Poisson–Lie identities and dualities of Bianchi cosmologies

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Abstract We investigate a special class of Poisson–Lie T-plurality transformations of Bianchi cosmologies invariant with respect to non-semisimple Bianchi groups. For six-dimensional semi-Abelian Manin triples $\mathfrak{b} \preceq \mathfrak{a}$ containing Bianchi algebras $\mathfrak{b}$ we identify general forms of Poisson–Lie identities and dualities. We show that these can be decomposed into simple factors, namely automorphisms of Manin triples, B-shifts, $\beta$-shifts, and “full” or “factorized” dualities. Further, we study effects of these transformations and utilize the decompositions to obtain new backgrounds which, supported by corresponding dilatons, satisfy Generalized Supergravity Equations.

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1 Introduction

Duality transformations play crucial role in the study of various aspects of string theory and related fields. They connect theories in different coupling regimes or, in the case of T-duality, backgrounds with distinct curvature properties. Both Abelian T-duality [1] and its non-Abelian generalization [2,3] rely on the presence of symmetries of sigma model backgrounds. Dual sigma model related to the original one by T-duality is obtained by gauging of the symmetry and introduction of Lagrange multipliers. However, the symmetries are not preserved in the non-Abelian case, meaning we may not be able to return to the original model by dualization. Despite this serious issue we see renewed interest in (non-) Abelian T-duality (NATD). The procedure was extended to RR fields in [4,5] and is used frequently to find new supergravity solutions, see e.g. [6,7] and references therein. It also applies in the study of integrable models [8–10]. In this paper we are going to present new solutions of the Generalized

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Supergravity Equations obtained by action of Poisson–Lie identities and Poisson–Lie dualities introduced in [11] on the Bianchi cosmologies [12]. This extends list of results obtained in [13].

Poisson–Lie T-duality [14] introduces Drinfel’d double as the underlying algebraic structure of T-duality and replaces symmetry of the sigma model background by the so-called Poisson–Lie symmetry [15]. This allows us to treat both models equally and solves the above mentioned problem. We shall use this formalism through the whole paper. In the case of (non-) Abelian T-duality the Lie group \( D \) of Drinfel’d double splits into Lie subgroups \( G \) and \( \tilde{G} \) of equal dimension, where the former represents symmetries of the original background while the latter is Abelian. In this paper we consider only these semi-Abelian Drinfel’d doubles as we focus on dualization of particular backgrounds and the presence of symmetries remains crucial in such case.\(^1\) Poisson–Lie duality exchanges roles of \( G \) and \( \tilde{G} \), and we understand it as a change of decomposition \((G|\tilde{G})\) of \( D \) to \((\tilde{G}|G)\) and vice versa. Beside \((G|\tilde{G})\) and \((\tilde{G}|G)\) there might be other decompositions \((\tilde{G}|\tilde{G})\), \((G|\tilde{G})\) of a Drinfel’d double \( D \) that can be used to construct mutually related sigma models. The corresponding transformation between sigma models was denoted Poisson–Lie T-plurality [17]. Decompositions of low-dimensional Drinfel’d doubles were classified in papers [18–20] in terms of Manin triples \((\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})\) that represent decompositions of Lie algebra \( \mathfrak{d} \) of the Drinfel’d double \( D \) into subalgebras \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) corresponding to subgroups \( G \) and \( \tilde{G} \).

In our recent paper [11] we noted that besides (non-) Abelian T-duality there exist other transformations that either preserve or exchange the algebras \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \) of the Manin triple \((\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})\). We shall call them Poisson–Lie identities and Poisson–Lie dualities. Similar transformations were studied in [21] to get insight into the structure of the so-called NATD group of T-duality transformations. Beside others, this group contains automorphisms of the algebras forming Manin triples, B-shifts, \( \beta \)-shifts,\(^2\) “factorized” dualities and their compositions. These, however, have to be understood as special cases of Poisson–Lie T-plurality. We continue the investigation of the NATD group probing its structure for low-dimensional Drinfel’d doubles, where general forms of Poisson–Lie identities and dualities can be identified. We show that all transformations are finite compositions of the special elements of NATD group that were mentioned earlier. It turns out that the effect of automorphisms and B-shifts on resulting backgrounds can be often eliminated by a change of coordinates, hence, we identify what parameters of the transformations are relevant.

Long-lasting problem appearing in discussion of non-Abelian T-duality is that dualization with respect to non-semisimple group \( G \) leads to mixed gauge and gravitational anomaly, see [24], proportional to the trace of structure constants of \( g \). Authors of paper [13] have found non-Abelian T-duals of Bianchi cosmologies [12] and have shown that instead of standard beta function equations the dual backgrounds satisfy the so-called Generalised Supergravity Equations containing Killing vector \( J \) whose components are given by the trace of structure constants. For Bianchi \( V \) cosmology this was observed already in [25]. Therefore, it is natural to ask if backgrounds and dilatons obtained from Bianchi cosmologies by Poisson–Lie identities and dualities satisfy Generalised Supergravity Equations as well and what Killing vectors have to be used. We show that components of \( J \) are determined by trace of structure constants, however, one must carefully inspect what groups (or subgroups of) \( G \) truly participate in the duality transformation. For Yang–Baxter deformations of \( AdS_5 \times S^5 \) the problem of finding \( J \) was addressed in [26, 27] and physical meaning of \( J \) was found in [28].

We start with a short description of Poisson–Lie T-plurality in Sect. 2, where necessary formulas are summarized and general forms of transformed backgrounds are presented. In Sects. 3–7 we investigate various transformations of Bianchi cosmologies focusing on groups that are not semisimple. Since calculations with general transformations often result in rather complicated backgrounds that cannot be displayed, detailed description is given only for special elements of the NATD group. Summary of transformed backgrounds can be found in the Appendix.

2 Basics of Poisson–Lie T-plurality

In the first two subsections we recapitulate Poisson–Lie T-plurality with spectators [14,17,31]. We follow the summary given in [11].

2.1 Sigma models

Let \( \mathcal{M} \) be \((n + d)\)-dimensional (pseudo-)Riemannian target manifold and consider sigma model on \( \mathcal{M} \) given by Lagrangian

\[
\mathcal{L} = \partial_\mu \phi^\mu \mathcal{F}_{\mu\nu}(\phi) \partial_\nu \phi^\nu, \quad \phi^\mu = \phi^\mu(\sigma_+, \sigma_-),
\]

\( \mu = 1, \ldots, n + d \)

where tensor field \( \mathcal{F} = \mathcal{G} + \mathcal{B} \) on \( \mathcal{M} \) defines metric and torsion potential (Kalb–Ramond field) of the target manifold. Assume that there is a \( d \)-dimensional Lie group \( G \) with free

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\(^1\) See [16] for discussion on this topic.

\(^2\) \( \beta \)-shifts are also referred to as TsT transformations, see [22,23].
action on \( \mathcal{M} \) that leaves the tensor invariant. The action of \( \mathcal{G} \) is transitive on its orbits, hence we may locally consider \( \mathcal{M} \approx (\mathcal{M}/\mathcal{G}) \times \mathcal{G} = N \times \mathcal{G} \), and introduce adapted coordinates 
\[ \{ s_\alpha, x_\alpha \}, \quad \alpha = 1, \ldots, n = \dim N, \quad \alpha = 1, \ldots, d = \dim \mathcal{G} \]
where \( s_\alpha \) label the orbits of \( \mathcal{G} \) and are treated as spectators, and \( x_\alpha \) are group coordinates. \(^3\) Dualizable sigma model on \( N \times \mathcal{G} \) is given by tensor field \( F \) defined by \((n+d) \times (n+d)\) matrix \( E(s) \) as
\[ F(s,x) = E(x) \cdot E(s) \cdot E^T(x), \quad E(x) = \begin{pmatrix} I_n & 0 \\ 0 & e(x) \end{pmatrix} \tag{1} \]
where \( e(x) \) is \( d \times d \) matrix of components of right-invariant Maurer–Cartan form \((dg)^{-1} \) on \( \mathcal{G} \).

Using non-Abelian \( T \)-duality one can find dual sigma model on \( N \times \mathcal{A} \), where \( \mathcal{A} \) is Abelian subgroup of semi-Abelian Drinfel’d double \( \mathcal{D} = (\mathcal{G}, \mathcal{A}) \). The necessary formulas will be given in the following subsection as a special case of Poisson–Lie T-plurality. In this paper the groups \( \mathcal{G} \) will be non-semisimple Bianchi groups. Bianchi cosmologies are defined on four-dimensional manifolds, hence \( d = 3 \), \( n = 1 \), and we denote the spectator as \( t := s_1 \). Elements of the group \( \mathcal{G} \) shall be parametrized as \( g = e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3} \) where \( e^{x_1 T_1} e^{x_2 T_2} e^{x_3 T_3} \) are normal subgroups of \( \mathcal{G} \). Similarly, elements of \( \mathcal{A} \) are parametrized as \( \hat{g} = e^{\hat{x}_1 \hat{T}_1} e^{\hat{x}_2 \hat{T}_2} e^{\hat{x}_3 \hat{T}_3} \).

### 2.2 Poisson–Lie T-plurality with spectators
For certain Drinfel’d doubles several decompositions may exist. Suppose that \( \mathcal{G} = (\mathcal{G}, \mathcal{A}) \) splits into another pair of subgroups \( \hat{\mathcal{G}} \) and \( \hat{\mathcal{A}} \). Then we can apply the full framework of Poisson–Lie T-plurality \([14,17]\) and find sigma model on \( N \times \hat{\mathcal{G}} \).

The 2\( d \)-dimensional Lie algebra \( \mathfrak{d} \) of the Drinfel’d double \( \mathcal{G} \) is equipped with an ad-invariant non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Let \( \mathfrak{d} = \mathfrak{g} \bowtie \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{d}} = \hat{\mathfrak{g}} \bowtie \mathfrak{g} \) be two decompositions (Manin triples \((\mathfrak{d}, \mathfrak{g}, \hat{\mathfrak{g}})\) and \((\hat{\mathfrak{d}}, \hat{\mathfrak{g}}, \mathfrak{g})\) of \( \mathfrak{d} \) into double cross sum of subalgebras \([34]\) that are maximally isotropic with respect to \( \langle \cdot, \cdot \rangle \). The pairs of mutually dual bases \( T_a \in \mathfrak{g}, \hat{T}_a \in \hat{\mathfrak{g}} \) and \( \hat{T}_a \in \hat{\mathfrak{g}}, T_a \in \mathfrak{g}, a = 1, \ldots, d \), satisfying
\[ \langle T_a, T_b \rangle = 0, \quad \langle \hat{T}_a, \hat{T}_b \rangle = 0, \quad \langle T_a, \hat{T}_b \rangle = \delta^b_a, \tag{2} \]
then must be related by transformation
\[ \begin{pmatrix} \hat{T} \\ T \end{pmatrix} = C \cdot \begin{pmatrix} T \\ \hat{T} \end{pmatrix} \tag{3} \]
where \( C \) is an invertible \( 2d \times 2d \) matrix. Due to ad-invariance of the bilinear form \( \langle \cdot, \cdot \rangle \) the algebraic structure of \( \mathfrak{d} \) is given both by
\[ [T_i, T_j] = f_{ij}^k T_k, \quad \hat{[T_i, T_j]} = \hat{f}_{ij}^k \hat{T}_k, \quad \hat{[\hat{T}_i, \hat{T}_j]} = \hat{f}_{ij}^k \hat{\hat{T}}_k, \eqref{4} \]
and
\[ [\hat{T}_i, \hat{T}_j] = \hat{f}_{ij}^k \hat{\hat{T}}_k, \quad [\hat{T}_i, T_j] = \hat{f}_{ij}^k \hat{T}_k, \quad \hat{[\hat{T}_i, T_j]} = \hat{f}_{ij}^k \hat{\hat{T}}_k. \tag{5} \]

Given the structure constants \( F_{ij}^{jk} \) of \( \mathfrak{d} = \mathfrak{g} \bowtie \hat{\mathfrak{g}} \) and \( \hat{F}_{ij}^{jk} \) of \( \hat{\mathfrak{d}} = \hat{\mathfrak{g}} \bowtie \mathfrak{g} \), the matrix \( C \) has to satisfy equation
\[ C_a^p C_b^q F_{pq}^{rs} = \hat{F}_a^p \hat{C}_b^r C_c^s. \tag{6} \]

To preserve the bilinear form \( \langle \cdot, \cdot \rangle \) and thus \( \eqref{2} \), \( C \) also has to satisfy
\[ C_a^p C_b^q (D_0)_{pq} = (D_0)_{ab} \tag{7} \]
where \( (D_0)_{ab} \) are components of matrix \( D_0 \) that can be written in block form as
\[ D_0 = \begin{pmatrix} 0_d & 1_d \\ 1_d & 0_d \end{pmatrix}. \tag{8} \]
In other words, \( C \) is an element of \( O(d,d) \) but, unlike the case of Abelian \( T \)-duality, not every element of \( O(d,d) \) is allowed in \( \eqref{3} \).

