Spectrum Analysis for the Vlasov–Poisson–Boltzmann System

HAI-LIANG LI*, TONG YANG & MINGYING ZHONG

Dedicated to Professor Ling Hsiao on the occasion of her eightieth birthday

Abstract

By identifying a norm capturing the effect of the forcing governed by the Poisson equation, we give a detailed spectrum analysis on the linearized Vlasov–Poisson–Boltzmann system around a global Maxwellian. It is shown that the electric field governed by the self-consistent Poisson equation plays a key role in the analysis so that the spectrum structure is genuinely different from the well-known one of the Boltzmann equation. Based on this, we give the optimal time decay rates of solutions to the equilibrium.

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1. Introduction

The Vlasov–Poisson–Boltzmann (VPB) system can be used to describe the motion of the dilute charged particles in plasma or semiconductor devices under
the influence of the self-consistent electric field [14]. In the present paper, we consider the following Cauchy problem for VPB system for one species:

\[
\begin{aligned}
\partial_t F + v \cdot \nabla_x F + \nabla_x \Phi \cdot \nabla_v F &= Q(F, F), \\
\Delta_x \Phi &= \int_{\mathbb{R}^3} Fdv - \bar{\rho}, \\
F(x, v, 0) &= F_0(x, v)
\end{aligned}
\] (1.1)

Here $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \in \mathbb{R}^+$ and $F = F(x, v, t)$ is the distribution function, and $\Phi(x, t)$ denotes the electrostatic potential. $\bar{\rho} > 0$ is the given doping density and assumed to be 1. As usual, the operator $Q(f, g)$ describing the binary elastic collision between particles takes the form

\[
Q(f, g) = \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, \omega) (f^*_g - f_g)dv_*d\omega,
\] (1.2)

where

\[
f^*_f = f(t, x, v'_*), \quad f^t = f(t, x, v'), \quad f_* = f(t, x, v_*), \quad f = f(t, x, v), \quad v' = v - [(v - v_*) \cdot \omega] \omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega] \omega, \quad \omega \in S^2.
\]

For monatomic gas, the collision kernel $B(|v - v_*|, \omega)$ is a non-negative function of $|v - v_*|$ and $|(v - v_*) \cdot \omega|$:

\[
B(|v - v_*|, \omega) = B(|v - v_*|, \cos \theta), \quad \cos \theta = \frac{|(v - v_*) \cdot \omega|}{|v - v_*|}, \quad \theta \in [0, \pi/2].
\]

In what follows, we consider both the hard sphere model and hard potential with angular cutoff. More precisely, for the hard sphere model,

\[
B(|v - v_*|, \omega) = |(v - v_*) \cdot \omega| = |v - v_*| \cos \theta,
\] (1.3)

and for the models of the hard potential with Grad angular cutoff assumption,

\[
B(|v - v_*|, \omega) = b(\cos \theta)|v - v_*|^{\gamma}, \quad 0 \leq \gamma < 1,
\] (1.4)

where we assume, for simplicity, that

\[
0 \leq b(\cos \theta) \leq C|\cos \theta|.
\]

There have been a lot of works on the existence and behavior of solutions to the Vlasov–Poisson–Boltzmann system. The global existence of renormalized solution for large initial data was proved in [15]. The first global existence result on classical solution in torus when the initial data is near a global Maxwellian was established in [9]. The global existence of classical solution in spatial 3D whole space was given in [3,20,21] for hard sphere collision, and then in [6,7] for hard potential or soft potential. The perturbation of a vacuum was investigated in [5,10].

However, in contrast to the works on Boltzmann equation [8,12,13,17–19], the spectrum of the linearized VPB system has not been given despite of its importance. On the other hand, an interesting phenomenon was shown recently in [2] on the time
asymptotic behavior of the solutions which shows that the global classical solution of one species VPB system tends to the equilibrium at $(1 + t)^{-\frac{1}{4}}$ in $L^2$-norm. This is slower than the rate for the two species VPB system, that is, $(1 + t)^{-\frac{3}{4}}$, obtained in [22]. Therefore, it is natural to investigate whether these rates are optimal.

Following the approach on the spectrum analysis on the Boltzmann equation [8,17], we now consider the VPB system as follows: the VPB system (1.1) has a stationary solution $(F^*, \Phi^*) = (M(v), 0)$ with the normalized Maxwellian $M(v)$ given by

\[ M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^3. \]

As usual, defining the perturbation $f(x, v, t)$ of $F(x, v, t)$ near $M$ by

\[ F = M + \sqrt{M} f, \]

the VPB system (1.1) for $F$ is reformulated in terms of $f$ into

\[
\begin{align*}
\partial_t f &+ v \cdot \nabla_x f - v \sqrt{M} \cdot \nabla_x \Phi - Lf = G(f), \\
\Delta_x \Phi &= \int_{\mathbb{R}^3} f \sqrt{M} \, dv, \\
f(x, v, 0) &= f_0(x, v) = (F_0 - M)M^{-\frac{1}{2}},
\end{align*}
\]

where the nonlinear term $G$ is given by

\[ G = \frac{1}{2} (v \cdot \nabla_x \Phi) f - \nabla_x \Phi \cdot \nabla f + \Gamma(f, f). \]  

(1.6)

The linearized collision operator $Lf$ and the nonlinear term $\Gamma(f, f)$ in (1.5) are defined by

\[ Lf = \frac{1}{\sqrt{M}} [Q(M, \sqrt{M} f) + Q(\sqrt{M} f, M)], \]  

(1.7)

\[ \Gamma(f, f) = \frac{1}{\sqrt{M}} Q(\sqrt{M} f, \sqrt{M} f). \]  

(1.8)

We have, cf [1],

\[
\begin{align*}
(Lf)(v) &= (Kf)(v) - v(v) f(v), \\
v(v) &= \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, \omega) M_* \omega dv_* dv, \\
(Kf)(v) &= \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, \omega) (\sqrt{M_*} f' + \sqrt{M} f_*' - \sqrt{M} f_*') \sqrt{M_*} \omega dv_* dv, \\
&= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) \, dv_*.
\end{align*}
\]  

(1.9)

where $v(v)$, the collision frequency, is a real function, and $K$ is a self-adjoint compact operator on $L^2(\mathbb{R}_v^3)$ with a real symmetric integral kernel $k(v, v_*)$.
space of the operator \( L \), denoted by \( N_0 \), is a subspace spanned by the orthonormal basis \( \{ \chi_j, \ j = 0, 1, \ldots, 4 \} \), with

\[
\chi_0 = \sqrt{M}, \quad \chi_j = v_j \sqrt{M} \quad (j = 1, 2, 3), \quad \chi_4 = \frac{|v|^2 - 3}{\sqrt{6}}. \tag{1.10}
\]

Let \( P_0 \) be the projection operator from \( L^2(\mathbb{R}^3_v) \) to the subspace \( N_0 \) and let \( P_1 = I - P_0 \), and let \( L^2(\mathbb{R}^3_v) \) be a Hilbert space of complex-value functions \( f(v) \) on \( \mathbb{R}^3 \) with the inner product and the norm

\[
(f, g) = \int_{\mathbb{R}^3} f(v)\overline{g(v)}dv, \quad \| f \| = \left( \int_{\mathbb{R}^3} |f(v)|^2dv \right)^{1/2}.
\]

From the Boltzmann’s H-theorem, the linearized collision operator \( L \) is non-positive, and moreover, \( L \) is locally coercive in the sense that there is a constant \( \mu > 0 \) such that

\[
(Lf, f) \leq -\mu \| P_1 f \|^2, \quad f \in D(L), \tag{1.11}
\]

where \( D(L) \) is the domain of \( L \) given by

\[
D(L) = \left\{ f \in L^2(\mathbb{R}^3_v) \mid \nu(v)f \in L^2(\mathbb{R}^3_v) \right\}.
\]

In addition, \( \nu(v) \) satisfies

\[
\nu_0(1 + |v|)^\gamma \leq \nu(v) \leq \nu_1(1 + |v|)^\gamma, \tag{1.12}
\]

with \( \gamma = 1 \) for the hard sphere and \( 0 \leq \gamma < 1 \) for the hard potential. Without the loss of generality, we assume in this paper that \( \nu(0) \geq \nu_0 \geq \mu > 0 \).

Introduce the macro-micro decomposition as follows:

\[
\begin{aligned}
f &= P_0 f + P_1 f, \\
P_0 f &= n \chi_0 + \sum_{j=1}^3 m_j \chi_j + q \chi_4. \tag{1.13}
\end{aligned}
\]

Here the density \( n \), the momentum \( m = (m_1, m_2, m_3) \) and the energy \( q \) are defined by

\[
(f, \chi_0) = n, \quad (f, \chi_j) = m_j, \quad (f, \chi_4) = q.
\]

From (1.5), we have the linearized Vlasov–Poisson–Boltzmann system

\[
\begin{aligned}
\partial_t f &= Bf, \quad t > 0, \\
f(0, x, v) &= f_0(x, v), \quad (x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v, \tag{1.14}
\end{aligned}
\]

where the operator \( B \) is defined by

\[
Bf = Lf - v \cdot \nabla_x f - v \sqrt{M} \cdot \nabla_x (-\Delta_x)^{-1} \left( \int_{\mathbb{R}^3} f \sqrt{M}dv \right). \tag{1.15}
\]
We take the Fourier transform in (1.14) with respect to $x$ to get
\[
\begin{align*}
\partial_t \hat{f}(\xi, v) &= B(\xi) \hat{f}(\xi), \quad t > 0, \\
\hat{f}(0, \xi, v) &= \hat{f}_0(\xi, v), \quad (\xi, v) \in \mathbb{R}_\xi^3 \times \mathbb{R}_v^3,
\end{align*}
\]
where the operator $B(\xi)$ is defined for $\xi \neq 0$ by
\[
B(\xi) = L - i(v \cdot \xi) - \frac{i(v \cdot \xi)}{|\xi|^2} P_d,
\]
with
\[
P_d f = \sqrt{M} \int_{\mathbb{R}^3} f \sqrt{M} dv, \quad f \in L^2(\mathbb{R}^3).
\]

**Notations:** Before stating the main results of this paper, we list some notations. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, denote
\[
\partial^\alpha_x = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad \partial^\beta_v = \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.
\]
Define the Fourier transform of $f = f(x, v)$ by
\[
\hat{f}(\xi, v) = \mathcal{F} f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x, v)e^{-ix \cdot \xi} \, dx,
\]
where and throughout this paper we denote $i = \sqrt{-1}$.

Set the weight function $w$ by
\[
w = w(v) = (1 + |v|^2)^{\frac{a}{2}}
\]
for the hard sphere model, or
\[
w = w(t, v) = (1 + |v|^2)^{\frac{a}{2}} e^{-a|v|^2}, \quad a, b > 0
\]
for the hard potential model, and set the Sobolev spaces $H^N = \{ f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) | \| f \|_{H^N} < \infty \}$ and $H^N_w = \{ f \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v) | \| f \|_{H^N_w} < \infty \}$ equipped with the norms
\[
\| f \|_{H^N} = \sum_{|\alpha| + |\beta| \leq N} \| \partial^\alpha_x \partial^\beta_v f \|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)}, \quad \| f \|_{H^N_w} = \sum_{|\alpha| + |\beta| \leq N} \| w \partial^\alpha_x \partial^\beta_v f \|_{L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)}.
\]
For $q \geq 1$, we also define
\[
L^{2,q} = L^2(\mathbb{R}^3_v, L^q(\mathbb{R}^3_x)), \quad \| f \|_{L^{2,q}} = \left( \int_{\mathbb{R}^3_v} \left( \int_{\mathbb{R}^3_x} |f(x, v)|^q \, dx \right)^{2/q} dv \right)^{1/2}.
\]
In what follows, we denote by $\| \cdot \|_{L^{2,q}_{x,v}}$ and $\| \cdot \|_{L^2_{x,v}}$ the norms of the function spaces $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ respectively, and denote by $\| \cdot \|_{L^2_{x,v}}$, $\| \cdot \|_{L^{2,q}_{x,v}}$, $\| \cdot \|_{L^2_{x,v}}$, and $\| \cdot \|_{L^{2,q}_{x,v}}$ the norms of the function spaces $L^2(\mathbb{R}^3_x), L^2(\mathbb{R}^3_v)$ and $L^2(\mathbb{R}^3_v)$ respectively. For any integer $m \geq 1$, we denote by $\| \cdot \|_{H^m}$ and $\| \cdot \|_{L^2_{x,v}(H^m)}$ the norms in the spaces $H^m(\mathbb{R}^3_x)$ and $L^2(\mathbb{R}^3_v, H^m(\mathbb{R}^3_x))$ respectively.

First, we have the following result about the spectrum structure and semigroup of the operator $B(\xi)$:
Theorem 1.1. Let \( \sigma(B(\xi)) \) denote the spectrum set of the operator \( B(\xi) \). We have that

(1) For any \( r_1 > 0 \) there exists a constant \( \alpha = \alpha(r_1) > 0 \) such that, for \( |\xi| \geq r_1 \),

\[
\sigma(B(\xi)) \subset \{ \lambda \in \mathbb{C} | \Re \lambda < -\alpha \}.
\]

(2) There exists a constant \( r_0 > 0 \) such that the spectrum \( \sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} | \Re \lambda > -\mu/2 \} \) consists of five points \( \{ \lambda_j(|\xi|), \ j = -1, 0, 1, 2, 3 \} \) for \( |\xi| \leq r_0 \). The eigenvalues \( \lambda_j(|\xi|) \) are \( C^\infty \) functions of \( |\xi| \) and satisfy

\[
\begin{align*}
\lambda_{\pm 1}(|\xi|) &= \pm i + (-a_1 \pm ib_1)|\xi|^2 + o(|\xi|^2), \quad \lambda_1 = \lambda_{-1}, \\
\lambda_0(|\xi|) &= -a_0|\xi|^2 + o(|\xi|^2), \\
\lambda_2(|\xi|) &= \lambda_3(|\xi|) = -a_2|\xi|^2 + o(|\xi|^2),
\end{align*}
\]

where \( a_j > 0, j = 0, 1, 2, \) and \( b_1 > 0 \) are constants defined by Theorem 2.11.

(3) The semigroup \( S(t, \xi) = e^{tB(\xi)} \) with \( \xi \neq 0 \) satisfies

\[
S(t, \xi)f = S_1(t, \xi)f + S_2(t, \xi)f, \quad f \in L^2_\xi(\mathbb{R}^3_v), \quad t > 0,
\]

where

\[
S_1(t, \xi)f = \sum_{j=-1}^3 e^{t\lambda_j(|\xi|)} \left( f, \psi_j(\xi) \right)_\xi \psi_j(\xi) 1_{{|\xi| \leq r_0}},
\]

with the inner product \((\cdot, \cdot)_\xi\) defined by (2.1), \((\lambda_j(|\xi|), \psi_j(\xi))\) being the eigenvalue and eigenfunction of the operator \( B(\xi) \) given in Theorem 2.11 for \( |\xi| \leq r_0 \), and \( S_2(t, \xi)f =: S(t, \xi)f - S_1(t, \xi)f \) satisfies for a constant \( \sigma_0 > 0 \) independent of \( \xi \) that

\[
\| S_2(t, \xi)f \|_\xi \leq C e^{-\sigma_0 t} \| f \|_\xi, \quad t > 0.
\]

Then, we have the following result on the large time behavior of the global solution to the IVP problem (1.5):

Theorem 1.2. Assume that \( f_0 \in H^N \cap L^{2,1} \) with \( N \geq 4 \), and \( \| f_0 \|_{H^N \cap L^{2,1}} \leq \delta_0 \) with \( \delta_0 > 0 \) being small enough, where \( w = w(v) \) is given by (1.19) for hard sphere model, and \( w = w(t, v) \) is given by (1.20) with \( a > 0, 0 < b < 1/4 \) for hard potential model. Let \( f \) be a solution of the VPB system (1.5). Then, it holds that, for \( |\alpha| = 0, 1, \)

\[
\begin{align*}
\| \partial_x^\alpha f(t, \chi_0) \|_{L^2_\xi} &\leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha f(t, v\chi_0) \|_{L^2_\xi} + \| \partial_x^\alpha \nabla_x \Phi(t) \|_{L^2_\xi} &\leq C \delta_0 (1 + t)^{-\frac{1}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha f(t, \chi_4) \|_{L^2_\xi} &\leq C \delta_0 (1 + t)^{-\frac{1}{4} - \frac{|\alpha|}{2}}, \\
\| P_1 f(t) \|_{H^N} + \| \nabla_x P_0 f(t) \|_{L^2_\xi(H^{N-1})} &\leq C \delta_0 (1 + t)^{-\frac{3}{4}}.
\end{align*}
\]
Moreover, if \((f_0, \chi_0) = 0\), then it holds that, for \(|\alpha| = 0, 1,\)
\[
\begin{align*}
\| \partial_x^\alpha (f(t), \chi_0) \|_{L^2_x} + \| \partial_x^\alpha P_1 f(t) \|_{L^2_{x,v}} & \leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), v \chi_0) \|_{L^2_x} + \| \partial_x^\alpha \nabla_x \Phi(t) \|_{L^2_x} & \leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), \chi_4) \|_{L^2_x} & \leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| P_1 f(t) \|_{H^N_x} + \| \nabla_x P_0 f(t) \|_{L^2_{x,v}(H^{N-1}_x)} & \leq C \delta_0 (1 + t)^{-\frac{5}{4}}.
\end{align*}
\]

Finally, we can establish the optimal time decay rates of the global solution in the following sense:

**Theorem 1.3.** Assume that \(f_0 \in H^N \cap L^{2,1} \) for \(N \geq 4\) satisfying \(\| f_0 \|_{H^N \cap L^{2,1}} \leq \delta_0\) with \(\delta_0 > 0\) being small enough, and that there exist two positive constants \(d_0\) and \(d_1\) such that \(\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0\) and \(\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1\). Then, the global solution \(f\) to the IVP problem \((1.5)\) satisfies
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), v \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| \nabla_x \Phi(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| P_1 f(t) \|_{L^2_{x,v}} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}
\]

for \(t > 0\) large enough and with \(C_2 \geq C_1 > 0\) being two constants.

If in addition \((f_0, \chi_0) = 0\) is assumed, and \(\inf_{|\xi| \leq r_0} |(\hat{f}_0, (v \cdot \frac{\xi}{|\xi|}) \sqrt{M})| \geq d_0\) and \(\inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_0\) for some constant \(d_0 > 0\). Then, it holds that
\[
C_1 \delta_0 (1 + t)^{-\frac{5}{4}} \leq \| (f(t), \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{5}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{5}{4}} \leq \| (f(t), v \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{5}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), \chi_4) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| \nabla_x \Phi(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}},
\]
\[
C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| P_1 f(t) \|_{L^2_{x,v}} \leq C_2 \delta_0 (1 + t)^{-\frac{5}{4}}
\]

for \(t > 0\) large enough.

**Remark 1.4.** Let us give an example of the initial function \(f_0\) which satisfies the assumptions of Theorem 1.3. For two positive constants \(d_0\) and \(d_1\), the initial data \(f_0(x, v)\) defined below satisfies the first assumption of Theorem 1.3:
\[
f_0(x, v) = d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}} \chi_0(v) + d_1 d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}} \chi_4(v).
\]

In addition, the second assumption of Theorem 1.3 is satisfied by
\[
f_0(x, v) = d_0 e^{\frac{r_0^2}{2}} (m \cdot v) \sqrt{M} + d_0 e^{\frac{r_0^2}{2}} e^{-\frac{|\xi|^2}{2}} \chi_4(v),
\]

with \(m(x) = \int_{\mathbb{R}^3} \frac{\xi}{|\xi|} e^{-\frac{|\xi|^2}{2}} e^{i x \cdot \xi} d\xi.\)
The rest of this paper will be organized as follows: in Section 2, we study the spectrum and resolvent of the linear VPB operator $B(\xi)$ and establish asymptotic expansions of the eigenvalues and eigenfunctions at low frequency. In Section 3, we decompose the semigroup $e^{tB(\xi)}$ generated by the linear operator $B(\xi)$ with respect to the low frequency and high frequency, and then establish the optimal time decay rates of the global solution to the linearized VPB system in terms of the semigroup $e^{tB}$. In Section 4, we prove the optimal time decay rates of the global solution to the original nonlinear VPB system. Some well-known results on the semigroup theory are recalled in Section 5 for the easy reference of the readers.

2. Spectral Analysis

In this section, we study the spectrum and resolvent sets of the linear operator $B(\xi)$ defined by (1.17) in order to obtain the optimal decay rate of the global solution to IVP (1.5).

