Mass generation of the lepton sector

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Abstract

We discuss the necessity of a right-handed Weyl neutrino due to the vector-like phenomenon of the regularized Standard Model. It is shown that this right-handed neutrino is decoupled from low energies as a free particle, and Dirac neutrino masses are very small. We suggest gauge-invariant couplings between down quarks and charged leptons so that charged leptons acquire masses without extra Goldstone modes. By examining Schwinger-Dyson equations for lepton self-energy functions, we show that the neutrinos get their Dirac masses via explicit symmetry breakings that attribute to the mixing between neutrinos and charged leptons. An analysis of these Dyson equations gives the four relationships between inter-generation mixing angles and lepton masses.

1st March, 1997
PACS 11.15Ha, 11.30.Rd, 11.30.Qc

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1 Introduction

Since their appearance neutrinos have always been extremely peculiar. In the sixty years of their life, their charge neutrality, their apparent masslessness, their left-handedness have been at the centre of a conceptual elaboration and an intensive experimental analysis that have played a major role in donating to mankind the beauty of the electroweak theory. V-A theory and Fermi universality would possibly have eluded us for a long time had the eccentric properties of neutrinos, all tied to their apparent masslessness, not captured the imagination of generations of experimentalists and theorists alike.

However, with the consolidation of the Standard Model (SM) and in particular with the general views on (spontaneous?) mass generation in the SM, the observed (almost) masslessness of the three neutrinos ($\nu_e, \nu_\mu, \nu_\tau$) has recently come to be viewed as a very problematic and bizarre feature of the mechanism(s) that must be at work to produce the very rich mass spectrum of the fundamental fields of the SM. Indeed, in the (somewhat worrying) proliferation of the Yukawa couplings of fermions to the Higgs fields that characterizes the generally accepted SM, no natural reason can be found why the charge-neutral neutrinos are the fundamental particles of the lowest mass; for in the generally accepted minimal Higgs mechanism, the actual values of the fermion masses are in direct relation with the strengths of their couplings to the Higgs doublet, and it appears rather bizarre that nature has chosen to create a very sophisticated mass pattern by the mere fine-tuning of a large number of parameters.

2 The “No-Go” theorem and high-dimension operators

For more than a decade and half, it has been known that the left-handed neutrino fields in the SM cannot consistently be defined in a lattice-regularized quantum field theory. This theoretical inconsistency was asserted by the rigorously demonstrated “no-go” theorem of Nielson and Ninomiya[1]. This theorem states that under certain conditions, there must exist exactly equal numbers of the left-handed neutrinos and right-handed neutrinos and both handed neutrinos couple to gauge fields with the same strength in the low-energy limit of a lattice-regularized quantum field theory, if one insists on
preserving chiral gauge symmetries. As a result, the neutrino spectrum and the $W^\pm$-gauge coupling are no longer chiral (left-handed), but rather vector-like (equally left- and right-handed). This vector-like phenomenon is the generic feature of lattice-regularized chiral gauge theories. In the basis of the ABJ anomaly, it was shown \[1\] that the absence of the left-handed neutrino in the lattice-regularized SM is not an artifact of lattice-regularization itself. In fact, the “No-Go” theorem shows a very generic feature concerning the inconsistency of regularizing chiral gauge theories in the high-energy region.

In the low-energy region, on the other hand, the successful Standard Model exhibits its very peculiar parity-violating features of purely left-handed gauge coupling of the $W^\pm$-boson and only the left-handed neutrinos. The “no-go” theorem seems to run into a paradox that the experimentally successful Standard model is in fact theoretically inconsistent. We do not consider this paradox to be an intrinsic problem of the Standards Model. Instead, we regard that this inconsistency may imply what are the Nature’s possible choices for the SM at short distances\[2, 3\].

Since one of the prerequisites of the “no-go” theorem requires the lagrangian to be bilinear in fermionic fields, as that of the SM, this theorem strongly implies that the possible extensions of the standard model in short distances are high-dimension operators in terms of fermionic fields,

$$L_{\text{effective}} = L_{SM} + \text{high-dimension-operators},$$

that we call the effective lagrangian at a certain physical cutoff $\Lambda$. This is to meant that if high-dimension operators are supplemented into the standard model at short distances, the “no-go” theorem and resultant paradox can be evaded \textit{in principle}. However, it is difficult to show how this “no-go” theorem and the paradox can be evaded \textit{in practice}. The reasons are the following: (i) there can be many high-dimension operators allowed by the chiral gauge symmetries of the SM\[1\] in the high-energy region (the cut off); (ii) it is a non-perturbative effort to determine where is the ultra-violet fixed point in the space of high-dimension operators (couplings), which high-dimension operators are relevant to meet with the needs of the phenomenologically successful SM at the low-energy region.

