Lower bounds for negative moments of $\zeta'(\rho)$

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Abstract
We establish lower bounds for the discrete $2k$th moment of the derivative of the Riemann zeta function at nontrivial zeros for all $k < 0$ under the Riemann hypothesis and the assumption that all zeros of $\zeta(s)$ are simple.

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1 | INTRODUCTION

It is an important subject to study various types of moments of the Riemann zeta function $\zeta(s)$ on the critical line as they have many interesting applications. Take for instance, the discrete $2k$th moment $J_k(T)$ of the derivative of $\zeta(s)$ at its nontrivial zeros $\rho$, given by

$$J_k(T) = \sum_{0 < \Im(\rho) \leq T} |\zeta'(\rho)|^{2k}.$$ 

Note that $J_k(T)$ is well defined for all real $k \geq 0$ with $J_0(T) = N(T)$, the number of zeros of $\zeta(s)$ in the rectangle with vertices $0, 1, 1+iT$ and $iT$. In this paper, we assume that all zeros of $\zeta(s)$ are simple which is widely believed to true. This allows one to extend the definition of $J_k(T)$ to hold for all real $k$. Then it is known (see [25, Theorem 14.27] and [20]) that the behavior of $J_k(T)$ for $k < 0$ is closely related to the size of the summatory function of the Möbius function.

Independently, S. M. Gonek [6] and D. Hejhal [11] conjectured that for any real $k$,

$$J_k(T) \approx T(\log T)^{(k+1)^2}. \quad (1.1)$$
Via the random matrix theory, C. P. Hughes, J. P. Keating, and N. O’Connell [12] conjectured an asymptotic formula for $J_k(T)$, which also suggests that (1.1) may not be valid for $k \leq -3/2$. This formula was recovered by H. M. Bui, S. M. Gonek, and M. B. Milinovich [1] using a hybrid Euler–Hadamard product.

Assuming the truth of the Riemann hypothesis (RH), S. M. Gonek [5] proved an asymptotic formula for $J_1(T)$ and N. Ng [21] established (1.1) for $k = 2$. Much progress has since been made toward establishing (1.1) for the case $k \geq 0$. In fact, based on the work in [4, 14, 16, 18], we know that (1.1) is valid for all real $k \geq 0$ on RH.

On the other hand, there are relatively fewer results for (1.1) with $k < 0$. Under RH and the assumption that all $\rho$’s are simple, S. M. Gonek [6] obtained the sharp lower bound for $J_{-1}(T)$. In [17], M. B. Milinovich and N. Ng further computed the implied constant to show that for any $\varepsilon > 0$,

$$J_{-1}(T) \geq (1 - \varepsilon) \frac{3}{2\pi^3} T.$$  \hfill (1.2)

The bound in (1.2) is consistent with (1.1) with the constant involved being half of that conjectured in [6, 12]. Under the same assumptions, W. Heap, J. Li, and J. Zhao [9] proved that the lower bounds implicit in the “$\asymp$” of (1.1) hold for all rational $k < 0$.

It is the aim of this paper to further extend the lower bounds for $J_k(T)$ obtained in [9] to all real $k < 0$. Our result is as follows.

**Theorem 1.1.** Assuming RH and zeros of $\zeta(s)$ are simple. For large $T$ and any $k < 0$, we have

$$J_k(T) \gg_k T (\log T)^{(k+1)^2}.$$  

We note that the approach in [9] follows the method developed by Z. Rudnick and K. Soundararajan in [23] and also makes use of ideas of V. Chandee and X. Li [2] when studying lower bounds for fractional moments of Dirichlet $L$-functions. The approach in this paper uses a variant of the lower bounds principle of W. Heap and K. Soundararajan [10] for proving lower bounds for general families of $L$-functions. This variant is motivated by a similar one used in [9]. In the proof of Theorem 1.1, we shall also adapt certain treatments of S. Kirila [14] concerning sharp upper bounds for $J_k(T)$ for $k \geq 0$. These treatments originate from a method of K. Soundararajan [24] and its refinement by A. J. Harper [8].

We also point out here that, as already mentioned above, the lower bounds for $J_k(T)$ given in Theorem 1.1 are only expected to be sharp for $k \geq -3/2$.

## 2 PRELIMINARIES

We now include some auxiliary results that are needed in the paper. Recall that $N(T)$ denotes the number of zeros of $\zeta(s)$ in the rectangle with vertices 0, 1, $1 + iT$ and $iT$. The Riemann–von Mangoldt formula asserts (see [3, Chapter 15]) that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$  \hfill (2.1)
As we assume the truth of RH, we may write each nontrivial zero \( \rho \) of \( \zeta(s) \) as \( \rho = \frac{1}{2} + i\gamma \), where we write \( \gamma \in \mathbb{R} \) for the imaginary part of \( \rho \). We set \( N(T, 2T) = N(2T) - N(T) \) so that (2.1) implies

\[
N(T, 2T) \ll T \log T. \tag{2.2}
\]

As usual, \( \Lambda(n) \) stands for the von Mangoldt function and we extend the definition of \( \Lambda \) to all real numbers \( x \) by defining \( \Lambda(x) = 0 \) for \( x \notin \mathbb{Z} \). We then have the following uniform version of Landau's formula [15], originally established by S. M. Gonek [7].

**Lemma 2.1.** Assume RH. We have for \( T \) large and any positive integers \( a, b \),

\[
\sum_{T < \gamma \leq 2T} \frac{(a/b)^i\gamma}{\sqrt{a/b}} = \begin{cases} 
N(T, 2T), & a = b, \\
-\frac{T}{2\pi} \frac{\Lambda(a/b)}{\sqrt{a/b}} + O\left(\sqrt{ab}(\log T)^2\right), & a > b, \\
-\frac{T}{2\pi} \frac{\Lambda(b/a)}{\sqrt{b/a}} + O\left(\sqrt{ab}(\log T)^2\right), & b > a.
\end{cases} \tag{2.3}
\]

Note that the cases in which \( a \neq b \) of Lemma 2.1 are given in [14, Lemma 5.1], while the case with \( a = b \) is trivial.

We reserve the letter \( p \) for a prime number in this paper and we recall the following result from parts (d) and (b) in [19, Theorem 2.7] concerning sums over primes.

**Lemma 2.2.** Let \( x \geq 2 \). We have for some constant \( b \),

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + b + O\left(\frac{1}{\log x}\right).
\]

Moreover, we have

\[
\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).
\]

We also include the following mean value estimate similar to [24, Lemma 3].

**Lemma 2.3.** Let \( m \) be a natural number and assume that \( y^m \ll T^{1-\varepsilon} \). Then we have for any \( a(p) \geq 0 \),

\[
\sum_{T < \gamma \leq 2T} \left| \sum_{p \leq y} \frac{a(p)p^{i\gamma}}{p^{1/2}} \right|^{2m} \ll \varepsilon T(\log T)m! \left( \sum_{p \leq y} \frac{a(p)}{p} \right)^m + (\log T)^2 \left( \sum_{p \leq y} a(p) \right)^{2m},
\]

and

\[
\sum_{T < \gamma \leq 2T} \left| \sum_{p \leq y} \frac{a(p)p^{2i\gamma}}{p} \right|^{2m} \ll \varepsilon T(\log T)m! \left( \sum_{p \leq y} \frac{a(p)}{p^2} \right)^m + (\log T)^2 \left( \sum_{p \leq y} a(p) \right)^{2m}.
\]
Proof. The proof is similar to that of [24, Lemma 3]. Hence, we only prove the first statement of the lemma. We write

\[
\left( \sum_{p \leq y} \frac{a(p)p^{iy}}{p^{1/2}} \right)^m = \sum_{f \leq y^m} a_{m,y}(f)f^{iy} \sqrt{f},
\]

where \( a_{m,y}(f) = 0 \) unless \( f \) is the product of \( m \) (not necessarily distinct) primes not exceeding \( y \). In that case, if we write the prime factorization of \( f \) as \( f = \prod_{j=1}^{m} p_j^{\alpha_j} \), then \( a_{m,y}(f) = \left( \prod_{j=1}^{m} a(p_j)^{\alpha_j} \right) \geq 0 \).

Now, as \( a_{m,y}(f) \geq 0 \), we apply Lemma 2.1. Note that the main terms in (2.3) for \( a \neq b \) are nonpositive and hence can be discarded in our upper bound computation. This leads to

\[
\sum_{T < \gamma \leq 2T} \left| \sum_{p \leq y} \frac{a(p)p^{iy}}{p^{1/2}} \right|^{2m} = \sum_{f, g \leq y^m} \frac{a_{m,y}(f)a_{m,y}(g)}{\sqrt{fg}} \sum_{T < \gamma \leq 2T} f^{iy} g^{-iy} \ll N(T, 2T) \sum_{f \leq y^m} \frac{a_{m,y}(f)^2}{f} + (\log T)^2 \sum_{f, g \leq y^m} a_{m,y}(f)a_{m,y}(g) \ll N(T, 2T) \sum_{f \leq y^m} \frac{a_{m,y}(f)^2}{f} + (\log T)^2 \left( \sum_{p \leq y} a(p) \right)^{2m}.
\]

We further estimate right-hand side expression above following exactly the treatments in [24, Lemma 3] and utilizing (2.2) to arrive at the desired result. \( \square \)

3 | PROOF OF THEOREM 1.1

3.1 | The lower bound principle

We assume that \( T \) is a large number and note here that throughout the proof, the implicit constants involved in various estimates in \( \ll \) and \( O \) notations depend on \( k \) only and are uniform with respect to \( \rho \). We recall the convention that the empty product is defined to be 1.

