NUMERICAL ANALYSIS OF A MODIFIED SMAGORINSKY MODEL

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Abstract. The classical Smagorinsky model’s solution is an approximation to a (resolved) mean velocity. Since it is an eddy viscosity model, it cannot represent a flow of energy from unresolved fluctuations to the (resolved) mean velocity. This model has recently been modified to incorporate this flow and still be well-posed. Herein we first develop some basic properties of the modified model. Next, we perform a complete numerical analysis of two algorithms for its approximation. They are tested and proven to be effective. 

Key words. Eddy Viscosity, Modified Smagorinsky, Complex turbulence, Backscatter

AMS subject classifications. 65M06, 65M12, 65M22, 65M60, 76M10

1. Introduction. Consider the Smagorinsky model [33], with prescribed body force \( f \), kinematic viscosity \( \nu \) in the regular and bounded flow domain \( \Omega \subset \mathbb{R}^d (d = 2, 3) \), which was later advanced independently by Ladyzhenskaya [19,20]:

\[
\nabla \cdot w = 0 \quad \text{and} \quad w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_s \delta)^2 |\nabla w| \nabla w \right) = f(x).
\]

Here \((w, q)\) approximate an ensemble average of Navier-Stokes solutions, \((u, p)\). This is an eddy viscosity model with turbulent viscosity, \( \nu_T = (C_s \delta)^2 |\nabla w| \), where \( C_s \approx 0.1 \), Lilly [24], \( \delta \) is a length scale (or grid scale). Like all eddy viscosity models, the Smagorinsky model represents a flow of energy from means to unresolved fluctuations \((u' = u - \bar{u})\), for a precise formula see Definition 2.14) and has errors by not representing any intermittent energy flow from fluctuations back to means. Corrections have recently been made representing this flow in Jiang and Layton [15] and Rong, Layton and Zhao [32]. Following their ideas, we develop a modified model in section 3. We also analyze and test numerical algorithms for effective approximation of the resulting modified model: 

\[
\nabla \cdot w = 0 \quad \text{and} \quad w_t - C_s^2 \delta^2 \mu^{-2} \Delta w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left( (C_s \delta)^2 |\nabla w| \nabla w \right) = f(x).
\]

Here \( \mu \) is a constant from Kolmogorov-Prandtl relation [18,31].

The main result of this paper is the complete numerical analysis and computational testing of effective algorithms for this model. This paper gives detailed numerical analyses in section 4 and section 5. This model is able to capture the phenomenon of transferring energy from fluctuation to means, which is tested numerically in subsection 6.2. There were few attempts made for extending model that represents flow at statistical equilibrium to non-equilibrium. For instance, in a previous work by Jiang and Layton [15], there was a fitting parameter \( \beta \) in the second term of (1.2) which is needlessly complicated. But in our paper, we use a different idea to model it, which results in a simpler model with no fitting parameter.

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1 The mechanically correct formulation is with the \( \nabla^s w \) instead of \( \nabla w \) in the term \( -\nabla \cdot \left( (C_s \delta)^2 |\nabla w| \nabla w \right) \) where \( \nabla^s \) is the symmetric part of the gradient tensor. But since the estimates are same and analyses are simpler with \( \nabla w \) due to Korn’s inequality \( \|v\|_{H^1(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)} + \|\nabla^s v\|_{L^2(\Omega)} \), we use \( \nabla w \) throughout the paper.

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1.1. Previous work. For simulating turbulent flow, there are different approaches, see [11, 12, 26, 29, 35, 36]. A summary of some recent work in eddy viscosity models of turbulence is presented in [16]. One of the recent approaches is by adding a term of Kelvin-Voigt form to the equations for the mean-field [1]. Recently, Rong, Layton and Zhao [32] and Berselli, Lewandowski and Nguyen [3] all studied the extension of the Baldwin & Lomax model [2] to non-equilibrium \( \frac{d}{dt} \|u'\|^2 \neq 0 \), for a precise definition see Remark 2.17) problems. A variant of the Smagorinsky model and detailed analysis is presented in the paper [7]. Jiang and Layton [15] derived a corrected eddy viscosity model for flow not at statistical equilibrium state.

2. Notation and Preliminaries. In this section, we introduce some of the notations and results used in this paper. We denote by \( \|\cdot\| \) and \((\cdot, \cdot)\) the \( L^2(\Omega) \) norm and inner product, respectively. We denote the \( L^p(\Omega) \) norm by \( \|\cdot\|_{L^p(\Omega)} \).

The solution spaces \( X \) for the velocity and \( Q \) for the pressure are defined as:

\[
X := \{ \mathbf{v} \in L^3(\Omega) : \nabla \mathbf{v} \in L^3(\Omega) \text{ and } \mathbf{v} = 0 \text{ on } \partial \Omega \},
\]

\[
Q := L^2(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \},
\]

and \( V := \{ \mathbf{v} \in X : (q, \mathbf{v} \cdot \mathbf{n}) = 0, \forall q \in Q \} \).

The space \( H^{-1}(\Omega) \) denotes the dual space of bounded linear functionals defined on \( H^1_0(\Omega) = \{ \mathbf{v} \in H^1(\Omega) : \mathbf{v} = 0 \text{ on } \partial \Omega \} \) and this space is equipped with the norm:

\[
\|f\|_{-1} = \sup_{0 \neq \mathbf{v} \in X} \frac{(f, \mathbf{v})}{\|\nabla \mathbf{v}\|} \quad \forall f \in H^{-1}(\Omega).
\]

The finite element method for this problem involves picking finite element spaces \[22\] \( X^h \subset X \) and \( Q^h \subset Q \). We assume that \((X^h, Q^h)\) satisfies the discrete inf-sup condition:

\[
\inf_{\lambda^h \in Q^h} \sup_{\mathbf{v}^h \in X^h} \frac{(\lambda^h, \mathbf{v}^h)}{\|\lambda^h\| \|\nabla \mathbf{v}^h\|} \geq \beta^h > 0,
\]

where \( \beta^h \) is bounded away from zero uniformly in \( h \).

**Definition 2.1.** (Trilinear Form) Define the trilinear form \( b^* : X \times X \times X \to \mathbb{R} \) as follows

\[
b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} (\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w}) - \frac{1}{2} (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}).
\]

**Lemma 2.2.** The nonlinear term \( b^*(\cdot, \cdot, \cdot) \) is continuous on \( X \times X \times X \) (and thus on \( V \times V \times V \)) which has the following skew-symmetry property,

\[
b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b^*(\mathbf{u}, \mathbf{w}, \mathbf{v}).
\]

As a consequence, we get

\[
b^*(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}), \quad \forall \mathbf{u} \in V \text{ and } \mathbf{v}, \mathbf{w} \in X,
\]

\[
b^*(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall \mathbf{u}, \mathbf{v} \in X.
\]

**Proof.** Proof of this lemma is standard, see p.114 of Girault and Raviart [13].
LEMMA 2.3. For any \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in X \) and \( p, q, r \) such that for \( 1 \leq p, q, r \leq \infty \), with \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \),

\[
\int_{\Omega} |\mathbf{u}| |\mathbf{v}| |\mathbf{w}| \, dx \leq \|\mathbf{u}\|_{L^p(\Omega)} \|\mathbf{v}\|_{L^q(\Omega)} \|\mathbf{w}\|_{L^r(\Omega)}.
\]

There is a constant \( C = C(\Omega) \) such that, in 2d and 3d,

\[
\left\| \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \, dx \right\| \leq C\|\mathbf{u}\|\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|, \quad \text{and}
\]

\[
\left\| \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} \, dx \right\| \leq C\|\mathbf{u}\|^{1/2}\|\nabla \mathbf{u}\|^{1/2}\|\nabla \mathbf{v}\|\|\nabla \mathbf{w}\|.
\]

LEMMA 2.4. (Polarization identity)

\[
(u, v) = \frac{1}{2} \|u\|^2 + \frac{1}{2} \|v\|^2 - \frac{1}{2} \|u - v\|^2.
\]

LEMMA 2.5. (The Poincaré-Friedrichs’ inequality) There is a positive constant \( C_{PF} = C_{PF}(\Omega) \) such that

\[
\|u\| \leq C_{PF}\|\nabla u\|, \quad \forall u \in X.
\]

LEMMA 2.6. (A Sobolev inequality) Let \( \Omega \) be a bounded open set and suppose \( \nabla u \in L^2(\Omega) \) with \( u = 0 \) on a subset of \( \partial \Omega \) with positive measure. Then there is a constant \( C = C(\Omega) \) such that

\[
\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|,
\]

for \( 1 \leq p^* < \infty \) in 2d and \( 1 \leq p^* \leq 6 \) in 3d.

LEMMA 2.7. (Young’s inequality) For any \( \epsilon, 0 < \epsilon < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1, \ 1 \leq p, q \leq \infty \),

\[
(u, v) \leq \frac{\epsilon}{p}\|u\|_{L^p(\Omega)}^p + \frac{\epsilon^{-a/p}}{q}\|v\|_{L^q(\Omega)}^q.
\]

LEMMA 2.8. (Hölder’s inequality) Let \( 1 \leq p, q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then if \( u \in L^p(\Omega) \) and \( v \in L^q(\Omega) \), the product \( uv \) belongs to \( L^1(\Omega) \), and

\[
(u, v) \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.
\]

Remark 2.9. For \( p = 2 \) and \( q = 2 \) in Hölder’s inequality, we can derive the Cauchy-Schwarz inequality, i.e., for any \( u, v \in L^2(\Omega) \),

\[
(u, v) \leq \|u\|\|v\|.
\]

Next is a Discrete Gronwall lemma see Lemma 5.1 p.369 [14].

LEMMA 2.10. Let \( \Delta t, B, a_n, b_n, c_n, d_n \) for integers \( n \geq 0 \) be nonnegative numbers such that for \( l \geq 1 \), if

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l-1} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + B, \quad \text{for } l \geq 0,
\]

then for all \( \Delta t > 0 \),

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp \left( \Delta t \sum_{n=0}^{l-1} d_n \right) \left( \Delta t \sum_{n=0}^{l} c_n + B \right), \quad \text{for } l \geq 0.
\]
In this paper, we will need this following well-known lemma, see, e.g., \cite{10, 17, 21}

**Lemma 2.11.** *(Strong Monotonicity (SM) and Local Lipschitz Continuity (LLC))*

There exists $C_1$, $C_2 > 0$ such that for all $u, v, w \in L^3(\Omega)$, $\nabla u, \nabla v, \nabla w \in L^3(\Omega)$

\begin{align}
(\text{SM}) & \quad (|\nabla u| \nabla u - |\nabla w| \nabla w, \nabla (u - w)) \geq C_1 \|\nabla (u - w)\|^2_{L^3(\Omega)}, \\
(\text{LLC}) & \quad (|\nabla u| \nabla u - |\nabla w| \nabla w, \nabla v) \leq C_2 r \|\nabla (u - w)\|_{L^3(\Omega)} \|\nabla v\|_{L^3(\Omega)},
\end{align}

where $r = \max\{\|\nabla u\|_{L^3(\Omega)}, \|\nabla w\|_{L^3(\Omega)}\}$.

