Quantum Cauchy problem in cosmology

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Abstract

We develop a general framework for effective equations of expectation values in quantum cosmology and pose for them the quantum Cauchy problem with no-boundary and tunneling wavefunctions. We apply this framework in the model with a big negative non-minimal coupling, which incorporates a recently proposed low energy (GUT scale) mechanism of the quantum origin of the inflationary Universe and study the effects of the quantum inflaton mode.

1. Introduction

Efim Samoilovich Fradkin created a scientific school that can be characterized by a peculiar style that combines great diversity of fundamental physical problems and efficient methods of their solution. He never worked in the area of quantum cosmology, but the generality of his methods, either it is the functional approach to QFT, the Euclidean QFT or the most advanced methods of gauge constrained theories, not only apply in this field but actually become indispensable. The authors of this paper had a happy opportunity to share a small piece of his scientific wisdom and, in this work, would like to show how these three methods pioneered by E.S.Fradkin fruitfully combine in quantum cosmological context.

The power of his approach is based on the fact that the consideration of the problem in question begins with the first fundamental principles followed by a set of clear mathematical transformations which unambiguously push the solution to its logical extreme. This strategy straightforwardly leads to the final result instead of unreliable hand waving and (very often erroneous) insight. In context of quantum cosmology this strategy implies a concrete setting of the Cauchy problem for the initial cosmological state, the unitary evolution from which gives rise to the present-day semiclassical Universe with its observable cosmological parameters – Hubble constant $H$, density parameter $\Omega$, anisotropy of microwave background, etc. Here we show how this strategy can be realized for a particular model of inflationary Universe.

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2. Quantum origin of the Universe as a low-energy phenomenon

The virtue of quantum cosmology is that it can give initial conditions for inflation, which in their turn determine main cosmological parameters of the observable Universe, including the density parameter $\Omega$. The requirement of $\Omega > 1$ in closed cosmology gives the bound on the e-folding number $N \geq 60$ – the logarithmic expansion coefficient for the cosmological scale factor $a$ during the inflation stage with a Hubble constant $H = \dot{a}/a$. In the chaotic inflation model $H = H(\varphi)$ is generated by the inflaton $\varphi$ and, therefore, all the parameters can be found as functions of initial $\varphi$. This quantity is subject to the quantum distribution $\rho(\varphi)$ determined by the cosmological wavefunction. If this distribution has a sharp probability peak at certain $\varphi = \varphi_I$, then this value serves as the initial condition for inflation.

Two known quantum states that lead in the semiclassical regime to the closed inflationary Universe are the no-boundary [1] and tunneling [2] wavefunctions. They both describe quantum nucleation of the Lorentzian quasi-DeSitter spacetime from the Euclidean hemisphere – the gravitational instanton responsible for the classically forbidden state of the gravitational field. Tree level wave functions are generally devoid of the observationally justified probability peaks. Beyond the tree level the distribution $\rho_{NB,T}(\varphi)$ becomes the diagonal element of the reduced density matrix obtained by tracing out all degrees of freedom but $\varphi$ [3, 4].

\[ \rho_{NB,T}(\varphi) \sim \exp[\mp I(\varphi) - I(\varphi)]. \] (2.1)

Here the classical action is amended by the Euclidean effective action $I(\varphi)$ of all quantum fields that are integrated out.

The model of [5] capable of generating the probability peak contains the graviton-inflaton sector with a big negative constant, $-\xi = |\xi| \gg 1$, of non-minimal curvature coupling,

\[ L(g_{\mu\nu}, \varphi) = g^{1/2} \left\{ \left( \frac{m^2}{16\pi} - \frac{1}{2} \xi \varphi^2 \right) R - \frac{1}{2} \nabla \varphi^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4} \varphi^4 \right\}, \] (2.2)

and generic GUT sector of Higgs $\chi$, vector gauge $A_\mu$ and spinor fields $\psi$ coupled to the inflaton via the interaction terms $\lambda_\chi \chi^2 \varphi^2$, $g_{\chi}^2 A_\mu \varphi^2$, $f_{\psi} \varphi \bar{\psi} \psi$ (non-derivative parts) with the coupling constants $\lambda_\chi, g_\chi, f_\psi$. Its peak-like distribution function can be approximated by the gaussian packet

