PERIODIC SOLUTIONS AND MULTIHARMONIC EXPANSIONS FOR THE WESTERVELT EQUATION

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ABSTRACT. In this paper we consider nonlinear time periodic sound propagation according to the Westervelt equation, which is a classical model of nonlinear acoustics and a second order quasilinear strongly damped wave equation exhibiting potential degeneracy. We prove existence, uniqueness and regularity of solutions with time periodic forcing and time periodic initial-end conditions, on a bounded domain with absorbing boundary conditions. In order to mathematically recover the physical phenomenon of higher harmonics, we expand the solution as a superposition of contributions at frequencies that are multiples of a fundamental excitation frequency. This multiharmonic expansion is proven to converge, in appropriate function spaces, to the periodic solution in time domain.

1. Introduction. While classical ultrasound imaging is based on linear sound propagation, more recently the appearance of higher harmonics due to nonlinearity is being explicitly made use of in order to enhance resolution and image quality [11, 1] Physical and mathematical models describing nonlinear acoustics – usually in time domain – classically range from the spatially one-dimensional Burgers equation, [6] via the unidirectional Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation [43] to the Westervelt [42] and the Kuznetsov [32, 31] equation. However, modeling of nonlinear sound propagation is a highly active field of ongoing research and we refer to [21, 22] for some account on enhanced models and further references. The mathematical analysis of both classical and enhanced models (except for the Burgers equation that can be found in many textbooks on PDEs) is more recent (see, e.g., [23, 25, 33, 34, 36] as well as further references in [22]) and to some extent concentrates on initial value problems for these typically second order wave type PDEs. Recently, some interesting results in the time periodic setting have been obtained in, e.g., [9, 10].

Our aim is to establish well-posedness of these models with time periodic (instead of initial) conditions and to establish multiharmonic approximations in case of a single frequency excitation, in view of the relevance of this setting for the above mentioned harmonic imaging applications. To the best of our knowledge, in the context of equations of nonlinear acoustics, this has so far only been done for the
Burgers equation (see [12, 15, 28, 35, 19, 13, 14]) which is inherently spatially one-dimensional and structurally very different from the second order wave type PDEs comprising the Westervelt and Kuzentsov equation as well as their more recent extensions.

To this end, in this first study we consider, as one of the most established classical model of nonlinear acoustics, the Westervelt equation in pressure formulation

\[ u_{tt} - c^2 \Delta u - b \Delta u_t = \frac{\beta_a}{\rho c^2} (u^2)_{tt} + g \text{ in } \Omega \]  

where \( u \) is the acoustic pressure, \( c \) the speed of sound, \( b \) the diffusivity of sound, and \( \beta_a \) a nonlinearity parameter, which, with regard to applicability of our results to ultrasound imaging techniques based on tissue dependence of this parameter [2, 7, 8, 18, 41, 44, 45], we allow to vary in space. We assume (1) to hold on a domain \( \Omega \subseteq \mathbb{R}^3 \) and equip it with absorbing / impedance boundary conditions

\[ \beta u_t + \gamma u + \partial_\nu u = 0 \text{ on } \partial \Omega \]  

to enable restriction to a bounded computational domain \( \Omega \), which we assume to be \( C^{1,1} \) smooth. The development and use of nonreflecting boundary conditions such as (2), which is the most basic first order absorbing boundary condition for a strongly damped wave equation, has turned out to be of high practical relevance, since such conditions allow to restrict numerical computations to a limited domain of interest, instead of having to simulate wave propagation on the whole free space \( \mathbb{R}^3 \). Reviews on absorbing boundary conditions for wave equations are, e.g., [16, 17, 40]; see also [37] for further references and an adaptation to a model of nonlinear acoustics. Alternatively, so-called perfectly matched layers may be used to avoid spurious reflections on the truncation boundary. Excitation by, e.g., an array of piezoelectric transducers is here modeled by interior sources \( g \). Usually this would be done by means of inhomogeneous Neumann boundary conditions on some two-dimensional manifold \( \Sigma \), which would make the analysis considerably more technical, though, see, e.g., [24] for an analysis of the initial value problem for the Westervelt equation with inhomogeneous Neumann boundary conditions. Also, since the transducer array lies in the interior of the computational domain, modeling excitation by means of interior sources appears to be justified in the sense that we can consider \( g \) as an approximation of a source \( \tilde{g} \cdot \delta \Sigma \) concentrated on \( \Sigma \), in view of the fact that formally

\[
\left\{ \begin{array}{l}
-\Delta u = 0 \text{ in } \Omega \\
\partial_\nu u = 0 \text{ on } \partial \Omega \\
\left[\partial_\nu u\right] = \tilde{g} \text{ on } \Sigma
\end{array} \right.
\]

\[
\Leftrightarrow \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Sigma \tilde{g} v \, ds \quad \text{for all } v \in H^1(\Omega)
\]

\[
\Leftrightarrow \left\{ \begin{array}{l}
-\Delta u = \tilde{g} \delta \Sigma \text{ in } \Omega \\
\partial_\nu u = 0 \text{ on } \partial \Omega
\end{array} \right.
\]

where \( \left[\partial_\nu u\right] \) denotes the jump of the normal derivative of \( u \) over the interface \( \Sigma \).

In this paper we assume the forcing \( g \) to be periodic with respect to time with period \( T \). A solution of the linear wave equation with such a source term (and homogeneous boundary conditions) would be periodic as well, with the same period \( T \). In a nonlinear setting such as (1), not only the fundamental frequency \( \omega = \frac{2\pi}{T} \) will be addressed but also higher harmonics, i.e., contributions at multiples of \( \omega \), will emerge.
Here we first of all investigate existence and uniqueness of $T$-periodic solutions to (1), (2) under periodic forcing. Then we focus on single frequency excitations $g(x,t) = \Re \left( \sum_{j=1}^{M} \exp(\imath j\omega t) \hat{g}(x) \right)$, derive a multiharmonic expansion for the solution of (1), and prove its convergence in an appropriate function space setting.

Note that due to the presence of a viscous damping term in the Westervelt equation, the situation differs substantially from the one in the classical references on periodic solutions of nonlinear wave equations such as [3]. Note that we do not require spatial periodicity here. Also the above mentioned references [9, 10] consider time periodic solutions for equations of nonlinear acoustics, namely the Kuznetsov and the Blackstock-Crighton equation, respectively. Their results are based on techniques related to maximal $L^p$ regularity for the linearized equation, as opposed to energy estimates in Hilbert Sobolev spaces in our paper. The periodicity conditions they assume only involve zero order time derivatives, while we impose periodic initial-end conditions on both $u$ and $u_t$. Moreover, the results in [9] include the all-space case $\Omega = \mathbb{R}^3$, while we here only work on bounded domains. On the other hand, we here also treat absorbing boundary conditions as practically relevant for numerical computations. Since these can also be tackled by maximal $L^p$ regularity techniques (see [38] in the non-periodic setting) possibly the results in [9] extended to absorbing boundary conditions as well. We also refer to [29] and the references therein, for some recent result on time periodic strongly damped nonlinear wave equations in a general but semilinear setting.