2.1. Spectrum and Resolvent

Introduce the weighted Hilbert space $L^2_{\xi}(\mathbb{R}^3_v)$ for $\xi \neq 0$ as

$$L^2_{\xi}(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3_v) \mid \| f \|_{\xi} = \sqrt{(f, f)_\xi} < \infty \},$$
equipped with the inner product

$$(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2} (P_d f, P_d g). \quad (2.1)$$

Since $P_d$ is a self-adjoint operator and satisfies $(P_d f, P_d g) = (P_d f, g) = (f, P_d g)$, we have

$$(f, g)_\xi = \left( f, g \frac{1}{|\xi|^2} P_d g \right) = \left( f + \frac{1}{|\xi|^2} P_d f, g \right). \quad (2.2)$$

We can regard $B(\xi)$ as a linear operator from the space $L^2_{\xi}(\mathbb{R}^3_v)$ to itself because

$$\| f \|^2_{\xi} \leq \| f \|^2 \leq (1 + |\xi|^{-2}) \| f \|^2, \quad \xi \neq 0.$$

The sign $\sigma(A)$ denotes the spectrum of the operator $A$. The discrete spectrum of $A$, denoted by $\sigma_d(A)$, is the set of all isolated eigenvalues with finite multiplicity. The essential spectrum of $A$, $\sigma_{ess}(A)$, is the set $\sigma(A) \setminus \sigma_d(A)$. We denote $\rho(A)$ to be the resolvent set of the operator $A$. In particular, we show

**Lemma 2.1.** The operator $B(\xi)$ generates a strongly continuous contraction semigroup on $L^2_{\xi}(\mathbb{R}^3_v)$ satisfying

$$\| e^{tB(\xi)} f \|_{\xi} \leq \| f \|_{\xi}, \quad \text{for } t > 0, \ f \in L^2_{\xi}(\mathbb{R}^3_v). \quad (2.3)$$
Lemma 5.2 that the operator \( (\lambda, \cdot) \) and let \( \lambda \) consists of discrete eigenvalues \( \lambda \).

Proof. First we show that both \( B(\xi) \) and \( B(\xi)^* \) are dissipative operators on \( L^2_\xi(\mathbb{R}^3_v) \).

By (2.2), we obtain for any \( f, g \in L^2_\xi(\mathbb{R}^3_v) \cap D(B(\xi)) \) that \( (B(\xi)f, g)_\xi = (f, B(\xi)^*g)_\xi \) because

\[
(B(\xi)f, g)_\xi = \left( f, \left( L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2} P_d \right) g \right)_\xi = (f, B(\xi)^*g)_\xi,
\]

with \( B(\xi)^* = B(-\xi) = L + i(v \cdot \xi) + \frac{i(v \cdot \xi)}{|\xi|^2} P_d \). Direct computation gives rise to the dissipation of both \( B(\xi) \) and \( B(\xi)^* \), namely, \( \text{Re}(B(\xi)f, f)_\xi = \text{Re}(B(\xi)^*f, f)_\xi = (Lf, f) \leq 0 \). Since \( B(\xi) \) is a densely defined closed operator, it follows from Lemma 5.2 that the operator \( B(\xi) \) generates a \( C_0 \)-contraction semigroup on \( L^2_\xi(\mathbb{R}^3_v) \).

Lemma 2.2. The following conditions hold for all \( \xi \neq 0 \):

1. \( \sigma_{ess}(B(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re} \leq -v_0 \} \) and \( \sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} | -v_0 < \text{Re} \leq 0 \} \subset \sigma_d(B(\xi)) \).
2. If \( \lambda(\xi) \) is an eigenvalue of \( B(\xi) \), then \( \text{Re}\lambda(\xi) < 0 \) for any \( \xi \neq 0 \).

Proof. Define

\[
c(\xi) = -v(v) - i(v \cdot \xi).
\]

It is obvious that \( \lambda - c(\xi) \) is invertible for \( \text{Re}\lambda > -v_0 \). Since \( K \) and \( \frac{i(v \cdot \xi)}{|\xi|^2} P_d \) are compact operators on \( L^2_\xi(\mathbb{R}^3_v) \) for any fixed \( \xi \neq 0 \), \( B(\xi) \) is a compact perturbation of \( c(\xi) \), and so, thanks to Theorem 5.35 in p.244 of [11], \( B(\xi) \) and \( c(\xi) \) have the same essential spectrum. Thus the spectrum of \( B(\xi) \) in the domain \( \text{Re}\lambda > -v_0 \) consists of discrete eigenvalues \( \lambda_j(\xi) \) with possible accumulation points only on the line \( \text{Re}\lambda = -v_0 \). This proves (1).

We claim that for any discrete eigenvalue \( \lambda(\xi) \) of \( B(\xi) \) on the domain \( \text{Re}\lambda > -v_0 \), it holds that \( \text{Re}\lambda(\xi) < 0 \) for \( \xi \neq 0 \). Indeed, set \( \xi = s\omega \) with \( \omega = \xi/|\xi| \in \mathbb{S}^2 \) and let \( (\lambda, h) \) be the eigenvalue and the corresponding eigenfunction of \( B(\xi) \) so that

\[
\lambda h = Lh - is(v \cdot \omega)
\left(
h + \frac{1}{s^2} P_d h \right).
\]

Taking the inner product between (2.5) and \( h + \frac{1}{s^2} P_d h \) and choosing the real part, we have

\[
(Lh, h) = \text{Re}\lambda
\left(\|h\|^2 + \frac{1}{s^2} \|P_d h\|^2\right),
\]

which, together with (1.11), implies \( \text{Re}\lambda \leq 0 \).

Furthermore, if there exists an eigenvalue \( \lambda \) with \( \text{Re}\lambda = 0 \), then it follows from the above that \( (Lh, h) = 0 \), namely, \( h \in N_0 \) and

\[

- is(v \cdot \omega)
\left(
h + \frac{1}{s^2} P_d h \right) = \lambda h,
\]

which, after being projected into the null space \( N_0 \) and its orthogonal complement \( N_0^\perp \), leads to

\[
P_0(v \cdot \omega)
\left(sh + \frac{1}{s} P_d h \right) = i\lambda h,
\]

(2.6)
By (2.7), we obtain that $h = C_0 \sqrt{M}$. Substituting this into (2.6), we have
\[
(v \cdot \omega) \left( s + \frac{1}{s} \right) C_0 \sqrt{M} = i \lambda C_0 \sqrt{M},
\]
which implies $C_0 = 0$. Therefore, we conclude that $h \equiv 0$. This is a contradiction, and thus $\text{Re} \lambda(\xi) < 0$ for all $\lambda(\xi) \in \sigma_d(B(\xi))$ and $\xi \neq 0$. \hfill \Box

Now denote by $T$ a linear operator on $L^2(\mathbb{R}^3_v)$ or $L^2_\xi(\mathbb{R}^3_\xi)$, and we define the corresponding norms of $T$ by
\[
\|T\| = \sup_{\|f\|_1=1} \|Tf\|, \quad \|T\|_\xi = \sup_{\|f\|_\xi=1} \|Tf\|. \tag{2.8}
\]
One can verify that \[
\frac{\|T\|}{(1+|\xi|^{-2}) \|f\|_\xi} \leq \frac{\sup \|Tf\|_\xi}{\|f\|_\xi} \leq \frac{(1+|\xi|^{-2}) \|Tf\|}{\|f\|_\xi}, \tag{2.9}
\]
which implies
\[
(1 + |\xi|^{-2})^{-1} \|T\| \leq \|T\|_\xi \leq (1 + |\xi|^{-2}) \|T\|. \tag{2.10}
\]

First, we study the spectrum and resolvent sets of $B(\xi)$ for $|\xi| > r_0$ with $r_0 > 0$. For $\text{Re} \lambda > -v_0$, we decompose $\lambda - B(\xi)$ into
\[
\lambda - B(\xi) = \lambda - c(\xi) - K + \frac{i(v \cdot \xi)}{|\xi|^2} P_d + \frac{i(v \cdot \xi) P_d (\lambda - c(\xi))^{-1}}{|\xi|^2} (\lambda - c(\xi)) \tag{2.11}
\]
and estimate the right hand terms of (2.9) as follows:

**Lemma 2.3.** There exists a constant $C > 0$ such that the following holds:

1. For any $\delta > 0$, if $x \geq -v_0 + \delta$, we have
\[
\|K(x + iy - c(\xi))^{-1}\| \leq C \delta^{-11/13}(1 + |\xi|)^{-2/13}. \tag{2.12}
\]

2. For any $\delta > 0$, if $x \geq -v_0 + \delta$ and $|y| \geq (2|\xi|)^{5/3} \delta^{-2/3}$, we have
\[
\|K(x + iy - c(\xi))^{-1}\| \leq C \delta^{-3/5}(1 + |y|)^{-2/5}. \tag{2.13}
\]

3. For any $\delta > 0$, $r_0 > 0$, if $x \geq -v_0 + \delta$ and $|\xi| \geq r_0$, we have
\[
\|(v \cdot \xi)|\xi|^{-2} P_d (x + iy - c(\xi))^{-1}\| \leq C (r_0^{-1} + 1)(|\xi| + |y|)^{-1}. \tag{2.14}
\]

**Proof.** The proof of (2.10) and (2.11) can be found in Lemma 2.2.6 in [19].

We prove (2.12) as follows: by the fact that $\|(v \cdot \xi)|\xi|^{-2} P_d\| \leq C |\xi|^{-1}$ and $\|(x + iy - c(\xi))^{-1}\| \leq \delta^{-1}$ for $x \geq -v_0 + \delta$, we have
\[
\|(v \cdot \xi)|\xi|^{-2} P_d (x + iy - c(\xi))^{-1}\| \leq C \delta^{-1} |\xi|^{-1}. \tag{2.15}
\]

Meanwhile, by the fact that $(v \cdot \xi)|\xi|^{-2} P_d (\lambda - c(\xi))^{-1} = \frac{1}{\lambda} (v \cdot \xi)|\xi|^{-2} P_d + \frac{1}{\lambda} (v \cdot \xi)|\xi|^{-2} P_d c(\xi)(\lambda - c(\xi))^{-1}$ and $\|(v \cdot \xi)|\xi|^{-2} P_d c(\xi)\| \leq C (r_0^{-1} + 1)$ for $|\xi| \geq r_0$, we have
\[
\|(v \cdot \xi)|\xi|^{-2} P_d (x + iy - c(\xi))^{-1}\| \leq C (\delta^{-1} + 1)(r_0^{-1} + 1)|y|^{-1}. \tag{2.16}
\]

The combination of (2.13) and (2.14) yields (2.12). \hfill \Box
With the help of Lemma 2.3, we have the spectral gap of the operator $B(\xi)$ for high frequency.

**Lemma 2.4.** (Spectral gap) For any $r_0 > 0$, there exists $\alpha = \alpha(r_0) > 0$ such that, for $|\xi| \geq r_0$,

$$\sigma(B(\xi)) \subset \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \leq -\alpha \}. \quad (2.15)$$

**Proof.** Let $\lambda(\xi) \in \sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta \}$. We first show that $\text{sup}_{|\xi| \geq r_0} |\lambda(\xi)| < +\infty$. Indeed, by (2.10), (2.12) and (2.8), there exists $r_1 = r_1(\delta) > r_0$ large enough so that for $\text{Re}\lambda \geq -\nu_0 + \delta$ and $|\xi| \geq r_1$,

$$\|K(\lambda - c(\xi))^{-1}\|_{\xi} \leq \frac{1}{4}, \quad \|(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}\|_{\xi} \leq \frac{1}{4}. \quad (2.16)$$

This implies that the operator $I + K(\lambda - c(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_d(\lambda - c(\xi))^{-1}$ is invertible on $L^2(\mathbb{R}^3_\nu)$, which together with (2.9) yield that $\lambda - B(\xi)$ is also invertible on $L^2(\mathbb{R}^3_\nu)$ for $\text{Re}\lambda \geq -\nu_0 + \delta$ and $|\xi| \geq r_1$ and it satisfies

$$(\lambda - B(\xi))^{-1} = (\lambda - c(\xi))^{-1} \left(I - K(\lambda - c(\xi))^{-1} + \frac{i(v \cdot \xi)}{|\xi|^2}P_d(\lambda - c(\xi))^{-1}\right)^{-1}, \quad (2.17)$$

namely, $\{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta \} \subset \rho(B(\xi))$ for $|\xi| \geq r_1$.

As for $r_0 \leq |\xi| \leq r_1$, by (2.11) and (2.12) there is a constant $\zeta = \zeta(r_0, r_1, \delta) > 0$ so that (2.16) still holds for $|\text{Im}\lambda| > \zeta$. This also implies the invertibility of $\lambda - B(\xi)$, namely, it holds that $\{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| > \zeta \} \subset \rho(B(\xi))$ for $r_0 \leq |\xi| \leq r_1$. Thus, we conclude that

$$\sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta \} \subset \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| \leq \zeta \}, \quad |\xi| \geq r_0. \quad (2.18)$$

Next, we prove that $\text{sup}_{|\xi| \geq r_0} \text{Re}\lambda(\xi) < 0$. Based on the above argument, it is sufficient to prove that $\text{sup}_{r_0 \leq |\xi| \leq r_1} \text{Re}\lambda(\xi) < 0$. If it does not hold, namely for some given $r_0 > 0$, there exist a sequence of $(\xi_n, \lambda_n, f_n)$ satisfying $|\xi_n| \in [r_0, r_1], f_n \in L^2(\mathbb{R}^3_\nu)$ with $\|f_n\| = 1$, and $\lambda_n \in \sigma(B(\xi_n))$ such that

$$Lf_n - i(v \cdot \xi_n)f_n - \frac{i(v \cdot \xi_n)}{|\xi_n|^2}P_d f_n = \lambda_n f_n, \quad \text{Re}\lambda_n \to 0, \quad n \to \infty.$$  

The above equation can be rewritten as $(\lambda_n + v + i(v \cdot \xi_n))f_n = Kf_n - \frac{i(v \cdot \xi_n)}{|\xi_n|^2}P_d f_n$. Since $K$ is a compact operator on $L^2(\mathbb{R}^3)$, there exists a subsequence $f_{n_j}$ of $f_n$ and $g_1 \in L^2(\mathbb{R}^3)$ such that

$$Kf_{n_j} \to g_1, \quad n \to \infty.$$  

Due to the fact that $|\xi_n| \in [r_0, r_1], P_d f_n = C_n \sqrt{M}$ with $|C_n| \leq 1$, there exists a subsequence of (still denoted by) $(\xi_{n_j}, f_{n_j})$, and $(\xi_0, C_0)$ with $|\xi_0| \in [r_0, r_1]$ and $|C_0| \leq 1$ such that $(\xi_{n_j}, C_{n_j}) \to (\xi_0, C_0)$ and

$$i(v \cdot \xi_{n_j})|\xi_{n_j}|^{-2}P_d f_{n_j} \to g_2 =: i(v \cdot \xi_0)|\xi_0|^{-2}C_0 \sqrt{M}, \quad n \to \infty.$$
Since $|\text{Im}\lambda_n| \leq \xi$ and $\text{Re}\lambda_n \to 0$, we can extract a subsequence of (still denoted by) $\lambda_{n_j}$ such that $\lambda_{n_j} \to \lambda_0$ with $\text{Re}\lambda_0 = 0$. Noting that $|\lambda_n + v + i(v \cdot \xi_n)| \geq \delta$, we have

$$
\lim_{j \to \infty} f_{n_j} = \lim_{j \to \infty} \frac{g_1 - g_2}{\lambda_{n_j} + v + i(v \cdot \xi_{n_j})} = \frac{g_1 - g_2}{\lambda_0 + v + i(v \cdot \xi_0)} := f_0 \text{ in } L^2(\mathbb{R}^3),
$$

and hence $Kf_0 = g_1$ and $i(v \cdot \xi_0)|\xi_0|^{-2}P_d f_0 = g_2$. It follows that $B(\xi_0)f_0 = \lambda_0 f_0$ and $\lambda_0$ is an eigenvalue of $B(\xi_0)$ with $\text{Re}\lambda_0 = 0$, which contradicts the fact $\text{Re}\lambda(\xi) < 0$ for $\xi \neq 0$ established by Lemma 2.2. The proof the lemma is then completed.

Then, we investigate the spectrum and resolvent sets of $B(\xi)$ at low frequency. To this end, we decompose $\lambda - B(\xi)$ as

$$
\lambda - B(\xi) = \lambda P_0 - A(\xi) + \lambda P_1 - Q(\xi) + i P_0(v \cdot \xi) P_1 + i P_1(v \cdot \xi) P_0,
$$

with the operators $A(\xi)$ and $Q(\xi)$ defined by

$$
A(\xi) = -i P_0(v \cdot \xi) P_0 - \frac{i(v \cdot \xi)}{|\xi|^2} P_d, \quad Q(\xi) = L - i P_1(v \cdot \xi) P_1.
$$

It is easy to verify that $A(\xi)$ is a linear operator from the null space $N_0$ to itself, and can be represented in the basis of $N_0$ as

$$
A(\xi) = \begin{pmatrix}
0 & -i\xi^T & 0 \\
-i\xi(1 + \frac{1}{|\xi|^2}) & 0 & -i\sqrt{\frac{2}{3}\xi} \\
0 & -i\sqrt{\frac{2}{3}\xi}^T & 0
\end{pmatrix},
$$

which admits five eigenvalues $\alpha_j(\xi)$ satisfying

$$
\alpha_j(\xi) = 0, \quad j = 0, 2, 3, \quad \alpha_{\pm 1}(\xi) = \pm i\sqrt{1 + \frac{5}{3}|\xi|^2}.
$$

**Lemma 2.5.** Let $\xi \neq 0$, we have for $A(\xi)$ and $Q(\xi)$ defined by (2.20) that

1. If $\lambda \neq \alpha_j(\xi)$, then the operator $\lambda P_0 - A(\xi)$ is invertible on $N_0$ and satisfies

$$
\| (\lambda P_0 - A(\xi))^{-1} \|_{\xi} = \max_{-1 \leq j \leq 3} \left( |\lambda - \alpha_j(\xi)|^{-1} \right),
$$

$$
\| P_1(v \cdot \xi) P_0(\lambda P_0 - A(\xi))^{-1} P_0 \|_{\xi} \leq C|\xi| \max_{-1 \leq j \leq 3} \left( |\lambda - \alpha_j(\xi)|^{-1} \right),
$$

where $\alpha_j(\xi)$, $j = -1, 0, 1, 2, 3$, are the eigenvalues of $A(\xi)$ defined by (2.22).

2. If $\text{Re}\lambda > -\mu$, then the operator $\lambda P_1 - Q(\xi)$ is invertible on $N_0^\perp$ and satisfies

$$
\| (\lambda P_1 - Q(\xi))^{-1} \| \leq (\text{Re}\lambda + \mu)^{-1},
$$

$$
\| P_0(v \cdot \xi) P_1(\lambda P_1 - Q(\xi))^{-1} P_1 \|_{\xi} \leq C(1 + |\lambda|)(\text{Re}\lambda + \mu)^{-1} + 1)(|\xi| + |\xi|^2).
$$
Proof. Since $\alpha_j(\xi)$ for $-1 \leq j \leq 3$ are the eigenvalues of $A(\xi)$, it follows that $\lambda P_0 - A(\xi)$ is invertible on $N_0$ for $\lambda \neq \alpha_j(\xi)$. By (2.2) we have for $f, g \in N_0$ that

$$
(iA(\xi)f, g)_\xi = \left( f + \frac{1}{|\xi|^2} P_d f, (v \cdot \xi)(g + \frac{1}{|\xi|^2} P_d g) \right) = (f, iA(\xi)g)_\xi. \quad (2.27)
$$

This means that the operator $iA(\xi)$ is self-adjoint with respect to the inner product $(\cdot, \cdot)_\xi$ so that

$$
\|(\lambda P_0 - A(\xi))^{-1}\|_\xi = \max_{-1 \leq j \leq 3} \left( |\lambda - \alpha_j(\xi)|^{-1} \right).
$$

This and the fact that $\|P_1(v \cdot \xi)P_0\| \leq C|\xi|$ imply (2.24).

Then, we show that for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > -\mu$, the operator $\lambda P_1 - Q(\xi) = \lambda P_1 - L + i P_1(v \cdot \xi)P_1$ is invertible from $N_0^\perp$ to itself. Indeed, by (1.11), we obtain for any $f \in N_0^\perp \cap D(L)$ that

$$
\text{Re}(\lambda P_1 - L + i P_1(v \cdot \xi)P_1)f = \text{Re}\lambda(f, f) - (Lf, f) \geq (\mu + \text{Re}\lambda)\|f\|^2,
$$

which implies that the operator $\lambda P_1 - Q(\xi)$ is an injective map from $N_0^\perp$ to itself so long as $\text{Re}\lambda > -\mu$, and its range $\text{Ran}[\lambda P_1 - Q(\xi)]$ is a closed subspace of $L^2(\mathbb{R}^3_0)$. It then remains to show that the operator $\lambda P_1 - Q(\xi)$ is also a surjective map from $N_0^\perp$ to $N_0^\perp$, namely, $\text{Ran}[\lambda P_1 - Q(\xi)] = N_0^\perp$. In fact, if it does not hold, then there exists a function $g \in N_0^\perp \setminus \text{Ran}[\lambda P_1 - Q(\xi)]$ with $g \neq 0$ so that for any $f \in N_0^\perp \cap D(L)$ that

$$
([\lambda P_1 - L + i P_1(v \cdot \xi)P_1]f, g) = (f, [\lambda P_1 - L - i P_1(v \cdot \xi)P_1]g) = 0,
$$

which yields $g = 0$ since the operator $\lambda P_1 - L - i P_1(v \cdot \xi)P_1$ is dissipative and satisfies the same estimate as (2.28). This is a contradiction, and thus $\text{Ran}[\lambda P_1 - Q(\xi)] = N_0^\perp$. The estimate (2.25) follows directly from (2.28).