\[ \rho_{NB,T}(\varphi) \simeq \frac{1}{2\pi} \exp \left[ -\left( \varphi - \varphi_I \right)^2 / 2\Delta^2 \right], \] (2.3)

where the parameters of the peak – mean value $\varphi_I = m_P(8\pi|1 + \delta/|\xi|A)^{1/2}$ and its quantum width $\Delta = (\varphi_I/\sqrt{12A})\sqrt{\lambda}/|\xi|$ are strongly suppressed by a small ratio $\sqrt{\lambda}/|\xi|$ known from the COBE normalization for $\Delta T/T \sim 10^{-5}$ [6] (because the CMBR anisotropy in this model is proportional to this ratio [6]). Here $A = 1/2\lambda \left( \sum_\chi \lambda_\chi^2 + 16 \sum_A g_{\chi}^4 - 16 \sum_\psi f_{\psi}^4 \right)$ is the universal combination of coupling constants above and

\[ \delta \equiv -8\pi |\xi| m^2 / \lambda m_P^2. \] (2.4)
For the no-boundary and tunneling states (±-signs respectively) the peak exists for positive $A$ and $±(1+δ) < 0$. The classical equations of motion in the slow roll approximation,

$$
\ddot{\varphi} + 3H(\varphi) \dot{\varphi} - F(\varphi) = 0,
$$

(2.5)

$H(\varphi) \simeq (\lambda/12|\xi|)^{1/2} \varphi$, $F(\varphi) \simeq -\lambda m_p^2 (1+\delta) \varphi/48\pi \xi^2$, show that the inflaton decreases from its initial value, $\dot{\varphi} \simeq F/H < 0$, only for $1+\delta > 0$, that is only for the tunneling state. Only in this case the duration of the inflationary epoch is finite with the e-folding number $N = \int dtH \simeq 48\pi^2/A$. Comparison with $N \geq 60$ yields the bound $A \sim 5.5$ which can be regarded as a selection criterion for particle physics models. This conclusion remains qualitatively true when taking into account the contribution of the inhomogeneous quantum modes to the radiation current of the effective equations. This contribution and its dynamical effect were obtained in \cite{8} by the method of the Euclidean effective action, however, the quantum fluctuations of the inflaton field itself have not been taken into account.

For the proponents of the no-boundary quantum states in a long debate on the wave-function discord \cite{3} \cite{10} this situation seems unacceptable. According to this result the no-boundary proposal does not generate realistic inflationary scenario, while the tunneling state does not satisfy important aesthetic criterion – the universal formulation of the initial conditions and dynamics in one concept – spacetime covariant path integral over geometries.

Thus, one of the motivations of considering the quantum mechanical sector of the inflaton mode is the hope that it can handle this difficulty. In view of the smallness of $\Delta$ the quantum fluctuations $\Delta \varphi \sim \Delta$ are expected to be negligible, but those of their quantum momenta $\Delta p_\varphi \sim 1/\Delta$ blow up for small $\Delta$. Therefore, apriori, it is hard to predict the overall magnitude of the quantum rolling force and its sign due to $\Delta \varphi(t)$. In what follows we carefully consider this problem.

3. Effective equations: setting the problem

Effective equations for expectation values of quantum fields, $\hat{g}(x) = \hat{\varphi}(x), \hat{\chi}(x), \hat{\psi}(x), \hat{A}_\mu(x), \hat{g}_{\mu\nu}(x), ...$, in the quantum state $|\Psi\rangle$, $g(x) = \langle \Psi | \hat{g}(x) | \Psi \rangle$, can be obtained by expanding the Heisenberg equations of motion for $\hat{g}(x) = g(x) + \Delta \hat{g}(x)$ in powers of $\Delta \hat{g}(x)$ and averaging them with respect to $|\Psi\rangle$:

$$
\frac{\delta S[g]}{\delta g(x)} + J(x) = 0.
$$

(3.1)

Here $S[g]$ is the classical action and $J(x)$ is the radiation current which accumulates all quantum corrections starting with the one-loop contribution

$$
J(x) = \frac{1}{2} \int dydz \frac{\delta^2 S[g]}{\delta g(x) \delta g(y) \delta g(z)} G(z, y) + \ldots = \frac{1}{2} \left\langle \left[ \frac{\delta S[g]}{\delta \hat{g}(x)} \right]_2 \right\rangle + \ldots,
$$

(3.2)

and $G(z, y) = \langle \Psi | \Delta \hat{g}(z) \Delta \hat{g}(y) | \Psi \rangle$ is the Wightman function of quantum disturbances in a given quantum state.

