THE POLETSKII AND VÄISÄLÄ INEQUALITIES FOR THE MAPPINGS WITH $(P, Q)$–DISTORTION

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Abstract

The present paper is devoted to the study of space mappings which are more general than quasiregular. Some modulus inequalities for this class of mappings are obtained. In particular, analogs of the well-known Poletskii and Väisälä inequalities were proved.

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1 Introduction

The present paper is devoted to the study of quasiconformal mappings and their generalizations, such as mappings with finite distortion intensively investigated last time, see [BGMV], [BGR], [Cr1]–[Cr2], [Gol1]–[Gol2], [IM], [IR], [KO], [MRSY1]–[MRSY2], [Mikl], [Pol], [Re1]–[Re2], [Ri], [RSY1]–[RSY2], [UV], [Va1]–[Va2].

Let us give some definitions. Everywhere below, $D$ is a domain in $\mathbb{R}^n$, $n \geq 2$, $m$ is the Lebesgue measure in $\mathbb{R}^n$, $m(A)$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$, $m_1$ is the linear Lebesgue measure in $\mathbb{R}$. A mapping $f : D \to \mathbb{R}^n$ is discrete if $f^{-1}(y)$ consists of isolated points for each $y \in \mathbb{R}^n$, and $f$ is open if it maps open sets onto open sets. The notation $f : D \to \mathbb{R}^n$ assumes that $f$ is continuous. In what follows, a mapping $f$ is supposed to be orientation preserving, i.e., the topological index $\mu(y, f, G) > 0$ for an arbitrary domain $G \subset D$ such that $\overline{G} \subset D$ and $y \in f(G) \setminus f(\partial G)$, see e.g. II.2 in [Re2]. Let $f : D \to \mathbb{R}^n$ be a mapping and suppose that there is a domain $G \subset D$, $\overline{G} \subset D$, for which $f^{-1}(f(x)) = \{x\}$. Then the quantity $\mu(f(x), f, G)$, which is referred to as the local topological index, does not depend on the choice of the domain $G$ and is denoted by $i(x, f)$. Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and a point $y \in \mathbb{R}^n$, we define the multiplicity function $N(y, f, E)$ as the number of pre-images of $y$ in $E$, i.e.,

$$N(y, f, E) = \text{card } \{x \in E : f(x) = y\}$$
and

\[ N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \]

Recall that a mapping \( f : D \to \mathbb{R}^n \) is said to have \( N \)-property (by Luzin) if \( m(f(S)) = 0 \) whenever \( m(S) = 0 \) for \( S \subset \mathbb{R}^n \). Similarly, \( f \) has the \( N^{-1} \)-property if \( m(f^{-1}(S)) = 0 \) whenever \( m(S) = 0 \).

A curve \( \gamma \) in \( \mathbb{R}^n \) is a continuous mapping \( \gamma : \Delta \to \mathbb{R}^n \) where \( \Delta \) is an interval in \( \mathbb{R} \). Its locus \( \gamma(\Delta) \) is denoted by \( |\gamma| \). Given a family \( \Gamma \) of curves \( \gamma \) in \( \mathbb{R}^n \), a Borel function \( \rho : \mathbb{R}^n \to [0, \infty] \) is called admissible for \( \Gamma \), abbr. \( \rho \in \text{adm} \Gamma \), if

\[ \int_\gamma \rho(x)|dx| \geq 1 \]

for each (locally rectifiable) \( \gamma \in \Gamma \). Given \( p \geq 1 \), the \( p \)-modulus of \( \Gamma \) is defined as the quantity

\[ M_p(\Gamma) := \inf_{\rho \in \text{adm} \Gamma} \int_{\mathbb{R}^n} \rho^p(x)dm(x) \]

interpreted as \( +\infty \) if \( \text{adm} \Gamma = \emptyset \). Note that \( M_p(\emptyset) = 0; M_p(\Gamma_1) \leq M_p(\Gamma_2) \) whenever \( \Gamma_1 \subset \Gamma_2 \), and \( M_p \left( \bigcup_{i=1}^\infty \Gamma_i \right) \leq \sum_{i=1}^\infty M_p(\Gamma_i) \), see Theorem 6.2 in [Va1].

We say that a property \( P \) holds for \( p \)-almost every (\( p \)-a.e.) curves \( \gamma \) in a family \( \Gamma \) if the subfamily of all curves in \( \Gamma \), for which \( P \) fails, has \( p \)-modulus zero.

If \( \gamma : \Delta \to \mathbb{R}^n \) is a locally rectifiable curve, then there is the unique nondecreasing length function \( l_\gamma \) of \( \Delta \) onto a length interval \( \Delta_\gamma \subset \mathbb{R} \) with a prescribed normalization \( l_\gamma(t_0) = 0 \in \Delta_\gamma \), \( t_0 \in \Delta \), such that \( l_\gamma(t) \) is equal to the length of the subcurve \( \gamma|_{[t_0, t]} \) of \( \gamma \) if \( t > t_0 \), \( t \in \Delta \), and \( l_\gamma(t) \) is equal to minus length of \( \gamma|_{[t, t_0]} \) if \( t < t_0 \), \( t \in \Delta \). Let \( g : |\gamma| \to \mathbb{R}^n \) be a continuous mapping, and suppose that the curve \( \tilde{\gamma} = g \circ \gamma \) is also locally rectifiable. Then there is a unique non-decreasing function \( L_{\gamma,g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}} \) such that \( L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t) \) for all \( t \in \Delta \). A curve \( \gamma \) in \( D \) is called here a (whole) lifting of a curve \( \tilde{\gamma} \) in \( \mathbb{R}^n \) under \( f : D \to \mathbb{R}^n \) if \( \tilde{\gamma} = f \circ \gamma \).

We say that a mapping \( f : D \to \mathbb{R}^n \) satisfies the \( L \)-property with respect to \((p,q)\)-modulus, iff the following two conditions hold:

1. \( \left( L_p^{(1)} \right) \) for \( p \)-a.e. curve \( \gamma \) in \( D \), \( \tilde{\gamma} = f \circ \gamma \) is locally rectifiable and the function \( L_{\gamma,f} \) has the \( N \)-property;
2. \( \left( L_q^{(2)} \right) \) for \( q \)-a.e. curve \( \tilde{\gamma} \) in \( f(D) \), each lifting \( \gamma \) of \( \tilde{\gamma} \) is locally rectifiable and the function \( L_{\gamma,f} \) has the \( N^{-1} \)-property.

The notation \((p,q)\)-modulus mentioned above means that the \( L_p^{(1)} \) and \( L_q^{(2)} \) properties hold with respect to \( p \) and \( q \)-moduli, respectively.

A mapping \( f : D \to \mathbb{R}^n \) is called a mapping with finite length \((p,q)\)-distortion, if \( f \) is differentiable a.e. in \( D \), has \( N \)– and \( N^{-1} \)–properties, and \( L \)-property with respect to \((p,q)\)-modulus. The mappings of finite length \((p,q)\)-distortion are natural generalization of the
mappings with finite length distortion introduced in the paper [MRSY1], see also monograph [MRSY2].

