Phantom Instability of Viscous Dark Energy in Anisotropic Space-Time

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Abstract

Phantom dark energy is a proposal that explains the current observations that mildly favor the equation of state of dark energy $\omega_{\text{de}}$ crossing $-1$ at 68% confidence level. However, phantom fields are generally ruled out by ultraviolet quantum instabilities. To overcome this discrepancy, in this paper we propose a mechanism to show that how the presence of bulk viscosity in the cosmic fluid can temporarily drive the fluid into the phantom region ($\omega < -1$). As time is going on, phantom decays and ultimately $\omega_{\text{de}}$ approaches to $-1$. Then we show these quintessence and phantom descriptions of non-viscous and viscous dark energy and reconstruct the potential of these two scalar fields. Also a diagnostic for these models are performed by using the statefinder pairs $\{s, r\}$. All results are obtained in an anisotropic space-time which is a generalization of FLRW universe.

Keywords : Bianchi Type I Model, Dark Energy, Phantom, Statefinder

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1 Introduction

It is a very well known fact that our universe is experiencing an accelerating expansion at the present time (Perlmutter et al. 1999; Riess et al. 1998, 2001; Tonry et al. 2003; Tegmark et al. 2004). It is believed that an exotic form of energy with negative pressure called dark energy is responsible for the current observed accelerating expansion of the universe (Tegmark et al. 2004; Bennett et al. 2003; Spergel et al. 2003; Abazajian et al. 2004). According to the recent observations we live in a nearly spatially flat Universe composed of approximately 4% baryonic matter, 22% dark matter and 74% dark energy. However, the observational data are far from being complete. It is not even known what is the current value of the dark energy effective equation of state (EoS) parameter $\omega_{\text{de}} = p_{\text{de}}/\rho_{\text{de}}$ which lies close to $-1$: it could be equal to $-1$ (standard $\Lambda$CDM cosmology), a little bit upper than $-1$ (the quintessence dark energy) or less than $-1$ (phantom dark energy). One of the main candidate for dark energy is cosmological constant $\Lambda$, which has pressure $p_{\text{de}} = -\rho_{\text{de}}$. Although, cosmological constant can explain the current acceleration phase of universe, it would suffer from many serious theoretical problems, such as the fine-tuning and the coincidence problems. Another candidate for dark energy is provided by introducing scalar fields. An important class of scalar fields are known as “quintessence” with $-\frac{1}{3} > \omega > -1$ (Ratra and Peebles 1988; Wetterich 1988; Turner and white 1997; Caldwell et al. 1998; Liddle and Scherrer 1999; Steinhardt et al. 1999) in which the scalar field mimics the perfect fluid and hence could lead to a solution for coincidence problem. However, quintessence scenario of dark energy is not in accurate consistent with recent observations as $\omega < -1$ has been favored by recent observations (Knop et al. 2003; Riess et al. 2004; Alam et al. 2004; Hannestad and E. Mortsell 2004). To get $\omega < -1$, a new class of scalar field models with negative kinetic energy, known as “phantom field” models have been suggested (Caldwell 2002). Nevertheless, in this case the universe shows some very strange properties (Carroll et al. 2003; Cline et al. 2004; Bunyi and Hsu 2006; Bunyi et al. 2006). For example, since the energy density of phantom field is unbounded from below, the vacuum becomes unstable against the production of positive energy fields hence these fields are generally ruled out by ultraviolet quantum instabilities (Carroll et al. 2003). Another problem is the future finite singularity called Big Rip (Caldwell et al. 2003) which leads to the occurrence of negative entropy (Brevik et al. 2004). Therefore, on the one hand observations mildly favors models with $\omega$ crossing $-1$ near the past and on another, models with $\omega < -1$ are unstable from theoretical point of view. In this paper we suggest a simple mechanism to overcome this discrepancy by introducing bulk viscosity in the cosmic fluid. First, viscosity causes dark energy which is
varying in quintessence to pass the phantom divided line (PDL) and drop it to phantom region. Next, since viscosity is a decreasing function of time, it will die out and $\omega$ will leave phantom region and tend to $-1$ at late time. Hence the problem of future singularity (big rip) will never occur in this scenario.