For the following formulas it will be convenient to introduce \( d \times d \) matrices \( P, Q, R, S \) as
\[ \begin{pmatrix} T \\ \hat{T} \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} T \\ \hat{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \cdot \begin{pmatrix} T \\ \hat{T} \end{pmatrix}. \tag{9} \]
and extend these to \( (n+d) \times (n+d) \) matrices
\[ P = \begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}, \quad Q = \begin{pmatrix} 0_n & 0 \\ 0 & Q \end{pmatrix}, \quad R = \begin{pmatrix} 0_n & 0 \\ 0 & R \end{pmatrix}, \quad S = \begin{pmatrix} I_n & 0 \\ 0 & S \end{pmatrix} \]
to accommodate the spectator fields. It is also advantageous to introduce block form of \( E(s) \) as
\[ E(s) = \begin{pmatrix} E_{ab}(s) & E_{ab}(s) \\ E_{ab}(s) & E_{ab}(s) \end{pmatrix}, \quad a, b = 1, \ldots, n \]
The sigma model on \( N \times \hat{\mathcal{G}} \) related to \( \eqref{1} \) via Poisson–Lie T-plurality is given by tensor

\[^3\] Detailed discussion of the process of finding adapted coordinates can be found e.g. in \([31-33]\).
\[ \mathcal{F}(s, \hat{x}) = \mathcal{E}(\hat{x}) \cdot \mathcal{E}(s, \hat{x}) \cdot \mathcal{E}^T(\hat{x}), \quad \mathcal{E}(\hat{x}) = \begin{pmatrix} I_n & 0 \\ 0 & \mathcal{c}(\hat{x}) \end{pmatrix} \] (10)

where \( \mathcal{c}(\hat{x}) \) is a \( d \times d \) matrix of components of right-invariant Maurer–Cartan form \( (dG)\mathcal{g}^{-1} \) on \( \mathcal{G} \).

\[ \mathcal{E}(s, \hat{x}) = (I_n + d + \mathcal{E}(s) \cdot \mathcal{F}(\hat{x}))^{-1} \cdot \mathcal{E}(s) \]

\[ \mathcal{F}(s, \hat{x}) = (\mathcal{P} + E(s) \cdot \mathcal{R})^{-1} \cdot (\mathcal{Q} + E(s) \cdot \mathcal{S}) \] (12)

so it is necessary that

\[ \text{det} (\mathcal{P} + E(s) \cdot \mathcal{R}) \neq 0 \neq \text{det} (\mathcal{Q} + E(s) \cdot \mathcal{S}). \]

Formulas (10)–(12) reduce to those for full Poisson–Lie dualities if we choose \( \mathcal{P} = S = 0 \) and \( \mathcal{Q} = R = 1 \). Furthermore, for a semi-Abelian Drinfeld’s double of the well-known Buscher rules for (non-) Abelian T-duality are restored. If there are no spectators the plurality is called atomic.

### 2.3 Poisson–Lie identities and Poisson–Lie dualities

Let us now restrict our considerations to mappings (3) that preserve the Manin triple, i.e., \( \hat{\mathcal{g}} = \mathcal{g}, \hat{\mathcal{g}} = \tilde{\mathcal{g}} \), and that satisfy (6) and (7). They are the Poisson–Lie identities. The Maurer–Cartan form \( (dG)\mathcal{g}^{-1} \) remains unchanged but \( E(s) \) transforms as in (12). Moreover, for non-Abelian T-duality the algebra \( \tilde{\mathcal{g}} \) is Abelian, i.e., \( \mathcal{g} = \alpha \), thus \( \tilde{b} \) and \( \tilde{\mathcal{F}} \) vanish, and we may write

\[ \mathcal{F}(s, x) = \mathcal{E}(\mathcal{x}) \cdot \mathcal{E}(s) \cdot \mathcal{E}^T(\mathcal{x}). \]

Let us note that both backgrounds \( \mathcal{F}(s, x) \) and \( \mathcal{F}(s, x) \) are invariant with respect to the group \( \mathcal{G} \).

For special transformations mentioned in the introduction we can further specify the resulting backgrounds. Namely, matrices

\[ I_A = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \] (13)

are always among the transformations (3) preserving the Manin triple \( \mathcal{d} = \mathcal{g} \Leftrightarrow a \) if \( A \) is an automorphism of \( \mathcal{g} \). Transformed \( \mathcal{E}(s) \) then reads

\[ \mathcal{E}(s) = A \cdot E(s) \cdot A^T, \quad A = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}. \]

Transformations (3) of the form

\[ I_B = \begin{pmatrix} 1_d & 0 \\ 0 & 1_d \end{pmatrix}, \quad B^T = -B \] (14)

are called B-shifts, since the background \( \mathcal{F}(s, x) \) obtained by this transformation is given by

\[ \mathcal{E}(s) = (E(s) - \tilde{B}), \quad \tilde{B} = \begin{pmatrix} 0_n & 0 \\ 0 & B \end{pmatrix}. \]

Tensor \( \mathcal{F} \) differs from the original one by an antisymmetric term \( B^T = -E(x) \cdot B \cdot E(x)^T \) that, however, for solvable Bianchi algebras does not produce supplementary torsion. Therefore, for all investigated Bianchi cosmologies \( \mathcal{F} \) is gauge equivalent to the initial tensor \( \mathcal{F} \).

\( \beta \)-shifts are generated by transformation matrices

\[ I_B = \begin{pmatrix} 0_d & 0 \\ 0 & B \end{pmatrix}, \quad B^T = -B \]

and the transformed \( \mathcal{F}(s, x) \) is given by

\[ \mathcal{E}(s) = (1 - E(s) \cdot \tilde{B})^{-1} \cdot E(s), \quad \tilde{B} = \begin{pmatrix} 0_n & 0 \\ 0 & B \end{pmatrix}. \]

For invertible \( E(s) \), we may write \( \mathcal{E}(s) = (E(s)^{-1} - \tilde{B})^{-1} \).

Beside these transformations we may also encounter mappings \( I_F \) that switch some of the basis vectors \( T_i \leftrightarrow \tilde{T}_i \) while preserving structure coefficients of the Manin triple. These “factorized” dualities can be interpreted as dualization with respect to subgroups of \( \mathcal{G} \). In general, these cannot be written concisely in block form and we do not discuss them here. We shall see many examples in the following sections.

Let us further investigate Poisson–Lie dualities, i.e. mappings (3) that change Manin triple \( \mathcal{d} = \mathcal{g} \leftrightarrow a \) to \( \mathcal{d} = a \leftrightarrow \mathcal{g} \).

Equation (3) implies that

\[ \left( \begin{array}{c} \mathcal{T} \\ \mathcal{T} \end{array} \right) = D_0 \cdot \left( \begin{array}{c} \mathcal{T} \\ \mathcal{T} \end{array} \right), \quad D_0 = D_0 \cdot I \cdot \left( \begin{array}{c} \mathcal{T} \\ \mathcal{T} \end{array} \right). \]

Therefore, Poisson–Lie dualities are composed of Poisson–Lie identities \( I \) and “full” T-duality \( D_0 \) that exchanges all generators of \( \mathcal{g} \) and \( \mathcal{a} \) as \( T_i \leftrightarrow \tilde{T}_i \) for \( i = 1, \ldots, d \). In this way we can define dual B-shifts, dual \( \beta \)-shifts and dual automorphisms.

\[ \text{Our approach differs from [21], where the authors consider vector space isomorphisms of Lie algebra of the Drinfeld’s double rather than automorphisms of a chosen Manin triple.} \]
Backgrounds on \( \hat{\mathcal{G}} = \mathcal{G} \) obtained from (1) by Poisson–Lie dualities have the form

\[
\hat{\mathcal{F}}(s, \hat{x}) = \left( \hat{E}^{-1}(s) + \hat{\Pi}(\hat{x}) \right)^{-1}, \quad \hat{\Pi}(\hat{x}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{b}(\hat{x}) \end{pmatrix}
\]

because \( \hat{e}(\hat{x}) = \hat{a}(\hat{x}) = 1_d \). For solvable groups \( \mathcal{G} \) we have \( \hat{b}_{ab}(\hat{x}) = f_{ac}^c \hat{x}_c \).

It is possible to be more specific when we restrict to specific elements of the NATD group. In the presence of spectators, however, the formulas for Poisson–Lie dualities are quite complicated. Fortunately, we do not need their full form since \( E_{ab}(s) \) and \( E_{ab}(s) \) in \( \hat{E}(s) \) vanish for the backgrounds discussed in the rest of the paper. Hence \( \hat{\mathcal{F}}_{ab}(s) = \mathcal{F}_{ab}(s) \) and plurality affects only \( \mathcal{F}_{ab}(s, x) \). The transformations we are interested in, therefore, concern only the \( \hat{\mathcal{F}}_{ab}(s, \hat{x}) \) block of the resulting background tensor \( \hat{\mathcal{F}} \).

For the dual B-shift

\[
D_B = D_0 \cdot I_B = \begin{pmatrix} 0 & 1_d \\ 1_d & B \end{pmatrix}, \quad B^T = -B
\]

and vanishing \( E_{ab}(s), E_{ab}(s) \) the matrix \( \left( \hat{E}^{-1}(s) + \hat{\Pi}(\hat{x}) \right) \) has the form

\[
\begin{pmatrix} E^{-1}_{ab}(s) & 0 \\ 0 & E_{ab}(s) - B + \hat{b}_{ab}(\hat{x}) \end{pmatrix}
\]

For solvable groups \( (\hat{b}(\hat{x}) - B)_{ab} = f_{ac}^c \hat{x}_c - B_{ab} \). As we shall see, for some groups this enables us to get rid of some parameters of \( B_{ab} \) by coordinate transformations.

General formulas for dual \( \beta \)-shift

\[
D_\beta = D_0 \cdot I_\beta = \begin{pmatrix} \beta & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \beta^T = -\beta
\]

in the presence of spectators are complicated and not particularly illuminating. For vanishing \( E_{ab}(s), E_{ab}(s) \) one gets

\[
\hat{E}(s) = \begin{pmatrix} E_{ab}(s) & 0 \\ 0 & E^{-1}_{ab}(s) - \beta \end{pmatrix}
\]

Let us focus on the role of automorphisms \( I_A \) and their duals now. As conjectured in [21], it turns out that all Poisson–Lie identities and Poisson–Lie dualities are generated by automorphisms of Manin triples, B-shifts, \( \beta \)-shifts and factorised dualities. Moreover, in most of the examples of Bianchi cosmologies discussed later we find that the general \( C \) matrix in (3) splits as

\[
C = I_{A_2} \cdot C' \cdot I_{A_1}
\]

where \( C' \) is either \( I_B, I_\beta, I_F \) or their duals. Transformed backgrounds then can be written as

\[
\hat{\mathcal{F}}(s, \hat{x}) = \hat{\mathcal{E}}(\hat{x}) \cdot \mathcal{A}_2 \cdot \hat{E}'(s, \hat{x}) \cdot \mathcal{A}'_2 \cdot \hat{\mathcal{E}}^T(\hat{x})
\]

where \( \hat{e}'(s, \hat{x}) \)

\[
= \left( (Q' + A_1 E(s) S')^{-1} (P' + A_1 E(s) R') + A_2 \hat{f}(\hat{x}) A_2 \right)^{-1}
\]

and \( P', Q', R', S' \) are found from \( C' \). As expected, these expressions can be interpreted as application of Poisson–Lie T-plurality on a background given by matrix \( E'(s) = A_1 E(s) A_1^T \). Important is that \( A_2 \) can be eliminated from (15) by suitable transformation of group coordinates. Using the transformation properties of the covariant tensor \( \hat{\mathcal{F}} \) we may try to integrate the Jacobi matrix

\[
J^\mu_\lambda = \frac{\partial \hat{e}'^\mu}{\partial \hat{e}^\lambda} = \hat{e}(\hat{x}) (A_2)^2_\lambda^\mu \hat{e}^{-1}(\hat{x})
\]

(16) to find coordinates \( \hat{x}' \) such that (15) simplifies to

\[
\hat{\mathcal{F}}(s, \hat{x}) = \hat{\mathcal{E}}(\hat{x}) \cdot \hat{E}'(s, \hat{x}) \cdot \hat{\mathcal{E}}^T(\hat{x})
\]

These transformations can be always found for Poisson–Lie dualities on semi-Abelian Drinfel’d double where \( \hat{g} = g \) and \( \hat{e}(\hat{x}) = 1 \). In this case \( J = A \), the transformation is linear and can be even combined with the coordinate shifts mentioned earlier for dual B-shifts.

2.4 Generalized supergravity equations and transformation of dilaton

One of the goals of this paper is to verify whether backgrounds obtained from Poisson–Lie identities and dualities satisfy Generalized Supergravity Equations of Motion. We adopt convention used in [13] so the equations read\(^7\)

\[
0 = R_{\mu \nu} - \frac{1}{4} H_{\rho \sigma} H_{\mu \nu}^{\rho \sigma} + \nabla_\mu X_\nu + \nabla_\nu X_\mu,
\]

(17)

\[
0 = -\frac{1}{2} \nabla_\rho H_{\mu \nu} + X_\rho H_{\mu \nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu,
\]

(18)

\[
0 = R - \frac{1}{12} H_{\rho \sigma \tau} H^{\rho \sigma \tau} + 4 \nabla_\mu X_\mu - 4 X_\mu X_\mu
\]

(19)

where \( \nabla \) is covariant derivative and

\[
X_\mu = \partial_\mu \Phi + J^\nu \mathcal{F}_{\nu \mu}.
\]

For vanishing vector \( J \) the usual one-loop beta function equations, i.e. conformal invariance conditions, are recovered.

