Formally Unimodular Packings for the Gaussian Wiretap Channel

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Abstract—This paper introduces the family of lattice-like packings, which generalizes lattices, consisting of packings possessing periodicity and geometric uniformity. The subfamily of formally unimodular (lattice-like) packings is further investigated. It can be seen as a generalization of the unimodular and isodual lattices, and the Construction A formally unimodular packings obtained from formally self-dual codes are presented. Recently, lattice coding for the Gaussian wiretap channel has been considered. A measure called the secrecy function was proposed to characterize the eavesdropper’s probability of correctly decoding. The aim is to determine the global maximum value of the secrecy function, called (strong) secrecy gain. We further apply lattice-like packings to coset coding for the Gaussian wiretap channel and show that the family of formally unimodular packings shares the same secrecy function behavior as unimodular and isodual lattices. We propose a universal approach to determine the secrecy gain of a Construction A formally unimodular packing obtained from a formally self-dual code. From the weight distribution of a code, we provide a necessary condition for a formally self-dual code such that its Construction A formally unimodular packing is secrecy-optimal. Finally, we demonstrate that formally unimodular packings/lattices can achieve higher secrecy gain than the best-known unimodular lattices.

Index Terms—Lattices, nonlattice packings, construction A, secrecy gain, Gaussian wiretap channel.

I. INTRODUCTION

In recent years, physical layer security, that only utilizes the resources at the physical layers of the transmission parties and provides information-theoretically unbreakable security, has been recognized as an appealing technique for safeguarding confidential data in 5G and beyond 5G wireless communication systems (see [2], [3], [4] and references therein). The root of this line of research goes back to the epochal work wiretap channel (WTC) model introduced by Wyner [5], which showed that reliable and secure communication can be achieved simultaneously without the need of an additional cryptographic layer on top of the communication protocol.

Since then, substantial research efforts have been devoted to developing practical codes for reliable and secure data transmission over WTCs. Potential candidates for practical wiretap code constructions include low density parity check (LDPC) codes [6], polar codes [7], [8], and lattices [9], [10], [11]. In [9] and [10] it was shown that a lattice-based coset coding approach can provide secure and reliable communication over the Gaussian WTC. In particular, it was shown that for Gaussian WTC, the so-called secrecy function expressed in terms of the theta series of a lattice (see the precise definition in Section IV) can be considered as a quality criterion of good wiretap lattice codes: to minimize the eavesdropper’s probability of correct decoding, one needs to maximize the secrecy function, and the corresponding maximum value is referred to as (strong) secrecy gain. Moreover, Ling et al. have also proposed another design criterion for wiretap lattice codes, called the flatness factor, to quantify how much confidential information can leak to Eve in terms of mutual information [11]. To guarantee secrecy-goodness, both the criteria of secrecy gain and the flatness factor require small theta series of the designed Eve’s lattice $\Lambda_e$ at a particular point.

In [12], Belfiore and Solé discovered that the secrecy functions of unimodular lattices have a symmetry point. The value of the secrecy function at this point is called the weak secrecy gain. Based on this, the authors conjectured that for unimodular lattices, the secrecy gain is achieved at the symmetry point of its secrecy function. I.e., the strong secrecy gain of a unimodular lattice is equivalent to its weak secrecy gain. Finding good unimodular lattices that attain large secrecy gain is of practical importance. The Belfiore and Solé conjecture has also been extended to isodual lattices in [10], which includes unimodular lattices as a special case. In [13], a novel technique was proposed to verify or disprove the Belfiore and Solé conjecture for a given unimodular lattice. Using this method, the conjecture is validated for all known extremal unimodular lattices in even dimensions less than 80.

In another work [14], the authors use a similar method as [13] to classify the best unimodular lattices in dimensions from 8 to 23. For unimodular lattices obtained by Construction A from binary doubly even self-dual codes (the so-called type II codes, where their weights of all the codewords are multiple of four) up to dimensions 40, their secrecy gains are also shown to be achieved at their symmetry points [15]. Recently, the analysis of secrecy gain has been extended to the family of...
isodual lattices [16], [17], [18], which is equivalent to the iso-
dual lattices for the case of \( \ell = 1 \). It is expected that one can achieve a better secrecy gain by using an \( \ell \)-modular lattice with a higher parameter \( \ell \).

In coding theory, the notion of formally self-dual codes apply for either linear or nonlinear codes, and it is known that this broader class of codes possesses several better features than isodual and self-dual linear codes. For instance, the Nordstrom-Robinson code is formally self-dual [19, Ch. 19], and it has a larger minimum Hamming distance than any isodual codes of length 16. In our earlier work [1], a new and wider family of lattices, referred to as formally unimodular lattices, that consists of lattices having the same theta series as their dual, has been introduced. It was shown that formally unimodular lattices have the same symmetry point as unimodular and isodual lattices. In this work, we further define a larger family of packings (either lattice or nonlattice packings), called formally unimodular packings, that contains formally unimodular lattices and also enjoys the same secrecy function properties as unimodular and isodual lattices. Our main contributions are summarized as follows:

- Due to the assumptions of periodicity and geometric uniformity for general packings (not necessarily lattices), we introduce the concept of lattice-like packings, where their volumes and theta series are well defined, and they have congruent Voronoi regions.

- We show how to construct the formally unimodular packings via Construction A from geometrically uniform formally self-dual codes that contain the all-zero vector (see Theorem 20). Moreover, we briefly discuss how to obtain Construction A lattice-like packings from codes through Gray map.

- The coset coding scheme, originally for two nested lattices \( \Lambda_c \subset \Lambda_b \), is generalized to the setup where \( \Lambda_b \) is a lattice and its subset \( \Gamma_c \) is a lattice-like packing. The challenge to construct such a coset decomposition for this setup is briefly discussed. Furthermore, we derive analytical results on the eavesdropper’s probability of correct decision. Under certain assumptions about our proposed lattice-like packings, we show that the performance of secrecy gain can be determined by the theta series of \( \Gamma_c \) (see Theorem 29), Construction A lattice-like packings do satisfy this particular assumption (see Remark 30).

- Hence, the secrecy gain criterion keeps unchanged even we consider Construction A lattice-like packings for Eve.

- We show that the formally unimodular packings possess the same secrecy function behavior as the isodual lattices (see Theorem 34). I.e., both formally unimodular packings and isodual lattices have the symmetry point at 1.

- A universal approach to determine the strong secrecy gain of formally unimodular packings is provided (see Theorem 40). For Construction A formally unimodular packings obtained from even formally self-dual codes, we also provide a sufficient condition to verify the Belfiore and Solé conjecture on the secrecy gain (see Theorem 45).

- By using the weight distribution of a code, we provide a new necessary condition to verify the secrecy-optimality of a Construction A formally unimodular packing obtained from a formally self-dual code (see Theorem 50). Naturally, one would anticipate that a formally self-dual code with a large minimum Hamming distance and a low number of low-weight codewords should lead to high secrecy gain of the corresponding Construction A formally unimodular packing. However, we present two counterexamples to disprove this observation.

- To have a more thorough secrecy performance comparison between Construction A formally unimodular packings obtained from formally self-dual codes, we provide a systematic approach through tail-biting the rate \( 1/2 \) binary convolutional codes to construct isodual codes (see Section VII).

Finally, numerical results are presented to validate our theoretical findings for strong secrecy gain of Construction A packings/lattices obtained from formally self-dual codes. For dimensions up to 70, we note that formally unimodular packings/lattices have better secrecy gain than the best-known unimodular lattices described in the literature, e.g., [14]. In addition to applying good formally self-dual codes from the literature, we also demonstrate high secrecy gains of the newly presented isodual codes by tail-biting the rate \( 1/2 \) convolutional codes.

II. DEFINITIONS AND PRELIMINARIES

A. Notation

We denote by \( \mathbb{Z}, \mathbb{Q}, \) and \( \mathbb{R} \) the set of integers, rationals, and reals, respectively. Moreover, \( \mathbb{Z}_{\geq 0} \) denotes the nonnegative integers, and \([a : b] \triangleq \{a, a + 1, \ldots, b\}\) for \( a, b \in \mathbb{Z}, \) \( a \leq b \). A binary field is denoted by \( \mathbb{F}_2 \triangleq \{0, 1\} \). Vectors are bold-faced, e.g., \( x, \langle x, y \rangle \) denotes the inner product between vectors \( x \) and \( y \) over \( \mathbb{F}_2 \) or \( \mathbb{R} \). \( \oplus \) represents the element-wise addition over \( \mathbb{F}_2 \). Matrices and sets are represented by capital sans serif letters and calligraphic uppercase letters, respectively, e.g., \( X \) and \( \mathcal{X} \). The Hamming, the Lee, and the Euclidean are the distance metrics attributed to codes over \( \mathbb{F}_2 \), codes over \( \mathbb{Z}_4 \triangleq \{0, 1, 2, 3\} \), and packings, respectively. Any binary \((n, M)\) or \((n, M, d)\) code \( C \) (linear or nonlinear) is a subset of \( \mathbb{F}_2^n \) with \( M \) codewords and minimum Hamming distance \( d \). In case \( C \) is a linear subspace of dimension \( k \) of \( \mathbb{F}_2^n \), then \( C \) is said to be linear, with parameters \([n, k]\) or \([n, k, d]\). The Hamming weight of a codeword \( x \in \mathbb{F}_2^n \) is denoted by \( w_H(x) \), and \( ||x|| \) represents the Euclidean norm of a vector \( x \in \mathbb{R}^n \). \( \phi: \mathbb{F}_2^n \rightarrow \mathbb{Z}_4^n \) is defined as the natural embedding, i.e., the elements of \( \mathbb{F}_2 \) are mapped to the respective integer by \( \phi \) element-wisely.

B. Codes and Packings

This paper focuses on the relation between binary codes and packings. These binary codes can be linear or not,
the packings can be lattices or not. We start by presenting the definitions relevant to our study.

The weight enumerator of an \((n, M)\) code \(C \subseteq \mathbb{F}_2^n\) is

\[
W_C(x, y) = \sum_{w \in C} w x^{n-w} y^w,
\]

\[
= \sum_{w=0} A_w(C) x^{n-w} y^w, \tag{1}
\]

where \(A_w(C) \triangleq |\{e \in C : w_1(e) = w\}|, w \in \{0 : n\}.

For an \([n, k]\) code \(C\), the dual code of \(C\) is the \([n, n-k]\) code \(C^\perp \triangleq \{u \in \mathbb{F}_2^n : \langle u, v \rangle = 0, \forall v \in C \}\).

**Definition 1 (Self-Dual and Isodual):** Let \(C\) be an \([n, k]\) code. Then

- \(C\) is said to be self-dual if \(C = C^\perp\).
- If there is a permutation \(\pi\) of coordinates such that \(C = \pi(C^\perp)\), \(C\) is called isodual.

For an \([n, k]\) code \(C\), the relation between \(W_C(x, y)\) and \(W_C^\perp(x, y)\) is characterized by the well-known MacWilliams identity (see, e.g., [19, Th. 1, Ch. 5]):

\[
W_C(x, y) = \frac{1}{|C|} W_{C^\perp}(x + y, x - y)
\]

\[
= \frac{1}{2^{n-k}} W_{C^\perp}(x + y, x - y). \tag{2}
\]

**Remark 2:** The MacWilliams identity as expressed in (2) relies on the notion of duality, which was defined above only for linear codes. This concept can be generalized for nonlinear codes in terms of the transform of a code. To simplify the presentation we omit further details on transforms, since our focus is on formally self-dual codes only, in which case the transform is implicitly defined. For the generalization, we refer the reader to [19, pp. 132–141].

**Definition 3 (Formally Self-Dual):** Given an \((n, M)\) code \(C\), we say it is formally self-dual if and only if

\[
W_C(x, y) = \frac{1}{M} W_{C^\perp}(x + y, x - y). \tag{3}
\]

Next, we prove that the code size \(M\) of a formally self-dual code should be \(2^{n/2}\).

**Proposition 4:** If an \((n, M)\) code is formally self-dual, then \(M = 2^{n/2}\).

**Proof:** Because \(C\) is formally self-dual, then

\[
W_C(x, y) \overset{(3)}{=} \frac{1}{M} \frac{1}{M} W_{C^\perp}(x + y, x - y)
\]

\[
= \frac{1}{M^2} W_{C^\perp}(2x + 2y, 2y) = \frac{2^n}{M^2} W_{C^\perp}(x, y),
\]

which implies that \(\frac{2^n}{M^2} = 1\) and \(M = 2^{n/2}\). \(\blacksquare\)

Thus, similar to self-dual and isodual codes, the weight enumerator \(W_C(x, y)\) of a formally self-dual code satisfies [19, eq. (7), p. 599]

\[
W_C(x, y) = W_{C^\perp} \left( \frac{x + y}{\sqrt{2}}, \frac{x - y}{\sqrt{2}} \right). \tag{4}
\]

In general, formally self-dual codes constitute a broader family of codes than isodual and self-dual codes.

**A packing** \(\Gamma \subset \mathbb{R}^n\) is an infinite discrete set. An isometry \(T\) is a transformation that preserves distance, with respect to a certain metric. We call a packing \(\Gamma\) geometrically uniform if for any two elements \(x, x' \in \Gamma\), there exists an isometry \(T_{x, x'}\) such that \(x' = T_{x, x'}(x)\) and \(T_{x, x'}(\Gamma) \triangleq \{T_{x, x'}(x) : x \in \Gamma\} = \Gamma\). A geometrically uniform packing \(\Gamma\) is also distance-invariant, which means that the number of elements of \(\Gamma\) at a distance \(d\) from an element \(x \in \Gamma\) is independent of \(x\).

The Voronoi region of a point \(x\) in a packing \(\Gamma \subset \mathbb{R}^n\) is defined as

\[
V_x(\Gamma) = \{y \in \mathbb{R}^n : \|y - x\|^2 \leq \|y - x'\|^2, \forall x' \in \Gamma\}.
\]

All Voronoi regions of geometrically uniform packings have the same shape [20], that we will refer to as \(V(\Gamma)\).