**Proposition 2.12.** *(see p.173 \cite{5})* Let $W^{m,p}(\Omega)$ denote the Sobolev space, let $p \in [1, +\infty]$ and $q \in [p, p']$, where $\frac{1}{p'} = \frac{1}{p} - \frac{1}{d}$ if $p < \dim(\Omega) = d$. There is a $C > 0$ such that

\begin{align}
\|u\|_{W^s(\Omega)} \leq C \|u\|_{W^{s/q,q}(\Omega)}^{1+d/q-d/p} \|u\|_{W^{p',d/q}(\Omega)}^{d/p-d/q}, \quad \forall u \in W^{1,p}(\Omega)
\end{align}

The weak formulation of (3.8) is: Find $(w,p) \in (X,Q)$ such that

\begin{align}
(w_t, v) + \frac{C_4^2}{\mu} \delta^2 (\nabla w_t, \nabla v) + (w \cdot \nabla w, v) + \nu (\nabla w, \nabla v) \\
= -(p, \nabla \cdot v) + ((Cs\delta)^2 \nabla w, \nabla v) = (f, v) \quad \text{for all } v \in X,
\end{align}

$q, \nabla \cdot w = 0$ for all $q \in Q = L^2(\Omega)$.

For the stationary Smagorinsky model, Du and Gunzburger \cite{9, 10} proved that the discrete solution converges to the continuous problem under minimal regularity assumptions. The existence and uniqueness of the strong solution of the Smagorinsky model (1.1) on a periodic domain have been studied \cite{23, 25}. Recently, the error estimates for Smagorinsky model have also been studied in \cite{6} and it showed that both the accuracy and the stability are enhanced for flows with high Reynolds number. Here we omit the proof for the existence of a strong solution. We assume the model has a solution in the following sense

**Definition 2.13.** A solution $w$ of the Modified Smagorinsky Model (3.8) is a strong solution if $w$ satisfies the following

1. $w \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;W^{1,3}(\Omega)) \cap L^2(0,T;L^6(\Omega))$,
2. $w(x,t) \to w_0(x)$ in $L^2(\Omega)$ as $t \to 0$,
3. $w$ satisfies the model’s weak form (2.11) for all $v \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;W^{1,3}(\Omega)) \cap L^2(0,T;L^6(\Omega))$.

**Definition 2.14.** *(Mean, Fluctuation and Variance)* The ensembles $u(x,t;\omega_j), j = 1, \ldots, J$ where $\omega$ is a a random variable, mean $\bar{u}$ and fluctuation $u'$ are defined as follows:

$$
\bar{u}(x,t) = \frac{1}{J} \sum_{j=1}^{J} u(x,t;\omega_j), \quad u'(x,t;\omega_j) = u(x,t;\omega_j) - \bar{u}(x,t).
$$

The variance of $u$ and $\nabla u$ are, respectively,

$$
V(u) := \int_{\Omega} |u|^2 - |\bar{u}|^2 \, dx, \quad V(\nabla u) := \int_{\Omega} |\nabla u|^2 - |\nabla \bar{u}|^2 \, dx.
$$
Definition 2.15. (Reynolds Stresses) The Reynolds stresses are
\[ R(u, u) := \nabla u \otimes u - u \otimes \nabla u = - u' \otimes u'. \]

Ensemble averaging satisfies the following properties, e.g., [27,28,30]
\[
\begin{align*}
\mathbb{u} &= u, \\
\mathbb{u}' &= 0, \\
\mathbb{W} \cdot \mathbb{v} &= \mathbb{W} \cdot v, \\
\mathbb{W} \otimes \mathbb{v} &= \mathbb{W} \otimes v, \\
\frac{\partial}{\partial t} \mathbb{u} &= \frac{\partial}{\partial t} u, \\
\frac{\partial}{\partial x} \mathbb{u} &= \frac{\partial}{\partial x} u.
\end{align*}
\]

Theorem 2.16. Suppose that each realization is a strong solution of the NSE. The ensemble is generated by different initial data and \( u(x, 0; \omega_j) \in L^2(\Omega), f(x, t) \in L^\infty(0, \infty; L^2(\Omega)). \) Then the following two properties are satisfied.

Property 1: (Time averaged dissipativity)
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\Omega} R(u, u) : \nabla \mathbb{u} \, dx \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{\Omega} \nu |\nabla u'|^2 \, dx \geq 0.
\]

Property 2: (Equation for the evolution of variance of fluctuations)
\[
\int_{\Omega} R(u, u) : \nabla \mathbb{u} \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u'|^2 \, dx + \int_{\Omega} \nu |\nabla u'|^2 \, dx. \tag{2.12}
\]

Proof. Proof of this theorem can be found in Section 2 of [15]. \(\Box\)

Remark 2.17. (Statistical steady state and statistical equilibrium, see [15]) Statistical steady state is \( P/\epsilon = 1 \) where
\[
\epsilon = \text{dissipation of turbulent kinetic energy (TKE)} = \nu \| \nabla u' \|^2,
\]
\[
P = \text{production of TKE} = \int_{\Omega} R(u, u) : \nabla \mathbb{u} \, dx.
\]

Hence \( \frac{P}{\epsilon} = 1 \) implies \( \int_{\Omega} R(u, u) : \nabla \mathbb{u} \, dx = \int_{\Omega} \nu |\nabla u'|^2 \, dx. \) Thus using the balance equation (2.12) we can conclude that for \( T \) large enough, \( \frac{d}{dt}\|u'(T)\|^2 = 0, \) and such flow is at statistical equilibrium.

3. Model derivation. In this section, we develop a model for turbulence not at statistical equilibrium unlike the Smagorinsky model (1.1).

Consider the Navier-Stokes Equations (NSE) which govern the flow of an incompressible fluid with velocity \( u(x, t) \), pressure \( p(x, t) \), prescribed body force \( f(x, t) \) and kinematic viscosity \( \nu \) in the regular and bounded flow domain \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)):
\[
u u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f(x) \text{ in } \Omega, \quad \text{and } \nabla \cdot u = 0 \text{ in } \Omega. \tag{3.1}
\]

To derive the Modified Smagorinsky model, following the work in [15], we begin with an ensemble of NSE solution \( u(x, t; w_j) \) with perturbed initial data \( u(x, 0; \omega_j) = u_0(x; \omega_j), j = 1, 2, \ldots, J, x \in \Omega. \)

The goal of a turbulent model solution of (1.1) and (1.2) is to approximate \( \mathbb{u}(x, t) \). By ensemble averaging the NSE gives a system that is not closed since \( \mathbb{u} \neq \mathbb{u} u. \)
Hence the Reynolds stress tensor, \( R(\textbf{u}, \textbf{u}) = \bar{\textbf{u}} \cdot \textbf{u} - \textbf{u} \cdot \textbf{u} \) which is accountable for all effects of the fluctuations on the mean flow must be modeled [32]. We rewrite \( \textbf{u} \cdot \textbf{u} \) as \( \bar{\textbf{u}} \cdot \textbf{u} = \bar{\textbf{u}} \cdot \bar{\textbf{u}} - R(\textbf{u}, \textbf{u}) \). Note that by using properties in (2.15), we get \( R = -\bar{\textbf{u}} \cdot \textbf{u} \).

Hence we get,

\[
\begin{align*}
\textbf{u}_t + \bar{\textbf{u}} \cdot \nabla \bar{\textbf{u}} - \nu \Delta \bar{\textbf{u}} + \nabla \bar{p} - \nabla \cdot R &= f(\textbf{x}) \text{ in } \Omega, \text{ and } \nabla \cdot \bar{\textbf{u}} = 0 \text{ in } \Omega.
\end{align*}
\]

Take the dot product of first and second equation in ((3.2)) with mean flow \( \bar{\textbf{u}} \) and \( \bar{p} \) respectively and doing integration by parts, we get the energy estimate as follows [15,32]

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\bar{\textbf{u}}\|^2 + \nu \|\nabla \bar{\textbf{u}}\|^2 &= \int_{\Omega} R(\textbf{u}, \textbf{u}) : \nabla \bar{\textbf{u}} \, dx = \langle f, \bar{\textbf{u}} \rangle. \tag{3.3}
\end{align*}
\]

In ((3.3)), if the term \( \int_{\Omega} R(\textbf{u}, \textbf{u}) : \nabla \bar{\textbf{u}} \, dx > 0 \), the effect of \( R(\textbf{u}, \textbf{u}) \) is dissipative while if \( \int_{\Omega} R(\textbf{u}, \textbf{u}) : \nabla \bar{\textbf{u}} \, dx < 0 \), fluctuations \( \textbf{u}' \) transfers energy back to mean \( \bar{\textbf{u}} \) which causes increased energy in mean flow.

Property 1 in Theorem 2.16 is consistent with the assumption of Boussinesq [4] that turbulent fluctuations are dissipative on the mean in the time averaged case. In property 2 of Theorem 2.16, the term \( \int_{\Omega} \nu |\nabla \textbf{u}'|^2 \, dx \) is clearly dissipative while \( \frac{d}{dt} \int_{\Omega} |\textbf{u}'|^2 \, dx = 0 \) for flows at statistical equilibrium. The idea of any EV model is based on three assumptions [15]. Firstly, the statistical equilibrium assumption that dissipativity holds at each instant time

\[
\int_{\Omega} R(\textbf{u}, \textbf{u}) : \nabla \bar{\textbf{u}} \, dx \simeq \int_{\Omega} \nu |\nabla \textbf{u}'|^2 \, dx.
\]

The second assumption is that \( \nabla \textbf{u}' \) aligns with \( \nabla \bar{\textbf{u}} \). Third, calibration [15] provides that the action of fluctuating velocities can be represented in terms of mean flow

\[
\text{action}(\nabla \textbf{u}') \simeq a(\bar{\textbf{u}}) \nabla \bar{\textbf{u}}.
\]

Combining all these three assumptions results in the eddy viscosity closure,

\[
-\nabla \cdot R(\textbf{u}, \textbf{u}) \Leftarrow -\nabla \cdot (\nu_T(\bar{\textbf{u}}) \nabla \bar{\textbf{u}}) + \text{terms incorporated in } \nabla \bar{p}.
\]

Here \( \nu_T \) denotes the turbulent viscosity. Thus we have the eddy viscosity (EV) model:

\[
\nabla \cdot \textbf{w} = 0 \text{ and } \tag{3.5}
\]

\[
\textbf{w}_t + \textbf{w} \cdot \nabla \textbf{w} - \nu \Delta \textbf{w} + \nabla q - \nabla \cdot (\nu_T(\textbf{w}) \nabla \textbf{w}) = f(\textbf{x}).
\]

The solution \((\textbf{w}, q)\) of ((3.5)) is an approximation of the ensemble average \((\bar{\textbf{u}}, \bar{p})\). In 1963, Smagorinsky [33] model \( \nu_T \) by

\[
\nu_T = (C_s \delta)^2 |\nabla \textbf{w}|, \tag{3.6}
\]

where \( C_s \approx 0.1, \text{ Lilly [24]. Let } \Delta x \text{ to be the mesh size and } \delta = \Delta x << 1 \text{ is the model length scale [34]. Thus we get the classic Smagorinsky model (1.1).}
By taking the dot product with $w$, here we have the energy equality for Smagorinsky model:

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + \nu \|\nabla w\|^2 + (C_s \delta)^2 \|\nabla w\|^3_L = (f, w).$$

$(C_s \delta)^2 \|\nabla w\|^3_L \geq 0$ approximates the average energy dissipated by fluctuation. Since it is positive, it prevents the energy from returning to the mean flow. Next, we introduce an approximation to the term $\frac{1}{2} \frac{d}{dt} \|u\|^2$ in our model to gain time accuracy and hence improve the existing Smagorinsky model.