Some helpful inequalities and notations that will be used in this paper are the following:

- spatial norms are denoted by single vertical lines;
- Poincaré-Friedrichs inequality:
  \[ |v|^2_{H^1(\Omega)} \leq C_{PF}^2 \left( |\nabla v|^2_{L^2(\Omega)} + |v|^2_{L^2(\partial\Omega)} \right); \]  \hfill (3)
- Trace Theorem:
  \[ |\text{tr}_{\partial\Omega} v|^2_{L^2(\partial\Omega)} \leq C_{tr} \|v\|^2_{H^1(\Omega)}; \]  \hfill (4)
- Sobolev embeddings, e.g., with $\Omega \subseteq \mathbb{R}^3$:
  \[ |v|_{H^s(\Omega)} \leq C_H^{H^s, L^s} |v|_{H^1(\Omega)} \quad \text{and} \quad |v|_{L^s(\partial\Omega)} \leq C_{H^s, L^s} |v|_{H^s(\Omega)} ; \]  \hfill (5)

2. Existence and uniqueness of periodic solutions. In fixed point form, with $\kappa = 2 \frac{\beta_a}{c^2}$, the Westervelt equation (1) reads

\[ (1 - \kappa u) u_{tt} - c^2 \Delta u - b \Delta u_t = \kappa u_t^2 + g , \]  \hfill (6)

This formulation also reveals the issue of potential degeneracy, which has to be counteracted by ensuring positivity of the coefficient $1 - \kappa u$ by controlling $u$ in $L^\infty(\Omega \times (0,T))$.

We first study the linearized equation

\[ \alpha u_{tt} - c^2 \Delta u - b \Delta u_t + \mu u_t = f , \]  \hfill (7)

with space- and time dependent $T$-periodic coefficients $\alpha$ and right hand side $f$, as well as positive constants $c$ and $b$, and follow the lines of [30] (where actually a nonlinear equation was analyzed) to prove existence and uniqueness of solutions in Section 2.1.
Based on these results, in Section 2.2 we employ a contraction fixed point argument (i.e., set \( \alpha = 1 - \kappa u, f = \kappa u^2 + g \) in (7)) to establish well-posedness of the nonlinear equation. This procedure is very much the same as in, e.g., [24], however, different energy estimates are required since the periodic setting does not allow direct access to \( L^\infty \) with respect to time estimates.

2.1. Periodic solutions of the linearized equation.

**Theorem 2.1.** For \( b, c, \beta, \gamma, T > 0, \) and
\[ \bullet \alpha \in L^\infty(0, T; L^\infty(\Omega)) \cap W^{1, \infty}(0, T; L^{3/2}(\Omega)), \quad \mu \in L^\infty(0, T; L^3(\Omega)) \] with
\[ 0 < \alpha \leq \alpha \text{ a.e. in } \Omega \times (0, T), \]
\[ \| \frac{1}{2} \alpha_t - \mu \|_{L^\infty(0, T; L^{3/2}(\Omega))} < \min \{ b, c^2 \beta + b \gamma \} \]
\[ \frac{C_2}{C_{PF}^2 (C_{H^2,L^2})^2}, \]
\[ \alpha(0) = \alpha(T), \quad \mu(0) = \mu(T), \]
\[ \bullet f \in L^2(0, T; L^2(\Omega)), \]
there exists a unique solution \( u \) of
\[ \begin{cases} \alpha u_{tt} - c^2 \Delta u - b \Delta u_t + \mu u_t = f & \text{in } \Omega \times (0, T), \\ \beta u_t + \gamma u + \partial_\nu u = 0 & \text{on } \partial \Omega \times (0, T), \\ u(0) = u(T), \quad u_t(0) = u_t(T) & \text{in } \Omega, \end{cases} \] (8)
with regularity
\[ u \in X := \{ v \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^{3/2}(\Omega)) \cap L^2(0, T; H^2(\Omega)) : \]
\[ \| \partial_\nu v \|_{H^1(0, T; L^2(\partial \Omega))} < \infty \text{ and } u(0) = u(T), \quad u_t(0) = u_t(T) \}
and the solution fulfills the estimate
\[ \| u \|_\Omega := \| u \|_{H^2(0, T; L^2(\Omega))} + \| u \|_{H^1(0, T; H^{3/2}(\Omega))} \]
\[ + \| u \|_{L^2(0, T; H^2(\Omega))} + \| \partial_\nu u \|_{H^1(0, T; L^2(\partial \Omega))}^2 \]
\[ \leq C^2 \| f \|_{L^2(0, T; L^2(\Omega))} \] (9)
for some constant \( C > 0 \) depending only on \( \alpha, b, c, \beta, \gamma, T, \) and the constants in (4), (5).

**Proof.** Galerkin approximation:

With a complete system of \( L^2 \) orthogonal eigenfunctions \( \{ w_i \}_{i \in \mathbb{N}} \) of the Laplacian (e.g., equipped with homogeneous Neumann boundary conditions) and \( V_n = \text{span}\{ w_1, \ldots, w_n \} \), so that \(- \Delta V_n = V_n \) and \( \bigcup_{n \in \mathbb{N}} V_n = H^1(\Omega), \) we define semidiscrete approximations \( u^n \in C^2(0, T; V_n) \) of the solution by
\[ \begin{cases} (\alpha u_{tt}^n + \mu u_t^n, \phi)_{L^2} + (c^2 \nabla u^n + b \nabla u_t^n, \nabla \phi)_{L^2(\Omega)} \\ + (\partial_\nu (b \beta u_t^n + (c^2 \beta + b \gamma) u^n_t + c^2 \gamma u^n), \partial_\nu \phi)_{L^2(\partial \Omega)} = (f, \phi)_{L^2}, \end{cases} \] (10)
for every \( \phi \in V_n \) pointwise a.e. in \( (0, T), \)
\[ u^n(0) = u^n(T), \quad u^n_t(0) = u^n_t(T) \text{ in } \Omega \]
which with
\[ u^n(x, t) = \sum_{i=1}^n \xi_i(t) w_i(x) \]
can be rewritten as a first order periodic system of ODEs for \( \zeta(t) = (\xi_1(t), \ldots, \xi_n(t), \xi_1'(t), \ldots, \xi_n'(t)) \)

\[
\zeta'(t) = \left( \begin{array}{cc}
I & 0 \\
0 & M(t) \\
\end{array} \right)^{-1} \left\{ \begin{array}{cc}
0 & I \\
-C & -B \\
\end{array} \right\} \zeta(t) + \left( \begin{array}{c}
0 \\
f \\
\end{array} \right)
\]

with

\[
M(t)_{i,j} = (\alpha(t)w_j, w_i)_{L^2} + b\beta(\text{tr}_{\partial\Omega}w_j, \text{tr}_{\partial\Omega}w_i)_{L^2(\partial\Omega)}
\]

\[
B_{i,j}(t) = (\mu(t)w_j, w_i)_{L^2} + b(\nabla w_j, \nabla w_i)_{L^2} + (c^2 + b\gamma)(\text{tr}_{\partial\Omega}w_j, \text{tr}_{\partial\Omega}w_i)_{L^2(\partial\Omega)}
\]

\[
C_{i,j} = c^2(\nabla w_j, \nabla w_i)_{L^2} + c^2\gamma(\text{tr}_{\partial\Omega}w_j, \text{tr}_{\partial\Omega}w_i)_{L^2(\partial\Omega)}
\]

\[f_i = (f(t), w_i)_{L^2},\]

where \( M(t) \) is positive definite and the spectral norm of its inverse is bounded by \( \frac{1}{2} \).