Since

$$
P_0(v \cdot \omega)P_1f, \sqrt{M} = (P_1f, (v \cdot \omega)\sqrt{M}) = 0, \quad \forall f \in L^2(\mathbb{R}^3_0),
$$

it follows that $P_d(P_0(v \cdot \xi)P_1) = 0$. This, together with (2.25) and the fact that $\|P_0(v \cdot \xi)P_1\| \leq C|\xi|$, leads to

$$
\|P_0(v \cdot \xi)P_1(\lambda P_1 - Q(\xi))^{-1}P_1f\|_\xi = \|P_0(v \cdot \xi)P_1(\lambda P_1 - Q(\xi))^{-1}P_1f\|_\xi
\leq C(\text{Re}\lambda + \mu)^{-1}|\xi|\|f\|.
$$

(2.29)

Meanwhile, we can decompose the operator $P_0(v \cdot \xi)P_1(\lambda P_1 - Q(\xi))^{-1}P_1$ as

$$
P_0(v \cdot \xi)P_1(\lambda P_1 - Q(\xi))^{-1}P_1 = \frac{1}{\lambda} P_0(v \cdot \xi)P_1 + \frac{1}{\lambda} P_0(v \cdot \xi)P_1 Q(\xi)(\lambda P_1 - Q(\xi))^{-1}P_1.
$$

This, together with (2.25) and the fact that $\|P_0(v \cdot \xi)P_1 Q(\xi)\| \leq C(|\xi| + |\xi|^2)$, gives

$$
\|P_0(v \cdot \xi)P_1(\lambda P_1 - Q(\xi))^{-1}P_1f\|_\xi \leq C|\lambda|^{-1}[(\text{Re}\lambda + \mu)^{-1} + 1](|\xi| + |\xi|^2)\|f\|.
$$

(2.30)

The combination of the two cases (2.29) and (2.30) yields (2.26).
By Lemmas 2.2–2.5, we are able to analyze the spectral and resolvent sets of the operator $B(\xi)$ as follows:

**Lemma 2.6.** For any $\delta_1 > 0$ and $\delta_2 > 0$, there exist two constants $y_1 = y_1(\delta_1) > 0$ and $r_2 = r_2(\delta_1, \delta_2) > 0$ such that the following holds:

1. It holds for all $\xi \neq 0$ that the resolvent set of $B(\xi)$ contains the following domain:
   \[
   \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1, |\text{Im}\lambda| \geq y_1 \} \cup \{ \lambda \in \mathbb{C} | \text{Re}\lambda > 0 \} \subset \rho(B(\xi)).
   \]  
   (2.31)

2. It holds for $0 < |\xi| \leq r_2$ that the spectrum set of $B(\xi)$ is located in the domain
   \[
   \sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1 \} \subset \sum_{j=-1}^{1} \{ \lambda \in \mathbb{C} | |\lambda - \alpha_j(\xi)| \leq \delta_2 \},
   \]  
   (2.32)

where $\alpha_j(\xi)$, $j = -1, 0, 1$, are the eigenvalues of $A(\xi)$ defined in (2.22).

**Proof.** By Lemmas 2.5, we have for $\text{Re}\lambda > -\mu$ and $\lambda \neq \alpha_j(\xi)$ ($-1 \leq j \leq 3$) that the operator $\lambda P_0 - A(\xi) + \lambda P_1 - Q(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$ and satisfies
   \[
   (\lambda P_0 - A(\xi) + \lambda P_1 - Q(\xi))^{-1} = (\lambda P_0 - A(\xi))^{-1} P_0 + (\lambda P_1 - Q(\xi))^{-1} P_1,
   \]  
   because the operator $\lambda P_0 - A(\xi)$ is orthogonal to $\lambda P_1 - Q(\xi)$. Therefore, we can re-write (2.19) as
   \[
   \lambda - B(\xi) = (I + Y_1(\lambda, \xi))((\lambda P_0 - A(\xi)) + (\lambda P_1 - Q(\xi))),
   \]
   \[
   Y_1(\lambda, \xi) = i P_1(v \cdot \xi) P_0(\lambda P_0 - A(\xi))^{-1} P_0 + i P_0(v \cdot \xi) P_1(\lambda P_1 - Q(\xi))^{-1} P_1.
   \]

As shown in the proof of Lemma 2.4, for any $\delta_1 > 0$ there exists $r_1 = r_1(\delta_1) > 0$ so that $\rho(B(\xi)) \supset \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta_1 \}$ for $|\xi| > r_1$. For the case $|\xi| \leq r_1$, by (2.24) and (2.26) we can choose $y_1 = y_1(\delta_1) > 0$ such that it holds for $\text{Re}\lambda \geq -\mu + \delta_1$ and $|\text{Im}\lambda| \geq y_1$ that
   \[
   \| P_1(v \cdot \xi) P_0(\lambda P_0 - A(\xi))^{-1} P_0 \|_{\xi} \leq \frac{1}{4}, \quad \| P_0(v \cdot \xi) P_1(\lambda P_1 - Q(\xi))^{-1} P_1 \|_{\xi} \leq \frac{1}{4}.
   \]  
   (2.33)

This implies that the operator $I + Y_1(\lambda, \xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$ and thus $\lambda - B(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$ and satisfies
   \[
   (\lambda - B(\xi))^{-1} = [(\lambda P_0 - A(\xi))^{-1} P_0 + (\lambda P_1 - Q(\xi))^{-1} P_1](I + Y_1(\lambda, \xi))^{-1}.
   \]  
   (2.34)

Therefore, $\rho(B(\xi)) \supset \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu + \delta_1, |\text{Im}\lambda| \geq y_1 \}$ for $|\xi| \leq r_1$. This and Lemma 2.2 lead to (2.31).

Assume that $\min_{-1 \leq j \leq 1} |\lambda - \alpha_j(\xi)| > \delta_2$ and $\text{Re}\lambda \geq -\mu + \delta_1$. Then, by (2.24) and (2.26) we can choose $r_2 = r_2(\delta_1, \delta_2) > 0$ so that estimates (2.33) still hold for $0 < |\xi| \leq r_2$, and the operator $\lambda - B(\xi)$ is invertible on $L^2_\xi(\mathbb{R}^3_v)$. Therefore, we have $\rho(B(\xi)) \supset \{ \lambda \in \mathbb{C} | \min_{-1 \leq j \leq 1} |\lambda - \alpha_j(\xi)| > \delta_2, \text{Re}\lambda \geq -\mu + \delta_1 \}$ for $0 < |\xi| \leq r_2$, which gives (2.32). \qed
2.2. Low Frequency Asymptotics of Eigenvalues

We study the low frequency asymptotics of the eigenvalues and eigenfunctions of the operator $B(\xi)$ in this subsection. In terms of (1.17), the eigenvalue problem $B(\xi)f = \lambda f$ can be written as

$$\lambda f = Lf - i(v \cdot \xi) f - \frac{i(v \cdot \xi)}{|\xi|^2} P_d f, \quad |\xi| \neq 0. \quad (2.35)$$

By macro-micro decomposition, the eigenfunction $f$ of (2.35) can be divided into

$$f = f_0 + f_1 =: P_0 f + P_1 f.$$ 

Hence (2.35) gives

$$\lambda f_0 = -P_0[i(v \cdot \xi)(f_0 + f_1)] - \frac{i(v \cdot \xi)}{|\xi|^2} P_d f_0, \quad (2.36)$$

$$\lambda f_1 = Lf_1 - P_1[i(v \cdot \xi)(f_0 + f_1)]. \quad (2.37)$$

By Lemma 2.5 and (2.37), the microscopic part $f_1$ can be represented by

$$f_1 = i[L - \lambda P_1 - i P_1(v \cdot \xi) P_1]^{-1} P_1(v \cdot \xi) f_0, \quad \text{Re} \lambda > -\mu. \quad (2.38)$$

Substituting (2.38) into (2.36), we obtain the eigenvalue problem for macroscopic part $f_0$ as

$$\lambda f_0 = -i P_0(v \cdot \xi) f_0 - \frac{i(v \cdot \xi)}{|\xi|^2} P_d f + P_0[(v \cdot \xi) R(\lambda, \xi) P_1(v \cdot \xi) f_0], \quad (2.39)$$

where

$$R(\lambda, \xi) = [L - \lambda P_1 - i P_1(v \cdot \xi) P_1]^{-1}, \quad \text{Re} \lambda > -\mu. \quad (2.40)$$

To solve the eigenvalue problem (2.39), we write $f_0 \in N_0$ as

$$f_0 = \sum_{j=0}^{4} W_j \chi_j \quad \text{with} \quad W_j = (f_0, \chi_j). \quad (2.41)$$

Taking the inner product between (2.39) and $\chi_j$ for $j = 0, 1, 2, 3, 4$, respectively, we have the equations about $\lambda$ and $(W_0, W, W_4)$ with $W = (W_1, W_2, W_3)$ for $\text{Re} \lambda > -\mu$:

$$\lambda W_0 = -i(W \cdot \xi) =: -i \sum_{k=1}^{3} W_k \xi_k, \quad (2.42)$$

$$\lambda W_j = -i W_0 \left( \frac{\xi_j}{|\xi|^2} \right) - i \sqrt{\frac{2}{3}} W_4 \xi_j$$

$$+ \sum_{k=1}^{4} W_k (R(\lambda, \xi) P_1(v \cdot \xi) \chi_k, (v \cdot \xi) \chi_j), \quad j = 1, 2, 3, \quad (2.43)$$

$$\lambda W_4 = -i \sqrt{\frac{2}{3}} (W \cdot \xi) + \sum_{k=1}^{4} W_k (R(\lambda, \xi) P_1(v \cdot \xi) \chi_k, (v \cdot \xi) \chi_4). \quad (2.44)$$

We apply the following transform so as to simplify the system (2.42)–(2.44):
Lemma 2.7. Let \( e_1 = (1, 0, 0) \), \( \xi = s \omega \) with \( s \in \mathbb{R} \), \( \omega = (\omega_1, \omega_2, \omega_3) \in S^2 \). Then, it holds for \( 1 \leq i, j \leq 3 \) and \( \text{Re} \lambda > -\mu \) that

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_j, (v \cdot \xi)\chi_i) = s^2(\delta_{ij} - \omega_i \omega_j)(R(\lambda, se_1)P_1(v_1 \chi_2), v_1 \chi_2) + s^2\omega_i \omega_j (R(\lambda, se_1)P_1(v_1 \chi_2), v_1 \chi_2),
\]

(2.45)

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_i) = s^2\omega_1 (R(\lambda, se_1)P_1(v_1 \chi_4), v_1 \chi_1),
\]

(2.46)

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_i, (v \cdot \xi)\chi_i) = s^2\omega_1 (R(\lambda, se_1)P_1(v_1 \chi_1), v_1 \chi_1),
\]

(2.47)

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_4) = s^2(R(\lambda, se_1)P_1(v_1 \chi_4), v_1 \chi_4).
\]

(2.48)

Proof. Let \( \mathcal{O} \) be an orthogonal transformation of \( \mathbb{R}^3 \), and denote \((\mathcal{O}f)(v) = f(\mathcal{O}v)\). Recalling the definition (1.9)

\[
(Lf)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_s|, \omega)(\sqrt{M_s}f' + \sqrt{M_s}f'_s - \sqrt{M_s}f - \sqrt{M_s}f'_s)\sqrt{M_s}d\omega dv_s,
\]

and making the variable transforms \( v \to \mathcal{O}v, v_s \to \mathcal{O}v_s \) and \( \omega \to \mathcal{O}\omega \), which imply \( v' \to \mathcal{O}v' \) and \( v'_s \to \mathcal{O}v'_s \), we can prove \((Lf)(\mathcal{O}v) = L(\mathcal{O}f)(v), P_0f(\mathcal{O}v) = P_0(\mathcal{O}f)(v)\) and \((R(\lambda, \xi)f)(\mathcal{O}v) = R(\lambda, \mathcal{O}^T \xi)f)(\mathcal{O}v)\) by straightforward computation.

For any given \( \xi \neq 0 \), we choose \( \mathcal{O} \) to be a rotation transform of \( \mathbb{R}^3 \) satisfying \( \mathcal{O}^T \xi = se_1 \), from which we have \( \mathcal{O}_i e_1 = \omega_i \). By changing variable \( v = \mathcal{O}u \) so that \( v \cdot \xi = u \cdot \mathcal{O}^T \xi = su_1 \), we have

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_j, (v \cdot \xi)\chi_i) = s^2 \sum_{k,l=1}^{3} \mathcal{O}_{jkl} \mathcal{O}_{i}P_1(u_1 \chi_k, u_1 \chi_l).
\]

(2.49)

Assume that \( l \neq 1 \) if \( k \neq l \). By changing variable \( w_l = -u_l, w_j = u_j \) (\( j \neq l \)), we have

\[
(R(\lambda, se_1)P_1(u_1 \chi_k), u_1 \chi_l) = -(R(\lambda, se_1)P_1(w_1 \chi_k), w_1 \chi_l),
\]

where we have used the fact that \( R(\lambda, se_1) \) is invariant under any rotation transform \( \mathcal{O} \) with \( \mathcal{O}e_1 = e_1 \). This implies

\[
(R(\lambda, se_1)P_1(u_1 \chi_k), u_1 \chi_l) = 0, \quad \text{for} \quad k \neq l.
\]

If \( k = l = 3 \), by changing variable \( w_2 = u_3, w_3 = u_2, w_1 = u_1 \), we have

\[
(R(\lambda, se_1)P_1(u_1 \chi_3), u_1 \chi_3) = (R(\lambda, se_1)P_1(w_1 \chi_2), w_1 \chi_2).
\]

The combination of the above two cases yields (2.45).

Applying the above variable transform \( v = \mathcal{O}u \) again, we have

\[
(R(\lambda, \xi)P_1(v \cdot \xi)\chi_4, (v \cdot \xi)\chi_i) = s^2 \sum_{k=1}^{3} \mathcal{O}_{ik} (R(\lambda, se_1)P_1(u_1 \chi_4), u_1 \chi_k),
\]

which leads to (2.46) since it can be shown that \( (R(\lambda, se_1)P_1(u_1 \chi_4), u_1 \chi_k) = 0 \) if \( k \neq 1 \) after changing variable \( u_k = -u_k \) and \( w_j = u_j \) with \( j \neq k \). Similar argument yields (2.47) and (2.48). \(\square\)
With the help of (2.45)–(2.48), the equations (2.42)–(2.44) can be simplified as

\[ \lambda W_0 = -is(W \cdot \omega), \quad (2.50) \]

\[ \lambda W_j = -iW_0 \left( s + \frac{1}{s} \right) \omega_j - is\sqrt{\frac{2}{3}} W_4 \omega_j + s^2 (W \cdot \omega) \omega_j R_{11} \]

\[ + s^2 (W_j - (W \cdot \omega) \omega_j) R_{22} + s^2 W_4 \omega_j R_{41}, \quad j = 1, 2, 3, \quad (2.51) \]

\[ \lambda W_4 = -is\sqrt{\frac{2}{3}} (W \cdot \omega) + s^2 (W \cdot \omega) R_{14} + s^2 W_4 R_{44}, \quad (2.52) \]

where

\[ R_{ij} = R_{ij}(\lambda, s) =: (R(\lambda, se_1) P_1(v_1 \chi_i), v_1 \chi_j). \quad (2.53) \]

Multiplying (2.51) by \( \omega_j \) and making the summation of resulted equations, we have

\[ \lambda (W \cdot \omega) = -iW_0 \left( s + \frac{1}{s} \right) - is\sqrt{\frac{2}{3}} W_4 + s^2 (W \cdot \omega) R_{11} + s^2 W_4 R_{41}. \quad (2.54) \]

Furthermore, we multiply (2.54) by \( \omega_j \) and subtract the resulted equation from (2.51) to get

\[ (W_j - (W \cdot \omega) \omega_j) (\lambda - s^2 R_{22}) = 0, \quad j = 1, 2, 3. \quad (2.55) \]

Denote by \( U = (W_0, W \cdot \omega, W_4) \) a vector in \( \mathbb{R}^3 \). The system (2.50), (2.52) and (2.54) can be written as \( \mathbb{M} U = 0 \) with the matrix \( \mathbb{M} \) defined by

\[ \mathbb{M} = \begin{pmatrix} \lambda & is & 0 \\ is & \lambda - s^2 R_{11} & is\sqrt{\frac{2}{3}} - s^2 R_{41} \\ 0 & is\sqrt{\frac{2}{3}} - s^2 R_{14} & \lambda - s^2 R_{44} \end{pmatrix}. \quad (2.56) \]

The equation \( \mathbb{M} U = 0 \) admits a non-trivial solution \( U \neq 0 \) if and only if \( \det(\mathbb{M}) = 0 \). Denote

\[ D_0(\lambda, s) = \lambda - s^2 R_{22}(\lambda, s), \quad D(\lambda, s) = \det(\mathbb{M}). \quad (2.57) \]

Then, by a direct computation and the implicit function theorem, we can show

**Lemma 2.8.** The equation \( D_0(\lambda, s) = 0 \) has a unique \( C^\infty \) solution \( \lambda = \lambda(s) \) for \( (s, \lambda) \in [-r_0, r_0] \times B_{r_1}(0) \) with \( r_0, r_1 > 0 \) being small constants that satisfies

\[ \lambda(0) = 0, \quad \lambda'(0) = 0, \quad \lambda''(0) = 2(L^{-1} P_1(v_1 \chi_2), v_1 \chi_2). \]

We have the following result about the solution of \( D(\lambda, s) = 0 \):

**Lemma 2.9.** There exist two small constants \( r_0 > 0 \) and \( r_1 > 0 \) so that the equation \( D(\lambda, s) = 0 \) admits three \( C^\infty \) solutions \( \lambda_j(s) \) (\( j = -1, 0, 1 \)) for \( (s, \lambda_j) \in [-r_0, r_0] \times B_{r_1}(ji) \) that satisfy

\[ \lambda_j(0) = ji, \quad \lambda_j'(0) = 0, \quad (2.58) \]

\[ \lambda_{-1}''(0) = (L + i P_1)^{-1} P_1(v_1 \chi_1), \quad (L + i P_1)^{-1} P_1(v_1 \chi_1). \]
\[\pm i \left( \| (L + i P_1)^{-1} P_1 (v_1 \chi_1) \|^2 + \frac{5}{3} \right), \tag{2.59}\]

\[\lambda''_0(0) = 2(L^{-1} P_1 (v_1 \chi_4), v_1 \chi_4). \tag{2.60}\]

Moreover, \(\lambda_j(s)\) is an even function and satisfies
\[\overline{\lambda_j(s)} = \lambda_{-j}(-s) = \lambda_{-j}(s), \quad j = -1, 0, 1. \tag{2.61}\]

In particular, \(\lambda_0(s)\) is a real function.

**Proof.** From (2.56),
\[D(\lambda, s) = \lambda^3 - \lambda^2 s^2 (R_{11} + R_{44}) + \lambda \left[ 1 + \frac{5}{3} s^2 + i \sqrt{\frac{2}{3}} s^3 (R_{41} + R_{14}) + s^4 R_{44} \right] - (s^2 + s^4) R_{44}, \tag{2.62}\]

where \(R_{ij} = R_{ij}(\lambda, s), i, j = 1, 2, 4\) are defined by (2.53). It follows that
\[D(\lambda, 0) = \lambda(\lambda^2 + 1) = 0\]

has three roots \(\lambda_j = ji\) for \(j = -1, 0, 1\). Since, for each \(ji, j = -1, 0, 1\) it holds that
\[\partial_s D(ji, 0) = 0, \quad \partial_{\lambda} D(ji, 0) = 1 - 3j^2 \neq 0, \tag{2.63}\]

the implicit function theorem implies that there exist small constants \(r_0, r_1 > 0\) and a unique \(C^\infty\) function \(\lambda_j(s): [-r_0, r_0] \to B_{r_1}(ji)\) so that \(D(\lambda_j(s), s) = 0\) for \(s \in [-r_0, r_0]\), and, in particular,
\[\lambda_j(0) = ji, \quad \lambda_j'(0) = -\frac{\partial_s D(ji, 0)}{\partial_{\lambda} D(ji, 0)} = 0, \quad j = -1, 0, 1. \tag{2.64}\]

A direct computation gives
\[\partial_s^2 D(ji, 0) = 2j^2 [(L - ji P_1)^{-1} P_1 (v_1 \chi_1), v_1 \chi_1] + (L - ji P_1)^{-1} P_1 (v_1 \chi_4), v_1 \chi_4) + \frac{10}{3} ji - 2((L - ji P_1)^{-1} P_1 (v_1 \chi_4), v_1 \chi_4),\]

which, together with (2.63), yields
\[
\begin{cases}
\lambda''_0(0) = -\frac{\partial_s^2 D(0, 0)}{\partial_{\lambda} D(0, 0)} = 2(L^{-1} P_1 (v_1 \chi_4), v_1 \chi_4), \\
\lambda''_{\pm 1}(0) = -\frac{\partial_s^2 D(\pm i, 0)}{\partial_{\lambda} D(\pm i, 0)} = ((L \mp i P_1)^{-1} P_1 (v_1 \chi_1), v_1 \chi_1) \pm \frac{5}{3} i.
\end{cases} \tag{2.65}\]

Thus, (2.58)–(2.60) follow from (2.64)–(2.65) and \(\overline{\lambda''_1(0)} = \lambda''_{-1}(0)\).

