phi\phi}(0) \right).
\end{cases}
\end{aligned}
\tag{3.18}
\]

We then have
\[
\langle g, T(h) \rangle = \int g_{\phi} v_{\phi} - \frac{1}{4} \int g v = \int \left( u_{\phi\phi\phi} + \frac{u_{\phi}}{4} \right) v_{\phi} - \frac{1}{4} \int \left( u_{\phi\phi} + \frac{u}{4} \right) v
\]
\[
= [u_{\phi\phi} v_{\phi} - u_{\phi} v_{\phi\phi}] \bigg|_{0}^{2\pi} + \int u_{\phi} \left( v_{\phi\phi\phi} + \frac{v_{\phi}}{4} \right)
\]
\[
- \frac{1}{4} [u_{\phi} v - w_{\phi}] \bigg|_{0}^{2\pi} - \frac{1}{4} \int u \left( v_{\phi\phi} + \frac{v}{4} \right)
\]
\[
= \left[ v_{\phi} \left( u_{\phi\phi} + \frac{u}{4} \right) - u_{\phi} \left( v_{\phi\phi\phi} + \frac{v}{4} \right) \right] \bigg|_{0}^{2\pi}
\]
\[
+ \int u_{\phi} h_{\phi} - \frac{1}{4} \int u h = \int u_{\phi} h_{\phi} - \frac{1}{4} \int u h = \langle T(g), h \rangle.
\]

The compactness of the operator follows easily from elliptic estimates. \(\square\)

### 3.4. Spectral analysis.

Consider now the eigenvalue problem on \(W\)
\[
\begin{aligned}
\begin{cases}
\frac{\sigma}{4} + \sigma_{\phi\phi} = -s\sigma \\
\sigma_{\phi}(0) = \frac{\pi}{2} \left( \frac{\sigma(0)}{4} + \sigma_{\phi\phi}(0) \right).
\end{cases}
\end{aligned}
\tag{3.19}
\]

**Definition 3.4.** We denote by \(\Sigma\) the set of \(\lambda \in \mathbb{R}\) for which there is a nontrivial \(\sigma \in W\) solving (3.19).

**Proposition 3.5.** \(\Sigma\) is a discrete set and we order its elements as \(0 = s_0 < s_1 < s_2 < \ldots < s_k < \ldots\) \(s_1 := \min\{\Sigma \setminus \{0\}\} > \frac{3}{4}\) and for each \(s_k \in \Sigma\) the corresponding eigenspace of solutions of (3.19) is 1-dimensional and generated by \(f_k(\phi) = c_k \sin \left( \sqrt{\frac{3}{4} + s_k} (\phi - \pi) \right)\). The normalization constant \(c_0\) is equal to 1 and for \(k \geq 1\) the \(c_k\) are chosen so that \(\langle f_k, f_k \rangle = 1\) for \(k \geq 1\).

**Proof.** The discreteness of the spectrum is an obvious consequence of Proposition 3.3. Consider now the equation
\[
\frac{\sigma}{4} + \sigma_{\phi\phi} = -s\sigma
\]
and set \(-\frac{1}{4} - s =: g\). To ease the ensuing computations we introduce \(\tilde{\sigma}(\cdot) := \sigma(\cdot + \pi)\). Being \(\sigma \in \mathcal{O}\), \(\tilde{\sigma}\) turns out to be odd w.r.t. \(\phi = 0\) and satisfying, in view of the boundary condition in (3.19),
\[
\tilde{\sigma}_{\phi}(-\pi) = \frac{\pi}{2} \left( \frac{\tilde{\sigma}(-\pi)}{4} + \tilde{\sigma}_{\phi\phi}(-\pi) \right).
\tag{3.20}
\]
If \( \rho = 0 \), then \( \tilde{\sigma}(\phi) = a\phi + b \). The constraint that \( \tilde{\sigma} \) is odd implies \( b = 0 \), and then we see at once that \( \tilde{\sigma} \) cannot satisfy the boundary condition. If \( \rho^2 > 0 \), without loss of generality we assume \( \rho > 0 \), then \( \tilde{\sigma}(\phi) = ae^{i\phi} + be^{-i\phi} \). The condition \( \tilde{\sigma} \) odd implies \( b = -a \). The boundary condition in (3.20) is then equivalent to

\[
\rho \left( e^{i\pi} + e^{-i\pi} \right) = -\frac{\pi}{2} \left( \rho^2 + \frac{1}{4} \right) \left( e^{i\pi} - e^{-i\pi} \right),
\]

which in turn rewrites as

\[
\left[ \frac{\pi}{2} \left( \rho^2 + \frac{1}{4} \right) + \rho \right] e^{2i\pi} = \frac{\pi}{2} \left( \rho^2 + \frac{1}{4} \right) - \rho.
\]

The latter equation has no positive solution.

If \( \rho = -\nu^2 \) with \( \nu > 0 \) we again easily conclude that \( \tilde{\sigma} \) must be a multiple of \( \sin \nu \phi \). We then want to show that \( \nu \) must be either \( 1/2 \) or strictly larger than 1. In the latter case, this would imply \( \rho < -1 \) and thus \( s_1 = \min \{ \Sigma \setminus \{0\} \} > 3/4 \), as desired.

Clearly, \( \nu = 1/2 \) corresponds to \( s = 0 \). Otherwise, the boundary condition becomes

\[
\nu \cos \nu \pi = \frac{\pi}{2} \left( \nu^2 - \frac{1}{4} \right) \sin \nu \pi.
\]

Introducing \( x = \nu \pi \) we then conclude that we are looking for positive solutions of

\[
\varpi(x) := \left( x^2 - \frac{\pi^2}{4} \right) \sin x - 2x \cos x = 0.
\]

We first show that \( \varpi \) has no zeros inside \( ]0, \frac{\pi}{2}[ \). Indeed, we compute its first and second derivatives to get

\[
\varpi'(x) = \left( x^2 - \frac{\pi^2}{4} - 2 \right) \cos x + 4x \sin x,
\]

\[
\varpi''(x) = 6x \cos x + \left( 6 + \frac{\pi^2}{4} - x^2 \right) \sin x.
\]

In particular, \( \varpi(0) = \varpi(\pi/2) = 0 \), and since \( \varpi'' \geq 0 \) on \( ]0, \pi/2[ \) and \( \varpi \) is not constant on such an interval we deduce that \( \varpi < 0 \) on \( ]0, \pi/2[ \). Moreover, it is straightforward to check that \( \varpi > 0 \) on \( ]\pi/2, \pi[ \). Thus, the first positive zero of the equation (3.21) is \( 1/2 \) and the next is strictly larger than 1. \( \square \)

**Remark 3.6.** In the second case one can get an explicit estimate of the smallest value \( \nu > 1/2 \) by taking into account that \( \varpi(x) = 0 \) is a linear trigonometric equation. We do not pursue this task here as we do not need such a piece of information.

**Corollary 3.7.** Let \( \Sigma = \{s_k\}_{k \in \mathbb{N}} \) and \( f_k \) be respectively the set and the functions in Proposition 3.5. There is a constant \( C \) such that for any \( \zeta \in \mathcal{O} \)
there is a unique decomposition of $\zeta$ as

$$\zeta(\phi) = \alpha_0 \cos \frac{\phi}{2} + \sum_{k=1}^{\infty} \alpha_k f_k(\phi) \quad (3.22)$$

with

$$C^{-1}\|\zeta\|^2_{W^{2,2}} \leq \sum_{k} \alpha_k^2 \leq C\|\zeta\|_{W^{2,2}}. \quad (3.23)$$

If in addition $\zeta \in W \cap W^{3,2}$ and satisfies the boundary condition

$$\zeta_\phi(0) = \frac{\pi}{2} \left( \frac{\zeta(0)}{4} + \zeta_{\phi\phi}(0) \right) \quad (3.24)$$

then we also have

$$\frac{\zeta(\phi)}{4} + \zeta_{\phi\phi}(\phi) = -\sum_{k=1}^{\infty} s_k \alpha_k f_k(\phi). \quad (3.25)$$

**Proof.** First of all $\alpha_0$ is chosen so that $\omega := \zeta - \alpha_0 \cos \phi/2 \in W$. From Proposition 3.3 and Proposition 3.5 it then follows that $\{f_k\}_k$ is an orthonormal Hilbert basis for $W$ endowed with the scalar product $\langle \cdot, \cdot \rangle$. We thus have the desired (unique) representation by setting

$$\alpha_k = \langle \omega, f_k \rangle. \quad (4.1)$$

The inequalities in Proposition 3.23 follow from the fact that the norm induced by $\langle \cdot, \cdot \rangle$ is equivalent to $\| \cdot \|_{W^{1,2}}$.