Under (non-) Abelian T-duality dilaton transforms as

\[
\tilde{\Phi} = \Phi + \frac{1}{2} \ln \det M
\]

(20)

where matrix \( M \) is given by "group block" \( E_{ab} \) of \( E(s) \) and submatrices of adjoint representation as

\[
M = (E_{ab}(s) + \tilde{b}(\tilde{x}) \cdot \tilde{a}^{-1}(\tilde{x}))^{-1} = (E^{-1}_{ab}(s) + \hat{b}(\hat{x}) \cdot \hat{a}^{-1}(\hat{x}))^{-1}.
\]
The formula (20) can be utilized not only for “full” duality
given by \(D_0\), but also for factorized dualities. However, for
successful application of this rule it is necessary to identify
the dualized directions, meaning we have to consider only
subgroups of \(\mathcal{G}\) and corresponding submatrices \(E_{ab}, \hat{a}, \hat{b}\).

For general Poisson–Lie T-plurality the dilaton transformation
rule was given in [17] and further studied in [37]. In
the current notation we can write it as
\[
\hat{\Phi}(s, \hat{x}) = \Phi(s, x) - \frac{1}{2} \ln \left| \det \left( (N + \tilde{\Pi}(\hat{x})M) \hat{a}(\hat{x}) \right) \right| + \frac{1}{2} \ln \left| \det \left( (I + \Pi(x)E(s)) a(x) \right) \right|,
\]
\[
M = S^T \cdot E(s) - Q^T, \quad N = P^T - R^T E(s).
\]
(21)

From the two possible decompositions of elements of Drin-
fel’d double
\[
l = g(x)\hat{h}(\hat{x}) = \hat{g}(\hat{x})\hat{h}(\hat{x}),
\]
l \(\in \mathcal{D}, \quad g \in \mathcal{G}, \quad \hat{h} \in \hat{\mathcal{G}}, \hat{g} \in \hat{\mathcal{G}}, \hat{h} \in \hat{\mathcal{G}}\)
we can in principle express coordinates \(x\) in terms of \(\hat{x}\)
and \(\hat{x}\). The expression is thus nonlocal in the sense that \(\Phi\)
may depend also on coordinates \(\hat{x}\) of \(\hat{\mathcal{G}}\). For Poisson–Lie
dentities we do not encounter this problem so it is plausible
to use (21) to calculate dilatons corresponding to B-shifts
and \(\beta\)-shifts.

For semi-Abelian Drinfel’d double we find that dilaton
does not change under B-shifts \(I_B\), while under \(I_{\beta}\) it transforms as
\[
\hat{\Phi}(s, x) = \Phi(s, x) - \frac{1}{2} \ln \left| \det 1 - \beta \cdot E_{ab}(s) \right|.
\]
(22)

For duals \(D_B = D_0 \cdot I_B\) and \(D_{\beta} = D_0 \cdot I_{\beta}\) we get the
correct dilaton by formula (20) applied on the dilaton and
background obtained earlier from identities \(I_B\) and \(I_{\beta}\).

### 3 Bianchi V cosmology

As a warm up we shall study the well-known Bianchi V
icosmology. Let us consider six-dimensional semi-Abelian
Drinfel’d double\(^8\) \(\mathcal{D} = (\mathcal{B}_V, \mathcal{A})\) whose Lie algebra \(\mathfrak{d} = \mathfrak{b}_V\Rightarrow \mathfrak{a}\) is
spanned by basis \((T_1, T_2, T_3, \hat{T}^1, \hat{T}^2, \hat{T}^3)\). The
non-trivial commutation relations of the generators of \(\mathfrak{b}_V\)
are
\[
\{T_1, T_2\} = T_2, \quad \{T_1, T_3\} = T_3.
\]
(23)
The group \(\mathcal{B}_V\) is not semisimple and trace of its structure
constants does not vanish.

\(^8\) By \(\mathcal{B}_V\), resp. \(\mathfrak{b}_V\), we denote the Bianchi \(V\) group, resp. its Lie algebra.
\(\mathcal{A}\) and \(\mathfrak{a}\) denote three dimensional Abelian group and its Lie algebra
respectively. Similar notation will be used in the following sections.

The sigma model background\(^9\) is given by metric \((B = 0\)
and \(F = G)\)
\[
F(t, x_1) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & t^2 & 0 & 0 \\
0 & 0 & e^{2x_1}t^2 & 0 \\
0 & 0 & 0 & e^{2x_1}t^2
\end{pmatrix}.
\]
(24)

Left-invariant vector fields that satisfy (23) and generate sym-
metries of this background are
\[
V_1 = \partial_{x_1} - x_2 \partial_{x_2} - x_3 \partial_{x_3}, \quad V_2 = \partial_{x_2}, \quad V_3 = \partial_{x_3}.
\]
In fact, the background is flat and torsionless so the standard
beta function equations are satisfied if we choose zero dilaton
\(\Phi = 0\). This background was studied already in [38], where
it was first noticed that duals with respect to non-semisimple
groups are not conformal. The related gravitational-gauge
anomaly was later investigated in [39].

#### 3.1 Poisson–Lie identities and dualities

Mappings \(C\) that preserve the algebraic structure of Manin
triple \((\mathcal{D}, b_V, a)\) and generate Poisson–Lie identities
are given by matrices
\[
I_1 = \begin{pmatrix}
1 & c_{12} & c_{13} & -c_{12}c_{15} - c_{13}c_{16} & c_{15} & c_{16} \\
0 & c_{22} & c_{23} & -c_{15}c_{22} - c_{16}c_{23} & 0 & 0 \\
0 & c_{32} & c_{33} & -c_{15}c_{32} - c_{16}c_{33} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -c_{13}c_{25} - c_{16}c_{25} & 0 & 0 \\
0 & 0 & 0 & c_{16}c_{35} + c_{13}c_{35} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & c_{16}c_{35} - c_{13}c_{35} & 0 & 0
\end{pmatrix}.
\]
\[
I_2 = \begin{pmatrix}
-1 & c_{12} & c_{13} & c_{15}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & c_{12}c_{25} + c_{13}c_{26} & 0 & 0 \\
0 & 0 & 0 & c_{15}c_{35} + c_{13}c_{36} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
\[
26 \end{pmatrix}.
\]
(26)

One can see that for \(c_{15} = c_{16} = 0\) the matrix \(I_1\) simplifies
to the block form (13) given by automorphisms of algebra
\(b_V\) that in general read
\[
A = \begin{pmatrix}
1 & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}.
\]
(25)

For \(c_{12} = c_{13} = c_{23} = 0\) and \(c_{22} = c_{33} = 1\) matrix
\(I_1\) reduces to B-shift (14) of the form
\[
I_B = \begin{pmatrix}
1 & 0 & 0 & c_{15} & c_{16} \\
0 & 1 & 0 & -c_{15} & 0 \\
0 & 0 & 1 & -c_{16} & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
(26)

\(^9\) \(E(s)\) is restored from \(F(s, x)\) by setting group coordinates to zero.
On the other hand, for \( c_{12} = c_{13} = c_{26} = c_{35} = 0, c_{25} = c_{36} = 1 \) matrix \( I_2 \) equals to

\[
I_F = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]  

(27)

Matrix \( I_F \) switches basis vectors \( T_2, T_3 \) and \( \tilde{T}^2, \tilde{T}^3 \). We identify its action as factorized duality with respect to Abelian subgroup generated by \( T_2, T_3 \). The change of sign of \( T_1 \) is necessary for being an automorphism of \( b_V \rightarrow a \).

To study models generated by \( I_1 \) and \( I_2 \) we decompose these matrices into product of special elements of NATD group. Namely, we note that \( I_1 \) can be written as

\[
I_1 = I_A \cdot I_B
\]

where \( I_A \) is given by (25) and \( I_B \) is the B-shift (26). Similarly, \( I_2 \) can be decomposed as

\[
I_2 = I_{A_2} \cdot I_F \cdot I_{A_1}
\]

for automorphisms \( A_1 \) and \( A_2 \) of the form (25). This decomposition is not unique. To identify relevant parameters of \( I_2 \) we choose the simplest possible \( I_A \), while including the rest of the parameters in \( I_{A_2} \) as follows:

\[
A_1 = \begin{pmatrix}
1 & -c_{12} & -c_{13} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
1 & c_{15} & c_{16} \\
0 & c_{25} & c_{26} \\
0 & c_{35} & c_{36}
\end{pmatrix}.
\]  

(28)

Matrices generating Poisson–Lie dualities can be obtained from those above by left-multiplication by matrix (8) representing canonical or “full” duality. This way we get dual automorphisms generated by

\[
D_A = D_0 \cdot I_A = (A_d (A^T)^{-1})_A
\]

dual B-shifts generated by

\[
D_B = D_0 \cdot I_B = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & c_{15} & c_{16} \\
0 & 1 & 0 & -c_{15} & c_{16} \\
0 & 0 & 1 & -c_{16} & 0 & 0
\end{pmatrix},
\]

and factorized duality

\[
D_F = D_0 \cdot I_F = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]  

(29)

that can be interpreted as Buscher duality with respect to \( T_1 \) accompanied by a change of sign in the dual coordinate.

3.2 Transformed backgrounds

3.2.1 B-shifts

Let us now apply Poisson–Lie identities on the sigma model (24). Plugging \( I_1 \) into formulas (9)–(12) we get rather complicated background tensor. Nevertheless, \( I_1 \) decomposes as \( I_1 = I_A \cdot I_B \) and we can get rid of the parameters that come from \( I_A \) by a change of coordinates found by integrating the Jacobi matrix (16). Indeed, after coordinate transformation

\[
y_1 = x_1, \quad y_2 = -c_{12} e^{-x_1} + c_{22} x_2 + c_{32} x_3, \\
y_3 = -c_{13} e^{-x_1} + c_{23} x_2 + c_{33} x_3
\]

we find that the symmetric part of \( \hat{\Phi} \) equals to the original metric (24). The antisymmetric part

\[
\hat{B}(y_1) = \begin{pmatrix}
0 & 0 & 0 & -e^{\eta_1} c_{15} & 0 & e^{\eta_1} c_{16} \\
0 & e^{\eta_1} c_{15} & 0 & 0 & 0 & 0 \\
e^{\eta_1} c_{16} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

generated by the B-shift represents a torsionless B-field. Up to coordinate transformations we would get the same background using \( I_B \) instead of the full \( I_1 \) so, from the point of view of Poisson–Lie identity, we consider these matrices equivalent. In other words, Poisson–Lie identity with respect to \( I_B \) and \( I_1 \) is just a gauge transformation of the original background, there is no change in the dilaton field, and \( \hat{\Phi} = \Phi \) satisfies beta function equations.

Background calculated using \( D_1 = D_0 \cdot I_1 \) is too extensive to be displayed. Nevertheless, a change of coordinates (16) simplifies it to the form that one would obtain using \( D_B \). Subsequent coordinate shift

\[
\tilde{x}_1 = c_{12} (\tilde{y}_2 - c_{15}) + c_{13} (\tilde{y}_3 - c_{16}) + \tilde{y}_1, \\
\tilde{x}_2 = c_{22} (\tilde{y}_2 - c_{15}) + c_{23} (\tilde{y}_3 - c_{16}), \\
\tilde{x}_3 = c_{32} (\tilde{y}_2 - c_{15}) + c_{33} (\tilde{y}_3 - c_{16})
\]

that agrees with the discussion in Sect. 2.3 eliminates the parameters of \( D_1 \) completely, producing tensor
The same background can be obtained via full duality using $D_0$, and, as discussed in [38, 39], it is not conformal. The standard beta function equations cannot be satisfied by any dilaton $\Phi$. On the other hand, dilaton

$$\tilde{\Phi}(t, \tilde{y}_2, \tilde{y}_3) = \frac{1}{2} \ln \left( r^2 \left( \frac{\tilde{y}_2^2}{\tilde{y}_3^2} + \frac{\tilde{y}_3^2}{\tilde{y}_2^2} + r^4 \right) \right)$$

together with background (30) satisfy Generalized Supergravity Equations (17)–(19) if we choose $J = (0, 2, 0, 0)$. Components of the Killing vector $\mathcal{J}$ are given by trace of structure constants of $\mathcal{S}$ as $\mathcal{J}^i = f_{ai}^i$. The dilaton agrees with the formula (20), and we conclude that up to a coordinate transformation the background (30) found using $D_0$ or $D_1$ is equivalent to non-Abelian T-dual investigated in [13].

