Relativistic fluctuating hydrodynamics

Esteban Calzetta*

IAFE and Physics Department, UBA, Buenos Aires, Argentina

We derive the formulae of fluctuating hydrodynamics appropriate to a relativistically consistent divergence type theory, obtaining Landau - Lifshitz fluctuating hydrodynamics as a limiting case.

I. INTRODUCTION

By applying the fluctuation - dissipation theorem [1] to the Navier - Stokes equation, Landau and Lifshitz developed long ago the standard theory of fluctuating hydrodynamics [2][3][4]. This theory is both appealing and successful, and it is consistent with the fluctuating hydrodynamics which is derived if fluctuation - dissipation theory is first applied to the Boltzmann equation [3], and then the hydrodynamic limit is taken by conventional means [3]. It has also been applied to black hole fluctuations and vacuum decay [7]. The simplest relativistic generalization of Navier - Stokes theory being the Eckart theory of dissipative fluids [3] (by simplicity, in this paper we shall not discuss the related Landau - Lifshitz theory [3], nor other ‘first order’ theories [9]), it is only natural to provide a relativistic version of fluctuating hydrodynamics by applying fluctuation - dissipation theory to it; this step has been taken, and the results applied to such fields as cosmology [10]. Trouble is, Eckart theory has strong drawbacks as a relativistic theory, among them lack of stable solutions and acausal propagation of perturbations [11][12]. This paper asks what changes must be introduced in Landau - Lifshitz fluctuating hydrodynamics if the Eckart theory is replaced by a suitable relativistic theory.

Unfortunately, there is not a single causal version of relativistic real fluid dynamics as compelling as the Eckart and Landau - Lifshitz theories [12][13], the main contenders

*E-mail address: calzetta@df.uba.ar
being the extended thermodynamics theories (ETTs) \[14\] \[15\] \[16\] and the divergence type theories (DTTs) \[17\] \[18\], both of which draw some support from relativistic kinetic theory \[19\] \[20\] \[21\] (see \[22\] for ETTs, \[23\] for DTTs). In particular the ETT framework has been extensively applied to cosmology \[24\]. The relationship between these two approaches is not yet well understood. For concreteness, we shall adopt the DTT framework, which is more adapted to a rigorous statement of results.

It ought to be said that the issues raised in this paper are to some extent academic, since there are strong reasons to believe that in any case the Eckart type theories are a good phenomenological description of any relativistic fluid \[25\] \[26\] \[27\]. However, it is to be expected that small differences will become important when the theory is pushed to its limits, such as the relativistic theory of dissipative superfluids (see \[28\] \[29\] \[30\] for a ETT approach to this problem, \[31\] for a DTT one, and \[32\] for a different perspective), the use of relativistic hydrodynamics as a phenomenological approximation to semiclassical cosmological models \[33\], or the emergence of hydrodynamic descriptions as a preferred set of decoherent histories \[34\] \[35\]. It is with these applications in mind that the following considerations were developed.

The next section provides a review of relativistic fluid dynamics; its aim, of course, is not to substitute the classic introductions such as refs. \[8\] \[12\] \[18\], but rather to have all the relevant results in one place and notation. We then give a primer on Landau - Lifshitz fluctuation theory, and develop the subject of fluctuations in DTTs.

II. RELATIVISTIC FLUID DYNAMICS

A. Basic thermodynamics

Consider a system described by extensive quantities like entropy, energy, volume, particle number and momentum \(S, U, V, N, \vec{P}\) and intensive ones like temperature, pressure, chemical potential and velocity \(T, p, \mu, \vec{v}\).
We build a covariant theory by adopting the following rules:

a) Intensive quantities \((T, p, \mu)\) are associated to scalars, which represent the value of the quantity at a given event, as measured by an observer at rest with respect to the fluid.

b) Extensive quantities \((S, V, N)\) are associated to vector currents \(S^a, u^a, N^a\) (we assume MTW conventions, \(c = 1\), signature -++++, and latin indexes go from 0 to 3 \([36]\)), such that given a time like surface element \(d\Sigma_a = n_a d\Sigma\), then \(-X^a d\Sigma_a\) is the amount of quantity \(X\) within the volume \(d\Sigma\) as measured by an observer with velocity \(n^a\). If further the quantity \(X\) is conserved, then \(X^a_{;a} = 0\). The quantity \(u^a\) associated to volume is the fluid four-velocity, and obeys the additional constraint \(u^2 = -1\).

c) Energy and momentum are combined into a single extensive quantity and associated to a tensor current \(T^{ab}\). The energy current, properly speaking, is \(U^a = -T^{ab} u_b\).

The entropy current \(S^a\) is given by

\[
T S^a = -T^{ab} u_b + pu^a - \mu N^a,
\]

which we rewrite as

\[
S^a = \Phi^a - \beta^b T^{ab} - \alpha N^a
\]

where we introduced the affinity \(\alpha = \mu/T\), the thermodynamic potential \(\Phi^a = p\beta^a\), and the inverse temperature vector \(\beta^a = T^{-1} u^a\). The Gibbs-Duhem relation becomes

\[
TdS^a = -d(T^{ab} u_b) + pdu^a - \mu dN^a
\]

We now introduce the concept of a ‘perfect fluid’, as a system whose energy-momentum tensor takes the form \(T^{ab} = \rho u^a u^b + p \Delta^{ab}\), where \(\Delta^{ab} = u^a u^b + g^{ab}\) (observe that \(\rho\) must be the energy density seen by an observer moving with the fluid). Since we must have \(u_a du^a = 0\), we conclude \(pdu^a - T^{ab} du_b = 0\). Thus for a perfect fluid, the Gibbs-Duhem relation reads

\[
dS^a = -\alpha dN^a - \beta_b dT^{ab}
\]

and
\[ d\Phi^a = N^a d\alpha + T^{ab} d\beta_b \]

which is how the thermodynamic potential got its name. Equivalently

\[ \frac{\partial \Phi^a}{\partial \alpha} = N^a; \quad \frac{\partial \Phi^a}{\partial \beta_b} = T^{ab} \quad (1) \]

The symmetry of the energy momentum tensor implies that the thermodynamic potential is itself a gradient

\[ \Phi^a = \frac{\partial \chi_p}{\partial \beta_a} \quad (2) \]

where \( \chi_p \) is the so called generating function. In a covariant theory, \( \chi_p \) may only depend on the scalars \( \alpha \) and \( \beta_a \); working out the corresponding derivatives, we conclude that necessarily \( N^a = nu^a \), where \( n \) is the particle number density seen by a comoving observer.

