Worst-Case Additive Noise in Wireless Networks
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Abstract

A classical result in Information Theory states that the Gaussian noise is the worst-case additive noise in point-to-point channels, meaning that, for a fixed noise variance, the Gaussian noise minimizes the capacity of an additive noise channel. In this paper, we significantly generalize this result and show that the Gaussian noise is also the worst-case additive noise in wireless networks with additive noises that are independent from the transmit signals. More specifically, we show that, if we fix the noise variance at each node, then the capacity region with Gaussian noises is a subset of the capacity region with any other set of noise distributions. We prove this result by showing that a coding scheme that achieves a given set of rates on a network with Gaussian additive noises can be used to construct a coding scheme that achieves the same set of rates on a network that has the same topology and traffic demands, but with non-Gaussian additive noises.

I. INTRODUCTION

The modeling of background noise in point-to-point wireless channels as an additive Gaussian noise is well supported from both theoretical and practical viewpoints. In practice, we have witnessed that current wireless systems that were designed based on the assumption of additive Gaussian noise perform quite well. This is intuitively explained by the fact that, from the Central Limit Theorem, the composite effect of many (almost) independent noise sources (e.g., thermal noise, shot noise, etc.) should approach a Gaussian distribution. From a theoretical point of view, Gaussian noise has been proven to be the worst-case noise for additive noise channels. This means that, given a variance constraint, the Gaussian noise minimizes the capacity of a point-to-point additive noise channel. This result follows mainly from the fact that the Gaussian distribution maximizes the entropy subject to a variance constraint. More precisely, from the Channel Coding Theorem [2], the capacity of a channel $f(y|x)$ is given by

$$C = \max_{f(x): E[X^2] \leq P} I(X;Y).$$

Thus, if we choose $X$ to be distributed as $\mathcal{N}(0,P)$, we have that

$$C \geq h(X) - h(X|Y) = \frac{1}{2} \log (2\pi e P) - h(X|Y).$$

As shown in [2], for an additive noise (AN) channel $Y = X + Z$, where $E[Z] = 0$ and $E[Z^2] = \sigma^2$, we have $h(X|Y) \leq \frac{1}{2} \log \left(\frac{2\pi e P}{P + \sigma^2}\right)$. We conclude that

$$C_{AN} \geq \frac{1}{2} \log \left(1 + \frac{P}{\sigma^2}\right) = C_{AWGN},$$

where $C_{AWGN}$ is the capacity of the AWGN channel, which is achieved by a Gaussian input distribution. Moreover, a more operational justification of the fact that Gaussian is the worst-case noise for additive noise channels was provided in [3], where it was shown that random Gaussian codebooks and nearest-neighbor decoding achieve the capacity of the corresponding AWGN channel on a non-Gaussian AN channel.

Worst-case noise characterizations in settings other than a simple scalar additive noise channel are few in the literature. One such example is [4], where the authors consider vector channels with additive noise subject to the
constraint that the noise covariance matrix lies in a convex set. It is shown that, in this setting, the worst-case noise is vector Gaussian with a covariance matrix that depends on the transmit power constraints. In [5], a scalar additive noise channel with binary input is considered. The probability mass function of the (discrete) worst-case noise is characterized, and the worst-case capacity (i.e., the capacity under the worst-case noise) is found. Once we go beyond point-to-point channels, Gaussian noise is only known to be the worst-case additive noise in some special wireless networks, such as the Multiple Access Channel, the Degraded Broadcast Channel and MIMO channels. In all such cases the capacity has been fully characterized and is known to be achievable with Gaussian inputs. Therefore, similar arguments to the one above can be used to show that, in these cases, Gaussian noise is indeed the worst-case additive noise. However, for more general wireless networks where the capacity is unknown, we lack the tools to make such an assertion. The recent constant-gap capacity approximations for the Interference Channel [6] and for single-source single-destination relay networks [7–9] can only be used to state that Gaussian noise is “approximately” the worst-case additive noise in these cases. Nonetheless, in a leap of faith, most of the research concerning such systems and many other wireless networks views the AWGN channel model as the standard wireless link model. In general, it remains unknown whether Gaussian noise is the worst-case additive noise in wireless networks.

In this work, we address this issue and show that the Gaussian noise is in fact the worst-case noise for arbitrary wireless networks with additive noises that are independent of the transmit signals. We consider wireless networks with unrestricted topologies and general traffic demands. We show that any coding scheme that achieves a given set of rates on a network with Gaussian additive noises can be used to construct a coding scheme that achieves the same set of rates on a network that has the same topology and traffic demands, but with non-Gaussian additive noises. It is also important to notice that our coding scheme construction only depends on the mean and variance of the noise distributions of our non-Gaussian network, and is oblivious to their precise statistics. This means that our approach also results in a framework to design codes for networks with unknown noise distributions with an asymptotic performance guarantee.

We prove that the Gaussian noise is the worst-case noise in wireless networks based on two main results. The first one is that, given a coding scheme with finite reading precision for an AWGN network, one can build a coding scheme that achieves the same rates on a non-Gaussian wireless network. A coding scheme is said to have finite reading precision if, for any node, its transmit signals only depend on its received signals read up to a finite number of digits after the decimal point. This result is proven in three main steps. We start by applying a transformation at the transmit signals and received signals of all nodes in the network in order to create an “approximately Gaussian” effective network. The technique resembles OFDM in that it uses the Discrete Fourier Transform in order to mix together multiple uses of the same channel. This mixing causes the additive noise terms from distinct network uses to be averaged over time and, by making use of Lindeberg’s Central Limit Theorem [10], it can be shown that the resulting effective noise is approximately Gaussian in the distribution sense. Thus, we create an approximately Gaussian network. However, this mixing causes distinct noise realizations at the same receiver to be dependent of each other. The second step is an interleaving technique, which allows us to handle this dependence between distinct noise realizations. The interleaving operation creates multiple blocks of network uses inside which the additive noises are i.i.d. and almost normally-distributed. Inside each of these blocks we are able to apply the original coding scheme that we have for the AWGN network. The third step involves evaluating the performance of our original coding scheme on this i.i.d. almost normally-distributed blocks. This can be done because we require the original coding scheme to have finite reading precision. For such coding schemes, the sets of noise realizations that cause the coding scheme to make an error can be shown to be continuity sets. It follows from the portmanteau Theorem [10] that the coding scheme’s performance on an almost-Gaussian network does not deviate much from its performance on an actual Gaussian network.

The second main result we need is that, for any wireless network, the capacity when we restrict ourselves to coding schemes with finite reading precision, and allow the precision to tend to infinity along the sequence of coding schemes, is the same as the unrestricted capacity. To prove this we show that, for any coding scheme with infinite precision, there exists a quantization scheme of the received signals which does not increase the error probability of the coding scheme too much. This is done by showing that a truncation of the bit expansion of the received signal followed by a random shift performs well; thus, there must exist a fixed shift for each node which guarantees the same performance. This quantization operation makes the coding scheme have finite reading precision, and the
result follows.

The paper is organized as follows. In Section II, we describe the network model and introduce the necessary terminology. We start by focusing on wireless networks with \( L \) unicast sessions, which makes the proofs simpler and easier to follow. In Section III, we state our main result (Theorem 1) and the two main theorems that are needed for it, in the context of \( L \)-unicast wireless networks. Theorem 2 states that coding schemes with finite reading precision can be used to construct coding schemes for non-Gaussian networks. Theorem 3 states that, for AWGN networks, coding schemes with infinite reading precision can be “quantized” yielding coding schemes with finite reading precision that perform almost as well. We then state our main result for networks with general traffic demands (Theorem 4). The proof of Theorem 2 is presented in Section IV, divided into three subsections as follows. We first describe the OFDM-like scheme in subsection IV-A. Then, in Section IV-B, we show that the additive noises obtained from the OFDM-like scheme in fact converge in distribution to Gaussian noises. In Section IV-C, we describe the interleaving technique and the outer code that are used to handle the dependence between the noises after the OFDM-like scheme, and we show how the requirement of finite reading precision can be used to show that our coding scheme designed for a Gaussian network can be applied to an almost-Gaussian network without much loss in performance. The proof of Theorem 3 is in Section IV-D. In Section V, we describe how we can modify the arguments in the previous Sections in order to consider, instead of \( L \)-unicast wireless networks, wireless networks with general traffic demands, proving Theorem 4. We conclude the paper in Section VI.

II. PROBLEM SETUP AND DEFINITIONS

In this work, we model wireless networks as follows.

Definition 1. An additive noise wireless network consists of a directed graph \( G = (V, E) \), where \( V \) is the vertex (or node) set and \( E \subseteq V \times V \) is the edge set, and a real-valued channel gain \( h_{u,v} \) associated with each edge \((u, v) \in E\). At time \( t = 0, 1, 2, ..., \) each node \( u \in V \) transmits a real-valued signal \( X_u[t] \). The signal received by node \( v \) at time \( t \) is given by

\[
Y_v[t] = \sum_{u \in \mathcal{I}(v)} h_{u,v}X_u[t] + N_v[t],
\]

where \( \mathcal{I}(v) = \{ u \in V : (u, v) \in E \} \), and the additive noise \( N_v \) is assumed to be i.i.d. over time and to satisfy \( E[N_v] = 0 \) and \( E[N_v^2] = \sigma_v^2 < \infty \). We also assume that the noise terms are independent from all transmit signals and from all noise terms at distinct nodes. If all the additive noises in the network are normal \( \mathcal{N}(0, \sigma_v^2) \), then we say the network is an AWGN network.

In order to define source-destination relationships in a wireless network, we introduce the following notion.

Definition 2. For a wireless network with graph \( G = (V, E) \), the traffic demand is described by a function \( \mathcal{T} : V \times \mathcal{P}(V) \rightarrow \{0, 1\} \), where \( \mathcal{P}(V) \) is the power set of \( V \). For \( s \in V \) and \( D \subseteq V \), \( \mathcal{T}(s, D) = 1 \) if \( s \) has a message that is required by all nodes in \( D \) and no node outside of \( D \), and \( \mathcal{T}(s, D) = 0 \) otherwise.

Example 1. An \( L \)-user multiple access channel is defined by a graph \( G = (V, E) \) with node set \( V = \{s_1, s_2, ..., s_L, d\} \), edge set \( E = \{(s_1, d), ..., (s_L, d)\} \), and traffic demands

\[
\mathcal{T}(v, U) = \begin{cases} 
1 & \text{if } v \in \{s_1, s_2, ..., s_L\} \text{ and } U = \{d\} \\
0 & \text{otherwise.} 
\end{cases}
\]

Example 2. An \( L \)-user broadcast channel with degraded message sets is defined by a graph \( G = (V, E) \) with node set \( V = \{s, d_1, d_2, ..., d_L\} \), edge set \( E = \{(s, d_1), ..., (s, d_L)\} \), and traffic demands

\[
\mathcal{T}(v, U) = \begin{cases} 
1 & \text{if } v = s \text{ and } U = \{d_1, d_2, ..., d_\ell\}, \text{ for } \ell = 1, 2, ..., L \\
0 & \text{otherwise.} 
\end{cases}
\]
Even though the results presented in this paper hold for wireless networks with any traffic demands, including the multiple access channel and the broadcast channel, we start by considering the following special class.