In closed cosmological model, the total metric and the scalar field (playing the role of an inflaton) is usually decomposed in the spatially homogeneous background and inhomogeneous perturbations $ds^2 = -N^2(t) dt^2 + a^2(t) \gamma_{ij} dx^i dx^j + h_{\mu\nu}(x) dx^\mu dx^\nu$, $\varphi(x) = \varphi(t) + \delta \varphi(x)$, where
\(a(t)\) is the scale factor, \(N(t)\) is the lapse function and \(\gamma_{ij}\) is the spatial metric of the 3-sphere of unit radius. Therefore, the full set of fields \(g(x)\) consists of the minisuperspace spatially homogeneous variables \(Q(t)\) and inhomogeneous fields \(f(x)\) depending on spatial coordinates \(x^i=x\), \(Q(t) = (a(t), \varphi(t), N(t))\), \(f(x) = (\delta \varphi(t, x), h_{\mu \nu}(t, x), \chi(t, x), \psi(t, x), A_\mu(t, x), ... )\).

From the structure of the initial quantum state, that will be discussed later, it follows that only minisuperspace variables have nonvanishing expectation values \(\langle \hat{Q}(t) \rangle \neq 0\), \(\langle \hat{f}(x) \rangle = 0\). Therefore, the full set of effective equations reduces to the following three equations in the minisuperspace sector

\[
\frac{\delta S[Q]}{\delta Q(t)} + J_Q(t) = 0, \quad J_Q = J_N, J_a, J_\varphi, \tag{3.3}
\]

their quantum radiation currents \(J_Q(t)\) containing the contribution of quantum fluctuations of minisuperspace modes themselves and those of the spatially inhomogeneous fields.

Splitting of the whole configuration space into minisuperspace and inhomogeneous sectors reflects the choice of the collective variables and their quantum states – they turn out to be very different for these two sectors of the theory. This results in different calculational strategies for the corresponding quantum averages. Inhomogeneous \(f\)-modes live in the Euclidean DeSitter invariant vacuum, so that their calculation is strongly facilitated by the Euclidean effective action method \([11]\). For the quantum mechanical \(\varphi\)-mode the situation is different.

In the tree-level approximation the wavefunction \(\exp[\mp I(\varphi)/2]\) of \(\varphi\) does not have good probability peaks and is even unnormalizable. Therefore, the tree-level quantum average \(\langle \Delta Q \Delta Q \rangle^\text{tree}\) is badly defined. Beyond the tree-level approximation the situation can be improved, because the quantum average should now be defined with the aid of the reduced density matrix \(\langle \Delta Q \Delta Q \rangle = \text{tr} [\hat{\Delta} \hat{Q} \hat{\Delta} \hat{\rho}]\),

\[
\hat{\rho} \equiv \rho(\varphi, \varphi') = \int df \Psi(\varphi, f) \Psi^*(\varphi', f), \tag{3.4}
\]

which originates from tracing the \(f\)-variables out and includes loop corrections. As shown in \([3, 12]\), the diagonal element of this density matrix – the distribution function of \(\varphi\), \(\rho(\varphi) = \rho(\varphi, \varphi)\), is given in the approximation of a gaussian integral by the effective action algorithm \((2.1)\) which can generate a sharp probability peak \((2.3)\). With this modification the quantum correlators become well defined, being expressed in terms of \(\langle \Delta \varphi \Delta \varphi \rangle \sim \Delta^2 < \infty\).

