Lipschitz continuity in the Hurst parameter of functionals of stochastic differential equations driven by a fractional Brownian motion

Alexandre Richard∗ Denis Talay†

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Abstract

Sensitivity analysis w.r.t. the long-range/memory noise parameter for probability distributions of functionals of solutions to stochastic differential equations is an important stochastic modeling issue in many applications.

In this paper we consider solutions \( \{X^H_t\}_{t \in \mathbb{R}_+} \) to stochastic differential equations driven by fractional Brownian motions. We develop two innovative sensitivity analyses when the Hurst parameter \( H \) of the noise tends to the critical Brownian parameter \( H = \frac{1}{2} \) from above or from below. First, we examine expected smooth functions of \( X^H \) at a fixed time horizon \( T \). Second, we examine Laplace transforms of functionals which are irregular with regard to Malliavin calculus, namely, first passage times of \( X^H \) at a given threshold.

In both cases we exhibit the Lipschitz continuity w.r.t. \( H \) around the value \( \frac{1}{2} \). Therefore, our results show that the Markov Brownian model is a good proxy model as long as the Hurst parameter remains close to \( \frac{1}{2} \).

Key words: Fractional Brownian motion, Malliavin calculus, first hitting time.

Contents

1 Introduction 2
2 Main results 4
3 Stochastic calculus for fractional Brownian motions and stochastic differential equations 6
  3.1 Elements of stochastic calculus for fractional Brownian motion 6
  3.2 Solutions to the SDE (2.2) 9
  3.3 The Lamperti process \( Y^H \) 10
  3.4 An Itô-Skorokhod formula for fractional Brownian motions with drift 13
  3.5 The Itô–Skorokhod formula for the Lamperti process \( Y^H \) 22
4 The smooth functional case: Sensitivity of time marginal distributions 24
  4.1 Preliminary results 25
  4.2 Proof of Proposition 4.1 (\( H > \frac{1}{2} \)) 25

∗Université Paris-Saclay, CentraleSupélec, MICS and CNRS FR-3487, France. E-mail: alexandre.richard@centralesupelec.fr.
†Equipe-projet ASCII, Centre Inria de Saclay and Ecole Polytechnique, France. E-mail: denis.talay@inria.fr.

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5 The irregular functional case: Sensitivity of Laplace transforms of first passage times

5.1 An ‘optimal’ extension of the Laplace transform for $H = \frac{1}{2}$ and related estimates

5.2 An error decomposition

5.3 Estimate on $I_1(\lambda)$ defined as in (5.14)

5.4 Estimate on $I_2(\lambda)$ defined as in (5.14)

5.5 An elementary proposition

5.6 $L^2$-estimate on $(\Upsilon_N^{(N)} - \Upsilon^{(N)}_t)$

5.7 The key lemma to estimate $D_{r,J}^{(1)}$

5.8 $L^p$-estimate on $(\Upsilon_N^{(N)} - \Upsilon^{(N)}_s)$

6 Conclusion and perspectives

Appendix A Representation of $K^*_H$ on $|\mathcal{H}_H|

Appendix B On various deterministic integrals depending on $K_H$ and $\partial_0 K_H$

B.1 Estimate on $\int_s^t \int_{t_0}^{t_1} (\mathcal{A}(v,r,t))^2 \, dr \, dv$: The regular case $\frac{1}{2} < H < 1$

B.2 Estimate on $\int_s^t \int_{t_0}^{t_1} (\mathcal{A}(v,r,t))^2 \, dr \, dv$: The irregular case $\frac{1}{4} < H < \frac{1}{2}$

B.3 Estimate on $\int_s^t \int_{t_0}^{t_1} (\mathcal{A}(v,r,t))^2 \, dr \, dv$

B.4 Estimate on $\int_s^t \int_{t_0}^{t_1} (\mathcal{A}(v,r,t))^2 \, dr \, dv$

B.5 Estimate on $\int_s^t \int_{t_0}^{t_1} (\mathcal{A}(v,r,t))^2 \, dr \, dv$

B.6 On variants of $I(v,r,t)$: $I(v,r,t), I^1(v,r,t)$ and $I^2(v,r,t)$

Appendix C Bounds on $W_\lambda$ and its derivatives (Proof of Proposition 5.6)

Appendix D Proof of Proposition 5.3

Appendix E Glossary

1 Introduction

In many applied situations where continuous-time stochastic differential equations are used, one chooses Markovian dynamics for many reasons: a huge literature exists on their analysis, calibration and simulation; their probability distributions can be characterized by partial differential equations or integro-differential equations; well developed techniques allow one to describe their asymptotic behaviours to homogenized systems, mean-field limits, equilibrium regimes.

However, Markov models may sometimes be seen as questionable idealizations of the reality. Empirical studies actually tend to show memory effects in biological, financial, physical data: see e.g. Rypdal and Rypdal [30] for a statistical evidence in climatology. Such empirical results justify to consider non-Markov models driven by noises with long-range memory such as fractional Brownian motions rather than by Lévy noises. But Jolis and Viles [18] emphasise that choosing a noise with long-range memory does not close the modeling problem since the parametric estimation of the model may be difficult and crude (see Berzin et al. [7] for the statistics of stochastic models with long-range memory).

Therefore, one often needs to balance tractable Markov models against more realistic but complex non-Markov models. A natural question then arises: Is it really worth introducing complex models?

To tackle this issue, we consider the case of solutions $X^H$ to one-dimensional stochastic differential equations driven by fractional Brownian motions $B^H$, where the Hurst constant $H$ parameterizes the covariance function of $B^H$. The case $H = \frac{1}{2}$ corresponds to the pure standard

2
Brownian case. The driving noise and corresponding solution are then respectively denoted by $B \equiv B^H$ and $X \equiv X^H$.

We develop a sensitivity analysis, w.r.t. $H$ around the reference value $H = \frac{1}{2}$, of probability distributions of certain functionals of the solutions $X^H$. We examine the two following cases which respectively are regular and strongly singular with regard to Malliavin calculus: on the one hand, the time marginal distributions of the solutions; on the other hand, the Laplace transform of first passage times.

We have three motivations to consider first passage times of $X^H$. First, the analysis of first passage times is an important issue in physical sciences (Metzler et al. [21]), in the evaluation of default risks and ruin probabilities (Jeanblanc et al. [16]), and in the study of neuron spike trains (Richard et al. [29]). Second, our discussion on Markovian or non-Markovian modeling applies in force to the study of hitting times. Indeed, the Markov property of the process is essential to calculate exact probability distributions of first passage times, characterize these distributions by means of partial differential equations, or construct numerical algorithms to simulate them: See e.g. Salminen and Yor [31], Alili and Patie [1], Deaconu and Herrmann [9] and citations therein. On the contrary, the long-range dependence leads to analytical and numerical difficulties: See e.g. Jeon et al. [17]. Last, it seems to us worthy of showing that an accurate sensitivity analysis is possible even in a case which is strongly singular with regard to Malliavin calculus.

For $H = \frac{1}{2}$ and $X^H$ reduced to the standard Brownian motion $B$, the Laplace transform of the first hitting time $\tau_B$ of the threshold 1 satisfies

$$\forall x_0 \leq 1, \quad \mathbb{E} \left( e^{-\lambda \tau_B} \mid B_0 = x_0 \right) = e^{-(1-x_0)\sqrt{2\lambda}}. \quad (1.1)$$

For $H \neq \frac{1}{2}$, even in the simple case where $X^H$ reduces to $B^H$, the probability distribution of $\tau^H_B$ is not explicitly known. Molchan [22] obtained the asymptotic behaviour of its tail distribution function and Decreusefond and Nualart [11] obtained an upper bound on the Laplace transform of $(\tau^H_B)^{2H}$. Related works are those of Nourdin and Viens [23] on the density of $\sup_{t \in [a,b]} B^H_t - \mathbb{E} \left( \sup_{t \in [a,b]} B^H_t \right)$ where $0 < a < b$, and of Baudoin et al. [6] on hitting probabilities of multidimensional fractional diffusions. The recent work of Delorme and Wiese [12] proposes an asymptotic expansion (in terms of $H - \frac{1}{2}$) of the density of $\sup_{t \in [0,b]} B^H_t$; this expansion is formally obtained by perturbation analysis techniques.

Below, we obtain an accurate estimate for

$$\left| \mathbb{E} \left( e^{-\lambda \tau^H_X} \right) - \mathbb{E} \left( e^{-\lambda \tau^H_X} \right) \right|,$$

with explicit rates in terms of $|H - \frac{1}{2}|$, $\lambda > 0$, and the distance from the initial condition $X^H_0$ to the threshold.

Our sensitivity analyses of time marginal and first passage time distributions tend to show that the Markov Brownian model is a good proxy model as long as the Hurst parameter remains close to $\frac{1}{2}$. This is an important information for modeling and simulation purposes: Whenever statistical or calibration procedures lead to estimated values of $H$ close to $\frac{1}{2}$, it then is reasonable to work with Brownian SDEs and standard stochastic integration theory.

**Why do we limit ourselves to a sensitivity analysis around $H = \frac{1}{2}$?** In this paper, contrary to Giordano et al. [14] and Jolis and Viles [18] we do not develop a sensitivity analysis of the model around $H' \neq \frac{1}{2}$. Our reasons are as follows. First, as already explained, it seems interesting to us to obtain as good as possible sensitivity estimates around a Markov proxy model: We actually get Lipschitz continuity properties. Second, the fact that the proxy model has the Markov property allows us to apply the Itô–Skorokhod formula proven in Section 3 to the
solution of a suitable ordinary differential equation, which allows us to transform the sensitivity analysis of the Laplace transform of $\tau^H_X$ around $H = \frac{1}{2}$ to the sensitivity analysis of Skorokhod integrals depending on $X^H$ and stopped at $\tau^H_X$. We thus can benefit from the fact that $X^H$ is a smooth functional on a suitable Wiener space. We do not see how to extend this strategy when the proxy model is not Markov. Finally, the equality

$$E\left( e^{-\lambda \tau^H_X} \right) - E\left( e^{-\lambda \tau^H_{X'}} \right) = E\left( e^{-\lambda \tau_X} \right) - E\left( e^{-\lambda \tau_X} \right) + E\left( e^{-\lambda \tau^H_X} \right) - E\left( e^{-\lambda \tau^H_{X'}} \right),$$

(1.2)

does not seem to help to get a sharp estimate in terms of $|H - H'|$: See Remark 5.10 below.

**Organization of the paper.** In Section 2 we state and comment our main results. In Section 3 we review elements of stochastic calculus for fractional Brownian motion and we prove an Itô formula for drifted fractional Brownian motions. In Section 4 we prove Proposition 4.1 which concerns the sensitivity w.r.t. $H$ of expected smooth functions of $X^H_t$ for every $t > 0$. Our proof of this proposition allows us to smoothly introduce our strategy to analyse Laplace transforms of first passage times. In Section 5 we prove our main result, namely, Theorem 5.2. Various technical lemmas are gathered in Appendix A, Appendix B and Appendix D. Estimates on the derivatives of the Laplace transform of $\tau^H$ are proven in Appendix C. Finally, the reader can find a glossary of our various processes, functions, etc. in Appendix E.

**Notations.** For any random variable in $L^p(\Omega)$ we set

$$\|F\|_p := \left\{ E(\|F\|^p) \right\}^{\frac{1}{p}}.$$

We denote by $C$ any constant which may change from line to line. It may depend on the Hurst parameter $H$ but, in that case, it is a bounded function of $H$ in $[\frac{1}{4}, 1]$.

We denote by $C_H$ any constant depending on $H$ which tends to infinity when $H$ tends to $\frac{1}{4}$ or $1$ and is bounded on any closed subinterval of $(\frac{1}{4}, 1)$. Such a constant may depend on various parameters except on the parameter $\lambda$ of the Laplace transform and the time horizon $N$ considered in Section 5.

### 2 Main results

We are given a fractional Brownian motion $\{B^H_t\}_{t \in \mathbb{R}_+}$ with Hurst parameter $H \in (0, 1)$. This process is the only Gaussian process with stationary increments which is self-similar of order $H$ (up to centering and normalization of the variance). Its covariance reads:

$$R_H(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right), \quad \forall s, t \in \mathbb{R}_+.$$  

(2.1)

We also are given two functions $b$ and $\sigma$ which satisfy:

(H1) $b$ belongs to $C^1_b(\mathbb{R})$ and $\sigma$ belongs to $C^2_b(\mathbb{R})$.

(H2) The function $\sigma$ satisfies the strong ellipticity condition: $\exists \sigma_0 > 0$ such that $|\sigma(x)| \geq \sigma_0$ for every $x \in \mathbb{R}$.

As it will be recalled in Subsection 3.2, the preceding hypotheses imply that for every $H \in (\frac{1}{4}, 1)$ and $x_0 \in \mathbb{R}$ there exists a unique solution $\{X^H_t\}_{t \in \mathbb{R}_+}$ to the stochastic differential equation driven by $\{B^H_t\}_{t \in \mathbb{R}_+}$:

$$X^H_t = x_0 + \int_0^t b(X^H_s) \, ds + \int_0^t \sigma(X^H_s) \circ dB^H_s.$$  

(2.2)
In particular, for \( H = \frac{1}{2} \), there exists a unique square integrable strong solution \( X \equiv X^{1/2} \) to the Brownian SDE in the Stratonovich sense

\[
    X_t = x_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \circ dB_s.
\]

(2.3)

Our first result is easy to prove but instructive. It will be proven in Section 4.2. It concerns the sensitivity w.r.t. \( H \) around the critical Brownian parameter \( H = \frac{1}{2} \) of \( \mathbb{E}_\mathcal{F}(X_t^H) \), where \( \varphi \) is a smooth function.

**Proposition 4.1.** Let \( X^H \) and \( X \) be as above. Suppose that \( b \) and \( \sigma \) satisfy (H1) and (H2), and that \( \varphi \) is bounded and Hölder continuous of order \( 2 + \beta \) for some \( \beta > 0 \). Then, for any \( T > 0 \), there exists \( C > 0 \) such that

\[
    \forall H \in (\frac{1}{4}, 1), \quad \sup_{t \in [0, T]} |\mathbb{E}_\mathcal{F}(X_t^H) - \mathbb{E}_\mathcal{F}(X_t)| \leq C |H - \frac{1}{2}|.
\]

Our second result concerns a highly singular functional of the paths, namely, the first passage time of \( X^H \) at a given threshold (1, say). Given \( x_0 < 1 \), set

\[
    T^H_x := \inf\{t \geq 0 : X_t^H = 1\}.
\]

(2.4)

The precise formulation of our result is obtained by combining the theorem 5.2 and the proposition 5.3. Part of the difficulties overcome in the lengthy proof of Theorem 5.2 come from the fact that we aim to get a sensitivity estimate which tends to 0 as fast as possible when \( H \) tends to \( \frac{1}{2} \) and decays to 0 at the same rates as in the exact formula (1.1) when \( \lambda \) and \( |1 - x_0| \) tend to infinity.

**Theorem 5.2.** Let \( X^H \) and \( X \) be the solutions to (2.2) and (2.3) respectively, both with initial condition \( x_0 < 1 \). Assume that \( b \) and \( \sigma \) satisfy (H1)-(H2). For the monotone function \( F \) defined as in Proposition 3.2 let \( \tilde{b} := \frac{b_0 F^{-1}}{\sigma_0 F^{-1}}, \tilde{\gamma} := F(1) \) and \( y_0 := F(x_0) \).

For any \( p \geq 1 \) and \( \lambda > |\tilde{b}|_\infty \) set

\[
    \mathcal{M}_p(\bar{Y} - y_0, \lambda) := \sup_{s \in \mathbb{R}_+} \left( e^{-\frac{1}{2}(\lambda - |\tilde{b}|_\infty)p s} \mathbb{E}_\mathcal{F} e^{-|\bar{Y} - y_0|^2/\rho(\lambda)} \right),
\]

where

\[
    \rho(\lambda) := \sqrt{2\lambda + \mu^2} - \mu \quad \text{with} \quad \mu := |\tilde{b}|_\infty.
\]

Suppose \( x_0 < 1 \) and \( \lambda > |\tilde{b}|_\infty \). Set \( \tilde{\lambda} := \lambda - |\tilde{b}|_\infty \). For any \( H \in (\frac{1}{4}, 1) \) we have

\[
    \mathbb{E}_\mathcal{F} \left( e^{-\lambda Y^H_{\tilde{t}}} \right) - \mathbb{E}_\mathcal{F} \left( e^{-\lambda T^H_{\bar{Y}} \tilde{t}} \right) \leq C_H |H - \frac{1}{2}| \left( \frac{1 + \lambda}{1 - \lambda} \right) \left( \mathcal{M}_1(\bar{Y} - y_0, \lambda) \right) \left( \lambda^H_{\tilde{t}} \right).
\]

**Proposition 5.3.** Let \( \lambda > |\tilde{b}|_\infty \). Let \( m := \bar{Y} - y_0, \mu := |\tilde{b}|_\infty, q := \rho(\lambda) \) and \( \tilde{\lambda} := \lambda - |\tilde{b}|_\infty \).

One has

\[
    \mathcal{M}_p(\bar{Y} - y_0, \lambda) \leq C \left( e^{-\frac{1}{2} m} + e^{-\frac{1}{2} \tilde{\lambda} Y^H_{\tilde{t}}(m)} + \exp \left( -2^{-\frac{1}{12}} m^{\frac{2}{12 + q}} \tilde{\lambda}^{\frac{1}{12 + q}} \right) \right),
\]

where

\[
    \psi^H_q(m) := \frac{m}{\mu + q} \left( \frac{m}{\mu + q} \right)^{\frac{2}{q} - 1} \left( \frac{m}{\mu + q} \right)^{\frac{1}{3q}} + \left( \frac{m}{\mu + q} \right)^{\frac{1}{3q}} \left( \frac{m}{\mu + q} \right)^{\frac{2}{q} - 1} \left( \frac{m}{\mu + q} \right)^{\frac{1}{3q}} \geq 1.
\]
The bound on $\mathcal{M}_p(\mathcal{Y} - y_0, \lambda)$ given in Proposition 5.3 yields the following asymptotic expressions when $\mu > 0$:

If $H > \frac{1}{2}$,

$$
\begin{align*}
    \mathcal{M}_p(\mathcal{Y} - y_0, \lambda) & \lesssim e^{-\frac{\mathcal{Y} - y_0}{\lambda^p} \sqrt{\frac{2}{\lambda}}}, \\
    \mathcal{M}_p(\mathcal{Y} - y_0, \lambda) & \lesssim e^{-\frac{1}{2} \left( \frac{\mathcal{Y} - y_0}{\lambda} \right) \frac{2}{\pi p} \lambda}, \\
    \mathcal{M}_p(\mathcal{Y} - y_0, \lambda) & \lesssim e^{-\frac{\mu R(\lambda)}{2} (\mathcal{Y} - y_0)} + e^{-\frac{\lambda}{2(\mu + p R(\lambda))} (\mathcal{Y} - y_0)}. 
\end{align*}
$$

If $H < \frac{1}{2}$,

$$
\begin{align*}
    \mathcal{M}_p(\mathcal{Y} - y_0, \lambda) & \lesssim e^{-\frac{1}{2} \left( \frac{\mathcal{Y} - y_0}{\lambda} \right) \frac{2}{\pi p} \lambda}, \\
    \mathcal{M}_p(\mathcal{Y} - y_0, \lambda) & \lesssim e^{-\frac{\mu R(\lambda)}{2} (\mathcal{Y} - y_0)} + e^{-\frac{\lambda}{2(\mu + p R(\lambda))} (\mathcal{Y} - y_0)}. 
\end{align*}
$$

3 Stochastic calculus for fractional Brownian motions and stochastic differential equations

3.1 Elements of stochastic calculus for fractional Brownian motion

In this section, we briefly review the definition of Skorokhod integrals w.r.t. fractional Brownian motions. The material mainly comes from [24].

Notational convention 3.1. In all this section we let the time horizon $T > 0$ be fixed. This parameter $T$ enters the definitions below of the operators $K_H$, $D^H$ and $\delta_H$, of the spaces $\mathcal{H}_H$ and $|\mathcal{H}_H|$, and of the corresponding norms on that spaces. For the sake of simplicity, the notation does not reflect the dependency on $T$. However, when necessary, we will change the notation e.g. from $\delta_H$ to $\delta_H^T$.

The integral kernels $K_H$. For any $H \in (0, 1) \setminus \{\frac{1}{2}\}$ define the kernel $K_H(s, u)$ as

$$
\begin{align*}
    \forall 0 < s \leq r, \quad K_H(s, r) := 0, \\
    \forall 0 < r < s, \quad K_H(s, r) := \chi_H \left\{ \left( \frac{s-r}{r} \right)^{\frac{H}{2} - \frac{1}{2}} - (H - \frac{1}{2}) \frac{1}{2} s^{-H} \int_r^s \theta^{-\frac{3}{2}} (\theta - r)^{H - \frac{1}{2}} d\theta \right\}, 
\end{align*}
$$

where $\chi_H$ is the bounded function of $H$ on $(\frac{1}{2}, 1)$ defined by

$$
\chi_H := \left( \frac{2H \Gamma(3/2 - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{\frac{1}{2}}.
$$

Recall that for any $H \in (0, 1)$, the covariance of the fractional Brownian motion is defined by (2.1). The next equality explains the reason for which the kernel $K_H$ is introduced:

$$
R_H(s, t) = \int_0^{s/t} K_H(s, u) K_H(t, u) du.
$$
**Useful properties of** $K_H$.  In the sequel we will need the following basic properties of the kernel $K_H$.

First, since $K_H(\theta,r) = 0$ for $r \geq \theta$, we have

$$\forall N > 0, \forall 0 < s < N, \forall 0 < t < N, \quad R_H(s,t) = \int_0^N K_H(s,r) K_H(t,r) \, dr.$$  (3.4)

In particular,

$$\forall 0 < \theta < N, \quad \theta^{2H} = \int_0^N K_H(\theta,v)^2 dv = \int_0^\theta K_H(\theta,v)^2 dv.$$  (3.5)

Second, in view of the preceding equality and (2.1), for any $0 < v < \theta < N$ one has

$$\int_0^N (K_H(\theta,r) - K_H(v,r))^2 dr = \int_0^N K_H(\theta,r)^2 dr + \int_0^N K_H(v,r)^2 dr - 2R_H(\theta,v)$$

$$\quad = \theta^{2H} + v^{2H} - (\theta^{2H} + v^{2H} - (\theta - v)^{2H})$$

$$\quad = (\theta - v)^{2H}.$$  (3.6)

The last property of $K_H$ we will need is obtained by an easy calculation:

$$\partial_s K_H(s,r) = \mathbb{I}_{\{s > r\}} \chi_H (H - \frac{1}{2}) \left( \frac{v}{s} \right)^{H-\frac{1}{2}} (s - r)^{-\frac{3}{2}}.$$  (3.7)

**The operators $K^*_H,s$ and the spaces $\mathcal{H}_H$, $|\mathcal{H}_H|$.**  Given $s > 0$ we define the operator $K^*_H,s$ as the dual in $L^2([0,s])$ of the integral operator with kernel $K_H$. For step functions on $[0,s]$ this operator is defined by

$$K^*_H,s \varphi(r) := K_H(s,r) \varphi(r) + \int_0^s \partial_\theta K_H(\theta,r) (\varphi(\theta) - \varphi(r)) \, d\theta.$$  

We now fix a time horizon $T$.  According to our notational convention 3.1, when no risk of confusion is possible we set $K^*_H \equiv K^*_H,T$. Denote by $\mathcal{H}_H$ the Hilbert space defined as the closure of the space of step functions w.r.t. the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_H} = \langle K^*_H \varphi, K^*_H \psi \rangle_{L^2([0,T])}.$$  

This extension of $K^*_H$ as an isometric operator from $\mathcal{H}_H$ to $L^2([0,T])$ is also denoted by $K^*_H$.  In particular, we have

$$\langle \mathbb{I}_{[0,s]}, \mathbb{I}_{[0,t]} \rangle_{\mathcal{H}_H} = R_H(s,t).$$

Note that $B^H \in \mathcal{H}_H$ iff $H > \frac{1}{2}$ (see Nualart [24, p.301]).

A natural subspace of $\mathcal{H}_H$ will be used in the sequel: $|\mathcal{H}_H|$ is the Banach space of measurable functions $\varphi$ on $[0,T]$ such that

- if $H < \frac{1}{2}$,

$$\|\varphi\|^2_{|\mathcal{H}_H|} := \int_0^T \varphi^2_H K_H(T,t)^2 \, dt + \int_0^T \left( \int_t^T \left| \varphi_s - \varphi_t \right| (s-t)^{H-\frac{3}{2}} \, ds \right)^2 \, dt < \infty,$$  (3.8)

- if $H > \frac{1}{2}$,

$$\|\varphi\|^2_{|\mathcal{H}_H|} := \alpha_H \int_0^T \int_0^T \left| \varphi_s \right| \left| \varphi_t \right| |s-t|^{2H-2} \, ds \, dt < \infty,$$  (3.9)

where

$$\alpha_H := 2H \left( H - \frac{1}{2} \right).$$  (3.10)
Useful properties of $K_H^*$. Below we will use the following properties of $K_H^*$ acting on $|\mathcal{H}_H|$.

We show in Appendix A that the following extension of $K_H^*$ from step functions to the space $|\mathcal{H}_H|$ is well defined:

$$\forall \varphi \in |\mathcal{H}_H|, \quad K_H^* \varphi(r) := K_H(T, r) \varphi(r) + \int_r^T \partial_b K_H(\theta, r) \left( \varphi(\theta) - \varphi(r) \right) d\theta. \quad (3.11)$$

One easily deduces the following from (3.11):

$$\begin{cases}
K_H^* \varphi(r) = K_H(t, r) \varphi(r) + \int_r^t \partial_b K_H(\theta, r) \left( \varphi(\theta) - \varphi(r) \right) d\theta \\
K_H^* \varphi(r) = K_H^* \varphi(r).
\end{cases} \quad (3.12)$$

When $H > \frac{1}{2}$ (and thus $H - \frac{3}{2} > -1$) and when $\varphi$ is in $|\mathcal{H}_H|$, it also comes from (3.11) that

$$K_H^* \varphi(r) = \int_r^T \partial_b K_H(\theta, r) \varphi(\theta) d\theta. \quad (3.13)$$

Finally, when $H \to \frac{1}{2}$, $\chi_H$ tends to 1 and thus $\frac{\partial}{\partial \theta} K_H(\theta, \sigma)$ converges in the distributional sense to the Dirac measure at point $\sigma$. Therefore, for any $\varphi \in |\mathcal{H}_H|$ and $0 < r < T$, $K_H^* \varphi(r)$ tends to $\varphi(r)$ when $H \to \frac{1}{2}$.

**Representation of fBm as non-anticipating stochastic integrals.** From the equality (3.3) one can deduce the following representation of the fBm $B^H$: for some standard Brownian motion $B \equiv B^{1/2}$,

$$\forall t \geq 0, \quad B^H_t = \int_0^t K_H(t, u) d\mathbf{B}_u. \quad (3.14)$$

**Malliavin calculus for fractional Brownian motion.** We are given a fBm $B^H$ and the corresponding Brownian motion $\mathbf{B}$ as in (3.14). Similarly to the Malliavin derivative $\mathbf{D}$ associated to the Brownian motion $\mathbf{B}$, the Malliavin derivative $D^H$ is defined as an operator acting on the smooth random variables with values in $\mathcal{H}_H$. The domain of $D^H$ in $L^p(\Omega)$ ($p > 1$) is denoted by $\mathbb{D}^{1,p}$ and is the closure of the space of smooth random variables with respect to the norm

$$\left\{ \mathbb{E}(|F|^p) + \mathbb{E}\left( \|D^H F\|_{\mathcal{H}_H}^p \right) \right\}^{\frac{1}{p}}.$$

Equivalently (cf [24, p.288]), $D^H$ can be defined as

$$D^H = (K_H^*)^{-1} \mathbf{D}. \quad (3.15)$$

In particular, for all $s, t \in [0, T]$ we have that

$$D^H_s B^H_t = \mathbb{1}_{[0,t]}(s) \quad \text{and} \quad D_s B^H_t = K_H^* \left( D^H B^H \right)(s) = K_H^* \left( \mathbb{1}_{[0,t]}(\cdot)(s) \right) = K_H(t, s).$$

We denote by $\mathbb{D}^{1,2}(\mathcal{H}_H)$ the set of the $|\mathcal{H}_H|$-valued random variables such that

$$\mathbb{E}\|\xi\|_{\mathcal{H}_H}^2 + \mathbb{E} \int_0^T \| D_r \xi \|_{\mathcal{H}_H}^2 dr < \infty \quad \text{if } H < \frac{1}{2}; \quad (3.16)$$

and

$$\mathbb{E}\|\xi\|_{\mathcal{H}_H}^2 + \mathbb{E} \int_{[0,T]^4} |D^H_{s \theta} \xi| |D^H_{s \eta} \xi| |s-r|^{2H-2} |\theta-\eta|^{2H-2} dr ds d\theta d\eta < \infty \quad \text{if } H > \frac{1}{2}. \quad (3.17)$$
See [3, Sec.3] and [24, p.295] when \( H < \frac{1}{2} \) and [24, p.288] when \( H > \frac{1}{2} \).

