Lax operator algebras and integrable systems

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Abstract. A new class of infinite-dimensional Lie algebras, called Lax operator algebras, is presented, along with a related unifying approach to finite-dimensional integrable systems with a spectral parameter on a Riemann surface such as the Calogero–Moser and Hitchin systems. In particular, the approach includes (non-twisted) Kac–Moody algebras and integrable systems with a rational spectral parameter. The presentation is based on quite simple ideas about the use of gradings of semisimple Lie algebras and their interaction with the Riemann–Roch theorem. The basic properties of Lax operator algebras and the basic facts about the theory of the integrable systems in question are treated (and proved) from this general point of view. In particular, the existence of commutative hierarchies and their Hamiltonian properties are considered. The paper concludes with an application of Lax operator algebras to prequantization of finite-dimensional integrable systems.

Bibliography: 51 titles.

Keywords: gradings of semisimple Lie algebras, Lax operator algebras, integrable systems, spectral parameter on a Riemann surface, Tyurin parameters, Hamiltonian theory, prequantization.

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This work is supported by the Russian Science Foundation under grant 14-50-00005.

AMS 2010 Mathematics Subject Classification. Primary 17B66, 17B67, 14H10, 14H15, 14H55, 30F30, 81R10, 81T40.

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1. Introduction

Methods of algebraic geometry, and methods from the theory of Lie groups and algebras represent two classical approaches in the theory of integrable systems, originating in the 1970s ([24], [14], [16], [26]). A broad and (so-called) horizonless literature is devoted to them. However, in our opinion the subject is sufficiently well characterized by the following fundamental surveys and monographs: [50], [4], [29], [27], [28], [30], [3]. In this paper we focus on finite-dimensional integrable systems.

The interrelation between algebro-geometric and group-theoretic approaches in the theory of finite-dimensional integrable systems was clearly manifested in the creation of the Hitchin systems [10]. This interrelation is based on applications of holomorphic bundles on Riemann surfaces with a simple Lie group as structure group, and ultimately on the remarkable interaction between the Riemann–Roch theorem and the structure of semisimple Lie algebras. The application of holomorphic $G$-bundles on elliptic curves has led to many results on Calogero–Moser systems (for example, see [23] and the references there).

In our survey we would like to elucidate a new direction connected with the concept of integrable systems with a spectral parameter on a Riemann surface [15], [51], and with the emergence of a new class of infinite-dimensional Lie algebras of an algebro-geometric nature called Lax operator algebras ([21], [39], [43], [42]).

In [51] it was shown that a naive generalization of Lax pairs with a spectral parameter to the case when the latter belongs to a Riemann surface results in overdetermined systems, as follows from the Riemann–Roch theorem. In [15] Krichever showed how to overcome this difficulty. There he applied the finite-zone integration technique previously developed for the Kadomtsev–Petviashvili equation [16] to the description of a certain class of Lax equations including Hitchin systems, some of their generalizations, and also zero curvature systems with one spatial variable. The description reduced to pointing out the specific form of the Laurent expansions of the Lax operators in terms of Tyurin parameters of holomorphic vector bundles on curves. For the class of equations introduced, Krichever constructed hierarchies of commuting flows, proved that they are Hamiltonian, and developed certain methods for algebro-geometric integration, namely, the method of Baker–Akhiezer functions and the method of deformation of the Tyurin parameters. In that paper Krichever actually formulated a programme for creating a general
theory of finite-dimensional integrable systems, which has by now already passed through several generalizations.

In [21] Krichever and the author showed that the Lax operators introduced in [15] form a Lie algebra generalizing the loop algebra for $\mathfrak{gl}(n)$, and then constructed analogues of it for orthogonal and symplectic algebras. Here the main point was that the form of the Laurent expansions of Lax operators in terms of the Tyurin parameters is invariant with respect to the commutation operation, and their codimension in the space of all formal $\mathfrak{g}$-valued Laurent expansions is equal to $k \dim \mathfrak{g}$, where $k$ is the pole order. It was shown that the infinite-dimensional Lie algebras obtained, which we called Lax operator algebras, possess an almost-graded structure and central extensions similar to those for loop algebras. In his next papers (see the survey and references in [39]) the present author proved an existence theorem for commuting flows and the Hamiltonian property for them, for the Lax operators belonging to these Lie algebras. The proofs were given for the classical Lie algebras and depended on their type, although the similarity between them was obvious. It was noticeable that all the integrability theorems ultimately appealed to the same relations on Tyurin parameters that implied the closedness of the space of Lax operators with respect to the commutator. These results, as well as others such as the connection between the Lax equations in question and conformal field theory, were summarized in the monograph [39].

In [40] and [41] Lax operator algebras for the Lie algebra of type $G_2$ were constructed, again in terms of Tyurin parameters. The classification of central extensions and Lax operator algebras for an arbitrary number of marked points on a Riemann surface were studied in [37] and [34], respectively.

In [42] and [43] a construction of Lax operator algebras in terms of root systems was proposed that is suitable for arbitrary semisimple Lie algebras. This came about as a result of discussions with È. B. Vinberg, who associated the Laurent expansions in the existing examples of Lax operator algebras with $\mathbb{Z}$-gradings of the corresponding finite-dimensional Lie algebras (see (2.1) below). With this observation as a starting point, the Lax operator algebra, its almost-graded structure, and central extensions corresponding to an arbitrary semisimple Lie algebra and a $\mathbb{Z}$-grading on it were constructed in [42] and [43]. It was shown how to retrieve the Tyurin parameters or analogues of them from a $\mathbb{Z}$-grading.

In the next papers [44]–[46] the basic questions of the theory of finite-dimensional integrable systems, that is, the existence of commutative hierarchies and their Hamiltonian properties, were considered at the same level of generality as in [42] and [43] (that is, for a Riemann surface of arbitrary genus and an arbitrary complex semisimple Lie algebra).

The present survey is devoted to a systematic treatment of the questions outlined above.

In §2 we give a general construction of Lax operator algebras and establish its correspondence with the construction in terms of Tyurin parameters that was proposed previously for classical Lie algebras. We introduce almost-graded structures and describe almost-graded central extensions of Lax operator algebras.

In §3 we define $M$-operators—the counterparts of $L$-operators in Lax pairs. Given a Lax operator, we construct a family of $M$-operators determining a hierarchy of commuting flows connected with $L$. 
For the systems in question we construct in §4 the Krichever–Phong symplectic structure \[39\], \[15\] and the family of Hamiltonians in involution corresponding to the flows constructed above. Each Hamiltonian is defined by an invariant polynomial on the Lie algebra, a point of a Riemann surface, and an integer. The same data define the \(M\)-operators constructed in §3. The relation between them is as follows: the principal part of an \(M\)-operator at the point is given by the gradient of the corresponding invariant polynomial. This relation is well known in the group-theoretic approach to Lax integrable systems \[30\], \[8\]. Further in §4 we consider examples: the Hitchin and Calogero–Moser systems for classical Lie algebras. It should be noted here that the results obtained are also applicable to many known systems with a rational spectral parameter, such as classical gyroscopes or integrable cases of a solid body in a fluid flow.

In the concluding §5 we follow \[38\] and \[39\] and treat the interrelationship between the integrable systems in question and 2-dimensional conformal field theories. Namely, with each integrable system we associate a unitary representation of its algebra of classical observables using Knizhnik–Zamolodchikov-type operators on the space of spectral curves. These results are relevant to questions concerning quantization of integrable systems considered in \[2\], \[11\], and \[5\] and are treated from the point of view of §§2–4 in the present paper.

The author is grateful to I.M. Krichever and M. Schlichenmaier for numerous discussions and collaboration over a span of years, and to È. B. Vinberg for discussions on the application of gradings of semisimple Lie algebras to the construction of Lax operator algebras.

2. Lax operator algebras

Adding to the Introduction, we underscore that the state of the theory of Lax operator algebras up to the end of 2012 was summarized in \[39\]. There a construction of Lax operator algebras was given for classical Lie algebras. The presentation was based on Tyurin parameters. Here we give a general construction suitable for an arbitrary semisimple Lie algebra. We introduce almost-graded structures and consider central extensions of Lax operator algebras, and we show how to retrieve the Tyurin parameters from \(Z\)-gradings for classical Lie algebras.

The two basic results in this section are given by Theorem 2.2 in §2.1 and Theorem 2.4 in §2.2. The former gives a description of the Lie algebra structure and the almost-graded structures of the Lax operator algebras, while the latter treats the construction and classification of their central extensions. In §2.3 we consider a number of examples, including all the classical Lie algebras and \(G_2\), in order to point out the correspondence with the previous approach and, in particular, to explain the emergence of the Tyurin parameters.

2.1. A current algebra and its almost-graded structures. Let \(\mathfrak{g}\) be a semisimple Lie algebra over \(\mathbb{C}\), \(\mathfrak{h}\) a Cartan subalgebra of it, and \(h \in \mathfrak{h}\) an element such that \(p_i = \alpha_i(h) \in \mathbb{Z}_+\) for every simple root \(\alpha_i\) of \(\mathfrak{g}\). Let

\[
\mathfrak{g}_p = \{ X \in \mathfrak{g} \mid (\text{ad} h)X = pX \} \quad \text{and} \quad k = \max \{ p \mid \mathfrak{g}_p \neq 0 \}.
\]

Then the decomposition \(\mathfrak{g} = \bigoplus_{p=-k}^{k} \mathfrak{g}_p\) gives a \(\mathbb{Z}\)-grading on \(\mathfrak{g}\). For the theory and classification results for such gradings we refer to \[49\]. We call \(k\) the depth of the
grading. Obviously,
\[ \mathfrak{g}_p = \bigoplus_{\alpha \in R, \alpha(h) = p} \mathfrak{g}_\alpha, \]
where \( R \) is the root system of \( \mathfrak{g} \). We also define the following filtration on \( \mathfrak{g} \):
\[ \tilde{\mathfrak{g}}_p = \bigoplus_{q = -k}^p \mathfrak{g}_q. \]
Then \( \tilde{\mathfrak{g}}_p \subset \tilde{\mathfrak{g}}_{p+1} (p > -k), \tilde{\mathfrak{g}}_{-k} = \mathfrak{g}_{-k}, \ldots, \tilde{\mathfrak{g}}_k = \mathfrak{g}, \) and \( \tilde{\mathfrak{g}}_p = \mathfrak{g} \) for \( p > k \).

Let \( \Sigma \) be a complex compact Riemann surface with two given finite sets of marked points: \( \Pi \) and \( \Gamma \). Let \( L \) be a meromorphic map \( \Sigma \to \mathfrak{g} \) which is holomorphic outside the marked points and may have poles of arbitrary order at the points in \( \Pi \), and which has an expansion of the following form at the points in \( \Gamma \):
\[ L(z) = \sum_{p=\text{even}}^{\infty} L_p z^p, \quad L_p \in \tilde{\mathfrak{g}}_p, \quad (2.1) \]
where \( z \) is a local coordinate in a neighbourhood of a point \( \gamma \in \Gamma \). In general, the grading element \( h \) may vary from one point of \( \Gamma \) to another. For simplicity, we assume that \( k \) is the same all over \( \Gamma \), though there would be no difference otherwise.

We denote by \( \mathcal{L} \) the linear space of all such maps. Since the relation (2.1) is preserved under the commutator, \( \mathcal{L} \) is a Lie algebra. Below we fix this important fact as assertion 1 of Theorem 2.2. The Lie algebra \( \mathcal{L} \), its almost-graded structures, and its central extensions make up the main topic of the present section. Sometimes we use the notation \( \tilde{\mathfrak{g}} \) instead of \( \mathcal{L} \), in order to stress the relationship of this algebra to a certain semisimple Lie algebra \( \mathfrak{g} \). We also retain the name Lax operator algebras for the current algebras just constructed, in order to stress that they come from those in [21] and [39]. The observation that the Laurent expansions of elements of the Lax operator algebras considered in [21] and [39] have the form (2.1) at points in \( \Gamma \) is due to Vinberg.\(^1\)

**Definition 2.1.** Given a Lie algebra \( \mathcal{L} \), we understand an almost-graded structure of it to be a system of finite-dimensional subspaces \( \mathcal{L}_m \) of it and two non-negative integers \( R, S \) such that
\[ \mathcal{L} = \bigoplus_{m=\text{even}}^{\infty} \mathcal{L}_m \quad \text{and} \quad [\mathcal{L}_m, \mathcal{L}_n] \subseteq \bigoplus_{r=m+n-R}^{m+n+S} \mathcal{L}_r \]
\((R \text{ and } S \text{ do not depend on } m \text{ or } n).\)

The concept of almost-graded structures on associative and Lie algebras was introduced by Krichever and Novikov in [17]. It was investigated in [21] for Lax operator algebras in the case when \( \Pi \) consists of two points (we distinguish the two-point and the many-point cases depending on the number of elements in \( \Pi \)). Under more general assumptions, almost-graded structures for Krichever–Novikov algebras and also for Lax operator algebras were considered by Schlichenmaier in [31], [32], and [34].

The Lie algebra \( \mathcal{L} \) defined above admits several almost-graded structures. To define such a structure we specify a splitting of \( \Pi \) into two disjoint subsets:
\[ \Pi = \{ P_i \mid i = 1, \ldots, N \} \cup \{ Q_j \mid j = 1, \ldots, M \}. \]
\(^1\)Private communication.
Following [31], [32], and [34], we consider three divisors for every \( m \in \mathbb{Z} \):

\[
D^P_m = -m \sum_{i=1}^{N} P_i, \quad D^Q_m = \sum_{j=1}^{M} (a_j m + b_{m,j}) Q_j, \quad D^\Gamma = k \sum_{\gamma \in \Gamma} \gamma, \tag{2.2}
\]

where \( a_j, b_{m,j} \in \mathbb{Q}, a_j > 0, a_j m + b_{m,j} \) is an increasing \( \mathbb{Z} \)-valued function of \( m \), and there exists a \( B \in \mathbb{R}_+ \) such that \( |b_{m,j}| \leq B \) for all \( m \in \mathbb{Z} \) and \( j = 1, \ldots, M \). We require that

\[
\sum_{j=1}^{M} a_j = N, \quad \sum_{j=1}^{M} b_{m,j} = N + g - 1. \tag{2.3}
\]

Let

\[
D_m = D^P_m + D^Q_m + D^\Gamma \tag{2.4}
\]

and

\[
\mathcal{L}_m = \{ L \in \mathcal{L} \mid (L) + D_m \geq 0 \}, \tag{2.5}
\]

where \( (L) \) is the divisor of the \( \mathfrak{g} \)-valued function \( L \). Regarding \( (L) \) we specify that the order of a meromorphic vector-valued function at a point is the minimum of the order of its coordinates.

We call \( \mathcal{L}_m \) a (homogeneous, grading) subspace of degree \( m \) of the Lie algebra \( \mathcal{L} \).

**Theorem 2.2.** 1. The subspace \( \mathcal{L} \) is closed with respect to the pointwise commutator \([L, L'](P) = [L(P), L'(P)] \) \( (P \in \Sigma) \).

2. \( \dim \mathcal{L}_m = \dim \mathfrak{g} \).

3. \( \mathcal{L} = \bigoplus_{m=-\infty}^{\infty} \mathcal{L}_m \).

4. \( [\mathcal{L}_m, \mathcal{L}_n] \subseteq \bigoplus_{r=m+n}^{m+n+S} \mathcal{L}_r \), where \( S \) is a positive integer depending on \( N, M, \) and \( g \) and independent of \( m \) and \( n \).

**Proof.** As noted above, the proof of 1 is obvious. For the proofs of 3 and 4 we refer to [34] (where they are given for classical Lie algebras, but in fact they hold under our assumptions). Only assertion 2 requires a special proof, and we give it now.

Let \( L(D_m) = \{ L \mid (L) + D_m \geq 0 \} \), where \( L : \Sigma \to \mathfrak{g} \) is a meromorphic function but not necessarily in \( \mathcal{L} \). Let \( l_m = \dim L(D_m) \). In the case of points in \( \Pi \) and \( \Gamma \) in general position the value of \( l_m \) is given by the Riemann–Roch theorem:

\[
l_m = (\dim \mathfrak{g})(\deg D_m - g + 1). \]

Note that

\[
\deg D_m = -m N + m \sum_{i=1}^{M} a_i + \sum_{i=1}^{M} b_{m,i} + k|\Gamma|,
\]

where \( |\Gamma| \) denotes the cardinality of \( \Gamma \). By (2.3) we get that \( \deg D_m = N + g - 1 + k|\Gamma| \). Hence,

\[
l_m = (\dim \mathfrak{g})(N + k|\Gamma|).
\]

By definition, \( \mathcal{L}_m \) is a subspace of \( L(D_m) \) distinguished by the conditions (2.1) which must hold at every \( \gamma \in \Gamma \). For any \( \gamma \in \Gamma \) the codimension of the expansions of the form (2.1) at \( \gamma \) in the space of all \( \mathfrak{g} \)-valued Laurent expansions with
respect to $z$ starting with $z^{-k}$ can be calculated as follows: $c_\gamma = \sum_{p=-k}^{k-1} \text{codim}_g \tilde{g}_p$ (taking into account that $\text{codim}_g \tilde{g}_p = 0$ for $p \geq k$). The grading $g = \bigoplus_{p=-k}^{k} g_p$ is symmetric in the sense that $\dim g_p = \dim g_{-p}$ [49] (moreover, $g_p$ and $g_{-p}$ are contragredient as $g_0$-modules). By definition, $\text{codim}_g \tilde{g}_p = \sum_{q=p+1}^{k} \dim g_q$, and by symmetry $\sum_{q=p+1}^{k} \dim g_q = \dim g_{-p-1}$, hence $\text{codim}_g \tilde{g}_p + \text{codim}_g \tilde{g}_{-p-1} = \dim g$. Obviously, $c_\gamma = \sum_{p=-k}^{k} (\text{codim}_g \tilde{g}_p + \text{codim}_g \tilde{g}_{-p-1})$. Therefore, $c_\gamma = k \dim g$.

Further, $\text{codim}_{L(D_m)} L_m = \sum_{\gamma \in \Gamma} c_\gamma = k(\dim g)(\dim \Gamma)$, and finally,

$$\dim L_m = l_m - k(\dim g)(\dim \Gamma) = N \dim g. \quad \Box$$

This calculation gives one more example of the non-trivial interaction between the Riemann–Roch theorem and the structure of semisimple Lie algebras (the first was mentioned in the Introduction). The underlying idea can be tracked back to [19] (via [34], [39] and [21]), where a similar calculation was carried out unconnected to Lie algebra theory.

