Triple Laplace Transform in Bicomplex Space with Application

Mahesh Puri Goswami¹*, Naveen Jha²

¹Department of Mathematics & Statistics, Mohanlal Sukhadia University, Udaipur-313001, India
²Department of Mathematics, Government Engineering College, Bharatpur-321303, India

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Abstract In this article, we investigate bicomplex triple Laplace transform in the framework of bicomplexified frequency domain with Region of Convergence (ROC), which is generalization of complex triple Laplace transform. Bicomplex numbers are pairs of complex numbers with commutative ring with unity and zero-divisors, which describe physical interpretation in four dimensional space and provide large class of frequency domain. Also, we derive some basic properties and inversion theorem of triple Laplace transform in bicomplex space. In this technique, we use idempotent representation methodology of bicomplex numbers, which play vital role in proving our results. Consequently, the obtained results can be highly applicable in the fields of Quantum Mechanics, Signal Processing, Electric Circuit Theory, Control Engineering, and solving differential equations. Application of bicomplex triple Laplace transform has been discussed in finding the solution of third-order partial differential equation of bicomplex-valued function.

Keywords Bicomplex functions, Triple Laplace transform and Bicomplex Laplace transform.

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1 Introduction

Enormous efforts have been done in past few years in the applications of bicomplex functions and fine research has been developed. The concept of bicomplex numbers was introduced by Segre [6], in order to compactly describe physical interpretation in four-dimensional space. Set of bicomplex numbers is a commutative ring with unity and zero divisors which contains the commutative ring of hyperbolic numbers and the field of complex numbers. In fact, bicomplex numbers are generalization of complex numbers and hyperbolic numbers [9]. Hence, adverse to quaternions, bicomplex numbers are commutative with some non-invertible elements interpolated on the null cone.

The concept of holomorphic functions of a bicomplex variable is generated by Futagawa [15, 16]. Some fundamental results in the theory of bicomplex holomorphic functions were given by Dragnoi [11] while Price [10] and Rönn [29] have evolved the bicomplex function theory and algebra.

Recently, enormous efforts have been done to expand the theory of integral transforms in bicomplex space and studied their applications by Agarwal et al. [19, 20, 21, 23]. However, in recent bicomplex Schrödinger equation and some of its properties studied by Rochon and Tremblay [8] and self adjoint operators were defined for finite and infinite dimensional bicomplex Hilbert spaces [9, 26, 27]. An analytical method to solve bicomplex version of Schrödinger equation corresponds to the Hamiltonian system was studied by Banerjee [2]. Lavoie et al. [28] examined the quantum harmonic oscillator problem in bicomplex numbers and obtained eigenvalues and eigenkets of the bicomplex harmonic oscillator. Kumar et al. [25] introduced the bicomplex version of topological vector spaces and topological modules were developed by Kumar and Saini [24] over the ring of bicomplex numbers. Cerejeiras et al. [17] reconstructed a bicomplex sparse signal with high probability from a reduced number of bicomplex random samples. Ghanmi and Zine [4] introduced bicomplex Segal-Bargmann and fractional Fourier transforms.

Double Laplace transform proposed by Van der Pol [31] and applied by Humbert [18] in the study of hypergeometric functions; by Jaeger [12] to solve boundary value problems in heat conduction. The complex double Laplace transform was expanded to multiple Laplace transform in n independent complex variables by Estrin and Higgins [30]. Applications of triple Laplace transform in solving third order partial differential Mboctara equation was discussed by Atangana [1]. Agar-
wal et al. [22] generalized double Laplace transform to bicomplex double Laplace transform and found some applications. For solving the large class of partial differential equations of bicomplex-valued function, we require integral transforms defined for large class. In this procedure we derive triple Laplace transform in bicomplex space with ROC that can be competent the transferring signals from real-valued \((x, y, z)\) domain to bicomplexified frequency \((\xi, \eta, \gamma)\) domain.

This article is organised as follows: In Section 3, we introduce triple Laplace transform in bicomplex space with ROC. Some fundamental properties of triple Laplace transform in bicomplex space are presented in Section 4. In Section 5, we introduce the inversion formula for triple Laplace transform in bicomplex space. In Section 6, we discuss application of bicomplex triple Laplace transform in finding the solution of third-order partial differential equations of bicomplex-valued functions and last Section 7 contains the conclusion.

