The resonant structure of Kink-Solitons in the Modified KP Equation

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Abstract

Using the Wronskian representation of $\tau$-function, one can investigate the resonant structure of kink-soliton and line-soliton of the modified KP equation. It is found that the resonant structure of the soliton graph is obtained by superimposition of the two corresponding soliton graphs of the two Le-Diagrams given an irreducible Schubert cell in a totally non-negative Grassmannian $Gr(N,M)_{\geq 0}$. Several examples are given.

Keywords: Wronskian, Grassmannian, Le-Diagram, Kink-Soliton
1 Introduction

Recently, the resonant theory of line solitons has attracted much attention, especially in the KP-(II) theory [1, 8, 11, 12, 13, 14, 15] and Novikov-Veselov (NV) equation [3, 4]. To study the resonant theory of integrable models, we use the Wronskian representation of the $\tau$-function for the solution structure. Given an $N \times M$-matrix ($N < M$), one can describe the $\tau$-function as a linear combination of exponential functions by Bitnet-Cauchy formula and their coefficients have to satisfy the Plucker relations. It is required that all these coefficients have to be non-negative to get non-singular solutions. Therefore, one introduces the totally non-negative Grassmannian to the resonant theory. Due to the success of resonant theory in the KP-(II) equation, one can also investigate the solitonic resonance of the Modified KP-(II) (MKP-(II)) equation, especially the resonant theory of kink-soliton and line-soliton. It is noticed that the solution of MKP-(II) equation is associated with the quotient of $\tau$-functions by the Wronskian representations, and then the parameters of the phases are also non-negative to obtain non-singular solutions (see below). Also, there is the Miura transformation between the KP-(II) equation and MKP-(II) equation.

The MKP-(II) equation is defined by [6, 16, 17, 19]

\[-4v_t + v_{xxx} - 6v^2v_x + 6v_x\partial_x^{-1}v_y + 3\partial_x^{-1}v_{yy} = 0.\]  

(1)

The equation (1) was introduced in [16] within the framework of gauge-invariant description of the KP equation. In [19], it appeared as the first member of modified KP hierarchy using the $\tau$-function theory. In [17], the inverse-scattering-transformation method is used to get the exact solutions for MKP equation (1), including rational solutions (lumps), line solitons and breathers.

Letting

\[v(x, y, t) = \partial_x \ln(P(x, y, t)/Q(x, y, t)),\]  

(2)

we have the Hirota bi-linear equation [7, 19]

\[(D_y - D_x^2)P \circ Q = 0\]  

(3)  

\[-4D_t + D_x^3 + 3D_xD_y)P \circ Q = 0,\]  

(4)

where the bi-linear operators $D_x^n$ and $D_y^n$ are defined by

\[D_x^n D_y^n P \circ Q = (\partial_x - \partial_x')^m(\partial_y - \partial_y')^n P(x, y)Q(x', y').\]

To construct the solutions of these Hirota equations [3] and [4], one defines the determinant

\[\tau_N^{(n)} = \det \begin{bmatrix} g_1^{(n)} & g_1^{(n+1)} & \cdots & g_1^{(n+N-1)} \\ g_2^{(n)} & g_2^{(n+1)} & \cdots & g_2^{(n+N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ g_N^{(n)} & g_N^{(n+1)} & \cdots & g_N^{(n+N-1)} \end{bmatrix},\]  

(5)
where the elements in the above determinant are defined by \((i = 1, 2, 3, \ldots, N)\)

\[
\frac{\partial g_i}{\partial x_m} = \frac{\partial^n g_i}{\partial x^m}, \quad x_1 = x, \ x_2 = y, \ x_3 = t,
\]

(6)

and \(g_i^{(n)}\) means the n-th order derivative with respect to \(x\), \(n = 0, 1, 2, 3, \ldots\). Also, we can write \(\tau_N^{(n)}\) as a Wronskian, i.e.,

\[
\tau_N^{(n)} = \text{Wr}(g_1^{(n)}, g_2^{(n)}, g_3^{(n)}, \ldots, g_N^{(n)}).
\]

It is shown that [7]

\[
P = \tau_N^{(1)}, \quad Q = \tau_N^{(0)}, \quad \text{or} \quad v(x, y, t) = \partial_x \ln \frac{\tau_N^{(1)}}{\tau_N^{(0)}},
\]

(7)

will be solutions of \([3]\) and \([4]\) for \(N = 1, 2, \ldots\).

