Approximation of multipartite quantum states and the relative entropy of entanglement

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Abstract

Special approximation technique for analysis of different characteristics of states of multipartite infinite-dimensional quantum systems is proposed and applied to study of the relative entropy of entanglement and its regularisation.

We prove several results about analytical properties of the multipartite relative entropy of entanglement and its regularization (the lower semicontinuity on wide class of states, the uniform continuity under the energy constraints, etc.).

We establish a finite-dimensional approximation property for the relative entropy of entanglement and its regularization that allows to generalize to the infinite-dimensional case the results proved in the finite-dimensional settings.

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1 Introduction

A specific feature of infinite-dimensional quantum systems is singular properties (discontinuity, infinite values, etc.) of basic characteristics of quantum states. So, for strict mathematical analysis of infinite-dimensional quantum systems it is necessary to apply special techniques, in particular, approximation techniques to overcome the problems arising from these singularities.

In this article we develop the approximation technique proposed in [25] in the one-party case to analysis of characteristics of multipartite infinite-dimensional quantum systems. A central notion of this technique is the finite-dimensional approximation property of a state (briefly called the FA-property). According to [25] a quantum state $\rho$ with the spectrum $\{\lambda_i\}$ has the FA-property if there exists of a sequence $\{g_i\}$ of nonnegative numbers such that

$$\sum_{i=1}^{+\infty} \lambda_i g_i < +\infty \quad \text{and} \quad \lim_{\beta \to 0^+} \left[ \sum_{i=1}^{+\infty} e^{-\beta g_i} \right]^\beta = 1. \quad (1)$$

It is shown in [25] that the FA-property of a state $\rho$ implies the finiteness of its entropy, but the converse implication remains an open question. There is a simple sufficient condition for the FA-property: it holds for a state $\rho$ with the spectrum $\{\lambda_i\}$ provided that

$$\sum_{i=1}^{+\infty} \lambda_i \ln^q i < +\infty \quad \text{for some} \quad q > 2.$$ 

This condition shows that the FA-property holds for all states whose eigenvalues tend to zero faster than $[i \ln^q i]^{-1}$ as $i \to +\infty$ for some $q > 3$, in particular, it holds for all Gaussian states playing essential role in quantum information theory.

Many characteristics of a state of a $n$-partite quantum system $A_1...A_n$ have the form of a function

$$f(\rho \mid p_1,...,p_l)$$

on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ of states of this system depending on some parameters $p_1,...,p_l$ (other states, quantum channels, quantum measurements, etc.). If $\rho$ is a state of infinite-dimensional $n$-partite quantum system $A_1...A_n$ then we may approximate it by the sequence of states

$$\rho_r = Q_r \rho Q_r [\text{Tr} Q_r \rho]^{-1}, \quad Q_r = P_r^1 \otimes ... \otimes P_r^n, \quad (2)$$

where $P_r^s$ is the spectral projector of $\rho_{A_s}$ corresponding to its r maximal eigenvalues (taking the multiplicity into account). Naturally, the question arises under what conditions

$$f(\rho_r \mid p_1,...,p_l) \quad \text{tends to} \quad f(\rho \mid p_1,...,p_l) \quad \text{as} \quad r \to \infty$$

uniformly on $p_1,...,p_l$. Our main technical result asserts that for given $m \leq n$ the FA-property of the marginal states $\rho_{A_1},...,\rho_{A_m}$ of a state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ guarantees
this uniform convergence for a wide class of functions $f$ (depending on $m$). In fact, the more general assertion is valid in which the states $\rho_r$ are defined by formula (2) via the operators $Q_r = P_{s_1}^r \otimes \ldots \otimes P_{s_l}^r \otimes I_R$, where $\{s_1, \ldots, s_l\}$ is any subset of $\{1, \ldots, n\}$ and $R = A_1 \ldots A_n \setminus A_{s_1} \ldots A_{s_l}$ (Theorem 1 in Section 3).

The above result is used in this article for analysis of the relative entropy of entanglement and its regularization in infinite-dimensional $n$-partite quantum systems – the basic entanglement measures used in quantum information theory [26, 27, 8]. Mathematically, particular problems of studying the relative entropy of entanglement in infinite dimensions are related to its definition involving the infimum of a lower semicontinuous function (the quantum relative entropy) over a noncompact set (the set of separable states). One of these problems is a proof of lower semicontinuity of the relative entropy of entanglement, which is a desirable property of an entanglement measure in infinite-dimensional composite systems.

The proposed approximation technique allows to establish the lower semicontinuity of the relative entropy of entanglement $E_R$ and its regularization $E_\infty^R$ on the subset $S_\times(\mathcal{H}_{A_1 \ldots A_n})$ of $S(\mathcal{H}_{A_1 \ldots A_n})$ consisting of states $\rho$ that have at least $n - 1$ marginal states with the FA-property. Moreover, it is proved that

\[
\liminf_{k \to +\infty} E^*_R(\rho_k) \geq E^*_R(\rho_0), \quad E^*_R = E_R, E_\infty^R,
\]

for arbitrary sequence $\{\rho_k\}$ converging to a state $\rho_0$ in $S_\times(\mathcal{H}_{A_1 \ldots A_n})$.

The approximation technique is used also to obtain several observations concerning definition of the relative entropy of entanglement. In particular, we prove that for all states in $S_\times(\mathcal{H}_{A_1 \ldots A_n})$ this quantity can be defined as the relative entropy distance to the set of all finitely decomposable separable states (despite the existence of finitely-non-decomposable and countably-non-decomposable separable states in $S(\mathcal{H}_{A_1 \ldots A_n})$).

We establish a finite-dimensional approximation property for the relative entropy of entanglement and its regularization that allows to generalize to the infinite-dimensional case the results proved in the finite-dimensional settings. Examples of using this property are presented.

Finally, we consider energy-constrained versions of the $n$-partite relative entropy of entanglement. It is proved, in particular, that for any state in $S(\mathcal{H}_{A_1 \ldots A_n})$ with finite energy the infimum in the definition of the relative entropy of entanglement can be taken over all finitely-decomposable separable states with finite energy (provided that the Hamiltonians of individual subsystems satisfy a particular condition).

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the operator norm $\| \cdot \|$ and $\mathcal{T}(\mathcal{H})$ the Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$. Let $S(\mathcal{H})$ be the set of quantum states (positive operators in $\mathcal{T}(\mathcal{H})$ with unit trace) [6, 13, 30].
Denote by $I_{\mathcal{H}}$ the unit operator on a Hilbert space $\mathcal{H}$ and by $\text{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathcal{S}(\mathcal{H})$.

Following [25] we will say that a state $\rho$ in $\mathcal{S}(\mathcal{H})$ with the spectrum $\{\lambda_i\}$ has the FA-property if there exists of a sequence $\{g_i\}$ of nonnegative numbers such that (1) holds. We will denote by $\mathcal{S}_{\text{FA}}(\mathcal{H})$ the set of all states in $\mathcal{S}(\mathcal{H})$ having the FA-property. It follows from Corollary 3 in [25] that $\mathcal{S}_{\text{FA}}(\mathcal{H})$ is a face of the convex set $\mathcal{S}(\mathcal{H})$.

The von Neumann entropy of a quantum state $\rho \in \mathcal{S}(\mathcal{H})$ is defined by the formula $H(\rho) = \text{Tr} \eta(\rho)$, where $\eta(x) = -x \ln x$ for $x > 0$ and $\eta(0) = 0$. It is a concave lower semicontinuous function on the set $\mathcal{S}(\mathcal{H})$ taking values in $[0, +\infty]$ [6, 10, 29]. The von Neumann entropy satisfies the inequality

$$H(p\rho + (1-p)\sigma) \leq pH(\rho) + (1-p)H(\sigma) + h_2(p)$$

valid for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ and $p \in (0, 1)$, where $h_2(p) = \eta(p) + \eta(1-p)$ is the binary entropy [13, 30]. Note that $H(\rho)$ is finite for any state $\rho$ in $\mathcal{S}_{\text{FA}}(\mathcal{H})$ [25].

We will use the Lindblad extension of the quantum relative entropy defined for any positive operators $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ as follows

$$H(\rho\|\sigma) = \sum_i \langle i | \rho \ln \rho - \rho \ln \sigma | i \rangle + \text{Tr} \sigma - \text{Tr} \rho,$$

where $\{ |i\rangle \}$ is the orthonormal basis of eigenvectors of the operator $\rho$ and it is assumed that $H(\rho\|\sigma) = +\infty$ if supp $\rho$ is not contained in supp $\sigma$ [10].

The quantum conditional entropy

$$H(A|B)_{\rho} = H(\rho) - H(\rho_B)$$

of a state $\rho$ of a bipartite quantum system $AB$ with finite marginal entropies $H(\rho_A)$ and $H(\rho_B)$ is essentially used in analysis of quantum systems [6, 30]. It can be extended to the set of all states $\rho$ with finite $H(\rho_A)$ by the formula

$$H(A|B)_{\rho} = H(\rho_A) - H(\rho\|\rho_A \otimes \rho_B)$$

proposed in [9], where it is shown that this extension possesses all basic properties of the quantum conditional entropy valid in finite dimensions.

The quantum mutual information (QMI) of a state $\rho$ of a multipartite quantum system $A_1 \ldots A_n$ is defined as (cf.[11, 5, 32])

$$I(A_1: \ldots : A_n)_\rho \doteq H(\rho\|\rho_{A_1} \otimes \cdots \otimes \rho_{A_n}) = \sum_{s=1}^{n} H(\rho_{A_s}) - H(\rho),$$

1It means that the set $\mathcal{S}_{\text{FA}}(\mathcal{H})$ is convex and contains any segment from $\mathcal{S}(\mathcal{H})$ provided that it contains at least one internal point of this segment.

2The support supp $\rho$ of a positive trace class operator $\rho$ is the closed subspace spanned by the eigenvectors of $\rho$ corresponding to its positive eigenvalues.
where the second formula is valid if $H(\rho) < +\infty$. It is easy to show (cf. [24]) that

$$I(A_1: \ldots : A_n)_\rho \leq 2 \sum_{s=1}^{n-1} H(\rho_{A_s}). \quad (7)$$

Similar upper bound holds with any other $n-1$ marginal entropies of the state $\rho$.

For any positive (semi-definite) densely defined operator $^3 G$ on a Hilbert space $\mathcal{H}$ and any positive operator $\rho$ in $\mathcal{T}(\mathcal{H})$ we assume that

$$\text{Tr} G \rho = \sup_n \text{Tr} P_n G \rho \leq +\infty, \quad (8)$$

where $P_n$ is the spectral projector of $G$ corresponding to the interval $[0, n]$. Then

$$\mathcal{C}_{G,E} = \{ \rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr} G \rho \leq E \} \quad (9)$$

is a closed convex nonempty subset of $\mathcal{S}(\mathcal{H})$ for any $E$ exceeding the infimum of the spectrum of $G$. If $G$ is treated as Hamiltonian of the quantum system associated with the space $\mathcal{H}$ then $\mathcal{C}_{G,E}$ is the set of states with the mean energy not exceeding $E$.