4. Quantum Cauchy problem: tree level approximation

In the one-loop approximation the radiation current can be calculated on the classical background – the lowest order approximation for the mean field. The initial conditions for this solution follow from the no-boundary and tunneling wavefunctions. We show this for the model of a minimally coupled inflaton field \(\phi\) with a generic potential \(V(\phi)\) (we reserve the notation \(\phi\) as opposed to the notation \(\varphi\) for the non-minimal inflaton\(^1\)). The Wheeler-DeWitt

\(^1\)This framework can be extended to the non-minimal model by reparametrizing the latter to the Einstein frame \([13]\), and this will be done below.
equation for the cosmological wavefunction on 2-dimensional minisuperspace, $\Psi(\phi, a)$, has two semiclassical solutions – the so-called no-boundary and tunneling wavefunctions. In the approximation of the inflationary slow roll (when the derivatives with respect to $\phi$ are much smaller than the derivatives with respect to $a$) they read 

$$
\Psi_{NB}(\phi, a) = C_{NB}(a^2 H^2(\phi) - 1)^{-1/4} e^{-I(\phi)/2} \cos \left[ S(a, \phi) + \frac{\pi}{4} \right],
$$

(4.1)

$$
\Psi_T(\phi, a) = C_T(a^2 H^2(\phi) - 1)^{-1/4} e^{I(\phi)/2} \exp \left[ i S(a, \phi) + \frac{i\pi}{4} \right]
$$

(4.2)

and describe the nucleation of the Lorentzian DeSitter spacetime with the Hamilton-Jacobi function $S(\phi, a) = -\pi m_p^2 (a^2 H^2(\phi) - 1)^{3/2}/2 H^2(\phi)$ from the gravitational half-instanton with the action $I(\phi)/2 = -\pi m_p^2/H(\phi)$. This nucleation takes place at $a = 1/H(\phi)$, $H(\phi) \equiv (\kappa V(\phi)/3)^{1/2}$ being the effective Hubble constant, driving the inflationary dynamics of the model, $\dot{a}/a \simeq H(\phi)$ ($\kappa \equiv 8\pi/m_p^2$ – the gravitational constant). This domain forms the one-dimensional curve in two-dimensional minisuperspace,

$$
\chi(\phi, a) = a - (3/\kappa V(\phi))^{1/2} = 0,
$$

(4.3)

which can be identified with the physical subspace, $\chi(\phi, a)$ being regarded as the corresponding gauge condition. If $\phi$ is chosen as a physical coordinate then, according to the formalism of [14], the physical wavefunction $\Psi(\phi)$ follows from the Dirac wavefunction $\Psi(\phi, a)$ by the transformation $\Psi(\phi) = J_\chi^{1/2} \Psi(\phi, a)|_{\chi(\phi, a)=0}$, where $J_\chi$ is the Faddeev-Popov determinant – the Poisson bracket of the gauge condition with the Hamiltonian constraint. Since $J_\chi^{1/2} \sim (H^2 a^2 - 1)^{1/4}$, it cancels the divergent preexponential factors in (4.1)-(4.2) and results in the initial physical wavefunctions $\Psi_{NB,T}(\phi) \sim \exp[\mp I(\phi)/2]$. Beyond the tree level they get replaced by the density matrix [3-4] which, for the model of the nonminimally coupled inflaton, even yields a sharp probability peak and, thus, generates the expectation value of the inflaton $\phi = \langle \hat{\phi} \rangle$ – the first initial condition of the above type.

The second initial condition arises from the expectation value of the physical momentum conjugated to $\phi$, $p_\phi = \langle \hat{p}_\phi \rangle$. In view of reality of the initial density matrix [3-4] this expectation value is vanishing, $\langle \hat{p}_\phi \rangle = 0$. From the Hamiltonian reduction of the symplectic form in the gauge $\chi(\phi, a) = 0$ it follows that the physical momentum expresses in terms of the original momenta, $\Pi_a da + \Pi_\phi d\phi = p_\phi d\phi$, $p_\phi = \Pi_\phi - \Pi_a \chi_\phi/\chi_a$, $\chi_\phi \equiv \partial_\phi \chi$, $\chi_a \equiv \partial_a \chi$. Therefore, for $p_\phi = 0$, $\Pi_\phi$ homogeneously expresses in terms of $\Pi_a$ and, in view of the Hamiltonian constraint this implies that at the initial Cauchy surface $\Pi_\phi = 0$ and $\Pi_a = 0$. Thus, the full set of initial conditions for the classical background reads as $\phi = \langle \hat{\phi} \rangle$, $a = 1/H(\phi)$, $\dot{\phi} = \dot{a} = 0$.

