THE BOSON STAR EQUATION WITH INITIAL DATA OF LOW REGULARITY

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Abstract. The Cauchy problem for the \(L^2\)-critical boson star equation with initial data of low regularity in spatial dimension \(d = 3\) is studied. Local well-posedness in \(H^s\) for \(s \geq 1/4\) is proved. Moreover, for radial initial data, local well-posedness is established in \(H^s\) for \(s > 0\). Both results are shown to be optimal by providing complementary ill-posedness results.

1. Introduction and Main Results

We consider the initial value problem

\[
\begin{aligned}
i \partial_t u &= \sqrt{-\Delta + m^2} u - (|x|^{-1} * |u|^2) u \quad \text{in } (-T, T) \times \mathbb{R}^3, \\
u(0, \cdot) &= \phi \in H^s(\mathbb{R}^3).
\end{aligned}
\] (1)

Here \(\sqrt{-\Delta + m^2}\) is defined via its symbol \(\sqrt{\xi^2 + m^2}\) in Fourier space, where the constant \(m \geq 0\) is a physical mass parameter, and the symbol \(\ast\) denotes convolution in \(\mathbb{R}^3\). The nonlinear dispersive evolution problem (1) arises as an effective equation describing the dynamics and gravitational collapse of relativistic boson stars; see [16, 7, 14, 8] and references therein. Given this physical background, we shall also refer to equation (1) as the boson star equation in the following.

We recall that the boson star equation exhibits the following conserved quantities of energy and \(L^2\)-mass, which are given by

\[
\begin{aligned}
E(u(t)) &= \frac{1}{2} \int_{\mathbb{R}^3} \overline{u} \sqrt{-\Delta + m^2} u \, dx - \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 \, dx, \\
M(u(t)) &= \int_{\mathbb{R}^3} |u|^2 \, dx.
\end{aligned}
\]

From these conservation laws, we see that the Sobolev space \(H^{1/2}(\mathbb{R}^3)\) serve as the energy space for problem (1). Furthermore, in the case of vanishing mass parameter \(m = 0\) in (1), we have the scaling symmetry

\[
u(t, x) \mapsto u_\lambda(t, x) = \lambda^{3/2} u(\lambda t, \lambda x)
\]
for fixed $\lambda > 0$. Clearly, the symmetry leaves the $L^2$-mass $M(u(t)) = M(u_\lambda(t))$ invariant. In this sense, the boson star equation exhibits the delicate feature of $L^2$-criticality.

The aim of this paper is to establish a local well-posedness theory for the boson star equation below $H^{1/2}$, which is sharp with respect to minimal regularity requirements on the initial data in the Sobolev scale $H^s(\mathbb{R}^3)$ and $H^s_{\text{rad}}(\mathbb{R}^3)$ (subspace of radial functions), respectively. Let us state our first main result.

**Theorem 1.1.** Let $m \geq 0$.

(i) Let $s \geq 1/4$. The initial value problem (1) is locally well-posed for initial data in $H^s(\mathbb{R}^3)$.

(ii) Let $s > 0$. The initial value problem (1) is locally well-posed for initial data in $H^s_{\text{rad}}(\mathbb{R}^3)$.

In both cases, the notion of local well-posedness includes existence for $T > 0$ sufficiently small and uniqueness of solutions in a certain subspace, persistence of initial regularity, and analytic dependence on the initial data.

A first well-posedness result for the boson star equation was obtained by the second author of this paper (see [12]) in $H^s$ for $s \geq 1/2$ by energy methods. Moreover, it was shown in [12] that global well-posedness holds in $H^{1/2}$ for initial data sufficiently small in $L^2$. In contrast to this, see [9] for the existence of finite-time blowup solutions with smooth initial data that are large in $L^2$. Moreover, we refer the reader to [4, 5, 6] for further well-posedness results for the boson star equation with initial data in $H^s$ with $s$ slightly below $1/2$.

Our second result shows that Theorem 1.1 is sharp in the following sense.

**Theorem 1.2.** Let $m \geq 0$.

(i) Let $s < 1/4$ and $T > 0$. If the flow map $\phi \mapsto u$ exists (in a small neighbourhood of the origin) as a map from $H^s(\mathbb{R}^3)$ to $C([0, T], H^s(\mathbb{R}^3))$, it fails to be $C^3$ at the origin.

(ii) Let $s < 0$ and $T > 0$. If the flow map $\phi \mapsto u$ exists (in a small neighbourhood of the origin) as a map from $H^s_{\text{rad}}(\mathbb{R}^3)$ to $C([0, T], H^s_{\text{rad}}(\mathbb{R}^3))$, it fails to be $C^3$ at the origin.

Ill-posedness results similar to Theorem 1.2 have been proved in [17, Theorem 2] for the Benjamin-Ono equation and in [3, pp. 155–158] for the Korteweg-de Vries equation. Compared to the $L^2$-critical NLS, it is interesting to note that the boson star equation (1) exhibits ill-posedness for non-radial data (in the sense given above) in a regularity...
class above the critical space $L^2$. Not so surprisingly, such a phenomenon of ill-posedness above scaling is much more akin to nonlinear wave equations (see [15]).

We remark that both Theorem 1.1 and Theorem 1.2 remain true in the defocusing case, i.e., we may replace $-|x|^{-1}$ by $+|x|^{-1}$ in (1).

For radial data in $L^2(\mathbb{R}^3)$ we prove the failure of uniform continuity on balls, similar to the results in [1] for nonlinear Schrödinger and generalized Benjamin-Ono equations:

**Theorem 1.3.** Let $m \geq 0$ and $T > 0$. If the flow map $\phi \mapsto u$ exists as a map from $L^2_{rad}(\mathbb{R}^3)$ to $C([0, T], L^2_{rad}(\mathbb{R}^3))$, it fails to be locally uniformly continuous.

We refer the reader to Propositions 3.3 and 3.7 for more precise statements. The proof of Theorem 1.3 utilizes the fact that (1) has solitary wave solutions $u(t, x) = e^{it\mu}Q_\mu(x)$, where $Q_\mu \in H^{1/2}(\mathbb{R}^3)$ solves the nonlinear elliptic equation

$$\sqrt{-\Delta + m^2}Q_\mu + \mu Q_\mu - (|x|^{-1} * |Q_\mu|^2)Q_\mu = 0,$$

with $\mu > 0$ given. In the case $m = 0$, the exact scaling properties simplify the analysis considerably and we can adapt an argument in [1] in order to prove Theorem 1.3. In contrast to this, the case $m > 0$ breaking exact scaling deserves an additional discussion of the elliptic problem for $Q_\mu$.

When studying (1), it might be convenient to add the linear term $mu$ by considering the function $e^{-itm}u$ instead of $u$, which turns the equation (1) into

$$i\partial_t u = (\sqrt{-\Delta + m^2} - m)u - (|x|^{-1} * |u|^2)u. \quad (2)$$

Furthermore, by rescaling $u$, it suffices to consider either $m = 0$ or $m = 1$ in the proofs.

The paper is organized as follows: In Section 2 we prove Theorem 1.1 concerning local well-posedness. In Section 3 we prove Theorems 1.2 and Theorem 1.3 on ill-posedness.

2. Well-posedness

Let us fix some notation. We denote the Fourier transform of a tempered distribution $f$ both by $\mathcal{F}f$ and $\hat{f}$, and to indicate a partial Fourier transform with respect to time and space variables we also write $\mathcal{F}_t f$ and $\mathcal{F}_x f$, respectively.

