Hidden symmetries in Dilaton–Axion Gravity

D.V. Gal'tsov and O.V. Kechkin
Department of Theoretical Physics, Moscow State University,
Moscow 119899, Russia

Abstract

Four–dimensional Einstein–Maxwell–dilaton–axion system restricted to space–times with one non–null Killing symmetry is formulated as the three–dimensional gravity coupled sigma–model. Several alternative representations are discussed and the associated hidden symmetries are revealed. The action of target space isometries on the initial set of (non–dualized ) variables is found. New multicenter solutions are obtained via generating technique based on the formulation in terms of the non–dualized variables.

PASC number(s): 97.60.Lf, 04.60.+n, 11.17.+y

---

1 Extended version of a talk at the JINR Workshop “Geometry and Integrable models”, JINR, Dubna, (Russia), October 1994, published in Proceedings, Published in Geometry and Integrable Models, P.N. Pyatov and S.N. Solodukhin (Eds.), World Scientific, 1996
1 Introduction

Einstein equations in General Relativity have a nice property to admit a three–dimensional gravity coupled sigma–model representation, being restricted to space–times possessing a non–null Killing symmetry. Moreover, it turns out that the target space of this sigma–model is a symmetric Riemannian space, namely, the coset $SL(2, R)/SO(2)$. This sigma–model may serve a convenient starting point for a number of solution–generating methods, most notable being an inverse scattering transform and Backlund transformations, which arise when further Killing symmetry commuting with the first one is imposed [1]. The existing knowledge of a great variety of analytic solutions to the Einstein equations is mostly due to this property [2].

Similar property is shared by other theories related to General Relativity: Einstein–Maxwell (EM) theory [3], higher dimensional vacuum Einstein equations [4], bosonic sectors of some dimensionally reduced supergravity theories [5], [6], [7]. Crucial feature of all these theories is the symmetric space property of the corresponding sigma–model in three dimensions (for a review see, e.g., Breitennlohner and Maison [6]). The symmetric space property in its turn requires that the target space possess a sufficient number of isometries. This convenient way to investigate the problem is due to the idea of potential space introduced by Neugebauer and Kramer [8].

Recent interest to such models is motivated by the string theory which is likely to provide a consistent description of gravity at the Planck scale. The most promising string model considered so far is that of the heterotic string. In the low–energy (classical) limit of this theory with compactified extra dimensions one gets in the bosonic sector the Einstein gravity coupled to massless vector and scalar fields. The simplest model of this kind — “dilaton–axion gravity” — incorporates basic features of the full effective action, and can be formulated directly in $D = 4$ where it includes one $U(1)$ vector and two scalar fields coupled in such a way that the theory possesses non–abelian $SL(2, R)$ duality symmetry. This theory was a subject of recent investigations [9] which are summarized and developed further in the present paper.

The Einstein–Maxwell–Dilaton–Axion (EMDA) coupled system contains a metric $g_{\mu\nu}$, a $U(1)$ vector field $A_{\mu}$, a Kalb-Ramond antisymmetric tensor field $B_{\mu\nu}$, and a dilaton $\phi$:

$$S = \frac{1}{16\pi} \int \left\{ -R + \frac{1}{3} e^{-4\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 2 \partial_{\mu} \phi \partial^{\mu} \phi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right\} \sqrt{-g} d^4 x, \quad (1.1)$$

where

$$H_{\mu\nu\lambda} = \partial_{\mu} B_{\nu\lambda} - A_{\mu} F_{\nu\lambda} \quad + \quad \text{cyclic},$$

$$F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}, \quad (1.2)$$

$\mu, \nu, \lambda = 0, 1, 2, 3$, and $\omega_{3L}, \omega_{3YM}$ are Lorentz and Yang–Mills Chern–Simons three–forms respectively. In four dimensions a Kalb–Ramond field is equivalent to the Peccei–Quinn pseudoscalar axion $\kappa$,

$$H^{\mu\nu\lambda} = \frac{1}{2} e^{4\phi} E^{\mu\nu\lambda\tau} \frac{\partial \kappa}{\partial x^\tau}, \quad (1.3)$$

and the corresponding Legendre transformation of the action (1.1) reads

$$S = \frac{1}{16\pi} \int \left\{ -R + 2 \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} e^{4\phi} \partial_{\mu} \kappa \partial^{\mu} \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \sqrt{-g} d^4 x, \quad (1.4)$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$. 
Our purpose is to investigate hidden symmetries associated with this model when an additional assumption of stationarity is imposed on the space–time metric. The plan of the papers is as follows. In Sec. 2 we outline the derivation of a three–dimensional sigma–model from the stationary EMDA system. In Sec. 3, complex potentials are introduced which make formulation similar to the Ernst formulation of vacuum and electrovacuum theories. The corresponding matrix formulation is given in Sec. 4. An alternative description in terms of non–dualized quantities is then suggested (Sec. 5) and used to generate EMDA multicenter solutions (Sec. 6).

