Integral Formulas and asymptotic behavior of lattice points in complex hyperbolic space

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Abstract

This paper deals with the $\Gamma$-lattice points problem associated to a discrete subgroup of motions $\Gamma$ in the complex hyperbolic space $\mathbb{C}H^n$. We give two integral formulas for the local average of the number $N(T, z, z')$ of $\Gamma$-lattice points in a sphere of radius $T$ in $\mathbb{C}H^n$. The first one is in terms of the solution of the $\Gamma$-automorphic wave equation on $\mathbb{C}H^n$ and the second is given in terms of the spectral function of the Laplace-Beltrami operator under $\Gamma$-automorphic boundary conditions. We use the obtained integral formulas to obtain an asymptotic behavior of the number $N(T, z, z')$ as $T \to \infty$, with an estimate of the remainder term. Our principal tools are the explicit solution of the wave equation on the complex hyperbolic space, special functions and spectral theory of the Laplace-Beltrami operator under $\Gamma$ automorphic boundary conditions.

1 Introduction

The lattice points are the orbit of the origin under the action of the group of integer translations on the Euclidean plane. The classical Gauss Circle Problem is to determine the best bound for the error between the number of lattice points inside a disk and that disk’s area, otherwise known as the lattice point discrepancy. Gauss had proved that the number of lattice points in a circle of large radius $T$, was equal to the area of the disc with a remainder term not exceeding the circumference of the circle (see Gauss [13] and Hafner [16]).
Since then lattice points counting problems in Euclidean plane and more generally in Euclidean space, have been considered by many authors, with various applications in number theory ([9, 22, 25, 26, 27, 28, 33, 38, 39, 42]). The Euclidean space can be replaced by any Riemannian space $X$ and the group of integer translations can be replaced by any discrete subgroup $\Gamma$ of the group of motions $G$ of the Riemannian space $X$.

The case of lattice points problem in the hyperbolic setting was first studied by Jean Delsarte who obtained in (Delsarte [10] and [11]) the following remarkable formula for the number of lattice points $N(T, z, z')$ in the hyperbolic circle of radius $T$.

$$N(T, z, z') = \pi z \sum_{n=0}^{+\infty} F((\alpha_n, \beta_n, 2, -z/4)\varphi_n(w_0), \varphi_n(w), \ (1.1)$$

where $z = 2a^2(\cosh T/a - 1)$ and $\alpha_n, \beta_n$ are solution of the equation $x^2 - x - \lambda_n a^2 = 0$, and $\varphi_n(w)$ are the eigenfunctions of the Laplace-Beltrami on a compact fundamental region of the hyperbolic plane and $\lambda_n$ are the corresponding eigenvalues, and $F(a, b, c, t)$ is the classical Gauss hypergeometric function $\text{2F}_1$, defined by (Magnus et al. [31] p. 37)

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{(b)_n}{n!} z^n, \quad |z| < 1, \quad (1.2)$$

$(a)_n$ is the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and $\Gamma$ is the classical Euler function.

Many other authors have studied the asymptotic behavior of the number $N(T, z, z')$ as $T \to \infty$, in the cases of non Euclidean spaces ([1, 2, 3, 5, 6, 12, 17, 20, 23, 34, 35, 36, 37, 40, 41, 43]).

For lattice points in non compact type symmetric spaces of rank one (see [7, 8, 15, 32]). Lax and Phillips [29] gave a formula expressing the average of number of lattice points, for the real hyperbolic case, in terms of the solution of real hyperbolic wave equation for a special initial function. Levitan [30] derived a formula which gives an expression for the average of number of lattice points in real hyperbolic space, in terms of the spectral function of the shifted Laplace-Beltrami operator on a fundamental region $F = R H^n/\Gamma$. He estimated the remainder term on the basis of an estimate for the spectral function of an elliptic operator for large frequencies.

Let $\Gamma$ be a discrete subgroup of motions of the complex hyperbolic space.
$\mathcal{H}^{n}$, for arbitrary points $z$ and $z'$ in $\mathcal{H}^{n}$, we count the number of $\Gamma$-lattice points in the complex hyperbolic space $\mathcal{H}^{n}$

\[ N(T, z, z') = \# \{ \gamma \in \Gamma, d(z, \gamma z') < T \}, \quad (1.3) \]

and we give the analogous of these results in the n-complex hyperbolic space. That is we investigate the local average of the number of lattice points in complex hyperbolic ball of radius $T$, we vary the center of the complex hyperbolic ball locally, and we study the function

\[ I(T, z, z', \alpha) = \int_{F} N(T, x, z') h(x) d\mu(x), \quad (1.4) \]

where $F = \mathcal{H}^{n}/\Gamma$ is the fundamental region of $\Gamma$, and $h(x)$ is a smooth compactly supported function satisfying the conditions:

1) $h(x) > 0$, 2) $\int h(x) d\mu(x) = 1$, 3) $h(x) = 0$ for $d(x, z) \geq \alpha$ and
4) $h(x) = O(\alpha^{-2n})$. (For the construction of the function $h$ see the Appendix).

