The best constant in the embedding of $W^{N,1}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$

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Abstract

We compute the best constant in the embedding of $W^{N,1}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$, extending a result of Humbert and Nazaret in dimensions one and two to any $N$. The main tool is the identification of $\log|x|$ as a fundamental solution of a certain elliptic operator of order $2N$.

1 Introduction

It is well known that the space $W^{N,1}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$. Actually, it is embedded in $C_0(\mathbb{R}^N)$ – the subspace of $C(\mathbb{R}^N)$ consisting of functions satisfying $\lim_{|x| \to \infty} u(x) = 0$. This follows from the density of $C^\infty_c(\mathbb{R}^N)$ in $W^{N,1}(\mathbb{R}^N)$ and the inequality

$$\|u\|_{L^\infty} \leq C \int_{\mathbb{R}^N} |\nabla^N u| \, dx, \forall u \in C^\infty_c(\mathbb{R}^N),$$

(1.1)

see e.g. Brezis [1, Remark 13, Chapter 9]. Above we used the notation

$$\nabla^N u = \left\{ \frac{\partial^N u}{\partial x_{i_1} \cdots \partial x_{i_N}} \right\}_{i_1, \ldots, i_N \in I_N},$$

(1.2)

where $I_N := \{1, \ldots, N\}$ (so that $\nabla^N u$ is a tensor of size $N^N$) and

$$|\nabla^N u| = \left\{ \sum_{i_1, \ldots, i_N \in I_N} \left( \frac{\partial^N u}{\partial x_{i_1} \cdots \partial x_{i_N}} \right)^2 \right\}^{1/2}.$$  

(1.3)
It is natural to look for the optimal constant $C$ in (1.1) (i.e., the smallest constant for which the inequality holds). Of course it is equivalent to consider the inequality for either $u \in C_c^\infty(\mathbb{R}^N)$ or $u \in W^{N,1}(\mathbb{R}^N)$. Following Humbert and Nazaret [4] we denote the optimal constant in (1.1) by $K_N$. The constant $K_N$ played an important role in their study of best constants in the inequality

$$\| u \|_\infty \leq A \int_M |\nabla_g^N u|_g \, dv_g + B \| u \|_{W^{N-1,1}(M)}, \quad \forall u \in C_c^\infty(M),$$

(1.4)

for a smooth compact Riemannian $N$–manifold $(M,g)$. In [4] they computed the value of $K_N$ for $N = 1, 2$, and left open the question of computing its value for $N \geq 3$. We answer this question in the current paper. Our argument is valid for every $N \geq 1$.

**Remark 1.** The particular choice of the norm as in (1.3) is important, since as we shall see below, it is invariant with respect to the orthogonal group $O(N)$.

In order to prescribe the value of the best constant we recall (see Morii, Sato and Sawano [2]) the following property of the function $\log|\cdot|$:

$$|\nabla^N \log|\cdot|| = \frac{\sqrt{l_N}}{|\cdot|^N},$$

(1.5)

for some positive constant $l_N$ (see also Corollary 2.2 below). Its explicit (and complicated!) value was calculated in [2] (see Remark 3 below). Our main result is the following ($\omega_m$ denotes the surface area of the unit sphere $S^m$):

**Theorem 1.1.** (i) The value of $K_N$ is $\left(\sqrt{\frac{l_N}{\omega_{N-1}}} \right)^{-1}$. That is,

$$\| u \|_\infty \leq \left(\sqrt{\frac{l_N}{\omega_{N-1}}} \right)^{-1} \int_{\mathbb{R}^N} |\nabla^N u| \, dx, \quad \forall u \in W^{N,1}(\mathbb{R}^N),$$

(1.6)

and one cannot replace $\left(\sqrt{\frac{l_N}{\omega_{N-1}}} \right)^{-1}$ by a smaller constant.

(ii) Moreover, for $N \geq 2$ there is no function in $W^{N,1}(\mathbb{R}^N)$ (except the zero function) for which equality holds in (1.6).