We will pay a special attention to the class of unbounded densely defined positive operators on $\mathcal{H}$ having discrete spectrum of finite multiplicity. In Dirac’s notations any such operator $G$ can be represented as follows

$$G = \sum_{i=1}^{+\infty} g_i |\tau_i\rangle \langle \tau_i| \quad (10)$$
on the domain $\mathcal{D}(G) = \{ \varphi \in \mathcal{H} \mid \sum_{i=1}^{+\infty} g_i^2 |\langle \tau_i| \varphi\rangle|^2 < +\infty \}$, where $\{\tau_i\}_{i=1}^{+\infty}$ is the orthonormal basis of eigenvectors of $G$ corresponding to the nondecreasing sequence $\{g_i\}_{i=1}^{+\infty}$ of eigenvalues tending to $+\infty$.

It is well known that the von Neumann entropy is continuous on the set $\mathcal{C}_{G,E}$ for any $E$ if (and only if) the operator $G$ satisfies the condition

$$\text{Tr} e^{-\beta G} < +\infty \quad \text{for all } \beta > 0 \quad (11)$$

and that the maximal value of the entropy on this set is achieved at the Gibbs state $\gamma(E) = e^{-\beta(E) G} / \text{Tr} e^{-\beta(E) G}$, where the parameter $\beta(E)$ is determined by the equality $\text{Tr} G e^{-\beta(E) G} = E \text{Tr} e^{-\beta(E) G}$ [29]. Condition (11) implies that $G$ is an unbounded operator having discrete spectrum of finite multiplicity, i.e. it has form (10). So, by the Lemma in [7] the set $\mathcal{C}_{G,E}$ defined in (9) is compact for any $E$.

We will often consider operators $G$ satisfying the condition

$$\lim_{\beta \to 0^+} \left[ \text{Tr} e^{-\beta G} \right]^{1/\beta} = 1, \quad (12)$$

$^3$We assume that a positive operator is a self-adjoint operator [17].
which is slightly stronger than condition (11). In terms of the sequence \( \{g_i\} \) of eigenvalues of \( G \) condition (11) means that \( \lim_{i \to \infty} g_i / \ln i = +\infty \), while condition (12) is valid if \( \lim \inf_{i \to \infty} g_i / \ln^q i > 0 \) for some \( q > 2 \) [21, Proposition 1]. By Lemma 1 in [21] condition (12) holds if and only if

\[
F_G(E) \overset{\text{def}}{=} \sup_{\rho \in G,E} H(\rho) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty.
\]

It is essential that condition (12) is valid for the Hamiltonians of many real quantum systems [1, 21].

3 Approximation of multipartite quantum states

In this section we will obtain the \( n \)-partite version of Theorem 2 in [25]. It turns out that many important characteristics of a state of a \( n \)-partite quantum systems belong to one of the classes \( \tilde{L}_n^m(C,D) \) and \( N_{n,n}^m(C,D) \) introduced in [24] and described below.

Let \( m \leq n \) and \( L_n^m(C,D) \) be the class of all functions \( f \) on the set\(^4\)

\[
\mathcal{G}_m(\mathcal{H}_{A_1...A_n}) \equiv \{ \rho \in \mathcal{G}(\mathcal{H}_{A_1...A_n}) \mid H(\rho_{A_1}), ..., H(\rho_{A_m}) < +\infty \}
\]

such that

\[
-a_f h_2(p) \leq f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \leq b_f h_2(p)
\]

for any states \( \rho \) and \( \sigma \) in \( \mathcal{G}_m(\mathcal{H}_{A_1...A_n}) \) and any \( p \in (0,1) \) and

\[
-c_f S_m(\rho) \leq f(\rho) \leq c_f^+ S_m(\rho), \quad S_m(\rho) = \sum_{s=1}^{m} H(\rho_{A_s}),
\]

for any state \( \rho \) in \( \mathcal{G}_m(\mathcal{H}_{A_1...A_n}) \), where \( h_2 \) is the binary entropy (defined after (3)), \( a_f, b_f \) and \( c_f, c_f^+ \) are nonnegative numbers such that \( a_f + b_f = D \) and \( c_f + c_f^+ = C \).

Let \( \tilde{L}_n^m(C,D) \) be the class containing all functions in \( L_n^m(C,D) \) and all the functions of the form

\[
f(\rho) = \sup_{\lambda} f_{\lambda}(\rho) \quad \text{and} \quad f(\rho) = \inf_{\lambda} f_{\lambda}(\rho),
\]

where \( \{f_{\lambda}\} \) is a family of functions in \( L_n^m(C,D) \).

Let \( N_{n,n}^m(C,D) \) be the class of all functions \( f \) on the set \( \mathcal{G}_m(\mathcal{H}_{A_1...A_n}) \) defined by the expression

\[
f(\rho) \overset{\text{def}}{=} \inf_{\hat{\rho} \in \mathcal{G}_m(\rho)} h(\hat{\rho})
\]

via particular function \( h \) in \( \tilde{L}_n^m(C,D) \) for some \( l > 0 \), where:

- \( \mathcal{M}_1(\rho) \) is the set of all extensions of \( \rho \) to a state of \( A_1...A_{n+l} \);

\(^4\)We assume that \( A_1,...,A_n \) are infinite-dimensional quantum systems.
• $\mathcal{M}_2(\rho)$ is the set of all extensions of $\rho$ having the form

$$\hat{\rho} = \sum_i p_i \rho_i \otimes |i\rangle\langle i|,$$

where $\{\rho_i\}$ is a collection of states in $\mathcal{S}(\mathcal{H}_{A_1..A_n})$, $\{p_i\}$ is a probability distribution and $\{|i\rangle\}$ is an orthonormal basis in $\mathcal{H}_{A_{n+1}}$ (in this case $l = 1$);

• $\mathcal{M}_3(\rho)$ is the set of all extensions of $\rho$ having the form (16) in which $\{\rho_i\}$ is a collection of pure states in $\mathcal{S}(\mathcal{H}_{A_1..A_n})$.

A noncomplete list of characteristics belonging to one of the classes $\hat{L}_{m}^{m}(C, D)$ and $N_{n,s}^{m}(C, D)$ includes the von Neumann entropy, the conditional entropy, the $n$-partite quantum (conditional) mutual information, the one way classical correlation, the quantum discord, the mutual and coherent informations of a quantum channel, the information gain of a quantum measurement with and without quantum side information, the $n$-partite relative entropy of entanglement, the quantum topological entropy and its $n$-partite generalization, the bipartite entanglement of formation, the $n$-partite squashed entanglement and c-squashed entanglement, the conditional entanglement of mutual information and other conditional entanglement measures obtained via some function from one of the classes $\hat{L}_{n+l}^{m}(C, D)$, $l > 0$ [24, Section 3].

To formulate our main technical result introduce the approximation map $\Lambda_{r}^{s_1,...,s_l}$ on the space $\mathcal{S}(\mathcal{H}_{A_1..A_n})$ determined by a set $\{s_1, ..., s_l\} \subseteq \{1, ..., n\}$ as follows

$$\Lambda_{r}^{s_1,...,s_l}(\rho) = Q_r \rho Q_r [\text{Tr}_r \rho]^{-1}, \quad Q_r = P_r^{s_1} \otimes \ldots \otimes P_r^{s_l} \otimes I_R,$$

where $P_r^{s}$ is the spectral projector of $\rho_{A_s}$ corresponding to its $r$ maximal eigenvalues (taking the multiplicity into account) and $R = A_1..A_n \setminus A_{s_1}..A_{s_l}$.
Theorem 1. Let \( \rho \) be a state in \( \mathcal{S}(\mathcal{H}_{A_1, A_n}) \) such that \( \rho_{A_s} \in \mathcal{S}_{FA}(\mathcal{H}_{A_s}) \) for \( s = 1, m \), \( m \leq n \). Let \( \{s_1, ..., s_l\} \) be a given subset of \( \{1, ..., n\} \). Then for given nonnegative numbers \( C \) and \( D \) there is a natural number \( r_0 \) and vanishing sequences \( \{Y^L_{C,D}(r)\}_{r \in \mathbb{N}} \) and \( \{Y^N_{C,D}(r)\}_{r \in \mathbb{N}} \) such that

\[
|f(\Lambda^{s_1, ..., s_l}_{r}(\rho)) - f(\rho)| \leq Y^L_{C,D}(r) \quad \text{and} \quad |f'(\Lambda^{s_1, ..., s_l}_{r}(\rho)) - f'(\rho)| \leq Y^N_{C,D}(r) \quad \forall r \geq r_0
\]

for any function \( f \) in \( \hat{L}_m^m(C, D) \) and any function \( f' \) in \( N^m_{n, s} (C, D) \).

The number \( r_0 \) and the sequences \( \{Y^L_{C,D}(r)\}_{r \in \mathbb{N}} \) and \( \{Y^N_{C,D}(r)\}_{r \in \mathbb{N}} \) are completely determined by the states \( \rho_{A_s}, s \in \{s_1, ..., s_l\} \cup \{1, ..., m\} \), and do not depend on \( n \).

Proof. For each natural \( s \) in \([1, m]\) let \( \{g^s_i\}_i \) be a sequence of nonnegative numbers such that

\[
\sum_{i=1}^{+\infty} \lambda^s_i g^s_i < +\infty \quad \text{and} \quad \lim_{\beta \to 0+} \left[ \sum_{i=1}^{+\infty} e^{-\beta g^s_i} \right]^\beta = 1,
\]

where \( \{\lambda^s_i\} \) is the spectrum of the state \( \rho_{A_s} \) taking in the non-increasing order. We may assume that all the sequences \( \{g^s_i\}_i \) are non-decreasing and that \( g^s_i = 0 \) for all \( s \).

For each \( s \) consider the positive operator

\[
G_{A_s} = \sum_{i=1}^{+\infty} g^s_i |\varphi^s_i\rangle \langle \varphi^s_i|
\]

on \( \mathcal{H}_{A_s} \), where \( \{\varphi^s_i\} \) is the basis of eigenvectors of \( \rho_{A_s} \) corresponding to the sequence \( \{\lambda^s_i\} \) of eigenvalues. Then the operator \( G_{A_s} \) satisfies condition (12) and

\[
E_s = \text{Tr} G_{A_s} \rho_{A_s} = \sum_{i=1}^{+\infty} \lambda^s_i g^s_i < +\infty.
\]

For each natural \( s \) in \([1, n]\) let \( P^s_r = \sum_{i=1}^r |\varphi^s_i\rangle \langle \varphi^s_i| \) be the spectral projector of \( \rho_{A_s} \) corresponding to its \( r \) maximal eigenvalues (taking the multiplicity into account) and \( \bar{P}^s_r = I_{A_s} - P^s_r \). Then

\[
\text{Tr} Q_r \rho \geq 1 - \sum_{j=1}^l \text{Tr} \bar{P}^s_{r_j} \rho_{A_{s_j}}, \quad (19)
\]

where \( Q_r \) is the projector defined in (17). To prove this inequality note that

\[
|\text{Tr} Q_r^{-1} I_{A_{s_1}} \otimes ... \otimes I_{A_{s_l}} - \text{Tr} Q_r^{-1} I_{A_{s_1}} \otimes ... \otimes I_{A_{s_l}}| \rho| \leq \text{Tr} \bar{P}^s_r \rho_{A_{s_j}} \quad j = 1, l,
\]

where \( Q^0_r = I_{A_1...A_n}, Q^j_r = P^s_{r^j_1} \otimes ... \otimes P^s_{r^j_l} \otimes I_{R_j}, R_j = A_1...A_n \setminus A_{s_1}...A_{s_l} \). Hence

\[
1 - \text{Tr} Q_r \rho \leq \sum_{j=1}^l |\text{Tr} Q_r^{-1} I_{A_{s_1}} \otimes ... \otimes I_{A_{s_l}} - \text{Tr} Q_r^{-1} I_{A_{s_1}} \otimes ... \otimes I_{A_{s_l}}| \rho| \leq \sum_{j=1}^l \text{Tr} \bar{P}^s_r \rho_{A_{s_j}}.
\]

\(^5\mathcal{S}_{FA}(\mathcal{H}) \) is the set of all states in \( \mathcal{S}(\mathcal{H}) \) having the FA-property (see Section 2).
By the construction of the state $\Lambda_r^{s_1,\ldots,s_l}(\rho)$ we have

$$c_r[\Lambda_r^{s_1,\ldots,s_l}(\rho)]_{A_s} \leq \rho_{A_s}, \quad c_r = \text{Tr}Q_r\rho,$$

for any $s$. Hence, it follows from (19) that

$$\sum_{s=1}^{m} \text{Tr}G_{A_s}[\Lambda_r^{s_1,\ldots,s_l}(\rho)]_{A_s} \leq c_r^{-1} \sum_{s=1}^{m} \text{Tr}G_{A_s}\rho_{A_s} = c_r^{-1}E_S \leq \frac{E_S}{1 - \sum_{j=1}^{l} \text{Tr}\bar{P}_r^{s_j}\rho_{A_{s_j}}}, \quad (20)$$

where $E_S = E_1 + \cdots + E_m$.

