On semilinear Tricomi equations in one space dimension

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Abstract

For 1-D semilinear Tricomi equation $\partial_t^2 u - t \partial_x^2 u = |u|^p$ with initial data $(u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x))$, where $t \geq 0$, $x \in \mathbb{R}$, $p > 1$, and $u_i \in C_0^\infty(\mathbb{R})$ $(i = 0, 1)$, we shall prove that there exists a critical exponent $p_{\text{crit}} = 5$ such that the small data weak solution $u$ exists globally when $p > p_{\text{crit}}$; on the other hand, the weak solution $u$, in general, blows up in finite time when $1 < p < p_{\text{crit}}$. We specially point out that for 1-D semilinear wave equation $\partial_t^2 v - \partial_x^2 v = |v|^p$, the weak solution $v$ will generally blow up in finite time for any $p > 1$. By this paper and \textsuperscript{[9]-[11]}, we have given a systematic study on the blowup or global existence of small data solution $u$ to the equation $\partial_t^2 u - t \Delta u = |u|^p$ for all space dimensions. One of the main ingredients in the paper is to establish a crucial weighted Strichartz-type inequality for 1-D linear degenerate equation $\partial_t^2 w - t \partial_x^2 w = F(t,x)$ with $(w(0,x), \partial_t w(0,x)) = (0,0)$, i.e., an inequality with the weight $(\frac{4}{3} t^3 - |x|^2)^\alpha$ between the solution $w$ and the function $F$ is derived for some real numbers $\alpha$.

Keywords: Tricomi equation, critical exponent, weighted Strichartz estimate, global existence, blowup.

Mathematical Subject Classification 2000: 35L70, 35L65, 35L67

1 Introduction

In our former papers \textsuperscript{[9][11]}, for the multi-dimensional semilinear Tricomi equation $\partial_t^2 u - t \Delta u = |u|^p$ with initial data $(u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x))$, where $t \geq 0$, $x \in \mathbb{R}^n$ with $n \geq 2$, $p > 1$, and $u_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 0, 1)$, we have given a systematic study on the blowup or global existence of small data solution $u$. In this paper, we focus on the 1-D semilinear Tricomi equation:

$$
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t^2 u - t \partial_x^2 u = |u|^p & \text{in } \mathbb{R}_+^{1+1}, \\
 u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), &
\end{array} \right.
\end{align*}
$$

where $p > 1$, $u_i(x) \in C_0^\infty(\mathbb{R})$ $(i = 0, 1)$ and supp $u_i \subset (-M,M)$ for some fixed constant $M > 1$. For the local existence and regularity of solution $u$ of (1.1) under weaker regularity assumptions on $(u_0, u_1)$, the reader may consult \textsuperscript{[18]-[20] and [24]-[25]}.

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Our present purpose is to determine a critical exponent $p_{\text{crit}} = 5$ such that the small data weak solution $u$ of (1.1) exists globally when $p > p_{\text{crit}}$; on the other hand, the weak solution $u$, in general, blows up in finite time when $1 < p < p_{\text{crit}}$. Since the local existence of weak solution $u$ to semilinear Tricomi equations with minimal regularities has been established in [20], without loss of generality, as in [10], we only focus on the global small data weak solution problem of (1.1) starting from some positive time $T_0 > 0$. Therefore, it is plausible that one utilizes the nonlinear function $F_p(t, u) = (1 - \chi(t))F_p(u) + \chi(t)|u|^p$ instead of $|u|^p$ in (1.1), where $F_p(u)$ is a $C^\infty$-smooth function with $F_p(0) = 0$ and $|F_p(u)| \leq C(1 + |u|^{p-1}|u|$, and $\chi(s) \in C^\infty(\mathbb{R})$ with $\chi(s) = \begin{cases} 1, & s \geq T_0, \\ 0, & s \leq T_0/2. \end{cases}$

Correspondingly, we shall study the following problem instead of (1.1)

$$\left\{ \begin{array}{ll}
\partial_t^2 u - t\partial_x^2 u = F_p(t, u) & \text{in } \mathbb{R}^{1+1}_+,
\vspace{1ex}
u(0, x) = \varepsilon u_0(x), \partial_t u(0, x) = \varepsilon u_1(x). \end{array} \right. \quad (1.2)$$

**Theorem 1.1 (Global existence for $p > p_{\text{crit}}$).** Assume that $p > p_{\text{crit}} \equiv 5$. Then there exists a constant $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, problem (1.2) admits a global weak solution $u$ such that

$$(1 + |\phi^2(t) - |x|^2|)\gamma u \in L^{p+1}(\mathbb{R}^{1+1}_+), \quad (1.3)$$

where $\phi(t) = \frac{2}{3}t^{\frac{3}{2}}$, and the positive constant $\gamma$ fulfills

$$0 < \gamma < \frac{1}{6} - \frac{5}{6(p+1)}. \quad (1.4)$$

With respect to the case of $1 < p < p_{\text{crit}}$, we have

**Theorem 1.2 (Blow up for $1 < p < p_{\text{crit}}$).** Let $1 < p < p_{\text{crit}} \equiv 5$. In addition, $u_i \geq 0$ and $u_i \neq 0$ for $i = 0, 1$ are assumed. Then problem (1.1) admits no global weak solution $u$ with $u \in C\left([0, \infty), H^1(\mathbb{R})\right) \cap C^1\left([0, \infty), L^2(\mathbb{R})\right)$.

**Remark 1.1.** For brevity, in the present paper we only study the semilinear Tricomi equation instead of the generalized semilinear Tricomi equation $\partial_t^2 u - t^m\Delta u = |u|^p$ ($m \in \mathbb{N}$) in problem (1.1). In fact, by the analogous methods in Theorem 1.1- Theorem 1.2 and [9]-[10], we can establish the same results to Theorem 1.1-Theorem 1.2 for the generalized semilinear Tricomi equation $\partial_t^2 u - t^m\partial_x^2 u = |u|^p$ ($m \in \mathbb{N}$) with the critical power $p_{\text{crit}}(m) = 1 + \frac{4}{m}$.

**Remark 1.2.** For the 1-D semilinear wave equation $\partial_t^2 v - \partial_x^2 v = |v|^p$ ($p > 1$), direct computation shows that the local weak solution $v$ will generally blow up in finite time, see for example [7]. However, for the 1-D semilinear Tricomi equation $\partial_t^2 u - t\partial_x^2 u = |u|^p$, by Theorem 1.1 we know that the global small data weak solution $u$ exists for $p > 5$, which is established through getting the decay property of solutions to the linear Tricomi equation (see (2.22) below) and through deriving some weighted Strichartz inequalities (see Theorem 2.1 and Theorem 3.2).

**Remark 1.3.** For the 1-D linear wave equation $\partial_t^2 v - \partial_x^2 v = 0$ with $(v(0, x), \partial_t v(0, x)) = (\varphi_0(x), 0)$, it follows from D’Alembert’s formula that $v(t, x) = \frac{1}{2}(\varphi_0(x + t) + \varphi_0(x - t))$. If $\varphi_0(x) \in H^1(\mathbb{R})$, one then has that $v \in L^q_p(0, \infty, L^p(\mathbb{R}))$ for $1 \leq p \leq \infty$ but $\varphi \notin L^q_p(0, \infty, L^p(\mathbb{R}))$ for any $q$ satisfying $1 \leq q < \infty$, namely, there is no global Strichartz-type inequality for the solution $v$ of 1-D wave equation. However, it is not the case for the 1-D linear Tricomi equation (see Theorem 2.1 and Theorem 3.2).
and motivated by [26], we can derive a Riccati-type ordinary differential inequality for Theorem 1.2 is established under the positivity assumption $s$ of $L^p$ phase method for treating the M-D problem in [10] and [12] is not applicable for the 1-D case, we to prove Theorem 1.2, we define the function $G(t) = \int_{\mathbb{R}} u(t,x) \, dx$. By an analysis similar to [9], and motivated by [26], we can derive a Riccati-type ordinary differential inequality for $G(t)$ through a delicate analysis of [11]. From this and Lemma 2.1 in [26], the blowup result for $1 < p < p_{\text{crit}}$ in Theorem 1.2 is established under the positivity assumptions of $u_0(x)$ and $u_1(x)$. 

**Remark 1.4.** In [9], [11], by the expression of the solution $w$ to the M-D linear equation $\partial_t^2 w - t\Delta w = F(t,x)$ with $w(0,x), \partial_t w(0,x) = (f,g)$, we have established such a weighted Strichartz inequality $\left\| \left( \frac{\sqrt{9} t^3 - |x|^2}{2} \right) \gamma_2 w \right\|_{L^9(\mathbb{R}^1)} \leq C \left\| f \right\|_{W^{\frac{1}{2}+\delta, \frac{1}{2}+\delta}(\mathbb{R}^n)} + \left\| g \right\|_{W^{\frac{1}{2}+\delta, \frac{1}{2}+\delta}(\mathbb{R}^n)} + \left\| \left( \frac{\sqrt{9} t^3 - |x|^2}{2} \right)^{\gamma_2} F \right\|_{L^9(\mathbb{R}^1)}$, for suitable positive numbers $\gamma_1, \gamma_2, \delta$ and $q > 1$. However, for the 1-D case of $\partial_t^2 w - t\partial_x^2 w = F(t,x)$ with $(w(0,x), \partial_t w(0,x)) = (0,0)$, we can derive such an inequality $\left\| \left( \frac{\sqrt{9} t^3 - |x|^2}{2} \right)^{\mu_1} w \right\|_{L^9(\mathbb{R}^1)} \leq C \left\| \left( \frac{\sqrt{9} t^3 - |x|^2}{2} \right)^{\mu_2} F \right\|_{L^9(\mathbb{R}^1)}$ for $q > 1$ and suitable real numbers $\mu_1$ and $\mu_2$ (here $\mu_1$ is negative and $\mu_2$ may be negative, see Theorem 3.1 below).

