Optimal Bounds of the Arithmetic Mean by Harmonic, Contra-harmonic and New Seiffert-like Means

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Authors’ contributions
This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract
We provide the optimal bounds for the arithmetic mean in terms of harmonic, contra-harmonic and new Seiffert-like means.

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1 Introduction
Seiffert [1, 2] introduced two means

$$P(a, b) = \frac{a - b}{2 \arcsin \left( \frac{a - b}{(a + b)} \right)},$$

$$T(a, b) = \frac{a - b}{2 \arctan \left( \frac{a - b}{(a + b)} \right)}.$$
These two means are called the first and second Seiffert means, respectively.

For two positive and unequal real numbers \(a\) and \(b\), Witkowski [3] introduced the Seiffert-like mean \(M_f(a, b)\) given by the formula

\[
M_f(a, b) = \frac{a - b}{2f[(a - b)/(a + b)]},
\]

(1.1)

where the function \(f : (0, 1) \rightarrow \mathbb{R}\) (called Seiffert function) satisfying

\[
\frac{x}{1 + x} \leq f(x) \leq \frac{x}{1 - x}.
\]

It was shown that every symmetric and homogeneous mean of two positive real numbers can be represented in the form (1.1) and that every Seiffert function produces a mean. The correspondence between means and Seiffert functions is given by the formula

\[
f(x) = \frac{x}{M_f(1 - x, 1 + x)}, \quad \text{where } x = \frac{|a - b|}{a + b}.
\]

(1.2)

Witkowski proved that the following conditions are equivalent:

\[
M_f(a, b) < M_g(a, b) \iff f(x) > g(x).
\]

(1.3)

The Neuman-Sándor mean \(NS(a, b)\) and logarithmic mean \(L(a, b)\) are the Seiffert-like means.

\[
NS(a, b) = \frac{a - b}{2\arcsinh[(a - b)/(a + b)]} := M_{\arcsinh}(a, b),
\]

\[
L(a, b) = \frac{a - b}{2\arctanh[(a - b)/(a + b)]} := M_{\arctanh}(a, b).
\]

Certainly, the first and second Seiffert means \(P(a, b)\) and \(T(a, b)\) can be denoted \(M_{\arcsin}(a, b)\) and \(M_{\arctan}(a, b)\). Further more, Witkowski extend the new Seiffert-like means by showing that also sine, tangent, hyperbolic sine and hyperbolic tangent are Seiffert functions, they are given as follows:

\[
M_{\sin}(a, b) = \frac{a - b}{2\sin[(a - b)/(a + b)]}, \quad M_{\tan}(a, b) = \frac{a - b}{2\tan[(a - b)/(a + b)]},
\]

\[
M_{\sinh}(a, b) = \frac{a - b}{2\sinh[(a - b)/(a + b)]}, \quad M_{\tanh}(a, b) = \frac{a - b}{2\tanh[(a - b)/(a + b)]}.
\]

(1.4)

(1.5)

In recent years, these Seiffert-like means and their inequalities have attracted attention of several researchers [3, 4, 5, 6]. Undoubtedly, the Seiffert-like means are studied always compared with some well-known symmetric and homogeneous means of positive arguments.

Let \(p \in \mathbb{R}\) and \(a, b > 0\) with \(a \neq b\), the \(p\)th Hölder mean \(H_p(a, b)\) are defined by

\[
H_p(a, b) = \left\{ \begin{array}{ll}
\left(\frac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0,
\end{array} \right.
\]

particularly

\[
H_{-1}(a, b) = \frac{2ab}{a + b} := H(a, b), \quad H_0(a, b) = \sqrt{ab} := G(a, b),
\]

\[
H_1(a, b) = \frac{a + b}{2} := A(a, b), \quad H_2(a, b) = \sqrt{\frac{a^2 + b^2}{2}} := Q(a, b),
\]

are the harmonic mean, geometric mean, arithmetic mean and quadratic mean, respectively.
It is well-known that the Hölder mean $H_p (a, b)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, hence the following inequalities hold

$$H(a, b) < G(a, b) < A(a, b) < Q(a, b) < C(a, b)$$

where $C(a, b) = (a^2 + b^2) / (a + b)$ is the contra-harmonic mean.

