Emergence of Geometric phase shift in Planar Non-commutative Quantum Mechanics

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Abstract

Appearance of adiabatic geometric phase shift in the context of non-commutative quantum mechanics is studied using an exactly solvable model of 2D simple harmonic oscillator (SHO) in Moyal plane, where momentum non-commutativity are also considered along with spatial non-commutativity. After finding a suitable Bopp’s shift, that bridges the non-commutative phase space operators with their effective commutative counterparts, having their dependence on the non-commutative parameters, we study the adiabatic evolution in the Heisenberg picture. An explicit expression for the geometric phase shift under adiabatic approximation is then found without using any perturbative technique. Lastly, this phase is found to be related to the Hannay’s angle of a classically analogous system, by studying the evolution of the coherent state of this system.

1 Introduction

Ever since M. Berry observed the occurrence of geometrical phase obtained in the adiabatic transport of a quantum system around a closed loop in the parameter space, the concept of Berry phase has attracted great interest both theoretically [1, 2] as well as experimentally [3, 4]. Particularly it has been observed in certain condensed matter system, that this phase can give rise to effective non-commutative structure among the coordinates and thereby impacting physics [5, 6, 7]. Besides, its occurrence in the lowest Landau level in the Landau problem is quite well-known. Apart from these effective low energy effects, there are very strong plausibility arguments due to Doplicher et al. [8, 9] that these noncommutative algebra satisfied by spatio-temporal coordinates, when elevated to the level of operators, can naturally serve as a "deterrent" against gravitational collapse, associated with the localisation of an event at the Planck scale. Here the status of the noncommutative parameters are more fundamental, as if they are new constant of Nature, like $\hbar, G, c$ etc [10] and possibly can play a vital role in the development of a future theory of quantum gravity [11].

This aspect was also corroborated in a separate study of low energy limit of string theory by Seiberg-Witten [12].
Besides this above mentioned non-commutative structure among the space-time co-
ordinates, it has also been proposed that along with the spatial components of space-
time, momenta components too can satisfy a non-commutative algebraic structure 
[13, 14]. This was indicated by the reciprocity theorem proposed by Max Born way 
back in 1938 [15]. Further, it was observed in [16] that, to maintain Bose-Einstein statist-
tics in noncommutative spaces one needs to introduce noncommutative momenta as well . In fact the sheer presence of momentum noncommutativity itself can have nontrivi-
al astrophysical consequences, like increasing Chandrasekhar mass limit for the white
dwarf stars, as has been shown by one of the authors very recently in [17]. The phase-
space noncommutative structure also shown to emerge naturally in certain system in 
an enlarged phase space analysis [18]. It is therefore quite natural to investigate the 
occurrence of Berry phase, if any, in a quantum mechanical system where both position 
and momentum operators satisfy noncommutative algebra. This will then serve, in some 
sense, as the converse of the case, where the occurrence of Berry phase can give rise to 
noncommutative algebra [5], as mentioned earlier. Indeed, as we shall see in this paper 
that the occurrence of both types of non-commutativity in a quantum system plays a 
vital and necessary role for the existence of non-vanishing geometric phase shift. In this 
context, we would like to mention that some authors [19] had considered this problem 
earlier, in a different system involving gravitational potential well and found no geo-
metrical phase shift. As we demonstrate below in the Appendix that this difference 
stands from a particular form of a Bopp shift (or Seiberg-Witten map in the parlance 
of [19]) which is a scaled version of (51) and its realisation (52). On the other hand, 
we make use of a generalised Bopp shift (54), as explained in the Appendix, thereby 
generating a crucial dilatation term in the Hamiltonian, whose presence is shown to be 
quite indispensable in getting the desired geometrical phase shift.

Furthermore, an interrelation between the extra quantal geometric phase, apart from 
the dynamical phase, in the wave function in the quantum description and the corre-
sponding angle shift at classical level was established by Berry through semiclassical 
torus quantization [1]. The change of the angle was found to be related to the rate of 
change of the extra phase with respect to the quantum number of the state which is be-
ing transported through adiabatically. This can be viewed as a manifestation of Bohr’s 
correspondence principle for phases arising through adiabatic transports in the respec-
tive quantum and classical systems. However, in the present note, we shall establish 
this classical correspondence by extending Berry’s analysis to non-stationary coherent 
states, in the spirit of [20], representing localized non-spread wave packets which are 
being transported along classical trajectories.

The paper is organised as follows: We begin by introducing in sec-II a noncommutative phase space operators where we 
consider momentum space noncommutativity, along with the spatial ones of Moyal 
type. Here we also compute the instantaneous energy spectrum of two dimensional har-
monic oscillator whose mass and frequency parameters are slowly varying with respect 
to time by making of a generalised non-canonical phase space transformation, the so 
called generalised Boop shift (see Appendix ), which maps the noncommutative phase 
space variables to their commutative counterparts. In sec-III we find out, in Heisenberg 
picture, the extra phase factor which is acquired by the of creation and annihilation 
operators under an adiabatic excursion in parameter space of the system. We then 
discuss the geometric phase shift in state space of the oscillator in sec-IV and provide
2 Planar Non-Commutative space:

We start by considering 2D harmonic oscillator on the Moyal plane, with time dependent coefficients $P(t), Q(t)$ varying adiabatically with period $T$:

$$\mathcal{H}(t) = P(t)(\hat{p}_1^2 + \hat{p}_2^2) + Q(t)(\hat{x}_1^2 + \hat{x}_2^2)$$  \hspace{1cm} (1)

such that $P(t)Q(t) > 0$ and these time-dependent parameters are assumed to subsume all other parameters like mass, frequency etc. We are further assuming that the momentum components also satisfy a non-commutative algebra $[21][22][23][24]$ in addition to the position coordinates, so that the entire non-commutative structure takes the following form:

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij}; [\hat{p}_i, \hat{p}_j] = i\eta\epsilon_{ij}; [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}; \theta\eta < 0$$ \hspace{1cm} (2)

Note that we need to enforce $\theta\eta$ to be negative for consistent quantization see for example $[25][26]$ and references therein.

