Integrable systems of quartic oscillators. II

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Abstract

Several completely integrable, indeed solvable, Hamiltonian many-body problems are exhibited, characterized by Newtonian equations of motion ("acceleration equal force"), with linear and cubic forces, in \( N \)-dimensional space (\( N \) being an arbitrary positive integer, with special attention to \( N = 2 \), namely motions in a plane, and \( N = 3 \), namely motions in ordinary three-dimensional space). All the equations of motion are written in covariant form ("\( N \)-vector equal \( N \)-vector"), entailing their rotational invariance. The corresponding Hamiltonians are of normal type, with the kinetic energy quadratic in the canonical momenta, and the potential energy quadratic and quartic in the canonical coordinates.

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1 Introduction

In a recent paper [1] we introduced several integrable, indeed solvable, nonlinear oscillators. The basic idea was to start from the integrable, indeed solvable, matrix ODE [2] [3] [4] [5]
\[ \ddot{U} = - (AU + UA + U^3) , \] (1)
and to transform it into a system of covariant, hence rotation-invariant, Newtonian equations ("acceleration equal force", with linear and cubic forces) for the time evolution of N-vectors, by parameterizing appropriately the matrix \( U \) in terms of these N-vectors. In the present paper we follow the same approach [1] [3] [4], but we exploit certain additional, convenient parameterizations of matrices in terms of vectors, which we presented recently [6] and which also transform the matrix evolution equation (1) into covariant, hence rotation-invariant, Newtonian equations of motions for N-vectors in N-dimensional space. We thereby obtain – and exhibit in the following Section 2 – the Newtonian equations of motion of several new integrable, indeed solvable, systems of oscillators subjected to linear and cubic forces. As shown in Section 3, all these systems are Hamiltonian, with standard Hamiltonians of normal form: the sum of a standard kinetic energy quadratic in the momenta and a potential energy depending (quadratically and quartically) only on the canonical coordinates. In view of the phenomenological relevance of such nonlinear oscillators, it is remarkable that several instances of such systems exist which are integrable indeed solvable.

Investigation of their detailed dynamics is postponed to future papers.

Notation: throughout this paper matrices are denoted by underlined characters, and vectors by superimposed arrows; their dimensionality will in each case be clear from the context. Superimposed dots indicate differentiations with respect to the independent variable ("time"), while dots sandwiched among two vectors denote the standard scalar product. As a rule, we write the equations of motion in the neatest form, forsaking any attempt to introduce additional constants by rescaling the independent variable and/or by redefining the dependent variables via linear transformations.

2 Newtonian equations of motion

A convenient way to obtain covariant ("vector equal vector") hence rotation-invariant Newtonian equations of motion is to insert in the matrix evolution equation (1) the parameterizations introduced in [6] (hereafter the equation (n) of this paper [6] is identified here as (I.n)). Since the parameterization P1 (I.3) was already exploited in [1], we exhibit here only the new dynamical systems yielded by the new parameterization P2 (I.4). Let us emphasize that, since the matrix evolution equation (1) is integrable, indeed solvable, all the dynamical systems obtained in this manner are as well integrable, indeed solvable.
Let us begin by writing down our basic result, obtained by inserting the formulas (1.4) into (1) and by then performing some appropriate, merely notational, changes:

\[ - \rightarrow r^{(\nu \lambda)}(n \ell) = - \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left\{ F^{(\nu \lambda)} \left[ a^{(\nu \lambda)}_{\lambda \ell} + \left( - \rightarrow \bar{r}^{(\nu \lambda)} \cdot - \rightarrow r^{(\nu \lambda)}(n \ell) \right) \right] \right\}, \quad (2a) \]

\[ - \rightarrow \bar{r}^{(\nu \lambda)}(n \ell) = - \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left\{ F^{(\nu \lambda)} \left[ a^{(\nu \lambda)}_{\nu n} + \left( - \rightarrow \bar{r}^{(\nu \lambda)} \cdot - \rightarrow \bar{r}^{(\nu \lambda)}(n \ell) \right) \right] \right\}, \quad (2b) \]