The Kolmogorov-Prandtl relation [18,31] for $\nu_T$ is

$$\nu_T = \mu \sqrt{k'},$$

where $\mu \approx 0.3$ to 0.55, $l$ = turbulent length scale and $k'$ is the turbulent kinetic energy: $k'(x,t) = \frac{1}{2} |u'(x,t)|^2$. Hence by comparing with (3.6), we get,

$$\mu \sqrt{k'} = (C_s \delta)^2 |\nabla w| = \mu \delta \left(\frac{C_k^2 \delta}{\mu} |\nabla w|\right).$$

This suggests that $k' = C_k^4 \delta^2 \mu^{-2} |\nabla w|^2$. Hence,

$$\frac{d}{dt} \int_{\Omega} |k' \, dx = \frac{d}{dt} C_k^4 \delta^2 \mu^{-2} (\nabla w, \nabla w) = C_k^4 \delta^2 \mu^{-2} (-\Delta w_t, w).$$

This suggests that by including $C_k^4 \delta^2 \mu^{-2} \Delta w_t$ in the model and taking the $L^2$ inner product with $w$, we can approximate $\frac{1}{2} \frac{d}{dt} \|u\|^2$.

As a result, the Modified Smagorinsky Model (MSM) is: $\nabla \cdot w = 0$ and

$$w_t - C_k^4 \delta^2 \mu^{-2} \Delta w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left((C_s \delta)^2 |\nabla w| \nabla w\right) = f(x).$$

Here we impose the no-slip boundary condition, $w = 0$ on $\partial \Omega$.

4. Basic Properties of the Model. In this section, we develop some basic properties of the model which are useful in numerical analysis. In particular, we derive the basic energy estimate, we prove a stability bound and uniqueness of the solution. We also analyze the modeling error and numerical error of the model.

4.1. Energy Estimate for the MSM. We will identify the model’s kinetic energy and energy dissipation in Theorem 4.1.

THEOREM 4.1. Let $w$ be a strong solution of the Modified Smagorinsky Model (3.8), then the following energy estimate holds

$$\frac{1}{2} \frac{d}{dt} \left( \|w\|^2 + \frac{C_k^4 \delta^2}{\mu^2} \|\nabla w\|^2 \right) + \nu \|\nabla w\|^2 + \frac{1}{\Omega} \|\nabla w\|^2 + \frac{1}{\Omega} (C_s \delta)^2 \|\nabla w\|^3_L = \frac{1}{\Omega} (f, w).$$

Proof. First, we take dot product in (3.8) with $w$ and do integration by parts which is shown below,

$$\int_{\Omega} \left( w_t - C_k^4 \delta^2 \mu^{-2} \Delta w_t + w \cdot \nabla w - \nu \Delta w + \nabla q - \nabla \cdot \left((C_s \delta)^2 |\nabla w| \nabla w\right) \right) \cdot w \, dx = \int_{\Omega} f \cdot w \, dx.$$
Here, \( \int_\Omega w \cdot w \, dx = \frac{d}{dt} \left( \frac{1}{2} \int_\Omega |w|^2 \, dx \right) \). By Lemma (2.2), \( \int_\Omega w \cdot \nabla w \cdot w \, dx = 0 \). Next, \(-\nu \int_\Omega \Delta w \cdot w \, dx = \int_\Omega \nu \|\nabla w\|^2 \, dx \). The next term, \( \int_\Omega \nabla q \cdot w \, dx = \int_{\partial\Omega} p w \cdot \hat{n} \, ds - \int_\Omega \beta \nabla \cdot w \, dx \), gives \( 0 \). The final term, \( \int_\Omega -\nabla \cdot (C_s \delta)^2 |\nabla w| \nabla w \cdot w \, dx = \int_\Omega (C_s \delta)^2 |\nabla w|^3 \, dx \).

Hence combining all these terms we get the following energy estimate per unit volume, \( 0 \)

\[
\frac{1}{2} \frac{d}{dt} \left( \|w\|^2 + \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w\|^2 \right) + \frac{1}{2} \nu \|\nabla w\|^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2 = \frac{1}{2} \Omega(f, w). 
\]

**Remark 4.2.** In (4.1), we can identify the following quantities:

1. **Model kinetic energy of mean flow per unit volume**
   
   \[ MKE := \frac{1}{2} \frac{d}{dt} \left( \|w\|^2 + \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w\|^2 \right) \]

   And the second term in MKE coming from the Modified Smagorinsky Model is the turbulent kinetic energy per unit volume.

2. **Rate of energy dissipation of mean flow per unit volume**

   \[ \dot{\varepsilon}_{\text{msm}}(t) := \frac{1}{\Omega} \left( \nu \|\nabla w\|^2 + (C_s \delta)^2 \|\nabla w\|_{L^2}^3 \right). \]

   This controls the time rate of change of kinetic energy. It’s always positive and it reduces the accumulation of kinetic energy.

3. **Rate of energy input to mean flow per unit volume**

   \[ \frac{1}{\Omega} (f, w). \]

**4.2. Stability.** Next, we give the stability bound of the Modified Smagorinsky Model (3.8) in **Theorem 4.3.** We prove the model kinetic energy is bounded uniformly in time and the time-averaged model energy dissipation rate is bounded as well in the same Theorem.

**Theorem 4.3.** (Stability of \( w \)) (3.8) is unconditionally stable. The solution \( w \) of (3.8) satisfies the following inequality

\[
\|w(T)\|^2 + \frac{C_s^4}{\mu^2} \delta^2 \|\nabla w(T)\|^2 \leq e^{-\alpha T} \left\{ \|w(0)\|^2 + \frac{C_s^4}{\mu^2} \delta^2 \|\nabla w(0)\|^2 + \frac{C}{\alpha} (e^{\alpha T} - 1) \right\} ,
\]

where \( \alpha = \min\left\{ \frac{\nu}{\sqrt{C_f p_f}}, \frac{\nu^2 p_f}{2 \sqrt{C_f}} \right\} \), and if \( f \in L^2(\Omega) \), we get

\[
\max_{0 \leq t \leq T} \left( \|w\|^2 + \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w\|^2 \right) \leq C' < \infty.
\]

and

\[
\mathcal{O}(\frac{1}{T} + \frac{1}{\Omega} \frac{1}{T} \int_0^T \left( \nu \|\nabla w\|^2 + (C_s \delta)^2 \|\nabla w\|_{L^2}^3 \right) \, dt \leq \frac{1}{\Omega} \frac{1}{T} \int_0^T \frac{C_s^2 \delta^2}{2 \nu} \|f\|^2 \, dt.
\]
Proof. Take $L^2$ inner product of (3.8) with $w$, we get the following energy equality,

$$
\frac{1}{2} \frac{d}{dt} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \nu \| \nabla w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^3} = (f, w).
$$

Consider the RHS of (4.2), $(f, w) \leq \frac{\epsilon}{2} \|w\|^2 + \frac{1}{2\epsilon} \|f\|^2$. Thus (4.2) implies

$$
\frac{d}{dt} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \nu \| \nabla w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^3} \leq \epsilon \|w\|^2 + \frac{1}{\epsilon} \|f\|^2.
$$

Using the inequality $\|w\| \leq C_{PF} \| \nabla w \|$, we have

$$
\frac{d}{dt} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \nu \| \nabla w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^3} \leq \epsilon \|w\|^2 + \frac{1}{\epsilon} \|f\|^2.
$$

Pick $\epsilon = \frac{\nu}{2C_{PF}}$ and drop the term $2(C_s \delta)^2 \| \nabla w \|^3_{L^3}$. We obtain

$$
\frac{d}{dt} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \alpha \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq 2 \frac{C_{PF}}{\nu} \|f\|^2.
$$

Let $\alpha = \min \left\{ \frac{\nu}{2C_{PF}}, \frac{\mu^2}{C_s \delta^2} \nu \right\}$, resulting in

$$
\frac{d}{dt} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) + \alpha \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq 2 \frac{C_{PF}}{\nu} \|f\|^2.
$$

Multiply by the integrating term $e^{\alpha t}$ and integrate from $t = 0$ to $t = T$, leading to

$$
\|w(T)\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(T) \|^2 \leq e^{-\alpha T} \left\{ \|w(0)\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(0) \|^2 + \frac{C}{\alpha} (e^{\alpha T} - 1) \right\},
$$

where $C = 2 \frac{C_{PF}}{\nu} \|f\|^2$.

This implies that kinetic energy is uniformly bounded, i.e. if $f \in L^2(\Omega)$, we get

$$
\max_{0 \leq t < \infty} \left( \|w\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w \|^2 \right) \leq C' < \infty.
$$

Integrate (4.2) from $t = 0$ to $t = T$ and divide by $|\Omega|$ and $T$, we have

$$
\frac{1}{|\Omega|} \frac{1}{2T} \left( \left( \|w(T)\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(T) \|^2 \right) - \left( \|w(0)\|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla w(0) \|^2 \right) \right)
$$

$$
+ \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \left( \nu \| \nabla w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^3} \right) dt = \frac{1}{|\Omega|} \frac{1}{2T} \int_0^T (f, w) dt.
$$

Consider the term on the right. Using the Poincaré-Friedrichs’ inequality (2.3), Cauchy Schwarz and Young’s inequality (2.5) gives

$$
\frac{1}{|\Omega|} \frac{1}{T} \int_0^T (f, w) dt \leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{1}{\sqrt{\nu}} \|f\|_{C_{PF} \sqrt{\nu}} \| \nabla w \| dt,
$$

$$
\leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{\nu}{2} \| \nabla w \|^2 dt + \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{C_{PF}^2}{2\nu} \|f\|^2 dt.
$$

The first term in (4.3) is bounded by the previous result. Thus,

$$
\mathcal{O} \left( \frac{1}{T} \right) + \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \left( \frac{\nu}{2} \| \nabla w \|^2 + (C_s \delta)^2 \| \nabla w \|^3_{L^3} \right) dt \leq \frac{1}{|\Omega|} \frac{1}{T} \int_0^T \frac{C_{PF}^2}{2\nu} \|f\|^2 dt.
$$

The time-averaged dissipation is bounded. \qed
4.3. Uniqueness. In this subsection, we prove the uniqueness of the strong solution to (3.8) in Theorem 4.4.

Theorem 4.4. Assume \( \nabla w \in L^3(0, T; L^3(\Omega)) \), the solution \( w \) of (3.8) is unique.