We conclude with a word on equilibrium states. Suppose the fluid departs from equilibrium by a fluctuation \( \delta N^a \), \( \delta T^{ab} \), consistent with the conservation laws but otherwise arbitrary. Then the change in entropy production is

\[ \delta S_{;a}^a = -\alpha_{;a} \delta N^a - \beta_{b;a} \delta T^{ab} \]

But for a true equilibrium state the entropy must be stationary, and so we must have \( \alpha_{;a} = \beta_{(b;a)} = 0 \) \( \square \) (here and henceforth, brackets stand for symmetrization). Thus the affinity must be constant, and the inverse temperature vector must be Killing.

**B. Eckart’s theory of real fluids**

We wish now to describe a weakly dissipative fluid. Following Eckart, we shall base our description on the same set of variables than for an ideal fluid, namely, \( T^{ab} \) and \( N^a \), only now the energy - momentum tensor is given as

\[ T^{ab} = \rho u^a u^b + p \Delta^{ab} + \tau^{ab} \quad (3) \]

where it is assumed that the viscous stress \( \tau^{ab} \) obeys
\[ \tau^{ab} u_a u_b = 0 \]

and \( p \) is defined to be the same function of \( \rho \) and \( n \) than in the ideal case. Assuming that the entropy current and thermodynamic potential are also the same, we get the entropy production rate

\[ S^a_{,a} = -\tau^{ab} \beta_{a;b} \]  

(4)

Now expand

\[ \beta_{a;b} = \left( \frac{1}{T^2} \right) \left[ -T_b u_a + Tu_{a;b} \right] \]

\[ \tau^{ab} = u^a q^b + u^b q^a + \pi \Delta^{ab} + \pi_{ab} \]

(5)

where

\[ u^a q_a = u_a \pi^{ab} = \pi_{a}^{a} = 0 \]

to get

\[ S^a_{,a} = \left( \frac{-1}{T} \right) \left[ \frac{q^a}{T} \left( T_{,a} + Tu^b_{a;b} \right) + \pi u^a_{,a} + \pi_{ab} u_{a;b} \right] \]  

(6)

The principle of positive entropy production is satisfied if

\[ q^a = -\kappa \Delta^{ab} (T_{,b} + Tu^c_{b;c}) \]  

(7)

\[ \pi = -\zeta_{BV} u^a_{,a} \]  

(8)

\[ \pi_{ab} = -\eta \Delta^{ac} \Delta^{bd} \left( u_{c;d} + u_{d;c} - \frac{2}{3} u^e_{;e} g_{cd} \right) \]  

(9)

representing heat flux, bulk and shear viscosity, respectively. With these constitutive relations, energy-momentum conservation yields a straightforward, covariant generalization of the Navier-Stokes theory.
\( \tau^{ab} \) can be written directly in terms of the covariant derivatives of the inverse temperature vector as

\[
\tau^{ab} = -B^{abcd} \beta_{(cd)}
\]

where

\[
B^{abcd} = -4\kappa T^2 P_v^{abcd} + 4\eta T P_T^{abcd} + 6\zeta B V P_S^{abcd}
\]

\[
P_s^{abcd} = \frac{1}{3} \Delta^{ab} \Delta^{cd}
\]

\[
P_v^{abcd} = \left( \frac{-1}{2} \right) \left[ u^a u^c \Delta^{bd} + u^a u^d \Delta^{bc} + u^b u^c \Delta^{ad} + u^b u^d \Delta^{ac} \right]
\]

\[
P_T^{abcd} = \left( \frac{1}{2} \right) \left[ \Delta^{ac} \Delta^{bd} + \Delta^{ad} \Delta^{bc} - \frac{2}{3} \Delta^{ab} \Delta^{cd} \right]
\]

Observe that \( B^{abcd} = B^{cdab} \), and that the \( P \)'s are actually projection operators, that is, \( P^2 = P \) in all three cases, while the product of different \( P \)'s vanishes.

Eckart’s theory is such a compelling generalization of non relativistic dissipative hydrodynamics that it is a pity it doesn’t work. The resulting equations allow for non causal propagation, and ipso facto all their solutions are unstable \([9][11]\).

While it is fairly clear that Eckart’s theory (and the closely related Landau’s theory as well) must be rejected, it is not clear at all what should be their replacement. The so-called Israel - Stewart or ‘second order’ type theories \([14][15][16]\) perform much better with regards to both causality and stability, while keeping much of the appeal of the Eckart framework, but still lack a rigorous proof of consistency.

In the following, we shall adopt instead the Geroch - Lindblom ‘divergence type’ description of a relativistic real fluid \([18]\). The resulting theory is further removed from direct thermodynamic intuition than the Eckart proposal, but does allow for a rigorous proof of causality and stability.
C. Divergence type real fluids

The failure of the Eckart approach to real fluids may be attributed to two unwarranted assumptions, namely, that the real fluid could be described within the same set of variables and with the same entropy current than its perfect counterpart. As a matter of fact, all that equilibrium thermodynamics suggests is that, whatever extra variables are brought in to describe the non equilibrium state, they must vanish in equilibrium, and the entropy current must match its equilibrium value up to first order in the deviations from equilibrium.

According to Geroch and Lindblom [18], description of a nonequilibrium state requires, besides the particle current and energy momentum tensor, a new third order tensor $A_{abc}$, obeying an equation of motion of divergence type

$$A^a_{bc;a} = I_{bc}$$

and

$$A_{abc} = A_{acb}; \quad A^a_{bb} = 0; \quad I^a_a = 0$$

The entropy current is enlarged to read

$$S^a = \Phi^a - \beta_b T^{ab} - \alpha N^a - A^{abc} \zeta_{bc}$$

$\zeta_{ab}$ vanishes identically in equilibrium, it is symmetric, and $\zeta^a_a = 0$. We further require entropy and the thermodynamic potential to be algebraic functions of their arguments.

