Distributed Event-triggered Consensus for Multi-agent Systems with Directed Topologies

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Abstract

In this paper, we study consensus problem in multi-agent system with directed topology by event-triggered feedback control. That is, at each agent, the diffusion coupling feedbacks are based on the information from its latest observations to its in-neighbours. We derive distributed criteria to determine the next observation time of each agent that are triggered by its in-neighbours’ information and its own states respectively. We prove that if the network topology is irreducible, then under the event-triggered coupling principles, the multi-agent system reach consensus. Then, we extend these results to the case of reducible topology with spanning tree. In addition, these results are also extended to the case of self-triggered control, in terms that the next triggering time of each agent is computed based on the current states, i.e., without observing the system’s states continuously. The effectiveness of the theoretical results are illustrated by numerical examples.

Key words: Consensus; multi-agent systems; event-triggered; distributed control; directed graph.

1 Introduction

In the past decades, consensus problem in multi-agent systems, i.e. a group of agents seeks to agree upon certain quantity of interest, has attracted many researchers. See Olfati-Saber, & Murray (2004); Moreau (2004); Ren, & Beard (2005); Cao, Zheng, & Zhou (2011); Liu, Lu, & Chen (2011). A fundamental assumption to realize consensus is that the underlying graph of the network system has a spanning tree, as proved by Olfati-Saber, & Murray (2004).

However, the results of all these papers are based on the continuous feedback of states as control to realize a consensus. In the future, agent could be equipped with embedded microprocessors with limited resources that will transmit and gather information, etc. Motivated by this, event-triggered control were proposed by Lu, & Chen (2004, 2007).

More related to the present work, consensus problem of multi-agent systems by event-triggered strategy were studied...
by Liu, Chen, & Yuan (2012) for in directed and weighted but balanced graph topologies. Directed graph topology was considered by Zhu, Jiang & Feng (2014), where each agent needs not only the information of its in-neighbours but also its in-neighbours’ latest controller update value to determine the next triggering time and also by Persis, Sailer, & Wirth (2013) with sufficient conditions with a large number of parameters.

In this paper, we study event-triggered and self-triggered principles for consensus in multi-agent system with directed and weighted topology. First, we proved that if the directed network topology is irreducible, then the event-triggered controller of distributed feedback, the multi-agent system can reach consensus, where the consensus value needs not only the information of its in-neighbors but also its in-neighbors’ latest controller update value to determine the next triggering time.

The paper is organized as follows: in Section 2, the preliminaries and problem formulation are given; in Section 3, the event-triggered consensus in multi-agent systems with directed topologies is discussed; in Section 4, the self-triggered formulation of the framework is presented; in Section 5, a numerical example is provided to show the effectiveness of the theoretical results; the paper is concluded in Section 6.

Notions: The notation \( \| \cdot \| \) represents the Euclidean 2-norm of vectors or the induced 2-norm of matrices. \( \mathbf{1} \) denotes a column vector with each component equal to 1 of an appropriate dimension. \( \rho(\cdot) \) stands for the spectral radius of matrices and \( \rho_2(\cdot) \) indicates the minimum positive eigenvalue for matrices. Given two symmetric matrices \( M \) and \( N \), \( M > N \) (or \( M \geq N \)) means \( M - N \) is a positive definite (or positive semi-definite) matrix.

2 Preliminaries and problem formulation

In this section, we firstly provide some definitions and results on algebraic graph theory, which will be used later. See the textbooks by Diestel (2005) and Horn, & Johnson (1987) for details.

For a weighted directed graph (digraph) \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \) of \( m \) agents (or nodes), where \( \mathcal{V} = \{v_1, \ldots, v_m\} \) is the set of agents, \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the set of links (edges), and \( \mathcal{A} = (a_{ij}) \) with nonnegative adjacency elements \( a_{ij} \) is the weighted adjacency matrix, a link of \( \mathcal{G} \) is denoted by \( e(i, j) = (v_i, v_j) \in \mathcal{E} \) if there is a directed link from agent \( j \) to agent \( i \) and the adjacency elements associated with the links of the graph are positive, i.e., \( e(i, j) \in \mathcal{E} \) if and only if \( a_{ij} > 0 \). We take \( a_{ii} = 0 \) for all \( i \in \mathcal{I} \). Moreover, the in- and out-neighbours set of agent \( v_i \) are defined as \( N^{in}_i = \{v_j \in \mathcal{V} | a_{ij} > 0\} \), \( N^{out}_i = \{v_j \in \mathcal{V} | a_{ji} > 0\} \). We define the in-degree of agent \( v_i \) as \( \text{deg}^{in}(v_i) = \sum_{j=1}^{m} a_{ij} \) and the (in-)degree matrix of digraph \( \mathcal{G} \) as \( D = \text{diag}[\text{deg}^{in}(v_1), \ldots, \text{deg}^{in}(v_m)] \). We also define the weighted Laplacian matrix associated with the digraph \( \mathcal{G} \) as \( L = A - D \). A directed path from agent \( v_0 \) to agent \( v_k \) in a directed graph is a sequence of agents \( v_0, \ldots, v_k \) and links \( e_0, \ldots, e_{k-1} \) such that \( e_i \) is a link from \( v_i \) to \( v_{i+1} \), for all \( i < k \). We say that a directed graph \( \mathcal{G} \) is strongly connected if for any pair of agents \( v_i \) and \( v_j \), there exits a directed path from \( v_i \) to \( v_j \).