**Remark 2.** An easy consequence of Theorem 1.1 is that the same result holds in the space $W^{N,1}_0(\Omega)$ (the closure of $C_c^\infty(\Omega)$ in $W^{N,1}(\Omega)$), for every $\Omega \subset \mathbb{R}^N$. More precisely,

$$\| u \|_\infty \leq \left(\sqrt{\frac{l_N}{\omega_{N-1}}} \right)^{-1} \int_{\Omega} |\nabla^N u| \, dx, \quad \forall u \in W^{N,1}_0(\Omega),$$

the constant $K_N = \left(\sqrt{\frac{l_N}{\omega_{N-1}}} \right)^{-1}$ is optimal, and the inequality is strict, unless $u$ is the zero function. This follows from the invariance of the $W^{N,1}$-seminorm with respect to scalings.
The main ingredient of the proof of Theorem 1.1 is another property of the function \( \log |x| \), namely, it is a fundamental solution for the elliptic operator

\[
\mathcal{L} = (-1)^N \sum_{i_1, \ldots, i_N \in I_N} \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \left( |x|^N \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \right),
\]

(see Proposition 3.1 below).

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2 Preliminaries

We will consider also the generalization of (1.2)–(1.3) for any \( m \geq 1 \). So for each \( u \in C^m(\Omega) \), \( m \geq 1 \), where \( \Omega \) is a domain in \( \mathbb{R}^N \), we set

\[
\nabla^m u = \left\{ \frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}} \right\}_{i_1, \ldots, i_m \in I_N},
\]

(2.1)

\[
|\nabla^m u| = \left\{ \sum_{i_1, \ldots, i_m \in I_N} \left( \frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}} \right)^2 \right\}^{1/2}.
\]

(2.2)

We will use the same notation for functions in \( W^{m,1}(\Omega) \).

A simple, yet important feature of our analysis is the invariance of the norm in (2.2) with respect to the orthogonal group \( O(N) \). This is the content of the next Lemma.

Lemma 2.1. For \( u \in C^{m}(B_R(0) \setminus \{0\}) \), \( m \geq 1 \), and \( A \in O(N) \) set \( u_A(x) = u(Ax) \). Then

\[
|\nabla^m u_A(x)| = |(\nabla^m u)(Ax)|, \quad \forall x \in B_R(0) \setminus \{0\}.
\]

(2.3)

The identity (2.3) holds a.e. on \( \mathbb{R}^N \) for \( u \in W^{m,1}(\mathbb{R}^N) \). In particular, for any \( u \in W^{m,1}(\mathbb{R}^N) \) and \( A \in O(N) \), \( u_A(x) := u(Ax) \) satisfies

\[
\int_{\mathbb{R}^N} |\nabla^m u_A| = \int_{\mathbb{R}^N} |\nabla^m u|.
\]

(2.4)

Proof. Put \( y = Ax \). From the basic formula

\[
\frac{\partial}{\partial x_i} (u(Ax)) = \sum_{j=1}^N a_{j,i} \frac{\partial u}{\partial y_j}(y)
\]

we deduce that for every \( m \)-tuple \( (i_1, \ldots, i_m) \in (I_N)^m \) we have

\[
\frac{\partial^m u_A}{\partial x_{i_1} \cdots \partial x_{i_m}}(x) = \sum_{j_1, \ldots, j_m \in I_N} a_{j_1,i_1} a_{j_2,i_2} \cdots a_{j_m,i_m} \frac{\partial^m u}{\partial y_{j_1} \cdots \partial y_{j_m}}(y).
\]

(2.5)
Taking the square of (2.5) and summing over all \( m \)-tuples yields

\[
|\nabla^m u(x)|^2 = \sum_{i_1,\ldots,i_m \in I_N} \left( \frac{\partial^m u}{\partial x_{i_1} \cdots \partial x_{i_m}}(x) \right)^2 = \\
\sum_{i_1,\ldots,i_m \in I_N} a_{j_1,i_1}a_{j_2,i_2} \cdots a_{j_m,i_m} a_{k_1,i_1}a_{k_2,i_2} \cdots a_{k_m,i_m} \left( \frac{\partial^m u}{\partial y_{j_1} \cdots \partial y_{j_m}}(y) \right) \left( \frac{\partial^m u}{\partial y_{k_1} \cdots \partial y_{k_m}}(y) \right)
\]