2.2. Central extensions. In this section we construct almost-graded central extensions of the Lie algebra $L$.

A central extension of a Lie algebra $L$ is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \overset{i}{\longrightarrow} \tilde{L} \overset{p}{\longrightarrow} L \longrightarrow 0,$$

(2.6)

where $\text{im}(i) = \ker(p)$ is the centre of $\tilde{L}$. The Lie algebra $\tilde{L}$ itself is also often called a central extension of the Lie algebra $L$.

Two central extensions $\tilde{L}$ and $\tilde{L}'$ are said to be equivalent if there is an isomorphism $e$ (equivalence) such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{i} & \tilde{L} & \xrightarrow{p} & L & \longrightarrow & 0 \\
 & & \downarrow{e} & & \downarrow{p'} \\
 & & \tilde{L}' & \xrightarrow{i'} & L & \longrightarrow & 0
\end{array}$$

(2.7)

By a 2-cocycle on $L$ we mean a bilinear skew-symmetric form $\eta$ satisfying the relation

$$\eta([f,g], h) + \eta([g,h], f) + \eta([h,f], g) = 0 \quad \forall f, g, h \in L. \quad (2.8)$$

If $\eta(f,g) = \phi([f,g])$, where $\phi \in L^*$, then $\eta$ is called a coboundary (and denoted by $\delta \phi$). If $\eta - \eta' = \delta \phi$, then $\eta$ and $\eta'$ are said to be cohomologous.

Every 2-cocycle $\eta$ on $L$ gives a central extension (and conversely) by means of the formulae $\tilde{L} = L \oplus \mathbb{C}t$ and

$$[f,g] = [f,g]_L + \eta(f,g) \cdot t, \quad [t,L] = 0, \quad (2.9)$$

where $f, g \in L$, $t$ is a formal generator of the 1-dimensional space $\mathbb{C}t$, and $[\cdot, \cdot]$ and $[\cdot, \cdot]_L$ are the commutators on $\tilde{L}$ and $L$, respectively. Two central extensions
are equivalent if and only if their defining cocycles are cohomologous. Thus, the equivalence classes of central extensions are in a one-to-one correspondence with the elements of the space $H^2(L, \mathbb{C})$.

**Example 2.3.** Regard $\mathcal{V} = \mathbb{C}^{2n}$ as a commutative Lie algebra, and let $\eta$ be a symplectic form on $\mathcal{V}$. Then $\mathcal{V}$ is the Heisenberg algebra.

A central extension is said to be almost graded if it inherits the almost-graded structure from the original Lie algebra while central elements are relegated to the degree-0 subspace.

Almost-graded central extensions are given by *local cocycles*. We recall ([17], [21], [37], [39], [34]) that a 2-cocycle $\eta$ on $L$ is said to be local if there exists an $M \in \mathbb{Z}_+$ such that for every $m, n \in \mathbb{Z}$ with $|m + n| > M$ and for every $L \in L_m$ and $L' \in L_n$ one has $\eta(L, L') = 0$.

Let $\langle \cdot, \cdot \rangle$ denote an invariant symmetric bilinear form on $g$. By abuse of notation, we use the same symbol to denote a natural extension of this form to $g$-valued functions and 1-forms on $\Sigma$. For example, for $L, L' \in L$ we denote by $\langle L, L' \rangle$ the scalar function on $\Sigma$ taking the value $\langle L(P), L'(P) \rangle$ at an arbitrary $P \in \Sigma$.

Finally, let $\omega$ be a $g_0$-valued 1-form on $\Sigma$ having an expansion of the form

$$\omega(z) = \left( \frac{h}{z} + \omega_0 + \cdots \right) dz$$

at any $\gamma \in \Gamma$, where $h \in \mathfrak{h}$ is the element giving the grading on $g$ at $\gamma$.

It is our main purpose in this section to give a proof of the following theorem.

**Theorem 2.4.** 1. For any $L, L' \in L$ the 1-form $\langle L, (d - \text{ad} \omega)L' \rangle$ is holomorphic outside the points $\{P_i \mid i = 1, \ldots, N\}$ and $\{Q_j \mid j = 1, \ldots, M\}$.

2. For any invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $g$, the relation

$$\eta(L, L') = \sum_{i=1}^{N} \text{res}_{P_i} \langle L, (d - \text{ad} \omega)L' \rangle$$

gives a local cocycle on $L$.

3. Up to equivalence, the almost-graded central extensions of the Lie algebra $L$ are in a one-to-one correspondence with invariant symmetric bilinear forms on $g$. In particular, if $g$ is simple, then the central extension given by the cocycle $\eta$ is unique in the class of almost-graded central extensions up to equivalence and normalization of the central generator.

**Proof of Theorem 2.4**, 1. In a neighbourhood of a point $\gamma \in \Gamma$ let

$$L = \sum_{p \geq -k} L_p z^p, \quad L' = \sum_{q \geq -k} L'_q z^q, \quad L_p \in \tilde{g}_p, \quad L'_q \in \tilde{g}_q,$$

where $\tilde{g}_p \subset g$ is a filtration subspace, that is, $\tilde{g}_p = \bigoplus_{s \leq p} g_s$, and the $g_s$ are grading subspaces. Then

$$dL' = \sum_{q \geq -k} qL'_q z^{q-1} dz,$$
and in a neighbourhood of $\gamma$

$$\langle L, dL' \rangle = \sum_{p,q \geq -k} q\langle L_p, L'_q \rangle z^{p+q-1} dz. \quad (2.10)$$

Observe that $(\text{ad } h)L_p = pL_p + \tilde{L}_{p-1}$, where $\tilde{L}_{p-1} \in \tilde{g}_{p-1}$. Thus,

$$\langle \text{ad } \omega \rangle L' = \text{ad}(hz^{-1} + \omega_0 + \cdots) \sum_{q \geq -k} L'_q z^q dz$$

$$= \sum_{q \geq -k} (qL'_q + \tilde{L}_{q-1})z^q dz + \sum_{q \geq -k, l \geq 0} L_{q,l} z^{q+l} dz,$$

where $L_{q,l} = (\text{ad } \omega_l) L'_q \in \tilde{g}_q$ (since $\omega_l \in g_0$). The terms $\tilde{L}_{q-1} z^{q-1}$ of the first sum have the same form as the terms of the second sum (with $l = 0$). By abuse of notation, we omit them, regarding them as having been moved to the second sum (where, however, we do not change the notation):

$$\langle \text{ad } \omega \rangle L' = \sum_{q \geq -k} qL'_q z^{q-1} dz + \sum_{q \geq -k, l \geq 0} L_{q,l} z^{q+l} dz.$$

Therefore,

$$\langle L, (\text{ad } \omega) L' \rangle = \sum_{p,q \geq -k} q\langle L_p, L'_q \rangle z^{p+q-1} dz + \sum_{p,q \geq -k, l \geq 0} \langle L_p, L_{q,l} \rangle z^{p+q+l} dz. \quad (2.11)$$

Subtracting (2.11) from (2.10), we find that

$$\langle L, (d - \text{ad } \omega) L' \rangle = -\sum_{p,q \geq -k, l \geq 0} \langle L_p, L_{q,l} \rangle z^{p+q+l} dz. \quad (2.12)$$

Let us show that the last expression is holomorphic in a neighbourhood of $z = 0$. Assume that $p + q + l < 0$. Since $l \geq 0$, this would imply that $p + q < 0$. By definition, $L_p \in \bigoplus_{i \leq p} g_i$ and $L_{q,l} \in \bigoplus_{j \leq q} g_j$. Therefore, $i + j \leq p + q < 0$, and hence $\langle g_i, g_j \rangle = 0$. This implies that $\langle L_p, L_{q,l} \rangle = 0$. Theorem 2.4, 1 is proved.

**Proof of Theorem 2.4, 2.** Here we give a proof of the localization of the cocycle. The technique we use for this has already become a kind of standard ([21], [37], [39], [40]).

As above, let $D_m$ be defined as follows:

$$D_m = -m \sum_{i=1}^{N} P_i + \sum_{j=1}^{M} (a_j m + b_{m,j})Q_j + k \sum_{s=1}^{K} \gamma_s. \quad (2.13)$$

We denote the divisor of a function or a 1-form using the notation for this function (respectively, 1-form) enclosed in brackets. Let $L \in \mathcal{L}_m$ and $L' \in \mathcal{L}_{m'}$. Then

$$((\langle L, dL' \rangle) \geq (m + m' - 1) \sum_{i=1}^{N} P_i - \sum_{j=1}^{M} (a_j (m + m') + b_{m,j} + b_{m',j} - 1) Q_j + D_{\Gamma},$$

where $D_{\Gamma}$ is a divisor supported on $\Gamma$. 

Assume that \( m_1^+, \ldots, m_N^+, m_1^-, \ldots, m_M^- \) are integers such that
\[
(\omega) \geq \sum_{i=1}^{N} m_i^+ P_i - \sum_{j=1}^{M} m_j^- Q_j - \sum_{s=1}^{K} \gamma_s.
\]
Then
\[
(\langle L, (\text{ad}\omega)L'\rangle) \geq \sum_{i=1}^{N} (m+m'+m_i^+) P_i - \sum_{j=1}^{M} \{a_j(m+m') + b_{m,j} + b_{m',j} + m_j^-\} Q_j + D'_{\Gamma},
\]
where \( D_{\Gamma} - D'_{\Gamma} \geq 0 \) by Theorem 2.4, 1.

In order for the 1-form \( \rho = \langle L, (d-\text{ad}\omega)L'\rangle \) to have a non-trivial residue at least at one of the points \( P_i \), it is necessary that
\[
\min_{i=1,\ldots,N} \{m+m' - 1, m+m' + m_i^+\} \leq -1;
\]
in other words,
\[
m + m' \leq -1 - \min_{i=1,\ldots,N} \{-1, m_i^+\}. \tag{2.14}
\]

Further, it is necessary that \( \sum_{i=1}^{N} \text{res}_P, \rho \neq 0 \), otherwise the cocycle \( \eta \) is zero. But then \( \sum_{j=1}^{M} \text{res}_{Q_j}, \rho \neq 0 \) also, and hence \( \rho \) has a non-trivial residue at least at one of the points \( Q_j \). This implies that
\[
\max_{j=1,\ldots,M} \{a_j(m+m') + b_{m,j} + b_{m',j} - 1, A_j(m+m') + b_{m,j} + b_{m',j} + m_j^-\} \geq 1.
\]
Since \( b_{j,m} \leq B \) for all \( j \) and \( m \) (as required in (2.2)), the last inequality implies that
\[
\max_{j=1,\ldots,M} \{a_j(m+m') + 2B - 1, A_j(m+m') + 2B + m_j^-\} \geq 1,
\]
and further,
\[
\max_{j=1,\ldots,M} \{a_j(m+m') + 2B - 1, A_j(m+m') + 2B + \max_{j=1,\ldots,M} m_j^-\} \geq 1,
\]
then
\[
\max_{j=1,\ldots,M} \{a_j(m+m')\} \geq 1 - \max\{2B - 1, 2B + \max_{j=1,\ldots,M} m_j^-\},
\]
and finally
\[
m + m' \geq \min_{j=1,\ldots,M} \{a_j^{-1}(1 - \max\{2B - 1, 2B + \max_{j=1,\ldots,M} m_j^-\})\}. \tag{2.15}
\]
By (2.14) and (2.15) we conclude that \( \eta(L, L') \) is a local cocycle. Theorem 2.4, 2 is proved.

Remark 2.5. By the \( g \)-invariance and symmetry of the bilinear form,
\[
\sum_{i=1}^{N} \text{res}_{P_i} \langle L, (\text{ad}\omega)L'\rangle = -\sum_{i=1}^{N} \text{res}_{P_i} \langle \omega_i[L, L']\rangle.
\]
Therefore, the corresponding part of the cocycle \( \eta \) is a coboundary. Hence, Theorem 2.4, 2 can be formulated in the following form: the standard cocycle given by the 1-form \( \langle L, dL'\rangle \) is local up to a coboundary.
2.3. Examples: gradings of classical Lie algebras and $G_2$. Here we consider gradings of depth 1 and 2 of classical Lie algebras, and of depth 2 and 3 for $G_2$. In these cases we reproduce the expansions of Lax operators obtained previously in [15, 21, 39], and then (in §2.4) their Tyurin parameters. We also point out new Lax operator algebras arising in these cases. We will focus on the gradings given by simple roots. For a simple root $\alpha$, such a grading is given by the element $h \in \mathfrak{h}$ such that $\alpha_i(h) = 1$ and $\alpha_j(h) = 0$ ($j \neq i$). Therefore, the grading subspace $\mathfrak{g}_p$ is a direct sum of the root subspaces $\mathfrak{g}_\alpha$ such that $\alpha_i$ is contained in the expansion of $\alpha$ over simple roots with multiplicity $p$. There is a dual grading when this multiplicity is taken to be $-p$. Note that the depth of the grading given by a simple root is equal to its multiplicity in the expansion of the highest root. Below we use the following conventions: $\mathfrak{g}_i \subset \mathfrak{g}$ is the eigenspace with the eigenvalue $-i$; in the figures, the lines extending outside matrices correspond to their medians. For detailed information on $\mathbb{Z}$-gradings of semisimple Lie algebras we refer to [49], Chapter 2, §3.5. Below, $e_1, \ldots, e_n$ denote the elements of the orthonormal (with respect to the Cartan–Killing form) basis in the space in which the root system is embedded.

2.3.1. The case of $A_n$. In this case $\mathfrak{g}$ has $[n/2]$ gradings of depth 1 (and has no grading of depth 2 or more, given by simple roots). The grading number $r$ ($1 \leq r \leq [n/2]$) is given by assigning the degree $-1$ to the simple root $\alpha_r = e_r - e_{r+1}$. To begin with, consider the grading number 1. The block structure of the grading subspaces in this case is given in Fig. 1 (a). The matrices corresponding to the subspace $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1}$ can be represented in the form $\alpha^\beta^{\dagger}$, where $\alpha \in \mathbb{C}^n$, the transposed vector for $\alpha$ is $\alpha^t = (1, 0, \ldots, 0)$, and $\beta \in \mathbb{C}^n$ is arbitrary. Such a matrix belongs to $\mathfrak{sl}(n)$ if $\beta^t \alpha = 0$. Further, an element $L_0 \in \mathfrak{g}$ belongs to the filtration subspace $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ if and only if $\alpha$ is an eigenvector of it. Since all the elements defining the grading on $\mathfrak{g}$ are determined up to an inner automorphism, the vector $\alpha = g(1, 0, \ldots, 0)^t$ ($g \in \text{GL}(n)$) can take an arbitrary value in $\mathbb{C}^n$, while the above relations between $\alpha$, $\beta$ and $L_0$ are preserved. In this way we arrive at the following expansion (first proposed in [15]) of the Lax operator for $\mathfrak{sl}(n)$ at the point $\gamma \in \Gamma$ (see also [21] and [39]):

$$L(z) = \alpha \beta^t z^{-1} + L_0 + \cdots,$$

(2.16)

where $\beta^t \alpha = 0$ and there exists a $\kappa \in \mathbb{C}$ such that $L_0 \alpha = \kappa \alpha$.

The grading number $r$ is given by the simple root $\alpha_r = e_r - e_{r+1}$. The corresponding matrix realization is shown in Fig. 1 (b). This root is contained in the expansions of the roots $e_i - e_j$ for $i = 1, \ldots, r$, $j = r + 1, \ldots, n$. The sum of the corresponding root subspaces gives the grading subspace $\mathfrak{g}_{-1}$. Therefore, the upper block $\mathfrak{g}_0$ has size $r \times r$ and the blocks $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have sizes $r \times (n-r)$ and $(n-r) \times r$, respectively. The subspace $\mathfrak{g}_{-1}$ consists of the matrices of the form $\tilde{\alpha}_1 \beta_1^t + \cdots + \tilde{\alpha}_r \beta_r^t$, where $\tilde{\alpha}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the $i$th position). The vectors $\tilde{\alpha}_i$ and $\beta_j$ are mutually orthogonal, and the linear span of the vectors $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ is an invariant subspace of the subalgebra $\tilde{\mathfrak{g}}_0$. 

Proof of Theorem 2.4, 3. For a proof of uniqueness in the case when $\mathfrak{g}$ is simple, we refer to [37], where this is proved for an arbitrary simple Lie algebra $\mathfrak{g}$ in the 2-point case ($N = M = 1$), and then to [34], where it is given for arbitrary $N$ and $M$. The remainder of the assertion 3 follows easily from this result. □
2.3.2. The case of $D_n$. The Dynkin diagram $D_n$ has the form

$$
\begin{align*}
\alpha_1 &\quad \alpha_2 &\quad \cdots &\quad \alpha_{n-2} &\quad \alpha_{n-1} \\
\alpha_1 &\quad \alpha_2 &\quad \cdots &\quad \alpha_{n-2} &\quad \alpha_{n-1}
\end{align*}
$$

where

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_{n-1} + e_n,$$

and a complete set of positive roots is given by $e_i \pm e_j$, $1 \leq i < j \leq n$. We will need expressions for the positive roots in terms of simple roots:

$$e_i - e_j = \alpha_i + \cdots + \alpha_{j-1} \quad (1 \leq i < j \leq n)$$

and

$$e_i + e_j = \begin{cases} 
\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, & i < j \leq n-2, \\
\alpha_i + \cdots + \alpha_{n-1} + \alpha_n, & i < j = n-1, \\
\alpha_i + \cdots + \alpha_{n-2} + \alpha_n, & i \leq n-2, \ j = n, \\
\alpha_n, & i = n-1, \ j = n.
\end{cases}$$

The highest root $\theta$ is equal to $\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, hence there are three simple roots $\alpha_1$, $\alpha_{n-1}$, and $\alpha_n$ giving gradings of depth 1, but the last two are equivalent under an outer automorphism.