It is well known that \( \mathcal{G} \) is strongly connected is equivalent to the corresponding Laplacian matrix \( L \) is irreducible. Also by Perron-Frobenius theorem (see Horn, & Johnson (1987)), we have

**Lemma 1** If \( L \) is irreducible, then \( \text{rank}(L) = m - 1 \); in addition, zero is an algebraically simple eigenvalue of \( L \) and there is a positive vector \( \xi \) such that \( L \xi = 0 \) and \( \sum_{i=1}^{m} \xi_i = 1 \). If the directed graph \( \mathcal{G} \) has a spanning tree then we should change the positive vector to nonnegative vector in above conclusion.

**Lemma 2** If \( L \) is irreducible, then \( \Xi L + L^\top \Xi \) is a symmetric matrix with all row sums equal to zeros and has zero eigenvalue with algebraic dimension one.

Let \( R = (1/2)(\Xi L + L^\top \Xi) \), denoted by \( R = [R_{ij}]_{i,j=1}^{m} \). Obviously, \( R \) is negative semi-definite. Let \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \) be the eigenvalue of \( -R \), counting the multiplicities. Let \( Q = \Xi L L^\top \Xi \), which obviously has a unique zero eigenvalue. Let \( 0 = \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \) be the eigenvalue of \( Q \), counting the multiplicities. Denote \( U = \Xi - \beta_1 \Xi \). It can also be seen that \( U \) has a unique zero eigenvalue and its eigenvalues (counting the multiplicities) can be arranged as \( 0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m \). Then, we have \( \lambda_m x^\top x \geq \min_{x \in \mathcal{I}} \{x^\top (-R) x \} \geq \lambda_2 x^\top x \), and \( \beta_m x^\top x \geq \min_{x \in \mathcal{I}} \{x^\top Q x \} \geq \beta_2 x^\top x \). Therefore, we have

\[
\frac{\lambda_2}{\beta_m} Q \leq -R \leq \frac{\lambda_m}{\beta_2} Q
\]

(1)

It is clear that \( L^\top L \) is positive semi-definite and has a unique zero eigenvalue. Similar to (1), we have

\[
-R \geq \frac{\lambda_2}{\rho_2(L^\top L)} L^\top L, \quad U \leq \frac{\mu_m}{\rho_2(L^\top L)} L^\top L
\]

(2)

Consider a continuous-time multi-agent system with discon-
continuous diffusions as follows
\[
\begin{align*}
\dot{x}_i(t) &= u_i(t) \\
u_i(t) &= \sum_{j=1}^{m} L_{ij} x_j(t_{k_j}^j(t)), \quad i = 1, \ldots, m
\end{align*}
\]
(3)
The increasing time sequences \(\{t^j_k\}_{k=1}^\infty, \quad j = 1, \ldots, m\), which is named \textit{triggering event time}, are agent-wise and \(t^0_k = 0\), for all \(j \in \mathcal{I}\). At each \(t\), each agent \(j\) pushes its state to its all out-neighbours with respect to an identical time point \(t^j_k(t)\) with \(k_j(t) = \arg\max_{k} \{t^j_k \leq t\}\).

Hereby, we highlight the idea of the coupling terms described above. Instead of employing continuous state observation to realize a consensus, an economic alternative for agent \(j\) is to push its constant state at the nearest time point \(t^j_k\) to its out-neighbours until some pre-defined event is triggered at time \(t^j_{k+1}\); then the received information from agent \(j\) to the incoming neighbour \(i\) is updated as its state at \(t^j_k\), until the next event is triggered, and so on. This process goes on for each agent in a parallel fashion.

Let \(x(t) = [x_1(t), \ldots, x_m(t)]^T\). State measurement error is defined as:
\[
e_i(t) = x_i(t_{k}^j) - x_i(t), \quad t \in [t_{k}^j, t_{k+1}^j), \quad k = 0, 1, 2, \ldots
\]
(4) and \(e(t) = [e_1(t), \ldots, e_m(t)]^T\). Furthermore, define the combinational state measurement \(q_i(t) = \sum_{j=1}^{m} L_{ij} x_j(t) = \sum_{j=1}^{m} L_{ij} x_j(t) - \bar{x}(t) = \sum_{j=1}^{m} L_{ij} x_j(t)\) and \(q(t) = [q_1(t), \ldots, q_m(t)]^T = L(x(t) - \bar{x}(t)) = Lx(t)\).

