$L^\infty$-uniqueness of Schrödinger operators restricted in an open domain

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Abstract

Consider the Schrödinger operator $\mathcal{A} = -\frac{\Delta}{2} + V$ acting on space $C_0^\infty(D)$, where $D$ is an open domain in $\mathbb{R}^d$. The main purpose of this paper is to present the $L^\infty(D, dx)$-uniqueness for Schrödinger operators which is equivalent to the $L^1(D, dx)$-uniqueness of weak solutions of the heat diffusion equation associated to the operator $\mathcal{A}$.

Key Words: $C_0$-semigroups; $L^\infty$-uniqueness of Schrödinger operators; $L^1$-uniqueness of the heat diffusion equation.

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1 Preliminaries

Let $D$ be an open domain in $\mathbb{R}^d$ with its boundary $\partial D$. We denote by $C_0^\infty(D)$ the space of all infinitely differentiable real functions on $D$ with compact support. Consider the Schrödinger operator $A = -\Delta + V$ acting on space $C_0^\infty(D)$, where $\Delta$ is the Laplace operator and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Borel measurable potential.

The essential self-adjointness of Schrödinger operator in $L^2(\mathbb{R}^d, dx)$, equivalent to the unique solvability of Schrödinger equation in $L^2(\mathbb{R}^d, dx)$, has been studied by Kato [Ka’84], Reed and Simon [RS’75], Simon [Si’82] and others because of its importance in Quantum Mechanics. In the case where $V$ is bounded, it is not difficult to prove that $(A, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint in $L^2(\mathbb{R}^d, dx)$. But in almost all interesting situations in quantum physics, the potential $V$ is unbounded. In this situation we need to consider the Kato class, used first by Schechter [Sch’71] and Kato [Ka’72]. A real valued measurable function $V$ is said to be in the Kato class $K^d$ on $\mathbb{R}^d$ if

$$\lim_{\delta \searrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \delta} |g(x-y)V(y)| \, dy = 0$$

where

$$g(x) = \begin{cases} \frac{1}{|x|^{d-2}}, & \text{if } d \geq 3 \\ \ln \frac{1}{|x|}, & \text{if } d = 2 \\ 1, & \text{if } d = 1. \end{cases}$$

If $V \in L^2_{\text{loc}}(\mathbb{R}^d, dx)$ is such that $V^-$ belongs to the Kato class on $\mathbb{R}^d$, it is well known that the Schrödinger operator $(A, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint and the unique solution in $L^2$ of the heat equation is given by the famous Feynmann-Kac semigroup.
\[
\{P_t^V\}_{t \geq 0} \\
\quad P_t^V f(x) := \mathbb{E}^x f(B_t) \exp \left( - \int_0^t V(B_s) \, ds \right)
\]

where \( f \) is a nonnegative measurable function, \((B_t)_{t \geq 0}\) is the Brownian Motion in \( \mathbb{R}^d \) defined on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}^d})\) with \( \mathbb{P}_x (B_0 = x) = 1 \) for any initial point \( x \in \mathbb{R}^d \) and \( \mathbb{E}^x \) means the expectation with respect to \( \mathbb{P}_x \).

In the case where \( D \) is a strict sub-domain, sharp results are known only when \( d = 1 \) or, in the multidimensional case, only in some special situations.

Consequently of an intuitive probabilistic interpretation of uniqueness, Wu \cite{Wu98} introduced and studied the uniqueness of Schrödinger operators in \( L^1(D, dx) \). On say that \((\mathcal{A}, C^\infty_0(D))\) is \( L^1(D, dx) \)-unique if \( \mathcal{A} \) is closable and its closure is the generator of some \( C_0 \)-semigroup on \( L^1(D, dx) \). This uniqueness notion was also studied in Arendt \cite{Ar86}, Eberle \cite{Eb97}, Djellout \cite{Dj97}, Röckner \cite{Ro98}, Wu \cite{Wu98} and \cite{Wu99} and others in the Banach spaces setting.