Set at points \( x \in D \) of differentiability of \( f \)

\[
l(f'(x)) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}, \|f'(x)\| = \max_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}, J(x, f) = \det f'(x),
\]

and define for any \( x \in D \) and fixed \( p, q, p, q \geq 1 \)

\[
K_{I,q}(x, f) = \begin{cases} 
\frac{|J(x, f)|}{|f'(x)|^p}, & J(x, f) \neq 0, \\
1, & f'(x) = 0, \\
\infty, & \text{otherwise}
\end{cases}
\]

\[
K_{O,p}(x, f) = \begin{cases} 
\frac{\|f'(x)\|^p}{|J(x,f)|}, & J(x, f) \neq 0, \\
1, & f'(x) = 0, \\
\infty, & \text{otherwise}
\end{cases}
\]

One of the main results proved in the paper is following.

**Theorem 1.1.** A mapping \( f : D \to \mathbb{R}^n \) with finite length \((p, q)\)-distortion satisfies the inequalities

\[
M_q(f(\Gamma)) \leq \int_D K_{I,q}(x, f) \cdot \rho^q(x) \, dm(x) \tag{1.1}
\]

for every family of curves \( \Gamma \) in \( D \) and \( \rho \in \text{adm} \Gamma \), and

\[
M_p(\Gamma) \leq \int_{f(E)} K_{I,p}(y, f^{-1}, E) \cdot \rho^p(y) \, dm(y) \tag{1.2}
\]

for every measurable set \( E \subset D \), every family \( \Gamma \) of curves \( \gamma \) in \( E \) and every function \( \rho_*(y) \in \text{adm} f(\Gamma) \), where

\[
K_{I,p}(y, f^{-1}, E) := \sum_{x \in E \cap f^{-1}(y)} K_{O,p}(x, f).
\tag{1.3}
\]

Remark that an analog of the Theorem 1.1 for \( p = q = n \) was proved in [MRSY2], see Theorems 8.5 and 8.6 (cf. [MRSY1]).

### 2 The proof of the main results

Further, we use the notation \( I \) for the segment \([a, b]\). Given a closed rectifiable path \( \gamma : I \to \mathbb{R}^n \), we define a length function \( l_\gamma(t) \) by the rule \( l_\gamma(t) = S(\gamma, [a, t]) \), where \( S(\gamma, [a, t]) \) is the length of the path \( \gamma|_{[a,t]} \). Let \( \alpha : [a, b] \to \mathbb{R}^n \) be a rectifiable curve in \( \mathbb{R}^n \), \( n \geq 2 \), and \( l(\alpha) \) be its length. A **normal representation** \( \alpha^0 \) of \( \alpha \) is defined as a curve \( \alpha^0 : [0, l(\alpha)] \to \mathbb{R}^n \) which can be got from \( \alpha \) by change of parameter such that \( \alpha(t) = \alpha^0(S(\alpha, [a, t])) \) for every \( t \in [0, l(\alpha)] \).
Suppose that $\alpha$ and $\beta$ are curves in $\mathbb{R}^n$. Then the notation $\alpha \subset \beta$ denotes that $\alpha$ is a subpath of $\beta$. In what follows, $I$ denotes either an open or closed or semi-open interval on the real axes. The following definition can be found in the section 5 of Ch. II in [Fe].

Let $f : D \to \mathbb{R}^n$ be a mapping such that $f^{-1}(y)$ does not contain a non-degenerate curve, $\beta : I_0 \to \mathbb{R}^n$ be a closed rectifiable curve and $\alpha : I \to D$ be such that $f \circ \alpha \subset \beta$. If the length function $l_\beta : I_0 \to [0, l(\beta)]$ is a constant on $J \subset I$, then $\beta$ is a constant on $J$ and, consequently, the curve $\alpha$ to be a constant on $J$. Thus, there is a unique function $\alpha^* : I_0 \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that $\alpha^*$ to be a $f$–representation of $\alpha$ with respect to $\beta$ if $\beta = f \circ \alpha$.

**Remark 2.1.** Given a closed rectifiable curve $\gamma : [a, b] \to \mathbb{R}^n$ and $t_0 \in (a, b)$, let $l_\gamma(t)$ denote the length of the subcurve $\gamma|_{[t_0, t]}$ of $\gamma$ if $t > t_0$, $t \in (a, b)$, and $l_\gamma(t)$ is equal to the length of $\gamma|_{[t_0, t]}$ with the sign ”$-$” if $t < t_0$, $t \in (a, b)$. Then we observe that properties of the $L_{\gamma, f}$ connected with the length functions $l_\gamma(t)$ and $l_\gamma(t)$, $\tilde{\gamma} = f \circ \gamma$, do not essentially depend on the choice of $t_0 \in (a, b)$. Moreover, we may consider that in this case $t_0 = a$ because given $t_0 \in (a, b)$, $S(\gamma, [a, t]) = S(\gamma, [a, t_0]) + l_\gamma(t)$. Hence further we choose $t_0 = a$ and use the notion $l_\gamma(t)$ for the length of the path $\gamma|_{[a, t]}$ whenever a curve $\gamma$ is closed.

The following statement gives the connection between $L_{1}^{(1)}$ and $L_{q}^{(2)}$–properties and some properties of curves meaning above.

**Proposition 2.1.** (i$_1$) A mapping $f : D \to \mathbb{R}^n$ has $L_{1}^{(1)}$–property if and only if the curve $f \circ \gamma^0$ is rectifiable and absolutely continuous for $p$–a.e. closed curve $\gamma$; (i$_2$) a mapping $f : D \to \mathbb{R}^n$ has $L_{q}^{(2)}$–property if and only if $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$, and the $f$–representation $\gamma^*$ is rectifiable and absolutely continuous for $q$–a.e. closed curve $\tilde{\gamma} = f \circ \gamma$.

**Proof.** (i$_1$) First assume that $f$ has $L_{1}^{(1)}$–property. Then for $p$–a.e. curve $\gamma$ the curve $f \circ \gamma$ is locally rectifiable and, thus, $f \circ \gamma^0$ is rectifiable for $p$–a.e. closed curve $\gamma$ because $(f \circ \gamma^0)^0 = (f \circ \gamma)^0$, see Theorem 2.6 in [Va$_1$]. By $L_{1}^{(1)}$–property $L_{\gamma, f}$ has $N$–property for $p$–a.e. curve $\gamma$ in $D$ that is equivalent to absolute continuity of $L_{\gamma, f}$, see Theorem 2.10.13 in [Fe]. By definition of the length function, we have that $\alpha(t) = \alpha^0 \circ l_\alpha(t)$ for each locally rectifiable curve $\alpha$ in $\mathbb{R}^n$. Thus, in particular, for $\alpha = f \circ \gamma^0$, we obtain

$$f \circ \gamma^0(s) = (f \circ \gamma^0)^0 \circ l_{f \circ \gamma^0}(s) = (f \circ \gamma)^0 \circ l_{f \circ \gamma^0}(s) = (f \circ \gamma)^0 \circ L_{\gamma, f}(s) \quad \forall \quad s \in [0, l(\gamma)].$$

Since $L_{\gamma, f}$ is absolutely continuous and $| (f \circ \gamma^0)^0 (s_1) - (f \circ \gamma^0)^0 (s_2) | \leq | s_1 - s_2 |$ for every $s_1, s_2 \in [0, l(f \circ \gamma)]$, the curve $f \circ \gamma^0$ is absolutely continuous for $p$–a.e. curve $\gamma$.

