Remarks on strong embeddability
for discrete metric spaces and groups*

Guoqiang Li and Xianjin Wang

Abstract

In this paper, we show that the strong embeddability has fibering permanence property and is preserved under the direct limit for the metric space. Moreover, we show the following result: let $G$ is a finitely generated group with a coarse quasi-action on a metric space $X$. If $X$ has finite asymptotic dimension and the quasi-stabilizers are strongly embeddable, then $G$ is also strongly embeddable.

1 Introduction

In [8], Gromov introduced the notion of coarse embeddability of metric spaces and suggested that a discrete finitely generated group that coarsely embeds into a Hilbert space, when equipped with a word length metric, would satisfy the Novikov conjecture [7, 8]. Subsequently, in [18], Yu proved the coarse Baum-Connes conjecture holds for bounded geometry discrete metric spaces which are coarsely embeddable in Hilbert space. In the same paper, Yu introduced a weak form of amenability that he called property A, which ensures the existence of a coarse embedding into Hilbert space. In later years, property A and coarse embeddability have been further studied [3, 5, 9, 13, 14]. In [5], the author showed that property A is preserved under group extensions. Unlike property A, coarse embeddability is not closed under group extensions [1]. In [10], Ji, Ogle and W. Ramsey introduced the notion of strong embeddability which is stable under arbitrary extensions.

Strong embeddability is an intermediate notion of strong coarse embeddability implied by Property A and implying coarse embeddability. In [16], J. Xia and X. Wang studied the permanence properties of strong embeddability. Moreover, they proved that a metric space is strongly embeddable if and only if it has weak finite decomposition complexity with respect to strong embeddability. In [17], J. Xia and X. Wang showed that a finitely generated group acting on a finitely asymptotic dimension metric space by isometries whose $k$-stabilizers are strongly embeddable is strongly embeddable. Now we extend this result to conclude as follows.

*The authors are supported by NSFC (No. 11231002).
Theorem 1.1. Assume that $G$ is a finitely generated group with a coarse quasi-action on a metric space $X$. If $X$ has finite asymptotic dimension and there exists a base point such that its all quasi-stabilizers are strongly embeddable, then $G$ is strongly embeddable.

The coarse quasi-action (see Definition 2.7) is designed to describe situations where elements of a group act on a metric space via coarse equivalence. And it is a significant and useful generalization of actions by isometries and quasi-isometries.

2 Preliminaries

A discrete metric space $X$ has bounded geometry if for all $R > 0$ there exists $N_R$ such that $|B(x, R)| \leq N_R$ for all $x \in X$, where $| \cdot |$ denotes the number of elements of the ball $B(x, R)$. $X$ is called uniformly discrete if there exists a constant $C > 0$ such that for any two distinct points $x, y \in X$ we have $d(x, y) \geq C$. Assume that all metric spaces in this paper are uniformly discrete with bounded geometry. This class includes many interesting examples, in particular, all countable and uniformly discrete groups. Let $B$ be a Banach space and $B_1 = \{ \eta \in B \mid \| \eta \| = 1 \}$. For any $R, \epsilon > 0$, a map $x \mapsto \xi_x$ from $X$ to $B$ will be said to have $(R, \epsilon)$ variation if $d(x, y) \leq R$ implies $\| \xi_x - \xi_y \| \leq \epsilon$.

Definition 2.1 (see [10]). Let $X$ be a metric space. Then $X$ is strongly embeddable if and only if for every $R, \epsilon > 0$ there exists a Hilbert space valued map $\beta : X \to (l^2(X))_1$ satisfying:

1. $\beta$ has $(R, \epsilon)$ variation;
2. $\lim_{S \to \infty} \sup_{x \in X} \sum_{w \notin B(x, S)} | \beta_x(w) |^2 = 0$.

We will need to make use of a family of strongly embeddable metric spaces, with some uniform control.

Definition 2.2 (see [10]). A family $(X_i)_{i \in I}$ of metric spaces is equi-strongly embeddable if for every $R, \epsilon > 0$ there exists a family of Hilbert space valued maps $\xi^i : X_i \to (l^2(X_i))_1$ satisfying:

1. for each $i \in I$, $\xi^i$ has $(R, \epsilon)$ variation;
2. $\lim_{S \to \infty} \sup_{i \in I} \sup_{x \in X_i} \sum_{w \notin B(x, S)} | \xi^i_x(w) |^2 = 0$.

