Continuous variable quantum teleportation with a finite-basis entanglement resource

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Entanglement is a crucial resource in quantum information theory. We investigate the use of different forms of entangled states in continuous variable quantum teleportation, specifically the use of a finite-basis entanglement resource. We also consider the continuous variable teleportation of finite-basis states, such as qubits, and present results that point to the possibility of an efficient conditional scheme for continuous variable teleportation of such states with near-unit fidelity using finite-basis entanglement.

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Quantum teleportation was first proposed by Bennett et al. [1] primarily for quantum states of finite-dimensional systems (e.g., two-state systems). Such “finite-basis” teleportation requires joint measurement in the Bell-operator basis of the system whose state is to be teleported and one component of an entangled bipartite system. This necessitates nonlinear quantum-quantum interactions and the ability to distinguish between $N^2$ states for an $N$-dimensional system [2, 3]. These are difficult and demanding requirements which have limited the efficiency (or fidelity) of teleportation achieved to date in experimental demonstrations involving the polarization states of single photons [4, 5, 6].

The procedure for teleportation has also been extended to continuous variable (i.e., infinite-dimensional) systems [7, 8] and a specific proposal for quadrature phase amplitudes of the electromagnetic field was put forward [8] and then demonstrated experimentally [9]. In this case, one requires (two-mode) squeezed states, linear mixers (i.e., beamsplitters), and quadrature phase measurements, all of which are well-established in the field of quantum optics and can be produced or implemented with reasonably high efficiency. Of course, any quantum state can in principle be teleported using the continuous variable quantum teleportation (CVQT) scheme, so it is clear that there could be some advantage to using this scheme for the teleportation of finite-basis states (such as qubit states) [10], in comparison to the original finite-basis teleportation scheme of Bennett et al. [1]. The states of finite-dimensional systems could be encoded straightforwardly onto number (Fock) states of modes of the electromagnetic field. However, perfect CVQT requires perfect entanglement between EPR (Einstein-Podolsky-Rosen) correlated modes, or, in other words, perfect squeezing. Unfortunately, this requires infinite energy, which can be seen from the number state expansion for the two-mode squeezed vacuum (2MSV) state,

$$|\Psi_{2\text{MSV}}\rangle_{23} = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle_2 |n\rangle_3$$

(1)

where $\lambda$, the squeezing parameter, approaches one in the limit of perfect squeezing, and the mean excitation in each of the modes (labelled 2 and 3) is given by $\lambda^2/(1 - \lambda^2)$. Nevertheless, non-maximally-entangled 2MSV states ($\lambda < 1$), as produced, for example, by non-degenerate optical parametric amplification, can still facilitate CVQT with a fidelity of teleportation higher than one can be achieved by classical means only [8, 9].

In this paper, we consider the teleportation procedure of Braunstein and Kimble [8] but with an alternative truncated entanglement resource. There has been some investigation into the improvement of CVQT by manipulating the 2MSV entanglement resource with conditional measurements [11, 12], and into the use of truncated entanglement resources in a CVQT scheme involving number-difference and phase-sum (as opposed to quadrature phase) measurements [13]. Here, we consider the case in which the entanglement resource, perhaps prepared also through manipulation of 2MSV states, or possibly by some other means (see below), takes the form of a maximally entangled state between $N$-dimensional subspaces (MEND state) of the (infinite-dimensional) Hilbert spaces of the entangled oscillator modes, i.e.,

$$|\Psi_{\text{MEND}}\rangle_{23} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_2 \otimes |k\rangle_3,$$

(2)

with $\{ |k\rangle \}$ the Fock basis. We show that with this entanglement resource the teleportation fidelity for given quadrature ("Bell-state") measurement results is directly proportional to the probability of obtaining those results and that, for sufficiently large $N$, there exists a broad and uniform region of measurement results over which the fidelity is essentially equal to one. This suggests the possibility of a highly efficient conditional teleportation scheme, particularly for quantum states in lower dimensional subspaces (e.g. qubits).

