THE HOMOTOPY THEORY OF COHERENTLY COMMUTATIVE MONOIDAL QUASI-CATEGORIES

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Abstract. The main objective of this paper is to construct a symmetric monoidal closed model category of coherently commutative monoidal quasi-categories.

Contents

1. Introduction 2
2. The Setup 2
2.1. Review of Γ-spaces 4
3. The strict JQ-model category structure on Γ-spaces 6
3.1. Enrichment of the JQ-model category 6
4. The JQ-model category 12
5. Equivalence with normalized Γ-spaces 14
Appendix A. Quillen Bifunctors 26
Appendix B. On local objects in a model category enriched over quasicategories 27
B.1. Introduction 27
B.2. Preliminaries 27
B.3. Function spaces for quasi-categories 29
B.4. Local objects 30
Appendix C. The strict JQ-model category of normalized Γ-spaces 32
C.1. The JQ-model category of normalized Γ-spaces 36
References 39

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1. Introduction

A symmetric monoidal category is a category equipped with a multiplicative structure which is associative, unital and commutative only up to natural (coherence) isomorphisms. A quasi-category is a simplicial set which satisfies the weak Kan condition namely every inner horn has a filler. In this paper we study quasi-categories which are equipped with a coherently commutative multiplicative structure and thereby generalize the notion of symmetric monoidal categories to higher categories. Such quasi-categories most commonly arise as (simplicial) nerves of simplicial model categories which are equipped a compatible symmetric monoidal structure see [NS17]. These quasi-categories played a prominent role in Jacob Lurie’s work on the cobordism hypothesis. Every stable quasi-category [Lurh] is equipped with a multiplicative structure which is coherently commutative. The coherence theorem for symmetric monoidal categories states that the category of (small) symmetric monoidal categories is equivalent to the category of algebras over the categorical Barrat-Eccles operad in \textbf{Cat}. We recall that the categorical Barrat-Eccles operad an $\mathcal{E}_\infty$-operad in \textbf{Cat}. In a subsequent paper we will prove a similar theorem for the quasi-categories equipped with a coherently commutative multiplicative structure. There are several different models present in the literature which were developed to encode a coherently commutative multiplicative structure on simplicial sets. The most commonly used model is based on operads. An $E_\infty$-simplicial set is a simplicial set equipped with a coherently commutative multiplicative structure which is encoded by an action of an $E_\infty$-operad. In other words an $E_\infty$-simplicial set is an algebra over an $E_\infty$-operad in the category of simplicial sets $\textbf{sSets}$. There are two model category structures on the category $\textbf{sSets}$ namely the standard or the Kan model category structure and the Joyal model category structure which is also referred to as the model category structure of quasi-categories. In this paper we will only be working with the later model category structure. The category of $E_\infty$-simplicial sets inherits a model category structure from the Joyal model category structure, see [BM03]. A fibrant object in this model category can be described as a quasi-category equipped with a coherently commutative multiplicative structure which is encoded by an action of an $E_\infty$-operad. However this model category is NOT symmetric monoidal closed. The main objective of this paper is to overcome this shortcoming by presenting a new model for coherently commutative monoidal quasi-categories based on $\Gamma$-spaces. Another model to encode a coherently commutative multiplicative structure on simplicial sets was presented by Jacob Lurie in his book [Lura] which he called symmetric monoidal quasi-categories. He modelled these objects as co-cartesian fibrations over a quasi-category which is the nerve of a skeletal category of based finite sets $\Gamma^{op}$ whose objects are $n^+ = \{0, 1, 2, \ldots, n\}$. In this paper we take a dual perspective namely we model these as functors from $\Gamma^{op}$ into ($\textbf{sSets}, \mathbb{Q}$). However Lurie does not construct a model category structure on his symmetric monoidal quasi-categories. Yet another model to encode a coherently commutative multiplicative structure on simplicial sets was presented by Kodjabachev and Sagave in the paper [KS15]. The authors present a rigidification of an $E_\infty$-quasi-category by replacing it with a commutative monoid in a symmetric monoidal functor category. They go on to construct a zig-zag of Quillen equivalences between a suitably defined model category structure on the category of commutative monoids mentioned
above and a model category of $E_\infty$-simplicial sets. However they were unable to show the existence of a symmetric monoidal closed model category structure.

A Γ-space is a functor from the category $\Gamma^{op}$ into the category of simplicial sets $sSets$. The category $\Gamma S$ of Γ-spaces is the category of functors and natural transformations $[\Gamma^{op}, sSets]$. A normalized Γ-space is a functor $X : \Gamma^{op} \rightarrow sSets_\bullet$ such that $X(0^+) = \ast$. The category of normalized Γ-spaces $\Gamma S_\bullet$ is the full subcategory of the functor category $[\Gamma^{op}; sSets_\bullet]$ whose objects are normalized Γ-spaces. In that paper [Seg74] Segal introduces a notion of normalized Γ-spaces and showed that they give rise to a homotopy category which is equivalent to the homotopy category of connective spectra. Segal’s Γ-spaces were renamed special Γ-spaces by Bousfield and Friedlander in [BF78] who constructed a model category structure on the category of all normalized Γ-spaces $\Gamma S_\bullet$. The two authors go on to prove that the homotopy category obtained by inverting stable weak equivalences in $\Gamma S_\bullet$ is equivalent to the homotopy category of connective spectra. In the paper [Sch99] Schwede constructed a symmetric monoidal closed model category structure on the category of normalized Γ-spaces which he called the stable Q-model category. The fibrant objects in this model category can be described as coherently commutative group objects in the category of (pointed) simplicial sets $sSets_\bullet$, where the latter category is endowed with the Kan model category structure. The objective of Schwede’s construction was to establish normalized Γ-spaces as a model for connective spectra. In this paper we extend the ideas in [Sch99] to study coherently commutative monoidal objects in the model category of quasi-categories and thereby generalizing the theory of symmetric monoidal categories. We construct a new symmetric monoidal closed model category structure on the category of $\Gamma$-spaces $\Gamma S$. Our model category is constructed along the lines of Schwede’s construction and we call it the JQ-model category structure. The fibrant objects in our model category structure can be described as coherently commutative quasi-categories. We will show that the JQ-model category is symmetric monoidal closed under the Day convolution product.

The category of (pointed) simplicial sets $sSets_\bullet$ inherits a model category structure from the Joyal model category. This model category is symmetric monoidal closed under the smash product of (pointed) simplicial sets, see [JT08]. We construct a new model category structure on the category of normalized Γ-spaces $\Gamma S\bullet$. The fibrant object of this model category can be described as strictly unital coherently commutative quasi-categories. We will refer to this model category as the normalized JQ-model category. In the paper [Lyd99] Lydakis constructed a smash product of Γ-spaces and showed that it endows $\Gamma S\bullet$ with a closed symmetric monoidal structure. We will show that the normalized JQ-model category structure is compatible with the smash product of Γ-spaces i.e. the normalized JQ model category is symmetric monoidal closed under the smash product. Another significant result of this paper is that the obvious forgetful functor $U : \Gamma S\bullet \rightarrow \Gamma S$ is the right Quillen functor of a Quillen equivalence between the JQ-model category and the normalized JQ-model category. This result is indicative of the presence of a weak semi-additive structure in the JQ-model category.

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2. The Setup

In this section we will collect the machinery needed for various constructions in this paper.

2.1. Review of \(\Gamma\)-spaces. In this subsection we will briefly review the theory of \(\Gamma\)-spaces. We begin by introducing some notations which will be used throughout the paper.

**Notation 2.1.** We will denote by \(n\) the finite set \(\{1, 2, \ldots, n\}\) and by \(n^+\) the based set \(\{0, 1, 2, \ldots, n\}\) whose basepoint is the element 0.

**Notation 2.2.** We will denote by \(\mathcal{N}\) the skeletal category of finite unbased sets whose objects are \(n\) for all \(n \geq 0\) and maps are functions of unbased sets. The category \(\mathcal{N}\) is a (strict) symmetric monoidal category whose symmetric monoidal structure will be denoted by +. For to objects \(k, l \in \mathcal{N}\) their tensor product is defined as follows:

\[ k + l := k + l. \]

**Notation 2.3.** We will denote by \(\Gamma^{op}\) the skeletal category of finite based sets whose objects are \(n^+\) for all \(n \geq 0\) and maps are functions of based sets.

**Notation 2.4.** We denote by \(\text{Inrt}\) the subcategory of \(\Gamma^{op}\) having the same set of objects as \(\Gamma^{op}\) and inert morphisms.

**Notation 2.5.** We denote by \(\text{Act}\) the subcategory of \(\Gamma^{op}\) having the same set of objects as \(\Gamma^{op}\) and active morphisms.

**Notation 2.6.** A map \(f : n \rightarrow m\) in the category \(\mathcal{N}\) uniquely determines an active map in \(\Gamma^{op}\) which we will denote by \(f^+ : n^+ \rightarrow m^+\). This map agrees with \(f\) on non-zero elements of \(n^+\).

**Notation 2.7.** Given a morphism \(f : n \rightarrow m\) in \(\Gamma^{op}\), we denote by \(\text{Supp}(f)\) the largest subset of \(\mathcal{N}\) whose image under \(f\) does not contain the basepoint of \(m^+\). The set \(\text{Supp}(f)\) inherits an order from \(\mathcal{N}\) and therefore could be regarded as an object of \(\mathcal{N}\). We denote by \(\text{Supp}(f)^+\) the based set \(\text{Supp}(f) \cup \{0\}\) regarded as an object of \(\Gamma^{op}\) with order inherited from \(\mathcal{N}\).

**Proposition 2.8.** Each morphism in \(\Gamma^{op}\) can be uniquely factored into a composite of an inert map followed by an active map in \(\Gamma^{op}\).

**Proof.** Any map \(f : n^+ \rightarrow m^+\) in the category \(\Gamma^{op}\) can be factored as follows:

\[
\begin{array}{ccc}
  n^+ & \xrightarrow{\text{Supp}(f)^+} & m^+ \\
  \downarrow{f_{\text{inrt}}} & & \downarrow{f_{\text{act}}} \\
  \text{Supp}(f)^+ & & \\
\end{array}
\]

where \(\text{Supp}(f) \subseteq n\) is the support of the function \(f\) i.e. \(\text{Supp}(f)\) is the largest subset of \(n\) whose elements are mapped by \(f\) to a non-zero element of \(m^+\). The map \(f_{\text{inrt}}\) is the projection of \(n^+\) onto the support of \(f\) and therefore \(f_{\text{inrt}}\) is an inert map. The map \(f_{\text{act}}\) is the restriction of \(f\) to \(\text{Supp}(f) \subseteq \mathcal{N}\), therefore it is an active map in \(\Gamma^{op}\). \(\square\)
Definition 2.9. A $\Gamma$-space is a functor from $\Gamma^{\text{op}}$ into the category of simplicial sets $\text{sSets}$.

Definition 2.10. A \textit{normalized} $\Gamma$-space is a $\Gamma$-space which satisfies the normalization condition namely $X(0^+) \cong *$. 
3. THE STRICT JQ-MODEL CATEGORY STRUCTURE ON Γ-SPACES

Schwede introduced two model category structures on the category of (normalized) Γ-spaces which he called the strict Q-model category structure and the stable Q-model category structure in [Sch99]. The strict Q-model category structure is obtained by restricting the projective model category structure on the functor category \([\Gamma^{op}, \mathbf{sSets}]\), where the codomain category \(\mathbf{sSets}\) is endowed with the Kan model category structure. In this section we study the projective model category structure on the category of Γ-spaces namely the functor category \([\Gamma^{op}, \mathbf{sSets}]\).

Following Schwede we will refer to this projective model category as the strict JQ-model category. We will show that the strict JQ-model category is a \(\mathbf{sSets}\)-model category, where \(\mathbf{sSets}\) is endowed with the Joyal model category structure. We go on further to show that the strict JQ model category is a symmetric monoidal closed model category. We begin by recalling the notion of a categorical equivalence of simplicial sets which is essential for defining weak equivalences of the desired model category structure.

**Definition 3.1.** A morphism of simplicial sets \(f : A \to B\) is called a categorical equivalence if for any quasi-category \(X\), the induced morphism on the homotopy categories of mapping spaces

\[
\text{ho}(\text{Map}_{\mathbf{sSets}}(f, X)) : \text{ho}(\text{Map}_{\mathbf{sSets}}(B, X)) \to \text{ho}(\text{Map}_{\mathbf{sSets}}(A, X)),
\]

is an equivalence of (ordinary) categories.

**Remark.** Categorical equivalences are weak equivalences in a cofibrantly generated model category structure on simplicial sets called the Joyal model category structure which we will denote by \((\mathbf{sSets}, Q)\), see [Joy08, Theorem 6.12] for the definition of the Joyal model category structure.

**Definition 3.2.** We call a map of Γ-spaces

1. A strict JQ-fibration if it is degreewise a pseudo-fibration i.e. a fibration of simplicial sets in the Joyal model category structure on simplicial sets.
2. A strict JQ-equivalence if it is degreewise a categorical equivalence i.e. a weak equivalence of simplicial sets in the Joyal model category structure on simplicial sets, see [Lur09].

**Theorem 3.3.** Strict JQ-equivalences, strict JQ-fibrations and JQ-cofibrations provide the category of Γ-spaces with a combinatorial, left-proper model category structure on the category of Γ-spaces \(\Gamma S\).

The model structure in the above theorem follows from [Lur09 Proposition A 3.3.2] and the left properness is a consequence of the left properness of the Joyal model category.

3.1. **Enrichment of the JQ-model category.** The goal of this section is to show that the JQ-model category is a (symmetric) monoidal model category which is enriched over itself in the sense of definition [C,J]. We will prove this in two steps, we first establish the existence of a Quillen bifunctor

\[- \times - : \Gamma S \times \mathbf{sSets} \to \Gamma S,
\]

where the category \(\Gamma S\) is endowed with the JQ-model category structure and \(\mathbf{sSets}\) is endowed with the Joyal model category structure \((\mathbf{sSets}, Q)\). Then we will use
this Quillen bifunctor to prove the desired enrichment. We begin by reviewing the notion of a monoidal model category.

