Inventory Model with Price-Dependent Demand Rate and No Shortages: An Interval-Valued Linear Fractional Programming Approach

Dr. Pavan Kumar
Assistant Professor, Department of Mathematics,
Faculty of Science and Technology,
IIS University, Jaipur-302020, India.

Abstract

In this paper, an interval-valued inventory optimization model is proposed. The model involves the price-dependent demand and no shortages. The input data for this model are not fixed, but vary in some real bounded intervals. The aim is to determine the optimal order quantity, maximizing the total profit and minimizing the holding cost subjecting to three constraints: budget constraint, space constraint, and budgetary constraint on ordering cost of each item. We apply the linear fractional programming approach based on interval numbers. To apply this approach, a linear fractional programming problem is modeled with interval type uncertainty. This problem is further converted to an optimization problem with interval-valued objective function having its bounds as linear fractional functions. Two numerical examples in crisp case and interval-valued case are solved to illustrate the proposed approach.

Keywords

Inventory model, Price-dependent demand, Interval-valued function, Fractional programming.

1. INTRODUCTION

For many practical inventory problems, some parameters are uncertain. Therefore to obtain more realistic results, the uncertainty in parameters must be considered, and the corresponding uncertain optimization methods must be constructed. Some techniques have been developed to solve uncertain optimization problems. The fuzzy and stochastic approaches are normally applied to describe imprecise characteristics. In these two types of techniques, the membership function and probability distribution play important roles.

This is, however, practically difficult to specify a precise probability distribution or membership function for some uncertain parameter. Over the past few decades, the interval number programming approach has been developed to deal with such type of problems in which the bounds of the uncertain parameters are only required. An optimization problem with interval coefficients is referred as the interval-valued optimization problem. In this case, the coefficients are assumed as closed intervals.

Charnes et al (1977) proposed a concept to solve the linear programming problems in which the constraints are considered as closed intervals. Steuer (1981) developed some algorithms to solve the linear programming problems with interval objective function coefficients. Alefeld and Herzberger (1983) introduced the concept of computations on interval-valued numbers. Ishibuchi and Tanaka (1990), and then Chinneck and Ramadan (2000) proposed the linear...
programming problem (LPP) with interval coefficients in objective function. Tong (1994) studied the interval number and fuzzy number linear programming, and considered the case in which the coefficients of the objective function and constraints are all interval numbers. The possible interval-solution was obtained by taking the maximum value range and minimum value range inequalities as constraint conditions.

Moore and Lodwick (2003) further extended the theory of interval numbers and fuzzy numbers. Mráz (1998) proposed an approach to determine the exact bounds of optimal values in LP with interval coefficients. In the interval number programming, the objective function and constraints may not always be linear, but are often nonlinear. Levin (1999) introduced the nonlinear optimization under interval uncertainty. Sengupta et al (2001) studied the Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming.

The optimality conditions of Karush–Kuhn–Tucker (KKT) possess a significant role in optimization theory. In literature, various approaches to interval-valued optimization problems have been proposed in details, while few papers studied the KKT conditions for interval-valued optimization problems. Wu (2007) used the KKT conditions to solve an optimization problem with interval-valued objective function. Wu (2008a) studied the interval-valued nonlinear programming problems, and then Wu (2008b) & (2010) established the Wolfe duality for optimization problems with interval-valued objective functions. Bhurjee and Panda (2012) suggested a technique to obtain an efficient solution of interval optimization problem.

In single objective optimization, the aim is to find the best solution which corresponds to the minimum or maximum value of a single objective function. As far as the applications of nonlinear programming problems are considered, a ratio of two functions is to be maximized or minimized under certain number of constraints. In some other applications, the objective function involves more than one ratio of functions. The problems of optimizing one or more ratios of functions are referred as fractional programming problems. Normally, most of the multi objective fractional programming problems are first converted into single objective fractional programming problems and then solved. The decision maker (DM) must altogether optimize these conflicting goals in a framework of fuzzy aspiration levels. Charnes and Cooper (1962) proposed that a linear fractional program with one ratio can be reduced to a linear program using a nonlinear variable transformation. Chanas and Kuchta (1996) generalized the solution concepts of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals.