Next we notice that since \( A \) is orthogonal we have

\[
\sum_{i=1}^N a_{j_s,i_s}a_{k_s,i_s} = \delta_{j_s,k_s}, \quad s = 1,\ldots,m.
\]

Using it in (2.6) gives

\[
|\nabla^m u_A(x)|^2 = \sum_{j_1,\ldots,j_m \in I_N} \left( \frac{\partial^m u}{\partial y_{j_1} \cdots \partial y_{j_m}}(y) \right)^2 = |(\nabla^m u)(y)|^2,
\]

and (2.3) follows.

The statement about \( u \in W^{m,1}(\mathbb{R}^N) \) and then the equality (2.4) follow from (2.3) and the density of \( C_0^\infty(\mathbb{R}^N) \) in \( W^{m,1}(\mathbb{R}^N) \).

Part (ii) of the next Corollary was proved in [2]; we present a more elementary proof. It is based on part (i) which in turn follows immediately from Lemma 2.1.

**Corollary 2.2.**

(i) If \( u \in C^m(B_R(0) \setminus \{0\}) \) or \( u \in W^{m,1}(B_R(0)) \), \( m \geq 1 \), is radial then the function \( x \mapsto |\nabla^m u(x)| \) is also radial.

(ii) For every \( m \geq 1 \) there exists a positive constant \( \ell_N^m \) such that

\[
|\nabla^m \log |x|| = \sqrt{\frac{\ell_N^m}{|x|^m}}, \quad x \in \mathbb{R}^N \setminus \{0\}. \tag{2.7}
\]

**Proof.** (i) This is a direct consequence of Lemma 2.1 which gives in our case

\[
|\nabla^m u(x)| = |\nabla^m u(Ax)|, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \forall A \in O(N).
\]

(ii) By (i), \( |\nabla^m \log |x|| \) is a radial function. Since each derivative \( \frac{\partial^m \log |x|}{\partial x_{i_1} \cdots \partial x_{i_m}} \) is clearly homogeneous of order \(-m\), the same is true for \( |\nabla^m \log |x|| \). Thus, the function \( |\nabla^m \log |x|| \) is necessarily of the form (2.7).

For the case \( m = N \) we use the shorthand \( \ell_N = \ell_N^N \) (see (1.5)).
Remark 3. The authors of [2] computed an explicit expression for $\ell_N^m$:

$$\ell_N^m = m! \sum_{l=0}^{\lfloor m/2 \rfloor} (m-2l)! \left( \frac{N-3}{2} + l \right)_l \left( \sum_{n=\lfloor m/2 \rfloor}^{m-l} \frac{(-1)^n}{2n} \binom{n}{m-n} \frac{l}{n} \right)^2,$$

(2.8)

with the notation

$$ (v)_k = \begin{cases} \frac{k-l}{j=0} (v-j) & \text{for } v \in \mathbb{R}, k \in \mathbb{N}, \\ 1 & \text{for } v \in \mathbb{R}, k = 0. \end{cases}$$

The first values of $\ell_N$ are: $\ell_1 = 1$, $\ell_2 = 2$, $\ell_3 = 28$, which by Theorem 1.1 imply

$$K_1 = \frac{1}{2}, K_2 = \frac{1}{2\pi \sqrt{2}}, K_3 = \frac{1}{4\pi \sqrt{28}}.$$

3 Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into two parts. In the first part we compute the value of $K_N$ and in the second we prove that equality cannot hold in (1.6), unless $u \equiv 0$.

