Entanglement of three-qubit pure states in terms of teleportation capability

Soojoon Lee,† †Jaewoo Joo,‡ and Jaewan Kim§§

1 Department of Mathematics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea
2 Blackett Laboratory, Imperial College London, Prince Consort Road, London, SW7 2BW, UK
3 School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-722, Korea

(Dated: April 1, 2022)

We define an entanglement measure, called the partial tangle, which represents the residual two-qubit entanglement of a three-qubit pure state. By its explicit calculations for three-qubit pure states, we show that the partial tangle is closely related to the faithfulness of a teleportation scheme over a three-qubit pure state.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Mn 03.67.Hk

Quantum entanglement has been considered to be one of the most crucial resources in quantum information processing, and hence has been studied intensively in various ways. Nevertheless, there are still a number of open problems for entanglement, such as what is the best way to quantify the amount of entanglement for bipartite or multipartite states.

For two-qubit states, the Wootters’ concurrence C 1,2 is known as a good measure of entanglement, since from it we can directly derive the explicit formula for the entanglement of formation as well as being readily calculable. On the other hand, in the multi-qubit cases, or even in the three-qubit case, no entanglement measure as good as the concurrence of two qubits has been found yet.

Coffman et al. 3 presented an inequality to explain the relation between bipartite entanglement in a three-qubit pure state. The inequality is called the Coffman-Kundu-Wootters (CKW) inequality, which is

$$C_{12}^2 + C_{13}^2 \leq C_{2(13)}^2,$$

(1)

where $C_{12} = C(\text{tr}_3(\Psi_{123}))$, $C_{13} = C(\text{tr}_2(\Psi_{123}))$, and $C_{2(13)} = C(\Psi_{2(13)}) = 2\sqrt{\det(\text{tr}_3(\Psi_{123}))}$ for a three-qubit pure state $\Psi_{123} = \langle \psi \rangle_{123}$. Here, the subscripts represent the indices of the qubits.

From the CKW inequality, an entanglement measure for three-qubit pure states was naturally derived 4, 5.

It is called the 3-tangle $\tau$, which is defined as

$$\tau = C_{1(23)}^2 - C_{12}^2 - C_{13}^2,$$

(2)

and represents the residual entanglement of the state. Here $\tau$ is invariant under any qubit taken as the focus qubit, that is, for any distinct $i$, $j$, and $k$ in $\{1,2,3\}$,

$$\tau = C_{i(jk)}^2 - C_{ij}^2 - C_{ik}^2.$$

(3)

Furthermore, it was shown that $\tau$ is an entanglement monotone 6, and it was also shown that $\tau$ can distinguish the Greenberger-Horne-Zeilinger (GHZ) class from the W class 7, where the GHZ class and the W class are the sets of all pure states with true three-qubit entanglement equivalent to the GHZ state $|GHZ\rangle$,

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

(4)

under stochastic local operations and classical communication (SLOCC), and equivalent to the W state,

$$|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle),$$

(5)

under SLOCC, respectively.

Even though the 3-tangle $\tau$ is a useful entanglement measure for three-qubit pure states, in this paper, we investigate another quantity similar to $\tau$, defined as

$$\tau_{ij} = \sqrt{C_{i(jk)}^2 - C_{jk}^2},$$

(6)

d for distinct $i$, $j$, and $k$ in $\{1,2,3\}$. We call the quantity the partial tangle. Then we clearly obtain the following equalities:

$$\tau_{12} = \sqrt{C_{1(23)}^2 - C_{13}^2} = \sqrt{\tau + C_{12}^2} = \tau_{21},$$

$$\tau_{23} = \sqrt{C_{2(13)}^2 - C_{21}^2} = \sqrt{\tau + C_{23}^2} = \tau_{32},$$

$$\tau_{31} = \sqrt{C_{3(12)}^2 - C_{32}^2} = \sqrt{\tau + C_{31}^2} = \tau_{13},$$

(7)

and hence

$$\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2 = 3\tau + C_{12}^2 + C_{23}^2 + C_{31}^2.$$

(8)

We clearly remark that $\tau_{ij} = C_{ij}$ if and only if a given state is contained in the W class, that is, $\tau = 0$.