**Definition 3.** An $L$-unicast wireless network has $L$ source nodes $s_1,\ldots,s_L \in V$ and $L$ destination nodes $d_1,\ldots,d_L \in V$ all of which are distinct nodes, and traffic demands given by

$$T(v,U) = \begin{cases} 
1 & \text{if } (v,U) = (s_\ell,\{d_\ell\}), \text{ for } \ell = 1,2,\ldots,L \\
0 & \text{otherwise.}
\end{cases}$$

Presenting our results for $L$-unicast wireless networks first has the advantage of making some of the proofs simpler and easier to follow. Later, in Section V, we describe how the same results can be extended to wireless networks with an arbitrary traffic demand $T$.

We point out that perfect (noiseless) feedback from a destination to a source is not allowed in our model. However, in Section V, we consider a generalization of Definition 4 that allows the sources’ transmit signals to depend on their previously received signals. Thus, noisy feedback links may exist between a destination and its corresponding source, and by setting the noise variance at the source to be very small, nearly perfect feedback can be simulated.

**Definition 4.** A coding scheme $C$ with block length $n \in \mathbb{N}$ and rate tuple $R = (R_1,\ldots,R_L) \in \mathbb{R}_+^L$ for an $L$-unicast additive noise wireless network consists of:

1. An encoding function $f_i : \{1,\ldots,2^{nR_i}\} \to \mathbb{R}^n$ for each source $s_i$, $i = 1,\ldots,L$, where each codeword $f_i(w_i)$, $w_i \in \{1,\ldots,2^{R_i}\}$, satisfies an average power constraint of $P$.
2. Relaying functions $r_i^{(t)} : \mathbb{R}^{t-1} \to \mathbb{R}$, for $t = 0,\ldots,n-1$, for each node $v \in V$ that is not a source, satisfying the average power constraint

$$\frac{1}{n} \sum_{t=0}^{n-1} \left[ r_i^{(t)}(y_0,\ldots,y_{t-1}) \right]^2 \leq P, \text{ for all } (y_0,\ldots,y_{n-1}) \in \mathbb{R}^n.$$

3. A decoding function $g_i : \mathbb{R}^n \to \{1,\ldots,2^{nR_i}\}$ for each destination $d_i$, $i = 1,\ldots,L$.

**Definition 5.** The error probability of a coding scheme $C$ (as defined in Definition 4), is given by

$$P_{\text{error}}(C) = \Pr \left[ \bigcup_{i=1}^{L} \{W_i \neq g_i(Y_{d_i[0]},\ldots,Y_{d_i[n-1]})\} \right],$$

where the message transmitted by source $s_i$, $W_i$, is assumed to be chosen uniformly at random from $\{1,\ldots,2^{nR_i}\}$, for $i = 1,\ldots,L$.

**Definition 6.** A rate tuple $R$ is said to be achievable for an $L$-unicast wireless network if there exists a sequence of coding schemes $C_n$ with rate tuple $R$ and block length $n$, for which $P_{\text{error}}(C_n) \to 0$, as $n \to \infty$. The sequence of coding schemes $C_n$, $n = 1,2,\ldots$, is then said to achieve rate tuple $R$. The capacity region of an $L$-unicast wireless network is the closure of the set of achievable rate tuples.

We will first focus on coding schemes that have *finite reading precision*. Then we will show that coding schemes with infinite reading precision can be converted into coding schemes with finite reading precision without much loss in performance.

**Definition 7.** For some $x \in \mathbb{R}$ and a positive integer $\rho$, let $\lfloor x \rfloor_\rho = 2^{-\rho} \lfloor 2^\rho x \rfloor$. A coding scheme $C$ is said to have finite reading precision $\rho \in \mathbb{N}$ if its relaying functions satisfy

$$r_i^{(t)}(y_1,\ldots,y_{t-1}) = r_i^{(t)}(\lfloor y_1 \rfloor_\rho,\ldots,\lfloor y_{t-1} \rfloor_\rho),$$

for any $(y_1,\ldots,y_{t-1}) \in \mathbb{R}^{t-1}$, any $v \in V - \{s_1,\ldots,s_L\}$, and any time $t$, and its decoding functions satisfy

$$g_i(y_1,\ldots,y_n) = g_i(\lfloor y_1 \rfloor_\rho,\ldots,\lfloor y_n \rfloor_\rho).$$
for any \((y_1, \ldots, y_n) \in \mathbb{R}^n\), and \(i \in \{1, \ldots, L\}\).

**Definition 8.** Rate tuple \(R\) is achievable by coding schemes with finite reading precision if we have a sequence of coding schemes \(C_n\), where coding scheme \(C_n\) has finite reading precision \(\rho_n\), which achieves rate tuple \(R\) according to Definition 6.

**Remark:** Notice that we allow the precision \(\rho_n\) to vary arbitrarily along the sequence of codes, and it may be the case that \(\rho_n \to \infty\) as \(n \to \infty\).

### III. Main Result

Our main result is to show that any rate tuple that is achievable on a network where each \(N_v\) is Gaussian for each \(v \in V\) is also achievable on a network where each \(N_v\) instead has any distribution with the same mean and variance. In the special case of \(L\)-unicast wireless networks, our main result is the following theorem.

**Theorem 1** (Worst-Case Noise for \(L\)-Unicast Networks). From a sequence of coding schemes that achieve rate tuple \(R\) on an AWGN \(L\)-unicast wireless network, it is possible to construct a single sequence of coding schemes that achieves arbitrarily close to \(R\) on the same \(L\)-unicast wireless network, where, for each relay \(v\), the distribution of \(N_v\) is replaced with any distribution satisfying \(E[N_v] = 0\) and \(E[N_v^2] = \sigma_v^2\). Therefore, if \(C_{AWGN}\) is the capacity region of the AWGN \(L\)-unicast wireless network, and \(C_{non-AWGN}\) is the capacity region of the same wireless network where, for each relay \(v\), the distribution of \(N_v\) is replaced with an arbitrary distribution satisfying \(E[N_v] = 0\) and \(E[N_v^2] = \sigma_v^2\), then

\[C_{AWGN} \subseteq C_{non-AWGN}.

We will prove Theorem 1 using the following two auxiliary results.

**Theorem 2.** Suppose a rate tuple \(R\) is achievable by coding schemes with finite reading precision on an AWGN \(L\)-unicast wireless network. Then it is possible to construct a single sequence of coding schemes that achieves arbitrarily close to \(R\) on the same \(L\)-unicast wireless network where, for each relay \(v\), the distribution of \(N_v\) is replaced with an arbitrary distribution satisfying \(E[N_v] = 0\) and \(E[N_v^2] = \sigma_v^2\).

**Theorem 3.** Suppose we have a sequence of coding schemes \(C_n\) achieving a rate tuple \(R\) on an AWGN network. Then it is possible to construct a sequence of coding schemes \(C^*_n\) with finite reading precision that also achieves \(R\) on the same AWGN network.

It is clear that by combining Theorems 2 and 3, Theorem 1 will follow. The proof of Theorems 2 and 3 will be presented in Section IV. The result in Theorem 1 can be generalized to networks with arbitrary traffic demands. By generalizing Definition 4 for the case of general traffic demands (which we do in Section V), we can state our main result as follows.

**Theorem 4** (Worst-Case Noise for Networks with General Traffic Demands). Suppose a rate tuple \(R\) is achievable on an AWGN wireless network with some arbitrary traffic demands \(T\). Then it is possible to construct a sequence of coding schemes that achieves arbitrarily close to \(R\) on the same additive noise wireless network where, for each relay \(v\), the distribution of \(N_v\) is replaced with an arbitrary distribution satisfying \(E[N_v] = 0\) and \(E[N_v^2] = \sigma_v^2\). Therefore, if \(C_{AWGN}\) is the capacity region of the AWGN wireless network, and \(C_{non-AWGN}\) is the capacity region of the same wireless network where, for each relay \(v\), the distribution of \(N_v\) is replaced with an arbitrary distribution satisfying \(E[N_v] = 0\) and \(E[N_v^2] = \sigma_v^2\), then

\[C_{AWGN} \subseteq C_{non-AWGN}.

In Section V, we describe how the proofs of Theorems 2 and 3 can be extended to the case of general traffic demands, in order to establish Theorem 4.
IV. PROOF OF MAIN RESULT FOR L-UNICAST WIRELESS NETWORKS

In this Section, we will prove Theorems 2 and 3, from which Theorem 1 will follow. To prove Theorem 2, we start by assuming that we have a sequence of coding schemes with finite reading precision designed to achieve a rate tuple $\mathbf{R}$ on an AWGN network. Then, through a series of steps, we will use this sequence of coding schemes to construct another sequence of coding schemes that achieves arbitrarily close to the rate tuple $\mathbf{R}$ on the corresponding network where the additive noises are not Gaussian.

A diagram illustrating the proof steps of Theorem 2 is shown in Fig. 2. We start by describing an OFDM-like scheme that is applied to all nodes in the network. The main idea is that, by applying an Inverse Discrete Fourier Transform (IDFT) to the block of transmit signals of each node, and a Discrete Fourier Transform (DFT) to the block of received signals of each node, we create effective additive noise terms that are weighted averages of the additive noise realizations during that block. We describe this procedure in detail in Section IV-A. Then, in Section IV-B, we show that this mixture of noises converges in distribution to a Gaussian additive noise term. This is done by showing that the weighted average of the noise realizations satisfies Lindeberg’s Central Limit Theorem Condition [10]. Therefore, the OFDM-like scheme effectively produces a network where the noises at each node are dependent across time and approximately Gaussian. The dependence across time is undesirable since our original coding scheme designed for the AWGN network assumed that the additive noise at each receiver is i.i.d. over time. To overcome this problem, in Section IV-C, we apply the OFDM-like scheme over multiple blocks, and then we interleave the effective network uses from distinct blocks. This effectively creates several blocks in which the network behaves as an Approximately AWGN network (with i.i.d. noises). Then our original code for the AWGN network can be applied to each approximately AWGN block. The fact that this code has finite reading precision guarantees that, when applied to the approximately AWGN block, its error probability is close to its error probability on the AWGN network. More formally, the error probability of a coding scheme with block length $k$, for a given choice of messages $\mathbf{w} \in \prod_{i=1}^{L} \{1, \ldots, 2^{kR_i}\}$, can be seen as the probability measure of the error set $A_{\mathbf{w}}$ (i.e., the set of noise realizations which causes an error to occur). As illustrated in Fig. 1, in general, this set could be arbitrarily ill shaped. However, if the coding scheme has finite reading precision, $A_{\mathbf{w}}$ can be shown to be a continuity set, which implies that its measure under similar probability measures cannot change much. Finally, we take care of the dependence between the noises of different blocks created in the interleaving operation by using a random outer code for each source-destination pair. This can be done if we view the coding scheme as creating a discrete channel between the message chosen at a given source and the decoded message at its corresponding destination. Then we can show via a mutual-information argument that we can use an outer code to achieve a rate tuple arbitrarily close to $\mathbf{R}$ on the non-Gaussian wireless network.

In Section IV-D, we prove Theorem 3. The main idea is to show that, given a coding scheme with infinite reading precision, there exists a set of quantization mappings, one for each node in the network, such that, if each node quantizes its received signal before applying the relaying or decoding function, the change in the error probability is arbitrarily small.