We consider the grading corresponding to the simple root $\alpha_1$. This simple root is contained in the expansions of the roots $e_1 \pm e_j$, $j = 2, \ldots, n$. The corresponding root subspaces form the grading subspace $g_{-1}$. The blocks corresponding to the matrix realization of $g = \mathfrak{so}(2n)$ with respect to the quadratic form $\sigma = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ are represented in Fig. 2 (a).
We note that $g_{-1}$ can be represented as the space of rank-2 matrices of the form $(\alpha \beta^t - \beta \alpha^t)\sigma$, where $\alpha, \beta \in \mathbb{C}^{2n}$, $\alpha^t = (1, 0, \ldots, 0)$, and $\beta^t \sigma \alpha = 0$. We note also that $\alpha$ is an eigenvector of the subalgebra $\widehat{g}_0$, and $\alpha^t \sigma \alpha = 0$. Hence, we obtain the following expansion (first found in [21]) for $L$ at $\gamma \in \Gamma$:

$$L(z) = (\alpha \beta^t - \beta \alpha^t)\sigma z^{-1} + L_0 + \cdots,$$

where $\alpha$ and $\beta$ satisfy the above conditions and there exists a $\kappa \in \mathbb{C}$ such that $L_0 \alpha = \kappa \alpha$.

Considering next the grading given by the simple root $\alpha_n$, we see that this root is contained in the expansions of the positive roots $e_i + e_j$ for $i < j$. The remaining simple roots form the Dynkin diagram $A_{n-1}$. The blocks corresponding to the grading subspaces are represented in Fig. 2 (b). In particular, the subspace $g_{-1}$ is represented by matrices of the form

$$(\bar{\alpha}_1 \beta_1^t - \beta_1 \bar{\alpha}_1^t)\sigma + \cdots + (\bar{\alpha}_n \beta_n^t - \beta_n \bar{\alpha}_n^t)\sigma,$$

where $\bar{\alpha}_i, \beta_i \in \mathbb{C}^{2n}$ for $i = 1, \ldots, n$.

the vector $\bar{\alpha}_i$ is given by its coordinates $\bar{\alpha}_j^i = \delta_j^i$ (where $j = 1, \ldots, 2n$, and $\delta_j^i$ is the Kronecker symbol) and $\beta_i = (\beta_1^i, \ldots, \beta_n^i, 0, \ldots, 0)$ (where the $\beta_i^j \in \mathbb{C}$ are arbitrary). Note that

$$\bar{\alpha}_i^t \sigma \beta_j = \bar{\alpha}_i^t \sigma \bar{\alpha}_j = \beta_i^t \sigma \beta_j = 0$$

for all $i, j = 1, \ldots, n$. Using these relations, we can independently show by methods from [21] that the property that $L$ has simple poles at the points $\gamma \in \Gamma$ is conserved under commutation.

2.3.3. The case of $C_n$. The Dynkin diagram $C_n$ is

\[ \begin{array}{cccccc}
\alpha_1 & & \cdots & & & \alpha_n \\
& \alpha_2 & & & & \\
& & \ddots & & & \\
& & & \alpha_{n-1} & & \\
& & & & \alpha_n & \\
\end{array} \]

where

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = 2e_n,$$
and all the positive roots are given by $e_i \pm e_j$, $1 \leq i < j \leq n$, and $2e_i$, $i = 1, \ldots, n$. The expressions for the positive roots in terms of simple roots are

$$e_i - e_j = \alpha_i + \cdots + \alpha_{j-1}, \quad 1 \leq i < j \leq n,$$

$$2e_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n, \quad i = 1, \ldots, n-1,$$

$$2e_n = \alpha_n,$$

and

$$e_i + e_j = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n, & i < j \leq n-1, \\ \alpha_i + \cdots + \alpha_{n-1} + \alpha_n, & i < j = n. \end{cases}$$

The highest root $\theta$ is equal to $2e_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. We consider here the grading of depth 2 given by the root $\alpha_1$, and the grading of depth 1 given by the root $\alpha_n$. The Lax operator algebra corresponding to the first of these roots was found in [21] (see also [39]). The algebra corresponding to the second of these roots was found in [43].

In the matrix realization of the Lie algebra $g = \mathfrak{sp}(2n)$ with respect to the symplectic form $\sigma = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, the blocks corresponding to the grading subspaces are shown in Fig. 3 (a). In particular, the 1-dimensional subspace $g_{-2}$ corresponding to the highest root consists of the matrices of the form $\nu \alpha^ t \sigma$, where $\alpha \in \mathbb{C}^{2n}$, $\alpha^ t = (1,0,\ldots,0)$, and $\nu \in \mathbb{C}$. The subspace $g_{-1}$ consists of the matrices of the form $(\alpha \beta^ t + \beta \alpha^ t)\sigma$, where $\alpha, \beta \in \mathbb{C}^{2n}$, $\alpha^ t = (1,0,\ldots,0)$, and $\beta^ t \sigma \alpha = 0$. Note also that $\alpha$ is an eigenvector of the subalgebra $\tilde{g}_0$, that is, for every $L_0 \in \tilde{g}_0$ there exists a $\gamma \in \mathbb{C}$ such that $L_0 \alpha = \gamma \alpha$. And note finally that for every $L_1 \in \tilde{g}_1$ we have $\alpha^ t \sigma L_1 \alpha = 0$. Therefore, we arrive at the following form for the expansion of the element $L$ at $\gamma \in \Gamma$:

$$L(z) = \nu \alpha^ t \sigma z^{-2} + (\alpha \beta^ t + \beta \alpha^ t)\sigma z^{-1} + L_0 + L_1 z + \cdots,$$  \hspace{1cm} (2.18)

where $\alpha$, $\beta$, $L_0$, and $L_1$ satisfy the above relations (see also [21] and [39]).

The matrix realization of the grading given by the simple root $\alpha_n$ is given in Fig. 3 (b). It is very similar to the one for $D_n$, with the only difference being that the matrices in $g_{-1}$ have the form $(\tilde{\alpha}_1 \beta^ t_1 + \beta_1 \tilde{\alpha}^ t_1)\sigma + \cdots + (\tilde{\alpha}_n \beta^ t_n + \beta_n \tilde{\alpha}^ t_n)\sigma$. As previously, $\tilde{\alpha}_i$ and $\beta_j$ satisfy the orthogonality relations which enable one to determine independently, using methods from [21], that the simple poles at the points $\gamma \in \Gamma$ remain simple after commutation (as in §2.3.2).

2.3.4. The case of $B_n$. The Dynkin diagram $B_n$ has the form

```
\begin{center}
\begin{tikzpicture}
\node (a1) at (0,0) {$\alpha_1$};
\node (a2) at (1,0) {$\alpha_2$};
\node (an) at (n,0) {$\alpha_n$};
\node (an-1) at (n-1,0) {$\alpha_{n-1}$};
\node (dots) at (n/2,0) {$\cdots$};
\draw (a1) -- (a2);
\draw (a2) -- (dots);
\draw (dots) -- (an-1);
\draw (an-1) -- (an);
\end{tikzpicture}
\end{center}
```

where

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{n-1} = e_{n-1} - e_n, \quad \alpha_n = e_n,$$
and all the positive roots are given by $e_i \pm e_j$, $1 \leq i < j \leq n$, and $e_i$, $i = 1, \ldots, n$. The expressions for the positive roots in terms of the simple roots are

\begin{align*}
e_i - e_j &= \alpha_i + \cdots + \alpha_{j-1}, \\
e_i &= \alpha_i + \cdots + \alpha_{n-1} + \alpha_n, \\
e_i + e_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + 2\alpha_n,
\end{align*}

$1 \leq i < j \leq n$.

The highest root $\theta$ is $\theta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$. We consider here the grading of depth 1 given by the simple root $\alpha_1$ and the grading of depth 2 corresponding to the simple root $\alpha_n$. The Lax operator algebra corresponding to the first of these roots was found in [21] (see also [39]). The algebra corresponding to the second was found in [43].

In the matrix realization of the Lie algebra $g = sl(2n+1)$ corresponding to the quadratic form

\[ \sigma = \begin{pmatrix} 0 & 0 & E \\ 0 & 1 & 0 \\ E & 0 & 0 \end{pmatrix} \]

the blocks corresponding to the grading subspaces are shown in Fig. 4 (a).

The subspace $g_{-1}$ consists of the matrices of the form $(\alpha\beta^t - \beta^t\alpha^t)\sigma$, where $\alpha, \beta \in \mathbb{C}^{2n+1}$, $\alpha^t = (1, 0, \ldots, 0)$, and $\beta^t\sigma\alpha = 0$ (note also that $\alpha^t\sigma\alpha = 0$). As above, $\alpha$ is an eigenvector of the subalgebra $g_0$. Therefore, we arrive at the following form for the Laurent expansion of the element $\tilde{L}$ at $\gamma \in \Gamma$:

\[ L(z) = (\alpha\beta^t - \beta^t\alpha^t)\sigma z^{-1} + L_0 + \cdots, \quad (2.19) \]

where $\alpha$ and $\beta$ satisfy the above relations, and there exists a $\nu \in \mathbb{C}$ such that $L_0\alpha = \nu\alpha$ (see also [21] and [39]).
The matrix realization of the grading given by the simple root $\alpha_n$ is shown in Fig. 4 (b). The subspace $\mathfrak{g}_{-1}$ is a direct sum of the root subspaces of the roots $e_i$ ($i = 1, \ldots, n$). The subspace $\mathfrak{g}_{-2}$ is a direct sum of the root subspaces of the roots $e_i + e_j$ for all $i, j = 1, \ldots, n$. The matrices in $\mathfrak{g}_{-1}$ have the form $\alpha_0^t \beta_0^t - \beta_0 \alpha_0^t$, where $\alpha_0, \beta_0 \in \mathbb{C}^{2n+1}$, $(\alpha_0)_i = \delta_{i,n+1}$ (Kronecker symbol), and $\beta_0 = (\beta_0^1, \ldots, \beta_0^n, 0, \ldots, 0)$ (the $\beta_0^i \in \mathbb{C}$ are arbitrary). The matrices in $\mathfrak{g}_{-2}$ have the form $(\alpha_1^t \beta_1^t - \beta_1 \alpha_1^t) \sigma + \cdots + (\alpha_n^t \beta_n^t - \beta_n \alpha_n^t) \sigma$, where $\alpha_i, \beta_i \in \mathbb{C}^{2n+1}$ for $i = 1, \ldots, n$, the vector $\alpha_i$ is given by its coordinates $\alpha_i^j = \delta_i^j$ (where $j = 1, \ldots, 2n+1$), and $\beta_i = (\beta_i^1, \ldots, \beta_i^n, 0, \ldots, 0)$ (where the $\beta_i^j \in \mathbb{C}$ are arbitrary).

2.3.5. The case of $G_2$. The Dynkin diagram $G_2$ is

$$
\begin{array}{c}
\alpha_1 \\
\alpha_2
\end{array}
$$

where $\alpha_1, \alpha_2 \in \mathbb{C}^2$,

$$
\alpha_1 = (1, 0), \quad \alpha_2 = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right),
$$

and all the positive roots are located at vertices of two regular hexagons with common centre at $(0, 0)$ that have vertices at the ends of the vectors $\alpha_1$ and $\alpha_2$, respectively. The Lie algebra $G_2$ has a faithful 7-dimensional representation by matrices of the form shown in Fig. 5 (b). In this figure the mutually dependent blocks have the same colour (bright grey, dark grey, or white). We let $[x]$ (where $x \in \mathbb{C}^3$, $x^t = (x_1, x_2, x_3)$) denote the skew-symmetric matrix

$$
[x] = \begin{pmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{pmatrix}.
$$

Below we give a complete list of positive roots, and their correspondence to matrix elements of the 7-dimensional representation. For each positive root we give the
value of an entry corresponding to this root and the list of all other corresponding entries (in the form \((i, j)\)):

\[
\begin{align*}
\alpha_1 & : \quad (a_1)_1 = \sqrt{2}, \quad (2, 1), (1, 5), (6, 4); \\
\alpha_2 & : \quad A_{21} = 1, \quad (3, 2), (5, 6); \\
\alpha_1 + \alpha_2 & : \quad (a_1)_2 = \sqrt{2}, \quad (3, 1), (1, 6), (5, 4); \\
2\alpha_1 + \alpha_2 & : \quad (a_2)_3 = \sqrt{2}, \quad (7, 1), (1, 4), (2, 6); \\
3\alpha_1 + \alpha_2 & : \quad A_{13} = 1, \quad (2, 4), (7, 5); \\
3\alpha_1 + 2\alpha_2 & : \quad A_{23} = 1, \quad (3, 4), (7, 6).
\end{align*}
\]

The highest root \(\theta\) is equal to \(3\alpha_1 + 2\alpha_2\). Consider first the grading of depth 2 given by the simple root \(\alpha_2\). The blocks corresponding to the grading subspaces in the matrix realization are shown in Fig. 5 (a).

![Diagram](a)

![Diagram](b)

Figure 5. Case of \(G_2\): depth 2.

It is easy to check that the subspace \(\mathfrak{g}_{-2}\) consists of the matrices of the form

\[
L_{-2} = \mu \begin{pmatrix}
0 & 0 & 0 \\
0 & \tilde{\alpha}_1 \tilde{\alpha}_2^t & 0 \\
0 & 0 & -\tilde{\alpha}_2 \tilde{\alpha}_1^t
\end{pmatrix}, \quad \mu \in \mathbb{C}, \tag{2.20}
\]

where \(\tilde{\alpha}_1 = (0, 1, 0)\) and \(\tilde{\alpha}_2 = (0, 0, 1)\), while the subspace \(\mathfrak{g}_{-1}\) consists of the matrices of the form

\[
L_{-1} = \begin{pmatrix}
0 & -\sqrt{2} \beta_0 \tilde{\alpha}_2 \tilde{\alpha}_1^t & -\sqrt{2} \beta_0 \tilde{\alpha}_1 \tilde{\alpha}_2^t \\
\sqrt{2} \beta_0 \tilde{\alpha}_1 \tilde{\alpha}_2^t & \tilde{\alpha}_1 \beta_2^t - \beta_1 \tilde{\alpha}_2^t & \beta_0 [\tilde{\alpha}_2] \\
\sqrt{2} \beta_0 \tilde{\alpha}_2 \tilde{\alpha}_1^t & \beta_0 [\tilde{\alpha}_1] & \tilde{\alpha}_2 \beta_1^t - \beta_2 \tilde{\alpha}_1^t
\end{pmatrix}, \tag{2.21}
\]

where \(\beta_{01}, \beta_{02} \in \mathbb{C}\) are arbitrary and \(\beta_1, \beta_2 \in \mathbb{C}^3\) satisfy the orthogonality relations \(\tilde{\alpha}_1^t \beta_2 = 0\) and \(\tilde{\alpha}_2^t \beta_1 = 0\). Observe also that \(\tilde{\alpha}_1^t \tilde{\alpha}_2 = 0\), and if \(L_0 \in \mathfrak{g}_0\) is as in Fig. 5 (b), then

\[
\tilde{\alpha}_1^t a_2 = 0, \quad \tilde{\alpha}_2^t a_1 = 0, \quad A \tilde{\alpha}_1 = \kappa_1 \tilde{\alpha}_1, \quad -A^t \tilde{\alpha}_2 = \kappa_2 \tilde{\alpha}_2, \tag{2.22}
\]

where \(\kappa_1, \kappa_2 \in \mathbb{C}\) are arbitrary.
where $\kappa_1, \kappa_2 \in \mathbb{C}$.

As a result we obtain the Lax operator algebra found in [40]. So we assert that the latter corresponds to the grading of depth 2 of $G_2$ given by the simple root $\alpha_2$.

In addition, $G_2$ has a grading of depth 3 given by the simple root $\alpha_1$. The matrix realization of this grading is shown in Fig. 6.

2.4. Tyurin parameters. A Lax operator algebra $\mathcal{L}$ is given by a choice of a Cartan subalgebra and a $\mathbb{Z}$-grading of a Lie algebra $\mathfrak{g}$ at every $\gamma \in \Gamma$. Such a pair of objects is defined up to an inner automorphism which depends on $\gamma$. In applying it to the indicated objects we actually apply it to local expansions (2.1) of elements $L \in \mathcal{L}$.

An inner automorphism given by a constant (with respect to $z$) element $g \in G$ is called a local inner automorphism of the Lie algebra of $\mathfrak{g}$-valued Laurent expansions at a point $\gamma \in \Gamma$.

Operating by local inner automorphisms, we deform the Lax operator algebra in general. In this section we indicate independent parameters, called Tyurin parameters, giving the Lax operator algebra for some of the examples considered in §2.3. For the case of $\mathfrak{g} = \mathfrak{gl}(n)$ they emerged earlier in the classification of holomorphic vector bundles on Riemann surfaces [47], [48], [16]. In the context of integrable systems with a spectral parameter on a Riemann surface and Lax operator algebras, they appeared in [15] and in [21], [39], respectively.

Consider the expansion (2.16) of an element $L$ in the case of $\mathfrak{g} = \mathfrak{gl}(n)$. A local automorphism operates on this expansion as follows:

$$L \rightarrow L', \quad \alpha \rightarrow \alpha', \quad \beta \rightarrow \beta', \quad L_0 \rightarrow L'_0,$$

where

$$L' = g^{-1}Lg, \quad \alpha' = g^{-1}\alpha, \quad \beta'^t = \beta^tg, \quad L'_0 = g^{-1}L'_0g. \quad (2.23)$$

It is clear that the expansion (2.1) and the relations $\beta^t\alpha = 0$ and $L_0\alpha = \kappa\alpha$ are preserved by this transformation.
The set $\Gamma$ and the set of parameters $\alpha'$ for all $\gamma \in \Gamma$ define the Lax operator algebra uniquely in this case. They are called Tyurin parameters. The Tyurin parameters are defined in the same way in the cases of the local expansions (2.17), (2.18), and (2.19) for orthogonal and symplectic algebras [21], [39].

In §3.1 we will need the parametrization of Lax operator algebras by means of local automorphisms (not only in the case of the existence of Tyurin parameters) in connection with the definition of Lax equations.

3. Lax equations and commutative hierarchies

In this section we define a certain class of finite-dimensional evolution systems corresponding to Lax operator algebras, and proceed to an investigation of their integrability. It is one of the definitions of integrability that there exists a complete (in a certain sense) set of flows with which the evolution system commutes (a commutative hierarchy of flows). Given a Lax operator of the class in question, we construct here a corresponding family of commuting flows.