Proof. Suppose \( (w_1, q_1) \) and \( (w_2, q_2) \) are two different solutions of (3.8) and let \( \phi, q \) denote the difference between two solutions: \( \phi = w_1 - w_2, q = q_1 - q_2 \), \( \phi \) satisfies \( \nabla \cdot \phi = 0 \) and

\[
\frac{\partial}{\partial t} \left( \phi - \frac{C^4}{\mu^2} \delta^2 \Delta \phi \right) + w_1 \cdot \nabla w_1 - w_2 \cdot \nabla w_2 - \nu \Delta \phi + \nabla q
\]

\[-(C_s \delta)^2 \nabla \cdot (|\nabla w_1| \nabla w_1 - |\nabla w_2| \nabla w_2) = 0.
\]

Take the \( L^2 \) inner product with \( \phi \) and let \( \tilde{w} \) represent either \( w_1 \) or \( w_2 \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \nu \| \nabla \phi \|^2 + (C_s \delta)^2 \int_\Omega [ |\nabla w_1| \nabla w_1 - |\nabla w_2| \nabla w_2 | \nabla (w_1 - w_2) \cdot \nabla (w_1 - w_2) \, dx = - \int_\Omega \phi \cdot \nabla \tilde{w} \cdot \phi \, dx.
\]

Using the Strong Monotonicity (2.8), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \nu \| \nabla \phi \|^2 + C_1(C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \leq - \int_\Omega \phi \cdot \nabla w_1 \cdot \phi \, dx.
\]

Consider the RHS, using (2.10) in 3D space,

\[
\left| - \int_\Omega \phi \cdot \nabla w_1 \cdot \phi \, dx \right| \leq \| \nabla w_1 \|_{L^2} \| \phi \|_{L^3}^2,
\]

\[
\leq C \| \nabla w_1 \|_{L^3} \| \phi \|^{1/2} \| \nabla \phi \|^{1/2}^2,
\]

\[
\leq \frac{\epsilon}{2} \| \nabla \phi \|^2 + C(\epsilon) \| \nabla w_1 \|^2_{L^3} \| \phi \|^2.
\]

Pick \( \epsilon = 2 \frac{C^4}{\mu^2} \delta^2 \), leading to

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \nu \| \nabla \phi \|^2 + C_1(C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \leq \frac{C^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 + C(\epsilon) \| \nabla w_1 \|^2_{L^3} \| \phi \|^2,
\]

\[
\leq \max\{1, C(\epsilon) \| \nabla w_1 \|^2_{L^3} \} \left( \frac{C^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 + \| \phi \|^2 \right).
\]

Here \( a(t) := \max\{1, C(\epsilon) \| \nabla w_1 \|^2_{L^3} \} \in L^1(0, T) \), because

\[
\int_0^T 1 \cdot \| \nabla w_1 \|^2_{L^3} \, dt \leq \left( \int_0^T 1^3 \, dt \right)^{1/3} \left( \int_0^T \| \nabla w_1 \|^2_{L^3} \, dt \right)^{2/3} = \left( \int_0^T 1^3 \, dt \right)^{1/3} \left( \int_0^T \| \nabla w_1 \|^3_{L^3} \, dt \right)^{2/3} < \infty.
\]
Then we can form its antiderivative

\[ A(T) := \int_0^T a(t) \, dt < \infty, \text{ for } \nabla w \in L^3(0, T; L^3(\Omega)). \]

Multiplying through by the integrating factor \( e^{-A(t)} \) gives

\[
\frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( \| \phi \|^2 + \frac{C_4^4}{\mu^2} \| \nabla \phi \|^2 \right) \right] + e^{-A(t)} [\nu \| \nabla \phi \|^2 + C_1 (C_4 \delta)^2 \| \nabla \phi \|^3_{L^3}] \leq 0.
\]

Then, integrating over \([0, T]\) and multiplying through by \( e^{A(t)} \) gives

\[
\frac{1}{2} \left( \| \phi(T) \|^2 + \frac{C_4^4}{\mu^2} \| \nabla \phi(T) \|^2 \right) + \int_0^T \left( \nu \| \nabla \phi \|^2 + C_1 (C_4 \delta)^2 \| \nabla \phi \|^3_{L^3} \right) \, dt \leq \frac{1}{2} \left( \| \phi(0) \|^2 + \frac{C_4^4}{\mu^2} \| \nabla \phi(0) \|^2 \right).
\]

4.4. Modelling error. In this subsection, we analyze the error between the solution to the NSE (3.1) and the Modified Smagorinsky Model (3.8) in Theorem 4.5.

**Theorem 4.5.** Assume \( \nabla u_0 \in L^2(\Omega) \) and \( \nabla w \in L^2(0, T; L^3) \), let \( \phi = u_{NSE} - w_{Smag} \) be the modelling error of Modified Smagorinsky, then \( \phi \) satisfies the following:

\[
\frac{1}{2} \left( \| \phi(T) \|^2 + \frac{C_4^4}{\mu^2} \| \nabla \phi(T) \|^2 \right) + \int_0^T \left( \nu \| \nabla \phi \|^2 + \frac{C_1}{2} (C_4 \delta)^2 \| \nabla \phi \|^3_{L^3} \right) \, dt \leq \frac{1}{2} \left( \| \phi(0) \|^2 + \frac{C_4^4}{\mu^2} \| \nabla \phi(0) \|^2 \right).
\]

Here \( C^* \) depends on \( \nu, T \), \( \int_0^T \| \nabla w \|^3_{L^3} \, dt \).

**Proof.** \( u_{NSE} \) satisfies \( \nabla \cdot u = 0 \) and the following equation

\[
u \Delta u - \nabla u + \nabla p - (C_4 \delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4^4}{\mu^2} \delta^2 \Delta u_t = f - (C_4 \delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4^4}{\mu^2} \delta^2 \Delta u_t.
\]

(4.4)

Subtract (3.8) from (4.4). We obtain, \( \nabla \cdot \phi = 0 \) and

\[
\phi_t - \frac{C_4^4}{\mu^2} \delta^2 \Delta \phi + u \cdot \nabla u - w \cdot \nabla w - \nu \Delta \phi + \nabla (p - q)
\]

\[
-(C_4 \delta)^2 \nabla \cdot (|\nabla u| \nabla u - |\nabla w| \nabla w) = -(C_4 \delta)^2 \nabla \cdot (|\nabla u| \nabla u) - \frac{C_4^4}{\mu^2} \delta^2 \Delta u_t.
\]

Here, \( u \cdot \nabla u - w \cdot \nabla w = u \cdot \nabla u - u \cdot \nabla w + u \cdot \nabla w - w \cdot \nabla w = u \cdot \nabla \phi + \phi \cdot \nabla w. \)

Take \( L^2 \) inner product with \( \phi \) gives

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C_4^4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \int_\Omega \phi \cdot \nabla w \cdot \phi \, dx + \nu \| \nabla \phi \|^2 \, dx
\]

\[+ (C_4 \delta)^2 \int_\Omega \left( |\nabla u| |\nabla u - |\nabla w| |\nabla w| \right) \cdot \nabla (u \cdot \nabla \phi) \, dx
\]

\[= (C_4 \delta)^2 \int_\Omega |\nabla u| |\nabla u : \nabla \phi | \, dx + \frac{C_4^4}{\mu^2} \delta^2 \int_\Omega \nabla u_t : \nabla \phi \, dx.
\]

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Using Strong Monotonicity (2.11), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \nu \| \nabla \phi \|^2 + C_1 (C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \leq - \int_{\Omega} \phi \cdot \nabla w \cdot \phi \, dx + (C_s \delta)^2 \int_{\Omega} |\nabla u| \nabla u : \nabla \phi \, dx + \frac{C_4}{\mu^2} \delta^2 \int_{\Omega} \nabla u_t : \nabla \phi \, dx.
\]

\hspace{1cm} (4.5)

Consider the first term in the RHS, similar to the previous steps\[ | - \int_{\Omega} \phi \cdot \nabla w \cdot \phi \, dx | \leq \frac{c_1}{2} \| \nabla \phi \|^2 + C(\epsilon_1) \| \nabla w \|^3_{L^3} \| \phi \|^2.\]

The second term in the RHS is\[ \left| (C_s \delta)^2 \int_{\Omega} |\nabla u| \nabla u : \nabla \phi \, dx \right| \leq \frac{c_2}{3} (C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \| \nabla u \|^2_{L^3},\]

\[ \leq \frac{c_2}{3} (C_s \delta)^2 \| \nabla \phi \|^3_{L^3} + C(\epsilon_2) (C_s \delta)^2 \| \nabla u \|^3_{L^3}.\]

The third term in the RHS satisfies\[ \left| \frac{C_4}{\mu^2} \delta^2 \int_{\Omega} \nabla u_t : \nabla \phi \, dx \right| \leq \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \| \| \nabla \phi \|,\]

\[ \leq \frac{c_3}{2} \left( C_4 \| \nabla w \|^2_{L^3}, \frac{c_4}{2} \right) \| \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2\]

\[ + C(\epsilon_3) (C_s \delta)^2 \| \nabla u \|^3_{L^3} + C(\epsilon_3) \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \|^2.\]

Pick \( \epsilon_1 = \nu, \epsilon_2 = \frac{3C_1}{2} \), collect all terms, (4.5) becomes

\[ \frac{1}{2} \frac{d}{dt} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) + \nu \| \nabla \phi \|^2 + \frac{C_1}{2} (C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \leq \max \left\{ \left( C(\epsilon_1) \| \nabla w \|^2_{L^3}, \frac{c_4}{2} \right) \right\} \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2.
\]

Denote \( a(t) := \max \left\{ C(\epsilon_1) \| \nabla w \|^2_{L^3}, \frac{c_4}{2} \right\} \) and its antiderivative is given by

\[ A(T) := \int_0^T a(t) \, dt < \infty \text{ for } \nabla w \in L^2(0, T; L^3).
\]

Multiplying through by the integrating factor \( e^{-A(t)} \) gives

\[ \frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( \| \phi \|^2 + \frac{C_4}{\mu^2} \delta^2 \| \nabla \phi \|^2 \right) \right] + e^{-A(t)} \left[ \nu \| \nabla \phi \|^2 + \frac{C_1}{2} (C_s \delta)^2 \| \nabla \phi \|^3_{L^3} \right] \leq e^{-A(t)} \left\{ C(\epsilon_2) (C_s \delta)^2 \| \nabla u \|^3_{L^3} + C(\epsilon_3) \frac{C_4}{\mu^2} \delta^2 \| \nabla u_t \|^2 \right\}.
\]
Then, integrating over \([0, T]\) and multiplying through by \(e^{A(t)}\) gives
\[
\frac{1}{2} \left( \|\phi(T)\|^2 + \frac{C_1}{\mu^2} \delta^2 \|\nabla \phi\|^2 \right) + \int_0^T \frac{\nu}{2} \|\nabla \phi\|^2 + \frac{C_1}{\nu} (C_s \delta)^2 \|\nabla \phi\|^2_{L^2} \, dt
\]
\[
\leq C(\nu, T, \|\nabla w\|_{L^2}) \left\{ \frac{1}{2} \left( \|\phi(0)\|^2 + \frac{C_1}{\nu} (C_s \delta)^2 \|\nabla \phi(0)\|^2 \right) + \int_0^T (C_s \delta)^2 \|\nabla u\|^2_{L^2} + \frac{C_1}{\nu} (C_s \delta)^2 \|\nabla u\|^2 \, dt \right\}.
\]
and \(C(\nu, T)\) depends on \(\nu, T, \int_0^T \|\nabla w\|^2_{L^2} \, dt\).