We point out that our results are not inconsistent with the intuition that, for a channel with a discrete output alphabet, the worst-case noise should be discrete. Theorems 2 and 3 do not imply that Gaussian noise is the worst-case noise if we restrict ourselves to coding schemes with finite precision, because, in Theorem 2, we may require coding schemes with infinite precision to achieve the same point in the capacity region in the non-AWGN network.
A. An OFDM-like scheme to mix the noises over time

We use an approach similar to OFDM in order to create an effective network with additive noises that are as close to normally-distributed as we wish. Essentially, each node in the network will apply transformations to its transmit signals and to its received signals, thus creating an effective network with new input-output relationships. If we focus on \( b \) uses of a single link of the network, then we convert the actual channel (i.e., a mapping from channel inputs \( X[0], X[1], ..., X[b-1] \) to channel outputs \( Y[0], Y[1], ..., Y[b-1] \)) into an effective channel that maps inputs \( d_0, d_1, ..., d_{b-1} \) into effective channel outputs \( \tilde{Y}_0, \Re[\tilde{Y}_1], \Im[\tilde{Y}_1], ..., \Re[\tilde{Y}_{b/2-1}], \Im[\tilde{Y}_{b/2-1}], \tilde{Y}_{b/2}, \) where \( \Re[z] \) and \( \Im[z] \) refer respectively to the real and imaginary parts of a complex number \( z \). The overall transformation, depicted in Fig. 3, can be described as follows. Assume that a node \( u \in V \) has \( b \) real numbers \( d_0, d_1, ..., d_{b-1} \) which

\[
\begin{align*}
\tilde{d}_0 &= d_0 \\
\tilde{d}_i &= d_{2i-1} + jd_{2i} & \text{for } i = 1, ..., \frac{b}{2} - 1 \\
\tilde{d}_{b/2} &= d_{b-1} \\
\tilde{d}_i &= d_{b-i}^* & \text{for } i = \frac{b}{2} + 1, ..., b - 1
\end{align*}
\]

are the inputs to the effective channels we intend to create. We assume that \( b \) is even, to simplify the expressions. Then node \( u \) “packs” these signals into \( b \) complex numbers \( \tilde{d}_0, ..., \tilde{d}_{b-1} \) as follows.

![Diagram of the steps that create the effective channel.](image_url)

From Lemma 2, codes with finite reading precision perform similarly (in fact we use coding schemes with infinite precision in our construction based on applying the OFDM-like scheme to the received signals first).

*(Fig. 2. Diagram of proof steps of Theorem 2. Thin arrows relate to steps in the construction of our new coding scheme, while the thick arrow indicates a conceptual connection established through Lemma 2)*

*Fig. 3. Diagram of the steps that create the effective channel.*
Next, node $u$ takes the IDFT of the vector $\tilde{d}_u = (\tilde{d}_0, ..., \tilde{d}_{b-1})$ to obtain the vector $X_u = \text{IDFT}(\tilde{d}_u)$. Throughout the paper, we assume that DFT and IDFT refer to the unitary version of the DFT and IDFT. Since $\tilde{d}_u$ is conjugate symmetric, $X_u$ is a real vector (in $\mathbb{R}^b$). Moreover, we will require the original real-valued signals to satisfy

$$\text{avg} \left[ d_i^2 \right] \leq P,$$

$$\text{avg} \left[ d_i^2 \right] \leq P/2, \text{ for } i = 1, ..., b - 2,$$

$$\text{avg} \left[ d_{b-1}^2 \right] \leq P,$$

where the avg operator refers to time average; i.e., if each $d_i$ is seen as a stream of signals $d_i[0], ..., d_i[k-1]$, then $\text{avg}(d_i) = \frac{1}{k} \sum_{t=0}^{k-1} d_i[t]$. Then we must have, by Parseval’s relationship,

$$\frac{1}{b} \text{avg} \left[ \|X_u\|^2 \right] = \frac{1}{b} \sum_{i=0}^{b-1} \text{avg} \left[ |\tilde{d}_i|^2 \right]$$

$$= \frac{1}{b} \left\{ \text{avg} \left[ d_0^2 \right] + \text{avg} \left[ d_{b-1}^2 \right] + 2 \sum_{i=1}^{b/2-1} \text{avg} \left[ d_{2i-1}^2 + d_{2i}^2 \right] \right\} \leq P.$$

Therefore, $u$ may transmit $k$ vectors $X_u$, each one over $b$ time-slots, and the average power constraint of $P$ over the block $n = kb$ will be satisfied. The parameter $k$ can be understood as the number of blocks of length $b$ to which we apply the OFDM-like scheme. A node $v$ will receive, over each sequence of $b$ time-slots,

$$Y_v = \sum_{u \in I(v)} h_{u,v} X_u + N_v.$$

By applying a DFT to each block of $b$ received signals, node $v$ will obtain

$$\tilde{Y}_v = \text{DFT}(Y_v) = \sum_{u \in I(v)} h_{u,v} \tilde{d}_u + \text{DFT}(N_v).$$

The transformation induced by the use of the IDFT on blocks of transmit signals and the DFT on blocks of received signals is illustrated in Fig. 4.

Next, by looking at each component of $\tilde{Y}_v$, we notice that we have effectively $b$ complex-valued received signals. The additive noise on the $\ell$th received signal is given by

$$\text{DFT}(N_v)_\ell = \frac{1}{\sqrt{b}} \sum_{i=0}^{b-1} N_v[i] e^{-j2\pi \frac{\ell i}{b}}$$

$$= \frac{1}{\sqrt{b}} \sum_{i=0}^{b-1} N_v[i] \cos \left( \frac{2\pi i \ell}{b} \right) - j \frac{1}{\sqrt{b}} \sum_{i=0}^{b-1} N_v[i] \sin \left( \frac{2\pi i \ell}{b} \right).$$

By considering the real and imaginary parts of each component $\tilde{Y}_{v,i}$ of $\tilde{Y}_v$, for $i = 0, ..., b - 1$, separately, we
obtain the following $2b - 2$ effective real-valued received signals:

$$
\begin{align*}
(\text{I}) & \quad \tilde{Y}_{v,0} = \sum_{u \in \Omega(v)} h_{u,v} d_{u,0} + \text{DFT}(N_v)_0 \\
(\text{II}) & \quad \Re \left[ \tilde{Y}_{v,i} \right] = \sum_{u \in \Omega(v)} h_{u,v} d_{u,2i-1} + \Re [\text{DFT}(N_v)_i] \quad \text{for } i = 1, \ldots, \frac{b}{2} - 1 \\
(\text{III}) & \quad \Im \left[ \tilde{Y}_{v,i} \right] = \sum_{u \in \Omega(v)} h_{u,v} d_{u,2i} + \Im [\text{DFT}(N_v)_i] \quad \text{for } i = 1, \ldots, \frac{b}{2} - 1 \\
(\text{IV}) & \quad \tilde{Y}_{v,b/2} = \sum_{u \in \Omega(v)} h_{u,v} d_{u,b-1} + \text{DFT}(N_v)_{b/2} \\
(\text{V}) & \quad \Re \left[ \tilde{Y}_{v,i} \right] = \sum_{u \in \Omega(v)} h_{u,v} d_{u,2(b-i)-1} + \Re [\text{DFT}(N_v)_i] \quad \text{for } i = \frac{b}{2} + 1, \ldots, b - 1 \\
(\text{VI}) & \quad \Im \left[ \tilde{Y}_{v,i} \right] = -\sum_{u \in \Omega(v)} h_{u,v} d_{u,2(b-i)} + \Im [\text{DFT}(N_v)_i] \quad \text{for } i = \frac{b}{2} + 1, \ldots, b - 1
\end{align*}
$$

However, from the conjugate symmetry of DFT$(N_v)$ (since $N_v$ is a real-valued vector), we have that $\Re [\text{DFT}(N_v)_i] = \Re [\text{DFT}(N_v)_{b-i}]$ and $\Im [\text{DFT}(N_v)_i] = -\Im [\text{DFT}(N_v)_{b-i}]$, for $i = 1, 2, \ldots, b - 1$, and all the received signals in (V) and (VI) are repetitions (up to a change of sign) of the received signals in (II) and (III). Therefore, we conclude that we have effectively $b$ distinct real-valued received signals with additive noise (i.e., the channels from (I), (II), (III) and (IV), which are the effective channel outputs shown in Fig. 3). It is important to notice that the additive noise terms are dependent across these $b$ received signals. We also point out that the stricter power constraint in (4) will not constitute a problem. The reason is that the effective received signals during the network uses corresponding to (4), given by (II) and (III), will be shown in the next Section to be subject to a noise with variance $\sigma_v^2/2$ as opposed to $\sigma_v^2$. Thus, the effective SNR is still $P/\sigma_v^2$.

### B. Noise mixture converges to Gaussian Noise

In this Section, we show that the additive noise terms of the effective received signals we obtained in the previous Section approximate a Gaussian distribution as $b$ gets large. In the remainder of the paper, we will write $X_n \xrightarrow{d} X$ to denote that the random variables $X_1, X_2, \ldots$ converge in distribution to $X$, and $X_n \xrightarrow{p} X$ to denote that the random variables $X_1, X_2, \ldots$ converge in probability to $X$. We will use the following classical result.

**Theorem 5** (Lindeberg’s Central Limit Theorem [11]). Suppose that for each $b = 1, 2, \ldots$, the random variables $Y_{b,1}, Y_{b,2}, \ldots, Y_{b,b}$ are independent. In addition, suppose that, for all $b$ and $i \leq b$, $E[Y_{b,i}] = 0$, and let

$$
\sigma_b^2 = \sum_{i=1}^{b} E \left[ Y_{b,i}^2 \right].
$$

Then, if for all $\varepsilon > 0$, Lindeberg’s condition

$$
\frac{1}{\sigma_b^2} \sum_{i=1}^{b} \sum_{i=1}^{b} E \left( Y_{b,i}^2 \mathbf{1}\{ |Y_{b,i}| \geq \varepsilon \sigma_b \} \right) \rightarrow 0 \text{ as } b \rightarrow \infty
$$

holds, we have that

$$
\frac{\sum_{i=1}^{b} Y_{b,i}}{\sigma_b} \xrightarrow{d} \mathcal{N}(0, 1).
$$

Lindeberg’s CLT can be used to prove the following lemma.

**Lemma 1.** Let $N[0], N[1], N[2], \ldots$ be i.i.d. random variables that are zero-mean, have variance $\sigma^2$ and let

$$
Z_b = \frac{1}{\sqrt{b}} \sum_{i=0}^{b-1} N[i] \cos \left( \frac{2\pi i \ell_b}{b} \right),
$$

for some $\ell_b \in \{1, \ldots, b-1\} \setminus \{b/2\}$. Then, $Z_b$ converges in distribution to $\mathcal{N}(0, \sigma^2/2)$ as $b \rightarrow \infty$. 