In the recent years, a kind of new applications of fractional programming was found in inventory optimization problems. Sadjadi et al (2005) proposed a fuzzy approach to solve a multi objective linear fractional inventory model. They considered two objective functions as fractional with two constraints: space capacity constraint, and budget constraint, and aimed to simultaneously maximize the profit ratio to holding cost and to minimize the back orders ratio to total ordered quantities. Carla Oliveira et al (2007) presented an overview of multiple objective linear programming models with interval coefficients. Some applications of multi objective linear fractional programming in inventory were proposed by Toksari (2008). Effati et al (2012) presented an approach to solve the interval-valued linear fractional programming problem.

Liu (2006) derived a computational method for the profit bounds of inventory model with interval demand and unit cost. Liu and Wang (2007) developed a method for numerical solution to interval quadratic programming. Liu (2008) discussed the posynomial geometric programming with interval exponents and coefficients.
This paper aims at extending the application of linear fractional programming to inventory optimization problems in interval uncertainty. We consider an interval-valued fractional objective function and interval-valued constraints for the proposed inventory problem. The paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, the notations and assumptions are given. In Section 4, a linear fractional programming inventory problem is proposed in crisp sense. In Section 5, we formulate above problem considering interval uncertainty. Two numerical examples are solved in Section 6. Finally, we conclude in Section 7.

2. PRELIMINARIES

Interval-Valued Number (Moore & Lodwick (2003)):

Let \( * \in \{ +, -, \ldots, / \} \) be a binary operation on real line \( \mathbb{R} \). Let \( A \) and \( B \) be two closed intervals. Then, a binary operation is defined as

\[
A * B = \{ a * b : a \in A, b \in B \}
\]

In the case of division, it is assumed that \( 0 \notin B \). The operations on intervals used in this paper are defined as:

a) \( kA = \begin{cases} k[a_L, k a_R] & \text{if } k \geq 0 \\ (k a_R, k a_L) & \text{if } k < 0 \end{cases} \)  \hspace{1cm} (1)

b) \( A + B = [a_L, a_R] + [b_L, b_R] = [a_L + b_L, a_R + b_R] \)  \hspace{1cm} (2)

c) \( A - B = [a_L, a_R] - [b_L, b_R] = [a_L - b_R, a_R - b_L] \)  \hspace{1cm} (3)

d) \( A \times B = \left[ \min\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}, \max\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\} \right] \)  \hspace{1cm} (4)

e) \( \frac{A}{B} = A \times \frac{1}{B} = [a_L, a_R] \times \left[ \frac{1}{b_R - b_L}, \frac{1}{b_R - b_L} \right] \), provided \( 0 \notin [b_L, b_R] \).

Alternatively, division of interval numbers can be defined as

\[
\frac{A}{B} = \left[ \frac{a_L}{b_L}, \frac{a_R}{b_R} \right] = \left[ \min\{\frac{a_L}{b_L}, \frac{a_L}{b_R}, \frac{a_R}{b_L}, \frac{a_R}{b_R}\}, \max\{\frac{a_L}{b_L}, \frac{a_L}{b_R}, \frac{a_R}{b_L}, \frac{a_R}{b_R}\} \right].
\]  \hspace{1cm} (5)

Alternatively, division of interval numbers can be defined as

\[
\frac{A}{B} = \left[ \frac{a_L}{b_L}, \frac{a_R}{b_R} \right], \quad \text{where } 0 \notin B, 0 \leq a_L \leq a_R, 0 < b_L \leq b_R.
\]  \hspace{1cm} (6)
Interval-Valued Function: Let $\mathbb{R}^n$ be an $n$-dimensional Euclidean space. Then a function

$$f : \mathbb{R}^n \rightarrow I$$

is called an interval valued function (because $f(x)$ for each $x \in \mathbb{R}^n$ is a closed interval in $\mathbb{R}$). Similar to interval notation, we denote the interval-valued function $f$ with

$$f(x) = [f^l(x), f^u(x)],$$

where for each $x \in \mathbb{R}^n$, $f^l(x)$, $f^u(x)$ are real valued functions defined on $\mathbb{R}^n$, called the lower and upper bounds of $f(x)$, and satisfy the condition:

$$f^l(x) \leq f^u(x).$$

**Proposition 1.** Let $f$ be an interval valued function defined on $\mathbb{R}^n$. Then $f$ is continuous at $c \in \mathbb{R}^n$ if and only if $f^l$ and $f^u$ are continuous at $c$.

