On Properties of Generalized Bi-$\Gamma$-Ideals of $\Gamma$-Semirings

Teerayut Chomchuen and Aiyared Iampan

Abstract — The notion of $\Gamma$-semirings was introduced by Murali Krishna Rao [10] as a generalization of the notion of $\Gamma$-rings as well as of semirings. We have known that the notion of $\Gamma$-semirings is a generalization of the notion of semirings. In this paper, extending Kaushik, Moin and Khan’s work, we generalize the notion of generalized bi-$\Gamma$-ideals of $\Gamma$-semirings and investigate some related properties of generalized bi-$\Gamma$-ideals.

Keywords — $\Gamma$-semiring, bi-$\Gamma$-ideal, generalized bi-$\Gamma$-ideal.

I. INTRODUCTION AND PRELIMINARIES

The notion of $\Gamma$-semirings was introduced and studied in 1995 by Murali Krishna Rao [10] as a generalization of the notion of $\Gamma$-rings as well as of semiring, and the notion of generalized bi-$\Gamma$-ideals was first introduced for rings in 1970 by Szász [12], [13] and then for semigroups by Lajos [8]. Many types of ideals on the algebraic structures were characterized by several authors such as: In 2000, Dutta and Sardar [3] studied the characterization of semiprime ideals and irreducible ideals of $\Gamma$-semirings. In 2004, Sardar and Dasgupta [11] introduced the notions of primitive $\Gamma$-semirings and primitive ideals of $\Gamma$-semirings. In 2008, Kaushik, Moin and Khan [7] introduced and studied bi-$\Gamma$-ideals in $\Gamma$-semirings, Pianskool, Sangwirotjanapat and Chinram [1] gave some properties of quasi-ideals in $\Gamma$-semirings and valuation $\Gamma$-ideals of a $\Gamma$-semiring, and Chomchuen and Aiyared Iampan [9] introduced and studied valuation $\Gamma$-semirings and valuation $\Gamma$-ideals.

To present the main results we first recall the definition of a $\Gamma$-semiring which is important here and discuss some elementary definitions that we use later.

Definition I.1. [10] Let $M$ and $\Gamma$ be two additive commutative semigroups. Then $M$ is called a $\Gamma$-semiring if there exists a mapping $\cdot : M \times \Gamma \times M \to M$ (the image of $(a, \alpha, b)$ to be denoted by $a\alpha b$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$) satisfying the following conditions:

1. $a\alpha (b + c) = a\alpha b + a\alpha c,$
2. $(a + b)\alpha c = a\alpha c + b\alpha c,$
3. $a(\alpha + \beta)b = a\alpha b + a\beta b,$
4. $a\alpha (b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Let $M$ be a $\Gamma$-semiring, $A$ and $B$ nonempty subsets of $M$, and $\Lambda$ a nonempty subset of $\Gamma$. Then we define $A + B := \{a + b \mid a \in A \text{ and } b \in B\}$ and

$A\Lambda B := \left\{ \sum_{i=1}^{n} \lambda_i a_i b_i \mid n \in \mathbb{Z}^+, a_i \in A, b_i \in B \text{ and } \lambda_i \in \Lambda \text{ for all } i \right\}.$

If $A = \{a\}$, then we also write $\{a\} + B$ as $a + B$, and $\{a\}\Lambda B$ as $a\Lambda B$, and similarly if $B = \{b\}$ or $\Lambda = \{\lambda\}$.

Example I.2. [6] Let $Q$ be set of rational numbers. Let $(S, +)$ be the commutative semigroup of all $2 \times 3$ matrices over $Q$ and $(\Gamma, \cdot)$ commutative semigroup of all $3 \times 2$ matrices over $Q$. Define $W\circ Y$ usual matrix product of $W$ and $Y$ for all $W, Y \in S$ and for all $\alpha \in \Gamma$. Then $S$ is a $\Gamma$-semiring but not a semiring.

Example I.3. [6] Let $\mathbb{N}$ be the set of natural numbers and $\Gamma = \{1, 2, 3\}$. Then $(\mathbb{N}, \max)$ and $(\Gamma, \max)$ are commutative semigroups. Define the mapping $\mathbb{N} \times \Gamma \times \mathbb{N} \to \mathbb{N}$, by $a\alpha b = \min\{a, \alpha, b\}$ for all $a, b \in \mathbb{N}$ and $\alpha \in \Gamma$. Then $\mathbb{N}$ is a $\Gamma$-semiring.