From the phenomenological point of view, one of relevant dimension-6 operators should be the four-fermion interaction for the $\bar{t}t$ condensate model\[6\].

\[1\]Some of these operators explicitly violate the global symmetries (e.g. the baryon number) that are anomalous in the standard model\[4, 5\].
of the third generation of the quark sector \((a, b)\) are the color indices),

\[
G Q^a_i(x) \cdot t^a_R(x) \bar{t}_R^b(x) \cdot Q^b_i(x); \quad Q_L = (t, b)_L,
\]

which undergoes the spontaneous breaking of the chiral gauge symmetries of the SM to generate the top quark mass that is much heavier than other quarks. In principle, other quarks could have the same interaction as (2) at the cutoff, we gave an interpretation\[7\] why the interaction (2) is the only one relevant in low energies.

Analogously, since the \(\tau\)-lepton is most heaviest in the lepton sector, we could have the following four-fermion interaction of the third generation of the lepton sector,

\[
G \bar{\psi}_i^L(x) \cdot \tau_R(x) \bar{\tau}_R(x) \cdot \psi_j^L(x); \quad \psi_j^L = (\nu_\tau, \tau)_L.
\]

If this operator undergoes the spontaneous symmetry breaking, there would be extra Goldstone bosons beside those from the \(\bar{t}t\)-condensate model. This situation is phenomenologically unacceptable. We make the following observations\[7\]:

(i) the four-fermion couplings \(G\) are equal in eqs.(2) and (3) for the reason of some underlying unification; (ii) the coupling \(G\) in eqs.(2) is enhanced by the color factor \(N_c = 3\); (iii) if one fine-tunes \(G\) in eq.(2) around \(\frac{N_c G_c A^2}{2\pi^2} = 4 + 0^+\), the spontaneous symmetry breaking takes place and the operator (2) is relevant, while the operator (3) is irrelevant since the lepton sector is colorless \((N_c = 1)\) and the effective four-fermion coupling is bellow the threshold of taking place the spontaneous symmetry breaking. This seems not to run into the problem of extra Goldstone modes. These discussions can be generalized to similar dimension-6 interaction for other charged leptons. The question is how \(\tau\) and other charged leptons acquire their masses?

Based on the chiral gauge symmetries of the SM, we are not only allowed to have the high-dimension operators (2,3), but also the four-fermion interaction between the quark and lepton sector,

\[
G \bar{\psi}_i^L(x) \cdot \tau_R(x) \bar{\tau}_R(x) \cdot Q^a_i(x).
\]

This operator, although it should be irrelevant in low energies, is clearly responsible for the \(\tau\)-lepton mass for any value of \(G\), once the bottom quark

\[\text{At this point, this operator receives anomalous dimension } \gamma_m = 2.\]
is massive. We can have gauge invariant operators similar to eq. (3) for the first and second families as well. Tuning the four-fermion coupling to the critical coupling $G \rightarrow G_c + 0^+$, we obtain, at the cutoff,

$$
\begin{align*}
    m_b &= m_\tau \\
    m_s &= m_\mu \\
    m_d &= m_e.
\end{align*}
$$

These are reminiscent of the predictions in the $SU(5)$ unification theory.

Obviously, we give no explanations, from theoretical point of view, why the operators should only be eqs. (2,3,4) of the third generation. One may conceives that the fermionic flavour symmetries of the effective lagrangian in the high-energy region should be exact because of the underlying physics that are quite possibly flavour blind, e.g. quantum gravity. For this reason, other possible dimension-6 operators comprising all fermionic flavours in various generations cannot be certainly precluded from the effective lagrangian, as far as the chiral gauge symmetries of the SM and naive dimensional counting are concerned. While, on the other hand, due to the fact that the fermionic flavour symmetries of the standard model is violently broken, and the relevant high-dimension operators upon the ultra-violet fixed point of the effective lagrangian are presumably not flavour symmetric. This circumstance is clearly governed by the properties and complexities of the flavour dynamics of the effective lagrangian and its ground states.

