Fully Distributed Nash Equilibrium Seeking in $N$-Cluster Games

Yipeng Pang and Guoqiang Hu

Abstract—Distributed optimization and Nash equilibrium (NE) seeking problems have drawn much attention in the control community recently. This paper studies a class of non-cooperative games, known as $N$-cluster game, which subsumes both cooperative and non-cooperative nature among multiple agents in the two problems: solving distributed optimization problem within the cluster, while playing a non-cooperative game across the clusters. Moreover, we consider a partial-decision information game setup, i.e., the agents do not have direct access to other agents’ decisions, and hence need to communicate with each other through a directed graph whose associated adjacency matrix is assumed to be non-doubly stochastic. To solve the $N$-cluster game problem, we propose a fully distributed NE seeking algorithm by a synthesis of leader-following consensus and gradient tracking, where the leader-following consensus protocol is adopted to estimate the other agents’ decisions and the gradient tracking method is employed to trace some weighted average of the gradient. Furthermore, the algorithm is equipped with uncoordinated constant step-sizes, which allows the agents to choose their own preferred step-sizes, instead of a uniform coordinated step-size. We prove that all agents’ decisions converge linearly to their corresponding NE so long as the largest step-size and the heterogeneity of the step-size are small. We verify the derived results through a numerical example in a Cournot competition game.

Index Terms—Nash equilibrium (NE) seeking, distributed methods, non-cooperative games.

I. INTRODUCTION

The collaboration and competition among multiple decision makers are respectively modeled by distributed optimization and non-cooperative games, which have received great attention in recent researches, due to the wide application range in the fields of economy systems, robotics networks, sensor networks, wireless communication networks, power networks, etc. However, in some practical scenarios, the collaboration and competition among the decision makers may exist at the same time. For example, the work in [1] studied two healthcare networks in Taiwan, and demonstrated how the cooperation and competition coexists between interacting networks. In fact, the coexistence of cooperation and competition also takes place in many other fields, such as business management, transportation systems, political science, sports, to list a few. In such cases, pure distributed optimization or non-cooperative game model may not be able to fit the behaviors of the engaged parties. Hence, a model that can capture both the cooperative and competitive behaviors of the interacting participants is of great demand. Motivated by this, a zero-sum game among two subnetworks was formulated in [2], where two networks of agents cooperatively optimize an opposing objective, respectively. If the two-subnetwork setup is extended to $N$ subnetworks, then it is known as $N$-cluster game, where the agents in the same cluster cooperatively minize a cluster-level cost function, and collectively act as a virtual player to play an $N$-player non-cooperative game across clusters. This paper aims to propose a fully distributed NE seeking strategy for the $N$-cluster game under a partial-decision information game setup.

A. Literature Review

Simultaneous social cost minimization and NE seeking among two networks of agents have been studied in [2], where a distributed saddle-point strategy was developed to achieve the convergence to the NE for both undirected and directed graphs. This work was extended to a switching communication topology in [3], where the convergence to the NE was established under both weight-balanced and unbalanced uniformly jointly strongly connected digraphs. Recently, researches on $N$-cluster games have been reported in [4]–[6]. Specifically, the work in [4] proposed a NE seeking algorithm based on a dynamic average consensus and the gradient play. This work was followed-up in [5] to reduce the communication and computation costs by introducing an interference graph among the agents in the same cluster, which achieved the convergence to a neighborhood of the NE. Then, the $N$-cluster game was solved by an extremum seeking-based approach in [6], where both local and non-local convergence results were derived under both local and global characterizations of the NE, respectively. It should be mentioned that all the aforementioned works for $N$-cluster games assume that each agent has direct access to all other agents’ decisions both within and across the clusters, either by observation or via a virtual central coordinator, i.e., a full-decision information setting. This game setup can be restrictive, especially in the scenarios where there is no central node with bidirectional communication with all agents. Different from these works, the work in [7] studied a multi-cluster game, where the decisions of the agents in the same cluster are modeled by a decision vector that needs to be agreed on. Hence, a leader-following mechanism was employed to achieve the consensus in the same cluster. However, the decisions of the agents in other clusters are still assumed to be directly accessible. Distributed NE seeking under partial-decision information over graphs have started to draw researchers’ attention very recently, see [8]–[10] for unconstrained or locally set constrained games, [11], [12] for aggregative games, and [13], [14] for generalized games. However, the partial-decision information game setup has received little attention in $N$-cluster games. The main challenge of considering partial-decision information setup in

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\(N\)-cluster games is the complexity brought by the estimation of all agents’ decisions. To be more specific, each agent needs to maintain an estimate of all other agents both within and across the clusters, which brings in substantial complexity in the convergence analysis.

This paper proposes a fully distributed NE seeking algorithm for the \(N\)-cluster game, based on a synthesis of leader-following consensus and gradient tracking. The leader-following consensus protocol is commonly adopted to achieve the consensus of the estimated and true decisions, see also \[15\]–\[17\]. The gradient tracking method is usually employed to track some aggregate of the gradient information of the agents. As such information is closely related to the optimality condition, gradient tracking techniques are widely explored in distributed optimization problems, such as \[18\]–\[21\]. Even though the optimization in each cluster in \(N\)-cluster games is non-trivial due to the differences in the problem setups. One technical challenge is the game mapping condition, which has no direct connection to the distribution optimization. The non-doubly stochastic condition on the adjacency matrices also brings in additional technical difficulties in the convergence analysis, since the gradient tracker does not trace the average but a weighted average of the agents’ gradient with the weights given by the right eigenvector of the adjacency matrix. By constructing a linear dynamical system of certain norms of the NE gap, total gradient tracking error, and total decision estimation error, the linear convergence of all agents’ decisions to their corresponding NE is established so long as the largest step-size and the heterogeneity of the step-size are small.

### C. Notations

We use \(1_m\) for an \(m\)-dimensional vector with all elements being 1, and \(I_m\) for an \(m \times m\) identity matrix. For a vector \(\pi\), we use \(\text{diag}(\pi)\) to denote a diagonal matrix formed by the elements of \(\pi\). For any two vectors \(u, v\), their inner product is denoted by \(\langle u, v \rangle\); their weighted inner product due to a positive vector (i.e., \(v\) with all elements being positive) \(\pi\) is denoted by \(\langle u, v \rangle_\pi = u^\top \text{diag}(\pi)^{-1} v\). The transpose of \(u\) is denoted by \(u^\top\). Moreover, we use \(\|u\|_2\) for its standard Euclidean norm, \(\|u\|_\pi = \text{diag}(\sqrt{\pi})^{-1} u\) for its weighted Euclidean norm due to \(\pi\), i.e., \(\|u\|_\pi = \|\text{diag}(\sqrt{\pi})^{-1} u\|_2\). For vector \(a\), we use \([a]_i\) to denote its \(i\)-th entry. The transpose and spectral norm of a matrix \(A\) are denoted by \(A^\top\) and \(\|A\|_2\), respectively. The matrix norm \(\|A\|_\pi\) induced by \(\|\cdot\|_\pi\) is defined as \(\|A\|_\pi = \|\text{diag}(\sqrt{\pi})^{-1} A \text{ diag}(\sqrt{\pi})\|_2\). We use \(\rho(A)\) to represent the spectral radius of a square matrix \(A\), and \(A_\infty\) to indicate its infinite power (if it exists) \(\lim_{k \to \infty} A^k\).