Next assume that $\zeta$ satisfies the boundary condition in (3.24). We then conclude that $\omega = T(\zeta_{\phi\phi} + \zeta/4)$. From the self-adjointness of the operator $T$ we thus get

$$-s_k \alpha_k = -s_k \langle T(\zeta_{\phi\phi} + \zeta/4), f_k \rangle = \langle \zeta_{\phi\phi} + \zeta/4, T(-s_k f_k) \rangle = \langle \zeta_{\phi\phi} + \zeta/4, f_k \rangle. \quad (4.2)$$

On the other hand, because of the boundary condition (3.24), the function $\zeta_{\phi\phi} + \zeta/4$ belongs to $W$ and thus the coefficients $-s_k \alpha_k$ give the unique representation

$$\zeta_{\phi\phi} + \frac{\zeta}{4} = \sum_{k} -s_k \alpha_k f_k. \quad (4.3)$$

This proves (3.25). \qed

### 4. The linear three annuli property

Let $a, b \in \mathbb{R}$, with $a < b$. Denote by $Y(a, b)$ the space of $L^2_\gamma(W^{2,2}_x)$ functions $w : ]0, 2\pi[ \times ]a, b[ \to \mathbb{R}$ and derivatives of $(W^{1,2}$ functions) $\lambda : ]a, b[ \to \mathbb{R}$ such that

$$\int w(\phi, t) \text{rad}(\phi) \, d\phi = 0 \quad (4.1)$$

$$\int_{a}^{b} \lambda(t) \, dt = 0 \quad (4.2)$$
In particular, we use the shorthand notation $Y$, when $a$ and $b$ are clear from the context.

In this section we establish suitable coercivity and growth properties of a functional equivalent to
$$
\int_a^b (||w(\cdot,t)||^2_{W^{2,2}} + \lambda^2(t)) \, dt \quad \text{for } (w, \lambda) \in Y(a,b).
$$

The functional will be evaluated often on solutions of (3.1). Observe that the estimates in Section 5 below and a simple bootstrap argument imply that any solution of (3.1) on $[0, 2\pi \times ]a, b]$ is necessarily smooth and we will therefore be allowed to differentiate in the classical sense. Moreover, since the functional shall depend only on $\dot{\lambda}$, condition (4.2) is only a normalization which does not influence any of our discussions.

To simplify the argument we shall make use of the odd-even splitting $w = w^o + w^e$ introduced in Subsection 3.1, and analyze first some functionals acting separately on $w^o$ and $w^e$.

4.1. The three annuli property for the odd part. In this paragraph we study a functional depending only on the odd part $w^o$ of $w$, for every $(w, \lambda) \in Y(a,b)$. Recall then that $(w^o, \lambda)$ is a solution on $[0, 2\pi \times ]a, b]$ of the system (3.2).

For all $(w, \lambda) \in Y(a,b)$ and for all $a, b \in \mathbb{R}$ with $a < b$ let
$$
\mathcal{L}^o(w^o, a, b) := \int_a^b ||w^o_{\phi}(\cdot,t)||^2_{L^2} dt. \quad (4.3)
$$

Lemma 4.1. There exist a constant $C_1 > 0$ such that, for all $(w, \lambda) \in Y$,
$$
C_1^{-1} \int_a^b ||w^o(\cdot,t)||^2_{L^2} dt \leq \mathcal{L}^o(w^o, a, b) \leq C_1 \int_a^b ||w^o(\cdot,t)||^2_{L^2} dt. \quad (4.4)
$$

Moreover, for every $T > 0$ there is $\eta_1 > 0$ such that, if $(w, \lambda)$ is a solution of (3.1), then
$$
\mathcal{L}^o(w^o, T, 2T) \geq (1 - \eta_1) \mathcal{L}^o(w^o, 0, T) \quad \Rightarrow \quad \mathcal{L}^o(w^o, 2T, 3T) \geq (1 + \eta_1) \mathcal{L}^o(w^o, T, 2T). \quad (4.5)
$$

Proof. Inequality (4.4) follows easily from the null boundary conditions in (3.2) and by taking into account that $w^o$ is odd.

We now establish (4.5). Recall that, by (4.1) $\int w(\phi, t) \cos \frac{k}{2} d\phi = 0$. Thus the Fourier decomposition of $w$ reads as
$$
w(\phi, t) = \sum_{k \geq 2} \alpha_k(t) \cos \left(\frac{k}{2} \phi\right),
$$
and the coefficients $\alpha_k$'s satisfy
$$
\ddot{\alpha}_k - \dot{\alpha}_k - \frac{k^2 - 1}{4} \alpha_k = 0.
$$
Thus
\[ \alpha_k(t) = C_k e^{\frac{k+1}{2}t} + D_k e^{\frac{k-1}{2}t} \]
for some \( C_k, D_k \in \mathbb{R} \). A simple calculation gives for all \( k \geq 2 \)
\[ \frac{d^2}{dt^2} (\alpha_k^2) \geq 0, \]
(4.6)
establishing the convexity of each \( \alpha_k^2 \).

We thus conclude
\[ L_0^\omega(w, a, b) = \frac{\pi}{16} \int_a^b \sum_{k \geq 2} k^4 \alpha_k^2(t) \, dt. \]

We now want to argue that, for each \( T > 0 \) there is a constant \( \eta > 0 \) with
the following property. If \( h \geq 0 \) is a nontrivial \( L^1 \) function such that \( h'' \geq h \)
on \( [0, 3T] \), then
\[ \int_T^{2T} h(t) \, dt \geq (1 - \eta) \int_0^T h(t) \, dt \implies \int_2^{3T} h(t) \, dt \geq (1 + \eta) \int_T^{2T} h(t) \, dt. \]

Indeed assume by contradiction this were not true and let \( h_j \) be a sequence of nontrivial functions such that \( h''_j \geq h_j \geq 0 \) and
\[ \int_T^{2T} h_j(t) \, dt \geq \max \left\{ (1 - 1/j) \int_0^T h_j(t) \, dt, (1 + 1/j)^{-1} \int_2^{3T} h_j(t) \, dt \right\}. \]

After multiplying by a suitable constant we can then assume
\[ \int_T^{2T} h_j(t) \, dt = 1. \]
The convexity of the \( h_j \) and the uniform bound on \( \|h_j\|_{L^1} \) implies easily a uniform bound on \( \|h_j\|_{L^\infty([a,b])} \) for any \( 0 < a < b < 3T \) and therefore (again by convexity) uniform Lipschitz bound any compact subset of \( [0, 3T] \). This ensures the local uniform convergence of a subsequence of \( h_j \) (not relabeled) to a nonnegative convex function \( h \), which is \( L^1 \) (and thus locally finite) on the open interval \( [0, 3T] \). In particular, \( \int_T^{2T} h(t) \, dt = 1 \). On the other hand it is also easy to see that
\[ \int_0^T h(t) \, dt \leq 1, \quad \text{and} \quad \int_2^{3T} h(t) \, dt \leq 1. \]

By the mean-value theorem this implies the existence of three points \( 0 < t_1 < T < t_2 < 2T < t_3 < 3T \) where \( h(t_2) \geq 1 \geq \max \{ h(t_1), h(t_3) \} \). But then the convexity of \( h \) implies that \( h \) must be constantly equal to 1 on \( [t_1, t_3] \). Since the inequality \( h'' \geq h \) is verified in the limit in the sense of distributions, this is a contradiction. \( \square \)
4.2. The three annuli property for the even part. We now deal with the functional depending on the even part \(w^e\). We first recall the results proved in Section 3, keeping the notation used there. Let \(z = w^e - \lambda \text{rad}_\phi\) and \(\zeta = z\phi\). The analysis in Section 3.4 leads to the expansion

\[
\zeta(\phi, t) = \alpha_{e,0}(t) \cos \frac{\phi}{2} + \sum_{k \geq 1} \alpha_{e,k}(t) f_k(\phi),
\]

cf. Corollary 3.7.

Integrating in space, we deduce

\[
w^e(\phi, t) = \beta_0(t) + \beta_1(t) \sin \frac{\phi}{2} + \sum_{k \geq 1} \alpha_{e,k}(t) \gamma_k(\phi), \quad \phi \in [0, 2\pi]
\]

where \(\gamma_k\) is the primitive of \(f_k\) such that \(\gamma_k(0) = \gamma_k(2\pi) = 0\), and

\[
\beta_1(t) := -2\alpha_{e,0}(t) - \frac{1}{2}\sqrt{\frac{2}{\pi}} \lambda(t).
\]

We are now ready to introduce our functional on \((w^e, \lambda)\). In particular we define

\[
\mathcal{L}^{e,0}(w^e, \lambda, a, b) := \max \left\{ \|\beta_0\|^2_{L^2([a,b])}, \|\beta_1\|^2_{L^2([a,b])}, \|\lambda\|^2_{L^2([a,b])} \right\},
\]

\[
\mathcal{L}^{e,1}(w^e, a, b) := \int_a^b \sum_{k \geq 1} \alpha_{e,k}^2(t) \, dt,
\]

and finally

\[
\mathcal{L}^{e}(w^e, \lambda, a, b) := \max \left\{ \mathcal{L}^{e,1}(w^e, a, b), \mathcal{L}^{e,0}(w^e, \lambda, a, b) \right\}.
\]

Lemma 4.2. There is a constant \(C_2(a, b) > 0\) such that for all \((w, \lambda) \in Y\)

\[
C_2^{-1} \int_a^b \left( \|w^e(\cdot, t)\|^2_{W^{1,2}} + \lambda^2(t) \right) \, dt \leq \mathcal{L}^e(w^e, \lambda, a, b)
\]

\[
\leq C_2 \int_a^b \left( \|w^e(\cdot, t)\|^2_{W^{1,2}} + \lambda^2(t) \right) \, dt,
\]

