NS Fivebrane and Tachyon Condensation

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We argue that a semi-infinite D6-brane ending on an NS5-brane can be obtained from the condensation of the tachyon on the unstable D9-brane of type IIA theory. The construction uses a combination of the descriptions of these branes as solitons of the worldvolume theory of the D9-brane. The NS5-brane, in particular, involves a gauge bundle which is operator valued, and hence is better thought of as a gerbe.
1. Introduction

In type IIA string theory a D6-brane can end on a Neveu-Schwarz fivebrane in a supersymmetric configuration. The simplest way to see this is to start from a fundamental string ending on a D5-brane in type IIB theory. Indeed this defines a Dirichlet fivebrane. Now by S-duality followed by T-dualities along all the spatial directions of the worldvolume of the resulting NS5-brane we reach the desired configuration. Recall that this system (and its T-dual cousins) are essential ingredients in ‘brane engineering’ of gauge theory dynamics following Ref. [2].

Naively a semi-infinite brane in a flat space cannot exist by charge conservation. There is a quantized charge of a D6-brane through a two-sphere enclosing it. However, in case of a semi-infinite brane this $S^2$ can just be ‘slipped off’ the end and collapsed, leading to an apparent contradiction. This argument fails because we are in a situation with a non-trivial Neveu-Schwarz $B$-field provided by the fivebrane. The gauge-invariant field strength is not simply the curvature of the RR one-form gauge field. The NS $B$-field also couples to the Chan-Paton gauge field modifying the Bianchi identity to $dF \sim H$.

We would like to obtain the semi-infinite D6-brane ending on a NS5-brane via tachyon condensation on the unstable D9-brane. According to Sen [3], all D-branes in type IIA string theory arise as solitons of the worldvolume theory on the D9-branes. In particular, the stable D6-brane is an ’t Hooft-Polyakov monopole of the gauge field-tachyon system on at least two D9-branes [4]. The situation is more complicated in the presence of the non-trivial $B$-field due to the fivebrane (see [5] where many configurations involving D-branes and the NS5-brane were discussed). Any configuration must satisfy the modified Bianchi identity. In our case, the relation $dF \sim H$ must hold for both the final D6-NS5-brane configuration after the tachyon condensation, as well as before it, for the D9-NS5-brane system. The problem of tachyon condensation in presence of an $H$ field whose quantized charges are $Z_n$ valued, was analyzed in Refs. [6]. This was generalized to the usual integrally quantized case in Ref. [7], which argues that in this situation one needs to consider the group of unitary operators in a Hilbert space as the gauge group on the D9-brane. Operator valued gauge fields appear in a natural way in the solitons of non-commutative gauge theory [9][10]. Indeed, Harvey and Moore [11] have suggested a configuration to describe an NS5-brane as a non-commutative soliton of open string theory (see also [12]). We present our arguments in this set-up.

It turns out that our construction is related to one version of what is called a gerbe [13], one in which it is described by operator valued gauge fields. (Let us note parenthetically
that in their study of the antisymmetric tensor gauge fields, Freund and Nepomechie discovered gerbes in string theory. Some recent applications to string theory may be found in [15,16].) As a matter of fact, the NS5-D6-brane configuration has been obtained as a stable solution of massive type IIA supergravity, in the language of gerbes. However, a different description of gerbes in terms of local U(1) bundles, was used in [15], which did not discuss tachyon condensation either.

2. Field theory analogue

It is instructive to look at a simpler field theory model, in four spacetime dimension, which share the essential features of the brane configuration we wish to obtain after tachyon condensation. This model consists of a semi-infinite Nielsen-Olesen vortex of the abelian Higgs model ending on a Dirac monopole. We can think of this monopole as a singular limit of the ’t Hooft-Polyakov monopole in the SO(3) Georgi-Glashaw model. With the Higgs field pointed radially outwards in the field space, this is a non-singular solution to the equations of motion with mass proportional to $M_W/g^2_{YM}$. The Nielsen-Olesen vortex, on the other hand, has constant finite energy per unit length. A semi-infinite vortex ending on a monopole is then an infinite energy configuration. To minimize its energy, the vortex will reduce its length thereby pulling the monopole along all the way to infinity. Hence the semi-infinite vortex string ending on a monopole is unstable.

There exists a remarkable way to stabilize this configuration by putting the monopole inside an accelerating black hole [17]. Here we begin with the abelian Higgs model coupled to gravity. This model has cosmic string, i.e., Nielsen-Olesen vortex solution as well as, say, a Schwarzschild black hole solution. Let us consider a configuration in which the vortex ends on a black hole. In this case one finds an axisymmetric metric with a conical singularity on the accelerating Schwarzschild black hole, whose metric up to a conformal factor is (see third reference in [18]),

$$ds^2 = \left(1 - \frac{2M}{r} - A^2 r^2 \right) dt^2 - \left(1 - \frac{2M}{r} - A^2 r^2 \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 (1-\alpha)^2 \sin^2 \theta d\phi^2, \quad (1)$$

where, $A$ is acceleration of the black hole. The deficit angle of the conical singularity is proportional to $\alpha$. This is a reflection of the fact that the vortex is piercing the black hole horizon. The Schwarzschild black hole horizon has the topology of a two sphere. Suppose in the frame of an asymptotic observer the vortex ends on the south pole of the horizon, then we can take a loop on the horizon around the south pole and measure the magnetic
flux of the vortex. For one vortex configuration, the \textit{angle valued} Higgs field $\Phi$ winds once around this loop. Now, by deforming this loop we can shrink it at the north pole. Since the vortex pierces the horizon only once, the shrinking of the loop at the north pole seems to lead to a contradiction. However, this result is misleading as the value of $\Phi$ is gauge dependent. The vortex is also accompanied by a topologically nontrivial gauge field configuration. A consistent solution corresponds to defining the Hopf fibration over $S^2$. This can be achieved by defining two charts on $S^2$

$$U_N = \{\theta, \phi : \theta < \pi\}, \quad U_S = \{\theta, \phi : \theta > 0\}. \quad (2)$$