In order to carry out the diagonalization, we first carry out the following linear transformation $(\hat{x}_i, \hat{p}_j) \rightarrow (q_i, p_j); i, j = 1, 2$, known in the literature as generalized Bopp’s shift (See Appendix for other kinds of Bopp shift $[19][27][28]$ and their relation to the one given below):

$$\begin{align*}
\hat{x}_i &= q_i - \frac{\theta}{2\hbar}\epsilon_{ij}p_j + \frac{\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}q_j \\
\hat{p}_i &= p_i + \frac{\eta}{2\hbar}\epsilon_{ij}q_j + \frac{\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}p_j,
\end{align*}$$ \hspace{1cm} (3)

where $q_i$ and $p_i$ are commuting coordinates and momenta respectively satisfying the usual Heisenberg algebra: $[q_i, q_j] = 0 = [p_i, p_j]; [q_i, p_j] = i\hbar\delta_{ij}$ and are distinguished by the absence of over head hats.. Although this transformation $[57]$ is not a canonical transformation it nevertheless helps us in diagonalising the Hamiltonian. Substituting (3) in (1) results in the following form of the Hamiltonian:

$$\mathcal{H}(t) = \alpha(t)(p_1^2 + p_2^2) + \beta(t)(q_1^2 + q_2^2) + \delta(t)(p_1q_i + q_ip_i) - \gamma(t)(q_1p_2 - q_2p_1);$$ \hspace{1cm} (4)

where the time dependent coefficients $\alpha, \beta, \gamma, \delta$ are given by,
\[\alpha(t) = P(t) \left\{ 1 + \left( \frac{\sqrt{-\eta \theta}}{2\hbar} \right)^2 \right\} + Q(t) \left( \frac{\theta}{2\hbar} \right)^2\]

\[\beta(t) = Q(t) \left\{ 1 + \left( \frac{\sqrt{-\eta \theta}}{2\hbar} \right)^2 \right\} + P(t) \left( \frac{\eta}{2\hbar} \right)^2\]

\[\gamma(t) = 2 \left[ P(t) \left( \frac{\eta}{2\hbar} \right) + Q(t) \left( \frac{\theta}{2\hbar} \right) \right]\]

\[\delta(t) = \left( \frac{\sqrt{-\theta \eta}}{2\hbar} \right) \left[ P(t) \left( \frac{\eta}{2\hbar} \right) - Q(t) \left( \frac{\theta}{2\hbar} \right) \right]\]

At this stage we recognise the Hamiltonian as a combination of three terms,

\[\mathcal{H}(t) = \mathcal{H}_{gho,1}(t) + \mathcal{H}_{gho,2}(t) + \mathcal{H}_{J3}(t)\]  \hspace{1cm} \text{(6)}

where \(\mathcal{H}_{gho,i}(t)'s \ (i=1 \text{ or } 2)\) are like generalised time dependent harmonic oscillator Hamiltonian along \(i^{th} - \text{direction}\),

\[\mathcal{H}_{gho,i}(t) = \alpha(t)(p_i^2) + \beta(t)(q_i^2) + \delta(t)(p_i q_i + q_i p_i) \quad \text{(no sum on } i\text{)} \hspace{1cm} \text{(7)}\]

and

\[\mathcal{H}_{J3}(t) = -\gamma(t)(q_1 p_2 - q_2 p_1)\]  \hspace{1cm} \text{(8)}

which is like a Zeeman term. In order to diagonalize the whole Hamiltonian, we first need to diagonalize \(\mathcal{H}_{gho,i}(t)\) of \(\text{(7)}\) for each \(i\), so that these Hamiltonians can be brought into the form \(\mathcal{H}_{gho,i}(t) = X(t)(a_i^\dagger a_i + 1) \quad \text{(no sum on } i\text{)}\). So we introduce \(a_1, a_2\) with the following structure, which indeed do this job:

\[a_j = \left( \frac{\beta}{2\hbar \sqrt{\alpha \beta - \delta^2}} \right)^{1/2} \left[ q_j + \left( \frac{\delta}{\beta} + i \frac{\sqrt{\alpha \beta - \delta^2}}{\beta} \right) p_j \right]; \quad j = 1, 2 \hspace{1cm} \text{(9)}\]

satisfying \([a_i, a_j^\dagger]\) = \(\delta_{ij}\). Note that, we have \(\beta > 0\) and \(\alpha \beta - \delta^2 = \left( \frac{P \eta}{2\hbar} - \frac{Q \theta}{2\hbar} \right)^2 + PQ > 0\), as follows from \(\text{(5)}\) and from the fact that \(PQ > 0\). The entire Hamiltonian \(\text{(4)}\) then takes the following form:

\[\mathcal{H}(t) = \hbar \omega \left( \sum_{j=1,2} a_j^\dagger a_j + 1 \right) + i\hbar \gamma \left( a_1^\dagger a_2 - a_2^\dagger a_1 \right); \quad \omega = 2 \sqrt{\alpha \beta - \delta^2} \hspace{1cm} \text{(10)}\]

Noting at this stage \(\text{(32)}\), that the second non-diagonal term, is like the Schwinger representation of \(J_2\) angular momentum operator \(\hat{\jmath} = a_i^\dagger (\sigma_i^+) a_j\), we carry out another additional unitary tranformation of the following form, which can bring the term into the exact diagonal form of \(J_3\), while retaining the diagonal form of the first term:

\[
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix} \rightarrow
\begin{bmatrix}
  a_+ \\
  a_-
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & -i \\
  i & -1
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix} \hspace{1cm} \text{(11)}
\]

\[\begin{array}{c}
[a_i, a_j^\dagger] = \delta_{ij};\ [a_i, a_j] = 0 \quad (i, j \in \{+, -\})
\end{array} \hspace{1cm} \text{(12)}\]
Finally, the diagonalized Hamiltonian in the standard quadratic form reads,

\[ \mathcal{H}(t) = \hbar \sum_{j=+,-} \omega_j a_j \dagger a_j + \hbar \omega; \ \omega_{\pm} = \omega \mp \gamma. \]  

Note that here we have identified two characteristic frequencies \( \omega_{\pm} \) of the system.

The eigenvalue equation of this Hamiltonian is,

\[ \mathcal{H}(t) |n_1, n_2(t)\rangle = E_{n_1 n_2}(t) |n_1, n_2(t)\rangle, \]  

whose solution spectrum can virtually be read-off from (13) as,

\[ E_{n_1 n_2}(t) = \hbar (n_1 + n_2 + 1) - \hbar \gamma (n_1 - n_2) \]

where \( n_1, n_2 \) are semipositive definite integers and \( a_{\pm}(t) |0, 0(t)\rangle = 0 \). This reproduces the spectrum obtained in [33]. Clearly the spectrum is non-degenerate and one can safely assume that there will not be any level crossing during the adiabatic process. In this context, we would like to point out that essentially the same system was analysed in [21] but using Bartelomi’s realisation. One can easily check that both the algebra and spectrum in [21] agrees with (2,15) respectively by making the following simple formal replacements in (51) and (52) : \( \theta \rightarrow \xi^{-1} \theta; \ \eta \rightarrow \xi^{-1} \eta; \ \hbar \rightarrow \hbar_{\text{eff}} = \hbar \xi^{-1} \).

Finally let us write down our previous operators \( a_1, a_2 \) of (9) in short as,

\[ a_i = A(t) q_i + (B(t) + iC(t)) p_i \ ( i \in \{1,2\}) \]

where \( A(t) = \left( \frac{\beta}{\hbar \omega} \right)^{1/2}; B(t) = \frac{\delta}{\sqrt{\beta \hbar \omega}}; C(t) = \frac{1}{2} \sqrt{\frac{\omega}{\hbar}} \)  

\[ \tag{16} \]

We see that \( a_{\pm}, a_{\pm}\dagger \) has explicit time dependence through the time dependence of \( A, B, C \).