**Definition 1.** We define a linear fractional function $F(x)$ as follows:

$$F(x) = \frac{cx + a_1}{dx + a_2}$$

(8)

Where $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $c = (c_1, c_2, ..., c_n) \in \mathbb{R}^n$, $d = (d_1, d_2, ..., d_n) \in \mathbb{R}^n$, and $a_1$, $a_2$ are real scalars.

**Definition 2.** Let $F : V \rightarrow I$ be an interval-valued function, defined on a real vector space $V$. Now, we consider the following interval-valued optimization problem

$$(IVP) \quad \text{Min}_{\leq} F(x) = [F^l(x), F^u(x)]$$

Subject to $x \in Y \subseteq V$, (9)

where $Y$ is the feasible set of problem (IVP) and $\leq$ is the partial ordering on the set of integers.

**Definition 3.** Let $\leq$ be a partial ordering on the set of integers. Then, for $a = [a^l, a^u]$, and $b = [b^l, b^u]$, we write

$$a \leq b$$

if and only if $a^l \leq b^l$ and $a^u \leq b^u$.

This means that $a$ is inferior to $b$ or $b$ is superior to $a$.

**Definition 4.** Let $F : V \rightarrow I$ be an interval-valued function, defined on a real vector space $V$, and let $Y$ be a subset of $V$. Suppose that we are going to maximize $F$. We say that $F(\bar{x})$ is a non-dominated objective value if and only if there exists no $x (\neq \bar{x}) \in Y$ such that $F(\bar{x}) < F(x)$.
Interval-Valued Linear Fractional Programming (IVLFP)

Minimize \( f(x) = \frac{c^T x + a_1}{d^T x + a_2} \)
Subject to \( A x = b \)
\( x \geq 0 \) \hspace{1cm} (10)

Suppose \( c = (c_1, c_2, ..., c_n) \), \( d = (d_1, d_2, ..., d_n) \), where \( c_j, d_j \in \mathbb{I}, j = 1, 2, ..., n \).
Define \( c^L = (c^L_1, c^L_2, ..., c^L_n), c^U = (c^U_1, c^U_2, ..., c^U_n) \), \( d^L = (d^L_1, d^L_2, ..., d^L_n), d^U = (d^U_1, d^U_2, ..., d^U_n) \), where \( c^L_j, c^U_j, d^L_j, d^U_j \) are real scalars for \( j = 1, 2, ..., n \). Also, \( \alpha_1 = [\alpha_1^L, \alpha_1^U] \), \( \alpha_2 = [\alpha_2^L, \alpha_2^U] \).

Then IVLFP (10) can be re-written as:

Minimize \( f(x) = \frac{p(x)}{q(x)} \)
Subject to \( A x = b \)
\( x \geq 0 \) \hspace{1cm} (11)

where \( p(x) \) and \( q(x) \) are interval-valued linear functions and are given by
\( p(x) = [p^L(x), p^U(x)] = [c^L x + \alpha_1^L, c^U x + \alpha_1^U] \),
\( q(x) = [q^L(x), q^U(x)] = [d^L x + \alpha_2^L, d^U x + \alpha_2^U] \).

Therefore, IVLFP (11) can re-written as

Minimize \( f(x) = \frac{[c^L x + \alpha_1^L, c^U x + \alpha_1^U]}{[d^L x + \alpha_2^L, d^U x + \alpha_2^U]} \)
Subject to \( A x = b \)
\( x \geq 0 \) \hspace{1cm} (12)

Further, using the concept of division of two interval-valued numbers, IVLFP (12) can re-written as

Minimize \( f(x) = [f^L(x), f^U(x)] \)
Subject to \( A x = b \)
\( x \geq 0 \) \hspace{1cm} (13)

where \( f^L(x) \) and \( f^U(x) \) are linear fractional functions.

**Definition 5. (Wu (2008))** Let \( x^* \) be a feasible solution of IVLFP (13). Then, \( x^* \) is a non-dominated solution of IVLFP (13) if there exist no feasible solution \( x \) such that \( f(x) \prec f(x^*) \). In this case, we say that \( f(x^*) \) is the non-dominated objective value of \( f \).

Corresponding to IVLFP (13), consider the following optimization problem:

Minimize \( g(x) = f^L(x) + f^U(x) \)
Subject to \( A x = b \)
\( x \geq 0 \) \hspace{1cm} (14)

To solve IVLFP (14), we use the following theorem (Wu (2008)).