We remark that after the Miura transformation [17], using the Hirota equation \([3]\), we have

\[
u = -\partial_x^{-1}v_y - v_x - v^2 = 2\partial_{xx} \ln \tau_N^{(0)},
\]

(8)

and then one can obtain the Hirota equation by \([4]\)

\[
(-4D_tD_x + D_x^4 + 3D_y^3)\tau_N^{(0)} \circ \tau_N^{(0)} = 0,
\]

or the KP-(II) equation

\[
-4u_t + u_{xxx} + 6uu_x + \partial_x^{-1}3uyy = 0.
\]

(9)

Next, we construct the resonant solutions of MKP-(II) equation using the totally non-negative Grassmannian (TNNG) in KP-(II) theory [13, 15]. Here one considers a finite dimensional solution

\[
g_i(x, y, t) = \sum_{j=1}^{M} k_{ij} H_j(x, y, t), \quad i = 1, 2, \ldots, N < M,
\]

\[
H_j(x, y, t) = e^{\theta_j}, \quad \theta_j = c_j x + c_j^2 y + c_j^3 t + \xi_j, \quad j = 1, 2, \ldots, M
\]

where \(c_j\) and \(\xi_j\) are real parameters. For simplicity, we take \(\xi_j = 0\) in this article.

Each \(H_j(x, y, t)\) satisfies the equations (6). Then each resonant solution of MKP-(II) equation can be parametrized by a full rank matrix

\[
K = \begin{bmatrix}
k_{11} & k_{12} & \cdots & k_{1M} \\
k_{21} & k_{22} & \cdots & k_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
k_{N1} & k_{N2} & \cdots & k_{NM}
\end{bmatrix} \in M_{N \times M}(\mathbb{R}).
\]

(10)
Using the Binet-Cauchy formula, the \( \tau \)-function \( \tau_{N}^{(0)} \) can be written as

\[
\tau_{N}^{(0)} = \tau_{K}^{(0)} = \tau_{N}^{(0)} = \text{Wr}(g_1, g_2, \ldots, g_N) = \det \begin{pmatrix}
g_1 & g_1' & \cdots & g_1^{(N-1)} 
g_2 & g_2' & \cdots & g_2^{(N-1)} 
\vdots & \vdots & \ddots & \vdots 
g_N & g_N' & \cdots & g_N^{(N-1)}
\end{pmatrix}
\]

\[
= \det \left[ \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1M} 
k_{21} & k_{22} & \cdots & k_{2M} 
\vdots & \vdots & \ddots & \vdots 
k_{N1} & k_{N2} & \cdots & k_{NM} \end{pmatrix} \right] \left[ \begin{pmatrix} H_1 & c_1H_1 & \cdots & c_1^{N-1}H_1 
H_2 & c_2H_2 & \cdots & c_2^{N-1}H_2 
\vdots & \vdots & \ddots & \vdots 
H_M & c_MH_M & \cdots & c_M^{N-1}H_M \end{pmatrix} \right]
\]

\[
= \sum_{J} \Delta_{J}(K) H_{J}(x, y, t), \quad (11)
\]

where \( \Delta_{J}(K) \) is the \( N \times N \) minor for the columns with the index set \( J = \{j_1, j_2, j_3, \ldots, j_N\} \), and \( H_{J} \) is the Wronskian

\[
H_{J} = \text{Wr}(H_{j_1}, H_{j_2}, H_{j_3}, \ldots, H_{j_N}) = \prod_{m<l} (c_{j_m} - c_{j_m}) H_{j_1} H_{j_2} H_{j_3} \cdots H_{j_N}. \quad (12)
\]

We notice that the coefficients \( \Delta_{J}(K) \) of \( \tau_{K}^{(0)} \) have to satisfy the Plucker relations. Similarly,

\[
\tau_{K}^{(1)} = \tau_{N}^{(1)} = \text{Wr}(g_1', g_2', \ldots, g_N') = \det \begin{pmatrix}
g_1' & g_1'' & \cdots & g_1^{(N)} 
g_2' & g_2'' & \cdots & g_2^{(N)} 
\vdots & \vdots & \ddots & \vdots 
g_N' & g_N'' & \cdots & g_N^{(N)}
\end{pmatrix}
\]

\[
= \det \left[ \begin{pmatrix} c_{1}k_{11} & c_{2}k_{12} & \cdots & c_{M}k_{1M} 
c_{1}k_{21} & c_{2}k_{22} & \cdots & c_{M}k_{2M} 
\vdots & \vdots & \ddots & \vdots 
c_{1}k_{N1} & c_{2}k_{N2} & \cdots & c_{M}k_{NM} \end{pmatrix} \right] \left[ \begin{pmatrix} H_1 & c_1H_1 & \cdots & c_1^{N-1}H_1 
H_2 & c_2H_2 & \cdots & c_2^{N-1}H_2 
\vdots & \vdots & \ddots & \vdots 
H_M & c_MH_M & \cdots & c_M^{N-1}H_M \end{pmatrix} \right]
\]

\[
= \sum_{J} \Delta_{J}(K)c_{j_1}c_{j_2}c_{j_3} \cdots c_{j_N} H_{J}(x, y, t). \quad (13)
\]