The divergence operator or Skorokhod integral \( \delta_H \) is defined by the following duality relation: for any \( F \) in \( \mathbb{D}^{1,2} \) and any \( \xi \) in the domain \( \text{dom}(\delta_H) \subset L^2(\Omega, \mathcal{H}_H) \) of \( \delta_H \), one has
\[
\mathbb{E}(\langle \xi, D^H F \rangle_{\mathcal{H}_H}) = \mathbb{E}(F \, \delta_H(\xi)).
\]
The Skorokhod integral \( \delta_H \) is related to the ordinary Skorokhod integral \( \delta \) w.r.t. the Brownian motion \( B \) as follows: for any \( \xi \) such that \( K_H^* \xi \in \text{dom}(\delta) \),
\[
\delta_H(\xi) = \delta(K_H^* \xi).
\]
It can be shown that \( \text{dom}(\delta_H) = (K_H^*)^{-1}(\text{dom}(\delta)) \) and that \( \text{dom}(\delta_H) \) contains \( \mathbb{D}^{1,2}(|\mathcal{H}_H|) \) (see Nualart [24, Sec.5.2.2 and p.295] and references therein).

We again emphasise that the preceding operators implicitly depend on \( T \). According to our notational convention 3.1 we will write \( \delta_H^{(T)} \) rather than \( \delta_H \) when it is necessary to take care of that dependency.

### 3.2 Solutions to the SDE (2.2)

Consider the SDE (2.2) in the Stratonovich sense under our hypotheses (H1) and (H2) on \( b \) and \( \sigma \).

For \( H > \frac{1}{2} \), we consider the unique solutions in the sense of Young [32] which are studied in Nualart and Rascuau [25]. They are based on the generalized Stieltjes integrals defined in Zähle [33]. They coincide with Stratonovich solutions since the regularity conditions for Stratonovich and Young integrals to coincide are met in our context. Their sample paths are Hölder continuous with order \( H - \epsilon \) for any \( 0 < \epsilon < H \).

For \( H \in (\frac{1}{4}, \frac{1}{2}) \) we deal with the notion of Stratonovich solution studied by Alòs et al [2]. In [2, Prop.6] it is shown that for \( b \in \mathcal{C}_b^1(\mathbb{R}) \) and \( \sigma \in \mathcal{C}_b^2(\mathbb{R}) \) there exists a pathwise unique solution to (2.2). This Stratonovich solution admits the Doss-Sussman representation:
\[
\begin{align*}
X_t^H &= \alpha(B_t^H, Z_t^H), \\
Z_t^H &= x_0 + \int_0^t b \circ \alpha(B_u^H, Z_u^H) \exp \left(-\int_0^t \sigma' \circ \alpha(u, Z_u^H)du\right) ds,
\end{align*}
\]
where \( \alpha(x, z) \) solves
\[
\begin{align*}
\frac{\partial \alpha}{\partial x}(x, z) &= \sigma \circ \alpha(x, z), \\
\alpha(0, z) &= z.
\end{align*}
\] (3.18)
The uniqueness results from Doss [13, Lem.2].

In both cases, \( H > \frac{1}{2} \) and \( H \in (\frac{1}{4}, \frac{1}{2}) \), we will need to apply an Itô type formula to processes of the type \( (\Phi(t, X_t^H)) \), where \( \Phi \) is a smooth function. For reasons which will be apparent in the sequel, we need that the formula involves stochastic integrals with zero expectation, and thus Skorokhod integrals rather than Stratonovich integrals.

A natural approach would consist in extending previous works, namely, the Itô-Skorokhod formula in [4, Thm.8] for \( H > \frac{1}{2} \) and the Itô-Stratonovich formula in [2, Thm.4] for \( \frac{1}{4} < H < \frac{1}{2} \). We would have to strnegthen our hypotheses to ensure that the process \( \sigma(X_t^H) \) belongs to \( \mathbb{D}^{2,2}(|\mathcal{H}_H|) \), however. We also would have to develop heavy calculations to get needed estimates on \( \mathbb{E}\sup_{s \leq t} |X_s^H| \) and \( \mathbb{E}|D^H X_t^H|^p \). In addition, the Itô-Skorokhod formula would involve integrals with algebraically complex integrands.

For all these reasons, we follow another way which is allowed by the ellipticity condition (H2).

First, we show that \( X^H \) is a one-to-one transform of a drifted fractional Brownian motion for any \( H \in (\frac{1}{4}, 1) \). That amounts to prove an Itô formula for a specific smooth map, namely, the
Lamperti transform. The formula is easy to prove since the dynamics of the transformed process does not involve any stochastic integral.

Second, we establish an Itô-Skorokhod formula for general functions of time and drifted fractional Brownian motions. We here benefit from the fact that the dynamics of the process under consideration does not involve a stochastic integral.

3.3 The Lamperti process $Y^H$

In this section we show that a one-to-one transform of $X^H$ is a fractional diffusion $Y^H$ with constant diffusion coefficient, and we prove regularity properties of $Y^H$.

**Proposition 3.2.** Let $H \in \left(\frac{1}{4}, 1\right)$. Assume that $b$ and $\sigma$ satisfy the hypotheses (H1) and (H2). Let $F(x) := \int_0^x \frac{1}{\sigma(z)} \, dz$ be the Lamperti transform. Set $\tilde{b} := \frac{bk_0}{\sigma_0 F^{-1}}$.

Then the process $Y^H := F(X^H)$ is the unique pathwise solution to the following SDE:

$$
\forall t \geq 0, \quad Y^H_t = F(x_0) + B^H_t + \int_0^t \tilde{b}(Y^H_s) \, ds. \tag{3.19}
$$

**Proof.** When $H > \frac{1}{2}$: The desired result is obtained by means of the classical chain rule since the Stratonovich integral coincides with a Stieltjes integral.

When $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$: To prove (3.19), fix an arbitrary time horizon $T > 0$ and for any $\epsilon > 0$ consider the regularised process

$$
B^H_{t,\epsilon} = \frac{1}{2\epsilon} \int_0^t \left( B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0} \right) \, ds. \tag{3.20}
$$

Set also $X^H_{t,\epsilon} = \alpha(B^H_{t,\epsilon}, Z_t)$, where $\alpha$ is defined by (3.18). Then the usual chain rule leads to

$$
F(X^H_{t,\epsilon}) = F(x_0) + \frac{1}{2\epsilon} \int_0^t \partial_x (F \circ \alpha)(B^H_{s,\epsilon}, Z_s) \times \left( B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0} \right) \, ds
$$

$$
+ \int_0^t \partial_z (F \circ \alpha)(B^H_{s,\epsilon}, Z_s) b \circ \alpha(B^H_{s,\epsilon}, Z_s^H) \exp \left( -\int_0^s \sigma' \circ \alpha(z, Z_s^H) \, dz \right) \, ds.
$$

The definition of $\alpha$ implies that $\partial_x (F \circ \alpha) = 1$ and $\partial_z \alpha(x, z) = \exp (\int_0^z \sigma' \circ \alpha(u, z) \, du)$. Thus

$$
F(X^H_{t,\epsilon}) = F(x_0) + \frac{1}{2\epsilon} \int_0^t \left( B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0} \right) \, ds + \int_0^t \frac{\partial_0}{\sigma^2} \left( B^H_{s,\epsilon}, Z_s \right) \times \frac{b \circ \alpha(B^H_{s,\epsilon}, Z_s^H)}{\sigma \circ \alpha(B^H_{s,\epsilon}, Z_s^H)} \, ds.
$$

As $B^H_0 = 0$ one can readily show that $\frac{1}{2\epsilon} \int_0^T \left( B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0} \right) \, ds$ almost surely converges to $B^H_0$ when $\epsilon$ tends to 0 and this convergence is uniform on $[0, T]$. The almost sure convergence of each side of the preceding equality yields Equation (3.19). Pathwise uniqueness of the solution results from the Lipschitz property of $\tilde{b}$: See Subsection 3.2.

**Properties of the Lamperti process $Y^H$.** We now state useful estimates on $Y^H$ and its Malliavin derivatives. We use the following representation which is valid for any $H \in \left(\frac{1}{4}, 1\right)$ (see Nualart and Saussereau [26]):

$$
\begin{align*}
\forall r > t, & \quad D^H_{r} Y^H_{t} = 0, \\
\forall r \leq t, & \quad D^H_{r} Y^H_{t} = 1 + \int_r^t D^H_{r} Y^H_{s} \tilde{b}'(Y^H_s) \, ds.
\end{align*} \tag{3.21}
$$

From (3.21) one readily gets

$$
\forall r > 0, \forall t > 0, \quad D^H_{r} Y^H_{t} = \mathbb{1}_{[0,t]}(r) \exp \left( \int_r^t \tilde{b}'(Y^H_u) \, du \right). \tag{3.22}
$$

The following proposition is an obvious consequence of (3.19) and (3.22).
Proposition 3.3. Let $b$ and $\sigma$ satisfy hypotheses (H1)-(H2). For any $\alpha < H$, it a.s. holds that

$$0 \leq D^H Y^H_t \leq \mathbb{I}_{[0,t]}(r) e^{\tilde{H}|r|^{(t-r)}}$$  \tag{3.23}

$$\forall 0 < r \leq s < t, \ |D^H Y^H_t - D^H Y^H_s| \leq e^{\tilde{H}|r|^{(t-r)}} \tilde{b}^\prime(t - s).$$  \tag{3.24}

In addition to the preceding estimates on $D^H Y^H$, we will need accurate estimates on $D_t Y^H$. The next proposition provides two such useful estimates. The upper bounds are expressed in terms of the kernel $K_H$ because, in the sequel, we will either use pointwise estimates on $K_H$ or the $L^2(0,T)$ properties (3.3) and (3.6) (see e.g. the proof of Proposition 3.7 and the calculations in Appendix B). Notice that (3.15) and (3.12) imply

$$\forall 0 < r \leq s < t \ D_t Y^H = K_H \left( D^H Y^H \right)(r) = K^H(t, r) \left( D^H Y^H \right)(r).$$  \tag{3.25}

Proposition 3.4. Let $H \in (0,1) \setminus \{\frac{1}{2}\}$. One then has

$$\left\{ \begin{array}{l} \forall r > t, \ |D_t Y^H| = 0, \\ \forall r \leq t, \ |D_t Y^H| \leq C e^{\tilde{H}|r|^{(t-r)}} \left\{ |K_H(t,r)| + (t-r)^H + \frac{1}{2} \mathbb{I}_{H<\frac{1}{2}} \right\} \end{array} \right.$$  \tag{3.26}

In addition, for any $r \leq s < t$ it holds that

$$|D_t Y^H - D_s Y^H| \leq C e^{\tilde{H}|r|^{(t-r)}} \left\{ |K_H(t,s) - K_H(s,r)| + (t-s) \left( |K_H(s,r)| + (s-r)^H + \frac{1}{2} \mathbb{I}_{H<\frac{1}{2}} \right) \right\}.$$  \tag{3.27}

Proof. (A) To prove (3.26) we successively examine the cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

The case $H > \frac{1}{2}$.

For $r > t$ we deduce (3.26) from (3.25) and (3.22).

For $r \leq t$ we start with using (3.25) and (3.13) to get

$$|D_t Y^H| = \left| \int_r^t \partial_\theta K_H(\theta, r) \ D^H Y^H d\theta \right|.$$  \tag{3.28}

Observe that (3.7) implies that $K_H$ and $\partial_\theta K_H$ are non-negative when $H > \frac{1}{2}$. By using (3.23) we deduce from the preceding that

$$|D_t Y^H| \leq e^{\tilde{H}|r|^{(t-r)}} \int_r^t \partial_\theta K_H(\theta, r) \ d\theta \leq e^{\tilde{H}|r|^{(t-r)}} K_H(t, r),$$

which is the desired inequality.

The case $H < \frac{1}{2}$.

For $r > t$ the inequality (3.26) follows from (3.22).

For $r \leq t$ we use (3.25) to get

$$|D_t Y^H| = \left| K_H(t, r) D^H Y^H_t + \chi_H \left( H - \frac{1}{2} \right) \int_r^t \left( \frac{\theta}{r} \right)^{-\frac{H}{2}} (\theta - r)^{H - \frac{3}{2}} \left( D^H Y^H_s - D^H Y^H_t \right) \ d\theta \right|.$$  \tag{3.29}

In view of (3.22) one thus has

$$|D_t Y^H| \leq e^{\tilde{H}|r|^{(t-r)}} \left\{ |K_H(t,r)| + \chi_H \ |\tilde{b}^\prime|_\infty \ |H - \frac{1}{2}| \int_r^t \left( \frac{\theta}{r} \right)^{-\frac{H}{2}} (\theta - r)^{H - \frac{3}{2}} \ d\theta \right\}.$$  \tag{3.30}

For $0 < r < t$ and $H < \frac{1}{2}$ one has \(\left( \frac{\theta}{r} \right)^{-\frac{H}{2}} < 1\). The inequality (3.26) follows.
(B) To prove (3.27), let $r \leq s < t$. We again successively examine the cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

The case $H > \frac{1}{2}$.

We use (3.25) and (3.13) to get
\[
\mathbf{D}_r Y_t^H - \mathbf{D}_r Y_s^H = K_H (D_t^H Y_t^H - D_s^H Y_s^H) (r)
\]
\[
= \int_r^t \partial_0 K_H (\theta, r) \left( D_0^H Y_t^H - D_0^H Y_s^H \right) d\theta
\]
\[
= \int_r^t \partial_0 K_H (\theta, r) \left( D_0^H Y_t^H - D_0^H Y_s^H \right) d\theta + \int_r^t \partial_0 K_H (\theta, r) D_0^H Y_t^H d\theta.
\]

We now combine (3.24), (3.23), and the non-negativity of $\partial_0 K_H$ when $H > \frac{1}{2}$.

It comes:
\[
|\mathbf{D}_r Y_t^H - \mathbf{D}_r Y_s^H| \leq e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} \left\{ \left| \tilde{\nu} \right|_{\infty} \int_r^s \partial_0 K_H (\theta, r) d\theta + \int_r^t \partial_0 K_H (\theta, r) d\theta \right\}
\]
\[
\leq C e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} \left\{ (t-s) K_H (s, r) + K_H (t, r) - K_H (s, r) \right\}.
\]

We thus have obtained (3.27) when $H > \frac{1}{2}$.

The case $H < \frac{1}{2}$.

In view of (3.25) one has
\[
\mathbf{D}_r Y_t^H - \mathbf{D}_r Y_s^H = K_H (D_t^H Y_t^H - D_s^H Y_s^H) (r)
\]
\[
= K_H (t, r) (D_t^H Y_t^H - D_s^H Y_s^H)
\]
\[
+ \int_r^s \partial_0 K_H (\theta, r) \left( D_0^H Y_t^H - D_0^H Y_s^H \right) d\theta
\]
\[
+ \int_r^t \partial_0 K_H (\theta, r) \left( D_0^H Y_t^H - D_0^H Y_s^H \right) d\theta
\]
\[
= \int_r^s \partial_0 K_H (\theta, r) \left( D_0^H Y_t^H - D_0^H Y_s^H \right) d\theta
\]
\[
+ \int_r^t \partial_0 K_H (\theta, r) D_0^H Y_t^H d\theta + K_H (s, r) (D_t^H Y_t^H - D_s^H Y_s^H)
\]
\[
=: A_1 + A_2 + A_3.
\]

Use (3.22) and apply the Mean Value theorem to the map
\[
v \in [r, \theta] \mapsto \exp \left( \int_v^t \tilde{\nu} (Y_u^H) du \right) - \exp \left( \int_v^s \tilde{\nu} (Y_u^H) du \right).
\]

For $r < \theta < s$ it comes
\[
|D_0^H Y_t^H - D_0^H Y_s^H | \leq \left| \tilde{\nu} \right|_{\infty} e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} (\theta - r) (t-s).
\]

By successively using (3.7) and $H < \frac{1}{2}$ we obtain
\[
|A_1| \leq C e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} (t-s) \int_r^s \left( \frac{\theta}{t} \right)^{H-\frac{1}{2}} (\theta - r)^{H-\frac{1}{2}} d\theta
\]
\[
\leq C e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} (t-s) \int_r^s (\theta - r)^{H-\frac{1}{2}} d\theta
\]
\[
\leq C e^{\left| \tilde{\nu} \right|_{\infty} (t-r)} (t-s) (s-r)^{H+\frac{1}{2}}.
\]

(3.29)
We now estimate $|A_2|$. The equality (3.7) shows that $\partial_b K_H(\theta, r) \leq 0$ when $H < \frac{1}{2}$. Therefore,

$$|A_2| \leq e^{\|\bar{b}\|_\infty (t-r)} \int_s^t (-\partial_b K_H(\theta, r)) \, d\theta = e^{\|\bar{b}\|_\infty (t-r)} |K_H(t, r) - K_H(s, r)|. \quad (3.30)$$

We finally consider $|A_3|$. In view of (3.24) we have

$$|A_3| \leq e^{\|\bar{b}\|_\infty (t-r)} |\bar{b}|_\infty (t-s) |K_H(s, r)|. \quad (3.31)$$

It now remains to combine (3.28), (3.29), (3.30) and (3.31). We deduce (3.27) for $H < \frac{1}{2}$. \qed

### 3.4 An Itô-Skorokhod formula for fractional Brownian motions with drift

In this subsection, we prove an Itô-Skorokhod formula for processes of the form

$$Y^H_t = y_0 + B^H_t + \int_0^t \beta_s \, ds, \quad t \in [0, T], \quad (3.32)$$

where $(\beta_s, s \in [0, T])$ is a smooth enough stochastic process. In our next section we will check that the formula (3.34) below applies to the solution $Y^H$ of the SDE (3.19). It involves the Trace term (3.35) which is related to the conversion formula from Stratonovich integrals w.r.t. $B^H$ to Skorokhod integrals (see e.g. [2]). However, as explained in Section 3.2, Stratonovich integrals are useless to our purpose.

**Proposition 3.5.** Let $(Y^H_t)_{t \in [0, T]}$ be a process of the form (3.32). Assume that $\beta$ is progressively measurable w.r.t. the Brownian filtration generated by $B$. Suppose also that for every $0 \leq s \leq T$ the random variable $\beta_s$ belongs to $D^{1,2}$ and the process $(\int_0^t \beta_s \, ds, \, t \in [0, T])$ belongs to $D^{1,2}([0, T] \times \mathbb{R})$.

Assume also that for any $0 \leq t, r \leq T$,

$$
\begin{cases}
\text{if } H > \frac{1}{2}, & \left( \int_0^t \mathbb{E} |D_r \beta_s|^2 \, ds \right)^{\frac{1}{2}} \leq C |t-r|^{\alpha} \quad \text{for some } \alpha > \frac{1}{2} - H, \\
\text{if } H < \frac{1}{2}, & \left( \int_0^t \mathbb{E} |D_r \beta_s|^2 \, ds \right)^{\frac{1}{2}} \leq C.
\end{cases}
$$

(3.33)

Then, for every $H \in (\frac{1}{4}, 1)$, the process $Y^H$ belongs to $\text{dom}(\delta_H)$ and for all $G \in C_b^{1,2}([0, T] \times \mathbb{R})$ and $0 \leq t \leq T$ one has

$$G(t, Y^H_t) = G(0, Y^H_0) + \int_0^t \left( \partial_s G(s, Y^H_s) + \partial_y G(s, Y^H_s) \beta_s \right) \, ds + \delta^T_H \left( 1_{[0,t]}(\cdot) \partial_y G(\cdot, Y^H) \right) + \text{Tr} \left[ D^H \partial_y G(\cdot, Y^H) \right]_t, \quad (3.34)$$

where

$$\text{Tr} \left[ D^H \partial_y G(\cdot, Y^H) \right]_t := \int_0^t \partial_y^2 G(s, Y^H_s) \left( H s^{2H-1} + \int_0^s \partial_y K_H(s, r) \int_0^r D_r \beta_v \, dv \, dr \right) ds. \quad (3.35)$$

**Proof.** The lengthy proof is divided in several steps. After having checked a preliminary result, in Step 1 we derive an Itô formula for smooth functions of a semi-martingale $Y^{H,\epsilon}$ which approximates $Y^H$. In Steps 2 and 3 we successively prove the convergence of each term which arises in the Itô formula for $Y^{H,\epsilon}$.

**Preliminary: A stochastic Fubini equality.** We start with proving a stochastic Fubini equality. Let the process $u(\cdot, \cdot)$ be such that

$$E \left[ \int_0^T \int_0^T |u(r, s)|^2 \, ds \, dr \right] + E \left[ \int_0^T \int_0^T |D_\theta u(r, s)|^2 \, d\theta \, ds \, dr \right] < \infty. \quad (3.36)$$
We use the notational convention 3.1 to define the operator $\delta^{(T)}$. Let us check that
\[
\delta^{(T)} \left( \int_0^T u(r, \cdot) \, dr \right) = \int_0^T \delta^{(T)}(u(r, \cdot)) \, dr.
\] (3.37)

Indeed, for any $F \in \mathbb{D}^{1,2}$ one has
\[
\mathbb{E} \left[ F \delta^{(T)} \left( \int_0^T u(r, \cdot) \, dr \right) \right] = \mathbb{E} \left[ \int_0^T D_s F \int_0^T u(r, s) \, dr \, ds \right]
\]
\[
= \int_0^T \mathbb{E} \left[ \int_0^T D_s F \, u(r, s) \, ds \right] \, dr
\]
\[
= \int_0^T \mathbb{E} \left[ F \delta^{(T)}(u(r, \cdot)) \right] \, dr
\]
\[
= \mathbb{E} \left[ F \int_0^T \delta^{(T)}(u(r, \cdot)) \, dr \right].
\]

In the preceding calculation, we used the classical Fubini Theorem in the second and fourth line, and the duality between the Skorokhod integral and the derivative operator in the third one.

We now proceed to the proof of (3.34).

*Step 1: An Itô formula for an approximation of $Y^H$.*

Recall the representation (3.14). By smoothing the kernel $K_H$, for any $\epsilon > 0$ we define the smoothened fBM $B^{H,\epsilon}$ by
\[
\forall t \geq 0, \quad B^{H,\epsilon}_t := \int_0^t K_H(t + \epsilon, s) \, dB_s.
\] (3.38)

Consider the following process:
\[
Y^{H,\epsilon}_t := y_0 + B^{H,\epsilon}_t + \int_0^t \beta_s \, ds.
\]

The process $B^{H,\epsilon}$ is not a martingale. As we plan to apply the standard Itô formula for continuous semimartingales we use (3.37) to rewrite $Y^{H,\epsilon}_t$ as
\[
Y^{H,\epsilon}_t = y_0 + \int_0^t K_H(s + \epsilon, s) \, dB_s + \int_0^t \left( \int_0^s \partial_s K_H(s + \epsilon, r) \, dB_r \right) \, ds + \int_0^t \beta_s \, ds.
\]

We thus can apply the Itô formula. It comes:
\[
G(t, Y^{H,\epsilon}_t) = G(0, Y^{H,\epsilon}_0) + \int_0^t \left\{ \partial_s G(s, Y^{H,\epsilon}_s) + \partial_y G(s, Y^{H,\epsilon}_s) \beta_s \right\} \, ds
\]
\[
+ \int_0^t \partial_y G(r, Y^{H,\epsilon}_r) \int_0^r \partial_r K_H(r + \epsilon, s) \, dB_s \, dr
\]
\[
+ \int_0^t \partial_y G(s, Y^{H,\epsilon}_s) K_H(s + \epsilon, s) \, dB_s + \frac{1}{2} \int_0^t \partial_y^2 G(s, Y^{H,\epsilon}_s) K_H(s + \epsilon, s)^2 \, ds.
\] (3.39)

In the preceding right-hand side, the second and third integrals diverge when $\epsilon$ tends to 0. We therefore are going to transform their sum. First, we use a standard property of Skorokhod integrals to get
\[
\delta^{(T)} \left( \mathbb{E} \left[ \mathbb{I}_{(0, r)}(\cdot) \, \partial_y G(r, Y^{H,\epsilon}_r) \partial_r K_H(r + \epsilon, \cdot) \right] \right) = \delta_y G(r, Y^{H,\epsilon}_r) \int_0^r \partial_r K_H(r + \epsilon, s) \, dB_s
\]
\[
- \int_0^r D_s \left( \partial_y G(r, Y^{H,\epsilon}_r) \right) \partial_r K_H(r + \epsilon, s) \, ds.
\] (3.40)
Second, we apply the Fubini formula (3.37) to the Skorokhod integral. This is allowed because
\[ u(r, s) \equiv \mathbb{I}_{[0,r]}(s) \partial_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \frac{\partial K_H}{\partial r}(r + \varepsilon, s) \]
satisfies (3.36) for the following reason. Since \( \partial_y G \) and \( s \mapsto \mathbb{I}_{[0,r]}(s) \frac{\partial K_H}{\partial r}(r + \varepsilon, s) \) are bounded, we have that \( u(\cdot, \cdot) \in L^2(\Omega; L^2([0,T]^2)) \). In addition, the assumption (3.33) implies that
\[
E \left[ \int_0^T \int_0^T \int_0^T |D^\theta u(r,s)|^2 \, d\theta \, ds \, dr \right] \leq C \sqrt{E \left[ \int_0^T \int_0^T \int_0^T |K_H(r + \varepsilon, \theta) + \int_0^r D^\theta \beta_v \, dv|^2 \, d\theta \, ds \, dr \right]} \leq C \left( 1 + \int_0^T \int_0^T \int_0^T \int_0^r E|D^\theta \beta_v|^2 \, dv \, d\theta \, ds \, dr \right) < \infty.
\]

Third, we observe that the assumption \( \int_0^T \beta_r \, dr \in \mathbb{D}^{1,2}(\mathcal{H}_H) \) implies that
\[
D^s (\partial_y G(r, \mathcal{Y}^H_{r, \varepsilon})) = \partial^2_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \left( K_H(r + \varepsilon, s) + \int_0^r D^s \beta_v \, dv \right).
\]
and we plug this equality into (3.40). Finally, we permute the variables \( r \) and \( s \) in the Lebesgue integral. It comes:
\[
\int_0^t \partial_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \int_0^r \partial_r K_H(r + \varepsilon, s) \, dB_s 
\]
\[
= \int_0^T \mathbb{I}_{(0,t)}(r) \delta^{(T)} \left[ \mathbb{I}_{(0,r)}(\cdot) \partial_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \partial_r K_H(r + \varepsilon, \cdot) \right] \, dr 
\]
\[
+ \int_0^t \partial^2_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \int_0^r \partial_r K_H(r + \varepsilon, s) \left( K_H(r + \varepsilon, s) + \int_0^r D^s \beta_v \, dv \right) \, ds \, dr 
\]
\[
= \delta^{(T)} \left[ \mathbb{I}_{(0,t)}(\cdot) \int_0^t \partial_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \partial_r K_H(r + \varepsilon, \cdot) \, dr \right] 
\]
\[
+ \int_0^t \partial^2_y G(s, \mathcal{Y}^H_{s, \varepsilon}) \int_0^s \partial_r K_H(s + \varepsilon, r) \left( K_H(s + \varepsilon, r) + \int_0^r D^r \beta_v \, dv \right) \, ds \, dr.
\]

Finally, we combine the preceding equality with (3.39). It results:
\[
G(t, \mathcal{Y}^H_{t, \varepsilon}) = G(0, \mathcal{Y}^H_{0, \varepsilon}) + \int_0^t \left\{ \partial_s G(s, \mathcal{Y}^H_{s, \varepsilon}) + \partial_y G(s, \mathcal{Y}^H_{s, \varepsilon}) \beta_s \right\} \, ds \tag{3.41}
\]
\[
\quad + \delta^{(T)} \left[ \mathbb{I}_{(0,t)}(\cdot) \partial_y G(\cdot, \mathcal{Y}^H_{s, \varepsilon}) K_H(\cdot + \varepsilon, \cdot) 
\quad + \mathbb{I}_{(0,t)}(\cdot) \int_0^t \partial_y G(r, \mathcal{Y}^H_{r, \varepsilon}) \partial_r K_H(r + \varepsilon, \cdot) \, dr \right] \tag{3.42}
\]
\[
+ \int_0^t \partial^2_y G(s, \mathcal{Y}^H_{s, \varepsilon}) \left( \frac{1}{2} K_H(s + \varepsilon, s)^2 + \int_0^s \partial_r K_H(s + \varepsilon, r) \, K_H(s + \varepsilon, r) \, dr 
\quad + \int_0^s \partial_r K_H(s + \varepsilon, r) \int_0^r D^r \beta_v \, dv \, dr \right) \, ds. \tag{3.43}
\]

**Step 2:** *Convergence of the terms in (3.41) and (3.43).*

By using (3.6) an easy calculation shows that \( \sup_{r \in [0,T]} E|\mathcal{Y}^H_{s, \varepsilon} - \mathcal{Y}^H_s|^2 \) converges to 0 as \( \varepsilon \to 0 \). The convergence in probability of the terms in (3.41) follows.

