TSIRELSON’S PROBLEM AND AN EMBEDDING THEOREM FOR GROUPS ARISING FROM NON-LOCAL GAMES

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Abstract. Tsirelson’s problem asks whether the commuting operator model for two-party quantum correlations is equivalent to the tensor-product model. We give a negative answer to this question by showing that there are non-local games which have perfect commuting-operator strategies, but do not have perfect tensor-product strategies. The weak Tsirelson problem, which is known to be equivalent to Connes embedding problem, remains open.

The examples we construct are instances of (binary) linear system games. For such games, previous results state that the existence of perfect strategies is controlled by the solution group of the linear system. Our main result is that every finitely-presented group embeds in some solution group. As an additional consequence, we show that the problem of determining whether a linear system game has a perfect commuting-operator strategy is undecidable.

CONTENTS

1. Introduction 2
2. Linear system games, hypergraphs, and solution groups 5
3. The embedding theorem and consequences 7
4. Presentations by involutions 8
5. The wagon wheel embedding 9
6. Pictures for groups generated by involutions 13
6.1. Pictures as planar graphs 15
6.2. Groups and labellings of pictures 19
7. Pictures over solution groups and hypergraphs 21
8. A category of hypergraphs 24
9. Cycles and outer faces 31
10. Pictures over suns 34
11. Stellar cycles and constellations 40
11.1. Structure of constellations 42
11.2. Stellar cycles 42
11.3. Covers versus copies 44
11.4. Cycles covered by other cycles 46
11.5. Proof of Theorem 11.4 47
12. Proof of the embedding theorem 49
References 56
1. Introduction

In a two-player non-local game, the players, commonly called Alice and Bob, are physically separated and unable to communicate. They each receive a question chosen at random from a finite question set, and reply with a response from a finite answer set. If the joint answers meet a predetermined winning condition dependent on the joint questions, then Alice and Bob win; otherwise they lose. The rules of the game, including the winning condition and distribution on questions, are completely known to Alice and Bob, and they can arrange in advance a strategy which will maximize their success probability. However, since they cannot communicate during the game, they may not be able to play perfectly, i.e. win with probability one.

Classically, Alice and Bob’s strategy for a non-local game is described by a local hidden variable model. Bell’s famous theorem states that Alice and Bob can achieve better results than is possible with a local hidden variable model if they share an entangled quantum state [Bel64]. Since Bell’s discovery, non-local games have been heavily studied in physics, mathematics, and computer science; see [CHSH69, FC72, Cir80, AGR82, CHTW04, BPA+08, NPA08, JPPG+10, KV11, KKM+11, HBD+15] for a small sample of results.

Despite this, a number of foundational questions remain open, chief among which is Tsirelson’s problem: A quantum strategy for a non-local game can be described as a set of measurement operators on Hilbert spaces $H_A$ and $H_B$, along with a quantum state in the joint space $H = H_A \otimes H_B$. We refer to this as the tensor-product model. While the tensor-product model is often the default, there is another choice: a quantum strategy can be described as a set of measurements and quantum state on a shared Hilbert space $H$, with the property that Alice’s measurement operators commute with Bob’s measurement operators. This commuting-operator model is used, for instance, in algebraic quantum field theory [HK64]. The observable consequences of a strategy in either model are captured by the correlation matrix of the strategy. If $H$ is finite-dimensional, every correlation matrix arising from a commuting-operator strategy can be realized using a tensor-product strategy. Tsirelson’s problem asks whether this is true for a general Hilbert space.

This problem has an interesting history. Tsirelson originally stated the problem in a survey on Bell inequalities [Ts93], and claimed without proof that the two models gave rise to the same set of correlation matrices. He later retracted this claim, and posted the question to a list of open problems in quantum information theory [Ts06]. Subsequent authors [NPA08, JNP+11, Fri12, NCPGV12, PT15, DP16, PSS+16] studying Tsirelson’s problem have

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1Usually under the name “Bell tests” or “Bell inequalities”. The term “non-local games” is more recent.
defined the set of tensor-product strategies in several different ways. Paulsen and Todorov [PT15] (see also Dykema and Paulsen [DP16]) observe that these variations lead to a hierarchy of sets of correlation matrices

\[ C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}, \]

where \( C_q \) is the set of correlations arising from tensor-product strategies on finite-dimensional Hilbert spaces, \( C_{qs} \) is the set of correlations arising from tensor-product strategies (on possibly infinite-dimensional Hilbert spaces) with a vector state, \( C_{qa} = C_q \) is the set of correlations which are limits of correlations in \( C_q \), and \( C_{qc} \) is the set of correlations arising from commuting-operator strategies. If we restrict to non-local games with question sets of size \( n \) and answer sets of size \( m \), then all these sets are convex subsets of \( \mathbb{R}^{m^2 n^2} \), and none of the inclusions were previously known to be strict. Thus for each \( t \in \{ q, qs, qa \} \) there is a Tsirelson problem asking whether \( C_{qc} = C_t \). Ozawa, building on the work of Junge, Navascués, Palazuelos, Pérez-García, Sholz, and Werner [JNP+11] and work of Fritz [Fri12], has shown that \( C_{qc} = C_{qa} \) if and only if Connes’ embedding conjecture is true [Oza13]. At the other end of the hierarchy, if \( C_q \) was equal to \( C_{qc} \) then every correlation matrix, whether commuting operator or tensor-product, would arise from a finite-dimensional Hilbert space. The “middle” version, which asks whether \( C_{qc} = C_{qs} \), seems closest to Tsirelson’s original problem statement.

The first main result of this paper is that there is a non-local game which can be played perfectly with a commuting-operator strategy, but which cannot be played perfectly using a tensor-product strategy with a vector state. Thus we resolve the middle version of Tsirelson’s problem by showing that \( C_{qc} \neq C_{qs} \).

This game is interesting from the perspective of quantum information and computation, where a non-local game is often regarded as a computational scenario in which better results can be achieved with entanglement as a resource. From this point of view, it is natural to ask how much entanglement is needed to play a game optimally, and in particular, whether every game can be played optimally on a finite-dimensional Hilbert space. The game we construct shows that this is not possible, at least if we allow commuting-operator strategies. Previously-known examples of this type have involved either quantum questions [LTW13, RV15] or infinite answer sets [MV14].

One reason we would have desired that every game have an optimal strategy on a finite-dimensional Hilbert space is that it would make it possible to determine the optimal winning probability of a non-local game over entangled strategies. At present the only known methods for this task, aside from brute-force search over strategies, are variants of the Navascués-Pironio-Acín (NPA) hierarchy [NPA08, DLTW08]. Given a non-local game, the NPA hierarchy provides a sequence of upper bounds which converge to the optimal winning
probability in the commuting-operator model. However, the hierarchy does not provide, outside of special cases, a stopping criterion, i.e. a way to tell if the value will fall below a given threshold. Our second main result is that it is undecidable to determine if a non-local game can be played perfectly with a commuting-operator strategy. In particular, this implies that there is no stopping criterion for the NPA hierarchy which applies to all games.

The games we consider are binary linear system games, so named because they arise from linear systems over $\mathbb{Z}_2$. Such games have been studied previously in [CM14, Ark12, Ji13]. Cleve and Mittal implicitly associate a certain group to every linear system over $\mathbb{Z}_2$, such that perfect tensor-product strategies for the game correspond to certain finite-dimensional representations of the group [CM14]. We call this group, which is analogous to the solution space of a linear system, the solution group. In [CLS16] it is shown that perfect commuting-operator strategies for a binary linear system game correspond to certain possibly-infinite-dimensional representations of the solution group. Solution groups form an interesting class of groups. They are finitely presented, but their presentations must satisfy a property which in [CLS16] is called local compatibility: if

$$x_1 \cdots x_n = 1$$

is a relation, where $x_i, 1 \leq i \leq n$, are not necessarily distinct generators of the group, then the presentation must also contain the relations $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. This condition is natural from the perspective of quantum mechanics, where two observables commute if and only if the observables correspond to quantities which can be measured (or known) simultaneously. Group relations of this exact type can be found in contextuality theorems of Mermin and Peres [Mer90, Per90, Mer93]. Local compatibility is a priori a strong constraint on group presentations. Our primary result, on which our other two results are based, is that any finitely-presented group can be embedded in a solution group. This embedding theorem allows us to extend results from combinatorial group theory, such as the existence of a non-residually-finite group, to solution groups. Using non-residually-finite groups to recognize infinite-dimensional state spaces was previously proposed in [Fri13].

A number of open questions remain, such as whether there are other separations between the correlation sets $C_t(m,n)$, and (assuming that $C_{qa} \neq C_{qc}$) whether it is decidable to determine the optimal value of a non-local game over $C_{qa}$. The argument used in this paper cannot distinguish between $C_{qs}$ and $C_q$, and we do not expect our methods to help with other separations between correlation sets.

The rest of this paper is structured as follows. In the next section, we give some background on binary linear system games, and introduce the solution group of a linear system. In Section 3 we state the embedding theorem for
solution groups, and prove our two main results as corollaries. The rest of the paper is concerned with the proof of the embedding theorem. The main technical tool used is pictures of groups. We give an overview of pictures tailored to solution groups in Sections 6 and 7; expert readers will want to skip or briefly review these sections on first reading. In dealing with pictures, it is more convenient to use hypergraphs instead of linear systems, and we introduce hypergraphs into our definitions immediately in Section 2. In Section 8 we develop a notion of morphisms between hypergraphs; the concept is similar to graph minors, but differs from standard notions of hypergraph minors as in [RS10]. The more technical aspects of the proof are contained in Sections 9-11; we suggest that the reader skim these sections on first reading and proceed to Section 12. For a second reading, Corollary 10.4 might provide a good initial target. One thing we do not provide is a definition of the correlation sets mentioned above, or a definition of binary linear system games; instead we defer to the excellent references listed above.

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2. Linear system games, hypergraphs, and solution groups

Binary linear system games are based on linear systems $Ax = b$ over $\mathbb{Z}_2$. It is convenient to think of linear systems in terms of hypergraphs. By a hypergraph, we mean a triple $\mathcal{H} = (V, E, A)$, where $V = V(\mathcal{H})$ and $E = E(\mathcal{H})$ are the sets of vertices and edges respectively, and $A \in \mathbb{Z}_2^{V \times E}$ is the incidence matrix between $V$ and $E$, so $A_{ve} \geq 0$ is the degree of incidence between edge $e$ and vertex $v$. We say that $v$ and $e$ are incident if $A_{ve} > 0$. If $v \in V$, then the degree of $v$ is $|v| = \sum_e A_{ve}$. Similarly if $e \in E$ then $|e| = \sum_v A_{ve}$. We say that $\mathcal{H}$ is simple if $A_{ve} \leq 1$ for all $v \in V$ and $e \in E$; $k$-regular if $|v| = k$ for all $v \in V$, and a graph if $|e| = 2$ for all $e \in E$.

Note that this definition of hypergraphs allows both isolated vertices and isolated edges, i.e. vertices (resp. edges) which are incident to no edges (resp. vertices). A $\mathbb{Z}_2$-vertex labelling of $\mathcal{H}$ is a function $b : V \to \mathbb{Z}_2$. With these conventions, there is a correspondence between linear systems $Ax = b$ and simple hypergraphs $\mathcal{H}$ with a vertex labelling $b$. From this point of view, the edges of a hypergraph correspond to the variables of a linear system, and the vertices correspond to constraints. Similarly, pairs $(\mathcal{H}, b)$ where $\mathcal{H}$ is not necessarily simple correspond to linear systems $Ax = b$ over $\mathbb{Z}_2$ with a choice of non-negative integer representatives for the coefficients $A_{ve}$. 
To any linear system \(Ax = b\), we can associate a linear system non-local game \(G\), and a group \(\Gamma\) \cite{CM14, CLS16}. The group \(\Gamma\) is the focus of attention of this paper, and is defined as follows.

**Definition 2.1.** Let \(\mathcal{H} = (V, E, I)\) be a (not necessarily simple) hypergraph and let \(b\) be a function \(V \to \mathbb{Z}_2\). The solution group \(\Gamma = \Gamma(\mathcal{H}; b)\) associated to \(\mathcal{H}\) and \(b\) is the group generated by \(\{x_e : e \in V\} \cup \{J\}\), subject to relations:

1. \(x_e^2 = 1\) for all \(e \in E\) and \(J^2 = 1\) (i.e. \(\Gamma\) is generated by involutions)
2. \([x_e, J] = 1\) for all \(e \in E\) (i.e. \(J\) is central),
3. \([x_e, x_{e'}] = 1\) if there is some vertex \(v\) incident to both \(e\) and \(e'\), and
4. \(\prod_e x_e^{A_{ve}} = J^{b_v}\) for all \(v \in V\).

The null solution group is the group \(\Gamma(\mathcal{H}) := \Gamma(\mathcal{H}, 0)\).

We call the last two types of relations *commuting relations* and *linear relations* respectively. The definition of the linear relations assumes that \(E\) is ordered, but the choice of order is irrelevant because of the commuting relations. Note that if \(v\) and \(e\) are incident and \(A_{ve}\) is even, then the linear relations

\[
\prod_{e'} x_{e'}^{A_{ve'}} = J^{b_v} \quad \text{and} \quad \prod_{e' \neq e} x_{e'}^{A_{ve'}} = J^{b_v}
\]

are equivalent. However, the fact that \(A_{ve} > 0\) might still lead to commuting relations that wouldn’t hold otherwise.

The one-dimensional representations \(\pi\) of \(\Gamma(\mathcal{H}; b)\) in which \(\pi(J) \neq 1\) correspond to the solutions of the linear system \(Ax = b\). Higher-dimensional representations of \(\Gamma(\mathcal{H}; b)\) with this property can be thought of as quantum solutions of \(Ax = b\). This is justified by the following theorem, which relates solution groups to non-local games.

**Theorem 2.2** \cite{CM14, CLS16}. Let \(Ax = b\) be a linear system over \(\mathbb{Z}_2\), where \(A\) is a non-negative integral matrix, let \(G\) be the associated linear system non-local game, and let \(\Gamma\) be the corresponding solution group. Then:

- \(G\) has a perfect quantum commuting-operator strategy if and only if \(J \neq 1\) in \(\Gamma\).
- \(G\) has a perfect quantum tensor-product strategy if and only if \(G\) has a perfect finite-dimensional quantum strategy, and this happens if and only if \(\Gamma\) has a finite-dimensional representation \(\pi\) with \(\pi(J) \neq 1\).
The first part of this theorem is due to [CLS16], while the second part is due to [CM14].

3. THE EMBEDDING THEOREM AND CONSEQUENCES

In light of Theorem 2.2, we would like to understand the structure (or lack thereof) of solution groups for linear system games. Recall that a presentation \( \langle S : R \rangle \) of a group \( G \) is a set \( S \) and subset \( R \) of the free group \( F(S) \) generated by \( S \), such that \( G = F(S)/(R) \), where \( (R) \) is the normal subgroup generated by \( R \). A group is finitely presented if it has a presentation \( \langle S : R \rangle \) where both \( S \) and \( R \) are finite. Our primary result is that understanding solution groups is as hard as understanding finitely presented groups.

**Theorem 3.1.** Let \( G \) be a finitely presented group, let \( J' \in G \) be a central element with \( (J')^2 = 1 \), and let \( w_1, \ldots, w_n, n \geq 0 \) be a sequence of elements in \( G \) such that \( w_i^2 = 1 \) for all \( 1 \leq i \leq n \). Then there is a hypergraph \( \mathcal{H} \), a vertex labelling function \( b : V(\mathcal{H}) \to \mathbb{Z}_2 \), a sequence of edges \( e_1, \ldots, e_n \) in \( \mathcal{H} \), and a homomorphism \( \phi : G \to \Gamma(\mathcal{H}, b) \) such that \( \phi \) is an embedding, \( \phi(J') = J \), and \( \phi(w_i) = x_{e_i} \) for all \( 1 \leq i \leq n \).

The hypergraph \( \mathcal{H} \), vertex labelling \( b \) and homomorphism \( \phi \) can be explicitly constructed from a presentation of \( G \). This is described in Sections 4 and 5. Theorem 3.1 is proved at the beginning of Section 5 via reduction to another embedding theorem. The proof of this latter embedding theorem (and hence the proof of Theorem 3.1) is completed in Section 12.

In the remainder of this section, we prove two consequences of Theorem 3.1. The first is an answer to Tsirelson’s problem.

**Corollary 3.2.** There is a linear system non-local game which has a perfect quantum commuting-operator strategy, but does not have a perfect quantum tensor-product strategy.

**Proof.** Suppose that \( G \) is a finitely-presented group with a central element \( J' \) of order two, such that \( \pi(J') = 1 \) for every finite-dimensional representation \( \pi \) of \( G \). By Theorem 3.1, there is an embedding of \( G \) in a solution group \( \Gamma \) which identifies \( J' \) with \( J \). In particular, this implies that \( J \neq 1 \) in \( \Gamma \), so the associated linear system non-local game \( G \) must have a perfect quantum commuting-operator strategy by the first part of Theorem 2.2. If \( \pi \) is a finite-dimensional representation of \( \Gamma \), then \( \pi(J) = \pi|_G(J') = 1 \). By the second part of Theorem 2.2, the associated linear system non-local game does not have a perfect quantum tensor-product strategy.

To finish the proof, we construct a group \( G \) with the above property. Consider Higman’s group

\[ H_0 = \langle a, b, c, d : aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle. \]
It is well-known that $H_0$ has no non-trivial linear representations, and that the generators $a, b, c, d$ of $H_0$ have infinite order \cite{Hig51,Ber94}. Let $H = H_0 \times \mathbb{Z}_2$, and let $J \in H$ denote the generator of the $\mathbb{Z}_2$-factor. Let $G$ be the HNN extension of $H$ by the automorphism of $\langle a, J \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$ sending $J \mapsto J$ and $a \mapsto aJ$. By the properties of the HNN extension, $H$ is a subgroup of $G$, and in particular $J$ is non-trivial in $G$. Furthermore, we can construct a presentation for $G$ from a presentation of $H$ by adding a generator $x$ and relations $[x, J] = 1$ and $[x, a] = J$. The former relation implies that $J$ is central in $G$. Finally, if $\pi$ is a finite-dimensional representation of $G$, then $\pi|_{H_0}$ is trivial, and in particular, $\pi(a) = 1$. But this implies that

$$\pi(J) = \pi([x, a]) = 1,$$

as required. \hfill \square

We note that any non-residually-finite group can be used in place of Higman’s group in the above proof.

It would be interesting to know the smallest linear system for which the corresponding game can be played perfectly only with commuting-operator strategies. No effort is made to reduce the size of the linear system in the proof of Theorem 3.1, and the main construction from Section 5 depends on the number of variables and the total length of the relations in the presentation of $G$. If we naively follow the proof through for the group in Corollary 3.2, we get a linear system with roughly 600 variables and 450 linear relations. Making some obvious improvements in Section 4 can get this down to 400 variables and 300 relations. It seems likely that this is far from the smallest possible example.

The second consequence concerns the difficulty of determining whether a non-local game has a perfect commuting-operator strategy.

**Corollary 3.3.** It is undecidable to determine if a binary linear system game has a perfect commuting-operator strategy.

*Proof.* By Theorem 2.2 determining if a binary linear system game has a perfect strategy is equivalent to determining if $J \neq 1$ in a solution group. Because Theorem 3.1 is constructive, this is in turn equivalent to the following decision problem: given a group presentation $G = \langle S : R \rangle$ and a word $J$ in the generators $S$ such that $J \in Z(G)$ and $J^2 = 1$, decide if $J = 1$ in $G$.

We claim that the word problem for groups can be reduced to this latter problem. Indeed, given a finitely presented group $K = \langle S : R \rangle$ and a word $w \in F(S)$ in the generators of $K$, it is possible to recursively construct a finitely presented group $L_w$ with the property that $K$ is a subgroup of $L_w$ if $w \neq 1$, and $L_w$ is trivial if $w = 1$ (see \cite{LS77} pg. 190), where this construction is attributed to Rabin). The presentation of $L_w$ can be constructed by adding
generators and relations to the presentation of $K$. Let $H_w$ be the result of applying this construction to the group $K \times \mathbb{Z}$, and let $z$ be the generator of $H_w$ corresponding to the $\mathbb{Z}$-factor in $K \times \mathbb{Z}$. If $w = 1$, then $H_w$ is trivial, and $z = 1$. If $w \neq 1$, then $K \times \mathbb{Z}$ is a subgroup of $H_w$, and the order of $z$ is infinite. Finally, construct a group $G_w$ by adding two generators $x$ and $J$ to the presentation of $H_w$, along with relations $J^2 = [x, J] = 1$, $[s, J] = 1$ for all generators $s$ of $H_w$, and $[x, z] = J$. As in Corollary 3.2, if $w \neq 1$ then $G_w$ is the HNN extension of the group $H_w \times \mathbb{Z}$ by the automorphism of the subgroup $\langle z, J \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$ sending $J \mapsto J$ and $z \mapsto zJ$. Thus, if $w \neq 1$ then $J \neq 1$ in $G_w$. If $w = 1$, then $z = 1$ in $H_w$, and consequently $J = [x, z] = 1$ in $G_w$. This completes the reduction. 