### II. Problem Statement

An \(N\)-cluster game, defined by \(\Gamma(\mathcal{N}, \{J^i\}, \Omega^i)\), is a multi-player non-cooperative game played among \(N\) clusters, where each cluster, indexed by \(i \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}\), consists of a group of agents, denoted by \(\mathcal{N}^i \triangleq \{1, 2, \ldots, n_i\}\), to cooperatively minimize a cluster-level cost function \(J^i\) with their decisions subject to a constraint set \(\Omega^i\). Denote \(n_i \triangleq \sum_{i=1}^N n_i\) and \(\Omega \triangleq \Omega^1 \times \cdots \times \Omega^N\). Then, the cluster-level cost function \(J^i : \Omega \to \mathbb{R}\) is defined as

\[
J^i(x^i, x^{-i}) \triangleq \frac{1}{n_i} \sum_{j=1}^{n_i} J^j_i(x^i, x^{-i}), \quad x^i \in \Omega^i, \quad \forall i \in \mathcal{N},
\]
where $J'_j(x^i, x^{-i})$ is a local cost function of agent $j$ in cluster $i$, $x^i \triangleq [x^{i1}, \ldots, x^{in_i}]^T \in \Omega_i = \Omega_1 \times \cdots \times \Omega_{n_i}$ is a collection of all agents' decisions in cluster $i$ with $x^i_j \in \Omega_j \subset \mathbb{R}^{p_j}$, being the action of agent $j$ in cluster $i$, and $x^{-i} \in \Omega_i \setminus \{x^i_j\}$ denotes a collection of all agents' decisions except cluster $i$. Denote $x \triangleq (x^1, x^{-1}) = [x^{11}, \ldots, x^{N1}]^T$. Without the loss of generality, we set $p^*_j = 1$, $j \in \mathcal{V}^i$, $i \in \mathcal{N}$ for simplicity.

Definition 1: (NE of $N$-Cluster Games) A vector $x^* \in (x^*, x^{-i}) \in \Omega$ is said to be an NE of the $N$-cluster non-cooperative game $\Gamma(\mathcal{N}, \{J^i\}, \{\Omega_i\})$, if and only if

$$J^i(x^i, x^{-i}) \leq J^i(x^*, x^{-i}), \quad \forall x^i \in \Omega_i, \quad \forall i \in \mathcal{N}.$$ 

In this paper, we consider the $N$-cluster non-cooperative game $\Gamma(\mathcal{N}, \{J^i\}, \{\Omega_i\})$ under a partial-decision information game setup. Motivated by the work in [25], we assume the agents are equipped with two level networks: a high-level network for agents' decisions exchange across clusters and $N$ low-level networks for agents' (partial) gradient exchange within the cluster. The following definitions for the two level networks are given.

For the low-level network in each cluster $i \in \mathcal{N}$, it is a directed graph consisting of all agents in the same cluster, denoted by $G_i(\mathcal{V}_i, \mathcal{E}_i)$ with an adjacency matrix $A^i \triangleq [a^i_{jk}] \in \mathbb{R}^{n_i \times n_i}$, $a^i_{jk} > 0$ if $(k, j) \in \mathcal{E}_i$ and $a^i_{jk} = 0$ otherwise. We assume $(k, k) \in \mathcal{E}_i, \forall k \in \mathcal{V}_i$.

For the high-level network, it is a directed graph consisting of all agents in all clusters, denoted by $\tilde{G}(\mathcal{V}, \tilde{\mathcal{E}})$ with an adjacency matrix $\tilde{A} \triangleq [\tilde{a}_{pq}] \in \mathbb{R}^{\sum n_i \times \sum n_i}$, $\tilde{a}_{pq} > 0$ if $(q, p) \in \tilde{\mathcal{E}}$ and $\tilde{a}_{pq} = 0$ otherwise. We assume $(p, p) \in \tilde{\mathcal{E}}, \forall p \in \mathcal{V}$.

The following two standard assumptions on the two level networks are imposed.

Assumption 1: For $i \in \mathcal{N}$, the digraph $G_i$ is strongly connected. The associated adjacency matrix $A^i$ is column stochastic, i.e., $1_{n_i} A^i = 1_{n_i}$.

Assumption 2: The digraph $\tilde{G}$ is strongly connected. Its associated adjacency matrix $\tilde{A}$ is row stochastic, i.e., $\tilde{A} 1_{\sum n_i} = 1_{\sum n_i}$.

Under Assumption 1, it is known that $A^i$ is primitive and column stochastic, then we denote its right eigenvector corresponding to the eigenvalue of 1 by $\pi^i \triangleq [\pi^i_1, \ldots, \pi^i_{n_i}]^T$, such that $1_{n_i} \pi^i = 1$. Then, $\pi^i$ corresponds to $A^i$'s non-$1_{n_i}$ Perron vector with eigenvalue 1, and hence all elements in $\pi^i$ are positive, and $A^i_{\infty} = \pi^i 1_{n_i}$. Define $\epsilon \triangleq \max_{i \in \mathcal{N}} \|\pi^i\|_2$, $\pi^i_{1+} \triangleq n_i \pi^i$, and $\pi \triangleq [\pi^1_{1+}, \ldots, \pi^N_{1+}]^T$. Denote the smallest and largest elements of $\pi$ by $\pi^\text{min}$ and $\pi^\text{max}$, respectively. With the above notations, the following results can be readily obtained based on the definitions of the weighted Euclidean norm introduced in Section I-C, and will be frequently applied in the subsequent analysis

$$\sqrt{\pi^\text{min}} \cdot \|\pi^\text{max}\|_{\pi^i} \leq \epsilon \leq \sqrt{\pi^\text{max}} \cdot \|\pi^\text{min}\|_{\pi^i}.$$ 

Then, the following contraction properties of the adjacency matrix $A^i$ can be deduced.

Lemma 1: (see [26, Lemma 1]) Under Assumption 1, the adjacency matrix $A^i$ holds that

$$\|A^i - A^i_{\infty}\|_{\pi^i} < 1, \quad \|I_{n_i} - A^i_{\infty}\|_{\pi^i} = 1.$$ 

Next, we make the following assumption on the agents' local cost functions.

Assumption 3: For each $j \in \mathcal{V}_i, i \in \mathcal{N}$, the set $\Omega_j \subset \mathbb{R}$; the local cost function $J^i_j(x^i, x^{-i})$ is convex, continuously differentiable in $x^i$, and jointly continuous in $x$; the partial gradient $\nabla_{x^i} J^i_j(x^i, x^{-i})$, $\forall x \in \mathcal{V}_i$, is $\mathcal{L}$-Lipschitz continuous in $x$, i.e., for any $x, x' \in \mathbb{R}^n$, we have $\|\nabla_{x^i} J^i_j(x^i, x^{-i}) - \nabla_{x^i} J^i_j(x^i)\|_2 \leq \mathcal{L}\|x - x'\|_2$.

The game mapping of $\Gamma(\mathcal{N}, \{J^i\}, \{\Omega_i\})$ is defined as

$$F(x) \triangleq [\nabla_{x^1} J^1_1(x^1)^T, \ldots, \nabla_{x^N} J^N_1(x^N)^T]^T.$$ 

Then, it follows from Assumption 3 that $F(x)$ is Lipschitz continuous, i.e., for any $x, x' \in \mathbb{R}^n$, we have $\|F(x) - F(x')\|_2 \leq \sqrt{\mathcal{L}}\|x - x'\|_2$. Next, the following assumption on the game mapping condition is supposed.

