Uniqueness Theorems for Subharmonic and Holomorphic Functions of Several Variables on a Domain

B. N. Khabibullin, N. R. Tamindarova

Abstract. We establish a general uniqueness theorem for subharmonic functions of several variables on a domain. A corollary from this uniqueness theorem for holomorphic functions is formulated in terms of the zero subset of holomorphic functions and restrictions on the growth of functions near the boundary of domain.

Key Words and Phrases: holomorphic function, zero set, uniqueness set, subharmonic function, Riesz measure, Jensen measure, potential, balayage

2000 Mathematics Subject Classifications: 32A10, 31B05

1. Introduction

1.1. Definitions and notations

We use an information and definitions from [1]–[5]. As usual, \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{R} \) and \( \mathbb{C} \) are the sets of all natural, real and complex numbers, resp. We set

\[
\mathbb{R}^+ := \{ x \in \mathbb{R} : x \geq 0 \}, \quad \mathbb{R}_- := \{ -\infty \} \cup \mathbb{R}, \quad \mathbb{R}_+ := \mathbb{R} \cup \{ +\infty \}, \quad \mathbb{R}_{\pm} := \mathbb{R}_- \cup \mathbb{R}_+ \tag{1}
\]

where the usual order relation \( \leq \) on \( \mathbb{R} \) is complemented by the inequalities \( -\infty \leq x \leq +\infty \) for all \( x \in \mathbb{R}_{\pm} \). Let \( f : X \to Y \) be a function. Given \( S \subset X \), we denote by \( f \big|_S \) the restriction of \( f \) to \( S \). For \( Y \subset \mathbb{R}_{\pm} \), \( g : X \to \mathbb{R}_{\pm} \) and \( S \subset X \), we write “\( f \leq g \) on \( S \)” if \( f(x) \leq g(x) \) for all \( x \in S \).

Let \( m \in \mathbb{N} \). Denote by \( \mathbb{R}^m \) the \( m \)-dimensional Euclidian real space. Then \( \mathbb{R}^\infty := \mathbb{R}^m \cup \{ \infty \} \) is the Alexandroff (a one-point) compactification of \( \mathbb{R}^m \). Given a subset \( S \) of \( \mathbb{R}^m \) (or \( \mathbb{R}^\infty \)), the closure \( \text{clos} S \) of \( S \), the interior \( \text{int} S \) and the boundary \( \partial S \) will always be taken relative \( \mathbb{R}^\infty \). Let \( S_0 \subset S \subset \mathbb{R}^\infty \). If the closure \( \text{clos} S_0 \) is a compact subset of \( S \) in the topology induced on \( S \) from \( \mathbb{R}^\infty \), then the set \( S_0 \) is a relatively compact subset of \( S \), and we write \( S_0 \subset S \).

Let \( n \in \mathbb{N} \). Denote by \( \mathbb{C}^n \) the \( n \)-dimensional Euclidian complex space. Then \( \mathbb{C}^\infty := \mathbb{C}^n \cup \{ \infty \} \) is the Alexandroff (a one-point) compactification of \( \mathbb{C}^n \). If it is necessary, we

\*Our research is supported by RFBR grant (project no. 16-01-00024).
identify $\mathbb{C}^n$ (or $\mathbb{C}^\infty_n$) with $\mathbb{R}^{2n}$ (or $\mathbb{R}^{2\infty}_n$). Given a subset $S$ of $\mathbb{C}^n$ (or $\mathbb{C}^\infty_n$), its closure $\text{clos} S$ and its boundary $\partial S$ will always be taken relative to $\mathbb{C}^\infty_n$.

Let $A, B$ are sets, and $A \subset B$. The set $A$ is a non-trivial subset of the set $B$ if the subset $A \subset B$ is non-empty ($A \neq \emptyset$) and proper ($A \neq B$).