Although not used in either of the above corollaries, Theorem 3.1 also allows us to embed a finitely-presented group $G$ in a solution group $\Gamma$ so that a given set of involutions of $G$ become generators of $\Gamma$. This can be used to prove that other tasks involving solution groups are undecidable. For instance, it is undecidable to determine if a generator $x_\epsilon$ of a null solution group is non-trivial.

4. Presentations by involutions

We are interested primarily in groups which (a) have a distinguished central element of order at most two, and (b) are generated by involutions. For clarity in subsequent sections, we encode these conditions in two formal definitions.

**Definition 4.1.** A group over $\mathbb{Z}_2$ is a group $G$ with a distinguished central element $J = J_G$ such that $J^2 = 1$.

A morphism $G_1 \to G_2$ over $\mathbb{Z}_2$ is a group homomorphism sending $J_{G_1} \mapsto J_{G_2}$. Similarly, an embedding over $\mathbb{Z}_2$ is an injective morphism over $\mathbb{Z}_2$.

Note that $J$ is allowed to be the identity in this definition. This is so that we can construct groups over $\mathbb{Z}_2$ by starting with some finite presentation, picking an element $J' \in \mathcal{F}(S)$, and adding relations $(J')^2 = 1$ and $[J', s] = 1$ for all $s \in S$. By allowing $J = 1$, we can do this even if $J'$ becomes trivial.

Elements of $\mathcal{F}(S)$ are represented by words over $\{s, s^{-1} : s \in S\}$. Every element $r \in \mathcal{F}(S)$ can be represented uniquely as $s_1^{a_1} \cdots s_n^{a_n}$, where $a_i \in \{\pm 1\}$, and $a_i = a_{i+1}$ whenever $s_i = s_{i+1}$. A word meeting these conditions is said to be reduced. The number $n$ is the length of $r$. The element $r$ is said to be cyclically reduced if, in addition, $s_n = s_1$ implies that $a_n = a_1$.

**Definition 4.2.** Given a set $S$, let $\mathcal{F}_2(S) = \langle S : s^2 = 1, s \in S \rangle$. A presentation by involutions over $\mathbb{Z}_2$ for a group $G$ is a set of generators $S$ and a set of relations $R \subset \mathcal{F}_2(S) \times \mathbb{Z}_2$ such that $G = \mathcal{F}_2(S) \times \mathbb{Z}_2/(R)$, where $(R)$ is
the normal subgroup generated by $R$. We denote presentations of this form by $\text{Inv}(S : R)$, and write $G = \text{Inv}(S : R)$ when the meaning is clear.

We use $J$ (written in multiplicative notation) to denote the generator of the $\mathbb{Z}_2$-factor in $F_2(S) \times \mathbb{Z}_2$. If $G = \text{Inv}(S : R)$, we can regard $G$ as a group over $\mathbb{Z}_2$ by letting $J = J_G$ be the image of $J \in F_2(S) \times \mathbb{Z}_2$ in $G$.

Elements of $F_2(S)$ are represented by words over $S$. Every element $r \in F_2(S) \times \mathbb{Z}_2$ can be represented uniquely as $J^a s_1 \cdots s_n$, where $a \in \mathbb{Z}_2$ and $s_1, \ldots, s_n$ is a sequence in $S$ with $s_i \neq s_{i+1}$. Again, a word of this form is said to be reduced, and $n$ is called the length of $r$. If, in addition, $s_n \neq s_1$ then we say that $r$ is cyclically reduced. We say that a set of relations $R$ is cyclically reduced if every element of $R$ is cyclically reduced.

If $R$ is a set of relations, the symmetrization of $R$ is the set of relations $R^{sym}$ containing all relations of the form

$$J^a s_i s_{i+1} \cdots s_n s_1 \cdots s_{i-1} \text{ and } J^a s_i s_{i-1} \cdots s_1 s_n \cdots s_{i+1}, 1 \leq i \leq n,$$

for every relation $J^a s_1 \cdots s_n$ in $R$.

Every group presented by involutions over $\mathbb{Z}_2$ has a presentation $\text{Inv}(S : R)$ where $R$ is cyclically reduced. The presentations $\text{Inv}(S : R)$ and $\text{Inv}(S : R^{sym})$ are equivalent (i.e. they define isomorphic groups), and if $R$ is cyclically reduced then $R^{sym}$ is cyclically reduced.

By definition, solution groups are examples of groups presented by involutions over $\mathbb{Z}_2$. Theorem 3.1 states that every finitely presented group over $\mathbb{Z}_2$ embeds (over $\mathbb{Z}_2$) in a solution group. The first step in proving Theorem 3.1 is showing that every finitely presented group embeds in a group presented by involutions.

**Proposition 4.3.** Suppose $(G, J)$ is a group over $\mathbb{Z}_2$ with finite presentation $(S : R)$, and $J \in F(S)$ is a representative of $J_G$. Let $T$ be the set of indeterminates $\{z_{s_1}, z_{s_2} : s \in S\}$, and choose integers $k_s \geq 1$ for all $s \in S$. Finally, let $\phi : F(S) \rightarrow F_2(T) \times \mathbb{Z}_2$ be the morphism sending $s \mapsto (z_{s_1} z_{s_2})^{k_s}$. Then the induced morphism

$$\phi : G \rightarrow K := \text{Inv}(T : R'), \text{ where } R' := \{\phi(r) : r \in R\} \cup \{J_K \phi(J')\}$$

is an embedding over $\mathbb{Z}_2$.

Furthermore, if $R \cup \{J'\}$ is cyclically reduced then $R'$ is cyclically reduced.

**Proof.** Let $s_1, \ldots, s_n$ be a list of the elements of $S$, and let $m_i$ be the order of $s_i$ in $G$. For convenience, we write $z_{i,j}$ in place of $z_{s_{i,j}}$. Let

$$D_i := \langle z_{i,1}, z_{i,2} : z_{i,1}^2 = z_{i,2}^2 = (z_{i,1} z_{i,2})^{k_{s_i} m_i} = 1 \rangle,$$

the dihedral group of order $k_{s_i} m_i$ (if $m_i$ is infinite, then the last relation is omitted, so that $D_i$ is the infinite dihedral group). We define an increasing
subgroup \( \langle Z \rangle \) cyclically reduced. If \( r \) is odd (resp. even), then a relation is odd (resp. even) if it is of the form \((z_1z_2)^{k_i} \in Z(D_i)\), we also get that \( J_G \) belongs to the centre of \( K_i \). When \( \langle s_i, J_G \rangle = Z_{m_i} \times Z_2 \), we let \( K_i \) be the amalgamated product of \( K_{i-1} \) and \( D_i \times Z_2 \), where we identify \( s_i \in K_{i-1} \) with \((z_1z_2)^{k_i} \) and \( J_G \in K_{i-1} \) with the generator \( J \) of \( Z_2 \) in \( D_i \times Z_2 \). Once again, the amalgamated product is well-defined and \( J_G \in Z(K_i) \).

Now it is not hard to see that \( K_m \) has presentation
\[
\langle S \cup \{z_{11}, z_{12}, \ldots, z_{m1}, z_{m2}, J\} : R \cup \{z_{ij}^2 = [z_{ij}, J] = 1 : 1 \leq i \leq m, j = 1, 2\}
\]
for all \( 1 \leq m \leq n \). For \( K_n \), this presentation is equivalent to the presentation \( \text{Inv}(T : R') \) of the group \( K' \), and the isomorphism \( K_n \cong K \) identifies the inclusion \( G \subseteq K_n \) with the morphism \( \phi : G \rightarrow K \).

Finally, it is easy to see that if \( r \in R \) is cyclically reduced, then \( \phi(r) \) is cyclically reduced. If \( J' \) is cyclically reduced, then \( J\phi(J') \) is also cyclically reduced. \( \square \)

**Definition 4.4.** A relation \( r = J^n s_1 \cdots s_n \in F_2(S) \times Z_2 \) is odd (resp. even) if \( a \) is odd (resp. even). Equivalently, a relation is odd (resp. even) if it is of the form \( r' = J \) (resp. \( r' = 1 \)) for some \( r' \in F_2(S) \).

The even part of the relation \( r = J^n s_1 \cdots s_n \) is \( r^+ = s_1 \cdots s_n \). If \( \text{Inv}(S : R) \) is a presentation by involutions over \( Z_2 \), then the corresponding even presentation over \( Z_2 \) is \( \text{Inv}(S : R^+) \), where \( R^+ = \{r^+ : r \in R\} \).

Similarly, if \( G \) is any group over \( Z_2 \), then the even quotient is \( G^+ := G/(J_G) \times Z_2 \). The group \( G^+ \) is regarded as a group over \( Z_2 \) with \( J_G^+ \) equal to the generator of the \( Z_2 \) factor.

It is easy to see that if \( G = \text{Inv}(S : R) \), then \( G^+ = \text{Inv}(S : R^+) \). For instance, the null solution group \( \Gamma(\mathcal{H}) = \Gamma(\mathcal{H}, 0) \) of a hypergraph \( \mathcal{H} \) is the even quotient \( \Gamma(\mathcal{H}, b)^+ \) of the solution group \( \Gamma(\mathcal{H}, b) \) for any \( b \).

**Definition 4.5.** Let \( J^n s_1 \cdots s_n \) be a reduced word for an element \( r \in F_2(S) \times Z_2 \). The multiplicity of \( s \in S \) in \( r \) is
\[
\text{mult}(s; r) := |\{1 \leq i \leq n : s_i = s\}|.
\]

We say that \( s \neq t \in S \) are adjacent in \( r \) if either \( \{s, t\} = \{s_i, s_{i+1}\} \) for some \( i = 1, \ldots, n-1 \) or \( \{s, t\} = \{s_1, s_n\} \).
The reason that we introduce numbers \( k_s \) in Proposition 4.3 is that, for the proof of Theorem 3.1, we would like to work with relations \( r \) where \( \text{mult}(s; r) \) is even for all \( s \). In fact, we will be able to handle slightly more general presentations, which we now define.

**Definition 4.6.** We say that a presentation \( \text{Inv}(S : R) \) by involutions over \( \mathbb{Z}_2 \) is collegial if

- (a) the presentation is finite and cyclically reduced,
- (b) \( R \cap \{1, J\} = R \cap S = \emptyset \), and
- (c) if \( \text{mult}(s; r_0) \) is odd for some \( r_0 \in R \), and \( t \) is adjacent to \( s \) in some \( r_1 \in R \), then \( \text{mult}(t; r') \) is even for all \( r' \in R \).

**Remark 4.7.** Note that if \( \text{Inv}(S : R) \) is collegial, then every relation \( r \in R \) must have length at least four, i.e. \( r = J^a s_1 \cdots s_n \) where \( n \geq 4 \). This is because relations of length zero and one are explicitly excluded by condition (b), relations of the form \( s^2 \), \( sts \), \( st^2 \), or \( ts^2 \) are not cyclically reduced, and relations \( st \) and \( str \), where \( s, t, r \) are distinct, do not satisfy condition (c).

**Corollary 4.8.** Let \( G \) be a finitely presented group over \( \mathbb{Z}_2 \), with a sequence of elements \( w_1, \ldots, w_n \in G \) such that \( w_i^2 = 1 \) for all \( 1 \leq i \leq n \). Then there is collegial presentation \( \text{Inv}(S : R) \), and an embedding \( \phi : G \to K := \text{Inv}(S : R) \) over \( \mathbb{Z}_2 \) such that \( \phi(w_i) \in S \subset K \) for all \( 1 \leq i \leq n \).

**Proof.** We can find a cyclically reduced presentation \( \langle S_0 : R_0 \rangle \) for \( G \) in which \( J_G \) is a generator, \( 1 \not\in R_0 \), and each \( w_i \) has a representative \( w'_i \in F(S_0) \setminus \{1\} \) (it is always possible to find such a presentation, since if necessary we can add an extra generator \( z \), along with the relation \( z = 1 \), and use this as a representative of the identity). In particular, this gives us a presentation where \( J_G \) is represented by a cyclically reduced non-identity element of \( F(S_0) \), namely itself.

Applying Proposition 4.3 to this presentation with \( k_s = 2 \) (or any other even number) for all \( s \in S_0 \) gives us an embedding \( \phi \) of \( G \) in a finite presentation \( \text{Inv}(T : R') \), where \( R' \) is cyclically reduced. Since all \( k_s \)'s are even, \( \text{mult}(t; \phi(r)) \) is even for every \( r \in F(S_0) \) and \( t \in T \), and every relation in \( R' \) has length \( \geq 4 \). We conclude that \( \text{Inv}(T : R') \) is collegial.

Now let

\[
S = T \cup \{\overline{w}_1, \ldots, \overline{w}_n\},
\]

where \( \overline{w}_1, \ldots, \overline{w}_n \) are new indeterminates, and set

\[
R = R' \cup \{\overline{w}_i \phi(w'_i) : 1 \leq i \leq n\},
\]

where \( \phi : F(S_0) \to F_2(T) \times \mathbb{Z}_2 \) as in Proposition 4.3. Since \( \overline{w}_i \) does not appear in \( \phi(w'_i) \), the relation \( r = \overline{w}_i \phi(w'_i) \) is cyclically reduced. Furthermore, none
of the $w_i$'s are adjacent, and $\text{mult}(s; r)$ is even for all $s \in T$ and $r \in R$, so $\text{Inv}(S : R)$ is collegial. But $\text{Inv}(S : R)$ is plainly equivalent to $\text{Inv}(T : R')$, so the corollary follows. \hfill \Box

5. The wagon wheel embedding

Using Corollary 4.8, the proof of Theorem 3.1 reduces to the following:

**Theorem 5.1.** Let $G$ be a group with a collegial presentation $I = \text{Inv}(S, R)$. Then there is a hypergraph $W := \mathcal{W}(I)$ and vertex labelling $b := b(I)$ such that $S \subset E(W)$, and the resulting map $F(S) \times \mathbb{Z}_2 \to \Gamma(W, b) : s \mapsto x_s$ descends to an embedding $G \hookrightarrow \Gamma(W, b)$ over $\mathbb{Z}_2$.

**Proof of Theorem 5.1 using Theorem 5.1.** Let $G$ be a group over $\mathbb{Z}_2$ with elements $w_1, \ldots, w_n$ such that $w_i^2 = 1$ for $i = 1, \ldots, n$. By Corollary 4.8 there is a collegial presentation $I := \text{Inv}(S, R)$ and an embedding $\phi_1 : G \to K := \text{Inv}(S, R)$ over $\mathbb{Z}_2$ with $\phi_1(w_i) \in S$ for all $i = 1, \ldots, n$.

By Theorem 5.1 there is an embedding $\phi_2 : K \to \Gamma(W(I), b(I))$ over $\mathbb{Z}_2$ with $\phi_2(s) = x_s$ for all $s \in S$. The composition $\phi_2 \circ \phi_1$ satisfies the conditions of Theorem 3.1. \hfill \Box

Although we are still very far from being able to prove Theorem 5.1 in this section we shall describe the hypergraph $W(I)$, which we call the **wagon wheel hypergraph** of $I$. The proof of Theorem 5.1 will be given in Section 12.

The wagon wheel hypergraph can be defined for any (not necessarily collegial) presentation $\text{Inv}(S, R)$. Let $R = \{r_1, \ldots, r_m\}$, let $n_i$ be the length of $r_i$, and write $r_i = J^{s_{i1}}s_{i2} \cdots s_{im}$, where $s_{ij} \in S$. The wagon wheel hypergraph is a simple hypergraph $W$ with vertex set

$$V := \{(i, j, k) : 1 \leq i \leq m, j \in \mathbb{Z}_{n_i}, 1 \leq k \leq 3\},$$

and edge set

$$E := S \sqcup \{a_{ij}, b_{ij}, c_{ij}, d_{ij} : 1 \leq i \leq m, j \in \mathbb{Z}_{n_i}\}.$$ 

As a result, if $M := \sum_{i=1}^{k} n_i$, then $W$ has $3M$ vertices and $4M + |S|$ edges. $W$ has the following incidence relations for every $1 \leq i \leq m$ and $1 \leq j \leq n_i$:

- $s \in S$ is incident with $(i, j, 1)$ if and only if $s_{ij} = s$,
- $a_{ij}$ is incident with $(i, j - 1, 2)$ and $(i, j, 1)$,
- $b_{ij}$ is incident with $(i, j, 1)$ and $(i, j, 2)$,
- $c_{ij}$ is incident with $(i, j, 2)$ and $(i, j, 3)$, and
- $d_{ij}$ is incident with $(i, j - 1, 3)$ and $(i, j, 3)$.
Figure 1. The portion of the wagon wheel hypergraph containing vertices $V_i$ and all incident edges. To save space, $(i, j, k)$ is written as $j, k$, and $s_{ij}, a_{ij}, \ldots$ are written as $s_j, a_j, \ldots$.

Note that the only edges incident with vertices

$$V_i := \{(i, j, k) : j \in \mathbb{Z}_n, 1 \leq k \leq 3\}$$

are the edges in

$$E_i := \{a_{ij}, b_{ij}, c_{ij}, d_{ij} : j \in \mathbb{Z}_n\}$$

and the edges $s_{i1}, \ldots, s_{im_i}$. Furthermore, all the edges in $E_i$ are incident with exactly two vertices, both belonging to $V_i$. The portion of the hypergraph $W$ incident with $V_i$ is shown in Figure 1. An example of a wagon wheel hypergraph for a small (non-collegial) presentation is shown in Figure 2.

We also need to define the vertex labelling in Theorem 5.1.

**Definition 5.2.** A $\mathcal{I}$-labelling of $W$ is a vertex labelling $b : V \to \mathbb{Z}_2$ such that $|b^{-1}(1) \cap V_i| = a_i \mod 2$ for all $1 \leq i \leq m$.

For Theorem 5.1 we can choose any $\mathcal{I}$-labelling. This is because all $\mathcal{I}$-labellings are equivalent in the following sense:

**Lemma 5.3.** Let $b$ and $b'$ be two $\mathcal{I}$-labellings of $W$. Then there is an isomorphism $\Gamma(W, b) \to \Gamma(W, b')$ which sends $x_s \mapsto x_s$ for all $s \in S$. 
Figure 2. An example of the wagon wheel hypergraph $\mathcal{W}(I)$ when $I = \text{Inv}(x, y, z, u, v : xyz = xuv = 1)$.

Proof. Suppose $\mathcal{H}$ is a hypergraph with incidence matrix $A(\mathcal{H})$ and vertex labelling $b^{(0)}$. Given $e \in E(\mathcal{H})$, let $b^{(1)}$ be the vertex labelling with $b^{(1)}_v = b^{(0)}_v + A(\mathcal{H})_{ve}$ (i.e. we toggle the sign of all vertices incident with $e$ according to multiplicity). Then there is an isomorphism

$$\Gamma(\mathcal{H}, b^{(0)}) \rightarrow \Gamma(\mathcal{H}, b^{(1)}): x_f \mapsto \begin{cases} x_f & f \neq e \\ Jx_e & f = e. \end{cases}$$

For $\mathcal{W}$, since $|b^{-1}(1) \cap V_i|$ and $|(b')^{-1}(1) \cap V_i|$ have the same parity, it is easy to see that $b|_{V_i}$ can be transformed to $b'|_{V_i}$ by toggling signs of vertices incident to edges $e \in E_i$ as necessary. The lemma follows. \[\Box\]

6. Pictures for Groups Generated by Involution

In this section we give an overview of the main technical tool used in the proof of Theorem 5.1: pictures of groups. These pictures, which are dual to the somewhat better known van Kampen diagrams, are a standard tool in combinatorial group theory. The purpose of pictures is to encode derivations of group identities from a set of starting relations; see [Sho07] for additional background. Here we introduce a variant adapted to groups generated by involutions.

6.1. Pictures as Planar Graphs. By a curve, we shall mean the image of a piecewise-smooth function $\gamma$ from a closed interval $[a, b]$ (where $a < b$) to either the plane or the sphere. A curve $\gamma$ is simple if $\gamma(s) \neq \gamma(t)$ for all $a \leq s < t \leq b$, except possibly when $s = a$ and $t = b$. The points $\gamma(a)$ and $\gamma(b)$ are called the endpoints of the curve. A curve has either one or two endpoints; if $\gamma(a) = \gamma(b)$
then the curve is said to be closed. A connected region (in the plane or on the
sphere) is simple if its boundary is a simple closed curve.