We follow the ideas of A. J. Harper in [8] and the notations of S. Kirila in [14] to define for a large number \( M \) depending on \( k \) only,

\[
\alpha_0 = 0, \quad \alpha_j = \frac{20^{j-1}}{(\log \log T)^2} \quad \forall j \geq 1, \quad J = J_{k,T} = 1 + \max\{j : \alpha_j \leq 10^{-M}\}. \quad (3.1)
\]

We define \( I_j = (T^{\alpha_j-1}, T^{\alpha_j}] \) for \( 1 \leq j \leq J \). With the above notations, we can infer from Lemma 2.2 that for \( 1 \leq j \leq J - 1 \) and \( T \) large enough,

\[
\sum_{p \in I_{j+1}} \frac{1}{p} = \log \alpha_{j+1} - \log \alpha_j + o(1) = \log 20 + o(1) \leq 10. \quad (3.2)
\]
We further define for any real number \( \alpha \) and any \( 1 \leq j \leq J \),

\[
P_j(s) = \sum_{p \in I_j} \frac{1}{p^s}, \quad \mathcal{N}_j(s, \alpha) = E_{e^{2\pi j / 4}}(\alpha P_j(s)), \quad \mathcal{N}(s, \alpha) = \prod_{j=1}^{J} \mathcal{N}_j(s, \alpha),
\]

(3.3)

where for any real number \( \ell' \) and any \( x \in \mathbb{R} \),

\[
E_{\ell'}(x) = \sum_{j=0}^{\lceil \ell' \rceil} \frac{x^j}{j!}.
\]

Here \( \lceil \ell' \rceil = \min\{m \in \mathbb{Z} : \ell' \leq m\} \) is the ceiling of \( \ell' \).

Now, applying a variant of the treatment given by W. Heap, J. Li, and J. Zhao in [9, (3)], which is based on the lower bounds principle of W. Heap and K. Soundararajan developed in [10], we have for \( k < 0 \), from Hölder’s inequality,

\[
\sum_{0 < \gamma \leq T} |\mathcal{N}(\rho, k)|^2 \leq \left( \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^2, |\mathcal{N}(\rho, k)|^{2(1 - 1/k)} \right)^{-k/(1-k)} \left( \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \right)^{1/(1-k)}.
\]

(3.4)

By RH, \( \overline{\rho} = 1 - \rho \). So,

\[
\sum_{0 < \gamma \leq T} |\mathcal{N}(\rho, k)|^2 = \sum_{0 < \gamma \leq T} \mathcal{N}(\rho, k) \mathcal{N}(\overline{\rho}, k) = \sum_{0 < \gamma \leq T} \mathcal{N}(\rho, k) \mathcal{N}(1 - \rho, k).
\]

(3.5)

Hence, from (3.4) and (3.5), in order to establish Theorem 1.1, it suffices to prove the following propositions.

**Proposition 3.2.** With notations as above, we have for \( k < 0 \),

\[
\sum_{0 < \gamma \leq T} \mathcal{N}(\rho, k) \mathcal{N}(1 - \rho, k) \gg T (\log T)^{k^2 + 1}.
\]

**Proposition 3.3.** With notations as above, we have for \( k < 0 \),

\[
\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1 - 1/k)} \ll T (\log T)^{k^2 + 3}.
\]

The remainder of the paper is devoted to the proofs of the above propositions.

### 4 PROOF OF PROPOSITION 3.2

We write \( \omega(n) \) for the number of distinct prime factors of \( n \) and \( \Omega(n) \) for the number of prime powers dividing \( n \). We also denote \( g(n) \) for the multiplicative function given on prime powers by \( g(p^r) = 1/r! \) and define functions \( b_j(n), 1 \leq j \leq J \) such that \( b_j(n) = 0 \) or \( 1 \) and that \( b_j(n) = 1 \) only if \( \Omega(n) \leq \ell_j := \lceil e^{2\pi j / 3^4} \rceil \) and all prime divisors of \( n \) lie in the interval \( I_j \). We can then rewrite
\( \mathcal{N}_j(s, \alpha) \) defined (3.3), using the above notations, as

\[
\mathcal{N}_j(s, \alpha) = \sum_{n_j} \frac{\alpha^{\Omega(n_j)} b_j(n_j)}{g(n_j)} \frac{1}{n_j^s}, \quad 1 \leq j \leq J. \tag{4.1}
\]

Note that each \( \mathcal{N}_j(s, \alpha) \) is a short Dirichlet polynomial of length not exceeding \( T^{\alpha_j} \lceil e^{2\alpha_j^{-3/4}} \rceil \). We also write for simplicity that

\[
\mathcal{N}(s, \alpha) = \sum_n \frac{a_\alpha(n)}{n^s}.
\]

Then, it follows from (3.3) and (4.1) that \( a_\alpha(n) \neq 0 \) only when \( n = \prod_{1 \leq j \leq J} n_j \) and in that case,

\[
a_\alpha(n) = \prod_{n_j} \frac{\alpha^{\Omega(n_j)}}{g(n_j)} b_j(n_j). \tag{4.2}
\]

Upon taking \( T \) large enough, we then deduce that \( \mathcal{N}(s, \alpha) \) is a Dirichlet polynomial not longer than

\[
T \sum_{i=1}^J \alpha_j \lceil e^{2\alpha_j^{-3/4}} \rceil \leq T^{40e^210^{-M/4}}.
\]

Moreover, applying the estimation that \( |\alpha^i/i!| \leq e^{|\alpha|} \) for any integer \( i \geq 0 \), we get that

\[
|a_\alpha(n)| \leq e^{|\alpha|} \omega(n), \quad a_\alpha(n) = 0, \text{ if } n > T^{40e^210^{-M/4}}. \tag{4.3}
\]

Note further the following estimation (see [19, Theorem 2.10]),

\[
\omega(n) \leq \frac{\log n}{\log \log n} \left( 1 + O\left( \frac{1}{\log \log n} \right) \right), \quad n \geq 3. \tag{4.4}
\]

It follows from (4.3) and (4.4) that, upon taking \( M \) large enough, we have

\[
\mathcal{N}(s, k) = \sum_{n \leq T^\vartheta} \frac{a_k(n)}{n^s},
\]

where \( \vartheta < 1 \) and \( |a_k(n)| \leq T^\varepsilon \).

We then apply [22, Proposition 3.1(iii)] to arrive at

\[
\sum_{0 < \gamma \leq T} \mathcal{N}(\rho, k) \mathcal{N}(1 - \rho, k) = N(T) \sum_n \frac{a_k^2(n)}{n} - \frac{T}{\pi} \sum_n \frac{(\Lambda * a_k)(n) \cdot a_k(n)}{n} + o(T), \tag{4.5}
\]

where \( f \ast g \) is the Dirichlet convolution of two arithmetic functions \( f(k), g(k) \).

We evaluate the first sum in (4.5) using (4.2) to obtain that

\[
\sum_n \frac{a_k^2(n)}{n} = \prod_{j=1}^J \left( \sum_{n_j} \frac{1}{n_j} \frac{k^{2\Omega(n_j)}}{g^2(n_j)} b_j(n_j) \right).
\]
Note that the factor \( b_j(n_j) \) restricts \( n_j \) to having all prime factors in \( I_j \) such that \( \Omega(n_j) \leq \ell_j \). If we remove this restriction on \( \Omega \), then the sum over \( n_j \) becomes

\[
\sum_{n_j} \frac{1}{n_j} \frac{k^{2\Omega(n_j)}}{g^2(n_j)} = \prod_{p \in I_j} \left( 1 + \frac{k^2}{p} + O \left( \frac{1}{p^2} \right) \right) =: P_j.
\]

Using Rankin’s trick by noticing that \( 2^{\Omega(n_j) - \ell_j} \geq 1 \) if \( \Omega(n_j) > \ell_j \), we see by the definition of \( \ell_j \) and (3.2) that the error introduced this way does not exceed

\[
\sum_{n_j} \frac{1}{n_j} \frac{k^{2\Omega(n_j)}}{g^2(n_j)} 2^{\Omega(n_j) - \ell_j} \leq 2^{-\ell_j} \prod_{p \in I_j} \left( \sum_{i=0}^{\infty} \frac{1}{p^i} \left( \frac{2k^2}{i!^2} \right) \right) \leq 2^{-\ell_j/2} P_j.
\]

We conclude from the above discussions, (2.1), and Lemma 2.2 that, by taking \( T \) large enough,

\[
N(T) \sum_n a_k^2(n) n \gg T \log T \prod_{j=1}^J \left( 1 - 2^{-\ell_j/2} \right) \prod_{j=1}^J P_j \gg T(\log T)^{k^2+1}. \tag{4.6}
\]

Next, we estimate

\[
\frac{T}{\pi} \sum_n \frac{\mathcal{A} \ast a_k(n)}{n} \cdot a_k(n) = \frac{T}{\pi} \sum_n \frac{\Lambda(n)}{n} \sum_m a_k(m)a_k(mn) \leq \frac{T}{\pi} \sum_n \frac{\Lambda(n)}{n} \left| \sum_m a_k(m)a_k(mn) \right|.
\]