Since \(R_{jj}(\lambda, -s) = R_{jj}(\lambda, s), R_{ij}(\lambda, -s) = -R_{ij}(\lambda, s), R_{ij}(\overline{\lambda}, -s) = R_{ij}(\overline{\lambda}, s), R_{ij}(\overline{\lambda}, s) = -R_{ij}(\lambda, s)\) for \(i, j = 1, 4\) and \(i \neq j\), we obtain by (2.62) that \(D(\lambda, s) = D(\lambda, -s), \overline{D(\lambda, s)} = D(\overline{\lambda}, -s)\). This together with the fact that \(\lambda_j(s) = ji + O(s^2), j = -1, 0, 1\) for \(|s| \leq r_0\) imply (2.61). \(\Box\)
Remark 2.10. In general, the electric potential equation in (1.5) takes the form
\[ \varepsilon^2 \Delta_x \Phi = \int_{\mathbb{R}^3} f \sqrt{M} dv \] with \( \varepsilon > 0 \). Then similar to the above lemma, we can prove that there exists a constant \( r_0(\varepsilon) > 0 \) such that the equation \( D(\lambda, s) = 0 \) has exactly three solutions \( \lambda_j(s) \) \( (j = 0, \pm 1) \) for \( |s| \leq r_0(\varepsilon) \), which satisfy \( \lambda_j(0) = \frac{iL}{\varepsilon} \to 0 \), as \( \varepsilon \to \infty \).

With the help of Lemmas 2.7–2.9, we are able to construct the eigenvalue \( \lambda_j(s) \) and the corresponding eigenfunction \( \psi_j(\xi) \) of \( B(\xi) \) at the low frequency. Indeed, we have

Theorem 2.11. There exists a constant \( r_0 > 0 \) such that the spectrum \( \sigma(B(\xi)) \cap \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda > -\mu/2 \} \) consists of five points \( \{ \lambda_j(s) \mid j = -1, 0, 1, 2, 3 \} \) for \( s = |\xi| \leq r_0 \). The eigenvalues \( \lambda_j(s) \) and the corresponding eigenfunction \( \psi_j(\xi) = \psi_j(s, \omega) \) with \( \omega = \xi/|\xi| \) are \( C^\infty \) functions of \( s \) for \( |s| \leq r_0 \). In particular, the eigenvalues \( \lambda_j(s) \) \( (j = -1, 0, 1, 2, 3) \) admit the following asymptotic expansion for \( |s| \leq r_0 \):

\[ \begin{align*}
\lambda_{\pm 1}(s) &= s + i(-a_1 \pm i b_1) s^2 + o(s^2), \\
\lambda_0(s) &= -a_0 s^2 + o(s^2), \\
\lambda_2(s) &= -a_2 s^2 + o(s^2) \quad \text{for} \quad \omega = \xi/|\xi|.
\end{align*} \]  

Here \( a_j > 0, \quad j = 0, 1, 2, \) and \( b_1 > 0 \) are defined by

\[ \begin{align*}
a_1 &= -\frac{1}{2} (L + i P_1)^{-1} P_1(v_1 \chi_1), \quad (L + i P_1)^{-1} P_1(v_1 \chi_1), \\
a_0 &= -(L^{-1} P_1(v_1 \chi_4), v_1 \chi_4), \quad a_2 = -(L^{-1} P_1(v_1 \chi_2), v_1 \chi_2), \\
b_1 &= \frac{1}{2} \left( \| (L + i P_1)^{-1} P_1(v_1 \chi_1) \|^2 + \frac{5}{3} \right).
\end{align*} \]  

The eigenfunctions \( \psi_j(\xi) = \psi_j(s, \omega) \) \( (j = -1, 0, 1, 2, 3) \) are orthogonal to each other and satisfy

\[ \begin{align*}
(\psi_j(s, \omega), \psi_k(s, \omega))_\xi &= \delta_{jk}, \\
\psi_j(s, \omega) &= \psi_{j, 0}(\omega) + \psi_{j, 1}(\omega)s + \psi_{j, 2}(\omega)s^2 + o(s^2), \quad |s| \leq r_0,
\end{align*} \]  

where the coefficients \( \psi_{j, n} \) are given by

\[ \begin{align*}
\psi_{0, 0} &= \chi_4, \quad \psi_{0, 1} = i L^{-1} P_1(v \cdot \omega) \chi_4, \quad (\psi_{0, 2}, \chi_0) = -\sqrt{\frac{2}{3}}, \\
\psi_{\pm 1, 0} &= \sqrt{\frac{2}{3}} \chi_0 \mp \sqrt{\frac{2}{3}} \chi_4 + i (L \mp i P_1)^{-1} P_1(v \cdot \omega)^2 \chi_0, \\
\psi_{\pm 1, 1} &= \mp \sqrt{\frac{2}{3}} \chi_0 \mp \sqrt{\frac{2}{3}} \chi_4 + i (L \mp i P_1)^{-1} P_1(v \cdot \omega)^2 \chi_0, \\
\psi_{j, 0} &= (v \cdot W^j) \chi_0, \quad (\psi_{j, n}, \chi_0) = (\psi_{j, n}, \chi_4) = 0 \quad (n \geq 0), \\
\psi_{j, 1} &= i L^{-1} P_1[(v \cdot \omega)(v \cdot W^j) \chi_0], \quad j = 2, 3.
\end{align*} \]  

Here, \( W^j \) \( (j = 2, 3) \) are orthonormal vectors satisfying \( W^j \cdot \omega = 0 \).
Remark 2.12. Different from the asymptotical behaviors of the eigenvalues of the linearized Vlasov–Poisson–Boltzmann operator shown in Theorem 2.11, the eigenvalues of the linearized Boltzmann operator $E(\hat{\xi}) := L - i(\hat{\nu} \cdot \hat{\xi})$ have the following expansions for $s = |\hat{\xi}| \leq r_1$ with $r_1 > 0$ a constant (cf. [8])

$$
\begin{aligned}
\lambda_{\pm 1}(s) &= \pm i \sqrt{\frac{5}{3} s - a_1 s^2 + o(s^2)}, \quad \lambda_1 = \lambda_{-1}, \\
\lambda_0(s) &= -a_0 s^2 + o(s^2), \\
\lambda_2(s) &= \lambda_3(s) = -a_2 s^2 + o(s^2),
\end{aligned}
$$

where $a_j > 0$, $j = 0, 1, 2$ are defined by

$$
\begin{aligned}
a_1 &= -\frac{1}{5}(L^{-1} P_1(v_1 \chi_4), v_1 \chi_4) - \frac{1}{2}(L^{-1} P_1(v_1 \chi_1), v_1 \chi_1), \\
a_0 &= -\frac{3}{5}(L^{-1} P_1(v_1 \chi_4), v_1 \chi_4), \quad a_2 = -(L^{-1} P_1(v_1 \chi_2), v_1 \chi_2).
\end{aligned}
$$

Remark 2.13. The operator $A(\hat{\xi})$ defined by (2.20) corresponds exactly to that of the linearized Euler-Poisson system after taking Fourier transform. Let $(\alpha_j(\hat{\xi}), \phi_j(\hat{\xi}))$, $-1 \leq j \leq 3$, be the eigenvalue and the corresponding eigenfunction of $A(\hat{\xi})$. Then we can obtain

$$
\begin{aligned}
\alpha_j(\hat{\xi}) &= 0, \quad j = 0, 2, 3, \quad \alpha_{\pm 1}(\hat{\xi}) = \pm i \sqrt{1 + \frac{5}{3} |\hat{\xi}|^2}, \\
\phi_0(\hat{\xi}) &= \frac{\sqrt{2} |\hat{\xi}|^2}{\sqrt{3 + 5 |\hat{\xi}|^2}} \chi_0 - \frac{\sqrt{3 + 3 |\hat{\xi}|^2}}{\sqrt{3 + 5 |\hat{\xi}|^2}} \chi_4, \\
\phi_{\pm 1}(\hat{\xi}) &= \sqrt{\frac{1}{2}} \left( \frac{\sqrt{3} |\hat{\xi}|}{\sqrt{3 + 5 |\hat{\xi}|^2}} \chi_0 \mp \frac{\hat{\nu} \cdot \hat{\xi}}{|\hat{\xi}|} \chi_0 + \frac{\sqrt{2} |\hat{\xi}|}{\sqrt{3 + 5 |\hat{\xi}|^2}} \chi_4 \right), \\
\phi_j(\hat{\xi}) &= \hat{\nu} \cdot W^j \chi_0, \quad j = 2, 3,
\end{aligned}
$$

where $W^j = (W^j_1, W^j_2, W^j_3)$ satisfies $W^j \cdot \hat{\xi} = 0$ and $W^i \cdot W^j = \delta_{ij}$ for $i, j = 2, 3$. It can be seen that $\phi_j(\hat{\xi})$ is not orthonormal to each other with the inner product $(\cdot, \cdot)$, but they obey the orthonormal relation as

$$(\phi_i(\hat{\xi}), \phi_j(\hat{\xi}))_{\hat{\xi}} = \delta_{ij}, \quad -1 \leq i, j \leq 3.$$

Proof. The eigenvalues $\lambda_j(s)$ and the eigenfunctions $\psi_j(s, \omega)$ can be constructed as follows: for $j = 2, 3$, we take $\lambda_j = \lambda(s)$ to be the solution of the equation $D_0(\lambda, s) = 0$ defined in Lemma 2.8, and choose $W_0 = 0$, $W_4 = 0$, and $W^j$ to be the linearly independent vector so that $W^2 \cdot \omega = 0$ and $W^2 \cdot W^3 = 0$. The corresponding eigenfunctions $\psi_2(s, \omega)$ and $\psi_3(s, \omega)$ are defined by

$$
\psi_j(s, \omega) = (W^j \cdot \nu) \chi_0 + is[L - \lambda_j P_1 - is P_1(\nu \cdot \omega) P_1]^{-1} P_1[(\nu \cdot \omega)(W^j \cdot \nu) \chi_0],
$$

which are orthonormal, i.e., $(\psi_2(s, \omega), \psi_3(s, \omega))_{\hat{\xi}} = 0$.

For $j = -1, 0, 1$, we choose $\lambda_j = \lambda_j(s)$ to be a solution of $D(\lambda, s) = 0$ given by Lemma 2.9, and denote by $\{a_j, b_j, d_j\} =: \{W^j_0, (W \cdot \omega)^j, W^j_4\}$ a solution of
system (2.50), (2.52), and (2.54) for \( \lambda = \lambda_j(s) \). Then we can construct \( \psi_j(s, \omega) \) (\( j = -1, 0, 1 \)) as

\[
\psi_j(s, \omega) = P_0 \psi_j(s, \omega) + P_1 \psi_j(s, \omega),
\]

\[
P_0 \psi_j(s, \omega) = a_j(s) \chi_0 + b_j(s) (v \cdot \omega) \chi_0 + d_j(s) \chi_4,
\]

\[
P_1 \psi_j(s, \omega) = i s [ L - \lambda_j P_1 - i s P_1 (v \cdot \omega) P_1 ]^{-1} P_1 [(v \cdot \omega) P_0 \psi_j(s, \omega)].
\]

We write

\[
\left( L - i s (v \cdot \omega) - \frac{i}{s} (v \cdot \omega) P_d \right) \psi_j(s, \omega) = \lambda_j(s) \psi_j(s, \omega), \quad -1 \leq j \leq 3.
\]

Taking the inner product \((\cdot, \cdot)_\xi\) of the above equation with \(\overline{\psi_j(s, \omega)}\) and using the facts that

\[
(B(\xi), f, g)_\xi = (f, B(-\xi) g)_\xi, \quad f, g \in D(B(\xi)),
\]

\[
B(-\xi) \psi_j(s, \omega) = \overline{\lambda_j(s)} \cdot \overline{\psi_j(s, \omega)},
\]

we have

\[
(\lambda_j(s) - \lambda_k(s)) (\psi_j(s, \omega), \overline{\psi_k(s, \omega)})_\xi = 0, \quad -1 \leq j, k \leq 3.
\]

For \( s \neq 0 \) being sufficiently small, \( \lambda_j(s) \neq \lambda_k(s) \) for \(-1 \leq j \neq k \leq 2\). Therefore, we have

\[
(\psi_j(s, \omega), \overline{\psi_k(s, \omega)})_\xi = 0, \quad -1 \leq j \neq k \leq 3.
\]

We can normalize them by taking \((\psi_j(s, \omega), \overline{\psi_j(s, \omega)})_\xi = 1\) for \(-1 \leq j \leq 3\).

The coefficients \( W_j = b_2(s) T_j(\omega) \) with \( T^j = (T^j_1, T^j_2, T^j_3) \in S^2 \) for \( j = 2, 3 \) defined in (2.72) are determined by the normalization condition as

\[
\begin{align*}
\frac{b_2(s)^2 (1 + s^2 D_2(s))}{T^2 \cdot \omega} &= 1, \\
T^2 \cdot \omega &= T^3 \cdot \omega = T^2 \cdot T^3 = 0,
\end{align*}
\]

(2.74)

where \( D_2(s) = (R(\lambda_2(s), se_1) P_1 v_1 \chi_1, R(\lambda_2(s), -se_1) P_1 v_1 \chi_1) \). Substituting (2.66) into (2.74), we obtain \( b_2(0) = 1 \) and \( b_2(-s) = b_2(s) \). This and (2.72) give the expansion of \( \Psi_j(s, \omega) \) for \( j = 1, 2 \), stated in (2.69).

To obtain expansion of \( \psi_j(s, \omega) \) for \( j = -1, 0, 1 \) defined in (2.73), we deal with its macroscopic part and microscopic part respectively. By (2.50), (2.52), and (2.54), the macroscopic part \( P_0(\psi_j(s, \omega)) \) is determined in terms of the coefficients \( \{a_j(s), b_j(s), d_j(s)\} \) that satisfy

\[
\begin{align*}
\lambda_j(s) a_j(s) + is b_j(s) &= 0, \\
(is^2 + 1) a_j(s) + (s \lambda_j(s) - s^3 R_1(\lambda_j, s)) b_j(s) + (is \sqrt{\frac{2}{3}} - s^3 R_4(\lambda_j, s)) d_j(s) &= 0,
\end{align*}
\]

(2.75)
Furthermore, we have the normalization condition

\[ 1 \equiv (\psi_j(s, \omega), \overline{\psi_j(s, \omega)}) = a_j(s)^2(1 + s^{-2}) + b_j(s)^2 + d_j(s)^2, \quad |s| \leq r_0. \]  

(2.76)

Assume

\[ a_j(s) = \sum_{n=0}^{2} a_{j,n}s^n + O(s^3), \quad b_j(s) = \sum_{n=0}^{2} b_{j,n}s^n + O(s^3), \quad d_j(s) = \sum_{n=0}^{2} d_{j,n}s^n + O(s^3). \]

Substituting the above expansion and (2.66) into (2.75) and (2.76), we have their expressions as

\[
O(1) \quad \begin{cases} 
  jia_{j,0} = ia_{j,0} = jid_{j,0} = 0, \\
  (b_{j,0})^2 + (d_{j,0})^2 + (a_{j,1})^2 = 1,
\end{cases}
\]

(2.77)

\[
O(s) \quad \begin{cases} 
  jia_{j,1} + ib_{j,0} = 0, \\
  ia_{j,1} + jib_{j,0} = 0, \\
  i\sqrt{3}b_{j,0} + jid_{j,1} = 0,
\end{cases}
\]

(2.78)

\[
O(s^2) \quad \begin{cases} 
  jia_{j,2} + ib_{j,1} = 0, \\
  ia_{j,2} + jib_{j,1} + i\sqrt{3}d_{j,0} = 0, \\
  i\sqrt{3}b_{j,1} + jid_{j,2} + \frac{1}{2}\lambda_j''(0)d_{j,0} - R_{44}(ji, 0)d_{j,0} = 0.
\end{cases}
\]

(2.79)

By a direct computation, we obtain from (2.77)–(2.79) that

\[
\begin{cases} 
  a_{0,0} = b_{0,0} = 0, & d_{0,0} = 1, & a_{0,1} = b_{0,1} = 0, & a_{0,2} = -\frac{2}{\sqrt{3}}, \\
  a_{\pm 1,0} = d_{\pm 1,0} = 0, & b_{\pm 1,0} = \mp a_{\pm 1,1} = \frac{\sqrt{3}}{2}, & d_{\pm 1,1} = \mp \frac{1}{\sqrt{3}}.
\end{cases}
\]

(2.80)

Since \( \lambda_k(s) = \lambda_k(-s), k = -1, 0, 1, \) and \( R_{jj}(-s) = R_{jj}(s), R_{ij}(-s) = -R_{ij}(s) \) with \( R_{ij}(-s) = R_{ij}(\lambda_k(s), s), i, j = 1, 4 \) and \( i \neq j, \) it follows from (2.75) and (2.80) that \( a_0(-s) = a_0(s), b_0(-s) = -b_0(s), d_0(-s) = d_0(s), a_{\pm 1}(-s) = -a_{\pm 1}(s), b_{\pm 1}(-s) = b_{\pm 1}(s), d_{\pm 1}(-s) = -d_{\pm 1}(s). \)

Base on the above argument, we can obtain the expansion of \( \psi_j(s, \omega) \) for \( j = -1, 0, 1 \) given in (2.69). This completes the proof of the theorem. \( \square \)

3. Optimal Time-Decay Rates of Linearized VPB

In this section, we consider the Cauchy problem (1.14) for the linearized Vlasov–Poisson–Boltzmann equations and establish the optimal time-decay rates of global solution based on the results obtained in Section 2.
3.1. Decomposition and Asymptotics of Linear Semigroup

We start by proving

**Lemma 3.1.** The operator \( Q(\xi) = L - i P_1 (v \cdot \xi) P_1 \) generates a strongly continuous contraction semigroup on \( N_{0}^\perp \) for any fixed \( \xi \in \mathbb{R}^3 \), which satisfies for any \( t > 0 \) and \( f \in N_{0}^\perp \) that

\[
\| e^{tQ(\xi)} f \| \leq e^{-\mu t} \| f \|. \tag{3.1}
\]

In addition, for any \( x > -\mu \) and \( f \in N_{0}^\perp \), it holds that

\[
\int_{-\infty}^{+\infty} \| (x + iy) P_1 - Q(\xi) \|^{-1} f \| dy \leq (x + \mu)^{-1} \| f \|^2. \tag{3.2}
\]

**Proof.** Since the operator \( Q(\xi) \) is a densely defined closed operator on \( N_{0}^\perp \), and both \( Q(\xi) \) and \( Q(\xi)^* = Q(-\xi) \) are dissipative on \( N_{0}^\perp \) and satisfy (2.25), it follows from Lemma 5.2 that \( Q(\xi) \) generates a strongly continuous contraction semigroup on \( N_{0}^\perp \) and satisfies (3.1).

The resolvent \( (\lambda - Q(\xi))^{-1} \) can be expressed for \( \lambda \in \rho(Q(\xi)) \) as

\[
[\lambda P_1 - Q(\xi)]^{-1} = \int_{0}^{\infty} e^{-\lambda t} e^{tQ(\xi)} dt, \quad \text{Re} \lambda > -\mu,
\]

which leads to

\[
[(x + iy) P_1 - Q(\xi)]^{-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iyt} \left[ \sqrt{2\pi} 1_{[t \geq 0]} e^{-xt} e^{tQ(\xi)} \right] dt,
\]

where the right hand side is the Fourier transform of the function \( \sqrt{2\pi} 1_{[t \geq 0]} e^{-xt} e^{tQ(\xi)} \) with respect to \( t \). By Parseval’s equality, we have for \( f \in N_{0}^\perp \) that

\[
\int_{-\infty}^{+\infty} \| [(x + iy) P_1 - Q(\xi)]^{-1} f \|^2 dy = \int_{-\infty}^{+\infty} \| (2\pi)^{\frac{1}{2}} 1_{[t \geq 0]} e^{-xt} e^{tQ(\xi)} f \|^2 dt \leq 2\pi \int_{0}^{\infty} e^{-2(x+\mu)t} dt \| f \|^2,
\]

which proves (3.2). This completes the proof of the lemma.