**Definition 6.1.** A picture is a collection \((V, E, D)\), where

(a) \(D\) is a closed simple region,

(b) \(V\) is a finite collection of points, called vertices, in \(D\),

(c) \(E\) is a finite collection of simple curves, called edges, in \(D\), and

(d) for all edges \(e \in E\) and points \(p\) of \(e\),

(i) if \(e\) is not closed and \(p\) is an endpoint of \(e\), then either \(p \in V\), or \(p\) belongs to the boundary of \(D\) and is not the endpoint of any other edge;

(ii) if \(e\) is closed and \(p\) is an endpoint of \(e\), then \(p\) does not belong to the boundary of \(D\);

(iii) if \(p\) is not an endpoint of \(e\), then \(p \notin V\), and \(p\) does not belong to any other edge or the boundary of \(D\).

If an edge \(e\) contains a vertex \(v\), then we say that \(e\) and \(v\) are incident. If \(e\) contains a point of the boundary of \(D\), then we say that \(e\) is incident with the boundary. A picture is closed if no edges are incident with the boundary of \(D\). The size of a \(G\)-picture \(P\) is the number of vertices in \(P\).

According to this definition, a picture is a type of planar embedding of a
graph, albeit a graph where we can have multiple edges between vertices, loops
at a vertex, and even closed loops which are not incident to any vertex. From
this point of view, the boundary of \(D\) can be regarded as a special type of
vertex; if we think of the picture as drawn on a sphere, then this vertex would
naturally be drawn at infinity. An illustration of these two equivalent points
of view is shown in Figure \ref{fig:2}. However, it is more convenient not to include the
boundary of \(D\) in the vertex set of a picture, and we stick with the convention
of treating the boundary separately. In particular, the picture in Figure \ref{fig:3} has
size 7.

There is one important exception where we want to forget the boundary of
\(D\), and that is when the picture is closed. If this happens, we often want to
think of the picture as embedded in the sphere, without the point at infinity
being marked. An example of a closed picture on the sphere, as seen from
two different positions, is given in Figure \ref{fig:4}. To handle this case, we allow
\(D\) to be the whole sphere, in which case the boundary is empty. Also note
that we consider two pictures equal if they differ up to isotopy, either in the
plane or on the sphere as appropriate. Such isotopies are allowed to move the
boundary of the simple region, as well as the location of endpoints of edges on the boundary, as long as endpoints are not identified.

If \( \mathcal{P} \) is a picture in \( \mathcal{D}_0 \) and \( \mathcal{D} \) is a closed simple subregion of \( \mathcal{D}_0 \), then the portion of \( \mathcal{P} \) contained in \( \mathcal{D}_0 \) can be interpreted as a picture inside \( \mathcal{D} \). We do, however, have to make sure that boundary edges of the picture inside \( \mathcal{D} \) do not have a common endpoint. This leads to two natural notions of the restriction of \( \mathcal{P} \) to \( \mathcal{D} \).

**Definition 6.2.** Let \( \mathcal{P} \) be a picture in \( \mathcal{D}_0 \), and let \( \mathcal{D} \) be a closed simple region in \( \mathcal{D}_0 \) with interior \( \mathcal{D}^0 \). Given \( \epsilon > 0 \), let \( \mathcal{D}^\epsilon \) denote the \( \epsilon \)-relaxation of \( \mathcal{D} \), and let \( \mathcal{D}^{-\epsilon} \) denote the \( \epsilon \)-contraction.

Recall that a curve intersects the boundary of \( \mathcal{D} \) transversally if, in every small disk around the intersection point, there are points of the curve which

---

**Figure 3.** A picture embedded in a disk (left) and on the plane with the exterior of the disk shrunk down to a special vertex at infinity (right).

**Figure 4.** A closed picture embedded in the sphere seen (up to isotopy) with two different choices for the location of the point at infinity. Faces are distinguished by different colours.
lie both on the interior and the exterior of $D$. We say that $D$ is transverse to $P$ if every edge which intersects the boundary of $D$ does so transversally, and the boundary of $D$ does not contain any vertices of $P$.

The restriction of $P$ to a transverse region $D$ is the picture $\text{res}(P, D)$ with vertex set $V(P) \cap D$, and whose edges are the closures of the connected components of $e \cap D^o$, for $e \in E(P)$. In other words, edges are cut off at the boundary, and edges which intersect the boundary at multiple points may be cut into multiple edges.

For a general region $D$, the contraction $D^{-\epsilon}$ will be transverse to $P$ for small enough $\epsilon > 0$. The restrictions $\text{res}(P, D^{-\epsilon})$ are thus well-defined, and can be identified via isotopy with pictures in $D$. These pictures belong to a single isotopy class $\text{res}(P, D)$, which we identify as the restriction of $P$ to $D$.

Similarly, the germ of $D$ in $P$ is the isotopy class $\text{germ}(P, D)$ of $\text{res}(P, D^\epsilon)$ for small $\epsilon > 0$.

Unless otherwise noted, we assume that subregions are closed. If $D$ is transverse to $P$, then $\text{res}(P, D)$ and $\text{germ}(P, D)$ agree. An example of this type of restriction is shown in Figure 5. In general, vertices in the boundary of $D$ will not appear in $\text{res}(P, D)$, but are preserved, along with all their outgoing edges, in $\text{germ}(P, D)$. One way to get a simple region is to take a region $D$ enclosed by a simple cycle in $P$. An example of this type, in which the germ is different than the restriction, is given in Figure 6.

**Definition 6.3.** Let $P$ be a picture in $D_0$. A simple cycle in $P$ is a collection of edges whose union is a simple closed curve.

A closed loop is an edge of $P$ which is not incident to any vertex or to the boundary, and thus forms a simple cycle by itself.

A face of $P$ is an open connected region $D$ of $D_0$ which does not contain any points of $P$, and such that the boundary of $D$ is a union of points of $P$ and
The germ of the region enclosed by a simple cycle is computed by first taking an $\epsilon$-relaxation of the region. In this case the cycle is facial, so the restriction would be empty.

points in the boundary of $D_0$. An outer face is a face whose boundary contains points of the boundary of $D_0$.

A simple cycle is facial if it is the boundary of a face.

Every simple cycle in the disk bounds a unique simple region (the interior of the cycle), while a simple cycle on the sphere bounds two simple regions. A face does not have to be simple, but a facial cycle always bounds a simple face by definition.

6.2. Groups and labellings of pictures.

Definition 6.4. Let $G = \text{Inv}(S : R)$. A $G$-picture is a picture $\mathcal{P}$ with every vertex $v$ labelled by a relation $r(v) \in R$ and every edge $e$ labelled by a generator $s(e) \in S$, such that if $e_1, \ldots, e_n$ is the sequence of edges incident to $v$, read in counter-clockwise order with multiplicity from some starting point, then $s(e_1)s(e_2)\cdots s(e_n) \in \{r(v)^+\}^\text{sym}$.

The boundary of $\mathcal{P}$ is the cyclic word $\text{bd}(\mathcal{P}) = s(e_1)\cdots s(e_n)$ over $S$, where $e_1, \ldots, e_n$ is the list of edges incident with the boundary, read in counter-clockwise order around the boundary of the disc, with multiplicity. If $\mathcal{P}$ is closed then we say that $\text{bd}(\mathcal{P}) = 1$, the empty word.

Two pictures $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent if $\text{bd}(\mathcal{P}_1) = \text{bd}(\mathcal{P}_2)$.

The sign of a picture $\mathcal{P}$ is $\text{sign}(\mathcal{P}) = |\{v \in V(\mathcal{P}) : r(v) \text{ is odd}\}| \mod 2$.

If $D$ is a simple region and $\mathcal{P}$ is a $G$-picture, then $\text{res}(\mathcal{P}, D)$ and $\text{germ}(\mathcal{P}, D)$ both inherit the structure of a $G$-picture from $\mathcal{P}$ via restricting the labelling functions.

 Typically pictures use directed edges to represent inverses of generators, but this is not necessary for groups generated by involutions. As previously mentioned, the point of pictures is that they capture relations in the group, in the following sense:
Proposition 6.5 (van Kampen lemma). Let $G = \text{Inv}(S : R)$, let $r$ be a word over $S$, and let $a \in \mathbb{Z}_2$. Then $r = J^a$ in $G$ if and only if there is a $G$-picture $P$ with $\text{bd}(P) = r$ and $\text{sign}(P) = a$.

The original version of the van Kampen lemma goes back to [VK33]. The proof of this version is not substantially different than the proof of the original version, and we omit it.

Example 6.6. A $G$-picture encodes a specific derivation of a group relation in the given presentation. Consider the Coxeter group $S_4 \times \mathbb{Z}_2$ over $\mathbb{Z}_2$. This group has presentation

$G = \text{Inv}(s_1, s_2, s_3 : s_1 s_3 = s_3 s_1, s_1 s_2 s_1 = s_2 s_1 s_2, s_2 s_3 s_2 = s_3 s_2 s_3)$,

and in particular is presented by involutions. An example of a $G$-picture showing that $s_1 s_2 s_3 s_2 s_1 = s_3 s_2 s_1 s_2 s_3$ is given in Figure 7.

![Figure 7](image-url)

**Figure 7.** A $G$-picture for $G = S_4 \times \mathbb{Z}_2$ showing that $s_1 s_2 s_3 s_2 s_1 = s_3 s_2 s_1 s_2 s_3$. Vertex labels are omitted, as they can be deduced from the edge labels.

Definition 6.7. Let $D$ be a simple region of a $G$-picture $P_0$, and let $P' = \text{res}(P, D)$ or $\text{germ}(P, D)$. If $P'$ is equivalent to a picture $P''$, then we can cut out $P'$ and glue in $P''$ in its place to get a new picture $P_1$. We refer to the process $P_0 \Rightarrow P_1$ as surgery.

If $P'$ has size zero then we call $P_0 \Rightarrow P_1$ a null surgery. We say that $P_0$ and $P_1$ are equivalent via null surgeries if there is a sequence of null surgeries transforming $P_0$ to $P_1$.

An example of a surgery is shown in Figure 8.

---

2Size zero pictures do not have vertices, but they can still have edges.
Figure 8. Surgery for an $S_4 \times \mathbb{Z}_2$-picture with boundary relation $s_1s_2s_3s_3s_1 = s_2s_1s_2$. Generators and relations are the same as in Example 6.6.

7. Pictures over solution groups and hypergraphs

Let $\mathcal{H}$ be a hypergraph with vertex labelling function $b : V(\mathcal{H}) \to \mathbb{Z}_2$. The solution group $\Gamma(\mathcal{H}, b)$ is finitely presented by involutions over $\mathbb{Z}_2$; to talk about $\Gamma$-pictures, we just need to pick a presentation of $\Gamma(\mathcal{H}, b)$. One candidate is the presentation from Definition 2.1. This presentation contains two types of relations: linear relations of the form $\prod x_e = J^a$, and commuting relations of the form $x_e x_{e'} = x_{e'} x_e$. However, it will be more convenient to use a presentation without commuting relations:

Definition 7.1. As a group presented by involutions over $\mathbb{Z}_2$, we let $\Gamma(\mathcal{H}, b) = \text{Inv}(S, R)$, where $S = \{x_e : e \in E(\mathcal{H})\}$ and

$$R = \{J^{b_v} x_{e_1} \cdots x_{e_n} : \text{all } v \in V \text{ and all orderings } e_1, \ldots, e_n \text{ of the edges incident to } v \text{ listed with multiplicity}\}$$

Clearly any linear relation in $R$ can be recovered from one linear relation and the commuting relations, while a commuting relation can be recovered from two linear relations as shown in Figure 9. Consequently, the relations in $R$ do
Figure 9. In a solution group with linear relation \( x_1 x_2 x_3 x_4 = J^a \), the commuting relation \( x_1 x_2 = x_2 x_1 \) can be represented pictorially in two different ways, depending on whether we use the presentation from Definition 2.1 (on the left), or the presentation from Definition 7.1 (on the right).

Indeed give a presentation for \( \Gamma(\mathcal{H}, b) \). We will always use the presentation in Definition 7.1 when working with \( \Gamma \)-pictures.

Definition 7.2. Let \( \mathcal{H} \) be a hypergraph. An \( \mathcal{H} \)-picture is a picture \( P \) with a pair of labelling functions \( h_V : V(P) \to V(\mathcal{H}) \) and \( h_E : E(P) \to E(\mathcal{H}) \), such that for all \( v \in V(P) \) and \( e' \in E(\mathcal{H}) \), if we list the edges \( e_1, \ldots, e_n \) of \( P \) incident to \( v \) with multiplicity then \( A_{h(v)e'} = \{ 1 \leq i \leq n : h(e_i) = e' \} \).

The boundary of an \( \mathcal{H} \)-picture \( P \) is the cyclic word \( \text{bd}(P) = h(e_1) \cdots h(e_n) \) over \( E(\mathcal{H}) \), where as before \( e_1, \ldots, e_n \) is the sequence of edges incident with the boundary, read counter-clockwise with multiplicity. The character of \( P \) is the vector \( \text{ch}(P) \in \mathbb{Z}_2^{V(\mathcal{H})} \) with \( \text{ch}(P)_v = |h^{-1}(v)| \mod 2 \).

It is easy to see that there is a one-to-one correspondence between \( \Gamma(\mathcal{H}, b) \)-pictures and \( \mathcal{H} \)-pictures. If \( P \) is an \( \mathcal{H} \)-picture, then the sign of the corresponding \( \Gamma \)-picture is the standard dot product \( \text{ch}(P) \cdot b \). The leads to the following restatement of the van Kampen lemma for \( \mathcal{H} \)-pictures.

Proposition 7.3 (van Kampen lemma). Let \( \Gamma(\mathcal{H}, b) \) be a solution group. Then \( x_{e_1} \cdots x_{e_n} = J^a \) in \( \Gamma(\mathcal{H}, b) \) if and only if there is an \( \mathcal{H} \)-picture \( P \) with \( \text{bd}(P) = e_1 \cdots e_n \) and \( \text{ch}(P) \cdot b = a \).

Example 7.4. Consider the solution group for the linear system

\[
\begin{align*}
(1) & \quad x + y + z = 1 \\
(2) & \quad x + y + z = 0
\end{align*}
\]
The underlying hypergraph $H$ of this system is shown below.

This drawing of the hypergraph is also a closed $H$-picture $P$ with character $\text{ch}(P) = (1, 1)$. Since $b = (1, 0)$ and $\text{ch}(P) \cdot b = 1$, van Kampen's lemma tells us that $J = 1$.

Remark 7.5. Given a hypergraph $H$ with incidence matrix $A$, let $X \subset \mathbb{Z}_2^V$ be the set of vertex labellings $b$ such that $J \neq 1$ in $\Gamma(H, b)$, and let $Y \subset \mathbb{Z}_2^V$ be the set of characters $\text{ch}(P)$ of $H$-pictures $P$. It is not hard to see that $X$ and $Y$ are subspaces of $\mathbb{Z}_2^V$. Proposition 7.3 states that $Y$ is the orthogonal subspace to $X$ with respect to the standard bilinear product. By Theorem 2.2, $X$ can be regarded as a quantum-information-theoretic analogue of the columnspace of $A$. From this point of view, $Y$ is then an analogue of the left nullspace of $A$.

If edge $e$ is incident to vertex $v$ in an $H$-picture $P$, then $h(e)$ will be incident to $h(v)$ in $H$. Consequently, the labelling function $h$ in Definition 7.2 can be seen as a type of weak hypergraph homomorphism. If $P$ is closed, and $H$ and $\mathcal{P}$ are simple loopless graphs, then $h$ will be an actual graph homomorphism if and only if $h(v) \neq h(v')$ for all adjacent vertices $v$ and $v'$ in $P$. Furthermore, if this happens then $h$ must be a planar graph cover. This motivates the following definition:

Definition 7.6. Let $H$ be a hypergraph. A closed $H$-picture $P$ is a cover of $H$ if every edge of $P$ is incident with two distinct vertices $v$ and $v'$ such that $h(v) \neq h(v')$.

Size, equivalence, restriction to a region, surgery, and null surgery are all defined for $H$-pictures via the correspondence with $\Gamma$-pictures. For instance, the size of an $H$-picture is simply the number of vertices in the picture.

Definition 7.7. Let $H$ be a hypergraph, and let $b : V(H) \to \mathbb{Z}_2$ be a vertex-labelling function. Two $H$-pictures $P_0$ and $P_1$ are $b$-equivalent (resp. character-equivalent) if $\text{bd}(P_0) = \text{bd}(P_1)$ and $\text{ch}(P_0) \cdot b = \text{ch}(P_1) \cdot b$ (resp. $\text{ch}(P_0) = \text{ch}(P_1)$).

An $H$-picture $P$ is $b$-minimal (resp. character-minimal) if $P$ has minimum size among all $b$-equivalent (resp. character-equivalent) pictures.

Two pictures are character-equivalent if and only if they are $b$-equivalent for all vertex-labelling functions $b$. Thus a $b$-minimal picture is also character-minimal.
If \( P_0 \to P_1 \) is a surgery in which a region \( P' \) is replaced by a \( b \)-equivalent (resp. character equivalent) region \( P'' \), then \( P_0 \) and \( P_1 \) will be \( b \)-equivalent (resp. character-equivalent). As a result, if \( \mathcal{P} \) is \( b \)-minimal (resp. character-minimal) then \( \text{res}(\mathcal{P}, \mathcal{D}) \) and \( \text{germ}(\mathcal{P}, \mathcal{D}) \) will be \( b \)-minimal (resp. character-minimal) for all simple regions \( \mathcal{D} \).

We can also remove closed loops without changing the character-equivalence class:

**Lemma 7.8.** Suppose \( \mathcal{P} \) is a picture, and let \( \mathcal{P}' \) be the same picture but with all closed loops deleted. Then \( \mathcal{P} \) and \( \mathcal{P}' \) are character-equivalent and have the same size.

### 8. A CATEGORY OF HYPERGRAPHS

**Definition 8.1.** Let \( \mathcal{H} = (V, E, A) \) be a hypergraph. A subhypergraph of \( \mathcal{H} \) is a hypergraph \( \mathcal{H}' = (V', E', A') \) with \( V' \subseteq V \), \( E' \subseteq E \), and \( A'_{ve} = A_{ve} \) for all \( v \in V' \) and \( e \in E' \).

In other words, a subhypergraph is simply a subset of \( V(\mathcal{H}) \cup E(\mathcal{H}) \). Although this definition is substantially less restrictive than other notions of subhypergraphs in the literature, it is natural in the context of hypergraphs with isolated edges.

**Definition 8.2.** If \( \mathcal{H}' \) is a subhypergraph of \( \mathcal{H} \), then the neighbourhood \( N(\mathcal{H}') \) of \( \mathcal{H}' \) is the subhypergraph with \( V(N(\mathcal{H}')) = V(\mathcal{H}') \), and \( E(N(\mathcal{H}')) = E(\mathcal{H}') \cup \{ e \in E(\mathcal{H}) : e \text{ is incident in } \mathcal{H} \text{ to some vertex } v \in V(\mathcal{H}') \} \).

We say that \( \mathcal{H}' \) is open if \( \mathcal{H}' = N(\mathcal{H}') \).

The proof of the following proposition is elementary, and we omit it.

**Proposition 8.3.** Let \( \mathcal{H} \) be a hypergraph. The collection of open subhypergraphs of \( \mathcal{H} \) forms a topology on \( V(\mathcal{H}) \cup E(\mathcal{H}) \). A subhypergraph \( \mathcal{H}' \) is closed in this topology if and only if, for all \( v \in V(\mathcal{H}) \), if \( v \) is incident to \( e \in E(\mathcal{H}') \) then \( v \in V(\mathcal{H}') \).

**Definition 8.4.** Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be hypergraphs. A generalized morphism \( \phi : \mathcal{H}_1 \to \mathcal{H}_2 \) consists of a pair of morphisms

\[
\phi_V : V(\mathcal{H}_1) \to V(\mathcal{H}_2) \cup \{ \varepsilon \} \quad \text{and} \quad \phi_E : E(\mathcal{H}_1) \to E(\mathcal{H}_2) \cup \{ \varepsilon \},
\]

such that, for all \( v \in V(\mathcal{H}_1) \),

(1) if \( \phi_V(v) \neq \varepsilon \), then

\[
\sum_{e \in \phi_E^{-1}(e')} A(\mathcal{H}_1)_{ve} = A(\mathcal{H}_2)_{\phi(v)e'},
\]

for all \( e' \in E(\mathcal{H}_2) \), and
(2) if \( \phi_V(v) = \varepsilon \), then \[
\sum_{e \in E(H_1) \setminus \phi^{-1}_E(\varepsilon)} A(H_1)_{ve}
\]
is even, and \( \phi_E(e_1) = \phi_E(e_2) \) for all edges \( e_1, e_2 \in E(H_1) \setminus \phi^{-1}_E(\varepsilon) \) incident to \( v \).