On the overlap, the fields are related by the transition function $g_{NS} = \exp(-i\phi)$ as

$$\exp(i\Phi_N) = g_{NS} \exp(i\Phi_S), \quad A_{N\mu} = A_{S\mu} + ig_{NS}^{-1}\partial_{\mu}g_{NS}. \quad (3)$$

Since the vortex is on the lower hemisphere, we can take $\Phi_N = 0$ and $A_N = 0$ on the northern hemisphere and connect it to the vortex configuration on the southern hemisphere via the topologically nontrivial transition function. This configuration makes sense as long as the $S^2$ horizon does not shrink to zero size. This is ensured by the fact that the extended Schwarzschild geometry in the Kruskal coordinates is a wormhole with topology $S^2 \times \mathbb{R}$, the minimum radius of the sphere being $2GM$. In the extended geometry, absence of the vortex in the northern hemisphere can be explained in the following manner. As the vortex approached the south pole of the horizon it goes down the throat of the wormhole and reappears through the horizon in the other asymptotic region.

Let us now get back to the winding number of $\Phi$. This quantity is not gauge invariant in the presence of a nontrivial gauge field configuration. The field $\Phi$ is also not single valued everywhere on the sphere. We can, however, define a quantity

$$\mathcal{A} = d\Phi - A, \quad (4)$$

which is both gauge invariant and single valued. In the northern hemisphere we have chosen $\Phi_N = 0$ and $A_N = 0$, which implies even in the southern hemisphere the net winding charge should vanish. Clearly $\mathcal{A}$ being single valued has zero winding charge. On the other hand, we saw that a vortex solution near the south pole has $d\Phi$ winding number one. Hence we conclude that the integral of $\mathcal{A}$ around the loop near the south pole also has unit winding charge.
3. Neveu-Schwarz fivebrane in open string theory

The NS5-brane is a soliton of the closed string theory. Therefore one does not expect to see detailed features of it in open string theory. Nevertheless, it turns out that some topological aspect of the NS5-brane can be captured in terms of open strings. Harvey and Moore \[11\] have a configuration with the right $H$-flux. In fact they have argued that the NS5-brane may be thought of as a particular soliton in the non-commutative gauge theory of the unstable D9-brane in type IIA theory. This is inspired by the idea of a non-commutative tachyon\[10\].

We will now review this construction. In \[11\], the spacetime topology is chosen to be $\mathbb{R}^{1,4} \times \mathbb{R}^2_{NC} \times S^2 \times S^1$ and the $H$-flux is through the 3-cycle $S^2 \times S^1$. This is based on an example in \[19\]. We will work, however, with $\mathbb{R}^{1,4} \times \mathbb{R}^2_{NC} \times S^3$, which is the spacetime topology, at least in the near horizon limit, of the NS5-brane\[20\]. There is a constant NS $B$-field along $\mathbb{R}^2_{NC}$. It should be emphasized that this is not the $B$-field that contributes to the $H$-flux, but has the effect of making (the $\mathbb{R}^2_{NC}$ part of) spacetime non-commutative. Therefore we may treat the tachyon, gauge and other fields as operator valued on $\mathbb{R}^{1,4} \times S^3$. Henceforth we will concentrate only on the $S^3$ part.

In Refs.\[8\] \[21\], it was argued that the gauge group in non-commutative gauge theory is $U_{cpt}(\mathcal{H})$, a subgroup of unitary operators in a Hilbert space $\mathcal{H}$ of the form $u = 1 + K$, where $K \in \mathcal{K}(\mathcal{H})$ is a compact operator. Conjugation by elements of the group $U(\mathcal{H})$ are automorphisms of the corresponding Lie algebra $\mathcal{K}(\mathcal{H})$ of compact operators. However, since the $U(1)$ centre of $U(\mathcal{H})$ acts trivially, the automorphism of $\mathcal{K}(\mathcal{H})$, (and hence of $U_{cpt}(\mathcal{H})$), is really $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$. The Lie algebra valued gauge field and the tachyon which transform in the adjoint representation are therefore valued in $\mathcal{K}(\mathcal{H})$. A non-trivial gauge bundle may be constructed with a twist by an element of $\text{Aut}(\mathcal{K}(\mathcal{H})) = PU(\mathcal{H})$. The proposal of Ref.\[11\] is that an appropriate non-trivial $PU(\mathcal{H})$ bundle on $S^3$, with the tachyon field at the maximum of the potential, represents a D9-brane and an NS5-brane. Moreover, when the tachyon condenses to a minimum of the potential, there is only an NS5-brane as the D9-brane ought to have disappeared according to Sen’s conjecture\[5\].