We denote dyadic numbers $\lambda = 2^l : l \in \mathbb{Z}$ by greek letters. Further, for $\lambda > 1$ we define dyadic annuli

$$\Delta_\lambda := \{ \xi \in \mathbb{R}^3 : \lambda/2 < |\xi| \leq 2\lambda \}, \text{ and } \Delta_1 := \{ \xi \in \mathbb{R}^3 : |\xi| \leq 1 \}. $$
We fix an even function $\beta_1 \in C_0^\infty((-2, 2))$ s.t. $\beta_1(s) = 1$ if $|s| \leq 1$, and define $\beta_\lambda(s) = \beta(s\lambda^{-1}) - \beta(2s\lambda^{-1})$ for $\lambda > 1$. Next, we define the (smooth) Fourier localization operators in the standard way: 

$$P_\lambda f = F^{-1}(\beta_\lambda(|\cdot|)Ff).$$

For measurable sets $S \subset \mathbb{R}^n$, let $\chi_S$ denote the sharp cutoff function, i.e. define $\chi_S(x) = 1$ if $x \in S$ and zero otherwise. We set 

$$P_S f = F^{-1}(\chi_S Ff).$$

For fixed $m \geq 0$ let $U(t)$ be the linear propagator defined by 

$$F_x(U(t)\phi)(\xi) = e^{it\varphi_m(\xi)}F_x\phi(\xi), \quad \varphi_m(\xi) = m - \sqrt{m^2 + |\xi|^2}.$$

**Definition 2.1.** Let $s, b \in \mathbb{R}$. We define the space $X_{s,b}$ of tempered distributions $u \in S(\mathbb{R}^{3+1})$ such that 

$$\|u\|_{s,b} := \left(\int_{\mathbb{R}^{3+1}} \langle \xi \rangle^{2s} \langle \tau + |\xi| \rangle^{2b} |\mathcal{F}u(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}} < \infty. \quad (3)$$

Further, $X_{s,b}^{\text{rad}}$ denotes the closed subspace of spatially radial distributions.

Concerning well-posedness, it suffices to consider the case $m = 0$ as long as we are considering short time scales only. Indeed, $\langle \tau + |\xi| \rangle \sim \langle \tau + \sqrt{m + |\xi|^2} - m \rangle$ and the corresponding $\|\cdot\|_{s,b}$-norms are equivalent.

We first recall Strichartz type estimates for the wave equation. They have been studied systematically in [18], see also references therein.

**Lemma 2.2.** Let $b > \frac{1}{2}$. 

(i) For all $\mu \geq 0$ and for all $u \in X_{\frac{1}{4},b}^{\frac{1}{4},b}$ it holds 

$$\|P_{B_\mu} u\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^{\frac{1}{4}} \|u\|_{\frac{1}{4},b}. \quad (4)$$

(ii) For all $u \in X_{\frac{1}{2},b}^{\frac{1}{2},b}$ it holds 

$$\|u\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|u\|_{\frac{1}{2},b}. \quad (5)$$

(iii) For all $\mu \geq 0$ and for all $u \in X_{\frac{1}{4},b}^{\frac{1}{4},b}$ it holds 

$$\|P_\mu(\tilde{u}_1 \tilde{u}_2)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^{\frac{1}{2}} \|u_1\|_{\frac{1}{4},b} \|u_2\|_{\frac{1}{4},b}, \quad (6)$$

where $\tilde{u}_j$ denotes either $u_j$ or $\bar{u}_j$.

**Proof.** Part i) Due to [18] Theorem 4.1] and the extension lemma [10] Lemma 2.3] we have

$$\|P_{B_\mu} P_\lambda u\|_{L^4(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^{\frac{1}{4}} \lambda^{\frac{1}{4}} \|P_\lambda u\|_{0,b}.$$
By Littlewood-Paley theory it follows that
\[ \|P_{B_{\mu}}u\|_{L^4} \lesssim \left( \sum_{\lambda \geq 1} \|P_{\lambda}P_{B_{\mu}}u\|_{L^4(\mathbb{R} \times \mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \]
\[ \lesssim \mu^{\frac{1}{2}} \left( \sum_{\lambda \geq 1} \lambda^{\frac{1}{2}} \|P_{\lambda}u\|_{0,b}^2 \right)^{\frac{1}{2}} \lesssim \mu^{\frac{1}{2}} \|u\|_{\frac{1}{4},b}. \]

**Part ii)** The Littlewood-Paley estimate and Part i) with \( \mu \sim \lambda \) imply
\[ \|u\|_{L^4} \lesssim \left( \sum_{\lambda \geq 1} \|P_{\lambda}u\|_{L^4(\mathbb{R} \times \mathbb{R}^3)}^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{\lambda \geq 1} \|P_{\lambda}u\|_{0,b}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{0,b}. \]

**Part iii)** We use almost orthogonality: Let us consider the collection of cubes \( C_z = \mu z + [0, \mu]^3, z \in \mathbb{Z}^3 \), which induce a disjoint covering of \( \mathbb{R}^3 \). We have
\[ \|P_{C_z}u_1 P_{C_z'}u_2\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|P_{C_z}u_1 P_{C_z'}u_2\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \]

For each \( z \in \mathbb{Z}^3 \), only those \( z' \in \mathbb{Z}^3 \) with \( |z - z'| \leq 1 \) yield a nontrivial contribution to the sum. Hence, by Cauchy-Schwarz and Part i) we obtain
\[ \|P_{\mu}(\tilde{u}_1 \tilde{u}_2)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^{\frac{1}{2}} \left( \sum_{z \in \mathbb{Z}^3} \|P_{B_{\sqrt{\mu}z}}u_1\|_{\frac{3}{4},b}^2 \right)^{\frac{1}{2}} \left( \sum_{z' \in \mathbb{Z}^3} \|P_{B_{\sqrt{\mu}z'}}u_2\|_{\frac{3}{4},b}^2 \right)^{\frac{1}{2}}, \]
and the claim follows from \( \sum_{z \in \mathbb{Z}^3} \chi_{B_{\sqrt{\mu}z}}^2(\mu z) \lesssim 1 \). \( \square \)

**Remark 1.** By (complex) interpolation of \( \|u\|_{L^4_t L^4_x} \lesssim \|u\|_{\frac{3}{2},b} \) and \( \|u\|_{L^4_t L^6_x} = \|u\|_{0,0} \) resp. \( \|u\|_{L^\infty_t L^2_x} = \|u\|_{0,b} \) we obtain
\[ \|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{\frac{3}{2},\frac{3}{2}} \]
\[ \|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{\frac{3}{2},b} \]
(7) (8)
if \( b > \frac{1}{2} \). The Sobolev embedding implies
\[ \|u\|_{L^4_t L^4_x} \lesssim \|D^{\frac{1}{2}} u\|_{L^4_t L^2_x} \lesssim \|u\|_{\frac{3}{4},\frac{3}{4}}. \]
By interpolation with \( \|u\|_{L^4_t L^4_x} \lesssim \|u\|_{\frac{1}{2},b} \) (for \( b > \frac{1}{2} \)) we obtain the following: For any \( \delta > 0 \) there exists \( b < \frac{1}{2} \) such that
\[ \|u\|_{L^4_t L^4_x} \lesssim \|u\|_{\frac{1}{2} + \delta,b}. \]
(9)
In the case of radial data there is the following improvement, which is an immediate consequence of [18, Theorem 2.6].
Lemma 2.3. Let $b > \frac{1}{2}$. For all $\lambda \geq \mu \geq 0$, and for all $u_1, u_2 \in X^{0,b}_{rad}$
\[
\|P_\mu (P_\lambda \tilde{u}_1 P_\lambda \tilde{u}_2)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu \|u_1\|_{0,b} \|u_2\|_{0,b},
\]
where $\tilde{u}_j$ denotes either $u_j$ or $\Pi_j$.

