LIFESPAN OF SOLUTIONS TO A PARABOLIC TYPE KIRCHHOFF EQUATION WITH TIME-DEPENDENT NONLINEARITY

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Abstract. In this paper, an initial boundary value problem for a parabolic type Kirchhoff equation with time-dependent nonlinearity is considered. A new blow-up criterion for nonnegative initial energy is given and upper and lower bounds for the blow-up time are also derived. These results partially generalize some recent ones obtained by Han and Li in [Threshold results for the existence of global and blow-up solutions to Kirchhoff equations with arbitrary initial energy, Computers and Mathematics with Applications, 75(2018), 3283-3297].

1. Introduction. In this paper, we consider the following initial boundary value problem for a parabolic type Kirchhoff equation

\[
\begin{aligned}
&u_t - M(\int_\Omega |\nabla u|^2 \, dx) \Delta u = k(t) f(u), & (x, t) \in \Omega \times (0, T), \\
&u = 0, & (x, t) \in \partial \Omega \times (0, T), \\
&u(x, 0) = u_0(x), & x \in \Omega,
\end{aligned}
\]

where \( M(s) = a + bs, \ a, b > 0, \ \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with smooth boundary \( \partial \Omega \), \( u_0 \in H^1_0(\Omega) \) and \( T \in (0, \infty] \) is the maximal existence time of the solution \( u(x, t) \). Moreover, we assume that \( k(t) \) and \( f(u) \) satisfy the following assumptions

(A1) \( k \in C^1[0, +\infty), \ k(0) > 0, \ k'(t) \geq 0, \ \forall \ t \in [0, +\infty); \)

(A2) \( sf'(s) \geq 0, \ \forall \ s \in \mathbb{R}; \)

(A3) \( f \in C^1(\mathbb{R}) \) and there exists a constant \( p \geq 3 \) such that

\[ s[f'(s) - pf(s)] \geq 0, \ \forall \ s \in \mathbb{R}; \]

(A4) there exist constants \( \alpha, \beta > 0 \) and \( q \in (0, 2^* - 1) \) such that

\[ |f(s)| \leq \alpha + \beta|s|^q, \ \forall \ s \in \mathbb{R}, \]

where \( 2^* = \infty \) for \( n = 1, 2 \) and \( 2^* = \frac{2n}{n - 2} \) for \( n \geq 3 \).

The main feature of the equation in (1) is that the coefficient of the diffusion term depends on the \( L^2(\Omega) \) norm of the gradient of the unknown \( u(x, t) \), which means...
that the equation is no longer a pointwise identity. Therefore, the equation is usually referred to as nonlocal equation or Kirchhoff equation. For a quick start, we refer the interested readers to \[1, 2, 7, 13\] and the references therein for the background of problem (1). When \(k(t) \equiv 1\) and \(f(u) = |u|^{p-1}u\), problem (1) was comprehensively studied in \[7\] at three different initial energy levels, i.e., subcritical initial energy level \(J(u_0) < d\), critical initial energy level \(J(u_0) = d\) and super-critical initial energy level \(J(u_0) > d\). For each initial energy level, sufficient conditions on the initial data are given for the solutions to exist globally or to blow up (in the sense of \(L^2\)-norm) in finite time. Here \(d > 0\) is the depth of the potential well and

\[
J(u_0) = \frac{a}{2} \| \nabla u_0 \|^2_2 + \frac{b}{4} \| \nabla u_0 \|^4_2 - \frac{1}{p+1} \| u_0 \|_{p+1}^{p+1}
\]

is the initial energy. Later, Han et. al \[6\] re-considered the finite time blow-up properties of this problem and obtained the upper and lower bounds for the blow-up time as well as a new blow-up criterion.

Motivated mainly by \[7\], we shall consider the blow-up phenomena for problem (1) and investigate what role the weight function \(k(t)\) plays in determining the blow-up condition and blow-up time for problem (1). To be a little more precise, we shall show, under the assumptions (A1)-(A4), that the solutions to problem (1) blow up in finite time if one of the following two assumptions holds:

(i) the initial energy is negative, i.e., \(J(u_0; 0) < 0\);

(ii) \(0 \leq J(u_0; 0) < \frac{a(p-1)\lambda_1}{2(p+1)}\| u_0 \|^2_2\).

