Classical and Quantum Gravity in 1+1 Dimensions

Part I: A Unifying Approach

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Abstract

We provide a concise approach to generalized dilaton theories with and without torsion and coupling to Yang-Mills fields. Transformations on the space of fields are used to trivialize the field equations locally. In this way their solution becomes accessible within a few lines of calculation only. In this first of a series of papers we set the stage for a thorough global investigation of classical and quantum aspects of more or less all available 2D gravity-Yang-Mills models.

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1 Introduction

One of the major open problems in theoretical physics is how to construct a consistent theory of quantum gravity. This long-standing issue has been approached in various different ways. Up to now it is not clear which of those approaches, ranging as far as from non-perturbative canonical quantum gravity \textsuperscript{1} to string theory, is most adequate. Also it may be that they all provide complementary, but legitimate viewpoints \textsuperscript{2}.

In all of these approaches some control over the classical theory seems indispensable. In a path integral, e.g., the leading contributions come from the local extremals of the action; or in the canonical approach the space of observables is correlated directly to the space of classical solutions modulo gauge transformations (provided always that there are no anomalies). In addition to this the relation between different approaches to quantum gravity might be illuminated through considerations on the classical level. However, in comparison to the vast infinity of possible solutions to the Einstein equations only a negligible number of (exact) solutions is known and the space of classical solutions (modulo gauge transformations) is maybe even less seizable than the corresponding quantum theory.

The situation changes drastically, if one regards gravity models in lower dimensions. Within the recent decade such models have attracted considerable interest – in three (cf., e.g., \cite{3, 4}) as well as in two spacetime dimensions (cf., e.g., \cite{5}). In the present treatise we will restrict ourselves to two spacetime dimensions with Minkowski signature. The claim of this and the following papers is that in the case of Lagrangians describing gravity without additional matter fields (a dilaton field is not considered as matter in this context) there is complete control over the space of classical solutions as well as on the space of quantum states. In Parts II,III \textsuperscript{3} of the present series we will provide a classification of all global solutions (modulo gauge transformations) of more or less all available gravitational models in 1+1 dimensions without matter couplings! We allow for all possible topologies of the spacetime manifold; and indeed there are models with solutions with basically arbitrarily complicated topologies. In Part IV \textsuperscript{4}, on the other hand, we will construct all quantum states of the considered two-dimensional models in a Hamiltonian approach à la Dirac.

What can we hope to learn from this? Is not the situation in two dimensions just too far away from Einstein gravity in four dimensions? In part this is true, certainly, but the point is that there are questions which are not sensitive to the dimensional simplification. One example is the issue of time (cf., e.g., \cite{8}), which arises in any diffeomorphism invariant theory.

Another such question could be the interplay between a path integral, a canonical Hamiltonian approach, and the topology of spacetime. Irrespective of the dimension of spacetime a Hamiltonian treatment implies a
restriction to topologies of the form $\Sigma \times I \mathbb{R}$, where $\Sigma$ is some (generically space-like) hypersurface. (In two dimensions obviously $\Sigma$ is either a circle or a real line). In this way one excludes, e.g., the interesting topic of topology changes. In a path integral, on the other hand, there is no restriction to particular topologies. Now, in two-dimensional Yang-Mills (YM) theories people invented some cut and paste technique so as to infer transition amplitudes of non-trivial spacetime topologies from the knowledge of the Hilbert space associated to cylindrical spacetimes \cite{9}. It is realistic to hope that something similar will be possible in the two-dimensional gravity theories. The result might provide a first idea of what may be expected of the somewhat analogous issue in four-dimensional canonical and path integral quantum gravity.

But there are also interesting technical issues that may be investigated: One of these is the role which degenerate metrics play in a canonical framework, another one is the construction of an inner product in a Dirac approach to a quantum theory, a third one a comparison of quantum theories for Minkowskian and Euclidean signatures of the (quantized) metric. All of these issues, taken up in Part IV of this work, are of current interest in 4D quantum gravity \cite{10}.

The purpose of the present paper is to set the stage for a thorough investigation of 1+1 gravity on the classical and quantum level. In Section 2 we introduce the models under study. They comprise all generalized dilaton theories \cite{11}, where, in its reformulated version \cite{12}, we allow also for non-trivial torsion \cite{13, 14}. In this way one captures theories such as $R^2$-gravity with \cite{15} and without \cite{16} torsion or spherically reduced 4D gravity \cite{17}; but it should be stressed that the class of considered theories is much more general. Not included in the present treatise is axially reduced 4D gravity \cite{18}. In Section 4 we further deal with dynamically coupled Yang-Mills gauge fields. A coupling to fermion or scalar matter fields (besides the dilaton), on the other hand, is mentioned briefly only, but cf. also \cite{19, 20} for instance.

The breakthrough in the analysis of 2+1 gravity (with Lagrangian $\int d^3x \sqrt{|\det g|} R$) came about with its identification as a Chern-Simons gauge theory of the 2+1 dimensional Poincare group \cite{3}. Similarly, in 1+1 dimensions there are two models that could be identified with standard gauge theories: The first of these is the Jackiw-Teitelboim (JT) model of 1+1 deSitter gravity with Lagrangian \cite{21}

$$L_{JT} = -\frac{1}{2} \int_M d^2x \sqrt{-\det g} \varphi (R - \Lambda),$$  \hspace{1cm} (1)

where $R$ is the Levi-Civita curvature scalar of the metric $g$ and $\varphi$ is a Lagrange multiplier field enforcing the field equation $R = \Lambda \equiv \text{const}$. Rewriting this action in an Einstein-Cartan formulation, it is found to coincide with a YM gauge theory of the BF-type \cite{22} (where the gauge group is the
universal covering group of $\text{SO}(2,1)$\cite{23}). Analogously, ordinary dilaton theory\cite{24} may be reformulated as a BF-theory of the 1+1 dimensional Poincaré group, if one promotes its cosmological constant to a dynamical field which becomes constant on-shell\cite{23}. In all of these cases the respective group structure greatly facilitated classical and especially quantum considerations.

Within the class of theories considered in this paper the JT- and the dilaton model are very particular (and comparatively simple). They are, e.g., the only ones which, in a Hamiltonian formulation, allow for global phase space coordinates such that the structure functions in the constraint algebra become constants (cf. Part IV). Still some of their features, such as a local degrees of freedom count, hold for more complicated 2D gravity theories, too. In a way the situation reminds one a bit of the Ashtekar formulation of 3+1 gravity with its similarities but also its differences to a 4D YM theory. The question arises: Given, say, spherically reduced gravity or the already somewhat more complicated $R^2$-theory, or even a generalized dilaton theory defined by a potential function $V(\cdot)$ or $W(\cdot,\cdot)$ (cf. Sec. 2 below): Is there some kind of gauge theory formulation for them, which is similarly helpful in the determination of the space of quantum states or the space of classical solutions modulo gauge transformations? In other words: Is there some unifying approach to all of these 2D gravity theories generalizing in a nonlinear way main features of YM-type gauge theories?

Indeed, this question may be answered in the affirmative, as is demonstrated at the beginning of Section 3. The key feature will be the identification of Poisson brackets on an appropriate target space associated with any generalized dilaton theory. The resulting point of view not only allows for a unified treatment of gravity-Yang-Mills systems in two dimensions, it also provides tools for their classical and quantum analysis, which are hardly accessible otherwise. This will be demonstrated first when employing the formalism to solve the field equations in a particularly efficient manner in the remainder of Section 3. There we will provide the general local solution to the field equations of the general model in the vicinity of arbitrary spacetime points. The extension of this to the case of dynamically coupled YM-fields is taken up in Section 4, finally. Section 4 includes also a brief summary of the results of this first part as well as a short outlook on Parts II–IV.