By using Winter’s gentle measurement lemma (cf.[30]) we obtain

$$\|\rho - \Lambda_r^{s_1,\ldots,s_l}(\rho)\|_1 \leq 2\sqrt{\text{Tr}Q_r\rho} \leq 2\sqrt{\sum_{j=1}^{l} \text{Tr}\bar{P}_r^{s_j}\rho_{A_{s_j}}}, \quad (21)$$

where $\bar{Q}_r = I_{A_1} \cdots I_{A_n} - Q_r$ and the last inequality follows from (19).

Let $A^m \doteq A_1 \cdots A_m$. Since all the operators $G_{A_1,\ldots,G_{A_m}}$ satisfy condition (12), the operator

$$G_{A^m} = G_{A_1} \otimes G_{A_2} \otimes \cdots \otimes G_{A_m} + \cdots + I_{A_1} \otimes \cdots \otimes I_{A_{m-1}} \otimes G_{A_m}.$$

satisfies the same condition by Lemma 2 in [24]. By Lemma 1 in [21] this implies that

$$F_{G_{A^m}}(E) \doteq \sup_{\text{Tr}G_{A^m}\sigma \leq E} H(\sigma) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty, \quad (22)$$

where the supremum is over all states $\sigma$ in $\mathcal{G}(\mathcal{H}_{A^m})$ such that $\text{Tr}G_{A^m}\sigma \leq E$.

Let $r_0$ be the minimal positive integer such that $\delta_r \doteq \sqrt{\sum_{j=1}^{l} \text{Tr}\bar{P}_r^{s_j}\rho_{A_{s_j}}} \leq 1/2$ for all $r \geq r_0$. The equality $\sum_{s=1}^{m} \text{Tr}G_{A_s}\rho_{A_s} = E_S$ and inequality (20) allow to apply Theorem 1 in [24] to the states $\rho$ and $\Lambda_r^{s_1,\ldots,s_l}(\rho)$ for all $r \geq r_0$. By this theorem it follows from (21) that

$$|f(\rho) - f(\Lambda_r^{s_1,\ldots,s_l}(\rho))| \leq C\sqrt{2\delta_r}F_{G_{A^m}}[\frac{4E_S}{3\delta_r}] + Dg(\sqrt{2\delta_r}) \quad \forall r \geq r_0 \quad (23)$$

for any function $f \in \hat{L}^m(C, D)$ and

$$|f'(\rho) - f'(\Lambda_r^{s_1,\ldots,s_l}(\rho))| \leq C\sqrt{2\delta'}F_{G_{A^m}}[\frac{4E_S}{3\delta'}] + Dg(\sqrt{2\delta'}) \quad \forall r \geq r_0, \quad (24)$$

where $\delta' = \sqrt{\delta_r(2 - \delta_r)}$, for any function $f' \in N_{m,s}^m(C, D)$.

Since $\delta_r$ tends to zero as $r \to +\infty$, by denoting the right hand sides of (23) and (24) by $Y_{E_D}(r)$ and $Y_{\delta_D}(r)$ correspondingly and taking (22) into account we obtain the main assertion of the theorem.
The last assertion follows from the above proof.

Example 1. Let $f_{\Phi_1...\Phi_n}(\rho) = I(B_1:...:B_n)_{\Phi_1\otimes...\otimes\Phi_n(\rho)}$, where $\Phi_1: A_1 \to B_1,..., \Phi_n: A_n \to B_n$ are arbitrary quantum channels. Inequality (10) in [24] (with trivial system $C$), upper bound (7), the nonnegativity of the quantum mutual information and its monotonicity under local channels imply that the function $f_{\Phi_1...\Phi_n}$ belongs to the class $L_{n-1}^n(2, n)$ for any channels $\Phi_1,..., \Phi_n$. Thus, if the marginal states $\rho_{A_1,...,A_{n-1}}$ have the FA-property then, by Theorem 1, for any subset $\{s_1,..., s_l\}$ of $\{1,..., n\}$ there exist a natural number $r_0$ and a vanishing sequence $\{Y(r)\}$ such that

$$|I(B_1:...:B_n)_{\Phi_1\otimes...\otimes\Phi_n(\rho)} - I(B_1:...:B_n)_{\Phi_1\otimes...\otimes\Phi_n(\sigma)}| \leq Y(r) \quad \forall r \geq r_0, \quad (25)$$

where $\rho_r = \Lambda_{s_1,...,s_l}(\rho)$, for any channels $\Phi_1,..., \Phi_n$.

The last assertion of Theorem 1 shows that inequality (25) remains valid with $\rho$ and $\rho_r = \Lambda_{s_1,...,s_l}(\rho)$ replaced by $\sigma$ and $\sigma_r = \Lambda_{s_1,...,s_l}(\sigma)$, where $\sigma$ is any state in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ such that $\sigma_{A_k} = \rho_{A_k}$ for each $k \in \{s_1,..., s_l\} \cup \{1,..., n-1\}$.

By using the above observation with $\{s_1,..., s_l\} = \{1,..., n\}$ it is easy to prove continuity of the function

$$(\Phi_1,..., \Phi_n) \mapsto I(B_1:...:B_n)_{\Phi_1\otimes...\otimes\Phi_n(\rho)}$$

on the set of all tuples $(\Phi_1,..., \Phi_n)$ of channels equipped with the topology of pairwise strong convergence provided that the FA-property holds for at least $n-1$ marginal states of the state $\rho$. By noting that the FA-property implies finiteness of the entropy, this assertion can be also obtained from Proposition 3 in [23] proved in a completely different way.

4 The relative entropy of entanglement and its regularization in infinite dimensions

The relative entropy of entanglement is one of the main entanglement measures in finite-dimensional multipartite quantum systems. For a state $\rho$ of a system $A_1...A_n$ it is defined as

$$E_R(\rho) = \inf_{\sigma \in \mathcal{S}_s(\mathcal{H}_{A_1...A_n})} H(\rho||\sigma), \quad (26)$$

where $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$ is the set of separable\(^7\) (nontangled) states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ defined as the convex hull of all product states $\rho_1 \otimes \cdots \otimes \rho_n$, $\rho_s \in \mathcal{S}(\mathcal{H}_{A_s})$, $s = \frac{1}{n}$ [26, 27, 8].\(^8\)

\(^6\)The obvious inequality $I(A_1:...:A_n)_{\rho} \leq \sum_{s=1}^n H(\rho_{A_s})$ implies that this function also belongs to the class $L_{n-1}^n(1,n)$.

\(^7\)Here and in what follows speaking about separable states we mean full separable states [8].

\(^8\)A detailed description of the works devoted to the relative entropy of entanglement and related entanglement measures in bipartite quantum systems is given in Section 5.8 in [31]. I would be grateful for any references concerning the relative entropy of entanglement in multipartite quantum systems.
The relative entropy of entanglement possesses basic properties of entanglement measures (convexity, LOCC-monotonicity, asymptotic continuity, etc.) and satisfies the inequality
\[ pE_R(\rho) + (1 - p)E_R(\sigma) \leq E_R(p\rho + (1 - p)\sigma) + h_2(p), \] (27)
valid for any states \( \rho \) and \( \sigma \) in \( \mathcal{S} \left( \mathcal{H}_A_1...A_n \right) \) and any \( p \in [0, 1] \), where \( h_2 \) is the binary entropy [33]. In [16] it is proved that
\[ E_R(\rho) \leq \sum_{s=1}^{n-1} H(\rho_{A_s}). \] (28)
Since this upper bound holds with arbitrary \( n - 1 \) subsystems of \( A_1...A_n \) (instead of \( A_1, ..., A_{n-1} \)), it is easy to show that
\[ E_R(\rho) \leq \frac{n-1}{n} \sum_{s=1}^{n} H(\rho_{A_s}). \] (29)

The relative entropy of entanglement is nonadditive. Its regularization is defined in the standard way:
\[ E_R^\infty(\rho) = \lim_{m \to +\infty} m^{-1} E_R(\rho^{\otimes m}), \] (30)
where \( \rho^{\otimes m} \) is treated as a state of the \( n \)-partite quantum system \( A_1^m...A_n^m \).

Below we consider two ways to generalize the finite-dimensional relative entropy of entanglement to states of an infinite-dimensional \( n \)-partite quantum system. Then, in Section 4.3, we analyse relations between these generalizations and consider their analytical properties. We also explore analytical properties of the regularized relative entropy of entanglement in the infinite-dimensional settings.

### 4.1 Direct definition of \( E_R \)

Definition (26) is valid in the case of infinite-dimensional quantum system \( A_1...A_n \). One should only to note that in this case the set \( \mathcal{S}_n(\mathcal{H}_A_1...A_n) \) is defined as the convex closure of all product states in \( \mathcal{S}(\mathcal{H}_A_1...A_n) \). It is essential that inequality (27) and upper bounds (28) and (29) remain valid in this case provided that all the involved quantities are finite. Inequality (27) in the infinite-dimensional settings is proved in [21, Lemma 6]. Upper bounds (28) and (29) can be easily obtained from Lemma 3 in the Appendix by using the \( \sigma \)-convexity of \( E_R \) (which directly follows from the joint convexity and lower semicontinuity of the quantum relative entropy, see Theorem 2 below).

Inequalities (27), (28) and (29) along with the convexity of \( E_R \) show that the function \( E_R \) belongs to the classes \( L_n^{n-1}(1,1) \) and \( L_n^n(C_n, 1) \), where \( C_n = (n - 1)/n. \)\(^9\)

\(^9\)These classes are described in Section 3.
Similar to other entanglement measures the relative entropy of entanglement is not continuous on the whole set of states of infinite-dimensional quantum system $A_1...A_n$ and may take the value $+\infty$ on this set.

