Remarks on the Dirac oscillator in \((2 + 1)\) dimensions

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Abstract – In this work the Dirac oscillator in \((2 + 1)\) dimensions is considered. We solve the problem in polar coordinates and discuss the dependence of the energy spectrum on the spin parameter \(s\) and angular-momentum quantum number \(m\). Contrary to earlier attempts, we show that the degeneracy of the energy spectrum can occur for all possible values of \(sm\). In an additional analysis, we also show that an isolated bound state solution, excluded from Sturm-Liouville problem, exists.

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The Dirac oscillator, first introduced in \([1]\) and after developed in \([2]\), has been a usual model for studying the physical properties of physical systems in various branches of physics. In the context of theoretical contributions, the Dirac oscillator has been analyzed under different aspects such as the study of the covariance properties and Foldy-Wouthuysen and Cini-Touschek transformations \([3]\), as a special case of a class of chiral solutions to the automorphism gauge field equations \([4]\), and hidden supersymmetry produced by the interaction \(iM\omega\gamma\mathbf{r}\), where \(M\) is the mass, \(\omega\) the frequency of the oscillator and \(\mathbf{r}\) is the position vector, when it plays a role of anomalous magnetic interaction \([5]\) (see also refs. \([3,6]\)).

Recently, the one-dimensional Dirac oscillator has had its first experimental realization \([7]\), which made the system more attractive from the point of view of applications. The Dirac oscillator in \((2 + 1)\) dimensions, when the third spatial coordinate is absent, has also been studied in refs. \([8–10]\). Additionally, this system was proposed in \([11]\) to describe some electronic properties of monolayer and bylayer graphene. For a detailed approach of the Dirac oscillator see refs. \([12,13]\).

In this letter, we address the Dirac oscillator in \((2 + 1)\) dimensions. In \([10]\), it was argued that the energy eigenvalues are degenerated only for negative values of \(k_\vartheta s\), where \(k_\vartheta\) represents the angular-momentum quantum number and \(s\) the spin projection parameter. This result, however, is not correct, as properly shown in this work. Additionally, an isolated bound state solution for the Dirac oscillator in \((2 + 1)\) is worked out.

We begin by writing the Dirac equation in \((2 + 1)\) dimensions \((\hbar = c = 1)\),

\[
(\beta\gamma \cdot \mathbf{p} + \beta M) \psi = E \psi,
\]

\((1)\)

where \(\mathbf{p} = (p_x, p_y)\) is the momentum operator and \(\psi\) is a two-component spinor. The Dirac oscillator is obtained through the following nonminimal substitution \([2]\):

\[
\mathbf{p} \rightarrow \mathbf{p} - iM\omega\beta\mathbf{r},
\]

\((2)\)

where \(\mathbf{r} = (x, y)\) is the position vector and \(\omega\) stands for the Dirac oscillator frequency. Thus, the relevant equation is

\[
[\beta\gamma \cdot (\mathbf{p} - iM\omega\beta\mathbf{r}) + \beta M] \psi = E \psi.
\]

\((3)\)

In three dimensions the \(\gamma\) matrices are conveniently defined in terms of the Pauli matrices \([14]\),

\[
\beta\gamma_x = \sigma_x, \quad \beta\gamma_y = s\sigma_y, \quad \beta = \sigma_z,
\]

\((4)\)

where \(s\) is twice the spin value, with \(s = +1\) for spin “up” and \(s = -1\) for spin “down”. In this manner, eq. \((3)\) can be written as

\[
(\sigma_x \pi_x + s\sigma_y \pi_y + \sigma_z M) \psi = E \psi,
\]

\((5)\)

where \(\pi_j = p_j - iM\omega\sigma_j r_j\).
As usual, we write eq. (5) in polar coordinates \((r, \phi)\),
\[
e^{i\sigma_z \phi} \left[ \sigma_x \partial_r + \sigma_y \left( \frac{s}{r} \partial_{\phi} - iM\omega r \right) \right] \psi = i(E - M\sigma_z) \psi.
\]
(6)