### 3.2.2 Factorized dualities

Using $I_2 = I_{A_2} \cdot I_F \cdot I_{A_1}$ in formulas (9)–(12) we get background that can be brought to the form

$$\tilde{\mathcal{F}}(t, y_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & r^2 & -e_{\gamma_1 \gamma_2} & -e_{\gamma_1 \gamma_3} \\ 0 & e_{\gamma_1 \gamma_2} & e_{\gamma_2 \gamma_3} & 0 \\ 0 & e_{\gamma_1 \gamma_3} & 0 & e_{\gamma_2 \gamma_3} \end{pmatrix}$$

by coordinate transformation

$$y_1 = x_1, \quad y_2 = -c_{15} e^{-x_1} + c_{25} x_2 + c_{35} x_3, \quad y_3 = -c_{16} e^{-x_1} + c_{26} x_2 + c_{36} x_3$$

whose Jacobi matrix (16) is determined by $A_2$ in (28). The background differs from $\mathcal{F}$ calculated using $I_F$ since $I_{A_1}$ changes $E(s)$ before the factorized duality is applied. However, the only difference is in the antisymmetric part $\tilde{\mathcal{B}}$. For $I_2$ there is a torsionless B-field, while for $I_F$ the B-field vanishes completely. The metric has vanishing scalar curvature but is not flat. Further coordinate transformation

$$t = \sqrt{-2u v + 2u + z_3^2 + z_4^2}, \quad y_2 = u z_3, \quad y_1 = \frac{1}{2} \ln \left( \frac{-2u v + 2u + z_3^2 + z_4^2}{u^2} \right), \quad y_3 = u z_4$$

brings it to the Brinkmann form of plane parallel wave [40] with

$$ds^2 = 2z_3^2 + z_4^2 du^2 + 2dv + dz_3^2 + dz_4^2.$$

Corresponding dilaton follows from the formula (20) if the factorized duality (27) is interpreted as Buscher duality [11] with respect to two-dimensional Abelian subgroup generated by left-invariant fields $V_2 = \partial_{y_2}, \ V_3 = \partial_{y_3}$. Metric (24) is written in coordinates adapted to the action of this subgroup and for the duality given by $I_F$ we can write

$$\tilde{\Phi}(t, x_1) = \frac{1}{2} \ln \det M = \frac{1}{2} \ln \det \begin{pmatrix} e_{\gamma_1 x_1} & 0 \\ 0 & e_{\gamma_2 x_1} \end{pmatrix} = -\ln t^2 + 2y_1 = -2 \ln u.$$

Dual dilaton for background given by $I_2$ is derived from the altered $E'(s) = A_1 E(s) A_1^T$ and differs from the previous expression by a constant. We again conclude that backgrounds found using $I_F$ and $I_2$ differ only by a coordinate and gauge transformation and can be considered equivalent. They satisfy beta function equations, or Generalized Supergravity Equations (17)–(19) where $\mathcal{J}$ is zero vector.

Background obtained by Poisson–Lie transformation using matrix $D_F$ has the form

$$\tilde{\mathcal{F}}(t, \tilde{y}_2, \tilde{y}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & r^2 \tilde{y}_2^2 + \tilde{y}_3^2 + 1 & \tilde{y}_2 & \tilde{y}_3 \\ 0 & \tilde{y}_2 & r^2 (\tilde{y}_2^2 + 1) & \tilde{y}_2 \tilde{y}_3 + \tilde{y}_3 \\ 0 & \tilde{y}_3 & \tilde{y}_2 \tilde{y}_3 + \tilde{y}_3 & \tilde{y}_3 \end{pmatrix}.$$ (31)

The same background is obtained using $D_2 = D_0 \cdot I_2 = D_0 \cdot I_{A_2} \cdot I_F \cdot I_{A_1}$ after change of coordinates

$$\tilde{x}_1 = c_{15} (\tilde{y}_2 - c_{12}) + c_{16} (\tilde{y}_3 - c_{13}) + \tilde{y}_1, \quad \tilde{x}_2 = c_{25} (\tilde{y}_2 - c_{12}) + c_{26} (\tilde{y}_3 - c_{13}), \quad \tilde{x}_3 = c_{35} (\tilde{y}_2 - c_{12}) + c_{36} (\tilde{y}_3 - c_{13}).$$

Thus, we are able to eliminate all parameters appearing in $D_2$. The background is torsionless and together with dilaton

$$\tilde{\Phi}(t, \tilde{y}_2, \tilde{y}_3) = -\frac{1}{2} \ln \left( r^2 \left( \frac{\tilde{y}_2^2}{\tilde{y}_3^2} + 1 \right) \right)$$

satisfies beta function equations, i.e. the Killing vector in the Generalized Supergravity Equations is zero. Explanation is that we can interpret the factorized duality (29) as Buscher duality of (24), this time with one-dimensional

---

[10] Compared to [13], in the present nomenclature the matrices $\mathcal{F}$ representing background tensors are transposed. This results in change of sign of $\mathcal{J}$.

[11] Followed by a change of sign in the spectator coordinate $x_1$. 

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Abelian subgroup generated by left-invariant field \( V_1 = \partial_{x_1} - x_2 \partial_{x_2} - x_3 \partial_{x_3} \). In adapted coordinates \( \{s_1, s_2, s_3, y_1\} \)
\( t = s_1, \quad x_1 = y_1, \quad x_2 = s_2 e^{-y_1}, \quad x_3 = s_3 e^{-y_1}, \)
where \( V_1 = \partial_{y_1} \), the tensor (24) is manifestly invariant with respect to shifts in \( y_1 \) since
\[
\mathcal{F}(s_1, s_2, s_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & s_1^2 & 0 & -s_1^3 s_2 \\ 0 & 0 & s_1^2 & -s_1^3 s_3 \\ 0 & -s_1^3 s_2 & -s_1^3 s_3 & s_1^2 (s_2^2 + s_3^2 + 1) \end{pmatrix}.
\]
Buscher duality with respect to \( y_1 \) then restores the tensor (31) and dilaton (32) agrees with formula (20).

To sum up, in this section we have shown that backgrounds emerging from general Poisson–Lie identities or dualities differ from those obtained from special elements of NATD group only by a coordinate or gauge transformation. From now on we shall display results for these special elements and only comment on the general cases.

4 Bianchi III cosmology

Several results for Bianchi III cosmology are similar to those for Bianchi V. The algebra \( \mathfrak{d} = b_{III} \bowtie a \) of six-dimensional semi-Abelian Drinfel’d double \( (\mathfrak{g}^{III} \bowtie \mathfrak{a}) \) is spanned by basis \( (T_1, T_2, T_3, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3) \). Non-trivial commutation relations of the generators of \( b_{III} \) are
\[
[T_1, T_3] = -T_3,
\]
while \( a \) is Abelian. The trace of structure constants does not vanish and group \( \mathfrak{g}^{III} \) is not semisimple. The background given by metric
\[
\mathcal{F}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t^2 e^{-2x_1} \end{pmatrix}
\]
is flat, torsionless, and invariant with respect to symmetries generated by left-invariant vector fields
\[ V_1 = \partial_{x_1} + x_3 \partial_{x_3}, \quad V_2 = \partial_{x_2}, \quad V_3 = \partial_{x_3} \]
satisfying (33). As the background is flat and torsionless the dilaton \( \Phi \) can be chosen zero. Authors of [41] mention this background in their analysis and note that its non-Abelian dual does not satisfy the standard beta function equations.

4.1 Poisson–Lie identities and dualities

Table 1 summarizes all eight types of solutions of equations (6) and (7) with structure constants \( F = \tilde{\Phi} \). These give rise to Poisson–Lie identities and dualities of \( (\mathfrak{g}^{III} \bowtie \mathfrak{a}) \). Nevertheless, all the identities are composed of automorphisms (13) with

\[
A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},
\]

B-shifts of the form (26), and factorized dualities\(^{12}\)
\[
I_{F_1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\]
and
\[
I_{F_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]
Matrices generating Poisson–Lie dualities can be again obtained from those above by left-multiplication by the matrix (8) representing full duality.

4.2 Transformed backgrounds

4.2.1 B-shifts

Using \( I_B \) (26) in the formulas (9)–(12) we find that the background \( \tilde{\Phi} \) has the same metric as the original model (34). In addition to that, a torsionless \( B \)-field
\[
\tilde{\Phi}(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{15} & -c_{16} e^{-x_1} \\ 0 & c_{15} & 0 & 0 \\ 0 & c_{16} e^{-x_1} & 0 & 0 \end{pmatrix}
\]
appears. This agrees with the interpretation of action of \( I_B \) as gauge transformation. There is no change in the dilaton and \( \Phi = \tilde{\Phi} \). With the full solutions \( I_4 \) and \( I_8 \) we get the same background as for \( I_B \). Indeed, both these matrices decompose as

\[ I_4 = I_A \cdot I_B, \quad I_8 = I_A \cdot I_B \]
with \( I_A \) given by (35). A linear change of coordinates (16) thus restores the metric (34) and torsionless \( B \)-field (38).\(^{13}\)

Dual background calculated using matrix \( D_B = D_0 \cdot I_B \) produces tensor

\[^{12}\]I_A and \( I_B \) appear as special cases of \( I_4 \) and \( I_8 \), factorized dualities \( I_{F_1}, I_{F_2} \) and their composition appear in \( I_1, I_2, I_3, I_5, I_6, I_7 \) and their duals.

\[^{13}\]The parameter \( c_{15} \) has to be replaced by \(-\frac{c_{15} + c_{16}}{c_{12}} \) for \( I_4 \).
Table 1  PLT-identities of Drinfel'd double $\mathcal{D}_{II}$

| $\mathcal{D}_{II}$ | $C$ matrix |
|---------------------|------------|
| $I_1$               | \[
\begin{pmatrix}
-1 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
0 & 0 & 0 & \frac{c_{14}}{c_{12}} & \frac{c_{15}}{c_{12}} & \frac{c_{16}}{c_{12}} \\
0 & 0 & 0 & \frac{c_{13}}{c_{12}} & 0 & c_{36} \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{15}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_2$               | \[
\begin{pmatrix}
-1 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
0 & 0 & 0 & \frac{c_{14}}{c_{12}} & \frac{c_{15}}{c_{12}} & \frac{c_{16}}{c_{12}} \\
0 & 0 & 0 & \frac{c_{13}}{c_{12}} & 0 & c_{36} \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{15}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_3$               | \[
\begin{pmatrix}
1 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
0 & 0 & 0 & \frac{c_{14}}{c_{12}} & \frac{c_{15}}{c_{12}} & \frac{c_{16}}{c_{12}} \\
0 & 0 & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{15}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_4$               | \[
\begin{pmatrix}
1 & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
0 & 0 & 0 & \frac{c_{14}}{c_{12}} & \frac{c_{15}}{c_{12}} & \frac{c_{16}}{c_{12}} \\
0 & 0 & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{15}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_5$               | \[
\begin{pmatrix}
-1 & 0 & c_{13} & c_{14}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & 0 & \frac{c_{14}}{c_{12}} & 0 \\
0 & 0 & 0 & \frac{c_{13}}{c_{12}} & 0 & c_{36} \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{16}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_6$               | \[
\begin{pmatrix}
-1 & 0 & c_{13} & c_{14}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & 0 & \frac{c_{14}}{c_{12}} & 0 \\
0 & 0 & 0 & \frac{c_{13}}{c_{12}} & 0 & c_{36} \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & c_{52} & 0 & \frac{c_{14}c_{16}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_7$               | \[
\begin{pmatrix}
1 & 0 & c_{13} & -c_{14}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & 0 & \frac{c_{14}}{c_{12}} & 0 \\
0 & 0 & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{c_{12}} & 0 \\
0 & c_{52} & 0 & \frac{-c_{14}c_{16}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
| $I_8$               | \[
\begin{pmatrix}
1 & 0 & c_{13} & -c_{14}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & 0 & \frac{c_{14}}{c_{12}} & 0 \\
0 & 0 & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{c_{12}} & 0 \\
0 & c_{52} & 0 & \frac{-c_{14}c_{16}}{c_{12}} & 0 & 0 \\
0 & 0 & \frac{1}{c_{36}} & 0 & 0 & 0 \\
\end{pmatrix}
\] |
\[ \tilde{F}(t, \tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 & 0 \\ 0 & \frac{1}{t^2(c_1 + c_3^2 + c_2)} & 0 & 0 \\ 0 & \frac{1}{t^2(c_1 + c_3^2 + c_2)} & \frac{1}{t^2(c_1 + c_3^2 + c_2)} & \frac{1}{t^2(c_1 + c_3^2 + c_2)} \end{pmatrix} \] (39)

whose curvature and torsion do not vanish. We can get rid of the parameter \( c_{16} \) by shift in \( \tilde{x}_3 \), but \( c_{15} \) remains. As earlier, backgrounds calculated using \( D_4 = D_0 \cdot I_4 = D_0 \cdot I_A \cdot I_B \) or \( D_8 = D_0 \cdot I_8 = D_0 \cdot I_A \cdot I_B \) differ from \( \tilde{F} \) only by a transformation of coordinates. For nonzero \( c_{15} \) the tensor \( \tilde{F} \) is not the same as non-Abelian dual of (34) that can be found using \( D_0 \). Nevertheless, if we understand the duality with respect to \( D_B = D_0 \cdot I_B \) as full duality applied to background changed by \( I_B \), the correct dilaton can be found from (20) as

\[ \tilde{\Phi}(t, \tilde{x}_3) = -\frac{1}{2} \ln \left( 1 + c_{16}^2 \frac{t^2}{(c_1 + c_3^2 + c_2)} \right) \] (40)

Such \( \tilde{\Phi} \) satisfies the Generalized Supergravity Equations for Killing vector \( J = (0, -1, 0, 0) \) whose components are given by trace of structure constants of \( b_{III} \) as suggested in [13].

4.2.2 Factorized dualities

Poisson–Lie identities (36) and (37) can be interpreted as Buscher dualities with respect to one-dimensional Abelian subgroups generated by left-invariant fields \( V_3 = \partial_{x_3} \) resp. \( V_2 = \partial_{x_2} \).

