Blow-up of critical norms for the 3-D Navier-Stokes equations

Wendong Wang† and Zhifei Zhang‡

†School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China
E-mail: wendong@dlut.edu.cn
‡School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China
E-mail: zfzhang@math.pku.edu.cn

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Abstract

Let $u = (u_h, u_3)$ be a smooth solution of the 3-D Navier-Stokes equations in $\mathbb{R}^3 \times [0, T)$. It was proved that if $u_3 \in L^\infty(0, T; \dot{B}_{p,q}^{-1+3/p}(\mathbb{R}^3))$ for $3 < p, q < \infty$ and $u_h \in L^\infty(0, T; BMO^{-1}(\mathbb{R}^3))$ with $u_h(T) \in VMO^{-1}(\mathbb{R}^3)$, then $u$ can be extended beyond $T$.

This result generalizes the recent result proved by Gallagher, Koch and Planchon, which requires $u \in L^\infty(0, T; \dot{B}_{p,q}^{-1+3/p}(\mathbb{R}^3))$. Our proof is based on a new interior regularity criterion in terms of one velocity component.

1 Introduction

We consider the 3-D incompressible Navier-Stokes equations

$$\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div} u &= 0, \\
u|_{t=0} &= u_0(x),
\end{align*}$$

$$\text{(NS)}$$

where $u(x,t)$ denotes the velocity of the fluid, and the scalar function $\pi(x,t)$ denotes the pressure.

The Navier-Stokes equations are scaling invariance in the sense: if $(u, \pi)$ is a solution of (NS), then so does $(u_\lambda, \pi_\lambda)$, where

$$u_\lambda = \lambda u(\lambda x, \lambda^2 t), \quad \pi_\lambda = \lambda^2 \pi(\lambda x, \lambda^2 t).$$

A Banach space $X$ is called a scaling invariant space with respect to the initial data if $\|u_0\|_X = \|u_{0,\lambda}\|_X$. Some classical examples are $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $L^3(\mathbb{R}^3)$, $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and $BMO^{-1}(\mathbb{R}^3)$. The local well-posedness of (NS) in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $L^3(\mathbb{R}^3)$, $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)(3 < p < \infty)$ and $BMO^{-1}(\mathbb{R}^3)$ was proved by Fujita-Kato [12], Kato [18], Cannone-Meyer-Planchon [3] and Koch-Tataru [21] respectively. However, whether local smooth solution can be extended to a global one is an outstanding open problem in the mathematical fluid mechanics.

Ladyzhenskaya-Prodi-Serrin type criterion states that if smooth solution $u$ satisfies

$$u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1 \quad \text{for} \quad p \geq 3,$$

$$2$$
then $u$ can be extended after $t = T$, see [15] [25] [27] for example. The difficult endpoint case (i.e. $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$) was solved by Escauriza-Seregin-Šverák [11]. This result in particular implies nonexistence of self-similar type singularity [23].

In this paper, we are concerned with the following interesting question:

**If** $u \in L^\infty(0, T; X)$ with $X$ a scaling invariant space, **then** $u$ can be extended beyond $t = T$?

The case of $X = L^3(\mathbb{R}^3)$ was proved by Escauriza-Seregin-Šverák (see [19] [13] for the alternative proofs by using the profile decomposition). The case of $X = \dot{B}^{-1+3/p}_p(\mathbb{R}^3)$ for $3 < p < \infty, q < 2p'$ was proved by Chemin-Planchon [8] by using self-improving bounds, and recently by Gallagher-Koch-Planchon [14] for the general case $3 < p, q < \infty$. Recall that the following inclusion relations (see Lemma 3.1)

$$\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}^{-1+3/p}_p(\mathbb{R}^3)(3 < p < \infty) \hookrightarrow BMO^{-1}(\mathbb{R}^3).$$

Thus, it is natural to ask whether the question holds for the case of $X = BMO^{-1}(\mathbb{R}^3)$.

In this paper, we prove a slightly weaker version, which requires that the horizontal velocity $u_h \in L^\infty(0, T; BMO^{-1}(\mathbb{R}^3))$ with $u_h(T) \in VMO^{-1}(\mathbb{R}^3)$, while the vertical velocity $u_3 \in L^\infty(0, T; \dot{B}^{-1+2/p}_p(\mathbb{R}^3))$.

**Theorem 1.1** Let $(u, \pi)$ be a smooth solution of (1) in $\mathbb{R}^3 \times [0, T)$. If $u$ satisfies

$$\|u_3\|_{L^\infty(0, T; \dot{B}^{-1+3/p}_p)} + \|u_h\|_{L^\infty(0, T; BMO^{-1})} = \tilde{M} < \infty,$$

for some $3 < p, q < \infty$ and $u_h(x, T) \in VMO^{-1}(\mathbb{R}^3)$, then $u$ can be extended after $t = T$.

**Remark 1.2** Under the assumption of the theorem, it holds that

$$u(t) \in C_w([0, T]; BMO^{-1}(\mathbb{R}^3)).$$

Hence, $u(T)$ is well-defined as a function in $VMO^{-1}(\mathbb{R}^3)$. Thanks to the inclusion $\dot{B}^{-1+3/p}_p(\mathbb{R}^3) \hookrightarrow VMO^{-1}(\mathbb{R}^3)$, our result improves the recent one proved by Gallagher, Koch and Planchon [14], which requires $u \in L^\infty(0, T; \dot{B}^{-1+3/p}_p(\mathbb{R}^3))$.