3 Event-triggered principles

In this section, by the techniques developed by Chen, Liu, & Lu (2007), we study event-triggered control for multi-agent systems with directed and weighted topology.

Consider the following candidate Lyapunov function:
\[
V(t) = \frac{1}{2} \sum_{i=1}^{m} \xi_i(x_i(t) - \bar{x}(t))^2
\]
(5)
where \(\bar{x}(t) = \sum_{i=1}^{m} \xi_i x_i(t)\) is the weighted average of \(x(t)\) by the left eigenvector \(\xi\) of \(L\) corresponding to the single eigenvalue zero, \(X(t) = [\bar{x}(t), \ldots, \bar{x}(t)]^T\) accordingly. Since \(\xi^T L = 0\), we have
\[
\dot{x}(t) = \sum_{i=1}^{m} \xi_i \dot{x}_i(t) = \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} L_{ij} x_j(t_{k_j}^j(t))
\]
\[
= \sum_{j=1}^{m} x_j(t_{k_j}^j(t)) \sum_{i=1}^{m} \xi_i L_{ij} = 0,
\]
(6) and
\[
\sum_{i=1}^{m} \bar{x}(t) \xi_i L_{ij} x_j(t_{k_j}^j(t)) = 0, \quad \sum_{i=1}^{m} \bar{x}(t) \xi_i L_{ij} x_j(t_{k_j}^j(t)) = 0.
\]
(7)

Then, the derivative of \(V(t)\) along (3) is
\[
\frac{d}{dt} V(t) = \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t))[L_{ij} x_j(t) - \dot{x}(t)]
\]
\[
+ \sum_{i=1}^{m} \xi_i (x_i(t) - \bar{x}(t)) \sum_{j=1}^{m} L_{ij} [x_j(t_{k_j}^j(t)) - x_j(t)]
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_i L_{ij} x_i(t) x_j(t)
\]
\[
+ \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_i L_{ij} x_i(t)[x_j(t_{k_j}^j(t)) - x_j(t)]
\]
(8)

Noting (4) and inequality (1), for any \(a > 0\), it holds that
\[
\frac{d}{dt} V(t) = \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_i L_{ij} x_i(t) x_j(t) + \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_i L_{ij} x_i(t) e_j(t)
\]
\[
= x^T R x + x^T \Xi L e \leq x^T R x + \frac{a}{2} x^T Q x + \frac{1}{2a} e^T e
\]
\[
\leq - (1 - \frac{a \beta m}{2 \lambda_2}) x^T(t)(-R)x(t) + \frac{1}{2a} e^T e
\]
(9)

Substituting fact (2) into (9), we have
\[
\frac{d}{dt} V(t) \leq - (1 - \frac{a \beta m}{2 \lambda_2}) \frac{\lambda_2}{\rho(L^T L)} x^T(t) L x(t) + \frac{1}{2a} e^T e
\]
\[
= - (1 - \frac{a \beta m}{2 \lambda_2}) \frac{\lambda_2}{\rho(L^T L)} q^T(t) q(t) + \frac{1}{2a} e^T e
\]
(10)

Then, we have

\textbf{Theorem 1} Suppose that \(G\) is strongly connected. Set
\[
t^j_{k+1} = \max \{\tau \geq t^j_k : |x_j(t^j_k) - x_j(t)| \leq \sqrt{2\gamma a b |q_j(t)|}, \forall t \in [t^j_k, \tau]\}
\]
(11)
with \(\gamma \in (0, 1), 0 < a < \frac{2 \lambda_2}{\rho(L^T L)}, \) and \(b = (1 - \frac{a \beta m}{2 \lambda_2}) \frac{\lambda_2}{\rho(L^T L)}\).

Then, system (3) reaches consensus exponentially; in addition, for all \(i\), \(\lim_{t \to \infty} x_i(t) = \sum_{j=1}^{m} \xi_j x_j(0)\).

\textbf{Proof}. By inequalities (10) and (2), and condition (11), we
have
\[
\frac{d}{dt} V(t) \leq - (1 - \gamma)(1 - \frac{a\beta_m}{2\lambda_2}) \frac{\lambda_2}{\rho(L^TL)} q^T(t)q(t) \\
\leq - (1 - \gamma)(1 - \frac{a\beta_m}{2\lambda_2}) \frac{\lambda_2}{\rho(L^TL)} \frac{2\rho_2(L^TL)}{\mu_m} V(t)
\]
for all \( t \geq 0 \). It means
\[
V(t) \leq \exp \left\{ - (1 - \gamma)(1 - \frac{a\beta_m}{2\lambda_2}) \frac{\lambda_2}{\rho(L^TL)} \frac{2\rho_2(L^TL)}{\mu_m} t \right\} V(0)
\]
(12)

This implies that system (3) reaches consensus exponentially. Combined with (6), which implies \( \bar{x}(t) = \bar{x}(0) \) for all \( t \geq 0 \), we have \( \lim_{t \to \infty} x_i(t) = \sum_{j=1}^{m} \xi_j x_j(t) \) exponentially for all \( i \). This completes the proof of this theorem. ■

Now, we will show that the Zeno behavior can be excluded (see Johansson, Egerstedt, Lygeros, & Sastry (1999)) by proving following theorem.

