Nonlinear theory of resonant slow waves in anisotropic and dispersive plasmas

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The solar corona is a typical example of a plasma with strongly anisotropic transport processes. The main dissipative mechanisms in the solar corona acting on slow magnetohydrodynamic waves are the anisotropic thermal conductivity and viscosity. Ballai et al. [Phys. Plasmas 5, 252 (1998)] developed the nonlinear theory of driven slow resonant waves in such a regime. In the present paper the nonlinear behaviour of driven magnetohydrodynamic waves in the slow dissipative layer in plasmas with strongly anisotropic viscosity and thermal conductivity is expanded by considering dispersive effects due to Hall currents. The nonlinear governing equation describing the dynamics of nonlinear resonant slow waves is supplemented by a term which describes nonlinear dispersion and is of the same order of magnitude as nonlinearity and dissipation. The connection formulae are found to be similar to their non-dispersive counterparts.

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I. INTRODUCTION

Resonances are ubiquitous, everyday magnetohydrodynamic (MHD) waves are driven in a transversally inhomogeneous (relative to the ambient magnetic field) plasma. From a mathematical point of view a resonance is equivalent to regular singular points in the equations describing the dynamics of waves, but these singularities can be removed by, for example, dissipation. At resonance interacting dynamical systems can transfer energy to each other. In this context, resonant absorption has been suggested as a method to create supplementary heating of fusion plasmas, but was, however, later rejected due to technical difficulties (see, e.g. Refs. [1]-[4]). Ionson [5] suggests, for the first time, that resonant MHD waves may be a means to heat magnetic loops in the solar corona. Since then, resonant absorption of MHD waves has become a popular and successful mechanism for providing some of the heating of the solar corona (see, e.g. Refs. [6], [7] and references therein). More recently resonant absorption has acquired a new applicability when the observed damping of waves and oscillations in coronal loops has been attributed to resonant absorption. Hence, resonant absorption has become a fundamental constituent block of one of the newest branches of solar physics, called coronal seismology (see, e.g. Refs. [5]-[10]).

A driven problem for resonant MHD waves occurs when there is an external (or internal) source of energy. For example, Ruderman et al. [11], [12] and Tirry et al. [13]. Secondly, in the case of indirect or lateral driving, the energy source can be either outside or inside the system. This energy source excites fast or slow magnetosonic waves which propagate across and along magnetic surfaces and reach the resonant magnetic surface where their energy is partly dissipated due to resonant coupling with localized Alfvén or slow waves. The lateral driving problem was studied by, for example, Davila [14] for planar geometry and by Erdélyi [15] for cylindrical geometry. In the present paper, we consider only the lateral driven case (for a comprehensive background to lateral driving see, e.g. Refs. [17]-[20]). An important property of these waves is that their damping rate is almost independent of the values of the dissipative coefficients, a situation characteristic for dissipative systems with large Reynolds numbers. As a result, the damping rate of near resonant MHD waves can be many orders of magnitudes larger than the damping rate of MHD waves with the same frequencies in homogeneous plasmas. The damping allows the waves energy to be converted into heat, which has made resonant absorption a subject of intense study.

The concept of connection formulae was introduced by Sakurai et al. [21], to determine the jumps in the normal component of velocity and perturbation of total pressure across the dissipative layer. The connection formulae avoid solving the full dissipative MHD equations when studying resonant MHD waves. Instead, the narrow layer embracing the resonant point can be considered as a surface of discontinuity and ideal MHD equations can be used to the left and right of this surface. Connection formulae are utilized in order to connect the ideal solution over the discontinuity.

Most studies on driven resonant MHD waves use isotropic viscosity and/or electrical resistivity. However, the solar corona is a well-known example of a plasma where viscosity is strongly anisotropic [22]. Hollweg and Yang [23] studied the laterally driven problem in the cold.
plasma approximation. They found that anisotropic viscosity does not remove the Alfvén singularity (if the Braginskii's viscosity tensor is approximated by its first term only). If Braginskii's full viscosity tensor is considered, the Alfvén singularity would be removed by the shear viscosity. The way the dissipative term appears in the governing equation is, however, identical to that of isotropic viscosity.

For the case of slow resonant waves the situation is different. The laterally driven linear slow resonant waves in plasmas with strongly anisotropic viscosity and thermal conductivity was studied first by Ruderman and Goossens [24]. They successfully showed that anisotropic viscosity and/or thermal conductivity removes the slow resonance present in ideal plasmas. They also obtained the explicit connection formulae, which are identical to those found in the case of plasmas with isotropic viscosity and finite electrical resistivity [21]. This fact supports the hypothesis that in weakly dissipative plasmas the connection formulae are independent of the exact form of dissipative processes present in the dissipative layer.

The laterally driven nonlinear slow resonant waves in plasmas with strongly anisotropic viscosity and thermal conductivity was studied first by Ballai et al. [25]. They found that nonlinearity was crucial in the dissipative layer. The governing equation for slow wave dynamics have only been found for linear theory and for the limit of strong nonlinearity [26]. The governing equation was almost identical to that found by Ruderman et al. [27], however the dissipative term was laterally dependent (θ) rather than normally dependent (ξ) dependent. The implicit connection formulae found coincide with those found in plasmas with isotropic viscosity and finite electrical resistivity [22].

A drawback of previous studies on resonant absorption is that even though anisotropy is considered, dispersion (by, e.g. Hall effect) is neglected. This approximation is acceptable only for lowest regions of the solar atmosphere. The solar corona is known to be strongly magnetized, hence the Hall term can be comparable with other effects considered in the process of resonance. The present paper will extend the nonlinear theory of resonant slow MHD waves in the dissipative layer with strongly anisotropic viscosity and thermal conductivity to include Hall dispersion and show that the effect of this new addition is of the same order of magnitude as nonlinearity and dissipation near resonance. The paper is organised as follows. In the next section we introduce the fundamental equations and discuss the main assumptions. In section III we derive the nonlinear governing equation for waves in the dissipative layer. Section IV is devoted to the derivation of the nonlinear analogues of the connection formulae. Finally, in section V we summarize and draw our conclusions pointing out a few further applications to be carried out in the future.

II. FUNDAMENTAL EQUATIONS

In what follows we use the visco-thermal MHD equations with strongly anisotropic viscosity and thermal conductivity. We assume that the plasma is strongly magnetized, so that the conditions \( \omega_e \tau_e \gg 1 \) and \( \omega_i \tau_i \gg 1 \) are satisfied, where \( \omega_e(i) \) is the electron (ion) gyrofrequency and \( \tau_e(i) \) is the mean electron (ion) collision time. Due to the strong magnetic field, transport processes are derived from Braginskii's expression for the viscosity tensor \( \hat{\eta} \) (see Refs. [12] and [28]). Under coronal conditions it is a good approximation to retain only the first term of Braginskii's expression for viscosity, [22], namely

\[
\hat{\eta} = \hat{\eta}_0 \left( b \otimes b - \frac{1}{3} \hat{I} \right) [3b \cdot (b \cdot \nabla) v - \nabla \cdot v],
\]

where \( v \) is the velocity, \( b = \mathbf{B}/B \) is the unit vector in the direction of magnetic field and \( \hat{\eta}_0 \) is the first Braginskii coefficient of viscosity (compressional viscosity). \( \hat{I} \) is the unit tensor and the symbol \( \otimes \) denotes the dyadic product of vectors.

In a strongly magnetized plasma the thermal conductivity parallel to the magnetic field lines dwarfs the perpendicular component so the heat flux becomes [29],

\[
q = -\kappa || (b \cdot \nabla T),
\]

where \( T \) is the temperature and \( \kappa || \) is the parallel coefficient of thermal conductivity (in the solar corona, \( \kappa || = 9 \times 10^{-12} T^{3/2} \text{Wm}^{-1}\text{K}^{-1} \)).

In the solar corona the finite electrical resistivity can be neglected as it is several orders of magnitude smaller than the dissipative coefficients considered here, see e.g. [30].