Inversely, let us assume that the curve $f \circ \gamma^0$ is rectifiable and absolutely continuous for $p$–a.e. closed curve $\gamma$. Note that $L_{\gamma, f} = l_{f \circ \gamma^0}$ for such $\gamma$. Let $\Gamma_1$ be a family of all closed curves $\gamma$ in $D$ such that $f \circ \alpha$ either is not rectifiable or $L_{\alpha, f}$ is not absolutely continuous. Let $\Gamma$ be a family of all curves $\gamma$ in $D$ such that $f \circ \gamma$ either is not locally rectifiable or $L_{\gamma, f}$ is not locally absolutely continuous. Then $\Gamma > \Gamma_1$ and, thus, $M_p(\Gamma) \leq M_p(\Gamma_1)$, i.e., $M_p(\Gamma) = 0$. 

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we observe that curve $L_\gamma$ whenever $\tilde{\gamma} = f \circ \gamma$ because $(\gamma^*)^0 = \gamma^0$, see Theorem 2.6 in [Va1]. Moreover, we observe that $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$ because $L_{\gamma,f}$ has $N^{-1}$–property for $q$–a.e. closed curve $\tilde{\gamma}$ and all $\gamma$ with $\tilde{\gamma} = f \circ \gamma$. Thus, $L_{\gamma,f}^{-1}$ is well–defined and, for such $\gamma$ and $\tilde{\gamma}$, we have

$$\gamma^* \circ l_{\tilde{\gamma}}(t) = \gamma(t) = \gamma^0 \circ l_{\gamma}(t) = \gamma^0 \circ L_{\gamma,f}^{-1}(l_{\tilde{\gamma}}(t))$$

and, denoting $s := l_{\tilde{\gamma}}(t)$, we obtain

$$\gamma^*(s) = \gamma^0 \circ L_{\gamma,f}^{-1}(s).$$

Thus $\gamma^*$ is absolutely continuous because $L_{\gamma,f}^{-1}(s)$ is absolutely continuous, see Theorem 2.10.13 in [Fe], and

$$|\gamma^0(s_1) - \gamma^0(s_2)| \leq |s_1 - s_2|$$

for all $s_1, s_2 \in [0, l(\gamma)].$

Inversely, let us assume that $f^{-1}(y)$ does not contain a nondegenerate curve for every $y \in \mathbb{R}^n$ and the curve $\gamma^*$ is rectifiable and absolutely continuous for $q$–a.e. closed curve $\tilde{\gamma} = f \circ \gamma$. Then $L_{\gamma,f}^{-1}$ is well–defined for $q$–a.e. closed curve $\tilde{\gamma}$ and all $\gamma$ with $\tilde{\gamma} = f \circ \gamma$. By Theorem 2.6 in [Va1] $\gamma^* \circ l_{\gamma}(t) = \gamma(t) = \gamma^0 \circ l_{\gamma}(t) = \gamma^0 \circ L_{\gamma,f}^{-1}(l_{\tilde{\gamma}}(t))$.

Moreover, for all such $\tilde{\gamma}$, $\gamma$ and $\gamma^*$, $L_{\gamma,f}(s) = L_{\gamma,f}^{-1}(s)$, and absolutely continuity of $L_{\gamma,f}^{-1}(s)$ follows from Theorem 1.3 in [Va1]. Let $\Gamma_1$ be the family of all closed curves $\tilde{\alpha} = f \circ \alpha$ in $f(D)$ such that $\alpha^*$ either is not rectifiable or $L_{\alpha, f}^{-1}(s)$ is not absolutely continuous. By the assumption $M_q(\Gamma_1) = 0$. Let $\Gamma$ be a family of all curves $\tilde{\gamma} = f \circ \gamma$ in $f(D)$ such that $\gamma$ either is not locally rectifiable or $L_{\gamma,f}^{-1}(s)$ is not locally absolutely continuous. Then $\Gamma > \Gamma_1$ and, thus, $M_q(\Gamma) \leq M_q(\Gamma_1)$ that implies the desired equality $M_q(\Gamma) = 0.$

A mapping $\varphi : X \to Y$ between metric spaces $X$ and $Y$ is said to be a Lipschitzian if

$$\text{dist} \left( \varphi(x_1), \varphi(x_2) \right) \leq M \cdot \text{dist}(x_1, x_2)$$

for some $M < \infty$ and for all $x_1$ and $x_2 \in X$. The mapping $\varphi$ is called bi–lipschitz if, in addition,

$$M^* \text{dist} \left( x_1, x_2 \right) \leq \text{dist} \left( \varphi(x_1), \varphi(x_2) \right)$$

for some $M^* > 0$ and for all $x_1$ and $x_2 \in X$. In what follows, $X$ and $Y$ are subsets of $\mathbb{R}^n$ with the Euclidean distance.

The following proposition can be found in [MRSY1], see Lemma 3.20, see also Lemma 8.3 in [MRSY2].

**Proposition 2.2.** Let $f : D \to \mathbb{R}^n$ be differentiable a.e. in $D$ and have $N$– and $N^{-1}$–properties. Then there is a countable collection of compact sets $C_k^* \subset D$ such that $m(B_0) = 0$ where $B_0 = D \setminus \bigcup_{k=1}^{\infty} C_k^*$ and $f|C_k^*$ is one–to–one and bi–lipschitz for every $k = 1, 2, \ldots$, and, moreover, $f$ is differentiable at points of $C_k^*$ with $J(x, f) \neq 0$. 


Given a set $E$ in $\mathbb{R}^n$ and a curve $\gamma: \Delta \to \mathbb{R}^n$, we identify $\gamma \cap E$ with $\gamma(\Delta) \cap E$. If $\gamma$ is locally rectifiable, then we set

$$l(\gamma \cap E) = m_1(E_\gamma),$$

where $E_\gamma = l_\gamma(\gamma^{-1}(E))$; here $l_\gamma: \Delta \to \Delta_\gamma$ as in the previous section. Note that $E_\gamma = \gamma_0^{-1}(E)$, where $\gamma_0: \Delta_\gamma \to \mathbb{R}^n$ is the natural parametrization of $\gamma$ and

$$l(\gamma \cap E) = \int_\Delta \chi_E(\gamma(t)) |dx| = \int_{\Delta_\gamma} \chi_{E_\gamma}(s) ds.$$

The following statement can be found in [MRSY], see Theorem 9.1 for $k = 1$.

