Chemical potential, microstructures and phase transition of the Schwarzschild-AdS black hole

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Abstract

In the AdS/CFT correspondence, a variable cosmological constant $\Lambda$ in the bulk corresponds to varying the number of colors $N$ in the boundary gauge theory, with chemical potential $\mu$ as its thermodynamic conjugate. In this work, within the context of $AdS_5 \times S^5$ and its dual $\mathcal{N} = 4$ SUSY Yang-Mills theory at large $N$, we investigate the nature of microstructures and phase transitions by studying the Ruppeiner curvature $R$. We find that the large black hole branch is associated with purely attractive microstructures and its behaviour bears some qualitative similarities with that of an ideal Bose gas ($R < 0$ and $\mu < 0$). It is seen that as the system’s fugacity approaches unity in the large black hole branch, $R$ takes increasingly negative values signifying long range correlations and strong quantum fluctuations. The small black hole branch on the other hand, is attractive at low temperatures and repulsive at high temperatures with a second order critical point which roughly separates the two regions. The scaling behaviour associated with the specific heat and the thermodynamic curvature around this second order critical point is obtained.

1 Introduction

The formulation of the laws of black hole mechanics in direct analogy with those of thermodynamics [1]-[6] has paved the way for a large body of work in this subject in last few decades. In particular, the Hawking-Page (HP) transition [7] in black holes asymptotic to AdS spacetimes have attracted much attention with interpretation involving AdS/CFT correspondence [8]-[11], as a confinement-deconfinement phase transition from the dual field theory side [11]. Particularly, the study of black hole phase transitions continues to provide a rich arena for exploring the connections between gravity and gauge theories (see [12, 13]).
The proposal of treating the negative cosmological constant of the AdS spacetime as dynamical (starting from earlier works in literature, e.g., [14]-[21]) has lead to a flurry of research activity over the last decade. However, the interpretation of the cosmological constant $\Lambda$ as a pressure $P$ and subsequently associating the notion of a thermodynamic volume with a black hole horizon has been a rather recent development [22] (see also [23, 24]), with at least two possible approaches. In the first approach, quite remarkably, for non-rotating black holes in four or higher dimensions, this thermodynamic volume $V$ exactly corresponds to the geometric volume of the black hole horizon. In this framework, most black holes have been associated with a $P - V$ critical behaviour similar to that of the van der Waals' fluid (see for example [25]-[28]) with enthalpy $H(S,P)$ playing the central role, where, $S$ is entropy. However, a dual field theoretic interpretation of such phase transitions is not developed yet and one could ask, what is the interpretation of the dynamical cosmological constant on the dual conformal field theory (CFT) at the boundary? To this end, there is a second approach which gives a more natural interpretation of a variable cosmological constant as arising from a variation of number of colours of the boundary CFT or equivalently, related to the number $N$ of branes within the AdS bulk [8] (also see [29, 30]). Progress in this approach has been slow, though the setting is very interesting as one can understand the physics of CFT a bit better. In this context, it is possible to envisage a situation where one introduces a chemical potential $\mu$ conjugate to a ‘variable’ $N$, to go with other thermodynamical potentials (such as, electric potential for charged systems and so on). The study of the chemical potential $\mu$ in this set up is expected to be interesting. For instance, if one invokes the microscopic interpretation from standard thermodynamics, $\mu$ is known to be negative for a classical ideal Bose gas and positive if there is a sufficiently strong repulsive interaction among particles or in fermionic systems at low temperatures. In particular, $\mu$ can be zero when the average thermal de Broglie wavelength of particles is comparable to the inter particle separation, signifying the onset of quantum effects in the system. Indeed, in a remarkable suggestion in the context of Schwarzschild black holes in AdS [30], Dolan noted that the temperature at which $\mu$ approaches zero is close to the point where the system undergoes Hawking-Page transition and proposed this to correspond to Bose condensation. This was improved further in [33], who showed that $\mu = 0$ might be reached in various black holes in AdS, above the HP transition temperature, if one makes use of densities instead of absolute quantities.