Assumption 4: The game mapping $F$ of game $\Gamma$ is strongly monotone on $\mathcal{V}$ with a constant $\chi > 0$, i.e., for any $x, x' \in \mathbb{R}^n$, we have $\langle F(x) - F(x'), x - x' \rangle \geq \chi\|x - x'\|_2^2$. Remark 1: Under Assumptions 3 and 4, game $\Gamma$ admits a unique NE $x^*$. Moreover, at NE, $F(x^*) = 0_n$, and hence $\langle F(x), x - x^* \rangle \geq \chi\|x - x^*\|_2^2$.

III. FULLY DISTRIBUTED NE SEEKING IN $N$-CLUSTER GAMES

In this section, we present a fully distributed NE seeking strategy for the $N$-cluster game under partial-decision information scenario, followed by the detailed convergence analysis.

For the notational convenience, agent $j \in \mathcal{V}_i$ in cluster $i \in \mathcal{N}$ of the low-level network is referred to agent $\sum_{l=0}^{n_i} n_l + j$ with $n_0 = 0$ in the high-level network. Hence, its action variable $x^i_j$ is relabeled by $y_{\sum_{l=0}^{n_i} n_l + j}$, i.e., $x^i_j = y_{\sum_{l=0}^{n_i} n_l + j}$. Then, each agent $j \in \mathcal{V}_i$, $i \in \mathcal{N}$ needs to maintain the action variable $x^i_j$, and gradient tracker variables $g^{i}_{jk}, \tilde{g}^{i}_{jk}$ for all $k \in \mathcal{V}_i$. Moreover, in the high-level network, each agent $p \in \mathcal{V}$ also needs to maintain an estimation variable $y^p_{n_i}$ for the action of agent $p \in \mathcal{V}$.

We use subscript $t$ to denote the values of all these variables at time-step $t$. The update laws are designed as follows.

high-level network:

$$y^p_{n_i,t+1} = \sum_{l=1}^{n_i} \tilde{a}_{pl}[y^p_{l,t} + \delta_p \tilde{a}_{pl}(y^p_{l,t} - y^p_{q,t})], \quad p, q \in \mathcal{V} \tag{1a}$$

low-level network:

$$x^i_{j,t+1} = x^i_{j,t} - \gamma^i_{j}g^i_{j,t}, \quad j \in \mathcal{V}_i, i \in \mathcal{N} \tag{1b}$$

$$g^i_{jk,t+1} = \sum_{l=1}^{n_i} \tilde{a}_{kl}[g^i_{jk,t} + \nabla_{x^i_j} J^i_j(y_{l,t} + \sum_{l=0}^{n_i} n_l)] \tag{1c}$$

with arbitrary $x^i_{j,0} \in \Omega_i, y^p_{n_i,0} \in \Omega$ and $g^i_{jk,0} = \nabla_{x^i_j} J^i_j(y^p_{n_i,0} + \sum_{l=0}^{n_i} n_l)$, where $\delta_p$ is a constant parameter for
agent $p \in \mathcal{V}$ to be determined later, and $\gamma^j > 0$ is a constant step-size sequence adopted by agent $j \in \mathcal{V}^i, i \in \mathcal{N}$. Denote the largest step-size by $\gamma^M = \max_{j \in \mathcal{V}^i, i \in \mathcal{N}} \gamma^j$ and the average of all step-sizes by $\bar{\gamma} = \frac{1}{|\mathcal{N}|} \sum_{j \in \mathcal{V}^i, i \in \mathcal{N}} \gamma^j$. Define the heterogeneity of the step-size as the following ratio, $\epsilon_i = \frac{\|\gamma - \bar{\gamma}\|_2}{\|\bar{\gamma}\|_2}$, where $\gamma = [\gamma^1, \ldots, \gamma^N]^T$ and $\bar{\gamma} = \bar{\gamma}1_N$. The proposed algorithm is summarized in Algorithm 1.

Algorithm 1 Fully distributed NE seeking in $N$-cluster games

1. **Initialize:** $j \in \mathcal{V}^i, i \in \mathcal{N}$
   set $x_{j,0} \in \mathbb{R}^n, y_0 = \mathbf{0}, x_1 = \mathbf{0}, n = \mathbb{R}^n$
   set $g^i_{j,k,0} = \nabla x_{j,1} J_j(y_0^{(\sum_{\ell=0}^{k-1} n+\ell)})$, $\forall k \in \mathcal{V}^i$

2. **Iteration** ($t \geq 0$; $j \in \mathcal{V}^i, i \in \mathcal{N}$)
   update $y^i_{q,t+1}$ based on (1a), $\forall q \in \mathcal{V}$
   update $x_{j,t+1}$ based on (1b)
   update $g^i_{j,k,t+1}$ based on (1c), $\forall k \in \mathcal{V}^i$

3. **Output:** $j \in \mathcal{V}^i, i \in \mathcal{N}$
   $x_{j,t} \rightarrow x^j_{j}$

IV. CONVERGENCE ANALYSIS

The following notations are made throughout the convergence analysis. For all $k \in \mathcal{V}^i, i \in \mathcal{N}$,

$$g^i_{k,t} = [g^i_{k,1,t}, \ldots, g^i_{k,n_i,t}]^T$$

$$g^i_{k,t} = [g^i_{1,1,t}, \ldots, g^i_{n_i,1,t}, \ldots, g^i_{1,n_i,t}, \ldots, g^i_{n_i,n_i,t}]$$

$$\nabla x^1 J_i(y^{(1:n_i)}) = \left[\nabla x^1 J_i(y^{(1:n_i+1)}), \ldots, \nabla x^{n_i} J_i(y^{(1:n_i)})\right]^T$$

$$\nabla x^1 J_i(y^{(1:n_i+1)}) = \frac{1}{n_i} \nabla x^1 J_i(y^{(1:n_i)})$$

Then, the concatenated form of (1) is given by

$$y_{q,t+1} = Ay_{q,t} + \text{diag}(\delta_{1,q}, \ldots, \delta_{n,q}) (1, n_{y,q,t} - y_{q,t})$$

$$x_{k,t+1} = x_{k,t} - \text{diag}(\gamma_{k,t}) g^i_{k,t}$$

$$g^i_{k,t+1} = A^i g^i_{k,t} + \nabla x^1 J_i(y^{(1:n_i)}) - \nabla x^{n_i} J_i(y^{(1:n_i)})$$

The convergence analysis of the proposed algorithm is conducted by establishing a linear system, which is composed of three major expressions: i) $\|x_t - x^*\|_2^2$, the gap between all agents’ decisions and the NE; ii) $\sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g^i_{k,t} - A^i g^i_{k,t}\|_2^2$, the total gradient tracking error; and iii) $\sum_{q=1}^{N} \|y_{q,t} - 1_n y_{q,t}\|_2^2$, the total decision estimation error, in terms of the inequality iterations of their past values.

A. Auxiliary Results

In this part, we derive the inequality iterations of the three major terms, respectively.

We first start by providing a bound on the stacked gradient estimator, $g^i$, in terms of the linear combinations of the three major terms, summarized in the following lemma.