We now prove the convergence of the trace term (3.43).
Lemma 3.6 proven below shows that there exists a positive function $\Psi$ with finite integral on $(0, t)$ such that

$$\sup_{0<\varepsilon<1} \left( \frac{1}{2} K_H(s+\varepsilon, s)^2 + \int_0^s \partial_s K_H(s+\varepsilon, r) K_H(s+\varepsilon, r) \, dr \right) \leq \Psi(s)$$

and

$$\forall s > 0, \ \frac{1}{2} K_H(s+\varepsilon, s)^2 + \int_0^s \partial_s K_H(s+\varepsilon, r) K_H(s+\varepsilon, r) \, dr \xrightarrow{\varepsilon \to 0} Hs^{2H-1}.$$ 

Therefore, Lebesgue’s Dominated Convergence theorem implies that the following a.s. convergence holds true:

$$\int_0^t \partial^2_y G(s, Y_s^{H,\varepsilon}) \left( \frac{1}{2} K_H(s+\varepsilon, s)^2 + \int_0^s \partial_s K_H(s+\varepsilon, r) K_H(s+\varepsilon, r) \, dr \right) \, ds \xrightarrow{\varepsilon \to 0} \int_0^t \partial^2_y G(s, Y_s^H) Hs^{2H-1} \, ds.$$

We now turn to the last term in (3.43). We have:

$$\mathbb{E} \left| \int_0^t \partial^2_y G(s, Y_s^H) \int_0^s \left( \int_0^s D_r \beta_v \, dv \right) (\partial_s K_H(s+\varepsilon, r) - \partial_s K_H(s, r)) \, dr \, ds \right| \leq C \int_0^t \int_0^s \left( \mathbb{E} \left| D_r \beta_v \right|^2 \, dv \right)^{\frac{1}{2}} |\partial_s K_H(s+\varepsilon, r) - \partial_s K_H(s, r)| \, dr \, ds.$$

Therefore, in view of Assumption (3.33), Equality (3.7) and Lebesgue’s Dominated Convergence theorem, the following a.s. convergence holds:

$$\int_0^t \partial^2_y G(s, Y_s^H) \int_0^s \partial_s K_H(s+\varepsilon, r) \left( \int_0^s D_r \beta_v \, dv \right) \, dr \, ds \xrightarrow{\varepsilon \to 0} \int_0^t \partial^2_y G(s, Y_s^H) \int_0^s \partial_s K_H(s, r) \left( \int_0^s D_r \beta_v \, dv \right) \, dr \, ds.$$

**Step 3: Convergence of the Skorokhod integral (3.42).**

Before proving the convergence of (3.42), we check that its potential limit is well defined. To this end, we notice that $Y^H \in \mathbb{D}^{1,2}([\mathcal{H}_t])$ since $B^H \in \mathbb{D}^{1,2}([\mathcal{H}_t])$ and $\int_0^t \beta_s \, ds \in \mathbb{D}^{1,2}([\mathcal{H}_t])$ by assumption. This implies that $K_{H,t, \mathbb{P}}(\partial_y G(\cdot, Y^H)) \in \text{dom}(\delta)$.

To prove the convergence, we consider an arbitrary random variable $F \in \mathbb{D}^{1,2}$. We have:

$$\mathbb{E} \left[ F \left( \delta \left( \mathbb{I}_{(0,t)}(\cdot) \{ \partial_y G(\cdot, Y^{H,\varepsilon}) K_H(\cdot + \varepsilon, \cdot) + \int_0^t \partial_y G(r, Y^{H,\varepsilon}) \partial_r K_H(r+\varepsilon, \cdot) \, dr \} \right) \right) - \delta \left( K_{H,t, \mathbb{P}}(\partial_y G(\cdot, Y^H)) \right) \right]$$

$$= \mathbb{E} \left[ F \left( \delta \left( \mathbb{I}_{(0,t)}(\cdot) \{ \partial_y G(\cdot, Y^{H,\varepsilon}) K_H(t+\varepsilon, \cdot) + \int_0^t (\partial_y G(r, Y^{H,\varepsilon}) - \partial_y G(\cdot, Y^{H,\varepsilon})) \partial_r K_H(r+\varepsilon, \cdot) \, dr \} \right) \right) - \delta \left( K_{H,t, \mathbb{P}}(\partial_y G(\cdot, Y^H)) \right) \right]$$

$$\leq \mathbb{E} \left[ F \delta \left( \mathbb{I}_{(0,t)}(\cdot) \{ \partial_y G(\cdot, Y^{H,\varepsilon}) K_H(t+\varepsilon, \cdot) - \partial_y G(\cdot, Y^H) K_H(t, \cdot) \} \right) \right] + \mathbb{E} \left[ F \delta \left( \mathbb{I}_{(0,t)}(\cdot) \left\{ \int_0^t (\partial_y G(r, Y^{H,\varepsilon}) - \partial_y G(\cdot, Y^{H,\varepsilon})) \partial_r K_H(r+\varepsilon, \cdot) \, dr \right\} \right) \right]$$

$$= A_1 + A_2.$$
Set
\[ \|DF\|_{L^2}^2 := \mathbb{E}\left[ \int_0^t (D_s F)^2 \, ds \right]. \]

From the duality formula and the Cauchy-Schwarz inequality it results that
\[ A_1^2 \leq \|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \partial_y G(y_s^{H,\varepsilon}) K_H(t + \varepsilon, s) - \partial_y G(y_s^{H}) K_H(t, s) \right\}^2 \, ds \right] \]
and
\[ A_2^2 \leq \|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \int_s^t (\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})) \partial_r K_H(r + \varepsilon, s) \, dr - \int_s^t (\partial_y G(r, Y_r^{H}) - \partial_y G(s, Y_s^{H})) \partial_r K_H(r, s) \, dr \right\}^2 \, ds \right]. \]

As for \( A_1 \), introduce the term \( \partial_y G(y_s^{H,\varepsilon}) K_H(t, s) \) to get
\[ A_1^2 \leq 2\|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \partial_y G(y_s^{H,\varepsilon})(K_H(t + \varepsilon, s) - K_H(t, s)) \right\}^2 \, ds \right] + 2\|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \int_s^t (\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})) \partial_r K_H(r + \varepsilon, s) \, dr \right\}^2 \, ds \right]. \]

For the first term in the right-hand side, we use the boundedness of \( \partial_y G \) and observe that
\[ \int_0^t (K_H(t + \varepsilon, r) - K_H(t, r))^2 \, dr \leq \int_0^{t+\varepsilon} (K_H(t + \varepsilon, r) - K_H(t, r))^2 \, dr = e^{2H}. \]

The Dominated Convergence theorem implies that the second term tends to 0. Therefore, \( A_1 \) tends to 0.

As for \( A_2 \), introduce the term \( \int_s^t (\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})) \partial_r K_H(r, s) \, dr \) to get
\[ A_2^2 \leq 2\|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \int_s^t (\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})) \left( \partial_r K_H(r + \varepsilon, s) - \partial_r K_H(r, s) \right) \right\}^2 \, ds \right] + 2\|DF\|_{L^2}^2 \mathbb{E}\left[ \int_0^t \left\{ \int_s^t (\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})) \partial_r K_H(r, s) \, dr \right\}^2 \, ds \right]. \]

For any bounded measurable process \((G_r, Y_r)\) and any deterministic positive integrable function \(K\), Minkowski’s integral inequality implies that
\[ \forall 0 < s < t, \quad \mathbb{E}\left( \int_s^t G_r K(r) \, dr \right)^2 \leq \left( \int_s^t \sqrt{\mathbb{E}(G_r^2)} \, K(r) \, dr \right)^2 \leq \left( \int_s^t \sqrt{\mathbb{E}(G_r^2)} \, K(r) \, dr \right)^2. \]

It follows that
\[ A_2^2 \leq 2\|DF\|_{L^2}^2 \int_0^t \left\{ \int_s^t \mathbb{E}|\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})|^2 \left| \partial_r K_H(r + \varepsilon, s) - \partial_r K_H(r, s) \right| \, dr \right\}^2 \, ds + 2\|DF\|_{L^2}^2 \int_0^t \left\{ \int_s^t \mathbb{E}|\partial_y G(r, Y_r^{H}) - \partial_y G(s, Y_s^{H})|^2 \left| \partial_r K_H(r, s) \right| \, dr \right\}^2 \, ds =: A_{2,1}^2 + A_{2,2}^2. \]

The Lipschitz property of \( \partial_y G \) implies that \( \mathbb{E}|\partial_y G(r, Y_r^{H,\varepsilon}) - \partial_y G(s, Y_s^{H,\varepsilon})|^2 \frac{1}{2} \leq C(\mathbb{E}|B_{r}^{H,\varepsilon} - B_{s}^{H,\varepsilon}|^2)^{\frac{1}{2}}. \)

In addition, the definition of \( B_{r}^{H,\varepsilon} \) yields that
\[ \mathbb{E}|B_{r}^{H,\varepsilon} - B_{s}^{H,\varepsilon}|^2 = \int_0^r (K_H(r + \varepsilon, u) - K_H(s + \varepsilon, u))^2 \, du \leq \int_0^{r+\varepsilon} (K_H(r + \varepsilon, u) - K_H(s + \varepsilon, u))^2 \, du = (r - s)^{2H}. \]
Therefore,

$$A_{2,1}^2 \leq C \int_0^t \left\{ \int_s^t (r-s)^H \left| \partial_r K_H(r, \varepsilon, s) - \partial_r K_H(r, s) \right| dr \right\}^2 ds.$$ 

By using the inequalities (B.3), (B.5) in Appendix and the Dominated Convergence theorem we conclude that $A_{2,1}$ converges to 0.

To prove the convergence of $A_{2,2}$, we first observe that

$$E|\partial_y G(r, \mathcal{H}_r^s) - \partial_y G(s, \mathcal{H}_s^r) - \partial_y G(r, \mathcal{H}_r^s, \varepsilon) + \partial_y G(s, \mathcal{H}_s^r, \varepsilon)|^2 \leq C E|B_r^H - B_r^{H, \varepsilon}|^2 + C E|B_s^H - B_s^{H, \varepsilon}|^2,$$
which obviously converges to 0 for any $r$ and $s$. Second, we notice that

$$E|\partial_y G(r, \mathcal{H}_r^s) - \partial_y G(s, \mathcal{H}_s^r) - \partial_y G(r, \mathcal{H}_r^s, \varepsilon) + \partial_y G(s, \mathcal{H}_s^r, \varepsilon)|^2 \leq C E|B_r^H - B_r^s|^2 + C E|B_r^{H, \varepsilon} - B_s^{H, \varepsilon}|^2 \leq C(r-s)^2 H.$$

By using (B.3), (B.5) and the Dominated Convergence theorem we conclude that $A_{2,2}$ converges to 0.

We have thus obtained that $A_1 + A_2$ converges to 0. Therefore, the term (3.42) weakly converges to $\delta(K^*_H)^{(1)(0,\varepsilon)}(\cdot, \mathcal{H}^\varepsilon_r))$.

In the second step of the preceding proof we have used the following result.

**Lemma 3.6.** It holds that

$$\sup_{0 < \varepsilon < 1} \left( \frac{1}{2} K_H(s + \varepsilon, s)^2 + \int_0^s \partial_s K_H(s + \varepsilon, r) K_H(s + \varepsilon, r) \, dr \right) \leq \begin{cases}
C (1 + s^{1-2H} + s^{2-H}) (s + 1)^{3H-\frac{3}{2}} & \text{when } H > \frac{1}{2}, \\
C (1 + s^{H-1} + s^{2H-1}) & \text{when } H < \frac{1}{2},
\end{cases} \tag{3.44}$$

and

$$\forall s > 0, \ \frac{1}{2} K_H(s + \varepsilon, s)^2 + \int_0^s \partial_s K_H(s + \varepsilon, r) K_H(s + \varepsilon, r) \, dr \xrightarrow{\varepsilon \to 0} H s^{2H-1}. \tag{3.45}$$

**Proof.** We have that

$$\frac{1}{2} K_H(s + \varepsilon, s)^2 + \int_0^s \partial_s K_H(s + \varepsilon, r) K_H(s + \varepsilon, r) \, dr = \frac{1}{2} \chi_H \left( \frac{s + \varepsilon}{s} \right)^{2H-1} \varepsilon^{2H-1} + \frac{1}{2} (H - \frac{1}{2})^2 \chi_H s^{1-2H} \left( \int_s^{s+\varepsilon} \theta^{H-\frac{3}{2}} (\theta - s)^{H-\frac{1}{2}} d\theta \right)^2$$

$$- \chi_H \left( H - \frac{1}{2} \right) \left( \frac{s + \varepsilon}{s} \right)^{H-\frac{1}{2}} \varepsilon^{H-\frac{1}{2}} s^{2-H} \int_s^{s+\varepsilon} \theta^{H-\frac{3}{2}} (\theta - s)^{H-\frac{1}{2}} d\theta$$

$$+ \left( H - \frac{1}{2} \right)^2 \chi_H \int_0^s \left( \frac{s + \varepsilon}{r} \right)^{2H-1} (s + \varepsilon - r)^{2H-2} dr$$

$$- \left( H - \frac{1}{2} \right)^2 \chi_H (s + \varepsilon)^{H-\frac{1}{2}} \int_s^{s+\varepsilon} \theta^{H-\frac{3}{2}} (\theta - s)^{H-\frac{1}{2}} d\theta dr$$

$$= A_1(\varepsilon) + A_2(\varepsilon) - A_3(\varepsilon) + A_4(\varepsilon) - A_5(\varepsilon).$$
Proof of (3.44): The case $H > \frac{1}{2}$.

In that case, for any $0 < \varepsilon < 1$ we obviously have

$$A_1(\varepsilon) + A_2(\varepsilon) \leq C \left( s^{1 - 2H} (s + 1)^{2H - 1} + C \right) s^{1 - 2H} \varepsilon^{2H - 1} \left( \int_s^{s + \varepsilon} \theta^{H - \frac{3}{2}} \, d\theta \right)^2 \leq C \left( s^{1 - 2H} (s + 1)^{2H - 1} \right).$$

Similarly,

$$A_3(\varepsilon) \leq C \left( \frac{s + 1}{s} \right)^H \varepsilon^{2H - 1} \int_s^{s + \varepsilon} \theta^{H - \frac{3}{2}} \, d\theta \leq C \left( s^{1 - 2H} (s + 1)^{2H - 1} \right).$$

As for $A_4(\varepsilon)$, we have

$$A_4(\varepsilon) \leq C \left( s + 1 \right)^{2H - 1} \int_0^s r^{1 - 2H} (s - r)^{2H - 2} \, dr$$

$$= C \left( s + 1 \right)^{2H - 1} s \int_0^1 (s\theta)^{1 - 2H} (s - s\theta)^{2H - 2} \, d\theta$$

$$= C \left( s + 1 \right)^{2H - 1}.$$

As for $A_5(\varepsilon)$ we have that

$$A_5(\varepsilon) \leq C \left( s + 1 \right)^{H - \frac{1}{2}} \int_0^s r^{1 - 2H} (s - r)^{H - \frac{3}{2}} \int_r^{s + \varepsilon} \theta^{H - \frac{3}{2}} \, d\theta \, dr$$

$$\leq C \left( s + 1 \right)^{3H - \frac{3}{2}} \int_0^s r^{1 - 2H} (s - r)^{H - \frac{3}{2}} \, dr$$

$$= C \left( s + 1 \right)^{3H - \frac{3}{2}} s \int_0^1 (s\theta)^{1 - 2H} (s - s\theta)^{H - \frac{3}{2}} \, d\theta$$

$$= C \left( s^{\frac{3}{2} - H} (s + 1)^{3H - \frac{3}{2}}. \right.$$

To summarize the preceding calculations, when $H > \frac{1}{2}$ one has

$$\sum_{i=1}^5 A_i(\varepsilon) \leq C \left( 1 + s^{1 - 2H} + s^{\frac{3}{2} - H} \right) (s + 1)^{3H - \frac{3}{2}}.$$

Proof of (3.44): The case $H < \frac{1}{2}$.

In that case, we estimate the sum $A_1(\varepsilon) + A_4(\varepsilon)$. Notice that both $A_1(\varepsilon)$ and $A_4(\varepsilon)$ are unbounded when $\varepsilon$ tends to 0 and that $A_4(\varepsilon)$ is negative when $H < \frac{1}{2}$.

From the equality

$$\frac{1}{2} \varepsilon^{2H - 1} = \frac{1}{2} (s + \varepsilon)^{2H - 1} - (H - \frac{1}{2}) \int_0^s (s + \varepsilon - r)^{2H - 2} \, dr$$

we get that

$$A_1(\varepsilon) + A_4(\varepsilon) = \chi_{H\varepsilon}^2 (s + \varepsilon)^{2H - 1} \left( \frac{1}{2} \left( \frac{s + \varepsilon}{s} \right)^{2H - 1} \right)$$

$$+(H - \frac{1}{2}) \int_0^s (s + \varepsilon - r)^{2H - 2} (r^{1 - 2H} - s^{1 - 2H}) \, dr.$$
For any $0 < \varepsilon < 1$ one thus has

$$|A_1(\varepsilon) + A_4(\varepsilon)| \leq C s^{2H-1} \left( 1 + \int_0^s (s-r)^{2H-2} (s^{1-2H} - r^{1-2H}) \, dr \right)$$

$$\leq C s^{2H-1} \left( 1 + s \int_0^1 (s-s\theta)^{2H-2} (s^{1-2H} - (s\theta)^{1-2H}) \, d\theta \right)$$

$$= C s^{2H-1} \left( 1 + \int_0^1 (1-\theta)^{2H-2} (1-\theta^{1-2H}) \, d\theta \right)$$

$$\leq C s^{2H-1},$$

since $0 < 1 - \theta^{1-2H} < 1 - \theta$ for all $0 < \theta < 1$ and $0 < H < \frac{1}{2}$.

As for $A_2(\varepsilon)$, we use the inequality

$$\forall 0 < H < \frac{1}{2}, \forall 0 < s < \theta, \quad \theta^{H-\frac{3}{2}} \leq \theta^{H-\frac{3}{2} + \frac{H}{2}} (\theta - s)^{-\frac{1}{2} \theta - \frac{H}{2}} \leq s^{\frac{H}{2} - 1} (\theta - s)^{-\frac{1}{2} \theta - \frac{H}{2}}$$

to get

$$A_2(\varepsilon) \leq C s^{H-1} \varepsilon^H < C s^{H-1}.$$

As for $A_3(\varepsilon)$ we rather use the inequality

$$\forall 0 < H < \frac{1}{2}, \forall 0 < s < \theta, \quad \theta^{H-\frac{3}{2}} \leq \theta^{H-\frac{3}{2} + H} (\theta - s)^{-H} \leq s^{2H-\frac{3}{2}} (\theta - s)^{-H}$$

to get

$$A_3(\varepsilon) \leq C \varepsilon^{2H-1} s^{\frac{1-H}{2}} \int_s^{s+\varepsilon} \theta^{H-\frac{3}{2}} \, d\theta \leq C s^{H-1} \varepsilon^H \leq C s^{H-1}.$$

As for $A_5(\varepsilon)$ we have that

$$A_5(\varepsilon) \leq C s^{H-\frac{1}{2}} \int_0^s r^{1-2H} (s + \varepsilon - r)^{H-\frac{3}{2}} \int_r^{s+\varepsilon} r^{H-\frac{1}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta \, dr$$

$$= C s^{H-\frac{1}{2}} \int_0^s r^{-H-\frac{1}{2}} (s + \varepsilon - r)^{2H-1} \, dr$$

$$\leq C s^{H-\frac{1}{2}} \int_0^s r^{-H-\frac{1}{2}} (s-r)^{2H-1} \, dr$$

$$= C s^{2H-1} \int_0^1 \theta^{-\frac{1}{2} - H} (1-\theta)^{2H-1} \, d\theta$$

$$= C s^{2H-1}.$$

To summarize the preceding calculations, when $H < \frac{1}{2}$ one has

$$\sum_{i=1}^5 A_i(\varepsilon) \leq C (1 + s^{H-1} + s^{2H-1}).$$

Proof of (3.45): The case $H > \frac{1}{2}$.
In that case, $A_i(\varepsilon)$ obviously tends to 0 with $\varepsilon$ for $i = 1, 2, 3$.

In addition, notice that $A_4(\varepsilon)$ tends to

$$(H - \frac{1}{2}) \chi_H^2 \int_0^s s^{2H-1} r^{1-2H} (s-r)^{2H-2} \, dr = (H - \frac{1}{2}) \chi_H^2 s^{2H-1} \int_0^1 \gamma^{1-2H} (1-\gamma)^{2H-2} \, d\gamma.$$

Now, observe that $A_5(\varepsilon)$ tends to

$$(H - \frac{1}{2})^2 \chi_H^2 s^{H-\frac{1}{2}} \int_0^s r^{1-2H} (s-r)^{H-\frac{3}{2}} \int_r^s \theta^{H-\frac{1}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta \, dr.$$
Use the change of variables $\theta = \frac{r}{\alpha}$. The above expression becomes

$$
(H - \frac{1}{2})^2 \chi_H^2 s^{H-\frac{1}{2}} \int_0^s (s-r)^{H-\frac{3}{2}} \int_{\frac{r}{\alpha}}^1 \alpha^{-2H} (1-\alpha)^{H-\frac{1}{2}} \, d\alpha \, dr.
$$

Now, use the change of variables $r = s\gamma$. It comes:

$$
(H - \frac{1}{2})^2 \chi_H^2 s^{2H-1} \int_0^1 (1-\gamma)^{H-\frac{3}{2}} \int_{\gamma}^1 \alpha^{-2H} (1-\alpha)^{H-\frac{1}{2}} \, d\alpha \, d\gamma.
$$

By integrating by parts the inner integral we finally get that $A_5(\varepsilon)$ tends to

$$
\frac{1}{2}(H - \frac{1}{2}) \chi_H^2 s^{2H-1} \left( \int_0^1 \gamma^{1-2H} (1-\gamma)^{2H-2} \, d\gamma - (H - \frac{1}{2}) \int_0^1 (1-\gamma)^{H-\frac{3}{2}} \int_{\gamma}^1 \alpha^{1-2H} (1-\alpha)^{H-\frac{1}{2}} \, d\alpha \, d\gamma \right).
$$

From

$$
\int_0^1 (1-\gamma)^{H-\frac{3}{2}} \int_{\gamma}^1 \alpha^{1-2H} (1-\alpha)^{H-\frac{1}{2}} \, d\gamma = \frac{1}{H - \frac{1}{2}} \left( \int_0^1 \alpha^{1-2H} (1-\alpha)^{H-\frac{3}{2}} \, d\alpha - \int_0^1 \gamma^{1-2H} (1-\gamma)^{2H-2} \, d\gamma \right)
$$

it results that $A_4(\varepsilon) - A_5(\varepsilon)$ tends to

$$
\frac{1}{2} \left( H - \frac{1}{2} \right) \chi_H^2 s^{2H-1} \int_0^1 \alpha^{1-2H} (1-\alpha)^{H-\frac{3}{2}} \, d\alpha.
$$

By using (3.2) and standard properties of Beta and Gamma functions we finally get that the preceding limit is equal to

$$
\frac{1}{2} \left( H - \frac{1}{2} \right) \chi_H^2 s^{2H-1} \frac{\Gamma(2-2H) \Gamma(H - \frac{1}{2})}{\Gamma(H + \frac{3}{2})} = H \, s^{2H-1}.
$$

Proof of (3.45): The case $H < \frac{1}{2}$.

In that case, $A_i(\varepsilon)$ and obviously tends to 0 with $\varepsilon$ for $i = 2, 3$.

In addition, notice that

$$
A_4(\varepsilon) + A_5(\varepsilon) \to \chi_H^2 s^{2H-1} \left(\frac{1}{2} + (H - \frac{1}{2}) \int_0^s (s-r)^{2H-2} \left(r^{1-2H} - s^{1-2H}\right) dr \right).
$$

By using the change of variables $r = s\theta$ and then integrating by parts we transform the right-hand side into

$$
\chi_H^2 s^{2H-1} \left(\frac{1}{2} + (H - \frac{1}{2}) \int_0^1 (1-u)^{2H-2} (u^{1-2H} - 1) \, du \right)
$$

$$
= \chi_H^2 s^{2H-1} \left(\frac{1}{2} + (H - \frac{1}{2}) \left( \frac{1}{1-2H} - \int_0^1 (1-u)^{2H-1} u^{-2H} \, du \right) \right)
$$

$$
= \chi_H^2 s^{2H-1} \left(\frac{1}{2} - H \right) \int_0^1 (1-u)^{2H-1} u^{-2H} \, du.
$$

We now observe that

$$
-A_5(\varepsilon) \to -(H - \frac{1}{2})^2 \chi_H^2 s^{H-\frac{1}{2}} \int_0^s s^{1-2H} (s-r)^{H-\frac{3}{2}} \int_r^s \theta^{H-\frac{3}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta \, dr.
$$

21
By using the change of variables \((r, \theta) = (su, s\gamma)\) we transform the right-hand side into
\[
-(H - \frac{1}{2})^2 \chi_H^2 s^{2H-1} \int_0^1 u^{1-2H} (1 - u)^{H - \frac{3}{2}} \int_u^1 \gamma^{H - \frac{3}{2}} (\gamma - u)^{H - \frac{1}{2}} \, d\gamma \, du.
\]
Now, use the change of variables \(v = \frac{u}{\gamma}\) and then integrate by parts to obtain the new value
\[
-(H - \frac{1}{2})^2 \chi_H^2 s^{2H-1} \int_0^1 (1 - u)^{H - \frac{3}{2}} \int_u^1 v^{-2H} (1 - v)^{H - \frac{1}{2}} \, dv \, du
= -(H - \frac{1}{2})^2 \chi_H^2 s^{2H-1} \left( \int_0^1 v^{-2H} (1 - v)^{H - \frac{1}{2}} \, dv - \int_0^1 u^{-2H} (1 - u)^{2H-2} \, du \right).
\]
Combining the preceding equality with \((3.48), (3.2)\) and standard properties of the beta function leads to
\[
A_1(\varepsilon) + A_4(\varepsilon) - A_5(\varepsilon) \rightarrow \chi_H^2 s^{2H-1} \left( \frac{1}{2} - H \right) \int_0^1 v^{-2H} (1 - v)^{H - \frac{1}{2}} \, dv
= s^{2H-1} \frac{2H \Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \left( \frac{1}{2} - H \right) \frac{\Gamma(1 - 2H) \Gamma(H + \frac{1}{2})}{\Gamma(\frac{3}{2} - H)}.
\]
It remains to use that \((1 - 2H) \Gamma(1 - 2H) = \Gamma(2 - 2H)\) to conclude that
\[
A_1(\varepsilon) + A_4(\varepsilon) - A_5(\varepsilon) \rightarrow H s^{2H-1}.
\]

3.5 The Itô–Skorokhod formula for the Lamperti process \(Y^H\)

Let \(Y^H\) be as above. In order to be in a position to apply the Itô–Skorokhod formula \((3.34)\) we need to check that \(Y^H\) belongs to \(D^{1,2}([\mathcal{H}_H])\). This property seems to be well-known for the fractional Brownian motion of Hurst parameter \(H > \frac{1}{4}\), but we could not find a proof.

**Proposition 3.7.** For any \(H \in (\frac{1}{4}, 1) \setminus \{\frac{1}{2}\}\), \(Y^H \in D^{1,2}([\mathcal{H}_H])\).

**Proof.** When \(H > \frac{1}{2}\) one obviously has \(\int_0^T \int_0^T |s - t|^{2H - 2} \, ds \, dt < \infty\). Therefore, in view of \((3.9)\) and Proposition 3.3, the inequality \((3.17)\) holds true, which means that \(Y^H \in D^{1,2}([\mathcal{H}_H])\).

We now treat the case \(H \in (\frac{1}{4}, \frac{1}{2})\). In view of \((3.8)\) and \((3.16)\) we need to check that
\[
\mathbb{E} \int_0^T \int_0^T \left( \int_t^T |D_r Y^H_t - D_r Y^H_s|^2 (s - t)^{H - \frac{3}{2}} \, ds \right)^2 \, dt \, dr < \infty.
\]

(3.49)

It suffices to prove that \(A\) and \(B\) are finite, where
\[
A := \int_0^T \int_0^T \left( \int_t^T |D_r Y^H_t - D_r Y^H_s|^2 (s - t)^{H - \frac{3}{2}} \, ds \right)^2 \, dt \, dr
\]
and
\[
B := \int_0^T \int_0^T \left( \int_t^T |D_r Y^H_t - D_r Y^H_s|^2 |s - t|^{H - \frac{3}{2}} \, ds \right)^2 \, dt \, dr.
\]

Use \((3.26)\) to get
\[
A = \int_0^T \int_0^r \left( \int_r^T (s - t)^{H - \frac{3}{2}} |K_H(s, r)| + (s - r)^{H + \frac{1}{2}} \, ds \right)^2 \, dt \, dr.
\]

\[
\leq C \int_0^T \int_0^r \left( \int_r^T (s - t)^{H - \frac{3}{2}} \left( |K_H(s, r)| + (s - r)^{H + \frac{1}{2}} \right) \, ds \right)^2 \, dt \, dr.
\]

\]

22
Observe that, for any $t < r < T$,
\[
\int_0^r \left( \int_r^T (s-t)^{H-\frac{3}{2}} (s-r)^{2H+1} \, ds \right)^2 \, dt \leq C \int_0^r \left( \int_r^T (s-t)^{H-\frac{3}{2}} \, ds \right)^2 \, dt \leq C \int_0^r (T-t)^{2H-1} \, dt \leq C.
\]
Therefore,
\[
A \leq C + C \int_0^T \int_0^r \left( \int_r^T (s-t)^{H-\frac{3}{2}} \, ds \right)^2 \, dt \, dr.
\]
As $(\frac{3}{2})^{H-\frac{1}{2}} < 1$ for $s > r$ and $H < \frac{1}{2}$, the change of variables $\gamma = \frac{t}{r}$ in (3.1) leads to
\[
|K_H(s,r)| \leq C \left\{ (s-r)^{H-\frac{1}{2}} + r^{H-\frac{1}{2}} \int_1^{\infty} \gamma^{H-\frac{3}{2}} (\gamma - 1)^{H-\frac{1}{2}} \, d\gamma \right\}
\]
\[
\leq C (s-r)^{H-\frac{1}{2}} + C H r^{H-\frac{1}{2}}.
\]
It follows that
\[
A \leq C_H + C \int_0^T \int_0^r \left( \int_r^T (s-t)^{H-\frac{3}{2}} (s-r)^{H-\frac{1}{2}} \, ds \right)^2 \, dt \, dr.
\]
Applying Hölder’s inequality with $p = \frac{4}{3}$ and $q = 4$ one gets
\[
A \leq C_H + C \left( \int_0^T \int_0^r (\int_r^T (s-t)^{\frac{4}{3}(H-\frac{3}{2})} \, ds)^3 \right)^{\frac{2}{3}} \left( \int_0^T (s-r)^{4H-2} \, ds \right)^{\frac{3}{2}} \, dt \, dr.
\]
The right-hand side is finite for $H > \frac{1}{4}$.