Let $X$ be a set. A partition of unity on $X$ is a collection of functions $\{ \phi_i \}_{i \in I}$, with $\phi_i : X \to [0, 1]$, and such that $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in X$. A partition of unity $\{ \phi_i \}_{i \in I}$ is said to subordinate to a cover $\mathcal{U} = \{ U_i \}_{i \in I}$ of $X$ if each $\phi_i$ vanishes outside $U_i$.

The following conclusion will be useful to prove the main theorem.
Theorem 2.3 (see [17]). Say $X$ is a metric space such that for any $R, \epsilon > 0$ there exists a partition of unity $\{\phi_i\}_{i \in I}$ on $X$ satisfying:

1. for all $x, y \in X$, if $d(x, y) \leq R$, then $\sum_{i \in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$;

2. $\{\phi_i\}_{i \in I}$ is subordinated to an equi-strongly embeddable cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$.

Then $X$ is strongly embeddable.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of a metric space $X$. We say the multiplicity of $\mathcal{U}$ is $k$ if each point of $X$ is contained in at most $k$ elements of $\mathcal{U}$. The $R$-multiplicity of $\mathcal{U}$ is the maximum number of elements of $\mathcal{U}$ that meet a common ball of radius $R$ in $X$ ($R > 0$). The Lebesgue number of $\mathcal{U}$ is $L$ if any ball of radius at most $L$ is contained in one element of $\mathcal{U}$. If $d(U, V) > L$ for all $U, V \in \mathcal{U}$ with $U \neq V$ then $\mathcal{U}$ is $L$-separated ($L > 0$). A cover $\mathcal{U}$ of $X$ is $(k, L)$-separated ($k \geq 0$ and $L > 0$) if there is a partition of $\mathcal{U}$ into $k + 1$ families $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_k$ such that each family $\mathcal{U}_i$ is $L$-separated. Note that a $(k, 2L)$-separated cover has $L$-multiplicity $\leq k + 1$. Let

$$\mathcal{U}_L = \{U(L) | U \in \mathcal{U} \}, \ U(L) = \{x \in X | d(x, U) \leq L\}.$$ 

Note then that: if a cover $\mathcal{U}$ of $X$ has $L$-multiplicity $\leq k + 1$ then the enlarged cover $\mathcal{U}_L$ has multiplicity $\leq k + 1$ and Lebesgue number $L$.

Definition 2.4 (see [15]). Let $X$ be a metric space. $X$ is said to have finite asymptotic dimension if there exists $k \geq 0$ such that for all $L \geq 0$ there exists a uniformly bounded cover of $X$ of Lebesgue number at least $L$ and multiplicity $k + 1$. The least possible such $k$ is the asymptotic dimension of $X$.

The following result is part of the folklore of the subject.

Lemma 2.5 (see [4]). Let $X$ be a metric space and $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of $X$ with multiplicity $k$ and Lebesgue number $L$. Then there is a partition of unity $\{\phi_i\}_{i \in I}$ subordinating to the cover such that

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)| \leq \frac{(2k + 2)(2k + 3)}{L} d(x, y)$$

for any $x, y \in X$.

Next we will recall the notion of coarse quasi-action which is a generalization of the notion of quasi-action central to the fundamental problem of quasi-isometry classification of finitely generated groups, see [6, 11, 12].
Definition 2.6 (see [13]). Let $X$ and $Y$ be metric space and $f : X \to Y$ be a map.

(1) The map $f$ is called proper if the inverse image, under $f$, of each bounded subset of $Y$, is a bounded subset of $X$.

(2) The map $f$ is called bornologous if there exists a real positive non-decreasing function $\ell : [0, +\infty) \to [0, +\infty)$ such that
\[ d(x, x') \leq r \Rightarrow d(f(x), f(x')) \leq \ell(r). \]

(3) The map $f$ is called coarse if it is both proper and bornologous.

We say that two maps $f, f' : X \to Y$ are close if $d(f, f') = \sup_{x \in X} d(f(x), f'(x))$ is bounded. $X$ and $Y$ are coarsely equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity maps on $Y$ and on $X$ respectively.

Definition 2.7 (see [2]). A coarse quasi-action of a group $G$ on a metric space $X$ is an assignment of a coarse self-equivalence $f_g : X \to X$ for each element $g \in G$ such that the following conditions are satisfied:

(1) all $f_g$ are coarse maps with respect to a uniform choice of the function $\ell$;

(2) there exists a number $A \geq 0$ such that $d(f_{id}, id_X) \leq A$;

(3) there exists a number $B \geq 0$ such that $d(f_g \circ f_h, f_{gh}) \leq B$ for all elements $g, h \in G$.