As far as preparation of states of the form (3) is concerned, there has been a proposal based on an entanglement purification scheme for 2MSV states, specifically for optical systems [14]. This technique can also be mapped onto other harmonic oscillator systems, such as the quantized motional modes of trapped atoms [15].

In addition, entangled collective spin states of atomic ensembles in the form (3) have recently been experimentally created [16], with a technique that should also enable the preparation of states of the form (3) for finite
Such entangled spin states can in principle be mapped onto light fields or Bell-state measurements as required for quantum teleportation can be implemented via dispersive coupling to laser fields. To begin our analysis, let us first outline the teleportation protocol of Braunstein and Kimble. Let the input quantum state be in system 1, while the entanglement resource is shared between systems 2 and 3, where system 3 is the final target system. The usual characters Alice (system 2) and Bob (system 3) share the entangled resource, and Alice is given the desired unknown quantum state, to be transferred to Bob. Alice combines this state with her portion of the entangled resource and makes joint measurements of the position difference, \( \tilde{x}_{12} = \tilde{x}_1 - \tilde{x}_2 \), and momentum sum \( \tilde{p}_{12} = \tilde{p}_1 + \tilde{p}_2 \), observables (which commute). She then communicates (classically) the results of these measurements to Bob, who makes an appropriate coherent displacement of his mode to complete the process.

To formulate the teleportation process mathematically, we use the transfer operator introduced in \(^{21, 22}\), which relates the teleported state \( |\psi_{\text{out}}\rangle \) to the input state as

\[
|\psi_{\text{out}}\rangle = \hat{T} |\psi_{\text{in}}\rangle. \tag{3}
\]

For perfect teleportation the transfer operator is simply the identity operator. The output state is not normalized, and its scalar product gives the probability of measuring a particular value \( \beta = x_{12} + ip_{12} \).

\[
P(\beta) = \langle \psi_{\text{out}} | \psi_{\text{out}} \rangle = \langle \psi_{\text{in}} | \hat{T}^\dagger \hat{T} | \psi_{\text{in}} \rangle. \tag{4}
\]

The teleportation fidelity is given by the overlap between the input and output state,

\[
F(\beta) = \frac{1}{P(\beta)} |\langle \psi_{\text{in}} | \psi_{\text{out}} \rangle|^2 = \frac{1}{P(\beta)} |\langle \psi_{\text{in}} | \hat{T} | \psi_{\text{in}} \rangle|^2. \tag{5}
\]

The average fidelity \( F_{\text{av}} \) is given by

\[
F_{\text{av}} = \int d^2 \beta P(\beta) F(\beta) = \int d^2 \beta \left| \langle \psi_{\text{in}} | \hat{T} | \psi_{\text{in}} \rangle \right|^2. \tag{6}
\]

Hence, the transfer operator holds all information about the teleportation process. For the 2MSV quantum channel (with unit gain on Bob’s displacement operator \(^{21, 22}\)) the transfer operator is given by

\[
\hat{T}_{\text{2MSV}} = \sqrt{\frac{1 - \lambda^2}{\pi}} \sum_{n=0}^{\infty} \lambda^n \tilde{D}(\beta) |n\rangle \langle n| \tilde{D}(-\beta). \tag{7}
\]

For the MEND quantum channel (and unit gain) the transfer operator is,

\[
\hat{T}_{\text{MEND}} = \frac{1}{\sqrt{\pi N}} \sum_{n=0}^{N-1} \tilde{D}(\beta) |n\rangle \langle n| \tilde{D}(-\beta). \tag{8}
\]

For a measurement of \( \beta = 0 \), the teleportation operator is essentially the identity operator in a finite basis, in which case we have perfect teleportation of states which involve \( N - 1 \) excitations or less.