5. Cauchy problem for linearized quantum fluctuations

In the Hamiltonian reduction to the physical sector for cosmological perturbations [13], scalar perturbations of metric and inflaton fields (which only contribute to the homogeneous sector) are defined by the ansatz: $ds_{\text{total}}^2 = a^2(\eta) [-(1 + 2A)dt^2 + (1 - 2\psi) \gamma_{ij} dx^i dx^j]$, $\phi_{\text{total}} = \phi + \delta \phi$. Their quadratic action reduces to the functional of only two invariant (with respect to linearized gauge transformations) combinations of canonical coordinates $(\psi, \delta \phi)$.
and their momenta \((\Pi_\psi, \Pi_\delta\phi)\), \(\Psi = \psi + \mathcal{H}\delta\phi/\phi', \quad \Pi_\psi = \Pi_\psi - 2a^2\sqrt{\gamma}D\delta\phi/\kappa\phi'\), where prime denotes derivatives with respect to the conformal time \(\eta\), \(\mathcal{H} = a'/a\) is the conformal time Hubble constant and \(D = \gamma^{ij}\nabla_i\nabla_j + 3\). With the choice of physical phase space variables, \(q = 2a\Psi/\kappa\phi' + \mathcal{H}D^{-1}\Pi_\psi/\phi' \sqrt{\gamma}, \quad p = -\phi'a\sqrt{\gamma}D\Psi/2\mathcal{H} + \kappa\phi'\Pi_\psi/4a\), the action acquires the final Lagrangian form

\[
S[q] = \frac{1}{2} \int d\eta \sqrt{\gamma}(-Dq) \left[ -a^2d^2/d\eta^2 + \phi'(1/\phi')'' + \kappa\phi'^2/2 + D \right]q. \tag{5.1}
\]

The field \(q\) is well known from the theory of cosmological perturbations as the Bardeen invariant \([13]\).

In the Newton gauge, widely used in the theory of cosmological perturbations, these perturbations read in terms of the Bardeen invariant \(q\) as \([13]\) \(A = \psi, \quad \psi = \kappa\phi' q/2a, \quad \delta\phi = (\phi' q')/a\phi'\). Similar relations in the minisuperspace gauge \([13]\) can be obtained by expressing \(da = -a\nu\) in terms of \(\delta\phi, \quad \psi = (\chi_\phi/a\chi_a)\delta\phi\), and finding the expression for the physical momentum \(p_{\delta\phi}\) conjugated to \(\delta\phi\), \(\Pi_\psi\psi' + \Pi_{\delta\phi}\delta\phi' = p_{\delta\phi}\delta\phi' + \cdots, \quad p_{\delta\phi} = \Pi_{\delta\phi} + \Pi_\psi\chi_\phi/a\chi_a\). Then, the linearized Hamiltonian constraint gives \(\Pi_\psi\) and \(\Pi_{\delta\phi}\) as functions of \((\delta\phi, p_{\delta\phi})\), and one directly proceeds to the transformation relating \((q, p)\) to \((\delta\phi, p_{\delta\phi})\). For \(\eta \to 0\) this transformation turns out to be singular (see \([11]\) for details). This singularity is, however, an artifact of the definition of the invariant variables non-analytic at \(\phi' \to 0\). To see this, we decompose the general solution for \(q(\eta)\) in the sum of two linearly independent solutions of the equation of motion for the action \([22]\), \(q(\eta) = c_+ q_+(\eta) + c_- q_- (\eta)\), one of them having a singular behaviour at \(\eta = 0\), \(q_+(\eta) = \eta^3/\phi'(1 + O(\eta^2)), \quad q_-(\eta) = 1/\phi'(1 - 3\eta^2/2 + O(\eta^3))\).

Then we show that the coefficients \(c_\pm\) have a regular solution in terms of the initial conditions for physical variables \((\delta\phi(0), p_{\delta\phi}(0))\) \([11]\), \(c_+ = -V_\phi p_{\delta\phi}(0)/9H\sqrt{\gamma}, \quad c_- = V_\phi\delta\phi(0)/3H^{3/2}\). This relation will be used throughout the rest of the paper to express the Heisenberg operators \(\Delta\hat{Q}_{\text{phys}}(\eta)\) and \(\Delta\hat{Q}(\eta)\) in terms of the Schrödinger ones, \(\delta\hat{\phi}(0) = \delta\phi, \quad \hat{p}_{\delta\phi}(0) = \partial/\partial\delta\phi\), and then find the quantum averages of their bilinear combinations in the \(\delta\phi\)-representation of the initial density matrix (see eq.\([23]\) with \(\delta\phi = \phi - \langle \phi \rangle\)).