The key to the construction of this $PU(\mathcal{H})$ bundle is the fact that $\pi_2(PU(\mathcal{H})) = \mathbb{Z}$. It is then possible to patch together trivial bundles on local coordinate charts of $S^3$ along their overlaps. This is a one higher dimensional generalization of the construction of a monopole on $S^2$, which used the fact that $\pi_1(U(1)) = \mathbb{Z}$. As we have mentioned before, the base space in \[11\] is $S^2 \times S^1$. The Hopf fibration $S^3 \to S^2$ and the covering space
of the fiber $\mathbf{R} \to S^1$ defines a natural $S^1 \times \mathbf{Z}$ bundle on it. This is embedded in $\text{PU}(\mathcal{H})$ by lifting the circle coordinate to an angle valued position operator $\hat{\Omega}$ and its (integrally quantized) conjugate momentum $\hat{L}$ satisfying a Heisenberg algebra

$$[\hat{\Omega}, \hat{L}] = i \hat{1}. \quad (5)$$

Let us review some details of this construction following Ref. [19]. One starts with a principle $\text{U}(1)$ bundle $P(\text{U}(1))$ over a base manifold $X$ and the universal covering space $\mathbf{R}$ of $S^1$. This defines a principle $P(\text{U}(1) \times \mathbf{Z})$ bundle over the base space $X \times S^1$. In the following we will consider the specific example of the Hopf bundle $X = S^2$ and $P(\text{U}(1)) = S^3$. Given a (vector) space $V$ on which the group $G$ acts, it is possible to define a fibre bundle $E_V \to X \times S^1$ associated to the principle $G$ bundle by the quotient $(P(G) \times V)/G$. The fibre of this bundle is isomorphic to $V$. The frame bundle and the tangent bundle to a manifold is an example of such a pair. Another natural association is one in which $V$ is the Lie algebra $\hat{\mathfrak{g}}$ of $G$ or any of its representations. The $G$ action on $V$, in turn, induces an adjoint action on the space of linear operators $\mathcal{L}(V)$ on $V$. One can, therefore, construct an associated ‘bundle’ whose fibre is $\mathcal{L}(V)$ and the transition functions $g_{ij}$ act by adjoint action, (with the centre acting trivially). In particular, our objective will be to construct a bundle whose fibre consists of operators in the Hilbert space $\mathcal{H}$ of square integrable functions $L^2(S^1)$ of a projective representation of the Heisenberg group $H$, which is a central extension of the group $S^1 \times \mathbf{Z}$.

In order to specify the bundle, it will be sufficient to give a local trivialization over open sets $\mathcal{U}_i$, in which we specify the Hilbert space of functions and provide the transition functions. Let $(x = (\psi, \theta), \phi)$ be points in $S^2 \times S^1$ and

$$p : S^3 \times \mathbf{R} \to S^2 \times S^1$$

be the projection map of the smaller bundle with which the construction proceeds. The fibre $p^{-1}(x)$ is a circle $S^1_x$ without any fixed base point. The universal cover $\tilde{S}^1_x \sim \mathbf{R}_x$ is ambiguous up to the cyclic group generated by $T$ which shifts the coordinate of $\mathbf{R}_x$ by $(2\pi$ times) an integer. Consider the Hilbert space of functions

$$\mathcal{H}_{(x, \phi)} = \left\{ f : \mathbf{R}_x \to \mathbf{R}_x \mid f(T(\xi)) = e^{i\phi} \cdot f(\xi), \tilde{\xi} \in \mathbf{R}_x \right\}. \quad (6)$$

---

1 Recall that in the associated bundle the sections coming from a quotient action of $G$ on $G \times V$ are identified as $(s, v) \sim (sg, g^{-1}v)$ providing a twist.
Of course, the space depends on the choice of coordinates on the universal cover, but the ambiguity is up to an action of $T$, which acts as multiplication by a scalar leading to a unique projective Hilbert space. Let $S^1$ be a circle which we can identify with the fibre $S^1_x$. With an abuse of notation we will use the same angular coordinate $\xi$ on both these circles. Consider the square integrable functions $f(\xi)$ on the circle satisfying the following property under the isomorphism $\lambda_{(x,\phi)} : L^2(S^1) \sim H_{(x,\phi)}$

$$\lambda_{(x,\phi)}(f(\xi)) = \exp\left(\frac{i}{2\pi} \xi \phi \right) f(\xi).$$

This specifies the local trivialization. It is easy to check that the above satisfies the property required of $H_{(x,\phi)}$ defined above.

The Lie algebra $\hat{h}$ of the group $H$ has generators $(\hat{L}, \hat{1}) = (-i \frac{d}{d\xi}, \hat{1})$. Since the spectrum of the angular momentum $\hat{L}$ is discrete, there is no Lie algebra associated with the generator $\hat{\Omega}$ of the Heisenberg algebra (5). Rather $\exp(2\pi i \hat{\Omega}) \sim T$ is the generator of the automorphism discussed earlier. The group $H$ acts by adjoint action on $\hat{h}$. While the action of the centre and the shift in $S^1$ generated by $\hat{L}$ is trivial, the $\mathbb{Z}$ acts nontrivially as follows:

$$e^{2\pi i \ell \hat{\Omega}} \hat{L} e^{-2\pi i \ell \hat{\Omega}} = \hat{L} - 2\pi \ell \hat{1}. \tag{7}$$

The sections of the associated PU($H$) bundle are vectors $v(x, \phi)$ in $\hat{h}$ satisfying

$$v(x, \phi + 2\pi \ell) = e^{2\pi i \ell \hat{\Omega}} v(x, \phi) e^{-2\pi i \ell \hat{\Omega}}.$$ 

Writing $v$ in terms of the basis elements as $v = v_1(x, \phi) \hat{1} + v_2(x, \phi) \hat{L}$, we get: $v_1(x, \phi + 2\pi \ell) = v_1(x, \phi) - 2\pi \ell v_2(x, \phi)$ and $v_2(x, \phi + 2\pi \ell) = v_2(x, \phi)$.