Here \(J(u_0; 0) = \frac{a}{2} \| \nabla u_0 \|^2_2 + \frac{b}{4} \| \nabla u_0 \|^4_2 - k(0) \int_\Omega F(u_0)dx\), \(F(s) = \int_0^s f(t)dt\) and \(\lambda_1 > 0\) is the first eigenvalue of \(-\Delta\) in \(\Omega\) with homogeneous Dirichlet boundary condition. Furthermore, the upper and lower bounds for the blow-up time are also derived. It is easily verified that a prototype of \(f(s)\) is a combination of some power type nonlinearities. In particular, our research includes the case \(f(s) = |s|^{p-1}s\), which was considered in \[7\].

The rest of this paper is organized as follows. In Section 2, we shall introduce some definitions and auxiliary lemmas as preliminaries. In Section 3, we give two sufficient conditions for the solutions to problem (1) to blow up in finite time, and derive an upper bound for the blow-up time for each case. A lower bound for the blow-up time will be derived in Section 4.

2. Preliminaries. We begin this section with some notations and definitions. Throughout this paper, we denote by \(\| \cdot \|_{r}\) the \(L^r(\Omega)\) norm of a Lebesgue function \(u \in L^r(\Omega)\) for \(r \geq 1\) and by \((\cdot, \cdot)\) the inner product in \(L^2(\Omega)\). We use \(H^1_0(\Omega)\) to denote the well-known Sobolev space such that both \(u\) and \(|\nabla u|\) are in \(L^2(\Omega)\) equipped with the norm \(\| u \|_{H^1_0(\Omega)} = \| \nabla u \|_2\), which, due to Poincaré’s inequality, is equivalent to the standard one.

In this paper, we consider weak solutions to problem (1), local existence of which can be obtained by combining the standard Galerkin’s approximation with Aubin-Lions compactness theorem. Interested readers may refer to \[3\] for the proof of local existence of weak solutions to a Kirchhoff equation with variable exponent, which contains (1) as a special case. If no confusion arises, we sometimes simply write \(u(t)\) to denote the weak solution \(u(x, t)\) to problem (1).

Definition 2.1. A function \(u = u(x, t) \in L^\infty(0, T; H^1_0(\Omega))\) with \(u_t \in L^2(0, T; L^2(\Omega))\) is called a weak solution to problem (1) on \(\Omega \times [0, T]\), if \(u(x, 0) = u_0 \in H^1_0(\Omega)\)
and satisfies
\[(u_t, \phi) + (a + b\|\nabla u\|_2^2)(\nabla u, \nabla \phi) = k(t)(f(u), \phi), \quad a.e. \ t \in (0, T), \quad (2)\]
for any $\phi \in H^1_0(\Omega)$. We say that $u(x, t)$ blows up at a finite time $T$ provided that
\[\lim_{t \to T^-} \|u(\cdot, t)\|_2^2 = +\infty.\]

Due to the appearance of $k(t)$, we define, for each $u \in H^1_0(\Omega)$ and $t \geq 0$, the time-dependent energy functional and Nehari functional, respectively, by
\[
J(u; t) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - k(t)\int_{\Omega} F(u)dx, \quad (3)
\]
\[
I(u; t) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - k(t)\int_{\Omega} uf(u)dx, \quad (4)
\]
where $F(s) = \int_0^s f(t)dt$ which is nonnegative for all $s \in \mathbb{R}$ by (A2). From (A4) one sees that both $J(\cdot; t)$ and $I(\cdot; t)$ are well defined and continuous in $H^1_0(\Omega)$ for any $t \geq 0$. Moreover, by applying a standard density argument, we can obtain the following conclusions
\[
J(u(t); t) + \int_0^t \left[\|u_\tau(\tau)\|_2^2 + k'(\tau)\int_{\Omega} F(u(\tau))dx\right]d\tau = J(u_0; 0), \quad t \in (0, T), \quad (5)
\]
\[
d\left(\frac{1}{2}\|u(t)\|_2^2\right) = (u, u_t) = -I(u(t); t), \quad t \in (0, T). \quad (6)
\]

**Remark 1.** From assumption (A3) we have
\[sf(s) \geq (p + 1)F(s), \quad s \in \mathbb{R}. \quad (7)\]

Therefore, for any $u \in H^1_0(\Omega)$ and $t \geq 0$, it is directly verified from (3), (4) and (7) that
\[
J(u; t) \geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{k(t)}{p + 1}\int_{\Omega} uf(u)dx
\]
\[= \frac{a(p - 1)}{2(p + 1)}\|\nabla u\|_2^2 + \frac{b(p - 3)}{4(p + 1)}\|\nabla u\|_2^4 + \frac{1}{p + 1}I(u; t) \quad (8)\]
\[\geq \frac{a(p - 1)}{2(p + 1)}\|\nabla u\|_2^2 + \frac{1}{p + 1}I(u; t). \quad (since \ p \geq 3)\]

At the end of this section, we recall a result on concave function, which will be needed to derive an upper bound for the blow up time when the initial energy is nonnegative.