Indeed, by the Floquet Theorem (see, e.g., [39, Theorem 3.15, as well as Problem 3.43]), this system has a unique periodic solution, since the homogeneous system (i.e., the one with vanishing \( f \)) only has the zero solution, as will become evident from the energy estimates (16), (17), (18) below.

**Energy estimates:**

Setting \( \phi = u^n(t), \phi = u^n(t), \) and \( \phi = u^n(t), \) respectively, in (10) yields the energy identities

\[
\frac{1}{2} \frac{d}{dt} \left( b|\nabla u^n|^2_{L^2(\Omega)} + (c^2 + b\gamma)|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)} \right) + c^2|\nabla u^n|^2_{L^2(\Omega)} + c^2\gamma|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)}
\]

\[= (f - \mu u^n, u^n)_{L^2(\Omega)} - \frac{d}{dt} (u^n, \alpha u^n)_{L^2(\Omega)} + (u^n, \alpha_t u^n + \alpha u^n)_{L^2(\Omega)} - b\beta(\text{tr}_{\partial\Omega}u^n_t, \text{tr}_{\partial\Omega}u^n)_{L^2(\partial\Omega)},\]

\[
\frac{1}{2} \frac{d}{dt} \left( |\sqrt{\alpha u^n}|^2_{L^2(\Omega)} + c^2|\nabla u^n|^2_{L^2(\Omega)} + b\beta|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)} + c^2\gamma|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)} \right) + b|\nabla u^n|^2_{L^2(\Omega)} + (c^2 + b\gamma)|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)} = (f + (\frac{1}{2} \alpha_t - \mu) u^n, u^n)_{L^2(\Omega)}
\]

\[
\frac{1}{2} \frac{d}{dt} \left( b|\nabla u^n|^2_{L^2(\Omega)} + (c^2 + b\gamma)|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)} \right) + |\sqrt{\alpha u^n}|^2_{L^2(\Omega)} + b\beta|\text{tr}_{\partial\Omega}u^n|_{L^2(\partial\Omega)}
\]

\[-\frac{d}{dt} (c^2|\nabla u^n, \nabla u^n|_{L^2(\Omega)} + c^2\gamma(\text{tr}_{\partial\Omega}u^n, \text{tr}_{\partial\Omega}u^n)_{L^2(\partial\Omega)})\]

where we have used

\[
\frac{d}{dt} (u^n, \alpha u^n)_{L^2(\Omega)} = (u^n_t, \alpha u^n)_{L^2(\Omega)} + (u^n, \alpha_t u^n + \alpha u^n_t)_{L^2(\Omega)}
\]

\[
\frac{1}{2} \frac{d}{dt} |u^n|^2_{L^2(\Omega)} = \frac{1}{2} \alpha_t (u^n)^2 + \alpha u^n u^n_t
\]

\[
\frac{d}{dt} (\nabla u^n, \nabla u^n)_{L^2(\Omega)} = |\nabla u^n|^2_{L^2(\Omega)} + (\nabla u^n, \nabla u^n)_{L^2(\Omega)}.
\]

Integrating with respect to time from 0 to \( T \) in (11) and using the fact that all terms appearing under \( \frac{d}{dt} \) are \( T \) periodic, i.e., have the same values at 0 and \( T \), we
get
\[
\frac{c^2 \min\{1, \gamma\}}{C_{PF}^2} \left\| u^n_t \right\|_{L^2(0,T;H^1(\Omega))}^2
\leq c^2 \| \nabla u^n_t \|_{L^2(0,T;L^2(\Omega))}^2 + c^2 \gamma \| \text{tr}_\Omega u^n_t \|_{L^2(0,T;L^2(\partial\Omega))}^2
\leq \int_0^T \left( (f - \mu u^n_t, u^n_t)_{L^2(\Omega)} + (u^n_t, \alpha_t u^n_t + \alpha u^n_t)_{L^2(\Omega)} + b\beta |\text{tr}_\Omega u^n_t|_{L^2(\partial\Omega)}^2 \right) dt
\leq \left\| f + (\alpha_t - \mu) u^n_t \right\|_{L^2(0,T;H^1(\Omega))} \| u^n_t \|_{L^2(0,T;H^1(\Omega))} + b\beta C_{PF}^2 \| u^n_t \|_{L^2(0,T;H^1(\Omega))}^2
\leq \left\| f + (\alpha_t - \mu) u^n_t \right\|_{L^2(0,T;H^1(\Omega))} \| u^n_t \|_{L^2(0,T;H^1(\Omega))} + b\beta \| \text{tr}_\Omega u^n_t \|_{L^2(0,T;L^2(\partial\Omega))}^2
\leq \frac{1}{\sqrt{C_{PF}^2}} \left( \| f \|_{L^2(0,T;L^2(\Omega))} + C_{H_1,L_6}^2 \| \mu \|_{L^\infty(0,T;L^2(\Omega))} \| u^n_t \|_{L^2(0,T;H^1(\Omega))} \right)
\times \left\| \sqrt{\alpha_t} u^n_t \right\|_{L^2(0,T;L^2(\Omega))} + c^2 (1 + \gamma C_{tr}) \| u^n_t \|_{L^2(0,T;H^1(\Omega))}^2.
\]
where we have used
\[
\left\| vw \right\|_{H^1(\Omega)^*} = \sup_{\phi \in H^1(\Omega) \setminus \{0\}} \left\langle \phi, \frac{1}{\| \phi \|_{H^1(\Omega)}} \int_\Omega vw \phi dx \right\rangle
\leq \sup_{\phi \in H^1(\Omega) \setminus \{0\}} \frac{1}{\| \phi \|_{H^1(\Omega)}} \| v \|_{L^3(\Omega)} \| w \|_{L^6(\Omega)} \| \phi \|_{L^6(\Omega)}
\leq (C_{H_1,L_6}^2)^2 \| v \|_{L^3(\Omega)} \| w \|_{H^1(\Omega)}.
\]
We first exploit inequality (13) as it is self-contained in the sense that it implies an estimate on its own, namely, again using (15),
\[
\| u^n_t \|_{L^2(0,T;H^1(\Omega))} \leq \frac{C_{PF}^2}{\min\{b, c^2 \beta + \gamma\}} \| f + (\frac{1}{2} \alpha_t - \mu) u^n_t \|_{L^2(0,T;H^1(\Omega))} + (C_{H_1,L_6}^2)^2 \left\| \frac{1}{2} \alpha_t - \mu \right\|_{L^\infty(0,T;L^2(\Omega))} \| u^n_t \|_{L^2(0,T;H^1(\Omega))},
\]
and using the implication
\[
az^2 \leq bz + c \quad \Rightarrow \quad z \leq \frac{b + \sqrt{b^2 + 4ac}}{2a}
\]
with
\[
C_1 = \frac{C_{PF}^2}{\min\{b, c^2 \beta + \gamma\} - C_{PF}(C_{H_1,L_6}^2)^2 \| \alpha_t \|_{L^\infty(0,T;L^2(\Omega))},
\]
Inserting this into (12), (14) and using the implication
\[
az^2 \leq bz + c \quad \Rightarrow \quad z \leq \frac{b + \sqrt{b^2 + 4ac}}{2a}
\]
for $a, b, c \geq 0$, as well as the estimate $\|f\|_{L^2(0,T;H^1(\Omega)^*)} \leq \|f\|_{L^2(0,T;L^2(\Omega))}$, we obtain
\[ \|u^n\|_{L^2(0,T;H^1(\Omega))} \leq C_0 \|f\|_{L^2(0,T;H^1(\Omega))} \] (17)
\[ \|\sqrt{\alpha}u^n\|_{L^2(0,T;L^2(\Omega))} \leq C_2 \|f\|_{L^2(0,T;L^2(\Omega))} \] (18)
\[ \|\text{tr}_{\partial\Omega} u^n\|_{L^2(0,T;L^2(\partial\Omega))} \leq \tilde{C}_2 \|f\|_{L^2(0,T;L^2(\Omega))} \] (19)
with appropriate constants $C_0, C_2, \tilde{C}_2 > 0$.