9
Proof: We start by letting $Y_{b,i+1} = N[i] \cos \left( \frac{2\pi ib}{b} \right)$, for $i = 0, 1, \ldots, b-1$. Then, by following (7), we have

$$s_b^2 = \sum_{i=1}^{b} E \left[ Y_{b,i}^2 \right] = \sum_{i=0}^{b-1} E \left[ N[i]^2 \right] \cos^2 \left( \frac{2\pi ib}{b} \right)$$

$$= \frac{\sigma^2}{4} \sum_{i=0}^{b-1} \left( e^{j2\pi ib} + e^{-j2\pi ib} \right)^2 = \frac{\sigma^2}{4} \sum_{i=0}^{b-1} \left( e^{4\pi ib} + e^{-4\pi ib} + 2 \right)$$

$$= \frac{\sigma^2}{2} + \frac{\sigma^2}{4} \sum_{i=0}^{b-1} \left( e^{4\pi ib} + e^{-4\pi ib} \right) = \frac{\sigma^2}{2} + \frac{\sigma^2}{4(1-e^{4\pi ib})} + \frac{\sigma^2}{4(1-e^{-4\pi ib})} = \frac{\sigma^2}{2}.$$ 

The last equality follows because $e^{-j4\pi ib} = 1$ and $e^{j4\pi ib} \neq 1$ for any $\ell_b \in \{1, \ldots, b-1\} \setminus \{b/2\}$. Next we let

$$U_{b,i} = Y_{b,i}^2 \mathbb{1} \{ |Y_{b,i}| \geq \varepsilon s_b \} = Y_{b,i}^2 \mathbb{1} \left\{ |Y_{b,i}| \geq \varepsilon \sigma \sqrt{b/2} \right\}.$$ 

Consider any sequence $i_b$, for $b = 1, 2, \ldots$, such that $i_b \in \{1, \ldots, b\}$, and any $\delta > 0$. Then we have that

$$\Pr \left( U_{b,i} \right) \geq \Pr \left( |Y_{b,i}| < \varepsilon \sigma \sqrt{b/2} \right)$$

$$\geq \Pr \left( |N[i_b - 1]| < \varepsilon \sigma \sqrt{b/2} \right)$$

$$= \Pr \left( |N[1]| < \varepsilon \sigma \sqrt{b/2} \right) \to 1,$$ 

as $b \to \infty$, which means that $U_{b,i_b} \xrightarrow{p} 0$ as $b \to \infty$. Moreover, we have that $|U_{b,i}| = U_{b,i} \leq N[i_b - 1]^2$ for all $b$, and $E \left[ N[i_b - 1]^2 \right] = \sigma^2 < \infty$. Next, we notice that $N[i_b - 1] \sim N[1]$ for all $i_b \geq 1$, which implies that, for any $\tau > 0$,

$$\Pr \left( |N[i_b]| \geq \tau \right) \leq \Pr \left( |N[i_b - 1]| \geq \tau \right) = \Pr \left( |N[1]| \geq \tau \right).$$ 

Thus, we can apply the version of the Dominated Convergence Theorem described in pages 338-339 of [11], to conclude that $E[U_{b,i}] \to 0$ as $b \to \infty$. We conclude that

$$\frac{1}{\sigma_b^2} \sum_{i=1}^{b} E \left( Y_{b,i}^2 \mathbb{1} \{ |Y_i| \geq \varepsilon s_b \} \right) = \frac{2}{\sigma_b^2} \sum_{i=1}^{b} E \left[U_{b,i}\right] \leq \frac{2}{\sigma^2} \max_{1 \leq i \leq b} E \left[U_{b,i}\right] \to 0$$

as $b \to \infty$, and Lindeberg’s condition (8) is satisfied for any $\varepsilon > 0$. Hence, from Theorem 5, we have that

$$\sum_{i=1}^{b} Y_{b,i} \xrightarrow{d} \mathcal{N}(0,1) \quad \Rightarrow \quad Z_b = \frac{\sigma}{\sqrt{\varepsilon}} \sum_{i=1}^{b} Y_{b,i} \xrightarrow{d} \mathcal{N}(0, \sigma^2/2).$$

Now consider the additive noise term in (II). It is the real part of (6), which, by Lemma 1, converges in distribution to $\mathcal{N}(0, \sigma_e^2/2)$, as $b \to \infty$. Moreover, it is easy to see that Lemma 1 can be restated with sines replacing the cosines, and the same result will hold. Thus, the additive noise in (III) also converges in distribution to $\mathcal{N}(0, \sigma_e^2/2)$. Finally, for the received signals in (I) and (IV), it is easy to see that the additive noise in (6) only has a real component, and by the usual Central Limit Theorem, it converges in distribution to $\mathcal{N}(0, \sigma_e^2)$. 

Notice that, since in (4) we restricted the power used in the network uses corresponding to (II) and (III) to $P/2$, all of our effective channels have the same SNR they would have if the transmit signals had power $P$ and the noise variance $\sigma_e^2$. Therefore, for the network uses corresponding to (II) and (III), we can instead assume that the power constraint is $P$, but all nodes divide their transmit signals by $\sqrt{2}$ prior to transmission, and multiply their received signals by $\sqrt{2}$. This yields the following $b$ effective channels,

(I) \quad \hat{Y}_{e,0} = \sum_{u \in \mathcal{I}(v)} h_{u,v} d_{u,0} + \text{DFT}(N_v)_0$

(II') \quad \sqrt{2} \cdot \Re \left[ \hat{Y}_{e,i} \right] = \sum_{u \in \mathcal{I}(v)} h_{u,v} d_{u,2i-1} + \sqrt{2} \cdot \Re \left[ \text{DFT}(N_v)_i \right] \quad \text{for } i = 1, \ldots, \frac{b}{2} - 1$

(III') \quad \sqrt{2} \cdot \Im \left[ \hat{Y}_{e,i} \right] = \sum_{u \in \mathcal{I}(v)} h_{u,v} d_{u,2i} + \sqrt{2} \cdot \Im \left[ \text{DFT}(N_v)_i \right] \quad \text{for } i = 1, \ldots, \frac{b}{2} - 1$

(IV) \quad \hat{Y}_{e,b/2} = \sum_{u \in \mathcal{I}(v)} h_{u,v} d_{u,b-1} + \text{DFT}(N_v)_{b/2}$

10
all of which have input power constraint $P$ and additive noise with variance $\sigma_v^2$. The diagram describing the steps that create the effective channel from Fig. 3 can then be updated as shown in Fig. 5. We notice that the transformation between the $b$ inputs to the effective channels and the $b$ inputs to the actual channel is in fact a 2-norm-preserving linear transformation, which we call $\mathcal{L}_T$. Similarly, the transformation between the $b$ outputs of the actual channel and the $b$ output of our effective channel is also a 2-norm-preserving linear transformation, which we call $\mathcal{L}_R$.

Now consider any sequence $\ell_b, b = 1, 2, \ldots$, where $\ell_b \in \{0, \ldots, b-1\}$. Let $Z_{b, \ell_b}$ now be the additive noise term of the $\ell_b$th effective channel above. The sequence indices $b \in \{1, 2, \ldots\}$ can be partitioned into four sets $J_1$, $J_2$, $J_3$ and $J_4$, according to whether $Z_{b, \ell_b}$ corresponds to the additive noise of an effective channel of type (I), (II'), (III') or (IV). According to Lemma 1, if $J_2$ or $J_3$ are infinite sets, the subsequence that they define $\{Z_{b, \ell_b}\}_{b \in J_2}$ or $\{Z_{b, \ell_b}\}_{b \in J_3}$ converge in distribution to $\mathcal{N}(0, \sigma_v^2)$ (after the multiplication by $\sqrt{2}$). Moreover, as we noticed above, from the usual Central Limit Theorem, it follows that if $J_1$ or $J_4$ are infinite sets, the subsequences defined by $\{Z_{b, \ell_b}\}_{b \in J_1}$ or $\{Z_{b, \ell_b}\}_{b \in J_4}$ also converge in distribution to $\mathcal{N}(0, \sigma_v^2)$. Therefore, we conclude that, for any arbitrary sequence $\ell_b, b = 1, 2, \ldots$, where $\ell_b \in \{0, \ldots, b-1\}$, $Z_{b, \ell_b}$ converges in distribution to $\mathcal{N}(0, \sigma_v^2)$.

C. Interleaving and Outer Code

In this Section, we address the fact that, as we mentioned before, the additive noise at node $v$ in the $b$ effective network uses are dependent of each other. In order to handle this dependence, we consider using the network for a total of $bk$ times, performing the OFDM-like approach from Section IV-A within each block of $b$ time steps. Then, by interleaving the symbols, it is possible to view the result as $b$ blocks of $k$ network uses. This idea is illustrated in Fig. 6. Notice that, within each block of $k$ network uses, the additive noises are i.i.d., but they are dependent.

![Diagram of the steps that create the effective channel](image)

![Interleaving the effective network uses obtained from the OFDM-like scheme](image)
among distinct blocks. Intuitively, this makes each of these blocks of \( k \) network uses suitable for the application of a coding scheme \( C_k \) with block length \( k \). The dependence between the noises of different blocks of length \( k \) will be handled at the end of this Section, through the application of a random outer code. Then, by considering a mutual-information argument, we will show that the performance of the resulting coding scheme on the wireless network with non-Gaussian noises is essentially the same as the performance of the original coding scheme \( C_k \) on the AWGN version of the network.

**Example 3.** Consider a simple relay channel, defined by a graph \( G = (V, E) \), where \( V = \{s, v, d\} \) and \( E = \{(s, v), (s, d), (v, d)\} \). Suppose we have a coding scheme \( \tilde{C}_k \) of block length \( k \) and rate \( R \) for this network. The operations performed by the nodes under this scheme at time \( t \) can be illustrated as in Fig. 7. Now suppose we want to apply the OFDM-like scheme and the interleaving procedure to this coding scheme \( \tilde{C}_k \). In essence, \( b \) versions of this coding scheme will be simultaneously used. Encoding, relaying and decoding functions are applied "in parallel" for each of the \( b \) coding schemes, as shown in Fig. 8(a) in detail. First, \( b \) codewords \( f(w_1), \ldots, f(w_b) \) are chosen at the source. At times \( bt, bt + 1, \ldots, bt + b - 1 \) for \( t = 0, \ldots, k - 1 \), the source transmits the signals obtained by applying \( L_T \) to the vector formed by the \( (t+1) \)th entries of these \( b \) codewords. Relay \( v \), in turn, applies \( L_R \) to the received signals at times \( bt, bt + 1, \ldots, bt + b - 1 \) for \( t = 0, \ldots, k - 1 \), can use the relaying function \( r_v^{(t+1)} \) a total of \( b \) times in order to obtain a length-\( b \) vector that goes through the transformation \( L_T \) to yield the \( b \) signals to be transmitted at times \( bt + 1, bt + 2, \ldots, bt + b - 1 \) for \( t = 0, \ldots, k - 2 \). The destination, after applying \( L_R \) to each block of \( b \) received signals, obtains \( b \) sequences of \( n \) received signals, and can apply its decoding function \( g \) to each of these sequences. As shown in Fig. 8(a), the application of the transformations \( L_T \) and \( L_R \) can be seen as creating \( b \) effective networks, where the transmit and received signals of the \( t \)th effective network are given by \( d[t], w_i \) and \( \tilde{Y}[t], w_i \) respectively.

The purpose of the interleaving procedure can be understood if we focus on what occurs to the signals in one of these effective networks, say the one indexed by \( w_1 \). By absorbing the transformations \( L_T \) and \( L_R \) into the network, and viewing the \( d[t], w_i \)s and \( \tilde{Y}[t], w_i \)s as inputs and outputs of the network, the network that is effectively experienced by the signals indexed by \( w_1 \) is shown in Fig. 8(b). Notice that the effective network in Fig. 8(b) is the same as the original network in Fig. 7 but with different additive noise terms \( Z_u[t] \) and \( \tilde{Z}_u[t] \). These effective noise terms are in fact i.i.d., since the operations \( L_T \) and \( L_R \) are applied to blocks of signals with different indices \( w_1, w_2, \ldots, w_b \), and this cannot create dependence between effective noises \( Z_u[t] \) and \( \tilde{Z}_u[t'] \) (or \( \tilde{Z}_d[t] \) and \( \tilde{Z}_d[t'] \)) for \( t \neq t' \), since they both correspond to received signals indexed by \( w_1 \). Therefore, we are essentially applying coding scheme \( C_k \) in \( b \) parallel effective relay channels, each of which has i.i.d. noises at \( v \) and \( d \).