2 σ–model in three dimensions

The system of equations corresponding to (1.4) consists of the dilaton–axion modified Maxwell equations

\[ \nabla_{\nu}(e^{-2\phi}F^{\mu\nu} + \kappa \tilde{F}^{\mu\nu}) = 0, \] (2.1)

dilaton and axion equations

\[ \nabla_{\mu}\nabla^{\mu}\phi = \frac{1}{2}e^{-2\phi}F^2 + \frac{1}{2}e^{4\phi}(\partial\kappa)^2, \] (2.2)

\[ \nabla_{\mu}(e^{4\phi}g^{\mu\nu}\partial_{\nu}\kappa) + F_{\mu\nu}\tilde{F}^{\mu\nu} = 0, \] (2.3)

and the Einstein equations

\[ R_{\mu\nu} = 2\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}e^{4\phi}\kappa_{,\mu}\kappa_{,\nu} + e^{-2\phi}(2F_{\mu\lambda}F^{\lambda\nu} + \frac{1}{2}F^2g_{\mu\nu}). \] (2.4)

Here \( F^2 \equiv F_{\mu\nu}F^{\mu\nu}, (\partial\kappa)^2 \equiv g^{\mu\nu}(\partial_{\mu}\kappa)(\partial_{\nu}\kappa), \) and \( \nabla_{\mu} \) is the covariant derivative with respect to the 4–dimensional metric \( g_{\mu\nu}. \)

Consider a space–time admitting (at least) one Killing vector field which we choose to be timelike. Then it is standard to write an interval using the three–metric \( h_{ij}, \) the rotation vector \( \omega_i, \) \( (i, j = 1, 2, 3) \) and the scalar \( f, \) depending on the space coordinates \( x^i, \) as follows

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(dt - \omega_i dx^i)^2 - \frac{1}{f}h_{ij}dx^i dx^j. \] (2.5)

To derive a three–dimensional σ–model one has to introduce the set of appropriate variables. It consists of the electric \( v \) and magnetic \( u \) potentials as well as the twist potential related to \( \omega_i \) through a certain curl equation. Electric and magnetic potentials are introduced through the relations

\[ F_{i\phi} = \frac{1}{\sqrt{2}}\partial_i v, \] (2.6)

\[ e^{-2\phi}F^{ij} + \kappa \tilde{F}^{ij} = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u. \] (2.7)

The first substitution solves the spatial part of the Bianchi identity

\[ \nabla_\mu \tilde{F}^{\mu\nu} = 0, \] (2.8)
while the second satisfies the spatial part of the modified Maxwell equations (2.1). From (2.6) and (2.7) the remaining Maxwell tensor components can be expressed through $v, u, f, \omega_i$ and $h_{ij}$. Introducing instead of $\omega_i$ a three vector

$$\tau^i = -f^2 \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_j \omega_k,$$  \hspace{1cm} (2.9)

one obtains from the remaining Maxwell equations the following set of three–covariant equations

$$\frac{1}{\sqrt{h}} \partial_i \left( \frac{1}{f} e^{-2\phi} \sqrt{h} h^{ij} \partial_j v \right) + \frac{1}{f^2} \tau^i \partial_i u - \frac{1}{\sqrt{h}} \partial_i \left[ \frac{1}{f} ke^{2\phi} \sqrt{h} h^{ij} (\partial_j u - \kappa \partial_j v) \right] = 0,$$  \hspace{1cm} (2.10)

$$\frac{1}{\sqrt{h}} \partial_i \left[ \frac{1}{f} e^{2\phi} \sqrt{h} h^{ij} (\partial_j u - \kappa \partial_j v) \right] - \frac{1}{f^2} \tau^i \partial_i v = 0.$$  \hspace{1cm} (2.11)

The equations for dilaton and axion fields (2.2), (2.3) with account for (2.5) will take the form

$$\frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j \phi) = \frac{1}{2f} \left[ e^{-2\phi} \partial_i v \partial^i v - e^{2\phi} (\partial_i u - \kappa \partial_i v)(\partial^i u - \kappa \partial^i v) \right] + \frac{e^{4\phi}}{2} \partial_i \kappa \partial^i \kappa,$$  \hspace{1cm} (2.12)

$$\frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} e^{4\phi} \partial_i \kappa) = \frac{2}{f} e^{2\phi} (\partial_i u - \kappa \partial_i v) \partial^i v,$$  \hspace{1cm} (2.13)

where (and in what follows) raising and lowering of the three–indices is understood with respect to the three–metric $h_{ij}$ and its inverse $h^{ij}$.

Remarkably, the mixed components of the Einstein equations (2.4) remain unaffected by neither a dilaton nor an axion:

$$\frac{1}{f} R^i_0 \equiv -\frac{1}{2\sqrt{h}} \epsilon^{ijk} \partial_k \tau_j = \frac{\epsilon^{ijk}}{\sqrt{h}} \partial_j v \partial_k a.$$  \hspace{1cm} (2.14)

This equation can be integrated giving the twist potential $\chi$

$$\tau_i = \partial_i \chi + v \partial_i u - u \partial_i v.$$  \hspace{1cm} (2.15)

An equation for $\chi$ can be found by taking a three–covariant divergence of (2.15) multiplied by $f^{-2}$ with account for the definition (2.9):

$$f(\Delta \chi + v \Delta u - u \Delta v) = 2(\partial_i \chi + v \partial_i u - u \partial_i v) \partial^i f,$$  \hspace{1cm} (2.16)

where

$$\Delta = \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j)$$  \hspace{1cm} (2.17)

is a three–dimensional Laplacian. Finally, an equation for $f$ follows from the (00)–component of the Einstein equations in a similar form

$$f \Delta f - \partial_i f \partial^i f + \tau_i \tau^i = f \left[ e^{2\phi} (\partial_i u - \kappa \partial_i v)(\partial^i u - \kappa \partial^i v) + e^{-2\phi} \partial_i v \partial^i v \right]$$  \hspace{1cm} (2.18)

where the definition (2.15) has to be used.

The system of 6 equations (2.10)–(2.13), (2.16), (2.18) constitutes the set of equations for a curved space $\sigma$–model with the 6–dimensional target space. The remaining (spatial)
Einstein equations from (2.4) may be consistently regarded as 3–dimensional Einstein equations for the metric \( h_{ij} \). Their source part is derivable from the same sigma–model as the 3–dimensional energy momentum tensor

\[
R_{ij} = \frac{1}{2f^2}(f_i f_j + \tau_i \tau_j) + 2\phi_i \phi_j + \frac{1}{2} e^{4\phi} \kappa_i \kappa_j - \frac{1}{f} \left[ e^{-2\phi} v_i v_j + e^{2\phi} (u_i - \kappa v_i)(u_j - \kappa v_j) \right],
\]

(2.19)

where \( \tau_i \) is given by (2.15), and \( R_{ij} \) is the three–dimensional Ricci tensor.