If one defines the spinor as
\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]
(7)
eq (6) leads to
\[
\begin{align*}
\left( \partial_r - i \frac{s}{r} \partial_{\phi} - M\omega r \right) \psi_2 &= i e^{is\phi} (E - M) \psi_1, \\
\left( \partial_r + i \frac{s}{r} \partial_{\phi} + M\omega r \right) \psi_1 &= i e^{-is\phi} (E + M) \psi_2.
\end{align*}
\]
(8a) and (8b)

We decomposes the spinor as
\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \sum_m f_m(r) e^{im\phi} \\ \sum_m g_m(r) e^{i(m+s)\phi} \end{pmatrix},
\]
(9)

where \(m = 0, \pm1, \pm2, \pm3, \ldots\) is the angular-momentum quantum number. The factor \(i\) on the lower spinor component is included for later convenience. By replacing eq. (9) into eq. (8), we can write the two coupled first-order radial equations,
\[
\begin{align*}
\left[ \frac{d}{dr} + \frac{s(m+s)}{r} - M\omega r \right] g_m(r) &= (E - M)f_m(r), \\
\left[ -\frac{d}{dr} + \frac{sm}{r} - M\omega r \right] f_m(r) &= (E + M)g_m(r).
\end{align*}
\]
(10a) and (10b)

Now the role of the \(i\) in the lower component of eq. (9) is apparent. It was inserted to ensure that the radial part of the spinors is manifestly real.

The problem of the Dirac oscillator in \((2+1)\) dimensions represented by eqs. (10a) and (10b) for \(E \neq \pm M\) can be mapped into a Sturm-Liouville problem for the upper and lower components of the Dirac spinor. In this manner, as we will show, the solutions can be found by solving a Schrödinger-like equation. An isolated solution for the problem, excluded from the Sturm-Liouville problem, can be obtained considering the particle at rest, i.e., \(E = \pm M\) directly in the first-order equations in (10a) and (10b). Such solution for the Dirac equation in \((1+1)\) dimensions was investigated in ref. [15] (see also refs. [16–21]). We are seeking bound state solutions subjected to the normalization condition,
\[
\int_0^{\infty} \left[ |f_m(r)|^2 + |g_m(r)|^2 \right] r dr = 1.
\]
(11)

Let us begin by determining the isolated bound states solutions. So, for \(E = -M\), we can write
\[
\begin{align*}
\left[ \frac{d}{dr} + \frac{s(m+s)}{r} - M\omega r \right] g_m(r) &= -2Mf_m(r), \\
\left[ -\frac{d}{dr} + \frac{sm}{r} - M\omega r \right] f_m(r) &= 0,
\end{align*}
\]
(12a) and (12b) whose general solutions are
\[
\begin{align*}
f_m(r) &= a_r r^{sm} e^{-M\omega r^2/2}, \\
g_m(r) &= [b_+ - 2Ma_+I_+(r)] r^{-sm-1} e^{M\omega r^2/2},
\end{align*}
\]
(13a) and (13b) where \(a_+\) and \(b_+\) are constants. In (13b), \(I_+(r)\) can be expressed in terms of the upper incomplete Gamma function [22],
\[
\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt, \quad \Re(a) > 0.
\]
(14)

In fact,
\[
I_+(r) = \frac{\Gamma(sm + 1, M\omega r^2)}{2(M\omega)^{sm+1}}.
\]
(15)

As \(M\omega > 0\), there are no integer values for \(sm\) that the functions (13a) and (13b) are square-integrable. Therefore, there is no bound state solution for \(E = -M\). In the other hand, for \(E = M\), from (10a) and (10b) we can write
\[
\begin{align*}
\left[ \frac{d}{dr} + \frac{s(m+s)}{r} - M\omega r \right] g_m(r) &= 0, \\
\left[ -\frac{d}{dr} + \frac{sm}{r} - M\omega r \right] f_m(r) &= 2Mg_m(r),
\end{align*}
\]
(16a) and (16b) whose general solutions are
\[
\begin{align*}
f_m(r) &= [b_- - 2Ma_-I_-(r)] r^{sm} e^{-M\omega r^2/2}, \\
g_m(r) &= a_+ r^{-1-sm} e^{M\omega r^2/2},
\end{align*}
\]
(17a) and (17b) where \(a_+\) and \(b_-\) are constants, and
\[
I_-(r) = \frac{(-M\omega)^{sm}}{2} \Gamma(-sm, -M\omega r^2).
\]
(18)