Since from the statement of Theorem 1, the rate tuple \( R \) is achievable by coding schemes with finite reading precision, we may assume that we have a sequence of coding schemes \( C_k \) (with block length \( k \) and rate tuple \( R_k \)) with finite reading precision \( \rho_k \), whose error probability when used on the AWGN network is \( \epsilon_k = P_{\text{error}}(C_k) \), and satisfies \( \epsilon_k \to 0 \) as \( k \to \infty \). Now, consider applying this code over each of the \( b \) blocks of length-\( k \) that we obtained from the interleaving, as demonstrated in Example 1. Over each block of length \( k \), the noises at all nodes are independent and i.i.d. over time, and, if \( b \) is chosen fairly large, they are very close to Gaussian in distribution,
Then, for each value of the receiving signals of a length-$b$ block and can apply $\mathcal{L}_R$ to them. At time $bt$, for $t = 1, ..., k - 1$, using all previously received effective signals, the relay can use relaying function $r^{(t)}(t)$ $b$ times to obtain $(d_s[t, w_1], ..., d_s[t, w_b])$. After applying $\mathcal{L}_T$ to this vector, the relay obtains the $b$ signals to be transmitted at times $bt, bt + 1, ..., bt + b - 1$. The destination, at time $bt + b - 1$, for $t = 0, ..., k - 1$, finishes receiving the signals of a length-$b$ block and can apply $\mathcal{L}_R$ to them. (b) Effective network experienced by the signals indexed by $w_1$.

and, intuitively, the error probability we obtain should be close to $\epsilon_k$. The actual distribution of the additive noise at each of these $b$ length-$k$ blocks is given by the noise terms in (I), (II'), (III') and (IV). For $\ell = 0, ..., b - 1$, we let $\epsilon_{k,b}^{(\ell)}$ be the error probability of coding scheme $C_k$ applied on the $(\ell + 1)$th such block, for which the i.i.d. additive noise at node $v$ is given by

$$Z_{v,b}^{(\ell)} = \begin{cases} 
\text{DFT}(\mathbf{N}_v)_0 & \text{for } \ell = 0 \\
\sqrt{2} \cdot \Re \left[ \text{DFT}(\mathbf{N}_v)_\ell \right] & \text{for } \ell = 1, ..., b - 1 \\
\sqrt{2} \cdot \Im \left[ \text{DFT}(\mathbf{N}_v)_{(1+\ell-b)/2} \right] & \text{for } \ell = b/2, ..., b - 2 \\
\text{DFT}(\mathbf{N}_v)_b/2 & \text{for } \ell = b - 1.
\end{cases}$$

Then, for each value of $b$, we let $\epsilon_{k,b} = \max_{0 \leq \ell \leq b - 1} \epsilon_{k,b}^{(\ell)}$, and $\ell_{b} = \arg \max_{0 \leq \ell \leq b - 1} \epsilon_{k,b}^{(\ell)}$, which defines a sequence $\ell_{b}$, $b = 1, 2, ...$ like the ones considered at the end of Section IV-B.

We let $\mathbf{Z}_b \in \mathbb{R}^{k|V|}$ be the random vector associated with the effective additive noises at all nodes in $V$ during
the $\ell_b$th length-$k$ block assuming that we performed the OFDM-like scheme in blocks of size $b$; i.e.,

$$Z_b = \left( Z_{\ell_b}^{(t)} \right)_{v \in V; 0 \leq t \leq k-1}. $$

Since each component of $Z_b$ is independent and they all converge in distribution to a zero-mean Gaussian random variable, we have that $Z_b$ converges in distribution to a Gaussian random vector. We let $Z$ be this limiting distribution, and we know that the component of $Z$, corresponding to node $v$ and time $t$ is distributed as $N(0, \sigma_v^2)$, for any $t \in \{0, \ldots, k-1\}$. Now notice that, if we fix the messages chosen at the sources to be $w = (w_1, w_2, \ldots, w_L) \in \prod_{i=1}^L \{1, \ldots, 2^{kR_i}\}$, then, whether $C_k$ makes an error is only a deterministic function of $Z_b$. Therefore, for each $w \in \prod_{i=1}^L \{1, \ldots, 2^{kR_i}\}$, we can define an error set $A_w$, corresponding to all realizations of $Z_b$ that cause coding scheme $C_k$ to make an error. It is important to notice that $A_w$ is independent of the actual joint distribution of the noise terms; it only depends on the coding scheme $C_k$. Then we can write

$$\epsilon_{k,b} = 2^{-k} \sum_{i=1}^L R_i \sum_w \Pr[Z_b \in A_w]$$

and also

$$\epsilon_k = 2^{-k} \sum_{i=1}^L R_i \sum_w \Pr[Z \in A_w].$$

Our first goal is to show that $\epsilon_{k,b} \to \epsilon_k$ as $b \to \infty$. Recall that a Borel set $A \subseteq \mathbb{R}^m$ is said to be a $\mu$-continuity set for some probability measure $\mu$ on $\mathbb{R}^m$, if $\mu(\partial A) = 0$, where $\partial A$ is the boundary of $A$ (see, for example, [11]). Next, we state the following classical result, which provides an alternative characterization of convergence in distribution.

**Theorem 6 (Portmanteau Theorem [10]).** Suppose we have a sequence of random vectors $Z_b \in \mathbb{R}^{k|V|}$ and another random vector $Z \in \mathbb{R}^{k|V|}$. Let $\mu_b$ and $\mu$ be the probability measures on $\mathbb{R}^{k|V|}$ associated to $Z_b$ and $Z$ respectively. Then $Z_b$ converges in distribution to $Z$ if and only if

$$\lim_{b \to \infty} \mu_b(A) = \mu(A)$$

for all $\mu$-continuity sets $A$.

Let $\mu$ be the probability measure on $\mathbb{R}^{k|V|}$ associated to $Z$. Then, if we show that $A_w$ is a $\mu$-continuity set for each choice of messages $w$, from Theorem 6, the fact that $Z_b \overset{d}{\to} Z$ will imply that

$$\lim_{b \to \infty} \Pr[Z_b \in A_w] = \Pr[Z \in A_w]$$

for each $w$, and from (10) and (11) we will conclude that $\epsilon_{k,b} \to \epsilon_k$ as $b \to \infty$. This is in fact what we do in the following Lemma.

**Lemma 2.** Suppose we have a coding scheme $C$ with block length $k$, rate tuple $R$, and finite reading precision $\rho$. Then, for any choice of messages $w \in \prod_{i=1}^L \{1, \ldots, 2^{kR_i}\}$, the error set $A_w$ is a $\mu$-continuity set.

**Proof:** Fix some choice of messages $w$. We will use the fact that $C$ has finite reading precision $\rho$ to show that our set $A_w$ and its complement $A_w^c = \mathbb{R}^{k|V|} \setminus A_w$ can be represented as a countable union of disjoint convex sets, which will then imply the $\mu$-continuity. Recall from Definition 7 that, in a coding scheme with finite reading precision $\rho$, a node $v$ only has access to $[Y_v]_{\rho}$. Thus, we will call $[Y_v]_{\rho}$ the effective received signal at $v$. The set

$$\mathcal{Y} = \left\{(y_1, \ldots, y_{k|V|}) \in \mathbb{R}^{k|V|} : y_i = [y_i]_{\rho}, i = 1, \ldots, k|V|\right\}$$

can be understood as the set of all possible values of the effective received signals at all nodes in $V$ during a length-$k$ block. It is clear that $\mathcal{Y}$ is a countable set for any finite $\rho$.

Notice that, for our fixed choice of messages $w$, the vector $y \in \mathcal{Y}$ corresponding to the effective received signals at all nodes during the length-$k$ block is a deterministic function of the value of all the noises in the network during the length-$k$ block, $z \in \mathbb{R}^{k|V|}$. Therefore, for each $y \in \mathcal{Y}$, we define $Q(y) \subseteq \mathbb{R}^{k|V|}$ to be the set of noise realizations $z$ that will result in $y$ being the effective received signals. In Lemma 5 in the Appendix, we prove that
$Q(y)$ is a convex set. We also prove that, for any convex set $S$, $\lambda(\partial S) = 0$, where $\lambda$ is the Lebesgue measure. Since our measure $\mu$ is absolutely continuous (as $Z$ is jointly Gaussian), it follows by definition [11] that

$$
\lambda(S) = 0 \implies \mu(S) = 0,
$$

for any Borel set $S$. Thus, since $\lambda(\partial Q(y)) = 0$, we have that $\mu(\partial Q(y)) = 0$. This, in turn, clearly implies that

$$
\mu(Q(y)^0) = \mu(\overline{Q(y)}) = \mu(Q(y)),
$$

(13)

where we use $S^o$ to represent the interior of a set $S$ and $\overline{S}$ to represent its closure. Next, let $\mathcal{Y}_{A_w} = \{y \in \mathcal{Y} : A_w \cap Q(y) \neq \emptyset\}$. Notice that all noise realizations $z \in Q(y)$ will cause all nodes and, in particular, the destination nodes to receive the exact same effective signals. Therefore, it must be the case that, if $A_w \cap Q(y) \neq \emptyset$, then $Q(y) \subseteq A_w$, which implies that

$$
\bigcup_{y \in \mathcal{Y}_{A_w}} Q(y) = A_w.
$$

Moreover, it is obvious that any noise realization must belong to exactly one set $Q(y)$, and we have

$$
\bigcup_{y \in \mathcal{Y} \setminus \mathcal{Y}_{A_w}} Q(y) = A_w^c.
$$

Finally, we obtain

$$
\mu(A_w^o) \geq (i) \mu\left(\bigcup_{y \in \mathcal{Y}_{A_w}} Q(y)^0\right)
$$

$$
= \sum_{y \in \mathcal{Y}_{A_w}} \mu(Q(y)^0) = \mu(Q(y))
$$

$$
\geq 1 - \mu\left(\bigcup_{y \in \mathcal{Y} \setminus \mathcal{Y}_{A_w}} Q(y)^0\right)
$$

$$
= 1 - \mu\left((A_w^c)^0\right)
= \mu\left((A_w^c)^c\right) = \mu(\overline{A_w}),
$$

(\text{where } (i) \text{ follows since, for sets } B_1, B_2, ..., (\cup_i B_i)^0 \supseteq \cup_i B_i^c, (ii) \text{ follows from the countability of } \mathcal{Y}_{A_w} \text{ and the fact that } Q(y_1) \cap Q(y_2) = \emptyset \text{ for } y_1 \neq y_2, \text{ and } (iii) \text{ follows from (13)}). \text{ We conclude that } \mu(\partial A_w) = \mu(\overline{A_w}) - \mu(A_w^o) = 0; \text{ i.e., } A_w \text{ is a } \mu\text{-continuity set.}

From our previous discussion, we conclude that $\epsilon_{k,b} \to \epsilon_k$ as $b \to \infty$. We then see that we can apply code $C_k$ within each of the $b$ blocks of length $k$ and obtain a probability of error (within that block) that tends to $\epsilon_k$ as $b \to \infty$. However, since we have a total of $b$ blocks of length $k$, we make an error if we make an error in any of the $b$ blocks of length $k$. It turns out that a simple union bound does not work here, since the error probability would be of the form $\epsilon_{b\epsilon_k,b}$ and we would not be able to guarantee that it tends to $0$ as $b$ and $k$ go to infinity. Instead we consider using an outer code for each source-destination pair.