2 Models of 1+1 Dimensional Gravity

It is well-known that in two dimensions the Einstein-Hilbert term, $\int \sqrt{|\text{det} g|} R d^2x$, does not provide a useful action for field equations, as it is a ‘boundary term’. However, a natural approach to find a gravity action in two dimensions is to dimensionally reduce the four-dimensional Einstein-
Hilbert action. Implementing, e.g., spherical symmetry, by plugging

\[(ds^2)_{(4)} = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu - \Phi^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad \mu, \nu \in \{0, 1\}, \tag{2}\]

into the four-dimensional action and integrating over the angle coordinates \(\theta\) and \(\phi\), one obtains the two-dimensional action \[L_{\text{spher}} = \int_M d^2x \sqrt{-\det g} \left[ \frac{1}{4} \Phi^2 R(g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{8} \Phi^2 \Lambda \right]. \tag{3}\]

Here \(R(g)\) denotes the Ricci scalar of the Levi-Civita connection of the (two-dimensional) metric \(g\) and \(\det g \equiv \det g_{\mu\nu}\). Note that the field \(\Phi(x^\mu)\) is restricted to positive values by definition.\footnote{This could be avoided by introducing a new field variable proportional to \(\ln \Phi\).} As a consistency check one may verify that the resulting field equations of this effective two-dimensional action provide solutions to the four-dimensional Einstein equations, and these are nothing but the Schwarzschild solutions, parametrized, according to Birkhoffs theorem, by the Schwarzschild mass \(m\):

\[g = \left(1 - \frac{2m}{r}\right) (dt)^2 - \left(1 - \frac{2m}{r}\right)^{-1} (dr)^2. \tag{4}\]

The currently most popular action for a two-dimensional gravity theory is, however, the CGHS-model \[L_{\text{CGHS}}(g, \phi, f_i) = L_{\text{dil}}(g, \phi) + L_{\text{Mat}}(f_i, g), \tag{5}\]

\[L_{\text{dil}}(g, \phi) = \int_M d^2x \sqrt{-\det g} \exp(-2\phi) \left[R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \Lambda\right], \tag{6}\]

\[L_{\text{Mat}}(f_i, g) = \int_M d^2x \sqrt{-\det g} \sum_{i=1}^N g^{\mu\nu} \partial_\mu f_i \partial_\nu f_i. \tag{7}\]

The first part of this action, \(L_{\text{dil}}\), is the so-called ‘string inspired’ or dilaton gravity action \[L_{\text{dil}}(g, \phi) = \int_M d^2x \sqrt{-\det g} \left[\frac{1}{8} \Phi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{8} \Phi^2 \Lambda\right]. \tag{8}\]

The vacuum solutions \((f_i \equiv 0)\) of (5) are of a similar form as (4) (with identical Penrose diagrams). Moreover, the classical model can be solved completely also when the \(f_i\) are present. The CGHS-model thus opened the possibility to discuss, e.g., the Hawking effect\footnote{This could be avoided by introducing a new field variable proportional to \(\ln \Phi\).} in a simplified two-dimensional framework. Also, motivated by the classical solvability of this model, one may hope for an exact quantum treatment of (5), allowing to test the semiclassical considerations leading to the Hawking effect. By means of the field redefinition \(\Phi := 2\sqrt{2} \exp(-\phi), \Phi > 0\), the gravitational or dilaton part of (5) may be put into the form

\[L_{\text{dil}}(g, \Phi) = \int_M d^2x \sqrt{-\det g} \left[\frac{1}{8} \Phi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{8} \Phi^2 \Lambda\right]. \tag{8}\]
Now (3) and (5) are found to show much similarity.

Both of the examples given above are a special case of the general action

$$L^{gdil}(g, \Phi) = \int_M d^2x \sqrt{-\det g} \left[ D(\Phi)R(g) + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - U(\Phi) \right].$$  (9)

This action, which we will call generalized dilaton action, was suggested first in [11]. It is the most general diffeomorphism invariant action yielding second order differential equations for the metric $g$ and a scalar dilaton field $\Phi$. In the following we will restrict ourselves to the case that $D$ has an inverse function $D^{-1}$ everywhere on its domain of definition. Also, for simplicity, we assume that $D$, $D^{-1}$, and $U$ are $C^\infty$.

With these assumptions we may use

$$X^3 := D(\Phi)$$  (10)

as a new field variable instead of $\Phi$. Introducing [12] instead of $g$

$$\bar{g} := \exp[\rho(\Phi)]g, \quad \rho = \frac{1}{2} \int^\Phi du \frac{dD(u)/du}{(D(u))'}, + const,$$  (11)

moreover, the action (6) takes the simplified form

$$L^{gdil}(\bar{g}, X^3) = \int_M d^2x \sqrt{-\det \bar{g}} \left[ X^3 R(\bar{g}) - V(X^3) \right].$$  (12)

Here we have put $V(z) := (U/\exp \rho)(D^{-1}(z))$ and different constants chosen for the definition of $\rho$ rescale only the potential $V$. Note that this field-dependent conformal transformation allowed to get rid of the kinetic term for the dilaton field.

In the case of $L^{dil}$ or $L^{CGHS}$, respectively, the transformation from $\phi$, $g$ to $X^3$, $\bar{g}$ proves specifically powerful [28]: With an appropriate choice of $\text{const}$ in $\rho$ one obtains $\exp \rho(\Phi) = D(\Phi), \ U(\Phi) = \Lambda D(\Phi)$ and thus $V(X^3) = \Lambda = \text{const}$. Since, moreover, $L^{Mat}$ is invariant under conformal transformations, the action (5) becomes

$$L^{CGHS}(:\bar{g}, X^3, f_i) = \int_M d^2x \sqrt{-\det \bar{g}} \left[ X^3 R(\bar{g}) - \Lambda \right] + L^{Mat}(f_i, \bar{g}).$$  (13)

In this formulation the classical solvability of the CGHS-model is most obvious: The variation with respect to $X^3$ yields $\partial R(\bar{g}) = 0$. This implies

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\footnote{At this point the chosen nomenclature might appear bizarre. In the sequel, however, $X^3$ will turn out to serve as the third target space coordinate of a useful $\sigma$-model formulation of (5).

\footnote{Mainly we use only one symbol for a function or functional, if it is represented in different coordinates (cf., e.g., (3) and (8)). To avoid misinterpretations we haven’t done so in the case of $V$.}
that up to diffeomorphisms the metric $\tilde{g}$ is Minkowskian. Thus the field equations resulting from the variation with respect to the $f_i$ reduce to the ones of $N$ massless scalar fields in Minkowski space. One then is left only to realize that due to the diffeomorphism invariance only one of the three field equations $\delta L^{\text{CGHS}} / \delta \tilde{g}^{\mu\nu}(x) = 0$ is independent \cite{2} and that this one may be solved always for the Lagrange multiplier field $X^3$ locally. Still, the representation of $L^{\text{CGHS}}$ in the form (13) does not imply that the scalar fields $f_i$ and the original metric $g$ decouple completely. Rather one should compare it to the introduction of normal coordinates for coupled harmonic oscillators. To trace the coupling explicitly, one notices that the transition from $\tilde{g}$ to $g$ involves $X^3$, which in turn is coupled directly (via $\tilde{g}$ in (13)) to the scalar fields $f_i$.

Let us represent $L^{\text{gdil}}$ in first order form. For this purpose we switch to the Cartan formulation of a gravity theory, implementing the zero-torsion condition by means of Lagrange multiplier fields $X^\pm$:

$$L^{\text{gdil}}(e^a, \omega, X^i) = -2 \int_M X_a De^a + X^3 d\omega + \frac{V(X^3)}{2} \varepsilon,$$

(14)

with

$$\tilde{g} = 2e^- e^+ \equiv e^- \otimes e^+ + e^+ \otimes e^-$$

(15)

and

$$De^a \equiv de^a + \varepsilon^a_{\ b} \omega \wedge e^b, \quad a \in \{-, +\}, \quad \varepsilon \equiv e^- \wedge e^+,$$

(16)

such that $e^\pm$ is the zweibein in a light cone basis of the frame bundle, $\omega$ (or $\omega^a_b \equiv \varepsilon^a_{\ b} \omega$) is the Lorentz or spin connection, and $\varepsilon_{++} = +1$. Here we have used $\varepsilon R = -2d\omega$.