Moreover, there are positive constants \(T\) and \(\eta_2\) such that, if \((w, \lambda)\) is a solution of (3.1), then

\[
\mathcal{L}^e(w^e, \lambda, T, 2T) \geq (1 - \eta_2) \mathcal{L}^e(w^e, \lambda, 0, T)
\]

\[
\implies \mathcal{L}^e(w^e, \lambda, 2T, 3T) \geq (1 + \eta_2) \mathcal{L}^e(w^e, \lambda, T, 2T).
\]

Proof. Observe that we know

\[
\alpha_{e,0}^2 + \sum_{k \geq 1} \alpha_{e,k}^2 \leq C \|w^e(\cdot, t) - \lambda \text{rad}_\phi\|^2_{W^{1,2}}.
\]
We therefore conclude that
\[
\mathcal{L}e.1(w^e, a, b) + \int_a^b \beta_1^2(t) \, dt \leq C \int_a^b \|w^e(\cdot, t)\|_{W^{2,2}}^2 \, dt + C \int_a^b \lambda^2(t) \, dt
\]
\[
\leq C \int_a^b \left( \|w^e(\cdot, t)\|_{W^{2,2}}^2 + \lambda^2(t)\right) \, dt, \quad (4.14)
\]
where in the last line we have used the Poincaré inequality (recall that \(\int_a^b \lambda(t) \, dt = 0\)). We therefore only need to bound \(\beta_0\) in order to show the second inequality in (4.12). Observe that
\[
2\pi \beta_0(t) = \int_0^{2\pi} w^e(\phi, t) \, d\phi = -\beta_1(t) \int_0^{2\pi} \sin \phi \, d\phi - \sum_{k \geq 1} \alpha_{e,k}(t) \int_0^{2\pi} \gamma_k(\phi) \, d\phi.
\]
We thus easily estimate
\[
\beta_0^2(t) \leq C \|w^e(\cdot, t)\|_{L^2}^2 + C \beta_1^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2 \|\gamma_k\|_{L^2}^2
\]
\[
\leq C \|w^e(\cdot, t)\|_{L^2}^2 + C \beta_1^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2 \|f_k\|_{L^2}^2
\]
\[
\leq C \|w^e(\cdot, t)\|_{L^2}^2 + C \beta_1^2(t) + \sum_{k \geq 1} \alpha_{e,k}^2(t).
\]
Integrating in time we then reach the desired estimate from (4.14). Next, we can use (4.8) to estimate
\[
\|w^e_\phi(\cdot, t)\|_{W^{1,2}}^2 \leq C \lambda^2(t) + C \|w^e_\phi(\cdot, t) - \lambda \text{rad}_\phi\|_{W^{1,2}}^2
\]
\[
\leq C \lambda^2(t) + C \alpha_{e,0}^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2(t)
\]
\[
\leq C \lambda^2(t) + C \beta_1^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2(t).
\]
Integrating and arguing as above we conclude
\[
\int_a^b \|w^e_\phi(\cdot, t)\|_{W^{1,2}}^2 \, dt \leq \mathcal{L}e(w^e, \lambda, a, b). \quad (4.15)
\]
To reach the first inequality in (4.12) we just need to bound \(\int_a^b \|w^e(\cdot, t)\|_{L^2}^2 \, dt\) which, using the Poincaré inequality and (4.15) can be reduced to bound
\[
A := \left( \int_a^b w^e(\phi, t) \, d\phi \right)^2.
\]
In turn we easily see that the latter can be estimated as follows
\[
A \leq C \beta_0(t)^2 + C \beta_1^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2(t) \|\gamma_k\|_{L^2}^2
\]
\[
\leq C \beta_0^2(t) + C \beta_1^2(t) + C \sum_{k \geq 1} \alpha_{e,k}^2(t).
\]
Let us now prove (4.13). We first observe that, by Corollary 3.7, for all $k \geq 1$, the coefficients $\alpha_{e,k}$ satisfy the ordinary differential equation

$$\ddot{\alpha}_{e,k} - \dot{\alpha}_{e,k} - s_k \alpha_{e,k} = 0.$$ 

Therefore, they take the form

$$\alpha_{e,k}(t) = C_k e^{\tilde{\rho}_k t} + D_k e^{-\rho_k t}$$

for some $C_k, D_k \in \mathbb{R}$, for all $k \geq 1$. In the latter formula

$$\tilde{\rho}_k = \frac{1}{2} (\sqrt{1 + 4s_k} + 1) > \frac{3}{2}, \quad \rho_k = \frac{1}{2} (\sqrt{1 + 4s_k} - 1) > \frac{1}{2},$$

as $s_k > s_1 > \frac{3}{2}$ for all $k \geq 1$ (cp. Proposition 3.5). A simple computation gives then $(\alpha_{e,k}^2)^{''} \geq \alpha_{e,k}^2 \geq 0$. We thus conclude as in Lemma 4.1 that, for every $T > 0$ there is $\eta_3 > 0$ such that

$$L_{e,1}(w, T, 2T) \geq (1 - \eta_3) L_{e,1}(w, 0, T) \implies L_{e,1}(w, 2T, 3T) \geq (1 + \eta_3) L_{e,1}(w, T, 2T). \quad (4.16)$$

We next claim that, if $T$ is suitably chosen, then there is a constant $\eta_4$ such that

$$L^{e,0}(w, \lambda, 2T, 3T) \geq (1 + \eta_4) L^{e,0}(w, \lambda, T, 2T), \quad (4.17)$$

which would obviously conclude the proof. Now observe that, since all the $\gamma_k$’s vanish on the extrema 0 and $\pi$, the functions $(\beta_0, \beta_1, \lambda)$ solve the following linear system of ODEs:

$$\begin{cases}
\dot{\beta}_0 = \beta_0 + \beta_0 \\
\dot{\beta}_1 = \beta_1 + \frac{1}{\sqrt{2\pi}} (\tilde{\lambda} - \dot{\lambda}) \\
\frac{1}{2} \beta_1 = -\frac{1}{\sqrt{2\pi}} \dot{\lambda} \\
\tilde{\lambda} - \dot{\lambda} = \sqrt{\frac{2}{\pi}} (\beta_0 - 2\dot{\beta}_0).
\end{cases} \quad (4.18)$$

Solving the first three equations yields

$$\begin{cases}
\beta_0(t) = (a_0 + b_0 t) e^{t/2} \\
\beta_1(t) = a_1 e^t + b_1 e^{t/2} \\
\dot{\lambda}(t) = -\sqrt{\frac{2}{\pi}} (a_1 e^t + b_1 e^{t/2}).
\end{cases} \quad (4.19)$$

In addition, the last equation in (4.18) implies $b_0 = -\frac{\sqrt{\pi}}{2} b_1$. We thus see that $\dot{\lambda}^2(t)$ is a constant multiple of $\beta_1^2(t)$. We now claim the existence of $T, \eta > 0$ such that

$$\int_{2T}^{3T} \beta_1^2(t) \geq (1 + \eta) \int_T^{2T} \beta_1^2(t) \, dt, \quad (4.20)$$
Since we can normalize the constants so that \( \int_T^{2T} \beta_i^2(t) \, dt = 1 \), by a simple compactness argument as already used in Lemma 4.1, it suffices to show that in fact
\[
\int_T^{3T} \beta_i^2(t) > \int_T^{2T} \beta_i^2(t) \, dt
\]
whenever the coefficients \( a_i \) and \( b_i \) are not both zero. Since in case \( a_0 = 0 \) or \( a_1 = 0 \) the statement would be obvious for both \( t^2 e^t \) and \( e^t \), which are strictly monotone on \([0, \infty[\), we can further assume that \( a_i \neq 0 \). Thus, if we normalize by multiplying by \( a_i^{-1} \), we just have to show the existence of a single \( T > 0 \) such that, no matter which coefficients \( b_i \) are chosen, we have
\[
\int_T^{3T} (1 + b_0 t)^2 e^t \, dt > \int_T^{2T} (1 + b_0 t)^2 e^t \, dt \quad (4.22)
\]
\[
\int_T^{3T} (e^t + b_1 e^{t/2})^2 \, dt > \int_T^{2T} (e^t + b_1 e^{t/2})^2 \, dt. \quad (4.23)
\]
We claim that this happens for \( T = 3 \). Observe that, denoting by \( h_i \) the integrand in both cases, the functions
\[
P_i(b_i) := \int_T^{3T} h_i(t) \, dt - \int_T^{2T} h_i(t) \, dt, \quad (4.24)
\]
are second order polynomials with positive quadratic coefficient. Our claim is therefore equivalent to the statement that the corresponding discriminants are strictly negative and we will show this fact when \( T = 3 \).