Theorem 2 For any initial condition, at any time \( t \geq 0 \), under the condition and the event-triggered principle in Theorem 1, there exists at least one agent \( v_j \), of which the next inter-event time is strictly positive before the system reach a consensus. In addition, \( \lim_{t \to \infty} x_i(t_k^n(t)) = \bar{x}(0) \) holds for all \( i = 1, \ldots, m \).

Proof. Suppose that there is no trigger event when \( t > T \). Then, we have
\[
\dot{x}_i(t) = \sum_{j=1}^{m} L_{ij} x_j(T_{kj}(T)), \quad t > T, \quad i = 1, \ldots, m
\]
which implies
\[
x_i(t) - x_i(T) = (t - T) \sum_{j=1}^{m} L_{ij} x_j(T_{kj}(T)).
\]
(14)

By Theorem 1, we have \( x_i(t) - \bar{x}(0) \to 0 \). Therefore, for all \( i = 1, \ldots, m \), \( \sum_{j=1}^{m} L_{ij} x_j(T_{kj}(T)) = 0 \) holds, which implies \( x_i(t) = x_i(T) \) for all \( t > T \). Therefore, \( x_i(T) = \bar{x}(0) \). It means consensus has reached at time \( T \). This implies that Zeno behavior can be excluded.

In addition, in case of \( \lim_{t \to \infty} t_k^n(t) = \infty \), then we have \( \lim_{t \to \infty} x_i(t_{k^n}(t)) = \lim_{t \to \infty} x_i(t) = \bar{x}(0) \); otherwise, if \( t_k^n(t) \) are bounded for some \( j \), letting \( \bar{t}_j = \sup_{t \geq 0} t_k^n(t) \), we have \( x_j(\bar{t}_j) = \lim_{t \to \infty} x_j(t) = \bar{x}(0) \). This completes the proof.

Theorem 3 Let \( 0 < \alpha < 2\lambda_2 \), \( c = (1 - \frac{a\beta_m}{2\lambda_2}) \frac{\lambda_2}{\rho(L^TL)} \), and \( \delta(t) = \delta_0 \exp(-\delta t) \) where \( \delta_0 > 0 \) and \( 0 < \delta < \frac{1}{2\lambda_2} \). Suppose that \( G \) is strongly connected. Set
\[
t_k^i = \max \left\{ \tau \geq t_k^i : |x_i(t_k^i) - x_j(t)| \leq \delta(t), \quad \forall t \in [t_k^i, \tau] \right\}
\]
(15)

Then, system (3) reaches a consensus exponentially and the inter-event time is strictly positive, namely \( t_k^i + 1 - t_k^{i+1} > 0 \) for all \( i = 1, \ldots, m \) and \( j = 1, 2, \ldots \): In addition, for all \( i \), \( \lim_{t \to \infty} x_i(t) = \sum_{j=1}^{m} \xi_j x_j(0) \).

Proof. By inequalities (9), under the condition (15), we have
\[
\frac{d}{dt} V(t) \leq - (1 - \frac{a\beta_m}{2\lambda_2}) \frac{\lambda_2}{\rho(L^TL)} \frac{2\rho_2(L^TL)}{\mu_m} V(t) + \frac{m}{2\alpha} \delta^2(t)
\]
By the Grönwall inequality, we can conclude
\[
V(t) \leq e^{-ct} V(0) + \frac{m}{2\alpha} \int_0^t \exp(-c(t - s)) \delta^2(s) ds
\]

\[
= e^{-ct} V(0) + \frac{m\delta^2}{2\alpha(c-2\delta)} \left[ \exp(-2\delta t) - \exp(-ct) \right]
\]
\[
\leq k_\delta \delta^2(t)
\]
(16)

where \( k_\delta = \frac{V(0)}{2\delta} + \frac{m}{2\alpha(c-2\delta)} > 0 \) is a constant. This implies that system (3) reaches a consensus exponentially and for all \( i \), one has \( \lim_{t \to \infty} x_i(t) = \sum_{j=1}^{m} \xi_j x_j(0) \).