On the otherhand, the use of Ruppeiner geometry in studying thermodynamics and phase transitions is well known, with the metric defined as [36],

$$g_{ij} = -\partial_i \partial_j S,$$  \hspace{1cm} (1.1)

where $S$ is the entropy of the system. At a finite temperature, the associated curvature scalar typically diverges when the system approaches a phase transition point. Ruppeiner geometry has been worked out for a wide range of thermodynamic systems including magnetic systems [37, 38] and quantum gases [39]. The magnitude of Ruppeiner curvature has been shown to provide physical insights on fluctuations and the stability of a given system [39]-[41]. Divergence of the Ruppeiner curvature indicates a strongly correlated behaviour among the microstructures (see for example [41]). In black hole thermodynamics, Ruppeiner geometry was first studied in ref [42] over two decades ago where it was applied to understand the thermodynamic behaviour.
of BTZ black holes. Subsequently, the Ruppeiner geometry for RN, Kerr and RN-AdS black holes has been studied in several works (see for example [43]-[46]; also see [47]) and it was also shown that the divergence of the curvature scalar is consistent with the Davies phase transition point [48] (also see [49, 50]). However, more recently there has been a considerable interest in the use of Ruppeiner geometry for probing the nature of microscopic interactions (see for example [51]-[67] for some selected recent works and references therein) in black holes, mostly from the dynamical cosmological constant picture. Since it is well known that black holes have a temperature and an entropy, it is suggestive to associate microstructures with them which share the degrees of freedom of the entropy. A positive Ruppeiner curvature has been shown to indicate dominance of repulsive interactions as is the case of charged and/or rotating BTZ black holes (see [64, 65] and references therein) whereas, a negative one indicates attractive interactions as in the case of Schwarzschild-AdS black holes [58]. Black holes in four or more dimensions carrying electric charge have been shown to be associated with both attractive and repulsive interactions among the microstructures [59]-[63]. However for AdS black holes, most of this work has been done in the setup where the cosmological constant is interpreted as the thermodynamic pressure. The primary motivation of this work is to fill the gap by studying the behaviour of the black hole microstructures in the alternate second approach, when the cosmological constant is related to the number of colours on the boundary CFT and we find a remarkable behaviour associated with both the small and large Schwarzschild-AdS black hole.

In this work, we set up Ruppeiner geometry in the novel \((s,N^2)\)-plane, where \(s\) is the entropy density, to probe the behaviour of the interacting black hole microstructures and also phase transitions in Schwarzschild-AdS black holes. Let us note that methods of Ruppeiner geometry have been applied earlier to studying phase transitions and critical behaviour of Schwarzschild and Reissner-Nordström black holes [31, 32, 34]. In particular, in [31], different metrics such as the ones given by Ruppeiner, Weinhold and Quevedo were used to check whether thermodynamic geometry captures the zeros and divergences of heat capacity at constant chemical potential. However, as mentioned above, the issue of microstructures and the nature of their interactions, along the lines of recent works on the \((T,V)\)- and \((S,P)\)-planes [57]-[67] has not been undertaken yet in the new plane being studied here, where the pairs \((S,N^2)\) or \((T,\mu)\) are chosen as fluctuation coordinates. As we show in this work, the study of thermodynamic geometry in these new planes is quite important as apart from capturing the divergences of specific heats, the classification of attraction and repulsion dominated regions is useful while probing the nature of phase transitions.

The plan of the rest of the paper is as follows. In the following section-[2], we set up the thermodynamics for the five dimensional Schwarzschild-AdS and present some results which would be required for our analysis of Ruppeiner geometry. In particular, we carefully scrutinise the behaviour of the chemical potential and the associated fugacity function. Section-[3], contains the main results of this work, with the expression and plots of the Ruppeiner curvature and its physical interpretation primarily in the context of black hole microstructures. It is shown that the large black hole branch qualitatively resembles an ideal Bose gas with fluctuations getting significant as the fugacity approaches unity. We also analyze the second order critical point in the small black hole branch. Finally, the work is concluded with some remarks in section-[4].
2 Thermodynamics with chemical potential

We shall for our purposes of this paper, consider the Schwarzschild black hole in $AdS_5$ coming from the $D = 10$ maximal supergravity which is compactified on $S^5$. More explicitly, one considers the type IIB supergravity on $AdS_5 \times S^5$ which corresponds to the thermal $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory at large $N$ on the dual side. The decoupled ten dimensional metric takes the form (see for example [12]),

$$ds_{(10)}^2 = ds_{(5)}^2 + l^2 \sum_{i=1}^{3} \left[ d\theta_i^2 + \theta_i^2 (d\phi_i + \frac{2}{\sqrt{3}} A_\nu dx^\nu)^2 \right],$$

where $\nu = 0, 1, 2, 3, 4$. The terms within the summation correspond to the $S^5$ part with $\{\theta_i\}$ being the direction cosines and $\{\phi_i\}$ being the rotation angles on $S^5$ whereas $ds_{(5)}^2$ is the Schwarzschild metric asymptotic to $AdS_5$ given as,

