On the Complexity of Computing Zero-Error and Holevo Capacity of Quantum Channels

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Abstract

One of the main problems in quantum complexity theory is that our understanding of the theory of QMA-completeness is not as rich as its classical analogue, the NP-completeness. In this paper we consider the clique problem in graphs, which is NP-complete, and try to find its quantum analogue. We show that, quantum clique problem can be defined as follows; Given a quantum channel, decide whether there are k states that are distinguishable, with no error, after passing through channel. This definition comes from reconsidering the clique problem in terms of the zero-error capacity of graphs, and then redefining it in quantum information theory. We prove that, quantum clique problem is QMA-complete.

In the second part of paper, we consider the same problem for the Holevo capacity. We prove that computing the Holevo capacity as well as the minimum entropy of a quantum channel is NP-complete. Also, we show these results hold even if the set of quantum channels is restricted to entanglement breaking ones.

1 Introduction

One of the basic results of complexity theory is Cook-Levin theorem. That is, SAT, the problem of whether a Boolean formula has a satisfying assignment or not, is NP-complete. In fact, the Cook-Levin theorem was the beginning of the theory of NP-completeness. After SAT, a series of natural problems in graph theory and combinatorics were shown to be NP-complete as well, see [27]. Hamiltonian cycle, clique problem, graph coloring, subset sum and vertex cover are examples of such problems. The rich theory of NP-completeness has been used in other parts of complexity theory as well. For instance, the basic ideas of such important results as IP = PSPACE [23], and the PCP theorem [4], are from this theory.

In quantum complexity theory, Kitaev’s result [13] is considered as the quantum analogue of Cook-Levin theorem. The complexity class QMA was defined by Watrous as the quantum analogue of NP. QMA is the class of problems that can be solved by a quantum polynomial time algorithm given a quantum witness. Then Kitaev defined a natural problem in physics, called the local Hamiltonian problem and showed it is QMA-complete, [13]. Indeed, the local Hamiltonian problem is considered as the quantum version of SAT problem, and also Kitaev’s result as the quantum analogue of Cook-Levin theorem. Thus, same as in classical case, one would guess that this is the beginning of a rich theory for QMA-complete problems.

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After Kitaev’s theorem few problems have been shown to be QMA-complete. For instance, circuit identity testing [9] and local consistency problem [29] are such problems. But this class is not as rich as the class of NP-complete problems.

One of the most natural ways of thinking of QMA-complete problems is to somehow define the quantum version of problems that are known to be NP-complete. An example of such a problem is the clique problem in graphs.

1.1 Quantum-clique problem

In a graph $G$ a clique is a subset of vertices every two of which are adjacent, and the size of a clique is the number of its vertices. The clique problem is that given a graph $G$ and an integer number $k$, decide whether $G$ contains a clique of size $k$ or not. It is well-known that clique problem is NP-complete [27].

We can think of this problem in $G^c$, the complement of graph $G$. In the complement of $G$ a clique is changed to an independent set. An independent set in a graph is a subset of vertices no two of which are adjacent, and the maximum size of an independent set is called the independence number of $G$, denoted $\alpha(G)$. So the clique problem in the complement graph reduces to decide whether $\alpha(G) \geq k$, and then it is NP-complete.

This reduction is important because the problem of computing $\alpha(G)$ is related to the problem of computing the zero-error capacity of a channel. Zero-error capacity of a classical (discrete memoryless) channel is the maximum rate of information that one can send through channel without error. This concept is first introduced by Shannon in his famous paper [25], and got much attention after that. For a survey on this topic see [14]. Also some interesting results on graph capacity are presented in [2, 8, 15].

Indeed, any classical channel corresponds to a graph $G$, and the zero-error capacity of the channel can be computed in terms of the independence number of $G$ and its powers. We will get to this correspondence latter on, but here we mention that the best way of coding messages in words of length one is to find an independent subset of size $\alpha(G)$. Therefore it is NP-complete to find such a codeword.

This way of thinking of clique problem is useful because it is in terms of channels and their zero-error capacity, the concepts that are already known in quantum information theory. The definition of a quantum channel is well-known in the theory. Also extending the definition of zero-error capacity to quantum channels is straightforward, [16]. That is, what is the maximum rate of classical information that can be sent through a quantum channel with zero-error and without using entanglement. Thus, the same question as in the classical case arises: How can we compute the zero-error capacity of a quantum channel?

As in classical case let us first try to code the messages in words of length one (not in product states). Then the question is that, are there $k$ states that after passing through channel can be recognized with no error? This is exactly the quantum analogue of the clique problem. So we call it the quantum clique problem. In this paper we prove that quantum clique is QMA-complete.

1.2 Computing the channel capacity

In the previous section we said that the problem of estimating the zero-error capacity of a classical channel (graph) is NP-complete, and its quantum analogue is QMA-complete. Both of these problems are about the channel capacities in the zero-error case. But, how hard is to compute the usual capacity of a channel?

In the classical case, there is an algorithm, called Arimoto-Blahut algorithm, that computes the capacity of a classical channel efficiently, [3, 5]. In the quantum case, the same idea as in Arimoto-Blahut algorithm leads us to an algorithm for computing the Holevo capacity,
The idea is to use Klein’s inequality, see [20], iteratively and find a sequence that converges to a local maximum of the expression in the Holevo capacity. Although finding this sequence can be done in polynomial time, it gives a local maximum not a global one, and unlike the classical case a local maximum is not a global one. So, the quantum version of Arimoto-Blahut algorithm fails. Here, we show that finding a bound for the Holevo capacity of a quantum channel is NP-complete.

To prove this result we first show that computing the minimum entropy of a quantum channel is NP-hard and then convert the problem of computing the Holevo capacity to this one.

2 Preliminaries

2.1 QMA

Definition 2.1 A language $L$ is said to be in QMA = QMA($2/3, 1/3$) if there exists a quantum polynomial time verifier $V$ such that

- **Completeness:** $\forall x \in L, \exists |\xi\rangle$, $\Pr(V(|x\rangle|\xi\rangle) \text{ accepts} \geq 2/3$.
- **Soundness:** $\forall x \not\in L, \forall |\xi\rangle$, $\Pr(V(|x\rangle|\xi\rangle) \text{ accepts} \leq 1/3$.

An easy amplification argument implies that the completeness and soundness bounds, $2/3$ and $1/3$, are not crucial and can be replaced by any functions $a(n)$ and $b(n)$ provided that, they are different from $0$ and $1$ by an inverse exponential function, and also there is an inverse polynomial gap between them. In other words, if $a(n)$ and $b(n)$ are two functions such that $0 < b(n) < a(n) < 1$, and for some constant $c$,

$$a(n) < 1 - e^{-n^c}, \quad b(n) > e^{-n^c},$$

and

$$a(n) - b(n) < n^{-c},$$

then QMA$(a, b) = \text{QMA}(2/3, 1/3) = \text{QMA}$, see [1].

Definition 2.2 $k$-Local Hamiltonian problem $(H_1, \ldots H_s, a, b)$

- **Input:** An integer $n$, real numbers $a, b$ such that $b - a > n^{-c}$, and polynomially many Hermitian non-negative semidefinite matrices $H_1, \ldots H_s$ with bounded norm, $\|H_i\| \leq 1$, such that each of them acts just on $k$ of $n$ qubits.
- **Promise:** The smallest eigenvalue of $H_1 + \ldots H_s$ is either less than $a$ or greater than $b$.
- **Output:** Decide which one is the case.

Kitaev proved that local Hamiltonian problem for $k = 5$ is QMA-complete [13], but latter this result was improved in [11] and [10].

