WELLPOSEDNESS OF THE NAVIER-STOKES-MAXWELL EQUATIONS

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Abstract. We study the local and global wellposedness of a full system of Magneto-Hydro-Dynamic equations. The system is a coupling of the forced (Lorentz force) incompressible Navier-Stokes equations with the Maxwell equations through Ohm’s law for the current. We show the local existence of mild solutions for arbitrarily large data in a space similar to the scale invariant spaces classically used for Navier-Stokes. These solutions are global if the initial data are small enough. Our results not only simplify and unify the proofs for the space dimensions two and three but also refine those in [10]. The main simplification comes from an a priori $L^2_t(L^\infty_x)$ estimate for solutions of the forced Navier-Stokes equations.

1. Introduction

The purpose of this paper is the study of the following full Magneto-Hydro-Dynamics system (MHD):

$$\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \nabla p &= j \times B \\
\partial_t E - \text{curl} \, B &= -j \\
\partial_t B + \text{curl} \, E &= 0 \\
\text{div} \, v &= \text{div} \, B = 0 \\
\sigma(E + v \times B) &= j
\end{align*}$$

subject to the initial data

$$v|_{t=0} = v^0, \quad B|_{t=0} = B^0, \quad E|_{t=0} = E^0.$$ 

Here, $v, E, B : \mathbb{R}^+_t \times \mathbb{R}^d_+ \rightarrow \mathbb{R}^3$ are vector fields defined on $\mathbb{R}^d$ ($d = 2$ or $3$). The vector field $v = (v_1, ..., v_d)$ is the velocity of the fluid, $\nu$ its viscosity and the scalar function $p$ stands for the pressure. The vector fields $E$ and $B$ are the electric and magnetic fields, respectively, and $j$ is the electric current given by Ohm’s law (the fifth equation of the system, where $\sigma$ is the electric resistivity). The force term $j \times B$ in the Navier-Stokes equations comes from the Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. Note that the pressure

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can be recovered from $v$ and $j \times B$ via an explicit Caldéron-Zygmund operator (see [5] for instance). The second equation in (1.1) is the Ampère-Maxwell equation for an electric field $E$. The third equation is nothing but Faraday’s law. For a detailed introduction to MHD, we refer to Davidson [8] and Biskamp [2].

Note that in the 2D case, the functions $v$, $E$, $B$, and $j$ are defined on the whole space $\mathbb{R}^2$ with values in $\mathbb{R}^3$. In this case, the operator $\nabla$ is given by

$$\nabla = (\partial_{x_1}, \partial_{x_2}, 0)^T.$$

Thus

$$\text{div } v := \partial_{x_1} v_1 + \partial_{x_2} v_2, \quad \nabla p := (\partial_{x_1} p, \partial_{x_2} p, 0)^T,$$

and

$$\text{curl } F := (\partial_{x_2} F_3, -\partial_{x_1} F_3, \partial_{x_1} F_2 - \partial_{x_2} F_1)^T.$$

In the following, we take $\sigma = \nu = 1$ to alleviate the notations.

Multiply the Navier-Stokes equations in (1.1) by $v$, the Ampère-Maxwell equations by $(B, E)^T$ and integrate (using the divergence free condition on the velocity); this gives the formal energy identity

$$\frac{1}{2} \frac{d}{dt} \left[ \|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2 \right] + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0.$$

This identity shows that the energy is dissipated by the viscosity and the electric resistivity. It also suggests that one should be able to construct a global finite energy weak solution (à la Leray) for data lying in $L^2(\mathbb{R}^d)$. However, this intuitive expectation remains an interesting open problem for (1.1) in both dimensions $d = 2, 3$. Indeed, given a standard approximating scheme, it is hard to obtain compactness of the solutions, especially for the magnetic field due to the hyperbolicity of Maxwell’s equations. In dimension 2, the equation is energy critical, but running a fixed point argument for data $(v^0, E^0, B^0)$ only in $L^2(\mathbb{R}^2)^3$ seems very difficult due to the term $E \times B$.

Imposing more regularity on the initial electro-magnetic field, existence results are known. Recently, for initial data $(v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with $s > 0$, Masmoudi in [14] proved the existence and uniqueness of global strong solutions to (1.1). His proof relies on the use of the energy inequality combined with a logarithmic inequality that enabled him to upper estimate the $L^\infty$ norm of the velocity field by the energy norm and higher Sobolev norms. It is also interesting to note that the proof in [14] does not use the divergence free condition of the magnetic field, nor the decay property of the linear part coming from Maxwell’s equations, namely

\[
\begin{cases}
\frac{\partial E}{\partial t} - \text{curl } B + E = f, \\
\frac{\partial B}{\partial t} + \text{curl } E = 0, \\
\nabla \cdot B = 0,
\end{cases}
\]

(1.2)
Another line of research was pursued by Ibrahim and Keraani [10] who considered data \((v_0, E_0, B_0) \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3))^2\) in dimension \(d = 3\), and \((v_0, E_0, B_0) \in \dot{B}^0_{2,1}(\mathbb{R}^2) \times (L^2_{\text{log}}(\mathbb{R}^2))^2\) in dimension \(d = 2\) (see below for the definition of these functional spaces). These authors built up strong solutions by using parabolic regularization arguments giving control of the \(L^\infty\) norm of the velocity field of the solution. More recently, Ibrahim and Yoneda constructed local in time solution for non-decaying initial data on the torus. See [11] for more details.

In this paper, we follow up on the work of Ibrahim and Keraani by running a fixed point argument to obtain mild solutions, but taking the initial velocity field in the natural Navier-Stokes space \(H^{d-1}_2\). Our main theorem extends the earlier results that were mentioned in many respects: the regularity of the initial velocity and electromagnetic fields is lowered, and we unify the proofs in the cases of space dimension 2 and 3. One of the key ingredients will be to use an \(L^2L^\infty\) estimate on the velocity field, which simplifies greatly the fixed point argument; in particular, the weak decay for the electromagnetic field is not needed any more in dimension 3.

Before stating our main result, we need a few definitions.