A packing that has a group structure is called a lattice, i.e., a (full rank) lattice \(\Lambda\) is a discrete additive subgroup of \(\mathbb{R}^n\), which is generated as

\[
\Lambda = \{\lambda = ul_n : u = (u_1, \ldots, u_n) \in \mathbb{Z}^n\},
\]

where the \(n\) rows of \(L\) form a lattice basis. The volume of \(\Lambda\) is \(\text{vol}(\Lambda) = |\det(L)|\). A lattice is an example of a geometrically uniform packing.

For lattices, the analogue of the weight enumerator is the theta series.

**Definition 5 (Theta Series):** Let \(\Lambda \subset \mathbb{R}^n\) be a lattice, its theta series is given by

\[
\Theta_\Lambda(z) = \sum_{\lambda \in \Lambda} q^{\|\lambda\|^2}, \tag{5}
\]

where \(q \triangleq e^{i\pi z}\) and \(\text{Im} \{z\} > 0\).

The theta series converges uniformly absolutely for all \(z\) such that \(\text{Im} \{z\} > 0\) [21, Lemma 2.2, pp. 39–40]. Note that sometimes the theta series of a lattice can be expressed in terms of the Jacobi theta functions defined as follows.

\[
\vartheta_2(z) \triangleq \sum_{m \in \mathbb{Z}} q^{(m+\frac{1}{2})^2} = \Theta_{2, \frac{1}{2}}(z),
\]

\[
\vartheta_3(z) \triangleq \sum_{m \in \mathbb{Z}} q^{m^2} = \Theta_{2, 0}(z), \quad \vartheta_4(z) \triangleq \sum_{m \in \mathbb{Z}} (-q)^{m^2}.
\]

**Example 6:** Consider \(\Lambda = \mathbb{Z}^n\). Then

\[
\Theta_{\mathbb{Z}^n}(z) = \sum_{\lambda \in \mathbb{Z}^n} q^{\|\lambda\|^2} = \sum_{\lambda \in \mathbb{Z}^n} q^{\lambda_1^2 + \cdots + \lambda_n^2}
\]

\[
= \left(\sum_{\lambda_1 \in \mathbb{Z}} q^{\lambda_1^2}\right) \cdots \left(\sum_{\lambda_n \in \mathbb{Z}} q^{\lambda_n^2}\right) = \vartheta_3^n(z). \quad \blacksquare
\]

Some theta series identities [22, p. 104] will be useful in the course of this paper, such as

\[
\vartheta_3 \left( \frac{1}{z} \right) = \vartheta_3(z), \quad \vartheta_4 \left( \frac{1}{z} \right) = \vartheta_4(z) \tag{6}
\]

\[
\vartheta_3(z) + \vartheta_2(z) = \vartheta_3 \left( \frac{z}{4} \right), \quad \vartheta_3(z) - \vartheta_2(z) = \vartheta_1 \left( \frac{z}{4} \right) \tag{7}
\]

\[
\vartheta_3^2(z) + \vartheta_4^2(z) = 2\vartheta_3 \vartheta_4(2z), \quad \vartheta_3^2(z) - \vartheta_4^2(z) = 2\vartheta_3^2(2z). \tag{8}
\]
For a lattice \( \Lambda \), we have similar concepts to self-dual and isodual codes. If a lattice \( \Lambda \) has generator matrix \( L \), then the lattice \( \Lambda^* \subset \mathbb{R}^n \) generated by \( (L^{-1})^T \) is called the dual lattice of \( \Lambda \).

**Remark 7:** \( \text{vol}(\Lambda^*) = \text{vol}(\Lambda)^{-1} \).

**Definition 8 (Unimodular and Isodual Lattices):** A lattice \( \Lambda \subset \mathbb{R}^n \) is said to be integral if the inner product of any two lattice vectors is an integer.

- An integral lattice such that \( \Lambda = \Lambda^* \) is a unimodular lattice.
- A lattice \( \Lambda \) is called isodual if it can be obtained from its dual \( \Lambda^* \) by (possibly) a rotation or reflection.

Analogously, the spirit of the MacWilliams identity can be captured by the Jacobi’s formula [22, eq. (19), Ch. 4] for a lattice \( \Lambda \):

\[
\Theta_{\Lambda}(z) = \text{vol}(\Lambda^*) \left( \frac{1}{z} \right)^2 \Theta_{\Lambda^*} \left( -\frac{1}{z} \right). \tag{9}
\]

A packing \( \Gamma \) is said to be periodic if it is a union of translates of a lattice, i.e., \( \Gamma = \bigcup_{j=1}^K (u_j + \Lambda) \), where \( \Lambda \) is a lattice and \( u_1, \ldots, u_K \subset \mathbb{R}^n \). Given a periodic packing, we define its volume as \( \text{vol}(\Gamma) = \text{vol}(\Lambda)/K \).

In general, there is no systematic way to derive the theta series of a nonlattice packing. However, it is known that the theta series of periodic packings \( \Gamma \) (not necessarily lattices) can be expressed as follows.

**Proposition 9 [23]:** Let \( \Gamma \) be a periodic packing. Then

\[
\Theta_{\Gamma}(z) = \frac{1}{K} \sum_{j=1}^K \sum_{\lambda \in \Lambda} \sum_{i=1}^K q^{||\lambda + u_j - u_i||^2} = \Theta_{\Lambda}(z) + \frac{2}{K} \sum_{j<l} \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} q^{||\lambda + u_j - u_i||^2}. \tag{10}
\]

If \( \Gamma \) is distance-invariant, then the theta series in (10) reduces to

\[
\Theta_{\Gamma}(z) = \sum_{j=1}^K \sum_{\lambda \in \Lambda} q^{||\lambda + u_j - u_i||^2}. \tag{11}
\]

Throughout this paper we work with lattices and a slightly larger class that we denote by lattice-like packings.

**Definition 10 (Lattice-Like Packing):** A packing \( \Gamma \) is said to be lattice-like if the following two properties hold:

- **Periodicity:** \( \Gamma \) is periodic, so it has its volume and theta series well defined.
- **Geometric uniformity:** \( \Gamma \) is assumed to be geometrically uniform, so that it has congruent Voronoi regions [20, Th. 1].

Lattice-like packings are illustrated by the shaded area in Fig. 1. Observe that a lattice is an example of a lattice-like packing. Consequently, the family of lattice-like packings can be seen as a generalization of lattices. In Example 11, we present an example of a lattice-like packing.

**Example 11:** Consider a \( \Gamma = \bigcup_{\ell \in \mathcal{C}} (\mathbf{e} + 4\mathbb{Z}^2) \subset \mathbb{R}^2 \), where \( \mathcal{C} = \{(0,0), (1,1)\} \). \( \Gamma \) is clearly periodic and it is also geometrically uniform according to [24, Th. 2]. Fig. 2 illustrates such a packing and its congruent Voronoi regions.

In Section II-C, we will show how to construct lattice-like packings.

The theta series of a lattice-like packing is obtained according to (11), due to the geometric uniformity. Mimicking the definition of formally self-dual codes, we define formally unimodular packings.

**Definition 12 (Formally Unimodular Lattice-Like Packings):**

We say that a lattice-like packing \( \Gamma \) is formally unimodular if and only if

\[
\Theta_{\Gamma}(z) = \text{vol}(\Gamma) \left( \frac{1}{z} \right)^\gamma \Theta_{\Gamma} \left( -\frac{1}{z} \right). \tag{12}
\]

The class is represented by an inner circle in Fig. 1, while the hashed area represents packings obtained via Construction A, to be discussed in a sequel. The latter are the ones we will work closely in this paper. Note that in our prior work [1], we have introduced a smaller class of formally unimodular lattices, where we only considered the lattices that have the same theta series as their dual and, thus, satisfy (12). Throughout out this paper, we will usually call formally unimodular lattice-like packings by formally unimodular packings for simplicity.

Analogous to formally self-dual codes, we show that \( \text{vol}(\Gamma) = 1 \) if \( \Gamma \) is formally unimodular.

**Proposition 13:** If a lattice-like packing \( \Gamma \) is formally unimodular, then \( \text{vol}(\Gamma) = 1 \).
TABLE I

| Code $\mathcal{C} \subseteq \mathbb{F}_2^n$ | Lattice-like Packing $\Gamma \subset \mathbb{R}^n$ |
|-----------------------------------------|---------------------------------------------|
| Linear code                             | Lattice                                     |
| Hamming distance                        | Euclidean distance                           |

**Definition 14 (Construction A [22, p. 137]):** Let $\mathcal{C}$ be an $(n, M)$ code, then a Construction A packing is defined as

$$\Gamma_\mathcal{C} \triangleq \frac{1}{\sqrt{2}} (\mathbf{c}(\mathcal{C}) + 2\mathbb{Z}^n) = \frac{1}{\sqrt{2}} \bigcup_{\mathbf{c} \in \mathcal{C}} (\mathbf{c} + 2\mathbb{Z}^n).$$

**Lemma 17 (22, Th. 3, Ch. 7):** Consider an $[n, k]$ code $\mathcal{C}$ with $W_{\mathcal{C}}(x, y)$ of a linear code $\mathcal{C}$ and a lattice $\Lambda_\mathcal{C}(\mathcal{C}).$

**Remark 15:**

- $\Lambda_\mathcal{C}(\mathcal{C})$ is a lattice if and only if the code $\mathcal{C}$ is an $[n, k]$ code. Also, $\text{vol}(\Lambda_\mathcal{C}(\mathcal{C})) = 2^{n-k}.$
- $\Lambda_\mathcal{C}(\mathcal{C})$ is, by definition, a periodic packing for any $(n, M)$ code $\mathcal{C}$, and $\text{vol}(\Gamma_\mathcal{C}(\mathcal{C})) = 2^{n/2}/M.$
- If and only if $\mathcal{C}$ is geometrically uniform (with respect to the Hamming metric), then $\Lambda_\mathcal{C}(\mathcal{C})$ is geometrically uniform (with respect to the Euclidean metric), and consequently, a lattice-like packing.

**Example 16:** Consider a formally unimodular packing $\mathcal{C}$, then $\Lambda_\mathcal{C}(\mathcal{C})$ is a composition of isometries, $\Gamma_\mathcal{C}(\mathcal{C})$ is geometrically uniform.

**III. Theta Series of Construction A Lattice-like Packings**

**Lemma 17** gives a connection between the weight enumerator $W_{\mathcal{C}}(x, y)$ of a linear code $\mathcal{C}$ and a lattice $\Lambda_\mathcal{C}(\mathcal{C}).$

**Remark 15:**

- $\Lambda_\mathcal{C}(\mathcal{C})$ is a lattice if and only if the code $\mathcal{C}$ is an $[n, k]$ code. Also, $\text{vol}(\Lambda_\mathcal{C}(\mathcal{C})) = 2^{n-k}.$
- $\Lambda_\mathcal{C}(\mathcal{C})$ is, by definition, a periodic packing for any $(n, M)$ code $\mathcal{C}$, and $\text{vol}(\Gamma_\mathcal{C}(\mathcal{C})) = 2^{n/2}/M.$
- If and only if $\mathcal{C}$ is geometrically uniform (with respect to the Hamming metric), then $\Lambda_\mathcal{C}(\mathcal{C})$ is geometrically uniform (with respect to the Euclidean metric), and consequently, a lattice-like packing.  

**Example 16:** Consider a formally unimodular packing $\mathcal{C}$, then $\Lambda_\mathcal{C}(\mathcal{C})$ is a composition of isometries, $\Gamma_\mathcal{C}(\mathcal{C})$ is geometrically uniform.

![Fig. 3. A lattice-like packing $\Gamma_A(\mathcal{C}) \subset \mathbb{R}^2$, where $\mathcal{C} = \{(0, 1), (1, 0)\}$.](image-url)
given by
\[ \Theta_{\Lambda}(\psi) (z) = W_{\psi}(\vartheta_3(2z), \vartheta_2(2z)). \]

**Remark 18:** It follows immediately from Lemma 17 that if an \([n,n/2]\) code \(C\) is formally self-dual then \(\Lambda(C)\) is a formally unimodular lattice.

We will now show that the property in Lemma 17 holds for lattice-like Construction A packings \(\Lambda(C)\), conditioned on some assumptions on the underlying \((n,M)\) code \(C\).

**Lemma 19:** Let \(C\) be an \((n,M)\) geometrically uniform code where \(0 \in C\). Also, let \(\Lambda(C) = \frac{1}{\sqrt{2}}(\phi(C) + 2Z^n)\) be a lattice-like packing generated via Construction A. Then its theta series is
\[ \Theta_{\Gamma}(\psi) (z) = W_{\psi}(\vartheta_3(2z), \vartheta_2(2z)). \]

**Proof:** We aim to apply the result from Proposition 9 to \(\Gamma\). Since \(C\) is geometrically uniform and \(0 \in C\), we can fix \(u_1 = 0\) as a representative. Since \(C\) is geometrically uniform, \(\Lambda(C)\) is also a geometrically uniform packing from Remark 15. Hence, we can apply (11). We will also disregard the scalar \(1/\sqrt{2}\) in \(\Gamma\). For now, we can rewrite
\[ \Theta_{\Gamma}(\psi) (z) = \sum_{k=1}^{M} \sum_{x \in Z^n} q^{\| \gamma + x_k \|^2} = \sum_{k=1}^{M} \sum_{x \in Z^n} q^{2z + x_k}. \]

The \(i\)-th coordinate of the vector \(2z + u_k\) can only assume the following values
\[ (2z + u_k)_i = \begin{cases} 2z_i + 1 & \text{if } u_k = 0, \\ 2z_i + \frac{1}{2} & \text{if } u_k = 1. \end{cases} \]

The theta series associated to each of these cases are \(\Theta_{2z}(z) = \vartheta_3(4z)\) and \(\Theta_2(z) = \vartheta_2(4z)\). Hence, if we fix a \(u_k = (u_{k_1}, \ldots, u_{k_n}) \in C\), we obtain
\[ \sum_{z \in Z^n} q^{2z + u_k} = \left( \sum_{z \in Z^n} q^{2z} \right)^n = \vartheta_3(4z)^n, \]
for \(z \in Z\) and \(w = w \cdot (u_k)\). Therefore, we can simply write,
\[ \Theta_{\Gamma}(\psi)(z) = \sum_{k=1}^{M} \sum_{x \in Z^n} q^{2z + u_k} = \sum_{k=1}^{M} A_w(\psi) \vartheta_3^{-w}(4z) \vartheta_2^{-w}(4z), \]
where \(A_w(\psi)\) is defined as in (1). Note that a property of theta series is that \(\theta_{\alpha}(z) = \Theta_{\psi}(\alpha^2z)\) for some \(\alpha > 0\). Hence, by considering the factor \(1/\sqrt{2}\), (13) becomes \(\Theta_{\Gamma}(\psi)(z) = W_{\psi}(\vartheta_3(2z), \vartheta_2(2z))\), as we wanted to demonstrate.

