ROW TRANSFER MATRIX FUNCTIONAL RELATIONS FOR
BAXTER’S EIGHT-VERTEX AND SIX-VERTEX MODELS WITH OPEN
BOUNDARIES VIA MORE GENERAL REFLECTION MATRICES

Yu-kui Zhou\textsuperscript{1,2}

Mathematics Department, The Australian National University,
Canberra, ACT 0200, Australia

Abstract

The functional relations of the transfer matrices of fusion hierarchies for six- and eight-vertex models with open boundary conditions have been presented in this paper. We have shown the su(2) fusion rule for the models with more general reflection boundary conditions, which are represented by off-diagonal reflection matrices. Also we have discussed some physics properties which are determined by the functional relations. Finally the intertwining relation between the reflection $K$ matrices for the vertex and SOS models is discussed.

to appear in Nucl. Phys. B.

Great progress has been made on the study of integrable models in statistical mechanics and quantum field theory. It is very clear that the important mathematical structure that ensures the exactly solvability of these models is governed by the Yang-Baxter equations if the models sit on square lattices with periodic boundary conditions. Recently there has been interest in studying integrable systems with open boundary conditions. The open boundary conditions are described by boundary reflection matrices satisfying reflection equations [1, 2] (boundary Yang-Baxter equations), which ensures the exactly solvability of the models with the open boundary conditions together with the Yang-Baxter equations. The boundary reflection matrices have been found for many integrable systems, in particular for the six-vertex and eight-vertex models in [3, 4, 1] (also see [5, 6] for related works). The eigenvalues of the transfer matrices have been solved for the six-vertex model or $A_1^{(1)}$ invariant chain [2, 8, 13] (also see [10] for related work), Izergin-Korepin vertex model or $A_2^{(2)}$ invariant chain [3, 10], $U_q(spl(2, 1))$-invariant t-J model [11, 12], $A_n^{(1)}$ invariant chain [3] and $A_{2n}^{(2)}$ invariant chain [14]. These exact solutions are constructed by the Bethe ansatz only for diagonal reflection matrices. The integrable

\textsuperscript{1}Email: zhouy@maths.anu.edu.au
\textsuperscript{2} On leave of absence from Institute of Modern Physics, Northwest University, Xian 710069, China
systems with non-diagonal reflection matrices are more difficult to solve and not many results have been obtained for such systems.

The fusion procedure has been shown very useful in studying two-dimensional integrable models. Similar to the method to fuse the $R$ matrix of the Yang-Baxter equation \cite{18}, the fusion procedure for the reflection matrix $K$ of the reflection equation has been presented in \cite{15}. The fused $R$- and $K$-matrices generate some new integrable models with the open boundaries based on the elementary model. The corresponding fused transfer matrices of fusion hierarchies are related through the functional relations, which can be shown by the fusion. In \cite{24} the functional relations of transfer matrices for the six-vertex model with diagonal reflection matrices have been shown and thus the finite size corrections to the transfer matrices have been obtained by solving the functional relations. The functional relations for the six-vertex model and eight-vertex model with non-diagonal reflection matrices have not been found even though the fusion procedure was described generally before.

In this paper we study the six-vertex model and eight-vertex model with the non-diagonal reflection matrices found in \cite{3,4}. We find the functional relations among the eigenvalues of the transfer matrices for whole fusion hierarchies of the models. This sheds light on the solution of the models with boundaries via the most general non-diagonal reflection matrices. In order to show the derivation of the functional relations of the six-vertex model and eight-vertex model with non-diagonal reflection matrices clearly we carry out the fusion of the models. Then the intertwining relation between vertex and SOS models for the boundary reflection $K$ matrices is also discussed and the reflection equations for the SOS models have been explained. From the functional relations we derive the crossing unitary conditions of bulk and surface free energies. For the six-vertex model the crossing unitary condition has been argued in terms of quantum field theory in \cite{30} and however it is not very clear for the (off-critical) eight-vertex model. Our crossing unitary conditions of bulk and surface free energies must be consistent with the argument given in \cite{30} for the six-vertex model. We believe the crossing unitary conditions for the off-critical model is a new result. Specially, some surface critical behaviour can be studied by solving the crossing unitary conditions. Also we present the eigen-spectra of the transfer matrix of the eight-vertex model with the diagonal reflection matrices, which solves the functional relations. The eigen-spectra has not been found before.

In section 1 we describe the fusion of the $R$ and $K$ matrices obviously. Then the functional equations are presented in section 2. For clarity all proofs are given in Appendices. In section 3 we present the functional equations for the six-vertex model and
eight-vertex model with some known boundary reflection matrices. In section 4 the intertwining relation between the vertex boundary \( K \) matrix and the SOS face boundary \( K \) matrix is discussed. Finally we conclude with a brief discussion at the crossing unitary conditions for the surface free energies and the eigen-spectra of the transfer matrix and discuss some physics consequence.

## 1. Some expressions for fusion

Let us begin by reviewing the fusion procedure for a given \( R \) matrix obeying the Yang-Baxter equation

\[
R^{12}(u)R^{13}(u + v)R^{23}(v) = R^{23}(v)R^{13}(u + v)R^{12}(u) \tag{1.1}
\]

where \( u, v \) are the spectral parameters. Suppose that \( R \)-matrix acting on \( \mathbb{C}^2 \times \mathbb{C}^2 \) satisfies the following properties, the \( PT \) symmetry

\[
P_{12}R^{12}(u)P_{12} = t_1 t_2 R^{12}(u) \tag{1.2}
\]

the unitary condition and the crossing unitary condition

\[
R^{12}(u)R^{12}(-u) = \rho(u) \tag{1.4}
\]

\[
R^{12}(u) \overset{t_1}{\rightarrow} R^{12}(-u - 2\lambda) = \tilde{\rho}(u) \tag{1.5}
\]

where \( t_1 \) denotes the transposition in the first space and \( \lambda \) is crossing parameter. These \( \rho(u), \tilde{\rho}(u) \) are some scalar \( u \)-dependent functions.

Fusion is the idea to build up new solutions of the Yang-Baxter equation. Define the projector

\[
Y^+_p = \frac{1}{p!}(P^{1,p} + \cdots + P^{p-1,p} + I) \cdots (P^{1,2} + I) \tag{1.6}
\]

the new solutions can be given by

\[
R_{(p,q)}(u) = Y^+_q R_{(p,q)}(u - q\lambda + \lambda) \cdots R_{(p,2)}(u - \lambda)R_{(p,1)}(u)Y^+_q \tag{1.7}
\]

\[
R_{(p,j)}(u) = Y^+_p R^{1,j}(u)R^{2,j}(u + \lambda) \cdots R^{p,j}(u + p\lambda - \lambda)Y^+_p \tag{1.8}
\]

where \( R_{(p,q)}(u) \) acts on \( \mathbb{C}^{p+1} \times \mathbb{C}^{q+1} \). These fused \( R \) matrices satisfy the fused unitary condition and the fused crossing unitary condition

\[
R^{12}_{(p,q)}(u)R^{12}_{(q,p)}(-u) = \rho_{q,p}(u)Y^+_p Y^+_q \tag{1.9}
\]

\[
R^{12}_{(p,q)}(u) \overset{t_1 t_2}{\rightarrow} R^{12}_{(q,p)}(-u - 2\lambda) = \tilde{\rho}_{q,p}(u)Y^+_p Y^+_q \tag{1.10}
\]
where
\[ \tilde{\rho}_{q,p}(u) = \prod_{k=0}^{q-1} \prod_{j=0}^{p-1} \tilde{\rho}(u + (k - j)\lambda). \]  

(1.11)

Note that the superscripts 1, 2 mean \( R_{(p,q)}^{12}(u) \in \mathbb{C}^{p+1} \otimes \mathbb{C}^{q+1} \). We suppress them if there is no confusion.