For \( i = 0 \) straightforward computations lead to the formula
\[
P_0(b) = e^3 [b^2(65e^6 - 52e^3 + 5) - 4b(1 - 5e^3 + 4e^6) + (e^3 - 1)^2].
\]
Our claim is therefore equivalent to the negativity of
\[
\Delta_0 = 4(4e^6 - 5e^3 + 1)^2 - (e^3 - 1)^2(65e^6 - 52e^3 + 5)
= -(e^3 - 1)^2(e^6 + 1 - 20e^3).
\]
As it can be easily checked with any electronic calculator, \( e^3 > 20 \) and thus \( e^6 - 20e^3 + 1 \) is positive.

For \( i = 1 \) we instead obtain
\[
P_1(b) = \frac{e^3}{6} [6(e^3 - 1)^2b^2 + 8e^{3/2}(e^{9/2} - 1)^2 + 3e^3(e^6 - 1)^2].
\]
We thus need to show that the number
\[
\Delta_1 = 8(e^{9/2} - 1)^4 - 9(e^3 - 1)^2(e^6 - 1)^2
\]
is negative, which can be readily checked with any electronic calculator. \( \square \)
4.3. The linear three annuli property. We are now ready to define the functional of interest in the linear case. For all \((w, \lambda) \in Y(a, b)\) set
\[
\mathcal{L}(w, \lambda, a, b) := \max \left\{ \mathcal{L}^o(w^o, a, b), \mathcal{L}^e(w^e, \dot{\lambda}, a, b) \right\}.
\] (4.25)

**Theorem 4.3.** For all \(a < b\) there is a constant \(C\) such that, for all \((w, \lambda) \in Y\),
\[
C^{-1} \int_a^b \left( \|w(\cdot, t)\|_{W^{2,2}}^2 + \dot{\lambda}^2(t) \right) dt \leq \mathcal{L}(w, \lambda, a, b)
\]
\[
\leq C \int_a^b \left( \|w(\cdot, t)\|_{W^{2,2}}^2 + \dot{\lambda}^2(t) \right) dt.
\] (4.26)
Moreover, there are constants \(T, \eta > 0\) such that, when \(w\) and \(\lambda\) solve (3.1),
\[
\mathcal{L}(w, \lambda, T, 2T) \geq (1 - \eta)\mathcal{L}(w, \lambda, 0, T)
\]
\[
\implies \mathcal{L}(w, \lambda, 2T, 3T) \geq (1 + \eta)\mathcal{L}(w, \lambda, T, 2T).
\] (4.27)

**Proof.** The coercivity estimate (4.26) follows easily from the corresponding estimates in Lemmas 4.1 and 4.2.

To establish (4.27) we set \(\eta := \min\{\eta_1, \eta_2\}\) and suppose by contradiction that we can find \((w, \lambda) \in Y\) such that
\[
\mathcal{L}(w, \lambda, 1, 2) \geq (1 - \eta)\mathcal{L}(w, \lambda, 0, 1), \quad (1 + \eta)\mathcal{L}(w, \lambda, 1, 2) > \mathcal{L}(w, \lambda, 2, 3).
\]
If \(\mathcal{L}(w, \lambda, 1, 2) = \mathcal{L}^o(w^o, 1, 2)\), as \(\eta \leq \eta_1\) by Lemma 4.1 we infer
\[
\mathcal{L}(w, \lambda, 2, 3) \geq \mathcal{L}^o(w^o, 2, 3) \geq (1 + \eta_1)\mathcal{L}^o(w^o, 1, 2) \geq (1 + \eta)\mathcal{L}(w, \lambda, 1, 2),
\]
a contradiction.

The other case follows similarly by means of Lemma 4.2. \(\square\)

5. Elliptic estimates

In this section we establish some elliptic estimates that will be employed in Section 6 to prove Theorem 2.4.

**Lemma 5.1.** Let \(\Omega\) be a \(C^\infty\)-open set diffeomorphic to the unit disk, and \(s \in \mathbb{R}\) be such that \(s \geq 2\). Let \(u \in H^s(\Omega)\), then
\[
\|u\|_{H^s(\Omega)} \leq C \left( \|u\|_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial \nu} \right\|_{H^{s-\frac{3}{2}}(\partial \Omega)} + \|\Delta u\|_{H^{s-2}(\Omega)} \right)
\] (5.1)
for some non negative constant \(C\) depending on \(s\) and \(\Omega\).

**Proof.** We introduce first some auxiliary functions and operators. Let \(\Phi\) be the solution of
\[
\begin{cases}
\Delta \Phi = 1 & \text{on } \Omega \\
\frac{\partial \Phi}{\partial \nu} = \frac{L^2(\Omega)}{H^1(\partial \Omega)} & \text{on } \partial \Omega,
\end{cases}
\]
and given $\varphi \in H^{s-2}(\Omega)$ and $\psi \in H^{s-3/2}(\partial \Omega)$ with

$$\int_{\Omega} \varphi = \int_{\partial \Omega} \psi = 0,$$

let $T(\varphi)$ be the (unique up to constants) solution of

$$\begin{cases}
\triangle v = \varphi & \text{on } \Omega \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}$$

and $S(\psi)$ be the (unique up to constants) solution of

$$\begin{cases}
\triangle v = 0 & \text{on } \Omega \\
\frac{\partial v}{\partial \nu} = \psi & \text{on } \partial \Omega.
\end{cases}$$

Let $u$ be as in the statement, then define

$$\tilde{u} := T(\triangle u - \int_{\Omega} \triangle u) + S\left(\frac{\partial u}{\partial \nu} - \int_{\partial \Omega} \frac{\partial u}{\partial \nu}\right) + \left(\int_{\Omega} \triangle u\right) \Phi + \int_{\Omega} u.$$

Note that

$$\begin{cases}
\triangle (\tilde{u} - u) = 0 & \text{on } \Omega \\
\frac{\partial}{\partial \nu}(\tilde{u} - u) = 0 & \text{on } \partial \Omega,
\end{cases}$$

and thus $\tilde{u} - u \equiv \Lambda$, $\Lambda \in \mathbb{R}$. Hence, by the very definition of $\tilde{u}$ we have that

$$\Lambda = \int_{\Omega} (\tilde{u} - u) = \int_{\Omega} T(\triangle u - \int_{\Omega} \triangle u)
\quad + \int_{\Omega} S\left(\frac{\partial u}{\partial \nu} - \int_{\partial \Omega} \frac{\partial u}{\partial \nu}\right) + \left(\int_{\Omega} \triangle u\right) \left(\int_{\Omega} \Phi\right).$$

Being

$$\left|\int_{\Omega} \triangle u\right| \leq \left|\int_{\partial \Omega} \frac{\partial u}{\partial \nu}\right| \leq \left(\mathcal{H}^1(\partial \Omega)\right)^{1/2} \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial \Omega)},$$

to conclude it is sufficient to recall that

$$\|T(\varphi)\|_{H^s(\Omega)} \leq C\|\varphi\|_{H^{s-2}(\Omega)}$$

and

$$\|S(\psi)\|_{H^s(\Omega)} \leq C\|\psi\|_{H^{s-3/2}(\Omega)},$$

for some non-negative constant $C$ depending on $s$ and $\Omega$ (cp. [20, Remarque 7.2 p. 202]). □