Using Eqs. (1) and (2), the visco-thermal MHD equations are

\[
\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\bar{\rho}v) = 0, \tag{3}
\]

\[
\frac{\partial \bar{v}}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho} \nabla \bar{P} + \frac{1}{\mu \rho} (\mathbf{B} \cdot \nabla) \mathbf{B}
+ \frac{1}{\rho} (\nabla \cdot b) \times (\hat{\eta}_0 Q), \tag{4}
\]

\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times (v \times \mathbf{B}) + \mathbf{H}, \tag{5}
\]

\[
\frac{\partial T}{\partial t} + v \cdot \nabla T + (\gamma - 1) T \nabla \cdot v = \frac{\gamma - 1}{R \bar{\rho}} \left\{ \nabla \cdot [\bar{\kappa} || (b \cdot \nabla T)] + \frac{1}{3} \hat{\eta}_0 Q^2 \right\}, \tag{6}
\]

\[
\mathbf{T} = \bar{\rho} + \frac{\mathbf{B}^2}{2 \mu} + \frac{\hat{\eta}_0}{3} Q, \tag{7}
\]
\[ \tilde{\rho} = \tilde{R} \rho \bar{T}, \quad (8) \]

\[ \nabla \cdot \mathbf{\bar{B}} = 0, \quad (9) \]

\[ Q = 3b \cdot (b \cdot \nabla) \mathbf{v} - \nabla \cdot \mathbf{v}. \quad (10) \]

In Eqs. (3)-(10), \( \tilde{\rho} \) is the kinematic pressure, \( \tilde{\rho} \) the density, \( \bar{T} \) the viscosity modified total pressure (kinetic and magnetic), \( \gamma \) the adiabatic exponent, \( \tilde{R} \) the gas constant and \( \mu \) the magnetic permeability. The term \( \mathbf{\bar{H}} \) in Eq. (5) is the Hall term given by

\[ \mathbf{\bar{H}} = \frac{1}{\mu e} \nabla \times \left( \frac{1}{n_e} \mathbf{\bar{B}} \times \nabla \times \mathbf{\bar{B}} \right), \quad (11) \]

where \( e \) is the electron charge and \( n_e \) is the electron number density. The propagation of compressional linear and nonlinear MHD waves in Hall plasmas has been studied by, for example, Baronov & Ruderman [31], Ruderman [32], Ballai et al. [33] and Miteva et al. [34]. As stated in an earlier study by Huba [35], Hall MHD is only relevant to plasma dynamics occurring on length scales of the order of the ion inertial length, \( d_i = c/\omega_i \), where \( c \) is the speed of light and \( \omega_i \) is the ion plasma frequency. For the present paper this would require that \( d_i = \theta(\delta_e) \), where \( \delta_e \) is the thickness of the dissipative layer. Indeed, starting from the upper chromosphere this condition is satisfied and the lengths involved in the problem are of the order of \( 10-100m \). Hall currents arise when considering off diagonal terms in the conductivity tensor.

We adopt a Cartesian coordinate system, and limit our analysis to a static background equilibrium (\( \mathbf{v}_0 = 0 \)). We assume that all equilibrium quantities depend on \( x \) only. The equilibrium magnetic field, \( \mathbf{B}_0 \), is unidirectional and lies in the \( yz \)-plane. The equilibrium quantities must satisfy the condition of total pressure balance,

\[ p_0 + \frac{B_0^2}{2\mu} \text{ = constant.} \quad (12) \]

In addition we assume that the wave propagation is independent of \( y \) (\( \partial/\partial y = 0 \)). In linear theory of driven waves all perturbed quantities oscillate with the same frequency \( \omega \), so they can be Fourier-analysed and taken to be proportional to \( \exp(i[kz - \omega t]) \), so solutions are sought in the form of propagating waves. All perturbations in these solutions depend on the combination \( \theta = z - Vt \), rather than \( z \) and \( t \) separately, with \( V = \omega/k \). In the context of resonant absorption the phase velocity, \( V \), must match the projection of the cusp velocity, \( c_T \), onto the \( z \)-axis when \( x = x_c \). To define the resonant position mathematically it is convenient to introduce the angle, \( \alpha \), between the \( z \)-axis and the direction of the equilibrium magnetic field, so that the components of the equilibrium magnetic field are

\[ B_{0y} = B_0 \sin \alpha, \quad B_{0z} = B_0 \cos \alpha. \quad (13) \]

The definition of the resonant position can now be written mathematically as

\[ V = c_T (x_c) \cos \alpha. \quad (14) \]

The square of the cusp speed is defined as

\[ c_T^2 = \frac{c_s^2 + v_A^2}{c_s^2 + v_A^2}. \quad (15) \]

where the squares of the sound and Alfvén speeds are given by

\[ c_s^2 = \frac{\gamma p_0}{\rho_0}, \quad v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}. \quad (16) \]

with all speeds being dependent on the coordinate \( x \).

In a nonlinear regime the perturbations cannot Fourier-analysed, however, in an attempt to adhere as close to linear theory as possible we look for travelling wave solutions and assume all perturbed quantities depend on \( \theta = z - Vt \), where \( V \) is given by Eq. (14). The perturbations of the physical quantities are defined by

\[ \rho = \rho_0 + \rho, \quad p = p_0 + p, \quad \bar{T} = T_0 + T \]

\[ \mathbf{B} = \mathbf{B}_0 + \mathbf{B}, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}, \quad \bar{T} = \bar{T}_0 + \bar{T} \quad (17) \]

It is clear from Equations (11) and (17) that \( \mathbf{H}_0 = 0 \).

The dominant dynamics of resonant slow waves, in linear MHD, resides in the components of magnetic field and velocity that are parallel to the equilibrium magnetic field (as well as in the compressional quantities \( \rho \), \( p \) and \( T \)). This dominant behaviour is created by an \( x^{-1} \) singularity in the spatial solution of these quantities at the cusp resonance. (21); these are known as large variables. The normal components of velocity, \( u \), and magnetic field, \( B_z \), are also singular however their singularity is proportional to \( \ln |x| \). In addition, the quantities \( \bar{T} \) and the components of \( \mathbf{B} \) and \( \mathbf{v} \) that are perpendicular to the equilibrium magnetic field are regular; these are known as small variables.

To make the mathematical analysis more concise and to make the physics more transparent we define the components of velocity and magnetic field that are parallel and perpendicular to the equilibrium magnetic field (as well as existing in the \( yz \)-plane):

\[ \left( \begin{array}{c} v_y \\ B_{\parallel} \end{array} \right) = \left( \begin{array}{cc} v & w \\ B_y & B_z \end{array} \right) \left( \begin{array}{c} \sin \alpha \\ \cos \alpha \end{array} \right), \]

\[ \left( \begin{array}{c} v_{\perp} \\ B_{\perp} \end{array} \right) = \left( \begin{array}{cc} v & -w \\ B_y & -B_z \end{array} \right) \left( \begin{array}{c} \cos \alpha \\ \sin \alpha \end{array} \right). \quad (18) \]

where \( u, w, B_y \) and \( B_z \) are the \( y \)- and \( z \)-components of the velocity and perturbation of magnetic field, respectively.

Let us introduce the characteristic scale of inhomogeneity, \( l_{inh} \). The classic Reynolds number, \( R_e \), and the Pecklet number, \( P_e \), are defined as

\[ R_e = \frac{\rho_0 V l_{inh}}{\eta_0}, \quad P_e = \frac{\rho_0 \bar{R} V l_{inh}}{\kappa ||}. \quad (19) \]
These two numbers determine the importance of viscosity and thermal conduction. We introduce the total Reynolds number as

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_s}. \quad (20)$$

The aim of this paper is to derive the governing equation for waves in the slow dissipative layers taking into account nonlinearity, dissipation and dispersion simultaneously. Since for the coronal plasma $R \gg 1$, we consider only weakly dissipative plasmas. We introduce $\epsilon$, as the dimensionless amplitude of variables far away from the resonance. Linear theory predicts that the characteristic scale of dissipation, $l_{\text{diss}}$, is of the order $R^{-1}l_{\text{inh}}$. The typical largest nonlinear term in the system of MHD equations is of the form $g \partial g / \partial z$ while the typical dissipative term if of the form $\eta_0 \partial^2 g / \partial z^2$, where $g$ is any ‘large’ variable. Linear theory shows that ‘large’ variables have an ideal singularity $(x - x_c)^{-1}$ in the vicinity of $x = x_c$. This implies that the ‘large’ variables have dimensionless amplitudes, inside the dissipative layer, of the order of $\epsilon R$. It is now straightforward to estimate the ratio of a typical nonlinear and dissipative term,

$$\phi = \frac{g \partial g / \partial z}{\eta_0 \partial^2 g / \partial z^2} = \Theta(\epsilon R^2), \quad (21)$$

where the quantity $\phi$ can be considered as a nonlinearity parameter. If the condition $\epsilon R^2 \ll 1$ is satisfied, linear theory is applicable. On the other hand, if $\epsilon R^2 \gtrsim 1$ then nonlinearity has to be taken into account when studying resonant waves in the dissipative layers. Using the same scalings, Ballai et al. [23] showed that nonlinearity has to be considered whenever slow wave resonant absorption is studied in the solar corona. This conclusion simply means that in the solar upper atmosphere resonant absorption is a nonlinear phenomena.