**3 Heisenberg evolution of Ladder operators:**

In this section we will solve the adiabatic evolution in Heisenberg picture to look for geometric phase shift. The equation of motion for a generic operator \( \hat{O} \) are given by \( \frac{d\hat{O}}{dt} = \frac{1}{i\hbar} [\hat{O}, \mathcal{H}] + \frac{\partial \hat{O}}{\partial t} \). Specifically, for the ladder operators, they take the following forms:

\[ \begin{bmatrix} \frac{da_+}{dt} \\ \frac{da_-}{dt} \\ \frac{da_+\dagger}{dt} \\ \frac{da_-\dagger}{dt} \end{bmatrix} = \begin{bmatrix} X_+ & 0 & 0 & Y \\ 0 & X_- & Y & 0 \\ 0 & Y^* & X_+ & 0 \\ Y^* & 0 & 0 & X^* \end{bmatrix} \begin{bmatrix} a_+ \\ a_- \\ a_+\dagger \\ a_-\dagger \end{bmatrix} \]  

\[ \tag{17} \]

\[ X_\pm = \frac{A}{4} \pm i \left( \gamma \mp 2C\beta \mp \frac{(B+iC)}{2C} \right), \ Y = -\frac{(B+iC)}{2C} \]

Uptill now all the expressions we have found are exact. However, here onwards we will start considering the adiabaticity of \( P(t) \) and \( Q(t) \). Note, from the dependence of \( A, B, C, \alpha, \beta, \gamma, \delta \) on \( P(t), Q(t) \), we anticipate that they also follow the same order.
of adiabaticity as $P$ and $Q$, i.e. if $\dot{P}, \dot{Q} \approx \epsilon$, $\ddot{P}, \ddot{Q} \approx \epsilon^2$, then $F \approx \epsilon$, $\dddot{F} \approx \epsilon^2$. We won’t omit any term under adiabatic approximation right now, but will only keep track of the order of various terms. Eventually, it will be clear, that it is the second or higher order terms which are ignorable [34, 35].

We can now decouple the four coupled equations occurring in (17) by taking the derivatives of these equations and combining them suitably to get:

$$
\frac{d^2 a_+}{dt^2} = \frac{da_+}{dt} \left( X_+ + \frac{\dot{Y}}{Y} + X^*_+ \right) + a_+ \left( X_+ - \frac{\dot{Y}}{Y} X_+ + Y Y^* - X_+ X^*_+ \right)
$$

We can make some important observations here, if $Y \approx \epsilon$, then $\dot{Y} \approx \epsilon^2$, and so $\dot{Y}Y \approx \epsilon$. Now substituting $X_+, X_-, Y$ from (17) and only retaining terms involving $\dot{Y}Y$, we deduce

$$
\frac{d^2 a_+}{dt^2} = \frac{da_+}{dt} \left( \mathfrak{P} + \dot{Y} \right) + a_+ \left( \Omega - \dot{Y} X_+ \right)
$$

(19)

where,

$$
\mathfrak{P} = \left( 2\frac{\dot{A}}{A} + 2i\gamma + \frac{\dot{C}}{C} \right)
$$

$$
\Omega = i \left( \dot{\gamma} - 2\frac{d}{dt}(C\beta) - \gamma \frac{\dot{C}}{C} - 2\frac{\dot{A}}{A}\gamma \right) + \left\{ \gamma^2 - 4C^2\beta^2 - 2B\beta \right\} + \mathcal{O}(\epsilon^2)
$$

(20)

As one can check that the differential equation satisfied by $a_-$ also has a similar form. To proceed further we now need to cast (19) into it’s, the so called, normal form. To that end, we define another time-dependent operator $b(t)$ as,

$$
a_+(t) = b(t)e^{\int t \left( \mathfrak{P} + \dot{Y} \right) dt},
$$

(21)

In terms of $b(t)$ the equation (19) can now be re-written as,

$$
\ddot{b} + b \left( \frac{\dot{\mathfrak{P}}}{2} - \frac{\dot{\Omega}}{4} - \Omega \right) + b \left( \frac{\dot{Y}}{Y} X_+ - \frac{\mathfrak{P}}{2Y} \dot{Y} + \mathcal{O}(\epsilon^2) \right) = 0
$$

(22)

Now, we have $\dot{Y}Y = B+iC \frac{\dot{C}}{C}$ as follows from (17). Let’s write it as

$$
\dot{Y}Y = Z + i\bar{Z} - \frac{\dot{C}}{C}
$$

(23)

here both $Z$ and $\bar{Z} \approx \mathcal{O}(\epsilon)$ and corresponds respectively to the real and imaginary part of $\frac{B+iC}{B+i\bar{C}}$.

Then using the expressions of $\mathfrak{P}$ and $\Omega$ from (20) we get,

$$
\ddot{b} + b(U + iV) = 0
$$

(24)
where,

\[ U = 4C^2 \beta^2 + 2 \dot{B} \beta + 2 \ddot{Z} C \beta + \mathcal{O} (\epsilon^2) \approx \mathcal{O} (\epsilon^0) \]

\[ V = 2 \frac{d}{dt} (C \beta) - 2C \beta \left( Z - \frac{\dot{C}}{C} \right) + \mathcal{O} (\epsilon^2) \approx \mathcal{O} (\epsilon) \]  \hfill (25)

Note that, since we are working in adiabatic regime, the functions \( U \) and \( V \) varies very very slowly with time. Hence, the formula for WKB approximation for complex potential \[36\] can be applied to get the general solution of the differential equation as,

\[
\begin{align*}
\mathbf{b}(t) &= \mathbf{b}(0) \left[ \frac{C_1}{\sqrt{\xi(t)}} \exp \left( \int_0^t (i \xi(\tau) - \phi(\tau)) d\tau \right) + \frac{C_2}{\sqrt{\xi(t)}} \exp \left( \int_0^t (-i \xi(\tau) + \phi(\tau)) d\tau \right) \right] \\
&= \mathbf{b}(0) \left[ \frac{1}{2} \exp \left( \int_0^t ( \xi(\tau) - \phi(\tau)) d\tau \right) + \frac{1}{2} \exp \left( \int_0^t ( -i \xi(\tau) + \phi(\tau)) d\tau \right) \right] \\
&\approx \mathbf{b}(0) \exp \left( \int_0^t \left( -i \mathcal{E} \right) d\tau \right)
\end{align*}
\]  \hfill (26)

where, \( \sqrt{U + iV} = \xi + i\phi \) and \( (C_1, C_2) \) are arbitrary coefficients. This result can also be derived by solving the differential equation using the method of successive approximation and considering the adiabatic variation of \( U \) and \( V \).