Proof. It suffices to consider $\lambda \gg \mu$. Decompose $u_j = u_{j,low} + u_{j,high}$ into low and high modulation, i.e.
\[
\text{supp} \mathcal{F} u_{j,low} \subseteq \{ (\tau, \xi) : |\tau + |\xi| | \leq \mu \}
\]
\[
\text{supp} \mathcal{F} u_{j,high} \subseteq \{ (\tau, \xi) : |\tau + |\xi| | > \mu \}
\]
Because of
\[
\text{supp} \mathcal{F} (P_\mu (P_\lambda u_{1,low} P_\lambda u_{2,low})) \subseteq \{ (\tau, \xi) : |\tau + |\xi| | \lesssim \mu \}
\]
the estimate of [18, Theorem 2.6] with $r \sim \mu$ readily implies
\[
\|P_\mu (P_\lambda \tilde{u}_1,low P_\lambda \tilde{u}_2,low)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu \|u_1\|_{0,b} \|u_2\|_{0,b}.
\]
On the other hand,
\[
\|P_\mu (P_\lambda \tilde{u}_1,low P_\lambda \tilde{u}_2,high)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^2 \|P_\lambda \tilde{u}_1,low P_\lambda \tilde{u}_2,high\|_{L^2_t L^2_x}
\]
\[
\lesssim \mu^2 \|P_\lambda \tilde{u}_1,low\|_{L^\infty_t L^2_x} \|P_\lambda \tilde{u}_2,high\|_{L^2_t L^2_x}
\]
\[
\lesssim \mu^{\frac{3}{2} - b} \|u_1\|_{0,b} \|u_2\|_{0,b}.
\]
The same argument applies in the remaining cases, since at least one factor has high modulation. \qed

Remark 2. By the Sobolev embedding, we obtain
\[
\|P_\mu (P_\lambda \tilde{u}_1 P_\lambda \tilde{u}_2)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^2 \|P_\mu (P_\lambda \tilde{u}_1 P_\lambda \tilde{u}_2)\|_{L^2_t L^4_x} \lesssim \mu^2 \|u_1\|_{0,\frac{1}{4}} \|u_2\|_{0,\frac{1}{4}}.
\]
Interpolation with (10) implies that for any $\delta > 0$ there exists $b < \frac{1}{2}$ such that
\[
\|P_\mu (P_\lambda \tilde{u}_1 P_\lambda \tilde{u}_2)\|_{L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \mu^{1+\delta} \|u_1\|_{0,b} \|u_2\|_{0,b}
\]
for radial $u_1, u_2$.

Next, we are prepared to prove

Proposition 2.4. Let $s \geq \frac{1}{4}$. There exists $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b' + 1$ and $\delta > 0$, such that
\[
\|x^{-1} \ast (u_1 \tilde{u}_2) u_3\|_{s,b} \lesssim T^\delta \|u_1\|_{s,b} \|u_2\|_{s,b} \|u_3\|_{s,b}
\]
for all $X^{s,b}$ with $\text{supp}(u_j) \subset \{ (t, x) : |t| \leq T \}$. 

Proof. Without loss we may assume $\frac{1}{4} \leq s < \frac{1}{2}$. We notice that in $\mathbb{R}^3$ convolution with $|x|^{-1}$ is (up to a multiplicative constant) the Fourier-multiplier $|D|^{-2}$ with symbol $|\xi|^{-2}$. By duality, it suffices to prove

$$I := \left| \int \int |D|^{-2} (u_1 \overline{u}_2) u_3 \langle D \rangle^s \overline{u}_4 dt dx \right|$$

$$\lesssim T^\delta \| u_1 \|_{s,b} \| u_2 \|_{s,b} \| u_3 \|_{s,b} \| u_4 \|_{0,-b'}.$$  

It suffices to consider $u_j$ with non-negative Fourier-transform. Then, we may split the left hand side into two terms:

$$I \lesssim \left| \int \int \langle D \rangle^s |D|^{-2} (u_1 \overline{u}_2) u_3 \overline{u}_4 dt dx \right| + \left| \int \int |D|^{-2} (u_1 \overline{u}_2) \langle D \rangle^s u_3 \overline{u}_4 dt dx \right|$$

$$=: I_1 + I_2.$$

First, we consider the contribution of $I_1$:

$$I_1 \leq \sum_{\mu \geq 1} \mu^{s-2} \| P_\mu (u_1 \overline{u}_2) \|_{L^4_t L^4_x} \| u_3 \overline{u}_4 \|_{L^\frac{4}{3} L^\frac{4}{3}}.$$

By Sobolev’s embedding and Hölder’s inequality we obtain

$$\| P_\mu (u_1 \overline{u}_2) \|_{L^4_t L^4_x} \lesssim \mu^{\frac{3}{4}} \| u_1 \overline{u}_2 \|_{L^4_t L^4_x} \lesssim \mu^{\frac{3}{4}} \| u_1 \|_{L^4_t L^4_x} \| \overline{u}_2 \|_{L^4_t L^4_x}.$$  

Due to (7), (8) and $\sum_{\mu \geq 1} \mu^{s-\frac{1}{2}} < \infty$ (dyadic summation) we obtain

$$I_1 \lesssim T^\delta \| u_1 \|_{s,b} \| u_2 \|_{s,b} \| u_3 \|_{s,b} \| u_3 \|_{\frac{1}{4},-b'}$$

if $b' < -1/4$.

Finally, we turn to $I_2$: By Cauchy-Schwarz,

$$I_2 \leq \sum_{\mu \geq 1} \mu^{-2} \| P_\mu (u_1 \overline{u}_2) \|_{L^2} \| P_\mu (\langle D \rangle^s u_3 \overline{u}_4) \|_{L^2}$$

$$\lesssim \| u_1 \|_{s,b} \| u_2 \|_{s,b} \sum_{\mu \geq 1} \mu^{-\frac{3}{2}} \| P_\mu (\langle D \rangle^s u_3 \overline{u}_4) \|_{L^2}$$

due to (3). Now, we use the Sobolev embedding and Hölder’s inequality

$$\| P_\mu (\langle D \rangle^s u_3 \overline{u}_4) \|_{L^2} \lesssim \mu^{g/8} \| \langle D \rangle^s u_3 \overline{u}_4 \|_{L^2_t L^{8/7}_x} \lesssim \mu^{g/8} \| \langle D \rangle^s u_3 \|_{L^8_t L^4_x} \| u_4 \|_{L^s_t L^{8/3}_x}$$

and due to $\sum_{\mu \geq 1} \mu^{-\frac{3}{2}} < \infty$ and (7) we obtain

$$\sum_{\mu \geq 1} \mu^{-\frac{3}{2}} \| P_\mu (u_3 \overline{u}_4) \|_{L^2} \lesssim T^{\delta - \frac{3}{4}} \| u_4 \|_{s,b} \| u_4 \|_{s,-b'}$$

if $b' < -\frac{1}{4}$.