The proof of [14] used the profile decomposition approach. Our proof still follows blow-up analysis and backward uniqueness method developed by Escauriza-Seregin-Šverák [11]. A key ingredient is to prove a new interior regularity criterion in terms of one velocity component, which is independent of interest. To state it, we introduce $G(f, p, q; r) \triangleq r^{1-\frac{2}{p}-\frac{2}{q}} \|f\|_{L^p_tL^q_x(Q_r)}$ and the scaling invariant quantities

$$C(u, r) \triangleq r^{-2} \int_{Q_r} |u|^3 dx dt, \quad D(\pi, r) \triangleq r^{-2} \int_{Q_r} |\pi|^2 dx dt,$$

where $Q_r = (-r^2, 0) \times B_r(0)$.
Theorem 1.3 Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1\). If \(u\) satisfies
\[
\sup_{0 < r < 1} G(u, p, q; r) \leq M < +\infty,
\]
for some \((p, q)\) with \(1 \leq \frac{3}{p} + \frac{2}{q} < 2\) and \(1 < q \leq \infty\), then there exists a positive constant \(\varepsilon\) depending on \(p, q, M\) such that \((0, 0)\) is a regular point if
\[
G(u_3, p, q; r^*) \leq \varepsilon
\]
for some \(r^*\) with \(0 < r^* < \min\{\frac{1}{2}, (C(u, 1) + D(\pi, 1))^{-2}\}\).

The second version is stated as follows.

Theorem 1.4 Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1\). If \((u, \pi)\) satisfies
\[
C(u, 1) + D(\pi, 1) \leq M < +\infty,
\]
then there exists a positive constant \(\varepsilon\) depending on \(M\) such that if
\[
\|u_3\|_{L^1(Q_1)} \leq \varepsilon,
\]
then \(u\) is regular in \(Q_{1/2}\).

Compared with classical interior regularity criteria (see Proposition 2.2 and Proposition 2.3), new regularity criterion allows two components of the velocity to be large in the local scaling invariant norm. The proof of Theorem 1.3 and Theorem 1.4 used blow-up analysis. The key point is that under the assumptions of the above theorems the blow-up limit \(v\) satisfies \(v_3 = 0\) and
\[
\begin{align*}
\partial_t v_h - \Delta v_h + v_h \cdot \nabla v_h + \nabla \pi &= 0, \\
\nabla v_h \cdot v_h &= 0, \\
\partial_{x_3} v_h &= 0, \\
\partial_{x_3} \pi &= 0,
\end{align*}
\]
which is similar to the situation considered by Cao-Titi [4]. Using the vorticity equation of the horizontal part \(v_h\), it can be proved that the blow-up limit \(v_h\) is regular (see Appendix for more details).

2 Classical interior regularity criterion

Let us first recall the definition of suitable weak solution introduced in [2].

Definition 2.1 Let \(\Omega \subset \mathbb{R}^3\) and \(T > 0\). The pair \((u, \pi)\) is called a suitable weak solution of (1) in \(Q_T = \Omega \times (-T, 0)\) if

1. \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega))\) and \(\pi \in L^{3/2}(Q_T)\);

2. \(u\) satisfies (1) in the sense of distribution;
3. \( u \) satisfies the local energy inequality

\[
\int_{\Omega} |u(x,t)|^2 \phi \, dx + 2 \int_{-T}^{t} \int_{\Omega} |\nabla u|^2 \phi \, dx \, ds \\
\leq \int_{-T}^{t} \int_{\Omega} |u|^2 (\partial_s \phi + \Delta \phi) + u \cdot \nabla \phi (|u|^2 + 2\pi) \, dx \, ds
\]

for a.e. \( t \in [-T,0] \) and any nonnegative \( \phi \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}) \) vanishing in a neighborhood of the parabolic boundary of \( Q_T \).

Let \( z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R} \). We define a weak solution \( u \) to be regular at \( z_0 \) if \( u \in L^\infty(Q_{r}(z_0)) \) for some \( r > 0 \), where \( Q_{r}(z_0) = (-r^2 + t_0, t_0) \times B_r(x_0) \).

The following quantities are invariant under the scaling (2):

\[
A(u, r, z_0) \triangleq \sup_{-r^2 + t_0 \leq t < t_0} r^{-1} \int_{B_r(x_0)} |u(y,t)|^2 \, dy,
\]

\[
C(u, r, z_0) \triangleq r^{-2} \int_{Q_r(z_0)} |u(y,s)|^3 \, dy \, ds,
\]

\[
E(u, r, z_0) \triangleq r^{-1} \int_{Q_r(z_0)} |\nabla u(y,s)|^2 \, dy \, ds,
\]

\[
D(\pi, r, z_0) \triangleq r^{-2} \int_{Q_r(z_0)} |\pi(y,s)|^\frac{3}{2} \, dy \, ds.
\]

We denote

\[
G(f, p, q; r, z_0) \triangleq r^{1-\frac{3}{p} - \frac{2}{q}} \|f\|_{L^p_t L^q_x(Q_{r}(z_0))},
\]

\[
H(f, p, q; r, z_0) \triangleq r^{2-\frac{3}{p} - \frac{2}{q}} \|f\|_{L^p_t L^q_x(Q_{r}(z_0))},
\]

where the space-time norm \( \| \cdot \|_{L^p_t L^q_x(Q_{r}(z_0))} \) is defined by

\[
\|f\|_{L^p_t L^q_x(Q_{r}(z_0))} \overset{\text{def}}{=} \left( \int_{t_0-r^2}^{t_0} \left( \int_{B_r(x_0)} |f(x,t)|^p \, dx \right)^{\frac{1}{p}} \, dt \right)^{\frac{1}{q}}.
\]

For simplicity, we denote

\[
A(u, r, (0,0)) = A(u, r), \quad G(f, p, q; r, (0,0)) = G(f, p, q; r)
\]

and so on.

Now we recall the following \( \varepsilon \)-regularity results from [2, 17].