It has been shown in refs (McInnes 2002; Barrow 2004) that, an ideal cosmic fluid, i.e. non-viscous, give raise to the occurrence of a singularity of the universe in the far future called big rip. The singularity problem can be modified or soften via following two methods. The first is the effect of quantum corrections due to the conformal anomaly (Brevik and Odintsov 1999; Nojiri and Odintsov 2003, 2004) and second, is to consider the bulk viscosity of the cosmic fluid (for example see (Misner 1968; Padmanabhan and Chitre 1987; Brevik and Hallanger 2004). The viscosity theory of relativistic fluids was first suggested by Eckart, Landau and Lifshitz (Eckart 1940; Landau and Lifshitz 1987). The introduction of viscosity into cosmology has been investigated from different view points (Gren 1990; Barrow 1986; Zimdahl 1996; Maartens 1996). The astrophysical observations also indicate some evidences that cosmic media is not a perfect fluid (Jaffe et al. 2005), and the viscosity effect could be concerned in the evolution of the universe (Brevik and Gorbunova 2005; Brevik et al. 2005; Cataldo et al. 2005). It was also argued that a viscous pressure can play the role of an agent that drives the present acceleration of the Universe (Zimdahl et al. 2001; Balakin et al. 2003). The possibility of a viscosity dominated late epoch of the Universe with accelerated expansion was already mentioned by Padmanabhan and Chitre (Padmanabhan and Chitre 1987). Brevik and Gorbunova (2005), Oliver et al (2011), Chen et al (2011), Jamil and Farooq (2010), Cai et al (2010), Setare (2007a, 2007b, 2007c), Setare et al (2007), Setare and Sardakis (2009), Setare et al (2009), Amirhashchi et al (2011 a, 2011 b, 2011 c, 2013), Pradhan et al (2011a, 2011b, 2011c), Saha et al (2012), and Sheykhi and Setare (2010) have studied viscous and non-viscous dark energy models in different contexts. Recently, viscous dark energy and generalized second law of thermodynamics has been studied by Setare and Sheykhi (2010).

To be general, we use generalized FLRW equations by considering an anisotropic metric as the line-element of the universe. The reason for this choice of metric is behind the fact that because of high symmetry, FLRW models are infinitely improbable in the space of all possible cosmologies. The high symmetry involved in FLRW models requires a very high degree of fine tuning of initial conditions which is extraordinary improbable. Moreover, we can always ask that does the universe necessarily have the same symmetries on very large scales outside the particle horizon or at early times?

The plan of our paper is as follows: In section 2 we give the metric and field equations. In section 3 we drive the generalized FLRW equations by solving the field equations of section 2. The general form of non-viscous and viscous dark energy equation of state parameter EoS are given in section 4. We suggest a correspondence between the non-viscous and viscous dark energy scenario and the quintessence and phantom dark energy model in section 5. In section 6, a statefinder diagnostic has been presented. In section 7 we apply our general results to a toy model in order to test the proposed mechanism. Our results are summarized in section 8.

# 2 The Metric and Field Equations

We consider the Bianchi type I space-time in the orthogonal form as

\[ ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2, \]  \hspace{1cm} (1)

where $A(t), B(t)$ and $C(t)$ are functions of time only.

The Einstein’s field equations ( in gravitational units $8\pi G = c = 1$) read as

\[ R^i_j - \frac{1}{2}g^i_j R = T^{(m)i}_j + T^{(de)i}_j, \]  \hspace{1cm} (2)

where $T^{(m)i}_j$ and $T^{(de)i}_j$ are the energy momentum tensors of barotropic matter and dark energy, respectively. These are given by

\[ T^{(m)i}_j = \text{diag}[-\rho^{(m)}, p^{(m)}, p^{(m)}, p^{(m)}], \]

\[ = \text{diag}[-1, \omega^{(m)}, \omega^{(m)}, \omega^{(m)}] \rho^{(m)}, \]  \hspace{1cm} (3)

and

\[ T^{(de)i}_j = \text{diag}[-\rho^{(de)}, p^{(de)}, p^{(de)}, p^{(de)}], \]
\[ = \text{diag}[-1, \omega^{(de)}, \omega^{(de)}, \rho^{(de)}], \quad (4) \]

where \( \rho^{(m)} \) and \( p^{(m)} \) are, respectively the energy density and pressure of the perfect fluid component or matter while \( \omega^{(m)} = p^{(m)}/\rho^{(m)} \) is its EoS parameter. Similarly, \( \rho^{(de)} \) and \( p^{(de)} \) are, respectively the energy density and pressure of the DE component while \( \omega^{(de)} = p^{(de)}/\rho^{(de)} \) is the corresponding EoS parameter. We assume the four velocity vector \( u^i = (1, 0, 0, 0) \) satisfying \( u^iu_j = -1. \)