Theorem 2.1 2-local Hamiltonian problem is QMA-complete.
2.2 QMA$_1$

We said that QMA$(a, b)$ is equal to QMA$(2/3, 1/3)$ if $a \neq 1$, $b \neq 0$, and there is a polynomial gap between them. But one may ask about the case where $a = 1$, or $b = 0$, i.e. perfect completeness or soundness. For instance QMA$(1, 1/3)$, is set of languages that have a protocol as in definition 2.1 with completeness bound 1 and soundness bound 1/3. Same as before, an amplification argument shows that if $b(n)$ is such that $e^{-n^c} < b(n) < 1 - n^{-c}$ for some constant $c$, then QMA$(1, b) = QMA(1, 1/3)$. Therefore, we get to a robust complexity class, denoted QMA$_1$, where the subscript 1 shows the perfect completeness.

This complexity class was first introduced by Bravyi [7]. He also defined the quantum $k$-SAT problem which is a special case of local Hamiltonian problem. To state this problem precisely we need to fix some notions. In the Hilbert space of $n$-qubits, a projection is a Hermitian operator $\Pi$ such that $\Pi^2 = \Pi$, i.e. eigenvalues of $\Pi$ are 0 and 1. Also, we say $\Pi$ is a $k$-projection if it acts just on $k$ qubits.

**Definition 2.3** Quantum $k$-SAT problem $(\Pi_1, \ldots, \Pi_s, \epsilon)$

- **Input:** An integer $n$, a real number $\epsilon > n^{-c}$, and polynomially many $k$-projections $\Pi_1, \ldots, \Pi_s$.
- **Promise:** Either there exists an $n$-qubit state $|\psi\rangle$ such that $\Pi_i|\psi\rangle = 0$ for all $i$, or $\sum_i \langle \psi | \Pi_i | \psi \rangle \geq \epsilon$ for all $|\psi\rangle$.
- **Output:** Decide which one is the case.

Given a witness $|\psi\rangle$, in polynomial time we can decide whether $\Pi_i|\psi\rangle = 0$ or not, and then $k$-SAT is in QMA$_1$. But we should be careful because we need a protocol that has no error if $|\psi\rangle$ is the right witness (the common eigenvector of $\Pi_i$’s). It means that the verifier should be able to check $\Pi_i|\psi\rangle = 0$ with no error, or equivalently, the verifier should be able to implement the projection $\Pi_i$ exactly.

This extra condition on the verifier for QMA$_1$ protocols arises naturally not only for quantum $k$-SAT, but also for any other problem in this class. If we can implement the gates just with an approximation, then our algorithm contains some error anyway. This problem can be resolve by emphasizing that the verifier can implement all quantum gates up to three-qubit gates, exactly. So in this paper, by a quantum verifier for QMA$_1$ protocols we mean the one that has all three-qubit quantum gates in hand. Bravyi pointed out this assumption in [7] and proved the following lemma.

**Lemma 2.1** Let $U$ be a unitary operator acting on $k$ qubits. Then $U$ can be exactly represented by a quantum circuit of size $\text{poly}(k)2^{2k}$ with three-qubit gates.

Also he showed the following theorem, that has almost the same proof as theorem 2.1.

**Theorem 2.2** Quantum 4-SAT is QMA$_1$-complete.

Both theorems 2.1 and 2.2 are important in the theory of QMA-complete problems. Because if we show some problem is in QMA$_1$ (QMA$_1$) and also find a reduction from local Hamiltonian (quantum SAT) to it then we conclude that it is QMA-complete (QMA$_1$-complete).
2.3 Zero-error channel capacity

A classical discrete memoryless channel consists of an input set $X$, an output set $Y$, and probability distributions $p(y|x)$ for every $x \in X$ and $y \in Y$, meaning that if we send $x$ through channel we get $y$ as output with probability $p(y|x)$. Since, we want to define the zero-error capacity of this channel, the exact value of $p(y|x)$ is not important for us, but whether it is zero or not. Therefore, to get a clearer representation, we correspond to the channel a graph $G$ on the vertex set $X$ in which two vertices $x,x' \in X$ are adjacent if there is $y \in Y$ such that $p(y|x), p(y|x')$ are both non-zero. It means that, $x,x'$ are adjacent in $G$ if they can be confused after passing through channel. Hence, some messages $x_1,\ldots,x_k$ can be sent through channel with no error iff there is no edges between them, i.e. $\{x_1,\ldots,x_k\}$ is an independent set.

**Definition 2.4** In a graph $G$, a subset of vertices no two of which are adjacent is called an independent set. Also $\alpha(G)$ denotes the maximum size of an independent set in $G$.

By the above discussion, if we want to code our messages in words of length one (one use of channel) then the best way is to code them in an independent set of maximum size. In this case we get to the rate $\alpha(G)$. But we may use words of length two. In this case we get to another graph, denoted $G \otimes G$.

**Definition 2.5** Assume $G$ and $H$ are two graphs on vertex sets $V$ and $U$, respectively. Then their tensor product $G \otimes H$ is a graph on the vertex set $V \times U$ such that $(v_1, u_1)$ and $(v_2, u_2)$ are adjacent if $v_1$, $v_2$ are either equal or adjacent in $G$, and also $u_1, u_2$ are either equal or adjacent in $H$.

It is not hard to see that the graph corresponding to words of length two is $G \otimes G$. Thus, the best way to code the messages in words of length two is to use an independent set in $G \otimes G$ of size $\alpha(G \otimes G)$. So we get to the rate $\alpha(G \otimes G)^{1/2}$ (square root is for normalization). Repeating this argument for higher products, we get to the following definition due to Shannon [24].

**Definition 2.6** $\Theta(G)$, the capacity of the graph $G$, is equal to

$$\Theta(G) = \lim_{n \to \infty} \alpha(G^\otimes n)^{1/n}.$$

These definitions all can be generalized for quantum channels, [16,17]. The zero-error capacity of a quantum channel is the maximum rate of classical information that one can send through a quantum channel without using entanglement. To get a closed form expression for this quantity, suppose $\Phi$ is a quantum channel, and we code $k$ messages in quantum states $\rho_1,\ldots,\rho_k$. If we want to decode the outputs of channel with no error, we should be able to identify states $\Phi(\rho_1),\ldots,\Phi(\rho_k)$ without error. It is well-known that some quantum states can be recognized with no error iff they have orthogonal supports.

**Definition 2.7** For a quantum channel $\Phi$, $\alpha(\Phi)$ is the maximum number of states $\rho_1,\ldots,\rho_k$ such that $\Phi(\rho_1),\ldots,\Phi(\rho_k)$ have orthogonal supports.

Also, $\alpha(\Phi^\otimes n)$ is the maximum number of product states $\rho_1 \otimes \cdots \otimes \rho_n, i = 1,\ldots,k$, such that all states $\Phi^\otimes n(\rho_1 \otimes \cdots \otimes \rho_n), i = 1,\ldots,k$, have orthogonal supports.

To make it clear we should highlight two points. First of all, if $\Phi(\rho)$ and $\Phi(\rho')$ have orthogonal supports then $\rho$ and $\rho'$ also have orthogonal supports. Therefore $\alpha(\Phi)$ is at most the dimension of input states, and is finite. Second, we emphasize that the input states of
Φ⊗n should be product states (because we do not want to use entanglement) and by abuse of notation we denote the maximum number of such product states by \( \alpha(\Phi^n) \).

By the above definition it is clear what the zero-error capacity of a quantum channel should be.