5. Numerical error. Consider the semi-discrete approximation of the MSM (3.8). Suppose \(w^h(x, 0)\) is approximation of \(w(x, 0)\). The approximate velocity and pressure are maps
\[
w^h : [0, T] \to X^h, \quad p^h : (0, T) \to Q^h
\]
satisfying
\[
(w^h, v^h) + \frac{C_4}{\mu^2} \delta^2 (\nabla w^h, \nabla v^h) + b^*(w^h, w^h, v^h) + \nu(\nabla w^h, \nabla v^h)
\]
\[
-(p^h, \nabla \cdot v^h) + \left( (C_s \delta)^2 |\nabla w^h| \nabla w^h, \nabla v^h \right) = (f, v^h) \text{ for all } v^h \in X^h,
\]
\[
(q^h, \nabla \cdot w^h) = 0 \text{ for all } q^h \in Q^h.
\]

In this section, we analyze the error between the strong solution to the MSM (3.8) and the semi-discrete solution to (5.1) in Theorem 5.1.

Theorem 5.1. (Numerical error of semi-discrete case) Let \(w\) be the strong solution of the MSM (3.8) (in particular \(\|w\| \in L^\infty(0, T), \|\nabla w\|_{L^2} \in L^2(0, T), w \in L^2(0, T; W^{1,3}(\Omega)) \cap L^3(0, T; L^6(\Omega))\)) and \(w^h\) be a solution to the semi-discrete problem (5.1). Let
\[
a(t) := C(\nu) \|\nabla w\|^2_{L^2} + \frac{1}{4} \|w\|^2_{L^2}.
\]
Then, for \(T > 0\) the error \(w - w^h\) satisfies
\[
\|(w - w^h)(T)\|^2 + \frac{C_4}{\mu^2} \delta^2 \|\nabla (w - w^h)(T)\|^2 + \int_0^T \left\{ \frac{\nu}{2} \|\nabla (w - w^h)\|^2 + \frac{C_4}{\nu} (C_s \delta)^2 \|\nabla (w - w^h)(0)\|^2 \right\} dt \leq \exp \left( \int_0^T a(t) \, dt \right) \left\{ \|(w - w^h)(0)\|^2 \right\}
\]
\[
+ \frac{C_4}{\mu^2} \delta^2 \|\nabla (w - w^h)(0)\|^2 + \inf_{v^h \in V^h} \|(w - v^h)(T)\|^2
\]
\[
+ \int_0^T C(\nu) \left\{ \inf_{v^h \in V^h} \left( \|w_t - v^h\|^2_{L^2} + (\frac{C_4}{\mu^2})^2 \|\nabla (w_t - v^h)\|^2 + \|\nabla (w - v^h)\|^2 \right) \right\}
\]
\[
+ \inf_{q^h \in Q^h} \|p - q^h\|^2 \right\} + C \inf_{v^h \in V^h} \left( (C_s \delta)^2 \|\nabla (w - v^h)\|_{L^2}^{3/2} + \delta^{-1} \|w - v^h\|_{L^2}^{3/2} \right) dt \right\}.
\]

Proof. Consider the variational problem of the MSM (3.8): Find \(w : [0, T] \to X = L^\infty(0, T; L^2(\Omega)) \cap L^3(0, T; W^{1,3}(\Omega))\) satisfying (2.11). Let \(v^h \in V^h = \{v^h \in

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$X^h : (\nabla \cdot v^h, q^h) = 0 \forall q^h \in Q^h$. Since $v \in X \& v^h \in V^h \subset X^h \subset X$, we restrict $v = v^h$ in continuous variational problem. Then subtract semi-discrete problem (5.1) from continuous problem (2.11). Let $e = error = w - w^h$. This gives,

$$\begin{align*}
(e_t, v^h) + (C_s^4 \delta^2 \mu^{-2} \nabla e_t, \nabla v^h) + b^*(w, w, v^h) - b^*(w^h, w^h, v^h) \\
+ \nu(\nabla e, \nabla v^h) + (C_s^4 \delta^2) \int_\Omega (|\nabla w|\nabla w - |\nabla w^h|\nabla w^h) : \nabla v^h \, dx \\
- (p - p^h, \nabla \cdot v^h) = 0.
\end{align*}
$$

We can write,

$$b^*(w, w, v^h) - b^*(w^h, w^h, v^h)$$

$$= b^*(w, w, v^h) - b^*(w^h, w, v^h) + b^*(w^h, w, v^h) - b^*(w^h, w^h, v^h),$$

$$= b^*(e, v, v^h) + b^*(w^h, e, v^h).$$

Also,

$$\begin{align*}
\int_\Omega (|\nabla w|\nabla w - |\nabla w^h|\nabla w^h) \cdot \nabla v^h \, dx \\
= \int_\Omega (|\nabla w|\nabla w - |\nabla w|\nabla w + \nabla w^h \cdot \nabla w^h) \cdot \nabla v^h \, dx.
\end{align*}$$

Pick $\tilde{w} \in V^h$. Let $\eta = w - \tilde{w}, \phi^h = w^h - \tilde{w}, \phi^h \in V^h$. This implies $e = (w - \tilde{w}) - (w^h - \tilde{w}) = \eta - \phi^h$. Then (5.2) becomes

$$\begin{align*}
(\phi_t^h, v^h) + (C_s^4 \delta^2 \mu^{-2} \nabla \phi_t^h, \nabla v^h) + b^*(e, v, v^h) + b^*(w^h, e, v^h) + \nu(\nabla \phi^h, \nabla v^h) \\
+ (C_s^4 \delta^2) \int_\Omega (|\nabla w^h|\nabla w^h - |\nabla \tilde{w}|\nabla \tilde{w}) : (\nabla v^h) \, dx \\
- (p - p^h, \nabla \cdot v^h) \\
= (\eta_t, v^h) + (C_s^4 \delta^2 \mu^{-2} \nabla \eta_t, \nabla v^h) + \nu(\nabla \eta, \nabla v^h) \\
+ (C_s^4 \delta^2) \int_\Omega (|\nabla w|\nabla w - |\nabla \tilde{w}|\nabla \tilde{w}) : (\nabla v^h) \, dx.
\end{align*}$$

Take $v^h = \phi^h$ and $\lambda^h \in Q^h$. Here $b^*(w^h, e, \phi^h) = b^*(w^h, \eta - \phi^h, \phi^h) = b^*(w^h, \eta, \phi^h)$ since $b^*(w^h, \phi^h, \phi^h) = 0$. Using strong monotonicity (2.11), we get

$$(C_s^4 \delta^2) \int_\Omega (|\nabla w^h|\nabla w^h - |\nabla \tilde{w}|\nabla \tilde{w}) : (\nabla \phi^h) \, dx \geq C_1(C_s^4 \delta^2)^2 \|\nabla \phi^h\|^3_{L^3}.$$ 

Using local Lipschitz continuity (2.11), we get

$$(C_s^4 \delta^2) \int_\Omega (|\nabla w|\nabla w - |\nabla \tilde{w}|\nabla \tilde{w}) : (\nabla \phi^h) \, dx \leq (C_s^4 \delta^2)C_2 \|\nabla \eta\|_{L^3}\|\nabla \phi^h\|_{L^3},$$

where $r = \max\{\|\nabla w\|_{L^3}, \|\nabla \tilde{w}\|_{L^3}\}.$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|\phi^h\|^2 + C_4 \delta^2 \mu^{-2} \|\nabla \phi^h\|^2 + \nu \|\nabla \phi^h\|^2 + b^*(\eta - \phi^h, w, \phi^h) + b^*(w^h, \eta, \phi^h)$$

$$+ C_1(C_s^4 \delta^2)^2\|\nabla \phi^h\|^3_{L^3} \leq (\eta_t, \phi^h) + (C_s^4 \delta^2 \mu^{-2} \nabla \eta_t, \nabla \phi^h) + \nu(\nabla \eta, \nabla \phi^h)$$

$$+ (C_s^4 \delta^2)C_2 r \|\nabla \eta\|_{L^3}\|\nabla \phi^h\|_{L^3} + (p - \lambda^h, \nabla \cdot \phi^h).$$

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We can rewrite it as

$$\frac{1}{2} \frac{d}{dt} \left\{ ||\phi^h||^2 + \frac{C_4}{\mu^2} \delta^2 ||\nabla \phi^h||^2 \right\} + \nu ||\nabla \phi^h||^2 + C_1(\nu \delta)^2 ||\nabla \phi^h||^3_{L^3}$$

$$\leq (\eta_t, \phi^h) + \frac{C_4}{\mu^2} \delta^2 (\nabla \eta_t, \nabla \phi^h) + \nu (\nabla \eta, \nabla \phi^h) + (p - \lambda^h, \nabla \cdot \phi^h)$$

$$+(\nu \delta)^2 C_2 \|\nabla \eta\|_{L^3} \|\nabla \phi^h\|_{L^3} - b^*(\eta, w, \phi^h) + b^*(\phi^h, w, \phi^h) - b^*(w, \eta, \phi^h).$$

Next we find the bounds for the terms in the RHS. For the first four terms on the right, use the Cauchy Schwarz (2.7) and Young’s inequality (2.5),

$$|(\eta_t, \phi^h)| \leq \|\eta_t\|_1 \|\nabla \phi^h\| \leq \frac{\nu}{2} \|\nabla \phi^h\|^2 + C(\nu)\|\eta_t\|_2.$$

$$\frac{C_4}{\mu^2} \delta^2 (\nabla \eta_t, \nabla \phi^h) \leq \|\nabla \phi^h\| \frac{C_4 \delta^2}{\mu^2} \|\nabla \eta_t\|,$$

$$\leq \frac{\nu}{16} \|\nabla \phi^h\|^2 + C(\nu)\left(\frac{C_4 \delta^2}{\mu^2}\right)^2 \|\nabla \eta_t\|^2.$$

$$\nu |(\nabla \eta, \nabla \phi^h)| \leq \nu \|\nabla \eta\| \|\nabla \phi^h\| \leq \frac{\nu}{16} \|\nabla \phi^h\|^2 + C(\nu)\|\nabla \eta\|^2.$$

(5.3) \[ ||(p - \lambda^h, \nabla \cdot \phi^h)| \leq ||p - \lambda^h|| \|\nabla \cdot \phi^h\| \leq \frac{\nu}{16} \|\nabla \phi^h\|^2 + C(\nu)\|p - \lambda^h\|^2.\]

For the fifth term on the right, use the Hölder’s inequality (2.6),

$$(\nu \delta)^2 C_2 \|\nabla \eta\|_{L^3} \|\nabla \phi^h\|_{L^3} \leq (\nu \delta)^2 \left\{ \frac{C_1}{3} \|\nabla \phi^h\|_{L^3}^3 + \frac{2}{3} C_1^{-1/2} r^{3/2} \|\nabla \eta\|_{L^3}^3 \right\}.$$