Note that by (4.3), we may assume that \( n \leq T^{\Delta(10^{-M/4})} \). We fix an integer \( n \) and write it as \( n = \prod n_j \) with \( p \mid n_j \) only if \( p \in I_j \). Then we have

\[
\left| \sum_m a_k(m)a_k(mn) \right| = \left| \prod_{j=1}^J \left( \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j) + \Omega(n_j)}}{g(m_j) g(m_j n_j) b_j(m_j n_j)} \right) \right| \leq \prod_{j=1}^J \left| \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j) + \Omega(n_j)}}{g(m_j) g(m_j n_j) b_j(m_j n_j)} \right|
\]

Observe that \( g(mn) \geq g(m)g(n) \) so that

\[
\prod_{j=1}^J \left| \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j) + \Omega(n_j)}}{g(m_j) g(m_j n_j) b_j(m_j n_j)} \right| \leq \prod_{j=1}^J \left| k \right|^{\Omega(n_j)} \left( \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j)}}{g(m_j) g(m_j n_j) b_j(m_j n_j)} \right) \leq \prod_{j=1}^J \left| k \right|^{\Omega(n_j)} \left( \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j)}}{g^2(m_j)} \right) = \left| k \right|^{\Omega(n)} \prod_{j=1}^J \left( \sum_{m_j} \frac{1}{m_j} \frac{k^{2\Omega(m_j)}}{g^2(m_j)} \right).
\]
We evaluate the last expression above using Lemma 2.2 to see that we have
\[
\frac{T}{\pi} \sum_{n} \frac{(\Lambda * a_k)(n) \cdot a_k(n)}{n} \ll T (\log T)^{k^2} \sum_{n} \frac{\Lambda(n)|\Omega(n)|}{n g(n)} \\
\ll T (\log T)^{k^2} \sum_{p \leq T^{4e^{-210-M/4}}} \frac{\log p}{p} \ll 10^{-M/4} T (\log T)^{k^2+1}.
\]

We then take $M$ large enough to deduce from the above, (4.5) and (4.6), that
\[
\sum_{0 < \gamma \leq T} \mathcal{N}(\rho, k) \mathcal{N}(1 - \rho, k) \gg T (\log T)^{k^2+1}.
\]

This completes the proof of Proposition 3.2.

5 | PROOF OF PROPOSITION 3.3

Upon dividing the range of $\gamma$ into dyadic blocks and replacing $T$ by $2T$, we see that it suffices to show for large $T$,
\[
\sum_{T < \gamma \leq 2T} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)} \ll T (\log T)^{k^2+3}.
\]

We now deduce from [14, (4.1)] the following upper bound for $\log |\zeta'(\rho)|$. For any $\gamma$ with $T < \gamma \leq 2T$ and any $1 \leq j \leq J$, assuming the truth of RH, we have
\[
\log |\zeta'(\rho)| \ll \Re \sum_{l=1}^{j} \sum_{n \in I_l} \frac{\Lambda(c)(n)}{\sqrt{n} n^{1/(\alpha_j \log T) \log n}} \frac{\log(p^{\alpha_j} / n)}{\log T^{\alpha_j}} + \log \log T + \alpha_j^{-1} + O(1),
\]

where we define $\mathcal{L} = \log T$ and
\[
\Lambda_{\mathcal{L}}(n) = \begin{cases} 
\Lambda(n), & \text{if } n = p \text{ or if } n = p^2 \text{ and } n \leq \mathcal{L}, \\
0, & \text{otherwise}.
\end{cases}
\]

Note that we have by Lemma 2.2 and the definition in (3.1),
\[
J \leq \log \log \log T, \quad \alpha_1 = \frac{1}{(\log \log T)^2}, \quad \text{and} \quad \sum_{p \leq T^{1/(\log \log T)^2}} \frac{1}{p} \leq \log \log T = \alpha_1^{-1/2}.
\]

It follows that $1/\alpha_j < 1/\alpha_1 \leq (\log T)^2$ and this, together with Lemma 2.2 and the mean value theorem in differential calculus, implies that for $j \geq 1$,
\[
\sum_{p \leq \log T} \frac{1}{p^{1+2/(\alpha_j \log T)}} \frac{\log p}{\log T^{\alpha_j}} \ll 1, \\
\sum_{p \leq \log T} \left( \frac{1}{p^{1+2/(\alpha_j \log T)}} - \frac{1}{p} \right) \ll \frac{1}{\alpha_j \log T} \sum_{p \leq \log T} \frac{\log p}{p} \ll 1.
\]
We then conclude that

$$\log |\zeta'(\rho)| \ll \Re \sum_{l=1}^{j} M_{l,j}(\gamma) + \Re \sum_{0 \leq m \leq \frac{\log \log T}{\log 2}} P_m(\gamma) + \log \log T + \alpha_j^{-1} + O(1),$$

(5.2)

where

$$M_{l,j}(\gamma) = \sum_{p \in I_l} \frac{p^{-iy}}{\sqrt{p}} \frac{1}{p^{1/(\alpha_j \log T)}} \frac{\log(T^{2j}/p)}{\log T^{2j}}, \quad 1 \leq l \leq j \leq J,$$

and

$$P_m(\gamma) = \sum_{2^m < p \leq 2^{m+1}} \frac{p^{-2iy}}{2p}, \quad 0 \leq m \leq \frac{\log \log T}{\log 2}.$$ 

We also define the following sets:

- $S(0) = \{T < \gamma \leq 2T : |M_{l,1}(\gamma)| > \alpha^{-3/4}_1 \text{ for some } 1 \leq l \leq J\},$
- $S(j) = \{T < \gamma \leq 2T : |M_{m,l}(\gamma)| \leq \alpha^{-3/4}_m \text{ for all } 1 \leq m \leq j, m \leq l \leq J, \text{ but } |M_{j+1,l}(\gamma)| > \alpha^{-3/4}_{j+1} \text{ for some } j+1 \leq l \leq J, \quad 1 \leq j \leq J,\}$
- $S(J) = \{T < \gamma \leq 2T : |M_{m,J}(\gamma)| \leq \alpha^{-3/4}_m \text{ for all } 1 \leq m \leq J\},$
- $P(m) = \left\{ T < \gamma \leq 2T : |P_m(\gamma)| > 2^{-m/10}, \text{ but } |P_n(\gamma)| \leq 2^{-n/10} \text{ for all } m + 1 \leq n \leq \frac{\log \log T}{\log 2} \right\}.$

Note that we have $|P_n(\gamma)| \leq 2^{-n/10}$ for all $n$ if $\gamma \notin P(m)$ for any $m$, which implies that

$$\sum_{p \leq \log T} \frac{p^{-2iy}}{2p} = O(1).$$

As the treatment for the case $\gamma \notin P(m)$ for any $m$ is easier compared to the other cases, we may assume that $\gamma \in P(m)$ for some $m$. Note that

$$\text{meas}(P(m)) \leq \sum_{T < \gamma \leq 2T} \left( 2^{m/10} |P_m(\gamma)| \right)^{2[2^m/2]},$$

where here and after $\text{meas}(X)$ denotes the cardinality of the set $X$.

We now apply Lemmas 2.2 and 2.3 to bound the last expression above. This gives that for $m \geq 10$,

$$\text{meas}(P(m)) \ll T(\log T)([2^{m/2}])^2(2^{m/5})^{[2m/2]} \left( \sum_{2^m < p} \frac{1}{4p^2} \right)^{[2^{m/2}] + (\log T)^2 \left( \sum_{p \leq \log T} 1 \right)^{2[2^{m/2}]}} \ll T2^{-2m/2},$$
where the last estimation above follows from Stirling’s formula (see [13, (5.12)]), which asserts that
\[ \left( \frac{m}{e} \right)^m \leq m! \ll \sqrt{m} \left( \frac{m}{e} \right)^m. \] (5.3)

Now Hölder’s inequality gives that if \( 2^m \geq (\log \log T)^3 \),
\[ \sum_{(\log T)^3 \leq 2^m \leq \log T} \sum_{\gamma \in \mathcal{P}(m)} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)} \leq \sum_{(\log T)^3 \leq 2^m \leq \log T} \left( \text{meas}(\mathcal{P}(m)) \right)^{1/4} \left( \sum_{T < \gamma < 2T} |\zeta'(\rho)|^8 \right)^{1/4} \left( \sum_{T < \gamma < 2T} |\mathcal{N}(\rho, k)|^{4(1-1/k)} \right)^{1/2}. \]

Similar to the proof of [4, Proposition 3.5], we have that
\[ \sum_{T < \gamma < 2T} |\mathcal{N}(\rho, k)|^{4(1-1/k)} \ll T(\log T)^{O(1)}. \] (5.4)

Also, note that by [14, Theorem 1.1], we have under RH, for any real \( k > 0 \)
\[ J_k(T) \ll_k T(\log T)^{(k+1)^2}. \]
Putting \( k = 4 \), we have
\[ \sum_{T < \gamma < 2T} |\zeta'(\rho)|^8 \ll T(\log T)^{25}. \] (5.5)

It thus follows that, as \( 2^m \geq (\log \log T)^3 \),
\[ \sum_{(\log T)^3 \leq 2^m \leq \log T} \sum_{\gamma \in \mathcal{P}(m)} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)} \ll \sum_{(\log T)^3 \leq 2^m \leq \log T} T \exp \left( -(\log 2)(\log \log T)^{3/2}/4 \right)(\log T)^{O(1)} \ll T(\log T)^{k^2}(\log \log T)^{-1}. \]

Hence, we may also assume that \( 0 \leq m \leq (3/ \log 2) \log \log \log T \). We further note that
\[ \text{meas}(S(0)) \leq \sum_{T < \gamma < 2T} \left( \alpha_1^{3/4} |\mathcal{M}_{i_1}(\gamma)| \right)^{2[1/(10\alpha_1)]}. \]

Now Lemma 2.3 applied to the last expression above yields
\[ \text{meas}(S(0)) \ll \mathcal{J} T(\log T)([1/(10\alpha_1)])^{1}(\alpha_1^{3/4})^{2[1/(10\alpha_1)]} \left( \sum_{p \leq T^{\alpha_1}} \frac{1}{p} \right)^{[1/(10\alpha_1)]} + \mathcal{J}(\log T)^2 \left( \sum_{p \leq T^{\alpha_1}} 1 \right)^{2[1/(10\alpha_1)]} \]
\[
\ll JT(\log T) \sqrt{\left[\frac{1}{1/(10\alpha_1)}\right] \left(\frac{[1/(10\alpha_1)]}{e}\right)^{1/[1/(10\alpha_1)]} \left(\sum_{p \in T^{\alpha_1}} \frac{1}{p}\right)^{[1/(10\alpha_1)]}}, \quad (22)
\]
where the last estimation above follows from Lemma 2.2 and (5.3).