We understand always the “positivity” or “positive” as $\geq 0$, where the symbol 0 denotes the number zero, the zero function, the zero measure, etc. So, a function $f : X \to \mathbb{R} \subset \mathbb{R}_{\pm \infty}$ is positive on $X$ if $f(x) \geq 0$ for all $x \in X$. In such case we write “$f \geq 0$ on $X$”.

The class of all Borel real measures on local compact space $X$ is denoted by $\mathcal{M}(X)$, $\mathcal{M}_{c}(X)$ is the subclass of all Borel measures $\mu$ on $X$ with compact support $\text{supp} \mu \subset X$, and $\mathcal{M}^+(X) \subset \mathcal{M}(X)$ is the subclass of all Borel positive measures on $X$, $\mathcal{M}^+_c(X) := \mathcal{M}^+(X) \cap \mathcal{M}_{c}(X)$. Given $\mu \in \mathcal{M}(X)$ and $S \subset X$, we denote by $\mu |_S$ the restriction of $\mu$ to $S$. For $\nu \in \mathcal{M}(X)$, we write “$\nu \geq \mu$ on $S$” if $(\nu |_S - \mu |_S) \in \mathcal{M}^+(S)$.

Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{R}^n_\infty$. We denote by $\text{sbh}(\mathcal{O})$ the class of all subharmonic functions $u : \mathcal{O} \to \mathbb{R}_{\pm \infty}$ on $\mathcal{O}$ for $m \geq 2$, and all (local) convex functions $u : \mathcal{O} \to \mathbb{R}_{\pm \infty}$ on $\mathcal{O}$ for $m = 1$. The class $\text{sbh}(\mathcal{O})$ contains the function $-\infty : x \mapsto -\infty$, $x \in \mathcal{O}$ (identical to $-\infty$); $\text{sbh}^+(\mathcal{O}) := \{ u \in \text{sbh}(\mathcal{O}) : u \geq 0 \text{ on } \mathcal{O} \}$. We set $\text{sbh}_+(\mathcal{O}) := \text{sbh}(\mathcal{O}) \setminus \{-\infty\}$. For $u \in \text{sbh}_+(\mathcal{O})$, the Riesz measure of $u$ is the Borel positive measure

$$
\nu_u := c_m \Delta u \in \mathcal{M}^+(\mathcal{O}), \quad c_m := \frac{\Gamma(m/2)}{2\pi^{m/2} \max\{1, (m-2)\}},
$$

where $\Delta$ is the Laplace operator acting in the sense of distribution theory, and $\Gamma$ is the gamma function. Such measures $\nu_u$ is Radon measures, i.e. $\nu_u(S) < +\infty$ for each subset $S \subset \mathcal{O}$. By definition, $\nu_{-\infty}(S) := +\infty$ for all $S \subset \mathcal{O}$.

Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{C}^\infty_n$. We denote by Hol$(\mathcal{O})$ and $\text{sbh}(\mathcal{O})$ the class of holomorphic and subharmonic functions on $\mathcal{O}$, resp. For $u \in \text{sbh}_+(\mathcal{O})$, the Riesz measure of $u$ is the Borel (and Radon) positive measure

$$
\nu_u := c_{2n} \Delta u \in \mathcal{M}^+(\mathcal{O}), \quad c_{2n} = \frac{(n-1)!}{2\pi^n \max\{1, 2n-2\}}.
$$

1.2. Main Theorem and Corollary

**Definition 1.** Let $D$ be a non-trivial open connected subset of $\mathbb{R}^n_\infty$, i.e. $D$ is a non-trivial domain in $\mathbb{R}^n_\infty$. Let $K$ be a non-trivial compact subset of $D$, i.e. $\emptyset \neq K = \text{clos} K \subset D$. A function $v \in \text{sbh}^+(D \setminus K)$ is called a test function for $D$ outside of $K$ if

$$
\lim_{D \ni x' \to x} v(x') = 0 \quad \text{for each } x \in \partial D \quad \text{and} \quad \sup_{x \in D \setminus K} v(x) < +\infty.
$$

The class of all such test functions for $D$ outside of $K$ is denoted by $\text{sbh}^0_+(D \setminus K)$.