The idea is to apply coding scheme $C_k$ to each of the $b$ length-$k$ blocks, and then view this as creating a discrete channel for each source-destination pair. More specifically, for each length-$bk$ block, source $s_j$ chooses a symbol (rather than a message) from $\{1, ..., 2^{kR_j}\}^b$ and transmits the $b$ corresponding codewords from $C_k$. Then destination $d_j$ will apply the decoder from code $C_k$ inside each length-$k$ block and obtain an output symbol also from $\{1, ..., 2^{kR_j}\}^b$. Notice that, by viewing the input to $bk$ network uses as a single input to this discrete channel, we make sure we have a discrete memoryless channel, and we can use the Channel Coding Theorem. We can view $W_j^b$ and $W_j$ as the discrete input and output of the channel between $s_j$ and $d_j$. We will then construct a code
(whose rate is to be determined) for this discrete channel between $s_j$ and $d_j$ by picking each entry uniformly at random from $\{1, ..., 2^{kR_j}\}$. Then, source-destination pair $(s_j, d_j)$ can achieve rate

$$\frac{1}{bk} I(W_j^b, \hat{W}_j^b) = \frac{1}{bk} \left( H(W_j^b) - H(W_j^b | \hat{W}_j^b) \right) \geq R_j - \frac{1}{bk} \sum_{\ell=0}^{b-1} H(W_j[\ell] | \hat{W}_j[\ell])$$

where $(i)$ follows from Fano’s Inequality, since, within the $\ell$th length-$k$ block, we are applying code $C_k$ and we have an average error probability of at most $\epsilon_k^{(\ell)}$ (it should in fact be less than $\epsilon_k^{(\ell)}$ since we are only considering the error event $W_j[\ell] \neq \hat{W}_j[\ell]$ and $\epsilon_k^{(\ell)}$ refers to the union of these events for all source-destination pairs).

We conclude that, by choosing $b$ and $k$ sufficiently large, it is possible for each source-destination pair to achieve arbitrarily close to rate $R_j$. Thus, our coding scheme can achieve arbitrarily close to the rate tuple $R$. This concludes the proof of Theorem 2.

**D. Optimality of Coding Schemes with Finite Reading Precision**

In this Section, we prove Theorem 3. This theorem implies that, if we restrict ourselves to coding schemes with finite reading precision, and allow the reading precision to tend to infinity along the sequence of coding schemes, we can achieve any point in the capacity region of an AWGN wireless network, thus characterizing the optimality of coding schemes with finite reading precision for AWGN networks. We start by considering a sequence of coding schemes $C_n$ (with infinite reading precision) that achieves rate tuple $R$ on an AWGN $L$-unicast wireless network. We will build a sequence of coding schemes $C_{n,m}$ with finite reading precision that also achieves rate tuple $R$ on the same $L$-unicast wireless network.

Let $\epsilon_n$ be the error probability of coding scheme $C_n$, which achieves rate tuple $R$ on the AWGN $L$-unicast wireless network. From Definition 6, we have that $\epsilon_n \to 0$ as $n \to \infty$. For any fixed $n$, we will first build a sequence of coding schemes with finite reading precision $C_{n,m}, m = 1, 2, ...,$ such that code $C_{n,m}$ has error probability $\epsilon_{n,m}$, where $\epsilon_{n,m} \to \epsilon_n$ as $m \to \infty$. This will allow us to choose a finite $m$ for which $\epsilon_{n,m}$ is arbitrarily close to $\epsilon_n$.

Notice that, from Definition 4, relaying and decoding functions should be deterministic. However, in order to construct coding scheme $C_{n,m}$, we will first assume that the relaying and decoding functions are allowed to be randomized, and later we will derandomize the constructed coding scheme. Recall that, from Definition 4, coding scheme $C_n$ is comprised of encoding functions $\{f_i : 1 \leq i \leq L\}$, relaying functions $\{r_v^{(t)} : v \in V, 1 \leq t \leq n\}$ and decoding functions $\{g_i : 1 \leq i \leq L\}$. We will build $C_{n,m}$ from $C_n$ by using the same encoding functions $f_i$, $i = 1, ..., L$, and replacing the relaying functions with

$$\hat{r}_v^{(t)}(Y_v[1], ..., Y_v[t-1]) \equiv r_v^{(t)}(Y_v^{(m)}[1], ..., Y_v^{(m)}[t-1])$$

for $1 \leq t \leq n$ and $v \in V$, and replacing the decoding functions with

$$\hat{g}_i(Y_v[1], ..., Y_v[n]) \equiv g_i(Y_v^{(m)}[1], ..., Y_v^{(m)}[n])$$

for $1 \leq i \leq L$, where we define

$$\hat{Y}_v^{(m)}[t] = [Y_v[t]]_m + U_v^{(m)}[t], \quad (14)$$

for $v \in V$ and $1 \leq t \leq n$, where $U_v^{(m)}[1], ..., U_v^{(m)}[n]$ are independent uniform random variables drawn from $(-2^{m-1}, 2^{m-1})$, independent from all signals and noises in the network. Notice that, since the relaying functions
\(r_v(t)\) satisfy the power constraint in Definition 4, so will the new relaying functions \(\tilde{r}_v(t)\). In order to relate the error probability of \(C_{n,m}^*\) to the error probability of \(C_n\), we will need the following lemma, whose proof is in the Appendix.

**Lemma 3.** Suppose \(Y\) is a random variable with density \(f\). Let \(\tilde{Y}^{(m)} = |Y|_m + U^{(m)}\), where \(U^{(m)}\) is uniformly distributed in \((-2^{m-1}, 2^{m-1})\) and independent from \(Y\). Then each \(\tilde{Y}^{(m)}\) has a density \(f^{(m)}\), and \(f^{(m)}\) converges pointwise almost everywhere to \(f\).

This lemma will be used to show that, by picking \(m\) sufficiently large, we can make the error probability of code \(C_{n,m}^*\) arbitrarily close to \(\epsilon_n\). Suppose we fix the message vector \(t \in \mathbb{Z}_{+}\) and let \(Y\) be the random vector of length \(n\) corresponding to all the received signals at all nodes during the \(n\) time steps in the block if code \(C_n\) is used. More precisely, we write \(Y = \{Y[0], ..., Y[n-1]\}\), where \(Y[t] = (Y_1[t], ..., Y|V|)[t]\) is the random vector of received signals at all \(|V|\) nodes at time \(t\), for \(0 \leq t \leq n - 1\). The received signal at node \(v\) at time \(t\), \(Y_v[t]\), is defined in (2). Notice that here we assume that the set of nodes \(V\) can be written as \(V = \{1, ..., |V|\}\), in order to simplify some expressions. We claim that the random vector \(Y\) conditioned on the choice of messages \(W = w\) has a density. To see this, we first notice that, conditioned on the received signals received up to time \(t - 1\), i.e., on \((Y[0], ..., Y[t-1]) = (y[0], ..., y[t-1])\), and on \(W = w\), the transmit signals at time \(t\), \(X_v[t]\) for \(v \in V\), are all deterministic. Thus, the received signals \(Y_v[t]\), for \(v \in V\), are conditionally independent and each one is normally-distributed, conditioned on \((Y[0], ..., Y[t-1]) = (y[0], ..., y[t-1])\) and \(W = w\). Therefore, the conditional pdf \(f_{Y_v[t]}(y_v[t]|y[0], ..., y[t-1], W)\) exists for each \(v \in V\). We conclude that, conditioned on \(W = w\), the random vector \(Y\) has a density given by

\[
f_{Y|W}(y|W) = \prod_{v=1}^{|V|} f_{Y_v[0]|W}(y_v[0]|W) \prod_{t=1}^{n-1} \prod_{v=1}^{|V|} f_{Y_v[t]|Y[0], ..., Y[t-1], W}(y_v[t]|y[0], ..., y[t-1], W). \tag{15}
\]

Similarly, we let \(\tilde{Y}^{(m)}\) be the vector of \(n|V|\) effective received signals (14) if code \(C_{n,m}^*\) is used instead, i.e., \(\tilde{Y}^{(m)} = \{\tilde{Y}^{(m)}[0], ..., \tilde{Y}^{(m)}[t-1]\}\), where \(\tilde{Y}[t] = (\tilde{Y}_1^{(m)}[t], ..., \tilde{Y}_{|V|}^{(m)}[t])\). By using similar arguments to those that led to (15), we see that, when we condition on \((\tilde{Y}^{(m)}[0], ..., \tilde{Y}^{(m)}[t-1]) = (y[0], ..., y[t-1])\), and on \(W = w\), the effective received signals \(\tilde{Y}_v^{(m)}[t]\), for \(v \in V\), are conditionally independent (although not normally-distributed). Then, using the fact that, from (14), \(Y_v^{(m)}[t]\) is the sum of two independent random variables and \(U_v^{(m)}[t]\) has a density (see page 266 in [11]), we conclude that, conditioned on \(W\), \(\tilde{Y}_v^{(m)}[t]\) has a conditional density given by

\[
f_{\tilde{Y}_v^{(m)}|W}(y|W) = \prod_{v=1}^{|V|} f_{\tilde{Y}_v^{(m)}[0]|W}(y_v[0]|W) \prod_{t=1}^{n-1} \prod_{v=1}^{|V|} f_{\tilde{Y}_v^{(m)}[t]|Y[0], ..., Y[t-1], W}(y_v[t]|y[0], ..., y[t-1], W). \tag{16}
\]

The random variables \(\tilde{Y}_v^{(m)} = [Y_v[t]]_m + U_v^{(m)}[t]\), for \(m = 1, 2, ..., \) conditioned on \((\tilde{Y}^{(m)}[0], ..., \tilde{Y}^{(m)}[t-1]) = (y[0], ..., y[t-1])\) and \(W = w\), satisfy the conditions of Lemma 3, and we have that

\[
f_{\tilde{Y}_v^{(m)}[0]|W}(y_v[0]|W) \rightarrow f_{Y_v[0]|W}(y_v[0]|W) \quad \text{and} \quad f_{\tilde{Y}_v^{(m)}[t]|\tilde{Y}^{(m)}[0], ..., \tilde{Y}^{(m)}[t-1], W}(y_v[t]|y[0], ..., y[t-1], W) \rightarrow f_{Y_v[t]|Y[0], ..., Y[t-1], W}(y_v[t]|y[0], ..., y[t-1], W),
\]

as \(m \rightarrow \infty\), for \(t = 2, ..., n\) and \(v \in V\), for almost all \(y \in \mathbb{R}^{|V|}\). Therefore, we conclude that \(f_{\tilde{Y}_v^{(m)}|W}(y|W) \rightarrow f_{Y_v|W}(y|W)\) as \(m \rightarrow \infty\) for almost all \(y \in \mathbb{R}^{|V|}\) and any \(w \in \mathbb{Z}_{+}^{2kR_i}\).