**Lemma 2.2.** (See [5, 8]) Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality
\[\psi''(t)\psi(t) - (1 + \theta)(\psi'(t))^2 \geq 0, \quad (9)\]
where $\theta > 0$. If $\psi(0) > 0$, $\psi'(0) > 0$, then $\psi(t) \to \infty$ as $t \to t^* \leq t^* = \frac{\psi(0)}{\theta\psi'(0)}$.

**Proof.** The proof of this lemma is more or less standard. We only sketch the outline for the convenience of the reader.

Set $\varphi(t) = \psi^{-\theta}(t)$. Then by direct computation,
\[\varphi'(t) = -\theta\psi^{-\theta-1}(t)\psi'(t), \quad (10)\]
\[\phi''(t) = (-\theta)(-\theta - 1)\psi^{-\theta - 2}(t)(\psi'(t))^2 - \theta \psi^{-\theta - 1}(t)\psi''(t)\]
\[= (-\theta)\psi^{-\theta - 2}(t)[\psi(t)\psi''(t) - (1 + \theta)(\psi'(t))^2]\]

\[\leq 0.\]  

It follows from (11) that \(\varphi(t)\) is convex and therefore

\[\varphi(t) \leq \varphi(0) + \varphi'(0)t,\]  

as long as it exists. Recalling that \(\varphi'(0) = -\theta \psi^{-\theta - 1}(0)\psi'(0) < 0\), (12) ensures that \(\varphi(t) \to 0\) as \(t \to t_*\). The proof is complete.

3. Upper bound for the blow up time. In this section, we shall give two blow up criteria for problem (1) and derive an upper bound for the blow up time, with the help of the preliminaries given in Section 2 and some differential inequalities.

Theorem 3.1. Let the assumptions (A1)-(A4) hold and \(u(t)\) be a weak solution to problem (1). Suppose that one of the following statements holds:

(i) \(J(u_0;0) < 0\);

(ii) \(0 \leq J(u_0;0) < C_0\|u_0\|_2^2 \defeq \frac{a(p - 1)\lambda_1}{2(p + 1)}\|u_0\|_2^2\), where \(\lambda_1 > 0\) is the first eigenvalue of \(-\Delta\) in \(\Omega\) with homogeneous Dirichlet boundary condition.

Then \(T < +\infty\), which implies that \(u(x,t)\) blows up in finite time. Moreover, an upper bound for \(T\) has the following form:

In case (i), \(T \leq \frac{\|u_0\|_2^2}{(1 - p^2) J(u_0;0)}\).

In case (ii), \(T \leq \frac{8p\|u_0\|_2^2}{(p - 1)^2[a(p - 1)\lambda_1\|u_0\|_2^2 - 2(p + 1)J(u_0;0)]}\).

Proof. (i) We shall prove this case by applying the first order differential inequality method, mainly inspired by [10]. Let \(u(t)\) be a weak solution to problem (1) and set \(\Phi(t) = \frac{1}{2}\|u(t)\|_2^2\), \(\Psi(t) = -J(u(t);t)\). Then \(\Phi(0) > 0\), \(\Psi(0) > 0\). Moreover, it follows from (A1), (A2) and (5) that

\[\Psi'(t) = -\frac{d}{dt}J(u(t);t) = \|u_t(t)\|_2^2 + k'(t)\int_{\Omega} F(u(t))dx \geq 0,\]

which implies \(\Psi(t) \geq \Psi(0) > 0\) for all \(t \in [0,T]\). In view of (3), (4), (8) and (6), we obtain, for any \(t \in [0,T]\), that

\[\Phi'(t) = (u_t(t),u_t(t)) = -I(u(t);t) \geq \frac{a(p - 1)}{2}\|\nabla u(t)\|_2^2 - (p + 1)J(u(t);t) \geq (p + 1)\Psi(t).\]