Dualization with respect to \( V_2 \) does not change \( F \) at all due to the form of the metric (34). The background is invariant with respect to \( I_{F_2} \). Its dual given by \( D_{F_2} = D_0 \cdot I_{F_2} \) needs to be understood as dual with respect to non-Abelian group generated by \( V_1, V_3 \) that is not semisimple. The background and dilaton are the same as for the full duality \( D_0 \). We can read them from (39), (40) setting \( c_{15} = c_{16} = 0 \). The same results, up to a coordinate or gauge transformation, are obtained for the full solutions \( I_3, I_7 \), see Table 1, and their duals \( D_3, D_7 \) since

\[ I_3 = I_{A_2} \cdot I_{F_2} \cdot I_B \cdot I_{A_1}, \quad I_7 = I_{A_2} \cdot I_{F_2} \cdot I_B. \]

Dualization with respect to \( V_3 \), i.e. Poisson–Lie identity \( I_{F_1} \), produces metric

\[ \tilde{F}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2x_1/t^2} \end{pmatrix} \] (41)

whose scalar curvature vanishes. In coordinates

\[ t = \sqrt{z_3^2 - 2u(v - 1)}, \quad x_2 = z_4, \]
\[ x_1 = -\frac{1}{2} \ln \left( \frac{z_3^2 - 2u(v - 1)}{u^2} \right), \quad x_3 = u z_3 \]

it acquires the Brinkmann form of a plane parallel wave with

\[ ds^2 = 2 \frac{z_3^2}{u^2} du^2 + 2 du \, dv + dz_3^2 + dz_4^2. \]

As expected, dilaton calculated via formula (20)

\[ \tilde{\Phi}(t, x_1) = \frac{1}{2} \ln \det M = \frac{1}{2} \ln \det \begin{pmatrix} 1 & 0 \\ 0 & e^{-2x_1/t^2} \end{pmatrix} \]

\[ = -\frac{1}{2} \ln t^2 - x_1 \]

satisfies beta function equations, or Generalized Supergravity Equations with \( J = 0 \), since we have dualized with respect to Abelian subgroup of \( b_{III} \). Poisson–Lie identities \( I_1, I_2, I_5 \) and \( I_6 \) decompose as

\[ I_1 = I_{A_2} \cdot I_{F_1} \cdot I_{F_2} \cdot I_{A_1}, \quad I_2 = I_{A_2} \cdot I_{F_1} \cdot I_B \cdot I_{A_1}, \]
\[ I_5 = I_{A_2} \cdot I_{F_1} \cdot I_{F_2} \cdot I_{A_1}, \quad I_6 = I_{A_2} \cdot I_{F_1} \cdot I_B \cdot I_{A_1}. \]

Resulting backgrounds differ from (41) only by a change of coordinates and torsionless B-field of the form (38) and can be found in Table 3 in the Appendix.

Dual background produced by \( D_{F_1} = D_0 \cdot I_{F_1} \) reads

\[ \tilde{F}(t, x_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t^2(z_3^2 + 1)} & 0 & -\frac{3}{z_3^2 + 1} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{3}{z_3^2 + 1} & 0 & \frac{1}{z_3^2 + 1} \end{pmatrix}. \]

Together with the dilaton

\[ \tilde{\Phi}(t, x_3) = -\frac{1}{2} \ln \left( t^2 \left( \frac{z_3^2}{z_3^2 + 1} \right) \right) \]

found from (20) this background satisfies beta function equations. Factorized duality given by \( D_{F_1} \) can be once again interpreted as Buscher duality with respect to symmetry generated by \( V_1, V_2 \). The same result is obtained for \( D_5 = D_0 \cdot I_5 \). For \( D_1, D_2, D_6 \) the tensor \( \tilde{F} \) and dilaton \( \tilde{\Phi} \) contain a parameter that cannot be eliminated by coordinate or gauge transformation. Interested reader may find its full form in Table 3 in the Appendix.
5 Bianchi $V I_e$ cosmology

Semi-Abelian Drinfel’d double $\mathcal{D} = (\mathcal{B}_V I_e | \mathcal{A})$ has Lie algebra $\mathcal{g} = \mathfrak{b}_V I_e \cong \alpha$ spanned by basis $(T_1, T_2, T_3, \tilde{T}^1, \tilde{T}^2, \tilde{T}^3)$ and the nontrivial commutation relations of $\mathfrak{b}_V I_e$ are

$$[T_1, T_2] = \kappa T_2, \quad [T_1, T_3] = T_3, \quad \kappa \neq -1. \quad (42)$$

Trace of structure constants does not vanish and group $\mathcal{B}_V I_e$ is not semisimple. In the parametrization used in [13] Bianchi $V I_e$ cosmology is given by metric

$$F(t, x_1) = \left(-e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the functions $a_i(t)$ are

$$a_1(t) = e^{\Phi(t)} \left( \frac{p_1}{\kappa + 1} \right) \kappa^{\frac{p_1}{\kappa + 1}} e^{\frac{(\kappa + 1)p_1}{\kappa + 1}} \sinh -\frac{\kappa}{(\kappa + 1)^2} (p_1 t),$$

$$a_2(t) = e^{\Phi(t)} \left( \frac{p_1}{\kappa + 1} \right) \kappa^{\frac{p_2}{\kappa + 1}} e^{\frac{p_2}{\kappa + 1}} \sinh -\frac{1}{\kappa + 1} (p_2 t),$$

$$a_3(t) = e^{\Phi(t)} \left( \frac{p_1}{\kappa + 1} \right) \kappa^{\frac{p_3}{\kappa + 1}} e^{-\frac{p_3}{\kappa + 1}} \sinh -\frac{1}{\kappa + 1} (p_3 t). \quad (44)$$

The background is invariant with respect to symmetry generated by left-invariant vector fields

$$V_1 = \partial_{x_1} - \kappa x_2 \partial_{x_2} - x_3 \partial_{x_3}, \quad V_2 = \partial_{x_2}, \quad V_3 = \partial_{x_3}$$

satisfying (42). For dilaton $\Phi(t) = c_1 t$ the beta function equations reduce to condition

$$c_1^2 = \frac{\kappa^2 + \kappa + 1}{\kappa + 1} \frac{p_1^2}{4} - \frac{p_2^2}{4}.$$ 

The background is torsionless and for $c_1 = 0$ also Ricci flat.

5.1 Poisson–Lie identities and dualities

Poisson–Lie identities of Drinfel’d double $(\mathcal{B}_V I_e | \mathcal{A})$ are given by matrices

$$I_1 = \begin{pmatrix} 1 & c_{12} & c_{13} & c_{12}c_{15} - c_{13}c_{16} & c_{15} & c_{16} \\ 0 & c_{22} & 0 & -c_{15}c_{16} & 0 & 0 \\ 0 & 0 & c_{33} & -c_{16}c_{33} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c_{12}}{\kappa} & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & -\frac{c_{25}}{\kappa} & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & -\frac{c_{36}}{\kappa} & \frac{1}{\kappa} & 0 \end{pmatrix},$$

$$I_2 = \begin{pmatrix} -1 & c_{12} & c_{13} & c_{12}c_{15} + c_{13}c_{16} & c_{15} & c_{16} \\ 0 & 0 & 0 & c_{12}c_{25} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} & \frac{1}{\kappa} & 0 \end{pmatrix}. $$

The algebra $\mathfrak{b}_V I_e$ admits automorphisms (35) and matrices $I_A$ of the form (13) are among the special cases of $I_1$. Clearly, $I_1$ is a product $I_1 = I_A \cdot I_B$ of automorphisms and B-shifts (26). Matrix $I_2$ can be written as $I_2 = I_{A_2} \cdot I_F \cdot I_{A_3}$ where $I_F$ is the factorized duality (27) and $I_{A_1}, I_{A_3}$ are given by automorphisms (28). Poisson–Lie dualities are obtained by multiplication by $D_0$.

5.2 Transformed backgrounds

5.2.1 B-shifts

Using $I_1$ directly in formulas (9)–(12) we get rather complicated background tensor. However, since $I_1$ splits as $I_1 = I_A \cdot I_B$, the dependence of $\tilde{F}$ on the parameters appearing in $I_A$ can be eliminated by transformation (16). The background obtained using $I_1$ is equivalent to that obtained by B-shift (26) and reads

$$\tilde{F}(t, x_1) = \left(-e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$-e^{x_1 c_{15}} c_{15} c_{16} - e^{x_1 c_{16}} c_{15} c_{16} - e^{x_1 c_{15}} c_{15} c_{16} \end{pmatrix}. \quad (44)$$

Beside the original metric (43) we have obtained a torsionless B-field. Together with the original dilaton $\Phi(t) = c_1 t$ the background satisfies beta function equations.

Dual background $\tilde{F}$ calculated using $D_1 = D_0 \cdot I_1 = D_0 \cdot I_A \cdot I_B$ is again too complicated to display. Nevertheless, linear transformation of coordinates (16) followed by shift in $\bar{y}_2$, $\bar{y}_3$ simplifies the background to

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14 Note that for $\kappa = 0$, or $\kappa = 1$, these are commutation relations of $\mathfrak{b}_{II/I}$, or $\mathfrak{b}_V$, respectively. The case $\kappa = -1$ will be treated separately in Sect. 6.
\[ \tilde{F}(t, \tilde{y}_2, \tilde{y}_3) = \begin{pmatrix} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 \\ 0 & a_2(t)^2a_3(t)^2 - e^{-2\kappa y_2} & 0 \\ 0 & 0 & a_3(t)^2 - e^{-2\kappa y_3} \end{pmatrix}, \]

where
\[ \Delta = a_1(t)^2a_2(t)^2a_3(t)^2 + \kappa^2 y_2^2a_3(t)^2 + a_2(t)^2y_3^2. \]

These results are the same as results obtained by full duality \( D_0 \). Dual dilaton
\[ \tilde{\Phi}(t, \tilde{y}_2, \tilde{y}_3) = c_1 t - \frac{1}{2} \ln \Delta \]

found from formula (20) satisfies the generalized supergravity equations (17)–(19) where components of Killing vector \( F = (0, \kappa + 1, 0, 0) \) correspond to trace of structure constants of \( b_{VI} \). Dualization with respect to \( D_1 \) can be treated as canonical duality in spite of the fact that it contains also B-shifts and automorphisms.

### 5.2.2 Factorized dualities

Poisson–Lie identity \( I_F \) in (27) can be interpreted as Buscher duality with respect to two-dimensional Abelian subgroup generated by left-invariant fields \( V_2 = \partial_{x_2}, \ V_3 = \partial_{x_3} \). Resulting curved background
\[ \tilde{F}(t, x_1) = \left( \begin{array}{ccc} -e^{-4\Phi(t)}a_1(t)^2a_2(t)^2a_3(t)^2 & 0 & 0 \\ 0 & a_2(t)^2 & 0 \\ 0 & 0 & e^{2\kappa x_1}a_3(t)^2 \end{array} \right), \]

and dilaton
\[ \tilde{\Phi}(t, x_1) = c_1 t + \frac{1}{2} \ln \left( e^{2(\kappa + 1)x_1}a_2(t)^2a_3(t)^2 \right) \]

may conclude that results of duality with respect to \( I_2 \) deviate from those obtained by \( I_F \) only by coordinate and gauge transformation.

After a suitable coordinate transformation we find that both matrices \( DF = D_0 \cdot I_F \) and \( D_2 = D_0 \cdot I_{A_2} \cdot I_F \cdot I_{A_1} \) produce background
\[ \tilde{F}(t, \tilde{y}_2, \tilde{y}_3) = \left( \begin{array}{ccc} -e^{-4\Phi(t)}a_1^2a_2^2 & 0 & 0 \\ 0 & \frac{1}{\Delta} & \frac{a_2}{a_1}a_1^2a_3^2 \frac{\kappa^2 y_2}{\Delta} \\ 0 & -\frac{a_2}{a_1}a_1^2a_3^2 & \frac{a_1^2a_3^2}{\Delta} \end{array} \right) \]

where
\[ \Delta = a_1(t)^2a_2(t)^2y_2^2 + a_3(t)^2y_3^2. \]

This background is the same as the one that would be obtained by performing Buscher duality with respect to symmetry generated by \( V_1 \). Dilaton
\[ \tilde{\Phi}(t, \tilde{y}_2, \tilde{y}_3) = c_1 t - \frac{1}{2} \ln \Delta \]

satisfies ordinary beta function equations.

Let us note that results of this section hold also for \( \kappa = 0, 1 \), i.e. for Bianchi III and Bianchi V. Dualities with respect to these groups were treated in Sects. 3 and 4 with different initial backgrounds.