Next, we consider the radial case.
Proposition 2.5. Let $s > 0$. There exists $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b' + 1$ and $\delta > 0$, such that

$$\|x|^{-1} \ast (u_1 \overline{u}_2) u_3\|_{s,b'} \lesssim T^\delta \|u_1\|_{s,b} \|u_2\|_{s,b} \|u_3\|_{s,b}$$

(14)

for all $u_j \in X_{rad}^{s,b}$ with $\text{supp}(u_j) \subset \{(t,x) : |t| \leq T\}$.

Proof. As above, by duality, it suffices to prove

$$I := \left| \iiint |D|^{-2}(u_1 \overline{u}_2) u_3 \langle D \rangle^s \overline{u}_4 dt dx \right| \lesssim T^\delta \|u_1\|_{s,b} \|u_2\|_{s,b} \|u_3\|_{s,b} \|u_4\|_{0,-b'}$$

(15)

for $0 < s < \frac{1}{2}$. It suffices to consider $u_j$ with non-negative Fourier-transform. Then, we may split the left hand side into two terms:

$$I \lesssim \iiint \langle D \rangle^s |D|^{-2}(u_1 \overline{u}_2) u_3 \overline{u}_4 dt dx + \left| \iiint |D|^{-2}(u_1 \overline{u}_2) \langle D \rangle^s u_3 \overline{u}_4 dt dx \right| =: I_1 + I_2.$$

First, we consider the contribution of $I_1$:

$$I_1 \leq \|\langle D \rangle^s |D|^{-1}(u_1 \overline{u}_2)\|_{L^2} \|\|D|^{-1}(u_3 \overline{u}_4)\|_{L^2} \lesssim I_{11} \cdot I_{12}$$

By symmetry, Hölder, (5) and (13) we have

$$I_{11} = \|\langle D \rangle^s |D|^{-1}(u_1 \overline{u}_2)\|_{L^2} \lesssim \sum_{\lambda_1 \ll \lambda_2} \Lambda_{\lambda_1}^{s-1} \|u_1, \lambda_1 \overline{u}_2, \lambda_2\|_{L^2} + \sum_{\mu \leq \lambda} \mu^{s-1} \|P_\mu(u_1, \lambda_1 \overline{u}_2, \lambda_2)\|_{L^2}$$

$$\lesssim \sum_{\lambda_1 \ll \lambda_2} \Lambda_{\lambda_1}^{s-\frac{1}{2}} \Lambda_{\lambda_2}^{\frac{s}{2}} \|u_1, \lambda_1\|_{0,b} \|u_2, \lambda_2\|_{0,b} + \sum_{\mu \leq \lambda} \mu^{s} \|u_1, \lambda\|_{0,b} \|u_2, \lambda\|_{0,b}$$

$$\lesssim \sum_{k=0}^\infty \sum_{0 \leq l \leq k} 2^{-l(\frac{1}{2} - s)} \|u_1, 2^k - l\|_{s,b} \|u_2, 2^k\|_{0,b} + \|u_1\|_{s,b} \|u_2\|_{0,b}$$

$$\lesssim \|u_1\|_{s,b} \|u_2\|_{0,b}$$

We decompose $I_{12} = J_1 + J_2 + J_3$. Using (9) with $\delta = s$ we see that

$$J_1 := \sum_{\lambda_3 \ll \lambda_4} \lambda_3^{-1} \|u_3, \lambda_3 \overline{u}_4, \lambda_4\|_{L^2} \lesssim \sum_{\lambda_3 \ll \lambda_4} \lambda_3^{\frac{s}{2}} \lambda_4^{\frac{1}{2} + s} \|u_3, \lambda_3\|_{0,b} \|u_4, \lambda_4\|_{0,-b'}$$

$$\lesssim \|u_3\|_{s,b} \|u_4\|_{0,-b'}$$

for some $b' > -\frac{1}{2}$. Similarly,

$$J_2 := \sum_{\lambda_3 \gg \lambda_4} \lambda_3^{-\frac{3}{2}} \lambda_4^{\frac{1}{2} + s} \|u_3, \lambda_3\|_{0,b} \|u_4, \lambda_4\|_{0,-b'} \lesssim \|u_3\|_{s,b} \|u_4\|_{0,-b'}.$$
Using (11) we obtain
\[ J_3 := \sum_{\mu \leq \lambda} \mu^{-1} \| P_{\mu}(u_{3,\lambda} \overline{\mu}_{4,\lambda}) \|_{L^2} \lesssim \sum_{\mu \leq \lambda} \mu^s \| u_{3,\lambda} \|_{b,0} \| u_{4,\lambda} \|_{0,-b'} \]
\[ \lesssim \| u_3 \|_{s,b} \| u_4 \|_{0,-b'} \]
for some $b' > -\frac{1}{2}$. All in all, we have proved
\[ I_1 \lesssim \| u_1 \|_{s,b} \| u_2 \|_{b,0} \| u_3 \|_{s,b} \| u_4 \|_{0,-b'} . \]
Concerning $I_2$, we obtain
\[ I_2 \leq \| \langle D \rangle^s |D|^{-1} (u_1 \overline{u}_2) \|_{L^2} \| \langle D \rangle^{-s} |D|^{-1} (\langle D \rangle^s u_3 \overline{u}_4) \|_{L^2} \lesssim I_{21} \cdot I_{22} \]
The estimate for $I_{11}$ above also applies to $I_{21}$. As above, we decompose $I_{22} = K_{1} + K_{2} + K_{3}$. Trivial modifications of the arguments above yield
\[ I_{22} \lesssim \| u_3 \|_{s,b} \| u_4 \|_{0,-b'} , \]
and altogether we obtain
\[ I_2 \lesssim \| u_1 \|_{s,b} \| u_2 \|_{b,0} \| u_3 \|_{s,b} \| u_4 \|_{0,-b'} . \]
By replacing $\| u_4 \|_{0,-b'}$ by $\| u_4 \|_{0,-b''}$ for some $-\frac{1}{2} < -b'' < b'$ shows that one can squeeze out a factor $T^\delta$.

Finally, we explain how Propositions 2.4 and 2.5 imply Theorem 1.1: The general idea – due to Bourgain [2] – is well-known by now, see e.g. [10] and references therein for more details. The basic idea is to solve the integral equation
\[ u(t) = \psi_T U(t) \phi + i \psi_T \int_0^t U(t - \tau) (|x|^{-1} * |U(\tau)\phi|^2 U(\tau) \phi) d\tau \quad (16) \]
with the smooth cutoff $\psi_T(t) = \beta_1(t/T)$ and given initial data $\phi$ by the contraction mapping principle. For $b > \frac{1}{2}$ [10] (2.19) implies
\[ \| \psi_T U(t) \phi \|_{s,b} \lesssim T^{\frac{1}{2} - b} \| \phi \|_{H^s} \]
Also, [10] Lemma 2.1 implies that
\[ \| \psi_T \int_0^t U(t - \tau) f(\tau) d\tau \|_{s,b} \lesssim T^{1-b+b'} \| f \|_{s,b'} . \]
if $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b' + 1$. Now, due to Propositions 2.4 and 2.5, it is an easy exercise to verify that the right hand side in (16) defines a contraction in an appropriate closed ball in $X^{s,b} \subset C([0, T]; H^s(\mathbb{R}^3))$ and $X^{s,b}_{rad} \subset C([0, T]; H^s_{rad}(\mathbb{R}^3))$, respectively. Hence, it has a fixed point. Uniqueness in $X^{s,b}$ resp. $X^{s,b}_{rad}$ and smooth (analytic) dependence on the initial data are immediate consequences.
3. ILL-POSEDNESS

3.1. Proof of Theorem 1.2. As explained in [17, Section 2.2] (for the quadratic Benjamin-Ono equation), it suffices to prove the following:

Proposition 3.1. For fixed $0 < t \leq 1$ and $s < \frac{1}{4}$ the inequality
\[
\left\| \int_0^t U(t - \tau) \left( |x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi \right) d\tau \right\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi\|^3_{H^s(\mathbb{R}^3)}
\] (17)
fails to hold for all $\phi \in H^s(\mathbb{R}^3)$ (in any neighborhood of the origin).