A desirable and physically motivated property of any entanglement measure $E$ in the infinite-dimensional case is a lower semicontinuity, which means that

$$\liminf_{k \to +\infty} E(\rho_k) \geq E(\rho_0)$$

for any sequence $\{\rho_k\}$ converging to a state $\rho_0$. This property corresponds to the natural assumption that the entanglement can not jump up in passing to the limit.

The lower semicontinuity of the entanglement of formation and of the squashed entanglement is proved on the set of states of a bipartite quantum system having at least one finite marginal entropy [19, 20]. In [23] it is shown that the $n$-partite squashed entanglement is lower semicontinuous on the set of states with finite marginal entropies.

Despite the fact that the quantum relative entropy is lower semicontinuous, we have not managed to prove global lower semicontinuity of the relative entropy of entanglement because of the non-compactness of the set of separable states. Nevertheless, by using Theorem 1 in Section 3 and the notion of universal extension of entanglement monotones considered below we will obtain in Section 4.3 a non-restrictive sufficient condition for local lower semicontinuity of the relative entropy of entanglement and its regularization.

### 4.2 Definition of $E_R$ via the universal extension

Assume that $E$ is a continuous entanglement monotone on the set of states of a $n$-partite quantum system $A_1...A_n$ composed of finite-dimensional subsystems, i.e. a continuous function on the set $\mathcal{S}(\mathcal{H}_A_1...A_n)$ with the following properties (cf. [28, 15]):

**EM1)** $\{E(\rho) = 0 \} \Leftrightarrow \{ \rho \text{ is a separable state} \}$;

**EM2)** monotonicity under selective unilocal operations:

$$E(\rho) \geq \sum_i p_i E(\rho_i), \quad p_i = \text{Tr} \Phi_i(\rho), \quad \rho_i = p_i^{-1} \Phi_i(\rho)$$

for any state $\rho \in \mathcal{S}(\mathcal{H}_{A_1...A_n})$ and any collection $\{\Phi_i\}$ of unilocal completely positive linear maps such that $\sum_i \Phi_i$ is a channel.

**EM3)** convexity:

$$E(p\rho_1 + (1-p)\rho_2) \leq pE(\rho_1) + (1-p)E(\rho_2)$$

for arbitrary states $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}_{A_1...A_n})$ and any $p \in (0,1)$.

If $A_1...A_n$ is a $n$-partite quantum system composed of infinite-dimensional subsystems then the function $E$ is well defined on the set of states $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ with finite rank marginal states $\rho_{A_s}, s = 1,...,n$. Thus, one can define the function

$$\hat{E}(\rho) \doteq \sup_{P_1,...,P_n} E(P_1 \otimes ... \otimes P_n \cdot \rho \cdot P_1 \otimes ... \otimes P_n)$$

(31)
on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$, where the supremum is over all finite-rank projectors $P_1 \in \mathcal{B}(\mathcal{H}_{A_1}),...,P_n \in \mathcal{B}(\mathcal{H}_{A_n})$ and it is assumed that $E(\rho) = cE(\rho/c)$ for any nonzero positive operator $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ such that $\text{rank}\rho_{A_s} < +\infty$, $s = 1,n$, where $c = \text{Tr}\rho$.

The above function $\widehat{E}$ is used in [20] in the bipartite case, where it is shown that $\widehat{E}$ is a lower semicontinuous entanglement monotone on $\mathcal{S}(\mathcal{H}_{A_1,A_2})$ inheriting many important properties of $E$ (the (sub)additivity for product states, the monogamy relation, etc). The results of applying this construction called the universal extension of $E$ to the bipartite squashed entanglement and the entanglement of formation are presented in [20].

By obvious generalization of the proof of Proposition 5 in [20] to the $n$-partite case one can prove the following

**Proposition 1.** Let $E$ be a continuous entanglement monotone on the set of states of $n$-partite quantum system composed of finite-dimensional subsystems. Let $A_1...A_n$ be an infinite-dimensional $n$-partite quantum system and $\mathcal{S}_l(\mathcal{H}_{A_1...A_n})$ the set of states $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ with finite rank marginal states $\rho_{A_s}$, $s = 1,n$.

A) $\widehat{E}$ is a unique lower semicontinuous entanglement monotone on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ such that $\widehat{E}(\rho) = E(\rho)$ for any state $\rho$ in $\mathcal{S}_l(\mathcal{H}_{A_1...A_n})$.

B) If $\{P_{A_s}^k\}_k \subset \mathcal{B}(\mathcal{H}_{A_s})$, $s = 1,n$, are arbitrary sequences of finite rank projectors strongly converging to the unit operators $I_{A_s}$ then

$$\widehat{E}(\rho) = \lim_{k \to +\infty} E(P_{A_1}^k \otimes ... \otimes P_{A_n}^k \cdot \rho \cdot P_{A_1}^k \otimes ... \otimes P_{A_n}^k) \quad (32)$$

for any state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$.

C) $\widehat{E}$ is the convex closure of $E$ - the maximal lower semicontinuous convex function on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ not exceeding the function $E$ on the set $\mathcal{S}_l(\mathcal{H}_{A_1...A_n})$.

The main advantage of the entanglement monotone $\widehat{E}$ is its global lower semicontinuity. It is this property that is used in [20] to show that $\widehat{E}_{sq}(\rho) = 0$ for any separable state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1,A_2})$, while the same property for the function $E_{sq}$ obtained by direct generalization of the finite-dimensional definition is hard to prove (until it’s proven that $\widehat{E}_{sq} = E_{sq}$) because of the existence of countably-non-decomposable separable states in infinite-dimensional bipartite systems (Remark 10 in [20]).

Since the relative entropy of entanglement $E_R$ is a continuous entanglement monotone on the set of states of a $n$-partite quantum system composed of finite-dimensional subsystems, Proposition 1 shows that

- the universal extension $\widehat{E}_R$ is a lower semicontinuous entanglement monotone on the set of states of an infinite-dimensional $n$-partite quantum system;

- $\widehat{E}_R(\rho) = E_R(\rho)$ for any state $\rho$ with finite rank marginal states;

- $\widehat{E}_R$ is the convex closure of $E_R$ (considered as a function on the set $\mathcal{S}_l(\mathcal{H}_{A_1...A_n}))$; it is explicitly determined for any state $\rho$ by the expression (32) with $E = E_R$ via arbitrary sequences $\{P_{A_s}^k\}_k \subset \mathcal{B}(\mathcal{H}_{A_s})$, $s = 1,n$ of finite rank projectors strongly converging to the unit operators $I_{A_s}$.  

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The regularization of $\tilde{E}_R$ is defined in the standard way:

$$\tilde{E}_R^\infty(\rho) = \lim_{m \to +\infty} m^{-1} \tilde{E}_R(\rho^m),$$  \hspace{5pt} (33)

where $\rho^m$ is treated as a state of the $n$-partite quantum system $A_1^m ... A_n^m$.

For an arbitrary state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ the monotonicity of $E_R$ under selective unilocal operations (which follows from Lemma 1 at the end of Section 4.3) implies that

$$\tilde{E}_R(\rho) \leq E_R(\rho)$$

and

$$\tilde{E}_R^\infty(\rho) \leq E_R^\infty(\rho)$$

($E_R$ and $E_R^\infty$ are the relative entropy of entanglement obtained by direct generalization of the finite-dimensional definition and its regularization). In the next subsection, we will obtain a simple sufficient condition for equality in these inequalities.

### 4.3 Analytical properties of $E_R$ and $E_R^\infty$

The following theorem contains some results about analytical properties of the relative entropy of entanglement $E_R$ directly defined by formula (26) for any state of a $n$-partite infinite-dimensional quantum system and its regularization $E_R^\infty$ defined by formula (30).

It also clarifies relations between the functions $E_R$ and $E_R^\infty$, the universal extension $\tilde{E}_R$ defined by formula (31) with $E = E_R$ and its regularization $\tilde{E}_R^\infty$ defined in (33).

**Theorem 2.** Let

$$\mathcal{S}_s(\mathcal{H}_{A_1...A_n}) = \{ \rho \in \mathcal{S}(\mathcal{H}_{A_1...A_n}) \mid \rho_{A_s} \in \mathcal{S}_{FA}(\mathcal{H}_{A_s}) \text{ for } n-1 \text{ indexes } s \}$$  \hspace{5pt} (34)

be the subset of $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ consisting of states $\rho$ such that the FA-property holds for at least $n-1$ marginal states of $\rho$.

A) The functions $E_R$ and $E_R^\infty$ are finite and lower semicontinuous on the set $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$. Moreover,

$$\liminf_{k \to +\infty} E_R^s(\rho_k) \geq E_R^s(\rho_0), \quad E_R^* = E_R, E_R^\infty,$$  \hspace{5pt} (35)

for arbitrary sequence $\{ \rho_k \} \subset \mathcal{S}(\mathcal{H}_{A_1...A_n})$ converging to a state $\rho_0 \in \mathcal{S}_s(\mathcal{H}_{A_1...A_n})$.

B) The functions $E_R$ and $E_R^\infty$ coincide on the set $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$ with the functions $\tilde{E}_R$ and $\tilde{E}_R^\infty$ correspondingly. $\tilde{E}_R$ is the convex closure of $E_R$ (considered as a function on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$).

C) For any state $\rho \in \mathcal{S}_s(\mathcal{H}_{A_1...A_n})$ the infimum in definition (26) of $E_R(\rho)$ can be taken only over all finitely-decomposable separable states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$, i.e. states having the form

$$\sigma = \sum_{i=1}^{m} p_i \alpha^i_1 \otimes ... \otimes \alpha^i_n, \quad \alpha^i_s \in \mathcal{S}(\mathcal{H}_{A_s}), \quad p_i > 0, \quad \sum_{i=1}^{m} p_i = 1, \quad m < +\infty.$$  \hspace{5pt} (36)
D) If the Hamiltonians $H_{A_1}, \ldots, H_{A_{n-1}}$ of subsystems $A_1, \ldots, A_{n-1}$ satisfy condition (12) then

- the functions $E_R$ and $E_R^\infty$ are uniformly continuous on the set of states $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ such that $\sum_{s=1}^{n-1} \text{Tr} H_{A_s} \rho_{A_s} \leq E$ for any $E > E_0^{A_1} + \ldots + E_0^{A_{n-1}}$;
- the function $E_R$ is uniformly continuous on the set

$$\left\{ \Lambda(\rho) \mid \rho \in \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}), \sum_{s=1}^{n-1} \text{Tr} H_{A_s} \rho_{A_s} \leq E \right\} \forall E > E_0^{A_1} + \ldots + E_0^{A_{n-1}},$$

where $\Lambda$ is any positive trace preserving linear transformation of $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ such that $\Lambda(\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})) \subseteq \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$;

- the functions $E_R$ and $E_R^\infty$ are asymptotically continuous in the following sense (cf. [4]): if $\{\rho_k\}$ and $\{\sigma_k\}$ are any sequences such that $\rho_k, \sigma_k \in \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$, $\text{Tr} H_{B_k} [\rho_k]_{B_k}, \text{Tr} H_{B_k} [\sigma_k]_{B_k} \leq k E$, $\forall k$, and $\lim_{k \to \infty} \|\rho_k - \sigma_k\|_1 = 0$, where $X^k$ denotes $k$ copies of a system $X$, $B = A_1 \ldots A_{n-1}$ and $H_{B_k}$ is the Hamiltonian of $B_k$, then

$$\lim_{k \to +\infty} \frac{|E_R^*(\rho_k) - E_R^*(\sigma_k)|}{k} = 0, \quad E_R^* = E_R, E_R^\infty.$$

E) The function $E_R$ is $\sigma$-convex on $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$, i.e.

$$E_R(\rho) \leq \sum_{i=1}^{+\infty} p_i E_R(\rho_i), \quad \rho = \sum_{i=1}^{+\infty} p_i \rho_i,$$

for any countable collection $\{\rho_i\} \subseteq \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ and any probability distribution $\{p_i\}$.