When there is no confusion, we write \( \phi \) for both \( \phi_V \) and \( \phi_E \). The composition \( \phi_2 \circ \phi_1 \) of two morphisms \( \phi_1 : H_1 \to H_2 \) and \( \phi_2 : H_2 \to H_3 \) is defined by setting \( \phi_2(\varepsilon) = \varepsilon \).

**Proposition 8.5.**

(a) If \( v \) and \( e \) are incident in \( H_1 \), and \( \phi : H_1 \to H_2 \) is a generalized morphism with \( \phi(v) \in V(H_2) \), \( \phi(e) \in E(H_2) \), then \( \phi(v) \) and \( \phi(e) \) are incident in \( H_2 \).

(b) If \( \phi_1 : H_1 \to H_2 \) and \( \phi_2 : H_2 \to H_3 \) are generalized morphisms, then \( \phi_2 \circ \phi_1 \) is a generalized morphism.

**Proof.** For part (a), let \( e' = \phi(e) \in E(H_2) \). If \( v \) and \( e \) are incident in \( H_1 \), and \( \phi(v) \neq \varepsilon \), then \( A(H_2)_{\phi(v)e'} \geq A(H_1)_{ve} > 0 \). So \( e' \) and \( \phi(v) \) are incident.

For part (b), let \( \phi = \phi_2 \circ \phi_1 \). If \( \phi(v) \neq \varepsilon \), then \( \phi_1(v) \neq \varepsilon \), so
\[
\sum_{e \in \phi^{-1}(e')} A(H_1)_{ve} = \sum_{e'' \in \phi_2^{-1}(e')} \sum_{e' \in \phi_1^{-1}(e'')} A(H_1)_{ve} = \sum_{e'' \in \phi_2^{-1}(e')} A(H_2)_{\phi_1(v)e''} = A(H_3)_{\phi(v)e'}
\]
for all \( e' \in E(H_3) \).

Next, suppose \( \phi(v) = \varepsilon \) and \( \phi_1(v) \neq \varepsilon \). If \( e_1, e_2 \in E(H_1) \setminus \phi^{-1}(\varepsilon) \) are both incident to \( v \), then \( \phi(e_1) \) and \( \phi(e_2) \) belong to \( E(H_2) \setminus \phi_2^{-1}(\varepsilon) \) and are incident to \( \phi_1(v) \) by part (a). Thus \( \phi(e_1) = \phi(e_2) \). Similarly,
\[
\sum_{e \in E(H_1) \setminus \phi^{-1}(\varepsilon)} A(H_1)_{ve} = \sum_{e' \in E(H_2) \setminus \phi_2^{-1}(\varepsilon)} \sum_{e' \in \phi_1^{-1}(e')} A(H_1)_{ve} = \sum_{e' \in E(H_2) \setminus \phi_2^{-1}(\varepsilon)} A(H_2)_{\phi_1(v)e'}
\]
is even, since \( \phi_2(\phi_1(v)) = \varepsilon \).

Finally, suppose \( \phi(v) = \phi_1(v) = \varepsilon \). If \( E(H_1) \setminus \phi^{-1}(\varepsilon) \) does not contain any edges incident with \( v \), then part (2) of Definition 8.4 is trivially satisfied. Suppose on the other hand that \( E(H_1) \setminus \phi^{-1}(\varepsilon) \) contains edges incident with \( v \). All such edges belong to \( E(H_1) \setminus \phi_1^{-1}(\varepsilon) \), and hence are sent by \( \phi_1 \) to
A generalized morphism $\phi$ between two hypergraphs. Hyperedges are drawn as shaded regions. The morphism sends an edge or vertex $x$ to $x'$, with the following exceptions: $\phi(i) = \phi(2) = \phi(4) = \varepsilon$, $\phi(b) = a'$, $\phi(d) = \phi(f) = \phi(h) = c'$, and $\phi(8) = 6'$.

some common edge $e' \in E(H_2)$, where $\phi_2(e') \neq \varepsilon$. Conversely, any edge of $E(H_1) \setminus \phi^{-1}_1(\varepsilon)$ incident with $v$ is sent by $\phi_1$ to $e'$, and hence belongs to $E(H_1) \setminus \phi^{-1}(\varepsilon)$. We conclude that all edges of $E(H_1) \setminus \phi^{-1}(\varepsilon)$ incident with $v$ are sent to $\phi_2(e')$, and that

$$
\sum_{e \in E(H_1) \setminus \phi^{-1}(\varepsilon)} A(H_1)_{ve} = \sum_{e' \in E(H_1) \setminus \phi^{-1}_1(\varepsilon)} A(H_1)_{ve}
$$

is even. Consequently $\phi$ is a generalized morphism.

**Example 8.6.** If $e$ is a hyperedge in a hypergraph $H$, we can delete $e$ to get a new hypergraph $H \setminus e$. There is a generalized morphism $H \to H \setminus e$ which sends $e \mapsto \varepsilon$. Similarly, we construct other generalized morphisms by identifying edges, deleting isolated vertices, and collapsing vertices incident to an even number of edges (deleting a vertex and identifying all incident edges). An example of a series of these operations is shown in Figure 10.

Another way we can construct generalized morphisms is through subhypergraphs.

**Proposition 8.7.** Let $H$ be a hypergraph.
Proof. If \( \mathcal{H}' \) is any subhypergraph, then part (1) of Definition 8.4 holds for \( r \) and part (2) holds vacuously for \( \iota \). If \( \mathcal{H}' \) is closed and \( r(v) = \varepsilon \), then \( r(e) = \varepsilon \) for all edges \( e \) incident to \( v \), so part (2) of Definition 8.4 holds for \( r \). If \( \mathcal{H}' \) is open, \( v \in V(\mathcal{H}') \), and \( e' \in E(\mathcal{H}) \) is incident to \( v \), then \( e' \) belongs to \( \mathcal{H}' \), and hence part (1) holds for \( \iota \).

**Definition 8.8.** A subhypergraph \( \mathcal{H}' \) of \( \mathcal{H} \) is a retract if there is a generalized morphism \( r : \mathcal{H} \to \mathcal{H}' \) such that \( r|_{\mathcal{H}'} \) is the identity.

Part (a) of Proposition 8.7 shows that every closed subhypergraph is a retract.

Morphisms can also be constructed by gluing together morphisms over open subhypergraphs.

**Proposition 8.9.** Let \( \{ \mathcal{H}_i \}_{i \in I} \) be a family of open subhypergraphs of \( \mathcal{H} \) such that \( \bigcup \mathcal{H}_i = \mathcal{H} \), and let \( \{ \phi_i \}_{i \in I} \) be a family of generalized morphisms \( \phi_i : \mathcal{H}_i \to \mathcal{H}' \) such that \( \phi_i|_{\mathcal{H}_i \cap \mathcal{H}_j} = \phi_j|_{\mathcal{H}_i \cap \mathcal{H}_j} \). Then there is a unique generalized morphism \( \phi : \mathcal{H} \to \mathcal{H}' \) such that \( \phi|_{\mathcal{H}_i} = \phi_i \).

**Proof.** Clearly \( \phi \) is uniquely defined as a function. Given \( v \in V(\mathcal{H}) \), find \( \mathcal{H}_i \) with \( v \in V(\mathcal{H}_i) \), so \( \phi(v) = \phi_i(v) \). Since \( \mathcal{H}_i \) is open, if \( A_{ve}(\mathcal{H}) > 0 \) then \( e \in \mathcal{H}_i \) and \( A(\mathcal{H}_i)_{ve} = A(\mathcal{H})_{ve} \). Consequently, if \( \phi(v) \neq \varepsilon \) then

\[
\sum_{e \in \phi^{-1}(e')} A(\mathcal{H})_{ve} = \sum_{e \in \phi_i^{-1}(e')} A(\mathcal{H}_i)_{ve} = \sum_{e \in \phi_i^{-1}(e')} A(\mathcal{H}_i)_{ve} = A(\mathcal{H'})_{\phi(v)e'}
\]

since \( \phi_i \) is a morphism. Similarly, if \( \phi(v) = \varepsilon \), then

\[
\sum_{e \in E(\mathcal{H}) \setminus \phi^{-1}(\varepsilon)} A(\mathcal{H})_{ve} = \sum_{e \in E(\mathcal{H}_i) \setminus \phi_i^{-1}(\varepsilon)} A(\mathcal{H}_i)_{ve} = \sum_{e \in E(\mathcal{H}_i) \setminus \phi_i^{-1}(\varepsilon)} A(\mathcal{H}_i)_{ve}
\]

is even, and \( \phi(e_1) = \phi_i(e_1) = \phi_i(e_2) = \phi(e_2) \) for all edges \( e_1, e_2 \in E(\mathcal{H}) \setminus \phi^{-1}(\varepsilon) \) incident to \( v \).

The functor from hypergraphs to null solution groups is natural with respect to generalized morphisms.
Proposition 8.10. Let \( \phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) be a generalized morphism. Then there is a morphism \( \phi : \Gamma(\mathcal{H}_1) \rightarrow \Gamma(\mathcal{H}_2) \) defined by

\[
\phi(x_e) = \begin{cases} 
1 & \phi_E(e) = \varepsilon \\
x_{\phi(e)} & \text{otherwise}
\end{cases}.
\]

Proof. The morphism, if it exists, will be uniquely determined by the values \( \phi(x_e), \ e \in E(\mathcal{H}_1) \). To show that the morphism is well-defined, we need to show that

\[
\prod_{i=1}^{n} \phi(x_{e_i}) = 1
\]

for every vertex \( v \) of \( \mathcal{H}_1 \) and ordering \( e_1, \ldots, e_n \) of the edges incident to \( v \), listed with multiplicity. Suppose that \( \phi(v) = \varepsilon \). If \( \phi(e_i) = \varepsilon \) for all \( 1 \leq i \leq n \), then \( \phi(x_{e_i}) = 1 \) for all \( i \), so Equation (8.1) holds. If \( \phi(e_i) \neq \varepsilon \) for some \( i \), then

\[
\prod_{j=1}^{n} \phi(x_{e_j}) = \prod_{\phi(e_j) \neq \varepsilon} x_{\phi(e_j)} = x_{\phi(e_i)}^{K} = 1,
\]

where the last equality holds because \( K = \sum_{e \in E(\mathcal{H}_1) \setminus \phi^{-1}(\varepsilon)} A_{ve} \) is even. Hence Equation (8.1) holds in this case as well.

If \( \phi(v) \neq \varepsilon \), then for all \( 1 \leq i \leq n \), either \( \phi(e_i) = \varepsilon \) or \( \phi(e_i) \) is incident to \( \phi(v) \). We conclude that \( \phi(x_{e_1}), \ldots, \phi(x_{e_n}) \) commute in \( \Gamma(\mathcal{H}_2) \). Consequently

\[
\prod_{i=1}^{n} \phi(x_{e_i}) = \prod_{e' \in E(\mathcal{H}_2)} \prod_{e \in \phi^{-1}(e')} \phi(x_{e})^{A_{ve}} = \prod_{e' \in E(\mathcal{H}_2)} x_{e'}^{A_{\phi(v)e'}} = 1.
\]

We conclude that the morphism \( \phi \) is well-defined.

As a consequence, open retracts are special:

Corollary 8.11. If \( \mathcal{H}' \) is an open subhypergraph of \( \mathcal{H} \) and a retract of \( \mathcal{H} \) then \( \Gamma(\mathcal{H}') \) is a (semidirect factor) subgroup of \( \Gamma(\mathcal{H}) \).

Proof. Let \( r \) be the retraction morphism \( \mathcal{H} \rightarrow \mathcal{H}' \), and let \( \iota \) be the inclusion \( \mathcal{H}' \rightarrow \mathcal{H} \). Then \( r \circ \iota \) is the identity on \( \mathcal{H}' \), so the composition \( r \circ \iota : \Gamma(\mathcal{H}') \rightarrow \Gamma(\mathcal{H}) \rightarrow \Gamma(\mathcal{H}') \) is the identity morphism, and consequently \( \iota : \Gamma(\mathcal{H}') \rightarrow \Gamma(\mathcal{H}) \) must be injective. It also follows immediately that \( \Gamma(\mathcal{H}) = N \rtimes \Gamma(\mathcal{H}') \), where \( N \) is the kernel of \( r : \Gamma(\mathcal{H}) \rightarrow \Gamma(\mathcal{H}') \).

Example 8.12. Consider the graph \( \mathcal{G} \) of a cube, shown on the left in Figure \( \Pi \) with vertices numbered for reference. The open neighbourhood of \( \{1, 2, 3, 4\} \) is the subhypergraph \( \mathcal{H} \) with vertex set \( \{1, 2, 3, 4\} \) and edge set
Figure 11. In the hypergraph of the cube (left), the open neighbourhood of the bottom face is a retract. In the hypergraph on the right, this is no longer the case.

$\{12, 23, 34, 14, 15, 26, 37, 48\}$. The morphism $G \rightarrow H$ which is the identity on $H$, and sends

\[
5 \mapsto 1, \quad 6 \mapsto 2, \quad 7 \mapsto 3, \quad 8 \mapsto 4, \text{ and }
\]

\[
56 \mapsto 12, \quad 67 \mapsto 23, \quad 78 \mapsto 34, \quad 58 \mapsto 14,
\]

is a retract of $G$ onto $H$.

On the other hand, if we subdivide the edge 12 with a vertex as shown on the right of Figure 11, then the open neighbourhood $H$ of $\{1, 2, 3, 4, 9\}$ is no longer a retract of $G$, since any retract must send $6 \mapsto 2$ and $7 \mapsto 3$, but 23 is not an edge of $H$. However, the open neighbourhood of $\{5, 6, 7, 8\}$ is still a retract, since we can define the morphism as above but with edges 29 and 39 sent to 67, and vertex 9 sent to $\varepsilon$.

Finally, we can also apply generalized morphisms to pictures.

**Proposition 8.13.** Let $\phi : H_1 \rightarrow H_2$ be a generalized morphism, and let $P$ be an $H_1$-picture with $bd(P) = e_1 \cdots e_n$. Construct a new picture $P'$ as follows:

1. If $e$ is an edge of $P$ such that $\phi(h(e)) = \varepsilon$, then delete $e$ from $P$.
2. For all remaining edges $e$ of $P$, change the label from $h(e) \in E(H_1)$ to $\phi(h(e)) \in E(H_2)$.
3. If $v$ is a vertex of $P$ such that $\phi(h(v)) = \varepsilon$, then at this point there is an even number of edges incident with $v$, and all have the same label. Delete $v$, and connect up the remaining incident edges so that no pair of edges cross.
4. For all remaining vertices $v$ of $P$, change the label from $h(v) \in E(H_1)$ to $\phi(h(v)) \in E(H_2)$. 
Then $\mathcal{P}'$ is an $\mathcal{H}_2$-picture of size less than or equal to the size of $\mathcal{P}$, with $bd(\mathcal{P}') = \phi(e_1) \cdots \phi(e_n)$, where $\varepsilon$ is regarded as the empty word.

**Proof.** To show that this picture is an $\mathcal{H}_2$-picture, we need to show that there are $A(\mathcal{H}_2)_{v' e'}$ edges labelled by $e'$ incident to any vertex labelled by $v'$ in $\mathcal{P}'$. But this follows immediately from the construction and part (1) of Definition 8.4. Since the construction does not add any vertices, the size of $\mathcal{P}'$ must be at most the size of $\mathcal{P}$. $\square$

An example is given in Figure 12. Since step (3) in Proposition 8.13 requires choosing a matching on edges, the graph $\mathcal{P}'$ we end up with is not unique. One exception is when $\phi$ is one of the morphisms $r$ or $\iota$ from Proposition 8.7; in this case, there is always a unique choice, and hence $\mathcal{P}'$ is uniquely determined. However, in general the different choices can defer by a sequence of null-surgeries.

**Definition 8.14.** If $\phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a generalized morphism and $\mathcal{P}$ is an $\mathcal{H}_1$-picture, we use $\phi(\mathcal{P})$ to denote either the null-surgery equivalence class of pictures constructed in Proposition 8.13, or some arbitrarily chosen representative of this class.

If $r : \mathcal{H} \to \mathcal{H}'$ is the retraction morphism onto a closed subhypergraph, and $\mathcal{P}$ is a $\mathcal{H}$-picture, then we also denote $r(\mathcal{P})$ by $\mathcal{P}[\mathcal{H}]$.

If $\iota$ is the inclusion of an open subhypergraph $\mathcal{H}'$ in $\mathcal{H}$, and $\mathcal{P}$ is an $\mathcal{H}'$-picture, then $\mathcal{P}$ and $\iota(\mathcal{P})$ are essentially identical. In this case, Proposition 8.13 states the obvious fact that every picture over $\mathcal{H}'$ can be regarded as a
picture over $\mathcal{H}$. Note that if such a picture is character (or b)-minimal as an $\mathcal{H}$-picture, then it is also minimal as an $\mathcal{H}'$-picture; however, the converse is not true.

9. Cycles and outer faces

In this section, we lay the foundation for the proof of Theorem 5.1 by looking at the interaction between pictures and cycles in hypergraphs.

**Definition 9.1.** A cycle in a hypergraph $\mathcal{H}$ is a closed subhypergraph $\mathcal{C}$ which is a simple connected 2-regular graph. A cycle $\mathcal{C}$ is cubic if every vertex $v \in V(\mathcal{C})$ has degree three in $\mathcal{H}$.

A $\mathcal{C}$-cycle in an $\mathcal{H}$-picture is a simple cycle $C$ such that every edge of $C$ is labelled by an edge of $\mathcal{C}$.

**Lemma 9.2.** Let $\mathcal{C}$ be a cycle in a hypergraph $\mathcal{H}$, and suppose $\mathcal{P}$ is an $\mathcal{H}$-picture such that the edges of $\mathcal{C}$ do not appear in $bd(\mathcal{P})$. Then every connected component of $\mathcal{P}[\mathcal{C}]$ is a $\mathcal{C}$-cycle.

**Proof.** $\mathcal{P}[\mathcal{C}]$ is a closed $\mathcal{C}$-picture in which every vertex has degree two. As a result, every connected component will be a simple closed curve. □

In several upcoming proofs, we will use the following measure of the complexity of $\mathcal{P}$.

**Definition 9.3.** If $\mathcal{C}$ is a cycle in $\mathcal{H}$, and $\mathcal{P}$ is a $\mathcal{H}$-picture, then we let $\#\text{Cycle}(\mathcal{P}, \mathcal{C})$ denote the number of $\mathcal{C}$-cycles in $\mathcal{P}$. If $\Phi$ is a collection of cycles, we let

$$\#\text{Cycle}(\mathcal{P}, \Phi) = \sum_{\mathcal{C} \in \Phi} \#\text{Cycle}(\mathcal{P}, \mathcal{C}).$$

If we include incident vertices, then a $\mathcal{C}$-cycle $C$ is a closed $\mathcal{C}$-picture, and thus the definition of cover from Definition 7.6 applies. If $C$ is both facial and a cover of $\mathcal{C}$, then we say that $C$ is a facial cover. Note that closed loops are not covers.

**Proposition 9.4.** If $\mathcal{C}$ is a cubic cycle in $\mathcal{H}$, and $\mathcal{P}$ is a character-minimal $\mathcal{H}$-picture with no closed loops, then every facial $\mathcal{C}$-cycle in $\mathcal{P}$ is a facial cover.

