ON THE COMPUTATION OF RATIONAL SOLUTIONS OF UNDERDETERMINED SYSTEMS OVER A FINITE FIELD

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ABSTRACT. We design and analyze an algorithm for computing solutions with coefficients in a finite field $\mathbb{F}_q$ of underdetermined systems defined over $\mathbb{F}_q$. The algorithm is based on reductions to zero-dimensional searches. The searches are performed on “vertical strips”, namely parallel linear spaces of suitable dimension in a given direction. Our results show that, on average, less than three searches suffice to obtain a solution of the original system, with a probability of success which grows exponentially with the number of searches. The analysis of our algorithm relies on results on the probability that the solution set (over the algebraic closure of $\mathbb{F}_q$) of a random system with coefficients in $\mathbb{F}_q$ satisfies certain geometric and algebraic properties which is of independent interest.

1. INTRODUCTION

Let $\mathbb{F}_q$ be the finite field of $q$ elements and let $\overline{\mathbb{F}}_q$ denote its algebraic closure. For $K = \mathbb{F}_q$ or $K = \overline{\mathbb{F}}_q$, we denote by $K[X_1, \ldots, X_r]$ the ring of polynomials in the indeterminates $X_1, \ldots, X_r$ and coefficients in $K$. For any $d \in \mathbb{N}$, we denote by $\mathcal{F}_d$ the set of elements of $\mathbb{F}_q[X_1, \ldots, X_r]$ of degree at most $d$. Given $1 < s < r$ and polynomials $F_1, \ldots, F_s \in \mathcal{F}_d$, in this paper we are concerned with the problem of finding a common $\mathbb{F}_q$-rational solution, namely a solution with coefficients in $\mathbb{F}_q$, of the underdetermined system $F_1 = 0, \ldots, F_s = 0$.

For $s = 1$ and $r = 2$, namely for the problem of finding $\mathbb{F}_q$-rational points of plane curves, this is considered in [vzGSS03]. For this purpose, the authors propose a combination of specialization and unidimensional search, which is called Search on a vertical strip (SVS). If the plane curve under consideration is defined by a polynomial $F \in \mathbb{F}_q[X_1, X_2]$, the idea consists of considering successive specializations $F(a, X_2)$, where the values $a \in \mathbb{F}_q$ are randomly chosen, until an $\mathbb{F}_q$-rational point of the plane curve $\{F = 0\}$ is obtained in a “vertical strip” $\{a\} \times \mathbb{F}_q$. As there are efficient algorithms for finding $\mathbb{F}_q$-rational zeros of univariate polynomials in $\mathbb{F}_q[X]$, the critical point consists of determining how many random choices must be done to have a “good” probability of finding a vertical strip $\{a\} \times \mathbb{F}_q$ with an $\mathbb{F}_q$-rational point of the plane curve under consideration. In [vzGSS03] this is done using an explicit version of the Weil estimate on the number of $\mathbb{F}_q$-rational points of absolutely irreducible (i.e., irreducible over $\mathbb{F}_q$) plane curves defined over $\mathbb{F}_q$.

The case $s = 1$ and arbitrary $r$, namely the problem of finding $\mathbb{F}_q$-rational points on hypersurfaces, is considered in [Mat10] and [MPP17]. Given an absolutely irreducible $F \in \mathbb{F}_q[X_1, \ldots, X_r]$, in [Mat10] it is shown that $\deg F$ random choices of $a \in \mathbb{F}_q^{r-1}$ suffice to have probability greater than $1/2$ of reaching a “vertical strip” $\{a\} \times \mathbb{F}_q \subset \mathbb{F}_q^r$ with $\mathbb{F}_q$-rational points of the hypersurface $\{F = 0\}$ under consideration. As “most” polynomials $F \in \mathbb{F}_q[X_1, \ldots, X_r]$ of a given degree are absolutely irreducible, this analysis comprises most of the elements of $\mathbb{F}_q[X_1, \ldots, X_r]$ (see [vzGZ15] for explicit estimates on the number of absolutely irreducible elements of $\mathbb{F}_q[X_1, \ldots, X_r]$). On the other hand, in [MPP17] there

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is an average-case analysis which considers all the elements of $\mathbb{F}_q[X_1, \ldots, X_r]$ of a given degree, assuming a uniform distribution of inputs. Such analysis shows that, on average, three random choices of $a \in \mathbb{F}_q^{r-1}$ suffice. This is further reflected by the fact that the average-case complexity of the corresponding procedure is asymptotically at most three times the cost of computing a specialization $F(a, X_r)$ and finding the $\mathbb{F}_q$-rational roots of $F(a, X_r)$.

Now, for the general case $1 < s < r$, we may distinguish two different approaches. On one hand, we have the so-called “rewriting” methods, particularly Gröbner basis methods and variants such as XL and MXL, which have been widely used for solving polynomial systems over finite fields. In \cite{AFI04} and \cite{ACFP12} it was shown that XL and MXL methods can be described essentially as variants of the F4 algorithm for computing Gröbner bases \cite{Fan99}. Therefore, we shall consider exclusively Gröbner basis algorithms. In particular, \cite{BFP09} considers the problem of finding $\mathbb{F}_q$-rational solutions of underdetermined systems over finite fields. An “hybrid” approach is applied, which consists on a mix of partial specialization and zero-dimensional solving using Gröbner bases. The amounts of partial specialization and zero-dimensional solving is determined by a heuristic analysis, which allows the authors to experimentally break several multivariate cryptographic schemes (TRMS, UOV, and others) with parameters assumed to be secure until then.

On the other hand, we have “geometric” methods, which compute a suitable description of the variety $V$ over $\mathbb{F}_q$ defined by the system under consideration (see, e.g., \cite{HW99} or \cite{CM06}). This description consists of one or several hypersurfaces, which represent the image of components of $V$ under suitable linear projections, together with some local inverses of these linear projections. After some random trials, which may be seen as zero-dimensional linear sections —or specializations— of the input system, an $\mathbb{F}_q$-rational point of one of these hypersurfaces is obtained, which yields an $\mathbb{F}_q$-rational solution of the input system. In both \cite{HW99} and \cite{CM06}, the proposed number of random trials is proportional to the degree of the hypersurfaces under consideration.

In this paper we propose a systematic treatment of these strategies which combine partial specialization with zero-dimensional solving over $\mathbb{F}_q$, together with an average-case analysis of the algorithm under consideration. As the algorithm we propose is reminiscent of the ones of \cite{vzGSS03} and \cite{MPP17}, for bivariate and $r$-variate polynomials respectively, we call it “Search on Vertical Strips” (SVS for short). The algorithm is the following.

\textbf{Algorithm 1.1.}

\begin{itemize}
\item Input: polynomials $F_s := (F_1, \ldots, F_s) \in \mathbb{F}_q^d$.
\item Output: either a zero $x \in \mathbb{F}_q^r$ of $F_s$, or “failure”.
\item Set $i := 1$ and \textbf{State} := \textbf{Failure}.
\item While $1 \leq i \leq r - s + 1$ and \textbf{State} = \textbf{Failure} do
  \begin{itemize}
  \item Choose at random $a_i \in \mathbb{F}_q^{-s} \setminus \{a_1, \ldots, a_{i-1}\}$
  \item Find a common $\mathbb{F}_q$-rational zero of $F_s(a_i, X_{r-s+1}, \ldots, X_r)$
  \item If $x \in \mathbb{F}_q^r$ is such a zero, then set \textbf{State} := \textbf{Success}
  \item $i := i + 1$
  \end{itemize}
\item End While
\item If \textbf{State} = \textbf{Success}, then return $(a_i, x)$; else, return “Failure”.
\end{itemize}

Ignoring the cost of random generation of elements of $\mathbb{F}_q^{-s}$, at the $i$th step of the main loop we compute the vector of coefficients of the polynomials $F_j(a_i, X_{r-s+1}, \ldots, X_r)$ for $1 \leq j \leq s$. Since an element of $\mathbb{F}_d$ has $D := \binom{d+r}{r}$ coefficients, the number of arithmetic operations in $\mathbb{F}_q$ required to compute such a vector is $O^\sim(sD)$, where the notation $O^\sim$ ignores logarithmic factors. Throughout this paper, all asymptotic estimates are valid for fixed $s$ and $r$, with $d \ll q$ growing to infinity. Then a common $\mathbb{F}_q$-rational zero of $F_s(a_i, X_{r-s+1}, \ldots, X_r)$ is computed, provided that there is such a zero. This can be done using one’s favorite zero-dimensional solver over $\mathbb{F}_q$. It may be based on Gröbner
bases as in [BFP09], or it may be a “geometric” solver as the ones of [HW99], [CM06a]
or [vdHL21]. As a solver based on Gröbner bases computations as F5 [Fan02], and the
“Kronecker” solver of [CM06a] or [vdHL21], seem to be the most efficient from a complexity
point of view, we shall express our costs in terms of the cost of these solvers exclusively.
Denote by $\tau(d, s, q)$ the maximum number of arithmetic operations required to compute
a specialization $F_s(a, -)$ and solve a zero-dimensional system $F_s(a, -) = 0$ as above
with any of these solvers. Given a choice $a := (a_1, \ldots, a_{r-s+1})$ for the vertical strips to be
considered, the whole procedure requires $O(\tau(s, d, q))$ arithmetic operations
in $\mathbb{F}_q$, where $\tau(s, d, q)$ is the least value of $i$ for which the zero-dimensional solver under
consideration succeeds to compute a common solution in $\mathbb{F}_q^r$ of the system $F_s(a, -) = 0$.
The average-case analysis of Algorithm 1.1 shows that this scheme, in which zero-
dimensional searches are performed only on a limited number of vertical strips, is efficient
and has a good probability of success. This is due to the fact that, as more vertical strips are
considered, the probability of finding $\mathbb{F}_q$-rational solutions decreases exponentially, and
therefore the additional computational effort becomes progressively useless.

The average-case analysis of Algorithm 1.1 is heavily based on the properties of a
specialization of a random system $F_s(a, -) = 0$ with $F_s \in F_d^s$ and $a \in \mathbb{F}_q^{-s}$. More
precisely, we show that such a specialization is defined by a reduced regular sequence in
$\mathbb{F}_q[X_{r-s+1}, \ldots, X_r]$ with high probability (see Section 2.1 for the definition of a reduced
regular sequence). This is crucial from the complexity point of view, as both Gröbner
basis methods and the Kronecker solver are shown to behave well on systems satisfying
this condition (see, e.g., [BFSS15] or [BCG+17], Chapter 26) for Gröbner bases and [CM06a]
or [vdHL21] for the Kronecker solver. We have the following result (see Corollary 3.3).

Theorem 1.2. Let $a \in \mathbb{F}_q^{-s}$, and let $N$ be the number of $F_s := (F_1, \ldots, F_s) \in F_d^s$
such that $F_s(a, -) := (F_1(a, -), \ldots, F_s(a, -))$ forms a reduced regular sequence. If $q > 2d^s(d + 1)^s$,
then
$$1 - \frac{2d^s(d + 1)^s}{q} \leq \frac{N}{|F_d^s|} \leq 1.$$  

Then we analyze the number of partial specializations which are necessary to reach to
a vertical strip where the system under consideration has an $\mathbb{F}_q$-rational solution. It turns
out that the probability that a high number $h$ of specializations are required decreases
exponentially with $h$. More precisely, we have the following result (see Theorems 4.2 and
4.9 and Corollaries 4.3 and 4.10 for precise statements).

Theorem 1.3. For any $n \in \mathbb{N}$ denote $\mu_n := \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j!}$. Let $C$ be the random variable
which counts the number of specializations required to obtain a vertical strip of a random
system $F_s = 0$ as above with an $\mathbb{F}_q$-rational solution. Suppose that $q^s > \max\{d, 6\}$ and
$1 < h \leq r - s + 1$. We have
$$P[C = h] - \mu_d(1 - \mu_d)^{h-1} \leq \frac{1}{(d+1)!}\left(\frac{\mu_d}{2}\right)^{h} + O\left(\frac{1}{d^h}\right) \quad \text{for } d \text{ odd,}$$
$$P[C = h] - \mu_{d+1}(1 - \mu_{d+1})^{h-1} \leq \frac{1}{(d+1)!}\left(\frac{\mu_{d+1}}{2}\right)^{h} + O\left(\frac{1}{d^h}\right) \quad \text{for } d \text{ even,}$$
where $P$ denotes probability and $e$ is the basis of the natural logarithm.

Observe that $\mu_d \approx 1 - e^{-1} = 0.6321 \ldots$ for large $d$. We remark that the quantity $\mu_d$
arises also in connection with a classical combinatorial notion over finite fields, that of the
value set of univariate polynomials (cf. [LN83], [MP13]). As $\mu_d(1 - \mu_d) \approx 0.2325 \ldots$ for
large $d > s$ and $q^s > d$, we may paraphrase Theorem 1.3 as saying that 2 specializations
will suffice with high probability to obtain a vertical strip with an $\mathbb{F}_q$-rational solution of
the system under consideration. We observe that the probabilistic algorithms of [HW99],
[CM06a] and [Mat10] propose a number of searches of order $O(d^s)$ to achieve a probability
of success greater than 1/2, while that of [BFP09] proposes $q$ specializations. Our result
suggests that these analyzes are somewhat pessimistic.
Finally, we analyze the average-case complexity and probability of success of Algorithm 1.1 (see Theorems 5.1 and 5.4 for precise statements).

**Theorem 1.4.** Let $h^* := r - s + 1$. For $q > 2d^*(d + 1)^s$ and $d > s$, the average-case complexity $E$ of Algorithm 1.1 is bounded in the following way:

$$
E \leq \begin{cases} 
\tau(d, s, q) \left( \mu_d^{-1} + h^*(1 - \mu_d)h^* + \frac{3h^*e^*}{(d+1)!} + O\left(\frac{(d+1)^{2r}}{q}\right) \right) & \text{for odd,} \\
\tau(d, s, q) \left( \mu_{d+1}^{-1} + h^*(1 - \mu_{d+1})h^* + \frac{3h^*e^*}{(d+1)!} + O\left(\frac{(d+1)^{2r}}{q}\right) \right) & \text{for even,}
\end{cases}
$$

where $\tau(d, s, q)$ is the cost of the search in a vertical strip and the constant underlying the $O$-notation is independent of $r$, $s$, $d$ and $q$. Further, the probability $P$ of failure of Algorithm 1.1 can be bounded as follows:

$$
|P - (1 - \mu_d)^{h^*}| \leq \frac{eh^*}{(d+1)!} + O\left(\frac{(d+1)^{2r}}{q}\right) \quad \text{for odd,}
$$

$$
|P - (1 - \mu_{d+1})^{h^*}| \leq \frac{eh^*}{(d+1)!} + O\left(\frac{(d+1)^{2r}}{q}\right) \quad \text{for even,}
$$

where the constant underlying the $O$-notation is independent of $r$, $s$, $d$ and $q$.

As $1/\mu_d \approx 1.58\ldots$ and $h^*(1 - \mu_d)h^* < 0.76$, this result suggests that, on average, at most $1/\mu_d + 0.76 \approx 2.34\ldots$ vertical strips must be searched to obtain an $\mathbb{F}_q$-rational zero of the polynomial system under consideration, for large $d$ and $q$ satisfying the hypotheses of Theorem 1.4.