3.1. Lax pairs. Elements of the Lie algebra $L$ discussed in the last section are here called $L$-operators. We supplement the notation for the algebra by incorporating a reference to the sets $\Gamma$ and $\{h\}$ defining it (see §2.1), and we now denote it by $L_{\Gamma,\{h\}}$. Next, we formulate an important property of $L$-operators generalizing Theorem 2.2, 2. Let $D$ be a non-special non-negative divisor on $\Sigma$. We associate with it the subspace

$$L_{\Gamma,\{h\}}^D = \left \{ L \in L_{\Gamma,\{h\}} \mid (L) + D + k \sum_{\gamma \in \Gamma} \gamma \geq 0 \right \},$$

where $(L)$ denotes the divisor of the map $L$. To be precise, we note that by the divisor of a vector-valued function we mean the pointwise minimum of the divisors of its components. Then the following dimension formula holds for $L$-operators:

$$\dim L_{\Gamma,\{h\}}^D = (\dim g)(\deg D - g + 1). \quad (3.1)$$

The proof of (3.1) is also very similar to the proof of Theorem 2.2, 2. Namely, the dimension of the space of all meromorphic $g$-valued functions with divisor $D$ outside $\Gamma$ and with poles of order $k$ at points $\gamma \in \Gamma$ is generically equal to

$$(\dim g)(\deg D + k|\Gamma| - g + 1)$$

by the Riemann–Roch theorem. But the $L$-operators satisfy the conditions (2.1), which, as shown in §2.1, give the codimension $k(\dim g)|\Gamma|$, leading to (3.1).

The relation (3.1) is applied below (in §4.3) to the calculation of the dimensions of the phase spaces of integrable systems.

We now allow the elements of $\Gamma$ to vary in such way that all the $\gamma \in \Gamma$ remain pairwise distinct and $\Gamma \cap \Pi = \emptyset$. Elements of the set $\{h\}$ are also assumed to be variables, with the following range. For each $\gamma \in \Gamma$ we fix an $h^0_\gamma \in h$ satisfying the integrality and non-negativity conditions formulated in §2.1 and assume that $h_\gamma = (\text{Ad} g_\gamma)h^0_\gamma$, $g_\gamma \in G$, where $G$ is a connected Lie group with Lie algebra $g$. By equipping the space of data sets obtained in this way with an appropriate topology
(and even a complex structure), we can consider on it a sheaf \( \mathcal{L} \) of Lax operator algebras \( \mathcal{L}_{\Gamma,\{h\}} \), and its subsheaf \( \mathcal{L}^D \) of subspaces \( \mathcal{L}_{\Gamma,\{h\}}^D \). The set \( \Pi \) is assumed to be fixed. The sheaf \( \mathcal{L}^D \) plays the role of the phase space of the Lax equations below.

We note that consideration of the variable sets \( \Gamma \) and \( \{h\} \) generalizes the method of deformation of Tyurin parameters, the application of which to the theory of the Kadomtsev–Petviashvili equation goes back to [16], and which is heavily used in [15].

More formally, we can define the base of the sheaves \( \mathcal{L} \) and \( \mathcal{L}^D \) as follows. Let \( H \) be the centralizer of the element \( \mathfrak{h}_0^0 \) in \( G \); then \( g_\gamma \in G/H \). For a fixed finite set \( \Gamma \), denote by \( G_H^\Gamma \) the set of all maps \( \Gamma \to G/H \), and by \( \Sigma^\Gamma \) the set of all embeddings \( \Gamma \to \Sigma \). Then we regard \( \Sigma^\Gamma \times G_H^\Gamma \) as the base.

A meromorphic map \( M : \Sigma \to \mathfrak{g} \) which is holomorphic outside \( \Pi \) and \( \Gamma \) is called an \( M \)-operator if at any \( \gamma \in \Gamma \) it has a Laurent expansion

\[
M(z) = \frac{\nu h}{z} + \sum_{i=-k}^{\infty} M_i z^i,
\]

where \( M_i \in \mathfrak{g}_i \) for \( i < 0 \), \( M_i \in \mathfrak{g} \) for \( i \geq 0 \), \( h \in \mathfrak{h} \) is the element giving the grading on \( \mathfrak{g} \) at the point \( \gamma \), and \( \nu \in \mathbb{C} \). We denote the collection of \( M \)-operators by \( \mathcal{M}_{\Gamma,\{h\}} \). Obviously, \( \mathcal{L}_{\Gamma,\{h\}} \subset \mathcal{M}_{\Gamma,\{h\}} \).

For an arbitrary non-special non-negative divisor \( D \) let

\[
\mathcal{M}_{\Gamma,\{h\}}^D = \left\{ M \in \mathcal{M}_{\Gamma,\{h\}} \mid (M) + D + k \sum_{\gamma \in \Gamma} \gamma \geq 0 \right\}.
\]

According to [44], we have the following dimension formula for \( M \)-operators:

\[
\dim \mathcal{M}_{\Gamma,\{h\}}^D = (\dim \mathfrak{g}) \left( \deg D + l - g + 1 \right),
\]

where \( l \in \mathbb{Z}_+ \) is determined by the number of points in \( \Gamma \) and the dimensions of the filtration spaces of the Lie algebra \( \mathfrak{g} \). We prove (3.3) in §3.2.

In particular cases \( M \)-operators, like \( L \)-operators, can be given by Tyurin parameters (see [39] for the details).

Similarly to what was done for \( L \)-operators, we will consider the sheaves \( \mathcal{M} \) and \( \mathcal{M}^D \) with the same base.

**Definition 3.1.** A Lax pair is a pair consisting of smooth sections of the sheaves \( \mathcal{L}^D \) and \( \mathcal{M}^D \). Here a section of the sheaf \( \mathcal{L}^D \) is called a Lax operator.

Let two smooth curves \( \Gamma(t) \) in \( \Sigma^\Gamma \) and \( h(t) \) in \( G_H^\Gamma \) be given. Giving a Lax pair defines a pullback of the curves to the sheaves \( \mathcal{L}^D \) and \( \mathcal{M}^D \). We obtain curves \( L(t) \) and \( M(t) \), where \( L(t) \in \mathcal{L}_{\Gamma(t),\{h(t)\}} \) and \( M(t) \in \mathcal{M}_{\Gamma(t),\{h(t)\}} \).

Let the curves \( L = L(t) \) and \( M = M(t) \) be obtained this way. The equation

\[
\dot{L} = [L, M],
\]

where \( \dot{L} = dL/dt \), is called the Lax equation. This is a system of ordinary differential equations for the curves \( \Gamma(t) \) and \( h(t) \) and for the principal parts of the meromorphic functions \( L(t) \) and \( M(t) \) at the points in \( \Pi \) and \( \Gamma \). For this system to be closed, it is necessary to give \( M \) as a function of \( L \).
Example 3.2. Let us consider an example which will be investigated in more detail later in §4.4, namely, the elliptic Calogero–Moser system for the root system $A_n$. From the physical point of view this is a system of pairwise interacting particles on a torus with coordinates $q_1, \ldots, q_n$ and momenta $p_1, \ldots, p_n$. Its Lax operator is a meromorphic function of $z$ on the torus taking values in $\mathfrak{gl}(n)$ and having matrix entries of the form

$$L_{ij} = f_{ij} \sigma(z + q_j - q_i)\sigma(z - q_j)\sigma(q_i)\sigma(z) - \sigma(q_i - q_j)\sigma(q_j)\sigma(z - q_i)\sigma(q_i)\sigma(z), \quad (i \neq j), \quad L_{jj} = p_j, \quad (3.5)$$

where $\sigma$ is the Weierstraß function. In this example $\Gamma = \{q_1, \ldots, q_n\}$, and at the point $q_i$ the element $h_i$ is given by a constant (independent of the time) diagonal matrix equal to $\text{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 at the $i$th position). The equation (3.4) gives a motion in the phase space $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$.

3.2. Dimension formula for $M$-operators. We choose an arbitrary non-special non-negative divisor

$$D = \sum_{i=1}^{N} m_i P_i, \quad m_i \geq 0 \quad (i = 1, \ldots, N).$$

and calculate the dimension of the space $\mathcal{M}^D_{\Gamma, \{h\}} = \{M \in \mathcal{M}_{\Gamma, \{h\}} \mid (M) + D \geq 0\}$. Taking into account the codimension of the expansions of $M$-operators at the points $\gamma \in \Gamma$ and the additional parameter $\nu$ at each of these points, we have by the Riemann–Roch theorem that

$$\dim \mathcal{M}^D_{\Gamma, \{h\}} = (\dim \mathfrak{g})(\deg D - \nu |\Gamma| - g + 1) - \sum_{\gamma \in \Gamma} \sum_{i=-k}^{1} \text{codim} \widetilde{\mathfrak{g}}_i^\gamma + |\Gamma|, \quad (3.6)$$

where $\mathfrak{g}_i^\gamma$ is the space of the grading at $\gamma$ (here we do not assume the gradings to be the same at different points) and $\widetilde{\mathfrak{g}}_i^\gamma$ is the corresponding filtration space. Replacing $\text{codim} \widetilde{\mathfrak{g}}_i^\gamma$ by $\dim \mathfrak{g} - \dim \widetilde{\mathfrak{g}}_i^\gamma$ in (3.6), we get that

$$\sum_{\gamma \in \Gamma} \sum_{i=-k}^{1} \text{codim} \widetilde{\mathfrak{g}}_i^\gamma = \sum_{\gamma \in \Gamma} \sum_{i=-k}^{1} (\dim \mathfrak{g} - \dim \widetilde{\mathfrak{g}}_i^\gamma)$$

$$= (\dim \mathfrak{g})k|\Gamma| - \sum_{\gamma \in \Gamma} \sum_{i=-k}^{1} \dim \widetilde{\mathfrak{g}}_i^\gamma.$$

Therefore,

$$\dim \mathcal{M}^D_{\Gamma, \{h\}} = (\dim \mathfrak{g})(\deg D - g + 1) + \sum_{\gamma \in \Gamma} \sum_{i=-k}^{1} \dim \widetilde{\mathfrak{g}}_i^\gamma + |\Gamma|,$$

and finally,

$$\dim \mathcal{M}^D_{\Gamma, \{h\}} = (\dim \mathfrak{g})(\deg D - g + 1) + \sum_{\gamma \in \Gamma} \left( \sum_{i=-k}^{1} \dim \widetilde{\mathfrak{g}}_i^\gamma + 1 \right). \quad (3.7)$$
Assume now that the gradings are the same up to inner automorphisms at all the points $\gamma \in \Gamma$. Then

$$\dim \mathcal{M}^D_{\Gamma,\{h\}} = (\dim \mathfrak{g})(\deg D - g + 1) + \left(\sum_{i=-k}^{-1} \dim \tilde{\mathfrak{g}}_i + 1\right)|\Gamma|. \tag{3.8}$$

Choose $|\Gamma|$ so that the last term is equal to $(\dim \mathfrak{g})l$, where $l \in \mathbb{Z}_+$. This is always possible, and can be done in several ways. Then (see [44])

$$\dim \mathcal{M}^D_{\Gamma,\{h\}} = (\dim \mathfrak{g})(\deg D + l - g + 1). \tag{3.9}$$

Below we always assume that $l - g + 1 \geq 0$.

For example, for the Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{sp}(2n)$, and $G_2$ and for a certain choice of gradings,

$$\dim \mathfrak{g} - \left(\sum_{i=-k}^{-1} \dim \tilde{\mathfrak{g}}_i + 1\right)n = 0, \tag{3.10}$$

and we can take $|\Gamma| = ng$, where $n = \text{rank} \mathfrak{g}$. Then $l = g$, and we obtain

$$\dim \mathcal{M}^D_{\Gamma,\{h\}} = (\dim \mathfrak{g})(\deg D + 1). \tag{3.11}$$

In this form the dimension formula for $\mathcal{M}^D_{\Gamma,\{h\}}$ was obtained by Krichever in [15] in the case $\mathfrak{g} = \mathfrak{gl}(n)$. For the classical Lie algebras it was obtained by the present author (see [39] and references there). The validity of (3.10) in the cases listed above is verified in the next four examples.

**Example 3.3.** $\mathfrak{g} = \mathfrak{gl}(n)$, and a grading of depth 1 is given by the simple root $\alpha_1$. Then $k = 1$ and

$$\dim \tilde{\mathfrak{g}}_{-1} = n - 1, \quad \dim \mathfrak{g} - (\dim \tilde{\mathfrak{g}}_{-1} + 1)n = \dim \mathfrak{g} - n^2 = 0.$$

**Example 3.4.** $\mathfrak{g} = \mathfrak{so}(2n)$, and a grading of depth 1 is given by the simple root $\alpha_1$. Then $k = 1$ and

$$\dim \tilde{\mathfrak{g}}_{-1} = 2n - 2, \quad \dim \mathfrak{g} - (\dim \tilde{\mathfrak{g}}_{-1} + 1)n = \dim \mathfrak{g} - (2n - 1)n = 0.$$

**Example 3.5.** $\mathfrak{g} = \mathfrak{sp}(2n)$, and a grading of depth 2 is given by the simple root $\alpha_1$. Then $k = 2$ and

$$\dim \mathfrak{g}_{-2} = 1, \quad \dim \mathfrak{g}_{-1} = 2n - 2, \quad \dim \tilde{\mathfrak{g}}_{-1} = \dim \mathfrak{g}_{-2} + \dim \mathfrak{g}_{-1} = 2n - 1, \quad \dim \mathfrak{g} - (\dim \tilde{\mathfrak{g}}_{-2} + \dim \tilde{\mathfrak{g}}_{-1} + 1)n = \dim \mathfrak{g} - (2n + 1)n = 0.$$

**Example 3.6.** $\mathfrak{g} = G_2$, and a grading of depth 2 is given by the simple root $\alpha_1$. Then $k = 2$, $n = 2$, and

$$\dim \mathfrak{g}_{-2} = 1, \quad \dim \mathfrak{g}_{-1} = 4, \quad \dim \tilde{\mathfrak{g}}_{-1} = \dim \mathfrak{g}_{-2} + \dim \mathfrak{g}_{-1} = 5, \quad \dim \mathfrak{g} - (\dim \tilde{\mathfrak{g}}_{-2} + \dim \tilde{\mathfrak{g}}_{-1} + 1)n = \dim \mathfrak{g} - 7 \cdot 2 = 0.
Consider the case \( g = 1 \). Then Example 3.3 corresponds to the Calogero–Moser system for the root system \( A_n \). The following two examples give the Calogero–Moser systems for the root systems \( D_n, C_n \), and \( B_n \).

**Example 3.7.** \( g = \mathfrak{so}(2n) \) or \( g = \mathfrak{sp}(2n) \), and \( |\Gamma| = 2n \). We consider the same gradings on \( g \) as in Examples 3.4 and 3.5, respectively. Since (3.10) still holds, we have \( l = 2 \). Then

\[
\dim \mathcal{M}^D_{\Gamma, \{h\}} = (\dim g)(\deg D + 2). \tag{3.12}
\]

**Example 3.8.** \( g = \mathfrak{so}(2n+1) \), the grading of depth 1 is given by the simple root \( \alpha_1 \), \( k = 1 \), \( \dim \widetilde{\mathfrak{g}}_{-1} = 2n-1 \), and \( \dim g = n(2n+1) \). Therefore, \( 2 \dim g = (\dim \widetilde{\mathfrak{g}}_{-1} + 1) \times (2n+1) \), and we can take \( |\Gamma| = 2n + 1 \) and \( l = 2 \). The relation (3.12) also holds in this case.

Note that neither the condition (3.10) nor the dimension formula (3.11) is satisfied for \( g = \mathfrak{so}(2n+1) \), nor for \( g = \mathfrak{sl}(n) \). In the latter case \( k = 1 \), \( \dim \mathfrak{g}_{-1} = n - 1 \), and \( \dim g = n^2 - 1 \), while \( (\dim \mathfrak{g}_{-1} + 1) \) rank \( \mathfrak{g} = n(n - 1) \).

For \( g = 1 \) and \( g = \mathfrak{so}(2n+1) \) there is an alternative to the dimension formula (3.11):

\[
\dim \mathcal{M}^D_{\Gamma, \{h\}} = (\dim g)(\deg D) + \dim g - \text{rank } g. \tag{3.13}
\]

This relation follows immediately, for example, from (3.7).

Below we make use of the dimension formula (3.3). This will be important beginning with §3.4.

### 3.3. M-operators and vector fields.

Consider the Lax equation

\[
\dot{L} = [L, M]. \tag{3.14}
\]

In order that the pair \( L, M \) be a solution of it, it is necessary that the following relations hold at every \( \gamma \in \Gamma \):

\[
\dot{z} = -\nu,
\]

\[
\dot{L}_p = \sum_{i+j=p} [L_i, M_j] + \nu \sum_{s=-k}^p (p + 1 - s)L_{p+1}^s, \quad p = -k, \ldots, 0, \tag{3.15}
\]

where \( L_{p+1}^s \) is the projection of \( L_{p+1} \) on \( \mathfrak{g}_s \). To obtain them, it is sufficient to compare the expansions

\[
\dot{L} = -kL_{-k}\dot{z}z^{-k-1} + \sum_{p=-k}^\infty \left( \dot{L}_p + (p + 1)L_{p+1}\dot{z} \right)z^p \tag{3.16}
\]

and

\[
[L, M] = kL_{-k}\nu z^{-k-1} + \sum_{p=-k}^\infty \left( \sum_{i+j=p} [L_i, M_j] + \nu \sum_{s=-k}^p (p + 1 - s)L_{p+1}^s - (p + 1)L_{p+1}\nu \right)z^p. \tag{3.17}
\]
By abuse of notation, we write here \( \dot{z} \) instead of \( \dot{z}_\gamma \). Formally, it would be necessary to write the expansions for \( L \) and \( M \) in powers of \( z - z_\gamma \), where \( z_\gamma \) is the coordinate of the point \( \gamma \), which is also assumed to depend on \( t \). Differentiation of such an expansion with respect to \( t \) would involve \( \dot{z}_\gamma \).

Note that the terms containing \( L_{p;+1}^s \) and \( L_{p+1}^s \) in (3.17) come from the commutator of \( L \) with \( \nu h/z \), the latter coming from the expansion of \( M \).

Note also that, due to the conditions \( M_i \in \tilde{\mathfrak{g}}_i \ (i < 0) \), the terms of degree \( p < -k - 1 \) vanish in the expansions of both sides of (3.14), that is, this equation does not give any new relations in these degrees.