**Definition 1.1.** First, let \(P\) denote the Leray projection on divergence-free vector fields.

A function \(\Gamma := (v, E, B)\) with \(\text{div}(v) = \text{div}(B) = 0\) is said to be a mild solution on a time interval \([0, T]\) of the full MHD problem [11] if \(\Gamma \in C([0, T], \dot{H}^{d-1}_2)\) and satisfies the integral equation

\[
\Gamma(t) = e^{tA}\Gamma(0) + \int_0^t e^{(t-t')A}\mathcal{N}(\Gamma(t'))\,dt',
\]

with

\[
A = \begin{pmatrix}
\Delta & 0 & 0 \\
0 & -I & \text{curl} \\
0 & -\text{curl} & 0
\end{pmatrix}
\]

and \(\mathcal{N}(\Gamma) = (P(-\nabla(v \otimes v) + E \times B + (v \times B) \times B), -v \times B, 0)^T\).

The functional analytic framework we will use is the following.

**Definition 1.2.** Let \(\Delta_q\) denote the dyadic frequency localization operator defined in section 2. For \(s, t \in \mathbb{R}\) and \(\alpha \geq 0\) define the space \(\dot{H}^{s,t}_{\alpha}\) by its norm

\[
\|\phi\|^2_{\dot{H}^{s,t}_{\alpha}} := \sum_{q \leq 0} 2^{2qs}\|\Delta_q\phi\|^2_{L^2} + \sum_{q > 0} q^\alpha 2^{2qt}\|\Delta_q\phi\|^2_{L^2}.
\]

We will also use the short-hands

\[
\dot{H}^s = \dot{H}^{0,s}, \quad \dot{H}^{s}_{\text{log}} := \dot{H}^{s,s}_{1} \quad \text{and} \quad \dot{H}^{s,t} := \dot{H}^{s,t}_{0}.
\]

Finally, define \(\tilde{L}^t_{\alpha}\dot{H}^{s,t}_{\alpha}\) by its norm

\[
\|\phi\|^2_{\tilde{L}^t_{\alpha}\dot{H}^{s,t}_{\alpha}} := \sum_{q \leq 0} 2^{2qs}\|\Delta_q\phi\|^2_{L^t_{\alpha}L^2} + \sum_{q > 0} q^\alpha 2^{2qt}\|\Delta_q\phi\|^2_{L^t_{\alpha}L^2},
\]

with obvious generalizations to \(\tilde{L}^t_{\alpha}\dot{H}^s\) etc...
The space $\dot{H}_{\log}^s$ is articulated on the standard homogeneous Sobolev space $\dot{H}^s$ with an extra logarithmic weight for the high frequency part. The space $\dot{H}^{s,t}$ is nothing but the usual Sobolev space $\dot{H}^t$ for high frequencies while it behaves like $\dot{H}^s$ for low frequencies. If $s > t$, it is not difficult to see that $\dot{H}^{s,t} = \dot{H}^s + \dot{H}^t$. The $\tilde{L}$ spaces were first used by Chemin and Lerner [7].

Our main result can be stated as follows.

**Theorem 1.3.**

- In dimension two and for any $\Gamma^0 := (v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2)$,
  
  there exists $T > 0$ and a unique mild solution $\Gamma = (v, E, B)$ of (1.1) with initial data $\Gamma^0$ and
  
  $v \in \tilde{L}^\infty(0, T; L^2) \cap L^2(0, T; \dot{H}^1 \cap L^\infty)$

  $E \in \tilde{L}^\infty(0, T; L^2_{\log}) \cap L^2(0, T; L^2_{\log})$

  $B \in \tilde{L}^\infty(0, T; L^2_{\log}) \cap L^2(0, T; \dot{H}^{1,0})$.

  Moreover, the solution is global (i.e. $T = \infty$) if the initial data is sufficiently small in $L^2 \times L^2_{\log} \times L^2_{\log}$.

- In dimension three and for any $\Gamma^0 := (v^0, E^0, B^0) \in \dot{H}^{3/2}(\mathbb{R}^3) \times \dot{H}^{3/2}(\mathbb{R}^3) \times \dot{H}^{3/2}(\mathbb{R}^3)$,
  
  there exists $T > 0$ and a unique mild solution $\Gamma = (v, E, B)$ of (1.1) with initial data $\Gamma^0$ and
  
  $v \in \tilde{L}^\infty(0, T; \dot{H}^{3/2}) \cap L^2(0, T; \dot{H}^{3/2} \cap L^\infty)$

  $E \in \tilde{L}^\infty(0, T; \dot{H}^{3/2}) \cap L^2(0, T; \dot{H}^{3/2})$

  $B \in \tilde{L}^\infty(0, T; \dot{H}^{3/2}) \cap L^2(0, T; \dot{H}^{3/2, 1/2})$.

  Moreover, the solution is global (i.e. $T = \infty$) if the initial data is sufficiently small in $\dot{H}^{3/2} \times \dot{H}^{3/2} \times \dot{H}^{3/2}$.

In dimension two, the extra logarithmic regularity is needed to estimate the term $E \times B$ appearing in the Navier-Stokes equations.

In dimension three, the control of $B$ in $L^2(0, T; \dot{H}^{3/2, 1/2})$ is not needed to close the fixed point estimate, but we added it for completeness.

The rest of this paper is organized as follows. In the next section we define some further tools needed in the proof. In Section 3, we detail the linear (parabolic regularity) and nonlinear (product law) estimates needed in the proof of the main theorem. The main theorem is then proved in Section 4. Finally, the proofs of some technical estimates are given in the appendix.

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Throughout this work we use the following notation.

1. For any positive $A$ and $B$ the notation $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leq CB$.
2. $c$ will always denote an absolute constant $0 < c < 1$.
3. For any tempered distribution $u$, both $\hat{u}$ and $\mathcal{F}u$ denote the Fourier transform of $u$.
4. For every $p \in [1, \infty]$, $\| \cdot \|_{L^p}$ denotes the norm in the Lebesgue space $L^p$.
5. For any normed space $X$, the mixed space-time Lebesgue space $L^p([0, T], X)$ denotes the space of functions $f$ such that for almost all $t \in (0, T)$, $f(t) \in X$ and $\|f(t)\|_X \in L^p(0, T)$. The notation $L^p([0, T], X)$ is often shortened to $L^p_T X$.