**Remark 1.5.** For the M-D semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = |u|^p$ with $(u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x))$ and $x \in \mathbb{R}^n$ ($n \geq 2$), in [9]–[11] we have shown that there exists a critical exponent $p_{\text{crit}}(m,n) > 1$ such that the weak solution $u$ generally blows up when $1 < p < p_{\text{crit}}(m,n)$ and meanwhile there exists a global small data weak solution $u$ when $p > p_{\text{crit}}(m,n)$, where $p_{\text{crit}}(m,n)$ is the positive root of the algebraic equation

\[
(m+2)\frac{n}{2} - 1)p^2 + ((m+2)(1-\frac{n}{2}) - 3)p - (m+2) = 0. \tag{1.5}
\]

If we formally let $m = 1$ and $n = 1$ in (1.5) (actually only holds for $n \geq 2$), then $p_{\text{crit}}(1,1) = \frac{3 + \sqrt{33}}{2}$. It is easy to know $p_{\text{crit}}(1,1) < 5$, which means that $p_{\text{crit}}(1,1)$ is strictly less than $p_{\text{crit}} = 5$ in Theorem 1.1 and Theorem 1.2.

The linear equation $\partial_t^2 u - t\partial_x^2 u = 0$ is the well-known Tricomi equation which arises from transonic gas dynamics (see [3] and [17]). There are extensive results for both linear and semilinear Tricomi equations in $n$ space dimensions ($n \in \mathbb{N}$). For instances, with respect to the linear Tricomi equation $\partial_t^2 u - t\Delta u = 0$, the authors in [11], [23] and [25] have computed its fundamental solution explicitly; with respect to the semilinear Tricomi equation $\partial_t^2 u - t\Delta u = f(t,x,u)$, under some certain assumptions on the function $f(t,x,u)$, the authors in [8] and [13]–[16] have obtained a series of interesting results on the existence and uniqueness of solution $u$ in bounded domains; with respect to the Cauchy problem of semilinear Tricomi equations, the authors in [2] and [18]–[20] established the local existence as well as the singularity structure of low regularity solutions in the degenerate hyperbolic region and the elliptic-hyperbolic mixed region, respectively. In addition, we have given a complete study on the blowup or global existence of small data solution $u$ to the semilinear Tricomi equation $\partial_t^2 u - t\Delta u = |u|^p$ for $n \geq 2$ (see [9], [11]). In the present paper, we shall systematically study the 1-D semilinear Tricomi equation $\partial_t^2 u - t\partial_x^2 u = |u|^p$.

We now comment on the proof of Theorem 1.1 and Theorem 1.2. To prove the global existence in Theorem 1.1, we require to establish some weighted Strichartz estimates for the Tricomi operator $\partial_t^2 - t\partial_x^2$ as in [10]. In this process, a series of inequalities are derived by applying an explicit formula for the solution $v$ of linear Tricomi equation $\partial_t^2 v - t\partial_x^2 v = f(t,x)$ and by utilizing a basic observation from [7] together with some delicate analysis. Here we point out that since the stationary phase method for treating the M-D problem in [10] and [12] is not applicable for the 1-D case, we cannot get a suitable $L^1 - L^\infty$ estimate of $v$ and then use interpolation between $L^1 - L^\infty$ estimate and $L^2 - L^2$ estimate to get the Strichartz-type estimate of $v$ as in [10]. Based on the resulting Strichartz inequalities and the contraction mapping principle, we complete the proof of Theorem 1.1. To prove Theorem 1.2, we define the function $G(t) = \int_{\mathbb{R}} u(t,x) \, dx$. By an analysis similar to [9], and motivated by [26], we can derive a Riccati-type ordinary differential inequality for $G(t)$ through a delicate analysis of [11]. From this and Lemma 2.1 in [26], the blowup result for $1 < p < p_{\text{crit}}$ in Theorem 1.2 is established under the positivity assumptions of $u_0(x)$ and $u_1(x)$.
This paper is organized as follows: In Section 2, some weighted Strichartz estimates for the linear homogeneous Tricomi equation are established. In Section 3, for the linear inhomogeneous Tricomi equation, the related weighted Strichartz estimates are derived. By applying the results in Section 2 and Section 3, Theorem 1.1 is proved in Section 4. In Section 5, we complete the proof of Theorem 1.2.

2 Mixed-norm estimate for homogeneous equation

In order to establish the global existence of weak solution $u$ to problem (1.1), we shall derive some mixed space-time norm estimates for the corresponding linear problem.

At first, we consider the following homogeneous problem

\[
\begin{cases}
\partial_t^2 v - t \partial_x^2 v = 0 & \text{in } \mathbb{R}^{1+1}, \\
v(0, x) = f(x), \quad \partial_t v(0, x) = g(x),
\end{cases}
\]

(2.1)

where $f, g \in C_0^\infty(\mathbb{R})$, $\text{supp}(f, g) \subseteq \{x : |x| \leq M\}$ for some fixed constant $M > 1$. We now derive a weighted space-time estimate of Strichartz-type for the solution $v$.

**Theorem 2.1.** For the solution $v$ of (2.1), one then has

\[
\left\| \left( \phi(t) + M \right)^2 - |x|^2 \right\|_{L^q(\mathbb{R}^{1+1})} \leq C \left( \left\| f \right\|_{W^{\frac{3}{2}, 1}} + \left\| g \right\|_{W^{\frac{3}{2}, 1}} \right),
\]

(2.2)

where $\phi(t) = \frac{2}{3} t^{\frac{3}{2}}$, $q = 1 + p$, $p > p_{\text{crit}}$, $\gamma < \frac{1}{6} - \frac{5}{6q}$, $0 < \delta < \frac{1}{6} - \gamma - \frac{5}{6q}$, and $C$ is a positive constant depending only on $q$, $\gamma$ and $\delta$.

**Proof.** It follows from [24] that the solution $v$ of (2.1) can be expressed as

\[
v(t, x) = V_1(t, D_x) f(x) + V_2(t, D_x) g(x),
\]

where the symbols $V_j(t, \xi)$ ($j = 1, 2$) of the Fourier integral operators $V_j(t, D_x)$ are

\[
V_1(t, |\xi|) = \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} \left[ e^{\frac{5}{6} \xi} H_+ \left(\frac{5}{6}, \frac{5}{3}; z \right) + e^{-\frac{5}{6} \xi} H_- \left(\frac{5}{6}, \frac{5}{3}; z \right) \right],
\]

(2.3)

and

\[
V_2(t, |\xi|) = \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \left[ e^{\frac{5}{6} \xi} H_+ \left(\frac{5}{6}, \frac{5}{3}; z \right) + e^{-\frac{5}{6} \xi} H_- \left(\frac{5}{6}, \frac{5}{3}; z \right) \right],
\]

(2.4)

here $z = 2i \phi(t) |\xi|$, $\xi \in \mathbb{R}$, $i = \sqrt{-1}$, and $H_\pm$ are smooth functions of the variable $z$. By [22], one knows that for $\beta \in \mathbb{N}_0$,

\[
\left| \partial_\xi^\beta H_+ (\alpha, \gamma; z) \right| \leq C (\phi(t) |\xi|)^{\alpha - \gamma} (1 + |\xi|^2)^{-\frac{|\beta|}{2}} \quad \text{if} \quad \phi(t) |\xi| \geq 1,
\]

(2.5)

and

\[
\left| \partial_\xi^\beta H_- (\alpha, \gamma; z) \right| \leq C (\phi(t) |\xi|)^{-\alpha} (1 + |\xi|^2)^{-\frac{|\beta|}{2}} \quad \text{if} \quad \phi(t) |\xi| \geq 1.
\]

(2.6)

To estimate $v$, it only suffices to deal with $V_1(t, D_x) f(x)$ since the treatment on $V_2(t, D_x) g(x)$ is similar. Indeed, if one just notices a simple fact of $t \phi(t)^{-\frac{3}{2}} = C_1 \phi(t)^{-\frac{3}{2}}$, it then follows from the
expressions of $V_1(t, \xi)$ and $V_2(t, \xi)$ that the orders of $t$ in $V_1(t, \xi)$ and $V_2(t, \xi)$ are the same. Choose a cut-off function $\chi(s) \in C^\infty(\mathbb{R})$ with $\chi(s) = \begin{cases} 1, & s \geq 2 \\ 0, & s \leq 1 \end{cases}$. Then

$$V_1(t, |\xi|) \hat{f}(\xi) = \chi(\phi(t)|\xi|)V_1(t, |\xi|) \hat{f}(\xi) + (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \hat{f}(\xi) =: \hat{v}_1(t, \xi) + \hat{v}_2(t, \xi).$$