The function $E_R$ is $\mu$-integrable w.r.t. any Borel probability measure $\mu$ on $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ with the barycenter $\bar{\rho}(\mu) = \int \rho \mu(d\rho)$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ and

$$E_R(\bar{\rho}(\mu)) \leq \int E_R(\rho) \mu(d\rho). \quad (37)$$

Remark 1. Theorem 2C implies, in particular, that we may define $E_R(\rho)$ for any state $\rho \in \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ by ignoring the existence of countably-non-decomposable

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$^{10}$ $E_0^{A_i}$ is the minimal eigenvalue of $H_{A_i}$.

$^{11}$ A convex nonnegative function on the set of quantum states may be not $\sigma$-convex [19, Ex.1].

$^{12}$ It means that $E_R$ is measurable w.r.t. the smallest $\sigma$-algebra on $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ that contains all Borel sets and all the subsets of the zero-$\mu$-measure Borel sets. The integral in the r.h.s. of (37) is over the corresponding completion of the measure $\mu$ [18]. We can not prove that $E_R$ is a Borel function on $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$, since it is defined via the infimum over the non-countable set of separable states.
separable states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$. This assertion is not trivial, since in general there are no reasons to assume that the infimum in definition (26) of $E_R(\rho)$ can be taken over the dense subset of $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$ consisting of countably decomposable separable states.

**Proof.** A) Let $\rho_0$ be a state in $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$. Assume that the FA-property holds for the states $[\rho_0]_{A_1},...,[\rho_0]_{A_{n-1}}$. The finiteness of $E_R(\rho_0)$ and $E_R^\infty(\rho_0)$ follows from inequality (28), since the FA-property of a state implies finiteness of its entropy [25].

For each natural $s$ in $[1,n]$ let $P_r^s$ be the spectral projector of $[\rho_0]_{A_s}$ corresponding to its $r$ maximal eigenvalues (taking the multiplicity into account). It is mentioned in Section 4.1 that the function $E_R$ belongs to the class $L_n^{-1}(1,1)$. Hence Theorem 1 implies that

$$\lim_{r \to +\infty} E_R(\rho_r) = E_R(\rho_0), \quad (38)$$

where $\rho_r = c_r^{-1}Q_r \rho_0 Q_r$, $Q_r = P_{r_1}^1 \otimes ... \otimes P_{r_n}^n$, $c_r = \text{Tr}Q_r \rho_0$. Since $c_r E_R(\rho_r)$ does not exceed $\hat{E}_R(\rho_0)$ and $c_r$ tends to 1 as $r \to +\infty$, the above limit relation implies that

$$\hat{E}_R(\rho_0) = E_R(\rho_0). \quad (39)$$

To prove relation (35) for $E_R^* = E_R$ it suffices to note that $\hat{E}_R$ is a lower semicontinuous function on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ not exceeding the function $E_R$ and to use (39).

To prove relation (35) for $E_R^* = E_R^\infty$ we apply the arguments from the proof of Theorem 1 to construct for each natural $s$ in $[1,n-1]$ a positive operator $G_{A_s}$ on $\mathcal{H}_{A_s}$ satisfying condition (12) such that $\text{Tr}G_{A_s} [\rho_0]_{A_s} < +\infty$. For each $s$ consider the quantum channel

$$\Phi_r^s(\rho) = P_r^s \rho P_r^s + [\text{Tr}(I_{A_s} - P_r^s) \rho] \tau_s$$

from $\mathcal{S}(\mathcal{H}_{A_s})$ to itself, where $P_r^s$ is the spectral projector of $G_{A_s}$ corresponding to its $r$ minimal eigenvalues and $\tau_s$ is a state in $\mathcal{S}(\mathcal{H}_{A_s})$ such that $\text{Tr}G_{A_s} \tau_s = 0$ and $P_r^s \tau_s P_r^s = \tau_s$ for all $r$. Since all the output states of $\Phi_r^s$ are supported by the finite-dimensional subspace $P_r^s(\mathcal{H}_{A_s})$, the function

$$f_r(\rho) = E_R^\infty(\Phi_r^1 \otimes ... \otimes \Phi_r^{n-1} \otimes \text{Id}_{A_n}(\rho))$$

is continuous on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ for each $r$ (this follows from the continuity bound (??) for $E_R^\infty$ obtained independently in Section 4.4 by generalizing the Winter arguments from [33]). Hence, $f_s = \sup_r f_r$ is a lower semicontinuous function on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$.

Since $\Phi_r^1 \otimes ... \otimes \Phi_r^{n-1} \otimes \text{Id}_{A_n}(\rho_0)$ tends to $\rho_0$ as $r \to +\infty$ and $\text{Tr}G_{A_s} \Phi_r^s([\rho_0]_{A_s}) \leq \text{Tr}G_{A_s} [\rho_0]_{A_s} < +\infty$ for all $r$ and $s = 1,n-1$, part D of the theorem implies that

$$\lim_{r \to +\infty} f_r(\rho_0) = E_R^\infty(\rho_0) \quad \text{and hence} \quad f_s(\rho_0) = E_R^\infty(\rho_0),$$

as $f_s$ does not exceed the function $E_R^\infty$ by the monotonicity of $E_R^\infty$ under local channels. This implies relation (35) for $E_R^* = E_R^\infty$, since $f_s$ is a lower semicontinuous function.

B) The properties of the function $\hat{E}_R$ stated in Section 4.2 show that $\hat{E}_R$ is the greatest lower semicontinuous convex function on the set $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ not exceeding the function $E_R$.  

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The arguments from the proof of part A of the theorem show that the functions $E_R$ and $\tilde{E}_R$ coincide on the set $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$. The same arguments show that $\tilde{E}_R(\rho^{\otimes m}) = E_R(\rho^{\otimes m})$ for any state $\rho$ in $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$ and arbitrary natural $m$, since for any such state the state $\rho^{\otimes m}$ belongs to the set $\mathcal{G}_s(\mathcal{H}_{A_1^{m} \ldots A_n^{m}})$ by Proposition 1B in [25]. This implies coincidence of the functions $E_R^\infty$ and $\tilde{E}_R^\infty$ on the set $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$.

C) Consider the function

$$\tilde{E}_R(\rho) = \inf_{\sigma \in \mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})} H(\rho \parallel \sigma),$$

where $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$ is the set of finitely-decomposable separable states in $\mathcal{G}(\mathcal{H}_{A_1 \ldots A_n})$. Since the set $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$ is convex, the joint convexity of the relative entropy and Lemma 6 in [21] imply that the function $\tilde{E}_R$ satisfies inequality (14) with $a_f = 1$ and $b_f = 0$. Since

$$\tilde{E}_R(\rho) \leq H(\rho \parallel \rho_{A_1} \otimes \ldots \otimes \rho_{A_n}) = I(A_1 : \ldots : A_n)_\rho \leq 2 \sum_{s=1}^{n-1} H(\rho_{A_s}),$$

where the second inequality follows from (7), the function $\tilde{E}_R$ satisfies inequality (15) with $m = n - 1$, $c^- = 0$ and $c^+ = 2$. It follows that the function $\tilde{E}_R$ belongs to the class $L_n^{-1}(2,1)$. Hence for any state $\rho_0$ in $\mathcal{G}_s(\mathcal{H}_{A_1 \ldots A_n})$ Theorem 1 implies that

$$\lim_{r \to +\infty} \tilde{E}_R(\rho_r) = \tilde{E}_R(\rho_0),$$

where the notation from the proof of part A of this theorem is used. Since $\tilde{E}_R(\rho_r) = E_R(\rho_r)$ for any $r$, this limit relation and (38) show that $\tilde{E}_R(\rho_0) = E_R(\rho_0)$.

D) The assertions about uniform continuity of the functions $E_R$ and $E_R^\infty$ follow directly from Propositions ?? and ?? in Section 4.4 below (proved independently).

To prove of the asymptotic continuity of the functions $E_R$ and $E_R^\infty$ note that $F_{H_{B_k}}(E) = kF_{H_B}(E/k)$ and $E_0^{B_k} = kE_0^B$ for each $k$ and hence $\tilde{F}_{H_{B_k}}(E) = k\tilde{F}_{H_B}(E/k)$, where the notation from Section 4.4 is used. So, Proposition ?? in Section 4.4 with $m = n - 1$ implies that

$$\frac{|E_R(\rho_k) - E_R^*(\sigma_k)|}{k} \leq \sqrt{2\varepsilon_k \tilde{F}_{H_B}(E/\varepsilon_k)} + (1/k)g(\sqrt{2\varepsilon_k}), \quad E_R^* = E_R^\infty, \quad E_R, \quad (40)$$

where $\varepsilon_k = \frac{1}{2}||\rho_k - \sigma_k||_1$ and $E = E - E_0^B$. Since the sequence $\{\varepsilon_k\}$ is vanishing by the condition and $\tilde{F}_{H_B}(E)$ is $o(\sqrt{E})$ as $E \to +\infty$ by Lemma 1 in [24], the r.h.s. of (40) tends to zero as $k \to +\infty$.

E) For arbitrary $\varepsilon > 0$ and each $i$ let $\sigma_i$ be a separable state such that $E_R(\rho_i) \geq H(\rho_i \parallel \sigma_i) - \varepsilon$. Let $\rho_k = c_k^{-1} \sum_{i=1}^{k} p_i \rho_i$ and $\sigma_k = c_k^{-1} \sum_{i=1}^{k} p_i \sigma_i$, where $c_k = \sum_{i=1}^{k} p_i$. By the joint convexity of the relative entropy we have

$$H(\rho_k \parallel \sigma_k) \leq c_k^{-1} \sum_{i=1}^{k} p_i H(\rho_i \parallel \sigma_i).$$
By using the lower semicontinuity of the relative entropy we obtain
\[ E_R(\rho) \leq H(\rho \| \sigma) \leq \sum_{i=1}^{+\infty} p_i H(\rho_i \| \sigma_i) \leq \sum_{i=1}^{+\infty} p_i E_R(\rho_i) + \varepsilon, \]
where \( \sigma = \sum_{i=1}^{+\infty} p_i \sigma_i \) and the first inequality follows from the separability of \( \sigma \).

Let \( \mu \) be a Borel probability measure on \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \) with the barycenter \( \rho = \bar{\rho}(\mu) \) in \( \mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n}) \). Assume that the FA-property holds for the states \( \rho_{A_1\ldots A_{n-1}} \). Following the proof of Theorem 1 for each natural \( s \) in \([1, n-1]\) we construct a positive operator \( G_{A_s} \) on \( \mathcal{H}_{A_s} \) satisfying condition \((12)\) such that \( \text{Tr} G_{A_s} \rho_{A_s} < +\infty \).