Let $u(t, z)$ be the solution of the Cauchy problem for the wave equation in the complex hyperbolic space $\mathcal{H}^{n}$

\[
\begin{align*}
\partial_{t}^{2}u(t, z) &= L_{n}u(t, z) \\
(t, z) &\in \mathbb{R} \times \mathcal{H}^{n} \\
u(0, z) &= 0 & \partial_{t}u(t, z) &= f(z) \in C_{0}^{\infty}(\mathcal{H}^{n}).
\end{align*}
\]

The main results of this paper are the following theorems.

**Theorem 1.1.** For sufficiently small $\alpha$, the following formula hold

i)

\[ I(T, z, z', \alpha) = \sum_{\gamma \in \Gamma} \int_{d(\gamma x, z') < T} h(x) d\mu(x), \quad (1.6) \]

ii)

\[ N(T - \alpha, z, z') \leq I(T, z, z', \alpha) \leq N(T + \alpha, z, z'). \quad (1.7) \]

iii) Let $u(t, z)$ be the solution of the Cauchy problem (1.5), with initial data

\[ f(x) = \sum_{\gamma \in \Gamma} h(\gamma^{-1} x), \quad (1.8) \]
then we have
\[ I(T, z, z', \alpha) = c_n \cosh^{1/2} T \int_0^T (\cosh T - \cosh t)^{n-3/2} \times \]
\[ F(-1/2, 3/2, n - 1/2, \frac{\cosh T - \cosh t}{2\cosh T}) \sinh tu(t, z')dt, \quad (1.9) \]
where \( c_n = (-1)^{n-1} \pi^{n-1/2} 2^{n+1/2}/\Gamma(n - 1/2), \) \( F(a, b, c, t) \) is the classical hypergeometric function \( _2F_1 \) given in (1.2).

**Theorem 1.2.** If \( \Gamma \) is a discrete subgroup of motions of the complex hyperbolic space \( \mathbb{CH}^n, n \geq 1, \) which we assume to be torsion free, then we have, for \( \alpha \) sufficiently small,
\[ I(T, z, z', \alpha) = \frac{\pi^n}{\Gamma(n + 1)} \sinh^{2n} T \times \]
\[ \int_{-n^2}^{+\infty} F((n - i\sqrt{\lambda})/2, (n + i\sqrt{\lambda})/2, n + 1, -\sinh^2 T) \]
\[ d_\lambda \int_F \theta_\Gamma(x, z', \lambda) h(x) d\mu(x), \quad (1.10) \]
where \( \theta_\Gamma(z, z', \lambda) \) is the spectral function of the Laplace-Beltrami operator on \( F, F(a, b, c, t) \) is the classical hypergeometric function \( _2F_1 \) given in (1.2).

**Theorem 1.3.** The hypotheses are the same as in Theorem 1.1, furthermore assume that \( \Gamma \) is cocompact or of finite covolume, then we have
\[ N(T, z, z') = A(T, z, z') + \begin{cases} O(e^{(2n-1-\frac{2n-2}{2n+1})T}) \\ O(e^{(2n-2-\frac{2n-4}{2n+1})T}) \end{cases} \quad (1.11) \]
where
\[ A(T, z, z') = \left( \frac{\pi}{2} \right)^n \sum_{j=1}^N \frac{2^{-\mu_j} \Gamma(n + \mu_j) e^{(n + \mu_j)T}}{\Gamma((n + \mu_j)/2) \Gamma((1 + (n + \mu_j)/2)} \varphi_j(z) \varphi_j(z') \quad (1.12) \]
where \( \mu_j = \sqrt{|\lambda_j|} \), and \( -n^2 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N < 0 \), are the eigenvalues of the shifted Laplace-Beltrami operator \( L_\Gamma \), on the interval \((−n^2, 0)\), and \( \varphi_j(z), 0 \leq j \leq N \) are the corresponding normalized eigenfunctions.