Together with (2.3), (2.5) and (2.6), we derive that

$$v_1(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x-x-\phi(t)|\xi|)} a_{11}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i(x-x-\phi(t)|\xi|)} a_{12}(t, \xi) \hat{f}(\xi) d\xi \right),$$

where $C > 0$ is a generic constant, and for $\beta \in \mathbb{N}_0$,

$$|\partial_\xi^\beta a_{11}(t, \xi)| \leq C_1 \beta |\xi|^{-|\beta|} (1 + \phi(t)|\xi|)^{-\frac{1}{\beta}}, \quad l = 1, 2.$$

Next we analyze $v_2(t, x)$. It follows from [4] or [24] that

$$V_1(t, |\xi|) = e^{-z \Phi}\left(\frac{1}{6}, \frac{1}{3}; z\right),$$

where $\Phi$ is the confluent hypergeometric function which is analytic with respect to the variable $z = 2i\phi(t)|\xi|$. Then

$$\left| \partial_\xi \left\{ (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \right\} \right| \leq C (1 + \phi(t)|\xi|)^{-\frac{1}{\beta}} |\xi|^{-1}.$$

Similarly, one has

$$\left| \partial_\xi^\beta \left\{ (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \right\} \right| \leq C (1 + \phi(t)|\xi|)^{-\frac{1}{\beta}} |\xi|^{-|\beta|}.$$

Thus we arrive at

$$v_2(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x-x-\phi(t)|\xi|)} a_{21}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i(x-x-\phi(t)|\xi|)} a_{22}(t, \xi) \hat{f}(\xi) d\xi \right),$$

where, for $\beta \in \mathbb{N}_0$,

$$|\partial_\xi^\beta a_{2\ell}(t, \xi)| \leq C l_\beta (1 + \phi(t)|\xi|)^{-\frac{1}{\beta}} |\xi|^{-|\beta|}, \quad l = 1, 2.$$

Substituting (2.8) and (2.9) into (2.7) yields

$$V_1(t, D_\xi)f(x) = C_1 \left( \int_{\mathbb{R}^n} e^{i(x-x-\phi(t)|\xi|)} a_{1}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i(x-x-\phi(t)|\xi|)} a_{2}(t, \xi) \hat{f}(\xi) d\xi \right),$$

where $a_l (l = 1, 2)$ satisfies

$$|\partial_\xi^\beta a_l(t, \xi)| \leq C l_\beta (1 + \phi(t)|\xi|)^{-\frac{1}{\beta}} |\xi|^{-|\beta|}. \quad (2.10)$$
To estimate $V_1(t, D_x) f(x)$, it only suffices to deal with \( \int_\mathbb{R} e^{i(x \xi + \phi(t) | \xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi \) since the term \( \int_\mathbb{R} e^{i(x \xi - \phi(t) | \xi|)} a_2(t, \xi) \hat{f}(\xi) d\xi \) can be analogously treated. Set

\[
(Af)(t, x) =: \int_\mathbb{R} e^{i(x \xi + \phi(t) | \xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi.
\]

Let \( \beta(\tau) \in C_0^\infty(\frac{1}{4}, 2) \) such that

\[
\sum_{j=\pm \infty} \beta(\frac{\tau}{2^j}) \geq 1 \quad \text{for} \quad \tau \in \mathbb{R}.
\]

(2.11)

To estimate \((Af)(t, x)\), we now study its corresponding dyadic operators

\[
(A_j f)(t, x) = \int_\mathbb{R} e^{i(x \xi + \phi(t) | \xi|)} \beta(\frac{|\xi|}{2^j}) a_1(t, \xi) \hat{f}(\xi) d\xi
\]

\[
= \int_\mathbb{R} e^{i(x \xi + \phi(t) | \xi|)} a_j(t, \xi) \hat{f}(\xi) d\xi,
\]

where \( j \in \mathbb{Z} \). Note that the kernel of operator \( A_j \) is

\[
K_j(t, x; y) = \int_\mathbb{R} e^{i(x - y \xi + \phi(t) | \xi|)} a_j(t, \xi) d\xi,
\]

where \( |y| \leq M \) because of \( \text{supp } f \subseteq \{ x : |x| \leq M \} \). By (3.29) of [12], we have that for any \( N \in \mathbb{R}^+ \),

\[
|K_j(t, x; y)| \leq C \lambda_j (1 + \phi(t) \lambda_j)^{-\frac{1}{6}} (1 + \lambda_j |x - y| - \phi(t))^{-N},
\]

(2.12)

where \( \lambda_j = 2^j \). Since the solution \( v \) of (2.1) is smooth and has compact support on the variable \( x \) for any fixed time, one easily knows that (2.2) holds in any fixed domain \([0, T] \times \mathbb{R}\). Therefore, in order to prove (2.2), it suffices to consider the case of \( \phi(t) \gg M \). At this time, the following two cases will be studied separately.

### 2.1 \( |x - y| - \phi(t)| \gg M \)

For this case, there exist two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 |x - y| - \phi(t)| \geq |x| - \phi(t) \geq C_2 |x - y| - \phi(t) | \gg M.
\]

If \( j \geq 0 \), we then take \( N = \frac{5}{6} + \delta \) in (2.12) and obtain

\[
|K_j(t, x; y)| \leq C \delta \lambda_j^{\frac{1}{6}} \phi(t)^{-\frac{5}{6}} \lambda_j^{-\frac{5}{6} - \delta} |x| - \phi(t)|^{-\frac{5}{6} - \delta}
\]

\[
\leq C \delta \lambda_j^{-\delta} (1 + \phi(t)^{-\frac{5}{6}} (1 + |x| - \phi(t))|^{-\frac{5}{6} + \delta}.
\]

For \( j < 0 \), taking \( N = \frac{5}{6} - \delta \) in (2.12) we arrive at

\[
|K_j(t, x; y)| \leq C \delta \lambda_j^{\frac{1}{6}} \phi(t)^{-\frac{5}{6}} \lambda_j^{-\frac{5}{6} + \delta} |x| - \phi(t)|^{-\frac{5}{6} + \delta}
\]

\[
\leq C \delta \lambda_j^{\frac{1}{6}} (1 + \phi(t))^{-\frac{5}{6}} (1 + |x| - \phi(t))|^{-\frac{5}{6} + \delta}.
\]
It follows from \( f(x) \in C_0^\infty(\mathbb{R}) \) and direct computation that
\[
|A_j f| \leq \begin{cases} 
C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{q}} (1 + |x| - \phi(t)) \| f \|_{L^1(\mathbb{R})}, & j < 0, \\
C_{\delta} \lambda_j^{-\delta} (1 + \phi(t))^{-\frac{1}{q}} (1 + |x| - \phi(t)) \| f \|_{L^1(\mathbb{R})}, & j \geq 0.
\end{cases}
\tag{2.13}
\]

Summing the right sides of (2.13), we get that for large \( \phi(t) \) and \( |x| - \phi(t)| \),
\[
|V_1(t, D_x) f| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{q}} (1 + |x| - \phi(t)) \| f \|_{L^1(\mathbb{R})}.
\tag{2.14}
\]

Analogously, we have
\[
|V_2(t, D_x) g| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{q}} (1 + |x| - \phi(t)) \| f \|_{L^1(\mathbb{R})} + \| g \|_{L^1(\mathbb{R})}.
\tag{2.15}
\]

\section*{2.2 \( ||x - y| - \phi(t)| \leq CM \)}

By the similar method as in 2.1, we can establish that for \( t > 1 \),
\[
\| v(t, \cdot) \|_{L^\infty(\mathbb{R})} \leq C_{\delta} \phi(t)^{-\frac{1}{q}} \left( \| f \|_{W^{\frac{5}{6}, 1}(\mathbb{R})} + \| g \|_{W^{\frac{5}{6}, 1}(\mathbb{R})} \right),
\tag{2.16}
\]

where \( 0 < \delta < \frac{5}{6} - \gamma - \frac{1}{q} \) is a constant.