**Weak limits:**
Due to boundedness (16)–(19) of the sequence $u_n$ in the reflexive (even Hilbert) space $X_0 := \{v \in H^2(0,T;L^2(\Omega)) \cap H^1(0;H^1(\Omega)) : \|\text{tr}_{\partial\Omega} v\|_{H^2(0,T;L^2(\partial\Omega))} < \infty \}$ and $u(0) = u(T), u_t(0) = u_t(T)$, we can extract a weakly convergent subsequence with weak limit $u \in X_0$. Based on linearity of the problem, it is readily checked that $u$ satisfies (8), first of all in a weak $H^1(\Omega)$ sense with respect to space, as well as the energy estimate
\[ \|u\|_{X_0} \leq C \|f\|_{L^2(0,T;L^2(\Omega))} \]

**Higher spatial regularity:**
To see that $u$ is in fact an element of $X$ and the PDE is satisfied in an $L^2(0,T;L^2(\Omega))$
\[ \Delta u(t) = -\alpha u(t) + \mu u_t(t) \] sense, we use the fact that the function $w$ defined by $w(t) := -\Delta (c^2 u(t) + bu_t(t)) = f(t) - \alpha u(t) - \mu u_t(t)$ is in $L^2(0,T;L^2(\Omega))$, as is $f - \alpha u - \mu u_t$ due to the above estimates. On the other hand, for any $\phi \in L^2(\Omega)$, the function $z(t) = b(-\Delta u(t), \phi)_{L^2(\Omega)}$ is uniquely determined as the periodic solution of the ODE
\[ z'(t) + \frac{c^2}{b} z(t) = (w(t), \phi)_{L^2(\Omega)}, \quad z(0) = z(T) \]
i.e.,
\[ z(t) = \frac{1}{1 - \exp(-\frac{c^2}{b} T)} \int_0^T \exp(-\frac{c^2}{b} (T - s)) (w(s), \phi) \, ds \]
\[ + \int_0^t \exp(-\frac{c^2}{b} (t - s)) (w(s), \phi) \, ds \]
and
\[ z'(t) = (w(t), \phi) - \frac{c^2}{b} z(t) \]
which implies
\[ \|\Delta u\|_{L^2(0,T;L^2(\Omega))} \leq C \|w\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|\alpha u\|_{L^2(0,T;L^2(\Omega))} + \|\mu u_t\|_{L^2(0,T;L^2(\Omega))}) \]
\[ \leq C(\|f\|_{L^2(0,T;L^2(\Omega))} + \|\alpha\|^{1/2}_{L^\infty(0,T;L^\infty(\Omega))} \|\sqrt{\alpha} u\|_{L^2(0,T;L^2(\Omega))} \]
\[ + C_2^{3/2} H_{L^2(0,T;L^2(\Omega))} \|u_t\|_{L^2(0,T;H^1(\Omega))} \]
\[ \leq \tilde{C}_3 \|f\|_{L^2(0,T;L^2(\Omega))} \cdot \]
We can thus apply elliptic regularity to obtain results on both $u$ and $u_t$. To do so,
we have to take into account the smoothness of the respective boundary conditions,
which turns out to be a limiting factor in case of $u_t$. More precisely, $u(t)$ and $u_t(t)$
satisfy the boundary conditions
\[ \partial_n u(t) = -\text{tr}(\beta u_t(t) + \gamma u(t)) \in H^{1/2}(\partial \Omega), \]
\[ \partial_n u_t(t) = -\text{tr}(\beta u_t(t) + \gamma u(t)) \in L^2(\partial \Omega), \] (20)
with
\[\|\text{tr}(\beta u_t + \gamma u)\|_{L^2(0,T; H^{1/2}(\partial \Omega))} + \|\text{tr}(\beta u_{tt} + \gamma u_t)\|_{L^2(0,T; L^2(\partial \Omega))} \leq \tilde{C}\|f\|_{L^2(0,T; L^2(\Omega))}\]
according to (16), (17) and the Trace Theorem, as well as (19). Therefore, elliptic regularity yields
\[\|u\|_{L^2(0,T; H^2(\Omega))} \leq C\|f\|_{L^2(0,T; L^2(\Omega))}, \quad \|u_t\|_{L^2(0,T; H^{3/2}(\Omega))} \leq C\|f\|_{L^2(0,T; L^2(\Omega))}.\]  
(21)

Remark 1. Note that \(H^1(0,T; H^{3/2}(\Omega)) \cap H^2(0,T; L^2(\Omega))\) embeds continuously into \(C([0,T]; H^{3/2}(\Omega)) \cap C^1([0,T]; L^2(\Omega))\), hence the periodicity conditions in the definition of \(X\) make sense in \(H^{3/2}(\Omega)\) and in \(L^2(\Omega)\), respectively (actually, by interpolation, in even stronger spaces).

Since we only need \(f \in L^2(0,T; L^2(\Omega))\), i.e., \(f\) is not necessarily continuous with respect to time, we also do not need to impose periodicity on \(f\). However, periodicity of \(\alpha\) was used several times in the energy estimates.

Remark 2. An inspection of the proof shows that Theorem 2.1 (and therewith also Theorem 2.2 below) remains valid with spatially varying \(W^{1,\infty}(\partial \Omega)\) coefficients \(\beta, \gamma\) as long as \(\beta, \gamma\) are bounded away from zero. In particular for the boundary regularity in (20) we apply the Trace Theorem to \(\tilde{\beta}u_t + \tilde{\gamma}u\) where \(\tilde{\beta}, \tilde{\gamma} \in W^{1,\infty}(\Omega)\) are extensions of \(\beta, \gamma\) to the interior or \(\Omega\) to still obtain \(\text{tr}(\beta u_t(t) + \gamma u(t)) \in H^{1/2}(\partial \Omega)\). Also pure Dirichlet in place of absorbing boundary conditions would be feasible; however, mixed Dirichlet - absorbing conditions might possibly spoil regularity in (20).

2.2. Periodic solutions of the nonlinear Westervelt equation. Applying Banach’s Fixed Point Theorem to the operator \(\mathcal{T}\) : \(v \mapsto u\) solving (8) with \(\alpha = 1 - \kappa\), \(f = \kappa (v)^2 + g\), \(\mu = 0\) on a suitable ball in \(X\) yields well-posedness of the time periodic problem for the Westervelt equation (1) with boundary conditions (2).