It is immediate from the above that $d(f_g \circ f_{-g}, id_X) \leq A + B$ for all $g \in G$.

3 Direct limits for the metric space

In this section, we will prove that the strong embeddability is closed under the direct limit for the metric space. Note that this result is rather different form ([16], Theorem 4.3). The group structure on the direct limit is essential to enable one to drop the boundedness assumptions necessary for this result.

First, we have the following fact.

Lemma 3.1. Let $\{X_i\}_{i \in I}$ be a collection of equi-strongly embeddable metric spaces. If $Y_i$ is the subspace of $X_i$ for each $i$, then $\{Y_i\}_{i \in I}$ is also equi-strongly embeddable.

Proof. For each $i \in I$, suppose $d_i$ is the metric on $X_i$. Let $p_i : X_i \to Y_i$ be a function defined by
\[
p_i(x) = \begin{cases} y, & \text{if } x \in X_i \setminus Y_i, \\ x, & \text{if } x \in Y_i, \end{cases}
\]
\[ d_i(x, y) \leq 2 d_i(x, Y_i); \]
\[ x, & \text{if } x \in Y_i. \]
Let $R > 0$ and $\epsilon > 0$, as in Definition 2.2, there exists a family of maps $\beta^i : X_i \to (\ell^2(X_i))_1$ satisfying:

1. for each $i \in I$, $\beta^i$ has $(R, \epsilon)$ variation;
2. $\lim_{S \to \infty} \sup_{i \in I, x \in X_i} \sum_{w \in \mathcal{B}(x, S)} |\beta^i_x(w)|^2 = 0$.

Then, for each $i \in I$ we define an isometry $\alpha^i : \ell^2(X_i) \to \ell^2(Y_i \times X_i)$ by

$$\alpha^i(\zeta)(y, x) = \begin{cases} \zeta(x), & \text{if } y = p_i(x); \\ 0, & \text{if otherwise.} \end{cases}$$

for each $\zeta \in \ell^2(X_i)$.

Define $\xi^i : Y_i \to \ell^2(Y_i \times X_i)$ by $\xi^i_y(t, s) = \alpha^i(\beta^i_y)(t, s)$ for $t \in Y_i, s \in X_i$. Note that for any $y \in Y_i$,

$$\|\xi^i_y\|^2_{\ell^2(Y_i \times X_i)} = \sum_{(t, s) \in Y_i \times X_i} |\xi^i_y(t, s)|^2 = \sum_{(t, s) \in Y_i \times X_i} |\alpha^i(\beta^i_y)(t, s)|^2 = \sum_{s \in X_i} |\beta^i_y(s)|^2 = \|\beta^i_y\|^2_{\ell^2(X_i)} = 1.$$

Similarly, for any $y, y' \in Y_i$,

$$\|\xi^i_y - \xi^i_{y'}\|^2_{\ell^2(Y_i \times X_i)} = \|\beta^i_y - \beta^i_{y'}\|^2_{\ell^2(X_i)}$$

Now we define $\eta^i : Y_i \to \ell^2(Y_i)$ by

$$\eta^i_y(t) = \|\xi^i_y(t, \cdot)\|_{\ell^2(X_i)}$$

for $t \in Y_i$.

Note that for any $y \in Y_i$,

$$\|\eta^i_y\|^2_{\ell^2(Y_i)} = \sum_{t \in Y_i} |\eta^i_y(t)|^2 = \sum_{t \in Y_i} \|\xi^i_y(t, \cdot)\|^2_{\ell^2(X_i)} = \|\xi^i_y\|^2_{\ell^2(Y_i \times X_i)} = 1.$$

Then for any $y, y' \in Y_i$ and $d_i(y, y') \leq R$, we have

$$\|\eta^i_y - \eta^i_{y'}\|^2_{\ell^2(Y_i)} = \sum_{t \in Y_i} |\eta^i_y(t) - \eta^i_{y'}(t)|^2$$
Then the claim follows.

$$= \sum_{i \in Y_i} \| \xi_i^y(t, \cdot) \|_{L^2(X_i)} - \| \xi_i^{y'}(t, \cdot) \|_{L^2(X_i)}^2$$

$$\leq \sum_{i \in Y_i} \| \xi_i^y(t, \cdot) - \xi_i^{y'}(t, \cdot) \|_{L^2(X_i)}^2$$

$$= \| \xi_i^y - \xi_i^{y'} \|_{L^2(X_i)}^2$$

$$\leq \| \beta_i^y - \beta_i^{y'} \|_{L^2(X_i)}^2$$

It follows that $\eta_i^y$ has $(R, \epsilon)$ variation.