The paper is organized as follows. In Section 2 we briefly recall the notions and notations of algebraic geometry and finite fields we use. Section 3 is devoted to estimate the dimensions of algebraic geometry and finite fields we use. Section 3 is devoted to estimate the dimension of the polynomial system under consideration, for large $d$ and $q$ satisfying the hypotheses of Theorem 1.4. Finally, in Section 5 we apply the results of Sections 4 and 3 to establish Theorem 1.4.

2. Preliminaries

We use standard notions and notations of commutative algebra and algebraic geometry as can be found in, e.g., [Har92], [Kun85] or [Sha94].

Let $K$ be any of the fields $\mathbb{F}_q$ or $\mathbb{F}_q$. We denote by $\mathbb{A}^r$ the $r$-dimensional affine space $\mathbb{F}_q^r$ and by $\mathbb{P}^r$ the $r$-dimensional projective space over $\mathbb{F}_q$. By a projective variety defined over $K$ (or a projective $K$–variety for short) we mean a subset $V \subset \mathbb{P}^r$ of common zeros of homogeneous polynomials $F_1, \ldots, F_m \in K[X_0, \ldots, X_r]$. Correspondingly, an affine variety of $\mathbb{A}^r$ defined over $K$ (or an affine $K$–variety) is the set of common zeros in $\mathbb{A}^r$ of polynomials $F_1, \ldots, F_m \in K[X_1, \ldots, X_r]$. We shall frequently denote by $V(F_m) = V(F_1, \ldots, F_m)$ or $\{F_m = 0\} = \{F_1 = 0, \ldots, F_m = 0\}$ the affine or projective $K$–variety consisting of the common zeros of the polynomials $F_m := (F_1, \ldots, F_m)$.

In what follows, unless otherwise stated, all results referring to varieties in general should be understood as valid for both projective and affine varieties. A $K$–variety $V$ is $K$–irreducible if it cannot be expressed as a finite union of proper $K$–subvarieties of $V$. Further, $V$ is absolutely irreducible if it is $\mathbb{F}_q$–irreducible as a $\mathbb{F}_q$–variety. Any $K$–variety $V$ can be expressed as an irredundant union $V = C_1 \cup \cdots \cup C_s$ of irreducible (absolutely irreducible) $K$–varieties, unique up to reordering, which are called the irreducible (absolutely irreducible) $K$–components of $V$.

For a $K$–variety $V$ contained in $\mathbb{P}^r$ or $\mathbb{A}^r$, we denote by $I(V)$ its defining ideal, namely the set of polynomials of $K[X_0, \ldots, X_r]$, or of $K[X_1, \ldots, X_r]$, vanishing on $V$. The coordinate ring $K[V]$ of $V$ is the quotient ring $K[X_0, \ldots, X_r]/I(V)$ or $K[X_1, \ldots, X_r]/I(V)$. The dimension $\dim V$ of $V$ is the length $n$ of the longest chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$ of nonempty irreducible $K$–varieties contained in $V$. We say that $V$ has pure dimension $n$ (or simply it is equidimensional) if all the irreducible $K$–components of $V$ are of dimension $n$. 
A $K$–variety in $\mathbb{P}^r$ or $\mathbb{A}^r$ of pure dimension $r − 1$ is called a $K$–hypersurface. A $K$–hypersurface in $\mathbb{P}^r$ (or $\mathbb{A}^r$) is the set of zeros of a single nonzero polynomial of $K[X_0, \ldots, X_r]$ (or of $K[X_1, \ldots, X_n]$).

The degree $\deg V$ of an irreducible $K$–variety $V$ is the maximum number of points lying in the intersection of $V$ with a linear space $L$ of codimension $\dim V$, for which $V \cap L$ is a finite set. More generally, following [Hei83] (see also [Ful84]), if $V = C_1 \cup \cdots \cup C_s$ is the decomposition of $V$ into irreducible $K$–components, we define the degree of $V$ as

$$\deg V := \sum_{i=1}^s \deg C_i.$$ 

We shall use the following Bézout inequality (see [Hei83], [Ful84], [Vog84]): if $V$ and $W$ are $K$–varieties of the same ambient space, then

$$\deg(V \cap W) \leq \deg V \cdot \deg W.$$ 

Let $V \subset \mathbb{A}^r$ be a $K$–variety and $I(V) \subset K[X_1, \ldots, X_r]$ its defining ideal. Let $x$ be a point of $V$. The dimension $\dim_x V$ of $V$ at $x$ is the maximum of the dimensions of the irreducible $K$–components of $V$ that contain $x$. If $I(V) = (F_1, \ldots, F_m)$, the tangent space $T_x V$ to $V$ at $x$ is the kernel of the Jacobian matrix $(\partial F_i/\partial X_j)_{1 \leq i \leq m, 1 \leq j \leq r}(x)$ of the polynomials $F_1, \ldots, F_m$ with respect to $X_1, \ldots, X_r$ at $x$. We have (see, e.g., [Sha94] page 94)

$$\dim T_x V \geq \dim_x V.$$ 

The point $x$ is regular if $\dim T_x V = \dim_x V$. Otherwise, the point $x$ is called singular. The set of singular points of $V$ is the singular locus $\text{Sing}(V)$ of $V$; a variety is called nonsingular if its singular locus is empty. For a projective variety, the concepts of tangent space, regular and singular point can be defined by considering an affine neighborhood of the point under consideration.

Let $V$ and $W$ be irreducible affine $K$–varieties of the same dimension and $f : V \to W$ a regular map for which $\overline{f(V)} = W$, where $\overline{f(V)}$ is the closure of $f(V)$ with respect to the Zariski topology of $W$. Such a map is called dominant. Then $f$ induces a ring extension $K[W] \hookrightarrow K[V]$ by composition with $f$, and thus a field extension $K(W) \hookrightarrow K(V)$ of the fraction fields $K(W)$ and $K(V)$ of $K[V]$ and $K[W]$ respectively. We define the degree $\deg f$ of $f$ as the degree $[K(V) : K(W)]$ of the (finite) field extension $K(W) \hookrightarrow K(V)$.

For $y \in W$ and $x \in f^{-1}(y)$, we say that $f$ is unramified at $x$ if the differential mapping $d_x f : T_x V \to T_y W$ is injective.

2.1. Reduced regular sequences. A set-theoretic complete intersection is a variety $V(F_1, \ldots, F_s) \subseteq \mathbb{A}^r$ defined by $s \leq r$ polynomials $F_1, \ldots, F_s \in K[X_1, \ldots, X_r]$ which is of pure dimension $r − s$. If $s \leq r+1$ homogeneous polynomials $F_1, \ldots, F_s \in K[X_0, \ldots, X_r]$ define a projective $K$–variety $V(F_1, \ldots, F_s) \subseteq \mathbb{P}^r$ which is of pure dimension $r − s$ (the case $r = s = 1$ meaning that $V(F_1, \ldots, F_s)$ is the empty set), then the variety is called a set-theoretic complete intersection. Elements $F_1, \ldots, F_s \in K[X_1, \ldots, X_r]$ form a regular sequence if $(F_1, \ldots, F_s)$ is a proper ideal of $K[X_1, \ldots, X_r]$, $F_1$ is nonzero and, for each $2 \leq i \leq s$, $F_i$ is neither zero nor a zero divisor in $K[X_1, \ldots, X_r]/(F_1, \ldots, F_{i-1})$. If in addition $(F_1, \ldots, F_s)$ is a radical ideal of $K[X_1, \ldots, X_r]$ for $1 \leq i \leq s$, then we say that $F_1, \ldots, F_s$ form a reduced regular sequence.

2.2. Rational points. We denote by $\mathbb{A}^r(\mathbb{F}_q)$ the $n$–dimensional $\mathbb{F}_q$–vector space $\mathbb{F}_q^r$ and by $\mathbb{P}^r(\mathbb{F}_q)$ the set of lines of the $(r + 1)$–dimensional $\mathbb{F}_q$–vector space $\mathbb{F}_q^{r+1}$. For a projective variety $V \subset \mathbb{P}^r$ or an affine variety $V \subset \mathbb{A}^r$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$–rational points of $V$, namely $V(\mathbb{F}_q) := V \cap \mathbb{P}^r(\mathbb{F}_q)$ in the projective case and $V(\mathbb{F}_q) := V \cap \mathbb{A}^r(\mathbb{F}_q)$ in the affine case. For an affine variety $V$ of dimension $n$ and degree $\delta$ we have the upper bound (see, e.g., [CM06b], Lemma 2.1)

$$|V(\mathbb{F}_q)| \leq \delta q^n.$$
3. Geometric properties satisfied by “most” systems

In the sequel, we fix $1 < s < r$ and $d \geq 2$. Denote by $F_d := F_{r,d}$ the set of nonzero polynomials in $k[x_1, \ldots, x_r]$ of degree at most $d$ and $F_d^s := F_d \times \cdots \times F_d$ ($s$ times). Given any $F \in k[x_1, \ldots, x_r]$ of degree $d$, we write $\text{coeffs}(F)$ for the corresponding $(d+r)$-tuple of coefficients (with respect to a given monomial order). Further, for any $s$-tuple $F_s := (F_1, \ldots, F_s) \in F_d^s$ we denote $\text{coeffs}(F_s) := (\text{coeffs}(F_1), \ldots, \text{coeffs}(F_s))$.

The probability of success of our algorithm relies heavily on the properties of a specialization $F_s(a, -)$ of a “random” system $F_s = 0$, with $F_s \in F_d^s$ and $a \in k^{r-s}$. For this reason, we devote this section to analyze the probability that a specialization $F_s(a, -) := (F_1(a, -), \ldots, F_s(a, -))$ of a random $F_s \in F_d^s$ satisfies certain critical properties for the analysis of our algorithm. In particular, we shall be interested in the following property:

(H) $F_s(a, -)$ forms a reduced regular sequence in $k[x_{r-s+1}, \ldots, x_r]$.

In this section we show that most specializations $F_s(a, -)$ satisfy condition (H). More precisely, we obtain a lower bound close to 1 for the probability that, for a random choice of $F_s := (F_1, \ldots, F_s)$ in $F_d^s$ and $a \in k^{r-s}$, the specialization $F_s(a, -)$ satisfies condition (H).

We start with the following result.

Lemma 3.1. Let $K$ be any field and let $G_1, \ldots, G_s$ be $s \leq n$ polynomials in $K[X_1, \ldots, X_n]$ with $\deg G_i \leq d$ for $1 \leq i \leq s$. If $V(G_1, \ldots, G_s) \subset \mathbb{A}^n(K)$ has degree $d^s$, then $G_1, \ldots, G_s$ form a reduced regular sequence.

Proof. Denote $G_i := (G_1, \ldots, G_i)$ for $1 \leq i \leq s$. By the Bézout inequality \[2.1\],

$$\deg V(G_i) \leq d \cdot \deg V(G_{i-1})$$

for $1 \leq i \leq s$. As a consequence, the hypothesis $\deg V(G_s) = d^s$ implies that $\deg V(G_i) = d^i$ for $1 \leq i \leq s$. Further, by considering the sequence $G_i, G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_s$, the previous argument shows that $\deg V(G_i) = d$ for $1 \leq i \leq s$. Since $\deg G_i \leq d$, it follows that $(G_i)$ is the vanishing ideal of the hypersurface $V(G_i)$ for $1 \leq i \leq s$.

Now suppose that $G_i$ is a zero divisor modulo $G_{i-1}$. Assume further that $i \geq 2$ is the minimum index with this property. Denote by $C_1, \ldots, C_t$ the irreducible components of $V(G_{i-1})$, which are all of pure dimension $n - i + 1$. By definition, we have $\sum_{j=1}^t \deg C_j = d^{i-1}$. As $G_i$ is a zero divisor modulo $G_{i-1}$, it vanishes identically over an irreducible component, say $C_1$. As a consequence,

$$\deg V(G_i) \leq \sum_{j=1}^t \deg C_j \cap V(G_i)$$

$$\leq \deg C_1 + \sum_{j=2}^t \deg C_j \cdot d < \sum_{j=1}^t \deg C_j \cdot d = d^i,$$

contradicting the fact that $\deg V(G_i) = d^i$. We conclude that $G_1, \ldots, G_s$ form a regular sequence.

We have already shown that $(G_i)$ is a radical ideal for $1 \leq i \leq s$. Now, let $2 \leq i \leq s$ and assume inductively that the ideal $(G_1, \ldots, G_{i-1})$ is radical. As $G_1, \ldots, G_i$ form a regular sequence, $V(G_{i-1})$ and $V(G_i)$ are varieties of pure dimension that intersect properly. On the other hand, we have

$$\deg V(G_{i-1}) \cdot \deg V(G_i) = \deg V(G_i) = \sum Z \deg Z,$$

where the sum runs over all irreducible components $Z$ of $V(G_i)$. By, e.g., [Har92 Corollary 18.4] we deduce that $V(G_{i-1})$ and $V(G_i)$ intersect transversely at a general point $p$ of
any component \(Z\) of \(V(G_i)\). Since, by the inductive hypothesis, \(G_1, \ldots, G_{i-1}\) generate the \(G_i\) vanishing ideal of \(V(G_{i-1})\) and \(G_i\) generates the vanishing ideal of \(V(G_i)\), the fact that \(V(G_{i-1})\) and \(V(G_i)\) intersect transversely at \(p\) implies that the Jacobian matrix of \(G_1, \ldots, G_i\) has rank equal to \(i\) at \(p\). As the ideal \((G_1, \ldots, G_i)\) has codimension \(i\), by [Eis95, Theorem 18.15] we conclude that \(G_1, \ldots, G_i\) generate a radical ideal. This concludes the proof of the lemma. 

Now we obtain an hypersurface containing all the \(F_s\) for which a specialization does not satisfy condition (H).

**Proposition 3.2.** There exists a nonzero \(P_S \in \mathbb{F}_q[\text{coeffs}(F_s)]\) with

\[\deg P_S \leq 2d^*(d + 1)^s,\]

such that for any \(F_s := (F_1, \ldots, F_s) \in \mathcal{F}_d(\mathbb{F}_q)\) with \(P_S(\text{coeffs}(F_s)) \neq 0\), the specialization \(F_s(a, -)\) satisfies condition (H).

**Proof.** Consider the incidence variety

\[W_a := \{(F_s, x) \in \mathcal{F}_d(\mathbb{F}_q) \times \mathbb{A}^s : F_s(a, x) = 0\}.\]

Let \(F_1, \ldots, F_s\) be formal polynomials of degree \(d\) in the variables \(X_1, \ldots, X_r\). It is easy to see that \(F_1(a, X_{r-s+1}, \ldots, X_r), \ldots, F_s(a, X_{r-s+1}, \ldots, X_r)\) form a regular sequence of \(\mathbb{F}_q[\text{coeffs}(F_s)], X_{r-s+1}, \ldots, X_r\). This implies that \(W_a\) is of pure dimension \(\dim \mathcal{F}_d(\mathbb{F}_q)\). Further, we may express \(W_a\) as the set of solutions of the following system:

\[\text{coeff}_0(F_i) = -(F_i - \text{coeff}_0(F_i))(a, X_{r-s+1}, \ldots, X_r) \quad (1 \leq i \leq s),\]

where \(\text{coeff}_0(F_i)\) denotes the coefficient of the monomial of degree zero of \(F_i\). We conclude \(W_a\) may be seen as the image of a regular mapping defined on the absolutely irreducible variety \(\mathcal{F}_d(\mathbb{F}_q) \times \mathbb{A}^s\), and therefore it is absolutely irreducible.