The reflection matrices \( K_-(u), K_+(u) \) satisfies the reflection equations [1, 2]

\[ R^{12}(u-v)K_-^1(u)R^{12}(u+v)K_-^2(v) = K_-^2(v)R^{12}(u+v)K_-^1(u)R^{12}(u-v) \]  

(1.12)

\[ R^{12}(v-u)K_+^{t_1}(u)R^{12}(-v-u-2\lambda)K_+^{t_2}(v) = K_+^{t_2}(v)R^{12}(-v-u-2\lambda)K_+^{t_1}(u)R^{12}(v-u). \]  

(1.13)

Similar to the fusion of the \( R \) matrix the reflection matrices \( K_-(u), K_+(u) \) acting on space \( \mathbb{C}^2 \) can be fused to give new solutions of the reflection equations. Following [15] the new solutions of the reflection matrices can be expressed by

\[ K_-^{(q)}(u) = \rho(u|q) Y_q [K_-^q(u)][R^{q,q-1}(2u + \lambda)K_-^{q-1}(u + \lambda)] \]

\[ \hspace{1cm} [R^{q,q-2}(2u + 2\lambda)R^{q-1,q-2}(2u + 3\lambda)K_-^{q-2}(u + 2\lambda)] \cdots \]

\[ \hspace{1cm} [R^{q,1}(2u + q\lambda - \lambda)R^{q-1,1}(2u + q\lambda)] \cdots \]

\[ \hspace{1cm} R^{2,1}(2u + 2q\lambda - 3\lambda)K_-^1(u + q\lambda - \lambda)] Y_q \]

\[ K_+^{(q)}(u) = Y_q [K_+^q(u)] R^{q,q-1,1}(-2u - 3\lambda) \cdots R^{1,q}(-2u - q\lambda) R^{1,1,q}(-2u - q\lambda - \lambda)] \cdots \]

\[ \hspace{1cm} [K_+^3(u + q\lambda - 3\lambda)] R^{3,2,3}(-2u - 2q\lambda + 3\lambda) \hspace{1cm} R^{1,3}(2u - 2q\lambda + 2\lambda)] \]

\[ \hspace{1cm} [K_+^2(u + q\lambda - 2\lambda)] R^{1,2}(2u - 2q\lambda + \lambda)[K_+^1(u + q\lambda - \lambda)] Y_q \]  

(1.14)

(1.15)

where \( K_-^{(q)}(u), K_+^{(q)}(u) \) act on \( \mathbb{C}^{q+1} \), \( R^{i,j} = R^{j,i} \) and

\[ \rho(u|q + 1) = \rho(u|q)/\tilde{\rho}_{q,1}(2u + q\lambda) \]

\[ \rho(u|1) = 1, \quad q = 1, 2, \cdots. \]  

(1.16)

The fused reflection equations become

\[ R_{(p,q)}(u-v)K_-^{(p)}(u)R_{(q,p)}(u+v-\lambda+p\lambda)K_-^{(q)}(v) = K_-^{(q)}(v)R_{(q,p)}(u+v-\lambda+p\lambda)K_-^{(p)}(u)R_{(p,q)}(u-v) \]  

(1.17)

\[ R_{(q,p)}(v-u)K_+^{t_1}(u)R_{(p,q)}(-v-u-\lambda-p\lambda)K_+^{t_2}(v) \]

\[ = K_+^{t_2}(v)R_{(p,q)}(-v-u-\lambda-p\lambda)K_+^{t_1}(u)R_{(q,p)}(v-u). \]  

(1.18)
There is an automorphism between the relation for $K^{(p)}_+$ and $K^{(p)}_-$:

\[
K^{(p)}_+(u)/\rho(u|p-1) = \frac{t^{(p)}}{t^{(p)}}K^{(p)}_+(-u-p\lambda) \quad (1.19)
\]
\[
K^{(q)}_+(u)/\rho(u|q-1) = \frac{t^{(q)}}{t^{(q)}}K^{(q)}_+(-u-q\lambda) \quad (1.20)
\]

and the automorphism for $R$-matrices

\[
R_{(q,p)}(u) = R_{(p,q)}(u + q\lambda - p\lambda) \quad (1.21)
\]

which can be seen directly from the fusion procedure.

The fused Yang-Baxter equation and the fused reflection equation for $R$ and $K$ matrices guarantee the following commuting families

\[
\left[ T^{(p,q)}(u), T^{(p,b)}(v) \right] = 0 \quad (1.22)
\]

where

\[
T^{(p,q)}(u) = \text{tr} \left( \frac{t^{(q)}}{t^{(q)}}K^{(q)}_+(u)U^{(p,q)}(u)K^{(q)}_-(u)\tilde{U}^{(p,q)}(u + q\lambda - \lambda) \right) \quad (1.23)
\]
\[
U^{(p,q)}(u) = R^{c,1}_{(q,p)}(u + \frac{1}{2}p\lambda - \frac{1}{2}\lambda)R^{c,2}_{(q,p)}(u + \frac{1}{2}p\lambda - \frac{1}{2}\lambda)\cdots R^{c,N}_{(q,p)}(u + \frac{1}{2}p\lambda - \frac{1}{2}\lambda) \quad (1.24)
\]
\[
\tilde{U}^{(p,q)}(u) = R^{N,c}_{(p,q)}(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)\cdots R^{2,c}_{(p,q)}(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda) \quad (1.25)
\]

and $q,p,b = 1,2,\cdots$. The proof of (1.22) can be done similarly to the study of the unfused six-vertex model given by Sklyanin [2] and is briefly described in Appendix A. using graph representation.

2. Functional equations for general reflection matrices

The last section has shown that fused models can be built up by fusion of the elementary $R$ and $K$ matrices. The fused models are integrable systems because we have the commuting families (1.22). In fact, the these fused transfer matrices are related each other by the groups of functional relations, which can be constructed by fusion. For the periodic boundary condition it was already well known that the transfer matrix

\[
T^{(p,q)}(u) = \text{tr} U^{(p,q)}(u) \quad (2.1)
\]

commutes

\[
\left[ T^{(p,q)}(z), T^{(p,q')}(y) \right] = 0 \quad (2.2)
\]

and the functional relations

\[
T^{(p,q)}(u) T^{(p,1)}(u + q\lambda) = T^{(p,q+1)}(u) + f^{p}_{q-1} T^{(p,q-1)}(u) \quad (2.3)
\]

\footnote{Some arbitrary parameters in $K^{(p)}_+$ are interchanged, e.g. $(\xi_+,\mu_+,\nu_+) \leftrightarrow (\xi_-,\mu_-,\nu_-)$ for the six vertex model (1.3).}
can be constructed \cite{[19]}, where $T^{(p,0)}(u) = I \in \mathbb{C}^{p+1}$ and the $u$-dependent function $f_q^p$ is generated from the antisymmetric fusion of the model. These relations, in fact, are the su(2) fusion rule and show the implication of the eigenvalues of transfer matrices of all fusion hierarchies. They have been applied successfully to derive the eigenvalues and the relevant Bethe ansatz equations of the fused or the elementary transfer matrices \cite{[19, 20]}. Also they could be solved directly in the thermodynamic limit to yield the finite size corrections to the eigenvalues of the transfer matrices \cite{[21, 22, 23, 24]}. 

For the models with open reflection boundary condition it has been noticed that the functional relations can be applied to obtain the eigenvalues of fused transfer matrices \cite{[13, 15]} and to find the finite size corrections to the eigenvalues of transfer matrices \cite{[24]}. These studies have concerned only models with diagonal reflections. For the off-diagonal reflection boundary solvable models we still expect the su(2) fusion rule. This is shown in this section.

Let us first prepare some notations as follows.

$$\phi_+^p(u) := \prod_{i=1}^{N} [Y_2^- R_{(1,p)}^{1,i}(u + \frac{1}{2} p \lambda - \frac{1}{2} \lambda) R_{(1,p)}^{2,i}(u + \frac{1}{2} p \lambda + \frac{1}{2} \lambda) Y_2^-] \in \otimes_{i=1}^{N} \mathbb{C}^{p+1}$$

$$\phi_-^p(u) := \prod_{i=1}^{N} [Y_2^- R_{(q,1)}^{q-1,i}(u + \frac{1}{2} p \lambda + \frac{1}{2} \lambda) R_{(q,1)}^{2,i}(u + \frac{1}{2} p \lambda + \frac{3}{2} \lambda) Y_2^-] \in \otimes_{i=1}^{N} \mathbb{C}^{p+1}$$

$$I_+(u|q) := Y_2^- R_{(q,1)}^{1,q-1}(q \lambda - 4 \lambda - 2 u) R_{(q,1-1)}^{2,q-1}(q \lambda - 3 \lambda - 2 u) Y_2^- \in \mathbb{C}^q$$ (2.4)

$$I_-(u|q) := Y_2^- R_{(q-1-1)}^{q-1,1}(2 u - q \lambda + \lambda) R_{(q-1-1)}^{2,q-1,2}(2 u - q \lambda + 2 \lambda) Y_2^- \in \mathbb{C}^q$$

where $Y_2^- = \frac{1}{2}(1 - P^{1,2})$ is an antisymmetric projector. The boundary dependent notations are defined by

$$\omega_+(u) := Y_2^- \tilde{K}_+^1(u + \lambda) R_{(q,1-1)}^{1,2}(-2 u - 3 \lambda) K_+^1(u) Y_2^-$$

$$\omega_-(u) := Y_2^- K_+^1(u) R_{(q,1-1)}^{1,2}(2 u + \lambda) K_+^1(u + \lambda) Y_2^-$$ (2.5)