A normalizable solution requires \(a_+ = 0\). In this case, the function \(f_m(r)\) is square-integrable only for \(sm \geq 0\). Therefore,
\[
\begin{pmatrix} f_m(r) \\ g_m(r) \end{pmatrix} = b_+ r^{sm} e^{-M\omega r^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad sm \geq 0.
\]
(19)

Now, for \(E \neq \pm M\), by the manipulation of eqs. (10a) and (10b), we can decouple them and obtain the following Schrödinger-like second-order differential equations for the components:
\[
\begin{align*}
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} &- \frac{[m + (s \mp s)/2]^2}{r^2} + 2M\omega s[m + (s \mp s)/2] \\
- M^2\omega^2 r^2 + (E^2 - M^2) \right) \begin{pmatrix} f_m(r) \\ g_m(r) \end{pmatrix} &= 0.
\end{align*}
\]
(20)

Our task now is to solve eq. (20). Using the change of variable \(\rho = M\omega r^2\), eq. (20) for the \(f_m(r)\) assumes the form
\[
\rho f''_m(\rho) + f'_m(\rho) - \left( \frac{m^2}{4\rho} + \frac{\rho}{4} - \frac{k^2}{4\gamma} \right) f_m(\rho) = 0,
\]
(21)
with $\gamma = M\omega$ and

$$k^2 = E^2 - M^2 + 2\gamma(sm + 1). \quad (22)$$

Studying the asymptotic limits of eq. (21), and the finiteness at the origin leads us to the following solution:

$$f_m(\rho) = \rho^{m/2} e^{-\rho/2} w(\rho). \quad (23)$$

Substitution of eq. (23) into eq. (21), results

$$\rho a''(\rho) + (1 + |m| - \rho) a'(\rho) - \left(\frac{1 + |m|}{2} - \frac{k^2}{4\gamma}\right) a(\rho) = 0. \quad (24)$$

Equation (24) is a confluent hypergeometric-like equation,

$$zw''(z) + (b - z)w'(z) - aw(z) = 0, \quad (25)$$

where $w(z)$ is the confluent hypergeometric function [22], with $a = (1 + |m|)/2 - k^2/4\gamma$ and $b = 1 + |m|$. In this manner, the only acceptable solution for eq. (24) is the confluent hypergeometric function of the first kind $\, _1F_1(a, b, z)$. As $b = 1 + |m| \geq 1$, the other linearly independent solution, the confluent hypergeometric function of the second kind $U(a, b, z)$, is rejected because it is irregular at the origin. Consequently,

$$f_m(\rho) = a_m \rho^{m/2} e^{-\rho/2} \, _1F_1(d, 1 + |m|, \rho), \quad (26)$$

where

$$d = \frac{1 + |m|}{2} - \frac{k^2}{4\gamma}. \quad (27)$$

In order to find the energy spectrum of the Dirac oscillator, we should establish as convergence criterion the condition $d = -n$, with $n$ a non-negative integer. Therefore, the energy levels are given by

$$E = \pm \sqrt{M^2 + 2M\omega(2n + |m| - sm)} - \sqrt{n}, \quad n = 0, 1, 2, \ldots, \quad (28)$$

and in this case $\, _1F_1(-n, 1 + |m|, \rho)$ can be written in terms of the generalized Laguerre polynomials $L_n^{|m|}(\rho)$ [22]. In this manner, the upper component of the bound state wave function is

$$f_m(\rho) = a_m \rho^{m/2} e^{-\rho/2} L_n^{|m|}(\rho). \quad (29)$$

The lower component is obtained in an analogous manner by directly solving eq. (20) and the result is

$$g_m(\rho) = b_m \rho^{m+s/2} e^{-\rho^2/2} L_n^{|m+s|}(\rho), \quad (30)$$

where $\Theta(x)$ is the Heaviside function, and $a_m$ and $b_m$ are constants subject to the normalization condition in (11).