**Proposition 2.3.** Let $E$ be a set in a domain $D \subset \mathbb{R}^n$, $n \geq 2$, $p \geq 1$. Then $E$ is measurable if and only if $\gamma \cap E$ is measurable for $p$–a.e. curve $\gamma$ in $D$. Moreover, $m(E) = 0$ if and only if

$$l(\gamma \cap E) = 0$$

on $p$–a.e. curve $\gamma$ in $D$.

The following result was proved in [MRSY]$^1$–[MRSY]$^2$ for a case $p = n$. Here we extend the study of this problem to arbitrary $p$, $p \geq 1$.

**Theorem 2.1.** Let $f: D \to \mathbb{R}^n$ be differentiable a.e. in $D$, have $N$ – and $N^{-1}$–properties and $L_\rho^{(1)}$–property. Then relation (1.2) holds for every measurable set $E \subset D$, every family $\gamma \in E$ of curves $\gamma$ in $E$ and any $\rho_*(y) \in \text{adm} f(\Gamma)$, where $K_{I,p}(y, f^{-1}, E)$ is defined by (1.3).

**Proof.** By Theorem III.6.6 $(iV)$ [Sa], $E = B \cup B_0$, where $B$ is a set of the class $F_\sigma$, and $m(B_0) = 0$. Consequently, $f(E)$ is measurable by $N$–property of $f$. Without loss of generality, we may assume that $f(E)$ is a Borel set and that $\rho_* \equiv 0$ outside of $f(E)$. In other case we can find a Borel set $G$ such that $f(E) \subset G$ and $m(G \setminus f(E)) = 0$, see (ii) of the Theorem III.6.6 in [Sa]. Now, a set $f^{-1}(G)$ is Borel and $E \subset f^{-1}(G)$. Note that in this case the function

$$\rho_*^G(y) = \begin{cases} \rho_*(y), & \text{for } y \in G, \\ 0, & y \in \mathbb{R}^n \setminus G \end{cases}$$

is a Borel function, as well. Now suppose that $f(E)$ is a Borel set. Let $B_0$ and $C^*_k$, $k = 1, 2, \ldots,$ be as in Proposition 2.2. Setting by induction $B_1 = C^*_1$, $B_2 = C^*_2 \setminus B_1$, $\ldots$, and

$$B_k = C^*_k \setminus \bigcup_{l=1}^{k-1} B_l$$

we obtain a countable covering of $D$ consisting of mutually disjoint Borel sets $B_k, k = 0, 1, 2, \ldots$ with $m(B_0) = 0, B_0 = D \setminus \bigcup_{k=1}^{\infty} B_k$. Remark that $\gamma^0(s) \notin B_0$ for a.e. $s$ and $p$–a.e. closed curve $\gamma \in \Gamma$, see Proposition 2.3; here $\gamma^0(s)$ denotes a normal representation of $\gamma$. By Proposition 2.1 a curve $f \circ \gamma^0$ is rectifiable and absolutely continuous for $p$–a.e. $\gamma \in \Gamma$. 
Let \(\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}\). Given \(\rho_* \in \text{adm } f(\Gamma)\), set
\[
\rho(x) = \begin{cases} 
\rho_*(f(x))\|f'(x)\|, & \text{for } x \in D \setminus B_0, \\
0, & \text{otherwise}.
\end{cases}
\]

By Theorem 5.3 in [Va] (see also Lemma II.2.2 in [Ri]) we obtain that
\[
\int_{\gamma} \rho(x)|dx| = \int_{\gamma} \rho_*(f(x))\|f'(x)\||dx| \geq \int_{f \circ \gamma} \rho_*(y)|dy| \geq 1
\]
for \(p\)-a.e. closed \(\gamma \in \Gamma\), i.e., \(\rho \in \text{adm } \Gamma\). The case of arbitrary \(\alpha \in \Gamma\) can be gotten from (2.2) by taking of sup over all closed subpaths \(\gamma \subset \alpha\). Therefore,
\[
M_p(\Gamma) \leq \int_D \rho^p(x)dm(x).
\]

Note that \(\rho = \sum_{k=1}^{\infty} \rho_k\), where \(\rho_k = \rho \cdot \chi_{B_k}\) have mutually disjoint supports. By 3.2.5 for \(m = n\) in [Fe] we obtain that
\[
\int_{f(B_k \cap E)} K_{O,p} \left(f_k^{-1}(y), f\right) \cdot \rho_k^p(y) dm(y) = \int_{B_k} K_{O,p}(x, f) \rho_k^p (f(x)) |J(x, f)|dm(x) =
\]
\[
= \int_{B_k} \|f'(x)\|^p \rho_k^p (f(x)) dm(x) = \int_D \rho_k^p(x)dm(x),
\]
where every \(f_k = |f|_{B_k}\), \(k = 1, 2, \ldots\) is injective by the construction.

Finally, by the Lebesgue positive convergence theorem, see e.g. Theorem I.12.3 in [Sa], we conclude from (2.3) and (2.4) that
\[
\int_{f(E)} K_{L,p}(y, f^{-1}, E) \cdot \rho_k^p(y)dm(y) = \int_D \sum_{k=1}^{\infty} \rho_k^p(x)dm(x) \geq M_p(\Gamma). \quad \Box
\]

The following result is a generalization of the known Poletskii inequality for quasiregular mappings, see Theorem 1 in [Pol] and Theorem II.8.1 in [Ri]. Its analog was also proved in [MRSY1]–[MRSY2] for the case \(q = n\).

**Theorem 2.2.** Let a mapping \(f : D \to \mathbb{R}^n\) be differentiable a.e. in \(D\), have \(N\)– and \(N^{-1}\)–properties, and \(L_q^{(2)}\)–property, too. Then relation (1.1) holds for every curve family \(\Gamma\) in \(D\) and any function \(\rho \in \text{adm } \Gamma\).
Proof. Let \( B_k, k = 0, 1, 2, \ldots \), be given as above by (2.1). By the assumption, \( f \) has \( N \)-property in \( D \) and, consequently, \( m(f(B_0)) = 0 \). Let \( \rho \in \text{adm}\Gamma \) and

\[
\tilde{\rho}(y) = \chi_{f(D \setminus B_0)} \cdot \sup_{x \in f^{-1}(y) \cap D \setminus B_0} \rho^*(x),
\]

where

\[
\rho^*(x) = \begin{cases} 
\rho(x)/l(f'(x)), & \text{for } x \in D \setminus B_0, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that \( \tilde{\rho}(y) = \sup_{k \in \mathbb{N}} \rho_k(y) \), where

\[
\rho_k(y) = \begin{cases} 
\rho^*(f_k^{-1}(y)), & \text{for } y \in f(B_k), \\
0, & \text{otherwise},
\end{cases}
\]

and every \( f_k = f|_{B_k}, k = 1, 2, \ldots \), is injective. Thus, the function \( \tilde{\rho} \) is Borel, see section 2.3.2 in \([Fe]\).