The product of all of these notations is expressed by

$$f^p(u) = \omega_-(u) \omega_+(u) \left( \phi_+^p(u) \phi_-^p(u) \right) \otimes \left( I_+(u|q) I_-(u|q) \right) \in \otimes_{i=1}^{N} \mathbb{C}^{p+1} \otimes \mathbb{C}^q ,$$ (2.6)

which is clearly a $q$-independent function if we have the relation

$$I_+(u|q) I_-(u|q) = I \tilde{\rho}_{q,1}^{-1}(2 u + \lambda) ,$$ (2.7)

where the identity matrix depends on the fusion level $q$ or $I \in \mathbb{C}^q$. The above relation (2.7) is just right for the six- and eight-vertex models, which will be seen in the next
Moreover $\phi$ and $I$ are the diagonal matrices and so is $f^p(u)$, which is written in the form

$$f^p(u) = I \cdot f^p(u)$$

$$f^p(u) = \omega_-(u)\omega_+(u)\phi_+(u)\phi_-(u)I_+(u|q)I_-(u|q)$$  \hspace{1cm} (2.8)

and $I \in \otimes_{i=1}^{N} \mathbb{C}^{p+1} \otimes \mathbb{C}^q$. Now we can express the functional relations for the six- and eight-vertex models with open reflection boundary conditions in the following theorem.

Theorem 1 ($su(2)$ Fusion Hierarchy) Let us define

$$f_q^p = f^p(u + q\lambda), \quad T^{(p,0)} = I \in \otimes_{i=1}^{N} \mathbb{C}^{p+1}$$

with the transfer matrices

$$T_q^{(q)} = T^{(p,q)}(u + k\lambda)$$

given by (1.23). Then the $su(2)$ fusion hierarchy follows as

$$T^{(q)}_0 T^{(1)}_q = T^{(q+1)}_0 + f_{q-1}^p T^{(q-1)}_0 \quad q = 1, 2, \cdots \hspace{1cm} (2.9)$$

if the reflection $K_-, K_+$ matrices satisfy the reflection equations (1.12)-(1.13).

The theorem is proved in Appendix B. Then following the standard procedure given by [21, 25] by introducing the inversion identity hierarchy (or $y$-system)

$$t_0^{(0)} = 0 \hspace{1cm} (2.10)$$

$$t_0^{(q)} = T^{(q+1)}_0 T^{(q-1)}_1 / \prod_{k=0}^{q-1} f_k^p, \hspace{1cm} (2.11)$$

the functional equations for $t$ can be derived, which are called thermodynamic Bethe ansatz (TBA) equations in [21, 27]. To see this let us consider the triple transfer matrices

$$T_0^{(p,q)}(T_1^{(p,q-1)} T_q^{(p,1)}) = (T_0^{(p,q)} T_q^{(p,1)}) T_1^{(p,q-1)}.$$ 

Inserting the functional relations (2.9) into the terms in parentheses this equation gives new functional equations,

$$T_0^{(q)} T_1^{(q)} = I \prod_{k=0}^{q-1} f_k^p + T^{(q+1)}_0 T^{(q-1)}_1. \hspace{1cm} (2.12)$$

This leads to the following theorem.
Theorem 2 (su(2) TBA) The inversion identity hierarchy $t^q$ satisfies the following su(2) TBA equations

$$t_0^{(q)} t_1^{(q)} = (I + t_0^{(q+1)})(I + t_1^{(q-1)})$$

for any reflection $K_-, K_+$ matrices satisfying the reflection equations (1.12)-(1.13), where $I \in \otimes_{i=1}^N \mathbb{C}^{p+1}$.

3. Examples

We have shown that the functional relations hold for the six- and eight-vertex models with any $K$ reflection matrices satisfying the reflection equations (1.12)-(1.13). The $su(2)$ fusion rule depends on the boundaries though the function $f^p(u)$ defined in (2.8). In this section we calculate explicitly the function for the six- and eight-vertex models. Here we use the general $K$ matrices presented in [3] and [4].

3.1 Six vertex model

The six vertex model is described by the following $R$ matrix,

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}$$

$$a(u) = \sin(u + \lambda), \quad b(u) = \sin(u), \quad c(u) = \sin(\lambda)$$

and the boundary reflection matrices are [3]

$$K_+(u) = \begin{pmatrix} \sin(\xi_+ - u - \lambda) & \nu_+ \sin(2u + 2\lambda) \\ \mu_+ \sin(2u + 2\lambda) & \sin(\xi_+ + u + \lambda) \end{pmatrix}$$

$$K_-(u) = \begin{pmatrix} \sin(\xi_- + u) & \mu_- \sin(2u) \\ \nu_- \sin(2u) & \sin(\xi_- - u) \end{pmatrix}$$

where $\xi_\pm, \mu_\pm, \nu_\pm$ are arbitrary parameters.

The unitary conditions can be checked by inserting the $R$ matrix (3.1) into (1.4) and (1.5) and we obtain

$$\rho(u) = \sin^2(u) - \sin^2(\lambda); \quad \tilde{\rho}(u) = \sin^2(\lambda) - \sin^2(u + \lambda) = -b(u)a(u + \lambda)$$

The quantities in (2.4) are given by calculation to be

$$\omega_+(u) = -\sin(2u + 4\lambda)\left(\sin(\xi_+ + u + \lambda) \sin(\xi_+ - u - \lambda) - \mu_+ \nu_+ \sin^2(2u + 2\lambda)\right)$$

$$\omega_-(u) = \sin(2u)\left(\sin(\xi_- + u + \lambda) \sin(\xi_- - u - \lambda) - \mu_- \nu_- \sin^2(2u + 2\lambda)\right)$$
\[ \phi_+^p(u) = \prod_{k=0}^{p-1} \left( -\tilde{\phi}_{1,1}(u - k\lambda + \frac{1}{2}p\lambda - \frac{1}{2}\lambda) \right)^N \]

\[ \phi_-^p(u) = \prod_{k=0}^{p-1} \left( -\tilde{\phi}_{1,1}(u + k\lambda - \frac{1}{2}p\lambda + \frac{1}{2}\lambda) \right)^N \] \hspace{1cm} (3.6)

\[ I_+(u|q) = \prod_{k=1}^{q-1} \left[ -\tilde{\phi}_{1,1}(2u + \lambda - k\lambda) \right] \]

\[ I_-(u|q) = -\prod_{k=1}^{q} \left[ -\tilde{\phi}_{1,1}(2u + 2\lambda - k\lambda) \right]^{-1}. \] \hspace{1cm} (3.7)

It is obvious that the relation (2.7) is satisfied.

The fusion procedure for the \( R \) matrix brings the extra zeros to the fused \( R \) matrices. The number of the zeros are only dependent on the fusion level \( p, q \). Some zeros of \( R_{(p,q)}(u) \) can be removed from \( U_{(p,q)}(u) \) and \( \tilde{U}_{(p,q)}(u + q\lambda) \) in (1.24) and (1.25) by the replacements

\[ R_{(q,p)}(u) \rightarrow R_{(q,p)}(u)/\prod_{j=0}^{q-1} \prod_{k=0}^{p-2} [b(u + j\lambda - k\lambda)] \]

\[ R_{(p,q)}(u) \rightarrow R_{(p,q)}(u)/\prod_{k=0}^{q-1} \prod_{j=1}^{p-1} [b(u + j\lambda - k\lambda)] \] \hspace{1cm} (3.8)

The replacement only changes \( \phi_+^p(u) \) to a simpler form,

\[ \phi_+^p(u) = \phi_-^p(u) = [b(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)a(u + \frac{1}{2}p\lambda + \frac{1}{2}\lambda)]^N \] \hspace{1cm} (3.9)

Inserting all of the quantities given in (3.3)-(3.9) into the definition (2.8) we have

\[ f^p(u) = \omega_+(u)\omega_-(u)\frac{[b(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)a(u + \frac{1}{2}p\lambda + \frac{1}{2}\lambda)]^{2N}}{b(2u + \lambda)a(2u + 2\lambda)} \] \hspace{1cm} (3.10)

For the simple case of diagonal \( K \) matrices the fusion rule (2.9) with (3.10) coincides with that obtained in [24].