Let us recall the well-known Littlewood-Paley decomposition and the corresponding cut-off operators. There exists a radial positive function $\varphi \in \mathcal{D}(\mathbb{R}^d\setminus\{0\})$ such that

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},$$

$$\text{Supp} \, \varphi(2^{-q} \cdot) \cap \text{Supp} \, \varphi(2^{-j} \cdot) = \emptyset, \quad \forall |q - j| \geq 2.$$

For every $q \in \mathbb{Z}$ and $v \in S'(\mathbb{R}^d)$ we set

$$\Delta_q v = \mathcal{F}^{-1} \varphi(2^{-q} \xi) \hat{v}(\xi) \quad \text{and} \quad S_q = \sum_{j=-\infty}^{q-1} \Delta_j.$$

Bony’s decomposition \cite{3} consists in splitting the product $uv$ into three parts\footnote{It should be said that this decomposition is true in the class of distributions for which $\sum_{q \in \mathbb{Z}} \Delta_q = I$. For example, polynomial functions do not belong to this class.}

$$uv = T_u v + T_v u + R(u, v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$
3. Linear and Nonlinear estimates

We will make an extensive use of Bernstein’s inequalities (see [5] for instance).

**Lemma 3.1** (Bernstein’s lemma). There exists a constant $C$ such that for any $q, k \in \mathbb{N}$, $1 \leq a \leq b$ and for $f \in L^a(\mathbb{R}^d)$,
\[
\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} \leq C^k 2^{q(k+\frac{d}{2}-\frac{d}{a})}\|S_q f\|_{L^a},
\]
and
\[
C^{-k} 2^{qk}\|\Delta f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta f\|_{L^a} \leq C^{k} 2^{qk}\|\Delta f\|_{L^a}.
\]

The parabolic regularity result we will need reads

**Lemma 3.2** (Parabolic regularization, see for example [1]). Let $u$ be a smooth divergence free vector field solving
\[
\begin{align*}
\partial_t u - \Delta u + \nabla p &= f, \\
u(t=0) &= u^0,
\end{align*}
\]
on some time interval $[0, T]$. Then, for every $p \geq r \geq 1$ and $s \in \mathbb{R}$ and $j \geq 1$,
\[
\|u\|_{C([0,T]; \dot{B}^{s+\frac{d}{2}}_{q,j}) \cap L^p_{\tau} \dot{B}^{s+\frac{d}{2}}_{q,j}^\prime} \lesssim \|u^0\|_{\dot{B}^{s+\frac{d}{2}}_{q,j}} + \|f\|_{L^p_{\tau} \dot{B}^{s-2+\frac{d}{2}}_{q,j}}.
\]

The following result is an $L^2_T L^\infty$ estimate which was originally proved in [12], [6] in dimension two.

**Lemma 3.3** ($L^2 T^\infty$ estimate). Let $d = 2, 3$ and $u$ be a smooth divergence free vector field solving
\[
\begin{align*}
\partial_t u - \Delta u + \nabla p &= f_1 + f_2, \\
u(t=0) &= u^0,
\end{align*}
\]
on some time interval $[0, T]$. Assume that $f_1 \in L^1_T \dot{H}^{\frac{d}{2}-1}$ and $f_2 \in \tilde{L}^2_T \dot{B}^{\frac{d}{2}-2}_{2,1}$. Then,
\[
\|u\|_{L^2_T L^\infty} \lesssim \|u^0\|_{\dot{H}^{\frac{d}{2}-1}} + \|f_1\|_{L^1_T \dot{H}^{\frac{d}{2}-1}} + \|f_2\|_{\tilde{L}^2_T \dot{B}^{\frac{d}{2}-2}_{2,1}}.
\]

**Proof.** Thanks to Lemma 3.2 and using the embeddings
\[
\tilde{L}^2_T \dot{B}^{d/2}_{2,1} \hookrightarrow L^2_T \dot{B}^{d/2}_{2,1} \hookrightarrow L^2_T L^\infty,
\]
we can assume that $f_2 = 0$. Duhamel’s formula gives
\[
u(t) = e^{t\Delta} u^0 + \int_0^t e^{(t-t')\Delta} \mathcal{P} f_1(t') \, dt',
\]
and thus
\[
\|u(t)\|_{L^2_T L^\infty} \lesssim \|e^{t\Delta} u^0\|_{L^2_T L^\infty} + \int_0^T \|e^{(t-t')\Delta} \mathcal{P} f_1(t')\|_{L^2_T L^\infty} \, dt'.
\]
Using the embedding $\dot{H}^{\frac{d}{2}-1} \hookrightarrow \dot{B}^{-1}_{\infty,2}$ and the characterization of Besov spaces of negative regularity (see for example [11]),
\[
\|u\|_{\dot{B}^{-1}_{\infty,2}} \sim \|e^{t\Delta} u\|_{L^\infty(0, \infty)},
\]
thus we obtain (3.1) as desired. \qed
Now we focus on Maxwell’s equations. The first result is an energy type estimate.

Lemma 3.4. Let $\alpha \geq 0$, $G_1 \in L^2_t\dot{H}^{-\frac{d}{2}}_\alpha$, and $(E, B)$ be a smooth solution of
\[
\begin{align*}
\partial_t E - \text{curl} \, B + E &= G \\
\partial_t B + \text{curl} \, E &= 0 \\
(E, B)|_{t=0} &= (E_0, B_0),
\end{align*}
\]
on some time interval $[0, T]$. Then, the following estimate holds (with constants independent of $T$)
\[
\begin{align*}
\|E\|_{L^\infty_t \dot{H}^{-\frac{d}{2}}_\alpha \cap L^2_t H^{-\frac{d}{2}}_\alpha} + \|B\|_{L^\infty_t \dot{H}^{-\frac{d}{2}}_\alpha \cap L^2_t H^{-\frac{d}{2}}_\alpha} &\lesssim \|(E_0, B_0)\|_{H^\frac{d}{2}_\alpha} + \|G\|_{L^2_t H^{-\frac{d}{2}}_\alpha}.
\end{align*}
\]
Moreover, $B$ satisfies the following decay estimate
\[
\begin{align*}
\|B\|_{L^2_t \dot{H}^{-\frac{d}{2}}_\alpha} &\lesssim \|(E_0, B_0)\|_{H^\frac{d}{2}_\alpha} + \|G\|_{L^2_t \dot{H}^{-\frac{d}{2}}_\alpha}.
\end{align*}
\]
We emphasize that for the existence and uniqueness part of Theorem 1.3 in dimension 3, estimate [3.4] is irrelevant.