Moreover,

$$\limsup_{S \to \infty} \sup_{i \in I} \sum_{y \in Y_i, y \not\in B(y, S)} \sum_{x \not\in B(y, S)} |\eta_i^y(z)|^2$$

$$= \limsup_{S \to \infty} \sup_{i \in I} \sum_{y \in Y_i, z \not\in B(y, S), x \in X_i} |\alpha^i(\beta_i^y)(z, x)|^2$$

$$= \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \in Y_i} \sum_{z \not\in B(y, S)} |\alpha^i(\beta_i^y)(z, x)|^2$$

$$= \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \in Y_i} \sum_{z \not\in B(y, S)} |\alpha^i(\beta_i^y)(z, x)|^2$$

$$+ \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \in Y_i} \sum_{z \not\in B(y, S)} |\alpha^i(\beta_i^y)(z, x)|^2$$

$$= \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \not\in B(y, S), y \in Y_i} |\beta_i^y(x)|^2 + \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \in Y_i, z \not\in B(y, S)} |\beta_i^y(x)|^2$$

$$\leq \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \not\in B(y, S), y \in Y_i} |\beta_i^y(x)|^2 + \limsup_{S \to \infty} \sum_{i \in I} \sum_{y \in Y_i, z \not\in B(y, S)} |\beta_i^y(x)|^2$$

$$= 0.$$

Then the claim follows.

Lemma 3.2. Let $c > 0$. Let $\{X_i\}_{i \in I}$ be a collection of metric spaces and $Y_i$ is a $c$-net of $X_i$ for each $i \in I$. If $\{Y_i\}_{i \in I}$ is equi-strongly embeddable, then so is $\{X_i\}_{i \in I}$.

Proof. This proof is analogous to ([16], Lemma 5.2), so we omit it.

Proposition 3.3. Let $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ be an increasing sequence of bounded metric space, and let $X = \bigcup_{n=1}^{\infty} X_n$. Assume also that any bounded subset of $X$ is contained in some $X_n$. If $\{X_n\}_{n=1}^{\infty}$ is equi-strongly embeddable, then $X$ is strongly embeddable.
Proof. Take $L > 0$. For given $X_{n_k}$, since each $X_n$ is bounded, we can choose $X_{n_{k+1}}$ satisfying $\bigcup_{x \in X_{n_k}} B(x, 3L) \subseteq X_{n_{k+1}}$. Thus we obtain a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that
\[
(\bigcup_{x \in X_{n_k}} \bar{B}(x, L)) \cap (\bigcup_{x \in X_{n_{k+2}} \setminus X_{n_{k+1}}} \bar{B}(x, L)) = \emptyset.
\]
Set
\[
U_k = \bigcup_{x \in X_{n_{k+1}} \setminus X_{n_k}} \bar{B}(x, L).
\]
for every $k \geq 1$.

Then we obtain a cover $\mathcal{U} = \{U_k\}_{k=1}^\infty$ of multiplicity at most 2 and Lebesgue number at least $L$. We claim that $\mathcal{U}$ is equi-strongly embeddable. Indeed, $\{X_{n_{k+1}} \setminus X_{n_k}\}$ are subspaces of the equi-strongly embeddable sequence $\{X_{n_k}\}$. It follows from Lemma 3.1 that $\{X_{n_{k+1}} \setminus X_{n_k}\}$ is equi-strongly embeddable. Note that for every $k$, $\{X_{n_{k+1}} \setminus X_{n_k}\}$ is a $L$-net of $U_k$. By lemma 3.2, $\{U_k\}_{k=1}^\infty$ is equi-strongly embeddable.

By choosing $L$ large enough, the result follows from Lemma 2.5 and Theorem 2.3.

4 Fibering permanence

In this section, we show an important property of strong embeddability which is called fibering permanence. The main motivating examples are fibre bundles $p : X \to Y$ with base space $Y$ and total space $X$. Moreover, fibering permanence is somewhat more subtle than the other permanence properties, and care must be taken to formulate it correctly for our other basic properties.

Proposition 4.1. Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ be an uniformly expansive map. Assume that $Y$ has Property A. If for every uniformly bounded cover $\{U_i\}_{i \in I}$ of $Y$, the collection $\{f^{-1}(U_i)\}_{i \in I}$ of subspaces of $X$ is equi-strongly embeddable, then $X$ is strongly embeddable.