**Lemma 3.2.** The operator \( A(\xi) = -i P_0 (v \cdot \xi) P_0 - \frac{i(v \cdot \xi)}{|\xi|^2} P_d \) generates a strongly continuous unitary group on \( N_0 \) for any fixed \( \xi \neq 0 \), which satisfies for \( t \in \mathbb{R} \) and \( f \in N_0 \) that

\[
\| e^{tA(\xi)} f \|_\xi = \| f \|_\xi. \tag{3.3}
\]

In addition, for any \( x \neq 0 \) and \( f \in N_0 \), it holds that

\[
\int_{-\infty}^{+\infty} \| [(x + iy) P_0 - A(\xi)]^{-1} f \|^2 \xi dy = \pi |x|^{-1} \| f \|^2_\xi. \tag{3.4}
\]
Proof. Since the operator \( iA(\xi) \) is self-adjoint on \( N_0 \) with respect to the inner product \((\cdot, \cdot)_\xi\) by (2.27), it follows from Lemma 5.3 that \( A(\xi) \) generates a strongly continuous unitary group on \( N_0 \) and satisfies (3.3).

By a similar argument for proving (3.2), we can obtain, for \( x > 0 \),

\[
[(x + iy)P_0 - A(\xi)]^{-1} = \int_0^\infty e^{-(x+iy)t} e^{tA(\xi)} \, dt,
\]

from which we get for, \( f \in N_0 \cap L^2_\xi(\mathbb{R}^3_v) \) that

\[
\int_{-\infty}^{+\infty} \|[(x + iy)P_0 - A(\xi)]^{-1} f\|_\xi^2 \, dy = 2\pi \int_0^\infty e^{-2xt} \|e^{tA(\xi)} f\|_\xi^2 \, dt = 2\pi \int_0^\infty e^{-2xt} \|f\|_\xi^2 \, dt.
\]

As for \( x < 0 \), we have

\[
[ - (x + iy)P_0 + A(\xi)]^{-1} = \int_0^\infty e^{(x+iy)t} e^{-tA(\xi)} \, dt,
\]

which also leads to (3.4). This completes the proof of the lemma. \( \square \)

By Lemma 2.3, we are able to show the following estimate about the operator \( I - K(\lambda - c(\xi))^{-1} + iv\cdot\xi|\xi|^2 P_d(\lambda - c(\xi))^{-1} \) by applying a similar arguments as for Lemma 2.4. And we omit the proof for brevity. Indeed, we have

Lemma 3.3. Given any constant \( r_0 > 0 \). Let \( \lambda = x + iy \) with \( x > -\alpha(r_0) \) and \( \alpha(r_0) \) defined in Lemma 2.4. Then, there exists a constant \( C > 0 \) such that

\[
\sup_{y \in \mathbb{R}, |\xi| \geq r_0} \|,[I - K(\lambda - c(\xi))^{-1} + iv\cdot\xi|\xi|^2 P_d(\lambda - c(\xi))^{-1}]^{-1}\| \leq C. \quad (3.5)
\]

With the help of Lemmas 3.1–3.3, we have the decomposition of the semigroup \( S(t, \xi) = e^{tB(\xi)} \) given by

Theorem 3.4. The semigroup \( S(t, \xi) = e^{tB(\xi)} \) with \( |\xi| \neq 0 \) satisfies

\[
S(t, \xi) f = S_1(t, \xi) f + S_2(t, \xi) f, \quad f \in L^2_\xi(\mathbb{R}^3_v), \quad t > 0,
\]

where

\[
S_1(t, \xi) f = \sum_{j=1}^{3} e^{t\lambda_j(|\xi|)} \left(f, \psi_j(\xi)\right)_\xi \psi_j(\xi) 1_{|\xi| \leq r_0},
\]

with \( (\lambda_j(|\xi|), \psi_j(\xi)) \) being the eigenvalue and eigenfunction of the operator \( B(\xi) \) given by Theorem 2.11 for \(|\xi| \leq r_0\), and \( S_2(t, \xi) f =: S(t, \xi) f - S_1(t, \xi) f \) satisfies for a constant \( \sigma_0 > 0 \) independent of \( \xi \) that

\[
\|S_2(t, \xi) f\|_\xi \leq C e^{-\sigma_0 t} \|f\|_\xi, \quad t > 0.
\]
Proof. By Lemma 5.5, it is sufficient to prove (3.6) for \( f \in D(B(\xi)^2) \) because the domain \( D(B(\xi)^2) \) is dense in \( L^2_{\xi}(\mathbb{R}^3_{\xi}) \). By Lemma 5.4, the semigroup \( e^{tB(\xi)} \) can be represented by

\[
e^{tB(\xi)} f = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} (\lambda - B(\xi))^{-1} f d\lambda, \quad f \in D(B(\xi)^2), \ \kappa > 0. \quad (3.9)
\]

It remains to analyze the operator \( (\lambda - B(\xi))^{-1} \) for \( \xi \neq 0 \) in order to obtain the decomposition (3.6) for the semigroup \( e^{tB(\xi)} \).

First of all, we investigate the formula (3.9) for \( |\xi| \leq r_0 \). By (2.34) we have

\[
(\lambda - B(\xi))^{-1} = [(\lambda P_0 - A(\xi))^{-1} P_0 + (\lambda P_1 - Q(\xi))^{-1} P_1] - Z_1(\lambda, \xi), \quad (3.10)
\]

with the operator \( Z_1(\lambda, \xi) \) defined by

\[
Z_1(\lambda, \xi) = [(\lambda P_0 - A(\xi))^{-1} P_0 + (\lambda P_1 - Q(\xi))^{-1} P_1][I + Y_1(\lambda, \xi)]^{-1} Y_1(\lambda, \xi),
\]

\[
Y_1(\lambda, \xi) = i P_1 (v \cdot \xi) P_0 (\lambda P_0 - A(\xi))^{-1} P_0 + i P_0 (v \cdot \xi) P_1 (\lambda P_1 - Q(\xi))^{-1} P_1.
\]

Substituting (3.10) into (3.9), we have the following decomposition of the semigroup \( e^{tB(\xi)} \)

\[
e^{tB(\xi)} f = e^{tA(\xi)} P_0 f + e^{tQ(\xi)} P_1 f - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_1(\lambda, \xi) f d\lambda, \quad |\xi| \leq r_0.
\]

To estimate the last term on the right hand side of (3.13), let us denote

\[
U_{\kappa,N} = \frac{1}{2\pi i} \int_{-N}^{N} e^{(k + iy) \lambda} Z_1(\kappa + iy, \xi) f 1_{|\xi| \leq r_0} dy,
\]

(3.14)

where the constant \( N > 0 \) is chosen large enough so that \( N > y_1 \) with \( y_1 \) defined in Lemma 2.6. Since \( Z_1(\lambda, \xi) \) is analytic on the domain \( \text{Re} \lambda > -\mu/2 \) with only finite singularities at \( \lambda = \lambda_j(|\xi|) \in \sigma(B(\xi)) \) and \( \lambda = \alpha_j(|\xi|) \in \sigma(A(\xi)) \) for \( j = -1, 0, 1, 2, 3 \), we can shift the integration (3.14) from the line \( \text{Re} \lambda = \kappa > 0 \) to \( \text{Re} \lambda = -\mu/2 \). Then by the Residue Theorem, we obtain

\[
U_{\kappa,N} = \sum_{j=-1}^{3} \text{Res} \left\{ e^{\lambda t} Z_1(\lambda, \xi) f; \lambda_j(|\xi|) \right\} + \text{Res} \left\{ e^{\lambda t} Z_1(\lambda, \xi) f; \alpha_j(|\xi|) \right\} + U_{-\frac{\mu}{2},N} + H_N.
\]

(3.15)

where \( \text{Res} \{ f(\lambda); \lambda_j \} \) means the residue of \( f(\lambda) \) at \( \lambda = \lambda_j \) and

\[
H_N = \frac{1}{2\pi i} \left( \int_{-\frac{\mu}{2} + iN}^{\kappa + iN} - \int_{-\frac{\mu}{2} - iN}^{\kappa - iN} \right) e^{\lambda t} Z_1(\lambda, \xi) f 1_{|\xi| \leq r_0} d\lambda.
\]

The right hand side of (3.15) is estimated as follows. By Lemma 2.5, it is easy to verify that

\[
\| H_N \|_{\xi} \to 0, \quad \text{as} \quad N \to \infty.
\]

(3.16)
Let
\[
\lim_{N \to \infty} U_{-\frac{\mu}{2}, N}(t) = U_{-\frac{\mu}{2}, \infty}(t) =: \int_{-\frac{\mu}{2} - i\infty}^{-\frac{\mu}{2} + i\infty} e^{it\lambda} Z_1(\lambda, \xi) f d\lambda. \tag{3.17}
\]

Since it follows from Lemma 2.5 that \(\|Y_1(-\frac{\mu}{2} + iy, \xi)\|_\xi \leq 1/2\) for \(y \in \mathbb{R}\) and \(|\xi| \leq r_0\) with \(r_0 > 0\) being sufficiently small, the operator \(I - Y_1(-\frac{\mu}{2} + iy, \xi)\) is invertible on \(L^2_\xi(\mathbb{R}^3_v)\) and satisfies \(\|\{I - Y_1(-\frac{\mu}{2} + iy, \xi)\}^{-1}\|_\xi \leq 2\) for \(y \in \mathbb{R}\) and \(|\xi| \leq r_0\). Thus, we have, for any \(f, g \in L^2_\xi(\mathbb{R}^3_v),\)
\[
|\langle U_{-\frac{\mu}{2}, \infty}(t) f, g \rangle_\xi| \leq e^{-\frac{\mu}{2}t} \int_{-\infty}^{+\infty} |\langle Z_1(\lambda, \xi) f, g \rangle_\xi| d\lambda \\
\leq C|\xi| e^{-\frac{\mu}{2}t} \int_{-\infty}^{+\infty} (\|\lambda P_1 - Q(\xi)\|^{-1} P_1 f + \|\lambda P_0 - A(\xi)\|^{-1} P_0 f \|_\xi) \\
\times (\|\lambda P_1 - Q(\xi)\|^{-1} P_1 g + \|\lambda P_0 - A(\xi)\|^{-1} P_0 g \|_\xi) \, d\lambda, \quad \lambda = -\frac{\mu}{2} + iy.
\]

This together with (3.2) and (3.4) yield \(\|\langle U_{-\frac{\mu}{2}, \infty}(t) f, g \rangle_\xi| \leq Cr_0 \mu^{-1} e^{-\frac{\mu}{2}t} \|f\|_\xi \|g\|_\xi\), and
\[
\|U_{-\frac{\mu}{2}, \infty}(t)\|_\xi \leq Cr_0 \mu^{-1} e^{-\frac{\mu}{2}t}. \tag{3.18}
\]

Since \(\lambda_j(|\xi|), \alpha_j(|\xi|) \in \rho(Q(\xi)), \lambda_j(|\xi|) \neq \alpha_j(|\xi|),\) and
\[
Z_1(\lambda, \xi) = (\lambda P_0 - A(\xi))^{-1} P_0 + (\lambda P_1 - Q(\xi))^{-1} P_1 - (\lambda - B(\xi))^{-1},
\]
we can prove
\[
\text{Res}\{e^{it\lambda} Z_1(\lambda, \xi) f; \lambda_j(|\xi|)\} = -\text{Res}\{e^{it\lambda} (\lambda - B(\xi))^{-1} f; \lambda_j(|\xi|)\} \\
= -e^{\lambda_j(|\xi|) t} f; \psi_j(\xi), \tag{3.19}
\]
\[
\text{Res}\{e^{it\lambda} Z_1(\lambda, \xi) f; \alpha_j(|\xi|)\} = \text{Res}\{e^{it\lambda} (\lambda P_0 - A(\xi))^{-1} P_0 f; \alpha_j(|\xi|)\} \\
= e^{\alpha_j(|\xi|) t} f; \phi_j(\xi) = e^{tA(\xi)} P_0 f. \tag{3.20}
\]

Indeed, by the spectral representation formula in [11], we have for \(|\xi| \leq r_0\) and \(|\lambda - \lambda_j(|\xi|)| \leq \delta\) with \(\delta > 0\) being small
\[
(\lambda - B(\xi))^{-1} = \sum_{j=-1}^{3} \left[ \frac{P_j}{\lambda - \lambda_j(|\xi|)} + \sum_{m=1}^{n_j} \frac{D_j^m}{(\lambda - \lambda_j(|\xi|))^{m+1}} \right] + S(\lambda), \tag{3.21}
\]
where \(P_j\) is the projection operator associated with \(\lambda_j(|\xi|),\) \(D_j\) is the nilpotent operator associated with \(\lambda_j(|\xi|),\) and the operator \(S(\lambda)\) is holomorphic on the domain \(|\lambda - \lambda_j(|\xi|)| < \delta|\) with \(\delta > 0\) sufficiently small.

We claim that \(D_j = 0\) for all \(j, -1 \leq j \leq 2,\). Indeed, if \(D_j \neq 0,\) then there exists \(n_j\) such that \(D_j^n \neq 0\) for \(m \leq n_j\) and \(D_j^{n_j+1} = 0.\) Thus \(D_j^{n_j} = (B(\xi) - \lambda_j(\xi))^{n_j} P_j \neq 0,\) \(D_j^{n_j+1} = (B(\xi) - \lambda_j(\xi))^{n_j+1} P_j = 0.\) Assume that \(n_j \geq 1\) and let \(g \in L^2_\xi(\mathbb{R}^3_v)\) be such that \(h = [B(\xi) - \lambda_j(\xi)]^{n_j} P_j g \neq 0.\) Then
$[B(\xi) - \lambda_j(\xi)]h = 0$. Hence $h = C\psi_j(\xi)$ for some constant $C \neq 0$. We may normalize $h$ so that $C = 1$. Then, we have

$$1 = (\psi_j(\xi), \overline{\psi_j(\xi)})(\xi) = (\{B(\xi) - \lambda_j(\xi)\}^{\nu_j} P_j g, \overline{\psi_j(\xi)})_{\xi}$$

$$= (\{B(\xi) - \lambda_j(\xi)\}^{\nu_j-1} P_j g, [B(-\xi) - \lambda_j(\xi)]\overline{\psi_j(\xi)})_{\xi} = 0,$$

because $\overline{\psi_j(\xi)}$ is the eigenfunction of $B(-\xi)$ with eigenvalue $\lambda_j(\xi)$. This is a contradiction. Thus it holds that $D_j = 0$, and then we obtain (3.19) by (3.21) and Cauchy integral theorem.

Therefore, we conclude from (3.13) and (3.14)–(3.20) that

$$e^{tB(\xi)}f = e^{tQ(\xi)}P_1 f + U_{-\frac{1}{2},\infty}(t) + \sum_{j=-1}^{3} e^{t\lambda_j(\xi)}(f, \overline{\psi_j(\xi)})_{\xi} \psi_j(\xi), \ |\xi| \leq r_0.$$  \hfill (3.22)

Next, we turn to investigate the formula (3.9) for $|\xi| > r_0$. By (2.17) we have

$$(\lambda - B(\xi))^{-1} = (\lambda - c(\xi))^{-1} + Z_2(\lambda, \xi),$$  \hfill (3.23)

with

$$Z_2(\lambda, \xi) = (\lambda - c(\xi))^{-1}[I - Y_2(\lambda, \xi)]^{-1}Y_2(\lambda, \xi),$$  \hfill (3.24)

$$Y_2(\lambda, \xi) =: (K - i(v \cdot \xi)|\xi|^{-2} P_d)(\lambda - c(\xi))^{-1}.$$  \hfill (3.25)

Substituting (3.23) into (3.9) yields

$$e^{tB(\xi)}f = e^{tQ(\xi)}P_1 f + \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_2(\lambda, \xi) f d\lambda, \ |\xi| > r_0.$$  \hfill (3.26)

Similarly, in order to estimate the last term on the right hand side of (3.26), let us denote

$$V_{\kappa,N} = \frac{1}{2\pi i} \int_{-\kappa-iN}^{\kappa+iN} e^{(\kappa + iy) t} Z_2(\kappa + iy, \xi) 1_{|\xi| > r_0} dy$$  \hfill (3.27)

for sufficiently large constant $N > 0$ as in (3.14). Since the operator $Z_2(\lambda, \xi)$ is analytic on the domain $\text{Re}\lambda \geq -\sigma_0$ for the constant $\sigma_0 = \frac{1}{2} \alpha(r_0)$ with $\alpha(r_0) > 0$ given by Lemma 2.4, we can again shift the integration of (3.27) from the line $\text{Re}\lambda = \kappa > 0$ to $\text{Re}\lambda = -\sigma_0$ to obtain

$$V_{\kappa,N} = V_{-\sigma_0,N} + I_N,$$  \hfill (3.28)

with

$$I_N = \frac{1}{2\pi i} \left( \int_{-\sigma_0+iN}^{-\kappa+iN} - \int_{-\sigma_0-iN}^{-\kappa-iN} \right) e^{\lambda t} Z_2(\lambda, \xi) f 1_{|\xi| > r_0} d\lambda.$$  

By Lemma 2.3 and Lemma 3.3, it holds that

$$\|I_N\| \to 0 \text{ as } N \to \infty, \ \sup_{|\xi| > r_0, y \in \mathbb{R}} \|\{I - Y_2(-\sigma_0 + iy, \xi)\}^{-1}\| \leq C.$$  \hfill (3.29)
By (3.24) and (3.29), we have for any $f, g \in L^2_{\xi}(\mathbb{R}^3)$

$$|(V_{-\sigma_0, \infty}(t) f, g)| \leq C e^{-\sigma_0 t} \int_{-\infty}^{+\infty} |(Z_2(\lambda, \xi) f, g)| dy$$

$$\leq C (\|K\| + r_0^{-1}) e^{-\sigma_0 t} \int_{-\infty}^{+\infty} \|\lambda - c(\xi))^{-1} f\| \|\bar{\lambda} - c(-\xi))^{-1} g\| dy$$

$$\leq C (\|K\| + r_0^{-1}) e^{-\sigma_0 t} (\nu_0 - \sigma_0)^{-1}\|f\|\|g\|, \quad \lambda = -\sigma_0 + iy,$$  

(3.30)

where we have used the fact (cf. Lemma 2.2.13 of [19])

$$\int_{-\infty}^{+\infty} \|(x + iy - c(\xi))^{-1} f\|^2 dy \leq \pi (x + \nu_0)^{-1}\|f\|^2, \quad x > -\nu_0.$$  

From (3.30) and the fact $\|f\|^2 \leq \|f\|_{\xi}^2 \leq (1 + r_0^{-2})\|f\|^2$ for $|\xi| > r_0$, we have

$$\|V_{-\sigma_0, \infty}(t)\|_{\xi} \leq C e^{-\sigma_0 t} (\nu_0 - \sigma_0)^{-1}. \quad (3.31)$$

Therefore, we conclude from (3.26) and (3.27)–(3.31) that

$$e^{tB(\xi)} f = e^{tc(\xi)} f + V_{-\sigma_0, \infty}(t), \quad |\xi| > r_0. \quad (3.32)$$

The combination of (3.22) and (3.32) gives rise to (3.6) with $S_1(t, \xi) f$ and $S_2(t, \xi) f$ defined by

$$S_1(t, \xi) f = \sum_{j=-1}^{3} e^{t\lambda_j(|\xi|)} \left( f, \psi_j(\xi) \right)_{\xi} \psi_j(\xi) 1_{|\xi| \leq r_0},$$

$$S_2(t, \xi) f = \left( e^{tQ(\xi)} P_1 f + U_{-\nu_0, \infty}(t) \right) 1_{|\xi| \leq r_0} + \left( e^{tc(\xi)} f + V_{-\sigma_0, \infty}(t) \right) 1_{|\xi| > r_0}. \quad (3.33)$$

In particular, $S_2(t, \xi) f$ satisfies (3.8) in terms of (3.1), (3.18), (3.31) and the estimate $\|e^{tc(\xi)} 1_{|\xi| > r_0}\|_{\xi} \leq C e^{-\nu_0 t}$ because (2.4) and (1.12). □

Theorem 1.1 follows from Lemma 2.4, Theorem 2.11 and Theorem 3.4.