Now, it can be easily checked that all 3–dimensional equations may be obtained by a variation of the action

\[
S = \int \left( R - \frac{1}{2f^2}[(\nabla f)^2 + (\nabla \chi + v \nabla u - u \nabla v)^2] - 2(\nabla \Phi)^2 - \frac{1}{2} e^{4\phi}(\nabla \kappa)^2 + \frac{1}{f} \left[ e^{2\phi}(\nabla u - \kappa \nabla v)^2 + e^{-2\phi}(\nabla v)^2 \right] \right) \sqrt{hd^3x},
\]

(2.20)

where \( R \equiv R^i_i \) and \( \nabla \) stands for 3–dimensional covariant derivative.

This action can be rewritten as the gravity coupled three–dimensional \( \sigma \)–model

\[
S = \int \left( R - G_{AB} \partial_i \varphi^A \partial_j \varphi^B h^{ij} \right) \sqrt{hd^3x},
\]

(2.21)

where \( \varphi^A = (f, \chi, v, u, \kappa, \phi) \) \( A = 1, \ldots, 6 \). The corresponding target space metric reads

\[
dl^2 = \frac{1}{2} f^{-2}[df^2 + (d\chi + vdu - udv)^2] - f^{-1}[e^{2\phi}(du - \kappa dv)^2 + e^{-2\phi}dv^2] + 2d\phi^2 + \frac{1}{2} e^{4\phi} d\kappa^2.
\]

(2.22)

This expression generalizes Neugebauer and Kramer potential space metric found for the stationary Einstein–Maxwell system in 1969 [8]. It is worth noting that the present \( \sigma \)–model does not reduce to the Einstein–Maxwell one if \( \kappa = \phi = 0 \), because the equations for a dilaton and an axion generate constraints \( F^2 = F \tilde{F} = 0 \). Hence, generally the solutions of the Einstein–Maxwell theory with one Killing symmetry are not related to solutions of the present theory by target space isometries (except for the case \( F^2 = F \tilde{F} = 0 \)). From the other hand, one can consistently set in the present \( \sigma \)–model \( f = \chi = \phi = \kappa = 0 \) reducing it to the Einstein vacuum sigma–model. Therefore all solutions to the vacuum Einstein equations with one non–null Killing symmetry are related to some solutions of the system in question by target space isometries.

### 3 Complex potentials

More concise formulation of the theory can be achieved in therm of the following complex variables

\[
z = \kappa + i e^{-2\phi}
\]

(3.1)

\[
\Phi = u - zv
\]

(3.2)

\[
E = if - \chi + v\Phi
\]

(3.3)
The first is the standard complex dilaton–axion field, while two other may be regarded as suitable generalizations of the Ernst potentials. In terms of them the target space metric reads

\[ dl^2 = \frac{1}{2f^2} \left| dE + \frac{2\text{Im}\Phi}{\text{Im}z} d\Phi - \left( \frac{\text{Im}\Phi}{\text{Im}z} \right)^2 dz \right|^2 - \frac{1}{f\text{Im}z} \left| d\Phi - \frac{\text{Im}\Phi}{\text{Im}z} dz \right|^2 + \frac{|dz|^2}{2(\text{Im}z)^2}. \] (3.4)

It can be shown that this three–dimensional complex manifold is Kähler. Indeed, in terms of the complex coordinates \( z^\alpha = (E, -z, \Phi) \), \( \bar{z}^\alpha = (\bar{E}, \bar{z}, \bar{\Phi}) \), where \( \bar{z}^\alpha \) is a shorthand for \( \bar{z}^\alpha \), one can write an Hermitian metric on the target space as

\[ dl^2 = K_{\alpha\beta} dz^\alpha dz^\beta. \] (3.5)

This implies the existence of a non–degenerate two–form

\[ \Omega = \frac{i}{2} K_{\alpha\beta} dz^\alpha \wedge dz^\beta. \] (3.6)

One can easily check that this form is closed:

\[ d\Omega = 0, \] (3.7)

what means that we have a Kähler manifold.

Two describe isometries of the target space it is convenient to take into account the following two discrete symmetries. The first is quite obvious

\[ z' = E, \quad E' = z, \quad \Phi' = \Phi. \] (3.8)

Another useful discrete symmetry operation, which will be called “double prime”, is

\[ E'' = \frac{z}{Ez + \Phi^2}, \quad z'' = \frac{E}{Ez + \Phi^2}, \quad \Phi'' = \frac{\Phi}{Ez + \Phi^2}. \] (3.9)

The set of continuous isometries consists of three gauge (electric \( e \), magnetic \( m \), and gravitational \( g \)) transformations, a scale transformation \( s \), \( SL(2, R) \) duality \( (d_1, d_2, d_3) \), two Harrison–type \( (H_1, H_2) \) and Ehlers–type \( (E) \) transformations. The Killing vectors corresponding to gauge transformations in terms of complex potentials read

\[ K_e = 2\Phi \partial_E - z \partial_\Phi + c.c., \] (3.10)

\[ K_m = \partial_\Phi + c.c., \] (3.11)

\[ K_g = \partial_E + c.c.. \] (3.12)

The scale generator is

\[ K_s = 2E \partial_E + \Phi \partial_\Phi + c.c.. \] (3.13)