**Proof.** Suppose that $C$ is a facial $\mathcal{C}$-cycle. Since $\mathcal{C}$ is not a loop at a vertex, and $\mathcal{P}$ has no closed loops, every edge of $C$ must be incident to two distinct vertices of $\mathcal{P}$. Suppose that $C$ has an edge $e$ connecting two vertices $u_0$ and $u_1$ with $h(u_0) = v = h(u_1)$. By hypothesis, $v$ has degree 3 in $\mathcal{H}$, and thus is incident with $e' = h(e)$, another edge $f'$ of $\mathcal{C}$, and an edge $g'$ which is not in $\mathcal{C}$. Hence each vertex $u_i$ is incident with an edge $f_i$ belonging to $\mathcal{C}$ with $h(f_i) = f'$, and an edge $g_i$ not in $\mathcal{C}$ with $h(g_i) = g'$. Note that the edges $f_0$
and $f_1$ could be equal, as could $g_0$ and $g_1$. Since $C$ is facial, $g_0$ and $g_1$ must lie on the same side of $C$. Thus there is a surgery which removes $u_0$ and $u_1$, connecting $g_0$ with $g_1$ and $f_0$ with $f_1$, as shown below:

But this means that $\mathcal{P}$ is not character-minimal. □

A connected closed cover $C$ of a cycle $\mathcal{C}$ is determined up to isotopy by an orientation of $C$, and the ply, that is, the size of $h^{-1}(v)$ for any $v \in V(\mathcal{C})$, where $h : C \rightarrow \mathcal{C}$ is the labelling function.

**Definition 9.5.** A $\mathcal{C}$-cycle $C$ is a copy of $\mathcal{C}$ if the labelling function $h : C \rightarrow \mathcal{C}$ is a graph isomorphism, or equivalently if $|h^{-1}(v)| = 1$ for all $v \in V(\mathcal{C})$.

We are also interested in how different cycles interact.

**Lemma 9.6.** Let $C$ be a cycle in $\mathcal{H}$, and let $D$ be a simple region in a picture $\mathcal{P}$. If $C$ is a facial $\mathcal{C}$-cycle in $\mathcal{P}$, then either $C$ is contained in $D$, or the edges of germ($\mathcal{P},D$) form the boundaries of simple outer faces of germ($\mathcal{P},D$).

**Proof.** Suppose $C$ is not contained in $D$. If germ($\mathcal{P},D$) is empty, then the lemma is vacuously true. Otherwise, choose $\epsilon > 0$ such that germ($\mathcal{P},D^\epsilon$) = res($\mathcal{P},D^\epsilon$), and in particular such that $C$ intersects the boundary of $D^\epsilon$ transversally. Then $C \cap D^\epsilon$ is divided into a number of segments which start and end on the boundary of $D^\epsilon$. Thus every face of germ($\mathcal{C},D$) will be a simple region whose boundary contains some (possibly disconnected) portion of the boundary of $D^\epsilon$. Each edge of germ($\mathcal{C},D$) will be incident to exactly two such faces, and one of these two faces will be contained in $\mathcal{F} \cap D^\epsilon$, where $\mathcal{F}$ is a face in $\mathcal{P}$ bounded by $C$. Since $\mathcal{F}$ is a face, any face of germ($\mathcal{C},D$) contained in $\mathcal{F} \cap D^\epsilon$ will be a face of germ($\mathcal{P},D$). □

**Lemma 9.7.** Let $C$ be a simple cycle in a picture $\mathcal{P}$, let $D$ be a simple region bounded by $C$, and let $\mathcal{F}$ be a simple outer face of germ($\mathcal{P},D$). Then the edges of germ($\mathcal{P},D$) in the boundary of $\mathcal{F}$ form a single simple path $P$ with at least two edges.

Furthermore, if we write the edges of $P$ in order as $e_1, \ldots, e_n$, then $e_1$ and $e_n$ are incident with the boundary of $D$, and $e_2, \ldots, e_{n-1}$ belong to $C$.

**Proof.** Every edge of bd(germ($\mathcal{P},D$)) is incident to a vertex in $C$. If $\mathcal{F}$ is simple, then bd(germ($\mathcal{P},D$)) must contain at least two edges. If these edges are incident with two or more vertices of $C$, then it follows from the definition of germ($\mathcal{P},D$) that every outer face is a simple region with boundary of the
required form. If all the edges of \( \text{bd}(\text{germ}(P, D)) \) are incident with a single vertex in \( C \), then all but one of the outer faces is simple, and the simple outer faces are bounded by a path formed by two edges of \( \text{bd}(\text{germ}(P, D)) \). \( \square \)

**Definition 9.8.** Let \( P \) be a picture in \( D_0 \), where \( D_0 \) is not the whole sphere. An outer quadrilateral of \( P \) is the closure of a simple outer face \( F \) of \( P \), whose boundary contains exactly three edges of \( P \), together forming a path of length three.

An outer quadrilateral must have a fourth side consisting of points in the boundary of \( D_0 \), hence the name. A picture with highlighted outer quadrilaterals is shown in Figure 13.

**Proposition 9.9.** Let \( C \) be a cycle in \( H \), and let \( C' \) be a cubic cycle such that \( |E(C) \cap E(C')| \leq 1 \). If \( D \) is a region bounded by a \( C' \)-cycle \( C' \), and \( C' \) is a facial \( C \)-cycle, then either \( C \) is contained in \( D \), or the edges of \( \text{germ}(C, D) \) form outer quadrilaterals of \( \text{germ}(P, D) \).

**Proof.** By Lemma 9.6, we can assume that \( F \) is a simple outer face of \( \text{germ}(P, D) \) formed by the edges of \( \text{germ}(C, D) \). By Lemma 9.7, the edges of \( \text{germ}(C, D) \) in the boundary of \( F \) form a single path \( e_0, e_1, \ldots, e_{n+1} \), where \( e_0 \) and \( e_{n+1} \) are incident with the boundary and \( e_1, \ldots, e_n \) belong to \( C' \), as shown below:

Since \( C' \) is cubic, it follows as well that \( n \geq 1 \). Since \( |E(C) \cap E(C')| \leq 1 \) and the edges \( e_1, \ldots, e_n \) belong to both \( C \) and \( C' \), we must have \( h(e_1) = \ldots = h(e_n) = e' \), the unique element of \( E(C) \cap E(C') \). Since \( C \) (and \( C' \)) are simple graphs, the edge \( e' \) is incident with exactly two vertices \( v \) and \( v' \), and both \( A_{vv'} = A_{v'v'} = 1 \). Thus \( n \) must be one, and we conclude that the closure of \( F \) is an outer quadrilateral. \( \square \)
10. Pictures over suns

In this section, we look at pictures over a specific family of hypergraphs:

**Definition 10.1.** The sun of size \( n \), where \( n \geq 3 \), is the hypergraph with vertex set \{1, \ldots, n\}, edge set \{e_i, f_i : 1 \leq i \leq n\}, and incidence relation

\[
A_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j
\end{cases} \quad \text{and} \quad A_{ie} = \begin{cases} 
1 & i \equiv j \text{ or } i \equiv j + 1 \pmod{n} \\
0 & \text{otherwise}
\end{cases}.
\]

Note that a sun has a unique cycle. The sun of size \( n = 6 \) is shown in Figure 14.

**Proposition 10.2.** Let \( H \) be a sun, and let \( P \) be a character-minimal \( H \)-picture such that \( bd(P) \) does not contain any edges from the cycle \( C \) of \( H \). Then \( P \) is character-equivalent to a character-minimal picture \( P' \) with no closed loops, in which every \( C \)-cycle is a facial cover. Furthermore, \( P' \) can be chosen so that every outer quadrilateral of \( P \) is an outer quadrilateral of \( P' \).

The last statement in this proposition implies in particular that if an edge \( e \) is part of an outer quadrilateral in \( P \), then \( e \) is also an edge of \( P' \), with the same label. The proof of Proposition 10.2 uses the following lemma.

**Lemma 10.3.** If \( P \) is a closed picture over a sun \( H \), then \( ch(P) = 0 \).

**Proof.** Let \( n \) be the size of \( H \), and let \( U \) be the closure of the complement of the cycle \( C \) of \( H \), or in other words, the subhypergraph with vertex set \{1, \ldots, n\} and edge set \{f_1, \ldots, f_n\}. Then every vertex in \( P[U] \) has degree one, so if \( P \) is closed then \( P[U] \) is a matching. Furthermore, if \( e \) is an edge of \( P[U] \) with
h(e) = f, and endpoints at vertices v and v', then h(v) = h(v') = i. We conclude that P has an even number of vertices labelled by i, so \( \text{ch}(P)_i = 0 \) for every i.

**Proof of Proposition 10.2.** Without loss of generality, we can assume that every picture lies in a bounded region, rather than a sphere. As a consequence, every simple closed curve will bound a unique simple region. Suppose that C is a C-cycle in some H-picture P, and let D be the simple region bounded by C. For the purpose of this proof, we let NE(C, P) be the number of edges e in res(P, D) with \( h(e) \in U \), where U is the closure of the complement of C, as in the proof of Lemma 10.3. We then set

\[
NE(P) := \sum_{C \text{ a C-cycle}} NE(C, P).
\]

Now suppose we start with some character-minimal picture P = P_0 with no edges from C in bd(P_0). Our strategy will be to reduce NE(P_0) via a sequence of surgeries. Let C be a C-cycle in P_0, and let D be the simple region bounded by C. Suppose e is an edge contained in D and incident with C, such that h(e) = f_i for some i. Clearly e cannot be incident with the boundary of P_0, and since e is not a closed loop, we conclude that e is incident with two distinct vertices v_0 and v_1 with h(v_0) = h(v_1) = i. Let e, a_0, a_1 be the sequence of edges incident with v_0, as they appear in counter-clockwise order, and let e, b_0, b_1 be the edges incident with v_1 as they appear in clockwise order. Then a_0, e, and b_0 all lie in the boundary of a common face, as do a_1, e, and b_1 (all these edges may lie in a common non-simple face, so we have to be careful with wording here). If h(a_0) = h(b_0) and h(a_1) = h(b_1), then we could delete e, v_0, and v_1, and connect a_i with b_i (shown in the diagram below, where we assume without loss of generality that h(a_0) = e_i)

![Diagram](image)

to get a character-equivalent picture of smaller size, contradicting minimality. We conclude that we must have \( \{h(e), h(a_j), h(b_j)\} = \{e_{i-1}, e_i, f_i\} \) for \( j = 0, 1 \). Thus we can perform the surgery where we make v_j incident to e, a_j, and b_j,
for $j = 0, 1$, (shown below, again with the assumption that $h(a_0) = e_i$)

![Diagram](image1)

...to get a new picture $P_1$. Since the surgery did not change the number or labels of vertices, $P_1$ will be character-minimal and character-equivalent to $P_0$. Since $bd(P_0)$ does not contain any edges from $C$, the boundary of an outer quadrilateral of $P_0$ must contain edges $e_1, e_2, e_3$, where $h(e_2) \in C$, and $e_1$ and $e_3$ are the two unique edges of $h^{-1}(U)$ which are incident with the endpoints of $e_2$. Since $e$ is not incident to the boundary, the edges $a_0, a_1, b_0,$ and $b_1$ and the vertices $v_0$ and $v_1$ do not belong to an outer quadrilateral of $P_0$, and hence every outer quadrilateral of $P_0$ will be an outer quadrilateral of $P_1$.

Recall that $e$ lies in $D$. Without loss of generality, we can assume that $v_0$ belongs to $C$. By Lemma 9.2, $v_1$ must also belong to a $C$-cycle. This creates two possible outcomes for $NE(P_1)$. First, suppose that both $v_0$ and $v_1$ lie on $C$. Then the surgery will “pinch” the region bounded by $C$, creating two $C$-cycles $C_0$ and $C_1$ connected by $e$, as shown below.

![Diagram](image2)

Any edge $e' \neq e$ in $D$ will end up in the region bounded by $C_0$ or the region bounded by $C_1$, so $NE(C, P_0) = NE(C_1, P_1) + NE(C_2, P_1) + 1$.

The other possibility is that $v_1$ lies on a different $C$-cycle $C'$, where (since $C$-cycles cannot cross) $C'$ lies in the interior of the region bounded by $C$. In this case, the surgery will connect $C$ and $C'$ to form a new cycle $C''$, as shown below.
The edge $e$, along with all the edges inside the cycle $C''$, will end up on the outside of $C''$. The only edges remaining in the region bounded by $C''$ are edges that belonged to $D$, so $NE(C'', P_1) = NE(C, P_0) - NE(C', P_0) - 1$. In both cases, no other cycles are changed by the surgery, so we conclude that $NE(P_0) > NE(P_1)$.

Iterating this procedure, we get a sequence $P_0, P_1, P_2, \ldots$ of character-minimal pictures, all character-equivalent, such that the outer quadrilaterals of $P_i$ are outer quadrilaterals of $P_{i+1}$, and $NE(P_i) > NE(P_{i+1})$. Since $NE(P_i)$ cannot decrease indefinitely, this process must terminate at a picture $P_n$ with the property that if $e \in h^{-1}(U)$ is incident with a $C$-cycle $C$, then $e$ is not contained in the region bounded by $C$. Equivalently, we can say that $res(P_n, D)$ is closed for every simple region $D$ bounded by a $C$-cycle.

Let $P'$ be the picture $P_n$ with all closed loops deleted. By Lemma 7.8, $P'$ is character-minimal and character-equivalent to $P_n$ (and hence $P_0$). In addition, it is easy to see that all outer quadrilaterals of $P_n$ will be outer quadrilaterals of $P'$, and that $res(P', D)$ will be closed for every simple region $D$ bounded by a $C$-cycle in $P'$. Suppose $D$ is a simple region bounded by a $C$-cycle such that $res(P', D)$ is non-empty. Since $P'$ does not contain any closed loops, $res(P', D)$ must contain a vertex. But since $res(P', D)$ is closed, we must have $ch(res(P', D)) = 0$ by Lemma 10.3 and hence we can delete $res(P', D)$ from $P'$ to get a character-equivalent picture of smaller size, contradicting the minimality of $P'$. We conclude that $res(P', D)$ must be empty for every simple region bounded by a $C$-cycle, and hence every $C$-cycle in $P'$ is facial. By Proposition 10.4, every $C$-cycle in $P'$ is a facial cover, as required. □

The following corollary is not needed for the proof of Theorem 5.1 but it does serve as a good example of how we will apply Proposition 10.2 in the following section.

**Corollary 10.4.** Let $\mathcal{H}$ be the sun of size $n$, let $b : V(\mathcal{H}) \to \mathbb{Z}_2$ be a vertex labelling function, and set $b_0 := \sum_{i=1}^{p} b_i$. Then the subgroup of $\Gamma = \Gamma(\mathcal{H}, b)$ generated by $S = \{x_{f_1}, \ldots, x_{f_n}\}$ is isomorphic to $K = \text{Inv}\langle f_1, \ldots, f_n : f_1 \cdots f_n = J^{b_0}\rangle$.

**Proof.** Clearly there is an $\mathcal{H}$-picture $P$ with $bd(P) = f_1 \cdots f_n$ and $ch(P) = (1, \ldots, 1)$, so $x_{f_1} \cdots x_{f_n} = J^{b_0}$ in $\Gamma$ by Proposition 7.3.

Conversely, suppose that $x_{f_{i_1}} \cdots x_{f_{i_k}} = J^c$ holds in $\Gamma$. By Proposition 7.3 again, there is an $\mathcal{H}$-picture $P$ with $bd(P) = f_{i_1} \cdots f_{i_k}$ and $ch(P) \cdot b = c$. Let $\mathfrak{P}$ be the set of character-minimal pictures which are character-equivalent to $P$, and in which every $C$-cycle is a facial cover. By Proposition 10.2, $\mathfrak{P}$ is non-empty, and since all elements of $\mathfrak{P}$ have the same number of vertices, $\#\text{Cycle}(P', C)$ is bounded for $P' \in \mathfrak{P}$. Let $P'$ be an element of $\mathfrak{P}$ which maximizes $\#\text{Cycle}(P', C)$. If $C$ is a $C$-cycle in $P'$ which is not a copy of $C$, then...
then it is possible to cut $C$ into two $C$-cycles as shown below, where the interior of $C$ is a face of $\mathcal{P}$.

Since this surgery does not change the character or modify any other $C$-cycle in $\mathcal{P}$, we get an element of $\mathcal{P}$ with more $C$-cycles than $\mathcal{P}'$, a contradiction. We conclude that every $C$-cycle in $\mathcal{P}'$ is a facial copy.

Since $C$ is cubic, each $C$-cycle in $\mathcal{P}'$ must bound a unique face, even if $\mathcal{P}$ is a picture in a sphere. Let $\mathcal{P}''$ be the picture constructed by contracting each one of these faces to a vertex, as shown below (for $n = 6$).

Since every edge in $\mathcal{P}'$ labelled by an $e_i$ must belong to a $C$-cycle by Lemma 9.2, every remaining in $\mathcal{P}''$ is labelled by an $f_i$ for some $1 \leq i \leq n$. If we label each vertex by the single relation $J^{b_0} f_1 \cdots f_n$, then $\mathcal{P}''$ is a $K$-picture with $\text{bd}(\mathcal{P}'') = f_{i_1} \cdots f_{i_k}$ and $\text{sign}(\mathcal{P}'') = c$. By Proposition 6.5, the relation $f_{i_1} \cdots f_{i_k} = J^c$ holds in $K$. □

The proof techniques of Proposition 10.2 and Corollary 10.4 are illustrated by the following example.

**Example 10.5.** Let $\mathcal{H}$ be the sun of size 4, and let $K = \langle f_1, \ldots, f_4 : f_1 f_2 f_3 f_4 = 1 \rangle$. It is not hard to see that the relation $w := f_1 f_2 f_3 f_4 (f_1 f_2 f_3)^2 = 1$ holds in $K$. Given a $K$-picture with boundary word $w$, we can replace every vertex with a $C$-cycle to get an equivalent $\mathcal{H}$-picture, for instance as shown in Figure 15a. However, there are $\mathcal{H}$-pictures with boundary word $w$, as shown in Figure 15a, which do not come from a $K$-picture in this way. Nonetheless, if $\mathcal{P}$ is a minimal $\mathcal{H}$-picture whose boundary contains only $f_i$'s, then the proofs of Proposition 10.2 and Corollary 10.4 provide a method to transform $\mathcal{P}$ to an equivalent picture which does come from a $K$-picture. The transformation process is shown in Figure 15b.

To finish the section, we prove one more technical lemma:
Figure 15. Using surgery, we turn a picture over the sun of size 4 into an equivalent picture where all $C$-cycles are facial copies.

Lemma 10.6. Let $\mathcal{P}$ be a character-minimal picture with no closed loops over the sun $\mathcal{H}$ of size $n$. Then there is no cycle in $\mathcal{P}$ with edge labels contained in $\{e_i, f_i, f_{i+1}\}$ for some fixed $1 \leq i \leq n$ (where $f_{n+1} := f_1$).

Proof. Suppose $C$ is a cycle of this form, and let $\mathcal{U}_i$ be the closure of the subhypergraph $\{f_i, f_{i+1}\}$. Since $\mathcal{P}$ has no closed loops, $C$ must have at least one vertex. Since $\mathcal{H}$ is simple, $C$ must have at least two vertices, and there cannot be two consecutive edges of $C$ with the same label. Thus $C[\mathcal{U}_i]$ is a (non-empty) matching between the vertices of $C$, where paired vertices must have the same label (either $i$ or $i+1$, depending on whether the edge connecting them is labelled by an $f_i$ or $f_{i+1}$). Since every vertex of $C$ occurs in $C[\mathcal{U}_i]$, $C$ must consist of a sequence of vertices $v_1, \ldots, v_{2n}, v_{2n+1} = v_1$, such that $v_{2i-1}$ and $v_{2i}$ are connected by an edge $b_i$ with $h(b_i) \in \{f_i, f_{i+1}\}$, and $v_{2i}$ and $v_{2i+1}$ are connected by an edge labelled by $e_i$, $i = 1, \ldots, n$. Let $a_i$ be the edge...
incident to \( v_i \) which is not in \( C \). Since every vertex \( v_i \) is incident to an edge of \( C \) labelled by \( e_i \), we conclude that

\[
h(a_i) = \begin{cases} e_{i-1} & h(v_i) = i \\ e_{i+1} & h(v_i) = i + 1 \end{cases}
\]

where \( e_0 := e_n \) and \( e_{n+1} := e_1 \). But since \( h(v_{2i-1}) = h(v_{2i}) \), this means that \( h(a_{2i-1}) = h(a_{2i}) \) for all \( i = 1, \ldots, n \). Thus if we delete all the vertices and edges of \( C \), we can connect \( a_{2i-1} \) and \( a_{2i} \) along the path previously taken by \( b_i \) to get a new \( \mathcal{H} \)-picture \( \mathcal{P}' \) which is character-equivalent to \( \mathcal{P} \). But the size of \( \mathcal{P}' \) would then be strictly smaller than the size of \( \mathcal{P} \), contradicting minimality.

\( \square \)

11. Stellar cycles and constellations

In this section, we extend the results for suns from the previous section to more complicated hypergraphs through the notion of a constellation. The result is the “constellation theorem”, which will be the key to the proof of Theorem 5.1.