6. Non-minimal model

The action of the model \([22]\) has a generic form

\[
S[g_{\mu\nu}, \varphi] = \int d^4x g^{1/2} \left\{ -V(\varphi) + U(\varphi)R - \frac{1}{2}G(\varphi)(\nabla \varphi)^2 \right\}, \tag{6.1}
\]

where the coefficient functions can be read off \([22]\). In the presence of spatial densities of one-loop radiation currents \(j_N \equiv J_N/a^3 \sqrt{\gamma}, \quad j_\varphi \equiv J_\varphi/Na^3 \sqrt{\gamma}\), the effective equation for the

\footnote{Note that the operator \(D\) is positive definite for the spatially homogeneous mode, \(D = +3\). Thus, this is a ghost variable signifying the classical instability. This instability at the linear level is the manifestation of inflation which is a huge instability phenomenon incorporating the runaway modes. In contrast with the S-matrix theory this instability should not be regarded as an irrecoverable flaw of the theory, because we know a nonlinear mechanism that provides a graceful exit from the inflation stage. In particular, no special measures like introducing the indefinite metric should be undertaken to eradicate this phenomenon. Homogeneous fluctuations of the inflaton field do not have a particle nature and one should not take care of guaranteeing the energy positivity of their excitations. Therefore, this mode can and should be quantized in the coordinate representation with positive metric in the Hilbert space.}
inflaton field reads \[11\]
\[\ddot{\phi} + \left(3 \frac{\dot{a}}{a} - \frac{a}{2} \ U \dot{\phi} \right) \dot{\phi} - F(\phi, a, \dot{\phi}) = 0, \tag{6.2}\]
\[F(\phi, a, \dot{\phi}) = \frac{2VU_{\phi} - UV_{\phi}}{U + 3U_{\phi}^2} - \frac{\dot{\phi}^2}{2} \frac{d}{d\phi} \ln(U + 3U_{\phi}^2) + F_{\text{loop}}(\phi, a, \dot{\phi}, \ddot{\phi}), \tag{6.3}\]
\[F_{\text{loop}}(\phi, a, \dot{\phi}, \ddot{\phi}) = \frac{1}{U + 3U_{\phi}^2} \left( U j_\phi - 2U \dot{\phi} j_N - \frac{a}{2\dot{a}} \frac{d j_N}{dt} \right). \tag{6.4}\]

It contains quantum contributions to the friction term and the rolling force, while the first two terms in (6.3) represent the classical part. As regards the quantum part, one-loop radiation currents split into the contributions of the quantum mechanical sector and the sector of spatially inhomogeneous modes, \(j_{\text{1-loop}} = j^q + j^f\). The \(f\)-part of the current can be generated by the effective action, which implies the replacement of the classical coefficient functions \(V(\phi), U(\phi), G(\phi)\), by their effective counterparts and truncation of the (generally infinite) series to the first three terms. This truncation is based on two assumptions – the smallness of inflaton derivatives due to the slow roll regime and smallness of \(R/m_{\text{part}}^2\) – the curvature to particle mass squared ratio \[8\]. Thus, with this approximation, the effective equations of motion in our non-minimal model take the form of (6.2) with \(V_{\text{eff}}(\phi), U_{\text{eff}}(\phi), G_{\text{eff}}(\phi)\) replacing \(V(\phi), U(\phi), G(\phi)\) and the radiation currents \(j_N, j_\phi\) saturated by the contribution of the quantum mechanical mode, \(j^q_\phi, j^q_N\). The resulting rolling force in the leading order of the slow roll expansion becomes the sum of the force induced by the effective action, \(F_{\text{eff}}\), and the quantum mechanical force, \(F^q = F_{\text{eff}} + F^q\),
\[F_{\text{eff}} = \frac{2V_{\text{eff}}U_{\phi} - U_{\text{eff}} V_{\phi}}{G_{\text{eff}} U_{\phi} + 3(U_{\text{eff}})^2}, \quad F^q = \frac{1}{U + 3U_{\phi}^2} \left( U j^q_\phi - 2U \dot{\phi} j^q_N - \frac{a}{2\dot{a}} \frac{d j^q_N}{dt} \right). \tag{6.5}\]