Denote by \(\pi_a : W_a \to \mathcal{F}_d(\mathbb{F}_q)\) its projection on the first coordinate. It is easy to see that there exists a zero-dimensional fiber. Indeed, let \(f \in \mathbb{F}_q[T]\) be a monic irreducible polynomial of degree \(d\), and let \(S := \{f\}\) be the set of zeros of \(f\). If \(F_s := (f(X_{r-s+1}), \ldots, f(X_r))\), then \(\pi_a^{-1}(F_s) = \{F_s\} \times S^s\). Then the theorem on the dimension of fibers (see, e.g., [Cut18, Theorem 8.13]) implies that \(\pi_a\) is dominant.

Let \(J(F_s(a, -))\) denote the Jacobian of \(F_i(a, X_{r-s+1}, \ldots, X_r)\) (1 \(\leq i \leq s\)) with respect to \(X_{r-s+1}, \ldots, X_r\). A straightforward computation shows that \(\pi_a\) is unramified at \((F_s, x) \in W_a\) if \(J(F_s(a, -))(x) \neq 0\). For \(F_i := f(X_{r-s+i})\) (1 \(\leq i \leq s\)) as above, \(J(F_s(a, -)) = \prod_{i=1}^s f'(X_{r-s+i})\). This shows that \(J(F_s(a, -))(x)\) does not vanish identically on \(W_a\). Taking into account that \(W_a\) is absolutely irreducible we conclude that the set of points at which \(\pi_a\) is unramified contains a nonempty Zariski open subset of \(W_a\).

[1ve73, Proposition T.8] shows that the field extension \(\mathbb{F}_q(\mathcal{F}_d(\mathbb{F}_q)) \hookrightarrow \mathbb{F}_q(W_a)\) is separable, and [Hei83, Proposition 1] proves that \#\(\pi_a^{-1}(F_s)\) \(\leq \deg \pi_a\) for any finite fiber \(\pi_a^{-1}(F_s)\), with equality in a nonempty Zariski open subset of \(\mathcal{F}_d(\mathbb{F}_q)\). We call any fiber unramified if it belongs to this Zariski open set, and ramified otherwise.

Observe that any zero-dimensional fiber of \(\pi_a\) consists of at most \(d^s\) points, due to the fact that it is defined by \(s\) polynomials of degree at most \(d\) (2.1). In particular, for \(F_i := f(X_{r-s+i})\) (1 \(\leq i \leq s\)) as above, the corresponding fiber has precisely \(d^s\) points. This proves that \(\deg \pi_a = d^s\).

Now [Sch03, Proposition 3] and its proof allows to describe more precisely an open subset for the condition \#\(\pi_a^{-1}(F_s)\) = \(\deg \pi_a\) = \(d^s\). Indeed, denote by \(G_i := F_i(a, X_{r-s+1}, \ldots, X_r)\) (1 \(\leq i \leq s\)) the polynomials of \(\mathbb{F}_q[\text{coeffs}(F_s)], X_{r-s+1}, \ldots, X_r\) which define the incidence variety \(W_a = \mathcal{F}_d(\mathbb{F}_q) \times \mathbb{A}^s(G_1, \ldots, G_s)\), and write \(J(G_s) \in \mathbb{F}_q[\text{coeff}(F_s), X_{r-s+1}, \ldots, X_r]\) for the Jacobian of \(G_1, \ldots, G_s\) with respect to \(X_{r-s+1}, \ldots, X_r\). Since \(J(G_s)(F_s, x) = J(F_s(a, -))(x)\) for any \((F_s, x) \in W_a\), it follows by the previous discussion that \(J(G_s)\) does not vanish identically on \(W_a\).
Let \( \mathcal{V} := V F^d_d(\mathbb{F}_q) \times \mathbb{A}^s((G_1, \ldots, G_s) : J(G_s) \infty) \) be the nonempty subvariety of \( W_a \) defined by the saturated ideal \((G_1, \ldots, G_s) : J(G_s) \infty)\). Denote \( \pi_a | \mathcal{V} \) the restriction of \( \pi_a \) to \( \mathcal{V} \). By the proof of [Sch03 Proposition 3] we deduce that exists a non-zero polynomial \( P_S \in \mathbb{E}_q[\text{coeffs}(F_s)] \) of total degree at most \( 2 \deg(\pi_a | \mathcal{V}) \deg \mathcal{V} \) with the following property: for any \( F_s \in F^d_d(\mathbb{F}_q) \) with \( P_S(F_s) \neq 0 \), we have that \( \pi_a^{-1}(F_s) \cap \mathcal{V} \) is a finite set containing \( \deg(\pi_a | \mathcal{V}) \) points. Now, as \( W_a \) is irreducible and \( J(G_s) \) does not vanish identically on \( W_a \), \( W_a \setminus V F^d_d(\mathbb{F}_q) \times \mathbb{A}^s((J(G_s)) \) is an open dense subset of \( W_a \). Thus

\[
\mathcal{V} = V F^d_d(\mathbb{F}_q) \times \mathbb{A}^s((G_1, \ldots, G_s) : J(G_s) \infty) = W_a \setminus V F^d_d(\mathbb{F}_q) \times \mathbb{A}^s((J(G_s)) = W_a.
\]

We conclude that for any \( F_s \in F^d_d(\mathbb{F}_q) \), the condition \( P_S(F_s) \neq 0 \) implies that the fiber \( \pi_a^{-1}(F_s) \) is finite and contains \( \deg(\pi_a | \mathcal{V}) = \deg \pi_a = d^s \) points. Further, observe that \( W_a \) is defined by \( s \) polynomials of degree \( d + 1 \). As a consequence, by (2.1) we conclude that \( \deg W_a \leq (d + 1)^s \). It follows that

\[
\deg P_S \leq 2d^s(d + 1)^s.
\]

Finally, for each \( F_s \in F^d_d(\mathbb{F}_q) \) with \( P_S(F_s) \neq 0 \), we have that \( \pi_a^{-1}(F_s) = \{ F_s \} \times V(A_s(a, -)) \) is zero-dimensional of degree \( d^s \). Therefore, Lemma 3.1 implies that \( F_s(a, -) \) satisfies condition (H). \( \square \)

Finally, we estimate the probability that a random specialization of a random \( F_s \) satisfies condition (H). For this purpose, we consider the random variable \( C_H := \mathbb{E}^{q^r - s} \times F^d_d \rightarrow \{1, \infty\} \) defined in the following way:

\[
C_H(a, F_s) := \begin{cases} 1, & \text{if } F_s(a, -) \text{ satisfies (H)}; \\ \infty, & \text{otherwise}. \end{cases}
\]

We consider the set \( \mathbb{E}^{q^r - s} \times F^d_d \) endowed with the uniform probability \( P_1 := P_{1,r,s,d} \) and study the probability of the set \( \{C_H = 1\} \).

**Corollary 3.3.** For \( q > 2d^s(d + 1)^s \) and \( S_H := \{C_H = 1\}, \) we have

\[
1 - \frac{2d^s(d + 1)^s}{q} \leq P_1(S_H) \leq 1.
\]

**Proof.** The upper bound being obvious, we prove the lower bound. For \( a \in \mathbb{E}^{q^r - s} \), let \( N \) be the number of \( F_s \in F^d_d \) such that \( F_s(a, -) \) satisfies condition (H), and let \( M \) be the number of \( F_s \in F^d_d \) such that \( P_S(\text{coeffs}(F_s)) = 0 \), where \( P_S \) is the polynomial of Proposition 3.2. Let \( e := 2d^s(d + 1)^s \) and \( D := s(d + r) \). According to (2.2),

\[
\frac{N}{|F^d_d|} \geq \frac{|F^d_d| - M}{|F^d_d|} \geq \frac{q^D - eq^{D-1}}{q^D} = 1 - \frac{e}{q}.
\]

This proves the corollary. \( \square \)

4. The number of specializations to have an \( \mathbb{F}_q \)-rational solution

For \( 1 < s < r \) and \( d \geq 2 \), given \( F_s := (F_1, \ldots, F_s) \) of \( F^d_d \), we are interested in finding an \( \mathbb{F}_q \)-rational solution of the system \( F_s = 0 \). For this purpose, we consider Algorithm 1.1 which performs searches for \( \mathbb{F}_q \)-rational solutions on the (hopefully) zero-dimensional systems arising from a number of specializations of \( F_s \). More precisely, for each choice of \( a \in \mathbb{E}^{q^r - s} \), we shall search for an \( \mathbb{F}_q \)-rational solution of the system \( F_s = 0 \) in the “vertical strip” \( \{a\} \times \mathbb{E}_q^r \) determined by \( a \). Thus our algorithm mainly consists of “searches on vertical strips” (SVS for short).

In Section 3 we estimate the probability that a specialization of the system \( F_s = 0 \) has “good” properties from the computational point of view. In this section we study the probability that \( h \) random specializations must be considered until a vertical strip with \( \mathbb{F}_q \)-rational solutions of the input system is attained.
4.1. The probability of having $\mathbb{F}_q$-rational solutions in the first specialization.

To analyze the probability of success of our algorithm in the first search, we introduce the random variable $C_1 := \mathbb{F}_q^{r-s} \times \mathcal{F}_d^s \to \{1, \infty\}$ defined in the following way:

$$C_1(a, F_s) := \begin{cases} 1, & \text{if } F_s(a, -) \text{ has a zero in } \mathbb{F}_q; \\ \infty, & \text{otherwise.} \end{cases}$$

We consider the set $\mathbb{F}_q^{r-s} \times \mathcal{F}_d^s$ endowed with the uniform probability $P_1 := P_{1,s,r,d}$ and study the probability of the set $\{C_1 = 1\}$. Explicitly, we aim to estimate the probability $P_1(S_1)$, for

$$S_1 := \{(a, F_s) \in \mathbb{F}_q^{r-s} \times \mathcal{F}_d^s : F_s(a, -) \text{ has a zero in } \mathbb{F}_q\}.$$

We shall need the following technical lemma.

**Lemma 4.1.** For $s \leq d+1$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{A}^r$ pairwise distinct, consider the $(s \times (d+r))$-matrix $A_{s,d} := (\alpha_i^j)_{1 \leq h \leq s; |i| \leq d}$, where $i$ runs over all exponents $i := (i_1, \ldots, i_r) \in \mathbb{Z}_{\geq 0}^r$ in some ordering and $|i| := i_1 + \cdots + i_s$. Then $A_{s,d}$ has maximal rank $s$.

**Proof.** Let $\Lambda_1, \ldots, \Lambda_r$ be new indeterminates over $\mathbb{F}_q$ and $\mathcal{L} := \Lambda_1 X_1 + \cdots + \Lambda_r X_r$. For any $c := (c_1, \ldots, c_r) \in \mathbb{A}^r$ we write $\ell_c := c_1 X_1 + \cdots + c_r X_r$. Since the points $\alpha_1, \ldots, \alpha_s$ are pairwise distinct,

$$P := \prod_{1 \leq j \leq s} \left(\mathcal{L}(\alpha_j) - \mathcal{L}(\alpha_k)\right)$$

is a nonzero polynomial in $\mathbb{F}_q[\Lambda_1, \ldots, \Lambda_r]$. It follows that there exists $\lambda := (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r$ such that $P(\lambda) \neq 0$. This means that $\ell\lambda(\alpha_j) \neq \ell\lambda(\alpha_k)$ for $1 \leq j < k \leq s$. We may also assume without loss of generality that $X_1, \ldots, X_{r-1}$, $\ell\lambda$ are $\mathbb{F}_q$-linearly independent.

Write $\alpha_h := (\alpha_{h,1}, \ldots, \alpha_{h,r})$ for $1 \leq h \leq s$. Observe that the $h$th row of $A_{s,d}$ consists of the specialization of the monomial basis $\{X^i := X_1^{i_1} \cdots X_r^{i_r} : |i| \leq d\}$ of $\mathcal{F}_{d,r}(\mathbb{F}_q) := \{F \in \mathbb{F}_q[X_1, \ldots, X_r] : \deg F \leq d\}$ at $\alpha_h$. Let $\ell := (X_1, \ldots, X_{r-1}, \ell\lambda)$. Since $\{\ell^i := X_1^{i_1} \cdots X_{r-1}^{i_{r-1}} \ell\lambda : |i| \leq d\}$ is another basis of $\mathcal{F}_{d,r}(\mathbb{F}_q)$ as $\mathbb{F}_q$-vector space, it turns out that

$$\text{rank } A_{s,d} = \text{rank } (\ell(\alpha_h))^t \big| 1 \leq h \leq s; |i| \leq d^r.$$

We may therefore assume without loss of generality that $\alpha_{h,r} \neq \alpha_{k,r}$ for $1 \leq h < k \leq s$.

Consider the submatrix $V_{s,d} := (\alpha_{h,r}^j)_{1 \leq h \leq s; 0 \leq j \leq d}$ of $A_{s,d}$. Observe that $V_{s,d}$ is a Vandermonde-like matrix of size $s \times (d+1)$. Since $s \leq d+1$, it is well known that our assumption implies that $V_{s,d}$ has rank $s$, and so does $A_{s,d}$. \hfill $\Box$

Now we are able to estimate the probability $P_1(S_1)$. From now on, for any $m \in \mathbb{N}$ we shall use the notations

$$s_m := \sum_{j=1}^{m} (-1)^{j-1} \binom{q^s}{j} q^{-sj}, \quad t_m := \binom{q^s}{m} q^{-sm}.$$

**Theorem 4.2.** For $s \leq d+1$ and $q^s > d$, we have

$$s_d - t_{d+1} \leq P_1(S_1) \leq s_d \quad \text{for } d \text{ odd}, \quad s_d \leq P_1(S_1) \leq s_d + t_{d+1} \quad \text{for } d \text{ even.}$$

**Proof.** For any $F_s \in \mathcal{F}_d^s$, we denote by $VS(F_s)$ the set of vertical strips where the system $F_s = 0$ has an $\mathbb{F}_q$-rational solution and by $NS(F_s)$ its cardinality, that is,

$$VS(F_s) := \{a \in \mathbb{F}_q^{r-s} : (\exists x \in \mathbb{F}_q^s) F_s(a, x) = 0\}, \quad NS(F_s) := |VS(F_s)|.$$