In our case, this boils down to

\[
\xi = \sqrt{\frac{\sqrt{U^2 + V^2} + U}{2}} \approx \sqrt{\frac{U + V^2}{4U}} \approx \sqrt{U} \approx 2C \beta + \frac{\dot{B} \beta + C \ddot{Z}}{2C \beta} \\
\phi = \sqrt{\frac{\sqrt{U^2 + V^2} - U}{2}} \approx \sqrt{\frac{V^2}{4U}} \approx \frac{2i}{4C \beta} (C \beta - 2C \beta \left( Z - \frac{\dot{C}}{C} \right))
\]  \hfill (27)

Note that we have ignored second and higher order terms. We now observe that the solution must satisfy the boundary condition: \( \mathbf{b}(t = 0) = \mathbf{b}(0) \). Also, the periodicity of the parameters imply \( \sqrt{\xi(0)} = \sqrt{\xi(T)} \). Finally, it can be observed that, only the second term with coefficient \( C_2 \) in the solution \[26\], yields the dynamical phase of \( a_+ \) with proper sign. This will eventually be clear as we calculate \( a_+(T) \). We therefore set \( C_1 = 0 \) in \[26\]. Now combining all these expressions, the particular solution of \[22\] is obtained as:

\[
\mathbf{b}(T) \approx \mathbf{b}(0) \exp \left( \int_0^T \left\{ -i \left( 2C \beta + \frac{\dot{B} \beta + C \ddot{Z}}{2C \beta} \right) + \phi \right\} d\tau \right)
\]  \hfill (28)

Now, we had, \( \tilde{\gamma} \gamma = Z + i\ddot{Z} - \frac{\dot{C}}{C} \). Being an exact differential we can write,

\[
\int_0^T \frac{\tilde{\gamma} \gamma}{Y} d\tau = \int_0^T \left( Z + i\ddot{Z} - \frac{\dot{C}}{C} \right) d\tau = \int_0^T Z d\tau + i \int_0^T \ddot{Z} d\tau = 0
\]  \hfill (29)

So, \( \int_0^T Z d\tau = \int_0^T \ddot{Z} d\tau = 0 \), implying that, \( \phi \) is also an exact differential.

Hence using \[21\], we can essentially drop the term involving just the exact derivatives and then split the respective dynamical and geometric phase shifts as,

\[
\mathbf{a}_+(T) = \mathbf{a}_+(0) \exp \left( -i \int_0^T \left( 2C \beta + \frac{\dot{B} \beta + C \ddot{Z}}{2C \beta} \right) d\tau + \frac{1}{2} \int_0^T \left( \frac{\dot{A}}{A} - 2i\gamma + \frac{\dot{C}}{C} + \frac{\tilde{\gamma} \gamma}{Y} \right) d\tau \right) = \mathbf{a}_+(0) \exp \left( -i \int_0^T \left( 2C \beta - \gamma \right) + \left( \frac{\dot{B} \beta + C \ddot{Z}}{2C \beta} \right) \right) \]  \hfill (30)
And the solution becomes,

\[
\mathbf{a}_+(T) = \mathbf{a}_+(0) \exp \left( -i \int_0^T \left[ (2C\beta - \gamma) + \left( \frac{\dot{B}}{2C} \right) \right] d\tau \right)
\]

\[
= \mathbf{a}_+(0) \exp \left( -i \int_0^T (\hbar \omega - \gamma \hbar) d\tau - i \int_0^T \frac{\beta}{\omega} \frac{d}{d\tau} \left( \frac{\delta}{\beta} \right) d\tau \right),
\]

with the two terms in the exponent representing the dynamical and the geometrical phases, respectively.

Finally, a close look into the decoupled evolution equation of \( \mathbf{a}_- \) in (18) tell us that, it is similar to the one for \( \mathbf{a}_+ \), except that \((+\gamma)\) is replaced by \((-\gamma)\). Also, \(\gamma\) is entering into the solution only through the substitution of (21). So, we get

\[
\mathbf{a}_-(T) = \mathbf{a}_-(0) \exp \left( -i \int_0^T (\hbar \omega + \gamma \hbar) d\tau - i \int_0^T \frac{\beta}{\omega} \frac{d}{d\tau} \left( \frac{\delta}{\beta} \right) d\tau \right),
\]

which gives the correct dynamical phase for \( \mathbf{a}_- \).

4 Geometric phases:

Now looking at the second phase factor in the expression of both the creation and annihilation operators \( \mathbf{a}_\pm(T) \) in (31) and (32), the additional factor over and above the dynamical phase, obtained by leading behaviour for adiabatic transport around a closed loop \( \Gamma \) in time \( T \) can be identified with the Berry phase or geometric phase (more precisely geometric phase shift) in the Heisenberg picture. As pointed out, the result obtained here can readily be converted to the more familiar form in terms of the phase gathered by the state vector by going over from the Heisenberg to the Schrodinger picture. The geometric phase shift \( \Phi_G \) found above can be written in a more familiar form by using the transformation \( d\tau = \frac{dR}{\beta} \nabla R \), where \( \mathbf{R} \) represents a vector in the parameter-space whose components are the time dependent. Then we can use Stoke's theorem to write \( \Phi_G \) as a line-integral over a closed loop \( \Gamma \) traced out in the parameter space as \( \tau \) varies from 0 to \( T \), i.e. a complete period and then convert it as a surface integral as,

\[
\Phi_G = \oint_{\Gamma} \frac{\beta}{\omega} \nabla R \left( \frac{\delta}{\beta} \right) \cdot d\mathbf{R} = \iint_S \nabla R \left( \frac{\beta}{\omega} \right) \times \nabla R \left( \frac{\delta}{\beta} \right) \cdot d\mathbf{S}
\]

Note that \( \mathbf{S} \) stand for any surface belong into the equivalence class of surfaces in the parameter space having the same boundary \( \Gamma \), and where any two such surfaces can be connected by smooth deformation without encountering any singularity. Now substituting \( \alpha, \beta, \gamma, \delta \) from (5), the geometric phase can now be expressed in terms of our original time dependent parameters \( P(t), Q(t) \) and the noncommutative parameters \( \theta \) and \( \eta \) as,
\[ \Phi_G = \int \int_S \nabla_R \left( \frac{Q \left( 1 + \left( \frac{\theta}{2\hbar} \right)^2 \right)}{2 \times \sqrt{P \left( \frac{\theta}{2\hbar} \right) - Q \left( \frac{\eta}{2\hbar} \right)^2 + PQ}} \right) \times \nabla_R \left( \frac{\left( \frac{\theta}{2\hbar} \right) \left( P \left( \frac{\eta}{2\hbar} \right) - Q \left( \frac{\eta}{2\hbar} \right) \right)}{Q \left( 1 + \left( \frac{\theta}{2\hbar} \right)^2 \right) + P \left( \frac{\eta}{2\hbar} \right)^2} \right) \cdot dS \]  

(34)

One can be rest assured at this stage, that the denominator never vanishes as PQ>0. Also it is worth noting that, in the absence of either of the two types of non-commutativity i.e. if \( \theta \) or \( \eta = 0 \), the geometric phase vanishes. So, it is the non-commutative nature of phase space, as a whole, alongside geometry of the parameter space trajectory, which plays crucial on the appearance of geometric phase shift for this 2D harmonic oscillator system.

The reason behind this appearance, can be argued by considering the time reversal symmetry of the Hamiltonian. In the commutative plane: \( \mathcal{H}_c(t) = P(t)(\hat{p}_1^2 + \hat{p}_2^2) + Q(t)(\hat{x}_1^2 + \hat{x}_2^2) \); with \( \hat{p}_1, \hat{p}_2, \hat{x}_1, \hat{x}_2 \) satisfying ordinary Heisenberg algebra. Time reversal transformation implemented by the operator \( \Theta \) as: \( \hat{p}_i \rightarrow \hat{p}_i' = \Theta \hat{p}_i \Theta^{-1} = -\hat{p}_i \) and \( \hat{x}_i \rightarrow \hat{x}_i' = \Theta \hat{x}_i \Theta^{-1} = \hat{x}_i \), shows that the Hamiltonian is invariant under time reversal: \( \Theta \mathcal{H}_c(t) \Theta^{-1} = \mathcal{H}_c(t) \).

