The dual properties of two-color QCD with baryon, chiral and isospin densities

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Abstract. The phase structure of dense quark matter in the two color case has been investigated with nonzero baryon $\mu_B$, isospin $\mu_I$ and chiral isospin $\mu_{I5}$ chemical potentials. It has been demonstrated that due to the duality properties the phase diagram is extremely symmetric and the whole phase diagram in the case of two colors can be obtained just by dualities from the phase structure of three color case. This shows that the dualities are rather useful tool.

1. Introduction

It is considered that QCD is the theory of strong interaction and it should be used for the studies of strongly interacting matter. However, due to the value of the coupling constant the perturbative methods cannot be applied in the conditions of compact stars or heavy ion collisions. Nonperturbative method such as lattice simulations of QCD, which is widely used for hot QCD matter, is not applicable in the case of non-zero (large) baryon density because of a notorious sign problem. Thus, in this regards there arise the interest to QCD-like models, for example, to Nambu–Jona-Lasinio (NJL)-like models [1, 2].

In addition to the non-zero baryon density, in quark matter there can be other important various quantities, for example, isospin imbalance that is an obvious feature of neutron stars. Quite recently, chiral imbalance (difference between densities of left and right-handed quarks) was understood to be also interesting possible property of quark matter [3, 4, 5, 6], connected with the so-called chiral magnetic effect [7]. As a rule, it is accounted for by chiral chemical potential $\mu_5$. There is another possibility of different chiral chemical potentials for $u$ and $d$ quarks, $\mu_{u5} \neq \mu_{d5}$, and chiral isospin chemical potential $\mu_{I5} = \mu_{u5} - \mu_{d5}$.

Recently, there appeared increasing attention to two-color QCD and QCD-like models [8, 9, 10, 11, 12, 13, 14, 15]. Although a two color and tree color cases are rather different, the investigations of SU(2) QCD can give us valuable insight into the properties of three color QCD at nonzero baryon density. As an additional motivation let us recall that in the QCD with two colors there is no sign problem and ab initio lattice simulations are possible. Also one can stress that QCD phase diagram in the case of two colors is rather rich and can be quite interesting in its own right.
2. Two color NJL model

In the chiral limit the two color QCD with two flavors of quarks possesses Pauli-Gursey symmetry, i.e. it is symmetric with respect to $SU(4)$ group [8, 9]. The Lagrangian of effective two-color four-fermion NJL model with baryon $\mu_B$, isospin $\mu_I$ and chiral isospin $\mu_{IS}$ chemical potentials is

$$L = \bar{q} \left[ i \partial - m_0 + M \gamma^0 \right] q + H \left[ (\bar{q}q)^2 + (\bar{q}i\gamma^5 \vec{r}q)^2 + (\bar{q}i\gamma^5 \sigma_2 \tau_2 q)^2 \right], \quad (1)$$

where $q \equiv q_\alpha$ is a flavor ($i = u, d$) and color ($\alpha = 1, 2$) doublet and a four-component Dirac spinor as well. $\partial \equiv \gamma^\mu \partial_\mu$ and charge-conjugated spinors are $q^\ast = Cq^T$, $\bar{q}^\ast = q^T C$, where $C = i\gamma^2 \gamma^0$. Chemical potentials are included in the term $M = \frac{\mu_B}{2} + \frac{\mu_I}{2} \tau_3 + \frac{\mu_{IS}}{2} \gamma^5 \tau_3$.

The Lagrangian (1) is invariant with respect to $SU(2)_c$ and $U(1)_B$ groups as well as Pauli-Gursey flavor $SU(4)$ group (as in two-color QCD case).