### 6 Bianchi \( V_{I-1} \) cosmology

For Bianchi \( V_{I-1} \) cosmology we shall consider Manin triple \((\mathbb{O}, b_{V_{I-1}}, a)\) whose algebraic structure is given by comutation relations (42) with \( \kappa = -1 \). Structure coefficients of Lie algebra \( b_{V_{I-1}} \) are traceless and the group \( \mathcal{B}_{V_{I-1}} \) is not semisimple. Metric has the form (43) with functions
\[ a_1(t) = \sqrt{p_1} \exp \left( \frac{e^{2p_1t} + p_1t}{2} + \Phi(t) \right), \]
\[ a_2(t) = a_3(t) = \sqrt{p_2} e^{\frac{p_2t}{2} + \Phi(t)} \]

and dilaton is again \( \Phi(t) = c_1 t \). The beta function equations are satisfied if
\[ c_1^2 = \frac{1}{4}(2p_1p_2 + p_2^2). \]
6.1 Poisson–Lie identities and dualities

Poisson–Lie identities of Drinfel’d double \( \mathcal{D}_{V_{L-1}}(\mathcal{A}) \) are given by matrices

\[
I_1 = \begin{pmatrix}
-1 & c_{12} & c_{13} & c_{12}c_{15} + c_{14}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & c_{15} & 0 & c_{36} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and

\[
I_2 = \begin{pmatrix}
1 & c_{12} & c_{13} & -c_{12}c_{15} - c_{13}c_{16} & c_{15} & c_{16} \\
0 & 0 & 0 & c_{33} & 0 & c_{35} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

As special cases we find two types of automorphisms \( I_A \) and \( I_{A'} \) given by

\[
A = \begin{pmatrix}
1 & a_{12} & a_{13} \\
0 & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix},
\quad
A' = \begin{pmatrix}
-1 & a_{12} & a_{13} \\
0 & 0 & a_{23} \\
0 & a_{32} & 0
\end{pmatrix},
\]

B-shifts generated by matrix

\[
I_B = \begin{pmatrix}
1 & 0 & 0 & b_{12} & b_{13} \\
0 & 1 & 0 & -b_{12} & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\beta\text{-shifts}
\]

\[
I_{\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

and factorized dualities (27). To analyze results following from application of Poisson–Lie identities \( I_1 \) and \( I_2 \) it is helpful to find their decomposition into products of special elements of NATD group. Depending on the values of parameters the matrices can be written as

\[
I_1 = \begin{cases}
I_{A'} \cdot I_B \cdot I_\beta & \text{for } c_{36} \neq 0 \\
I_A \cdot I_B \cdot I_\beta & \text{for } c_{36} = 0, c_{32} \neq 0 \\
I_A \cdot I_B \cdot I_F & \text{for } c_{36} = c_{32} = 0
\end{cases}
\]

and

\[
I_2 = \begin{cases}
I_A \cdot I_B \cdot I_\beta & \text{for } c_{35} \neq 0 \\
I_A \cdot I_B \cdot I_\beta & \text{for } c_{35} = 0, c_{32} \neq 0 \\
I_A \cdot I_B \cdot I_F & \text{for } c_{35} = c_{32} = 0
\end{cases}
\]

for some \( I_A, I_A', I_B, I_\beta \) and \( I_F \). The parameters rising from \( I_A \) and \( I_A' \) can be again eliminated by coordinate transformation (16). It is thus sufficient to discuss backgrounds obtained from \( I_B, I_\beta, I_F \) and their products. Multiplying these matrices by \( D_0 \) we get Poisson–Lie dualities.

6.2 Transformed backgrounds

6.2.1 B-shifts

Transformed background

\[
\tilde{F}(t, x_1) = \begin{pmatrix}
-e^{-\Phi(t)} a_1(t)^2 a_2(t)^4 & 0 & 0 & 0 \\
0 & b_{12} e^{-x_1} & -b_{12} e^{-x_1} & -b_{13} e^{x_1} \\
0 & b_{13} e^{x_1} & b_{23} & -b_{23}
\end{pmatrix}.
\]

(49)

given by B-shift differs from original \( F \) by a torsionless B-field and together with the dilaton \( \Phi = c_1 t \) satisfies beta function equations.

Coordinate shifts eliminate \( b_{12}, b_{13} \) in the dual obtained from \( D_0 \cdot I_B \) so it reads

\[
\tilde{F}(t, y_2, y_3) = \begin{pmatrix}
-e^{-\Phi(t)} a_1^4 a_2^2 & a_1^2 a_2 & -\frac{a_1^2 + a_2}{\Delta} & a_2 a_1 \\
0 & a_1^2 a_2 & a_1^2 a_2 + a_1 & a_1 a_2 \\
0 & a_1^2 a_2 & a_1^2 a_2 + a_1 & a_1 a_2 \\
0 & a_1 & a_2 & \frac{a_1^2}{\Delta}
\end{pmatrix}
\]

where

\[
\Delta = \left( a_2(t)^4 + b_{23}^2 \right) a_1(t)^2 + a_2(t)^2 \left( y_2^2 + y_3^2 \right).
\]

The constant \( b_{23} \) remains. For dilaton

\[
\tilde{\Phi}(t, y_2, y_3) = c_1 t - \frac{1}{2} \ln \Delta
\]

beta function equations are satisfied. Vanishing of vector \( J \) corresponds to the fact that structure constants of \( B_{V_{L-1}} \) are traceless.

6.2.2 Beta-shifts

Background given by \( \beta \)-shift is

\[
\tilde{F}(t, x_1) = \begin{pmatrix}
-e^{-\Phi(t)} a_1(t)^2 a_2(t)^4 & 0 & 0 & 0 \\
0 & a_1(t)^2 & 0 & 0 \\
0 & 0 & \frac{e^{-2a_1(t) a_1(t)^2} a_1^2}{\beta_{a_1(t)}^2} & 0 \\
0 & 0 & \frac{e^{-2a_1(t) a_1(t)^2} a_1^2}{\beta_{a_1(t)}^2} & \beta_{a_1(t)}^2 + 1
\end{pmatrix}
\]

(50)

and together with the dilaton calculated by formula (22)

\[
\tilde{\Phi}(t) = c_1 t - \frac{1}{2} \ln \left( \beta_{a_1(t)}^2 a_2(t)^4 + 1 \right)
\]
satisfy beta function equations. Although matrices $I_B$ and $I_\beta$ do not commute, backgrounds obtained from $I_B \cdot I_\beta$ and $I_\beta \cdot I_B$ are the same and differ from $\mathcal{F}$ in (50) only by a torsionless B-field.

The dual obtained from $D_0 \cdot I_\beta$ is

$$\mathcal{F}(t, \hat{x}_2, \hat{x}_3) = \begin{pmatrix}
  -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\
  0 & a_2^2 - \frac{1}{\Delta} & a_2^2(a_1^2 + a_2^2) & \frac{1}{\Delta} a_2^2 \frac{a_1^2}{a_2} \\
  0 & \frac{1}{\Delta} a_2^2 \frac{a_1^2}{a_2} & a_2^2(1 + a_2^2) & \frac{1}{\Delta} a_2^2(a_1^2 + a_2^2) \\
  0 & a_2^2(1 + a_2^2) & \frac{1}{\Delta} a_2^2(a_1^2 + a_2^2) & a_2^2(1 + a_2^2)
\end{pmatrix}$$

and with dilaton

$$\tilde{\Phi}(t, \hat{x}_2, \hat{x}_3) = c_1 t - \frac{1}{2} \ln \left( a_2^2(t)^2 \left( a_1(t)^2 + a_2(t)^2 \right)^2 + \hat{x}_2^2 + \hat{x}_3^2 \right)$$

they satisfy beta function equations. Tensors $\mathcal{F}$ arising from $D_0 \cdot I_B$ and $D_0 \cdot I_\beta$ are too extensive to be displayed here. Nevertheless, it is straightforward to calculate them and verify that together with corresponding dilatons they satisfy beta function equations.

$$\mathcal{F}(t, \hat{y}_2, \hat{y}_3) = \begin{pmatrix}
  -e^{-4\Phi(t)}a_1^2a_2^4 & 0 & 0 & 0 \\
  0 & \frac{b_2^2a_2 + 1}{\Delta} & \frac{a_2^2(b_2a_2^2 + \hat{y}_2)}{\Delta} & \frac{a_2^2(b_2a_2^2 + \hat{y}_3)}{\Delta} \\
  0 & \frac{a_2^2(b_2a_2^2 + \hat{y}_2)}{\Delta} & \frac{a_2^2(a_1^2 + a_2^2)}{\Delta} & \frac{a_2^2(b_2a_2^2 + \hat{y}_3)}{\Delta} \\
  0 & \frac{a_2^2(b_2a_2^2 + \hat{y}_3)}{\Delta} & \frac{a_2^2(a_1^2 + a_2^2)}{\Delta} & \frac{a_2^2(b_2a_2^2 + \hat{y}_3)}{\Delta}
\end{pmatrix}$$

where

$$\Delta = a_1^2 + a_2^2 \left( \hat{x}_2^2 + \hat{x}_3^2 \right)$$

This background and dilaton

$$\tilde{\Phi}(t, \hat{y}_2, \hat{y}_3) = c_1 t - \frac{1}{2} \ln \Delta$$

satisfy beta function equations. Background

6.2.3 Factorized dualities

Poisson–Lie identity (27), interpreted as Buscher duality with respect to symmetry generated by $V_2$ and $V_3$, produces metric (45) with $\kappa = -1$ and functions $a_2(t) = a_3(t)$ given by (47). Dilaton calculated by the formula (20)

$$\tilde{\Phi}(t) = c_1 t - \frac{1}{2} \ln a_2(t)^2 = -(c_1 + p_2) t + const.$$ 

satisfies beta function equations. Background obtained from $I_B \cdot I_F$ differs from this $\mathcal{F}$ only by a torsionless B-field that is the same as in (49). Let us note that for $c_1 = -p_2$ the metric is Ricci flat.

Dual background produced by $D_0 \cdot I_F$ reads

$$\Delta = \left( b_2 a_2(t)^4 + 1 \right) a_1(t)^2 + a_2(t)^2 \left( \hat{y}_2^2 + \hat{y}_3^2 \right)$$

is obtained from $D_0 \cdot I_B \cdot I_F$ and with dilaton

$$\tilde{\Phi}(t, \hat{y}_2, \hat{y}_3) = c_1 t - \frac{1}{2} \ln \Delta$$

it satisfies beta function equations.

7 Bianchi II cosmology

Lie algebra $\mathfrak{d} = \mathfrak{b}_{11}$ of the Drinfel’d double $\mathcal{D} = \mathfrak{B}_{11}(\mathcal{A})$ is spanned by basis $(T_1, T_2, T_3, \bar{T}_1, \bar{T}_2, \bar{T}_3)$ where nontrivial commutation relations of $\mathfrak{b}_{11}$ are

$$[T_2, T_3] = T_1.$$ 

(51)

Trace of structure constants is zero and group $\mathcal{B}_{11}$ is not semisimple.
Cosmology invariant with respect to symmetry generated by left-invariant vector fields

\[ V_1 = \partial_{x_1}, \quad V_2 = -x_3 \partial_{x_1} + \partial_{x_2}, \quad V_3 = \partial_{x_3} \]

satisfying (51) is given by the metric

\[ \mathcal{F}(t, x_2) = \begin{pmatrix}
-e^{-4\Phi(t)} a_1(t)^2 a_2(t)^2 a_3(t)^2 & 0 & 0 \\
0 & a_1(t)^2 & 0 \\
0 & 0 & a_2(t)^2 \\
0 & a_1(t)^2 x_2 & 0
\end{pmatrix} \]

where the functions \( a_i(t) \) are

\[ a_1(t) = e^{\Phi(t)} \sqrt{\frac{p_1}{\cosh(p_1 t)}}, \]

\[ a_2(t) = e^{\Phi(t) + \frac{p_2}{2}} \sqrt{\cosh(p_1 t)}, \]

\[ a_3(t) = e^{\Phi(t) + \frac{p_2}{2}} \sqrt{\cosh(p_1 t)} \]

as in [13]. For dilaton \( \Phi(t) = c_1 t \) the beta function equations reduce to

\[ 4c_1^2 = p_3 p_2 - p_1. \]

The background is torsionless and for \( c_1 = 0 \) also Ricci flat.

There are no factorized dualities satisfying (6) and (7). Poisson–Lie dualities can be obtained from identities by left-multiplication by the matrix (8) representing full duality.

7.2 Transformed backgrounds

We already know that if Poisson–Lie identity decomposes as in (54), coordinate transformations can eliminate parameters of \( I_A \) in the resulting backgrounds. Thus it is sufficient to investigate the effects of \( I_B, I_B \) and their products.

7.2.1 B-shifts

Structure coefficients of Manin triple \((\delta, b_{IJ}, a)\) remain invariant under B-shift (48) that transforms the background (52) to

\[ \widetilde{\mathcal{F}}(t, x_2) = \begin{pmatrix}
-e^{-4\Phi(t)} a_1^2 a_2^2 a_3^2 & 0 & 0 \\
0 & a_1^2 & -b_{12} a_2^2 - b_{13} a_3^2 \\
0 & b_{12} & b_{23} - b_{12} x_2 \\
0 & x_2 a_1^2 + b_{13} & a_3^2 + a_1^2 x_2
\end{pmatrix}. \]

7.1 Poisson–Lie identities and dualities

Unfortunately, we are not able to display general forms of matrices generating Poisson–Lie identities of Manin triple \((\delta, b_{IJ}, a)\) because the expressions are too extensive. However, we were able to decompose them into products of automorphisms, B-shifts and \( \beta \)-shifts. To be more specific, all the solutions can be written in one of the two forms

\[ I_1 = I_A \cdot I_B \cdot I_\beta, \quad I_2 = I_A \cdot I_\beta \cdot I_B \]

(54)

where automorphisms have the form (13) with

\[ A = \begin{pmatrix}
\Lambda & 0 & 0 \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}, \quad \Lambda = \det \begin{pmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{pmatrix}, \]

B-shifts are generated by matrix (48), and \( \beta \)-shifts are given by

\[ \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \beta_{12} & \beta_{13} & 1 & 0 \\
-\beta_{12} & 0 & 0 & 1 & 0 \\
-\beta_{13} & 0 & 0 & 0 & 1
\end{pmatrix}. \]

Up to gauge transformation of the antisymmetric part it is equivalent to (52). Together with dilaton \( \tilde{\Phi}(t) = c_1 t \) the background satisfies beta function equations.