Our main result for subharmonic functions is the following
Theorem 1 (see [6, Corollary 1.1] for the case \(m = 2\)). Let \(D\) be a non-trivial domain in \(\mathbb{R}^m_\infty\), \(K\) a compact subset of \(D\) with non-empty interior \(\text{int} \ K \neq \emptyset\). Let \(M \in \text{sbh}_*(D)\) be a function with the Riesz measure \(\nu_M \in \mathcal{M}^+(D)\), \(v \in \text{sbh}^+_0(D \setminus K)\) a test function for \(D\) outside of \(K\). Assume that

\[
\int_{D \setminus K} v \, d\nu_M < +\infty. \tag{4}
\]

If \(u \in \text{sbh}(D)\) is a function with the Riesz measure \(\nu_u \in \mathcal{M}^+(D)\) such that

\[
\nu_u \geq \nu \in \mathcal{M}^+(D \setminus K) \quad \text{on} \quad D \setminus K, \tag{5a}
\]

\[
\int_{D \setminus K} v \, d\nu = +\infty, \tag{5b}
\]

\[
u_u \leq M + \text{const} \quad \text{on} \quad D, \tag{5c}
\]

where \(\text{const}\) is a constant, then \(u = -\infty\).

We denote by \(\sigma_{2n-2}\) the \((2n-2)\)-dimensional surface (\(\ast\)-Hausdorff) measure on \(\mathbb{C}^n\) and its restrictions to subsets of \(\mathbb{C}^n\). So, if \(n = 1\), i.e. \(2n - 2 = 0\), then \(\sigma_0(S) = \sum_{z \in S} 1\) for each \(S \subset \mathbb{C}\), i.e. \(\sigma_0(S)\) is equal to the number of points in the set \(S \subset \mathbb{C}\).

Below we identify \(\mathbb{C}^n\) (or \(\mathbb{C}^n_\infty\)) with \(\mathbb{R}^m\) (or \(\mathbb{R}^m_\infty\)) where \(m = 2n\).

Our main result for holomorphic functions is the following

**Corollary 1.** Let all conditions of Theorem 1 are fulfilled including \((4)\). Let \(f \in \text{Hol}(D)\) be a holomorphic function on \(D\) and \(\text{Zero}_f := \{z \in D : f(z) = 0\}\). If

\[
Z \subset (D \setminus K) \cap \text{Zero}_f, \tag{6a}
\]

\[
\int_Z v \, d\sigma_{2n-2} = +\infty, \tag{6b}
\]

\[
|f| \leq \text{const} \quad e^M \quad \text{on} \quad D, \tag{6c}
\]

where \(\text{const}\) is a constant, then \(f \equiv 0\) on \(D\), i.e. \(\text{Zero}_f = D\).

**Proof.** Under the conditions of Corollary 1, suppose that \(f \neq 0\). Then we have \(\log |f| \in \text{sbh}_*(D)\) with the Riesz measure \(\nu_{\log |f|} \in \mathcal{M}^+(D)\). Let \(n_f : D \to \{0\} \cup \mathbb{N}\) be the multiplicity function of \(f\) [8, 4]. It is known that \(\text{supp} n_f = \text{Zero}_f\). By the classical Poincaré–Lelong formula [9] we have \(n_f \, d\sigma_{2n-2} = d\nu_{\log |f|}\) on \(D\). Hence, if the condition \((6a)\) is fulfilled then we get

\[
\int_Z v \, d\sigma_{2n-2} \overset{(6a)}{\leq} \int v \, n_f \, d\sigma_{2n-2} \leq \int v \, d\nu_{\log |f|}. \tag{7}
\]

If the condition \((6c)\) is also fulfilled, then for \(u := \log |f|\) we have \((5c)\) together with \((5a)\) for \(\nu = \nu_u = \nu_{\log |f|}\). Since \(u \neq -\infty\), by Theorem 1 we obtain the negation of the equality \((5b)\). Therefore we get

\[
\int_Z v \, d\sigma_{2n-2} \overset{(7)}{=} \int v \, d\nu_{\log |f|} = \int v \, d\nu_u < +\infty
\]

what contradicts \((6b)\). Corollary 1 is proved. ▲
2. Main results

2.1. Gluing Theorem for \( m \in \mathbb{N} \)

The next result shows how two subharmonic functions can be glued together.