In order to study the phase structure one starts from an equivalent semi-bosonized Lagrangian $\tilde{L}$ with auxiliary bosonic fields $\sigma(x), \pi, \Delta(x)$ and $\Delta^*(x)$

$$\tilde{L} = \bar{q} \left[ i \partial - m_0 + M \gamma^0 - \sigma - i\gamma^5 \vec{r} \pi \right] q - \frac{\sigma^2 + \pi^2 + \Delta^2}{4H} - \frac{\Delta}{2} \left[ \bar{q}i\gamma^5 \sigma_2 \tau_2 q \right] - \frac{\Delta^*}{2} \left[ \bar{q}i\gamma^5 \sigma_2 \tau_2 q \right], \quad (2)$$

The equations of motion of auxiliary fields are

$$\sigma(x) = -2H(q\bar{q}), \quad \Delta(x) = -2H(\bar{q}i\gamma^5 \sigma_2 \tau_2 q) = -2H \left[ \bar{q}q \gamma^T C \gamma^5 \sigma_2 \tau_2 q \right]. \quad (3)$$

Bosonic fields $\pi_3(x), \pi_{\pm}(x) = (\pi_1(x) \mp i\pi_2(x))/\sqrt{2}$ could be identified with the pion fields. If $\langle \sigma(x) \rangle \neq 0$ or $\langle \pi_0(x) \rangle \neq 0$, then one can see that chiral symmetry is broken down and we will call this phase chiral symmetry breaking (CSB) one. If $\langle \pi_{1,2}(x) \rangle \neq 0$, this phase is called charged pion condensation phase (PC) and as the isospin as well as the electromagnetic symmetries are spontaneously broken. If $\langle \Delta(x) \rangle \neq 0$, then the baryon symmetry gets broken down spontaneously and this phase will be called baryon superfluid phase (BSF).

Thermodynamic potential (TDP) can be defined from the effective action $S_{\text{eff}}(\sigma, \pi, \Delta, \Delta^*)$ in the mean-field approximation

$$S_{\text{eff}} |_{\sigma, \pi, \Delta, \Delta^* = \text{const}} = -\Omega(\sigma, \pi, \Delta, \Delta^*) \int d^4x.$$

The ground state expectation values of $\langle \sigma(x) \rangle \equiv \sigma, \langle \pi(x) \rangle \equiv \pi, \langle \Delta(x) \rangle \equiv \Delta, \langle \Delta^*(x) \rangle \equiv \Delta^*$, can be found as solutions of the following equations (the so-called gap equations)

$$\frac{\partial \Omega}{\partial \pi_a} = 0, \quad \frac{\partial \Omega}{\partial \sigma} = 0, \quad \frac{\partial \Omega}{\partial \Delta} = 0, \quad \frac{\partial \Omega}{\partial \Delta^*} = 0. \quad (4)$$

It is assumed that $\sigma, \pi, \Delta, \Delta^*$ do not depend on the space coordinates $x$.

The expressions for the TDP $\Omega(\sigma, \pi, \Delta, \Delta^*)$ can be obtained analytically. In general the TDP depends on all condensates, $M, \pi, \Delta$ and $\Delta^*$. But if one uses the symmetries of the model this number could be reduced and, without loss of generality, the TDP can be considered as a function of $M, \pi_1$ and $|\Delta|$. 


3. Duality properties

It can be shown with the use of any analytical calculation program that the expression for the TDP is invariant under the so-called dual transformation $D_1$,

$$D_1 : \quad \mu \leftrightarrow \nu, \quad \pi_1 \leftrightarrow |\Delta|.$$  \hspace{1cm} (5)

This dual property of the TDP was noticed for the first time in framework of two color NJL model in [9, 13] but for the case of $\nu_5 = 0$. In particular, it was shown that the PC and BSF phases are arranged on the $(\mu, \nu)$-phase diagram symmetrically.