In order to obtain nonlinearity and dissipation of equal order we assume that $\epsilon R^2 = \Theta(1)$, i.e. $R \sim \epsilon^{-1/2}$, when deriving the nonlinear governing equations for waves in the slow dissipative layer. Accordingly, we can re-scale the coefficients of viscosity and thermal conductivity as

$$\eta_0 = \epsilon^{1/2} \eta_0, \quad \kappa = \epsilon^{1/2} \kappa. \quad (22)$$

We also consider the coefficient of Hall conduction, defined as $\chi = \mathbf{\Gamma} \omega_e \tau_e$, where $\mathbf{\Gamma}$ is the coefficient of magnetic diffusivity (although $\mathbf{\Gamma}$ is small enough, in the solar corona, to be neglected in comparison to $\eta_0$, here it is multiplied by the product $\omega_e \tau_e$ which is very large under coronal conditions). Similar to the previous dissipative coefficients, we introduce the scaling

$$\chi = \epsilon^{1/2} \chi. \quad (23)$$

Using Eq. (22), we can rewrite Eqs. (3)-(7) in the form

$$\rho_0 V \frac{\partial u}{\partial \theta} - \frac{\partial (\rho_0 u)}{\partial x} - \rho_0 \frac{\partial w}{\partial \theta} = \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho w)}{\partial \theta}, \quad (24)$$

$$\frac{\partial}{\partial \theta} \left( \rho_0 V v_\parallel + \rho_0 V v_\perp \sin \alpha + B_\parallel \frac{\rho_0 \cos \alpha}{\mu} \right) = \rho \left( u \frac{\partial v_\parallel}{\partial x} + w \frac{\partial v_\perp}{\partial \theta} \right) - \frac{\rho}{\mu} \frac{\partial B_\parallel}{\partial x} - \frac{\partial B_\perp}{\partial \theta}$$

$$- \epsilon^{1/2} \left( \frac{\partial}{\partial x} b_x + \frac{\partial}{\partial \theta} b_z \right) (\eta_0 b_y Q), \quad (25)$$

$$\frac{\partial}{\partial \theta} \left( \rho_0 V v_\parallel - P \cos \alpha + \frac{\rho_0}{\mu} \frac{B_\parallel}{B_\perp} \right) + B_\perp \frac{\rho_0 \cos \alpha}{\mu} \frac{\partial B_\perp}{\partial x} =$$

$$\rho \left( u \frac{\partial v_\parallel}{\partial x} + w \frac{\partial v_\perp}{\partial \theta} \right) - \frac{\rho}{\mu} \frac{\partial B_\parallel}{\partial x} - \frac{\partial B_\perp}{\partial \theta}$$

$$- \epsilon^{1/2} \left( \frac{\partial}{\partial x} b_x + \frac{\partial}{\partial \theta} b_z \right) (\eta_0 b_y Q), \quad (26)$$

$$V B_x + B_0 u \cos \alpha = w B_x - u B_z$$

$$- \epsilon^{1/2} \chi \frac{\partial B_\parallel}{\partial \theta} \cos \alpha \sin \alpha, \quad (28)$$

$$\frac{\partial}{\partial \theta} \left( V B_\perp + B_0 v_\perp \cos \alpha \right) = \frac{\partial (u B_\perp)}{\partial x} + \frac{\partial (w B_\perp)}{\partial \theta}$$

$$- B_\perp \frac{\partial v_\parallel}{\partial x} - \frac{\partial B_\parallel}{\partial \theta} - \epsilon^{1/2} \chi \frac{\partial^2 B_\parallel}{\partial x \partial \theta} \cos \alpha, \quad (29)$$

$$\frac{\partial}{\partial \theta} \left( V B_\parallel + B_0 v_\parallel \cos \alpha \right) - \frac{\partial (0 u B_\parallel)}{\partial x} - \frac{\partial w}{\partial \theta} =$$

$$\frac{\partial (w B_\parallel)}{\partial x} - B_\perp \frac{\partial v_\parallel}{\partial x} - B_\parallel \frac{\partial v_\parallel}{\partial \theta} - \epsilon^{1/2} \chi \frac{\partial B_\parallel}{\partial x} \frac{\partial \rho}{\rho_0} \frac{\partial \rho}{\partial \theta} \sin \alpha, \quad (30)$$

$$\frac{\partial B_\parallel}{\partial x} + \frac{\partial B_\perp}{\partial \theta} = 0, \quad (31)$$

$$V \frac{\partial T}{\partial \theta} - u \frac{dT_0}{dx} - (\gamma - 1) T_0 \frac{\partial u}{\partial x} + \frac{\partial w}{\partial \theta} =$$

$$u \frac{\partial T}{\partial x} + \frac{\partial T}{\partial \theta} + (\gamma - 1) T \frac{\partial u}{\partial x} \frac{\partial w}{\partial \theta}$$

$$- \epsilon^{1/2} \gamma \frac{1}{\rho R} \left\{ \frac{1}{3} \eta_0 Q^2 + \left( \frac{\partial}{\partial x} b_x + \frac{\partial}{\partial \theta} b_z \right) \right\} \left[ b_x \left( \frac{dT_0}{dx} + \frac{\partial T}{\partial x} \right) + b_z \frac{\partial T}{\partial \theta} \right], \quad (32)$$
\[ \bar{P} = p + \frac{1}{2\mu} \left( B_z^2 + B_\perp^2 + B_\parallel^2 + 2B_0B_\parallel \right) + \frac{1}{3}\epsilon^{1/2} \eta_0 Q, \]  
(33)

\[ \frac{\gamma T_0 \rho}{c_s^2} - T_0 \rho - \rho_0 T = \rho T, \]  
(34)

\[ Q = 3b_x \left( b_x \frac{\partial u}{\partial x} + b_z \frac{\partial u}{\partial \theta} \right) + 3b_y \left( b_x \frac{\partial u}{\partial x} + b_z \frac{\partial u}{\partial \theta} \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial \theta} \right). \]  
(35)

We should state that in Eqs. (28)-(30) we have used the coefficient of Hall conduction, \( \chi \), which does not contribute to the total Reynolds number because it is the multiplier of dispersive terms rather than dissipative ones. The derivation of the expressions of the Hall terms in the induction equation can be found in the Appendix.

The set of Eqs. (24)-(35), will be used in the next section to derive the governing equation for wave motion in the dissipative layer.

### III. DERIVING THE GOVERNING EQUATION IN THE DISSIPATIVE LAYER

In order to derive the governing equation for wave motions in the slow dissipative layer we employ the method of matched asymptotic expansions (25). This method requires us to find the so-called outer and inner expansions and then match them in the overlap regions. This nomenclature is ideal for our situation. The outer expansion corresponds to the solution outside the dissipative layer and the inner expansion corresponds to the solution inside the dissipative layer. A simplified version of the method of matched asymptotic expansions, developed by Ballai et al. [25], is adopted here.