$$ds_{(5)}^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega_3^2,$$

with $d\Omega_3^2$ being the line element on the maximally symmetric Einstein space in three dimensions with a spherical topology and enclosing a unit volume. The lapse function $f(r)$ is,

$$f(r) = 1 - \frac{8G_{(5)}M}{3\pi r^2} + \frac{r^2}{l^2},$$

where $l$ is the radius of the AdS$_5$ spacetime and $G_{(5)}$ is Newton’s constant in five dimensions. The black hole mass is calculated from the equation, $f(r_+) = 0$ where $r_+$ is the event horizon radius. Subsequently, the thermodynamics of the five dimensional Schwarzschild-AdS black hole can be discussed setting the mass to be equal to an appropriate thermodynamic potential, as is done in black hole thermodynamics.

We now briefly discuss some thermodynamic features of the five dimensional Schwarzschild-AdS black hole. The black hole mass is obtained from the metric to be,

$$M = \frac{3\pi^4 r_+^4 l^5}{8G_{(10)}} \left( 1 + \frac{r_+^2}{l^2} \right),$$

where $G_{(10)}$ is the fixed Newton’s constant in ten dimensions which we from now on set equal to unity, i.e. $G_{(10)} = 1$. The entropy of the black hole is given by the Bekenstein-Hawking entropy formula which in $d = 5$ gives

$$S = \frac{\pi^2 r_+^3}{2G_{(5)}} = \frac{\pi^3 l^5 r_+^3}{2}.$$

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1. Being direction cosines, they are not all independent and satisfy $\sum_{i=1}^{3} \theta_i^2 = 1$.
2. Some of these discussions have appeared earlier in the literature. See [30]-[34] for some earlier works.
3. The Newton’s constant in five dimensions is related to that in ten dimensions as, $G_{(5)} = G_{(10)}/\pi^3 l^5$. Therefore, $G_{(5)}$ depends on $l$ and is not strictly constant.
4. We shall set $\hbar = k_B = 1$ for all subsequent discussions. Consequently, the (fixed) ten dimensional Planck length, $l_P = \hbar G_{(10)} = 1$ and $G_{(5)} = G_{(10)}/\pi^3 l^5 = 1/\pi^3 l^5$. 

4
Further, we note that the number of D3-branes $N$ is related to $l$ as,

$$N = \frac{\pi^2 l^4}{\sqrt{2}}. \quad (2.6)$$

This means that a dynamical $\Lambda$ leads to the notion of varying the number of branes in the bulk. On the boundary CFT however, which in this case is the $\mathcal{N} = 4$ SYM theory, $N$ is the rank of the gauge group $SU(N)$. Noting that the number of degrees of freedom in the large $N$ limit scales as $N^2$ rather than $N$ \cite{35}, we would consider $N^2$ as the thermodynamic variable associated with the dynamical cosmological constant.

The black hole mass in eqn (2.4) can be written down as a function of $S$ and $N^2$ using eqns (2.5) and (2.6) as: $M = M(S, N^2)$. The first law of black hole thermodynamics reads,

$$dM = TdS + \mu dN^2, \quad (2.7)$$

where $\mu$ is a suitable thermodynamic conjugate to $N^2$ and bears its interpretation as a chemical potential. Noting however that $N^2$ corresponds to the number of degrees of freedom which are associated with an effective energy $\mu N^2$, we define the internal energy of the black hole as $U = M - \mu N^2$, which bears a close resemblance with the identification of the black hole mass with its enthalpy as is done in conventional extended black hole thermodynamics \cite{22}. We would, following \cite{32, 33} consider for our purposes, the densities of the extensive thermodynamic quantities $M$ (and hence all other thermodynamic potentials defined via Legendre transformations of $M$) and $S$ by scaling them down by a factor of $l^3$ since for the five dimensional bulk of $AdS_5$ its dual CFT is associated with a volume which apart from constant factors goes as, $V \sim l^3$. The mass density $\rho$ is then given as,

$$\rho = \frac{3s^{2/3} \left( N^{5/6} + 2\sqrt{2}s^{2/3} \right)}{4 \times 2^{2/3}\pi N^{2/3}}, \quad (2.8)$$

which satisfies the first law of black hole thermodynamics expressed in terms of densities \cite{5},

$$d\rho = Tds + \mu dN^2, \quad (2.9)$$

where $s$ is the entropy density obtained from eqn (2.5). The Hawking temperature is given by,

$$T = \left( \frac{\partial \rho}{\partial s} \right)_{N^2} = \frac{N^{5/6} + 4\sqrt{2}s^{2/3}}{2 \times 2^{2/3}\pi N^{2/3}\sqrt{s}}. \quad (2.10)$$

The Hawking temperature has a minimum value, $T_{min}$ which occurs at $s_0 = 0.0883883$. For any temperature above $T_{min}$, there are two values of $s$ with the same temperature: $s < 0.0883883$ corresponds to the small black hole branch where as $s > 0.0883883$ corresponds to the large black hole branch.