**Definition 3.4.** A monoidal model category is a closed monoidal category $C$ with a model category structure, such that $C$ satisfies the following conditions:

1. The monoidal structure $\otimes: C \times C \to C$ is a Quillen bifunctor.
2. Let $QS \xrightarrow{q} S$ be the cofibrant replacement for the unit object $S$, obtained by using the functorial factorization system to factorize $0 \to S$ into a cofibration followed by a trivial fibration. Then the natural map
   $$QS \otimes X \xrightarrow{q\otimes 1} S \otimes X$$
   is a weak equivalence for all cofibrant $X$. Similarly, the natural map $X \otimes QS \xrightarrow{1\otimes q} X \otimes S$ is a weak equivalence for all cofibrant $X$.

**Example 3.5.** The model category of simplicial sets with the Joyal model category structure, $(sSets, Q)$ is a monoidal model category.

**Example 3.6.** The stable $Q$-model category is a monoidal model category with respect to the smash product defined in [Lyd99].

**Definition 3.7.** Let $S$ be a monoidal model category. An $S$-enriched model category is an $S$-enriched category $A$ equipped with a model category structure (on its underlying category) such that

1. The category $A$ is tensored and cotensored over $S$.
2. There is a Quillen adjunction of two variables, (see definition C.7),
   $$(\otimes,\text{hom}_A,\text{Map}_A,\phi,\psi): A \times S \to A.$$ When $A$ is itself a monoidal model category which is also an $A$-enriched model category, we will say that $A$ is enriched over itself as a model category.

**Example 3.8.** Both strict and stable $Q$-model category structures, constructed in [Sch99], on the category $\Gamma S$ are simplicial, i.e. both strict and stable $Q$-model categories are $(sSets, Kan)$-enriched model categories.

**Remark.** The strict $JQ$-model category structure is NOT simplicial.

For each pair $(F, K)$, where $F \in \text{Ob}(\Gamma S)$ and $K \in \text{Ob}(sSets)$, one can construct a $\Gamma$-space which we denote by $F \otimes K$ and which is defined as follows:

$$(F \otimes K)(n^+) := F(n^+) \times K,$$

where the product on the right is taken in that category of simplicial sets. This construction is functorial in both variables. Thus we have a functor

$$- \otimes -: \Gamma S \times sSets \to \Gamma S.$$ Now we will define a couple of function objects for the category $\Gamma S$. The first function object enriches the category $\Gamma S$ over $sSets$ i.e. there is a bifunctor

$$\text{Map}_{\Gamma S}(-, -): \Gamma S^{op} \times \Gamma S \to sSets$$

which assigns to each pair of objects $(X, Y) \in \text{Ob}(\Gamma S) \times \text{Ob}(\Gamma S)$, a simplicial set $\text{Map}_{\Gamma S}(X, Y)$ which is defined in degree zero as follows:

$$\text{Map}_{\Gamma S}(X, Y)_0 := \Gamma S(X, Y)$$
and the simplicial set is defined in degree \( n \) as follows:

\[
\mathcal{M}ap_{\Gamma S}(X, Y)_n := \Gamma S(X \otimes \Delta[n], Y)
\]

For any \( \Gamma \)-space \( X \), the functor \( X \otimes - : \mathbf{sSets} \to \Gamma S \) is left adjoint to the functor \( \mathcal{M}ap_{\Gamma S}(X, -) : \Gamma S \to \mathbf{sSets} \). The counit of this adjunction is the evaluation map \( ev : X \otimes \mathcal{M}ap_{\Gamma S}(X, Y) \to Y \) and the unit is the obvious simplicial map \( K \to \mathcal{M}ap_{\Gamma S}(X, X \otimes K) \).

To each pair of objects \((K, X) \in \text{Ob}(\mathbf{sSets}) \times \text{Ob}(\Gamma S)\) we can define a \( \Gamma \)-space \( X^K \), in degree \( n \), as follows:

\[
(X^K)(n^+) := [K, X(n^+)]
\]

This assignment is functorial in both variable and therefore we have a bifunctor

\[
- \otimes - : \mathbf{sSets} \times \Gamma S \to \Gamma S.
\]

For any \( \Gamma \)-space \( X \), the functor \( X^- : \mathbf{sSets} \to \Gamma S^{op} \) is left adjoint to the functor \( \mathcal{M}ap_{\Gamma S}(-, X) : \Gamma S^{op} \to \mathbf{sSets} \). The following proposition summarizes the above discussion.

**Proposition 3.9.** There is an adjunction of two variables

(2) \((- \otimes -, -, \mathcal{M}ap_{\Gamma S}(-, -)) : \Gamma S \times \mathbf{sSets} \to \Gamma S\).

**Theorem 3.10.** The strict model category of \( \Gamma \)-spaces, \( \Gamma S \), is a \((\mathbf{sSets}, Q)\)-model category.

**Proof.** We will show that the adjunction of two variables (2) is a Quillen adjunction for the strict \( JQ \)-model category structure on \( \Gamma S \) and the model category \((\mathbf{sSets}, Q)\). In order to do so, we will verify condition (2) of Lemma C.8. Let \( g : K \to L \) be a cofibration in \( \mathbf{sSets} \) and let \( p : Y \to Z \) be a strict fibration of \( \Gamma \)-spaces, we have to show that the induced map

\[
\text{hom}_{\Gamma S}(g, p) : Y^L \to Z^L \times Z_K^Y
\]

is a fibration in \( \Gamma S \) which is acyclic if either of \( g \) or \( p \) is acyclic. It would be sufficient to check that the above morphism is degreewise a fibration in \((\mathbf{sSets}, Q)\), i.e. for all \( n^+ \in \Gamma^{op} \), the morphism

\[
\text{hom}_{\Gamma S}(g, p)(n^+) = \text{hom}_{\mathbf{sSets}}(g, p(n^+)) : Y(n^+)^L \to Z(n^+)^L \times Z_{(n^+)}^Y(n^+)^K
\]

is a fibration in \((\mathbf{sSets}, Q)\). This follows from the observations that the simplicial morphism \( p(n^+) : Y(n^+) \to Z(n^+) \) is a fibration in \((\mathbf{sSets}, Q)\) and the model category \((\mathbf{sSets}, Q)\) is a cartesian closed model category whose internal Hom is provided by the bifunctor \(- \otimes - : \mathbf{sSets} \times \mathbf{sSets} \to \mathbf{sSets}\). \( \square \)

Let \( X \) and \( Y \) be two \( \Gamma \)-spaces, the Day convolution product of \( X \) and \( Y \) denoted by \( X \ast Y \) is defined as follows:

(3) \( X \ast Y(n^+) := \int_{(k^+, l^+) \in \Gamma^{op}} \Gamma^{op}(k^+ \wedge l^+, n^+) \times X(k^+) \times Y(l^+) \).
Equivalently, one may define the Day convolution product of $X$ and $Y$ as the left Kan extension of their external tensor product $X \boxtimes Y$ along the smash product functor $- \land - : \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$.

we recall that the external tensor product $X \times Y$ is a bifunctor $X \times Y : \Gamma^{op} \times \Gamma^{op} \to \mathbf{sSets}$ which is defined on objects by $X \times Y(m^+, n^+) = X(m^+) \times Y(n^+)$. It follows from [1, Thm.] that the functor $- \ast \Gamma^n$ has a right adjoint which we denote by $- \ast (n^+ \land -) : \Gamma S \to \Gamma S$. We will denote the $\Gamma$-space $-(n^+ \land -)(X)$ by $X(n^+ \land -)$ and define it by the following composite:

\[ \Gamma^{op} \to \Gamma^{op} \xrightarrow{n^+ \land -} X \to \mathbf{sSets}. \]

The following proposition sums up this observation:

**Proposition 3.11.** There is a natural isomorphism

\[ \phi : -(n^+ \land -) \cong \mathbf{Map}_{\Gamma S}(\Gamma^n, -). \]

In particular, for each $\Gamma$-space $X$ there is an isomorphism of $\Gamma$-spaces

\[ \phi(X) : X(n^+ \land -) \cong \mathbf{Map}_{\Gamma S}(\Gamma^n, X). \]

**Proof.** Consider the functor $n^+ \land - : \Gamma^{op} \to \Gamma^{op}$. We observe that a Left Kan extension of $\Gamma^1 : \Gamma^{op} \to \mathbf{sSets}$ along $n^+ \land -$ is the $\Gamma$-space $\Gamma^n : \Gamma^{op} \to \mathbf{sSets}$. This implies that we have the following bijection

\[ \Gamma S(\Gamma^n, X) \cong \Gamma S(\Gamma^1, X(n^+ \land -)). \]

We observe that this natural bijection extends to a natural isomorphism of $\Gamma$-spaces:

\[ \mathbf{Map}_{\Gamma S}(\Gamma^n, X) \cong \mathbf{Map}_{\Gamma S}(\Gamma^1, X(n^+ \land -)). \]

\[ \Box \]

**Proposition 3.12.** The category of all $\Gamma$-spaces $\Gamma S$ is a symmetric monoidal category under the Day convolution product. The unit of the symmetric monoidal structure is the representable $\Gamma$-space $\Gamma^1$.

Next we define an internal function object of the category $\Gamma S$ which we will denote by

\[ \mathbf{Map}_{\Gamma S}(-, -) : \Gamma S^{op} \times \Gamma S \to \Gamma S. \]

Let $X$ and $Y$ be two $\Gamma$-spaces, we define the $\Gamma S \mathbf{Map}_{\Gamma S}(X, Y)$ as follows:

\[ \mathbf{Map}_{\Gamma S}(X, Y)(n^+) := \mathbf{Map}_{\Gamma S}(X \ast \Gamma^n, Y). \]

**Proposition 3.13.** The category $\Gamma S$ is a closed symmetric monoidal category under the Day convolution product. The internal Hom is given by the bifunctor $\mathbf{Map}_{\Gamma S}$ defined above.

The above proposition implies that for each $n \in \mathbb{N}$ the functor $- \ast \Gamma^n : \Gamma S \to \Gamma S$ has a right adjoint $\mathbf{Map}_{\Gamma S}(\Gamma^n, -) : \Gamma S \to \Gamma S$.

The next theorem shows that the strict model category $\Gamma S$ is compatible with the Day convolution product.
Theorem 3.14. The strict JQ-model category $\Gamma S$ is a symmetric monoidal closed model category under the Day convolution product.

Proof. Using the adjointness which follows from proposition 3.13 one can show that if a map $f : U \to V$ is a (acyclic) cofibration in the strict JQ-model category $\Gamma S$ then the induced map $f * \Gamma^n : U * \Gamma^n \to V * \Gamma^n$ is also a (acyclic) cofibration in the strict JQ-model category for all $n \in \mathbb{N}$. By (3) of Lemma C.8 it is sufficient to show that whenever $f$ is a cofibration and $p : Y \to Z$ is a fibration then the map

$$\text{Map}_{\Gamma S}(f, p) : \text{Map}_{\Gamma S}(V, Y) \to \text{Map}_{\Gamma S}(V, Z) \times_{\text{Map}_{\Gamma S}(U, Z)} \text{Map}_{\Gamma S}(U, Y).$$

is a fibration in $\Gamma S$ which is acyclic if either $f$ or $p$ is a weak equivalence. The above map is a (acyclic) fibration if and only if the simplicial map

$$\text{Map}_{\Gamma S}(f * \Gamma^n, p)(n^+) : \text{Map}_{\Gamma S}(V * \Gamma^n, Y) \to \text{Map}_{\Gamma S}(V * \Gamma^n, Z) \times_{\text{Map}_{\Gamma S}(U * \Gamma^n, Z)} \text{Map}_{\Gamma S}(U * \Gamma^n, Y)$$

is a (acyclic) fibration in $(\text{sSets}, \mathbf{Q})$ for all $n \in \mathbb{N}$. Since $f * \Gamma^n$ is a cofibration (which is acyclic whenever $f$ is acyclic as observed above) therefore it follows from theorem 3.10 that the simplicial map $\text{Map}_{\Gamma S}(f * \Gamma^n, p)(n^+)$ is an (acyclic) fibration of simplicial sets for all $n \in \mathbb{N}$. □

The following corollary is an easy consequence of the above theorem and we leave the proof as an exercise for the interested reader.

Corollary 3.15. Let $F'$ be a $Q$-cofibrant $\Gamma$-space and $p : F \to G$ be a strict JQ-fibration. Then the morphism induced by $p$ on the function objects

$$\text{Map}_{\Gamma S}(F', p) : \text{Map}_{\Gamma S}(F', F) \to \text{Map}_{\Gamma S}(F', G)$$

is a strict JQ-fibration.

Definition 3.16. A morphism in $\Gamma S$ is called a trivial fibration of $\Gamma$-spaces if it has the right lifting property with respect to all maps in the following class of maps

$$\{ \Gamma^n \times f : f \text{ is a simplicial monomorphism and } n \geq 0 \}$$

Proposition 3.17. A trivial fibration is a strict JQ equivalence.

Proof. Let $p : X \to Y$ be a trivial fibration of $\Gamma$-spaces and $f : A \to B$ be a simplicial monomorphism then whenever the outer diagram commutes in the following diagram:

$$\begin{array}{c}
\Gamma^n \times A \\
\downarrow \Gamma^n \times f \\
\Gamma^n \times B \\
\end{array} \xrightarrow{p} X \xleftarrow{X}$$

there exists a dotted arrow which makes the whole diagram commutative, for each $n \geq 0$. By adjointness, we get the following commutative diagram in the category of simplicial sets:

$$\begin{array}{c}
A \xrightarrow{f} \text{Map}_{\Gamma S}(\Gamma^n, X) \\
\downarrow \text{Map}_{\Gamma S}(\Gamma^n, X) \\
B \xrightarrow{g} \text{Map}_{\Gamma S}(\Gamma^n, Y)
\end{array}$$
We observe that the map
\[ \text{Map}_{\Gamma S}(\Gamma^n, X) : \text{Map}_{\Gamma S}(\Gamma^n, X) \to \text{Map}_{\Gamma S}(\Gamma^n, Y) \]
is the same as the simplicial map \( p(n^+) : X(n^+) \to Y(n^+) \) up to isomorphism namely we have the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Map}_{\Gamma S}(\Gamma^n, X)} & X(n^+) \\
\downarrow f & \text{isomorphism} & \downarrow \text{Map}_{\Gamma S}(\Gamma^n, p) \\
B & \xrightarrow{\sim} & Y(n^+) \\
\end{array}
\]

This observation and the above simplicial commutative diagram together imply that for each \( n \geq 0 \), the simplicial map \( p(n^+) \) has the right lifting property with respect to all simplicial monomorphisms, in other words \( p(n^+) \) is a trivial fibration of simplicial sets. By [Joy08, Prop. 1.22], this implies that the simplicial map \( p(n^+) \) being a weak equivalence in the Joyal model category of simplicial sets. Thus we have shown that \( p \) is a strict \( JQ \)-equivalence of \( \Gamma \)-spaces.

\[ \square \]

**Proposition 3.18.** A strict \( JQ \)-fibration is a trivial fibration of \( \Gamma \)-spaces if and only if it is a strict \( JQ \) equivalence.
4. The JQ-model category

The objective of this section is to construct a new model category structure on the category $\Gamma S$. This new model category is obtained by localizing the strict JQ-model category defined above. We will refer to this new model category structure as the model category structure of coherently commutative monoidal quasi-categories. The guiding principle of this new model structure is to endow its homotopy category with a semi-additive structure. In other words we want this new model category structure to have finite homotopy biproducts. We go on further to show that this new model category is symmetric monoidal closed with respect to the Day convolution product, see [Day70].