We only consider weakly dissipative plasmas, so viscosity and thermal conductivity are only important in the narrow dissipative layer (here dissipation and nonlinearity are of the same order) embracing the resonant position. Far away from the dissipative layer the amplitudes of perturbations are small, so we use linear ideal MHD equations in order to describe the wave motion far away from the dissipative layer. The full set of nonlinear dissipative MHD equations are used for describing wave motion inside the dissipative layer where the amplitudes are much larger than those far away from the dissipative layer. We therefore look for solutions in the form of asymptotic expansions. The equilibrium quantities change only slightly across the dissipative layer so it is possible to approximate them by the first non-vanishing term in their Taylor series expansion with respect to \( x \). Similar to linear theory, we assume the expansions of equilibrium quantities are valid in a region embracing the ideal resonant position, which is assumed to be much wider than the dissipative layer, \( l_{\text{inh}}/l_{\text{diss}} \sim R \). This implies there are two overlap regions, one to the left and one to the right of the dissipative layer, where both the outer (the solution to the linear ideal MHD equations) and inner (the solution to the nonlinear dissipative MHD equations) solutions are valid. Therefore, both solutions must coincide in the overlap regions which provides the matching conditions.

Before deriving the nonlinear governing equation we ought to make a note. In linear theory, perturbations of physical quantities are harmonic functions of \( \theta \) and their mean values vanish over a period. In nonlinear theory, however, the perturbations of variables can have nonzero values as a result of nonlinear interaction of different harmonics. Due to the nonlinear absorption of wave momentum, a mean shear flow is generated outside the dissipative layer, as shown by Ruderman et al. [27] in Cartesian geometry. In our scenario a mean shear flow is created outside the dissipative layer, but as there is no perpendicular component to viscosity oscillating plasma can slide past each other without friction. This produces a mean flow with infinite amplitude. However, boundaries can prevent such oscillations. Therefore, we will assume such boundaries exist and will not consider the generation of mean shear flow.

The first step in our mathematical description is the derivation of governing equations outside the dissipative layer where the dynamics is described by ideal (\( \eta_0 = \kappa_0 = 0 \)) and linear MHD. Since dispersion is assumed to act over scales comparable to dissipative scales, dispersion will only be taken into account inside the dissipative layer. The linear form of Eqs. (24)-(35) can be obtained by assuming a regular expansion of variables of the form

\[ f = \epsilon f^{(1)} + \epsilon^{3/2} f^{(2)} \ldots, \]  
(36)

and collect only terms proportional to the small parameter \( \epsilon \). This leads to a system of linear equations for the variables with superscript ‘1’. All variables can be eliminated, with the exception of \( u^{(1)} \) and \( P^{(1)} \), leading to the system

\[ \frac{\partial u^{(1)}}{\partial x} = \frac{V}{F} \frac{\partial P^{(1)}}{\partial \theta}, \quad \frac{\partial P^{(1)}}{\partial x} = \frac{\rho_0 A}{V} \frac{\partial u^{(1)}}{\partial \theta}, \]  
(37)

where

\[ F = \frac{\rho_0 AC}{V^4 - V^2 \left( v_A^2 + c_S^2 \right) + v_A^2 c_S^2 \cos^2 \alpha}, \]  
(38)

\[ A = V^2 - v_A^2 \cos^2 \alpha, \]  
(39)

\[ C = \left( v_A^2 + c_S^2 \right) \left( V^2 - c_S^2 \cos^2 \alpha \right). \]  
(39)

The reason for using \( P \) instead of \( \bar{P} \) is that outside the dissipative layer the plasma is ideal, so there is no viscous addition to the total pressure. The remaining variables can all be expressed in terms of \( u^{(1)} \) and \( P^{(1)} \),

\[ u^{(1)} = -\frac{V \sin \alpha}{\rho_0 A} P^{(1)}, \quad v^{(1)} = \frac{V c_S^2 \cos \alpha}{\rho_0 C} P^{(1)}, \]  
(40)
\[ B^{(1)}_e = - \frac{B_0 \cos \alpha}{V} u^{(1)}_e, \quad B^{(1)}_v = \frac{B_0 \cos \alpha \sin \alpha}{\rho_0 A} P^{(1)}, \quad \frac{\partial B^{(1)}_e}{\partial \theta} = B_0 \frac{V^2 - c_s^2 \cos \alpha}{\rho_0 C} \frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}_e dB_0}{V \, dx}, \]

\[ \frac{\partial p^{(1)}}{\partial \theta} = \frac{V^2 c_s^2}{C} \frac{\partial P^{(1)}}{\partial \theta} - \frac{u^{(1)}_e dB_0}{\mu V} \frac{dB_0}{dx}, \]

\[ \frac{\partial \rho^{(1)}}{\partial \theta} = \frac{V^2 \rho^{(1)}}{C} \frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}_e d\rho_0}{V} \frac{d\rho_0}{dx}, \]

\[ \frac{\partial T^{(1)}}{\partial \theta} = \frac{(\gamma - 1) T_0 V^2}{\rho_0 C} \frac{\partial P^{(1)}}{\partial \theta} + \frac{u^{(1)}_e d\xi}{\gamma V R} \frac{d\xi}{dx}. \]

The differential Eq. (37) have regular singularities at the resonant position, therefore the solutions can be obtained in terms of Frobenius series with respect to \( x \) for details see, e.g. [23, 27] of the form

\[ P^{(1)} = P^{(1)}_1(\theta) + 2(\theta)x \ln |x| + P^{(1)}_3(\theta) + \ldots, \]

\[ u^{(1)} = u^{(1)}_1(\theta) \ln |x| + u^{(1)}_2(\theta) + u^{(1)}_3(\theta)x \ln |x| + \ldots. \]

The coefficient functions depending on \( \theta \) in the above expansions are generally different for \( x < 0 \) and \( x > 0 \). The salient property of these solutions is that the perturbation of the total pressure is regular at the ideal resonant position \( x = x_c \). From Eqs. (40)-(45), we see that the quantities \( v^{(1)}_e \) and \( B^{(1)}_v \) are also regular at \( x = x_c \). All other quantities are singular. The quantities \( u^{(1)}_e \) and \( B^{(1)}_v \) behave as \( \ln |x| \), while \( v^{(1)}_v, B^{(1)}_v, p^{(1)}, \rho^{(1)} \) and \( T^{(1)} \) behave as \( x^{-1} \), so they are the most singular. This property can be extended and will hold to all higher order approximations. [27]

Now let us concentrate on the solution in the dissipative layer. The thickness of the dissipative layer is of the order \( l_{inh} R^{-1} \). We have assumed that \( R \sim \epsilon^{-1/2} \) so we obtain \( l_{inh} R^{-1} = \theta^\epsilon (1/l_{inh}) \). The implication of this scaling is that we have to introduce a new stretched variable to replace the transversal coordinate in the dissipative layer, so we are going to use \( \xi = \epsilon^{-1/2} x \). Equations (24)-(33) are not rewritten here as they are easily obtained by the substitution of

\[ \frac{\partial}{\partial x} = \epsilon^{-1/2} \frac{\partial}{\partial \xi}, \]

for all derivatives. The equilibrium quantities still depend on \( x \), not \( \xi \) (their expression is valid in a wider region than the characteristic scale of dissipation). All equilibrium quantities are expanded around the ideal resonant position, \( x = x_c \), as

\[ f_0 = f_{0c} + \xi \left( \frac{\partial f_0}{\partial \xi} \right) + \ldots \]

\[ \approx f_{0c} + \epsilon^{1/2} \xi \left( \frac{df_0}{dx} \right) c, \]

where \( f_0 \) is any equilibrium quantity and the subscript ‘c’ indicates the equilibrium quantity has been evaluated at the resonant point (we can always make \( x_c = 0 \) by proper translation of the coordinate system).