It is easy to see that $S_1 = \bigcup_{F_s \in \mathcal{F}_d} VS(F_s) \times \{F_s\}$. Since this is a disjoint union of $\mathbb{F}_q^{r-s} \times \mathcal{F}_d^s$, it follows that

$$P_1(S_1) = \frac{1}{q^{r-s}|\mathcal{F}_d|} \sum_{F_s \in \mathcal{F}_d^s} NS(F_s).$$
Fix $F_s \in \mathcal{F}_d^s$. Observe that
\[
V S(F_s) = \bigcup_{x \in \mathbb{F}_{q^s}^d} \{a \in \mathbb{F}_{q^s}^r : F_s(a, x) = 0\}. 
\]
Assume that $d$ is odd. By the Bonferroni inequalities we deduce that
\[
NS(F_s) \geq \sum_{j=1}^{d+1} (-1)^{j-1} \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} |\{a \in \mathbb{F}_{q^s}^r : (\forall x \in X_j) F_s(a, x) = 0\}|,
\]
\[
NS(F_s) \leq \sum_{j=1}^d (-1)^{j-1} \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} |\{a \in \mathbb{F}_{q^s}^r : (\forall x \in X_j) F_s(a, x) = 0\}|.
\]
For $1 \leq j \leq d + 1$, denote
\[
\mathcal{N}_j = \frac{1}{q^{r-s}|\mathcal{F}_d^s|} \sum_{F_s \in \mathcal{F}_d^s} \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} |\{a \in \mathbb{F}_{q^s}^r : (\forall x \in X_j) F_s(a, x) = 0\}| 
= \frac{1}{q^{r-s}|\mathcal{F}_d^s|} \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} \sum_{a \in \mathbb{F}_{q^s}^r} |\{F_s \in \mathcal{F}_d^s : (\forall x \in X_j) F_s(a, x) = 0\}|.
\]
We conclude that
\[
(4.1) \quad \sum_{j=1}^{d+1} (-1)^{j-1} \mathcal{N}_j \leq P(S_1) \leq \sum_{j=1}^d (-1)^{j-1} \mathcal{N}_j.
\]
Next we obtain an explicit expression for each $\mathcal{N}_j$. We have a direct-product decomposition of vector spaces
\[
\{F_s \in \mathcal{F}_d^s : (\forall x \in X_j) F_s(a, x) = 0\} = \prod_{i=1}^s \{F_i \in \mathcal{F}_d : (\forall x \in X_j) F_i(a, x) = 0\}.
\]
Further, all the factors in the product in the right-hand side are isomorphic $\mathbb{F}_{q^s}$-vector spaces. As a consequence, we may write
\[
\mathcal{N}_j = \frac{1}{q^{r-s}} \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} \sum_{a \in \mathbb{F}_{q^s}^r} \left( \frac{1}{|\mathcal{F}_d|} \right) |\{F \in \mathcal{F}_d : (\forall x \in X_j) F(a, x) = 0\}|^s.
\]
Let $j \leq d + 1$ and $a \in \mathbb{F}_{q^s}^r$ be fixed. The set of equalities $F(a, x) = 0$ ($x \in X_j$) constitute $j$ linear conditions on the coefficients of $F$, which can be expressed by a $j \times (d+r)$-matrix of rank $j$ by Lemma 4.1. We conclude that
\[
\mathcal{N}_j = \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} \left( \frac{1}{|\mathcal{F}_d|} \right) |\{F \in \mathcal{F}_d : (\forall x \in X_j) F(a, x) = 0\}|^s
= \sum_{X_j \subseteq \mathbb{F}_{q^s}^d} \left( \frac{q^{s \dim F_{r,d} - j} \cdot 1}{|\mathcal{F}_d|} \right) = \left( \frac{q^s}{j} \right)^{q^{-sj}}.
\]
By (4.1) it follows that
\[
\sum_{j=1}^{d+1} (-1)^{j-1} \left( \frac{q^s}{j} \right)^{q^{-sj}} \leq P(S_1) \leq \sum_{j=1}^d (-1)^{j-1} \left( \frac{q^s}{j} \right)^{q^{-sj}},
\]
which proves the statement for $d$ odd.

The statement for $d$ even follows by a similar argument mutatis mutandis.

We now discuss the asymptotic behavior of the probability $P_1(S_1) = P_1[C_1 = 1]$. 


Corollary 4.3. For \( s \leq d + 1 \) and \( q^s > d \), we have
\[
\mu_{d+1} - \frac{2}{q^s} \leq P_1(S_1) \leq \mu_d + \frac{2}{q^s} \quad \text{for } d \text{ odd},
\]
\[
\mu_d - \frac{2}{q^s} \leq P_1(C_1 = 1) \leq \mu_{d+1} + \frac{2}{q^s} \quad \text{for } d \text{ even}.
\]

Proof. For positive integers \( k, j \) with \( k \leq j \), we denote by \( \binom{j}{k} \) the unsigned Stirling number of the first kind, namely the number of permutations of \( j \) elements with \( k \) disjoint cycles. The following properties of Stirling numbers are well-known (see, e.g., [FS09, Section A.8])
\[
\binom{j}{j} = 1, \quad \binom{j}{j-1} = \frac{j}{2}, \quad \sum_{k=0}^{j} \binom{j}{k} = j!.
\]
We shall also use the following well-known identity (see, e.g., [GKP94, (6.13)]):
\[
\binom{q^s}{j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} q^{sk}.
\]

According to (4.2), for any positive integer \( m \) we have that
\[
s_m = \sum_{j=1}^{m} (-1)^{j-1} \binom{q^s}{j} q^{-sj} = \sum_{j=1}^{m} (-1)^{j-1} \sum_{k=0}^{j} \frac{(-1)^{j-k}}{j!} \binom{j}{k} q^{s(k-j)}
\]
\[
= \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j!} \binom{j}{j} + \sum_{j=1}^{m} \frac{(-1)^{j}}{j!} \binom{j}{j-1} q^{-s}
\]
\[
+ \sum_{j=1}^{m} \sum_{k=0}^{j-2} \frac{(-1)^{k-1}}{j!} \binom{j}{k} q^{s(k-j)}.
\]
It follows that
\[
s_m = \nu_m + \frac{1}{q^s} \sum_{j=1}^{m} \frac{(-1)^{j}}{j!} \binom{j}{2} - \sum_{j=1}^{m} \sum_{k=0}^{j-2} \frac{(-1)^{k}}{j!} \binom{j}{k} q^{s(k-j)}.
\]
As a consequence, for \( m > 2 \) we obtain
\[
|s_m - \nu_m| \leq \frac{1}{q^s} \sum_{j=1}^{m} \frac{(-1)^{j}}{j!} \binom{j}{2} + \sum_{j=1}^{m} \sum_{k=0}^{j-2} \frac{1}{j!} \binom{j}{k} q^{2s} \leq \frac{1}{4q^s} + \frac{m}{q^s}.
\]
For \( m = 2 \), this inequality is obtained by a direct calculation. Combining this for \( m = d \) and \( m = d + 1 \) with Theorem 4.2 we readily deduce the statement of the corollary. \( \square \)

4.2. The probability that \( h \) specializations are required. Now we study the probability that multiple specializations are required to attain a vertical strip with \( \mathbb{F}_q \)-rational solutions of the input system \( F_s = 0 \). More precisely, for \( 2 \leq h \leq r - s + 1 \), we analyze the probability that a random choice \( \mathbf{a}_1, \ldots, \mathbf{a}_h \in \mathbb{F}_q^{r-s} \) yields systems \( F_s(\mathbf{a}_i, -) = 0 \) for \( 1 \leq i \leq h - 1 \) with no \( \mathbb{F}_q \)-rational solutions, such that \( F_s(\mathbf{a}_h, -) = 0 \) has \( \mathbb{F}_q \)-rational solutions.

Let \( \mathbf{a} := (\mathbf{a}_1, \ldots, \mathbf{a}_h) \) be such that \( \mathbf{a}_i \neq \mathbf{a}_j \) for \( i \neq j \) and
\[ S^*_{h, \mathbf{a}} := \{ F_s \in F^s_d : Z(F_s(\mathbf{a}_i, -)) = 0 \ (1 \leq i \leq h - 1), \ Z(F_s(\mathbf{a}_h, -)) > 0 \}, \]
where \( Z(G_s) \) denotes the number of zeros in \( \mathbb{F}_q^s \) of \( G_s \in \mathbb{F}_q[X_{r-s+1}, \ldots, X_r]^s \). We may express \( S^*_{h, \mathbf{a}} \) in the following way:
\[
S^*_{h, \mathbf{a}} = S^*_{1, \mathbf{a}_h} \setminus \bigcup_{j=1}^{h-1} S^*_{2, (\mathbf{a}_j, \mathbf{a}_h)}.
\]
where \( S_{2,(a_j,a_h)} := \{ F_s \in F^s_d : Z(F_s(a_j, -)) > 0, Z(F_s(a_h, -)) > 0 \} \). Therefore, by the inclusion-exclusion principle, we obtain

\[
|S_{h,a}^*| = |S_{1,a_h}^*| - \sum_{j=1}^{h-1} |S_{2,(a_j,a_h)}|
\]

\[
= |S_{1,a_h}^*| + \sum_{k=1}^{h-1} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k < h} |S_{2,(a_{i_1},a_{i_k},a_h)}|,
\]

(4.4)

where \( S_{k,(b_1,...,b_k)} := \{ F_s \in F^s_d : Z(F_s(b_j, -)) > 0 \text{ for } 1 \leq j \leq k \} \).

To estimate \( |S_{h,a}^*| \), we establish a condition on \( a \) which implies that \( |S_{k+1,(a_1,...,a_k,a_h)}| \) has a predictable behavior. Then we estimate \( |S_{k+1,(a_1,...,a_k,a_h)}| \) for \( 1 \leq k \leq h-1 \), which will lead us to an estimate for \( |S_{h,a}^*| \). Finally, we shall be able to estimate the probability that \( h \) random choices \( a_1,...,a_h \) are required to have an \( F_q \)-rational solution of the system under consideration.

4.2.1. A condition on \( a \). We start with the condition on \( a \) mentioned above. Given positive integers \( d \geq j_1 \geq j_2 \geq \cdots \geq j_h \geq 1 \) and sets \( X_{j_i} \subset F_q^d \) of cardinality \( j_i \) for \( 1 \leq i \leq h \), we shall be interested in the number of \( F_s := (F_1,...,F_s) \in F^s_d \) satisfying

\[
F_s(a_i, x) = 0 \text{ for all } x \in X_{j_i} \text{ and } 1 \leq i \leq h.
\]

We have the following result.

**Proposition 4.4.** Denote \( a_i := (a_{i,1},...,a_{i,r-s}) \) for \( 1 \leq i \leq h \) and let

\[
M := \begin{bmatrix}
1 & a_{1,1} & \cdots & a_{1,h-1} \\
1 & a_{2,1} & \cdots & a_{2,h-1} \\
\vdots & \vdots & & \vdots \\
1 & a_{h,1} & \cdots & a_{h,h-1}
\end{bmatrix}.
\]

If \( M \) is invertible, then

\[
|\{ F_s \in F^s_d : F_s(a_i, x) = 0 \forall x \in X_{j_i} \text{ and } 1 \leq i \leq h \}| = q^{s(\dim F_d - (j_1+\cdots+j_h))}.
\]

**Proof.** Observe that

\[
\{ F_s \in F^s_d : F_s(a_i, x) = 0 \forall x \in X_{j_i} \text{ and } 1 \leq i \leq h \} = \prod_{i=1}^{s} \{ F \in F_d : F(a_i, x) = 0 \forall x \in X_{j_i} \text{ and } 1 \leq i \leq h \}.
\]

(4.5)

As a consequence, we shall analyze the number of \( F \in F_d \) satisfying

\[
F(a_i, x) = 0 \text{ for all } x \in X_{j_i} \text{ and } 1 \leq i \leq h.
\]

(4.6)

We may consider \( F(a_i, x) \) as a linear system consisting on \( j_1 + \cdots + j_h \) equations on the \((d+r)\) coefficients of \( F \), expressing \( F \) in the monomial basis \( \{ X^\alpha := X_1^{\alpha_1} \cdots X_r^{\alpha_r} : |\alpha| \leq d \} \). We claim that the matrix of this system has full rank.

Up to a change of the monomial basis under consideration, we may assume without loss of generality that \( X_r \) separates the points of \( X_{j_i} \) for each \( i \). To prove the claim, it suffices to show that the \((j_1 + \cdots + j_h) \times (j_1 + \cdots + j_h)\)-submatrix \( A \) consisting of the \( j_1 + \cdots + j_h \) rows of this matrix and the columns corresponding to the monomials \( \{ X_r, X_{j_i-1}X_r, \cdots, X_{j_i-1}X_r^{j_i-1} \} \) has full rank \((j_1 + \cdots + j_h)\).
Denote

\[ V(\mathcal{X}_k, l) := \begin{pmatrix} 1 & x_1 & \cdots & x_1^{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \cdots & x_k^{l-1} \end{pmatrix}, \]

where \( x_1, \ldots, x_k \) are the \( r \)-th coordinates of the elements of the set \( \mathcal{X}_k \) (of cardinality \( k \)). Then the matrix \( A \) can be expressed in the following way:

\[
A = \begin{pmatrix}
V(\mathcal{X}_{j_1}, j_1) & a_{1,1}V(\mathcal{X}_{j_1}, j_2) & \cdots & a_{1,h-1}V(\mathcal{X}_{j_1}, j_h) \\
V(\mathcal{X}_{j_2}, j_1) & a_{2,1}V(\mathcal{X}_{j_2}, j_2) & \cdots & a_{2,h-1}V(\mathcal{X}_{j_2}, j_h) \\
\vdots & \vdots & \ddots & \vdots \\
V(\mathcal{X}_{j_h}, j_1) & a_{h,1}V(\mathcal{X}_{j_h}, j_2) & \cdots & a_{h,h-1}V(\mathcal{X}_{j_h}, j_h)
\end{pmatrix}.
\]

For positive integers \( k, l \), let \( 1_{k,l} \) be the \((k \times l)\)-matrix having 1 on the diagonal elements and 0 elsewhere. Then the matrix \( A \) admits the following factorization:

\[
A = \begin{pmatrix}
V(\mathcal{X}_{j_1}, j_1) \\
V(\mathcal{X}_{j_2}, j_1) \\
\vdots \\
V(\mathcal{X}_{j_h}, j_1)
\end{pmatrix}
\begin{pmatrix}
1_{j_1,j_1} & a_{1,1}1_{j_1,j_2} & \cdots & a_{1,h-1}1_{j_1,j_h} \\
1_{j_1,j_1} & a_{2,1}1_{j_1,j_2} & \cdots & a_{2,h-1}1_{j_1,j_h} \\
\vdots & \vdots & \ddots & \vdots \\
1_{j_1,j_1} & a_{h,1}1_{j_1,j_2} & \cdots & a_{h,h-1}1_{j_1,j_h}
\end{pmatrix}.
\]

The fact that \( j_1 \geq j_i \) and \( \mathcal{X}_{j_i} \) consists of \( j_i \) pairwise-distinct elements implies that \( V(\mathcal{X}_{j_1}, j_1) \) has full rank \( j_i \) for \( 1 \leq i \leq h \). It follows that the left-hand matrix in the factorization of \( A \) has full rank, and thus

\[
\text{rank} A = \text{rank} \begin{pmatrix}
1_{j_1,j_1} & a_{1,1}1_{j_1,j_2} & \cdots & a_{1,h-1}1_{j_1,j_h} \\
1_{j_1,j_1} & a_{2,1}1_{j_1,j_2} & \cdots & a_{2,h-1}1_{j_1,j_h} \\
\vdots & \vdots & \ddots & \vdots \\
1_{j_1,j_1} & a_{h,1}1_{j_1,j_2} & \cdots & a_{h,h-1}1_{j_1,j_h}
\end{pmatrix}.
\]

Up to a row permutation in the matrix \( M \) of the statement of the proposition, we may assume that we can perform Gauss elimination without permutations to obtain a row echelon form of \( M \). By hypothesis, such a row echelon form is upper triangular and invertible. Similarly, we can perform a process of block Gauss elimination on the right-hand matrix in the factorization of \( A \) mutatis mutandis. As a consequence, we obtain a block matrix which is block upper triangular, with invertible full-rank blocks on the diagonal. It follows that this matrix has full rank \( j_1 + \cdots + j_j \), and thus so does \( A \). This finishes the proof of our claim.