Note that the summation in (1.12) is meaningful as long as
\[ n + \mu_j > \begin{cases} 2n - 1 - \frac{2n-2}{2n+1} & n \geq 2 \\ 2n - 2 - \frac{2n-4}{2n+1} & n > 2 \end{cases} \]
and consequently

\[-n^2 \leq \lambda_j < -(n - 1)^2 \left(\frac{2n - 1}{2n + 1}\right)^2, \quad n \geq 2\]
\[-n^2 \leq \lambda_j < -(n - 2)^2 \left(\frac{2n - 1}{2n + 1}\right)^2, \quad n > 2.\]

In this paper we take as a model of the complex hyperbolic space the ball
\[\mathcal{H}H^n = \{z = (z_1, z_2, ..., z_n) \in \mathcal{D}^n, |z| < 1\}\] where \(\mathcal{D}\) is the the complex field and \(|z|^2 = z\overline{z}\), equipped with the metric \(ds\) given by.

\[ds^2 = (1 - |z|^2)^{-2} \sum_{i,j=1}^{n} [(1 - |z|^2)\delta_{ij} + z_i\overline{z_j}]dz_i \otimes dz_j.\]

Recall that \((\mathcal{H}H^n, ds)\) is a complete Riemannian manifold with negative sectional curvature, in which the group of motions is \(SU(n, 1)\). In this case, the volume element is given by

\[d\mu(z) = \frac{dz}{(1 - |z|^2)^{n+1}},\]

where \(dz\) is the Lebesgue volume element on \(C^n\). The \(SU(n, 1)-\)invariab Laplace-Beltrami operator associated to the metric \(ds\) is of the form

\[L_n = 4(1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - |z|^2) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} + n^2.\] (1.13)

It is known that the operator \(-L_n\) is elliptic selfadjoint non-negative and has a continuous spectrum represented by the positive real axis.

The remaining of the paper is organized as follows, Section 2 is devoted to the integral formulas for the number of lattice points \(N(T, z, z')\). That is we prove the Theorems 1.1 and 1.2. Section 3 is devoted to the proof of the Theorem 1.3 which gives an asymptotic formula for the number of lattice point \(N(T, z, z')\) with an estimate of the remainder term.

2 Integral formulas for the number of lattice points \(N(T, z, z')\)

We prove the following lemmas
Lemma 2.1. Let $\Gamma$ be a discrete subgroup of motions of the complex hyperbolic space, for $h(x)$ is a smooth function satisfying the above conditions 1), 2) and 3), then

$$\int_{d(\gamma x, z') < T} h(x) d\mu(x) = \begin{cases} 1, & \text{if } d(z', \gamma z) \leq T - \alpha \\ 0, & \text{if } d(z', \gamma z) \geq T + \alpha \end{cases},$$ (2.1)

and lies between 0 and 1 otherwise.

Proof. If $d(z', \gamma z) \leq T - \alpha$ and $d(x, z) \leq \alpha$, then

$$d(\gamma x, z') \leq d(\gamma x, \gamma z) + d(\gamma z, z') \leq \alpha + T - \alpha = T,$$

this shows that the support of $h$ is contained in the ball $\{z', d(\gamma x, z') < T\}$. Now if $d(z', \gamma z) \geq T + \alpha$ then

$$d(x, z) = d(\gamma x, \gamma z) \geq d(z', \gamma z) - d(\gamma x, z') \geq T + \alpha - T = \alpha,$$

and therefore the above integral is equal to zero because $d(x, z) > \alpha$ and the proof of the lemma is finished. \qed

Lemma 2.2. Set $K(T, t) = \left(\frac{\partial}{\partial \sinh t}\right)^{n-1} (\cosh T - \cosh t)^{n-3/2} \times F(-1/2, 3/2, n - 1/2, (\cosh T - \cosh t)/2 \cosh T)$, (2.2)

then the following formula holds

$$K(T, t) = c_n^1 \cosh^{-1/2} T \cosh t (\cosh^2 T - \cosh^2 t)^{-1/2}$$ (2.3)

with $c_n^1 = (-1)^{n-1} \Gamma(n - 1/2) \frac{\sqrt{2}}{\sqrt{\pi}}$.