Let $a > b > 0$ and $x = (a - b) / (a + b) \in (0, 1)$, Witkowski [3, Lemma 3.1-3.2] proved the following chains of inequalities

$$\arctan(x) > \tan(x) > \sinh(x) > x > \arcsinh(x) > \sin(x) > \arctan(x) > \tanh(x)$$

hold for all $x \in (0, 1)$. From (1.3) the following chains inequalities of means

$$M_{\arctan} (a, b) < M_{\tan} (a, b) < M_{\sinh} (a, b) < A$$

$$< M_{\arcsinh} (a, b) < M_{\sin} (a, b) < M_{\arctan} (a, b) < M_{\tanh} (a, b)$$

(1.6)

hold for $a, b > 0$ with $a \neq b$.

From the formula (1.2), we can get the Serret functions of the harmonic, geometric, arithmetic, quadratic and contra-harmonic means, they are listed as follows:

$$h(x) = \frac{x}{1 - x^2}, \quad g(x) = \frac{x}{\sqrt{1 - x^2}}, \quad a(x) = x, \quad q(x) = \frac{x}{\sqrt{1 + x^2}}$$

(1.7)

Note that

$$h(0) - \arctan(0) = 0, \quad [h(x) - \arctan(x)]' = \frac{2x}{(1 - x^2)^2} > 0 \iff h(x) > \arctan(x),$$

$$\cosh^2(x) > \left(1 + x^2/2\right)^2 > 1 + x^2 \iff \frac{1}{\cosh^2(x)} < \frac{1}{1 + x^2} \iff \tanh(x) > \frac{x}{\sqrt{1 + x^2}},$$

for $x \in (0, 1)$.

Therefore,

$$H(a, b) < M_{\arctan} (a, b), \quad M_{\tanh} (a, b) < Q(a, b), \quad (1.8)$$

hold for all $a, b > 0$ with $a \neq b$. From (1.6), (1.8) we obtain chains inequalities

$$H(a, b) < M_{\arctan} (a, b) < M_{\tan} (a, b) < M_{\sinh} (a, b) < A(a, b)$$

$$< M_{\arcsinh} (a, b) < M_{\sin} (a, b) < M_{\arctan} (a, b) < M_{\tanh} (a, b) < Q(a, b) < C(a, b),$$

(1.9)

Y.-M.Chu [7] et al. find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha T(a, b) + (1 - \alpha) G(a, b) < A(a, b) < \beta T(a, b) + (1 - \beta) G(a, b),$$

(1.10)

hold for all $a, b > 0$ with $a \neq b$.

F.Yang [8] et al. find the greatest value $\alpha$ and the least value $\beta$ such that the double inequality

$$\alpha NS(a, b) + (1 - \alpha) H(a, b) < A(a, b) < \beta NS(a, b) + (1 - \beta) H(a, b),$$

(1.11)

hold for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (1.9)-(1.11), we will present the best possible parameters $\alpha_i, \beta_i \in \mathbb{R} (i = 1, 2, 3, 4)$ such that the double inequalities

$$\alpha_1 C(a, b) + (1 - \alpha_1) M_{\tan} (a, b) < A(a, b) < \beta_1 C(a, b) + (1 - \beta_1) M_{\tanh} (a, b),$$

$$\alpha_2 C(a, b) + (1 - \alpha_2) M_{\sinh} (a, b) < A(a, b) < \beta_2 C(a, b) + (1 - \beta_2) M_{\sinh} (a, b),$$

$$\alpha_3 M_{\sin} (a, b) + (1 - \alpha_3) H(a, b) < A(a, b) < \beta_3 M_{\sin} (a, b) + (1 - \beta_3) H(a, b),$$

$$\alpha_4 M_{\tanh} (a, b) + (1 - \alpha_4) H(a, b) < A(a, b) < \beta_4 M_{\tanh} (a, b) + (1 - \beta_4) H(a, b)$$

hold for all $a, b > 0$ with $a \neq b$.
2 Lemmas

In order to prove our main results we need some lemmas, which we present in this section.