In addition, noting \( \dot{e}_i(t) = - \sum_{j=1}^{m} L_{ij} x_j(t_k^n(t)) = - \sum_{j=1}^{m} L_{ij} (x_j(t) + e_j(t)) \) and (16), we have
\[
|\dot{e}_i(t)| \leq 2|L_{ii}| \delta(t) + \frac{\rho(L^TL)}{\mu_2} V(t) \leq 2|L_{ii}| \delta(t) + \frac{\rho(L^TL)}{\mu_2} k_\delta \delta(t) = \omega_i \delta(t)
\]
where \( \omega_i = 2|L_{ii}| + \sqrt{\frac{\rho(L^TL)}{\mu_2} k_\delta} \). Thus \( |e_i(t)| \leq \int_{t_k^n(t)}^{t} |\dot{e}_i(s)| ds \leq \omega_i \delta(t_{k^n}(t)) (t - t_k^n(t)) \). When the event is triggered, i.e., the equality of (15) holds at time \( t \), which implies \( e_i(t) = \delta(t) \). Hence, \( \omega_i \delta(t_{k^n}(t)) (t - t_k^n(t)) \geq \delta(t) = \delta(t_k^n(t)) \exp(-\delta_0(t - t_k^n(t))) \). Therefore, \( t_{k+1}^n - t_k^n \geq \exp(-\delta(t_k^n(t) - t_k^n(t))/\omega_i), \) which implies that \( t_{k+1}^n - t_k^n \) is strictly positive. This completes the proof. ■

The results in Theorem 1 and Theorem 3 can be extended to the case of directed and reducible topology without efforts. We assume that the graph of \( L \) has spanning tree. Without
loss of generality, $L$ can be written as the following Perron-Frobenius form:

$$
L = \begin{bmatrix}
L^{1,1} & L^{1,2} & \cdots & L^{1,K} \\
0 & L^{2,2} & \cdots & L^{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L^{K,K}
\end{bmatrix}
$$

(17)

where $L^{k,k}$, with dimension $n_k$, associated with the $k$-th strongly connected component (SCC) of $G$, denoted by $SCC_k$, $k = 1, \ldots, K$, and for each $k = 2, \ldots, K$, there exists some $j > k$, such that $L^{k,j} \neq 0$.

Due to the limit of space, for simplicity, hereby we only consider the case of $K = 2$. The case $K > 2$ can be treated in the same way. For $SCC_2$, $k = 1, 2$, define $x^k(t) = [x^1_1(t), \ldots, x^k_{n_k}(t)]^\top$, $e^k(t) = [e^1_1(t), \ldots, e^k_{n_k}(t)]^\top$, where $e^k_1(t) = x^k_1(t) - x^1_1(t)$, and $q^k(t) = [q^k_1(t), \ldots, q^k_{n_k}(t)]^\top$, where $q^k_i(t) = \sum_{j=1}^{n_k} L^{k,k}_{ij} x^k_j(t) + \sum_{j=1}^{n_k} L^{k,k+1}_{ij} x^{k+1}_j(t)$. Define an auxiliary matrix $L^{k,k} = [L^{k,k}_{ij}]_{i,j=1}^{n_k}$ as

$$
\tilde{L}^{k,k}_{ij} = \begin{cases}
L^{k,k}_{ij} & i \neq j \\
-\sum_{p=1, p \neq i}^{n_k} L^{k,k}_{ip} & i = j
\end{cases}
$$

then each $\tilde{L}^{k,k}$, $k = 2, \ldots, K$, is irreducible or is of dimension one. Lemma 1 implies that we can find $\xi^k \top$ to be the left eigenvector of $\tilde{L}^{k,k}$ corresponding to the eigenvalue zero and has the sum of components equal to 1.

First, applying Theorem 1 to the subsystem corresponding to the $SCC_2$, one can conclude that $x^2_1(t)$, and $x^2_{l_{p+1}+n_1}(t)$, $j = 1, \ldots, n_2$, converge to $\nu(0)$ exponentially, where $\nu(t) = \sum_{p=1}^{n_2} \xi^2_{p}x^2_p(t)$.

Then, for the subsystem of the $SCC_1$, we will prove that $\lim_{t \to \infty} |x^1_j(t) - \nu(t)| = 0$, for all $p = 1, \ldots, n_1$. Define $V_1(t) = \frac{1}{2} \sum_{i=1}^{n_1} \xi^1_{i}(x^1_i(t) - \nu(t))^2$. Then

$$
\frac{d}{dt} V_1(t) = \sum_{i=1}^{n_2} \xi^1_i (x^1_i(t) - \nu(t)) \left\{ \sum_{j=1}^{n_1} L^{1,1}_{i,j} [x^1_j(t) - \nu(t)]
+ \sum_{p=1}^{n_2} L^{1,2}_{i,p} [x^2_p(t_{p+1}+n_1) - \nu(t)]
+ \sum_{j=1}^{n_1} L^{1,1}_{i,j} [x^1_j(t_{p+1}) - x^1_j(t)] \right\}
= Q^1_1(t) + Q^1_2(t) + Q^1_3(t)
$$