Combining (5) with (13), recalling the nonnegativity of \(k'(t)\) and \(F(s)\) and making use of Cauchy-Schwarz inequality, we arrive at

\[\Phi(t)\Psi'(t) \geq \frac{1}{2}\|u_t(t)\|_2^2\|u_t(t)\|_2^2 \geq \frac{1}{2}(u_t(t),u_t(t))^2 = \frac{1}{2}(\Phi'(t))^2 \geq \frac{p + 1}{2}\Phi'(t)\Psi(t).\]  

(14)

Rewrite (14) as follows

\[\left(\Psi(t)\Phi^{-\frac{p+1}{2}}(t)\right)' = \Phi^{-\frac{p+1}{2}}(t)\left(\Phi(t)\Psi'(t) - \frac{p + 1}{2}\Phi'(t)\Psi(t)\right) \geq 0.\]
Therefore, \( \Psi(t) \Phi^{-\frac{p+1}{p}}(t) \) is monotone increasing on \((0, T)\), which then guarantees that
\[
0 < \eta := \Psi(0) \Phi^{-\frac{p+1}{p}}(0) \leq \Psi(t) \Phi^{-\frac{p+1}{p}}(t) \leq \frac{2}{1-p^2} \left( \Phi^{-\frac{p}{2}}(t) - \Phi^{-\frac{p}{2}}(0) \right),
\] (15)
Integrating (15) over \([0, t]\) for any \(t \in (0, T)\) and noticing that \(p \geq 3\), one has
\[
\eta t \leq \frac{2}{1-p^2} \left( \Phi^{-\frac{p}{2}}(t) - \Phi^{-\frac{p}{2}}(0) \right),
\]
or equivalently
\[
0 \leq \Phi^{-\frac{p}{2}}(t) \leq \Phi^{-\frac{p}{2}}(0) - \frac{p^2 - 1}{2} \eta t, \quad t \in (0, T).
\] (16)
Obviously, (16) cannot hold for all \(t > 0\). Therefore, \(T < +\infty\) and it can be inferred from (16) that
\[
T \leq \frac{2}{(p^2 - 1)\eta} \Phi^{-\frac{p}{2}}(0) = \frac{2\Phi(0)}{(1-p^2)J(u_0;0)} = \frac{||u_0||_2^2}{(1-p^2)J(u_0;0)}.
\]
(ii) With the help of (7), we can show that the solutions to problem (1) blow up in finite time when \(0 \leq J(u_0) < C_0||u_0||_2^2\), by applying the concavity arguments due to Levine [8]. Similar treatments can also be found in [4, 11, 12] to deal with other evolution problems.
First, it follows from (8), the assumption (ii) and the definition of \(\lambda_1\) that
\[
I(u_0;0) \leq (p+1)J(u_0;0) - \frac{a(p-1)}{2} ||\nabla u_0||_2^2 = (p+1) \left[ J(u_0;0) - C_0||u_0||_2^2 \right] - \frac{a(p-1)}{2} \left[ ||\nabla u_0||_2^2 - \lambda_1||u_0||_2^2 \right] < 0.
\]
We claim that \(I(u(t); t) < 0\) for all \(t \in [0, T)\). If not, there would exist a \(t_0 \in (0, T)\) such that \(I(u(t); t) < 0\) for all \(t \in [0, t_0)\) and \(I(u(t_0); t_0) = 0\). On one hand, it follows from (13) that \(||u(t)||_2^2\) is strictly increasing in \(t\) for \(t \in [0, t_0)\), and therefore
\[
0 \leq J(u_0;0) < C_0||u_0||_2^2 < C_0||u(t_0)||_2^2.
\] (17)
On the other hand, from the monotonicity of \(J(u(t); t)\), (8) and the variational characteristic of \(\lambda_1\), we obtain
\[
J(u_0;0) \geq J(u(t_0); t_0) \geq \frac{a(p-1)}{2(p+1)} ||\nabla u(t_0)||_2^2 + \frac{1}{q+1} I(u(t_0); t_0) \geq \frac{a(p-1)\lambda_1}{2(p+1)} ||u(t_0)||_2^2 = C_0||u(t_0)||_2^2,
\]
which contradicts (17). Therefore, \(I(u(t); t) < 0\) for all \(t \in [0, T)\) as claimed, and \(||u(t)||_2^2\) is strictly increasing on \([0, T)\).
For any \(T^* \in (0, T), \beta > 0\) and \(\sigma > 0\), define
\[
G(t) = \int_0^t ||u(\tau)||_2^2 d\tau - (T^* - t)||u_0||_2^2 + \beta(t + \sigma)^2, \quad t \in [0, T^*].
\] (18)
Taking derivative with respect to $t$, one obtains

