Enumeration of lozenge tilings of hexagon with three holes

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Abstract

We enumerate lozenge tilings of a hexagon on the triangular lattice with three bowtie-shaped holes on non-consecutive sides. The result generalizes a Propp’s problem on enumeration of tilings of a hexagon of side-lengths $2n, 2n+3, 2n, 2n+3, 2n, 2n+3$ with the central unit triangles on $(2n+3)$-sides removed. We also consider a related result on $q$-enumeration of plane partitions fitting in a connected union of several boxes.

Keywords: perfect matching, lozenge tiling, dual graph, graphical condensation.

1 Introduction

A plane partition is a rectangular array of non-negative entries so that all rows are weakly decreasing from left to right and all columns are weakly decreasing from top to bottom. A plane partition having $a$ rows and $b$ columns with entries at most $c$ is identified with its 3-D interpretation—a stack of unit cubes fitting in an $a \times b \times c$ box. MacMahon’s classical theorem [12] on the number of plane partitions that fit in an $a \times b \times c$ box is equivalent to the fact that the number of lozenge tilings of a semi-regular hexagon of side-lengths $a, b, c, a, b, c$ (in cyclic order) on the triangular lattice is

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{H(a) H(b) H(c) H(a+b+c)}{H(a+b) H(b+c) H(c+a)},$$

where the hyperfactorial function $H(n)$ is defined by $H(n) := 0! \cdot 1! \cdot 2! \cdots (n-1)!$. Here, a lozenge is a union of any two unit equilateral triangles sharing an edge; and a lozenge tiling of a region $R$ is a covering of $R$ by lozenges so that there are no gaps or overlaps.

A large body of work followed (see e.g. [1], [2], [4], [5], [6], [7], [8]), centered on enumeration of lozenge tilings of a hexagon with defects.

In 1999, James Propp [13] published a list of 32 open problems in the field of enumeration of tilings (equivalently, perfect matchings). Problem 3 on the list asks for the number of lozenge...
of tilings of a hexagon of side-lengths $2n, 2n + 3, 2n, 2n + 3, 2n, 2n + 3$ on the triangular lattice, where three central unit triangles have been removed from its long sides (see Figure 1.1 for the case $n = 2$). Theresia Eisenkölbl solved (and generalized) this problem by computing the number of lozenge tilings of a hexagon with sides $a, b + 3, c, a + 3, b, c + 3$, where an arbitrary unit triangle is removed from each of the $(a + 3)$-, $(b + 3)$- and $(c + 3)$-sides.

One can view the unit triangles removed in the Propp’s problem as triangular holes of size 1. In this paper, we consider a more general situation when our hexagon has three triangular holes of arbitrary sizes on non-consecutive sides. Moreover, these triangular holes can be extended to bowtie-shaped holes, which consist of one up-pointing and one down-pointing triangular holes sharing a vertex.

Assume that $a, b, c, d, e, f, x, y, z$ are nine non-negative integers. We consider the hexagon of side-lengths $z + x + a + b + c, x + y + d + e + f, y + z + a + b + c, z + x + d + e + f, x + y + a + b + c, y + z + d + e + f$ on the triangular lattice. Next, we remove three bowties along the northeast, south and northwest sides of the hexagon at the locations specified as in Figure 1.2. We denote by $F \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}$ the resulting region.

We call the common vertex of two triangles in a bowtie its center. We notice that the centers of three bowties removed in our region are always the vertices of a down-pointing equilateral triangle with side $x + y + z + d + e + f$ (indicated by the triangle with dotted sides in Figure 1.2).

We use the notation $M(R)$ for the number of lozenge tiling of a region $R$. The number of lozenge tilings of a hexagon with three bowties removed from non-consecutive sides is given by the theorem stated below.

\footnote{From now on, we always list the side-lengths of a hexagon on the triangular lattice in clockwise order, starting from the northwest side.}
Figure 1.2: Hexagon with three bowtie holes on three non-consecutive sides.

**Theorem 1.1.** For non-negative integers $a, b, c, d, e, f, x, y, z$

\[
M \begin{pmatrix}
  x & y & z \\
  a & b & c \\
  d & e & f
\end{pmatrix} = \\
\frac{H(x) H(y) H(z) H(a)^2 H(b)^2 H(c)^2 H(d) H(e) H(f) H(d + e + f + x + y + z)^4}{H(a + d) H(b + e) H(c + f) H(d + e + x + y + z) H(e + f + x + y + z) H(f + d + x + y + z)} \\
\times \frac{H(A + 2x + 2y + 2z) H(A + x + y + z)^2}{H(A + 2x + y + z) H(A + x + 2y + z) H(A + x + y + 2z)} \\
\times \frac{H(a + d + x + y) H(b + e + y + z) H(c + f + z + x)}{H(a + d + x + y + z) H(a + e + f + x + y + z)^2 H(b + d + e + f + x + y + z)^2 H(c + d + e + f + x + y + z)^2} \\
\times \frac{H(A + a + x + y + z) H(A - b + 2x + y + z) H(A - c + x + 2y + z)}{H(b + c + e + f + x + y + z) H(c + a + d + f + 2x + y + z) H(a + b + d + e + x + 2y + z)},
\]

(1.2)

where $A = a + b + c + d + e + f$.

One readily sees that, Theorem 1.1 solves Propp’s Problem 3 [13] by setting $x = y = z = n$, $a = b = c = 1$ and $d = e = f = 0$.

Let $q$ be an indeterminate. The $q$-integer $[n]_q$ is defined as $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$. The $q$-factorial $[n]_q!$ is the product $[1]_q [2]_q \cdots [n]_q$, and the $q$-hyperfactorial is defined by $H_q(n) := [0]_q! [1]_q! \cdots [n - 1]_q!$. 

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Figure 1.3: Viewing a tiling of a hexagon with three dents as a stack of cubes in a special box. Picture (b) shows the tiling corresponding to the empty stack.

MacMahon [12] actually proved a stronger version of the formula (1.1) as follows:

\[ \sum_{\pi} q^{\vert \pi \vert} = \frac{H_q(a) H_q(b) H_q(c) H_q(a + b + c)}{H_q(a + b) H_q(b + c) H_q(c + a)}, \]

where the sum is taken over all plane partitions \( \pi \) fitting in an \( a \times b \times c \) box, and \( \vert \pi \vert \) is the volume of \( \pi \) (i.e. the number of unit cubes in \( \pi \)). It is easy to see that the \( q = 1 \) specialization of (1.3) deduces (1.1).

Similar to the case of semi-regular hexagons, the lozenge tilings of our \( F \)-type region can also be viewed as a stack of unit cubes fitting in a special box (see Figure 1.3(a)). Figure 1.3(b) shows the empty box. We denote by \( \mathcal{B} := \mathcal{B} \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} \) the box (the precise definition of the box \( \mathcal{B} \) will be shown in Section 4). One readily sees that our stacks satisfy the same monotonicity as the ordinary plane partitions: the levels of the tops of the columns are weakly decreasing along \( i \)- and \( j \)-directions. In the view of this, we call our stacks generalized plane partitions (GPP’s). The next theorem shows that the generating function of the volume of GPP’s fitting in the box \( \mathcal{B} \) is given by a simple product formula, which is obtained from the expression on the right-hand side of (1.2) in Theorem 1.1 by replace each hyperfactorial by the corresponding \( q \)-hyperfactorial.
Theorem 1.2. For non-negative integers $a, b, c, d, e, f, x, y, z$

$$
\sum_{\pi} q^{\pi} = \frac{H_q(x) H_q(y) H_q(z) H_q(a)^2 H_q(b)^2 H_q(c)^2 H_q(d) H_q(e) H_q(f) H_q(d + e + f + x + y + z)^4}{H_q(a + d) H_q(b + e) H_q(c + f) H_q(d + e + x + y + z) H_q(e + f + x + y + z) H_q(f + d + x + y + z)} \\
\times \frac{H_q(A + 2x + 2y + 2z) H_q(A + x + y + z)^2}{H_q(A + 2x + y + z) H_q(A + x + 2y + z) H_q(A + x + y + 2z)} \\
\times \frac{H_q(a + b + d + e + x + y + z) H_q(a + c + d + f + x + y + z) H_q(b + c + e + f + x + y + z)}{H_q(a + d + e + f + x + y + z) H_q(b + d + e + f + x + y + z) H_q(c + d + e + f + x + y + z)^2} \\
\times \frac{H_q(a + d + x + y) H_q(b + e + y + z) H_q(c + f + z + x)}{H_q(a + b + y) H_q(b + c + z) H_q(c + a + z)} \\
\times \frac{H_q(A - a + x + y + 2z) H_q(A - b + 2x + y + z) H_q(A - c + x + 2y + z) H_q(A - d + e + x + 2y + z)}{H_q(b + c + e + f + x + y + 2z) H_q(c + a + d + f + 2x + y + z) H_q(a + b + d + e + x + 2y + z)} \tag{1.4}
$$

where the sum is taken over all GPP’s $\pi$ fitting in the box $B \left(\begin{array}{ccc}x & y & z \\ a & b & c \\ d & e & f\end{array}\right)$ and where $A = a + b + c + d + e + f$.

The paper is organized as follows. In Section 2, we introduce two new families of hexagons with holes on boundary. The exact enumeration of these regions will be employed in the proof of Theorem 1.1 in Section 3. We will use Kuo’s graphical method [9] in our proof. In Section 4, we investigate precisely the bijection between lozenge tilings of our regions and GPP’s fitting in a certain box. Section 5 introduces two simple weight assignments, where each lozenge is weighted by a power of $q$. Section 6 is devoted to $q$-enumerations of the regions introduced in Section 2. We present the proof of Theorem 1.2 in Section 7. Section 8 shows a nice application of our result in enumeration of (ordinary) plane partitions with certain constraints. Several conclusion remarks will be given in Section 9 at the end of the paper.