Proof. Let $1 \leq \mu \ll \lambda$. We will choose $\mu = \mu(\lambda) = \delta \lambda^4$ for fixed $0 < \delta \ll 1$. Define the cube
\[ W^\pm = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1 \mp \lambda| \leq \mu, |\xi_2|, |\xi_3| \leq \mu \}, \]
which is centered at $\pm \lambda e_1$ with sidelength $2\mu$. Let $\phi$ be the inverse Fourier transform of the characteristic function $\chi_{W^+_\lambda}$ of $W^+_\lambda$. Obviously,
\[ \|\phi\|_{H^s(\mathbb{R}^3)} \approx \mu \frac{3}{2} \lambda^s. \]

Next, we consider
\[ F_t(\xi) := \mathcal{F}_x \left( \int_0^t U(t - \tau) \left( |x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi \right) d\tau \right)(\xi). \]

Our aim is to prove that for $0 < t \ll 1$ and all $\xi \in \frac{1}{4} W^+_\lambda$
\[ |F_t(\xi)| \gtrsim |t|\mu^s. \] (18)
Assuming that (18) holds, the claim follows since the validity of (17) implies
\[ |t|\mu^\frac{1}{2} \lambda^s \lesssim \|\langle \xi \rangle^s F_t(\xi)\|_{L^2(\mathbb{R}^3)} \lesssim \mu^\frac{3}{2} \lambda^{2s} \]
which for $\mu = \delta \lambda^4$ is equivalent to
\[ |t| \delta \lesssim \lambda^{2s-\frac{1}{2}}, \]
which can hold for fixed $t, \delta > 0$ and $\lambda \to \infty$ only if $s \geq \frac{1}{4}$. Hence, it suffices to establish (18). Similarly to [17, p.985] we compute
\[ F_t(\xi) = ce^{it\varphi_m(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{itr_m(\xi_1, \xi_2, \xi)} \frac{\chi_{W^+_\lambda}(\xi_1) \chi_{W^+_\lambda}(\xi_2)}{\langle \xi_1 + \xi_2 \rangle^2} \chi_{W^+_\lambda}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2, \]
where
\[ r_m(\xi_1, \xi_2, \xi) = \varphi_m(\xi_1) - \varphi_m(\xi_2) + \varphi_m(\xi - \xi_1 - \xi_2) - \varphi_m(\xi). \]
We notice that in the domain of integration we have
\[ |\xi_1| - |\xi_2| + |\xi - \xi_1 - \xi_2| - |\xi| \lesssim \frac{\mu^2}{\lambda}, \]
hence, for each fixed $m \geq 0$,
\[ |r_m(\xi_1, \xi_2, \xi)| \lesssim \frac{1}{\lambda} + \frac{\mu^2}{\lambda} \ll 1 \text{ (by choosing } \delta > 0 \text{ small enough)}. \]

Therefore, if $\xi \in \frac{1}{4}W^+_{\lambda}$ we have
\[ |F_t(\xi)| \geq |t| \int_{\frac{1}{4}W^+_{\lambda}} \int_{\frac{1}{4}W^-_{\lambda}} \langle \xi_1 + \xi_2 \rangle^{-2} d\xi_1 d\xi_2 \gtrsim |t| \mu^4, \]
and the proof is complete. Notice that (by multiplying the above function $\phi$ by a small fixed parameter) we can provide such a counterexample in any neighbourhood of the origin. □

**Proposition 3.2.** For fixed $T > 0$ and $s < 0$ the inequality
\[ \sup_{t \in [0, T]} \left\| \int_0^t U(t - \tau) \left( |x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\right) d\tau \right\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi\|_{H^s(\mathbb{R}^3)}^3 \tag{19} \]
fails to hold for all radial $\phi \in H^s(\mathbb{R}^3)$ (in any neighborhood of the origin).

**Proof.** Let $T > 0$ be fixed. For $\lambda \gg 1$ we define the annulus
\[ A_{\lambda} = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \lambda \leq |\xi| \leq 2\lambda \}, \]
and let $\phi$ be the inverse Fourier transform of the characteristic function $\chi_{A_{\lambda}}$ of $A_{\lambda}$. Obviously, $\phi$ is radial and $\|\phi\|_{H^s(\mathbb{R}^3)} \approx \lambda^{s+\frac{9}{2}}$. As above we consider
\[ F_t(\xi) := F_x \left( \int_0^t U(t - \tau) \left( |x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\right) d\tau \right)(\xi). \]
For $t = \delta \lambda^{-1}$ with $0 < \delta \ll 1$ (such that $t < T$) and $\xi \in \frac{1}{4}A_{\lambda}$ we will prove
\[ |F_t(\xi)| \gtrsim \delta \lambda^3. \tag{20} \]
The estimate (19) in conjunction with (20) implies
\[ \delta \lambda^{s+\frac{9}{2}} \lesssim \|\phi\|^3_{H^s(\mathbb{R}^3)} \lesssim \lambda^{3s+\frac{9}{2}} \]
which can hold for fixed $\delta > 0$ and $\lambda \to \infty$ only if $s < 0$. It remains to prove (20): As above (cp. [17, p.985]) we compute
\[
F_t(\xi) = e^{it\varphi_m(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{itr_m(\xi_1, \xi_2, \xi)} - 1}{itr_m(\xi_1, \xi_2, \xi)} \chi_{A_{\lambda}}(\xi_1) \chi_{A_{\lambda}}(\xi_2) \chi_{A_{\lambda}}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2;
\]
where
\[ r_m(\xi_1, \xi_2, \xi) = \varphi_m(\xi_1) - \varphi_m(\xi_2) + \varphi_m(\xi - \xi_1 - \xi_2) - \varphi_m(\xi). \]
Obviously, in the domain of integration we have

$$|t r_m(\xi_1, \xi_2, \xi)| \lesssim |t\lambda| \ll 1.$$  

Therefore, if $\xi \in \frac{1}{4} A_\lambda$ and $t = \delta \lambda^{-1}$ we have

$$|F_t(\xi)| \gtrsim \delta \lambda^{-1} \int_{\frac{1}{4} A_\lambda} \int_{A_\lambda} (\xi_1 + \xi_2)^{-2} d\xi_1 d\xi_2 \gtrsim \delta \lambda^3,$$

and the proof is complete. As above, we can provide such a counterexample in any neighbourhood of the origin by multiplying $\phi$ by a small parameter. □

3.2. Proof of Theorem 1.3: Failure of Uniform Continuity. By exploiting the fact that (1) exhibits solitary wave solutions, we show failure of (local) uniform continuity of the solution map $\phi \mapsto u(t)$ in $L^2$. In the case $m = 0$, the exact scaling symmetry of (1) simplifies the analysis considerably. From [12, Appendix A.2] (see also [14]) we recall the existence of radial ground state solutions $Q \in H^{\frac{1}{2}}(\mathbb{R}^3)$ of

$$\sqrt{-\Delta} Q + Q - (|x|^{-1} * |Q|^2)Q = 0. \quad (21)$$

We divide this subsection by treating first the massless case $m = 0$, followed by a discussion of the more complicated situation when $m > 0$ hold in (1). The following result concerns the massless case. To treat the case $m > 0$, we need more elaborate arguments worked out below.