Indeed, note that
\[
|A_j f| = \left| \int_{\mathbb{R}} e^{i(x - y)(\xi + \phi(t))} \frac{a_j(t, \xi)}{\xi^\alpha} |D_x|^{\alpha} f(\xi) d\xi \right|,
\]

where \( \alpha = \frac{5}{6} + \delta \). Then by direct computation, we have that for \( j \geq 0 \),
\[
|A_j f| \leq C_{\delta} \lambda_j^{-\alpha} \lambda_j (1 + \phi(t) \lambda_j)^{-\frac{1}{q}} \| f \|_{W^{\frac{5}{6}, 1}(\mathbb{R})}
\leq C_{\delta} \lambda_j^{-\delta} (1 + \phi(t))^{-\frac{1}{q}} \| f \|_{W^{\frac{5}{6}, 1}(\mathbb{R})}.
\tag{2.17}
\]

Similarly, for \( j < 0 \), we have
\[
|A_j f| \leq C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{q}} \| f \|_{W^{\frac{5}{6}, 1}(\mathbb{R})}.
\tag{2.18}
\]

Summing all the terms in (2.17) and (2.18) yields (2.16) for \( g = 0 \). Note that \( ||x - y| - \phi(t)| \leq CM \) and \( \text{supp } f \subseteq [-M, M] \), we then have
\[
||x| - \phi(t)| \leq CM.
\]

This together with (2.16) for \( g = 0 \) yields
\[
|V_1(t, D_x) f| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{q}} (1 + |x| - \phi(t)) \| f \|_{W^{\frac{5}{6}, 1}(\mathbb{R})},
\tag{2.19}
\]
Analogously, we have

\[ |V_2(t, D_x)g| \leq C_\delta (1 + \phi(t))^{-\frac{1}{6}} (1 + |x| - \phi(t))^{-\frac{1}{6} + \delta} \|g\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})}. \]  

(2.20)

Therefore, it follows from (2.19) and (2.20) that

\[ |v| \leq C_\delta (1 + \phi(t))^{-\frac{1}{6}} (1 + |x| - \phi(t))^{-\frac{1}{6} + \delta} \left( \|f\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})} + \|g\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})} \right). \]  

(2.21)

Combining (2.16) with (2.21) and noting the compact supports of \(f, g\), we have

\[ |v| \leq C_\delta (1 + \phi(t))^{-\frac{1}{6}} (1 + |x| - \phi(t))^{-\frac{1}{6} + \delta} \left( \|f\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})} + \|g\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})} \right). \]  

(2.22)

Next we derive (2.2) from (2.22). Set

\[ R = \|f\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})} + \|g\|_{W^{\frac{1}{6} + \delta, 1}(\mathbb{R})}. \]

Then

\[
\left\| \left( (\phi(t) + M)^2 - |x|^2 \right) \gamma v \right\|_{L^q(\mathbb{R}^{1+1})}^q 
\leq C_\delta R \int_0^\infty \int_{\mathbb{R}} \left( (\phi(t) + M)^2 - |x|^2 \right)^\gamma (1 + \phi(t))^{-\frac{1}{6}} (1 + |x| - \phi(t))^{-\frac{1}{6} + \delta} \, dx \, dt 
\leq C_\delta R \int_0^\infty \int_0^\infty \left( (\phi(t) + M + r)^\gamma (\phi(t) + M - r)^\gamma \right.
\times \left. (1 + \phi(t))^{-\frac{1}{6}} (1 + |r - \phi(t)|)^{-\frac{1}{6} + \delta} \right)^q \, dr \, dt 
\leq C_{m, \delta} R \int_0^\infty \int_0^\infty \left( (1 + \phi(t))^{-\frac{1}{6} + \gamma} (1 + |r - \phi(t)|)^{-\frac{1}{6} + \delta} \right)^q \, dr \, dt 
\leq C_{m, \delta} R \int_0^\infty (1 + \phi(t))^{-\frac{1}{6} + 2\gamma + \delta} q \, dt.
\]  

(2.23)

Notice that by our assumption, \( \gamma - \frac{1}{6} < -\frac{5}{6q} \) holds. Then we can choose a constant \( \delta > 0 \) such that

\[
\left( \left( -\frac{1}{3} + 2\gamma + \delta \right) q + 1 \right)^\frac{3}{2} < -1.
\]

Hence, for some positive constant \( \sigma > 0 \), the integral in the last line of (2.23) can be controlled by

\[
\int_0^\infty (1 + \phi(t))^{-\frac{1}{6} + 2\gamma + \delta} q \, dt 
\leq C \int_0^\infty (1 + t)^{-\sigma} \, dt 
\leq C.
\]

This, together with (2.23), yields (2.2). Namely, we complete the proof of Theorem 2.1. \( \square \)
3 Mixed-norm estimate for inhomogeneous equation

In this section we turn to the inhomogeneous Tricomi equation:

\[
\begin{cases}
\partial_t^2 w - t \partial_x^2 w = F(t, x), & \text{in } \mathbb{R}_+^{1+1}, \\
w(0, x) = 0, & \partial_t w(0, x) = 0.
\end{cases}
\] (3.1)

Since the stationary phase method is not applicable for the solution \(w\) in the case of \(n = 1\), we cannot get a suitable \(L^1 - L^\infty\) estimate and then use interpolation to get the Strichartz-type estimate as in [10]. To overcome this difficulty, we shall cite a conclusion from [7] and subsequently use the representation formula of \(w\) to establish the space-time mixed norm estimate by delicate analysis.

**Lemma 3.1.** (see (1.16) of [7]) If

\[
f(u) = \int_0^u \frac{g(\xi)}{|u - \xi|^\beta |\xi|^\gamma} d\xi,
\]

then

\[
\|f\|_{L^q((0, \infty))} \leq C \|g\|_{L^r((0, \infty))},
\] (3.2)

where

\[
1 < r < q < \infty, \alpha + \beta + \delta = 1 - \left( \frac{1}{r} - \frac{1}{q} \right), \alpha + \beta \geq 0, \quad \text{and} \quad \alpha + \delta > \frac{1}{q}.
\]

**Theorem 3.2.** For problem (3.1), if \(F(t, x) \equiv 0\) when \(|x| > \phi(t) - 1\), then there exist some constants \(\alpha\) and \(\beta\) satisfying

\[
\alpha + \frac{1}{6} + \beta = \frac{5}{3q}, \quad \beta < \frac{1}{q},
\] (3.3)

such that

\[
\|w(\phi(t)^2 - |x|^2)^{-\alpha} w\|_{L^q(\mathbb{R}_+^{1+1})} \leq C \|w(\phi(t)^2 - |x|^2)^{\beta} F\|_{L^{q-1}(\mathbb{R}_+^{1+1})},
\] (3.4)

where \(q = 1 + p, p_{\text{crit}} < p < p_0 = 9\) and \(C > 0\) is a constant depending on \(m, q, \alpha\) and \(\beta\).

**Remark 3.1.** Recall that in [9] we have defined the conformal exponent \(p_{\text{conf}}(n)\) for \(n\)-dimensional semilinear Tricomi equation \(\partial_t^2 u - t \Delta u = |u|^p\)

\[
p_{\text{conf}}(n) = \frac{N + 2}{N - 2} = \frac{3n + 6}{3n - 2}.
\]

Set \(n = 1\), we then have \(p_{\text{conf}}(1) = p_0 = 9\).

**Proof.** By the formula in Theorem 2.4 of [25], the solution \(w\) of (3.1) satisfies

\[
w(t, x) = C \int_0^t \int_{x - \phi(t) + \phi(s)}^{x + \phi(t) - \phi(s)} \left( \phi(t) + \phi(s) + x - y \right)^{-\gamma} \left( \phi(t) + \phi(s) - (x - y) \right)^{-\gamma} \\
\times H(\gamma, \gamma, 1, z) F(s, y) dy ds,
\] (3.5)
where \( z = \frac{(-x+y+\phi(t)-\phi(s))(-x+y-(\phi(t)+\phi(s)))}{(-x+y-\phi(t)+\phi(s))(-x+y-(\phi(t)+\phi(s)))} \), \( H(\gamma, \gamma, 1, z) \) is the hypergeometric function and \( \gamma = \frac{1}{6} \). By Page 59 of [4],

\[
H(\gamma, \gamma, 1, z) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_{0}^{1} t^{1-\gamma}(1-t)^{-\gamma}(1-zt)^{-\gamma} dt
\]

\[
\leq \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_{0}^{1} t^{1-\gamma}(1-t)^{-\gamma}(1-t)^{-\gamma} dt
\]

\[
= \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} B(\gamma, 1-2\gamma)
\]

\[= C.
\]

Thus we have

\[
|w(t, x)| \leq C \int_{0}^{t} \int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)} (\phi(t) + \phi(s) + x - y)^{-\gamma} (\phi(t) + \phi(s) - (x - y))^{-\gamma} |F(s, y)| dy ds.
\]  

By (3.4), we need to estimate

\[
\left\| (\phi^2(t) - |x|^2)^{-\alpha} w \right\|_{L^q(R^{1+1}_+)} = \sup_{K \in L^q_t(R^{1+1}_+)} \left| \int_{0}^{\infty} \int_{-\infty}^{\infty} K(t, x)(\phi^2(t) - |x|^2)^{-\alpha} w(x, t) dx dt \right|.
\]

Note that

\[
I =: \int_{0}^{\infty} \int_{-\infty}^{\infty} K(t, x)(\phi^2(t) - |x|^2)^{-\alpha} w(t, x) dx dt
\]

\[
= \int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{t} \int_{x-\phi(t)+\phi(s)}^{x+\phi(t)-\phi(s)} \frac{K(t, x)F(s, y)}{(\phi^2(t) - |x|^2)^{\alpha}}
\]

\[
\times \frac{1}{(\phi(t) + \phi(s) + x - y)^2(\phi(t) + \phi(s) - (x - y))^2} dy ds dx dt.
\]