We apply the bounds in (5.1) to the last expression in (5.6), getting

\[
\text{meas}(S(0)) \ll JT(\log T) \sqrt{\left[\frac{1}{1/(10\alpha_1)}\right]} e^{-1/[1/(10\alpha_1)]} \ll T e^{-(\log \log T)^2/20}. \quad (5.7)
\]

Now Hölder’s inequality renders

\[
\sum_{\gamma \in S(0)} |\xi'(\rho)|^2 |\mathcal{N} (\rho, k)|^{2(1-1/k)} 
\leq (\text{meas}(S(0)))^{1/4} \left( \sum_{T < \gamma \leq 2T} |\xi'(\rho)|^8 \right)^{1/4} \left( \sum_{T < \gamma \leq 2T} |\mathcal{N} (\rho, k)|^{4(1-1/k)} \right)^{1/2}. \quad (5.8)
\]

We use the bounds (5.4), (5.5), and (5.7) in (5.8) to conclude that

\[
\sum_{\gamma \in S(0)} |\xi'(\rho)|^2 |\mathcal{N} (\rho, k)|^{2(1-1/k)} \ll T(\log T)^{k^2+3}.
\]

Similarly, we define

\[
\mathcal{T} = \{T < \gamma \leq 2T : |k P_1 (\rho)| \leq \frac{\alpha_1^{-3/4}}{1 - 1/k}\}.
\]

We further write \(\mathcal{T}^c\) for the complementary of \(\mathcal{T}\) in the set \(\{T < \gamma \leq 2T\}\). Then similar to our approach above, we have

\[
\text{meas}(\mathcal{T}^c) \ll T e^{-(\log \log T)^2/20}.
\]

Our arguments above allow us to deduce

\[
\sum_{\gamma \in \mathcal{T}^c} |\xi'(\rho)|^2 |\mathcal{N} (\rho, k)|^{2(1-1/k)} \ll T(\log T)^{k^2+3}.
\]

Thus, we may further assume that \(j \geq 1\) and \(\gamma \in \mathcal{T}\). Note that

\[
\{\gamma \in \mathcal{T}, \gamma \in P(m), \gamma \in S(j), 0 \leq m \leq (3/ \log 2) \log \log T, 1 \leq j \leq J\} = \bigcup_{m=0}^{(3/ \log 2) \log \log T} \bigcup_{j=1}^{J} \left(S(j) \cap P(m) \cap \mathcal{T}\right).
\]

Thus, it suffices to show that

\[
\sum_{m=0}^{(3/ \log 2) \log \log T} \sum_{j=0}^{J} \sum_{\gamma \in S(j) \cap P(m) \cap \mathcal{T}} |\xi'(\rho)|^2 |\mathcal{N} (\rho, k)|^{2(1-1/k)} \ll T(\log T)^{k^2+3}. \quad (5.9)
\]
Now, we consider the sum of $|\xi'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)}$ over $S(j) \cap P(m) \cap T$ by fixing an $m$ such that $0 \leq m \leq (3/\log 2) \log \log \log T$ and fixing a $j$ with $1 \leq j \leq J$. We then deduce from (5.2) that

$$|\xi'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)} \ll (\log T)^2 \exp \left( \frac{2}{\alpha_j} \right) \exp \left( 2 \Re \sum_{l=1}^{j} \mathcal{M}_{l,j}(\gamma) + 2 \Re \sum_{m=0}^{\log \log T/2} P_m(\gamma) \right) |\mathcal{N}(\rho, k)|^{2(1-1/k)}$$

$$\times \prod_{l=2}^{J} |\mathcal{N}_l(\rho, k)|^{2(1-1/k)}. \quad (5.10)$$

Now, for any $z \in \mathbb{C}$, by the Taylor formula with integral remainder, we have

$$\left| e^z - \sum_{j=0}^{d-1} \frac{z^j}{j!} \right| = \left| \frac{1}{(d-1)!} \int_0^z e^t (z-t)^{d-1} dt \right| = \left| \frac{e^z}{(d-1)!} \int_0^1 e^{zs} (1-s)^{d-1} ds \right|$$

$$\leq \frac{|z|^d}{d!} \max(1, e^{\Re z}) \leq \frac{|z|^d}{d!} e^z \max(e^{-z}, e^{\Re z - z}) \leq \frac{|z|^d}{d!} e^z e^{\Re |z|}. $$

The above computation implies that

$$\sum_{j=0}^{d-1} \frac{z^j}{j!} = e^z \left( 1 + O \left( \frac{|z|^d}{d!} e^{\Re |z|} \right) \right). \quad (5.11)$$

As $\gamma \in T$, we apply the above formula that, by setting $z = k P_1(\rho)$, $d = [e^2 \alpha^{-3/4}]$ (5.11). In a manner similar to the bound after [14, (5.2)] and applying (5.3), we get

$$|\mathcal{N}_1(\rho, k)|^2 = \exp \left( 2k \Re P_1 \left( \frac{1}{2} + i\gamma \right) \right) \left( 1 + O \left( e^{-\alpha^{-3/4}} \right) \right). \quad (5.12)$$

Now applying (5.12) in (5.10) yields

$$|\xi'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)} \ll (\log T)^2 \times \exp \left( \frac{2}{\alpha_j} \right) \exp \left( 2 \Re \mathcal{M}_{1,j}(\gamma) + 2(k-1) \Re P_1 \left( \frac{1}{2} + i\gamma \right) + 2 \Re \sum_{l=2}^{J} \mathcal{M}_{l,j}(\gamma) + 2 \Re \sum_{m=0}^{\log \log T/2} P_m(\gamma) \right)$$

$$\times \prod_{l=2}^{J} |\mathcal{N}_l(\rho, k)|^{2(1-1/k)}. $$
We want to now separate the sums over $p \leq 2^{m+1}$ on the right-hand side of the above expression from those over $p > 2^{m+1}$. To do so, we note that if $\gamma \in P(m)$, then

$$
\left| 2\Re \sum_{p \leq 2^{m+1}} \frac{p^{-iy}}{\sqrt{p}} \frac{1}{p^{1/(\alpha_j \log T)}} \frac{\log(T^{\alpha_j}/p)}{\log T^{\alpha_j}} + 2(k - 1)\Re \sum_{p \leq 2^{m+1}} \frac{p^{-iy}}{\sqrt{p}} + 2\Re \sum_{p \leq \log T} \frac{p^{-iy}}{2p} \right| 
$$

$$
\leq 2 \left| \sum_{p \geq 2^{m+1}} \frac{p^{-iy}}{\sqrt{p}} \frac{1}{p^{1/(\alpha_j \log T)}} \frac{\log(T^{\alpha_j}/p)}{\log T^{\alpha_j}} \right| + (1 - k) \left| \sum_{p \leq 2^{m+1}} \frac{p^{-iy}}{\sqrt{p}} \right| + \left| \sum_{p \leq \log T} \frac{p^{-iy}}{2p} \right| 
$$

$$
+ O(1) \leq (4 - 2k)2^{m/2+3} + O(1),
$$

where we note by Lemma 2.2 that (using $\log(1 + x) \leq x$ and $x \leq 2^x$ (this last inequality is valid for $x \geq 1$ and $x = 1/2$))

$$
\left| \sum_{p \leq 2^{m+1}} \frac{p^{-iy}}{2p} \right| \leq \sum_{p \leq 2^{m+1}} \frac{1}{2p} = \frac{1}{2} \log(m + 1) + O(1) \leq \frac{m}{2} + O(1) \leq 2^{m/2} + O(1).
$$

We deduce from the above that

$$
\sum_{\gamma \in S(j) \cap P(m) \cap T} |\zeta'(\rho)|^2 |N(\rho, k)|^{2(1-1/k)} 
$$

$$
\ll (\log T)^2 e^{(2-k)2^{m/2+4}} \exp \left( \frac{2}{\alpha_j} \sum_{\chi \in S(j) \cap P(m)} \exp \left( 2\Re M'_{1,j}(\gamma) + 2\Re \sum_{i=2}^j M_{i,j}(\gamma) \right) \right) 
$$

$$
\times \prod_{l=2}^J |N_l(\rho, k)|^{2(1-1/k)} \ll (\log T)^2 e^{(2-k)2^{m/2+4}} \exp \left( \frac{2}{\alpha_j} \right) 
$$

$$
\times \sum_{\gamma \in S(j)} \left( \frac{2^m}{10} |p_m(\gamma)| \right)^{2[2^{m/2}]} \exp \left( 2\Re M'_{1,j}(\gamma) + 2\Re \sum_{i=2}^j M_{i,j}(\gamma) \right) \prod_{l=2}^J |N_l(\rho, k)|^{2(1-1/k)},
$$

where we define

$$
M'_{1,j}(\gamma) = \sum_{2^{m+1} < p < T^{\alpha_1}} \frac{p^{-iy}}{\sqrt{p}} c_j(p, k),
$$

and set for any $1 \leq j \leq T$ and $n \in \mathbb{Z},$

$$
c_j(p, n) = \frac{1}{p^{1/(\alpha_j \log T)}} \frac{\log(T^{\alpha_j}/p)}{\log T^{\alpha_j}} + n - 1.
$$

