A GENERALIZATION OF NIL-CLEAN RINGS

ALEKSANDRA KOSTIĆ, ZORAN Z. PETROVIĆ, ZORAN S. PUCANOVIĆ,
AND MAJA ROSLAVCEV

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Abstract. The conditions that allow an element of an associative, unital, not necessarily commutative ring $R$, to be represented as a sum of (commuting) idempotents and one nilpotent element are analyzed. Some applications to group rings are also presented.

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1. INTRODUCTION

An element $a$ in an associative unital ring $R$ is called clean if it can be represented as a sum $a = e + u$, where $e$ is an idempotent element and $u$ is a unit. This notion was introduced by Nicholson in [8]. If one can find such elements $e$ and $u$ such that $a = e + u$ and $eu = ue$, the element $a$ is called strongly clean. The ring $R$ itself is called (strongly) clean if every element in $R$ is (strongly) clean.

Many families of clean rings were investigated in previous decades. In recent years, a particular attention has been paid to the nil-clean rings and its relatives. A nil-clean ring (see [5]) is a ring in which every element is nil-clean, which means that every element can be written as a sum of an idempotent element and a nilpotent one. Analogously, we have a notion of strongly nil-clean elements (and rings). For some of the results concerning this class of rings and some of the related classes of rings, the reader may wish to consult also [1, 3, 7, 10].

A class of strongly 2-nil-clean rings was introduced in [4]. Namely, an element $a$ in a ring $R$ is called strongly 2-nil-clean if it can be represented in the form $a = e + f + n$, where $e$ and $f$ are idempotents, $n$ is a nilpotent element and they all commute with each other.

In this paper we analyze elements of a ring which can be written as a sum of finitely many idempotents and one nilpotent element which are pairwise commutative. If the number of idempotents which appear in this sum is $s$, we call these elements strongly

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s-nil-clean. It turns out that if every element in a ring is strongly s-nil-clean for some s, this ring has finite characteristic and every element in this ring is strongly \((p - 1)\)-nil-clean, where \(p\) is the largest prime dividing the characteristic of this ring (see Theorem 1). These rings are naturally called strongly \((p - 1)\)-nil-clean rings and they are all strongly clean (see Corollary 3 and the discussion preceding it). There are many examples of strongly \((p - 1)\)-nil-clean rings. Theorem 2 shows that in every commutative ring \(R\) of finite characteristic \(k\), elements which are strongly s-nil-clean for some \(s\), form a subring which is \((p - 1)\)-nil clean (where \(p\) is the largest prime dividing \(k\)) and if this ring contains idempotents or nilpotents not belonging to \(\mathbb{Z}_k\) (which is necessarily contained in \(R\)) we have a non-trivial example of such a ring. Proposition 5 provides examples of finite commutative local rings which are \((p - 1)\)-nil-clean.

The plan of the paper is as follows. In Section 2, we analyze sums of idempotents and one nilpotent element and derive our main criteria for strongly \(s\)-nil-clean elements in a ring. Section 3 deals with some structure theorems. It is shown that in analyzing strongly \((p - 1)\)-nil-clean rings, we may reduce this analysis to the case when \(p\) is a nilpotent element in a ring under investigation. We also show in this section that strongly \((p - 1)\)-nil-clean rings are strongly \(\pi\)-regular and, consequently, strongly clean.

Section 4 deals with the investigation of group rings \(RG\) where \(R\) is a \((p - 1)\)-nil-clean commutative ring and \(G\) is a group. For example, we show that, for the ring \(RG\) to be strongly \((p - 1)\)-nil-clean, when the characteristic of \(R\) is of the form \(p^s\), for a prime integer \(p\), it is necessary that the order of any element of a group \(G\) is of the form \(dp^k\), for some \(d\mid (p - 1)\) and \(k \geq 0\) (see Lemma 3). In the case of a commutative group \(G\) this condition is also sufficient (see Theorem 4).

We emphasize that we work in associative, unital rings which need not be commutative. When a ring is commutative, we drop the adjective “strongly” since it is unnecessary. We use the same symbol \(k\) to denote the integer \(k\) and to denote the ring element \(k1_R\). It will always be clear what we mean. Finally, we denote the field with \(p\) elements by \(\mathbb{Z}_p\), the Jacobson radical by \(J(R)\) and the set of nilpotent elements by \(N(R)\).

2. Basic results

A ring in which every element is a sum of certain number of idempotents and one nilpotent element, that commute with each other, is a generalization of strongly nil-clean rings and strongly 2-nil-clean rings. In view of this, we introduce the following definition.

**Definition 1.** An element \(a\) of a ring \(R\) is \(s\)-nil-clean if it can be written in the following form:

\[
a = e_1 + \cdots + e_s + n,
\]  

(2.1)
where elements $e_1, \ldots, e_s$ are idempotents and $n$ is nilpotent. If an element $a$ can be written in the form (2.1) so that elements in this sum are pairwise commutative, we say that this element is strongly $s$-nil-clean. If every element in $R$ is (strongly) $s$-nil clean, we say that $R$ is a (strongly) $s$-nil-clean ring.