Furthermore, it can be shown that the TDP is also invariant with respect to the following dual transformations $D_2$ and $D_3$

$$D_2 : \quad \mu \leftrightarrow \nu_5, \quad M \leftrightarrow |\Delta|; \quad D_3 : \quad \nu \leftrightarrow \nu_5, \quad M \leftrightarrow \pi_1.$$  \hspace{1cm} (6)

There are rather cogent arguments for the absence of mixed phases (meaning and one can assume that the global minimum point (GMP) $(M, \pi_1, |\Delta|)$ of the TDP has only one nonzero coordinate) and it is based on dual properties discussed above. Concerning chiral symmetry breaking and charged pion condensation phenomena the phase diagram has the same structure in two color and three color cases in framework of NJL models. These phenomena in the three color case have been investigated in [16, 17, 18] and it has been displayed that there is no mixed phase. Hence, this holds also for two color NJL model. Then, in order to show that there is no such a region, where chiral and diquark condensates are non-zero simultaneously, the duality $D_2$ should be used. The absence of mixed phase with non-zero pion and diquark condensates can be shown applying either $D_2$ or the $D_3$. This leaves the possibility of a phase with all three non-zero condensates, but it seems quite unlikely. Without dualities showing the absence of mixed phases would be rather hard and can be done only numerically.

The possible phases in the system are the following: (i) the chiral symmetry breaking (CSB) phase: GMP has the form $(M \neq 0, \pi_1 = 0, |\Delta| = 0)$. (ii) charged pion condensation (PC) phase: GMP of the form $(M = 0, \pi_1 \neq 0, |\Delta| = 0)$. (iii) baryon superfluid (BSF) phase: GMP is $(M = 0, \pi_1 = 0, |\Delta| \neq 0)$. (iv) symmetrical phase: GMP has the form $(M = 0, \pi_1 = 0, |\Delta| = 0)$.

4. Phase diagrams with one or two chemical potentials

The phase diagram of the two color NJL model with only nonzero chemical potential $\mu$ (although with nonzero temperature $T$) has been investigated in [10]. It was shown that at non-zero $\mu$ (though not for too large) there is the BSF phase. If one applies $D_1$ to this phase diagram
one would get the phase structure with only nonzero $\nu$ and at $\nu > 0$ one would observe the charged PC phase. Now if one acts by $D_2$ upon the phase diagram with nonzero $\mu$ (original one) one would observe CSB phase at $\nu_5 > 0$. One can conclude that there is in a way one-to-one correspondence between different phenomena and various chemical potentials. Chiral symmetry breaking phenomenon is triggered by chiral isospin chemical potential $\nu_5$, $\mu$ leads to appearance of BSF phase in the system and pion condensation is promoted by isospin density (chemical potential $\nu$).

Now let us discuss the case of two chemical potentials. The $(\mu, \nu)$-phase diagram at $\nu_5 = 0$ has been already considered in [9, 13]. It is depicted in figure 1(a) and one can note that it is self-dual, i.e. BSF and PC phases are symmetric with respect to each other. This is due to the duality $D_1$ of TDP. If one employs the $D_2$ and act on this diagram, one obtain the $(\nu_5, \nu)$-phase diagram at $\mu = 0$ (it is presented in figure 1(b)). In a similar fashion, acting on the diagram of figure 1(a) with $D_3$, one can get the $(\mu, \nu_5)$-phase diagram at $\nu = 0$ (see figure 1(c)). Moreover, one can note CSB and BSF phases are located mirror symmetrically w.r.t the line $\nu_5 = \mu$.

5. Phase structure of the general case: $\mu \neq 0$, $\nu \neq 0$ and $\nu_5 \neq 0$

In this section let us consider the general $(\mu, \nu, \nu_5)$-phase diagram of the model. One can use the assumption that there is no mixed phase and perform numerical calculations only for projections of the TDP on the axes of condensates ($M$, $\pi$).