**Definition 4.1.** Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure $L_{\mathcal{S}} \mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

1. The class of cofibrations of $L_{\mathcal{S}} \mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.
2. A map $f : A \rightarrow B$ is a weak equivalence in $L_{\mathcal{S}} \mathcal{M}$ if it is an $\mathcal{S}$-local equivalence, namely, for every fibrant $\mathcal{S}$-local object $X$, the induced map on homotopy function complexes

   $$f^* : \text{Map}_{\mathcal{M}}^h(B, X) \rightarrow \text{Map}_{\mathcal{M}}^h(A, X)$$

   is a homotopy equivalence of simplicial sets. Recall that an object $X$ is called fibrant $\mathcal{S}$-local if $X$ is fibrant in $\mathcal{M}$ and for every element $g : K \rightarrow L$ of the set $\mathcal{S}$, the induced map on homotopy function complexes

   $$g^* : \text{Map}_{\mathcal{M}}^h(L, X) \rightarrow \text{Map}_{\mathcal{M}}^h(K, X)$$

   is a weak homotopy equivalence of simplicial sets.

where $\text{Map}_{\mathcal{M}}^h(-, -)$ is the simplicial function object associated with the strict model category $\mathcal{M}$, see [DK80a], [DK80c] and [DK80b].

We want to construct a left Bousfield localization of the strict model category of $\Gamma$-spaces. For each pair $k^+, l^+ \in \Gamma^{\text{op}}$, we have the obvious projection maps in $\Gamma S$

$$\delta_{k+}^l : (k + l)^+ \rightarrow k^+ \quad \text{and} \quad \delta_{l+}^k : (k + l)^+ \rightarrow l^+.$$

The maps

$$\Gamma^{\text{op}}(\delta_{k+}^l, -) : \Gamma^k \rightarrow \Gamma^{k+l} \quad \text{and} \quad \Gamma^{\text{op}}(\delta_{l+}^k, -) : \Gamma^l \rightarrow \Gamma^{k+l}$$

induce a map of $\Gamma$-spaces on the coproduct which we denote as follows:

$$h_{k+}^l : \Gamma^l \sqcup \Gamma^l \rightarrow \Gamma^{l+k}.$$

We now define a class of maps $\mathcal{E}_{\infty} \mathcal{S}$ in $\Gamma S$:

$$\mathcal{E}_{\infty} \mathcal{S} := \{h_{k+}^l : \Gamma^l \sqcup \Gamma^l \rightarrow \Gamma^{l+k} : l, k \in \mathbb{Z}^+\}$$

**Definition 4.2.** We call a $\Gamma$-space $X$ a $(\Delta \times \mathcal{E}_{\infty} \mathcal{S})$-local object if it is a fibrant object in the strict $JQ$-model category and for each map $h_{k+}^l \in \mathcal{E}_{\infty} \mathcal{S}$, the induced simplicial map

$$\text{Map}_{\Gamma S}^h(\Delta[n] \times h_{k+}^l X) : \text{Map}_{\Gamma S}^h(\Delta[n] \times \Gamma^{k+l}, X) \rightarrow \text{Map}_{\Gamma S}^h(\Delta[n] \times (\Gamma^l \sqcup \Gamma^l), X),$$

The maps

$$\Gamma^{\text{op}}(\delta_{k+}^l, -) : \Gamma^k \rightarrow \Gamma^{k+l} \quad \text{and} \quad \Gamma^{\text{op}}(\delta_{l+}^k, -) : \Gamma^l \rightarrow \Gamma^{k+l}$$

induce a map of $\Gamma$-spaces on the coproduct which we denote as follows:
is a homotopy equivalence of simplicial sets for all $n \geq 0$ where $Map^b_{\Gamma S}(\dash, \dash)$ is the simplicial function complexes associated with the strict model category $\Gamma S$, see [DK80a], [DK80c] and [DK80b].

Appendix B tell us that a model for $M_{\Gamma S}(X, Y)$ is the Kan complex $J(\text{Map}_{\Gamma S}(X, Y))$ which is the maximal Kan complex contained in the quasicategory $\text{Map}_{\Gamma S}(X, Y)$.

The following proposition gives a characterization of $\mathcal{E}_{\infty} S$-local objects

**Proposition 4.3.** A $\Gamma$-space $X$ is a $(\Delta \times \mathcal{E}_{\infty} S)$-local object in $\Gamma S$ if and only if it satisfies the Segal condition namely the functor

$$(X(\delta^{(k+1)}_k), X(\delta^{(k+1)}_l)) : X((k + l)^+) \rightarrow X(k^+) \times X(l^+)$$

is an equivalence of categories for all $k^+, l^+ \in \text{Ob}(\Gamma^{\text{op}})$.

**Proof:** We begin the proof by observing that each element of the set $\mathcal{E}_{\infty} S$ is a map of $\Gamma$-spaces between cofibrant $\Gamma$-spaces. Theorem B.6 implies that $X$ is a $(\Delta \times \mathcal{E}_{\infty} S)$-local object if and only if the following simplicial map

$$\text{Map}_{\Gamma S}(h^k_l, X) : \text{Map}_{\Gamma S}(\Gamma^{(k+1)}(l^+), X) \rightarrow \text{Map}_{\Gamma S}(\Gamma^k \sqcup \Gamma^l, X)$$

is a categorical equivalence of quasi-categories.

We observe that we have the following commutative square in $(s\text{Sets}, Q)$

$$\begin{array}{ccc}
\text{Map}_{\Gamma S}(\Gamma^{(k+1)}(l^+), X) & \xrightarrow{\text{Map}_{\Gamma S}(h^k_l, X)} & \text{Map}_{\Gamma S}(\Gamma^k \sqcup \Gamma^l, X) \\
X((k + l)^+) & \xrightarrow{(X(\delta^{(k+1)}_k), X(\delta^{(k+1)}_l))} & X(k^+) \times X(l^+)
\end{array}$$

This implies that the functor $(X(\delta^{(k+1)}_k), X(\delta^{(k+1)}_l))$ is an equivalence of categories if and only if the functor $\text{Map}_{\Gamma S}(h^k_l, X)$ is an equivalence of categories. \(\square\)

**Definition 4.4.** We will refer to a $(\Delta \times \mathcal{E}_{\infty} S)$-local object as a coherently commutative monoidal quasi-category.

**Definition 4.5.** Let $X$ be a coherently commutative monoidal quasi-category. We will refer to the homotopy category of the quasi-category $X(1^+), \text{ho}(X(1^+))$, as the homotopy category of $X$ and denote it by $\text{ho}(X)$.

**Proposition 4.6.** The homotopy category of a coherently commutative monoidal quasi-category is a permutative category.

**Definition 4.7.** A morphism of $\Gamma$-spaces $F : X \rightarrow Y$ is a $(\Delta \times \mathcal{E}_{\infty} S)$-local equivalence if for each coherently commutative monoidal category $Z$ the following simplicial map

$$\text{Map}_{\Gamma S}^b(F, Z) : \text{Map}_{\Gamma S}^b(Y, Z) \rightarrow \text{Map}_{\Gamma S}^b(X, Z)$$

is a homotopy equivalence of simplicial sets.

**Proposition 4.8.** A morphism between two cofibrant $\Gamma$-spaces $F : X \rightarrow Y$ is an $(\Delta \times \mathcal{E}_{\infty} S)$-local equivalence if and only if the simplicial map

$$\text{Map}_{\Gamma S}(F, Z) : \text{Map}_{\Gamma S}(Y, Z) \rightarrow \text{Map}_{\Gamma S}(X, Z)$$

is an equivalence of quasi-categories for each coherently commutative monoidal quasi-category $Z$. 
By Proposition B.5, we have quasi-categories:

is a homotopy equivalence of Kan complexes. We observe that for each condition.

which the first map is an equivalence of quasi-categories satisfies the Segal condition, see C.14, namely we have the following diagram in which the first map is an equivalence of quasi-categories

This implies that for each \( n > 0 \) the following simplicial map is an equivalence of quasi-categories:

By adjointness we have the following isomorphisms in the category of arrows of simplicial sets:

By Proposition B.5 we have

By adjointness we have the following isomorphisms in the category of arrows of simplicial sets:

Since the map \( J(\text{Map}_{\Gamma S}(F, Z \Delta^\infty)) \) is a homotopy equivalence of Kan complexes, the above isomorphisms imply that so is the simplicial map \( J(\text{Map}_{\Gamma S}(F, Z \Delta^\infty)) \).

Now Lemma B.6 says that the simplicial map \( \text{Map}_{\Gamma S}(F, Z) \) is an equivalence of quasi-categories.

Conversely, let us assume that the simplicial map \( \text{Map}_{\Gamma S}(F, Z) \) is an equivalence of quasi-categories. Since the functor \( J \) takes equivalences of quasi-categories to homotopy equivalences of Kan complexes, therefore \( J(\text{Map}_{\Gamma S}(F, Z)) = \text{Map}_{\Gamma S}(F, Z) \) is a homotopy equivalence of Kan complexes. Thus we have shown that \( F : X \to Y \) is a \( (\Delta \times \mathcal{E}_\infty \mathcal{S}) \)-local object.

\[ \square \]

Definition 4.9. We will refer to a \( (\Delta \times \mathcal{E}_\infty \mathcal{S}) \)-local equivalence either as an equivalence of coherently commutative monoidal categories or as a JQ-equivalence.

The main result of this section is about constructing a new model category structure on the category \( \Gamma \mathcal{S} \), by localizing the strict model category of \( \Gamma \)-spaces with respect to morphisms in the set \( \mathcal{E}_\infty \mathcal{S} \). We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [?] but a proof was later provided by Barwick in [Bar13].

Theorem 4.10. [Bar13, Theorem 2.11] If \( \mathcal{M} \) is a combinatorial model category and \( \mathcal{S} \) is a small set of homotopy classes of morphisms of \( \mathcal{M} \), the left Bousfield localization \( L_\mathcal{S} \mathcal{M} \) of \( \mathcal{M} \) along any set representing \( \mathcal{S} \) exists and satisfies the following conditions.

1. The model category \( L_\mathcal{S} \mathcal{M} \) is left proper and combinatorial.
2. As a category, \( L_\mathcal{S} \mathcal{M} \) is simply \( \mathcal{M} \).
3. The cofibrations of \( L_\mathcal{S} \mathcal{M} \) are exactly those of \( \mathcal{M} \).
4. The fibrant objects of \( L_\mathcal{S} \mathcal{M} \) are the fibrant \( \mathcal{S} \)-local objects \( Z \) of \( \mathcal{M} \).
5. The weak equivalences of \( L_\mathcal{S} \mathcal{M} \) are the \( \mathcal{S} \)-local equivalences.
**Theorem 4.11.** There is a closed, left proper, combinatorial model category structure on the category of $\Gamma$-spaces, $\Gamma S$, in which

1. The class of cofibrations is the same as the class of $JQ$-cofibrations of $\Gamma$-spaces.
2. The weak equivalences are equivalences of coherently commutative monoidal quasi-categories.

An object is fibrant in this model category if and only if it is a coherently commutative monoidal quasi-category. A fibration between two coherently commutative monoidal quasi-categories is a strict $JQ$-equivalence.

**Proof.** The strict model category of $\Gamma$-spaces is a combinatorial model category therefore the existence of the model structure follows from the theorem 4.10 stated above. The last statement follows from (1). □

**Notation 4.12.** The model category constructed in theorem 4.11 will be called the model category of coherently commutative monoidal quasi-categories.

The rest of this section is devoted to proving that the model category of coherently commutative monoidal quasi-categories is a symmetric monoidal closed model category. In order to do so we will need some general results which we state and prove now.

**Proposition 4.13.** A cofibration, $f : A \to B$, between cofibrant objects in a model category $\mathcal{C}$ is a weak equivalence in $\mathcal{C}$ if and only if it has the right lifting property with respect to all fibrations between fibrant objects in $\mathcal{C}$.

**Proof.** The unique terminal map $B \to *$ can be factored into an acyclic cofibration $\eta_B : B \to R(B)$ followed by a fibration $R(B) \to *$. The composite map $\eta_B \circ f$ can again be factored as an acyclic cofibration followed by a fibration $R(f)$ as shown in the following diagram:

$$
\begin{array}{c}
A \xrightarrow{\eta_A} R(A) \\
\downarrow f \\
B \xrightarrow{\eta_B} R(B)
\end{array}
$$

Since $B$ is fibrant and $R(f)$ is a fibration, therefore $R(A)$ is a fibrant object in $\mathcal{C}$. Thus $R(f)$ is a fibration between fibrant objects in $\mathcal{C}$ and now by assumption, the dotted arrow exists which makes the whole diagram commutative. Since both $\eta_A$ and $\eta_B$ are acyclic cofibrations, therefore the two out of six property of model categories implies that the map $F$ is a weak-equivalence in the model category $\mathcal{C}$. □

**Proposition 4.14.** Let $X$ be a coherently commutative monoidal quasi-category, then for each $n \in \mathbb{N}$, the $\Gamma$-space $X(n^+ \wedge -)$ is also a coherently commutative monoidal quasi-category.

**Proof.** We begin by observing that $X(n^+ \wedge -(1^+)) = X(n^+)$ and since $X$ is fibrant, the pointed category $X(n^+)$ is equivalent to $\prod_1^\infty X(1^+)$. Notice that the isomorphisms $(n^+ \wedge (k+l)^+) \cong \big(\big(\big((\big(n^+ k^+\big) \wedge (\big(n^+ l^+)\big)\big)\big)\big)\big)$.