2 Two new families of hexagons with holes

Consider a hexagon $H$ of side-lengths $z + a, x + y, t + a, z, x + y + a, t$. We remove an equilateral triangle of side $a$ from the south side of the region as in Figure 2.1. Denote by $K_q(x, y, z; t)$ the resulting region. One readily sees that the region has the same number of tilings as the region obtained from the hexagon $H$ by removing $a$ consecutive up-pointing unit triangles from the
Figure 2.2: The hexagon with two dents $Q_{2,3}(2,2,3,2)$.

south side. However the number of lozenge tilings of this type of regions is given by Proposition 2.1 in [6]. Thus, the following lemma can be considered as a consequence of the later proposition.

Lemma 2.1. For non-negative integers $a, x, y, z, t$

\[
M \left( K_a(x, y, z, t) \right) = \frac{H(a) \cdot H(x) \cdot H(y) \cdot H(z) \cdot H(t)}{H(a + x) \cdot H(a + y) \cdot H(y + z) \cdot H(t + x)} \times \frac{H(a + x + y) \cdot H(a + y + z) \cdot H(a + t + x) \cdot H(a + x + y + z + t)}{H(a + x + y + z) \cdot H(a + x + y + t) \cdot H(a + z + t)}. \tag{2.1}
\]

Next, we introduce a new family of hexagons with two triangular holes. To precise, we consider the hexagon of side-lengths $z + a + b, x + y, y + t + a + b, z, x + y + a + b, y + t$. Remove an a-triangle along the south side and a b-triangle along the northeast side of the hexagon as in Figure 2.2. Denote by $Q_{a,b}(x, y, z, t)$ the resulting region. We note that the left sides of the two triangular holes in our region are on the same lattice line (illustrated by the dotted line in Figure 2.2).

Theorem 2.2. For non-negative integers $a, b, x, y, z, t$

\[
M \left( Q_{a,b}(x, y, z, t) \right) = \frac{H(x) \cdot H(y) \cdot H(z) \cdot H(t) \cdot H(a) \cdot H(b)}{H(a + x) \cdot H(b + t) \cdot H(a + b + y)} \times \frac{H(a + b + x + 2y + z + t) \cdot H(a + b + x + y + t)}{H(a + b + y + z + t) \cdot H(a + b + x + y + z)} \times \frac{H(a + x + y) \cdot H(b + y + t) \cdot H(a + b + y + z)^2}{H(x + y + t) \cdot H(a + y + z) \cdot H(b + y + z)}. \tag{2.2}
\]

A perfect matching of a graph $G$ is a collection of disjoint edges covering all vertices of $G$. The dual graph of a region $R$ is the graph whose vertices are unit triangles in $R$ and whose edges connect precisely two unit triangles sharing an edge. The tilings of a region can be identified with the perfect matchings of its dual graph. In the view of this, we use the notation $M(G)$ for the number of perfect matchings of a graph $G$.

One readily sees that if a region $R$ admits a tiling, then the numbers of up-pointing and down-pointing unit triangles in $R$ are equal. If a region $R$ satisfies the above balancing condition, we say that $R$ is balanced.
We will employ the following lemma in our proofs.

**Lemma 2.3** (Region Splitting Lemma). Let \( R \) be a balanced region. Assume that a sub-region \( Q \) of \( R \) satisfies the following two conditions:

(i) (Separating Condition) There is only one type of unit triangles (up-pointing or down-pointing unit triangles) running along each side of the border between \( Q \) and \( R - Q \).

(ii) (Balancing Condition) \( Q \) is balanced.

Then
\[
M(R) = M(Q) M(R - Q). \tag{2.3}
\]

*Proof.* Let \( G \) be the dual graph of \( R \), and \( H \) the dual graph of \( Q \). Then \( H \) satisfy the conditions in Lemma 3.6(a) in [10], so
\[
M(G) = M(H) M(G - H). \tag{2.4}
\]

Then (2.3) follows. \( \square \)

The next two Kuo’s theorems are the key of our proofs in this paper.

**Theorem 2.4** (Theorem 5.1 [9]). Let \( G = (V_1, V_2, E) \) be a (weighted) bipartite planar graph in which \( |V_1| = |V_2| \). Assume that \( u, v, w, s \) are four vertices appearing in a cyclic order on a face of \( G \) so that \( u, w \in V_1 \) and \( v, s \in V_2 \). Then
\[
M(G) M(G - \{u, v, w, s\}) = M(G - \{u, v\}) M(G - \{w, s\}) + M(G - \{u, s\}) M(G - \{v, w\}). \tag{2.5}
\]

**Theorem 2.5** (Theorem 5.3 [9]). Let \( G = (V_1, V_2, E) \) be a (weighted) bipartite planar graph in which \( |V_1| = |V_2| + 1 \). Assume that \( u, v, w, s \) are four vertices appearing in a cyclic order on a face of \( G \) so that \( u, v, w \in V_1 \) and \( s \in V_2 \). Then
\[
M(G - \{v\}) M(G - \{u, w, s\}) = M(G - \{u\}) M(G - \{v, w, s\}) + M(G - \{w\}) M(G - \{v, w, s\}). \tag{2.6}
\]

*Proof of Theorem 2.2.* We prove (2.2) by induction on \( y + t + 2b \). Our base cases are the situations when \( b = 0, y = 0 \) or \( t = 0 \).

If \( b = 0 \), then our region becomes a \( K \)-type region in Lemma 2.1, and (2.2) follows. If \( y = 0 \), using Region-splitting Lemma 2.3, we can split our region into two semi-regular hexagons as in Figure 2.3(a) and obtain
\[
M(Q_{a,b}(x,0,z,t)) = M(\text{Hex}(z + a + b, x, t)) M(\text{Hex}(z,b,a)), \tag{2.7}
\]
where \( \text{Hex}(a,b,c) \) denotes the (semi-regular) hexagon of sides \( a, b, c, a, b, c \). Then (2.2) follows from MacMahon’s theorem (1.1). If \( t = 0 \), there are several lozenges that are forced to be in any tilings of our region. Removing these forced lozenges, we get a new region having the same number of tilings as the original region (see the region restricted by the bold contour in Figure 2.3(b)). It is easy to see that the new region is a \( K \)-type region in Lemma 2.1. In particular, we get
\[
M(Q_{a,b}(x,y,z,0)) = M(K_{a}(x, y + b, z, y)), \tag{2.8}
\]
and (2.2) follows from Lemma 2.1.
Figure 2.3: The base cases when (a) $y = 0$ and (b) $t = 0$ in the proof of Theorem 2.2.

Figure 2.4: How we apply Kuo condensation to a $Q$-type region.
Figure 2.5: Obtaining recurrence of the numbers of tilings of $Q$-type regions.
Our induction step is based on Kuo's Condensation Theorem 2.5.

For the induction step, we assume that $b, y, t \geq 1$ and that (2.2) holds for any $Q$-type regions in which the sum of the $y$-parameter, $t$-parameter and two times $b$-parameter is strictly less than $y + t + 2b$.

We apply Kuo Condensation Theorem 2.5 to the dual graph $G$ of the region $R$ obtained from $Q_{a,b}(x, y, z, t)$ by adding a band of $2b - 1$ unit triangles along the bottom of the $b$-hole (see Figure 2.4 for the case $a = 2$, $b = 3$, $x = 2$, $y = 2$, $z = 3$, $t = 2$). Each vertex of $G$ corresponds to a unit triangle in $R$. The shaded unit triangles in Figure 2.4 indicate the ones corresponding to the four vertices $u, v, w, s$ in Theorem 2.5. In particular, the lowest shaded unit triangle corresponds to $u$; and $v, w$ and $s$ correspond respectively to the next shaded unit triangles as we move counter-clockwise from the lowest one.

Consider the region corresponding to $G - \{v\}$. By removing forced lozenges along the bottom of the $b$-hole, we get back the original region $Q_{a,b}(x, y, z, t)$ (see the region restricted by the bold contour in Figure 2.5(a)). Thus,

$$M(G - \{v\}) = M(Q_{a,b}(x, y, z, t)).$$

(2.9)

Similarly, by removing all lozenges forced by shaded unit triangles, we obtain five more equalities, which are illustrated in Figures 2.5(b)–(f), respectively:

$$M(G - \{u, w, s\}) = M(Q_{a,b-1}(x, y, z + 1, t - 1)),$$

(2.10)

$$M(G - \{u\}) = M(Q_{a,b-1}(x, y, z + 1, t)),$$

(2.11)

$$M(G - \{v, w, s\}) = M(Q_{a,b}(x, y, z, t - 1)),$$

(2.12)

$$M(G - \{v\}) = M(Q_{a,b-1}(x, y + 1, z, t - 1)),$$

(2.13)

and

$$M(G - \{u, v, s\}) = M(Q_{a,b}(x, y - 1, z + 1, t)).$$

(2.14)

Plugging the above six identities (2.9)–(2.14) into the equation (2.6) in Kuo's Theorem 2.5, we get

$$M(Q_{a,b}(x, y, z, t)) M(Q_{a,b-1}(x, y, z + 1, t - 1)) = M(Q_{a,b-1}(x, y, z + 1, t)) M(Q_{a,b}(x, y, z, t - 1)) + M(Q_{a,b-1}(x, y + 1, z, t - 1)) M(Q_{a,b}(x, y - 1, z + 1, t)).$$

(2.15)

One readily sees that all regions in (2.15), except for the first one, have the sum of their $y$-parameter, $t$-parameter and two times $b$-parameter strictly less than $y + t + 2b$. Thus, by induction hypothesis, those regions have all the numbers of lozenge tilings given by (2.2). Substituting these formulas into (2.15) and simplifying, one gets that $M(Q_{a,b}(x, y, z, t))$ is exactly the expression on the right-hand side of (2.2). This finishes our proof.