Proof. Only the estimate of $\|B\|_{L^2_t \dot{H}^{-\frac{d}{2}}_\alpha}$ requires a proof, which is given in the Appendix. All other estimates can be derived by a standard energy estimate; apply $\Delta_q$ to the system, derive an energy inequality, multiply both members of that inequality by $2^{(\frac{d}{2}-1)}\sqrt{\max(1, q^2)}$ and take the $\ell^2(\mathbb{Z})$-norm. \(\square\)

The following is a series of nonlinear estimates needed for the contraction argument.

Proposition 3.5. For all smooth functions $u$, $E$ and $B$ defined on some interval $[0, T]$, we have the following estimates, with constants independent of $T$: in space dimension 2,
\[
\begin{align*}
\|\nabla (u \otimes v)\|_{L^1_t L^2(\mathbb{R}^2)} &\lesssim \|u\|_{L^2_t (L^\infty(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2))} \|v\|_{L^2_t (L^\infty(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2))} \\
\|E \times B\|_{L^1_t L^2(\mathbb{R}^2) \cap L^2_t L^2(\mathbb{R}^2)} &\lesssim \|E\|_{L^2_t L^2(\mathbb{R}^2)} \|B\|_{L^2_t L^2(\mathbb{R}^2) \cap \dot{H}^{1,0}(\mathbb{R}^2)} \\
\|u \times B\|_{L^2_t L^2(\mathbb{R}^2)} &\lesssim \|u\|_{L^2_t (L^\infty(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2))} \|B\|_{L^2_t L^2(\mathbb{R}^2)},
\end{align*}
\]
and in space dimension 3,
\[
\begin{align*}
\|\nabla (u \otimes v)\|_{L^1_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)} &\lesssim \|u\|_{L^2_t (L^\infty(\mathbb{R}^3) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} \|v\|_{L^2_t (L^\infty(\mathbb{R}^3) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} \\
\|E \times B\|_{L^1_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)} &\lesssim \|E\|_{L^2_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)} \|B\|_{L^2_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)} \\
\|u \times B\|_{L^2_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)} &\lesssim \|u\|_{L^2_t (L^\infty(\mathbb{R}^3) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))} \|B\|_{L^2_t \dot{H}^{\frac{3}{2}}_\alpha(\mathbb{R}^3)}. 
\end{align*}
\]
Estimates (3.5) and (3.8) enable us to control the advection term in the Navier-Stokes component of the system in dimension two and three, respectively. Estimates (3.6) and (3.9) are needed to control the Maxwell part in the Navier-Stokes component. To estimate the term \((u \times B) \times B\), we use (3.6), (3.7) in two space dimensions and (3.9), (3.10) in three space dimensions.

**Remark 3.6.** Ignoring the time variable, estimate (3.9) is a particular case of the product law

\[
H^{s_1}(\mathbb{R}^d) \cdot H^{s_2}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1+s_2-d/2}_{2,1}(\mathbb{R}^d)
\]

with \(s_1, s_2 \in ]-d/2, d/2[\) and \(s_1 + s_2 > 0\). Indeed, it corresponds to the admissible choice \(s_1 = s_2 = \frac{1}{2}\). However, this product law becomes critical in two space dimensions. Estimate (3.6) shows that an extra logarithmic loss is needed in this case.

We give the proof of the above proposition in the Appendix.

### 4. Proof of Theorem 1.3

4.1. **Small data, global existence.** Let \(\alpha = 1\) if \(d = 2\) and \(\alpha = 0\) if \(d = 3\). Let \(\mathcal{Z}\) be the set of \(\Gamma := (u, E, B)^T\) such that

- \(u \in \mathcal{Z}^u := L^2(0, \infty, \dot{H}^{d/2} \cap L^\infty) \cap \dot{L}^\infty(0, \infty, \dot{H}^{d/2-1})\)
- \(E \in \mathcal{Z}^E := (\dot{L} \cap L^2)(0, \infty, \dot{H}_\alpha^{d/2-1})\)
- \(B \in \mathcal{Z}^B := \dot{L}^\infty(0, \infty, \dot{H}_\alpha^{d/2-1}) \cap L^2(0, \infty, \dot{H}_\alpha^{d/2-1}).\)

Endow \(\mathcal{Z}, \mathcal{Z}^u, \mathcal{Z}^E\) and \(\mathcal{Z}^B\) with the natural norms. Recall that we seek a solution to (1.1) in the integral form

\[
\Gamma(t) = e^{tA} \Gamma(0) + \int_0^t e^{(t-t')A} \mathcal{N}(\Gamma(t')) \, dt',
\]

with

\[
A = \begin{pmatrix}
\Delta & 0 & 0 \\
0 & -I & \text{curl} \\
0 & \text{curl} & 0
\end{pmatrix}.
\]

and \(\mathcal{N}(\Gamma) = (\mathcal{P}(-\nabla(u \otimes u) + E \times B + (u \times B) \times B), -u \times B, 0)^T\). Let \(B_\delta\) be the ball of the space \(\mathcal{Z}_\infty\) centered at zero and with radius \(\delta > 0\) to be chosen. Define the map \(\Phi\) on that ball as follows

\[
\Phi : B_\delta \subset \mathcal{Z} \rightarrow \mathcal{Z}
\]

(4.1) 

\[\Gamma \mapsto \Phi(\Gamma) := \int_0^t e^{(t-t')A} \mathcal{N}(e^{t'A} \Gamma(0) + \Gamma(t')) \, dt'.\]

**Claim 4.1.** If \(\|\Gamma(0)\|_{\dot{H}_\alpha^{d/2-1} \times \dot{H}_\alpha^{d/2-1} \times \dot{H}_\alpha^{d/2-1}} \leq \kappa \delta\), with \(\delta > 0\) and \(\kappa > 0\) sufficiently small, then the map \(\Phi\) is a contraction on \(B_\delta\).
The theorem follows immediately from the claim: Picard’s theorem gives the existence of a fixed point of \( \Phi \), call it \( \Gamma \). Then \( e^{tA} \Gamma_0 + \Gamma(t) \) is the desired solution.