The two projection maps $\delta^{k+l}_k : (k+l)^+ \to k^+$ and $\delta^{k+l}_l : (k+l)^+ \to l^+$ induce an
equivalence of categories \( X((\vee_1^0 k^+ + \vee_1^0 l^+)) \to X((\vee_1^0 k^+) \times X((\vee_1^0 l^+)). \) Composing with the isomorphisms above, we get the following equivalence of pointed simplicial sets 
\( X(n^+ \land -)((k + l)^+) \to X(n^+ \land -(k^+) \times X(n^+ \land -(l^+). \)

**Corollary 4.15.** For each coherently commutative monoidal category \( X \), the mapping object \( \text{Map}_{\Gamma S}(\Gamma^n, X) \) is also a coherently commutative monoidal category for each \( n \in \mathbb{N} \).

**Proof.** The corollary follows from the above proposition and proposition 3.11. □

The category \( \Gamma^{op} \) is a symmetric monoidal category with respect to the smash product of pointed sets. In other words the smash product of pointed sets defines a bi-functor \(- \land -\) : \( \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op} \). For each pair \( k^+, l^+ \in \text{Ob}(\Gamma^{op}) \), there are two natural transformations

\[ \delta_{k^+ l^+}^- : ((k + l)^+) \land - \Rightarrow k^+ \land - \text{ and } \delta_{l^+}^- : (k^+ \land -) \Rightarrow l^+ \land -. \]

Horizontal composition of either of these two natural transformations with a \( \Gamma \)-space \( X \) determines a morphism of \( \Gamma \)-spaces

\[ \text{id}_X \circ (\delta_{k^+ l^+}^-) = X(k^+ \land -) : X((k + l)^+ \land -) \to X(k^+ \land -). \]

**Proposition 4.16.** Let \( X \) be a coherently commutative monoidal quasi-category, then for each pair \( (k, l) \in \mathbb{N} \times \mathbb{N} \), the following morphism

\[ (X(\delta_{k^+ l^+}^-), X(\delta_{l^+}^-)) : X((k + l)^+ \land -) \to X(k^+ \land -) \times X(l^+ \land -) \]

is a strict equivalence of \( \Gamma \)-spaces.

Using the previous two propositions, we now show that the mapping space functor \( \text{Map}_{\Gamma S}(-, -) \) provides the homotopically correct function object when the domain is cofibrant and codomain is fibrant.

**Lemma 4.17.** Let \( W \) be a Q-cofibrant \( \Gamma \)-space and let \( X \) be a coherently commutative monoidal quasi-category. Then the mapping object \( \text{Map}_{\Gamma S}(W, X) \) is also a coherently commutative monoidal quasi-category.

**Proof.** We begin by recalling that

\[ \text{Map}_{\Gamma S}(W, X)((k + l)^+) = \text{Map}_{TS}(W \ast \Gamma^{k+l}, X). \]

We recall that the \( \Gamma^{k+l} \) is a cofibrant \( \Gamma \)-space and by assumption \( W \) is also a cofibrant \( \Gamma \)-space therefore it follows from Theorem 3.14 that \( W \ast \Gamma^{k+l} \) is also a cofibrant \( \Gamma \)-space. Since \( X \) is a coherently commutative monoidal quasi-category i.e., a fibrant object in the model category of coherently commutative monoidal quasi-categories, therefore it follows from Theorem 3.14 that the mapping object \( \text{Map}_{\Gamma S}(W \ast \Gamma^{k+l}, X) \) is a quasi-category, for all \( k, l \geq 0 \).

We recall that the map \( h^+_k : \Gamma^k \lor \Gamma^l \to \Gamma^{k+l} \) is a weak equivalence in the model category of coherently commutative monoidal quasi-categories, therefore the top
The proposition 4.16 tells us that the map \((X(\delta_k^{k+l} \land -), X(\delta_l^{k+l} \land -))\) is a strict equivalence of \(\Gamma\)-spaces. Now Theorem 3.14 implies that the following induced functor on the mapping (pointed) categories

\[
\text{Map}_{\Gamma S}(W, X((k+l)^+ \land -)) \times \text{Map}_{\Gamma S}(W, X((l)^+ \land -))
\]

is an equivalence of categories.

\[\square\]

Finally we get to the main result of this section. All the lemmas proved above will be useful in proving the following theorem:

**Theorem 4.18.** The model category of coherently commutative monoidal quasi-categories is a symmetric monoidal closed model category under the Day convolution product.

**Proof.** Let \(i : U \to V\) be a \(\text{JQ}\)-cofibration and \(j : Y \to Z\) be another \(\text{JQ}\)-cofibration. We will prove the theorem by showing that the following pushout product morphism

\[i \Box j : U \ast Z \coprod_{U \ast Y} V \ast Y \to V \ast Z\]

is a \(\text{JQ}\)-cofibration which is also a \(\text{JQ}\)-equivalence whenever either \(i\) or \(j\) is a \(\text{JQ}\)-equivalence. We first deal with the case of \(i\) being a generating \(\text{JQ}\)-cofibration. The closed symmetric monoidal model structure on the strict \(\text{JQ}\)-model category, see theorem 3.14, implies that \(i \Box j\) is a \(\text{JQ}\)-cofibration. Let us assume that \(j\) is an acyclic \(\text{JQ}\)-cofibration \(i.e\). the \(\text{JQ}\)-cofibration \(j\) is also a \(\text{JQ}\)-equivalence of coherently commutative monoidal categories. According to proposition 4.13 the \(\text{JQ}\)-cofibration \(i \Box j\) is a \(\text{JQ}\)-equivalence if and only if it has the left lifting property with respect to all strict \(\text{JQ}\)-fibrations of \(\Gamma\)-spaces between coherently commutative monoidal quasi-categories. Let \(p : W \to X\) be a strict \(\text{JQ}\)-fibration between two coherently commutative monoidal quasi-categories. By adjointness, a (dotted) lifting arrow would exists in the following diagram

\[
\begin{array}{ccc}
U \ast Z \coprod_{U \ast Y} V \ast Y & \to & W \\
\downarrow & & \downarrow p \\
V \ast Z & \to & Y
\end{array}
\]
if and only if a (dotted) lifting arrow exists in the following adjoint commutative diagram

\[
\begin{array}{ccc}
Y & \rightarrow & \text{Map}_{\Gamma S}(V, W) \\
\downarrow & & \downarrow (j^*, p^*) \\
Z & \rightarrow & \text{Map}_{\Gamma S}(U, X) \times \text{Map}_{\Gamma S}(U, Y) \rightarrow \text{Map}_{\Gamma S}(V, Y)
\end{array}
\]

The map \((j^*, p^*)\) is a strict \(JQ\)-fibration of \(\Gamma\)-spaces by lemma C.8 and theorem 3.14. Further the observation that both \(V\) and \(U\) are \(JQ\)-cofibrant and the above lemma 4.17 together imply that \((j^*, p^*)\) is a strict \(JQ\)-fibration between coherently commutative monoidal categories and therefore a fibration in the \(JQ\)-model category. Since \(j\) is an acyclic cofibration in the \(JQ\)-model category by assumption therefore the (dotted) lifting arrow exists in the above diagram. Thus we have shown that if \(i\) is a \(JQ\)-cofibration and \(j\) is a \(JQ\)-cofibration which is also a weak equivalence in the \(JQ\)-model category then \(i \Box j\) is an acyclic cofibration in the \(JQ\)-model category. Now we deal with the general case of \(i\) being an arbitrary \(JQ\)-cofibration. Consider the following set:

\[S = \{i : U \rightarrow V \mid i \Box j \text{ is an acyclic cofibration in } \Gamma S\}\]

where \(\Gamma S\) is endowed with the \(JQ\)-model structure. We have proved above that the set \(S\) contains all generating \(JQ\)-cofibrations. We observe that the set \(S\) is closed under pushouts, transfinite compositions and retracts. Thus \(S\) contains all \(JQ\)-cofibrations. Thus we have proved that \(i \Box j\) is a cofibration which is acyclic if \(j\) is acyclic. The same argument as above when applied to the second argument of the Box product (i.e. in the variable \(j\)) shows that \(i \Box j\) is an acyclic cofibration whenever \(i\) is an acyclic cofibration in the \(JQ\)-model category.

\(\square\)
5. Equivalence with normalized Γ-spaces

In this section we will establish a Quillen equivalence between the model category of coherently commutative monoidal quasi-categories and the model category of strictly unital coherently commutative monoidal quasi-categories which is constructed in appendix C.1. The category of normalized Γ-spaces is equipped with a forgetful functor

\[ U : \Gamma S \rightarrow \Gamma S. \]

This functor maps a normalized Γ-space \( X \) to the following composite

\[ \Gamma^{op} X \xrightarrow{\sim} sSets_{\bullet}^+ \xrightarrow{U} sSets \]

where the second functor is the obvious forgetful functor which forgets the basepoint of a simplicial set. The forgetful functor \( U \) has some very desirable homotopical properties: We will show in this section that \( U \) preserves weak-equivalences namely it maps \( JQ \)-equivalences of normalized Γ-spaces to \( JQ \)-equivalences. This functor also preserves cofibrations even though it is a right Quillen functor.

Proposition 5.1. The forgetful functor \( U : \Gamma S \rightarrow \Gamma S \) preserves acyclic fibrations.

Proof. A morphism of normalized Γ-spaces \( p : X \rightarrow Y \) is an acyclic fibration in the \( JQ \)-model category of normalized Γ-spaces if and only if there is a lifting arrow in the following commutative diagram for each \( n \in \mathbb{N} \)

\[
\begin{array}{ccc}
\Gamma^n \land \partial \Delta[n]^+ & \rightarrow & X \\
\downarrow & & \downarrow \\
\Gamma^n \land \Delta[n]^+ & \rightarrow & Y
\end{array}
\]

because the collection \( I_{\bullet} = \{ \Gamma^n \land \partial \Delta[n]^+ \rightarrow \Gamma^n \land \Delta[n]^+ : n \in \mathbb{N} \} \) is a set of generating cofibrations for the combinatorial \( JQ \)-model category of normalized Γ-spaces \( \Gamma S_{\bullet} \). By adjointness the lifting arrow exists in the above diagram if and only if a lifting arrow exists in the following (adjunct) commutative diagram in \( sSets_{\bullet} \)

\[
\begin{array}{ccc}
\partial \Delta[n]^+ & \rightarrow & \text{Map}_{\Gamma S_{\bullet}}(\Gamma^n, X) \cong X(n^+) \\
\downarrow & & \downarrow \\
\Delta[n]^+ & \rightarrow & \text{Map}_{\Gamma S_{\bullet}}(\Gamma^n, Y) \cong Y(n^+)
\end{array}
\]

We recall the adjunction \((-)^+ : sSets \cong sSets_{\bullet} : U_{sSets}\) and observe that \( U(X)(n^+) = U_{sSets}(X(n^+)) \). This implies that the lifting arrow in the above commutative diagram of pointed simplicial sets will exist if and only if a lifting arrow exists in the following (adjunct) commutative diagram in \( sSets \)

\[
\begin{array}{ccc}
\partial \Delta[n] & \rightarrow & U_{sSets}(X(n^+)) \\
\downarrow & & U_{sSets}(p_{n^+}) \\
\Delta[n] & \rightarrow & U_{sSets}(Y(n^+))
\end{array}
\]

We observe that for any normalized Γ-space \( Z \), \( U_{sSets}(Z(n^+)) \cong U(Z)(n^+) \). Therefore a lifting arrow exists in the above diagram if and only if a lifting arrow exists
in the following commutative diagram:
\[
\begin{array}{c}
\partial \Delta[n] \\ \downarrow \downarrow \\
\Delta[n] \\ \downarrow \\
\end{array}
\begin{array}{c}
\rightarrow \rightarrow \\
\rightarrow \rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\text{Map}_{\Gamma S}(\Gamma^n, U(X)) \\
\text{Map}_{\Gamma S}(\Gamma^n, U(p_n)) \\
\text{Map}_{\Gamma S}(\Gamma^n, U(Y))
\end{array}
\]

By adjointness, this lifting arrow would exist if and only if there exists a lifting arrow in the following (adjunct) commutative diagram:
\[
\begin{array}{c}
\Gamma^n \times \partial \Delta[n] \\ \downarrow \downarrow \\
\Gamma^n \times \Delta[n] \\
\downarrow \\
\end{array}
\begin{array}{c}
\rightarrow \rightarrow \\
\rightarrow \rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
U(X) \\
U(Y)
\end{array}
\]

The collection \( I_\bullet = \{ \Gamma^n \land \partial \Delta[n]^+ \to \Gamma^n \land \Delta[n]^+ : n \in \mathbb{N} \} \) is a set of generating cofibrations for the combinatorial model category \( \Gamma S \). Thus we have shown that the map of \( \Gamma \)-spaces \( U(p) \) has the right lifting property with respect to the set of generating cofibrations of the \( JQ \)-model category and hence \( U(p) \) is an acyclic fibration.

\[\square\]

A similar argument as in the proof of the above proposition when applied to the collection of generating acyclic cofibrations of the strict \( JQ \)-model category of normalized \( \Gamma \)-spaces \( \Gamma S_\bullet \) gives a proof of the following proposition:

**Proposition 5.2.** The forgetful functor \( U : \Gamma S_\bullet \to \Gamma S \) preserves strict \( JQ \)-fibrations.

We would like to construct a left adjoint of the functor \( U \). For a given \( \Gamma \)-space \( X \) we will construct another \( \Gamma \)-space \( X[0] \) which is equipped with a map \( \iota : X[0] \to X \).

**Definition 5.3.** Let \( X \) be a \( \Gamma \)-space, the **unital part** of \( X \) is the constant \( \Gamma \)-space \( X[0] \) which is defined by
\[
X[0](n^+) := X(0^+)
\]
for all \( n^+ \in \text{Ob}(\Gamma^\text{op}) \). The map \( \iota \) is defined, in degree \( n \) by the following simplicial map:
\[
\iota(n^+) := X(0_n) : X(0^+) \to X(n^+),
\]
where \( 0_n : 0^+ \to n^+ \) is the unique map in \( \Gamma^\text{op} \) between \( 0^+ \) and \( n^+ \).