**Theorem 2** (see [4, Corollary 2.4.5], and [2, Theorem 2.4.5] for \( m = 2 \)). Let \( O, O_0 \) are open sets in \( \mathbb{R}^m_\infty \), and \( O \subset O_0 \). Let \( v_0 \in \text{sbh}(O_0) \), and \( v \in \text{sbh}(O) \). If

\[
\limsup_{O \ni x' \to x} v(x') \leq v_0(x) \quad \text{for all points } x \in O_0 \cap \partial O,
\]

then the function

\[
\tilde{v} := \begin{cases} 
\max\{v, v_0\} & \text{on } O, \\
v_0 & \text{on } O_0 \setminus O,
\end{cases}
\]

belong to the class \( \text{sbh}(O_0) \).

**Remark 1.** A similar gluing theorem true also for classes of plurisubharmonic functions [4, Corollary 2.9.5].

2.2. Jensen measures and potentials

Let \( m \in \mathbb{N} \). Given \( t \in \mathbb{R}_\ast := \mathbb{R} \setminus \{0\} \), we set

\[
h_m(t) := \begin{cases} 
|t| & \text{for } m = 1, \\
\log |t| & \text{for } m = 2, \\
\frac{-1}{|t|^{m-2}} & \text{for } m \geq 3.
\end{cases}
\]

For simplicity, we consider only domains \( D \) in \( \mathbb{R}^m \subset \mathbb{R}^m_\infty \), i.e. \( \infty \notin D \).

**Definition 2** ([10]–[14]). Let \( D \subset \mathbb{R}^m \) be a subdomain, \( x_0 \in D \). A measure \( \mu \in \mathcal{M}^+_c(D) \) is called the Jensen measure for \( \text{sbh}(D) \) at \( x_0 \in D \) if

\[
u(x_0) \leq \int u \, d\mu \quad \text{for all } u \in \text{sbh}(D).
\]

By \( J_{x_0}(D) \) we denote the class of all Jensen measures for \( D \) at \( x_0 \). Each Jensen measure \( \mu \in J_{x_0}(D) \) is a probability measure, i.e. \( \mu(D) = 1 \).

For \( \mu \in J_{x_0}(D) \) we shall say that the function

\[
V_\mu(x) := \int h_m(x - y) \, d\mu(y) - h_m(x - x_0), \quad x \in \mathbb{R}^m_\infty \setminus \{x_0\},
\]

is a potential of the Jensen measure \( \mu \in J_0(D) \).

A function \( V \in \text{sbh}^+(\mathbb{R}^m_\infty \setminus \{x_0\}) \) is called the Jensen potential inside of \( D \) with pole at \( x_0 \in D \) if the following two conditions hold:
(i) there is a compact subset $K_V \subset D$ such that $V \equiv 0$ on $\mathbb{R}_\infty^m \setminus K_V$ (finiteness),

(ii) and $\limsup_{x_0 \neq x \to x_0} \frac{V(x)}{|h_m(x - x_0)|} \leq 1$ (semi-normalization at $x_0$).

By $PJ_{x_0}(D)$ we denote the class of all Jensen potentials inside of $D$ with pole at $x_0 \in D$.