We seek the solution to the set of equations obtained from Eqs. (24)-(33) by substitution of \( x = \epsilon^{1/2} \xi \) into variables in the form of power series of \( \epsilon \). These equations contain powers of \( \epsilon^{1/2} \), so we use this quantity as an expansion parameter. To derive the form of the inner expansions of different quantities we have to analyze the outer solutions. First, since \( v_\| \) and \( B_\| \) are regular at \( x = x_c \), we can write their inner expansions in the form of their outer expansions, namely Eq. (36). The quantity \( \tilde{P} \) is the sum of the perturbation of total pressure \( P \), which is regular at \( x = x_c \), and the dissipative term proportional to \( Q \). From Eq. (35) it is obvious that \( Q \) behaves as \( x^{-1} \) in the vicinity of \( x = x_c \). Far away from the dissipative layer, \( Q \) is of the order \( \epsilon \). Since the thickness of the dissipative layer is of the order \( \epsilon^{1/2} l_{inh} R \), \( Q \) is of the order \( \epsilon^{1/2} \) in the dissipative layer. However, Eq. (43) clearly shows the term proportional to \( Q \) contains a multiplier \( \epsilon^{1/2} \), which implies the contribution of \( \tilde{P} \) supplied by the dissipative term is of the order \( \epsilon \). As a consequence, we write the inner expansion of \( \tilde{P} \) in the form of its outer expansion, Eq. (36). The amplitudes of large variables in the dissipative layer are of the order \( \epsilon^{1/2} \), so the inner expansion of the variables \( v_\|, B_\|, p, \rho \) and \( T \) is

\[ g = \epsilon^{1/2} g^{(1)} + \epsilon g^{(2)} + \ldots. \]

The quantities \( u \) and \( B_x \) behave as \( \ln |x| \) in the vicinity of \( x = x_c \), which suggests that they have expansions with terms of the order \( \ln \epsilon \) in the dissipative layer. Ruderman et al. [27] showed that, strictly speaking, the inner expansions of all variables have to contain terms proportional to \( \ln \ln \epsilon \) and \( \epsilon^{1/2} \ln \epsilon \). In the simplified version of matched asymptotic expansions, [23], we utilize the fact that \( |\ln \epsilon| \ll \epsilon^{-\kappa} \) for any positive \( \kappa \) and \( \epsilon \to +0 \), and consider \( \ln \epsilon \) as a quantity of the order of unity. This enables us to write the inner expansions for \( u \) and \( B_x \) in the form of Eq. (36).

We now substitute the expansion (36) for \( u, B_x, \tilde{P}, v_\perp, B_\perp \) and the expansion given by (30) for \( v_\|, B_\|, p, \rho, T \) into the set of equations obtained from Eqs. (24)-(33) after substitution of \( x = \epsilon^{1/2} \xi \). The first order approximation (terms proportional to \( \epsilon \)), yields a linear homogeneous system of equations for the terms with superscript ‘1’. The important result that follows from this set of equations is that

\[ \tilde{P}^{(1)} = \tilde{P}^{(1)}(\theta), \]

that is to say \( \tilde{P}^{(1)} \) does not change across the dissipative layer. This result parallels the result found in linear theory [21; 39] and nonlinear theory [23; 27]. Subsequently, all remaining variables can be expressed in
In addition, we find that the equation that relates $u^{(1)}$, $v^{(1)}$ and $\tilde{P}(1)$ as

$$
\frac{c^2}{\sqrt{s}} \sin \alpha \tilde{P}(1)(\theta), \quad B_z = -\frac{B_0}{V} \cos \alpha u^{(1)},
$$

$$
B_{\perp} = -\frac{B_0 c^2}{\rho_0 V^2 v_A^2} \tilde{P}(1)(\theta),
$$

$$
B_{||} = -\frac{B_0 V}{\rho_0 v_A^2} \cos \alpha v^{(1)}, \quad \rho^{(1)} = \frac{\rho_0 V}{c^2} \cos \alpha v^{(1)}, \quad T^{(1)} = \frac{(\gamma - 1)B_0 V}{c^2} v^{(1)}.
$$

In addition, we find that the equation that relates $u^{(1)}$ and $v^{(1)}$ is

$$
\frac{\partial u^{(1)}}{\partial \xi} + \frac{V^2}{v_A^2} \cos \alpha \frac{\partial v^{(1)}}{\partial \theta} = 0.
$$

In the second order approximation we use only the expressions obtained from Eqs. (24), (27), (29), (32), (33) and (33). Employing Eqs. (51)-(56), we replace the variables in the first order approximation. The equations obtained in the second order approximation are

$$
\rho_0 \left( \frac{\partial u^{(2)}}{\partial \xi} + \frac{\partial v^{(2)}}{\partial \theta} \cos \alpha \right) - V \frac{\partial v^{(2)}}{\partial \theta} =
$$

$$
- u^{(1)} \left( \frac{d \rho_0}{d x} \right) \frac{V}{v_A^2} \cos \alpha \frac{d \rho_0}{d x} \frac{d v^{(1)}}{d \theta} + \frac{c^2}{\sqrt{s}} \sin \alpha \frac{d \tilde{P}}{d \theta} - \rho_0 V \frac{u^{(1)}}{c^2} \cos \alpha \frac{d v^{(1)}}{d \theta} + \frac{2v_A^2}{c^2} + \frac{c^2}{c^2 + v_A^2} \frac{d v^{(1)}}{d \theta},
$$

$$
\frac{\partial}{\partial \theta} \left( \frac{V}{v_A^2} + \frac{B_0 \cos \alpha}{\mu \rho_0} \frac{B^{(2)}}{\rho_0} \right) = \frac{\cos \alpha}{\rho_0} \frac{d \tilde{P}}{d \theta} + \frac{B_0 \cos \alpha}{\mu \rho_0} \left( \frac{d \rho_0}{d x} \right) \frac{B^{(2)}}{\rho_0} \frac{d \rho_0}{d x} + V \frac{d \rho_0}{d x} \frac{d \rho_0}{d x} + \frac{\rho_0 c^2}{\rho_0} \frac{d v^{(1)}}{d \theta},
$$

$$
\frac{\partial B^{(2)}}{\partial \theta} - B_0 \frac{\partial u^{(2)}}{\partial \xi} = u^{(1)} \left( \frac{d B_0}{d x} \right) \frac{d \rho_0}{d x} + \frac{V^2}{v_A^2} \cos \alpha \left( \frac{d B_0}{d x} \right) \frac{d v^{(1)}}{d \theta} - \frac{B_0 c^2 \sin \alpha}{\rho_0} \frac{d \tilde{P}}{d \theta} + \frac{B_0 V^2}{v_A^2} \left( \frac{d v^{(1)}}{d \xi} \cos \alpha - \frac{V^2}{v_A^2} \frac{d v^{(1)}}{d \theta} \right) + \frac{\lambda}{v_A^2} \frac{\sin \alpha \frac{d v^{(1)}}{d \theta} \frac{d v^{(1)}}{d \theta}}{\cos \alpha \frac{d \theta}{d \theta}}.
$$

In deriving the above system we have used the fact that

$$
\frac{\partial}{\partial \theta} \left( \frac{V}{v_A^2} + \frac{B_0 \cos \alpha}{\mu \rho_0} \frac{B^{(2)}}{\rho_0} \right) = \frac{\cos \alpha}{\rho_0} \frac{d \tilde{P}}{d \theta} + \frac{B_0 \cos \alpha}{\mu \rho_0} \left( \frac{d \rho_0}{d x} \right) \frac{B^{(2)}}{\rho_0} \frac{d \rho_0}{d x} + V \frac{d \rho_0}{d x} \frac{d \rho_0}{d x} + \frac{\rho_0 c^2}{\rho_0} \frac{d v^{(1)}}{d \theta},
$$

obtained from Eq. (33) in the first order of approximation. With the exception of Eq. (59), which has the addition of the Hall term, these equations are identical to those found by Ballai et al. [25].

