THE SPACE-TIME MANIFOLD AS A CRITICAL SOLID

Dedicated to the memory of Robert E. Marshak

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Abstract

It is argued that the problems of the cosmological constant, stability and renormalizability of quantum gravity can be solved if the space-time manifold is not fundamental, but arises through spontaneous symmetry breaking. A “pre-manifold” model is presented in which many points are connected by random bonds. A set of $D$ real numbers is assigned to each point. These numbers are coupled between points connected by bonds. It is then found that the dominant configuration of bonds is a flat $D$-dimensional manifold, on which there is a massless matter field. Adjusting the parameters of the model leads to fluctuations which at large distances describe quantized massless gravity if $D = 4, 6, 8,\ldots$. These fluctuations do not destabilize the manifold. An approach to include Lorentzian signature is presented.

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1 Introduction

Any attempt to quantize gravity faces challenging questions. How can the theory be renormalized, but still give Einstein’s equations in the infrared? In the Euclidean formulation, how can the conformal instability of the action be cured? If the instability can indeed be cured, why don’t space-time fluctuations lead to a creased or branched-polymer phase like that in random surface models, destroying the interpretation of the manifold as space-time? What makes the manifold nearly flat, in other words, why should the cosmological constant be zero?

The most popular approach to quantum gravity has been through supergravity theories and, more recently, superstrings. Another program, begun by Ashtekhar, is to canonically quantize the theory in a new way. Unfortunately difficult, perhaps insurmountable technical difficulties face such programs. Numerical methods have been used to study functional integrals for lattice Euclidean gravity. It seems that a continuum limit exists, but one in which the vacuum curvature does not vanish. While anyone working on the problem of quantum gravity needs an optimistic temperament, no attempt has met with even qualified success.

The issue taken up here is not whether the approaches above are mathematically adequate to solve the problems of quantum gravity. Rather it is that instead of beginning by formally defining or generalizing Einstein’s theory, one should try a more intuitive tack and look to a physical example for guidance. Such an example exists; the melting of a solid.

In solids and monatomic incompressible liquids it is natural to view each pair of nearest neighbor atoms being joined a bond. A discrete manifold is then defined by the graph of points (atoms) connected by links (bonds). In solids held together by covalent or hydrogen bonds, these bonds have a genuine physical meaning (such as a shared electron). In liquids they do not exist between different molecules; however they are still a useful fiction. The statistical mechanics of a liquid membrane is described by a covariant string model for precisely this reason. The dominant manifolds are triangular in the partition function. By viewing each bond as being of length \(a\), a metric is defined, thereby leading to an effective model such as that discussed in ref. If the number of points diverges at the critical point, random surface models, and possibly models of random simplicial lattice manifolds in higher dimensions will polymerize. This means that the manifolds collapse to one-dimensional structures which may branch.

It is suggested here that a gravitational partition function free of the branched polymer disease with vanishing vacuum curvature is not a sum over liquid manifolds at all, but must be a sum over the manifolds of a “critical solid”. The partition function of an ordinary simplicial lattice crystal solid is dominated by flat manifolds, so polymers do not form. The variations of the manifold are short-range in ordinary three-dimensional solids. This means that the effective theory of gravity is massive. It is argued here that under certain circumstances that the solid-liquid phase boundary can become second-order or critical. The mass of the gravitational fluctuations in the crystal phase near this phase boundary can then be made as small as desired. Furthermore, the presence of long-range order will guarantee that the manifold will tend to be flat in the infrared. The sum over manifolds is then indistinguishable from that of a liquid phase at distances shorter than the correlation length. This length is the Compton wavelength of the graviton.

In conventional models of random manifolds, it appears that crystal formation is ruled out. It therefore seems the most sensible way a space-time manifold can stay approximately...
flat is the same way the manifold of a solid does. The manifold should not be put into the theory at all in the beginning, but should appear dynamically.