### 3.2. Optimal Time-Decay Rates of Linearized VPB

Let us introduce a Sobolev space of function $f = f(x, v)$ by $H^1_P = \{ f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) : \|f\|_{H^1_P} < \infty \}$ ($L^2_P = H^0_P$) with the norm $\| \cdot \|_{H^1_P}$ defined by

$$\|f\|_{H^1_P} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \|\hat{f}\|^2_{2\xi} d\xi \right)^{1/2}$$

$$= \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^l \left( \int_{\mathbb{R}^3} |\hat{f}|^2 dv + \frac{1}{|\xi|^2} \left( \int_{\mathbb{R}^3} \hat{f} \sqrt{M} dv \right)^2 \right) d\xi \right)^{1/2},$$

where $\hat{f} = \hat{f}(\xi, v)$ is the Fourier transform of $f(x, v)$ with respect to $x \in \mathbb{R}^3$. It holds that

$$\|f\|^2_{H^1_P} = \|f\|^2_{L^2(V_{-\xi})} + \|\nabla_x \Delta_x^{-1}(f, \chi_0)\|_{H^1_x}^2.$$
For any $f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, set

$$e^{tB} f_0 = (\mathcal{F}^{-1} e^{tB(\xi)} \mathcal{F}) f_0.$$  \hfill (3.33)

By Lemma 2.1, we have

$$\|e^{tB} f_0\|_{H^l_p} = \int_{\mathbb{R}^3} (1 + |\xi|^2)\|e^{tB(\xi)} \hat{f}_0\|_{\xi}^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)\|\hat{f}_0\|_{\xi}^2 d\xi = \|f_0\|_{H^l_p}.$$  

This means that the linear operator $B$ generates a strongly continuous contraction semigroup $e^{tB}$ in $H^l_p$, and therefore, $f(x, v, t) = e^{tB} f_0(x, v)$ is a global solution to the linear VPB system (1.14) for any $f_0 \in H^l_p$. We are now going to establish the time-decay rates of the global solution.

First of all, we have the upper bounds of the time decay rates of the global solutions to the system (1.14) as follows:

**Theorem 3.5.** Set $\nabla_x \Phi(t) = \nabla_x \Delta_x^{-1} (e^{tB} f_0, \chi_0)$. If $f_0 \in L^2_v(H^l_x) \cap L^{2,1} \ for \ l \geq 0$, then

$$\| (\partial^\alpha e^{tB} f_0, \chi_0) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.34)$$

$$\| (\partial^\alpha e^{tB} f_0, v\chi_0) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.35)$$

$$\| (\partial^\alpha e^{tB} f_0, \chi_4) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.36)$$

$$\| \partial^\alpha \nabla_x \Phi(t) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.37)$$

$$\| P_1 (\partial^\alpha e^{tB} f_0) \|_{L^2_{x,v}} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.38)$$

where $k = |\alpha - \alpha'|$, $\alpha' \leq \alpha$ and $|\alpha| \leq l$. Moreover, if $f_0 \in L^2_v(H^l_x) \cap L^{2,1} \ for \ l \geq 0$ and $(f_0, \chi_0) = 0$, then

$$\| (\partial^\alpha e^{tB} f_0, \chi_0) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.39)$$

$$\| (\partial^\alpha e^{tB} f_0, v\chi_0) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.40)$$

$$\| (\partial^\alpha e^{tB} f_0, \chi_4) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.41)$$

$$\| \partial^\alpha \nabla_x \Phi(t) \|_{L^2_x} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}), \hfill (3.42)$$

$$\| P_1 (\partial^\alpha e^{tB} f_0) \|_{L^2_{x,v}} \leq C (1 + t)^{-\left(\frac{|\alpha|}{4} + \frac{1}{2}\right)} (\|\partial^\alpha f_0\|_{L^2_v} + \|\partial^\alpha f_0\|_{L^{2,1}}). \hfill (3.43)$$

**Proof.** We prove (3.34)–(3.38) first. It follows from (3.6) and the Plancherel’s equality that

$$\| \partial^\alpha (e^{tB} f_0, \chi_j) \|_{L^2_x} = \| \hat{\xi}^\alpha (S(t, \xi) \hat{f}_0, \chi_j) \|_{L^2_x} \leq \| \hat{\xi}^\alpha (S_1(t, \xi) \hat{f}_0, \chi_j) \|_{L^2_x} + \| \hat{\xi}^\alpha S_2(t, \xi) \hat{f}_0 \|_{L^2_{x,v}}, \hfill (3.44)$$
\[
\| \frac{\partial}{\partial x} \nabla_x \Phi \|_{L^2} = \| \xi^\alpha |\xi|^{-1} (S(t, \xi) \hat{f}_0, \chi_0) \|_{L^2} \\
\leq \| \xi^\alpha |\xi|^{-1} (S_1(t, \xi) \hat{f}_0, \chi_0) \|_{L^2} + \| \xi^\alpha |\xi|^{-1} (S_2(t, \xi) \hat{f}_0, \chi_0) \|_{L^2}.
\]

(3.45)

By (3.8) and the fact that
\[
\int_{\mathbb{R}^3} \frac{(\xi^\alpha)^2}{|\xi|^2} |(\hat{f}_0, \chi_0)|^2 \, d\xi \leq \sup_{|\xi| \leq 1} \left| \xi^\alpha (\hat{f}_0, \chi_0) \right|^2 \int_{|\xi| \leq 1} \frac{1}{|\xi|^2} \, d\xi + \int_{|\xi| > 1} (\xi^\alpha)^2 |(\hat{f}_0, \chi_0)|^2 \, d\xi \\
\leq C (\| \partial_x \xi (f_0, \chi_0) \|_{L^1_x}^2 + \| \partial_x f_0 \|_{L^2_x,v}^2), \quad \alpha' \leq \alpha.
\]

we can estimate the high frequency terms on the right hand side of (3.44)–(3.45) as follows:

\[
\| \xi^\alpha S_2(t, \xi) \hat{f}_0 \|_{L^2_{x,v}}^2 + \| \xi^\alpha |\xi|^{-1} (S_2(t, \xi) \hat{f}_0, \chi_0) \|_{L^2_x}^2 \\
= \int_{\mathbb{R}^3} (\xi^\alpha)^2 \| S_2(t, \xi) \hat{f}_0 \|_{L^2_{x,v}}^2 \, d\xi \leq C \int_{\mathbb{R}^3} e^{-2s_0t} (\xi^\alpha)^2 \| \hat{f}_0 \|_{L^2_{x,v}}^2 \, d\xi \\
\leq C e^{-2s_0t} \left( \| \partial_x \xi (f_0, \chi_0) \|_{L^1_x}^2 + \| \partial_x f_0 \|_{L^2_x,v}^2 \right), \quad \alpha' \leq \alpha.
\]

(3.46)

By (3.7), we have, for \(|\xi| \leq r_0\), that
\[
S_1(t, \xi) \hat{f}_0 = \sum_{j=-1}^{3} e^{i\lambda_j(|\xi|)} P_j(\xi) \hat{f}_0.
\]

where
\[
P_j(\xi) \hat{f}_0 = (\hat{f}_0, \psi_j(\xi))_\xi \psi_j(\xi) \chi_{1(|\xi| \leq r_0)}.
\]

According to (2.68) and (2.69), we can decompose \(P_j(\xi) \hat{f}_0\) for \(|\xi| \leq r_0\) as

\[
P_j(\xi) \hat{f}_0 = (\hat{m}_0 \cdot W^j)(W^j \cdot v) \chi_0 + |\xi| T_j(\xi) \hat{f}_0, \quad j = 2, 3,
\]

(3.47)

\[
P_0(\xi) \hat{f}_0 = (\hat{q}_0 - \sqrt{\frac{2}{3}} \hat{n}_0) \chi_4 + |\xi| T_0(\xi) \hat{f}_0.
\]

(3.48)

\[
P_{\pm 1}(\xi) \hat{f}_0 = \frac{1}{2} \left( \hat{m}_0 \cdot \omega \mp \frac{1}{|\xi|} \hat{n}_0 \right) (v \cdot \omega) \chi_0 + \frac{1}{2} \hat{n}_0 \left( \chi_0 + \sqrt{\frac{2}{3}} \chi_4 \right) \\
\mp \frac{i}{2} \hat{n}_0 (L \mp i P_1)^{-1} P_1 (v \cdot \omega)^2 \chi_0 + |\xi| T_{\pm 1}(\xi) \hat{f}_0,
\]

(3.49)

where \((\hat{n}_0, \hat{m}_0, \hat{q}_0) = ((\hat{f}_0, \chi_0), (\hat{f}_0, v \chi_0), (\hat{f}_0, \chi_4)), W^j\) is given by (2.69), and \(T_j(\xi), -1 \leq j \leq 3,\) is the linear operators with the norm \(\| T_j(\xi) \|\) being uniformly bounded for \(|\xi| \leq r_0\). Thus, it follows from (3.47)–(3.49) that
\begin{align}
(S_1(t, \xi) \hat{f}_0, \chi_0) &= \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|) t} \hat{n}_0 + |\xi| (T(t, \xi) \hat{f}_0, \chi_0), \\
(S_1(t, \xi) \hat{f}_0, v \chi_0) &= \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|) t} (\hat{n}_0 \cdot \omega - \frac{j}{|\xi|} \hat{n}_0) \omega + \sum_{j=2, 3} e^{\lambda_j(|\xi|) t} (\hat{n}_0 \cdot W^j) W^j \\
&\quad + |\xi| (T(t, \xi) \hat{f}_0, v \chi_0), \\
(S_1(t, \xi) \hat{f}_0, \chi_4) &= \sqrt{\frac{1}{6}} \sum_{j=\pm 1} e^{\lambda_j(|\xi|) t} \hat{n}_0 + e^{\lambda_0(|\xi|) t} \left( \hat{q}_0 - \sqrt{\frac{2}{3}} \hat{n}_0 \right) + |\xi| (T(t, \xi) \hat{f}_0, \chi_4),
\end{align}

\begin{align}
P_1(S_1(t, \xi) \hat{f}_0) &= - \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|) t} \hat{n}_0 j i (L - ji P_1)^{-1} P_1 (v \cdot \omega)^2 \chi_0 \\
&\quad + |\xi| P_1 (T(t, \xi) \hat{f}_0),
\end{align}

where

\begin{align}
T(t, \xi) &= \sum_{j=-1}^{3} e^{\lambda_j(|\xi|) t} T_j(\xi).
\end{align}

Since

\begin{align}
\text{Re} \lambda_j(|\xi|) &= a_j |\xi|^2 (1 + O(|\xi|)) \leq -\beta |\xi|^2, \quad |\xi| \leq r_0,
\end{align}

for some constant $\beta > 0$, it follows from (3.50)–(3.55) that

\begin{align}
\| \xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_0) \|^2_{L^2_x} &\leq C \int_{|\xi| \leq r_0} \left( \xi^\alpha \right)^2 e^{-2\beta |\xi|^2 t} \left( |\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2_x} \right) d\xi \\
&\leq C (1 + t)^{-1} \left( \| \partial_x^\alpha n_0 \|^2_{L^1_t} + \| \partial_x^\alpha f_0 \|^2_{L^2_t} \right),
\end{align}

\begin{align}
\| \xi^\alpha (S_1(t, \xi) \hat{f}_0, v \chi_0) \|^2_{L^2_x} &\leq C \int_{|\xi| \leq r_0} \left( \xi^\alpha \right)^2 e^{-2\beta |\xi|^2 t} \left( |\hat{n}_0|^2 + |\xi|^{-2} |\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2_x} \right) d\xi \\
&\leq C (1 + t)^{-1} \left( \| \partial_x^\alpha n_0 \|^2_{L^1_t} + \| \partial_x^\alpha m_0 \|^2_{L^1_t} + \| \partial_x^\alpha f_0 \|^2_{L^2_t} \right),
\end{align}

\begin{align}
\| \xi^\alpha (S_1(t, \xi) \hat{f}_0, \chi_4) \|^2_{L^2_x} &\leq C \int_{|\xi| \leq r_0} \left( \xi^\alpha \right)^2 e^{-2\beta |\xi|^2 t} \left( |\hat{q}_0|^2 + |\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2_x} \right) d\xi \\
&\leq C (1 + t)^{-1} \left( \| \partial_x^\alpha n_0 \|^2_{L^1_t} + \| \partial_x^\alpha q_0 \|^2_{L^1_t} + \| \partial_x^\alpha f_0 \|^2_{L^2_t} \right),
\end{align}

\begin{align}
\| \xi^\alpha P_1(S_1(t, \xi) \hat{f}_0) \|^2_{L^2_x} &\leq C \int_{|\xi| \leq r_0} \left( \xi^\alpha \right)^2 e^{-2\beta |\xi|^2 t} \left( |\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2_x} \right) d\xi \\
&\leq C (1 + t)^{-1} \left( \| \partial_x^\alpha n_0 \|^2_{L^1_t} + \| \partial_x^\alpha f_0 \|^2_{L^2_t} \right),
\end{align}

for $k = |\alpha - \alpha'|$, $\alpha' \leq \alpha$. These, together with (3.44), (3.45) and (3.46), lead to (3.34)–(3.38).

Similarly, we can obtain the time decay rates (3.39)–(3.43) from (3.56)–(3.59) for the case $\hat{n}_0 = 0$ (which is true if $(f_0, \chi_0) = 0$).

Then, we show that the above time-decay rates of the global solutions are optimal. Indeed, we have
Theorem 3.6. Assume that \( f_0 \in L^2_{x,v} \cap L^{2,1} \), and that there exist two constants \( d_0 > 0 \) and \( d_1 > 0 \) such that \( \inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)| \geq d_0 > 0 \) and \( \inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_1 \sup_{|\xi| \leq r_0} |(\hat{f}_0, \chi_0)|. \) Then, it holds for \( t > 0 \) being large enough that

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| (e^{tB} f_0, \chi_0) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.60}
\]

\[
C_1(1 + t)^{-\frac{1}{4}} \leq \| (e^{tB} f_0, v\chi_0) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{1}{4}}, \tag{3.61}
\]

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| (e^{tB} f_0, \chi_4) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.62}
\]

\[
C_1(1 + t)^{-\frac{1}{4}} \leq \| \nabla_x \Phi(t) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{1}{4}}, \tag{3.63}
\]

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| P_1(e^{tB} f_0) \|_{L^2_{x,v}} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.64}
\]

where \( C_2 \geq C_1 > 0 \) are two generic constants.

If \( (f_0, \chi_0) = 0, \) \( \inf_{|\xi| \leq r_0} |(\hat{f}_0, (v \cdot \frac{\xi}{|\xi|}) \chi_0)| \geq d_0 \) and \( \inf_{|\xi| \leq r_0} |(\hat{f}_0, \chi_4)| \geq d_0, \) then it holds for \( t > 0 \) being large enough that

\[
C_1(1 + t)^{-\frac{5}{4}} \leq \| (e^{tB} f_0, \chi_0) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{5}{4}}, \tag{3.65}
\]

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| (e^{tB} f_0, v\chi_0) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.66}
\]

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| (e^{tB} f_0, \chi_4) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.67}
\]

\[
C_1(1 + t)^{-\frac{3}{4}} \leq \| \nabla_x \Phi(t) \|_{L^2_x} \leq C_2(1 + t)^{-\frac{3}{4}}, \tag{3.68}
\]

\[
C_1(1 + t)^{-\frac{5}{4}} \leq \| P_1(e^{tB} f_0) \|_{L^2_{x,v}} \leq C_2(1 + t)^{-\frac{5}{4}}. \tag{3.69}
\]

Proof. By Theorem 3.5, we only need to show the lower bounds of the time-decay rates for the solution \( e^{tB} f_0 \) under the assumptions of Theorem 3.6. Let us prove (3.60)–(3.64) first. Indeed, in terms of Theorem 3.4, we can verify that

\[
\| (e^{tB} f, \chi_j) \|_{L^2_x} \geq \| (S_1(t, \xi) \hat{f}_0, \chi_j) \|_{L^2_x} - \| S_2(t, \xi) \hat{f}_0 \|_{L^2_{x,v}},
\]

\[
\geq \| (S_1(t, \xi) \hat{f}_0, \chi_j) \|_{L^2_x} - C e^{-\sigma_0 t} (\| f_0 \|_{L^{2,1}} + \| f_0 \|_{L^2_{x,v}}), \tag{3.70}
\]

\[
\| \nabla_x \Phi(t) \|_{L^2_x} \geq \| |\xi|^{-1} (S_1(t, \xi) \hat{f}_0, \chi_0) \|_{L^2_x} - \| |\xi|^{-1} (S_2(t, \xi) \hat{f}_0, \chi_0) \|_{L^2_x}
\]

\[
\geq \| |\xi|^{-1} (S_1(t, \xi) \hat{f}_0, \chi_0) \|_{L^2_x} - C e^{-\sigma_0 t} (\| f_0 \|_{L^{2,1}} + \| f_0 \|_{L^2_{x,v}}), \tag{3.71}
\]

\[
\| P_1(e^{tB} f) \|_{L^2_{x,v}} \geq \| P_1(S_1(t, \xi) \hat{f}_0) \|_{L^2_{x,v}} - \| P_1(S_2(t, \xi) \hat{f}_0) \|_{L^2_{x,v}}
\]

\[
\geq \| P_1(S_1(t, \xi) \hat{f}_0) \|_{L^2_{x,v}} - C e^{-\sigma_0 t} (\| f_0 \|_{L^{2,1}} + \| f_0 \|_{L^2_{x,v}}), \tag{3.72}
\]

where we have used (3.46) for \( \alpha = 0. \)
By (3.50) and \( \lambda_{-1}(|\xi|) = \frac{1}{\lambda_1(|\xi|)} \), we have
\[
|\langle \hat{S}(t, \xi) \hat{f}_0, \chi_0 \rangle |^2 = \left| e^{i \Re \lambda_1(|\xi|) t} \cos(\Im \lambda_1(|\xi|) t) \hat{n}_0 + |\xi|(T(t, \xi) \hat{f}_0, \chi_0) \right|^2 \\
\geq \frac{1}{2} e^{2 \Re \lambda_1(|\xi|) t} \cos^2(\Im \lambda_1(|\xi|) t) |\hat{n}_0|^2 - C |\xi|^2 e^{-2\beta |\xi|^2 t} \| \hat{f}_0 \|^2_{L^2}.
\]
(3.73)

Since
\[
\cos^2(\Im \lambda_1(|\xi|) t) \geq \frac{1}{2} \cos^2[(1 + b_1 |\xi|^2) t] - O(|\xi|^3 t^2),
\]
and
\[
\Re \lambda_j(|\xi|) = a_j |\xi|^2 (1 + O(|\xi|)) \geq -\eta |\xi|^2, \quad |\xi| \leq r_0,
\]
for some constant \( \eta > 0 \), we obtain by (3.73) that
\[
\| \langle \hat{S}(t, \xi) \hat{f}_0, \chi_0 \rangle \|^2_{L^2} \geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2 t} \cos^2(t + b_1 |\xi|^2 t) d\xi \\
- C \int_{|\xi| \leq r_0} e^{-2\beta |\xi|^2 t} \left[ (|\xi|^3 t)^2 |\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2} \right] d\xi \\
\geq \frac{d_0^2}{4} \int_{|\xi| \leq r_0} e^{-2\eta |\xi|^2 t} \cos^2(t + b_1 |\xi|^2 t) d\xi - C (1 + t)^{-5/2}
=: I_1 - C (1 + t)^{-5/2}.
\]
(3.74)

It holds for \( t \geq t_0 =: \frac{L^2}{b_1} \) with the constant \( L \geq \sqrt{\frac{4\pi}{b_1}} \) that
\[
I_1 \geq \pi d_0^2 t^{-3/2} \int_0^L r^2 e^{-2\eta r^2} \cos^2(t + b_1 r^2) dr \\
\geq (1 + t)^{-3/2} \pi d_0^2 L e^{-2\eta L^2} \int_{L/2}^L r \cos^2(t + b_1 r^2) dr \\
\geq (1 + t)^{-3/2} \pi d_0^2 L e^{-2\eta L^2} \int_0^{\pi/2} \cos^2 y dy \geq C_3 (1 + t)^{-3/2},
\]
(3.75)

where \( C_3 > 0 \) denotes a generic positive constant. We can substitute (3.74) and (3.75) into (3.70) with \( j = 0 \) to prove (3.60) for \( t > 0 \) being large enough.

By (3.51), we can decompose
\[
\langle S_1(t, \xi) f_0, v \chi_0 \rangle = - \frac{i \xi}{|\xi|^2} e^{i \Re \lambda_1(|\xi|) t} \sin(\Im \lambda_1(|\xi|) t) \hat{n}_0 + T_1(t, \xi) \hat{f}_0,
\]
with \( T_1(t, \xi) \hat{f}_0 =: \langle S_1(t, \xi) f_0, v \chi_0 \rangle + \frac{i \xi}{|\xi|^2} e^{i \Re \lambda_1(|\xi|) t} \sin(\Im \lambda_1(|\xi|) t) \hat{n}_0 \) being the remainder terms on right hand side of (3.51). Then
\[ |(S_1(t, \xi) \hat{f}_0, v \chi_0)|^2 \geq \frac{1}{2} \| e^{2\Re \lambda_1(|\xi|) t} \sin^2(\Im \lambda_1(|\xi|) t)|\hat{n}_0|^2 - 2\| T_1(t, \xi) \hat{f}_0\|^2 \]
\[ \geq \frac{1}{2} \| e^{2\Re \lambda_1(|\xi|) t} \sin^2(\Im \lambda_1(|\xi|) t)|\hat{n}_0|^2 \]
\[ - Ce^{-2\beta|\xi|^2t} (|\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2}). \]

Similarly to (3.75), we get
\[
\| (S_1(t, \xi) \hat{f}_0, v \chi_0) \|^2_{L^2(E)} \geq \frac{d}{4} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2} \sin^2(t + b_1|\xi|^2) d\xi
\[ - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2t} (|\xi|^4 t^2 |\hat{n}_0|^2 + |\xi|^2 |\hat{f}_0|^2_{L^2}) d\xi \]
\[ \geq C_3(1 + t)^{-1/2} - C(1 + t)^{-3/2}, \tag{3.76} \]

which, together with (3.70) for \( j = 1, 2, 3 \), leads to (3.61) for \( t > 0 \) being large enough.