Observing the definition of $\tau_{ij}$ in Eq. (6), $\tau_{ij}$ seems to represent the residual two-qubit entanglement of a three-qubit pure state. However, we cannot say that $\tau_{ij}$ represents only the entanglement for two qubits in the compound system $ij$ since $\tau_{ij}$ is not equivalent to $C_{ij}$ in general as in Eq. (7). Therefore, in order to understand the entanglement of three-qubit states more evidently, it would be important to investigate the meaning of $\tau_{ij}$.
One-qubit orthogonal measurement

Two-qubit orthogonal measurement

Unitary operation

(i) One-qubit orthogonal measurement

(ii) Two-qubit orthogonal measurement (iii) Unitary operation

FIG. 1: Our modified teleportation scheme over a three-qubit state: The dotted boxes and ellipse represent performing the orthogonal measurements and applying the unitary operation, respectively. The arrows represent sending classical information corresponding to the measurement results.

In this paper, we explicitly calculate the partial tangle for three-qubit pure states so as to investigate its meaning, and we show that the partial tangle is closely related to a teleportation scheme over three-qubit pure states as a relation between the concurrence and the fully entangled fraction for two-qubit pure states.

We note that any three-qubit pure state $|\psi\rangle$ can be written in the form

\begin{equation}
|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\theta} |010\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle,
\end{equation}

where $i = \sqrt{-1}$, $0 \leq \theta \leq \pi$, $\lambda_j \geq 0$, and $\sum_j \lambda_j^2 = 1$. Thus, in order to calculate the partial tangles for three-qubit pure states, it suffices to consider the states for the cases in Eq. (9). By somewhat tedious but straightforward calculations, we obtain the following results on the partial tangles $\tau_{ij}$ for $|\psi\rangle$:

\begin{align*}
\tau_{12} &= 2\lambda_0 \sqrt{\lambda_2^2 + \lambda_4^2}, \\
\tau_{23} &= 2\sqrt{\lambda_0^2 \lambda_2^2 + \lambda_1^2 \lambda_4^2 + \lambda_2^2 \lambda_4^2 - 2\lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos \theta}, \\
\tau_{31} &= 2\lambda_0 \sqrt{\lambda_2^2 + \lambda_3^2}. 
\end{align*}

Since one of the most important practical features of entanglement is the teleportation capability, we now consider a teleportation scheme over a three-qubit state in the compound system 123, which is a modification of the splitting and reconstruction of quantum information over the GHZ state, introduced by Hillery et al. [9]. The modified scheme is illustrated in Fig. 1 and is described as follows: Let $i$, $j$, and $k$ be distinct in {1, 2, 3}. (i) Make a one-qubit orthogonal measurement on the system $i$. (ii) Prepare an arbitrary one-qubit state, and then make a two-qubit orthogonal measurement on the one qubit and the system $j$. (iii) On the system $k$, apply a proper unitary operation related to the 3-bit classical information of the two above measurement results.

We note that this scheme is nothing but a teleportation over the two-qubit state on the systems $j$ and $k$ after the measurement of the system $i$, and that the faithfulness of this teleportation completely depends on the probabilities corresponding to the one-qubit measurement results in step (i) and the resulting state of the systems $j$ and $k$ after the one-qubit measurement.

We remark that any observable for a one-qubit measurement can be described as

\begin{equation}
U^\dagger \sigma_3 U = U^\dagger |0\rangle \langle 0| U - U^\dagger |1\rangle \langle 1| U,
\end{equation}

where $\sigma_3 = |0\rangle \langle 0| - |1\rangle \langle 1|$ is one of Pauli matrices, and $U$ is a $2 \times 2$ unitary matrix. Thus, after the step (i) of the teleportation scheme over $|\psi\rangle$, the resulting 2-qubit state of the compound system $jk$ becomes

\begin{equation}
g_{jk}^t = \frac{\text{tr} \left( U^\dagger_1 |t\rangle \langle t| U_1 \otimes I_j |\psi\rangle \langle \psi| U^\dagger_1 |t\rangle \langle t| U_1 \otimes I_j \right)}{\text{tr} \left( |t\rangle \langle t| U_1 \otimes I_j |\psi\rangle \langle \psi| U^\dagger_1 |t\rangle \langle t| I_j \right)}
\end{equation}

with probability $\langle t| U_1 \rho U^\dagger_1 |t\rangle$ for $t = 0$ or 1, where $U_1$ is a $2 \times 2$ unitary matrix of the system $i$, and $\rho_i = \text{tr}_{jk}(|\psi\rangle \langle \psi|)$. Since $g_{jk}^t$ is the resulting state after the orthogonal measurement, it must be a 2-qubit pure state. For example, if $i = 1$, $j = 2$, $k = 3$, and