**Proof of the claim:** First, notice that \( \Phi \left( -e^{tA} \Gamma_0 \right) = 0 \), while by Lemmas 3.2, 3.3 and 3.4

\[
\| e^{tA} \Gamma_0 \|_Z \leq C \| \Gamma_0 \|_{H^{\frac{d}{2}} + \frac{d}{2} \times H^{\frac{d}{2}}_0 + \delta \cdot \frac{d}{2}} \leq C \kappa \delta \leq \frac{\delta}{2}
\]

for \( \kappa \) small enough. On the other hand, we will prove below that, if \( \Gamma_1 \) and \( \Gamma_2 \) belong to \( B_\delta \),

\[
\| \Phi(\Gamma_1) - \Phi(\Gamma_2) \|_Z \leq \frac{1}{2} \| \Gamma_1 - \Gamma_2 \|_Z
\]

under the assumptions of the claim.

The estimates (4.2) and (4.3) easily yield the claim.

To prove (4.3), let \( \Gamma_j := (u_j, E_j, B_j)^T \in B_\delta \) for \( j = 1, 2 \). Write further

\[
e^{tA} \Gamma_0 + \Gamma_j(t) = (\tilde{u}_j, \tilde{E}_j, \tilde{B}_j)
\]

and set \( \Gamma := \Gamma_1 - \Gamma_2 := (u, E, B)^T \), and \( \Phi(\Gamma_j) := \tilde{\Gamma}_j = (\tilde{u}_j, \tilde{E}_j, \tilde{B}_j)^T \) be given by (4.1). Let \( \tilde{\Gamma} := \tilde{\Gamma}_1 - \tilde{\Gamma}_2 := (\tilde{u}, \tilde{E}, \tilde{B})^T \). We decompose \( \tilde{u} \) into \( \tilde{u} = \tilde{u}^{NS} + \tilde{u}^M \), with \( \tilde{u}^{NS} \) accounting for the convection term

\[
\tilde{u}^{NS} := -\int_0^t e^{(t-t')A} \mathcal{P} \nabla (u \otimes \tilde{u}_1 + \tilde{u}_2 \otimes u) \, dt',
\]

and \( \tilde{u}^M \) for the Lorentz force

\[
\tilde{u}^M : = \int_0^t e^{(t-t')A} \mathcal{P} (E \times \tilde{B}_1 + \tilde{E}_2 \times B) \, dt' + \int_0^t e^{(t-t')A} \mathcal{P} \left( (u \times \tilde{B}_1) \times \tilde{B}_1 + [\tilde{u}_2 \times B] \times \tilde{B}_1 + [\tilde{u}_2 \times \tilde{B}_2] \times B \right) \, dt'.
\]

Moreover, the electro-magnetic field \( (\tilde{E}, \tilde{B}) \) satisfies

\[
\partial_t \tilde{E} - \text{curl} \, \tilde{B} + \tilde{E} = u \times \tilde{B}_1 + \tilde{u}_2 \times B
\]

\[
\partial_t \tilde{B} + \text{curl} \, \tilde{E} = 0
\]

with 0 data. First, by Lemma 3.2 Lemma 3.3 and the embedding \( L^1 \dot{H}^{\frac{d}{2}-1} \hookrightarrow \dot{L}^1 \dot{H}^{\frac{d}{2}-1} \), we have

\[
\| \tilde{u}^{NS} \|_Z \lesssim \| \mathcal{P} \nabla (u \otimes \tilde{u}_1 + \tilde{u}_2 \otimes u) \|_{L^1 \dot{H}^{\frac{d}{2}-1}}
\]

and

\[
\| \tilde{u}^M \|_Z \lesssim \| \mathcal{P} (E \times \tilde{B}_1 + \tilde{E}_2 \times B) \|_{L^2 \dot{B}^{\frac{d}{2}-2}_2 + L^1 \dot{H}^{\frac{d}{2}-1}}
\]

\[
+ \| \mathcal{P} \left( (u \times \tilde{B}_1) \times \tilde{B}_1 + [\tilde{u}_2 \times B] \times \tilde{B}_1 + [\tilde{u}_2 \times \tilde{B}_2] \times B \right) \|_{L^2 \dot{B}^{\frac{d}{2}-2}_2 + L^1 \dot{H}^{\frac{d}{2}-1}}.
\]
Second, applying estimates (3.5) and (3.8), we obtain for the convection term

\[
\|\tilde{u}^{NS}\|_{Zu} \lesssim \|u\|_{L^2(L^\infty \cap H^{\frac{d}{2}})} \sum_{j=1,2} \|\tilde{u}_j\|_{L^2(L^\infty \cap H^{\frac{d}{2}})}
\]

(4.6)

\[
\lesssim \|\Gamma\|_Z \sum_{j=1,2} \left(\|\Gamma_j\|_Z + \|e^{tA}\Gamma^0\|_Z\right),
\]

whereas the Lorentz force term can be estimated by (3.6), (3.7), (3.9), and (3.10)

(4.7)

\[
\|\tilde{u}^{M}\|_{Zu} \lesssim \|E\|_{L^2H^{\frac{d}{2}}}, \|\tilde{B}_1\|_{L^\infty H^{\frac{d}{2}}}, \|\tilde{E}_2\|_{L^2H^{\frac{d}{2}} \cap L^2H^{\frac{d}{2}}}, \|\tilde{B}\|_{L^\infty H^{\frac{d}{2}} \cap L^2H^{\frac{d}{2}}},
\]

\[
+ \|u \times \tilde{B}_1\|_{L^2H^{\frac{d}{2}}}, \|\tilde{B}_1\|_{L^\infty H^{\frac{d}{2}} \cap L^2H^{\frac{d}{2}}}, \|\tilde{u}_2 \times \tilde{B}\|_{L^2H^{\frac{d}{2}} \cap L^2H^{\frac{d}{2}}},
\]

\[
\lesssim \|E\|_{Zu}, \|\tilde{B}_1\|_{Zu} + \|\tilde{E}_2\|_{Zu} \|B\|_{Zb} + \|u\|_{Zu} \|\tilde{B}_1\|_{Zu} + \|\tilde{u}_2\|_{Zu} \|B\|_{Zb} \|\tilde{B}_1\|_{Zb}
\]

\[
\lesssim \|\Gamma\|_Z \sum_{j=1,2} \left(\|e^{tA}\Gamma^0\|_Z + \|e^{tA}\Gamma^0\|_Z^2 + \|\Gamma_j\|_Z + \|\Gamma_j\|_Z^2\right).
\]