Next, consider the nonlinear terms

$$|b^*(\eta, w, \phi^h)| = \frac{1}{2} |(\eta \cdot \nabla w, \phi^h) - (\eta \cdot \nabla \phi^h, w)|,$$

$$\leq \frac{1}{2} \left\{ ||\eta||_{L^6} \|\nabla w||_{L^3} \|\phi^h\| + ||\eta||_{L^6} \|\nabla \phi^h||_{L^3} \|w\| \right\},$$

$$\leq \frac{1}{4} \|\nabla w\|^2_{L^3} \|\phi^h\|^2 + \frac{1}{4} \|\eta\|^2_{L^6} + (\epsilon_1^{1/3})^3 \|\nabla \phi^h\|^3_{L^3}$$

$$+ C(\epsilon_1^{-1/3})^{3/2} \|w\|^{3/2} ||\eta||^{3/2}_{L^6}.$$

$$|b^*(\phi^h, w, \phi^h)| \leq \|\nabla w\|_{L^3} \|\nabla \phi^h\|^2_{L^3},$$

$$\leq \|\nabla w\|_{L^3} (\|\phi^h\|^{1/2} \|\nabla \phi^h\|^{1/2})^2,$$

$$\leq \frac{\nu}{16} \|\nabla \phi^h\|^2 + C(\nu)\|\nabla w\|^2_{L^3} \|\phi^h\|^2.$$

$$|b^*(w, \eta, \phi^h)| = \frac{1}{2} |(w^h \cdot \nabla \eta, \phi^h) - (w^h \cdot \nabla \phi^h, \eta)|,$$

$$\leq \frac{1}{2} \left\{ \|w^h\|_{L^6} \|\nabla \eta\|_{L^3} \|\phi^h\| + \|w^h\| \|\nabla \phi^h\|_{L^3} \|\eta\|_{L^6} \right\},$$

$$\leq \frac{1}{4} \|w\|^2_{L^3} \|\phi^h\|^2 + \frac{1}{4} \|\nabla \eta\|^2_{L^3} + \epsilon_2 \|\nabla \phi^h\|^3_{L^3} + C\epsilon_2^{-1/2} \|w\|^{3/2} \|\eta\|^{3/2}_{L^6}. $$
Setting \( \epsilon_1 = \epsilon_2 = \frac{1}{6} C_1 (C_s \delta)^2 \) and collecting all the terms gives
\[
\frac{1}{2} \frac{d}{dt} \left( \| \phi^h \|^2 + \frac{C_4^4 s^2}{\mu^2} \| \nabla \phi^h \|^2 \right) + \frac{\nu}{4} \| \nabla \phi^h \|^2 + \frac{1}{3} C_1 (C_s \delta)^2 \| \nabla \phi^h \|^2_{L^2}
\leq \left[ C(\nu) \| \nabla w \|^2_{L^2} + \frac{1}{4} \| \nabla w \|^2_{L^6} + \frac{1}{4} \| w \|^2_{L^6} \right] \| \phi^h \|^2
\]
\[
+ \left\{ C(\nu) \left[ \| \eta \|^2_{L^2} + \left( \frac{C_4^4 \delta^2}{\mu^2} \right)^2 \| \nabla \eta \|^2 + \| \nabla \eta \|^2 + \| p - \lambda h \|^2 \right] + (C_s \delta)^2 r^{3/2} \| \nabla \eta \|^2_{L^2}
\]
\[
+ \frac{1}{4} \| \eta \|^2_{L^6} + \delta^{-1} \| w \|^2 \| \eta \|^2_{L^6} + \frac{1}{4} \| \nabla \eta \|^2_{L^6} + \delta^{-1} \| w \|^3 \| \eta \|^3_{L^6} \right\}.
\]
Denote \( a(t) := C(\nu) \| \nabla w \|^2_{L^2} + \frac{1}{4} \| \nabla w \|^2_{L^6} + \frac{1}{4} \| w \|^2_{L^6} \) and its antiderivative is
\[
A(t) := \int_0^t a(s) \, ds < \infty \text{ for } w \in L^2(0, T; W^{1,3}(\Omega)) \cap L^2(0, T; L^6(\Omega)).
\]
Multiplying through by the integrating factor \( e^{-A(t)} \) gives
\[
\frac{d}{dt} \left[ \frac{1}{2} e^{-A(t)} \left( \| \phi^h \|^2 + \frac{C_4^4 s^2}{\mu^2} \| \nabla \phi^h \|^2 \right) \right] + e^{-A(t)} \left[ \frac{\nu}{4} \| \nabla \phi^h \|^2 + \frac{1}{3} C_1 (C_s \delta)^2 \| \nabla \phi^h \|^2_{L^2} \right]
\leq e^{-A(t)} \left\{ C(\nu) \left[ \| \eta \|^2_{L^2} + \left( \frac{C_4^4 \delta^2}{\mu^2} \right)^2 \| \nabla \eta \|^2 + \| \nabla \eta \|^2 + \| p - \lambda h \|^2 \right]
\right.
\[
\left. + (C_s \delta)^2 r^{3/2} \| \nabla \eta \|^2_{L^2} + \frac{1}{4} \| \eta \|^2_{L^6} + \delta^{-1} \| w \|^2 \| \eta \|^2_{L^6} \right\}
\]
Integrating over \([0, T] \) and multiplying through by \( e^{A(t)} \) gives
\[
\frac{1}{2} \left\{ \| \phi^h(T) \|^2 + \frac{C_4^4 s^2}{\mu^2} \| \nabla \phi^h(T) \|^2 \right\} + \int_0^T \left[ \frac{\nu}{4} \| \nabla \phi^h \|^2 + \frac{1}{3} C_1 (C_s \delta)^2 \| \nabla \phi^h \|^2_{L^2} \right] \, dt
\leq \exp \left( \int_0^T a(t) \, dt \right) \left\{ \frac{1}{2} \left( \| \phi^h(0) \|^2 + \frac{C_4^4 \delta^2}{\mu^2} \| \nabla \phi^h(0) \|^2 \right) \right.
\[
\left. + \int_0^T \left[ C(\nu) \left[ \| \eta \|^2_{L^2} + \left( \frac{C_4^4 \delta^2}{\mu^2} \right)^2 \| \nabla \eta \|^2 + \| \nabla \eta \|^2 + \| p - \lambda h \|^2 \right]
\right.
\[
\left. + (C_s \delta)^2 r^{3/2} \| \nabla \eta \|^2_{L^2} + \frac{1}{4} \| \eta \|^2_{L^6} + \delta^{-1} \| w \|^2 \| \eta \|^2_{L^6} \right\}
\]
Apply the Hölder’s inequality gives
\[
\int_0^T r^{3/2} \| \nabla \eta \|^2_{L^2} \, dt \leq \left( \int_0^T r^3 \, dt \right)^{1/2} \| \nabla \eta \|^2_{L^2(0, T; L^2)},
\]
\[
\int_0^T \| w \|^2 \| \eta \|^2_{L^6} \, dt \leq \| w \|^2 \| \eta \|^2_{L^6(0, T; L^6)},
\]
\[
\int_0^T \| w \|^3 \| \eta \|^3_{L^6} \, dt \leq \| w \|^3 \| \eta \|^3_{L^6(0, T; L^6)}.
\]
\[ \|w\|_{L^2(0,T;L^2)} \text{ and } \|w^h\|_{L^2(0,T;L^2)} \text{ are bounded by problem data by stability bound.} \]

Here \( r = \max\{\|\nabla w\|_{L^2}, \|\nabla w^h\|_{L^2}\} \) and \( \left( \int_0^T r^3 \, dt \right)^{1/2} = \|\nabla w\|_{L^2(0,T;L^2)}^{3/2} \) or \( \|\nabla w^h\|_{L^2(0,T;L^2)}^{3/2} \) also bounded. Using triangle inequality: \( \|e\| \leq \|\phi^h\| + \|\eta\|. \)

We obtain,

\[ \begin{aligned}
\| (w - w^h)(T) \|^2 &+ \frac{C\delta^2}{\mu^2} \| \nabla (w - w^h)(T) \|^2 + \int_0^T \left\{ \frac{\nu}{2} \| \nabla (w - w^h) \|^2 
+ \frac{2}{3} C(Cs\delta)^2 \| \nabla (w - w^h) \|_{L^3}^3 \right\} dt 
\leq \exp \left( \int_0^T a(t) dt \right) \left\{ \| (w - w^h)(0) \|^2 
+ \frac{C\delta^2}{\mu^2} \| \nabla (w - w^h)(0) \|^2 
+ \int_0^T \left[ C(\nu) \left( \inf_{\nabla h} \left( \| w_t - v_t^h \|_{L^2}^2 + \frac{C^4\delta^2}{\mu^2} \| \nabla (w_t - v_t^h) \|^2 + \| \nabla (w - v^h) \|^2 \right) \right) 
+ \inf_{\nabla^h} \| p - q^h \|^2 \right) \right] dt \right\}. \end{aligned} \]

**Remark 5.2.** If \( \hat{w} \) is taken to be the Stokes projection, then \( \| \nabla \eta \|^2 \) does not occur at the RHS.

**Remark 5.3.** If we use the following estimate for the term \( (p - \lambda^h, \nabla \cdot \phi^h) \) [17],

\[ \| (p - \lambda^h, \nabla \cdot \phi^h) \| \leq \| p - \lambda^h \|_{L^2} \| \nabla \cdot \phi^h \|_{L^2} \leq C \| \nabla \phi^h \|_{L^2} \| p - \lambda^h \|_{L^2}, \]

and set \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{9} C(Cs\delta)^2 \), the term \( C(\nu) \| p - \lambda^h \|_{L^2} \) in the RHS of (5.3) will become \( \delta^{-1} \| p - \lambda^h \|_{L^2} \) which does not depend on the Reynolds number but has lower order.

**Remark 5.4.** If we can replace the assumption \( \nu_{\text{turb}} = (Cs\delta)^2 \| \nabla w \| \) with new assumption \( \nu_{\text{turb}} = a_0(\delta) + (Cs\delta)^2 \| \nabla w \| \), then following [17], we can estimate \( (p - \lambda^h, \nabla \cdot \phi^h) \) as

\[ \begin{aligned}
\| (p - \lambda^h, \nabla \cdot \phi^h) \| &\leq \| p - \lambda^h \| \| \nabla \cdot \phi^h \|,
\leq a_0 \| \nabla \phi^h \|^2 + C(a_0) \| p - \lambda^h \|^2.
\end{aligned} \]

### 5.1. Time discretization of Modified Smagorinsky model

This subsection presents the unconditionally stable, linearly implicit, full discretization of (3.8).

Let the time-step and other quantities be denoted by

- time-step \( = k \), \( t_n = nk \), \( f_n(x) = f(x, t_n) \),
- \( w^h_n(x) \) = approximation to \( w(x, t_n) \),
- \( p^h_n(x) \) = approximation to \( p(x, t_n) \).

We perform the finite element spatial discretization and the first-order Backward Euler scheme for time discretization to get the following full discretization: Given \( (w^h_n, p^h_n) \in (X^h, Q^h) \), find \( (w^h_{n+1}, p_{n+1}^h) \in (X^h, Q^h) \) satisfying
\[
\left( \frac{w_{n+1}^h - w_n^h}{k}, v^h \right) + \frac{C_s^4 \delta^2}{\mu^2} \left( \nabla w_{n+1}^h - \nabla w_n^h, \nabla v^h \right) + b^*(w_n^h, w_{n+1}^h, v^h)
\]

\[
+ \nu(\nabla w_{n+1}^h, \nabla v^h) + (C_s \delta)^2(\nabla w_n^h \nabla w_{n+1}^h, \nabla v^h)
\]

\[
- (p_{n+1}^h, \nabla \cdot v^h) = (f_{n+1}(x), v^h) \forall v^h \in X^h,
\]

\[
(\nabla \cdot w_{n+1}^h, q^h) = 0 \forall q^h \in Q^h.
\]

This method is semi-implicit. We shall prove it is unconditionally stable in Theorem 5.5.