$$G'(t) = \|u(t)\|^2_2 - \|u_0\|^2_2 + 2\beta(t + \sigma) = \int_0^t \frac{d}{dt}\|u(\tau)\|^2_2 d\tau + 2\beta(t + \sigma)$$

$$= 2\int_0^t (u, u_\tau)d\tau + 2\beta(t + \sigma),$$

(19)

Taking derivative again and recalling (8) and (5), we have

$$G''(t) = 2\langle u, u_t \rangle + 2\beta = -2I(u(t); t) + 2\beta$$

$$\geq -2(p + 1)J(u(t); t) + a(p - 1)\|\nabla u(t)\|^2_2 + 2\beta$$

$$= -2(p + 1)J(u_0; 0) + 2(p + 1)\int_0^t \left[\|u_\tau(\tau)\|^2_2 + k'(\tau)\int_\Omega F(u(\tau))d\tau\right]d\tau$$

$$+ a(p - 1)\|\nabla u(t)\|^2_2 + 2\beta$$

$$\geq -2(p + 1)J(u_0; 0) + 2(p + 1)\int_0^t \|u_\tau(\tau)\|^2_2 d\tau + a(p - 1)\|u(t)\|^2_2 + 2\beta.$$  

(20)

For $t \in [0, T^\star]$, set

$$\kappa(t) = \left(\int_0^t \|u(\tau)\|^2_2 d\tau + \beta(t + \sigma)^2\right) \left(\int_0^t \|u_\tau(\tau)\|^2_2 d\tau + \beta\right) - \left(\int_0^t (u, u_\tau)d\tau + \beta(t + \sigma)\right)^2,$$

which is nonnegative on $[0, T^\star]$, by Cauchy-Schwarz inequality and Hölder’s inequality.

In view of (18)-(20) and recalling the monotonicity of $\|u(t)\|^2_2$, we obtain

$$G(t)G''(t) - \frac{p + 1}{2}(G'(t))^2$$

$$= G(t)G''(t) - 2(p + 1)\left(\int_0^t (u, u_\tau)d\tau + \beta(t + \sigma)\right)^2$$

$$= G(t)G''(t) + 2(p + 1)\left[\kappa(t) - (G(t) - (T^\star - t)\|u_0\|^2_2)(\int_0^t \|u_\tau\|^2_2 d\tau + \beta)\right]$$

$$\geq G(t)G''(t) - 2(p + 1)G(t)(\int_0^t \|u_\tau\|^2_2 d\tau + \beta)$$

$$\geq G(t) \left[-2(p + 1)J(u_0; 0) + 2(p + 1)\int_0^t \|u_\tau\|^2_2 d\tau + a(p - 1)\|u(t)\|^2_2 + 2\beta\right]$$

$$+ 2\beta - 2(p + 1)\int_0^t \|u_\tau\|^2_2 d\tau - 2(p + 1)\beta$$

$$\geq G(t) \left[-2(p + 1)J(u_0; 0) + a(p - 1)\|u_0\|^2_2 - 2p\beta\right]$$

$$= 2(p + 1)G(t)\left[C_0\|u_0\|^2_2 - J(u_0; 0) - \frac{p\beta}{p + 1}\right].$$

Therefore, for any $t \in [0, T^\star]$ and $\beta \in \left(0, \frac{p + 1}{p}(C_0\|u_0\|^2_2 - J(u_0; 0))\right]$, we have

$$G(t)G''(t) - \frac{p + 1}{2}(G'(t))^2 \geq 0.$$
An application of Lemma 2.2 to $G(t)$ guarantees that
\[ T^* \leq \frac{2G(0)}{(p-1)G'(0)} = \frac{2(T^*\|u_0\|^2 + \beta \sigma^2)}{2(p-1)\beta \sigma} = \frac{\|u_0\|^2}{(p-1)\beta \sigma} T^* + \frac{\sigma}{p-1}, \]
or equivalently
\[ T^* \left(1 - \frac{\|u_0\|^2}{(p-1)\beta \sigma} \right) \leq \frac{\sigma}{p-1}. \tag{22} \]
for any $\beta \in \left(0, \frac{p+1}{p}(C_0\|u_0\|^2 - J(u_0;0))\right)$ and $\sigma > 0$.