Lemma 2.1. (See [5]) Let \(-\infty < a < b < +\infty\), and let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable on \((a, b)\), and \(g'(x) \neq 0\) on \((a, b)\). If \(f'(x) / g'(x)\) is increasing (decreasing) on \((a, b)\), then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If \(f'(x) / g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. The function

\[
f(x) = \tan(x) - x
\]

is strictly increasing from \((0, 1)\) onto \((1/4, [\tan(1) - 1]/[2\tan(1) - 1])\).

Proof. Let \(f_1(x) = \tan(x) - x, f_2(x) = (1 + x^2)\tan(x) - x\). Then elaborated computations lead to

\[
f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0)}{f_2(x) - f_2(0)},
\]

\[
f_1'(x) \quad f_2'(x) = -\frac{\tan^2(x)}{2x \tan(x)} + (x^2 + 1)(1 + \tan^2(x)) - 1 := \varphi(x),
\]

\[
\varphi'(x) = \frac{2 \tan(x)(x^2 \tan^2(x) - \tan^2(x) + x^2)}{(x^2 \tan^2(x) + \tan^2(x) + 2x \tan(x) + x^2)^2} = \frac{2 \tan(x)(x^2 - \sin^2(x))}{(x^2 \tan^2(x) + \tan^2(x) + 2x \tan(x) + x^2)^2} \cos^2(x) > 0,
\]

and

\[
f(0^+) = 1/4, f(1^-) = [\tan(1) - 1]/[2\tan(1) - 1] = 0.2635\ldots.
\]

Therefore, Lemma 2.2 follows easily from (2.1)-(2.3) and Lemma 2.1.

Lemma 2.3. The function

\[
g(x) = \sinh(x) - x
\]

is strictly decreasing from \((0, 1)\) onto \(([\sinh(1) - 1]/[2\sinh(1) - 1], 1/7)\).

Proof. Let \(g_1(x) = \sinh(x) - x, g_2(x) = (1 + x^2)\sinh(x) - x\). Then elaborated computations lead to

\[
g(x) = \frac{g_1(x)}{g_2(x)} = \frac{g_1(x) - g_1(0)}{g_2(x) - g_2(0)},
\]

\[
g_1'(x) = \cosh(x) - 1, \quad g_2'(x) = 2x \sinh(x) + (x^2 + 1) \cosh(x) - 1,
\]

\[
g_1''(x) = \frac{g_1''(x)}{g_2''(x)} = \frac{\sinh(x)}{4x \cosh(x) + 2 \sinh(x) + (x^2 + 1) \sinh(x)} = \frac{1}{4x / \tanh(x) + 2 + (x^2 + 1)}.
\]
It follows from (2.4) C(2.6) and together with the fact that the function $x \mapsto x/\tanh(x)$ is positive and strictly increasing on $(0,1)$, we clearly see that $g_1''(x)/g_2''(x)$ is strictly decreasing on $(0,1)$. Note that
\[
g(0^+) = \frac{1}{7}, \quad g(1^-) = [\sinh(1) - 1]/[2\sinh(1) - 1] = 0.1297 \cdots.
\] (2.7)
Therefore, Lemma 2.3 follows easily from (2.7) and the monotonicity of $g(x)$. \hfill \square

**Lemma 2.4.** The function
\[
h(x) = \frac{x^2 \sin(x)}{x - (1 - x^2) \sin(x)}
\]
is strictly decreasing from $(0,1)$ onto $(\sin(1), 6/7)$.