Proof. Set $z = (\cosh T - \cosh t)/2 \cosh T$, $\frac{\partial}{\partial \sinh t} = \frac{-1}{2 \cosh T} \frac{\partial}{\partial z}$, $K(T, t) = (-1)^{n-1} (2 \cosh T)^{-1/2} \times$

$$\left(\frac{\partial}{\partial z}\right)^{n-1} [z^{n-3/2} F(-1/2, 3/2, n - 1/2, z)].$$ (2.4)

Using the formula in (Magnus et al. [31] p. 41)

$$\frac{d^m}{dy^m} y^{c-1} F(a, b, c; y) = (c - m)_m y^{c-m-1} F(a, b, c - m, y)$$ (2.5)
we can write $K(T, t) = (1/2)_{n-1}(-1)^{n-1}(2 \cosh T)^{-1/2} \times$
\[ z^{-1/2} F(-1/2, 3/2, 1/2, z). \] (2.6)

Using the following relation (Magnus et al. [31] p. 50)
\[ F(a, 1 - a, c, y) = (1 - y)^{c-1} F((c - a)/2, (c + a - 1)/2, c, 4y - 4y^2), \]

$K(T, t) = (1/2)_{n-1}(-1)^{n-1}(2 \cosh T)^{-1/2} \times$
\[ z^{-1/2} F(1/2, -1/2, 1/2, 4z - 4z^2). \] (2.7)

and the formula [31] p. 38) $F(a, b, b, z) = (1 - z)^{-a}$,

$K(T, t) = (1/2)_{n-1}(-1)^{n-1}\sqrt{2}(\cosh T)^{-1/2} \cosh t(\cosh^2 T - \cosh^2 t)^{-1/2}$

$K(T, t) = c_n^{-1}(\cosh T)^{-1/2} \cosh t(\cosh^2 T - \cosh^2 t)^{-1/2}$ as stated, and the proof of Lemma 2.2 is finished.

Lemma 2.3. Set $I(T, z') = c_n \cosh^{1/2} T \int_0^T (\cosh T - \cosh t)^{n-3/2} \times$
\[ F(-1/2, 3/2, n - 1/2, \frac{\cosh T - \cosh t}{2 \cosh T}) \sinh tu(t, z') dt, \] (2.8)

where $u(t, z')$ is the solution of the Cauchy problem for the wave equation on the complex hyperbolic space, $c_n = (-1)^{n-1} \pi^{n-1/2} 2^{n+1/2}/\Gamma(n-1/2)$, $F(a, b, c, t)$ is the classical hypergeometric function $\pFq21$ given in (1.2),

then we have
\[ I(T, z') = \int_{d(z', x) < T} f(x) d\mu(x). \] (2.9)

Proof. Recall that the Cauchy problem (1.3) has a unique solution given by (Intissar-Ould Moustapha [21]):
\[ u(t, z') = (2\pi)^{-n} \left( \frac{\partial}{\sinh t \partial t} \right)^{n-1} \int_{d(z', x) < t} \frac{f(x) d\mu(x)}{\sqrt{\cosh^2 t - \cosh^2 d(z', x)}}. \] (2.10)

where $d(z', x)$ is the geodesic distance between $z'$ and $x$ in $\mathcal{H}^n$ and $d\mu(x)$ is the volume element on $\mathcal{H}^n$.

Inserting the expression of $u(t, z')$ given by (2.10) in the integral (2.8) and
integrating by parts \((n - 1)\)-times, we obtain

\[
I(T, z') = c_n(2\pi)^{-n}(\cosh T)^{1/2} \int_0^T K(T, t) \times \int_{d(z', x) < t} \frac{f(x)}{\sqrt{\cosh t - \cosh^2 d(z', x)}} d\mu(x) \sinh t dt,
\]

(2.11)

where \(K(T, t)\) is as in (2.2) Using the formula (2.3) and changing the order of integration we obtain

\[
I(T, z') = c_n(2\pi)^{-n} \int_0^T (\cosh^2 T - \cosh^2 t)^{-1/2} \times \int_{d(z', x) < t} \frac{f(x)}{\sqrt{\cosh^2 t - \cosh^2 d(z', x)}} d\mu(x) \sinh t dt 
\]

(2.12)

where

\[
j(T, r) = \int_r^T (\cosh^2 T - \cosh^2 t)^{-1/2}(\cosh^2 t - \cosh^2 r)^{-1/2} \sinh t \cosh t dt,
\]

putting, \(s = \cosh^2 t - \cosh^2 r\) we obtain

\[
j(T, r) = \int_r^T (\cosh^2 T - \cosh^2 t)^{-1/2}(\cosh^2 t - \cosh^2 r)^{-1/2} \sinh t \cosh t dt
\]

and

\[
j(T, r) = \int_0^{\cosh^2 T - \cosh^2 t} (\cosh^2 T - \cosh^2 r - s^2)^{-1/2}s^{-1/2} ds
\]