Here, and throughout, the index \( n \) (as well of course as \( \nu \)) ranges from 1 to \( N \), and the index \( \ell \) (as well of course as \( \lambda \)) ranges from 1 to \( L \), with \( N \) and \( L \) two arbitrary positive integers; the \( 2NL \) time-dependent quantities \(- \rightarrow r^{(\nu \lambda)}(n \ell) \equiv - \rightarrow r^{(\nu \lambda)}(n \ell)(t)\), \(- \rightarrow \bar{r}^{(\nu \lambda)}(n \ell) \equiv - \rightarrow \bar{r}^{(\nu \lambda)}(n \ell)(t)\) are \( N \)-vectors (identifying the positions of moving point particles in \( N \)-dimensional space); the dots sandwiched among \( N \)-vectors denote the standard scalar product; and the \( (NL)^2 \) quantities \( a^{(\nu \lambda)}_{n \ell} \) are arbitrary scalar constants. Because of the way they have been obtained, these \( 2NL \) Newtonian equations of motion, featuring linear and cubic forces, are integrable indeed solvable; and they are clearly covariant (\( N \)-vector equal \( N \)-vector), hence rotation-invariant in \( N \)-dimensional space.

It would seem that this system always involves a number of \( N \)-vectors which is an even multiple of the number of dimensions \( N \) of the vector space. But this is not the case, due to the possibility of reductions, which may be enforced (with an appropriate choice of initial conditions) provided the constants \( a^{(\nu \lambda)}_{n \ell} \) satisfy appropriate restrictions. For instance if these constants satisfy the symmetry restriction

\[ a^{(\nu \lambda)}_{n \ell} = a^{(\nu n)}_{\ell \lambda}, \quad (3a) \]

it is clearly possible to impose the corresponding reduction

\[ - \rightarrow r^{(n \ell)} = - \rightarrow \bar{r}^{(n \ell)}, \quad (3b) \]

which obviously halves (from \( 2NL \) to \( NL \)) the number of evolving vectors, as well as the number of \( N \)-vector Newtonian equations of motion, which clearly read then (see (2c))

\[ - \rightarrow r^{(n \ell)} = - \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left\{ F^{(\nu \lambda)} \left[ a^{(\nu n)}_{\nu \ell} + \left( - \rightarrow \bar{r}^{(\nu \lambda)} \cdot - \rightarrow \bar{r}^{(\nu \lambda)}(n \ell) \right) \right] \right\}. \quad (3c) \]

Other (or additional) reductions of the Newtonian equations of motion are moreover possible, in which some of the \( N \)-vectors vanish identically, since clearly, provided for some value of the indices \( n \) and \( \ell \) the constants \( a^{(\nu \lambda)}_{n \ell} \) vanish unless \( \nu = n \) and \( \lambda = \ell \) (\( a^{(\nu \lambda)}_{n \ell} = \delta_{n \nu} \delta_{\ell \lambda} a^{(n)}_{\ell} \) for \( \nu = 1, ..., N; \lambda = 1, ..., L \) with \( n, \ell \) fixed), then it is consistent, see (2d), to set \(- \rightarrow r^{(n \ell)} = 0\); and likewise, provided for some value of the indices \( n \) and \( \ell \) the constants \( a^{(\nu \lambda)}_{n \ell} \) vanish unless \( \nu = n \).
and \( \lambda = \ell \) (\( a_{\ell \lambda}^{(n \nu)} = \delta_{n \nu} \delta_{\lambda \ell} a_{\ell}^{(n)} \)) for \( \nu = 1, \ldots, N; \) \( \lambda = 1, \ldots, L \) with \( n, \ell \) fixed), then it is consistent, see \( 20 \), to set \( \vec{r}^{(n \ell)} = 0 \).

Let us also point out that, if all the constants \( a_{\ell \lambda}^{(n \nu)} \) vanish, so that the Newtonian equations of motion \( 2 \) only involve cubic forces, by applying an argument analogous to that used in \( 7 \) one concludes that all the nonsingular solutions of the following (\( \omega \)-deformed, complex) Newtonian equations of motion,

\[
\ddot{\vec{r}}^{(n \ell)} - 3 i \omega \dot{\vec{r}}^{(n \ell)} - 2 \omega^2 \vec{r}^{(n \ell)} = c \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left[ \vec{p}^{(\nu \lambda)} \left( \vec{p}^{(\nu \lambda)} \cdot \vec{r}^{(n \ell)} \right) \right],
\]