As a consequence of the claim, the \( j_1 + \cdots + j_j \) equations of (4.6) are linearly independent, which implies that the set of solutions of (4.6) has codimension \( j_1 + \cdots + j_j \). It follows that (4.6) has \( q^{\dim F_d - (j_1 + \cdots + j_j)} \) solutions in \( F_d \). Finally, taking into account (4.5) we readily deduce the statement of the proposition.

**Remark 4.5.** With notations as in Proposition 4.4 there are

\[
(q^h - q)(q^h - q^2) \cdots (q^h - q^{h-1})q^{h(r-s-h+1)} = (q^{h-1} - 1)(q^{h-2} - 1) \cdots (q - 1)q^{h(r-s-h+1)}
\]

elements \( a_1, \ldots, a_h \) satisfying the condition of Proposition 4.4. Indeed, it is easy to see that there are \( (q^h - q)(q^h - q^2) \cdots (q^h - q^{h-1}) \) possible choices of the \( h \times (h-1) \) right submatrix of \( M \) such that \( M \) is invertible. Then the factor \( q^{h(r-s-h+1)} \) takes into account that the last \( r - s - (h - 1) \) coordinates of \( a_1, \ldots, a_h \) can be arbitrarily chosen.

4.2.2. Estimates for \( \|S_k, a\| \). Let \( a := (a_1, \ldots, a_h) \in (\mathbb{F}_q^{r-s})^h \) be such that \( (a_1, \ldots, a_k) \) satisfies the hypothesis of Proposition 4.4 for \( 2 \leq k \leq h \). As asserted before, we aim to estimate the number of \( F_s \) with \( Z(F_s(a_k, -)) = 0 \) for \( 1 \leq k \leq h - 1 \) and \( Z(F_s(a_h, -)) > 0 \). According to (4.4), this can be reduced to estimate the number of elements of the set

\[
S_k, a := \{ F_s \in F_d^a : Z(F_s(a_j, -)) > 0 \text{ for } 1 \leq j \leq k \}
\]
for $2 \leq k \leq r - s + 1$. Assume that $d$ is odd. Since
\[
S_{k,a} = \bigcup_{x^1 \in \mathbb{F}_q^*} \cdots \bigcup_{x^k \in \mathbb{F}_q^*} \{ F_s \in \mathcal{F}_d^*: F_s(a_j, x^j) = 0 \text{ for } 1 \leq j \leq k \},
\]
by the Bonferroni inequalities, we have
\[
\sum_{j_1=1}^{d+1} (-1)^{j_1-1} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}} \leq |S_{k,a}| \leq \sum_{j_1=1}^{d} (-1)^{j_1-1} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}},
\]
with
\[
A_{X_{j_1}} := A_{X_{j_1}}^{ja} = \left| \bigcup_{x^1 \in \mathbb{F}_q^*} \cdots \bigcup_{x^k \in \mathbb{F}_q^*} \{ F_s \in \mathcal{F}_d^*: F_s(a_j, X_{j_1}) = F_s(a_j, x^j) = 0 \text{ for } 2 \leq j \leq k \} \right|.
\]
We may rewrite the previous inequalities in the following form:
\[
(4.7) \quad \sum_{j_1=1}^{d+1} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}} - \sum_{j_1=1}^{d+1} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}} \leq |S_{k,a}| \leq \sum_{j_1=1}^{d} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}} - \sum_{j_1=1}^{d} \sum_{X_{j_1} \subseteq \mathbb{F}_q^*} A_{X_{j_1}}.
\]
Next, we apply the Bonferroni inequalities to each $A_{X_{j_1}}$. More precisely, we have
\[
(4.8) \quad A_{X_{j_1}} \leq \sum_{j_2=1}^{d+1} \sum_{X_{j_2} \subseteq \mathbb{F}_q^*} A_{X_{j_1}, X_{j_2}} - \sum_{j_2=1}^{d+1} \sum_{X_{j_2} \subseteq \mathbb{F}_q^*} A_{X_{j_1}, X_{j_2}} \leq \sum_{j_2=1}^{d} \sum_{X_{j_2} \subseteq \mathbb{F}_q^*} A_{X_{j_1}, X_{j_2}},
\]
where
\[
A_{X_{j_1}, X_{j_2}} := \left| \bigcup_{x^1 \in \mathbb{F}_q^*} \cdots \bigcup_{x^k \in \mathbb{F}_q^*} \{ F_s \in \mathcal{F}_d^*: F_s(a_1, X_{j_1}) = F_s(a_2, X_{j_2}) = F_s(a_j, x^j) = 0 \text{ for } 3 \leq j \leq k \} \right|.
\]
Proceeding in this way, we obtain upper and lower bounds for $|S_{k,a}|$ in terms of signed double sums of terms of the form
\[
A_{X_{j_1}, \ldots, X_{j_k}} := |\{ F_s \in \mathcal{F}_d^*: F_s(a_1, X_{j_1}) = \cdots = F_s(a_k, X_{j_k}) = 0 \}|.
\]
According to Proposition 4.4, $A_{X_{j_1}, \ldots, X_{j_k}} = q^{s \dim \mathcal{F}_d - s(j_1 + \cdots + j_k)}$, depending only of the cardinality of the sets $X_{j_1}, \ldots, X_{j_k}$ under consideration. Therefore, rearranging sums we may express the upper and lower bounds for $|S_{k,a}|$ as $2^k$ signed sums over $j_1, \ldots, j_k$ of terms of the form
\[
\sum_{X_{j_1} \subseteq \mathbb{F}_q^*} \cdots \sum_{X_{j_k} \subseteq \mathbb{F}_q^*} A_{X_{j_1}, \ldots, X_{j_k}} = \left( q^{s} \right)^{j_1} \cdots \left( q^{s} \right)^{j_k} q^{s \dim \mathcal{F}_d - s(j_1 + \cdots + j_k)}.
\]
Denote by $U_k$ and $L_k$ the expressions of the upper and lower bounds for $\left| S_{k,a} \right|_{\mathcal{F}_d^*}$ obtained in this way. According to (4.7), for $k = 1$ one has
\[
L_1 \leq \frac{|S_{1,a_1}|_{\mathcal{F}_d^*}}{|\mathcal{F}_d^*|} \leq U_1,
\]
where

\[U_1 := \sum_{j_1=1, j_1 \text{ odd}}^{d} \left( \frac{q^s}{j_1} \right) q^{-sj_1} - \sum_{j_1=1, j_1 \text{ even}}^{d} \left( \frac{q^s}{j_1} \right) q^{-sj_1} = \sum_{j_1=1}^{d} (-1)^{j_1-1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} =: s_d,\]

\[L_1 := \sum_{j_1=1, j_1 \text{ odd}}^{d+1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} - \sum_{j_1=1, j_1 \text{ even}}^{d+1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} = \sum_{j_1=1}^{d+1} (-1)^{j_1-1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} =: s_{d+1}.\]

For \(m \in \mathbb{N}\), denote

\[s_{m,\text{even}} := \sum_{i=1, i \text{ even}}^{m} \left( \frac{q^s}{i} \right) q^{-si}, \quad s_{m,\text{odd}} := \sum_{i=1, i \text{ odd}}^{m} \left( \frac{q^s}{i} \right) q^{-si}.\]

We have the following result.

**Lemma 4.6.** For \(2 \leq k \leq r - s + 1\) and \(d\) odd, we have

\[U_k = U_{k-1} \sum_{i=1, i \text{ odd}}^{d} \left( \frac{q^s}{i} \right) q^{-si} - L_{k-1} \sum_{i=1, i \text{ even}}^{d} \left( \frac{q^s}{i} \right) q^{-si} = U_{k-1}s_{d,\text{odd}} - L_{k-1}s_{d,\text{even}},\]

\[L_k = L_{k-1} \sum_{i=1, i \text{ odd}}^{d+1} \left( \frac{q^s}{i} \right) q^{-si} - U_{k-1} \sum_{i=1, i \text{ even}}^{d+1} \left( \frac{q^s}{i} \right) q^{-si} = L_{k-1}s_{d+1,\text{odd}} - U_{k-1}s_{d+1,\text{even}}.\]

On the other hand, for \(d\) even,

\[U_k = U_{k-1} \sum_{i=1, i \text{ odd}}^{d+1} \left( \frac{q^s}{i} \right) q^{-si} - L_{k-1} \sum_{i=1, i \text{ even}}^{d+1} \left( \frac{q^s}{i} \right) q^{-si} = U_{k-1}s_{d+1,\text{odd}} - L_{k-1}s_{d+1,\text{even}},\]

\[L_k = L_{k-1} \sum_{i=1, i \text{ odd}}^{d} \left( \frac{q^s}{i} \right) q^{-si} - U_{k-1} \sum_{i=1, i \text{ even}}^{d} \left( \frac{q^s}{i} \right) q^{-si} = L_{k-1}s_{d,\text{odd}} - U_{k-1}s_{d,\text{even}}.\]

**Proof.** Assume that \(d\) is odd. Arguing by induction on \(k\), for \(k = 2\) we have \(A_{\mathcal{X}_{j_1}, \mathcal{X}_{j_2}} = q^s \dim \mathcal{F}_{d-s(j_1+j_2)} = |\mathcal{F}_{d}|q^{-s(j_1+j_2)}\). Therefore, by (1.8) we have

\[q^{-sj_1} \left( \sum_{j_2=1, j_2 \text{ odd}}^{d+1} \left( \frac{q^s}{j_2} \right) q^{-sj_2} - \sum_{j_2=1, j_2 \text{ even}}^{d+1} \left( \frac{q^s}{j_2} \right) q^{-sj_2} \right) \leq \frac{A_{\mathcal{X}_{j_1}}}{|\mathcal{F}_{d}|} \leq q^{-sj_1} \left( \sum_{j_2=1, j_2 \text{ odd}}^{d} \left( \frac{q^s}{j_2} \right) q^{-sj_2} - \sum_{j_2=1, j_2 \text{ even}}^{d} \left( \frac{q^s}{j_2} \right) q^{-sj_2} \right).\]

As a consequence, by the definition of \(L_1\) and \(U_1\) it follows that

\[q^{-sj_1} L_1 \leq \frac{A_{\mathcal{X}_{j_1}}}{|\mathcal{F}_{d}|} \leq q^{-sj_1} U_1.\]

Using these bounds in (1.7) we obtain

\[\sum_{j_1=1, j_1 \text{ odd}}^{d+1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} L_1 - \sum_{j_1=1, j_1 \text{ even}}^{d+1} \left( \frac{q^s}{j_1} \right) q^{-sj_1} U_1 \leq \frac{|S_{k,\mathcal{A}}|}{|\mathcal{F}_{d}|} \leq \sum_{j_1=1, j_1 \text{ odd}}^{d} \left( \frac{q^s}{j_1} \right) q^{-sj_1} U_1 - \sum_{j_1=1, j_1 \text{ even}}^{d} \left( \frac{q^s}{j_1} \right) q^{-sj_1} L_1.\]

This proves the assertion for \(k = 2\).
Similarly, recall the notations Proposition 4.7. This proves the assertion on $U_k$. The assertion on $L_k$ is showed by a similar argument. Finally, for $d$ even the bounds are obtained arguing as above mutatis mutandis. □

For $d$ odd, we may express the recursive relation between the $U_k, L_k$ in the following form:

$$
\begin{pmatrix}
U_k \\
L_k
\end{pmatrix} = A_d \begin{pmatrix}
U_{k-1} \\
L_{k-1}
\end{pmatrix}, \quad A_d := \begin{pmatrix}
s_{d,\text{odd}} & -s_{d,\text{even}} \\
-s_{d+1,\text{even}} & s_{d+1,\text{odd}}
\end{pmatrix} = \begin{pmatrix}
s_{d,\text{odd}} & -s_{d,\text{even}} \\
-s_{d+1,\text{even}} & s_{d,\text{odd}}
\end{pmatrix}.
$$

We observe that

$$
C_d := \begin{pmatrix}
s_{d+1,\text{odd}} & -s_{d+1,\text{even}} \\
-s_{d+1,\text{even}} & s_{d+1,\text{odd}}
\end{pmatrix} \leq A_d \leq \begin{pmatrix}
s_{d,\text{odd}} & -s_{d,\text{even}} \\
-s_{d,\text{even}} & s_{d,\text{odd}}
\end{pmatrix} =: B_d,
$$

where the signs $\leq$ are understood componentwise. It follows that

$$
C_d^{k-1} \begin{pmatrix}
s_d \\
s_{d+1}
\end{pmatrix} \leq \begin{pmatrix}
U_k \\
L_k
\end{pmatrix} = A_d^{k-1} \begin{pmatrix}
U_1 \\
L_1
\end{pmatrix} = A_d^{k-1} \begin{pmatrix}
s_d \\
s_{d+1}
\end{pmatrix} \leq B_d^{k-1} \begin{pmatrix}
s_d \\
s_{d+1}
\end{pmatrix}.
$$

$B_d$ is a symmetric positive-definite matrix. By considering its spectral decomposition, one readily deduces the following expression for its $(k-1)$th power, which is expressed in terms of its eigenvalues $s_d^\pm := s_{d,\text{odd}} + s_{d,\text{even}}$ and $s_d = s_{d,\text{odd}} - s_{d,\text{even}}$:

$$
B_d^{k-1} = \begin{pmatrix}
\frac{1}{2} (s_d^+)^{k-1} + \frac{1}{2} s_d^{-1} & \frac{1}{2} (s_d^+)^{k-1} - \frac{1}{2} s_d^{-1} \\
\frac{1}{2} (s_d^{-1})^{k-1} + \frac{1}{2} s_d^+ & \frac{1}{2} (s_d^{-1})^{k-1} - \frac{1}{2} s_d^+
\end{pmatrix}.
$$

We conclude that

$$
U_k \leq \left( \frac{1}{2} (s_d^+)^{k-1} + \frac{1}{2} s_d^{-1} \right) s_d + \left( -\frac{1}{2} (s_d^+)^{k-1} + \frac{1}{2} s_d^{-1} \right) s_{d+1}
$$

$$
= \frac{1}{2} (s_d^+)^{k-1} (s_d - s_{d+1}) + \frac{1}{2} s_d^{-1} (s_d + s_{d+1})
$$

$$
= s_d^k + \frac{1}{2} (q_d^+)^{s(d+1)} (s_d^+)^{k-1} - s_d^{-1}.
$$

Similarly,

$$
L_k \geq \frac{1}{2} (s_d^+)^{k-1} (s_d+1 - s_d) + \frac{1}{2} s_d^{-1} (s_d + s_{d+1})
$$

$$
= s_d^{k+1} - \frac{1}{2} (q_d^+)^{s(d+1)} (s_d^+)^{k-1} - s_d^{-1}.
$$

Next we obtain simpler upper and lower bounds for $|S_k(a_1,\ldots,a_k)|$.