In the context of the SM, the most crucial observation is that the fermionic flavour symmetries should be broken explicitly rather than spontaneously, otherwise we would have extra Goldstone modes, which are not observed and not energetically favourable in the ground state. Thus, we stipulate that the ultra-violet fixed point is such that other dimension-6 high-dimension operators except eq. (2), which develops the spontaneous symmetry breaking, are irrelevant in the low-energy limit.

Even though the “no-go” theorem does not tell us what are the relevant high-dimension operators of the SM in the low-energy limit, it really suggests an existence of the right-handed neutrino and necessary high-dimension

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3It is discussed in ref.[8] that quark (except top) masses are generated by explicit symmetry breakings.

4As for this point, some discussions are presented in refs.[8]
operators at short distances without going into the details of a concrete unification model.

3 On the smallness of neutrino masses

As discussed, the right-handed neutrino $\nu_R$ is theoretically forced to exist in the SM by the “no-go” theorem. However, it is experimentally illusive and its couplings to the left-handed neutrinos, namely Dirac neutrino masses, are very small contrasted sharply with other fermion masses. These two points implies us that the right-handed Weyl neutrinos $\nu_R$ should be almost free particles, weakly couple to the left-handed neutrinos and other particles. This means that each external right-handed neutrino line of all interacting (1PI, one particle irreducible) operators should be significantly suppressed in the low-energy limit. For this observation, we stipulate that in all high-dimension operators of the effective lagrangian (1), the right-handed neutrino fields appear as a “high-dimension” field defined as,

$$\Delta \nu_R(x) \equiv \sum_{\mu} \left[ \nu_R(x + \mu) + \nu_R(x - \mu) - 2\nu_R(x) \right],$$

where the operator “$\Delta$” is written as a discrete differentiation, and actually is a Dalambert’s operator. Thus, the effective lagrangian (1) exactly is invariant under the transformations,

$$\bar{\nu}_R(x) \rightarrow \bar{\nu}_R(x) + \bar{\epsilon}, \quad \nu_R(x) \rightarrow \nu_R(x) + \epsilon,$$

where $\epsilon$ is independent of space-time. As a result, in the effective lagrangian (1) there not exists dimension-6 operators involving $\nu_R$ analogous to (2,3) for the $\tau$-lepton.

In the most simplest case, we introduce only a single right-handed neutrino $\nu_R$ that is a singlet of the chiral gauge symmetries of the SM. We discuss this problem in the third lepton generation, and it can be easily generated into the first and second lepton generations.

In the basis of similar arguments for eq.(4) in the previous section, except its kinetic term, this right-handed neutrino $\nu_R$ couples to the left-handed neutrino through the gauge invariant dimension-8 operator given by

$$G \bar{\psi}_L^i(x) \cdot [\Delta \nu_R(x)] \bar{t}_R^a(x) \cdot Q^a_{L}^i(x),$$

It can be possible that each generation has its own right-handed neutrino.
which could be responsible for neutrino’s Dirac masses. There are other possible high-dimension (at least dimension-10) gauge invariant operators, which are relevant for doublers residing at the cut-off. The details of discussions presented in ref.\cite{10} are out of the scope of this paper.

We turn to discuss the peculiar properties that Dirac neutrino masses are very small, the right-handed neutrino $\nu_R$ is a almost free particle and decouple from all physical particles in the low-energy limit. These properties can be demonstrated by Ward identities of the $\nu_R$-shift-symmetry (7). The Ward identity in terms of the primed fields corresponding to the $\nu_R$-shift-symmetry of the action (7) is given as

$$
\gamma_\mu \partial^\mu \nu'_R(x) + G\langle \Delta \left( \bar{Q}^a_L(x) \cdot t^a_R(x) \psi'^i_L(x) \right) \rangle - \frac{\delta \Gamma}{\delta \nu'_R(x)} = 0,
$$

where "$\Gamma$" is the effective potential with non-vanishing external sources; the prime field $\nu'_R \equiv \langle \nu_R \rangle$, where $\langle \cdots \rangle$ is the expectation value respect to generating function $Z(J, \eta)$. Based on this Ward identity, one can get all one-particle irreducible (1PI) vertices containing at least one external $\nu_R$.