\section{\( L^\infty(D, dx) \)-uniqueness of Schrödinger operators}

Our purpose is to study the \( L^\infty(D, dx) \)-uniqueness of the Schrödinger operator \((\mathcal{A}, C^\infty_0(D))\) in the case where \( D \) is a strict sub-domain on \( \mathbb{R}^d \). But how we can define the uniqueness in \( L^\infty(D, dx) \)? One can prove rather easily that the killed Feynmann-Kac semigroup

\[
\{P_t^{D,V}\}_{t \geq 0} \\
\quad P_t^{D,V} f(x) := \mathbb{E}^x 1_{\{t < \tau_D\}} f(B_t) \exp \left( - \int_0^t V(B_s) \, ds \right)
\]

where \( \tau_D := \inf\{t > 0 : B_t \notin D\} \) is the first exiting time of \( D \), is a semigroup of bounded operators on \( L^p(D, dx) \) for any \( 1 \leq p \leq \infty \), which is strongly continuous for
$1 \leq p < \infty$, but never strongly continuous in $(L^\infty(D, dx), \| \cdot \|_\infty)$. Moreover, a well known result of LOTZ [Lo’86, Theorem 3.6, p. 57] says that the generator of any strongly continuous semigroup on $(L^\infty(D, dx), \| \cdot \|_\infty)$ must be bounded.

To obtain a correct definition of $L^\infty(D, dx)$-uniqueness, we should introduce a weaker topology of $L^\infty(D, dx)$ such that \( \{ P^D,V_t \} \) becomes a strongly continuous semigroup with respect to this new topology. Remark that the natural topology for studying $C_0$-semigroups on $L^\infty(D, dx)$ used first by Wu and Zhang [WZ’06] is the topology of uniform convergence on compact subsets of $L^1(D, dx)$, denoted by $C(L^\infty, L^1)$. More precisely, if we denote

$$\langle f, g \rangle := \int_D f(x)g(x)dx$$

for all $f \in L^1(D, dx)$ and $g \in L^\infty(D, dx)$, then for an arbitrary point $g_0 \in L^\infty(D, dx)$, a basis of neighborhoods with respect to $C(L^\infty, L^1)$ is given by

$$N(g_0; K, \varepsilon) := \left\{ \begin{array}{l} g \in L^\infty(D, dx) : \sup_{f \in K} |\langle f, g \rangle - \langle f, g_0 \rangle| < \varepsilon \end{array} \right\}$$

where $K$ runs over all compact subsets of $L^1(D, dx)$ and $\varepsilon > 0$.

Remark that $(L^\infty(D, dx), C(L^\infty, L^1))$ is a locally convex space and if \( \{ T(t) \}_{t \geq 0} \) is a $C_0$-semigroup on $L^1(D, dx)$ with generator $\mathcal{L}$, by [WZ’06, Theorem 1.4, p. 564] it follows that \( \{ T^\ast(t) \}_{t \geq 0} \) is a $C_0$-semigroup on $(L^\infty(D, dx), C(L^\infty, L^1))$ with generator $\mathcal{L}^\ast$.

Now we can introduce the uniqueness notion in $L^\infty(D, dx)$. Let $A$ be a linear operator on $L^\infty(D, dx)$ with domain $D$ which is assumed to be dense in $L^\infty(D, dx)$ with respect to the topology $C(L^\infty, L^1)$.

**Definition 2.1.** The operator $A$ is said to be a pre-generator on $L^\infty(D, dx)$ if there exists some $C_0$-semigroup on $(L^\infty(D, dx), C(L^\infty, L^1))$ such that its generator $\mathcal{L}$ extends
A. We say that $A$ is $L^\infty(D, dx)$-unique if $A$ is closable and its closure with respect to the topology $C(L^\infty, L^1)$ is the generator of some $C_0$-semigroup on $(L^\infty(D, dx), C(L^\infty, L^1))$.