Of course, if $a$ is (strongly) $s$-nil-clean and $s < t$, $a$ is also $t$-nil-clean — we simply add $t - s$ zeroes to the presentation of $a$ as a (strongly) $s$-nil-clean element.

**Remark 1.** It is clear that, if $f : R \to S$ is a ring homomorphism and an element $a \in R$ is (strongly) $s$-nil-clean, then $f(a) \in S$ is (strongly) $s$-nil-clean. Similarly, an element $a = (a_1, \ldots, a_l) \in R_1 \times \cdots \times R_l$ is (strongly) $s$-nil-clean iff $a_i$ is (strongly) $s$-nil-clean for all $i$. So, homomorphic images and finite direct products of (strongly) $s$-nil-clean rings are itself (strongly) $s$-nil-clean. Also, a subring of a strongly $s$-nil-clean ring has the same property, as we shall see later. However, this does not hold for $s$-nil-clean rings. Namely, if we have a ring which is $s$-nil-clean, but it is not strongly $s$-nil-clean, it is enough to take an element $a$ which is not strongly $s$-nil-clean and look at the subring generated by this element. This subring is commutative, so it cannot be $s$-nil-clean — if it were, this element would also be strongly $s$-nil-clean.

We begin our analysis with a useful result concerning sum of several idempotents and one nilpotent element.

**Proposition 1.** Let $R$ be a ring and suppose that element $a \in R$ is strongly $s$-nil clean. Then $a(a - 1) \cdots (a - s)$ is nilpotent.

**Proof.** Let $a = e_1 + e_2 + \cdots + e_s + n$, where $e_1, e_2, \ldots, e_s$ are idempotents and $n$ is nilpotent that commute with each other. Observe that

$$1 = ((1 - e_1) + e_1)((1 - e_2) + e_2) \cdots ((1 - e_s) + e_s).$$

After multiplication, we get a sum of products of the form

$$e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}),$$

where $1 \leq k \leq s$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{s-k}$ and \{i_1, \ldots, i_k, j_1, \ldots, j_{s-k}\} = \{1, \ldots, s\}.

Next, we get that

$$(k - a)e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}})$$

$$= ((1 - e_{j_1}) + \cdots + (1 - e_{j_k}) - e_{j_1} - \cdots - e_{j_{s-k}} - n)e_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}})$$

$$= -ne_{i_1} \cdots e_{i_k} (1 - e_{j_1}) \cdots (1 - e_{j_{s-k}}).$$

This follows from the fact that $(1 - e)e = e - e^2 = 0$, for an idempotent $e$. Since $n$ is nilpotent, so is this product. Thus, when 1 is multiplied by $a(a - 1) \cdots (a - s)$, we get a sum of nilpotent elements that commute with each other. Therefore, $a(a - 1) \cdots (a - s)$ is nilpotent.

The following corollary is simple, but important.
Corollary 1. a) If a ring \( R \) is such that \(-1\) is strongly \( s \)-nil-clean for some \( s \geq 1 \), then this ring has finite characteristic.

b) If \( \text{char}(R) = k \), then \(-1\) is strongly \((p-1)\)-nil-clean, where \( p \) is the largest prime dividing \( k \).

Proof. a) From the previous proposition we conclude that \((-1)(-2) \cdots (-\(s+1\)) = (-1)^{s+1}(s+1)!\) is nilpotent, so \((-1)^{m}(s+1)! = 0\) for some \( m \geq 1 \) and the characteristic of the ring \( R \) is not 0.

b) It is enough to show that \(-1\) is \((p-1)\)-nil-clean in the ring \( \mathbb{Z}_k \), which is contained in \( R \). If \( k = p_1^{a_1} \cdots p_l^{a_l} \) is the prime factorization of \( k \), where \( p_1 < \cdots < p_l = p \), then \( \mathbb{Z}_k \cong \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_l^{a_l}} \). Since \(-1 \mapsto (-1, \ldots, -1)\) under this isomorphism, this reduces the proof to show that \(-1\) is \((p-1)\)-nil-clean in \( \mathbb{Z}_{p_i^{a_i}} \). This is clear since \(-1 = p_i^{a_i} - 1 = 1 + \cdots + 1 + p_i(p_i^{a_i-1} - 1)\), and \( p_i(p_i^{a_i-1} - 1) \) is nilpotent in \( \mathbb{Z}_{p_i^{a_i}} \).