Theorem 2.2. For \(b, c^2, \beta, \gamma, T > 0\), \(\kappa \in L^\infty(\Omega)\), there exists \(\rho > 0\) such that for all \(g \in L^2(0,T; L^2(\Omega))\) with \(\|g\|_{L^2(0,T; L^2(\Omega))} \leq \rho\) there exists a unique solution \(u \in X\) of
\[\begin{align*}
  u_{tt} - c^2 \Delta u - b \Delta u_t &= \frac{\kappa}{2} (u^2)_{tt} + g \quad \text{in } \Omega \times (0,T), \\
  \beta u_t + \gamma u + \partial_n u &= 0 \quad \text{on } \partial \Omega \times (0,T), \\
  u(0) = u(T), \quad u_t(0) = u_t(T) \quad \text{in } \Omega,
\end{align*}\]
(22)
and the solution fulfills the estimate
\[\|u\|_X \leq \tilde{C}\|g\|_{L^2(0,T; L^2(\Omega))}\]
(23)
for some constant \(\tilde{C} > 0\) depending only on \(\alpha, b, c^2, \beta, \gamma, T, \|\kappa\|_{L^\infty(\Omega)}\), and the constants in (4), (5).

Proof. To prove that \(\mathcal{T}\) is a self-mapping on the ball \(B^X_r(0)\) in \(X\) with center zero and appropriately chosen radius \(r\), we estimate, for any \(v \in B^X_r(0)\), using
interpolation,
\[ \|1 - \alpha\|_{L^\infty(0,T;L^\infty(\Omega))} = \|\kappa\|_{L^\infty(0,T;L^\infty(\Omega))} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{3/4} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T} \|\nu\|_{H^{3/4}(0,T;H^{13/8}(\Omega))} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{3/4} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T} \|\nu\|_{H^1(0,T;H^{3/2}(\Omega))} \|\nu\|_{L^2(0,T;H^2(\Omega))}^{1/4} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{3/4} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T} \|\alpha\|_{L^\infty(0,T;L^{3/2}(\Omega))} = \|\kappa\nu\|_{L^\infty(0,T;L^{3/2}(\Omega))} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{1} \to L^\infty}^{0,T} C_{L^2 \to L^{3/2}} \|\nu\|_{H^2(0,T;L^2(\Omega))} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{1} \to L^\infty}^{0,T} C_{L^2 \to L^{3/2}} \|\nu\|_{L^2(0,T;H^{3/2}(\Omega))}^{1/2} \]

and,
\[ \|f - g\|_{L^2(0,T;L^2(\Omega))} = \|\kappa\nu\|_{L^4(0,T;L^4(\Omega))} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} (C_{H^{1} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T})^2 (\|\nu\|_{H^{1/2}(0,T;H^{9/8}(\Omega))})^2 \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} (C_{H^{1} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T})^2 (\|\nu\|_{H^{1/4}(0,T;L^2(\Omega))})^{3/4} \]
\[ \leq \|\kappa\|_{L^\infty(\Omega)} (C_{H^{1} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T})^2 (\|\nu\|_{L^2(0,T;H^{3/2}(\Omega))}) \]

To obtain \( \mathcal{T}(v) \in B^\kappa_{r^2}(0) \) from (9), with \( \alpha := \frac{1}{2} \), we therefore impose the following requirements on \( r \) and \( \rho \).

Contractivity of \( \mathcal{T} \) follows from the fact that for \( v_1, v_2 \in B^\kappa_r(0) \), the difference \( \hat{u} := v_1 - v_2 = \mathcal{T}(v_1) - \mathcal{T}(v_2) \) solves (8) with \( \alpha = 1 - \kappa v_1, f = \kappa((v_1 - v_2)u_2 + (v_1 - v_2)(v_1 + v_2)) \), where, similarly to above,

\[ \|(v_1 - v_2)u_2 + (v_1 - v_2)(v_1 + v_2)\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C_{H^{3/4} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T} \|v_1 - v_2\|_{H^{3/4}(0,T;H^{13/8}(\Omega))} \|u_2\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C_{H^{3/4} \to L^\infty}^{0,T} C_{H^{13/8} \to L^\infty}^{0,T} \|v_1 - v_2\|_{L^\infty} \|u_2\|_{L^\infty} \]

and

\[ \|(v_1 - v_2)(v_1 + v_2)\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq \|v_1 - v_2\|_{L^4(0,T;L^4(\Omega))} \|v_1 + v_2\|_{L^4(0,T;L^4(\Omega))} \]
\[ \leq (C_{H^{1/2} \to L^4}^{0,T})^2 (\|v_1 - v_2\|_{H^1(0,T;L^2(\Omega))})^{1/2} \]
\[ \leq (C_{H^{1/2} \to L^4}^{0,T})^2 (\|v_1 - v_2\|_{L^2(0,T;H^2(\Omega))})^{1/2} \]

hence

\[ \|\mathcal{T}(v_1) - \mathcal{T}(v_2)\| \leq C\|f\|_{L^2(0,T;L^2(\Omega))} \leq q(r)\|v_1 - v_2\|_x \]
with \( q(r) = C \left( C_{H^{5/2} \to L^\infty}^{(0,T)} C_{H^{13/8} \to L^\infty}^{(\Omega)} + 2(C_{H^{1/4} \to L^1}^{(0,T)})^2 (C_{H^{9/8} \to L^1}^{\Omega})^2 \right) r \), so that contractivity can be achieved by possibly decreasing \( r, \rho \) so that additionally to (24)

\[
C \left( C_{H^{5/2} \to L^\infty}^{(0,T)} C_{H^{13/8} \to L^\infty}^{(\Omega)} + 2(C_{H^{1/4} \to L^1}^{(0,T)})^2 (C_{H^{9/8} \to L^1}^{\Omega})^2 \right) r < 1
\]

holds.

**Remark 3.** Alternatively to using the fixed point operator \( T : v \mapsto u \) solving (8) with \( \alpha = 1 - \kappa v, f = \kappa (v_t)^2 + g, \mu = 0 \), we could do the analysis based on the fixed point operator \( T_1 : v \mapsto u \) solving (8) with \( \alpha = 1 - \kappa v, f = g, \mu = -\kappa v_t \). The estimates for proving the self-mapping property of \( T_1 \) remain the same for \( 1 - \alpha \) and are trivial for \( f \). For \( \mu \) we estimate

\[
\|\mu\|_{L^\infty(0,T;L^2(\Omega))} = \|\kappa v_1\|_{L^\infty(0,T;L^2(\Omega))}
\leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{1/3} \to L^\infty}^{(0,T)} C_{H^{1/2} \to L^3}^{\Omega}(v_1) \|v_1\|_{H^{2/3}(0,T;L^2(\Omega))}^{1/3}
\leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{1/3} \to L^\infty}^{(0,T)} C_{H^{1/2} \to L^3}^{\Omega}(v_1) \|v_1\|_{L^2(0,T;H^{3/2}(\Omega))}^{1/3}
\leq \|\kappa\|_{L^\infty(\Omega)} C_{H^{1/3} \to L^\infty}^{(0,T)} C_{H^{1/2} \to L^3}^{\Omega}(\kappa v_1)
\]

so that by a slight modification of (24) we can achieve that \( T_1 \) maps \( B_r^2(0) \) into itself for \( r, \rho \) sufficiently small. Contractivity of \( T_1 \) on \( B_r^2(0) \) with a potentially reduced radius \( r > 0 \) follows from the fact that for \( v_1, v_2 \in B_r^2(0) \), the difference \( \tilde{u} := u_1 - u_2 = T_1(v_1) - T_1(v_2) \) solves (8) with \( \alpha = 1 - \kappa v_1, f = \kappa ((v_1 - v_2)u_{2t} + \kappa(v_1 - v_2)v_t), \mu = -\kappa v_1 \).

