From the Complete Yang Model to Snyder’s Model, de Sitter Special Relativity and Their Duality*

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By means of Dirac procedure, we re-examine Yang’s quantized space-time model, its relation to Snyder’s model, the $dS$ special relativity and their UV-IR duality. Starting from a dimensionless $dS_5$-space in a 5+1-d Mink-space a complete Yang model at both classical and quantum level can be presented and there really exist Snyder’s model, the $dS$ special relativity and the duality.

Keywords: Yang model, Snyder’s model, de Sitter special relativity, duality

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Recently, based on Yang’s quantized space-time model [1], we present a complete Yang model [2] from a 5+1-d space with Mink-signature mainly by projective geometry method. We have shown that in the model, there are Snyder’s model [3], the $dS$ special relativity [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and their UV-IR duality [4].

In this letter, we re-examine the same topic from a slightly different angle: Dirac procedure for the constrained systems [17]. Starting from a dimensionless $dS_5$-space of radius $R/a$ embedded in a dimensionless 5+1-d Mink-space, by means of Dirac procedure, Yang’s algebra can be given at both classical and quantum level. We show that Snyder’s model, the $dS$ special relativity and their duality can also be given from such a complete Yang model in a slightly different way with Dirac procedure.

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The dimensionless $dS_5$-space reads

$$\mathcal{H} : \quad \eta_{AB} \zeta^A \zeta^B = -\frac{R^2}{a^2}, \quad A, B = 0, 1, \cdots, 5,$$

$$d\chi^2 = \eta_{AB} d\zeta^A d\zeta^B$$

(1)\hspace{1cm} (2)

with $\eta_{AB} = \text{diag}(+, -, -, -, -, -)$, where $a$ and $R$ are of length dimension.

For a fictitious particle moving on $\mathcal{H}$ with a Lagrangian function

$$\mathcal{L} = -\sqrt{\eta_{AB} \dot{\zeta}^A \dot{\zeta}^B}, \quad \dot{\zeta}^A = \frac{d\zeta^A}{d\chi},$$

(3)

the ‘canonical momentum’ and the 6-d ‘angular momentum’ read, respectively

$$N_A = \frac{\partial \mathcal{L}}{\partial \dot{\zeta}^A} = -\eta_{AB} \frac{d\zeta^B}{d\chi},$$

$$\mathcal{L}^{AB} = \zeta^B N^A - \zeta^A N^B.$$  

(4)\hspace{1cm} (5)

Thus, there is a ‘phase space’ with $(\zeta^A, N_A)$ and Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \zeta^B} \frac{\partial g}{\partial N_A} - \frac{\partial f}{\partial N_B} \frac{\partial g}{\partial \zeta^B}.$$  

(6)

The Hamiltonian $H = N_B \dot{\zeta}^B - \mathcal{L}$ for the particle vanishes.

Note that there are two primary constraints: $\phi_1 = \eta_{AB} \zeta^A \zeta^B + \frac{R^2}{a^2} = 0$ from Eq. (1), and $\phi_2 = \eta^{AB} N_A N_B - 1 = 0$ as the ‘mass-shell’ condition from Eqs. (3) and (4). According to Dirac [17], the total Hamiltonian can be introduced as $H_T = H + \mu^m \phi_m, (m = 1, 2)$, and the dynamical consistency condition leads to

$$0 = \dot{\phi}_1 = \{\phi_1, H_T\} = 4\mu^2 N_B \zeta^B,$$

$$0 = \dot{\phi}_2 = \{\phi_2, H_T\} = -4\mu^1 N_B \zeta^B.$$  

(7)

Let $\phi_3 = N_B \zeta^B = 0$, then we get a secondary constraint $\phi_3$. Its dynamical consistency condition results in $\mu^2 = -\frac{R^2}{a^2} \mu^1$. Thus, the total Hamiltonian becomes $H_T = \mu^1 \left( \phi_1 - \frac{R^2}{a^2} \phi_2 \right)$. Set $\phi_0 = \phi_1 - \frac{R^2}{a^2} \phi_2 = 0$. Obviously, $\phi_0$ is of the first class and satisfies the consistency condition. Hence, there are one first-class constraint $\phi_0$ and two second-class constraints $\phi_1, \phi_3$ with Dirac bracket

$$\{f, g\}_{DB} := \{f, g\} - \{f, \phi_m\} C^{-1}_{mk} \{\phi_k, g\}, \quad m, k = 1, 3,$$