We now aim to prove that $B < \infty$.

By using (3.27) we get
\[
B \leq C \int_0^T \int_0^r \left( \int_t^T (s-t)^{H-\frac{1}{2}} \left( |K_H(s,r)| + (s-r)^{H+\frac{1}{2}} \right) \, ds \right)^2 \, dt \, dr
\]
\[
+ C \int_0^T \int_0^r \left( \int_t^T (s-t)^{H-\frac{1}{2}} |K_H(s,r) - K_H(t,r)| \, ds \right)^2 \, dt \, dr
\]
\[
:= B_1 + B_2.
\]
The inequality (3.50) implies that
\[
B_1 \leq C + C \int_0^T \int_0^r \left( \int_t^T (s-t)^{H-\frac{1}{2}} (s-r)^{H-\frac{1}{2}} \, ds \right)^2 \, dt \, dr
\]
\[
\leq C + C \int_0^T \int_0^r \left( \int_t^T (s-t)^{2H-1} \, ds \right)^2 \, dt \, dr < \infty.
\]

Now, as $(\frac{3}{2})^{H-\frac{1}{2}} < 1$ for $r < t \leq \theta \leq s$ and $H < \frac{1}{2}$, in view of (3.7) we have
\[
|K_H(s,r) - K_H(t,r)| \leq C \int_t^s (\theta-r)^{H-\frac{3}{2}} \, d\theta = C \int_t^s (\theta-r)^{H-\frac{3}{2}} \, d\theta \leq (t-r)^{H-\frac{3}{2}} \int_t^s (\theta-r)^{-\frac{3}{2}} \, d\theta.
\]
By using the Hölder continuity of the function $x^{\frac{3}{4}}$ we get
\[
\int_t^T (s-t)^{H-\frac{3}{2}} |K_H(s,r) - K_H(t,r)| \, ds \leq C (t-r)^{H-\frac{3}{2}} \int_t^T (s-t)^{H-\frac{3}{2}} \, ds.
\]
As $H > \frac{1}{4}$ we deduce that $B_2$ is finite and thus $B$ also is finite. □
To conclude this section, we combine the propositions 3.5 and 3.7 to get the Itô–Skorokhod formula for functions of $Y_t^H$.

**Theorem 3.8.** For all $H \in \left(\frac{1}{4}, 1\right) \setminus \{\frac{1}{2}\}$, $0 \leq t \leq T$ and $G \in C_b^{1,2}[0, T] \times \mathbb{R}$ one has

$$G(t, Y_t^H) = G(0, Y_0^H) + \int_0^t \left(\partial_s G(s, Y_s^H) + \tilde{b}(Y_s^H) \partial_y G(s, Y_s^H)\right) ds + \delta_H^{(T)}(\mathbb{I}_{[0,t]}(\cdot) \partial_y G(\cdot, Y_\cdot^H)) + \text{Tr} \left[D^H \partial_y G(\cdot, Y_\cdot^H)\right]_t,$$

where

$$\text{Tr} \left[D^H \partial_y G(\cdot, Y_\cdot^H)\right]_t = \int_0^t \partial_y^2 G(s, Y_s^H) \left(H s^{2H-1} + \int_0^s \partial_s K_H(s, r) \int_0^r \mathbb{D}_r \tilde{b}(Y_r^H) dv dr\right) ds.$$

**Proof.** This is a direct consequence of Proposition 3.5 applied to $\beta = b(Y^H)$, provided that $(\int_0^t b(Y_s^H) ds, t \in [0, T])$ is in $\mathbb{D}^{1,2}([H_H])$ and that (3.33) is satisfied. By Proposition 3.7, the processes $Y^H$ and $B^H$ belong to $\mathbb{D}^{1,2}([H_H])$. Hence, so does $(\int_0^t b(Y_s^H) ds, t \in [0, T])$. Finally, one easily deduce the inequality (3.33) from (3.26).

\[\square\]

4 The smooth functional case: Sensitivity of time marginal distributions

The aim of this section is to prove the following proposition which precises the weak convergence result of [18] for the process $X^H$ when $H \to \frac{1}{2}$ by giving a convergence rate.

**Proposition 4.1.** Let $X^H$ and $X$ be the solutions to (2.2) and (2.3) respectively. Suppose that $b$ and $\sigma$ satisfy the hypotheses (H1) and (H2), and $\varphi \in C^{2+\beta}$ for some $\beta > 0$. Then, for any $T > 0$, there exists $C > 0$ such that

$$\forall H \in \left(\frac{1}{4}, 1\right), \sup_{t \in [0, T]} \mathbb{E}|\varphi(X_t^H) - \mathbb{E}\varphi(X_t)| \leq C |H - \frac{1}{2}|.$$

**Remark 4.2.** The convergence rate in Proposition 4.1 is optimal. Indeed, a Taylor expansion of the function $H \mapsto t^{2H-1}$ shows that there exists $C > 0$ satisfying

$$\forall t > 0, \forall H \in \left(\frac{1}{4}, 1\right), \mathbb{E}(\mathbb{E}(B_t^H)^2) - \mathbb{E}(\mathbb{E}(B_t)^2) - 2 \mathbb{E}(B_t)^2 \leq C (H - \frac{1}{2}) t \log(t) \leq C (H - \frac{1}{2})^2 (1 + t^2).$$

Therefore, by means of a suitable truncation of the function $x^2$ one can easily construct a bounded smooth function $\varphi$ such that

$$\forall H \in \left(\frac{1}{4}, 1\right), \sup_{t \in [0, T]} \mathbb{E}|\varphi(X_t^H) - \mathbb{E}\varphi(X_t)| = C |H - \frac{1}{2}| + o(H - \frac{1}{2}).$$

One can prove Proposition 4.1 by applying the Lamperti transform and using estimates on $\mathbb{E}(B_t^H - B_t)$ in terms of $|H - \frac{1}{2}|$ (see Decreusefond [10] or Richard [27, p.1404]) and proceed with a Gronwall argument. However, for pedagogical reasons we follow another way in subsections 4.1 and 4.2. We thus softly introduce our methodology to study the sensitivity of Laplace transforms of hitting times. Laplace transforms of hitting times involve irregular functionals of the paths of $X^H$ and thus arguments based upon Gronwall’s lemma cannot work.

Our strategy is based upon the following observation: when $H = \frac{1}{2}$ the process $X$ is Markovian. Thus, integrals w.r.t. its time marginal probability distributions can be expressed in terms of elliptic or parabolic PDEs. Whenever the coefficients of the generator of $X$ are smooth enough to allow it, the key argument consists in applying Itô’s formula to the solution of the suitable PDE and then using that the resulting Itô integral is a martingale and thus has zero expectation.
We thereby apply Itô-Skorokhod’s formula to the solution of the suitable PDE and the fractional diffusion. We thus transform our sensitivity problem to the comparison between stochastic integrals driven by standard Brownian motions and, respectively, by fractional Brownian motions. The resulting estimates reflect that the larger is \(|H - \frac{1}{2}|\), the bigger is the loss of the Markov property.

As explained in Section 3.2, to be in a position to apply Itô-Skorokhod’s formula we use the Lamperti process \(Y^H\) rather than \(X^H\).

### 4.1 Preliminary results

Let \(Y := Y^\frac{1}{2}\) be the solution to (3.19) in the pure Brownian case. Let \(F\) be the Lamperti transform of Proposition 3.2. We need to prove that

\[
\forall H \in \left(\frac{1}{2}, 1\right), \quad \sup_{t \in [0,T]} |\mathbb{E}\varphi \circ F^{-1}(Y^H_t) - \mathbb{E}\varphi \circ F^{-1}(Y_t)| \leq C |H - \frac{1}{2}|.
\]

Arbitrarily fix a time \(t \in (0,T]\) and consider the following parabolic PDE with terminal condition \(\varphi \circ F^{-1}\) at time \(t\):

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial}{\partial t} u(s,x) + \tilde{b}(x) \frac{\partial}{\partial x} u(s,x) + \frac{1}{2} \beta^2 u(s,x) = 0, & (s,x) \in [0,t) \times \mathbb{R}, \\
u(t,x) = \varphi \circ F^{-1}(x), & x \in \mathbb{R}.
\end{array} \right.
\end{align*}
\]

We prove Proposition 4.1 in the case \(H > \frac{1}{2}\) only. The necessary additional arguments to handle the case \(H < \frac{1}{2}\) can be found in the technically demanding Section 5. We will need the following integrability result.

**Lemma 4.3.** Let \(\varphi \in C^{2+\beta}_0(\mathbb{R})\) for some \(0 < \beta < 1\). Suppose that \(b, \sigma\) satisfy the hypotheses (H1)-(H2). There exists a unique solution \(u(s,x)\) in \(C^{1,2+\beta}_b([0,T] \times \mathbb{R})\) to the PDE (4.1). For any \(x \in \mathbb{R}\), the functions \(\partial_s u(\cdot, x)\) and \(\partial_x u(\cdot, x)\) are bounded. In addition, for any \(H > \frac{1}{2}\) one has

\[
\int_0^t \int_0^t |y - s|^{2H-2} |D_H^s (\partial_x u(s,Y^H_s(x)))| \, ds \, ds < \infty \text{ a.s.} \tag{4.2}
\]

**Proof.** Notice that \(\varphi \circ F^{-1} \in C^{2+\gamma}_b(\mathbb{R})\) and \(\tilde{b} \in C^1_b(\mathbb{R})\). The existence and uniqueness of \(u(s,x)\) in \(C^{1,2+\beta}_b([0,T] \times \mathbb{R})\) is a classical result: See e.g., Lunardi [20, p.189]. From Feynman-Kac’s formula, there exists a locally bounded positive function \(C(t)\) such that

\[
|u|_{C^{1,2+\beta}_b([0,T] \times \mathbb{R})} \leq C(t) \cdot |\varphi \circ F^{-1}|_{C^{2+\beta}_b(\mathbb{R})}.
\]

As \(D^H_s (\partial_x u(s,Y^H_s)) = D^H_s Y^H_s \partial_{xx}^2 u(s,Y^H_s)\), the desired inequality (4.2) follows from Proposition 3.3 when \(H > \frac{1}{2}\).

We now are in a position to prove Proposition 4.1. As already said, we limit ourselves to the case \(H > \frac{1}{2}\).

### 4.2 Proof of Proposition 4.1 \((H > \frac{1}{2})\)

Let \(0 < t \leq T\) be arbitrarily fixed. We start with representing \(\mathbb{E}\varphi(Y^H_t) - \mathbb{E}\varphi(Y_t)\) in an integral form by using the solution \(u\) of the PDE (4.1). The properties of the function \(u\) recalled in Lemma 4.3 imply that the process \((u(s,Y^H_s))_{0 \leq s \leq t}\) is a martingale.

By using the Itô–Skorokhod formula (3.51) we get

\[
u(t,Y^H_t) = u(0,F(x_0)) + \int_0^t \left( \partial_s u(s,Y^H_s) + \partial_x u(s,Y^H_s) \tilde{b}(Y^H_s) \right) \, ds + \delta_H \left(\mathbb{I}_{[0,t]} \partial_x u(\cdot,Y^H_s)\right) + \operatorname{Tr} \left[D^H \partial_y G(\cdot,Y^H_s)\right]_t.
\]
Let us explicit the Trace term. Since we consider the case $H > \frac{1}{2}$, by using (3.52), (3.15) and (3.19) we get

$$\text{Tr} \left[D^H \partial_y G(\cdot, Y^H_t)\right] = \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \left( H s^{2H-1} + \int_0^s \partial_s K_H(s, r) K^*_H[D^H Y^H_s - \mathcal{I}_{[0, s]}(\cdot)](r) \, dr \right) ds.$$  

As $H > \frac{1}{2}$ we also have $K_H(s, s) = 0$. Therefore,

$$\int_0^s \partial_s K_H(s, r) K^*_H[\mathcal{I}_{[0, s]}(\cdot)](r) \, dr = \int_0^s \partial_s K_H(s, r) K_H(s, r) \, dr \quad = \frac{1}{2} \int_0^s \partial_s (K_H(s, r))^2 \, dr = \frac{1}{2} \partial_s \left( \int_0^s (K_H(s, r))^2 \, dr \right) \quad = H s^{2H-1}.$$  

Combine this equality with (3.13) and $K_H(\theta, \theta) = 0$ for any $0 < \theta$ to obtain

$$\text{Tr} \left[D^H \partial_y G(\cdot, Y^H_t)\right] = \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \int_0^s \partial_s K_H(s, r) K^*_H[D^H Y^H_s](r) \, dr \, ds$$  

$$= \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \int_0^s \partial_s K_H(s, r) \int_0^r \partial_y K_H(\theta, r) D^H Y^H_s \, d\theta \, dr \, ds$$  

$$= \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \int_0^s D^H Y^H_s \left( \int_0^0 \partial_s K_H(s, r) \partial_y K_H(\theta, r) \right) \, d\theta \, ds$$  

$$= \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \int_0^s D^H Y^H_s \partial_s \theta \left( \int_0^0 K_H(s, r) K_H(\theta, r) \right) \, d\theta \, ds.$$  

It remains to use (3.3) and (3.10) to get the following explicit formula for the Trace term:

$$\text{Tr} \left[D^H \partial_y G(\cdot, Y^H_t)\right] = \alpha_H \int_0^t \int_0^s \partial_y^2 G(s, Y^H_s) \int_0^s D^H Y^H_s (s - \theta)^{2H-2} \, d\theta \, ds.$$  

Now, use the definition of $u$ and the fact that the Skorokhod integral has zero expectation to get

$$\mathbb{E} \varphi \circ F^{-1}(Y^H_t) - \mathbb{E} \varphi \circ F^{-1}(Y_t) = \mathbb{E} u(t, Y^H_t) - u(0, F(x_0))$$  

$$= -\frac{1}{2} \mathbb{E} \int_0^t \partial_y^2 u(s, Y^H_s) \, ds$$  

$$+ \alpha_H \mathbb{E} \int_0^t \partial_y^2 u(s, Y^H_s) \int_0^s D^H Y^H_s (s - \theta)^{2H-2} \, d\theta \, ds$$  

$$= \mathbb{E} \int_0^t \partial_y^2 u(s, Y^H_s) \left( H s^{2H-1} - \frac{1}{2} \right) ds$$  

$$+ \alpha_H \mathbb{E} \int_0^t \partial_y^2 u(s, Y^H_s) \int_0^s (D^H Y^H_s - 1) (s - \theta)^{2H-2} \, d\theta \, ds$$  

$$= \Delta^1_H + \Delta^2_H.$$  

For any $y > 0$ one has $e^y - 1 \leq y e^y$ and $1 - e^{-y} \leq y$, from which

$$\forall x > 0, \forall \alpha \in (-\frac{1}{2}, \frac{1}{2}), \quad |x^\alpha - 1| \leq |\alpha \log(x)| (1 \vee x^\alpha) \leq |\alpha \log(x)| (1 + x^\alpha). \quad (4.3)$$  

By using the preceding inequality with $x = s^2$ and $\alpha = H - \frac{1}{2}$ we get

$$|\Delta^1_H| \leq C \left( H - \frac{1}{2} \right) |\partial_y^2 u|_{\infty} \int_0^t |\log(s)| (1 + s^{2H-1}) \, ds \leq C \left( H - \frac{1}{2} \right).$$  

26
In view of (3.24) we also have
\[
|\Delta_H^2| \leq C \alpha_H \int_t^t |\partial_y^2 u(s, Y^H_s)| \int_0^s (s - \theta)^{2H-2} (s - \theta) \, d\theta \, ds
\]
\[
\leq C |\partial_y^2 u|_\infty \alpha_H \int_t^t \int_0^s (s - \theta)^{2H-1} \, d\theta \, ds
\]
\[
\leq C (H - \frac{1}{2}).
\]
That ends the proof for \( H > \frac{1}{2} \).

Remark 4.4. We come back to the discussion initiated in the Introduction to justify the choice of the Markov model as the proxy model. If the proxy model were driven by a fractional Brownian motion with Hurst index \( H' \neq \frac{1}{2} \), by applying the Itô-Skorokhod formula (3.51) we would be led to estimate
\[
E \text{Tr} \left[ D^H \phi'(Y^H) \right]_t - E \text{Tr} \left[ D^{H'} \phi'(Y^{H'}) \right]_t.
\]
Therefore, in the case of smooth test functionals such as \( \varphi(X^H_t) \) it seems possible to get an accurate sensitivity estimate in terms of \( |H - H'| \). We do not develop here the calculations and prefer to concentrate on the case of irregular functionals.

5 The irregular functional case: Sensitivity of Laplace transforms of first passage times

The aim of this section is to estimate the sensitivity w.r.t. the Hurst parameter \( H \) of the Laplace transform of \( \tau_X^H \) defined as in (2.4).

Our sensitivity analysis on Laplace transforms is based on the PDE representation of first hitting time Laplace transforms in the pure Markov case \( H = \frac{1}{2} \). Our strategy starts as in Section 4.2: Apply Itô’s formula to the solution of the suitable PDE in order to transform the sensitivity problem into a comparison between stochastic integrals w.r.t. standard Brownian motions and, respectively, fractional Brownian motions. As explained in Section 3.2 we need to consider the Lamperti process \( Y^H \).

Observe that the first hitting time \( \tau_X^H \) of 1 by \( X^H \) started from \( x_0 < 1 \) is equal to the first hitting time \( \tau_Y^H \) of \( F(1) \) by \( Y^H \) started from \( F(x_0) \).

Before stating our main result in this section we recall that the notation \( C_H \) has been defined at the end of Section 1 and we introduce the following new notation.

Notational convention 5.1. In all the sequel we set

\[ \Upsilon := F(1) \quad \text{and} \quad y_0 := F(x_0) < \Upsilon. \]

Theorem 5.2. Let \( X^H \) and \( X \) be the solutions to (2.2) and (2.3) respectively, both with initial condition \( x_0 < 1 \). Assume that \( b \) and \( \sigma \) satisfy (H1)-(H2). Let the function \( \tilde{b} \) be defined as in Proposition 3.2.

For any \( p \geq 1 \) and \( \lambda > |\tilde{b}'|_\infty \) set

\[
\mathcal{M}_p(\Upsilon - y_0, \lambda) := \sup_{s \in \mathbb{R}_+} \left( e^{-\frac{1}{2}(\lambda - |\tilde{b}'|_\infty)ps} \mathbb{E} e^{-|\Upsilon - Y^H_s|_p \mathcal{R}(\lambda)} \right),
\]

(5.1)

where

\[
\mathcal{R}(\lambda) := \sqrt{2\lambda + \mu^2} - \mu \quad \text{with} \quad \mu := |\tilde{b}|_\infty.
\]

(5.2)

Suppose \( x_0 < 1 \) and \( \lambda > |\tilde{b}'|_\infty \). Set \( \tilde{\lambda} := \lambda - |\tilde{b}'|_\infty \). For any \( H \in \left( \frac{1}{2}, 1 \right) \) we have
\[
\left| \mathbb{E}\left( e^{-\lambda \gamma H} \right) - \mathbb{E}\left( e^{-\lambda \gamma x} \right) \right| 
\leq C_H |H - \frac{1}{2}| \left( \frac{(1 + \lambda)}{1 \wedge \lambda^3} \left( M_1(Y - y_0, \lambda) + (M_2(Y - y_0, \lambda))^\mu \right) + (M_4(Y - y_0, \lambda))^\mu \right).
\]

(5.3)

The following proposition precises the convergence rate in (5.3). It is proven in Appendix D.

**Proposition 5.3.** Let \( \lambda > \tilde{b}'_1 \). Let \( m := Y - y_0, \mu := \tilde{b}'_1, q := \rho R(\lambda) \) and \( \tilde{\lambda} := \lambda - \tilde{b}'_1 \).

One has

\[
M_p(Y - y_0, \lambda) \leq C \left( e^{-\frac{\tilde{\lambda}}{2} m} e^{-\frac{\tilde{\lambda}}{2} \psi^H_q(m)} + \exp \left( -2^{-\frac{\xi}{2}} m^{\frac{2}{1+2\xi}} \tilde{\lambda}^{\frac{2}{1+2\xi}} \right) + \exp \left( -\tilde{\lambda} \mu \right) \right),
\]

(5.4)

where

\[
\psi^H_q(m) := \frac{m}{\mu + q} \mathbb{I} \left[ \frac{m}{\mu + q} < 1 \right] + \left( \frac{m}{\mu + q} \right)^{\frac{1}{\xi}} \mathbb{I} \left[ \frac{m}{\mu + q} \geq 1 \right].
\]

(5.5)

**Remark 5.4.** In Theorem 5.2 the convergence rates w.r.t. \( |H - \frac{1}{2}| \) and \( \lambda \) are optimal. See remarks 5.9 and 5.12 below.

**Remark 5.5.** Theorem 5.2 provides a sensitivity estimate with a constant which explodes when \( \tilde{\lambda} \) tends to 0. In [28] we conjecture that, by using the joint probability distribution of \( B^H \) and its running maximum, one might be able to show that the constants are uniform w.r.t. \( 0 < \tilde{\lambda} = \lambda < 1 \) when \( X^H \) is reduced to be the fractional Brownian motion \( B^H \).

As explained in the Introduction, the proof of Theorem 5.2 is technically demanding because we desire a bound from above which tends to 0 as fast as possible when \( H \) tends to \( \frac{1}{2} \) and decays at the same exponential rate when \( \lambda \) or \( |1 - x_0| \) tends to infinity as in the exact formula (1.1). This proof is split into Subsections 5.1 to 5.8:

- In Subsection 5.1 we remind the differential equation solved by the function \( \mathbb{E}\left( e^{-\lambda \gamma Y} \right| Y_0 = y \) . We suitably define an extension \( W_\lambda \) to the whole real line of that function and get estimates on \( W_\lambda^{(i)}(Y_0^H) \) (\( i = 0, 1, 2 \)).

- In Subsection 5.2 we adopt the same strategy as in Subsection 4.2. The difference \( \left| \mathbb{E}\left( e^{-\lambda \gamma H} \right) - \mathbb{E}\left( e^{-\lambda \gamma x} \right) \right| \) is split into the sum of a stopped Lebesgue integral and a stopped Skorokhod integral, the integrands being expressed in terms of the function \( W_\lambda \).

- In Subsection 5.3 we get an accurate estimate on the stopped Lebesgue integral.

- In Subsection 5.4 we get an accurate estimate on the stopped Skorokhod integral.

- In Subsections 5.5, 5.6, 5.7 and 5.8 we prove technical intermediate results.

### 5.1 An ‘optimal’ extension of the Laplace transform for \( H = \frac{1}{2} \) and related estimates

Denote by \( Y \) the Lamperti process solution to the SDE (3.19) driven by a standard Brownian motion \( B \):

\[
\forall t \geq 0, \quad Y_t = y_0 + B_t + \int_0^t b(Y_s) \, ds.
\]

(5.6)
Let \( \tau_Y \) be the first hitting time of \( Y \) by \( Y \). For any \( \lambda > 0 \) the function \( W_\lambda(y) := \mathbb{E}\left( e^{-\lambda \tau_Y} \mid Y_0 = y \right) \) defined on the interval \( (-\infty, Y] \) solves the following ODE:

\[
\begin{aligned}
\dot{b}(y)W_\lambda'(y) + \frac{1}{2}W_\lambda''(y) &= \lambda W_\lambda(y), \quad y < Y, \\
W_\lambda(Y) &= 1, \\
\lim_{y \to -\infty} W_\lambda(y) &= 0.
\end{aligned}
\]

(5.7)

In the sequel we will need to consider ‘stopped’ Skorokhod integrals of the type

\[
\delta_H^{(N)}\left( \mathbb{1}_{[0,t]}(\cdot) e^{-\lambda W_\lambda'(Y^H)} \right) \big|_{t = \tau_H \wedge N}.
\]

These ‘stopped’ integrals can only be defined by considering the parameterized family

\[
\delta_H^{(N)}\left( \mathbb{1}_{[0,t]}(\cdot) e^{-\lambda W_\lambda'(Y^H)} \right)
\]

which cannot be defined without extending the domain of the function \( W_\lambda \) to the whole real line. Of course, we have to choose an extension which allows us to get sharp estimates: We discuss this important issue in the remarks 5.9 and 5.12 below.

By abuse of notation we denote our extension below by \( W_\lambda \). For any \( \lambda > 0 \), \( W_\lambda \) is the non-negative \( C^2_b(\mathbb{R}) \) function defined as follows:

\[
\begin{aligned}
\forall y \leq Y, \quad W_\lambda(y) &:= \mathbb{E}\left( e^{-\lambda \tau_Y} \mid Y_0 = y \right), \\
\forall y \geq Y, \quad W_\lambda(y) &:= \phi(y) W_\lambda(2Y - y),
\end{aligned}
\]

(5.8)

where \( \phi(z) \) is a non-negative function in \( C^2_b(\mathbb{R}) \) with \( \phi(0) = 1 \), uniformly bounded w.r.t. \( \lambda \) and such that the first and second derivatives at \( Y \) of the map \( \phi(y) W_\lambda(2Y - y) \) respectively coincide with the left derivatives \( W_\lambda'(Y^-) \) and \( W_\lambda''(Y^-) = 2\lambda - \dot{b}(Y^-) W_\lambda'(Y^-) \). For example, one can choose

\[
\phi(y) = \Psi \left( 2(W_\lambda'(Y))^2(y - Y)^2 + 2W_\lambda'(Y)(y - Y) + 1 \right),
\]

where \( \Psi \) is any non-negative function in \( C^2_b(\mathbb{R}) \) such that \( \Psi(x) = x \) on \([\frac{1}{2}, 2]\) and \( \Psi(x) = 0 \) on \(( -\infty, 0 ) \cup ( 3, +\infty ) \).

In the Brownian motion case, the Laplace transform of the first hitting time at the threshold 1 is explicitly given by (1.1). One easily deduce that the derivatives w.r.t. \( x_0 \) of this Laplace transform tend exponentially fast to 0 when \( \lambda \) or \( (1 - x_0) \) tends to infinity. The following proposition shows that the two first derivatives of the function \( W_\lambda \) defined as in Section 5.1 satisfy similar exponential convergence rates. We postpone to Appendix C its easy proof.

**Proposition 5.6.** For any \( \lambda > 0 \), let \( W_\lambda(y) \) be defined as in (5.8). Under the assumptions (H1) and (H2) on \( b \) and \( \sigma \) one has

\[
\forall y \in \mathbb{R}, \quad 0 \leq W_\lambda(y) \leq e^{-|Y - y| R(\lambda)},
\]

(5.9)

where \( R(\lambda) \) is defined as in (5.2): \( R(\lambda) := \sqrt{2\lambda + \mu^2} - \mu \).

In addition, the two first derivatives of \( W_\lambda \) satisfy the following estimates: There exists \( C > 0 \) depending on \( \mu \) only such that, for all real numbers \( y \) and \( \tilde{y} \),

\[
|W_\lambda'(y)| \leq C(1 + \lambda) e^{-|Y - y| R(\lambda)},
\]

(5.10)

\[
|W_\lambda''(y)| \leq C(1 + \lambda) e^{-|Y - y| R(\lambda)},
\]

(5.11)

\[
|W_\lambda''(y) - W_\lambda''(\tilde{y})| \leq C(1 + \lambda)^2 |y - \tilde{y}| (e^{-|Y - y| R(\lambda)} + e^{-|Y - \tilde{y}| R(\lambda)}).
\]

(5.12)
5.2 An error decomposition

Proposition 5.7. Set

\[ \Delta(s, H) := H s^{2H-1} - \frac{1}{2} + \int_0^s \partial_s K_H(s, r) \int_0^r D_r \tilde{b}(Y_v^H) \ dv \ dr. \]  

(5.13)

For any \( \lambda > 0 \) it holds that

\[ E\left(e^{-\lambda \tau_H^N}\right) - E\left(e^{-\lambda \tau_H}\right) = E\left[\int_0^{T_H} \Delta(s, H) e^{-\lambda s} W_{\lambda}''(Y_s^H) \ ds\right] \]

\[ + \lim_{N \to +\infty} E\left[\delta_H^{(N)}\left(\|_{[0,t]} e^{-\lambda} W_{\lambda}'(Y_t^H)\right)\right]_{t = \tau_H \wedge N}\]

\[ =: I_1(\lambda) + I_2(\lambda). \]

(5.14)

**Proof.** Let \( N > 0 \). All the stochastic integrals below are well-defined and integrable in view of the bounds on \( |W_{\lambda}'| \) and \( |W_{\lambda}''| \) in Proposition 5.6.