By part D of the theorem the function \( E_R \) is continuous on the closed subset \( \mathcal{C}_E \) of \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \) consisting of states \( \rho \) such that \( \sum_{s=1}^{n-1} \text{Tr} G_{A_s} \rho_{A_s} \leq E \) for any \( E > 0 \). It follows that the function \( f_E \) coinciding with \( E_R \) on the set \( \mathcal{C}_E \) and equal to \(+\infty\) on the set \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \setminus \mathcal{C}_E \) is a Borel function on \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \) for each \( E > 0 \).

Since
\[ \int \left[ \sum_{s=1}^{n-1} \text{Tr} G_{A_s} \varrho_{A_s} \right] \mu(d\varrho) = \sum_{s=1}^{n-1} \text{Tr} G_{A_s} \varrho_{A_s} < +\infty, \]
we have \( \mu(\mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \setminus \mathcal{C}_s) = 0 \), where \( \mathcal{C}_s = \bigcup_{E>0} \mathcal{C}_E \). Hence the function \( E_R \) is \( \mu \)-integrable as it coincides with the Borel function \( f_* = \inf_{E>0} f_E \) on the set \( \mathcal{C}_s \).

The function \( \hat{E}_R \) defined in \((31)\) with \( E = E_R \) is a convex and lower semicontinuous function on \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \) not exceeding the function \( E_R \). Hence, by using the validity of Jensen’s inequality for \( \hat{E}_R \) (cf. [19, the Appendix]) we obtain
\[ E_R(\bar{\rho}(\mu)) = \hat{E}_R(\bar{\rho}(\mu)) \leq \int \hat{E}_R(\rho) \mu(d\rho) \leq \int E_R(\rho) \mu(d\rho), \]
where the equality follows from part B of the theorem. \( \square \)

In the following corollary we present several sufficient conditions for convergence of the relative entropy of entanglement and its regularization used below.

**Corollary 1.** Let \( \{\rho_k\} \) be a sequence of states in \( \mathcal{S}(\mathcal{H}_{A_1\ldots A_n}) \) converging to a state \( \rho_0 \) from the set \( \mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n}) \) defined in \((34)\). Then the relation
\[ \lim_{k \to +\infty} E_R(\rho_k) = E_R(\rho_0) < +\infty \]
holds provided that one of the following conditions is valid:

a) all the states \( \rho_k \) are obtained from the state \( \rho_0 \) by LOCC;

b) \( \rho_k = \Phi_k(\rho_0)/\text{Tr} \Phi_k(\rho_0) \), where \( \Phi_k \) is a positive trace-non-increasing linear transformation of \( \mathcal{I}(\mathcal{H}_{A_1\ldots A_n}) \) such that \( \Phi_k(\mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n})) \subseteq \text{conv}\{0, \mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n})\} \)
\[ \Phi_k(\mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n})) \subseteq \text{conv}\{0, \mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n})\} \]
for each \( k \) and \( \text{Tr} \Phi_k(\rho_0) \) tends to \( 1 \) as \( k \to +\infty \);

\( \Phi_k(\mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n})) \) is the set of all operators of the form \( c\rho \), where \( \rho \in \mathcal{S}_s(\mathcal{H}_{A_1\ldots A_n}) \) and \( c \leq 1 \). Condition \((42)\) does not imply that \( \Phi_k \) is a separable transformation (Definition 3.38 in [31]), see Example 2 below.
c) $c_k \rho_k \leq \sigma_k$ for all $k$, where $\{c_k\}$ is a sequence of positive numbers tending to 1 and $\{\sigma_k\}$ is a sequence of states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ converging to the state $\rho_0$ such that

$$\lim_{k \to +\infty} E_R(\sigma_k) = E_R(\rho_0).$$

The above condition a) implies that

$$\lim_{k \to +\infty} E_R^\infty(\rho_k) = E_R^\infty(\rho_0) < +\infty. \quad (43)$$

The above condition b) also implies relation (43) provided that $\Phi_k$ is a completely positive trace-non-increasing linear transformation of $\mathcal{E}(\mathcal{H}_{A_1...A_n})$ such that the map $\Phi_k^{\otimes m}$ satisfies the $m$-partite version of condition (42) for each natural $m$ and all $k$.\(^{14}\)

Proof. a) The LOCC monotonicity of $E_R$ and $E_R^\infty$ implies that $E_R^*(\rho_k) \leq E_R^*(\rho_0)$, $E_R^\infty = E_R, E_R^\infty$, for all $k$. So, relations (41) and (43) follow from Theorem 2A.

b) Lemma 1 below implies that $\text{Tr}\Phi_k(\rho_0)E_R(\rho_k) \leq E_R(\rho_0)$ for all $k$. So, in this case (41) also follows from Theorem 2A.

If all the maps $\Phi_k$ are completely positive and the maps $\Phi_k^{\otimes m}$ satisfy the $m$-partite version of condition (42) for each natural $m$ then Lemma 1 below implies that $\text{Tr}\Phi_k^{\otimes m}(\rho_0^{\otimes m})E_R(\rho_k^{\otimes m}) \leq E_R(\rho_0^{\otimes m})$ for all $k$. Proposition 1B in [25] implies that the state $\rho_0^{\otimes m}$ belongs to the set $\mathcal{S}_s(\mathcal{H}_{A_1^m...A_n^m})$. So, it follows from the above inequality and Theorem 2A that

$$\lim_{k \to +\infty} E_R(\rho_k^{\otimes m}) = E_R(\rho_0^{\otimes m}) \quad \forall m.$$ 

Since $E_R^\infty(\rho_k) = \inf_m m^{-1}E_R(\rho_k^{\otimes m})$, $k = 0, 1, 2,...$, the validity of the last relation for all $m$ shows that

$$\lim_{k \to +\infty} \sup E_R^\infty(\rho_k) \leq E_R^\infty(\rho_0).$$

This limit relation and Theorem 2A imply (43).

c) Since the function $E_R$ satisfies inequality (14) with $a_f = 1$ and $b_f = 0$, we have

$$E_R(\sigma_k) \geq c_k E_R(\rho_k) + (1-c_k)E_R((\sigma_k - c_k \rho_k)/(1-c_k)) - h_2(c_k) \geq c_k E_R(\rho_k) - h_2(c_k) \quad \forall k.$$ 

It follows that

$$\lim_{k \to +\infty} \sup E_R(\rho_k) \leq \lim_{k \to +\infty} E_R(\sigma_k) = E_R(\rho_0).$$

This relation and Theorem 2A imply (41). $\square$

Consider simple applications of continuity conditions in Corollary 1.

**Example 2.** Let $\rho_0$ be a state from the set $\mathcal{S}_s(\mathcal{H}_{A_1...A_n})$ and $\{\sigma_k\}$ an arbitrary sequence of separable states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$. Then

$$\lim_{k \to +\infty} E_R^*(((1-p_k)\rho_0 + p_k \sigma_k)) = E_R^*(\rho_0), \quad E_R^* = E_R, E_R^\infty, \quad (44)$$

\(^{14}\)It means that $\Phi_k^{\otimes m}(\mathcal{S}_s(\mathcal{H}_{A_1^m...A_n^m})) \subseteq \text{conv}\{0, \mathcal{S}_s(\mathcal{H}_{A_1^m...A_n^m})\}$, where $A_s^m$ denotes the composition of $m$ copies of the system $A_s^m$. 

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for any vanishing sequence \( \{p_k\} \in [0, 1] \). Indeed, let \( \Phi_k(\rho) = (1 - p_k)\rho + p_k\sigma_k \) be a channel from \( \mathcal{T}(\mathcal{H}_1, \ldots, \mathcal{H}_n) \) to itself. It is easy to see that the channel \( \Phi_k^{\otimes m} \) satisfies the \( m \)-partite version of condition (42) for each natural \( m \) and all \( k \). So, both limit relations in (44) follow from condition b) in Corollary 1.

**Example 3.** Let \( \rho_0 \) be a state from the set \( \mathcal{S}_+(\mathcal{H}_1, \ldots, \mathcal{H}_n) \) and \( \{\rho_k\} \) a sequence of states converging to the state \( \rho_0 \) such that \( c_k\rho_k \leq \rho_0 \) for all \( k \), where \( \{c_k\} \) is a sequence of positive numbers tending to 1. Then

\[
\lim_{k \to +\infty} E^*_R(\rho_k) = E^*_R(\rho_0), \quad E^*_R = E_R, E^*_R.
\]

Indeed, relation (45) with \( E_R^* = E_R \) directly follows from condition c) in Corollary 1. Since \( c_k^m \rho_k^{\otimes m} \leq \rho_0^{\otimes m} \) for each natural \( m \) and the state \( \rho_0^{\otimes m} \) belongs to the set \( \mathcal{S}_+(\mathcal{H}_1, \ldots, \mathcal{H}_n) \) by Proposition 1B in [25], this condition also implies that

\[
\lim_{k \to +\infty} E_R(\rho_k^{\otimes m}) = E_R(\rho_0^{\otimes m}) \quad \forall m.
\]

Since \( E_R^\infty(\rho_k) = \inf_m m^{-1} E_R(\rho_k^{\otimes m}), k = 0, 1, 2, \ldots \), the validity of the last relation for all \( m \) shows that

\[
\limsup_{k \to +\infty} E_R^\infty(\rho_k) \leq E_R^\infty(\rho_0).
\]

This limit relation and Theorem 2A imply (45) with \( E_R^* = E_R^\infty \).

Continuity condition c) in Corollary 1 will be essentially used in the proof of Proposition 2 in Section 4.

**Lemma 1.** Let \( \{\Phi_i\} \) be a collection of positive linear maps from \( \mathcal{T}(\mathcal{H}) \) to itself such that the map \( \sum_i \Phi_i \) is trace preserving. Let \( \rho \) and \( \sigma \) be any states in \( \mathcal{S}(\mathcal{H}) \). Then

\[
\sum_i p_i H(\rho_i \| \sigma_i) \leq H(\rho \| \sigma),
\]

where \( p_i = \text{Tr}\Phi_i(\rho), \rho_i = p_i^{-1}\Phi_i(\rho), q_i = \text{Tr}\Phi_i(\sigma) \) and \( \sigma_i = q_i^{-1}\Phi_i(\sigma) \).