**Theorem 20:** Consider an \((n,2^{n/2})\) formally self-dual code \(C \subseteq \mathbb{F}_2^n\) that is geometrically uniform and \(0 \in C\), then
\[ \Theta_{\Gamma}(\psi)(z) = W_{\psi}(\vartheta_3(2z), \vartheta_2(2z)) = \left( \frac{1}{z} \right)^n \Theta_{\Gamma}(\psi) \left( \frac{1}{z} \right), \]
and \(\Gamma(C)\) is a formally unimodular packing.

**Proof:** Expanding the left-hand side of (14), considering that \(C\) is formally self-dual and (3) holds, we get
\[ W_{\psi}(\vartheta_3(2z), \vartheta_2(2z)) = \left( \frac{1}{2} \right)^{2z} W_{\psi}(\vartheta_3(2z) + \vartheta_2(2z), \vartheta_3(2z) - \vartheta_2(2z)). \]

Therefore, the results of both Lemma 19 and Theorem 20 apply.

We present now how to construct geometrically uniform codes where \(0 \in C\).

Consider \(Z_4 = \{0,1,2,3\}\), the ring of integers modulo 4, and let \(C_4 \subseteq Z_4\) be a linear code over \(Z_4\), i.e., \(C_4\) is an additive subgroup of \(Z_4^n\). Then, we construct the binary code \(C_g\) as the binary image of \(C_4\) under the Gray map \(\psi : Z_4^n \to \mathbb{F}_2^2 \times \mathbb{F}_2^2\), which maps
\[ 0 \mapsto (0,0), \quad 1 \mapsto (0,1), \quad 2 \mapsto (1,1), \quad 3 \mapsto (1,0). \]

This mapping can be naturally extended such that \(\psi : Z_4^n \to \mathbb{F}_2^n\) and we define \(C_g = \psi(C_4)\).

We review several useful properties of binary codes obtained from Gray map.

**Remark 21:**
- The code \(C_g\) is not necessarily linear, but it is geometrically uniform with respect to the Hamming metric [25], which implies that the set of Hamming distances from a fixed codeword in \(C_g\) coincides with the set of Lee distances from a fixed codeword in \(C_g\).
- If \(C_4\) is also formally self-dual over \(Z_4\) with respect to the Lee metric, then \(C_g\) is formally self-dual with respect to the Hamming metric [26, Th. 1].

Hence, we can then draw the following conclusions regarding the packing \(\Gamma(C_g) = \frac{1}{\sqrt{2}}(\phi(C_g) + 2Z^n)\) i) when \(C_g\) is nonlinear, \(\Gamma(C_g)\) is a nonlattice packing (see Definition 14) and ii) since \(C_g\) is geometrically uniform and \(0 \in C_g\), then \(\Gamma(C_g)\) is also a geometrically uniform packing. Therefore, the results of both Lemma 19 and Theorem 20 apply.

The Nordstrom-Robinson code [19, Ch. 2. §8] is a well known example of a nonlinear code and it can be constructed from the \(Z_4\)-linear octacode through Gray map.
Example 22: The octacode $\mathcal{O}_8 \subseteq \mathbb{Z}_2^8$ has the following generator matrix

$$
\mathbf{G} = \begin{pmatrix}
1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 2 & 1 & 3 \\
0 & 0 & 1 & 0 & 1 & 3 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 & 2
\end{pmatrix}.
$$

The code $\mathcal{N}_{16} = \psi(\mathcal{O}_8)$ of length 16 is the Nordstrom-Robinson code, which is the unique (up to translation or permutation) (16,256,6) binary code and it is optimal in the sense that no (16,M,6) binary code can have $M > 256$ codewords [27]. Moreover, since the octacode $\mathcal{O}_8$ is linear and self-dual over $\mathbb{Z}_4$, $\mathcal{N}_{16} = \psi(\mathcal{O}_8)$ is geometrically uniform, formally self-dual, and contains 0. Its weight enumerator is [19, p. 74]

$$W_{\mathcal{N}_{16}}(x,y) = 2^{16} + 112x^{10}y^6 + 30x^8y^8 + 112x^6y^{10} + y^{16}.$$ 

The packing $\Gamma_{\Lambda}(\mathcal{N}_{16})$ is lattice-like and has the following theta series, according to Lemma 19,

$$\Theta_{\Gamma_{\Lambda}(\mathcal{N}_{16})}(z) = W_{\mathcal{N}_{16}}(\vartheta_3(2z), \vartheta_2(2z)) = 1 + 32q^2 + 7168q^3 + 8160q^4 + 258048q^5 + 127360q^6 + \cdots.$$

Example 23: Consider $\mathcal{C} = \{c_1, c_2, c_3, c_4\} = \{(0,0,0,0), (1,1,0,0), (1,0,1,0), (1,0,0,1)\} \subseteq \mathbb{F}_2^4$. Observe that $\mathcal{C}$ is non-linear, but geometrically uniform. Indeed, given any $c, c' \in \mathcal{C}$, there exists an isometry $T_{c,c'} = P_{c,c'} \circ Q_{c,c'}$, where $P_{c,c'}$ is a permutation and $Q_{c,c'}$ is a translation, such that $T_{c,c'}(c) = c'$ and $T_{c,c'}(c') = c'$. Hence, we can write $\Gamma_c = \Gamma_{\Lambda}(\mathcal{C}) = \bigcup_{c} (c + \mathbb{Z}_2^4)$ (disregarding the scalar $1/\sqrt{2}$) as a lattice-like packing. We want to write $\Gamma_b$ as a union of cosets of $\Gamma_c$ like (16), i.e.,

$$\Gamma_b = \bigcup_{j=1}^{2^k} (b_j + \Gamma_c) = \bigcup_{j=1}^{2^k} (b_j + c_\ell + \mathbb{Z}_2^4)$$

where $a_i = b_j + c_\ell, \ell \in [1 : 2^k]$, $j \in [1 : 4]$. The choices of $b_j$ would impact the characteristics of $\Gamma_b$ and the possibilities are, for example:

i) Lattice: Let $\mathcal{R} = \{b_1, b_2, b_3, b_4\} = \{(0,0,0,0), (1,1,1,0), (0,0,0,1), (1,1,1,0)\} \subseteq \mathbb{F}_2^4$. Then, $\Gamma_b = \bigcup_{i=1}^{16} (a_i + \mathbb{Z}_2^4)$, where $\mathcal{R} = \{a_i\}_{i=1}^{16}$ is the linear code $\mathbb{F}_2^4$, and $\Lambda_b = \Gamma_{\Lambda}(\mathcal{R})$ is a lattice.

ii) Nonlattice: Let $\mathcal{R} = \{b_1, b_2\} = \{(0,0,0,0), (1,1,1,0)\}$. Then, $\Gamma_b = \bigcup_{i=1}^{8} (a_i + \mathbb{Z}_2^4)$, where $\mathcal{R} = \{a_i\}_{i=1}^{8} = \{(0,0,0,0), (1,1,0,0), (1,0,1,0), (0,0,1,0), (1,1,0,0), (0,0,1,0), (0,1,0,0), (0,1,1,1)\}$. Because $\mathcal{R}$ is non-linear, $\Gamma_{\Lambda}(\mathcal{R})$ is not a lattice. Moreover, since $\mathcal{R}$ is not distance-invariant, and therefore, not geometrically uniform, $\Gamma_{\Lambda}(\mathcal{R})$ is also not a lattice-like packing.

iii) Indecomposable: Let $\mathcal{R} = \{b_1, b_2\} = \{(0,0,0,0), (0,1,1,0)\}$. Then, $\Gamma_b = \bigcup_{i=1}^{2} (b_i + \Gamma_c)$ cannot be written as a union of disjoint cosets since, for example, a binary vector $(1,0,1,0) \in \mathcal{R} = \{a_i\}_{i=1}^{8}$, can be represented in two different ways as follows.

$$(1,0,1,0) = (0,0,0,0) + (1,0,1,0) + (1,1,0,0) \quad \in \mathcal{C}$$

Hence, there exists a vector $a \in \mathcal{R}$ that cannot be uniquely written as $a = b + c$ over $\mathbb{F}_2$, $b \in \mathcal{R}, c \in \mathcal{C}$. Therefore, this leads to that $\bigcup_{i=1}^{2} (b_i + \Gamma_c) \neq \emptyset$, which prevents this pair of packings from being used in our context.

Our main objective is the following: given a lattice-like packing $\Gamma_c$, we aim to find a lattice $\Lambda_b$ such that $\Gamma_b \subseteq \Lambda_b$. Examples are described below with our previous examples of lattice-like packings.

Example 24: We consider here the lattice-like packings presented in Examples 11 and 16. For the first construction, $\Lambda_b = \bigcup_{i=1}^{4} (a_i + \mathbb{Z}_2^4)$, where $a_i \in \{(0,0), (1,1), (2,2), (3,3)\}$. For the second one, $\Lambda_b = \Gamma_{\Lambda}(\mathcal{C})$, where $\mathcal{C} = \{(0,0), (1,0), (0,1), (1,1)\}$. Their corresponding Voronoi regions are represented in Fig. 4. The points in black represent $\Gamma_c$, while the lattice points of $\Lambda_b$ are the union of the black and orange points.
Our work is focused on Construction A packings. Hence, consider a geometrically uniform code \( \mathcal{C} \subseteq \mathbb{F}_2^M \) where \(|\mathcal{C}| = M \) and \( \Gamma_e = \Gamma_e(\mathcal{C}) \subseteq \bigcup_{\ell=1}^{M} (c_\ell + 2\mathbb{Z}^n) \). \( c_\ell \in \mathcal{C} \). We are interested in some conditions on \( b_j \), \( j \in [1 : 2^k] \), such that

\[
\Gamma_b = \bigcup_{j=1}^{2^k} \bigcup_{\ell=1}^{M} (b_j + c_\ell + 2\mathbb{Z}^n)
\]

\[
= \bigcup_{i=1}^{M-2^k} (a_i + 2\mathbb{Z}^n) = \Gamma_A(\mathcal{A}) \tag{17}
\]

is a lattice, and those cosets \( \{a_i + 2\mathbb{Z}^n\} \) are disjoint, where \( \mathcal{A} = \{a_i: a_i = b_j + c_\ell, j \in [1 : 2^k], \ell \in [1 : M]\} \subseteq \mathbb{F}_2^M \). Since Construction A packings are lattices if and only if the underlying code is linear, we will work directly with binary codes of \( \mathcal{A} \) and \( \mathcal{C} \), such that \( \Gamma_e = \Gamma_A(\mathcal{C}) \) is a lattice-like packing and \( \Lambda_b = \Gamma_A(\mathcal{A}) \) is a lattice. Here, to simplify the analysis further, \( \mathcal{B} \) is assumed to be linear.

Now, consider a geometrically uniform code \( \mathcal{C} = \{c_1, \ldots, c_M\} \subseteq \mathbb{F}_2^n \). Let us choose \( \mathcal{A} = \text{span}(\mathcal{C}) \), the smallest linear code that contains \( \mathcal{C} \). We first show that it is always possible to find a code \( \mathcal{B} \) such that \( (\mathcal{B} - \mathcal{B}) \cap (\mathcal{C} - \mathcal{C}) = \{0\} \). We can simply define \( \mathcal{B} \) as the operations are defined over \( \mathbb{F}_2 \).

Proposition 25: \[30, \text{Prop. 1}] \) Given \( \mathcal{A} \subseteq \mathbb{F}_2^M \) a linear code and \( \mathcal{B}, \mathcal{C} \subseteq \mathbb{F}_2^n \) two codes (linear or not) where \( \mathbf{0} \in \mathcal{B}, \mathcal{C} \). Then, every element \( a \in \mathcal{A} \) can be uniquely written as \( a = b + c \), with \( b \in \mathcal{B} \) and \( c \in \mathcal{C} \), if and only if \( (\mathcal{B} - \mathcal{B}) \cap (\mathcal{C} - \mathcal{C}) = \{0\} \) and \(|\mathcal{A}| = |\mathcal{B}| \).  

Example 26: We continue with Example 23. Let us first choose

\[
\mathcal{A} = \text{span}\{(1,1,0,0),(1,0,1,0),(1,0,0,1)\}
\]

\[
= \{(0,0,0,0),(1,1,0,0),(0,0,1,1),(1,0,0,1),(0,1,0,1),(0,1,1,0),(1,1,1,1)\}.
\]

Then, \( \mathcal{B} \) can be \( [1,1,0,0] \oplus [1,0,1,0] \oplus [1,0,0,1] = (1,1,1,1) \). Since we want \( \mathcal{B} \) to be linear, \( \mathcal{B} = \{(0,0,0,0),(1,1,1,1)\} \). In addition, it can be seen that

\[
(\mathcal{B} - \mathcal{B}) \cap (\mathcal{C} - \mathcal{C}) = \{(0,0,0,0),(1,1,0,0),(1,0,1,0),(0,1,0,1),(1,0,0,1),(0,0,1,1)\} = \{0\},
\]

which satisfies the condition of Proposition 25. Let \( \mathcal{A} = \mathbb{F}_2^2 \) span \( \mathcal{C} \), and \( \mathcal{B} \) be \( (1,1,1,1) \) or \( (1,1,0,0) \). Then, \( \mathcal{B} = \{(0,0,0,0),(1,1,1,1),(1,1,1,0),(0,0,1,0)\} \), and one can check that this \( \mathcal{B} \) also satisfies Proposition 25.

We remark here that the construction for the coset decomposition as in (17) is an interesting direction to study. However, in this work, we mainly focus on the analysis of secrecy gain of lattice-like packings.