3.1 The value of $K_N$

The main ingredient of the proof of Theorem 1.1 is the identification of $\log |x|$ as a fundamental solution of a certain operator of order $2N$:

Proposition 3.1. For all $N \geq 2$ we have, in the sense of distributions:

$$(-1)^N \sum_{i_1, \ldots, i_N \in I_N} \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \left( |x|^N \frac{\partial^N \log |x|}{\partial x_{i_1} \cdots \partial x_{i_N}} \right) = -\ell_N \omega_{N-1} \delta_0.$$

(3.1)

Proof. The function

$$F(x) := (-1)^N \sum_{i_1, \ldots, i_N \in I_N} \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \left( |x|^N \frac{\partial^N \log |x|}{\partial x_{i_1} \cdots \partial x_{i_N}} \right)$$

(3.2)

clearly belongs to $C^\infty(\mathbb{R}^N \setminus \{0\})$. We claim that $F$ is a radial function. Indeed, A similar computation to the one used in the proof of Lemma 2.1 shows that for a smooth $u$ on either $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$ and $A \in O(N)$, the function $u_A(x) = u(Ax)$ satisfies

$$\sum_{i_1, \ldots, i_N \in I_N} \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \left( |x|^N \frac{\partial^N u_A}{\partial x_{i_1} \cdots \partial x_{i_N}} \right)(x) = \sum_{i_1, \ldots, i_N \in I_N} \frac{\partial^N}{\partial y_{i_1} \cdots \partial y_{i_N}} \left( |y|^N \frac{\partial^N u}{\partial y_{i_1} \cdots \partial y_{i_N}} \right)(y),$$

with $y = Ax$. Since $\log |x|$ is radial, this implies that $F(Ax) = F(x)$, whence $F$ is radial.
From (3.2) it is clear that $F$ is homogenous of degree $-N$. Therefore it must be of the form

$$F(x) = c|x|^{-N},$$  \hspace{1cm} (3.3)

for some constant $c \in \mathbb{R}$. We claim that $c = 0$.

Assume by contradiction that $c \neq 0$. Fix a function $\varphi \in C^\infty(0,\infty)$ taking values in $[0,1]$ and satisfying

$$\varphi(t) = \begin{cases} 0 & \text{for } t \in [0,1/2], \\ 1 & \text{for } t \geq 1, \end{cases}$$  \hspace{1cm} (3.4)

and then, for any $\delta \in (0,1)$ let $\varphi_\delta(t) = \varphi(t/\delta)$. Fix also a function $\zeta \in C^\infty_c(\mathbb{R}^N)$, taking values in $[0,1]$ and satisfying $\zeta = 1$ on $B_1(0)$ and $\zeta = 0$ on $\mathbb{R}^N \setminus B_2(0)$. Finally, define $v_\delta(x) = \zeta(x)\varphi_\delta(|x|)$ which belongs to $C^\infty_c(\mathbb{R}^N)$. Since $v_\delta \equiv 1$ on $\{\delta < |x| < 1\}$ and $|\nabla^N v_\delta(|x|)| \leq C/\delta^N$ for $|x| \leq \delta$ we have

$$\int_{\mathbb{R}^N} |\nabla^N v_\delta| \leq C, \text{ uniformly in } \delta. \hspace{1cm} (3.5)$$

On the other hand, by (2.7),

$$|x|^N \left| \frac{1}{\sqrt{\ell}} \nabla^N \log |x| \right| = 1, \hspace{1cm} (3.6)$$

whence

$$\int_{\mathbb{R}^N} |\nabla^N v_\delta| \geq \int_{|\delta/2 < |x| < 2|} |\nabla^N v_\delta| \geq \frac{1}{\sqrt{\ell}} \int_{|\delta/2 < |x| < 2|} |\nabla^N v_\delta| \left| |x|^N \nabla^N \log |x| \right| \ {.} \hspace{1cm} (3.7)$$