Denote \( \tilde{K}(T, x) = K(t, x) \) and \( \tilde{F}(S, y) = F(s, y) \) with \( T = \phi(t) \) and \( S = \phi(s) \). Then

\[
\left\| K \right\|_{L^q_t(R^{1+1}_+)} = \left( \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} |K(t, x)|^{\frac{q}{T}} dx \right)^{\frac{q-1}{q}} \right)^{\frac{q}{q-1}}
\]

\[
= \left( \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} |T^{-\frac{1}{6}} \frac{q-1}{q} \tilde{K}(T, x)|^{\frac{q}{T}} dx \right)^{\frac{q-1}{q}} \right)^{\frac{q}{q-1}}
\]

and

\[
\left\| (\phi^2(t) - |x|^2)^{\beta} F \right\|_{L^r_t(R^{1+1}_+)} = \left( \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} |T^{-\frac{1}{6}} \frac{q-1}{q} (T^2 - |x|^2)^{\beta} \tilde{F}(T, x)|^{\frac{q}{T}} dx \right)^{\frac{q-1}{q}} \right)^{\frac{q}{q-1}}.
\]
With (3.9) and (3.10), we can further write

\[
I = \int_0^\infty \int_{-\infty}^\infty \int_{x-(T-S)}^{x+T-S} \frac{\tilde{K}(T, x) \tilde{F}(S, y)}{(T^2 - x^2)^\alpha} \times \frac{1}{(T + S + x - y)^{\frac{1}{6}} (T + S - (x - y))^{\frac{1}{6}}} \, dS \, ds \, dT
\]

\[
= \int_0^\infty \int_{-\infty}^\infty \int_{x-(T-S)}^{x+T-S} \frac{T \cdot \frac{1}{6} \cdot \frac{1}{6} |\tilde{K}(T, x)| S \cdot \frac{1}{6} \cdot \frac{1}{6} (S^2 - y^2)^\beta |\tilde{F}(S, y)| (S^2 - y^2)^\beta (T^2 - x^2)^\alpha s^{\frac{1}{6}} T^{\frac{1}{6}}}{(T + S + x - y)^{\frac{1}{6}} (T + S - (x - y))^{\frac{1}{6}}} \, dS \, ds \, dT.
\]

(3.11)

Let

\[
\begin{align*}
\{ & u = T + x, \\
& v = T - x, \\
\{ & \xi = S + y, \\
& \eta = S - y.
\end{align*}
\]

By the assumption of supp\(F(t, x)\), we know that

\[
0 \leq \xi \leq u, \quad 0 \leq \eta \leq v.
\]

Set

\[
G(\xi, \eta) = S^{-\frac{1}{6}} \cdot \frac{1}{6} (S^2 - y^2)^\beta |\tilde{F}(S, y)|, \\
H(u, v) = T^{-\frac{1}{6}} \cdot \frac{1}{6} |\tilde{K}(T, x)|.
\]

By \(T \geq S \geq \phi(1)\), we then have

\[
I \leq \int_0^u \int_{0 \leq \xi \leq u} \int_{0 \leq \eta \leq v} \frac{G(\xi, \eta)H(u, v)}{\xi^\beta \eta^\beta u^\alpha s^\alpha (u + \xi)^{\frac{1}{6}} (v + \eta)^{\frac{1}{6}} s^{\frac{1}{6}}} \, dS \, ds \, dT
\]

\[
\leq \int_0^u \int_{0 \leq \xi \leq u} \int_{0 \leq \eta \leq v} \frac{G(\xi, \eta)H(u, v)}{\xi^\beta \eta^\beta u^\alpha |u - \xi|^{\frac{1}{6} + \frac{1}{6}} |v - \eta|^{\frac{1}{6} + \frac{1}{6}}} \, d\eta \, d\xi \, du.
\]

(3.12)

By conditions (3.3) in Theorem 3.2, we require

\[
1 < p < 2 < q < \infty
\]

and

\[
\alpha + \beta = \frac{2}{q} - \frac{1}{3q} - \frac{1}{6} \geq 0, \quad \beta < \frac{1}{q}.
\]

(3.13)

Choosing \(\beta = -p\alpha\) in (3.13), then by \(\beta < \frac{1}{q}\) and \(q = p + 1\) one has that

\[
-p\alpha < \frac{1}{p + 1} \Rightarrow \alpha > -\frac{1}{p(p + 1)}.
\]

(3.14)

Substitute (3.14) into (3.13), we get

\[
\frac{p - 1}{p(p + 1)} + \frac{1}{6} \left( \frac{1}{2} + \frac{1}{q} \right) > \frac{2}{p + 1} \iff p^2 - 3p - 6 > 0 \iff p > p_1 = \frac{3 + \sqrt{33}}{2}.
\]
On the other hand,
\[
\frac{2}{q} - \frac{1}{3q} - \frac{1}{6} \geq 0 \iff q \leq 10 \iff p \leq p_0 = 9.
\]
Thus Lemma 3.1 is applicable for \( \delta = \frac{1}{3q} + \frac{1}{6} \) and \( p = \frac{q}{q-1} \), we then estimate the integral in (3.12) as follows
\[
\int_0^{\xi_0} \int_0^{\eta_0} \frac{G(\xi, \eta) H(u, v)}{|\xi|^\beta |u|^{\alpha} |v| - |\xi|^\beta |v|^{\alpha} |u| - \eta} \, d\eta \, d\xi \, du
\]
\[
= \int_0^{\xi_0} \int_0^{u} \frac{1}{|\xi|^\beta |u|^{\alpha} |u| - \xi} \int_0^{\xi} H(u, v) \int_0^{v} \frac{G(\xi, \eta)}{|\eta|^\beta |v|^{\alpha} |v| - \eta} \, d\eta \, d\xi \, du
\]
\[
\leq \int_0^{\xi_0} \int_0^{u} \frac{1}{|\xi|^\beta |u|^{\alpha} |u| - \xi} \|H(u, \cdot)\|_{L^p} \int_0^{v} \frac{G(\xi, \eta)}{|\eta|^\beta |v|^{\alpha} |v| - \eta} \, d\eta \, d\xi \, du
\]
\[
\leq \|H\|_{L^p_{\xi, u}} \|G\|_{L^p_{\xi, u}}.
\]
This, together with (3.7) and (3.12), yields the proof of Theorem 3.2.

Based on Theorem 3.2, we are able to prove such a crucial result:

**Theorem 3.3.** For problem (3.1), if \( F(t, x) \equiv 0 \) when \( |x| > \phi(t) + M - 1 \) and \( F \in C^\infty([0, T_0] \times \mathbb{R}) \) for some fixed number \( T_0 > 0 \) (\( T_0 < 1 \)), then there exist some constants \( \alpha \) and \( \beta \) satisfying \( \alpha + 1 + \beta = \frac{q}{3q} \), \( \beta > \frac{1}{q} \), such that
\[
\left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{-\alpha} w \right\|_{L^q(\{x \in \mathbb{R}^d : \frac{q}{2q}, |x| \leq \phi(t) + M - 1 \})} \leq C \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\beta F \right\|_{L^p(\{x \in \mathbb{R}^d : \frac{q}{2q}, |x| \leq \phi(t) + M - 1 \})},
\]
(3.15)

where \( q = 1 + p \), \( p_{\text{crit}} < p < 9 \), and \( C > 0 \) is a constant depending on \( q \), \( \alpha \) and \( \beta \).

**Proof.** To prove (3.15), at first we focus on a special case of \( F(t, x) \equiv 0 \) when \( |x| > \phi(t) - \phi(T_0) \).

By the finite propagation speed property for the hyperbolic equation (3.1), we know that the integral domain in (3.15) is just only \( Q = \{(t, x) : t \geq \frac{T_0}{2}, |x| \leq \phi(t) + M - 1 \} \). Note that \( Q \) can be covered by a finite number of angular domains \( \{Q_j\}_{j=1}^{N_0} \), where the curved cone \( Q_j \) (\( j \geq 2 \)) is a shift in the \( x \) variable with respect to the angular domain
\[
Q_1 = \{(t, x) : t \geq \frac{T_0}{2}, |x| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right) \}.
\]

Set
\[
F_1 = \chi_{Q_1} F,
\]
\[
F_2 = \chi_{Q_2} (1 - \chi_{Q_1}) F,
\]
\[
\ldots
\]
\[
F_{N_0} = \chi_{Q_{N_0}} (1 - \chi_{Q_1} - \chi_{Q_2} (1 - \chi_{Q_1}) - \cdots - \chi_{Q_{N_0-1}} (1 - \chi_{Q_{N_0-2}})) F,
\]
and
where \( \chi_{Q_j} \) stands for the characteristic function of \( Q_j \), and \( \sum_{j=1}^{N_0} F_j = F \). Let \( w_j \) solve

\[
\begin{cases}
\partial_t^2 w_j - t \Delta w_j = F_j(t, x), \\
w_j(0, x) = 0, \quad \partial_t w_j(0, x) = 0.
\end{cases}
\]

Then \( \text{supp } w_j \subseteq Q_j \). Since the Tricomi equation is invariant under the translation with respect to the variable \( x \), it follows from Theorem 3.2 that

\[
\| (\phi(t)^2 - |x - \nu_j|^2)^{\gamma_1} w_j \|_{L^2(Q_j)} \leq C \| (\phi(t)^2 - |x - \nu_j|^2)^{\gamma_2} F_j \|_{L^2(Q_j)}^{\gamma_3},
\]

where \( \nu_j \in \mathbb{R}^n \) corresponds to the coordinate shift of the space variable \( x \) from \( Q_1 \) to \( Q_j \), and \( Q_j = \{(t, x) : t \geq \frac{T_0}{2}, |x - \nu_j| \leq \phi(t) - \phi\left(\frac{\nu_j}{2}\right)\} \).