However, there is another way that is much simpler, one can use the dual properties of the model. Let us first note that chiral symmetry breaking and pion condensation phenomena in the framework of NJL model are very similar for three and two color cases, what is different is diquark condensation. The $(\mu, \nu, \nu_5)$ phase diagram, where only chiral symmetry breaking and pion condensation phenomena were accounted for, have been investigated in the case of three colors in [16, 17, 18]. Therefore, one knows the behavior of corresponding phases (PC and CSB) in two color case. Then one can use duality transformations $D_1$, $D_2$ and $D_3$ to get the information about BSF phase and hence obtain the whole phase diagram. It is quite remarkable that the whole phase structure of two color NJL model can be obtained from some results of three color case. Dualities constrain the phase portrait and make it so highly symmetric that this becomes possible.

Now let us discuss the phase structure itself. One can start with several $(\mu, \nu)$ phase diagrams at various values of $\nu_5$ that are not that large (see figure 1 (a), figure 2 (a) and figure 3 (a, b)). One can note that BSF phase is realized in the domain, where $\mu > \nu, \nu_5$, whereas if $\nu > \mu, \nu_5$ the charged PC phase is present. $\nu_5$ is not very large and if for simplicity one puts it $\nu_5 \approx 0$, then qualitatively this behavior can be explained in a similar way as in [13] (where the case $\nu_5 = 0$
was considered). If $\nu > \mu$, $u$ and $\bar{d}$ quarks form Fermi seas and the condensation of Cooper pairs $u\bar{d}$ is possible, and one can see that it is PC phase. Whereas, if $\mu > \nu$, then $u$ and $d$ quarks form Fermi seas and in this case the formation of Cooper pairs $u\bar{d}$ and their condensation can happen and this leads to BSF phase.

Now if the values of $\nu_5$ is rather large (figure 3 (c)) the BSF phase is realized mainly in the domain, where $\mu < \nu \approx \nu_5$, and the charged PC phase in the domain, where $\nu < \mu \approx \nu_5$). This is a different behavior in comparison with the one discussed above for rather small values of $\nu_5$.

This difference can be qualitatively explained by the following arguments. The Fermi surfaces of the left and right-handed $u$ and $d$ quarks have the form

$$\mu_d = \mu + \nu + \nu_5, \quad \mu_d = \mu - \nu - \nu_5.$$ 

At $\mu < \nu \approx \nu_5$ for the quarks $d_R$ the chemical potential $\mu_{d_R}$ is negative, hence the Fermi sea of charge conjugated $\bar{d}_R$ quarks can be formed, so as the Fermi sea of right-handed $u_R$ quarks, which $\mu_{u_R}$ is also greater than zero. The formation of particle-hole Cooper pair $\bar{u}_Rd^c_R$, which has quantum numbers of $\Delta^*(x)$, is possible and its condensation leads to appearance of BSF phase.

6. Summary
Let us now sum up the central results of our studies of dense quark matter with isospin and chiral isospin imbalances in the framework of the two-color NJL model.

- It has been shown in framework of NJL model that $(\mu, \nu, \nu_5)$ phase structure of quark matter in the two color case is highly symmetric. Behind this symmetry lies the dual properties (dualities) between CSB, charged PC and BSF phenomena. One of these dualities is similar to the one found in the case of three colors.

In the dualities the phase diagram in the three color case looks as if it is not complete and only one side of highly symmetric two color phase diagram. It has been found that the phase structure of (two color) dense quark matter is quite rich.

- Using the duality properties one can show that there is no mixed phase (phase with several non-zero condensates simultaneously). This conclusion drastically simplifies corresponding numerical calculations.

- The dualities are very powerful tool and can greatly simplify the studies of the phase structure. Employing only duality properties the whole phase diagram of two color NJL model can be obtained from the results of previous studies of NJL model in the three color case.
• The phenomena of chiral symmetry breaking, charged pion condensation and diquark condensation have been shown to be in a way connected with the corresponding chemical potentials, $\nu_5$, $\mu_\pi$ and $\mu_B$, respectively.

• It has been revealed that in dense (nonzero baryon density) quark matter the generation by chiral imbalance $\nu_5$ of the charged PC phase is not impeded in any way by diquark condensation at least in two color case.

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8. References
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