**Proposition 4.7.** Recall the notations $s_d^+ := s_{d,\text{odd}} + s_{d,\text{even}}$, $s_d := s_{d,\text{odd}} - s_{d,\text{even}}$ and $t_{d+1} := (q_d^+)^{s(d+1)}$. For $2 \leq k \leq r - s + 1$, we have

$$
\left| \frac{S_k(a_1,\ldots,a_k)}{F_d^k} \right| - s_d^k \leq \frac{t_{d+1}}{2} (s_d^+)^{k-1} + (2k-1) s_d^{-1} \quad \text{for } d \text{ odd},
$$

$$
\left| \frac{S_k(a_1,\ldots,a_k)}{F_d^k} \right| - s_d^k \leq \frac{t_{d+1}}{2} (s_d^+)^{k-1} + (2k-1) s_d^{-1} \quad \text{for } d \text{ even}.
$$
Proof. Suppose that \( d \) is odd. By the lower bound for \( L_k \) above, we have
\[
\frac{|S_{k,(a_1,\ldots,a_k)}|}{|F_d^k|} \geq s_d^k + s_{d+1}^k - s_d^k - \frac{t_{d+1}}{2}((s_{d+1}^+)^{k-1} - s_{d+1}^{k-1})
\]
\[
\geq s_d^k - t_{d+1}(s_{d+1}^+ + \cdots + s_{d}^{k-1}) - \frac{t_{d+1}}{2}((s_{d+1}^+)^{k-1} - s_{d+1}^{k-1})
\]
\[
= s_d^k - \frac{t_{d+1}}{2}((s_{d+1}^+)^{k-1} + s_{d+1}^{k-1} + 2s_{d+1}^{k-2}s_d + \cdots + 2s_d^{k-1})
\]
\[
\geq s_d^k - \frac{t_{d+1}}{2}((s_{d+1}^+)^{k-1} + (2k - 1)s_d^{k-1}).
\]

On the other hand,
\[
\frac{|S_{k,(a_1,\ldots,a_k)}|}{|F_d^k|} \leq |S_{1,a_k}| + \sum_{k=1}^{h-1} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k < h} |S_{k+1,(a_1,\ldots,a_k,a_h)}|.
\]

We have the following result.

4.2.3. An estimate for \( |S_{h,a}^*| \). Now we are able to estimate the probability that \( h \) random choices are made until we reach a vertical strip with \( E_q \)-rational solutions of the system under consideration.

Recall that, given \( a := (a_1,\ldots,a_h) \) which satisfies the hypothesis of Proposition 4.4 we aim to estimate the probability of the set
\[
(4.9) \quad S_{h,a}^* := \{ F_s \in F_d^k : Z(F_s(a_i,-)) = 0 \ (1 \leq i \leq h-1), \ Z(F_s(a_h,-)) > 0 \}.
\]

For this purpose, according to (4.4), we have
\[
|S_{h,a}^*| = |S_{1,a_h}| + \sum_{k=1}^{h-1} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k < h} |S_{k+1,(a_1,\ldots,a_k,a_h)}|.
\]

We have the following result.

Theorem 4.8. Let \( s < d \) and denote
\[
s_d := \sum_{i=1}^{d} (-1)^{i-1} (q^+)_i q^{-s_i}, \quad s_{d+1}^+ := \sum_{i=1}^{d+1} (q^+)_i q^{-s_i} \text{ and } \ t_{d+1}^+ := (q^+_{d+1}) q^{-s(d+1)}.
\]

For \( 2 \leq h \leq r - s \), we have
\[
|S_{h,a}^*| - s_d(1 - s_d)^{h-1} \leq t_{d+1}^+ ((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) \quad \text{for } d \text{ odd},
\]
\[
|S_{h,a}^*| - s_d(1 - s_d)^{h-1} \leq t_{d+1}^+ ((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) \quad \text{for } d \text{ even}.
\]

Proof. Assume that \( d \) is odd. Observe that
\[
s_d(1 - s_d)^{h-1} = s_d + \sum_{k=1}^{h-1} \binom{h-1}{k} s_d^{k+1} = s_d + \sum_{k=1}^{h-1} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k < h} s_d^{k+1}.
\]

According to Theorem 4.2 and Proposition 4.7, we have
\[
|S_{k+1,(a_1,\ldots,a_k,a_h)}| - s_d^{k+1} \leq \frac{t_{d+1}^+}{2} ((s_{d+1}^+)^{k} + (2k + 1)s_d^{k})
\]

for any \( 1 \leq i_1 < \cdots < i_k < h \). It follows that
\[
|S_{h,a}^*| - s_d(1 - s_d)^{h-1} \leq \frac{t_{d+1}^+}{2} \sum_{k=0}^{h-1} \binom{h-1}{k} ((s_{d+1}^+)^{k} + (2k + 1)s_d^{k}).
\]
Now we bound the sums in the right-hand side of the previous inequality. First, we have
\[
\sum_{k=0}^{h-1} \binom{h-1}{k} s_{d+1}^k = (1 + s_{d+1}^+)^{h-1}.
\]
On the other hand,
\[
\sum_{k=0}^{h-1} \binom{h-1}{k} (2k + 1) s_d^k = 2 s_d \sum_{k=1}^{h-1} \binom{h-1}{k} k s_d^{k-1} + \sum_{k=0}^{h-1} \binom{h-1}{k} s_d^k
\]
\[
= 2 s_d (h - 1)(1 + s_d)^{h-2} + (1 + s_d)^{h-1}
\]
\[
= ((2h - 1)s_d + 1)(1 + s_d)^{h-2}.
\]
We conclude that
\[
\left| \frac{|S_{h,a}^s|}{|F_d^s|} - s_d(1 - s_d)^{h-1} \right| \leq \frac{t_{d+1}}{2} \frac{(1 + s_{d+1}^+)^{h-1}}{1 + s_d}(2h - 1)(1 + s_d)^{h-2} + (1 + s_d)^{h-1}.
\]
We claim that
\[
-1 + ((2h - 1)s_d + 1)(1 + s_d)^{h-2} \leq (1 + s_{d+1}^+)^{h-1}.
\]
Indeed, for \(d = 3\) and for \(h = 2, 3, 4, 5\) this can be checked by a direct computation. For \(d \geq 5\) and \(h \geq 4\), the claim is a consequence of the inequality \((2h - 1)(1 + s_d)^{h-1} \leq (1 + s_{d+1}^+)^{h-1}\), which is equivalent to
\[
2h - 1 \leq \frac{(1 + s_{d+1}^+)^{h-1}}{1 + s_d^+} = \left(1 + \frac{s_{d+1}^+ - s_d^+}{1 + s_d^+}\right)^{h-1}.
\]
Now, as \(d \geq 5\), we have
\[
1 + \frac{s_{d+1}^+ - s_d^+}{1 + s_d^+} \geq 1 + \frac{2s_d^{even} + t_{d+1}}{1 + 1/e} \geq 1 + \frac{2(t_2 + t_4) + t_6}{1 + 1/e} \geq 1 + \frac{2(1 + 1/e) + 1}{1 + 1/e} = 1 + \frac{43}{1 + 1/e},
\]
and the inequality \((1 + \frac{43}{1 + 1/e})^{h-1} \geq 2h - 1\) holds for \(h \geq 6\), which proves the claim.

By the claim we deduce that
\[
\left| \frac{|S_{h,a}^s|}{|F_d^s|} - s_d(1 - s_d)^{h-1} \right| \leq t_{d+1} \left(1 + s_{d+1}^+\right)^{h-1} - \frac{t_{d+1}}{2},
\]
which implies the bound in the theorem for \(d\) odd. The bound for \(d\) even follows by a similar argument.

\[4.2.4. \text{Probability that } h \text{ specializations are required.} \]

Now we are able to complete the analysis of the probability that \(h\) specializations are required. For this purpose, we fix \(h\) with \(2 \leq h \leq r - s + 1\), denote
\[
F_h := \{(a_1, \ldots, a_h) \in \mathbb{F}_q^{r-s} \times \cdots \times \mathbb{F}_q^{r-s} : a_i \neq a_j \text{ for } i \neq j\}, \quad N_h := |F_h|,
\]
and introduce the random variable \(C_h := F_h \times F_d^s \rightarrow \{1, \ldots, h, \infty\}\) defined as follows:
\[
(4.10) \quad C_h((a_1, \ldots, a_h), F_s) := \begin{cases} \min \{j : Z(F_s(a_j, -)) > 0\} & \text{if } \exists j \text{ with } Z(F_s(a_j, -)) > 0; \\ \infty, & \text{otherwise.} \end{cases}
\]
We consider the set \(F_h \times F_d^s\) endowed with the uniform probability \(P_h := P_{h,r,s,d}\) and study the probability of the set \(\{C_h = h\}\), that is, we aim to estimate the probability \(P_h(S_h^s)\), where
\[
S_h^s := \{(a_1, \ldots, a_h) : (a_1, \ldots, a_h, F_s) \in F_h \times F_d^s : Z(F_s(a_i, -)) = 0 \ (1 \leq i < h), \ Z(F_s(a_h, -)) > 0\}.
\]
We have the following result.
Theorem 4.9. With assumptions as in Theorem 4.8, suppose further that \( q^s > d \) and 
\( 1 < h \leq r - s + 1 \). We have
\[
|P_h[C_h = h] - s_d(1 - s_d)^{h-1}| \leq t_d + 1((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) + \frac{2}{q} \quad \text{for } d \text{ odd,}
\]
\[
|P_h[C_h = h] - s_d(1 - s_d)^{h-1}| \leq t_d + 1((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) + \frac{2}{q} \quad \text{for } d \text{ even.}
\]

Proof. Assume that \( d \) is odd. Denote by \( \mathcal{G}_d \subset F_h \) the set of \( \alpha \) satisfying the hypothesis of Proposition 4.4. We have
\[
P_h[C_h = h] = \frac{|S_h^*|}{N_h|F_d^*|} = \frac{1}{N_h|F_d^*|} \left( \sum_{\alpha \in F_h \setminus \mathcal{G}_d} |S_h^*| + \sum_{\alpha \in \mathcal{G}_d} |S_h^*| \right),
\]
where \( S_h^* \) is defined as in (4.9).

From Remark 4.5 we see that
\[
\frac{|N_h - |\mathcal{G}_d|/N_h|}{N_h} = 1 - \frac{q^{h(r-s)}}{N_h} \left( 1 - \frac{1}{q} \right) \cdots \left( 1 - \frac{1}{q^{h-1}} \right) \leq \frac{1}{q} + \cdots + \frac{1}{q^{h-1}} \leq \frac{2}{q},
\]
\[
\frac{|\mathcal{G}_d|/N_h}{N_h} = \frac{q^{h(r-s)}}{N_h} \left( 1 - \frac{1}{q} \right) \cdots \left( 1 - \frac{1}{q^{h-1}} \right) \geq 1 - \frac{1}{q} - \cdots - \frac{1}{q^{h-1}} \geq 1 - \frac{2}{q}.
\]

It follows that
\[
\frac{1}{N_h|F_d^*|} \sum_{\alpha \in F_h \setminus \mathcal{G}_d} |S_h^*| \leq \frac{1}{N_h} \sum_{\alpha \in F_h \setminus \mathcal{G}_d} |S_h^*| \leq \frac{N_h - |G_d|}{N_h} \leq \frac{2}{q}.
\]

On the other hand, by Theorem 4.8 we obtain
\[
\frac{1}{N_h} \sum_{\alpha \in \mathcal{G}_d} \frac{|S_h^*|}{|F_d^*|} \geq s_d(1 - s_d)^{h-1} + t_d + 1((1 + s_{d+1}^+)^{h-1} + \frac{1}{2})
\]
\[
\frac{1}{N_h} \sum_{\alpha \in \mathcal{G}_d} \frac{|S_h^*|}{|F_d^*|} \geq \left( s_d(1 - s_d)^{h-1} - t_d + 1((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) \right)(1 - \frac{2}{q}).
\]

Putting these estimates together, and taking into account that \( s_d(1 - s_d)^{h-1} \leq \frac{1}{q} \) for \( h > 1 \), we readily deduce the statement of the theorem. The case \( d \) even follows similarly. \( \Box \)

Finally, we express the estimates of Theorem 4.9 asymptotically for large \( q \).

Corollary 4.10. With notations and hypotheses as in Theorem 4.9, assume further that \( q^s > 6 \). We have
\[
|P_h[C_h = h] - \mu_d(1 - \mu_d)^{h-1}| \leq \frac{1}{(d+1)!} (e^{h-1} + \frac{1}{2}) + \frac{2}{q} + \frac{5}{q^r} (2 - \mu_d)^{h-1} \quad \text{for } d \text{ odd,}
\]
\[
|P_h[C_h = h] - \mu_d + 1(1 - \mu_d)^{h-1}| \leq \frac{1}{(d+1)!} (e^{h-1} + \frac{1}{2}) + \frac{2}{q} + \frac{5}{q^r} (2 - \mu_d)^{h-1} \quad \text{for } d \text{ even.}
\]

Proof. Since \( (q^s)^{-ji} \leq \frac{1}{j} \) for every \( j \geq 0 \), we have \( s_{d+1}^+ \leq e - 1 \) and \( t_d + 1 \leq \frac{1}{(d+1)!} \). It follows that
\[
(4.11) \quad t_d + 1((1 + s_{d+1}^+)^{h-1} + \frac{1}{2}) + \frac{2}{q} \leq \frac{1}{(d+1)!} (e^{h-1} + \frac{1}{2}) + \frac{2}{q}.
\]

Now assume that \( d \) is odd. According to (4.3),
\[
|s_d - \mu_d| \leq \frac{1}{4q^s} + \frac{d}{q^2s} < 2^{-2s}.
\]

As a consequence,
\[
s_d(1 - s_d)^{h-1} < (\mu_d + \frac{2}{q})(1 - \mu_d + \frac{2}{q})^{h-1}
\]
\[
= (\mu_d + \frac{2}{q^r}) \left( (1 - \mu_d)^{h-1} + \sum_{j=0}^{h-2} (h-1)(1 - \mu_d)^j (\frac{2}{q^r})^{h-1-j} \right).
\]
As \( \frac{2}{q} < 1 \), we have
\[
\sum_{j=0}^{h-2} (\mu_d)^j (\frac{2}{q})^{h-1-j} \leq \frac{2}{q} \sum_{j=0}^{h-2} (\mu_d)^j (1-\mu_d)^j \leq \frac{2}{q} (2-\mu_d)^{h-1}.
\]

Thus,
\[
s_d(1-s_d)^{h-1} \leq (\mu_d + \frac{2}{q})(1-\mu_d)^{h-1} + \frac{2}{q} (2-\mu_d)^{h-1} \leq \mu_d(1-\mu_d)^{h-1} + \frac{5(2-\mu_d)^{h-1}}{q^2}.
\]

Combining (1.11) with this inequality and Theorem 4.9 we obtain
\[
P_h[C_h = h] \leq \mu_d(1-\mu_d)^{h-1} + \frac{5(2-\mu_d)^{h-1}}{q^2} + \frac{1}{(d+1)!} (e^{h-1} + \frac{1}{2}) + \frac{2}{q}.
\]

To prove the lower bound for \( P_h[C_h = h] \) we observe that, as \( d > 1 \) is odd, then \( d \geq 3 \). Since \( \mu_d = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \), it is easy to see that \( \frac{1}{2} < \mu_d < \frac{3}{4} \), and the hypothesis \( q^s > 6 \) implies \( \mu_d - \frac{2}{q^s} > 0 \) and \( 1 - \mu_d - \frac{2}{q^s} > 0 \). We conclude that
\[
s_d(1-s_d)^{h-1} \geq \left( \mu_d - \frac{2}{q^s} \right) \left( 1 - \mu_d - \frac{2}{q^s} \right)^{h-1}.
\]

Arguing as above we show that
\[
\left( \mu_d - \frac{2}{q^s} \right) \left( 1 - \mu_d - \frac{2}{q^s} \right)^{h-1} \geq \mu_d(1-\mu_d)^{h-1} - \frac{4}{q^2} (2-\mu_d)^{h-1},
\]
which proves the lower bound. A similar analysis works for \( d \) even.