We notice here that the functions \( \tau_{N}^{(0)} \) and \( \tau_{K}^{(1)} \) in (11) and (13) are invariant under the transformations \( K \rightarrow \Gamma K \) in (10), \( \Gamma \) being a constant \( N \times N \) matrix, due to the solution structure (7). As a result, the K-matrix in (10) can be chosen as a reduced row echelon form (RREF). Also, we can define a K-matrix to be irreducible in RREF if

(1) in each column, there is at least one non-zero element;
(2) in each row, there is at least one more non-zero element in addition to the pivot.
On the other hand, using the Plucker embedding \[11, 15\], one can consider the real Grassmannian \(\text{Gr}(N, M) \cong \text{GL}_N(R) \setminus M_{N \times M}(R)\). The Schubert decomposition of \(\text{Gr}(N, M)\) is

\[
\text{Gr}(N, M) = \bigcup_I \Omega_I,
\]

where \(\Omega_I\) (a Schubert cell) in RREF is defined by the set of all matrices whose pivots are given by \(I = \{i_1, i_2, i_3, \ldots, i_N\}\). For each Schubert cell \(\Omega_I\), one introduces the Young diagram to express the index set \(I\) for an alternative parametrization. To obtain non-singular solutions of MKP-(II), from (7), (11) and (13), it can be seen that \(\Delta_J(K) \geq 0\) for all \(J\), i.e., \(K\) is an element of the TNNG \(\text{Gr}(N, M)_{\geq 0}\), i.e., the corresponding Plucker coordinates of each \(\Omega_I\) are non-negative. And we assume the ordering in the \(c\)-parameters,

\[
0 \leq c_1 < c_2 < c_3 < \cdots < c_M.
\] (14)

In the KP-(II) equation, we have the following Classification Theorem for the unbounded line solitons as \(y \to \pm \infty\)

**Theorem [1]:** Let \(\{e_1, e_2, e_3, \ldots, e_N\}\) be the pivot indices, and let \(\{f_1, f_2, \ldots, f_{M-N}\}\) be the non-pivot indices for an irreducible and totally non-negative K-matrix. Then the soliton solution associated with the K-matrix has

(a) \(N\)-line solitons of \([e_n, j_n]\)-type for \(n = 1, 2, \ldots, N\) as \(y \to \infty\)

(b) \((M - N)\)-line solitons of \([i_m, f_m]\)-type for \(m = 1, 2, \ldots, M - N\) as \(y \to -\infty\).

The set of those unbounded line solitons \([e_n, j_n]\) and \([i_m, f_m]\) are expressed by an unique derangement (a permutation of \(\{1, 2, 3, \ldots, M\}\) without any fixed point).

For an irreducible \(\Omega_I \in \text{Gr}(N, M)_{\geq 0}\), we can find its associated Le-diagram through the pipe dreams \[13, 22\]. A Le-Diagram is a Young diagram filling with + or ◦ in each box, which has the Le-property: there is no ◦ which has + above it AND its left. Then we have

**Theorem [22]:** There is a bijection between the set of irreducible Le-Diagrams and the set of derangements.

Therefore one can represent an irreducible Schubert cell \(\Omega_I \in \text{Gr}(N, M)_{\geq 0}\) with a Le-Diagram, and conversely. Then the corresponding soliton graph is obtained from the Le-Diagram. Much more detail can be found in \[13, 15\].

This paper is organized as follows. In next section, one describes the resonant structure by the superimposition of soliton graphs of Le-Diagrams. In section 3, several examples are given to illustrate the method in section 2. In section 4, we conclude the paper with several remarks.

## 2 Resonant Structure

In this section, one constructs the resonant structure. It is the superimposition of soliton graphs of the Le-Diagrams. For basic solutions of interaction between line
solitons and multi-kinks solitons, one refers to [5].

As in the KP-(II) case [1, 8], from the form of $\tau$-function (11) and (13), each line soliton is obtained by the balance between $H_J$ and $H_J'$ in the $xy$-plane and happens only at the boundaries of the dominant regions. The line solitons are obtained in adjacent regions of $xy$-plane contain $N - 1$ common phases and differ by only one single phase. Now, suppose $H_{i,j_2,j_3,\ldots,j_N}$ and $H_{j,j_2,j_3,\ldots,j_N}$ in (12) are adjacent regions. Then one has the boundary, i.e., the line soliton $[i, j]$-soliton [5]

$$v \approx \frac{c_j - c_i}{2} \left( \frac{\Omega_j - \Omega_i + \ln \frac{c_j}{c_i}}{2} - \ln \frac{\Omega_j - \Omega_i}{2} \right) \geq 0,$$

where

$$\Omega_j = \theta_j + \ln \left| \prod_{m=2}^{N} (c_j - c_{jm}) \right|, \quad \Omega_i = \theta_i + \ln \left| \prod_{m=2}^{N} (c_i - c_{jm}) \right|.$$