The condition that the conservation laws and Eq. (15) imply positive entropy production demands that Eqs. (11) hold; in particular, the thermodynamic potential derives from a generating function as in Eq. (2). The thermodynamic potential is also allowed to depend on the new tensor $\zeta_{ab}$; actually

$$\frac{\partial \Phi^a}{\partial \zeta_{bc}} = A^{abc}$$

Thus the entropy production
\[ S_{ab}^a = -I^{bc} \zeta_{bc} \]  

(18)

We obtain positivity by making the former linearly dependent on the latter.

The Eckart theory is actually a particular case of Geroch and Lindblom’s [18]. Write the generating functional as

\[ \chi_E = \chi_p + \left( \frac{1}{2} \right) \zeta_{ab} u^a u^b \]  

(19)

where \( \chi_p \) has the same form than for a perfect fluid.

Recalling

\[ \frac{\partial u^a}{\partial \beta_b} = \beta^{-1} \Delta^{ab}; \quad \frac{\partial T}{\partial \beta_a} = T^2 u^a \]  

(20)

we get the thermodynamic potential

\[ \Phi^a_E = \Phi^a_p + \beta^{-1} \Delta^{ab} \zeta_{bc} u^c \]  

(21)

The particle current is the same than for a perfect fluid, and the energy-momentum tensor is again of the form \( T^{ab} = T_p^{ab} + \tau^{ab} \), where the dissipative tensor

\[ \tau^{ab} = C^{abcd} \zeta_{cd} \]  

(22)

\[ C^{abcd} = T^2 \left\{ -P^{abcd}_V + P^{abcd}_{TT} + \frac{1}{6} \Delta^{ab} \left[ 4u^c u^d + g^{cd} \right] \right\} \]

(in writing this equation, we have used that \( \zeta_a^a = 0 \)). Observe that \( C \) is not symmetric. This equations allows us to define the viscous stresses in terms of \( \zeta_{ab} \), namely

\[ \eta^a = T^2 \Delta^{ab} \zeta_{bc} u^c; \quad \pi = \left( \frac{4}{3} \right) T^2 \zeta_{ab} u^a u^b; \]

\[ \pi^{ab} = T^2 \left[ \Delta^{ac} \Delta^{bd} + \Delta^{ad} \zeta_{bc} \Delta^{cd} - \frac{2}{3} \Delta^{ab} \Delta^{cd} \right] \zeta_{cd} \]  

(23)

The new tensor is given by

\[ A_{E}^{abc} = \left( \frac{T}{2} \right) \left[ \Delta^{ab} u^c + \Delta^{ac} u^b \right] \]  

(24)

and its divergence
\[ A_{E; a}^{abc} = \left( \frac{1}{2} \right) \left( T_{a} + T u^d u_{a;d} \right) \left[ \Delta^{ab} u^c + \Delta^{ac} u^b \right] + \left( \frac{T}{2} \right) \left[ \frac{2}{3} u^a_{;a} \left( 4 u^b u^c + g^{bc} \right) + \Delta^{ab} \left( u_{d;a} + u_{a;d} - \frac{2}{3} u^e_{;e} \Delta_{ad} \right) \Delta^{dc} \right] \] (25)

Observe that we can write

\[ A_{E; e}^{ecd} = C^{T cdab} \beta_{(a;b)} \] (26)

where \( C^{T cdab} = C^{abcd} \).

Let us write Eqs. (15, 22 and 26) in shorthand as

\[ \nabla A_E = I \]
\[ \tau = C \zeta \]
\[ \nabla A_E = C^T \nabla \beta \]

These three equations ought to be equivalent to Eq. (10)

\[ \tau = -B \nabla \beta \]

To obtain this, we must provide a linear relationship

\[ I^{ab} = -D^{abcd} \zeta_{cd} \] (27)

or in shorthand

\[ I = -D \zeta \] (28)

where

\[ D = C^T B^{-1} C \] (29)

The inverse must exist, since \( B \) is positive definite. This equation may be inverted, to yield
\[ B = CD^{-1}C^T \]  

(30)

where we assume that all matrices have inverses, even the non symmetric ones.

Written in full

\[ D^{abcd} = \frac{4T^3}{3\zeta_{BV}}P_{ST}^{abcd} - \frac{T^2}{2\kappa}P_V^{abcd} + \frac{T^3}{\eta}P_{TT}^{abcd} \]  

(31)

where

\[ P_{ST}^{abcd} = \left( \frac{1}{12} \right) \left[ g_{ab} + 4u^a u^b \right] \left[ g_{cd} + 4u^c u^d \right] \]  

(32)

and the other projectors are defined in eqs. (13 and 14). Observe that \( D \) is a symmetric operator.

This concludes the setting of Eckart’s theory in a DTT framework.

**D. Causality and divergence type theories**

While the Eckart theory belongs to the DTT class, it is not a ‘good’ theory, as we shall presently see. To investigate what conditions must a suitable DTT theory satisfy, we must discuss further the issues of causality and stability.

The main advantage of the dissipative type theory framework is that the condition of causality may be expressed in a remarkably simple form. Let us introduce the symbol \( \zeta^A \) to denote the triad \((\alpha, \beta^a, \zeta^{ab})\), \( A^a_B \) the triad \((N^a, T^{ab}, A^{abc})\), and \( I_B \) the triad \((0, 0, I_{ab})\). Then the theory is summed up in the equations (cfr. Eqs. (1,7,8,13))

\[ A^a_B = \frac{\partial \Phi^a}{\partial \zeta^B} \]

\[ S^a = -I_B \zeta^B \]

\[ A^a_{B,\alpha} = I_B \]

The equations of motion can also be written as
\[ M_{BC}^a \zeta_a^C = I_B \]

where

\[ M_{BC}^a = M_{CB}^a = \frac{\partial^2 \Phi}{\partial \zeta^B \partial \zeta^C} \quad (33) \]

Then the causality condition is that the quadratic form \( M_{BC}^a w_a \) be negative definite for all future directed timelike vectors \( w_a \), or, equivalently, that for any displacement \( \delta \zeta^A \) from an equilibrium state, the vector \( Q^a = M_{BC}^a \delta \zeta^B \delta \zeta^C \) be timelike and future oriented [18].