4.2.5. Probability that more specializations are required. Theorems 4.2 and 4.9 provide estimates on the probability that \( h \) specializations are required for \( 1 \leq h \leq r-s+1 \). These estimates show that such a probability decreases exponentially with \( h \). For more specializations this exponential behavior might be altered, as further conditions imposed on the elements of \( F^*_d \) might not be linearly independent. For this reason, in this section we estimate the probability that more than \( r-s+1 \) specializations are required.

Similarly to previously, for any \( h \geq 3 \) we denote
\[
\mathcal{F}_h := \{(a_1, \ldots, a_h) \in \mathbb{F}_q^{r-s} \times \cdots \times \mathbb{F}_q^{r-s} : a_i \neq a_j \text{ for } i \neq j\}, \quad N_h := |\mathcal{F}_h|,
\]
and consider the random variable \( C_h := C_{h,r,d} : \mathcal{F}_h \times F^*_d \to \{1, \ldots, h, \infty\} \) defined for \( a := (a_1, \ldots, a_h) \in \mathcal{F}_h \) and \( F_s \in F^*_d \) in the following way:
\[
C_{h}(a, F_s) := \begin{cases} 
\min\{j : Z(F_s(a_j, -)) > 0\} \text{ if } \exists j \text{ with } Z(F_s(a_j, -)) > 0, \\
\infty \text{ otherwise.}
\end{cases}
\]

We consider the set \( \mathcal{F}_h \times F^*_d \) as before endowed with the uniform probability \( P_h := P_{h,r,d} \) and analyze the probability \( P_h[C_h = h] \). Up to now, all the probability estimates concerned the probability spaces \( \mathcal{F}_h \times F^*_d \) separately. To link the probabilities \( P_h \) for \( 1 \leq h \leq q^{r-s} \), we have the following result.

Lemma 4.11. Let \( h > 1 \) and let \( \pi_h : \mathcal{F}_h \times F^*_d \to \mathcal{F}_{h-1} \times F^*_d \) be the mapping induced by the projection \( \pi_h : \mathcal{F}_h \to \mathcal{F}_{h-1} \) on the first \( h-1 \) coordinates. If \( S \subset \mathcal{F}_{h-1} \times \mathcal{F}^*_d \), then \( P_h[\pi_h^{-1}(S)] = P_{h-1}[S] \).

Proof. Note that
\[
\pi_h^{-1}(S) = \bigcup_{F_s \in F^*_d} \left\{ (a_1, \ldots, a_h) \in F_h : (a_1, \ldots, a_{h-1}, F_s) \in S \right\} \times \{ F_s \}
\]
\[
= \bigcup_{F_s \in F^*_d} \bigcup_{(a_1, \ldots, a_{h-1}) \in \mathcal{F}_{h-1} : (a_1, \ldots, a_{h-1}, F_s) \in S} \{ (a_1, \ldots, a_{h-1}) \} \times (\mathbb{F}_q^{r-s} \setminus \{ a_1, \ldots, a_{h-1} \}) \times \{ F_s \}.
\]
It follows that
\[
P_h[\pi_h^{-1}(S)] = \frac{1}{N_{h}[F_d]} \sum_{F_s \in F_d} \sum_{\mathcal{a} \in F_{h-1}(\mathcal{a}, F_s) \in S} (q^{r-s} - h + 1)
= \frac{1}{N_{h-1}[F_d]} \sum_{F_s \in F_d} \left| \{\mathcal{a} \in F_{h-1} : (\mathcal{a}, F_s) \in S\} \right| = P_{h-1}[S].
\]
This proves the lemma. \(\square\)

According to the Kolmogorov extension theorem (see, e.g., [Fel71, Chapter IV, Section 5, Extension Theorem]), the conditions of “consistency” of Lemma 4.11 imply that the probabilities \(P_h(1 \leq h \leq q^{r-s})\) can be put in a unified framework. More precisely, we define \(\mathcal{F} := \mathcal{F}_{q^{r-s}}\) and \(P := P_{q^{r-s}}\). Then the probability measure \(P\) defined on \(\mathcal{F}\) allows us to interpret consistently all the results of this paper. In the same vein, the variables \(C_h(1 \leq h \leq q^{r-s})\) can be naturally extended to a random variable \(C : \mathcal{F} \times F_d \rightarrow \mathbb{N} \cup \{\infty\}\). Consequently, in what follows we shall drop the subscript \(h\) from the notations \(P_h\) and \(C_h\).

Now we are able to state and prove our result on the probability that more than \(r-s+1\) random choices are required.

**Corollary 4.12.** With assumptions as in Theorem 4.8, suppose further that \(q^s > d\) and denote \(h^* := r-s + 1\). We have
\[
|P[C > h^*] - (1 - \mu_d)^{h^*}| \leq \frac{e^{h^*}}{(d+1)!} + \frac{2h^*}{q} + \frac{15(2-\mu_d)^{h^*}}{q^2} \quad \text{for } d \text{ odd},
\]
and
\[
|P[C > h^*] - (1 - \mu_{d+1})^{h^*}| \leq \frac{e^{h^*}}{(d+1)!} + \frac{2h^*}{q} + \frac{15(2-\mu_d)^{h^*}}{q^2} \quad \text{for } d \text{ even}.
\]

**Proof.** Suppose that \(d\) is odd. Observe that \(P[C > h^*] = 1 - \sum_{h=1}^{h^*} P[C = h]\) and
\[
(1 - \mu_d)^{h^*} = 1 - \sum_{h=1}^{h^*} \mu_d(1 - \mu_d)^{h-1}.
\]
According to Corollaries 4.3 and 4.10
\[
|P[C > h^*] - (1 - \mu_d)^{h^*}| \leq \sum_{h=1}^{h^*} |P[C = h] - \mu_d(1 - \mu_d)^{h-1}|
\leq \frac{1}{(d+1)!} \left( \frac{h^*-1}{2} + \sum_{h=1}^{h^*} e^{h-1} \right) + \frac{2(h^*-1)}{q} + \frac{5}{q} \sum_{h=1}^{h^*} (2 - \mu_d)^{h-1}
\leq \frac{e^{h^*}}{(d+1)!} + \frac{2h^*}{q} + \frac{15(2-\mu_d)^{h^*}}{q^2}.
\]
This proves the corollary. \(\square\)

5. **Average-case complexity and probability of success**

Now we are finally able to provide estimates on the average-case complexity and the probability of success of Algorithm 1.1. For \(1 < s < r\) and \(d \geq 2\), we recall that this algorithms takes as input an element of \(F_d\), namely an \(s\)-tuple \(F_s := (F_1, \ldots, F_s)\) of elements of \(F_d := \{F \in \mathbb{F}_q[X_1, \ldots, X_r] : \deg F \leq d\}\), and outputs an \(\mathbb{F}_q\)-rational solution \(x \in \mathbb{F}_q^r\) of the system \(F_s = 0\), or “Failure”.

For this purpose, Algorithm 1.1 successively generates a sequence \(\mathcal{a} := (a_1, a_2, \ldots, a_r) \in F_{h^*}\), where \(h^* := r-s+1\), and searches for \(\mathbb{F}_q\)-rational zeros of \(F_s\) in the vertical strips \(\{a_i\} \times \mathbb{F}_q^s\) for \(1 \leq i \leq h^*\), until a zero of \(F_s(a_{i, -})\) is found or all the vertical strips are exhausted. As discussed in Section 1, given \(\mathcal{a} \in F_{h^*}\), the whole procedure requires at most \(C_\mathcal{a}(F_s) \cdot \tau(d, s, q)\) arithmetic operations in \(\mathbb{F}_q\), where \(\tau(d, s, q)\) is the maximum number of arithmetic operations in \(\mathbb{F}_q\) necessary to perform a search in an arbitrary vertical strip,
provided that \( i^* := C(a, F_s) \leq h^* \), \( F_s(a_{r^*}, -) \) satisfies hypothesis (H) and the choice of parameters \( \omega \) for the probabilistic algorithm performing the zero-dimensional search on the specialized system \( F_s(a_{r^*}, -) = 0 \) is accurate. The analysis of the probability that all these hold is essential to determine the average-case complexity and probability of success of Algorithm 1.1 as we shall see in the next sections.

5.1. Average-case complexity. To perform a search in a vertical strip \( \{a\} \times \mathbb{F}_q^s \) we shall use Gröbner-basis technology or a Kronecker-like algorithm. In the first case, we shall use a deterministic algorithm, combined with a probabilistic routine for computing \( \mathbb{F}_q \)-rational roots of univariate polynomials. In the second case, we shall use a probabilistic algorithm. Therefore, in both cases we rely on \( r_d \) random choices of elements of \( \mathbb{F}_q^s \) for certain \( r_d \in \mathbb{N} \). We denote by \( \Omega_d := \mathbb{F}_q^{r_d} \) the set of all such random choices and consider \( \Omega_d \) endowed with the uniform probability, \( F_{h^*} \times \mathbb{F}_q^s \) with the probability \( P \) of Section 4.2.5 and \( F_{h^*} \times \mathbb{F}_q^s \times \Omega_d \) with the product probability. Therefore, we may represent the cost of Algorithm 1.1 either using Gröbner bases or Kronecker-like algorithms by the random variable

\[
X := X_{s, r, d} : F_{h^*} \times \mathbb{F}_q^s \times \Omega_d \to \mathbb{N}
\]

which counts the number \( X(a, F_s, \omega) \) of arithmetic operations performed on input \( F_s \in \mathbb{F}_q^s \), with the choice of vertical strips defined by \( a \) and the choice \( \omega \) for the parameters of the routine for zero-dimensional solving. We shall use versions of these routines having a probability of failure of order \( O(q^{-1}) \). This can be achieved by a modification of the algorithm for computing \( \mathbb{F}_q \)-rational roots of, e.g., [vzGG99, Algorithm 14.15], and the Kronecker-like algorithm in [CM06a], which consists of performing the random choices on a set \( \Omega_d \) suitably augmented, with an increase of the number of arithmetic operations in \( \mathbb{F}_q \) of the corresponding algorithms by a factor \( O(\log q) \).

We also remark that the search for \( \mathbb{F}_q \)-rational solutions in an arbitrary vertical strip \( \{a\} \times \mathbb{F}_q^s \) will be truncated when \( \tau(s, d, q) \) arithmetic operations in \( \mathbb{F}_q \) are performed, so that at most \( \tau(s, d, q) \) arithmetic operations in \( \mathbb{F}_q \) are performed during all the searches of Algorithm 1.1. For Gröbner-basis technology, this can be done by establishing bounds on the degrees of the polynomials arising in the underlying Macaulay matrix, while for Kronecker algorithms it can be done by controlling the degrees in the intermediate varieties arising during the execution of the algorithm.

Having made the algorithmic model precise, we proceed to bound from above the asymptotic behavior of the expected value \( E[X] \) of \( X \), namely

\[
E[X] := \frac{1}{|F_{h^*}| |\mathbb{F}_q^s| |\Omega_d|} \sum_{(a, F_s, \omega)} X(a, F_s, \omega).
\]

**Theorem 5.1.** Let \( h^* := r - s + 1 \). For \( q > 2d^s(d+1)^s \) and \( d > s \), the average-case complexity of Algorithm 1.1 is bounded in the following way:

\[
E[X] \leq \tau(d, s, q) \left( \mu_d^{-1} + h^*(1 - \mu_d)^{h^*} + \frac{3h^*e^{h^*}}{(d+1)!} + O\left( \frac{h^*d^s(d+1)^s}{q} + \frac{h^*(2 - \mu_d)^{h^*}}{q^2} \right) \right) \quad \text{for } d \text{ odd},
\]

\[
E[X] \leq \tau(d, s, q) \left( \mu_d^{-1} + h^*(1 - \mu_d+1)^{h^*} + \frac{3h^*e^{h^*}}{(d+1)!} + O\left( \frac{h^*d^s(d+1)^s}{q} + \frac{h^*(2 - \mu_d)^{h^*}}{q^2} \right) \right) \quad \text{for } d \text{ even},
\]

where \( \tau(d, s, q) \) is the cost of the search in a vertical strip and the constant underlying the \( O \)-notation is independent of \( r, s, d, q \).

**Proof.** We first note that to each \( (a, F_s) \in F_{h^*} \times \mathbb{F}_q^s \) and \( i^* := C_{h^*}(a, F_s) \leq h^* \), such that \( F_s(a_{r^*}, -) \) satisfies condition (H), there corresponds a subset \( \Omega_d(a, F_s) \) of “bad” choices of parameters \( \omega \) for the zero-dimensional solver under consideration, that is, choices for which the solver fails to find an \( \mathbb{F}_q \)-rational solution of the system \( F_s(a_{r^*}, -) = 0 \). According to our previous remarks, the probability of failure is \( |\Omega_d(a, F_s)|/|\Omega_d| = O(q^{-1}) \).
We decompose $F_h^* \times F_d^* \times \Omega_d$ as the disjoint union
\[
F_h^* \times F_d^* \times \Omega_d = C \cup D \cup E \cup F,
\]
where
\[
C := \{ (a, F_s, \omega) : h^* (a, F_s) = \infty \},
D := \{ (a, F_s, \omega) : i^* := h^* (a, F_s) \leq h^* \text{ and } F_s (a_i, -) \text{ does not satisfy } (H) \},
E := \{ (a, F_s, \omega) : i^* := h^* (a, F_s) \leq h^*, F_s (a_i, -) \text{ satisfies } (H) \text{ and } \omega \in \Omega_d (a, F_s) \},
F := \{ (a, F_s, \omega) : i^* := h^* (a, F_s) \leq h^*, F_s (a_i, -) \text{ satisfies } (H) \text{ and } \omega \notin \Omega_d (a, F_s) \}.
\]

For each $(a, F_s, \omega) \in F$, when the zero–dimensional solver under consideration is applied to the system $F_s (a_i, -) = 0$, it succeeds. Then Algorithm 4.1 stops at the $i^*$th vertical strip and outputs an $E_q$–rational solution of the system $F_s (a_i, -) = 0$, and we have
\[
X (a, F_s, \omega) \leq \tau (s, d, q) C_{h^*} (a, F_s).
\]

On the other hand, for any $(a, F_s, \omega) \in C \cup D \cup E$, Algorithm 4.1 does not stop searching for $E_q$–rational solutions at the $i^*$th vertical strip, and the search continues until an $E_q$–rational solution of a system $F_s (a_i, -) = 0$ with $i^* < i \leq h^*$ is obtained, or the remaining vertical strips are searched. In this case we have $X (a, F_s, \omega) \leq \tau (s, d, q) h^*$. We conclude that
\[
(5.1) \quad E[X] \leq \frac{\tau (s, d, q)}{|F_h^*| |F_d^*| |\Omega_d|} \left( \sum_{(a, F_s, \omega) \in F} C_{h^*} (a, F_s) + h^* (|C| + |D| + |E|) \right)
\]