It remains to estimate the electro-magnetic field components of \(\Gamma\). Applying the energy and the decay estimates (3.3) to the system (1.4), we get

(4.8)

\[
\|\tilde{E}\|_{Ze} + \|\tilde{B}\|_{Zb} \lesssim \|\Gamma\|_Z \sum_{j=1}^2 \left(\|e^{tA}\Gamma^0\|_Z + \|\Gamma_j\|_Z\right).
\]

Gathering the estimates (1.6), (1.7) and (1.8) gives

(4.9)

\[
\|\tilde{\Gamma}\|_Z \lesssim \|\Gamma\|_Z \left(\|e^{tA}\Gamma^0\|_Z + \|e^{tA}\Gamma^0\|_Z^2 + \|\Gamma_j\|_Z + \|\Gamma_j\|_Z^2\right).
\]

Choosing \(\delta\) small enough gives (1.3).

4.2. The local existence. Decompose the initial data \((u^0, E^0, B^0) = (u^0_s, E^0_s, B^0_s) + (u^0_r, E^0_r, B^0_r)\) where \((u^0_s, E^0_s, B^0_s)\) is regular (say in \(H^2\)) and \((u^0_r, E^0_r, B^0_r)\) is small in \(H^{\frac{d}{2}-1} \times H^{\frac{d}{2}-1} \times H^{\frac{d}{2}-1}\) (this can be done by a Fourier cut-off). We look for a solution \(\Gamma\) of (1.4) of the form \((u, E, B) = (u_s, E_s, B_s) + (u_r, E_r, B_r)\) with

\[
\left\{
\begin{array}{l}
\frac{\partial u_r}{\partial t} + u_r \cdot \nabla u_r - \Delta u_r + \nabla p_r = j_r \times B_r \\
\partial_t E_r - \text{curl} B_r = -j_r \\
\partial_t B_r + \text{curl} E_r = 0 \\
\text{div} u_r = \text{div} B_r = 0 \\
(E_r + u_r \times B_r) = j_r
\end{array}
\right.
\]

subject to the initial data

\[
u_r|_{t=0} = u^0_r, \quad B_r|_{t=0} = B^0_r, \quad E_r|_{t=0} = E^0_r.
\]
Arguing as in [11], we know that (4.10) has a unique regular solution. Now we solve for \((u_s, E_s, B_s)\). We have
\[
\frac{\partial u_s}{\partial t} + u_s \cdot \nabla u_s - \Delta u_s + u_s \cdot \nabla u_r + u_r \cdot \nabla u_s + \nabla p_s = j \times B - j_r \times B_r \\
\partial_t E_s - \text{curl } B_s = j - j_r \\
\partial_t B_s + \text{curl } E_s = 0 \\
\text{div } u_s = \text{div } B_s = 0
\]
subject to the initial data
\[
u_s|_{t=0} = u_s^0, \quad B_s|_{t=0} = B_s^0, \quad E_s|_{t=0} = E_s^0.
\]
Proceeding similarly as for the small data result, set
\[
\Phi(\Gamma) := \int_0^t e^{(t-t')A} \left( \mathcal{P} \left[ -u_s \cdot \nabla u_s - u_s \cdot \nabla u_r - u_r \cdot \nabla u_s + j \times B - j_r \times B_r \right] \right) dt'
\]
where \(\Gamma\) is defined by
\[
(u_s, B_s, E_s) = e^{tA}(u_s^0, B_s^0, E_s^0) + \Gamma.
\]

Applying the same proof as for the small data existence, we can show that the map \(\Phi\) is a contraction if we choose a time of existence \(T\) sufficiently small. The main difference is that new linear terms (in \(\Gamma\)) appear in \(\Phi\). These linear terms need to be small (as linear maps) for \(\Phi\) to be a contraction; this can be achieved by using the smoothness of \(B\) and by choosing \(T\) small enough.

For instance:
\[
\left\| \int_0^t e^{(t-t')\Delta} \mathcal{P} E_s \times B_r \ dt' \right\|_{Z_T^H} \lesssim \| E_s \times B_r \|_{L^2_T \dot{H}_{d-1}^{\frac{d}{2}}} \lesssim \| E_s \parallel_{L^2_T \dot{H}_{d-1}^{\frac{d}{2}}} \| B_r \|_{L^1_T \dot{H}_{d-2}} \\
\lesssim \| E_s \parallel_{Z_T^H} \| B_r \|_{L^1_T \dot{H}_{d-2}}.
\]
The key point is of course that \(\| B_r \|_{L^1_T \dot{H}_{d-2}}\) can be made arbitrarily small by choosing \(T\) small enough.

5. Appendix: Proof of the decay for \(B\) and Proposition 3.5

Recall the main part of Lemma 3.3. We emphasize on the fact that this property is an extra information about a weak decay that the magnetic field satisfies and it has no impact on the well posedness result.