Proof. Let $R, \epsilon > 0$ be given. Since $f$ is uniformly expansive, there exists $S > 0$, such that $d(f(x), f(y)) \leq S$ whenever $d(x, y) \leq R$. Since $Y$ has property A, by one of the equivalent definitions of property A (see [15]), there exists an uniformly bounded cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $Y$, together with a partition of unity $\{\phi_i\}_{i \in I}$ subordinated to $\mathcal{U}$, such that
\[
\sum_{i \in I} |\phi_i(y) - \phi_i(y')| \leq \epsilon
\]
for $y, y' \in Y$, $d(y, y') \leq S$. 

7
We define \( \varphi_i = \phi_i \circ f \) for each \( i \in I \). Clearly, \( \{\varphi_i\}_{i \in I} \) is a partition of unity on \( X \) subordinated to \( \{f^{-1}(U_i)\}_{i \in I} \) and satisfying the assumptions of Theorem 2.3. Thus \( X \) is strongly embeddable.

**Corollary 4.2.** Let \( X \) and \( Y \) be two metric spaces and \( f : X \to Y \) be a Lipschitz map of metric spaces. Suppose that a group \( G \) acts by isometries on both \( X \) and \( Y \), that the action on \( Y \) is transitive and that \( f \) is \( G \)-equivariant. Suppose that \( Y \) has property A. If there exists \( y_0 \in Y \) satisfying for every \( n \in \mathbb{N} \) the inverse image \( f^{-1}(B(y_0, n)) \) is strongly embeddable, then \( X \) is strongly embeddable.

**Proof.** Let \( \{U_i\}_{i \in I} \) be a uniformly bounded cover of \( Y \). Note that the action of \( G \) on \( Y \) is isometrical and transitive, there exists \( n \in \mathbb{N} \) and \( g_i \in G \) such that \( g_i U_i \subseteq B(y_0, n) \) for all \( i \in I \). Note also that \( g_i f^{-1}(U_i) = f^{-1}(g_i U_i) \subseteq f^{-1}(B(y_0, n)) \), we have that the collection \( \{f^{-1}(U_i)\}_{i \in I} \) is isometric to a collection of subspaces of \( f^{-1}(B(y_0, n)) \).

Continuing, note that \( f^{-1}(B(y_0, n)) \) is strongly embeddable, then we have that \( \{f^{-1}(U_i)\}_{i \in I} \) is equi-strongly embeddable. Now the result immediately follows from Theorem 4.1.

A metric space of finite asymptotic dimension has property A, also is strongly embeddable. Now we prove a natural generalization of this result, where uniform boundedness of the cover is replaced by the appropriate uniform version of strong embeddability.

**Theorem 4.3.** Let \( X \) be a metric space. If for any \( \sigma > 0 \) there exists a \((k, L)\)-separated cover \( \mathcal{U} \) of \( X \) with \( k^2 + 1 \leq L \sigma \) and \( \mathcal{U} \) is equi-strongly embeddable, then \( X \) is strongly embeddable.

**Proof.** Let \( R, \epsilon > 0 \). Take a number \( \sigma \), such that \( 0 < \sigma < 1/20R \). Then for any integer \( k \geq 0 \), we have

\[
k^2 + 1 \geq 2(2k + 2)(2k + 3) R \sigma.
\]

It follows from the assumption that there exists a \((k, 2L)\)-separated cover \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} \) is equi-strongly embeddable and \( k^2 + 1 \leq 2L \sigma \). Note that the cover \( \mathcal{U}_L \) has multiplicity \( \leq k + 1 \) and Lebesgue number \( L \). Moreover, \( \mathcal{U}_L \) is equi-strongly embeddable, as it is coarsely equivalent to \( \mathcal{U}_L \). By Lemma 2.5, there is a partition of unity \( \{\phi_{\mathcal{U}_L}\}_{\mathcal{U}_L} \subseteq \mathcal{U}_L \) subordinated to \( \mathcal{U}_L \) such that for all \( x, y \in X \)

\[
\sum_{\mathcal{U}_L} |\phi_{\mathcal{U}_L}(x) - \phi_{\mathcal{U}_L}(y)| \leq \frac{(2k + 2)(2k + 3)}{L} d(x, y)
\]

\[
\leq \frac{k^2 + 1}{2RL \sigma} d(x, y).
\]
In particular, if \( d(x, y) \leq R \), we have that
\[
\sum_{\upsilon(L) \in \upsilon_L} |\phi_{\upsilon(L)}(x) - \phi_{\upsilon(L)}(y)| \leq \frac{k^2 + 1}{2L\sigma} \leq \epsilon.
\]
This shows that \( X \) satisfies the conditions in Theorem 2.3. Hence, \( X \) is strongly embeddable.