We derived (14) from the general action (9) (for the case that (10) is a diffeomorphism). In this way the zweibein and spin connection are interpreted as quantities corresponding to the auxiliary metric $\tilde{g}$, which in turn is related to the ‘true’ metric via (11). In the following we will argue that (14), or its generalization

$$L^{\text{grav}} = \int_M X_a De^a + X^3 d\omega + W((X)^2, X^3) \varepsilon,$$

(17)

$$X^2 \equiv X_a X^a \equiv 2X^- X^+,$$

(18)

may be regarded also as a gravity theory with metric

$$g = 2e^- e^+.$$  

(19)

First we note that $L^{\text{grav}}$ is invariant with respect to the standard gravity symmetries, which are diffeomorphisms and local frame rotations. Second, more or less by construction the action (17) is in first order form. It is not difficult to see then that the $X^i$, $i = +, -, 3$, are precisely the generalized
momenta canonically conjugate to the one-components of the zweibein and
the spin connection (the corresponding zero-components serve as Lagrange
multipliers for the constraints of the theory, cf. also Part IV). Obviously
there is a need for momenta in any first order formulation of a gravity
theory, so the $X^i$ appear very natural from this point of view.

Last but not least, (17) may be seen to yield further already accepted
models of 2D gravity for some specific choices of the potential $W$. Let
us choose, e.g., $2W[(X)^2, X^3] = V(X^3) = (X^3)^\gamma \Lambda$ where $\Lambda \neq 0$ and $\gamma$
are some real constants. For $\gamma = 1$ we immediately recognize the good
old Jackiw-Teitelboim model of two-dimensional deSitter gravity [21]. For
$\gamma \neq 1$, on the other hand, we may eliminate the field $X^3$ by means of its
equation of motion. Implementing, furthermore, the zero-torsion constraint
by hand again, the resulting Lagrangian is found to be of the form
$L \propto \int M d^2 x \sqrt{-\det g (R^2/16 + \Lambda)}$ (20)
of two-dimensional $R^2$-gravity.

Similarly, the potential
$W^{KV} = -\alpha (X)^2/2 - (X^3)^2 + \Lambda/\alpha^2$ (21)
leads, upon elimination of the $X$-coordinates, to
$L^{KV} = \int \left[ -\frac{1}{4} \omega \wedge * \omega - \frac{1}{2\alpha} D e^a \wedge * D e_a + \frac{\Lambda}{\alpha^2} \epsilon \right]$, (22)
proposed first in [15]. This Lagrangian is the most general (diffeomorphism
and frame invariant) Lagrangian yielding second order differential equations
for zweibein and spin connection. It is noteworthy that, in contrast to four
dimensions [20], it contains only three terms. Here one allowed for nontrivial
torsion. All torsion-free theories described by (17) have a potential $W$
that is independent of $(X)^2$, or, equivalently, by Lagrangians of the form (19)
(with $\tilde{g} \to g$).

Before we close this section, let us return to the case of spherical sym-
metry (3). An appropriate choice of the integration constant in (11) yields
$g = \tilde{g}/\sqrt{X^3}$ and the potential $W$ becomes $W = V/2 = 1/4\sqrt{X^3}$ in this
case. Thus, the space of solutions to (3) will be reproduced from (17) with
this potential, if, according to (15), $g := 2e^{-e^+}/\sqrt{X^3}$. Alternatively, as we
will find in the following section, the positive mass solutions (4) may be
described also by (17) with potential $W = 1/(X^3)^2$, if we use the simpler
identification (19), $g := 2e^{-e^+}$.
In this section we have shown that (17) is a universal action for gravity theories in two dimensions. In the following section we will find it to be a special case of a $\sigma$-model defined by a Poisson structure on a target space, an observation that allows to solve the theory in an elegant and efficient manner.

3 The Local Solutions of the Field Equations

With the notational convention
\[ A_- \equiv e_- \equiv e^+, \quad A_+ \equiv e_+ \equiv e^-, \quad A_3 \equiv \omega \] (23)
we can rewrite the action (17) up to a boundary term as
\[ L = \int_M A_i \wedge dX^i + \frac{1}{2} P^{ij}(X(x)) A_i \wedge A_j \] (24)
with
\[ (P^{ij})(X) = \begin{pmatrix} 0 & -W & -X^- \\ W & 0 & X^+ \\ X^- & -X^+ & 0 \end{pmatrix} \quad i, j \in \{-, +, 3\}, \] (25)
where, as before, $W$ is a function of $(X)^2 \equiv 2X^-X^+ + X^3$. The first decisive observation is that in this form the action is not only covariant with respect to diffeomorphisms on the spacetime manifold $M$, but also with respect to diffeomorphisms on the space of values of the fields $X_i$, i.e. on a ‘target space’ $N = \mathbb{R}^3$; we only have to define the transformation of the $A_i$ and of $P^{ij}$ as those of one-forms and bivectors on $N$, respectively. (The term $A_i \wedge dX^i$ in (24) may then be interpreted as the pullback of a one-one-form $A = A_\mu dx^\mu \wedge dX^i$ on $M \times N$ under the map of $M$ into the space of fields and likewise the second term in (24) as the pullback of the twofold contraction of $A$ with the two-tensor $P = (1/2)P^{ij}\partial_i \wedge \partial_j$ on $N$, cf. [31, 32]). The second decisive observation is that the matrix $P$ obeys the following identity:
\[ \frac{\partial P^{ij}}{\partial X^l} P^{lk} + \text{cycl.}(ijk) = 0. \] (26)
It establishes that $P$ is a Poisson structure on $N = \mathbb{R}^3$. To see this one defines
\[ \{F, G\}_N = P^{ij}(X) \frac{\partial F(X)}{\partial X^i} \frac{\partial G(X)}{\partial X^j} \] (27)
for any two functions $F$ and $G$ on $N$; the Jacobi identity for the Poisson brackets $\{\cdot, \cdot\}_N$ on $N$ is then found to be equivalent to (26). Vice versa, the

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4The study of actions of the form (24) where $P^{ij}$ satisfies (26) has been proposed also in [33]. However, the implications of the identity (26), recapitulated in what follows, have been realized only in [13, 14, 31, 32].
Leibniz rule and antisymmetry of Poisson brackets ensures that they can be written in the form (27) with a skew-symmetric bivector $\mathcal{P}$ on $N$. Thus we see that (24), and therefore also (17), may be interpreted as a $\sigma$-model, where the world sheet $M$ is the spacetime manifold and the target space $N$, which in the present case equals $\mathbb{R}^3$, is a Poisson manifold [34, 35].

Note, however, that in our case the Poisson tensor (25) is degenerate necessarily, as it is skew-symmetric and $N$ is three-dimensional here. Obviously at points of $N$ where

$$X^- = X^+ = W = 0$$

(28)

$\mathcal{P}$ has rank zero – we will call these points ‘critical’ further on –, everywhere else it has rank two. Furthermore (as a consequence of the Jacobi identity for $\mathcal{P}$) in the neighbourhood of generic (i.e. non-critical) points there exists a foliation of $N$ into two-dimensional submanifolds $\mathcal{S}$, which are integral manifolds of the set of Hamiltonian vector fields (we will label them with the coordinate function $\tilde{X}^1$). Clearly, these leaves $\mathcal{S}$ are symplectic (the restriction of $\mathcal{P}$ onto them is nondegenerate), and one may choose coordinates $\tilde{X}^2, \tilde{X}^3$ such that on each of the leaves $\mathcal{P}|_{T^*\mathcal{S}}$ (or its inverse) is in Darboux-form. In such an adapted coordinate system $\tilde{X}^i$, which we will call Casimir-Darboux (CD) coordinate system $\mathcal{P} \in \Lambda^2(TN)$ takes the simple form $\mathcal{P} = \frac{\partial}{\partial \tilde{X}^2} \wedge \frac{\partial}{\partial \tilde{X}^3}$.