**Definition 2.8** Θ(Φ), the zero-error capacity of the quantum channel Φ, is

\[
\Theta(\Phi) = \lim_{n \to \infty} \alpha(\Phi^n)^\frac{1}{n}.
\]

### 2.4 How to compute \( \alpha(\Phi) \)

We do not repeat all the known properties of \( \alpha(\Phi) \) and refer the reader to [10]. Here, we need just two basic properties. Suppose \( \alpha(\Phi) = n \), and \( \rho_1, \ldots, \rho_n \) are states such that the supports of \( \Phi(\rho_1), \ldots, \Phi(\rho_n) \) are orthogonal. For \( i = 1, \ldots, n \), let \( |\psi_i\rangle \) be a pure state in the support of \( \rho_i \). Then, since the support of \( \Phi(|\psi_i\rangle) \) is a subspace of the support of \( \Phi(\rho_i) \), the states \( \Phi(|\psi_1\rangle), \ldots, \Phi(|\psi_n\rangle) \) have orthogonal supports as well. It means that, to compute \( \alpha(\Phi) \) it suffices to restrict ourselves to pure states.

Now assume that the operator sum representation of \( \Phi \) is

\[
\Phi(\rho) = \sum_{k=1}^r E_k \rho E_k^\dagger,
\]

where \( \sum_{k=1}^r E_k^\dagger E_k = I \). Then the support of \( \Phi(|\psi_i\rangle) \) is spanned by vectors \( E_1|\psi_i\rangle, \ldots, E_r|\psi_i\rangle \). Therefore \( \Phi(|\psi_1\rangle), \ldots, \Phi(|\psi_n\rangle) \) have orthogonal supports if these vectors, for different indices \( i \) and \( j \), are orthogonal. Summarizing these two statements, we get to the following proposition.

**Proposition 2.1** For a quantum channel \( \Phi \) with operator sum representation (1), we have \( \alpha(\Phi) \geq n \) if and only if there exist pure states \( |\psi_1\rangle, \ldots, |\psi_n\rangle \) such that \( \langle \psi_i|E_k^\dagger E_l|\psi_j\rangle = 0 \), for every \( k, l \) and \( i, j \), where \( i \neq j \).

### 2.5 Quantum clique problem

We know that deciding whether a given graph has a clique of size \( k \) is NP-complete. Considering this problem in the complement graph we find that, deciding whether \( \alpha(G) \geq k \) is NP-complete. In our notion, it means that having a classical channel, deciding whether by coding messages in words of length one, we can get to the rate \( k \) for transmitting information with zero-error, is NP-complete. Since we have all these notions for the quantum case we can define the quantum clique problem.

Basically, the quantum version of clique problem is also to decide whether \( \alpha(\Phi) \geq k \), for a given quantum channel \( \Phi \). It is equivalent to decide whether there exist quantum states \( \rho^1, \ldots, \rho^k \) such that \( \Phi(\rho^1), \ldots, \Phi(\rho^k) \) have orthogonal supports or not. Note that, for any two states \( \sigma^1, \sigma^2 \), we have \( tr(\sigma^1 \sigma^2) \geq 0 \) and equality holds if and only if \( \sigma^1, \sigma^2 \) have orthogonal supports.

Let \( \sigma^{1,2} = \sigma^1 \otimes \sigma^2 \) then \( tr(\sigma^1 \sigma^2) = tr(S \sigma^{1,2}) \), where \( S \) is the swap gate \( (S|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle) \). Therefore by applying the swap gate we can estimate \( tr(\sigma^1 \sigma^2) \). But notice that if \( \sigma^{1,2} \) is not separable then this equality does not hold and the orthogonality of \( \sigma^1 \) and \( \sigma^2 \) is not implied by \( tr(S \sigma^{1,2}) = 0 \). To resolve this problem we can restrict ourselves to entanglement breaking channels to ensure that the output states of the channel are not entangled.
A quantum channel $\Phi$ is called entanglement breaking if there are POVM $\{M_i\}$ and states $\sigma_i$ such that
$$\Phi(\rho) = \sum_i \text{tr}(M_i \rho) \sigma_i.$$ 
In this case, $\Phi \otimes^2 (\rho^{1,2})$ is always separable, $\text{tr}(S \Phi \otimes^2 (\rho^{1,2})) \geq 0$ and equality implies $\Phi(\rho^1)$ and $\Phi(\rho^2)$ are orthogonal.

**Definition 2.9** Quantum clique problem $(\Phi, k, a, b)$

- **Input** Integer numbers $n$ and $k$, non-negative real numbers $a$, $b$ with an inverse polynomial gap $b - a > n^{-c}$, and an entanglement breaking channel $\Phi$ that acts on $n$-qubit states.
- **Promise** Either there exists $\rho^1 \otimes \cdots \otimes \rho^k$ such that $\sum_{i,j} \text{tr}(S \Phi(\rho^i) \otimes \Phi(\rho^j)) \leq a$ or for any state $\rho^{1,2,\ldots,k}$ we have $\sum_{i,j} \text{tr}(S \Phi \otimes^2 (\rho^{i,j})) \geq b$.
- **Output** Decide which one is the case.

Notice that, if we let $a = 0$ we get to the exact orthogonality assumption that is a special case of quantum clique, and in general is a simpler problem. Indeed, we show that quantum clique problem is QMA-complete, and in the special case where $a = 0$ and $\Phi$ is restricted to quantum-classical channels, it is QMA$_1$-complete.

### 2.6 Holevo capacity

The Holevo capacity of a quantum channel is the maximum rate of classical information that can be sent through a quantum channel without using entanglement, [21]. Assume that $\Phi$ is a quantum channel. Then $\chi(\Phi)$, the Holevo capacity of $\Phi$, is equal to
$$\chi(\Phi) = \max_{p_i, \rho_i} H\left(\sum_i p_i \Phi(\rho_i)\right) - \sum_i p_i H(\Phi(\rho_i)),$$  
(2)
where $H(\rho) = -\text{tr}(\rho \log \rho)$ denotes the von Neumann entropy, and the maximum is taken over probability distributions $\{p_i\}$ and quantum states $\{\rho_i\}$. Using the convexity of von Neumann entropy, we can assume that states $\rho_i$ are pure. Also, if $\Phi$ acts on an $n$-dimensional Hilbert space then we may assume that number of $\rho_i$'s is at most $n^2$, [21]. However, these are not enough information on what the maximum point is, and how we can compute $\chi(\Phi)$.

There is an algorithm called the Arimoto-Blahut algorithm that given a classical discrete memoryless channel computes its capacity, see [3, 5]. Indeed, computing the capacity of a classical channel involves maximization of some mutual information. In the Arimoto-Blahut algorithm this maximization problem is converted to an alternating maximization one, that tends to the channel capacity and is more tractable. Using the same idea, Nagaoka in [20] proposed the same algorithm to compute the Holevo capacity. But, the point is that in [2] there can be a local maximum which is not a global one. So that, in the quantum Arimoto-Blahut algorithm the alternate maximum value may tend to a local maximum, not to $\chi(\Phi)$.