Also, when $\Omega_j - \Omega_i = -\frac{1}{2} \ln \frac{c_j}{c_i}$, this $[i, j]$-soliton has maximal value $(\sqrt{c_j} - \sqrt{c_i})^2$. When $c_1 \neq 0$, the line $[i, j]$-soliton graph (15) of MKP equation is similar to the KP equation. Similar to the case KP-(II) [11], it can be seen that the $[i, j]$-line soliton solution (15) has the wave vector

$$\vec{\Sigma}_{[i,j]} = (c_j - c_i, c_j^2 - c_i^2),$$

and can be measured in the counterclockwise sense from the $y$-axis, i.e.,

$$\tan \Phi_{[i,j]} = \frac{c_j^2 - c_i^2}{c_j - c_i} = c_i + c_j,$$

moreover, its velocity is given by

$$\vec{V}_{[i,j]} = \frac{c_j^2 + c_i c_j + c_j^2}{1 + (c_i + c_j)^2} (1, c_i + c_j),$$

and the frequency is given by

$$\Delta_{i,j} = c_j^3 - c_i^3 = (c_j - c_i)(c_i^2 + c_i c_j + c_j^2).$$

Similar to the KP-(II) equation [11], three line solitons can interact to form a trivalent vertex and satisfy the resonant conditions for wave number and frequency by (16) and (19) ($i < m < j$)

$$\vec{\Sigma}_{[i,j]} = \vec{\Sigma}_{[i,m]} + \vec{\Sigma}_{[m,j]}, \quad \Delta_{[i,j]} = \Delta_{[i,m]} + \Delta_{[m,j]}.$$

On the other hand, a multi-kinks solution can be obtained by $c_1 = 0$ [5]. In this case, each $[1, j]$-line soliton in (15) becomes kink front, i.e.,

$$v \approx \frac{c_j}{2} (1 - \tanh \frac{\Omega_j - \ln \prod_{m=2}^{N} c_{1m}}{2}) \rightarrow \begin{cases} c_j, & x \to -\infty, \\ 0, & x \to \infty, \end{cases}$$

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and forms the boundary of the multi-kinks solution. The front of the multi-kinks solution of (21) is defined as

$$\Omega_j - \ln \prod_{m=2}^{N} c_{1m} = 0. \tag{22}$$

We notice that in the MKP equation line solitons can pierce fronts of the multi-kinks solutions to form new boundaries by resonance when \(c_1 = 0\) \([5]\), which is different from the KP equation. The resonant structure of line solitons and the multi-kinks solutions of the MKP equation is the main purpose for our investigation.

To this end, one considers the equation (13). When \(c_1 = 0\), it is equivalent to deleting the first column vector of \(K\)-matrix. Therefore, we have to consider the

$$TNNG Gr(N,M-1) \geq 0.$$  

Also from (7) it’s known that

$$v = \partial_x \ln \tau_1^{(1)} - \partial_x \ln \tau_0^{(0)}.$$  

Our main observation is that when \(c_1 = 0\) the resonant structure of MKP equation is

$$Gr(N, M - 1)_{\geq 0} + Gr(N, M)_{\geq 0}, \tag{23}$$

i.e, the soliton graph is the superimposition of the soliton graphs of the Le-Diagrams of the corresponding totally non-negative Grassmannians \(Gr(N, M - 1)_{\geq 0}\) and \(Gr(N, M)_{\geq 0}\). We remark that \(Gr(N, M - 1)_{\geq 0}\) here may be not irreducible and then the corresponding permutation is not of derangement due to deleting the first column vector of \(K\)-matrix, i.e., it has some fixed points. Also, one can think the Le-Diagram of the corresponding \(Gr(N, M - 1)_{\geq 0}\) as the soliton graph for \(c_1 = 0\) obtained from (21) and the resonance (20) of line solitons (15) piercing the front (22).

Next, we describe how to construct the soliton graph given a \(K\)-matrix (irreducible in RREF) \(\in Gr(N, M)_{\geq 0}\). Firstly, one constructs the Le-Diagrams \(L_+^{(M)}\) (\(L_-^{(M)} = L\) and \(L_+^{(M)} = L^*\) in [13]) corresponding to \(Gr(N, M)_{\geq 0}\) for \(c_1 \neq 0\) when \(t \to \pm\infty\). The soliton graph is obtained similarly to the KP equation [11] [13]. Secondly, one constructs the soliton graphs for \(c_1 = 0\) as follows when \(t \to \pm\infty\).

- Delete the first column of the Le-Diagram \(L_+^{(M)}\) to get the Le-Diagram \(L_+^{(M-1)}\) corresponding to \(Gr(N, M - 1)_{\geq 0}\) and its corresponding permutation of \(\{2, 3, 4, \cdots, M\}\)

- Then from the corresponding permutation \(\pi_+ [N, M - 1] = (\pi_+ [N, M - 1])^{-1}\) one obtains the Le-Diagram \(L_-^{(M-1)}\) also by pipe dreams.