**Theorem 5.5.** (5.4) is unconditionally energy stable. For any \(N \geq 1\),

\[
\left( \frac{1}{2} \|w_N^h\|^2 + \frac{1}{2} \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_N^h\|^2 \right) + \sum_{n=0}^{N-1} \frac{1}{2} \left( \|w_{n+1}^h - w_n^h\|^2 \right)
\]

\[
+ \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2 \right) + k \sum_{n=0}^{N-1} \int_{\Omega} \left[ \nu + (C_s \delta)^2 \|\nabla w_n^h\| \|\nabla w_{n+1}^h\|^2 \right] dx
\]

\[
= \left( \frac{1}{2} \|w_0^h\|^2 + \frac{1}{2} \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_0^h\|^2 \right) + k \sum_{n=0}^{N-1} (f_{n+1}, w_{n+1}^h).
\]

**Proof.** Multiply (5.4) by \(k\) and take \(v^h = w_{n+1}^h\). Use Lemma (2.2) to get

\[
b^*(w_n^h, w_{n+1}^h, w_{n+1}^h) = 0. \text{ Hence,}
\]

\[
\|w_{n+1}^h\|^2 - (w_{n+1}^h, w_n^h) + \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_{n+1}^h\|^2 - \frac{C_s^4 \delta^2}{\mu^2} (\nabla w_{n+1}^h, \nabla w_n^h)
\]

\[
+ k \int_{\Omega} \left[ \nu + (C_s \delta)^2 \|\nabla w_n^h\| \|\nabla w_{n+1}^h\|^2 \right] dx = k(f_{n+1}, w_{n+1}^h).
\]

For the second and fourth term, apply the polarization identity (2.2),

\[
(w_{n+1}^h, w_n^h) = \frac{1}{2} \|w_{n+1}^h\|^2 + \frac{1}{2} \|w_n^h\|^2 - \frac{1}{2} \|w_{n+1}^h - w_n^h\|^2,
\]

\[
(\nabla w_{n+1}^h, \nabla w_n^h) = \frac{1}{2} \|\nabla w_{n+1}^h\|^2 + \frac{1}{2} \|\nabla w_n^h\|^2 - \frac{1}{2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2.
\]

Collecting terms and summing from \(n = 0\) to \(N - 1\), we get the result. \(\square\)

**Remark 5.6.** (5.5) is an energy equality, we can identify the following quantities:

1. Model kinetic energy = \(\frac{1}{2} \|w_N^h\|^2 + \frac{1}{2} \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_N^h\|^2\).
2. Eddy viscosity dissipation = \(\int_{\Omega} (C_s \delta)^2 \|\nabla w_N^h\| \|\nabla w_{n+1}^h\|^2 \) dx.
3. Numerical diffusion = \(\frac{1}{2} (\|w_{n+1}^h - w_n^h\|^2 + \frac{C_s^4 \delta^2}{\mu^2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2)\). This numerical diffusion arises due to the Backward Euler scheme.

**Remark 5.7.** The energy equality (5.5) can be also written as

\[
\frac{1}{2k} (\|w_{n+1}^h\|^2 - \|w_n^h\|^2) + \frac{1}{2k} \|w_{n+1}^h - w_n^h\|^2 + \nu\|\nabla w_n^h\|^2
\]

\[
+ \left\{ \frac{C_s^4 \delta^2}{2k\mu^2} (\|\nabla w_{n+1}^h\|^2 - \|\nabla w_n^h\|^2) + \frac{C_s^4 \delta^2}{2k\mu^2} \|\nabla w_{n+1}^h - \nabla w_n^h\|^2 \right. \int_{\Omega} (C_s \delta)^2 \|\nabla w_n^h\| \|\nabla w_{n+1}^h\|^2 \) dx = (f_{n+1}, w_{n+1}^h).
Line one and the RHS are from the backward Euler discretization of usual NSE. The bracketed term is a discretized form of model dissipation at \( t = t_{n+1} \). Here the term model dissipation in the paper can be positive or negative. When it is positive, it aggregates energy from mean to fluctuations. And when it is negative, energy is being transferred from fluctuations back to mean.

**Remark 5.8.** For (5.4), the model dissipation is

\[
\text{MD}^{n+1} = \frac{C^4 \delta^2}{2k \mu_2} (\| \nabla w_{n+1}^h \|^2 - \| \nabla w_n^h \|^2) + \frac{C^4 \delta^2}{2k \mu_2} \| \nabla w_{n+1}^h - \nabla w_n^h \|^2 \\
+ \int_{\Omega} (C_\delta)^2 |\nabla w_n^h| |\nabla w_{n+1}^h| \, dx.
\]

In this Test 8.2, we test use both Backward Euler and Crank-Nicolson to see the difference. We perform the finite element spatial discretization and the linearly implicit Crank-Nicolson (also called CNLE-CN with Linear Extrapolation) scheme for time discretization to get the following full discretization: for function \( w \), we denote the implicit Crank-Nicolson (also called CNLE-CN with Linear Extrapolation) scheme for time discretization to get the following full discretization: for function \( w \), we denote

\[
w_{n+\frac{1}{2}}^h = \frac{w_n^h + w_{n+1}^h}{2}, \quad w_n^h = \frac{3w_n^h - w_{n-1}^h}{2}.
\]

Given \((w_n^h, p_n^h) \in (X^h, Q^h)\), find \((w_{n+1}^h, p_{n+1}^h) \in (X^h, Q^h)\) satisfying

\[
\binom{w_{n+1}^h - w_n^h}{k} + \frac{C^4 \delta^2}{\mu^2} \binom{\nabla w_{n+1}^h - \nabla w_n^h}{k} \nabla v^h + b^*(\tilde{w}_{n+\frac{1}{2}}^h, w_{n+\frac{1}{2}}^h, v^h)

+ \nu (\nabla w_{n+\frac{1}{2}}^h \cdot \nabla v^h) + (C_\delta)^2 (|\nabla w_{n+\frac{1}{2}}^h| |\nabla w_{n+\frac{1}{2}}^h| \nabla v^h)

- (p_{n+\frac{1}{2}}^h, \nabla \cdot v^h) = (f_{n+\frac{1}{2}}(x), v^h) \quad \forall \, v^h \in X^h,

(\nabla \cdot w_{n+\frac{1}{2}}^h, q^h) = 0 \quad \forall \, q^h \in Q^h.
\]

We will prove it is unconditionally stable in Theorem 5.9.

**Theorem 5.9.** (5.6) is unconditionally energy stable. For any \( N \geq 1 \),

\[
\left( \frac{1}{2} \|w_N^h\|^2 + \frac{1}{2} C^4 \delta^2 |\nabla w_N^h|^2 \right) + k \sum_{n=0}^{N-1} \int_{\Omega} [\nu + (C_\delta)^2 |\nabla w_{n+\frac{1}{2}}^h| |\nabla w_{n+\frac{1}{2}}^h|^2 \, dx
\]

\[
= \left( \frac{1}{2} \|w_0^h\|^2 + \frac{1}{2} C^4 \delta^2 |\nabla w_0^h|^2 \right) + k \sum_{n=0}^{N-1} (f_{n+\frac{1}{2}}, w_{n+\frac{1}{2}}^h).
\]

**Proof.** Multiply (5.6) by \( k \) and take \( v^h = w_{n+\frac{1}{2}}^h \). Use Lemma (2.2) to get

\[
b^*(w_{n+\frac{1}{2}}^h, w_{n+\frac{1}{2}}^h, w_{n+\frac{1}{2}}^h) = 0.
\]

Hence,

\[
\frac{1}{2} \|w_{n+1}^h\|^2 - \frac{1}{2} \|w_n^h\|^2 + \frac{1}{2} C^4 \delta^2 |\nabla w_{n+1}^h|^2 - \frac{1}{2} C^4 \delta^2 |\nabla w_n^h|^2

+ k \int_{\Omega} [\nu + (C_\delta)^2 |\nabla w_{n+\frac{1}{2}}^h| |\nabla w_{n+\frac{1}{2}}^h|^2 \, dx = k (f_{n+\frac{1}{2}}, w_{n+\frac{1}{2}}^h).
\]

Collecting terms and summing from \( n = 0 \) to \( N - 1 \), we get the result.

**Remark 5.10.** (5.7) is an energy equality, we can identify the following quantities:
1. Model kinetic energy = \( \frac{1}{2} \|w_h^N\|^2 + \frac{1}{2} C_s^4 \delta^2 \|\nabla w_h^N\|^2 \).
2. Eddy viscosity dissipation = \( \int_{\Omega} (C_s\delta)^2 |\nabla w_{n+\frac{1}{2}}^h| \|\nabla w_{n+\frac{1}{2}}^h\|^2 \, dx \).
3. No Numerical diffusion.

Remark 5.11. The energy equality can be also written as
\[
\frac{1}{2k} (\|w_{n+1}^h\|^2 - \|w_n^h\|^2) + \nu \|\nabla w_{n+\frac{1}{2}}^h\|^2 \\
+ \left\{ \frac{C_s^4 \delta^2}{2k\mu^2} (\|\nabla w_{n+1}^h\|^2 - \|\nabla w_n^h\|^2) + \int_{\Omega} (C_s\delta)^2 |\nabla w_{n+\frac{1}{2}}^h| \|\nabla w_{n+\frac{1}{2}}^h\|^2 \, dx \right\} \\
= (f_{n+\frac{1}{2}}, w_{n+\frac{1}{2}}).
\]

Line one and line three are from the CNLE discretization of usual NSE. The bracketed term in the second line is a discretized form of model dissipation at \( t = t_{n+1} \).

Remark 5.12. For (5.6), the model dissipation is
\[
MD_{n+1} = \frac{C_s^4 \delta^2}{2k\mu^2} (\|\nabla w_{n+1}^h\|^2 - \|\nabla w_n^h\|^2) + \int_{\Omega} (C_s\delta)^2 |\nabla w_{n+\frac{1}{2}}^h| \|\nabla w_{n+\frac{1}{2}}^h\|^2 \, dx.
\]

6. Numerical Tests. In this section, we perform two numerical tests. In the first test, we show the numerical error and the rate of convergence of the Backward Euler scheme. In the second test, we show among Backward Euler (BE) and Crank-Nicolson with Linear Extrapolation (CNLE), CNLE exhibits intermittent backscatter.

6.1. A test with exact solution. (taken from V. DeCaria, W. J. Layton and M. McLaughlin [8]) The first experiment tests the accuracy of Modified Smagorinsky Model (3.8) and convergence rate of (5.4). The following test has an exact solution for the 2D Navier Stokes problem. Let the domain \( \Omega = (-1, 1) \times (-1, 1) \). The exact solution is as follows:
\[
\begin{align*}
  u(x, y, t) &= \pi \sin t (\sin 2\pi y \sin^2 \pi x, -\sin 2\pi x \sin^2 \pi y), \\
  p(x, y, t) &= \sin t \cos \pi x \sin \pi y.
\end{align*}
\]

This is inserted into the NSE and the body force \( f(x, t) \) calculated.
Uniform meshes were used with 270 nodes per side on the boundary. The mesh is fine enough compared to the time-step so that the main error from time-steps is only considered here. Taylor-Hood elements (P2-P1) were used in this test. We ran the test up to \( T = 10 \).