The notation of (24) allows to derive and depict the gravity field equations in a concise manner:

$$dX^i + \mathcal{P}^{ij} A_j = 0$$

(29)

$$dA_i + \frac{1}{2} \frac{\partial \mathcal{P}^{lm}}{\partial X^i} A_l \wedge A_m = 0$$

(30)

But what is more important, in order to solve these equations of motion, the considerations above suggest the use of CD coordinates on the target space $N$. As $L$ is written in an $N$-covariant manner, the field equations will still have the form (29, 30), only now $\mathcal{P}$ is in Casimir-Darboux form. Explicitly this reads

$$d\tilde{X}^1 = 0, \quad dA_{\tilde{1}} = 0$$

(31)

$$A_2 = d\tilde{X}^3, \quad A_3 = -d\tilde{X}^2,$$

(32)

while the remaining two field equations $dA_2 = dA_3 = 0$ are redundant obviously. In this form the solution of the field equations becomes a triviality: Locally (31) is equivalent to

$$\tilde{X}^1 = \text{const}, \quad A_{\tilde{1}} = df$$

(33)

\footnote{In some textbooks (cf., e.g., [35]) such coordinates are called simply ‘Darboux coordinates’; however, we prefer the more suggestive term above.}
where \( f \) is some arbitrary function on \( M \), while (32) determines \( A_2 \) and \( A_3 \) in terms of the otherwise unrestricted functions \( \tilde{X}^2, \tilde{X}^3 \).

Now we have to transform this solution back to the gravity variables (23) only. Let us do this for the torsion-free case \( W = V(X^3)/2 \) first, Eqs. (14, 12). Here

\[
\tilde{X}^i := \left( \frac{1}{2} \left[ (X)^2 - \int V(z)dz \right], \ln |X^+|, X^3 \right)
\]

forms a CD coordinate system on \( N \) on patches with \( X^+ \neq 0 \). To verify this we merely have to check \{\( \tilde{X}^1, \cdot \}_N \neq 0 \) and \{\( \tilde{X}^2, \tilde{X}^3 \}_N = 1 \, \) using the definition (25, 27) of the brackets. From \( A_i = \frac{\partial \tilde{X}^j}{\partial X^i} \) we then infer

\[
e^+ \equiv A_- = X^+ A_1, \quad e^- \equiv A_+ = \frac{1}{X^+} A_2 + X^- A_1.
\]

By means of (32, 33) the metric \( g = 2e^+e^- \) thus becomes:

\[
g = 2dx^3df + (X)^2 df df,
\]

where \( (X)^2 = \int X^3 V(z)dz + const =: h(X^3) \) according to (34) and (33). Using \( X^3 \) and \( f \) as coordinates \( x^0 \) and \( x^1 \) on \( M \), this may be rewritten as

\[
g = 2dx^0dx^1 + h(x^0)dx^1dx^1
\]

with the function \( h \) as defined above. In the case of \( R^2 \)-gravity (21), e.g.,

\[
h^{R^2} = -\frac{2}{3}(x^0)^3 + 2\Lambda x^0 + C,
\]

where \( h \) depends on one integration constant only, which is a specific function of the total mass (at least in cases where the latter may be defined in a sensible way), reobtaining what has been called ‘generalized Birkhoff theorem’ in various special cases (cf. [16, 36]).

Maybe at this point it is worth mentioning that the coordinate transformation

\[
r := x^0, \quad t := x^1 + \int^{x^0} \frac{dz}{h(z)},
\]

well-defined wherever \( h \neq 0 \), brings the generalized Eddington-Finkelstein form of the metric, Eq. (37), into the ‘Schwarzschild form’

\[
g = h(r)(dt)^2 - \frac{1}{h(r)}(dr)^2,
\]

with the same function \( h \). This confirms also that the Schwarzschild case (with positive \( m \)) may be described by (17) also with the identification \( g = 2e^+e^- \): The potential \( W = V/2 = 1/(X^3)^2 \) yields \( h(r) = C - 1/r, \)

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of the shapes of the Penrose diagrams are different. Eqs. (40,42) and \( \tilde{g} \) at implications on the global structure of the resulting theory, namely if due to a divergent conformal factor the domain of \( g \) behaved for \( m \) leading to the identification \( X \) Although of the local charts obtained above (cf., e.g., Part II), is of Schwarzschild-type. Let us illustrate this by means of the dilaton theory (8). The re transformation \( x \) above coordinate transformation \( x \) brings \( g \) into the form (37) again, where now \( h(x^0) = e^{-\rho(x^0)\tilde{h}(x^0(x^0))} \). From Eqs. (11,11) we have \( \tilde{g} = \exp[-\rho(D^{-1}(x^0))]\tilde{g} \). Here the coordinate transformation \( x^0(x^0) := \int x^0 \exp[-\rho(D^{-1}(x^0))]dx \) (41) brings \( g \) into the form (37) again, where now \( h(x^0) = e^{-\rho(x^0)\tilde{h}(x^0(x^0))} \). Let us illustrate this by means of the dilaton theory (8). There \( \tilde{h}^{\text{dil}}(x^0) = \Lambda \tilde{x}^0 + C \), \( C \) denoting the integration constant, and \( \tilde{x}^0 = X^3 = D(\Phi) = \frac{1}{8} \Phi^2 = \exp(-2\phi) \in \mathbb{IR}^+ \). As \( \rho = \ln \tilde{x}^0 \), Eq. (11) becomes \( g = \tilde{g}/\tilde{x}^0 \). The above coordinate transformation \( x^0 = \ln \tilde{x}^0 \) yields

\[
\tilde{h}^{\text{dil}}(x^0) = \Lambda + C \exp(-x^0) \tag{42}
\]

with \( x^0 \in \mathbb{IR} \). For (8) the analogous procedure yields \( h^{\text{SS}}(x^0) = 1 + 2C/x^0 \), leading to the identification \( m = -C \) in this case (cf. Eqs. (13) and (8)).

The transition from \( \tilde{g} \) to \( g \), although conformal, may have important implications on the global structure of the resulting theory, namely if due to a divergent conformal factor the domain of \( g \) is only part of the maximally extended domain of \( \tilde{g} \). For instance, in the dilaton theory the maximal extension of \( \tilde{g} \) is Minkowski space with its diamond-like Penrose-diagram, whereas the Penrose diagram of \( g \), found by studying the universal coverings of the local charts obtained above (cf., e.g., Part II), is of Schwarzschild-type. Although \( X^3 = \tilde{x}^0 \) was defined for positive values only, \( \tilde{g} \) remains well-behaved for \( \tilde{x}^0 \in \mathbb{IR} \) and may be extended to that values. The conformal factor in the relation between \( g \) and \( \tilde{g} \), \( 1/\tilde{x}^0 \), on the other hand, blows up at \( \tilde{x}^0 = 0 \). Correspondingly \( R(g) \) is seen to diverge at \( \tilde{x}^0 \equiv \exp x^0 = 0 \), cf. Eqs. (40,42) and \( g \) cannot be extended in the same way as \( \tilde{g} \). As a result the shapes of the Penrose diagrams are different.