The left-hand sides of the set of Eqs. (57) - (62) could be obtained from the left-hand sides of the first order approximation by substituting variables with the superscript ‘2’ for those with superscript ‘1’. The first order of approximation possesses a non-trivial solution, so Eqs.
are compatible only if the right-hand sides of Eqs. [57]–[62] satisfy a compatibility condition. To derive the compatibility condition we express \( \rho \) and \( B \) in terms of \( u \), \( v \), \( u^{(1)} \), \( v^{(1)} \) and \( \tilde{P} \), using Eqs. [58] and [60]–[62]. Subsequently, we substitute these expressions into Eqs. [57]–[62] to obtain

\[
\frac{\partial u^{(2)}}{\partial \xi} + \frac{V^2}{v_A^2} \frac{\partial v^{(2)} ||}{\partial \theta} = \frac{V (v_A^2 + c_S^2 \sin^2 \alpha)}{\rho_0 v_A^2} \frac{d \tilde{P}^{(1)}}{d \theta} + \frac{V}{v_A^2 + c_S^2} \frac{\partial u^{(1)} ||}{\partial \theta} 
\]

\[
+ \frac{V^2}{v_A^2} \cos \alpha \left[ \frac{2}{B_0} \frac{d B_0}{d x} \right] \left( \frac{d \rho_0}{d x} \right) \frac{\partial v^{(1)} ||}{\partial \theta} + \frac{\eta V \cos \alpha}{\rho_0 v_A^2} \left( \frac{2 v_A^2 + 3 c_S^2}{v_A^2 + c_S^2} \right) \frac{\partial^2 u^{(1)}}{\partial \theta^2} - \frac{V}{c_S^2 \cos \alpha} \frac{u^{(1)} ||}{\partial \xi} \partial v^{(1)} || = \frac{\chi \sin \alpha}{v_A^2 + c_S^2} \frac{\partial v^{(1)} ||}{\partial \theta} \left( \frac{\partial v^{(1)} ||}{\partial \theta} \right), \quad \text{(64)}
\]

It can be seen that Eqs. [64] and [65] have identical left-hand sides. Extracting these two equations we derive the compatibility condition, which is the equation connecting \( v^{(1)} || \) and \( \tilde{P}^{(1)} \)

\[
\frac{\partial u^{(2)}}{\partial \xi} + \frac{V^2}{v_A^2} \frac{\partial v^{(2)} ||}{\partial \theta} = \frac{c_S^2 \sin^2 \alpha}{\rho_0 v_A^2} \frac{d \tilde{P}^{(1)}}{d \theta} 
\]

\[
- \frac{V^2}{T_0 c_S^3} \cos \alpha \left( \frac{d T_0}{d x} \right) \frac{\xi \partial v^{(1)} ||}{\partial \theta} - \frac{V}{c_S^2 \cos \alpha} \frac{u^{(1)} ||}{\partial \xi} \partial v^{(1)} || 
\]

\[
+ \frac{V \cos \alpha}{\gamma \rho_0 c_S^3} \left[ \frac{2 \gamma \rho_0}{3} \left( \frac{2 v_A^2 + 3 c_S^2}{v_A^2 + c_S^2} \right) + \frac{(\gamma - 1) \kappa_{\parallel}}{R} \right] \frac{\partial^2 u^{(1)}}{\partial \theta^2} \quad \text{(65)}
\]

where we have used the notation

\[
a = \frac{V ((\gamma + 1) v_A^2 + 3 c_S^2)}{(v_A^2 + c_S^2)} \frac{v_A^2 \cos \alpha}{R} \quad \text{(67)}
\]

\[
\lambda = \frac{\eta (2 v_A^2 + 3 c_S^2)^2}{3 \rho_0 (v_A^2 + c_S^2)} + \frac{(\gamma - 1) \kappa_{\parallel} (v_A^2 + c_S^2)}{\gamma \rho_0 R c_S^3} \quad \text{(68)}
\]

\[
\Omega = \frac{\chi c_S^2 v_A^2}{(v_A^2 + c_S^2)} \cos \alpha \sin \alpha \quad \text{(69)}
\]

\[
\Delta = \left( \frac{d c_T^2}{d x} \right) \quad \text{(70)}
\]

Equation [66] differs from its counterpart found by Ballai et al. [23] only by the last term of the left-hand side representing Hall dispersion. If we linearize Eq. [66] and take \( v^{(1)} || \) proportional to \( \exp(ik \theta) \), we arrive at the linear equation for the parallel velocity obtained in linear theory by Ruderman and Goossens [24].

Equation [66] is the complete nonlinear governing equation for the parallel velocity in the slow dissipative layer. The function \( \tilde{P}^{(1)} \) in this equation is determined by the solution outside the dissipative layer, and is thought to be the driving term. We should note here that the dispersion (the last term on the left-hand side) appears as a nonlinear term (nonlinear dispersion).

**IV. NONLINEAR CONNECTION FORMULAE**

In linear dissipative MHD it was assumed that when dissipative effects are weak they are only important in the thin dissipative layer that embraces the ideal resonant position (see, e.g., [21]; [23]; [39]; [40]). Outside this layer, ideal MHD can be employed to describe the plasma motion. The dissipative layer is treated as a surface of discontinuity. In order to solve Eq. [37] boundary conditions are needed for the variables \( u \) and \( P \) at this surface of discontinuity. In linear theory these conditions are described by the explicit connection formulae that determine the jumps in the quantities \( u \) and \( P \). In order to derive the nonlinear counterpart of connection formulae we first define the jump of a function, \( f(x) \), across the dissipative layer as

\[
[f] = \lim_{x \to 0} \{ f(x) - f(-x) \} \quad \text{(71)}
\]

where the coordinate system has been translated such that \( x_c = 0 \). The thickness of this dissipative layer, \( \delta_c \), is determined by the condition that the first and third terms in Eq. [66] are of the same order, i.e.

\[
\delta_c = \frac{V^3 k \lambda}{(v_A^2 + c_S^2) |\Delta|} \quad \text{(72)}
\]

where \( k \) is the wavenumber of waves. It is instructive to introduce a new, dimensionless variable, \( \sigma = x / \delta_c \), in the
dissipative layer. Let $x_0$ be the characteristic width of the overlap regions of the dissipative layer (where both the linear ideal MHD equations and the nonlinear dissipative MHD equations are valid). One of the main reasons we have introduced the variable $\sigma$ is the property that $\sigma = \mathcal{O}(1)$ in the dissipative layer, while $|\sigma| \to \infty$. This provides us with the second definition of the jump in the function $f(x)$ across the dissipative layer,

$$[f] = \lim_{\sigma \to +\infty} \{f(\sigma) - f(-\sigma)\}. \quad (73)$$

The first connection formula can be obtained in a straightforward way by taking into account that the variable $\tilde{P}^{(1)}$ does not change across the dissipative layer, so there cannot be any jump in the total pressure,

$$[\tilde{P}] = 0. \quad (74)$$

This connection formula is the same as obtained previously by linear and nonlinear theories.

In order to derive the second connection formula we use the approximate relations $u \approx \epsilon u^{(1)}$, $v_\parallel \approx \epsilon^{1/2} v^{(1)}$, $\tilde{P} \approx \epsilon \tilde{P}^{(1)}$ and introduce the new dimensionless variable, $q$, defined as

$$q = \epsilon^{1/2} kV \delta_c \cos \alpha \frac{v_\parallel}{v_\parallel^{(1)}}. \quad (75)$$

In the new variable, Eqs. (56) and (66) are rewritten as

$$\frac{\partial u}{\partial \sigma} = -\frac{V}{k \cos^2 \alpha} \frac{\partial q}{\partial \theta}, \quad (76)$$

$$\text{sign}(\Delta) \frac{\partial q}{\partial \theta} - \Lambda q \frac{\partial q}{\partial \theta} + k^{-1} \frac{\partial^2 q}{\partial \theta^2} + \Psi \frac{\partial^2 q}{\partial \sigma \partial \theta} = \frac{kV^4}{\rho_0 \epsilon v_{A_\parallel}^2 |\Delta|} \frac{d\tilde{P}}{d\theta}, \quad (77)$$

where

$$\Lambda = R^2 v_{A_\parallel}^2 |\Delta| \left[(\gamma + 1) v_{A_\parallel}^2 + 3cS_c^2\right], \quad (78)$$

$$\Psi = R^2 |\Delta| v_{A_\parallel}^2 v_{A_\parallel}^2 |\Delta| \left(c^2 + cS_c^2\right) \sin \alpha \frac{kV^4}{kV^4}. \quad (79)$$