B. Probability Analysis

Based on the discussion of the previous subsection, we consider two packings, \( \Gamma_e \subseteq \Gamma_b \), where \( \Gamma_e \) is a lattice-like packing and \( \Gamma_b = \Lambda_b \) is a lattice. This will be our assumption from now on. \( \Lambda_b \) is designed to ensure reliability for a legitimate receiver Bob and required to have a good coding gain.\(^4\) The packing \( \Gamma_e \) is aimed to increase the eavesdropper Eve’s confusion, so it should be chosen to minimize \( P_{e|x} \), the eavesdropper’s success probability of correctly guessing the transmitted message, or (almost) equivalently, to maximize Eve’s equivocation conditioned on the channel output.

We start by writing \( \Lambda_b = \bigcup_{j=1}^{2^k} \mathbf{u}_j + \Gamma_e \). A lattice coset coding scheme works as follows: Alice wants to transmit a message \( s = (s_1, \ldots, s_k) \in \{0,1\}^k \) to Bob. The message \( s \) is mapped to a coset \( \mathbf{u}_j = \mathbf{u}_{j(s)} \). Alice selects a random vector \( \mathbf{r} \in \Gamma_e \) and transmits a codeword \( \mathbf{x} = \mathbf{u}_j + \mathbf{r} \) over the Gaussian WTC. Bob, the legitimate receiver, is assumed to have a channel of sufficient quality to enable correct decoding, while the eavesdropper Eve, has an inferior SNR, i.e., \( \sigma_b^2 \ll \sigma_e^2 \).

When a message \( \mathbf{x} \in \mathbb{R}^n \) is transmitted over an additive Gaussian channel with variance \( \sigma_e^2 \), the decoder makes the

\(^4\) Quantified in terms of the Hermite’s constant, defined as \( \gamma_\mathcal{A} = \frac{\operatorname{det}_\mathcal{A}(I)}{\operatorname{vol}(I)} \) [31, p. 20].
correct decision if and only if the received vector $y = x + h$ at the destination lies in $\mathcal{V}(x)$, where $h$ is a vector that has $n$ independent and identically distributed Gaussian random variables, each has mean 0 and variance $\sigma^2$. This gives the probability of correct decoding:

$$\frac{1}{(2\pi\sigma^2)^{n/2}} \int_{\mathcal{V}(x)} e^{-\frac{\|y - x\|^2}{2\sigma^2}} dy.$$ 

Suppose a lattice vector $x = u_j + r \in \Lambda_0$, $j \in [1 : 2^k]$, is transmitted, where a random vector $r$ is chosen from $\Gamma_c$. Then, the probability of correctly guessing the transmitted message, $P_c$, can be shown to be bounded from above by [10, Appendix B]

$$P_c \leq \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{\mathbf{r} \in \Gamma_c} \int_{\mathcal{V}(x)} e^{-\frac{\|y - x\|^2}{2\sigma^2}} dy. \quad (18)$$

Since all the Voronoi regions of a lattice are independent on the choice of $x \in \Lambda_0$ and are equal to $\mathcal{V}(\Lambda_0)$, (18) indicates that $P_{c,e}$ in an additive Gaussian channel with variance $\sigma^2$ is bounded from above by

$$P_{c,e} \leq \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{\mathbf{r} \in \Gamma_c} \int_{\mathcal{V}(\Lambda_0)} e^{-\frac{\|y + \mathbf{r}\|^2}{2\sigma^2}} dy,$$

where (19) holds by using the change of variable $w = y - x - r$.

In [10, eq. (45)], the authors use the Poisson summation formula for lattices to further bound $P_{c,e}$ for the setting of a pair of nested lattices $\Lambda_c \subset \Lambda_0$. However, we need to generalize this bound in order to adapt it to our setup, where $\Lambda_0$ is a lattice and its subset $\Gamma_c$ is a lattice-like packing. To this end, we first introduce the following two useful lemmas.

**Lemma 27 (Poisson Summation Formula)** [21, Th. 2.3, 10, Appendix C]: Let $A \subset \mathbb{R}^n$ be an arbitrary lattice and $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be a function satisfying

1) $\int_{\mathbb{R}^n} |f(t)| dt < \infty$,
2) The infinite series $\sum_{x \in A} |f(x + u)|$ converges uniformly for all $u$ belonging to a compact subset of $\mathbb{R}^n$,
3) The infinite series $\sum_{x \in A} \phi(x) \psi(x)$ converges, where $\phi(x) \triangleq \int_{\mathbb{R}^n} f(t)e^{-2\pi i \langle t, x \rangle} dt$.

Then,

$$\sum_{x \in A} f(x) = \frac{1}{\text{vol}(A)} \sum_{x \in A} \phi(x). \quad (20)$$

**Lemma 28:** [23, p. 462]: For any $\alpha \in \mathbb{C}$ such that its real part $\text{Re} \{\alpha\} > 0$, we have

$$\frac{1}{(2\pi\alpha)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{\|t\|^2}{2\alpha}} e^{-2\pi i \langle t, \lambda \rangle} dt = e^{-\pi^2 \alpha \|\lambda\|^2}. \quad (21)$$

**Theorem 29:** Let $\Gamma_c = \bigcup_{j=1}^K (u_j + \Lambda_c)$ be a lattice-like packing such that for all $j \in [1 : K]$, $(u_j, \lambda) \in \frac{1}{Z} \mathbb{Z}^n, \forall \lambda \in \Lambda_c$. Then,

$$P_{c,e} \leq \frac{\text{vol}(\Lambda_0)}{(2\pi\sigma^2)^{n/2}} \sum_{\mathbf{r} \in \Gamma_c} e^{-\frac{\|\mathbf{r}\|^2}{2\sigma^2}}, \quad (22)$$

**Proof:** Since by definition, $\Gamma_c$ is periodic, it can be expressed as $\Gamma_c = \bigcup_{j=1}^K (u_j + \Lambda_c)$ for a lattice $\Lambda_c \subset \mathbb{R}^n$, and some translating vectors $u_j \in \mathbb{R}^n, j \in [1 : K]$. From (19), one needs to calculate

$$\frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{\mathbf{r} \in \Gamma_c} \int_{\mathcal{V}(\Lambda_0)} e^{-\frac{\|y + \mathbf{r}\|^2}{2\sigma^2}} dy,$$

$$= \int_{\mathcal{V}(\Lambda_0)} \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{\mathbf{r} \in \Gamma_c} e^{-\frac{\|y + \mathbf{r}\|^2}{2\sigma^2}} dy,$$

$$= \int_{\mathcal{V}(\Lambda_0)} \frac{1}{(2\pi\sigma^2)^{n/2}} \sum_{j=1}^K e^{-\frac{\|y + u_j + \mathbf{r}\|^2}{2\sigma^2}} dy. \quad (22)$$

Define $z \triangleq \frac{1}{2\pi\sigma^2}$ with $\text{Im} \{\frac{1}{2\pi\sigma^2}\} = \text{Im} \{z\} > 0$. We will apply Lemma 27 (Poisson summation formula) twice to (22). We first let $f(\lambda) \triangleq e^{\pi i \langle \mathbf{w} + u_j + \mathbf{r}, \lambda \rangle}$ for fixed vectors $\mathbf{w}$ and $u_j$. Then, according to Lemma 27, we have

$$\sum_{\lambda \in \Lambda_c} e^{\pi i \|\mathbf{w} + u_j + \mathbf{r}\|^2} = \sum_{\lambda \in \Lambda_c} f(\lambda) \quad (20)$$

$$= \frac{1}{\text{vol}(\Lambda_c)} \sum_{\lambda \in \Lambda_c} \int_{\mathbb{R}^n} e^{\pi i \|\mathbf{w} + u_j + \mathbf{r}, \lambda\|^2} e^{-2\pi i \langle \mathbf{r}, \lambda \rangle} d\mathbf{r} \quad (a)$$

$$= \frac{1}{\text{vol}(\Lambda_c)} \sum_{\lambda \in \Lambda_c} \int_{\mathbb{R}^n} e^{\pi i \|\mathbf{w} + u_j\|^2} e^{-2\pi i \langle \mathbf{r}, \lambda \rangle} d\mathbf{r} \quad (b) \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c)$$

$$= \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c) \sum_{\lambda \in \Lambda_c} e^{2\pi i \langle \mathbf{w}, \lambda \rangle} e^{\pi i \|\lambda\|^2} \quad (c) \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c)$$

$$= \frac{1}{\text{vol}(\Lambda_c)} \sum_{\lambda \in \Lambda_c} \left[ \cos \left(2\pi \langle \mathbf{w}, \lambda \rangle \right) + i \sin \left(2\pi \langle \mathbf{w}, \lambda \rangle \right) \right] e^{2\pi i \langle u_j, \lambda \rangle} e^{\pi i \|\lambda\|^2} \quad (d) \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c)$$

$$= \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c) \sum_{\lambda \in \Lambda_c} \cos \left(2\pi \langle \mathbf{w}, \lambda \rangle \right) e^{2\pi i \langle u_j, \lambda \rangle} \quad (e) \left(2\pi\sigma^2\right)^{n/2} \text{vol}(\Lambda_c)$$

where (a) follows by making the change of variable $\rho = \mathbf{w} + u_j + \mathbf{r}$; (b) holds by choosing $\alpha = \frac{1}{2\pi\sigma^2} = \sigma^2 > 0$ in (21);
where \( g(\lambda^*) = e^{2\pi i \langle u_j, \lambda^* \rangle} \). Moreover, and by Lemma 27, we obtain
\[
\sum_{\lambda^* \in \Lambda^*} e^{2\pi i \langle u_j, \lambda^* \rangle} e^{\frac{\pi}{2} \| \lambda^* \|^2} = \sum_{\lambda^* \in \Lambda^*} g(\lambda^*)
\]
for all \( i \geq 0 \) and \( j \leq m \).

Subsequently, let \( g(\lambda^*) = e^{2\pi i \langle u_j, \lambda^* \rangle} \) for all \( \lambda^* \in \Lambda^* \). Then, applying Lemma 27 again, we obtain
\[
\sum_{\lambda^* \in \Lambda^*} e^{2\pi i \langle u_j, \lambda^* \rangle} e^{\frac{\pi}{2} \| \lambda^* \|^2} = \sum_{\lambda^* \in \Lambda^*} g(\lambda^*)
\]
where (i) follows by choosing \( \alpha = \frac{2}{\sqrt{2}} > 0 \) in (21).

Finally, using (23) and (24) together, (22) becomes
\[
\int_{\mathcal{V}(\Lambda^*_e)} \frac{1}{(2\pi \sigma_e^2)^{n/2}} \sum_{\lambda^* \in \Lambda^*_e} e^{-\frac{\| \lambda^* \|^2}{2\sigma_e^2}} \, d\lambda^*
\]
and (ii) is trivial as \( \cos (2\pi \langle w, \lambda^* \rangle) \leq 1 \).

Therefore, minimizing the upper bound on \( P_{c,e} \) is equivalent to minimizing
\[
\sum_{\lambda^* \in \Lambda^*} e^{-\frac{\| \lambda^* \|^2}{2\sigma_e^2}} = \Theta_{\Gamma_e}(z),
\]
subject to \( |\Lambda|/|\Gamma| = 2^k \). Note that since \( \Gamma_e \) is a lattice-like packing, its theta series \( \Theta_{\Gamma_e}(z) \) is well-defined. Moreover, since \( \text{Im}(z) > 0 \), we consider only the positive values of \( \tau = -iz = 1/2\pi \sigma_e^2 > 0 \) for \( \Theta_{\Gamma_e}(z) \). In summary, the lattice coset coding scheme is aimed at finding a lattice-like packing \( \Gamma_e \) such that \( \Theta_{\Gamma_e}(z) \) is minimized, which motivates the definition of the secrecy function below. It is worth mentioning that in [11], the authors also pointed out that minimizing the theta series of \( \Gamma_e \) leads to a small flatness factor, a criterion that directly relates to the mutual information leakage to the eavesdropper, instead of the success probability. Therefore, the optimization of \( \Theta_{\Gamma_e}(z) \) is of interest in both scenarios.

Remark 30: Construction A packings \( \Gamma_e = \Gamma(\mathcal{E}) = \bigcup_{\phi \in \mathcal{E}} \phi(\mathcal{E}) + 2\mathbb{Z}^n \) satisfy the condition imposed by Theorem 29. Notice that \( u_j = \phi(c) \in \mathcal{E} \subseteq \mathbb{F}_2^n \), \( \Lambda_e = 2\mathbb{Z}^n \) and \( \Lambda^*_e = \frac{1}{2}\mathbb{Z}^n \). Therefore, it can be verified that \( 2 \langle u_j, \lambda^* \rangle = \langle 2u_j, \lambda^* \rangle \in \mathbb{Z} \), and the error probability analysis holds.

For 2-level Construction C lattice-like packings of the form \( \Gamma_e = \phi(\mathcal{E}_1) + 2\phi(\mathcal{E}_2) + 4\mathbb{Z}^n [24], [32] \), where \( \mathcal{E}_1, \mathcal{E}_2 \subseteq \mathbb{F}_2^n \) are linear, the assumption required by Theorem 29 does not hold in general. Indeed, we have \( \Lambda_e = 4\mathbb{Z}^n \), \( \Lambda^*_e = \frac{1}{2}\mathbb{Z}^n \), and for \( u_j = \phi(c_1) + 2\phi(c_2) \) with \( c_1 \in \mathcal{E}_1, c_2 \in \mathcal{E}_2 \), one can observe that it does not necessarily hold that \( \langle u_j, \lambda^* \rangle \in \mathbb{Z}^n \).

This is the case of Example 11.

For general lattice-like packings, it is unknown whether the result of Theorem 29 is still valid. For the sake of brevity, we will always refer to a lattice-like packing under the premise of Theorem 29 unless otherwise specified.