- When \(t \to \infty\), the soliton graph is obtained by superimposition of the soliton graphs of the Le-Diagrams \(L_+^{(M)}\) and \(L_+^{(M-1)}\).
• When $t \to -\infty$, the soliton graph is also obtained by superimposition of the soliton graphs of the Le-Diagrams $L_{(M)}^M$ and $L_{(M-1)}^M$.

One has two remarks here.

(1) The permutation $\pi_+^{[N, M - 1]}$ may be not a derangement, i.e., it has some fixed points, as mentioned previously. Therefore, the permutation $\pi_-^{[N, M - 1]}$ is not a derangement, either.

(2) If K-matrix is an element of the TNNG, then every $N \times N$ sub-determinant is non-negative. As a result, the matrix obtained by deleting the first column of K-matrix is still an element of the TNNG.

3 Examples

In this section, we give several examples to illustrate the procedure described in last section. For simplicity, we focus on the TNNG $Gr(2, 4)$ [11].

• One considers the $\tau$-function

$$\tau_5 = H_1 + aH_2 + bH_3 + cH_4 + dH_5,$$

where $a, b, c, d > 0$ and it corresponds the TNNG $Gr(1, 5)$. Then

$$\tau_{5x} = k_1 H_1 + k_2 aH_2 + k_3 bH_3 + k_4 cH_4 + k_5 dH_5.$$

So we have the solution of MKP-(II)

$$u = \partial_x \ln \frac{\tau_{5x}}{\tau_5} = \partial_x \ln \frac{k_1 H_1 + k_2 aH_2 + k_3 bH_3 + k_4 cH_4 + k_5 dH_5}{H_1 + aH_2 + bH_3 + cH_4 + dH_5}. \quad (24)$$

Please see the figure 1 for the soliton graphs and the corresponding Le-diagrams (or the permutations) when $t \to \pm \infty$. For $t \to \pm \infty$, one obtains unbounded line solitons (from left to right):

- for $y >> 0$, [2,5], [1,5]
- for $y << 0$, [1,2], [2,3], [3,4], [4,5].

The kink-fronts are [1, 2] and [1, 5], and the line solitons [3, 4], [4, 5] and [3, 5] form a resonant trivalent vertex [21]. Also, the line soliton [2, 5] pierce the front [1, 2] and then with the front [1, 5] form an unbounded terrace of height $k_5 = 1$. For $t << 0$, the unbounded line solitons are the same as $t >> 0$; moreover, we see that [1, 2], [1, 3], [1, 4] and [1, 5] form the multi-kink fronts and look like a refraction of light. On the other hand, the kink-front [2, 4] is the resonance of line solitons [3, 4] and [2, 3], and then form the quadrilateral of height $k_4 = 0.7$ with the kink-front [1, 4] and the line solitons [3, 4] and [4, 5].
piercing the fronts.
When comparing with the unbound line solitons in KP-(II) equation, i.e., \(|y| \to \infty\), we know that the line soliton \([2, 5]\) is the only difference, i.e., there is no such unbounded kink front in KP-(II) equation. It is explained by the corresponding Le-Diagrams. The permutations corresponding to \(t \to \infty\) are \(\pi_+[1, 5] = (23451)\) and \(\pi_+[1, 4] = (3452)\); moreover, the permutations corresponding to \(t \to -\infty\) are \(\pi_-[1, 5] = (51234) = (\pi_+[1, 5])^{-1}\) and \(\pi_-[1, 4] = (5234) = (\pi_+[1, 4])^{-1}\).

For general case, it is similar. One considers \((M \geq 4)\)

\[
\tau_M = H_1 + a_2 H_2 + a_3 H_3 + \cdots + a_M H_M,
\]

where \(a_2, a_3, a_4, \ldots, a_M > 0\) and it corresponds the TNNG \(Gr(1, M)\). Then

\[
\tau_{Mx} = k_1 H_1 + a_2 k_2 H_2 + a_3 k_3 H_3 + \cdots + a_M k_M H_M.
\]

Then we have the solution of MKP-(II)

\[
\frac{\partial}{\partial x} \ln \frac{\tau_{Mx}}{\tau_M} = \frac{\partial}{\partial x} \ln \frac{k_1 H_1 + a_2 k_2 H_2 + a_3 k_3 H_3 + \cdots + a_M k_M H_M}{H_1 + a_2 H_2 + a_3 H_3 + \cdots + a_M H_M}.
\]

(25)

For \(t \gg 0\), one obtains unbounded line solitons (from left to right):

- for \(y \gg 0\), \([2, M], [1, M]\)
- for \(y \ll 0\), \([1, 2], [2, 3], [3, 4], [4, 5], \ldots, [M - 1, M - 2], \ldots, [M - 1, M]\).