where

$$
Q^1_1(t) = (x^1(t) - \nu(t)) \top \Xi^1 L^{1,1} (x^1(t) - \nu(t))
$$

$$
Q^1_2(t) = \sum_{i=1}^{n_1} \xi^1_i (x^1_i(t) - \nu(t)) \sum_{p=1}^{n_2} L^{1,2}_{i,p} [x^2_p(t_{p+1}+n_1) - \nu(t)]
$$

$$
Q^1_3(t) = (x^1(t) - \nu(t)) \top \Xi^1 L^{1,1} (x^1(t))
$$

By Cauchy inequality, for any $\kappa_1 > 0$, we have

$$
Q^1_2(t) \leq \kappa_1 V_1(t) + F_1(t)
$$

(19)

where $F_1(t) = \frac{1}{\xi^1_{i}} \sum_{i=1}^{n_1} \xi^1_i \left\{ \sum_{p=1}^{n_2} L^{1,2}_{i,p} [x^2_p(t_{p+1}+n_1) - \nu(t)] \right\}^2$.

Because $\lim_{t \to \infty} x^2_p(t_{p+1}+n_1) = \nu(0) = \nu(t)$, $p = 1, \ldots, n_2$, exponentially, we have $\lim_{t \to \infty} F_1(t) = 0$ exponentially.

Denote $Q^1 = \frac{1}{2} [\Xi^1 L^{1,1} (\Xi^1 L^{1,1})^\top]$, $\hat{Q}^1 = \Xi^1 L^{1,1} [\Xi^1 L^{1,1}]^\top$, for any $\alpha > 0$, we have

$$
\frac{d}{dt} V_1(t) \leq \frac{\alpha_1}{2} (x^1(t) - \nu(t))^\top \hat{Q}^1 (x^1(t) - \nu(t))
+ \frac{1}{2\alpha_1} [e^1(t)]^\top e^1(t) + F_1(t)
\leq (1 - \frac{\alpha_1 \rho(Q^1)}{2\rho \rho(Q^1)}) Q^1_1(t)
+ \frac{1}{2\alpha_1} [e^1(t)]^\top e^1(t) + \kappa_1 V_1(t) + F_1(t)
$$

(20)

Then we have

**Corollary 1** Suppose that $G$ has spanning tree and $L$ is written in the form of (17) with $K = 2$. For $v_{p+n_1} \in SCC_2$, the event time sequence $\{t_{p+1}^{n_1}\}$ is determined by the rule given in Theorem 1. For $v_p \in SCC_1$ the event triggering time $\{t^p\}$ is given by

$$
t_{p+1}^p = \max \left\{ \tau \geq t^p : |x^1_p(t) - x^1_{p+1}^p(t)| \leq \sqrt{\gamma a_1 b_1 [q^1_p(t)]}, \forall t \in [t^p, \tau] \right\}
$$

(21)

where $\gamma \in (0, 1)$, $0 < a_1 < \frac{2\rho \rho(Q^1)}{\rho(Q^1)}$ and $b_1 = (1 - \frac{\alpha_1 \rho(Q^1)}{2\rho \rho(Q^1)}) \rho(L^{1,1})^{-1} L^{1,1}$. Then, system (3) reaches a consensus: In addition, for all $i$, $\lim_{t \to \infty} x_i(t) = \sum_{p=1}^{n_2} \xi^2_{p}x^2_p(0)$.

**Proof.** We only need to discuss the components $v_p \in$
SCC. From (21) and (20), we have

\[ \frac{d}{dt} V_1(t) \leq (1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) Q_1^1(t) + \kappa_1 V_1(t) + F_1(t) \\
+ \frac{\gamma b_1}{2} \| L^{1,1}(x^1(t) - \nu(t)\mathbf{1}) + L^{1,2}(x^2(t) - \nu(t)\mathbf{1}) \|^2 \\
\leq (1 - \gamma)(1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) Q_1^1(t) + \kappa_1 V_1(t) \\
+ F_1(t) + \gamma b_1 \| L^{1,1} \|^2 \| x^2(t) - \nu(t)\mathbf{1} \|^2 \\
\leq - (1 - \gamma)(1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) \rho(\tilde{Q}^1) V_1(t) + \kappa_1 V_1(t) \\
+ F_1(t) + \gamma b_1 \| L^{1,1} \|^2 \| x^2(t) - \nu(t)\mathbf{1} \|^2 \\
\]

Denoting \( F_2(t) = F_1(t) + \gamma b_1 \| L^{1,1} \|^2 \| x^2(t) - \nu(t)\mathbf{1} \|^2 \) and picking sufficiently small \( \kappa_1 \), there exists some \( \kappa_1 > 0 \) such that \( \frac{d}{dt} V_1(t) \leq -\kappa_1 V_1(t) + F_2(t) \), which implies

\[ V_1(t) \leq \exp(-\kappa t) \left\{ V_1(0) + \int_0^t \exp(-\kappa s) F_2(s) ds \right\} \]

and \( \lim_{t \to \infty} F_2(t) = 0 \) exponentially, since \( \lim_{t \to \infty} F_1(t) = 0 \) and \( \lim_{t \to \infty} \| x^2(t) - \nu(t)\mathbf{1} \| = 0 \) exponentially. This completes the proof for the part SCC.