Apply the Itô–Skorokhod formula (3.51) to \( e^{-\lambda t} W_{\lambda}(Y_t^H) \) and use the convention of writing 3.1: For any \( 0 < t \leq N \),

\[ e^{-\lambda t} W_{\lambda}(Y_t^H) - W_{\lambda}(y_0) = \int_0^t e^{-\lambda s} \left( b(Y_s^H) W_{\lambda}'(Y_s^H) - \lambda W_{\lambda}(Y_s^H) \right) \ ds \]

\[ + \delta_H^{(N)}\left(e^{-\lambda t} \|_{[0,t]} W_{\lambda}'(Y_t^H)\right) + \text{Tr}\left[D^H e^{-\lambda t} W_{\lambda}'(Y_t^H)\right]_{t = N \wedge \tau_H}. \]

Using the ODE (5.7) satisfied by \( W_{\lambda} \) we get:

\[ e^{-\lambda (N \wedge \tau_H)} W_{\lambda}(Y_{N \wedge \tau_H}^H) - W_{\lambda}(y_0) = \frac{1}{2} \int_0^{N \wedge \tau_H} e^{-\lambda s} W_{\lambda}''(Y_s^H) \ ds + \delta_H^{(N)}\left(\|_{[0,t]} W_{\lambda}'(Y_t^H) e^{-\lambda}\right)_{t = N \wedge \tau_H} \]

\[ + \text{Tr}\left[D^H e^{-\lambda t} W_{\lambda}'(Y_t^H)\right]_{t = N \wedge \tau_H}. \]

We now use the equality (3.52) and get:

\[ E\left(e^{-\lambda (N \wedge \tau_H)} W_{\lambda}(Y_{N \wedge \tau_H}^H)\right) - W_{\lambda}(y_0) = E\left[\int_0^{N \wedge \tau_H} \Delta(s, H) W_{\lambda}''(Y_s^H) e^{-\lambda s} \ ds\right] \]

\[ + E\left[\delta_H^{(N)}\left(\|_{[0,t]} W_{\lambda}'(Y_t^H) e^{-\lambda}\right)_{t = N \wedge \tau_H}\right]. \]

The dominated convergence theorem and the inequality (5.9) imply that the left-hand side converges when \( N \) tends to infinity. We claim that the first integral in the right-hand side also converges in the same limit. Actually, we combine the dominated convergence theorem with the inequality (5.11) and the estimate (5.15) which will be proven below.

In the preceding we have used the following technical lemma which will also be needed in the proof of Proposition 5.11.

**Lemma 5.8.** One has

\[ \forall H \in (\frac{1}{2}, 1), \quad \left|\int_0^s \partial_s K_H(s, r) \int_0^r D_r \tilde{b}(Y_v^H) \ dv \ dr\right| \leq C_H \left|H - \frac{1}{2}\right| e^{s|v|^s} (1 + s^2) \text{ a.s.} \]

Therefore,

\[ |\Delta(s, H)| \leq |H s^{2H-1} - \frac{1}{2}| + C_H \left|H - \frac{1}{2}\right| e^{s|v|^s} (1 + s^2) \text{ a.s.} \]

(5.15)
Proof. In view of Proposition 3.4 we have
\[ \int_0^s |\mathbf{D}_r \tilde{b}(Y^H_v)| \, dv \leq C \, e^{s|\tilde{b}|_\infty} \int_r^s \left\{ |K_H(v, r)| + (v - r)^{H + \frac{1}{2}} \, \mathbb{I}_{\{H < \frac{1}{2}\}} \right\} \, dv. \]

Then use (3.1) to get
\[ \int_0^s |\mathbf{D}_r \tilde{b}(Y^H_v)| \, dv \leq C \, e^{s|\tilde{b}|_\infty} \int_r^s \left\{ \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (v - r)^{H-\frac{1}{2}} + |H - \frac{1}{2}| \, r^{\frac{1}{2} - H} \int_r^v \theta^{H-\frac{1}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta 
+ (v - r)^{H + \frac{1}{2}} \, \mathbb{I}_{\{H < \frac{1}{2}\}} \right\} \, dv. \]

(5.16)

We now distinguish two cases.

The case $H < \frac{1}{2}$.
In this case, the change of variables $\gamma = \frac{\theta}{r}$ leads to
\[ \int_r^v \theta^{H-\frac{3}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta \leq r^{2H-1} \int_1^\infty \gamma^{H-\frac{3}{2}} (\gamma - 1)^{H-\frac{1}{2}} \, d\gamma \leq r^{2H-1} (C + \int_2^\infty \gamma^{H-\frac{3}{2}} \, d\gamma) \leq \frac{C \, r^{2H-1}}{|H - \frac{1}{2}|}. \]

In view of (5.16) and $\left( \frac{2}{r} \right)^{H-\frac{1}{2}} < 1$ for $0 < r < v$ and $H < \frac{1}{2}$ we deduce that
\[ \int_0^s |\mathbf{D}_r \tilde{b}(Y^H_v)| \, dv \leq C \, e^{s|\tilde{b}|_\infty} ((s - r)^{H + \frac{1}{2}} + r^{H-\frac{1}{2}} (s - r) + (s - r)^{H + \frac{1}{2}}). \]

Recall (3.7). It comes:
\[ \left| \int_0^s \partial_s K_H(s, r) \int_0^s \mathbf{D}_r \tilde{b}(Y^H_v) \, dv \, dr \right| \leq C \, |H - \frac{1}{2}| \, e^{s|\tilde{b}|_\infty} \int_0^s \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (s - r)^{H-\frac{3}{2}} ((s - r)^{H + \frac{1}{2}} + r^{H-\frac{1}{2}} (s - r) + (s - r)^{H + \frac{1}{2}}) \, dr. \]

It now remains to observe that
\[ \int_0^s \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (s - r)^{2H-1} \, dr \leq \int_0^s (s - r)^{2H-1} \, dr \leq C \, s^{2H}, \]
\[ \int_0^s \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (s - r)^{H-\frac{1}{2}} \, r^{H-\frac{1}{2}} \, dr = C \, s^{2H}, \]

and
\[ \int_0^s \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (s - r)^{2H} \, dr \leq \int_0^s (s - r)^{2H} \, dr \leq C \, s^{2H+1} \leq C \, (1 + s^2). \]

The case $H > \frac{1}{2}$.
In this case,
\[ \int_r^v \theta^{H-\frac{3}{2}} (\theta - r)^{H-\frac{1}{2}} \, d\theta \leq \int_r^v (\theta - r)^{2H-2} \, d\theta = \frac{(v - r)^{2H-1}}{2H - 1}. \]

In view of (5.16) we deduce that
\[ \int_0^s |\mathbf{D}_r \tilde{b}(Y^H_v)| \, dv \leq C \, e^{s|\tilde{b}|_\infty} \int_r^s \left\{ \left( \frac{v}{r} \right)^{H-\frac{1}{2}} (v - r)^{H-\frac{1}{2}} + r^{\frac{1}{2} - H} (v - r)^{2H-1} \right\} \, dv \]
\[ \leq C \, e^{s|\tilde{b}|_\infty} s^{H-\frac{1}{2}} \, r^{\frac{1}{2} - H} (s - r)^{H + \frac{1}{2}} + r^{\frac{1}{2} - H} (s - r)^{2H}. \]
As $H < \frac{1}{2}$ and $\int$ By using the change of variable Brownian motion with Hurst index choice of the Markov model as the proxy model. If the proxy model were driven by a fractional estimate Remark 5.10. However, the convergence rates to 0 w.r.t. $I$ below explains why the obtained convergence rates to 0 w.r.t. $\lambda \geq 0$. Consequently, up to the choice of the constants, the estimate (5.3) seems sharp.

Remark 5.9. We emphasize that $I_1(\lambda)$ depends on the path of $Y^H$ up to time $\tau_H$. Therefore, the value of $I_1(\lambda)$ does not depend on the way the original function $W_\lambda$ is extended. The particular choice (5.8) allows us to obtain estimates on $I_1(\lambda)$ in terms of $\mathcal M_1(Y - y_0, \lambda)$. The remark 5.12 below explains why the obtained convergence rates to 0 w.r.t. $\lambda \geq 0$, and $(Y - y_0)$ seem sharp. Similarly, the constants in our estimate of $I_2(\lambda)$ below depend on the particular extension (5.8). However, the convergence rates to 0 w.r.t. $\lambda \geq 0$ and $(Y - y_0)$ are similar to those obtained for $I_1(\lambda)$.

Consequently, up to the choice of the constants, the estimate (5.3) seems sharp.

Remark 5.10. We again come back to the discussion initiated in the Introduction to justify the choice of the Markov model as the proxy model. If the proxy model were driven by a fractional Brownian motion with Hurst index $H' \neq \frac{1}{2}$, in view of (5.14), the equality (1.2) would lead to estimate

$$
E \left[ \delta^{(N)}_H \left( \mathbb I_{[0,\tau]}(\cdot) e^{-\lambda \cdot} W_\lambda \left( Y^{H'} \right) \right) \bigg| \mathcal I_{\tau = \tau_H \wedge \tau_N} \right] - E \left[ \delta^{(N)}_H \left( \mathbb I_{[0,\tau]}(\cdot) e^{-\lambda \cdot} W_\lambda' \left( Y^{H'} \right) \right) \bigg| \mathcal I_{\tau = \tau_H \wedge \tau_N} \right],
$$

in terms of $|H - H'|$. We do not see how to solve this issue.

5.3 Estimate on $I_1(\lambda)$ defined as in (5.14)

Applying Fubini’s theorem we get

$$
I_1(\lambda) = \int_0^\infty e^{-\lambda s} E(\Delta(s, H) \mathbb I_{\tau_H \geq s} W_\lambda' \left( Y^{H'} \right)) \, ds,
$$

where $\Delta(s, H)$ is defined by (5.13).

**Proposition 5.11.** As in Proposition 5.3 set $\tilde{\lambda} := \lambda - |\tilde{b}|_\infty$. Suppose $\tilde{\lambda} > 0$. One has

$$
|I_1(\lambda)| \leq C_H \frac{1 + \lambda}{1 \wedge \tilde{\lambda}^3} |H - \frac{1}{2}| \mathcal M_1(Y - y_0, \lambda).
$$
Proof. In view of Inequalities (5.15) and (5.11) one has

\[ |I_1(\lambda)| = \left| \int_0^{\infty} e^{-\lambda s} \Delta(s, H) \mathbb{E} \left( \mathbb{I}_{\{ \tilde{Y}_\lambda \geq s \}} \mathbb{W}_s^H(Y_s^H) \right) \, ds \right| \]

\[ \leq C \left( 1 + \lambda \right) \int_0^{\infty} e^{-\lambda s} \mathbb{E} \left( e^{-e^{[\tilde{Y}_s^H(\mathcal{R}(\lambda))]}} |Hs^{2H-1} - \frac{1}{2}| \right) \, ds \]

\[ + C_H |H - \frac{1}{2}| \left( 1 + \lambda \right) \int_0^{\infty} e^{-\lambda |\tilde{Y}_s^H|} \mathbb{E} \left( e^{-e^{[\tilde{Y}_s^H(\mathcal{R}(\lambda))]}} (1 + s^2) \right) \, ds \]

\[ \leq C \left( 1 + \lambda \right) \mathcal{M}_1(\mathbb{Y} - y_0, \lambda) \int_0^{\infty} e^{-\frac{1}{2} \lambda s} |Hs^{2H-1} - \frac{1}{2}| \, ds \]

\[ + C_H |H - \frac{1}{2}| \left( 1 + \lambda \right) \mathcal{M}_1(\mathbb{Y} - y_0, \lambda) \int_0^{\infty} e^{-\frac{1}{2} s} (1 + s^2) \, ds. \]

Split the integral

\[ \int_0^{\infty} e^{-\frac{1}{2} \lambda s} |Hs^{2H-1} - \frac{1}{2}| \, ds \]

into integrals from 0 to \( \alpha := \left( \frac{1}{2H} \right) \left[ \frac{1}{2} \right] \) and from \( \alpha \) to \( +\infty \). This leads one to consider

\[ I_{11} := \text{sign}(H - \frac{1}{2}) \int_0^\alpha e^{-\frac{1}{2} \lambda s} \left( \frac{1}{2} - Hs^{2H-1} \right) \, ds \quad \text{and} \quad I_{12} := \text{sign}(H - \frac{1}{2}) \int_\alpha^{+\infty} e^{-\frac{1}{2} \lambda s} \left( Hs^{2H-1} - \frac{1}{2} \right) \, ds. \]

As for \( I_{11} \), integrate by parts and use that \( 1 - \alpha^{2H-1} = \frac{1}{H} (H - \frac{1}{2}) \). It comes:

\[ I_{11} = \frac{1}{2} \text{sign}(H - \frac{1}{2}) \alpha \left( \frac{1}{H} (H - \frac{1}{2}) \right) e^{-\frac{1}{2} \lambda \alpha} + \frac{1}{2} \int_0^\alpha e^{-\frac{1}{2} \lambda s} (s - s^{2H}) \, ds. \]

Observe that \( \alpha \) is a bounded function of \( H \in (\frac{1}{4}, 1) \). In addition, for any \( s \in [0, \alpha] \) apply the Mean Value theorem to the map \( H \in (\frac{1}{4}, 1) \mapsto s - s^{2H} = s - s^{1+2(H - \frac{1}{2})} \) around the point \( H = \frac{1}{2} \). It comes:

\[ I_{11} \leq C \left| H - \frac{1}{2} \right| + \left| H - \frac{1}{2} \right| \sup_{s \in [0, \alpha]} \sup_{\gamma \in (-\frac{1}{4}, \frac{1}{2})} (|\log(s)| \, s^{1+2\gamma}) \, \tilde{\lambda} \int_0^\alpha e^{-\frac{1}{2} \lambda s} \, ds \]

\[ \leq C \left| H - \frac{1}{2} \right|. \quad \text{(5.18)} \]

As for \( I_{12} \), we proceed as above. In addition, we use that

\[ \exists C > 0, \forall s > 0, \forall \gamma \in (-\frac{1}{4}, \frac{1}{2}), \quad |\log(s)| \, (s^{1+2\gamma}) \leq C \left( 1 + s^2 \right). \]

We get:

\[ I_{12} \leq C \left| H - \frac{1}{2} \right| + C \left| H - \frac{1}{2} \right| \tilde{\lambda} \int_\alpha^{\infty} (1 + s^2) \, e^{-\frac{1}{2} \lambda s} \, ds \]

\[ \leq C \left| H - \frac{1}{2} \right| (1 + \frac{1}{\lambda^2}). \quad \text{(5.19)} \]

To conclude, it remains to gather the inequalities (5.18) and (5.19) with

\[ \forall H \in (\frac{1}{4}, 1), \quad \int_0^{\infty} e^{-\frac{1}{2} \gamma s} (1 + s^2) \, ds \leq C \left( 1 + \frac{1}{\lambda^3} \right). \]

\[ \square \]
Remark 5.12. When \( Y^H \) reduces to \( B^H \), that is, when \( \tilde{b} \equiv 0 \), \( F(y) = y \) and \( y_0 \equiv 0 \), in view of (5.17) and (5.7) one has

\[
I_1(\lambda) = 2\lambda \int_0^\infty e^{-\lambda s} \left( H s^{2H-1} - \frac{1}{2} \right) \mathbb{E} \left( I_{\{\tau_H \geq s\}} \mathbf{W}_\lambda(B^H_s) \right) \, ds.
\]

One cannot compute the exact value of the right-hand side since the joint law of \( (\tau^H, Y^H) \) is unknown. The preceding proof consists in replacing the function \( \mathbf{W}_\lambda(y) \) with a continuous extension on the whole real line which decays fast to 0 when \( y \) tends to +\( \infty \). In view of (1.1), a natural choice is \( e^{-|1-y|\sqrt{2\lambda}} \). It leads to estimate

\[
2\lambda \int_0^\infty e^{-\lambda s} \left( H s^{2H-1} - \frac{1}{2} \right) \mathbb{E} \left( I_{\{\tau_H \geq s\}} \mathbf{e}^{-|1-B^H_s\sqrt{2\lambda}} \right) \, ds \simeq 2\lambda \int_0^\infty e^{-\lambda s} \left( H s^{2H-1} - \frac{1}{2} \right) \mathbf{e}^{-|1-B^H_s\sqrt{2\lambda}} \, ds.
\]

The calculation done above to estimate \( I_{11} \) and \( I_{12} \) shows that the preceding quantity is of the order \( |H - \frac{1}{2}| \frac{1}{\sqrt{\lambda \Delta}} \mathcal{M}_4(\mathbb{Y}, \lambda) \).

5.4 Estimate on \( I_2(\lambda) \) defined as in (5.14)

Recall that

\[
I_2(\lambda) := \lim_{N \to +\infty} \mathbb{E} \left[ \delta_H^{(N)} \left( \mathbb{I}_{[0,t]}(\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \right]_{t=\tau_H \wedge N}.
\]

To obtain Theorem 5.2 we aim to prove that for every \( \bar{\lambda} = \lambda - |\tilde{b}'|_\infty > 0 \) one has

\[
|I_2(\lambda)| \leq C_H \left| H - \frac{1}{2} \right| \left( \frac{1 + \lambda^2}{1 \wedge \lambda^3} \left( \mathcal{M}_2(\mathbb{Y} - y_0, \lambda) \right)^{\frac{H \wedge \lambda}{2}} + \mathcal{M}_4(\mathbb{Y} - y_0, \lambda)^{\frac{H \wedge \lambda}{2}} \right). \tag{5.20}
\]

We emphasize that the optional stopping theorem does not hold true for the Skorokhod integrals \( \delta_H^{(N)} \) when \( H \neq \frac{1}{2} \). However, applying this theorem to standard Itô integrals provides

\[
\forall N > 0, \quad \mathbb{E} \left[ \delta^{(N)} \left( \mathbb{I}_{[0,t]}(\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \right]_{t=\tau_H \wedge N} = 0.
\]

We thus are led to introduce the centering term \( \delta^{(N)} \left( \mathbb{I}_{[0,t]}(\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \left|_{t=\tau_H \wedge N} \right. \), which is crucial to get an estimate on \( I_2(\lambda) \) of the order \( |H - \frac{1}{2}| \frac{1}{2} \wedge \lambda^3 \):

\[
\left| \mathbb{E} \left[ \delta_H^{(N)} \left( \mathbb{I}_{[0,t]}(\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \right]_{t=\tau_H \wedge N} - \delta^{(N)} \left( \mathbb{I}_{[0,t]}(\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \right|_{t=\tau_H \wedge N} = \left| \mathbb{E} \left[ \delta^{(N)} \left( \{ K_{H,N}^* - \text{Id} \} \left( \mathbb{I}_{[0,t] \wedge N} \right) e^{-\lambda} \mathbf{W}_\lambda'(Y^H) \right) \right]_{t=\tau_H \wedge N} \right|
\]

Define the field \( \{ U^{(N)}_t(v), v \geq 0, t > 0 \} \) and the process \( \{ T^{(N)}_t, t > 0 \} \) by

\[
U^{(N)}_t(v) := \{ K_{H,N}^* - \text{Id} \} \left( \mathbb{I}_{[0,t] \wedge N} \right) (\cdot) e^{-\lambda} \mathbf{W}_\lambda'(Y^H)(v) \tag{5.21}
\]

and

\[
\Upsilon^{(N)}_t := \delta^{(N)}(U^{(N)}_t). \tag{5.22}
\]

Let \( [t] \) denote the integer part of \( t \). As \( \Upsilon^{(N)}_0 = 0 \) for any \( t > 0 \) we have

\[
\Upsilon^{(N)}_t = \Upsilon^{(N)}_t - \Upsilon^{(N)}_{[t]} + \sum_{n=1}^{[t]} (\Upsilon^{(N)}_n - \Upsilon^{(N)}_{n-1}) \mathbb{I}_{t \geq 1}.
\]

34
Therefore,

\[ |I_2(\lambda)| \leq \lim_{N \to \infty} \sum_{n=0}^{N-1} \mathbb{E} \sup_{t \in [n, n+1]} \left[ \left| \mathcal{Y}_t^{(N)} - \mathcal{Y}_n^{(N)} \right| \right]. \tag{5.23} \]

In order to estimate the right-hand side of the preceding inequality we now apply the following corollary of Garsia-Rodemich-Rumsey’s lemma:

**Lemma 5.13** (Garsia-Rodemich-Rumsey). Let \( \{X_t, t \in [a, b]\} \) be an \( \mathbb{R} \)-valued continuous stochastic process. Then, for \( p \geq 1 \) and \( q > 0 \) such that \( pq > 2 \),

\[
\mathbb{E} \left( \sup_{t \in [a,b]} |X_t - X_a| \right) \leq C \frac{pq}{pq - 2} (b-a)^{q-\frac{2}{p}} \mathbb{E} \left[ \left( \int_a^b \int_a^b \frac{|X_s - X_t|^p}{|s-t|^{pq}} \, ds \, dt \right)^{\frac{1}{p}} \right] 
\]

\[
\leq C \frac{pq}{pq - 2} (b-a)^{q-\frac{2}{p}} \left( \int_a^b \int_a^b \frac{\mathbb{E} \left( |X_s - X_t|^p \right)}{|s-t|^{pq}} \, ds \, dt \right)^{\frac{1}{p}},
\]

provided the right-hand side in each line is finite.

**Proof.** With the notations of [24, p.353-354], apply the general Garsia-Rodemich-Rumsey lemma with \( \psi(x) = x^p \) and \( p(x) = x^q \) to obtain the first line. The second line results from Hölder’s inequality. \( \square \)

We thus obtain:

\[
|I_2(\lambda)| \leq \lim_{N \to \infty} \sum_{n=0}^{N-1} C \frac{pq}{pq - 2} \left( \int_n^{n+1} \int_n^{n+1} \mathbb{E} \left( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}|^p \right) \, ds \, dt \right)^{\frac{1}{p}}, \tag{5.24} \]

for any \( p \geq 1 \) and \( q > 0 \) such that \( pq > 2 \).

We now need to estimate moments of \( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}| \) with two different constraints. On the one hand, to get finiteness of the right-hand side of (5.24) it is natural to choose the value of \( pq \) large to allow the \( p \)-th moment of \( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}| \) to be of order \( (t-s)^\gamma(p) \) with a large enough power \( \gamma(p) \). On the other hand, to get a convergence rate of \( |I_2(\lambda)| \) in terms of \( |H - \frac{1}{2}|, \lambda \) and \( \|\mathcal{Y} - y_0\| \) it is convenient to consider the second moment of \( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}| \) whose convergence rate to 0 can be obtained by using the explicit value of \( (K_H(t, v) - 1)^2 \) (see the term \( J_1 \) in the proof of Lemma 5.16 and Lemma 5.17), whereas the estimation of other moments of \( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}| \) would necessarily involve the hardly tractable terms \( (K_H(t, v) - 1)^7 \) with \( \gamma > 2 \).

The preceding leads us to use the obvious inequality

\[
\forall p > 2, \quad \mathbb{E} \left( |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}|^p \right) \leq \left( \mathbb{E} |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}|^{2(p-1)} \right)^{\frac{1}{2}} \times \left\| \mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)} \right\|_2.
\]

In Subsections 5.6 and 5.8 we respectively prove that for any \( 0 < s < t < N \) with \( 0 < t - s < 1 \) we have

\[
\left\| \mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)} \right\|_2 \leq C_H |H - \frac{1}{2}| (t-s)^{H^\frac{1}{2}} (1 + |\log(t-s)|) (1 + \lambda)^2 (1 + t^2) e^{-\frac{1}{2} \lambda s} \times \left( (M_2(\mathcal{Y} - y_0, \lambda))^{\frac{1}{2}} + (M_4(\mathcal{Y} - y_0, \lambda))^{\frac{1}{2}} \right)
\]

and, for any \( p \geq 2, \)

\[
\left( \mathbb{E} |\mathcal{Y}_t^{(N)} - \mathcal{Y}_s^{(N)}|^{2(p-1)} \right)^{\frac{1}{2}} \leq C_H |H - \frac{1}{2}|^{p-1} (t-s)^{(p-1)(H^\frac{1}{2})} (1 + |\log(t-s)|)^{p-1} (1 + \lambda)^{2(p-1)} (1 + t^2)^{p-1} e^{-(p-1)\lambda s}.
\]
Coming back to (5.24) and for instance choosing $p = \frac{3}{H^2}$ and $q = H \wedge \frac{1}{2}$ we get

$$\left| I_2(\lambda) \right| \leq C_H \left| H - \frac{1}{2} \right| (1 + \lambda)^2 \left( (\mathcal{M}_2(\mathbb{W} - y_0, \lambda))^{\frac{H^2}{q}} + (\mathcal{M}_4(\mathbb{W} - y_0, \lambda))^{\frac{H^2}{q}} \right)$$

$$\times \lim_{N \to \infty} \sum_{n=0}^{N-1} (1 + (n + 1))^2 e^{-\lambda n} \left( \int_n^{n+1} \int_n^{n+1} (1 + |\log(t-s)|)^{\frac{3}{H^2}} ds \, dt \right)^{\frac{H^2}{q}}$$

$$\leq C_H \left| H - \frac{1}{2} \right| \frac{(1 + \lambda)^2}{1 \wedge \lambda^3} \left( (\mathcal{M}_2(\mathbb{W} - y_0, \lambda))^{\frac{H^2}{q}} + (\mathcal{M}_4(\mathbb{W} - y_0, \lambda))^{\frac{H^2}{q}} \right).$$

5.5 An elementary proposition

In this subsection we prove the following elementary result which will be often used in the sequel.

**Proposition 5.14.** Let $0 \leq S < T$. Let $(\xi_\theta)$ be a square integrable process on $[S, T]$.

(i) Let $f$ be an integrable function on $[S, T]$. One has

$$\mathbb{E}\left( \int_{S}^{T} \xi_\theta \, f(\theta) \, d\theta \right)^2 \leq \sup_{S \leq \theta \leq T} \mathbb{E}(\xi_\theta)^2 \left( \int_{S}^{T} |f(\theta)| \, d\theta \right)^2. \quad (5.25)$$

(ii) Suppose in addition that $0 < T - S < 1$ and $\sup_{S \leq \theta \leq T} \mathbb{E}(\xi_\theta^4) < \infty$. Let $\frac{1}{4} < H < 1$. Let $\phi(\cdot, S)$ be a Lebesgue measurable function such that $(\cdot - S)^H \, |\phi(\cdot, S)|$ is integrable on $[S, T]$.

Then

$$\mathbb{E} \left( \int_{S}^{T} \xi_\theta \, (Y^H_\theta - Y^H_S) \, \phi(\theta, S) \, d\theta \right)^2 \leq C \sup_{S \leq \theta \leq T} \sqrt{\mathbb{E}(\xi_\theta^4)} \left( \int_{S}^{T} (\theta - S)^H \, |\phi(\theta, S)| \, d\theta \right)^2. \quad (5.26)$$

**Proof.** By Cauchy-Schwarz formula,

$$\left( \int_{S}^{T} \xi_\theta \, f(\theta) \, d\theta \right)^2 \leq \int_{S}^{T} |f(\theta)| \, d\theta \int_{S}^{T} (\xi_\theta)^2 \, |f(\theta)| \, d\theta.$$

This provides (5.25).

Similarly,

$$\left( \int_{S}^{T} \xi_\theta \, \frac{(Y^H_\theta - Y^H_S)}{(\theta - S)^H} \, \phi(\theta, S) \, d\theta \right)^2 \leq \int_{S}^{T} (\xi_\theta)^2 \, \frac{(Y^H_\theta - Y^H_S)^2}{(\theta - S)^{2H}} \, (\theta - S)^H \, |\phi(\theta, S)| \, d\theta \int_{S}^{T} (\theta - S)^H \, |\phi(\theta, S)| \, d\theta.$$

From (3.19) and (2.1) we deduce that

$$\mathbb{E}(\xi_\theta^2 \, (Y^H_\theta - Y^H_S)^2) \leq C \sqrt{\mathbb{E}(\xi_\theta^4)} \, ((\theta - S)^{2H} + (\theta - S)^2).$$

To get (5.26) it remains to use that $(\theta - S)^{2-2H} \leq 1$ since $0 < T - S < 1$ by hypothesis. \qed
5.6 $L^2$-estimate on $(\Upsilon^N_t - \Upsilon^N_s)$

Lemma 5.15. Suppose $\frac{1}{4} < H < 1$ and $H \neq \frac{1}{2}$.