We note that as $0 \leq m \leq (3 / \log 2) \log \log \log T$ and $T$ is large, we have

$$
\left| \sum_{p \leq 2^{m+1}} \frac{p^{-iy}}{\sqrt{p}} c_j(p, k) \right| \leq (2 - k) \sum_{p \leq 2^{m+1}} \frac{1}{\sqrt{p}} \leq \frac{100(2 - k)(\log \log T)^{3/2}}{\log \log \log T}.
$$
It follows that for \( \gamma \in S(j) \) and large \( T \),
\[
|M'_{1,j}(\gamma)| \leq 100(2 - k)(\log \log T)^{3/2}(\log \log \log T)^{-1} + |M_{1,j}(\gamma)| + \left(1 - \frac{1}{k}\right)|kP_1(\rho)| \leq 2.01\alpha^{-3/4}_1
\]
\[= 2.01(\log \log T)^{3/2}.\]

We apply (5.11) with \( z = M'_{1,j}(\gamma) \), \( d = [e^{2\alpha^{-3/4}_1}] \) and argue as before to see that
\[
\exp \left(2\Re M'_{1,j}(\gamma)\right) \ll \left|E e^{2\alpha^{-3/4}_1}(M'_{1,j}(\gamma))\right|^2. \tag{5.15}
\]

As we also have \( |M_{l,j}| \leq \alpha^{-3/4}_l \) when \( \gamma \in S(j) \), we repeat our arguments to deduce, if
\[
\left|kP_1\left(\frac{1}{2} + iy\right)\right| \leq \frac{\alpha^{-3/4}_l}{1 - 1/k},
\]
then
\[
\exp \left(2\Re M_{l,j}(\gamma)\right) \ll \left|N'_{l}(\rho, k)\right|^{2(1 - 1/k)} \ll \left(1 + O\left(e^{-\alpha^{-3/4}_l}\right)\right)\left|E e^{2\alpha^{-3/4}_l}(M'_{1,j}(\gamma))\right|^2, \tag{5.16}
\]
where we define, as before, for \( 2 \leq l \leq j \),
\[
M'_{l,j}(\gamma) = \sum_{p \in I_l} \frac{p^{-i\gamma}}{\sqrt{p}} c_j(p, k).
\]

On the other hand, when \( |kP_1(\frac{1}{2} + iy)| > \alpha^{-3/4}_l/(1 - 1/k) \), we have
\[
|N'_{l}(\rho, k)|
\]
\[
\leq \sum_{r=0}^{\ell_l} \frac{|kP_1(\rho)|^r}{r!} \leq |kP_1(\rho)|^{\ell_l} \sum_{r=0}^{\ell_l} \left(1 - \frac{1}{k}\right)\alpha_l^{3/4} \frac{1}{r!} \leq (k - 1)\alpha^{3/4}_l P_1(\rho)|^{\ell_l} e^{\alpha^{3/4}_l/(1-1/k)}
\]
\[
\leq |(k - 1)e_1\alpha^{3/4}_l P_1(\rho)|^{\ell_l}. \tag{5.17}
\]

Note that we also have
\[
\exp \left(2\Re M_{l,j}(\gamma)\right) \leq \exp(2\alpha^{-3/4}_l) \leq 2^{\ell_l}. \tag{5.16}
\]

We now define for all \( 1 \leq l \leq J \),
\[
Q_l(\rho, k) := \begin{cases}
2(1 - k)e_{1-1/(\sqrt{k})}^{\alpha^{-3/4}_l P_1(\rho)} & l \neq j + 1, \\
4(1 - k)\alpha_l^{3/4} P_1(\rho) & l = j + 1.
\end{cases}
\]
It follows from the above discussion that if $|kP(\frac{1}{2} + iy)| > \alpha_i^{-3/4}/(1 - 1/k)$ and $2 \leq l \leq j$, then

$$
\exp \left( 2R \mathcal{M}_{l,j}(y) \right) |\mathcal{N}_l(\rho, k)|^{2(1-1/k)} \leq |Q_l(\rho, k)|^2.
$$

(5.18)

Our arguments above also allow us to deduce that for $j + 2 \leq l \leq J$,

$$
|\mathcal{N}_l(\rho, k)|^{2(1-1/k)} \leq (1 + O(e^{-\alpha_i^{-3/4}})) |\mathcal{N}_l(\rho, k - 1)|^2 + |Q_l(\rho, k)|^2.
$$

(5.19)

We apply the bounds given in (5.15), (5.16), (5.18), and (5.19) in (5.13) to arrive at

$$
\sum_{\gamma \in S(j) \cap P(m) \cap T} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)}
$$

$$
\ll (\log T)^2 e^{(2-k)2^m/2+4} \exp \left( \frac{2}{\alpha_j} \right) \sum_{\gamma \in S(j)} \left( 2^{2m/10} |P_m(\gamma)| \right)^{2[2^{m/2}]^2} \left| E_{e^{2\alpha_i^{-3/4}}(\mathcal{M}'_{l,j}(y))} \right|^2
$$

$$
\times \prod_{l=2}^j \left( 1 + O(e^{-\alpha_i^{-3/4}}) \right) \left| E_{e^{2\alpha_i^{-3/4}}(\mathcal{M}'_{l,j}(y))} \right|^2 + \left| Q_l(\rho, k) \right|^2
$$

$$
\times |\mathcal{N}_{j+1}(\rho, k)|^{2(1-1/k)} \prod_{l=j+2}^J \left( 1 + O(e^{-\alpha_i^{-3/4}}) \right) |Q_l(\rho, k) - 1|^2 + |Q_l(\rho, k)|^2
$$

$$
\ll (\log T)^2 e^{(2-k)2^m/2+4} \exp \left( \frac{2}{\alpha_j} \right) \sum_{\gamma \in S(j)} \left( 2^{2m/10} |P_m(\gamma)| \right)^{2[2^{m/2}]^2} \left| E_{e^{2\alpha_i^{-3/4}}(\mathcal{M}'_{l,j}(y))} \right|^2
$$

$$
\times \prod_{l=2}^j \left( |\mathcal{N}_l(\rho, k - 1)|^2 + |Q_l(\rho, k)\right|^2
$$

$$
\times \prod_{l=j+2}^J \left( |\mathcal{N}_l(\rho, k - 1)|^2 + |Q_l(\rho, k)|^2 \right)^2,
$$

where the last estimation above follows by virtue of the bound

$$
\prod_{l=2}^j \left( 1 + O \left( e^{-\alpha_i^{-3/4}} \right) \right) \ll 1.
$$

We also deduce from the description on $S(j)$ and above that when $j \geq 1$,

$$
\sum_{\gamma \in S(j) \cap P(m) \cap T} |\zeta'(\rho)|^2 |\mathcal{N}(\rho, k)|^{2(1-1/k)}
$$

$$
\ll (\log T)^2 e^{(2-k)2^m/2+4} \exp \left( \frac{2}{\alpha_j} \right) \sum_{u=j+1}^j \sum_{\gamma \in S(j)} \left( 2^{2m/10} |P_m(\gamma)| \right)^{2[2^{m/2}]^2} \left| E_{e^{2\alpha_i^{-3/4}}(\mathcal{M}'_{l,j}(y))} \right|^2
$$

$$
\times \prod_{l=2}^j \left( 1 + O \left( e^{-\alpha_i^{-3/4}} \right) \right) \ll 1.
$$
\[
\times \prod_{l=2}^{j} \left( |E_{\alpha_{j+1}} \mathcal{M}_{l,j}(\gamma)|^{2} + |Q_{l}(\rho, k)|^{2} \right) \prod_{l=j+2}^{J} \left( |\mathcal{N}_{l}(\rho, k-1)|^{2} + |Q_{l}(\rho, k)|^{2} \right)
\]
\[
\times \left| \alpha_{j+1}^{3/4} \mathcal{M}_{j+1,u}(\gamma) \right|^{2 \left[1/(10\alpha_{j+1}) \right]} |\mathcal{N}_{j+1}(\rho, k)|^{2(1-1/k)}. \tag{5.20}
\]

We further simplify the right-hand expression above by noting that if \(|k P_{j+1}(\rho)| \leq \alpha_{j+1}^{-3/4} (1 - 1/k)\), then similar to (5.12),
\[
|\mathcal{N}_{j+1}(\rho, k)|^{2(1-1/k)} \ll \exp \left( 2(2 - k) \Re P_{j+1}(\rho) \right) \ll \exp \left( 2\alpha_{j+1}^{-3/4} \right) \ll 2^{2[1/(10\alpha_{j+1})]}. \tag{5.21}
\]

While when \(|k P_{j+1}(\rho)| > \alpha_{j+1}^{-3/4} (1 - 1/k)\), we have by (5.17),
\[
|\mathcal{N}_{j+1}(\rho, k)|^{2(1-1/k)} \leq |Q_{j+1}(\rho, k)|^{2}. \tag{5.22}
\]