### 3.2 Eight-vertex model

The eight-vertex model is the generalization of the six-vertex model [26]. The \( R \) matrix for the eight-vertex model is given by

\[ R(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix} \] \hspace{1cm} (3.11)

These non-zero elements are given by

\[ a(u) = H(u + \lambda)\Theta(u)\Theta(\lambda), \quad b(u) = \Theta(u + \lambda)H(u)\Theta(\lambda), \]

\[ c(u) = \Theta(u + \lambda)\Theta(u)H(\lambda), \quad d(u) = H(u + \lambda)H(u)H(\lambda). \] \hspace{1cm} (3.12)
where $H(u), \Theta(u)$ are the $theta$ functions. The relevant boundary reflection matrices are given by [4]

\[
K_-(u) = \begin{pmatrix}
H(\xi_- + u)\Theta(\xi_- - u) & \mu_- k^{1/2} \text{sn}(2u) \text{sn}^2(u)/f(u;\xi_-) \\
-\mu_- k^{-1/2} \text{sn}(2u)/f(u;\xi_-) & \Theta(\xi_- + u)H(\xi_- - u)
\end{pmatrix}
\]

(3.13)

\[
K_+(u) = \begin{pmatrix}
H(\xi_+ - u - \lambda)\Theta(\xi_+ + u + \lambda) & \mu_+ k^{-1/2} \text{sn}(2u + 2\lambda)/f(u + \lambda;\xi_+) \\
-\mu_+ k^{1/2} \text{sn}(2u + 2\lambda)\text{sn}^2(u + \lambda)/f(u + \lambda;\xi_+) & \Theta(\xi_+ - u - \lambda)H(\xi_+ + u + \lambda)
\end{pmatrix}
\]

(3.14)

with

\[
f(u;\xi) = [1 - k^2 \text{sn}^2(u) \text{sn}^2(\xi)]/[\Theta(\xi + u)\Theta(\xi - u)],
\]

and

\[
\text{sn}(u) = k^{-1/2}H(u)/\Theta(u)
\]

where $\xi_\pm, \mu_\pm$ are arbitrary parameters. The scalar functions in the unitary relations (1.4) and (1.7) for the eight vertex models are

\[
\rho(u) = h(\lambda + u)h(\lambda - u)
\]

(3.15)

\[
\tilde{\rho}(u) = h(u)h(u + 2\lambda)
\]

(3.16)

\[
h(u) = H(u)\Theta(u)\Theta(0)
\]

(3.17)

The expressions of the quantities in (3.7) with (3.16) are still correct except the $\omega_\pm(u)$, or

\[
\omega_-(u) = h(\xi_- + u + \lambda)h(\xi_- - u - \lambda)H(2u)\Theta(2u + 2\lambda)/\Theta(0)
\]

+ $K_-(u)_{21}K_-(u + \lambda)_{21}H(2u + 2\lambda)H^2(u + \lambda)\Theta(2u)\Theta(0)/\Theta^2(u)$

(3.18)

\[
\omega_+(u) = -h(\xi_+ - u - \lambda)h(\xi_+ + u + \lambda)H(2u + 4\lambda)\Theta(2u + 2\lambda)/\Theta(0)
\]

(3.19)

\[
-K_+(u + \lambda)_{12}K_+(u)_{12}H(2u + 2\lambda)H^2(u + \lambda)\Theta(2u + 4\lambda)\Theta(0)/\Theta^2(u + 2\lambda)
\]

\[
\phi_+^p(u) = \prod_{k=0}^{p-1}[-\tilde{\rho}_{1,1}(u - k\lambda + \frac{i\pi p\lambda}{2}) - \frac{1}{2}]^{-N}
\]

(3.20)

\[
\phi_-^p(u) = \prod_{k=0}^{p-1}[-\tilde{\rho}_{1,1}(u + k\lambda - \frac{i\pi p\lambda}{2}) - \frac{1}{2}]^{-N}
\]

(3.20)

\[
I_+(u|q) = \prod_{k=1}^{q-1}[-\tilde{\rho}_{1,1}(2u + \lambda - k\lambda)]^{-1}
\]

(3.21)

From (3.21) it can be seen that (2.7) is satisfied.

Similar to the six-vertex model discussed in the last subsection, removing the zeros generated by fusion from $U_{(p,q)}(u)$ and $\check{U}_{(p,q)}(u + q\lambda)$ in (1.24) and (1.25)

\[
R_{(q,p)}(u) \rightarrow R_{(q,p)}(u)/\prod_{j=0}^{q-1} \prod_{k=0}^{p-2}[h(u + j\lambda - k\lambda)]
\]
\[ R_{(p,q)}(u) \rightarrow R_{(p,q)}(u) / \prod_{k=0}^{q-1} \prod_{j=1}^{p-1} [h(u + j\lambda - k\lambda)] \]  

(3.22)

from the transfer matrix (1.23), the quantities \( \phi^p_\pm(u) \) become

\[
\begin{align*}
\phi_+^p(u) &= [h(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)h(u + \frac{1}{2}p\lambda + \frac{1}{2}\lambda + \lambda)]^N \\
\phi_-^p(u) &= [h(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)h(u + \frac{1}{2}p\lambda + \frac{1}{2}\lambda + \lambda)]^N .
\end{align*}
\]

(3.23)

Inserting all of the quantities given in (3.5)-(3.9) into the definition (2.8) we have

\[ f^p(u) = \omega_+(u)\omega_-(u)\frac{[h(u - \frac{1}{2}p\lambda + \frac{1}{2}\lambda)h(u + \frac{1}{2}p\lambda + \frac{1}{2}\lambda + \lambda)]^{2N}}{h(2u + \lambda)h(2u + 3\lambda)} . \]

(3.24)

4. SOS models and intertwiner

Intertwining relations are the correspondence between the \( R \)-matrix (the Boltzmann weights of the vertex model) and \( W \)-matrix (the Boltzmann weights of the relevant face model) \[27, 29\]. Baxter introduced the intertwining relation for the eight-vertex and the relevant face model to solve the eigenvalues of transfer matrices of the eight-vertex model. The related face model is called the solid-on-solid (SOS) model. The similar formulation can be generalized to the boundary \( K \) matrices (boundary Boltzmann weights). This however has not been done before. We show briefly in this section the correspondence between the boundary \( K \) matrices of the eight-vertex and the relevant SOS models. For clarity the corresponding graph representations are briefly given in appendix C.

At first let us review Baxter’s intertwining relation between the eight-vertex and the SOS model. Choose arbitrary constants \( s^\pm \) and integers \( a, b \in \mathbb{Z} \) and set the nonzero intertwiners by

\[
\varphi_{a,b}(u) = \left( \frac{H(s^\pm + a - \epsilon u)}{\Theta(s^\pm + a - \epsilon u)} \right) / \sqrt{S(b)} \quad \text{if} \quad \epsilon = b - a = \pm 1
\]

(4.1)

where the factors \( S(b) \) is given by

\[ S(b) = h(w_0 + b\lambda) \]

(4.2)

The \( R \) matrix is given by (3.11) with (3.12). Then Baxter’s intertwining relation is given by

\[ R(u - v) \varphi_{d,c}(u) \otimes \varphi_{c,b}(v) = \sum_a \varphi_{d,a}(v) \otimes \varphi_{a,b}(u) \ W\left( \begin{array}{cc} d & c \\ a & b \end{array} \right | u - v ) , \]

(4.3)

where \( W\left( \begin{array}{cc} d & c \\ a & b \end{array} \right | u \right ) \) are the face weights of the SOS model satisfying the star-triangle relation

\[ \sum_g W\left( \begin{array}{cc} f & g \\ a & b \end{array} \right | u \right ) W\left( \begin{array}{cc} e & d \\ f & g \end{array} \right | v \right ) W\left( \begin{array}{cc} d & c \\ g & b \end{array} \right | v - u \right ) . \]
\[
W \left( \begin{array}{c|c} e & g \\ \hline f & a \\ \end{array} \right) v - u \right) W \left( \begin{array}{c|c} g & c \\ \hline a & b \\ \end{array} \right) W \left( \begin{array}{c|c} e & d \\ \hline g & c \\ \end{array} \right) u \right) \tag{4.4}
\]

for any \( a, b, c, d, e, f \). The nonzero weights are given by

\[
W \left( \begin{array}{c|c c} a & a \pm 1 \\ \hline a \pm 1 & a \pm 2 \\ \end{array} \right) u \right) = \frac{h(\lambda + u)}{h(\lambda)} \\
W \left( \begin{array}{c|c c} a & a \pm 1 \\ \hline a \pm 1 & a \\ \end{array} \right) u \right) = \sqrt{\frac{S(a+1)S(a-1)h(u)}{S(a)^2 h(\lambda)}} \\
W \left( \begin{array}{c|c c} a & a \pm 1 \\ \hline a \pm 1 & a \\ \end{array} \right) u \right) = \frac{h(w_0 + a\lambda \mp u)}{h(w_0 + a\lambda)} \tag{4.5}
\]