The main result of this paper is

**Theorem 2.2.** Let $V \in L^\infty_{\text{loc}}(D, dx)$ such that $V^- \in \mathcal{K}^d$. Then the Schrödinger operator $(A, C^\infty_0(D))$ is $(L^\infty(D, dx), C(L^\infty, L^1))$-unique.

**Proof.** First, we must remark that the existence assumption of pre-generator in [WZ’06 Theorem 2.1, p. 570] is satisfied. Indeed, if consider the killed Feynman-Kac semigroup $\{P_{D,V}^t\}_{t \geq 0}$ on $L^\infty(D, dx)$ and for any $p \in [1, \infty]$ we define

\[ \left\| P_{D,V}^t \right\|_p := \sup_{\|f\|_p \leq 1} \left\| P_{D,V}^t f \right\|_p, \]

next lemma show that $A$ is a pre-generator on $(L^\infty(D, dx), C(L^\infty, L^1))$, i.e. $A$ is contained in the generator $L_{(\infty)}^{D,V}$ of the killed Feynmann-Kac semigroup $\{P_{D,V}^t\}_{t \geq 0}$.

**Lemma 2.3.** Let $V \in L^\infty_{\text{loc}}(D, dx)$ such that $V^- \in \mathcal{K}^d$ and let $\{P_{D,V}^t\}_{t \geq 0}$ be the killed Feynman-Kac semigroup on $L^\infty(D, dx)$. If $\left\| P_{D,V}^t \right\|_{\infty}$ is bounded over the compact intervals, then $\{P_{D,V}^t\}_{t \geq 0}$ is a $C_0$-semigroup on $(L^\infty(D, dx), C(L^\infty, L^1))$ and its generator $L_{(\infty)}^{D,V}$ is an extension of $(A, C^\infty_0(D))$.

**Proof.** The proof is close to that of [Wu’98 Lemma 2.3, p. 288]. Let $\{P_{D,V}^t\}_{t \geq 0}$ be the killed Feynman-Kac semigroup on $L^\infty(D, dx)$. Remark that

\[ \left| P_{D,V}^t f(x) \right| \leq P_{D,V}^t |f|(x) \leq P_{D,-V}^t |f|(x) \leq P_{-V}^t |f|(x) \]

from where we deduce that

\[ \sup_{0 \leq t \leq 1} \left\| P_{D,V}^t \right\|_{\infty} \leq \sup_{0 \leq t \leq 1} \left\| P_{-V}^t \right\|_{\infty} < \infty \]
since \( \left\| P_t^{-V} \right\|_\infty \) is uniformly bounded by the assumption that \( V^- \in K^d \) (see [AS'82]).

Since \( \left\| P_t^{D,V} \right\|_1 = \left\| P_t^{D,V} \right\|_\infty \) is bounded for \( t \) in compact intervals of \([0, \infty)\), using [Wu'01] Lemma 2.3, p. 59 it follows that \( \left\{ P_t^{D,V} \right\}_{t \geq 0} \) is a \( C_0 \)-semigroup on \( L^1(D, dx) \).

By [WZ'06] Theorem 1.4, p. 564 we find that \( \left\{ P_t^{D,V} \right\}_{t \geq 0} \) is a \( C_0 \)-semigroup on \( L^\infty(D, dx) \) with respect to the topology \( C(L^\infty, L^1) \). We have only to show that its generator \( L_{(\infty)}^{D,V} \) is an extension of \( (\mathcal{A}, C_0^\infty(D)) \).