\[
\ddot{\vec{r}}^{(n \ell)} - 3 i \omega \dot{\vec{r}}^{(n \ell)} - 2 \omega^2 \vec{r}^{(n \ell)} = c \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left[ \vec{p}^{(\nu \lambda)} \left( \vec{p}^{(\nu \lambda)} \cdot \vec{r}^{(n \ell)} \right) \right].
\]

are completely periodic with period \( 2 \pi / \omega \), provided \( \omega \) is a nonvanishing real (without loss of generality, positive) constant. Note that these equations of motion are complex (see the second term in their left-hand sides); but real equations can be obtained from these by introducing the real and imaginary parts of the \( N \)-vectors \( \vec{r}^{(n \ell)} \) and \( \vec{r}^{(n \ell)} \), and of the "coupling constant" \( c \) (which has been introduced in \( 4 \)) via a trivial rescaling of the dependent variables, and by then considering separately the real and imaginary parts of these equations. There thus results a doubling (from \( 2 N L \) to \( 4 N L \)) of the number of evolving real \( N \)-vectors; but this number can be cut down via the same kind of reductions discussed above. No additional elaboration is reported below on these systems of nonlinear harmonic oscillators \( 7 \).

In the rest of this Section 2 we display explicitly, in the physically more interesting cases with \( N = 2 \) (evolutions in the plane) and \( N = 3 \) (evolutions in ordinary, three-dimensional, space), some simple examples involving few moving points. For the sake of completeness, and for the reader’s enlightenment, we also display, in self-evident notation \( 9 \), the parameterizations that relate the equations of motion we report to the basic matrix evolution equation \( 1 \).

### 2.1 Oscillators in the plane

Let us consider the following parameterization of a \( 4 \otimes 4 \) matrix \( U \) (see \( 1 \)) in terms of four \( 2 \)-vectors (see \( 1.6 \)):

\[
U = \left( \vec{r}^{(1)}, \vec{r}^{(2)}, \vec{p}^{(1)}, \vec{p}^{(2)} \right), \tag{5a}
\]

\[
\vec{p}^{(n)} \equiv \left( x^{(n)}, y^{(n)} \right), \quad \vec{r}^{(n)} \equiv \left( \dot{x}^{(n)}, \dot{y}^{(n)} \right), \quad n = 1, 2, \tag{5b}
\]

\[
U = \begin{pmatrix}
0 & x^{(1)} & 0 & x^{(2)} \\
\dot{x}^{(1)} & 0 & \dot{x}^{(2)} & 0 \\
0 & y^{(1)} & 0 & y^{(2)} \\
\dot{y}^{(2)} & 0 & \dot{y}^{(2)} & 0
\end{pmatrix}, \tag{5c}
\]
as well as the following parameterization of the constant $4 \otimes 4$ matrix $A$:

$$A = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & a^{(11)} - \alpha & 0 & a^{(21)} \\
0 & 0 & \alpha & 0 \\
0 & a^{(12)} & 0 & a^{(22)} - \alpha
\end{pmatrix}.$$ (5d)

Note that we are considering a case with $N = 2$, $L = 1$.

We then obtain the following integrable, indeed solvable, system of four coupled linear plus cubic oscillators in the plane:

$$\ddot{\mathbf{r}}^{(1)} + a^{(11)} \mathbf{r}^{(1)} + a^{(21)} \mathbf{r}^{(2)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(1)} \right) - \mathbf{r}^{(2)} \left( \mathbf{r}^{(2)} \cdot \mathbf{r}^{(1)} \right),$$ (6a)

$$\ddot{\mathbf{r}}^{(2)} + a^{(12)} \mathbf{r}^{(1)} + a^{(22)} \mathbf{r}^{(2)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)} \right) - \mathbf{r}^{(2)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)} \right),$$ (6b)

$$\ddot{\mathbf{r}}^{(1)} + a^{(11)} \mathbf{r}^{(1)} + a^{(12)} \mathbf{r}^{(2)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(1)} \right) - \mathbf{r}^{(2)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)} \right),$$ (6c)

$$\ddot{\mathbf{r}}^{(2)} + a^{(21)} \mathbf{r}^{(1)} + a^{(22)} \mathbf{r}^{(2)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(2)} \cdot \mathbf{r}^{(1)} \right) - \mathbf{r}^{(2)} \left( \mathbf{r}^{(2)} \cdot \mathbf{r}^{(2)} \right).$$ (6d)