Definition 31 (Secrecy Function and Secrecy Gain [10, Defs. 1 and 2]): Let \( \Gamma \) be a lattice-like packing with volume \( \text{vol}(\Gamma) = V^n \). The secrecy function of \( \Gamma \) is defined by
\[
\Xi(\tau) = \frac{\Theta_{\Lambda(\mathcal{E})}(i\tau)}{\Theta_{\Gamma}(i\tau)},
\]
for \( \tau = -iz > 0 \). The (strong) secrecy gain of a lattice is given by
\[
\xi(\Gamma) = \sup_{\tau > 0} \Xi(\tau).
\]

Ideally, the goal is to determine \( \xi(\Gamma) \). However, since the global maximum of a secrecy function of an arbitrary \( \Gamma \) is
in general not always easy to calculate, a weaker definition is introduced. We start by defining the symmetry point.

**Definition 32 (Symmetry Point):** A point \( \tau_0 \in \mathbb{R} \) is said to be a symmetry point if for all \( \tau > 0 \),

\[
\Xi(\tau_0 \cdot \tau) = \Xi\left(\frac{\tau_0}{\tau}\right).
\]  

(25)

**Definition 33 (Weak Secrecy Gain [10, Def. 3]):** If the secrecy function of a lattice-like packing \( \Gamma \) has a symmetry point \( \tau_0 \), then the weak secrecy gain \( \chi_\Gamma \) is defined as

\[
\chi_\Gamma = \Xi_\Gamma(\tau_0).
\]

V. WEAK SECRECY GAIN OF FORMALLY UNIMODULAR PACKINGS

This section shows that formally unimodular packings also hold the same secrecy function property as unimodular, isodual lattices, and formally unimodular lattices [11, 10]. Theorem 34 gives a necessary and sufficient condition for a packing \( \Gamma \) to achieve its weak secrecy gain at \( \tau = 1 \).

**Theorem 34:** Consider a lattice-like packing \( \Gamma \) with \( \text{vol}(\Gamma) = 1 \). Then, \( \Gamma \) achieves its weak secrecy gain at \( \tau = 1 \), if and only if \( \Gamma \) is formally unimodular.

**Proof:** By the definition of the secrecy function and weak secrecy gain, we have

\[
\frac{\Theta_{\Xi}(i\tau)}{\Theta(1)} = \Xi(\tau) = \Xi\left(\frac{1}{\tau}\right) = \frac{\Theta_{\Xi}(i/\tau)}{\Theta(1/\tau)},
\]  

(26)

Consider that \( z = i\tau \). Since the cubic lattice \( \mathbb{L}^n \) is formally unimodular, it follows from (12) that

\[
\Theta_{\Xi}(i\tau) = \Theta_{\Xi}(z) = \left(\frac{1}{z}\right)^{2} \Theta_{\Xi}\left(-\frac{1}{z}\right) = \left(\frac{1}{\tau}\right)^{2} \Theta_{\Xi}\left(\frac{1}{\tau}\right),
\]  

(27)

Therefore, given that \( \text{vol}(\Gamma) = 1 \), we can obtain from (26) and (27) that

\[
\Theta(\tau) = \frac{\Theta_{\Xi}(i\tau)}{\Xi(\tau)} = \frac{\Theta_{\Xi}(i/\tau)\Theta(1/\tau)}{\Theta_{\Xi}(1/\tau)},
\]  

(27)

which also gives (12) for \( z = i\tau \). Hence, \( \Gamma \) is formally unimodular.

Conversely, if

\[
\Theta(\tau) = \left(\frac{1}{\tau}\right)^{2} \Theta(-1/\tau),
\]

then for \( z = i\tau \), we have

\[
\Xi(\tau) = \frac{\Theta_{\Xi}(i\tau)}{\Theta(1)} = \left(\frac{1}{\tau}\right)^{2} \Theta_{\Xi}(1/\tau) = \Xi\left(\frac{1}{\tau}\right).
\]

Thus, \( \Gamma \) achieves its weak secrecy gain at \( \tau = 1 \).

Note that Theorem 34 holds for isodual lattices as well, which yields to [10, Prop. 1].

**Corollary 35:** Consider a lattice-like packing \( \Gamma \) with \( \text{vol}(\Gamma) = \nu^n \). Then, \( \Gamma \) achieves its weak secrecy gain at \( \tau = \nu^{-2} \), if and only if \( \nu^{-1}\Gamma \) is formally unimodular.

**Proof:** Consider a scaled packing \( \tilde{\Gamma} = \nu^{-1}\Gamma \). Then, we have \( \text{vol}(\tilde{\Gamma}) = 1 \). Now, observe that

\[
\Xi(\tau) = \frac{\Theta_{\Xi}(i\tau)}{\Theta_{\Xi}(1)} = \frac{\Theta_{\Xi}(\nu^{-2} \cdot i\tau)}{\Theta_{\Xi}(\nu^{-2} \cdot 1)} = \Xi(\nu^{-2} \cdot \tau),
\]

and

\[
\Xi(\frac{1}{\tau}) = \frac{\Theta_{\Xi}(1/\tau)}{\Theta_{\Xi}(1)} = \frac{\Theta_{\Xi}(\nu^{-2} \cdot 1/\tau)}{\Theta_{\Xi}(\nu^{-2} \cdot 1)} = \Xi(\nu^{-2} \cdot \tau).
\]

A direct application of Theorem 34 completes the proof. ■

Equation (25) with \( \tau_0 = \nu^{-2} \) holds for a lattice equivalent to its dual. See [10, Prop. 2].

VI. SECRECY GAIN OF FORMALLY UNIMODULAR PACKINGS

Our goal in this section is to investigate the following conjecture.

**Conjecture 36 (The Belfiore-Solé Secrecy Function Conjecture [11, 12]):** The secrecy function of a formally unimodular lattice \( \Lambda \) achieves its maximum at \( \tau = 1 \), i.e., \( \xi_\Lambda = \Xi_\Lambda(1) \).

By Theorem 34, a formally unimodular packing \( \Gamma \) also achieves its weak secrecy gain at \( \tau = 1 \). Hence, we focus on a generalized version of this conjecture, as follows.

**Conjecture 37:** The secrecy function of a formally unimodular packing \( \Gamma \) achieves its maximum at \( \tau = 1 \), i.e., \( \xi_\Gamma = \Xi_\Gamma(1) \).

Although we cannot completely prove Conjectures 36 and 37, we proceed to study the secrecy gain for formally unimodular packings obtained from formally self-dual codes via Construction A (see Remark 18). Note that for linear codes, it is known that formally self-dual codes that are not self-dual can outperform self-dual codes in some cases, as they comprise a wider class and hence may allow a better minimum Hamming distance or an overall more favorable weight enumerator. This leads us to look for improved results on the secrecy gain compared to unimodular lattices [13, 14, 15].

**Lemma 38:** Let \( s(\tau) \triangleq \vartheta_3(i\tau)/\vartheta_3(i) \). Then, \( 0 < s(\tau) < 1 \), and \( s(\tau) \) is strictly increasing and a bijection for \( \tau > 0 \).

**Proof:** By definition, the fact that \( 0 < s(\tau) < 1 \) is trivial. Next, consider the following product representation of Jacobi theta functions [22, p. 105].

\[
\vartheta_3(\tau) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2,
\]

\[
\vartheta_4(\tau) = \prod_{m=1}^{\infty} (1 - q^{2m-1})(1 - q^{2m-1})^2,
\]

where \( q = e^{-\pi \tau} \) for \( \tau > 0 \). Hence,

\[
s(\tau) = \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)} = \prod_{m=1}^{\infty} (1 - e^{-(2m-1)\pi \tau})^2 = \prod_{m=1}^{\infty} \left( e^{(m-\frac{1}{2})\pi \tau} - e^{-(m-\frac{1}{2})\pi \tau} \right)^2
\]

\[
= \prod_{m=1}^{\infty} \tanh^2 \left( \left( m - \frac{1}{2} \right) \pi \tau \right).
\]
Observe that for any $m \in \mathbb{N}$, $\tanh^2\left(\left(m - \frac{1}{2}\right)\pi \tau\right)$ is continuous and strictly increasing from 0 to 1 on $\tau > 0$. Then, $s(\tau)$ is an injection on $\tau > 0$. Therefore, we have for any $s \in (0, 1)$, $s(\tau)$ must pass through the value $s$ exactly once while $\tau$ increases, which shows that $s(\tau)$ is surjective.

**Remark 39:** Let $t(\tau) \triangleq s(\tau)^2$. Then, $0 < t(\tau) < 1$ and $t(\tau)$ is also an increasing function for $\tau > 0$. Hence, according to Lemma 38, given any $t \in (0, 1)$, there always exists a unique $\tau > 0$ such that $t(\tau) = \sigma^2(\tau)/\sigma^2(\tau)$. Moreover, we have $t(1) = 1/\sqrt{2}$ by using the identity of $\sigma_3(i) = 2^{1/4}\sigma_4(i)$ from [33].

Now, we are able to give a new universal approach to derive the strong secrecy gain of a Construction A packing obtained from a formally self-dual code. We define $f_{s}(t) \triangleq W_{s}(\sqrt{1 + t}, \sqrt{1 - t})$ for $0 < t < 1$.

**Theorem 40:** Consider a geometrically uniform $(n, M)$ code $\mathcal{C}$ where $0 \in \mathbb{C}$, and its Construction A packing has vol $\Gamma_\mathcal{A}(\mathcal{E}) = 1$. Then

$$
\Xi_{\Gamma_\mathcal{A}(\mathcal{E})}(\tau)^{-1} = W_{\mathcal{E}}\left(\sqrt{\sqrt{1 + t(\tau)}}, \sqrt{1 - t(\tau)}\right) = \frac{f_\mathcal{E}(t(\tau))}{2^2},
$$

where $0 < t(\tau) = \sigma^2(\tau)/\sigma^2(\tau) < 1$. Moreover, maximizing the secrecy function $\Xi_{\Gamma_\mathcal{A}(\mathcal{E})}(\tau)$ is equivalent to determining the minimum of $f_\mathcal{E}(t)$ on $t \in (0, 1)$.

**Proof:** From Lemma 19 and (8), the theta series $\Theta_{\Gamma_\mathcal{A}(\mathcal{E})}$ becomes

$$
\Theta_{\Gamma_\mathcal{A}(\mathcal{E})}(z) = W_{\mathcal{E}}\left(\sqrt{\sigma^2_{3}(z) + \sigma^2_{4}(z)}/2, \sqrt{\sigma^2_{3}(z) - \sigma^2_{4}(z)}/2\right),
$$

where (a) follows by the definition of weight enumerator (1). From Definition 31, the secrecy function of $\Gamma_\mathcal{A}(\mathcal{E})$ with volume 1 becomes

$$
\Xi_{\Gamma_\mathcal{A}(\mathcal{E})}(\tau)^{-1} = \frac{\Theta_{\Gamma_\mathcal{A}(\mathcal{E})}(z)}{\Theta_{\mathcal{E}}(z)},
$$

where \(b\) holds because of $\Theta_{\mathcal{E}}(z) = \sigma^2_3(z)$ and (28).

Lastly, the second part of the theorem follows directly from Remark 39.

We remark that it turns out that our universal approach is similar to the technique proposed by [13] and adapted for [14] and [15], but it is fundamentally different. The two fundamental differences are: i) We work on formally unimodular packings, which are not necessarily unimodular and isodual lattices. ii) The technique that the previous works rely on is to express the theta series of a unimodular lattice in terms of $\sigma^2(\tau)/\sigma^2(\tau)$. However, we derive the theta series of formally unimodular packings depending on $t(\tau) = \sigma^2(\tau)/\sigma^2(\tau)$.

We can apply Theorem 40 for any formally unimodular packing as its volume is always equal to 1.

**Example 41:** Consider a [6, 3, 3] odd formally self-dual code $\mathcal{C}$ with $W_{\mathcal{E}}(x, y) = x^5 + 4x^3y^3 + 5x^2y^4$ [34]. Thus, we can get $f_{\mathcal{E}}(t) = W_{\mathcal{E}}(\sqrt{1 + t}, \sqrt{1 - t}) = 4[1 + t^2 + (1 - t^2)^{1/2}]$ and $f'_{\mathcal{E}}(t) = 12t(1 - t^2)$ by performing some simple calculations. Observe that for $0 < t < 1/\sqrt{2}$, we have $t(1 - t^2) > 1/\sqrt{2}$. Then, $t - \sqrt{1 - t^2} < 1/\sqrt{2} - 1/\sqrt{2} = 0$. This indicates that the derivative $f'_{\mathcal{E}}(t) < 0$ on $t \in (0, 1/\sqrt{2})$.

Similarly, one can also show that $f_{\mathcal{E}}(t) > 0$ on $t \in (1/\sqrt{2}, 1)$, and $t = 1/\sqrt{2}$ is the minimizer of $f_{\mathcal{E}}(t)$. Hence, Remark 39 and Theorem 40 indicate that the maximum of $\Xi_{\Gamma_\mathcal{A}(\mathcal{E})}(\tau)$ is achieved at $t = 1/\sqrt{2}$, which equals 1.127.

**Example 42:** Applying the result of Theorem 40 to $\mathcal{C}_{13}$ (see Example 22), we have that $f_{\mathcal{C}_{13}}(t) = -64(-4 + 8t - 5t^4 - 6t^6 + 5t^8)$, $f'_{\mathcal{C}_{13}}(t) = -256t(-4 - 5t^2 - 9t^4 + 6t^6)$. The value of $t$ that minimizes $f_{\mathcal{C}_{13}}(t)$ and consequently maximizes the secrecy function is $t = 1/\sqrt{2}$, yielding a strong secrecy gain of 2.207.

**Example 43:** A similar approach can be carried out for the nonlinear but formally self-dual codes obtained from [35, Tab. VII] in dimensions 12 and 20 (these codes are constructed under the Gray map $\psi$, thus are geometrically uniform and contain 0). For dimension 12, among the four weight enumerators presented,

$$
W_{\mathcal{E}_{12}}(x, y) = x^{12} + 6x^8y^4 + 24x^7y^5 + 16x^6y^6 + 9x^4y^8 + 8x^3y^9
$$

gives the highest secrecy gain. Here, we have

$$
f_{\mathcal{E}}(t) = W_{\mathcal{E}_{12}}(\sqrt{1 + t}, \sqrt{1 - t}) = 8[4 - 6t^2 + 4t^4 + 6t^4 + (1 - t)^{2/3} + 3(1 - t)^{2/2}(1 + t)^{2/2}]
$$

and $f'_{\mathcal{E}}(t) = 96t \cdot h_1(t)$, where $h_1(t) = -1 + t + 2t^2 + \sqrt{1 - t^2}(2t^2 - 1 - t)$, $1 - h_1(0) = -2$, and $h_1'(t) > 0$ on $t \in (0, 1)$. This unique root is $t = 1/\sqrt{2}$, which gives $\xi_{\Gamma_\mathcal{A}(\mathcal{E}_{12})} \approx 1.657$ and coincides with the best in Table II below.