By (3.52) and the fact that \( \lambda_0(|\xi|) \) is real, we have
\[
|\langle S_1(t, \xi) \hat{f}_0, \chi_4 \rangle|^2 \geq \frac{1}{2} e^{2\Re \lambda_1(|\xi|) t}|\hat{q}_0|^2 - \frac{2}{3} e^{\Re \lambda_1(|\xi|) t} |\Im \lambda_1(|\xi|) t| \bigg( |\hat{n}_0|^2 \bigg)
\[ - C |\xi|^2 e^{-2\beta|\xi|^2t} \| \hat{f}_0 \|^2_{L^2}. \]

It follows that
\[
\| (S_1(t, \xi) \hat{f}_0, \chi_4) \|^2_{L^2(E)} \geq \frac{1}{2} \int_{|\xi| \leq r_0} e^{-2\eta|\xi|^2} |\hat{q}_0|^2 d\xi - C \int_{|\xi| \leq r_0} e^{-2\beta|\xi|^2t} (|\hat{n}_0|^2 + |\xi|^2 \| \hat{f}_0 \|^2_{L^2}) d\xi
\[ \geq C_3 \bigg( \inf_{|\xi| \leq r_0} |\hat{q}_0|^2 - d_1 \sup_{|\xi| \leq r_0} |\hat{n}_0|^2 \bigg) (1 + t)^{-3/2} - C(1 + t)^{-5/2}. \]

This and (3.70) with \( j = 4 \) lead to (3.62) for \( t > 0 \) being large enough.

By (3.53), we have
\[
P_1(S_1(t, \xi) \hat{f}_0) = e^{\Re \lambda_1(|\xi|) t} \hat{n}_0 \left[ \sin(\Im \lambda_1(|\xi|) t) L \Psi + \cos(\Im \lambda_1(|\xi|) t) \Psi \right]
\[ + |\xi| P_1(T(t, \xi) \hat{f}_0), \]

where \( \Psi \in N_0^\perp \) is a non-zero real function given by
\[
\Psi = (L - i P_1)^{-1}(L + i P_1)^{-1}P_1(v \cdot \omega)^2 \sqrt{M} \neq 0. \]
Thus, a direct computation yields

\[
\| P_1(S(t, \xi) \hat{f}_0) \|_{L_2}^2 \geq \frac{d_2^2}{4} \int_{|\xi| \leq r_0} e^{-2|\xi|^2r^2} \| \sin(t + b_1|\xi|^2t)L\Psi + \cos(t + b_1|\xi|^2t)\|_{L_2}^2 d\xi
\]

\[
- C \int_{|\xi| \leq r_0} e^{-2|\xi|^2r^2} \left( |\xi|^2|\hat{\mathbf{n}}_0|^2 + |\xi|^2\|\hat{f}_0\|_{L_2}^2 \right) d\xi
\]

\[
=: I_2 - C(1 + t)^{-5/2}.
\]

(3.77)

We obtain for time \( t \geq t_0 =: \frac{L^2}{r_0^2} \) with the constant \( L \geq \sqrt{\frac{4\pi}{r_0^2}} \) that

\[
I_2 \geq \pi d_2^2 t^{-3/2} \int_0^L r^2 e^{-2\nu^2r^2} \| \sin(t + b_1r^2)L\Psi + \cos(t + b_1r^2)\|_{L_2}^2 dx
\]

\[
\geq \frac{\pi d_2^2 L}{2} e^{-2\eta L^2t^{-3/2}} \int_0^\pi \| \sin(t + b_1r^2)L\Psi + \cos(t + b_1r^2)\|_{L_2}^2 r dr
\]

\[
\geq \frac{\pi d_2^2 L}{4b_1} e^{-2\eta L^2t^{-3/2}} \int_0^\pi \| L\Psi \sin y + \Psi \cos y \|_{L_2}^2 dy
\]

\[
\geq \frac{\pi d_2^2 L}{4b_1} e^{-2\eta L^2t^{-3/2}} \int_0^\pi \left( \| L\Psi \|_{L_2}^2 \sin^2 y + \| \Psi \|_{L_2}^2 \cos^2 y + (L\Psi, \Psi) \sin 2y \right) dy
\]

\[
\geq \frac{\pi d_2^2 L}{4b_1} e^{-2\eta L^2t^{-3/2}} \int_0^\pi \cos^2 y dy = \frac{\pi^2 d_0^2 L}{8b_1} e^{-2\eta L^2} \| L\Psi \|_{L_2}^2 (1 + t)^{-3/2},
\]

which together with (3.77) and (3.72) imply (3.64) for \( t > 0 \) sufficiently large.

Next, we turn to deal with (3.65)–(3.69) for the case \( \hat{f}_0, \chi_0 = 0 \). Indeed, by (3.50)–(3.52) we can obtain

\[
(S_1(t, \xi) \hat{f}_0, \chi_0) = -\frac{1}{2} |\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)^{1/2}} j (\hat{m}_0 \cdot \omega) + |\xi|^2 (T_3(t, \xi) \hat{f}_0, \chi_0),
\]

(3.78)

\[
(S_1(t, \xi) \hat{f}_0, v\chi_0) = \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|\xi|)^{1/2}} (\hat{m}_0 \cdot \omega)\omega + \sum_{j=2,3} e^{\lambda_j(|\xi|)^{1/2}} (\hat{m}_0 \cdot W^j) W^j
\]

\[
+ |\xi| (T_2(t, \xi) \hat{f}_0, v\chi_0),
\]

(3.79)

\[
(S_1(t, \xi) \hat{f}_0, \chi_4) = e^{\lambda_0(|\xi|)^{1/2}} \hat{q}_0 + |\xi| (T_2(t, \xi) \hat{f}_0, \chi_4),
\]

(3.80)

\[
P_1(S_1(t, \xi) \hat{f}_0) = i \frac{1}{2} |\xi| \sum_{j=\pm 1} e^{\lambda_j(|\xi|)^{1/2}} (\hat{m}_0 \cdot \omega)(L - ji P_1)^{-1} P_1 (v \cdot \omega) \sqrt{M}
\]

\[
+ i |\xi| \sum_{j=2,3} e^{\lambda_j(|\xi|)^{1/2}} (\hat{m}_0 \cdot W^j) L^{-1} P_1 (v \cdot \omega)(v \cdot W^j) \sqrt{M}
\]

\[
+ i |\xi| e^{\lambda_0(|\xi|)^{1/2}} \hat{q}_0 L^{-1} P_1 (v \cdot \omega) \sqrt{M} + |\xi|^2 P_1 (T_2(t, \xi) \hat{f}_0),
\]

(3.81)

where \( T_j(t, \xi) \hat{f}_0 \) for \( j = 2, 3 \) is the remainder term satisfying \( \| T_j(t, \xi) \hat{f}_0 \|_{L_2}^2 \leq C e^{-2|\xi|^2t} \| \hat{f}_0 \|_{L_2}^2 \). Since the vectors \( W^2, W^3 \) and \( \omega \) are orthogonal to each other,
and hence \((L \pm i P_1)^{-1} P_1 (v \cdot \omega)^2 \sqrt{M}, L^{-1} P_1 (v \cdot \omega) \chi_4, L^{-1} P_1 (v \cdot \omega) (v \cdot W^2)^2 \sqrt{M}\) and \(L^{-1} P_1 (v \cdot \omega) (v \cdot W^3)^2 \sqrt{M}\) are orthogonal, it follows from (3.78)–(3.81) that
\[
|\langle S_1(t, \xi) \hat{f}_0, \chi_0 \rangle|^2 \geq \frac{1}{2} |\xi|^2 e^{2 \Re \lambda_0 (|\xi|) t} \sin^2 \left( \Im \lambda_1 (|\xi|) t \right) |(\hat{m}_0 \cdot \omega)|^2 \\
- C |\xi|^4 e^{-2 \beta |\xi|^2 t} \|\hat{f}_0\|_{L^2_v}^2,
\]
\[
|\langle S_1(t, \xi) \hat{f}_0, v \chi_0 \rangle|^2 \geq \frac{1}{2} e^{2 \Re \lambda_0 (|\xi|) t} \cos^2 \left( \Im \lambda_1 (|\xi|) t \right) |(\hat{m}_0 \cdot \omega)|^2 \\
- C |\xi|^2 e^{-2 \beta |\xi|^2 t} \|\hat{f}_0\|_{L^2_v}^2,
\]
\[
|\langle S_1(t, \xi) \hat{f}_0, \chi_4 \rangle|^2 \geq \frac{1}{2} e^{2 \lambda_0 (|\xi|) t} |\hat{q}_0|^2 - C |\xi|^2 e^{-2 \beta |\xi|^2 t} \|\hat{f}_0\|_{L^2_v}^2,
\]
\[
\|P_1 \langle S_1(t, \xi) \hat{f}_0 \rangle\|_{L^2_v} \geq \frac{1}{2} \|L^{-1} P_1 (v_1 \chi_4)\|_{L^2_v}^2 |\xi|^2 e^{2 \lambda_0 (|\xi|) t} |\hat{q}_0|^2 \\
- C |\xi|^2 e^{-2 \beta |\xi|^2 t} \|\hat{f}_0\|_{L^2_v}^2.
\]
This together with the assumptions that \(\inf_{|\xi| \leq r_0} |\hat{m}_0 \cdot \omega| \geq d_0\) and \(\inf_{|\xi| \leq r_0} |\hat{q}_0| \geq d_0\) give
\[
\|\langle S_1(t, \xi) \hat{f}_0, \chi_0 \rangle\|_{L^2_v}^2 \geq C_4 (1 + t)^{-5/2}, \quad \|P_1 \langle S_1(t, \xi) \hat{f}_0 \rangle\|_{L^2_v}^2 \geq C_4 (1 + t)^{-5/2},
\]
\[
\|\langle S_1(t, \xi) \hat{f}_0, v \chi_0 \rangle\|_{L^2_v}^2 \geq C_4 (1 + t)^{-3/2}, \quad \|\langle S_1(t, \xi) \hat{f}_0, \chi_4 \rangle\|_{L^2_v}^2 \geq C_4 (1 + t)^{-3/2}.
\]
With this, (3.70), (3.71) and (3.72) imply (3.65)–(3.68). The proof is then completed. \(\square\)

4. The Original Nonlinear Problem

In this section, we prove the long time decay rates of the solution to the nonlinear Vlasov–Poisson–Boltzmann system (1.5) with the help of the asymptotic behaviors of linearized problem established in Section 3.

4.1. Hard Sphere Case

For hard sphere case, we define the weighted function \(w(v)\) by
\[
w(v) = (1 + |v|^2)^{1/2},
\]
and the energy norms by
\[
\|f\|_{H^N} = \sum_{|\alpha| + |\beta| \leq N} \|\partial_\alpha \partial_\beta f\|_{L^2_v}, \quad \|f\|_{H^N_w} = \sum_{|\alpha| + |\beta| \leq N} \|w(v) \partial_\alpha \partial_\beta f\|_{L^2_v}.
\]
For the hard sphere model, we will prove
Theorem 4.1. Assume that $f_0 \in H^N \cap L^{2,1}$ with $N \geq 4$, and $\|f_0\|_{H^N \cap L^{2,1}} \leq \delta_0$ with $\delta_0 > 0$ being small enough. Let $f$ be a solution of the VPB system (1.5). Then, it holds that, for $|\alpha| = 0, 1,$

$$
\begin{align*}
\| \partial_x^\alpha (f(t), \chi_0) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), v \chi_0) \|_{L^2_t} + \| \partial_x^\alpha \nabla_x \Phi(t) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), \chi_4) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\| P_1 f(t) \|_{H^N_t} + \| \nabla_x P_0 f(t) \|_{L^2_t(H^{N-1}_t)} &\leq C \delta_0 (1 + t)^{-\frac{3}{4}}.
\end{align*}
$$

(4.1)

Moreover, if $(f_0, \chi_0) = 0,$ then it holds that, for $|\alpha| = 0, 1,$

$$
\begin{align*}
\| \partial_x^\alpha (f(t), \chi_0) \|_{L^2_t} + \| \partial_x^\alpha P_1 f(t) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{5}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), v \chi_0) \|_{L^2_t} + \| \partial_x^\alpha \nabla_x \Phi(t) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{5}{4} - \frac{|\alpha|}{2}}, \\
\| \partial_x^\alpha (f(t), \chi_4) \|_{L^2_t} &\leq C \delta_0 (1 + t)^{-\frac{5}{4} - \frac{|\alpha|}{2}}, \\
\| P_1 f(t) \|_{H^N_t} + \| \nabla_x P_0 f(t) \|_{L^2_t(H^{N-1}_t)} &\leq C \delta_0 (1 + t)^{-\frac{5}{4}}.
\end{align*}
$$

(4.2)

Proof. First, we deal with (4.1). Let $f$ be a solution to the IVP problem (1.5) for $t > 0.$ We can represent this solution in terms of the semigroup $e^{tB}$ as

$$
f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} G(s) ds,
$$

(4.3)

where the nonlinear term $G$ is given by (1.6). For this global solution $f,$ we define a functional $Q(t)$ for any $t > 0$ as

$$
Q(t) = \sup_{0 \leq s \leq t} \sum_{|\alpha| = 0, 1} \left\{ (1 + s)^{\frac{3}{4} + \frac{|\alpha|}{2}} \| \partial_x^\alpha (f(s), \chi_0) \|_{L^2_t} + (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \| \partial_x^\alpha (f(s), v \chi_0) \|_{L^2_t}
\right.
\left. + (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \| \partial_x^\alpha (f(s), \chi_4) \|_{L^2_t} + (1 + s)^{\frac{3}{4}} \| \nabla_x \Phi(s) \|_{L^2_t}
\right.
\left. + (1 + s)^{\frac{3}{4}} (\| P_1 f(s) \|_{H^N_t} + \| \nabla_x P_0 f(s) \|_{L^2_t(H^{N-1}_t)}) \right\}.
$$

We claim that it holds under the assumptions of Theorem 4.1 that

$$
Q(t) \leq C \delta_0.
$$

(4.4)

It is easy to verify that the estimate (4.1) follows from (4.4).

To estimate the second term in the right hand side of (4.3), we shall use the following estimates: since the term $\Gamma(f, g)$ satisfies (cf. [4, 19])

$$
\| \Gamma(f, g) \|_{L^2_v} \leq C (\| \partial_x f \|_{L^2_t} \| v g \|_{L^2_v} + \| v f \|_{L^2_v} \| g \|_{L^2_v}),
$$

(4.5)

we can estimate the nonlinear term $G(s)$ given by (1.6) for $0 \leq s \leq t$ in terms of $Q(t)$ as

$$
\| G(s) \|_{L^2_{x,v}} \leq C (\| \nabla_x f \|_{L^2_{x,v}} + \| \nabla_x \Phi \|_{L^2_{x,v}} (\| w f \|_{L^2_{x,v}} + \| \nabla_v f \|_{L^2_{x,v}})).
$$
\[ \leq C(1 + s)^{-1} Q(t)^2, \quad (4.6) \]
\[ \|G(s)\|_{L^2_{x,v}} \leq C \left\{ \|f\|_{L^2_{x,v}} \left\| w f \right\|_{L^2_{x,v}} + \|\nabla_x \Phi\|_{L^2_{x,v}} \left( \|w f\|_{L^2_{x,v}} + \|\nabla_v f\|_{L^2_{x,v}} \right) \right\} \]
\[ \leq C(1 + s)^{-1/2} Q(t)^2, \quad (4.7) \]

and similarly,
\[ \|\nabla_x G(s)\|_{L^2_{x,v}} \leq C(1 + s)^{-3/2} Q(t)^2, \quad (4.8) \]
\[ \|\nabla_x G(s)\|_{L^2_{1,1}} \leq C(1 + s)^{-1} Q(t)^2. \quad (4.9) \]

In the case of \((f_0, \chi_0) = 0\), we can obtain by (3.78) that
\[ \|\partial_x^\alpha (e^{tB} f_0, \chi_0)\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{1}{2} - \frac{\alpha}{2}} \left( \|\partial_x^\alpha f_0\|_{L^2_{x,v}} + \|(f_0, v \chi_0)\|_{L^2_{x,v}} + \|\nabla_x f_0\|_{L^2_{x,v}} \right), \quad (4.10) \]
\[ \|\partial_x^\alpha (e^{tB} f_0, \chi_0)\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{3}{2} - \frac{\alpha}{2}} \left( \|\partial_x^\alpha f_0\|_{L^2_{x,v}} + \|\nabla_x f_0\|_{L^2_{1,1}} \right) \quad (4.11) \]

for \(|\alpha| \geq 0\).

Noting that \((G, \chi_0) = 0\), we obtain by (3.34), (3.39), (4.10), (4.11), and (4.6)–(4.9) the long time decay rate of the macroscopic density \((f(t), \chi_0)\) as
\[ \|(f(t), \chi_0)\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{3}{2}} \left( \|f_0\|_{L^2_{x,v}} + \|f_0\|_{L^2_{1,1}} \right) \]
\[ + C \int_0^{t/2} (1 + t - s)^{-\frac{3}{2}} \left( \|G(s)\|_{L^2_{x,v}} + \|G(s)\|_{L^2_{1,1}} \right) ds \]
\[ + C \int_{t/2}^t (1 + t - s)^{-\frac{3}{2}} \left( \|G(s)\|_{L^2_{x,v}} + \|\nabla_x G(s)\|_{L^2_{1,1}} \right) ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{3}{4}} + C \int_0^{t/2} (1 + t - s)^{-\frac{5}{2}} (1 + s)^{-\frac{1}{2}} Q(t)^2 ds \]
\[ + C \int_{t/2}^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-1} Q(t)^2 ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{3}{4}} + C(1 + t)^{-\frac{3}{2}} Q(t)^2, \quad (4.12) \]

and
\[ \|\nabla_x f(t), \chi_0\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{5}{4}} \left( \|\nabla_x f_0\|_{L^2_{x,v}} + \|f_0\|_{L^2_{1,1}} \right) \]
\[ + C \int_0^{t/2} (1 + t - s)^{-\frac{7}{4}} \left( \|\nabla_x G(s)\|_{L^2_{x,v}} + \|G(s)\|_{L^2_{1,1}} \right) ds \]
\[ + C \int_{t/2}^t (1 + t - s)^{-1} \left( \|(G(s), v \chi_0)\|_{L^2_{x,v}} + \|\nabla_x G(s)\|_{L^2_{1,1}} \right) ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{5}{4}} + C(1 + t)^{-\frac{5}{4}} Q(t)^2, \quad (4.13) \]

where we have used the following estimates:
\[ \|(G(s), v \chi_0)\|_{L^2_{x,v}} + \|\nabla_x G(s)\|_{L^2_{1,1}} \leq C(1 + s)^{-\frac{3}{2}} Q(t)^2, \quad 0 \leq s \leq t. \]
Similarly, in terms of (3.35), (3.40) and (4.6)–(4.9) we can establish the long time decay rates of the macroscopic momentum \((f(t), v\chi_0)\) as

\[
\| (f(t), v\chi_0) \|_{L^2_t} \leq C(1 + t)^{-\frac{1}{4}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2}) + C \int_0^t (1 + t - s)^{-\frac{3}{4}} (\| G(s) \|_{L^2_{x,v}} + \| G(s) \|_{L^2}) ds
\]

\[
\leq C\delta_0(1 + t)^{-\frac{1}{4}} + C(1 + t)^{-\frac{1}{4}} Q(t)^2,
\]

(4.14)

and

\[
\| \nabla_x (f(t), v\chi_0) \|_{L^2_t} \leq C(1 + t)^{-\frac{3}{2}} (\| \nabla_x f_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2}) + C \int_0^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x G(s) \|_{L^2_{x,v}} + \| G(s) \|_{L^2}) ds
\]

\[
+ C \int_{t/2}^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x G(s) \|_{L^2_{x,v}} + \| \nabla_x G(s) \|_{L^2}) ds
\]

\[
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}} Q(t)^2.
\]

(4.15)

In terms of (3.36), (3.41) and (4.6)–(4.9), we can estimate the macroscopic energy \((f(t), \chi_4)\) as

\[
\| (f(t), \chi_4) \|_{L^2_t} \leq C(1 + t)^{-\frac{1}{4}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2}) + C \int_0^t (1 + t - s)^{-\frac{3}{4}} (\| G(s) \|_{L^2_{x,v}} + \| G(s) \|_{L^2}) ds
\]

\[
\leq C\delta_0(1 + t)^{-\frac{1}{4}} + C(1 + t)^{-\frac{1}{4}} Q(t)^2,
\]