Further reductions to systems of less than four 2-vectors are easy to obtain by appropriate reductions. For instance by setting to zero the coefficient $a^{(21)}$ and the 2-vector $\mathbf{r}^{(2)}$ (and setting for notational convenience $a^{(11)} = a$, $a^{(12)} = b$, $a^{(22)} = c$ and $\mathbf{r}^{(1)} = \mathbf{r}^{(3)}$) we get the following system of three coupled oscillators in the plane:

$$\ddot{\mathbf{r}}^{(1)} + a \mathbf{r}^{(1)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(1)} \cdot \mathbf{r}^{(3)} \right) - \mathbf{r}^{(2)} \left( \mathbf{r}^{(2)} \cdot \mathbf{r}^{(1)} \right),$$ (7a)

$$\ddot{\mathbf{r}}^{(2)} + b \mathbf{r}^{(1)} + c \mathbf{r}^{(2)} = - \mathbf{r}^{(1)} \left( \mathbf{r}^{(2)} \cdot \mathbf{r}^{(3)} \right),$$ (7b)

$$\ddot{\mathbf{r}}^{(3)} + a \mathbf{r}^{(3)} = - \mathbf{r}^{(3)} \left( \mathbf{r}^{(3)} \cdot \mathbf{r}^{(1)} \right).$$ (7c)

By setting moreover to zero $\mathbf{r}^{(2)}$ and $b$, and setting $\mathbf{r}^{(1)} = \mathbf{r}$, $\mathbf{r}^{(3)} = \mathbf{\tilde{r}}$, we obtain the following system of two coupled oscillators in the plane:

$$\ddot{\mathbf{r}} + a \mathbf{r} = - \mathbf{r} \left( \mathbf{r} \cdot \mathbf{\tilde{r}} \right),$$ (8a)

5
\[ \ddot{\vec{r}} + a \vec{r} = - \vec{r} \left( \vec{r} \cdot \vec{r} \right) . \]  

(8b)

And clearly the choice \( \vec{r} = \vec{r} \) reduces this system to the (trivially solvable) single equation

\[ \ddot{\vec{r}} + a \vec{r} = - \vec{r}^2 . \]  

(9)

A more general system of two coupled oscillators than (8) obtains from (6) by setting

\begin{align*}
\alpha^{(1)} & = a, \\
\alpha^{(2)} & = a, \\
\alpha^{(3)} & = b, \\
\alpha^{(4)} & = c, \\
\alpha^{(5)} & = \mu^{(1)}, \\
\alpha^{(6)} & = \mu^{(2)};
\end{align*}

\[ \ddot{\vec{r}}^{(1)} + a \vec{r}^{(1)} + b \vec{r}^{(2)} = - \vec{r}^{(1)} \left( \vec{r}^{(1)} \cdot \vec{r}^{(1)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(1)} \right) , \]  

(10a)

\[ \ddot{\vec{r}}^{(2)} + b \vec{r}^{(1)} + c \vec{r}^{(2)} = - \vec{r}^{(1)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(2)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(2)} \right) . \]  

(10b)

And of course from this system the single equation (9) can be easily obtained via an additional, obvious, reduction.

### 2.2 Oscillators in ordinary (3–dimensional) space

Let us consider the following parameterization of a 6 \( \otimes \) 6 matrix \( U \) (see (1)) in terms of six 3–vectors (see (I.7)):

\[ U \equiv \left( \overrightarrow{r}^{(1)}, \overrightarrow{r}^{(2)}, \overrightarrow{r}^{(3)}; \overline{\vec{r}}^{(1)}, \overline{\vec{r}}^{(2)}, \overline{\vec{r}}^{(3)} \right) , \]  

(11a)

\[ \overrightarrow{r}^{(n)} \equiv \left( x^{(n)}, y^{(n)}, z^{(n)} \right) , \quad \overline{\vec{r}}^{(n)} \equiv \left( \bar{x}^{(n)}, \bar{y}^{(n)}, \bar{z}^{(n)} \right) , \quad n = 1, 2, 3 ; \]  