In this paper a simple model with no a priori space-time manifold is presented. This model consists of a set of points randomly connected by bonds. To each point is assigned a set of $D$ real numbers. It is found that a flat $D$-dimensional simplicial manifold can arise spontaneously, much in the way that a crystal does. Furthermore if $D = 4, 6, 8 \ldots$ the melting transition can be second order, that is the solid has a critical region. Fluctuations in the metric of the manifold lead to a discretized formulation of Riemannian quantum gravity coupled to a scalar field at large distances, if the solid is close to the melting transition. Since the system is critical, it can be renormalized, though not by perturbation theory. At the critical point, dimensionful quantities scale in such a way that the cut-off can be removed, leaving a renormalized field theory (for an example of the renormalization of a perturbatively non-renormalizable theory see [5]). It was first suggested by Weinberg that a formally non-renormalizable theory of gravity might have a critical surface [11]. It is quite striking that the smallest value of $D$ for which a critical solid exists is 4 (as the criterion of renormalizable field theory implies $D \leq 4$, the space-time dimension is completely fixed). Since the fluctuations are around a flat manifold, the cosmological constant is automatically zero.

Fluctuations in the basic structure of space-time beyond the Planck scale have been speculated about for some time [12]. Fluctuations specifically in topology were considered by Hawking [13] in the context of Regge calculus [14]. It has been suggested that Regge calculus or a related formulation might be the physically correct description of the space-time manifold above the Planck mass [16]. Topological fluctuations were suggested to have important dynamical effects by Coleman [15]. If the picture presented here is correct, notions like topology, dimension, etc. begin to make sense only below the Planck scale. It is my own suspicion that critical solid formation could arise in other pre-manifold models which are not based on the notion of a discrete set of points, such as string theory (in which the appearance of space-time is not yet understood) [3] or some other (true?) theory of gravity yet hidden from the imagination.

A second model is discussed in which gravitation with Lorentzian signature might arise. The motivation lies in the fact that a Lorentz metric can be constructed from a Riemannian metric and a vector field [17]. Unfortunately, the validity of the simple picture of the spontaneous appearance of a flat manifold with gravitational fluctuations is not as obvious as for the first model.

Some suggestions for further numerical and analytic work are presented in the conclusion.

2 Discretized Gravity

The purpose of this section is to briefly familiarize the reader with the regularized formalism of gravity discussed in references [3], [6], [18]. The power of this formalism is that there is no need for explicit coordinates. The law of gravitation can be expressed in terms of a network of points, guaranteeing general covariance.

Suppose $\mathcal{M}$ is a $D$-dimensional simplicial manifold (usually called a “simplicial complex” by mathematicians) [18] of fixed topology. This means that $\mathcal{M}$ is a network of points (which will be called “sites”) connected by line segments (which will be called “links”) which can be constructed by gluing together $D$-dimensional simplices $K_D$ along their $D-1$-dimensional sub-simplices $K_{D-1}$. If $D = 2$, triangles, $K_2$, are glued together by identifying links, $K_1$, if
$D = 3$ tetrahedra, $K_3$, are connected by identifying triangles, etc. If $M$ has no boundary, each $D - 1$-dimensional subsimplex is shared by two $D$-dimensional simplices. At boundaries $D - 1$-dimensional simplices are not shared.

In the original formalism of Regge [14] the network of sites and links (Regge called the links “bones”) is fixed and each link is assigned a variable length. A simplex is regular if and only if the lengths of all its links are the same. In the alternative approach of references [5], [6], [16] the links are instead have a fixed length $a$. The quantum functional integral is defined as a sum over different manifolds (possibly of fixed topology) $M$,

$$Z = \sum_M e^{-S(M)}.$$  

(1)

The sum in (1) becomes a sum over metrics in the continuum limit $a \to 0$. The metric is defined to be flat inside each $D$-dimensional simplex, consistent with links having length $a$. Curvature can exist on the boundaries between simplices. Each simplex has the fixed volume it would possess in flat space.