Lemma 5.1 and a cut-off argument imply the following result. We set first some notation: for $0 < a < b < 2 < b $ let $I :=]a, b[ be an interval, $\Omega :=]0, 2\pi[ \times I$ and $\Omega_d :=]0, 2\pi[ \times ]a + d, b - d[, d \in ]0, 1[$. 
Lemma 5.2. Let $\alpha, \beta \in H^{(k+1)/2}(I)$ and $F, G \in H^{k/2}(\Omega)$, $k \in \mathbb{N}$. If $u \in H^{k/2+2}(\Omega)$ is a solution of
\[
\begin{cases}
\Delta u = F + \partial_t G & \text{on } \Omega \\
u_\phi(0, \cdot) = \alpha \\
u_\phi(2\pi, \cdot) = \beta,
\end{cases}
\]
then for every $d \in [0, 1]$ there exists a constant $C > 0$ depending on $k, d$ and $\Omega$ such that
\[
\|u\|_{H^{k/2+1}(\Omega_d)} \leq C \left( \|u\|_{L^2(\Omega)} + \|\alpha\|_{H^{(k+1)/2}(I)} + \|\beta\|_{H^{(k+1)/2}(I)} + \|F\|_{H^{k/2}(\Omega)} + \|G\|_{H^{k/2}(\Omega)} \right). \tag{5.4}
\]
Proof. Let $\Lambda$ be a smooth open set diffeomorphic to the unit disk such that
\[
\Omega_{d/2} \subset \Lambda \subset \Omega_{d/4}
\]
and $\varphi \in C^\infty_c \left( [a + d/2, b - d/2] \right)$ be a cut-off function with $\varphi|_{[a+3/4d,b-3/4d]} \equiv 1$. Introduce functions $v$ and $w$ to be, respectively, weak solutions of
\[
\begin{cases}
\Delta v = \varphi F & \text{on } \Lambda \\
v_\phi(0, \cdot) = \varphi \alpha + (1 - \varphi) \delta & \text{on } \{0\} \times [a + \frac{d}{2}, b - \frac{d}{2}] \\
v_\phi(2\pi, \cdot) = \varphi \beta + (1 - \varphi) \delta & \text{on } \{2\pi\} \times [a + \frac{d}{2}, b - \frac{d}{2}] \\
\frac{\partial v}{\partial t} = (1 - \varphi) \delta & \text{otherwise on } \partial \Lambda
\end{cases}
\]
and of
\[
\begin{cases}
\Delta w = \varphi (G + \gamma) & \text{on } \Lambda \\
\frac{\partial w}{\partial t} = 0 & \text{on } \partial \Lambda,
\end{cases}
\]
with the constants $\delta$ and $\gamma \in \mathbb{R}$ chosen, depending on $\varphi$, in a way that the corresponding problem is solvable. Observe that $\{0\} \times [a + \frac{d}{2}, b - \frac{d}{2}]$ and $\{2\pi\} \times [a + \frac{d}{2}, b - \frac{d}{2}]$ are subsets of the boundary of $\Lambda$, where the $\phi$-derivative is in fact the normal derivative. Therefore, the conditions on $\delta$ and $\gamma$ are that
\[
\int_{\{0\} \times [a + \frac{d}{2}, b - \frac{d}{2}]} \varphi(t) \alpha(t) \, dt + \int_{\{2\pi\} \times [a + \frac{d}{2}, b - \frac{d}{2}]} \varphi(t) \beta(t) \, dt
\]
\[
+ \delta \int_{\partial \Lambda \setminus \{0, 2\pi\} \times [a + \frac{d}{2}, b - \frac{d}{2}]} (1 - \varphi(t)) = \int_{\Lambda} \varphi(t) F(\phi, t) \, d\phi \, dt
\]
and
\[
\int_{\Lambda} \varphi(t) G(\phi, t) \, d\phi \, dt = -\gamma \int_{\Lambda} \varphi(t) \, d\phi \, dt.
\]
We therefore conclude easily
\[
|\delta| \leq C(\varphi) \left( \|F\|_{L^2(\Omega)} + \|\alpha\|_{L^2(I)} + \|\beta\|_{L^2(I)} \right)
\]
\[
|\gamma| \leq C(\varphi) \|G\|_{L^2(\Omega)}.
\]
Furthermore, we assume that the mean value of the trace of \( v \) on \( \partial \Lambda \) and of \( w \) on \( \Lambda \) are null. Thus, Poincaré type inequalities and an integration by parts yield for some positive constant depending on \( d \) and \( \Lambda \)

\[
\|v\|_{L^2(\Lambda)} \leq C\left(\|\alpha\|_{L^2(I)} + \|\beta\|_{L^2(I)} + \|F\|_{L^2(\Omega)}\right). \tag{5.5}
\]

and

\[
\|w\|_{L^2(\Lambda)} \leq C\|G\|_{L^2(\Omega)}. \tag{5.6}
\]

We may apply Lemma 5.1 on \( \Lambda \) to \( v \) and \( w \) to get

\[
\|v\|_{H^{k+2}(\Lambda)} \leq C\left(\|v\|_{L^2(\Lambda)} + \|\alpha\|_{H^{(k+1)/2}(I)} + \|\beta\|_{H^{(k+1)/2}(I)} + \|F\|_{H^{k/2}(\Lambda)}\right) \tag{5.5}
\]

\[
\leq C\left(\|\alpha\|_{H^{(k+1)/2}(I)} + \|\beta\|_{H^{(k+1)/2}(I)} + \|F\|_{H^{k/2}(\Lambda)}\right), \tag{5.7}
\]

and

\[
\|w\|_{H^{k+2}(\Lambda)} \leq C\left(\|w\|_{L^2(\Lambda)} + \|G\|_{H^{k/2}(\Lambda)}\right) \tag{5.6}
\]

\[
\leq C\|G\|_{H^{k/2}(\Lambda)}, \tag{5.8}
\]

for some positive constant \( C \) depending on \( k, \Omega, \Lambda \) and \( d \).

Notice that \( u - v - w_t \) is an harmonic function on \( \Omega_{4d} \) with null normal derivatives on \( \{0\} \times [a + 3d/4, b - 3d/4] \) and \( \{2\pi\} \times [a + 3d/4, b - 3d/4] \). By odd extension across 0 and 2\( \pi \) we get an harmonic function in \( H^1(-2\pi, 4\pi \times [a + 3d/4, b - 3d/4]) \), hence \( u - v - w_t \in C^\infty(\Omega_d) \). Thanks to (5.7), (5.8) and the interior estimates for harmonic functions we may conclude that

\[
\|u\|_{H^{k+1}(\Omega_d)} \leq \|v + w_t\|_{H^{k+1}(\Omega_d)} + \|u - v - w_t\|_{H^{k}(\Omega_d)}
\]

\[
\leq C\left(\|\alpha\|_{H^{(k+1)/2}(I)} + \|\beta\|_{H^{(k+1)/2}(I)} + \|F\|_{H^{k/2}(\Omega)} + \|G\|_{H^{k/2}(\Omega)} + \|u - v - w_t\|_{L^2(\Lambda)}\right)\]

\[
\leq C\left(\|u\|_{L^2(\Omega)} + \|\alpha\|_{H^{(k+1)/2}(I)} + \|\beta\|_{H^{(k+1)/2}(I)} + \|F\|_{H^{k/2}(\Omega)} + \|G\|_{H^{k/2}(\Omega)}\right). \tag{6.1}
\]

\[
6. \text{ Proof of the nonlinear three annuli property}
\]

We extend to the nonlinear framework the results established in Section 4 by proving Theorem 2.4: the claims of the Theorem will be proved with the constants \( T \) and \( \eta \) of Theorem 4.3. Given any couple \((f, \vartheta)\) solution of system (2.11) in Section 2 we decompose \( f \) along its components parallel and orthogonal to rad:

\[
f(\phi, t) = (1 + \gamma(t)) \ \text{rad}(\phi) + v(\phi, t), \tag{6.1}
\]

with

\[
\int v(\phi, t) \ \text{rad}(\phi) \ d\phi = 0. \tag{6.2}
\]
We will then prove the statement of Theorem 2.4 with the very same functional $\mathcal{L}$ of Theorem 4.3. We first set some notation: let $I := [0, 3T]$, $\Omega := [0, 2\pi I]$ and, for $d \in [0, 1]$, $I_d := [dT, (3 - d)T]$ and $\Omega_d := [0, 2\pi I_d]$.

In what follows we shall denote by $C$ a positive constant that may vary from line to line, without highlighting the parameters on which each constant depends.

**Proof of Theorem 2.4.** The growth estimates in (2.24) are proved in Theorem 4.3, cf. (4.26).

Let us now take $T, \eta > 0$ to be the constants in Theorem 4.3 and suppose, by contradiction, that we can find sequences $(f_j, \vartheta_j)$, solutions of (2.11) for all $j \in \mathbb{N}$, such that

$$\lim_{j} (\|f_j - \text{rad}\|_{C^2} + \|\vartheta_j\|_{C^2}) = 0$$

and violating (2.25), i.e.

$$\mathcal{L}(v_j, \vartheta_j, T, 2T) \geq \max \{(1 - \eta)\mathcal{L}(v_j, \vartheta_j, 0, T), (1 + \eta/2)^{-1}\mathcal{L}(v_j, \vartheta_j, 2T, 3T)\}.$$  

By taking into account (6.1) we have

$$f_j(\phi, t) = (1 + \gamma_j(t)) \text{rad}(\phi) + v_j(\phi, t),$$

with $v_j$ satisfying (6.2) and, by (6.3), with

$$\lim_{j} \|v_j\|_{C^2} = \lim_{j} \|\gamma_j\|_{C^2} = 0.$$  

Moreover, without loss of generality we can also assume

$$\int_0^{3T} \vartheta_j(t) \, dt = 0.$$  

In addition, we define

$$\mathcal{L}(v_j, \vartheta_j, T, 2T) =: \varepsilon_j^2 \downarrow 0.$$  

Therefore, by introducing $(v_j, \vartheta_j) = \varepsilon_j(w_j, \lambda_j)$ and rescaling we get

$$1 = \mathcal{L}(w_j, \lambda_j, T, 2T) \geq \max \{(1 - \eta)\mathcal{L}(w_j, \lambda_j, 0, T), (1 + \eta/2)^{-1}\mathcal{L}(w_j, \lambda_j, 2T, 3T)\}.$$  

We claim that, up to subsequences not relabeled for convenience, for all $d > 0$

$$(w_j, \lambda_j) \rightarrow (w, \lambda) \quad C^k(\Omega_d) \text{ for all } k \in \mathbb{N}.$$  

This claim follows by studying the asymptotic behaviour of system (2.11) for $(f_j, \vartheta_j)$ as $j \uparrow \infty$. 
Given (6.7) for granted we conclude as follows. In proving (6.7) we shall also show that \((w, \lambda)\) solves (3.1), i.e.,

\[
\begin{aligned}
\frac{w_t}{4} &= w + w\phi_t + w_{H} - (\ddot{\lambda} - \lambda)\text{rad}_1 \\
\frac{w_\phi(0, \cdot)}{2\pi} &= -\frac{1}{\sqrt{2\pi}} \ddot{\lambda} \\
\frac{w_\phi(2\pi, \cdot)}{2\pi} &= \frac{1}{\sqrt{2\pi}} \ddot{\lambda} \\
\ddot{\lambda} - \ddot{\lambda} &= \frac{1}{\sqrt{2\pi}} (w(0, \cdot) + w(2\pi, \cdot) - 2w_t(0, \cdot) + w_t(2\pi, \cdot)),
\end{aligned}
\]

and

\[
\int w(\phi, t) \text{rad}_1(\phi) d\phi = \int_0^{3T} \lambda(t) dt = 0.
\]

The latter equality holds true in view of (6.2) for each \(v_j\) and (6.5). Therefore, we conclude that the couple \((w, \lambda)\) belongs to \(Y\).