**Proof.** Consider the trace preserving positive linear map

\[
\mathcal{T}(\mathcal{H}) \ni \rho \mapsto \tilde{\Phi}(\rho) = \sum_i \Phi_i(\rho) \otimes |i\rangle\langle i| \in \mathcal{T}(\mathcal{H} \otimes \mathcal{H}_R),
\]

where \( \{|i\rangle\} \) is a basis in appropriate Hilbert space \( \mathcal{H}_R \). By using the monotonicity of the relative entropy under action of the map \( \tilde{\Phi} \) (proved in [12]) and the properties of the Lindblad extension (4) of the relative entropy (presented in [10]) we obtain

\[
H(\rho \| \sigma) \geq H(\tilde{\Phi}(\rho) \| \tilde{\Phi}(\sigma)) = \sum_i H(p_i \rho_i \| q_i \sigma_i) = \sum_i p_i H(\rho_i \| q_i \sigma_i / p_i)
\]

\[
= \sum_i p_i H(\rho_i \| \sigma_i) + D(\{p_i\} \| \{q_i\}) \geq \sum_i p_i H(\rho_i \| \sigma_i),
\]

where the last inequality follows from the nonnegativity of the Kullback–Leibler divergence \( D(\{p_i\} \| \{q_i\}) \) between the probability distributions \( \{p_i\} \) and \( \{q_i\} \). □
4.4 Finite-dimensional approximation of $E_R$ and $E_R^\infty$

Many results describing properties of the relative entropy of entanglement and its regularization in finite-dimensional multipartite quantum systems remain valid in the infinite-dimensional case under some additional conditions. A simple way to establish validity of these results is given by the following

**Proposition 2.** Let $\rho$ be a state from the set $\mathcal{S}_*([H_{A_1}^{\otimes n}])$ defined in (34).\footnote{$\mathcal{S}_*([H_{A_1}^{\otimes n}])$ is the subset of $\mathcal{S}([H_{A_1}^{\otimes n}])$ consisting of states $\rho$ such that the FA-property holds for at least $n - 1$ marginal states of $\rho$.} Let $\rho_k = Q_k \rho Q_k [\text{Tr} Q_k \rho]^{-1}, Q_k = P_k^1 \otimes \cdots \otimes P_k^n$, where $\{P_k^1\} \subset \mathcal{B}([H_{A_1}]), \ldots, \{P_k^n\} \subset \mathcal{B}([H_{A_n}])$ are arbitrary sequences of projectors strongly converging to the unit operators $I_{A_1}, \ldots, I_{A_n}$ correspondingly. Then

$$E^*_R(\rho_{A_1^{\otimes n}}) = \lim_{k \to +\infty} E^*_R([\rho_k]_{A_1^{\otimes n}}), \quad E^*_R = E_R, E_R^\infty,$$

(46)

for any $m = 2, 3, \ldots, n$, where $E_R$ and $E_R^\infty$ denote, respectively, the relative entropy of entanglement and its regularization of a state of the system $A_1^{\otimes n}$.

**Proof.** For each $k$ let

$$\sigma_k = c_{k,m}^{-1} Q_k, m \rho_{A_1^{\otimes n}} Q_k, m, \quad c_{k,m} = \text{Tr} Q_k, m \rho_{A_1^{\otimes n}}, \quad Q_k, m = P_k^1 \otimes \cdots \otimes P_k^n,$$

be a state of the system $A_1^{\otimes n}$. Then for any natural $u$ we have

$$\sigma_u \otimes u = c_{k,m}^{-u} [Q_k, m]^{-u} \rho_{A_1^{\otimes n}} [Q_k, m]^{-u}.$$

By the monotonicity of $E_R$ under selective unilocal operations we have

$$c_{k,m}^u E_R(\sigma_k^{\otimes u}) \leq E_R(\rho_{A_1^{\otimes n}}).$$

Since the state $\rho_{A_1^{\otimes n}}$ belongs to the set $\mathcal{S}_*([H_{A_1}^{\otimes n}])$, Proposition 1B in [25] implies that the state $\rho_{A_1^{\otimes n}}^{\otimes u}$ belongs to the set $\mathcal{S}_*([H_{A_1}^{\otimes n}])$. So, the above inequality and Theorem 2A show that

$$\lim_{k \to +\infty} E_R(\sigma_k^{\otimes u}) = E_R(\rho_{A_1^{\otimes n}}^{\otimes u}) < +\infty, \quad u = 1, 2, \ldots$$

Since $c_{k,m}^u \rho_{A_1^{\otimes n}}^{\otimes u} \leq c_{k,m}^{\otimes u}$ for each natural $u$, where $c_k = \text{Tr} Q_k, m \rho_{A_1^{\otimes n}}$ is a number tending to 1 as $k \to +\infty$, the last limit relation implies, by Corollary 1 with condition c), that

$$\lim_{k \to +\infty} E_R([\rho_k]_{A_1^{\otimes n}}^{\otimes u}) = E_R(\rho_{A_1^{\otimes n}}^{\otimes u}), \quad u = 1, 2, \ldots$$

This relation with $u = 1$ means (46) for $E^*_R = E_R$, while its validity for all $u$ shows that

$$\lim_{k \to +\infty} \sup E_R^\infty([\rho_k]_{A_1^{\otimes n}}) \leq E_R^\infty(\rho_{A_1^{\otimes n}}).$$
Since the state \( \rho_{A_1...A_m} \) belongs to the set \( \mathcal{S}_s(\mathcal{H}_{A_1...A_m}) \), this relation and Theorem 2A imply (46) for \( E_R^* = E_R^\infty \). □

**Remark 2.** If \( \rho \) is a state in \( \mathcal{S}_s(\mathcal{H}_{A_1...A_n}) \) and \( \{\rho_k\} \) is the sequence defined in Proposition 2 by means of the sequences \( \{P_r^i\}, \{P_r^n\} \) consisting of the spectral projectors of the states \( \rho_{A_1...A_n} \) corresponding to their \( r \) maximal eigenvalues then by using Proposition ?? in Section 4.4 one can show that

\[
E_R(\Lambda([\rho_k]_{A_1...A_m})) \Rightarrow \frac{E_R(\Lambda(\rho_{A_1...A_m}))}{A} \quad \text{as} \quad k \to +\infty,
\]

where \( \Rightarrow \) denotes the uniform convergence on the set of all positive trace-non-increasing linear transformations of \( \mathcal{S}(\mathcal{H}_{A_1...A_n}) \) satisfying condition (??).

Below we consider examples of using Proposition 2.

**Example 4.** Lemma 5 in [14] and the additivity of the entropy imply that

\[
E_R^\infty(\rho) \geq -H(A_i|A_j)_\rho, \quad (i, j) = (1, 2), (2, 1),
\]

for any state \( \rho \) of a bipartite finite-dimensional quantum system \( A_1A_2 \).

Proposition 2 allows to show that the inequalities in (47) remain valid for any state \( \rho \) of a bipartite infinite-dimensional system provided that the state \( \rho_A \) has the FA-property\(^{16} \) and \( H(\cdot|\cdot) \) is the extended conditional entropy defined in (5). Indeed, it suffices to show that

\[
H(A_1|A_2)_\rho = \lim_{k \to +\infty} H(A_1|A_2)_{\rho_k}, \quad \rho_k = Q_k \rho Q_k [\text{Tr} Q_k \rho]^{-1}, \quad Q_k = P_k^1 \otimes P_k^2,
\]

where \( P_k^s \) is the spectral projector of \( \rho_A \) corresponding to its \( r \) maximal eigenvalues, \( s = 1, 2 \). This can be done by using Theorem 1, since the function \( \rho \mapsto H(A_1|A_2)_\rho \) belongs to the class \( L_1^2(2, 1) \) (see Section 3).

Note that direct proof of (47) in the infinite-dimensional case requires technical efforts (especially, if \( H(\rho) = H(\rho_{A_j}) = +\infty \)).

**Example 5.** It is shown in [16] that

\[
E_R^*(\rho) \geq E_R^*(\rho_{A_iA_j}) + H(\rho_{A_iA_j}), \quad E_R^* = E_R, E_R^\infty, \quad (i, j) = (1, 2), (2, 3), (3, 1),
\]

for any pure state \( \rho \) in a tripartite finite-dimensional quantum system \( A_1A_2A_3 \).

Proposition 2 allows to show that all the inequalities in (48) remain valid for any pure state \( \rho \) of a tripartite infinite-dimensional system provided that any two of the states \( \rho_{A_1}, \rho_{A_2} \) and \( \rho_{A_3} \) have the FA-property.\(^{17} \) Indeed, it suffices to show that

\[
H(\rho_{A_1A_2}) = \lim_{k \to +\infty} H([\rho_k]_{A_1A_2}), \quad \rho_k = Q_k \rho Q_k [\text{Tr} Q_k \rho]^{-1}, \quad Q_k = P_k^1 \otimes P_k^2 \otimes P_k^3,
\]

\(^{16}\)This assumption implies, by Theorem 1 in [25], the finiteness of the extended conditional entropy \( H(A_i|A_j)_\rho \) defined in (5).

\(^{17}\)This assumption and the purity of the state \( \rho \) imply, by Theorem 1 in [25], the finiteness of all the entropies \( H(\rho_{A_iA_j}) \) involved in (48).
where \( P^k_s \) is the spectral projector of \( \rho_A \), corresponding to its \( r \) maximal eigenvalues, \( s = 1, 2, 3 \). This can be done by noting that \( H(\rho_{A_1A_2}) = H(\rho_A) \) and \( H([\rho_k]_{A_1A_2}) = H([\rho_k]_{A_2}) \) as the states \( \rho \) and \( \rho_k \) are pure. Since \( c_k[\rho_k]_{A_2} \leq \rho_{A_2} \) for each \( k \), where \( c_k \) is a number tending to 1 as \( k \to +\infty \), the required limit relation can be proved easily by using concavity and lower semicontinuity of the entropy.

### 5 On energy-constrained versions of \( E_R \)

Dealing with infinite-dimensional quantum systems we have to take into account the existence of quantum states with infinite energy which can not be produced in a physical experiment. This motivates an idea to define the relative entropy of entanglement of a state \( \rho \) with finite energy by formula (26) in which the infimum is taken over all separable states with the energy not exceeding some (sufficiently large) bound \( E \) [3]. This gives the following energy-constrained version of the relative entropy of entanglement

\[
E^{H_{A^n}}_R(\rho|E) = \inf_{\sigma \in \mathcal{S}_s(H_{A_1...A_n}), \text{Tr} H_{A^n}\sigma \leq E} H(\rho\|\sigma), \tag{49}
\]

where \( H_{A^n} \) is the Hamiltonian of system \( A^n = A_1...A_n \). This definition looks attractive from mathematical point of view, since the subset of \( \mathcal{S}_s(H_{A_1...A_n}) \) satisfying the inequality \( \text{Tr} H_{A^n}\sigma \leq E \) is compact provided that the Hamiltonian \( H_{A^n} \) has discrete spectrum of finite multiplicity. This implies, by the lower semicontinuity of the relative entropy, that the infimum in (49) is attainable (this advantage was exploited in [3]).

An obvious drawback of definition (49) is its dependence of the bound \( E \). Note also that \( E^{H_{A^n}}_R(\rho|E) \neq 0 \) for any separable state \( \rho \) such that \( \text{Tr} H_{A^n}\rho > E \) and that the monotonicity of \( E^{H_{A^n}}_R(\rho|E) \) under local operations can be shown only under the appropriate restrictions on the energy amplification factors of such operations.

A less restrictive way to take the energy constraints into account is to take the infimum in formula (26) over all separable states with finite energy, i.e. to consider the following quantity

\[
E^{H_{A^n}}_R(\rho) = \inf_{\sigma \in \mathcal{S}_s(H_{A_1...A_n}), \text{Tr} H_{A^n}\sigma < +\infty} H(\rho\|\sigma) = \lim_{E \to +\infty} E^{H_{A^n}}_R(\rho|E), \tag{50}
\]

where the limit can be replaced by the infimum over all \( E > 0 \), since the function \( E \mapsto E^{H_{A^n}}_R(\rho|E) \) is non-increasing. If the Hamiltonian \( H_{A^n} \) of the system \( A_1...A_n \) has the form

\[
H_{A^n} = H_{A_1} \otimes I_{A_2} \otimes ... \otimes I_{A_n} + \cdots + I_{A_1} \otimes ... \otimes I_{A_{n-1}} \otimes H_{A_n} \tag{51}
\]

then the set of separable states with finite energy is dense in \( \mathcal{S}_s(H_{A_1...A_n}) \). So, in this case it is reasonable to assume the coincidence of \( E^{H_{A^n}}_R(\rho) \) and \( E_R(\rho) \) for any state \( \rho \) with finite energy.\(^{18}\) In the following proposition we establish the validity of this assumption under the particular condition on the Hamiltonians \( H_{A_1},...,H_{A_n} \).