**Proof.** Let $h_1(x) = x^2 \sin(x), \quad h_2(x) = x - (1 - x^2) \sin(x)$. Then simple computations lead to
\[
h(x) = \frac{h_1(x)}{h_2(x)} = \frac{h_1(x) - h_1(0)}{h_2(x) - h_2(0)}, \quad h_1'(x) = 2x \sin(x) + x^2 \cos(x),\quad h_2'(x) = 1 + (x^2-1) \cos(x) + 2x \sin(x),
\] (2.8)
\[
h_1''(x) = \frac{4x \cos(x) + 2 \sin(x) - x^2 \sin(x)}{-(x^2 - 1) \sin(x) + 4x \cos(x) + 2 \sin(x)}
\]
\[
\phi(x) = \frac{\sin(x)}{4x \cos(x) + 2 \sin(x) - x^2 \sin(x)} = \frac{1}{4x/\tan(x) + 2 - x^2}.
\] (2.10)
It is easy to verify the function $x \mapsto x/\tan(x)$ is positive and strictly decreasing on $(0,1)$, which imply that the function $\phi(x)$ is increasing on $(0,1)$. Follow from (2.8)-(2.9) lead to the conclusion that $h_1''(x)/h_2''(x)$ is strictly decreasing on $(0,1)$.

Note that
\[
h(0^+) = \frac{6}{7}, \quad h(1^-) = \sin(x) = 0.8414 \cdots.
\] (2.11)
Therefore, Lemma 2.4 follows easily from (2.11) and Lemma 2.1 together with the monotonicity of $h(x)$. \hfill \square

**Lemma 2.5.** The function
\[
k(x) = \frac{x^2}{x/\tanh(x) - (1 - x^2)}
\]
is strictly increasing from $(0,1)$ onto $(3/4, \tanh(1))$.

**Proof.** Let $k_1(x) = x^2, k_2(x) = x/\tanh(x) - (1 - x^2)$. Then elaborated computations lead to
\[
k(x) = \frac{k_1(x)}{k_2(x)} = \frac{k_1(x) - k_1(0)}{k_2(x) - k_2(0^+)},
\] (2.12)
\[
k_1'(x) = \frac{2x \sinh^2(x)}{2x \cosh^2(x) + \cosh(x) \sinh(x) - 3x}.
\] (2.13)
Let $k_3(x) = 2x \sinh^2(x), k_4(x) = 2x \cosh^2(x) + \cosh(x) \sinh(x) - 3x$, one has
\[
k_1'(x) = \frac{k_3(x)}{k_4(x)} = \frac{k_3(x) - k_3(0)}{k_4(x) - k_4(0)},
\] 34
The double inequalities
\[ k_1'(x) = \frac{2x \cosh(x) \sinh(x) + \sinh^2(x)}{2x \cosh(x) \sinh(x) + 2 \sinh^2(x)} = 1 - \frac{1}{2x/\tanh(x) + 2}. \] (2.14)

By (2.14) and the function \( x \mapsto x/\tanh(x) \) is positive and strictly increasing on \((0,1)\), we clearly see that \( k_1'(x)/k_2'(x) \) is strictly increasing on \((0,1)\). Note that
\[ k(0^+) = \frac{3}{4}, k(1^-) = \tanh(1) = 0.7615 \ldots \] (2.15)

Therefore, Lemma 2.5 follows easily from (2.15) and Lemma 2.1 together with the monotonicity of \( k(x) \).

\[ \square \]

3 Main Results

**Theorem 3.1.** The double inequalities
\[ \alpha_1 C(a, b) + (1 - \alpha_1) M_{\text{tanh}}(a, b) < A(a, b) < \beta_1 C(a, b) + (1 - \beta_1) M_{\text{tanh}}(a, b), \] (3.1)
\[ \alpha_2 C(a, b) + (1 - \alpha_2) M_{\text{sinh}}(a, b) < A(a, b) < \beta_2 C(a, b) + (1 - \beta_2) M_{\text{sinh}}(a, b), \] (3.2)
hold for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_1 \leq 1/4, \beta_1 \geq [\tan(1) - 1]/[2 \tan(1) - 1] = 0.2635 \ldots, \alpha_2 \leq [\sinh(1) - 1]/[2 \sinh(1) - 1] = 0.1297 \ldots \) and \( \beta_2 \geq 1/7 \).