Next we notice that, conditioned on the message vector \(W = w\), whether we make an error or not is a function of the received signals at all nodes during the \(n\) time steps (it is in fact only a function of the received signals at the destinations). Thus, there exists a set \(E_w \subseteq \mathbb{R}^{|V|}\) of received signals during the \(n\) time steps which cause a decoding error (at any of the decoders). We will let \(\mu_w^{(n)}\) be the probability measure on \(\mathbb{R}^{|V|}\) corresponding to \(Y\) (the received signals when using coding scheme \(C_n\)) conditioned on \(W = w\) and \(\mu_w^{(m,n)}\) be the probability measure
on $\mathbb{R}^{|V|}$ corresponding to $\tilde{Y}^{(m)}$ (the effective received signals when we use coding scheme $C_{m,n}^{*}$) conditioned on $W = w$. By Scheffé’s Theorem [11], we have that

$$\sup_{A \in \mathcal{B}} |\mu_{w}^{(n)}(A) - \mu_{w}^{(m,n)}(A)| \leq \int_{\mathbb{R}^{|V|}} \left| f_{\tilde{Y}|W}(y|w) - f_{\tilde{Y}^{(m)}|W}(y|w) \right| d\lambda \to 0, \text{ as } m \to \infty,$$

where $\mathcal{B}$ is the Borel $\sigma$-field on $\mathbb{R}^{|V|}$, and $\lambda$ is the Lebesgue measure. This, in turn, implies that for any choice of messages $w$, we must have $\lim_{m \to \infty} f_{w}^{(m,n)}(E_w) = \mu_{w}^{(n)}(E_w)$. We conclude that

$$\epsilon_{n,m} = 2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \Pr \left[ \tilde{Y}^{(m)}(w) \in E_w \mid W = w \right] = 2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \mu_{w}^{(m,n)}(E_w) \to \frac{2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \mu_{w}^{(n)}(E_w)}{2^{-n} \sum_{i=1}^{L} R_i} = \epsilon_n. \quad (18)$$

Therefore, we can choose, for each $n$, $m_n$ sufficiently large such that the probability of error of code $C_{m,n}^{*}$, $\epsilon_{m,n}$, is at most $2\epsilon_n$. Finally, we need to take care of the fact that $C_{m,n}^{*}$ uses randomized relaying and decoding functions. First, we notice that if we let $U_{m} \in \mathbb{R}^{n|V|}$ be the random vector corresponding to the $n|V|$ samples from $(-2^{-(m+1)}, 2^{-(m+1)})$ drawn at the $|V|$ nodes during $n$ time steps, then we can write

$$\epsilon_{m,n} = 2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \Pr \left[ \tilde{Y}^{(m)}(w) \in E_w \mid W = w \right] = E \left[ 2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \Pr \left[ \tilde{Y}^{(m)}(w) \in E_w \mid W = w, U_{m} = u \right] \right].$$

Therefore, there must exist some $u \in \mathbb{R}^{n|V|}$ for which

$$2^{-n} \sum_{i=1}^{L} R_i \sum_{w} \Pr \left[ \tilde{Y}^{(m)}(w) \in E_w \mid W = w, U_{m} = u \right] \leq \epsilon_{m,n}.$$

Thus, we define the coding scheme $C_{n}^{*}$ by having each node $v$ at time $t$ quantize its received signal with resolution $m_n$, add to it $u_v[t]$ (i.e., the entry of $u$ corresponding to node $v$ and time $t$) and then apply the relaying/decoding function from code $C_{n}^{*}$. It is then clear that $C_{n}^{*}$ has deterministic relaying/decoding functions, and its error probability is at most $\epsilon_{m,n} \leq 2\epsilon_n$. Therefore, the sequence of codes $C_{n}^{*}$, $n = 1, 2, \ldots$, has finite reading precision and achieves the rate tuple $R$.

V. Extension to General Traffic Demands

One immediate extension of the result in Theorem 1 is to consider wireless networks with general traffic demands. These could include non-unicast flows such as multicast and broadcast flows. We again consider an additive noise wireless network described by a directed graph $G = (V, E)$. This time, we will assume that traffic demands are given by $T(v, U) = 1$, for all $v \in V$ and $U \subseteq V$. This way, every node has a message for every subset of the remaining nodes.

By proving the worst-case noise result for a wireless network with such traffic demands, the result is also proved for any other traffic demand $T'$. To see this, notice that, if $C_T \subseteq \mathbb{R}^{V \times P(V)}$ is the capacity region of a wireless network with traffic demand $T(v, U) = 1$, for all $v \in V$ and $U \subseteq V$, then the capacity region of a wireless network with traffic demand $T'$ can be written as

$$C_{T'} = \{R \in C_T : R(v, U) = 0 \text{ if } T'(v, U) = 0\}.$$

Hence, if we prove that

$$C_{T', \text{AWGN}} \subseteq C_{T', \text{non-AWGN}},$$

we also prove that, for any traffic demand $T'$,

$$C_{T', \text{AWGN}} \subseteq C_{T', \text{non-AWGN}}.$$
We can now replace Definition 4 with the following.

**Definition 9.** A coding scheme \( C \) with block length \( n \in \mathbb{N} \) and rate tuple \( R \in \mathbb{R}^V \times \mathcal{P}(V) \) for an additive noise wireless network consists of:

1. Encoding/relaying functions \( r_v^{(t)} : \mathbb{R}^{t-1} \times \prod_{D \in \mathcal{P}(V)} \{1, \ldots, 2^{nR(v,D)}\} \rightarrow \mathbb{R} \), for \( t = 0, \ldots, n-1 \), for each node \( v \in V \), satisfying the average power constraint
   \[
   \frac{1}{n} \sum_{t=1}^{n} \left[ r_v^{(t)}(y_1, \ldots, y_{t-1}, w_v) \right]^2 \leq P,
   \]
   for all \( (y_1, \ldots, y_{t-1}) \in \mathbb{R}^{t-1} \) and \( w_v \in \prod_{D \in \mathcal{P}(V)} \{1, \ldots, 2^{nR(v,D)}\} \).
2. A decoding function \( g_u : \mathbb{R}^n \rightarrow \prod_{v \in V, D \in \mathcal{P}(V) : u \in D} \{1, \ldots, 2^{nR(v,D)}\} \) for each node \( u \in V \).

With this definition of a coding scheme, it is straightforward to extend Definitions 6, 7 and 8 to this setting. We can then generalize Theorem 1 as stated in Theorem 4.

Theorem 4 can be proved using essentially the same steps in the proof of Theorem 1. From the previous discussion, it suffices to prove this result for traffic demands given by \( T(v, U) = 1 \), for all \( v \in V \) and \( U \subseteq V \). To re-prove Theorem 2 in this new setting, we start by applying the OFDM-like scheme to the transmit and received signals of every node exactly as done in Section IV-A. Thus, the convergence in distribution of the effective additive noise terms to Gaussian, proved in Section IV-B, still holds. Therefore, we may assume that, as in the beginning of Section IV-C, we have \( k \) blocks of \( b \) network uses each, and we apply the OFDM-like scheme inside each length-\( b \) block. Next, by interleaving the network uses, we obtain \( b \) blocks of length \( k \) inside which the network is approximately AWGN. Furthermore, since we start off with a sequence of coding schemes \( C_k \) with finite reading precision, the proof of Lemma 2 holds verbatim, except that \( w \), the vector of messages chosen, is now a vector in \( \prod_{v \in V, D \in \mathcal{P}(V)} \{1, \ldots, 2^{kR(v,D)}\} \). Thus, within each length-\( k \) block, the probability that any node decodes any of its messages incorrectly (assuming all messages are chosen independently and uniformly at random) is upper bounded by \( \epsilon_{k,b} \), where \( \epsilon_{k,b} \rightarrow \epsilon_k \) as \( b \rightarrow \infty \) and \( \epsilon_k \rightarrow 0 \) as \( k \rightarrow \infty \).

In order to deal with the dependence between the noise realizations of different length-\( k \) blocks, we will again consider employing outer codes. This time, however, instead of having one outer code for each source-destination pair, we will have one outer code for each message \( w(v, D) \) (i.e., one outer code for each \( v \in V \) and \( D \in \mathcal{P}(V) \)). Thus, for each \( v \in V \) and \( D \in \mathcal{P}(V) \), we will define a broadcast discrete channel with input and output alphabet \( \{1, \ldots, 2^{kR(v,D)}\}^b \), where \( v \) is the source and all nodes in \( D \) are the destinations, which are all interested in the same message. We construct each code by sampling \( \{1, \ldots, 2^{kR(v,D)}\}^b \) uniformly at random. Let \( W(v,d)^b \) correspond to a random symbol chosen by \( v \) uniformly at random from \( \{1, \ldots, 2^{kR(v,D)}\}^b \), and \( W(v,d)_u^b \) be the corresponding output symbol at each node \( u \in D \). For the outer code associated with \( v \) and \( D \), we can achieve rate
\[
\frac{1}{bk} \min_{u \in D} I \left( W(v,D)^b; \hat{W}(v,D)_u^b \right) = \frac{1}{bk} \min_{u \in D} \left( H(W(v,D)^b) - H(W(v,D)^b | \hat{W}(v,D)_u^b) \right) \\
\geq R(v,D) - \max_{u \in D} \frac{1}{bk} \sum_{\ell=0}^{b-1} H(W(v,D)[\ell]|\hat{W}(v,D)_u[\ell]) \\
\overset{(i)}{\geq} R(v,D) - \max_{u \in D} \frac{1}{bk} \left( 1 + \epsilon_{k,b} k R(v,D) \right) \\
= R(v,D)(1 - \epsilon_{k,b} - \frac{1}{k}),
\]
where \( (i) \) follows from Fano’s Inequality, since, within each length-\( k \) block, we apply code \( C_k \) and we have an average error probability of at most \( \epsilon_{k,b} \). Therefore, by choosing \( b \) and \( k \) sufficiently large, our constructed code achieves arbitrarily close to \( R \) on the non-Gaussian additive noise wireless network.

The proof of Theorem 3 holds in this new setting almost verbatim. The only difference is that we now have one rate for each source \( s \in V \) and destination set \( D \subseteq V \) and the message vector \( w \) has size \( V \times \mathcal{P}(V) \); thus, the expressions for the error probability in (17) must be modified accordingly. This concludes the proof of Theorem 4.
VI. CONCLUDING REMARKS

In this work, we proved that the Gaussian noise is the worst-case noise in additive noise wireless networks. This extends the classical result that Gaussian noise is the worst-case noise for point-to-point additive noise channels, which is commonly used as a justification for the modeling of the noise in wireless systems as Gaussian noise. Thus, we provide formal evidence that this modeling is indeed justified beyond the point-to-point setting.

It is important to highlight the fact that we prove our result by actually constructing a coding scheme that performs well on a non-Gaussian network from a coding scheme designed to perform well on an AWGN network. This is different from the mutual-information-based proof for point-to-point channels, described in Section I, which relies on the Channel Coding Theorem, and, thus, in random coding arguments.

Another important point about the techniques we introduce is that the only information about the actual noise distributions required for the coding scheme construction are the mean and the variance. This means that, given a wireless network with unknown noise distributions where only the mean and variance can be measured, it is possible to construct a sequence of coding schemes that achieves the capacity of the corresponding AWGN network.

One simple extension of this work is to consider MIMO wireless networks; i.e., wireless networks where each node can have multiple antennas. It is not difficult to see that the same arguments will hold in this case, and the Gaussian noise can also be seen to be worst-case. But the tools we developed are in fact also useful for establishing several other worst-case results in different classes of problems. In particular, the same DFT-based linear transformation followed by an interleaving procedure was used in [12] in order to show that the Gaussian sources are worst-case data sources for distributed compression of correlated sources over rate-constrained, noiseless channels, with a quadratic distortion measure (i.e., in the context of the quadratic k-encoder source coding problem). A similar approach was also taken in [13], where the authors consider the problem of communicating a distributed correlated memoryless source over a memoryless network, under quadratic distortion constraints. In this setting they show that, (a) for an arbitrary memoryless network, among all distributed memoryless sources with a particular correlation, Gaussian sources are the worst compressible, that is, they admit the smallest set of achievable distortion tuples, and (b) for any arbitrarily distributed memoryless source to be communicated over a memoryless additive noise network, among all noise processes with a fixed correlation, Gaussian noise admits the smallest achievable set of distortion tuples.