## 2 Preliminaries on Bicomplex Numbers

We start with an unconventional interpretation of the set of complex numbers \(\mathbb{C}\) which is ordered pair of two real numbers in complex plane with a non-real unit \(i_1\) s.t. \(i_1^2 = -1\) as follows

\[
\mathbb{C} = \{z = x + i_1 y : x, y \in \mathbb{R}\},
\]

where \(\mathbb{R}\) is the set of real numbers. In similar way, the set of bicomplex numbers \(\mathbb{C}_2\) which is ordered pair of two complex numbers with non-real units \(i_1\) and \(i_2\) s.t. \(i_1^2 = i_2^2 = -1\), \(i_1 i_2 = i_2 i_1 = j\), \(j^2 = 1\) as follows

\[
\mathbb{C}_2 = \{\xi = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}\}. \tag{2}
\]

Or

\[
\mathbb{C}_2 = \{\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\}. \tag{3}
\]

An important characteristic of bicomplex numbers is the unambiguous representation using the idempotent elements \(e_1 = \frac{1+i_1 j}{2}\) and \(e_1 = \frac{1+i_2 j}{2}\) with \(e_1 + e_2 = 1\) and \(e_1 e_2 = e_2 e_1 = 0\). In fact, for every \(\xi = z_1 + i_2 z_2 \in \mathbb{C}_2\), we get

\[
z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2 = P_1(\xi) e_1 + P_2(\xi) e_2,
\]

where the projections \(P_1: \mathbb{C}_2 \rightarrow \mathbb{C}\) and \(P_2: \mathbb{C}_2 \rightarrow \mathbb{C}\) are defined as \(P_1(z_1 + i_2 z_2) = z_1 - i_1 z_2\) and \(P_1(z_1 + i_2 z_2) = z_1 + i_1 z_2\), respectively. \(\{e_1, e_2\}\) is idempotent basis of bicomplex numbers. For further details of bicomplex holomorphic functions and bicomplex numbers you can refer [7, 10, 14].

## 3 Bicomplex Triple Laplace Transform

Let \(f(x, y, z)\) be a bicomplex-valued function of three variables \(x, y, z > 0\), which is piecewise continuous and has exponential order \(K_1, K_2,\) and \(K_3\) w.r.t. \(x, y,\) and \(z\), respectively.

The bicomplex Laplace transform (see, Kumar and Kumar [5]) w.r.t. \(x\) is

\[
L_x[f(x, y, z); \xi] = \int_0^{\infty} e^{-\xi x} f(x, y, z)dx
\]

\[
= \tilde{f}(\xi, y, z), \quad \xi \in \Omega_1 \subset \mathbb{C}_2 \tag{4}
\]

where

\[
\Omega_1 = \{\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2 : \text{Re}(P_1 : \xi) > K_1
\]

\[
\text{and Re}(P_2 : \xi) > K_1\} \tag{5}
\]

or

\[
\Omega_1 = \{\xi \in \mathbb{C}_2 : \text{Re}(\xi) > K_1 + |\text{Im}_j(\xi)|\} \tag{6}
\]

where \(\text{Im}_j(\xi)\) denotes the imaginary part of \(\xi\) w.r.t. \(j\). The integral in (4) is convergent and bicomplex holomorphic in \(\Omega_1\). Similarly, bicomplex Laplace transform of \(f(x, y, z)\) w.r.t. \(y\) is

\[
L_y[f(x, y, z); \eta] = \int_0^{\infty} e^{-\eta y} f(x, y, z)dy
\]

\[
= \tilde{f}(x, \eta, z), \quad \eta \in \Omega_2 \subset \mathbb{C}_2 \tag{7}
\]

where

\[
\Omega_2 = \{\eta = \eta_1 e_1 + \eta_2 e_2 \in \mathbb{C}_2 : \text{Re}(P_1 : \eta) > K_2
\]

\[
\text{and Re}(P_2 : \eta) > K_2\} \tag{8}
\]

