Transition from Quantum to Classical Information in a Superfluid

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Abstract

Whereas the entropy of any deterministic classical system described by a principle of least action is zero, one can assign a "quantum information" to quantum mechanical degree of freedom equal to Hausdorff area of the deviation from a classical path. This raises the question whether superfluids carry quantum information. We show that in general the transition from the classical to quantum behavior depends on the probing length scale, and occurs for microscopic length scales, except when the interactions between the particles are very weak. This transition explains why, on macroscopic length scales, physics is described by classical equations.

1 Introduction

In this letter we consider the following two questions:

1) Can information be carried by a superfluid order parameter $\psi$ in the same way that information is carried by, say, a radio wave?

and

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2) Since the superfluid order parameter $\psi$ depends on entanglement \textit{a la} Bogoluibov, is there any difference between information carried by a superfluid order parameter and quantum information?

The answer to the first question is pretty clearly yes, but the answer to the second is not so obvious. It was shown by Bogoluibov \cite{1} that superfluids depend on existence of EPR-like correlations between particles of opposite momenta. Since EPR correlations play an important role in quantum computing \cite{2}, one might conjecture that spatial variations in the order parameter qualify as "quantum informations".

As was first pointed out by Planck \cite{3}, the Clausius-Boltzmann entropy of any classical system obeying a principle of least action is zero. This serves as a hint that in a search of the answer to the second question one should turn to the path integral formulation of quantum mechanics where the principle of least action is not valid anymore. One feature of the path integral formulation which can serve our purposes is an existence of quantum-mechanical paths \cite{4} that are continuous but non-differentiable everywhere.

Therefore a transition to a classical regime is characterized by smoothing out the "irregularities" of a quantum-mechanical path. This smoothing was ascribed to the process of averaging "over a reasonable length of time to produce ... an 'average' velocity" \cite{4}. In fact, the emergence of a classical path is due to a decreased resolution used in measuring the path’s length.

This is clearly seen if one would use for the description of a quantum-mechanical path the concept of Hausdorff length and dimension \cite{5}. It was found that independently of the length definition, the Hausdorff dimension of a quantum-mechanical path is $D_H = 2$ as compared to the classical dimension $D_H = 1$. The transition from one regime to another can be demonstrated by explicitly evaluating the Hausdorff length in \cite{5} for any value of the Hausdorff dimension.

It turns out that for

\begin{equation}
\langle \Delta l \rangle = \int_{\mathbb{R}^3} d^3 x |x| |\psi(x, \Delta t)|^2
\end{equation}

where $\langle \Delta l \rangle$ is the average distance a particle travels in a time $\Delta t$, its path
Figure 1: Log of Hausdorff length according to Eq.(1) as a function of the Hausdorff dimension $D_H$ and the resolution $\Delta x$ for Abbot-Wise analysis [5] with $f(|k|) = \exp(-|k|^2)$

length is

$$< L > \sim (\Delta x)^{D-1}\left\{\sqrt{\frac{\pi}{2}} \Phi\left(\frac{\sqrt{2(\Delta x)}}{\sqrt{1+4(\Delta x)^2}}\right) + \frac{\sqrt{1+4(\Delta x)^2}}{\Delta x} \exp[-\frac{(\Delta x)^2}{\sqrt{1+4(\Delta x)^2}}]\right\}$$

(1)

Here $\Phi(y) = (2/\sqrt{\pi}) \int_0^y e^{-y^2}$, $\Delta x = \Delta x/\lambda$, $\Delta x$ is the spatial resolution, $\lambda = \hbar/pav$ is the de Broglie wavelength and $pav$ is particle’s average momentum. Fig.1 illustrates the dependence given by (1)

If we use another definition of length

$$b) < \Delta l >= \sqrt{\int_{R^3} d^3x |x|^2 |\psi_{\Delta x}(x, \Delta t)|^2}$$

then the resulting path length is

$$< L > \sim (\Delta x)^{D-2} \sqrt{16(\Delta x)^2 + 3}$$

(2)

The respective graph is shown in Fig.2.
Figure 2: Log of Hausdorff length according to Eq.(2) as a function of the Hausdorff dimension $D_H$ and the resolution $\Delta x$ for Abbot-Wise analysis \cite{5} with $f(|k|) = \exp(-|k|^2)$;

In both cases the parameter describing a transition from a classical to quantum regime (and vice versa) is the dimensionless spatial resolution

$$\Delta x = \frac{\Delta x}{\hbar/\text{p}_{av}}$$

(3)

It has a suggestive physical meaning: a ratio of a resolution (physically implementable by some measuring classical device) used to measure length (a probing scale) and the "characteristic" length scale intrinsic to quantum process, which can be viewed as De Broglie wave length. In that we made full circle on a spiral, referring to the earlier view of quantum mechanics, but this time on a higher level. Roughly speaking, the magnitude of this parameter indicates how strong(weak) are quantum effects as viewed from the classical world. As will be seen below, this parameter has a universal character, and it will emerge in our discussion of the
2 Bose Condensate and the transition from Classical to Quantum Behavior

To illustrate this point we consider a Bose condensate of interacting bosons at zero temperature \[6\]. Small perturbations to the superfluid order parameter satisfy the linear Schrödinger-like equation:

\[
\frac{\partial^2 \phi}{\partial t^2} = v_s^2 \nabla^2 \phi - \left( \frac{\hbar}{2M} \right)^2 \nabla^4 \phi
\]

(4)

where \(v_s\) is the speed of sound and \(M\) is the mass of the fluid.