**Proposition 2.2** Let \( (u, \pi) \) be a suitable weak solution of (1) in \( Q_1(z_0) \). There exists an \( \varepsilon_0 > 0 \) such that if

\[
\int_{Q_1(z_0)} |u(x,t)|^3 + |\pi(x,t)|^{3/2} \, dx \, dt \leq \varepsilon_0,
\]

then \( u \) is regular in \( Q_{1/2}(z_0) \). Moreover, \( \pi \) can be replaced by \( \pi - (\pi)_{B_r} \) in the integral.
**Proposition 2.3** Let \((u, \pi)\) be a suitable weak solution of \((\mathcal{I})\) in \(Q_1(z_0)\). There exists an \(\varepsilon_1 > 0\) such that if one of the following two conditions holds

1. \(G(u, p, q; r, z_0) \leq \varepsilon_1\) for any \(0 < r < \frac{1}{2}\), where \(1 \leq \frac{3}{p} + \frac{2}{q} \leq 2\);

2. \(H(\nabla \times u, p, q; r, z_0) \leq \varepsilon_1\) for any \(0 < r < \frac{1}{2}\), where \(2 \leq \frac{3}{p} + \frac{2}{q} \leq 3\) and \((p, q) \neq (1, \infty)\);

then \(u\) is regular at the point \(z_0\).

Let us conclude this section by recalling the following lemmas from [28].

**Lemma 2.4** Let \((u, \pi)\) be a suitable weak solution of \((\mathcal{I})\) in \(Q_1\) and \(p > 2, q \geq 2\). Then it holds that

\[
H(\pi, p/2, q/2; r) \leq C \left( \frac{\rho}{r} \right)^{\frac{1}{4} + \frac{2}{p} - 2} G(u, p, q; \rho)^2 + C \left( \frac{\rho}{r} \right)^{2 - \frac{1}{q}} H(\pi, 1, q/2; \rho)
\]

for any \(0 < 4r < \rho < 1\), where \(C\) is a constant independent of \(r, \rho\).

**Lemma 2.5** Let \((u, \pi)\) be a suitable weak solution of \((\mathcal{I})\) in \(Q_1\). If

\[
\sup_{0 < r < 1} G(u, p, q; r) \leq M \quad \text{with} \quad 1 \leq \frac{3}{p} + \frac{2}{q} < 2, 1 < q \leq \infty,
\]

then there holds

\[
A(u, r) + E(u, r) + D(\pi, r) \leq C \left( r^{1/2} (C(u, 1) + D(\pi, 1)) + 1 \right)
\]

for any \(0 < r < 1/2\), where \(C\) is a constant depending on \(p, q, M\).

### 3 BMO space and inclusion relations

Recall that a local integrable function \(f \in BMO(\mathbb{R}^3)\) if it satisfies

\[
\sup_{x_0 \in \mathbb{R}^3, R > 0} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| dx < \infty.
\]

We denote by \(VMO(\mathbb{R}^3)\) the complete space of \(C_0^\infty(\mathbb{R}^3)\) under the norm of \(BMO(\mathbb{R}^3)\). Thus, it holds that

\[
\limsup_{R \to 0} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| dx = 0,
\]

\[
\limsup_{|x_0| \to \infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| dx = 0.
\]

We say that a function \(u \in BMO^{-1}(\mathbb{R}^3)\) if there exist \(U^j \in BMO(\mathbb{R}^3)\) such that \(u = \sum_{j=1}^3 \partial_j U^j\). \(VMO^{-1}(\mathbb{R}^3)\) is defined similarly. A remarkable property of \(BMO\) function is

\[
\sup_{x_0 \in \mathbb{R}^3, R > 0} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}|^q dx < \infty. \quad (3)
\]
for any $1 \leq q < \infty$.

We also need the following Carleson measure characterization of BMO space. We say that the tempered distribution $v$ is in $BMO(\mathbb{R}^3)$ if

$$
\|v\|_{BMO} = \sup_{x_0 \in \mathbb{R}^3, R > 0} \left( |B(x_0, R)|^{-1} \int_{B(x_0, R)} \int_0^R \left| \nabla e^{t\triangle} v \right|^2 dt \right)^{1/2} < \infty,
$$

(4)

and $v$ is in $BMO^{-1}(\mathbb{R}^3)$ if

$$
\|v\|_{BMO^{-1}} = \sup_{x_0 \in \mathbb{R}^3, R > 0} \left( |B(x_0, R)|^{-1} \int_{B(x_0, R)} \int_0^R |e^{t\triangle} v|^2 dt \right)^{1/2} < \infty.
$$

(5)

We refer to [16] and [26] for more introductions.

For the homogeneous Besov space $\dot{B}^{s}_{p,q}(\mathbb{R}^3)$ for $s < 0, p, q \in [1, \infty]$, we have the following characterization by the heat flow:

$$
\|f\|_{\dot{B}^{s}_{p,q}} \overset{\text{def}}{=} \|\|\tau^{-s/2} e^{\tau\triangle} f\|_{L^p(\mathbb{R}^3)}\|_{L^q(\mathbb{R}^3 \frac{dt}{\tau^2})},
$$

(6)

Let us prove the following inclusion lemma.