where \( a \in \mathbb{Z} \) and \( w_0 = \frac{1}{2}(s^+ + s^-)\lambda - K \). These face weights satisfy the following unitary condition

\[
\sum_c W \left( \begin{array}{c|c c} d & c \\ \hline a & b \\ \end{array} \right) \rho(u) \delta_{aa'} = \sum_c W \left( \begin{array}{c|c c} a & d \\ \hline b & c \\ \end{array} \right) \rho(u) \delta_{aa'} \tag{4.6}
\]

and the crossing unitary condition

\[
\sum_c W \left( \begin{array}{c|c c} a & d \\ \hline b & c \\ \end{array} \right) \rho(u) \delta_{aa'} = \sum_c W \left( \begin{array}{c|c c} c & d \\ \hline b & a \\ \end{array} \right) \frac{S(a)S(c)}{S(b)S(d)} \rho(u) \delta_{aa'} , \tag{4.7}
\]

where the scalar function is given by

\[
\rho(u) = \frac{h(\lambda - u) h(\lambda + u)}{h(\lambda)^2} . \tag{4.8}
\]

The intertwining relation for the boundaries is set out by

\[
K_-(u) \varphi_{b,a}(u) = \sum_c \varphi_{b,c}(-u) K_-(b \left. \begin{array}{c} a \\ c \end{array} \right| u) \\
K_+(u) \varphi_{b,a}(-u - \lambda) = \sum_c S(c) S(b) \varphi_{b,c}(u + \lambda) K_+(c \left. \begin{array}{c} a \\ b \end{array} \right| u) , \tag{4.9}
\]

where \( K_-(b \left. \begin{array}{c} a \\ c \end{array} \right| u) \) and \( K_+(c \left. \begin{array}{c} a \\ b \end{array} \right| u) \) are the reflection matrices (\( K \) matrices) of the SOS model. It is obvious that the face \( K \) matrix elements are nonzero only for \(|b - a| = 1\) and \(|b - c| = 1\). The nonzero elements are found to be

\[
K_-(b \left. \begin{array}{c} a \\ c \end{array} \right| u) = \varphi_{b,c}^{-1}(-u) K_-(u) \varphi_{b,a}(u) \\
K_+(c \left. \begin{array}{c} a \\ b \end{array} \right| u) = \varphi_{b,a}^{-1}(u + \lambda) K_+(u) \varphi_{b,c}(-u - \lambda) \frac{S(b)}{S(a)} \tag{4.10}
\]

\[12\]
The intertwiners are dependent on two arbitrary parameters \( s^\pm \) and thus \( \varphi_{a,a+1}(u) \) and \( \varphi_{a,a-1}(u) \) can be treated as independent vectors. We therefore can define the inverse vectors of them by

\[
\sum_{b=a\pm 1} \varphi_{a,b}(u) \varphi_{a,b}^{-1}(u) = \delta_{i,j}
\]

\[
\sum_{i=1,2} \varphi_{a,b}^{-1}(u) \varphi_{a,c}(u) = \delta_{a,c}
\]  

where

\[
\varphi_{a,b}^{-1}(u) = S(b) \left( \epsilon \Theta(s^{-} + a + \epsilon u), -\epsilon H(s^{-} + a + \epsilon u) \right)/\det_a[\varphi(u)]
\]  

(4.11)

with \( \epsilon = b - a = \pm 1 \) and

\[
\det_a[\varphi(u)] = \frac{2H(x - K)\Theta(x - K)H(y)\Theta(y)}{H(K)\Theta(K)}
\]

with \( x = \frac{1}{2}(s^- + s^+) + a\lambda \) and \( y = \frac{1}{2}(s^- - s^+) + u \).

Applying the intertwining relations (1.3) and (1.9) to the reflection equations for the vertex model we are able to obtain the reflection equations for the SOS model,

\[
\sum_{f,g} W \left( \begin{array}{ccc} a & b & | \ u - v \\ g & c & | \ u \\ f & \end{array} \right) K_-(\begin{array}{ccc} g & c & | \ u \\ f & | \ u + v \\ d & e & \end{array}) W \left( \begin{array}{ccc} a & g & | \ u + v \\ d & f & | \ v \\ e & \end{array} \right) K_-(\begin{array}{ccc} d & f & | \ v \\ e & | \ u - v \\ f & \end{array}) K_+(\begin{array}{ccc} f & d & | \ v \\ e & | \ u \end{array}) S(f)S(a) \right) \frac{S(d)S(g)}{S(b)S(g)}
\]

(4.12)

(4.13)

This SOS analogue of the reflection equations, directly following from the reflection equations of the vertex model, has not been given before [32]. Similar to the vertex model, we have the commuting transfer matrix \( V(u) \) defined by the following elements

\[
\langle a \vert V(u) \vert b \rangle = \sum_{\{c_0, \ldots, c_N\}} K_+ \left( \begin{array}{ccc} a_0 & b_0 & | \ u \\ c_0 & | \ u \end{array} \right) \prod_{k=0}^{N-1} W \left( \begin{array}{ccc} c_k & c_{k+1} & | \ u + v_k \\ b_k & b_{k+1} & \end{array} \right) \times W \left( \begin{array}{ccc} c_{k+1} & a_k & | \ u - v_k \\ c_k & a_{k+1} & \end{array} \right) K_+ \left( \begin{array}{ccc} a_N & \ \ | \ \ \\ c_N & b_N & \end{array} \right)
\]

(4.14)

which satisfies

\[
[ \ V(u) , \ V(v) ] = 0
\]

(4.15)

(4.16)

where \( a = \{a_0, a_1, \ldots, a_N\} \), \( b = \{b_0, b_1, \ldots, b_N\} \) and these \( v_k \) are some arbitrary parameters. The proof of (4.16) can be done similarly to the vertex models. Also the su(2) type fusion rule in theorem (1) still works for the SOS models with open reflection boundaries.
**Theorem 3 (su(2) Fusion Hierarchy)** Let us define that

\[ V^{(q)}_k = V^{(p,q)}(u + k\lambda), \quad V^{(p,0)} = I \]

are the fused transfer matrices of the SOS model with the fusion levels \( p \) and \( q \). Then the su(2) fusion hierarchy follows that

\[ V^{(q)}_0 V^{(1)}_q = V^{(q+1)}_0 + f^p_{q-1} V^{(q-1)}_0 \quad q = 1, 2, \ldots \] (4.17)

if the reflection \( K_-, K_+ \) matrices satisfy the reflection equations (4.13)-(4.14). The function \( f^p_q = f^p(u + q\lambda) \) is given by the antisymmetric fusion.

The proof can be done similarly to the case of vertex models. In appendix C we give the proof for the simpler case of \( q = 1 \) and \( p = 1 \). It is interesting to notice that the functional relations (4.17) of the SOS model with the open boundaries and periodic boundaries have the same form except the \( u \)-dependent function \( f^p_{q-1} \) being different.

### 5. Discussion

In the previous sections we have studied the fusion hierarchies of the six and eight vertex models with the open boundaries. The functional relations of the fused transfer matrices of the models with the non-diagonal reflection matrices have been obtained. It has been shown that the functional relations can be solved to obtained the finite size corrections to transfer matrices for the six-vertex model with the diagonal reflection matrices [24]. It should be interesting in the near future to solve the fusion hierarchy and thermodynamic Bethe ansatz equations for the six-vertex model with the general non-diagonal reflection matrices by similar techniques. In this way it should be possible to obtain the central charges and scaling dimensions in terms of Rogers dilogarithms and their analytic continuations and hence completely elucidate the critical behaviour of the model and their fusion hierarchies.

The free energies and the eigen-spectra of the transfer matrices of the models with the open boundaries can be determined by the functional relations. To show this the discussion can be divided into a number of points.

#### 5.1. Crossing unitary condition for boundaries:

The crossing symmetry for boundary \( K \) matrices have been expressed in [30]. Using the crossing symmetries and the unitary conditions of the boundary \( K \) matrices the surface free energy \( f_s \) can be fixed. In [30] the argument is given in terms of quantum field theory. According to this argument it is not very clear how to express the crossing and unitary relation of the surface free energy for an off-critical model. However, like the \( R \) matrix the crossing
symmetry of the $K$ matrix can be given by the fusion procedure in statistical mechanics. To see this let us first consider the crossing symmetry of the $R$ matrix.