In this paper, we prove that computing $\chi(\Phi)$ is NP-complete. In fact, a more strong theorem holds: computing the Holevo capacity of entanglement breaking channels is NP-complete.
2.7 Minimum entropy of a quantum channel

Minimum entropy of a quantum channel is equal to the minimum entropy of its output states,

\[ \min_{\rho} H(\Phi(\rho)). \]  

(3)

Again, using the convexity of von Neumann entropy, the minimum is achieved on pure states. The minimum entropy is an important invariant of quantum channels. Indeed, it is proved that the famous additivity conjecture of Holevo capacity is equivalent to the additivity of minimum entropy, [26]. This result is important for us because it somehow expresses the minimum entropy in terms of Holevo capacity, and using this idea we convert the problem of computing the minimum entropy to the problem of computing Holevo capacity. Indeed, to prove that computing Holevo capacity is \( \text{NP} \)-complete, we first state the \( \text{NP} \)-completeness of computing minimum entropy.

2.8 SWAP test

SWAP test is a well-known protocol for deciding whether two given quantum states are the same or not. The protocol is as follows. Given two states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) and an ancilla qubit \( |0\rangle \), first apply the Hadamard gate on the ancilla, then the controlled-swap gate on two registers, and again, Hadamard on the ancilla. At the end, measure the ancilla qubit in the computational basis. It is easy to see that this protocol computes the channel

\[ \Phi_{\text{swap}}(|\psi_1\rangle|\psi_2\rangle) = \frac{1}{2}(1 + |\langle \psi_1 | \psi_2 \rangle|^2)|0\rangle\langle 0| + \frac{1}{2}(1 - |\langle \psi_1 | \psi_2 \rangle|^2)|1\rangle\langle 1|. \]  

(4)

In fact, in the measurement we get \( |0\rangle \) with probability \( \frac{1}{2}(1 + |\langle \psi_1 | \psi_2 \rangle|^2) \) and \( |1\rangle \) with probability \( \frac{1}{2}(1 - |\langle \psi_1 | \psi_2 \rangle|^2) \). Therefore, if we correspond the output \( |0\rangle \) to +1 and output \( |1\rangle \) to −1 then the expected value of this number is equal to |\langle \psi_1 | \psi_2 \rangle|^2. In general, when the input state is \( \sigma^{12} \) we can compute \( \text{tr}(S \sigma^{12}) \), where \( S \) is the swap gate (\( S|\psi_1\rangle|\psi_2\rangle = |\psi_2\rangle|\psi_1\rangle \)).

3 Complexity of quantum clique problem

3.1 Quantum clique is QMA-complete

Here is the main theorem of this section.

**Theorem 3.1** The quantum clique problem \( (\Phi, k, a, b) \) where \( \Phi \) is an entanglement breaking channel on \( n \)-qubit states and has the operator sum representation

\[ \Phi(\rho) = \sum_{i=1}^{r} E_i \rho E_i^\dagger, \]  

(5)

where \( \sum_i E_i^\dagger E_i = I \) and \( r = \text{poly}(n) \), is QMA-complete.

**Proof:** First we show that \( (\Phi, k, a, b) \) is in QMA. Note that, \( \Phi \) can be written as \( \Phi(\rho) = \text{tr}_2(U \rho \otimes |1\rangle\langle 1| U^\dagger) \), where \( U \) is a unitary operator and

\[ U|\psi\rangle|1\rangle = \sum_{i=1}^{r} E_i |\psi\rangle |i\rangle. \]  

(6)
Since \( r = \text{poly}(n) \), a polynomial time verifier can implement \( U \) and then \( \Phi \), with arbitrary small error. Therefore, given witness \( \rho^{1,...,k} \), verifier can randomly choose \( i, j, 1 \leq i, j \leq k \), compute \( \Phi^{\otimes 2}(\rho^{i,j}) \), and then apply the SWAP test. As we said in section 2.2, the expected value of the outcome of SWAP test for fixed \( i, j \), is equal to
\[
\frac{1}{k} \sum_{i,j} \text{tr} (S \Phi^{\otimes 2}(\rho^{i,j})),
\]
which is either less than \( \frac{2}{k(k-1)} a \) or greater than \( \frac{2}{k(k-1)} b \). Hence, there is an inverse polynomial gap between them and the verifier can recognize them in polynomial time. Thus, quantum clique problem is in QMA.

To prove the hardness, we establish a polynomial time reduction from local Hamiltonian problem to quantum clique. Let \( (H_1, \ldots, H_s, a, b) \) be an instance of local Hamiltonian problem. Since \( \|H_i\| \leq 1 \), then \( \frac{1}{2} H \leq I \), where \( H = \sum H_i \). Thus, \( M = I - \frac{1}{2} H \) is a positive operator and we can define the following quantum channel
\[
\Phi(\rho) = \frac{1}{s} \text{tr}(H \otimes I \rho) |00\rangle \langle 00| + \text{tr}(M \otimes |0\rangle \langle 0| \rho) |11\rangle \langle 11| + \text{tr}(M \otimes |1\rangle \langle 1| \rho) |10\rangle \langle 10|.
\]
Note that, \( s = \text{poly}(n) \) and then \( \Phi \) is of the form of \([3]\), \((r = \text{poly}(n))\). So, we can consider \( (\Phi, k = 2, \frac{1}{s^2} a^2, \frac{1}{s^2} b^2) \) as an instance of quantum clique. We prove that \((H_1, \ldots, H_s, a, b)\) is a “yes” instance of local Hamiltonian if and only if \((\Phi, k = 2, \frac{1}{s^2} a^2, \frac{1}{s^2} b^2)\) is a “yes” instance of quantum clique.

Suppose \((H_1, \ldots, H_s, a, b)\) is a ”no” instance. Then for any state \( \sigma \), \( \text{tr}(H \sigma) \geq b \), and then, for any state \( \rho^{1,2} \) we have
\[
\text{tr}(S \Phi^{\otimes 2}(\rho^{1,2})) \geq \frac{1}{s^2} \text{tr}(H \otimes I \rho^1) \text{tr}(H \otimes I \rho^2) \geq \frac{1}{s} b^2.
\]
So, \((\Phi, k = 2, \frac{1}{s^2} a^2, \frac{1}{s^2} b^2)\) is also a ”no” instance. Now, assume that there is \(|\psi\rangle\) such that \( \langle \psi|H|\psi\rangle \leq a \). Let \( \rho^1 = |\psi\rangle \langle \psi| \otimes |0\rangle \langle 0| \), and \( \rho^2 = |\psi\rangle \langle \psi| \otimes |1\rangle \langle 1| \). We have
\[
\text{tr}(S \Phi(\rho^1) \otimes \Phi(\rho^2)) = \text{tr}\left(\left(\frac{1}{s} \langle \psi|H|\psi\rangle |00\rangle \langle 00| + \langle \psi|M|\psi\rangle |11\rangle \langle 11|\right)\left(\frac{1}{s} \langle \psi|H|\psi\rangle |00\rangle \langle 00| + \langle \psi|M|\psi\rangle |10\rangle \langle 10|\right)\right)
\]
\[
= \frac{1}{s^2} (\langle \psi|H|\psi\rangle)^2 \leq \frac{1}{s^2} a^2.
\]
Therefore, \((\Phi, k = 2, \frac{1}{s^2} a^2, \frac{1}{s^2} b^2)\) is also a ”yes” instance.

\[
\square
\]

### 3.2 Channels that can be implemented exactly

Theorem 3.1 says that quantum clique problem \((\Phi, k, a, b)\) is QMA-complete. In this QMA protocol since, in general, \( a \) is a positive number, we are allowed to have some probability of error. But one may consider the case \( a = 0 \) and try to find a protocol with no error. Recall that, if \( a = 0 \) then \((\Phi, k, a = 0, b)\) exactly says that whether \( a(\Phi) \geq k \) or not. Here, we show that this problem is QMA₁-complete.