We use (5.21) and (5.22) in (5.20) to deduce that
\[
\sum_{\gamma \in S(j) \cap \mathcal{M}(m) \cap \mathcal{T}} |\mathcal{N}(\rho, k)|^{2(1-1/k)}
\ll (\log T)^{2} e^{(2-k)2m/2+4} \exp \left( \frac{2}{\alpha_{j}} \right) \sum_{u=j+1}^{J} \sum_{\gamma \in S(j)} \left( 2^{m/10} |P_{m}(\gamma)| \right)^{2/2^{m/2}} \left| E_{e^{2\alpha_{j}} \mathcal{M}_{1,j}(\gamma)} \right|^{2}
\]
\[
\times \prod_{l=2}^{j} \left( |E_{e^{2\alpha_{j}} \mathcal{M}_{l,j}(\gamma)}|^{2} + |Q_{l}(\rho, k)|^{2} \right) \prod_{l=j+2}^{J} \left( |\mathcal{N}_{l}(\rho, k-1)|^{2} + |Q_{l}(\rho, k)|^{2} \right)
\]
\[
\times \left( 2\alpha_{j+1}^{3/4} \mathcal{M}_{j+1,u}(\gamma) \right)^{2 \left[1/(10\alpha_{j+1}) \right]} + |Q_{j+1}(\rho, k)|^{2} \left| \alpha_{j+1}^{3/4} \mathcal{M}_{j+1,u}(\gamma) \right|^{2 \left[1/(10\alpha_{j+1}) \right]}
\]
\[
\ll (\log T)^{2} e^{(2-k)2m/2+4} \exp \left( \frac{2}{\alpha_{j}} \right) \sum_{u=j+1}^{J} \sum_{\gamma \in S(j)} \left( 2^{m/10} |P_{m}(\gamma)| \right)^{2/2^{m/2}} \left| E_{e^{2\alpha_{j}} \mathcal{M}_{1,j}(\gamma)} \right|^{2}
\]
\[
\times \prod_{l=2}^{j} \left( |E_{e^{2\alpha_{j}} \mathcal{M}_{l,j}(\gamma)}|^{2} + |Q_{l}(\rho, k)|^{2} \right) \prod_{l=j+2}^{J} \left( |\mathcal{N}_{l}(\rho, k-1)|^{2} + |Q_{l}(\rho, k)|^{2} \right)
\]
\[
\times \left( 2\alpha_{j+1}^{3/4} \mathcal{M}_{j+1,u}(\gamma) \right)^{2 \left[1/(10\alpha_{j+1}) \right]} + |Q_{j+1}(\rho, k)|^{4} \left| \alpha_{j+1}^{3/4} \mathcal{M}_{j+1,u}(\gamma) \right|^{4 \left[1/(10\alpha_{j+1}) \right]}
\]
As the treatments are similar, it suffices to estimate the following expression given by
\[
S := (\log T)^{2} e^{(2-k)2m/2+4} \exp \left( \frac{2}{\alpha_{j}} \right) \sum_{u=j+1}^{J} \sum_{\gamma \in S(j)} \left( 2^{m/10} |P_{m}(\gamma)| \right)^{2/2^{m/2}} \left| E_{e^{2\alpha_{j}} \mathcal{M}_{1,j}(\gamma)} \right|^{2}
\]
\[
\times \prod_{l=2}^{j} \left( |E_{e^{2\alpha_{j}} \mathcal{M}_{l,j}(\gamma)}|^{2} + |Q_{l}(\rho, k)|^{2} \right) \]
\[
\times \prod_{l=j+2}^{J} \left( |N'_l(\rho, k - 1)|^2 + |Q_l(\rho, k)|^2 \right) \left| 2\alpha_l^{3/4} M_{j+1, u}(\gamma) \right|^2 [1/(10\alpha_{j+1})] \]
\[
:= \sum_{u=j+1}^{J} S_u. \tag{5.23}
\]

It remains to evaluate \( S_u \) for a fixed \( u \). To that end, we define a totally multiplicative function \( c_j(n, k) \) such that \( c_j(p, k) \) is defined in (5.14). This allows us to write for each \( 1 \leq l \leq j \),

\[
E e^{2\alpha_{l}^{-3/4}}(\mathcal{M}'_{l,j}(\gamma)) = \sum_{n_l \leq T_{\alpha_{l}^{2}\alpha_l^{-3/4}}} v_l(n_l) n_l^{-iy}, \quad \text{where } v_l(n_l) = \begin{cases} c_j(n_l, k) b_{l,m}(n_l) g_{l}(n_l) \sqrt{n_l}, & l = 1, \\ c_j(n_l, k) b_l(n_l) g_l(n_l) \sqrt{n_l}, & l \neq 1, \end{cases}
\]

and where the function \( b_{1,m}(n) \) is defined to be take values 0 or 1 and \( b_{1,m}(n) = 1 \) if and only if \( n \) is composed of at most \( \ell_1 \) primes, all from the interval \((2^m+1, 2^m] \).

Note that for \( 1 \leq l \leq j \) and integers \( n_l \) satisfying \( b_l(n_l) = 1 \) when \( 2 \leq l \leq j \) and \( b_{1,m}(n_1) = 1 \) when \( l = 1 \),

\[
|c_j(n_l, k)| \leq \left| 2 - k \right| \ell_l. \tag{5.24}
\]

We also write for each \( j + 1 \leq l \leq J \),

\[
\mathcal{N}_l(\rho, k - 1) = \sum_{n_l \leq T_{\alpha_l^{2}\alpha_l^{-3/4}}} v_l(n_l) n_l^{-iy},
\]

where by (4.1),

\[
v_l(n_l) = \frac{(k - 1)^{\Omega(n_l)}}{g(n_l) \sqrt{n_l}} b_l(n_l). \tag{5.25}
\]

We define the functions \( q_l(n) \) for \( 0 \leq l \leq J \) such that \( q_l(n) = 0 \) or 1, and \( q_l(n) = 1 \) if and only if \( n \) is composed of exactly \( \lfloor 1 - 1/k \rfloor \ell_l \) primes (counted with multiplicity), all from the interval \( I_l \) when \( l \neq 0, j + 1 \). While we define \( q_0(n) = 1 \) (resp. \( q_{j+1}(n) = 1 \)) if and only if \( n \) is composed of exactly \( \lfloor 2^{m/2} \rfloor \) (resp. \( \lfloor 1/(10\alpha_{j+1}) \rfloor \)) primes (counted with multiplicity), all from the interval \((2^m, 2^{m+1}] \) (resp. \( I_{j+1} \)). Further, we define for \( 0 \leq l \leq J \),