Next we derive (3.15) by utilizing (3.16) and the condition of \( t \geq \frac{T_0}{4} \). At first, we illustrate that there exists a constant \( \delta > 0 \) such that for \( (t, x) \in Q_j \),

\[
\phi(t)^2 - |x - \nu_j|^2 \geq \delta \left( (\phi(t) + M)^2 - |x|^2 \right).
\]

To prove (3.17) for \( 1 \leq j \leq N_0 \), it only suffices to consider the two extreme cases: \( \nu_j = 0 \) (corresponding to \( j = 1 \)) and \( |\nu_{j_0}| = M - 1 + \phi\left(\frac{T_0}{4}\right) \) (choosing \( j_0 \) such that \( |\nu_{j_0}| = \max_{1 \leq j \leq N_0} |\nu_j| = M - 1 + \phi\left(\frac{T_0}{4}\right) \)). Note that \( |\nu_{j_0}| > M - 1 \) holds so that the domain \( Q \) can be covered by \( \bigcup_{j=1}^{N_0} Q_j \).

For \( \nu_j = 0 \), (3.17) is equivalent to

\[
\phi(t)^2 \geq (1 - \delta)|x|^2 + \delta(\phi(t) + M)^2.
\]

We now illustrate that (3.18) is correct. By \( |x| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right) \) for \( (t, x) \in Q_1 \), then in order to show (3.18) it suffices to prove

\[
\phi(t)^2 \geq (1 - \delta)\left( \phi(t) - \phi\left(\frac{T_0}{4}\right) \right)^2 + \delta(\phi(t) + M)^2.
\]

This is equivalent to

\[
\left\{ 2(1 - \delta)\phi\left(\frac{T_0}{4}\right) - 2\delta M \right\} \phi(t) \geq (1 - \delta)\phi^2\left(\frac{T_0}{4}\right) + \delta M^2.
\]

Obviously, this is easily achieved by \( t \geq \frac{T_0}{4} \) and the smallness of \( \delta \).

For \( \nu_{j_0} = M - 1 + \phi\left(\frac{T_0}{4}\right) \), the argument on (3.17) is a little involved. First, note that for fixed \( t > 0 \), the domain \( Q \) is symmetric with respect to the variable \( x \), thus we can assume \( \nu_{j_0} = \nu = M - 1 + \phi\left(\frac{T_0}{4}\right) \). In this case, (3.17) is equivalent to

\[
\phi(t)^2 \geq |x - \nu|^2 + \delta((\phi(t) + M)^2 - |x|^2)
\]

\[
= (1 - \delta)x^2 - 2\nu x + \nu^2 + \delta(\phi(t) + M)^2 \quad \text{(3.19)}
\]

For fixed \( t > 0 \), \( G(t, x) \) is a quadratic function of the variable \( x \), and takes minimum at the point \( x = \frac{\nu}{1 - \delta} \). Thus for the same fixed \( t > 0 \), the maximum of \( G(t, x) \) in the domain \( Q^t = \{ x : |x - \nu| \leq \frac{\nu}{1 - \delta} \} \).
For this end, it is only enough to consider the case that \( T \)

\[ \phi(t) - \phi\left(\frac{T_0}{4}\right) \]

must be achieved on the boundary \( \partial Q^t =: \{ x : |x - \nu| = \phi(t) - \phi\left(\frac{T_0}{4}\right) \} \). Then in order to show \( (3.19) \), our task is to prove

\[
\phi(t)^2 \geq \left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right)^2 + \delta\left(\phi(t) + M\right)^2 - |x|^2. \tag{3.20}
\]

For this end, it is only enough to consider the case that \( |x|^2 \) takes its minimum on \( \partial Q^t \). Note that on \( \partial Q^t \), we have

\[
|x|^2 = \left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right)^2 + 2\nu x - \nu^2. \tag{3.21}
\]

Therefore, without loss of generality, we can take

\[
x = \nu - \phi(t) + \phi\left(\frac{T_0}{4}\right). \tag{3.22}
\]

Substituting \( (3.22) \) and \( (3.21) \) into \( (3.20) \), we are left to prove

\[
\phi(t)^2 \geq \left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right)^2 + \delta\left(\phi(t) + M\right)^2 - \frac{\delta}{2}\phi\left(\frac{T_0}{4}\right)^2 - 2\nu\left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right) - \nu^2 \tag{3.23}
\]

For fixed \( T_0 > 0 \) and \( M > 1 \), if \( \delta > 0 \) is small enough, one then has

\[
2\delta\left(\phi\left(\frac{T_0}{4}\right) + M + \nu\right) \leq \frac{1}{2}\phi\left(\frac{T_0}{4}\right), \tag{3.24}
\]

\[
(1 - \delta)\phi\left(\frac{T_0}{4}\right)^2 + \delta M^2 \leq \frac{3}{2}\phi\left(\frac{T_0}{4}\right)^2.
\]

By \( (3.24) \) and \( (3.23) \), in order to derive \( (3.20) \), one should derive

\[
-\frac{3}{2}\phi\left(\frac{T_0}{4}\right)\phi(t) + \frac{3}{2}\phi^2\left(\frac{T_0}{4}\right) \leq 0.
\]

Obviously, this holds true by \( t \geq \frac{T_0}{4} \). Then \( (3.20) \) is proved.

Consequently, for \( (t, x) \in \bigcup_{j=1}^{N_0} Q_j \), there exists a fixed positive constant \( c > 0 \) such that for \( 1 \leq j \leq N_0 \),

\[
c\left(\phi(t) + M\right)^2 - |x|^2 \leq \phi(t)^2 - |x - \nu_j|^2. \tag{3.25}
\]

On the other hand, note that by \( |x| \leq \phi(t) + M - 1 \) for \( (t, x) \in Q \), one has

\[
2\left\{\left(\phi(t) + M\right)^2 - |x|^2\right\} - \left\{\phi^2(t) - |x - \nu_j|^2\right\}
\geq (|x| + 1)^2 - |x|^2 + \left(\phi(t) + M\right)^2 - |x|^2 - \phi(t)^2 + |x - \nu_j|^2
\geq 2M\phi(t) + M^2 + |\nu_j|^2 + 1 + 2(1 - |\nu_j||x|). \tag{3.26}
\]
Therefore, On the other hand, if \( |\nu_j| > 0 \), then by \( |\nu_j| \leq M - 1 + \phi \left( \frac{3T_0}{8} \right) \) and the smallness of \( T_0 \), the last line in (3.26) is bounded from below by

\[
2M\phi(t) + M^2 + |\nu_j|^2 + 1 + 2 \left\{ 2 - M - \phi \left( \frac{3T_0}{8} \right) \right\} \{ \phi(t) + M - 1 \} \\
= 4\phi(t) - M^2 + 6M - 3 + |\nu_j|^2 - 2\phi \left( \frac{3T_0}{8} \right) \phi(t) - 2(M - 1)\phi \left( \frac{3T_0}{8} \right) \\
\geq 2\phi(t) - M^2 + 1;
\]

while in the case of \( 1 - |\nu_j| \geq 0 \), it follows from (3.26) that

\[
2 \left\{ (\phi(t) + M)^2 - |x|^2 \right\} - \{ \phi(t)^2 - |x - \nu_j|^2 \} \geq M^2 + 1 > 0.
\] (3.28)

Substituting (3.27)-(3.28) into (3.26) yields that for \( 2\phi(t) \geq M^2 - 1 \),

\[
\phi(t)^2 - |x - \nu_j|^2 \leq C \left( (\phi(t) + M)^2 - |x|^2 \right).
\] (3.29)

On the other hand, if \( 2\phi(t) < M^2 - 1 \), then

\[
\phi(t)^2 - |x - \nu_j|^2 \leq \phi(t)^2 \leq C_M \leq C_M \left( (\phi(t) + M)^2 - |x|^2 \right).
\] (3.30)

Therefore,

\[
\left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\frac{\gamma_1}{2} w \right\|_{L^q \left( \frac{2\alpha}{\gamma_2}, \infty \times \mathbb{R} \right)} \\
\leq C \sum_{j=1}^{N_0} \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\frac{\gamma_1}{2} w_j \right\|_{L^q (Q_j)} \\
\leq C \sum_{j=1}^{N_0} \left\| \left( \phi(t)^2 - |x - \nu_j|^2 \right)^\frac{\gamma_1}{2} w_j \right\|_{L^q (Q_j)} \quad \text{(by (3.25))} \\
\leq C \sum_{j=1}^{N_0} \left\| \left( \phi(t)^2 - |x - \nu_j|^2 \right)^\frac{\gamma_2}{2} F_j \right\|_{L^q \left( \frac{\alpha}{\gamma_2}, \infty \right) (Q_j)} \quad \text{(by (3.16))} \\
\leq C \sum_{j=1}^{N_0} \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\frac{\gamma_2}{2} F_j \right\|_{L^q \left( \frac{\alpha}{\gamma_2}, \infty \right) (Q_j)} \quad \text{(by (3.30))} \\
\leq C N_0 \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\frac{\gamma_2}{2} F \right\|_{L^q \left( \frac{\alpha}{\gamma_2}, \infty \right) \left( \frac{2\alpha}{\gamma_2}, \infty \times \mathbb{R} \right)},
\]

which derives (3.15).

\[\square\]

### 4 Proof of Theorem 1.1

To establish the global existence, we shall choose \( q = p + 1 \) in Theorem 2.1 and Theorem 3.3, and subsequently consider the following two cases:

**Case I.** \( p_{\text{crit}} < p < p_0 = 9 \)
By the local existence and regularity of weak solution $u$ to (1.2) (for examples, one can see [18] or references therein), one knows that $u \in C^{\infty}([0, T_0] \times \mathbb{R})$ exists for any fixed constant $T_0 < 1$ and $u$ has a compact support on the variable $x$. Moreover, for any $N \in \mathbb{N}$,

$$\|u\left(\frac{T_0}{2}, \cdot\right)\|_{C^N} + \|\partial_t u\left(\frac{T_0}{2}, \cdot\right)\|_{C^N} \leq C_N \varepsilon. \tag{4.1}$$

Then we can take $\left(u\left(\frac{T_0}{2}, x\right), \partial_t u\left(\frac{T_0}{2}, x\right)\right)$ as the new initial data to solve (1.2) from $t = \frac{T_0}{2}$.