On the other hand, in the non-commutative plane, the dynamics is given by the Hamiltonian \( \mathcal{H} \) with non-commutative coordinates and momenta satisfying algebra or equivalently by the Hamiltonian with the mathematically commuting coordinates and momentum, transforming like, \( p_i \rightarrow -p_i \), \( q_i \rightarrow q_i \), under time reversal. Hence, the Hamiltonian \( \mathcal{H}(t) = \alpha(t)(\hat{p}_1^2 + \hat{p}_2^2) + \beta(t)(\hat{q}_1^2 + \hat{q}_2^2) + \delta(t)(p_1q_1 + q_1p_1) - \gamma(t)(q_1p_2 - q_2p_1) \) is not time reversal symmetric: \( \Theta \mathcal{H}(t) \Theta^{-1} \neq \mathcal{H}(t) \); the presence of the dilatation term and the Zeeman like term breaks this symmetry. Particularly, the dilatation term is primarily responsible for the breaking of time reversal symmetry. In fact it has been shown in that this time reversal symmetry breaking is a necessary, but not sufficient, condition for the appearance of non-vanishing Berry phase. And it is because of this broken time reversal symmetry that a possibility of obtaining a non-vanishing geometrical phase shift in our system of 2D SHO in noncommutative phase space arises.

Before we proceed further, let us pause for a while and make some pertinent observations:

- The Berry connection one-form \( A \) on the loop \( \Gamma \) can be directly read-off from and as:

\[ A = \frac{\beta}{\omega} d\left( \frac{\delta}{\beta} \right) = -\frac{\alpha}{\omega} d\left( \frac{\delta}{\alpha} \right) - d[tan^{-1}(\sqrt{\frac{\alpha\beta}{\delta^2} - 1})] \]  

(35)

Showing that, upto a non-singular gauge transformation, the Berry connection can also be written as

\[ A := -\frac{\alpha}{\omega} d\left( \frac{\delta}{\alpha} \right) \]  

(36)
This particular feature of this connection one-form is indeed quite gratifying as the symmetry between $\alpha$ and $\beta$ the coefficients of $p^2$ and $q^2$ in the Hamiltonian (4) is some how restored with this. In fact, this form of the connection one form (36) occurred earlier in [1, 41, 42, 43], where a Hamiltonian of the same form as (4) was used to describe 1D parametric generalized harmonic oscillator.

- Although, last dilation term involving $\delta(t)$ in (4) can be gotten rid-off by canonical transformation, as in the case of [44, 43], thereby eliminating the Berry’s phase completely , it was shown in [42] that this is only apparent; it re-appears through the dynamical part-retaining it’s geometrical characteristics.

Now returning back to the main remaining issue, let us try to relate this geometric phase shift obtained in Heisenberg picture, with the more familiar form of Berry phases acquired by state vectors, we revert back to the Schrodinger picture. First let’s rewrite (31) and (32) as,

$$a_{\pm}(T) = a_{\pm}(0) \exp(-i\Theta_{\pm,d} - i\Phi_G)$$  \hspace{1cm} (37)

where

$$\Theta_{\pm,d} = \int_0^T (\hbar \omega \pm \gamma \hbar) ; \Phi_G = \int_0^T \frac{\beta}{\omega} \frac{d}{d\tau} \left( \frac{\delta}{\beta} \right) d\tau$$  \hspace{1cm} (38)

are the dynamical and geometric phases respectively.

Let $U(0,t)$ be the Schrodinger evolution operator of our concerned system, generated by the Hamiltonian (4). Then, $a_{\pm}(t) = U^\dagger(0,t) a_{S\pm}(t) U(0,t)$, where $a_{S\pm}(t)$ are the ladder operators in Schrodinger picture. Note that the time dependence is not entirely frozen here, even in this Schrodinger picture; it creeps in through the time dependent parameters.

We can therefore write,

$$\begin{align*}
\left( a_+^\dagger(T) \right)^{n_1} \left( a_-^\dagger(T) \right)^{n_2} &\sqrt{n_1! n_2!} |0,0(t=0)\rangle_S \\
&= U^\dagger(0,T) \left( a_{S+}^\dagger(T) \right)^{n_1} \left( a_{S-}^\dagger(T) \right)^{n_2} \sqrt{n_1! n_2!} U(0,T) |0,0(t=0)\rangle_S \\
&= U^\dagger(0,T) \left( a_{S+}^\dagger(T) \right)^{n_1} \left( a_{S-}^\dagger(T) \right)^{n_2} \sqrt{n_1! n_2!} e^{-i\phi_{0,0}} |0,0(t=T)\rangle_S \\
&= e^{i(\phi_{n_1,n_2}-\phi_{0,0})} |n_1,n_2(t=0)\rangle_S e^{-i\phi_{0,0}}
\end{align*}$$  \hspace{1cm} (39)

where, $\phi_{n_1,n_2}$ represents the total adiabatic phase acquired by $|n_1,n_2(t=0)\rangle_S$ after evolving by $\mathcal{H}(t)$ over it’s complete period $T$. Further using (37) we also find,
\[
\left( a_+^\dagger(T) \right)^{n_1} \left( a_-^\dagger(T) \right)^{n_2} |0,0(t=0)\rangle_S
\]
\[
= e^{i n_1 (\Theta_{+,d} + \Phi_G)} e^{i n_2 (\Theta_{-,d} + \Phi_G)} \frac{\left( a_+^\dagger(0) \right)^{n_1} \left( a_-^\dagger(0) \right)^{n_2}}{\sqrt{n_1! \sqrt{n_2!}}} |0,0(t=0)\rangle_S
\]
\[
= e^{i n_1 (\Theta_{+,d} + \Phi_G)} e^{i n_2 (\Theta_{-,d} + \Phi_G)} |n_1, n_2(t=0)\rangle_S.
\]
Note that here we have made use of the fact that \( a_\pm(t=0) = a_\pm S(t=0) \). Now comparing the above two equations (39) and (40), we get
\[
\phi_{n_1,n_2} = \phi_{0,0} + [n_1(\Theta_{+,d} + \Phi_G) + n_2(\Theta_{-,d} + \Phi_G)]
\]
So, the Berry phase acquired by the state \( |n_1,n_2(t=0)\rangle_S \) is given by,
\[
\phi_B^{(n_1,n_2)} = \phi_B^{(0,0)} + (n_1 + n_2)\Phi_G
\]
This kind of linear nature in the Berry phases of different eigenstates, is a general result [45] for any Hamiltonian with equally spaced discrete spectrum. And in our case, the total Hamiltonian (13) is partitioned into two commuting parts corresponding to \( a_+ \) and \( a_- \), where each part produces its own equally spaced spectrum in their respective sub-Hilbert spaces \( H_\pm \), whose tensor product forms the total Hilbert space: \( H = H_+ \otimes H_- \).