For dimension 20, consider the weight enumerator

$$
W_{\mathcal{E}_{20}}(x, y) = x^{20} + 90x^{14}y^6 + 255x^{12}y^8 + 332x^{10}y^{10} + 255x^8y^{12} + 90x^6y^{14} + y^{20}.
$$

We have $f_{\mathcal{E}}(t) = W_{\mathcal{E}_{20}}(\sqrt{1 + t}, \sqrt{1 - t}) = -64(5t^8 - 10t^6 - 35t^4 + 40t^2 - 16)$ and $f'_{\mathcal{E}}(t) = -1280h_2(t)$, where $h_2(t) = 2t^6 - 3t^4 - 7t^2 + 4$. Note that $h_2(t)$ has only one root in $[0, 1]$, as $h_2(0) = -4, h_2(1) = 4,$ and $h_2'(t) < 0$ on $t \in (0, 1)$. This unique root is $t = 1/\sqrt{3}$, which yields $\xi_{\Gamma_\mathcal{A}(\mathcal{E}_{20})} \approx 2.813$, and coincides with the second best value tabulated in Table II.
We need to determine the coefficients $a_r$ such that

$$\sum_{r=0}^{\lfloor n/2 \rfloor} a_r t^r = 2^n \sum_{r=0}^{\lfloor n/2 \rfloor} a_r (t^2 - 1)^r,$$  \hfill (29)

where $a_r \in \mathbb{Q}$ and $\sum_{r=0}^{\lfloor n/2 \rfloor} a_r = 1.$

**Proof:** Using the **invariant theory** (cf. [19, Ch. 19]), one can show that the weight enumerator $W_\varnothing(x, y)$ of any even code $\varnothing$ (linear or nonlinear) that satisfies (4) can be expressed as

$$W_\varnothing(x, y) = \sum_{r=0}^{\lfloor n/2 \rfloor} a_r g_1(x, y)^r g_2(x, y)^r,$$  \hfill (30)

where $g_1(x, y) \equiv x^2 + y^2,$ $g_2(x, y) \equiv x^8 + 14x^4y^4 + y^8,$ $a_r \in \mathbb{Q},$ and $\sum_{r=0}^{\lfloor n/2 \rfloor} a_r = 1.$

Then, by performing some simple calculations, we obtain

$$g_1(\sqrt{1 + t}, \sqrt{1 - t}) = 2,$$

$$g_2(\sqrt{1 + t}, \sqrt{1 - t}) = 16(t^4 - t^2 + 1).$$

Therefore, (29) follows from (30).

Consider the weight enumerator $W_\varnothing(x, y)$ as in (1). We need to determine the coefficients $a_r$ in (30) in terms of $A_w(\varnothing), w \in [0 : n],$ if the coefficients $A_w(\varnothing)$ are known, and will provide numerical details in Example 46. First, we expand $g_1(x, y)^\frac{n}{2} - 4r$ and $g_2(x, y)^r.$ Observe that

$$g_1(x, y)^{\frac{n}{2} - 4r} = \sum_{j=0}^{\frac{n}{2} - 4r} \binom{\frac{n}{2} - 4r}{j} (x^2)^j (y^2)^{r-j},$$

and

$$g_2(x, y)^r = \sum_{h=0}^{r} \left( \sum_{\ell=0}^{2r-2h-\ell} x^{2r-2h-\ell} y^{2\ell} \right).$$

Given $w \in [0 : n],$ by collecting the terms of $y^{2j+8h+4\ell}$ for $2j + 8h + 4\ell = w,$ we get

$$g_1(x, y)^{\frac{n}{2} - 4r} g_2(x, y)^r$$

$$= \sum_{j, h, \ell \in \mathbb{Z}_{\geq 0}} \binom{n/2 - 4r}{j} \binom{2r-2h}{\ell} x^{2j+8h+4\ell} y^{2j+8h+4\ell},$$

where we define $\binom{p}{q} = 0,$ if $p < q.$

By comparing the coefficients of (1) and (30), we get

$$A_w(\varnothing) = \sum_{r=0}^{\lfloor n/2 \rfloor} a_r \sum_{j, k, \ell \in \mathbb{Z}_{\geq 0}} \binom{n}{j} \binom{2r-2h}{\ell} x^{n-2j-8h-4\ell} y^{2j+8h+4\ell},$$

(31)

For an even formally self-dual code, according to [36, p. 378], we know that $A_w(\varnothing) = A_{n-w}(\varnothing)$ for $w$ even.
and $A_w(\mathcal{C}) = 0$ for $w$ odd, in (1). Thus, there are at most $\left\lfloor \frac{v}{2} \right\rfloor + 1$ nonzero coefficients $A_w(\mathcal{C})$. For instance, if we want to determine the coefficients of the term corresponding to $A_3$, this would only be possible if we set $j = 2$, $h = \ell = 0$ or $j = h = 0$, $\ell = 1$ in (31), which yields

$$
A_4 = \left( \sum_{r=0}^{\left\lfloor \frac{v}{2} \right\rfloor} a_r \left( \left\lfloor \frac{n}{2} - 4r \right\rfloor + \frac{14r}{2} \right) \right)
$$

$$
= a_0 \left( \frac{n}{2} \right) + a_1 \left( \frac{n}{2} - 4 \right) + a_2 \left( \frac{n}{2} - 8 \right) + a_3 \left( \frac{n}{2} - 12 \right) + \cdots
$$

For ease of illustration, we compute more terms of (31):

$$
A_0 = \sum_{r=0}^{\left\lfloor \frac{v}{2} \right\rfloor} a_r, \quad A_2 = \sum_{r=0}^{\left\lfloor \frac{v}{2} \right\rfloor} a_r \left( \frac{n}{2} - 4r \right),
$$

$$
A_4 = \sum_{r=0}^{\left\lfloor \frac{v}{2} \right\rfloor} a_r \left( \frac{n}{2} - 4r \right) + 14r \left( \frac{n}{2} - 4r \right),
$$

$$
A_8 = \sum_{r=0}^{\left\lfloor \frac{v}{2} \right\rfloor} a_r \left( \frac{n}{2} - 4r \right) + 14r \left( \frac{n}{2} - 4r \right) + 49 \left( \frac{2r}{2} - 48r \right).
$$

As a result, we can obtain the $\left\lfloor \frac{v}{2} \right\rfloor + 1$ unknown coefficients $a_r$, $r \in \{0 : \left\lfloor \frac{v}{2} \right\rfloor\}$ by solving the system of $\left\lfloor \frac{v}{2} \right\rfloor + 1$ linear equations in (31). The uniqueness of the set of coefficients $a_r$ follows from Gleason’s Theorem [36, Th. 9.2.1].

Next, we provide a sufficient condition for a Construction A packing obtained from an even formally self-dual code to achieve its strong secrecy gain at $\tau = 1$, or, equivalently, $t = 1/\sqrt{2}$. Note that unless otherwise specified, for the rest of the paper, the formally self-dual codes we consider are geometrically uniform and contain $\mathcal{C}$. We seek to establish a necessary condition for a formally self-dual code $\mathcal{C}$ with weight enumerator $W_\mathcal{C}(x, y) = x^{18} + 102x^{12}y^6 + 153x^{10}y^8 + 153x^8y^{10} + 102x^6y^{12} + y^{18}$.

By solving $f_\mathcal{C}(t) = W_\mathcal{C}(\sqrt{1 + t}, \sqrt{1 - t})$ with (29), we find that $a_0 = -29/16, a_1 = 27/8$ and $a_2 = -9/16$. Observe that the right-hand side of (32) gives $\sum_{r=1}^{\left\lfloor \frac{v}{2} \right\rfloor} \Gamma_r h(t)^{r-1} = -\left( \frac{t^4}{t^2} - 2 \right) - \left( \frac{1}{8} \right) \left( \frac{t^2}{t^2} - 1 \right) + \left( \frac{1}{32} \right) \left( \frac{81}{32} \right)$,

$$
\sum_{r=1}^{\left\lfloor \frac{v}{2} \right\rfloor} \Gamma_r h(t)^{r-1} = -\left( \frac{t^4}{t^2} - 2 \right) - \left( \frac{1}{8} \right) \left( \frac{t^2}{t^2} - 1 \right) + \left( \frac{1}{32} \right) \left( \frac{81}{32} \right),
$$

which implies that $f_\mathcal{C}(t)$ is decreasing in $t \in (0, 1/\sqrt{2})$ and increasing in $t \in (1/\sqrt{2}, 1)$. This completes the proof.

**Example 46:** Consider an $[18, 9, 6]$ even formally self-dual code $\mathcal{C}$ with weight enumerator $W_\mathcal{C}(x, y) = x^{18} + 102x^{12}y^6 + 153x^{10}y^8 + 153x^8y^{10} + 102x^6y^{12} + y^{18}$.

As in general, it is hard to confirm whether a Construction A packing obtained from a formally self-dual code $\mathcal{C}$ achieves its strong secrecy gain at $\tau = 1$, or, equivalently, $t = 1/\sqrt{2}$. We then use the secrecy function $\Xi_{\Gamma_\mathcal{C}(\mathcal{C})}(\tau)$ to give a slightly weaker definition of a secrecy-optimal formally self-dual code.

**Definition 49:** A formally self-dual code $\mathcal{C}$ of length $n$ is called strongly secrecy-optimal if

$$
\mathcal{C}^* = \argmax_{\mathcal{C} : \text{formally self-dual}} \Xi_{\Gamma_\mathcal{C}(\mathcal{C})}(\tau).
$$

Based on Conjecture 37, Remark 39, and Theorem 40, we further conjecture the following condition for a formally self-dual code to be secrecy-optimal.

**Conjecture 48:** For a given formally self-dual code $\mathcal{C}$ of length $n$, if

$$
\mathcal{C}^* = \argmin_{\mathcal{C} : \text{formally self-dual}} f_\mathcal{C} \left( \frac{1}{\sqrt{2}} \right),
$$

then the code $\mathcal{C}^*$ is strongly secrecy-optimal.

As in general, it is hard to confirm whether a Construction A packing obtained from a formally self-dual code $\mathcal{C}$ achieves its strong secrecy gain at $t = 1/\sqrt{2}$, we then use the secrecy function $\Xi_{\Gamma_\mathcal{C}(\mathcal{C})}(\tau)$ to give a slightly weaker definition of a secrecy-optimal formally self-dual code.

**Definition 49:** A formally self-dual code $\mathcal{C}$ of length $n$ is said to be weakly secrecy-optimal if for all $\tau > 0$,

$$
\Xi_{\Gamma_\mathcal{C}(\mathcal{C})}(\tau) \geq \Xi_{\Gamma_\mathcal{C}(\mathcal{C})}(\tau)
$$

for any $n$-dimensional formally self-dual code $\mathcal{C}$.
Theorem 50: Given a length $n \geq 2$, if $C^\circ$ is weakly secrecy-optimal, then

$$C^\circ = \arg\min_{C: \text{formally self-dual}} \left\{ \sum_{w=0}^{n} \frac{A_w(C)}{w + 1} \right\}.$$ 

Proof: By Definition 49 and applying Theorem 40, one can see that if a formally self-dual code $C^\circ$ is weakly secrecy-optimal, then for all $t \in (0, 1)$, $f_{C^\circ}(t) - f_C(t) \leq 0$ for any $n$-dimensional formally self-dual code $C$. Expressed in terms of the weight enumerators, we can obtain

$$f_{C^\circ}(t) - f_C(t) = \sum_{w=0}^{n} A_w(C^\circ)(\sqrt{1 + t})^{n-w}(\sqrt{1 - t})^w - \sum_{w=0}^{n} A_w(C)(\sqrt{1 + t})^{n-w}(\sqrt{1 - t})^w = (\sqrt{1 + t})^n \sum_{w=0}^{n} A_w(C) \left( \frac{1 - t}{1 + t} \right)^w - \sum_{w=0}^{n} A_w(C^\circ) \left( \frac{1 - t}{1 + t} \right)^w \geq 0.$$ 

(33)

Now, we define $u(t) \triangleq \frac{1 - t}{1 + t}$ over $t \in (0, 1)$. It can be shown that $u(t)$ is a decreasing function for $0 < t < 1$, and we have $0 < u(t) < 1$. Hence, (33) implies that for any formally self-dual code $C$,

$$\Delta(u) \triangleq n \sum_{w=0}^{n} \frac{A_w(C)u^w - A_w(C^\circ)u^w}{w + 1} \geq 0.$$ 

Integrating $\Delta(u)$ over $u \in (0, 1)$ results in

$$\sum_{w=0}^{n} \left( \frac{A_w(C^\circ) - A_w(C)}{w + 1} \right) \geq 0$$

for any formally self-dual code $C$. This completes the proof.

Example 51: For $n = 18$, we consider the Construction A lattices obtained from the codes listed in Table II below. We use the function $\varphi_{C^\circ}(u) \triangleq \sum_{w=0}^{n} A_w(C^\circ)u^w$ for a formally self-dual code $C^\circ$, to indicate the secrecy performance. Note that $u(t) = \sqrt{\frac{1 - t}{1 + t}}$, $f_{C^\circ}(t) = (\sqrt{1 + t})^n \sum_{w=0}^{n} A_w(C^\circ)u^w(t)$, and $u(1/\sqrt{2}) = \sqrt{\frac{1}{2}} - 1 \approx 0.4142$. Fig. 5 indicates that $\Xi_{\Lambda(\varphi_{C^\circ}(\tau))}(\tau) > \Xi_{\Lambda(\varphi_{C^\circ}(\tau))}(\tau) > \Xi_{\Lambda(\varphi_{C^\circ}(\tau))}(\tau)$ for all $\tau > 0$, and one can verify that $\varphi_{C^\circ}(6) = \arg\min_{\varphi \in \{\varphi_{C^\circ(3)}, \varphi_{C^\circ(4)}, \varphi_{C^\circ(5)}, \varphi_{C^\circ(6)}\}} \left\{ \sum_{w=0}^{n} A_w(C^\circ) \right\}$ (see Appendix A for the corresponding weight enumerators).