In a co-moving coordinate system \( (u^i = \delta^i_0) \), Einstein’s field equations \((3)\) with \((1)\) subsequently lead to the following system of equations:

\[
\ddot{B} - \dot{C} + \ddot{C} + \dot{B} = -\omega^m \rho^m - \omega^{de} \rho^{de},
\]

\[
\dot{A} + \dot{C} + \ddot{A} = -\omega^m \rho^m - \omega^{de} \rho^{de},
\]

\[
\dot{A} - \dot{B} + \ddot{A} = -\omega^m \rho^m - \omega^{de} \rho^{de},
\]

\[
\dot{A} + \dot{B} + \ddot{A} = \rho^m + \rho^{de}.
\]

If we consider \( a = (ABC)^{\frac{1}{3}} \) as the average scale factor of Bianchi type I model, then the generalized mean Hubble’s parameter \( H \) defines as

\[
H = \frac{\dot{a}}{a} = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right).
\]

The Bianchi identity \( G^j_{ij} = 0 \) leads to \( T^j_{ij} = 0 \). Therefore, the continuity equation for dark energy and baryonic matter can be written as

\[
\dot{\rho}^m + 3H(1 + \omega^m) \rho^m + \dot{\rho}^{de} + 3H(1 + \omega^{de}) \rho^{de} = 0.
\]

## 3 Friedmann-Like Equations

In this section, we derive the general solution for the Einstein’s field equations \((5)-(8)\).

Subtracting Eq. \((3)\) from Eq. \((6)\), Eq. \((5)\) from Eq. \((7)\), and Eq. \((5)\) from Eq. \((7)\) we obtain

\[
\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = 0,
\]

\[
\frac{\dot{B}}{B} - \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = 0,
\]

and

\[
\frac{\dot{A}}{A} - \frac{\dot{C}}{C} + \frac{\dot{B}}{B} + \left( \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) = 0.
\]

First integral of Eqs. \((11)\), \((12)\) and \((13)\) leads to

\[
\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_1}{ABC},
\]

\[
\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = \frac{k_2}{ABC},
\]

and

\[
\frac{\dot{A}}{A} - \frac{\dot{C}}{C} = \frac{k_3}{ABC}.
\]
where \( k_1, k_2 \) and \( k_3 \) are constants of integration. By taking integral from Eqs. (14), (15) and (16) we get

\[
\frac{\dot{A}}{B} = d_1 \exp[k_1 \int (ABC)^{-1} dt],
\]

(17)

\[
\frac{\dot{B}}{C} = d_2 \exp[k_2 \int (ABC)^{-1} dt],
\]

(18)

and

\[
\frac{\dot{A}}{C} = d_3 \exp[k_3 \int (ABC)^{-1} dt]
\]

(19)

where, \( d_1, d_2 \) and \( d_3 \) are constants of integration.

Now, we can find all metric potentials from Eqs. (17), (19) as follow

\[
A(t) = a_1 e^{b_1 \int a^{-3} dt},
\]

(20)

\[
B(t) = a_2 e^{b_2 \int a^{-3} dt},
\]

(21)

and

\[
C(t) = a_3 e^{b_3 \int a^{-3} dt}.
\]

(22)

Here

\[
a_1 = (d_1 d_2)^{\frac{1}{3}}, \quad a_2 = (d_1^{-1} d_3)^{\frac{1}{3}}, \quad a_3 = (d_2 d_3)^{-\frac{1}{3}}, \quad b_1 = \frac{k_1 + k_2}{3}, \quad b_2 = \frac{k_3 - k_1}{3}, \quad b_3 = -\frac{k_2 + k_3}{3},
\]

where

\[
a_1 a_2 a_3 = 1, \quad b_1 + b_2 + b_3 = 0.
\]

Therefore, one can write the general form of Bianchi type I metric as

\[
ds^2 = -dt^2 + a^2 \left[ a_1^2 e^{2b_1 \int a^{-3} dt} dx^2 + a_2^2 e^{2b_2 \int a^{-3} dt} dy^2 + a_3^2 e^{2b_3 \int a^{-3} dt} dz^2 \right].
\]

(23)

Using eqs. (20)–(22) in eqs. (5)-(8) we can write the analogue of the Friedmann equation as

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3} + Ka^{-6},
\]

(24)

and

\[
2 \left( \frac{\ddot{a}}{a} \right) = -\frac{1}{3}(\rho + 3p).
\]

(25)

Here \( \rho = \rho^m + \rho^{de}, p = p^m + p^{de} \) and \( K = b_1 b_2 + b_1 b_3 + b_2 b_3 \). Note that \( K \) denotes the deviation from isotropy e.g. \( K = 0 \) represents flat FLRW universe. Thus, when the universe is sufficiently large, almost at the present time, the space-time behaves like a flat FLRW universe.