\[
\beta_l(n_l) = \begin{cases} (2^{m/10-1})^{\lfloor 2^{m/2} \rfloor} [2^{m/2}]!, & l = 0, \\ (2\alpha_l^{3/4})^{\lfloor 1/(10\alpha_l) \rfloor} [1/(10\alpha_l)]! c_u(n_l, 1), & l = j + 1, \\ (2(1-k)e^{(e(1-k))-1})^{\lfloor 1/(1-1/k) \ell_l \rfloor} \alpha_l^{3/4} \lfloor 1/(1/k) \ell_l \rfloor! ((1 - 1/k) \ell_l)! , & 1 \leq l \leq J, \ l \neq j + 1. \end{cases}
\]
With the above notations, we write
\[
\left(2^{m/10} P_{m}(y)\right)^{[2m/2]} = \sum_{n_0} w_0(n_0) n_0^{-iy}, \quad \left(2^{3/4} \alpha_{j+1} \mathcal{M}_{j+1,u}(y)\right)^{[1/(10\alpha_{j+1})]} = \sum_{n_{j+1}} w_{j+1}(n_{j+1}) n_{j+1}^{-iy},
\]
\[Q_l(\rho, k) = \sum_{n_l} w_l(n_l) n_l^{-iy}, \quad 1 \leq l \leq J, \quad l \neq j + 1,
\]
where
\[
w_0(n_0) = \frac{q_0(n_0) \beta_0(n_0)}{g(n_0)}, \quad w_{j+1}(n_{j+1}) = \frac{q_{j+1}(n_{j+1}) \beta_{j+1}(n_{j+1})}{\sqrt{n_{j+1}} g(n_{j+1})},
\]
\[
w_l(n_l) = \frac{q_l(n_l) \beta_l(n_l)}{\sqrt{n_l} g(n_l)}, \quad 1 \leq l \leq J, \quad l \neq j + 1.
\]
We note that by (5.3), we have for any integer \(n_0\),
\[
\beta_0(n_0) \leq \left[2^{m/2}\right]\left(\frac{2^{m/10-1} \left[2^{m/2}\right]}{e}\right) = e^{(\log \log T)^3}, \quad (5.26)
\]
since we have \(0 \leq m \leq (3 / \log 2) \log \log T\). Also, for any integer \(n_{j+1}\) such that \(q_{j+1}(n_{j+1}) = 1\), we note that the estimation given (5.24) continues to hold for \(k = 1\) and \(j = s\). This together with (5.3) implies that
\[
\beta_{j+1}(n_{j+1}) \leq \left[1/(10\alpha_{j+1})\right]\left(\frac{2^{3/4} \alpha_{j+1} \left[1/(10\alpha_{j+1})\right]}{e}\right) = e^{(\log \log T)^3}. \quad (5.27)
\]
Moreover, for any integer \(n_l\) such that \(q_l(n_l) = 1\) for \(1 \leq l \leq J, l \neq j + 1\), applying (5.3), we get
\[
\beta_l(n_l) \leq [1 - 1/k] e^\left(\frac{2(1 - k)e^{(e(1-1/k))^{-1}} \alpha_l^{3/4} \left[1 - 1/k\right] e_l^{1-1/k} e_l}{e}\right) \leq [1 - 1/k] e^\left(2(1 - k)e^{(e(1-1/k))^{-1} + 2 \left[1 - 1/k\right]} e_l\right). \quad (5.28)
\]
It follows from this, (5.24)–(5.25), that we can write \(|\mathcal{N}_l(\rho, k - 1)|^2 + |Q_l(\rho, k)|^2\) for \(j + 1 \leq l \leq J\) as a Dirichlet polynomial of the form
\[
\sum_{n_l, n_l' \leq T^{1-1/k}[\alpha_l e^{2\alpha_l^{3/4}}]} a_{n_l} a_{n_l'} b_l(n_l) b_l(n_l') n_l^{-iy} n_l'^{iy} \sqrt{n_l n_l'},
\]
where for some constant \(B(k)\), whose value depends on \(k\) only,
\[
|a_{n_l}|, |a_{n_l'}| \leq B(k)^{\epsilon_j} \quad (5.29)
\]
We may also recast
\[
\left(2^{m/10}|P_m(\gamma)|\right)^2 [2^{m/2}], \quad \left|\mathcal{E}_{e^{2\alpha_1^{-3/4}}} (\mathcal{M}_{1,j}(\gamma))\right|^2, \quad \left|2\alpha_{j+1}^{3/4} \mathcal{M}_{j+1,l}(\gamma)\right|^{2[1/(10\alpha_{j+1})]},
\]
and
\[
\left|\mathcal{E}_{e^{2\alpha_1^{-3/4}}} (\mathcal{M}_{l,j}(\gamma))\right|^2 + \left|Q_l(\rho, k)\right|^2, \quad 2 \leq l \leq j
\]
into similar Dirichlet polynomials of the form
\[
\sum_{n,n'} \frac{a_n a_{n'}}{n n'} n^{-2i\gamma} n'^{2i\gamma} \quad \text{or} \quad \sum_{n,n'} \frac{a_n a_{n'}}{\sqrt{n n'}} n^{-i\gamma} n'^{i\gamma}.
\]
We may enlarge \(B(k)\) to see by (5.24) and (5.26)–(5.28),
\[
|a_n|, |a_{n'}| \leq B(k) e^{(\log \log T)^3} \ll e^{2(\log \log T)^3}.
\]
Also, the lengths of the four Dirichlet polynomials in (5.30) and (5.31) are bounded by, respectively,
\[
2^{((3/\log 2)\log \log T + 1)[2^{(3/\log 2)\log \log T}/2]}, \quad T^{\alpha_1 [e^{2\alpha_1^{-3/4}}]}, \quad T^{\alpha_{j+1} [1/(10\alpha_{j+1})]}, \quad T^{[1-1/k]\alpha_1 [e^{2\alpha_1^{-3/4}}]}.
\]
We apply the above discussions to write \(S_u\) for simplicity as
\[
S_u = (\log T)^2 e^{(2-k)2^{m/2+4}} \exp\left(\frac{2}{\alpha_j}\right) \sum_{a,b,c,d} \frac{A_{a,b,c,d}}{\sqrt{abcd}} a^b b^{-i\gamma} c^{2i\gamma} d^{-2i\gamma},
\]
where by (5.29), (5.32), and (5.33),
\[
A_{a,b,c,d} \ll e^{4T(\log \log T)^3} \ll e^{4(\log \log T)^4},
\]
\[
a, b \leq T^{\sum_{j=1}^{j'} [1-1/k] \alpha_j [e^{2\alpha_j^{-3/4}}] + \alpha_{j+1} [1/(10\alpha_{j+1})]} \leq T^{40e^2 10^{-M/4} + 1/5},
\]
\[
c, d \leq 2^{((3/\log 2)\log \log T + 1)[2^{(3/\log 2)\log \log T}/2]}, \quad \ll T^c.
\]
It follows from this that if we apply Lemma 2.1 to evaluate
\[
\sum_{T < \gamma \leq 2T} \sum_{a,b,c,d} \frac{A_{a,b,c,d}}{\sqrt{abcd}} a^b b^{-i\gamma} c^{2i\gamma} d^{-2i\gamma},
\]
the error term in (2.3) contributes
\[
\ll (\log T)^2 \sum_{a,b,c,d} |A_{a,b,c,d}| \ll T^{1-\varepsilon}.
\]
Thus, we only need to focus on the contribution from the main terms in Lemma 2.1 in the process. As the estimations are similar, it suffices to treat the sum

$$
\frac{T}{2\pi} \sum_{a,b,c,d} \left( \frac{A_{a,b,c,d} \Lambda((ac^2)/(bd^2))}{\sqrt{abcd}} \sqrt{(ac^2)/(bd^2)} + \frac{A_{a,b,c,d} \Lambda((bd^2)/(ac^2))}{\sqrt{abcd}} \sqrt{(bd^2)/(ac^2)} \right)
$$

(5.35)

As \(\Lambda(n)\) is supported on positive prime powers, we consider the case \(\Lambda((ac^2)/(bd^2))\) or \(\Lambda((bd^2)/(ac^2))\) takes the value \(\log q\) for a fixed prime \(q\). Without loss of generality, we may assume that \(q \in I_2\) and \(j > 2\). It is then easy to see that we must have \(c = d\) and \(a, b\) can be written in the form:

$$
a = (n_2 q^{l_1}) \prod_{0 \leq l < J, l \neq 2} n_l, \quad b = (n_2 q^{l_2}) \prod_{0 \leq l < J, l \neq 2} n_l,
$$

where \(l_1, l_2 \geq 0, l_1 \neq l_2, (n_2, q) = 1\) and the integers \(n_l\) satisfy (note that there is no contribution from the terms in the expansion of \(|Q_2(\rho, k)|^2\))

$$
q_0(n_0) = b_{1,m}(n_1) = b_2(n_2 q^{l_1}) = b_2(n_2 q^{l_2}) = q_{j+1}(n_{j+1}) = 1,
$$

$$
b_1^2(n_l) + q_1^2(n_l) = 1, \quad 2 \leq l \leq J, \quad l \neq j + 1.
$$

It follows that the contribution from a fixed \(q \in I_2\) to the expression in (5.35) equals

$$
\frac{T}{2\pi} \sum_{l_1 \geq 0, l_2 \geq 0, l_1 \neq l_2} \frac{\log q}{q^{l_1-l_2}/2} \left( \sum_{n_2} v_{n_2 q^{l_1}} v_{n_2 q^{l_2}} \right) \sum_{n_0} w_0^2(n_0) \sum_{n_1} v_1^2(n_1) \sum_{n_{j+1}} w_{j+1}^2(n_{j+1})
$$

$$
\times \prod_{3 \leq l < J \atop l \neq j + 1} \sum_{n_l} (v_l^2(n_l) + w_l^2(n_l))
$$

$$
= \frac{T}{2\pi} \sum_{l_1 \geq 0, l_2 \geq 0, l_1 \neq l_2} \frac{\log q}{q^{l_1-l_2}/2} \frac{c_{j}(q, k)^{l_1+l_2}}{l_1! l_2!} \left( \sum_{n_2} c_j^2(n_2, k) b_2(n_2 q^{l_1}) b_2(n_2 q^{l_2}) \right) \prod_{n_0} w_0^2(n_0) \sum_{n_1} v_1^2(n_1) \sum_{n_{j+1}} w_{j+1}^2(n_{j+1}) \prod_{3 \leq l < J \atop l \neq j + 1} \sum_{n_l} (v_l^2(n_l) + w_l^2(n_l)).
$$

(5.36)

As in the proof of Proposition 3.2, we remove the restriction in \(b_2(n_2 q^{l_1})\) on \(\Omega(n_2 q^{l_1})\) and in \(b_2(n_2 q^{l_2})\) on \(\Omega(n_2 q^{l_2})\) to get that the sum on the right-hand side in (5.36) over \(n_2\) becomes

$$
\sum_{n_2} \frac{c_j^2(n_2, k)}{g^2(n_2) n_2} = \prod_{p \in I_2} \left( 1 + \frac{k^2}{p} + O\left( \frac{1}{p^2} \right) \right) \ll \exp \left( \sum_{p \in I_2} \frac{k^2}{p} + O\left( \sum_{p \in I_2} \frac{1}{p^2} \right) \right).
$$
Moreover, if \( \Omega(n_2 q^l) \geq \ell_2 \) for \( i = 1, 2 \), then \( 2^{\Omega(n_2)} l_i - \ell_2 \geq 1 \). Thus, we apply Rankin’s trick and the error introduced in the relaxation of the conditions on \( b_2 \) is

\[
\ll (2^{l_1} + 2^{l_2}) 2^{-\ell_2} \sum_{n_2} 2^{\Omega(n_2)} \frac{|A(n_2)|^2}{g^2(n_2)n_2} \ll (2^{l_1} + 2^{l_2}) 2^{-\ell_2/2} \exp \left( \sum_{p \in \mathcal{L}_2} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_2} \frac{1}{p^2} \right) \right).
\]