**Corollary 2** Suppose that \( G \) has spanning tree and \( L \) is written in the form of (17) with \( K = 2 \). For \( v_{i,n+1} \in SCC \), the event time sequence \( \{t_n^{i,n+1}\} \) is determined by the same rule in Theorem 3. For \( v_p \in SCC \) the triggering time \( \{t_p^p\} \) is given by

\[ t_{p+1}^p = \max \left\{ \tau \geq t_p^p : |x_p^1(\tau) - x_p^1(t_p^p)| \leq \delta(\tau), \forall t \in [t_p^p, \tau] \right\} \]  \( (22) \)

where \( 0 < a_1 < \frac{2 \rho_2(-\tilde{Q}^1)}{\rho(\tilde{Q}^1)} \), \( c_1 = (1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) \rho_2(-\tilde{Q}^1) \), and \( \delta(\tau) = \delta_0 \exp(-\delta_1 \tau) \) where \( \delta_0 > 0 \) and \( 0 < \delta_1 < \frac{c_1}{\rho^2} \).

Then, system (3) reaches a consensus; In addition, for all \( i \),

\[ \lim_{t \to \infty} x_i(t) = \sum_{k=1}^{n_2} \xi^2 \xi^2(0) \]

**Remark 1** It can be seen that the case of \( K > 2 \) can treated and corresponding event-triggering principles can be formulated iteratively by the same fashion.

### 4 Distributed self-triggered principles

Continuous monitoring of the system states are required according to the principles in Theorems 1 and 3, as well as Corollaries 1 and 2. To avoid this sort of costly monitoring, in this section, we consider an alternative principle of predicting the timing when inequalities (11) and (15) do not hold and update the event timing accordingly. This is named the self-triggered principle.

\begin{align*}
\frac{d}{dt} V_1(t) &\leq (1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) Q_1^1(t) + \kappa_1 V_1(t) + F_1(t) \\
&+ \frac{\gamma b_1}{2} \| L^{1,1}(x^1(t) - \nu(t)\mathbf{1}) + L^{1,2}(x^2(t) - \nu(t)\mathbf{1}) \|^2 \\
&\leq (1 - \gamma)(1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) Q_1^1(t) + \kappa_1 V_1(t) \\
&+ F_1(t) + \gamma b_1 \| L^{1,1} \|^2 \| x^2(t) - \nu(t)\mathbf{1} \|^2 \\
&\leq - (1 - \gamma)(1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) \rho(\tilde{Q}^1) V_1(t) + \kappa_1 V_1(t) \\
&+ F_1(t) + \gamma b_1 \| L^{1,1} \|^2 \| x^2(t) - \nu(t)\mathbf{1} \|^2 \\
\end{align*}

For any \( p = 1, \ldots, m \), \( x_p(t) \) can be rewritten as

\[ x_p(t) = x_p(t_{k_p}(t)) + (t - t_{k_p}(t)) \sum_{j=1}^{m} L_{pj} x_j(t_{k_p}(t)) \]  \( (23) \)

for all \( t > t_{k_p}^*(t) \) and less than the next even triggering times of its in-neighbors, where \( t_{k_p}^*(t) = \max_{e(p,j) \in E} t_{k_p}^*(t) \) is the newest timing of the events of all its in-neighbours agents. For any \( i \) with \( e(p,i) \in E^m \), we have

\[ x_i(t) = x_i(t_{k_p}(t)) + (t - t_{k_p}(t)) \sum_{j=1}^{m} L_{ij} x_j(t_{k_p}(t)) \]  \( (24) \)

For agent \( v_p \), Theorems 1 and 3 imply that solving the following maximization problems

\[ \tau_{p+1}^p = \max \left\{ \tau > t_{k_p}(t) : |x_p(\tau) - x_p(t_{k_p}(\tau))| \leq \sqrt{\delta q_p(\tau)} \right\} \]

or

\[ \tau_{p+1}^p = \max \left\{ \tau > t_{k_p}(t) : |x_p(\tau) - x_p(t_{k_p}(\tau))| \leq \delta(\tau) \right\} \]  \( (26) \)

can predict triggering event time. In addition, when the agent \( v_i \) updates its observation time of the triggering event, the triggering event time predictions of \( v_i \)'s out-neighbours will be affected. Therefore, besides the event prediction rule given before, each agent should take their triggering event time whenever any of its out-neighbours renews its event timing. In other words, when one agent updates its event timing, it is mandatory to inform all its out-neighbours.