We notice that if \( X \) is a normalized \( \Gamma \)-space then the unital part of \( U(X)[0] \) is the terminal \( \Gamma \)-space. We want to use the above construction to associate with a \( \Gamma \)-space a normalized \( \Gamma \)-space which is equipped with a map from the original \( \Gamma \)-space.

**Definition 5.4.** Let \( X \) be a \( \Gamma \)-space, we define another \( \Gamma \)-space \( U(X^{\text{nor}}) \) by the following pushout square:
\[
\begin{array}{c}
X[0] \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
X \\
\downarrow \pi_X \\
U(X^{\text{nor}})
\end{array}
\]
where $1$ is the terminal $\Gamma$-space. The bottom horizontal arrow in the above pushout square is an object of the category $\mathcal{S}$. Since $U(X^{\text{nor}})(0^+) = \ast$ therefore the image is an object of the category $(\mathcal{S}^\ast)$, see (C.12). Its image under the isomorphism of categories from the remark following (C.12) determines a normalized $\Gamma$-space which we denote by $X^{\text{nor}}$ and call it the normalization of $X$.

The above construction is functorial in $X$ and hence we have defined a functor $(-)^{\text{nor}} : \mathcal{S} \to \mathcal{S}^\ast$.

**Proposition 5.5.** For any $\Gamma$-space $X$ the map $\iota : X[0] \to X$ defined above is a degreewise monomorphism of simplicial sets.

**Proof.** We want to show that for each $n^+ \in Ob(\Gamma^{\text{op}})$ the simplicial map $\iota(n^+) : X(0^+) \to X(n^+)$ is a monomorphism. We observe that the object $0^+$ is the zero object in $\Gamma^{\text{op}}$ therefore the unique composite arrow $0^+ \to n^+ \to 0^+$ is the identity map of $0^+$, for all $n^+ \in Ob(\Gamma^{\text{op}})$. This implies that the simplicial map $X(0^+) \to X(n^+) \to X(0^+)$ is the identity map of $X(0^+)$, in other words the simplicial map $\iota(n^+)$ has a left inverse which implies that the map $\iota(n^+) : X(0^+) \to X(n^+)$ is a monomorphism. □

**Proposition 5.6.** For each coherently commutative monoidal quasi-category $X$ the map of $\Gamma$-spaces $\eta_X : X \to U(X^{\text{nor}})$ defined in (7) is a (strict) $JQ$-equivalence.

**Proof.** Since $X$ is a coherently commutative monoidal quasi-category by assumption therefore the unique terminal map $X[0] \to 1$ is a strict $JQ$-equivalence because $X(0^+)$ is homotopy equivalent to the terminal simplicial set in the Joyal model category. The $\Gamma$-space $U(X^{\text{nor}})$ is defined as a pushout, see (7), and pushouts in the category $\Gamma$-space are degreewise therefore we have the following pushout diagram in the category $\mathbf{sSets}$:

$$
\begin{array}{ccc}
X[0](n^+) & \xrightarrow{\iota(n^+)} & X(n^+) \\
\downarrow & & \downarrow_{\eta_X(n^+)} \\
1 & \xrightarrow{} & U(X^{\text{nor}})(n^+)
\end{array}
$$

for each $n^+ \in Ob(\Gamma^{\text{op}})$. By proposition 5.5 the simplicial map $\iota(n^+)$ is a monomorphism. Since monomorphisms are cofibrations in the Joyal model category which is a left proper model category therefore a pushout of a weak equivalence along a monomorphism is a weak equivalence in Joyal model category. Thus we have shown that the map $\iota(n^+) : X(n^+) \to U(X^{\text{nor}})(n^+)$ is a weak equivalence in the Joyal model category which proves that the unit map $\eta_X : X \to U(X^{\text{nor}})$ is a strict $JQ$-equivalence whenever $X$ is a coherently commutative monoidal quasi-category. □

**Corollary 5.7.** The functor $(-)^{\text{nor}}$ takes coherently commutative monoidal quasi-categories to strictly unital coherently commutative monoidal quasi-categories.

**Corollary 5.8.** The functor $U : \mathcal{S}^\ast \to \mathcal{S}$ preserves weak equivalences.

**Proof.** A map $f : X \to Y$ is a $JQ$-equivalence of normalized $\Gamma$-spaces if and only if each cofibrant replacement if also the same so we may assume that $f$ is a map
between cofibrant objects. It would be sufficient to show that for each coherently commutative monoidal quasi-category $Z$ the following simplicial map is an equivalence of quasi-categories:

$$\text{Map}_{\Gamma S}(U(f), Z) : \text{Map}_{\Gamma S}(U(Y), Z) \to \text{Map}_{\Gamma S}(U(X), Z)$$

We have the following commutative diagram of simplicial mapping objects:

\[
\begin{array}{ccc}
\text{Map}_{\Gamma S}(U(Y), Z) & \xrightarrow{\text{Map}_{\Gamma S}(U(f), Z)} & \text{Map}_{\Gamma S}(U(X), Z) \\
\downarrow & & \downarrow \\
\text{Map}_{\Gamma S}(U(Y), U(Z^{\text{nor}})) & \xrightarrow{\cong} & \text{Map}_{\Gamma S}(U(X), U(Z^{\text{nor}})) \\
\downarrow & & \downarrow \\
U(\text{Map}_{\Gamma S}(Y, Z^{\text{nor}})) & \xrightarrow{\cong} & U(\text{Map}_{\Gamma S}(X, Z^{\text{nor}}))
\end{array}
\]

Since $f$ is a $JQ$-equivalence of normalized $\Gamma$-spaces by assumption and $Z^{\text{nor}}$ is a strictly unital coherently commutative monoidal quasi-category therefore the simplicial map $U(\text{Map}_{\Gamma S}(f, Z^{\text{nor}}))$ is an equivalence of quasi-categories. The vertical arrows in the bottom rectangle are isomorphisms by corollary [C.5](#note1). The vertical arrows in the top rectangle are equivalences of quasi-categories because $U(X)$ and $U(Y)$ are cofibrant and $U(Z^{\text{nor}})$ is fibrant. Now the two-out-of-three property of weak equivalences in a model category tells us that the top horizontal map in the above diagram namely $\text{Map}_{\Gamma S}(U(f), \eta_Z)$ is a weak equivalence in the Joyal model category. By lemma [4.8](#note2) we have shown that the map $U(f)$ is a $JQ$-equivalence. □

We claim that $(-)^{\text{nor}}$ is a left adjoint of the forgetful functor $U : \Gamma S_\bullet \to \Gamma S$. The unit of this adjunction is given by the quotient map $\eta_X : X \to U(X^{\text{nor}})$. For a normalized $\Gamma$-space $Y$ we have a canonical isomorphism (depicted by the dotted arrow) in the following diagram

\[
\begin{array}{ccc}
1 & \to & U(Y) \\
\downarrow & & \downarrow \\
1 & \to & U(U(Y)^{\text{nor}})
\end{array}
\]

The diagram [9](#note3) is a composite arrow in the category $(1/\Gamma S)_\bullet$. The image of the map $\epsilon_Y$ under the isomorphism from the remark following [C.12](#note4) gives us the counit map which we also denote by $\epsilon_Y$.

The next proposition verifies our claim made above:

**Proposition 5.9.** The functor $(-)^{\text{nor}} : \Gamma S \to \Gamma S_\bullet$ is a left adjoint to the forgetful functor $U : \Gamma S_\bullet \to \Gamma S$.

**Proof.** We will prove this proposition by showing that the unit map $\eta_X$ constructed above is universal. Let $X$ be a $\Gamma$-space and $Y$ be a normalized $\Gamma$-space and $f : X \to
Let $U(Y)$ be a map in $\Gamma S$. We will show the existence of a unique map $g : X^{\text{nor}} \to Y$ in $\Gamma S_\bullet$ such that the following diagram commutes in the category $\Gamma S$:

(10)

The map $1 \to U(Y)$ in the diagram below is the image of the normalized $\Gamma$-space $Y$ under the isomorphism of categories in remark following (C.12):

Since $U(Y)(0^+) = \ast$ therefore $f$ maps $X(0^+)$ to a point. This implies that the outer solid diagram in the figure above commutes. Since the square in the above diagram is a pushout square therefore there exists a unique (dotted) arrow which makes the whole diagram commutative. The lower commutative triangle in the diagram above is a map in the category $(1/\Gamma S)_\bullet$. The image of this map under the isomorphism of categories from the remark following (C.12) is a map $g : X^{\text{nor}} \to Y$ in $\Gamma S_\bullet$ whose image under the forgetful functor $U(g)$ makes the diagram (10) commute.

This proposition has the following consequence:

**Corollary 5.10.** The forgetful functor $U : \Gamma S_\bullet \to \Gamma S$ maps $JQ$-cofibrations of normalized $\Gamma$-spaces to $JQ$-cofibrations.

**Proof.** Let $i : V \to W$ be a $JQ$-cofibration of normalized $\Gamma$-spaces, we will show that $U(i)$ is a $JQ$-cofibration. By adjointness $U(i)$ is a cofibration if and only if $U(i)^{\text{nor}}$ is a $JQ$-cofibration of normalized $\Gamma$-spaces. The following commutative square in $\Gamma S_\bullet$ shows that $U(i)$ is a cofibration because $i$ is one by assumption:

```
U(V)^{\text{nor}} \cong V
\downarrow \cong \downarrow i
U(i)^{\text{nor}} \cong \cong U(W)^{\text{nor}} \cong W
```

Next we show that the adjunction $(-)^{\text{nor}} \dashv U$ is compatible with the model category structures i.e. it is a Quillen adjunction.

**Lemma 5.11.** The pair of adjoint functors $((-)^{\text{nor}}, U)$ is a Quillen pair.

**Proof.** A pair of adjoint functors between two model categories is a Quillen pair if and only if the left adjoint preserves cofibrations and the right adjoint preserves
fibrations between fibrant objects, see [JT06, Prop. 7.15]. Let \( i : A \to B \) be a cofibration in \( \Gamma S \) and let \( p : X \to Y \) be an acyclic fibration in \( \Gamma S \), then by proposition 5.1, there is a lifting arrow in the following (outer) commutative diagram:

\[
\begin{array}{c}
A & \xrightarrow{i} & U(X) \\
\downarrow & & \downarrow U(p) \\
B & \xrightarrow{U(p)} & U(Y)
\end{array}
\]

By adjointness this lifting arrow exists if and only if there exists a lifting arrow in the following (adjunct) commutative diagram:

\[
\begin{array}{c}
A_{nor} & \xrightarrow{i_{nor}} & X \\
\downarrow & & \downarrow p \\
B_{nor} & \xrightarrow{U(p)} & Y
\end{array}
\]

Thus we have shown that for each cofibration \( i \) in \( \Gamma S \), its image \( i_{nor} \) in \( \Gamma S \), has the left lifting property with respect to acyclic fibrations in the \( JQ \)-model category of normalized \( \Gamma \)-spaces \( \Gamma S \). Hence we have shown that the left adjoint preserves cofibrations.

We recall that \( JQ \)-fibrations between \( JQ \)-fibrant normalized \( \Gamma \)-spaces are just strict \( JQ \)-fibrations of normalized \( \Gamma \)-spaces. Now proposition 5.2 tells us that \( U \) preserves fibrations between fibrant normalized \( \Gamma \)-spaces. Hence by [JT06, Prop. 7.15] the adjunction in context is a Quillen pair.

By definition, the counit map \( \epsilon_Y : U(Y)^{nor} \to Y \) of the adjunction \( ((-)^{nor}, U) \) is an isomorphism for each normalized \( \Gamma \)-space \( Y \). Now we want to show that the unit of the same adjunction is a \( JQ \)-equivalence.

**Lemma 5.12.** The unit map \( \eta_X : X \to U(X^{nor}) \) is a \( JQ \)-equivalence for each \( \Gamma \)-space \( X \).

**Proof.** We have already seen in Proposition 5.6 that this result holds when the \( \Gamma \)-space \( X \) is a coherently commutative monoidal quasi-category. Now we tackle the general case wherein \( X \) is an arbitrary \( \Gamma \)-space. Since the unit map \( \eta \) is a natural transformation therefore we have the following commutative diagram in the category \( \Gamma S \):

\[
\begin{array}{c}
X & \xrightarrow{\eta_X} & R(X) \\
\downarrow & & \downarrow \eta_{R(X)} \\
U(X^{nor}) & \xrightarrow{U(R(X)^{nor})} & U(R(X)^{nor})
\end{array}
\]

where \( X \to R(X) \) is a fibrant replacement of \( X \) and therefore it is an acyclic \( JQ \)-cofibration and \( R(X) \) is a coherently commutative monoidal quasi-category. Thus we have shown that the top and right vertical arrow in the commutative diagram above are \( JQ \)-equivalences. Now we want to show that the bottom horizontal arrow is also a \( JQ \)-equivalence. The functor \( (-)^{nor} \) is a left Quillen functor, see (5.11), therefore it preserves acyclic \( JQ \)-cofibrations. Now proposition 5.10 says that \( U \) preserves weak equivalences which implies that the bottom horizontal map is a \( JQ \)-equivalence.
An easy consequence of the above lemma and the fact that the counit of the quillen pair \((-)^{nor}, U\) is a natural isomorphism is the following theorem:

**Theorem 5.13.** The Quillen pair \((-)^{nor}, U\) is a Quillen equivalence.
Appendix A. Quillen Bifunctors

The objective of this section is to recall the notion of Quillen Bifunctors. In order to do so, we begin with the definition of a two variable adjunction:

**Definition A.1.** Suppose $C$, $D$ and $E$ are categories. An adjunction of two variables from $C \times D$ to $E$ is a quintuple $(\otimes, \text{hom}_C, \text{Map}_C, \phi, \psi)$, where

$\otimes : C \times D \to E$, \hspace{0.5cm} $\text{hom}_C : D^{\text{op}} \times E \to C$, \hspace{0.5cm} and \hspace{0.5cm} $\text{Map}_C : C^{\text{op}} \times E \to D$

are functors and $\phi$, $\psi$ are the following natural transformations

$C(C, \text{hom}_C(D, E)) \xrightarrow{\phi^{-1}} E(\otimes D, E) \xrightarrow{\psi} D(\text{Map}_C(C, E))$.

The following definition is based on Quillen’s SM7 axiom, see [Qui67], and is also found in [?].