Now we use the standard Picard iteration to prove Theorem 1.1. Let $u_{-1} \equiv 0$, and for $k = 0, 1, 2, 3, \ldots$, let $u_k$ be the solution of the following equation

$$\begin{cases}
\partial_t^2 u_k - t \partial_x^2 u_k = F_p(t, u_{k-1}), & (t, x) \in \left(\frac{T_0}{2}, \infty\right) \times \mathbb{R}, \\
u_k\left(\frac{T_0}{2}, x\right) = u\left(\frac{T_0}{2}, x\right), & \partial_t u_k\left(\frac{T_0}{2}, x\right) = \partial_t u\left(\frac{T_0}{2}, x\right).
\end{cases}$$

For $p > p_{\text{crit}}$, we can fix a number $\gamma$ satisfying

$$\gamma < \frac{1}{p(p+1)},$$

$$\gamma < \frac{1}{6} - \frac{5}{6(p+1)},$$

and

$$(p - 1) \gamma + \frac{1}{6} > \frac{5}{3(p+1)}.$$

Set

$$M_k = \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\gamma u_k \right\|_{L^q\left(\left[\frac{T_0}{2}, \infty\right) \times \mathbb{R}\right)},$$

$$N_k = \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^\gamma (u_k - u_{k-1}) \right\|_{L^q\left(\left[\frac{T_0}{2}, \infty\right) \times \mathbb{R}\right)},$$

where $q = p + 1$. By (4.1) and Theorem 2.1, we know that there exists a constant $C_0 > 0$ such that

$$M_0 \leq C_0 \varepsilon.$$

Notice that for $j, k \geq 0$,

$$\begin{cases}
\partial_t^2 (u_{k+1} - u_{j+1}) - t \partial_x^2 (u_{k+1} - u_{j+1}) = V(u_k, u_j)(u_k - u_j), \\
u_{k+1} - u_{j+1}\left(\frac{T_0}{2}, x\right) = 0, & \partial_t (u_{k+1} - u_{j+1})\left(\frac{T_0}{2}, x\right) = 0,
\end{cases}$$

where

$$|V(u_k, u_j)| \leq \begin{cases}
C(|u_k| + |u_j|)^{p-1} & \text{if } t \geq T_0, \\
C(1 + |u_k| + |u_j|)^{p-1} & \text{if } \frac{T_0}{2} \leq t \leq T_0.
\end{cases}$$
Then applying Theorem 3.2 and Hölder’s inequality yields that for \( q = p + 1 \),
\[
\left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} (u_{k+1} - u_j) \right\|_{L^q([\frac{T_0}{2}, \infty) \times \mathbb{R})}
\leq C \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} V(u_k, u_j) (u_k - u_j) \right\|_{L^{\frac{q}{p}}([\frac{T_0}{2}, \infty) \times \mathbb{R})}
\leq C \left\{ \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} (1 + |u_k| + |u_j|) \right\|_{L^q([T_0, \infty) \times \mathbb{R})}
+ \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} (|u_k| + |u_j|) \right\|_{L^{q}(\mathbb{R} \times \mathbb{R})} \right\}^{p-1}
\times \left\{ \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} (u_k - u_j) \right\|_{L^q([\frac{T_0}{2}, \infty) \times \mathbb{R})} \right\}^{p-1}
\leq C (C_1 T_0^\frac{1}{q} + M_k + M_j)^{p-1} \left\| \left( (\phi(t) + M)^2 - |x|^2 \right)^{\gamma} (u_k - u_j) \right\|_{L^q([\frac{T_0}{2}, \infty) \times \mathbb{R})}.
\] (4.2)

If \( j = -1 \), then \( M_j = 0 \), and we conclude that from (4.2)
\[
M_{k+1} \leq M_0 + \frac{M_k}{2} \quad \text{for} \quad C (C_1 T_0^\frac{1}{q} + M_k)^{p-1} \leq \frac{1}{2}.
\]
This yields that
\[
M_k \leq 2M_0 \quad \text{if} \quad C (C_1 T_0^\frac{1}{q} + C_0 \varepsilon)^{p-1} \leq \frac{1}{2}.
\]
Thus we get the boundedness of \( \{u_k\} \) in the space \( L^q(\mathbb{R}^{1+1}) \) when the fixed constant \( T_0 \) and \( \varepsilon > 0 \) are sufficiently small. Similarly, we have
\[
N_{k+1} \leq \frac{1}{2} N_k,
\]
which derives that there exists a function \( u \in L^q(\left[\frac{T_0}{2}, \infty) \times \mathbb{R}\right) \) such that \( u_k \to u \in L^q(\left[\frac{T_0}{2}, \infty) \times \mathbb{R}\right) \). In addition, by the uniform boundedness of \( M_k \) and the computations above, one easily obtains
\[
\| F_p(t, u_{k+1}) - F_p(t, u_k) \|_{L^{\frac{q}{p}}([\frac{T_0}{2}, \infty) \times \mathbb{R})}
\leq C \| u_{k+1} - u_k \|_{L^q([\frac{T_0}{2}, \infty) \times \mathbb{R})}
\leq C \phi \left( \frac{T_0}{4} \right)^{-\gamma} N_k
\leq C 2^{-k}.
\]
Therefore \( F_p(t, u_k) \to F_p(t, u) \) in \( L^{\frac{q}{p}}([\frac{T_0}{2}, \infty) \times \mathbb{R}) \) and hence \( u \) is a weak solution of (1.2) in the sense of distributions.

**Case II.** \( p \geq p_0 = 9 \)

In this case, Theorem 1.1 can be achieved by completely analogous method in Theorem 1.2 of [9]. we omit it here.

Combining **Case I** and **Case II**, we complete the proof of Theorem 1.1.
5 Proof of Theorem 1.2.

In this section, we shall show Theorem 1.2. Motivated by [26], we introduce the function $G(t) = \int_{\mathbb{R}} u(t, x) \, dx$. By some estimates from [9], we can obtain a Riccati-type differential inequality for $G(t)$ so that blowup of $G(t)$ is deduced from the following result (see Lemma 4 of [21]):

**Lemma 5.1.** Suppose that $G \in C^2([a, b]; \mathbb{R})$ and, for $a \leq t < b$,

\[
G(t) \geq C_0 (R + t)^\alpha, \quad (5.1)
\]

\[
G''(t) \geq C_1 (R + t)^{-q} G(t)^p, \quad (5.2)
\]

where $C_0$, $C_1$, and $R$ are some positive constants. Suppose further that $p > 1$, $\alpha \geq 1$, and $(p - 1) \alpha \geq q - 2$. Then $b$ is finite.

In view of supp $u_i \subseteq (-M, M)$ $(i = 0, 1)$ and the finite propagation speed of solutions to hyperbolic equations, one has that, for any fixed $t > 0$, the support of $u(t, \cdot)$ with respect to the variable $x$ is contained in the interval $(-M - \phi(t), M + \phi(t))$, where $\phi(t) = \frac{2}{3} t^{\frac{3}{2}}$. Then it follows from an integration by parts that

\[
G''(t) = \int_{\mathbb{R}} |u(t, x)|^p \, dx \geq \left( \int_{|x| \leq M + \phi(t)} |u(t, x)| \, dx \right)^p \geq C(M + t)^{-\frac{3}{2} (p-1)} |G(t)|^p,
\]

which means that $G(t)$ fulfills inequality (5.2) with $q = \frac{3}{2} (p - 1)$ (once inequality (5.1) has been verified, we then know that $G$ is positive).