\(^{18}\)This coincidence is not obvious, since the infima of a lower semicontinuous function over a closed set and over a dense subset if this set may be different.
Proposition 3. Let $H_{A^n}$ be the Hamiltonian of a composite system $A_1 \ldots A_n$ expressed by formula (51) via the Hamiltonians $H_{A_1}, \ldots, H_{A_n}$ of the subsystems $A_1, \ldots, A_n$. If at least $n - 1$ of them satisfy condition (12) then

$$E_{R}^{H_{A^n}}(\rho) = E_R(\rho)$$

(52)

for any state $\rho$ such that $\text{Tr} H_{A^n} \rho = \sum_{s=1}^{n} \text{Tr} H_{A_s} \rho_{A_s} < +\infty$. Moreover, for any such state $\rho$ the infimum in definition (26) of $E_R(\rho)$ can be taken only over all finitely-decomposable separable states $\sigma$ (i.e. the states having form (36)) such that $\text{Tr} H_{A^n} \sigma = \sum_{s=1}^{n} \text{Tr} H_{A_s} \sigma_{A_s} < +\infty$.

Proof. Assume that the Hamiltonians $H_{A_1}, \ldots, H_{A_{n-1}}$ satisfy condition (12). Let $S_0$ be the convex subset of $\mathcal{S}(\mathcal{H}_{A_1, \ldots, A_n})$ consisting of all states $\rho$ with finite value of $\text{Tr} H_{A^n} \rho = \sum_{s=1}^{n} \text{Tr} H_{A_s} \rho$. On the set $S_0$ consider the function

$$\tilde{E}_{R}^{H_{A^n}}(\rho) = \inf_{\sigma \in \mathcal{S}(\mathcal{H}_{A_1, \ldots, A_n}) \cap S_0} \text{H}(\rho \| \sigma),$$

where $\mathcal{S}(\mathcal{H}_{A_1, \ldots, A_n})$ is the set of finitely-decomposable separable states in $\mathcal{S}(\mathcal{H}_{A_1, \ldots, A_n})$. Since the set $\mathcal{S}(\mathcal{H}_{A_1, \ldots, A_n}) \cap S_0$ is convex, the joint convexity of the relative entropy and Lemma 6 in [21] imply that the function $\tilde{E}_{R}^{H_{A^n}}$ satisfies inequality (14) with $a_f = 1$ and $b_f = 0$ on the set $S_0$. Since

$$\tilde{E}_{R}^{H_{A^n}}(\rho) \leq H(\rho \| \rho_{A_1} \otimes \ldots \otimes \rho_{A_n}) = I(A_1 : \ldots : A_n) \rho \leq 2 \sum_{s=1}^{n-1} H(\rho_{A_s}) \quad \forall \rho \in S_0,$$

where the second inequality follows from (7), the function $\tilde{E}_{R}^{H_{A^n}}$ satisfies inequality (15) with $m = n - 1$, $c_f = 0$ and $c_f = 2$ on the set $S_0$. In terms of Remark 3 in [24] it means that the function $\tilde{E}_{R}^{H_{A^n}}$ belongs to the class $L_{n}^{n-1}(2,1|S_0)$. It is clear that $E_R(\rho) \leq E_{R}^{H_{A^n}}(\rho) \leq \tilde{E}_{R}^{H_{A^n}}(\rho)$ for any state $\rho$ in $S_0$.

For each natural $s$ in $[1, n - 1]$ let $P^s_r$ be the spectral projector of the operator $H_{A_s}$ corresponding to its $r$ minimal eigenvalues (taking the multiplicity into account). Let $P^s_r = \sum_{i=1}^{s} |i \rangle \langle i |$, where $\{|i\rangle\}$ is a basis in $H_{A_s}$ such that $\langle i | H_{A_s} | i \rangle < +\infty$ for all $i$.

By the proof of Theorem 1 in [21] the set $S_0$ has the invariance property stated in Remark 3 in [24]. By this remark Theorem 1 in [24] is generalized to all the functions from the class $L_{n}^{n-1}(2,1|S_0)$ containing the function $\tilde{E}_{R}^{H_{A^n}}$. Since the function $E_R$ belongs to the class $L_{n}^{n-1}(1,1)$, Theorem 1 in [24], its generalization mentioned before and the arguments from the proof of Theorem 1 in Section 3 show that

$$\lim_{r \to +\infty} E_{R}(\rho_r) = E_{R}(\rho) \quad \text{and} \quad \lim_{r \to +\infty} \tilde{E}_{R}^{H_{A^n}}(\rho_r) = \tilde{E}_{R}^{H_{A^n}}(\rho), \quad \forall \rho \in S_0,$$

(53)

where $\rho_r = c_r^{-1} Q_r \rho Q_r$, $Q_r = P^1_r \otimes \ldots \otimes P^n_r$, $c_r = \text{Tr} Q_r \rho$. Hence, to prove that $\tilde{E}_{R}^{H_{A^n}}(\rho) = E_{R}^{H_{A^n}}(\rho) = E_{R}(\rho)$ it suffices to show that $\tilde{E}_{R}^{H_{A^n}}(\rho_r) = E_{R}(\rho_r)$ for any $r$. This can be done by using Lemma 2 below and by noting that $\text{Tr} H_{A^n} \sigma = \sum_{s=1}^{n} \text{Tr} H_{A_s} \sigma_{A_s} < +\infty$ for any separable state $\sigma$ supported by the subspace $P^1_r \otimes \ldots \otimes P^n_r(\mathcal{H}_{A_1, \ldots, A_n})$. □
Lemma 2. For an arbitrary state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ the infimum in (26) can be taken over the set of all separable states in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ supported by the subspace $\mathcal{H}_1^0 \otimes \ldots \otimes \mathcal{H}_n^0$, where $\mathcal{H}_s^0 = \text{supp}\rho_{A_s}$, $s = 1, n$.

Proof. Consider the channel

$$\Phi(\rho) = Q\rho Q + [\text{Tr}(I_{A_1 \ldots A_n} - Q)\rho]\tau, \quad Q = P_1 \otimes \ldots \otimes P_n,$$

where $P_s$ is the projector on the subspace $\mathcal{H}_s^0$ and $\tau$ is any separable state in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ supported by the subspace $\mathcal{H}_1^0 \otimes \ldots \otimes \mathcal{H}_n^0$. By monotonicity of the relative entropy we have

$$H(\rho||\Phi(\sigma)) = H(\Phi(\rho)||\Phi(\sigma)) \leq H(\rho||\sigma)$$

for any separable state $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$. Since $\Phi(\sigma)$ is a separable state supported by the subspace $\mathcal{H}_1^0 \otimes \ldots \otimes \mathcal{H}_n^0$, the above inequality implies the assertion of the lemma. □

Appendix

The following lemma is a $n$-partite generalization of the observation in [27].

Lemma 3. For any pure state $\omega$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ there is a countably decomposable separable state $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1 \ldots A_n})$ such that $\sigma_{A_s} = \omega_{A_s}$ for $s = 1, n$ and

$$H(\omega||\sigma) \leq \sum_{s=1}^{n-1} H(\omega_{A_s}).$$

Proof. Let $\omega = |\Omega\rangle\langle\Omega|$. The Schmidt decomposition w.r.t. the subsystems $A_1$ and $A_2 \ldots A_n$ implies

$$|\Omega\rangle = \sum_{i_1} \sqrt{p_{1i_1}} |\varphi_{1i_1}^1\rangle \otimes |\psi_{1i_1}^1\rangle,$$

where $\{\varphi_{1i_1}^1\}$ and $\{\psi_{1i_1}^1\}$ are orthogonal sets of unit vectors in $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2 \ldots A_n}$ correspondingly and $\{p_{1i_1}^1\}$ is a probability distribution. By using the Schmidt decomposition of any vector of the set $\{\psi_{1i_1}^1\}$ w.r.t. the subsystems $A_2$ and $A_3 \ldots A_n$ we obtain

$$|\Omega\rangle = \sum_{i_1, i_2} \sqrt{p_{1i_1}^1 p_{1i_12}^2} |\varphi_{1i_12}^1\rangle \otimes |\varphi_{1i_12}^2\rangle \otimes |\psi_{1i_12}^2\rangle,$$

where $\{\varphi_{1i_12}^2\}_{i_2}$ and $\{\psi_{1i_12}^2\}_{i_2}$ are orthogonal sets of unit vectors in $\mathcal{H}_{A_2}$ and $\mathcal{H}_{A_3 \ldots A_n}$ correspondingly and $\{p_{1i_12}^2\}_{i_2}$ is a probability distribution for any given $i_1$.

By repeating this process we get

$$|\Omega\rangle = \sum_{i_1, i_2, \ldots, i_{k-1}} \sqrt{p_{1i_1}^1 p_{1i_12}^2 \ldots p_{1i_12 \ldots i_{k-1}}^{k-1}} |\varphi_{1i_12 \ldots i_{k-1}}^{-1}\rangle \otimes \ldots \otimes |\varphi_{1i_12 \ldots i_{k-1}}^{k-1}\rangle \otimes |\psi_{1i_12 \ldots i_{k-1}}^{k-1}\rangle,$$
for any $k \leq n$, where $\{p_{i_1i_2...i_s}^s\}_{i_s}$ and $\{\varphi_{i_1i_2...i_s}^s\}_{i_s}$, $s < k$, are, respectively, a probability distribution and an orthogonal set of unit vectors in $\mathcal{H}_{A_s}$ for any $i_1, i_2, ..., i_{s-1}$ and $\{\psi_{i_1i_2...i_{k-1}}^{k-1}\}_{i_{k-1}}$ is an orthogonal set of unit vectors in $\mathcal{H}_{A_{i_1}...A_{i_{k-1}}}$ for any $i_1, i_2, ..., i_{k-1}$.

Consider the countably decomposable separable state

$$
\sigma = \sum_{i_1,i_2,...,i_{n-1}} p_{i_1}^1 p_{i_1i_2}^2 ... p_{i_1i_2...i_{n-1}}^{n-1} \otimes \rho_{i_1i_2...i_{n-1}}^{n-1} \otimes \varrho_{i_1i_2...i_{n-1}}^{n-1}
$$

(54)

in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$, where $\rho_{i_1i_2...i_s}^s = |\varphi_{i_1i_2...i_s}^s\rangle \langle \varphi_{i_1i_2...i_s}^s|$, $s < n$, is a state in $\mathcal{S}(\mathcal{H}_{A_s})$ for any $i_1, i_2, ..., i_s$ and $\varrho_{i_1i_2...i_{n-1}}^{n-1} = |\psi_{i_1i_2...i_{n-1}}^{n-1}\rangle \langle \psi_{i_1i_2...i_{n-1}}^{n-1}|$ is a state in $\mathcal{S}(\mathcal{H}_{A_{n-1}})$ for any $i_1, i_2, ..., i_{n-1}$.

By noting that all the summands in (54) are mutually orthogonal pure states one can show that the state $\sigma$ has the required properties. □

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