**Definition 11.1.** Let \( \mathcal{H} \) be a hypergraph with vertex labelling function \( b : V(\mathcal{H}) \to \mathbb{Z} \). A cycle \( C \) in \( \mathcal{H} \) is \( b \)-stellar if

(a) the neighbourhood \( \mathcal{N}(C) \) is isomorphic to a sun,

(b) \( \mathcal{N}(C) \) is a retract of \( \mathcal{H} \), and

(c) \( b_v = 0 \) for all \( v \in V(C) \).

**Definition 11.2.** Let \( \mathcal{H} \) be a hypergraph with vertex labelling \( b : V(\mathcal{H}) \to \mathbb{Z}_2 \). A \( b \)-constellation is a collection \( \Phi \) of subhypergraphs of \( \mathcal{H} \) satisfying the following properties:

(a) If \( C \in \Phi \), then \( C \) is a cycle, the neighbourhood \( \mathcal{N}(C) \) is isomorphic to a sun, and \( C \) is either:

(i) \( b \)-stellar, or

(ii) a sequence of edges \( e_1 e_2 \cdots e_n \) (in order), \( n \geq 3 \), such that \( e_k \) belongs to a \( b \)-stellar cycle \( C' \in \Phi \) for all \( 3 \leq k \leq n \).

(b) For every element \( C \in \Phi \), either:

(i) there is an edge \( e \) in \( C \) which does not belong to any cycle in \( \Phi \setminus \{C\} \),

or

(ii) there is another cycle \( C' \in \Phi \) such that \( E(C) \cap E(C') \neq \emptyset \), and \( C' \) contains an edge \( e \) which does not belong to any cycle in \( \Phi \setminus \{C'\} \).
Figure 16. A hypergraph based on Figure 11, but where the open neighbourhood of each face is a sun.

(c) If \( C_0, C_1 \in \Phi \), where \( C_0 \neq C_1 \), then \(|E(C_0) \cap E(C_1)| \leq 1 \), and if neither \( C_0 \) or \( C_1 \) is \( b \)-stellar, then \( E(C_0) \cap E(C_1) = \emptyset \).

If \( \Phi \) is a \( b \)-constellation, then a cycle \( C \) in a picture \( \mathcal{P} \) is a \( \Phi \)-cycle if \( C \) is a \( C \)-cycle for some \( C \in \Phi \).

Roughly speaking, property (a) in Definition 11.2 says that every cycle \( C \) in \( \Phi \) is either stellar or (mostly) covered by other stellar cycles, while properties (b) and (c) state that the cycles in \( \Phi \) do not overlap too much.

Example 11.3. Consider the hypergraph \( \mathcal{H} \) shown in Figure 16, and let \( b \) be the vertex labelling function with \( b_9 = 1 \) and \( b_v = 0 \) for \( v \neq 9 \). Let \( C_1, C_2, \) and \( C_3 \) be the cycles with vertex sets \( \{1, 2, 5, 6\} \), \( \{1, 4, 5, 8\} \), and \( \{3, 4, 7, 8\} \) respectively, and let \( C_4 \) be the cycle with vertex set \( \{1, 2, 3, 4, 9\} \). As in Example 8.12, the neighbourhoods \( \mathcal{N}(C_i), i = 1, 2, 3 \), are retracts of \( \mathcal{H} \), and hence \( C_1, C_2, \) and \( C_3 \) are \( b \)-stellar. The cycle \( C_4 \) is not a retract, nor is \( b_v = 0 \) for all \( v \in V(C_4) \), so \( C_4 \) is not \( b \)-stellar. But \( \mathcal{N}(C_4) \) is a sun, and the edges 12, 14, and 34 belong to \( C_1, C_2, \) and \( C_3 \) respectively. Thus \( \Phi = \{C_1, \ldots, C_4\} \) is a \( b \)-constellation.

The cycle with vertex set \( \{5, 6, 7, 8\} \) is also \( b \)-stellar, and could be added to this constellation, but the cycle with vertex set \( \{2, 3, 6, 7, 9\} \) cannot be added since it is not \( b \)-stellar and shares edges with \( C_4 \).

We can now state the main theorem of this section:

Theorem 11.4 (Constellation theorem). Let \( \mathcal{H} \) be a hypergraph with vertex labelling \( b \), and let \( \Phi \) be a \( b \)-constellation. Let \( \mathcal{P} \) be an \( \mathcal{H} \)-picture such that:

(p.1) \( \text{bd}(\mathcal{P}) \) does not contain any edges from any cycle \( C \in \Phi \), and

(p.2) either \( b = 0 \) or \( \mathcal{P} \) is closed.
Then $P$ is $b$-equivalent to a picture $P'$ such that all $\Phi$-cycles in $P'$ are facial copies.

The rest of this section is concerned with the proof of Theorem 11.4. To aid the reader, the proof is split into a number of lemmas, which are in turn grouped into subsections. In all the lemmas, $\mathcal{H}$ will be a hypergraph and $b$ will be a vertex labelling. We refer to the hypotheses (p.1) and (p.2) of Theorem 11.4 as necessary.

11.1. Structure of constellations. We start by proving two lemmas about constellations.

**Lemma 11.5.** If $\Phi$ is a $b$-constellation in $\mathcal{H}$, and $e$ is an edge of $\mathcal{H}$, then there are at most two cycles in $\Phi$ containing $e$.

**Proof.** Suppose $C$ is an element of $\Phi$ containing $e$. Since $N(C)$ is a sun, $C$ is cubic. Thus if $v$ is an endpoint of $e$, then there is a unique edge $f$ incident to $v$ and not contained in $C$. If $C' \in \Phi \setminus \{C\}$ contains $e$, then $C'$ must also contain $f$, since $|E(C) \cap E(C')| \leq 1$. Hence if $C'$ and $C'' \in \Phi \setminus \{C\}$ both contain $e$, then $\{e, f\} \subset E(C') \cap E(C'')$, and consequently $C' = C''$. □

**Lemma 11.6.** Let $\Phi$ be a $b$-constellation in $\mathcal{H}$, and let $\Phi' \subseteq \Phi$. If all elements of $\Phi'$ are $b$-stellar or $\Phi'$ contains all $b$-stellar elements of $\Phi$, then $\Phi'$ is a $b$-constellation.

**Proof.** Every subset of $\Phi$ satisfies part (c) of Definition 11.2. If all elements of $\Phi'$ are $b$-stellar, or $\Phi'$ contains all $b$-stellar cycles in $\Phi$, then $\Phi'$ also satisfies part (a). If $e$ is an edge of $C \in \Phi'$ such that $e$ does not belong to any element of $\Phi \setminus \{C\}$, then clearly $e$ does not belong to any element of $\Phi' \setminus \{C\}$. Suppose every edge of $C \in \Phi'$ belongs to some element of $\Phi \setminus \{C\}$. Then by definition there is a cycle $C' \in \Phi$ such that $E(C) \cap E(C') \neq \emptyset$ and $C'$ contains an edge $e$ which does not belong to any cycle in $\Phi \setminus \{C'\}$. If $C'$ belongs to $\Phi'$, then we are done. If $C'$ does not belong to $\Phi'$, then let $e'$ be the unique edge of $E(C) \cap E(C')$. By Lemma 11.5, the only cycles of $\Phi$ which contain $e'$ are $C$ and $C'$, and hence no element of $\Phi' \setminus \{C\}$ contains $e'$. □

11.2. Stellar cycles. Now we turn to the core of the argument: showing that $C$-cycles can be turned into facial covers if $C$ is stellar. The proof relies on hypothesis (p.2) in the following way:

**Lemma 11.7.** Suppose $C$ is a $b$-stellar cycle in $\mathcal{H}$, and $P$ is a $b$-minimal $\mathcal{H}$-picture satisfying hypothesis (p.2). Then every $C$-cycle in $P$ bounds a simple region $D$ such that $\text{ch(germ}(P, D)) \cdot b = 0$. 
Proof. If \( b = 0 \), then the lemma is vacuously true. If \( \mathcal{P} \) is closed and \( \text{ch}(\mathcal{P}) \cdot b = 0 \), then \( \mathcal{P} \) is \( b \)-equivalent to the empty picture. But since \( \mathcal{P} \) is \( b \)-minimal, in this case \( \mathcal{P} \) must have size zero, and again the lemma is vacuously true.

Suppose that \( \mathcal{P} \) is closed and \( \text{ch}(\mathcal{P}) \cdot b = 1 \). Since \( \mathcal{P} \) is closed, we can think of \( \mathcal{P} \) as a picture in the sphere, in which case every \( C \)-cycle bounds two simple regions \( D_1 \) and \( D_2 \). Every vertex of \( \mathcal{P} \) appears in one of \( \text{germ}(\mathcal{P}, D_1) \) or \( \text{germ}(\mathcal{P}, D_2) \), with only the vertices of \( C \) appearing in both. Hence

\[
\text{ch}(\mathcal{P}) = \text{ch}(\text{germ}(\mathcal{P}, D_1)) + \text{ch}(\text{germ}(\mathcal{P}, D_2)) - \text{ch}(C).
\]

Since \( C \) is \( b \)-stellar, \( b_v = 0 \) for all \( v \in V(C) \), so \( \text{ch}(C) \cdot b = 0 \). Consequently,

\[
\text{ch}(\text{germ}(\mathcal{P}, D_i)) \cdot b + \text{ch}(\text{germ}(\mathcal{P}, D_2)) \cdot b = \text{ch}(\mathcal{P}) \cdot b = 1
\]

and we conclude that one of \( \text{ch}(\text{germ}(\mathcal{P}, D_i)) \cdot b, \quad i = 1, 2 \) must be 0. \( \square \)

Lemma 11.8. Let \( \Phi \) be a \( b \)-constellation in which every cycle \( C \in \Phi \) is \( b \)-stellar. Let \( \mathcal{P} \) be an \( H \)-picture satisfying \([p.1]\) and \([p.2]\). Then \( \mathcal{P} \) is \( b \)-equivalent to a \( b \)-minimal picture \( \mathcal{P}' \) with no closed loops, such that every \( \Phi \)-cycle in \( \mathcal{P}' \) is facial.

Proof. The proof is by induction on the size of \( \Phi \). If \( \Phi \) is empty, then the lemma is true by Definition 7.7 and Lemma 7.8. Suppose the lemma is true for all \( b \)-constellations of size \( m \), where \( m \geq 0 \), and let \( \Phi \) be a \( b \)-constellation of size \( m + 1 \) in which every cycle is \( b \)-stellar. If we pick an element \( C \in \Phi \), then \( \Phi' := \Phi \setminus \{C\} \) will be a \( b \)-constellation of size \( m \) by Lemma 11.6 so that every \( H \)-picture satisfying \([p.1]\) and \([p.2]\) (with respect to \( \Phi \)) will be \( b \)-equivalent to some \( b \)-minimal picture \( \mathcal{P} \), also satisfying \([p.1]\) and \([p.2]\) such that every \( \Phi' \)-cycle in \( \mathcal{P} \) is facial. Thus, to show that the lemma holds for \( \Phi \), we can assume that we start with a picture \( \mathcal{P} \) of this form.

Let \( NF(\mathcal{P}) \) denote the number of non-facial \( C \)-cycles in \( \mathcal{P} \), where \( C \in \Phi \) is the cycle chosen above. Similarly to the proof of Proposition 10.2, our strategy will be to perform a sequence of surgeries, starting from \( \mathcal{P}_0 := \mathcal{P} \), each of which decreases \( NF(\mathcal{P}) \). Suppose that \( \mathcal{P} \) has a non-facial \( C \)-cycle \( C \). By Lemma 11.7, there is a simple region \( D \) bounded by \( C \) such that \( \text{ch}(\text{germ}(\mathcal{P}, D)) \cdot b = 0 \). Let \( f : \mathcal{H} \to \mathcal{N}(C) \) be a retract onto \( \mathcal{N}(C) \), so that \( \hat{\mathcal{P}} := f(\text{germ}(\mathcal{P}, D)) \) is an \( \mathcal{N}(C) \)-picture. Recall that, since \( \mathcal{N}(C) \) is open, an \( \mathcal{N}(C) \)-picture like \( \hat{\mathcal{P}} \) can be regarded as an \( H \)-picture. By the definition of germ, the labels of edges and vertices in the boundary of the outer faces of \( \text{germ}(\mathcal{P}, D) \) must belong to \( \mathcal{N}(C) \). Since \( f \) is a retract, we conclude that the closures of the outer faces of \( \text{germ}(\mathcal{P}, D) \) will be identical to the closures of the outer faces of \( \hat{\mathcal{P}} \), and in particular, \( \text{bd}(\hat{\mathcal{P}}) = \text{bd}(\text{germ}(\mathcal{P}, D)) \). The construction in Proposition 8.13 relabels or deletes vertices not in \( \mathcal{N}(C) \), so \( \hat{\mathcal{P}} \) might not be character-equivalent to \( \text{germ}(\mathcal{P}, D) \). But since \( C \) is \( b \)-stellar,
$$b_v = 0$$ for all $$v \in V(C)$$, and consequently, $$\text{ch} (\hat{P}) \cdot b = 0$$. We conclude that $$\hat{P}$$ is $$b$$-equivalent to $$\text{germ}(P, D)$$. Since the size of $$\hat{P}$$ is less than or equal to the size of $$\text{germ}(P, D)$$, and $$\text{germ}(P, D)$$ is $$b$$-minimal, we also conclude that $$\hat{P}$$ is $$b$$-minimal as an $$H$$-picture. It follows that $$\hat{P}$$ is $$b$$-minimal (and hence character-minimal) as an $$N(C)$$-picture. From, again, the definition of germ, we know that $$\text{bd} (\text{germ}(P, D)) = \text{bd}(\hat{P}))$$ does not contain any edges from $$C$$. Finally, $$N(C)$$ is a sun, so we can apply Proposition 10.2 to $$\hat{P}$$ to get an $$N(C)$$-picture $$\hat{P}'$$ such that every $$\hat{P}$$ is character-equivalent to and of the same size as $$\hat{P}$$, every $$C$$-cycle in $$\hat{P}'$$ is facial, every outer quadrilateral of $$\hat{P}$$ is an outer quadrilateral of $$\hat{P}'$$, and $$\hat{P}'$$ has no closed loops. In particular, $$\hat{P}'$$ is $$b$$-minimal and $$b$$-equivalent to $$\text{germ}(P, D)$$, and every outer quadrilateral of $$\text{germ}(P, D)$$ is an outer quadrilateral of $$\hat{P}'$$.

Let $$P_1$$ be the result of replacing $$\text{germ}(P, D)$$ with $$\hat{P}'$$. Clearly $$P_1$$ is $$b$$-minimal and $$b$$-equivalent to $$P$$. Suppose $$C'$$ is a $$C'$$-cycle in $$P_1$$, where $$C' \in \Phi$$. Now $$\text{bd}(P) = \text{bd}(P_1)$$ does not contain any edges from the cycles of $$\Phi$$, so by Lemma 9.2 every edge of $$\hat{P}$$ which is labelled by an edge of $$C'$$ belongs to a unique $$C'$$-cycle, and the same holds for $$P_1$$. If $$C'$$ contains a boundary edge $$e$$ of $$\hat{P}'$$, then there is a $$C'$$-cycle $$C''$$ in $$P$$ which also contains $$e$$. By hypothesis, $$C''$$ is facial, and hence by Proposition 9.9 the edges of $$\text{germ}(C'', D)$$ form outer quadrilaterals in $$\text{germ}(P, D)$$. But since the outer quadrilaterals of $$\text{germ}(P, D)$$ are also outer quadrilaterals of $$\hat{P}'$$, the edges of $$C''$$ are unchanged in $$P_1$$. Since $$e$$ belongs to a unique $$C'$$-cycle in $$P_1$$, we must have $$C' = C''$$, and hence $$C'$$ is facial in $$P_1$$. On the other hand, if $$C''$$ does not contain a boundary edge of $$\hat{P}'$$, then $$C'$$ either does not intersect $$D$$, or lies entirely inside of $$D$$. In the former case, $$C'$$ will also be a facial $$C'$$-cycle in $$P$$, and will remain facial in $$P_1$$. In the latter case, $$C'$$ would have to consist only of edges labelled by $$E' = E(C') \cap E(N(C))$$. Since $$N(C)$$ is a sun and $$|E(C') \cap E(C)| \leq 1$$, the intersection $$E'$$ is either empty, or is equal to (in the notation of Definition 10.1) $$\{e_i, f_i, f_{i+1}\}$$ for some $$i$$. But since $$\hat{P}'$$ is character-minimal and has no closed loops, Lemma 10.6 implies that $$\hat{P}'$$ does not contain any cycles of this form. We conclude that every $$C'$$-cycle in $$P_1$$ remains facial. Finally, every $$C$$-cycle in $$P_1$$ belongs either to $$\hat{P}'$$ or is inherited unchanged from $$P$$, with the consequence that $$NF(P_1) = NF(P)$$.

Iterating this procedure, we get a sequence $$P_0 = P, P_1, \ldots, P_k$$ of $$b$$-minimal pictures, all $$b$$-equivalent, such that all $$\Phi$$-cycles in $$P_i$$ are facial for $$1 \leq i \leq k$$, and such that all $$\Phi$$-cycles in $$P_k$$ are facial. Deleting all closed loops from $$P_k$$ finishes the proof. \(\square\)

11.3. Covers versus copies. For the next lemma, we show that if all $$\Phi$$-cycles are facial covers, then we can turn $$\Phi$$-cycles into facial copies.
Lemma 11.9. Let $\Phi$ be a $b$-constellation. Suppose that $\mathcal{P}$ is an $\mathcal{H}$-picture satisfying (p.1), and such that all $\Phi$-cycles in $\mathcal{P}$ are facial covers. Then $\mathcal{P}$ is character-equivalent to a picture $\mathcal{P}'$ in which all $\Phi$-cycles are facial copies.

Proof. The proof is similar to the proof of Corollary 10.4. Let $\mathfrak{P}$ be the set of pictures which are character-equivalent to $\mathcal{P}$, have the same size as $\mathcal{P}$, and in which all $\Phi$-cycles are facial covers. Since all elements of $\mathfrak{P}$ have the same number of vertices, $\# \text{Cycle}(\mathcal{P}', \Phi)$ is bounded across $\mathcal{P}' \in \mathfrak{P}$. Let $\mathcal{P}'$ be an element of $\mathfrak{P}$ which maximizes $\# \text{Cycle}(\mathcal{P}', \Phi)$.

Suppose that $C$ is a $C$-cycle in $\mathcal{P}'$ which is not a copy of $C$, where $C \in \Phi$ has an edge $e$ which does not belong to any other cycle in $\Phi$. Since $C$ is a cover, there are two distinct edges $e_1$ and $e_2$ in $C$ with $h(e_1) = h(e_2) = e$. As in the proof of Corollary 10.4, we can cut $C$ at $e_1$ and $e_2$ to get a new picture $\mathcal{P}''$, character-equivalent to $\mathcal{P}'$, in which $C$ has been replaced by two facial covers. By Lemma 9.2 and the hypothesis on $e$, the edges $e_1$ and $e_2$ are not contained in any other $\Phi$-cycle. Thus all other $\Phi$-cycles in $\mathcal{P}'$ are unchanged in $\mathcal{P}''$, and $\mathcal{P}''$ will be an element of $\mathfrak{P}$ with $\# \text{Cycle}(\mathcal{P}', \Phi) < \# \text{Cycle}(\mathcal{P}'', \Phi)$, a contradiction. We conclude that if $C \in \Phi$ has an edge which does not belong to any other cycle in $\Phi$, then all $C$-cycles in $\mathcal{P}'$ are facial copies.

Now suppose that $C$ is a $C$-cycle in $\mathcal{P}'$ which is not a copy of $C$, where every edge of $C \in \Phi$ belongs to some other cycle of $\Phi$. By Definition 11.2, part (b), there is another cycle $C' \in \Phi$ such that $E(C) \cap E(C') = \{e\}$, and $C'$ contains an edge $e'$ which does not belong to any cycle in $\Phi \setminus \{C'\}$. As above, there are

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{surgery.png}
\caption{Surgery in the proof of Lemma 11.9. Edges labelled by $e$ and $f$ are cut and reconnected to turn three $\Phi$-cycles into four $\Phi$-cycles. The edges of $C$ are dashed, while the edges of $C_1$ and $C_2$ are dotted. Interiors of cycles are faces in $\mathcal{P}'$ and $\mathcal{P}''$.}
\end{figure}
distinct edges \(e_1\) and \(e_2\) in \(C\) such that \(h(e_1) = h(e_2) = e\). By Lemma 9.2, each edge \(e_i\) belongs to a unique \(C_i\)-cycle \(C_i\), \(i = 1, 2\). Since \(C_1\) and \(C_2\) are copies of \(C_i\), we must have \(C_1 \neq C_2\). Let \(f_i\) be the unique edge in \(C_i\) with \(h(f_i) = e_i\). Thus we can construct a new picture \(\mathcal{P}''\) by cutting and reconnecting \(e_i\) and \(f_i\), \(i = 1, 2\), as shown in Figure 17.