In order to see the meaning of this condition, it is interesting to observe that under any such displacement, the change in the entropy current is precisely \( \delta S^a = -2Q^a \), and therefore the change in entropy density, as seen by an observer with velocity \( v_a \), is \( \delta s = 2v_a Q^a \). Thus the causality condition states that the entropy will be reduced by any displacement from equilibrium. This goes beyond mere thermodynamic stability, which only requires entropy reduction at constant energy and particle number densities.

For imperfect fluids, it is necessary to consider the full quadratic form, that is, as a functional of \( \delta \alpha, \delta \beta^a \) and \( \delta \zeta^{ab} \). It is easy to see that the Eckart theory cannot possibly meet the test, since a whole diagonal block is missing. Geroch and Lindblom [18] have suggested a simple way of constructing causal theories close to Eckart’s. The idea is to write down a generating functional of the form (cfr. Eq. (19))

\[ \chi = \chi_E + \gamma(T) Q(u^a, \zeta^{ab}) \quad (34) \]

where \( Q \) is a positive definite scalar quadratic form on the \( \zeta^{ab} \). Then the thermodynamic potential

\[ \Phi^a = \Phi^a_E + u^a Q T^2 \frac{d \gamma}{dT} + \Delta^{ab} T \frac{\partial Q}{\partial u^b} \quad (35) \]

and, provided

\[ \frac{d \gamma}{dT} \gg \left| \frac{\gamma}{T} \right| > 0, \quad (36) \]
the second term dominates and guarantees the positivity of the corresponding vector \( Q^a \).

In what follows we shall assume one such extension of Eckart’s theory has been adopted.

This closes our review of causal relativistic hydrodynamics. We now proceed to our main subject, namely, the form the fluctuation - dissipation theorem takes in this context.

III. FLUCTUATION THEORY

A. The classical fluctuation - dissipation theorem

The simplest possible application of fluctuation - dissipation theory relates to an homogeneous system described by time - dependent variables \( x^i \). Let

\[
X_i = -\frac{\delta S}{\delta x^i}
\]

Then the entropy production rate is given by

\[
\dot{S} = -X_i \dot{x}^i
\]

Following common usage, we refer to the \( X \)'s as thermodynamic forces, and the \( \dot{x} \)'s as thermodynamic fluxes. Then the principle of positive entropy production on average is satisfied by posing a linear relationship

\[
\dot{x}^i(t) = -\int dt' \gamma^{ij}(t, t') X_j(t') + \xi^i(t)
\]

where \( \gamma \) is positive definite (Onsager’s reciprocity principle further asserts that \( \gamma^{ij} = \pm \gamma^{ji} \) according to whether \( x^i \) behaves as \( x^j \) under time reversal or not, and we also assume causality). Under equilibrium, we have the average

\[
\langle x^i(t)X_j(t') \rangle = \frac{\delta x^i(t)}{\delta x^j(t')}
\]

which follows from Einstein’s formula relating entropy to fluctuations, and averages are time translation invariant. Therefore

\[
\langle x^i \dot{x}^i \rangle + \langle x^i \dot{x}^j \rangle = 0
\]
If the noise $\xi$ is Gaussian, then

$$\langle x^i \xi^i \rangle = \int dt' \langle \xi^i(t) \xi^k(t') \rangle \frac{\delta x^j(t)}{\delta x^k(t')}$$

From the equations of motion, we conclude that for a given initial condition $x^j(t')$, the solution is written in terms of a single Green function as

$$x^i(t) = G^i_j(t, t') x^j(t') + \int_{t'}^t d\tau \, G^i_j(t, \tau) \xi^j(\tau)$$

where $G^i_j(t^+, t) = \delta^i_j$. Therefore

$$\frac{\delta x^i(t)}{\delta \xi^j(t')} = \frac{\delta x^i(t)}{\delta x^j(t')}$$

(which expresses in symbols Onsager’s insight that the regression of microscopic fluctuations follows the same rules than macroscopic ones)

Thus

$$\langle \xi^i \xi^j \rangle = \left( \frac{1}{2} \right) [\gamma^{ij} + \gamma^{ji}]$$

which is our basic result.

The relationship of forces to fluxes may be inverted to yield

$$X_i = G_{ij} \dot{x}^j + \theta_i$$

(40)

Then it is immediate that

$$\langle \theta_i(t) \theta_j(t') \rangle = \left( \frac{1}{2} \right) [G_{ij} + G_{ji}]$$

We may also wish to introduce new fluxes

$$y = \dot{X}$$

conjugated to forces

$$Y = M^{-1} X.$$

The relationship between fluxes and forces becomes

$$y = RY + \sigma; \quad R = M \gamma M : \quad \sigma = M \xi$$

Then

$$\langle \sigma \sigma \rangle = R$$

This generalized fluctuation dissipation theorem will be relevant in what follows.

We shall now apply this basic framework to Eckart’s theory and to a causal relativistic theory.
B. Fluctuations in Eckart’s theory

In order to derive the spontaneous fluctuations of a Eckart type real fluid, we first regard a fluid theory as an instance of the above example, with the index \( i \) now running over a continuous as well as a discrete range. We also go back to the entropy production rate Eq. (4) and identify \( \tau^{ab} \) as a ‘flux’ multiplying the corresponding ‘force’ \( \beta_{(a,b)} \) (which, as it should, vanishes in equilibrium) (since we shall presently derive these results from a DTT viewpoint, we only give here the somewhat sketchy original Landau argument [2]; the derivation is spelled out in full in Fox and Uhlembeck [4]). We then postulate a Langevin-type relationship [3]

\[
\tau^{ab}(x) = -\int d^4x' \sqrt{-g(x')} B^{abc'd'} \delta^{(4)}(x, x') \beta_{(c'd')(x')} + s^{ab}(x) \tag{42}
\]

where \( \delta^{(4)}(x, x') \) is the covariant four dimensional delta function. The deterministic part is chosen to reproduce the Eckart constitutive relationships.