5 Groups acting on metric spaces

In this section, we are ready to complete the proof of our main result.

First, we recall the notion of \( T \)-quasi-stabilizer (\( T > 0 \)). Let \( G \) be a finitely generated group with a coarse quasi-action on a metric space \( X \) and \( x_0 \) be a chosen base point in \( X \). The \( T \)-quasi-stabilizer \( W_T(x_0) \) is defined to be the subset of all elements \( g \) in \( G \) such that \( d(gx_0, x_0) \leq T \). Moreover, we can view \( G \) as a metric space with a word length metric (see [13]).

**Theorem 5.1.** Assume that \( G \) is a finitely generated group with a coarse quasi-action on a metric space \( X \). If \( X \) has finite asymptotic dimension and there exists a base point \( x_0 \in X \) such that \( W_T(x_0) \) is strongly embeddable for any \( T > 0 \). Then \( G \) is strongly embeddable.

**Proof.** Since the orbit \( Gx_0 \) is a subset of \( X \), \( Gx_0 \) has finite asymptotic dimension. Without loss of generality, we can assume that the action of \( G \) on \( X \) is transitive. Let \( S \) be the finite symmetric generating set in the definition of the word length metric for \( G \) and take \( \lambda = \max\{d(sx_0, x_0) \mid s \in S\} \).

Then there exists a map \( \pi : G \to X \) given by \( \pi(g) = gx_0 \) for all \( g \in G \). If the action of \( G \) on \( X \) is by isometries, \( \pi \) is \( \lambda \)-Lipschitz. In our case, we show that \( \pi \) is \( \ell(\lambda) \)-Lipschitz.

Indeed, \( d(\pi(g), \pi(gs)) = d(gx_0, gsx_0) \leq \ell(d(x_0, sx_0)) \leq \ell(\lambda) \) for any \( g \in G \) and \( s \in S \).

Suppose \( X \) has asymptotic dimension \( \leq k \). Let \( L > 0 \) be given. By definition 2.4 there exists a uniformly bounded cover \( \mathcal{U} = \{U_i\}_{i \in I} \) of \( X \) with Lebesgue number \( L \) and multiplicity \( k + 1 \) such that the \( L \)-neighbourhood \( \mathcal{V} = \{V_i\}_{i \in I} \) of \( \mathcal{U} \) is also a cover of multiplicity \( k + 1 \).

Since \( \mathcal{V} \) is uniformly bounded, there exist \( T > 0 \) and \( x_i \in X \) such that \( V_i \subseteq B(x_i, T) \) for each \( i \in I \). On the other hand, we can take \( g_i \in G \) such that \( x_i = g_i x_0 \).

By Definition 2.7 we have that \( d(g_i^{-1}x_i, x_0) \leq A + B \), then
\[
g_i^{-1}(B(x_i, T)) \subseteq B(g_i^{-1}x_i, \ell(T)) \subseteq B(x_0, A + B + \ell(T)).
\]
It follows from the definition of $\pi$,

$$g_i^{-1}\pi^{-1}(B(x_i, T)) \subseteq \pi^{-1}(B(x_0, A + 2B + \ell(T))) = W_{A+2B+\ell(T)}(x_0)$$

Then we get

$$g_i^{-1}(V_i) \subseteq W_{A+2B+\ell(T)}(x_0).$$

Note that $\{\pi^{-1}(V_i)\}_{i \in I}$ is isometric to a family of subspaces of $W_{A+2B+\ell(T)}(x_0)$. Since $W_{A+2B+\ell(T)}(x_0)$ is strongly embeddable, we have that $\{\pi^{-1}(V_i)\}_{i \in I}$ is equi-strongly embeddable. By the same argument, $\{\pi^{-1}(U_i)\}_{i \in I}$ is also equi-strongly embeddable and covers $G$.

We will use Theorem 2.3 to complete the proof. Let $R > 0$ and $\epsilon > 0$. Take $L \geq \frac{2k(\lambda)R(2k+2)(2k+3)}{\epsilon}$. Since $\mathcal{U}$ is a uniformly bounded cover of $X$ with Lebesgue number $L$ and multiplicity $k + 1$, it follows from Lemma 2.5 that there exists a partition of unity $\{\phi_{U_i}\}_{u_i \in \mathcal{U}}$ subordinated to the cover satisfying

$$\sum_{U_i \in \mathcal{U}} |\phi_{U_i}(x) - \phi_{U_i}(y)| \leq \frac{(2k+2)(2k+3)}{L}d(x, y)$$

for any $x, y \in X$.