Next, we consider a new family of hexagons with three holes.

Let $a, b, c, d, x, y, z, t$ be eight non-negative integers. Our base region is a hexagon of side-lengths $z + b + c + d, x + y + z + t + a, t + b + c + d, z + a, x + y + z + t + b + c + d, t + a$. Next, we remove a bowtie from the north side, and two triangles from the south side. The locations and the sizes of the holes are indicated as in Figure 2.6 (for the case $a = 2$, $b = 2$, $c = 3$, $d = 2$, $x = 1$, $y = 2$, $z = 2$, $t = 1$). Denote by $B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right)$ the resulting region.
Figure 2.6: The dented magnet bar.

Figure 2.7: Base cases of Theorem 2.6 when (a) \( z = 0 \) and (b) \( t = 0 \).

**Theorem 2.6.** For non-negative integers \( a, b, c, d, x, y, z, t \)

\[
M \left( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \right) = \frac{H(x) H(y) H(z) H(t) H(a)}{H(a + c) H(b + y) H(d + x)} \frac{H(b + c + d + x + z + t + 2) H(a + b + c + d + y + z + t + 2) H(a + b + c + d + y + z + t + 2)}{H(a + b + c + d + z + t + 2)}

\times \frac{H(a + b + c + d + x + y + z + t) H(a + b + c + d + x + y + z + t)}{H(a + b + c + d + x + y + z + t)} \frac{H(a + b + c + y + z + t) H(a + b + c + x + z + t)}{H(a + b + c + d + z + t + 2)} \frac{H(a + b + c + y + z + t) H(b + c + d + y + z + t) H(b + c + d + x + z + t)}{H(b + c + d + z + t)}.

(2.16)

**Proof.** We prove (2.16) by induction on \( z + t \). The base cases are the situations when \( z = 0 \) or \( t = 0 \).

If \( z = 0 \), then we can split the region into two subregions by Region-splitting Lemma 2.3 as in Figure 2.7(a). The left subregion is the semi-regular hexagon \( Hex(a, y, t + b + c + d) \) and the
Figure 2.8: Obtaining recurrence of the numbers of tilings of $B$-regions.
right subregion is \( Q_{d,c}(x, t, b, a) \). Thus, we get

\[
M \left( B \left( \begin{array}{cccc}
 x & y & 0 & t \\
a & b & c & d
\end{array} \right) \right) = M \left( H_{a,y,t+b+c+d} \right) M \left( Q_{d,c}(x, t, b, a) \right),
\]

and (2.16) follows from MacMahon Theorem (1.1) and Theorem 2.2. The case \( t = 0 \) can be treated by a completely analogous manner, based on Figure 2.7(b).

For the induction step, we assume that \( z, t \geq 1 \) and that (2.16) holds for any \( B \)-type regions having the sum of their \( z \)- and \( t \)-parameters strictly less than \( z + t \).

Apply Kuo Theorem 2.4 to the dual graph \( G = B \left( \begin{array}{cccc}
x & y & z & t \\
a & b & c & d
\end{array} \right) \) with the four vertices \( u, v, w, s \) chosen as in Figure 2.8(b). In particular, the shaded unit triangles indicate the unit triangles corresponding to the vertices \( u, v, w, s \). The shaded unit triangle on the bottom corresponds to \( u \), and \( v, w, s \) correspond respectively to the next shaded unit triangles as we move clockwise from the lowest one.

Next, we consider the region corresponding to \( G - \{u, v, w, s\} \). By removing all lozenges forced by the four shaded unit triangles, we get the region \( B \left( \begin{array}{cccc}
x & y & z - 1 & t - 1 \\
a & b & c + 1 & d + 1
\end{array} \right) \) (see the region restricted by the bold contour in Figure 2.8(b)). Thus, we get

\[
M(G - \{u, v, w, s\}) = M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t - 1 \\
a & b & c + 1 & d + 1
\end{array} \right) \right).
\]

Similarly, we get the following four identities:

\[
M(G - \{u, v\}) = M \left( B \left( \begin{array}{cccc}
x & y & z & t - 1 \\
a & b & c + 1 & d + 1
\end{array} \right) \right) \quad \text{(see Figure 2.8(c))},
\]

\[
M(G - \{w, s\}) = M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t \\
a & b & c + 1 & d
\end{array} \right) \right) \quad \text{(see Figure 2.8(d))},
\]

\[
M(G - \{u, s\}) = M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t \\
a & b & c & d + 1
\end{array} \right) \right) \quad \text{(see Figure 2.8(e))},
\]

\[
M(G - \{v, w\}) = M \left( B \left( \begin{array}{cccc}
x & y & z & t - 1 \\
a & b & c + 1 & d
\end{array} \right) \right) \quad \text{(see Figure 2.8(f))}.
\]

Substituting (2.18)-(2.22) into the equality (2.5) in Kuo Theorem 2.4, we obtain the following recurrence

\[
M \left( B \left( \begin{array}{cccc}
x & y & z & t \\
a & b & c & d
\end{array} \right) \right) M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t - 1 \\
a & b & c + 1 & d + 1
\end{array} \right) \right) =
M \left( B \left( \begin{array}{cccc}
x & y & z & t - 1 \\
a & b & c & d + 1
\end{array} \right) \right) M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t \\
a & b & c + 1 & d
\end{array} \right) \right)
+ M \left( B \left( \begin{array}{cccc}
x & y & z - 1 & t \\
a & b & c & d + 1
\end{array} \right) \right) M \left( B \left( \begin{array}{cccc}
x & y & z & t - 1 \\
a & b & c + 1 & d
\end{array} \right) \right).
\]

It is easy to see that all the regions in (2.23), except for the first one, have the sum of their \( z \)- and \( t \)-parameters strictly less than \( z + t \).
Denote by \( \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d} \right) \) the expression on the right-hand side of (2.16). To prove (2.16), we only need to show that \( \Phi \) satisfies the same recurrence (2.23) as the number of tilings of \( B \)-type regions. Equivalently, we need to verify that
\[
\frac{\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d+1} \right)}{\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d} \right)} = 1,
\]
\[
\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d+1} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d} \right) + \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d+1} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d} \right) = 1.
\]

Let us simplify the first term on the left-hand side of (2.15). We note that the numerator and the denominator of the first fraction in the first term are different only at their \( d \)- and \( t \)-parameters. By cancelling out all terms without \( d \)- or \( t \)-parameters and using the simple fact \( H(n+1) = n! H(n) \), we can evaluate the first fraction as
\[
\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d+1} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c}, \frac{t}{d} \right) = \frac{d! (a+b+c+x+y+z+2t-1)!}{(t-1)! (a+b+c+d+x+y+z+2t-1)!} \times \frac{(a+c+d+x+z+2t-1)!(a+b+c+y+2z+t-1)!(b+c+z+t-1)!(a+x+t-1)!}{(a+b+c+d+x+y+2z+2t-1)!(a+b+c+y+z+t-1)!(a+c+z+t-1)!}.
\]

Similarly, the second fraction of the first term can be written as
\[
\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d+1} \right) = \frac{(t-1)! (a+b+c+x+z+2t-1)!}{d! (a+b+c+d+x+y+z+2t-1)!} \times \frac{(a+b+c+d+x+y+2z+2t-2)!(a+b+c+y+z+t-1)!(a+c+z+t-1)!(a+x+t-1)!}{(a+c+d+x+z+2t-1)!(a+b+c+y+2z+t-2)!(b+c+z+t-1)!(a+x+t-1)!}.
\]

Thus, the first term is simplified as
\[
\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d+1} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d} \right) = \frac{a+b+c+y+2z+t-1}{a+b+c+d+x+y+2z+2t-1}.
\]

By a completely analogous manner, we can simplify the second term on the left-hand side of (2.15) as
\[
\Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d+1} \right) \Phi \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c+1}, \frac{t}{d} \right) = \frac{d+x+t}{a+b+c+d+x+y+2z+2t-1}.
\]
and (2.15) becomes the following obvious equation

\[
\frac{a + b + c + y + 2z + t - 1}{a + b + c + d + x + y + 2z + 2t - 1} + \frac{d + x + t}{a + b + c + d + x + y + 2z + 2t - 1} = 1.
\]

(2.29)

This completes our proof. \qed

\section{Proof of Theorem 1.1}

In this section, we prove Theorem 1.1 using Kuo’s Theorem 2.5 and Theorem 2.6.

\begin{proof}[Proof of Theorem 1.1] We prove (1.2) by induction on \( y + z \). We have two base cases: \( y = 0 \) and \( z = 0 \).

If \( y = 0 \), we apply of Region-splitting Lemma 2.3 to split our regions into two parts as in Figure 3.1(a). The right part is the hexagon \( Hex(x + z + d + e + f, b, a) \) and the left part is the region \( B \left( \begin{array}{cccc} a & b & z & x \\ c & e & f & d \end{array} \right) \) rotated 60° counter-clockwise. Thus, we get

\[
M \left( \begin{array}{cccc} x & 0 & z & \end{array} \right) = M \left( Hex(x + z + d + e + f, b, a) \right) M \left( \begin{array}{cccc} a & b & z & x \\ c & e & f & d \end{array} \right).
\]

(3.1)

Then (1.2) follows from MacMahon’s Theorem (1.1) and Theorem 2.6. The case \( z = 0 \) can be treated similarly, based on Figure 3.1(b).