3. Multiharmonic expansion of periodic solutions. In this section we consider the special case \( g(x,t) = \Re(\sum_{j=0}^M \exp(\imath j \omega t)\hat{g}_j(x)) \) for \( \omega = \frac{2\pi}{a}, \hat{g} \in L^2(\Omega; \mathbb{C}) \). Existence of a periodic solution to the Westervelt equation follows from the previous section. To transform (1) into frequency domain, the strategy is to insert the Ansatz

\[
u(x,t) = 1/2 \sum_{k=0}^\infty \left( \exp(\imath k \omega t)\hat{u}_k(x) + \exp(-\imath k \omega t)\overline{\hat{u}}_k(x) \right) = \Re \left( \sum_{k=0}^\infty \exp(\imath k \omega t)\hat{u}_k(x) \right)
\]

into (1) and use the Cauchy product formula for two series

\[
\left( \sum_{i=0}^\infty a_i \right) \left( \sum_{j=0}^\infty b_j \right) = \sum_{k=0}^\infty \sum_{\ell=0}^k a_j b_{k-j}.
\]

The latter holds provided both sums converge absolutely; however we will only consider finite sums anyway.

3.1. A multiharmonic approximation scheme. More precisely, we will consider approximation by finite sums (skipping the hats for simplicity)

\[
u_N(x,t) = 1/2 \sum_{k=0}^N \left( \exp(\imath k \omega t)\hat{u}_k^N(x) + \exp(-\imath k \omega t)\overline{\hat{u}}_k^N(x) \right)
\]

along with the corresponding finite Cauchy product formula

\[
\left( \sum_{i=0}^N a_i \right) \left( \sum_{j=0}^N b_j \right) = \sum_{k=0}^{2N} \sum_{\ell=\max\{k-N,0\}}^{\min\{k,N\}} a_j b_{k-j}.
\]

(25)
Lemma 3.1. For any \( u^N = \frac{1}{2} \sum_{k=0}^{N} \left( \exp(ik\omega t)u^N_k(x) + \exp(-ik\omega t)\overline{u^N_k(x)} \right) \), \( v^N = \frac{1}{2} \sum_{k=0}^{N} \left( \exp(ik\omega t)v^N_k(x) + \exp(-ik\omega t)\overline{v^N_k(x)} \right) \) \( \in X_N \) we have

\[
\text{Proj}_{X_N}(u^N v^N) = \frac{1}{2} \Re \left\{ \sum_{m=1}^{N} \left( \sum_{\ell=0}^{2N-m} u^N_m v^N_{m-\ell} + \sum_{k=2}^{2N-m} \left( \frac{u^N_{k-m} v^N_{k+m}}{2} + \frac{v^N_{k-m} u^N_{k+m}}{2} \right) \right) \exp(im\omega t) \right\}
\]

This expression is indeed an element of \( X_N \), due to the fact that \( H^2(\Omega; \mathbb{C}) \) is a Banach algebra, i.e., products of \( H^2(\Omega; \mathbb{C}) \) functions are contained in \( H^2(\Omega; \mathbb{C}) \).

Proof. The finite Cauchy product formula (25) yields

\[
u^N v^N = \frac{1}{2} \Re \left\{ \sum_{k=0}^{N} \min(k,N) \left( u^N_k v^N_{k-\ell} \exp(ik\omega t) + \overline{u^N_k} v^N_{k-\ell} \exp(i(k-2\ell)\omega t) \right) \right\}
\]

where the first term after projection just becomes

\[
\Re \left\{ \sum_{m=0}^{N} \sum_{k=0}^{N} u^N_m v^N_{m-k} \exp(i(k-2\ell)\omega t) \right\}
\]

and for the second one we compute

\[
\Re \left\{ \sum_{m=-N}^{N} \sum_{k=-N}^{N} u^N_{k-m} v^N_{k+m} \exp(im\omega t) \right\}
\]

\[
= \Re \left\{ \sum_{m=1}^{2N-|m|} \sum_{k=-N}^{N} \frac{u^N_{k-m} v^N_{k+m}}{2} \exp(im\omega t) \right\}
\]

\[
= \Re \left\{ \sum_{m=-N}^{N} \sum_{k=-N}^{N} u^N_{k-m} v^N_{k+m} \exp(im\omega t) \right\}
\]

since

\[
\Re \left\{ \sum_{m=-N}^{N} \sum_{k=-N}^{N} u^N_{k-m} v^N_{k+m} \exp(im\omega t) \right\}
\]

\[
= \Re \left\{ \sum_{m=1}^{N} \sum_{k=-m}^{N} u^N_{k+m} v^N_{k-m} \exp(-im\omega t) \right\}
\]

\[
= \Re \left\{ \sum_{m=1}^{N} \sum_{k=-m}^{N} u^N_{k+m} v^N_{k-m} \exp(im\omega t) \right\}
\]

\( \square \)
Hence we have

\[
\text{Proj}_{X_N}(u^N)^2 = \frac{1}{2} \Re \left\{ \left| u^N_N \right|^2 + \sum_{j=0}^{N} |u^N_j|^2 \right\}
\]

\[
+ \frac{1}{2} \Re \left\{ \sum_{m=1}^{N} \left( \sum_{\ell=0}^{m} u^N_{\ell} u^N_{m-\ell} + 2 \sum_{k=m+2}^{2N-m} \frac{u^N_{k-m} u^N_{k+m}}{2} \right) \exp(\im \omega t) \right\}
\]

and

\[
\text{Proj}_{X_N}((u^N)^2)_{tt} = -\omega^2 \frac{1}{2} \Re \left\{ \sum_{m=1}^{N} \left( \sum_{\ell=0}^{m} u^N_{\ell} u^N_{m-\ell} + 2 \sum_{k=m+2}^{2N-m} \frac{u^N_{k-m} u^N_{k+m}}{2} \right) m^2 \exp(\im \omega t) \right\}
\]

Inserting this into (1) and using linear independence of the functions \( t \mapsto \exp(\im \omega t) \), we end up with a completely coupled nonlinear system for the functions \( \{u^N_0, \ldots, u^N_N\} \).

\[
m = 0 \quad - (c^2 + \im \omega b) \Delta u^N_0 = 0
\]

\[
m = 1 \quad - \omega^2 \underbrace{u^N_1 - (c^2 + \im \omega b) \Delta u^N_1}_{= \hat{g}_1 - \frac{\kappa}{2} \omega^2 \left( u^N_0 u^N_1 + \sum_{k=1:2}^{2N-1} \frac{u^N_{k-1} u^N_{k+1}}{2} \right)}
\]

\[
m \in \{2, \ldots, N\} : \quad - \omega^2 m^2 u^N_m - (c^2 + \im \omega b) \Delta u^N_m = \hat{g}_m - \frac{\kappa}{4} \omega^2 m^2 \left( \sum_{\ell=0}^{m} u^N_{\ell} u^N_{m-\ell} + 2 \sum_{k=m+2}^{2N-m} \frac{u^N_{k-m} u^N_{k+m}}{2} \right)
\]

where we set \( \hat{g}_m = 0 \) for \( m > M \). Additionally to these PDEs on \( \Omega \), each of the individual functions \( u^N_j \) is supposed to satisfy the impedance boundary condition

\[
(\im \omega \beta + \gamma) u^N_j + \partial_t u^N_j = 0 \quad \text{on } \partial \Omega.
\]

