From Small Space to Small Width in Resolution

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Proof Complexity and Resolution

**Proof complexity**: Hardness of reasoning in propositional logic

**Resolution**

- **Input**: CNF formula $F$
  \[
  (x \lor \overline{y} \lor z) \land (\overline{y} \lor \overline{z}) \land (x \lor y) \land (\overline{x} \lor \overline{z}) \land (\overline{x} \lor z)
  \]

- **Resolution rule**: 
  \[
  \frac{C \lor x \quad D \lor \overline{x}}{C \lor D}
  \]

- **Goal**: Proof of unsatisfiability (refutation) $= \text{Derive empty clause } \bot$

Refer to clauses of formula as **axioms**
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

1. \( x \lor \overline{y} \lor z \) Axiom
2. \( \overline{y} \lor \overline{z} \) Axiom
3. \( x \lor \overline{y} \) Res(1, 2)
4. \( x \lor y \) Axiom
5. \( x \) Res(3, 4)
6. \( \overline{x} \lor \overline{z} \) Axiom
7. \( \overline{x} \lor z \) Axiom
8. \( \overline{x} \) Res(6, 7)
9. \( \bot \) Res(5, 8)
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
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Example:

Size: 9
Space: # clauses before used after
Width: size of the largest clause

Example:

\[
\begin{align*}
&x \lor \overline{y} \lor z \\
&\overline{y} \lor \overline{z} \\
&x \lor \overline{y} \\
&x \lor y \\
&x \\
&\overline{x} \lor \overline{z} \\
&\overline{x} \lor z \\
&\overline{x} \\
&\bot
\end{align*}
\]
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

Size: number of steps in proof
Space: memory usage (at step $t$: # clauses before $t$ used after $t$)
Width: size of the largest clause

Example:
Size
Space
Width
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
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**Size**: number of steps in proof

**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example:**

|   |   |   |
|---|---|---|
| 1 | $x \lor \overline{y} \lor z$ | Axiom |
| 2 | $\overline{y} \lor z$ | Axiom |
| 3 | $x \lor \overline{y}$ | Res(1, 2) |
| 4 | $x \lor y$ | Axiom |
| 5 | $x$ | Res(3, 4) |
| 6 | $\overline{x} \lor z$ | Axiom |
| 7 | $\overline{x} \lor z$ | Axiom |
| 8 | $\overline{x}$ | Res(6, 7) |
| 9 | $\bot$ | Res(5, 8) |

Size 9
Space
Width
Resolution Size, Space, and Width

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**Size:** number of steps in proof

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**Example:**
- Size 9
- Space
- Width
Can represent refutation as

- annotated list or
- DAG

**Size:** number of steps in proof

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**

|   | Size | Space | Width |
|---|------|-------|-------|
|   | 9    | 0     |       |

Space at current step 0
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in proof
**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)
**Width**: size of the largest clause

**Example**:

Size 9
Space 1
Width

Space at current step 1
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
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**Size:** number of steps in proof
**Space:** memory usage (at step \( t \): 
  \# clauses before \( t \) used after \( t \))
**Width:** size of the largest clause

**Example:**
- Size 9
- Space 2
- Width

Space at current step 2
Resolution Size, Space, and Width

Can represent refutation as
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  - DAG

Size: number of steps in proof
Space: memory usage (at step $t$: # clauses before $t$ used after $t$)
Width: size of the largest clause

Example:
  - Size 9
  - Space 3
  - Width

Space at current step 3
Resolution Size, Space, and Width

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**Size**: number of steps in proof

**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size 9
- Space 3
- Width

Space at current step 2
Resolution Size, Space, and Width

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**Size:** number of steps in proof

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**Width:** size of the largest clause

**Example:**
- Size 9
- Space 3
- Width

Space at current step 3
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Size: number of steps in proof
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Example:
- Size 9
- Space 3
- Width

Space at current step 2
Resolution Size, Space, and Width

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**Size**: number of steps in proof