But \([2, M]\) and \([1, M]\) form a smaller unbounded terrace of height \(k_M\). For \(t \ll 0\), similarly, the unbounded line solitons are the same as \(t \gg 0\) and \([1, 2], [1, 3], [1, 4], \ldots, [1, M - 1]\) and \([1, M]\) form the multi-kink fronts. Also, there are \((M - 3)\) quadrilaterals of different heights coming from the resonance condition \((20)\), composed of line solitons piercing the fronts. The Le-Diagrams look like the figure 1. The permutations corresponding to \(t \to \infty\) are \(\pi_+[1, M] = (2345 \cdots M1)\) and \(\pi_+[1, M - 1] = (34 \cdots M2)\); moreover, the permutations corresponding to \(t \to -\infty\) are \(\pi_-[1, M] = (M12 \cdots M - 1) = (\pi_+[1, M])^{-1}\) and \(\pi_-[1, M - 1] = (M23 \cdots M - 1) = (\pi_+[1, M - 1])^{-1}\).

One notices here that in [23] there are also have the soliton graphs of interaction between kink-soliton and line soliton but their solution structure is different from the one \((7)\) represented by the Wronskian representations \((11)\) and \((13)\).

- The Grassmannian of T-type has the RREF

\[
K_T = \begin{bmatrix}
1 & 0 & -c & -d \\
0 & 1 & a & b
\end{bmatrix},
\]

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where $a, b, c$ and $d$ are positive numbers, and $ad - bc > 0$. It is of the most complicated TNNG in $Gr(2, 4)$ and the total number of non-zero minors $\Delta_J$ is six. The corresponding $\tau$-functions are

$$
\tau_T^{(0)} = (c_2 - c_1)H_1H_2 + a(c_3 - c_1)H_3H_1 + b(c_4 - c_1)H_4 + c(c_3 - c_2)H_2H_3 + d(c_4 - c_2)H_2H_4 + (ad - bc)(c_4 - c_3)H_3H_4,
$$

and

$$
\tau_T^{(1)} = (c_2 - c_1)c_1c_2H_1H_2 + a(c_3 - c_1)c_1c_3H_1H_3 + b(c_4 - c_1)c_1c_4H_4 + c(c_3 - c_2)c_2c_3H_2H_3 + d(c_4 - c_2)c_2c_4H_2H_4 + (ad - bc)(c_4 - c_3)c_3c_4H_3H_4.
$$

Please see the figure 2 for the soliton graphs and the corresponding Le-diagrams (or the permutations) when $t \rightarrow \pm \infty$. For $t \rightarrow \pm \infty$, one obtains unbounded line solitons or fronts (from left to right):

- for $y \gg 0$, $[3, 4], [2, 4], [2, 3], [1, 3]$;
- for $y \ll 0$, $[1, 3], [2, 4]$.

When $t \rightarrow \infty$, the kink-fronts are $[1, 2], [1, 3]$ and $[1, 4]$, and the line solitons $[3, 4], [2, 4]$ and $[2, 3]$ form a resonant trivalent vertex $[20]$. Also, the line solitons $[3, 4]$ and $[2, 3]$ pierce the fronts $[1, 4]$ and $[1, 3]$, and then with the line soliton $[2, 4]$ and front $[1, 3]$ form two unbounded terraces of heights $k_4 = 3.5$ and $k_3 = 2$, respectively. For $t \ll 0$, we also see that $[1, 2], [1, 3]$ and $[1, 4]$ form the multi-kink fronts. On the other hand, the kink-front $[3, 4]$ is the resonance of kink fronts $[2, 4]$ and $[2, 3]$, and then form the terrace of height $k_4 = 3.5$. We notice the front $[1, 2]$ has the height $k_2 = 1$.

When comparing with the unbound line solitons in KP-(II) equation, i.e., $y \rightarrow \pm \infty$, one knows that the fronts $[2, 3]$ and $[3, 4]$ in figure 2 are different, i.e., there are no such unbounded kink fronts in KP-(II) equation. It can be read off by the corresponding Le-Diagrams. The permutations corresponding to $t \rightarrow \infty$ are $\pi_+[2, 4](K_T) = (3412)$ and $\pi_+[2, 3](K_T) = (342)$; moreover, the permutations corresponding to $t \rightarrow -\infty$ are $\pi_-[2, 4](K_T) = (3412) = (\pi_+[2, 4](K_T))^{-1} = (3412)^{-1}$ and $\pi_-[2, 3](K_T) = (423) = (\pi_+[2, 3](K_T))^{-1} = (342)^{-1}$.