\footnote{Note that the $\mu$ appearing in eqns (2.7) and (2.9) cannot have the same functional form because we haven’t scaled down $N^2$ by a factor of $l^3$. They are naively labelled the same just for notational ease and for the rest of the paper, we would be considering the chemical potential $\mu$ to be the one defined from eqn (2.9).}
2.1 Chemical potential

Since, a density of \( N^2 \) has not been considered, then for the first law \([\text{eqn}(2.9)]\) to hold good one has to define the chemical potential as,

\[
\mu =: \left( \frac{\partial \rho}{\partial N^2} \right)_s = \frac{\sqrt{2}N^{5/6}s^{2/3} - 8 \times 2^{2/3}s^{4/3}}{32\pi N^{8/3}}.
\]

(2.11)

Figure-(1) shows the variation of the chemical potential as a function of the temperature. The large and small black hole branches are indicated. The chemical potential is negative definite in the large black hole branch and monotonically goes to negative infinity as \( T \to \infty \) (or equivalently \( s \to \infty \)). It can be checked that the chemical potential goes to zero exactly at the Hawking-Page temperature\(^6\) \( T_{HP} = 0.477465 \) as also noted in \([33]\). However, one has to be careful since although, the Hawking-Page transition happens in the large black hole branch at entropy density \( s_{HP} = 0.25 \) whereas, the point \( \mu = 0 \) lies in the small black hole branch with \( s_{\mu=0} = 0.0312 \). They are therefore not the same physical point. The chemical potential being negative definite, i.e. \( -\infty > \mu > 0 \) for the large black hole branch is a characteristic of an ideal non-relativistic Bose gas. It shall subsequently be shown below through the study of microstructures that the large black hole branch interact in a purely attractive manner, similar to the case of an ideal Bose gas (see \([39]\)). It is helpful to define fugacity of the system for later purpose as:

\[
z = e^{\beta \mu},
\]

(2.12)

where \( \beta = 1/T \) is the inverse temperature factor. For a quantum gas, the limit \( z \to 0 \) is the classical limit where quantum effects become completely inconsequential. The classical limit is reached by the large black hole as \( T \to \infty \) or equivalently \( \mu \to -\infty \). The other extreme limit for an ideal gas of Bosons is \( z \to 1 \) where quantum effects become the most significant and the gas is macroscopic.

\(^6\)For general \( N \), one has \( T_{HP} = 0.477465 N^{-1/4} \). We will set \( N^2 = 1 \) in subsequent plots.
undergoes a Bose-Einstein condensation. The fugacity for the large black hole is plotted as a function of temperature in figure-(2) and it is clear that $z \in (0, 1)$. We note that since the point

\[ \mu = 0 \]

does not lie in the large black hole branch, the fugacity does not exactly reach unity. Consequently, there is no Bose condensation-like phenomena for the large black hole. Rather, the limit $z \to 1$ can be reached by the black hole and it will be argued in the next section that such a situation corresponds to strong fluctuations among the microstructures of the large black hole.

The small black hole branch shows a much more interesting behaviour of the chemical potential. There is a zero crossing at $s_{\mu=0} = 0.0312$ such that for black holes smaller than that point, i.e. $s < s_{\mu=0}$, the chemical potential is positive and vice versa. In terms of the Hawking temperature, black holes colder than the $\mu = 0$ case are associated with negative chemical potential whereas the hotter ones have positive chemical potentials. The fugacity is plotted in figure-(3) and reaches unity exactly at the point where $\mu = 0$ which corresponds to the temperature $T_{\mu=0} = 0.477465$. For $T > T_{\mu=0}$, the fugacity is positive and close to unity while if $T_{\text{min}} < T < T_{\mu=0}$, fugacity can get negative (not shown in the plot) but is still close to unity.