For the induction step, we assume that \( y, z \geq 1 \) and that (1.2) holds for any \( F \)-type regions whose sum of \( y \)- and \( z \)-parameters strictly less than \( y + z \).

We apply Kuo Theorem 2.5 to the dual graph \( G \) of the region \( R \) obtained from the region

\[
F := F \left( \begin{array}{cccc} x & y & z \\ a & b & c & d & e & f \end{array} \right)
\]

by adding a strip of \( 2(y+z+d+e+f) - 1 \) unit triangles along the southwest
side. The positions of the four vertices $u, v, w, s$ are indicated by the four shaded unit triangles as in Figure 3.2. In particular, the shaded unit triangle corresponding to $u$ is the highest one; and if we move counter-clockwise from the later shaded unit triangle, we get the shaded unit triangles corresponding to $v, w, s$, respectively. Figure 3.3 shows how we obtain the recurrence on the number of lozenge tilings of our region. In particular, after removing all lozenges forced by shaded unit triangles, we get back some $F$-type regions. The equation (2.6) in Kuo’s Theorem 2.5 tells us that the product of the numbers of tilings of the two regions on the top of Figure 3.3 is equal to the product of the numbers of tilings of the two regions in the middle, plus the product of the numbers of tilings of the two regions on the bottom. To precise, we get

\[
M \left( F \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} \right) M \left( F \begin{pmatrix} x + 1 & y - 1 & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix} \right) = \\
M \left( F \begin{pmatrix} x + 1 & y & z - 1 \\ a & b & c \\ d & e & f \end{pmatrix} \right) M \left( F \begin{pmatrix} x & y - 1 & z \\ a & b & c \\ d & e + 1 & f \end{pmatrix} \right) + M \left( F \begin{pmatrix} x + 1 & y - 1 & z \\ a & b & c \\ d & e & f \end{pmatrix} \right) M \left( F \begin{pmatrix} x & y & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix} \right).
\]

(3.2)

It is easy to see that all the regions in (3.2), except for the first one, have the sum of their $y$- and $z$-parameters strictly less than $y + z$.

Denote by $\Psi \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}$ the expression on the right-hand side of (1.2). To verify (1.2), we need to show that the function $\Psi$ satisfies also the recurrence (3.2). This is equivalent to
Figure 3.3: Obtaining recurrence of the numbers of tilings of $F$-type regions.
verifying the following equation

\[
\begin{align*}
\Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) & \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \right) = 1.
\end{align*}
\]

(3.3)

Similar to the proof of Theorem 2.6, we simplify two terms on the left-hand side of (3.3). First, we evaluate the first fraction in the first term with the notice that its numerator and denominator are different only at \(x\)- and \(t\)-parameters. Cancelling out all terms without \(x\)- or \(t\)-parameter and using the trivial fact \(H(n + 1) = n! H(n)\), we get

\[
\begin{align*}
\Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) & \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \right) = \frac{x!}{(z-1)!} \frac{(A+x+y+2z)!}{(A+2x+y+z)!} \frac{(a+d+x+y)!}{(b+c+z-1)!} \frac{(b+c+z-1)!}{(a+c+x)!}
\end{align*}
\]

(3.4)

Similarly, one can evaluate the second fraction of the first term as

\[
\begin{align*}
\Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) & \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \right) = \frac{(z-1)!}{(x-1)!} \frac{(A+x+y+2z)!}{(A+x+y+2z-1)!} \frac{(a+d+x+y)!}{(a+c+z-1)!} \frac{(b+c+z-1)!}{(b+c+z-1)!}
\end{align*}
\]

(3.5)

By (3.4) and (3.5), the first term can be simplified as

\[
\begin{align*}
\Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) & \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \right) = \frac{a+d+x+y}{a+c+d+f+2x+y+z}.
\end{align*}
\]

(3.6)

We simplify the second term in the same way

\[
\begin{align*}
\Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) & \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} \Psi \left( \begin{array}{c} x+1 \\ a \\ d \\
\end{array} \right) + \Psi \left( \begin{array}{c} y+1 \\ b \\ e \\
\end{array} \right) \\
\Psi \left( \begin{array}{c} z+1 \\ c \\ f \\
\end{array} \right) & \Psi \left( \begin{array}{c} a \\ b \\ c \\
\end{array} \right) \right) = \frac{c+f+x+z}{a+c+d+f+2x+y+z}.
\end{align*}
\]

(3.7)
and (3.3) becomes the following trivial equation
\[
\frac{a + d + x + y}{a + c + d + f + 2x + y + z} + \frac{c + f + x + z}{a + c + d + f + 2x + y + z} = 1.
\]

This completes our proof. \(\square\)

4 Generalized plane partitions fitting in a compound box

As shown in Figure 1.3, the lozenge tilings of region \(F := \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}\) can be viewed as stacks of unit cubes fitting in a certain box \(\mathcal{B}\), which is defined below.

A **compound box** is obtained from several non-overlapping smaller boxes, called **rooms**, as follows. Each room has four **walls** (faces perpendicular to \(\overrightarrow{O_1}\) or \(\overrightarrow{O_1}\)), **floor** and **ceiling** (faces perpendicular to \(\overrightarrow{O_k}\)). If two rooms share a portion of their walls, we remove this portion to make them connected and say that the two rooms are **adjacent**.

Our box \(\mathcal{B}\) is now a compound pound box consisting of 10 rooms (see Figure 3.4(c) where the floors of the rooms are labeled by 1, 2, \ldots, 10).

To present the connectivity of the rooms in a compound box, we use its **connected graph** \(G\) whose vertices are the rooms and whose edges connect precisely two adjacent rooms. The connected graph of the box \(\mathcal{B}\) is illustrated in Figure 3.4(b).

We also consider the **floor plan** a compound box (i.e. the projective diagram of the box on the \(\text{Oij}\) plane). In the floor plan, each room is represented by a rectangle of the same sides as its floor. In each rectangle, we record a pair of integers \((a, b)\), where \(a\) is the level of the floor and \(b\) is the height of the room. We always assume that the floor of room 1 is on the level 0. The floor plan of the box \(\mathcal{B}\) is shown in Figure 3.4(a). One readily sees that the floor plan determines uniquely the box \(\mathcal{B}\). In other words, we can define officially the box \(\mathcal{B} = \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix}\) to be the compound box determined by the floor plan in Figure 3.4(a).

Notice that there are two overlapped rectangles corresponding the rooms 3 and 6 in the floor plan of the box \(\mathcal{B}\). The intersection is indicated by the dashed rectangle in Figure 3.4(a). However, this does not violate the non-overlapping property of a compound box. The explanation is that the floor of the room 6 is higher than the ceiling of the room 3 (as \(y + a + c \geq y + a\)).

Next, we consider GPP’s as the stacks of unit cubes corresponding to the tilings of the region \(F\). A GPP consists of several columns of unit cubes. We say that two columns are **adjacent** if they are in the same room or in two adjacent rooms so that their projections on the \(\text{Oij}\) plane are two unit squares sharing an edge. The levels of the tops of columns in our stack are weakly decreasing along \(\overrightarrow{O_1}\) and \(\overrightarrow{O_3}\) in the sense that: the top of each column does not exceed the tops of its adjacent columns on the left and behind.

We call a stack of unit cubes satisfying the above monotonicity a **generalized plane partition**. One readily sees that if \(a = b = c = d = e = f = 0\), then the box \(\mathcal{B}\) consists of only one room; and our GPP’s become exactly the ordinary plane partitions fitting in the same box.

In summary, we have a bijection between lozenge tilings of the region \(F\) and the GPP’s that fit in the compound box \(\mathcal{B}\).
Figure 3.4: (a) The floor plan of the box. (b) Connected graph. (c) The location of the rooms in the 3-D picture of the box.
Similarly, the lozenge tilings of the region \( Q_{a,b}(x,y,z,t) \) are in bijection with the GPP’s fitting in the compound box \( C := C_{a,b}(x,y,z,t) \), which is determined by the floor plan in Figure 4.1(c). The box \( C \) consists of 4 rooms.

Finally, the lozenge tilings of the region \( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \) can be viewed as the GPP’s fitting in the 7-room compound box \( D := D \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \) illustrated in Figure 4.2.

5 Two \( q \)-weight assignments

We would like to use Kuo condensation to prove Theorem 1.2. In order to do so, we introduce two simple weight assignments for lozenges in our regions. Consider the lozenges in a \( F \)-type region. There are three orientations of the lozenges: left, right and vertical (see Figure 5.1).

In both weight assignments, all left and vertical lozenges are weighted by 1. The right lozenges are weighted as below:

1. **Assignment 1.** Each right lozenge is weighted by \( q^l \), where \( l \) is the distance between its left side and the southeast side of the region.

2. **Assignment 2.** Each right lozenge is weighted by \( q^k \), where \( k \) is the distance between its top and the south side of the region.

We denote the above weight assignment \( \text{wt}_1 \) and \( \text{wt}_2 \), respectively. Figure 5.2 shows the two weight assignments for a tiling of \( F \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \). The similar weight assignments can be applied to the regions \( Q_{a,b}(x,y,z,t) \), \( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \), or any hexagon regions with holes.

Given a weight assignment on lozenges of a region \( R \). The weight a tiling \( T \) of \( R \) is the product of weights of all lozenges in \( T \). \( M(R) \) is now the sum of weights of all lozenge tilings of \( R \). We call \( M(R) \) the tiling generating function of \( R \). Similarly, we can define the matching generating function of a weighted graph \( G \). We also note that each edge in the dual graph \( G \) of a region \( R \) carries the same weight as its corresponding lozenge in \( R \).