Let \( \tilde{\gamma} \) be a closed rectifiable curve such that \( \tilde{\gamma} = f \circ \gamma, \tilde{\gamma}^0 \) be a normal representation of \( \tilde{\gamma} \) and \( \gamma^* \) be \( f \)-representation of \( \gamma \) by the respect to \( \tilde{\gamma} \), see above. Since \( m(f(B_0)) = 0 \), \( \tilde{\gamma}^0(s) \notin f(B_0) \) for \( q \)-a.e. curve \( \tilde{\gamma} \) and for a.e. \( s \in [0, l(\tilde{\gamma})] \), see Proposition 2.3. For \( q \)-a.e. paths \( \tilde{\gamma} \) and all \( \gamma \) with \( \tilde{\gamma} = f \circ \gamma \) we have

\[
\int_{\tilde{\gamma}} \tilde{\rho}(y)|dy| = \int_0^{l(\tilde{\gamma})} \tilde{\rho}(\tilde{\gamma}^0(s))\, ds = \int_0^{l(\tilde{\gamma})} \sup_{x \in f^{-1}(\tilde{\gamma}^0(s)) \cap D \setminus B_0} \rho^*(x)\, ds \geq \int_0^{l(\tilde{\gamma})} \frac{\rho(\gamma^*(s))}{l(f'(\gamma^*(s)))}\, ds.
\]

(2.5)

Since \( \tilde{\gamma}^0 \) is rectifiable, \( \tilde{\gamma}^0(s) \) is differentiable a.e. Besides that, a curve \( \gamma^* \) is absolutely continuous for \( q \)-a.e. \( \tilde{\gamma} \) by Proposition 2.1. Since \( \tilde{\gamma}^0(s) \notin f(B_0) \) for a.e. \( s \in [0, l(\tilde{\gamma})] \) and \( q \)-a.e. curve \( \tilde{\gamma} \), we have \( \gamma^*(s) \notin B_0 \) for a.e. \( s \in [0, l(\tilde{\gamma})] \). Thus, the derivatives \( f'(\gamma^*(s)) \) and \( \gamma''(s) \) exist for a.e. \( s \). Taking into account the formula of the derivative of the superposition of functions, and that the modulus of the derivative of the curve by the natural parameter equals to 1, we have that

\[
1 = |(f \circ \gamma^*)'(s)| = |f'(\gamma^*(s)) \gamma''(s)| = |f'(\gamma^*(s))| \cdot \frac{\gamma''(s)}{|\gamma''(s)|} \cdot |\gamma''(s)| \geq l(f'(\gamma^*(s))) \cdot |\gamma''(s)|.
\]

(2.6)

It follows from (2.6) that a.e.

\[
\frac{\rho(\gamma^*(s))}{l(f'(\gamma^*(s)))} \geq \rho(\gamma^*(s)) \cdot |\gamma''(s)|.
\]

(2.7)
By absolute continuity of $\gamma^*$, definition of $\rho$ and Theorem 4.1 in [Vä1] we obtain

$$1 \leq \int_\gamma \rho(x)|dx| = \int_0^{l(\tilde{\gamma})} \rho(\gamma^*(s)) \cdot |\gamma^{**}(s)| \, ds. \quad (2.8)$$

It follows from (2.5), (2.7) and (2.8) that $\int \tilde{\rho}(y)|dy| \geq 1$ for $q$–a.e. closed curve $\tilde{\gamma}$ in $f(\Gamma)$.

The case of the arbitrary path $\tilde{\gamma}$ can be got from the taking of $\sup$ in $\int \tilde{\rho}(y)|dy| \geq 1$ over all closed subpaths $\tilde{\gamma}'$ of $\tilde{\gamma}$. Thus, $\tilde{\rho}(y) \in \text{adm} f(\Gamma) \setminus \Gamma_0$, where $M_q(\Gamma_0) = 0$. Hence

$$M_q(f(\Gamma)) \leq \int_{f(D)} \tilde{\rho}^q(y) dm(y). \quad (2.9)$$

Further, by 3.2.5 for $m = n$ in [Fe] we have that

$$\int_{B_k} K_{I,q}(x,f) \cdot \rho^q(x) dm(x) = \int_{B_k} \frac{|J(x,f)|}{(l(f'(x)))^q} \cdot \rho^q(x) dm(x) =$$

$$= \int_{f(B_k)} \rho^q \left( f_k^{-1}(y) \right) \frac{1}{(l(f'(f_k^{-1}(y))))^q} dm(y) = \int_{f(D)} \rho_k^q(y) dm(y). \quad (2.10)$$

Finally, by the Lebesgue theorem, see Theorem 12.3 § 12 of Ch. I in [Sa], we obtain from (2.9) and (2.10) the desired inequality

$$\int_D K_{I,q}(x,f) \cdot \rho^q(x) dm(x) = \sum_{k=1}^{\infty} \int_{B_k} K_{I,q}(x,f) \cdot \rho^q(x) dm(x) =$$

$$= \int_{f(D)} \sum_{k=1}^{\infty} \rho_k^q(y) dm(y) \geq \int_{f(D)} \sup_{k \in \mathbb{N}} \rho_k^q(y) dm(y) =$$

$$= \int_{f(D)} \tilde{\rho}^q(y) dm(y) \geq M_q(f(\Gamma)). \quad \square$$

The proof of Theorem [L.7] directly follows from Theorems [2.1] and [2.2]. \square

3 The analog of the Väisälä inequality

The following result generalizes the well–known Väisälä inequality for the mappings with bounded distortion, see § 9 of Ch. II in [Ri] and Theorem 3.1 in [Vä3]; see also Theorem 4.1 in [KO].

**Theorem 3.1.** Let $f : D \to \mathbb{R}^n$ be differentiable a.e. in $D$, have $N$– and $N^{-1}$–properties, and $L^2_q$–property. Let $\Gamma$ be a curve family in $D$, $\Gamma'$ be a curve family in $\mathbb{R}^n$ and $m$ be a positive integer such that the following is true. Suppose that for every curve $\beta : I \to D$ in
there are curves \( \alpha_1, \ldots, \alpha_m \) in \( \Gamma \) such that \( f \circ \alpha_j \subset \beta \) for all \( j = 1, \ldots, m, \) and for every \( x \in D \) and all \( t \in I \) the equality \( \alpha_j(t) = x \) holds at most \( i(x, f) \) indices \( j. \) Then

\[
M_q(\Gamma') \leq \frac{1}{m} \int_D K_{I,q}(x, f) \cdot \rho^q(x) \, dm(x)
\]

for every \( \rho \in \text{adm} \Gamma. \)

**Proof.** Let \( B_0 \) and \( C_k^* \) be as in Proposition 2.2. Setting by induction \( B_1 = C_1^*, B_2 = C_2^* \setminus B_1 \ldots, \)

\[
B_k = C_k^* \setminus \bigcup_{t=1}^{k-1} B_t,
\]

we obtain a countable covering of \( D \) consisting of mutually disjoint Borel sets \( B_k, k = 1, 2, \ldots, \) with \( m(B_0) = 0, B_0 := D \setminus \bigcup_{k=1}^{\infty} B_k. \) By the construction and \( N\)–property, \( m(f(B_0)) = 0. \)

Thus \( \tilde{\gamma}^0(s) \not\in f(B_0) \) for a.e. \( s \) and \( q\)–a.e. closed curves \( \tilde{\gamma} \) in \( f(D) \) by Proposition 2.3; here \( \tilde{\gamma}^0(s) \) is a normal representation of \( \tilde{\gamma}(s). \) Besides that a curve \( \gamma^* \), which is the \( f\)–representation of \( \gamma, \) is absolutely continuous for \( q\)–a.e. \( \tilde{\gamma} = f \circ \gamma. \) Here the \( f\)–representation \( \gamma^* \) of \( \gamma \) is well–defined for \( q\)–a.e. curves \( \tilde{\gamma} = f \circ \gamma \) (see Proposition 2.1).