The final ingredient is a linear function from the PU($H$) bundle to $\mathbb{R}$. In order to motivate this, let us start with the exact sequence of vector spaces $V_C$ (generated by $\hat{1}$), $\hat{h}$ and $V_L$ (generated by $\hat{L}$). The exact sequence of bundles

$$E_R \rightarrow E_{\hat{h}} \rightarrow E_{V_L}$$

follows from it, moreover, $E_R \sim \mathbb{R} \times (S^2 \times S^1)$ is a trivial bundle. However, while in the former sequence $\hat{h}$ cannot be written as a direct sum of $V_C$ and $V_L$, it is possible to do so in the latter (although the Lie algebra will not be respected in the process). The linear function, which we will call ‘tr’

$$\text{tr} : E_{\hat{h}} \rightarrow \mathbb{R},$$
provides this decomposition. For a vector \( v \in \mathfrak{h} \), which can be written in terms of the basis elements as \( v = v_1(x, \phi) \hat{I} + v_2(x, \phi) \hat{L} \), we define
\[
\text{tr} (v) = (v_1(x, \phi) - \phi v_2(x, \phi)).
\tag{8}
\]
It is clear from (7) that the function tr is well defined on the \( \text{PU}(\mathcal{H}) \) bundle. In particular, when \( v_1 = 0 \), \( v_2 = 1 \), we have
\[
\text{tr} (\hat{L}) = \phi.
\]
We will use this in a moment.

In order to specify a connection on this bundle, we start with a connection on the principle \( \text{U}(1) \) bundle \( p : S^3 \to S^2 \), which is the familiar monopole gauge field configuration \( A^{(M)} \). The gauge field of this bundle is a 1-form on \( S^2 \) valued in the Lie algebra generated by \( -i \frac{d}{dx} \). The gauge connection of the \( \text{PU}(\mathcal{H}) \) bundle is a 1-form valued in \( \mathfrak{h} \) and is taken to be \( A^{(M)} \). Since \( \mathfrak{h} \) is abelian, the curvature of this connection is a 2-form
\[
F^{(M)} = dA^{(M)} \hat{L},
\]
where we have displayed the \( \mathfrak{h} \) dependent part explicitly.\(^2\) Acting with the linear function tr, we obtain \( \text{tr} F^{(M)} = dA^{(M)} \phi \). This is called the ‘scalar curvature’ in [19]. It is a 2-form which is not closed, rather \( d \text{tr} F^{(M)} \sim \text{vol}(S^2 \times S^1) \).

In case of the \( S^3 \) base, once again we use the Hopf fibration, however, this time, following [22] the \( \text{U}(1) \) action along the fibre is lifted to a \( \text{U}(1) \) action in \( \text{PU}(\mathcal{H}) \) with the help of a cocycle. Let us consider \( S^3 \) as the unit sphere defined by \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \), which may be written as
\[
|z_0|^2 + |z_1|^2 = 1,
\]
in terms of complex coordinates of \( \mathbb{C}^2[x_1 + ix_2, x_3 + ix_4] \). Let us cover \( S^3 \) by charts
\[
\mathcal{U}_0 = \{(z_0, z_1) \in S^3 : |z_0| \geq |z_1|\}
\]
\[
\mathcal{U}_1 = \{(z_0, z_1) \in S^3 : |z_0| \leq |z_1|\}.
\tag{9}
\]
Each \( \mathcal{U}_i \) is topologically a disc times \( S^1 \) and they overlap on a two-torus
\[
\mathcal{U}_0 \cap \mathcal{U}_1 = T^2 = \{(z_0, z_1) \in S^3 : |z_0| = |z_1| = 1/\sqrt{2}\}.
\tag{10}
\]
\(^2\) This is same as writing \( F = F^a T^a \) in YM theories.
The simplest way to see that $S^3$ has this topological structure is to think $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Now, if we remove a solid torus from $\mathbb{R}^3$, what remains, together with the point at infinity, is also a solid torus. Introduce coordinates $(\psi, \theta, \phi)$ on $S^3$ such that $z_0 = \cos \frac{\psi}{2} e^{i(\phi+\theta)/2}$ and $z_1 = \sin \frac{\psi}{2} e^{i(\phi-\theta)/2}$. The bundle structure of Hopf fibration is given by the local trivializations

$$
U_0 \sim \left( \frac{z_1}{z_0} \right), \quad U_1 \sim \left( \frac{z_0}{z_1} \right),
$$

along with the transition function $z_0/z_1$ on the overlap. The $U(1)$ action

$$
(z_0, z_1) \rightarrow (e^{-i\omega} z_0, e^{-i\omega} z_1)
$$

along the fibre is an isometry$^3$ of $S^3$.