**Lemma 5.1.** Let \(\alpha \geq 0\) and \(G \in L^2_T H_{d-1}^{\frac{d}{2}}\), and \((E, B)\) be a solution of
\[
\partial_t E - \text{curl } B + E = G, \\
\partial_t B + \text{curl } E = 0
\]
on some time interval \([0, T]\). Then, the following estimate hold with constants not depending upon time
\[
\parallel B \parallel_{L^2_T H_{d-1}^{\frac{d}{2}}} \lesssim \| (E_0, B_0) \|_{H_{d-1}^{\frac{d}{2}}} + \| G \|_{L^2_T H_{d-1}^{\frac{d}{2}}}.
\]
Proof. Because of the divergence free property of $B$, we have

$$
\partial_t B - \Delta B + \partial_t B = \text{curl } G, \quad (B, \partial_t B)_{t=0} = (B^0, B^1).
$$

Thus, the magnetic field $B$ satisfies an inhomogeneous damped wave equation \textbf{(5.2)}. In the sequel we denote by $L_1(t)$ and $L_2(t)$ the propagators associated to the Fourier multiplier functions

$$
\Phi_1(t, \xi) = e^{-t/2} \cosh(\sqrt{1/4 - |\xi|^2 t}), \quad \Phi_2(t, \xi) = e^{-t/2} \frac{\sinh(\sqrt{1/4 - |\xi|^2 t})}{\sqrt{1/4 - |\xi|^2 t}}.
$$

A direct computation gives the following Duhamel formula

$$
B(t, x) = L_1(t)B^0(x) + L_2(t)(B^0/2 + B^1)(x) + \int_0^t L_2(t-s)\text{curl } G(s, x) ds,
$$

with $B^1 = \partial_t B(t = 0) = -\text{curl } (E(t = 0)) = -\text{curl } (E^0)$. As this was observed in \textbf{[10]}, there exists $0 < c < 1$ such that we have the following bounds

- For $|\xi| \geq 2$

$$
|\Phi_1(t, \xi)| \lesssim e^{-ct}, \quad |\Phi_2(t, \xi)| \lesssim e^{-ct} |\xi|.
$$

- For $1/4 \leq |\xi| < 2$

$$
|\Phi_1(t, \xi)| + |\Phi_2(t, \xi)| \lesssim e^{-ct}.
$$

- For $2^{q-1} \leq |\xi| \leq 2^{q+1}$, with $q \leq -3$

$$
|\Phi_1(t, \xi)| \leq \Phi_1^q(t) := e^{-\frac{t}{2}} \cosh(\sqrt{1/4 - 2^{2(q-1)}}),
$$

$$
|\Phi_2(t, \xi)| \leq \Phi_2^q(t) := e^{-\frac{t}{2}} \frac{\sinh(\sqrt{1/4 - 2^{2(q-1)})}}{\sqrt{1/4 - 2^{2(q-1)}}}.
$$

On the one hand, for $q \geq -1$, one has

$$
\|\Delta_q B(t)\|_{L^2} \lesssim e^{-ct} \|\Delta_q B^0\|_{L^2} + e^{-ct} 2^{-q} (\|\Delta_q B_0\|_{L^2} + \|\Delta_q B^1\|_{L^2})
$$

$$
+ \int_0^t e^{-c(t-s)} \|\Delta_q G\|_{L^2} ds.
$$

Multiplying both sides of \textbf{(5.3)} by $2^{q(2q-1)}$, applying Young’s inequality (in time) and summing in $q$ yields

$$
\|(I - S_0)B\|_{L^2_t H^\frac{q}{2} \alpha^{-1}} \lesssim \|(I - S_0)B^0\|_{H^\frac{q}{2} \alpha^{-1}} + \|(I - S_0)B^1\|_{H^\frac{q}{2} \alpha^{-2}}
$$

$$
+ \|(I - S_0)G\|_{L^2_t H^\frac{q}{2} \alpha^{-1}}
$$
On the other hand, for \( q \leq 0 \), one has
\[
\|\Delta_q B(t)\|_{L^2} \leq \Phi_1^q(t)\frac{\|\Delta_q B^0\|_{L^2}}{2^q} + \Phi_2^q(t)\left(\frac{\|\Delta_q B^1\|_{L^2} + \|\Delta_q B^0\|_{L^2}}{2^q}\right)
\]
\[+ 2^q \int_0^t \Phi_2^q(t-s)\|\Delta_q G(s)\|_{L^2} ds.
\]
Taking the \( L^2_T \) norm in time and applying Young’s inequality we get
\[
\|\Delta_q B\|_{L^2 L^2} \lesssim \|\Phi_1^q\|_{L^2(R^+)}\|\Delta_q B^0\|_{L^2}
\]
\[+ \|\Phi_2^q\|_{L^2(R^+)}\left(\|\Delta_q B^1\|_{L^2} + \|\Delta_q B^0\|_{L^2}\right)
\]
\[+ 2^q \|\Phi_2^q\|_{L^1(R^+)}\|\Delta_q G\|_{L^2 L^2}.
\]
But since for every \( q \leq 0 \) and \( r \in [1, +\infty) \), \( \Phi_i^q \) satisfies
\[
\|\Phi_i^q\|_{L^r(R^+)} \lesssim 2^{-\frac{\alpha}{2}q}, \quad i = 1, 2.
\]
Multiplying both sides by \( 2^q \frac{\alpha}{2} q^\alpha/2 \) and taking the \( \ell^2 \) norm gives
\[
\sum_{\ell \in \mathbb{Z}} \|S_0 B\|_{L^2(\dot{H}^{\frac{\alpha}{2}})^2} \lesssim \|S_0 B^0\|_{\dot{H}^{\frac{\alpha}{2}-1}} + \|S_0 B^1\|_{\dot{H}^{\frac{\alpha}{2}-2}} + \|S_0 G\|_{L^2(\dot{H}^{\frac{\alpha}{2}-1})}.
\]
Putting together (5.4) and (5.5) gives (5.3) as desired.

**Proof of Proposition 3.5** The proof is based on the paraproduct decomposition. We choose to prove in details only estimates (3.6) and (3.7). The other estimates are easier or classical and left to the reader.