Now let \( \Gamma_1 \) be a family of all (locally rectifiable) curves \( \gamma_1 \in \Gamma' \) for which there exists a closed subcurve \( \beta_1, \beta_1 = f \circ \alpha_1, \) such that the \( f\)–representation \( \alpha_1^* \) of \( \alpha_1 \) either is not rectifiable or is not absolutely continuous. Denote by \( \Gamma_2 \) the family of all closed curves \( \gamma_2, \gamma_2 = f \circ \alpha_2, \) such that the \( f\)–representation \( \alpha_2^* \) of \( \alpha_2 \) either is not rectifiable or is not absolutely continuous. We have proved that \( M_q(\Gamma_2) = 0. \) On the other hand \( \Gamma_1 > \Gamma_2, \) consequently, \( M_q(\Gamma_1) \leq M_q(\Gamma_2) = 0. \)

Let \( \rho \in \text{adm} \Gamma \) and

\[
\tilde{\rho}(y) = \frac{1}{m} \cdot \chi_{f(D \setminus B_0)}(y) \sup_{C} \sum_{x \in C} \rho^*(x), \tag{3.1}
\]

where

\[
\rho^*(x) = \begin{cases} 
\rho(x)/l(f'(x)), & x \in D \setminus B_0, \\
0, & x \in B_0, 
\end{cases}
\]

and \( C \) runs over all subsets of \( f^{-1}(y) \) in \( D \setminus B_0 \) such that \( \text{card} \, C \leq m. \) Note that

\[
\tilde{\rho}(y) = \frac{1}{m} \cdot \sup_{s} \sum_{i=1}^{s} \rho_{k_i}(y), \tag{3.2}
\]

where \( \sup \) in 3.2 is taken over all \( \{k_i, \ldots, k_s\} \) such that \( k_i \in \mathbb{N}, \) \( k_i \neq k_j \) if \( i \neq j, \) all \( s \leq m \) and

\[
\rho_{k_i}(y) = \begin{cases} 
\rho^* \left( f_{k_i}^{-1}(y) \right), & y \in f(B_k), \\
0, & y \not\in f(B_k)
\end{cases}
\]

where \( f_k = f|B_k, k = 1, 2, \ldots \) is injective and \( f(B_k) \) is Borel. Thus, the function \( \tilde{\rho}(y) \) is Borel, see e.g. 2.3.2 in [Fe].
Suppose that $\beta$ is a curve in $\Gamma'$. There exist curves $\alpha_1, \ldots, \alpha_m$ in $\Gamma$ such that $f \circ \alpha_j \subset \beta$ and for all $x \in D$ and $t$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices $j$. We show that \( \tilde{\rho} \in \text{adm} \Gamma' \setminus \Gamma_0 \), where $M_0(\Gamma_0) = 0$. Without loss of generality we may consider that all of the curves $\beta$ of $\Gamma'$ are locally rectifiable, see Section 6 in [Vaj]. p. 18. Now we suppose that $\beta$ is closed. We may consider that $\beta^0(t) \notin f(B_0)$ for a.e. $t \in [0, l(\beta)]$, where $\beta^0: [0, l(\beta)] \rightarrow \mathbb{R}^n$ is a normal representation of $\beta$, i.e., $\beta(t) = \beta^0 \circ l_\beta(t)$. Denote by $\alpha_j^*(t) : J_j \rightarrow D$ the corresponding $f$–representation of $\alpha_j$ with respect to $\beta$, i.e., $\alpha_j(t) = \alpha_j^* \circ l_\beta(t)$, $f \circ \alpha_j^* \subset \beta^0$. Let

$$h_j(t) = \rho^* \left( \alpha_j^*(t) \right) \chi_{I_j} (t), \quad t \in [0, l(\beta)], \quad J_j := \{ j : t \in I_j \}.$$ 

Since $\beta^0(t) \notin f(B_0)$ for a.e. $t \in [0, l(\beta)]$, the points $\alpha_j^*(t) \in f^{-1}(\beta^0(t))$, $j \in J_t$, are distinct for a.e. $t$. By the definition of $\tilde{\rho}$ in (3.1),

$$\tilde{\rho}(\beta^0(t)) \geq \frac{1}{m} \cdot \sum_{j=1}^{m} h_j(t)$$

(3.3)

for a.e. $t \in [0, l(\beta)]$. By (3.3) we have that

$$\int_{\beta} \tilde{\rho}(y)dy = \int_{0}^{l(\beta)} \tilde{\rho}(\beta^0(t))dt \geq$$

$$\geq \frac{1}{m} \cdot \sum_{j=1}^{m} \int_{0}^{l(\beta)} h_j(t)dt = \frac{1}{m} \cdot \sum_{j=1}^{m} \int_{I_j} \rho^* \left( \alpha_j^*(t) \right) dm_1(t).$$

Now we show that

$$\int_{I_j} \rho^* \left( \alpha_j^*(t) \right) dm_1(t) \geq 1$$

(3.4)

for $q$–a.e. curve $\beta \in \Gamma'$. Since $\beta^0(t)$ is rectifiable, $\beta^0(t)$ is differentiable for a.e. $t \in I$. Besides that, the curve $\alpha_j^*$ from the $f$–representation of $\beta$ is absolutely continuous for $q$–a.e. $\beta$ by Proposition 2.1. Since $\beta^0(t) \notin f(B_0)$ for a.e. $t \in [0, l(\beta)]$, we have $\alpha_j^*(t) \notin B_0$ at a.e. $t \in I_j$. Thus, the derivatives $f'(\alpha_j^*(t))$ and $\alpha_j''(t)$ exist for a.e. $t \in I_j$. Taking into account the formula of the derivative of the superposition of functions, and that the modulus of the derivative of the curve by the natural parameter equals 1, we have

$$1 = \left| (f \circ \alpha_j^*)'(t) \right| = \left| f'(\alpha_j^*(t)) \alpha_j''(t) \right| =$$

$$= \left| f'(\alpha_j^*(t)) \cdot \frac{\alpha_j''(t)}{|\alpha_j''(t)|} \right| \cdot |\alpha_j''(t)| \geq l \left( f'(\alpha_j^*(t)) \right) \cdot |\alpha_j''(t)|. \quad (3.5)$$