**Definition A.2.** Given model categories $C$, $D$ and $E$, an adjunction of two variables, $(\otimes, \text{hom}_C, \text{Map}_C, \phi, \psi) : C \times D \to E$, is called a Quillen adjunction of two variables, if, given a cofibration $f : U \to V$ in $C$ and a cofibration $g : W \to X$ in $D$, the induced map

$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$

is a cofibration in $E$ that is trivial if either $f$ or $g$ is. We will refer to the left adjoint of a Quillen adjunction of two variables as a Quillen bifunctor.

The following lemma provides three equivalent characterizations of the notion of a Quillen bifunctor. These will be useful in this paper in establishing enriched model category structures.

**Lemma A.3.** [Hov99, Lemma 4.2.2] Given model categories $C$, $D$ and $E$, an adjunction of two variables, $(\otimes, \text{hom}_C, \text{Map}_C, \phi, \psi) : C \times D \to E$. Then the following conditions are equivalent:

1. $\otimes : C \times D \to E$ is a Quillen bifunctor.
2. Given a cofibration $g : W \to X$ in $D$ and a fibration $p : Y \to Z$ in $E$, the induced map

$\text{hom}_C(g, p) : \text{hom}_C(X, Y) \to \text{hom}_C(X, Z) \times \text{hom}_C(W, Y)_{\text{hom}_C(W, Z)}$

is a fibration in $C$ that is trivial if either $g$ or $p$ is a weak equivalence in their respective model categories.
3. Given a cofibration $f : U \to V$ in $C$ and a fibration $p : Y \to Z$ in $E$, the induced map

$\text{Map}_C(f, p) : \text{Map}_C(V, Y) \to \text{Map}_C(V, Z) \times \text{Map}_C(W, Y)_{\text{Map}_C(W, Z)}$

is a fibration in $C$ that is trivial if either $f$ or $p$ is a weak equivalence in their respective model categories.
Appendix B. On local objects in a model category enriched over quasicategories

A very detailed sketch of this appendix was provided to the author by Andre Joyal. This appendix contains some key results which have made this research possible.

B.1. Introduction. A model category $E$ is enriched over quasicategories if the category $E$ is simplicial, tensored and cotensored, and the functor $[-; -] : E^{op} \times E \to sSets$ is a Quillen functor of two variables, where $sSets = (sSets, Qcat)$ is the model structure for quasi-categories. The purpose of this appendix is to introduce the notion of local object with respect to a map in a model category enriched over quasi-categories.

B.2. Preliminaries. Recall that a Quillen model structure on a category $E$ is determined by its class of cofibrations together with its class of fibrant objects. For examples, the category of simplicial sets $sSets = ([\Delta^{op}, Set]$ admits two model structures in which the cofibrations are the monomorphisms: the fibrant objects are the Kan complexes in one, and they are the quasi-categories in the other. We call the former the model structure for Kan complexes and the latter the model structure for quasi-categories. We shall denote them respectively by $(sSets, Kan)$ and $(sSets, QCat)$. Recall that a simplicial category is a category enriched over simplicial sets. There is a notion of simplicial functor between simplicial categories, and a notion of strong natural transformation between simplicial functors. If $E = (E, [-; -])$ is a simplicial category, then so is the category $\text{SFunc}(E; sSets)$ of simplicial functors $E \to sSets$. A simplicial functor $F : E \to sSets$ isomorphic to a simplicial functor $[A, -] : E \to sSets$ is said to be representable. Recall Yoneda lemma for simplicial functors: if $F : E \to sSets$ is a simplicial functor and $A \in E$, then the map $y : \text{Nat}([A, -], F) \to F(A)_0$ defined by putting $y(\alpha) = \alpha(A)(1_A)$ for a strong natural transformation $\alpha : [A, -] \to F$ is bijective. The simplicial functor $F$ is said to be represented by a pair $(A, a)$, with $a \in F(A)_0$, if the unique strong natural transformation $\alpha : [A, -] \to F$ such that $\alpha(A)(1_A) = a$ is invertible. We say that a simplicial category $E = (E, [-, -])$ is tensored by $\Delta$ if the simplicial functor

$$[A, -]^{\Delta[n]} : E \to sSets$$

is representable (by an object denoted $\Delta[n] \times A$) for every object $A \in E$ and every $n \geq 0$. If $E$ has finite colimits and is tensored by $\Delta$, then it is tensored by finite simplicial sets: the simplicial functor is representable (by an object $K \times A$) for every object $A \in E$ and every finite simplicial set $K$. Dually, we say that a simplicial category $E$ is cotensored by $\Delta$ if the simplicial functor

$$[-, X]^{\Delta[n]} : E^{op} \to sSets$$

is representable (by an object denoted $X^{\Delta[n]}$) for every object $X \in E$ and every $n \geq 0$. If $E$ has finite limits and is cotensored by $\Delta$, then it is cotensored by finite simplicial sets: the simplicial functor

$$[-, X]^K : E^{op} \to sSets$$

is representable by an object $X^K$ for every object $X \in E$ and every finite simplicial set $K$. Recall that a model category $E$ is said to be simplicial if the category $E$ is simplicial, tensored and cotensored by $\Delta$ and the functor $[-, -] : E^{op} \times E \to sSets$
is a Quillen functor of two variables, where \textbf{sSets} = (sSet, Kan). The last condition implies that if \( A \in E \) is cofibrant and \( X \in E \) is fibrant, then the simplicial set \([A,X]\) is a Kan complex. For this reason, we shall say that a simplicial model category is enriched over Kan complexes.

**Definition B.1.** We shall say that a model category \( E \) is enriched over quasi-categories if the category \( E \) is simplicial, tensored and cotensored over \( \Delta \) and the functor \([-,-] : E^{op} \times E \to \textbf{sSets} \) is a Quillen functor of two variables, where \textbf{sSets} = (sSets, Qcat).

The last condition of definition B.1 implies that if \( A \in E \) is cofibrant and \( X \in E \) is fibrant, then the simplicial set \([A,X]\) is a quasi-category. If \( E \) is a category with finite limits than so is the category \([\Delta^{op}, E]\) of simplicial objects in \( E \). The evaluation functor \( ev_0 : [\Delta^{op}, E] \to E \) defined by putting \( ev_0(X) = X_0 \) has a left adjoint \( sk^0 \) and a right adjoint \( cosk^0 \). If \( A \in E \), then \( sk^0(A)_n = A \) and \( cosk^0(A)_n = A^{[n]} = A^{n+1} \) for every \( n \geq 0 \) (the simplicial object \( sk^0(A) \) is the constant functor \( cA : \Delta^{op} \to E \) with values \( A \)). The category \([\Delta^{op}, E]\) is simplicial. If \( X, Y \in [\Delta^{op}, E] \) then we have

\[
[X,Y]_n = Nat(X \circ p_n, Y \circ p_n)
\]

for every \( n \geq 0 \), where \( p_n \) is the forgetful functor \( \Delta/[n] \to \Delta \). If \( A \in E \) and \( cA := sk^0(A) \), then

\[
[cA,X]_n = E(A,X_n)
\]

for every \( n \geq 0 \). The simplicial category \([\Delta^{op}, E]\) is tensored and cotensored by \( \Delta \). By construction, if \( X \in [\Delta^{op}, E] \) and \( K \) is a finite simplicial set, then

\[
(K \times X)_n = k_n \times X_n \quad (X^K)_n = \int_{[k] \in [n]} X^K_k
\]

The object \( M_n(X) := (X \partial [n])_n \) is called the \( n \)-th matching object of \( X \). If \( S(n) \) denotes the poset of non-empty proper subsets of \([n]\) then we have

\[
M_n(X) = \lim_{S(n)} X \circ s(n)
\]

where \( s(n) : S(n) \to \Delta \) is the canonical functor. From the inclusion \( \partial [n] \subset [n] \) we obtain a map \( X^{\Delta[n]} \to X^{\partial \Delta[n]} \) hence also a map \( X_n \to M_n(X) \).

If \( E \) is a model category, then a map \( f : X \to Y \) in \([\Delta^{op}, E]\) is called a **Reedy fibration** if the map \( X_n \to Y_n \times M_n(X) \) obtained from the square

\[
\begin{array}{ccc}
X_n & \to & M_n(X) \\
\downarrow f_n & & \downarrow M_n(f) \\
Y_n & \to & M_n(Y)
\end{array}
\]

is a fibration for every \( n \geq 0 \). There is then a model structure on the category \([\Delta^{op}, E]\) called the **Reedy model structure** whose fibrations are the Reedy fibrations and whose weak equivalences are the level-wise weak equivalences. A simplicial object \( X : \Delta^{op} \to E \) is Reedy fibrant if and only if the canonical map \( X_n \to M_n(X) \) is a fibration for every \( n \geq 0 \). The Reedy model structure is simplicial. If \( X \) is Reedy fibrant and \( A \in E \) then the simplicial set \( E(A,X) := [cA,X] \) is a Kan complex.
Definition B.2. Let $E$ be a model category. Then a simplicial object $Z : \Delta^{op} \to E$ is called a frame (see [Hov99]) if the following two conditions are satisfied:

1. $Z$ is Reedy fibrant;
2. $Z(f)$ is a weak equivalence for every map $f \in \Delta$.

The frame $Z$ is cofibrant if the canonical map $sk^0 Z_0 \to Z$ is a cofibration in the Reedy model structure. A coresolution of an object $X \in E$ is a frame $Fr(X) : \Delta^{op} \to E$ equipped with a weak equivalence $X \to Fr(X)_0$. Every fibrant object $X \in E$ has a (cofibrant) coresolution $Fr(X) : \Delta^{op} \to E$ with $Fr(X)_0 = X$. Let $E$ be a model category. If $A, X \in E$, then the homotopy mapping space $\mathcal{M}ap^h_E(A, X)$ is defined to be the simplicial set

$$\mathcal{M}ap^h_E(A, X) = E(A^c, Fr(X))$$

where $A^c \to A$ is a cofibrant replacement of $A$ and $Fr(X)$ is a coresolution of $X$. The simplicial set $E(A^c, Fr(X))$ is a Kan complex and it is homotopy unique. If $E$ is enriched over Kan complexes, if $A$ is cofibrant and $X$ is fibrant, then the simplicial set $\mathcal{M}ap^h_E(A, X)$ is homotopy equivalent to the simplicial set $[A, X]$ (see [Hir02]).

B.3. Function spaces for quasi-categories. If $C$ is a category, we shall denote by $J(C)$ the sub-category of invertible arrows in $C$. The sub-category $J(C)$ is the largest sub-groupoid of $C$. More generally, if $X$ is a quasi-category, we shall denote by $J(X)$ the largest sub- Kan complex of $X$. By construction, we have a pullback square

$$\begin{array}{ccc}
J(X) & \longrightarrow & X \\
\downarrow & & \downarrow h \\
J(\tau_1(X)) & \longrightarrow & \tau_1(X)
\end{array}$$

where $\tau_1(X)$ is the fundamental category of $X$ and $h$ is the canonical map. The function space $X^A$ is a quasi-category for any simplicial set $X$. We shall denote by $X^{(A)}$ the full sub-simplicial set of $X^A$ whose vertices are the maps $A \to X$ that factor through the inclusion $J(X) \subseteq X$. The simplicial set $X^{(\Delta[1])}$ is a path-space for $X$.

Lemma B.3. If $X$ is a quasi-category, then the simplicial object $P(X) \in [\Delta^{op}, sSet]$ defined by putting $P(X)_n = X^{(\Delta[n])}$ for every $n \geq 0$ is a cofibrant coresolution of $X$.

Proposition B.4. If $X$ is a quasi-category and $A$ is a simplicial set, then

$$\mathcal{M}ap^h_{sSets}(A, X) \simeq J(X^A).$$

Proof. Proof. By Lemma [B.3] we have

$$\mathcal{M}ap^h_{sSets}(A, X)_n = sSets(A, P(X)_n) = sSets(A, X^{(\Delta[n])})$$

But a map $f : A \to X^{\Delta[n]}$ factors through the inclusion $X^{(\Delta[n])} \subseteq X^{\Delta[n]}$ if and only if the transposed map $f^t : \Delta[n] \to X^A$ factors through the inclusion $J(XA) \subseteq X^A$. Thus, $sSets(A, X^{(\Delta[n])}) = sSets(\Delta[n], J(X^A)) = J(X^A)_n$ and this shows that $\mathcal{M}ap^h_{sSets}(A, X) \simeq J(X^A)$. □
Proposition B.5. Let $E$ be a model category enriched over quasi-categories. If
$A \in E$ is cofibrant and $X \in E$ is fibrant, then the function space $\text{Map}^h_{E}(A, X)$ is
equivalent to the Kan complex $J([A, X])$.

Proof. The functor $[A, -] : E \to \text{sSets}$ is a right Quillen functor with values in
the model category $(\text{sSets}, \text{Qcat})$, since $A$ is cofibrant. It thus takes a coresolution
$Fr(X)$ of $X \in E$ to a coresolution $[A, Fr(X)]$ of the quasi-category $[A, X]$. We have
$\text{Map}^h_{\text{sSets}}([A, X]) \simeq \text{sSets}(1, P([A, X]))$, since the simplicial set 1 is cofibrant.
By Lemma B.3 the quasi-category $[A, X]$ has a cofibrant coresolution $P([A, X])$.
We have $\text{Map}^h_{\text{sSets}}([A, X]) \simeq \text{sSets}(1, [A, Fr(X)])$, since the simplicial set 1 is cofibrant. There exists a level-wise weak categorical equivalence $\phi : P([A, X]) \to
[A, Fr(X)]$ such that the map $\phi(0)$ is the identity, since the coresolution $P([A, X])$
is cofibrant. Moreover, the map
$$\text{sSets}(1, \phi) : \text{sSets}(1, P([A, X])) \to \text{sSets}(1, [A, Fr(X)])$$
is a weak homotopy equivalence. But we have $\text{sSets}(1, P([A, X])) = J([A, X])$ by
lemma B.3. Moreover, $\text{sSets}(1, [A, Fr(X)]) = E(A, Fr(X))$, since
$$\text{sSets}(1, [A, Fr(X)])_n = \text{sSets}(1, [A, Fr(X)]_n) =$$
$$\text{sSets}(1, [A, Fr(X)]_n) = E(A, Fr(X)_n)$$
for every $n \geq 0$.