**Step 1: the case \( V \geq 0 \).** For \( n \in \mathbb{N} \) we consider \( V_n := V \wedge n \). By a theorem of bounded perturbation (see [Da'80] Theorem 3.1, p. 68]) it follows that

\[
\mathcal{A}_n = -\frac{\Delta}{2} + V_n
\]

is the generator of a \( C_0 \)-semigroup \( \left\{ P_t^{D,V_n} \right\}_{t \geq 0} \) on \( (L^\infty(D, dx), C(L^\infty, L^1)) \). So for any \( f \in C^\infty_0(D) \) we have

\[
P_t^{D,V_n} f - f = \int_0^t P_s^{D,V_n} \mathcal{A}_n f \, ds , \quad \forall t \geq 0.
\]

Letting \( n \to \infty \), we have pointwisely on \( D \):

\[
P_t^{D,V_n} f \to P_t^{D,V} f
\]

and

\[
P_t^{D,V_n} \mathcal{A}_n f \to P_t^{D,V} \mathcal{A} f .
\]

Moreover, for any \( x \in D \) we have:

\[
\left| P_t^{D,V_n} f(x) \right| \leq P_t^{D,V} |f|(x)
\]

and

\[
\left| P_t^{D,V_n} \mathcal{A}_n f(x) \right| \leq P_t^{D,V} \left( \left| \frac{\Delta}{2} \right| + |V f| \right)(x) .
\]
Hence by the dominated convergence we derive that

\[ P^D,V_t f - f = \int_0^t P^D,V_s A f ds \quad \forall t \geq 0. \]

It follows that \( f \) is in the domain of the generator \( \mathcal{L}^D,V_{(\infty)} \) of \( C_0 \)-semigroup \( \{ P^D,V_t \}_{t \geq 0} \).

**Step 2: the general case.** Setting \( V^n = V \vee (-n) \), for \( n \in \mathbb{N} \), and denoting by

\[ A^n = -\frac{\Delta}{2} + V^n \]

the generator of the \( C_0 \)-semigroup \( \{ P^D,V^n_t \}_{t \geq 0} \) on \( (L^\infty(D), \mathcal{C}(L^\infty, L^1)) \), we have by Step 1

\[ P^D,V^n_t f - f = \int_0^t P^D,V^n_s A^n f ds \quad t \geq 0. \]

Notice that

\[ |P^D,V^n_s A^n f(x)| \leq P^D,V_s \left( \left| \frac{\Delta}{2} f \right| + |V f| \right)(x) \]

which is uniformly bounded in \( L^\infty(D, dx) \) over \([0, t]\). By Fubini’s theorem we have

\[ \int_0^t P^D,V_s \left( \left| \frac{\Delta}{2} f \right| + |V f| \right)(x) ds < \infty \text{ dx-a.e. on } D. \]

On the other hand, for any \( x \in D \) fixed such that

\[ P^D,V_s \left( \left| \frac{\Delta}{2} f \right| + |V f| \right)(x) < \infty \]

then by dominated convergence we find

\[ P^D,V^n_s \left( -\frac{\Delta}{2} + V^n \right) f(x) \rightarrow P^D,V_s \left( -\frac{\Delta}{2} + V \right) f(x). \]

Thus by dominated convergence we have dx-a.e. on \( D \),

\[ \int_0^t P^D,V^n_s \left( -\frac{\Delta}{2} + V^n \right) f ds \rightarrow \int_0^t P^D,V_s \left( -\frac{\Delta}{2} + V \right) f ds \quad \forall t \geq 0. \]
The same argument shows that

\[ P_t^{D,V} f - f \to P_t^{D,V} f - f \]

By consequence

\[ P_t^{D,V} f - f = \int_0^t P_s^{D,V} \left( -\frac{\Delta}{2} + V \right) f \, ds , \quad \forall t \geq 0. \]

Hence \( f \) is in the domain of generator \( \mathcal{L}_{(\infty)}^{D,V} \) of semigroup \( \{ P_t^{D,V} \}_{t \geq 0} \). So \( \mathcal{L}_{(\infty)}^{D,V} \) is an extension of the operator \((A, C_0^\infty(D))\) and the lemma is proved.