Making use of the substitution \(s = (\cosh^2 T - \cosh^2 t)z\), we have

\[
j(T, r) = \int_1^1 (1 - z)^{-1/2} z^{-1/2} dz = \frac{\beta(1/2, 1/2)}{2} = \frac{\pi}{2},
\]

and the proof of the Lemma is finished.

\[\Box\]

**Lemma 2.4.** Set

\[
H_n(\lambda, T) = \sinh^{2n} T F((n - i\sqrt{\lambda})/2, (n + i\sqrt{\lambda})/2, n + 1, -\sinh^2 T)
\]

then we have

\[
H_n(\lambda, T) = \int_0^T (\cosh T - \cosh t)^{n-1/2} \times \frac{\cosh T - \cosh t}{2\cosh T} \cos \sqrt{\lambda} dt.
\]

(2.13)
Proof. This lemma is a direct consequence of the generalized Mehler Fuchs formula for the Jacobi function (Koornwinder [24])

\[ [\Gamma(\alpha + 1)]^{-1} \Delta(T) \varphi_{\sqrt{\lambda}}^{(\alpha, \beta)}(T) = \pi^{-1/2} \int_0^T \cos \sqrt{\lambda t} E(t, T) dt, \]  

(2.14)

where \( \Delta(T) = \sinh^{2\alpha + 1} T \cosh^{2\beta + 1} \) and

\[ E(t, T) = c_\alpha \sinh 2T \cosh^{\beta - 1/2} T (\cosh T - \cosh t)^{\alpha - 1/2} \times F(1/2 + \beta, 1/2 - \beta, \alpha + 1/2, (\cosh T - \cosh t)/2 \cosh T), \]  

(2.15)

and \( c_\alpha = 2^{\alpha - 1/2} [\Gamma(\alpha + 1/2)]^{-1} \) and

\[ \varphi_{\lambda}^{(\alpha, \beta)}(x) = 2F_1 \left( \frac{\alpha + \beta + 1 - i\lambda}{2}, \frac{\alpha + \beta + 1 + i\lambda}{2}, \alpha + 1, -\sinh^2 x \right). \]

\[ \square \]

Proof of the Theorem 1.1 To prove i)

\[ N(T, z, z') = \sum_{\gamma \in \Gamma} \chi_B(z, T)(\gamma z') \]

where \( \chi_B(z, T) \) is the characteristic function of the ball \( B(z, T) \) is the complex hyperbolic ball of center \( z \) and radius \( T \)

\[ I(T, z, z', \alpha) = \int_F N(T, x, z') h(x) d\mu(x) = \int_F \sum_{\gamma \in \Gamma} \chi_B(x, T)(\gamma z') h(x) d\mu(x) \]

\[ I(T, z, z', \alpha) = \sum_{\gamma \in \Gamma} \int_{d(x, \gamma z') < T} h(x) d\mu(x) \]

\[ I(T, z, z', \alpha) = \sum_{\gamma \in \Gamma} \int_{d(\gamma x, z') < T} h(x) d\mu(x) \]

and hence

\[ I(T, z, z', \alpha) = \sum_{\gamma \in \Gamma} \int_{d(\gamma x, z') < T} h(x) d\mu(x) \]

and this prove i).

The part ii) is a consequence of the i) and the Lemma 2.1.
Finally iii) is a consequence of the Lemma 2.3 and the part i). The proof of Theorem 1.1 is finished.

Proof of the Theorem 1.2

We note that if the initial data \( f \) of the problem (1.5) is \( \Gamma \)– automorphic, then the solution \( u(t, z') \) has the same property. This follows from the uniqueness of the solution and the invariance of Laplace-Beltrami operator under motions group \( SU(n, 1) \).

On the other hand the solution of the Cauchy problem (1.5) for the initial data \( f(x) = \sum_{\gamma \in \Gamma} h(\gamma^{-1}x) \), can be represented for a sufficiently small \( \alpha \) in the form

\[
u(t, z') = \int_{-\frac{1}{2}n^2}^{+\infty} \frac{\sin \sqrt{\lambda t}}{\sqrt{\lambda}} d\lambda \int_F h(x) \theta_T(x, z', \lambda) d\mu(x).
\]

Substituting this in (2.8), we obtain after integrating by parts and changing the order of integration:

\[
I(T, z, z', \alpha) = c_n \cosh^{1/2} T \int_{-\frac{1}{2}n^2}^{+\infty} \int_0^T (\cosh T - \cosh t)^{n-1/2} \times \\
\int_{-\frac{1}{2}n^2}^{+\infty} \frac{\sin \sqrt{\lambda t}}{\sqrt{\lambda}} d\lambda \int_F h(x) \theta_T(x, z', \lambda) d\mu(x),
\]

and the proof of the theorem 1.2 is finished. Note that iii) is analogous of the formulas in the Lemma 2.5 of (Lax-Phillips [29] p.318).