Dependence on \( b_{23} \) can be eliminated in background obtained from \( D_B = D_0 \cdot I_B \) and we have

\[ \tilde{\mathcal{F}}(t, y_1) = \begin{pmatrix}
-e^{-4\Phi(t)} a_1^2 a_2^2 a_3^2 & 0 & 0 \\
0 & b_{12} a_2^2 - b_{13} a_3^2 & 0 \\
0 & b_{23} - b_{12} x_2 & a_3^2 + a_1^2 x_2
\end{pmatrix}. \]

where

\[ \Delta = a_1(t)^2 \left( a_2(t)^2 a_3(t)^2 + \frac{\gamma_1^2}{2} \right) + b_{12} a_2(t)^2 + b_{13} a_3(t)^2. \]

With dilaton

\[ \tilde{\Phi}(t, \gamma_1) = c_1 t - \frac{1}{2} \ln \Delta \]
given by (20) the beta function equations are satisfied. Results for the dual B-shift differ from the canonical dual obtained by $D_0$ not only by a shift in $y_1$ but also by other terms depending on $b_{12}, b_{13}$.

7.2.2 $\beta$-shifts

Let us now investigate the transformation of metric (52) given by $\beta$-shift (55). This Poisson–Lie identity generates

$$
\tilde{F}(t, x_2) = \begin{pmatrix}
-e^{-4\Phi(t)} a_1^2 a_2^2 a_3^2 & 0 & 0 & a_1^2 a_2^2 \beta_{12} & a_1^2 a_2^2 (\beta_{13} a_1^2 + a_2^2) \\
0 & a_1^2 a_2^2 \beta_{12} & a_2^2 (a_1^2 a_2^2 + a_1^2 a_2^2) & 0 & a_1^2 a_2^2 (\beta_{13} a_1^2 + a_2^2) \\
0 & a_1^2 (x_2 - a_1^2 \beta_{13}) & a_2^2 \beta_{12} (x_2 - a_1^2 \beta_{13}) & a_2^2 (a_1^2 a_2^2 + a_1^2 a_2^2) & a_2^2 (\beta_{13} a_1^2 + a_2^2) \\
0 & a_1^2 (x_2 - a_1^2 \beta_{13}) & a_2^2 \beta_{12} (x_2 - a_1^2 \beta_{13}) & a_2^2 (a_1^2 a_2^2 + a_1^2 a_2^2) & a_2^2 (\beta_{13} a_1^2 + a_2^2)
\end{pmatrix}
$$

where

$$
\Delta = \left( a_2(t)^2 \beta_{12} + a_3(t)^2 \beta_{13} \right) a_1(t)^2 + 1.
$$

Together with corresponding dilaton

$$
\tilde{\Phi}(t) = c_1 t - \frac{1}{2} \ln \Delta
$$

the background satisfies beta function equations. Poisson–Lie identity $I_B$ acting on this background adds a torsionless B-field, so we conclude that backgrounds obtained from $I_B \cdot I_B$ and $I_B$ can be considered equivalent. Despite the fact that $I_B$ and $I_B$ do not commute, $\tilde{F}$ obtained from $I_B \cdot I_B$ is exactly the same as for $I_B \cdot I_B$.

Dual background resulting from $D_\beta = D_0 \cdot I_B$ is

$$
\tilde{F}(t, \tilde{x}_1) = \begin{pmatrix}
-e^{-4\Phi(t)} a_1^2 a_2^2 a_3^2 & 0 & a_1^2 a_2^2 (\beta_{13} \tilde{x}_1^2 - \beta_{12} \tilde{x}_1^2) & -a_1^2 a_2^2 (\beta_{13} \tilde{x}_1^2 + \beta_{12} \tilde{x}_1^2) & 0 \\
0 & a_1^2 a_2^2 (\beta_{13} \tilde{x}_1^2 + \beta_{12} \tilde{x}_1^2) & a_2^2 (\beta_{13} \tilde{x}_1^2 - \beta_{12} \tilde{x}_1^2) & -a_2^2 (\beta_{13} \tilde{x}_1^2 + \beta_{12} \tilde{x}_1^2) & 0 \\
0 & a_1^2 a_2^2 (\beta_{13} \tilde{x}_1^2 + \beta_{12} \tilde{x}_1^2) & a_2^2 (\beta_{13} \tilde{x}_1^2 - \beta_{12} \tilde{x}_1^2) & -a_2^2 (\beta_{13} \tilde{x}_1^2 + \beta_{12} \tilde{x}_1^2) & 0
\end{pmatrix}
$$

The dilaton is

$$
\tilde{\Phi}(t, \tilde{x}_1) = c_1 t - \frac{1}{2} \ln \left( a_1(t)^2 \left( a_2(t)^2 a_3(t)^2 + \tilde{x}_1^2 \right) \right)
$$

and it is interesting that it does not depend on $\beta_{12}$ and $\beta_{13}$. Together they satisfy beta function equations.

Dual backgrounds and dilatons found from $D_0 \cdot I_B \cdot I_B$ and $D_0 \cdot I_B \cdot I_B$ are too complicated to display and not particularly illuminating so we omit them here. Nevertheless, one can check that they satisfy beta function equations.

8 Conclusions

We have identified general forms of Poisson–Lie identities and Poisson–Lie dualities for six-dimensional semi-Abelian Manin triples $\mathcal{D} = \mathcal{B} + \mathcal{A}$ where $\mathcal{B}$’s are Bianchi algebras that generate isometries of Bianchi cosmologies. Subsequently we have managed decomposing both the Poisson–Lie identities and Poisson–Lie dualities into simple factors, namely automorphisms of Manin triples, B-shifts, $\beta$-shifts and “full” or “factorized” dualities. This supports the conjecture posed in [21] that NATD group is generated by these elements. Finally, we have used these decompositions to transform Bianchi cosmologies supplemented by dilaton fields [12]. For these transformations we used Poisson–Lie T-plurality and dilaton formula described in Sect. 2.

In this way we have obtained many new backgrounds and corresponding dilatons that solve the Generalized Supergravity Equations, and confirmed that the Killing vector $\mathcal{F}$ in Generalized Supergravity Equations is given by trace of structure constants [13]. One must, however, carefully evaluate what groups, more precisely what subgroups of Drinfel’d double, truly participate in the transformation since it influences resulting Killing vector. For factorized dualities these subgroups often become Abelian and the Generalized Supergravity Equations reduce to standard beta function equations. New backgrounds, dilatons and corresponding Killing vectors are summarized in the Tables in the Appendix. The backgrounds obtained by Poisson–Lie identities are again invariant with respect to Bianchi groups.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Paper is self-contained, all results can be reproduced using information contained in the text.]

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Appendix

For reader’s convenience we recapitulate backgrounds and dilatons yielded from Poisson–Lie identities and dualities in the following tables. We add vector $\mathcal{J}$ as well to indicate whether the backgrounds satisfy standard beta function equations (in which case $\mathcal{J} = 0$) or Generalized Supergravity Equations (17)–(19). In the first column we display which one of the special transformations was used to get the result. Automorphisms $IA$ are not mentioned. Nevertheless, since we want to include results obtained from general Poisson–Lie identities and dualities, some parameters appearing in the tensors may arise from automorphisms. We recommend to check details in previous sections (Tables 2, 3, 4, 5, 6).

### Table 2 Results for Poisson–Lie identities and dualities of Bianchi $II$ cosmology. Functions $a_i(t)$ are given by (53) and $\Phi(t) = c_1 t$

| $\mathcal{B}_{II}$ | Transformed backgrounds, dilatons and vectors $\mathcal{J}$ |
|---------------------|----------------------------------------------------------|
| $IB$                | $\tilde{F}(t, x_2) = \begin{pmatrix} e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -b_{12} & a_1x_2 - b_{13} \\ 0 & b_{12} & a_3^2 & b_{12}x_2 - b_{23} \\ 0 & x_2a_1^2 + b_{13} & b_{23} - b_{12}x_2 & a_3^2 + a_1^2x_2^2 \end{pmatrix}$ |
| $\tilde{\Phi}(t) = c_1 t$, $\mathcal{J} = 0$ |
| $DB$                | $\tilde{F}(t, y_1) = \begin{pmatrix} e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & b_{12}a_1^2 - b_{13}a_2^2 & b_{12}a_1a_2 + b_{13}a_1a_3 \\ 0 & b_{12}a_1^2 + b_{13}a_2^2 & a_3^2 & b_{12}a_2a_3 + b_{13}a_2a_3 \\ 0 & a_1^2b_{12}a_1^2 + b_{13}a_2^2 & a_3^2 & b_{12}a_2a_3 + b_{13}a_2a_3 \end{pmatrix}$ |
| $\Delta = a_1(t)^2 (a_3(t)^2 a_3(t)^2 + \tilde{y}_1^2) + b_{12}^2 a_2(t)^2 + b_{13}^2 a_3(t)^2$ |
| $\tilde{\Phi}(t, \tilde{y}_1) = c_1 t \ln \Delta$, $\mathcal{J} = 0$ |
| $IB$                | $\tilde{F}(t, x_2) = \begin{pmatrix} e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 \Delta & a_1^2a_2^2 & a_1^2(a_2^4 + a_1^4) \\ 0 & a_1^2a_2^2 & a_3^2 \Delta & a_1^2(a_3^4 + a_1^4) \\ 0 & a_1^2(a_3^4 + a_1^4) & a_3^2(a_3^4 + a_1^4) & a_3^2(a_3^4 + a_1^4) \end{pmatrix}$ |
| $\Delta = (a_2(t)^2 a_1^2 + a_3(t)^2 a_3^2) a_1(t)^2 + 1$ |
| $\tilde{\Phi}(t) = c_1 t - \frac{1}{2} \ln \Delta$, $\mathcal{J} = 0$ |
| $DB$                | $\tilde{F}(t, x_1) = \begin{pmatrix} e^{-4\Phi(t)}a_1^2a_2^2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2(a_3^4 + a_1^4) & a_1^2(a_2^4 + a_1^4) & a_1^2(a_3^4 + a_1^4) \\ 0 & a_1^2(a_3^4 + a_1^4) & a_1^2(a_2^4 + a_1^4) & a_1^2(a_3^4 + a_1^4) \\ 0 & a_1^2(a_3^4 + a_1^4) & a_1^2(a_2^4 + a_1^4) & a_1^2(a_3^4 + a_1^4) \end{pmatrix}$ |
| $\tilde{\Phi}(t, x_1) = c_1 t - \frac{1}{2} \ln \left(a_1(t)^2 a_2(t)^2 a_3(t)^2 + \tilde{x}_1^2\right)$, $\mathcal{J} = 0$ |
Table 3: Results for Poisson–Lie identities and dualities of Bianchi III cosmology and dilaton $\Phi = 0$

| $I_{III}$ | Transformed backgrounds, dilatons and vectors $\mathcal{J}$ |
|-----------|-------------------------------------------------------------|
| $I_B$     | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & -c_{15} & -c_{16} e^{-x_1} \\ 0 & c_{16} e^{-x_1} & 0 & 0 \\ \Phi = 0, & \mathcal{J} = 0 \end{pmatrix}$ |
| $D_B$     | $\tilde{\mathcal{F}}(t, \tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & -c_{12} & -c_{13} e^{-x_1} \\ 0 & c_{13} e^{-x_1} & 0 & 0 \\ \tilde{\Phi}(t, \tilde{x}_3) = \frac{1}{2} \ln \left(t^2 + c_{15} t^2 + x_3^2\right), & \mathcal{J} = 0 \end{pmatrix}$ |
| $I_{F_1}$ | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & c_{15} e^{x_1} & -c_{16} e^{x_1} \\ 0 & c_{16} e^{x_1} & 0 & 0 \\ \tilde{\Phi}(t, x_1) = -\ln t - x_1, & \mathcal{J} = 0 \end{pmatrix}$ |
| $D_{F_1}$ | $\tilde{\mathcal{F}}(t, \tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & c_{15} e^{x_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{\Phi}(t, \tilde{x}_3) = \frac{1}{2} \ln \left(t^2 + c_{15} t^2 + x_3^2\right), & \mathcal{J} = 0 \end{pmatrix}$ |