**Lemma 3.1** Let $p \in (3, \infty)$ and $q \in [1, \infty)$. It holds that

$$
\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^3) \hookrightarrow VMO^{-1}(\mathbb{R}^3).
$$

**Proof.** Due to $\dot{B}^{-1+\frac{2}{p}}_{p,q_1}(\mathbb{R}^3) \subset \dot{B}^{-1+\frac{2}{p}}_{p,q_2}(\mathbb{R}^3)$ for $q_1 \leq q_2$, we may assume $q > 2$. Let

$$
I(x_0, R) = \left( |B(x_0, R)|^{-1} \int_{B(x_0, R)} \int_0^R |e^{t\triangle} f|^2 dt \right)^{1/2},
$$

and $s = -1 + \frac{2}{p}$. By Hölder inequality, we get

\[
I(x_0, R) \leq C \left[ R^{-\frac{2}{p}} \int_0^R \left( \int_{B(x_0, R)} |e^{t\triangle} f|^p dx \right)^{\frac{2}{p}} dt \right]^{1/2} \\
\leq C \left[ R^{-\frac{2}{p}} \int_0^R \left( \int_{B(x_0, R)} |e^{t\triangle} f|^q dx \right)^{\frac{2}{q}} t^{-\frac{s}{q} + \frac{s-1}{q}} dt \right]^{1/q} \left[ \int_0^R t^{(s+\frac{2}{q})-\frac{s-1}{q}} dt \right]^{\frac{q-2}{2q}} \\
\leq C \|f\|_{\dot{B}^{-1+\frac{2}{p}}_{p,q}}
\]

which gives rise to $\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3)$ by [5] and [6]. Since $C^\infty_0(\mathbb{R}^3)$ is dense in $\dot{B}^{-1+\frac{2}{p}}_{p,q}(\mathbb{R}^3)$ for $p, q < \infty$, $f \in VMO^{-1}(\mathbb{R}^3)$. \qed
Lemma 3.2 If \( u \in \dot{B}^{1+3/p}_{p,\infty}(\mathbb{R}^3) \cap H^1(B_2(x_0)) \) for \( p \in (3, \infty) \) and some \( x_0 \in \mathbb{R}^3 \), then we have

\[
\|u\|_{L^2(B_1(x_0))} \leq C(p)\|u\|_{\dot{B}^{1+3/p}_{p,\infty}}^{\theta} \|u\|_{H^1(B_2(x_0))}^{1-\theta}
\]

for \( \theta = \frac{p}{2p-3} \).

Proof. For any \( \varphi(x) \in C_0^\infty(\mathbb{R}^3) \), we have

\[
\|\varphi u\|_{\dot{B}^{1+3/p}_{p,\infty}} \leq C(\varphi)\|u\|_{\dot{B}^{1+3/p}_{p,\infty}}.
\]

Thus, we may assume that \( u \in H^1(\mathbb{R}^3) \) and it suffices to show that

\[
\|u\|_{L^2(B_1(x_0))} \leq C(p)\|u\|_{\dot{B}^{1+3/p}_{p,\infty}}^{\theta} \|u\|_{H^1(B_2(x_0))}^{1-\theta}.
\]

We decompose \( u \) into the low frequency part and high frequency part, i.e.,

\[
u = u_N + u^N, \quad u_N = \mathcal{F}^{-1}(\chi(\xi/2^N)\hat{u}(\xi)),\]

where \( \chi(\xi) \) is a smooth cut-off function with \( \chi(\xi) = 1 \) for \( |\xi| \leq 1 \). For the low frequency, it can be proved by using the Littlewood-Paley characterization of Besov space [1] that

\[
\|u_N\|_{L^2(B_1(x_0))} \leq C\|u_N\|_{L^p(B_1(x_0))} \leq C2^N(1-3/p)\|u\|_{\dot{B}^{1+3/p}_{p,\infty}},
\]

while for the high frequency,

\[
\|u^N\|_{L^2} \leq C2^{-N}\|u\|_{H^1}.
\]

Optimizing \( N \) gives the lemma. \( \Box \)

4 Blow-up of the critical norm

In this section, we prove Theorem 1.1 under the assumption of Theorem 1.3 and Theorem 1.4. By translation and scaling, we may assume that \((u, \pi)\) is a smooth solution of (1) in \( \mathbb{R}^3 \times [-1,0) \) satisfying

\[
\|u_3\|_{L^\infty(-1,0;\dot{B}^{1+3/p}_{p,q})} + \|u_h\|_{L^\infty(-1,0;BMO^{-1})} = M < \infty,
\]

and \( u_h(x,0) \in VMO^{-1}(\mathbb{R}^3) \).

Let us first give a bound of local scaling invariant energy.

Lemma 4.1 Let \((u, \pi)\) be a smooth solution of (7) in \( \mathbb{R}^3 \times [-1,0) \) satisfying (7). Then for \( z_0 \in \mathbb{R}^3 \times (-\frac{1}{2},0) \) and \( 0 < r < \frac{1}{2} \), there holds

\[
A(u, r, z_0) + E(u, r, z_0) + D(\pi, r, z_0) \leq C(M, C(u, \frac{1}{2}, z_0), D(\pi, \frac{1}{2}, z_0)).
\]
Proof. Without loss of generality, we may assume $z_0 = (0, 0)$. Let $\zeta(x, t)$ be a smooth function with $\zeta \equiv 1$ in $Q_r$ and $\zeta = 0$ in $Q_r^\circ$. Since $u \in L^\infty(-1, 0; BMO^{-1}(\mathbb{R}^3))$, there exists $U_i^j(x, t) \in L^\infty(-1, 0; BMO(\mathbb{R}^3))$ such that $u_i = \partial_j \cdot U_i^j$ for $i, j = 1, 2, 3$.