It follows from (3.5) that

$$\rho^* (\alpha_j^*(t)) = \frac{\rho(\alpha_j^*(t))}{l \left( f'(\alpha_j^*(t)) \right)} \geq \rho(\alpha_j^*(t)) \cdot |\alpha_j''(t)|. \quad (3.6)$$
From (3.6) by absolutely continuously of \( \alpha^*_j \), definition of \( \rho \) and Theorem 4.1 in [Va1] we have
\[
1 \leq \int_{\alpha_j} \rho(x) \, dx = \int_{I_j} \rho \left( \alpha^*_j(t) \right) \cdot |\alpha^*_j(t)| \, dm_1(t) \leq \int_{I_j} \rho^* \left( \alpha^*_j(t) \right) \, dm_1(t). \tag{3.7}
\]

Now inequality (3.4) directly follows from (3.7). Next we have that
\[
\int_{\alpha} \rho(y) \, dy = 1 \quad \text{for each } \alpha.
\]

From (3.6) by absolutely continuously of \( \alpha^*_j \), definition of \( \rho \) and Theorem 4.1 in [Va1] we have
\[
1 \leq \int_{\alpha_j} \rho(x) \, dx = \int_{I_j} \rho \left( \alpha^*_j(t) \right) \cdot |\alpha^*_j(t)| \, dm_1(t) \leq \int_{I_j} \rho^* \left( \alpha^*_j(t) \right) \, dm_1(t). \tag{3.7}
\]

Now inequality (3.4) directly follows from (3.7). Next we have that \( \int_{\beta} \rho(y) \, dy \geq 1 \) for \( q \)-a.e. closed curve \( \beta \) of \( \Gamma' \). The case of the arbitrary curve \( \beta \) can be got from the taking of sup in \( \int_{\beta} \rho(y) \, dy \geq 1 \) over all closed subcurves \( \beta' \) of \( \beta \). Thus, \( \rho \in \text{adm } \Gamma' \setminus \Gamma_0 \), where \( M_q(\Gamma_0) = 0 \). Hence
\[
M_q(\Gamma') \leq \int_{f(D)} \rho^q(y) \, dm(y). \tag{3.8}
\]

By 3.2.5 for \( m = n \) in [Fe], we obtain that
\[
\int_{B_k} K_{I,q}(x, f) \cdot \rho^q(x) \, dm(x) = \int_{f(D)} \rho^q_k(y) \, dm(y). \tag{3.9}
\]

By Hölder inequality for series,
\[
\left( \frac{1}{m} \cdot \sum_{i=1}^s \rho_k,(y) \right)^q \leq \frac{1}{m} \cdot \sum_{i=1}^s \rho_k^q(y) \tag{3.10}
\]
for each \( 1 \leq s \leq m \) and every \( k_1, \ldots, k_s, k_i \in \mathbb{N}, i = 1, 2, \ldots, k_i \neq k_j \) if \( i \neq j \).

Finally, by Lebesgue positive convergence theorem, see Theorem I.12.3 in [Sa], we conclude from (3.8)–(3.10) that
\[
\frac{1}{m} \cdot \int_{D} K_{I,q}(x, f) \cdot \rho^q(x) \, dm(x) = \frac{1}{m} \cdot \int_{f(D)} \sum_{k=1}^\infty \rho_k^q(y) \, dm(y) \geq \frac{1}{m} \cdot \int_{f(D)} \sup_{k_i \neq k_j \text{ if } i \neq j} \sum_{i=1}^s \rho_k^q(y) \, dm(y) \geq \int_{f(D)} \rho^q(y) \, dm(y) \geq M_q(\Gamma').
\]
The proof is complete. \( \square \)

4 Applications

Let \( f : D \to \mathbb{R}^n \) be a discrete open mapping, \( \beta : [a, b] \to \mathbb{R}^n \) be a curve and \( x \in f^{-1}(\beta(a)) \).

A curve \( \alpha : [a, c] \to D \) is called a maximal \( f \)-lifting of \( \beta \) starting at \( x \), if

(1) \( \alpha(a) = x \);

(2) \( f \circ \alpha = \beta|_{[a, c]} \);

(3) if \( c < c' \leq b \), then there is no curve \( \alpha' : [a, c'] \to D \) such that \( \alpha = \alpha'|_{[a, c]} \) and \( f \circ \alpha' = \beta|_{[a, c']} \). By assumption on \( f \) one implies that every curve \( \beta \) with \( x \in f^{-1}(\beta(a)) \) has a maximal \( f \)-lifting starting at the point \( x \), see Corollary II.3.3 in [Ri].

Let \( x_1, \ldots, x_k \) be \( k \) different points of \( f^{-1}(\beta(a)) \) and let
\[
m = \sum_{i=1}^k i(x_i, f).
\]
We say that the sequence $\alpha_1, \ldots, \alpha_m$ is a maximal sequence of $f$–liftings of $\beta$ starting at points $x_1, \ldots, x_k$, if

(a) each $\alpha_j$ is a maximal $f$–lifting of $\beta$,
(b) card $\{ j : a_j(a) = x_i \} = i(x_i, f), \ 1 \leq i \leq k$,
(c) card $\{ j : a_j(t) = x \} \leq i(x, f)$ for all $x \in D$ and for all $t$.

Let $f$ be a discrete open mapping and $x_1, \ldots, x_k$ be distinct points in $f^{-1}(\beta(a))$. Then $\beta$ has a maximal sequence of $f$–liftings starting at the points $x_1, \ldots, x_k$, see Theorem II.3.2 in [3].

A domain $G \subset D, \overline{G} \subset D$, is said to be a normal domain of $f$, if $\partial f(G) = f(\partial G)$. If $G$ is a normal domain, then $\mu(y, f, G)$ is a constant for $y \in f(G)$. This constant will be denoted by $\mu(f, G)$. Let $f : D \to \mathbb{R}^n$ be a discrete open mapping, then $\mu(f, G) = N(f, G)$ for every normal domain $G \subset D$, see e.g. Proposition I.4.10 in [3]. We need bellow the following statement, see Corollary II.3.4 in [3].

**Lemma 4.1.** Let $f : G \to \mathbb{R}^n$ be a discrete open mapping, $G$ be a normal domain, $m = N(f, G)$, $\beta : [a,b) \to f(G)$ be a curve. Then there exist curves $\alpha_j : [a,b) \to G, 1 \leq j \leq m$, such that:

1) $f \circ \alpha_j = \beta$,
2) card $\{ j : \alpha_j(t) = x \} = i(x, f)$ for $x \in G \cap f^{-1}(\beta(t))$,
3) $|\alpha_1| \cup \ldots \cup |\alpha_m| = G \cap f^{-1}(|\beta|)$.