Example 1. The ring \( \mathbb{Z}_k \) is (strongly) \((p-1)\)-nil-clean, where \( p \) is the largest prime integer dividing \( k \). Namely, using the notation from the previous corollary, this reduces to show that \( \mathbb{Z}_{p_i^{a_i}} \) is \((p-1)\)-nil-clean. Since any element in \( \mathbb{Z}_{p_i^{a_i}} \) can be written in the form \( 1 + \cdots + 1 + p_i t \), for some \( s \in \{0, \ldots, p_i - 1\} \) and \( t \in \{0, \ldots, p_i^{a_i-1} - 1\} \), and since \( p_i \) is nilpotent in \( \mathbb{Z}_{p_i^{a_i}} \), we are done.

Lemma 1. If \( k = \text{char}(R) = p_1^{a_1} \cdots p_l^{a_l} \) is the prime factorization of the characteristic of the ring \( R \), then \( R \cong R_1 \times \cdots \times R_l \), where \( R_i = R/p_i^{a_i}R \). In particular, \( \text{char}(R_i) = p_i^{a_i} \).

Proof. This follows easily from the Chinese remainder theorem taking into account the fact that elements \( p_i^{a_i} \) are central.

Since the products of the form \( a(a-1) \cdots (a-s) \) are important for our investigation, we introduce the symbol \((a)_k := a(a-1) \cdots (a-(k-1))\) (falling factorial, as is known in combinatorics), where \( k \) is a positive integer.

We have the following corollary.

Corollary 2. Let \( R \) be a ring. Suppose that element \( a \in R \) is strongly \( s \)-nil clean and \( k(<s) \) is nilpotent. Then \((a)_k\) is a nilpotent element.

Proof. Clearly

\[
(a)_k = a(a-1) \cdots (a-(s-1)) = a^{t_0}(a-1)^{t_1} \cdots (a-(k-1))^{t_{k-1}} + kq(a),
\]

for some non-negative integers \( t_i \), such that \( \sum_{i=0}^{k-1} t_i = s \) and polynomial \( q(X) \in \mathbb{Z}[X] \). Taking into account that \( k \) is nilpotent, the result follows.
In order to see what the fact that \((a)_s\) is nilpotent implies, we start with a simple lemma.

**Lemma 2.** Let \(p\) be a prime integer and \(r, m \geq 1\) arbitrary positive integers. Then the element \(x\) in the ring \(\mathbb{Z}_{pr} \langle X \rangle / (X^m(X-1)^m \cdots (X-(p-1))^m)\) is (strongly) \((p-1)\)-nil-clean if \(x\) is the class of \(X\) in this quotient ring.

**Proof.** Ideals \((X-i)\) and \((X-j)\) are coprime in \(\mathbb{Z}_{pr} \langle X \rangle\), for all \(0 \leq i < j < p\), since \((X-i)-(X-j) = j-i\) and \(j-i\) is invertible in the ring \(\mathbb{Z}_{pr} \langle X \rangle\). So, \((X-i)^m\), \((X-j)^m\) are coprime as well. By applying the Chinese remainder theorem we obtain the isomorphism
\[
\mathbb{Z}_{pr} \langle X \rangle / (X^m \cdots (X-(p-1))^m) \cong \mathbb{Z}_{pr} \langle X \rangle / (X^m) \times \cdots \times \mathbb{Z}_{pr} \langle X \rangle / ((X-(p-1))^m),
\]
such that \(x \mapsto (x, \ldots, x)\). Thus, it is enough to show that \(x\) has the desired presentation in all factors and since \(x = 1 + \cdots + 1 + (x-i)\) in the factor \(\mathbb{Z}_{pr} \langle X \rangle / ((X-i)^m)\), this is true. \(\square\)

**Proposition 2.** Let \(R\) be a ring. Suppose that the element \(p\) is nilpotent, where \(p\) is a prime integer, and let \(a \in R\) be such that \((a)_p\) is nilpotent. Then \(a\) is strongly \((p-1)\)-nil-clean.

**Proof.** Consider the homomorphism \(f: \mathbb{Z}[X] \rightarrow R\), given by \(f(X) = a\). An immediate consequence of the fact that \((a(a-1) \cdots (a-(p-1)))^m = 0\) and that \(p'^r = 0\) in \(R\), for some \(m, r \geq 1\), is the existence of an induced homomorphism
\[
\tilde{f}: \mathbb{Z}_{pr} \langle X \rangle / (X^m(X-1)^m \cdots (X-(p-1))^m) \rightarrow R,
\]
such that \(x \mapsto a\). Since \(x\) is strongly \((p-1)\)-nil-clean, so is its image \(a\). \(\square\)

**Proposition 3.** Let \(R\) be a ring of characteristic \(k(>0)\). If \(p\) is the largest prime dividing \(k\) and \(a \in R\) is such that \((a)_s\) is nilpotent for some \(s \geq 1\), then \(a\) is strongly \((p-1)\)-nil-clean.