B.4. Local objects. Let $\Sigma$ be a set of maps in a model category $E$. An object
$X \in E$ is said to be $\Sigma$-local if the map
$$\text{Map}^h_{E}(u, X) : \text{Map}^h_{E}(A', X) \to \text{Map}^h_{E}(A, X)$$
is a homotopy equivalence for every map $u : A \to A'$ in $\Sigma$. Notice that if an
object $X$ is weakly equivalent to a $\Sigma$-local object, then $X$ is $\Sigma$-local. If the model
category $E$ is simplicial (=enriched over Kan complexes) and $\Sigma$ is a set of maps
between cofibrant objects, then a fibrant object $X \in E$ is $\Sigma$-local iff the map
$[u, X] : [A', X] \to [A, X]$ is a homotopy equivalence for every map $u : A \to A'$ in $\Sigma$.

Lemma B.6. Let $E$ be a model category enriched over quasi-categories. If $u : A \to B$ is a map between cofibrant objects, then the following conditions on a
fibrant object $X \in E$ are equivalent

1) the map $[u, X] : [B, X] \to [A, X]$ is a categorical equivalence;
2) the object $X$ is local with respect to the map $\Delta[n] \times u : \Delta[n] \times A \to \Delta[n] \times B$
for every $n \geq 0$.

Proof. (1 $\Rightarrow$ 2) The map $[u, X]|_{\Delta[n]} : [B, X]|_{\Delta[n]} \to [A, X]|_{\Delta[n]}$ is a categorical equivalence for every $n \geq 0$, since the map $[u, X]$ is a categorical equivalence by
the hypothesis. Hence the map $[\Delta[n] \times u, X]$ is a categorical equivalence, since $[\Delta[n] \times u, X] = [u, X]|_{\Delta[n]}$. It follows that the map $J([\Delta[n] \times u, X])$ is a homotopy
equivalence, since the functor $J : \text{QCat} \to \text{Kan}$ takes a categorical equivalences
to homotopy equivalences by [Joy08]. But we have $\text{Map}^h_{E}([\Delta[n] \times u, X]) =
J([\Delta[n] \times u, X])$ by Proposition B.3, since $\Delta[n] \times u$ is a map between cofibrant
objects. Hence the map $\text{Map}^h_{E}([\Delta[n] \times u, X)$ is a homotopy equivalence for
every $n \geq 0$. This shows that the object $X$ is local with respect to the map $\Delta[n] \times u$ for
every $n \geq 0$.

(1 $\Leftarrow$ 2) By Proposition B.3, we have $\text{Map}^h_{E}([\Delta[n] \times u, X]) = J([\Delta[n] \times u, X])$
for every $n \geq 0$, since $\Delta[n] \times u$ is a map between cofibrant objects. Hence the
map \( J([\Delta[n] \times u, X]) \) is a homotopy equivalence for every \( n \geq 0 \). But we have 
\[ \Delta[n] \times u, X = [u, X][\Delta[n]]. \]
Hence the map \( J([u, X][\Delta[n]]) \) is a homotopy equivalence for every \( n \geq 0 \). By Theorem 4.11 and Proposition 4.10 of [JT08] a map between 
quasi-categories \( f : U \rightarrow V \) is a categorical equivalence if and only if the map 
\( J(f[\Delta[n]]) : J(U[\Delta[n]]) \rightarrow J(V[\Delta[n]]) \) is a homotopy equivalence for every \( n \geq 0 \). This 
shows that the map \([u, X]\) is a categorical equivalence. \( \square \)
Appendix C. The strict JQ-model category of normalized Γ-spaces

A normalized Γ-space is a functor \( X : \Gamma^{op} \to s\text{Sets}_\bullet \) such that \( X(0^+) = 1 \). The category of all (small) normalized Γ-spaces \( \Gamma S_\bullet \) is the category whose objects are normalized Γ-spaces. This category is defined by the following equilizer diagram in \( \text{Cat} \):

\[
\begin{array}{ccc}
\Gamma S_\bullet & \longrightarrow & [\Gamma^{op}; s\text{Sets}_\bullet] \\
\downarrow & & \downarrow \\
0 & \longrightarrow & [1; s\text{Sets}_\bullet]
\end{array}
\]

where \([0; s\text{Sets}_\bullet]\) is the functor which precomposes a functor in \([\Gamma^{op}; s\text{Sets}_\bullet]\) with the unique (pointed) functor \( 1 \to \Gamma^{op} \) whose image is \( 0^+ \in \Gamma^{op} \) and the upward diagonal functor \( 0 \) maps the terminal category \( 1 \) to the identity functor on the terminal simplicial sets. In this appendix we will describe a model category structure on the category \( \Gamma S_\bullet \), which is a version of the projective model category structure for the category of basepoint preserving functors.

Definition C.1. A morphism \( F : X \to Y \) of Γ-spaces is called

1. a strict JQ equivalence of normalized Γ-spaces if it is degreewise weak equivalence in the Joyal model category structure on \( s\text{Sets}_\bullet \) i.e. \( F(n^+) : X(n^+) \to Y(n^+) \) is a weak categorical equivalence of (pointed) simplicial sets.
2. a strict JQ fibration of normalized Γ-spaces if it is degreewise a fibration in the Joyal model category structure on \( s\text{Sets}_\bullet \) i.e. \( F(n^+) : X(n^+) \to Y(n^+) \) is a pseudo-fibration of (pointed) simplicial sets.
3. a JQ-cofibration of normalized Γ-spaces if it has the left lifting property with respect to all morphisms which are both strict JQ weak equivalence and strict JQ fibrations of normalized Γ-spaces.

In light of proposition ?? we observe that a map of Γ-spaces \( F : X \to Y \) is a strict acyclic fibration of Γ-spaces if and only if it has the right lifting property with respect to all maps in the set

\[
\mathcal{I} = \{ \Gamma^n \otimes_{s\text{Sets}_\bullet} \partial_0, \Gamma^n \wedge_{s\text{Sets}_\bullet} \partial_1, \Gamma^n \wedge_{s\text{Sets}_\bullet} \partial_2 \mid \forall n \in Ob(\mathcal{N}) \}.
\]

We further observe, in light of proposition ??, that \( F \) is a strict fibration if and only if it has the right lifting property with respect to all maps in the set

\[
\mathcal{J} = \{ \Gamma^n \wedge_{s\text{Sets}_\bullet} i_0, \Gamma^n \wedge_{s\text{Sets}_\bullet} i_1 \mid \forall n \in Ob(\mathcal{N}) \}.
\]

Remark. The category Γ-space is a locally presentable category. The small object argument (for presentable categories), [Lur09, Proposition A.1.2.5], implies that the sets \( \mathcal{I} \) and \( \mathcal{J} \) provide two functorial factorization systems on the category Γ-space. The first one is the strict acyclic fibration of Γ-spaces followed by a strict acyclic fibration of Γ-spaces and the second functorial factorization system factors each morphism in Γ-space into a composite of a strict acyclic cofibration of Γ-spaces followed by a strict fibration of Γ-spaces.
The main aim of this subsection is to construct a model category structure on the category of all $\Gamma$-spaces $\Gamma$-space whose three classes of morphisms are the ones defined above. We will refer to this model structure as the \textit{strict model category structure} on $\Gamma$-space and will refer to the model category as the \textit{strict model category} of $\Gamma$-spaces.

**Theorem C.2.** Strict JQ equivalences, strict JQ fibrations and JQ-cofibrations of $\Gamma$-spaces provide the category $\Gamma S_{\bullet}$ with a combinatorial model category structure.

**Proof.** The category of all functors from $\Gamma^{\text{op}}$ to $sSets_{\bullet}$, namely $[\Gamma^{\text{op}}, sSets_{\bullet}]$ has a model category structure, called the \textit{projective model category structure}, in which a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) degreewise, see [Lur09, Prop. A.3.3.2] for a proof. The category $\Gamma$-space $= [\Gamma^{\text{op}}, sSets_{\bullet}]$ is a subcategory of $[\Gamma^{\text{op}}, sSets_{\bullet}]$ this implies that the axioms $CM(2), CM(3)$ and $CM(4)$, see [Qui67], [GJ99] Chap. 2) are satisfied by $\Gamma$-space because they are satisfied by the projective model category $[\Gamma^{\text{op}}, sSets_{\bullet}]$. Finally, $CM(5)$ follows from remark C above. The category $\Gamma$-space is locally presentable. The sets $I$ and $J$ defined above form the sets of generating cofibrations and generating acyclic cofibrations respectively of the strict model category structure. $\square$

**Notation C.3.** We will refer to the above model category as the \textit{strict JQ model category of normalized $\Gamma$-spaces} and we denote it by $\Gamma S_{\bullet}^{\text{str}}$.

We recall that the \textit{smash product} of two (pointed) simplicial sets $(X,x)$ and $(Y,y)$, where the simplicial maps $x : 1 \to X$ and $y : 1 \to Y$ specify the respective basepoints, is defined by the following pushout square:

$$
\begin{array}{ccc}
X \vee Y & \longrightarrow & X \times Y \\
\downarrow & & \downarrow \\
1 & \longrightarrow & X \wedge Y
\end{array}
$$

where the top horizontal arrow is the canonical map between the coproduct and product of the two (pointed) simplicial sets. To any pair of objects $(X,C) \in \text{Ob}(\Gamma S_{\bullet}) \times \text{Ob}(sSets_{\bullet})$ we can assign a $\Gamma$-space $X \otimes_{sSets_{\bullet}} C$ which is defined in degree $n$ as follows:

$$(X \otimes_{sSets_{\bullet}} C)(n^+) := X(n^+) \wedge C,$$

where the pointed category on the right is the smash product of (pointed) simplicial sets, see [13]. This assignment is functorial in both variables and therefore we have a bifunctor

$$- \otimes_{sSets_{\bullet}} - : \Gamma S_{\bullet} \times sSets_{\bullet} \to \Gamma S_{\bullet}.$$ 

Next, we define a couple of function objects for the category $\Gamma$-space. The first function object enriches the category $\Gamma S_{\bullet}$ over $sSets_{\bullet}$, i.e. there is a bifunctor

$$\text{Map}_{\Gamma S_{\bullet}}(-, -) : \Gamma\text{-space}^{op} \times \Gamma S_{\bullet} \to sSets_{\bullet},$$

which assigns to any pair of objects $(X,Y) \in \text{Ob}(\Gamma S_{\bullet}) \times \text{Ob}(\Gamma S_{\bullet})$, a pointed simplicial set $\text{Map}_{\Gamma S_{\bullet}}(X,Y)$ which is defined in degree zero as follows:

$$\text{Map}_{\Gamma S_{\bullet}}(X,Y)_0 := \Gamma S_{\bullet}(X,Y).$$
The mapping simplicial set is defined in degree $n$ as follows:

$$\text{Map}_{\Gamma_S}(X,Y)_n := \Gamma_S(X \wedge \Delta[n]^+, Y)$$

For any $\Gamma$-space $X$, the functor $X \otimes_{\text{sSets}_*} - : \text{sSets}_* \to \Gamma_S$ is left adjoint to the functor $\text{Map}_{\Gamma_S}(X, -) : \Gamma_S \to \text{sSets}_*$. The counit of this adjunction is the evaluation map $ev : X \otimes_{\text{sSets}_*} \text{Map}_{\Gamma_S}(X,Y) \to Y$ and the unit is the obvious functor $C \to \text{Map}_{\Gamma_S}(X, X \otimes_{\text{sSets}_*} C)$, where $Y$ is a normalized $\Gamma$-space and $C$ is a pointed simplicial set.

The mapping object $\text{Map}_{\Gamma_S}(X,Y)$ is a (pointed) simplicial set whose base-point is the composite map $X \to \Gamma_0 \to Y$, where $\Gamma_0$ is the zero object in $\Gamma_S$.

Let $U(\text{Map}_{\Gamma_S}(X,Y))$ denote the simplicial set obtained by forgetting the base-point of $\text{Map}_{\Gamma_S}(X,Y)$. We also recall the forgetful functor $U$ which forgets the normalization of a $\Gamma$-space, see (6).

**Lemma C.4.** Let $X$ and $Y$ be two normalized $\Gamma$-spaces. The mapping simplicial set $U(\text{Map}_{\Gamma_S}(X,Y))$ is an equilizer of the following diagram:

$$\begin{array}{ccc}
\text{Map}_{\Gamma_S}(U(X),U(Y)) & \longrightarrow & \text{Map}_{\Gamma_S}(\Gamma^0,U(Y)) \\
\downarrow & & \downarrow 1 \\
0 & & 0
\end{array}$$

**Proof.** Each normalized $\Gamma$-space $X$ uniquely determines a morphism $0_X : \Gamma^0 \to U(X)$. It is sufficient to observe that a morphism $f : U(X) \to U(Y)$ lies in the image of the forgetful functor $U$ if and only if the following diagram commutes:

$$\begin{array}{ccc}
\Gamma^0 & \xrightarrow{0_X} & U(X) \\
\downarrow f & & \downarrow 0_Y \\
U(Y) & \xleftarrow{0_Y} & U(Y)
\end{array}$$

□

**Corollary C.5.** For each pair of normalized $\Gamma$-spaces $X$ and $Y$ we have the following canonical isomorphism of mapping simplicial sets

$$U(\text{Map}_{\Gamma_S}(X,Y)) \cong \text{Map}_{\Gamma_S}(U(X),U(Y)).$$

**Proof.** It is sufficient to observe that for any normalized $\Gamma$-space $Y$, the Yoneda’s lemma tells us that the mapping simplicial set $\text{Map}_{\Gamma_S}(\Gamma^0,U(Y)) \cong 1$.

□

To each pair of objects $(C, X) \in \text{Ob(\text{sSets}_*)} \times \text{Ob(\Gamma\text{-space})}$ we can assign a $\Gamma$-space $X^C$ which is defined in degree $n$ as follows:

$$(X^C)_n := X(n)^+_C$$

where the (pointed) simplicial set on the right is defined by the following equilizer diagram:

$$\begin{array}{ccc}
\text{Map}_{\Gamma_S}(\Delta[n]^+,Y) & \longrightarrow & \text{Map}_{\Gamma_S}(\Gamma^0,Y) \\
\downarrow & & \downarrow 1 \\
0 & & 0
\end{array}$$

This assignment is functorial in both variable and therefore we have a bifunctor

$$- : \text{sSets}_{*\times} \times \Gamma_S \to \Gamma_S.$$
For any $\Gamma$-space $X$, the functor $X^- : \mathbf{sSets}_\bullet \to \Gamma$-space$^{op}$ is left adjoint to the functor $\mathcal{M}ap_{\Gamma}$-space$(-,X) : \Gamma$-space$^{op} \to \mathbf{sSets}_\bullet$.