7. Radiation currents and their effect on inflationary dynamics

The role of \(F_{\text{eff}}\) in the inflationary evolution has been studied in \[8\]. For the no-boundary and tunneling states in the one-loop approximation it equals
\[F_{\text{eff}}^{\text{NB,T}} = -\frac{\lambda m_{\text{eff}}^2 (1 + \delta)}{48\pi \xi^2} \dot{\phi} \left(1 + \frac{\phi^2}{\xi^2}\right) + O(1/|\xi|^3) \tag{7.1}\]
and leads to different conclusions. The no-boundary peak is realized for \(1 + \delta < 0\), therefore the point \(\phi_1\) is an attractor – quantum terms in \(F_{\text{eff}}\) lock the inflaton at its constant initial value and give rise to infinitely long inflationary scenario with exactly DeSitter spacetime. In the tunneling case, the probability peak exists in the opposite range of the parameter \(\delta > -1\), and the rolling force has the quantum term which initially doubles the negative classical part. Therefore, the inflaton decreases under the influence of this force, and the tunneling state generates a finite inflation stage with the estimated e-folding number \(N \simeq 48\pi^2 \ln 2/\Lambda\) leading to the estimate on \(\Lambda, \Lambda \leq 5.5\). As we shall now see, the quantum mechanical force \(F^q\) does not qualitatively change these predictions.
To obtain radiation currents in (6.5) we expand the first order variations of (5.1) to the second order in perturbations \((A, \psi, \delta \varphi)\) and make their quantum averaging with respect to the initial state. For this purpose we, first, need the initial reduced density matrix of the inflaton field \(\rho(\varphi, \varphi')\) in the non-minimal model and, second, the expressions for the Heisenberg operators \(\Delta Q(\eta)\) in terms of quantum initial data, \(\delta \dot{\varphi} = \delta \varphi, \ \hat{p}_{\varphi} = \partial / i \partial (\delta \varphi)\). In [12] it was shown that for the model with a big \(|\xi|\) the initial density matrix describes practically pure quantum state and expresses in terms of the distribution function \(\rho(\varphi, \varphi') \simeq \rho^{1/2}(\varphi) \rho^{1/2}(\varphi')\), \(|\xi| \gg 1\). The effective pure quantum state, \(\Psi_{NB,T}(\varphi) \simeq \rho^{1/2}_{NB,T}(\varphi)\), in the vicinity of the probability maximum, which is located at \(\varphi_I\), can, thus, be approximated by the gaussian packet of small quantum width \(\Delta = \sqrt{2 (2.3)}\).

The operators of quantum disturbances in the \(\delta \varphi\)-representation can be found by using the results of Sects.4-5 obtained for the minimal model because it can be regarded as the Einstein frame for the non-minimal one. The Einstein frame for (5.1) with \(\bar{U} = m_p^2 / 16\pi\) and \(\bar{G} = 1\) arises by a special conformal transformation and reparameterization of the inflaton field [13]: \((g_{\mu \nu}, \varphi) \rightarrow (\bar{g}_{\mu \nu}, \bar{\varphi})\). We denote the objects in the Einstein frame by bars and identify them with those of the minimal model considered in Sects.4-5. In this way we reduce all the calculations, Cauchy data setting, gauge fixing, reduction to the physical sector, etc. to those of the minimal model. This makes the further calculations of radiation currents straightforward. Below we separately consider the beginning of the inflation epoch, \(t = 0\), and late stationary stage of inflation.

At the onset of inflation \(t = 0\) the radiation currents equal [11]

\[
j_N^q(0) = -\frac{\lambda^2 \varphi_I^4}{384 \pi^2 |\xi|^2} \left(1 - \frac{1}{3} f\right), \quad j_\varphi^q(0) = -\frac{\lambda^2 \varphi_I^3}{96 \pi^2 |\xi|^2} \left(1 - \frac{1}{6} f\right),
\]