Finally, notice that by (6.7) we get

\[
\begin{aligned}
\mathcal{L}(w, \lambda, 1, 2) &= \lim_j \mathcal{L}(w_j, \lambda_j, T, 2T) = 1, \\
\mathcal{L}(w, \lambda, 0, 1) &= \liminf_j \mathcal{L}(w_j, \lambda_j, 0, T), \\
\mathcal{L}(w, \lambda, 2, 3) &= \liminf_j \mathcal{L}(w_j, \lambda_j, 2T, 3T),
\end{aligned}
\]

in turn implying by (6.6)

\[
\mathcal{L}(w, \lambda, T, 2T) \geq \max \left\{ (1 - \eta) \mathcal{L}(w, \lambda, 0, T), (1 + \eta/2)^{-1} \mathcal{L}(w, \lambda, 2T, 3T) \right\}.
\]

This is clearly a contradiction to Theorem 4.3.

In the rest of the proof we address the compactness issue claimed in (6.7). To simplify our discussion we introduce the notation \(w - X\) and \(s - X\) to denote, respectively, weak convergence and strong convergence in any Hilbert space \(X\).

In view of (2.24) and (6.6) we deduce that

\[
\max_{k \in \{0,1,2\}} \|\partial_\phi^k w_j\|_{L^2(\Omega)} + \|\dot{\lambda}_j\|_{L^2(I)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T) \leq C,
\]

Hence, up to subsequences not relabeled for convenience, we may suppose that

\[
w_j \to w \quad \text{w-L}^2(\Omega), \quad \lambda_j \to \lambda \quad \text{w-H}^1(I) \quad \text{and} \quad \text{s-L}^2(I).
\]

In addition, (6.8) yields

\[
\|\partial_{\phi} w_j\|_{H^{-1}(\Omega)} + \|\partial_{\phi} w_j\|_{H^{-1}(\Omega)} + \|\dot{\lambda}_j\|_{H^{-1}(I)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T),
\]

and by the trace theory, from (6.8) we can also infer that for \(k \in \{0,1\}\)

\[
\begin{aligned}
\|\partial_{\phi}^k w_j(2\pi, \cdot)\|_{L^2(I)} + \|\partial_{\phi}^k w_j(0, \cdot)\|_{L^2(I)} \\
\leq \|\partial_{\phi}^k w_j(2\pi, \cdot)\|_{H^{1/2}(I)} + \|\partial_{\phi}^k w_j(0, \cdot)\|_{H^{1/2}(I)} \\
\leq C (\|\partial_{\phi}^k w_j\|_{L^2(\Omega)} + \|\partial_{\phi}^{k+1} w_j\|_{L^2(\Omega)}).
\end{aligned}
\]
In what follows we shall rewrite the whole system (2.11) satisfied by \((f_j, \vartheta_j)\) in terms of \(\gamma_j, w_j\) and \(\lambda_j\). We start off with the first equation,

\[
\partial \phi w_j + \partial \kappa w_j = \underbrace{\frac{w_j}{4} - \lambda_j \, \text{rad}_\phi}_{F_j :=} + \underbrace{\partial t w_j + \lambda_j \, \text{rad}_\phi}_{G_j :=},
\]

where

\[
I_j := -\frac{1}{\varepsilon_j} (\bar{\gamma}_j - \gamma_j) \, \text{rad}
\]

and

\[
I_j^{(2)} := 2\lambda_j \bar{\gamma}_j \, \text{rad}_\phi + (\bar{\lambda}_j - \lambda_j) \gamma_j \, \text{rad}_\phi
\]

\[
- \varepsilon_j \left( \lambda_j^2 (f_j)_{\phi\phi} + \lambda_j \left( -2 \partial \phi w_j + \partial \phi w_j \right) - \bar{\lambda}_j \partial \phi w_j \right)
\]

We claim that

\[
\lim_j \left( \|I_j\|_{L^2(\Omega)} + \|I_j^{(2)}\|_{L^2(\Omega)} \right) = 0.
\]

To establish (6.13), first note that the convergences in (6.9) together with (6.3) and (6.4) yield

\[
\|I_j^{(2)}\|_{L^2(\Omega)} \leq C \|\lambda_j\|_{L^2(I)} \cdot \left( \|\gamma_j\|_{C^1(I)} + \|\vartheta_j\|_{C^2(I)} \|f_j\|_{C^2(\Omega)} + \|\partial \phi w_j\|_{L^2(\Omega)} \right)
\]

Thus, we deduce that

\[
\|I_j^{(2)}\|_{L^2(\Omega)} = o\left(L^{1/2}(w_j, \lambda_j, 0, 3T)\right) \quad j \uparrow \infty,
\]

finally implying

\[
\lim_j \|I_j^{(2)}\|_{L^2(\Omega)} = 0.
\]

On the other hand, multiplying (6.12) by rad, an integration by parts and the \(L^2\)-orthogonality of \(v_j\) to rad (cp. (6.2)) yield

\[
- \int_0^{2\pi} I_j(\phi, t) \, \text{rad}(\phi) \, d\phi = \frac{2}{\varepsilon_j} (\bar{\gamma}_j - \gamma_j)(t)
\]

\[
= \int_0^{2\pi} I_j^{(2)}(\phi, t) \, \text{rad}(\phi) \, d\phi + \int_0^{2\pi} \partial \phi w_j \, \text{rad}(\phi) \, d\phi
\]

\[
= \int_0^{2\pi} I_j^{(2)}(\phi, t) \, \text{rad}(\phi) \, d\phi + \sqrt{\frac{2}{\pi}} \left( \partial \phi w_j(2\pi, t) + \partial \phi w_j(0, t) \right).
\]

Summing up the second and third equations in (2.11) gives

\[
\partial \phi w_j(2\pi, \cdot) + \partial \phi w_j(0, \cdot) = \frac{\lambda_j}{1 + \vartheta_j^2}
\]

\[
\cdot \left( 2\sqrt{\frac{2}{\pi}} \bar{\gamma}_j - \frac{\varepsilon_j}{2} \left( w_j(2\pi, \cdot) + \bar{w}_j(0, t) + 2\partial w_j(2\pi, \cdot) + 2\partial w_j(0, \cdot) \right) \right)
\]
in turn implying that
\[ \| \partial_\phi w_j(2\pi, \cdot) + \partial_\phi w_j(0, \cdot) \|_{L^2(I)} \leq C \| \dot{\lambda}_j \|_{L^2(I)} \left( \| \dot{\gamma}_j \|_{L^2(I)} + \| v_j(2\pi, \cdot) \|_{C^0(I)} + \| v_j(0, \cdot) \|_{C^0(I)} + \| (v_j)_t(2\pi, \cdot) \|_{C^0(I)} + \| (v_j)_t(0, \cdot) \|_{C^0(I)} \right). \]

Hence, \( \partial_\phi w_j(2\pi, \cdot) + \partial_\phi w_j(0, \cdot) \to 0 \) in \( L^2(I) \), by the decay properties of \( \gamma_j \) and \( \vartheta_j \) in (6.3) and (6.4), and by (6.8), (6.10). Thus, since
\[ \| I_j \|_{L^2(\Omega)} \leq \frac{C}{\varepsilon_j} \| \gamma_j - \dot{\gamma}_j \|_{L^2(I)} \]
\[ \leq C \left( \| I_j^{(2)} \|_{L^2(\Omega)} + \| \partial_\phi w_j(2\pi, \cdot) + \partial_\phi w_j(0, \cdot) \|_{L^2(I)} \right) \]
\[ = o(\mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T)) \quad j \uparrow \infty, \quad (6.15) \]
we have shown that
\[ \lim_j \| I_j \|_{L^2(\Omega)} = 0, \]
and (6.13) then follows at once. Therefore, we conclude that
\[ w_{\phi\phi} + w_{tt} = w_t - \frac{w}{4} + (\bar{\lambda} - \dot{\lambda}) \text{rad}_\phi. \quad (6.16) \]

Hence, from (6.12) and by taking into account (6.8), (6.10), (6.14) and (6.15) we deduce that
\[ \| F_j \|_{L^2(\Omega)} + \| G_j \|_{L^2(\Omega)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T). \quad (6.17) \]

Recalling (6.11), by Lemma 5.2 applied with \( k = 0 \) we infer for all \( d \in [0, 1] \]
\[ \| w_j \|_{H^1(\Omega_d)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T) \leq C. \quad (6.18) \]