We will consider the classical-quantum transition with the help of 2 methods which, as will be seen later, turn out to be equivalent:

i) Numerical Calculation

In this approach the average distance \(<\Delta l>\) the particle travels in time \(\Delta t\) can be written as follows:

\[
<\Delta l> = \int_{\mathbb{R}^3} d^3x |x||\Psi(x, \Delta t)|^2
\]

(5)

Here

\[
\Psi(x, \Delta t) = \frac{(\Delta x)^{3/2}}{\hbar^3} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^{3/2}} f(\frac{|p|\Delta x}{\hbar}) e^{ip\cdot x/\hbar - i\alpha\sqrt{p^2/\Delta x^2 + 1/2(k/\Delta x)^2}}
\]

(6)

and \(E\) is given in \[6\] as

\[
E = \sqrt{(pv_s)^2 + \frac{p^4}{4M^2}}
\]

(7)

By introducing the dimensionless quantities

\[
k = \frac{p\Delta x}{\hbar}, \quad \overline{\Delta x} = \Delta x \frac{Mv_s}{\hbar}, \quad y = \frac{x}{\Delta x}, \quad \alpha = \frac{Mv_s^2 \Delta t}{\hbar}
\]

(8)

and using (7) and (6) we rewrite the expression (5):

\[
<\Delta l> = \Delta x \int_{\mathbb{R}^3} d^3y |y| \int_{\mathbb{R}^3} d^3k f(k) e^{ik\cdot y - i\alpha\sqrt{(k/\overline{\Delta x})^2 + 1/2(k/\Delta x)^2}}
\]

(9)

Interestingly enough, the dimensionless quantities \(\alpha\) and \(\overline{\Delta x}\) have a very simple physical meaning: \(\overline{\Delta x}\) is the ratio of the classical and quantum momenta
(here the classical quantity appears naturally and not introduced by hand as in [5]) and $\alpha$ is the ratio of the respective energies.

If we take the function $f|\mathbf{k}|$ to be Gaussian, that is $f|\mathbf{k}| = e^{-|\mathbf{k}|^2}$ then (9) yields the following Hausdorff length $<L>$

$$<L> \sim \Delta x^D \int_{\mathbb{R}^3} d^3y |y||\int_{\mathbb{R}^3} d^3ke^{-|\mathbf{k}|^2+i\mathbf{k}\cdot\mathbf{y}-i\alpha\sqrt{(k/\Delta x)^2+\frac{1}{2}(k/\Delta x)^2}}^2$$ (10)

In 2 limiting cases of

1) a purely classical regime ($\Delta x \gg k$) and
2) a purely quantum regime ($\Delta x \ll k$)

the above expression can be evaluated analytically. For simplicity sake (and without any loss of generality) we consider a 1-D realization of (10).

1) After some algebra, the first case yields the following expression for the Hausdorff length $<L>$:

$$<L> \sim (\Delta x)^D \left\{ \sqrt{\frac{\pi}{v_{av}}} v_s \Phi \left( \frac{1}{\sqrt{2}v_{av}} \right) - 2e^{-\frac{1}{2}(v_s/v_{av})^2} \right\}$$ (11)

where we denote $v_{av} = \Delta x/\Delta t$. The result is rather trivial, since in this case in the limit of $\Delta x \to 0$ the Hausdorff length is the conventional length, whose dimension is $D = 1$.

2) Quite analogously we find that in this case the Hausdorff length is

$$<L> \sim \Delta x^D \sqrt{1 + \left( \frac{\hbar\Delta t}{M\Delta x^2} \right)^2}$$ (12)

In the limit $\Delta x \to 0$ Eq. (12) yields

$$L \sim \Delta x^{D-2}$$ (13)

which is the same result as was obtained for quantum case in [5].
Figure 3: The Hausdorff length according to Eq.(9) as a function of the Hausdorff dimension $D_H$, resolution $\Delta x$, and sound speed $v_s = 0, 2, 4, 6, 8$ (bottom to top).