**Theorem. Wu (2008)** If \( x^* \) is an optimal solution of IVLFP(14), then \( x^* \) is a non-dominated solution of IVLFP(13).

**Proof.** Let us refer to Wu (2008).
3. NOTATIONS AND ASSUMPTIONS

To develop the proposed model, we considered the following notations and assumptions:

**Notations**

- \( \lambda \): Fixed cost per order
- \( B \): Maximum available budget for all products
- \( W \): Maximum available space for all products
- \( Q_i \): Ordering quantity of product \( i \), for \( i = 1, 2, \ldots, n \)
- \( h_i \): Holding cost per product per unit time for \( i \)-th product
- \( P_i \): Purchasing price of \( i \)-th product
- \( S_i \): Selling price of \( i \)-th product
- \( D_i \): Demand quantity per unit time of \( i \)-th product
- \( f_i \): Space required per unit for \( i \)-th product
- \( OC_i \): Ordering cost of \( i \)-th product.

**Assumptions**

1. Multiple products are considered in this model.
2. Infinite time horizon is considered. Further, there is only one period in the cycle time.
3. Lead time is zero, and so rate of replenishment is infinite.
4. Holding cost is known and constant for each product.
5. Demand is taken here as inversely related to the selling price of the product, that is,
   \[ D_i = D_i(S_i) = m_1S_i^{-m_2}, \]
   where \( m_1 > 0 \) is a scaling constant, and \( m_2 > 1 \) is price-elasticity coefficient. For notational simplicity, we will use \( D_i \) and \( D_i(S_i) \) interchangeably.
6. Shortages are not allowed.
7. Purchase price is constant for each product, i.e. no discount is available.

4. CRISP LINEAR FRACTIONAL PROGRAMMING INVENTORY PROBLEM

A multi-product inventory model under resource constraints is introduced as a linear fractional programming problem. This model refers to a multi-product inventory problem, with limited capacity of warehouse and constraints on investment in inventories. For each product, we impose the constraint on ordering cost.

We practically observe the inventory models with more than one objective functions. These objectives may be in conflict with each other, or may not be. In such type of inventory models, the decision maker is interested to maximize or minimize two or more objectives simultaneously over a given set of decision variables. We call this model as linear fractional inventory model. Without loss of generality, we assume there is only one period in the cycle time. Then,

\[
\text{Total Profit} = \sum_{i=1}^{n} (S_i - P_i)Q_i \quad (15a)
\]

\[
\text{Holding cost} = \sum_{i=1}^{n} \frac{h_iQ_i}{2} \quad (15b)
\]
Ordering cost = \( \sum_{i=1}^{n} \frac{\lambda D_i}{Q_i} = \sum_{i=1}^{n} \frac{\lambda m_i S_i^{-m_2}}{Q_i} \) (15c)

Back ordered quantity = \( \sum_{i=1}^{n} (D_i - Q_i) = \sum_{i=1}^{n} (m_i S_i^{-m_2} - Q_i) \) (15d)

Total ordered quantity = \( \sum_{i=1}^{n} Q_i \) (15e)

In this paper, the crisp linear fractional programming inventory model is formulated as

Maximize: \( Z = \frac{\text{Total Profit}}{\text{Holding cost}} \),

Subject to, (i) Total Budget Limit,
(ii) Storage Space Limit,
(iii) Limit on Ordering Cost of Each Product.
(iv) Non-Negative Restriction.

Formulation of Constraints

(i) **Total Budget Limit:** A budget constraint is all about the combinations of goods and services that a consumer may purchase given current prices within his or her given income. For the proposed model, the budget constraint is \( \sum_{i=1}^{n} P_i Q_i \leq B \) (16a)

(ii) **Storage Space Limit:** \( \sum_{i=1}^{n} f_i Q_i \leq W \) (16b)

(iii) **Limit on Ordering Cost of Each Product:** We impose the upper limit of ordering cost as a constraint. Since \( OC_1, OC_2, \ldots, OC_n \) are the ordering cost of 1\(^{st}\) product, 2\(^{nd}\) product, ..., \( n^{th}\) product, we can express the concerned constrained as follows:

For first product, \( \frac{\lambda D_1}{Q_1} \leq OC_1 \)
\[ \Rightarrow \lambda D_1 - OC_1 Q_1 \leq 0 \]