As the first example, one can obtain an identity for the self-energy function $\Sigma^i(p)$, which is Dirac mass for the $\tau$-neutrino. Performing a functional derivative of eq. (9) with respect to the prime field $\psi'^i_L(0)$ and then putting external sources $\eta = 0$ and $J = 0$, and we obtain

$$
G\langle \Delta \left( \bar{Q}^a_L(0) \cdot t^a_R(0) \right) \rangle = 4w(p)m_t, \quad i = \frac{1}{2}
$$

where $\langle \cdots \rangle_o$ is the expectation value with vanishing external sources $\eta$ and $J_\mu$. Transforming into momentum space, we obtain

$$
\frac{1}{2}\Sigma^i(p) = 2Gw(p)\langle \bar{Q}^a_L_i(0) \cdot t^a_R(0) \rangle_o = 4w(p)m_t, \quad i = \frac{1}{2}
$$

where the well-known Wilson factor and the top-quark mass are,

$$
w(p) \equiv \frac{1}{2} \int d^4xe^{-ipx}\Delta(x) = \sum_\mu \left( 1 - \cos(p_\mu a) \right), \quad \frac{\pi}{a} = \text{cutoff}
$$

$$
m_t = \frac{G}{2}\langle \bar{Q}^a_L_i(0) \cdot t^a_R(0) \rangle_o.
$$
where $m_t$ is the top-quark mass of the $\bar{t}t$-condensate model \( \theta \). This clearly shows that in the low-energy limit, the self-energy function of Dirac $\tau$-neutrinos vanishes at the order of

$$\Sigma(p) \to O\left( \left( \frac{m_t}{\Lambda} \right)^2 m_t \right), \quad p \to m_t.$$  \hspace{1cm} (13)

This could be one of possible reasons for the smallness of Dirac neutrino masses.

As the second example, taking the functional derivative of eq. (9) with respect to $\nu_R'(0)$ and then putting external sources $\eta = 0$ and $J = 0$, we derive

$$\left( \gamma_\mu P_R \right)^{\beta \alpha} \partial_\mu \delta(x) - \frac{\delta^2 \Gamma}{\delta \nu_R'(0) \delta \bar{\nu}_R'(x)} = 0.$$ \hspace{1cm} (14)

Thus, the two-point function in eq. (14) is given as,

$$\int_x e^{-ipx} \frac{\delta^{(2)} \Gamma}{\delta \psi_R'(x) \delta \bar{\psi}_R'(0)} = i \gamma_\mu P^\mu,$$ \hspace{1cm} (15)

indicating that $\nu_R$ does not receive wave-function renormalization $Z_3$.

The third example is of the four-fermion interaction vertex. Analogously, one takes functional derivatives of the Ward identity (9) with respect to $\psi_i^L(0)$, $Q_{ai}^a(y)$ and $t_{ai}^a(z)$ and obtains 4-points interacting vertex involving an external $\nu'_R$,

$$\int_{xyz} e^{-i(q - p + q')x - i(p - q')y - i(p' - q')z} \frac{\delta^{(4)} \Gamma}{\delta \psi_i^L(0) \delta Q_{ai}^a(y) \delta t_{ai}^a(z) \delta \bar{\nu}_R'(x)} = 2Gw(p + q, \frac{q}{2}),$$ \hspace{1cm} (16)

where $p + \frac{q}{2}$ are the momenta of the $\nu_R(x)$ field and $p' + \frac{q}{2}$ are the momenta of the $t_{ai}^a(x)$ field; $p - \frac{q}{2}$ and $p' - \frac{q}{2}$ are the momenta of $\psi_i^L(x)$ field and $Q_{ai}^a(x)$ ($q$ is the momentum transfer.). This interacting vertex vanishes in the low-energy limit ($p, q \to 0$) for the same reason (13). Further, as the consequence of the Ward identity (9), all 1PI n-point vertices ($n > 4$) containing $\nu'_R$'s are just identical to zero.

$$\frac{\delta^{(n)} \Gamma}{\delta^{(n-1)}(\cdots) \delta \bar{\nu}_R'(x)} = 0, \quad n > 4.$$ \hspace{1cm} (17)

where $\delta^{(n-1)}(\cdots)$ indicates $(n - 1)$ derivatives with respect to other prime (external) fields.
These four identities eqs. (13, 15, 16) and (17) show us two conclusions owing to the $\nu_R$-shift-symmetry:

- the Dirac neutrino masses due to high-dimensions operators are extremely small;
- the right-handed neutrino $\nu_R(x)$ in low-energies is a free particle and decouples from other physical particles.

These conclusions do not change if the gauge interactions of the SM are taken into account, since the right-handed neutrino introduced is a gauge singlet and the $\nu_R$-shift-symmetry must not be violated by gauge interactions.