Example I.4. [6] Let $Q$ be set of rational numbers and $\Gamma = \mathbb{N}$ the set of natural numbers. Then $(Q, +)$ and $(\mathbb{N}, \cdot)$ are commutative semigroups. Define the mapping $Q \times \Gamma \times Q \to Q$, by $a\alpha b$ usual product of $a, \alpha, b$; for all $a, b \in Q$ and $\alpha \in \Gamma$. Then $Q$ is a $\Gamma$-semiring.

Example I.5. [2] For consider the additively abelian groups $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\Gamma = \{2, 4, 6\}$. Let $\cdot : Z_8 \times \Gamma \times Z_8 \to Z_8, (y, \alpha, s) = y \cdot s$. Then $Z_8$ is a $\Gamma$-semiring.
Definition I.6. A nonempty subset $A$ of a $\Gamma$-semiring $M$ is called
(1) a sub-$\Gamma$-semiring of $M$ if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$.
(2) a $\Gamma$-ideal of $M$ if $(A, +)$ is a subsemigroup of $(M, +)$, and $x\gamma a \in A$ and $a\gamma x \in A$ for all $a \in A, x \in M$ and $\gamma \in \Gamma$.
(3) a quasi-$\Gamma$-ideal of $M$ if $A$ is a sub-$\Gamma$-semiring of $M$ and $\mathcal{A}M \cap \mathcal{M}A \subseteq A$.
(4) a bi-$\Gamma$-ideal of $M$ if $A$ is a sub-$\Gamma$-semiring of $M$ and $\mathcal{A}M \cap \mathcal{M}A \subseteq A$.
(5) a generalized bi-$\Gamma$-ideal of $M$ if $\mathcal{A}M \cap \mathcal{M}A \subseteq A$.

Lemma II.1. Let $M$ be a $\Gamma$-semiring. We have the following:
(1) Every quasi-$\Gamma$-ideal of $M$ is a bi-$\Gamma$-ideal.
(2) Every bi-$\Gamma$-ideal of $M$ is a generalized bi-$\Gamma$-ideal.

Definition I.8. A $\Gamma$-semiring $M$ is called a GB-simple $\Gamma$-semiring if $M$ is the unique generalized bi-$\Gamma$-ideal of $M$.

II. MAIN RESULTS

Before the characterizations of generalized bi-$\Gamma$-ideals of $\Gamma$-semirings for the main results, we give some auxiliary results which are necessary in what follows. By Lemma I.7 (2) and [7], we have the following lemma.

Lemma II.1. Let $M$ be a $\Gamma$-semiring and $a \in M$. Then $a\Gamma M$ and $M\Gamma a$ are generalized bi-$\Gamma$-ideals of $M$.

Lemma II.2. Let $M$ be a $\Gamma$-semiring and $\{B_i \mid i \in I\}$ a nonempty family of generalized bi-$\Gamma$-ideals of $M$ with $\bigcap_{i \in I} B_i \neq \emptyset$. Then $\bigcap_{i \in I} B_i$ is a generalized bi-$\Gamma$-ideal of $M$.

Proof: For all $i \in I$, we have
$$\left(\bigcap_{i \in I} B_i\right)\Gamma M \left(\bigcap_{i \in I} B_i\right) \subseteq B_i \Gamma M B_i \subseteq B_i.$$
Thus
$$\left(\bigcap_{i \in I} B_i\right)\Gamma M \left(\bigcap_{i \in I} B_i\right) \subseteq \bigcap_{i \in I} B_i.$$
Hence $\bigcap_{i \in I} B_i$ is a generalized bi-$\Gamma$-ideal of $M$.

Lemma II.3. Let $M$ be a $\Gamma$-semiring and $\emptyset \neq A \subseteq M$. Then
$$A \cup \mathcal{A}M \cap \mathcal{M}A$$
(1)
is the smallest generalized bi-$\Gamma$-ideal of $M$ containing $A$.