In general, the integrals in Eq.(10) cannot be found in a closed form. Instead we evaluate them numerically. The result is shown in Fig.3 where the Hausdorff length is presented as a function of the Hausdorff dimension $D_H$, resolution $\Delta x$ and the following values of sound speed $v_s = 0, 2, 4, 6, 8$.

ii) Heuristic (De Broglie) Construction

To compare the results of the previous section with the calculations based on an heuristic (De Broglie) picture, we begin with the dispersion relation which follows from (4):

$$\omega_k = kv_s \sqrt{1 + \left( \frac{\hbar k}{2Mv_s} \right)^2}$$  \hspace{1cm} (14)

where $k = 2\pi/\Lambda$ and $\Lambda$ is the wave length of the small perturbations which we take as the probing length scale, that is $\Delta x = \Lambda$.

In the De Broglie picture the group velocity is considered as a velocity of a quantum "particle" which in turn would allow us to introduce an analogue of the path length $< L >$ travelled by such a particle in a time $T$. Moreover, since we are dealing with the quantum path, this path length must be
understood in Hausdorff sense:

\[ < L_H > = (\Delta x)^{D-1} < L > \]

From (14) we find the group (particle) velocity \( v_g \)

\[ v_g = v_s \frac{2 + \left( \frac{\hbar/Mv_s}{\Delta x} \right)^2}{2\sqrt{1 + \left( \frac{\hbar/Mv_s}{2\Delta x} \right)^2}} \] (15)

The path length \( < l > \) (in its conventional sense) travelled by this particle in a time interval \( T \) is then

\[ < l > = v_g T = v_s \frac{2 + \left( \frac{\hbar/Mv_s}{\Delta x} \right)^2}{2\sqrt{1 + \left( \frac{\hbar/Mv_s}{2\Delta x} \right)^2}} \] (16)

We notice that in this case the dimensionless quantity

\[ \Delta x \equiv \frac{Mv_s \Delta x}{\hbar} \] (17)

emerges which is exactly the same as in the previous (numerical) case (cf. Eq.8). As a result we obtain from (16) the following expression for the Hausdorff length \( < L >_H \):

\[ < L >_H \sim (\Delta x)^D \frac{1 + 2(\Delta x)^2}{\sqrt{1 + (2\Delta x)^2}} \] (18)

From Eq.(18) follows that for the limiting case of \( v_s = 0 \) (that is purely quantum case) the Hausdorff dimension is \( D = 2 \). On the other hand, for another limiting case of \( \Delta x \gg 1 \), that is \( Mv_s \gg \hbar/\Delta x \) the Hausdorff dimension is \( D = 1 \), corresponding to the classical limit.

The graph of the general dependence \( \log(< L_H >) = f(D, \Delta x, v_s) \) according to Eq.(18), that is according to De Broglie picture, is shown in Fig.4. A comparison of Figs.(3) and (4) shows a remarkable qualitative similarity between the Hausdorff length \( < L >_H \) calculated on the basis of the De Broglie picture and the same length found with the help of the numerical analysis. However, this should not be very surprising, if we take into account that the integrand in Eq.(10) contains as a power of the exponent the dispersion relation (5).
It is seen (both from Eq.18 and Figs.3 and 4) that the transition from quantum to classical regime (characterized by a change of Hausdorff dimension) is a continuous process, such that a change from a classical to quantum regime is governed by the dimensionless parameter (dimensionless length) \( \Delta x \) \( \text{(Eq.17)} \) whose physical meaning was given earlier as the ratio of the quantum "momentum" (in the De Broglie sense) and the classical momentum of a particle moving with the speed of sound. A gradual increase of \( \Delta x \) signals a continuous transition from a purely classical \( (\Delta x \to \infty) \) to purely quantum \( (\Delta x = 0) \) regime.

### 3 Conclusion

The emerging picture allows us to answer the second question posed at the beginning of this letter. For small perturbations a superfluid order parameter \( \psi \) is a function of the parameter \( \Delta x \). On the other hand, a continuous change of this parameter from \( \Delta x \to \infty \) to \( \Delta x = 0 \) describes a transition from the classical regime in a superfluid to the quantum regime. Therefore in the limit \( \Delta x \to 0 \) there is no difference between the information carried by the superfluid order parameter and the quantum information.
Amazingly enough, the above process describes a continuous transition from a fluid-like coherent state to a coherent state of weakly interacting particles. For ordinary sound velocities this transition occurs for microscopic probing length scales. However, if we are near a quantum critical point where $v_s \to 0$, then this transition will occur for macroscopic length scales.

Our results run contrary to the conventional point of view regarding the transition from classical to quantum physics as being necessarily due to decoherence [7]. Indeed, in our view decoherence plays essentially no role in the transition from ordinary classical physics to quantum physics. To the contrary our results strongly support the view [6] that the validity of classical equations of motion for macroscopic length scales is a consequence of having a vacuum state with a 'stiff' order parameter. In fact, identifying ordinary spacetime with a superfluid-like quantum state with a small value of $\hbar/Mv_s$ would be a natural result in almost any physically reasonable quantum theory of gravity [8].

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5 Addendum

After completion of this paper we became aware of a recent paper by Y. Shi [9], that also shows that the transition from quantum to classical behavior in a superfluid is not due to decoherence, but using rather different arguments.

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