We have used a slightly different Reynolds number, $R$, to the one used in previous sections, here it is defined as

$$R = \frac{V}{k\lambda}. \quad (80)$$

It is easy to show that the estimations

$$q = \mathcal{O}(\epsilon^{1/2} k l_{inh} R^{-1}),$$

$$\delta_c = \mathcal{O}(l_{inh} R^{-1}),$$

$$\Lambda = \mathcal{O}(R^2 k l_{inh}^{-1}),$$

$$\Psi = \mathcal{O}(R^2 k l_{inh}^{-2}),$$

are valid. The ratios of the nonlinear to dissipative term, dispersive to dissipative term, and dispersive to nonlinear term in Eq. (77), $N, D_d$ and $D_n$ respectively, are

$$N = \mathcal{O}(\epsilon^{1/2} R l_{inh}^{1}), \quad (81)$$

$$D_d = \mathcal{O}(\epsilon^{1/2} R^2 k l_{inh}^{-1}), \quad (82)$$

$$D_n = \mathcal{O}(R l_{inh}^{2}). \quad (83)$$

The parameters $N$ and $D_d$ can be considered as nonlinearity and dispersive parameters, respectively. Nonlinearity (dispersion) is important if $N \gtrsim 1$ ($D_d \gtrsim 1$). When $N \ll 1$ ($D_d \ll 1$) the nonlinear (dispersive) term in Eq. (77) can be neglected. Dispersion dominates nonlinearity if $D_n \gg 1$. In the opposite case, $D_n < 1$, nonlinearity dominates dispersion. When $k l_{inh} = \mathcal{O}(1)$ (in fact our analysis is valid when $\epsilon^{1/2} \ll k l_{inh} \ll \epsilon^{-1/2}$), the total Reynolds number, $R$, used in this section is of the same order of magnitude as that used in the previous sections, and the criterion of nonlinearity coincides with that obtained in section II from the qualitative analysis. With the above scalings in mind, it is obvious that the physical background of this paper is applicable for relatively short inhomogeneity scales.

Following linear studies, the outer solution reveals that $v_\parallel = \mathcal{O}(\epsilon^{(x-1)})$ as $x \to 0$. Thus, to match the outer and inner solutions in the overlap regions the asymptotic relation $q = \mathcal{O}(\sigma^{-1})$ as $|\sigma| \to \infty$ must be valid. It then directly follows from Eq. (77) that

$$q \simeq \frac{kV^4 \tilde{P}_c(\theta)}{\rho_0 \epsilon v_{A_\parallel}^2 \Delta \sigma}, \quad (84)$$

for $|\sigma| \to \infty$. From Eqs. (70) and (84), we obtain that

$$u = -\frac{V cS_c^2 \cos^2 \alpha}{\rho_0 \epsilon (v_{A_\parallel}^2 + cS_c^2)} \frac{d\tilde{P}_c}{d\theta} \ln |\sigma| + u_+(\theta) + \mathcal{O}(\sigma^{-1}) \quad (85)$$

for $\sigma \to \pm \infty$. The functions $u_+$ and $u_-$ are related by

$$u_+(\theta) - u_-(\theta) = -\frac{V}{k \cos^2 \alpha} \mathcal{P} \int_{-\infty}^{\infty} \frac{\partial q}{\partial \theta} d\sigma. \quad (86)$$

Equation (86) uses the symbol of Cauchy principal part, $\mathcal{P}$, because the integral is divergent at infinity. So, in accordance with Eq. (73), we obtain the implicit jump condition for the normal component of velocity

$$[u] = -\frac{V}{k \cos^2 \alpha} \mathcal{P} \int_{-\infty}^{\infty} \frac{\partial q}{\partial \theta} d\sigma. \quad (87)$$

Equation (87) is the nonlinear analog for the implicit connection formula for the normal component of velocity. The main difference between the linear and the nonlinear connection formula is that while in the linear version the jumps are expressed explicitly in terms of $\tilde{P}^{(1)}$ and equilibrium quantities, in the nonlinear version the jump in
the normal component of velocity, $u$, is expressed implicitly in terms of an unknown quantity $q$. The connection formula (54) is identical to that found by Ballai et al. [25] and Ruderman et al. [27] in the limit of weak nonlinearity for non-dispersive plasmas. To find solutions in the dissipative layer we have to use Eqs. (74) and (77) simultaneously. The boundary conditions for the outer solution are provided by Eqs. (74) and (77).

The governing equation for $q$, Eq. (77), differs to that derived by Ballai et al. [25] only in the appearance of the last term in the left-hand side which is related to the consideration of the Hall term in the generalized Ohm’s law.

V. CONCLUSIONS

In the present paper we have further developed the nonlinear theory of resonant slow MHD waves in the dissipative layer in one-dimensional planar geometry in plasmas with strongly anisotropic viscosity and thermal conductivity by considering dispersive effects. The plasma motion outside the dissipative layer is described by the set of linear, ideal MHD equations. This set of equations can be reduced to Eq. (37) for the component of the velocity in the direction of the inhomogeneity, $u$, and the perturbation of total pressure, $P$. The wave motion in the dissipative layer is governed by Eq. (74) for the quantity $q$, which is the dimensionless component of the velocity parallel to the equilibrium magnetic field, defined by Eq. (75). The dissipative layer is considered as a surface of discontinuity when solving Eq. (57) to describe the wave motion outside the dissipative layer. The jumps across the dissipative layer are given by Eqs. (74) and (77), thus providing the boundary conditions at the surface of discontinuity. In stark contrast to linear theory, the jump in $u$ is not solvable analytically, as it given in terms of a infinite integral of $q$ - which in turn is determined by Eq. (77). Since this equation has not been solved analytically we must solve Eqs. (57) and (77) simultaneously when studying resonant slow waves that are nonlinear in the dissipative layer.

The reader should note that in the upper solar corona the plasma $\beta$ (ratio of kinetic to magnetic pressure) is very small, hence the significance of slow resonance is dramatically reduce (in the limit $\beta = 0$, slow waves cease to exist). The best applicability for the present paper is in the regions of the chromosphere and lower solar corona.

It is interesting to note that the dispersion at the slow resonance appears in the form of a nonlinear term (nonlinear dispersion). It is expected that if the Hall term is included in the Alfvén resonance, this will appear as a linear term. The governing equations and the jump conditions will be used later in studying the absorption of an external driver in the limit of weak and strong nonlinearity. It is intended that the authors will investigate the possibility of describing the propagation of solitary and shock waves in the slow dissipative layer. It remains to be investigated how the inclusion of the dispersive term will influence the conclusions drawn when resonant absorption was used to explain the damping of waves and oscillations in the coronal seismology framework.

Earlier studies (see, e.g. [16]; [41]), show that nonlinearity decreases the effective absorption of waves in the limit of weak nonlinearity. The effect of dispersion on the absorption of waves in slow dissipative layers will be addressed in the near future.

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APPENDIX: THE DERIVATION OF THE HALL TERM IN THE INDUCTION EQUATION FOR SLOW RESONANT WAVES IN THE DISSIPATIVE LAYER

In this appendix we will derive the components of the Hall term in the induction equations and study the conditions under which this extra effect is important. The parallel component of the magnetic field perturbation dominates the other components in the slow dissipative layer. The Hall term contains the first derivative of this parallel component of the magnetic field perturbation with respect to $z$. The first term of Braginskii’s viscosity tensor contains the second derivative of the parallel component of the magnetic field perturbation with respect to $z$. As a result the Hall term can be of the same order or larger than the dissipative term.

The generalized Ohm’s law including the Hall term can be written as (see, e.g., [29])

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma} \mathbf{j} + \frac{1}{\epsilon n_e} \mathbf{j} \times \mathbf{B},$$  \hspace{1cm} (A1)

where $\mathbf{E}$ is the electric field, $\mathbf{j}$ the density of the electrical current, $n_e$ the electron number density, $\epsilon$ the electron charge and $\sigma$ the electrical conductivity. The density of electrical current and magnetic induction, $\mathbf{B}$, are related by Ampère’s law

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B},$$  \hspace{1cm} (A2)

with the electrical conductivity given by

$$\sigma = \frac{n_e e^2 m_e^{-1}}{\tau_e + \tau_n},$$  \hspace{1cm} (A3)
Here \( m_e \) is the electron mass, \( \tau_e \) the electron collision time and \( \tau_n \) the neutral collision time.