It is important to note that the energy levels expressed in (28) are spin dependent and, contrary to the results of [10], the energy expression yield an infinity degeneracy for all possible values of $sm$. Indeed, for $sm < 0$, we have: for $s = 1$ and $m < 0$ all levels with $n \pm q$ and $m \pm q$ have the same energy, while for $s = -1$ and $m > 0$ the equal energy levels are those with $n \pm q$ and $m \mp q$, being $q$ an integer. Moreover, for $sm \geq 0$, all the energy levels are independent of the angular-quantum number $m$. This behavior is depicted in fig. 1 and represented in table 1.

![Fig. 1: (Color online) The positive energy spectrum, eq. (28), for the Dirac oscillator in (2 + 1) dimensions for different values of $n$ and $m$ with $M = \omega = 1$ and for: (a) $s = 1$ and (b) $s = -1$. Notice that levels with quantum numbers $n \pm q$ for $s = 1$ ($s = -1$) have the same energy as levels with $m \pm q$ ($m \mp q$), with $q$ an integer. For $sm \geq 0$, it is clear that the energy spectrum is independent of $m$.](image-url)
Table 1: The energy spectrum for the Dirac oscillator in (2+1) dimensions as a function of the good quantum numbers \( m \) and \( n \) for \( s = 1 \). For convenience the values correspond to \((E^2 - M^2)/2M\).\\n
\[\begin{array}{cccccccccc}
  m \Rightarrow & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
  n \downarrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  0 & 10 & 8 & 6 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 12 & 10 & 8 & 6 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
  2 & 14 & 12 & 10 & 8 & 6 & 4 & 4 & 4 & 4 & 4 & 4 \\
  3 & 16 & 14 & 12 & 10 & 8 & 6 & 6 & 6 & 6 & 6 & 6 \\
  4 & 18 & 16 & 14 & 12 & 10 & 8 & 8 & 8 & 8 & 8 & 8 \\
  5 & 20 & 18 & 16 & 14 & 12 & 10 & 10 & 10 & 10 & 10 & 10
\end{array}\]

For \( s = 1 \), agreed with the expression found in eq. (A2) of ref. [8] and eqs. (9) and (22) of ref. [23], respectively.

In order to investigate the role played by the nonminimal substitution in (2) as well as its physical implications, we should evaluate the nonrelativistic limit of eq. (31). In this case, by writing \( E = M + \mathcal{E} \), with \( M \gg \mathcal{E} \), we get

\[2Mc^2 \left( \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 + \hbar \omega - s\omega L_z \right) \psi_2 = (E^2 - M^2c^4)\psi_2. \quad (32b)\]

Equations (32a) and (32b), for \( s = 1 \), agreed with the expressions found in eq. (2) of ref. [12] and eqs. (9) and (22) of ref. [23], respectively.

The first two terms on the left side of eq. (33) are those that appear in the Hamiltonian of the nonrelativistic circular harmonic oscillator [24], explaining why this system is called Dirac oscillator. The third term is a constant which shifts all energy levels. The last term is the spin-orbit coupling, which (restoring the factor \( \hbar \)) shifts all energy levels. Indeed, the shifted energy levels are

\[\mathcal{E} + \omega = (1 + 2n + |m| - sn)\omega. \quad (34)\]

As for the relativistic case, the infinity degeneracy is also present.

In summary, we have shown that the energy spectrum of the Dirac oscillator in (2+1) dimensions depends on the value of the spin projection parameter \( s \). On the other hand, it has also been shown that energy spectrum is degenerated for all possible values of \( sn \), a behavior not discussed before in the literature. Additionally, an isolated bound state solution, excluded from the Sturm-Liouville problem, was discussed.

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