4 Dark Energy Equation of State

In this section we obtain the general form of the equation of state (EoS) for the viscous and non viscous dark energy (DE) \( \omega^{de} \) in Bianchi type I space-time when there is no interaction between dark energy and Cold Dark Matter (CDM) with \( \omega_m = 0 \). In this case the conservation equation (10) for dark and barotropic fluids can be written separately as

\[
\dot{\rho}^{de} + 3H(1 + \omega^{de})\rho^{de} = 0,
\]

(26)

and

\[
\dot{\rho}^m + 3H\rho^m = 0.
\]

(27)

Eq. (27) leads to

\[
\rho^m = \rho_0^m a^{-3}.
\]

(28)
Using eqs. (24), (28) in eqs. (7), (8) we obtain the energy density and pressure of dark fluid as
\[ \rho_{de} = 3H^2 - 3Ka^{-6} - \rho_0^m a^{-3} \]  
(29)
and
\[ p_{de} = -2\frac{\dot{a}}{a} - H^2 - La^{-6}, \]
(30)
respectively. Therefore, the equation of state parameter (EoS) of DE in its general form is given by
\[ \omega_{pf}^\ell = \frac{p_{de}}{\rho_{de}} = \frac{2q - 1 - La^{-6}H^{-2}}{3 + 3La^{-6}H^{-2} - 3\Omega_0^m a^{-3}}, \]
(31)
where \( q = -\frac{3}{aH^2} \) is the deceleration parameter, \( \Omega_0^m \) is the current value of matter density and \( L = b_2^2 + b_3^2 + b_2b_3 \) is a positive constant (Note that \( K + L = 0 \)).

From eq. (31) we see that at present time (i.e \( L = 0, q = -0.55, \Omega_0^m = 0.24, a = 1 \)), approximately, \( \omega_{pf}^\ell = -0.92 \). At late time, EoS parameter is given by
\[ \omega_{pf}^\ell \sim \frac{2q - 1}{3}, \]
(32)
here subscript ‘\( pf \)’ refers to “perfect fluid”.

According to the observations deceleration parameter is restricted as \(-1 \leq q < 0 \). Therefore, from eq. (32) we observe that at the best approximation the minimum value of \( \omega_{pf}^\ell \) is \(-1 \) i.e EoS of non-viscous DE can not cross phantom divided line (PDL). In another word, non-viscous dark energy can be described by quintessence \( \omega \). On thermodynamical grounds, in conventional physics (cross phantom divided line (PDL)). In another word, non-viscous dark energy can be described by quintessence \( \omega \).

In Eckart’s theory (Eckart 1940) a viscous dark energy EoS is specified by
\[ p_{vf}^\ell = p_{pf}^\ell + \Pi. \]
(33)
Here \( \Pi = -\xi(\rho_{de})u_i^i \) is the viscous pressure and \( H = \frac{u_i^i}{3} \) is the Hubble’s parameter and subscript ‘\( vf \)’ referees to “viscous fluid”. On thermodynamical grounds, in conventional physics \( \xi \) has to be positive. This is a consequence of the positive sign of the entropy change in an irreversible process (Nojiri and Odintsov 2003). In general, \( \xi(\rho_{de}) = \xi_0(\rho_{de})^\tau \), where \( \xi_0 > 0 \) and \( \tau \) are constant parameters. Note that, here we have to assume \( \tau > 0 \) since for negative \( \tau \) this form of bulk viscosity does not allow our models to cross PDL. A power-law expansion for the scale factor can be achieved for \( \tau = \frac{1}{2} \) (Barrow 1987, 1988). It has been shown by Goliath and Ellis (1999) that some Bianchi models isotropise due to inflation.