The Ehlers–Harrison–type generators \([3]\) can be obtained by “priming” the gauge generators

\[ K_{H_1} = K'_e = 2\Phi \partial_z - E \partial_\Phi + c.c., \] (3.14)

\[ K_{H_2} = K''_m = (Ez - \Phi^2) \partial_\Phi - 2\Phi(z \partial_z + E \partial_E) + c.c., \] (3.15)

\[ K_E = K''_g = \Phi^2 \partial_z - E(E \partial_E + \Phi \partial_\Phi) + c.c.. \] (3.16)
Two generators of the $S$–duality subgroup correspond to the primed $g$ and $s$ transformations

$$K_{d_1} = K'_g = \partial_z + c.c.,$$  \hspace{1cm} (3.17)

$$K_{d_3} = K'_s = 2z\partial_z + \Phi\partial\Phi + c.c.,$$  \hspace{1cm} (3.18)

while the remaining one may be obtained by priming the Ehlers generator

$$K_{d_2} = K'_E = \Phi^2\partial_E - z(\partial_z + \Phi\partial\Phi) + c.c.$$  \hspace{1cm} (3.19)

It can be checked that these 10 Killing vectors form the algebra $sp(4, R)$, while the target space is isomorphic to the coset $Sp(4, R)/U(2)$. Finite $Sp(4, R)$ transformations may also be obtained starting with the gravitational

$$E = E_0 + \lambda, \quad \Phi = \Phi_0, \quad z = z_0,$$  \hspace{1cm} (3.20)

electric

$$E = E_0 - 2\lambda\Phi_0 - \lambda^2 z_0, \quad \Phi = \Phi_0 + \lambda z_0, \quad z = z_0,$$  \hspace{1cm} (3.21)

the magnetic

$$E = E_0, \quad \Phi = \Phi_0 + \lambda, \quad z = z_0,$$  \hspace{1cm} (3.22)

gauge as well as the scale transformation

$$E = e^{2\lambda} E_0, \quad \Phi = e^{\lambda} \Phi_0, \quad z = z_0,$$  \hspace{1cm} (3.23)

(which can be easily integrated from the infinitesimal form). The remaining six group elements can be found using prime operations. The $SL(2, R)$–duality subgroup read

$$d_1 : \quad E = E_0, \quad \Phi = \Phi_0, \quad z = z_0 + \lambda,$$

$$d_2 : \quad E = E_0 + \lambda - \Phi_0^2 / (1 + \lambda z_0), \quad \Phi = \Phi_0 / (1 + \lambda z_0), \quad z = z_0 / (1 + \lambda z_0),$$  \hspace{1cm} (3.24)

$$d_3 : \quad E = E_0, \quad \Phi = e^{\lambda} \Phi_0, \quad z = e^{2\lambda} z_0,$$

the electric and magnetic Harrison transformations are

$$E = E_0, \quad \Phi = \Phi_0 + \lambda E_0, \quad z = z_0 - 2\lambda\Phi_0 - \lambda^2 E_0,$$  \hspace{1cm} (3.25)

$$E = \frac{E_0}{(1 + \lambda\Phi_0)^2 + \lambda^2 E_0 z_0}, \quad \Phi = \frac{\Phi_0 (1 + \lambda\Phi_0) + \lambda E_0 z_0}{(1 + \lambda\Phi_0)^2 + \lambda^2 E_0 z_0}, \quad z = \frac{z_0}{(1 + \lambda\Phi_0)^2 + \lambda^2 E_0 z_0},$$  \hspace{1cm} (3.26)

while the Ehlers–type transformation reads

$$E = \frac{E_0}{1 + \lambda E_0}, \quad \Phi = \frac{\Phi_0}{1 + \lambda E_0}, \quad z = z_0 + \frac{\lambda\Phi_0^2}{1 + \lambda E_0}.$$  \hspace{1cm} (3.27)

The isometry group acts transitively on the target space which is a symmetric Riemannian space. The direct way to show this is to introduce matrix representation in terms of symmetric symplectic $4 \times 4$ matrices forming a coset $Sp(4, R)/U(2)$. 
4 Matrix representation

For any matrix $G \in Sp(4, R)$ one can perform a Gauss decomposition

$$G = G_L G_S G_R,$$

where

$$G_R = \begin{pmatrix} I & R \\ O & I \end{pmatrix}, \quad G_S = \begin{pmatrix} S^{-1}T & O \\ O & S \end{pmatrix}, \quad G_L = \begin{pmatrix} I & O \\ L & I \end{pmatrix},$$

and $S, R, L$ are real $2 \times 2$ matrices, $R, L$ being symmetric, $R^T = R$, $L^T = L$. Useful relations are

$$G_R G_R = G_R + R, \quad G_S G_S = G_S G_S, \quad G_L G_L = G_L + L,$$

so that for inverse matrices one has $G^{-1}_R = G_{-R}$, $G^{-1}_S = G_{-S}$, $G^{-1}_L = G_{-L}$. With this parametrization

$$G = \begin{pmatrix} S^{-1}T & S^{-1}R \\ L S^{-1}T & S + L S^{-1}R \end{pmatrix}.$$

A subset of symmetric symplectic matrices $M \in Sp(4, R)$, $M^T = M$ represents a coset $Sp(4, R)/U(2)$. The corresponding Gauss decomposition reads

$$M = Q^T P Q,$$

where

$$Q = \begin{pmatrix} I & Q \\ O & I \end{pmatrix}, \quad P = \begin{pmatrix} P^{-1} & O \\ O & P \end{pmatrix},$$

and two real symmetric $2 \times 2$ matrices are introduced $Q^T = Q$, $P^T = P$. Multiplying matrices we obtain

$$M = \begin{pmatrix} P^{-1} & P^{-1}Q \\ Q P^{-1} & P + Q P^{-1}Q \end{pmatrix}.$$