A general coordinate invariant action, $S(M)$, should be local, at least as $a \to 0$. It will contain covariant terms depending on the metric and the Riemann tensor, if coordinates are chosen. The discrete version of the cosmological term is

$$S_{\text{cosm}}(M) = \Lambda \sum_K V(K_D) \to \int d^D x g^{1/2},$$  

(2)

where $V(K_D)$ is the volume of a $D$-dimensional simplex (a fixed constant). The discrete Einstein action is

$$S_{\text{Einstein}}(M) = -\frac{1}{16\pi^2} \sum_{K_{D-2}} \delta(K_{D-2}) V(K_{D-2}) \to -\int d^D x g^{1/2} R.$$  

(3)

In equation (3), $\delta(K_{D-2})$ is the deficit angle associated with the $D - 2$-dimensional simplex $K_{D-2}$. Terms in the action which are quadratic in the Riemann tensor can also be accommodated, for which the reader is referred to reference [6].

### 3 A Pre-Manifold Model

Consider a set of points $j = 1, ..., N$. To each of these points a value of the “pre-phonon field” $\phi_j \epsilon \mathbb{R}^D$ is assigned. Later on, when it is explained how the space-time manifold forms, it will become clear that $D$ is in fact the dimension of this manifold. To each pair of points $j \neq k$ is assigned a “link variable” $s_{jk} = s_{kj} = 0, 1$. The partition function of the model is

$$Z = \sum_{N=0}^{\infty} \zeta^N \prod_{j=1}^{N} \int d^D \phi_j \prod_{j<k} \sum_{s_{jk}=0,1} g(s_{jk}) \right) \times \exp \left[ \frac{c}{2} \sum_{j<k} s_{jk}(\phi_j - \phi_k)^2 + K \sum_{j<k} \theta(R - |\phi_j - \phi_k|) \right],$$  

(4)

where $\theta$ is the usual step function, $g(0) = 1$ and $g(1) = g$, $\zeta$, $c$, $K$ and $R$ are all real constants. The link variable $s_{jk}$ is quite similar to that used in a non-perturbative formulation of strings [17]. The number of points, $N$, is eventually taken to infinity.
In (4) a natural metric exists. The length if each link is defined to be $a$. The the distance between any two points is defined to be equal to the length of the shortest path connecting them; if no such path exists, the distance between them is infinite. Thus, even if the set of points together with this distance function do not form a manifold, they satisfy the conditions of a metric space. It for this reason (4) is called a “pre-manifold” rather than a “pre-geometric” model.

Why should this be a theory of gravity? Specifically:

1. Is there a saddle point of the sum in (4) for which $s_{jk}$ describes a discrete manifold of small constant curvature, i.e. a nearly regular simplicial lattice? If so, there will be a massless covariant phonon field $\psi_j = \phi_j - <\phi_j>$ on this discrete space-time.

2. Are the fluctuations in $s_{jk}$ near this saddle point correctly describing quantum gravity in the infrared? Allowed fluctuations can be dislocations and disclinations, but there should be no tearing of the lattice. Since the graviton is massless, these fluctuations must have an infinite correlation length. There must be no instabilities destroying the manifold.

It is not hard to see, using a little intuition garnered from the physics of crystal formation that the answer to the first question is yes under a wide variety of choices of the constants, provided $D > 2$. Furthermore, $D$ turns out to be the dimension of the dominant manifold. I will also argue on the basis of this theory that the answer to the second question is also affirmative, provided the couplings are tuned correctly and $D$ is even. The resulting theory of gravity can therefore be only in $D = 4, 6, 8, ...$ dimensions. The cosmological constant is zero, as the saddle point is a flat manifold. The fluctuations are not strong enough to destroy the manifold, so there is no conformal instability, nor is there a pathological branched polymer phase. Furthermore the theory exists at a critical point, and so is well behaved in the ultraviolet.

4 Flat Manifold Formation

To get some understanding of what is going on, it is easy to sum over $s_{jk}$ in (4) to get

$$Z = \sum_{N=1}^{\infty} \zeta^N \prod_{j=1}^{N} \int d^D \phi_j \exp -[V(|\phi_j - \phi_k|)] ,$$

where

$$V(|\phi_j - \phi_k|) = -\log[1 + e^{-\frac{g}{2}(\phi_j - \phi_k)^2}] + K_{\theta}(R - |\phi_j - \phi_k|) .$$

Now $V(|\phi_j - \phi_k|)$ can be interpreted as a two-particle potential for a system of atoms. Assume that the repulsive core radius, $R$, is smaller than $c^{-1/2}$. This potential is attractive for $R < |\phi_j - \phi_j| < c^{-1/2}$, but quickly falls off to zero for $|\phi_j - \phi_j| > c^{-1/2}$. Thus $Z$ can be interpreted as the partition function of such a collection of atoms with $\beta = 1/\kappa T = 1$.