or

\[
\Omega_2 = \{\eta \in \mathbb{C}_2 : \text{Re}(\eta) > K_2 + |\text{Im}_j(\eta)|\} \tag{9}
\]

where (7) is convergent and bicomplex holomorphic in \(\Omega_2\). And bicomplex Laplace transform of \(f(x, y, z)\) w.r.t. \(z\) is

\[
L_z[f(x, y, z); \gamma] = \int_0^{\infty} e^{-\gamma z} f(x, y, z)dz
\]

\[
= \tilde{f}(x, y, \gamma), \quad \gamma \in \Omega_3 \subset \mathbb{C}_2 \tag{10}
\]

where

\[
\Omega_3 = \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \in \mathbb{C}_2 : \text{Re}(P_1 : \eta) > K_3
\]

\[
\text{and Re}(P_2 : \eta) > K_3\} \tag{11}
\]

or

\[
\Omega_3 = \{\gamma \in \mathbb{C}_2 : \text{Re}(\eta) > K_3 + |\text{Im}_j(\gamma)|\} \tag{12}
\]

Now, taking the bicomplex Laplace transform of (4) w.r.t. \(y\) and \(z\) and using (7) and (10), we have

\[
L_{x, y, z}[f(x, y, z); \xi, \eta, \gamma]
\]

\[
= \int_0^{\infty} \int_0^{\infty} e^{-(\xi x + \eta y + \gamma z)} f(x, y, z)dxdydz
\]

\[
= \tilde{f}(\xi, \eta, \gamma), \quad (\xi, \eta, \gamma) \in \Omega \tag{13}
\]

the integral on right hand side is convergent and bicomplex holomorphic in

\[
\Omega = \{(\xi, \eta, \gamma) \in \mathbb{C}_2^3 : \xi \in \Omega_1, \eta \in \Omega_2, \text{and } \gamma \in \Omega_3\}. \tag{14}
\]

Now, we define the bicomplex triple Laplace transform as follows:
Definition 3.1. Let \( f(x, y, z) \) be a bicomplex-valued function of three variables \( x, y, z > 0 \), which is piecewise continuous and has exponential order \( K_1, K_2, \) and \( K_3 \) w.r.t. \( x, y, \) and \( z \), respectively. We call the transform in (13) bicomplex triple Laplace transform.

4 Properties of Bicomplex Triple Laplace Transform

In this section, we present some fundamental properties of bicomplex triple Laplace transform.

Proposition 4.1 (Linearity Property). Let \( \tilde{f}(\xi, \eta, \gamma) \) and \( \bar{g}(\xi, \eta, \gamma) \) be bicomplex triple Laplace transforms of two bicomplex-valued functions \( f(x, y, z) \) and \( g(x, y, z) \) of \( x, y, z > 0 \) with ROC \( \Omega_1 \) and \( \Omega_2 \), respectively. Then

\[
L_{x,y,z}[c_1 f(x, y, z) + c_2 g(x, y, z)] = c_1 \tilde{f}(\xi, \eta, \gamma) + c_2 \bar{g}(\xi, \eta, \gamma), \quad (\xi, \eta, \gamma) \in \Omega = \Omega_1 \cap \Omega_2.
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Proof. From the definition (13), we have

\[
L_{x,y,z}[c_1 f(x, y, z) + c_2 g(x, y, z)] = \frac{1}{\alpha_1 \alpha_2 \alpha_3} \int_{\alpha_1}^{\infty} \int_{\alpha_2}^{\infty} \int_{\alpha_3}^{\infty} e^{-\xi z + \eta y + \gamma x} f(x, y, z) dx dy dz,
\]

\[
= c_1 \tilde{f}(\xi, \eta, \gamma) + c_2 \bar{g}(\xi, \eta, \gamma).
\]

Thus,

\[
L_{x,y,z}[c_1 f(x, y, z) + c_2 g(x, y, z)] = c_1 \tilde{f}(\xi, \eta, \gamma) + c_2 \bar{g}(\xi, \eta, \gamma).
\]

Proposition 4.2 (Change of Scale Property). Let \( f(x, y, z) \) be bicomplex-valued function of \( x, y, z > 0 \) such that \( L_{x,y,z}[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma) \). Then

\[
L_{x,y,z}[e^{ax + by + cz} f(x, y, z)] = \tilde{f}(\xi - a, \eta - b, \gamma - c), \quad (\xi - a, \eta - b, \gamma - c) \in \Omega
\]

where \( \Omega \) defined in (14).