and \((a/\dot{a})d_j^q / dt(0) = 0\), where \(f = (A / 16\pi^2)|\xi|/|1 + \delta|\). These quantities are strongly suppressed as compared to their classical values, by a very small factor \(\lambda/|\xi|^2 \sim \Delta T^2/T^2 \sim 10^{-10}\) related to the CMBR anisotropy [7]. Their sign crucially depends on the magnitude of the parameter \(f\), which in our model is likely to be very big, \(f \gg 1\) (in view of the estimate \(N \geq 60\) on the e-folding number and the value of \(|\xi| \sim 10^4 [4]\)). In this case, the terms proportional to \(f \sim 1/\Delta^2\), generated by the kinetic terms of the radiation currents, \(\langle \Delta Q' \Delta Q' \rangle\), dominate and, in particular, lead to the positive energy density, \(\varepsilon^q(0) = -j_N(0) \simeq m_p^4 \lambda^2 |1 + \delta|/64 \pi^2 |\xi|^3 \ll m_p^4\), and, as one can check [11], negative pressure \(p^q(0) = -\varepsilon^q(0)\) (DeSitter equation of state). Interestingly, the sign of the quantum rolling force due to the homogeneous mode is independent of the magnitude of \(f\), \(F^q(0) \simeq (\lambda^2 \varphi_I^3/36) f / 96 \pi^2 |\xi|^3 > 0\). In view of the expression for \(f\) the actual magnitude of this force is again much smaller than its classical counterpart \(F^q(0) \sim |F^{\text{class}}(0)| / |\xi| \ll |F^{\text{class}}(0)|\). Therefore, for the tunneling state it gives a negligible contribution to the full rolling force. For the no-boundary state, the initial effective force \((\tau, I)\) vanishes, but the only effect that the positive \(F^q(0)\) can produce is that it shifts the equilibrium point from \(\varphi_I\) to slightly higher value, at which again the system undergoes endless inflation.

At late stationary stage of inflation the dynamics of the classical background can be approximated by the ansatz \(a = \cosh[H(\varphi)t] / H(\varphi), \ \varphi \simeq \varphi_I\), and the resulting radiation currents are dominated by the contribution of the growing mode \(q_+ \simeq \sinh Ht/3a(t) \phi \rightarrow \).
const, $Ht \gg 1$:

\[ j^q_N = \frac{\lambda \varphi^4}{4} \frac{\lambda}{864\pi^2|\xi|^2} f, \quad j^q_\varphi = \frac{\lambda \varphi^3}{2} \frac{\lambda}{864\pi^2|\xi|^2} f, \quad Ht \gg 1. \]  

(7.3)

Similarly to the onset of inflation, they are strongly suppressed relative to the classical values by the factor $\lambda/|\xi|^2 \sim 10^{-10}$. The energy density of the quantum mechanical mode, $\varepsilon^q = -j^q_N \simeq -\lambda^2[1 + \delta m^4_p/54(16\pi^2)|\xi|^3]$, $|\varepsilon^q| \ll m^4_p$, is negative, while the pressure is positive $p^q = -\varepsilon^q$ (anti-DeSitter case). Apparently, this is a manifestation of the ghost nature of the mode $q$ whose kinetic term enters the action (5.1) with the wrong sign. Radiation currents (7.3) lead to the quantum rolling force (6.4), $F^q \simeq 0$, which vanishes in the leading order of the slow roll expansion.

8. Conclusion

The dynamical contribution of the quantum mechanical mode to effective equations turned out to be very small – it is strongly dominated by the effective rolling force. This property was actually conjectured in [8], and now it is quantitatively confirmed. Thus, this mode cannot change the dynamical predictions in spatially closed model with strong non-minimal coupling. As a model of the low-energy quantum origin of the Universe only the tunneling state remains observationally justified, because the no-boundary wavefunction generates infinitely long inflationary stage. The role of this mode should not, however, be underestimated, because its effect is model dependent, and might be important in other models generating initial conditions for inflation. Moreover, the inflaton mode simulates the DeSitter and Anti-DeSitter effective equations of state, $\varepsilon + p = 0$, respectively at the onset of inflation and at late times. The sign of its energy density contribution can change depending on the balance of the potential and kinetic terms of this ghost mode. Therefore, it is not quite clear at the moment, what can the role of this mode be at post inflationary epoch.

A natural question arises if this mode can be responsible for the present day observable acceleration of the Universe as an alternative to quintessence or be capable of inducing DeSitter-anti-DeSitter phase transitions in cosmology? This question is subject to further studies.

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