In [20], the authors constructed the solutions of MKP equation using the Kaup-Newell hierarchy, and then the resonance line solitons different from the Wronskian representation [7] were obtained by the Hirota method. Their Hirota constraint plays the similar role as the T-type. The relations between these T-type solitons of the MKP equation are not clear and need further investigation.
The Grassmannian of reduced T-type has the same RREF as $K_T$ but $ad - bc = 0$. Then the corresponding $\tau$-functions are
\[
\tau_T^{(0)} = (c_2 - c_1)H_1H_2 + a(c_3 - c_1)H_3H_1 + b(c_4 - c_1)H_4H_1 + c(c_3 - c_2)H_2H_3
+ d(c_4 - c_2)H_2H_4,
\]
and
\[
\tau_T^{(1)} = (c_2 - c_1)c_1c_2H_1H_2 + a(c_3 - c_1)c_1c_3H_3H_1 + b(c_4 - c_1)c_1c_4H_4H_1
+ c(c_3 - c_2)c_2c_3H_2H_3 + d(c_4 - c_2)c_2c_4H_2H_4.
\]

Please see the figure 3 for the soliton graphs and the corresponding Le-diagrams when $t \to \pm \infty$. For $t \to \pm \infty$, one obtains unbounded line solitons or fronts (from left to right):
- for $y >> 0$, $\{3,4\}, \{2,4\}, \{1,2\}$;
- for $y << 0$, $\{1,3\}, \{3,4\}$

When $t \to \infty$, the kink-fronts are $\{1,2\}, \{1,3\}$ and $\{1,4\}$, and the fronts $\{1,2\}$, $\{1,4\}$ and $\{2,4\}$ form a resonant trivalent vertex (20). Also, the line soliton $\{3,4\}$ pierces the front $\{1,4\}$, and then with the front $\{2,4\}$ forms an unbounded terrace of heights $k_4 = 3.5$. For $t << 0$, the fronts $\{1,2\}, \{1,3\}$ and $\{2,3\}$ form a resonant trivalent vertex, and then we see the kink-front $\{2,4\}$ is the resonance of kink front $\{2,3\}$ and the line soliton $\{3,4\}$ piercing the front $\{1,2\}$. We notice the interaction between the front $\{1,2\}$ and the line soliton $\{3,4\}$ is of O-type [5].

Comparing with the unbound line solitons in KP-(II) equation for $y \to \pm \infty$, we know that only the front $\{3,4\}$ in figure 3 is different. It also can be read off by the corresponding Le-Diagrams. The permutations corresponding to $t \to \infty$ are $\pi_+ [2,4] (K_T^1) = (3142)$ and $\pi_+ [2,3] (K_T^1) = (243)$; moreover, the permutations corresponding to $t \to -\infty$ are $\pi_- [2,4] (K_T^1) = (2413) = (\pi_+ [2,4] (K_T^1))^{-1}$ and $\pi_- [2,3] (K_T^1) = (243) = (\pi_+ [2,3] (K_T^1))^{-1} = (243)^{-1}$. We notice here that the permutations $\pi_+ [2,3] (K_T^1) = (243)$ and $\pi_- [2,3] (K_T^1) = (243)$ both have the fixed point ”2”.

The Grassmannian of Mach-type has the RREF
\[
K_M = \begin{bmatrix}
1 & a & 0 & -c \\
0 & 0 & 1 & b
\end{bmatrix},
\]
where $a, b$ and $c$ are positive numbers. The corresponding $\tau$-functions are
\[
\tau_M^{(0)} = (c_3 - c_1)H_3H_1 + b(c_4 - c_1)H_4H_1 + a(c_3 - c_2)H_2H_3
+ ab(c_4 - c_2)H_2H_4 + c(c_4 - c_3)H_3H_4,
\]
\[ \tau_M^{(1)} = (c_3 - c_1)c_1c_3H_1H_3 + b(c_4 - c_1)c_1c_4H_1H_4 + a(c_3 - c_2)c_2c_3H_2H_3 + ab(c_4 - c_2)c_2c_4H_2H_4 + c(c_4 - c_3)c_3c_4H_3H_4. \]

Please see the figure 4 for the soliton graphs and the corresponding Le-diagrams when \( t \to \pm \infty \). For \( t \to \pm \infty \), one obtains unbounded line solitons or fronts (from left to right):

- for \( y >> 0 \), \([3, 4], [2, 3], [1, 3]\);
- for \( y << 0 \), \([1, 2], [2, 4]\).

When \( t \to \infty \), the kink-fronts are \([1, 2]\) and \([1, 3]\), and the line solitons \([3, 4]\), \([2, 4]\) and \([2, 3]\) form a resonant trivalent vertex \([20]\). Also, the line soliton \([2, 3]\) pierces the front \([1, 3]\), and then with the front \([1, 3]\) forms an unbounded terrace of height \( k_3 = 1.4 \); moreover, the interaction between the front \([1, 2]\) and the line soliton \([3, 4]\) is of O-type \([5]\).