Remark 52:

- In Example 51, only the three best-known formally self-dual codes are verified for Theorem 50. However, to exactly determine the weakly secrecy-optimal code $C^\circ$, we need to check all the possible weight enumerators of formally self-dual codes for a given dimension $n$.

- It is intuitive to believe that the secrecy gain can be improved with higher minimum Hamming distance and lower kissing number (i.e., the number of codewords with minimum weight $d$) of a given dimension $n$. However, Theorem 50 indicates that one needs to rely on the entire weight enumerators to determine the weakly secrecy-optimal code. In the following, we present two counterexamples.

To generate the two counterexamples, we use the direct sum construction [19, p. 76] of codes. Given two binary codes $C_1$, $C_2$, with parameters $(n_1, M_1, d_1)$ and $(n_2, M_2, d_2)$ respectively, the direct sum $C \subseteq F_2^{n_1+n_2}$ consists of the concatenation of vectors $c = (c_1 | c_2) \in C$, where $c_1 \in C_1$ and $c_2 \in C_2$. This idea can be generalized for more codes, and next we apply this construction for two and three codes. Since the underlying codes to be considered are formally self-dual, geometrically uniform, and contain 0, it can be shown that their direct sum also preserves such properties.

Example 53: Consider two formally self-dual codes, the $[2, 1, 2]$ repetition code and the $(16, 256, 8)$ Nordstrom-Robinson code $A_{16}$. Their direct sum, denoted by $C_{dsum}$, is an $(18, 2^n, 2)$ code, which is also formally self-dual. This code has weight enumerator

$$W_{C_{dsum}}(x, y) = x^{18} + 16x^{16}y^2 + 112x^{12}y^6 + 142x^{10}y^8 + 142x^8y^{10} + 112x^6y^{12} + x^2y^{16} + y^{18}.$$
The secrecy gain of its respective Construction A packing is $\xi_{\text{A}}(\mathcal{C}_{\text{isd}}^{(2)}) = 2.207$.

Now, we consider three [6, 3, 3] formally self-dual codes, their direct sum gives an [18, 9, 3] odd formally self-dual code $\mathcal{C}_{\text{ofsd}}^{(3)}$ with weight enumerator

$$W_{\mathcal{C}_{\text{ofsd}}^{(3)}}(x, y) = 2^{18} + 12x^{15}y^3 + 9x^{14}y^4 + 48x^{12}y^6$$

$$+ 72x^{11}y^7 + 27x^{10}y^8 + 64x^9y^9$$

$$+ 144x^8y^{10} + 108x^7y^{11} + 27x^6y^{12},$$

which induces a Construction A lattice such that $\xi_{\text{A}}(\mathcal{C}_{\text{isd}}^{(2)}) = 1.608$. Hence, comparing the two direct sum formally self-dual codes $\mathcal{C}_{\text{ofsd}}^{(2)}$ and $\mathcal{C}_{\text{ofsd}}^{(3)}$, we obtain a better secrecy gain from a formally self-dual code with a smaller minimum Hamming distance. However, by verifying the condition of Theorem 50, it can be seen that $\mathcal{C}_{\text{ofsd}}^{(2)}$ outperforms $\mathcal{C}_{\text{ofsd}}^{(3)}$ in terms of secrecy gain.

VII. CONSTRUCTION OF ISODUAL CODES FROM RATE $\frac{1}{2}$ BINARY CONVOLUTIONAL CODES

Before presenting the numerical results, we introduce in this session the construction of some even and odd formally self-dual codes obtained via convolutional codes. The construction allows us to consider codes of parameters larger than those found in the current literature on formally self-dual codes. The performance of the corresponding Construction A lattices is presented in Table II.

An $(n, k, m)$ binary convolutional code $\mathcal{C}$ is a $k$-dimensional subspace of $\mathbb{F}_2(D)^n$, where $D$ is an indeterminate variable, $\mathbb{F}_2(D)$ consists of all rational functions in $D$, and $m$ is the memory, i.e., the maximum degree of the generator polynomials for $\mathcal{C}$. For a background on convolutional codes, please see, e.g., [45]. It is well known [46] that tail-biting convolutional codes often produce very competitive linear codes. We point out the following property of the linear block codes obtained by tail-biting technique applied to convolutional codes of rate $\frac{1}{2}$.

**Proposition 54:** Let $\mathcal{C}$ be a $(2, 1, m)$ binary convolutional code. Then, any $[2k, k]$ linear code $\mathcal{C}_{\text{tb}}$ obtained from $\mathcal{C}$ by tail-biting is isodual, where $k \geq (m + 1)$.

**Proof:** For brevity, we prove this by an example of the convolutional code generated by the minimal generator matrix

$$G(D) = \begin{pmatrix} g_1(D) & g_2(D) \\ b + dD + fD^2 & a + cD + eD^2 \end{pmatrix}$$

and its associated $[2 \times 5, 1 \times 5] = [10, 5]$ linear code $\mathcal{C}_{\text{tb}}$ by tail-biting for $k = 5$. The proof is easily adapted to other tail-biting codes for different code dimensions $k$ and to other convolutional codes with different memory length, but the matrices involved tend to not fit nicely in a page.

It is well known [47, 48, p. 107] that the matrix $G_{\text{tb}}$ is a generator matrix of the linear code $\mathcal{C}_{\text{tb}}$, where

$$G_{\text{tb}} = \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \\ a & b & c & d & e & f \end{pmatrix}, \quad (34)$$

and that the matrix $H_{\text{tb}}$ is a parity check matrix for $\mathcal{C}_{\text{tb}}$, where

$$H_{\text{tb}} = \begin{pmatrix} b & a & f & e & d & c \\ d & c & b & a & f & e \\ f & e & d & c & b & a \\ f & e & d & c & b & a \\ f & e & d & c & b & a \end{pmatrix}.$$
**Improvements:** For most dimensions \( n > 8 \), the secrecy gain of formally unimodular lattices that are not unimodular outperform the unimodular lattices (obtained from self-dual codes), presented in [14, Tables I and II]. Also, to highlight the comparison with unimodular lattices, the second column refers to the upper bound on the secrecy gain of unimodular lattices obtained from Construction A in [37, Tab. III] and not all of the values are known to be achieved. Improvements can be observed in dimensions 10, 12, 14, 20, and 22.

**Dimension 32:** It is known that the Barnes-Wall lattice of dimension 32, denoted by \( B W_{32} \), achieves its secrecy gain of \( 64/9 \approx 7.11 \) [10, Sec. IV-C], which is better than all the tabulated values in dimension 32. However, because \( B W_{32} \) is not obtained via Construction A, we did not include its secrecy gain here.

**Nonlattice packings:** We have presented nonlattice packings, generated via Construction A from nonlinear formally self-dual codes, in dimensions 12, 16, and 20 (see Examples 42 and 43). Among these, a gain is observed only in dimension 16, illustrated in Fig. 6. The functions being represented are, respectively, the best formally unimodular lattice from Table II, the Construction A packing generated from the Nordstrom-Robinson code \( N_{16} \), and the Barnes-Wall lattice \( BW_{16} \), whose theta series and construction can be found at [22, pp. 129–131]. In terms of comparison, we have \( \xi_{\Gamma_A(N_{16})} = 2.2069 > \xi_{BW_{16}} = 2.0256 > \xi_{\Lambda_A(\psi_{sd})} = 2.141 \).

**IX. Conclusion and Future Work**

A new class consisting of nonlattice packings that are periodic and geometrically uniform was introduced, namely the lattice-like packings. Its subclass, called formally unimodular (lattice-like) packings, which is analogous to isodual and unimodular lattices, was further studied. We showed several fundamental properties of formally unimodular packings and built a coset coding scheme for a lattice containing a lattice-like packing as a subset. Their secrecy function behavior over the Gaussian WTC was shown to be the same as that of unimodular and isodual lattices.

In particular, we investigated the Construction A formally unimodular packings obtained from formally self-dual codes and gave a universal approach to determine their secrecy gain. Furthermore, we provided a necessary condition for Construction A formally unimodular packings to be weakly secrecy-optimal of a given dimension. The necessary condition is based on the weight distribution of the underlying formally self-dual code. We found formally unimodular packings/lattices of better secrecy gain than the best-known unimodular lattices from the literature.

Note that we mainly focus on the secrecy performance comparison between formally unimodular packings of the same dimension. The desired properties of the weight enumerator of Theorem 50 showed that the rough rule-of-thumb suggesting that codes with larger minimum Hamming distances yield a better secrecy gain is generally not true. However, even comparing the secrecy gains of formally unimodular packings for different dimensions, this simple rule can fail. To illustrate this, consider two pairs of code examples in Appendix A: the [72, 36, 16] code \( C_{sd}^{(14)} \) versus the [78, 39, 14] code \( C_{sd}^{(14)} \) and the [104, 52, 20] code versus the [108, 54, 14] code. It can be seen that the [72, 36, 16] self-dual code with a larger minimum Hamming distance is worse than the [78, 39, 14] even formally self-dual code in terms of secrecy gain (\( \xi_{\Gamma_A(\psi_{sd})} = 146.844 < \xi_{\Gamma_A(\psi_{sd})} = 241.042 \)). The same observation can be made for the [104, 52, 20] self-dual code and the [108, 54, 14] even formally self-dual code. These two examples illustrate that providing sufficient or necessary conditions to verify the optimality of secrecy gain for different dimensions remains an interesting avenue for future work. It would also be interesting to investigate whether the result of Theorem 29 applies to more general lattice-like packings.

Finally, in order to limit the scope of the paper, we have chosen not to address several interesting issues. We save these for future research. First of all, we plan to address the asymptotic behavior of the secrecy gain of lattice-like packings, as well as bounds on the mutual information leakage to the eavesdropper. Likewise, the practical application of the kind of schemes addressed in this work necessitates a discussion of the complexity of encoding and decoding. Schemes
| $\mathcal{C}$ | Type | Reference | $\mathcal{W}_C(x, y)$ | $\xi_{\mathcal{C}_\lambda}(x)$ |
|---------|------|-----------|----------------------|-------------------|
| [6, 3, 2] | efsd | [36] | $x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ | 1 |
| [6, 3, 3] | ofsd | [34] | $x^6 + 4x^2y^3 + 3x^2y^4$ | 1.172 |
| [8, 4, 4] | sd | [36] | $x^8 + 14x^2y^2 + y^8$ | 1.333 |
| [8, 4, 3] | ofsd | [34] | $x^8 + 3x^5y^3 + 7x^4y^4 + 4x^3y^5 + xy^7$ | 1.282 |
| [8, 4, 3] | ofsd | [34] | $x^8 + 4x^5y^3 + 5x^4y^4 + 4x^3y^5 + 2x^2y^6$ | 1.264 |
| [10, 5, 4] | efsd | [36] | $x^{10} + 15x^6y^4 + 15x^2y^6 + y^{10}$ | 1.455 |
| [10, 5, 4] | ofsd | [34] | $x^{10} + 10x^6y^4 + 16x^2y^6 + 5x^2y^8$ | 1.478 |
| [12, 6, 4] | sd | [14] | $x^{12} + 15x^8y^4 + 32x^4y^6 + 15x^2y^8 + y^{12}$ | 1.6 |
| [12, 6, 4] | efsd | [39] | $x^{12} + 15x^8y^4 + 32x^4y^6 + 15x^2y^8 + y^{12}$ | 1.6 |
| [12, 6, 4] | ofsd | [34] | $x^{12} + 6x^8y^4 + 24x^4y^6 + 16x^2y^8 + 9x^4y^8 + 8x^2y^9$ | 1.657 |
| (12, 64, 4) | ofsd | [35] | $x^{12} + 6x^8y^4 + 24x^4y^6 + 16x^2y^8 + 9x^4y^8 + 8x^2y^9$ | 1.657 |
| [14, 7, 4] | sd | [14] | $x^{14} + 14x^{10}y^4 + 49x^8y^6 + 49x^6y^8 + 14x^4y^{10} + y^{14}$ | 1.778 |
| [14, 7, 2] | efsd | [39] | $x^{14} + 12x^{12}y^4 + 15x^{10}y^6 + 47x^8y^8 + 47x^6y^8 + 15x^4y^{10} + x^2y^{12} + y^{14}$ | 1.6 |
| [14, 7, 4] | ofsd | [34] | $x^{14} + 3x^{10}y^4 + 24x^8y^6 + 36x^6y^8 + 16x^2y^{10} + 11x^6y^8 + 24x^4y^6 + 12x^2y^{10} + x^2y^{12}$ | 1.875 |
| [16, 8, 4] | sd | [14] | $x^{16} + 12x^{12}y^4 + 64x^{10}y^6 + 102x^8y^8 + 64x^6y^10 + 12x^4y^{12} + y^{16}$ | 2 |
| [16, 8, 4] | efsd | [40] | $x^{16} + 4x^{12}y^4 + 96x^{10}y^6 + 54x^8y^8 + 96x^6y^{10} + 4x^4y^{12} + y^{16}$ | 2.133 |
| [16, 8, 5] | ofsd | [34] | $x^{16} + 24x^{10}y^6 + 44x^8y^8 + 40x^6y^{10} + 45x^4y^8 + 40x^2y^{10} + 28x^2y^{12} + 24x^2y^{11} + 10x^4y^{12}$ | 2.141 |
| (16, 256, 6) | efsd | [50, p. 74] | $x^{16} + 112x^{10}y^6 + 302x^8y^8 + 112x^6y^{10} + y^{16}$ | 2.207 |
| (18, 512, 2) | efsd | Ex. 53 | $x^{18} + x^{16}y^2 + 112x^{12}y^6 + 142x^{10}y^8 + 142x^8y^{10} + 112x^6y^{12} + x^2y^{16} + y^{18}$ | 2.207 |
| (18, 9, 3) | ofsd | Ex. 53 | $x^{18} + 12x^{15}y^3 + 9x^{14}y^4 + 48x^{12}y^6 + 72x^{11}y^7 + 27x^{10}y^8 + 64x^9y^9 + 144x^8y^{10} + 108x^7y^{11} + 27x^6y^{12}$ | 1.608 |
| (18, 9, 4) | sd | [14] | $x^{18} + 9x^{14}y^4 + 75x^{12}y^6 + 171x^{10}y^8 + 171x^8y^{10} + 75x^6y^{12} + 9x^4y^{14} + y^{18}$ | 2.286 |
| (18, 9, 6) | efsd | [41] | $x^{18} + 102x^{12}y^6 + 153x^{10}y^8 + 153x^8y^{10} + 102x^6y^{12} + y^{18}$ | 2.485 |
| (18, 9, 5) | ofsd | tb | $x^{18} + 18x^{13}y^5 + 48x^{12}y^6 + 63x^{11}y^7 + 81x^{10}y^8 + 100x^9y^9 + 72x^8y^{10} + 54x^7y^{11} + 54x^6y^{12} + 18x^5y^{13} + 3x^3y^{15}$ | 2.424 |
| [20, 10, 4] | sd | [14] | $x^{20} + 5x^{16}y^4 + 80x^{14}y^6 + 250x^{12}y^8 + 352x^{10}y^{10} + 250x^8y^{12} + 80x^6y^{14} + 5x^2y^{16} + y^{20}$ | 2.667 |
| [20, 10, 6] | efsd | [42] | $x^{20} + 90x^{14}y^6 + 255x^{12}y^8 + 332x^{10}y^{10} + 255x^8y^{12} + 90x^6y^{14} + y^{20}$ | 2.813 |
| (20, 1024, 6) | efsd | [35] | $x^{20} + 90x^{14}y^6 + 255x^{12}y^8 + 332x^{10}y^{10} + 255x^8y^{12} + 90x^6y^{14} + y^{20}$ | 2.813 |
| [20, 10, 6] | ofsd | [39] | $x^{20} + 40x^{14}y^6 + 160x^{13}y^7 + 130x^{12}y^8 + 176x^{10}y^{10} + 320x^9y^{11} + 120x^8y^{12} + 40x^6y^{14} + 32x^5y^{15} + 5x^4y^{16}$ | 2.868 |