To distinguish between the two weight assignments \( \text{wt}_1 \) and \( \text{wt}_2 \), we denote by \( \text{wt}_1(T) \) and \( \text{wt}_2(T) \) the weights of a tiling \( T \) with respect to \( \text{wt}_1 \) and \( \text{wt}_2 \). Moreover, we define two tiling generating functions \( M_1 \) and \( M_2 \) by

\[
M_1(R) := \sum_T \text{wt}_1(T) \quad \text{and} \quad M_2(R) := \sum_T \text{wt}_2(T),
\]

where the sums are both taken over all tilings \( T \) of the region \( R \).

If we apply the above weight assignments to a semi-regular hexagon, then the next lemma tells us that the \( M_1 \)- and \( M_2 \)-tiling generating functions of the hexagon are not too “far” from the formula on the right-hand side of (1.3).

**Lemma 5.1** (Proposition 4.1 in [11]). For any non-negative integers \( a, b, c \)

\[
M_1 \left( \text{Hex}(a,b,c) \right) = q^{ab(b+1)/2} \frac{H_q(a) H_q(b) H_q(c) H_q(a+b+c)}{H_q(a+b) H_q(b+c) H_q(c+a)}
\]

(5.1)
Figure 4.1: (a) Viewing a lozenge tiling of the region $Q_{a,b}(x, y, z, t)$ as a stack of unit cubes. (b) The 3-D picture of the box $C$. (c) The floor plan of the box $C$. (d) The connected graph of the box $C$. 
Figure 4.2: (a) Viewing a lozenge tiling of a $B$-type region as a stack of unit cubes. (b) The 3-D picture of the box $D$. (c) The floor plan of the box $D$. (d) The connected graph of the box $D$.

Figure 5.1: Three orientations of lozenges.
Figure 5.2: The two weight assignments of a $F$-type region: (a) Assignment 1, (b) Assignment 2. The right lozenges with label $l$ have weight $q^l$.

and

$$M_2 \left( Hex(a, b, c) \right) = q^{ba(a+1)/2} \frac{H_q(a) H_q(b) H_q(c) H_q(a + b + c)}{H_q(a + b) H_q(b + c) H_q(c + a)}$$

(5.2)

Moreover, in the proof of Proposition 4.1 [11], we showed that for any tiling $T$ of the hexagon $Hex(a, b, c)$

$$wt_1(T)/q^{\lvert \pi_T \rvert} = q^{ab(b+1)/2} \quad \text{and} \quad wt_2(T)/q^{\lvert \pi_T \rvert} = q^{ba(a+1)/2},$$

where $\pi_T$ is the plane partition corresponding to $T$.

The next lemma was also proven in [11].

**Lemma 5.2.** For any non-negative integers $a, x, y, z, t$

$$M_1(K_a(x, y, z, t)) = q^{g(x^{y+1})+y(a+x+y)+y+z(a+y+z)} \frac{H_q(a) H_q(x) H_q(y) H_q(z) H_q(t)}{H_q(a+x) H_q(a+y) H_q(y+z) H_q(t+x)} \times \frac{H_q(a+x+y) H_q(a+y+z) H_q(a+t+x) H_q(a+x+y+z+t)}{H_q(a+x+y+z+t) H_q(a+x+y+z+t)}$$

(5.3)

and

$$M_2(K_a(x, y, z, t)) = q^{g(x^{y+1})+y(a+x+y+z)} \frac{H_q(a) H_q(x) H_q(y) H_q(z) H_q(t)}{H_q(a+x) H_q(a+y) H_q(y+z) H_q(t+x)} \times \frac{H_q(a+x+y) H_q(a+y+z) H_q(a+t+x) H_q(a+x+y+z+t)}{H_q(a+x+y+z+t) H_q(a+x+y+z+t)}.$$

(5.4)

**Proof.** The second equation was proven explicitly in Corollary 5.2 [11]. The first equation follows from the second equation and Proposition 4.2 in [11].

Moreover, following the argument in the proof of Proposition 4.2 [11], we are able to show that the ratios $wt_1(T)/q^{\lvert \pi_T \rvert}$ and $wt_2(T)/q^{\lvert \pi_T \rvert}$ do not depend on the choice of the tiling $T$ of a $F$-type region.
Define two functions

\[ f := f \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) := (x+z+d+f) \left( \begin{array}{c} y+b+1 \\ 2 \end{array} \right) + e \left( \begin{array}{c} b+1 \\ 2 \end{array} \right) + a(c+x)(a+b+y) \\
+ a \left( \begin{array}{c} x+c+1 \\ 2 \end{array} \right) + (x+d)(a+b+y)(f+x+z) + (x+z+f) \left( \begin{array}{c} d+x+1 \\ 2 \end{array} \right) + b(d+e+x+y)(a+b+y) \\
+ b \left( \begin{array}{c} x+y+d+e+1 \\ 2 \end{array} \right) + f(z+b)(a+b+d+e+x+2y+z) + (z+b) \left( \begin{array}{c} f+1 \\ 2 \end{array} \right) \\
+ xc(a+b+d+x+y) + x \left( \begin{array}{c} c+1 \\ 2 \end{array} \right) \right) \]  

and

\[ g := g \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) := b \left( \begin{array}{c} x+z+d+e+f+1 \\ 2 \end{array} \right) + y \left( \begin{array}{c} x+z+d+f+1 \\ 2 \end{array} \right) + (x+c) \left( \begin{array}{c} a+1 \\ 2 \end{array} \right) \\
+ xc(a+d) + c \left( \begin{array}{c} x+1 \\ 2 \end{array} \right) + (x+d) \left( \begin{array}{c} x+z+f+1 \\ 2 \end{array} \right) + (x+d)(x+z+f)(a+d) + f \left( \begin{array}{c} b+z+1 \\ 2 \end{array} \right) \\
+ f(z+b)(x+y+z+a+d+e+f) + b(x+y+z+a+d+e+f)(x+y+d+e) + (x+y+d+e) \left( \begin{array}{c} b+1 \\ 2 \end{array} \right). \]  

**Proposition 5.3.** For any non-negative integers \( a, b, c, d, e, f, x, y, z \)

\[ M_1 \left( F \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) \right) = q^f \sum_{\pi} q^{\pi} \]  

and

\[ M_2 \left( F \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) \right) = q^g \sum_{\pi} q^{\pi}, \]  

where the sums are taken over all GPP’s \( \pi \) fitting in the compound box \( B := B \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) \).

**Proof.** We assume that the room \( i \) \((1 \leq i \leq 10)\) in the box \( B \) has size \( a_i \times b_i \times c_i \). The floor of the room \( i \) is pictured as an \( a_i \times b_i \) parallelogram \( P_i \) in Figure 3.4(c). Assume that the southeast side of \( P_i \) is \( x_i \) units to the left of the southeast side of \( F := F \left( \begin{array}{ccc} x & y & z \\ a & b & c \\ d & e & f \end{array} \right) \) and that the bottom of \( P_i \) is \( y_i \) units above the bottom of the region \( F \). We call \( x_i \) and \( y_i \) the relative positions of \( P_i \) to the boundary of the region \( F \). Following the arguments in the proof of Proposition 4.2 in [11], we get

\[ \text{wt}_1(T)/q^{\pi_f} = q^{\sum_{i=1}^{10} a_i b_i x_i + a_i b_i (b_i+1)/2} \]  

and

\[ \text{wt}_2(T)/q^{\pi_f} = q^{\sum_{i=1}^{10} a_i b_i y_i + b_i a_i (a_i+1)/2}, \]  

(5.9)
for any tiling $T$ of the region $F$. From Figure 3.4, one gets the formulas for $a_i, b_i, x_i, y_i$ in terms of 9 parameters $a, b, c, d, e, f, x, y, z$. Plugging these formulas into the above two equations, we get

$$\text{wt}_1(T)/q^{\pi_T} = q^E$$ and $$\text{wt}_2(T)/q^{\pi_T} = q^F.$$

Summing over all tilings $T$ of $F$, we get (5.7) and (5.8), respectively.  

## 6 $q$-enumerations of $Q$- and $B$-type regions

In this section, we ($q$-)enumerate the lozenge tilings of the regions $Q_{a,b}(x, y, z, t)$ and $B_{a,b}(x, y, z, t)$.

**Theorem 6.1.** For non-negative integers $a$, $b$, $x$, $y$, $z$, $t$

$$q^{E_1} \sum_{\pi} q^{\pi} = q^{E_2} M_1(Q_{a,b}(x, y, z, t)) = M_2(Q_{a,b}(x, y, z, t)) = \sum_{\pi} q^{\pi}$$

where

$$E_1 = z \left( \frac{y + b + 1}{2} \right) + b(x + y)(a + b + y) + b \left( \frac{x + y + 1}{2} \right) + (a + z)(a + b + y) + (a + z) \left( \frac{x + 1}{2} \right)$$

and

$$E_2 = (y + b) \left( \frac{z + 1}{2} \right) + x \left( a + z + 1 \right) + b(x + y)(a + y + z) + (x + y) \left( \frac{b + 1}{2} \right).$$

**Proof.** First, by the same spirit as the proof of Proposition 5.3, we have for any tiling $T$ of the region $Q := Q_{a,b}(x, y, z, t)$:

$$\text{wt}_1(T)/q^{\pi_T} = \frac{\sum_{i=1}^{4} a_i b_i x_i + a_i b_i (b_i + 1)/2}{q}$$

and

$$\text{wt}_2(T)/q^{\pi_T} = \frac{\sum_{i=1}^{4} a_i b_i y_i + a_i b_i (a_i + 1)/2}{q},$$

where $a_i, b_i, x_i, y_i$ are the sides and the relative coordinates of the floor of the room $i$ in the box $C$ (given in Figure 4.1). Obtaining the formulas for the later parameters in terms of $a, b, x, y, z, t$, and summing each of the equation (6.2) and (6.3) over all tilings $T$’s of $Q_{a,b}(x, y, z, t)$, we get

$$M_1(Q)/\sum_{\pi} q^{\pi} = q^{E_1}$$ and $$M_2(Q)/\sum_{\pi} q^{\pi} = q^{E_2}.$$

This means that we only need to prove the third equal sign in (6.1). This can be treated in the same way as Theorem 2.2 with some extra care of the weights of forced lozenges.