Let $\Upsilon^N$ be defined as in (5.22). For any $0 < s < t < N$ with $0 < t - s < 1$ we have

$$\left\| \Upsilon^N_t - \Upsilon^N_s \right\|_2 \leq C_H \left| H - \frac{1}{2} \right| (t - s)^{H^\lambda + \frac{1}{2}} (1 + \lfloor \log(t - s) \rfloor) (1 + \lambda)^2 (1 + t^2) e^{-\frac{1}{2}\bar{\lambda}s} \times \left( (\mathcal{M}_2(Y - y_0, \lambda))^{\frac{1}{2}} + (\mathcal{M}_4(Y - y_0, \lambda))^{\frac{1}{2}} \right).$$  \hspace{1cm} (5.27)

Proof. Recall (5.21) and (3.12). For any $0 \leq s \leq t \leq N$ and $v$ in $[0, N]$ one has

$$U^N_t(v) - U^N_s(v) = \mathbb{I}_{(s, t]}(v) \left( K_H(t, v) - 1 \right) W^\alpha(Y^H) e^{-\lambda v}$$

$$+ \int_t^s \partial_\theta K_H(\theta, v) \left( \mathbb{I}_{(s, t]}(\theta) \ W^\alpha(Y^H) e^{-\lambda \theta} - \mathbb{I}_{(s, t]}(v) \ W^\alpha(Y^H) e^{-\lambda v} \right) d\theta$$

$$- \mathbb{I}_{(s, t]}(v) \ W^\alpha(Y^H) e^{-\lambda v}.$$  \hspace{1cm} (5.28)

Therefore,

$$U^N_t(v) - U^N_s(v) = \mathbb{I}_{(s, t]}(v) \left( K_H(t, v) - 1 \right) W^\alpha(Y^H) e^{-\lambda v}$$

$$+ \int_t^s \partial_\theta K_H(\theta, v) \left( e^{-\lambda \theta} - e^{-\lambda v} \right) d\theta$$

$$+ \int_t^s \partial_\theta K_H(\theta, v) \left( W^\alpha(Y^H) - W^\alpha(Y^H) \right) e^{-\lambda \theta} d\theta$$

$$+ \int_t^s \partial_\theta K_H(\theta, v) \left( W^\alpha(Y^H) - W^\alpha(Y^H) \right) e^{-\lambda \theta} d\theta$$

$$=: J^{(1)} + J^{(2)} + J^{(3)} + J^{(4)}.$$  \hspace{1cm} (5.29)

In view of Meyer’s inequalities ([24, Prop.3.2.1]) we have

$$\left\| \Upsilon^N_t - \Upsilon^N_s \right\|_2 \leq C \sum_{i=1}^{4} \left\{ \int_0^N \mathbb{E}(|J^{(i)}|^2) dv \right\}^{\frac{1}{2}} + C \sum_{i=1}^{4} \left\{ \mathbb{E} \int_{0}^{N} \int_{0}^{N} |D_r J^{(i)}|^2 dr dv \right\}^{\frac{1}{2}}.$$  \hspace{1cm} (5.30)

The first term in the right-hand side is simpler than the second one and leads to even better estimates. We thus only detail the calculations which concern the second term. We will use the two following inequalities which result from (3.26) and (5.11):

$$|D_r W^\alpha(Y^H)| e^{-\lambda \theta} \leq \mathbb{I}_{\{r \leq \theta\}} |W^\alpha(Y^H)| e^{-\lambda \theta + [\theta]} \left( K_H(\theta, r) + C (\theta - r)^{H^\lambda + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right)$$

$$\leq C \mathbb{I}_{\{r \leq \theta\}} (1 + \lambda)^2 \mathcal{M}_2(Y - y_0, \lambda) \left( |K_H(\theta, r)|^2 + (\theta - r)^{2H + 1} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) e^{-\bar{\lambda} \theta}.$$  \hspace{1cm} (5.31)

from which

$$\mathbb{E}(|D_r W^\alpha(Y^H)|^2) e^{-2\lambda \theta}$$

$$\leq C \mathbb{I}_{\{r \leq \theta\}} (1 + \lambda)^2 \mathcal{M}_2(Y - y_0, \lambda) \left( |K_H(\theta, r)|^2 + (\theta - r)^{2H + 1} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) e^{-\bar{\lambda} \theta}.$$  \hspace{1cm} (5.32)

A bound from above for $\int_0^N \int_0^N \mathbb{E}(|D_r J^{(1)}|^2) dr dv$.

We have

$$D_r J^{(1)} = \mathbb{I}_{(s, t]}(v) \mathbb{I}_{r \leq v} D_r (W^\alpha(Y^H)) (K_H(t, v) - 1) e^{-\lambda v},$$

$$\text{Hence,}$$

$$\mathbb{E}(|D_r J^{(1)}|^2) \leq C \mathbb{I}_{\{r \leq \theta\}} (1 + \lambda)^2 \mathcal{M}_2(Y - y_0, \lambda) \left( |K_H(\theta, r)|^2 + (\theta - r)^{2H + 1} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) e^{-\bar{\lambda} \theta}.$$
We thus have obtained:
\[
\int_0^N \int_0^N \mathbb{E}(|D_r J(1)|^2) \, dr \, dv = \int_s^t \left( \int_0^v \mathbb{E}(|D_r W^H_\lambda (Y_v^H)|^2) \, dr \right) (K_H(t, v) - 1)^2 e^{-2\lambda v} \, dv.
\]

Now, successively use (5.30) and (3.5) to get
\[
\int_s^t \int_0^v \mathbb{E}(|D_r J(1)|^2) \, dr \, dv \leq C (1 + \lambda)^2 \mathcal{M}_2(\mathcal{Y} - y_0, \lambda) \int_s^t (v^{2H} + v^{2H+2}) (K_H(t, v) - 1)^2 e^{-\tilde{\lambda} v} \, dv.
\]

Bound \( v^{2H+2} \) in the right-hand side by \( t^2 v^{2H} \) and use (5.38) (see Lemma 5.16 below) to conclude that
\[
\begin{align*}
\int_0^N \int_0^N \mathbb{E}(|D_r J(1)|^2) \, dr \, dv & \leq C \left| H - \frac{1}{2} \right|^2 (t - s)^{(2H+1)} \left( 1 + (\log(t - s))^2 \right) (1 + \lambda)^2 \mathcal{M}_2(\mathcal{Y} - y_0, \lambda) \left( 1 + t^4 \right) e^{-\tilde{\lambda} s}. \quad (5.31)
\end{align*}
\]

**A bound from above for \( \int_0^N \int_0^N \mathbb{E}(|D_r J(2)|^2) \, dr \, dv \).**

We have
\[
D_r J(2) = \mathbb{I}_{(s,t)}(v) \mathbb{I}_{r \leq v} D_r (W^H_\lambda (Y_v^H)) \int_v^t \partial_\theta K_H(\theta, v) \left( e^{-\lambda \theta} - e^{-\lambda v} \right) d\theta,
\]
from which
\[
\begin{align*}
\int_0^N \int_0^N \mathbb{E}(|D_r J(2)|^2) \, dr \, dv & = \int_s^t \left( \int_0^v \mathbb{E}(D_r W^H_\lambda (Y_v^H))^2 \right) \left( \int_v^t \partial_\theta K_H(\theta, v) \left( e^{-\lambda \theta} - e^{-\lambda v} \right) d\theta \right)^2 \, dv.
\end{align*}
\]

Notice that \( |e^{-\lambda \theta} - e^{-\lambda v}| \leq (\theta - v) e^{-\lambda v} \) for \( 0 < v \leq \theta \). Combine this inequality with (B.3) and (B.5) in the appendix to get
\[
\int_v^t |\partial_\theta K_H(\theta, v)| \left| e^{-\lambda \theta} - e^{-\lambda v} \right| d\theta \leq C \left| H - \frac{1}{2} \right| (t - v)^{H+\frac{1}{2}} \left( 1 + \frac{t^{H-\frac{1}{2}}}{v^{H-\frac{1}{2}}} \mathbb{I}_{\frac{1}{2} < H < 1} \right) e^{-\lambda v}.
\]

Bound \( (t - v) \) by \( (t - s) \) in the right-hand side. Then, as above, successively use (5.30) and (3.5) to get
\[
\begin{align*}
\int_0^N \int_0^N \mathbb{E}(|D_r J(2)|^2) \, dr \, dv & \leq C \left| H - \frac{1}{2} \right|^2 (1 + \lambda)^2 \mathcal{M}_2(\mathcal{Y} - y_0, \lambda) (t - s)^{2H+1} \\
& \quad \int_s^t (v^{2H} + v^{2H+2}) \left( 1 + \frac{t^{2H-1}}{v^{2H-1}} \mathbb{I}_{\frac{1}{2} < H < 1} \right) e^{-\tilde{\lambda} v} \, dv.
\end{align*}
\]

Notice that
\[
\forall s \leq v \leq t, \quad (v^{2H} + v^{2H+2}) \left( 1 + \frac{t^{2H-1}}{v^{2H-1}} \right) = (v + v^3) \left( v^{2H-1} + t^{2H-1} \right) \leq C (1 + t^{2H+2}).
\]

We thus have obtained:
\[
\begin{align*}
\int_0^N \int_0^N \mathbb{E}(|D_r J(2)|^2) \, dr \, dv & \leq C \left| H - \frac{1}{2} \right|^2 (t - s)^{2H+2} (1 + \lambda)^2 \mathcal{M}_2(\mathcal{Y} - y_0, \lambda) (1 + t^4) e^{-\tilde{\lambda} s}. \quad (5.32)
\end{align*}
\]

**A bound from above for \( \int_0^N \int_0^N \mathbb{E}(|D_r J(3)|^2) \, dr \, dv \).**

38
In view of \((5.29)\) we have

\[
\int_{t}^{r} \frac{d}{d\theta} e^{-\lambda \theta} d\theta \leq C \int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

Now, in view of \((5.29)\) we have

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

Apply \((5.25)\) with

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

It comes:

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

Now, in view of \((B.3)\) we have

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

By using the definitions \((B.1)\) and \((B.20)\) we get

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

In view of \((B.6)\), \((B.11)\) and \((B.21)\) the right-hand side is bounded from above by

\[
\begin{cases}
C_H |H - \frac{1}{2}|^2 (t-s)^{3-2H} \theta^H e^{-\lambda s} & \text{when } \frac{1}{2} < H < 1, \\
C_H |H - \frac{1}{2}|^2 ((t-s)^{4H} + (t-s)^{4H+2}) e^{-\lambda s} & \text{when } \frac{1}{4} < H < \frac{1}{2}.
\end{cases}
\]

As \(H > \frac{1}{2}\) and \(0 < t - s < 1\) we have thus obtained

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

A bound from above for \(\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta\)

We have

\[
\int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]

Insert and subtract \(W''(Y_v^{\theta}) Y_v^{\theta} H\) in the right-hand side. For \(r \leq v\) set

\[
D_r J^{(3)} = \int_{0}^{\infty} \frac{d}{d\theta} e^{-\lambda \theta} d\theta.
\]
Therefore, in view of (B.1) and (B.22), we have

\[ D_r J_2^{(3)} := \mathbb{I}_{(s,t)}(v) \int_v^t \partial_\theta K_H(\theta, v) \, W^\prime_\lambda(Y_v^H) \left( D_r Y_\theta^H - D_r Y_v^H \right) e^{-\lambda \theta} \, d\theta. \]

(i) A bound for \( D_r J_1^{(3)} \). In view of (C.4) and (3.26), for \( r \leq v \) we have

\[ |D_r J_1^{(3)}| \leq C \mathbb{I}_{(s,t)}(v) (1 + \lambda)^2 \int_v^t |\partial_\theta K_H(\theta, v)| \left( e^{-|Y^H_\theta - Y^H_v|} R(\lambda) + e^{-|Y^H_\theta - Y^H_v|} R(\lambda) \right) |Y^H_\theta - Y^H_v| \left\{ K_H(\theta, r) + (\theta - r)^{H+\frac{1}{2}} \mathbb{I}_{\{H<\frac{3}{2}\}} \right\} e^{-\tilde{\lambda} \theta} \, d\theta. \]

Apply (3.26) with

\[ \xi_\theta \equiv e^{-\frac{1}{2} \tilde{\lambda} \theta} \left( e^{-|Y^H_\theta - Y^H_v|} R(\lambda) + e^{-|Y^H_\theta - Y^H_v|} R(\lambda) \right) \]

and

\[ \phi(\theta, v) \equiv \mathbb{I}_{(s,t)}(v) |\partial_\theta K_H(\theta, v)| \left\{ K_H(\theta, r) + (\theta - r)^{H+\frac{1}{2}} \mathbb{I}_{\{H<\frac{3}{2}\}} \right\} e^{-\frac{1}{2} \tilde{\lambda} \theta}. \]

It comes:

\[ \mathbb{E}_r \left( |D_r J_1^{(3)}|^2 \right) \leq C (1 + \lambda)^4 \mathcal{M}_4(\mathbb{Y} - y_0, \lambda). \]

Notice that

\[ \sup_{v \leq \theta \leq t} \mathbb{E}(\xi^6) \leq \mathcal{M}_4(\mathbb{Y} - y_0, \lambda). \]

In addition, in view of (B.3), for any \( \theta > v \) one has

\[ \phi(\theta, v) \leq \mathbb{I}_{(s,t)}(v) \left\{ |\partial_\theta K_H(\theta, v)| \left( K_H(\theta, r) + C |H - \frac{1}{2}| (\theta - r)^{H+\frac{1}{2}} \right) \right\} e^{-\frac{1}{2} \tilde{\lambda} \theta}. \]

Therefore, in view of (B.1) and (B.22), we have

\[ \int_s^t \int_0^t \left( \int_v^t (\theta - v)^H \phi(\theta, v) \, d\theta \right)^2 \, dv \, dr \leq C \int_s^t e^{-\tilde{\lambda} v} \int_0^v (A^v_t(v, r, t))^2 \, dv \, dr + C \mathbb{I}_{\{H<\frac{3}{2}\}} \int_s^t e^{-\tilde{\lambda} v} \int_0^v (A^v_t(v, r, t))^2 \, dv \, dr. \]

We now use (B.12), (B.13) and (B.23). As we are in the case \( 0 < t - s < 1 \) the right-hand side of the preceding inequality is bounded from above by

\[ C_H |H - \frac{1}{2}|^2 (t - s)^{4H \land (2H+1)} (1 + t^{2H} + t^{4H-1} + t^{2H+2}) e^{-\tilde{\lambda} s}. \]

We thus have obtained that

\[ \int_0^N \int_0^v \mathbb{E}(\xi^{(3)})^2 \, dv \, dr \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H \land (2H+1)} (1 + \lambda)^4 \mathcal{M}_4(\mathbb{Y} - y_0, \lambda)(1 + t^{4}) e^{-\tilde{\lambda} s}. \]

(ii) We now turn to \( D_r J_2^{(3)} \). In view of (C.3) and (3.27), for \( r \leq v \) we have

\[ |D_r J_2^{(3)}| \leq C \mathbb{I}_{(s,t)}(v) (1 + \lambda) \int_v^t |\partial_\theta K_H(\theta, v)| \, e^{-\lambda \theta} e^{-|Y^H_\theta - Y^H_v|} R(\lambda) \left\{ |K_H(\theta, r) - K_H(v, r)| + (\theta - v) \left( K_H(v, r) + (v - r)^{H+\frac{1}{2}} \mathbb{I}_{\{H<\frac{3}{2}\}} \right) \right\} d\theta. \]
In order to be in a position to again use our estimate on $\mathcal{A}^2(v, r, t)$ and $\mathcal{T}^2(v, r, t)$ we replace $K_H(v, r)$ by $K_H(v, r) - K_H(\theta, r) + K_H(\theta, r)$ and we bound $|v-r|^H + \frac{1}{2}$ from above by $(\theta - r)^{H + \frac{1}{2}}$. In addition, we use (B.3) and the obvious inequalities $\theta - v \leq (\theta - v)^H$ and $(\theta - v)^{H - \frac{1}{2}} \leq (\theta - v)^{2H - \frac{1}{2}}$ for any $0 < \theta - v < t - s < 1$ and $\frac{1}{4} < H < 1$. This leads us to apply (5.25) with

$$\xi_\theta \equiv e^{-\frac{1}{2} \lambda \theta} e^{-|v-Y_\theta^H|} \mathcal{R}(\lambda)$$

and

$$f(\theta) \equiv C \left[ \partial_\theta K_H(\theta, v) \right] \mathbb{I}_{r \leq t} \left[ |K_H(\theta, r) - K_H(v, r)| + |\partial_\theta K_H(\theta, v)| \right] K_H(\theta, r) (\theta - v)^H + |H - \frac{1}{2}| (\theta - v)^{2H - \frac{1}{2}} (\theta - r)^{H + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right] e^{-\frac{1}{2} \lambda \theta}.$$ 

It comes:

$$\mathbb{E}(|D_r J_2^{(3)}|^2) \leq C (1 + \lambda)^2 \sup_{v < \hat{\theta} \leq t} \mathbb{E}((\xi_\theta)^2) \left( \int_v^t f(\theta) \, d\theta \right)^2.$$ 

Notice that

$$\sup_{v < \hat{\theta} \leq t} \mathbb{E}((\xi_\theta)^2) \leq \mathcal{M}_2(\mathcal{Y} - y_0, \lambda).$$

In view of (B.1) and (B.22) we also have

$$\int_s^t \int_0^v \left( \int_v^t f(\theta) \, d\theta \right)^2 \, dr \, dv \leq C \int_s^t e^{-\lambda \theta} \int_0^v (\mathcal{A}^2(v, r, t))^2 \, dr \, dv + C \int_s^t e^{-\lambda \theta} \int_0^v (\mathcal{T}^2(v, r, t))^2 \, dr \, dv$$

$$+ C \mathbb{I}_{\{H < \frac{1}{2}\}} |H - \frac{1}{2}|^2 \int_s^t e^{-\lambda \theta} \int_0^v (\mathcal{T}^2(v, r, t))^2 \, dr \, dv.$$ 

We now use (B.12), (B.13), (B.16), (B.17) and (B.23). As we are in the case $0 < t - s < 1$ the right-hand side of the preceding inequality is bounded from above by

$$C_H |H - \frac{1}{2}|^2 (t-s)^{4H \wedge (2H + 1)} (1 + t^{2H + 2}) e^{-\lambda s}.$$ 

We thus have obtained that

$$\int_0^N \int_0^v \mathbb{E}(|D_r J_2^{(3)}|^2) \, dr \, dv \leq C_H |H - \frac{1}{2}|^2 (t-s)^{4H \wedge (2H + 1)} (1 + \lambda)^2 \mathcal{M}_2(\mathcal{Y} - y_0, \lambda) (1 + t^4) e^{-\lambda s}. \tag{5.36}$$

**A bound from above for** $\int_0^N \int_0^v \mathbb{E}(|D_r J_4^{(3)}|^2) \, dr \, dv$. 

In view of (5.29) we have

$$|D_r J_4^{(4)}| = \mathbb{I}_{(0,s)}(v) \left| \int_s^t \partial_\theta K_H(\theta, v) \, D_r \left( \mathcal{W}_\lambda(Y_\theta^H) \right) e^{-\lambda \theta} \, d\theta \right|$$

$$\leq C \mathbb{I}_{(0,s)}(v) (1 + \lambda) \left| \int_s^t \mathbb{I}_{r \leq \theta} \partial_\theta K_H(\theta, v) e^{-\lambda \theta - |v-Y_\theta^H| \mathcal{R}(\lambda)} \left( K_H(\theta, r) + (\theta - r)^{H + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) d\theta \right|.$$ 

Apply (5.25) with

$$\xi_\theta \equiv e^{-\frac{1}{2} \lambda \theta} e^{-|v-Y_\theta^H|} \mathcal{R}(\lambda)$$

and

$$f(\theta) \equiv \mathbb{I}_{r \leq \theta} \left( K_H(\theta, r) + (\theta - r)^{H + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) \partial_\theta K_H(\theta, v) e^{-\frac{1}{2} \lambda \theta}.$$
In view of (B.1) and (B.24) we have considered the following maps

We thus are led to consider

Proof. Notice that

We now use (B.18) and (B.25) and get

It comes:

Lemma 5.16. For any $0 < s < t < N$ with $0 < t - s < 1$ and $H \in (\frac{1}{4}, 1)$ it holds that

Proof. Notice that $\chi_H$ is a bounded function of $H \in (\frac{1}{4}, 1)$. We therefore have

We thus are led to consider

and

As for $R_1$ we have

We aim to use Taylor expansions of functions of $H - \frac{1}{2}$ around $H - \frac{1}{2} = 0$. This leads us to consider the following maps $\Psi_1(z)$ and $\Psi_2(z)$ for $z \in (-\frac{1}{4}, 1/2)$:

$$
\Psi_1(z) := \frac{1}{1 + z} (t(s - t))^{1+z} - \frac{2}{1 + z} (t(s - t))^{1+z} + t(s - t),
$$

$$
\Psi_2(z) := \frac{1}{2} (t^2 - s^2) + \frac{1}{2t + 2z} (t^{2z+2} - s^{2+z}) - \frac{1}{1 + z} (t^{2+z} - s^{2+z}).
$$
Observe that $\Psi_t'(0) = \Psi'_s(0) = 0$ for $i = 1, 2$. In addition, an easy calculation shows that $\Psi''_t(z)$ is a sum of terms of the type
\[
\frac{1}{(1 + 2z)^i}((\log(t(t - s)))^j (t(t - s))^{1 + z}) \quad \text{or} \quad \frac{1}{(1 + z)^i}((\log(t(t - s)))^j (t(t - s))^{1 + z})
\]
with $i \in \{1, 2, 3\}$ and $j \in \{0, 1, 2\}$. Consequently, as $0 < t - s < 1$,
\[
\text{for } H < \frac{1}{2}, \quad \sup_{z \in (H - \frac{1}{4}, 0)} |\Psi''_t(z)| \leq C (1 + (\log(t - s))^2) (t - s)^{2H} (1 + t^2)
\]
and
\[
\text{for } H > \frac{1}{2}, \quad \sup_{z \in (0, H - \frac{1}{2})} |\Psi''_t(z)| \leq C (1 + (\log(t - s))^2) (t - s) (1 + t^2).
\]
Similarly, $\Psi''_s(z)$ is a sum of terms of the type
\[
\frac{1}{(1 + 2z)^i}((\log(t))^j t^{2 + z} - (\log(s))^j s^{2 + z}) \quad \text{or} \quad \frac{1}{(1 + z)^i}((\log(t))^j t^{2 + z} - (\log(s))^j s^{2 + z})
\]
with $i \in \{1, 2, 3\}$ and $j \in \{0, 1, 2\}$. Consequently,
\[
\sup_{z \in (-\frac{1}{2}, \frac{1}{2})} |\Psi''_s(z)| \leq C (t - s) (1 + t^2).
\]
It therefore results from Taylor expansions of $\Psi_t$ that
\[
R_1 \leq C (\Psi_1(H - \frac{1}{2}) + \Psi_2(H - \frac{1}{2})) \leq C (H - \frac{1}{2})^2 (1 + (\log(t - s))^2) (t - s)^{(2H)\wedge 1} (1 + t^2).
\]
(5.39)

As for $R_2$ we observe that
\[
\theta^{H - \frac{3}{2}} \leq \frac{v^{H-1}}{\sqrt{\theta}} \leq \frac{v^{H-1}}{\sqrt{\theta - v}},
\]
from which
\[
R_2 \leq C (H - \frac{1}{2})^2 \int_s^t v^{2H-1} \left( \int_v^{t} (\theta - v)^{-H} d\theta \right)^2 dv.
\]
We thus get
\[
R_2 \leq C (H - \frac{1}{2})^2 (t - s)^{2H} \int_s^t v^{2H} dv. \quad (5.40)
\]

\[\square\]

5.8 $L^p$-estimate on $(T_t^{(N)} - T_s^{(N)})$

In this section we prove $L^p$-estimates on $\delta^{(N)}_H(U_t^{(N)}(\cdot) - U_s^{(N)}(\cdot))$. In the calculations below it will suffice to use the following estimate which results from (5.29):
\[
\forall 0 \leq r, \quad |D_r(W_\lambda(Y^H_v))| \leq C \mathbb{I}_{r \leq v} (1 + \lambda) \left( |K_H(v, r)| + (v - r)^{H + \frac{1}{2}} \right) e^{\theta^\wedge v}. \quad (5.41)
\]

Lemma 5.17. Suppose $\frac{1}{4} < H < 1$ and $H \neq \frac{1}{2}$.

Let $\Upsilon^{(N)}$ be defined as in (5.22). For any $p \geq 2$ and $0 < s < t < N$ with $0 < t - s < 1$ we have
\[
\|T_t^{(N)} - T_s^{(N)}\|_p \leq C_H |H - \frac{1}{2}| (t - s)^{H \wedge \frac{3}{2}} (1 + \lambda)^2 (1 + \log(t - s)) (1 + t^2) e^{-3s}. \quad (5.42)
\]
Proof. We again consider (5.28). In view of Meyer’s inequalities ([24, Prop.3.2.1]) we have
\[ \|Y^{(N)}_t - Y^{(N)}_s\|_p \leq C \sum_{i=1}^4 \left\{ \int_0^N \mathbb{E}(|J^{(i)}|^2) \, dv \right\}^{\frac{1}{2}} + C \sum_{i=1}^4 \left\{ \mathbb{E}\left( \left( \int_0^N \int_0^N |D_r J^{(i)}|^2 \, dr \, dv \right)^2 \right) \right\}^{\frac{1}{2}}. \]
As in the proof of Lemma 5.15 we limit ourselves to treat the second term. We start with applying Minkowski’s inequality to get for \( p \geq 2 \)
\[ \left\{ \mathbb{E}\left( \left( \int_0^N \int_0^N |D_r J^{(i)}|^2 \, dr \, dv \right)^2 \right) \right\}^{\frac{1}{2}} \leq \left( \int_0^N \int_0^N \mathbb{E}(|D_r J^{(i)}|^p) \left\{ \mathbb{E}(|D_r J^{(i)}|^2) \right\}^{\frac{p}{2}} \, dr \, dv \right)^{\frac{1}{2}}. \]
In most of the calculations below we use (5.41) and exhibit a deterministic upper bound \( \mathcal{D}^{(i)}_v \) for \( |D_r J^{(i)}| \). We are thus reduced to use the \( L^2 \)-estimates obtained in Section 5.6 to get suitable upper bounds for
\[ \int_0^N \int_0^N |D_r^{(i)}|^2 \, dr \, dv \quad (1 \leq i \leq 4). \]

A bound from above for \( \int_0^N \int_0^N \mathbb{E}(|D_r J^{(1)}|^p) \left\{ \mathbb{E}(|D_r J^{(1)}|^2) \right\}^{\frac{p}{2}} \, dr \, dv \)
Recall that
\[ D_r J^{(1)} = \mathbb{I}_{[s,t]}(v) \mathbb{I}_{r \leq v} \, D_r(W^t_\lambda(Y^v_x)) \left( K_H(t,v) - 1 \right) e^{-\lambda v} \]
and, by hypothesis, \( \tilde{\lambda} > 0 \). By using (5.41) we thus are in a position to choose
\[ \mathcal{D}^{(1)}_v := C \mathbb{I}_{[s,t]}(v) \mathbb{I}_{r \leq v} (1 + \lambda) \left( |K_H(v,r)| + (v-r)^{H+\frac{1}{2}} \right) (K_H(t,v) - 1) e^{-\tilde{\lambda} s}. \]
As in the proof of (5.31) we use (3.5) and get
\[ \int_0^N \int_0^N \mathbb{E}(|D_r J^{(1)}|^p) \left\{ \mathbb{E}(|D_r J^{(1)}|^2) \right\}^{\frac{p}{2}} \, dr \, dv \leq C (1 + \lambda)^2 e^{-2\tilde{\lambda} s} \int_s^t (v^{2H} + v^{2H+2}) (K_H(t,v) - 1)^2 \, dv. \]
In view of (5.38) we deduce that
\[ \int_0^N \int_0^N \mathbb{E}(|D_r J^{(1)}|^p) \left\{ \mathbb{E}(|D_r J^{(1)}|^2) \right\}^{\frac{p}{2}} \, dr \, dv \leq C (H - \frac{1}{2})^2 (t-s)^{(2H+1)(1+\log(t-s))^2} (1+\lambda)^2 (1+t^4) e^{-2\tilde{\lambda} s}. \]

A bound from above for \( \int_0^N \int_0^N \mathbb{E}(|D_r J^{(2)}|^p) \left\{ \mathbb{E}(|D_r J^{(2)}|^2) \right\}^{\frac{p}{2}} \, dr \, dv \)
We have
\[ D_r J^{(2)} = \mathbb{I}_{[s,t]}(v) \mathbb{I}_{r \leq v} \, D_r(W^t_\lambda(Y^v_x)) \int_v^t \partial_\theta K_H(\theta,v) \left( e^{-\tilde{\lambda} \theta} - e^{-\tilde{\lambda} v} \right) d\theta. \]
By using (5.41) and the hypothesis \( \tilde{\lambda} > 0 \) we are in a position to choose
\[ \mathcal{D}^{(2)}_v := C \mathbb{I}_{r \leq v} (1 + \lambda) \left( |K_H(v,r)| + (v-r)^{H+\frac{1}{2}} \right) \int_v^t \partial_\theta K_H(\theta,v) \left( e^{-\tilde{\lambda} \theta} - e^{-\tilde{\lambda} v} \right) d\theta. \]
Now proceed as in the proof of (5.32). It comes:
\[ \int_0^N \mathbb{E}(|D_r J^{(2)}|^p) \left\{ \mathbb{E}(|D_r J^{(2)}|^2) \right\}^{\frac{p}{2}} \, dr \leq C \mathbb{I}_{[s,t]}(v) \left| H - \frac{1}{2} \right|^2 (t-s)^{2H+1} (1 + \lambda)^2 \times (v^{2H} + v^{2H+2}) \left( 1 + t^{2H-1} \mathbb{I}_{\frac{1}{2} < H < 1} e^{-2\tilde{\lambda} v} \right), \]
\[ \leq C \mathbb{I}_{[s,t]}(v) \left| H - \frac{1}{2} \right|^2 (t-s)^{2H+1} (1 + \lambda)^2 (1 + t^{2H+2}) e^{-2\tilde{\lambda} s}, \]
From Minkowski’s inequality it results that

\[ \int_0^N \int_0^N \left\{ \mathbb{E}(|D_r J^{(3)|^p}|) \right\}^\frac{2}{p} \, dr \, dv \leq C \left| H - \frac{1}{2} \right|^2 (t - s)^{2H + 2} (1 + \lambda)^2 (1 + t^3) e^{-2\bar{\lambda}s}. \] (5.44)

**A bound from above for** \( \int_0^N \int_0^N \left\{ \mathbb{E}(|D_r J^{(3)|^p}|) \right\}^\frac{2}{p} \, dr \, dv \)

We proceed as above. From

\[ |I_{v \leq r} D_r J^{(3)}| = I_{(s,t)}(v) I_{v \leq r} \left| \int_v^t D_r W'_\lambda(Y^H_{\theta}) \partial_\theta K_H(\theta, v) e^{-\bar{\lambda}\theta} \, d\theta \right| \]

we deduce that we can choose

\[ D_r^{(3)} := C I_{(s,t)}(v) I_{v \leq r} (1 + \lambda) e^{-\bar{\lambda}s} \int_v^t \left( K_H(\theta, r) + (\theta - r)^H + \frac{1}{2} \right) I_{\{H < \frac{1}{2}\}} \left| \partial_\theta K_H(\theta, v) \right| \, d\theta. \]