(8)

where $C$ is a reversible matrix, defined by

$$C = 2 \frac{R^2}{a^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

(9)
The basic non-vanishing Dirac brackets with Jacobi identity read

\[ \{ N^A, N^B \}_{DB} = -\frac{a^2}{R^2} \mathcal{L}^{AB}, \]

\[ \{ \zeta^A, N_B \}_{DB} = \delta^A_B + \frac{a^2}{R^2} \zeta^A \zeta_B. \]

Further, the 6-d ‘angular-momentum’ \( \mathcal{L}^{AB} \) forms an \( so(1, 5) \) algebra

\[ \{ \mathcal{L}^{AB}, \mathcal{L}^{CD} \}_{DB} = \eta^{AD} \mathcal{L}^{BC} + \eta^{BC} \mathcal{L}^{AD} - \eta^{AC} \mathcal{L}^{BD} - \eta^{BD} \mathcal{L}^{AC}, \]

and is conserved under the ‘canonical equations’

\[ \dot{\zeta}^A = \{ \zeta^A, H_T \}_{DB} = -2\mu^1 \frac{R^2}{a^2} \eta^{AB} N_B, \]

\[ \dot{N}_A = \{ N_A, H_T \}_{DB} = -2\mu^1 \eta_{AB} \zeta^B. \]

From the definition of \( N_A \), it follows \( \mu^1 = \frac{a^2}{2R^2} \).

Introduce a set of variables \((c = 1)\):

\[ x_0 = -a(\zeta^5 N_0 + \zeta^0 N_5) = a \mathcal{L}^{50}, \]

\[ x_i = a(\zeta^i N_5 - \zeta^5 N_i) = -a \mathcal{L}^{5i}, \]

\[ p_0 = -\frac{\hbar}{R}(\zeta^4 N_0 + \zeta^0 N_4) = \frac{\hbar}{R} \mathcal{L}^{40}, \]

\[ p_i = \frac{\hbar}{R}(\zeta^i N_4 - \zeta^4 N_i) = \frac{\hbar}{R} \mathcal{L}^{4i}, \]

\[ M_i = -\hbar(\zeta^0 N_i + \zeta^i N_0) = -\hbar \mathcal{L}^{0i}, \]

\[ L_i = -\hbar \epsilon_{ijk} (\zeta^j N^k) = \frac{1}{2} \hbar \epsilon_{ijk} \mathcal{L}^{jk}, \]

\[ \psi = \frac{a}{R}(\zeta^4 N^5 - \zeta^5 N^4) = \frac{a}{R} \mathcal{L}^{45}. \]

\( \epsilon_{ijk} \) is an anti-symmetric tensor with \( \epsilon_{123} = 1 \). The index is raised by \( \eta^{ij}, i, j = 1, 2, 3. \)
They satisfy Yang’s algebra under Dirac bracket:

\[
\begin{align*}
\{p^\mu, p^\nu\}_{DB} &= \frac{\hbar}{R^2} l^{\mu\nu}, \\
\{l^{\mu\nu}, p^\rho\}_{DB} &= \eta^{\rho\nu} p^\mu - \eta^{\mu\rho} p^\nu, \\
\{x^\mu, x^\nu\}_{DB} &= a^2 \hbar^{-1} l^{\mu\nu}, \\
\{l^{\mu\nu}, x^\rho\}_{DB} &= \eta^{\nu\rho} x^\mu - \eta^{\mu\rho} x^\nu, \\
\{x^\mu, p^\nu\}_{DB} &= \hbar \eta^{\mu\nu} \psi, \\
\{\psi, x^\mu\}_{DB} &= -a^2 \hbar^{-1} p^\mu, \\
\{\psi, p^\mu\}_{DB} &= \frac{\hbar^2}{R^2} x^\mu, \\
\{\psi, l^{\mu\nu}\}_{DB} &= 0,
\end{align*}
\]

with \(x^\mu = \eta^{\mu\nu} x_\nu\), \(p^\mu = \eta^{\mu\nu} p_\nu\), \(\mu, \nu = 0, \cdots, 3\), \(l^{0i} = -M_i\), and \(\frac{1}{2} \epsilon_{ijk} l^{jk} = L_i\).