Applying integration by parts to the integral on the R.H.S. of (3.7) and using (3.3) gives

$$\frac{1}{\sqrt{\ell}} \int_{|\delta/2 < |x| < 2|} \left| \nabla^N v_\delta \right| \left| |x|^N \nabla^N \log |x| \right| = \frac{1}{\sqrt{\ell}} \int_{|\delta/2 < |x| < 2|} F v_\delta$$

$$= \frac{c}{\sqrt{\ell}} \int_{|\delta/2 < |x| < 2|} v_\delta \frac{1}{|x|^N} + O(1) = \frac{c}{\sqrt{\ell}} \omega_{N-1} \log(1/\delta) \ + \ O(1), \hspace{1cm} (3.8)$$

where $O(1)$ denotes a bounded quantity, uniformly in $\delta$. Combining (3.7)–(3.8) with (3.5) leads to a contradiction for $\delta$ small enough, whence $c = 0$ as claimed.

From the above we deduce that the distribution

$$\mathcal{F} := (-1)^N \sum_{i_1,\ldots,i_N \in I} \frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} \left( |x|^N \frac{\partial^N \log |x|}{\partial x_{i_1} \cdots \partial x_{i_N}} \right) \in \mathcal{D}'(\mathbb{R}^N) \hspace{1cm} (3.9)$$

satisfies $\text{supp}(\mathcal{F}) \subset \{0\}$. By a celebrated theorem of L. Schwartz [5] it follows that

$$\mathcal{F} = \sum_{j=1}^L c_j \mathcal{D}_j \delta_0, \hspace{1cm} (3.10)$$
for some multi-indices $\alpha_1, \ldots, \alpha_L$. But by (2.7) each term on the R.H.S. of (3.9) can be written as

$$\frac{\partial^N}{\partial x_{i_1} \cdots \partial x_{i_N}} (|x|^N \frac{\partial^N \log |x|}{\partial x_{i_1} \cdots \partial x_{i_N}}) = \frac{\partial}{\partial x_{i_1}} \left( \frac{\partial^{N-1}}{\partial x_{i_2} \cdots \partial x_{i_N}} (|x|^N \frac{\partial^N \log |x|}{\partial x_{i_1} \cdots \partial x_{i_N}}) \right),$$

with $g$ satisfying $|g(x)| \leq C/|x|^{N-1}$, so that $g \in L^1_{\text{loc}}(\mathbb{R}^N)$. It follows that $\mathcal{F}$ in (3.9) is a sum of first derivatives of functions in $L^1_{\text{loc}}$, whence for some $\mu \in \mathbb{R},$

$$\mathcal{F} = \mu \delta_0. \quad (3.11)$$

It remains to determine the value of $\mu$ in (3.11). For that matter we will use a family of test functions $\{u_\varepsilon\}$, for small $\varepsilon > 0$. Let $\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$ with $\varphi$ given by (3.4) and define on $[0, \infty)$, $f_\varepsilon(t) = (-\log \varepsilon) - \int_\varepsilon^t \frac{\varphi_\varepsilon(s)}{s} \, ds$. Finally, let $u_\varepsilon(x) = \zeta(x)f_\varepsilon(|x|)$ on $\mathbb{R}^N$, where $\zeta$ is the same function used in the proof of Proposition 3.1 (recall $\zeta = 1$ on $B_1(0)$ while $\zeta = 0$ outside $B_2(0)$). It is easy to verify that $u_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ (in fact, $\text{supp}(u_\varepsilon) \subset B_2(0)$) and it satisfies:

$$\|u\|_{L^\infty(\mathbb{R}^N)} = u_\varepsilon(0) = \log(1/\varepsilon) + O(1), \quad (3.12)$$

$$u_\varepsilon(x) = \log(1/|x|) \text{ on } B_1 \setminus B_\varepsilon, \quad (3.13)$$

$$\|\nabla^k u_\varepsilon\|_{L^\infty(B_1)} = O(1) \cdot \varepsilon^{-k}, \quad 1 \leq k \leq N, \quad (3.14)$$

$$\|\nabla^k u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus B_1)} = O(1), \quad 1 \leq k \leq N, \quad (3.15)$$

where $O(1)$ stands for a quantity which is bounded uniformly in $\varepsilon$.