The transfer operator for the MEND quantum channel given in equation (8) has the following property,

\[
\hat{T}^\dagger_{\text{MEND}} \hat{T}_{\text{MEND}} = \frac{1}{\sqrt{\pi N}} \hat{T}_{\text{MEND}}. \tag{9}
\]

Therefore, the probability distribution shown in equation (8) may be written as

\[
P_{\text{MEND}}(\beta) = \frac{1}{\sqrt{\pi N}} \langle \psi_{\text{in}} | \hat{T}_{\text{MEND}} | \psi_{\text{in}} \rangle. \tag{10}
\]

The fidelity distribution defined in (8) can then be written in terms of \( P_{\text{MEND}}(\beta) \) as

\[
F_{\text{MEND}}(\beta) = \pi N P_{\text{MEND}}(\beta), \tag{11}
\]

which shows that, independent of the state we are teleporting, the fidelity and probability distribution have the same functional form, differing only by the constant \( \pi N \). It shows a direct connection between the amount of information gained from the measurement and the disturbance to the teleportation process. Note, \( F(\beta) \) must be bounded by the set of real numbers \([0, 1]\), which means \( P(\beta) \) must be bounded by the set \([0, 1/(\pi N)]\). Since \( P(\beta) \) will have non-zero values and must be normalized, it is of compact support, that is, bounded and closed for some region. It follows that \( F(\beta) \) will also be of compact support and as \( N \) is increased the maximum probability is reduced, and consequently \( P(\beta) \) becomes broader, as does \( F(\beta) \). These comments are general but the precise nature of the distributions depends on the type of state being teleported, so we now consider several popular examples.

Suppose the input state to teleport is a coherent state, \( |\psi_{\text{in}}\rangle = |\alpha\rangle \). For the 2MSV quantum channel, using the transfer operator in equation (8) we obtain the usual Gaussian fidelity distribution

\[
F(\beta) = e^{-(1-\lambda^2)|\alpha-\beta|^2}, \tag{12}
\]

with the probability for measuring \( \beta \) given by

\[
P(\beta) = \frac{1 - \lambda^2}{\pi} e^{-(1-\lambda^2)|\alpha-\beta|^2}. \tag{13}
\]

For the MEND quantum channel, the fidelity distribution found using the transfer operator (8) is

\[
F(\beta) = e^{-|\alpha-\beta|^2} \sum_{n=0}^{N-1} \frac{|\alpha-\beta|^{2n}}{n!}, \tag{14}
\]

with, as shown before, the probability of measuring \( \beta \) given by \( P(\beta) = 1/(\pi N) F(\beta) \). It follows from \(^{14}\) that high fidelities close to one occur when the Taylor expansion to order \( N \) of \( \exp(|\alpha-\beta|^2) \) is a good approximation.
to \( \exp(|\alpha - \beta|^2) \). Hence, for sufficiently large \( N \), the fidelity distribution has the significant feature of being flat and essentially equal to one over the region where the residual of the Taylor expansion, \( \sum_{n=N}^{\infty} \frac{|\alpha - \beta|^{2n}}{n!} \) is small. This is shown in Fig. 1 for different values of truncation number \( N \) in the MEND quantum channel. In contrast, with the 2MSV resource the fidelity is only equal to one for \( \beta = \alpha \) and drops off rapidly for \( \beta \neq \alpha \) (unless \( \lambda \) is close to one).

Another standard example in the context of CVQT is the “Schrödinger Cat” state, \( |\psi_{in}\rangle = N_\alpha(|\alpha| + |-\alpha|) \) \( (N_\alpha = \frac{2 + 2e^{-2|\alpha|^2} - 2}{1 + |\alpha|^2}) \) [8, 11, 12]. The fidelity and probability distributions for this state are shown in Fig. 2 for both the MEND \( (N = 21) \) and 2MSV \( (\lambda = 0.85) \) entanglement resources. Again, we observe a distinct “flat-top” region for the MEND resource where the fidelity is equal to one, as opposed to isolated maxima for the 2MSV resource.