Since the coefficients are already symmetric, the fluctuation dissipation theorem suggests

\[
\langle s_s^{ab} s_s^{cd} \rangle = B^{abcd} \delta^{(4)}(x, x') \tag{43}
\]

On the other hand, if we decompose the stochastic energy-momentum tensor as in Eq. (44)

\[
s^{ab} = u^a q^b_s + u^b q^a_s + \pi_s \Delta^{ab} + \pi_s^{ab} \tag{44}
\]

we find

\[
\langle q_s^a q_s^c \rangle = 2\kappa T^2 \Delta^{ac} \delta^{(4)}(x, x')
\]

\[
\langle \pi_s^{ab} \pi_s^{cd} \rangle = 2T \eta \left[ \Delta^{ac} \Delta^{bd} + \Delta^{ad} \Delta^{bc} - \frac{2}{3} \Delta^{ab} \Delta^{cd} \right] \delta^{(4)}(x, x')
\]

\[
\langle \pi_s \pi_s \rangle = 2T \zeta_{BV} \delta^{(4)}(x, x')
\]

\[
\langle \pi_s \pi_s^{ab} \rangle = 0
\]
\[ \left\langle q^a \left( \pi_s \Delta^{cd} + \pi^{cd}_s \right) \right\rangle = 0 \quad (45) \]

Which generalize the Landau–Lifshitz formulae to the relativistic regime [3] [10].

We are finally prepared to consider the issue of fluctuations in a relativistically consistent, divergence type theory.

C. Fluctuations in divergence type theories

In a divergence type theory, the entropy flux takes the form (cfr. Eq. (48))

\[ S^a = \Phi^a - \zeta^B A^a_B \]

The definition

\[ A^a_B = \frac{\partial \Phi^a}{\partial \zeta^B} \]

implies the entropy production

\[ S^a_{;a} = -\zeta^B A^a_{B;a} \]

The Langevin-type equations of motion read

\[ A^a_{B;a} = I_B + j_B \quad (46) \]

Where \( j_B \) is the stochastic noise. So far, all formulae in this section have been ultralocal, meaning that \( \Phi^a \) and \( I_B \) are algebraic functions of the field variables at the same point (as opposed, e.g., to depending on their derivatives).

A satisfactory theory must predict vanishing mean entropy production in equilibrium, namely

\[ \left\langle \zeta^B(x) I_B(x) \right\rangle + \left\langle \zeta^B(x) j_B(x) \right\rangle = 0 \quad (47) \]

However, because the coincidence limit of the correlation functions in the left hand side may not be well defined, we must impose this condition in a smeared form, or else, coupling it to an elementary causality consideration, request the stronger condition
\[
\langle \zeta^B(x) I_B(x') \rangle + \langle \zeta^B(x) j_B(x') \rangle = 0
\] (48)

for every spacelike pair \((x, x')\).

In the linear approximation, the first term yields

\[
\langle \zeta^B(x) I_B(x') \rangle = I_{(B,C)} \langle \zeta^B(x) \zeta^C(x') \rangle
\] (49)

We now appeal to the general theory above to make the assumption that the noise is Gaussian, and, since the dynamics is purely local, \textit{white}, meaning that

\[
\langle j_B(x) j_C(x') \rangle = \sigma_{BC} \delta^{(4)}(x, x')
\] (50)

where \(\sigma_{BC}\) is a given matrix. Under these hypothesis we obtain

\[
\langle \zeta^B(x) j_B(x') \rangle = \sigma_{BC} \frac{\partial \zeta^B(x)}{\partial j_C(x')}
\]

Our goal is to derive the matrix \(\sigma_{BC}\), and our result shall be that indeed \(\sigma_{BC} = -I_{(B,C)}\).

To arrive to this end, we must show that, at equal times,

\[
\langle \zeta^B(x) \zeta^C(x') \rangle = \frac{\partial \zeta^B(x)}{\partial j_C(x')}
\] (51)

Let us multiply both sides of this equation by the non singular matrix

\[
M_{AB} = n_a \frac{\partial^2 \Phi^a}{\partial \zeta^A \partial \zeta^B}
\]

Where \(n_a\) is the unit normal to some Cauchy surface containing both \(x\) and \(x'\) (henceforth, \textit{the surface}). In the linear approximation

\[
M_{AB} \zeta^B = \frac{\partial S^0}{\partial \zeta^A}
\]

where \(S^0 = -n_a S^a\) is the entropy density in an adapted coordinate system. In equilibrium, we may apply Einstein’s formula (cfr. Eq. (38)) above, to conclude

\[
M_{AB} \langle \zeta^B(x) \zeta^C(x') \rangle = -\delta^C_A \delta(x, x')
\] (52)
where $\delta(x, x')$ is the three dimensional covariant delta function on the Cauchy surface.

On the other hand

$$M_{AB} \frac{\partial \zeta^B(x)}{\partial j_C(x')} = - \frac{\partial A^0_a(x)}{\partial j_C(x')} \tag{53}$$

where $A^0_a(x) = -n_a A^a_A(x)$. Let us write the equations of motion Eq. (46) as

$$\frac{\partial A^0_A(x)}{\partial t} + L = j_A(x)$$

where $L$ involves the field variables on the surface, but not their normal derivatives, and $\partial/\partial t \equiv n^a \partial/\partial x^a$. Indeed

$$\frac{\partial A^0_A(x)}{\partial j_C(x')} = \delta^C_A \delta(x, x') \tag{54}$$

(at equal times). From Eqs. (52), (53) and (54), we obtain the desired result

$$\langle j_A j_B \rangle = \left( -\frac{1}{2} \right) \left[ \frac{\delta I_A}{\delta \zeta^B} + \frac{\delta I_B}{\delta \zeta^A} \right] \tag{55}$$

Eqs. (46) and (55) are the basic equations incorporating fluctuations to the DTT framework, and are our main result.