4 Composite vector-like phenomenon

In the effective lagrangian (1), the high-dimension operators implied by the “no-go” theorem should in principle be all possible operators allowed by the chiral gauge symmetries of standard model and the $\nu_R$-shift-symmetry (5). In section 2 and 3, we only discussed the dimension-6 and dimension-8 operators as far as the mass generation of quark and lepton sectors is concerned. Beside, there must be operators whose dimension are larger than 8. It is shown in ref. [10] that we need the dimension-10 operators to gauge-invariantly decouple unwanted “doublers” and avoid the vector-like phenomenon in low energies. This means that some dimension-10 operators should be relevant for “doublers” in the low-energy limit. To be more specific, the effective couplings of these dimension-10 operators certainly are momentum dependent. When these effective couplings are larger than a certain threshold $\epsilon$ in high-energies, three-fermion Weyl states with appropriate chiral quantum numbers are bound[4, 10]. These composite Weyl fermions couple to elementary Weyl fermions to form gauge-invariantly massive Dirac fermions and all 1PI vertices are vector-like consistently with the chiral gauge symmetries of the SM. We call this scenario composite vector-like phenomenon.

We will not enter into the details of this issue to show all vector-like vertices and composite spectra in the high-energy region. Instead, inspired by this composite vector-like phenomenon due to high-dimension operators in the high energy region, we postulate an extension (model) of the standard model beyond a certain energy scale $\epsilon$. 

right-handed three-fermion Weyl states possessing definite chiral
gauge quantum numbers of the $SU_L(2) \otimes U_Y(1)$ group are bound,
and the threshold $\epsilon$ associating with the binding energy is larger
than the weak scale $\Lambda_w (\sim 250 \text{GeV})$;

- for given a conserved quantum number of the $SU_L(2) \otimes U_Y(1)$
gauge symmetries, the number of these composite Weyl states
is equal to the number of the elementary Weyl states, and they
couple together to form massive Dirac fermions;

- the spectra and vertices are vector-like consistently with the $SU_L(2) \otimes$
$U_Y(1)$ symmetries, and the $W^\pm$ gauge bosons possesses vector-like
coupling to these composite Dirac fermions.

Within the context of the third lepton generation, we explicitly discuss
these three assumptions. It is assumed there is an intermediate energy-
threshold $\epsilon$ between the cutoff and the weak scale ($v \sim 250 \text{GeV}$) of the
spontaneous symmetry breaking,

$$250 \text{GeV} < \epsilon < \Lambda,$$  \hspace{1cm} (18)

above this energy-threshold, the effective high-dimension operators are strong
enough to form the three-fermion bound states that are given by\footnote{We do not discuss baryon number violating case, where three-fermion states are anti-proton and anti-neutron.}

$$\nu_R^3 \sim (\bar{\nu}_R \cdot \nu_L) \nu_R, \quad \tau_R^3 \sim (\bar{\tau}_R \cdot \tau_L) \tau_R,$$ \hspace{1cm} (19)

which are right-handed Weyl fermions with the appropriate gauge quantum
number of the $SU_L(2) \otimes U_Y(1)$ symmetries. Whereas, to coincide with the
parity-violating gauge coupling observed in low-energies, at the threshold
$\epsilon$ (18), these three-fermion bound states turn to three-fermion cuts, where
they dissolve to their constituents (18) because of vanishing their binding
energy. This intermediated scale $\epsilon$ should be determined by effective high-
dimension operators (couplings) and vanishing the binding energy of three-
fermion states.

These three-fermion Weyl states (19) couple to the elementary Weyl fields
$\nu_L, \tau_L$ to form gauge invariant massive Dirac fermions,

$$\{\nu_L, \nu_R^3\}; \quad \{\tau_L, \tau_R^3\}.$$ \hspace{1cm} (20)
These massive Dirac fermions carry appropriate quantum numbers of the $SU_L(2)$ symmetry and couple to the $W^\pm$ boson. The chiral gauge symmetries of the SM is exact in high-energies, if we do not consider the soft spontaneous symmetry breaking of the Higgs mechanism.