Proof: Let $B = A \cup \mathcal{A}M \cap \mathcal{M}A$. Then $A \subseteq B$. Therefore
$$B\Gamma M B = (A \cup \mathcal{A}M \cap \mathcal{M}A)\Gamma M (A \cup \mathcal{A}M \cap \mathcal{M}A) \subseteq \mathcal{A}(\mathcal{A}M \cap \mathcal{M}A)\Gamma M (A \cup \mathcal{A}M \cap \mathcal{M}A) \subseteq \mathcal{A}M \cap \mathcal{M}A \subseteq A.$$
Hence $C$ is a sub-$\Gamma$-semiring of $M$.

**Proposition II.7.** Let $M$ be a $\Gamma$-semiring and $T$ a $\Gamma$-ideal of $M$. Then every subset of $T$ containing $MT \cup TM$ is a $\Gamma$-ideal of $M$.

**Proof:** Let $B$ be a subset of $T$ such that $MT \cup TM \subseteq B$. Then
\[
MT \subseteq MT \cup TM \subseteq B
\]
and
\[
TM \subseteq TM \cup MT \subseteq B.
\]
Hence $B$ is a $\Gamma$-ideal of $M$.

**Proposition II.8.** Let $M$ be a $\Gamma$-semiring and $T$ a quasi-$\Gamma$-ideal of $M$. Then every subset of $T$ containing $TTM \cap MTT$ is a quasi-$\Gamma$-ideal of $M$.

**Proof:** Let $C$ be a subset of $T$ such that $TTM \cap MTT \subseteq C$. Then
\[
CTC \subseteq TTM \cap MTT \subseteq C
\]
and
\[
CTM \cap MTC \subseteq TTM \cap MTT \subseteq C.
\]
Hence $C$ is a quasi-$\Gamma$-ideal of $M$.

**Proposition II.9.** Let $M$ be a $\Gamma$-semiring and $T$ a $\Gamma$-ideal of $M$. Then every subset of $T$ containing $TTMT$ and all of its images is a $\Gamma$-ideal of $M$.

**Proof:** Let $D$ be a subset of $T$ such that $TTMT \subseteq D$ and $DG \subseteq D$. Then
\[
TGM \subseteq TTM \subseteq D.
\]
Hence $D$ is a $\Gamma$-ideal of $M$.

**Proposition II.10.** Let $M$ be a $\Gamma$-semiring and $T$ a generalized $\gamma$-ideal of $M$. Then every subset of $T$ containing $TTMT$ is a generalized $\gamma$-ideal of $M$.

**Proof:** Let $E$ be a subset of $T$ such that $TTMT \subseteq E$. Then
\[
TGM \subseteq TTM \subseteq E.
\]
Hence $E$ is a generalized $\gamma$-ideal of $M$.

**Theorem II.11.** Let $M$ be a $\Gamma$-semiring. Then the following statements are equivalent.

1. $M$ is a GB-simple $\gamma$-semiring.
2. $a \Gamma MTa = M$ for all $a \in M$.
3. $(a) = M$ for all $a \in M$.

**Proof:** (1) $\Rightarrow$ (2) Assume that $M$ is a GB-simple $\gamma$-semiring and $a \in M$. By Lemma II.5, we have $a \Gamma MTa$ is a generalized $\gamma$-ideal of $M$. Since $M$ is a GB-simple $\Gamma$-semiring, we have $a \Gamma MTa = M$.

(2) $\Rightarrow$ (3) Assume that $a \Gamma MTa = M$ for all $a \in M$ and let $a \in M$. Then, by (2), we have
\[
(a) = \{a\} \cup a \Gamma MTa = \{a\} \cup M = M.
\]

(3) $\Rightarrow$ (1) Assume that $(a) = M$ for all $a \in M$, and let $A$ be a generalized $\gamma$-ideal of $M$ and $a \in A$. Then $(a) \subseteq A$. By assumption, we have
\[
M = (a) \subseteq A \subseteq M.
\]

Thus $M = A$. Therefore $M$ is a GB-simple $\gamma$-semiring.

**Lemma II.12.** Let $B$ be a generalized $\gamma$-ideal of a $\Gamma$-semiring $M$ and $T$ a sub-$\Gamma$-semiring of $M$. If $T$ is a GB-simple $\gamma$-semiring such that $T \cap B \neq \emptyset$, then $T \subseteq B$.