Such a behavior is generic also in the case that (11) maps \( \Phi \in \mathbb{IR} \) to a part of \( \mathbb{IR} \) only, say, e.g., to \( (a, \infty) \) with an increasing \( D \). The conformal exponent

\[
C = \text{const}, \text{ from which one finds } m = C^{-\frac{3}{2}}, \text{ after rescaling coordinates according to } r \rightarrow \sqrt{C} r \text{ and } t \rightarrow t/\sqrt{C} \text{ (} C > 0 \). \text{ In fact, we learn that given any metric } g \text{ in (effectively) } 1+1 \text{ dimensions with (at least) one Killing field, (7) with } g = 2e^\pm e^- \text{ and } 2W := V(X^3) = h'(X^3) \text{ will provide an action which has } g \text{ within its space of solutions.} \text{ We remark, finally, that in the present case of a torsion-free connection the Ricci scalar is just:}

\[
R = h''(x^0). \tag{40}
\]

For a reformulated dilaton theory (8) Eq. (37) gives \( \tilde{g} \) only, which we will write as \( \tilde{g} = 2d\tilde{x}^0 dx^1 + \tilde{h}(\tilde{x}^0)(dx^1)^2 \) with \( \tilde{h}(\tilde{x}^0) \equiv \int \tilde{x}^0 V(z)dz + \text{const.} \), from which one finds

\[
\tilde{h}^{\text{dil}}(x^0) = \Lambda \tilde{x}^0 + C, \text{ } C \text{ denoting the integration constant, and } \tilde{x}^0 = X^3 = D(\Phi) = \frac{1}{8} \Phi^2 = \exp(-2\phi) \in \mathbb{IR}^+ \]. As \( \rho = \ln \tilde{x}^0 \), Eq. (11) becomes \( g = \tilde{g}/\tilde{x}^0 \). The above coordinate transformation \( x^0 = \ln \tilde{x}^0 \) yields

\[
\tilde{h}^{\text{dil}}(x^0) = \Lambda + C \exp(-x^0) \tag{42}
\]

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The transition from \( \tilde{g} \) to \( g \), although conformal, may have important implications on the global structure of the resulting theory, namely if due to a divergent conformal factor the domain of \( g \) is only part of the maximally extended domain of \( \tilde{g} \). For instance, in the dilaton theory the maximal extension of \( \tilde{g} \) is Minkowski space with its diamond-like Penrose-diagram, whereas the Penrose diagram of \( g \), found by studying the universal coverings of the local charts obtained above (cf., e.g., Part II), is of Schwarzschild-type. Although \( X^3 = \tilde{x}^0 \) was defined for positive values only, \( \tilde{g} \) remains well-behaved for \( \tilde{x}^0 \in \mathbb{IR} \) and may be extended to that values. The conformal factor in the relation between \( g \) and \( \tilde{g} \), \( 1/\tilde{x}^0 \), on the other hand, blows up at \( \tilde{x}^0 = 0 \). Correspondingly \( R(g) \) is seen to diverge at \( \tilde{x}^0 \equiv \exp x^0 = 0 \), cf. Eqs. (40,42) and \( g \) cannot be extended in the same way as \( \tilde{g} \). As a result the shapes of the Penrose diagrams are different.

Such a behavior is generic also in the case that (11) maps \( \Phi \in \mathbb{IR} \) to a part of \( \mathbb{IR} \) only, say, e.g., to \( (a, \infty) \) with an increasing \( D \). The conformal exponent

\[\text{This holds because any metric with a Killing field } v \text{ may be brought into the form (37) locally. Coordinate independently } h \text{ may be characterized as the norm squared of } v \text{ as a function of an affine parameter along a null-line, furthermore. For more details cf. Part II.}\]
\( \rho (11) \) will diverge at \( \tilde{x}^0 = a \), as \( D \) was required to be a diffeomorphism, and for potentials \( U \) that do not diverge too rapidly for \( \phi \rightarrow -\infty \) the Penrose diagrams of \( \tilde{g} \) and \( g \) differ.

Let us now discuss the solution to the field equations for a general potential \( W \) in (17), using this opportunity to present also a more systematic construction of a CD-coordinate system.

By definition a Casimir coordinate \( \tilde{X}^1 \) is characterized by the equation

\[
\mathcal{P}(d\tilde{X}^1, \cdot) = 0 \iff (\partial \tilde{X}^1/\partial X^i)\mathcal{P}^{ij} = 0.
\]

For \( j = 3 \) the latter implies that \( \tilde{X}^1 \) has to be a Lorentz invariant function of \( X^\pm \), i.e.

\[
\tilde{X}^1 = \frac{1}{2} C \left[ (X^2)^2, X^3 \right] \quad (43)
\]

for some two-argument function \( C = C(u, v) \). For \( j = \pm \) we then obtain

\[
2W(u, v)C_{,u} + C_{,v} = 0 \quad (44)
\]

where the comma denotes differentiation with respect to the corresponding argument of \( C \). As (44) is a first order differential equation, it may be solved for any given potential \( W \) locally, illustrating the general feature of a local foliation of Poisson manifolds for the special case (23). An important consequence of (44) is the relation \( C_{,u} \neq 0 \). This follows as on the target space (!) we have \( dC \neq 0 \) (by definition of a target space coordinate function), and, according to Eq. (17), \( C_{,u} = 0 \) at some point implies that there also \( C_{,v} = 0 \) and thus \( dC = 0 \). Certainly, as \( C_{,u} \) is a function on \( N \), in contrast to \( dC \) it remains non-zero also upon restriction to the submanifolds \( C = \text{const} \).

We may verify this also explicitly at the torsion-free example: (34) yields \( C_{,u}(u, v) \equiv 1 \).

Using the method of characteristics, (44) may be reduced to an ordinary first order differential equation: We may express the lines of constant values of the function \( C \) in the form

\[
\frac{du}{dv} = 2W(u, v) \quad .
\]

The constant of integration of this equation is a function of \( C \) in general; however, as clearly any function of a Casimir is a Casimir again, we may just identify the integration constant with \( C \). To illustrate this, we choose

\[
2W(u, v) := V(v) + T(v)u \quad .
\]

We then obtain from (13)

\[
u = \left[ \int^v V(z) \exp \left( - \int^z T(y)dy \right) dz + \text{const}(C) \right] \exp \left( \int^v T(x)dx \right),
\]

\[
(47)
\]
where the lower boundaries in the integrations over $T$ coincide. Upon the choice $\text{const}(C) := C$ (47) gives

$$C(u, v) = u \exp \left( -\int T(x)dx \right) - \int u \exp \left( -\int T(y)dy \right) dz.$$ (48)

The integrations on the right-hand side should be understood as definite integrals with somehow fixed, $C$-independent lower boundaries. Different choices for these boundaries rescale $C$ linearly.

Let us specialize (48) to some cases of particular interest. In the case $T \equiv 0$, describing torsionless gravity and discussed already above, it gives

$$C = (X)^2 - \int X^3 V(z) dz ,$$ (49)

in coincidence with the first entry of (34). For the Katanaev-Volovich (KV) model of 2D gravity with torsion (22), as a second example, (48) and (21) yield upon appropriate choices for the constants of integration and a rescaling by $\alpha^3$

$$C_{KV} = -2\alpha^2 \exp(\alpha X^3) \left(W_{KV} + \frac{2X^3}{\alpha} - \frac{2}{\alpha^2} \right).$$ (50)

Here $C, u = \alpha^3 \exp(\alpha v)$.

The remaining task is to find Darboux coordinates. On patches with either $X^+ \neq 0$ or $X^- \neq 0$ this is a triviality almost: According to the defining relations (25) of the Poisson brackets we have $\{X^\pm, X^3\}_N = \pm X^\pm$, so obviously $\pm \ln |X^\pm|$ and $X^3$ are conjugates. Altogether therefore

$$\tilde{X}^i := (\frac{1}{2} C, \pm \ln |X^\pm|, X^3)$$ (51)

forms a CD coordinate system on regions of $N$ with $X^\pm \neq 0$, respectively.