If the parameters $g$, $c$, $k$, $R$ are adjusted properly a crystal phase should form in which the points are close together in $\phi$-space. For other choices of these parameters thermal fluctuations will destabilize such a crystal forming a liquid. For the application to quantum gravity, it is desirable that when the crystal forms, it defines a regular simplicial lattice (this defines a manifold with a flat metric according to the discussion of section 2). In one, two,
three, four or five dimensions, the most “close packed” arrangements of spheres, called $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$, respectively, are all simplicial lattices \[20\]. More precisely, if a site is placed at the center of each sphere, and whenever two spheres touch a link connects their respective sites, the resulting lattice of sites and links is periodic and built entirely of $D$-simplices. There is an elementary prescription for constructing $L_3$, $L_4$ and $L_5$. One simply takes a regular square lattice of points $(n_1, n_2, \ldots, n_D)$, where $n_\mu$ is an integer, and removes all the points for which $n_1 + \ldots + n_D$ is an odd integer. Then one draws a sphere of radius $1/\sqrt{2}$ around each of the remaining points. The reader can easily verify that the maximal number of spheres each of which touches the others is $D + 1$. However, in higher dimensions, there are other lattices which are more close packed. For $D \leq 5$ it must be true that for $R$ less than $c$, the simplicial lattice is a solution of lower free energy than the less closely packed lattices. It is probably also the case that simplicial lattices could form for $D \geq 6$, but verifying this would take some work. Notice that this scenario can work only if $D > 2$, since long-ranged crystalline order is forbidden for $D \leq 2$ by Peierls’ theorem \[21\].

To actually determine the range of parameters for which a solid crystal forms with a simplicial lattice will require an application of density functional theory \[22\]. I intend to apply this technique in a later paper.

Now if a simplicial crystal configuration arises spontaneously in (4), then the dominant configuration of $s_{jk}$ will describe a $D$-dimensional simplicial manifold. The point is that $s_{jk}$ is strongly correlated with the separation $|\phi_j - \phi_k|$, i.e. for $j < k$ and $l < m$, the correlation between the two behaves the following way:

\[
C_{j,k;l,m} \equiv <(\phi_j - \phi_k)^2 s_{lm}> \approx \Phi^2 \delta_{j,l} \delta_{k,m} .
\]

This means if $j, k$ are not nearest neighbors in $\phi$-space, hence separated by $|\phi_j - \phi_k| \approx f$, $f \gg c^{-1/2}$, it is much less probable that $s_{jk} = 1$ (by a factor of $e^{-cf^2}$) than $s_{jk} = 0$, while if $j, k$ are nearest neighbors, it is more probable that $s_{jk} = 1$ than $s_{jk} = 0$. This means that nearest neighbors in $\phi$-space are nearly always joined by links, whereas other pairs of sites are almost never joined by links.

The massless Goldstone Boson of the spontaneous symmetry breaking in $\phi$-space is $\psi_k = a^{d/2-1} \langle \phi_k - <\phi_k> \rangle$, where $a$ is the length of a link. This can be thought of as a phonon field in the crystal.

To summarize, it has been shown that the model (4) is dominated by a flat simplicial manifold under certain choices of the couplings, provided the number of components of $\phi$, namely $D$, is greater than two. This manifold has formed through spontaneous breaking of translation and rotation symmetry in $\phi$-space. The resulting manifold has a dimension equal to $D$.

5 Metric Fluctuations and the Critical Solid

Showing that a flat simplicial manifold forms is a hollow victory. It is necessary to see what the nature of the fluctuations around this manifold are. What is desirable is that these fluctuations do not shear or form cracks in the crystal, so that the effective theory resembles one the formulations of gravity discussed by Lehto et al. \[16\], Ambjørn et. al. and Migdal et. al. \[5\].