We observe that establishing the worst-case noise for wireless networks can also be a useful tool in determining the relationship between the capacity regions of the same network under different channel models. For example, in [14], an additive uniform noise network is used as a way to connect the capacity region of Gaussian networks with the capacity region of truncated deterministic networks (first introduced in [7]). The worst-case noise result is used first to establish that the capacity region of a Gaussian network with noises distributed as $N(0, 1/12)$ is a subset of the capacity region of the same network with noises distributed uniformly in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Then, by noticing that the uniform noise network can be emulated on a network with truncated deterministic channels, it is shown that the capacity region of the truncated deterministic network where the nodes have slightly more power contains the capacity region of the corresponding Gaussian network.

Finally, we point out that the result in Theorem 3 is interesting in itself, since it implies that the capacity region when we restrict ourselves to coding schemes with finite reading precision and allow the precision to tend to infinity along the sequence of coding schemes is equal to the unrestricted capacity. In fact, it is not difficult to change the proof of the theorem in order to prove that $C^{(m)}$, the capacity region when we restrict ourselves to coding schemes where only $m$ bits after the decimal point are available, converges to the unrestricted capacity region $C$, as $m \to \infty$. Since in any practical wireless system the analog received signals must go through an analog-to-digital converter, this result essentially implies that by increasing the resolution of the analog-to-digital converters used in a wireless network, the capacity region of the practical system is indeed approaching the capacity region of the usual infinite-precision models used in the study of wireless networks.

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APPENDIX

Lemma 4. Let \( \lambda \) denote the Lebesgue measure. Then, for any convex set \( S \), \( \lambda(\partial S) = 0 \).

**Proof:** Consider any point \( p \in \partial S \). Clearly, \( p \notin S^o \), and by the Supporting Hyperplane Theorem [15], there exists a hyperplane that passes through \( p \) and contains \( S \) in one of its closed half-spaces. Let \( H \) be such a closed half-space. Since \( H \) is closed, it is clear that \( \partial S \subseteq H \). Then, for any closed ball \( B_r(p) \) centered at \( p \), it is clear that

\[
\frac{\lambda(B_r(p) \cap \partial S)}{\lambda(B_r(p))} \leq \frac{\lambda(B_r(p) \cap H)}{\lambda(B_r(p))} = 1/2.
\]

By Lebesgue’s Density Theorem, the set

\[
P = \left\{ p \in \partial S : \liminf_{\epsilon \to 0} \frac{\lambda(B_r(p) \cap \partial S)}{\lambda(B_r(p))} < 1 \right\}
\]

should have Lebesgue measure zero. But since \( P = \partial S \), we conclude that \( \lambda(\partial S) = 0 \). \( \blacksquare \)

Lemma 5. In the proof of Lemma 2, for each \( y \in \mathcal{Y} \), \( Q(y) \) is a convex set.

**Proof:** Consider two noise realizations \( z, z' \in Q(y) \) and fix some \( \alpha \in [0, 1] \). We will show that if we replace one of the components of \( z \) with the corresponding component of \( \alpha z + (1-\alpha)z' \), the resulting noise realization \( z'' \) is still in \( Q(y) \). Then, by using the same argument with \( z'' \) instead of \( z \), another component of \( z'' \) is replaced with a component \( \alpha z + (1-\alpha)z' \), and by repeating this argument, it follows that \( \alpha z + (1-\alpha)z' \) is itself in \( Q(y) \). Let us focus on the component corresponding to node \( v \) at time \( t \). Let \( y_v[\ell]^* \) be the noiseless version of the received signal at \( v \) at time \( t \) with its complete binary expansion. Since \( z \) and \( z' \) result in the same \( y \), we have that

\[
y_v[\ell] = [y_v[\ell]^* + z_v[\ell]]_\rho = [y_v[\ell]^* + z_v'[\ell]]_\rho.
\]

Now, if we assume wlog that \( z_v[\ell] \leq z_v'[\ell] \), we have

\[
[y_v[\ell]^* + z_v[\ell]]_\rho \leq [y_v[\ell]^* + \alpha z_v[\ell] + (1-\alpha)z_v'[\ell]]_\rho \leq [y_v[\ell]^* + z_v'[\ell]]_\rho.
\]

Thus, it follows that \( y_v[\ell] = [y_v[\ell]^* + \alpha z_v[\ell] + (1-\alpha)z_v'[\ell]]_\rho \), and by replacing \( z_v[\ell] \) with \( \alpha z_v[\ell] + (1-\alpha)z_v'[\ell] \), we obtain a noise realization \( z'' \) that is still in \( Q(y) \), and the lemma follows. \( \blacksquare \)

Lemma 3. Suppose \( Y \) is a random variable with density \( f \). Let \( \hat{Y}_m = [Y]_m + U_m \), where \( U_m \) is uniformly distributed in \( (-2^{-m-1}, 2^{-m-1}) \) and independent from \( Y \). Then each \( \hat{Y}_m \) has a density \( f_m \), and \( f_m \) converges pointwise almost everywhere to \( f \).

**Proof:** Since the density of \( U \left( -2^{-m-1}, 2^{-m-1} \right) \) is \( g(x) = 2^m I \{ x \in (-2^{-m-1}, 2^{-m-1}) \} \), \( \hat{Y}_m \) will have a density \( f_m \) that can be written, for almost all \( y \), as

\[
f_m(y) = E \{ g(y - [Y]_m) \mid [Y]_m = 2^m \\Pr \left[ y - [Y]_m \in (-2^{-m-1}, 2^{-m-1}) \right]
= 2^m \Pr \left[ y - [Y]_m \in (-2^{-m-1}, 2^{-m-1}) \right]
= 2^m \Pr \left[ [Y]_m \in (y - 2^{-m-1}, y + 2^{-m-1}) \right]
= 2^m \Pr \left[ [2^m Y] \in (y 2^{-m-1} - 1/2, y 2^{-m-1} + 1/2) \right]
= 2^m \Pr \left[ [y 2^{m-1} - 1/2, [y 2^{m-1} + 1/2]] \right]
= 2^m \Pr \left[ Y \in \left( 2^{-m}[y 2^{m-1} - 1/2], 2^{-m}[y 2^{m-1} + 1/2] \right) \right]
= 2^m \int_{a_m}^{b_m} f(x)dx,
\]

where \( a_m = 2^{-m}[y 2^{m-1} - 1/2] \) and \( b_m = 2^{-m}[y 2^{m-1} + 1/2] \). Notice that we can write \( b_m = a_m + 2^{-m} \). Moreover, we have that

\[
y - 2^{-(m+1)} \leq a_m < y + 2^{-(m+1)},
\]

(20)
from which we have \( a_m \to y \) as \( m \to \infty \). If we let \( F(y) \) be the cdf of \( Y \), then (19) can be written as

\[
\frac{F(b_m) - F(a_m)}{2^{-m}} = \frac{F(a_m + 2^{-m}) - F(a_m)}{2^{-m}} \triangleq q_m.
\] (21)

Our goal is to show that \( q_m \) converges to \( f(y) \) as \( m \to \infty \) for almost all \( y \). Since by assumption \( Y \) has an absolutely continuous distribution, \( F(y) \) is differentiable almost everywhere, so it suffices to show that \( q_m \) converges to \( f(y) \) as \( m \to \infty \) wherever \( F(y) \) is differentiable and the derivative is \( f(y) \). Thus, we focus on a \( y \) where \( F(y) = f(y) \).

Suppose by contradiction that \( q_m \) does not converge to \( f(y) \). Then there must be an \( \epsilon > 0 \) and a subsequence \( \{q_{m_i}\}_{i=1}^{\infty} \), such that one of the following

\[
q_{m_i} > f(y) + \epsilon \quad (22)
\]
\[
q_{m_i} < f(y) - \epsilon \quad (23)
\]

holds for all \( i \geq 1 \). Suppose wlog that we have a subsequence \( \{q_{m_i}\}_{i=1}^{\infty} \) for which (22) holds for all \( i \geq 1 \). We will now pick a further subsequence of \( \{q_{m_i}\}_{i=1}^{\infty} \) in the following way. First, we choose \( K \in \mathbb{Z}_+ \) large enough so that \( f(y)/K < \epsilon \), and we define \( K \) subsets of \( \{1, 2, \ldots\} \) as

\[
S_j = \left\{ i \geq 1 : y - 2^{-(m_i+1)} + \frac{j}{K} 2^{-m_i} \leq a_{m_i} < y - 2^{-(m_i+1)} + \frac{j}{K} 2^{-m_i} \right\},
\]

for \( j = 1, 2, \ldots, K \). From (20), the sets \( S_1, \ldots, S_K \) partition \( \{1, 2, \ldots\} \), and we must be able to find some \( S_j \) that is infinite. Suppose \( |S_j| = \infty \). Then we have a subsequence \( \{q_{m_i}\}_{i=1}^{\infty} \), which we re-index as \( \{q_{\ell_i}\}_{i=1}^{\infty} \). For each of the elements in this subsequence we have

\[
q_{\ell_i} = \frac{F(a_{\ell_i} + 2^{-\ell_i}) - F(a_{\ell_i})}{2^{-\ell_i}} = \frac{F(a_{\ell_i} + 2^{-\ell_i}) - F(y) + F(y) - F(a_{\ell_i})}{2^{-\ell_i}}
\]
\[
= \frac{a_{\ell_i} + 2^{-\ell_i} - y F(a_{\ell_i} + 2^{-\ell_i}) - F(y) + y - a_{\ell_i} F(y) - F(a_{\ell_i})}{2^{-\ell_i} y - a_{\ell_i}}
\]
\[
\leq \frac{2^{-\ell_i} y - a_{\ell_i}}{2^{-\ell_i}} \frac{F(a_{\ell_i} + 2^{-\ell_i}) - F(y)}{a_{\ell_i} + 2^{-\ell_i} - y} + \frac{2^{-\ell_i} (1/2 - (t-1)/K) F(y) - F(a_{\ell_i})}{2^{-\ell_i} y - a_{\ell_i}}
\]
\[
= \frac{(t/K + 1/2) F(a_{\ell_i} + 2^{-\ell_i}) - F(y)}{a_{\ell_i} + 2^{-\ell_i} - y} + (1/2 - (t-1)/K) \frac{F(y) - F(a_{\ell_i})}{y - a_{\ell_i}},
\] (24)

where (i) follows since \( F(y) \) is non-decreasing and \( \ell_i \in S_t \). Now, notice that the right-hand side in (24) has a limit, and, by taking the lim sup, we obtain

\[
\limsup_{i \to \infty} q_{\ell_i} \leq \left( \frac{t}{K} + \frac{1}{2} \right) F(y) + \left( \frac{1}{2} - (t-1)/K \right) f(y)
\]
\[
= \left( 1 + \frac{1}{K} \right) f(y) < f(y) + \epsilon.
\]

But this is a contradiction because all \( q_{m_i} \) satisfied \( q_{m_i} > f(y) + \epsilon \), and \( \{q_{\ell_i}\}_{i=1}^{\infty} \subseteq \{q_{m_i}\}_{i=1}^{\infty} \). We conclude that we must have

\[
\lim_{m \to \infty} q_m = f(y),
\]

which implies that \( f_m(y) \to f(y) \) as \( m \to \infty \). \( \blacksquare \)

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