3 The asymptotic behavior of the number \( N(T, z, z') \)

It is clear, from the inequalities ii) of the theorem 1.2, which equivalent to

\[
I(T - \alpha, z, z') \leq N(T, z, z', \alpha) \leq I(T + \alpha, z, z', \alpha),
\]

that is in order to study the asymptotic behavior of the number \( N(T, z, z') \) it suffices to study that of \( I(T, z, z', \alpha) \).

For this set

\[
I(T, z, z', \alpha) = I_1 + I_2
\]
where

\[ I_1 = \frac{\pi^n}{\Gamma(n+1)} \sinh^{2n} T \times \]

\[ \sum_{j=1}^{N} F\left(\frac{n + \mu_j}{2}, \frac{n + \mu_j}{2}, n + 1, -\sinh^2 T\right) \varphi_j(z') \int_{F} \varphi_j(x) h(x) d\mu(x) \]  
\[ (3.3) \]

with \( \mu_j = \sqrt{|\lambda|} \) and

\[ I_2 = \frac{\pi^n}{\Gamma(n+1)} \sinh^{2n} T \int_0^{+\infty} F((n - i\sqrt{\lambda})/2, (n + i\sqrt{\lambda})/2, n + 1, -\sinh^2 T) \times \]

\[ d\lambda \int_{F} \theta_{T}(x, z', \lambda) h(x) d\mu(x). \]  
\[ (3.4) \]

To estimate \( I_1 \) we use the formula relating hypergeometric functions of arguments \( z \) and \( 1/z \) (Magnus et al. [31]) p.48

\[ F(a, b, c, z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a, a-c+1, a-b+1, 1/z) + \]

\[ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b, b-c+1, b-a+1, 1/z) \]  
\[ (3.5) \]

and we obtain for \( T \to +\infty : \)

\[ F\left(\frac{n + \mu_j}{2}, \frac{n - \mu_j}{2}, n + 1, -\sinh^2 T\right) = c_j(n)e^{(\mu_j-n)T}[1 + O(e^{-2T})] \]  
\[ (3.6) \]

\[ c_j(n) = \frac{\Gamma(n+1)\Gamma(\mu_j)2^{-\mu_j+n}}{\Gamma((n + \mu_j)/2)\Gamma(1 + (n + \mu_j)/2)} \]  
\[ (3.7) \]

replacing in (2.2) we get:

\[ I_1 = \left(\frac{\pi}{2}\right)^n \sum_{j=1}^{N} \frac{\Gamma(\mu_j)2^{-\mu_j}}{\Gamma((n + \mu_j)/2)\Gamma(1 + (n + \mu_j)/2)} e^{(n+\mu_j)T} \varphi_j(z') \]
\[ \int_{F} \varphi_j(x) h(x) d\mu(x) + O(\alpha e^{(2n-2)T}) \]  
\[ (3.8) \]

In order to estimate \( I_2 \) we need the following lemmas
Lemma 3.1. Let \( H_n(\lambda, T) \) be as in Lemma [2.4] then we have for \( \lambda \to \infty \) and \( T \to \infty \), the following estimates hold

\[
H_n(\lambda, T) = O(\lambda^{-(2n+1)/4} e^{nT})
\] (3.9)

Proof. We remark that for \( 0 \leq \frac{\cosh T - \cosh t}{\cosh T} \leq \frac{1}{2} \) and hence the hypergeometric in the R.H.S. of (2.13) is bounded,

\[
H_n(\lambda, T) = O(\int_0^T (\cosh T - \cosh t)^{(2n-1)/2} \cos t \sqrt{\lambda} dt) \tag{3.10}
\]

using the formula (2.14) in (Levitan [30], p. 27)

\[
\int_0^T (\cosh T - \cosh t)^{(m-1)/2} \cos t \sqrt{\lambda} dt = O(\lambda^{-(m+1)/4} e^{(m-1)/4T})
\]

we have the result of the Lemma.