It is useful to combine $P$ and $Q$ into one complex symmetric matrix

$$Z = Q + iP,$$

which transforms under an action of $G_R, G_L, G_S$ as follows:

$$Z \rightarrow Z + R,$$

for $G = G_L$:

$$Z^{-1} \rightarrow Z^{-1} - L,$$

while for $G = G_S$:

$$Z \rightarrow S^T Z S.$$

Here the following transformation law for $M$ is assumed

$$M \rightarrow G^T M G.$$

These transformations are precisely the matrix analog of the $SL(2, R)$ transformations which hold for the usual scalar dilaton–axion field (3.1) under S–duality generated by $K_{d1}$, $K_{d2}$, $K_{d3}$.
It can be shown that the explicit expression for $Z$ in terms of the complex potentials introduced in the previous section is very simple

$$Z = \begin{pmatrix} E & \Phi \\ \Phi & -z \end{pmatrix}.$$  \hspace{1cm} (4.13)

Sigma–model of the Sec. 3 can now be rewritten as a coset model for the matrix $M$ (4.7). Let us introduce a matrix current

$$J^M = \nabla M M^{-1}.$$ \hspace{1cm} (4.14)

Using (4.8) into we get

$$J^M = \begin{pmatrix} -P^{-1} \nabla P - P^{-1} \nabla Q P^{-1} Q & P^{-1} \nabla Q P^{-1} \\ \nabla Q - Q P^{-1} \nabla P - \nabla P P^{-1} Q - Q P^{-1} \nabla Q P^{-1} Q & \nabla P P^{-1} + Q P^{-1} \nabla Q P^{-1} \end{pmatrix}.$$ \hspace{1cm} (4.15)

In this notation, the three–dimensional sigma–model action can now be written as

$$S = \int \left( -\mathcal{R} + \frac{1}{4} \text{Tr}(J^M)^2 \right) \sqrt{h} d^3 x,$$ \hspace{1cm} (4.16)

while the corresponding field equations read

$$\nabla J^M = 0,$$ \hspace{1cm} (4.17)

$$\mathcal{R}_{mn} = \frac{1}{4} \text{Tr}(J^M_m J^M_n).$$ \hspace{1cm} (4.18)

There exists also a concise representation of the same model directly by $2 \times 2$ matrices. Indeed, from three $2 \times 2$ matrix equations following the present representation

$$\nabla \left( P^{-1} \nabla Q P^{-1} \right) = 0,$$ \hspace{1cm} (4.19)

$$\nabla \left( Q P^{-1} \nabla Q P^{-1} + \nabla P P^{-1} \right) = 0,$$ \hspace{1cm} (4.20)

only first two are independent. The last can be rewritten using first two as

$$P \nabla \left( P^{-1} \nabla Q P^{-1} \right) P = 0.$$ \hspace{1cm} (4.21)

Introducing two $2 \times 2$ matrix currents

$$J_1 = \nabla P P^{-1}, \quad J_2 = \nabla Q P^{-1},$$ \hspace{1cm} (4.22)

we can rewrite the equations of motion as

$$\nabla J_1 + J_2^2 = 0,$$ \hspace{1cm} (4.23)

$$\nabla J_2 - J_1 J_2 = 0,$$ \hspace{1cm} (4.24)

$$\mathcal{R}_{mn} = \frac{1}{2} \text{Tr} \left( J_{1m} J_{1n} + J_{2m} J_{2n} \right).$$
It can also be checked that
\[ Tr \left( J_1^2 + J_2^2 \right) = \frac{1}{2} Tr J^2. \] (4.24)

Further simplification comes from the introduction of a complex matrix dilaton–axion \( Z \) instead of its real and imaginary parts (from an explicit form of \( Z \)) it is clear that real and imaginary parts are more involved because of the complicated structure of the denominator. Then instead of two real \( 2 \times 2 \) currents one can build one complex matrix current
\[ J_Z = \nabla Z (Z - \bar{Z})^{-1}, \] (4.25)
and cast the action into the following form
\[ S = \int \frac{1}{2} \left( -\mathcal{R} + 2 Tr J_Z \bar{J}_Z \right) \sqrt{h} d^3x. \] (4.26)
The corresponding equations read
\[ \nabla J_Z = J_Z (J_Z + \bar{J}_Z), \]
\[ \mathcal{R}_{mn} = 2 Tr J_{Zm} \bar{J}_{Zn}. \] (4.27)

The metric of the target space can be presented in terms of the matrix–valued one–forms corresponding to the above currents
\[ \omega = dM M^{-1}, \quad \omega_1 = dP P^{-1}, \quad \omega_2 = dQ P^{-1}, \quad \omega_Z = dZ (Z - \bar{Z})^{-1}, \] (4.28)
by three alternative ways
\[ ds^2 = \frac{1}{4} Tr (\omega^2) = \frac{1}{2} Tr (\omega_1^2 + \omega_2^2) = 2 Tr (\omega_Z \bar{\omega}_Z). \] (4.29)

5 Back to non–dualized variables

To achieve sigma–model description of the system in three dimensions it was essential to use the dualized variables for the rotational part of a metric and for the magnetic part of a vector field, as well as to introduce the Peccei–Quinn pseudoscalar counterpart to the antisymmetric Kalb–Ramond field. This description was particularly useful in deriving hidden symmetries of the dimensionally reduced system. Once they are found, one can go back to the initial formulation and to reveal their action on the non–dualized set of variables. Apart from purely formal interest, this also opens a way to obtain an infinite symmetry algebra in the further two–dimensional reduction.