The antisymmetric fusion of the $R$ matrix

$$
Y_2^+ R^{1,1}(u - \lambda) R^{2,1}(u) Y_2^- = -\tilde{\rho}_{1,1}(u - \lambda) = \rho_{1,1}(u)
$$

is equivalent to the crossing unitary condition (1.5). This relation leads to the crossing unitary condition for the bulk free energy

$$
f_b(u) f_b(u + \lambda) = [\rho_{1,1}(u)]^{2N}
$$

Similarly, the crossing unitary condition of the $K$ matrices is related to the antisymmetric fusion of the $K$ matrices (see (2.5)). More precisely the factors $\omega_{\pm}$ determine the crossing unitary conditions of the $K_{\pm}$ matrices. Thus the surface free energies $f_s$ are determined by

$$
f_s(u) f_s(u + \lambda) = \frac{\omega_-(u - \lambda) \omega_+(u - \lambda)}{h(2u - \lambda) h(2u + \lambda)}
$$

The denominator $h(2u - \lambda) h(2u + \lambda)$ or $\sin(2u - \lambda) \sin(2u + \lambda)$ in the function $f^1(u - \lambda)$ can be removed by normalization of the transfer matrix $T^{(1,1)}$. Thus the functions $f^p(u)$ in (3.10) or (3.24) contain the unitary conditions and crossing symmetries of the bulk $R$ and the boundary $K$ matrices.

The bulk free energy of the eight-vertex model has been obtained by Baxter [28]. By the similar method we can solve the the (5.3) to obtain the surface free energy for the off-critical model. Therefore we should have the surface specific heats and the related critical exponents. This study does not rely on the exact diagonalization of the transfer matrix of the model with the non-diagonal reflection matrices, which has not yet solved. Thus we have presented a possible method to find the crossing unitary conditions for solving the surface free energy of an exact solvable model with open boundary conditions.

5.2. Ansatz of eigen-spectra: We have found the functional relations of the transfer matrices for all fusion hierarchies of the six- and eight-vertex models with the most general reflection matrices. The su(2) fusion rule still hold for the models with any $K$ matrices satisfying the reflection equations. It is not very clear to extract the exact eigen-spectra of transfer matrices from the functional relations for the models with general reflection matrices. However we may solve the eigen-spectra of transfer matrix from the functional relations for the eight-vertex model with the diagonal reflection matrices, which, in fact, has not been given before.

For the special case of diagonal $K$ matrices, the transfer matrix $T(u) = T^{(1,1)}(u)$ of the six vertex model has been diagonalized using algebraic Bethe ansatz [2] and the
eigen-spectra is given by

\[ T(u) = \frac{\Lambda_1(u) + \Lambda_2(u)}{g(u)} \]  

\[ \Lambda_1(u) = \omega_1(u) \phi(u + \lambda) Q(u - \lambda) / Q(u) \]  

\[ \Lambda_2(u) = \omega_2(u) \phi(u) Q(u + \lambda) / Q(u) \]  

\[ g(u) = \sin(2u + \lambda) \]  

where

\[ Q(u) = \prod_{m=1}^{M} \{ \sin(u - v_m) \sin(u + v_m + \lambda) \} \]  

\[ \phi(u) = \sin^{2N}(u) \]  

\[ \omega_2(u) = \omega_1(-u - \lambda) = \sin(\xi_+ + u + \lambda) \sin(\xi_- - u - \lambda) \sin(2u) . \]  

These \( v_1, v_2, \ldots, v_M \) satisfy the Bethe ansatz equations

\[ T(v_j) = 0 . \]  

It is easy to see that

\[ \Lambda_1(u + \lambda) \Lambda_2(u) = f^1(u) . \]  

Thus ansatz (5.4) solves the functional relations (2.9). If we use the same ansatz (5.4) with all trig-functions \( \sin \) replaced with elliptic function \( h \),

\[ Q(u) = \prod_{m=1}^{M} \{ h(u - v_m) h(u + v_m + \lambda) \} \]  

\[ \phi(u) = h^{2N}(u) \]  

\[ \omega_2(u) = \omega_1(-u - \lambda) = h(\xi_+ + u + \lambda) h(\xi_- - u - \lambda) \times H(2u) \Theta(2u + 2\lambda) , \]  

\[ g(u) = h(2u + \lambda) \Theta(0) \]  

then (5.12) is still correct for the transfer matrix \( T(u) = T^{(1,1)}(u) \) of the eight-vertex model with diagonal \( K \) matrices, which are given by setting \( \mu_\pm = 0 \). Thus the ansatz solves the functional relations of the eight-vertex model. Moreover it is reduces to the solution of the six-vertex model as the elliptic nome \( p \to 0 \). These facts, following Reshetikhin [31], support that the above ansatz could indeed be the eigenvalues of the eight-vertex transfer matrix. Therefore the thermodynamics of the models could be determined by solving the Bethe ansatz solutions (5.4) and (5.11).

5.3. RSOS model with open boundaries: In [17] the functional relations for the restricted SOS model or ABF model with open boundaries have been constructed. Here
we have shown the functional relations of the unrestricted SOS model and the correspondence between vertex $K$- and face $K$-matrices. But we have not given the explicit form of the face $K$ matrices by this correspondence. This however is a very interesting problem. We may obtain the face $K$ matrices of the ABF model by restricting the SOS model. Also whether or not the face $K$ matrices following from this correspondence are the same as the face $K$ matrices presented in [17] is an open question and is presently under investigation. It is obvious that the same idea can be applied to higher rank JMO IRF models and Belavin’s $Z_n$ vertex models.

The definition of the SOS transfer matrix is not unique. The definition given in (4.15) is different from the one presented in [17] because it seems that we cannot get one from another one by using simple symmetries like crossing symmetry of the face weights. However, the definition given in (4.15) follows directly from the original formulation given by Sklyanin [2] and is good to study a square lattice rotated by $45^0$ (see [10]), which is the natural geometry for an integrable loop model with open boundaries in statistical mechanics. This further study will be helpful to generalize the study of Baxter’s eight-vertex model with periodic boundaries in [27] to the model with open boundaries and to well understand the critical behaviour of the face models [33, 34].

Acknowledgements

This research has been supported by the Australian Research Council. The author thanks Bo-Yu Hou and Paul A. Pearce for discussions and is also grateful to Murray Batchelor for a critical reading of the manuscript.

Appendix A: Transfer matrices and their graphs

Here we prove the commuting relation (1.22) for the fused models graphically following the Sklyanin’s arguments in the study of the unfused six-vertex model [4]. For the sake of clarity let us represent the $R$- and $K$-matrices by the graphs,

\[
R^{12}(u) = \begin{array}{c}
1 \\
\scriptstyle 2 \\
\scriptstyle u
\end{array}, \quad K_-(u) = \begin{array}{c}
\scriptstyle u
\end{array}, \quad K_+(u) = \begin{array}{c}
\scriptstyle u
\end{array}
\]  

(A.1)
Therefore we can represent the Yang-Baxter equation (1.1) graphically by

\[
\begin{align*}
1 \xrightarrow{u} 2 & \xrightarrow{v} 3 \quad \text{or} \quad 2 \xrightarrow{u} 3 & \xrightarrow{v} 1 \\
\end{align*}
\]

(A.2)

the reflection equations (1.12) and (1.13) by

\[
\begin{align*}
1 \xrightarrow{u} 2 \xrightarrow{v} 1 & = \rho(u) \\
1 \xrightarrow{-u-2} 2 \xrightarrow{v} 1 & = \tilde{\rho}_{1,1}(u)
\end{align*}
\]

(A.3)

and the unitary and crossing unitary relations by

\[
\begin{align*}
1 & \xrightarrow{u} 2 \\
2 & \xrightarrow{-u-2} 1
\end{align*}
\]

(A.4)

(A.5)

Using the graphs the fused R matrix \( R_{(p,q)}(u) \) (1.7) is given by the elements

\[
\begin{align*}
\sum_{j,l} l_{1j} l_{2j} & = \rho(u) \\
\sum_{k} l_{1k} l_{2k} & = \tilde{\rho}_{1,1}(u)
\end{align*}
\]

(A.6)

and the fused K matrices \( K^{-}(q)(u) \) (1.14) and \( K^{+}(q)(u) \) (1.15) are given, respectively, by
the elements

\[ u \uparrow = \rho(u|q) \sum_j \]

and

\[ u \downarrow = \sum_j \]

where \( u_{r,s} = u + r\lambda - s\lambda \) and sum over \( j, l \) is over all possible spins \( j_1, j_2, \ldots, j_p \) and \( l_1, l_2, \ldots, l_q \) satisfying

\[ j = \sum_{s=1}^{p} j_s \quad \text{and} \quad l = \sum_{s=1}^{q} l_s . \]

The spins \( i \) and \( k \) are

\[ i = \sum_{s=1}^{p} i_s \quad \text{and} \quad k = \sum_{s=1}^{q} k_s . \]

The indices \( i, j = -\frac{1}{2}p, -\frac{1}{2}p + 1, \ldots, \frac{1}{2}p \) and \( k, l = -\frac{1}{2}q, -\frac{1}{2}q + 1, \ldots, \frac{1}{2}q \).