We take \( C_s = 0.1, \mu = 0.4, \delta \) is taken to be the shortest edge of all triangles \( \approx 0.104757 \). The norms used in the table heading are defined as follows,
\[
\|w\|_{\infty,0} := \text{ess sup}_{0 < t < T} \|w\|_{L^2(\Omega)} \quad \text{and} \quad \|w\|_{0,0} := \left( \int_0^T \|w(\cdot, t)\|_{L^2(\Omega)}^2 \, dt \right)^{1/2}.
\]
From the Table 1, we see the convergence rate is 1 which is expected from Backward Euler (5.4) discretization.

6.2. Test2. Flow between offset cylinder. (taken from N. Jiang and W. J. Layton [15]) This flow problem is tested to show the transfer of energy from fluctuations back to means in the turbulent flow using the Modified Smagorinsky Model (3.8).

The domain is a disk with a smaller off center obstacle inside. Let \( r_1 = 1, r_2 = 0.1, c = (c_1, c_2) = (1/2, 0) \), then the domain is given by

\[
\Omega = \{(x, y) : x^2 + y^2 < r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 > r_2^2\}.
\]

The flow is driven by a counterclockwise rotational body force

\[
f(x, y, t) = (-4y*(1-x^2-y^2), 4x*(1-x^2-y^2))^T,
\]

with no-slip boundary conditions on both circles. We discretize in space using Taylor-Hood elements. There are 80 mesh points around the outer circle and 60 mesh points around the inner circle. The flow is driven by a counterclockwise force (\( f=0 \) on the outer circle). Thus, the flow rotates about the origin and interacts with the immersed circle.

We start the initial condition by solving the Stokes problem. We compute up to final time \( T_{\text{final}} = 3 \). Take \( C_s = 0.1, \mu = 0.3, \delta \) is taken to be the shortest edge of all triangles \( \approx 0.0112927, \text{Re}=10,000 \). For Backward Euler (5.4), we compute the following quantities:

Model dissipation \( MD = \int_\Omega \left( \frac{C_s^4 \delta^2}{\mu^2} \nabla w_{n+1}^h - \nabla w_n^h \cdot \nabla w_n^h \right) dx \),

Effect of new term from MSM, \( MSMD = \int_\Omega \frac{C_s^4 \delta^2}{\mu^2} \nabla w_{n+1}^h - \nabla w_n^h \cdot \nabla w_n^h dx \),

Eddy viscosity dissipation \( EVD = \int_\Omega (C_s \delta)^2 |\nabla w_n^h||\nabla w_n^h|^2 dx \),

Viscous dissipation \( VD = \nu \|
\n\nTable 1: Numerical error and convergence rate \( Re = 5,000, T_{final} = 10, C_s = 0.1, \mu = 0.4, \delta = 0.104757 \).

| \( \Delta t \) | \( \| w - w^h \|_\infty,0 \) | rate | \( \| \nabla (w - w^h) \|_{0,0} \) rate | \( \| p - p^h \|_{0,0} \) rate |
|---|---|---|---|---|
| 0.05 | 3.269 | - | 5.2456 | - | 0.640426 | - |
| 0.02 | 0.82211 | 1.506 | 1.57800 | 1.311 | 0.235829 | 1.090 |
| 0.01 | 0.34689 | 1.245 | 0.70611 | 1.160 | 0.108197 | 1.124 |
| 0.005 | 0.16150 | 1.103 | 0.33166 | 1.090 | 0.0470288 | 1.202 |
For Crank-Nicolson CNLE (5.6), we compute the following quantities:

Model dissipation \( MD \) = \[ \int_{\Omega} \left( \frac{C_s^4 \delta^2}{\mu^2} \frac{\nabla w_{n+1}^h - \nabla w_n^h}{k} \cdot \nabla w_{n+\frac{1}{2}}^h \right. \]
\[ \left. + \left( C_s \delta \right)^2 \left| \nabla \tilde{w}_{n+\frac{1}{2}}^h \right| \left| \nabla w_{n+\frac{1}{2}}^h \right|^2 \right) \, dx. \]

Effect of new term from MSM, \( MSMD \) = \[ \int_{\Omega} \left( \frac{C_s^4 \delta^2}{\mu^2} \frac{\nabla w_{n+1}^h - \nabla w_n^h}{k} \cdot \nabla w_{n+\frac{1}{2}}^h \right) \, dx. \]

Eddy viscosity dissipation \( EVD \) = \[ \int_{\Omega} \left( C_s \delta \right)^2 \left| \nabla \tilde{w}_{n+\frac{1}{2}}^h \right| \left| \nabla w_{n+\frac{1}{2}}^h \right|^2 \, dx. \]

Viscous dissipation \( VD \) = \( \nu \left| \nabla w_{n+\frac{1}{2}}^h \right|^2 \).

Fig. 1: Comparison of Backward Euler (5.4) and linearized Crank-Nicolson (5.6) with \( \Delta t = 0.01, Re = 10,000, T_{final} = 3, C_s = 0.1, \mu = 0.4, \delta = 0.0112927. \)

It can be seen from the Figure Figure 1, model dissipation MD becomes negative sometimes for linearized Crank-Nicolson (5.6) and MD are all positive for Backward Euler (5.4). Only CNLE for Modified Smagorinksy has backscatter, which is consistent with the purpose of this model. Backward Euler has too much numerical diffusion, which makes it harder to see the backscatter from BE.
Fig. 2: Streamline plot using CNLE (5.6) There are 270 mesh points around the outer circle and 180 mesh points around the inner circle.

In the Figure 2, we notice the flow becomes smoother as it approaches statistical equilibrium.

7. Conclusion and future prospects. It was demonstrated that the Smagorinsky Model could be extended to non-equilibrium turbulence. In addition to that, we were able to show statistical backscatter without using negative turbulent viscosities. The stability of the model, uniqueness of the model’s solution, modeling error, and numerical error were analyzed in the paper. Since BE has numerical diffusion while CNLE does not, we can clearly observe backscatter from CNLE in the second numerical test.

In the next step, we can incorporate the penalty method with this model to get the desired result more efficiently.

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REFERENCES
[1] C. Amrouche, L. C. Berselli, R. Lewandowski, and D. D. Nguyen, Turbulent flows as generalized Kelvin–Voigt materials: Modeling and analysis, Nonlinear Analysis, 196 (2020), p. 111790.

[2] B. Baldwin and H. Lomax, Thin-layer approximation and algebraic model for separated turbulent flows, in 16th aerospace sciences meeting, 1978, p. 257.

[3] L. C. Berselli, R. Lewandowski, and D. D. Nguyen, Rotational forms of large eddy simulation turbulence models: Modeling and mathematical theory, Chinese Annals of Mathematics, Series B, 42 (2021), pp. 17–40, https://doi.org/10.1007/s11401-021-0243-z, https://doi.org/10.1007/s11401-021-0243-z.

[4] J. Boussinesq, "Essai sur la théorie des eaux courantes", Mémoires présentés par divers savants à l’Académie des Sciences 23 (1): 1-680, 1877.

[5] F. Boyer and P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models, vol. 183, Springer Science & Business Media, 2012.

[6] E. Burman, P. Hansbo, and M. G. Larson, Error estimates for the smagorinsky turbulence model: enhanced stability through scale separation and numerical stabilization, arXiv preprint arXiv:2102.00043, (2021).

[7] N. Chorfi, M. Abdelwahed, and L. C. Berselli, On the analysis of a geometrically selective turbulence model, Advances in Nonlinear Analysis, 9 (2020), pp. 1402–1419, https://doi.org/10.1515/anona-2020-0057, https://doi.org/10.1515/anona-2020-0057.

[8] V. DeCari, W. J. Layton, and M. McLaughlin, A conservative, second order, unconditionally stable artificial compression method, Computer Methods in Applied Mechanics and Engineering, 325 (2017), pp. 733–747.

[9] Q. Du and M. D. Gunzburger, Finite-element approximations of a Ladyzhenskaya model for stationary incompressible viscous flow, SIAM journal on numerical analysis, 27 (1990), pp. 1–19.

[10] Q. Du and M. D. Gunzburger, Analysis of a Ladyzhenskaya model for incompressible viscous flow, Journal of Mathematical Analysis and Applications, 155 (1991), pp. 21–45, https://doi.org/10.1016/0022-247X(91)90024-T, https://www.sciencedirect.com/science/article/pii/0022247X9190024T.

[11] P. A. Durbin and B. A. Pettersson-Reif, Statistical Theory and Modeling for Turbulent Flows, Second Edition, Wiley, Chichester, 2011.

[12] W. K. GEORGE, Lectures in turbulence for the 21st century. Chalmers University of Technology, available at http://www.turbulence-online.com, 2013.

[13] V. Girault and P. A. Raviart, Finite element approximation of the Navier-Stokes equations, vol. 749 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1979.

[14] J. G. Heywood and R. Rannacher, Finite-element approximation of the nonstationary Navier–Stokes problem. Part IV: Error analysis for second-order time discretization, SIAM Journal on Numerical Analysis, 27 (1990), pp. 353–384, https://doi.org/10.1137/0727022, https://doi.org/10.1137/0727022, https://arxiv.org/abs/https://doi.org/10.1137/0727022.

[15] N. Jiang and W. J. Layton, Algorithms and models for turbulence not at statistical equilibrium, Computers & Mathematics with Applications, 71 (2016), pp. 2352–2372, https://doi.org/10.1016/j.camwa.2015.10.004, https://www.sciencedirect.com/science/article/pii/S0898122115004861. Proceedings of the conference on Advances in Scientific Computing and Applied Mathematics. A special issue in honor of Max Gunzburger’s 70th birthday.

[16] N. Jiang, W. J. Layton, M. McLaughlin, Y. Rong, and H. Zhao, On the foundations of eddy viscosity models of turbulence, Fluids, 5 (2020), https://doi.org/10.3390/ Fluids5040167, https://www.mdpi.com/2311-5521/5/4/167.

[17] V. John and W. J. Layton, Analysis of numerical errors in large eddy simulation, SIAM Journal on Numerical Analysis, 40 (2002), pp. 995–1020, https://doi.org/10.1137/S0036142900375554, https://doi.org/10.1137/S0036142900375554, https://arxiv.org/abs/https://doi.org/10.1137/S0036142900375554.

[18] A. N. Kolmogorov, Equations of turbulent motion of an incompressible fluid, Izv. Acad. Sci. USSR, Physics, 6 (1942), p. 2.

[19] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, vol. 2, Gordon and Breach New York, 1969.

[20] O. A. Ladyzhenskaya, Modification of the Navier–Stokes equations for large velocity gradients, in Seminars in Mathematics VA Stheklov Mathematical Institute, vol. 2, 1970.

[21] W. J. Layton, A nonlinear, subgrid-scale model for incompressible viscous flow problems, SIAM Journal on Scientific Computing, 17 (1996), pp. 347–357, https://doi.org/10.1137/S1064827594262303, https://doi.org/10.1137/S1064827594262303, https://arxiv.org/abs/
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