\begin{equation}
U_1 = \begin{pmatrix}
  u_{00} & u_{01} \\
  u_{10} & u_{11}
\end{pmatrix} \in U(2),
\end{equation}

then

\begin{equation}
g_{jk}^t |t\rangle \langle t| U_1^\dagger \rho_i U_1 |t\rangle = |\psi_{jk}^t\rangle \langle \psi_{jk}^t|,
\end{equation}

where

\begin{equation}
|\psi_{jk}^t\rangle = \frac{(\lambda_0 u_{0t} + \lambda_1 e^{i\theta} u_{1t})|00\rangle + \lambda_2 u_{1t}|01\rangle + \lambda_3 u_{1t}|10\rangle + \lambda_4 u_{1t}|11\rangle}{\text{tr} \left( |t\rangle \langle t| U_1 \rho U^\dagger_1 |t\rangle \langle t| I_j \right)}.
\end{equation}

For the moment, we shall review the properties of the faithfulness of a teleportation over a 2-qubit state. This faithfulness is naturally provided by teleportation’s fidelity [10],

\begin{equation}
F(\Lambda_\rho) = \int d\xi |\xi\rangle |\Lambda_\rho(|\xi\rangle \langle \xi|)|\xi\rangle,
\end{equation}

where $\Lambda_\rho$ is a given teleportation scheme over a 2-qubit state $\rho$, and the integral is performed with respect to the uniform distribution $d\xi$ over all one-qubit pure states. We also consider the fully entangled fraction [4, 11, 12, 13] of $\rho$ defined as

\begin{equation}
f(\rho) = \max \langle e|\rho|e\rangle,
\end{equation}

where $e$ is a maximally entangled two-qubit state.
where the maximum is over all maximally entangled states $|e\rangle$ of 2 qubits. It has been shown that the maximal fidelity achievable from a given bipartite state $\rho$ is

$$F(\Lambda_\rho) = \frac{2f(\rho) + 1}{3},$$

where $\Lambda_\rho$ is the standard teleportation scheme over $\rho$ to provide the maximal fidelity. Furthermore, for any two-qubit pure state $|\phi\rangle = \sqrt{\alpha}|00\rangle + \sqrt{\beta}|11\rangle$ with $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$, we can readily obtain that

$$f(|\phi\rangle\langle\phi|) = \frac{1}{2} + \sqrt{\alpha\beta},$$

$$C(|\phi\rangle\langle\phi|) = 2\sqrt{\alpha\beta},$$

and hence

$$C(|\phi\rangle\langle\phi|) = 2f(|\phi\rangle\langle\phi|) - 1 = 3F(\Lambda_{|\phi\rangle\langle\phi|}) - 2$$

for any two-qubit pure state $|\phi\rangle$.

Let us define $F_i$ as the maximal teleportation’s fidelity over the resulting 2-qubit state in the systems $j$ and $k$ after the measurement of the system $i$. Then, from the above review, it is straightforward to obtain that for $i \in \{1, 2, 3\}$

$$F_i = \frac{2f_i + 1}{3},$$

(21)

where

$$f_i = \max_{U_i} \left[ |\langle 0|U_i\rho_iU_i^\dagger|0 \rangle f(\rho_{jk}^0) + \langle 1|U_i\rho_iU_i^\dagger|1 \rangle f(\rho_{jk}^1) \right].$$

(22)

Here, the maximum is over all $2 \times 2$ unitary matrices. Since $\rho^0_{jk}$ is pure, $f_i$ can be rewritten as

$$f_i = \frac{1}{2} \max_{U_i} \left[ |\langle 0|U_i\rho_iU_i^\dagger|0 \rangle (1 + C(\rho_{jk}^0)) + \langle 1|U_i\rho_iU_i^\dagger|1 \rangle (1 + C(\rho_{jk}^1)) \right].$$

(23)

After tedious calculations [14], we get the following results:

$$f_1 = \frac{1}{2} + \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2\lambda_3\lambda_4 \cos \theta},$$

$$f_2 = \frac{1}{2} + \lambda_0 \sqrt{\lambda_3^2 + \lambda_4^2},$$

$$f_3 = \frac{1}{2} + \lambda_0 \sqrt{\lambda_1^2 + \lambda_2^2}. $$

(24)