Moreover, note that $\{\pi^{-1}(V_i)\}_{i \in I}$ is equi-strongly embeddable, where there exists a collection of maps $\beta^i : \pi^{-1}(V_i) \to (\ell^2(\pi^{-1}(V_i)))_1$ such that $\beta^i$ has $\left( R, \frac{\epsilon}{4} \right)$ variation for every $i \in I$.

Define for each $i \in I$ a map $\varphi_i : G \to [0, 1]$ by setting

$$\varphi_i(g) = \sum_{h \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(g))|\beta^i_g(h)|^2$$

for $g \in G$. We claim that $\{\varphi_i\}_{i \in I}$ is partition of unity on $G$ satisfying the conditions of Theorem 2.3. First, for any $g \in G$,

$$\sum_{i \in I} \varphi_i(g) = \sum_{i \in I} \sum_{h \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(g))|\beta^i_g(h)|^2$$

$$= \sum_{i \in I} \phi_{U_i}(\pi(g)) \sum_{h \in \pi^{-1}(V_i)} |\beta^i_g(h)|^2$$

$$= \sum_{i \in I} \phi_{U_i}(\pi(g))$$

$$= 1.$$
Note also that each $\varphi_i$ vanishes outside $\pi^{-1}(U_i)$, so $\{\varphi_i\}_{i \in I}$ is a partition of unity on $G$ and subordinates to $\{\pi^{-1}(U_i)\}_{i \in I}$. Second, for any $g, g' \in G$ with $d(g, g') \leq R$. If $g \in \pi^{-1}(U_i)_{i \in I}$, then $\pi(g')$ belongs to the $\ell(\lambda)R$-neighbourhood of $U_i$. The latter space is a subspace of $V_i$ as $L$ is large enough. Then $g' \in \pi^{-1}(V_i)$. We have the following estimates

$$\sum_{i \in I} |\varphi_i(g) - \varphi_i(g')|$$

$$= \sum_{i \in I} \left| \sum_{h \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(g))|\beta_g^i(h)|^2 - \sum_{h \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(g'))|\beta_{g'}^i(h)|^2 \right|$$

$$= \sum_{i \in I} \left| \sum_{h \in \pi^{-1}(V_i)} (\phi_{U_i}(\pi(g))|\beta_g^i(h)|^2 - \phi_{U_i}(\pi(g'))|\beta_{g'}^i(h)|^2) \right|$$

$$\leq \sum_{i \in I} \sum_{h \in \pi^{-1}(V_i)} |\phi_{U_i}(\pi(g))|\beta_g^i(h)|^2 - \phi_{U_i}(\pi(g'))|\beta_{g'}^i(h)|^2|$$

$$\leq \sum_{i \in I} \sum_{h \in \pi^{-1}(V_i)} \phi_{U_i}(\pi(g))|\beta_g^i(h)|^2 - \phi_{U_i}(\pi(g'))|\beta_{g'}^i(h)|^2|$$

$$= \sum_{i \in I} \phi_{U_i}(\pi(g)) \sum_{h \in \pi^{-1}(V_i)} |\beta_g^i(h) + \beta_{g'}^i(h)| \cdot |\beta_g^i(h) - \beta_{g'}^i(h)|$$

$$+ \frac{(2k + 2)(2k + 3)}{L}d(\pi(g), \pi(g'))$$

$$\leq \sum_{i \in I} \phi_{U_i}(\pi(g)) \left[ \sum_{h \in \pi^{-1}(V_i)} |\beta_g^i(h) + \beta_{g'}^i(h)|^2 \right] \cdot \left( \sum_{h \in \pi^{-1}(V_i)} |\beta_g^i(h) - \beta_{g'}^i(h)|^2 \right)^{\frac{1}{2}}$$

$$+ \frac{(2k + 2)(2k + 3)}{L}\ell(\lambda)R$$

$$\leq \sum_{i \in I} \phi_{U_i}(\pi(g)) \| \beta_g^i + \beta_{g'}^i \| \cdot \| \beta_g^i - \beta_{g'}^i \| + \frac{\epsilon}{2}$$

$$\leq \sum_{i \in I} \phi_{U_i}(\pi(g)) \cdot 2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{2}$$

$$= \epsilon.$$  

Thus $\{\varphi_i\}_{i \in I}$ has the properties in Theorem 2.3, whence $G$ is strongly embeddable.  