The $PU(H)$ bundle on $S^3$ is specified by a map

$$
g_{01} : \mathcal{U}_0 \cap \mathcal{U}_1 = T^2 \rightarrow PU(H).
$$

Local trivializations

$$f_0 : \mathcal{U}_0 \rightarrow \mathcal{K}(H) \quad f_1 : \mathcal{U}_1 \rightarrow \mathcal{K}(H)$$

are related by $f_0 = g_{01} f_1 g_{01}^{-1}$ on the overlap. The topological properties of the bundle are characterized by the homotopy class of $g_{01}$, an element of maps from $T^2$ to $PU(H)$. Now since $\pi_n(U(H)) = 0$ for all $n$, we have $\pi_{n-1}(U(1)) = \pi_n(PU(H))$, hence the only non-vanishing homotopy group of $PU(H)$ is $\pi_2$ and this is $\mathbb{Z}$. The homotopy classes of maps $g_{01}$ of interest is therefore isomorphic to $H^2(T^2, \mathbb{Z})$ (see, for example, chapter 1). Now, using the relation between the differential complexes on $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_0 \cup \mathcal{U}_1 = S^3$ and $\mathcal{U}_0 \cap \mathcal{U}_1 = T^2$, we have $H^2(T^2, \mathbb{Z}) = H^3(S^3, \mathbb{Z}) = \mathbb{Z}$.

Let us define the $U(1)$ action on the $PU(H)$ bundle as

$$\Omega_\omega : (f_0, f_1) \rightarrow (f_0^\omega, f_1^\omega)$$

where,

$$f_0^\omega(z_0, z_1) = f_0(e^{-i\omega} z_0, e^{-i\omega} z_1)$$

and

$$f_1^\omega(z_0, z_1) = h(\omega, z_0, z_1) (f_1(e^{-i\omega} z_0, e^{-i\omega} z_1)),
$$

for a function $h(\omega, z_0, z_1) : S^1 \times \mathcal{U}_1 \rightarrow PU(H)$ which satisfy

$$h(\omega, z_0, z_1) = g_{01}^{-1}(z_0, z_1) g_{01}(e^{-i\omega} z_0, e^{-i\omega} z_1) \quad \text{for} \quad (z_0, z_1) \in \mathcal{U}_0 \cap \mathcal{U}_1,$$

$$h(\omega_1 + \omega_2, z_0, z_1) = h(\omega_1, z_0, z_1) h(\omega_2, e^{-i\omega_1} z_0, e^{-i\omega_1} z_1) \quad \text{for} \quad (z_0, z_1) \in \mathcal{U}_1.$$

---

$^3$ The metric in these coordinates is $ds^2 = \frac{1}{4} \left( d\psi^2 + \sin^2 \psi d\theta^2 + (d\phi + \cos \psi d\theta)^2 \right)$. 

8
The first of the conditions (13) ensures that the patching condition given by (13) is respected. In other words, \( \Omega_{\omega} \) is an automorphism of the triple \((f_0, f_1; g_{01})\) used to define the \( \text{PU}(\mathcal{H}) \) bundle. The second condition is a group homomorphism that identifies an \( S^1 \) in \( \text{PU}(\mathcal{H}) \). The existence of \( \Omega_{\omega} \) with the required properties is proven in Ref. [22].

We will use the above to propose a construction along the lines of [11], with \( \Omega_{\omega} \) playing the role of \( \hat{\Omega} \) in (5). We cannot identify \( \hat{L} \) explicitly in \( \text{PU}(\mathcal{H}) \), but proceed with the assumption that there is one such \( \hat{L} \) such that (3) is true. This assumption is not untenable since the size of the fibre in Hopf fibration is fixed, namely \( 4\pi \), therefore the spectrum of \( \hat{L} \) is discrete. Motivated by the construction in \( S^2 \times S^1 \) and the Hopf bundle description of \( S^3 \) outlined above, we propose the following expression for the gauge fields using the two charts (11) of \( S^2 \).

\[
A_0 = +\frac{i}{2} (1 - \cos \psi) d\theta \cdot \hat{L} \\
A_1 = -\frac{i}{2} (1 + \cos \psi) d\theta \cdot \hat{L},
\]

where we have displayed the algebra generator explicitly. Recall the charts overlap for \( \psi = \frac{\pi}{2} \), where the transition functions \((\psi, \theta) \to (\pi - \psi, -\theta)\) of the Hopf bundle ensures that \( A_0 \) and \( A_1 \) differ by a gauge transformation.

The ‘scalar curvature’ of the gauge field (16) is obtained by taking the tr:

\[
\text{tr} F = \frac{i}{2} \sin \psi d\psi \wedge d\theta (\phi + \theta),
\]

where, in analogy with (8), we have used \( \text{tr} \hat{L} = \phi + \theta \), the value of the coordinate of the Hopf fibre of the base space. Notice that this is a well defined 2-form. Finally,

\[
d\text{tr} F = \frac{i}{2} \sin \psi d\psi \wedge d\theta \wedge d\phi,
\]

is the volume form on \( S^3 \) yielding a unit 3-form flux through it. It is assumed here that the tachyon is trivial, it is zero corresponding to the maximum of the potential. In other words, this gauge field configuration describes the unstable D9-brane in presence of the NS5-brane.

\[
\text{PU}(\mathcal{H}) \] bundle on \( S^3 \) has also been defined through local trivializations \( S^3 = D^3_+ \cup D^3_- \), \( D^3_+ \cap D^3_- = S^2 \) at the equator [20]. The topological properties of the bundle are characterized by maps from \( S^2 \) to \( \text{PU}(\mathcal{H}) \). This, however, does not seem suitable for our purpose as there is no natural U(1) action.
The NS5-brane so constructed has its worldvolume along $\mathbb{R}^{1,4}$ as well as along one of the non-commutative dimensions in $\mathbb{R}^{2}_{NC}$. We refer to [1] for some subtleties with this description.