Under the ‘canonical quantization’, i.e. replacing \(\zeta^A\) and \(N_B\) with the corresponding operators and the Dirac bracket \(\{ , \}_{DB}\) with \(-i[ , , ]\), the quantized commutator of \(\hat{\zeta}^A\) and \(\hat{N}_B\) reads

\[
[\hat{\zeta}^A, \hat{N}_B] = i(\delta_B^A + \frac{a^2}{R^2} \hat{\zeta}^A \hat{\zeta}_B).
\]

In the ‘coordinate picture’,

\[
\hat{N}_A = -i \left( \frac{\partial}{\partial \zeta^A} + \frac{a^2}{R^2} \zeta_A \zeta_B \frac{\partial}{\partial \zeta^B} \right),
\]

and an \(\mathfrak{so}(1,5)\) in commutator for the 6-d ‘angular-momentum’ follows

\[
[\hat{L}^{AB}, \hat{L}^{CD}] = \eta^{AD} \hat{L}^{BC} + \eta^{BC} \hat{L}^{AD} - \eta^{AC} \hat{L}^{BD} - \eta^{BD} \hat{L}^{AC}.
\]

Obviously, Yang’s algebra can be derived by replacing the variables in Eqs. (15) with corresponding operators.

Note that in Yang’s algebra at both the classical and quantum level there are two \(dS\) subalgebras for \(x^\mu\) (or \(\hat{x}^\mu\)) and \(p^\mu\) (or \(\hat{p}^\mu\)) with a common \(\mathfrak{so}(1,3)\) for \(l^{\mu\nu}\) (or \(\hat{l}^{\mu\nu}\)), respectively. There is an important property: Yang’s algebra with respect to the 6-d ‘angular momentum’ (5) is invariant under a \(\mathbb{Z}_2 = \{e, \tau \mid \tau^2 = e\}\) \(\{2\}\) with \(\tau\)

\[
\tau : \quad a \leftrightarrow \frac{\hbar}{R}, \quad x^\mu \leftrightarrow p^\mu, \quad \psi \leftrightarrow -\psi.
\]
If the universal constants $a$ and $R$ are of the Planck length $\ell_p$ and the cosmological radius $R = \sqrt{3/\Lambda}$, the $\mathbb{Z}_2$-duality is a UV-IR duality.

As was shown in [2], it is obvious that $dS_5$-space $\mathcal{H}$ contains two $dS$ sub-manifolds and both Snyder’s model and the $dS$ special relativity can be derived from the complete Yang model in $\mathcal{H}$.

In the subspace $\mathcal{I}_1$ of $\mathcal{H} \subset \mathcal{M}^{1,5}$ as an intersection

$$\mathcal{I}_1 = \mathcal{H}|_{\zeta^4=0} : \mathcal{H} \cap \mathcal{P}_1 \subset \mathcal{M}^{1,5},$$

where $\mathcal{P}_1$ is the hyperplane defined by $\zeta^4 = 0$, let

$$\eta_\mu = \frac{\hbar}{R^2} \zeta^\mu \quad \text{and} \quad \eta_4 = \frac{\hbar}{R^5} \zeta^5,$$

then a $dS$-space of momentum with $\eta^{AB} = \text{diag}(+,-,-,-,-)$, $A, B = 0, \ldots, 4$, follows

$$\mathcal{H}_s : \quad \eta^{AB} \eta_A \eta_B = -\frac{\hbar^2}{a^2},$$

$$ds^2_s = \eta^{AB} \eta_A d\xi^B = \frac{\hbar^2}{R^2} d^2 \chi.$$  \hspace{1cm} (22)