Proof of Theorem 1.3 From (2.16) and Lemma 2.4, we have:

\[
I_2 = O(e^{nT} \int_F h(x) \int_0^1 |d\lambda \theta_T(x, z', \lambda)| d\mu(x)) +
\]

\[+O(e^{nT} \int_1^{+\infty} \lambda^{-\frac{2n+1}{4}} |d\lambda \int_F \theta_T(x, z', \lambda) h(x) d\mu(x)|)\]

\[
I_2 = O(e^{nT}) + O(e^{nT} \int_1^{+\infty} \lambda^{-\frac{2n+1}{4}} |d\lambda \int_F \theta_T(x, z', \lambda) h(x) d\mu(x)|)
\]

set :

\[
J = \int_1^{+\infty} \lambda^{-\frac{2n+1}{4}} |d\lambda \int_F \theta_T(x, z', \lambda) h(x) d\mu(x)|
\]

let \( \delta > 0 \), set :

\[
J = J_1 + J_2 \tag{3.11}
\]

where

\[
J_1 = \int_1^{e^{\delta T}} \lambda^{-\frac{2n+1}{4}} |d\lambda \int_F \theta_T(x, z', \lambda) h(x) d\mu(x)|
\]

and

\[
J_2 = \int_{e^{\delta T}}^{+\infty} \lambda^{-\frac{2n+1}{4}} |d\lambda \int_F \theta_T(x, z', \lambda) h(x) d\mu(x)|
\]

12
Hence we have

$$I_2 = O(e^{nT}) + O(e^{nT}J_1) + O(e^{nT}J_2).$$ \hfill (3.12)

Next we use the inequality as in [30] page 29:

$$|\Delta \theta(x, z', \lambda)| \leq \frac{1}{2} [\Delta \theta(x, x, \lambda) + \Delta \theta(z', z', \lambda)]$$ \hfill (3.13)

to obtain:

$$J_1 \leq \frac{1}{2} \int_{1}^{e^{\delta T}} \lambda^{-\frac{2n+1}{4}} d\theta(x, x, \lambda) + \frac{1}{2} \int_{1}^{e^{\delta T}} \lambda^{-\frac{2n+1}{4}} d\theta(z', z', \lambda)$$

and by the estimate [18]:

$$\theta(x, x, \lambda) = O(\lambda^n)$$ \hfill (3.14)

we have:

$$J_1 = O(e^{(\frac{2n-1}{4})\delta T}).$$ \hfill (3.15)

For the estimation of $J_2$, we recall the expansion of the spectral function of an elliptic operator in terms of its eigenfunctions:

$$\theta_{\Gamma}(x, z', \lambda) = \int_{-n^2}^{\lambda} \sum_{j=1}^{N(\lambda)} \varphi_j(\nu, z') \overline{\varphi_j(\nu, x)} \, d\rho(\nu)$$ \hfill (3.16)

where $1 \leq N(\nu) \leq \infty, \varphi_j(x, \nu)$ is a $\Gamma-$ automorphic solution of the equation $L \varphi = \nu \varphi_j$ and $d\rho(\nu)$ is a non decreasing function (see Berezanskii [4] and Gel’fand- Kostyuchenko [14]).

Recall that in this case, Parseval’s equation is given by:

$$\int_{F} |f(x)|^2 d\mu(x) = \int_{-n^2}^{+\infty} \sum_{j=1}^{N(\lambda)} |\int_{F} f(z') \overline{\varphi_j(z', \nu)} d\mu(z')|^2 d\rho(\lambda)$$ \hfill (3.17)

From (3.16) we have:

$$J_2 \leq \int_{e^{\delta T}}^{+\infty} \lambda^{-\frac{2n+1}{4}} \sum_{j=1}^{N(\lambda)} |\varphi_j(z', \lambda)||\int_{F} h(x) \overline{\varphi_j(x, \lambda)} d\mu(x)| d\rho(\lambda).$$