We note that in the weighted case, removal of some forced lozenges changes the tiling generating function of the region by a factor equal to the product of weights of the forced lozenges.
We now prove the third equal sign in (6.1) by induction on $y+t+2b$. Our base cases are still the situations when at least one of the three parameters $b, y, t$ is equal to 0.

Assume that our region $Q_{a,b}(x, y, z, t)$ is weighted by $wt_{2}$. If $b = 0$, our region is exactly the regions in Lemma 5.2, and the third equal sign in (6.1) follows. If $y = 0$, Region-splitting Lemma 2.3 gives us

$$M_{2}(Q_{a,b}(x, 0, z, t)) = M_{2}(Hex(z + a + b, x, t)) M_{2}(Hex(z, b, a))$$

(6.5)

(see Figure 2.3(a)). Then the third equal sign of (6.1) follows from Lemmas 5.1 and 5.2. If $t = 0$, there are several forced right lozenges on top of our region as in Figure 2.3(b). Collecting the weights of those forced lozenges, we get

$$M_{2}(Q_{a,b}(x, y, z, 0)) = q^{(b+1)}(x+y)(y+z+a) + (x+y)(z+b) M_{2}(K_{a}(x, y + b, z, y)),$$

(6.6)

and (6.1) follows from Lemma 5.2 again.

For the induction step, we apply Kuo Theorem 2.4 to the dual graph $G$ of the region $R$ weighted by $wt_{2}$, where $R$ is still the region obtained from the region $Q_{a,b}(x, y, z, t)$ by adding a band of $2b - 1$ unit triangles along the bottom of the $b$-hole (see again Figure 2.4). Our process is also based on Figure 2.5. The only difference is that the forced lozenges have weights. In particular, by investigating the weights of forced lozenges, we get

$$M(G - \{v\}) = M_{2}(Q_{a,b}(x, y, z, t)).$$

(6.7)

$$M(G - \{u, w\}) = q^{(x+y-1)}(y+z+t+a+b) M_{2}(Q_{a,b-1}(x, y, z+1, t-1)),$$

(6.8)

$$M(G - \{u\}) = M_{2}(Q_{a,b-1}(x, y, z+1, t)),$$

(6.9)

$$M(G - \{v, w\}) = q^{(x+y-1)}(y+z+t+a+b) M_{2}(Q_{a,b}(x, y, z, t-1)),$$

(6.10)

$$M(G - \{w\}) = q^{(x+y)}(y+z+t+a+b) M_{2}(Q_{a,b-1}(x, y+1, z, t-1)),$$

(6.11)

and

$$M(G - \{u, v, s\}) = M_{2}(Q_{a,b}(x, y - 1, z + 1, t)).$$

(6.12)

Substituting the above six identities (6.7) – (6.12) into the equation (2.6) in Kuo’s Theorem 2.5, we get

$$M_{2}(Q_{a,b}(x, y, z, t)) M_{2}(Q_{a,b-1}(x, y, z + 1, t - 1)) =$$

$$M_{2}(Q_{a,b-1}(x, y, z + 1, t)) M_{2}(Q_{a,b}(x, y, z, t - 1))$$

$$+ q^{y+z+t+a+b} M_{2}(Q_{a,b-1}(x, y + 1, z, t - 1)) M_{2}(Q_{a,b}(x, y - 1, z + 1, t)).$$

(6.13)

By induction hypothesis, all regions in (6.13), except for the first one, have the tiling generating function given by (6.1). Substituting these formulas into (6.13) and simplifying, one gets that $M_{2}(Q_{a,b}(x, y, z, t))$ is given exactly by the expression on the rightmost side of (6.13). We finish our proof here.

To prove Theorem 1.2, we need also the following variation of Theorem 6.1.
**Theorem 6.2.** Assume that all right and vertical lozenges of the region \( Q_{a,b}(x,y,z,t) \) have weight 1. We now assign a weight \( q^l \) to any left lozenge, where \( l \) is the distance from the top of the lozenge to the bottom of the region. Denote by \( M_3(Q_{a,b}(x,y,z,t)) \) the tiling generating function of \( Q_{a,b}(x,y,z,t) \) with respect to this new weight assignment. Then

\[
M_3(Q_{a,b}(x,y,z,t)) = q^{b(\binom{x+1}{2}+y\binom{x+1}{2})+x(\binom{x+1}{2}+1)} \frac{H_q(x) H_q(y) H_q(z) H_q(t) H_q(a) H_q(b)}{H_q(a+x) H_q(b+t) H_q(a+b+y)} H_q(a+b+x+2y+z+t) H_q(a+b+x+y+t) H_q(a+b+x+y+z) H_q(a+x+y) H_q(b+y+t) H_q(a+b+y+t)^2 H_q(x+y+t) H_q(a+y+z) H_q(b+y+z).
\]

(6.14)

**Proof.** This theorem can be treated similarly to Theorem 6.1. We also prove (6.2) by induction on \( y + t + 2b \) with the base cases are: \( b = 0, y = 0, \) and \( t = 0 \).

Consider the region \( Q_{a,b}(x,y,z,t) \) associated with the new weight assignment.

If \( b = 0 \), we reflect our region about a vertical line to get the region \( K_a(y,x,y+t,z) \) weighted by \( wt_2 \). Then \( 6.14 \) follows from Lemma 5.2. If \( y = 0 \), we also slit the region into two parts by using Region-splitting Lemma 2.3 as in Figure 2.3(a). However, we need to reflect these parts about a vertical line to get back the weight assignment \( wt_2 \). In particular, we get

\[
M_3(Q_{a,b}(x,0,z,t)) = M_2( Hex(t,x,z+a+b)) M_2( Hex(a,b,z))
\]

then \( 6.14 \) follows from Lemma 5.1. Finally, if \( t = 0 \), by removing the forced right lozenges as in Figure 2.3(b) (which have all weight 1 in the new weight assignment), we get a \( K \)-type region. However, we also need to reflect this region about a vertical line to get back the weight assignment \( wt_2 \). To precise, we get

\[
M_3(Q_{a,b}(x,y,z,0)) = M_2(K_a(y+b,x,y,z)),
\]

and \( 6.14 \) is implied by Lemma 5.2 again.

The induction step is treated similarly to that of the proof of Theorem 6.1, the only difference is that the forced lozenges have different weights. In particular, by collecting new weights of forced lozenges, we get the following new recurrence for \( M_3 \)-tiling generating functions:

\[
M_3(Q_{a,b}(x,y,z,t)) M_3(Q_{a,b-1}(x,y,z+1,t-1)) =
M_3(Q_{a,b-1}(x,y,z+1,t)) M_3(Q_{a,b}(x,y,z,t-1)) + M_3(Q_{a,b-1}(x,y+1,z,t-1)) M_3(Q_{a,b}(x,y-1,z+1,t)),
\]

(6.17)

and by the induction hypothesis and some simplifications, one gets that \( M_3(Q_{a,b}(x,y,z,t)) \) is given exactly by the expression on the right-hand side of \( 6.14 \). 

Similarly, the lozenge tilings of a \( B \)-type region are \( q \)-enumerated by the following theorem.
Theorem 6.3. For non-negative integers \(a, b, c, d, x, y, z, t\)

\[
q A_2 \sum_{\pi} q^{\pi} = q^{A_2-A_1} M_1 \left( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \right) = M_2 \left( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \right) =
\]

\[
q A_2 \frac{H_q(x) H_q(y) H_q(z) H_q(t) H_q(a)^2 H_q(b) H_q(c) H_q(d)}{H_q(a+c) H_q(b+x) H_q(d+y)} H_q(a+b+c+d+y+z+2t) H_q(a+b+c+d+x+2z+t) \\
\times H_q(a+b+c+d+x+y+z+t) H_q(a+b+c+d+x+y+2z+t) \\
\times H_q(a+b+c+x+z+t) H_q(a+b+c+d+y+z+t) \\
\times \frac{H_q(a+b+c+d+y+z+2t) H_q(a+b+c+x+2z+t) H_q(b+c+d+x+z+t) H_q(b+c+d+y+z+t)}{H_q(d+y+t) H_q(a+c+z+t) H_q(a+c+d+z+t) H_q(a+y+t) H_q(a+d+z+t)},
\]

where

\[
A_1 = A_1 \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) =
\]

\[
(a+z) \left( \frac{y+1}{2} \right) + (b+y)(b+z)(c+z+t) + (z+b) \left( \frac{c+z+t+1}{2} \right) + az(b+y) + a \left( \frac{z+1}{2} \right) \\
+ x(b+d+z)(b+c+d+y+z+t) + (b+d+z) \left( \frac{x+1}{2} \right) + c(x+t)(b+c+d+y+z+t) + c \left( \frac{x+t+1}{2} \right)
\]

and

\[
A_2 = A_2 \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) =
\]

\[
y \left( \frac{a+z+1}{2} \right) + (c+z+t) \left( \frac{z+b+1}{2} \right) + x \left( \frac{b+d+z+1}{2} \right) + az(b+z) \\
+ z \left( \frac{a+1}{2} \right) + c(x+t)(b+d+z+t) + (x+t) \left( \frac{c+1}{2} \right).
\]

**Proof.** Similar to the proof of Theorem 6.1, we have

\[
M_1 \left( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \right) = q \sum_{i=1}^{7} a_i b_i x_i + a_i b_i (b_i + 1)/2
\]

and

\[
M_2 \left( B \left( \begin{array}{cccc} x & y & z & t \\ a & b & c & d \end{array} \right) \right) = q \sum_{i=1}^{7} a_i b_i y_i + b_i a_i (a_i + 1)/2
\]

where \(a_i, b_i, x_i, y_i\) are the sides and the relative coordinates of the floor of room \(i\) in the compound box \(D\) defined in Figure 4.2, for \(i = 1, 2, \ldots, 7\). Writing \(a_i, b_i, x_i, y_i\) in terms of \(a, b, c, d, x, y, z, t\) from Figure 4.2, we get

\[
\sum_{i=1}^{7} a_i b_i x_i + a_i b_i (b_i + 1)/2 = A_1.
\]
and
\[ \sum_{i=1}^{7} a_i b_i y_i + b_i a_i (a_i + 1)/2 = A_2. \]  

(6.24)

Thus, we only need to prove the third equal sign in (6.18).