Transition to the initial variables may now be regarded as the formal solution of the matrix equation
\[ \nabla J^M = 0, \quad J^M = \nabla M M^{-1}. \] (5.1)

Since its divergence is zero, the matrix current can be presented as a curl of some vector matrix \( \vec{N} \)
\[ \nabla \times \vec{N} = J^M. \] (5.2)

It can be checked that an alternative form of the equation \( \nabla J^M = 0 \) reads
\[ \nabla \times J^M - J^M \times J^M = 0. \] (5.3)
In terms of $2 \times 2$ symmetric matrices $P$ ($Q$ one gets the following set of equations
\[
\nabla \times \tilde{\Omega} = P^{-1} \nabla Q P^{-1}, \quad (5.4)
\]
\[
\nabla \times \tilde{U} = \nabla P P^{-1} + Q P^{-1} \nabla Q P^{-1}, \quad (5.5)
\]
\[
\nabla \times \tilde{V} = \nabla Q - \nabla P P^{-1} Q - Q P^{-1} \nabla P - Q P^{-1} \nabla Q P^{-1} Q, \quad (5.6)
\]
where $\tilde{\Omega}$, $\tilde{V}$ and $\tilde{U}$ are symmetric real $2 \times 2$ matrices forming $\tilde{N}$ as follows
\[
\tilde{N} = \left( \begin{array}{cc}
-\tilde{U}^T & \tilde{\Omega} \\
\tilde{V} & \tilde{U}
\end{array} \right). \quad (5.7)
\]

From explicit expressions for $P$ and $Q$
\[
P = \left( \begin{array}{cc}
f - e^{-2\phi} v^2 & -e^{-2\phi} v \\
-e^{-2\phi} v & -e^{-2\phi}
\end{array} \right),
\]
\[
Q = \left( \begin{array}{cc}
wv - \chi & w \\
w & -\kappa
\end{array} \right)
\]
it is clear that they provide a natural decomposition of the full set of sigma–model coordinates into non–dualized $(f, \phi, v)$ and dualized $(\chi, u, \kappa)$ subsets. Hence, it can be anticipated that the matrix $P$ will remain unchanged, while $Q$ will be replaced by $\tilde{\Omega}$. After some calculations one can express it through the non–dualized variables as follows:
\[
\tilde{\Omega} = \left( \begin{array}{cc}
\bar{\omega} & -(\bar{A} + A_0 \omega) \\
-(\bar{A} + A_0 \omega) & -\bar{B} + A_0 (\bar{A} + A_0 \omega)
\end{array} \right), \quad (5.8)
\]
where $A_0 = v, B_i = B_{0i}, \omega_i$ is the metric function and $A_i$ is three–dimensional vector–potential defined as $\sqrt{2} F_{ij} = (\partial_i A_j - \partial_j A_i)$. Two other matrices $\tilde{U}$ and $\tilde{V}$ can be found once $\tilde{\Omega}$ is known (all three–dimensional vector operations in the above formulas have to be effected using the metric $h_{ij}$.

It is worth noting that the spatial components of the Kalb–Ramond field do not enter into the above set of variables. In fact, in the stationary case they can be expressed through other variables. Introducing a three–dimensional dual to $B_{ij}$
\[
C^{ij} = \frac{1}{2} E^{ijk} B_{jk}, \quad (5.9)
\]
from stationarity $\kappa_0 = 0$ one can find after some rearrangement
\[
\nabla \tilde{\nabla} = \nabla (\bar{B} \times \bar{\omega}) + [\bar{B} - A_0 (\bar{A} + A_0 \bar{\omega})] \nabla \times \bar{\omega} + (\bar{A} + A_0 \bar{\omega}) \nabla \times (\bar{A} + A_0 \bar{\omega}). \quad (5.10)
\]

One can derive the three–dimensional action variation of which gives the equations for matrices $P$ and $\tilde{\Omega}$. First, from the Eq.(5.4) one gets
\[
\nabla \times [P (\nabla \times \tilde{\Omega}) P] = 0, \quad (5.11)
\]
while (5.5) takes the form
\[
\nabla (\nabla P P^{-1}) + P (\nabla \times \tilde{\Omega}) P \nabla \times \tilde{\Omega} = 0. \quad (5.12)
\]
Together with the Einstein equations
\[ R_{ij} = 2Tr(J_i^P J_j^P + J_i^\Omega J_j^\Omega), \tag{5.13} \]
where
\[ J^P = \nabla P P^{-1}, \quad J^\Omega = P \nabla \times \vec{\Omega} \tag{5.14} \]
we have the complete description of the system in terms of \( P \) and \( \vec{\Omega} \). The Eqs. (5.11-12) can be rewritten as
\[ \nabla J^P + (J^\Omega)^2 = 0, \quad \nabla \times J^\Omega - J^\Omega \times J^P = 0, \tag{5.15} \]
while the corresponding action read
\[ S = \int d^3x h^{1/2}(-R + 2Tr[(J^P)^2 - (J^\Omega)^2]). \tag{5.16} \]

The equations for \( \vec{U} \) and \( \vec{V} \) still contain undesired matrix \( Q \). However two “Legendre transformed” quantities
\[ \vec{U}' = \vec{U} - Q\vec{\Omega}, \quad \vec{V}' = \vec{V} + Q\vec{\Omega}Q + \vec{U}' Q + Q\vec{U}'^T \tag{5.17} \]
satisfy the equations depending only on \( P \) (\( \vec{\Omega} \)):
\[ \nabla \times \vec{U}' = \nabla P P^{-1} - P(\nabla \times \vec{\Omega})P \times \vec{\Omega} \tag{5.18} \]
\[ \nabla \times \vec{V}' = P(\nabla \times \vec{\Omega})P + P(\nabla \times \vec{\Omega})P \times \vec{U}'^T - \vec{U}' \times P(\nabla \vec{\Omega})P. \tag{5.19} \]
Similarly, a non–dynamical variable \( \vec{C} \) may be replaced by
\[ \vec{C}' = \vec{C} - \frac{1}{2}[\vec{B} + A_0(\vec{A} + A_0\vec{\omega})] \times \vec{\omega}, \tag{5.20} \]
the equation for which contains only “allowed” variables:
\[ \nabla \vec{C}' = \frac{1}{2}Tr(\vec{\Omega} \epsilon \nabla \times \vec{\Omega} \epsilon). \tag{5.21} \]
where
\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Target space isometries induce certain transformations of the matrix \( \vec{\Omega} \). These can be found by a direct computation using the Eqs. (5.4–6). The subgroup \( R \) leaves \( \nabla P \) and \( \nabla \times \vec{\Omega} \) unchanged, and now some more general transformation of the same kind can be found
\[ P \rightarrow P, \quad \vec{\Omega} \rightarrow \vec{\Omega} + \nabla R', \tag{5.22} \]
where \( R' \) is an arbitrary non–constant symmetric matrix.
Now, for the \( S \)–transformation it is convenient to redefine
\[ S' = S^{-1T}, \tag{5.23} \]
so that we get the following counterpart