So, now we just have to repeat the steps following Eq. (34) in the more general setting of a potential $W(u, v)$. However, as we do not want to restrict ourselves to potentials e.g. purely linear in $u$, Eq. (50), we do not know $C$ in explicit form. Still it is nice to find that also in the case of a completely general Lagrangian (17) the metric takes the form (37) locally. Moreover, $h$ may be determined in terms of the Casimir function $C(u, v)$ and the Killing field $\partial/\partial_1$ will be shown to be a symmetry direction of all of the solution. For the sake of brevity we display the calculation for both of the sets (51) of CD-coordinates simultaneously. This means that in the following we restrict our attention to local solutions on $M$ that map into regions of $N$ with $X^\pm \neq 0$ or $X^- \neq 0$.

This is not a triviality as, e.g., for nonvanishing torsion the connection $\omega$ is not determined (up to Lorentz transformations) by the metric $g$ already.
Inserting into the analogue of Eqs. (35) the solutions (32, 3 3) and re-expressing thereby the $\tilde{X}^i$ in terms of the original fields $X^i$ again, (51), we obtain (the upper/lower signs being valid for the charts $X^+$ resp. $X^- \neq 0$)

$$
eq \equiv A^\pm = C_{u^i} X^\pm df$$
$$e^\mp \equiv A_\pm = \pm \frac{1}{X^\pm} dX^3 + C_{u^i} X^\mp df$$
$$\omega \equiv A_3 = \mp d\ln |X^\pm| - C_{u^i} W df \ .$$

where in the last line we used (44). For the metric $g = 2e^+e^-$ this yields

$$g = \pm 2C_{u^i} dX^3 df + \tilde{h}(X^3, C) df df \ ,$$

with

$$\tilde{h}(X^3, C) := C_{u^i} \cdot (X)^2 \ .$$

Here $(X)^2$ is a function of $X^3$ and $C$ by inverting the field equation $C((X)^2, X^3) = const$, whereas $C_{u^i}$ in (54) and (53) is, more explicitly, $C_{u^i} ((X)^2(X^3, C), X^3)$. Note that according to the context $C$ either denotes a function of $X^i$ or the constant which it equals due to the first equation (31).

Now again we want to fix a gauge. From (53) we learn that $C_{u^i} dX^3 \wedge df \neq 0$, because otherwise det$g = 0$. This implies that we may choose $C_{u^i} dX^3$ and $df$ as coordinate differentials on $M$. Let us therefore fix the diffeomorphism invariance of the underlying gravity theory by setting

$$\int^{X^3} C_{u^i} [(X)^2(z, C), z] dz := x^0 \ ,$$

$$f := \pm x^1 \ .$$

In this gauge $g$ is seen to take the form (37) again with

$$h(x^0) = \tilde{h}(X^3(x^0), C) \ ,$$

where $X^3(x^0)$ denotes the inverse of (55). The local Lorentz invariance may be fixed by means of

$$X^\pm := \pm 1 \ ,$$

finally. Besides (58, 55) the complete set of fields then takes the form

$$e^\pm = C_{u^i} dx^1 \ , \quad e^\mp = \frac{dx^0}{C_{u^i}} + \frac{1}{2} (X)^2 C_{u^i} dx^1 \ , \quad \omega = \mp W C_{u^i} dx^1 \ ,$$

and $X^\pm = \pm \frac{1}{2} (X)^2$. Here again $(X)^2$, $C_{u^i}$, and $W$ depend on $X^3(x^0)$ and $C = const$ only, $C$ being the only integration constant left in the local solutions.
In the torsion-free case, considered already above, \( C_{\nu u} = 1, X^3(x^0) = x^0 \), and, cf. Eqs. (54, 57, 49), \( (X)^2 = h(x^0) = \int x^0 V(z)dz + C \), thus reproducing the results obtained there. For the KV-model \( \| \) as an example for a theory with torsion, on the other hand, the above formulas yield \( x^0 = \alpha^2 \exp(\alpha X^3), x^0 \in \mathbb{R}^+ \), and

\[
\begin{align*}
h^{KV}(x^0) &= \frac{1}{\alpha} \left\{ C x^0 - 2x^0 \left[ (\ln x^0 - 1)^2 + 1 - \Lambda \right] \right\} \\
&= \frac{1}{\alpha} \left\{ C x^0 - 2\left( x^0 \right)^2 \left[ (\ln x^0 - 1)^2 + 1 - \Lambda \right] \right\}
\end{align*}
\]

for instance. Certainly \( \| \) does not hold any more, instead we find \( R = -\frac{4}{\alpha \ln \left( \frac{x^0}{\alpha^2} \right) } \). Here this is obtained most easily by concluding \( R = -4X^3 \) from (the Hodge dual of) the three-component of \( \| \).

In the above we captured the solutions within regions of \( M \) where either \( X^+ \neq 0 \) or \( X^- \neq 0 \). Clearly, in regions where \( (X)^2 = 2X^+X^- \neq 0 \) the two charts \( \| \) must be related to each other by a gauge transformation, i.e., up to a Lorentz transformation, by a diffeomorphism. However, one of these two charts extends smoothly into regions with only \( X^+ \neq 0 \) (but possibly with zeros of \( X^- \)), and \( (+ \leftrightarrow -) \) for the other chart. In this way the above mentioned diffeomorphism may serve as a gluing diffeomorphism, allowing to extend the generically just local solution \( \| \) to one that applies wherever \( X^+ \) and \( X^- \) do not vanish simultaneously. Let us remark here, furthermore, that the two representatives \( \| \) are mapped into each other by

\[
\begin{align*}
e^+ &\leftrightarrow e^-, \quad \omega \leftrightarrow -\omega, \quad X^+ \leftrightarrow -X^-, \quad X^3 \leftrightarrow X^3
\end{align*}
\]

This transformation reverses the sign of the action integral \( \| \) only and therefore does not affect the equations of motion. From \( \| \) it is obvious that the gluing diffeomorphism (cf. above) maps one set of null lines onto the respective other one, leaving the form \( \| \) of \( g = 2e^+e^- \) unchanged. It corresponds to a discrete symmetry of \( \| \) (called ‘flip’ in Part II), which is independent from the continuous one generated by \( \partial/\partial_1 \). Further details shall be provided in Part II.

We are left with finding the local shape of the solutions in the vicinity of points on \( M \) that map to \( X^+ = X^- = 0 \). Here we have to distinguish between two qualitatively different cases: First, \( C = C_{\text{crit}} \equiv C(0, X^3_{\text{crit}}) \), where \( X^3_{\text{crit}} \) has been chosen to denote the zeros of \( W(0, X^3) \), and second, \( C \neq C_{\text{crit}} \). The special role of the critical values of \( C \) is due to the fact that the Poisson structure \( \mathcal{P} \) vanishes precisely at the points \( X^+ = X^- = 0 \) on the two-surfaces \( C = C_{\text{crit}}, \) cf. Eq. \( \| \).