We infer from Hölder inequality that
\[
    r^{-2} \int_{Q_{2r}} |u_i|^3 \zeta^2 dxdt = r^{-2} \int_{Q_{2r}} \sum_{j=1}^{3} \partial_j U_i^j \cdot u_i |u| \zeta^2 dxdt
    \leq C r^{-2} \int_{Q_{2r}} |U - U_{B_{2r}}| (||\nabla u|| + |u|^2|\nabla \zeta|) dxdt,
\]
which gives rise to
\[
    C(u, r) \leq C r^{-2} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^6 dxdt \right)^{1/6} \left( \int_{Q_{2r}} |\nabla u|^2 dxdt \right)^{1/2} \left( \int_{Q_{2r}} |u|^3 dxdt \right)^{1/3}
    + C r^{-3} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^3 dxdt \right)^{1/3} \left( \int_{Q_{2r}} |u|^3 dxdt \right)^{2/3},
\]
from which and (3), it follows that
\[
    C(u, r) \leq C(M) \left( E(u, 2r)^{1/2} C(u, 2r)^{1/3} + C(u, 2r)^{2/3} \right)
    \leq C(M) \left( 1 + E(u, 2r) + C(u, 2r) \right)^{5/6}.
\]
Using the local energy inequality, we can deduce that
\[
    A(u, r) + E(u, r) \leq C \left( C(u, 2r)^{2/3} + C(u, 2r) + C(u, 2r)^{1/3} D(\pi, 2r)^{2/3} \right). \tag{8}
\]
This concludes that
\[
    C(u, r) \leq C(M) \left( 1 + C(u, 4r) + D(\pi, 4r) \right)^{5/6}. \tag{9}
\]
Lemma (2.3) yields that for $0 < 4r < \rho < 1$,
\[
    D(\pi, r) \leq C \left( \frac{r}{\rho} D(\pi, \rho) + \left( \frac{\rho}{r} \right)^2 C(u, \rho) \right). \tag{10}
\]
Let $F(r) = C(u, r) + D(\pi, r)$. We infer from (9–10) that for $0 < 16r < 4\rho < 1$,
\[
    F(r) \leq C(M) \left( 1 + C(u, 4r) + D(\pi, 4r) \right)^{5/6} + C \left( \frac{r}{\rho} \right) D(\pi, \rho)
    + C(M) \left( \frac{\rho}{r} \right)^2 \left( 1 + C(u, 4\rho) + D(\pi, 4\rho) \right)^{5/6}
    \leq C(M) \left( \frac{r}{\rho} \right) F(4\rho) + C(M) \left( \frac{\rho}{r} \right)^{17}.
\]
Choosing $\theta$ such that $C(M) \theta < \frac{1}{2}$, we deduce that for any $0 < r < 1$,
\[
    F(\theta r) \leq \frac{1}{2} F(r) + C(M, \theta).
\]
Then a standard iterative scheme ensures that for $0 < r < \frac{1}{2}$,
\[
    F(r) = C(u, r) + D(\pi, r) \leq C \left( M, C(u, \frac{1}{2}), D(\pi, \frac{1}{2}) \right),
\]
which along with (8) implies the desired result. \(\square\)
Proof of Theorem 1.1. Following [11], the proof is based on blow-up analysis and backward uniqueness of parabolic equations. It suffices to prove that $u \in L^\infty(\mathbb{R}^3 \times (-1, 0))$ by Ladyzhenskaya-Prodi-Serrin type criteria. For this, we assume that $(0, 0)$ is a singular point of $u$ without loss of generality.

Step 1. Blow-up analysis

It follows from Lemma 4.1 that for $z_0 \in \mathbb{R}^3 \times (-\frac{1}{4}, 0)$ and $0 < r < \frac{1}{2}$,

$$A(u, r, z_0) + E(u, r, z_0) + D(\pi, r, z_0) \leq C(M, C(u, 1), D(\pi, 1)). \quad (11)$$

By the interpolation inequality, we get

$$C(u, r, z_0) \leq C(M, C(u, 1), D(\pi, 1)).$$

If the point $(0, 0)$ is not regular, Theorem 1.3 ensures that there exists a sequence $r_k \downarrow 0$ such that

$$r_k^{-2} \int_{Q_{r_k}} |u_3|^3 \, dx \, dt \geq \varepsilon^3. \quad (12)$$

Let

$$u^k(x, t) = r_k u(r_k x, r_k^2 t), \quad \pi^k(x, t) = r_k^2 \pi(r_k x, r_k^2 t).$$

For any $a > 0, T > 0$ and $z_0 \in \mathbb{R}^3 \times (-T, 0)$, it follows from (11) that if $k$ large enough, we have

$$A(u^k, a, z_0) + E(u^k, a, z_0) + C(u^k, a, z_0) + D(\pi^k, a, z_0) \leq C(M, C(u, 1), D(\pi, 1)). \quad (13)$$

Then Aubin-Lions Lemma ensures that there exists $(v, \pi')$ such that for any $a, T > 0$ (up to subsequence)

$$u^k \to v \quad \text{in} \quad L^3(B_a \times (-T, 0)),$$
$$u^k \to v \quad \text{in} \quad C([-T, 0]; L^{9/8}(B_a)),$$
$$\pi^k \to \pi' \quad \text{in} \quad L^4(B_a \times (-T, 0)),$$

as $k \to +\infty$. The lower semi-continuity of the norm gives that

$$\|v_3\|_{L^\infty(-a^2, 0; \dot{B}^{-1+3/p}_{p,q})} \leq M, \quad \|v\|_{L^\infty(-a^2, 0; \dot{B}^{-1+3/p}_{p,q})} \leq M, \quad (14)$$

$$A(v, a, z_0) + E(v, a, z_0) + C(v, a, z_0) + D(\pi', a, z_0) \leq C(M, C(u, 1), D(\pi, 1)). \quad (15)$$