For second product, \( \frac{\lambda D_2}{Q_2} \leq OC_2 \)
\[ \Rightarrow \lambda D_2 - OC_2 Q_2 \leq 0 \]

Similarly, for \( n^{th}\) product, \( \lambda D_n - OC_n Q_n \leq 0 \)
\[ \Rightarrow \lambda m_n S_n^{-m_2} - OC_n Q_n \leq 0 \] (16c)

(iv) **Non-Negative Restriction:**
\( Q_1, Q_2, \ldots, Q_i \geq 0 \) (16d)

5. **INTERVAL-VALUED LINEAR FRACTIONAL PROGRAMMING INVENTORY PROBLEM**

In real life cases, it is observed that the selling price is uncertain due to dynamic behavior of the market. So it may be considered to vary in an interval. Let the selling price be represented by the interval \([S_{il}, S_{iu}]\), where \( S_{il} \) & \( S_{iu} \) are the left limit & right limit of interval for \( S_i \). Mathematically, we can express as

\( S_i = [S_{il}, S_{iu}] \), with \( S_{il} \leq S_{iu} \) (17)
Non-Negative Restriction:

coefficients are crisp numbers. Hence, the interval-valued optimization problem (IVP1), transformed into multi-objective functions given by (27) where, in each objective function, the single-objective function consisting of interval-valued number coefficients is

Using the arithmetic operations of intervals, the interval-valued fractional objective function can

Accordingly, the constraints transform to the interval-valued form as follows

Total Budget Limit: \[ \sum_{i=1}^{n} [P_{il}, P_{iu}] Q_i \leq B \] (21)

Storage Space Limit: \[ \sum_{i=1}^{n} f_i Q_i \leq W \] (22)

Limit on Ordering Cost of each Product:

For \( n \)th product, we have

Non-Negative Restriction:

\[ Q_1, Q_2, \ldots, Q_i \geq 0 \] (24)

In this case, we define the following interval-valued fuzzy linear fractional programming problem for the proposed inventory model,

Maximize: \[ Z = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} h_{il} Q_i}, \] (25)

Subject to

\[ \sum_{i=1}^{n} [P_{il}, P_{iu}] Q_i \leq B, \]
\[ \sum_{i=1}^{n} f_i Q_i \leq W, \]
\[ Q_i [OC_{il}, OC_{iu}] \geq \lambda [m_1 S_{il}^{-m_2}, m_1 S_{iu}^{-m_2}] \] (for \( i = 1, 2, \ldots, n \))
\[ Q_i \geq 0, \] (for \( i = 1, 2, \ldots, n \))

Using the arithmetic operations of intervals, the interval-valued fractional objective function can be written as

\[
Z = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} h_{il} Q_i},
\]

\[
= \frac{\sum_{i=1}^{n} h_{il} Q_i}{\sum_{i=1}^{n} h_{il} Q_i},
\]

\[
= \frac{\sum_{i=1}^{n} h_{il} Q_i}{\sum_{i=1}^{n} h_{il} Q_i},
\]

\[
= \left[ Z_L(Q_1, Q_2, \ldots, Q_n), Z_U(Q_1, Q_2, \ldots, Q_n) \right].
\] (26)

where

\[
Z_L(Q_1, Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} h_{il} Q_i},
\]

and

\[
Z_U(Q_1, Q_2, \ldots, Q_n) = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} h_{il} Q_i}.
\] (27)

Thus the single-objective function consisting of interval-valued number coefficients is transformed into multi-objective functions given by (27) where, in each objective function, the coefficients are crisp numbers. Hence, the interval-valued optimization problem (IVP1), transforms to following optimization problem as follows,
Operations Research and Applications: An International Journal (ORAJ), Vol. 2, No. 4, November 2015

(IVP2) Maximize \[ Z_L(Q_1, Q_2, ..., Q_n), Z_U(Q_1, Q_2, ..., Q_n) \] \[ \text{Subject to } \sum_{i=1}^{n} P_{il} Q_i \leq B, \]
\[ \sum_{i=1}^{n} P_{il} Q_i \leq B, \]
\[ \sum_{i=1}^{n} f_i Q_i \leq W, \]
\[ Q_i [OC_{il}, OC_{il}'] \geq \lambda [m_1 S_{il}^{-m_2}, m_2 S_{il}^{-m_2}], \text{ (for } i = 1, 2, ..., n), \]
And \[ Q_i \geq 0, \text{ (for } i = 1, 2, ..., n). \]

To solve this problem, the techniques of classical linear fractional programming cannot be applied if and unless the above interval-valued structure of the problem be reduced into a standard linear fractional programming problem.