The first step toward proving such a result is to show that if \( a = 0 \) then quantum clique is in QMA₁. But, QMA₁ consists of problems that have a quantum Merlin-Arthur protocol with one sided error, and in fact perfect completeness. So, we should be able to check the orthogonality of two quantum states without error. But, in general, the SWAP-test, that we applied in theorem 3.1, contains some non-zero probability of error.
The idea to resolve this problem is to restrict ourselves to the special case of quantum-classical channels (q-c channels). A channel $\Phi$ is called a q-c channel if it can be written in the form
\[
\Phi(\rho) = \sum_{i=1}^{r} \text{tr}(M_i \rho) |i\rangle \langle i|,
\]
where $\{M_1, \ldots, M_r\}$ is a POVM and $|1\rangle, \ldots, |r\rangle$ are orthogonal states. Checking orthogonality of two outcome states of these channels is easy. Given two such states $\Phi(\rho)$, $\Phi(\rho')$, we can measure them in the basis $|1\rangle, \ldots, |r\rangle$. If the outcome of the measurements were the same then their supports are not orthogonal.

Another restriction that we should keep in mind is that, the verifier should be able to compute $\Phi(\rho)$, exactly. If the verifier could just implement $\Phi$ with some approximation, then all the computation contains some probability of error. In fact, this is the same kind of restriction that we mentioned for the quantum 4-SAT problem. So, we should restrict the set of channels to the quantum channels that can be implemented exactly by a polynomial time quantum verifier. In section 2.2 we pointed out that by a quantum verifier for QMA$_1$ protocols we mean the one that can implement all 3-qubit quantum gates, exactly. But, it does not mean that we can implement any channel with no error. So, we should clarify that, in this case, by a quantum channel we mean the one the can be implemented exactly by a quantum verifier. We do not need to classify all of these channels. We just need to show that this class of channels is enough rich.

**Lemma 3.1**

(i) Any quantum channel that acts just on a constant number of qubits and has an operator sum representation of the form (5) where $r$ is a constant can be implemented with no error.

(ii) For polynomially many channels $\Phi_1, \ldots, \Phi_s$ that can be implemented exactly, $\frac{1}{s} \sum_i \Phi_i$ can be implemented with no error.

(iii) If $\Pi$ is a $k$-projection where $k$ is a constant, then the following channel can be implemented exactly
\[
\Phi(\rho) = \text{tr}(\Pi \otimes I \rho) |00\rangle \langle 00| + \text{tr}((I - \Pi) \otimes |0\rangle \langle 0| \rho) |11\rangle \langle 11| + \text{tr}((I - \Pi) \otimes |1\rangle \langle 1| \rho) |10\rangle \langle 10|.
\]

(iv) Suppose $\Pi_1, \ldots, \Pi_s$ are polynomially many $k$-projections, where $k$ is a constant. Let $\Pi = \sum_i \Pi_i$, and $M = I - \frac{1}{s} \Pi$. Then the following channel can be implemented exactly.
\[
\Phi(\rho) = \frac{1}{s} \text{tr}(\Pi \otimes I \rho) |00\rangle \langle 00| + \text{tr}(M \otimes |0\rangle \langle 0| \rho) |11\rangle \langle 11| + \text{tr}(M \otimes |1\rangle \langle 1| \rho) |10\rangle \langle 10|.
\]

**Proof:** (i) The idea is same as what we did in the proof of theorem 3.1. In fact, such a channel can be written of the form $\Phi(\rho) = \text{tr}_2(U \rho U^\dagger)$, where $U$ is given by equation (6). In this special case, $U$ acts just on constant number of qubits, and then by lemma 2.1 it can be implemented efficiently and with no error.

(ii) Pick a random $i$, $1 \leq i \leq s$, and apply $\Phi_i$.

(iii),(iv) are easy consequences of (i) and (ii).
3.3 \( a = 0 \)

**Theorem 3.2**  Quantum clique problem \((\Phi, k, a = 0, b)\), where \(\Phi\) is a q-c channel that can be implemented exactly by a polynomial time verifier is QMA\(_1\)-complete.

**Proof:** First we show that this problem is in QMA\(_1\). Given a channel \(\Phi\) of the form (7), if \(\alpha(\Phi) \geq k\), then there are states \(\rho^1, \ldots, \rho^k\) such that \(\Phi(\rho^1), \ldots, \Phi(\rho^k)\) have orthogonal supports. Hence, \(\rho^1 \otimes \cdots \otimes \rho^k\) is a witness, and verifier can randomly choose two indices \(i, j\), \(1 \leq i, j \leq k\), apply \(\Phi\) on \(\rho^i\) and \(\rho^j\), and then check whether \(\Phi(\rho^i)\) and \(\Phi(\rho^j)\) are orthogonal or not. Since \(\Phi\) is a q-c channel, \(\Phi(\rho^i)\) and \(\Phi(\rho^j)\) are orthogonal iff the outcome of their measurement in the basis \(|1\rangle, \ldots, |r\rangle\), never be the same. Note that, conditioned on \(i, j\), the probability of a collision in the measurement is equal to \(\text{tr}(S \Phi \otimes (\rho^i \rho^j))\), and in general it is

\[
\frac{1}{\binom{k}{2}} \sum_{i,j} \text{tr}(S \Phi \otimes (\rho^i \rho^j)).
\]

Thus, if \((\Phi, k, a = 0, b)\) is a "yes" instance, (8) is equal to zero and we get to the right answer with probability 1. On the other hand, if it is a "no" instance then

\[
\frac{1}{\binom{k}{2}} \sum_{i,j} \text{tr}(S \Phi \otimes (\rho^i \rho^j)) \geq \frac{1}{\binom{k}{2}} b,
\]

and with probability at least \(\frac{1}{\binom{k}{2}} b\) which is greater than an inverse polynomial, we get to a collision. Therefore, \((\Phi, k, a = 0, b)\) is in QMA\(_1\).

It remains to show that quantum clique, in the special case stated in the theorem, is QMA\(_1\)-hard. By theorem 2.2, quantum 4-SAT is QMA\(_1\)-complete. Thus, if we establish a polynomial time reduction from quantum 4-SAT to quantum clique, we are done. Let \((\Pi_1, \ldots, \Pi_s, \epsilon)\) be an instance of quantum 4-SAT problem. Define the channel \(\Phi\) as follows

\[
\Phi(\rho) = \frac{1}{s} \text{tr}(\Pi \otimes I \rho) |00\rangle \langle 00| + \text{tr}(M \otimes |0\rangle \langle 0| \rho) |11\rangle \langle 11| + \text{tr}(M \otimes |1\rangle \langle 1| \rho) |10\rangle \langle 10|,
\]

where \(\Pi = \sum_i \Pi_i\), and \(M = I - \frac{1}{s} \Pi\), and consider the instance \((\Phi, k = 2, a = 0, \frac{1}{s} \epsilon)\) of the quantum clique problem. Note that, \(\Phi\) is a q-c channel and by lemma 3.1 it can be implemented exactly by a quantum verifier. So \((\Phi, k = 2, a = 0, \frac{1}{s} \epsilon)\) satisfies the conditions of theorem. The other parts of proof, that \((\Pi_1, \ldots, \Pi_s, \epsilon)\) is a "yes" instance if and only if \((\Phi, k = 2, a = 0, \frac{1}{s} \epsilon)\), are exactly same as in the proof of theorem 3.1.

\[\square\]

4  **Complexity of computing Holevo capacity**

Here is the main theorem of this section.