Substituting eq. (33) in eq. (31) by considering the above description we obtain the EoS parameter of viscous DE as
\[ \omega_{vf}^\ell = \frac{p_{vf}^\ell}{\rho_{vf}^\ell} + \Pi = \frac{2q - 1 - La^{-6}H^{-2}}{3 + 3La^{-6}H^{-2} - 3\Omega_0^m a^{-3}} - 3\xi_0 H^{-2\alpha} \]
(34)
where \( \Omega_{de} = \frac{\rho_{de}}{\rho_0^m} \) and \( \alpha = 1 - \tau \).

From eq. (34) we observe that the EoS of viscous DE at present time (i.e \( L = 0, q = -0.55, H_0 = 70, \Omega_0^m = 0.24, \Omega_{de}^m = 0.76, a = 1 \)), approximately is
\[ \omega_{vf}^\ell \sim -0.92 - \frac{213\xi_0}{(12501.68)^\alpha}, \]
(35)
which clearly cross the PDL for appropriate values of \( \alpha \) and \( \xi_0 \). As mentioned before, phantom fields are generally plagued by ultraviolet quantum instabilities. Naively, any phantom model with \( \omega_{de} < -1 \) should
decay to $\omega_{de} = -1$ at late time. As mentioned in (Carroll et al. 2003), this ensures that there is no future singularity (Big Rip); rather, the universe eventually settles into a de Sitter phase. Here we highlight since $\xi(\rho_{de}) = \xi_0(\rho_{de})^7$, and $\rho_{de}$ is a decreasing function of time in an expanding universe we conclude that the bulk viscosity dies out as time goes on and viscous phantom DE is an unstable state (as expected) and EoS of DE tends to $-1$ at late time (de-Sitter Universe).

5 Correspondence Between Dark Energy And Scalar Fields

It is believed that the current accelerated expansion is driven by a dynamical scalar field $\phi$ with potential $V(\phi)$. These models introduce a scalar field $\phi$ that is minimally coupled to gravity. As it is shown in previous section, one can generate quintessence and phantom fields from non-viscous and viscous fluids in an anisotropic universe respectively.

Quintessence and phantom fields are generally given by the action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \epsilon (\nabla \phi)^2 - V(\phi) \right]. \quad (36)$$

The energy density and pressure of scalar field (DE) are given by

$$\rho_\phi = \frac{1}{2} \epsilon \dot{\phi}^2 + V(\phi) \quad (37)$$

and

$$p_\phi = \frac{1}{2} \epsilon \dot{\phi}^2 - V(\phi), \quad (38)$$

where $\epsilon = \pm 1$. $\epsilon = 1$ is referred to as quintessence whereas $\epsilon = -1$ is referred to as phantom. From eqs. (29), (30) and eqs. (37), (38) we find the general form of the scalar field $\phi$ and potential $V(\phi)$ as

$$\dot{\phi}^2 = 2\epsilon \left[ H^2 (1 + q) + 3a^{-6} - \frac{3}{2} H^2 \Omega_m^0 a^{-3} - \frac{\xi_0}{2} \sqrt{3} \Omega_{de} \right], \quad (39)$$

and

$$V(\phi) = 2 \left[ H^2 (1 - q) - 4a^{-6} - \frac{3}{2} H^2 \Omega_m^0 a^{-3} + \frac{\xi_0}{2} \sqrt{3} \Omega_{de} \right]. \quad (40)$$

Note that putting $\xi_0 = 0$ and $\epsilon = 1$ in eqs. (39), (40) we get the scalar field and potential of quintessence. Also for sufficiently large time, the asymptotic behavior of $\phi$ and $V(\phi)$ is given by

$$\phi \sim \left(-\epsilon \xi_0 \sqrt{3}\right)^{1/2} t + \text{constant}, \quad (41)$$

and

$$V(\phi) \sim \xi_0 \sqrt{3}, \quad (42)$$

respectively. Eq. (41) clearly shows that the only possible scenario at far future is the phantom scenario as $\epsilon = 1$ (quintessence) gives an imaginary $\phi$. It is worth to mention that at late time i.e $a \to \infty$ which implies $\xi_0 \to 0$, the potential asymptotically tends to vanish and $\phi = \text{constant}$.