**Proposition 3.3.** Suppose that $m = 0$ holds in (1) and let $t > 0$. Then the map $\phi \mapsto u(t)$ fails to be uniformly continuous for initial data in the set

$$M = \{ \phi \in L^2(\mathbb{R}^3) : \phi \text{ radial}, \|\phi\|_{L^2}^2 \geq \|Q\|_{L^2}^2 \}$$

with respect to the $L^2$-norm.

**Proof.** We adapt the arguments in [1] to our setting here. Suppose that $m = 0$ holds in (1) and suppose that $Q$ is a ground state solution (21). By scaling, we have that $Q_\mu(x) = \mu^\frac{5}{2}Q(\mu x)$ with $\mu > 0$ solves

$$\sqrt{-\Delta} Q_\mu + \mu Q_\mu - (|x|^{-1} * |Q_\mu|^2)Q_\mu = 0.$$  

Note that $\|Q_\mu\|_{L^2} = \|Q\|_{L^2}$ reflecting the $L^2$-criticality of (1).

For $\mu_1, \mu_2 > 0$ we consider the solutions

$$u_{\mu_1}(t, x) = e^{it\mu_1}Q_{\mu_1}(x), \quad u_{\mu_2}(t, x) = e^{it\mu_2}Q_{\mu_2}(x).$$

and we set

$$I_{\mu_1, \mu_2}(t) = \|u_{\mu_1}(t) - u_{\mu_2}(t)\|_{L^2}^2.$$
Note that
\[ I_{\mu_1,\mu_2}(t) = \|Q_{\mu_1}\|_{L^2}^2 + \|Q_{\mu_2}\|_{L^2}^2 - 2\text{Re} \langle e^{it\mu_1} Q_{\mu_1}, e^{it\mu_2} Q_{\mu_2} \rangle \]
\[ = 2\|Q\|_{L^2}^2 - 2\text{Re} \left( e^{it(\mu_1 - \mu_2)} \left( \frac{\mu_2}{\mu_1} \right)^{3/2} \int_{\mathbb{R}^3} Q(x) Q \left( \left( \frac{\mu_1}{\mu_2} \right) x \right) \, dx \right) \]
\[ = 2\|Q\|_{L^2}^2 - 2\cos(t(\mu_1 - \mu_2)) \left( \frac{\mu_2}{\mu_1} \right)^{3/2} \int_{\mathbb{R}^3} Q(x) Q \left( \left( \frac{\mu_1}{\mu_2} \right) x \right) \, dx \]

Let \( t > 0 \) be fixed and choose the sequences
\[ \mu_1(n) = \frac{\pi}{2t}(n + 1)^2, \quad \mu_2(n) = \frac{\pi}{2t}n^2, \quad \text{with } n \in \mathbb{N}. \]
This choice implies that \( t(\mu_1(n) - \mu_2(n)) = \frac{\pi}{2}(2n+1) \) and thus \( \cos(t(\mu_1(n) - \mu_2(n))) = 0. \) Therefore,
\[ I_{\mu_1(n),\mu_2(n)}(t) = 2\|Q\|_{L^2}^2 \quad \text{for all } n \in \mathbb{N}. \]
On the other hand, since \( \mu_1(n)/\mu_2(n) \to 1 \) as \( n \to +\infty, \) we easily see that
\[ \lim_{n \to +\infty} I_{\mu_1(n),\mu_2(n)}(0) = 0. \]
But this show that the solution \( u(t) \) cannot depend in a uniformly continuous way on the initial datum \( \phi \) in the \( L^2 \)-topology.

This completes the proof of Proposition 3.3. \( \square \)

We now turn to the case when \( m > 0 \) holds in (1). Without loss of generality, we assume that \( m = 1 \) throughout the following. To derive an illposedness result that is analogue to Proposition 3.3, we utilize solitary wave solutions \( u(t,x) = e^{it\mu} Q_{\mu}(x), \) where \( Q_{\mu} \in H^{1/2}(\mathbb{R}^3) \) are radial positive solutions to
\[ \sqrt{-\Delta + 1} Q_{\mu} + \mu Q_{\mu} - (|x|^{-1} \ast |Q_{\mu}|^2) Q_{\mu} = 0. \] (22)
Here \( \mu > 0 \) can be chosen arbitrarily; in contrast to [14], where \( \mu \) arises as a Lagrange multiplier. However, we need some auxiliary results about a class of \( Q_{\mu} \) that arise from a suitable variational problem. In particular, we are ultimately interested in the limit of large \( \mu > 0 \) and we will show that
\[ R_{\mu}(x) = \mu^{-3/2} Q_{\mu}(\mu^{-1}x) \]
converges strongly in \( L^2 \) (up to subsequences) as \( \mu \to \infty \) to some positive radial solution \( R \in H^{1/2} \) that solves \( \sqrt{-\Delta} R + R - (|x|^{-1} \ast |R|^2) R = 0. \)

Note that \( R_{\mu} \in H^{1/2} \) solves
\[ \sqrt{-\Delta + \mu^{-2}} R_{\mu} + R_{\mu} - (|x|^{-1} \ast |R_{\mu}|^2) R_{\mu} = 0 \] (23)
Lemma 3.4. For any \( \mu > 0 \), there exists a radial positive minimizer \( T_\mu \in H^{1/2}(\mathbb{R}^3) \) for problem (24). Moreover, the function \( T_\mu \) satisfies
\[
\sqrt{-\Delta + \mu^{-2}} T_\mu + T_\mu - \theta(|x|^{-1} * |T_\mu|^2) T_\mu = 0,
\]
where \( \theta = \theta(T_\mu) > 0 \) is some Lagrange multiplier. In particular, the function \( R_\mu = \theta^{1/2} T_\mu \in H^{1/2}(\mathbb{R}^3) \) is a positive radial solution of (23).

Proof. This follows from standard variational arguments. We provide a brief sketch of the main steps, where we use rearrangement inequalities to gain compactness of minimizing sequences.