**Lemma 2:** Under Assumptions 1 and 3, the stacked gradient tracker, $g^i$, holds that

$$\|g^i_t\|_2^2 \leq 3\pi^2 L^2 \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|x_t - x^*\|_2^2 + 3\pi \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g^i_{k,t}\|_2^2$$

$$- A^i g^i_{k,t} \|_{\pi^2} + \frac{3n_i^2 \pi^2 L^2}{\bar{\gamma}^2} \sum_{q=1}^{N} \|y_{q,t} - 1_n y_{q,t}\|_2^2$$

**Proof:** It is noted that for $k \in \mathcal{V}^i, i \in \mathcal{N}$

$$\|g^i_{k,t}\|_2 \leq \|g^i_{k,t} - v^i_{1,n} g^i_{k,t}\|_2 + \|v^i_{1,n} g^i_{k,t}\|_2$$

$$\leq \sqrt{\pi} \|g^i_{k,t} - A^i g^i_{k,t}\|_2 + n_i \|g^i_{k,t} - A^i g^i_{k,t}\|_2$$

$$+ n_i \|x^1 J_i(y_t^{(1:n_i)}) - \nabla x^1 J_i(x_t)\|_2$$

$$+ n_i \|x^1 J_i(x_t) - \nabla x^1 J_i(y_t^{(1:n_i)})\|_2,$$

where $\nabla x^1 J_i(y^*) = 0$ has been applied. As $A^i$ is column stochastic, it follows from (2c) that

$$g^i_{k,t+1} = g^i_{k,t} + \nabla x^1 J_i(y_t^{(1:n_i)}) - \nabla x^1 J_i(y_t^{(1:n_i)}).$$

Since $g^i_{k,0} = \nabla x^1 J_i(y_t^{(0:n_i)})$ due to the initial conditions, then

$$\bar{g}^i_{k,t} = \nabla x^1 \bar{J}_i(y_t^{(1:n_i)}),$$

which gives

$$\|g^i_{k,t}\|_2 \leq 3\pi \|g^i_{k,t} - A^i g^i_{k,t}\|_2$$

$$+ \frac{n_i^2 \pi^2 L^2}{\bar{\gamma}^2} \sum_{q=1}^{N} \|y_{q,t} - 1_n y_{q,t}\|_2,$$

Thus, the above two results, and noting that

$$\sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g^i_{k,t}\|_2^2 \leq \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g^i_{k,t}\|_2^2,$$

combining the above two results, and noting that

$$\sum_{i=1}^{N} \sum_{k=1}^{n_i} \|\nabla x^1 J_i(y_t^{(1:n_i)}) - \nabla x^1 J_i(x_t)\|_2^2$$

$$\leq \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|\nabla x^1 J_i(y_t^{(1:n_i)}) - \nabla x^1 J_i(x_t)\|_2^2$$

$$= \sum_{i=1}^{N} \sum_{k=1}^{n_i} \sum_{j=1}^{n_i} \|\nabla x^1 J_i(y_t^{(1:n_i)}) - \nabla x^1 J_i(x_t)\|_2^2$$

$$\leq \frac{n_i^2 \pi^2 L^2}{\bar{\gamma}^2} \sum_{q=1}^{N} \|y_{q,t} - 1_n y_{q,t}\|_2^2,$$

we obtain the desired result.

Next, we establish the inequality iterations of the three major terms in Lemmas 4, 5, and 6, respectively. We first start with an important property of the adjacency matrix $A$ in the following lemma.
Lemma 3: Under Assumption 2, let $\delta_q > 0, p \in \mathcal{V}$ be chosen such that $0 \leq \delta_p \alpha_{pq} < 2\alpha_{pp} \forall q \in \mathcal{V}$. Then, the matrix $\tilde{A}_q \triangleq [\tilde{a}_{pm}]$, $q \in \mathcal{V}$ with its entry given by

$$
\tilde{a}_{pm} = \begin{cases} 
\tilde{a}_{pm} & \text{if } p \neq m \\
\alpha_{pp} - \delta_p \alpha_{pq} & \text{if } p = m 
\end{cases}
$$

holds that $\rho(\tilde{A}_q) < 1$. Moreover, there exists a matrix norm $\| \cdot \|_E$ such that $\| \tilde{A}_q \|_E < 1$ for $\forall q \in \mathcal{V}$.

Proof: See Appendix A for the proof.

We denote the vector norm which is compatible with the matrix norm $\| \cdot \|_E$, $\| \cdot \|_e$, $\| \cdot \|_{e} \leq \| \cdot \|_{E} \| \cdot \|_e$, for a matrix $A$ and a vector $v$ with compatible size. Due to the equivalence of all norms in a finite-dimensional vector space, there exists $C > 0$ and $C > 0$ such that

$$
\|v\|_2 \leq \|v\|_e \leq C\|v\|_2.
$$

Now we are ready to derive a bound on the total action estimation error characterized by $\sum_{q=1}^{n} \|n_{t}y_{q,t} - y_{q,t}\|_e^2$ in the following lemma.

Lemma 4: Under Assumptions 1, 2 and 3, the total action estimation error satisfies:

$$
\sum_{q=1}^{n} \|n_{t}y_{q,t+1} - y_{q,t+1}\|_e^2 
\leq \left[ \frac{1 + \tilde{\sigma}_2^2}{2} + \frac{3n^2C^2\gamma^2_{M}L^2\gamma^2_{M}}{C^2} \right] \sum_{q=1}^{n} \|n_{t}y_{q,t} - y_{q,t}\|_e^2 
+ \frac{3n^{2}C^2\gamma^2_{M}}{C^2} \sum_{i=1}^{N} \|g_{k,t}^i - A_{k}^{i}\|_e^2, 
+ \frac{3n^{2}C^2\gamma^2_{M}}{C^2} \sum_{i=1}^{N} \|n_{i,\gamma^2_{M}}\|_e^2, 
$$

where $\tilde{\sigma}_2 \triangleq \max_{q \in \mathcal{V}} \tilde{\sigma}_q$ and $\tilde{\sigma}_q \triangleq \max_{q \in \mathcal{V}} \frac{1 + \alpha_{q}}{1 - \alpha_{q}}$ and $\sigma_{\tilde{A}_q} \triangleq \| \tilde{A}_q \|_e$.