Fix $\beta_0 \in \left(0, \frac{p+1}{p}(C_0\|u_0\|^2 - J(u_0;0))\right)$. Then for any $\sigma \in \left(\frac{\|u_0\|^2}{(p-1)\beta_0}, +\infty\right)$, we have $0 < \frac{\|u_0\|^2}{(p-1)\beta_0} < 1$, which, together with (22), implies that
\[ T^* \leq \frac{\sigma}{p-1} \left(1 - \frac{\|u_0\|^2}{(p-1)\beta_0} \right)^{-1} = \frac{\beta_0 \sigma^2}{(p-1)\beta_0 \sigma - \|u_0\|^2}. \tag{23} \]
Minimizing the right hand side term in (23) first for $\sigma \in \left(\frac{\|u_0\|^2}{(p-1)\beta_0}, +\infty\right)$ and then for $\beta_0 \in \left(0, \frac{p+1}{p}(C_0\|u_0\|^2 - J(u_0;0))\right)$, one finally obtains
\[ T^* \leq \frac{8\sigma}{(p-1)^2|a(p-1)|\lambda_1\|u_0\|_2^2 - 2(p+1)J(u_0;0)}. \]
By the arbitrariness of $T^* < T$, it follows that
\[ T \leq \frac{8\sigma}{(p-1)^2|a(p-1)|\lambda_1\|u_0\|_2^2 - 2(p+1)J(u_0;0)}. \]
The proof is complete. $\square$

**Remark 2.** When $k(t) \equiv 1$ and $f(u) = |u|^{p-1}u$ with $3 < p < 2^* - 1$, it was shown in [7], by using the theory of dynamical systems and some variational tricks, that the solutions to problem (1) blow up in finite time provided that $\frac{4(p+1)}{p-3}|\Omega|^{\frac{p-1}{p}}J(u_0;0) \leq \|u_0\|_2^{2^*+1}$. Since we do not know the measure of $\Omega$ and $J(u_0;0)$, this condition and condition (ii) in Theorem 3.1 cannot be compared with each other. Therefore, we obtain a new blow-up criterion that cannot be included in [7]. Moreover, the case $p = 3$ is permitted in our research.

4. Lower bound for the blow up time. In this section, we shall derive a lower bound for the blow-up time $T$ when blow-up occurs.

**Theorem 4.1.** Assume that (A1)-(A3) hold and that (A4) holds with $q \in (1, \frac{2^*}{2}]$. Let $u(t)$ be a weak solution to problem (1) that blows up at $T$. Then
\[ T \geq \int_{N(0)}^\infty \frac{ds}{C_1 + C_2 s^q}, \]
where $N(0)$, $C_1$ and $C_2$ are positive constants that will be determined in the proof.

**Proof.** We aim to determine a time interval $(0, T_0)$ on which the quantity $\|\nabla u(t)\|_2^2$ is bounded. Clearly $T_0$ is a lower bound for $T$ since $\|u\|_2^2 \leq \lambda_1^{-1}\|\nabla u\|_2^2$ for any $u \in H_0^1(\Omega)$.

Set
\[ N(t) := \gamma(k(t))\|\nabla u(t)\|_2^2, \quad t \in [0, T). \tag{24} \]
where \( \gamma(k) := e^{2k/(1-q)k^2/q} \) for \( k \in C^1[0, +\infty) \). Then
\[
\lim_{t \to T^-} N(t) = +\infty. \tag{25}
\]