We present interrelations between Jensen measures and potentials. The first is

**Proposition 1** ([14, Proposition 1.4, Duality Theorem]). The map

$$
P: J_{x_0}(D) \to PJ_{x_0}(D), \quad P(\mu) := V_\mu, \quad \mu \in J_{x_0}(D),
$$

is the bijection from $J_{x_0}(D)$ to $PJ_{x_0}(D)$ such that $P(t\mu_1 + (1-t)\mu_2) = tP(\mu_1) + (1-t)P(\mu_2)$ for all $t \in [0,1]$ and for all $\mu_1, \mu_2 \in J_{x_0}(D)$. Besides,

$$
P^{-1}(V) = c_m \Delta V \bigg|_{D \setminus \{x_0\}} + \left(1 - \limsup_{x_0 \neq x \to x_0} \frac{V(x)}{|h_m(x - x_0)|}\right) \cdot \delta_{x_0}, \quad V \in PJ_{x_0}(D),
$$

where $\delta_{x_0}$ is the Dirac measure at the point $x_0$, i.e. $\operatorname{supp} \delta_{x_0} = \{x_0\}$ and $\delta_{x_0}(\{x_0\}) = 1$.

The second is a generalized Poisson–Jensen formula.

**Proposition 2** ([14, Proposition 1.2]). Let $\mu \in J_{x_0}(D)$. For each function $u \in \operatorname{shh}(D)$ with $u(x_0) \neq -\infty$ and the Riesz measure $\nu_u \in \mathcal{M}^+(D)$ we have the equality

$$
u_u(x_0) + \int_{D \setminus \{x_0\}} V_\mu \, d\nu_u = \int_D u \, d\mu.
$$

Given $x \in \mathbb{R}^m$ and $r \in \mathbb{R}^+$, we set $B(x, r) := \{x' \in \mathbb{R}^m : |x' - x| < r\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$; $B_+(x, r) := B(x, r) \setminus \{x\}$; $\overline{B}(x, r) := \operatorname{clos} B(x, r)$.

By $\operatorname{shh}(O)$ denote the class of all harmonic functions on an open subset $O \subset \mathbb{R}^m$.

**Corollary 2.** Let $D$ be a domain in $\mathbb{R}^m$, $x_0 \in D$, $r_0 > 0$, and $B(x_0, r_0) \subset D$; $b \in \mathbb{R}^+$. If functions $u \in \operatorname{shh}_u(D)$ with the Riesz measure $\nu_u$ and $M \in \operatorname{shh}_u(D)$ with the Riesz measure $\nu_M$ satisfy the inequality

$$u \leq M + \text{const} \quad \text{on } D,
$$

then there is a constant $C \in \mathbb{R}$ such that

$$
\int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_u \leq \int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_M + C
$$

for all functions $V \in PJ_{x_0}(D)$ satisfying the following three conditions:

$$V \big|_{B_+(x_0, r_0)} \in \operatorname{har}(B_+(x_0, r_0)),
$$

$$\limsup_{x_0 \neq x \to x_0} \frac{V(x)}{|h_m(x - x_0)|} \equiv 1 \quad \text{(normalization at } x_0),$$

$$\sup_{x \in \partial B(x_0, r_0)} V(x) \leq b.$$

**Proof.**
Proof. The technique of balayage out from the ball $B(x_0, r_0)$ gives two functions $u_0 \in \text{sbh}_s(D)$ with the Riesz measure $\nu_{u_0}$ and $M_0 \in \text{sbh}_s(D)$ with the Riesz measure $\nu_{M_0}$ such that

$$u_0 \mid_{B(x_0, r_0)} \in \text{har}(B(x_0, r_0)), \quad u_0 = u \text{ on } D \setminus B(x_0, r_0), \quad (17u)$$

$$M_0 \mid_{B(x_0, r_0)} \in \text{har}(B(x_0, r_0)), \quad M_0 = M \text{ on } D \setminus B(x_0, r_0), \quad (17v)$$

$$u_0 \leq M_0 + C_0 \quad \text{on } D \quad \text{where } C_0 \in \mathbb{R}^+ \text{ is a constant}, \quad (17b)$$

$$\text{supp } \nu_{M_0} \subset D \setminus B(x_0, r_0), \quad \nu_{M_0}(\partial B(x_0, r_0)) = \nu_M(\overline{B}(x_0, r_0)) \quad (17n)$$