Table 4: Results for Poisson–Lie identities and dualities of Bianchi V cosmology and dilaton $\Phi = 0$

| $I_B$ | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & e^{x_1} c_{15} & -e^{x_1} c_{16} \\ 0 & e^{x_1} c_{16} & 0 & 0 \\ \Phi = 0, & \mathcal{J} = 0 \end{pmatrix}$ |
| $D_B$ | $\tilde{\mathcal{F}}(t, \tilde{x}_2, \tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & -e^{y_1} c_{12} & -e^{y_1} c_{13} \\ 0 & e^{y_1} c_{13} & 0 & 0 \\ \tilde{\Phi}(t, \tilde{x}_2, \tilde{x}_3) = \frac{1}{2} \ln \left(t^2 + x_2^2 + x_3^2 + r^2\right), & \mathcal{J} = 0 \end{pmatrix}$ |
| $I_F$ | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & e^{x_1} c_{12} & -e^{x_1} c_{13} \\ 0 & e^{x_1} c_{13} & 0 & 0 \\ \tilde{\Phi}(t, x_1) = -2 \ln t + 2 x_1, & \mathcal{J} = 0 \end{pmatrix}$ |
| $D_F$ | $\tilde{\mathcal{F}}(t, \tilde{x}_2, \tilde{x}_3) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & -e^{y_1} c_{12} & -e^{y_1} c_{13} \\ 0 & e^{y_1} c_{13} & 0 & 0 \\ \tilde{\Phi}(t, \tilde{x}_2, \tilde{x}_3) = \frac{1}{2} \ln \left(t^2 + x_2^2 + x_3^2 + r^2\right), & \mathcal{J} = 0 \end{pmatrix}$ |
Table 5 Results for Poisson–Lie identities and dualities of Bianchi $V_{Ic}$ cosmology. Functions $a_1(t)$ are given by (44), $\kappa \neq -1$ and $\Phi(t) = c_1 t$

| $\mathcal{B}_{V_{Ic}}$ | Transformed backgrounds, dilatons and vectors $\mathcal{J}$ |
|-------------------------|--------------------------------------------------|
| $I_B$                   | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -e^{x_1}c_{15} & -e^{x_1}c_{16} \\ 0 & e^{x_1}c_{15} & e^{2x_1}a_2^2 & 0 \\ 0 & e^{x_1}c_{16} & 0 & e^{3x_1}a_3^2 \end{pmatrix}$ |
|                         | $\tilde{\Phi}(t) = c_1 t,$ $\mathcal{J} = 0$ |
| $D_B$                   | $\tilde{\mathcal{F}}(t, \tilde{x}_2, \tilde{x}_3) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & 0 & a_1^2 \frac{\kappa}{\Delta} \\ 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} & a_1^2 \frac{1}{\Delta} & 0 \\ 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} & 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} \end{pmatrix}$ |
|                         | $\mathcal{J} = 0$ |
| $I_F$                   | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -e^{x_1}c_{12} & -e^{x_1}c_{13} \\ 0 & e^{x_1}c_{12} & a_1^2 & 0 \\ 0 & e^{x_1}c_{13} & 0 & a_1^2 \right)$ |
|                         | $\tilde{\Phi}(t, x_1) = c_1 t + (\kappa + 1)x_1 - \ln (a_2(t)a_3(t)),$ $\mathcal{J} = 0$ |
| $D_F$                   | $\tilde{\mathcal{F}}(t, \tilde{x}_2, \tilde{x}_3) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta} & \frac{\kappa a_1^2}{\Delta} & \frac{a_1^2}{\Delta} \\ 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} & a_1^2 & 0 \\ 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} & 0 & -\frac{a_1^2 + \kappa a_1^2}{\Delta} \end{pmatrix}$ |
|                         | $\mathcal{J} = 0$ |

Table 6 Results for Poisson–Lie identities and dualities of Bianchi $V_{I_{-1}}$ cosmology. Functions $a_1(t)$ are given by (47) and $\Phi(t) = c_1 t$

| $\mathcal{B}_{V_{I_{-1}}}$ | Transformed backgrounds, dilatons and vectors $\mathcal{J}$ |
|-----------------------------|--------------------------------------------------|
| $I_B$                       | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & -b_{12}e^{-x_i} & -b_{15}e^{x_i} \\ 0 & b_{12}e^{-x_i} & e^{-2x_i}a_2^2 & -b_{23} \\ 0 & b_{15}e^{x_i} & b_{23} & e^{2x_i}a_2^2 \end{pmatrix}$ |
|                            | $\tilde{\Phi}(t) = c_1 t,$ $\mathcal{J} = 0$ |
| $D_B$                       | $\tilde{\mathcal{F}}(t, \tilde{y}_2, \tilde{y}_3) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & \frac{a_1^2 + b_{12}^2}{\Delta} & \frac{\gamma_2a_2 + b_{23}a_3}{\Delta} & \frac{a_{21}^2 - b_{23}^2}{\Delta} \\ 0 & \frac{a_1^2 + b_{15}^2}{\Delta} & \frac{\gamma_2a_2 + b_{23}a_3}{\Delta} & \frac{a_{21}^2 - b_{23}^2}{\Delta} \\ 0 & -\frac{\gamma_1a_2 + b_{23}a_3}{\Delta} & \frac{\gamma_1a_2 + b_{23}a_3}{\Delta} & \frac{a_{21}^2 + \frac{a_1^2}{2}}{\Delta} \\ 0 & -\frac{\gamma_1a_2 + b_{23}a_3}{\Delta} & \frac{\gamma_1a_2 + b_{23}a_3}{\Delta} & \frac{a_{21}^2 + \frac{a_1^2}{2}}{\Delta} \end{pmatrix}$ |
|                            | $\mathcal{J} = 0$ |
| $I_F$                       | $\tilde{\mathcal{F}}(t, x_1) = \begin{pmatrix} -e^{-4t}a_1^2d_2a_3^2 & 0 & 0 & 0 \\ 0 & a_1^2 & 0 & 0 \\ 0 & 0 & e^{2x_1}a_2^2 & \frac{\gamma_2a_2 + b_{23}a_3}{\Delta} \\ 0 & 0 & -\frac{\gamma_1a_2 + b_{23}a_3}{\Delta} & e^{2x_1}a_3^2 \end{pmatrix}$ |
|                            | $\tilde{\Phi}(t) = c_1 t - \frac{1}{2} \ln (b_{12}a_2(t)^4 + 1),$ $\mathcal{J} = 0$ |
Table 6 continued

| $\mathcal{F}_{VL-1}$ | Transformed backgrounds, dilatons and vectors $\mathcal{F}$ |
|----------------------|---------------------------------|
| $D_B$                | $\tilde{\mathcal{F}}(t, \tilde{x}_2, \tilde{x}_3) = \left(\begin{array}{cccc}
0 & a_2^d & 0 & 0 \\
-a_2^d & 0 & a_2^d & 0 \\
a_2^d & -a_2^d & 0 & a_2^d \\
a_2^d & -a_2^d & -a_2^d & 0
\end{array}\right)$ |
| $I_F$                | $\tilde{\Phi}(t, \tilde{x}_2, \tilde{x}_3) = c_1t - \frac{1}{2} \ln \left( a_2(t)^2 \left( a_2(t)^2 + x_2^2 + x_3^2 \right) \right)$, $\mathcal{J} = 0$ |
| $D_F$                | $\mathcal{F}(t, x_4) = \left(\begin{array}{cc}
0 & 0 \\
-a_2^d & a_2^d
\end{array}\right)$ |

References

1. T.H. Buscher, A symmetry of the string background field equations. Phys. Lett. B 194, 51 (1987)
2. X.C. de la Ossa, F. Quevedo, Duality symmetries from non-abelian isometries in string theories. Nucl. Phys. B 403, 377 (1993). arXiv:hep-th/9210053
3. M. Roček, E. Verlinde, Duality, quotients, and currents. Nucl. Phys. B 373, 630 (1992). arXiv:hep-th/9110053
4. K. Sfetsos, D.C. Thompson, On non-abelian T-dual geometries with Ramond fluxes. Nucl. Phys. B 846, 21 (2011). arXiv:1012.1320
5. Y. Lozano, E.O. Colgáin, K. Sfetsos, D.C. Thompson, Non-abelian T-duality, Ramond fields and coset geometries. JHEP 06, 106 (2011). arXiv:1104.5196
6. G. Itsios, Y. Lozano, J. Montero, C. Núñez, The $AdS_5$ non-abelian T-dual of Klebanov–Witten as a $N = 1$ linear quiver from M5-branes. JHEP 09, 38 (2017). arXiv:1705.09961
7. G. Itsios, H. Nastase, C. Núñez, K. Sfetsos, S. Zacarías, Penrose limits of Abelian and non-Abelian T-duals of $AdS_5 \times S^5$ and their field theory duals. JHEP 01, 71 (2018). arXiv:1711.09911
8. R. Borsato, L. Wulff, Integrable deformations of T-dual sigma-models. Phys. Rev. Lett. 117, 251602 (2016). arXiv:1609.09834
9. B. Hoare, A.A. Tseytlin, Homogeneous Yang–Baxter deformations as non-abelian duals of the $AdS_5$ sigma-model. J. Phys. A 49, 494001 (2016). arXiv:1609.02550v3
10. R. Borsato, L. Wulff, On non-abelian T-duality and deformations of supercoset string sigma-models. JHEP 10, 24 (2017). arXiv:1706.10169
11. L. Hlaváč, J. Petri, Poisson-Lie T-plurality revisited. Is T-duality unique? JHEP 04, 157 (2019). arXiv:1811.12235
12. N.A. Bataikis, A.A. Kehagias, Anisotropic space-times in homogeneous string cosmology. Nucl. Phys. B 449, 248 (1995). arXiv:hep-th/9502007
13. M. Honga, Y. Kima, E.O. Colgáin, On non-Abelian T-duality for non-semisimple groups. Eur. Phys. J. C 78, 1025 (2018). arXiv:1801.09567
14. C. Klimek, P. Severa, Dual non-abelian duality and the Drinfeld double. Phys. Lett. B 351, 455 (1995). arXiv:hep-th/9502122
15. C. Klimek, Poisson-Lie T-duality. Nucl. Phys. Proc. Suppl. 46, 116 (1996). arXiv:hep-th/9509095
16. P. Bouwknegt, M. Burgdörfer, C. Klimek, K. Wright, Hidden isometry of “T-duality without isometry”. JHEP 08, 116 (2017). arXiv:1705.09254
17. R. von Unge, Poisson-Lie T-plurality. JHEP 07, 014 (2002). arXiv:hep-th/0205245
18. X. Gomez, Classification of three-dimensional Lie bialgebras. J. Math. Phys. 41, 4939 (2000)
19. L. Hlavatý, L. Šnobl, Classification of Poisson–Lie T-dual models with two-dimensional targets. Mod. Phys. Lett. A 17, 429 (2002). arXiv:hep-th/0110139
20. L. Šnobl, L. Hlavatý, Classification of 6-dimensional linear Drinfeld’s doubles. Int. J. Mod. Phys. A 17, 4043 (2002). arXiv:math.QA/0202209
21. D. Lüst, D. Osten, Generalised fluxes, Yang–Baxter deformations and the $(d, d)$ structure of non-abelian T-duality. JHEP 05, 165 (2018). arXiv:1803.03971
22. S. Frolov, Lax pair for strings in Lunnin–Maldacena background. JHEP 05, 069 (2005). arXiv:hep-th/0503201
23. D. Osten, S.J. van Tongeren, Abelian Yang–Baxter deformations and T-fold transformations. Nucl. Phys. B 915, 184 (2017). arXiv:1608.08504
24. E. Álvarez, L. Álvarez-Gaumé, Y. Lozano, On non-abelian duality. Nucl. Phys. B 424, 155 (1994). arXiv:hep-th/9403155v4
25. J.J. Fernandez-Melgarejo, J. Sakamoto, Y. Sakatani, K. Yoshida, T-folds from Yang–Baxter deformations. JHEP 12, 108 (2017). arXiv:1710.06849
26. T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari, K. Yoshida, Yang-Baxter $\sigma$-models, conformal twists,
and noncommutative Yang–Mills theory. Phys. Rev. D 95 (2017).

27. T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari, K. Yoshida, Conformal Twists, Yang-Baxter $\sigma$-models and Holographic Noncommutativity. J. Phys. A 51 (2018). arXiv:1705.02063

28. T. Araujo, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari, K. Yoshida, I in generalized supergravity. Eur. Phys. J. C 77 (2017). arXiv:1710.03163

29. S. Demulder, F. Hassler, D.C. Thompson, Doubled aspects of generalised dualities and integrable deformations. JHEP 02, 189 (2019). arXiv:1810.11446

30. Y. Sakatani, Type II DFT solutions from Poisson–Lie T-duality/plurality. arXiv:1903.12175

31. L. Hlavatý, I. Petr, V. Štěpán, Poisson–Lie T-plurality with spectators. J. Math. Phys. 50, 043504 (2009)

32. L. Hlavatý, F. Petrášek, On uniqueness of T-duality with spectators. Int. J. Mod. Phys. A 31, 1650143 (2016). arXiv:1606.02522

33. F. Petrášek, L. Hlavatý, I. Petr, Plane-parallel waves as duals of the flat background II: T-duality with spectators. Class. Quantum Gravity 34, 155003 (2017). arXiv:1612.08015

34. S. Majid, Foundations of quantum group theory (Cambridge University Press, Cambridge, 1995)

35. G. Arutyunov, S. Frolov, B. Hoare, R. Roiban, A.A. Tseytlin, Scale invariance of the $\eta$-deformed $AdS_5 \times S^5$ superstring. T-duality and modified type II equations. Nucl. Phys. B 903, 262 (2016). arXiv:1511.05795

36. C.G. Callan Jr., E.J. Martinec, M.J. Perry, D. Friedan, Strings in background fields. Nucl. Phys. B 262, 593 (1985)

37. L. Hlavatý, L. Šnobl, Poisson–Lie T-plurality of three-dimensional conformally invariant sigma models II: nondiagonal metrics and dilaton puzzle. JHEP 10, 045 (2004). arXiv:hep-th/0408126

38. M. Gasperini, R. Ricci, G. Veneziano, A problem with non-Abelian duality? Phys. Lett. B 319, 438 (1993). arXiv:hep-th/9308112

39. S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer, G. Veneziano, Remarks on non-Abelian duality. Nucl. Phys. B 435, 147 (1995). arXiv:hep-th/9409011

40. G. Papadopoulos, J.G. Russo, A.A. Tseytlin, Solvable model of strings in a time-dependent plane-wave background. Class. Quantum Gravity 20, 969 (2003). arXiv:hep-th/0211289

41. M. Gasperini, R. Ricci, Homogeneous conformal string backgrounds. Class. Quantum Gravity 12, 677 (1995). arXiv:hep-th/9501055