Similar to the Theorem 2.6, we prove the third equal sign in (6.18) by induction on \( z + t \). Our base cases are still the cases when \( z = 0 \) or \( t = 0 \).

Assume that our region are weighted by \( \text{wt}_2 \). If \( z = 0 \), we use Region-splitting Lemma 2.3 to separate our region into two parts as in Figure 2.27(a). The right part is the hexagon \( \text{Hex}(a, y, t + b + c + d) \) and the left part is \( Q_{d,c}(x, t, b, a) \), both parts are weighted by \( \text{wt}_2 \). Then, we get

\[ M_2 \left( B \left( \begin{array}{c} x \\ a \\ y \\ b \\ 0 \\ c \\ d \\ t \end{array} \right) \right) = M_2 \left( \text{Hex}(a, y, t + b + c + d) \right) M_2 \left( Q_{d,c}(x, t, b, a) \right), \]  

(6.25)

and the third equal sign in (6.18) follows from Lemma 5.1 and Theorem 6.1. If \( t = 0 \), we also split our region into two subregions as in Figure 2.7(b). The left subregion is the hexagon \( \text{Hex}(z + b + c + d, x, a) \) weighted by \( \text{wt}_2 \). For the right subregion, we reflect it about a vertical line, and get the region \( Q_{b,c}(y, z, d, a) \) weighted by the weight assignment in Theorem 6.2. Therefore, we get

\[ M_2 \left( B \left( \begin{array}{c} x \\ a \\ y \\ b \\ z \\ c \\ 0 \\ d \end{array} \right) \right) = M_2 \left( \text{Hex}(z + b + c + d, x, a) \right) M_3 \left( Q_{b,c}(y, z, d, a) \right), \]  

(6.26)

and the third equal sign in (6.18) is implied by Lemma 5.1 and Theorem 6.2.

The induction step can be treated similarly to that of the proof of Theorem 2.6 (based on Figure 2.8). We still apply Kuo’s Theorem 2.4 to the dual graph \( G \) of \( B \left( \begin{array}{c} x \\ y \\ z \\ t \\ a \\ b \\ c \\ d \end{array} \right) \) weighted by \( \text{wt}_2 \). One readily sees that all lozenges, which are forced by the shaded unit triangles corresponding to \( u, v, w, s \), have weight 1. Thus, the \( M_2 \)-tiling generating functions of \( B \)-type regions satisfy also the recurrence (2.24) in the proof of Theorem 2.6. Finally, we only need to show that the expression on the rightmost side of (6.18) satisfies the same recurrence.

By definition of the exponent \( A_2 \), we have

\[ A_2 \left( \begin{array}{c} x \\ y \\ z \\ t \\ a \\ b \\ c \\ d \end{array} \right) + A_2 \left( \begin{array}{c} x \\ y \\ z-1 \\ t-1 \\ a \\ b \\ c+1 \\ d+1 \end{array} \right) = A_2 \left( \begin{array}{c} x \\ y \\ z \\ t-1 \\ a \\ b \\ c+1 \\ d+1 \end{array} \right) + A_2 \left( \begin{array}{c} x \\ y \\ z-1 \\ t \\ a \\ b \\ c+1 \\ d \end{array} \right) - d - x - t \]  

(6.27)

\[ A_2 \left( \begin{array}{c} x \\ y \\ z \\ t \\ a \\ b \\ c \\ d \end{array} \right) + A_2 \left( \begin{array}{c} x \\ y \\ z-1 \\ t-1 \\ a \\ b \\ c+1 \\ d+1 \end{array} \right) = A_2 \left( \begin{array}{c} x \\ y \\ z-1 \\ t \\ a \\ b \\ c+1 \\ d+1 \end{array} \right) + A_2 \left( \begin{array}{c} x \\ y \\ z \\ t-1 \\ a \\ b \\ c+1 \\ d \end{array} \right) \]  

(6.28)

We denote by \( \Phi' \left( \begin{array}{c} x \\ y \\ z \\ t \\ a \\ b \\ c \\ d \end{array} \right) \) the expression on the rightmost side of (6.18) divided by \( q^{A_2} \).

By (6.27) and (6.27), we need to show that

\[ q^{d+x+t} \frac{\Phi' \left( \begin{array}{c} x \\ y \\ z \\ t-1 \\ a \\ b \\ c+1 \\ d+1 \end{array} \right)}{\Phi' \left( \begin{array}{c} x \\ y \\ z \\ t \\ a \\ b \\ c \\ d \end{array} \right)} \Phi' \left( \begin{array}{c} x \\ y \\ z-1 \\ t \\ a \\ b \\ c+1 \\ d+1 \end{array} \right) + \frac{\Phi' \left( \begin{array}{c} x \\ y \\ z \\ t-1 \\ a \\ b \\ c+1 \\ d+1 \end{array} \right)}{\Phi' \left( \begin{array}{c} x \\ y \\ z-1 \\ t \\ a \\ b \\ c \\ d \end{array} \right)} = 1. \]  

(6.29)
Applying the same simplifying process as in the proof of Theorem 2.6 (with hyperfactorials are replaced by the corresponding $q$-hyperfactorials), we get

\[
\Phi \left( \begin{array}{cccc}
\frac{x}{a} & \frac{y}{b} & \frac{z}{c} & \frac{t-1}{d+1} \\
\end{array} \right) + \Phi \left( \begin{array}{cccc}
\frac{x}{a} & \frac{y}{b} & \frac{z}{c} & \frac{t}{d+1} \\
\end{array} \right) = \frac{[a+b+c+y+2z+t-1]_q}{[a+b+c+d+x+y+2z+2t-1]_q}
\]

(6.30)

and

\[
\Phi \left( \begin{array}{cccc}
\frac{x}{a} & \frac{y}{b} & \frac{z}{c} & \frac{t-1}{d+1} \\
\end{array} \right) + \Phi \left( \begin{array}{cccc}
\frac{x}{a} & \frac{y}{b} & \frac{z}{c} & \frac{t}{d+1} \\
\end{array} \right) = \frac{[d+x+t]_q}{[a+b+c+d+x+y+2z+2t-1]_q}
\]

(6.31)

Then (6.29) becomes the following equation

\[
\frac{q^{d+x+t}[a+b+c+y+2z+t-1]_q}{[a+b+c+d+x+y+2z+2t-1]_q} + \frac{[d+x+t]_q}{[a+b+c+d+x+y+2z+2t-1]_q} = 1,
\]

(6.32)

which follows directly from the definition of $q$-integer. This completes our proof.

\[\Box\]

7 Proof of Theorem 1.2

From (5.3), we only need to show that

\[
M_2(F) = q^\sum \frac{H_q(x)H_q(y)H_q(z)H_q(a)^2H_q(b)^2H_q(c)^2H_q(d)H_q(e)H_q(f)H_q(d+e+f+x+y+z)^4}{H_q(a+d)H_q(b+e)H_q(c+f)H_q(d+e+x+y+z)H_q(e+f+x+y+z)H_q(f+d+x+y+z)}
\]

\[
\times \frac{H_q(A+2x+2y+2z)H_q(A+x+y+z)^2}{H_q(A+2x+y+z)H_q(A+x+2y+z)H_q(A+x+y+2z)}
\]

\[
\times \frac{H_q(a+b+d+e+x+y+z)H_q(a+c+d+f+x+y+z)}{H_q(a+d+e+f+x+y+z)H_q(b+d+e+f+x+y+z)^2H_q(c+d+e+f+x+y+z)^2}
\]

\[
\times \frac{H_q(a+d+x+y)H_q(b+e+y+z)H_q(c+f+z+x)}{H_q(a+b+y)H_q(b+c+z)H_q(c+a+z)}
\]

\[
\times \frac{H_q(A-a+x+y+2z)H_q(A-b+2x+y+z)H_q(A-c+x+2y+z)}{H_q(b+c+e+f+x+y+2z)H_q(c+a+d+f+2x+y+z)H_q(a+b+d+e+x+2y+z)}
\]

(7.1)

Similar to the Theorem 1.1, we prove (7.1) by induction on $y+z$, with the base cases are $y = 0$ and $z = 0$.