Let \( T_L \) denote a tangent space at the point \( L \), and let \( M \in \mathcal{M} \). We stress that it is not assumed above that \( \hat{L} \in T_L \mathcal{L}^D_{\Gamma,\{h\}} \). We cannot tell anything about \( L^p \) either, except that it is an element of \( \mathfrak{g} \). This is because the element \( h \) giving the grading depends on \( t \).

**Theorem 3.9.** Assume that \( \hat{L} \) and \( M \) satisfy (3.15) for every \( \gamma \in \Gamma \). Then \( [L, M] \in T_L \mathcal{L}^D \) if and only if \( ([L, M]) + D \geq 0 \) outside \( \Gamma \).

*Proof.* The plan of the proof is as follows. First, we construct an auxiliary subspace \( \mathcal{D} \), and, applying the Riemann–Roch theorem, we deduce that \( \dim \mathcal{D} = \dim T_L \mathcal{L}^D \). Then we note that \( T_L \mathcal{L}^D \subseteq \mathcal{D} \), and hence these spaces coincide. Finally, \( [L, M] \in \mathcal{D} \) if and only if \( ([L, M]) + D \geq 0 \) outside \( \Gamma \).

We define \( \mathcal{D} \) as the sheaf of subspaces \( \mathcal{D}_{\Gamma,\{h\}} \) of meromorphic maps \( T : \Sigma \to \mathfrak{g} \) which are holomorphic outside the sets \( \Pi \) and \( \Gamma \), satisfy the condition

\[(T) + D \geq 0\]

outside \( \Gamma \), and have an expansion

\[T = \sum_{i=-k-1}^{\infty} T_i z^i\] (3.18)

at each \( \gamma \in \Gamma \), where \( T_i \in \mathfrak{g} \), \( i = -k - 1, k, \ldots, \infty \),

\[T_i = \dot{L}_i + (i + 1)L_{i+1} \dot{z}, \quad i = -k, \ldots, -1,\]
\[T_0 = \dot{L}_0 + L_1 \dot{z},\] (3.19)

and the \( L_i \in \tilde{\mathfrak{g}}_i \) are fixed and the \( \dot{L}_i \in \mathfrak{g} \) are free parameters. Comparing the right-hand sides of (3.16) and (3.19), we conclude that \( T_L \mathcal{L}^D \subseteq \mathcal{D} \).

Next, we calculate the dimension of the space \( \mathcal{D}_{\Gamma,\{h\}} \). If we relax the conditions (3.19), then the dimension of the space \( \mathcal{D} \) of meromorphic functions obtained can be calculated using the Riemann–Roch theorem:

\[\dim \mathcal{D} = (\dim \mathfrak{g})(\deg D + (k + 1)|\Gamma| - g + 1).\]

The conditions (3.19) give \((\dim \mathfrak{g})(k + 1)\) relations at each point \( \gamma \in \Gamma \), that is, \((\dim \mathfrak{g})(k + 1)|\Gamma|\) relations in total. The total dimension of the parameters \( \dot{L}_i, \ i = -k, \ldots, 0 \), is also equal to \((\dim \mathfrak{g})(k + 1)|\Gamma|\), but they satisfy the same number of relations (3.15). Therefore, the subspace \( \mathcal{D}_{\Gamma,\{h\}} \) is distinguished in the space \( \mathcal{D} \) by \((\dim \mathfrak{g})(k + 1)|\Gamma|\) effective relations, and hence

\[\dim \mathcal{D}_{\Gamma,\{h\}} = \dim \mathcal{D} - (\dim \mathfrak{g})(k + 1)|\Gamma| = (\dim \mathfrak{g})(\deg D - g + 1).\]
But the dimension $\dim T_{L,\mathcal{L}_{\Gamma,\{h\}}}^D$ is the same, because

$$\dim T_{L,\mathcal{L}_{\Gamma,\{h\}}}^D = \dim \mathcal{L}_{\Gamma,\{h\}}^D$$

(these are finite-dimensional spaces) and because of (3.1). Therefore, we have proved that $\dim T_{L,\mathcal{L}_{\Gamma,\{h\}}}^D = \dim \mathcal{F}_{\Gamma,\{h\}}^D$; hence $T_{L,\mathcal{L}}^D = \dim \mathcal{F}^D$, and these spaces coincide.

Comparing (3.17) and (3.19), we see that under the conditions (3.15) the expansion (3.17) satisfies the conditions (3.19). Therefore, for $[L, M] \in \mathcal{F}^D$ it is necessary and sufficient that $([L, M]) + D \geq 0$ (outside $\Gamma$). □

We proceed to the next property of $M$-operators. For any $M$-operator we write its Laurent expansion in a neighbourhood of $\gamma \in \Gamma$ in the form

$$M(z) = \frac{\nu h}{z} + \sum_{i=-k}^{\infty} M_i z^i,$$

and also in the form

$$M = M^- + \frac{\nu h}{z} + M^+,$$

where $M^- = \sum_{i=-k}^{-1} M_i z^i$ and $M^+ = \sum_{i=0}^{\infty} M_i z^i$.

Lemma 3.10. Given two $M$-operators $M_a$ and $M_b$, consider two tangent vector fields $\partial_a$ and $\partial_b$ given on $\mathcal{L}^D$ by the equalities $\partial_a L = [L, M_a]$ and $\partial_b L = [L, M_b]$ and then continued to $\mathcal{M}^D$ so that

$$\partial_a M_b^- = [M_b^-, M_a^-] + L_a^-, \quad \partial_b M_a^- = [M_a^-, M_b^-] + L_b^-$$

(3.20)

and

$$\partial_a z = -\nu_a, \quad \partial_a h = [h, M_a,0] + O(z),$$
$$\partial_b z = -\nu_b, \quad \partial_b h = [h, M_b,0] + O(z),$$

(3.21)

where $L_{a,i}, L_{b,i} \in \tilde{\mathcal{G}}_i$ ($i = -k, \ldots, -1$), and the upper minus on a commutator means its principal part.\(^2\) Then

$$\partial_a M_b - \partial_b M_a + [M_a, M_b] \in \mathcal{M}.$$

Proof. The lemma will be proved by straightforward calculation of the principal part of the expression $\partial_a M_b - \partial_b M_a + [M_a, M_b]$ at an arbitrary point $\gamma \in \Gamma$.

We write the Laurent expansions of the operators $M_a$ and $M_b$ in a neighbourhood of $\gamma \in \Gamma$ in the form

$$M_a = M_a^- + \frac{\nu_a h}{z} + \sum_{i\geq0} M_{a,i} z^i, \quad M_b = M_b^- + \frac{\nu_b h}{z} + \sum_{i\geq0} M_{b,i} z^i,$$

(3.22)

where $M_a^- = \sum_{i=-k}^{-1} M_{a,i} z^i$ and $M_b^- = \sum_{i=-k}^{-1} M_{b,i} z^i$.

\(^2\)The relations for $h$ in (3.21) are an additional condition on the continuation of the vector fields, a condition that does not follow from the foregoing, because $h$, as a coefficient of $z^{-1}$, appears only in the $M$-operators. The equations for $h$ are made more precise below by (4.17).
At the first step we have
\[ \partial_a M_b = \partial_a \left( M_b^{-} + \frac{\nu_b h}{z} + \sum_{i \geq 0} M_{b,i} z^i \right) \]
\[ = \partial_a M_b^{-} + \frac{(\partial_a \nu_b) h}{z} + \frac{\nu_b(\partial_a h)}{z} - \frac{\nu_b h(\partial_a z)}{z^2} + O(1). \]

Here \( O(1) \) is an expansion in non-negative powers with coefficients in \( g \). We make the substitutions \( \partial_a z = -\nu_a \) and \( \partial_a h = [h, M_{a,0}] \) (in view of (3.21)). At the next step we replace \( \partial_a M_b^{-} \) by \( [M_b^{-}, M_a^{-}] + [M_b^{-}, \nu_a h + M_a^{+}] + L_a^{-} + O(1) \) in view of (3.20). The second commutator cancels with \( [\nu_a h + M_a^{+}, M_b^{-}] \) in \([M_a, M_b], \) as is clear from (3.22). Note that the term \( \nu_b h(\partial_a z)z^{-2} = -\nu_a \nu_b h z^{-2} \) is symmetric with respect to \( a \) and \( b \) and cancels with the corresponding term in \( \partial_b M_a \). Therefore, replacing by dots the terms which turn out to cancel in the result, we have
\[ \partial_a M_b = [M_b^{-}, M_a^{-}] + L_a^{-} + \frac{(\partial_a \nu_b) h}{z} + \frac{\nu_b(\partial_a h)}{z} + \cdots + O(1). \]
Calculating \( \partial_b M_a \) similarly, we find that
\[ \partial_a M_b - \partial_b M_a = 2[M_b^{-}, M_a^{-}] + \frac{(\partial_a \nu_b - \partial_b \nu_a) h}{z} + \frac{\nu_b(\partial_a h) - \nu_a(\partial_b h)}{z} + L_a^{-} - L_b^{-} + \cdots + O(1). \] (3.23)

Next, we calculate \([M_a, M_b]\) starting from (3.22):
\[ [M_a, M_b] = [M_a^{-}, M_b^{-}] + \sum_{i=0}^{\infty} \nu_a [h, M_{b,i}] z^{i-1} + \sum_{i=0}^{\infty} \nu_b [M_{a,i}, h] z^{i-1} + \cdots + O(1). \] (3.24)

Note that by (3.21) the term \( \nu_b(\partial_a h)z^{-1} \) in (3.23) cancels with the term with \( i = 0 \) in the second sum in (3.24), and similarly \(-\nu_a(\partial_b h)z^{-1}\) cancels with the zeroth term of the first sum. What remains of these two sums in (3.24) is \( O(1) \).

Therefore,
\[ \partial_a M_b - \partial_b M_a + [M_a, M_b] = [M_b^{-}, M_a^{-}] + L_a^{-} - L_b^{-} + \frac{(\partial_a \nu_b - \partial_b \nu_a) h}{z} + O(1). \]

Since \( M_{a,i} \in \mathfrak{g}_i \) and \( M_{b,i} \in \mathfrak{g}_i \) for \( i < 0 \), \([M_b^{-}, M_a^{-}]\) also has this property, and by assumption so do \( L_a^{-} \) and \( L_b^{-} \), hence \( \partial_a M_b - \partial_b M_a + [M_a, M_b] \) is an \( M \)-operator. □

We note that the continuations of vector fields from \( \mathcal{L}^D \) to \( \mathcal{M} \) that were constructed in [39] for classical Lie algebras by means of Tyurin parameters satisfy the conditions (3.20). Since this is not completely obvious, we demonstrate it for the example of \( g = \mathfrak{gl}(n) \).

**Example 3.11.** For \( g = \mathfrak{gl}(n) \), in a neighbourhood of a point \( \gamma \) we have
\[ M_b^{-} = \frac{\alpha \mu_b^i}{z}, \quad M_a = \frac{\alpha \mu_a^i}{z} + M_{a0} + \cdots, \]

where \( \alpha \) denotes the Tyurin parameters, and \( \mu^t_b \alpha = 0 \) (this is not true for \( \mu_a \), because the matrix \( \alpha \mu^t_a \) contains \( \nu_a h \)). Therefore,

\[
[M^-_b, M_a]^- = -\frac{(\mu^t_b \alpha) \alpha \mu^t_b}{z^2} + \frac{\alpha \mu^t_b M_a 0 - M_a 0 \alpha \mu^t_b}{z} .
\] (3.25)

If we calculate \((\partial_a M_b)^-\) in another way, formally applying the derivation \(\partial_a\) and using the Leibnitz formula, then we obtain

\[
(\partial_a M_b)^- = -\frac{(\partial_a z) \alpha \mu^t_b}{z^2} + \frac{(\partial_a \alpha) \mu^t_b + \alpha (\partial_a \mu^t_b)}{z} .
\] (3.26)

The equations of motion of the Tyurin parameters [15], [39] give

\[
\partial_a z = \mu_a \alpha, \quad \partial_a \alpha = -M_a 0 \alpha + \lambda \alpha \quad (\lambda \in \mathbb{C})
\] (3.27)

(for details of how to obtain them see also Example 4.7 in §4.5). By virtue of these equations

\[
(\partial_a M_b)^- - [M^-_b, M_a]^- = \frac{\lambda \alpha \mu^t_b + \alpha (\partial_a \mu^t_b - \mu^t_b M_a 0)}{z} .
\]

The first term in the numerator belongs to the space \( g_{-1} \) by assumption. To show that the second term also belongs there, we check that \((\partial_a \mu^t_b - \mu^t_b M_a 0) \alpha = 0 \). This can be derived by differentiating the relation \( \mu^t_b \alpha = 0 \) and using the second relation in (3.27).

### 3.4. Hierarchies of Lax equations.

Here the indices \( a \) and \( b \) introduced in the last subsection denote triples of the form \( \{\chi, P \in \Pi, m > -m_P\} \), where \( \chi \) is an invariant polynomial on the Lie algebra \( g \) and \( m_P \) is the multiplicity of the point \( P \) in the divisor \( D \).

**Example 3.12.** For \( g = gl(n) \) we can take \( \chi(L) = tr L^P, \ p \in \mathbb{Z}_+ \). In the cases \( g = so(n) \) and \( g = sp(2n) \) we can proceed in the same way, assuming that \( p \in 2\mathbb{Z}_+ \).

Also, we introduce \( l - g + 1 \) fixed points \( P_j \notin (\Pi \cup \Gamma), j = 1, \ldots, l - g + 1 \), to normalize \( M \)-operators (\( l \) is the same here as in (3.3)).

The following fragment, up to the end of the proof of Lemma 3.13, is inspired by [8]. The only obstacle to establishing the equivalence between the results here and in [8] is that the map taking the logarithm on a Lie group is multivalued.

We define the gradient \( \delta \chi(L) \in g \) of the polynomial \( \chi \) at the point \( L \in g \) by starting from the equality

\[
d\chi(L) = \langle \delta \chi(L), \delta L \rangle,
\] (3.28)

where \( d\chi \) is the differential of \( \chi \) as a function on \( g \), and \( \langle \cdot, \cdot \rangle \) is a non-degenerate invariant bilinear form on \( g \). If \( L \in \mathcal{L} \), that is, if it is considered as a meromorphic function on \( \Sigma \) taking values in \( g \), then so will \( \delta \chi(L) \) be also. If this function is considered as a function of a local coordinate \( w \) on \( \Sigma \), then we write \( \delta \chi(w) \).

**Lemma 3.13.** Let \( \chi \) be an invariant polynomial on the Lie algebra \( g \). Then

\[
[\delta \chi(L), L] = 0.
\]
Proof. By definition, the invariance of $\chi$ means that for any $g \in \exp g$

$$\chi((\text{Ad} \, g)L) = \chi(L).$$

Taking the differential of both sides of the last equality, we get by its invariance with respect to changes of variables and by (3.28) that

$$\langle \delta \chi((\text{Ad} \, g)L), \delta((\text{Ad} \, g)L) \rangle = \langle \delta \chi(L), \delta L \rangle.$$

Since $\delta((\text{Ad} \, g)L) = (\text{Ad} \, g)\delta L$, the invariance of the bilinear form (which implies that $\text{Ad} \, g$ is an orthogonal operator) gives us that

$$\langle (\text{Ad} \, g^{-1})\delta \chi((\text{Ad} \, g)L), \delta L \rangle = \langle \delta \chi(L), \delta L \rangle.$$

The last equality holds for every $\delta L \in g$, and the bilinear form is non-degenerate, so $\delta \chi(L)$ is equivariant:

$$\delta \chi((\text{Ad} \, g)L) = (\text{Ad} \, g)\delta \chi(L). \quad (3.29)$$

Put $g = \exp(tL)$ here, and differentiate the equality obtained at $t = 0$. Since the left-hand side is equal to $\delta \chi(L)$ and does not depend on $t$, we arrive at the statement of the lemma. □

**Lemma 3.14.** Given $L \in \mathcal{L}$ and $M \in \mathcal{M}$, let $\xi_M$ be the vector field on $\mathcal{L}$ defined by

$$\xi_M L = [L, M],$$

and let $\chi$ be an invariant polynomial on the Lie algebra $g$. Then

$$\xi_M \delta \chi(L) = [\delta \chi(L), M].$$

**Proof.** Let $g = \exp(-tM)$ in (3.29). The assertion of the lemma is proved by differentiating both sides of the equality obtained at $t = 0$. □

**Lemma 3.15.** For any $a = \{\chi, P, m\}$ and $L \in \mathcal{L}$ there is a unique $M$-operator $M_a$ with a single pole outside $\Gamma$, namely, at $P$, where the following relation holds:

$$M_a(w) = w^{-m} \delta \chi(L(w)) + O(1) \quad (3.30)$$

(where $w$ is a local parameter in a neighbourhood of $P$), and $M_a(P_j) = 0$, $j = 1, \ldots, l - g + 1$. If $L \in \mathcal{L}^D$, then $[[L, M_a]] + D \geq 0$ outside $\Gamma$.

**Proof.** Outside $\Gamma$ the divisor of $M_a$ consists of the single point $P$ and has order $d = \text{ord}_P w^{-m} \delta \chi(L(w))$ there. By (3.3) the dimension of the space of such operators is equal to $(\dim g)(d+l-g+1)$. The equality (3.30) fixes $(\dim g)d$ degrees of freedom, and $(\dim g)(l-g+1)$ more relations are given by the normalization conditions. Thus, the existence and the uniqueness of $M_a$ are proved.

By (3.30) and Lemma 3.13, $[[L, M_a]] = [L, O(1)]$ in a neighbourhood of $P$. The same is true at the other points in $\Pi$, because $M_a$ is holomorphic there. For this reason, $[[L, M_a]] \geq (L)$ outside $\Gamma$. The relation $(L) + D \geq 0$ implies that $[[L, M_a]] + D \geq 0$, which proves the second assertion of the lemma. □
Lemma 3.15 determines $M_a$ as a function of $L$, which we denote by $M_a(L)$.

A point $P \in \Sigma$ is said to be a regular point for a Lax operator $L$ if $L(P)$ is defined and is a regular element of the Lie algebra $\mathfrak{g}$, and $P$ is said to be non-regular if this value is defined but not regular. The set of non-regular points of a given Lax operator is finite. This implies that the set of Lax operators for which the set of non-regular points has empty intersection with $\Pi$ is open.