6 Statefinder Diagnostic

V. Sahni and coworkers (2003) have recently introduced a pair of parameters $\{r, s\}$ called “statefinders”, which are useful to distinguish different types of dark energy. The statefinders were introduced to characterize primarily flat universe models with cold dark matter (dust) and dark energy. They were defined as

$$r \equiv \frac{\dot{a}}{aH}, \quad s \equiv \frac{r - \Omega}{3(q - \Omega)}. \quad (43)$$

Here the formalism of Sahni and coworkers is extended to permit curved universe models. If we suppose that dark energy does not interact with dark matter (as we assumed), then the statefinder pair can be further expressed as

$$r = \Omega_m + \frac{9\omega_{de}}{2} \Omega_{de} (1 + \omega_{de}) - \frac{3}{2} \Omega_{de} \frac{\dot{\omega}_{de}}{H}, \quad (44)$$

$$s = \frac{r - \Omega}{3(q - \Omega)}. \quad (45)$$
\[
\omega_{de} = \frac{\dot{\phi}^2 - 2\epsilon V(\phi)}{\dot{\phi}^2 + 2\epsilon V(\phi)},
\]

by taking differentiation we get
\[
\dot{\omega}_{de} \rho_{de} = \frac{2\epsilon \dot{\phi}(2\ddot{\phi}V - \dot{\phi}^2 \dot{V})}{\dot{\phi}^2 + 2\epsilon V(\phi)} \tag{47}
\]

Using the equation of motion for the scalar field
\[
\ddot{\phi} + 3H \dot{\phi} + \epsilon \dot{V} = 0, \tag{48}
\]
in eq. (47) and inserting the result into (44) we obtain (note that \(\dot{V} = V' \dot{\phi}, V' = dV(\phi) \frac{d\phi}{d\phi} \))
\[
r = \Omega + \frac{3}{2} \dot{\phi}^2 + \epsilon \frac{\dot{V}}{H^3} \tag{49}
\]
Furthermore, from Raychaudhuri’s equation
\[
\frac{\ddot{a}}{a} = \frac{3}{2} \xi_0 H (\rho^{de})^\tau - \frac{1}{6} \rho^{de}(1 + 3\omega^{de}) - \frac{1}{6} \rho^m(1 + 3\omega^m) - \frac{2}{3} \sigma^2, \tag{50}
\]
we find
\[
q - \frac{\Omega}{2} = \frac{\xi_0}{2} H^{2\tau - 1} (3\Omega^{de})^\tau - \frac{2}{3} \sigma^2 + \frac{1}{2H^2} \left( \frac{1}{2} \rho^{de} \dot{\phi}^2 - V \right), \tag{51}
\]
where \(\sigma_{ij}\) is the shear tensor which is given by
\[
\sigma_{ij} = u_{i;k}u^k u_{j} + u_{j;k}u^k u_{i} + \frac{1}{3} \theta(g_{ij} + u_i u_j). \tag{52}
\]
Therefore, the statefinder \(s\) is also obtained as
\[
s = \frac{\dot{\phi}^2 + \frac{2}{3} \dot{\phi}}{\frac{3}{2} H^3 - 2\alpha (3\Omega^{de})^{1-\alpha} - (\frac{2\epsilon \dot{\phi}^2}{3})^2 + \left( \frac{1}{2} \rho^{de} \dot{\phi}^2 - V \right)} \tag{53}
\]
To study the behavior of viscous DE more precisely we consider a toy model in the next section.

### 7 Test Model

To examine our above general results we present a worked example in this section. For this propose we assume the following scale factor
\[
a(t) = \sinh(t). \tag{54}
\]
By assuming a time varying deceleration parameter one can generate such a scale factor (Amirhashchi et al. 2011). It has also been shown that this scale factor is stable under metric perturbation (Chen and Kao 2001). In terms of redshift the above scale factor is
\[
a = \frac{1}{1 + z}, \quad z = \frac{1}{\sinh(t)} - 1. \tag{55}
\]
In this case one can find the DE energy density \(\rho^{de}\), the bulk viscosity \(\xi(\rho^{de})\), deceleration parameter \(q\), and average anisotropy parameter \(A_m\) as
\[
\rho^{de} = 3\coth^2(t) + 3L \sinh^{-b}(t) - \rho^m_0 \sinh^{-3}(t)
\]
\[ \rho_{\text{de}}(z) = 3\xi_0 \frac{\cosh(t)}{3 \cosh^2(t) + 3L \sinh^{-6}(t) - \rho_m^0 \sinh^{-3}(t)} \]  
(56)

\[ \xi(\rho_{\text{de}}) = 3\xi_0 \coth(t) \left[ 3 \coth^2(t) + 3L \sinh^{-6}(t) - \rho_m^0 \sinh^{-3}(t) \right]^{1-\alpha} \]
(57)

\[ q = -\tanh^2(t) = -\frac{1}{1+(1+z)^2} \]  
(58)

Figure 1: The plot of the DE energy density $\rho_{\text{de}}$, average anisotropy parameter $A_m$, and the bulk viscosity $\xi(\rho_{\text{de}})$ versus redshift $z$ for $\rho_m^0 = 0.24$, $L = 0.1$, $\xi_0 = 0.1$.