In the subspace $\mathcal{I}_1$ with the variables $[21]$, the operators $\hat{x}^0$, $\hat{x}^i$, $\hat{M}_i$, and $\hat{L}_i$ in Yang’s algebra become

$$\hat{x}_0 = ia \left( \eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4} \right), \quad \hat{x}_i = ia \left( \eta_i \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4} \right),$$

$$\hat{M}_i = i\hbar \left( \eta_0 \frac{\partial}{\partial \eta_i} + \eta_i \frac{\partial}{\partial \eta_0} \right), \quad \hat{L}_i = i\hbar \epsilon_{ijk} \left( \eta^j \frac{\partial}{\partial \eta_k} \right),$$

which form an $\mathfrak{so}(1,4)$ under Lie bracket and keep Snyder’s $dS$-space of momentum $[22]$ invariant. Thus, Snyder’s model is contained in the complete Yang model.

On the other hand, in the subspace $\mathcal{I}_2$ of $\mathcal{H} \subset \mathcal{M}^{1,5}$ as another intersection

$$\mathcal{I}_2 = \mathcal{H}|_{\zeta^5=0} : \mathcal{H} \cap \mathcal{P}_2 \subset \mathcal{M}^{1,5},$$

where $\mathcal{P}_2$ is the hyperplane defined by $\zeta^5 = 0$, let

$$\xi^\mu = a\zeta^\mu \quad \text{and} \quad \xi^4 = a\zeta^4,$$

then a $dS$-spacetime follows

$$\mathcal{H}_R : \quad \eta_{AB} \xi^A \xi^B = -R^2,$$

$$ds^2_R = \eta_{AB} d\xi^A d\xi^B = a^2 d^2 \chi.$$  \hspace{1cm} (26)
In the subspace $\mathcal{S}_2$ with the variables (25), the $\mathcal{A}$, $\mathcal{B} \neq 5$ components of the 6-d ‘angular momentum’ consist of the 5-d angular momentum in the $dS$ special relativity. Now, the classical variables $p_0$, $p_i$, $M_i$, and $L_i$ defined in (14) become

\begin{equation}
\begin{aligned}
p_0 &= \frac{1}{R/a} \left( \xi^4 \frac{d\xi^0}{ds_R} - \xi^0 \frac{d\xi^4}{ds_R} \right), \\
p_i &= \frac{1}{R/a} \left( \xi^i \frac{d\xi^4}{ds_R} - \xi^4 \frac{d\xi^i}{ds_R} \right), \\
M_i &= \frac{\hbar}{a} \left( \xi^i \frac{d\xi^0}{ds_R} - \xi^0 \frac{d\xi^i}{ds_R} \right), \\
L_i &= \frac{\hbar}{a} \epsilon_{ijk} \left( \xi^j \frac{d\xi^k}{ds_R} \right),
\end{aligned}
\end{equation}

which are the 5-d formalism of the conserved quantities for a free particle with mass $\frac{\hbar}{a}$ in the $dS$ special relativity. Thus, the $dS$ special relativity is also contained in the complete Yang model.

As for the duality between Snyder’s model and the $dS$ special relativity [4], it can also be studied in the complete Yang model. In fact, the $dS_4$-spaces for both Snyder’s model and the $dS$ special relativity are given by the sub-manifolds with $\zeta^4 = 0$ and $\zeta^5 = 0$, respectively, from the $dS_5$-space $\mathcal{H}$ in the model. Therefore, the duality is related to a one-to-one correspondence between $\zeta^4$ and $\zeta^5$. Since these variables are only related to $\psi$ (or $\zeta$ in [1]), and $\psi$ becomes $-\psi$ when $\zeta^4$ and $\zeta^5$ are interchanged, the duality of two theories is the correspondence of $\psi$ and $-\psi$. Moreover, the duality can also be considered as the correspondence of $x_\mu$ and $p_\mu$ in addition to $a$ and $R$ in the complete Yang model. Thus, this correspondence is just the duality given by the $\mathbb{Z}_2 := \{e, \tau\}$ with (19).

By studying the motion of fictitious particle in a dimensionless $dS_5$-space with the radius $R/a$, we obtain a complete Yang model at classical level in a slightly different way under Dirac bracket. The canonical quantization leads to the model at quantum level. We also re-examine why the complete Yang model contains both Snyder’s model, the $dS$ special relativity, and their duality as a UV-IR duality [2].

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