To establish (5.1), we will introduce a suitable test function. The modified Bessel function is

\[
K_\nu(t) = \int_0^\infty e^{-t \cosh z} \cosh(\nu z) \, dz, \quad \nu \in \mathbb{R},
\]

which satisfies

\[
\left( t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - (t^2 + \nu^2) \right) K_\nu(t) = 0, \quad t > 0.
\]

From page 24 of [5], we have

\[
K_\nu(t) = \sqrt{\frac{\pi}{2t}} e^{-t} \left( 1 + O(t^{-1}) \right) \quad \text{as } t \to \infty,
\]

provided that Re $\nu > -1/2$. Set

\[
\lambda(t) = C_1 t^{\frac{1}{2}} K_{\frac{3}{2}} \left( \frac{2}{3} t^{\frac{3}{2}} \right), \quad t > 0,
\]

where the constant $C_1 > 0$ is chosen so that $\lambda(t)$ satisfies

\[
\begin{cases}
\lambda''(t) - t \lambda(t) = 0, & t \geq 0 \\
\lambda(0) = 1, \quad \lambda(\infty) = 0.
\end{cases}
\]

Introduce the test function $\psi$ with

\[
\psi(t, x) = \lambda(t) \varphi(x),
\]
where $\varphi = e^x$. Let

$$G_1(t) = \int_{\mathbb{R}} u(t, x) \psi(t, x) \, dx.$$  \hspace{1cm} (5.7)

Then

$$G''(t) = \int_{\mathbb{R}} |u(t, x)|^p \, dx \geq \frac{|G_1(t)|^p}{\left( \int_{|x| \leq M + \phi(t)} \psi(t, x) \frac{p}{p-1} \, dx \right)^{p-1}}.$$  \hspace{1cm} (5.8)

Since the function $\varphi(x) > 0$ holds for all $x \in \mathbb{R}$, one can repeat the proof of Lemma 2.3 in [9] with little modification to get

**Lemma 5.2.** Under the assumptions of Theorem 1.2 there exists a $t_0 > 0$ such that

$$G_1(t) \geq Ct^{-\frac{1}{4}}, \quad t \geq t_0.$$  \hspace{1cm} (5.9)

Based on Lemma 5.2, we are now able to prove Theorem 1.2.

**Proof of Theorem 1.2.** By (5.3) and (5.4), we have that

$$\lambda(t) \sim t^{-\frac{1}{4}} e^{-\phi(t)} \quad \text{as} \quad t \to \infty.$$  

Next we estimate the denominator $\left( \int_{|x| \leq M + \phi(t)} \psi(t, x) \frac{p}{p-1} \, dx \right)^{p-1}$ in (5.8). Note that

$$\left( \int_{|x| \leq M + \phi(t)} \psi(t, x) \frac{p}{p-1} \, dx \right)^{p-1} = \lambda(t)^p \left( \int_{|x| \leq M + \phi(t)} \varphi(x) \frac{p}{p-1} \, dx \right)^{p-1}$$

and

$$|\varphi(x)| \leq Ce^{|x|}.$$  

Then

$$\int_{|x| \leq M + \phi(t)} \varphi(x) \frac{p}{p-1} \, dx \leq C \int_0^{M+\phi(t)} \rho \frac{p}{p-1} \, d\rho + C \int_{M+\phi(t)}^{M+\phi(t)} \rho \frac{p}{p-1} \, d\rho$$

$$\leq C e^{M+\phi(t)} \rho \frac{p}{p-1} (M+\phi(t)) \leq C e^{M+\phi(t)} (M+\phi(t))$$

and

$$\left( \int_{|x| \leq M + \phi(t)} \psi(t, x) \frac{p}{p-1} \, dx \right)^{p-1} \leq C t^{-\frac{p}{4}} e^{-p\phi(t)} e^{p(M+\phi(t))}$$

$$\leq C t^{-\frac{p}{4}}.$$  \hspace{1cm} (5.10)

Therefore, it follows from (5.8) and (5.10) that, for $t \geq t_0$,

$$G''(t) \geq ct^{-\frac{p}{4}}.$$  \hspace{1cm} (5.11)

If $-\frac{p}{4} > -1$, then $p < 4$. In this case, we have

$$G(t) \geq (M + t)^{2-\frac{p}{4}}.$$
and further get $\alpha = 2 - \frac{p}{4}$ in (5.1). Hence the condition $(p - 1) \alpha \geq q - 2$ is written as

$$(p - 1) \left(2 - \frac{p}{4}\right) > \frac{3}{2}(p - 1) - 2,$$

which is equivalent to

$$p^2 - 3p - 6 < 0 \quad \text{(5.12)}$$

The positive root of $p^2 - 3p - 6 = 0$ is

$$p_1 = \frac{3 + \sqrt{33}}{2}.$$

It follows from (5.12) that $p < p_1$. Direct computation shows $p_1 > 4$. Thus, by Lemma 5.1 we know that the weak solution $u$ of problem (1.1) will blow up in finite time when $1 < p < 4$.

While if $-\frac{p}{4} \leq -1$ or $p \geq 4$, we have

$$G(t) \geq C(M + t),$$

then we can take $\alpha = 1$ in (5.1) and further obtain

$$(p - 1) > \frac{3}{2}(p - 1) - 2 \implies p < p_2 = 5,$$

which means that the solution $u$ of problem (1.1) will blow up for $4 \leq p < 5$ in terms of Lemma 5.1.

Collecting these results above, we complete the proof of Theorem 1.2. □

References

[1] J. Barros-Neto, I.M. Gelfand, *Fundamental solutions for the Tricomi operator. I, II, III*, Duke Math. J. 98 (1999), 465-483; 111 (2002), 561-584; 117 (2003), 385-387.

[2] M. Beals, *Singularities due to cusp interactions in nonlinear waves*, in: Nonlinear Hyperbolic Equations and Field Theory, Lake Como, 1990, in: Pitman Res. Notes Math. Ser., vol. 253, Longman Sci. Tech., Harlow, 1992, 36-51.

[3] L. Bers, *Mathematical aspects of subsonic and transonic gas dynamics*, Surveys in Applied Mathematics, 3. Wiley, New York; Chapman and Hall, London, 1958.

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions, Vol. 1*, McGraw-Hill, New York, 1953.

[5] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions, Vol. 2*, McGraw-Hill, New York, 1953.

[6] R.T. Glassey, *Blow-up theorems for nonlinear wave equations*, Math. Z. 132 (1973), 183-203.

[7] V. Georgiev, H. Lindblad, C.D. Sogge, *Weighted Strichartz estimates and global existence for semi-linear wave equations*, Amer. J. Math. 119 (1997), 1291-1319.

[8] D.K. Gvazava, *The global solution of the Tricomi problem for a class of nonlinear mixed differential equations*, Differential Equations 3 (1967), 1-4.
[9] Daoyin He, Ingo Witt, Huicheng Yin, *On the global solution problem of semilinear generalized Tricomi equations, I*, Calc. Var. Partial Differential Equations 56 (2017), No. 2, 1-24.

[10] Daoyin He, Ingo Witt, Huicheng Yin, *On the global solution problem of semilinear generalized Tricomi equations, II*, arXiv:1611.07606, Preprint, 2016.

[11] Daoyin He, Ingo Witt, Huicheng Yin, *On semilinear Tricomi equations with critical exponents or in two space dimensions*, J. Differential Equations 263 (2017), no. 12, 8102-8137.

[12] H. Lindblad, C.D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. 130 (1995), 357-426.

[13] D. Lupo, C.S. Morawetz, K.R. Payne, *On closed boundary value problems for equations of mixed elliptic-hyperbolic type*, Comm. Pure Appl. Math. 60 (2007), no. 9, 1319-1348.

[14] D. Lupo, K.R. Payne, *Spectral bounds for Tricomi problems and application to semilinear existence and existence with uniqueness results*, J. Differential Equations 184 (2002), no. 1, 139-162.

[15] D. Lupo, K.R. Payne, *Conservation laws for equations of mixed elliptic-hyperbolic and degenerate types*, Duke Math. J. 127 (2005), no. 2, 251-290.

[16] D. Lupo, K.R. Payne, *Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types*, Comm. Pure Appl. Math. 56 (2003), no. 3, 403-424.

[17] C.S. Morawetz, *Mixed equations and transonic flow*, J. Hyperbolic Differ. Eqs. 1 (2004), no. 1, 1-26.

[18] Zhuoping Ruan, Ingo Witt, Huicheng Yin, *On the existence and cusp singularity of solutions to semilinear generalized Tricomi equations with discontinuous initial data*, Commun. Contemp. Math. 17 (2015), 1450028 (49 pages).

[19] Zhuoping Ruan, Ingo Witt, Huicheng Yin, *On the existence of low regularity solutions to semilinear generalized Tricomi equations in mixed type domains*, J. Differential Equations 259 (2015), 7406-7462.

[20] Zhuoping Ruan, Ingo Witt, Huicheng Yin, *On the existence of solutions with minimal regularity for semilinear generalized Tricomi equations*, Pacific J. Math. 296 (2018), No.1, 181-226.

[21] T. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations 52 (1984), 378-406.

[22] K. Taniguchi, Y. Tozaki, *A hyperbolic equation with double characteristics which has a solution with branching singularities*, Math. Japon 25 (1980), 279-300.

[23] K. Yagdjian, *A note on the fundamental solution for the Tricomi-type equation in the hyperbolic domain*, J. Differential Equations 206 (2004), 227-252.

[24] K. Yagdjian, *Global existence for the n-dimensional semilinear Tricomi-type equations*, Comm. Partial Diff. Equations 31 (2006), 907-944.

[25] K. Yagdjian, *The self-similar solutions of the Tricomi-type equations*, Z. angew. Math. Phys. 58 (2007), 612-645.
[26] B. Yordanov, Q.-S. Zhang. *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. 231 (2006), 361-374.