Figure 2: The plot of deceleration parameter $q$ versus redshift $z$.

By using eq. (54) in eqs. (34), (39), and (40) and after simplification the EoS of viscous dark energy $\omega_{\nu f}$, scalar field $\phi$ and the potential $V(\phi)$ are obtained as

\[ \omega_{\nu f} = -\frac{1}{3} \left[ \frac{2 \tanh^2(t) + 1 + L \sinh^{-2}(t) \cosh^{-2}(t)}{1 + L \sinh^{-2}(t) \cosh^{-2}(t) - \Omega_m^0 \sinh^{-3}(t)} \right] \]

\[ -3^{1-\alpha} \xi_0 \coth^{1-2\alpha}(t) \]  
(60)

\[ \dot{\phi}^2 = 2\epsilon \left[ \sinh^{-2}(t) + L \sinh^{-6}(t) - \frac{3}{2} \Omega_m^0 \sin^{-5}(t) \cosh^2(t) - \frac{\xi_0}{2} \sqrt{3\Omega_{\text{de}}} \right] \]  
(61)
V = 2 \left[ 2 \tanh^{-2}(t) + 1 - 4L \sinh^{-6}(t) - \frac{3}{2} \Omega_0^m \sinh^{-5}(t) \cosh^2(t) + \frac{\xi_0}{2} \sqrt{3 \Omega^{de}} \right]. \quad (62)

The behavior of EoS parameter, $\omega_{de}$, in terms of redshift $z$ is shown in Fig. 3. It is observed that the EoS parameter is a decreasing function of $z$ and the rapidity of its decrease depends on the value of $\xi_0$. We see that in absence of bulk viscosity the EoS always varying in quintessence region (red line/solid line) whereas in presence of viscosity EoS cross PDL and varying in phantom region. But at the later stage of evolution it tends to the same constant value i.e $\omega_{de} = -1$ independent of the value of $\xi_0$. This behavior clearly shows that the phantom phase i.e $\omega_{de} < -1$ is an unstable phase and there is a transition from phantom to the cosmological constant phase at late time. As we mention above, the phantom phase instability of the universe is because of the fact that the viscosity dies out as time is passing.

We can re-write eqs. (60)-(62) in term of redshift as

$$\omega_{de} = -\frac{1}{3} \left( \frac{1 + \frac{2}{1 + (1 + z)^2} + L \frac{(1 + z)^6}{1 + (1 + z)^2} - \Omega_0^m (1 + z)^{-3}}{1 + L \frac{(1 + z)^6}{1 + (1 + z)^2} - \Omega_0^m (1 + z)^{-3}} \right) - 3^{1 - n} \xi_0 \frac{[(1 + z)^{-4} + (1 + z)^{-2}]^{1 - 2\alpha}}{(\Omega^{de})^\alpha}, \quad (63)$$

$$\dot{\varphi}^2 = 2\epsilon \left[ (1 + z)^2 + L(1 + z)^6 - \frac{3}{2} \Omega_0^m (1 + z)^3(1 + (1 + z)^2) - \frac{\xi_0}{2} \sqrt{3 \Omega^{de}} \right], \quad (64)$$

$$V = 2 \left[ 1 + \frac{2}{1 + (1 + z)^2} - 4L(1 + z)^6 - \frac{3}{2} \Omega_0^m (1 + z)^3(1 + (1 + z)^2) + \frac{\xi_0}{2} \sqrt{3 \Omega^{de}} \right]. \quad (65)$$