By Lemmas 9.2 and 11.3, \(C\) and \(C_i\) are the only \(\Phi\)-cycles containing \(e_i\), while \(C_i\) is the only \(\Phi\)-cycle containing \(f_i\). Thus all \(\Phi\)-cycles in \(\mathcal{P}''\), aside from those shown in Figure 17, are \(\Phi\)-cycles in \(\mathcal{P}'\), and hence will be facial covers. Because \(C\) is a cover and \(C_1\) and \(C_2\) are copies, the new cycles created in Figure 17 will be facial covers. We conclude again that \(\mathcal{P}''\) will be an element of \(\mathfrak{P}\) with \(\#\text{Cycle}(\mathcal{P}'', \Phi) < \#\text{Cycle}(\mathcal{P}'', \Phi)\), a contradiction. Thus every \(\Phi\)-cycle in \(\mathcal{P}'\) is a facial copy.

11.4. Cycles covered by other cycles. Finally, we prove two lemmas that will allow us to handle non-stellar cycles.

Lemma 11.10. Let \(\Phi' \subseteq \Phi\) be a pair of \(b\)-constellations. Suppose that \(\mathcal{P}\) is an \(\mathcal{H}\)-picture satisfying (p.1) with respect to \(\Phi\), such that all \(\Phi'\)-cycles in \(\mathcal{P}\) are facial covers. Let \(\mathcal{C}_0\) be a connected closed subhypergraph of \(\mathcal{C} \in \Phi \setminus \Phi'\) such that every edge of \(\mathcal{C}_0\) is contained in an element of \(\Phi'\). If \(C\) is a \(\mathcal{C}\)-cycle in \(\mathcal{P}\), then:

(a) \(\mathcal{C}[\mathcal{C}_0]\) is a cover of \(\mathcal{C}_0\), and

(b) if \(\mathcal{C}_0\) is a connected component of \(\mathcal{C}[\mathcal{C}_0]\), then all edges not contained in \(\mathcal{C}\) and incident with a vertex of \(\mathcal{C}_0\) lie on the same side of \(\mathcal{C}\).

Proof. Let \(e\) be an edge of \(\mathcal{C}\) with \(h(e) \in E(\mathcal{C}_0)\). By hypothesis and Lemma 9.2, \(e\) belongs to a \(C_i\)-cycle \(C_i\), where \(C_i \in \Phi'\). Since \(C_i\) is a facial cover, \(e\) has two distinct endpoints \(a\) and \(b\) with \(h(a) \neq h(b)\). We conclude that \(\mathcal{C}[\mathcal{C}_0]\) is a cover of \(\mathcal{C}_0\).

Now let \(x\) and \(y\) be the edges of \(\mathcal{P}\) incident to \(a\) and \(b\) respectively, but not contained in \(\mathcal{C}\). Since \(|E(\mathcal{C}) \cap E(\mathcal{C}')| \leq 1\) and \(\mathcal{C}\) is simple, we conclude that \(x\) and \(y\) belong to \(C_i\). Let \(\mathcal{D}\) be a simple region bounded by \(C_i\). By Proposition 9.9 since \(C_i\) is facial the edges \(x, y\), and \(e\) either belong to \(\mathcal{D}\), or form an outer quadrilateral in germ(\(\mathcal{P}, \mathcal{D}\)). It follows that \(x\) and \(y\) lie on the same side of \(\mathcal{C}\). Since \(\mathcal{C}_0\) is connected, all edges not contained in \(\mathcal{C}\) and incident to \(\mathcal{C}_0\) lie on the same side of \(\mathcal{C}\).

Lemma 11.11. Let \(\Phi' \subseteq \Phi\) be a pair of \(b\)-constellations. Suppose that \(\mathcal{P}\) is a \(b\)-minimal picture with no closed loops satisfying (p.1) and (p.2), such that every \(\Phi'\)-cycle is a facial cover. If every edge of \(\mathcal{C} \in \Phi\) is contained in some element of \(\Phi'\), then every \(\mathcal{C}\)-cycle in \(\mathcal{P}\) is a facial cover.

\(^3\)In other words, \(\mathcal{C}_0\) is either equal to \(\mathcal{C}\), or a path in \(\mathcal{C}\).
Proof. Let \( C \) be a \( C \)-cycle. Applying Lemma 11.10 with \( C_0 = C \), we get immediately that \( C \) is a cover, and that the edges incident to \( C \) all lie on the same side of \( C \).

Suppose \( b = 0 \), and let \( \mathcal{D} \) be a simple region bounded by \( C \). If all the edges incident to \( C \) lie in \( \mathcal{D} \), then germ\((\mathcal{P}, \mathcal{D})\) is closed, and \( \text{ch}(\text{germ}(\mathcal{P}, \mathcal{D})) \cdot b = 0 \).

Since \( \mathcal{P} \) is \( b \)-minimal, germ\((\mathcal{P}, \mathcal{D})\) must be \( b \)-minimal, and this implies that germ\((\mathcal{P}, \mathcal{D})\) must have size zero, in contradiction of the fact that germ\((\mathcal{P}, \mathcal{D})\) contains \( C \). Thus all the edges incident to \( C \) lie outside the interior of \( C \), and we conclude that res\((\mathcal{P}, \mathcal{D})\) is closed. But once again, this implies that res\((\mathcal{P}, \mathcal{D})\) must have size zero, and since \( \mathcal{P} \) has no closed loops, this implies that res\((\mathcal{P}, \mathcal{D})\) is empty. We conclude that \( C \) is facial.

Now suppose that \( \mathcal{P} \) is closed. Similarly to the proof of Lemma 11.7, the fact that \( \mathcal{P} \) is \( b \)-minimal with size greater than zero implies that \( \text{ch}(\mathcal{P}) \cdot b = 1 \). Now \( C \) bounds two simple regions \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) in the sphere, where we assume that all edges incident to \( C \) are contained in \( \mathcal{D}_1 \), so res\((\mathcal{P}, \mathcal{D}_2)\) is closed. If \( \text{ch}(\text{res}(\mathcal{P}, \mathcal{D}_2)) \cdot b = 1 \), then \( \text{res}(\mathcal{P}, \mathcal{D}_2) \) is \( b \)-equivalent to \( \mathcal{P} \), contradicting the \( b \)-minimality of \( \mathcal{P} \). Thus \( \text{ch}(\text{res}(\mathcal{P}, \mathcal{D}_2)) \cdot b = 0 \), and as above, res\((\mathcal{P}, \mathcal{D}_2)\) must be empty. We conclude again that \( C \) is facial. \( \square \)

11.5. **Proof of Theorem 11.4** Given a \( b \)-constellation \( \Phi \), let \( \Phi_0 \) be the set of \( b \)-stellar cycles in \( \Phi \), and let \( \Phi_1 \) be the set of cycles \( \mathcal{C} \in \Phi \) such that every edge of \( \mathcal{C} \) belongs to a cycle in \( \Phi_0 \). Then \( \Phi_0 \subseteq \Phi_1 \), and by Lemma 11.6, \( \Phi_0 \) and \( \Phi_1 \) are \( b \)-constellations.

Given a picture \( \mathcal{P}_0 \) satisfying \([p.1]\) and \([p.2]\). Lemma 11.8 states that we can find a \( b \)-equivalent picture \( \mathcal{P}_1 \) which is \( b \)-minimal and has no closed loops, such that every \( \Phi_0 \)-cycle in \( \mathcal{P}_1 \) is facial. By Proposition 11.4, every \( \Phi_0 \)-cycle in \( \mathcal{P}_1 \) is a facial cover. By Lemma 11.11 every \( \Phi_1 \)-cycle in \( \mathcal{P}_1 \) is also a facial cover.

This leaves the cycles in \( \Phi \setminus \Phi_1 \). By definition, any element of \( \Phi \setminus \Phi_1 \) is non-stellar. If a non-stellar cycle \( \mathcal{C} \in \Phi \) shares an edge with another cycle \( \mathcal{C}' \in \Phi \), then \( \mathcal{C}' \) must be \( b \)-stellar. Hence \( \Phi \setminus \Phi_1 \) consists of the non-stellar cycles \( \mathcal{C} \in \Phi \) which have an edge \( e \) not contained in any element of \( \Phi \setminus \{ \mathcal{C} \} \).

For the purpose of this proof, we say that such an edge \( e \in \mathcal{C} \) is independent. By part (a) of Definition 11.2 every element \( \mathcal{C} \in \Phi \setminus \Phi_1 \) has either one or two independent edges. In the latter case, the two edges will be incident with a common vertex of \( \mathcal{C} \).

Suppose that \( \mathcal{C} \) is a \( \mathcal{C} \)-cycle in \( \mathcal{P}_1 \), where \( \mathcal{C} \in \Phi \setminus \Phi' \). Let \( \mathcal{C}_0 \) be the path containing the non-independent edges of \( \mathcal{C} \), regarded as a closed subhypergraph. By Lemma 11.10, \( \mathcal{C}[\mathcal{C}_0] \) is a cover of \( \mathcal{C}_0 \), and since \( \mathcal{C}_0 \) is a path rather than a cycle, all connected components of \( \mathcal{C}[\mathcal{C}_0] \) are copies of \( \mathcal{C}_0 \). If \( \mathcal{C}[\mathcal{C}_0] \) is non-empty, we can write \( \mathcal{C} \) as a sequence \( C_1C_2 \cdots C_{2k-1}C_{2k} \) of paths \( C_i \), where
$k \geq 1$, the path $C_{2i-1}$ is a connected component of $C[C_0]$ for all $i = 1, \ldots, k$, and the edges of $C_{2i}$ are labelled by independent edges of $C$ for all $i = 1, \ldots, k$.

Let $v_1$ (resp. $v_2$) be the first (resp. last) vertex of $C_1$, and let $f_1$ (resp. $f_2$) be the edge of $C_{2k}$ (resp. $C_2$) which is incident to $v_1$ (resp. $v_2$). The labels $a_i = h(v_i)$ are the endpoints of the path $C_0$ in $C$. If $C$ has one independent edge $e$, then $e$ will join $a_1$ and $a_2$, and $h(f_1) = h(f_2) = e$. By Lemma [11.10] every edge incident to $C_1$ lies on the same side of $C$, so we can cut and reconnect $f_1$ and $f_2$ as shown in Figure 18a. If $C$ has two independent edges, then we can do something similar. In this case, there will be an independent edge $e_i = h(f_i)$ incident to $a_i$, $i = 1, 2$, and both edges will be incident with a third vertex $b$. Let $g$ be the edge incident to $b$ not in $C$. Since $C$ is cubic, any cycle containing $g$ must also contain either $e_1$ or $e_2$. Since these edges are independent, $g$ is not contained in any element of $\Phi$. We can then cut $f_1$ and $f_2$ and reconnect them by adding two vertices labelled by $b$, and a edge labelled by $g$, as shown in Figure 18b.

These surgeries are not as well behaved as those considered previously: When $C$ has two independent edges, we end up increasing the size, so the result will no longer be $b$-minimal. When $C$ has a single independent edge, it
is possible that \( f_1 = f_2 \), in which case we create a closed loop. However, in both cases we split \( C \) into two cycles \( C^{(1)} \) and \( C^{(2)} \), where \( C^{(1)} \) is a facial copy of \( C \), and \( C^{(2)}[C_0] \) has fewer connected components than \( C[C_0] \). Furthermore, we do not change any other \( C \)-cycle. And since we only change or add edges whose labels are not contained in any element of \( \Phi \setminus \{ C \} \), we conclude that \( C \) is the only \( \Phi \)-cycle changed by this surgery. As a result we may repeat this type of surgery to get a picture \( P_2 \) (not necessarily \( b \)-minimal, and possibly containing closed loops) which is character-equivalent to \( P_1 \), and in which every \( \Phi \)-cycle is either a facial cover, or labelled only by independent edges.

Let \( \mathcal{P} \) be the collection of pictures which are character-equivalent to \( P_2 \), and in which every \( \Phi \)-cycle is either a facial cover or labelled only by independent edges. Let \( P_3 \) be an element of \( \mathcal{P} \) of minimum size, and let \( P_4 \) be the picture \( P_3 \) with all closed loops deleted. Clearly \( P_4 \) is also an element of \( \mathcal{P} \) of minimum size. Suppose \( P_4 \) has a \( C \)-cycle \( C \) which is not a facial cover for some \( C \in \Phi \). By definition, \( C \) is labelled by independent edges of \( C \). Since \( P_4 \) has no closed loops and \( C \) is simple, \( C \) has at least two edges. Consequently \( C \) must have two independent edges, say \( e_1 \) and \( e_2 \). As above, let \( b \) be the vertex incident to both \( e_1 \) and \( e_2 \), and let \( g \) be the edge incident to \( b \) and not in \( C \). Since every edge of \( C \) is labelled by \( e_1 \) or \( e_2 \), every vertex of \( C \) must be labelled by \( b \). We can now argue similarly to Lemma 10.6: \( C \) must consist of a sequence of edges \( f_1, \ldots, f_{2k} \), where \( h(f_{2i-1}) = e_1 \) and \( h(f_{2i}) = e_2 \) for all \( 1 \leq i \leq k \). Let \( v_1, \ldots, v_{2k} \) be the vertices of \( C \) in order, so \( f_i \) has endpoints \( v_i \) and \( v_{i+1} \), where \( v_{2k+1} := v_1 \), and let \( g_i \) be the edge of \( P_4 \) incident to \( v_i \) with \( h(g_i) = g \). Let \( P_5 \) be the picture \( P_4 \) with all the edges and vertices of \( C \) deleted, and \( g_{2i-1} \) and \( g_{2i} \) joined into a single edge along the path taken by \( f_{2i-1} \). Since the edges \( e_1 \), \( e_2 \), and \( g \) do not belong to any element of \( \Phi \setminus \{ C \} \), this process does not create or change any other \( \Phi \)-cycle. Thus \( P_5 \in \mathcal{P} \), in contradiction of the minimality of \( P_4 \). We conclude that every \( \Phi \)-cycle in \( P_4 \) is a facial cover, and the theorem follows from Lemma 11.9.

12. Proof of the embedding theorem

In this section we finish the proof of Theorem 5.1 (and thus complete the proof of Theorem 3.1). We continue with the notation from Section 5, so in particular \( \mathcal{I} := \text{Inv}(S : R) \) is a presentation by involutions over \( \mathbb{Z}_2 \), \( G \) is the group with presentation \( \mathcal{I} \), \( \mathcal{W} := \mathcal{W}(\mathcal{I}) \) is the corresponding wagon wheel hypergraph, and \( R = \{ r_1, \ldots, r_m \} \) is an ordered set of relations. Although we do not yet assume that \( \mathcal{I} \) is collegial, for convenience we assume that the length \( n_i \) of the relation \( r_i \) is at least 4 for all \( 1 \leq i \leq m \) (by Remark 4.7, this assumption holds if \( \mathcal{I} \) is collegial). In addition, we make the following definitions:
Figure 19. To retract $\mathcal{N}(\mathcal{W}_i)$ onto $\mathcal{N}(B_i)$, we remove intermediary vertices to get a simplified wagon wheel shape, and then fold this wagon wheel onto the central cycle. In the example shown above, $n_i = 4$.

- Let $\mathcal{W}_i$ be the closed subhypergraph of $\mathcal{W}$ containing vertices $V_i$ and edges $E_i$. (The open neighbourhood $\mathcal{N}(\mathcal{W}_i)$ is shown in Figure 1)
- Let $\mathcal{A}_i$, $1 \leq i \leq m$ be the cycle containing edges $a_{i1}, b_{i1}, a_{i2}, b_{i2}, \ldots, a_{im_i}, b_{im_i}$.
- Let $\mathcal{B}_i$, $1 \leq i \leq m$, be the cycle containing edges $d_{i1}, \ldots, d_{im_i}$.
- Let $\mathcal{C}_{ij}$, $1 \leq i \leq m$, $j \in \mathbb{Z}_{n_i}$, be the cycle containing edges $a_{ij}, b_{ij}, c_{ij}, d_{ij}$, and $c_{i,j-1}$.
- Let $\Phi = \{\mathcal{C}_{ij} : 1 \leq i \leq m, j \in \mathbb{Z}_{n_i}\} \cup \{\mathcal{B}_i : 1 \leq i \leq m\}$.

Before we can prove Theorem 5.1, we need some preliminary lemmas.

Lemma 12.1. If $b$ is an $I$-labelling of $\mathcal{W}$, then there is a well-defined morphism $G \rightarrow \Gamma(\mathcal{W}, b)$ over $\mathbb{Z}_2$ sending $s \mapsto x_s$ for all $s \in S$.

**Proof.** There is a well-defined morphism $\mathcal{F}(S) \times \mathbb{Z}_2 \rightarrow \Gamma(\mathcal{W}, b)$ over $\mathbb{Z}_2$ sending $s \mapsto x_s$. As can be seen from Figure 1, there is a $\mathcal{N}(\mathcal{W}_i)$-picture $\mathcal{P}$ with $\text{bd}(\mathcal{P}) = s_{i1} \cdots s_{im_i}$ and $\text{ch}(\mathcal{P}) \cdot b = \sum_{v \in V_i} b_v = a_i$. By Proposition 7.3, the relation $r_i$ holds in $\Gamma(\mathcal{W}, b)$ for all $1 \leq i \leq n$. □

Lemma 12.2. $\mathcal{N}(B_i)$ is a retract of $\mathcal{W}$ for all $1 \leq i \leq n$.

**Proof.** Define $r : \mathcal{W} \rightarrow \mathcal{N}(B_i)$ by

$$r((i', j, k)) = \begin{cases} 
\varepsilon & i' \neq i \\
\varepsilon & i = i' \text{ and } k = 1 \\
(i, j, 3) & i = i' \text{ and } k = 2, 3
\end{cases},$$

and
It is clear that \( r \) is the identity on \( \mathcal{N}(B_i) \). The only vertices mapped to \( \varepsilon \) which are incident with edges in \( E(\mathcal{W}) \setminus r^{-1}(\varepsilon) \) are the vertices \((i, j, 1)\) for \( j \in \mathbb{Z}_{n_i} \). For these vertices, the incident edges \( a_{ij} \) and \( b_{ij} \) are identified as required by condition (2) of Definition 8.4.

Since \( r \) is the identity on \( \mathcal{N}(B_i) \), condition (1) of Definition 8.4 holds for the vertices \((i, j, 1), 1 \leq j \leq n_i\). The vertices \((i, j, 2)\) are incident with three edges of \( E(\mathcal{W}) \setminus r^{-1}(\varepsilon) \), namely \( a_{i,j+1}, c_{ij}, \) and \( b_{i,j} \), and these edges are mapped to the three edges \( d_{i,j+1}, c_{ij}, \) and \( d_{i,j} \) incident to \((i, j, 3)\). We conclude that condition (1) of Definition 8.4 also holds for the vertices \((i, j, 2)\), and hence \( r \) is a generalized morphism.

The map \( r \) can be visualized as deleting everything outside of \( \mathcal{W}_i \) to get a simplified wagon wheel shape, and then folding this wagon wheel onto \( \mathcal{N}(B_i) \). This is depicted in Figure 19.

**Lemma 12.3.** Let \( s = s_{ij} \) for some \( 1 \leq i \leq m \) and \( j \in \mathbb{Z}_{n_i} \). If \( R \) is cyclically reduced, and \( \text{mult}(s; r') \) is even for all \( 1 \leq i' \leq m \), then \( \mathcal{N}(C_{ij}) \) is a retract of \( \mathcal{W} \).