Assume our region is having the weight assignment $wt_2$. If $y = 0$, we also split up the region into two parts by using Region-splitting Lemma 2.3 as in Figure 3.1(a). The right part is isomorphic to $Hex(x+z,d+e+f,b,a)$ weighted by $wt_2$. For the left part, we rotate it $60^\circ$ clockwise and reflect by a vertical line. This way, we get the region $B \left( \frac{b}{c} \frac{a}{d} \frac{x}{f} \frac{z}{e} \right)$ weighted

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by \( \text{wt}_1 \). In particular, we get
\[
M_2 \left( F \begin{pmatrix} x & 0 & z \\ a & b & c \\ d & e & f \end{pmatrix} \right) = M_2 \left( \text{Hex}(x + z + d + e + f, b, a) \right) M_1 \left( B \begin{pmatrix} b & a & x & z \\ c & d & f & e \end{pmatrix} \right). \tag{7.2}
\]
Then (7.1) follows from Lemma 5.1 and Theorem 6.3.

If \( z = 0 \), we split the region into two parts as in Figure 3.1(b). The upper part is the hexagon \( \text{Hex}(b, x + y + d + e + f, c) \) in which each right lozenge is weighted by \( q^{x+y+a+d+e+f+l} \), where \( l \) is the distance between the top of the lozenge and the bottom of the hexagon. Dividing the weight of each right lozenge by \( q^{x+y+a+d+e+f} \), we get the hexagon \( \text{Hex}(b, x + y + d + e + f, c) \) weighted by \( \text{wt}_2 \). For the bottom part, we rotate it \( 180^0 \) counter-clockwise, and get the region \( B \begin{pmatrix} b & c & x & y \\ a & f & d & e \end{pmatrix} \) in which each right lozenge is weighted by \( q^{x+y+a+d+e+f+1-l} \), where \( l \) is the distance between its top and the bottom of the \( B \)-type region. Dividing the weights of all right lozenges by \( q^{x+y+a+d+e+f+1} \), we get back the weight \( \text{wt}_2 \), where \( q \) is replaced by \( q^{-1} \). By Lemma 5.1, Theorem 6.3 and the simple fact \([n]_{q^{-1}} = [n]_q/q^{n-1}\), we get (7.1).

Our induction step is completely analogous to that of the proof of Theorem 1.1. Apply Kuo’s Theorem 2.5 to the dual graph \( G \) of the region \( R \) weighted by \( \text{wt}_2 \), where \( R \) is the region obtained from \( F \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} \) by adding a band of unit triangles along the southwest side (see Figure 3.2). Our process is still based on Figure 3.3. Notice that all lozenges, which are forced by shaded unit triangles corresponding to the four vertices \( u, v, w, s \), are weighted by 1 here. Therefore, we get the same recurrence (3.2) for the \( M_2 \)-tiling generating functions of \( F \)-type regions. Our final job is checking that the expression on the right-hand side of (7.1) satisfies also the recurrence (3.2).

By definition of the exponent \( g \), we have
\[
g \begin{pmatrix} x + 1 & y & z - 1 \\ a & b & c \\ d & e & f \end{pmatrix} + g \begin{pmatrix} x & y - 1 & z \\ a & b & c \\ d & e + 1 & f \end{pmatrix} = g \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} + g \begin{pmatrix} x + 1 & y - 1 & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix}. \tag{7.3}
\]
and
\[
g \begin{pmatrix} x + 1 & y - 1 & z \\ a & b & c \\ d & e & f \end{pmatrix} + g \begin{pmatrix} x & y & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix} = a + d + x + y + g \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} + g \begin{pmatrix} x + 1 & y - 1 & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix}. \tag{7.4}
\]
Denote by \( \Psi' \begin{pmatrix} x & y & z \\ a & b & c \\ d & e & f \end{pmatrix} \) the expression on the right-hand side of (7.1) \textit{divided by } \( q^g \) (i.e. the expression on the right-hand side of (1.4)). We need to show that
\[
\Psi' \begin{pmatrix} x + 1 & y & z - 1 \\ a & b & c \\ d & e & f \end{pmatrix} + \Psi' \begin{pmatrix} x & y - 1 & z \\ a & b & c \\ d & e + 1 & f \end{pmatrix} + q^{a+d+x+y} \Psi' \begin{pmatrix} x + 1 & y - 1 & z \\ a & b & c \\ d & e & f \end{pmatrix} + q^{a+d+x+y} \Psi' \begin{pmatrix} x & y & z - 1 \\ a & b & c \\ d & e + 1 & f \end{pmatrix} = 1. \tag{7.5}
\]
Now, we can apply the simplifying process in the proof of Theorem 1.1, where each hyperfactorial is replaced by the corresponding $q$-hyperfactorial. Similar to (3.6) and (3.7), we get

$$\Psi'\left(\begin{array}{ccc} x+1 & y & z-1 \\
 & a & b & c \\
 & d & e & f \end{array}\right) = \frac{[a+d+x+y]_q}{[a+c+d+f+2x+y+z]_q}$$  \hfill (7.6)

and

$$\Psi'\left(\begin{array}{ccc} x+1 & y-1 & z \\
 & a & b & c \\
 & d & e & f \end{array}\right) = \frac{[c+f+x+z]_q}{[a+c+d+f+2x+y+z]_q}.$$  \hfill (7.7)

Therefore, the equation (7.5) is equivalent to the following equation

$$\frac{[a+d+x+y]_q}{[a+c+d+f+2x+y+z]_q} + q^{a+d+x+y}\frac{[c+f+x+z]_q}{[a+c+d+f+2x+y+z]_q} = 1,$$  \hfill (7.8)

which is obviously true by the definition of $q$-integer. This finishes our proof.

8 Plane partitions with constraints

In this section, we present a consequence of our result on (ordinary) plane partitions with some constraints.

Let us assume that $d$-, $e$- and $f$-parameters in a $F$-type region are equal to 0. The region $F\left(\begin{array}{ccc} x & y & z \\
 & a & b & c \\
 & 0 & 0 & 0 \end{array}\right)$ becomes a hexagon with three triangular holes on non-consecutive sides. We use the notation $N_{a,b,c}(x,y,z)$ for this special $F$-type region. By adding the forced lozenges as in Figure 8.1(a) to each tiling of $N_{a,b,c}(x,y,z)$, we get a lozenge tiling of the semi-regular hexagon $Hex(z+x+a+b,x+y+b+c,y+z+c+a)$. The later lozenge tiling in turn corresponds to a plane partition (in tabular form) having $z+x+a+b$ rows and $x+y+b+c$ columns with entries at most $y+z+c+a$ (or a stack of unit cubes fitting in a $(z+x+a+b) \times (x+y+b+c) \times (y+z+c+a)$ box as in 8.1(b)). The forced lozenges give certain constraints on this plane partition. In particular, the forced lozenge on the northwest side implies that:

(i). The first $z+b$ entries of the columns $1,2,\ldots,c$ are all $y+z+c+a$. Moreover, the remaining entries of these columns are at most $y+z+a$.

The forced lozenges on the northeast side says:

(ii). The last $b$ entries of the columns $x+y+b+1,x+y+b+2,\ldots,x+y+b+c$ are all $y+a$.

Finally, the forced lozenges along the south side is equivalent to the fact that:

(iii). The last $y+b$ entries of the rows $z+x+b+1,z+x+b+2,\ldots,z+x+b+a$ are all 0; and the remaining entries of these rows are at least $a$. 

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Figure 8.1: (a) Adding forced lozenges along the northeast, the northwest and the south sides of the region $N_{3,2,2}(2, 2, 2)$ (the shaded region with the bold contour) (b) Viewing a the lozenge tiling of $N_{3,2,2}(2, 2, 2)$, where forced lozenges have been added, as a stack of unit cubes fitting in a $9 \times 8 \times 9$ box.

Figure 8.2: The plane partitions satisfying all the three constraints. The entries in the vertically stripped area are at most $y + z + a$, entries in the horizontally stripped area are at least $a$; and the entries in the crossed area are between $a$ and $y + z + a$. 
Figure 8.2 illustrates the plane partitions with the above constraints (i), (ii) and (iii).

Thus by Theorem 1.1, we get the following results.

**Corollary 8.1.** Let $a, b, c, x, y, z$ be non-negative integers. The number of plane partitions having $z + x + a + b$ rows and $x + y + b + c$ columns with entries at most $y + z + c + a$, where the constraints (i), (ii) and (iii) hold, is equal to

$$H(x) H(y) H(z) H(a) H(b) H(c) H(x + y + z)$$

$$\times \frac{H(a + b + c + 2x + 2y + 2z) H(a + b + c + x + y + z)}{H(a + b + c + 2x + y + z) H(a + b + c + x + 2y + z) H(a + b + c + x + y + 2z)}$$

$$\times \frac{H(a + b + x + y + z) H(a + c + x + y + z) H(b + c + x + y + z)}{H(a + x + y + z) H(b + x + y + z) H(c + x + y + z)}$$

$$\times \frac{H(a + x + y) H(b + y + z) H(c + z + x)}{H(a + b + y) H(b + c + z) H(c + a + x)}.$$  \hfill (8.1)

9 Concluding remarks

Our method in proving Theorem 1.2 can be used to give a new proof of MacMahon $q$-theorem (1.3).

Besides giving a generalization for Propp’s Problem 3 in [13], the present paper (together with [11]) gives several examples where the MacMahon’s $q$-enumeration can be extended to GPP’s fitting in a compound box. We would like to find more such examples, especially large families of examples. This would help us find exactly the structure of the compound boxes in which the generating function of the volume of the corresponding GPP’s is given by a simple product formula.

There are still many interesting aspects of GPP’s that we would like to understand. For example, the number of symmetric GPP’s. However, this will be investigated in a separate paper.

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