Lattice fluctuations around the flat simplicial manifold are made by removing or adding links. They are of two types, which will be referred to as “geometric” and “non-geometric”
fluctuations. The latter include cracks or holes (naked singularities?), local lattice connectivities not consistent with the lattice being a manifold (see for example the book by Seifert and Triefal [8]) as well as $D$-dimensional polyhedra which are not simplices. The former are fluctuations which change deficit angles but preserve the simplicial character of the lattice.

In order for cracks or holes not to form in a lattice, it must be able to flow like a liquid at short distances. Yet at the same time, translation invariance must be broken and the lattice must be rigid macroscopically. The only way for this to be the case is if the system is near a second-order phase boundary between the solid and liquid phase. In other words the crystal is a critical solid and the correlation length approaches infinity. Thus if a theory of gravity arises from the model (4), it will automatically be a massless theory of gravity. This rules out the possibility of asymptotically free theories, which are massive in the infrared. The vacuum expectation value of the Riemann tensor must be zero. Therefore the effective cosmological constant must vanish. The only possible form for the action describing the fluctuations of the simplicial manifold in the continuum limit is

$$S = \frac{1}{16m_p^2} \int d^4x \sqrt{g} (-R + M_{\mu\nu\lambda\sigma} R_{\alpha\beta\gamma\epsilon} R^{\mu\nu\lambda\sigma} + O(R^4)) + \int d^4x \sqrt{g} \partial_\mu \psi \partial^\mu \psi, \quad (8)$$

where $m_p$ is the Planck mass, and the constant tensor $M$ is numerically small compared to $m_p^2$, and far from the perturbative fixed points of the $R^2$ theory (at which the theory is asymptotically free [8]). Notice that the second term stabilizes the conformal instability of the first term. Beyond the second-order phase transition, $M = 0$ and the manifold is no longer stable, becoming a liquid.

Under what circumstances can the continuum action be of the form (8), i.e. the solid-liquid transition be second order? It is well known that this transition is always first order in three dimensions. The reason for this is the following. The natural order parameter is the density

$$\rho(\phi) = \sum_j \delta^D(\phi - \phi_j), \quad (9)$$

which can be approximated by a smooth function. The Ginzburg-Landau free energy is of the form [24]

$$F = \int \frac{d^Dk}{(2\pi)^D} \alpha(k) \rho(-k) \rho(k) + \int \frac{d^Dk_1}{(2\pi)^D} \ldots \int \frac{d^Dk_D}{(2\pi)^D} t(k_1, \ldots, k_D) \delta^D(k_1 + \ldots + k_D) \rho(k_1) \ldots \rho(k_D) + \ldots, \quad (10)$$

where

$$\rho(\phi) = \int \frac{d^Dk}{(2\pi)^D} \rho(k) e^{ik\cdot\phi} \quad (11)$$

The reason the next to leading order term is of order $\rho^D$ is that the crystal must defined by $D$ different wave vectors which add up to zero. Thus the transition is second order if $D$ is even and first order if $D$ is odd. Therefore, in order for a continuum theory of gravity to arise for small $a$, the integer $D$ can only be an even integer greater than two, $D = 4, 6, 8, \ldots$. It is rather nice that the smallest number of possible dimensions is four.
6 Boundary Conditions

Thus far, boundary conditions in $\phi$-space and those of the manifold in physical space have not been discussed. If the range of $\phi$ is restricted to a finite box with “hard-walls”, the resulting manifold $\mathcal{M}$ will have a boundary with a Neumann boundary condition. This Neumann condition results from minimizing the potential (6) for points on the boundary (just as Neumann boundary conditions appear automatically in the continuum limit of open lattice strings [19]) and are

$$n^\mu \partial_\mu g^{\alpha\beta}|_{\partial \mathcal{M}} = n^\mu \Gamma^\alpha_{\mu \gamma} g^{\gamma \beta}|_{\partial \mathcal{M}}.$$  

(12)

Such boundary conditions seem rather unphysical from the point of view of general relativity. On the other hand, by compactifying $\phi$ on a region with no boundary will result in a compact boundaryless space-time $\mathcal{M}$. In general, the topology of space-time above the Planck length (though not the metric!) will be the same as that of $\phi$-space. This suggests that the topology of the universe at distances above the Planck length is not an arbitrary consequence of initial conditions, but is determined by physical law.