**Proof.** Under isomorphism \(R \cong R_1 \times \cdots \times R_t\), where \(R_i = R / p_i^{a_i} R\) and \(k = p_1^{a_1} \cdots p_t^{a_t}\), implied by Lemma 1, the element \(a\) goes to \((a_1, \ldots, a_t)\). Also, \(a\) is strongly \((p-1)\)-nil-clean iff \(a_i\) is such for all \(i\). However, from the fact that \((a)_s\) is nilpotent, it follows that \((a_i)_s \in R_i\) is nilpotent for all \(i\). From this, it easily follows that \((a_i)_p\) is also nilpotent for all \(i\). Namely, if \(s < p_i\), this follows since \((a_i)_p = (a_i)_s(a_i-s)_p\) and if \(s > p_i\), it follows from Corollary 2. Since \(p_i\) is nilpotent in \(R_i\), Proposition 2 gives that \(a_i \in R_i\) is strongly \((p_i - 1)\)-nil-clean for all \(i\). From the fact that \(p_i \leq p\) for all \(i\), it follows that these elements \(a_i\) are all \((p-1)\)-nil-clean and so is \(a\). \(\square\)

**Theorem 1.** Let \(R\) be a ring such that every element \(a \in R\) is strongly \(s\)-nil-clean for some \(s\). Then \(R\) has finite characteristic and \(R\) is strongly \((p-1)\)-nil-clean, where \(p\) is the largest prime dividing \(\text{char}(R)\).
Proof. Corollary 1 shows that $R$ has finite characteristic and from Proposition 1 and Proposition 3 it follows that every element is strongly $(p - 1)$-nil-clean, where $p$ is the largest prime dividing this characteristic.

Theorem 1 shows that one needs only to investigate strongly $(p - 1)$-nil-clean rings, where $p$ is a prime integer. For example, the class of all strongly 3-nil-clean rings is the same as the class of strongly 2-nil-clean rings, and the class of strongly 9-nil-clean-rings is the same as the class of strongly 6-nil-clean rings. Namely, if a ring is, say, strongly 9-nil-clean, then $a^10$ is nilpotent for all $a \in R$. So, this is true for $a = 10$. Consequently, $10! = (10)^{10}$ is nilpotent, so char$(R) | (10)^m$, for some $m \geq 1$. We conclude that the largest prime dividing char$(R)$ is at most 7 (it may be smaller), so our ring is strongly 6-nil-clean.

Proposition 4. A subring of a strongly $(p - 1)$-nil-clean ring is also strongly $(p - 1)$-nil-clean.

Proof. Let $S$ be a subring of a strongly $(p - 1)$-nil-clean ring $R$ and let $a \in S$. Since $a \in R$, $a$ is strongly $(p - 1)$-nil-clean, and according to Proposition 1 element $(a)_p$ is nilpotent. $R$ is of finite characteristic, say $k$, which means that char$(S) = k$, with $p$ being the largest prime dividing $k$. When we apply Proposition 3, we get that $a$ is strongly $(p - 1)$-nil-clean in $S$.

Theorem 2. Let $R$ be a commutative ring of finite characteristic $k$ and $p$ the largest prime dividing $k$. Let $S = \{a \in R : a$ is $s$-nil-clean for some $s\}$. Then $S$ is the largest subring of $R$ which is $(p - 1)$-nil-clean.

Proof. Since $k = 0$ in $R$, we have: $-1 = 1 + \cdots + 1$, so $-1$ is $(k - 1)$-nil-clean and $-1 \in S$. Also, if $a, b \in S$, then $a = e_1 + \cdots + e_s + n, b = f_1 + \cdots + f_t + n'$ and we get $ab = \sum_{i,j} e_i f_j + N$, where $N$ is nilpotent and all $e_i f_j$ are idempotents ($R$ is a commutative ring). So, $ab \in S$. Similarly, $a + b = e_1 + \cdots + e_s + f_1 + \cdots + f_t + n + n' \in S$. Finally, since $a - b = a + (-1)b$ we conclude that $a - b \in S$ as well. So, $S$ is a ring in which every element is $s$-nil-clean for some $s$. From Theorem 1, we conclude that $S$ is actually $(p - 1)$-nil-clean ring. It is clear that $S$ is the largest such subring.

Remark 2. From the Theorem 2 it is clear that, in order to show that a commutative ring of finite characteristic $k$ is $(p - 1)$-nil-clean, it is enough to check only its ring generators over $\mathbb{Z}_k$. For example, the ring $\mathbb{Z}_{p^r}[X]/(X^m(X - 1)^m \cdots (X - (p - 1))^m)$, appearing in Lemma 2, is $(p - 1)$-nil-clean, since it is generated by $x$ and this element is $(p - 1)$-nil-clean.

The following proposition provides us with a lot of examples of commutative $(p - 1)$-nil-clean rings.
Proposition 5. Let $R$ be a finite commutative local ring, $M$ its maximal ideal and $R/M \cong \mathbb{Z}_p$. Then $R$ is $(p - 1)$-nil-clean.