Since $u_\varepsilon \in C_c^\infty(\mathbb{R}^N)$, we get by the definition of $\mathcal{F}$ (see (3.9)) and (3.11) that

$$\mu u_\varepsilon(0) = \int_{\mathbb{R}^N} |x|^N \left( \nabla^N u_\varepsilon \right) \cdot \left( \nabla^N \log |x| \right). \quad (3.16)$$

By (3.13)–(3.15) we get for the R.H.S. of (3.16),

$$\int_{\mathbb{R}^N} |x|^N \left( \nabla^N u_\varepsilon \right) \cdot \left( \nabla^N \log |x| \right) = -\int_{|x| < 1} |x|^N \left( \nabla^N \log |x| \right) \cdot \left( \nabla^N \log |x| \right) + O(1)$$

$$= -\ell_N \int_{|x| < 1} \frac{dx}{|x|^N} + O(1) = -\ell_N \omega_{N-1}(-\log \varepsilon) + O(1), \quad (3.17)$$

where we also used (3.6). Using (3.16)–(3.17) in conjunction with (3.12) yields $\mu = -\ell_N \omega_{N-1}$, as claimed.

**Proof of part (i) of Theorem 1.1.** Clearly it is enough to consider $u \in C_c^\infty(\mathbb{R}^N)$ and without loss of generality we may assume $u(0) = \|u\|_\infty$. By (3.1),

$$\ell_N \omega_{N-1} u(0) = -\int_{\mathbb{R}^N} |x|^N \left( \nabla^N u \right) \cdot \left( \nabla^N \log |x| \right). \quad (3.18)$$
Using (3.6) to bound the R.H.S. of (3.18) from above (as in (3.7)) yields
\[ \ell_N \omega_{N-1} u(0) \leq \sqrt{I_N} \int_{\mathbb{R}^N} |\nabla^N u|, \]
whence
\[ K_N \leq \left( \sqrt{\ell_N} \omega_{N-1} \right)^{-1}. \] (3.19)

To prove that equality holds in (3.19) it suffices to consider \( u_\varepsilon \) constructed in the course of the proof of Proposition 3.1. Indeed, the arguments used there yield
\[ \hat{R}_N |\nabla^N u_\varepsilon| = \hat{\{ \varepsilon < |x| < 1 \}} |\nabla^N u_\varepsilon| + O(1) = \hat{\{ \varepsilon < |x| < 1 \}} |\nabla^N \log |x|| + O(1) = \sqrt{\ell_N} \omega_{N-1} (-\log \varepsilon) + O(1), \] (3.20)
which in conjunction with (3.12) gives
\[ \lim_{\varepsilon \to 0} \frac{\int_{\mathbb{R}^N} |\nabla^N u_\varepsilon|}{u_\varepsilon(0)} = \sqrt{\ell_N} \omega_{N-1}. \]
This clearly implies equality in (3.19).

3.2 Nonexistence of an optimizer in (1.6)

Proof of Theorem 1.1(ii). Looking for contradiction, assume that for some \( N \geq 2 \) there exists \( u \in W^{N,1}(\mathbb{R}^N), u \neq 0 \), for which equality holds in (1.6). We may assume without loss of generality that \( u(0) = \|u\|_\infty \).