\[ P^{-1} \to S'^T P^{-1} S', \quad (5.24) \]
\[ \vec{\Omega} \to S'^T \vec{\Omega} S'. \quad (5.25) \]

Finally, for the \( L \)-transformations one has

\[ P^{-1} = P_0^{-1} + L Q_0 P_0^{-1} + P_0^{-1} Q_0 L + L(P_0 + Q_0 P_0^{-1} Q_0)L, \quad (5.26) \]

while \( \vec{\Omega} \) can be shown to transform as

\[ \vec{\Omega} \to \vec{\Omega} + L \vec{U} + \vec{U}^T L + L \vec{V} L. \quad (5.27) \]

For generating purposes it is convenient to combine the action of \( S, L \) transformations. Omitting primes, one gets

\[ P^{-1} = S^T [P_0^{-1} + L Q_0 P_0^{-1} + P_0^{-1} Q_0 L + L(P_0 + Q_0 P_0^{-1} Q_0)L] S \quad (5.28) \]
\[ \vec{\Omega} = S^T [\vec{\Omega}_0 + L \vec{U}_0 + \vec{U}_0^T L - L \vec{V}_0 L] S. \quad (5.29) \]

It is worth noting that for the three–dimensional description in terms of the non–dualized quantities we have introduced more variables than in the previous sigma–model formulation. The excessive variables could in principle be eliminated by fixing the gauge, but then three–dimensional covariance would be lost.

### 6 Multicenter solutions

Here we give some examples of the application of symmetries to derive multicenter solutions to dilaton–axion gravity. Let us specify the transformation formulas for asymptotically flat solutions with zero asymptotic values of material fields. In this case

\[ P_0^{-1} = \sigma_3, \quad Q_0 = \vec{\Omega}_0 = 0. \quad (6.1) \]

Demanding that the \( L \)-transformation leave unchanged asymptotic values of \( f, \phi, v \), one obtains for the matrix \( L \) the following expression

\[ L = l \Sigma, \quad (6.2) \]

where

\[ \Sigma = \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}. \]

and \( \sigma = \pm 1, \ l = const. \) We will assume that the \( S \)-transformation leads to a non–trivial asymptotic for the dilaton

\[ P^{-1}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\phi_\infty} \end{pmatrix}. \]

These assumptions leave one free parameter in \( S \):

\[ S = \begin{pmatrix} ch\theta & e^{\phi_\infty} sh\theta \\ sh\theta & e^{\phi_\infty} ch\theta \end{pmatrix}. \]
As a whole we get two parameter ($l$ and $\theta$) family of solutions applying $S, L$–transformation to a chosen asymptotically flat seed solution.

As a first example we take as a seed

$$P_0^{-1} = \sigma_3, \quad \vec{\Omega}_0 = \vec{\Lambda}\Sigma'$$

(6.3)

where $\Sigma'$ corresponds to the replacement of $\sigma$ to $\sigma'$ in $()$. A (non–constant) matrix vector $\vec{\Lambda}$ is introduced according to

$$\nabla \times \vec{\Lambda} = \nabla \lambda, \quad \lambda = \sum_{i=1}^{N} \frac{\lambda_i}{|\vec{r} - \vec{r}_i|},$$

(6.4)

where $\vec{r}_i$ are positions of the sources. Then three–metric turns out to be flat

$$dl^2 = d\vec{r}^2.$$  

(6.5)

Physically this solution corresponds to an equilibrium system of an arbitrary number of point–like magnetic monopoles endowed with axion charges (magnetic repulsion is balanced by axion attraction). This field configuration generates twist, so that the resulting metric is a massless multi–Taub–NUT.

Now we apply to this solution a combined $S, L$ transformation described above $()$ setting $\sigma' = -\sigma$ (only in this case $L$–transformation gives non–trivial result). One gets

$$P^{-1} = (1 + 4l\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\phi_\infty} \end{pmatrix},$$

$$\vec{\Omega} = \vec{\Lambda} \begin{pmatrix} e^{-2\sigma\theta} & \frac{1}{-\sigma e^{\phi_\infty}} & -\sigma e^{\phi_\infty} \\ -\sigma e^{\phi_\infty} & e^{2\phi_\infty} & 4l^2 e^{2\sigma\theta} \end{pmatrix}.$$

In physical terms

$$f^{-1} = 1 + 4l\lambda, \quad e^{2\phi} = e^{2\phi_\infty} f^{-1}, \quad A_0 = 0$$

(6.6)

$$\vec{\omega} = (e^{-2\sigma\theta} - 4l^2 e^{2\sigma\theta})\vec{\Lambda}, \quad \vec{A} = \sigma e^{\phi_\infty} (e^{-2\sigma\theta} + 4l^2 e^{2\sigma\theta})\vec{\Lambda}, \quad \vec{B} = e^{2\phi_\infty} \vec{\omega}.$$  

(6.7)

In the case

$$e^{-4\sigma\theta} = 4l^2,$$

(6.8)

after redefinitions $2l\lambda \rightarrow \lambda, \sigma l/|l| \rightarrow \sigma$ the solution can be written as

$$f^{-1} = e^{2(\phi - \phi_\infty)} = 1 + 2\lambda, \quad \vec{A} = 2\sigma e^{\phi_\infty} \vec{\Lambda}, \quad A_0 = \vec{\omega} = \vec{B} = 0.$$  

(6.9)

and therefore describes the system of point massive magnetic charges endowed with dilaton charges. In this case (no axion field) twist is not generated.