(11b)

\[ U = \begin{pmatrix}
0 & x^{(1)} & 0 & x^{(2)} & 0 & x^{(3)} \\
\bar{x}^{(1)} & 0 & \bar{y}^{(1)} & 0 & \bar{z}^{(1)} & 0 \\
0 & y^{(1)} & 0 & y^{(2)} & 0 & y^{(3)} \\
\bar{x}^{(2)} & 0 & \bar{y}^{(2)} & 0 & \bar{z}^{(2)} & 0 \\
0 & z^{(1)} & 0 & z^{(2)} & 0 & z^{(3)} \\
\bar{x}^{(3)} & 0 & \bar{y}^{(3)} & 0 & \bar{z}^{(3)} & 0
\end{pmatrix} , \]  

(11c)

as well as the following parameterization of the constant 6 \( \otimes \) 6 matrix \( A \):

\[ A = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha^{(1)} - \alpha & 0 & a^{(12)} & 0 & a^{(13)} \\
0 & 0 & \alpha & 0 & 0 & 0 \\
0 & a^{(21)} & 0 & \alpha^{(2)} - \alpha & 0 & a^{(23)} \\
0 & 0 & 0 & 0 & \alpha & 0 \\
0 & a^{(31)} & 0 & a^{(32)} & 0 & \alpha^{(3)} - \alpha
\end{pmatrix} . \]  

(11d)
Note that we are considering a case with $N = 3$, $L = 1$.

We then obtain the following integrable, indeed solvable, system of six coupled linear plus cubic oscillators in ordinary (3-dimensional) space:

$$\dot{\vec{r}}^{(n)} = -\sum_{\nu=1}^{3} \left\{ \vec{p}^{(\nu)} \left[ a^{(\nu n)} + \left( \vec{r}^{(\nu)} \cdot \vec{p}^{(n)} \right) \right] \right\}, \quad n = 1, 2, 3 \quad (12a)$$

$$\dot{\vec{r}} = -\sum_{\nu=1}^{3} \left\{ \vec{p}^{(\nu)} \left[ a^{(n \nu)} + \left( \vec{r}^{(n)} \cdot \vec{p}^{(\nu)} \right) \right] \right\}, \quad n = 1, 2, 3 \quad (12b)$$

It is plain that if $a^{(n \nu)} = a^{(\nu n)}$ one can set $\vec{p}^{(n)} = \vec{r}^{(n)}$, obtaining thereby the following system of three coupled linear plus cubic oscillators:

$$\dot{\vec{r}}^{(n)} = -\sum_{\nu=1}^{3} \left\{ \vec{p}^{(\nu)} \left[ a^{(\nu n)} + \left( \vec{r}^{(\nu)} \cdot \vec{r}^{(n)} \right) \right] \right\}, \quad n = 1, 2, 3 \quad (13)$$

Reductions of the system (12) to a number of oscillators smaller than six can be simply obtained by setting to zero some vectors and, for consistency, some of the constants $a^{(n \nu)}$. For instance, by setting $a^{(12)} = a^{(13)} = a^{(32)} = a^{(23)} = 0$ (and, for notational convenience, $a^{(11)} = a$, $a^{(21)} = b$, $a^{(12)} = c$, $a^{(22)} = d$) and $\vec{p}^{(3)} = \vec{r}^{(3)} = 0$, one gets the following system of four coupled linear plus cubic oscillators in ordinary (3-dimensional) space:

$$\dot{\vec{r}}^{(1)} + a \vec{r}^{(1)} + b \vec{r}^{(2)} = -\vec{r}^{(1)} \left( \vec{r}^{(1)} \cdot \vec{r}^{(1)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(1)} \cdot \vec{r}^{(2)} \right), \quad (14a)$$

$$\vec{r}^{(2)} + c \vec{r}^{(1)} + d \vec{r}^{(2)} = -\vec{r}^{(1)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(1)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(2)} \right), \quad (14b)$$

$$\dot{\vec{r}}^{(1)} + a \vec{r}^{(1)} + b \vec{r}^{(2)} = -\vec{r}^{(1)} \left( \vec{r}^{(1)} \cdot \vec{r}^{(1)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(1)} \cdot \vec{r}^{(2)} \right), \quad (14c)$$

$$\vec{r}^{(2)} + c \vec{r}^{(1)} + d \vec{r}^{(2)} = -\vec{r}^{(1)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(1)} \right) - \vec{r}^{(2)} \left( \vec{r}^{(2)} \cdot \vec{r}^{(2)} \right). \quad (14d)$$