Importantly, it is the difference of the Berry phases of different eigenstates, which contributes in the expectation value of any operator at time \( t \) in a state obtained from any initial state and evolving under an adiabatic Hamiltonian, 
\[
\langle \hat{O} \rangle(t) = \langle \psi(t) \mid \hat{O}(t) \mid \psi(t) \rangle,
\]
where the ground state contribution \( \phi_B^{(0,0)} \) cancels out. And most experiments concerning Berry’s phase [46] employ this idea only. Our derivation certainly provides complete information which, in principle, may facilitate the predictions of such cases.

5 Classical analogue Hannay angles:

We now take-up the study of classical analogue of this quantal geometric phase, namely the Hannay angles [41]. To clinch the correspondence we will exploit the correspondence principle of quantum mechanics with classical mechanics, using coherent states [47] and some suitable chosen quantum operators that represents the classical action and angle variables. In doing so, we make use of the diagonalized form of our Hamiltonian (14) into the usual quadratic form using ladder operators \( a_+ \) and \( a_- \). We can now introduce a set of canonically conjugate position and momentum operators, the so called quadrature variables, from these ladders as,
\[
\hat{q}_\pm = \sqrt{\frac{\hbar}{2\omega_\pm}}(a_\pm^\dagger + a_\pm)
\]
\[
\hat{p}_\pm = i \sqrt{\frac{\hbar \omega_\pm}{2}}(a_\pm^\dagger - a_\pm)
\]
This transforms canonically our original Hamiltonian of (11) into a standard 2D harmonic oscillator Hamiltonian:

\[
\mathcal{H}(t) = \frac{\omega_+^2}{2} \hat{q}_+^2 + \frac{\omega_-^2}{2} \hat{q}_-^2 + \hat{p}_+^2 + \hat{p}_-^2,
\]

as can be seen by making use of (11). Hence, for the quantum case we are able to find a suitable canonical variables which reduces the generalized Hamiltonian into the standard form.

Likewise, the classical counterpart \(\mathcal{H}_{\text{Clu}}(R(t))\) of the Hamiltonian (11) too, can be transformed into a simple 2D harmonic oscillator like Hamiltonian using a similar canonical transformation, for every fixed tuple of time-dependent parameters \(R(t) = (\alpha(t), \beta(t), \gamma(t), \delta(t))\), which of course give rise to a periodic motion in the phase-space. So, in this classical case [17], let \(\{C(I, R)\}\) denotes a continuous family of periodic trajectories \(C(I, R)\) in the phase space associated with the classical Hamiltonians \(\mathcal{H}(R)\) and let \(\omega(I, R)\) be the angular velocity on \(C(I, R)\), where each curve is equipped with a definite origin so that the angle variable \(\theta\) which is conjugate to the action variable \(I\) can be defined. Now, during the adiabatic evolution, a point in phase space follows a trajectory of constant action, and that its angular coordinate \(\theta(t)\) at time \(t\) is given by

\[
\theta(t) = \theta(0) + \int_0^t \omega(I, R(s)) ds + \theta_i(0) \tag{45}\]

and involves as an integration along the curve \(C(I, R(t))\), and also contains, apart from the usual dynamical contribution, a geometrical one also-the so-called Hannay’s angle \(\theta_i(0)\). Note that, since our classical Hamiltonian \(\mathcal{H}_{\text{Clu}}(R)\) has two degrees of freedom, so we expect two sets of action-angle coordinates \(\{(I_i, \theta_i) : i = 1, 2\}\).

Now let’s consider the coherent states [48, 49, 50] of our two-dimensional harmonic oscillator (13), which are supposed to be the best approximations to a classical state. The coherent states analogous to [51] in this case are the tensor product of two independent Glauber-Klauder-Sudarshan (GKS)-coherent states [50], which are simultaneous (normalised) eigen states of the two mutually commuting annihilation operators \(a_+\) and \(a_-\):

\[
\left| z_1, z_2; R \right\rangle = \left| z_1(R) \right\rangle \otimes \left| z_2(R) \right\rangle
\]

\[
a_+ \left| z_1(R) \right\rangle = z_1 \left| z_1(R) \right\rangle
\]

\[
a_+ \left| z_2(R) \right\rangle = z_2 \left| z_2(R) \right\rangle
\]

\[
\left| z_1, z_2; R \right\rangle = e^{-\frac{1}{2}((|z_1|^2+|z_2|^2))/2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} z_1^{n_1} z_2^{n_2} \frac{\sqrt{n_1! \sqrt{n_2!}}}{\sqrt{n_1!} \sqrt{n_2!}} \left| n_1, n_2; R \right\rangle
\]

Further it has been shown in [17], that a suitable quantum operator for classical action \(I_i\) is \(\hat{I}_i(R) = \hbar \hat{N}_i(R)\), where \(\hat{N}_i(R)\) is the \(i\)-th number operator. with \(i \in \{+, -\}\) in our case. Now, let \(\hat{U}_i(R)\’s\) be the unitary operators defined through their action, \(\hat{U}_1(R) \left| n_1, n_2(R) \right\rangle = \left| n_1 - 1, n_2(R) \right\rangle\); \(\hat{U}_1(R) \left| 0, n_2(R) \right\rangle = 0\) and similarly for \(\hat{U}_2(R)\). They essentially correspond to the well known polar decompositions of the operators like \(a\) into the so-called number \(\hat{N}\) and phase operators \(\hat{\theta}\): \(a_\pm = \sqrt{N_\pm} e^{i \hat{\theta}_\pm}\), with \(\hat{U}_i(R)\) can be thought of the operator corresponding to \(e^{-i \hat{\theta}}\). It is also shown that the expectation
values of these operators in the state $|z_1, z_2(R)\rangle$ are given by, $I_i = \langle \hat{I}_i(R) \rangle = |z_i|^2 \hbar$ and $\langle \hat{U}_i(R) \rangle = e^{i\chi_{arg}(z_i)}$, so that in the classical limit, we can identify $z_j = \sqrt{\frac{T}{\hbar}} e^{-i\theta_j}$.

Finally we consider an initial wave packet,

$$|z_1, z_2; R(0)\rangle = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1!} \sqrt{n_2!}} |n_1, n_2; R(0)\rangle$$  \hspace{1cm} (47)

and evolve it adiabatically over a complete cycle, to get, using (41)

$$\mathcal{U}(0, T) |z_1, z_2; R(0)\rangle = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1!} \sqrt{n_2!}} e^{-i\phi_{0,0}} |n_1, n_2; R(T)\rangle \times$$

$$e^{-in_1(\Theta_{+,d} + \Phi_G)} e^{-in_2(\Theta_{-,d} + \Phi_G)}$$

$$= e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \times e^{-i\phi_{0,0}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left( z_1 e^{-i(\Theta_{+,d} + \Phi_G)} \right)^{n_1} \times$$

$$\left( z_2 e^{-i(\Theta_{-,d} + \Phi_G)} \right)^{n_2} |n_1, n_2; R(T)\rangle$$

Therefore, up to a global phase factor, the evolved state of a coherent state, associated with the initial Hamiltonian $\mathcal{H}(0)$, is also a coherent state $|z_1(T), z_2(T), R(T)\rangle$ associated with the Hamiltonian $\mathcal{H}(T)$ at time $T$, where $z_1(T) = z_1 e^{-i(\Theta_{+,d} + \Phi_G)}$ and $z_2(T) = z_2 e^{-i(\Theta_{-,d} + \Phi_G)}$.