While our derivation does not apply to Eckart’s theory, since the matrix $M$ above fails to be non singular, we may consider it as a limiting case of causal DTTs and still apply the final result. Since $I_\alpha = I_\beta = 0$, and in equilibrium $I_{\zeta,\alpha} = I_{\zeta,\beta} = 0$, we get the reassuring result that particle number and energy momentum conservation are not violated. For the remaining equation we have, in the shorthand notation of the previous section

$$C^T \nabla \beta = \nabla A = I + j = -D \zeta + j$$

$$\langle jj \rangle = D$$

The first equation is an algebraic equation for the variables $\zeta^{ab}$. Its solution reads $\zeta = \zeta_{\text{old}} + \zeta_s$, where $\zeta_{\text{old}}$ corresponds to the usual expression in terms of the gradients in temperature and velocity (which may themselves be stochastic) and $\zeta_s = D^{-1} j$ is a purely stochastic component with self correlation $D^{-1}$. For the random viscous stresses we find (cfr. Eq. (22))
\[ s = C\zeta_s \]

and therefore

\[ \langle ss \rangle = CD^{-1}CT \]

But according to Eq. (30), \( CD^{-1}CT \equiv B \), and these are just the Landau - Lifshitz fluctuation formulae Eq. (43).

This deceptively simple derivation should not disguise the fact that the Eckart theory is pathological. The fluctuation formulae Eqs. (44) and (53) can only be expected to yield sensible results in the context of truly causal theories, like those described in Eqs. (34, 35 and 36).

**D. Fluctuations in causal theories**

We finally arrive at our designated goal, namely, the characterization of hydrodynamic fluctuations in a truly consistent relativistic dissipative theory. The problem is that we do not have a unique generalization of Eckart’s theory to rely on, but rather a wide family of seemingly plausible extensions. We shall therefore appeal to Occam’s razor, and concentrate on the simplest possible example, namely, the family of generalizations of Eckart’s theory introduced in equations (34, 35 and 36) above. While we shall appeal to rather drastic approximations to keep things simple, it should be clear that this serves illustration purposes only, and does not affect the validity of the noise correlation formula Eq. (55) above.

A DTT theory is defined by its generating functional and the driving forces \( I^{ab} \) in the equations of motion Eq. (15). Since the shortcomings of Eckart’s theory concern only the first, it is simplest to assume that the driving forces of the full theory remain the same, namely, they are still given by Eqs. (27 and 31) above. The generating functional is modified by adding a new term depending on a positive definite scalar \( Q \) (cfr. eq. (34)) and a function \( \gamma \) of temperature. The simplest choice for the latter is a scale - free power law, \( \gamma \sim T^n \). Since
the restriction eq. (30) excludes the exponent \( n = 1 \), we shall settle for the next available alternative, \( n = 2 \).

Concerning \( Q \), we notice that the Eckart’s theory already contains such a scalar, namely the entropy production rate \( -T^{ab} \zeta_{ab} \), and it is simplest to assume that both are related. We therefore seek a \( Q \) scalar with the structure (cfr. eq. (31), and note that this \( Q \) depends explicitly on \( T \))

\[
Q \sim \left( \frac{1}{4} \right) \left[ \frac{4 \tau_{BV}}{3 \zeta_{BV}} (\zeta P_{ST} \zeta) - \frac{\tau_{HC}}{\kappa T} (\zeta P_{V} \zeta) + \frac{\tau_{SV}}{\eta} (\zeta P_{TT} \zeta) \right]
\]

(56)

The physical meaning of the new \( \tau \) coefficients will become clear soon; for the time being we only remark that all three are assumed to be positive. As a matter of fact, using the Eckart’s theory results as a guide, we know that in the end, contrasting eqs. (22) and (23) with eqs. (8, 9 and 10), we shall obtain \( P_{ST} \zeta \sim \zeta_{BV} \), \( P_{V} \zeta \sim \kappa T \) and \( P_{TT} \zeta \sim \eta \). Now in actual applications, shear viscosity effects are often more important than either bulk viscosity or heat conduction (cfr. ref. [8]). We could therefore retain only the third term, although that would mean our \( Q \) would be merely non negative, rather than positive definite.

We shall appeal to the ”2 \( \gg \) 1 approximation” (see Appendix) to retain only the first of the two new terms in the thermodynamic potential, eq. (53), obtaining

\[
\Phi^a = \Phi_E^a + T^3 u^a \left( 2Q + T \frac{\partial Q}{\partial T} \right)
\]

(57)

where the Eckart potential is still given by eq. (21). Assuming for simplicity that the \( \tau \)’s are independent of \( \alpha \) (we shall of course assume that they are simply constant) the new addition to the thermodynamic potential does not affect the expression for the particle current in terms of field variables. The energy momentum tensor gets a new term, quadratic in the \( \zeta^{ab} \)’s, but we know that this term is small and can be neglected, at least in a first run (see below). The real change comes in the non equilibrium current \( A^{abc} \), which now reads

\[
A^{abc} = A_E^{abc} + T^3 u^a \left[ \frac{4 \tau_{BV}}{3 \zeta_{BV}} (P_{ST} \zeta)^{bc} - \frac{\tau_{HC}}{2 \kappa T} (P_{V} \zeta)^{bc} + \frac{\tau_{SV}}{\eta} (P_{TT} \zeta)^{bc} \right]
\]

(58)

where \( A_E^{abc} \) is the Eckart result eq. (24). Its divergence becomes
\[ \dot{A}_{ia}^{abc} = A_{Eia}^{abc} + T^3 \left[ \frac{4}{3} \tau_{BV} \left( P_{ST} \dot{\zeta} \right)^{bc} - \frac{\tau_{HC}}{2kT} \left( P_{V} \dot{\zeta} \right)^{bc} + \frac{\tau_{SV}}{\eta} \left( P_{TT} \dot{\zeta} \right)^{bc} \right] + R^{bc} \]  

(59)

where \( \dot{\zeta}_{de} = u^a \zeta_{de;a} \). The Eckart term is given in eq. (23), and it is independent of the \( \zeta \)'s. The remainder \( R^{bc} \) is a complicated expression involving products of \( \zeta^{ab} \)'s and derivatives of the temperature and four velocity; it is therefore of second order in departure from equilibrium and can be neglected within present accuracy.