**Theorem 3.16.** The relations

$$\partial_a L = [L, M_a],$$

(3.31)

where $M_a = M_a(L)$, define a set of commuting vector fields on an open subset of $\mathcal{L}^D$ consisting of Lax operators for which the set of non-regular points has empty intersection with $\Pi$.

**Proof.** By Lemma 3.15 and Theorem 3.9, $\partial_a$ is a tangential vector field on $\mathcal{L}^D$.

The commutator $[\partial_a, \partial_b]$ is calculated as follows:

$$\partial_a \partial_b L = \partial_a [L, M_b] = [\partial_a L, M_b] + [L, \partial_a M_b] = [[L, M_a], M_b] + [L, \partial_a M_b].$$

Therefore,

$$(\partial_a \partial_b - \partial_b \partial_a)L = [L, \partial_a M_b - \partial_b M_a] + [[L, M_a], M_b] - [[L, M_b], M_a].$$

By the Jacobi identity,

$$[\partial_a, \partial_b]L = [L, \partial_a M_b - \partial_b M_a + [M_a, M_b]].$$

For $\partial_a$ and $\partial_b$ to commute, it suffices that $\partial_a M_b - \partial_b M_a + [M_a, M_b] = 0$. We verify that this equality does hold. First of all, we show that the expression on the left-hand side is holomorphic at any $P \in \Pi$. We assume first that $a$ and $b$ correspond to the same point $P \in \Pi$, that is, $a = (\chi_a, P, m)$ and $b = (\chi_b, P, m')$. Denote $M_a - w^{-m} \delta \chi_a(L) \text{ by } M_a^+$ and $M_b - w^{-m'} \delta \chi_b(L) \text{ by } M_b^+$. Then by (3.30) $M_a^+$ and $M_b^+$ are holomorphic at $P$. We have

$$\partial_a M_b = w^{-m'} \partial_a \delta \chi_b(L) + \partial_a M_b^+,$$

from which, by Lemma 3.14,

$$\partial_a M_b = w^{-m'}[\delta \chi_b(L), M_a] + \partial_a M_b^+ = w^{-m'}[\delta \chi_b(L), w^{-m} \delta \chi_a(L) + M_a^+] + \partial_a M_b^+.$$

By Lemma 3.13, $\delta \chi_a(w)$ and $\delta \chi_b(w)$ commute with $L(w)$ for any $w$, where, as above, $w$ is a local coordinate in a neighbourhood of $P$. But according to the assumptions of the theorem, $L(w) \in \mathfrak{g}$ is a regular element for sufficiently small $w$. Then its centralizer coincides with the Cartan subalgebra containing it and hence is commutative. Therefore,

$$[\delta \chi_a(w), \delta \chi_b(w)] = 0.$$  

(3.32)

Applying (3.32) to the previous relation, we find that

$$\partial_a M_b = w^{-m'}[\delta \chi_b(L), M_a^+] + \partial_a M_b^+,$$
and a similar formula holds for $\partial_b M_a$.

Further,

$$
[M_a, M_b] = [M^+_a + w^{-m} \delta \chi_a(L), M^+_b + w^{-m'} \delta \chi_b]
$$

$$
= [M^+_a, M^+_b] + [M^+_a, w^{-m'} \delta \chi_b] + [w^{-m} \delta \chi_a, M^+_b].
$$

Therefore,

$$
\partial_a M_b - \partial_b M_a + [M_a, M_b] = \partial_a M^+_b - \partial_b M^+_a + [M^+_a, M^+_b].
$$

The function on the right-hand side of this equality is holomorphic in a neighbourhood of $P$.

By Lemma 3.10, $\partial_a M_b - \partial_b M_a + [M_a, M_b]$ is an $M$-operator. By what we have just proved, this $M$-operator has zero divisor outside $\Gamma$. According to (3.3) the dimension of the space of such $M$-operators is equal to $(\dim g)(l-g+1)$. Because $M_a$ and $M_b$ satisfy the normalizing condition $M_a(P_j) = M_b(P_j) = 0$, $j = 1, \ldots, l-g+1$, our operator also satisfies this condition. Therefore, $\partial_a M_b - \partial_b M_a + [M_a, M_b] = 0$.

The proof is similar in the case when $a$ and $b$ correspond to distinct points. $\square$

**Corollary 3.17.** The Lax equations $\dot{L} = [L, M_a]$ give commuting flows on $L^D$.

Indeed, these equations can be rewritten as $\dot{L} = \partial_a L$, where the $\partial_a$ are commuting tangential vector fields on $L^D$.

**Example 3.18.** In the case of the classical Lie algebras in Example 3.12 the space of invariant polynomials is generated by the polynomials of the form $\chi_p(L) = \operatorname{tr} L^p$, $p \in \mathbb{Z}_+$, where $p$ is even for orthogonal and symplectic algebras. Then $\delta \chi_p(L) = pL^{p-1}$. Lemma 3.13 and the relation (3.32) become obvious, and Lemma 3.14 can easily be proved by induction.

### 4. Hamiltonian theory

In this section, we construct a symplectic structure and an involutive system of Hamiltonians, and these give a commutative hierarchy of flows $\dot{L} = [L, M_a]$ as constructed in the previous section. As above, all constructions are given in terms of semisimple Lie algebras and their invariants.

It should be stressed that the basic principles of the Hamiltonian theory of Lax equations with a spectral parameter on a Riemann surface are established in [15]. Here we actually show that these principles work in a more general context.

The symplectic structure we use below was introduced by Krichever [15] for $g = \mathfrak{gl}(n)$ and is carried over here without changes to the case of an arbitrary semisimple Lie algebra (for the classical Lie algebras see [39], and in the broader context of soliton theory see [20]). We call it the *Krichever–Phong symplectic structure*. For the classical Lie algebras, everything presented here was formulated in terms of Tyurin parameters in [39], so we omit here some characteristic details for that approach (for example, the contribution of the points $\gamma \in \Gamma$ to the symplectic structure in terms of the Tyurin parameters).

In this section we adopt the following convention regarding notation. We do not distinguish between the notation for elements of Lie groups and Lie algebras and the
notation for the operators of their action: an element and its operator are denoted by the same letter. In other words, we retain the conventional notation for the case of classical groups, where the operator is the same as the action of the element in the standard representation. In general, we could instead consider the action in the adjoint representation (see [46], for example): the exposition below does not depend on this.

4.1. Symplectic structure. Following [15], we now introduce a closed skew-symmetric 2-form (the Krichever–Phong form) on $\mathcal{L}^D$, which becomes non-degenerate (that is, a symplectic structure) on some submanifold $\mathcal{P}^D \subset \mathcal{L}^D / G$ (where $G$ is a connected Lie group with the Lie algebra $\mathfrak{g}$).

The Krichever–Phong form, like an invariant scalar product on $\mathcal{L}^\Gamma / \{ \hbar \}$, depends on the choice of a holomorphic differential $\wp$ on the Riemann surface. In the context of the theory of integrable systems the non-uniqueness of the invariant scalar product in the case of loop algebras was noted in [30], p. 66, for example. In [9] the necessity of the choice of $\wp$ was explained from the point of view of the Seiberg–Witten theory.

Let $\Psi : \Sigma \to G$ be the function diagonalizing $L$ at a generic point of the Riemann surface, and let $\delta L$ and $\delta \Psi$ be the corresponding exterior differentials, which are 1-forms on $\mathcal{L}^D$. Similarly, we consider the function $K : \Sigma \to \mathfrak{h}$ defined by

$$\Psi L = K \Psi,$$

and the $\mathfrak{h}$-valued 1-form $\delta K$. The functions $K$ and $\Psi$ are defined up to the action of the Weyl group, and this does not affect anything in what follows. Let $\Omega$ be a 2-form on $\mathcal{L}^D$ taking values in the space of meromorphic functions on $\Sigma$ and defined by

$$\Omega = \text{tr}(\Psi^{-1} \delta \Psi \wedge \delta L - \delta K \wedge \delta \Psi \cdot \Psi^{-1}).$$

We choose a holomorphic differential $\wp$ on $\Sigma$ and define a scalar 2-form $\omega$ on $\mathcal{L}^D$ by

$$\omega = \sum_{\gamma \in \Gamma} \text{res}_\gamma \Omega \wp + \sum_{p \in \Pi} \text{res}_p \Omega \wp.$$

The 2-form $\Omega$ also has another representation:

$$\Omega = 2 \delta \text{ tr}(\delta \Psi \cdot \Psi^{-1} K),$$

which obviously implies that $\omega$ is closed. As we pointed out above, we do not consider here the question of non-degeneracy of $\omega$, but refer to [15] for a corresponding discussion. We note only that one of the conditions defining symplectic submanifolds of $\mathcal{L}^D$ is the holomorphy of the 1-form $\delta K \wp$ on $\Pi$.

4.2. System of Hamiltonians in involution. We establish here that the hierarchies of commuting flows defined by Theorem 3.16 are Hamiltonian with respect to the Krichever–Phong symplectic structure.

Let $a = \{ \chi, P, m \}$ be as introduced in § 3.4, and let

$$H_a(L) = \text{res}_P w^{-m} \chi(L(w)) \wp(w).$$
For a vector field $e$ on $\mathcal{L}^D$, let $i_e\omega$ be the 1-form defined by $i_e\omega(X) = \omega(e, X)$ (where $X$ is an arbitrary vector field). By definition, $e$ is Hamiltonian if $i_e\omega = \delta H$, where $H$ is a function called the Hamiltonian of $e$. The following theorem asserts that the vector fields $\partial_a$ are Hamiltonian. For the classical Lie algebras this was proved in [15] and [39].

**Theorem 4.1.** Let $\partial_a$ be the vector field defined by (3.31). Then

$$i_{\partial_a}\omega = \delta H_a.$$ 

Before proving this theorem, we note the following. The operators $L$ and $\partial_a + M_a$ commute by virtue of the Lax equation, and hence they can be diagonalized by the same function $\Psi$. The diagonal forms of these two operators are

$$K = \Psi L \Psi^{-1} \quad \text{and} \quad F_a = \Psi (\partial_a + M_a) \Psi^{-1}. \quad (4.1)$$

An important role in the proof of the theorem is played by the holomorphy of $K$ and $F_a$ as functions on $\Sigma$. It follows from the next two lemmas, which we prove below in §4.5.

**Lemma 4.2.** The poles of the elements $L \in \mathcal{L}$ at the points of the set $\Gamma$ can be eliminated via conjugation by a local holomorphic function on $\Sigma$ taking values in $\exp \mathfrak{h}$. In a neighbourhood of a point $\gamma \in \Gamma$ this function has the form $e^{-h \log z}$, where $h$ gives the grading at $\gamma$.

**Lemma 4.3.** The spectrum $F_a$ of the operator $\partial_a + M_a$ is holomorphic at the points of $\Gamma$ along solutions of the equation $\partial_a L = [L, M_a]$.

We also assume that $\Psi$ is holomorphic and has a holomorphic inverse on $\Pi$, which is a general position requirement for $L$.

**Proof of Theorem 4.1.** By definition,

$$i_{\partial_a}\omega = \omega(\partial_a, \cdot) = -\frac{1}{2} \left( \sum_{\gamma \in \Pi} \text{res}_\gamma \Lambda + \sum_{p \in \Pi} \text{res}_p \Lambda \right),$$

where $\Lambda = \Omega(\partial_a, \cdot)$. Since $\delta \Psi(\partial_a) = \partial_a \Psi$ and $\delta L(\partial_a) = \partial_a L$ (the evaluation of a differential on a vector field is equal to the derivative along the vector field), we obtain

$$\Lambda = \text{tr}(\partial_a \Psi \cdot \delta L \cdot \Psi^{-1} - \delta \Psi \cdot \partial_a L \cdot \Psi^{-1} - \partial_a K \cdot \delta \Psi \cdot \Psi^{-1} + \delta K \cdot \partial_a \Psi \cdot \Psi^{-1}).$$

By the Lax equation,

$$\Lambda = \text{tr}(\Psi M_a - F_a \Psi) \delta L \cdot \Psi^{-1} - \delta \Psi[L, M_a] \Psi^{-1} + \delta K(\Psi M_a - F_a \Psi) \Psi^{-1})$$

$$= \text{tr}(M_a \delta L - F_a \Psi \delta L \cdot \Psi^{-1} - \delta \Psi[L, M_a] \Psi^{-1} + \delta K \Psi M_a \Psi^{-1} - \delta K F_a).$$

Next we transform the middle term. From $\Psi L = K \Psi$ we deduce that $\delta \Psi \cdot L = -\Psi \delta L + \delta K \Psi + K \delta \Psi$. Therefore,

$$\text{tr} \delta \Psi[L, M_a] \Psi^{-1} = \text{tr}(\delta \Psi \cdot L) M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1})$$

$$= \text{tr}((-\Psi \delta L + \delta K \Psi + K \delta \Psi) M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1})$$

$$= \text{tr}(-\Psi \delta LM_a \Psi^{-1} + \delta K \Psi M_a \Psi^{-1}$$

$$+ K \delta \Psi M_a \Psi^{-1} - \delta \Psi M_a L \Psi^{-1}).$$
The last two terms cancel with each other, because
\[ \text{tr}(\delta \Psi M_a L \Psi^{-1}) = \text{tr}(\delta \Psi M_a (\Psi L \Psi^{-1})) = \text{tr}(\delta \Psi M_a \Psi^{-1} K), \]
and we get that
\[ \text{tr} \delta \Psi [L, M_a] \Psi^{-1} = \text{tr}(-\delta L M_a + \delta K \Psi M_a \Psi^{-1}). \]
Substituting this into the last expression for \( \Lambda \), we have
\[ \Lambda = \text{tr}(2M_a \delta L - F_a \Psi \delta L \cdot \Psi^{-1} - \delta K F_a). \]
The last two terms in parentheses are equal under the trace, as can be seen by replacing \( \Psi \delta L \) by \(-\delta \Psi L + \delta K \Psi + K \delta \Psi \) and using the commutativity of \( K \) and \( F_a \). Finally,
\[ \Lambda = \text{tr}(2M_a \delta L - 2\delta K F_a). \]
This implies that
\[ i_{\partial_a} \omega = \sum_{p \in \Pi} \text{res}_p \text{tr}(\delta K F_a) \varpi - R_a, \] (4.2)
where
\[ R_a = \sum_{\gamma \in \Gamma} \text{res}_\gamma \text{tr}(\delta L M_a) \varpi + \sum_{p \in \Pi} \text{res}_p \text{tr}(\delta L M_a) \varpi. \] (4.3)
In general, the sum of the residues of the 1-form \( \text{tr}(\delta K F_a) \varpi \) over the points \( \gamma \in \Gamma \) should be present in (4.2), but it vanishes because \( \delta K \) and \( F_a \) are holomorphic there. Note that the function \( F_a \) has singularities outside \( \Pi \cup \Gamma \). Indeed, \( F_a = -\partial_a \Psi \cdot \Psi^{-1} + \Psi M_a \Psi^{-1} \), and \( \Psi^{-1} \) has poles at the branch points of the spectrum of \( L \).

On the other hand, \( L \) and \( M_a \) are holomorphic everywhere except at points in \( \Pi \cup \Gamma \). Therefore, \( R_a = 0 \) as the sum of the residues of a meromorphic 1-form over all its poles. Moreover, by the construction of \( M_a \), the function \( F_a \) is holomorphic at all points in \( \Pi \) except \( P \). Thus,
\[ i_{\partial_a} \omega = \text{res}_P \text{tr}(\delta K F_a) \varpi. \]

So far we have followed literally the lines of the proof of the theorem in [15] (and in [39]). The rest of the proof will be given in the more general setting of this paper, that is, for the \( M_a \) defined by an arbitrary invariant polynomial \( \chi \) (Lemma 3.15). We recall that \( \delta \chi \) denotes the gradient of the invariant polynomial \( \chi \) on the Lie algebra \( g \) with respect to the Cartan–Killing form, while \( \delta K \) and \( \delta L \) denote the differentials of the corresponding functions on \( \mathcal{L}^D \).

In a neighbourhood of \( P \) we have, first,
\[ F_a = -\partial_a \Psi \cdot \Psi^{-1} + \Psi M_a \Psi^{-1} = \Psi M_a \Psi^{-1} + O(1) \]
by the holomorphy of \( \Psi \) and \( \Psi^{-1} \), and second, \( M_a = w^{-m} \delta \chi(L(w)) + O(1) \) by definition. In view of (3.29)
\[ \Psi \delta \chi(L) \Psi^{-1} = \delta \chi(\Psi L \Psi^{-1}) = \delta \chi(K). \]
Therefore, $F_a = w^{-m} \delta \chi(K) + O(1)$. Since $\delta K \varpi$ is also holomorphic at $P$,

\[
\partial_a \omega = \text{res}_P \text{tr}(w^{-m} \delta \chi(K) \delta K \varpi) = \delta \text{res}_P w^{-m} \chi(K) \varpi = \delta H_a.
\]

The theorem is proved. \(\square\)

**Corollary 4.4.** The Hamiltonians $H_a$ are in involution.

Regarding the obvious independence of the Hamiltonians $H_a$ for the basis invariants $\chi_i$ ($i = 1, \ldots, r$, where $r = \text{rank } g$), we count their number below. First of all, if $\chi$ is an invariant polynomial on the Lie algebra $g$, then the function $\chi(L(P))$ has no poles on $\Gamma$. This follows immediately from Lemma 4.2: by the invariance of the polynomial $\chi$,

\[
\chi(L(z)) = \chi(\text{Ad } e^{-h \log z} L(z)).
\]

By Lemma 4.2, $\text{Ad } e^{-h \log z} L(z)$ has no pole on $\Gamma$, which implies the corresponding assertion about $\chi(L(z))$.