**Space**: memory usage (at step $t$: 
$\#$ clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:

| Size   | 9    |
|--------|------|
| Space  | 3    |
| Width  |      |

Space at current step 3
Can represent refutation as

- annotated list or
- DAG

**Size:** number of steps in proof

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**

| Size | Space | Width |
|------|-------|-------|
| 9    | 4     |       |

Space at current step 4
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in proof

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**

Size 9
Space 4
Width

Space at current step 3

$\text{Example clause: } x \lor \neg y \lor z$

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STACS '14
Resolution Size, Space, and Width

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**Size:** number of steps in proof

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**

Size 9
Space 4
Width

Space at current step 0
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in proof

**Space:** memory usage (at step $t$: 
# clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size 9
- Space 4
- Width 3

Space at current step 0
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in proof

**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size: 9
- Space: 4
- Width: 3

Space at current step: 0
Relation Between Width and Size/Space

**Width** helps us understand size and space
Makes most sense for small width formulas — focus on $k$-CNFs

**Size:** Ben-Sasson and Wigderson ’99

\[ \log(\text{Size}) \gtrsim \text{Width} \]

Proof by syntactically manipulating short refutation into narrow refutation

**Space:** Atserias and Dalmau ’03

\[ \text{Space} \geq \text{Width} \]

More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games
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More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games

**Our result:** Simple purely syntactic proof
Refutation presented on whiteboard
Refutation presented on whiteboard

- Write down axioms
Refutation presented on whiteboard

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- Write down axioms
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
Refutation presented on whiteboard

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Whiteboard Interpretation of Space

Refutation presented on whiteboard
- Write down axioms
- Use resolution rule
- Erase clause
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Whiteboard Interpretation of Space

Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause

**Space:** max # clauses on board
No finite model theory and no Ehrenfeucht-Fraïssé games

Want to turn small-height whiteboard into small-width one

\[
\begin{align*}
x \lor \overline{y} \lor \overline{z} \lor v \lor w \\
\overline{y} \lor \overline{z} \lor \overline{w} \lor x \\
x \lor \overline{y}
\end{align*}
\]
Proof Idea in One Slide

No finite model theory and no Ehrenfeucht-Fraïssé games

Want to turn small-height whiteboard into small-width one

Rotate whiteboard and get narrow whiteboard

\[ x \lor \neg y \lor \neg z \lor v \lor w \]
\[ \neg y \lor \neg z \lor \neg w \lor x \]
\[ x \lor \neg y \]
In Somewhat More Detail... 

\[
x \lor \overline{y} \lor z \\
\overline{y} \lor \overline{z}
\]
In Somewhat More Detail...

\[ \neg((x \lor \overline{y} \lor z) \land (\overline{y} \lor \overline{z})) \]

- View clauses on whiteboard as CNF and negate
In Somewhat More Detail...

- View clauses on whiteboard as CNF and \textbf{negate}
- Apply DeMorgan’s rules
In Somewhat More Detail...

\[ \neg(((x \lor \overline{y} \lor z) \land \overline{(y \lor \overline{z})}) \land ((\overline{x} \lor y \lor \overline{z}) \lor (y \lor z)) \lor ((\overline{x} \lor y) \lor (\overline{y} \lor y) \lor (\overline{y} \lor z) \lor (\overline{z} \lor y) \lor (\overline{z} \lor z)) \]

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses
In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses
- Write CNF on whiteboard

\[ x \lor \bar{y} \lor z \]
\[ \bar{y} \lor \bar{z} \]
\[ (\bar{x} \land y \land \bar{z}) \lor (y \land z) \]
\[ (\bar{x} \lor y) \land (x \lor z) \land (y \land z) \land (\bar{y} \lor z) \land (\bar{z} \lor y) \land (\bar{z} \lor z) \]
\[ \bar{x} \lor z \]

Original  
Negated

Space (\# clauses) of Original \( \geq \) Width of Negated
Consequences of Negation