As a consequence of Eq. (59), the third set of equations of motion is no longer merely algebraic; it now reads (cfr. eqs. (27 and 31))

\[ \dot{A}_{Eia}^{abc} + T^3 \left[ \frac{4}{3} \tau_{BV} \left( P_{ST} \dot{\zeta} \right)^{bc} - \frac{\tau_{HC}}{2kT} \left( P_{V} \dot{\zeta} \right)^{bc} + \frac{\tau_{SV}}{\eta} \left( P_{TT} \dot{\zeta} \right)^{bc} \right] = -T^3 \left[ \frac{4}{3} \zeta_{BV} \left( P_{ST} \zeta \right)^{bc} - \frac{1}{2kT} \left( P_{V} \Zeta \right)^{bc} + \frac{1}{\eta} \left( P_{TT} \zeta \right)^{bc} \right] + j^{bc} \]  

(60)

Using the orthogonality properties of the \( P \)'s it is possible to decouple these equations; the simplest case is when all \( \tau \)'s are equal, whereby we simply get

\[ \tau \dot{\zeta}_{ab} + \zeta_{ab} = \left( \frac{-1}{T^3} \right) \left[ \frac{3}{4} \zeta_{BV} P_{ST}^{abcd} - 2kT P_{V}^{abcd} + \eta P_{TT}^{abcd} \right] \{ A_{Ead,e} - j_{ed} \} \]  

(61)

We may also use eqs. (23) and (27) to transform eqs. (60) into a set of Maxwell - Cattaneo equations for the viscous stresses [38]. In any case, the \( \tau \)'s represent the characteristic times in which the \( \zeta^{ab} \)'s (and therefore the viscous stresses as well) relax to their Eckart values. The apparition of new coefficients with the physical meaning of relaxation times is also characteristic of the ETT approach (cfr. [14,15]) and to this extent we may conjecture that both approaches are physically equivalent within this accuracy.

We can see now how the theory works. Because the \( I^{ab} \) are the same, the noise statistics has not changed, and in particular we still have Gaussian white noise. The energy momentum tensor has not changed either, at least as regards the expression of the viscous stresses in terms of the \( \zeta_{ab} \)'s (there is a systematic contribution coming from the nonvanishing expectation value of the new quadratic terms which ought to be included, though). What has changed is that, if the noise - noise correlation is ultralocal, then the stress - stress correlation cannot be, because the solution to eq. (60) will be necessarily non local in the noises.
Of course, we expect these correlations will decay exponentially on a time of order $\tau$, and in the approximation in which these characteristic times are neglected, we recover the Landau - Lifshitz results above.

IV. ACKNOWLEDGMENTS

This work is a spin-off from a larger project involving Oscar Reula and Gabriel Nagy. It is a pleasure to acknowledge the comments and suggestions received from both of them.

This work has been partially supported by the European project CI1-CT94-0004, CONICET, UBA and Fundación Antorchas.

V. APPENDIX

In this appendix, we shall take a closer view into the approximations leading to Eq. (57). The starting point is the generating functional (cfr. Eq. (34))

$$\chi = \chi_E + \gamma (T) Q \left( T, u^a, \zeta^{ab} \right)$$

(62)

where $Q$ is a positive definite scalar quadratic form on the $\zeta^{ab}$. Then the thermodynamic potential (cfr. Eq. (35))

$$\Phi^a = \Phi_E^a + u^a T^2 \frac{d}{dT} (\gamma Q) + \Delta^{ab} T \gamma \frac{\partial Q}{\partial u^b}$$

(63)

If $\gamma = T^2$ and $Q$ is given by Eq. (56), we get

$$\Phi^a = \Phi_E^a + \left( \frac{u^a T^3}{2} \right) \left[ \frac{4 \tau_{BV}}{3 \kappa_{BV}} (\zeta P_{ST} \zeta) - \frac{\tau_{HC}}{2 \kappa T} (\zeta P_V \zeta) + \frac{\tau_{SV}}{\eta} (\zeta P_{TT} \zeta) \right] + \Delta^{ab} T \gamma \frac{\partial Q}{\partial u^b}$$

(64)

Recall also Eqs. (32, 13 and 14).

The basic idea is that the timelike second term will dominate the spacelike third term, thus ensuring negativity of $M_B C w_a$, where $w_a$ is an arbitrary future oriented timelike vector, and $M_{BC}^a$ is the Hessian Eq. (33). If this is true, then the third term in Eq. (64) may be
discarded. Since the second term is larger than the third roughly by a factor 2, we have called this the "2 ≫ 1 approximation".

(This argument does not hold for the heat conduction terms, which in fact are linear in $T$; for them, we must rely directly on the argument below, or else assume that $\tau_{HC}$ has a hidden temperature dependence.)

Of course, it is not hard to compute the third term explicitly, but it is not very illuminating either. Our real interest lies in finding the tensor $A^{abc}$ and its divergence; when we do this, we find that the terms proportional to $u^a$ contribute time derivatives of $\zeta^{ab}$ as well as several extra terms, called $R^{ab}$ in the text, Eq. (59), while the third term will only contribute space derivatives and cross terms containing both the tensor $\zeta^{ab}$ and derivatives of the temperature and fluid velocity. Thus, in physical terms, the approximation involved is that evolution in time is more important than inhomogeneity in space. In the end we recover essentially the ETT equations of motion, which have their own physical motivation, independent of the formal procedure above.

[1] H. Callen and T. Welton, Phys. Rev. 83, 34 (1951). M. S. Green, J. Chem. Phys. 19, 1036 (1951) R. Kubo, J. Phys. Soc. Japan 12, 570 (1957); Rep. Prog. Phys. 29, 255 (1966); D. W. Sciama (1979) "Thermal and Quantum Fluctuations in Special and General Relativity: an Einstein Synthesis" in Centenario di Einstein (Editrici Giunti Barbera Universitaria) (1979); P. Candelas and D. W. Sciama, Phys. Rev. Lett. 38, 1372 (1977).

[2] L. Landau and E. Lifshitz, J. Exptl. Theoret. Phys. (USSR) 32, 618 (1957) (Engl. Trans. Sov. Phys. JETP 5, 512 (1957)).

[3] L. Landau and E. Lifshitz, Fluid Mechanics (Pergamon Press, Oxford, 1959).