Differentiating (24), making use of Green’s second identity and the arithmetic-geometric inequality, we have
\[
N'(t) = (\gamma(k(t))' \|\nabla u(t)\|^2_2 + 2\gamma(k(t)) \int_\Omega \nabla u \nabla u dx
\]
\[
= (\gamma(k(t))' \|\nabla u(t)\|^2_2 - 2\gamma(k(t)) \int_\Omega u_t \Delta u dx
\]
\[
= c\|\nabla u(t)\|^2_2 - 2\alpha \gamma(k(t)) \|\Delta u(t)\|^2_2 - 2b \gamma(k(t)) \|\nabla u(t)\|^2_2 \|\Delta u(t)\|^2_2
\]
\[
- 2e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) \int_\Omega f(u) \Delta u dx
\]
\[
\leq (\gamma(k(t))' \|\nabla u(t)\|^2_2 - 2\alpha \gamma(k(t)) \|\Delta u(t)\|^2_2 - 2e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) \int_\Omega f(u) \Delta u dx
\]
\[
\leq (\gamma(k(t))' \|\nabla u(t)\|^2_2 + \frac{1}{2a} \frac{e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t)}{k^{\frac{q+1}{2}} (t)} \int_\Omega f^2(u) dx
\]
\[
= \left[ \frac{2}{q-1} \frac{e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t)}{k^{\frac{q+1}{2}} (t)} \left( 1 - \frac{k(t)}{k(0)} \right) \right] \|\nabla u(t)\|_2^2
\]
\[
+ \frac{1}{2a} \frac{e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t)}{k^{\frac{q+1}{2}} (t)} \int_\Omega f^2(u) dx
\]
\[
\leq \frac{1}{2a} \frac{e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t)}{k^{\frac{q+1}{2}} (t)} \int_\Omega f^2(u) dx,
\]
where we use the fact that \( k'(t)[1 - \frac{k(t)}{k(0)}] \leq 0 \) to obtain the last inequality, since \( k(t) \) is monotonically increasing.

From (A4) it follows that
\[
f^2(u) \leq 2\alpha^2 + 2\beta^2 |u|^{2q}. \tag{26}
\]

Substituting (26) into the above differential inequality, one obtains
\[
N'(t) \leq \frac{\alpha^2}{a} |\Omega| e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) + \frac{\beta^2}{a} e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) \|u(t)\|_2^{2q}
\]
\[
\leq \frac{\alpha^2}{a} |\Omega| e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) + \frac{\beta^2}{a} S_q^2 e^{\frac{2k(T)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (T) \|\nabla u(t)\|_2^{2q}
\]
\[
= \frac{\alpha^2}{a} |\Omega| e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) + \frac{\beta^2}{a} S_q^2 e^{\frac{2k(T)}{1-q}qk^2/q} N^q(t)
\]
\[
\leq \frac{\alpha^2}{a} |\Omega| e^{\frac{2k(t)}{1-q}qk^2/q} k^{\frac{q+1}{2}} (t) + \frac{\beta^2}{a} S_q^2 e^{\frac{2k(T)}{1-q}qk^2/q} N^q(t)
\]
\[
:= C_1 + C_2 N^q(t), \tag{27}
\]
where \( T' \) is any finite upper bound for \( T \) and \( S_q > 0 \) is the embedding constant from \( H^1_0(\Omega) \) to \( L^{2q}(\Omega) \), i.e.,
\[
\|u\|_{2q} \leq S_q \|\nabla u\|_2, \quad \forall u \in H^1_0(\Omega).
\]

Integrating (27) over \([0, t]\), we have
\[
t \geq \int_0^t \frac{N'(s)}{C_1 + C_2 N^q(s)} ds. \tag{28}
\]
Letting $t \to T^-$ and recalling (25), we obtain
\[
T \geq \int_{N(0)}^{\infty} \frac{ds}{C_1 + C_2 s^q}.
\] (29)

The proof is complete. \(\square\)

**Remark 3.** It is worth pointing out that a lower bound for the blow-up time is obtained by Philippin [10] for a fourth order parabolic equation with $k(t)|u|^{p-1}u$ as the nonlinearity. Our assumptions on $k(t)$ are in some sense weaker than that in [10], due to the newly constructed auxiliary functional. In fact, it is easy to see that $k(t) = e^{t^2}$ satisfies (A1), while $k'(t)/k(t)$ is not bounded from above on $[0, \infty)$ (which is required in [10]). Therefore, the method used in [10] cannot be applied to this case to derive the lower bound for the blow-up time.

**Remark 4.** In [9], the authors considered a counterpart of problem (1), which is sometimes called $p$-Kirchhoff problem. The blow-up results in Section 3 can be extended to $p$-Kirchhoff problem, provided that the assumptions (A3) and (A4) are changed accordingly. The lower bound for the blow-up time may be derived by using the famous Gagliardo-Nirenberg’s interpolation inequality, as was done in [5]. Interested readers may check it.

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