Since the 0th of these boundary value problems implies that \( u^N_0 \) vanishes, we can reduce the system to

\[
m = 0 : \quad u^N_0 = 0
\]

\[
m = 1 : \quad - \omega^2 \underbrace{u^N_1 - (c^2 + \im \omega b) \Delta u^N_1}_{= \hat{g}_1 - \frac{\kappa}{2} \omega^2 \sum_{k=1:2}^{2N-1} \frac{u^N_{k-1} u^N_{k+1}}{2}}
\]

\[
m \in \{2, \ldots, N\} : \quad - \omega^2 m^2 u^N_m - (c^2 + \im \omega b) \Delta u^N_m = \hat{g}_m - \frac{\kappa}{4} \omega^2 m^2 \left( \sum_{\ell=1}^{m-1} u^N_{\ell} u^N_{m-\ell} + 2 \sum_{k=m+2}^{2N-m} \frac{u^N_{k-m} u^N_{k+m}}{2} \right)
\]

Note that without the underbraced terms, the system would be triangular and could be solved by substitution. As a matter of fact, this triangular system is obtained by neglecting the fact that \( u \) should be real valued and hence skipping \( \Re \), see Section 3.3 below.
3.2. A multilevel multiharmonic iteration scheme. In order to linearize these equations and moreover, to prove convergence of the sequence \( u^N \) to the unique periodic solution \( u \) of (1), we will modify the above approach to a multilevel fixed point method. I.e., we will consider the sequence of linearized projected equations

\[
\text{Proj}_{X^N} \left( (1 - \kappa u^{N-1}) u_{tt}^N - c^2 \Delta u^N - b \Delta u_t^N - \kappa (u_{t}^{N-1})^2 - g \right) = 0 \quad (31)
\]

with boundary conditions (2) and employ energy estimates as those in Section 2.1 as well as a fixed point (or rather an induction) argument to prove well-posedness.

Before doing so, we rewrite (31) as a multiharmonic system, equipped with boundary conditions (29). Analogously to above, formally setting \( u_{tt}^{N-1} = 0 \), we get

\[
\text{Proj}_{X^N} (u_{tt}^{N-1}) = -\omega^2 \frac{1}{2} \Re \left\{ \sum_{j=1}^{N} j^2 u_j^{N-1} \sum_{m=1}^{m-1} (m-\ell)^2 u_{m-\ell}^{N-1} \exp(im\omega t) \right\}
\]

and

\[
\text{Proj}_{X^N} (u_{tt}^{N-1})^2 = -\omega^2 \frac{1}{2} \Re \left\{ \sum_{j=1}^{N-1} j^2 |u_j^{N-1}|^2 + \sum_{m=1}^{N-1} (m-\ell)^2 u_{m-\ell}^{N-1} \exp(im\omega t) \right\} + \sum_{m=1}^{N-1} \sum_{k=m+2}^{2N-2} \frac{N+1}{4} u_{N+1}^{N-1} \exp(im\omega t) \}
\]

Thus, computation of \( u^N \) according to (31) amounts to solving the following set of coupled linear Helmholtz type equations

\[
m = 0: \quad \kappa \omega^2 \sum_{j=1}^{N} j^2 u_j^{N-1} u_j^N - (c^2 + \omega b) \Delta u_0^N = -\omega^2 \sum_{j=1}^{N} j^2 |u_j^{N-1}|^2
\]

\[
m = 1: \quad -\omega^2 (1 - \kappa u_0^{N-1}) u_1^N - (c^2 + \omega b) \Delta u_1^N
\]

\[
+ \kappa \omega^2 \sum_{k=1}^{2N-1} \left( \frac{(k+1)^2 u_{N+1}^{N-1} u_k^{N-1}}{2} + \frac{(k-1)^2 u_{N-1}^{N-1} u_k^{N-1}}{2} \right)
\]

\[
= \hat{g}_1 - \kappa \omega^2 \sum_{k=3}^{2N-1} \frac{(k+1)(k-1)}{4} u_{N+1}^{N-1} u_k^{N-1}
\]

(32)
where due to the fact that 

\[ u \rightarrow \| u \|_{\alpha, f} \leq \| u \|_{\alpha, f} \]

and therefore

\[ \| u \|_{\alpha, f} \leq \| u \|_{\alpha, f} \]

for every

\[ \int_{0}^{T} (\alpha u_{t}^{N} + (c^{2} \beta + b \gamma) u_{t}^{N} + c^{2} \gamma u_{t}^{N}, \phi)_{L^{2}(\Omega)} - (f, \phi)_{L^{2}} \] \( dt = 0 \)

(34)

with

\[ \alpha = 1 - \kappa u^{N-1}, \quad f = \kappa(u^{N-1})^{2} + g. \]

Existence and uniqueness of a solution to (34) for general \( \alpha, f \) satisfying the conditions of Theorem 2.1 as well as the estimate (using the analogous test functions \( \alpha, \xi, \xi' \), \( \kappa, \xi, \xi' \), which are indeed contained in \( X_{N} \))

\[ \| u^{N} \|_{X} \leq C \| f \|_{L^{2}(0, T; L^{2}(\Omega))} \]

follows analogously to the proof of Theorem 2.1. Hence, as in the proof of Theorem 2.2 we obtain existence of a solution to (34) with (35), for each \( N \in \mathbb{N} \), and satisfying \( \| u^{N} \|_{X} \leq r \) provided \( \| u^{N-1} \|_{X} \leq r \) and \( \| g \|_{L^{2}(0, T; L^{2}(\Omega))} \leq \rho \), with \( r, \rho \) chosen according to (24).

To estimate the error \( u^{N} - u \) between the so defined approximation and the solution \( u \) to (22), we use the fact that \( \hat{u}^{N} := u^{N} - \text{Proj}_{X_{N}}^{X} u \in X_{N} \) solves (34) with

\[ \alpha = 1 - \kappa u^{N-1}, \]

\[ (f, \phi) = \kappa(u^{N-1} - u)_{11} + \kappa(u^{N-1} - u)_{11}(u^{N-1} + u)_{11} + \kappa(u^{N-1} + u)_{11}, \phi) \]

where due to the fact that \( \bigcup_{N \in \mathbb{N}} X_{N} \) is dense in \( X \), the term \( \text{Proj}_{X_{N}}^{X} u - u \) tends to zero as \( N \to \infty \) in \( X \) and therefore

\[ \| A(\alpha)(\text{Proj}_{X_{N}}^{X} u - u) \|_{X_{N}} \leq \| A(\alpha) \|_{X_{N} \to X_{N}} \| \text{Proj}_{X_{N}}^{X} u - u \|_{X} \to 0 \) as \( N \to \infty. \)

As in the proof of contractivity of \( T \) for Theorem 2.2, we therefore get

\[ \| u^{N} - \text{Proj}_{X_{N}}^{X} u \|_{X} \leq q \| u^{N-1} - u \|_{X} + C \| \text{Proj}_{X_{N}}^{X} u - u \|_{X} \]
for some $q \in (0, 1)$, $C > 0$, hence, by the triangle inequality $\|u^N - u\|_X \leq q\|u^{N-1} - u\|_X + (C + 1)\|\text{Proj}_X^N u - u\|_{X^*}$ and therefore the error estimate

$$\|u^N - u\|_X \leq q^N\|u^0 - u\|_X + C N \sum_{j=1}^{N} q^{N-j} \|\text{Proj}_X^j u - u\|_X \to 0 \quad (36)$$

as $N \to \infty$.