A diagonalization argument then implies, again up to subsequences not relabeled, that for all \( d \in ]0, 1[ \) it holds true
\[
\begin{cases}
  w_j \to w & \text{w-}H^1(\Omega_d) \text{ and s-}L^2(\Omega_d) \\
  \partial_\phi w_j(0, \cdot) \to w_\phi(0, \cdot) & \text{s-}L^2(I_d) \\
  \partial_\phi w_j(2\pi, \cdot) \to w_\phi(2\pi, \cdot) & \text{s-}L^2(I_d) \\
  \partial_t w_j(0, \cdot) \to w_t(0, \cdot) & \text{s-}L^2(I_d) \\
  \partial_t w_j(2\pi, \cdot) \to w_t(2\pi, \cdot) & \text{s-}L^2(I_d).
\end{cases} \quad (6.19)
\]

Next, we analyze the second and third equations in (2.11). We start with the former. We have
\[ \partial_\phi w_j(0, \cdot) = \frac{\lambda_j}{1 + \vartheta_j^2} \left( -\frac{1 + \gamma_j}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} \dot{\gamma}_j - \frac{\varepsilon_j}{2} \dot{w}_j(0, \cdot) + \varepsilon_\vartheta \partial_t w_j(0, \cdot) \right), \]
then by (6.9), (6.19) and by the decay properties of \( \vartheta_j \) and \( \dot{\gamma}_j \) (cp. with (6.3), (6.4)) we get
\[ w_\phi(0, \cdot) = -\frac{1}{\sqrt{2\pi}} \dot{\lambda}. \quad (6.20) \]
Analogously, the third equation gives

$$w_\phi(2\pi, \cdot) = \frac{1}{\sqrt{2\pi}} \hat{\lambda}.$$  

(6.21)

Eventually, we deal with the fourth equation in (2.11) that we rewrite as follows,

$$\lambda_j - \bar{\lambda}_j = -\frac{\varepsilon_j^2 \lambda_j^3}{I_j^{(3)}} + (1 + \hat{\phi}_j^2)^{3/2} \frac{1}{\varepsilon_j} \left[ \frac{f_j}{2} + \hat{\phi}_j \partial_\phi f_j - \partial_\varepsilon f_j \right]^2 + (\partial_\phi f_j)^2 \right] \frac{2\pi}{0}.$$  

(6.22)

By taking into account (6.9) and the equality $I_j^{(3)} = \hat{\phi}_j^2 \bar{\lambda}_j$, we infer from (6.19)

$$\|I_j^{(3)}\|_{L^2(I)} \leq \|\hat{\phi}_j\|_{C^0(I)}^2 \|\bar{\lambda}_j\|_{L^2(I)} \leq \|\hat{\phi}_j\|_{C^0(I)}^2 \mathcal{L}^{1/2}(w, \lambda_j, 0, 3T),$$  

(6.23)

and thus by (6.3)

$$\lim_j \|I_j^{(3)}\|_{L^2(I)} = 0.$$  

(6.24)

Recalling that $\text{rad}_\phi(0) = \text{rad}_\phi(2\pi) = 0$ we find

$$\frac{1}{\varepsilon_j} \left[ (\partial_\phi f_j)^2 \right]^{2\pi} = \varepsilon_j \left( (\partial_\phi w_j(2\pi, t))^2 - (\partial_\phi w_j(0, t))^2 \right)$$

implying

$$\frac{1}{\varepsilon_j} \left[ (\partial_\phi f_j)^2 \right]^{2\pi} \leq C \|v_j\|_{C^0(I)} \left( \|\partial_\phi w_j(2\pi, \cdot)\|_{L^2(I)} + \|\partial_\phi w_j(0, \cdot)\|_{L^2(I)} \right),$$

so that by (6.4)

$$\lim_j \frac{1}{\varepsilon_j} \left[ (\partial_\phi f_j)^2 \right]^{2\pi} = 0 \quad \text{in} \quad L^2(I).$$

In addition, using $\text{rad}_\phi(0) = \text{rad}_\phi(2\pi) = 0$ and $\text{rad}(0) = -\text{rad}(2\pi)$ we get

$$\frac{1}{\varepsilon_j} \left[ \left( \frac{f_j}{2} + \hat{\phi}_j \partial_\phi f_j - \partial_\varepsilon f_j \right) \right]^{2\pi} \left. \right|_0$$

$$= \frac{1}{\varepsilon_j} \left[ \left( \frac{1 + \gamma_j - \gamma_j}{2} \right) \text{rad} - \varepsilon_j \left( \frac{w_j(\cdot, t)}{2} - \partial_t w_j(\cdot, t) + \varepsilon_j \hat{\lambda}_j \partial_\phi w_j(\cdot, t) \right) \right]^{2\pi} \left. \right|_0$$

$$= \left( \frac{w_j(0, t) + w_j(2\pi, t)}{2} - (\partial_t w_j(0, t) + \partial_t w_j(2\pi, t)) \right.$$

$$+ \varepsilon_j \hat{\lambda}_j(\partial_\phi w_j(0, t) + \partial_\phi w_j(2\pi, t))) \cdot \left( \sqrt{\frac{2}{\pi}} (1 + \gamma_j - 2\gamma_j) \right.$$

$$+ \varepsilon_j \hat{\lambda}_j(\partial_\phi w_j(0, t) - \partial_\phi w_j(2\pi, t)) \right) \left( \sqrt{\frac{2}{\pi}} \right) \left( \frac{2}{\pi} (1 + \gamma_j - 2\gamma_j) \right. \right.$$  

(6.25)

$$+ \varepsilon_j \hat{\lambda}_j(\partial_\phi w_j(0, t) - \partial_\phi w_j(2\pi, t)) \right).$$  

(6.26)
In particular, by taking into account (6.24), (6.19) gives by passing to the limit in (6.26) as \( j \uparrow \infty \)
\[
\dot{\lambda} - \dot{\lambda} = \frac{1}{\sqrt{2\pi}} \left( w(0, t) + w(2\pi, t) - 2w_t(0, t) - 2w_t(2\pi, t) \right). \tag{6.27}
\]
By collecting (6.16), (6.20), (6.21) and (6.27) we have established that \((w, \lambda)\) solves (3.1).

Furthermore, (6.18) yields for all \( d \in [0, 1]\)
\[
\| I_j^{(4)} \|_{L^2(I_d)} \leq C \left( 1 + \| \dot{\gamma}_j \|_{C^1(I)} + \| \dot{\phi}_j \|_{C^1(I)} \right) \| w_j \|_{C^2(\Omega)} \cdot \left( \| w_j(0, \cdot) \|_{L^2(I)} + \| w_j(2\pi, \cdot) \|_{L^2(I)} + \| \partial_t w_j(0, \cdot) \|_{L^2(I)} + \| \partial_t w_j(2\pi, \cdot) \|_{L^2(I)} + \| \partial_{\phi} w_j(0, \cdot) \|_{L^2(I)} + \| \partial_{\phi} w_j(2\pi, \cdot) \|_{L^2(I)} \right),
\]
from which, in view of (6.3), (6.4), (6.11) and (6.17), we deduce
\[
\| I_j^{(4)} \|_{L^2(I_d)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T). \tag{6.28}
\]
Hence, plugging (6.23) and (6.28) into (6.22) we conclude that
\[
\| \dot{\lambda}_j \|_{L^2(I_d)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T). \tag{6.29}
\]
Therefore, the estimates in (6.18) and (6.29) yield
\[
\| w_j \|_{H^1(\Omega_d)} + \| \dot{\lambda}_j \|_{H^1(I_d)} \leq C \mathcal{L}^{1/2}(w_j, \lambda_j, 0, 3T). \tag{6.30}
\]
In estimate (6.30) above we have gained an exponent 1 in the Sobolev norms both of \( w_j \) and of \( \lambda_j \) with respect to the initial bounds in (6.8). A bootstrap argument (using Lemma 5.2) then gives \( H^k \)-estimates, for every \( k \in \mathbb{N} \), both for \((w_j)_{j \in \mathbb{N}}\) and for \((\lambda_j)_{j \in \mathbb{N}}\). Thus, the \( C^k \)-convergence claimed in (6.7) follows at once. \( \square \)