By Proposition 1 the measure $\mu := \mathcal{P}^{-1}(V) \in J_{x_0}(D)$ satisfies the following conditions:

$$\text{supp } \mu \subset D \setminus B(x_0, r_0), \mu(D) = 1. \quad \text{The inequality } (17b) \text{ entails the inequality}$$

$$\int u_0 \, d\mu \leq \int M_0 \, d\mu + C_0$$

Hence by Proposition 2

$$\int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_u \leq \int_D V \, d\nu_{u_0} \leq \int_D V \, d\nu_{M_0} + (C_0 - u_0(x_0) + M_0(x_0)) \quad (17q)$$

Put $C_1 := C_0 - u_0(x_0) + M_0(x_0) \in \mathbb{R}$. We continue this inequality as

$$\int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_u \leq \int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_{M_0} + \int_{B(x_0, r_0)} V \, d\nu_{M_0} + C_1 \quad (17n)$$

$$\equiv \int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_M + \int_{\partial B(x_0, r_0)} V \, d\nu_{M_0} + C_1 \quad (17r)$$

$$\leq \int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_M + b \nu_{M_0}(\overline{B}(x_0, r_0)) + C_1 \quad \text{by (16)}$$

$$\equiv \int_{D \setminus \overline{B}(x_0, r_0)} V \, d\nu_M + b \nu_M(\overline{B}(x_0, r_0)) + C_1. \quad (17n)$$

We choose $C := b \nu_M(\overline{B}(x_0, r_0)) + C_1$ and obtain the inequality (15). \hfill \Box

2.3. Continuation of test functions

**Proposition 3.** Let $v \in \text{sbh}_s^+(D \setminus K)$ be a test function for $D$ outside of $K$, $r_0 > 0$, and $B(x_0, 2r_0) \subset K$. Then there are subdomains $D_0 \subset D_1 \subset D$, a number $r_0 > 0$ and a function $\tilde{v} \in \text{sbh}_s^+(D \setminus \{x_0\})$ such that

$$\overline{B}(x_0, 2r_0) \subset K \subset D_0, \quad (18a)$$
\[
\tilde{v} \big|_{B_*(x_0,2r_0)} \in \text{har}(B_*(x_0,2r_0)), \quad (18b) \\
\lim_{x_0 \neq x \to x_0} \frac{\tilde{v}(x)}{h_m(x-x_0)} \in (0, +\infty), \quad (18c) \\
\tilde{v} = v \text{ on } D \setminus D_1. \quad (18d)
\]

**Proof.** Obviously, there is a subdomain \(D_0 \Subset D\) satisfying (18a). There is a subdomain \(D_1\) that regular for Dirichlet problem and \(D_0 \Subset D_1 \Subset D\). Let \(g_{D_1}(\cdot, x_0)\) be the Green’s function of \(D_1\) with the pole \(x_0\) where \(g_{D_1}(x, x_0) \equiv 0\) for all \(x \in D \setminus D_1\). Then \(g_{D_1}(\cdot, x_0) \in \text{sh}(D \setminus \{x_0\})\). Put

\[
q := \sup_{x \in \partial D_0} v(x), \quad a := \inf_{x \in \partial D_0} g_{D_1}(x, x_0) > 0, \quad v_0 := \frac{q}{a} g_{D_1}(\cdot, x_0),
\]
and \(O := D \setminus \text{clos} D_0, O_0 := D \setminus \{x_0\}\). Then the condition (8) is fulfilled. By Theorem 2 the function (9) is required according to the known properties of the Green’s function. \(\blacksquare\)