**Proof.** We start by showing that \( \mathcal{N}(C_{ij}) \) is a retract of \( \mathcal{N}(W_i) \). Since \( \mathcal{W}_i \) depends only on the cyclic order of \( s_{i1} \cdots s_{in_i} \), we can assume without loss of generality that \( j = 1 \). Suppose that \( \text{mult}(s; r_i) = 2k \), and let \( 1 = j_1 < j_2 < \cdots < j_{2k} \leq n_i \) be a list of the indices \( 1 \leq l \leq n_i \) such that \( s_l = s \). Since \( R \) is cyclically reduced, \( j_{i+1} > j_i + 1 \) for all \( i = 1, \ldots, 2k - 1 \), and \( j_{2k} < n_i \). For convenience, let

\[
\begin{align*}
\mathcal{J}_r &= \{1, j_2, \ldots, j_{2k}\}, \quad \mathcal{J}_l = \{0, j_2 - 1, \ldots, j_{2k} - 1\}, \\
\mathcal{J}_r^{\text{odd}} &= \{1, j_3, j_5, \ldots, j_{2k-1}\}, \quad \mathcal{J}_l^{\text{odd}} = \{0, j_3 - 1, j_5 - 1, \ldots, j_{2k-1} - 1\}, \\
\mathcal{J}_r^{\text{even}} &= \{j_2, j_4, \ldots, j_{2k}\}, \quad \text{and} \quad \mathcal{J}_l^{\text{even}} = \{j_2 - 1, j_4 - 1, \ldots, j_{2k} - 1\}.
\end{align*}
\]

These sets represent the indices of vertices and edges on the right and left of the cycles \( C_{ij} \). To talk about edges which do not belong to these cycles, we also define

\[
\begin{align*}
\overline{\mathcal{J}_r} &= \{j_1 + 1, j_1 + 2, \ldots, j_2 - 1, j_3 + 1, \ldots, j_4 - 1, \ldots, j_{2k-1} + 1, \ldots, j_{2k} - 1\} \quad \text{and} \\
\overline{\mathcal{J}_l} &= \{j_2 + 1, j_2 + 2, \ldots, j_3 - 1, j_4 + 1, \ldots, j_5 - 1, \ldots, j_{2k} + 1, \ldots, n_i\}.
\end{align*}
\]
Define \( q_i : \mathcal{N}(\mathcal{W}_i) \rightarrow \mathcal{N}(\mathcal{C}_{i1}) \) by
\[
q_i((i, j, k)) = \begin{cases} 
\varepsilon & k = 1 \text{ and } j \notin \mathcal{J}_r \\
(i, 1, 1) & k = 1 \text{ and } j \in \mathcal{J}_r \\
(i, 0, k) & k = 2, 3 \text{ and } j \notin \mathcal{J}_r \cup \mathcal{J}_t \ \\n(i, 1, k) & k = 2, 3 \text{ and } j \in \mathcal{J}_t^{\text{odd}} \cup \mathcal{J}_t^{\text{even}}
\end{cases},
\]

\[
q_i(a_{ij}) = \begin{cases} 
a_{i1} & j \in \mathcal{J}_r^{\text{odd}} \\
b_{i1} & j \in \mathcal{J}_t \ \\ba_{i2} & j \in \mathcal{J}_t
\end{cases},
q_i(b_{ij}) = \begin{cases} 
a_{i1} & j \in \mathcal{J}_r^{\text{even}} \\
b_{i1} & j \in \mathcal{J}_t^{\text{odd}} \\
a_{i2} & j \in \mathcal{J}_t
\end{cases},
\]

\[
q_i(c_{ij}) = \begin{cases} 
c_{i0} & j \in \mathcal{J}_t^{\text{odd}} \cup \mathcal{J}_t^{\text{even}} \\
c_{i1} & j \in \mathcal{J}_t^{\text{odd}} \cup \mathcal{J}_t^{\text{even}} \\
\varepsilon & j \notin \mathcal{J}_r \cup \mathcal{J}_t
\end{cases},
q_i(d_{ij}) = \begin{cases} 
d_{i1} & j \in \mathcal{J}_r \\
d_{i0} & j \in \mathcal{J}_t \\
d_{i2} & j \in \mathcal{J}_t
\end{cases},
\]

\[
q_i(s') = \begin{cases} 
s & s' = s \\
\varepsilon & s' \in S \setminus \{s\}
\end{cases}.
\]

The map \( q_i \) can be visualized as deleting vertices and edges \( c_{ij} \) (the spokes of the wagon wheel) not in the cycles \( \mathcal{C}_{ijp} \), and then folding up the cycles \( \mathcal{C}_{ijp} \) onto \( \mathcal{C}_{i1} \), alternating the directions of the folds after each \( \mathcal{C}_{ijp} \) like a napkin. The smallest example, when \( k = 1 \), is depicted in Figure 20. As in the proof of Lemma 12.2, it follows that \( q_i \) is a retract.

Now we look at \( \mathcal{W}_i \) for \( i' \neq i \). If \( \text{mult}(s; r_{i'}) = 0 \), then define
\[
q_{i'} : \mathcal{N}(\mathcal{W}_{i'}) \rightarrow \mathcal{N}(\mathcal{C}_{ij}) : x \mapsto \varepsilon.
\]

If \( \text{mult}(s; r_{i'}) > 0 \), then find \( j' \) such that \( s_{ij'} = s \), and let \( f : \mathcal{N}(\mathcal{W}_{i'}) \rightarrow \mathcal{N}(\mathcal{C}_{i'j'}) \) be the retract defined above onto \( \mathcal{N}(\mathcal{C}_{i'j'}) \). Now \( \mathcal{N}(\mathcal{C}_{i'j'}) \) and \( \mathcal{N}(\mathcal{C}_{ij}) \) are both sums, so there is an isomorphism \( g : \mathcal{N}(\mathcal{C}_{i'j'}) \rightarrow \mathcal{N}(\mathcal{C}_{ij}) \) with \( g(s) = s \), and we let \( q_{i'} = g \circ f \).

The morphisms \( q_{i'}, 1 \leq i' \leq m \) all send \( s' \in S \) to either \( \varepsilon \) if \( s' \neq s \), or to \( s \) if \( s' = s \). If \( i' \neq i'' \), then the open subhypergraphs \( \mathcal{N}(\mathcal{W}_{i'}) \) and \( \mathcal{N}(\mathcal{W}_{i''}) \) have no vertices in common. All common edges of \( \mathcal{N}(\mathcal{W}_{i'}) \) and \( \mathcal{N}(\mathcal{W}_{i''}) \) belong to \( S \), so \( q_{i'} \) and \( q_{i''} \) agree on the intersection. Every vertex of \( \mathcal{W} \) belongs to some \( \mathcal{W}_i \). There may be elements \( s' \) of \( S \) which do not appear in any relation \( r_{i'} \), and hence do not belong to any \( \mathcal{N}(\mathcal{W}_i) \); for these edges, we can add additional morphisms which send \( s' \mapsto \varepsilon \). By Proposition 8.9, there is a
Figure 20. To retract $\mathcal{N}(W_i)$ onto $\mathcal{N}(C_{ij})$, we remove intermediary vertices and edges, and then fold up the remaining cycles like a napkin. In the example above, $n_i = 4$ and $\text{mult}(s; r_i) = 2$. In general, if $\text{mult}(s; r_i) = 2k$ then we make $k$ folds.

morphism $q : W \to \mathcal{N}(C_{ij})$ which agrees with $q_i'$ on $\mathcal{N}(W_i')$, and in particular is a retract.

Lemma 12.4. Suppose $I$ is collegial. Then there is an $I$-labelling $b$ of $W$ such that $\Phi$ is a $b$-constellation.

Proof. Let $b$ be any $I$-labelling such that

- $|b^{-1}(1) \cap V(W_i)| \leq 1$ for all $1 \leq i \leq m$,
- $b((i, j, 2)) = b((i, j, 3)) = 0$ for all $1 \leq i \leq m$ and $j \in \mathbb{Z}_{n_i}$, and
- if $b((i, j, 1)) = 1$, then either $\text{mult}(s_{ij}, r_i)$ is odd for some $1 \leq i' \leq m$, or $\text{mult}(s_{ij'}, r_{i'})$ is even for all $j' \in \mathbb{Z}_{n_i}$ and $1 \leq i' \leq m$.

We will show that $\Phi$ is a $b$-constellation. First, we observe that $B_i$ is $b$-stellar for all $1 \leq i \leq n$. Indeed, $\mathcal{N}(B_i)$ is a sun, and by Lemma 12.2, $\mathcal{N}(B_i)$ is a retract of $W$. Finally, $b|_{B_i} = 0$, so $B_i$ is $b$-stellar. Similarly, Lemma 12.3 implies that $C_{ij}$ will be $b$-stellar for $1 \leq i \leq m$ and $j \in \mathbb{Z}_{n_i}$ as long as $b((i, j, 1)) = 0$ and $\text{mult}(s_{ij}; r_i)$ is even for all $1 \leq i' \leq m$.

Suppose $C_{ij}$ is not $b$-stellar. If $\text{mult}(s_{ij}; r_i)$ is odd for some $1 \leq i' \leq m$, then by Definition 4.6, $\text{mult}(s_{ij+1}; r_{i'})$ and $\text{mult}(s_{ij-1}; r_{i'})$ are even for all $1 \leq i'' \leq m$. By the definition of $b$, we must have $b((i, j + 1, 1)) = b((i, j - 1, 1)) = 0$. Thus $C_{ij+1}$ and $C_{ij-1}$ are $b$-stellar, so $C_{ij}$ (which consists of edges $a_{ij}, b_{ij}, c_{ij}, d_{ij}, c_{i,j-1}$) shares edges $c_{ij}$, $d_{ij}$, and $c_{i,j-1}$ with $b$-stellar cycles. If $\text{mult}(s_{ij}; r_i)$ is even for all $1 \leq i' \leq m$, then we must have $b((i, j, 1)) = 1$. By the definition of $b$, this means that $\text{mult}(s_{ij'}, r_{i'})$ is even for all $1 \leq i' \leq m$ and $j' \in \mathbb{Z}_{n_i}$, and $b((i, j + 1, 1)) = b((i, j - 1, 1)) = 0$. It follows that $C_{i,j+1}$ and
\( C_{i,j-1} \) are \( b \)-stellar, and once again \( C_{ij} \) will share edges \( c_{ij}, \ d_{ij}, \) and \( c_{i,j-1} \) with \( b \)-stellar cycles. Thus \( \Phi \) satisfies condition (a) of Definition 11.2.

The cycle \( C_{ij} \) is the only cycle in \( \Phi \) containing edges \( a_{ij} \) and \( b_{ij} \). Every edge \( d_{ij} \) of \( B_i \) is also contained in \( C_{ij} \), so \( \Phi \) satisfies condition (b) of Definition 11.2.

Finally, it is easy to see that \(|E(C_{ij}) \cap E(D_i')| \leq 1\) for all \(1 \leq i, i' \leq m\) and \(1 \leq i' \leq m\), and (since \( n_i \geq 4\) for all \(1 \leq i \leq m\)) that \(|E(C_{ij}) \cap E(C_{i'j'})| \leq 1\) for all distinct \((i, j)\) and \((i', j')\). Thus \( C_{ij} \) is a facial copy, so every \( \Phi \)-cycle in \( \mathcal{B} \) is also a facial copy, and hence consists of edges ˆ\( a_j \) which will consist of edges ˆ\( d_j \), ˆ\( c_j \), and ˆ\( c_{i,j-1} \), and two additional edges ˆ\( a_j \) and ˆ\( b_j \) with \( h(\hat{d}_j) = a_{ij} \) and \( h(b_j) = b_{ij} \). Let

\[
\mathcal{P}'_0 = B \cup \bigcup_{j \in \mathbb{Z}_{n_i}} C_j.
\]

The common endpoint of ˆ\( a_j \) and ˆ\( b_j \) has degree two in \( \mathcal{P} | \mathcal{W} \setminus S \), while the endpoints of ˆ\( c_j \) have degree three and are incident with ˆ\( d_j \), ˆ\( b_j \), ˆ\( c_{i,j-1} \), ˆ\( c_j \), and ˆ\( c_{i,j+1} \), ˆ\( c_{i,j+1} \), ˆ\( c_{i,j} \), respectively. Hence \( \mathcal{P}'_0 \) contains every edge of \( \mathcal{P} | \mathcal{W} \setminus S \) incident to a vertex of \( \mathcal{P}'_0 \). Thus \( \mathcal{P}'_0 \) is a maximal connected subgraph of \( \mathcal{P} | \mathcal{W} \setminus S \), so \( \mathcal{P}'_0 = \mathcal{P}_0 \). In particular, we conclude that \( |h^{-1}(v) \cap V(\mathcal{P}_0)| = 1\) for all \( v \in V(\mathcal{W}_i) \). Since \( b \) is an \( \mathcal{I} \)-labelling, it follows that \( \text{ch}(\mathcal{P}_0) \cdot b = a_i \).

Lemma 12.5. Let \( \mathcal{P} \) be a \( \mathcal{W} \)-picture in which all \( \Phi \)-cycles are facial copies, and such that all edges in \( \text{bd}(\mathcal{P}) \) belong to \( S \). Let \( G \) be the group with presentation \( \mathcal{I} = \text{Inv}(S; R) \), and let \( b \) be an \( \mathcal{I} \)-labelling. Then there is a \( G \)-picture \( \mathcal{P}' \) with \( \text{bd}(\mathcal{P}') = \text{bd}(\mathcal{P}) \) and \( \text{sign}(\mathcal{P}') = \text{ch}(\mathcal{P}) \cdot b \).

Proof. Let \( \mathcal{W} \setminus S \) denote the closed subhypergraph containing all vertices of \( \mathcal{W} \) and all edges except those in \( S \). Equivalently, \( \mathcal{W} \setminus S \) is the subhypergraph with connected components \( \mathcal{W}_i \), \( 1 \leq i \leq m \). Suppose that \( \mathcal{P}_0 \) is a connected component of \( \mathcal{P} | \mathcal{W} \setminus S \). Since \( \text{bd}(\mathcal{P}) \) does not contain any edges from outside \( S \), \( \mathcal{P}_0 \) must be closed. Since \( \mathcal{P}_0 \) is connected, there must be some \( 1 \leq i \leq m \) such that \( \mathcal{P}_0 \) is a \( \mathcal{W}_i \)-picture. By Lemma 9.2, \( \mathcal{P}_0 \) contains a \( \Phi \)-cycle. By hypothesis, every \( \Phi \)-cycle in \( \mathcal{P}_0 \) is a facial copy, so every \( C_{ij} \)-cycle will contain an edge labelled by \( d_{ij} \). Thus \( \mathcal{P}_0 \) contains an edge labelled by \( d_{ij} \) for some \( j \in \mathbb{Z}_{n_i} \), and by Lemma 9.2 again, \( \mathcal{P}_0 \) contains a \( B_i \)-cycle \( B \). Since \( B_i \in \Phi \), \( B \) is also a facial copy, and hence consists of edges \( \hat{d}_j \), \( j \in \mathbb{Z}_{n_i} \), such that \( h(\hat{d}_j) = d_{ij} \). Let \( \hat{c}_j \), \( j \in \mathbb{Z}_{n_i} \), be the third edge incident to the common endpoint of \( \hat{d}_j \) and \( d_{ij} \). Every edge \( \hat{d}_j \) is contained in a unique \( C_{ij} \)-cycle \( C_j \), which will consist of edges \( \hat{c}_j \), \( \hat{d}_j \), \( \hat{c}_{j-1} \), and two additional edges \( \hat{a}_j \) and \( \hat{b}_j \) with \( h(\hat{a}_j) = a_{ij} \) and \( h(\hat{b}_j) = b_{ij} \). Let

\[
\mathcal{P}'_0 = B \cup \bigcup_{j \in \mathbb{Z}_{n_i}} C_j.
\]
Now let $D(B)$ be the closure of the face bounded by $B$, let $D(C_j)$ be the closure of the face bounded by $C_j$, $j \in \mathbb{Z}_{n_1}$, and let

$$D(P_0) := D(B) \cup \bigcup_{j \in \mathbb{Z}_{n_1}} D(C_j).$$

Clearly $P_0$ is contained in $D(P_0)$, and conversely every edge or vertex of $P$ in $D(P_0)$ belongs to $P_0$. Since $\hat{a}_j$ and $\hat{b}_j$ belong to the boundary of $D(C_j)$ and are not contained in $B$ or any $C_j$, $j' \neq j$, we conclude that $\hat{a}_j$ and $\hat{b}_j$ belong to the boundary of $D(P_0)$. Every other edge of $C_j$ belongs either to $B$, to $C_{j+1}$, or to $C_{j-1}$, so the boundary of $D(P_0)$ does not contain any other edges of $C_j$. Similarly, the boundary does not contain any edges of $B$, or any of the vertices $(i, j, 3)$, $j \in \mathbb{Z}_{n_1}$. We conclude that $D(P_0)$ is bounded by the $A_i$-cycle $A = \hat{a}_1 \hat{b}_1 \cdots \hat{a}_n \hat{b}_n$. In particular, $D(P_0)$ is a simple region, and $bd(\text{germ}(P, D(P_0))) = s_{i_1} \cdots s_{i_n}$ or $s_{i_n} \cdots s_{i_1}$ depending on the orientation of $A$.

Since $D(P_0)$ contains only edges and vertices of $P_0$, if $P_1$ is another connected component of $P[W \setminus S]$, then $D(P_0)$ and $D(P_1)$ are completely disjoint. Thus, as in the proof of Corollary 10.4, we can collapse each region $D(P_0)$ to a single vertex labelled by $r_i = J^{a_i} s_{i_1} \cdots s_{i_n}$ to form a $G$-picture $P'$. Since the edges of $S$ will be unchanged, $bd(P) = bd(P')$. Since every vertex of $P$ belongs to a unique connected component of $P[W \setminus S]$, we conclude that $\text{sign}(P') = \text{ch}(P) \cdot b$. □

Proof of Theorem 5.4. Let $G$ be the group with presentation $\text{Inv}\langle S : R \rangle$, and suppose that $I = \text{Inv}\langle S : R \rangle$ is collegial. Let $I^+ = \text{Inv}\langle S : R^+ \rangle$, and let $G^+$ be the even quotient of $G$, as in Definition 4.4. By Lemma 12.4, we can choose an $I$-labelling $b$ of $W$ such that $\Phi$ is a $b$-constellation. Every $b$-stellar cycle is also 0-stellar, so $\Phi$ is also a 0-constellation by Definition 11.2. By Lemma 12.1, there are morphisms $\phi : G \rightarrow \Gamma(W,b)$ and $\phi^+ : G^+ \rightarrow \Gamma(W,0)$, both sending $s \mapsto x_s$ (note that 0 is an $I^+\setminus b$-labelling).

To show that $\phi$ is injective, we start with $\phi^+$. If $\phi^+(w) = 1$ for some $w \in F_2(S)$, then by Proposition 7.3 there is a $W$-picture $P$ with $bd(P) = w$. By Theorem 11.4 ($b = 0$ case), we can choose $P$ so that all $\Phi$-cycles in $P$ are facial copies. By Lemma 12.5, there is a $G^+$-picture $P'$ such that $bd(P') = bd(P)$, so $w = 1$ in $G^+$. Since $\phi^+(J^aw) = 1$ for $a \in \mathbb{Z}_2$ and $w \in F_2(S)$ if and only if $a = 0$ and $\phi^+(w) = 1$, it follows that $\phi^+$ is injective.
Now there is a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & \Gamma(\mathcal{W}, b) \\
q_1 \downarrow & & \downarrow q_2 \\
G^+ & \xrightarrow{\phi^+} & \Gamma(\mathcal{W}, 0)
\end{array}
\]

where \(q_1\) and \(q_2\) are the quotient maps \(G \to G^+ = (G/(J_G)) \times \mathbb{Z}_2\) and \(\Gamma(\mathcal{W}, b) \to \Gamma(\mathcal{W}, 0) = (\Gamma(\mathcal{W}, b)/(J_\Gamma)) \times \mathbb{Z}_2\) by \(J_G\) and \(J_\Gamma := J_{\Gamma(\mathcal{W}, b)}\) respectively. Since \(J_G\) is central of order \(\leq 2\), we conclude that \(\ker q_1 = \{1_G, J_G\}\).

Since \(\phi^+\) is injective, if \(\phi(w) = 1\) for \(w \in G\), then \(q_1(w) = 1\), and consequently \(w \in \{1, J_G\}\). Thus it remains only to show that \(\phi(J_G) = 1\) if and only if \(J_G = 1\) in \(G\). By definition, \(\phi(J_G) = J_\Gamma\), and if \(J_\Gamma = 1\), then by Proposition 7.3 there is a closed \(\mathcal{W}\)-picture \(\mathcal{P}\) with \(\text{ch}(\mathcal{P}) \cdot b = 1\). Since \(\mathcal{P}\) is closed, Theorem 11.4 again implies that we can choose \(\mathcal{P}\) so that all \(\Phi\)-cycles in \(\mathcal{P}\) are facial copies. By Lemma 12.5, there is a closed \(G\)-picture \(\mathcal{P}'\) such that \(\text{sign}(\mathcal{P}') = 1\), and we conclude that \(J_G = 1\) in \(G\). Thus \(\phi\) is injective. \(\square\)

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