Proof. From the definition (13), we have

\[
L_{x,y,z}[e^{ax + by + cz} f(x, y, z)] = \frac{1}{\alpha_1 \alpha_2 \alpha_3} \int_{\alpha_1}^{\infty} \int_{\alpha_2}^{\infty} \int_{\alpha_3}^{\infty} e^{-\xi x - \eta y - \gamma z} e^{ax + by + cz} f(x, y, z) dx dy dz,
\]

\[
= e^{-a\xi x - b\eta y - c\gamma z} \frac{1}{\alpha_1 \alpha_2 \alpha_3} \int_{\alpha_1}^{\infty} \int_{\alpha_2}^{\infty} \int_{\alpha_3}^{\infty} e^{-\xi x - \eta y - \gamma z} f(x, y, z) dx dy dz,
\]

\[
= \tilde{f}(\xi - a, \eta - b, \gamma - c).
\]

Thus,

\[
L_{x,y,z}[e^{ax + by + cz} f(x, y, z)] = \tilde{f}(\xi - a, \eta - b, \gamma - c).
\]
Proposition 4.4 (Triple Laplace Transform of Derivatives). Let $f(x, y, z)$ be bicomplex-valued function of $x, y, z > 0$ such that $L_{x,y,z}[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma)$. Then

$$L_{x,y,z}[f_{xyz}(x, y, z)] = L_{x,y,z}\left[xyz f(x, y, z)\right] = -\frac{\partial^3}{\partial \xi \partial \eta \partial \gamma} \tilde{f}(\xi, \eta, \gamma), \quad (\xi, \eta, \gamma) \in \Omega$$

where $\Omega$ defined in (14) and $f_{xyz}(x, y, z) = \frac{\partial^3}{\partial x \partial y \partial z} f(x, y, z)$.

Proof. From the definition (13), we have

$$L_{x,y,z}[f_{xyz}(x, y, z)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\eta y + \gamma z)} \left[e^{-\xi f_{xyz}(x, y, z)dx} dy dz \right.$$ 
$$+ \xi \int_0^\infty e^{-\xi f_{xyz}(x, y, z)dx} dy dz$$ 
$$= \int_0^\infty \int_0^\infty e^{-(\eta y + \gamma z)} \left[-f_{xyz}(0, 0, y, z) + \xi f_{xyz}(\xi, \eta, y, z)\right] dy dz$$

$$= \int_0^\infty e^{-\gamma z} \left[-\int_0^\infty e^{-\eta y} f_{xyz}(0, y, z)dy\right.$$
$$+ \xi \int_0^\infty e^{-\eta y} f_{xyz}(\xi, y, z)dy \right] dz$$
$$= \int_0^\infty \int_0^\infty \left[-f_{xyz}(0, 0, 0, z) - \eta f_{xyz}(0, 0, z) - \xi f_{xyz}(0, 0, z)\right]$$
$$+ \xi f_{xyz}(\xi, \eta, 0)\right] dz$$

$$= \int_0^\infty \int_0^\infty e^{-\eta y} f_{xyz}(0, 0, z)dy dz$$

$$= \eta \int_0^\infty e^{-\eta y} f_{xyz}(0, 0, z)dy dz$$

$$- \xi \int_0^\infty e^{-\eta y} f_{xyz}(\xi, 0, z)dy dz + \xi \eta \int_0^\infty e^{-\xi f_{xyz}(\xi, 0, z)dz}$$

$$= [e^{-\eta y} f_{xyz}(0, 0, z)dz] + \gamma \int_0^\infty e^{-\eta y} f_{xyz}(0, 0, z)dz$$

$$- \eta \left[e^{-\eta y} f_{xyz}(0, 0, z)\right]_{t=0}^\infty - \gamma \int_0^\infty e^{-\eta y} f_{xyz}(0, 0, z)dz$$

$$= [e^{-\eta y} f_{xyz}(0, 0, z)dz] + \gamma \int_0^\infty e^{-\eta y} f_{xyz}(0, 0, z)dz$$