7 A Lorentzian Model

No model of quantum gravity can be taken entirely seriously unless the dominant manifolds in the configuration space are of Lorentzian signature ($-{}, {}, {}, {}, {}$) and the sum over this space is a Feynman (rather than Weiner) path integral. It is possible to define a Lorentzian analogue of (4), though understanding its behavior is harder, as there is not yet a physical picture of crystallization.

A model similar to (4) will now be discussed for which spontaneous formation of a manifold with Lorentzian signature could occur. The basic modification is to introduce two types of links, which separate points by a time-like or space-like displacement. It is well known that any manifold with a Riemannian metric $g^+_{\mu \mu}$ and a non-vanishing vector field $\xi^\alpha$ can be interpreted as a manifold with a Lorentz metric given by [17]

$$g_{\mu \nu} = g^+_{\mu \nu} - 2g^+_{\mu \alpha} \xi^\alpha g^+_{\nu \beta} \xi^\beta g^+_{\rho \sigma} \xi^\rho \xi^\sigma.$$  

(13)

Then the vector field $\xi^\alpha$ is time-like. The converse, that any manifold which admits a Lorentz metric also admits a time-like vector field, is obvious.

The model now has a three-valued link variable $s_{jk} = -1, 0, 1$ between points $j$ and $k$. As before, $s_{jk} = 0$ means that no link connects these two points. If $s_{jk} = -1$, the link is time-like, while if $s_{jk} = 1$ the link is space-like. The difference between a time-like and a space-like link is in the sign of the coupling between $\phi_j$ and $\phi_k$. A time-like vector field is defined on the collection of points by requiring that exactly two time-like link are connected to each point, i.e., for any point $j$

$$\sum_{k=1}^{N} s_{jk}(s_{jk} - 1) = 4.$$  

(14)

The model is then

$$Z = \sum_{N=0}^{\infty} \zeta^N \left[ \prod_{j=1}^{N} \int d^D \phi_j \right] \left[ \prod_{j<k} \sum_{s_{jk}=-1,0,1} g(s_{jk}) \right] \left[ \prod_{j} \delta \left( \sum_{k} s_{jk}(s_{jk} - 1), 4 \right) \right].$$
\[
\times \exp\left[-\frac{c}{2} \sum_{j<k} (is_jk - \epsilon)(\phi_j - \phi_k)^2 + i \sum_{j<k} \theta(R - |\phi_j - \phi_k|)\right].
\]

(15)

Here \(\delta(l, m)\) is a convenient way of writing the Kronecker delta, \(g(0) = 1\) and \(g(1), g(-1), \zeta, c, K\) and \(R\) are all real constants. Notice that an \(i\epsilon\) prescription is built into (15) to assure convergence to a unique answer. Unfortunately, this Lorentzian model is not some simple analytic continuation of (4). The link variable cannot be simply summed out as before. The model is therefore more difficult to understand physically.

8 Conclusions

A simple physical picture of how the space-time manifold of the universe can arise spontaneously has been presented, and it has been argued that this picture solves the fundamental problems of quantized general relativity. The discussion in this paper has been qualitative, though the arguments are built on a strong physical foundation.

It is important to investigate the specific form of crystal structures which can arise for different choices of the fundamental parameters. The best way of studying this question is density functional theory [22]. It should be possible to roughly locate the critical region this way. It might also be possible to apply density functional methods to the more interesting model (15). If so, what would be particularly important is the global structure of the resulting space-time, e.g. whether closed time-like curves and naked singularities are absent [17]. For the simpler model (6) numerical calculations using molecular dynamics or Monte-Carlo methods seem quite feasible.

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