To deal with interval inequality constraints, we use the Tong’s Approach [Tong (1994) & Sengupta et al (2001)]. Tong deals with interval inequality constraints in a separate way.

According to Tong’s Approach, each interval inequality constraint is transformed into \( 2^{n+1} \) crisp inequalities. Let us take a simple inequality constraint with a single variable:

\[ [10, 20] \leq [5, 35]. \]

According to Tong (1994), the interval inequality generates \( 2^{n+1} \) crisp inequalities:

\[ 10x \leq 5 \implies x \leq 0.5 \]
\[ 10x \leq 35 \implies x \leq 3.5 \]
\[ 20x \leq 5 \implies x \leq 0.25 \]
\[ 20x \leq 35 \implies x \leq 1.75 \]

Hence, the interval-valued linear fractional inventory problem (IVP2) can be transformed to a non-interval optimization problem as follows:

\[ \text{(IVP3) Maximize } [Z_L(Q_1, ..., Q_n), Z_U(Q_1, ..., Q_n)] \] \[ \text{where } Z_L(Q_1, Q_2, ..., Q_n) = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} S_{il} Q_i} \]
\[ Z_U(Q_1, Q_2, ..., Q_n) = \frac{\sum_{i=1}^{n} (S_{il} - P_{il}) Q_i}{\sum_{i=1}^{n} S_{il} Q_i} \]

\[ \text{Subject to: } \sum_{i=1}^{n} P_{il} Q_i \leq B \]
\[ \sum_{i=1}^{n} P_{il} Q_i \leq B \]
\[ \sum_{i=1}^{n} f_i Q_i \leq W \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ Q_i \geq 0, \text{ (for } i = 1, 2, ..., n). \]

Now, let us write the corresponding optimization problem of (IVP3) as follows:

\[ \text{(IVP4) Maximize } g(x) = Z_L(Q_1, Q_2, ..., Q_n) + Z_U(Q_1, Q_2, ..., Q_n) \] \[ \text{subject to } \sum_{i=1}^{n} P_{il} Q_i \leq B \]
\[ \sum_{i=1}^{n} P_{il} Q_i \leq B \]
\[ \sum_{i=1}^{n} f_i Q_i \leq W \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ OC_{il} Q_i \geq \lambda m_1 S_{il}^{-m_2}, \text{ (for } i = 1, 2, ..., n) \]
\[ Q_i \geq 0, \text{ (for } i = 1, 2, ..., n). \]
To solve the (IVP3), theorem (Wu (2008)) is conveniently used. According to this theorem, the non-dominated solution of (IVP3), would be determined by solving its corresponding optimization problem (IVP4).

6. NUMERICAL EXAMPLES

To illustrate the proposed approach, we solve the following two numerical examples in crisp and interval-value cases:

**Example1. Crisp Case**  Consider an inventory model with following input data (in proper units):

Table 1: Input data (Crisp Case)

| Product | \( h_i \) | \( P_i \) | \( S_i \) | \( C_i \) | \( f_i \) | \( m_1 \) | \( m_2 \) | \( \lambda \) | \( W \) | \( B \) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| i = 1   | 12      | 125     | 200     | 80      | 2       |         |         |         |         |         |
| i = 2   | 16      | 160     | 220     | 90      | 4       | 80,000  | 1.20    | 7       | 300     | 90,000  |

Maximize: \( Z = \frac{\sum_{i=1}^{n}(s_i - p_i)Q_i}{\sum_{i=1}^{n} d_i} \) \( = \frac{75Q_1 + 60Q_2}{6Q_1 + 90Q_2} \)

Subject to: \( \sum_{i=1}^{n} P_i Q_i \leq B \) \( \Rightarrow 125Q_1 + 160Q_2 \leq 90000, \)
\( \sum_{i=1}^{n} f_i Q_i \leq W \) \( \Rightarrow 2Q_1 + 4Q_2 \leq 300, \)
\( OC_{C_i} Q_i \geq \lambda m_1 S_{i}^{-m_2} \) \( \Rightarrow 80Q_1 \geq 970.4027, \) and \( 90Q_2 \geq 865.5273. \)

The optimal solution is \( Q_1^* = 130.7658, Q_2^* = 9.6170, Z^* = 12.0535, \)

Maximum profit = 10384.455.