7. Proof of Theorem 1.2

Proof. Let \( \eta, \delta \) and \( k \in \mathbb{N} \) be as in Theorem 2.4. Then, by Lemma 2.3 we can find \( \varepsilon_0 > 0 \) such that if \( \sup_r |\alpha'(r)| \leq \varepsilon_0 \), then
\[
\| f - \text{rad} \|_{C^k} + \| \vartheta \|_{C^k} < \delta. \tag{7.1}
\]
In particular, Theorem 2.4 and an induction argument yield for all \( j \in \mathbb{N} \)
\[
\mathcal{L}(v, \vartheta, jT, (j + 3)T) < 3(1 - \eta)^j \mathcal{L}(v, \vartheta, 0, T). \tag{7.2}
\]
Indeed, suppose by contradiction that for some \( \ell \in \mathbb{N} \)
\[
\mathcal{L}(v, \vartheta, (\ell + 1)T, (\ell + 2)T) \geq (1 - \eta) \mathcal{L}(v, \vartheta, \ell T, (\ell + 1)T),
\]
then Theorem 2.4 would imply that
\[
\mathcal{L}(v, \vartheta, (\ell + 2)T, (\ell + 3)T) \geq (1 + \eta/2) \mathcal{L}(v, \vartheta, (\ell + 1)T, (\ell + 2)T).
\]
Therefore, by induction, for all \( j \geq \ell + 1 \) we would infer that
\[
\mathcal{L}(v, \vartheta, jT, (j + 1)T) \geq (1 - \eta)(1 + \eta/2)^{j-\ell-1} \mathcal{L}(v, \vartheta, \ell T, (\ell + 1)T).
\]
This is in contradiction to the equi-boundedness of the energy on each interval \((j T, (j + 1)T)\) that follows from (2.24), (7.1) and by taking into account that for some universal positive constant \(C\)

\[ \|v\|_{C^k} \leq C \|f - \text{rad}\|_{C^k}. \]

Therefore, for all \(j \in \mathbb{N}\)

\[ \mathcal{L}(v, \vartheta, (j + 1)T, (j + 2)T) < (1 - \eta) \mathcal{L}(v, \vartheta, j T, (j + 1)T), \]

and thus

\[ \mathcal{L}(v, \vartheta, (j + 1)T, (j + 2)T) < (1 - \eta)^{j + 1} \mathcal{L}(v, \vartheta, 0, T), \]

from which (7.2) follows at once.

By setting \(f_j(\phi, t) := f(\phi, t + j) = (1 + \gamma_j(t)) \text{rad}(\phi) + v_j(\phi, t)\) and \(\vartheta_j(t) := \vartheta(t + jT)\), the growth condition (2.24) implies that for all \(j \in \mathbb{N}\) we have

\[ \max_{k \in \{0, 1, 2\}} \| \partial_k \vartheta_j \|_{L^2(\Omega)} + \| \dot{\vartheta}_j \|_{L^2(I_j)} \leq \mathcal{L}^{1/2}(v, \vartheta, j T, (j + 3)T). \]  

(7.3)

Note also that (7.1) rewrites as

\[ \sup_j \left( \|f_j - \text{rad}\|_{C^k} + \|\vartheta_j\|_{C^k} \right) < \delta, \]

and thus for some positive constant \(C\) we have

\[ \sup_j \|\gamma_j\|_{C^k} < C \delta. \]

Therefore, with fixed \(d \in [0, 1]\), thanks to (7.3) we can argue as in the proof of the compactness issue in Theorem 2.4, without rescaling by \(\varepsilon_j\), to infer for some constant \(C > 0\) depending on \(d\) and \(\delta\)

\[ \|\dot{\vartheta}_j\|_{H^1(I_d)} \leq C \mathcal{L}^{1/2}(v, \vartheta, j T, (j + 3)T) \leq C (1 - \eta)^{j/2} \mathcal{L}^{1/2}(v, \vartheta, 0, T), \]  

(7.4)

(cp. with (6.30) and the paragraph afterwards).

Let us first prove that \(\lim_{n \uparrow \infty} \vartheta^{(i)}(nT) = 0, \ i \in \{1, 2\}\). We have for all \(m, n \in \mathbb{N}\) with \(m < n\)

\[ |\vartheta^{(i)}(nT) - \vartheta^{(i)}(mT)| \leq \sum_{j = m}^{n-1} |\vartheta^{(i)}((j + 1)T) - \vartheta^{(i)}(jT)| \]

\[ \leq T^{1/2} \sum_{j = m}^{n-1} \|\dot{\vartheta}^{(i+1)}\|_{L^2([jT, (j+1)T])} \]

\[ = T^{1/2} \sum_{j = m}^{n-1} \|\dot{\vartheta}^{(i+1)}\|_{L^2([jT, (j+1)T])} \]

\[ \leq C T^{1/2} \eta^{-1} (1 - \eta)^{m/2} \mathcal{L}^{1/2}(v, \vartheta, 0, T). \]
Therefore, the limit of \((\vartheta^{(i)}(nT))_{n \in \mathbb{N}}\) exists, and being \(\vartheta^{(i)} \in L^2([0, +\infty[)\) it is necessarily equal to 0. In particular, from the latter we can actually deduce that for \(i \in \{1, 2\}\) and for all \(t > 0\)
\[
|\vartheta^{(i)}(t)| \leq |\vartheta^{(i)}(T/t) + 1 - \vartheta^{(i)}(t)| + \sum_{j \geq \lfloor t/T \rfloor + 1} |\vartheta^{(i)}((j + 1)T) - \vartheta^{(i)}(jT)|
\leq CT^{1/2} \eta^{-1} (1 - \eta)^{\lfloor t/T \rfloor/2} \mathcal{L}^{1/2}(\vartheta, 0, T),
\]
where \(\lfloor t/T \rfloor\) denotes the integer part of \(t/T\). The latter inequality straightforwardly implies
\[
\sup_t e^{\kappa t} |\vartheta^{(i)}(t)| \leq CT^{1/2} (\eta(1 - \eta)^{1/2})^{-1} \mathcal{L}^{1/2}(\vartheta, 0, T)
\]
with \(\kappa = \frac{1}{2\pi} |\ln(1 - \eta)| > 0\). In conclusion, (1.4) follows at once recalling the identities
\[
\dot{\vartheta}(t) = -e^{-t} \alpha'(e^{-t}), \quad \ddot{\vartheta}(t) + \dot{\vartheta}(t) = e^{-2t} \alpha''(e^{-t}).
\]

8. Proof of Theorem 1.1

Proof. By the usual simple rescaling argument we can assume that \(r = 1\) and \(x_0 = 0\). It then follows from [13, Theorem 69.29] that, if \(\varepsilon\) is sufficiently small, there is a point \(y_0 \in B_{1/\varepsilon}(0)\) such that the set \(S_u \cap B_{1/\varepsilon}(y_0)\) is given by
\[
\{ y_0 + r(\cos \alpha(r), \sin \alpha(r)) \}
\]
for some smooth function \(\alpha : [0, 1/2[ \to \mathbb{R}\) which satisfies
\[
|\alpha'(r)| \leq \frac{1}{8r} \quad \text{for all } r \in [0, 1/2[. \quad (8.1)
\]
However, as observed in the first line of page 484 of [13], 8 can be replaced by any other large constant, provided \(\varepsilon\) is chosen sufficiently small. In particular, it can be chosen small enough so to imply \(\sup_{r \in [0, 1/2[} |\alpha'(r)| \leq \varepsilon_0\) for the positive constant \(\varepsilon_0\) of Theorem 1.2.

As observed in (1.5) this gives a \(C^{0, \kappa}\) estimate on the tangent \(\tau(r)\) to \(S_u\) at the point \(r(\cos \alpha(r), \sin \alpha(r))\). It follows easily that there is an \(\eta > 0\), depending only upon \(C\) and \(\kappa\), such that \(B_\eta(y_0) \cap S_u\) is a graph \(\{ y_0 + t \nu_1 + \psi(t) \nu_2 \cap B_\eta(y_0)\) for some orthonormal basis \(\nu_1, \nu_2\) and some \(C^{1, \kappa}\) function \(\psi : [0, \eta] \to \mathbb{R}\) with \(\psi(0) = \psi'(0) = 0\) and \(\|\psi\|_{C^{1, \kappa}} \leq \tilde{C}\), where \(\tilde{C}\) depends only on \(C\). On the other hand, if \(\varepsilon_0\) is sufficiently small, since \(|\alpha'(r)| \leq \varepsilon_0/\eta\) for \(r \in [\eta, 1/\varepsilon[\), we conclude that \(S_u \cap B_{1/\varepsilon}(y_0)\) is graphical in the coordinates induced by the orthonormal base \(\{\nu_1, \nu_2\}\). Finally, since such graph will have to be sufficiently close to the line \(\{(s, 0) : s \geq 0\}\), we conclude that \(\nu_1\) must be close to \((1, 0)\). Therefore, \(S_u \cap B_{1/\varepsilon}(y_0)\) is in fact a graph in the standard coordinates, as claimed in Theorem 1.1. □
Remark 8.1. Combining Theorem 1.1 with the results of [13] we can infer also a further version of the regularity result. There are universal constants \( \varepsilon, \alpha > 0 \) with the following property. Let \( u \) be a local minimizer of the Mumford-Shah functional in the ball \( B_r(x) \) such that, for some line \( \ell \) passing through \( x \), \( S_u \) is contained in the \( \varepsilon r \)-neighborhood of \( \ell \). Then one of the following alternatives hold:

- Either \( B_{r/2}(x) \cap S_u \) is empty;
- or \( B_{r/2}(x) \cap S_u \) is a \( C^{1,\alpha} \) embedded arc with endpoints lying in \( \partial B_{r/2}(x) \);
- or \( B_{3r/4}(x) \cap S_u \) is a \( C^{1,\alpha} \) embedded arc with one endpoint in \( \partial B_{3r/4}(x) \) and one endpoint \( y_0 \in B_{r/2}(x) \).

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