The above discussions are straightforwardly generalized to the first and second generations. This scenario of the composite vector-like phenomenon above the intermediate scale (18) is reminiscent of the “left-right” symmetric extensions ($SU_L(2) \otimes SU_R(2) \otimes U_{B-L}(1)$) of the Standard model [11]. However, comparing with the “left-right” symmetric model, it should be noted that in this model, (i) the gauge symmetries are still $SU_L(2) \otimes U_Y(1)$; (ii) there are no needs of new elementary fermions and gauge bosons accommodated by the $SU_R(2)$ gauge group; (iii) the intermediate scale $\epsilon$ is not due to spontaneous symmetry breakings, no Goldstone bosons associate with the form of three-fermion states at the scale $\epsilon$.

In this early stage, we make no attempt to give a complete description of various effective vertices (1PI) in this model. In this section, we wish to reconsider the self-energy functions (Dirac masses) of neutrinos $\Sigma_{\nu_i}(p)$ and charged leptons $\Sigma_{\ell_i}(p)$ by taking into account the possible relevant 1PI vertices function raised in the high-energy region of this model. The right-handed fermion states are the mixing states comprising the elementary state $\nu_R(\tau_R)$ and the composite state $\nu_R^3(\tau_R^3)$:

$$\Psi_R^\nu = (\nu_R, \nu_R^3); \quad \Psi_R^\tau = (\tau_R, \tau_R^3). \tag{21}$$

The composite Dirac particle instead of (20) are then given,

$$\Psi_D^\nu = \{\nu_L, \nu_R^\nu\}; \quad \Psi_D^\tau = \{\tau_L, \tau_R^\tau\}. \tag{22}$$

If the soft spontaneous breaking of chiral gauge symmetries is introduced, the self-energy functions $\Sigma_{\nu_i}(p)$ ($\Sigma_{\ell_i}(p)$) are coupling between $\nu_L(\tau_L)$ and mixing right-handed fermion states $\Psi_R^\nu(\Psi_R^\tau)$ given by (21). This clearly modifies the self-energy functions $\Sigma_{\nu_i}(p)$ ($\Sigma_{\ell_i}(p)$) of neutrinos and charged leptons, which were the couplings (mass operators) between $\nu_L(\tau_L)$ and $\nu_R(\tau_R)$ in the SM.

According to eq.(22), the modified self-energy functions are the effective vertices coupling between the $\nu_L(\tau_L)$ and mixing states $\psi_R^\nu(\psi_R^\tau)$. The effective gauge coupling of $W^\pm$-bosons to composite Dirac fermions is vector-like in the high-energy region. Together with $W^\pm$’s purely left-handed gauge coupling
observed in the low-energy region, one can write an effective gauge coupling as,

$$\Gamma_{ij}^\mu(q) = i \frac{g_2}{2\sqrt{2}} V_{ij} \gamma_\mu (P_L + f(q))$$

(23)

$$f(q) \neq 0, \quad q \geq \epsilon,$$

(24)

where $g_2$ is the $SU_L(2)$ coupling. In eq.(23), the non-vanishing of the vector-like vertex function $f(q)$ in the high-energy region $\epsilon < q < \Lambda$ is clearly related to the existence of the three-fermion states (19). Upon the energy threshold (18) where the three-fermion states turn to three-fermion cuts and dissemble into their constituents, the effective vertex function $f(q)$ must vanishes,

$$f(q)|_{q\to\epsilon+0} \to 0.$$  

(25)

The $V_{ij}$ in eq.(23) is the CKM-matrix [12], since all fermionic states discussed are not the eigenstates of the chiral gauge symmetries of the SM.

Because of the effective gauge coupling (23), we find that the $W^\pm$ bosons have the contributions to the Schwinger-Dyson equations for the self-energy functions $\Sigma_{\nu}(p)$ and $\Sigma_{l}(p)$ for $(p \geq \epsilon)$. We can approximately write the $W^\pm$-boson’s contributions:

$$W_{\nu i}(p) = \left( \frac{g_2}{2\sqrt{2}} \right)^2 |V_{ij}|^2 \int_{|p'|\geq\epsilon} \frac{f(p' - p)}{(p - p')^2 + M^2_{w} p^2 + \Sigma_{\nu}(p'^2)},$$

$$W_{l j}(p) = \left( \frac{g_2}{2\sqrt{2}} \right)^2 |V_{ji}|^2 \int_{|p'|\geq\epsilon} \frac{f(p' - p)}{(p - p')^2 + M^2_{w} p^2 + \Sigma_{l}(p'^2)},$$

(26)

where the integration of the internal momentum $p'$ starts from the intermediate threshold $\epsilon$ to the cut-off $\Lambda$.