Thanks to (14), there exists a sequence $v_3^0 \in C_0^\infty(\mathbb{R}^3 \times (-a^2, 0))$ so that for any $\ell \in [1, \infty)$,

$$\|v_3^n - v_3\|_{L^\ell(-T, 0; \dot{B}^{-1+3/p}_{p,q})} \to 0 \quad \text{as} \quad n \to \infty,$$

and by (15), we have $\|v\|_{L^{10/3}(Q_a(z_0))} + \|\nabla v\|_{L^2(Q_a(z_0))} < \infty$. Thus, we have

$$\int_{Q_1(z_0)} |v_3|^3 \, dx \, dt \to 0 \quad \text{as} \quad |z_0| \to \infty. \quad (16)$$
Indeed, for any $\epsilon > 0$, we first take $n$ large enough so that

$$\|v^n_3 - v_3\|_{L^2(-T,0;B_{p,q}^{-1+3/p})} < \epsilon.$$ 

Then take $|z_0|$ big enough so that $v^n_3 = 0$ in $Q_2(z_0)$. By Lemma 3.2 we have

$$\int_{Q_1(z_0)} |v^3_3|^2 dx dt = \int_{Q_1(z_0)} |v^n_3 - v_3|^2 dx dt \leq C \|v^n_3 - v_3\|_{L^2(-T,0;B_{p,q}^{-1+3/p})}^{2\theta} \|v_3\|_{L^2_t H^1_z(Q_2(z_0))}^{2(1-\theta)} \leq C \epsilon^{2\theta},$$

which implies (10) by the interpolation.

Thanks to (15) and (16), Theorem 1.4 ensures that there exist $s$ such that $v$ is regular in $(R^3 \setminus B_R) \times (-T,0)$ with the bound

$$|v(x,t)| + |\nabla v(x,t)| \leq C \quad \text{for} \quad (t,x) \in (R^3 \setminus B_R) \times (-T,0). \quad (17)$$

**Step 2.** Prove $v(x,0) = 0$ in $R^3$

Due to $u(x,0) \in VMO^{-1}(R^3)$, $u = \nabla \cdot U$ with $U(x,0) \in VMO(R^3)$. Then for any $\varphi \in C_c^\infty(B_a)$,

$$\left| \int_{B_a} v(x,0) \varphi dx \right| \leq C \int_{B_a} |v(x,0) - u^k(x,0)| dx + \int_{B_a} u^k(x,0) \varphi dx \leq C \int_{B_a} |v(x,0) - u^k(x,0)| dx + Cr_k^3 \int_{B_{2r_k}} |U(y,0)(\nabla \varphi)(y/R_k) dy| \leq C \int_{B_a} |v(x,0) - u^k(x,0)| dx + Cr_k^3 \int_{B_{2r_k}} |U(y,0) - U_{B_{2r_k}}| dy \to 0 \quad \text{as} \quad k \to \infty.$$ 

This shows that $v(x,0) = 0$.

**Step 3.** Backward uniqueness and unique continuation argument

Let $w = \nabla \times v$. By (17) and $v(x,0) = 0$, we get $w(x,0) = 0$ and

$$|\partial_t w - \Delta w| \leq C (|w| + |\nabla w|) \quad \text{in} \quad (R^3 \setminus B_R) \times (-T,0).$$

By the backward uniqueness of parabolic operator [11], we deduce that $w = 0$ in $(R^3 \setminus B_R) \times (-T,0)$. Similar arguments as in [11], by using spatial unique continuation, give $w = 0$ in $R^3 \times (-T,0)$. Hence, we get

$$\Delta v \equiv 0 \quad \text{in} \quad R^3 \times (-T,0).$$

This implies $v_3 = 0$ in $R^3 \times (-T,0)$, since $v_3(t) \in B_{p,q}^{-1+3/p}(R^3)$. This leads to an obvious contradiction with the fact [12]. Thus, $(0,0)$ is a regular point.  \[\square\]
5 New interior regularity criterion

In this section, we first prove two new interior regularity criterion in terms of one velocity component. Then we present an applications to type I singularity.

5.1 Interior regularity criterion

In this subsection, we prove Theorem 1.3 and Theorem 1.4 under the assumption of Proposition 6.1.

Proof of Theorem 1.3 The proof uses blow-up argument. Assume that the statement of the theorem is false. Then there exist constants $p, q, M$ and a sequence of suitable weak solutions $(u^k, \pi^k)$ of $1$ in $Q_1$, which are singular at $(0,0)$ and satisfy
\[
G(u^k, p, q; r) \leq M \quad \text{for all } 0 < r < 1, \\
G(u^k, p, q; r_k) \leq \frac{1}{k},
\]
where
\[
0 < r_k < \min \left\{ \frac{1}{2}, \left( C(u^k, 1) + D(\pi^k, 1) \right)^{-2} \right\}.
\]
Then it follows from Lemma 2.5 that
\[
A(u^k, r) + E(u^k, r) + D(\pi^k, r) \leq C_0(M, p, q)
\]
for any $0 < r < r_k$. Set
\[
v^k(x,t) = r_k u^k(r_k x, r_k^2 t), \quad q^k(x,t) = r_k^2 \pi^k(r_k x, r_k^2 t).
\]
Thus, it holds that
\[
A(v^k, r) + E(v^k, r) + D(\pi^k, r) \leq C_0(M, p, q)
\]
for any $0 < r < 1$. Lions-Aubin’s lemma ensures that there exists a suitable weak solution $(v, \pi)$ of $1$ such that (up to subsequence),
\[
v^k \to v \quad \text{in} \quad L^3(Q_{1/2}), \quad \pi^k \to \pi \quad \text{in} \quad L^2(Q_{1/2}), \\
v^k \to 0 \quad \text{in} \quad L^3((-\frac{1}{4}, 0); L^p(B_{1/2})),
\]
as $k \to +\infty$. So, the blow-up limit $v$ satisfies $v_3 = 0, \partial_3 \pi = 0$ and
\[
\partial_t v^h + v^h \cdot \nabla_h v^h + \nabla_h \pi - \Delta v^h = 0, \quad \nabla_h \cdot v^h = 0.
\]
The limit $v_h$ is bounded in $Q_{1/2}$, by Proposition 6.1 which will contradicts with the fact that $(0,0)$ is a singular point of $v^k$. Indeed by Proposition 2.2 and Lemma 2.4 for any $0 < r < 1/4$,
\[
\varepsilon_0 \leq \liminf_{k \to \infty} r^{-2} \int_{Q_r} |v^k|^3 + |\pi^k|^3 \, dx \, dt
\]
\[
\leq C \liminf_{k \to \infty} \left( C(v, r) + \frac{r}{\rho} D(\pi^k, \rho) + \left( \frac{\rho}{r} \right)^2 C(v^k, \rho) \right)
\]
\[
\leq C \left( C(v, r) + \frac{r}{\rho} + \left( \frac{\rho}{r} \right)^2 C(v, \rho) \right)
\]
\[
\leq Cr^{1/2} \quad \text{(by choosing } \rho = r^{1/2}),
\]
which is a contradiction if we take $r$ small enough. \qed
To prove Theorem 1.4, we need the following lemma from [28].