Proof: From (1b), we have for $q \in \mathcal{V}$

$$
1_{n}y_{q,t+1} = 1_{n}y_{q,t} - \gamma_{q}1_{n}g_{q,t},
$$

where $\gamma_{q}$ and $g_{q,t}$ denote the step-size and gradient tracker of agent $q$, respectively. That is, $\gamma_{j}$ and $g_{j,t}$ correspond to $\gamma_{q}$ and $g_{q,t}$ with $q = \sum_{j=1}^{i}n_{i} + j$, if agent $j \in \mathcal{V}^i$ in cluster $i \in \mathcal{N}$ is considered. Subtracting the above relation by (2a) and taking the vector norm $\| \cdot \|_e$ on both sides, we have

$$
\|1_{n}y_{q,t+1} - y_{q,t+1}\|_e \leq \| (A - \nabla \text{diag}(\delta_{1,\alpha_{q}, \ldots, \delta_{n,\alpha_{n}}}) \| \| 1_{n}y_{q,t} - y_{q,t}\|_e + \gamma_{q}1_{n}g_{q,t}\|_e 
\leq \| \tilde{A}_q \|_e \| 1_{n}y_{q,t} - y_{q,t}\|_e + n\gamma_{q}\| g_{q,t}\|_e 
$$

Define $\sigma_{\tilde{A}_q} \triangleq \| \tilde{A}_q \|_e$ and square both sides, we obtain

$$
\|1_{n}y_{q,t+1} - y_{q,t+1}\|_e^2 \leq \sigma_{\tilde{A}_q}^2 \| 1_{n}y_{q,t} - y_{q,t}\|_e^2 + n^2\gamma^2_{M}\| g_{q,t}\|_e^2 
+ \frac{1 - \sigma_{\tilde{A}_q}^2}{2} \| 1_{n}y_{q,t} - y_{q,t}\|_e^2 + \frac{2\sigma_{\tilde{A}_q}^2n^2C^2\gamma^2_{M}}{1 - \sigma_{\tilde{A}_q}^2} \| g_{q,t}\|_e^2 
$$

where we denote $\hat{\sigma}_2 \triangleq \max_{q \in \mathcal{V}} \sigma_{\tilde{A}_q}$, $\hat{\sigma}_2 \triangleq \max_{q \in \mathcal{V}} \frac{1 + \alpha_{q}^2}{1 - \alpha_{q}}$.

Summing over $q = 1$ to $n$ gives

$$
\sum_{q=1}^{n} \|1_{n}y_{q,t+1} - y_{q,t+1}\|_e^2 \leq \frac{1 + \hat{\sigma}_2^2}{2} \sum_{q=1}^{n} \|1_{n}y_{q,t} - y_{q,t}\|_e^2 
+ n^2\gamma^2_{M}\| \mathcal{G}_e\|_e^2.
$$

Substituting the result in Lemma 2 completes the proof.

Next, we bound the gap between all agents’ decisions and the NE, characterized by $|x_{t} - x^*|_e^2$.

Lemma 5: Under Assumptions 1, 2, 3 and 4, the agents’ decisions $x_{t}$ satisfies that

$$
\|x_{t+1} - x^*\|_e^2 \leq \frac{3\pi_{\gamma}^2 + 3\pi_{\gamma}^2}{\pi} \sum_{i=1}^{N} \|g_{i,t}^e - A_{i}^{e}g_{i,t}^e\|_e^2 
+ \left[ 1 - \pi \chi \gamma^2 + \frac{3\pi \gamma^2 L^2\gamma^2_{M}}{\pi} \sum_{i=1}^{N} \gamma_{i}^3 + 2\pi \sqrt{\gamma n L e \gamma^2} \right] \|x_{t} - x^*\|_e^2 
+ \left[ \frac{3n^2\gamma^2 L^2\gamma^2_{M}}{\pi C^2} + \frac{3n^2\gamma^2 L^2\gamma^2_{M}}{\pi C^2} \right] \sum_{q=1}^{n} \|y_{q,t} - 1_{n}y_{q,t}\|_e^2.
$$

Proof: It follows from (2b) that

$$
x_{t+1} - x^* = x_{t} - \text{diag}(\gamma)g_{t} - x^*.
$$

Taking the norm on both sides gives

$$
\|x_{t+1} - x^*\|_e^2 = \|x_{t} - \text{diag}(\gamma)g_{t} - x^*\|_e^2 
\leq \|x_{t} - x^*\|_e^2 + \gamma_{M}^2 \| g_{t}\|_e^2 
- 2\langle x_{t} - x^*, \text{diag}(\gamma)(g_{t} - \text{diag}(\pi)F(x_{t})) \rangle \pi 
- 2\langle x_{t} - x^*, \text{diag}(\gamma \gamma)(\pi)F(x_{t}) \rangle \pi 
- 2\hat{\gamma}_{M} \langle x_{t} - x^*, \text{diag}(\pi)F(x_{t}) \rangle \pi.
$$

For (7a), it follows that

$$
- 2\langle x_{t} - x^*, \text{diag}(\gamma)(g_{t} - \text{diag}(\pi)F(x_{t})) \rangle \pi 
= - 2\sum_{i=1}^{N} \sum_{k=1}^{n_{i}} \gamma_{i}^k \langle x_{k,t}^i - x_{k,t}^i, g_{k,t}^i - n_{i} \pi_{k}^i \nabla_{x_{k,t}^i} \nabla f_{i}(x_{t}) \rangle n_{i} \pi_{k}^i 
= - 2\sum_{i=1}^{N} \sum_{k=1}^{n_{i}} \gamma_{i}^k \langle x_{k,t}^i - x_{k,t}^i, g_{k,t}^i - n_{i} \pi_{k}^i \nabla_{x_{k,t}^i} \nabla f_{i}(x_{t}) \rangle n_{i} \pi_{k}^i 
- 2\sum_{i=1}^{N} \sum_{k=1}^{n_{i}} \gamma_{i}^k \langle x_{k,t}^i - x_{k,t}^i, n_{i} \pi_{k}^i \nabla_{x_{k,t}^i} \nabla f_{i}(x_{t}) \rangle n_{i} \pi_{k}^i 
- 2\sum_{i=1}^{N} \sum_{k=1}^{n_{i}} \gamma_{i}^k \langle x_{k,t}^i - x_{k,t}^i, n_{i} \pi_{k}^i \nabla_{x_{k,t}^i} \nabla f_{i}(x_{t}) \rangle n_{i} \pi_{k}^i 
- 2\hat{\gamma}_{M} \langle x_{t} - x^*, \text{diag}(\pi)F(x_{t}) \rangle \pi.
The first part holds that
\[
-2 \sum_{i=1}^{N} \sum_{k=1}^{n_i} \gamma_k (x_{k,t}^i - x_{k,t}^* + g_{k,t}^i - n_i \pi_k^i g_{k,t}^i) n_i \pi_k^i
- 2 \sum_{i=1}^{N} \sum_{k=1}^{n_i} \gamma_k (x_{k,t}^i - x_{k,t}^* + g_{k,t}^i - n_i \pi_k^i g_{k,t}^i) n_i \pi_k^i
\leq 2 \gamma_M \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|x_{k,t}^i - x_{k,t}^*\|_n \pi_k^i \|g_{k,t}^i - n_i \pi_k^i g_{k,t}^i\|_n \pi_k^i
\leq \frac{3 \pi \gamma}{\|x_t - x^*\|_\pi^2} + \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g_{k,t}^i - n_i \pi_k^i g_{k,t}^i\|_n \pi_k^i
\leq \frac{3 \pi \gamma}{\|x_t - x^*\|_\pi^2} + \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g_{k,t}^i - A_i^* g_{k,t}^i\|_\pi^2.
\]

For the second part, it follows from (3) that
\[
\langle x_{k,t}^i - x_{k,t}^*, n_i \pi_k^i (g_{k,t}^i - \nabla x_k^i J^i(y_t^{(1:n_i)})) \rangle_{n_i \pi_k^i} = 0.
\]