Now let us suppose that the input state is a general finite-basis pure state encoded on harmonic oscillators,

\[
|\psi_{in}\rangle = \sum_{m=0}^{M} c_m |m\rangle ,
\]

and let us also assume a general entangled state resource

\[
|\psi_{EPR}\rangle_{23} = \sum_{n=0}^{\infty} d_n |n\rangle_2 |n\rangle_3 ,
\]

which has a transfer operator given by,

\[
\hat{T} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} d_n \hat{D}(\beta)|n\rangle \langle n| \hat{D}(-\beta) .
\]

The action of the displacement operator on a Fock state can be expressed as [23]

\[
\hat{D}(\beta)|n\rangle = e^{-|\beta|^2} \sum_{k=0}^{\infty} \frac{|n\rangle \langle k|}{\sqrt{k!}} \sqrt{1 - |\beta|^2} \sqrt{|\beta|^2} \beta^{k-n}|k\rangle ,
\]

where \( L_n^\alpha \) are the generalized Laguerre polynomials [24]. The probability distribution is then

\[
P(\beta) = e^{-|\beta|^2} G(\beta, \{|d_n|\}) ,
\]

and the fidelity distribution becomes,

\[
F(\beta) = e^{-|\beta|^2} G(\beta, \{|d_n|\}) / G(\beta, \{|d_n|^2\}) ,
\]

where

\[
G(\beta, \{f_n\}) = e^{-|\beta|^2} \times \sum_{m,l=0}^{M} \sum_{n=0}^{\infty} c_m f_n \frac{n!}{\sqrt{m!}} L_m^{n-m} (|\beta|^2) L_l^{n-l} (|\beta|^2) \frac{\beta^m \beta^l}{|\beta|^{2n}} .
\]
FIG. 4: The probability of obtaining a fidelity greater than 0.99 against truncation number $N$ of the MEND quantum channel for coherent state, cat state, and qubit inputs.

Fig. 3 shows the results for the most fundamental case of a finite-basis state, the qubit, defined as

$$|\psi_{in}\rangle = a|0\rangle + b|1\rangle, \quad a^2 + b^2 = 1.$$  \hfill (22)

The same general features are exhibited as for the previous examples. Despite the broader range of high fidelities for the MEND resource, an important point to note in the comparison between the MEND and the 2MSV resources is that the average teleportation fidelity is actually very similar for the case shown – 0.86 for the 2MSV resource with $\lambda = 0.8$, and 0.85 for the MEND resource with $N = 21$. This is because the probability distribution for the 2MSV resource is narrower than its fidelity distribution, thus giving more weight to the central regions of higher fidelity.

However, the significant feature that we wish to emphasize is that for the MEND resource with $N = 21$ there is a 48% chance of obtaining Bell-state measurement results for which the corresponding fidelity of teleportation is greater than 0.99. For the 2MSV resource, the maximum fidelity, occurring only at the peak of the distribution shown in Fig. 3(c), is 0.99.

This points to the possibility of a very efficient conditional teleportation scheme using the MEND resource, where, in particular, the conditioning is upon obtaining Bell-state measurement results that lie within the region for which the fidelity of teleportation is known to be very close to one. That is, given some knowledge of the general class of input state (e.g., coherent state, qubit) and of the truncation number $N$ of the MEND state, one could select out, or choose to pursue, only those events where Alice’s measurements fall within the (broad) region that gives near-perfect teleportation. The potential efficiency of such a conditional teleportation scheme is characterized in Fig. 3 where we plot, as a function of $N$ (and for each of the examples considered), the probability of Alice obtaining a measurement result for which the teleportation operation, upon completion, occurs with a fidelity greater than 0.99. As one can see, this probability can be substantial for only moderately large values of $N$.

In conclusion, the maximally entangled state in a finite-basis subspace of the full infinite-dimensional Hilbert space gives a direct proportionality between the Bell-state measurement probability and the fidelity of teleportation. This relationship constrains the form of the distribution of fidelities over the entire measurement domain to one which has a very flat region of unit fidelity followed by a quick cut off to zero fidelity. Using the measurement result, one can in principle post-select or pursue only those teleportation events which occur in the flat region of high fidelity to implement an efficient conditional teleportation scheme.

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