The matter density $\Omega^m$ and dark energy density $\Omega^{de}$ can be easily calculated as

$$\Omega^m = \Omega_0^m \sinh^{-3}(t) = \Omega_0^m (1 + z)^3, \quad (66)$$

$$\Omega^{de} = 1 + L \sinh^{-4}(t) \cosh^{-2}(t) - \Omega_0^m \sinh^{-3}(t) = 1 + L \frac{(1 + z)^6}{(1 + (1 + z)^2)^2} - \Omega_0^m (1 + z)^3. \quad (67)$$

Also from above two equations we obtain the total energy density as

$$\Omega = \Omega^m + \Omega^{de} = 1 + L \frac{(1 + z)^6}{(1 + (1 + z)^2)^2}. \quad (68)$$
The variation of density parameters $\Omega^m$ and $\Omega^{de}$ with redshift $z$ have been shown in Fig. 4. Here, we observe that $\Omega^{de}$ increases as redshift decreases and approaches to 1 at late time whereas $\Omega^m$ decreases as $z$ decreases and approaches to zero at late time.

For our model, the parameters $\{r, s\}$ can be explicitly written in terms of cosmic time $t$ or redshift $z$ as

$$r = \tanh^2(t) = \frac{1}{1 + (1 + z)^2}$$

and

$$s = \frac{1 + L \sinh^{-4}(t) \cosh^{-2}(t) - \tanh^2(t)}{\frac{3}{2} \left[ 1 + 2 \tanh^2(t) + L \sinh^{-4}(t) \cosh^{-2}(t) \right]} = \frac{(1 + z)^2 (1 + L (1 + z)^4)}{\frac{3}{2} \left[ 2 + L (1 + z)^2 (1 + (1 + z)^4) \right]}$$

Figure 5 shows the values of $\Omega^{de}_0$ and $\Omega^m_0$ which are permitted by our model. From this figure we observe that for case $L = 0$ which represents a spatially flat universe ($\Omega = 1$), $\Omega^{de}_0 \approx 0.76$ and $\Omega^m_0 \approx 0.24$. These results are in good agreement with the CMB results, the supernova results, and the computed density of matter in clusters. Other models with $L \neq 0$, represent open universes with $\Omega < 1$.

Trajectories in $s - r$ plane corresponding to different cosmological models are shown in figure 6. The dots in the diagram locate the current values of the statefinder pairs $\{s, r\}$. From this figure we see explicitly that the ingredient parameter $L$ (or $K$) makes the model evolve along different trajectories on the $s - r$ plane. It is worth to mention that the cold dark matter with a cosmological constant ($\Lambda$CDM) diagrams (spatially flat) corresponds to the fixed point $\{s, r\}_{\Lambda$CDM} = \{0, 1\}. From eqs. (69) and (70) we obviously see that $\{s, r\} = \{0, 1\}$ at late time i.e $z = -1$.

8 Concluding Remarks

Phantom field models have been suggested in order to provide a theoretical support for the recent observation that mildly favor the EoS of DE crossing $-1$ near the past. A lot of studies have been done in this regard and many phantom field models have been proposed. Some of these models are evolving from quintessence to phantom called quintom. However, these models suffer from two major problems i.e. (1) Instability of phantom field and (2) finite future singularity (big rip). In this paper we proposed a simple mechanism to alleviate these problems by introducing a special form of bulk viscosity i.e. $\Pi = -3\xi_0 H (\rho^{de})^\tau$ in the cosmic fluid. In
this mechanism first, viscosity causes dark energy which is varying in quintessence to pass phantom divided line (PDL) and drop it to the phantom region but since viscosity is a decreasing function of time, as time is passing it dies out and $\omega$ leaves phantom region and tends to $-1$ at late time. Hence the problem of future singularity (big rip) does not occur in this scenario. To test the impact of the anisotropy parameter ($L$), we perform a statefinder diagnostic on this scenario. This diagnostic shows that the statefinder parameters can probe the anisotropy of the model. May be future SNAP would be capable of probing this effect. In summary, The general form of the EoS parameter of viscous and non-viscous dark energy has been investigated in this paper. It is found that the presence of bulk viscosity causes our universe to get to the darker region i.e phantom temporarily. It is worth to mention that since our anisotropic model behaves as isotropic FLRW universe at late time, as a result, the phantom does not survive in isotropic universe as well. Our results fulfil the theoretical requirement argued by Carroll et all (2003) which state that, to avoid the big rip problem, all phantom models should decay to cosmological constant at late time. Moreover, since we have not restricted our study to the maximally symmetric FLRW space-times, our results seems to be more general than those obtained on the bases of this isotropic universes.

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