**Lemma 5.1** Let \((u, \pi)\) be a suitable weak solution of (7) in \(Q_1\) and \(D(\pi, 1) \leq M\). Then \(u\) is regular at \((0, 0)\) if

\[
C(u, r_0) \leq c\varepsilon_0^{9/5} r_0^{8/5} \text{ for some } 0 < r_0 \leq 1.
\]

Here \(c\) is a small constant depending on \(M\).

**Proof of Theorem 1.3** We use the contradiction argument. Assume that the statement of the theorem is false. Then there exist \(M\) and a sequence of suitable weak solutions \((u^k, \pi^k)\) of (1) in \(Q_1\), which are singular at a point \(z_0 \in Q_{1/4}\) (we assume \(z_0 = (0, 0)\)) and satisfy

\[
C(u^k, 1) + D(\pi^k, 1) \leq M, \quad \|u^k_3\|_{L^1(Q_1)} \leq \frac{1}{k}.
\]

Then by the local energy inequality, it is easy to show that

\[
A(u^k, 3/4) + E(u^k, 3/4) \leq C(M).
\]

By using Lions-Aubin’s lemma, there exists a suitable weak solution \((v, \pi')\) of (1) such that (up to subsequence),

\[
u^k \to v, \quad u^k_3 \to 0 \text{ in } L^3(Q_{1/4}), \quad \pi^k \to \pi' \text{ in } L^{3/2}(Q_{1/4}),
\]

as \(k \to +\infty\). Note that \(v_3 = 0\) implies \(\partial_3\pi' = 0\), hence

\[
\partial_t v_h + v_h \cdot \nabla_h v_h + \nabla_h \pi' - \Delta v_h = 0, \quad \nabla_h \cdot v_h = 0.
\]

The limit \(v_h\) is bounded in \(Q_{1/4}\) by Proposition 6.1. Since \((0, 0)\) is a singular point of \(u^k\), we have by Lemma 5.1 that for any \(0 < r < 1/4\),

\[
c\varepsilon_0^{9/5} r^{8/5} \leq \lim_{k \to \infty} r^{-2} \int_{Q_r} |u^k_3|^3 dx dt \leq \lim_{k \to \infty} C(v, r) \leq C(M)r^3,
\]

which is a contradiction by letting \(r \to 0\). \(\square\)

### 5.2 Application to type I singularity

In two important works [7, 20], the authors excluded the existence of type I singularity (i.e., \(\|u(t)\|_{L^\infty} \leq \frac{C}{\sqrt{-t}}\)) for the axisymmetric Navier-Stokes equations. The general case remains open. As an application of interior regularity criterion, we prove nonexistence of type I singularity in the case when one velocity component has a better control.

**Theorem 5.2** Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1\) and satisfy

\[
|u_h(x, t)| \leq \frac{M}{\sqrt{-t}}.
\]

If \(u_3 \in L^p_t L^q_x(Q_1)\) with \(\frac{2}{p} + \frac{2}{q} \leq 1\) for \(3 < p \leq \infty\) or there exists \(\epsilon > 0\) depending on \(M\) so that \(|u_3(x, t)| \leq \frac{\epsilon}{\sqrt{-t}}\), then \(u\) is regular at \((0, 0)\).
Proof. In the case when \( u_3 \in L_t^qL_x^p(Q_1) \), Lemma 2.5 ensures that there exists \( r_0 > 0 \) so that for any \( 0 < r < r_0 \),

\[
C(u, r) + D(\pi, r) \leq M, \quad \|u_3\|_{L_t^qL_x^p(Q_\tau)} \leq \epsilon,
\]

which implies that \( u \) is regular at \( (0, 0) \) by Theorem 1.3. The case of \( |u_3(x, t)| \leq \frac{\epsilon}{\sqrt{-t}} \) follows from Theorem 1.4 with \( p = \infty \) and \( q = \frac{3}{2} \).

Recently, there are many interesting works devoted to the regularity criterions involving one velocity component, see [5, 6, 9, 10, 22, 24] and references therein. However, the obtained regularity conditions are not scaling invariant in these works except [9, 10], where they showed that if \( u_3 \in L^p(0, T; H^{\frac{3}{2} + \frac{2}{p}}(\mathbb{R}^3)) \) for \( p \in (4, \infty) \), then the solution can be extended after \( t = T \). The question of whether the solution is regular in the class \( u_3 \in L^q(0, T; L^p(\mathbb{R}^3)) \) with \( \frac{3}{p} + \frac{2}{q} = 1 \) for \( 3 < p \leq \infty \) is still open.