In general

$$L_{x,y,z}\left[x^m y^n z^m f(x, y, z)\right] = \frac{\partial^{m+n}}{\partial \xi \partial \eta \partial \gamma} \tilde{f}(\xi, \eta, \gamma).$$

Proposition 4.5. Let $f(x, y, z)$ be bicomplex-valued function of $x, y, z > 0$ such that $L_{x,y,z}[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma)$. Then

$$L_{x,y,z}[xyz f(x, y, z)] = \frac{\partial^3}{\partial \xi \partial \eta \partial \gamma} \tilde{f}(\xi, \eta, \gamma), \quad (\xi, \eta, \gamma) \in \Omega$$

where $\Omega$ defined in (14) and provided the integral on right hand exists.

Proposition 4.6 (Division by $xyz$). Let $f(x, y, z)$ be bicomplex-valued function of $x, y, z > 0$ such that $L_{x,y,z}[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma)$. Then

$$L_{x,y,z}\left[\frac{f(x, y, z)}{xyz}\right] = \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma,$$

where $\Omega$ defined in (14) and provided the integral on right hand exists.
Proof. From the definition (13), we have

\[
\tilde{f}(\xi, \eta, \gamma) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi_1 x - \eta_1 y - \gamma_1 z} f(x, y, z) dx dy dz,
\]
(15)

where \( \xi = \xi_1 e_1 + \xi_2 e_2, \eta = \eta_1 e_1 + \eta_2 e_2, \gamma = \gamma_1 e_1 + \gamma_2 e_2, \) and \( f(x, y, z) = f_{e_1}(x, y, z)e_1 + f_{e_2}(x, y, z)e_2. \) Therefore,

\[
\tilde{f}(\xi, \eta, \gamma) = \tilde{f}_{e_1}(\xi_1, \eta_1, \gamma_1)e_1 + \tilde{f}_{e_2}(\xi_2, \eta_2, \gamma_2)e_2
\]
\[
= \left( \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi_1 x - \eta_1 y - \gamma_1 z} f_{e_1}(x, y, z) dx dy dz \right) e_1
\]
\[
+ \left( \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi_2 x - \eta_2 y - \gamma_2 z} f_{e_2}(x, y, z) dx dy dz \right) e_2.
\]
(16)

Integrating (16) w.r.t. \( \xi, \eta, \) and \( \gamma \) from \( \xi \) to \( \infty, \eta \) to \( \infty, \) and \( \gamma \) to \( \infty, \) respectively, we have

\[
\int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma
\]
\[
= \left( \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \tilde{f}_{e_1}(\xi_1, \eta_1, \gamma_1) d\xi_1 d\eta_1 d\gamma_1 \right) e_1
\]
\[
+ \left( \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \tilde{f}_{e_2}(\xi_2, \eta_2, \gamma_2) d\xi_2 d\eta_2 d\gamma_2 \right) e_2
\]
\[
= \left[ \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \left( \int_0^\infty \int_0^\infty \int_0^\infty \right.ight.
\]
\[
\times e^{-\xi_1 x - \eta_1 y - \gamma_1 z} f_{e_1}(x, y, z) dx dy dz d\xi_1 d\eta_1 d\gamma_1 \left. \right] e_1
\]
\[
+ \left[ \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \left( \int_0^\infty \int_0^\infty \int_0^\infty \right.ight.
\]
\[
\times e^{-\xi_2 x - \eta_2 y - \gamma_2 z} f_{e_2}(x, y, z) dx dy dz d\xi_2 d\eta_2 d\gamma_2 \left. \right] e_2.
\]