Proof. We know that every element in $M$ is nilpotent. If $x \in R$, then $x + M = s + M$, for some $s \in \{0, \ldots, p - 1\}$. So, $x = \underbrace{1 + \cdots + 1 + m}_s$, where $m$ is nilpotent. □

Remark 3. In the case of non-commutative rings, the set of all elements which are strongly $s$-nil-clean for some $s$ do not necessarily form a subring. Such examples will be given in Section 3 and Section 4. However, a simple application of Zorn’s lemma shows that there exist maximal subrings which are strongly $(p - 1)$-nil-clean.

The following proposition gives another characterization of strongly $(p - 1)$-nil-clean elements.

Proposition 6. Let $R$ be a ring of finite characteristic $k$, $p$ the largest prime dividing $k$ and $a \in R$. The following conditions are equivalent.

1. $(a)_p$ is nilpotent.
2. $a$ is strongly $(p - 1)$-nil-clean.
3. $a = b + n$, where $b \in R$ is such that $(b)_p = 0$, $n$ is nilpotent and $bn = nb$.

Proof. (1) $\implies$ (2). This is contained in Proposition 3.
(2) $\implies$ (3). Let $a = e_1 + \cdots + e_{p-1} + n$ be a $(p - 1)$-nil-clean decomposition of $a$. Take $b := e_1 + \cdots + e_{p-1}$. The proof of Proposition 1 shows that $(b)_p = 0$.
(3) $\implies$ (1). Assume that $a = b + n$. So, we have

$$(a)_p = (b + n)((b - 1) + n)\cdots((b - p + 1) + n).$$

Therefore, since $n$ and $b$ commute, $(a)_p = (b)_p + n q(n, b) = n q(n, b)$, for some polynomial $q(X, Y) \in \mathbb{Z}[X, Y]$. Since $n$ is nilpotent, so is $(a)_p$. □

3. Structure theorems

The purpose of this section is to discuss the structure of (strongly) $(p - 1)$-nil-clean rings, for prime number $p$.

The following proposition sums up the discussion from the previous section.

Proposition 7. Suppose that $\text{char}(R) = k = p_1^{a_1} \cdots p_l^{a_l}$, where $p_1 < \cdots < p_l = p$. Then $R$ is strongly $(p - 1)$-nil-clean if and only if $R_i$ is strongly $(p_i - 1)$-nil-clean, where $R_i = R/p_i^{a_i} R$ and $1 \leq i \leq l$.

This shows that in investigation of strongly $(p - 1)$-nil-clean rings, for $p$ prime, we can reduce our analysis to the case when $p$ is nilpotent (equivalently, when the characteristic of the ring is a power of a prime).

Let us recall that a ring $R$ is called strongly $\pi$-regular if for every element $a \in R$ there exists $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1} x$.

Theorem 3. Every strongly $(p - 1)$-nil-clean ring is strongly $\pi$-regular.
Proof. It is enough to consider the case when \( p \) is a nilpotent element. Then \( (p - 1)! \) is invertible. Let \( a \in R \). Since \( (a)_p \) is nilpotent, we have that \((a)_p^s = 0\) for some \( s \). But

\[
0 = ((a)_p)^s = (a(a - 1) \cdots (a - (p - 1)))^s = a^s((p - 1)!)^s + a^{s+1} y,
\]

for some \( y \in R \). Since \( (p - 1)! \) is invertible, we get that \( a^s = a^{s+1} x \), for some \( x \in R \) and we are done. \( \square \)

It is well known that strongly \( \pi \)-regular ring is strongly clean (see [2, Proposition 2.6], [9, Theorem 1], [5, Corollary 2.4]). Also, Jacobson radical of a strongly \( \pi \)-regular ring is nil and commutative strongly \( \pi \)-regular rings have Krull dimension 0 (see [2]). So, we have the following corollary.

**Corollary 3.** Every strongly \((p - 1)\)-nil-clean ring is strongly clean.

The following proposition is rather useful.

**Proposition 8.** Let \( R \) be a ring, \( a \in R \) and let \( p \) be a nilpotent element, where \( p \) is prime. Then \( a^p - a \) is nilpotent if and only if \( (a)_p \) is nilpotent.

*Proof.* It is well-known that \( X^p - X = (X)_p \) in \( \mathbb{Z}[X] \). So, \( a^p - a - (a)_p = pr(a) \), for some polynomial \( r(X) \in \mathbb{Z}[X] \). From this fact, the proof follows immediately. \( \square \)

For future reference, we formulate the following corollary which directly follows from Proposition 6 and Proposition 8.

**Corollary 4.** Let \( R \) be a ring. If \( p \) is nilpotent, then \( R \) is a strongly \((p - 1)\)-nil-clean ring if and only if \( a^p - a \) is nilpotent for every \( a \in R \).

Let us proceed with some of the special properties of \((p - 1)\)-nil-clean rings.

**Proposition 9.** Let \( R \) be a ring and let \( I \) be any nil ideal of \( R \). Then \( R \) is \((p - 1)\)-nil-clean if and only if \( R/I \) is \((p - 1)\)-nil-clean.