**Interval-Values Case**  Consider an inventory model with following input data:

Table 2: Input data (Interval-Valued Case)

| Product | \( h_i \) | \( P_i \) | \( S_i \) | \( C_i \) | \( f_i \) | \( m_1 \) | \( m_2 \) | \( \lambda \) | \( W \) | \( B \) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| i = 1   | [10,14] | [120,130] | [190,210] | [70,90] | 2       |         |         |         |         |         |
| i = 2   | [14,18] | [150,170] | [210,230] | [80,100] | 4       | 80,000  | 1.20    | 7       | 300     | 90,000  |
The corresponding optimization problem would be:

Maximize  \[ g(x) = \frac{50Q_1 + 40Q_2}{7Q_1 + 9Q_2} + \frac{90Q_1 + 80Q_2}{5Q_1 + 7Q_2} \]

Subject to:
- \[ \sum_{i=1}^{n} P_i Q_i \leq B \Rightarrow 120Q_1 + 150Q_2 \leq 90000, \]
- \[ \sum_{i=1}^{n} P_i Q_i \leq B \Rightarrow 130Q_1 + 170Q_2 \leq 90000, \]
- \[ \sum_{i=1}^{n} f_i Q_i \leq W \Rightarrow 2Q_1 + 4Q_2 \leq 300, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 70Q_1 \geq 1032.0095, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 90Q_1 \geq 1032.0095, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 70Q_1 \geq 915.2187, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 90Q_1 \geq 915.2187, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 80Q_2 \geq 820.5681, \]
- \[ OC_{il} Q_i \geq \lambda m_{i} s_{i}^{m_{i}} \Rightarrow 80Q_2 \geq 820.5681, \]

The optimal solution is \( Q_1^* = 127.1197, \quad Q_2^* = 11.4402 \) with optimum value \( g(Q_1^*, Q_2^*) = 25.4081. \)

Maximum Profit = \( ([190, 210] - [120, 130])Q_1^* + ([210, 230] - [150, 170])Q_2^* \)
\[ = [60, 90]Q_1^* + ([40, 80]Q_2^* \]
\[ = [60Q_1^* + 40Q_2^*, 90Q_1^* + 80Q_2^*] \]
\[ = [8084.79, 12355.989]. \]

Hence, the bounds of profit are 8084.79 (lower bound), and 12355.989 (upper bound).

**Example 2. Crisp Case**  Consider an inventory model with following input data (in proper units):

| Product | \( h_i \) | \( P_i \) | \( S_i \) | \( OC_i \) | \( \bar{f}_i \) | \( m_1 \) | \( m_2 \) | \( \lambda \) | \( W \) | \( B \) |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| i = 1 | 12  | 125 | 180 | 70  | 1   | 15,000 | 1.10 | 5   | 400 | 60,000 |
| i = 2 | 15  | 150 | 220 | 80  | 2   | 15,000 | 1.10 | 5   | 400 | 60,000 |

Maximize:  \[ Z = \frac{\sum_{i=1}^{n} (S_i - P_i) Q_i}{\sum_{i=1}^{n} \frac{h_i Q_i}{h_i}} = \frac{55Q_1 + 70Q_2}{6Q_1 + 7.5Q_2} \]

Subject to:
- \[ \sum_{i=1}^{n} P_i Q_i \leq B \Rightarrow 125Q_1 + 150Q_2 \leq 60000, \]
- \[ \sum_{i=1}^{n} f_i Q_i \leq W \Rightarrow Q_1 + 2Q_2 \leq 400, \]
- \[ OC_{n}, Q_n \geq \lambda m_i s_{i}^{m_{i}} \Rightarrow 70Q_1 \geq 247.8914, \]

The optimal solution is \( Q_1^* = 3.5413, \quad Q_2^* = 198.2293, \quad Z^* = 9.3310, \)

Maximum profit = 14070.8225.
**Interval-Values Case**  Consider an inventory model with following input data:

Table 4: Input data (Interval-Valued Case)

| Product | $h_i$ | $P_i$ | $s_i$ | $OC_i$ | $f_i$ | $m_1$ | $m_2$ | $\lambda$ | $W$ | $\beta$ |
|---------|-------|-------|-------|--------|-------|-------|-------|--------|-----|-------|
| i = 1   | [8, 16] | [100, 150] | [160, 200] | [60, 80] | 1     |       |       |        |     |       |
| i = 2   | [12, 18] | [125, 175] | [200, 240] | [70, 90] | 2     | 15,000 | 1.10  | 5       | 400 | 60,000 |

Objective: Maximize  

$$Z(Q_i) = \sum_{i=1}^{n} \left( \left( \frac{s_i - p_i}{h_i} \right) + \frac{s_i}{h_i} \right) Q_i$$

$$Z_L(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left( \frac{s_i - p_i}{h_i} \right) Q_i = \frac{160 - 150}{8} Q_1 + \frac{150 - 200}{8} Q_2 = \frac{10 Q_1 + 25 Q_2}{8 Q_1 + 9 Q_2}$$

$$Z_U(Q_1, Q_2, \ldots, Q_n) = \sum_{i=1}^{n} \left( \frac{s_i - p_i}{h_i} \right) Q_i = \frac{100 - 100}{4} Q_1 + \frac{240 - 125}{6} Q_2 = \frac{100 Q_1 + 115 Q_2}{4 Q_1 + 6 Q_2}$$

The corresponding optimization problem would be:

Maximize  

$$g(x) = \frac{100 Q_1 + 25 Q_2}{8 Q_1 + 9 Q_2} + \frac{100 Q_1 + 115 Q_2}{4 Q_1 + 6 Q_2}$$

Subject to

$$\sum_{i=1}^{n} P_i Q_i \leq B$$  \quad \Rightarrow \quad 100 Q_1 + 125 Q_2 \leq 60,000, $$

$$\sum_{i=1}^{n} P_i Q_i \leq B$$  \quad \Rightarrow \quad 150 Q_1 + 175 Q_2 \leq 60,000, $$

$$\sum_{i=1}^{n} f_i Q_i \leq W$$  \quad \Rightarrow \quad Q_1 + 2 Q_2 \leq 400, $$

$$OC_{nL} Q_n \geq \lambda m_1 s_{II}$$  \quad \Rightarrow \quad 60 Q_1 \geq 282.1819, \quad 70 Q_2 \geq 220.7640, $$

$$OC_{nU} Q_n \geq \lambda m_1 s_{II}$$  \quad \Rightarrow \quad 80 Q_1 \geq 282.1819, \quad 90 Q_2 \geq 220.7640, $$

$$OC_{nL} Q_n \geq \lambda m_1 s_{II}$$  \quad \Rightarrow \quad 60 Q_1 \geq 220.7640, \quad 70 Q_2 \geq 180.6462, $$

$$OC_{nU} Q_n \geq \lambda m_1 s_{II}$$  \quad \Rightarrow \quad 80 Q_1 \geq 220.7640, \quad 90 Q_2 \geq 180.6462, $$

and  \quad Q_1 \geq 0, \quad Q_2 \geq 0.$$

The optimal solution is  \( Q_1 = 393.6925, \quad Q_2 = 3.1538 \) with optimum value  \( g(Q_1, Q_2) = 26.1944.\)

Maximum Profit  \( = ((160, 200) - (100, 150)) Q_1 + ((200, 240) - (125, 175)) Q_2 \)

\( = [10, 100] Q_1 + [75, 115] Q_2 \)

\( = [10 Q_1 + 75 Q_2, 100 Q_1 + 115 Q_2] \)

\( = [4173.46, 39731.937] \)

Hence, the bounds of profit are 4173.46 (lower bound), and 39731.937 (upper bound).

7. CONCLUSIONS

In this paper, the author presented an approach to solve interval-valued inventory optimization problem based on linear fractional programming. The uncertainty in inventory parameters is represented by interval-valued numbers. Using interval arithmetic, the interval-valued optimization problem is changed into a crisp multi-objective linear fractional programming problem. LINGO package is used to solve the subsequent optimization problem. The optimal order quantity and profit bounds are determined.
For future directions, the paper can be extended to inventory model with shortages case and inventory model with price-discount. Moreover, it can also be extended to interval-valued multi-objective optimization case.

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