For a fully-ionized, collision-dominated, plasma Eq. (A3) reduces to

\[
\sigma \approx \frac{n_e e^2 \tau_e}{m_e}.
\]

(A4)

In accordance with Spitzer [42] the electron collision time is given by

\[
\tau_e = 2.66 \times 10^5 \frac{T^{3/2}}{n_e \ln \Lambda} \text{s},
\]

(A5)

where \( T \) is the temperature and \( \ln \Lambda \) is the Coulomb logarithm (here taken to be 22). From Eq. (A5), \( \tau_e \) changes from \( 9.4 \times 10^{-8} \text{s} \) in the upper photosphere to \( 1.4 \times 10^{-2} \text{s} \) in the solar corona. On the other hand, \( \omega_c \), changes from 1.8 \( \times 10^{10} \text{s}^{-1} \) in the upper photosphere to 1.8 \( \times 10^{8} \text{s}^{-1} \) in the solar corona. As a consequence the Hall parameter, \( \omega_c \tau_e \), changes from 1.69 \( \times 10^{3} \) in the upper photosphere to 2.52 \( \times 10^{6} \) in the solar corona. Since \( \omega_c \tau_e \gg 1 \), the Hall term cannot be neglected in the upper photosphere nor the solar corona.

In order to estimate the relative importance of the Hall term and viscous term in the dissipative layer we must employ a more sophisticated analysis similar to the analysis presented by Ruderman et al. [27]. The generalized induction equation (neglecting finite electrical resistivity) is

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_e} \nabla \times \left( \frac{1}{n_e} \nabla \times \nabla \times \mathbf{B} \right).
\]

(A6)

In what follows we assume that the ionization coefficient is constant, so that \( n_e \) is proportional to \( \rho \), and in particular \( n_e^{-1} \nabla n_e = \rho^{-1} \nabla \rho \). Equations (30) and (50) provide the following estimations in the dissipative layer:

\[
\begin{align*}
\mathbf{u} &= \mathbf{O}(\epsilon), \\
\mathbf{v}_\perp &= \mathbf{O}(\epsilon), \\
\rho &= \mathbf{O}(\epsilon^{1/2}), \\
\mathbf{v}_\parallel &= \mathbf{O}(\epsilon^{1/2}), \\
\mathbf{B}_\parallel &= \mathbf{O}(\epsilon^{1/2}),
\end{align*}
\]

(A7)

where \( \epsilon \) still denotes the dimensionless amplitude of oscillations far away from the dissipative layer.

The thickness of the dissipative layer divided by the characteristic scale of inhomogeneity is \( \delta_e/l_{inh} = \mathbf{O}(\epsilon^{1/2}) \). This gives rise to

\[
\begin{align*}
l_{inh} \frac{\partial h}{\partial x} &= \mathbf{O}(\epsilon^{-1/2} h), \\
l_{inh} \frac{\partial h}{\partial z} &= \mathbf{O}(h), \\
l_{inh}^2 \frac{\partial^2 h}{\partial x^2} &= \mathbf{O}(h),
\end{align*}
\]

(A8)

where \( h \) denotes any of the quantities \( u, \rho, \mathbf{v}_\parallel \) or \( B_\parallel \).

Since the first term in the expansion of \( B_\perp \) is independent of \( x \), it follows that

\[
\begin{align*}
l_{inh} \frac{\partial B_\perp}{\partial x} &= \mathbf{O}(B_\perp), \\
l_{inh} \frac{\partial B_\perp}{\partial z} &= \mathbf{O}(B_\perp), \\
l_{inh}^2 \frac{\partial^2 B_\perp}{\partial x^2} &= \mathbf{O}(\epsilon^{-1/2} B_\perp),
\end{align*}
\]

(A9)

(the same applies to the variable \( v_\perp \)).

We now need to calculate the components of the vectors of the Braginskii’s viscosity tensor and the Hall term from Eq. (A9) normal to the magnetic surfaces (the \( x \)-direction) and in the magnetic surfaces parallel and perpendicular to the equilibrium magnetic field lines. We use Eqs. (A8) and (A9) in order to estimate all the terms and we only retain the largest. The components of the Braginskii tensor acting in the normal and perpendicular directions relative to the equilibrium magnetic field are the second and third ones, describing shear viscosity (even though in the paper we only consider Eq. (11) we need the second and third terms of Braginskii’s viscosity tensor to complete our scalings). Since they are of the same order, for the purpose of our estimations it is enough to consider \( \tau_1 \) only. Braginskii’s viscosity tensor simplifies to,

\[
\eta_1 (\nabla \cdot S_1) = \eta_1 \frac{\partial^2 u}{\partial x^2} + \ldots,
\]

(A10)

\[
\eta_1 (\nabla \cdot S_1) = \eta_1 \frac{\partial^2 v_\parallel}{\partial x^2} + \ldots,
\]

(A11)

\[
\eta_0 (\nabla \cdot S_0) = \eta_0 \cos \alpha \left( 2 \cos \alpha \frac{\partial^2 v_\parallel}{\partial z^2} - \frac{\partial^2 u}{\partial z \partial x} \right) + \ldots
\]

(A12)

It should be stated that \( \tau_0 \gg \tau_1 \) and \( \tau_0 (\nabla \cdot S_1) = \tau_0 (\nabla \cdot S_1) = 0 \). The components of the Hall term in the induction equation reduce to

\[
\begin{align*}
H_x &= \frac{B_0 \cos \alpha \sin \alpha}{\mu n_e} \frac{\partial^2 B_\parallel}{\partial z^2} + \ldots, \\
H_\perp &= \frac{B_0 \cos \alpha}{\mu n_e} \frac{\partial B_\parallel}{\partial x} + \ldots, \\
H_\parallel &= \frac{B_0 \sin \alpha}{\rho_0 n_e \cos \alpha} \frac{\partial B_\parallel}{\partial x} + \ldots.
\end{align*}
\]

(A13)

(A14)

(A15)

With the aid of Eqs. (A12), (A8) and (A9) we obtain the ratios

\[
\begin{align*}
\frac{H_x}{\eta_1 (\nabla \cdot S_1)} &\sim \epsilon^{1/2} \frac{\tau}{\tau_1}, \\
\frac{H_\perp}{\eta_1 (\nabla \cdot S_1)} &\sim \epsilon^{-1/2} \frac{\tau}{\tau_1}, \\
\frac{H_\parallel}{\eta_0 (\nabla \cdot S_0)} &\sim \frac{\tau}{\rho_0 \eta_0}.
\end{align*}
\]

(A16)

(A17)

(A19)

Where \( \tau = \tau_0 \omega_c \tau_e \) is the coefficient of Hall conduction and \( \tau = 1/\sigma \mu \) is the magnetic diffusivity. Strictly speaking, even the diffusivity is anisotropic in the solar corona, but the parallel and perpendicular components only differ by a factor of 2. It has been noted that magnetic diffusivity is much much smaller that the compressional viscosity in the solar corona. However, in the coefficient of Hall conduction, \( \tau = \tau_0 \omega_c \tau_e \), we observe that the magnetic diffusivity is multiplied by the product \( \omega_c \tau_e \), which is very large in the solar corona (\( 10^3 - 10^6 \)). Moreover, if
we look at Eq. (A19) we see that the coefficient of Hall conduction is divided by the density, which is very small under solar coronal conditions. Therefore, the parallel component of the Hall term in the induction equation becomes very important in the slow dissipative layer.

The Hall terms in the normal and perpendicular direction relative to the background magnetic field are included for completeness in the paper, but they do not play a role in the governing equation (i.e. can be left out completely and will not alter the result shown). This is attributed to the fact that the dominant dynamics of resonant slow waves is in the parallel direction relative to the ambient magnetic field.

In summary, the Hall term in the parallel direction relative to the ambient magnetic field, $H_\parallel$, must be included in the slow dissipative layer when $\omega_e \tau_e \gg 1$ because it is the same order of magnitude (or larger) than the compressional viscous term.

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