Note that, denoting by $\pi$ the plane identified by the requirement that the two vectors $\vec{r}^{(1)}$ and $\vec{r}^{(2)}$ lie initially (namely, at $t = 0$) in it, if the initial velocities $\vec{r}^{(1)}(0)$ and $\vec{r}^{(2)}(0)$ also lie in the same plane $\pi$, then the two vectors $\vec{r}^{(1)}(t)$ and $\vec{r}^{(2)}(t)$ remain in this same plane $\pi$ throughout their time evolution – independently of the behavior of the two vectors $\vec{r}^{(1)}$ and $\vec{r}^{(2)}$, which of course might as well also always remain on some (possibly different) plane, say $\tilde{\pi}$, if their initial velocities lie in the same plane defined by their initial positions.

Further reductions to two, or just one, oscillators yield equations (for 3-vectors) analogous to those written above for 2-vectors (see (8), (9) and (10)).
3 Hamiltonians

The matrix evolution equation (14) is Hamiltonian: it obtains, in the standard manner, from the Hamiltonian

\[ H(P, L) = \text{trace} \left[ \frac{1}{2} P^2 + L \Delta L + \frac{1}{4} U^4 \right], \]

where the canonical coordinates are the components of the matrix \( L \) and the corresponding canonical momenta are the components of the matrix \( P \). (Indeed, the integrability of this matrix evolution equation, (14), is due to its being a reduction \([3] [2] [4]\) of the Non-Abelian Toda Lattice \([8] [5]\), the Hamiltonian structure of which was already exploited in \([9]\)). This clearly entails that all the Newtonian equations of motion reported in this paper (except (4)) are as well Hamiltonian.

We display here only the Hamiltonian function that, via the standard Hamiltonian equations, yields the basic Newtonian equations of motion (2). It reads:

\[ H \left( \vec{p}^{(n\ell)}, \vec{p}^{(\tilde{n}\ell)}; \vec{r}^{(n\ell)}, \vec{r}^{(\tilde{n}\ell)} \right) = \sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{\nu, \lambda=1}^{L} \left( \vec{p}^{(\nu\lambda)} \cdot \vec{p}^{(\nu\lambda)} \right) + \sum_{n=1}^{N} \sum_{\ell=1}^{L} \sum_{\nu, \lambda=1}^{L} \left( \vec{r}^{(\nu\lambda)} \cdot \vec{r}^{(\nu\lambda)} \right) \]

In this Hamiltonian, the canonical coordinates are the \( 2N^2L \) components of the \( 2NL \) \( N \)-vectors \( \vec{r}^{(n\ell)}, \vec{r}^{(n\ell)}, n = 1, 2, \ldots, N, \ell = 1, 2, \ldots, L \), and the corresponding canonical momenta are the \( 2N^2L \) components of the \( 2NL \) \( N \)-vectors \( \vec{p}^{(n\ell)}, \vec{p}^{(n\ell)}, n = 1, 2, \ldots, N, \ell = 1, 2, \ldots, L \). Note that this Hamiltonian entails that

\[ \vec{r}^{(n\ell)} = \vec{p}^{(n\ell)} = \vec{p}^{(\tilde{n}\ell)}. \]

Also note that this Hamiltonian is of normal type, namely it features a kinetic energy term depending quadratically only on the momenta and a potential energy term depending (quadratically and quartically) only on the coordinates.

4 Final remarks

As we already emphasized above, the Newtonian equations of motion considered in this paper are all covariant hence rotation-invariant, this being a rather essential condition to interpret them as describing a many-body problem. They are, however, not invariant under translations; but variants of them that do possess this additional property, without forsaking the property to be integrable
indeed solvable, can be easily manufactured (for a technique to do so see [4]). This generalization, as well as detailed explorations of the actual behavior of the solutions of these models, are left as tasks for the future; as well as the application of these findings to phenomenologically interesting situations (in physics or elsewhere).

A more challenging task for the future – hopefully appealing to physicists and to mathematicians – will be the study of the quantal counterparts of the models discussed in this paper.
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