**Proof:** Assume that $T$ is a GB-simple $\gamma$-semiring such that $T \cap B \neq \emptyset$ and let $a \in T \cap B$. By Lemma II.3, we have $\{a\} \cup a \Gamma TMa$ is a generalized $\gamma$-ideal of $T$. Since $T$ is a GB-simple $\gamma$-semiring, we have $\{a\} \cup a \Gamma TMa = T$. Thus
\[
T = \{a\} \cup a \Gamma TMa \subseteq B \cup B \Gamma M TB \subseteq B \cup B \subseteq B.
\]
Hence $T \subseteq B$.

**Theorem II.13.** Let $M$ be a $\gamma$-semiring. $B$ a generalized $\gamma$-ideal of $M$ and $\emptyset \neq A \subseteq M$. Then $B \Gamma A$ and $A \Gamma B$ are generalized $\gamma$-ideals of $M$.

**Proof:** Since $B$ is a generalized $\gamma$-ideal of $M$, we have
\[
(B \Gamma A) \Gamma M \Gamma (B \Gamma A) = (B \Gamma (A \Gamma M) \Gamma B) \Gamma A \subseteq (B \Gamma M \Gamma B) \Gamma A \subseteq B \Gamma A
\]
and
\[
(A \Gamma B) \Gamma M \Gamma (A \Gamma B) = A \Gamma (B \Gamma (M \Gamma A) \Gamma B) \subseteq A \Gamma (B \Gamma M \Gamma B) \subseteq A \Gamma B.
\]
Therefore $B \Gamma A$ and $A \Gamma B$ are generalized $\gamma$-ideals of $M$.

**Theorem II.14.** Let $M$ be a $\gamma$-semiring and $B$ a $\gamma$-ideal of $M$. Then $B$ is a minimal generalized $\gamma$-ideal of $M$ if and only if $B$ is a GB-simple $\gamma$-semiring.

**Proof:** Assume that $B$ is a minimal generalized $\gamma$-ideal of $M$. By assumption, $B$ is a $\gamma$-semiring. Let $C$ be a generalized $\gamma$-ideal of $B$. Then
\[
CTBC \subseteq C \subseteq B.
\]
Since $B$ is a generalized $\gamma$-ideal of $M$ and by Theorem II.13, we have $CTBC$ is a generalized $\gamma$-ideal of $M$. Since $B$ is a minimal generalized $\gamma$-ideal of $M$, we get $CTBC = B$. Thus, by (3), we have $B = C$. Hence $B$ is a GB-simple $\gamma$-semiring.

Conversely, assume that $B$ is a GB-simple $\gamma$-semiring. Let $C$ be a generalized $\gamma$-ideal of $M$ such that $C \subseteq B$. Then
\[
CTBC \subseteq C \subseteq B.
\]
Thus $C$ is a generalized $\gamma$-ideal of $B$. Since $B$ is a GB-simple $\gamma$-semiring, we have $B = C$. Hence $B$ is a minimal generalized $\gamma$-ideal of $M$.

**Theorem II.15.** Let $M$ be a $\gamma$-semiring having a proper generalized $\gamma$-ideal. Then every proper generalized $\gamma$-ideal of $M$ is minimal if and only if the intersection of any two distinct proper generalized $\gamma$-ideals is empty.
Proof: Assume that every proper generalized bi-$\Gamma$-ideal of $M$ is minimal and let $B_1$ and $B_2$ be two distinct proper generalized bi-$\Gamma$-ideals of $M$. By assumption, we have $B_1$ and $B_2$ are minimal. We shall show that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cap B_2 \neq \emptyset$. By Lemma II.2, we have $B_1 \cap B_2$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$, thus $B_1 = B_2$ which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

Conversely, assume that the intersection of any two distinct proper generalized bi-$\Gamma$-ideals is empty. Let $B$ be a proper generalized bi-$\Gamma$-ideal of $M$ and $C$ a generalized bi-$\Gamma$-ideals of $M$ such that $C \subseteq B$. Suppose that $C \neq B$. Then $C$ is a proper generalized bi-$\Gamma$-ideal of $M$. Since $C \subset B$ and by assumption, we have $C = C \cap B = \emptyset$ which is a contradiction. Therefore $C = B$, so $B$ is minimal.

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