[4] R. Fox and G. Uhlembeck, Phys. Fluids 13, 1893, 2881 (1970).

[5] M. Bixon and R. Zwanzig, Phys. Rev. 187, 267 (1969); M. Kac and J. Logan, “Fluctuations”,
in *Fluctuation Phenomena*, edited by E. W. Montroll and J. L. Lebowitz (Elsevier, New York, 1979), p.1; *Phys. Rev.* **A13**, 458 (1976);

[6] S. Chapman and T. Cowling, *The mathematical theory of non-uniform gases* (Cambridge University Press, Cambridge (England), 1939 (reissued 1990)).

[7] D. Pavón and J. M. Rubi, *Phys. Rev.* **D37**, 2052 (1988); D. Pavón, *Phys. Rev.* **D43**, 375 (1991), **D43**, 2495 (1991).

[8] S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972)

[9] W. Hiscock and L. Lindblom, *Phys. Rev.* **D31**, 725 (1985)

[10] W. Zimdahl, *Physica A* **182**, 197 (1992); *Phys. Rev.* **D48**, 2431 (1993)

[11] W. Hiscock and L. Lindblom, *Ann. Phys.* (NY) **151**, 466 (1983)

[12] W. Israel, *Covariant fluid mechanics and thermodynamics: an introduction*, in A. Anile and Y. Choquet-Bruhat (eds.) *Relativistic fluid dynamics* (Springer, New York, 1988).

[13] R. Geroch and L. Lindblom, *Ann. Phys.* (NY) **207**, 394 (1991)

[14] D. Jou, G. Leblon and J. Casas Vasquez, *Extended thermodynamics* (Springer, New York, 1993)(2nd. edition Springer, Heidelberg (1996)).

[15] W. Israel, *Ann. Phys.* (NY) **100**, 310 (1976)

[16] W. Israel and J. Stewart, *Gen. Rel. Grav.* **2**, 491 (1980)

[17] I. Liu, I. Muller and T. Ruggeri, *Ann. Phys.* (NY) **169**, 191 (1986)

[18] R. Geroch and L. Lindblom, *Phys. Rev.* **D41**, 1855 (1990)

[19] W. Israel, *J. Math. Phys.* **4**, 1163 (1963)

[20] W. Israel, *The relativistic Boltzmann equation*, in L. O’Raifeartaigh (ed.) *General relativity: papers in honour of J. L. Synge* (Clarendon Press, Oxford, 1972), p. 201.
[21] W. Israel and H. Kandrup, *Ann. Phys. (NY)* **152**, 30 (1984)

[22] W. Israel and J. Stewart, *Ann. Phys. (NY)* **118**, 341 (1979)

[23] G. Nagy, Ph. D. Thesis, University of Córdoba (1995); G. Nagy and O. Reula, *J. Phys. A* **30**, 1695 (1997)

[24] V. Belinsky, E. Nikomarov and I. Khalatnikov, *Zh. Eksp. Teor. Fiz* **77**, 417 (1979) (Engl. trans. *Sov. Phys. JETP* **50**, 213 (1979)); W. Hiscock and J. Salmonson, *Phys. Rev.* **D43**, 3249 (1991); V. Romano and D. Pavon, *Phys. Rev.* **D47**, 1396 (1993); M. Zakari and D. Jou, *Phys. Rev.* **D48**, 1597 (1993); W. Zimdahl and D. Pavon, *Mon. Not. R. Astron. Soc* **266**, 872 (1994); *Gen. Rel. Grav.* **26**, 1259 (1994); W. Zimdahl, D. Pavon and D. Jou, *Class. Q. Grav.* **10**, 1775 (1993).

[25] R. Geroch, *J. Math. Phys.* **36**, 4226 (1995)

[26] H. Kreiss, O. Ortiz and O. Reula, e-print gr-qc/9702008

[27] L. Lindblom, *Ann. Phys. (NY)* **247**, 1 (1996)

[28] W. Israel, *Phys. Lett.* **86A**, 79 (1981)

[29] W. Israel, *Thermodynamics and field statistics of a relativistic superfluid*, in M Markov, V. Berezin and V. Frolov (eds.) *Proceedings of the third Seminar on quantum gravity* (World Scientific, Singapore, 1985), p. 246.

[30] W. Hiscock and L. Lindblom, *Phys. Lett A* **131**, 280 (1988).

[31] E. Calzetta, G. Nagy and O. Reula, to appear.

[32] B. Carter and I. Khalatnikov, *Ann. Phys. (NY)* **219**, 243 (1992); *Phys. Rev* **D45**, 4536 (1992); G. Comer and D. Langlois, *Class. Q. Grav.* **11**, 709 (1994); B. Carter, *Class. Q. Grav.* **11**, 2013 (1994); B. Carter and D. Langlois, *Phys. Rev* **D51**, 5855 (1995); *Phys. Rev* **D52**, 4640 (1995).
[33] B. L. Hu, Phys. Lett. A90, 375 (1982); A97, 368 (1983)

[34] R. B. Griffiths, J. Stat. Phys. 36, 219 (1984); R. Omnés, J. Stat Phys. 53, 893, 933, 957 (1988); Ann. Phys. (NY) 201, 354 (1990); Rev. Mod. Phys. 64, 339 (1992); The interpretation of quantum mechanics, (Princeton University Press, Princeton, 1994). M. Gell-Mann and J. B. Hartle, in Complexity, Entropy and the Physics of Information, ed. by W. H. Zurek (Addison-Wesley, Reading, 1990); J. B. Hartle and M. Gell-Mann, Phys. Rev. D47, 3345 (1993). J. B. Hartle, “Quantum Mechanics of Closed Systems” in Directions in General Relativity Vol. 1, eds B. L. Hu, M. P. Ryan and C. V. Vishveswara (Cambridge Univ., Cambridge, 1993).

[35] E. Calzetta and B. L. Hu, Hydrodynamics as a consistent set of histories (to appear).

[36] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[37] L. Landau, E. Lifshitz and L. Pitaevsky (1980) Statistical Physics, Vol I (Pergamon press, London)

[38] D. Joseph and L. Preziosi, Rev. Mod. Phys. 61, 41 (1989); 62, 375 (1990).