5 Mass generation of the lepton sector

With the $W$-boson’s contributions (26), the Schwinger-Dyson equations for the self-energy functions of the neutrinos and charged leptons respectively turn out to be highly non-trivial and coupled,

$$\Sigma_{\nu}(p) = W_{\nu i}(p);$$

$$\Sigma_{l j}(p^2) = \Sigma_{q j}(\Lambda) + W_{l j}(p) + 3e^2 \int_{p'}^\Lambda \frac{1}{(p - p')^2 + \Sigma_{l j}(p'^2)},$$

(27)

(28)
where the bare down quark masses are given as,
\[ \Sigma_{q_j}(\Lambda) = m_d(\Lambda), m_s(\Lambda), m_b(\Lambda), \] (29)
are due to the four-fermion interaction \([1]\) and the contribution of \([8]\) to neutrinos masses is neglected. This \(W(p)\)'s contributions are perturbative additions to the original Schwinger-Dyson equations of the SM. One can see that eq.(26) mixes up the Schwinger-Dyson equations for fermionic self-energy functions of different generations and charge sectors. We have no reason to put the CKM-matrix \(V_{ij} = \delta_{ij}\), since the mixing between generations could be very large.

To solve the integral equations (27,28), one way is to divide them into two integral equations corresponding to the regions \(p \in (0, \epsilon)\) and \(p \in (\epsilon, \Lambda)\) respectively, and use the continuation of self-energy functions \(\Sigma(p)\) at the scale \(\epsilon\) to match two solutions. Here, we alternatively adopt a simple and approximate way to solve these coupled integral equations. Assuming the scale \(\epsilon\) is large enough and \(p' > \epsilon \gg 1\), we approximate eqs.(26) to be,
\[ W_{\nu_i}(p) \simeq \alpha_w(p)|V_{ij}|^2\Sigma_{l_j}(\Lambda), \quad W_{l_j}(p) \simeq \alpha_w(p)|V_{ji}|^2\Sigma_{\nu_i}(\Lambda), \] (30)
where
\[ \alpha_w(p) \simeq \left( \frac{g_2}{2\sqrt{2}} \right)^2 \int_{p' \geq \epsilon} \frac{f(p' - p)}{(p - p')^2 + M_w^2 p'^2} \frac{1}{p'}. \] (31)
For the low-energy \(p \ll \epsilon\), assuming \(f(p') \simeq f\), as a slow-varying (small) function of \(p'(p' > \epsilon)\), we get
\[ \alpha(p) = \frac{\alpha_2}{16\pi} f\ell n\frac{\Lambda}{\epsilon}, \quad \alpha_2 = \frac{g_2^2}{4\pi}. \] (32)
For the high-energy \(p > \epsilon \gg 1\), we approximately set
\[ \alpha_w(p) \simeq \alpha_w(\Lambda), \] (33)
as an unknown constant.

Using eqs.(27,32), we obtain the three relations (gap-equations) between neutrino masses and charged lepton masses,
\[ \Sigma_{\nu_i}(p) = \frac{\alpha_2}{16\pi} f\ell n\frac{\Lambda}{\epsilon}|V_{ij}|^2\Sigma_{l_j}(\Lambda), \quad p \ll \epsilon. \] (34)
We find that neutrino masses are related to charged lepton masses via flavour
mixing, and neutrino masses are zero, if the intermediated scale $\epsilon = \Lambda$, which
means no vector-like phenomenon described in the previous section.

We can straightforwardly solve the coupled integral equation (28) of
charged leptons in the high-energy region. In the ultraviolet region ($x =
p^2 \gg 1$), the nonlinearity is negligible and the integral eq.(28) can be con-
verted to the following boundary value problem:

$$\frac{d}{dx} \left( x^2 \Sigma'_{ij}(x) \right) + \frac{\alpha}{4@\alpha_c} \Sigma_{ij}(x) = 0,$$

$$\Lambda^2 \Sigma'_{ij}(\Lambda^2) + \Sigma_{ij}(\Lambda^2) = \Sigma_{qj}(\Lambda) + \alpha_w(\Lambda)|V_{ji}|^2 \Sigma_{\nu_i}(\Lambda),$$

These are differential equations with the coupled inhomogeneous boundary
conditions at the cutoff. Those inhomogeneous terms act as bare mass terms
in the integral equation (28).