Use (5.33). It comes:

\[ \int_0^N \int_0^N \left\{ \mathbb{E}(|D_r J^{(3)|^p}|) \right\}^\frac{2}{p} \, dr \, dv \leq C_H \left| H - \frac{1}{2} \right|^2 (t - s) (1 + \lambda)^2 (1 + t^3) e^{-2\bar{\lambda}s}. \]

**A bound from above for** \( \int_0^N \int_0^N \left\{ \mathbb{E}(|D_r J^{(3)|^p}|) \right\}^\frac{2}{p} \, dr \, dv \)

As in the proof of (5.35), for \( r \leq v \) we consider

\[ D_r J^{(3)}_1 := I_{(s,t)}(v) \int_v^t \partial_\theta K_H(\theta, v) (W''_\lambda(Y^H_{\theta}) - W''_\lambda(Y^H_{\theta})) D_r Y^H e^{-\lambda\theta} \, d\theta \]

and

\[ D_r J^{(3)}_2 := I_{(s,t)}(v) \int_v^t \partial_\theta K_H(\theta, v) W''_\lambda(Y^H_{\theta}) (D_r Y^H - D_r Y^H) e^{-\lambda\theta} \, d\theta. \]

We have

\[ |D_r J^{(3)}_1| \leq C I_{(s,t)}(v) (1 + \lambda)^2 e^{-\bar{\lambda}s} \int_v^t |\partial_\theta K_H(\theta, v)| \left| Y^H_{\theta} - Y^H_{\theta} \right| \left\{ K_H(\theta, r) + (\theta - r)^H + \frac{1}{2} I_{\{H < \frac{1}{2}\}} \right\} \, d\theta. \]

From Minkowski’s inequality it results that

\[ \left\{ \mathbb{E}(|D_r J^{(3)}_1|^p) \right\}^\frac{1}{p} \]

\[ \leq C (1 + \lambda)^2 e^{-\bar{\lambda}s} \int_v^t \left\{ \mathbb{E}(|Y^H_{\theta} - Y^H_{\theta}|^p) \right\}^\frac{1}{p} \left| \partial_\theta K_H(\theta, v) \right| \left\{ K_H(\theta, r) + (\theta - r)^H + \frac{1}{2} I_{\{H < \frac{1}{2}\}} \right\} \, d\theta \]

\[ \leq C (1 + \lambda)^2 e^{-\bar{\lambda}s} \int_v^t (\theta - v)^H \left| \partial_\theta K_H(\theta, v) \right| \left\{ K_H(\theta, r) + (\theta - r)^H + \frac{1}{2} I_{\{H < \frac{1}{2}\}} \right\} \, d\theta. \]

We now use our estimate on the weighted \( L^2 \)-norm of the function \( \phi(\theta, v) \) chosen in the proof of (5.35) to get

\[ \int_0^N \int_0^N \left\{ \mathbb{E}(|D_r J^{(3)}_1|^p) \right\}^\frac{2}{p} \, dr \, dv \leq C_H \left| H - \frac{1}{2} \right|^2 (t - s)^{4H + (2H + 1)} (1 + \lambda)^4 (1 + t^4) e^{-2\bar{\lambda}s}. \] (5.45)
Similarly, for $r \leq v$ we have
\[
|D_r J_2^{(3)}(v)| \leq C \mathbb{I}_{\{s,t\}}(v) (1 + \lambda) e^{-\overline{\lambda} v} \int_v^t |\partial_\theta K_H(\theta, v)| \left\{ |K_H(\theta, r) - K_H(v, r)| + (\theta - v) \left( K_H(v, r) + (v - r)^{H + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) \right\} d\theta.
\]
We now use our estimate on the weighted $L^2$-norm of the function $f(\theta)$ chosen in the proof of (5.36) to get
\[
\int_0^N \int_0^v \left\{ \mathbb{E}(|D_r J_2^{(3)}|^p) \right\}^{\frac{2}{p}} dr dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H\wedge(2H+1)} (1 + \lambda)^2 (1 + t^4) e^{-2\overline{\lambda} s}. \tag{5.46}
\]

A bound from above for $\int_0^N \int_0^v \left\{ \mathbb{E}(|D_r J_2^{(3)}|^p) \right\}^{\frac{2}{p}} dr dv$

We obviously can choose
\[
D_r^{(4)} := C \mathbb{I}_{\{s,t\}}(v) (1 + \lambda) e^{-\overline{\lambda} v} \int_s^t \mathbb{I}_{r \leq \theta} \partial_\theta K_H(\theta, v) \left( K_H(\theta, r) + (\theta - r)^{H + \frac{1}{2}} \mathbb{I}_{\{H < \frac{1}{2}\}} \right) d\theta.
\]

Proceed as in the proof of (5.37) to obtain
\[
\int_0^N \int_0^v \left\{ \mathbb{E}(|D_r J_2^{(4)}|^p) \right\}^{\frac{2}{p}} dr dv \leq C |H - \frac{1}{2}|^2 (t - s)^{2H} (1 + \lambda)^2 (1 + t^{2H+1}) e^{-2\overline{\lambda} s}. \tag{5.47}
\]

To conclude the proof of (5.42), it remains to gather (5.43), (5.44), (5.45), (5.46), (5.47).

6 Conclusion and perspectives

In this paper we have developed a sensitivity analysis w.r.t. the Hurst parameter of the driving noise for the probability distribution of functionals of solutions to stochastic differential equations, including the probability distribution of first hitting times, when the Hurst parameter is close to $1/2$, that is, when the noise is close to the pure Brownian case. Our estimates seem accurate. As explained in the introduction, in practice they tend to justify the use of Markov Brownian models when estimated Hurst parameters remain close to $\frac{1}{2}$. In principle, by using similar analytical tools as above, it should be possible to get expansions in terms of $|H - \frac{1}{2}|$. However, the calculations would be still much more lengthy and heavy than above. The following open questions deserve future works.

It would be interesting to extend our results to SDEs driven by a Gaussian noise with general kernel $K$ and to estimate the sensitivity of first hitting time Laplace transforms in terms of the $L^2$ distance between $K$ and $K_{\frac{1}{2}}$.

The ellipticity condition (H2) may seem restrictive but it seems difficult to get rid of it. A natural attempt is as follows. When $H = \frac{1}{2}$ the SDE (2.2) can be written in the following Itô’s form:
\[
X_t = x_0 + \int_0^t (b(X_s) + \frac{1}{2} \sigma(X_s)\sigma'(X_s)) \, ds + \int_0^t \sigma(X_s) \, dW_s.
\]
Let $u$ be the solution to the following parabolic PDE
\[
\begin{aligned}
\left\{ \frac{\partial}{\partial t} u(s, x) + (b(x) + \frac{1}{2} \sigma(x)\sigma'(x)) \frac{\partial}{\partial x} u(s, x) + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} u(s, x) = 0, \quad (s, x) \in [0, t] \times \mathbb{R}, \\
u(t, x) = \varphi(x), \quad x \in \mathbb{R}.
\end{aligned}
\]
Recall the calculation made in Section 4.2. For $\frac{1}{2} < H < 1$ Itô’s formula applied to $u(t, X_t^H)$ leads to

$$
E \left( u(t, X_t^H) \right) - u(0, x_0) = 
- E \int_0^t \frac{1}{2} (\sigma') (X_s^H) \partial_x u(s, X_s^H) \, ds + \alpha_H E \int_0^t \int_0^T |r-s|^{2H-2} \sigma(X_r^H) \sigma'(X_s^H) \partial_x u(s, X_s^H) \, dr \, ds
- E \int_0^t \frac{1}{2} \sigma^2 (X_s^H) \partial_{xx} u(s, X_s^H) \, ds + \alpha_H E \int_0^t \int_0^T |r-s|^{2H-2} \sigma(X_r^H) \sigma(X_s^H) \partial_{xx}^2 u(s, X_s^H) \, dr \, ds
+ \alpha_H E \int_0^t \int_0^T |r-s|^{2H-2} (D_r X_s^H - \sigma(X_r^H)) (\partial_{xx}^2 u(s, X_s^H) \sigma(X_s^H) + \partial_x u(s, X_s^H) \sigma'(X_s^H)) \, dr \, ds.
$$

First, note that the hypothesis (H2) helps to get sharp estimates on derivatives of $u$. Second, without this hypothesis we have not succeeded to obtain accurate enough bounds on the sup and $H$ norms of $D_r X_s^H$ to deduce relevant sensitivity estimates w.r.t. $H$. When $H > \frac{1}{2}$, the estimates obtained in [15] on the supremum and Hölder norm of $X$ and $DX$ do not require the ellipticity of $\sigma$. However, they depend on the Hölder norm $|B|^H_{0,0,T}$, where $\alpha \in (\frac{1}{2}, H)$, which tends to infinity when $H \to \frac{1}{2}$.

In our calculations we often used our assumption $\lambda > |\tilde{b}|_\infty$. However, as already noticed in Remark 5.5, Theorem 5.2 should extend to every positive $\lambda$.

A sensitivity analysis of the density of $\tau^H_X$ would certainly be useful for applications. Our estimate on the Laplace transform of $\tau^H_X$ gives information on the robustness of this density around time $0$ when $H$ is close to $\frac{1}{2}$. This seems interesting since the simulations in [12] suggest that, when $H > \frac{1}{2}$, the density of sup$_{t\in[0,1]} B^H_t$ is unbounded around $0$. To go further, one should compute the inverse Laplace transform of the formula for $E \left( e^{-\lambda \tau^H_X} \right) - E \left( e^{-\lambda \tau^H_{X/2}} \right)$ given in Proposition 5.7 below. Handling technical issues raised by the singularity of the inverse Laplace transform and by terms whose Malliavin derivatives are highly singular is out of the scope of the present paper.

The extension of our analysis to multidimensional SDEs and first exit times of domains is another interesting further direction of research.

Finally, as explained at the end of the Introduction, sharp sensitivity analyses around $H \neq \frac{1}{2}$ seems to be a challenging problem.

**Appendices**

**A Representation of $K^*_H$ on $|H_H|$**

Denote by $E$ the set of simple functions on $[0, T]$. We recall that $|H_H| \subset H_H$ is the completion of $E$ with respect to the norm (3.9) (respectively, (3.8)) when $H > \frac{1}{2}$ (respectively, $H < \frac{1}{2}$).

Consider the operator $\tilde{K}^*_H : E \to L^2[0, T]$ defined by

$$
\tilde{K}^*_H \varphi(t) = K_H(T, t) \varphi(t) + \chi_H (H + \frac{1}{2}) \int_0^T \left( \frac{\theta}{H} \right)^{-H-\frac{1}{2}} (\theta - t)^{H-\frac{3}{2}} (\varphi(\theta) - \varphi(t)) \, d\theta.
$$

In view of (3.12) this operator coincides with $K^*_H$ on $E$. As $|H_H| \subset H_H$ is a continuous embedding we have

$$
\forall \varphi \in E, \quad \|\tilde{K}^*_H \varphi\|_{L^2[0, T]} = \|\varphi\|_{H_H} \leq C \|\varphi\|_{|H_H|}.
$$
which implies that $\tilde{K}_H^*$ can be continuously extended to an operator from $|H_H|$ to $L^2[0,T]$.

Let us now prove that $\tilde{K}_H^*$ and $K_H^*$ coincide on $|H_H|$. Let $\varphi \in |H_H|$ and let $(\varphi_n) \in \mathcal{E}^N$ be a sequence which converges to $\varphi$ in $|H_H|$ (and thus also converges in $H_H$). We have

$$\|\tilde{K}_H^* \varphi - K_H^* \varphi\|_{L^2[0,T]} \leq \|\tilde{K}_H^* (\varphi - \varphi_n)\|_{L^2[0,T]} + \|K_H^*(\varphi_n - \varphi)\|_{L^2[0,T]} \leq C\|\varphi - \varphi_n\|_{H_H} + \|\varphi - \varphi_n\|_{H_H}.$$ 

Since the right-hand side of the preceding inequality converges to 0 when $n$ tends to infinity we conclude that $\tilde{K}_H^* \varphi = K_H^* \varphi$. Therefore the representation (3.12) of $K_H^*$ holds true for any $\varphi \in |H_H|$.

B On various deterministic integrals depending on $K_H$ and $\partial_0 K_H$

Given $\frac{1}{4} < H < 1$ and $0 < r < t$, set

$$\begin{align*}
A(v, r, t) &:= \int_r^t \partial_0 K_H(\theta, v) \left| K_H(\theta, r) \right| d\theta \quad \text{for} \ 0 < s < v < t, \\
A^r(v, r, t) &:= \int_r^t \partial_0 K_H(\theta, v) \left| K_H(\theta, r) \right| (\theta - v)^H d\theta \quad \text{for} \ 0 < s < v < t, \\
A^v(v, r, t) &:= \int_v^t \partial_0 K_H(\theta, v) \left| K_H(\theta, r) \right| (\theta - v^H) d\theta \quad \text{for} \ 0 < s < v < t, \\
A^t(v, r, t) &:= \int_s^t \partial_0 K_H(\theta, v) \left| K_H(\theta, r) \right| d\theta \quad \text{for} \ 0 < s < v < t.
\end{align*}
$$

(B.1)

In many calculations we need to consider time intervals $0 < s < t$ with $0 < t - s < 1$ and the integrals

$$\int_s^t \int_v^t (A(v, r, t))^2 \, dr \, dv, \quad \int_s^t \int_0^v (A^r(v, r, t))^2 \, dr \, dv, \quad \text{etc.}$$

We need to bound these integrals from above by a constant of the type $C_H \left| H - \frac{1}{2} \right|^2 (t - s)^k$ for some $\delta > 0$ and $k \geq 0$. Getting such an estimate requires totally different arguments according as $\frac{1}{2} < H < 1$ or $\frac{1}{4} < H < \frac{1}{2}$ because of the difference of behaviour of the kernel $K_H$ in these two cases:

- **In the singular case** $H < \frac{1}{2}$ we have $(\frac{\theta}{r})^{H - \frac{1}{2}} < 1$ for any $\theta > r$. The formulae (3.1) and (3.7) respectively lead to

$$0 \leq K_H(\theta, r) \leq C \mathbb{1}_{\{\theta > r\}} \left( (\theta - r)^{H - \frac{1}{2}} + |H - \frac{1}{2}| r^{1 - H} \int_r^\theta \xi^{H - \frac{3}{2}} (\xi - r)^{H - \frac{1}{2}} d\xi \right)$$

and

$$- \partial_0 K_H(\theta, v) = |\partial_0 K_H(\theta, v)| \leq \mathbb{1}_{\{\theta > v\}} C \left| H - \frac{1}{2} \right| (\theta - v)^{H - \frac{3}{2}}.$$ 

(B.2)

- **In the regular case** $H > \frac{1}{2}$ we have

$$0 \leq K_H(\theta, r) \leq C \mathbb{1}_{\{\theta > r\}} \left( \frac{\theta}{r} \right)^{H - \frac{1}{2}} (\theta - r)^{H - \frac{1}{2}}$$

and

$$0 \leq \partial_0 K_H(\theta, v) \leq C \mathbb{1}_{\{\theta > v\}} \left| H - \frac{1}{2} \right| \left( \frac{\theta}{v} \right)^{H - \frac{1}{2}} (\theta - v)^{H - \frac{1}{2}}.$$ 

(B.3)

(B.5)
B.1 Estimate on $\int_s^t \int_v^t (A(v, r, t))^2 \, dr \, dv$: The regular case $\frac{1}{2} < H < 1$

**Proposition B.1.** Let $A(v, r, t)$ be defined as in (B.1). For any $\frac{1}{2} < H < 1$ it holds that

$$
\int_s^t \int_v^t (A(v, r, t))^2 \, dr \, dv \leq C_H \left| H - \frac{1}{2} \right|^2 (t-s)^{3-2H} t^{6H-3}. \quad (B.6)
$$

**Proof.** Since $H > \frac{1}{2}$, in view of (B.4) and (B.5) we have

$$
|A(v, r, t)| \leq C \left| H - \frac{1}{2} \right| \left( \frac{t^2}{v} \right)^{H-\frac{1}{2}} \int_r^t (\theta - v)^{H-\frac{3}{2}} (\theta - r)^{H-\frac{1}{2}} d\theta \leq C \left| H - \frac{1}{2} \right| \frac{t^{3H-\frac{3}{2}}}{(v^r)^{H-\frac{1}{2}}} \int_r^t (\theta - v)^{H-\frac{3}{2}} d\theta.
$$

We thus are led to consider

$$
\int_v^t \frac{1}{r^{2H-1}} \left( \int_r^t (\theta - v)^{H-\frac{3}{2}} d\theta \right)^2 dr \leq \int_v^t \frac{1}{(r-v)^{2H-1}} \left( \int_r^t (\theta - v)^{H-\frac{3}{2}} d\theta \right)^2 dr.
$$

We need some care to get a bound on the right-hand side of the preceding inequality which does not explode when $H$ tends to $\frac{1}{2}$. By two successive integrations by parts we get that the right-hand side equals to

$$
\frac{1}{3-2H} \int_v^t (r - v)^{2-2H} \int_r^t (\theta - v)^{H-\frac{3}{2}} d\theta (r - v)^{H-\frac{3}{2}} dr = C_H \int_v^t (r - v)^{\frac{1}{2}} \int_r^t (\theta - v)^{H-\frac{3}{2}} d\theta \, dr = C_H (t - v).
$$

As $t - v \leq t - s$ we deduce that

$$
\int_s^t \int_v^t (A(v, r, t))^2 \, dr \, dv \leq C_H \left| H - \frac{1}{2} \right|^2 (t-s)^{6H-3} \int_s^t \frac{1}{v^{2H-1}} \, dv \\
\leq C_H \left| H - \frac{1}{2} \right|^2 (t-s)^{6H-3} (t^{2-2H} - s^{2-2H}).
$$

As $\frac{1}{2} < H < 1$ one has $t^{2-2H} - s^{2-2H} \leq (t-s)^{2-2H}$. That ends the proof. \qed

B.2 Estimate on $\int_s^t \int_v^t (A(v, r, t))^2 \, dr \, dv$: The irregular case $\frac{1}{4} < H < \frac{1}{2}$

In the irregular case the calculations are longer than in the regular case. We start with an easy lemma.

**Proposition B.2.** For any $T > 0$ we have

$$
\forall \gamma > -1, \quad \int_0^T \theta^\gamma (\log(\theta))^2 \, d\theta \leq \frac{C}{\gamma + 1} T^{\gamma+1} (\log(T))^2 + \frac{C}{(\gamma + 1)^3} T^{\gamma+1}. \quad (B.7)
$$

For any $0 < s < t$ we have

$$
\forall \gamma > -1, \quad \int_s^t \int_v^t (r-v)^\gamma \left( \log\left( \frac{t-v}{r-v} \right) \right)^2 \, dr \, dv \leq \frac{C}{(\gamma + 1)^2} (t-s)^{\gamma+2} (\log(t-s))^2 + \frac{C}{(\gamma + 1)^4} (t-s)^{\gamma+2}. \quad (B.8)
$$

**Proof.** Two successive integrations by parts lead to

$$
\int_0^T \theta^\gamma (\log(\theta))^2 \, d\theta \leq \frac{1}{\gamma + 1} T^{\gamma+1} (\log(T))^2 + \frac{2}{(\gamma + 1)^2} T^{\gamma+1} |\log(T)| + \frac{2}{(\gamma + 1)^3} T^{\gamma+1} \\
\leq \frac{1}{\gamma + 1} T^{\gamma+1} (|\log(T)| + \frac{1}{\gamma + 1} \log(T))^2.
$$

49
Therefore, as \( z \) is less

and involves constants which do not explode when

\[ z \]

For any

\[ z \]

As for the integral over

\[ z \]

we observe that

\[ \int_z^t \phi (t - v) \phi (v) \phi (r) \phi (v) \phi (r) \phi (v) \phi (r) \phi (v) \phi (r) \phi (v) \phi (r) \phi (v) \phi (r) \phi (v) \]

The inequality (B.7) follows.

To prove (B.8) we start with deducing from (B.7) that

\[
\int_v^t (r - v)^\gamma (\log(r - v))^2 \, dr = \int_0^{t - v} r^\gamma (\log(r))^2 \, dr \leq \frac{C}{(\gamma + 1)^3} (t - v)^{\gamma + 1} \left((\gamma + 1)^2 \log(t - v)^2 + 1\right).
\]

In addition,

\[
\int_s^t \int_v^t (r - v)^\gamma (\log(t - v))^2 \, dr \, dv = \frac{1}{\gamma + 1} \int_s^t (t - v)^{\gamma + 1} (\log(t - v))^2 \, dv.
\]

Therefore, as \( \gamma + 2 > 1 \) the left-hand side of (B.8) is bounded from above by

\[
\frac{C}{\gamma + 1} \int_0^{t - s} \theta ^{\gamma + 1} (\log(\theta))^2 \, d\theta + \frac{C}{(\gamma + 1)^2} (t - s)^{\gamma + 2}.
\]

It remains to again use (B.7) (with \( T = t - s \)) to get (B.8).

We will need to consider the integral \( I(v, r, t) \) defined for \( 0 \leq v < r < t \) by

\[
I(v, r, t) := \int_v^t (\xi - v)^{H - \frac{3}{2}} (\xi - r)^{H - \frac{3}{2}} \, d\xi.
\]

It will be decisive to bound \( I(v, r, t) \) from above by a function of \( r \) and \( v \) which is square integrable and involves constants which do not explode when \( H \) tends to \( \frac{1}{2} \).

**Proposition B.3.** For any \( \frac{1}{4} < H < \frac{1}{2} \) and \( 0 \leq v < r < t \), let \( I(v, r, t) \) be defined as in (B.9). One has

\[
I(v, r, t) \leq C \left(r - v \right)^{2H - 1} \left(\log\left(\frac{t - v}{r - v}\right) + 1\right).
\]

**Proof.** Use the change of variable \( \xi = r + \frac{v - r}{\alpha} \). It comes:

\[
I(v, r, t) = (r - v)^{2H - 1} \int_{\frac{t - v}{r - v}}^{\infty} (1 + \alpha)^{H - \frac{3}{2}} \alpha^{-2H} \, d\alpha.
\]

Now, split the integration interval into \( \left(\frac{t - v}{r - v}, 1, 1\right) \) and \( (1, +\infty) \). As for the integral over \( (1, +\infty) \) we observe that

\[
\int_1^\infty (1 + \alpha)^{H - \frac{3}{2}} \alpha^{-2H} \, d\alpha \leq \int_1^{\infty} \alpha^{-H - \frac{3}{2}} \, d\alpha \leq C.
\]

As for the integral over \( \left(\frac{t - v}{r - v}, 1, 1\right) \), for any \( 0 < z < 1 \) and \( \frac{1}{4} < H < \frac{1}{2} \) one has

\[
\int_z^1 (1 + \alpha)^{H - \frac{3}{2}} \alpha^{-2H} \, d\alpha \leq \int_z^1 \alpha^{-2H} \, d\alpha \leq \int_z^1 \alpha^{-1} \, d\alpha = \log(z).
\]

For \( z = \frac{t - v}{r - v} \) and \( 1 \) one has \( \log(z) \leq \log(\frac{t - v}{r - v}) \). The desired result follows.

We now are in a position to get the main result in this subsection.

**Proposition B.4.** Let \( A(v, r, t) \) be defined as in (B.1). For any \( \frac{1}{4} < H < \frac{1}{2} \) one has

\[
\int_v^t \int_v^t (A(v, r, t))^2 \, dr \, dv \leq C_H \left|H - \frac{1}{2}\right|^2 \left((t - s)^{2H} ((\log(t - s))^2 + 1)\right).
\]
Proof. In view of (B.2), (B.3) and (B.9) one has

\[ |A(v, r, t)| \leq C |H - \frac{1}{2}| I(v, r, t) + C |H - \frac{1}{2}|^2 \int_r^t (\theta - v)^{H - \frac{3}{2}} r^{\frac{1}{2} - H} I(0, r, \theta) \, d\theta. \]

First, we use (B.10) to bound \( I(v, r, t) \) from above. Second, we notice that (B.10) implies

\[ I(0, r, \theta) \leq C r^{2H - 1} \left( \log \left( \frac{r}{\theta - v} \right) + 1 \right), \]

from which

\[ |H - \frac{1}{2}|^2 \int_r^t (\theta - v)^{H - \frac{3}{2}} r^{\frac{1}{2} - H} I(0, r, \theta) \, d\theta \leq C |H - \frac{1}{2}| r^{H - \frac{3}{2}} \left( \log \left( \frac{r}{\theta - v} \right) + 1 \right) (r - v)^{H - \frac{3}{2}} \]

\[ \leq C |H - \frac{1}{2}| (r - v)^{2H - 1} \left( \log \left( \frac{r}{\theta - v} \right) + 1 \right). \]

We thus have

\[ A(v, r, t) \leq C |H - \frac{1}{2}| (r - v)^{2H - 1} \left( \log \left( \frac{r}{\theta - v} \right) + 1 \right). \]

The inequality (B.11) then results from the inequality (B.8) with \( \gamma = 4H - 2 \).

\[ \Box \]

**B.3 Estimate on** \( \int_s^t \int_v^r (A^2(v, r, t))^2 \, dr \, dv \)**

**Proposition B.5.** For \( 0 < r < v < t \) let \( A^2(v, r, t) \) be defined as in (B.1).

(i) In the regular case \( \frac{1}{2} < H < 1 \) it holds that

\[ \int_s^t \int_0^v (A^2(v, r, t))^2 \, dr \, dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^{2H + 1} t^{4H - 1}. \]

(ii) In the irregular case \( \frac{1}{4} < H < \frac{1}{2} \) it holds that

\[ \int_s^t \int_0^v (A^2(v, r, t))^2 \, dr \, dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H} t^{2H}. \]

Proof. Notice that

\[ (A^2(v, r, t))^2 = 2 \int_v^t A^2(v, r, \theta) |\partial_{\theta} K_H(\theta, v)| K_H(\theta, r) (\theta - v)^H \, d\theta \]

\[ = 2 \int_v^t \int_0^\theta \left| \partial_\alpha K_H(\alpha, v) \right| K_H(\alpha, r) (\alpha - v)^H \, d\alpha \left| \partial_\theta K_H(\theta, v) \right| K_H(\theta, r) (\theta - v)^H \, d\theta. \]

Now, in view of (3.4), for any \( v < \alpha < \theta < t \) we have

\[ \int_0^v K_H(\alpha, r) K_H(\theta, r) \, dr \leq \int_0^t K_H(\alpha, r) K_H(\theta, r) \, dr = \frac{1}{2} (\alpha^{2H} + \theta^{2H} - (\theta - \alpha)^{2H}) \leq t^{2H}. \]

We deduce:

\[ \int_0^v (A^2(v, r, t))^2 \, dr \leq 2 t^{2H} \int_v^t \int_0^\theta \left| \partial_\alpha K_H(\alpha, v) \right| (\alpha - v)^H \, d\alpha \left| \partial_\theta K_H(\theta, v) \right| (\theta - v)^H \, d\theta \]

\[ = t^{2H} \left( \int_v^t \left| \partial_\theta K_H(\theta, v) \right| (\theta - v)^H \, d\theta \right)^2. \]
In the regular case \( \frac{1}{2} < H < 1 \) we therefore can use (B.5) to get
\[
\int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \leq C_H |H - \frac{1}{2}|^2 (t - v)^{4H-1} \frac{t^{4H-1}}{v^{2H-1}} \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H-1} \frac{t^{4H-1}}{v^{2H-1}},
\]
from which
\[
\int_s^t \int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \, dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H-1} (t^{2-2H} - s^{2-2H}) t^{4H-1}.
\]
It then remains to use \( t^{2-2H} - s^{2-2H} \leq (t - s)^{2-2H} \) to obtain (B.12).