Since Theorem 2.3 also holds for coarse embeddability and exactness (it is equivalent to property A for the metric spaces with bounded geometry) [5], a similar proof shows
that the same hypotheses, with 'coarse embeddability' (resp. exactness) replacing 'strong embeddability', imply that $G$ is coarsely embeddable (resp. exact), as described next.

**Theorem 5.2** (see [5]). Say $X$ is a metric space such that for any $R, \epsilon > 0$ there exists a partition of unity $\{\phi_i\}_{i \in I}$ on $X$ satisfying:

1. for all $x, y \in X$, if $d(x, y) \leq R$, the $\sum_{i \in I} |\phi_i(x) - \phi_i(y)| \leq \epsilon$;
2. $\{\phi_i\}_{i \in I}$ is subordinated to an equi-coarsely embeddable (resp. equi-exact) cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$.

Then $X$ is coarsely embeddable (resp. exact).

**Theorem 5.3.** Assume that $G$ is a finitely generated group with a coarse quasi-action on a metric space $X$. If $X$ has finite asymptotic dimension and there exists a base point $x_0 \in X$ such that $W_T(x_0)$ is coarsely embeddable (resp. exact) for any $T > 0$. Then $G$ is coarsely embeddable (resp. exact).

**References**

[1] G. Arzhantseva and R. Tessera, *Admitting a coarse embedding is not preserved under group extensions*, arXiv:1605.01192.

[2] S. Beckhardt and B. Goldfarb, *Extension properties of asymptotic property C and finite decomposition complexity*, arXiv:1607.00445.

[3] G. Bell, *Property A for groups acting on metric spaces*, Topology Appl. 130(2003), 239C251.

[4] G. Bell, *Asymptotic properties of groups acting on complexes*, Proc. Amer. Math. Soc. 133(2005), no. 2, 387C396.

[5] M. Dadarlat and E. Guentner, *Uniform embeddability of relatively hyperbolic groups*, J. Reine Angew. Math. 612(2007), 1-15.

[6] C. Drutu, *Quasi-isometry rigidity of groups*. in Géométries à courbure négative ou nulle, groupes discrets et rigidités, in Sémin. Congr., 18, Soc. Math. France, Paris, 2009, 321-371.

[7] S. Ferry, A. Ranicki, and J. Rosenberg (eds.), *Novikov conjectures, index theorems and rigidity*, London Mathematical Society Lecture Notes, no. 226, 227, Cambridge University Press, 1995.
[8] M. Gromov, *Asymptotic Invariants of Infinite Groups*, Volume 2 of “Geometry Group Theory, Sussex 1991”, G. A. Niblo and M. A. Roller Eds, Cambridge Univ. Press, 1993.

[9] N. Higson and J. Roe, *Amenable group actions and the Novikov conjecture*, J. Reine Angew. Math. **519**(2000), 143-153.

[10] R. Ji, C. Ogle and B. W. Ramsey, *Strong embeddability and extensions of groups*, arXiv:1307.1935.

[11] M. Kapovich, *Lectures on quasi-isometric rigidity, in Geometric Group Theory*, volume 21 of Publications of IAS/Park City Summer Institute, Amer. Math. Soc., Providence, RI, 2014, 127-172.

[12] L. Mosher, M. Sageev and K. Whyte, *Quasi-actions on trees I. Bounded valence*, Annals Math. **158**(2003), 115-164.

[13] P. Nowak and G. Yu, *Large Scale Geometry*, European Mathematical Society, 2012.

[14] Jean-Louis Tu, *Remarks on Yu’s property A for discrete metric spaces and groups*, Bull. Soc. Math. France. **129**(2001), no. 1, 115-139.

[15] R. Willett, *Some notes on property A*, EPFL Press, Lansanne, 2009.

[16] J. Xia and X. Wang, *On strong embeddability and finite decomposition complexity*, Acta Math. Sin., Engl. Ser. **33**(2017), 403-418.

[17] J. Xia and X. Wang, *Strong embeddability for groups acting on metric spaces*, In press, 2017.

[18] G. Yu, *The coarse Baum-connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139**(2000), 201-240.

Guoqiang Li
College of Mathematics and Statistics,
Chongqing University (at Huxi Campus),
Chongqing 401331, P. R. China
E-mail: guoqiangli@cqu.edu.cn

Xianjin Wang
College of Mathematics and Statistics,
Chongqing University (at Huxi Campus),
Chongqing 401331, P. R. China
E-mail: xianjinwang@cqu.edu.cn