By change of order of integration, we have

\[
\int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma
\]
\[
= \left[ \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \right.ight.
\]
\[
\times e^{-\xi_1 x - \eta_1 y - \gamma_1 z} f_{e_1}(x, y, z) dx dy dz d\xi_1 d\eta_1 d\gamma_1 \left. \right] e_1
\]
\[
+ \left[ \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_\xi^\infty \int_\eta^\infty \int_\gamma^\infty \right.ight.
\]
\[
\times e^{-\xi_2 x - \eta_2 y - \gamma_2 z} f_{e_2}(x, y, z) dx dy dz d\xi_2 d\eta_2 d\gamma_2 \left. \right] e_2
\]
\[
= \left[ \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\xi_1 x - \eta_1 y - \gamma_1 z}}{xy} f_{e_1}(x, y, z) dx dy dz \right] e_1
\]
\[
+ \left[ \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-\xi_2 x - \eta_2 y - \gamma_2 z}}{xy} f_{e_2}(x, y, z) dx dy dz \right] e_2
\]
\[
= \left[ \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi_1 x - \eta_1 y - \gamma_1 z} f(x, y, z) \frac{1}{xy} d\xi d\eta d\gamma \right] e_1
\]
\[
+ \left[ \int_0^\infty \int_0^\infty \int_0^\infty e^{-\xi_2 x - \eta_2 y - \gamma_2 z} f(x, y, z) \frac{1}{xy} d\xi d\eta d\gamma \right] e_2
\]
\[
= \left[ \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y, z)}{xy} d\xi d\eta d\gamma \right] e_1
\]
\[
+ \left[ \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y, z)}{xy} d\xi d\eta d\gamma \right] e_2.
\]

Thus,

\[
L_{x,y,z} \left[ \frac{f(x, y, z)}{xy} \right] = \int_0^\infty \int_0^\infty \int_0^\infty \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma.
\]

Proposition 4.7 (Triple Laplace Transform of Integrals). Let \( f(x, y, z) \) be bicomplex-valued function of \( x, y, z > 0 \) such that \( L_{x,y,z}[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma). \) Then

\[
L_{x,y,z} \left[ \int_0^x \int_0^y \int_0^z \frac{f(u, v, w)}{du dv dw} \right] = \int_0^\infty \int_0^\infty \int_0^\infty \frac{f(x, y, z)}{xy} d\xi d\eta d\gamma.
\]
\[
Re(\xi) > |Im_j(\xi)|, \ Re(\eta) > |Im_j(\eta)|, \ Re(\gamma) > |Im_j(\gamma)|.
\]

Proof. Let

\[
g(x, y, z) = \int_0^x \int_0^y \int_0^z f(u, v, w) du dv dw.
\]

Hence, we have

\[
g_{xyz}(x, y, z) = f(x, y, z) \quad \text{and} \quad g(0, 0, 0) = 0.
\]

\[ L_{x,y,z} \left[ g_{xyz}(x, y, z) \right] = L[f(x, y, z)] = \tilde{f}(\xi, \eta, \gamma). \]

Now, from the Proposition 4.4, we have

\[
L_{x,y,z} \left[ g_{xyz}(x, y, z) \right] = \xi \eta \gamma \tilde{g}(\xi, \eta, \gamma) - \xi \eta \tilde{g}(\xi, \eta, 0)
\]
\[
- \eta \gamma \tilde{g}(0, \eta, \gamma) - \gamma \tilde{g}(\xi, 0, \gamma) + \xi \tilde{g}(\xi, 0, 0) + \eta \tilde{g}(0, \eta, 0)
\]
\[
+ \gamma \tilde{g}(0, 0, \gamma) - g(0, 0, 0)
\]
\[
\Rightarrow \tilde{f}(\xi, \eta, \gamma) = \xi \eta \gamma \tilde{g}(\xi, \eta, \gamma) - \xi \eta \tilde{g}(\xi, \eta, 0)
\]
\[
- \eta \gamma \tilde{g}(0, \eta, \gamma) - \gamma \tilde{g}(\xi, 0, \gamma) + \xi \tilde{g}(\xi, 0, 0) + \eta \tilde{g}(0, \eta, 0)
\]
\[
+ \gamma \tilde{g}(0, 0, \gamma) - g(0, 0, 0)
\]
Therefore,
\[ \tilde{g}(\xi, \eta, \gamma) = \frac{\tilde{f}(\xi, \eta, \gamma)}{\xi \eta \gamma} + \frac{\tilde{g}(\xi, \eta, 0)}{\xi \gamma} + \frac{\tilde{g}(0, \eta, \gamma)}{\eta \xi \gamma} + \frac{\tilde{g}(0, 0, \gamma)}{\xi \eta} \]
\[ - \frac{\tilde{g}(\xi, 0, 0)}{\xi \gamma} - \frac{\tilde{g}(0, \eta, 0)}{\eta \xi \gamma} - \frac{\tilde{g}(0, 0, \gamma)}{\xi \eta} . \]