2.2 Behaviour of the specific heats

We shall next briefly comment on the behaviour of the specific heats. From the first law [eqn (2.9)], one can obtain the specific heat density at constant $N^2$ as,

\[ C_{N^2} = \left( \frac{\partial \rho}{\partial T} \right)_{N^2} = -\frac{3s\left(N^{5/6} + 4\sqrt{2} s^{2/3}\right)}{N^{5/6} - 4\sqrt{2} s^{2/3}}. \]  

(2.13)

\[ ^{7} \text{Incidentally, this temperature happens to have the same numerical value as } T_{HP}. \]

\[ ^{8} \text{Since we are working with densities of the thermodynamic potentials, the specific heat so defined is essentially the specific heat density. Having said that, we will from now on refer to them simply as specific heats expecting the reader to understand without confusion that a density it is implied.} \]
Fugacity (z) vs T: AdS Schwarzschild (D=5)

Figure 3: Fugacity of the five dimensional Schwarzschild-AdS large small hole as a function of the Hawking temperature.

which is plotted as a function of the temperature in figure-(4). It has a divergence at $T = T_{\text{min}}$ and is positive for large black hole while it is negative for the small black hole as also noted in [7]. The point of divergence therefore separates the positive and negative branches of the specific heat $C_{N^2}$. Moreover, since from eqn (2.9), one can also define $C_{N^2}$ as the second derivative of the Gibbs free energy density $g = \rho - Ts$ as,

$$C_{N^2} = -T \left( \frac{\partial^2 g}{\partial T^2} \right)_{N^2},$$

(2.14)

one concludes that the divergence of $C_{N^2}$ is reminiscent of a second order phase transition, much like that pointed out by Davies [18] which separates the positive and negative regions of the specific heat. Particularly of interest in our case is the specific heat at constant chemical potential, i.e. $C_\mu$. We define the internal energy density of the black hole from its mass density

Figure 4: Specific heat at constant $N^2$ for the five dimensional Schwarzschild-AdS black hole as a function of the Hawking temperature.
as, \( u = \rho - \mu N^2 \) and then the first law [eqn (2.9)] is modified to,

\[
du = T ds - N^2 d\mu,
\]

leading to the straightforward definition of \( C_\mu \) as,

\[
C_\mu = \left( \frac{\partial u}{\partial T} \right)_\mu,
\]

or equivalently,

\[
C_\mu = -T \left( \frac{\partial^2 f}{\partial T^2} \right)_\mu,
\]

where \( f = u - Ts \) is the Helmholtz free energy density. Subsequently, \( C_\mu \) can be calculated to give,

\[
C_\mu = -\frac{\frac{512}{3} N^{2/3} s^{7/3}}{N^{5/6}} + 11 N^{5/6} s - 84 \sqrt{2} s^{5/3} \frac{3 N^{5/6} - 36 \sqrt{2} s^{2/3}}{3 N^{5/6} - 36 \sqrt{2} s^{2/3}}.
\]

The variation of \( C_\mu \) as a function of temperature is shown in figure-(5). We should bear in mind that \( C_\mu \) is not the standard specific heat of the black hole defined directly from the mass (in this case, the mass density) and therefore, its behaviour is expected to be atypical. As can be seen from figure-(5), it is almost always negative for both the branches. Further, exists a point of divergence at \( T_{C_\mu=0} = 0.519798 \) in the small black hole branch which might be taken to be an indication of a phase transition. It will be shown subsequently that such a point exactly corresponds to a second order critical point of the Schwarzschild-AdS small black hole.
3 Ruppeiner geometry, interacting microstructures and phase transitions

We shall now study the Ruppeiner geometry for the five dimensional Schwarzschild-AdS black hole whose thermodynamics was briefly explored in the previous section. We note that the Ruppeiner metric is defined as the negative Hessian of the entropy (here, entropy density) so that the corresponding line element becomes,

\[ dl_R^2 = - \frac{\partial^2 s}{\partial x^i \partial x^j} dx^i dx^j, \]  

(3.1)

where \( i, j \in \{1, 2, \ldots, n\} \) and \( \{x^i\} \) are independent coordinates in the thermodynamic phase space or more precisely the subspace of thermodynamic equilibrium states which are allowed to fluctuate. For our purposes, we shall require \( n = 2 \), i.e. two fluctuation coordinates in the thermodynamic phase space. In that case, the metric shall have components denoted by \( g_{11}, g_{12}, g_{21} \) and \( g_{22} \) with \( g_{12} = g_{21} \) as a consequence of the symmetry of the metric tensor. The line element can then be expressed as,

\[ dl_R^2 = g_{11}(dx^1)^2 + 2g_{12}(dx^1)(dx^2) + g_{22}(dx^2)^2, \]  

(3.2)

with,

\[ g_{ij} = - \frac{\partial^2 s}{\partial x^i \partial x^j}, \quad i, j = 1, 2. \]  