*Proof.* \( (\implies) \) As observed before, this is trivial since \( R/I \) is a homomorphic image of \( R \).

\( (\impliedby) \) Suppose that \( R/I \) is \((p - 1)\)-nil-clean. Let \( x \in R \). Then \( x + I \) is \((p - 1)\)-nil-clean. Thus, \( x + I = (x_1 + I) + (x_2 + I) + \cdots + (x_{p-1} + I) + (y + I) \), where \( x_i + I \) are idempotents, \( 1 \leq i \leq p - 1 \), and \( y + I \) is nilpotent. It is well known that idempotents lift modulo nil ideals (see, e.g., [6, Theorem 21.28]) so there are idempotents \( e_i \) such that \( x_i + I = e_i + I \). So, \( x - e_1 - e_2 - \cdots - e_{p-1} - y \in I \), i.e., \( x = e_1 + e_2 + \cdots + e_{p-1} + y + n \), for some \( n \in I \). Element \( y + n \) is nilpotent. Indeed, since \( y + I \) is nilpotent, \( y^k \in I \) for some \( k \in \mathbb{N} \). Every element different from \( y^k \) in the sum one gets in the expansion of \((y + n)^k\), is in \( I \) and we can conclude that \((y + n)^k \in I \), so \( y + n \) is nilpotent. Therefore \( R \) is \((p - 1)\)-nil-clean. \( \square \)

An analogous result holds for the strongly \((p - 1)\)-nil-clean rings.
Proposition 10. Let $R$ be a ring and let $I$ be any nil ideal of $R$. Then $R$ is strongly $(p - 1)$-nil-clean if and only if $R/I$ is strongly $(p - 1)$-nil-clean.

Proof. ($\implies$) Again, this is trivial since $R/I$ is a homomorphic image of $R$.

($\impliedby$) Let $a \in R$. Since $R/I$ is strongly $(p - 1)$-nil-clean, one has $((a + I)p)^k = I$, for some $k \in \mathbb{N}$. Consequently, $((a)\cdot p)^k \in I$. As $I$ is nil ideal, $((a)\cdot p)^k \in N(R)$. So, $(a)\cdot p$ is nilpotent. Since $(p)\cdot p$ is also nilpotent, the characteristic $l$ of $R$ is finite. The characteristic $l$ of $R/I$ has the property that $p$ is the largest prime dividing this characteristic, but this also holds for $k$. Namely, $l \in I$ and therefore $l$ is nilpotent in $R$. So we have that $k | l^s$ for some $s$, and also $l | k$. It follows that the sets of primes dividing $k$ and $l$ are the same. Now the result follows from Proposition 6. □

The following corollary follows directly from the fact that $J(R)$ is nil for a strongly $(p - 1)$-nil-clean ring and Proposition 10.

Corollary 5. A ring $R$ is strongly $(p - 1)$-nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is strongly $(p - 1)$-nil-clean.

4. Group rings

Let us recall the notion of a group ring. Let $G$ be a group, written multiplicatively, and let $R$ be a commutative ring. The group ring of $G$ over $R$, denoted by $RG$, is a free $R$-module with generating set $G$, i.e.:

$$RG = \bigoplus_{g \in G} Rg.$$ 

So, elements of $RG$ are formal finite sums of the form $\sum_i r_i g_i$, with $r_i \in R$, $g_i \in G$, while the multiplication is implied by multiplication in $G$. The identity of this ring is $1_{RE}$, where $1_R$ is the identity in $R$ and $e$ is the neutral element of $G$. We denote the identity simply by $1$.

Our main interest here is focused on strongly $(p - 1)$-nil-clean group rings $RG$. It is obvious that if $RG$ is strongly $(p - 1)$-nil-clean, so is $R$. Since we assume that the coefficient ring $R$ is commutative, we refrain from using adjective “strongly” when referring to $R$, we use it only for $RG$ when appropriate. We begin by discussing rings $R$, such that char($R$) is a power of a prime.

Lemma 3. Let $R$ be a $(p - 1)$-nil-clean ring such that char($R$) = $p^s$, for a prime $p$ and some $s \geq 1$ and let $G$ be a group. For the list of conditions:

1. $RG$ is strongly $(p - 1)$-nil-clean;
2. For each $g \in G$, the element $g^{p^{s-1}} - 1$ is nilpotent;
3. For each $g \in G$, there exists $k \geq 0$ and $d | (p - 1)$ such that $\omega(g) = dp^k$,

the following holds: (1) $\implies$ (2) and (2) $\iff$ (3). Here, $\omega(g)$ denotes the order of $g$ in $G$. 
Proof. (1) $\implies$ (2). Let $g \in G$. Since $RG$ is strongly $(p-1)$-nil-clean, from Corollary 4 it follows that $g^p - g$ is nilpotent. Since $g$ is invertible, we get that $g^{p-1} - 1$ is nilpotent.