Thus, from the evolution of $z_1$ and $z_2$ in (48) through a complete period of the adiabatic Hamiltonian, we can identify $\Theta_{\pm, d} = \int_0^T \omega_\pm(t') dt'$, where $\omega_i = \frac{1}{\hbar} \frac{\partial E_{n_1, n_2}}{\partial n_i}$ (Like $\frac{\partial H_{\text{cl}}}{\partial I_i}$), with the dynamical phases and $\Phi_G$ (from (33)) with the angular shift which was obtained classically by Hannay.

The expectation values of the new set of position-momentum i.e. the quadrature operators (13) are found to be given by (13),

$$\langle \hat{q}_\pm \rangle = \sqrt{2\hbar/\omega_\pm} Re(z_i)$$

$$\langle \hat{p}_\pm \rangle = \sqrt{2\hbar \omega_\pm} Im(z_i)$$  \hspace{1cm} (49)

On adopting the parametrization of $z_1$ and $z_2$, introduced above, the expectation values of these phase space operators in the transported state are obtained as,

$$\langle \hat{q}_\pm \rangle_T = \sqrt{2I_i/\omega_\pm} Cos(\theta_i(0) + \Theta_{\pm, d} + \Phi_G)$$

$$\langle \hat{p}_\pm \rangle_T = - \sqrt{2I_i/\omega_\pm} Sin(\theta_i(0) + \Theta_{\pm, d} + \Phi_G).$$  \hspace{1cm} (50)

showing that the corresponding classically canonical conjugate phase space i.e the quadrature operators $\hat{q}_\pm, \hat{p}_\pm$ undergo oscillatory motion. Thus the geometric phase $\Phi_G$, entering into the non-stationary coherent-state through all of it’s stationary components i.e. the energy eigenstates, generates in the classical limit ($\hbar \to 0$, $|z_i| \to \infty$, $\sqrt{I_i} = \sqrt{\hbar}|z_i| \to \text{finite}$), the angle variable conjugate to the action $I_i$ and are given
by the phase of $z_i$, and therefore the additional phase of $z_i$ i.e apart from the dynamical one, can be identified with Hannay angle, which can clearly be understood from classical arguments.

6 Conclusions:

We have considered the system of harmonic oscillator in Moyal plane, but with the additional feature that there is noncommutativity among momentum components also like the spatial ones and other parameters are varying slowly with time. Although periodicity, rather that adiabaticity, is more relevant in the computation of geometrical phase, as shown by Aharanov and Anandan [52], we nevertheless find the original adiabatic approach due to Berry convenient to execute. For that we introduce a novel form of Bopp shift, which is more general in nature and does not involve any effective plank constant $\hbar_{eff}$, as has been done in the literature. Through this Bopp shift, we can generate a certain dilatation term involving commutative phase space variable $([q_i, q_j] = 0 = [p_i, p_j]; [q_i, p_j] = i\hbar\delta_{ij})$, which plays an indispensable role in generating this geometrical i.e Berry phase. We computed this shift initially in Heisenberg picture and then related it with the conventional Berry phase in the Schrodinger picture. Finally, the classical analogue of Hannay angle was also computed using Glauber-Sudarshan coherent states. We finally observe that the emergent Berry phase (geometrical phase shift) depends on both types of noncommutative parameters ($\theta$ and $\eta$) and it will be vanish in the situation if either one of these parameters were to vanish. Thus we can conclude that, the noncommutative phase space structure induces a suitable geometry on the circuit $\Gamma$ in parameter space of the 2D time dependent harmonic oscillator system, which manifests in the appearance of the associated geometrical phase shift, when a circuital adiabatic excursion in the parameter space is considered.

Finally, we briefly mention some interesting directions in which our present work can be extended. The first is to construct coherent state Euclidean path integral [53, 54] formulation invoking adiabatic iterative prescription [55, 56] for calculating non-adiabatic corrections on Berry phase in non-commutative phase space. Apart from 2D oscillator, one also can think of other exactly integrable system where the partition function in the coherent state path integral method can be computed so that one can eventually obtain the Gibb’s entropy of the system and also try to make a possible connection with von Neumann entropy [57, 58] of the system in presence of the modified geometric phase.

The analysis presented here can perhaps also be extented to compute quantum information metric and Berry curvature [59] from the effective action [54] corresponding to the above mentioned path integral, so that one can try to connect some feature in noncommutative quantum mechanics with quantum information science. In light of the present paper, there could appear many surprises in this area. We hope to return to some of these issues in a future work.

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7 Appendix

7.1 Comments on unitary equivalence of different realization of NC phase space algebra:

In a different realization, other than (3), given by Zhang and also Bertolami [27, 28] no Berry phase was observed, albeit in a different system involving gravitational potential well [19]. What we would like to show here is that our realization is not quite equivalent to the one in [27, 28] in the generic case and it is only in realization (3) that we get a certain dilatation term (4,3) in the Hamiltonian which plays an indispensable role for the occurrence of Berry’s phase or geometric phase. To the end, consider the following structure of noncommutativity among the phase space variables:

\[ [\hat{x}_i, \hat{x}_j] = i\xi \epsilon_{ij}; [\hat{p}_i, \hat{p}_j] = i\eta \epsilon_{ij}; [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} ; \theta \eta < 0; \]

\( (51) \)

Where \( \theta \) and \( \eta \) are constant parameters; \( \epsilon_{ij} \) is an anti-symmetric constant tensor and \( \xi \) being a scaling parameter, related here to \( \theta \) and \( \eta \). We then introduce the commuting coordinates \( q_i \) and momenta \( p_i \) respectively satisfying the usual Heisenberg algebra;

\[ [q_i, q_j] = 0 = [p_i, p_j]; [q_i, p_j] = i\hbar \delta_{ij}. \]

Note that in our notation, these \( q_i \)'s and \( p_i \)'s carry no over-head hats, in contrast to their noncommutative counterparts (\( \hat{x}_i \)'s and \( \hat{p}_i \)'s).