Finally, although we have been talking about a PU($\mathcal{H}$) bundle, the above construction is not a bundle in the usual sense. In Appendix A, we show how it satisfies the conditions required of a gerbe.

4. Semi-infinite D6-brane and NS fivebrane

In the previous section we have constructed a configuration of the operator valued gauge fields in the noncommutative worldvolume theory of the unstable D9-brane of type IIA theory. This configuration carries a unit $H$-flux through $S^3$, and satisfies the modified Bianchi $d \text{tr} F = H$. It is argued[11] that for the tachyon at the maximum of the potential, this configuration describes the D9-NS5-brane system, while at a minimum there is only an NS5-brane. One expects that, when the tachyon is non-trivial, we should have a configuration of the NS5-brane together with a D-brane of appropriate codimension.

Recall, that in the absence of any $H$-flux, (no NS5-brane), all the stable D$p$-branes may be obtained as odd codimension soliton solutions of the tachyon and gauge field theory. In particular the D6-brane is the ’t Hooft-Polyakov monopole of the U(2) theory on two D9-branes[6]. Let $(x_1, x_2, x_3)$ be the space transverse to the would be D6-brane. Identifying $\text{SU}(2) \subset \text{U}(2)$ with the (covering space) of the SO(3) group of rotations, the configuration is

$$
T \sim x_i \sigma^i \\
A_i \sim \epsilon_{ijk} x^j \sigma^k,
$$

where $\sigma^i$ are the Pauli matrices. There should also be some convergence factors on the right hand side. One important feature of this construction is that it is local, i.e. it relies only on coordinates in a small neighbourhood of the origin where the D6-brane is located.

The configuration we would like to obtain is that of a semi-infinite D6-brane that ends on the NS5-brane. The fivebrane shares all its worldvolume dimensions with the D6-brane, whose additional dimension has a boundary on which the NS5-brane lies. Let us put the NS-brane at the origin of its transverse $\mathbb{R}^4$ directions. This space is foliated by $S^3$ of varying radii, with the size finally saturating to give the ‘throat’ geometry[20]. The D6-brane appears to be a string which pierces the $S^3$’s at, say, the south pole. Although, to an observer far away from the origin, it would seem that the D6-brane ends on the
NS5-brane at the origin, it would be more correct to say that it goes down the ‘throat’. Actually for our case, where there are two non-commutative dimensions and the D6-brane worldvolume extends along both; this picture is an extrapolation from the commutative limit. In particular, the radial direction in $\mathbb{R}^4$, \textit{i.e.}, the direction transverse to $S^3$ is one of the non-commutative directions.

We would like to argue that the situation is different \textit{after} tachyon condensation. The operator $\hat{\Omega}$ is a shift along the Hopf fibre of $S^3$. The process of tachyon condensation selects a special point, which we may choose to be the south pole. The fields are localized around this point, in particular, also along the Hopf fibre through it. This in turn, determines the value of the operator $\hat{\Omega}$ to be, say, zero, to an accuracy $\Delta \Omega \sim \varepsilon$. As a result, there is a large uncertainty in the value of the conjugate variable $\hat{L}$: $\Delta L \sim 1/\varepsilon$. This, in effect makes the spectrum of $\hat{L}$ continuous for sufficiently small $\varepsilon$. Therefore, after the tachyon has condensed, it should be possible to shift $\hat{L}$ by an arbitrary amount. This is in contrast to the previous section in which the fields are not localized in $S^3$.

We can follow the field theory example in Sec. 2 and use continuous gauge transformation of the form $\exp(ix\hat{\Omega})$ (for any real $x$) available now to propose the following gauge field configurations:

$$A_0 = +\frac{i}{2}(1 - \cos \psi)d\theta \cdot \hat{L}$$
$$A_1 = -\frac{i}{2}(1 + \cos \psi)d\theta \cdot (\hat{L} + (\phi - \theta)\hat{1}) .$$

(19)

At the overlap, an operator valued U(1) gauge transformation

$$A_1 = e^{-i(\phi - \theta)\hat{\Omega}} A_0 e^{i(\phi - \theta)\hat{\Omega}},$$

(20)

relates the gauge configurations from the chart $\mathcal{U}_0$ to $\mathcal{U}_1$.

An operator valued U(1) gauge transformation is in fact equivalent to the gauge transformation of the $B$ field (see Appendix A and [10]). The field configuration (19) has the property that in the chart $\mathcal{U}_0$, $d\text{tr} F_0 \sim \text{vol}(S^3)$ as before, but in $\mathcal{U}_1$, $\text{tr} F_1$ and hence $d\text{tr} F_1$ vanish. Hence there is an NS $H$-flux through $\mathcal{U}_0$ but none through $\mathcal{U}_1$.

Continuing to follow the field theory example, we now need to show that the operator valued U(1) gauge field configuration (19) arises from a configuration of the tachyon and gauge fields (in the $S^3$ part of the worldvolume of the non-BPS D9-brane). This ought to localize the energy around the south pole of $S^3$, which we assume is at the origin of $\mathcal{U}_0$. 