But \( \tilde{g}(\xi, \eta, 0) = 0, \tilde{g}(0, \eta, \gamma) = 0, \tilde{g}(0, 0, \gamma) = 0, \tilde{g}(\xi, 0, 0) = 0, \) and \( \tilde{g}(0, 0, \gamma) = 0. \) Therefore,
\[ \tilde{g}(\xi, \eta, \gamma) = \frac{\tilde{f}(\xi, \eta, \gamma)}{\xi \eta \gamma} , \]
\[ \therefore \quad L_{x,y,z}[g(x,y,z)] = \frac{\tilde{f}(\xi, \eta, \gamma)}{\xi \eta \gamma} . \]

Hence,
\[ L_{x,y,z} \left[ \int_0^x \int_0^y \int_0^z f(u,v,w) dw \right] = \frac{\tilde{f}(\xi, \eta, \gamma)}{\xi \eta \gamma} . \]

**Proposition 4.8.** Let \( f(x,y,z) \) be a periodic function of period \( T_1, T_2, \) and \( T_3 \) w.r.t. \( x, y, \) and \( z \) respectively. Then the bicomplex triple Laplace transform is given by
\[ L_{x,y,z}[f(x,y,z)] = \frac{\int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-\xi x - \eta y - \gamma z} f(x,y,z) dx dy dz}{(1 - e^{-T_1 \xi}) (1 - e^{-T_2 \eta}) (1 - e^{-T_3 \gamma})} , \]
\[ Re(\xi) > |Im_1(\xi)|, Re(\eta) > |Im_1(\eta)|, \text{ and } Re(\gamma) > |Im_1(\gamma)| . \]

**Proof.** Let \( f(x,y,z) \) be a periodic function with period \( T_1 \) w.r.t. \( x. \) Then for \( \xi \in \mathbb{C}_2 \) and \( Re(\xi) > |Im_1(\xi)|, \) see Agarwal et al. [21]
\[ L_x[f(x,y,z)] = \frac{\int_0^{T_1} e^{-\xi x} f(x,y,z) dx}{1 - e^{-T_1 \xi}} = \tilde{f}(\xi, \eta, \gamma) \quad (17) \]

Similarly, for \( \eta \in \mathbb{C}_2 \) and \( Re(\eta) > |Im_1(\eta)| \) taking the bicomplex Laplace transform of (17) w.r.t. \( y, \) we have
\[ L_y \left[ \tilde{f}(\xi, \eta, \gamma) \right] = \tilde{f}(\xi, \eta, \gamma) = \frac{\int_0^{T_2} e^{-\eta y} \tilde{f}(\xi, \eta, \gamma) dy}{1 - e^{-T_2 \eta}} \]
\[ = \frac{\int_0^{T_2} e^{-\eta y} \int_0^{T_1} e^{-\xi x} f(x,y,z) dx dy}{1 - e^{-T_2 \eta}} \]
\[ = \frac{\int_0^{T_1} \int_0^{T_2} e^{-\xi x - \eta y} f(x,y,z) dx dy}{(1 - e^{-T_1 \xi}) (1 - e^{-T_2 \eta})} \quad (18) \]

And for \( \gamma \in \mathbb{C}_2 \) and \( Re(\gamma) > |Im_1(\gamma)| \) taking the bicomplex Laplace transform of (18) w.r.t. \( z, \) we have
\[ L_z \left[ \tilde{f}(\xi, \eta, \gamma) \right] = \tilde{f}(\xi, \eta, \gamma) = \frac{\int_0^{T_3} e^{-\gamma z} \tilde{f}(\xi, \eta, \gamma) dz}{1 - e^{-T_3 \gamma}} \]
\[ = \frac{\int_0^{T_3} e^{-\gamma z} \int_0^{T_2} e^{-\eta y} \tilde{f}(\xi, \eta, \gamma) dy dz}{1 - e^{-T_3 \gamma}} \]
\[ = \frac{\int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-\xi x - \eta y - \gamma z} f(x,y,z) dx dy dz}{(1 - e^{-T_1 \xi}) (1 - e^{-T_2 \eta}) (1 - e^{-T_3 \gamma})} . \]