6 Appendix

In this appendix, we study the regularity of the blow-up limit introduced in the proof of interior regularity criterion, which satisfies

\[
\begin{align*}
\partial_t v_h - \Delta v_h + v_h \cdot \nabla_h v_h + \nabla_h \pi &= 0, \\
\nabla_h \cdot v_h &= 0, \\
\partial_{x_3} \pi &= 0.
\end{align*}
\] (18)

Here \( \nabla_h = (\partial_{x_1}, \partial_{x_2}) \).

Proposition 6.1 Let \((v_h, \pi)\) be a suitable weak solution of (18) in \( Q_1 \). Assume that

\[
\|v_h\|_{L_t^\infty L_x^2(Q_1)} + \|\nabla v_h\|_{L^2(Q_1)} \leq M.
\]

Then it is regular in \( Q_{\frac{3}{4}} \) and \( \|v_h\|_{L^\infty(Q_{\frac{3}{4}})} \leq C(M) \).

Proof. By Proposition 2.3 we only need to prove that \( \|\nabla v_h\|_{L_t^\infty L_x^2(Q_{\frac{3}{4}})} \) is finite. Using the fact that the \( \frac{1}{2} \)-dimensional Hausdorff measure of the set of singular time is zero (see [2]), without loss of generality we may assume \( v_h \in C^\infty(Q_1) \). Let \( w_h = \partial_1 v_2 - \partial_2 v_1 \) be the vorticity of \( v_h \). Then \( w_h \) satisfies

\[
\partial_t w_h - \Delta w_h + v_h \cdot \nabla_h w_h = 0 \quad \text{in} \quad Q_1.
\] (19)

While, \( d = \partial_3 v_h \) satisfies

\[
\partial_t d - \Delta d + v_h \cdot \nabla_h d + d \cdot \nabla_h v_h = 0 \quad \text{in} \quad Q_1.
\] (20)

Step 1. Estimate of \( \|w_h\|_{L_t^\infty L_x^2(Q_{\frac{3}{4}})} \)

Let \( \zeta \) be a cutoff function, which vanishes outside of \( Q_1 \) and equals 1 in \( Q_{\frac{3}{4}} \), and satisfies

\[
|\nabla \zeta| + |\partial_t \zeta| + |\nabla^2 \zeta| \leq C.
\]
Acting on $2uζ^4$ on both sides of (19), we obtain

$$I ≜ \sup_t \int_{B_1} |w_h|^2ζ^4dx + 2\int_{Q_1} |\nabla(w_hζ^2)|^2dxdt$$

$$\leq \int_{Q_1} [\partial_t(ζ^4) + 2|\nabla(ζ^2)|^2]|w_h|^2dxdt + \int_{Q_1} v_h \cdot \nabla(ζ^4)|w_h|^2dxdt$$

$$\triangleq I_1 + I_2.$$ 

Obviously, $I_1 \leq C(M)$. For $I_2$, we get by Sobolev inequality that

$$I_2 \leq C\int_{Q_1} |v_h||w_h|^2|ζ^2|^2dxdt = C\|v_h\|_{L^\infty_t L^2_x(Q_1)}\|w_h\|_{L^2_t L^2_y(Q_1)} \|ζ^2\|_{L^2_t L^2_y(Q_1)} \leq C(M)I^2.$$ 

This shows that

$$I \leq C(M) + C(M)I^2,$$

which implies that $\|w_h\|_{L^\infty_t L^2_x(Q_1)} < \infty$.

**Step 2.** Estimate of $\|∇_h v_h\|_{L^\infty_t L^2_x(Q_1)}$

Since $w_h$ is the vorticity of $v_h$ and $v_h$ is divergence-free, the elliptic estimate gives

$$\|∇_h v_h\|_{L^2_x(B_1)} \leq C(\|w_h\|_{L^2_x(B_1)} + \|v_h\|_{L^2_x(B_1)}) \leq C(M).$$

**Step 3.** Estimate of $\|d\|_{L^\infty_t L^2_x(Q_1)}$.

Let $η$ be a cutoff function, which vanishes outside of $Q_1$ and equals 1 in $Q_1$, and satisfies

$$|\nabla η| + |\partial_t η| + |\nabla^2 η| \leq C_0.$$ 

Acting $2dη^4$ on both sides of (20), we obtain

$$II ≜ \sup_t \int_{B_1} |d|^2η^4dx + 2\int_{Q_1} |\nabla(dη^2)|^2dxdt$$

$$\leq \int_{Q_1} [\partial_t(η^4) + 2|\nabla(η^2)|^2]|d|^2dxdt + \int_{Q_1} v_h \cdot \nabla(η^4)|d|^2dxdt$$

$$- \int_{Q_1} d \cdot \nabla_h v_h \cdot (2dη^4)dxdt$$

$$\triangleq II_1 + II_2 + II_3.$$ 

Obviously, $II_1 \leq C(M)$. Similar to $I_2$, we have

$$II_2 \leq C(M)I^2.$$ 

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Thanks to $\|\nabla_h v_h\|_{L^\infty_t L^2_x(Q_\frac{1}{2})} \leq C(M)$, we have

$$II_3 \leq C\|\nabla_h v_h\|_{L^\infty_t L^2_x(Q_\frac{1}{2})}\|d\|_{L^\frac{1}{2}_t L^6_x(Q_1)}^\frac{1}{2}\|d\eta\|_{L^2_t L^6_x(Q_1)}^\frac{3}{2} \leq C(M)II_3^\frac{3}{4}.$$ 

This shows that

$$II \leq C(M) + C(M)II_3^\frac{3}{4},$$

which implies that $\|d\|_{L^\infty_t L^2_x(Q_\frac{1}{2})} \leq C(M)$. □

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