11
Unfortunately, we are not able to write this explicitly. However, the operator corresponding to the tachyon field is expected to be of the form

$$T \sim \int d\varphi \langle \varphi| e^{-\varphi^2/\varepsilon^2} |\varphi\rangle,$$

which is a projection operator. Interestingly, in [6], it is shown that the configuration (18) can also be thought of as a two step process of a vortex and a kink. This seems more natural in the present situation as the local symmetry group around the south pole is $SO(2) \times \mathbb{R}$, since $U_0$ has the topology of a cylinder $D_0^3 \times S^1$, which naturally accommodates this break up.

5. Bianchi identities and charge conservation

In Ref.[13] the NS5-D6-brane configuration described in the previous section was shown, following a construction in [13], to be a solution of type IIA supergravity using a description of gerbes as local line bundles. There the various charge conservation conditions are discussed in details. Briefly, consider an $S^2$ surrounding the D6-brane. There is a flux of the RR one-form gauge field $C^{(1)}_{RR}$ through it. In the absence of any $H$-flux, this measures the quantized RR charge of the D6-brane. This arises from the Chern-Simons couplings, $(dT \wedge F_{CP} \wedge C^{(7)}, dT \wedge dT \wedge dT \wedge C^{(7)},$ etc), on the D9-brane. When an NS $B$-field is present, the correct gauge invariant field strength for this field is

$$G^{(2)}_{RR} = dC^{(1)}_{RR} + mB,$$

where, $B$ is the NS 2-form and $m$ is the mass parameter. Similarly,

$$\mathcal{F} = dA_{CP} - B,$$

is a gauge invariant combination of the field strength involving the Chan-Paton gauge field.

In our description, the flux is through a 2-cycle $T^2$ at the overlap of $U_0$ and $U_1$. This is assumed to enclose D6-brane at the south pole of $S^3$. Since the D6-brane is semi-infinite, there is no Chan-Paton gauge field flux through the chart $U_1$. We can also choose to set $B = 0$ here. This means $\mathcal{F} = 0$ in $U_1$, and in particular, there is no $\mathcal{F}$-flux through it. Notice that $\mathcal{F}$ is gauge invariant and therefore $\mathcal{F}$-flux must vanish everywhere. The overlap $T^2$ is the boundary of a 3-space $U_0$, which is that part of the $S^3$ through which there is a nontrivial $H$-flux. The anomalous Bianchi identity[3] from (22) ensures that there is no net six-brane charge. By drawing analogy with the field theory example, it now follows that
the net monopole charge through a $T^2$ (at the overlap of $U_0$ and $U_1$ or any deformation of it in $U_0$), enclosing the south pole of $S^3$ should also vanish. Since $\mathcal{F}$-flux through $T^2$ vanishes and the flux of $dA_{CP}$ does not, we conclude that the monopole charge evaluated by integrating $dA_{CP}$ over a $T^2$ enclosing the south pole is equal to the boundary value of the $H$-flux through the 3-space $D_0^2 \times S^1$ enclosed by $T^2$, i.e.,

$$\oint_{T^2} dA_{CP} = \int_{D_0^2 \times S^1} H = \oint_{T^2} B. \quad (24)$$

This in effect implies a modified Bianchi identity

$$dF_{CP} = H \quad (25)$$

for the NS5-D6-brane configuration.

6. Discussion

We have argued how to realize a configuration in which a semi-infinite D6-brane ends on an NS5-brane via condensation of the tachyon field on the worldvolume of unstable D9-branes. Both the five- as well as the six-brane are solitonic configurations in the non-commutative field theory on the D9-brane. Let us emphasize that although the six-brane by itself is a solution of this field theory, the NS5-brane is only a configuration. In the framework of open string theory, it is as yet unclear in what sense a solitonic object of closed string theory can be realized as a solution. Some topological aspect of the NS5-brane can, however, be reproduced. Our intersecting brane configuration, in which we have combined features of [6] and [11], is also not a solution of the open string equations of motion.

The geometrical description we have used is strictly valid for large values of NS5-brane charge, or far away from the core of the fivebrane. Moreover, two of the longitudinal directions of our D6-brane carry a constant $B$ field. This is the $B$ field introduced to have a non-commutative worldvolume theory. The Chern-Simons coupling $B \wedge C_{RR}^{(5)}$ therefore results in an induced D4-brane charge in the configuration.

Let us end by some speculative remarks on D3-branes in SU(2) WZW model. In this case, it is well-known that the symmetries allow only D2-branes and D0-branes along conjugacy classes of the group manifold [27]. These are $S^2$’s at some fixed ‘latitudes’ of $S^3$. On the other hand, Ref. [28] argued in favour of D3-branes which wrap almost all of $S^3$ except for a set of points. More recently, based on consistency with T-duality, Ref. [29]
showed that there should be D3-branes which are ‘fat’ D-strings. These have the topology of a cylinder reminiscent of the coordinate charts we have used in our construction of the NS5-D9-brane configuration. While a single fat string cannot cover the entire group manifold without having a singularity, it seems possible for a configuration of two fat D-strings ‘linked’ together to do so. A nontrivial linking should capture the fact that this configuration is a gerbe. This may be possible with operator valued gauge fields on the fat strings.

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Appendix A. Gerbes

Gerbes are generalization of U(1) bundles (more generally line bundles) on a manifold. This appendix contains a quick description of gerbes. Further details and references can be found in the expository article by Hitchin[13]. Ref.[16] is an incomplete list of their applications in string theory.