We also adopt the following conventions. Given $q \geq 1$, a family of curves $\Gamma$ in $\mathbb{R}^n$, denote

$$M_{q,K_{I,q}}(\cdot,f)(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \rho^q(x) K_{I,q}(x,f) \, dm(x).$$

The following result holds.

**Corollary 4.1.** Let $f : D \to \mathbb{R}^n$ be an open discrete mapping, which is differentiable a.e. in $D$, have $N^-$ and $N^{-1}$–properties, and $L^{(2)}_q$–property. Suppose that $G$ is a normal domain for $f$, $\Gamma'$ is a curve family in $G' = f(G)$ and $\Gamma$ is a family consisting of all curves $\alpha$ in $G$ such that $f \circ \alpha \subset \Gamma'$ and $m = N(f, G)$. Then

$$M_q(\Gamma') \leq \frac{1}{N(f,G)} \int_{G} K_{I,q}(x,f) \cdot \rho^q(x) \, dm(x)$$

for every $\rho \in \text{adm } \Gamma$. Moreover,

$$M_q(\Gamma') \leq \frac{1}{N(f,G)} \, M_{q,K_{I,q}}(\cdot,f)(\Gamma).$$

The proof directly follows from Theorem 3.1 and Lemma 4.1. □

Following section II.10 in [3], a condenser is a pair $E = (A, C)$ where $A \subset \mathbb{R}^n$ is open and $C$ is non–empty compact set contained in $A$. A condenser $E = (A, C)$ is said to be in a
domain $G$ if $A \subset G$. For a given condenser $E = (A, C)$ and $q \geq 1$, we set

$$\text{cap}_q E = \text{cap}_q (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^q dm(x)$$

where $W_0(E) = W_0(A, C)$ is the family of all non-negative functions $u : A \to \mathbb{R}^1$ such that (1) $u \in C_0(A)$, (2) $u(x) \geq 1$ for $x \in C$, and (3) $u$ is ACL. In the above formula $|\nabla u| = \left( \sum_{i=1}^n (\partial_i u)^2 \right)^{1/2}$, and $\text{cap}_q E$ is called $q$–capacity of the condenser $E$, see Section II.10 in [Ri].

Let $E = (A, C)$ be a condenser and $\omega$ be a nonnegative measurable function. We define the $\omega$–weighted capacity of $E$ by setting

$$\text{cap}_{q, \omega} E = \text{cap}_{q, \omega} (A, C) = \inf \int_A |\nabla u(x)|^q \omega(x) dm(x) , \quad (4.1)$$

where $\inf$ in (4.1) is taken over all functions $u \in C_0^\infty(A)$ and $u \geq 1$ on $C$.

Given a mapping $f : D \to \mathbb{R}^n$ and a condenser $E = (A, C)$, we call

$$M(f, C) = \inf_{y \in f(C)} \sum_{x \in f^{-1}(y) \cap C} i(x, f)$$

the minimal multiplicity of $f$ on $C$.

We need the following statement, see Proposition II.10.2 in [Ri].

**Lemma 4.2.** Let $E = (A, C)$ be a condenser in $\mathbb{R}^n$ and let $\Gamma_E$ be the family of all curves of the form $\gamma : [a, b] \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \emptyset$ for every compact $F \subset A$. Then $\text{cap}_q E = M_q (\Gamma_E)$.

**Theorem 4.1.** Let $f : D \to \mathbb{R}^n$ be an open discrete mapping, which is differentiable a.e. in $D$, have $N$– and $N^{-1}$–properties, and $L_q^{(2)}$–property. Suppose that $E = (A, C)$ is a condenser in $D$. Then

$$\text{cap}_q f(E) \leq \frac{1}{M(f, C)} \text{cap}_{q, K_{i, q}(\cdot, f)} E . \quad (4.2)$$

**Proof.** Since $E = (A, C)$ is a condenser in $D$, then $f(E) = (f(A), f(C))$ is a condenser in $f(D)$. Let $\Gamma_E$ and $\Gamma_{f(E)}$ be curve families such as in Lemma 4.2. Set $m = M(f, C).$ Let $\beta : [a, b] \to f(A)$ be a curve in $\Gamma_{f(E)}$. Then $C \cap f^{-1}(\beta(a))$ contains points $x_1, \ldots, x_k$ such that

$$m' = \sum_{l=1}^k i(x_l, f) \geq m .$$

By Theorem II.3.2 in [Ri], there is a maximal sequence of $f|A$–liftings $\alpha_j : [a, c_j] \to D$ of $\beta$, $1 \leq j \leq m'$, starting at the points $x_1, \ldots, x_k$. Then each $\alpha_j$ belongs to $\Gamma_E$. It follows that $\Gamma = \Gamma_E$ and $\Gamma' = \Gamma_{f(E)}$ satisfy the Theorem 3.1. Hence by Lemma 4.2

$$\text{cap}_q f(E) \leq \frac{1}{M(f, C)} \text{M}_{q, K_{i, q}(\cdot, f)}(\Gamma_E). \quad (4.3)$$
Finally, (4.2) follows from (4.3) because
\[ M_{q,K} (\cdot, f) (\Gamma_E) \leq \text{cap}_{q,K} (\cdot, f) E, \]
as it is easily seen by considering \( \rho(x) = |\nabla u(x)| \) for a given test function \( u \) in \( \text{cap}_{q,K} (\cdot, f) E \).

In fact, let \( \gamma \in \Gamma_E \) be a locally rectifiable curve, \( s \) be a natural parameter on \( \gamma \), \( \gamma^0 \) be a normal representation of \( \gamma \) and let \( s_0 \in (0, l(\gamma)) \), where \( l(\gamma) \) denotes the length of the curve \( \gamma \). Using a geometrical sense of the gradient, for the function \( \rho(x) = |\nabla u(x)| \), we have
\[
\int_{\gamma} \rho(x) \, dx = \int_{\gamma} |\nabla u(x)| \, dx = \int_{0}^{l(\gamma)} |\nabla u(\gamma^0(s))| \, ds \geq \int_{0}^{l(\gamma)} \left| \frac{du(\gamma^0(s))}{ds} \right| \, ds \geq \int_{0}^{s_0} \left| \frac{du(\gamma^0(s))}{ds} \right| \, ds \geq \int_{0}^{s_0} \left| \frac{du(\gamma^0(s))}{ds} \right| \, ds = |u(\gamma^0(s_0)) - u(\gamma^0(0))| \rightarrow |u(\gamma^0(l(\gamma))) - u(\gamma^0(0))| = 1
\]
as \( s_0 \to l(\gamma) \). Here we have used that \( r(s) = u(\gamma^0(s)) \) is absolutely continuous by parameter \( s \) for every rectifiable closed curve \( \gamma \) in \( D \) because a function \( u \in C_0^\infty (A) \) is locally Lipschitzian. Thus, \( \rho(x) = |\nabla u(x)| \in \text{adm} \Gamma_E \), and we have the desired conclusion (4.2).

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