The fused Yang-Baxter equation (1.1) and reflection equations (1.17)-(1.18) are ex-

19
pressed by graphs

\[
\begin{align*}
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = 
\end{align*}
\]

for (1.17) for (1.18)

The unitary (1.9) and crossing unitary conditions (1.10) for the fused \( R \) matrix become

\[
\begin{align*}
p & \quad u \quad q \\
q & \quad -u-2 \quad p
\end{align*}
\]

\[
\begin{align*}
p & \quad u \quad q \\
q & \quad -u-2 \quad p
\end{align*}
\]

To prove (1.22) consider \( T^{(p,q)}(u) T^{(p,b)}(v) \)

Inserting the operators \( Y^+_q \) and \( Y^+_b \) respectively into the positions \( a \) and \( b \) and using the crossing unitary condition (A.12) and then using the Yang-Baxter equation (A.9) to push the vertex with spectra parameter \( u + v - \lambda + b\lambda \) through to the most right, we are able to obtain

\[
\frac{1}{\rho_{q,b}(u+v-\lambda+b\lambda)} \times \\
\begin{align*}
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = \\
\quad & \quad = 
\end{align*}
\]

(A.14)
Then inserting the operators $Y^+_q$ and $Y^+_b$ again into the positions $a$ and $c$ respectively and employing the unitary condition (A.11) and then using the Yang-Baxter equation (A.9) to push the vertex with spectra parameter $v - u$ through again to the most right, we have

\[
\frac{1}{\rho_{q,b}(u+v-\lambda+b\lambda)\rho_{b,q}(u-v)} \times \begin{pmatrix}
K^{(q)}_+(u) & K^{(q)}_+(v) & K^{(q)}_+(u) \\
K^{(q)}_-(v) & K^{(q)}_-(u) & K^{(q)}_-(v) \\
K^{(q)}_-(u) & K^{(q)}_+(u) & K^{(q)}_-(v)
\end{pmatrix}
\]

(A.15)

By the reflection equations (A.10) we obtain

\[
\frac{1}{\rho_{q,b}(u+v-\lambda+b\lambda)\rho_{b,q}(u-v)} \times \begin{pmatrix}
K^{(q)}_+(u) & K^{(q)}_+(v) & K^{(q)}_+(u) \\
K^{(q)}_-(v) & K^{(q)}_-(u) & K^{(q)}_-(v) \\
K^{(q)}_-(u) & K^{(q)}_+(u) & K^{(q)}_-(v)
\end{pmatrix}
\]

(A.16)

Using the Yang-Baxter equation to push the vertex with spectral parameter $v - u$ back through to the left and removing the vertices with spectral parameters $v - u$ and $u - v$ by the unitary condition (A.11), then pushing the vertex with spectral parameter $v + u - \lambda + b\lambda$ through back to the left and absorbing the vertices with spectral parameters $-v - u - \lambda - b\lambda$ and $u + v + b\lambda$ by the crossing unitary condition (A.12), we finally obtain (A.13) with $u$ and $v$ exchanged. So we have shown that $\left[ T^{(p,b)}(v), T^{(p,q)}(u) \right] = 0$ as required.

**Appendix B: Functional equations and their graphs**

In this appendix we prove the functional equations expressed in Theorem 1. Let us consider $T_0^{(q)}$ $T_q^{(1)}$

\[
\begin{pmatrix}
K_+(u+\lambda) \\
K^{(q)}_+(u) \\
K^{(q)}_-(u)
\end{pmatrix}
\]

(B.1)

Inserting the operator $Y^+_q$ into the position $a$ and the identity operator into the position $b$ and then using the unitary condition (A.11) and the Yang-Baxter equation (A.9), we
are able to obtain

\[
\frac{1}{\tilde{\rho}_{q,1}(2u+q\lambda)} \times K_{+}^{(q)}(u) \times K_{-}^{(q)}(u) \times K_{-}^{(q+1)}(u) \times K_{+}^{(q+1)}(u)
\]

(B.2)

It is easy to see that the equation can be split into the sum of the following two parts,

\[
K_{+}^{(q+1)}(u) \times \left( K_{+}^{(q+1)}(u) \right)
\]

(B.3)

and

\[
\bar{J}_{q-1}^{p}(u) \times \left( I_{+}^{(q)}(u) \times I_{-}^{(q)}(u) \right)
\]

(B.4)

where the \( u \)-dependent function \( \bar{J}_{q-1}^{p}(u) \) is generated from the antisymmetric fusion of the model. The underlying models are \( su(2) \) type and the antisymmetric fusion gives only one independent nonzero element for the \( R \) or \( K \) matrices. Therefore \( \bar{J}_{q-1}^{p}(u) \) can be factorized as

\[
\bar{J}_{q-1}^{p}(u) = \omega_- (u + q\lambda - \lambda) \omega_+ (u + q\lambda - \lambda) \phi_+^p (u + q\lambda - \lambda) \times \phi_- (u + q\lambda - \lambda) I_+ (u + q\lambda - \lambda|q) I_- (u + q\lambda - \lambda|q).
\]

(B.5)

For the six- and eight-vertex models we have shown

\[
I_+ (u|q) I_- (u|q) = I_+ (u|1) I_- (u|1) = \tilde{\rho}_{1,1}^{-1}(2u + \lambda).
\]

(B.6)

Therefore \( f_q^p(u) \) depends on \( q \) only through the spectra parameter shift \( u + q\lambda - \lambda \). We suppress the subscript \( q \) and define \( f^p(u) = \bar{J}_{q-1}^{p}(u - q\lambda + \lambda) \). Graphically \( f^p(u) \) is given by

\[
\frac{1}{\tilde{\rho}_{1,1}(2u+\lambda)} \times K_{+}^{(u+\lambda)}(u) \times K_{-}^{(u+\lambda)}(u)
\]

(B.7)
or
\[
    f^P(u) = \omega_-(u)\omega_+(u)\phi_+^P(u)\phi_-^P(u)I_+(u|q)I_-(u|q)
\]  
(B.8)

where
\[
    \omega_+(u) = \frac{K_+(u+\lambda)}{K_+(u)}
\]
\[
    \omega_-(u) = \frac{K_-(u+\lambda)}{K_-(u)}
\]
(B.9)
\[
    \phi_+^P(u-\frac{1}{2} p\lambda + \frac{1}{2} \lambda) = \frac{2\lambda}{q-2\lambda} \times \tilde{\rho}_{q,1,1}^{-1}(2t-q\lambda+\lambda)
\]  
(B.10)
\[
    \phi_-^P(u+\frac{1}{2} p\lambda - \frac{1}{2} \lambda) = \frac{2\lambda}{q-2\lambda} \times \tilde{\rho}_{q,1,1}^{-1}(2t-q\lambda+2\lambda)
\]  
(B.11)

where the cross with a circle plays the role of antisymmetric operator,
\[
    \frac{i}{j} \times \frac{k}{l} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}).
\]

The theorem 1 can be seen from (B.3), (B.4) and (B.8).