**Theorem 4.1**  Suppose \(\Phi\) is a quantum channel that acts on an \(n\)-dimensional Hilbert space, and is given by \(\text{poly}(n)\) number of bits. Also, let \(c\) be a real number. Then deciding whether \(\chi(\Phi) > c\), is NP-complete.

To prove this theorem we show that this problem is "harder" that the problem of computing the minimum entropy of quantum channels, and then prove computing the minimum entropy is NP-complete. In fact, the minimum entropy of channel, equation (5), seems to be more tractable than the Holevo capacity. Then proving the NP-completeness of this problem is simpler.
Theorem 4.2 Assume $\Phi$ is a quantum channel acting on an $n$-dimensional Hilbert space, and is given by polynomially many bits. Also, let $c$ be a real number. Then deciding whether the minimum entropy of $\Phi$ is less than $c$ is $\text{NP}$-complete.

First, using theorem 4.2 we prove theorem 4.1 and then get to the $\text{NP}$-completeness of computing minimum entropy.

Proof of theorem 4.1 First of all if $\Phi$ is a channel and $\chi(\Phi) \geq c$, then there are probability distribution $\{p_i\}$ and states $\rho_1, \ldots, \rho_s$ such that
\[
H(\sum_i p_i \Phi(\rho_i)) - \sum_i p_i H(\Phi(\rho_i)) \geq c.
\]
(9)
The point is that, we may assume $s \leq n^2$, see [21]. Therefore, given this probability distribution and the quantum states, the verifier can check whether (9) holds or not. So, it is a problem in $\text{NP}$.

To prove the hardness, since, by theorem 4.2 computing the minimum entropy is $\text{NP}$-complete, if we establish a reduction from minimum entropy to computing Holevo capacity we are done.

Let $(\Phi, c)$ be an instance of minimum entropy problem as in theorem 4.2. Let $|1\rangle, \ldots, |n\rangle$ be an orthonormal basis for the Hilbert space. Also, let $X_0, \ldots, X_{n^2-1}$ be the $n$-dimensional generalized Pauli matrices. That is, $X_{mn+d} = T^m R^d$, where $T|j\rangle = |j + 1 \text{ mod } n\rangle$ and $R|j\rangle = e^{2ij\pi/n}|j\rangle$. Define the channel $\Psi$ such that
\[
\Psi(\rho \otimes |i\rangle\langle i|) = X_i \Phi(\rho) X_i^\dagger.
\]
It is obvious that
\[
\chi(\Psi) = \max_{\rho, \rho_i} \left( \sum_i p_i (\rho \otimes |i\rangle\langle i|) - \sum_i p_i H(\Psi(\rho_i)) \right) \leq \log n - \min_{\rho} H(\Psi(\rho)).
\]
(10)
Also, it is easy to see that the minimum entropy of $\Psi$ is equal to the minimum entropy of $\Phi$. On the other hand, if the minimum entropy of $\Phi$ is taken on $|\phi\rangle$, and we let $\rho_i = |\phi\rangle\langle \phi| \otimes |i\rangle\langle i|$ and $p_i = 1/n$, for $i = 1, \ldots, n$, then equality holds in (10). It means that, the minimum entropy of $\Phi$ is less than $c$ if and only if the Holevo capacity of $\Psi$ is greater than $\log n - c$.

We are done

4.1 Complexity of computing the minimum capacity of a quantum channel

The only remaining step is the proof of theorem 4.2. To get a clearer proof it would be helpful to first state some lemmas.

4.1.1 Some lemmas on the minimum entropy of channels

In this section we study some properties of the points that a channel achieves its minimum entropy. Before stating the lemmas, remember that the von Neumann entropy is convex, and then the minimum entropy of a channel is attained on pure states.

Lemma 4.1 Suppose $\Phi_1, \ldots, \Phi_k$ are $k$ channels with the same input and output state spaces. Also, assume that the output states of every two of them are orthogonal. In other words, for any $i, j$, $1 \leq i < j \leq k$, and any states $\rho$, $\rho'$,
\[
\text{tr}(\Phi_i(\rho)\Phi_j(\rho')) = 0.
\]
Then
\[
\min_\rho H\left( \sum_{i=1}^k p_i \Phi_i(\rho) \right) \geq \sum_{i=1}^k p_i \min_\rho H(\Phi_i(\rho)) + H(p_1, \ldots, p_k),
\]
where, \( \{p_1, \ldots, p_k\} \) is a probability distribution and \( H(p_1, \ldots, p_k) \) is its entropy. In particular, the minimum entropy of \( \sum_{i=1}^k p_i \Phi_i \) is at least \( H(p_1, \ldots, p_k) \), and equality holds iff there is \( \rho \) such that all states \( \Phi_i(\rho) \) are pure.

**Proof:** Since \( \Phi_i(\rho) \)'s have orthogonal supports
\[
\min_\rho H\left( \sum_{i=1}^k p_i \Phi_i(\rho) \right) = \min_\rho \sum_{i=1}^k -p_i \text{tr} \left( \Phi_i(\rho) \log \Phi_i(\rho) \right) = \min_\rho \sum_{i=1}^k p_i H(\Phi_i(\rho)) - \sum_{i=1}^k p_i \log p_i \\
\geq \sum_{i=1}^k p_i \min_\rho H(\Phi_i(\rho)) + H(p_1, \ldots, p_k).
\]

\[\square\]

**Lemma 4.2** Let \( \Phi_{\text{trace}} \) be the channel that acts on the Hilbert space \( \mathcal{H} \otimes \mathcal{H} \), and traces out the second register:
\[
\Phi_{\text{trace}}(\rho^{12}) = \text{tr}_2(\rho^{12}) = \rho^1. \tag{11}
\]
Then, the minimum entropy of \( \Phi_{\text{trace}} \) is zero, and it is achieved at the product states \( |\psi_1\rangle |\psi_2\rangle \).

**Proof:** Let \( |\psi_{12}\rangle \) be a pure state in \( \mathcal{H} \otimes \mathcal{H} \). By the Schmidt decomposition [21], there are orthonormal bases \( \{|i\rangle\}, \{|i'\rangle\} \), and real non-negative numbers \( \lambda_i \), such that
\[
|\psi_{12}\rangle = \sum_i \lambda_i |i\rangle |i'\rangle. \tag{12}
\]
Hence, \( \Phi(|\psi_{12}\rangle) = \sum_i \lambda_i^2 |i\rangle \langle i| \), and it is a pure state if only if just one of \( \lambda_i \)'s is non-zero, or equivalently \( |\psi_{12}\rangle \) is a product state.

The next lemma is on the minimum entropy of the SWAP test, described in section 2.8.

**Lemma 4.3** Let \( \Phi_{\text{swap}} \) be the channel defined in equation (4). Then the minimum entropy of the channel
\[
\Phi(\rho) = \frac{1}{2} \Phi_{\text{trace}}(\rho) \otimes |u\rangle \langle u| \otimes |000\rangle \langle 000| + \frac{1}{2} |u'_{12}\rangle \langle u'_{12}| \otimes |10\rangle \langle 10| \otimes \Phi_{\text{swap}}(\rho),
\]
where \( |u\rangle \in \mathcal{H} \) and \( |u'_{12}\rangle \in \mathcal{H} \otimes \mathcal{H} \) are arbitrary states, is equal to \( H(2) = 1 \), and is attained at the pure states of the form \( |\psi\rangle \langle \psi| \).

**Proof:** By lemma 4.1 it is sufficient to show that states of form \( |\psi\rangle \langle \psi| \) are the only states \( \rho \) such that \( \Phi_{\text{trace}}(\rho) \) and \( \Phi_{\text{swap}}(\rho) \) are simultaneously pure.