### 2.4. Proof of Theorem 1

**Proof.** Let \(B(x_0, 2r_0) \subset K\) where \(r_0 > 0\). Suppose that the properties (5a), (5c) are fulfilled for \(u \neq -\infty\). We must prove that the integral from (5b) with \(\nu := \nu_u\) is finite. Consider the function \(\tilde{v}\) from Proposition 3 where

\[
0 < c < \lim_{x_0 \neq x \to x_0} \frac{\tilde{v}(x)}{h_m(x-x_0)} < +\infty.
\]

Then the function \(V := \frac{1}{c} \tilde{v}\) satisfies the following conditions

\[
\begin{align*}
V \big|_{B_*(x_0,2r_0)} &\in \text{har}(B_*(x_0,2r_0)), \quad (19b) \\
\lim_{x_0 \neq x \to x_0} \frac{V(x)}{h_m(x-x_0)} &\in (0, 1], \quad (19c) \\
V &\equiv 1 \text{ on } D \setminus D_1, \quad (19d) \\
\lim_{D \ni x' \to x} V(x') &\equiv 0 \quad \text{for each } x \in \partial D, \quad (19e) \\
\sup_{x \in \partial B(x_0,r_0)} V(x) &=: b < +\infty. \quad (19f)
\end{align*}
\]

We put \(V(x) \equiv 0\) at \(x \in \mathbb{R}^m_\infty \setminus D\). Then \(V \in \text{sh}(\mathbb{R}^m_\infty \setminus \{x_0\})\). Consider the sequence of functions \(V_n := \max\{0, V - 1/n\}, \quad n \in \mathbb{N}\). If \(n_0 \in \mathbb{N}\) is a sufficiently large number, then every function \(V_n\), \(n \geq n_0\), is a Jensen function inside of \(D\) with the pole at \(x_0 \in D\) such that

\[
V_n \big|_{B_*(x_0,r_0)} \in \text{har}(B_*(x_0,r_0)), \quad \lim_{x_0 \neq x \to x_0} \frac{V_n(x)}{h_m(x-x_0)} = 1, \quad \sup_{x \in \partial B(x_0,r_0)} V_n(x) \leq b. \quad (19g)
\]
\[
\frac{1}{c} v(x) = V(x) \geq V_n(x) \geq \frac{1}{c} v(x), \quad \text{for all } x \in D \setminus D_1 \text{ when } n \to \infty. \quad (20)
\]

Hence by Corollary 2 there is a constant \( C \in R^+ \) such that
\[
\int_{D \setminus \overline{B}(x_0,r_0)} V_n \, d\nu_u \overset{(15)}{\leq} \int_{D \setminus \overline{B}(x_0,r_0)} V_n \, d\nu_M + C \quad \text{for all } n \geq n_0.
\]

Therefore, for all \( n \geq n_0 \),
\[
\int_{D \setminus \overline{B}(x_0,r_0)} V_n \, d\nu_u \overset{(15)}{\leq} \int_{D \setminus \overline{B}(x_0,r_0)} V_n \, d\nu_M + C \overset{(20)}{\leq} \int_{D \setminus \overline{B}(x_0,r_0)} V \, d\nu_M + C
\]
\[
\leq \int_{D \setminus D_1} V \, d\nu_M + \int_{D_1 \setminus \overline{B}(x_0,r_0)} V \, d\nu_M + C \overset{(19)}{\leq} \int_{D \setminus D_1} V \, d\nu_M + b \nu_M(D_1 \setminus \overline{B}(x_0,r_0)) + C
\]

where we use the maximum principle for \( V \in \text{sbh}(\mathbb{R}^m \setminus \{x_0\}) \) in \( \mathbb{R}^m \setminus B(x_0,r_0) \). Further, we put \( C_1 := b \nu_M(D_1 \setminus \overline{B}(x_0,r_0)) + C \in \mathbb{R} \) and continue as
\[
\int_{D \setminus \overline{B}(x_0,r_0)} V_n \, d\nu_u \overset{(15)}{\leq} \int_{D \setminus D_1} V \, d\nu_M + C_1 \overset{(20)}{=} \frac{1}{c} \int_{D \setminus D_1} v \, d\nu_M + C_1 \overset{(4)}{\leq} \frac{1}{c} \int_{D \setminus K} v \, d\nu_M + C_1.
\]