**Theorem 3.2.** Under the conditions of Theorem 2.2 the multiharmonic approximations defined by (32), (29) converge to the solution $u$ to (22) in $X$ as $N \to \infty$, with the error estimate (36) for some $q \in (0, 1)$, $C > 0$.

Note that the 0th equation in (32) is fully coupled. This can be amended by modifying (31) to

$$\text{Proj}_X^N \left( (1 - \kappa u^{N-1})u^N_{tt} - c^2 \Delta u^N - b \Delta u^N - \kappa u^{N-1}_t u^N_t - g \right) = 0 \quad (37)$$

since the approximation to the quadratic terms altogether can be written as a time derivative $-\kappa u^{N-1}_t u^N_t - \kappa u^{N-1}_t u^N_t = -\kappa (u^{N-1} u^N)_t$, hence its $\exp(\im \omega t)$ term vanishes. More precisely, by Lemma 3.1, we have

$$\text{Proj}_X^N (u^{N-1} u^N_t) = \frac{\omega^2}{2} \Re \left\{ \sum_{j=0}^{N} \im j u^{N-1}_j u^N_j \right\}$$

$$+ \frac{\omega^2}{2} \Re \left\{ \sum_{m=1}^{N} i \sum_{\ell=0}^{m-1} (m-\ell) u^N_{\ell} u^{N-1}_{m-\ell} \right.$$ \n
$$\left. + \sum_{k=m:2}^{2N-m} \left( \frac{k+m}{2} u^{N-1}_{k+m} u^N_{k+m} - \frac{k-m}{2} u^{N-1}_{k-m} u^N_{k-m} \right) \exp(\im m \omega t) \right\}$$

and, after differentiation with respect to time

$$\text{Proj}_X^N (u^{N-1} u^N_t)_t$$

$$= -\frac{\omega^2}{2} \Re \left\{ \sum_{m=1}^{N} m \sum_{\ell=0}^{m-1} (m-\ell) u^{N-1}_{\ell} u^N_{m-\ell} \right.$$ \n
$$\left. + \sum_{k=m:2}^{2N-m} \left( \frac{k+m}{2} u^{N-1}_{k+m} u^N_{k+m} - \frac{k-m}{2} u^{N-1}_{k-m} u^N_{k-m} \right) \exp(\im m \omega t) \right\}$$
Theorem 3.3. Under the conditions of Theorem 2.2 the multiharmonic approximations defined by (38), (22) converge to the solution \( u \) to (22) in \( X \) as \( N \to \infty \), with the error estimate (36) for some \( q \in (0, 1) \), \( C > 0 \).

3.3. A simplified setting. Ignoring the fact that the excitation and the solution should be real valued, i.e., simply setting \( g(x, t) = \exp(\omega t)\hat{g}(x) \) and making the ansatz \( u(x, t) = \sum_{k=0}^{\infty} \exp(ik\omega t)\hat{u}_k(x) \) one ends up with a lower triangular system

\[
\begin{align*}
m = 0 : & \quad - (\varepsilon^2 + \omega b) \Delta \hat{u}_0 = 0 \quad (\Rightarrow \hat{u}_0 = 0) \\
m = 1 : & \quad - \omega^2 \hat{u}_1 - (\varepsilon^2 + \omega b) \Delta \hat{u}_1 = \hat{g}_1 \\
m = 2, 3, \ldots : & \quad - \omega^2 m^2 \hat{u}_m - (\varepsilon^2 + \omega b) \Delta \hat{u}_m = \hat{g}_m - \frac{\kappa}{2} \omega^2 m^2 \left( \sum_{\ell=1}^{m-1} \hat{u}_\ell \hat{u}_{m-\ell} \right)
\end{align*}
\]

The functions defined by the partial sums

\[
u^N(x, t) = \sum_{k=0}^{N} \exp(ik\omega t)\hat{u}^N_k(x)
\]
can be easily verified to solve (26), with the redefined space $X_N$ of complex valued functions $X_N := \{\sum_{k=0}^{N} \exp(ik\omega t) v_k(x) : v_k \in H^2(\Omega; \mathbb{C})\}$. Analogously to Theorems 3.2, 3.3 we get convergence of the iterates from the multilevel schemes defined by (31), (37), i.e., by

\[
\begin{align*}
    m = 0 : & \quad u_0^N = 0 \\
    m = 1 : & \quad -\omega^2 u_1^N - (c^2 + \kappa \omega b) \Delta u_1^N = \hat{g}_1 \\
    m = 2, 3, \ldots : & \quad -\omega^2 m^2 u_m^N - (c^2 + \kappa \omega b) \Delta u_m^N \\
                                & \quad = \hat{g}_m - \kappa \omega^2 \sum_{\ell=1}^{m-1} \left( (m-\ell)^2 u_{\ell-1}^{N-1} u_{m-\ell}^N + \ell(m-\ell) u_{\ell-1}^{N-1} u_{m-\ell}^N \right)
\end{align*}
\]

and

\[
\begin{align*}
    m = 0 : & \quad u_0^N = 0 \\
    m = 1 : & \quad -\omega^2 u_1^N - (c^2 + \kappa \omega b) \Delta u_1^N = \hat{g}_1 \\
    m = 2, 3, \ldots : & \quad -\omega^2 m^2 u_m^N - (c^2 + \kappa \omega b) \Delta u_m^N \\
                                & \quad = \hat{g}_m - \kappa \omega^2 \sum_{\ell=1}^{m-1} m(m-\ell) u_{\ell-1}^{N-1} u_{m-\ell}^N
\end{align*}
\]

respectively, with boundary conditions (29), to the solution $u$ of the periodic Westervelt equation. Note that these are complex valued approximations to the real valued function $u$, though, i.e., convergence takes place in the complex valued counterpart of the space $X$ and is therefore only of limited practical use. In particular, taking only the real part of these functions $u^N$ in general does not yield a convergent approximation to $u$.

4. Conclusion and outlook. In this paper we have established periodic solutions and multiharmonic expansions for the Westervelt equation, a classical second order wave type model of nonlinear acoustics. In this first study we assume the excitation to emerge from an interior source. An excitation model based on Neumann boundary conditions on a surface lying inside the domain $\Omega$ will be subject of future research.

Moreover, we intend to study related problems (existence of periodic solutions as well as mutiharmonic expansions thereof) for enhanced models of nonlinear acoustics, see, e.g., [4, 20, 26, 27].

We here allowed to $\kappa$ to vary in space since this inhomogeneity is key for certain imaging applications. In particular, our assumption $\kappa \in L^\infty(\Omega)$ allows for the practically relevant setting of coefficients with jumps in their values. However, also the coefficients $b, c^2$, might actually be space-dependent functions. Due to the fact that these coefficients appear within second order space derivative terms, cf. e.g., [5], either sufficient smoothness of these coefficients will be needed, or, if only $b, c^2 \in L^\infty(\Omega)$ is assumed, the Westervelt equation will have to be modified, e.g., as in [5].

Finally, as the multiharmonic expansions can also be viewed as numerical tools for approximating periodic solutions, we plan to carry out an implementation as well as numerical tests with this approach.
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