To sum up, we have the following result.

**Theorem 4** Suppose that \( G \) is strongly connected. Using the following event triggered strategy:

1. For each agent \( v_p \), initialize at \( t_0^p = 0; \)
2. Pick \( \gamma \in (0, 1) \), \( 0 < \alpha < 2\frac{\delta}{2\delta} \), \( c = (1 - \frac{\alpha_1 \rho(\tilde{Q}^1)}{2 \rho_2(-\tilde{Q}^1)}) \rho_2(-\tilde{Q}^1) \), and \( \delta_0 > 0 \) and \( 0 < \delta_1 < \frac{c_1}{\rho^2} \) at time \( t \). For any agent \( v_p \), let \( t_{k_p}^* = t_{k_p}^*(t) \), search \( \tau_{p+1}^p \) by the rule (25) or (26);
3. In case that no triggering events occur in all \( v_p \)'s in-neighbors during \( (t_{k_p}^*(t) + \tau_{p+1}^p) \), i.e., the agent \( v_p \) does not receive any renewed information form its in-neighbors during \( (t_{k_p}^*(t) + \tau_{p+1}^p) \), then \( v_p \) triggers at time \( t_{k_p}^* = t_{k_p}^*(t) + \tau_{p+1}^p \). The agent \( v_p \) renews its state at \( t = t_{k_p}^* \) and sends the renewed information to all its out-neighbours simultaneously;
(4) In case that some in-neighbors of agent $v_p$ triggers at time $t \in (t_p^k(t), t_{k+1}^p)$, then updating $t_{k+1}^p(t)$ in (23) and go to step (2);

Then, system (3) reaches a consensus exponentially. In addition, \( \lim_{t \to \infty} x_i(t) = \sum_{j=1}^m \xi_j x_j(0) \) for all $i$.

**Proof.** Following steps 1-4, under the maximizing (25) or (26), by the same arguments as in the proof of Theorem 1 or Theorem 3, this theorem can be proved. \( \blacksquare \)

**Remark 2** Theorem 4 can be extended to the case that the directed graph $G$ has spanning trees as we did above without much effort. Due to the space limit, we neglect the details.

5 Examples

In this section, a numerical example is given to demonstrate the presented results. Consider a network of four agents, of which the Laplacian matrix is

$$L = \begin{bmatrix}
-7 & 3 & 0 & 4 \\
1 & -3 & 0 & 2 \\
0 & 2 & -7 & 5 \\
0 & 0 & 4 & -4
\end{bmatrix},$$

which is a directed strongly connected weighted network described by Figure 1. The initial value of each agent is randomly selected within the interval \([-5, 5]\) in the simulation. We select the fourth agent for illustration. Figure 2 shows dynamics of the fourth agent under the triggered principle provided in Theorem 4 using (25) with $\gamma = 0.9$, $\alpha = \lambda_2 / \beta_m = 0.0226$, with initial value $[2.5320, 4.7160, -4.1310, 1.2830]^T$, in comparison to continuous feedback control, i.e. $\dot{x}(t) = Lx(t)$. Figure 3 shows the dynamics of the fourth agent under the triggered principle provided in Theorem 4 using (26) with $\alpha = \lambda_2 / \beta_m = 0.0226$, $c = \lambda_2 / 2 \mu_m = 0.4305$, $\delta_0 = 10$, $\delta_1 = 0.1$, and initial value $[2.5320, 4.7160, -4.1310, 1.2830]^T$. The agreement value is $\bar{x}(0) = 0.4814$. It can be seen that under all event-triggering principles, agents reach consensus as the same value as that of the continuous feedback control.

6 Conclusion

In conclusion, we presented event-triggered and self-triggered principles in distributed formulations for multi-agent systems with directed and reducible topologies. First, we derived event-triggered principle in the case of directed strongly connected graph. The triggering time of each agent are determined by inequality (11) only depending on the states of each agent’s in-neighbours or by inequality (15) only depending on the states of itself. Both event-triggered principles are distributed. It was shown that by these principles, consensus is reached exponentially, and Zeno behavior is excluded. Then, we extended these results to the cases of directed and reducible topology with spanning tree. Second, we proposed self-triggered principles, which resulted in an easily-computable law for predicting the next triggering time without continuous monitoring the system states. The effectiveness the theoretical results were verified by
numerical examples.

Acknowledgements

This work is jointly supported by the Marie Curie International Incoming Fellowship from the European Commission (FP7-PEOPLE-2011-IIF-302421), the National Natural Sciences Foundation of China (Nos. 61273211 and 61273309), and the Program for New Century Excellent Talents in University (NCET-13-0139).

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