So, when \( n \to \infty \), we obtain in view of (20)
\[
\frac{1}{c} \int_{D \setminus D_1} v \, d\nu_u \overset{(20)}{=} \int_{D \setminus D_1} V \, d\nu_u \leq C_2 := \frac{1}{c} \int_{D \setminus K} v \, d\nu_M + C_1 \in \mathbb{R}.
\]

Hence
\[
\int_{D \setminus K} v \, d\nu_u \leq \int_{D_1 \setminus K} v \, d\nu_u + cC_2 \leq \nu_u(D_1 \setminus K) \sup_{x \in D \setminus K} v(x) + cC_2 \overset{(3)}{<} +\infty.
\]

This completes proof of Theorem 1. \( \blacksquare \)

**Remark 2.** Our results show that the construction of test functions in the sense of Definition 2 is important. For \( m = 2 = 2n \) such constructions developed in [6]. We consider the case \( m \neq 2 \) in another place.

**References**

[1] W. K. Hayman, P. B. Kennedy, *Subharmonic functions*, Vol. 1, Acad. Press, London etc., 1976.

[2] Th. Ransford, *Potential Theory in the Complex Plane*, Cambridge: Cambridge University Press, 1995.

[3] J. L. Doob *Classical Potential Theory and Its Probabilistic Counterpart*, Springer-Verlag, N.-Y., 1984.
[4] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[5] L. Hörmander, *Notions of Convexity*, Progress in Mathematics, Boston: Birkhäuser, 1994.

[6] Bulat Khabibullin, Nargiza Tamindarova, *Distribution of zeros and masses for holomorphic and subharmonic functions. I. Hadamard- and Blaschke-type conditions*, http://arxiv.org/pdf/1512.04610v2.pdf [math.CV], 12/2015–03/2016, 70 pages, list of references 71. This work submitted to the journal “Sbornik: Mathematics” in 2015 (Matematicheskii sbornik, in Russian).

[7] N. Bourbaki, *Integration I (Chapters 1-6) and II (Chapters 7-9).* 2 volumes, Springer-Verlag, Berlin, Heidelberg, 2004.

[8] Chee Pak Soong, *The Blaschke Condition for Bounded Holomorphic Functions*, Transactions of the American Mathematical Society, 148(1), 1970, 249–263.

[9] P. Lelong, *Propriétés métriques des variétés analytiques complexes définies par une équation*, Ann. Sci. Ec. Norm. Sup. 67, 1950, 393–419.

[10] T.W. Gamelin, *Uniform Algebras and Jensen Measures*, Cambridge Univ. Press, Cambridge, 1978.

[11] B. N. Khabibullin, *Estimates for the volume of null sets of holomorphic functions*, Russian Mathematics (Izvestiya VUZ. Matematika), 36(3), 1992, 56–62.

[12] B. J. Cole, T. J. Ransford, *Subharmonicity without upper semicontinuity*, J. Func. Anal. 147, 1997, 420–442.

[13] B. J. Cole, T. J. Ransford, *Jensen measures and harmonic measures*, J. Reine Angew. Math. 541, 2001, 29–53.

[14] B. N. Khabibullin, *Criteria for (sub-)harmonicity and continuation of (sub-)harmonic functions*, Siberian Mathematical Journal (Sibirsk. Mat. Zh.), 44(4), 2003, 713–728.

Bulat N. Khabibullin
*Bashkir State University, Ufa, Russian Federation*
*E-mail: khabib-bulat@mail.ru*

Nargiza R. Tamindarova
*Bashkir State University, Ufa, Russian Federation*
*E-mail: nargiza89@gmail.com*

Received .........
Accepted .........