13
Using the Cauchy-Shwarz inequality twice, we obtain:

\[ J_2 \leq \int_{e^{\delta T}}^{+\infty} \lambda^{-\frac{2n+1}{2}} \left( \sum_{j=1}^{N(\lambda)} |\varphi_j(z', \lambda)|^2 \right)^{1/2} \left( \sum_{j=1}^{N(\lambda)} |\int_{F} h(x) \varphi_j(x, \lambda) d\mu(x)|^2 \right)^{1/2} d\rho(\lambda) \]

\[ \leq \left( \int_{e^{\delta T}}^{+\infty} \lambda^{-\frac{2n+1}{2}} \sum_{j=1}^{N(\lambda)} |\varphi_j(z', \lambda)|^2 d\rho(\lambda) \right)^{1/2} \left( \int_{e^{\delta T}}^{+\infty} \sum_{j=1}^{N(\lambda)} |\int_{F} h(x) \varphi_j(x, \lambda) d\mu(x)|^2 d\rho(\lambda) \right)^{1/2} \]

and by using (3.16) and (3.17) we obtain:

\[ J_2 \leq \left( \int_{e^{\delta T}}^{+\infty} \lambda^{-\frac{2n+1}{2}} d\lambda \right)^{1/2} \left( \int_{F} h^2(x) d\mu(x) \right)^{1/2}. \]

Using the estimates (3.14) and the formula \( \int h^2(x) d\mu(x) = O(\alpha^{-2n}) \), we have:

\[ J_2 = O(e^{-\delta T/4} \alpha^{-n}) \] (3.18)

It follows from (3.2), (3.8), (3.12), (3.15) and (3.18) that:

\[ I(T \pm \alpha, z_0, z, \alpha) = A(T, z_0, z) + O(\alpha e^{2(n-1)T}) + O(e^{n+\frac{2n-1}{2}}) + O(e^{\frac{n-\delta}{2} T_\alpha^{-n}}) \]

Putting \( \alpha = e^{-\epsilon}, \epsilon > 0 \) the best ratio between \( \epsilon \) and \( \delta \) is obtained by satisfying the equation:

\[ n + \frac{2n - 1}{4} \delta = (n - \delta/4) + n \epsilon = 2n - 1 - \epsilon \]

from which it follows that for \( n \geq 2 \)

\[ \delta = 2\epsilon, \quad \epsilon = \frac{2(n - 1)}{2n + 1}, \quad 2n - 1 - \epsilon = 2n - 1 - \frac{2(n - 1)}{2n + 1} \]

and for \( n > 2 \)

\[ \delta = 2\epsilon, \quad \epsilon = \frac{2(n - 2)}{2n + 1}, \quad 2n - 1 - \epsilon = 2n - 2 - \frac{2(n - 2)}{2n + 1} \]

This completes the proof of Theorem 1.3.
4 Appendix

Let \( z \) be a fixed point in the complex hyperbolic space \( \mathcal{H}^n \), and let \( h_1 \) be a smooth function whose support is concentrated in the complex hyperbolic ball of radius 1 and centre \( z \), and suppose that

\[
\int h_1(x) d\mu(x) = 1 \quad (4.1)
\]

Using the geodesic polar coordinates on the the complex hyperbolic space \( \mathcal{H}^n \), \( x = \tanh r \omega, r \geq 0 \) and \( \omega \in S^{2n-1} \), for \( \alpha \) a small positive number, put

\[
h(x) = c_n(\alpha) h_1(\frac{r}{\alpha}, \theta), r = d(x, z), \quad (4.2)
\]

where the constant \( c_n(\alpha) \) is chosen so that

\[
\int h(x) d\mu(x) = 1. \quad (4.3)
\]

We show that the constant \( c_n(\alpha) \) satisfies the condition

\[
\lim_{\alpha \to 0} \alpha^{2n} c_n(\alpha) = 1.
\]

\[
1 = c_m(\alpha) \int_{S^{2n-1}} d\theta \int_0^\alpha h_1(\frac{r}{\alpha}, \theta) \sinh^{2n-1} r \cosh r dr \quad (4.4)
\]

\[
1 = c_m(\alpha) \alpha \int_{S^{2n-1}} d\theta \int_0^1 h_1(t, \theta) \sinh^{2n-1} \alpha t \cosh \alpha t dt \quad (4.5)
\]

\[
1 = \alpha^{2n} c_m(\alpha) \int_{S^{2n-1}} d\theta \int_0^1 h_1(t, \theta) \left( \frac{\sinh \alpha t}{\alpha \sinh t} \right)^{2n-1} \sinh^{2n-1} t \cosh t dt \quad (4.6)
\]

\[
1 = \alpha^{2n} c_m(\alpha) \left[ \int_{S^{2n-1}} d\omega \int_0^1 h_1(t, \omega) \sinh^{2n-1} t \cosh t dt + o(1) \right] \quad (4.7)
\]

\[
1 = \alpha^{2n} c_n(\alpha) [1 + o(1)]. \quad (4.8)
\]
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