Therefore, it follows from Eqs. (20), (24) and (22) that

$$\tau_{ij} = 2f_k - 1 = 3f_k - 2.$$  

(25)

We remark that $f_i \geq 1/2$ and $F_i \geq 2/3$ for three-qubit pure states, and that the above result in Eq. (25) is surprisingly of the same form as that in Eq. (20). Thus, we could say that $\tau_{ij}$ is a three-qubit version of the concurrence with respect to a teleportation over a three-qubit pure state. Moreover, it could be meaningful that a kind of mathematical quantity, $\tau_{ij}$, is closely concerned with $f_k$ and $F_k$ as the quantities derived from physical information processing, as in the two-qubit case.

In conclusion, we have considered the so-called partial tangle $\tau_{ij}$ as an entanglement measure for three-qubit pure states. We have also considered the quantities $f_k$ and $F_k$ obtained from the maximal fidelity of a teleportation scheme over a three-qubit pure state. By their explicit calculations for three-qubit pure states, we have shown that there exists a close relation between the mathematical quantity $\tau_{ij}$ related to the three-qubit entanglement and the physical quantities $f_k$ and $F_k$ related to the teleportation capability, as in the two-qubit case.

S.L. acknowledges V. Bužek and M. Horodecki for encouraging discussions, and J.J. thanks M.B. Plenio for useful advices. J.J. was supported by the Overseas Research Student Award Program for financial support, and J.K. by a Korea Research Foundation Grant (KRF-2002-070-C00029).

[1] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1996).
[2] S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022 (1997).
[3] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[4] V. Coffman, J. Kundu, and W.K. Wootters, Phys. Rev. A 61, 052306 (2000).
[5] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000).
[6] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.
[7] A. Acín, A. Andrianov, L. Costa, E. Jané, J.I. Latorre, and R. Tarrach, Phys. Rev. Lett. 85, 1560 (2000).
[8] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001).
[9] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A 59, 1829 (1999).
[10] S. Popescu, Phys. Rev. Lett. 72, 797 (1994).
[11] R. Horodecki, M. Horodecki, and P. Horodecki, Phys. Lett. A 222, 21 (1996).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).
[13] P. Badziąg, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 62, 012311 (2000).
[14] For example, we consider the case that $i = 1$, $j = 2$, $k = 3$, and $U_1$ is the same as that in Eq. (18), and hence $\rho_{23}^{12}$ is the same as that in Eq. (18) for $t = 0$ or 1. Since $C(\rho_{23}^{12}) = 2\sqrt{\det(\rho_{23}^{12})}$ and $\sum_i (t|U_i\rho_iU_i^\dagger|t) = 1$, we
obtain

\[ f_1 = \frac{1}{2} + \max_{U_1} \sum_{t=0}^{1} |u_{1t}| \left| \lambda_4 \left( \lambda_0 u_{0t} + \lambda_1 u_{1t} e^{i\theta} \right) - \lambda_2 \lambda_3 \lambda_4 u_{1t} \right| \]

\[ \leq \frac{1}{2} + \max_{U_1} \sqrt{\sum_{t=0}^{1} |\lambda_4 (\lambda_0 u_{0t} + \lambda_1 u_{1t} e^{i\theta}) - \lambda_2 \lambda_3 \lambda_4 u_{1t}|^2} \]

\[ = \frac{1}{2} + \sqrt{\lambda_0^2 \lambda_4^2 + \lambda_1^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - 2\lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos \theta}, \quad (26) \]

where the inequality is a consequence of the Cauchy-Schwarz inequality, and the last equality is independent of the above maximum, that is,

\[ \sum_{t=0}^{1} |\lambda_4 (\lambda_0 u_{0t} + \lambda_1 u_{1t} e^{i\theta}) - \lambda_2 \lambda_3 \lambda_4 u_{1t}|^2 \]

\[ = \lambda_0^2 \lambda_4^2 + \lambda_1^2 \lambda_4^2 + \lambda_2^2 \lambda_3^2 - 2\lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos \theta, \quad (27) \]

for any 2 × 2 unitary \( U_1 \). Since we can readily check that there exists a 2 × 2 unitary matrix providing the equality of the Cauchy-Schwarz inequality, we can prove the first equality in Eq. (24). The other equalities in Eq. (26) can be obtained in the same way.