Using lemma 4.2 such a state \( \rho \) should be a product state \( |\psi_1\rangle |\psi_2\rangle \). On the other hand, by equation (4), it is clear that \( \Phi_{\text{swap}}(|\psi_1\rangle |\psi_2\rangle) \) is pure if \( |\langle \psi_1| \psi_2\rangle| = 1 \), or equivalently \( |\psi_1| = |\psi_2| \).
For the next lemma, it is helpful to fix some notations. Let $H$ an $n$-dimensional Hilbert space with the orthonormal basis $\{ |1\rangle, \ldots, |n\rangle \}$. For any $1 \leq i < j \leq n$, let $\Pi_{ij}$ be the projection over $\frac{1}{\sqrt{2}}(|i\rangle + |j\rangle)$,

$$\Pi_{ij} = \frac{1}{2}(|i\rangle + |j\rangle)(\langle i| + \langle j|).$$

Also, let $\Pi'_{ij}$ be the projection over $\frac{1}{\sqrt{2}}(|i\rangle - |j\rangle)$,

$$\Pi'_{ij} = \frac{1}{2}(|i\rangle - |j\rangle)(\langle i| - \langle j|).$$

$\Pi_{ij} \otimes \Pi'_{ij}$ is a projection and its operator norm is equal to one. Therefore, $\sum_{ij} \Pi_{ij} \otimes \Pi'_{ij}$ is a hermitian matrix and its norm is at most $\binom{n}{2}$. Thus,

$$M = I \otimes I - \frac{1}{n(n-1)} \sum_{ij} \Pi_{ij} \otimes \Pi'_{ij},$$

is a positive semidefinite matrix, and does not have zero eigenvalue. It means that, $|v'_{12}\rangle$ is always in the support of the following channel.

$$\Phi_{cube}(\rho) = \frac{1}{n(n-1)} \sum_{ij} tr(\Pi_{ij} \otimes \Pi'_{ij}\rho) |v_{12}\rangle\langle v_{12}| + tr(M\rho)|v'_{12}\rangle\langle v'_{12}|.$$

(13)

Lemma 4.4 Let $|v_{12}\rangle, |v'_{12}\rangle \in H \otimes H$ be two orthogonal states, and define $\Phi_{cube}$ as in equation (13). Then the minimum entropy of the channel $\Phi$ is equal to $H(3) = \log(3)$ and is attained at the states $|v_{12}\rangle = |\psi\rangle |\psi\rangle$, where

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i |i\rangle,$$

(14)

and $x_i \in \{+1, -1\}$.

Proof: Again, using lemma 4.1, it suffices to show that the only states $\rho$ such that $\Phi_{trace}(\rho)$, $\Phi_{swap}(\rho)$ and $\Phi_{cube}(\rho)$ are pure, are the states $|\psi\rangle |\psi\rangle$ where $|\psi\rangle$ is of the form (14).

In lemma 4.3, we showed if $\Phi_{trace}(\rho)$ and $\Phi_{swap}(\rho)$ are pure then $\rho$ is a pure state of the form $|\psi\rangle |\psi\rangle$. So, it remains to show that if $\Phi_{cube}(|\psi\rangle |\psi\rangle)$ is pure then $|\psi\rangle$ is of the form (14).

As we mentioned, $|v'_{12}\rangle$ is always in the support of $\Phi_{cube}(\rho)$. Hence, if $\Phi_{cube}(|\psi\rangle |\psi\rangle)$ is pure then it is equal to $|v'_{12}\rangle$. It means that, $\Phi_{cube}(\rho)$ is pure if and only if

$$tr(\Pi_{ij} \otimes \Pi'_{ij}\rho) = 0,$$

for any $i, j$. Let $|\psi\rangle = \sum_\lambda \lambda |i\rangle$, and suppose $\Phi_{cube}(|\psi\rangle |\psi\rangle)$ is pure. We have

$$0 = \langle \psi|\langle \psi|\Pi_{ij} \otimes \Pi'_{ij}|\psi\rangle |\psi\rangle = \langle \langle \psi|\Pi_{ij}|\psi\rangle |\psi\rangle |\Pi'_{ij}| |\psi\rangle = \frac{1}{2} |\lambda_i + \lambda_j|^2 |\lambda_i - \lambda_j|^2.$$

In other words, for any $i, j$, either $\lambda_i = \lambda_j$ or $\lambda_i = -\lambda_j$. So, $|\psi\rangle$ should be of the form (14).
4.1.2 Proof of theorem 4.2

To prove the NP-hardness of the problem of computing minimum entropy, we should find a reduction from an NP-complete problem to this one. The most comfortable such problem for us is the 2-Out-of-4-SAT problem [12]. We can formulate this problem as follows. Given $m = poly(n)$ vectors of the form

$$|A_k⟩ = ∑_{i=1}^{n} a_k^i |i⟩,$$

where for each $k$, $1 ≤ k ≤ m$, there are exactly four non-zero $a_k^i$, and $a_k^i$ is zero or $±\frac{1}{2}$, decide whether there exists a vector $|ψ⟩$ of the form (14) orthogonal to all $|A_k⟩$'s, $⟨A_k|ψ⟩ = 0$.

Now we are ready to prove the theorem. Given a witness state $ρ$, we can check whether $H(Φ(ρ)) < c$, in polynomial time. Therefore, this problem is in NP.

To prove hardness, let $|A_1⟩, \ldots, |A_m⟩$ be an instance of 2-Out-of-4-SAT. Let

$$H = \frac{1}{m} ∑_{k=1}^{m} |A_k⟩⟨A_k| ∘ |A_k⟩⟨A_k|$$

and define

$$Φ_H(ρ) = \frac{1}{2} tr(H |w⟩⟨w| + tr((I ∘ I - \frac{1}{2} H) ρ |w⟩⟨w|),$$

where $|w⟩$ and $|w′⟩$ are two orthogonal states in $ℋ ∘ ℋ$. Since the norm of $\frac{1}{2} H$ is less than or equal to 1/2, $|w⟩$ is always in the support of $Φ_H$. Then the minimum entropy of $Φ_H$ is zero and is achieved at the states $|ψ⟩$ that are orthogonal to all $|A_k⟩$'s, $k = 1, \ldots, m$.

Now, define the channel

$$Φ(ρ) = \frac{1}{4} Φ_{trace}(ρ) ∘ |u⟩⟨u| ∘ |000⟩⟨000| + \frac{1}{4} |u′⟩⟨u′| ∘ |10⟩⟨10| ∘ Φ_{swap}(ρ) + \frac{1}{4} Φ_{cube}(ρ) ∘ |110⟩⟨110| + \frac{1}{4} Φ_H(ρ) ∘ |111⟩⟨111|.$$ 

By lemma 4.4, the minimum entropy of $Φ$ is at least $H(4) = 2$, and equality holds if there exists $|ψ⟩$ such that all the states $Φ_{trace}(ψ), Φ_{swap}(ψ), Φ_{cube}(ψ)$ and $Φ_H(ψ)$ are pure. By lemma 4.3, such a state should be of the form $|ψ⟩ = |ψ⟩ |ψ⟩$, where $|ψ⟩$ is of the form (14). Also, for such a state $Φ_H(ψ)$ is pure iff $⟨ψ|A_k⟩ = 0, k = 1, \ldots, m$. Therefore, the minimum entropy of $Φ$ is $H(4)$, if and only if $⟨ψ|A_k⟩ = 0, k = 1, \ldots, m$. Thus, computing the minimum entropy is in NP-complete.