The generic solution to eq.(35) for ($x \gg 1$) is given:

$$\Sigma_{ij}(x) \simeq \frac{A_{ij} \mu^2}{\sqrt{x}} \sinh \left( \frac{1}{2} \sqrt{1 - \frac{\alpha}{\alpha_c} \ln(x^2/\mu^2)} \right),$$

where $A_{ij}$ are arbitrary constants, and $\mu$ is an inferred scale. Thus, we obtain
the gap-equation of this coupled system from the boundary condition (36):

$$\alpha_w(\Lambda)|V_{ji}|^2 \Sigma_{\nu_i}(\Lambda) = \Sigma_{ij}(\Lambda) - \frac{1}{4 \alpha_c} \Sigma_{ij}(\Lambda) - \Sigma_{qj}(\Lambda),$$

where

$$\theta = \frac{1}{2} \sqrt{1 - \frac{\alpha}{\alpha_c} \ln(\Lambda^2/\mu^2)},$$

The first conclusion can be derived from these gap-equations (34,38) is that
if the down quarks are massive, the self-energy functions of the neutrinos,
charged leptons must be non-trivial

$$\Sigma_{\nu_i}(\Lambda) \neq 0; \ \text{and} \ \Sigma_{ij}(\Lambda) \neq 0,$$

they are generated by the explicit symmetry breaking (34,38).

We turn to find the solution of the gap-equation (38) in the low-energy
limit ($\mu \ll \Lambda$). Using eq.(37) for $\Sigma_{ij}(\Lambda)$, we obtain the gap-equations for
$\mu \ll \Lambda$:

$$\alpha_w(\Lambda)|V_{ji}|^2 \Sigma_{\nu_i}(\Lambda) = \Sigma_{ij}(\Lambda) - \frac{1}{4 \alpha_c} \Sigma_{ij}(\Lambda) - \Sigma_{qj}(\Lambda).$$
Since down quark and charged lepton masses at the cutoff are equal (5) if gauge interactions are turned off, we have a cancellation in the RHS of gap-equation (41), as a result,

\[ \alpha_w(\Lambda)|V_{ij}|^2\Sigma_l(\Lambda) = \Sigma_{\nu_i}(\Lambda); \] (42)

\[ \alpha_w(\Lambda)|V_{ji}|^2\Sigma_{\nu_j}(\Lambda) = -\frac{1}{4}\alpha_c\Sigma_l(\Lambda). \] (43)

These are three gap-equations relating neutrino and charged lepton masses at the cutoff.

6 Lepton masses and mixing angles

In previous section, we obtained six gap-equations (14) relating neutrino and charged lepton masses at the cutoff. Noticing the mass ratio of fermions in the same charge sector (but different generations) should be scaling invariant (renormalization group invariant), we take ratios between two equations of the gap-equations (34), and two equations of the gap-equations (43). We arrive at:

\[ m_{\nu_e} = |V_{\nu e}\nu_e|^2m_e + |V_{\nu e}\mu|^2m_\mu + |V_{\nu e}\tau|^2m_\tau; \] (44)

\[ m_{\nu_\mu} = |V_{\nu\mu}\nu_e|^2m_e + |V_{\nu\mu}\mu|^2m_\mu + |V_{\nu\mu}\tau|^2m_\tau; \] (45)

\[ m_{\nu_\tau} = |V_{\nu\tau}\nu_e|^2m_e + |V_{\nu\tau}\mu|^2m_\mu + |V_{\nu\tau}\tau|^2m_\tau; \] (46)

and

\[ m_e = |V_{ee}\nu_e|^2m_e + |V_{e\mu}\nu_e|^2m_\mu + |V_{e\tau}\nu_e|^2m_\tau; \] (47)

\[ m_\mu = |V_{\mu e}\nu_e|^2m_e + |V_{\mu\mu}\nu_e|^2m_\mu + |V_{\mu\tau}\nu_e|^2m_\tau; \] (48)

\[ m_e = |V_{ee}\nu_e|^2m_e + |V_{e\mu}\nu_e|^2m_\mu + |V_{e\tau}\nu_e|^2m_\tau; \] (49)

In these equations, all fermion masses are defined at the same low-energy scale. There are only four independent equations that completely determine the four CKM mixing angles in terms of six lepton masses.
The analysis of these four equations to find explicit relations between masses and mixing angles will be presented in the coming paper soon. This paper is written for the proceeding of the 1997 Shizuoka workshop on masses and mixings of quarks and leptons (March 19-21). I thank Prof. Yoshio Koide and other organizers for providing me the financial support to participate this workshop.

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