(3.3)

For such a two dimensional metric, the scalar curvature can be written down as [68],

\[ R = - \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial x^1} \left( g_{12} \frac{\partial g_{11}}{\sqrt{g}} \frac{\partial^2 s}{\partial x^2} - \frac{1}{\sqrt{g}} \frac{\partial g_{22}}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left( \frac{2}{\sqrt{g}} \frac{\partial g_{12}}{\partial x^2} - \frac{1}{\sqrt{g}} \frac{\partial g_{11}}{\partial x^2} - \frac{g_{12}}{g_{11} \sqrt{g}} \frac{\partial^2 s}{\partial x^1 \partial x^1} \right) \right]. \]  

(3.4)

Here \( g \) is the determinant of the metric tensor. A straightforward comparison of the first law of conventional extended black hole thermodynamics, i.e. \( dM = TdS + VdP \) with eqn (2.7) leads to the correspondence \((\mu, N^2) \rightarrow (V, P)\). We shall perform our calculations on the direct analogue of the \((S, P)\)-plane, which in this case and in terms of densities is the \((s, N^2)\)-plane, i.e. with fluctuation coordinates \( s \) and \( N^2 \). The fundamental thermodynamic potential is \( \rho = \rho(s, N^2) \) and the line element can be expressed without much difficulty as,

\[ dl_R^2 = \frac{1}{C_{N^2}} (ds)^2 + 2 \left( \frac{\partial T}{\partial N^2} \right)_s (ds)(dN^2) + \frac{1}{T} \left( \frac{\partial \mu}{\partial N^2} \right)_s (dN^2)^2. \]  

(3.5)

The analogue of the \((T, V)\)-plane in our case is the \((T, \mu)\)-plane on which the thermodynamic potential is \( f(T, \mu) = u - Ts = \rho - \mu N^2 - Ts \). The line element is obtained to be,

\[ dl_R^2 = - \frac{C_{N^2}}{T^2} (dT)^2 - 2 \left( \frac{\partial N^2}{\partial T} \right) (dT)(d\mu) - \frac{1}{T} \left( \frac{\partial N^2}{\partial \mu} \right)_T (d\mu)^2. \]  

(3.6)

\[ ^9 \text{More geometrically, the thermodynamic phase space is five dimensional in this case and spaces the thermodynamic equilibrium states corresponding to the system are all two dimensional Legendre submanifolds (see for example [63]).} \]
One can similarly consider the other two planes, i.e. with \((s, \mu)\) and \((T, N^2)\) as fluctuation coordinates. The curvature scalars can be shown to be equivalent by direct calculation and hence we use the notation \(R\) without explicit reference to the plane on which it is calculated. The divergence of the Ruppeiner curvature can be shown to signal critical behaviour (or extremality if one considers charged black holes) if it exists for a black hole. In the limit that \(R \to \infty\), it is therefore expected that the system gets strongly correlated. Indeed, it is now known [41] that \(|R| \sim \xi^d\) for a system of \(d\) spatial dimensions with \(\xi\) being the correlation length meaning that the divergence of the Ruppeiner curvature can therefore be used as a test probe find out possible critical points in a black hole system.

It has long been observed that \(R = 0\) for non-interacting systems such as the ideal gas whereas, a non-zero Ruppeiner curvature indicates presence of non-trivial interactions. For systems with dominant attraction, such as a van der Waals gas or an ideal Bose gas, one gets \(R < 0\) whereas for a repulsive system such as an ideal Fermi gas, one has \(R > 0\). This allows one to probe using the sign of \(R\), the nature of dominant interactions among black hole microstructures. Further, since it is well known that the Ruppeiner metric is closely related to thermodynamic fluctuations, it has been argued (see for example [39] although their sign convention is opposite to ours) that the Ruppeiner curvature \(R\) be used as a measure of the stability of a thermodynamic system against fluctuations. We shall next closely examine the physical implications of \(R\) for the five dimensional Schwarzschild-AdS black hole.