**Lemma 4.5.** Let $N$ be the number of independent Hamiltonians of the form $H_a$. Then for a semisimple Lie algebra $g$ and a non-special divisor $D$

\[
2N = \dim g \deg D + r(\deg D - \deg \mathcal{X}). \quad (4.4)
\]

**Proof.** For the proof, we make use of the same identity that is central in the proof of Hitchin’s theorem on the integrability of his celebrated systems [10]. Namely, let $d_1, \ldots, d_r$ be the set of degrees of the invariant basis polynomials of the Lie algebra $g$ ($r = \text{rank } g$). Then

\[
\sum_{i=1}^r (2d_i - 1) = \dim g. \quad (4.5)
\]

This implies that

\[
\sum_{i=1}^r d_i = \frac{\dim g + r}{2}. \quad (4.6)
\]

The basis Hamiltonians are obtained by expanding functions of the form $\chi_i(L)$ with respect to the basis meromorphic functions on $\Sigma$, which are holomorphic (as follows from Lemma 4.2) on $\Gamma$, where $\chi_i$ runs over the basis invariants. The divisors of the spaces of such functions are as follows: $d_iD, i = 1, \ldots, r$. By the Riemann–Roch theorem, the total dimension of these spaces (equal to the number of Hamiltonians) is

\[
N = \sum_{i=1}^r h^0(d_iD) = \sum_{i=1}^r (d_i \deg D - g + 1)
\]

\[
= (\deg D) \sum_{i=1}^r d_i - r(g - 1)
\]

\[
= (\deg D) \frac{\dim g + r}{2} - r(g - 1), \quad (4.7)
\]

which immediately implies the assertion of the lemma (taking into account that $\deg \mathcal{X} = 2(g - 1)$). \(\square\)
4.3. Example: Hitchin systems. Here we show that in the case when $D = \mathcal{K}$ and $\mathfrak{g}$ is one of the classical Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{so}(2n)$, $\mathfrak{so}(2n + 1)$, or $\mathfrak{sp}(2n)$, the number of the integrals found above is exactly what is required for the integrability of the system. The Lax pairs are considered in the standard representations of the corresponding classical Lie algebras. We note that the case $D = \mathcal{K}$ corresponds to Hitchin systems. We will verify this fact only with regard to dimension, leaving a complete verification for another occasion. For $\mathfrak{g} = \mathfrak{gl}(n)$ it was proved in [15].

We will specify the space $\mathcal{L}^D$ by means of Tyurin parameters (see (2.16)–(2.18)). Elements of $\mathcal{L}^D$ are operators in the standard representation of the Lie algebra $\mathfrak{g}$. Note that the set $\{h\}$ defining the gradings is specified up to an action of the group $\text{Ad} G$, while the group acting in the standard module is rather the classical group $G$ itself that corresponds to $\mathfrak{g}$. The relation between $G$ and $\text{Ad} G$ is known: $\text{Ad} G \cong G/\mathcal{L}(G)$, where $\mathcal{L}(G)$ is the centre of $G$. Therefore, in the cases when $\mathcal{L}(G)$ is non-trivial (these are $\mathfrak{g} = \mathfrak{gl}(n)$ with centre $\mathcal{L}(G) = CE$, and $\mathfrak{g} = \mathfrak{so}(2n)$ with centre $\mathcal{L}(G) = \{\pm E\}$) it is necessary to projectivize the standard module (the Tyurin parameters $\alpha$) in order to obtain the correct action of $\text{Ad} G$.

Taking into account these remarks, we show that for the simple classical Lie algebras $\mathfrak{so}(2n)$, $\mathfrak{so}(2n + 1)$, and $\mathfrak{sp}(2n)$ the following relation holds:

$$\dim \mathcal{L}^D = (\dim \mathfrak{g})(\deg D + 1). \quad (4.8)$$

The dimension of $\mathcal{L}^D$ is equal to the dimension of $\mathcal{L}^D_{\Gamma,\{h\}}$ plus the number of Tyurin parameters $\alpha$ and $\gamma$. We take $|\Gamma|$ to be equal to $ng$. In each case we check that the number of Tyurin parameters is equal to $(\dim \mathfrak{g})g$. Since

$$\dim \mathcal{L}^D_{\Gamma,\{h\}} = (\dim \mathfrak{g})(\deg D - g + 1),$$

the relation (4.8) will follow.

For $\mathfrak{g} = \mathfrak{so}(2n)$ we have $\alpha \in \mathbb{C}^{2n}$, and $\alpha \in \mathbb{CP}^{2n-1}$ in view of the projectivization. These are $2n - 1$ parameters for each $\gamma$, and the point $\gamma$ itself gives one more parameter. We also have one relation: $\alpha^t \alpha = 0$. In total, we have $(2n - 1) \cdot ng = (\dim \mathfrak{g})g$ parameters.

For $\mathfrak{g} = \mathfrak{so}(2n + 1)$ we have $\alpha \in \mathbb{C}^{2n+1}$, there is one relation $\alpha^t \alpha = 0$, and $\gamma$ itself gives one more parameter. In total, we have $(2n + 1) \cdot ng = (\dim \mathfrak{g})g$ parameters.

For $\mathfrak{g} = \mathfrak{sp}(2n)$ we have $\alpha \in \mathbb{C}^{2n}$, and $\gamma$ itself gives one more parameter. In total, we have $(2n + 1) \cdot ng = (\dim \mathfrak{g})g$ parameters.

This calculation proves (4.8) in all three cases. Taking the quotient of $\mathcal{L}^D$ by $G$ defines a quotient space $\mathcal{L}^D_0$ of dimension $\dim \mathfrak{g} \deg D$. For $D = \mathcal{K}$ we have $\dim \mathcal{L}^D_0 = 2(\dim \mathfrak{g})(g - 1)$. The number of integrals for $D = \mathcal{K}$ is equal to $(\dim \mathfrak{g})(g - 1)$, that is, the system is integrable.

In the case of $\mathfrak{g} = \mathfrak{gl}(n)$ we have $\dim \mathcal{L}^D_0 = 2(n^2(g - 1) + 1)$, and the number of integrals is equal to half this dimension. Indeed, in this case $\dim \mathcal{L}^D_{\Gamma,\{h\}} = n^2(g - 1) + 1$. The anomalous 1 in this relation is due to the following. The elements $L \in \mathcal{L}$ taking values in $\mathfrak{sl}(n)$ contribute $(n^2 - 1)(g - 1)$ to the dimension, as above. The elements taking values in the subalgebra of scalar matrices have the property that they are holomorphic outside the divisor $D$. Indeed, for them $\text{tr} L_{-1} = 0$ (cf. (2.16)), and since they are scalar, $L_{-1} = 0$. Therefore, if the divisor $D$ is special, then a certain dimensional ‘anomaly’ emerges. Namely,
for $D = \mathcal{H}$ the dimension of the space of such elements is equal to $h^0(\mathcal{H}) = g$, which together with $(n^2 - 1)(g - 1)$ gives exactly the expression for $\dim \mathcal{L}_{\Gamma, \{k\}}$. The number of Tyurin parameters is equal to $n^2g$, as above (by the projectivization, we have $\alpha \in \mathbb{C}P^{n-1}$; this gives $n - 1$ parameters for each $\gamma$, which, with the points $\gamma$ themselves taken into account, gives $n \cdot ng$ parameters). Therefore,

$$\dim \mathcal{L}_{\mathcal{H}} = n^2(g - 1) + 1 + n^2g = 2n^2(g - 1) + n^2 + 1.$$ 

Taking the quotient by $\mathrm{GL}(n)$ decreases the dimension by $n^2 - 1$ (the centre is not counted, since it does not affect the Lax equation, that is, we actually consider $\mathcal{L}_{\mathcal{H}}/\mathrm{SL}(n)$). Finally,

$$\dim \mathcal{L}_0^{\mathcal{H}} = \dim \mathcal{L}_{\mathcal{H}} - (n^2 - 1) = 2(n^2(g - 1) + 1).$$

As for the number of integrals, for $\mathfrak{g} = \mathfrak{gl}(n)$ the quantity $h^0(D)$ also appears in (4.7) (there is an invariant of degree 1, namely, the trace, while for the semisimple Lie algebras the degrees of the basis invariants are always at least 2). For $D = \mathcal{H}$ this gives exactly the same ‘anomalous’ 1.

4.4. Example: the Calogero–Moser systems. We consider here one more important series of examples: the elliptic Calogero–Moser systems for classical Lie algebras.

Let us start with the example considered in [15], the elliptic Calogero–Moser system for the root system $A_n$. We define a $\mathfrak{gl}(n)$-valued Lax operator by giving its matrix entries as follows:

$$L_{ij} = f_{ij} \frac{\sigma(z + q_j - q_i)\sigma(z - q_j)\sigma(q_i)}{\sigma(z)\sigma(z - q_i)\sigma(q_i - q_j)\sigma(q_j)} \quad (i \neq j), \quad L_{jj} = p_j, \quad (4.9)$$

where $\sigma$ (and $\phi$ below) is the Weierstrass function and the $f_{ij} \in \mathbb{C}$ are constants. Up to the constants $f_{ij}$ this form of the operator $L$ is determined by the requirement that it is elliptic and has simple poles at the points $z = q_i$ ($i = 1, \ldots, n$) and $z = 0$. The last point is the unique element of the set $\Pi$. By reduction of the remaining gauge degrees of freedom it was shown in [15] that $f_{ij} f_{ji} = 1$. According to [39], for the second-order Hamiltonian corresponding to the pole at $z = 0$ we have

$$H = \text{res}_{z=0} z^{-1} \left( -\frac{1}{2} \sum_{j=1}^n p_j^2 - \sum_{i<j} L_{ij} L_{ji} \right)$$

to within a normalization. By the addition theorem for the Weierstrass functions,

$$-L_{ij} L_{ji} = \frac{\sigma(z + q_i - q_j)\sigma(z + q_j - q_i)}{\sigma(z)^2 \sigma(q_i - q_j)^2} = \phi(q_i - q_j) - \phi(z),$$

and hence

$$H = -\frac{1}{2} \sum_{j=1}^n p_j^2 + \sum_{i<j} \phi(q_i - q_j).$$

Let us now consider the cases $\mathfrak{g} = \mathfrak{so}(2n)$ and $\mathfrak{g} = \mathfrak{sp}(2n)$. We set

$$\Gamma = \{ \pm q_1, \ldots, \pm q_n \} \quad (|\Gamma| = 2n)$$

and use the results in Example 3.7.
To obtain the Calogero–Moser system corresponding to the series $D_n$, we take the Lax operator in the form

$$L = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix},$$

(4.10)

where $A$, $B$, and $C$ are $n \times n$ matrices, $B = -B^t$, and $C = -C^t$. This is the standard form of the elements of the Lie algebra $\mathfrak{so}(2n)$.

Let $A$ be given by (4.9). For $i > j$ we set

$$B_{ji} = f^B_{ji} \frac{\sigma(z - q_j - q_i)\sigma(z + q_i)}{\sigma(z)\sigma(z - q_j)\sigma(q_i + q_j)} \quad \text{and} \quad C_{ij} = f^C_{ij} \frac{\sigma(z + q_j + q_i)\sigma(z - q_j)}{\sigma(z)\sigma(z + q_i)\sigma(q_i + q_j)},$$

(4.11)

where the $f^B_{ij}, f^C_{ij} \in \mathbb{C}$ are constants. These relations completely determine the matrices $B$ and $C$ in view of their skew-symmetry. Similarly to the case of $\mathfrak{gl}(n)$ we get that $f^B_{ij}f^C_{ji} = -1$, reducing the remaining gauge degrees of freedom. For the Hamiltonian we have

$$H = - \text{res}_{z=0} z^{-1} \left( \sum_{i=1}^{n} p_i^2 + 2 \sum_{i<j} A_{ij} A_{ji} + 2 \sum_{i<j} B_{ij} C_{ji} \right)$$

$$= - \sum_{i=1}^{n} p_i^2 + 2 \sum_{i<j} \varphi(q_i - q_j) + 2 \sum_{i<j} \varphi(q_i + q_j),$$

which is a familiar form of the second-order Hamiltonian for the elliptic Calogero–Moser model for $D_n$.

Turning to the Calogero–Moser system for the root system $C_n$, we take $L$ in the form (4.10), where $B = B^t$ and $C = C^t$. The corresponding matrix entries $A_{ij}$ are defined by the right-hand sides of (4.9), and $B_{ij}$ and $C_{ij}$ are defined by (4.11). The relations (4.11) also make sense for $i = j$, and in this case their contribution to the second-order Hamiltonian is equal to

$$B_{ii} C_{ii} = f^B_{ii} f^C_{ii} \left( \varphi(2q_i) - \varphi(z) \right),$$

where we can set $f^B_{ii} f^C_{ii} = 2$. Therefore,

$$H = - \sum_{i=1}^{n} p_i^2 + 2 \sum_{i<j} \varphi(q_i - q_j) + 2 \sum_{i<j} \varphi(q_i + q_j) + 2 \sum_{i=1}^{n} \varphi(2q_i),$$

which is the conventional form of the second-order Hamiltonian of the elliptic Calogero–Moser system in the symplectic case.

Finally, for the root system $B_n$ we take $\Gamma = \{ \pm q_1, \ldots, \pm q_n, q_0 \}$ (see Example 3.8), and the Lax operator in the form

$$L = \begin{pmatrix} A & a & B \\ -b^t & 0 & -a^t \\ C & b & -A^t \end{pmatrix},$$

(4.12)
where $A, B,$ and $C$ are $n \times n$ matrices, $B = -B^t$, $C = -C^t$, and $a, b \in \mathbb{C}^n$. This is the standard form of an element of the Lie algebra $so(2n+1)$. Let $A, B,$ and $C$ be given as for $D_n$, and let $a$ and $b$ be given by their coordinates as follows:

$$a_i = f_i^a \frac{\sigma(z - q_0 - q_i)\sigma(z)}{\sigma(z - q_0)\sigma(z - q_i)\sigma(q_i)} \quad \text{and} \quad b_i = f_i^b \frac{\sigma(z - q_0 + q_i)\sigma(z - q_i)}{\sigma(z)\sigma(z - q_0)\sigma(q_i)}. \quad (4.13)$$

By the addition formulae, $a_i b_i = f_i^a f_i^b (\varphi(q_i) - \varphi(z - q_0))$. For the Hamiltonian we have

$$H = -\text{res}_{z=0} z^{-1} \left( \sum_{i=1}^n p_i^2 + 2 \sum_{i<j} A_{ij} A_{ji} + 2 \sum_{i<j} B_{ij} C_{ji} + 2 \sum_{i=1}^n a_i b_i \right)$$

$$= -\sum_{i=1}^n p_i^2 + 2 \sum_{i<j} \varphi(q_i - q_j) + 2 \sum_{i<j} \varphi(q_i + q_j) + 2 \sum_{i=1}^n (\varphi(q_i) - \varphi(z - q_0)).$$

We note that $p_0 = 0$, and therefore $q_0 = \text{const}$ gives an invariant submanifold of the hierarchy of commuting flows (in the Lax form of the equations of motion this corresponds to the relation $v_0 = 0$, which follows from the fact that $M$ is an element of $so(2n+1)$). Taking the restriction of the system to this invariant submanifold, and omitting $\varphi(q_0)$ as a non-essential constant, we arrive at the conventional form of the second-order Hamiltonian of the elliptic Calogero–Moser system for $B_n$.

**4.5. Holomorphy of spectra and the method of deformation of the Tyurin parameters.** Here we give proofs of Lemmas 4.2 and 4.3 and discuss the relationship of the latter to the equations of motion of the element $h$, which lead to the equations of deformation of the Tyurin parameters in the case of classical Lie algebras.

**Proof of Lemma 4.2.** We consider the adjoint action on $L$ of the local holomorphic function $e^{-h \log z}$, where $z$ is a local coordinate in a neighbourhood of a $\gamma \in \Gamma$ and $h$ is the element of the Cartan subalgebra used above to give the grading on $\mathfrak{g}$ (the choice of the branch of $\log z$ does not matter). On the homogeneous subspace of degree $s$ of this grading, that is, on $\mathfrak{g}_s \subset \mathfrak{g}$, the operator $-\text{ad} h$ acts as multiplication by $-s$. Therefore, the operator $\text{Ad} e^{-h \log z}$ acts as multiplication by $z^{-s}$. In the Laurent expansion of the element $L \in \mathcal{L}$ at the point $\gamma \in \Gamma$, the coefficient of $z^i$ is an element of $\widetilde{\mathfrak{g}}_i = \bigoplus_{-k \leq s \leq i} \mathfrak{g}_s$. Under the action of $\text{Ad} e^{-h \log z}$ a sum of powers $z^{i-s}$ with $-k \leq s \leq i$ (with certain coefficients) appears at this position in the Laurent expansion. Obviously, these are non-negative powers. The lemma is proved. □

**Remark 4.6.** For $M$-operators a similar argument does not lead to a similar result, because for $0 \leq i < k$ the components $M_i$ have non-zero projections on the homogeneous subspaces $\mathfrak{g}_s$ with $s > i$, and negative powers of $z$ will appear at those places.

A prototype of Lemma 4.2 for $\mathfrak{g} = \mathfrak{gl}(n)$ was formulated in [15]. The proof given there is also equivalent to our proof in this particular case. First, $L_{-1}$ is taken by conjugation to the form in Fig. 1. This corresponds to $h = \text{diag}(-1, 0, \ldots, 0)$. Then we conjugate by the matrix $\text{diag}(z, 0, \ldots, 0)$, which exactly coincides with $e^{-h \log z}$. 

Proof of Lemma 4.3. In accordance with §3.1, we represent the grading element in a neighbourhood of a $\gamma \in \Gamma$ in the form $h = g^{-1}h_0g$, where $h_0$ gives the grading at the point $\gamma$, and $g$ is a local holomorphic function taking values in $G$ such that $g(\gamma) \in G_0$, $G_0$ being the centralizer of $h_0$ in $G$ (that is, $\text{Lie}(G_0) = \mathfrak{g}_0$). Then it follows from Lemma 4.2 that

$$
\Psi = \Psi_0 e^{-h_0 \log z} g,
$$

(4.14)

where $\Psi_0$ is holomorphic and has a holomorphic inverse in a neighbourhood of $\gamma$. From this and the fact that $\partial_a z = -\nu_a$ (see (3.21)), we get that

$$
F_a = -\partial_a \Psi \cdot \Psi^{-1} + \Psi M_a \Psi^{-1}
$$

$$
= -\partial_a \Psi_0 \cdot \Psi_0^{-1} - \Psi_0 \frac{\nu_a h_0}{z} \Psi_0^{-1} - \Psi_0 e^{-h_0 \log z} \partial_a g \cdot g^{-1} e^{h_0 \log z} \Psi_0^{-1}
$$

$$
+ \Psi_0 e^{-h_0 \log z} g^{-1} \left( \frac{\nu_a h_0}{z} + \sum_{i=-k}^{\infty} M_{a,i} z^i \right) g e^{h_0 \log z} \Psi_0^{-1}.
$$

(4.15)

Taking into account that $\partial_a \Psi_0 \cdot \Psi_0^{-1}$ is holomorphic, we find that

$$
F_a = \Psi_0 e^{-h_0 \log z} g A g^{-1} e^{h_0 \log z} \Psi_0^{-1} + O(1),
$$

where

$$
A = -g^{-1} \partial_a g + \frac{\nu_a}{z} (h_0 - g^{-1} h_0 g) + \sum_{i=-k}^{\infty} M_{a,i} z^i.
$$

Next, we find $g$ from the equation

$$
g^{-1} \partial_a g = \frac{\nu_a}{z} (h_0 - g^{-1} h_0 g) + \sum_{0 \leq i < k} M_{a,i} z^i.
$$

(4.16)

Then $e^{-h_0 \log z} g A g^{-1} e^{h_0 \log z}$ is holomorphic by the same argument as for $L$ (the terms mentioned in Remark 4.6 are absent). The lemma is proved. \[\Box\]

From $h = g^{-1} h_0 g$ as a starting point, we find the equations of motion of $h$. To this end we differentiate the equality and obtain

$$
\partial_a h = [h, g^{-1} \partial_a g].
$$

(4.17)

By (4.16), for the initial condition $g(z)|_{z=0} = \text{id}$ we get in the zeroth approximation with respect to $z$ that $g^{-1} \partial_a g = M_{a,0}$ (note that $h_0 - g^{-1} h_0 g = 0$ for $z = 0$). This results in the following form of (4.17):

$$
\partial_a h = [h, M_{a,0}],
$$

(4.18)

which coincides exactly with the equation (3.21) for $h$.