As a second example consider an $(S, L)$–transformation of the electric Majumdar–Papapetrou–like solution

$$P_0 = \sigma_3 + \lambda \Sigma, \quad \vec{\Omega}_0 = 0, \quad dl^2 = d\vec{r}^2,$$

(6.10)

corresponding to

$$Q_0 = \vec{\Omega}_0 = \vec{V}_0 = 0,$$

(6.11)

$$\vec{U}_0 = \vec{\Lambda} \begin{pmatrix} 1 & -\sigma \\ \sigma & -1 \end{pmatrix},$$
With no additional assumptions for $\sigma'$ ($\sigma$, one obtains

$$P^{-1} = \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{-2\phi_\infty} \end{array} \right) - \lambda \left( e^{-2\sigma\theta} \left( \begin{array}{cc} 1 & -\sigma e^{\phi_\infty} \\ -\sigma e^{\phi_\infty} & e^{2\phi_\infty} \end{array} \right) - 2l^2 (1 + \sigma\sigma') e^{2\sigma\theta} \left( \begin{array}{cc} 1 & \sigma e^{\phi_\infty} \\ \sigma e^{\phi_\infty} & e^{2\phi_\infty} \end{array} \right) \right),$$

$$\vec{\Omega} = 2l\lambda \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{-2\phi_\infty} \end{array} \right) + \sigma\sigma' \left( \begin{array}{cc} ch2\theta & e^{\phi_\infty}sh2\theta \\ e^{\phi_\infty}sh2\theta & e^{2\phi_\infty}ch2\theta \end{array} \right).$$

Extracting physical parameters, one can see that we have generated magnetic, axion and NUT charges.

References

[1] R. Geroch, Journ. Math. Phys. 13, 394 (1972); V.A. Belinskii and V.E. Zakharov. Sov. Phys. JETP, 48, 985 (1978); 50, 1 (1979); G. Neugebauer, Journ. Phys. A: 12, L67; 1, L19 (1979). D. Maison, Journ. Math. Phys. 21, 871 (1979). I. Hauser and F. J. Ernst, Phys. Rev. D 20, 362, 1783 (1979).

[2] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, Exact Solutions of the Einstein’s field equations, CUP, 1980.

[3] W. Kinnersley, Journ. Math. Phys. 14, 651 (1973); 18, 1529 (1977); W. Kinnersley and D. Chitre, Journ. Math. Phys. 18, 1538 (1977); 19, 1926, 2037 (1978). G.A. Alekseev, Pis’ma Zh. Eksp. Teor. Fiz. 32, 301 (1980); P.O. Mazur, Acta Phys. Polon. B14, 219 (1983); A. Eris, M. Güurses, and A. Karasu, Journ. Math. Phys. 25, 1489 (1984).

[4] D. Maison, Gen. Rel. and Grav. 10, 717 (1979); V. Belinskii and R. Ruffini, Phys. Lett. B89, 195 (1980).

[5] B. Julia, in: “Proceedings of the John Hopkins Workshop on Particle Theory”, Baltimore (1981); in “Superspace and Supergravity”, Eds. S. Hawking and M. Rocek, Cambridge, 1981; in “Unified Theories of More than Four Dimensions” Eds. V. De Sabbata and E. Shmutzer, WS, Singapore 1983.

[6] P. Breitenlohner and D. Maison, in “Solutions of the Einstein’s Equations: Techniques and Results”, Eds. C. Hoenselaers, W. Dietz, Lectures Notes in Physics, 205 (1984) 276; P. Breitenlohner, D. Maison, and G. Gibbons, Comm. Math. Phys. 120, 253 (1988).

[7] H. Nicolai, Phys. Letts. B194 (1987) 402.

[8] G. Neugebauer and D. Kramer, Ann. der Physik (Leipzig) 24, 62 (1969); in Galaxies, Axisymmetric systems and Relativity, ed. by M. MacCallum, CUP, 1986, p.149.

[9] D.V. Gal’tsov and O.V. Kechkin, Phys. Rev. D50 (1994) 7394; (hep-th/9407153)); D.V. Gal’tsov, “Symmetries of Heterotic String Effective Theory in Three and Two Dimensions”, a talk given at the International Workshop “Heat Kernel Techniques and Quantum Gravity”, Winnipeg, Canada, 2—6 August, 1994), published in Heat Kernel Techniques and Quantum Gravity, ed. by S. A. Fulling, Discourses in Mathematics and Its Applications, No. 4, ©Department of Mathematics, Texas A&M University, College Station, Texas, 1995, pp. 423–449; D.V. Gal’tsov, Phys. Rev. Lett. 74 (1995) 2863,
D.V. Gal’tsov and O.V. Kechkin, *U–Duality and Symplectic Formulation of Dilaton–Axion gravity*, Preprint IC/95/155, hep-th/9507005. Gal’tsov D.V., Garcia A.A., Kechkin O.V., J. Math. Phys., 1995, v. 36, n. 9, p. 1 – 19. Gal’tsov D.V., Kechkin O.V. Matrix Dilaton–Axion for Heterotic String in Three Dimensions, preprint DTP–MSU 95/26, hep-th/9507164.