In [27], the realisation in terms of the above \( q_i \)'s and \( p_i \)'s are given by the following linear transformation:

\[ \hat{x}_i^{(1)} = \sqrt{\xi} (q_i - \frac{\theta}{2\hbar} \epsilon_{ij} p_j) \]

\[ \hat{p}_i^{(1)} = \sqrt{\xi} (p_i + \frac{\eta}{2\hbar} \epsilon_{ij} q_j), \]

\( (52) \)

Which holds only if

\[ \xi = (1 + \frac{\theta \eta}{4\hbar^2})^{-1}; 4\hbar^2 + \theta \eta > 0 \]

\( (53) \)

But this realization of the algebra (51) is not unique [60]. Indeed, we provide below another possible realization of (51) in terms of another linear transformation, as

\[ \hat{x}_i^{(2)} = q_i - \frac{\xi \theta}{2\hbar} \epsilon_{ij} p_j + \frac{\xi \sqrt{-\theta \eta}}{2\hbar} \epsilon_{ij} q_j \]

\[ \hat{p}_i^{(2)} = p_i + \frac{\xi \eta}{2\hbar} \epsilon_{ij} q_j + \frac{\xi \sqrt{-\theta \eta}}{2\hbar} \epsilon_{ij} p_j, \]

\( (54) \)

The merit of this realisation is that it is valid for any value of \( \xi \) and need not be fixed to the value given in (53). This is unlike the one in (52). Clearly, neither of the transformations (52) or (54) represent canonical transformations, as they change the basic commutator algebra. It is, however, quite obvious that for the value of \( \xi \)
parameter, fine tuned to value in (53), the realisations should be unitarily equivalent. We now construct this unitary transformation explicitly, which map the realisation (52) to the other one (54). To that end, let us make the following ansatz of the unitary operator:

$$U = \exp[-i\frac{\sigma D}{\hbar}] \exp[-i\frac{\beta L}{\hbar}] \exp[-i\alpha_2 \hat{p}^2] \exp[-i\alpha_1 \hat{q}^2],$$  \hspace{1cm} (55)$$

where $D = \hat{q} \cdot \hat{p} + \hat{p} \cdot \hat{q}$ and $L = \hat{q} \times \hat{p}$ are respectively the dilatation and angular momentum operators\(^1\), and relates these two representations as,

$$\hat{x}_i^{(2)} = U \hat{x}_i^{(1)} U^\dagger, \quad \hat{p}_i^{(2)} = U \hat{p}_i^{(1)} U^\dagger$$ \hspace{1cm} (56)$$

Note that we have taken the parameters $\sigma$ and $\beta$ to be dimension-less, in contrast to the parameters $\alpha_1$ and $\alpha_2$ which are dimensionful. Now making use of Hadamard identity we can easily show that,

$$\hat{x}_i^{(2)} = Aq_i - B\epsilon_{ij} p_j + C\epsilon_{ij} q_j + Dp_i$$

$$\hat{p}_i^{(2)} = Ep_i + F\epsilon_{ij} q_j + G\epsilon_{ij} p_j - Hq_i,$$  \hspace{1cm} (57)$$

where

$$A = \lambda \sqrt{\xi} [\cos(\beta) + \alpha_1 \theta \sin(\beta)], \quad B = \frac{\sqrt{\xi}}{\lambda} \left[ (\frac{\theta}{2\hbar}) - 2\alpha_1 \alpha_2 \theta \hbar \cos(\beta) + 2\alpha_2 \hbar \sin(\beta) \right]$$

$$C = \sqrt{\xi} \lambda \left[ \sin(\beta) - \alpha_1 \theta \cos(\beta) \right], \quad D = \frac{\sqrt{\xi}}{\lambda} \left[ (\frac{\theta}{2\hbar}) - 2\alpha_1 \alpha_2 \theta \hbar \sin(\beta) - 2\alpha_2 \hbar \cos(\beta) \right]$$

$$E = \sqrt{\xi} \left[ (1 - 4\alpha_1 \alpha_2 \hbar^2) \cos(\beta) + \eta \alpha_2 \sin(\beta) \right], \quad F = \lambda \sqrt{\xi} \left[ \frac{\eta}{2\hbar} \cos(\beta) + 2\alpha_1 \hbar \sin(\beta) \right]$$

$$G = \sqrt{\xi} \left[ (1 - 4\alpha_1 \alpha_2 \hbar^2) \sin(\beta) - \eta \alpha_2 \cos(\beta) \right], \quad H = \lambda \sqrt{\xi} \left[ \frac{\eta}{2\hbar} \sin(\beta) - 2\alpha_1 \hbar \cos(\beta) \right].$$

$$\lambda = \exp(-\sigma).$$ \hspace{1cm} (58)$$

On the other hand, all these eight coefficients $A - H$ in (58) can be determined easily by comparing (57) with (54) and is provided below in two segregated clusters:

$$H = 0, \quad D = 0, \quad A = 1, \quad C = \xi \frac{\sqrt{-\eta \theta}}{2\hbar}$$ \hspace{1cm} (59)$$

and,

$$B = \xi \frac{\theta}{2\hbar}, \quad E = 1, \quad F = \frac{\xi \eta}{2\hbar}, \quad G = \xi \frac{\sqrt{-\eta \theta}}{2\hbar},$$ \hspace{1cm} (60)$$

The reason for this segregation is that a simple inspection suggests that we can make use of first three equations in (58) to solve for $\alpha_1$, $\alpha_2$, and $\lambda$ in terms of the single parameter $\beta$ as,

\(^1\)While former represents scalar operator, latter represents a pseudo scalar operator in a commutative plane and generates appropriate transformations. It is also quite well-known that the three scalar generators ($D, \hat{p}^2, \hat{q}^2$) form a closed $SO(1,2)$ algebra [44], while $L$ commutes with all of them: $[L, \hat{q}^2] = [L, \hat{p}^2] = [L, D] = 0.$
\[ \alpha_1 = \frac{\eta}{4\hbar} \tan(\beta) \]

\[ \alpha_2 = \frac{\eta}{4\hbar^2} \left[ \frac{\tan(\beta)}{1 + \frac{\eta}{4\hbar^2}\tan^2(\beta)} \right] \]

\[ \lambda = \left[ \sqrt{\xi}(\cos(\beta) + \alpha_1\theta\sin(\beta)) \right]^{-1} \]

and then this \( \beta \), can be determined by first making use of the fourth equation in (57) to get the following quadratic equation:

\[ \frac{\xi}{4\hbar^2}(\theta\eta)^2\tan^2(\beta) - \left( \frac{\theta\eta}{2\hbar} - 2\hbar \right)\tan(\beta) - \xi\sqrt{\theta\eta} = 0, \]

yielding the following two roots for \( \beta \):

\[ \beta_1 = \tan^{-1}\left( \sqrt{-\frac{\theta\eta}{4\hbar^2}} \right) \quad \beta_2 = -\tan^{-1}\left( \left( -\frac{\theta\eta}{4\hbar^2} \right)^{\frac{1}{2}} \right) \]

It is now a matter of a lengthy but straightforward computation to verify that only when \( \beta_1 \) from (63) along with \( \alpha_1, \alpha_2, \) and \( \lambda \) from (61) are substituted to the set of expressions of \( B, E, F \) and \( G \) in (58) they readily yield the corresponding expression given in (60). This therefore provides an explicit demonstration of the unitary equivalence of two realizations (52,54) for specific values of \( \xi \) in (53). For other values of \( \xi \) the realisation (52) will not hold, in contrast to the realisation (54) which persists to hold. In this sense the realisation (54) is more general and in this paper, we are basically working with the algebra (2) and its realisation (3), which are nothing but the equations (54,51) themselves with \( \xi = 1 \).

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