We then deduce that the contribution to (5.35) from the case \( \Lambda(ac^2/bd^2) = \Lambda(bd^2/ac^2) = \log q \) for a fixed prime \( q \in I_2 \) is

\[
\sum_{l_1 \geq 0, l_2 \geq 0, l_1 \neq l_2} \frac{\log q}{q} \frac{c_j(q, k)^{l_1+l_2}}{l_1! l_2!} (1 + O \left( (2^{l_1} + 2^{l_2}) 2^{-\ell_2/2} \right)) \exp \left( \sum_{p \in \mathcal{L}_2} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_2} \frac{1}{p^2} \right) \right)
\]

\[
\ll \left( 1 + O \left( 2^{-\ell_2/2} \right) \right) \left( \frac{\log q}{q} + O \left( \frac{\log q}{q^2} \right) \right) \exp \left( \sum_{p \in \mathcal{L}_2} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_2} \frac{1}{p^2} \right) \right).
\]

Similar estimations carry over to the sums over \( v^2_i(n_i) \) for \( 3 \leq l \leq J, l \neq j + 1 \) in (5.36) so that we have

\[
\sum_{n_i} v^2_i(n_i) = \left( 1 + O \left( 2^{-\ell_i/2} \right) \right) \exp \left( \sum_{p \in \mathcal{L}_1} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_1} \frac{1}{p^2} \right) \right).
\]

To treat the other sums, we note that

\[
\sum_{n_0} w^2_0(n_0) \ll \frac{(2m/10 - 1)^{2[2m/2]} [2m/2]!}{[2m/2]!} \left( \sum_{2m < p} \frac{1}{p^2} \right)^{[2m/2]},
\]

\[
\sum_{n_{j+1}} w^2_{j+1}(n_{j+1}) \ll \frac{(2\alpha_{j+1}^3/4)^{2[1/(10\alpha_{j+1})]} [1/(10\alpha_{j+1})]!}{[1/(10\alpha_{j+1})]!} \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{c^2_u(p, 1)}{p} \right)^{[1/(10\alpha_{j+1})]},
\]

\[
\sum_{n_{j}} w^2_{j}(n_{j}) \ll \frac{(2(1 - k)e^{(1-1/k)})^{-1} \alpha_j^{3/4}}{(1 - 1/k)^{\ell_i}} \left( \sum_{p \in \mathcal{L}_j} \frac{1}{p} \right)^{[1-1/k] \ell_i},
\]

\( l \neq j + 1 \).

We apply (3.2) and (5.3) and note that \( \alpha_{j+1} = 2\alpha_j \), deducing from the above that

\[
\sum_{n_0} w^2_0(n_0) \ll e^{-m_2^{m/2}/5} \exp \left( \sum_{2m < p \leq 2m+1} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{1}{p^2} \right) \right),
\]

\[
\sum_{n_{j+1}} w^2_{j+1}(n_{j+1}) \ll e^{-\frac{30}{\alpha_{j+1}}} \exp \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{1}{p^2} \right) \right) = e^{-\frac{4}{5}} \exp \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_{j+1}} \frac{1}{p^2} \right) \right),
\]

\[
\sum_{n_{j}} w^2_{j}(n_{j}) \ll 2^{-\ell_i/2} \exp \left( \sum_{p \in \mathcal{L}_j} \frac{k^2}{p} + O \left( \sum_{p \in \mathcal{L}_j} \frac{1}{p^2} \right) \right).
\]
The bounds above enable us to derive that (5.36) is

\[ \ll e^{-m^{2m/2}/5} e^{-\frac{4}{\alpha_j}} \prod_{l=1}^{J} \left( 1 + O\left(2^{-\ell/2}\right) \right) \exp \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} k^2 + O \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} \frac{1}{p^2} \right) \right) \log \frac{q}{q} + O \left( \log \frac{q}{q^2} \right) \]

\[ \ll e^{-m^{2m/2}/5} e^{-\frac{4}{\alpha_j}} \exp \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} k^2 + O \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} \frac{1}{p^2} \right) \right) \log \frac{q}{q} + O \left( \log \frac{q}{q^2} \right). \]

Summing over \( q \) renders that (5.35) is

\[ \ll T e^{-m^{2m/2}/5} e^{-\frac{4}{\alpha_j}} \exp \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} k^2 + O \left( \sum_{p \in \mathcal{J}_{l=1}^{J} I_l} \frac{1}{p^2} \right) \right) \sum_{q \leq T} \left( \log \frac{q}{q} + O \left( \log \frac{q}{q^2} \right) \right) \]

\[ \ll 10^{-M T (\log T)^{k^2+1}} e^{-m^{2m/2}/5} e^{-\frac{4}{\alpha_j}}, \]

utilizing Lemma 2.2. We then conclude from this and (5.34) that

\[ S_u \ll T (\log T)^{k^2+3} e^{(2-k)2m^{2/2}+4-m^{2m/2}/5} e^{-\frac{2}{\alpha_j}}. \]

Now summing over \( u \) in (5.23) gives rise to the bound

\[ S \ll T (\log T)^{k^2+3} e^{(2-k)2m^{2/2}+4-m^{2m/2}/5} e^{-\frac{2}{\alpha_j}} (J - j) \ll T (\log T)^{k^2+3} e^{(2-k)2m^{2/2}+4-m^{2m/2}/5} e^{-\frac{1}{\alpha_j}}, \]

where the last estimation above follows by observation that for \( 1 \leq j \leq J \),

\[ J - j \leq \frac{\log(1/\alpha_j)}{\log 20} \leq \frac{1}{\alpha_j}. \]

Finally, the bound in (5.9) emerges from summing over \( j \) and \( m \), thus completing the proof of the proposition.

We conclude the paper with the remark here that it may be possible to bound \( S_u \), defined in (5.23), unconditionally using the approach of Theorem 2 of [9]. However, this would not be an easy or simple adaption. Moreover, our proof of Proposition 3.3 provides a different approach and the truth of RH is necessary elsewhere in the paper and hence indispensable in our main result Theorem 1.1 anyway.

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LOWERBOUNDSFORNEGATIVEMOMENTSOF\(\zeta'(\rho)\)

**JOURNAL INFORMATION**

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**REFERENCES**

1. H. M. Bui, S. M. Gonek, and M. B. Milinovich, *A hybrid Euler-Hadamard product and moments of\(\zeta'(\rho)\)*, Forum Math. **27** (2015), no. 3, 1799–1828.
2. V. Chandee and X. Li, *Lower bounds for small fractional moments of Dirichlet L-functions*, Int. Math. Res. Not. **19** (2013), 4349–4381.
3. H. Davenport, *Multiplicative number theory*, 3rd ed., Graduate Texts in Mathematics, vol. 74, Springer, Berlin, 2000.
4. P. Gao, *Sharp lower bounds for moments of \(\xi'(\rho)\)*, arXiv:2106.03057.
5. S. M. Gonek, *Mean values of the Riemann zeta function and its derivatives*, Invent. Math. **75** (1984), no. 1, 123–141.
6. S. M. Gonek, *On negative moments of the Riemann zeta-function*, Mathematika **36** (1989), no. 1, 71–88.
7. S. M. Gonek, *An explicit formula of Landau and its applications to the theory of the zeta-function*, Contemp. Math. **143** (1993), 395–413.
8. A. J. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, arXiv:1305.4618.
9. W. Heap, J. Li, and J. Zhao, *Lower bounds for discrete negative moments of the Riemann zeta function*, Algebra Number Theory **16** (2022), no. 7, 1589–1625.
10. W. Heap and K. Soundararajan, *Lower bounds for moments of zeta and L-functions revisited*, Mathematika **68** (2022), no. 1, 1–14.
11. D. A. Hejhal, *On the distribution of \(\log |\zeta'(1/2 + it)|\)*, Number theory, trace formulas and discrete groups (Oslo, 1987), Academic Press, Boston, MA, 1989, pp. 343–370.
12. C. P. Hughes, J. P. Keating, and N. O’Connell, *Random matrix theory and the derivative of the Riemann zeta function*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **456** (2000), no. 2003, 2611–2627.
13. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.
14. S. Kirila, *An upper bound for discrete moments of the derivative of the Riemann zeta-function*, Mathematika **66** (2020), no. 2, 475–497.
15. E. Landau, *Über die Nullstellen der Zetafunktion*, Math. Ann. **71** (1912), no. 4, 548–564.
16. M. B. Milinovich, *Upper bounds for moments of \(\xi'(\rho)\)*, Bull. Lond. Math. Soc. **42** (2010), no. 1, 28–44.
17. M. B. Milinovich and N. Ng, *A note on a conjecture of Gonek*, Funct. Approx. Comment. Math. **46** (2012), no. part 2, 177–187.
18. M. B. Milinovich and N. Ng, *Lower bounds for moments of \(\xi'(\rho)\)*, Int. Math. Res. Not. **12** (2014), 3190–3216.
19. H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.
20. N. Ng, *The distribution of the summatory function of the Möbius function*, Proc. Lond. Math. Soc. (3) **89** (2004), no. 2, 361–389.
21. N. Ng, *The fourth moment of \(\xi'(\rho)\)*, Duke Math. J. **125** (2004), no. 2, 243–266.
22. N. Ng, *Extreme values of \(\xi'(\rho)\)*, J. Lond. Math. Soc. (2) **78** (2008), no. 2, 273–289.
23. Z. Rudnick and K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Natl. Acad. Sci. USA **102** (2005), no. 19, 6837–6838.
24. K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), no. 2, 981–993.
25. E. C. Titchmarsh, *The theory of the Riemann Zeta-function*, 2nd ed., Edited and with a preface by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.