We first show that such \( u \) can be assumed radial. Indeed, notice that for every \( A_1, A_2 \in O(N) \), the function \( v(x) := (u(A_1 x) + u(A_2 x))/2 \) satisfies \( v(0) = \|v\|_\infty = \|u\|_\infty \) and
\[ \int_{\mathbb{R}^N} |\nabla^N v| \leq \frac{1}{2} \left( \int_{\mathbb{R}^N} |\nabla^N u(A_1 x)| + \int_{\mathbb{R}^N} |\nabla^N u(A_2 x)| \right) = \int_{\mathbb{R}^N} |\nabla^N u|, \]
where in the last equality we used (2.4). It follows that \( v \) too realizes equality in (1.6). We can apply the same principle also for continuous averaging. Indeed, the function
\[ \tilde{u}(x) := \int_{SO(N)} u(A x) dA, \] (3.21)
where the integration is with respect to the (normalized) Haar measure on \( SO(N) \) (see [3]), belongs to \( W^{N,1}(\mathbb{R}^N) \) and satisfies \( \tilde{u}(0) = \|\tilde{u}\|_\infty = \|u\|_\infty \) and
\[ \int_{\mathbb{R}^N} |\nabla^N \tilde{u}| \leq \int_{\mathbb{R}^N} |\nabla^N u|. \]
Hence, equality must hold in the last inequality and \(\tilde{u}\) is a (nontrivial) radial function for which equality holds in (1.6).

Let \(\{u_\varepsilon\}_{\varepsilon>0} \subset C^\infty_c(\mathbb{R}^N)\) satisfy \(u_\varepsilon \rightharpoonup \tilde{u}\) in \(W^{N,1}(\mathbb{R}^N)\), whence also in the uniform norm on \(\mathbb{R}^N\). Since (3.18) holds for \(u = u_\varepsilon\), passing to the limit yields

\[
\ell_N \omega_{N-1} \tilde{u}(0) = -\int_{\mathbb{R}^N} |x|^N \left( \nabla^N \tilde{u} \right) \cdot \left( \nabla^N \log |x| \right).
\]

(3.22)

Applying the Cauchy-Schwarz inequality to the integrand on the R.H.S. of (3.22) in conjunction with (1.5) and our assumption that equality holds in (1.6) for \(\tilde{u}\) yields

\[
\ell_N \omega_{N-1} \tilde{u}(0) = -\int_{\mathbb{R}^N} |x|^N \left( \nabla^N \tilde{u} \right) \cdot \left( \nabla^N \log |x| \right) \leq \sqrt{\ell_N} \int_{\mathbb{R}^N} |\nabla^N \tilde{u}| = \ell_N \omega_{N-1} \tilde{u}(0).
\]

(3.23)

It follows from (3.23) that \(\nabla^N \tilde{u}(x) \parallel \nabla^N \log |x|\), a.e. in \(\mathbb{R}^N\). Since both \(|\nabla^N \log |x||\) and \(|\nabla^N \tilde{u}(x)|\) are radial (see Corollary 2.2), it follows that there exists a function \(\alpha(t)\) such that

\[
\nabla^N \tilde{u}(x) = \alpha(|x|) \nabla^N \log |x|, \text{ a.e. on } \mathbb{R}^N.
\]

(3.24)

For a smooth radial function \(u\) on \(\mathbb{R}^N\) we compute, introducing the variable \(s = r^2/2 = |x|^2/2\),

\[
\begin{align*}
u_{x_i} &= \left(\frac{du}{ds}\right) x_i, \\
u_{x_i x_j} &= \left(\frac{d^2u}{ds^2}\right) x_i^2 + \left(\frac{du}{ds}\right) x_i x_j (i \neq j).
\end{align*}
\]

Here and in the sequel, with a slight abuse of notation, we will consider a radial function \(u(x)\) also as a function of \(s\). A simple induction shows that for any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_N)\) with \(|\alpha| = \sum_{j=1}^N \alpha_j = m\) we have

\[
\frac{\partial^m u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \left( \frac{d^{m-i} u}{ds^{m-i}} \right) P^{(\alpha)}_{m-2i}(x_1, \ldots, x_N),
\]

(3.25)

where each \(P^{(\alpha)}_{m-2i}\) is either an homogenous polynomial of degree \(m - 2i\) with positive integer coefficients, or the zero polynomial. It follows from (3.25) that the tensor \(\nabla^m u\) can be written as

\[
\nabla^m u(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} \left( \frac{d^{m-i} u}{ds^{m-i}} \right) \mathcal{P}_{m-2i}(x_1, \ldots, x_N),
\]