For the third part, we have
\[
-2 \sum_{i=1}^{N} \sum_{k=1}^{n_i} \gamma_k (x_{k,t}^i - x_{k,t}^* + g_{k,t}^i - n_i \pi_k^i (\nabla x_k^i J^i(y_t^{(1:n_i)}))
- \nabla x_k^i J^i(y_t^{(1:n_i)}))_{n_i \pi_k^i} \leq 2 \gamma_M \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|x_{k,t}^i - x_{k,t}^*\|_n \pi_k^i \times \sqrt{n_i \pi_k^i} \|\nabla x_k^i J^i(y_t^{(1:n_i)}) - \nabla x_k^i J^i(x_t^i)\|_2
\leq \frac{3 \pi \gamma}{\|x_t - x^*\|_\pi^2} + \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g_{k,t}^i - A_i^* g_{k,t}^i\|_\pi^2.
\]
The term follows the same derivation as in (4) that
\[
\sum_{i=1}^{N} \sum_{k=1}^{n_i} \|\nabla x_k^i J^i(y_t^{(1:n_i)}) - \nabla x_k^i J^i(x_t^i)\|_2 \leq \frac{n \mathcal{L}^2}{2} \sum_{q=1}^{n} \|y_{q,t} - 1_n y_{q,t}\|_e^2.
\]
Hence, the third part can be further obtained that
\[
-2 \sum_{i=1}^{N} \sum_{k=1}^{n_i} \gamma_k (x_{k,t}^i - x_{k,t}^* + g_{k,t}^i - n_i \pi_k^i (\nabla x_k^i J^i(y_t^{(1:n_i)}))_{n_i \pi_k^i} \leq \frac{3 \pi \gamma}{\|x_t - x^*\|_\pi^2} + \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|y_{q,t} - 1_n y_{q,t}\|_e^2.
\]
Combining the above three parts, we obtain that
\[
-2 \gamma (x_t - x^*, g_{t} - \text{diag}(\pi) F(x_t))_{\pi}
\leq \frac{3 \pi \gamma}{\|x_t - x^*\|_\pi^2} + \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|g_{k,t}^i - A_i^* g_{k,t}^i\|_\pi^2
+ \frac{3 \gamma_M^2}{\pi \gamma} \sum_{i=1}^{N} \sum_{k=1}^{n_i} \|y_{q,t} - 1_n y_{q,t}\|_e^2.
\]
Hence, from Lemma 1, then

\[
\|g_k^{i+1} - A_i g_k^i\|_2^2 + \|\nabla x^i_k \mathbf{J}(y_{t+1}) - \mathbf{J}^t(y_{t+1})\|_p^2 \\
\leq \sigma_{A_i}^2 \|g_k^i\|_p^2 + \|g_k^i\|_p^2 + A_i \|\nabla x^i_k \mathbf{J}(y_{t+1}) - \mathbf{J}^t(y_{t+1})\|_p^2 \\
\leq \sigma_{A_i}^2 \|g_k^i\|_p^2 + A_i \|\nabla x^i_k \mathbf{J}(y_{t+1}) - \mathbf{J}^t(y_{t+1})\|_p^2 \\
+ \|\nabla x^i_k \mathbf{J}(y_{t+1}) - \nabla x^i_k \mathbf{J}(y_t)\|_p^2 \]

Thus, we have

\[
\sum_{i=1}^N \sum_{k=1}^{n_i} \|\nabla x^i_k \mathbf{J}(y_{t+1}) - \mathbf{J}^t(y_t)\|_p^2 \\
\leq \frac{3n(3 + \sigma^2)}{2\pi C^2} \sum_{q=1}^n \|y_{q,t} - 1_n y_{q,t}\|_c^2 \\
+ \frac{3n^2(1 + \eta_2 C^2) L^2 \gamma_M^2}{2\pi} \|g_k^i\|^2.
\]

Summing over \(k = 1\) to \(n_i\), \(i = 1\) to \(N\) (11) and substituting the above result gives

\[
\sum_{i=1}^N \sum_{k=1}^{n_i} \|g_k^{i+1} - A_i g_k^i\|_2^2 \leq \frac{1 + \sigma_i^2}{2} \sum_{i=1}^N \sum_{k=1}^{n_i} \|g_k^i\|_2^2 \\
+ \frac{3n^2(1 + \eta_2 C^2) L^2 \gamma_M^2}{2\pi} \|g_k^i\|^2.
\]

The proof is completed by substituting Lemma 2.

\[\square\]

**B. Main Results**

Now, we are ready for the analysis on the convergence of the proposed algorithm. With the inequality derivations presented in Lemmas 4, 5 and 6, we can establish the following linear dynamical system

\[
u_{t+1} = \mathbf{T} u_t,
\]

where

\[
\begin{align*}
\mathbf{u}_t &\triangleq \left[ \frac{\|x_t - x^*\|_2}{\pi} \right] \\
\mathbf{T} &\triangleq \left[ \begin{array}{c}
1 - k_1 \gamma + k_2 \gamma^2 \\
k_4 - k_5 \gamma^2 + k_6 \gamma^3 + k_7 \gamma^4 \\
k_8 - k_9 \gamma^5 + k_{10} \gamma^6 + k_{11} \gamma^7 + k_{12} \gamma^8 \\
k_{13} - k_{14} \gamma^9 + k_{15} \gamma^{10} + k_{16} \gamma^{11}
\end{array} \right] \\
k_1 &\triangleq \frac{\pi}{\pi - \chi}, \quad k_2 \triangleq \frac{3\pi^2 C^2}{\pi} \sum_{i=1}^n n_i^3, \quad k_3 \triangleq \frac{2\pi \sqrt{nL}}{C^2}, \\
k_4 &\triangleq \frac{3\pi^2 C^2}{\pi} \sum_{i=1}^n n_i^3, \quad k_5 \triangleq \frac{3\pi^2 C^2}{\pi}, \quad k_6 \triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3, \\
k_7 &\triangleq \frac{3\pi^2 C^2}{\pi}, \quad k_8 \triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3, \quad k_9 \triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3, \quad k_{10} \triangleq \frac{1 - \sigma_2}{\pi}, \\
k_{11} &\triangleq \frac{3n(3 + \sigma_2)}{2\pi C^2} \sum_{i=1}^n n_i^3, \quad k_{12} \triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3, \\
k_{13} &\triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3, \quad k_{14} \triangleq \frac{3n^2 C^2L^2}{\pi} \sum_{i=1}^n n_i^3.
\end{align*}
\]

Then, the convergence results of all agents’ decisions to the NE \(x^*\) can be established based on the convergence of the dynamical system (12), which are summarized in the following theorem.

**Theorem 1:** Suppose Assumptions 1, 2, 3 and 4 hold. Generate the agent’s action \(\{x^i_{t+1}\}_{t\geq0}\), gradient tracker \(\{g_k^i\}_{k,t\geq0}\) and estimation variable \(\{y_{q,t}^i\}_{t\geq0}\) by Algorithm 1 with the uncoordinated constant step-size \(\gamma^t\) satisfying

\[
0 \leq \gamma^t \leq \frac{k_1}{k_3}, \quad 0 < \gamma_M < \min \left\{ \frac{1}{k_1}, \gamma_1^*, \gamma_2^*, \gamma_3^* \right\}.
\]
where $\gamma^*_1$, $\gamma^*_2$ and $\gamma^*_3$ are some constants related to the heterogeneity $\epsilon_c$. Then, we have $\rho(T)<1$, and $\sup_{t\geq T} \|x_t-x^*\|^2_\pi$ (respectively, $\sup_{t\geq T} \sum_{i=1}^N \sum_{k=1}^n \|g_{i,k,t} - A^\infty g_{i,k,t}\|^2_\pi$, and $\sup_{t\geq T} \sum_{q=1}^n \|1_n y_{q,t} - y_{q,t}\|^2_\pi$) linearly converges to 0 at a rate of $\rho(T)$.