4.2 Computing Holevo capacity of entanglement breaking channels

We proved that computing the minimum entropy, and then, Holevo capacity are NP-complete. In these two theorems, we considered general quantum channels. But one suspects that if we restrict ourselves to a special class of quantum channels then we get to simpler problems.

For example, let $Φ$ be a classical-quantum channel (c-q channel)

$$Φ(ρ) = ∑_{i=1}^{n} ⟨i|ρ|i⟩ σ_i, \quad (15)$$
where $|1\rangle, \ldots, |n\rangle$ is an orthonormal basis and $\sigma_1, \ldots, \sigma_n$ are arbitrary states. Then, obviously, the minimum entropy of $\Phi$ is equal to

$$\min_i H(\sigma_i),$$

and then, can be computed in polynomial time. Also, it is easy to see that computing the Holevo capacity of $\Phi$ is a convex optimization problem and can be solved efficiently.

Therefore, to get a non-trivial problem we should consider a more general class of quantum channels. Indeed, c-q channels that we considered in equation (15), and also q-c channels, equation (7), are special cases of Entanglement breaking channels. An entanglement breaking channel is a channel $\Phi$ of the form

$$\Phi(\rho) = \sum_{i=1}^{r} \text{tr}(M_i \rho) \sigma_i,$$

(16)

where $\{M_i\}$ is a POVM and $\sigma_1, \ldots, \sigma_r$ are arbitrary states. Although, it seems that the problem of computing the minimum entropy and Holevo capacity of entanglement breaking channels is simpler than the general case, we prove that these are also NP-complete.

**Theorem 4.3** Assume $\Phi$ is an entanglement breaking channel of the form (16) acting on an $n$-dimensional Hilbert space, and is given by polynomially many bits. Also let $c$ be a real number. Then the questions of bounding the Holevo capacity and the minimum entropy of $\Phi$,

$$\chi(\Phi) > c$$

and

$$\min_{\rho} H(\Phi(\rho)) < c,$$

are NP-complete.

Note that, if we show that computing the minimum entropy for entanglement breaking channels is NP-hard, then by the same argument as in the proof of theorem 4.1 we can prove the hardness of computing the Holevo capacity. Also, recall that, in the proof of theorem 4.2 all the channels that we used were entanglement breaking except $\Phi_{\text{trace}}$. Therefore, if we replace $\Phi_{\text{trace}}$ with an entanglement breaking channel that captures the same properties then we are done.

The key idea is the following observation first appeared in [6]. Suppose $\rho$ is the density matrix of a two-qubit state. Let $\sigma_0 = I, \sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices. Also, for $i = 1, 2, 3$ let $P_i^\pm = \frac{1}{2}(I \pm \sigma_i)$ be density matrices of the +1 and -1 eigenstates of $\sigma_i$. For $0 \leq i, j \leq 3$ define $c_{ij} = \text{tr}(\sigma_i \otimes \sigma_j \rho)$. Then, we know that

$$\rho = \frac{1}{4} \sum_{i,j=0}^{3} c_{ij} \sigma_i \otimes \sigma_j.$$

If we rewrite this equation in terms of $P_i^\pm$, we get to

$$\rho = \frac{1}{4} \sum_{i,j=1}^{3} \left( \frac{1}{4} c_{i0} + \frac{1}{4} c_{0j} + c_{ij} \right) P_i^+ \otimes P_j^+$$

$$+ \left( \frac{1}{4} - \frac{1}{4} c_{i0} + \frac{1}{4} c_{0j} - c_{ij} \right) P_i^- \otimes P_j^+$$

$$+ \left( \frac{1}{4} + \frac{1}{4} c_{i0} - \frac{1}{4} c_{0j} - c_{ij} \right) P_i^+ \otimes P_j^-$$

$$+ \left( \frac{1}{4} - \frac{1}{4} c_{i0} - \frac{1}{4} c_{0j} + c_{ij} \right) P_i^- \otimes P_j^-.$$
Suppose all the coefficients in this expression are non-negative. Then

\[
tr_2(\rho) = \frac{1}{4} \sum_{i,j=1}^{3} \left( \frac{1}{9} + \frac{1}{3}c_{i0} + \frac{1}{3}c_{0j} + c_{ij} \right) P_i^+ \\
+ \left( \frac{1}{9} - \frac{1}{3}c_{i0} + \frac{1}{3}c_{0j} - c_{ij} \right) P_i^- \\
+ \left( \frac{1}{9} + \frac{1}{3}c_{i0} - \frac{1}{3}c_{0j} - c_{ij} \right) P_i^+ \\
+ \left( \frac{1}{9} - \frac{1}{3}c_{i0} - \frac{1}{3}c_{0j} + c_{ij} \right) P_i^-. 
\]

In other words, \( tr_2(\rho) \) can be written as a linear combination of states \( P_i^\pm \) with coefficients of the form \( tr(M\rho) \), where \( \{M\} \) is some POVM.

It means that, if the coefficients were always non-negative then \( \rho \mapsto tr_2(\rho) \) was an entanglement breaking channel. To satisfy this extra assumption we can replace \( \rho \) with \( \rho = \frac{1}{4}(1-\epsilon)I + \epsilon\rho \), where \( 0 < \epsilon < 1/16 \), and observe that the coefficients for \( \rho \) are all non-negative. In general, we have the following lemma, proved in [6].

**Lemma 4.5** Let \( \rho \) be a state in \( \mathcal{H} \otimes \mathcal{H} \) where \( \mathcal{H} \) is an \( n \)-dimensional Hilbert space. Also let \( 1/n^2I \) be the maximally mixed state in \( \mathcal{H} \otimes \mathcal{H} \) and \( 0 < \epsilon < 1/n^2 \). Then \( 1/n^2(1-\epsilon)I + \epsilon\rho \) is an un-entangled state. Also, as a consequence,

\[
\Phi'_\text{trace}(\rho) = tr_2(1/n^2(1-\epsilon)I + \epsilon\rho) \tag{17}
\]

is an entanglement breaking channel.

Using this lemma, the proof of theorem 4.3 follows immediately.

**Proof of theorem 4.3** All the steps of the proof are same as in theorem 4.2 except that we replace the channel \( \Phi_\text{trace} \) with \( \Phi'_\text{trace} \), which is an entanglement breaking one. The only property we should check is that the minimum entropy of \( \Phi'_\text{trace} \) is achieved at product states. It holds because \( tr_2(1/n^2(1-\epsilon)I + \epsilon\rho) = 1/n(1-\epsilon)I + \epsilon tr_2(\rho) \), and \( H(tr_2(1/n^2(1-\epsilon)I + \epsilon\rho)) \) is minimum if and only if \( H(tr_2(\rho)) \) is minimum.

\( \square \)

## 5 Conclusion

In this paper we prove that the quantum clique problem is \textbf{QMA}-complete. This is obtained by considering an \textbf{NP}-complete problem, and somehow translating it to the language of quantum information theory. The key point is that clique problem in graphs can be stated in terms of zero error capacity. So, this translation is straightforward. Now the question is that whether this method can lead us to other \textbf{QMA}-complete problems. Note that this idea is first captured in the \textbf{QMA}-completeness of local Hamiltonian problem by considering \textbf{NP}-harness of SAT.

In the second part of paper, we consider the problem of computing the Holevo capacity, and then, minimum entropy of a quantum channel, and prove that they are \textbf{NP}-complete. Since, there are few results on the computational complexity of invariants of quantum channel, it would be a natural question to consider the complexity of other such quantities for channels as well as quantum states.
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