(3) $\implies$ (2). Let $g \in G$. Then $\omega(g) = dp^k$, where $k \geq 0$ and $d \mid (p-1)$. Let $p - 1 = ds$. Since $\omega(g^d) = p^k$ and $\gcd(s, p^k) = 1$, we have that $\omega(g^{p-1}) = \omega((g^d)^s) = p^k$.

Therefore,

$$(g^{p-1} - 1)^p^k = \underbrace{(g^{p-1})^p^k - 1 + p \cdot u}_{0}$$

for some $u \in RG$.

Since $p$ is nilpotent, the element $g^{p-1} - 1$ is nilpotent as well.

(2) $\implies$ (3). Let $g \in G$. The order of $g$ cannot be infinite — in that case, it would not be possible for $g^{p-1} - 1$ to be nilpotent. Namely, the element $g^{(p-1)s}$ in the sum $(g^{p-1} - 1)^s = g^{(p-1)s} + \cdots + (-1)^s$ cannot be cancelled out.

So, let us suppose that $\omega(g) = tp^k$, for some $k \geq 0$ and $t$ such that $p \nmid t$ and $\gcd(t, p-1) = d \neq t$. Also, let $t = dt_1$, $p - 1 = dz_1$ and $h = g^{p-1}$. Since $\omega(g^d) = t_1p^k$ and $\gcd(z_1, t_1p^k) = 1$, it follows that

$$\omega(h) = \omega(g^{p-1}) = \omega((g^d)^{z_1}) = t_1p^k.$$  

Since $h - 1$ is nilpotent, $h^{t_1} - 1$ is nilpotent as well. Let $h_1 = h^{t_1}$. Then $\omega(h_1) = t_1$. Let us focus on the polynomial $f(X) = (X - 1)^t_1 - (X^{t_1} - 1)$, which is clearly divisible by $X - 1$:

$$f(X) = (X - 1)((X - 1)^{t_1-1} - (X^{t_1-1} + \cdots + X + 1)) = (X - 1)f_1.$$  

This follows from the fact that $t_1 \neq 1$ (which also implies that $h_1 - 1 \neq 0$). We can see that

$$f(X) = (X - 1)(-t_1 + (X - 1)q(X)),$$

for some polynomial $q \in \mathbb{Z}[X]$, since $f_1(1) = -t_1$. We can conclude that

$$f(h_1) = (h_1 - 1)^{t_1} - (h_1^{t_1-1}) = h_1 - 1)(-t_1 + (h_1 - 1)q(h_1)).$$

We know that $h_1 - 1$ is nilpotent. As $p \nmid t_1$ and $p$ is nilpotent, element $t_1$ is invertible in $R$. So

$$(h_1 - 1)^{t_1} = u(h_1 - 1),$$

for an invertible $u \in RG$. So

$$(h_1 - 1)((h_1 - 1)^{t_1-1} - u) = 0,$$

and since $(h_1 - 1)^{t_1-1} - u$ is invertible, we have that $h_1 - 1 = 0$. That is a contradiction.

It is easy to check that the proof of 2 $\iff$ 3 is valid, although shorter, even for $p = 2$.  \(\square\)
In the previous lemma, $G$ was an arbitrary group. If we add commutativity, we actually get equivalence (1) $\iff$ (2).

**Theorem 4.** Let $R$ be a $(p-1)$-nil-clean ring such that $\text{char}(R) = p^s$, for a prime $p$ and some $s \geq 1$ and let $G$ be an Abelian group. Then $RG$ is $(p-1)$-nil-clean iff $g^{p^s-1} = 1$ is nilpotent for every $g \in G$.

**Proof.** We only need to prove the “if” part. It follows directly from the Remark following Theorem 2 since elements of the group $G$ form a generating set for $RG$ over $R$. □

**Example 2.** The previous theorem does not hold for non-commutative groups. Let us take $R = \mathbb{Z}_5$ and $G = \mathbb{D}_4$, the dihedral group of order 8 generated by elements $s$ and $r$ such that $s^2 = 1 = r^4$, $sr = r^3s$. In this group, $g^4 = 1$ for all $g \in G$, so the condition that $g^4 = 1$ is nilpotent is trivially satisfied. However, direct computation shows that $$
(s + sr)^5 = 2s + 2r + 3sr^2 + 3r^3 $$ $$((s + sr)^5)^8 = 3 + 2r^2 $$ $$(3 + 2r^2)^2 = 3 + 2r^2,$$
so $(s + sr)^5$ is not nilpotent and the group ring $\mathbb{Z}_5 \mathbb{D}_4$ is not strongly $4$-nil-clean.

Let us concentrate now on the general case.

**Proposition 11.** Let $R$ be a $(p-1)$-nil-clean ring and $\text{char}(R) = p_1^{e_1} \cdots p_l^{e_l}$, so that $l > 1$, $p_1 < \cdots < p_l = p$.