(4.16)

and

\[
\| \nabla_x (f(t), \chi_4) \|_{L^2_t} \leq C(1 + t)^{-\frac{3}{2}} (\| \nabla_x f_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2}) + C \int_0^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x G(s) \|_{L^2_{x,v}} + \| G(s) \|_{L^2}) ds
\]

\[
+ C \int_{t/2}^t (1 + t - s)^{-\frac{3}{2}} (\| \nabla_x G(s) \|_{L^2_{x,v}} + \| \nabla_x G(s) \|_{L^2}) ds
\]

\[
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}} Q(t)^2.
\]

(4.17)

Moreover, the electricity potential \(\nabla_x \Phi(t)\) is bounded by

\[
\| \nabla_x \Phi(t) \|_{L^2_t} \leq C(1 + t)^{-\frac{1}{4}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{L^2}) + C \int_0^t (1 + t - s)^{-\frac{3}{4}} (\| G(s) \|_{L^2_{x,v}} + \| G(s) \|_{L^2}) ds
\]

\[
\leq C\delta_0(1 + t)^{-\frac{1}{4}} + C(1 + t)^{-\frac{1}{4}} Q(t)^2.
\]

(4.18)
In a fashion similar to Lemma 4.6 in [2], we claim that there are two functionals, $H(f)$ and $D(f)$, related to the global solution $f$ as follows:

$$
H(f) \sim \sum_{|\alpha|+|\beta| \leq N} \| w \partial_\alpha^\alpha \partial_\beta^\beta P_1 f \|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \| \partial_\alpha^\alpha \nabla_x P_0 f \|_{L^2_{x,v}}^2 + \| P_d f \|_{L^2_{x,v}}^2,
$$
\[H(f), D(f)\sim \sum_{|\alpha|+|\beta| \leq N} \| w \partial_\alpha^\alpha \partial_\beta^\beta P_1 f \|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \| \partial_\alpha^\alpha \nabla_x P_0 f \|_{L^2_{x,v}}^2 + \| P_d f \|_{L^2_{x,v}}^2,\]

such that

$$\frac{d}{dt} H(f(t)) + \mu D(f(t)) \leq C \| \nabla_x P_0 f(t) \|_{L^2_{x,v}}^2. \quad (4.19)$$

This, together with $(4.13), (4.15)$ and $(4.17)$, leads to

$$H(f(t)) \leq e^{-\mu t} H(f_0) + \int_0^t e^{-\mu (t-s)} \| \nabla_x P_0 f(s) \|_{L^2_{x,v}}^2 \, ds$$

$$\leq C \delta_0^2 e^{-\mu t} + \int_0^t e^{-\mu (t-s)} (1 + s)^{-\frac{3}{2}} (\delta_0 + Q(t)^2)^2 \, ds$$

$$\leq C (1 + t)^{-\frac{3}{2}} (\delta_0 + Q(t)^2)^2. \quad (4.20)$$

Taking the summation of $(4.12)-(4.18)$ and $(4.20)$, we have

$$Q(t) \leq C\delta_0 + CQ(t)^2,$$

from which the claim $(4.4)$ can be verified, provided that $\delta_0 > 0$ is small enough. Similarly, we can prove $(4.2)$ in the case of $(f_0, \chi_0) = 0$; the details are omitted. □

Finally, we can establish the optimal time decay rates of the global solution in the following sense:

**Theorem 4.2.** Assume that $f_0 \in H^N \cap L^{2,1}$ for $N \geq 4$ satisfying $\| f_0 \|_{H^N \cap L^{2,1}} \leq \delta_0$ with $\delta_0 > 0$ being small enough, and that there exist two positive constants $d_0$ and $d_1$ so that $\inf_{|\xi| \leq r_0} |\hat{f}_0, \chi_0| \geq d_0$ and $\inf_{|\xi| \leq r_0} |\hat{f}_0, \chi_4| \geq d_1 \sup_{|\xi| \leq r_0} |\hat{f}_0, \chi_0|$. Then, for time $t > 0$ large enough the global solution $f$ to the IVP problem (1.5) satisfies

$$C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}, \quad (4.21)$$

$$C_1 \delta_0 (1 + t)^{-\frac{1}{4}} \leq \| (f(t), v \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{1}{4}}, \quad (4.22)$$

$$C_1 \delta_0 (1 + t)^{-\frac{1}{4}} \leq \| \nabla_x \Phi(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{1}{4}}, \quad (4.23)$$

$$C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| P_1 f(t) \|_{L^2_{x,v}} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}, \quad (4.24)$$

with $C_2 \geq C_1 > 0$ being two constants.

If, in addition, $(f_0, \chi_0) = 0$ is assumed, and $\inf_{|\xi| \leq r_0} |\hat{f}_0, (v \cdot \xi) \sqrt{M}| \geq d_0$ and $\inf_{|\xi| \leq r_0} |\hat{f}_0, \chi_4| \geq d_0$ for some constant $d_0 > 0$, then it holds for time $t > 0$ large enough that

$$C_1 \delta_0 (1 + t)^{-\frac{5}{4}} \leq \| (f(t), \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{5}{4}}, \quad (4.25)$$
\[ C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), \chi_0) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}, \quad (4.26) \]

\[ C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| (f(t), \chi_4) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}, \quad (4.27) \]

\[ C_1 \delta_0 (1 + t)^{-\frac{3}{4}} \leq \| \nabla \Phi(t) \|_{L^2_x} \leq C_2 \delta_0 (1 + t)^{-\frac{3}{4}}, \quad (4.28) \]

\[ C_1 \delta_0 (1 + t)^{-\frac{5}{4}} \leq \| P_1 f(t) \|_{L^2_{x,v}} \leq C_2 \delta_0 (1 + t)^{-\frac{5}{4}}. \quad (4.29) \]

**Proof.** By (4.3), Theorem 3.6 and Theorem 4.1, we can establish the lower bounds of the time decay rates of macroscopic density, momentum and energy of the global solution \( f \) and its microscopic part for \( t > 0 \) large enough that

\[
\| (f(t), \chi_0) \|_{L^2_{x}} \geq \| (e^B f_0, \chi_0) \|_{L^2_{x}} - \int_0^t \| (e^{(t-s)B} G(s), \chi_0) \|_{L^2_{x}} ds \\
\geq C_1 \delta_0 (1 + t)^{-3/4} - C_2 \delta_0^2 (1 + t)^{-3/4},
\]

\[
\| (f(t), \chi_4) \|_{L^2_{x}} \geq \| (e^B f_0, \chi_4) \|_{L^2_{x}} - \int_0^t \| (e^{(t-s)B} G(s), \chi_4) \|_{L^2_{x}} ds \\
\geq C_1 \delta_0 (1 + t)^{-1/4} - C_2 \delta_0^2 (1 + t)^{-1/4},
\]

\[
\| f(t) \|_{L^2_{x,v}} \geq \| P_1 (e^B f_0) \|_{L^2_{x,v}} - \int_0^t \| P_1 (e^{(t-s)B} G(s)) \|_{L^2_{x,v}} ds \\
\geq C_1 \delta_0 (1 + t)^{-3/4} - C_2 \delta_0^2 (1 + t)^{-3/4}.
\]

This gives rise to (4.21)–(4.24) for sufficiently large \( t > 0 \) and small \( \delta_0 > 0 \). (4.26)–(4.29) can be proved similarly, so we omit the details for brevit. \( \square \)

### 4.2. Hard Potential Case

For the hard potential case, we can use a mixed time-velocity weight function introduced in [6] defined by

\[ w_l(t, v) = (1 + |v|^2)^{\frac{l}{2}} e^{\frac{a|v|}{(1 + |v|^2)^b}}, \]

where \( l \in \mathbb{R}, a > 0 \) and \( b > 0 \), and the energy norms

\[
\| f(t) \|_{N,l} = \sum_{|\alpha| + |\beta| \leq N} \| w_l(t, v) \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2_{x,v}}, \quad \| f_0 \|_{N,l} = \sum_{|\alpha| + |\beta| \leq N} \| w_l(0, v) \partial_x^\alpha \partial_v^\beta f_0 \|_{L^2_{x,v}}
\]

(4.30)

to prove
Theorem 4.3. Let $N \geq 4$, $l \geq 1$, $a > 0$ and $0 < b \leq 1/4$. Assume that $\|f_0\|_{N,l} + \|f_0\|_{L^2,1} \leq \delta_0$ with $\delta_0 > 0$ small. Let $f$ be a solution of the VPB system (1.5). Then, it holds that, for $|\alpha| = 0, 1$,

\[
\begin{align*}
\|\partial^\alpha_x (f(t), \chi_0)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}, \\
\|\partial^\alpha_x (f(t), v\chi_0)\|_{L^2_x} + \|\partial^\alpha_x \nabla_x \Phi(t)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{1}{2} - \frac{|\alpha|}{2}}, \\
\|\partial^\alpha_y (f(t), \chi_4)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{1}{2} - \frac{|\alpha|}{2}}, \\
P_1 f(t)\|_{N,l} + \|\nabla_x P_0 f(t)\|_{L^2_w(H^N_{-1})} &\leq C\delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}}.
\end{align*}
\]

(4.31)

Moreover, if $(f_0, \chi_0) = 0$, then it holds that, for $|\alpha| = 0, 1$,

\[
\begin{align*}
\|\partial^\alpha_x (f(t), \chi_0)\|_{L^2_x} + \|\partial^\alpha_x P_1 f(t)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{5}{4} - \frac{|\alpha|}{2}}, \\
\|\partial^\alpha_x (f(t), v\chi_0)\|_{L^2_x} + \|\partial^\alpha_x \nabla_x \Phi(t)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{3}{2} - \frac{|\alpha|}{2}}, \\
\|\partial^\alpha_y (f(t), \chi_4)\|_{L^2_x} &\leq C\delta_0 (1 + t)^{-\frac{3}{2} - \frac{|\alpha|}{2}}, \\
P_1 f(t)\|_{N,l} + \|\nabla_x P_0 f(t)\|_{L^2_w(H^N_{-1})} &\leq C\delta_0 (1 + t)^{-\frac{5}{2} - \frac{|\alpha|}{2}}.
\end{align*}
\]

(4.32)

Proof. We prove (4.31) first. For the global solution $f$ to the IVP problem (1.5), we define a functional $Q_1(t)$ for any $t > 0$ by

\[
Q_1(t) = \sup_{0 \leq s \leq t, |\alpha| = 0, 1} \left\{ (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \|\partial^\alpha_x (f(s), \chi_0)\|_{L^2_x} + (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \|\partial^\alpha_x (f(s), v\chi_0)\|_{L^2_x} \\
+ (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \|\partial^\alpha_y (f(s), \chi_4)\|_{L^2_x} + (1 + s)^{\frac{1}{4} + \frac{|\alpha|}{2}} \|\nabla_x \Phi(s)\|_{L^2_x} \\
+ (1 + s)^{\frac{3}{4}} \left( \|P_1 f(s)\|_{N,l} + \|\nabla_x P_0 f(s)\|_{L^2_w(H^N_{-1})} \right) \right\}.
\]

(4.33)

We claim that it holds under the assumptions of Theorem 4.3 that

\[
Q_1(t) \leq C\delta_0.
\]

It is easy to verify that the estimate (4.31) follows from (4.33).

Since $v(v) \leq w_l(t, v)$ for all $l \geq 1$ and $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^3$, it follows from (4.5) that

\[
\|\Gamma(f, g)\|_{L^2_x} \leq C \left( \|f\|_{L^2_x} \|w_l g\|_{L^2_x} + \|w_l f\|_{L^2_x} \|g\|_{L^2_x} \right).
\]

Then, we can obtain, by using arguments similar to those of (4.6)–(4.9), to obtain

\[
\begin{align*}
\|\partial^\alpha_x G(s)\|_{L^2_x} &\leq C (1 + s)^{-1 - \frac{|\alpha|}{2}} Q_1(t)^2, \\
\|\partial^\alpha_y G(s)\|_{L^2_x} &\leq C (1 + s)^{-\frac{1}{2} - \frac{|\alpha|}{2}} Q_1(t)^2
\end{align*}
\]

for $|\alpha| = 0, 1$. In a manner similar to the proof of (4.12)–(4.18), we have

\[
\|\partial^\alpha_x (f(t), \chi_0)\|_{L^2_x} \leq C\delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}} + C(1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}} Q_1(t)^2,
\]

(4.34)
\[ \| \tilde{\alpha}^\alpha(f(t), v \chi(0)) \|_{L^2_x} \leq C\delta_0 (1 + t)^{-\frac{1}{2} - \frac{|\alpha|}{2}} + C(1 + t)^{-\frac{1}{4} - \frac{|\alpha|}{2}} Q_1(t)^2, \quad (4.35) \]
\[ \| \tilde{\alpha}^\alpha(f(t), \chi_4) \|_{L^2_x} \leq C\delta_0 (1 + t)^{-\frac{3}{4} - \frac{|\alpha|}{2}} + C(1 + t)^{-\frac{1}{2} - \frac{|\alpha|}{2}} Q_1(t)^2 \quad (4.36) \]
for \(|\alpha| = 0, 1,\) and
\[ \| \nabla_x \Phi(t) \|_{L^2_x} \leq C\delta_0 (1 + t)^{-\frac{1}{4}} + C(1 + t)^{-\frac{3}{4}} Q_1(t)^2. \quad (4.37) \]

By Lemma 4.4 in [6], there are two functionals \(H_{N,1}(f)\) and \(D_{N,1}(f)\) defined by
\[ H_{N,1}(f) \sim \sum_{|\alpha| + |\beta| \leq N} \| \alpha! \beta! P_1 f \|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \| \tilde{\alpha}^\alpha \nabla_x P_0 f \|_{L^2_{x,v}}^2 + \| P_d f \|_{L^2_{x,v}}^2, \]
\[ D_{N,1}(f) \sim \sum_{|\alpha| + |\beta| \leq N} \| v^{1/2} \alpha! \beta! P_1 f \|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \| \tilde{\alpha}^\alpha \nabla_x P_0 f \|_{L^2_{x,v}}^2 + \| P_d f \|_{L^2_{x,v}}^2, \]
such that
\[ \frac{d}{dt} H_{N,1}(f(t)) + \kappa D_{N,1}(f(t)) \leq C \| \nabla_x P_0 f(t) \|_{L^2_{x,v}}^2. \quad (4.38) \]
This, together with (4.34), (4.35) and (4.36), leads to
\[ H_{N,1}(f(t)) \leq e^{-\kappa t} H_{N,1}(f_0) + \int_0^t e^{-\kappa(t-s)} \| \nabla_x P_0 f(s) \|_{L^2_{x,v}}^2 \, ds \]
\[ \leq e^{-\kappa t} H_{N,1}(f_0) + \int_0^t e^{-\kappa(t-s)} (1 + s)^{-3/2} (\delta_0 + Q_1(t)^2)^2 \, ds \]
\[ \leq C(1 + t)^{-3/2} (\delta_0 + Q_1(t)^2)^2. \quad (4.39) \]

Summing up (4.34)–(4.37) and (4.39), we have
\[ Q_1(t) \leq C\delta_0 + C Q_1(t)^2, \]
from which the claim (4.33) can be verified provided that \(\delta_0 > 0\) is small enough. Similarly, we can prove (4.32) in the case of \((f_0, \chi_0) = 0.\) The detail is omitted for brevity.

Note that the same statements on the optimal decay rates given Theorem 4.2 for the hard sphere model also hold for the hard potential case.

Theorem 1.2 follows from Theorems 4.1 and 4.3, and Theorem 1.3 follows from Theorem 4.2 for both the hard sphere and the hard potential.

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Appendix

Let us list the following results on the semigroup theory (cf. [16]) for the easy reference by the readers. Let $H$ be a Hilbert space with the inner product denoted by $(\cdot, \cdot)$.

**Definition 5.1.** A linear operator $A$ is dissipative if $Re(Af, f) \leq 0$ for every $f \in D(A) \subset H$.

**Lemma 5.2.** Let $A$ be a densely defined closed linear operator on $H$. If both $A$ and its adjoint operator $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$-semigroup on $H$.

**Lemma 5.3.** (Stone) The operator $A$ is the infinitesimal generator of a continuous unitary group on a Hilbert space $H$ if and only if the operator $iA$ is self-adjoint.

**Lemma 5.4.** Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\kappa t}$. Then, it holds for $f \in D(A^2)$ and $\sigma > \max(0, \kappa)$ that

$$T(t)f = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} (\lambda - A)^{-1} f d\lambda. \quad (5.40)$$

**Lemma 5.5.** Let $A$ be the infinitesimal generator of the $C_0$ semigroup $T(t)$. If $D(A^n)$ is the domain of $A^n$, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in $X$.

References

1. Cercignani, C., Illner, R., Pulvirenti, M.: *The Mathematical Theory of Dilute Gases*. Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994
2. Duan, R.J., Strain, R.M.: Optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$. *Arch. Ration. Mech. Anal.* 199(1) 291–328, 2011.
3. Duan, R.J., Yang, T.: Stability of the one-species Vlasov–Poisson–Boltzmann system. *SIAM J. Math. Anal.* 41, 2353–2387, 2010.
4. Duan, R.J., Ukai, S., Yang, T., Zhao, H.J.: Optimal decay estimates on the linearized Boltzmann equation with time-dependent forces and their applications. *Commun. Math. Phys.* 277(1), 189–236, 2008.
5. Duan, R.J., Yang, T., Zhu, C.J.: Boltzmann equation with external force and Vlasov–Poisson–Boltzmann system in infinite vacuum. *Discrete Contin. Dyn. Syst.* 16, 253–277, 2006.
6. Duan, R.J., Yang, T., Zhao, H.J.: The Vlasov–Poisson–Boltzmann system in the whole space: the hard potential case. *J. Differ. Equ.* 252, 6356–6386, 2012.
7. Duan, R.J., Yang, T., Zhao, H.J.: The Vlasov–Poisson–Boltzmann system for soft potentials. *Math. Models Methods Appl. Sci.* 23(6), 979–1028, 2013.
8. Ellis, R.S., Pinsky, M.A.: The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pure Appl.* 54, 125–156, 1975.
9. Guo, Y.: The Vlasov–Poisson–Boltzmann system near Maxwellians. *Commun. Pure Appl. Math.* 55(9), 1104–1135, 2002.
10. Guo, Y.: The Vlasov–Poisson–Boltzmann system near vacuum. *Commun. Math. Phys.* 218(2), 293–313, 2001.
11. Kato, T.: *Perturbation Theory of Linear Operator*. Springer, New York 1996
12. **Liu, T.-P., Yu, S.-H.:** The Green’s function and large-time behavior of solutions for the one-dimensional Boltzmann equation. *Commun. Pure Appl. Math.* 57, 1543–1608, 2004

13. **Liu, T.-P., Yang, T., Yu, S.-H.:** Energy method for the Boltzmann equation. *Phys. D* **188**(3–4), 178–192, 2004.

14. **Markowich, P.A., Ringhofer, C.A., Schmeiser, C.:** *Semiconductor Equations.* Springer-Verlag, Vienna, 1990.

15. **Mischler, S.:** On the initial boundary value problem for the Vlasov–Poisson–Boltzmann system. *Commun. Math. Phys.* **210**, 447–466, 2000.

16. **Pazy, A.:** *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983

17. **Ukai, S.:** On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Jpn. Acad.* **50**, 179–184, 1974.

18. **Ukai, S., Yang, T.:** The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: Global and time-periodic solutions. *Anal. Appl.* **4**, 263–310, 2006

19. **Ukai, S., Yang, T.:** *Mathematical Theory of Boltzmann Equation.* Lecture Notes Series No. 8, Hong Kong: Liu Bie Ju Center for Mathematical Sciences, City University of Hong Kong, March 2006.

20. **Yang, T., Yu, H.J., Zhao, H.J.:** Cauchy problem for the Vlasov–Poisson–Boltzmann system. *Arch. Ration. Mech. Anal.* **182**, 415–470, 2006.

21. **Yang, T., Zhao, H.J.:** Global existence of classical solutions to the Vlasov–Poisson–Boltzmann system. *Commun. Math. Phys.* **268**, 569–605, 2006.

22. **Yang, T., Yu, H.J.:** Optimal convergence rates of classical solutions for Vlasov–Poisson–Boltzmann system. *Commun. Math. Phys.* **301**, 319–355, 2011

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**Hai-Liang Li**
School of Mathematical Science and Academy for Multidisciplinary Studies, Capital Normal University, Beijing China.

e-mail: hailiang.li.math@gmail.com

and

**Tong Yang**
Department of Mathematics, City University of Hong Kong, Hong Kong China.

e-mail: matyang@cityu.edu.hk

and

**Mingying Zhong**
College of Mathematics and Information Sciences, Guangxi University, Nanning China.

e-mail: zhongmingying@sina.com

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