### 3.1 Ruppeiner geometry for the \(D = 5\) Schwarzschild-AdS black hole

For the present case, the Ruppeiner curvature can be calculated to be,

\[
R = \frac{8 \left( 40 \frac{2^{2/3} N^{5/3} s^{2/3}}{3N^{5/6} \sqrt{s}} + 160 N^{5/6} s^{4/3} - \frac{5^{3/2} N^{5/2}}{\sqrt{2}} + \frac{768}{\sqrt{2}} s^2 \right)}{N^{5/6} \frac{\sqrt{3}}{6} \left( N^{5/6} - 12 \frac{\sqrt{2}}{s^{2/3}} \right)^2} 
\]

(3.9)

It has been plotted as a function of \(s\) is figure-(6). The curvature initially starts out as positive for small values of \(s\) within the small black hole branch and has an infinite discontinuity at the point \(s_{R=\infty} = 0.0170103\) which happens to be the same point at which \(C_\mu\) diverges. The Ruppeiner curvature then has a zero crossing at \(s_{R=0} = 0.0214457\) and then becomes

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10On the \((s, \mu)\)-plane, the thermodynamic potential is \(u(s, \mu) = \rho - \mu N^2\) and the line element reads,

\[
dl^2_R = \frac{1}{C_\mu} (ds)^2 + \frac{2}{T} \left( \frac{\partial T}{\partial \mu} \right)_s (ds)(d\mu) - \frac{1}{T} \left( \frac{\partial N^2}{\partial \mu} \right)_s (d\mu)^2, \tag{3.7}
\]

whereas, on the \((T, N^2)\)-plane the thermodynamic potential is, \(g(T, N^2) = \rho - Ts\) and the corresponding line element can be shown to be given as,

\[
dl^2_R = -\frac{C_{N^2}}{T^2} (dT)^2 - \frac{2}{T} \left( \frac{\partial s}{\partial N^2} \right)_T (dT)(dN^2) + \frac{1}{T} \left( \frac{\partial \mu}{\partial N^2} \right)_T (dN^2)^2. \tag{3.8}
\]

11In fact, the notion of the Ruppeiner metric can be derived by inverting the well known relation, \(S = \ln \Omega\). See [62] for instance.
negative for higher values of $s$. All of this happens in the small black hole branch while the chemical potential is positive. For an even higher entropy density $s_{\mu=0} = 0.0312$, the chemical potential vanishes and further takes negative values. The large black hole branch begins at $s_0 = 0.0883883$ and the Hawking-Page transition happens at $s_{HP} = 0.25$ which corresponds to the $r_+ = l$ point. The Ruppeiner curvature can also be plotted with respect to the temperature in which the small and large black hole branches can be seen separately, as is shown in figure-(7).

As can be easily seen, nothing very remarkable happens in the large black hole branch as far as Ruppeiner curvature is concerned even though the Hawking-Page transition happens in this
The Ruppeiner curvature is negative for the large black hole indicating presence of attractive interactions among the black hole microstructures. Moreover, the chemical potential is negative definite in this branch tempting us to interpret the Schwarzschild-AdS large black hole as the black hole analogue of the ideal Bose gas. The curvature scalar asymptotically goes to zero as $T \to \infty$ (equivalently $s \to \infty$) which results in the fugacity, $z \to 0$ which is the limit in which the black hole behaves as a classical ideal gas. This is consistent because the classical ideal gas is non-interacting and hence, $R = 0$. In the low temperature limit of the large black hole, one has $z \to 1$ which in principle should indicate strong quantum mechanical behaviour for an ideal Bose gas. The Ruppeiner curvature is plotted in figure-(8) as a function of the fugacity and it can be seen that as $z$ approaches unity, $R$ rapidly takes large negative values. This can be taken to indicate the presence of strong quantum fluctuations leading to long range correlations within the system as was argued in ref [39]. However, the case $\mu = 0$ is not reached by the large black hole branch and therefore there is no Bose condensation involved.

We shall next discuss the small black hole branch, which seems to be more interesting. As far as nature of interactions among the microstructures is concerned, the small black hole starts out as a repulsive system at very small $s$, i.e. high $T$. Exactly at $s_{R=\infty} = 0.0170103$, the Ruppeiner curvature shows a divergence and so does the specific heat $C_\mu$. Since the system has no extremal point, such a divergence is expected to signal phase transition of the black hole. Such a point can be therefore be interpreted as a second order critical point. It should be remarked that there are no first order lines which terminate at the second order critical point.