Example 4.7. Let $\mathfrak{g} = \mathfrak{gl}(n)$. It follows easily from §2.3.1 that $h = -\alpha \mu^t$ in this case, where $\alpha = g(1,0,\ldots,0)^t$ and $\mu^t = (1,0,\ldots,0) g^{-1}$. Differentiating the relation $h = \alpha \mu^t$ and applying (4.18), we get that

$$
\partial_a \alpha \cdot \mu^t + \alpha \partial_a \mu^t = -M_{a,0} \alpha \mu^t + \alpha \mu^t M_{a,0}.
$$
Next, we multiply this equality on the right by a vector \( \vartheta \) such that \( \mu^t \vartheta = 1 \). Then

\[
\partial_a \alpha = -M_{a,0} \alpha + \lambda \alpha, \tag{4.19}
\]

where \( \lambda = (-\partial_a \mu^t + \mu^t M_{a,0}) \vartheta \in \mathbb{C} \).

Equation (4.19) is nothing but the equation of motion for the Tyurin parameters that was found in [15]. In addition, the following equation holds: \( \partial_a z = \mu_a \alpha \), where

\[
M_a = \alpha \mu^t_a z^{-1} + O(1).
\]

Indeed, it is easy to see that \( \nu_a = -\mu_a^t \alpha \). Then the equation \( \partial_a z = -\nu_a \) (3.21) gives what is required.

5. Lax integrable systems and conformal field theory

In this section we briefly sketch results in [38] (see also [39]) which let us assign to each integrable system of the type discussed above a unitary projective representation of the corresponding Lie algebra of Hamiltonian vector fields. To the family of spectral curves over the phase space of a system we apply the technique developed earlier for the tautological bundle over the moduli space of curves ([35], [36], [39], [33]). This enables us to construct a Knizhnik–Zamolodchikov-type connection on the phase space, and to represent the Hamiltonian vector fields by covariant derivatives with respect to this connection. From the physical point of view this is a Dirac-type prequantization. For the Hitchin systems, the idea of their quantization by means of the Knizhnik–Zamolodchikov connection has often been used, or at least mentioned, in the physical literature ([6], [12], [22], [25]), with various restrictions, for example, for second-order Hamiltonians, or on elliptic curves. In [38], [39] this idea was realized in complete generality in the (finite-dimensional) context of Lax equations with a spectral parameter on a Riemann surface, and for Hamiltonians of all orders.

5.1. The centralizer of an element and its vacuum representation. Let \( \mathcal{L} \) be a Lax operator algebra, let \( L \in \mathcal{L} \), and let \( \Psi \) diagonalize \( L \) as defined in §4.1, that is,

\[
\Psi L = K \Psi,
\]

where \( K = \text{diag}(\kappa_1, \ldots, \kappa_n) \). The diagonal elements of the matrix \( K \) are roots of the characteristic equation \( \det(L(z) - \kappa) = 0 \). The curve given by this equation is called the spectral curve of the element \( L \). We denote it by \( \Sigma_L \). It is an \( n \)-sheeted branch covering of the curve \( \Sigma \). Lemma 4.2 immediately implies that the meromorphic function \( K \) is holomorphic on \( \Gamma \) and has poles only at points \( P \in \Pi \).

Below we follow the assumptions and notation of §4.1. In particular, we consider only classical Lie algebras and use the Tyurin parametrization.

If we denote by \( \mathcal{A} \) the algebra of scalar functions on \( \Sigma \) which are holomorphic except possibly at points \( P \in \Pi \), then \( K \in \mathfrak{h} \otimes \mathcal{A} \), where \( \mathfrak{h} \subset \mathfrak{g} \) is a diagonal subalgebra. The algebra \( \mathcal{A} \) is called the Krichever–Novikov function algebra, and \( \mathfrak{k} = \mathfrak{h} \otimes \mathcal{A} \) is called the Krichever–Novikov current algebra (only commutative current algebras arise in our case, although in general any reductive Lie algebra could be used instead of \( \mathfrak{h} \)). For a detailed presentation of Krichever–Novikov algebras, see [39] and [33].

Conversely, for any \( h \in \mathfrak{k} \) we have \( \Psi^{-1} h \Psi \in \mathcal{L} \) [38], [39], and therefore we arrive at the following assertion.
Lemma 5.1. The Lie subalgebra of elements commuting with an $L \in \mathcal{L}$ is isomorphic to $\Psi \bar{\mathfrak{h}} \Psi^{-1}$.

Since the diagonal elements of the matrices $K \in \bar{\mathfrak{h}}$ correspond to sheets of the covering $\Sigma_L \to \Sigma$, each element $K \in \bar{\mathfrak{h}}$ can be pulled back to $\Sigma_L$ and will give a meromorphic scalar function on $\Sigma_L$ having poles only in the pre-image of $\Pi$. We denote the algebra of such functions by $\mathcal{A}_L$. The inverse map $\mathcal{A}_L \to \bar{\mathfrak{h}}$ is given by the construction of the direct image: a function $A \in \mathcal{A}_L$ is associated with a $K \in \bar{\mathfrak{h}}$ such that

$$K(P) = \text{diag}(A(P_1), \ldots, A(P_n)),$$

where $P_1, \ldots, P_n$ are the pre-images of the point $P \in \Sigma$.

Consider an element $A \in \mathcal{A}_L$ and its direct image $K \in \bar{\mathfrak{h}}$. Let $L_A = \Psi K \Psi^{-1}$. Each sheet of the curve $\Sigma_L$ is associated with a certain row in $K$. At branch points the eigenvalues of $K$ may coincide. The order of the diagonal entries of the matrix $K$ depends on the order of the sheets in the covering. The ambiguity here is the same as in the definition of the matrix $\Psi$: the permutation of the rows of $\Psi$ which corresponds to an element $w$ of the Weyl group reduces to the transformation $\Psi \to w\Psi$ (this can easily be checked for transpositions), and $K$ transforms as follows: $K \to wKw^{-1}$. Hence, $L_A = \Psi^{-1}K\Psi$ does not depend on $w$, and $L_A$ is well defined.

Lemma 5.2. The map $\mathcal{A}_L \to \mathcal{L}$ sending $A$ to $L_A$ establishes an isomorphism between $\mathcal{A}_L$ and the Lie subalgebra of $\mathcal{L}$ consisting of elements commuting with $L$.

The Lie algebra $\mathcal{L}$ has a canonical representation on the space $\mathcal{F}$ of meromorphic vector-functions taking values in $\mathbb{C}^n$, holomorphic outside the sets $\Pi$ and $\Gamma$, and having expansions of the form

$$\psi(z) = \nu \frac{\alpha}{z} + \psi_0 + \cdots$$

at points $\gamma \in \Gamma$, where $\alpha$ is the vector of Tyurin parameters. The space $\mathcal{F}$ is an almost-graded $\mathcal{L}$-module [39], [33]. The invariance of $\mathcal{F}$ with respect to $\mathcal{L}$ for classical Lie algebras is easily derived from (2.16)–(2.19). We denote the space of semi-infinite exterior forms of fixed charge on $\mathcal{F}$ by $\mathcal{F}^{\infty/2}$ [13]. It is a vacuum $\mathcal{L}$-module. Due to the above morphism $\mathcal{A}_L \to \mathcal{L}$ (Lemma 5.2), we can also regard $\mathcal{F}^{\infty/2}$ as an $\mathcal{A}_L$-module.

We conclude this section with the Sugawara representation. For a vacuum module of a Krichever–Novikov current algebra, the Sugawara construction gives a canonical representation of the corresponding Lie algebra of vector fields on the same space. We apply it to the $\mathcal{A}_L$-module $\mathcal{F}^{\infty/2}$, where $\mathcal{A}_L$ is regarded as a commutative Lie algebra, and we obtain a representation of the Lie algebra of meromorphic vector fields on $\Sigma_L$ that are holomorphic outside the pre-image of $\Pi$ under the covering. We denote this representation by $T$ below. We do not give a definition of the Sugawara representation here. It is presented in several places: [13] for loop algebras, [39] and [33] for Krichever–Novikov algebras. In particular, for commutative Krichever–Novikov current algebras the construction was originally proposed in [18].

We note that the Sugawara construction does not exist (and, perhaps, cannot exist; see the discussion in [39]) for Lax operator algebras, and this is one of the reasons for addressing the algebra $\mathcal{A}_L$ in this context.
Thus, each point of the phase space $\mathcal{P}^D$ is associated with an element $L \in \mathcal{L}$ (where $\mathcal{L}$ is specific to each point), its spectral curve $\Sigma_L$, and the $\mathcal{L}$-module $\mathcal{F}^{\infty/2}$ over it. As a result, we obtain a family of spectral curves on $\mathcal{P}^D$ and a bundle with an infinite-dimensional fibre over it. This whole picture is represented symbolically in Fig. 7. Below we construct a finite-rank quotient sheaf of this bundle, and a projective flat connection on it. In this way we transform our object into a conformal field theory in the sense of [7].

Figure 7. The family of spectral curves on the phase space.

5.2. Knizhnik–Zamolodchikov connection. Let $X$ be a tangent vector to $\mathcal{P}^D$ at a point $L$. We consider a deformation of the complex structure of the corresponding spectral curve in the direction of $X$. It is given by a Kodaira–Spencer class in $H^1(\Sigma_L, T\Sigma_L)$, where $T\Sigma_L$ is the tangent sheaf on $\Sigma_L$. The cocycle $\rho(X)$ representing this class can be constructed as follows. We fix a local section of the sheaf of spectral curves. In Fig. 7 its intersection with the fibre over $L$ is denoted by the marked point $P_\infty$. Let us consider a family of gluing functions (giving a smooth structure on spectral curves) defined in an annulus with ‘centre’ at the marked point. We denote such a gluing function by $d_L$. Let

$$\rho(X) = d_L^{-1} \partial_X d_L.$$ 

Since $d_L$ can be regarded as a diffeomorphism of the annulus, $\rho(X)$ is a local vector field defined there.

It was shown in [35], [36], and [39] that the Kodaira–Spencer cocycle can be represented by a Krichever–Novikov vector field (that is, by a global meromorphic vector field on $\Sigma_L$ which is holomorphic outside $\Pi$), and the ambiguity arising in the construction is compensated by passing to a quotient sheaf to be defined below. Corresponding to this, we regard $\rho(X)$ below as an element of the space $\mathcal{V}_L^{\text{reg}} \setminus \mathcal{V}_L/\mathcal{V}_L^{(1)}$, where $\mathcal{V}_L$ is the Lie algebra of Krichever–Novikov vector fields on $\Sigma_L$ (with $\Pi$ as the set of allowed poles), $\mathcal{V}_L^{(1)}$ is the direct sum of its homogeneous
subspaces of non-negative degree, and \( \mathcal{V}_L^{\text{reg}} \subset \mathcal{V}_L \) is the subspace of vector fields vanishing at \( P_\infty \).\(^3\) Both these subspaces are Lie subalgebras of \( \mathcal{V}_L \).

Whenever \( \rho(X) \) is a Krichever–Novikov vector field, we can consider \( T(\rho(X)) \), where \( T \) is the Sugawara representation, and then define the operators

\[
\nabla_X = \partial_X + T(\rho(X)).
\]

Next, we consider the sheaf of \( \mathcal{A}_L \)-modules \( \mathcal{F}_\infty^/2 \) on \( \mathcal{P}^D \). Let \( \mathcal{A}_L^{\text{reg}} \subset \mathcal{A}_L \) be the subalgebra of functions that are regular at the point \( P_\infty \). The sheaf of quotient spaces \( \mathcal{F}_\infty^/2 \mathcal{F}_\infty^/2 \mathcal{A}_L^{\text{reg}} \) on \( \mathcal{P}^D \) is called the sheaf of co-invariants, or the sheaf of conformal blocks in another terminology.

**Theorem 5.3.** The operators \( \nabla_X \) define a projective flat connection \( \nabla \) on the sheaf of co-invariants; in particular,

\[
[\nabla_X, \nabla_Y] = \nabla_[X,Y] + \lambda(X,Y) \cdot \text{id},
\]

where \( \lambda \) is a cocycle on the Lie algebra of tangent vector fields on \( \mathcal{P}^D \), and \( \text{id} \) is the identity operator.

In [36], [39], and [33] Theorem 5.3 is formulated and proved for the conformal field theory on the moduli space of curves with marked points and fixed (up to a certain order) jets at these points. We assert that here (as well as in [39]), the situation is completely similar and the same proofs work. In view of this analogy, we call the projective flat connection defined by Theorem 5.3 the Knizhnik–Zamolodchikov connection.

The horizontal sections of the Knizhnik–Zamolodchikov connection are also called conformal blocks.

### 5.3. A representation of the Lie algebra of Hamiltonian vector fields.

By Theorem 5.3 the map \( X \rightarrow \nabla_X \) is a projective representation of the Lie algebra of vector fields on \( \mathcal{P}^D \) on the space of sections of the sheaf of co-invariants. Denote this representation by \( \nabla \). The restriction of \( \nabla \) to the Lie subalgebra of Hamiltonian vector fields gives a projective representation of the latter. It is remarkable because it is unitary, and the representation operators of Hamiltonians which are spectral invariants (that is, correspond to invariants of the Lie algebra \( \mathfrak{g} \) as described in § 4.2) commute with each other.

**Theorem 5.4** ([38], [39]). If \( X \) and \( Y \) are Hamiltonian vector fields and their Hamiltonians are in involution, then

\[
[\nabla_X, \nabla_Y] = \lambda(X,Y) \cdot \text{id}.
\]

If the Hamiltonians are spectral invariants, then

\[
[\nabla_X, \nabla_Y] = 0.
\]

**Proof.** The first assertion follows immediately from Theorem 5.3, since \([X,Y] = 0\).

The coefficients of the spectral curve are first integrals for the Lax equation. Therefore, if \( X \) is a Hamiltonian vector field, and its Hamiltonian is a spectral invariant, then the complex structure, and hence the gluing functions, are invariant along the phase trajectories of the vector field \( X \). Let \( \{d_L\} \) be the family of these

\(^3\)It is convenient to think that \( P_\infty \in \Pi \) and use the splitting \( \Pi = (\Pi \setminus \{P_\infty\}) \cup \{P_\infty\} \) of the set \( \Pi \) to define an almost-graded structure; cf. § 2.1.
gluing functions (depending on $L$). The invariance along the trajectories immediately implies that $\partial_X d_L = 0$, so $\rho(X) = d_L^{-1} \partial_X d_L = 0$, and therefore $\nabla_X = \partial_X$. Let $H_X$ and $H_Y$ be Hamiltonians depending only on the spectrum of the Lax operator. Then according to the foregoing, $[\nabla_X, \nabla_Y] = [\partial_X, \partial_Y] = \partial_{[X,Y]}$, and $[X,Y] = 0$. This implies that $[\nabla_X, \nabla_Y] = 0$. □

Let $\mathcal{G}$ be a Lie algebra with an antilinear anti-involution $^\dagger$ and let $T$ be a representation of it on the linear space $V$. A Hermitian scalar product in $V$ is said to be contravariant if $T(X)^\dagger = T(X^\dagger)$, where on the left-hand side $^\dagger$ denotes Hermitian conjugation of operators. According to [13], the pair consisting of the representation $T$ and a contravariant scalar product on $V$ is called a unitary representation of the Lie algebra $\mathcal{G}$. In such a case the restriction of $T$ to the Lie subalgebra of elements $X$ such that $X^\dagger = -X$ (that is, to the real subalgebra) is unitary in the usual sense, that is, $T(X)^\dagger = -T(X)$.

In the fibres of the sheaf of co-invariants we can define the scalar product induced by the standard scalar product of semi-infinite monomials [13], [18], [39], when semi-infinite monomials consisting of basis vectors as cofactors are declared to be orthogonal.

Let $\omega$ be the symplectic form on $\mathcal{P}^D$, that is, the restriction of the Krichever–Phong form to $\mathcal{P}^D$, and let $\omega^p/p!$ be the corresponding volume form on $\mathcal{P}^D$. Let $L^2(\omega^p/p!)$ be the space of sections of the sheaf of co-invariants which are square integrable with respect to the measure given by the volume form. By the square of a section we mean the scalar square with respect to the scalar product introduced above.

**Theorem 5.5** ([38], [39]). The representation $\nabla : X \rightarrow \nabla_X$ of the Lie algebra of Hamiltonian vector fields on $\mathcal{P}^D$ on the space of smooth sections in $L^2(C, \omega^p/p!)$ is unitary.

We refer to [38] and [39] for the proof. Here we only note that, by the Poincaré theorem on absolute integral invariants, the symplectic form and its powers are absolute integral invariants of the Hamiltonian phase flows [1]. Hence, the volume form $\omega^p/p!$ defines an invariant measure on $\mathcal{P}^D$ with respect to the Hamiltonian phase flows, and averaging with respect to the invariant measure gives a unitary representation.

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Received 14/JAN/16
Translated by THE AUTHOR