For \( \sigma \geq 1 \) given, we embed (24) into the family of variational problems
\[
I_\mu(\sigma) = \inf\{ F_\mu(u) : u \in H^{1/2}, V(u) = \sigma \}.
\]
Clearly, we have that \( I_\mu(\sigma) \geq 0 \). Moreover, by scaling, we readily check that \( I_\mu(\sigma) = \sigma^{1/2} I_\mu(1) \) and hence \( I_\mu(\sigma) \geq I_\mu(1) \) for \( \sigma \geq 1 \). Now, by rearrangement inequalities (see, e.g., [13]) we have that \( F_\mu(u^*) \leq F_\mu(u) \) and \( V(u^*) \geq V(u) \), where \( u^* \) denotes the symmetric-decreasing rearrangement of \( u \in H^{1/2} \). Therefore, any minimizing sequence \( (u_n) \) for problem (24) can be replaced by \( (u_n^*) \) without loss of generality. Hence we assume \( u_n^* = u_n \) from now on. Since \( F_\mu(u_n) \to F_\mu^* \), we have that \( \|u_n\|_{H^{1/2}} \lesssim 1 \). Thus, by passing to a subsequence, we can assume that \( u_n \rightharpoonup u_* \) weakly in \( H^{1/2} \) and \( u_n \to u_* \) strongly in \( L^p_{\text{loc}} \) for \( p \in [1,3) \). Furthermore, by using that \( \|u_n\|_{L^2} \lesssim 1 \) and the fact that \( u_n = u_n(|x|) \) are radial and monotone decreasing in \(|x|\), we deduce the pointwise bound \( |u_n(x)| \lesssim |x|^{-3/2} \). Using this decay bound together with \( u_n \to u_* \) strongly in \( L^{12/5}_{\text{loc}} \), we deduce that \( u_n \to u_* \) strongly in \( L^{12/5} \). Thus, by the Hardy-Littlewood-Sobolev inequality, this implies that \( V(u_n) \to V(u_*) = 1 \). In particular, we have that \( u_* \neq 0 \) holds. Next, by the
lower semi-continuity of $F_\mu$ with respect to weak convergence in $H^{1/2}$, we deduce that $\lim_{n \to \infty} F_\mu(u_n) \geq F_\mu(u_s)$. Hence $T_\mu := u_s \geq 0$ is a radial and nonnegative minimizer for problem (24) and satisfies the corresponding Euler-Lagrange equation with some multiplier $\theta \in \mathbb{R}$. The positivity of $\theta > 0$ follows from integrating the Euler-Lagrange equation against $T_\mu$. In fact, we easily see that $\theta \leq 0$ is not possible for $T_\mu \neq 0$. Finally, we note that $T_\mu(x) > 0$ is in fact positive by adapting an argument in [12].

Next we derive uniform bounds for the minimizers $T_\mu$ given above when taking in the limit $\mu \to \infty$. Hence, without loss generality, we consider the case $\mu \geq 1$, say. We have the following uniform $L^2$ bounds.

**Lemma 3.5.** Suppose that $\mu \geq 1$ and let $R_\mu = \theta^{1/2}T_\mu \in H^{1/2}(\mathbb{R}^3)$ be given as in Lemma 3.4 above. Then we have the uniform bounds

$$1 \lesssim \|R_\mu\|_{L^2} \leq \|R_\mu\|_{H^{1/2}} \lesssim 1.$$  

**Proof.** We derive appropriate bounds for $T_\mu \in H^{1/2}$ and $\theta = \theta(T_\mu) > 0$ as follows.

First, we claim that

$$\|T_\mu\|_{H^{1/2}} \lesssim 1. \tag{25}$$

To see this, we simply note that $T_\mu$ minimizes problem (24). Therefore, by taking some fixed function $w \in H^{1/2}$ such that $V(w) = 1$, we deduce that

$$F_\mu(T_\mu) \leq F_\mu(w) \leq \int_{\mathbb{R}^3} \pi \sqrt{-\Delta} w \, dx + (1 + \mu^{-1}) \int_{\mathbb{R}^3} |w|^2 \, dx \lesssim 1 + \mu^{-1} \lesssim 1,$$

using the operator inequality $\sqrt{-\Delta + \mu^{-2}} \leq \sqrt{-\Delta} + \mu^{-1}$, which directly follows in Fourier space. From this we deduce that the bound (25) holds true.

Next, we derive an upper bound for $\theta > 0$ as follows. By integrating the Euler-Lagrange equation for $T_\mu$ against $T_\mu$, we deduce that

$$\theta = F_\mu(T_\mu) \lesssim 1, \tag{26}$$

using that $V(T_\mu) = 1$. Combining (25) and (26), we find that $R_\mu = \theta^{1/2}T_\mu$ satisfies

$$\|R_\mu\|_{L^2} \leq \|R_\mu\|_{H^{1/2}} \lesssim \theta^{1/2}\|T_\mu\|_{H^{1/2}} \lesssim 1.$$

It remains to show the uniform lower

$$\|R_\mu\|_{L^2} \gtrsim 1. \tag{27}$$
Indeed, this can be seen by exploiting a Hardy type inequality as follows. Let \( H \) denote the self-adjoint operator
\[
H = \sqrt{-\Delta + \mu - 2 - \left| x \right|^{-1} \ast |R_{\mu}|^2}.
\]
Note that \(-1\) is an eigenvalue of \( H \), since \( HR_{\mu} = -R_{\mu} \) holds. Furthermore, by radiality of \( R_{\mu} \) and Newton’s theorem (see, e.g., [13]), we have the pointwise bound
\[
\int_{\mathbb{R}^3} \frac{|R_{\mu}(y)|^2}{|x-y|} \, dy \leq \frac{\|R_{\mu}\|_{L^2}}{|x|}.
\]
Suppose now that \( \|R_{\mu}\|_{L^2}^2 \leq \left(\frac{2}{\pi}\right) \) was true. In view of the simple fact that \( \sqrt{-\Delta + \mu - 2} \geq \sqrt{-\Delta} \), we obtain the operator inequality
\[
H \geq \sqrt{-\Delta} - \frac{\|R_{\mu}\|_{L^2}^2}{|x|} \geq 0,
\]
by Hardy’s inequality \( |x|^{-1} \leq \frac{\pi}{2} \sqrt{-\Delta} \), see [11]. But the nonnegativity of \( H \) contradicts the fact that \(-1\) is an eigenvalue of \( H \). Therefore we deduce that (27) holds, which completes the proof of Lemma 3.5. □

Next, we derive the following strong convergence result for the family \( R_{\mu} \).

**Lemma 3.6.** Suppose that \( \mu_n \to \infty \) as \( n \to \infty \) and let \( T_{\mu_n} \in H^{1/2}(\mathbb{R}^3) \) be a sequence of minimizers as given by Lemma 3.4. Furthermore, we denote \( R_{\mu_n} = \theta_n^{1/2} T_{\mu_n} \in H^{1/2}(\mathbb{R}^3) \) with \( \theta_n = \theta(T_{\mu_n}) \) be as above. Then, after passing to a subsequence if necessary, we have that
\[
R_{\mu_n} \to R \text{ strongly in } L^2(\mathbb{R}^3) \text{ as } n \to \infty,
\]
where \( R \in H^{1/2}(\mathbb{R}^3) \) is a positive radial solution of
\[
\sqrt{-\Delta} R + R - (|x|^{-1} \ast |R|^2) R = 0,
\]
and it holds that \( \|R\|_{L^2}^2 \geq \|Q\|_{L^2}^2 \) (i.e. the critical \( L^2 \)-mass for the boson star equation, see (21)).

**Proof.** For notational convenience, we write \( R_n = R_{\mu_n} \) in what follows.

By the bounds in Lemma 3.5 we can assume that \( R_n \to R \) weakly in \( H^{1/2} \). Moreover, by local Rellich compactness, we can have that \( R_n \to R \) strongly in \( L^2_{\text{loc}} \). We will upgrade this to strong convergence in \( L^2 \), by deriving a uniform decay estimate for \( R_n \) as follows.