We start treating the non-critical case \( C \neq C_{\text{crit}} \): For this, one could attempt to construct a CD coordinate system valid in a neighborhood of a

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\[\text{In Part II regions with } (X)^2 \neq 0 \text{ will be called ‘sectors’, while patches with merely } X^+ \neq 0 \text{ or } X^- \neq 0, \text{ generically containing several sectors, will be our ‘building blocks’ for the global extension.}\]
(non-critical) point $X^+ = X^- = 0$. At least in the torsionless case this may be done in an explicit way, but while e.g. for ordinary dilaton gravity ($W \equiv \text{const}$) Darboux coordinates are provided already by rescaling merely $X^+$ and $X^-$ (since $\{X^+, X^-\}_N = W$), in the more general case the formulas are somewhat cumbersome. We therefore follow a somewhat less systematic, but simpler route here: From the three-component of \eqref{29} it is straightforward to infer that points with $X^+ = X^- = 0$ are saddle points of $X^3$. This suggests to replace the gauge conditions \eqref{55,56} by an ansatz of the form

$$\int^3 C_{u a} [(X)^2(z; C), z] dz := xy + a,$$  \hspace{1cm} \text{(62)}

$$f := b \ln x,$$  \hspace{1cm} \text{(63)}

where $a$ and $b$ are constants to be determined below. As a first justification of \eqref{62,63} we find $C_{u a} dX^3 \wedge df$ to be finite and non-vanishing on $(x, y) \in \mathbb{R}^2$. Implementing the above conditions in \eqref{53}, the metric becomes

$$g = 2b dx dy + b \frac{2xy + bh(xy + a)}{x^2} (dx)^2,$$  \hspace{1cm} \text{(64)}

where $h$ is the function defined in \eqref{57}. For generic values of $a$ and $b$ \eqref{64} is singular at $x = 0$. However, the choice

$$a := h^{-1}(0) \quad b := -2/h' \left(h^{-1}(0)\right)$$  \hspace{1cm} \text{(65)}

is seen readily to yield a smooth $g!$

The singularity of the gauge choice \eqref{53} at $x = 0$ was devised such that it compensated precisely the singularity of the CD coordinate system at $X^\pm = 0$, used to derive \eqref{53}. Indeed, the lines of vanishing $x$ or $y$ in \eqref{64} may be seen to correspond to lines of vanishing $X^+$ and $X^-$, respectively. Also, they are Killing horizons: According to \eqref{54} zeros of $(X)^2$ coincide precisely with the zeros of $h(x^0)$, indicating that the Killing-field $\partial/\partial x^1$ (in charts \eqref{57}) becomes null on those lines (cf. also \eqref{59}). The charts \eqref{64} provide a simple alternative to a generalized Kruskal extension (cf. Part II). For Schwarzschild, $h(r) = 1 - 2m/r$, it is a global chart (as $h^{-1}(0)$ is single-valued), in the more complicated Reissner-Nordstrom case $h(r) = 1 - 2m/r + q^2/r^2$ the constant $a$ may take one of the two values $r_{\pm} = m \pm \sqrt{m^2 - q^2}$ and \eqref{64} provides a local chart in the vicinity of the respective value of $r = x^0$. A generalization of the right-hand sides of \eqref{62} and \eqref{63} to $F(x)y + G(x)$ and $\int^x dz/F(z)$, respectively, with appropriate functions $F$ and $G$, allows even for global charts of (two-dimensional) spacetimes of the form $\mathbb{R} \times \{\text{null-lines}\}$. For more details on this confer \cite{37}, where also a more systematic approach to these charts is presented, illustrating the considerations by Schwarzschild and Reissner-Nordstrøm.
For (64) to exist it is decisive that the corresponding zero \( a \) of \( h \) is simple, cf. the second Eq. (65). For non-critical values of \( C \), e.g., all zeros of \( h \) are simple. This is particularly obvious for torsionless theories, where \( h = 2X^+X^- \) and \( h'(X^3) = V(X^3) \), but holds also in general. If \( C \in \{C_{\text{crit}}\} \), on the other hand, there exist zeros of \( h \) of higher degree. Then the spacetime manifolds \( M \) with varying \( X^i \) do not contain the critical points \( X^i = (0,0,X^3_{\text{crit}}) \). This may be seen in two different ways: First, studying extremals running towards such a point, one finds the point to be infinitely far away, cf. Part II. Second, from the field equations point of view: Taking successive derivatives \( \partial/\partial x^\mu \) of the Eqs. (29) and evaluating them at the critical points, we find

\[
X^- \equiv 0, \quad X^+ \equiv 0, \quad X^3 \equiv X^3_{\text{crit}} = \text{const.} \quad (66)
\]

(66) corresponds to additional, separate solutions of the field equations not treated before. Actually, they come as no surprise. More or less by definition the critical points (25) of the target space constitute zero dimensional symplectic leaves. It is a general feature of Poisson \( \sigma \)-models, verified here explicitly in (66) and the first Eq. of (33), that the image \( X(x) \) of the map from the worldsheet or space time \( M \) into the target space \( N \) has to lie entirely within a symplectic leaf \( S \subset N \).

The remaining field equations (30), which are, more explicitly,

\[
De^a = 0, \quad d\omega = -W_{uv}(0,X^3_{\text{crit}})\varepsilon, \quad (67)
\]

show that the solutions (67) have vanishing torsion and constant curvature all over \( M \). The metric for such a solution can be brought into the form (57), too, with \( h(x^0) = W_{uv}(0,X^3_{\text{crit}}) \cdot (x^0)^2 + 1 \). This in turn determines the zweibein and spin connection up to Lorentz transformations.

4 Summary and Extension to Gravity-Yang-Mills

We demonstrated that any of the 2D gravity models introduced in Section 4 is of Poisson \( \sigma \)-form (24). Exploiting some fundamental facts of Poisson structures, namely the local existence of what we called Casimir-Darboux coordinates, the field equations reduced to (31,32), the solution of which is immediate. The relation between the original field variables and the transformed ones, Eq. (32), provided the general solution in terms of the metric then. With the choice of a gauge the latter took the form (57), where \( h \)

---

9 The employed method may be viewed as a perfected generalization of what has been done previously in ordinary dilaton theory [28] or the KV-model [38]; it is, however, not inspired by that works, but self-evident from the Poisson \( \sigma \) point of view. Let us note on this occasion that with appropriate gauge conditions the field equations of (14) may be solved in a maybe less elegant, but almost as straightforward manner, too (cf. also [3]).
was parametrized by a single meaningful constant (the value of the Casimir-function $C$).

(37) provided the local solution on strips with either $X^+ \neq 0$ or $X^- \neq 0$. For non-critical values of the Casimir constant $C$ (guaranteeing $h'|_{h=0} \neq 0$) the metric could be brought into the form (65) in the vicinity of points $X^+ = X^- = 0$. For critical values of $C$, finally, we obtained the deSitter solutions (67) in addition to the solutions (37) (which in this case may not be extended to points of simultaneous zeros of $X^+$ and $X^-$).

The Poisson $\sigma$-model formulation (24) of 2D gravity theories provides the proper generalization of Yang-Mills (YM) gauge theories advocated in the introduction. Actually (24) is able to describe 2D YM-theories with arbitrary gauge group. Identifying the target space $N$ of a Poisson $\sigma$-model with (the dual of) some Lie algebra with structure constants $f^{ijk}$ and defining $P_{ij} := f^{ijk}X^k$, the action (24) is seen to become

$$\int X^i F_i$$

(68)

after a partial integration, where $F = dA + A \wedge A$ is the standard Lie algebra valued YM-curvature two-form. The local symmetries of the BF-YM action (68) are the standard ones: $A \rightarrow g^{-1}Ag + g^{-1}dg$, $X \rightarrow g^{-1}Xg$. The symmetries of the general model (24) are a straightforward, nonlinear generalization of this:

$$\delta_\epsilon X^i = \epsilon_j(x)P_{ji}, \quad \delta_\epsilon A_i = d\epsilon_i + P^{jk}iA_j\epsilon_k.$$  

(69)

These symmetries are the Lagrangian analogues of what is generated by the constraints in a Hamiltonian formulation (cf. Part IV). In the present context of (17), where $N = \mathbb{R}^3$ with Poisson bracket (25), the symmetry transformations (69) are equivalent to diffeomorphisms and local Lorentz transformations on-shell. This equivalence holds only under the assumption of non-degenerate metrics $g = 2A_+A_-$, however, a feature shared also by the Ashtekar formulation of 3+1 gravity. We will see in Part IV (cf. also [23]) how this seemingly irrelevant restriction will lead to different factor spaces (even if chosen representatives of gauge equivalence classes are restricted to non-degenerate metrics $g$).