For \( t << 0 \), we see that \([1, 2], [1, 3]\) and \([1, 4]\) form the multi-kink fronts. On the other hand, the kink-front \([1, 4]\) is the resonance of kink front \([1, 2]\) and the line soliton \([2, 4]\), and then form the bounded terrace of height \( k_4 = 2.5 \) which has the highest amplitude. We notice the front \([1, 2]\) has the height \( k_2 = 1 \).

Comparing with the unbound line solitons in KP-(II) equation for \( y \to \pm \infty \), one knows that the front \([2, 3]\) in figure 4 is different. It can be read off by the corresponding Le-Diagrams. The permutations corresponding to \( t \to \infty \) are \( \pi_+ [2, 4](K_M) = (2413) \) and \( \pi_+ [2, 3](K_M) = (423) \); moreover, the permutations corresponding to \( t \to -\infty \) are \( \pi_- [2, 4](K_M) = (3142) = (\pi_+ [2, 4](K_T))^{-1} = (2413)^{-1} \) and \( \pi_- [2, 3](K_T) = (342) = (\pi_+ [2, 3](K_M))^{-1} = (423)^{-1} \).

### 4 Concluding Remarks

In this article, using the Wronskian structure of \( \tau \)-functions, we investigate the resonant structure of kink-solitons and line solitons by the Le-Diagrams given an irreducible Schubert cell \( \Omega_I \in Gr(N, M)_{\geq 0} \). It turns out that the soliton graph comes from the superimposition of the soliton graphs of the Le-Diagrams of the corresponding totally non-negative Grassmannians \( Gr(N, M - 1)_{\geq 0} \) and \( Gr(N, M)_{\geq 0} \). To illustrate the construction, one gives several examples focusing on \( Gr(2, 4)_{\geq 0} \) for simplicity. Also, we make a comparison with the KP equation when \( c_1 = 0 \), and the main difference can be described by the Le-Diagram of \( Gr(N, M - 1)_{\geq 0} \). Some line solitons may pierce fronts to form bounded terraces and unbounded terraces with the kink solitons or by resonance with the kink solitons.
In addition, the Mach type kink soliton $K_M$ could be interesting \cite{3,11,14}. It is known that the Mach stem wave, the bounded terrace of height $k_4$ in figure 4, has the highest amplitude and it is the resonance of kink soliton \cite{1,3} (incidence wave) and line soliton \cite{3,4} (reflected wave) \cite{14}. There is a critical angle between O-type soliton and Mach type soliton in the KP-(II) equation \cite{14} or the Veselov-Novikov equation \cite{3}. The critical angle depends on the initial angle of V-shape. It could be noteworthy to find the critical angle and its relation with the amplitude. Besides, given a portion of a permutation or a partial chord diagram as the initial value problem, one has to consider a completion of this partial chord diagram for convergence \cite{11}. Completion means adding other solitons (kink solitons) or chords completes the initial diagram. The completion may not be unique. In \cite{11}, it is proposed a concept of minimal completion in the sense that the completed diagram has the minimal length of the chords and the corresponding TNNG cell has the minimal dimension. It is very interesting to investigate the minimal completion for the line solitons or kink solitons in the MKP equation even though there is the Miura transformation between KP equation and MKP equation. Also, the condition \cite{14} can’t be used to get the solitons for the Modified KdV (MKdV) equation \cite{10}. On the other hand, by the complex conjugate pairs of $c_i$, breather solutions in Wronskian form for MKdV equation are also constructed in \cite{24,25}, and there exist breather solutions for MKP equation \cite{17,18}. One can hope the complex Grassmannian theory could be generalized to the breather solutions using the correspondence between the Wronskian and Grammian forms of $\tau$-function \cite{2,9}. Finally, we notice that the two-solitons solution of Kaup-Newell equation, a reduction of MKP equation, corresponds to the T-type soliton of MKP equation \cite{20}. In \cite{21}, the constrained MKP flows is constructed and then one has the the reduction of MKP hierarchy. The solitons solution can be constructed using the Hirota bilinear method but may be not in Wronskian representation. On the other hand, the T-type soliton is generalized in \cite{12} (self-dual $\tau$-function). The relations between the reduction in \cite{21} and the corresponding Schubert cell of self-dual $\tau$-function in \cite{12} would be interesting. These issues will be published elsewhere.

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Figure 1: Refraction \( k_1 = 0, k_2 = 0.2, k_3 = 0.5, k_4 = 0.7, k_5 = 1, a = 10, b = 40, c = 20, d = 30 \)
Figure 2: $a = 10, b = 20, c = 30, d = 70, k_1 = 0, k_2 = 1, k_3 = 2, k_4 = 3.5$
Figure 3: $a = 10, b = 20, c = 30, d = 60, k_1 = 0, k_2 = 1, k_3 = 2, k_4 = 3.5$
Figure 4: \( a = 100, b = 400, c = 500, k_1 = 0, k_2 = 1, k_3 = 1.4, k_4 = 2.5 \)