**Proof.** Since all the bivariate means concerned in Theorem 3.1 are symmetric and homogeneous of degree one, we assume that \( a > b > 0 \). Let \( x = (a - b)/(a + b) \in (0,1) \). Then we making use of (1.4)-(1.5) and (1.7) lead to the conclusion that inequalities (3.1) and (3.2) are respectively equivalent to
\[ \alpha_1 < \frac{A(a, b) - M_{\text{tanh}}(a, b)}{C(a, b) - M_{\text{tanh}}(a, b)} = \frac{\tan(x) - x}{(1 + x^2) \tan(x) - x} := f(x) < \beta_1, \] (3.3)
\[ \alpha_2 < \frac{A(a, b) - M_{\text{sinh}}(a, b)}{C(a, b) - M_{\text{sinh}}(a, b)} = \frac{\sinh(x) - x}{(1 + x^2) \sinh(x) - x} := g(x) < \beta_2, \] (3.4)
where \( f(x) \) and \( g(x) \) are defined as in Lemmas 2.2 and 2.3.

Therefore, Theorem 3.1 follows easily from (3.3), (3.4) together with Lemmas 2.2 and 2.3.

\[ \square \]

**Theorem 3.2.** The double inequalities
\[ \alpha_3 M_{\text{sin}}(a, b) + (1 - \alpha_3) H(a, b) < A(a, b) < \beta_3 M_{\text{sin}}(a, b) + (1 - \beta_3) H(a, b), \] (3.5)
\[ \alpha_4 M_{\text{tanh}}(a, b) + (1 - \alpha_4) H(a, b) < A(a, b) < \beta_4 M_{\text{tanh}}(a, b) + (1 - \beta_4) H(a, b), \] (3.6)
hold for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_3 \leq \sin(1) = 0.8414 \ldots, \beta_3 \geq 6/7, \alpha_4 \leq 3/4 \) and \( \beta_4 \geq \tanh(1) = 0.7615 \ldots \).

**Proof.** Since all the bivariate means concerned in Theorem 3.2 are symmetric and homogeneous of degree one, we assume that \( a > b > 0 \). Let \( x = (a - b)/(a + b) \in (0,1) \). Then we making use of (1.4)-(1.5) and (1.7) lead to the conclusion that inequalities (3.5) and (3.6) are respectively equivalent to
\[ \alpha_3 < \frac{A(a, b) - M_{\text{sin}}(a, b)}{M_{\text{sin}}(a, b) - H(a, b)} = \frac{x^2 \sin(x)}{x - (1 - x^2) \sin(x)} := h(x) < \beta_3, \] (3.7)
\[ \alpha_4 < \frac{A(a, b) - M_{\text{tanh}}(a, b)}{M_{\text{tanh}}(a, b) - H(a, b)} = \frac{x^2}{x/\tanh(x) - (1 - x^2)} := k(x) < \beta_4, \] (3.8)
where \( h(x) \) and \( k(x) \) are defined as in Lemmas 2.4 and 2.5.

Therefore, Theorem 3.2 follows easily from (3.7), (3.8) together with Lemmas 2.4 and 2.5.

\[ \square \]
4 Conclusion

In this paper, we used mathematical analysis method and the monotonicity of the functions to study the arithmetic mean of some Seiffert-like functions, and obtained some optimal bounds of these arithmetical means.

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Competing Interests

The authors declare that they have no competing interests.

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