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12 The Ruppeiner curvature doesn’t seem to capture the physics of the Hawking-Page transition.
13 A similar remark was made earlier in ref [31]. However, no further investigation on the nature of this point was pursued.
point as is the case for the familiar liquid-gas transition. The temperature at which \( R \) diverges is given by \( T_c = 0.519798 \) and shall be called the critical temperature for the small black hole. Re-expressing the specific heat \( C_\mu \) given in eqn (2.18) in terms of the temperature as,

\[
C_\mu = -\frac{256 \frac{2^{2/3}}{N^{3/2}} N^5}{384 (3 2^{2/3} A^{2/3} - 2 N^{5/6})},
\]

where,

\[
A = -N^{5/4} \sqrt{\left( \pi^2 \sqrt{NT^2} - 2 \right) \left( 2\pi^2 \sqrt{NT^2} - 1 \right)^2 - 3\pi N^{3/2}T + 2\pi N^2T^3},
\]

it can be verified that upon defining \( t = T/T_c - 1 \), the specific heat diverges at \( t = 0 \) as,

\[
C_\mu \sim |t|^{-1},
\]

which gives the value of the critical exponent \( \alpha = 1 \). It is also straightforward to show that the Ruppeiner curvature exhibits the following scaling behaviour around the critical point \( t = 0 \),

\[
R \sim |t|^{-2}.
\]

which is the same as that noted in refs 34, 60, 69, 70 in slightly different contexts. Consequently, we expect that the correlation length also scales with temperature around the critical point. The Schwarzschild-AdS small black hole therefore, admits a second order critical point around which both the specific heat \( C_\mu \) and the Ruppeiner curvature \( R \) admit scaling behaviour and diverge exactly at the critical temperature. Further, at a slightly lower entropy density \( s_{R=0} = 0.0214457 \) corresponding to the temperature \( T_{R=0} = 0.501256 \), the Ruppeiner curvature crosses zero and then becomes negative indicating dominance of attractive interactions (see figure -6). Therefore, the second order critical point roughly separates the repulsion dominated and attraction dominated regimes for the small black hole. The two temperatures differ by,

\[
\frac{T_c - T_{R=0}}{T_c} = 0.03567,
\]

i.e. approximately by 3.56%. For small black holes with temperature \( T_{R=0} < T < T_{\text{min}} \), attractive interactions are dominant. Therefore, the small black hole is repulsive at low temperatures while it is attractive at high temperatures with a switching between the two regions taking place exactly at \( T = T_{R=0} \). While all of this happens in the \( \mu > 0 \) region, one can qualitatively explain this behaviour by considering the small black hole to be associated with both attractive and repulsive microstructures which share the degrees of freedom of the total black hole entropy (see also 59, 63). At a given temperature (and hence at a particular \( s \)), the relative number densities of microstructures of the two kinds would dictate which kind of interaction would be dominant in the system explaining the presence of both repulsion and attraction dominated regions.

\[\text{The critical exponent } \alpha \text{ is defined as, } C \sim |T - T_c|^{-\alpha}.\]
4 Remarks

In this work, we have studied the thermodynamic geometry of the five dimensional Schwarzschild-AdS black hole in the context where the dynamical cosmological constant is treated to correspond to number of colours in the dual gauge theory. The main focus was to get an understanding of the nature of interactions of microstructures through the behaviour of Ruppeiner curvature and comparing it with the behaviour of chemical potential, for large and small black hole branches of the black hole, respectively. Our results are summarised as follows,

- The large black hole is associated with a chemical potential $\mu$ which can only assume negative values. This means that the fugacity $z$ always lies between 0 and 1. We have shown that in the limit $z \to 0$, the Ruppeiner curvature vanishes indicating non-interacting behaviour among the microstructures which is consistent with the interpretation that it is a classical limit for the large black hole.

- The large black hole is associated with attractive interactions among the microstructures ($R < 0$) in general away from the quantum limit. In the past, it has been argued in the context of quantum gases, that the divergence of Ruppeiner curvature could indicate an instability among the microstructures and possibly correspond to Bose condensation. In the context of black holes in AdS, the limit $z \to 1$ has been proposed to correspond to Bose condensation point as well [30]. In the present case though $R$ takes increasingly negative values, if one considers the limit $z \to 1$, indicating the dominance of long range correlations and quantum mechanical fluctuations similar to the case of an ideal gas of Bosons. However, since no divergence of $R$ (and hence $\xi$) is observed where $z \approx 1$, it seems more appropriate to understand the $z \to 1$ limit as a strongly correlated state of the large black hole.

- The small black hole branch is associated with both attraction and repulsion dominated regions approximately separated by a second order critical point at which the specific heat $C_\mu$ diverges.

- Around the second order critical point of the small black hole, both the specific heat and the Ruppeiner curvature (hence, the correlation length) exhibit scaling behaviours, $C_\mu \sim |t|^{-1}$ and $R \sim |t|^{-2}$ where $t$ is the reduced temperature.

It would further be interesting to see the effects of charge on the black holes as well as higher curvature terms in the action on the microstructures of the large and small black holes in this set up, although we can expect that some qualitative features shall remain the same as discussed in this paper.

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