**Proof of (3.6)** We decompose \( EB \) into
\[
EB = T_E B + T_B E + S_2 R(E, B) + (I - S_2) R(E, B),
\]
and will show the following estimates:
\[
\|T_E B + T_B E\|_{\dot{L}^{2, \dot{B}^{\frac{\alpha}{2}-1}}(\mathbb{R}^2)} \lesssim \|E\|_{L^2(\dot{L}^2)^2} \|B\|_{L^2(\dot{L}^2)^2}
\]
\[
\|S_2 R(E, B)\|_{L^2 L^2} \lesssim \|E\|_{L^2(\dot{L}^2)^2} \|B\|_{L^2(\dot{H}^1)^2}
\]
\[
\|\Delta_q R(E, B)\|_{L^2 L^2} \lesssim \|E\|_{L^2(\dot{L}^2)^2} \|B\|_{L^2(\dot{L}^2)^2}
\]
First, we prove (5.6). Since the term \( T_B E \) can be treated in a very similar way, we focus on \( T_E B \). First,
\[
\Delta_q(T_E B) = \sum_{|\tilde{q} - q| \leq 1} \Delta_q(\Delta_{\tilde{q}} BS_{\tilde{q}} E).
\]
Since \( \Delta_q \) is uniformly bounded on \( L^2 \), we have
\[
\sum_q 2^{-q} \|\Delta_q(T_E B)\|_{L^2 L^2} \lesssim \sum_q 2^{-q} \sum_{|\tilde{q} - q| \leq 1} \|\Delta_{\tilde{q}} BS_{\tilde{q}} E\|_{L^2 L^2}.
\]
We are going to deal with the term \( \tilde{q} = q \) only (the two other terms \( \tilde{q} = q \pm 1 \) can be estimated similarly). Applying successively Hölder’s inequality (in the variables
t and \(x\), Bernstein’s lemma, Young’s inequality (in the variable \(q\)), and Hölder’s inequality (in the variable \(q\)) gives

\[
\sum_{q} 2^{-q} \| \Delta_q BS_q E \|_{L^2 \dot{H}^1} \leq \sum_{q} 2^{-q} \| \Delta_q B \|_{L^\infty} \| S_q E \|_{L^2 \dot{H}^1} \leq \sum_{q} 2^{-q} \sum_{j \leq q} \| \Delta_q B \|_{L^\infty} \| \Delta_j E \|_{L^2 \dot{H}^1} \leq \sum_{q} \sum_{j \leq q} 2^{j-q} \| \Delta_q B \|_{L^\infty} \| \Delta_j E \|_{L^2 \dot{H}^1} \leq \left( \sum_{q} \| \Delta_q B \|_{L^\infty}^{2} \right)^{1/2} \left( \sum_{j} \| \Delta_j E \|_{L^2 \dot{H}^1}^{2} \right)^{1/2}.
\]

Next we prove (3.7). Applying Bernstein’s Lemma 3.1 and Cauchy-Schwarz (in \(j\)) gives

\[
\| S_2 R(B, E) \|_{L^1 \dot{H}^1} \leq \sum_{q \leq 0} \| \Delta_q R(B, E) \|_{L^1 \dot{H}^1} \leq \sum_{q \leq 0} 2^q \| \Delta_q R(B, E) \|_{L^1 \dot{H}^1} \leq \sum_{q \leq 0} 2^q \sum_{j \geq q-2} \| \Delta_j B \|_{L^2 \dot{H}^1} \| \Delta_j E \|_{L^2 \dot{H}^1} \leq \sum_{j \geq 0} \sum_{q \leq \inf(0, j+2)} 2^q \| \Delta_j B \|_{L^2 \dot{H}^1} \| \Delta_j E \|_{L^2 \dot{H}^1} \leq \sum_{j \geq 0} 2^j \| \Delta_j B \|_{L^2 \dot{H}^1} \| \Delta_j E \|_{L^2 \dot{H}^1} + \sum_{j \geq 0} \| \Delta_j B \|_{L^2 \dot{H}^1} \| \Delta_j E \|_{L^2 \dot{H}^1} \leq \| E \|_{L^1 \dot{H}^1} \| B \|_{L^2 \dot{H}^1}.
\]

To estimate (3.8), Hölder’s inequality (in \(t, x\)) and Cauchy-Schwarz (in \(j\)) gives

\[
\| (I - S_2) R(E, B) \|_{L^2_B \dot{B}^{-1}_{2,1}} \leq \sum_{q \geq 0} \sum_{j \geq q-2} \| \Delta_j E \|_{L^2 \dot{H}^1} \| \Delta_j B \|_{L^\infty} \| L^2 \dot{B}^{-1}_{2,1} \|
\leq \sum_{j \geq -2} \sum_{0 \leq q \leq j+2} \| \Delta_j E \|_{L^2 \dot{H}^1} \| \Delta_j B \|_{L^\infty} \| L^2 \dot{B}^{-1}_{2,1} \|
\leq \sum_{j \geq -2} \max(j, 1) \| \Delta_j E \|_{L^2 \dot{H}^1} \| \Delta_j B \|_{L^\infty} \| L^2 \dot{B}^{-1}_{2,1} \|
\leq \| E \|_{L^2 \log L^2} \| B \|_{L^\infty \log L^2}.
\]

Proof of (3.7). As for the proof of (3.6), we split \(uB\) following the paraproduct decomposition:

\[
uB = T_B u + T_u B + R(u, B).
\]
We shall only estimate here $T_B u$, the estimate of $R(u, B)$ being similar, and that of $T_u B$ easier. By H"older's inequality,
\[
\|T_B u\|_{L^2_T L^2_{\log}}^2 = \sum_q \max(1, q) \|S_q B \Delta_q u\|_{L^2_T L^2}^2 \\
\lesssim \sum_q \max(1, q) \|\Delta_q u\|_{L^2_T L^2}^2 \|S_q B\|_{L^\infty_T L^\infty}^2.
\]
Now observe that Bernstein’s lemma and Cauchy-Schwarz’ inequality (in $j$) give
\[
\|S_q B\|_{L^\infty_T L^\infty} \lesssim \sum_{j < q} 2^j \|\Delta_j B\|_{L^\infty_T L^2} \\
\lesssim \left( \sum_{j < q} \frac{2^{2j}}{\max(1, q)} \right)^{1/2} \|B\|_{L^\infty_T L^2_{\log}} \\
\lesssim \frac{2^q}{\sqrt{\max(1, q)}}.
\]
Coming back to the bound for $\|T_B u\|_{L^2_T L^2_{\log}}^2$, this gives
\[
\|T_B u\|_{L^2_T L^2_{\log}}^2 \lesssim \sum_q 2^{2q} \|\Delta_q u\|_{L^2_T L^2}^2 \|B\|_{L^\infty_T L^2_{\log}}^2.
\]

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