In the irregular case \( \frac{1}{4} < H < \frac{1}{2} \) we can use (B.3) to get
\[
\int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \leq C_H |H - \frac{1}{2}|^2 (t - v)^{4H-1} t^{2H} \leq C_H |H - \frac{1}{2}|^2 (t - s)^{4H-1} t^{2H},
\]
from which (B.13) follows.

\[\square\]

### B.4 Estimate on \( \int_s^t \int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \, dv \)

**Proposition B.6.** For \( 0 < r < v < t \) let \( \mathcal{A}_t^\delta(v,r,t) \) be defined as in (B.1).

(i) In the regular case \( \frac{1}{2} < H < 1 \) it holds that
\[\int_s^t \int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \, dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^{2H+1} t^{2H-1}. \tag{B.16}\]

(ii) In the irregular case \( \frac{1}{4} < H < \frac{1}{2} \) it holds that
\[\int_s^t \int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \, dv \leq C |H - \frac{1}{2}|^2 (t - s)^{4H}. \tag{B.17}\]

**Proof.** Notice that
\[
(\mathcal{A}_t^\delta(v,r,t))^2 = 2 \int_v^t \mathcal{A}_t^\delta(v,r,\theta) \partial_\theta K_H(\theta,v) \, |K_H(\theta,r) - K_H(v,r)| \, d\theta
\]
\[
\leq 2 \int_v^t \int_0^\theta \partial_\alpha K_H(\alpha,v) \, |K_H(\alpha,r) - K_H(v,r)| \, d\alpha \, \partial_\theta K_H(\theta,v) \, |K_H(\theta,r) - K_H(v,r)| \, d\theta.
\]
Now, in view of (3.5), for any \( v < \alpha < \theta < t \) we have
\[
\int_0^v \left| (K_H(\alpha,r) - K_H(v,r)) \, (K_H(\theta,r) - K_H(v,r)) \right| \, dr
\]
\[
\leq \left( \int_0^t (K_H(\alpha,r) - K_H(v,r))^2 \, dr \right)^{\frac{1}{2}} \left( \int_0^t (K_H(\theta,r) - K_H(v,r))^2 \, dr \right)^{\frac{1}{2}}
\]
\[
\leq (\alpha - v)^H (\theta - v)^H.
\]
We deduce:
\[
\int_0^v (\mathcal{A}_t^\delta(v,r,t))^2 \, dr \leq 2 \int_v^t \int_0^\theta \partial_\alpha K_H(\alpha,v) \, (\alpha - v)^H \, d\alpha \, \partial_\theta K_H(\theta,v) \, (\theta - v)^H \, d\theta
\]
\[
= \left( \int_v^t \partial_\theta K_H(\theta,v) \, (\theta - v)^H \, d\theta \right)^2.
\]
In view of (B.15) we deduce the desired inequalities by dividing the right-hand side of (B.12) and (B.13) by \( t^{2H} \). \(\square\)
B.5 Estimate on $\int_0^s \int_0^t (A^2(v, r, t)) dr dv$

**Proposition B.7.** For $v$ and $r$ in $(0, t)$ let $A^2(v, r, t)$ be defined as in (B.1).

For any $\frac{1}{4} < H < 1$ it holds that

$$\int_0^s \int_0^t (A^2(v, r, t))^2 dr dv \leq C_H |H - \frac{1}{2}|^2 (t - s)^2H t^{2H}.$$  \hspace{1cm} (B.18)

**Proof.** Notice that

$$(A^2(v, r, t))^2 = 2 \int_0^t A^2(v, r, \theta) \partial_{\delta} K_H(\theta, v) K_H(\theta, r) d\theta$$

$$= 2 \int_s^t \int_s^\theta \left| \partial_{\alpha} K_H(\alpha, v) \right| K_H(\alpha, r) d\alpha \left| \partial_{\delta} K_H(\theta, v) \right| K_H(\theta, r) d\theta.$$  

We again use (B.14) and (3.7) to get

$$\int_0^t (A^2(v, r, t))^2 dr \leq 2 t^{2H} \int_s^t \int_s^\theta \left| \partial_{\alpha} K_H(\alpha, v) \right| d\alpha \left| \partial_{\delta} K_H(\theta, v) \right| d\theta$$

$$\leq 4 \chi_H t^{2H} (H - \frac{1}{2})^2 \int_s^t \int_s^\theta (\theta \alpha)^{H-\frac{1}{2}} v^{-1-2H} (\theta - \alpha)^{H-\frac{3}{2}} v^{-1-2H} (\alpha - v)^{H-\frac{3}{2}} d\alpha d\theta.$$  

By changing the variable $v$ into $z = 1 - \frac{\theta}{\alpha} \cdot \frac{\theta - \alpha}{\theta - \alpha}$ one gets

$$\int_0^s v^{1-2H} (\theta - v)^{H-\frac{3}{2}} (\alpha - v)^{H-\frac{3}{2}} dv = (\alpha \theta)^{\frac{3}{2}-H} (\theta - \alpha)^{2H-2} \int_0^s \frac{\theta - \alpha}{\theta - \alpha} z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz.$$  

Therefore,

$$\int_0^s \int_0^t (A^2(v, r, t))^2 dr dv \leq C (H - \frac{1}{2})^2 t^{2H} \int_s^t \int_s^\theta (\theta - \alpha)^{2H-2} \int_0^s \frac{\theta - \alpha}{\theta - \alpha} z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz d\alpha d\theta.$$  

We now combine the inequality $\frac{\alpha}{\alpha} \cdot \frac{\theta - \alpha}{\theta - \alpha} \leq \frac{\theta - \alpha}{\theta - \alpha}$ with the change of variables $x = \frac{\theta - \alpha}{\theta - \alpha}$ to get

$$\int_0^s \int_0^t (A^2(v, r, t))^2 dr dv \leq C(H - \frac{1}{2})^2 t^{2H} \int_s^t (\theta - s)^{2H-1} \int_0^1 x^{2H-2} \int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} d\alpha d\theta.$$

To end the proof of Inequality (B.18) it remains to prove that

$$\int_0^1 x^{2H-2} \int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz dx \leq C_H.$$  \hspace{1cm} (B.19)

We have:

$$\int_0^1 x^{2H-2} \int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz dx = \int_0^1 \frac{1}{x} x^{2H-2} \int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz dx$$

$$+ \int_0^1 \frac{1}{x} x^{2H-2} \int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz dx$$

$$=: I_1 + I_2.$$  

We have $I_1 \leq C_H$ since for any $x \leq \frac{1}{2}$,

$$\int_0^x z^{1-2H} (1 - z)^{H-\frac{3}{2}} dz \leq C \int_0^x z^{1-2H} dz = C_H x^{2-2H}.$$  

53
We now turn to $I_2$ which we split into the sum of

$$I_{21} := \int_{\frac{1}{2}}^{1} x^{2H-2} \int_{0}^{\frac{1}{2}} z^{1-2H} (1-z)^{\frac{3}{2}} \, dz \, dx \quad \text{and} \quad I_{22} := \int_{\frac{1}{2}}^{1} x^{2H-2} \int_{\frac{1}{2}}^{x} z^{1-2H} (1-z)^{\frac{3}{2}} \, dz \, dx.$$ 

On the one hand, the bound on $I_1$ leads to

$$I_{21} \leq 2^{2-2H} C_H.$$ 

On the other hand, we have

$$I_{22} \leq 2^{2-2H} \int_{\frac{1}{2}}^{x} z^{1-2H} (1-z)^{\frac{3}{2}} \, dz \, dx \leq 2^{2-2H} (2^{2H-1} \vee 1) \int_{\frac{1}{2}}^{1} (1-x)^{-\frac{1}{2}} \int_{\frac{1}{2}}^{x} (1-z)^{-1} \, dz \, dx \leq C.$$ 

We thus have obtained (B.19).

**B.6 On variants of $I(v, r, t)$:** $I(v, r, t), I^\dagger(v, r, t)$ and $I^\natural(v, r, t)$

**Proposition B.8.** Let $I(v, r, t)$ be defined for $0 < v < r < t$ by

$$I(v, r, t) := \int_{r}^{t} (\theta - v)^{H-\frac{3}{2}} (\theta - r)^{H+\frac{1}{2}} \, d\theta. \quad (B.20)$$

For any $\frac{1}{4} < H < \frac{1}{2}$ one has

$$\int_{s}^{t} \int_{v}^{t} (I(v, r, t))^2 \, dr \, dv \leq C (t - s)^{4H+2}. \quad (B.21)$$

**Proof.** As above, use the change of variable $\theta = r + \frac{v-r}{\alpha}$. It comes:

$$I(v, r, t) = (r - v)^{2H} \int_{\frac{v-r}{v}}^{\infty} (1 + \alpha)^{H-\frac{3}{2}} \alpha^{-2H-1} \, d\alpha.$$ 

Notice that

$$\forall z > 0, \quad \int_{z}^{\infty} (1 + \alpha)^{H-\frac{3}{2}} \alpha^{-2H-1} \, d\alpha \leq \int_{z}^{\infty} \alpha^{-2H-1} \, d\alpha \leq \frac{C}{z^{2H}}.$$ 

For $z = \frac{v-r}{v-r} \vee 1$ one has $\frac{1}{z^{2H}} \leq \max(1, \frac{(t-r)^{2H}}{(r-v)^{2H}})$. It follows that

$$I(v, r, t) \leq C ((r - v)^{2H} + (t - r)^{2H}).$$

The inequality (B.21) then results from

$$\int_{s}^{t} \int_{v}^{t} ((r - v)^{4H} + (t - r)^{4H}) \, dr \, dv \leq C \int_{s}^{t} (t - v)^{4H+1} \, dv.$$ 

**Proposition B.9.** Let $I^\dagger(v, r, t)$ be defined for $0 < r < v < t$ by

$$I^\dagger(v, r, t) := \int_{v}^{t} (\theta - v)^{2H-\frac{3}{2}} (\theta - v)^{H+\frac{1}{2}} \, d\theta. \quad (B.22)$$

For any $\frac{1}{4} < H < \frac{1}{2}$ one has

$$\int_{s}^{t} \int_{0}^{v} (I^\dagger(v, r, t))^2 \, dr \, dv \leq C_H (t - s)^{4H} t^{2H+2}. \quad (B.23)$$
Proof.

\[
\int_v^t (\theta - v)^{2H - \frac{3}{2}} (\theta - r)^{H + \frac{1}{2}} \, d\theta \leq C_H (t - r)^{H + \frac{1}{2}} (t - v)^{2H - \frac{1}{2}} \leq C_H \, t^{H + \frac{1}{2}} (t - s)^{2H - \frac{1}{2}}.
\]

\[\Box\]

**Proposition B.10.** Let \( \mathcal{I}^2(v, r, t) \) be defined for \( 0 < v < s < t \) by

\[
\mathcal{I}^2(v, r, t) := \int_s^t \left| \partial_\theta K_H(\theta, v) \right| (\theta - r)^{H + \frac{1}{2}} \, d\theta.
\]

For any \( \frac{1}{4} < H < \frac{1}{2} \) one has

\[
\int_0^s \int_0^t (\mathcal{I}^2(v, r, t))^2 \, dr \, dv \leq (t - s)^{2H + 1}.
\]

Proof. We again notice that the map \( \partial_\theta K_H(\theta, v) \) is either positive or negative (see (B.3) and (B.5)). Therefore,

\[
\int_0^t (\mathcal{I}^2(v, r, t))^2 \, dr \leq t^{2H + 1} \left( \int_s^t \partial_\theta K_H(\theta, v) \, d\theta \right)^2 = t^{2H + 1} \left( K_H(t, v) - K_H(s, v) \right)^2.
\]

By again using (3.5) we deduce:

\[
\int_0^s \int_0^t (\mathcal{I}^2(v, r, t))^2 \, dr \, dv \leq \int_0^t (\mathcal{I}^2(v, r, t))^2 \, dr \, dv \leq (t - s)^{2H} t^{2H + 1}.
\]

\[\Box\]

### C Bounds on \( W_\lambda \) and its derivatives (Proof of Proposition 5.6)

The aim of this section is to show the following proposition.

**Proposition 5.6.** For any \( \lambda > 0 \), let \( W_\lambda(y) \) be defined as in (5.8). Under the assumptions (H1) and (H2) on \( b \) and \( \sigma \) one has

\[
\forall y \in \mathbb{R}, \ 0 \leq W_\lambda(y) \leq e^{-|y-y|} \mathcal{R}(\lambda), \tag{C.1}
\]

where \( \mathcal{R}(\lambda) \) is defined as in (5.2): \( \mathcal{R}(\lambda) := \sqrt{2\lambda + \mu^2} - \mu \).

In addition, the two first derivatives of \( W_\lambda \) satisfy the following estimates: There exists \( C > 0 \) depending on \( \mu \) only such that, for all real numbers \( y \) and \( \tilde{y} \),

\[
|W'_\lambda(y)| \leq C(1 + \lambda) e^{-|y-y|} \mathcal{R}(\lambda), \tag{C.2}
\]

\[
|W''_\lambda(y)| \leq C(1 + \lambda) e^{-|y-y|} \mathcal{R}(\lambda), \tag{C.3}
\]

\[
|W''_\lambda(y) - W''_\lambda(\tilde{y})| \leq C (1 + \lambda)^2 |y - \tilde{y}| \left( e^{-|y-y|} \mathcal{R}(\lambda) + e^{-|y-\tilde{y}|} \mathcal{R}(\lambda) \right). \tag{C.4}
\]

Proof. We successively consider \( y < \mathcal{Y} \) and \( y \geq \mathcal{Y} \).

The case \( y < \mathcal{Y} \).

Let the Lamperti process \( Y \) be defined as in (5.6). Let \( Y^\uparrow \) be defined as: \( Y^\uparrow = y + B_t + \mu t \)
where, as above, \( \mu := |\hat{b}|_\infty \). Denote by \( \tau_\mathcal{Y}^+ \) the first time \( Y^\uparrow \) hits \( \mathcal{Y} \). As \( Y_t \leq Y^\uparrow_t \) a.s. for every \( t \geq 0 \) one has \( \tau_\mathcal{Y}^+ \leq \tau_\mathcal{Y} \) a.s., from which

\[
\mathbb{E} \left( e^{-\lambda \tau_\mathcal{Y}} \right) \leq \mathbb{E} \left( e^{-\lambda \tau_\mathcal{Y}^+} \right) = e^{\mu(Y-y) - (Y-y)\sqrt{2\lambda + \mu^2}},
\]

55
where the last equality can be found in e.g. [8]. The inequality (C.1) follows.

Let us now prove the estimate on $W^\prime_\lambda$. We use a trick provided to us by P-E. Jabin. In view of (5.7) we have

$$\forall \tilde{y} \leq y \leq Y, \quad W^\prime_\lambda(y) = W^\prime_\lambda(\tilde{y}) - 2 \int_{\tilde{y}}^{y} \tilde{b}(z) W^\prime_\lambda(z) \, dz + 2\lambda \int_{\tilde{y}}^{y} W_\lambda(z) \, dz. \tag{C.5}$$

Integrate w.r.t. $\tilde{y}$ between $y-1$ and $y$ to obtain

$$W^\prime(y) = W_\lambda(y) - W_\lambda(y-1) + \int_{y-1}^{y} \left( -2 \int_{\tilde{y}}^{y} \tilde{b}(z) W^\prime_\lambda(z) \, dz + 2\lambda \int_{\tilde{y}}^{y} W_\lambda(z) \, dz \right) \, d\tilde{y}. \tag{C.1}$$

From (5.8) it results that the function $W_\lambda$ is positive and increasing on the interval $(-\infty, Y)$. Consequently,

$$0 \leq W^\prime_\lambda(y) \leq W_\lambda(y) + 2\mu \int_{y-1}^{y} \int_{\tilde{y}}^{y} W^\prime_\lambda(z) \, dz \, d\tilde{y} + 2\lambda \int_{y-1}^{y} \int_{\tilde{y}}^{y} W_\lambda(z) \, dz \, d\tilde{y} \leq C(1 + \lambda) W_\lambda(y),$$

The desired inequality (C.2) follows from (C.1).

The inequality (C.3) follows from (C.1), (C.2), and the differential equation (5.7).

Finally, to get (C.4) we start from (5.7):

$$W^\prime_\lambda(y) - W^\prime_\lambda(\tilde{y}) = 2\lambda(W_\lambda(y) - W_\lambda(\tilde{y})) - (\tilde{b}(y) - \tilde{b}(\tilde{y})) W^\prime_\lambda(y) - \tilde{b}(\tilde{y})(W^\prime_\lambda(y) - W^\prime_\lambda(\tilde{y})).$$

First, for any $\tilde{y} < y$, in view of (C.2) we have

$$W_\lambda(y) - W_\lambda(\tilde{y}) \leq C(1 + \lambda) (y - \tilde{y}) e^{-(Y-y)} R(\lambda).$$

Second, from (C.2) we deduce that

$$|(\tilde{b}(y) - \tilde{b}(\tilde{y})) W^\prime_\lambda(y)| \leq C(1 + \lambda) (y - \tilde{y}) e^{-(Y-y)} R(\lambda).$$

Finally, again use that $W^\prime_\lambda$ is positive and satisfies (C.5) to get

$$|W^\prime_\lambda(y) - W^\prime_\lambda(\tilde{y})| \leq C \int_{\tilde{y}}^{y} W^\prime_\lambda(z) \, dz + 2\lambda \int_{\tilde{y}}^{y} W_\lambda(z) \, dz \leq C(W_\lambda(y) - W_\lambda(\tilde{y})) + 2\lambda(y - \tilde{y}) e^{-(Y-y)} R(\lambda) \leq C(1 + \lambda) (y - \tilde{y}) e^{-(Y-y)} R(\lambda).$$

It then remains to exchange the roles of $y$ and $\tilde{y}$ to obtain (C.4).

The case $y \geq Y$.

In that case, we have that $Y - (2Y - y) = |Y - y|$. The desired estimates follow from the definition of $W^\prime_\lambda$ on the interval $(Y, +\infty)$ (see (5.8)) and the calculations for the case $y < Y$ which imply that $|W^\prime_\lambda(Y)| \leq C(1 + \lambda)$ and $|W^\prime_\lambda(Y)| \leq C(1 + \lambda)$.

$\square$

D Proof of Proposition 5.3

The proof of Proposition 5.3 relies on the following elementary lemma.

**Lemma D.1.** Set $m := Y - y_0$ and $\mu := |\tilde{b}|_{\infty}$. Let $q > 0$.  

56
(i) Let $Y^H_t$ be the process defined as

$$Y^H_t = y_0 + \mu t + B^H_t.$$  \hspace{1cm} (D.1)

One has

$$\mathbb{E} \left( e^{-q(Y^H_t - y_0)} \mathbb{1}_{Y^H_t \leq y} \right) \leq C \exp \left( -\frac{1}{2} \frac{(m - \mu s)^2}{s^{2H}} \mathbb{1}_{\frac{m - \mu s}{s^{2H}} \leq q} - \frac{q}{2} (m - \mu s) \mathbb{1}_{\frac{m - \mu s}{s^{2H}} > q} \right).$$ \hspace{1cm} (D.2)

(ii) Let $G$ be any standard Gaussian random variable. One has

$$\mathbb{E} \left( e^{-q|Y^H_t|} \mathbb{1}_{Y^H_t \leq y} \right) \leq C \exp \left( -\frac{1}{2} \frac{(m - \mu s)^2}{s^{2H}} \mathbb{1}_{\frac{m - \mu s}{s^{2H}} \leq q} - \frac{q}{2} (m - \mu s) \mathbb{1}_{\frac{m - \mu s}{s^{2H}} > q} \right) + C \mathbb{P}(G \geq \frac{m - \mu s}{s^{2H}}).$$ \hspace{1cm} (D.3)

Proof. We start with proving (D.2).

Define the decreasing function $f$ on $\mathbb{R}_+$ by

$$f(q) := \mathbb{E} \left( e^{-q(Y^H_t - y_0)} \mathbb{1}_{Y^H_t \leq y} \right) = \mathbb{E} \left( \exp(-q(m - B^H_t - \mu s)) \mathbb{1}_{B^H_t + \mu s \leq m} \right).$$

Notice that

$$\int_{-\infty}^{\frac{m - \mu s}{s^{2H}}} e^{qs^2 y - \frac{q}{2}} dy = e^{\frac{1}{2} q^2 s^{2H}} \int_{-\infty}^{\frac{m - \mu s}{s^{2H}}} e^{-\frac{1}{2} (y - q s^H)^2} dy = e^{\frac{1}{2} q^2 s^{2H}} \int_{-\infty}^{\frac{m - \mu s}{s^{2H}}} e^{-\frac{q}{2} z} dz.$$

Therefore,

$$f(q) = \frac{1}{\sqrt{2\pi}} \exp(-q(m - \mu s) + \frac{1}{2} q^2 s^{2H}) \int_{-\infty}^{\frac{m - \mu s}{s^{2H}}} e^{-\frac{q}{2} z} dz.$$

When $\frac{m - \mu s}{s^{2H}} \leq q$: As $f$ is decreasing, one has

$$f(q) \leq f\left( \frac{m - \mu s}{s^{2H}} \right) = C \exp \left( -\frac{(m - \mu s)^2}{s^{2H}} + \frac{1}{2} \frac{(m - \mu s)^2}{s^{2H}} \right) = C \exp \left( -\frac{1}{2} \frac{(m - \mu s)^2}{s^{2H}} \right).$$

When $\frac{m - \mu s}{s^{2H}} > q$: One then has

$$f(q) \leq \exp(-q(m - \mu s) + \frac{1}{2} q(m - \mu s)) = \exp \left( -\frac{1}{2} q(m - \mu s) \right).$$

We therefore have obtained (D.2).

We now turn to (D.3). Observe that

$$\mathbb{E} e^{-q|Y^H_t|} \leq \mathbb{E} \left( e^{-q(Y^H_t - y_0)} \mathbb{1}_{Y^H_t \leq y} \right) + \mathbb{E} \left( e^{-q|Y^H_t - y_0|} \mathbb{1}_{Y^H_t \geq y} \right) \leq \mathbb{E} \left( e^{-q(Y^H_t - y_0)} \mathbb{1}_{Y^H_t \leq y} \right) + \mathbb{P}(Y^H_t \geq Y).$$

Letting $G$ be defined as in the statement of the proposition we thus have

$$\mathbb{E} e^{-q|Y^H_t|} \leq \mathbb{E} \left( e^{-q(Y^H_t - y_0)} \mathbb{1}_{Y^H_t \leq y} \right) + \mathbb{P} \left( G \geq \frac{m - \mu s}{s^{2H}} \right).$$

It then remains to use (D.2).

We now are in a position to prove Proposition 5.3 that we recall here.
Proposition 5.3. Let $\lambda > |\tilde{b}|_\infty$. Let $m := Y - y_0$, $\mu := |\tilde{b}|_\infty$, $q := pR(\lambda)$ and $\tilde{\lambda} := \lambda - |\tilde{b}|_\infty$.

One has
\[
\mathcal{M}_p(Y - y_0, \lambda) \leq C \left( e^{-\frac{m}{\mu + q}} + e^{-\frac{1}{2} \Psi_H(m)} + \exp \left( -2^{-\frac{1}{2}} m \frac{2H}{2H+1} \tilde{\lambda} \frac{2H}{2H+1} \right) \right),
\] (D.4)
where
\[
\Psi_H(m) := \frac{m}{\mu + q} \left( 2^{H-1} \right) + \left( \frac{m}{\mu + q} \right)^{\frac{1}{2H}} \left( \left( \frac{m}{\mu + q} \right)^{2H-1} \right) \geq 1.
\] (D.5)

Proof. In view of (D.3) we have
\[
\sup_{s \in \mathbb{R}_+} e^{-\tilde{\lambda}s} \mathbb{E} e^{-q|Y - Y_s^H|} \leq J_1(\tilde{\lambda}) + J_2(\tilde{\lambda}),
\]
where
\[
J_1(\tilde{\lambda}) := C \sup_{s \in \mathbb{R}_+} e^{-\tilde{\lambda}s} \exp \left( -\frac{1}{2} \frac{(m - \mu s)^2}{s^{2H}} \mathbb{I}_{\frac{m - \mu s}{s^{2H}} \leq q} - \frac{q}{2} (m - \mu s) \mathbb{I}_{\frac{m - \mu s}{s^{2H}} > q} \right)
\]
and
\[
J_2(\tilde{\lambda}) := C \sup_{s \in \mathbb{R}_+} e^{-\tilde{\lambda}s} \mathbb{P}(G \geq \frac{m - \mu s}{s^H}).
\]

We start with estimating $J_1(\tilde{\lambda})$. Observe that the map
\[
\phi(s) := qs^{2H} + \mu s - m
\]
is increasing, which implies that there exists a unique $s_*$ such that $\phi(s_*) = 0$, that is, such that
\[
\left[ \frac{m - \mu s}{s^{2H}} \leq q \right] \iff s \geq s_*.
\]
It comes:
\[
J_1(\tilde{\lambda}) \leq C \sup_{s \in [0, s_*)} e^{-\tilde{\lambda}s - \frac{1}{2} q(m - \mu s)} + C \sup_{s \in (s_*, \infty)} e^{-\tilde{\lambda}s}.
\] (D.6)

In order to bound the preceding expression from above we bound $s_*$ from above and below as follows.

**An upper bound for $s_*$.** Noticing that $\phi\left( \frac{m}{\mu} \right) > 0$ we get $s_* < \frac{m}{\mu}$.

**A lower bound for $s_*$.** We aim to get a $s$ such that $\phi(s) \leq 0$. We distinguish two cases:

- If $\left( \frac{m}{\mu + q} \right)^{2H-1} < 1$ one has
  \[
  \phi \left( \frac{m}{\mu + q} \right) \leq q \frac{m}{\mu + q} + \mu \frac{m}{\mu + q} - m = 0.
  \]

- If $\left( \frac{m}{\mu + q} \right)^{2H-1} \geq 1$ one has
  \[
  \phi \left( \left( \frac{m}{\mu + q} \right)^{\frac{1}{2H}} \right) = q \frac{m}{\mu + q} + \mu \left( \frac{m}{\mu + q} \right) \frac{1}{2H} - m \leq q \frac{m}{\mu + q} + \mu \frac{m}{\mu + q} - m = 0.
  \]
To summarize, \[ s_\ast \geq \Psi^H_q(m), \]
where \( \Psi^H_q(m) \) is defined as in (D.5).

We now come back to (D.6) and observe that
\[
\sup_{s \in [0,s_\ast]} e^{-\tilde{\lambda}s - \frac{1}{2}q(m-\mu s)} \leq \sup_{s \in [0,\frac{m}{2q}]} e^{-\tilde{\lambda}s - \frac{1}{2}q(m-\mu s)} + \sup_{s \in (\frac{m}{2q},s_\ast]} e^{-\tilde{\lambda}s - \frac{1}{2}q(m-\mu s)} \\
\leq e^{-\frac{1}{2}qm} + e^{-\frac{1}{2}q(m-\mu s_\ast) - \tilde{\lambda} \frac{m}{2q}} \\
\leq e^{-\frac{1}{2}qm} + e^{-\frac{1}{2}\tilde{\lambda}\Psi^H_q(m)},
\]
and
\[
\sup_{s \in (s_\ast, \infty)} e^{-\tilde{\lambda}s} \leq e^{-\tilde{\lambda}\Psi^H_q(m)}.
\]

We are in a position to conclude that
\[
J_1(\tilde{\lambda}) \leq Ce^{-\frac{m}{2q}} + Ce^{-\frac{1}{2}\tilde{\lambda}\Psi^H_q(m)}.
\] (D.7)

**An upper bound for \( J_2(\tilde{\lambda}) \).**

\[
J_2(\tilde{\lambda}) \leq C \sup_{s \in [0,\frac{m}{2q}]} e^{-\tilde{\lambda}s} P\left( G \geq \frac{m - \mu s}{sH} \right) + C \sup_{s \in (\frac{m}{2q}, \infty)} e^{-\tilde{\lambda}s} P\left( G \geq \frac{m - \mu s}{sH} \right).
\]

Observe that
\[
\sup_{s \in (\frac{m}{2q}, \infty)} e^{-\tilde{\lambda}s} P\left( G \geq \frac{m - \mu s}{sH} \right) \leq \exp\left(-\tilde{\lambda}m \right).
\]

In addition,
\[
\sup_{s \in [0,\frac{m}{2q}]} e^{-\tilde{\lambda}s} P\left( G \geq \frac{m - \mu s}{sH} \right) \leq \sup_{s \in [0,\frac{m}{2q}]} \exp\left(-\tilde{\lambda}s - \frac{1}{2} \frac{(m-\mu s)^2}{s^2H} \right) \\
\leq \sup_{s \in [0,\frac{m}{2q}]} \exp\left(-\tilde{\lambda}s - \frac{m^2}{8s^2H} \right).
\]

The function \( s \mapsto -\tilde{\lambda}s - \frac{m^2}{8s^2H} \) reaches its maximum at \( s = \left( \frac{H m^2}{4 \tilde{\lambda}} \right)^{\frac{1}{1+2H}} \). Therefore,
\[
\forall s > 0, \quad -\tilde{\lambda}s - \frac{m^2}{8s^2H} \leq -\tilde{\lambda}\left( \frac{H m^2}{4 \tilde{\lambda}} \right)^{\frac{1}{1+2H}} - \frac{m^2}{8} \left( \frac{H m^2}{4 \tilde{\lambda}} \right)^{-\frac{2H}{1+2H}} \leq -c m^{\frac{2}{1+2H}} \tilde{\lambda}^{\frac{2H}{1+2H}}
\]
where \( c := \min_{\frac{1}{3} < H < 1} \left( \frac{H}{4} \right)^{\frac{1}{1+2H}} = 2^{-\frac{8}{3}} \), from which
\[
\sup_{s \in [0,\frac{m}{2q}]} e^{-\tilde{\lambda}s} P\left( G \geq \frac{m - \mu s}{sH} \right) \leq \exp\left(-2^{-\frac{8}{3}} \frac{m^{\frac{2}{1+2H}} \tilde{\lambda}^{\frac{2H}{1+2H}}}{m^{\frac{2}{1+2H}}} \right).
\]

We conclude:
\[
J_2(\tilde{\lambda}) \leq C \exp\left(-2^{-\frac{8}{3}} \frac{m^{\frac{2}{1+2H}} \tilde{\lambda}^{\frac{2H}{1+2H}}}{m^{\frac{2}{1+2H}}} \right) + C \exp\left(-\tilde{\lambda}\frac{m}{2q} \right).
\] (D.8)

That ends the proof of (D.4).  \( \square \)
E Glossary

- The process $Y^H$ is defined in Proposition 3.2.
- In the statement of Theorem 5.2 one defines $R(\lambda) := \sqrt{2\lambda + \mu^2} - \mu$ and $\mu := |\hat{\beta}|_\infty := |\frac{\beta}{\sigma_\infty}|_\infty$.
- The constants $\Theta$ and $y_0$ are defined at the beginning of the subsection 5.2. In that subsection we set $m := \Theta - y_0$.
- The function $W_\lambda$ is defined by (5.8). It satisfies the ODE (5.7) on the interval $(-\infty, \Theta)$.

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