TABLE III
Codes and Their Weight Enumerators
| Code     | sd & [14]          | $x^{22} + 77x^{16}y^{6} + 330x^{14}y^{8} + 616x^{12}y^{10} + 616x^{10}y^{12}$  |
|----------|---------------------|----------------------------------------------------------------------------------|
| [22, 11, 6] | ofsd & [34]         | $x^{22} + 44x^{16}y^{6} + 121x^{12}y^{8} + 143x^{14}y^{8} + 231x^{10}y^{9}$       |
| [22, 11, 7] | ofsd & [34]         | $x^{22} + 176x^{15}y^{7} + 330x^{14}y^{8} + 672x^{11}y^{11} + 616x^{10}y^{12}$  |
| [24, 12, 8] | sd & [19]           | $x^{24} + 759x^{16}y^{6} + 2576x^{12}y^{12} + 759x^{2}y^{16} + y^{24}$            |
| [24, 12, 6] | ofsd & [31]         | $x^{24} + 64x^{18}y^{6} + 375x^{16}y^{8} + 960x^{14}y^{10} + 1296x^{12}y^{12}$  |
| [30, 15, 6] | sd & [19]           | $x^{30} + 19x^{24}y^{6} + 393x^{22}y^{8} + 1848x^{20}y^{10} + 5192x^{18}y^{12}$  |
| [30, 15, 8] | ofsd & [43]         | $x^{30} + 30x^{24}y^{8} + 1848x^{20}y^{10} + 5040x^{18}y^{12} + 9045x^{16}y^{14}$ |
| [30, 15, 7] | ofsd & [14]         | $x^{30} + 60x^{23}y^{7} + 210x^{21}y^{8} + 500x^{19}y^{9} + 930x^{20}y^{10} + 1560x^{19}y^{11}$ |
| [31, 16, 8] | sd & [41]           | $x^{32} + 62x^{24}y^{8} + 1388x^{20}y^{12} + 36518x^{16}y^{16} + 1388x^{12}y^{20}$ |
| [32, 16, 8] | ofsd & [43]         | $x^{32} + 364x^{24}y^{8} + 2048x^{22}y^{10} + 6720x^{20}y^{12} + 14336x^{18}y^{14}$ |
| [32, 16, 8] | ofsd & [31]         | $x^{32} + 348x^{21}y^{7} + 2176x^{22}y^{10} + 672x^{20}y^{12} + 15232x^{18}y^{14}$ |
| [32, 16, 7] | ofsd & [41]         | $x^{32} + 64x^{25}y^{7} + 176x^{24}y^{8} + 384x^{21}y^{9} + 984x^{22}y^{10} + 2096x^{21}y^{11}$ |
| [31, 20, 8] | sd & [31]           | $x^{40} + 285x^{32}y^{8} + 21280x^{28}y^{12} + 239970x^{24}y^{16} + 525504x^{20}y^{20}$ |
| [40, 20, 8] | sd & [31]           | $x^{40} + 125x^{32}y^{8} + 1664x^{28}y^{10} + 10720x^{24}y^{12} + 44160x^{20}y^{14}$ |
| [40, 20, 8] | ofsd & [31]         | $x^{40} + 150x^{32}y^{8} + 1564x^{28}y^{10} + 10770x^{24}y^{12} + 44460x^{20}y^{14}$ |
| [40, 20, 8] | ofsd & [31]         | $x^{40} + 360x^{31}y^{9} + 922x^{29}y^{10} + 2060x^{27}y^{11} + 5778x^{28}y^{12}$   |
| [40, 20, 9] | ofsd & [31]         | $x^{40} + 360x^{31}y^{9} + 922x^{29}y^{10} + 2060x^{27}y^{11} + 5778x^{28}y^{12}$   |
| [42, 21, 10] | efsd | tb | \( x^{42} + 1722 x^{32} y^{10} + 10619 x^{30} y^{12} + 49815 x^{28} y^{14} + 157563 x^{26} y^{16} + 341530 x^{24} y^{18} + 487326 x^{22} y^{20} + 487326 x^{20} y^{22} + 341530 x^{18} y^{24} + 157563 x^{16} y^{26} + 49815 x^{14} y^{28} + 10619 x^{12} y^{30} + 1722 x^{10} y^{32} + 9 y^{42} \) | 14.482 |
| [56, 28, 12] | efsd | tb | \( x^{56} + 4634 x^{44} y^{12} + 44828 x^{42} y^{14} + 307650 x^{40} y^{16} + 157592 x^{38} y^{18} + 586538 x^{36} y^{20} + 15906960 x^{34} y^{22} + 3240013 x^{32} y^{24} + 49502068 x^{30} y^{26} + 57033513 x^{28} y^{28} + 49502068 x^{26} y^{30} + 3240013 x^{24} y^{32} + 15906960 x^{22} y^{34} + 586538 x^{20} y^{36} + 157592 x^{18} y^{38} + 307650 x^{16} y^{40} + 44828 x^{14} y^{42} + 4634 x^{12} y^{44} + y^{56} \) | 42.838 |
| [70, 35, 12] | sd | [44] | \( x^{70} + 832 x^{58} y^{12} + 10770 x^{56} y^{14} + 142279 x^{54} y^{16} + 1353320 x^{52} y^{18} + 9437352 x^{50} y^{20} + 49957193 x^{48} y^{22} + 204165154 x^{46} y^{24} + 650426976 x^{44} y^{26} + 1627816992 x^{42} y^{28} + 3221537512 x^{40} y^{30} + 5066102223 x^{38} y^{32} + 6348918576 x^{36} y^{34} + 6348918576 x^{34} y^{36} + 5066102223 x^{32} y^{38} + 3221537512 x^{30} y^{40} + 1627816992 x^{28} y^{42} + 650426976 x^{26} y^{44} + 204165154 x^{24} y^{46} + 49957193 x^{22} y^{48} + 9437352 x^{20} y^{50} + 1353320 x^{18} y^{52} + 142279 x^{16} y^{54} + 10770 x^{14} y^{56} + 832 x^{12} y^{58} + y^{70} \) | 127.712 |
| [70, 35, 13] | ofsd | tb | \( x^{70} + 455 x^{58} y^{12} + 11235 x^{56} y^{14} + 145985 x^{54} y^{16} + 1348130 x^{52} y^{18} + 9403974 x^{50} y^{20} + 4992695 x^{48} y^{22} + 204318835 x^{46} y^{24} + 650297655 x^{44} y^{26} + 1627628010 x^{42} y^{28} + 3221888194 x^{40} y^{30} + 5066104095 x^{38} y^{32} + 6348862502 x^{36} y^{34} + 6348862502 x^{34} y^{36} + 5066104095 x^{32} y^{38} + 3221888194 x^{30} y^{40} + 1627628010 x^{28} y^{42} + 650297655 x^{26} y^{44} + 204318835 x^{24} y^{46} + 4992695 x^{22} y^{48} + 9403974 x^{20} y^{50} + 1348130 x^{18} y^{52} + 145985 x^{16} y^{54} + 11235 x^{14} y^{56} + 455 x^{12} y^{58} + y^{70} \) | 128.073 |
| [72, 36, 16] | sd | [41] | \( x^{72} + 2982 x^{60} y^{12} + 214065 x^{58} y^{14} + 18303516 x^{56} y^{16} + 462300915 x^{54} y^{18} + 4398818490 x^{52} y^{20} + 1660354155 x^{50} y^{22} + 2575947648 x^{48} y^{24} + 1600354155 x^{46} y^{26} + 4398818490 x^{44} y^{28} + 462300915 x^{42} y^{30} + 18303516 x^{40} y^{32} + 214065 x^{38} y^{34} + 2982 x^{36} y^{36} + y^{72} \) | 146.844 |
| [78, 39, 14] | efsd | tb | \( x^{78} + 3471 x^{66} y^{14} + 63363 x^{64} y^{16} + 772980 x^{62} y^{18} + 7219368 x^{60} y^{20} + 51527346 x^{58} y^{22} + 287551706 x^{56} y^{24} + 1266693912 x^{54} y^{26} + 4442835540 x^{52} y^{28} + 12510913844 x^{50} y^{30} + 2845316744 x^{48} y^{32} + 5249394668 x^{46} y^{34} + 7882380270 x^{44} y^{36} + 9653904862 x^{42} y^{38} + 9653904862 x^{40} y^{40} + 7882380270 x^{38} y^{42} + 5249394668 x^{36} y^{44} + 2845316744 x^{34} y^{46} + 12510913844 x^{32} y^{48} + 4442835540 x^{30} y^{50} + 1266693912 x^{28} y^{52} + 287551706 x^{26} y^{54} + 51527346 x^{24} y^{56} + 7219368 x^{22} y^{58} + 772980 x^{20} y^{60} + 63363 x^{18} y^{62} + 3471 x^{16} y^{64} - y^{78} \) | 241.042 |
TABLE III  
(Continued.) CODES AND THEIR WEIGHT ENUMERATORS

| C   | CODES AND THEIR WEIGHT ENUMERATORS |
|-----|-----------------------------------|
| 104, 52, 20 | \(x^{104} + 1138150 x^{84} y^{20} + 206232780 x^{80} y^{24} + 1590698064 x^{76} y^{28} + 567725836992 x^{72} y^{32} + 991518501320 x^{68} y^{36} + 883557097889052 x^{64} y^{40} + 413543821457520 x^{60} y^{44} + 1036378989344140 x^{56} y^{48} + 140604530291756 x^{52} y^{52} + 1036378989344140 x^{48} y^{56} + 415343821457520 x^{44} y^{60} + 883557097889052 x^{40} y^{64} + 991518501320 x^{36} y^{68} + 567725836992 x^{32} y^{72} + 1590698064 x^{28} y^{76} + 206232780 x^{24} y^{80} + 1138150 x^{20} y^{84} + 5022 x^{16} y^{90} + 3035 x^{12} y^{94} + 3035 x^{8} y^{96} + 3035 x^{4} y^{98} + 1885.06 |
| 108, 54, 14 | \(x^{108} + 756 x^{94} y^{14} + 5022 x^{90} y^{16} + 3035 x^{86} y^{18} + 371223 x^{88} y^{20} + 5418846 x^{86} y^{22} + 71058978 x^{84} y^{24} + 765738684 x^{82} y^{26} + 6738702390 x^{80} y^{28} + 48969093384 x^{78} y^{30} + 29643892362 x^{76} y^{32} + 150587581558 x^{74} y^{34} + 645619068648 x^{72} y^{36} + 2347380436104 x^{70} y^{38} + 7267868868432 x^{68} y^{40} + 192289983824460 x^{66} y^{42} + 436005471914253 x^{64} y^{44} + 849263560631748 x^{62} y^{46} + 1423721087648100 x^{60} y^{48} + 205713311131674 x^{58} y^{50} + 25644330382478 x^{56} y^{52} + 275097610146792 x^{54} y^{54} + 25644330382478 x^{52} y^{56} + 205713311131674 x^{50} y^{58} + 1423721087648100 x^{48} y^{60} + 849263560631748 x^{46} y^{62} + 436005471914253 x^{44} y^{64} + 192289983824460 x^{42} y^{66} + 7267868868432 x^{40} y^{68} + 2347380436104 x^{38} y^{70} + 645619068648 x^{36} y^{72} + 150587581558 x^{34} y^{74} + 29643892362 x^{32} y^{76} + 48969093384 x^{30} y^{78} + 6738702390 x^{28} y^{80} + 765738684 x^{26} y^{82} + 71058978 x^{24} y^{84} + 5418846 x^{22} y^{86} + 371223 x^{20} y^{88} + 3035 x^{18} y^{90} + 5022 x^{16} y^{92} + 756 x^{14} y^{94} + 3035 x^{12} y^{96} + 1885.06 |

Appendix A  
WEIGHT ENUMERATORS OF CODES FOR TABLE II

See Table III.

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