Next we prove the \( L^\infty(D, dx) \)-uniqueness of \( A \). By [WZ'06, Theorem 2.1, p. 570], we deduce that the operator \((A, C_0^\infty(D))\) is \( L^\infty(D, dx) \)-unique if and only if for some \( \lambda \), the range \((\lambda I - A)(C_0^\infty(D))\) is dense in \((L^\infty(D, dx), C(L^\infty, L^1))\). It is enough to show that for any \( h \in L^1(D, dx) \) which satisfies the equality

\[ \langle h, (\lambda I + A) f \rangle = 0 , \quad \forall f \in C_0^\infty(D) \]

it follows \( h = 0 \).

Let \( h \in L^1(D, dx) \) be such that for some \( \lambda \) one have

\[ \langle h, (\lambda I + A) f \rangle = 0 , \quad \forall f \in C_0^\infty(D) \]

or

\[ (\lambda I + A)h = 0 \quad \text{in the sense of distribution}. \]

Since \( V \in L^\infty_{\text{loc}}(D, dx) \), by applying [AS'82, Theorem 1.5, p. 217] we can see that \( h \) is a continuous function. By the mean value theorem due to AIZENMANN and SIMON [AS'82, Corollary 3.9, p. 231], there exists some constant \( C > 0 \) such as

\[ |h(x)| \leq C \int_{|x-y| \leq 1} |h(y)| \, dy , \quad \forall x \in D. \]
As \( V^- \in \mathcal{K}^d \), \( C \) may be chosen independently of \( x \in D \). Since \( h \in L^1(D, dx) \), it follows that \( h \) is bounded and, consequently, \( h \in L^2(D, dx) \). Now by the \( L^2(D, dx) \)-uniqueness of \((A, C_0^\infty(D))\) and [WZ'06 Theorem 2.1, p. 570], \( h \) belongs to the domain of the generator \( \mathcal{L}_{(2)}^{D,V} \) of \( \{P_t^{D,V}\}_{t \geq 0} \) on \( L^2 \) and

\[
\mathcal{L}_{(2)}^{D,V} h = \left( -\frac{\Delta}{2} + V \right) h = -\lambda h .
\]

Hence

\[
P_t^{D,V} h = e^{-\lambda t} h , \quad \forall t \geq 0.
\]

Let

\[
\lambda(D, V) := \inf_{f \in C_0^\infty(D)} \left\{ \frac{1}{2} \int_D |\nabla f|^2 dx + V f^2 dx : \|f\|_2 \leq 1 \right\} .
\]

be the lowest energy of the Schrödinger operator. If we take \( \lambda < \lambda(D, V) \), then the last equality is possible only for \( h = 0 \), because \( \left\| P_t^{D,V} \right\|_2 = e^{-\lambda(D,V)t} \) (see Albeverio and Ma [AM'91 Theorem 4.1, p. 343]).

**Remarque 2.4.** Intuitively, to have \( L^1(D, dx) \)-uniqueness, the repulsive potential \( V^+ \) should grow rapidly to infinity near \( \partial D \), this means

\[
(C_1) \quad \mathbb{P}_x \left( \int_0^{\tau_D} V^+(B_s) ds + \tau_D = \infty \right) = 1 \quad \text{for a.e. } x \in D
\]

so that a particle with starting point inside \( D \) can not reach the boundary \( \partial D \) (see [Wu'98 Theorem 1.1, p. 279]).

By analogy with the uniqueness in \( L^1(D, dx) \), the \( L^\infty(D, dx) \)-uniqueness of \((A, C_0^\infty(D))\) means that a particle starting from the boundary \( \partial D \) can not enter in \( D \). Unfortunately, here we have a problem: \( L^\infty(D, dx) \)-uniqueness of \( A \) is equivalent to the existence of a unique boundary condition for \( A^* \). It is well known that there are many boundary conditions (Dirichlet, Newmann, etc.). Remark that in the case of \( L^1(D, dx) \)-uniqueness...
of $A$, the effect of the boundary condition for $A^*$ is eliminated by the condition $(C_1)$ for potential. To find such condition in the case of $L^\infty(D, dx)$-uniqueness is very difficult. In this moment we can present here an interesting result from $[WZ'06]$:

**Proposition 2.5.** Let $D$ be a nonempty open domain of $\mathbb{R}^d$. If the Laplacian $(\Delta, C_0^\infty(D))$ is $L^\infty(D, dx)$-unique, then $D^C = \emptyset$ or $D = \mathbb{R}^d$.