Negate every whiteboard and run refutation in reverse

Note: Empty whiteboard turns into contradiction and vice versa

Small space refutation is transformed into narrow one
Missing Details

Need two things

1. **Prove** we have backbone of resolution refutation
2. **Fill in** missing details (without blowing up width)

Proof by *case analysis* over derivation steps:

- Axiom download
- Resolution rule application
- Clause erasure
Original: Erasure weakens whiteboard

Right board weaker than left board
Original: Erasure weakens whiteboard

Right board weaker than left board

Negated: Negation inverts relation

Left board weaker than right board

Negated refutation run in reverse!
Can skip weaker whiteboards
Resolution Rule Application

Original refutation

Original: No change in semantic content

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Resolution Rule Application

Original refutation: $x \lor \neg y \lor z \land \neg y \lor z \land x \lor \neg y$

Original: No change in semantic content

Negated refutation: $\neg x \lor z \land y$

Negated: No change in syntactic content (after pruning redundant clauses)

Whiteboard stays the same!
Resolution Rule Application

Original refutation

\[ x \lor \overline{y} \lor z \]
\[ \overline{y} \lor z \]

Negated refutation

Original refutation

\[ x \lor \overline{y} \lor z \]
\[ \overline{y} \lor z \]
\[ x \lor \overline{y} \]

Negated refutation

\[ \overline{x} \lor z \]
\[ y \]

Original: No change in semantic content

Negated: No change in syntactic content (after pruning redundant clauses)

Whiteboard stays the same!

Note: No work done thus far!
**Original**: Add axiom $A$ to whiteboard

Original refutation

$x \lor \overline{y} \lor z$

$x \lor \overline{y} \lor z$

$\overline{y} \lor \overline{z}$
**Original:** Add axiom $A$ to whiteboard

**Negated:** For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \lor \overline{a}$ and $A$ to derive $C$
**Original:** Add axiom $A$ to whiteboard

**Negated:** For every literal $a \in A$ add $\bar{a}$ to all clauses of whiteboard

Use clauses $C \lor \bar{a}$ and $A$ to derive $C$

$$
\begin{align*}
\bar{y} \lor \bar{z} & \quad \bar{x} \lor y \\
\bar{x} \lor \bar{z} & \quad \bar{x} \lor \bar{z} \\
\end{align*}
$$

Adds constant width to derivation
**Original**: Add axiom $A$ to whiteboard

**Negated**: For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \lor \overline{a}$ and $A$ to derive $C$

$$
\frac{\overline{y} \lor \overline{z} \quad \overline{x} \lor y}{\overline{x} \lor \overline{z} \quad \overline{x} \lor z}
$$

Adds constant width to derivation

**Theorem**

Space $\geq$ Width
Open Problem: Similar Problem for Polynomial Calculus

**Polynomial calculus**
Stronger proof system based on *algebraic* reasoning
Lines are *polynomial equations* instead of clauses
Degree of refutation analogous to width in resolution

**Size:** Impagliazzo, Pudlák, and Sgall ’99

\[ \log(\text{Size}) \gtrapprox \text{Degree} \]
Polynomial calculus
Stronger proof system based on algebraic reasoning
Lines are polynomial equations instead of clauses
Degree of refutation analogous to width in resolution

Size: Impagliazzo, Pudlák, and Sgall ’99

\[ \log(\text{Size}) \gtrapprox \text{Degree} \]

Open Problem
Is Space \( \geq \) Degree in polynomial calculus?

Original motivation for our work
We show our approach is unlikely to work (see paper for details)
Concluding Remarks

- Space upper bounds width in resolution [Atserias and Dalmau '03]
- This work: New simple proof of this theorem
- Open problem: Space-degree relation in polynomial calculus?
Concluding Remarks

- Space upper bounds width in resolution [Atserias and Dalmau ’03]
- **This work:** New simple proof of this theorem
- **Open problem:** Space-degree relation in polynomial calculus?

Thank you for your attention!