Consider a manifold $\mathcal{M}$ and a set of open charts $\{\mathcal{U}_i\}$ that covers it: $\mathcal{M} = \bigcup_i \mathcal{U}_i$. We will assume, for simplicity, that each $\mathcal{U}_i$ is contractible. A 1-gerbe $\mathcal{G}$ on $\mathcal{M}$ is defined by a set of U(1) bundles $\mathcal{L}_{ij}$ on each (ordered) overlap $\mathcal{U}_i \cap \mathcal{U}_j$, satisfying the following conditions

(i) $\mathcal{L}_{ji} = \mathcal{L}_{ij}^*$ (where $\mathcal{L}^*$ is the bundle dual to $\mathcal{L}$),

(ii) on triple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, the tensor product bundle $\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ has a nowhere vanishing section $s_{ijk}$,

(iii) on quadruple overlaps $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l$, the section $s_{ijk} \otimes s_{ijl}^* \otimes s_{ikl} \otimes s_{jkl}^* = 1$.

Notice that in the last condition, the tensor product of the sections is that of a trivial bundle $\mathcal{M} \times \text{U}(1)$, as follows from the other two conditions. Let us also note that if we take $\mathcal{L}_{ij} = \mathcal{L}_i \otimes \mathcal{L}_j^*$, where $\mathcal{L}_i$ are U(1) bundles on $\mathcal{U}_i$, then all the conditions are trivially satisfied. Therefore this is called a trivial gerbe.

The above may be generalized to $k$-gerbes by defining line bundles on $(k + 1)$-fold overlaps with appropriate conditions. An ordinary line bundle is a 0-gerbe from this point of view. It should be noted that, (except for $k = 0$), the ‘total space’ of a gerbe is not a manifold, as the definition involves conditions on more than two overlaps.
A connection on a 1-gerbe is specified by connections $A_{ij}$ for each $L_{ij}$ and a two-form $B_i$ on each chart $U_i$, such that

(i) $A_{ij} = -A_{ji}$,
(ii) $s_{ijk}$ is flat with respect to the induced connection,
(iii) on $U_i \cap U_j$, we have $B_i - B_j = dA_{ij}$.

The ‘curvature’ $H = dB$ of this connection is independent of the chart and hence makes sense globally. The cohomology of $H$ is characterized by $H^3(\mathcal{M}, \mathbb{Z})$. The quantization is analogous to the case of usual U(1) gauge fields [30].

As an example, let us describe the ‘NS5-brane’. Consider spacetime of the form $\mathbb{R}^{1,5} \times \mathbb{R} \times S^3$, the geometry of the NS5-brane. The only relevant part of it is $S^3$, on which we will construct a gerbe such that it carries an $H$-flux. This example is due to Hitchin [13] (and has been used in [15]). First, we cover $S^3$ with two open 3-discs $D^3_\pm$, which overlap around a region around the equatorial $S^2$. The overlap $D^3_+ \cap D^3_-$ has the topology of a ‘cylinder’ $S^2 \times \mathbb{R}$. For the U(1) bundle on the overlap, we take the monopole bundle on $S^2$ (more precisely, the pull-back of this bundle). Let $A_{+-}$ be the gauge field and $F = dA_{+-}$ be its curvature. In order to give their concrete forms, let us introduce coordinates $(\alpha, \beta, \gamma)$ on $S^3$, such that the metric is

$$ds^2 = d\alpha^2 + \sin^2 \alpha \left(d\beta^2 + \sin^2 \beta d\gamma^2\right).$$

The overlap region is $\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2} + \epsilon$, and $F_{+-} \sim \sin \beta d\beta \wedge d\gamma$. We need to specify the gerbe connections $B_\pm$. To this end, consider a partition of unity $\varphi_\pm$. Recall that these are functions with supports respectively in $D^3_\pm$, such that $0 \leq \varphi_\pm \leq 1$ and $\varphi_+ + \varphi_- = 1$ at each point. We write,

$$B_\pm = \pm \varphi_\pm F_{+-},$$

which satisfy the condition $B_+ - B_- = F_{+-}$. It is easy to check that the curvature $H = dB$ is independent of the chart. In fact it equals $F_{+-} \wedge d\varphi_+$, which is supported on the overlap. Therefore, using the quantization of the monopole field $F_{+-}$, we see that the $H$-flux through $S^3$ is integrally quantized. It is curious that the gerbe defining an NS5-brane is roughly like a monopole ($F$ part) times a kink ($\varphi$ part), quite similar to, say, the soliton description of D6-brane in [6]. Ref. [15] describes the NS5- and semi-infinite D6-brane configuration in this language.
Finally, let us show how the $\mathcal{K}(\mathcal{H})$ valued gauge fields patched together by $\text{Aut}(\mathcal{K}(\mathcal{H})) = \text{PU}(\mathcal{H})$ satisfy the axioms of a gerbe. In the construction Sec.3, we have only two coordinate charts, so there is not much to check. Consider, instead a general set up where, we have local trivializations given by maps

$$f_i : U_i \to \mathcal{K}(\mathcal{H}),$$

which satisfy $f_i = g_{ij}(f_j) = g_{ij}f_jg_{ij}^{-1}$, for $g_{ij} \in \text{PU}(\mathcal{H})$, on twofold overlaps. Hence, on triple overlaps $h_{ijk} = g_{ij}g_{jk}g_{ki}$ must be an element of $U(1)$, (since action of the $U(1)$ centre of $U(\mathcal{H})$ is trivial). These $h_{ijk}$’s may be taken as the sections $s_{ijk}$’s of (trivial) $U(1)$ bundles on threefold overlaps. It is also easy to check that the conditions on fourfold overlaps is satisfied.
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