For the heat diffusion equation we can formulate the next result

**Corollary 2.6.** If $V \in L^\infty_{\text{loc}}(\mathbb{R}^d, dx)$ and $V^- \in \mathcal{K}^d$, then for every $h \in L^1(\mathbb{R}^d, dx)$, the heat diffusion equation

\[
\begin{aligned}
\partial_t u(t, x) &= (-\frac{\Delta}{2} + V) u(t, x) \\
 u(0, x) &= h(x)
\end{aligned}
\]

has one $L^1(\mathbb{R}^d, dx)$-unique weak solution which is given by $u(t, x) = P_t^V h(x)$.

**Proof.** The assertion follows by $[WZ'06$, Theorem 2.1, p. 570] and Theorem 2.2.

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**References**

[AM’91] Albeverio, S., Ma, Z.M. Perturbation of Dirichlet form: lower boundedness, closability and form cores. *J. Funct. Anal.*, 99(1991), 332-356.
[AS’82] Aizenman, M., Simon, B. Brownian motion and Harnack’s inequality for Schrödinger Operators. Comm. Pure Appl. Math., 35(1982), 209-271.

[Ar’86] Arendt, W. The abstract Cauchy problem, special semigroups and perturbation. One Parameter Semigroups of Positive Operators (R. Nagel, Eds.), Lect. Notes in Math., 1184, Springer, Berlin, 1986.

[Da’80] Davies, E.B. One-parameter semigroups. Academic Press, London, New York, Toronto, Sydney, San Francisco, 1980.

[Dj’97] Djellout, H. Unicité dans $L^p$ d’opérateurs de Nelson. Prépublication, 1997.

[Eb’97] Eberle, A. Uniqueness and non-uniqueness of singular diffusion operators. Doctor-thesis, Bielefeld, 1997.

[Ka’84] Kato, T. Perturbation theory for linear operators. Springer Verlag, Berlin, Heidelberg, New York, 1984.

[Ka’72] Kato, T. Schrödinger operators with singular potentials. Israel J. Math., 13(1972), 135-148.

[Lo’86] Lotz, H.P. The abstract Cauchy problem, special semigroups and perturbation. One Parameter Semigroups of Positive Operators (R. Nagel, Eds.), Lect. Notes in Math., 1184, Springer, Berlin, 1986.

[RS’75] Reed, M., Simon, B. Methods of Modern Mathematical Physics, II, Fourier Analysis, Self-adjointness. Academic Press, New York, 1975.

[Rö’98] Röckner, M. $L^p$-analysis of finite and infinite dimensional diffusion operators. Lect. Notes in Math., 1715(1998), 65-116.
[Sch’71] Schechter, M. *Spectra of partial differential operators*. North-Holland, Amsterdam, 1971.

[Si’82] Simon, B. Schrödinger Semigroups. *Bull. Amer. Math. Soc.* (3) **7**(1982), 447-526.

[Wu’98] Wu, L. Uniqueness of Schrödinger Operators Restricted in a Domain. *J. Funct. Anal.*, (2) **153**(1998), 276-319.

[Wu’99] Wu, L. Uniqueness of Nelson’s diffusions. *Probab. Theory Relat. Fields*, **114**(1999), 549-585.

[Wu’01] Wu, L. $L^p$-uniqueness of Schrödinger operators and the capacitary positive improving property. *J. Funct. Anal.*, **182**(2001), 51-80.

[WZ’06] Wu, L., Zhang, Y. A new topological approach to the $L^\infty$-uniqueness of operators and $L^1$-uniqueness of Fokker-Planck equations. *J. Funct. Anal.*, **241**(2006), 557-610.