**Proof:** For the dynamical system (12), if $\rho(T)<1$, then $T$ converges to 0 at a geometric rate with exponent $\rho(T)$ [27], which implies that $\sup_{t\geq T} \|x_t-x^*\|^2_\pi$, $\sup_{t\geq T} \sum_{i=1}^N \sum_{k=1}^n \|g_{i,k,t} - A^\infty g_{i,k,t}\|^2_\pi$, and $\sup_{t\geq T} \sum_{q=1}^n \|1_n y_{q,t} - y_{q,t}\|^2_\pi$, respectively, converge to 0 with the same rate. The following lemma provides a sufficient condition to quantify the spectral radius of a non-negative matrix:

**Lemma 7:** (see [27, Cor. 8.1.29]) Let $A \in \mathbb{R}^{m \times m}$ be a matrix with non-negative entries and $\theta \in \mathbb{R}^m$ be a vector with positive entries. If there exists a constant $\lambda > 0$ such that $A \theta < \lambda \theta$, then $\rho(A)<\lambda$.

To invoke Lemma 7, the matrix $T$ has to be non-negative. Thus, it suffices to have

$$0 < \bar{\gamma} \leq \frac{1}{k_1}. \quad (13)$$

According to Lemma 7, to ensure $\rho(T) < 1$, one needs to seek for some positive vector $\theta \triangleq [\theta_1, \theta_2, \theta_3]^\top$, where $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 > 0$, such that $A \theta < \lambda \theta$, i.e.,

$$\begin{cases}
(1-k_1 \bar{\gamma} + k_2 \gamma^*_M + k_3 \epsilon_c) \theta_1 \\
+ (k_4 \gamma^*_M + k_5 \gamma^*_M / \bar{\gamma}) \theta_2 \\
+ (k_6 \gamma^*_M + k_7 \gamma^*_M / \bar{\gamma}) \theta_3 < \theta_1, \\
(1-k_1 \bar{\gamma} + k_2 \gamma^*_M + k_3 \epsilon_c) \theta_1 \\
+ (k_4 \gamma^*_M + k_5 \gamma^*_M / \bar{\gamma}) \theta_2 \\
+ (k_6 \gamma^*_M + k_7 \gamma^*_M / \bar{\gamma}) \theta_3 < \theta_2, \\
(1-k_1 \bar{\gamma} + k_2 \gamma^*_M + k_3 \epsilon_c) \theta_1 \\
+ (k_4 \gamma^*_M + k_5 \gamma^*_M / \bar{\gamma}) \theta_2 \\
+ (k_6 \gamma^*_M + k_7 \gamma^*_M / \bar{\gamma}) \theta_3 < \theta_3.
\end{cases} \quad (14)$$

Without the loss of generality, we can set $\theta_3 = 1$. Then these inequalities can be further simplified as

$$\begin{cases}
(k_2 \theta_1 + k_4 \theta_2 + k_6) \bar{\gamma} \\
< (k_1 - k_3 \epsilon_c) \theta_1 \gamma^*_M / \bar{\gamma} - (k_5 \theta_2 + k_7), \\
(k_8 \theta_1 + k_9 \theta_2 + k_{12}) \gamma^*_M < (k_1 \theta_2 - k_{11}), \\
(k_{13} \theta_1 + k_1 \theta_2 + k_{15}) \gamma^*_M < k_{16}.
\end{cases} \quad (14)$$

Therefore, we would like to find the range of the step-size such that (14) hold simultaneously for some $\theta_1 > 0$, $\theta_2 > 0$. To ensure the existence of solution $\gamma_M$ and $\bar{\gamma}$, the right-hand-side of (14) needs to be positive

$$\epsilon_c < \frac{k_1}{k_3}, \theta_1 > \left(\frac{(k_5 \theta_2 + k_7) \gamma^*_M / \bar{\gamma}^2}{k_1 - k_3 \epsilon_c}, \theta_2 > \frac{k_{11}}{k_{10}}.\right.$$  

Thus, we can set

$$\theta_1 = \frac{2(k_5 k_{11} + k_7 k_{10}) \gamma^*_M / \bar{\gamma}^2}{k_{11}(k_1 - k_3 \epsilon_c)}, \theta_2 = \frac{k_{11}}{k_{10}}.$$

Then, we can solve the three inequalities in (14) respectively. For the first inequality, we have

$$\bar{\gamma} < \frac{k_7 k_{10}(k_1 - k_3 \epsilon_c)}{2 k_2 (k_5 k_{11} + k_7 k_{10}) \gamma^*_M / \bar{\gamma}^2} + (2k_3 k_{11} + k_6 k_{10})(k_1 - k_3 \epsilon_c).$$

Noting that $\bar{\gamma} \leq \gamma_M$ and $\gamma_M / \bar{\gamma} < n$, for the above inequality to hold, it suffices to have

$$\gamma_M < \gamma^*_1,$$

where

$$\gamma^*_1 \triangleq \frac{k_7 k_{10}(k_1 - k_3 \epsilon_c)}{k_1 k_{10}}.$$  

$$\gamma^*_2 \triangleq \frac{2k_2 k_5 k_{11} + 2k_2 k_7 k_{10} + 2k_1 k_4 k_{11} + k_1 k_6 k_{10}}{k_1 k_{10}}.$$  

$$\gamma^*_3 \triangleq \sqrt{\frac{2k_5 k_{11} + 2k_7 k_{10} + 2k_1 k_4 k_{11} + k_1 k_{10}}{k_1 k_{10}}},$$

Similarly, for the second and third inequalities to hold, it suffices to have

$$\gamma_M < \gamma^*_2, \gamma_M < \gamma^*_3,$$

where

$$\gamma^*_2 \triangleq \frac{k_{10} k_{11}(k_1 - k_3 \epsilon_c)}{q^*_2}, \gamma^*_3 \triangleq \frac{k_{10} k_{16}(k_1 - k_3 \epsilon_c)}{q^*_3},$$  

$$q^*_2 \triangleq 2n^2 k_8 (k_5 k_{11} + k_7 k_{10}) + (2k_9 k_{11} + k_{10} k_{12})(k_1 - k_3 \epsilon_c),$$  

$$q^*_3 \triangleq 2n^2 k_{13} (k_5 k_{11} + k_7 k_{10}) + (2k_{11} k_{14} + k_{10} k_{15})(k_1 - k_3 \epsilon_c).$$

Thus, to ensure both (13) and (14) hold simultaneously, it suffices to have

$$0 \leq \epsilon_c < \frac{k_1}{k_3}, 0 < \gamma_M < \min \left\{ \frac{1}{k_1}, γ^*_1, γ^*_2, γ^*_3 \right\},$$

which completes the proof.

**Remark 2:** Theorem 1 shows that the linear convergence of all agents’ decisions to the NE is guaranteed when both the largest step-size and its heterogeneity are sufficiently small. Besides, it should be remarked that the bounds on both the largest step-size and heterogeneity are only sufficient but not necessary conditions for the convergence results. That implies, the bound conditions may be conservative, and the dynamical system may still converge even though the bound conditions are not satisfied.