The following proposition summarizes the above discussion.

**Proposition C.6.** There is an adjunction of two variables

$$(\mathbf{sSets}_\bullet, -, -) : \mathcal{M}ap_{\Gamma}$-space$(-,X) : \Gamma$-space$^{op} \to \mathbf{sSets}_\bullet$$

**Definition C.7.** Given model categories $C$, $D$ and $E$, an adjunction of two variables, $\left(\otimes, \text{hom}_{C}, \mathcal{M}ap_{C}, \phi, \psi\right) : C \times D \to E$, is called a *Quillen adjunction of two variables*, if, given a cofibration $f : U \to V$ in $C$ and a cofibration $g : W \to X$ in $D$, the induced map

$$f \Box g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration in $E$ that is trivial if either $f$ or $g$ is. We will refer to the left adjoint of a Quillen adjunction of two variables as a *Quillen bifunctor*.

The following lemma provides three equivalent characterizations of the notion of a Quillen bifunctor. These will be useful in this paper in establishing enriched model category structures.

**Lemma C.8.** [Hov99, Lemma 4.2.2] Given model categories $C$, $D$ and $E$, an adjunction of two variables, $\left(\otimes, \text{hom}_{C}, \mathcal{M}ap_{C}, \phi, \psi\right) : C \times D \to E$. Then the following conditions are equivalent:

1. $\otimes : C \times D \to E$ is a Quillen bifunctor.
2. Given a cofibration $g : W \to X$ in $D$ and a fibration $p : Y \to Z$ in $E$, the induced map

$$\text{hom}_{C}(g,p) : \text{hom}_{C}(X,Y) \to \text{hom}_{C}(X,Z) \times \text{hom}_{C}(W,Y)$$

is a fibration in $C$ that is trivial if either $g$ or $p$ is a weak equivalence in their respective model categories.
3. Given a cofibration $f : U \to V$ in $C$ and a fibration $p : Y \to Z$ in $E$, the induced map

$$\mathcal{M}ap_{C}(f,p) : \mathcal{M}ap_{C}(V,Y) \to \mathcal{M}ap_{C}(V,Z) \times \mathcal{M}ap_{C}(W,Y)$$

is a fibration in $C$ that is trivial if either $f$ or $p$ is a weak equivalence in their respective model categories.

**Definition C.9.** Let $S$ be a monoidal model category. An *$S$-enriched model category* is an $S$ enriched category $A$ equipped with a model category structure (on its underlying category) such that there is a Quillen adjunction of two variables, see definition $\left(\otimes, \text{hom}_{A}, \mathcal{M}ap_{A}, \phi, \psi\right) : A \times S \to A$.

**Theorem C.10.** The strict model category of $\Gamma$-spaces, $\Gamma$-space, is an $\mathbf{sSets}_\bullet$-enriched model category.

**Proof.** We will show that the adjunction of two variables $\left(\mathbf{sSets}_\bullet, -, -\right)$ is a Quillen adjunction for the strict model category structure on $\Gamma$-space and the natural model category structure on $\mathbf{sSets}_\bullet$. In order to do so, we will verify condition (2) of Lemma C.8.
Let \( g : C \to D \) be a cofibration in \( \mathbf{sSets}_\bullet \) and let \( p : Y \to Z \) be a strict fibration of \( \Gamma \)-spaces, we have to show that the induced map

\[
\text{hom}_{\Gamma\text{-space}}(g,p) : Y^X \to Z^D \times_{Z^C} Y^C
\]

is a fibration in \( \mathbf{sSets}_\bullet \) which is acyclic if either of \( g \) or \( p \) is acyclic. It would be sufficient to check that the above morphism is degreewise a fibration in \( \mathbf{sSets}_\bullet \), i.e. for all \( n^+ \in \Gamma^{op} \), the morphism

\[
\text{hom}_{\Gamma\text{-space}}(g,p)(n^+) : [D,Y(n^+)]_\bullet \to [D,Z(n^+)]_\bullet \times_{[C,Z(n^+)]_\bullet} [C,Y(n^+)]_\bullet
\]

is a fibration in \( \mathbf{sSets}_\bullet \). This follows from the observations that the functor \( p(n^+) : Y(n^+) \to Z(n^+) \) is a fibration in \( \mathbf{sSets}_\bullet \) and the natural model category \( \mathbf{sSets}_\bullet \) is a \( \mathbf{sSets}_\bullet \)-enriched model category whose enrichment is provided by the bifunctor \([−,−]_\bullet\). □

The adjunction \(-^+ : \mathbf{Cat} \rightleftarrows \mathbf{sSets}_\bullet : U\) provides us with an enrichment of the strict model category of \( \Gamma \)-spaces, \( \Gamma\text{-space} \), over the natural model category of all (small) categories \( \mathbf{Cat} \).

**Corollary C.11.** The strict model category of \( \Gamma \)-spaces, \( \Gamma\text{-space} \), is a \( \mathbf{Cat} \)-enriched model category.

**C.1. The JQ-model category of normalized \( \Gamma \)-spaces.** The objective of this subsection is to construct a new model category structure on the category \( \Gamma\mathcal{S}_\bullet \). This new model category is obtained by localizing the strict JQ-model category of normalized \( \Gamma \)-spaces (see section C) and we refer to it as the \( \text{JQ-model category of normalized } \Gamma \text{-spaces} \). We go on further to show that this new model category is symmetric monoidal closed with respect to the smash product which is a categorical version of the smash product constructed in [Lyd99].

**Notation C.12.** We denote by \( 1/\Gamma\mathcal{S} \) the overcategory whose objects are maps in \( \Gamma\mathcal{S} \) having domain the terminal \( \Gamma \)-space \( 1 \). We denote by \( (1/\Gamma\mathcal{S})_\bullet \) the subcategory of \( 1/\Gamma\mathcal{S} \) whose objects are those maps \( 1 \to X \) in \( \Gamma\mathcal{S} \) whose codomain \( \Gamma \)-space satisfies the following normalization condition:

\[
X(0^+) = *.
\]

**Remark.** We observe that the category of normalized pointed objects \( (1/\Gamma\mathcal{S})_\bullet \) is isomorphic to the category of normalized \( \Gamma \)-spaces \( \Gamma\mathcal{S}_\bullet \).

We want to construct a left Bousfield localization of the strict model category of \( \Gamma \)-spaces. For each pair \( k^+, l^+ \in \Gamma^{op} \), we have the obvious projection maps in \( \Gamma^{op} \)

\[
\delta_k^{k+l} : (k+l)^+ \to k^+ \text{ and } \delta_l^{k+l} : (k+l)^+ \to l^+.
\]

The following two inclusion maps between representable \( \Gamma \)-spaces

\[
\Gamma^{op}(\delta_k^{k+l}, -) : \Gamma^k \to \Gamma^{k+l} \text{ and } \Gamma^{op}(\delta_l^{k+l}, -) : \Gamma^l \to \Gamma^{k+l}
\]

induce a pair of maps of \( \Gamma \)-spaces on the coproduct which we denote as follows:

\[
h_k^l : \Gamma^l \vee \Gamma^l \to \Gamma^{l+k}.
\]

We now define a set of maps \( \mathcal{E}_{\infty}\mathcal{S}_\bullet \) in \( \Gamma\mathcal{S}_\bullet \):

\[
\mathcal{E}_{\infty}\mathcal{S}_\bullet := \{ h_k^l : \Gamma^l \vee \Gamma^l \to \Gamma^{l+k} : l, k \in \mathbb{Z}^+ \}
\]
Next we define the set of arrows in $\Gamma S_\bullet$ with respect to which we will localize the strict $JQ$-model category of normalized $\Gamma$-spaces:

$$\Delta \times E_\infty S_\bullet := \{\Delta[n]^{+} \otimes h_k^i : h_k^i \in \Delta \times E_\infty S_\bullet\}$$

**Definition C.13.** We call a $\Gamma$-space $X$ a $(\Delta \times E_\infty S_\bullet)$-local object if it is a fibrant object in the strict $JQ$-model category of normalized $\Gamma$-spaces and for each map $\Delta[n]^{+} \otimes h_k^i \in \Delta \times E_\infty S_\bullet$, the induced simplicial map

$$\map_{\Gamma S_\bullet}^h(\Delta[n]^{+} \otimes h_k^i, X) : \map_{\Gamma S_\bullet}^h(\Delta[n]^{+} \otimes \Gamma^{k+l}, X) \to \map_{\Gamma S_\bullet}^h(\Delta[n]^{+} \otimes (\Gamma^k \lor \Gamma^l), X),$$

is a homotopy equivalence of simplicial sets for all $n \geq 0$ where $\map_{\Gamma S_\bullet}^h(-, -)$ is the simplicial function complexes associated with the strict model category $\Gamma S_\bullet$, see [DK80a], [DK80c] and [DK80b].

Remark (??) above and appendix [B] tell us that a model for $\map_{\Gamma S_\bullet}^h(X,Y)$ is the Kan complex $J(\map_{\Gamma S_\bullet}(X,Y))$ which is the maximal Kan complex contained in the quasicategory $\map_{\Gamma S_\bullet}(X,Y)$.

The following proposition gives a characterization of $(\Delta \times E_\infty S_\bullet)$-local objects

**Proposition C.14.** A normalized $\Gamma$-space $X$ is a $(\Delta \times E_\infty S_\bullet)$-local object if and only if it satisfies the Segal condition namely the functor

$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^{+}) \to X(k^{+}) \times X(l^{+})$$

is an equivalence of (pointed) quasi-categories for all $k^{+}, l^{+} \in Ob(\Gamma^{op})$.

**Proof.** We begin the proof by observing that each element of the set $\Delta \times E_\infty S_\bullet$ is a map of $\Gamma$-spaces between cofibrant $\Gamma$-spaces. Theorem [B.6] implies that $X$ is a $(\Delta \times E_\infty S_\bullet)$-local object if and only if the following map of simplicial sets

$$\map_{\Gamma S_\bullet}(h_k^i, X) : \map_{\Gamma S_\bullet}(\Gamma^{k+l}, X) \to \map_{\Gamma S_\bullet}(\Gamma^k \lor \Gamma^l, X)$$

is an equivalence of quasi-categories.

We observe that we have the following commutative square in $(sSets, Q)$

$$\begin{array}{ccc}
\map_{\Gamma S_\bullet}(\Gamma^{k+l}, X) & \xrightarrow{\map_{\Gamma S_\bullet}(h_k^i, X)} & \map_{\Gamma S_\bullet}(\Gamma^k \lor \Gamma^l, X) \\
\simeq & \quad \downarrow \simeq \\
X((k+l)^{+}) & \xrightarrow{(X(\delta^{k+l}_k), X(\delta^{k+l}_l))} & X(k^{+}) \times X(l^{+})
\end{array}$$

By the two out of three property of weak equivalences in a model category the simplicial map $(X(\delta^{k+l}_k), X(\delta^{k+l}_l))$ is an equivalence of quasi-categories if and only if the map $\map_{\Gamma S_\bullet}(h_k^i, X)$ is an equivalence of quasi-categories. $\square$

**Definition C.15.** We will refer to a $(\Delta \times E_\infty S_\bullet)$-local object as a normalized coherently commutative monoidal quasi-category.
Definition C.16. A morphism of normalized Γ-spaces \( F : X \to Y \) is a \((\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)\)-local equivalence if for each normalized coherently commutative monoidal quasi-category \( Z \) the following simplicial map

\[
\text{Map}_{\Gamma \mathcal{S}_\bullet}(F, Z) : \text{Map}_{\Gamma \mathcal{S}_\bullet}(Y, Z) \to \text{Map}_{\Gamma \mathcal{S}_\bullet}(X, Z)
\]

is a homotopy equivalence of simplicial sets. We may sometimes refer to a \((\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)\)-local equivalence as an equivalence of normalized coherently commutative monoidal quasi-categories.

An argument similar to the proof of proposition 4.8 proves the following proposition:

Proposition C.17. A morphism between two \( JQ \)-cofibrant normalized Γ-spaces \( F : X \to Y \) is an \((\Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet)\)-local equivalence if and only if the simplicial map

\[
\text{Map}_{\Gamma \mathcal{S}_\bullet}(F, Z) : \text{Map}_{\Gamma \mathcal{S}_\bullet}(Y, Z) \to \text{Map}_{\Gamma \mathcal{S}_\bullet}(X, Z)
\]

is an equivalence of quasi-categories for each normalized coherently commutative monoidal quasi-category \( Z \).

The main objective of the current subsection is to construct a new model category structure on the category of normalized Γ-spaces \( \Gamma \mathcal{S}_\bullet \) by localizing the strict \( JQ \)-model category of normalized Γ-spaces with respect to morphisms in the set \( \Delta \times \mathcal{E}_\infty \mathcal{S}_\bullet \). The desired model structure follows from theorem ??.

Theorem C.18. There is a closed, left proper, combinatorial model category structure on the category of normalized Γ-spaces \( \Gamma \mathcal{S}_\bullet \), in which

1. The class of cofibrations is the same as the class of \( JQ \)-cofibrations of normalized Γ-spaces.
2. The weak equivalences are equivalences of normalized coherently commutative monoidal quasi-categories.

An object is fibrant in this model category if and only if it is a normalized coherently commutative monoidal quasi-category. Further, this model category structure makes \( \Gamma \mathcal{S}_\bullet \) a closed symmetric monoidal model category under the smash product.

Proof. The strict \( JQ \)-model category of normalized Γ-spaces is a combinatorial model category therefore the existence of the model structure follows from theorem 4.10. The statement characterizing fibrant objects also follows from theorem 4.10. An argument similar to the proof of theorem 4.18 using the enrichment of the strict \( JQ \)-model category of normalized Γ-spaces over the \((\mathcal{sSets}_\bullet, \mathcal{Q})\) established in proposition C.6 shows that the localized model category has a symmetric monoidal closed model category structure under the smash product.

Notation C.19. The model category constructed in theorem C.18 will be referred to either as the \( JQ \)-model category of normalized Γ-spaces or as the model category of normalized coherently commutative monoidal quasi-categories.
COHERENTLY COMMUTATIVE MONOIDAL QUASI-CATEGORIES

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