**Appendix C: Intertwining relations and their graphs**

According to the graphs presented in Appendix B and Appendix C the intertwining relations between the vertex and face models (2.3) can be represented as
\[
    \begin{array}{c}
    \begin{array}{c}
        \text{d} \quad \text{u} \quad \text{c} \\
        \text{i} \quad \text{u-v} \quad \text{v-b} \quad \text{j} \\
    \end{array}
    \\
    \begin{array}{c}
        \text{d} \quad \text{u-v} \\
        \text{i} \quad \text{j} \quad \text{u} \quad \text{v-b} \quad \text{j} \\
    \end{array}
\end{array}
\]  
(C.1)

where the solid dots mean summation (as do other solid dots in this section). The Star-triangle relation (Yang-Baxter equation) of these face weights is graphically
\[
    \begin{array}{c}
    \begin{array}{c}
        \text{e} \quad \text{v} \quad \text{d} \\
        \text{f} \quad \text{v-u} \quad \text{c} \\
    \end{array}
    \\
    \begin{array}{c}
        \text{e} \quad \text{v} \\
        \text{f} \quad \text{v-u} \quad \text{c} \\
    \end{array}
\end{array}
\]  
(C.2)
For the boundaries the correspondence (4.9) are given by

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 i \\
 u \\
 u \\
 i \\
\end{array}
\end{array}
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 i \\
 u \\
 -u \\
 c
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.3)

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 u \\
 b \\
 a' = -u - \lambda
\end{array}
\end{array}
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 c \\
 u + \lambda \\
 i
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.4)

The unitary conditions of the face weights (4.6) is

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 a' \\
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.5)

and the crossing unitary condition (4.7) is

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a < -u - \lambda \\
 b \\
 a' = -u - \lambda
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.6)

The reflection equations for the SOS model (4.14) and (4.13) can be presented graphically by

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 i \\
 v - u \\
 b \\
 a \\
 u + v \\
 d \\
 e \\
 f \\
 e
\end{array}
\end{array}
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
 b \\
 i \\
 v - u \\
 b \\
 a \\
 u + v \\
 d \\
 e \\
 f \\
 e
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.7)

and

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 d \\
 c \\
 u + v \\
 a \\
 v \\
 d \\
 e \\
 f \\
 e \\
 f
\end{array}
\end{array}
\end{array}
\end{array}
\]  
(C.8)
The transfer matrix of the SOS model with the open boundaries (4.13) is presented graphically by

\[
\langle a|V(u)|b \rangle = K_+(u)
\]

where the boundaries are given by

\[
K_+ \left( \begin{array}{c} a \\ b \\ u \end{array} | \begin{array}{c} a \\ b \\ u \end{array} \right) := \frac{S(b)}{S(a)} \varphi_{b,a}^{-1}(u + \lambda) \quad K_+ (u) \varphi_{b,c} (-u - \lambda) = K_+(u)
\]

and

\[
K_- \left( \begin{array}{c} b \\ a \\ u \end{array} | \begin{array}{c} a \\ b \\ u \end{array} \right) := \varphi_{b,c}^{-1}(-u) K_- (u) \varphi_{b,a} (u) = K_-(u)
\]

Let us consider \( V(u) V(u + \lambda) \) graphically,

Inserting the crossing unitary condition (4.7) into above term we have

\[
\sum_a c, a, c = \frac{c, c - 1, c}{c, -1, c} + \frac{c, c + 1, c}{c, +1, c}
\]

\[
= \left( \sqrt{\frac{S(c+1)}{S(c+1)}} \right) \frac{c, c - 1, c}{c, c + 1, c} + \frac{c, c + 1, c}{c, +1, c}
\]

\[
\oplus \left( \frac{c, c - 1, c}{c, c - 1, c} - \sqrt{\frac{S(c+1)}{S(c+1)}} \right)
\]

where we have not written down the ratio of the functions \( S \), which come from the crossing unitary condition and we have to take care of them. The sum over \( a \) can be divided into the symmetric and antisymmetric ones for \( c = d \),

\[
\sum_a c, a, c = \frac{c, c - 1, c}{c, -1, c} + \frac{c, c + 1, c}{c, +1, c}
\]

\[
= \left( \sqrt{\frac{S(c+1)}{S(c+1)}} \right) \frac{c, c - 1, c}{c, c + 1, c} + \frac{c, c + 1, c}{c, +1, c}
\]

\[
\oplus \left( \frac{c, c - 1, c}{c, c - 1, c} - \sqrt{\frac{S(c+1)}{S(c+1)}} \right)
\]
Therefore (C.13) is divided into two terms, which are

\[
\sum_b g_s(c, b, d) = \sum_b \begin{cases} 
\frac{S(a-1)}{S(a+1)}, & b = a-1, a = c \\
1, & \text{otherwise}
\end{cases}
\]

(C.14)

where \(a = c + 1\) for \(c = d\) and \(a = (c + d)/2\) otherwise and

\[
\sum_a g_a(c, a, d) = \sum_a \begin{cases} 
-\frac{S(a-1)}{S(a+1)}, & b = a+1, a = c \\
\delta_{a,c}, & \text{otherwise}
\end{cases}
\]

(C.15)

where \(c = d\) and \(b = c - 1\). The factors \(g_s\) and \(g_a\) are given by

The first term is the fused transfer matrix of the model at level 2, which is expressed by \(V^{(2)}(u) = V^{(2)}_0\). The second term is the antisymmetric fusion of the model, which gives exactly the factor \(f_0^1 = f^1(u)\). Note that the function \(f^1(u)\) is dependent the face boundary \(K_{\pm}\) matrices. So we finally have the functional equations (4.17).

References

[1] I. Cherednik, Theor. Mat. Fiz. 61 (1984) 977.
[2] E. K. Sklyanin, J. Phys. A21 (1988) 2375.
[3] H. J. de Vega and A. González Ruiz, J. Phys. A26 (1993) L519.
[4] B. Y. Hou and R. H. Yue, Phys. Lett. A183 (1993) 169.
[5] T. Inami and H. Konno, J. Phys. A27 (1994) L913.
[6] H. J. de Vega and A. González Ruiz, *Nucl. Phys.* **B417** (1994) 553; *Phys. Lett.* **B332** (1994) 123.

[7] B.-Y. Hou, K.-J. Shi, H. Fan and Z. X. Yang, *Commun. Theor. Phys.* **23**(1995) 163.

[8] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys.* **A20** (1987) 6397.

[9] L. Mezincescu and R. I. Nepomechie, *Nucl. Phys.* **B372** (1992) 579.

[10] C. M. Yung and M. T. Batchelor, *Nucl. Phys.* **B435** (1995) 430.

[11] A. Foerster and M. Karowski, *Nucl. Phys.* **B408** (1993) 512.

[12] A. González Ruiz, preprint F.T./U.C.M.-94/1; [hep-th/9401118](http://arxiv.org/abs/hep-th/9401118).

[13] L. Mezincescu, R. I. Nepomechie and V. Rittenberg, *Phys. Lett.* **B147** (1990) 70.

[14] S. Artz, L. Mezincescu and R. I. Nepomechie, Bonn-Th-94-19, UMTG-178.

[15] L. Mezincescu and R. I. Nepomechie, *J. Phys.* **A25** (1992) 2533.

[16] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa, *Nucl. Phys.* **B441** (1995) 437.

[17] P. A. Pearce and R. E. Behrend, *Commuting row transfer matrices and functional equations for lattice spin models with fixed boundary conditions.*

After this paper was submitted, two relevant preprints appeared in the HEP-TH bulletin-board:

P. P. Kulish “Yang-Baxter and reflection equations in integrable models” [hep-th/9507070](http://arxiv.org/abs/hep-th/9507070).

R. E. Behrend, P. A. Pearce and D. L. O’Brien “Interaction-Round-a-Face models with fixed boundary conditions: the ABF fusion hierarchy” [hep-th/9507118](http://arxiv.org/abs/hep-th/9507118).

[18] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, *Lett. Math. Phys.* **5** (1981) 393.

[19] A. N. Kirillov and N. Yu. Reshetikhin, *J. Phys.* **A20** (1987) 1565.

[20] V. V. Bazhanov and N. Yu. Reshetikhin, *Int. J. Mod. Phys.* **B4** (1989) 115.

[21] A. Klümper and P. A. Pearce, *Physica A* **183** (1992) 304.

[22] A. Kuniba, T. Nakanishi and J. Suzuki, *Int. J. Mod. Phys.* **A9** (1994) 5267.

[23] Y. K. Zhou and P. A. Pearce, *Nucl. Phys.* B (1995).
[24] Y. K. Zhou, Nucl. Phys. B (1995).

[25] A. Kuniba and T. Nakanishi, Int. J. Mod. Phys. (Proc. Suppl.) A3 (1993) 419.

[26] R. J. Baxter, J. Stat. Phys. 28 (1982) 1.

[27] R. J. Baxter, Ann. Phys. 76 (1973) 1; 25; 48.

[28] R. J. Baxter, Phys. Rev. Lett. 26 (1971) 832; Ann. Phys. 70 (1972) 193.

[29] M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 14 (1987) 123.

[30] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A 21 (1994) 3841.

[31] N. Yu Reshetikhin, Sov. Phys. JETP. 57 (1983) 691.

[32] In [17] the reflection equations do not follow from the original formulation given by Sklyanin in [2] and thus they are set up differently.

[33] J. Cardy, Nucl. Phys. B324 (1989) 581.

[34] H. Saleur and Bauer, Nucl. Phys. B320 (1989) 591.