Thus,
\[ L_{x,y,z}[f(x,y,z)] = \frac{\int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-\xi x - \eta y - \gamma z} f(x,y,z) dx dy dz}{(1 - e^{-T_1 \xi}) (1 - e^{-T_2 \eta}) (1 - e^{-T_3 \gamma})} . \]

**5 Inversion**

In this section, we derive the inversion formula for bicomplex triple Laplace transform.

**Proposition 5.1.** Let \( f(x,y,z) \) be bicomplex-valued function of \( x, y, z > 0 \) such that \( L_{x,y,z}[f(x,y,z)] = \tilde{f}(\xi, \eta, \gamma) . \) Then
\[ f(x,y,z) = - \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} e^{\xi x + \eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma , \]
\[ (\xi, \eta, \gamma) \in \Omega \]

where \( \Omega \) defined in (14) and \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) are Bromwich closed contours in bicomplex space.

**Proof.** Taking the inverse bicomplex Laplace transform [3] of \( \tilde{f}(\xi, \eta, \gamma) \) w.r.t. \( \gamma, \) we get
\[ L_{\gamma}^{-1} \left[ \tilde{f}(\xi, \eta, \gamma) \right] = \tilde{f}(\xi, \eta, z) = \frac{1}{2\pi i} \int_{\Gamma_3} e^{\gamma z} \tilde{f}(\xi, \eta, \gamma) d\gamma \]
\[ = \frac{1}{(2\pi i)^2} \int_{\Gamma_2} \int_{\Gamma_3} e^{\eta y} \left( \int_{\Gamma_1} e^{\gamma z} \tilde{f}(\xi, \eta, \gamma) d\gamma \right) d\eta \]
\[ = - \frac{1}{4\pi^2} \int_{\Gamma_2} \int_{\Gamma_3} e^{\eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\eta d\gamma \quad (19) \]

Similarly, taking inverse bicomplex Laplace transform of (19) w.r.t. \( \eta, \) we have
\[ L_{\eta}^{-1} \left[ \tilde{f}(\xi, \eta, z) \right] = f(x,y,z) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\eta y} \tilde{f}(\xi, \eta, z) d\xi \]
\[ = \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} e^{\xi x + \eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma \]
\[ = \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} e^{\xi x + \eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma . \]

And, taking inverse bicomplex Laplace transform of (20) w.r.t. \( \xi, \) we have
\[ L_{\xi}^{-1} \left[ \tilde{f}(\xi, \eta, \gamma) \right] = f(x,y,z) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\xi x} \tilde{f}(\xi, \eta, z) d\xi \]
\[ = - \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} e^{\xi x + \eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma . \]

Hence,
\[ f(x,y,z) = - \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_3} e^{\xi x + \eta y + \gamma z} \tilde{f}(\xi, \eta, \gamma) d\xi d\eta d\gamma . \]

**6 Application**

In this section, we present application of bicomplex triple Laplace transform in finding the solution of some kind of third-order partial differential equation of bicomplex-valued function.

Consider the third order partial differential equation of the type
\[ \frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} + u(x,y,z) = f(x,y,z) , \quad x,y,z > 0 \quad (21) \]
where \( u : \mathbb{R}^3 \rightarrow \mathbb{C}_2 \) and \( f : \mathbb{R}^3 \rightarrow \mathbb{C}_2 \) can be expressed explicitly as
\[
u(x, y, z) = u_0(x, y, z) + i_1u_1(x, y, z) + i_2u_2(x, y, z) + i_1i_2u_3(x, y, z) \quad \text{and} \quad f(x, y, z) = f_0(x, y, z) + i_1f_1(x, y, z) + i_2f_2(x, y, z) + i_1i_2f_3(x, y, z).
\]

Taking bicomplex triple Laplace transform of (21) on both sides and constructive suggestions to improve this article.

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