(3.26)

where each \(\mathcal{P}_{m-2i}\) is a tensor whose nonzero elements are taken from the set

\[
\{ P^{(\alpha)}_{m-2i} : |\alpha| = m, 0 \leq i \leq \lfloor m/2 \rfloor \}.
\]

We claim that none of the tensors \(\{ \mathcal{P}_{m-2i} \}_{i=0}^{\lfloor m/2 \rfloor}\) is the zero tensor. Indeed, this follows from the simple observation that for each fixed \(j \in I_N\) we have

\[
\frac{\partial^m u}{\partial x_j^m}(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} b_i \left( \frac{d^{m-i} u}{ds^{m-i}} \right) x_j^{m-2i},
\]
with positive integer coefficients \(b_i\). Now we can rewrite (3.26) as
\[
\nabla^m \bar{u}(x) = \frac{\partial u}{\partial x^m} = \frac{m!}{m!}\left(\frac{d^m}{dx^m}\bar{u}(x)\right) = \frac{m!}{m!}\left(\frac{d^m}{dx^m}\bar{u}(x)\right),
\]
with \(\bar{u}(x_1, \ldots, x_N) = (1/|x|)(x_1, \ldots, x_N)\in \mathbb{S}^{N-1}\). Of course, the above formulas continue to hold when we replace the smooth \(u\) by a function belonging to \(W^{m,1}\).

Going back to (3.24), using (3.27) for \(u = \tilde{u}\) and \(u = \log|x|\), we conclude that
\[
0 = \sum_{i=0}^{[N/2]} \left(\frac{d^{N-i}\tilde{u}(x)}{ds^{N-i}} - a(|x|)\frac{d^{N-i}\log|x|}{ds^{N-i}}\right)(2s)^{N/2-i}\mathcal{P}_{N-2i}\tilde{u}(x), \text{ a.e. on } \mathbb{R}^N.
\]
(3.28)
Since for \(i_1 \neq i_2\) the monomials in the components of \(\mathcal{P}_{N-2i_1}\) and \(\mathcal{P}_{N-2i_2}\) have different degrees, it follows from (3.28) that
\[
\frac{d^{N-i}\tilde{u}(x)}{ds^{N-i}} = a(|x|)\frac{d^{N-i}\log|x|}{ds^{N-i}}, \text{ a.e. on } \mathbb{R}^N, \text{ for } i = 0, \ldots, [N/2].
\]
(3.29)
Using (3.29) for \(i = i_0 := [N/2]\) and \(i = i_0 - 1\) yields that \(\tilde{v} := \frac{d^{N-i_0}\tilde{u}}{ds^{N-i_0}}\) satisfies
\[
\frac{d\tilde{v}}{ds} = \frac{d^{N-i_0+1}\log|x|}{d^{N-i_0+1}x}s^{N-i_0+1} = -\frac{N - i_0}{s}, \text{ a.e. for } s \in (0, \infty).
\]
(3.30)
Integrating (3.30) gives \(\tilde{v} = \frac{c}{s^{N-i_0}}\) for some constant \(c\), whence
\[
\tilde{u} = \tilde{c}\log s + Q_{N-i_0-1}(s),
\]
(3.31)
where \(Q_{N-i_0-1}\) is a polynomial of degree less or equal to \(N - i_0 - 1\) and \(\tilde{c}\) is another constant. For \(\tilde{u}\) as in (3.31), the requirements \(\tilde{u}, |\nabla^m \tilde{u}| \in L^1(\mathbb{R}^N)\) clearly impose \(\tilde{u} = 0\). Contradiction.
\[\square\]

**Remark 4.** It was shown in [4] that for \(N = 1\) the function \(u(x) = e^{-|x|}\) satisfies \(u(0) = \|u\|_\infty = (1/2)\int_\mathbb{R}|u'|\), that is, equality holds in (1.6). In fact, this is true for any \(u \in W^{1,1}(\mathbb{R})\) satisfying \(\text{sgn } u'(x) = -\text{sgn } x\), a.e. on \(\mathbb{R}\).

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