We rewrite the equation satisfied by \( R_n \) as
\[
\sqrt{-\Delta + \mu_n^{-2}} R_n = f_n
\]
with \( f_n(x) = (V_n(x) - 1) R_n(x) \) and \( V_n(x) = (|x|^{-1} \ast |R_n|^2)(x) \). Since \( R_n \) are radial functions and \( \|R_n\|_{L^2} \lesssim 1 \), we derive from Newton’s theorem
The uniform pointwise bound $V_n(x) \leq \|R_n\|_{L^2}^2 |x|^{-1} \lesssim |x|^{-1}$. Moreover, by the fact that $R_n(x) > 0$ is positive, we deduce that

$$f_n^+(x) := \max\{0, f_n(x)\} \equiv 0 \text{ for } |x| \gtrsim 1.$$ 

Now, let $G_\mu(x, y)$ be the convolution kernel associated to $(-\Delta + \mu^{-2})^{-1/2}$. From well-known facts (see e.g. [13, p. 183, formula (11)]) we have the explicit formula

$$G_\mu(x) = \int_0^{\infty} e^{-t\sqrt{-\Delta + \mu^{-2}}(x, y)} \, dt = \frac{\mu^{-4}}{2\pi^2} \int_0^{\infty} \frac{t}{t^2 + |x-y|^2} K_2 \left( \mu^{-2} \sqrt{|x-y|^2 + t^2} \right) \, dt,$$

where $K_2$ is the modified Bessel function of the third kind. From the fact that $K_\nu(z) \lesssim |z|^{-\nu}$ for $\Re \nu > 0$, we easily deduce the uniform bound

$$0 < G_\mu(x, y) \lesssim |x-y|^{-4}.$$

Hence, by using the positivity of $R_n$ and the fact that $f_n^+$ have compact support in a fixed large ball independent of $n$, we obtain from $R_\mu = (-\Delta + \mu^{-2})^{-1/2}f_n$ that

$$0 < R_n(x) \leq \int_{\mathbb{R}^3} G_{\mu_n}(x-y)f_n^+(y) \, dy \leq \int_{|y|\leq 1} G_{\mu_n}(x-y)f_n^+(y) \, dy$$

$$\leq |x|^{-4} \int_{|y|\leq 1} f_n^+(y) \, dy \lesssim |x|^{-4} \text{ for } |x| \gtrsim 1.$$ 

In the last step, we used that

$$\int_{|y|\leq 1} f_n^+(y) \, dy \lesssim \int_{|y|\leq 1} \frac{1}{|y|} R_n(y) \, dy \lesssim 1,$$

thanks to Newton’s theorem and $\|R_n\|_{L^2} \lesssim 1$ again.

In summary, we have shown that

$$R_n(x) \lesssim |x|^{-4} \text{ for } |x| \gtrsim 1 \text{ and } n \geq 1.$$ 

Using this decay bound (which is square integrable at infinity), we easily see that $R_n \to R$ strongly in $L^2_{\text{loc}}$ implies that

$$R_n \to R \text{ strongly in } L^2(\mathbb{R}^3) \text{ as } n \to \infty.$$

In view of the lower bound in Lemma 3.5, we deduce that $R \neq 0$ holds. By passing to the limit in the equation satisfied by $R_n$, we deduce that $R$ satisfies the equation displayed in Lemma 3.6.

Finally, integrating the equation satisfied by $R$ against

$$\Lambda R = x \cdot \nabla R + \frac{3}{2} R = \frac{d}{da} a^\frac{2}{3} R(ax) \bigg|_{a=1}$$
we obtain that
\[
\int_{\mathbb{R}^3} R \sqrt{-\Delta} R \, dx = \frac{1}{2} \int_{\mathbb{R}^3} (|x|^{-1} * |R|^2)|R|^2 \, dx.
\]
Recall the interpolation estimate
\[
\int_{\mathbb{R}^3} (|x|^{-1} * |f|^2)|f|^2 \, dx \leq C_{\text{opt}} \left( \int_{\mathbb{R}^3} f \sqrt{-\Delta} f \right) \left( \int_{\mathbb{R}^3} |f|^2 \right)
\]
for all \( f \in H^{1/2} \), where the optimal constant is given by \( C_{\text{opt}} = 2/\|Q\|_{L^2}^2 \); see, e.g., [12, Appendix] for this fact. Hence we deduce that \( \|R\|_{L^2}^2 \geq \|Q\|_{L^2}^2 \). This completes the proof of Lemma 3.6. \( \Box \)

Having the result of Lemma 3.6 at hand, we can now prove the following illposedness following in the case of positive mass parameter \( m > 0 \).

**Proposition 3.7.** Suppose that \( m > 0 \) holds in (1), and let \( t > 0 \). Then the map \( \phi \mapsto u(t) \) fails to be uniformly continuous for initial data in the set
\[
M = \{ \phi \in L^2 : \phi \text{ radial}, \|\phi\|_{L^2}^2 \geq K \}
\]
with respect to the \( L^2 \)-norm, where \( K \geq \|Q\|_{L^2}^2 \) is some universal constant.

**Proof.** Recall that we can assume \( m = 1 \) without loss of generality. For \( \mu_1, \mu_2 > 0 \) given, we consider the solitary wave solutions
\[
u_{\mu_1}(t, x) = e^{i\mu_1 t} Q_{\mu_1}(t, x), \quad \nu_{\mu_2}(t, x) = e^{i\mu_2 t} Q_{\mu_2}(t, x),
\]
where \( Q_{\mu}(x) \) are radial positive solutions to (22) that are given by \( Q_{\mu} = \mu^{3/2} R_{\mu}(x) \) with \( R_{\mu} \) taken from Lemma 3.4. Following the proof of Proposition 3.3 we define
\[
I_{\mu_1, \mu_2}(t) = \|u_{\mu_1}(t) - u_{\mu_2}(t)\|_{L^2}^2.
\]
Similarly as above, we find that
\[
I_{\mu_1, \mu_2}(t) = \|Q_{\mu_1}\|_{L^2}^2 + \|Q_{\mu_2}\|_{L^2}^2 - 2 \cos(t(\mu_1 - \mu_2)) \left( \frac{\mu_1}{\mu_2} \right)^{3/2} \int_{\mathbb{R}^3} R_{\mu_1}(x) R_{\mu_2} \left( \left( \frac{\mu_1}{\mu_2} \right) x \right) \, dx
\]
Now let \( t > 0 \) be given. Define the sequence \( \mu(n) = \frac{\pi}{2n} n^2 \) for \( n \in \mathbb{N} \), which ensure that \( \cos(t(\mu(n + 1) - \mu(n))) = 0 \) for all \( n \in \mathbb{N} \). By Lemma 3.6 and after possibly passing to a subsequence, we have that \( R_{\mu(n)} \to R \) strongly in \( L^2 \) with a radial positive function \( R \neq 0 \). Hence, we conclude
\[
\lim_{n \to \infty} I_{\mu(n+1), \mu(n)}(t) = 2\|R\|_{L^2}^2 \neq 0.
\]
On the other hand, we have
\[
\lim_{n \to \infty} I_{\mu(n+1),\mu(n)}(0)
\]
\[
= 2\|R\|_{L^2}^2 - 2 \lim_{n \to \infty} \left( \frac{\mu(n+1)}{\mu(n)} \right)^{\frac{3}{2}} \int_{\mathbb{R}^3} R_{\mu(n)}(x) R_{\mu(n+1)} \left( \frac{\mu(n+1)}{\mu(n)} x \right) dx
\]
\[
= 2\|R\|_{L^2}^2 - 2 \int_{\mathbb{R}^3} |R|^2 \, dx = 0.
\]
This completes the proof of Proposition 3.7.

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