With the addition of one more term not spoiling the symmetries (69) the model (24) is capable also of describing YM actions $\int F \wedge *F$ [14, 11] or even G/G gauged WZW theories [10] (cf. [32] for a pedagogical exposition). Of more interest for our present intentions are, however, dynamically coupled 2D gravity-YM-systems. Allowing for a dilaton- and also $(X)^2$-dependent coupling constant $\alpha((X)^2, X^3)$, the action for such a system has the form

$$L^{gravYM} = L^{grav} + L^{YM}$$

$$L^{YM} = \int \frac{1}{4\alpha((X)^2, X^3)}tr(F \wedge *F),$$

(70)  

(71)
where the trace is taken in some matrix representation of the chosen Lie algebra and $\ast$ is the Hodge dual operation with respect to the dynamical metric $g = 2e^+ e^-$ of $L^{grav}$, Eq. (17). The special case with an abelian YM-part (gauge group $U(1)$) and with $\alpha$ and $W$ depending on $X^3$ only has received some attention in [41] recently (but cf. also [38, 42]). Let us show in the following that the general combined system (70) is of Poisson $\sigma$-form again! As a byproduct many of the results of [41] may be obtained as a lemma to the general theory of Poisson $\sigma$-models (including an exact Dirac quantization, cf. Part IV as well as [12, 3, 14]).

To begin with we bring (71) into first order form:

$$L^{YM} \sim L^{YM'} = \int (E^i F_i + \alpha((X)^2, X^3) e^i e_i)$$

where the indices $i$ are raised and lowered by means of the Killing metric and $\varepsilon \equiv e^- \land e^+$. The equivalence of $L^{YM}$ with $L^{YM'}$ is seen by integrating out the ‘electrical’ fields $E$ (either on the path integral level or just by implementing the equations of motion for the $E_i$ back into the action, in complete analogy to how we obtained (22) from (17)). To avoid notational confusion let us rename $X^\pm, X^3$ into $\phi^\pm, \phi$ and denote the YM-connection by a small letter $a$. Then $L^{grav} + L^{YM'}$ reads

$$L^{gravYM'} = \int \phi^a D e_a + \phi d \omega + E^j F_j + \left[ W((\phi)^2, \phi) + \alpha((\phi)^2, \phi) E^j E_j \right] \varepsilon,$$

where $F_j \equiv da_j + f^{klj} a_k \land a_l$ and the indices $a$ run over $+$ and $-$ while the indices $j$ run from 1 to $n$, $n$ being the dimension of the chosen Lie group. After partial integrations (dropping the corresponding surface terms) and the identifications

$$X^i := (\phi^a, \phi, E^j), \quad A_i := (e_a, \omega, a_j),$$

(73) becomes of Poisson $\sigma$-form (24) on an $n + 3$-dimensional target space with Poisson brackets:

$$\{\phi^+, \phi^-\} = W + \alpha E^j E_j, \quad \{\phi^\pm, \phi\} = \pm \phi^\pm, \quad \{\phi^\pm, E^j\} = 0 = \{\phi, E^j\}, \quad \{E^j, E^k\} = f^{jk} E^l.$$

As any Poisson tensor also the one defined in (73) (note $P^{ij} \equiv \{X^i, X^j\}$) allows for Casimir-Darboux coordinates in the neighbourhood of generic points. If the rank of the chosen Lie algebra is $r$ then the rank of the Poisson tensor is $n - r + 2$ and there will be $r + 1$ such Casimir coordinates. Correspondingly there will be $r + 1$ field equations (31) and $n - r + 2$ field equations (32). In the CD-coordinates the symmetries (69) take a very simple form and it is a triviality to realize that again the local solutions are
parametrized by a number of integration constants which coincides with the
number of independent Casimir functions. Note that in the present con-
text (69) entail diffeomorphisms, Lorentz transformations, and non-abelian
gauge transformations all at once. So, more or less without performing any
calculation, we obtain the result that the local solutions are parametrized
by \( r + 1 \) constants now. In the case of \[ n = 1, r = 1, \text{ and } r + 1 = 2. \] (Here
we ignored additional exceptional solutions of the type (50), corresponding
to maps into lower dimensional symplectic leaves).

The Poisson structure (75) has a very particular form: First the \( E^j \) span
an \( n \)-dimensional Poisson submanifold of \( N = \mathbb{R}^{n+3} \). Second, the Poisson
brackets between the gravity coordinates \((\varphi^a, \phi)\) and the coordinates \( E^i \)
of this submanifold vanish. And, last but not least, the Poisson brackets
between the ‘gravity’-coordinates close among themselves up to a Casimir
function of the \( E \)-submanifold. With this observation it is a triviality to
infer the local form of \( g \) of \( \text{L}^{\text{gravYM}} \) from the results of Section 3: On-shell
\( E^j E_j \) is some constant \( C^E \). So in all of the formulas of Section 3 we merely
have to replace \( W \) by \( W + C^E \alpha \). Thus the metric \( g \) again takes the form
(57) locally, where now \( h \) is parametrized by \( \text{two constants } C^E \) and \( C^{\text{grav}} \)
(‘generalized Birkhoff theorem for 2D gravity-YM-systems’). For torsionless
theories, e.g.,

\[
h(x^0) = \int_c^{x^0} V(z)dz + \frac{1}{2}C^E \int_c^{x^0} \alpha(z)dz + C^{\text{grav}},
\]

where \( c \) is some fixed constant (and again \( 2W(u, v) = V(v) \)).

The addition of a YM-part still rendered a theory with finitely many
‘physical’ degrees of freedom. There is at least one generalization of the
gravity action (17) that yields a (classically) solvable theory with an \( \text{infinite} \)
number of degrees of freedom. This is obtained when coupling fermions of
one chirality to (17) [43], as was observed first in [44] in the context of the
KV-model (22).

On the level of local considerations the models introduced in Section 2
all look quite alike. They all have a one-parameter family of solutions of the
form (37) (in the neighbourhood of generic points). As will be seen in Parts
II,III of this work, this changes drastically, if one turns to global considera-
tions. The richness and complexity of the global solutions is encoded in the
kind in which the target space \( N = \mathbb{R}^3 \) foliates (more precisely ‘stratifies’) into symplectic leaves. E.g., a potential \( V \) in (17) with many zeros will
lead to a topologically complicated stratification of \( N \) into two- and zero-
dimensional symplectic submanifolds. Correspondingly, as we will find in
Part III, there will exist smooth solutions on space-times \( M \) with relatively
complicated topologies. If, on the other hand, \( V \) has no zeros (such as in
the JT- or the ordinary dilaton theory), the foliation (stratification) of \( N \)
is quite simple and the most complicated topologies of smooth space-time
solutions $M$ occurring are cylinders and Möbius strips.

On the global level the solutions will be parametrized no longer by a single parameter only (or, in the context of (70), no longer by $r + 1$ parameters). Instead there will be $m + 1$ (resp. $(m + 1)$ times $(r + 1)$) such continuous parameters labeling them (in addition to further discrete parameters), where $m$ is the number of independent non-contractible loops on $M$. Here it should be noted that given some particular model (via specifying $V$ or $W$) there will be solutions with different topologies of $M$. Consequently the solution space (defined as the space of all possible globally smooth solutions modulo gauge transformations) of the chosen model will have different components differing in dimension.\footnote{Not to be mixed up with the dimension of $M$, which certainly always remains two. It is the fundamental group of a possible $M$ which determines the dimension of the respective component of the solution space.}

As at present there is no classification of the solution space of a general Poisson $\sigma$-model yet, in Parts II, III we will approach this issue in our specific gravity context from a more traditional, pedestrian point of view. First we will construct the universal coverings to the local solutions studied in this paper. Thereafter we then investigate the possible smooth factor solutions, keeping track of all arising parameters.

In Part IV, finally, we will turn to the quantum theory of (17) or (70) and compare the result to the classical solution space. It is in this quantum regime where the full power of the Poisson $\sigma$-formulation will become particularly apparent.

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