Next, we analyze the convergence of the algorithm when all agents adopt uniform constant step-size, summarized in the following corollary.

**Corollary 1:** Suppose Assumptions 1, 2, 3 and 4 hold. Generate the agent’s action $\{x^i_t\}_{t \geq 0}$, gradient tracker $\{g^i_{j,k,t}\}_{t \geq 0}$ and estimation variable $\{y^i_{q,t}\}_{t \geq 0}$ by Algorithm 1 with uniform constant step-size $\gamma$ satisfying

$$0 < \gamma < \min \left\{ \frac{1}{k_1}, γ^*_1, γ^*_2, γ^*_3 \right\},$$

where $γ^*_1, γ^*_2$ and $γ^*_3$ are some constants. Then, $\sup_{t \geq T} \|x_t - x^*\|^2_\pi$ (respectively, $\sup_{t \geq T} \sum_{i=1}^N \sum_{k=1}^n \|g^i_{i,k,t} - A^\infty g^i_{i,k,t}\|^2_\pi$, and $\sup_{t \geq T} \sum_{q=1}^n \|1_n y_{q,t} - y_{q,t}\|^2_\pi$) linearly converges to 0.

**Proof:** The result directly follows from Theorem 1 by noting that $\epsilon_c = 0$ and $\gamma_M / \bar{\gamma} = 1$, which gives

$$γ^*_1 \triangleq \frac{k_1 k_{10} k_{11}}{2k_2 k_5 k_{11} + 2k_2 k_7 k_{10} + 2k_1 k_4 k_{11} + k_1 k_6 k_{10}},$$  

$$γ^*_2 \triangleq \sqrt{\frac{1}{k_1 k_{10} k_{11}}}.$$  

$$γ^*_3 \triangleq \frac{2k_5 k_{11} + 2k_7 k_{10} + 2k_1 k_4 k_{11} + k_1 k_{10}}{k_1 k_{10} k_{15}},$$
Then, following the same arguments in Theorem 1, when $0 < \gamma < \min \left\{ \frac{1}{\kappa^1_i}, \gamma^1_{i,c}, \gamma^2_{i,c}, \gamma^3_{i,c} \right\}$, the convergence of the algorithm is guaranteed.

V. NUMERICAL SIMULATIONS

In this section, we validate the performance of the proposed algorithm by a Cournot competition game. In particular, we consider $N$ firms, and each firm $i \in \mathcal{N}$ consists of $n_i$ branches to help produce goods. For $j \in \mathcal{V}^i$, $i \in \mathcal{N}$, let $x^i_j$ be the quantity of goods produced by branch $j$ of firm $i$, then its local cost function $J^i_j(x)$ is modeled by the following function

$$J^i_j(x) = c^i_j(x^i_j) - p^i_j(x)x^i_j,$$

where $c^i_j(x^i_j) = a^i_j(x^i_j)^2 + b^i_j(x^i_j)$ models the cost incurred by generating $x^i_j$ quantity of goods, $p^i_j(x) = d^i_j - w^i_j x$ models the selling price of such goods, $a^i_j, b^i_j, d^i_j \in \mathbb{R}$ and $w^i_j \in \mathbb{R}^n$ are constant parameters.

As a numerical setting, we set $N = 3$, $n_i = 3, 4$ and 5, respectively. For constant parameters, we let $a^i_j = 1$, $d^i_j = 10 + i + j$, $b^i_j$ and each element of $w^i_j$ be uniformly drawn from $[0,1]$, respectively. The two level networks are given in Fig. 1, which are strongly connected. The corresponding row stochastic and column stochastic weights of the adjacency matrices are set as $a^i_{jk} = 1/|\mathcal{N}^k_{\text{out}}|$ and $\bar{a}_{pq} = 1/|\mathcal{N}^p_{\text{in}}|$, where $\mathcal{N}^k_{\text{out}}$ (respectively, $\mathcal{N}^p_{\text{in}}$) denotes the set of out-neighbors (respectively, in-neighbors) of agent $k$ (respectively, $p$), and $|\cdot|$ denotes the number of elements in the corresponding set. The initial conditions of $x$ and $y_q$ are set to zero, while the rest of the parameters remain the same as in Sec. V-A. The NE gaps generated by Algorithm 1 under these three network topologies are plotted in Fig. 4. As can be seen, faster rate of convergence is obtained for denser edges, which is as expected since more information can be exchanged per iteration.

B. Influence of the network topology on the convergence

In our second experiment, we study the influence of the network topology of the high-level network on the rate of convergence. We consider three different network topologies, shown in Fig. 3, where the edges are from sparse to dense. The initial conditions of $x$ and $\bar{y}_q$ are set to zero, while the rest of the parameters remain the same as in Sec. V-A. The NE gaps generated by Algorithm 1 under these three network topologies are plotted in Fig. 4. As can be seen, faster rate of convergence is obtained for denser edges, which is as expected since more information can be exchanged per iteration.

C. Influence of step-size on the convergence

In our third experiment, we investigate the influence of the step-size including the heterogeneity on the convergence. Specifically, we let the agents’ step-sizes be selected within $(0, 0.1]$. The initial conditions of $x$ and $\bar{y}_q$ are set to zero, while the rest of the parameters are kept the same as in Sec. V-A. Fig. 5 plots the NE gaps under various step-size cases with different averaged step-size and different heterogeneity. As can be seen, smaller heterogeneity of the step-size and larger averaged step-size lead to a faster rate of convergence.
sizes when the largest step-size and the heterogeneity of the synthesis of leader-following consensus and gradient tracking. A fully distributed Nash equilibrium (NE) seeking algorithm has been proposed based on a decision information settings. A fully distributed Nash equilibrium in two-network zero-sum games,” Automatica, vol. 103, pp. 20–26, 2019.

Fig. 4. Influence of the network topology of the high-level network on the rate of convergence.

Fig. 5. Influence of the step-size and heterogeneity on the rate of convergence.

VI. CONCLUSIONS
This paper has considered the $N$-cluster game under partial-decision information settings. A fully distributed Nash equilibrium (NE) seeking algorithm has been proposed based on a synthesis of leader-following consensus and gradient tracking. It has been shown that all agents’ decisions linearly converge to their corresponding NE with uncoordinated constant step-sizes when the largest step-size and the heterogeneity of the step-size are small. The derived result has been validated through a numerical example in a Cournot competition game.

APPENDIX

A. Proof of Lemma 3
The first part of the result (i.e., $\rho(\tilde{A}_q) < 1$) can be readily proved based on [10, Lemma 3]. For the second part of the result, the following lemma is invoked to facilitate the proof.

Lemma 8: (see [27, Lemma 5.6.10]) Let $\rho(A)$ be the spectral radius of a (square) matrix $A$. For any given $\rho > 0$, there exists a matrix norm $\| \cdot \|_E$ such that $\rho(A) \leq \|A\|_E \leq \rho(A) + \rho$.

Based on the above lemma, we can choose $\rho \in (0, 1 - \max_{q \in \bar{V}} \rho(\bar{A}_q))$, then there exists a matrix norm $\| \cdot \|_E$ such that $\|\bar{A}_q\|_E \leq \rho(\bar{A}_q) + \rho < 1, \forall q \in \bar{V}$, which completes the proof.

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