1. If $G$ is an elementary Abelian 2-group, then $RG$ is strongly $(p-1)$-nil-clean.
2. If $G$ is an elementary Abelian group in which every element has order $q$ and $q \mid \gcd(p_1-1, \ldots, p_l-1)$, then $RG$ is strongly $(p-1)$-nil-clean.

**Proof.** (1) As we know, $R \cong R_1 \times \cdots \times R_l$, where $p_i$ is nilpotent in $R_i$, and consequently $RG \cong R_1G \times \cdots \times R_lG$. So, all the rings $R_i$ are strongly $(p-1)$-nil-clean, and since $p_i$ is nilpotent in $R_i$, $R_i$ is actually strongly $(p_i-1)$-nil-clean. We will use Theorem 4. Since $2 \mid (p_i-1)$ for all $i \geq 2$ and $g^2 = 1$ for all $g \in G$, we have that $g^{p_i-1} = 1 = 0$ for all $g \in G$. If $p_1 > 2$, the same holds for $p_1$. If $p_1 = 2$, then $(g-1)^2 = g^2 - 2g + 1 = 2(1-g)$. Since 2 is nilpotent in $R_1G$, $g-1$ is also nilpotent in $R_1G$. In this case also, from $G$ being Abelian, we can conclude that $R_1G$ is $(p_1-1)$-nil-clean. Therefore, $RG$ is $(p-1)$-nil-clean. (2) Similarly, from $g^q = 1$, and $q \mid \gcd(p_1-1, \ldots, p_l-1)$, we conclude that $g^{p_i-1} = 1 = 0$ for all $g \in G$, and proceed as in (1). □

**Theorem 5.** Let $\text{char}(R) = p_1^{e_1} \cdots p_l^{e_l}$, where $l > 1$, $p_1 < \cdots < p_l = p$ and let $G$ be a group. Suppose that $RG$ is strongly $(p-1)$-nil-clean ring.

1. For all $g \in G$ the following holds: $\omega(g) \mid \gcd(p_2-1, \ldots, p_l-1)$ and $\omega(g) = d_1 p_1^{e_1}$, such that $d_1 \mid (p_1-1)$ and $s \geq 0$. 

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Lemma 3 shows that $g$. Since $g_i$, we are done.

We conclude that $\omega(g) = d_1 p_i^{k_i}$ for some $k_i \geq 0$ and $d_i \mid (p_i - 1)$. So,

$$d_1 p_i^{k_i} = d_2 p_2^{k_2} = \cdots = d_l p_l^{k_l}.$$  
Since $d_1 < p_1$, it is clear that $d_1 p_i^{k_i}$ cannot be divisible by any prime greater than $p_1$. So $k_i = 0$ for $i \geq 2$. Therefore,

$$\omega(g) = d_1 p_i^{k_i} = d_2 = \cdots = d_l,$$
for all $g \in G$, where $d_i \mid (p_i - 1)$. Hence, $\omega(g) \mid (p_2 - 1), \ldots, \omega(g) \mid (p_l - 1)$, and we are done.

(2) If $h \in G$ is such that $\omega(h) \geq p_1$, we have $\omega(h) = d_1 p_i^{k_i} = d_2 = \cdots = d_l$, where $k_1 \geq 1$. Since $d_1 \mid (p_1 - 1)$, it follows that $p_1 \mid (p_i - 1)$.

(3) Under this assumption, we get that $k_1 = 0$ in (4.1), hence for all $g \in G$

$$\omega(g) = d_1 = d_2 = \cdots = d_l.$$  
We conclude that $\omega(g) \mid \gcd(p_1 - 1, \ldots, p_l - 1)$.

(4) The fourth assertion follows easily. Namely, in this case $p_1 = 2$, so $\omega(g) = d_1 2^{k_1}$, where $d_1 \mid (2 - 1)$. So, $\omega(g) = 2^{k_1}$.

(5) It is enough to show that there are no elements of order 4 in $G$. If it were, then for an element $g \in G$, we would have equalities

$$4 = d_2 = \cdots = d_l,$$
where $d_i \mid (p_i - 1)$, for $2 \leq i \leq l$. This would imply that $4 \mid (p_i - 1)$, that is $p_i \equiv 1 \pmod{4}$, for $2 \leq i \leq l$, which is a contradiction. Hence, we can conclude that $G$ is an elementary Abelian 2-group.

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Authors’ addresses

Aleksandra Kostić
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
*E-mail address*: alex@matf.bg.ac.rs

Zoran Z. Petrović
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
*E-mail address*: zoranp@matf.bg.ac.rs

Zoran S. Pucanović
University of Belgrade, Faculty of Civil Engineering, Bulevar kralja Aleksandra 73, Belgrade, Serbia
*E-mail address*: pucanovic@grf.bg.ac.rs

Maja Roslavcev
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
*E-mail address*: roslavcev@matf.bg.ac.rs