Now we study the sum in the right–hand side of (5.1). We have
\[
\frac{\tau (s, d, q)}{|F_h^*| |F_d^*| |\Omega_d|} \sum_{(a, F_s, \omega) \in F} C_{h^*} (a, F_s) \leq \frac{\tau (s, d, q)}{|F_h^*| |F_d^*|} \sum_{(a, F_s) : C_{h^*} (a, F_s) \leq h^*} C_{h^*} (a, F_s)
\]
\[
= \frac{\tau (s, d, q)}{|F_d^*|} \sum_{F_s \in F_d^*} h^* \sum_{h = 1}^{h^*} \frac{\{ a \in F_h^* : C_{h^*} (a, F_s) = h \}}{|F_h^*|}.
\]

From the conditions of consistency of Lemma 4.11 it follows that
\[
\frac{1}{|F_h^*| |F_d^*|} \sum_{(a, F_s) : C_{h^*} (a, F_s) \leq h^*} C_{h^*} (a, F_s) = \frac{1}{|F_d^*|} \sum_{F_s \in F_d^*} \sum_{h = 1}^{h^*} \frac{\{ a \in F_h^* : C_{h^*} (a, F_s) = h \}}{|F_h^*|}
\]
\[
= \sum_{h = 1}^{h^*} \frac{h}{|F_d^*|} \sum_{F_s \in F_d^*} \frac{\{ a \in F_h : C_h (a, F_s) = h \}}{|F_h|}
\]
\[
= \sum_{h = 1}^{h^*} h P_h [C_h = h].
\]

Assuming that $d$ is odd, by Corollary 4.10 we have
\[
\frac{1}{|F_h^*| |F_d^*|} \sum_{(a, F_s) : C_{h^*} (a, F_s) \leq h^*} C_{h^*} (a, F_s)
\]
\[
\leq \sum_{h = 1}^{h^*} h \mu_d (1 - \mu_d)^{h-1} + \frac{2h^* e^{h^*}}{(d+1)!} + \frac{h^* (h^* + 1)}{q} + O \left( \frac{h^* (2 - \mu_d) h^*}{q^2} \right)
\]
\[
\leq \mu_d \sum_{h = 1}^{h^*} h (1 - \mu_d)^{h-1} + \frac{2h^* e^{h^*}}{(d+1)!} + \frac{2h^*}{q} + O \left( \frac{h^* (2 - \mu_d) h^*}{q^2} \right).
\]
Taking into account that $\sum_{n \geq 1} nz^{n-1} = 1/(1-z)^2$ for any $|z| < 1$, we conclude that

$$
\tau(s, d, q) \leq \frac{\tau(s, d, q)}{|F_d^*|} \sum_{(a, F_s) \in F} C_{h^*}(a, F_s) \leq \tau(s, d, q) \left( \frac{1}{\mu_d} + \frac{2h^*}{q} + \frac{h^*}{q^2} + \mathcal{O}\left( \frac{h^*(2-\mu_d)h^*}{q^2} \right) \right).
$$

Next we consider the remaining terms in the right–hand side of (5.1). We have

$$\frac{|C|}{|F_h^*||F_d^*|} = \left\{ \frac{C_{h^*}(a, F_s) = \infty}{|F_h^*||F_d^*|} \right\} = P_{h^*}[C_{h^*} = \infty] = P[C > h^*].$$

On the other hand,

$$\frac{|D|}{|F_h^*||F_d^*|} = \left\{ \frac{C_{h^*}(a, F_s) \leq h^*}{|F_h^*||F_d^*|} \right\} \leq \sum_{j=1}^{h^*} \left\{ \frac{C_{h^*}(a, F_s) = j}{|F_h^*||F_d^*|} \right\} \leq \sum_{j=1}^{h^*} \left\{ \frac{\pi_j^{-1}(\{C_H = \infty\})}{|F_h^*||F_d^*|} \right\},$$

where $\pi_j : F_h^* \times F_d^* \to \mathbb{F}^*_q \times F_d^*$ is the map $\pi_j(a, F_s) := (a_j, F_s)$. Since

$$|\pi_j^{-1}(\{C_H = \infty\})| = (q^{r-s} - 1) \cdots (q^{r-s} - h^* + 1)$$

for every $(a, F_s) \in \mathbb{F}^*_q \times F_d^*$, it follows that

$$|\pi_j^{-1}(\{C_H = \infty\})| = (q^{r-s} - 1) \cdots (q^{r-s} - h^* + 1)|\{C_H = \infty\}|.$$ 

As a consequence,

$$\frac{|D|}{|F_h^*||F_d^*|} \leq h^*(q^{r-s} - 1) \cdots (q^{r-s} - h^* + 1)|\{C_H = \infty\}| = h^* P_1[C_H = \infty].$$

Finally, to estimate $|E|$, note that for each $(a, F_s) \in F_h^* \times F_d^*$ with $i^* := C_{h^*}(a, F_s) \leq h^*$, such that $F_s(a_i, -)$ satisfies $(H)$, we have $|\Omega(a, F_s)| = O(q^{-1})$. It follows that

$$\frac{|E|}{|F_h^*||F_d^*|} = \left\{ \frac{(a, F_s) : i^* := C_{h^*}(a, F_s) \leq h^* \text{ and } F_s(a_{i^*}, -) \text{ satisfies } (H)}{|F_h^*||F_d^*|} \right\} = O(q^{-1}).$$

Combining the estimates for $C$, $D$ and $E$ with Corollaries 3.3 and 4.12 we obtain

$$\tau(s, d, q) h^* \left| F_h^* \right| \left| F_d^* \right| \left| \Omega_d \right| \leq \tau(s, d, q) h^* \left( P[C > h^*] + P_1[C_{h^*} = \infty] + O(q^{-1}) \right)$$

$$\leq \tau(s, d, q) h^* \left( 1 - \mu_d \right)^h + \frac{h^*}{(d+1)!} + \frac{2h^*}{q} + \frac{13h^*(2-\mu_d)h^*}{q^2} + O\left( \frac{d^{r+1}}{q} \right).$$

Taking into account (5.2) and this inequality we readily deduce the theorem for $d$ odd. The case $d$ even follows with a similar argument.

We briefly mention the average-case complexity of Algorithm 1.1 performing the zero-dimensional searches with Gröbner bases and Kronecker-like algorithms.

**Corollary 5.2.** Let notations and assumptions be as in Theorem 5.1. Denote by $E_{GB}(X)$ and $E_K(X)$ the average–case complexities of Algorithm 1.1 performing zero-dimensional
searches with Gröbner bases and the Kronecker algorithm respectively. We have
\[
E_{\text{GB}}[X] = \mathcal{O}^\sim \left( \left( \frac{(d+r)}{r} \right) + d \left( \frac{sd+1}{s} \right)^{\omega} + d^3s + d^8 \log q \right) \left( 1 + \frac{h^s h^*}{(d+1)^s} + \frac{d^3s}{q} + \frac{h^s (2 - \mu_d) h^*}{q^{s^2}} \right),
\]
\[
E_{K}[X] = \mathcal{O}^\sim \left( \left( \frac{(d+r)}{r} \right) + \left( \frac{d^4s}{s} \right) d^{2s} + d^8 \log q \right) \left( 1 + \frac{h^s h^*}{(d+1)^s} + \frac{d^3s}{q} + \frac{h^s (2 - \mu_d) h^*}{q^{s^2}} \right),
\]
where the notation \( \mathcal{O}^\sim \) ignores logarithmic factors and \( \omega \) is the exponent of the complexity of multiplication of square matrices with coefficients in \( \mathbb{F}_q \).

**Proof.** Denote by \( \tau_{\text{GB}}(d, s, q) \) the number of arithmetic operations in \( \mathbb{F}_q \) required to perform a zero-dimensional search using Gröbner bases, assuming that hypothesis (H) holds. As explained in the introduced, if \( D := \left( \frac{(d+r)}{r} \right) \), then \( \mathcal{O}(sD) \) arithmetic operations in \( \mathbb{F}_q \) are performed to compute the partial specialization \( F_s(a, -) \), where \( F_s \in F_d^* \) is the input system and \( a \in \mathbb{F}_q^{-s} \) is the vertical strip under consideration. Then solving the zero-dimensional system \( F_s(a, -) = 0 \) for the degree reverse lexicographic order requires \( \mathcal{O}(s^2 d \left( \frac{sd+1}{s} \right)^{\omega}) \) arithmetic operations in \( \mathbb{F}_q \) (see, e.g., [BFS15] or [BCG+17, Chapter 26]). Then the FGLM algorithm (see [FGLM93]) is applied to solve the system \( F_s(a, -) = 0 \) for the lexicographical order, with \( \mathcal{O}(sd^{\omega}) \) arithmetic operations in \( \mathbb{F}_q \). Finally, we apply a routine for computing the \( \mathbb{F}_q \)-rational roots of the resulting univariate polynomial with \( \mathcal{O}^\sim (d^8 \log q) \) operations in \( \mathbb{F}_q \) (see, e.g., [vzGG99, Algorithm 14.15]). We conclude that \( \tau_{\text{GB}}(d, s, q) \in \mathcal{O}^\sim \left( sD + s^2 d \left( \frac{sd+1}{s} \right)^{\omega} + sd^{3s} + d^8 \log q \right) = \mathcal{O}^\sim \left( D + d \left( \frac{sd+1}{s} \right)^{\omega} + d^3s + d^8 \log q \right) \).

On the other hand, if \( \tau_{K}(d, s, q) \) is the number of arithmetic operations in \( \mathbb{F}_q \) required to perform a zero-dimensional search using the Kronecker algorithm, assuming that hypothesis (H) holds, then, according to, e.g., [CM06a], one has \( \tau_{K}(d, s, q) \in \mathcal{O}^\sim \left( sD + \left( \frac{d^4s}{s} \right) d^{2s} + d^8 \log q \right) = \mathcal{O}^\sim \left( D + \left( \frac{d^4s}{s} \right) d^{2s} + d^8 \log q \right) \).

This proves the estimates in the corollary. \( \square \)

Taking into account the bound \( \left( \frac{e^t}{t} \right) \leq \frac{3}{2} e^t \), which holds for \( e \geq 2 \), we may simplify the bounds in the corollary in the following way:
\[
E_{\text{GB}}[X] \in \mathcal{O}^\sim \left( \left( d^r + (sd)^{\omega} + d^8 \log q \right) \left( 1 + \frac{h^s h^*}{(d+1)^s} + \frac{d^3s}{q} + \frac{h^s (2 - \mu_d) h^*}{q^{s^2}} \right) \right),
\]
\[
E_{K}[X] \in \mathcal{O}^\sim \left( \left( d^r + d^3s + d^8 \log q \right) \left( 1 + \frac{h^s h^*}{(d+1)^s} + \frac{d^3s}{q} + \frac{h^s (2 - \mu_d) h^*}{q^{s^2}} \right) \right),
\]

### 5.2. Probability of success.

We end with an estimate on the probability of success of our algorithm. For this purpose, we first analyze the probability that, given \( h \) with \( 1 \leq h \leq h^* \), after exactly \( h \) random choices \( a_1, \ldots, a_h \) in \( \mathbb{F}_q^{-s} \) we obtain a system \( F_s(a_h, -) = 0 \) with \( \mathbb{F}_q \)-rational solutions which satisfies hypothesis (H). More precisely, we consider the random variable \( C_h := F_h \times F_d^* \to \{1, \ldots, h, \infty\} \) defined as in (4.10) and analyze the probability of the set
\[
\{C_h = h\} \cap S_h^0, \quad S_h^0 := \{(a, F_s) \in F_h \times F_d^* : F_s(a_h, -) \text{ satisfies hypothesis (H)}\}.
\]
We have the following result.

**Lemma 5.3.** For \( q > 2d^3(d+1)^s \), \( s < d \) and \( 1 \leq h \leq h^* \) we have
\[
|P_h|\{C_h = h\} \cap S_h^0| - \mu_d(1 - \mu_d)^{h-1} \leq \frac{h^s - 1}{(d+1)^s} + \frac{2d^3(d+1)^s + 2}{q} + \frac{5d(2 - \mu_d)^{h-1}}{q} \quad \text{for } d \text{ odd},
\]
\[
|P_h|\{C_h = h\} \cap S_h^0| - \mu_d(1 - \mu_d+1)^{h-1} \leq \frac{h^s - 1}{(d+1)^s} + \frac{2d^3(d+1)^s + 2}{q} + \frac{5d(2 - \mu_d)^{h-1}}{q} \quad \text{for } d \text{ even}.
\]

**Proof.** For \( d \) odd, according to the Fréchet inequalities,
\[
\max\{P_h|C_h = h\} + P_h|S_h^0| - 1, 0) \leq P_h|\{C_h = h\} \cap S_h^0| \leq \min\{P_h|C_h = h\}, P_h|S_h^0|\).
It is easy to see that \( P_h [S_h] = P_1 [S_H] \). Therefore, by Corollary 3.3 it follows that
\[
P_h [C_h = h] - \frac{2d(d+1)^s}{q} \leq P_h [(C_h = h) \cap S_h] \leq P_h [C_h = h].
\]
As a consequence, from Corollaries 4.3 and 4.10 we easily deduce the lemma. The case \( d \) even follows similarly. 

Now we are able to estimate the probability of success of Algorithm [1]\dagger

**Theorem 5.4.** Let \( q > 2d(d+1)^s \) and \( s < d \). If \( P \) denotes the probability of failure of Algorithm [1]\dagger then
\[
|P - (1 - \mu_d)^h| \leq \frac{h^*}{(d+1)!} + O\left(\frac{h^* d^s (d+1)^s}{q} + \frac{(2 - \mu_d)^*}{q}\right)
\]
for \( d \) odd,
\[
|P - (1 - \mu_{d+1})^h| \leq \frac{h^*}{(d+1)!} + O\left(\frac{h^* d^s (d+1)^s}{q} + \frac{(2 - \mu_d)^*}{q}\right)
\]
for \( d \) even.

**Proof.** Suppose that \( d \) is odd. Observe that
\[
\sum_{h=1}^{h^*} \mu_d (1 - \mu_d)^{h-1} = 1 - (1 - \mu_d)^{h^*}.
\]
By Lemma 5.3, the probability \( P^* \) that, for a random choice of \( \mathbf{a} := (a_1, \ldots, a_h) \in \mathbb{F}_q^s \) and \( \mathbf{F}_s \in \mathbb{F}_d^s \), there is \( h \) with \( 1 \leq h \leq h^* \) such that \( C(\mathbf{F}, \mathbf{a}) = h \) and \( \mathbf{F}_s(\mathbf{a}_h, -) \) satisfies condition (H) can be estimated as follows:
\[
|P^* - (1 - (1 - \mu_{d+1})^h)| \leq \sum_{h=1}^{h^*} |P_h [(C_h = h) \cap S_h] - \mu_d (1 - \mu_d)^{h-1}|
\]
\[
\leq \sum_{h=1}^{h^*} \left( \frac{2^h h^* d^s (d+1)^s}{q} + \frac{5 (2 - \mu_d)^{h-1}}{q^s} \right) + \frac{2 h^* d^s (d+1)^s+1}{q}
\]
\[
\leq \frac{h^*}{(d+1)!} + \frac{5 (2 - \mu_d)^{h^*}}{q^s} + \frac{2 h^* d^s (d+1)^s+1}{q}.
\]
It remains to take into account the probability that the routine for zero-dimensional search is successful. Taking into account that such a routine has a probability of success of order \( O(q^{-1}) \), applying the Fréchet inequalities as in the proof of Lemma 5.3 we readily deduce the statement of the theorem. The case \( d \) even follows similarly. 

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