ZERO-DIVISOR GRAPHS OF AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL

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ABSTRACT. Let $R$ be a commutative ring with identity and let $I$ be an ideal of $R$. Let $R \rtimes I$ be the subring of $R \times R$ consisting of the elements $(r, r+i)$ for $r \in R$ and $i \in I$. We study the diameter and girth of the zero-divisor graph of the ring $R \rtimes I$.

1. INTRODUCTION

Let $R$ be a commutative ring with non-zero unity. The concept of the graph of the zero divisors of $R$ was first introduced by Beck [4], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph. This investigation of colorings of a commutative ring was then continued by D. D. Anderson and Naseer in [3]. Let $Z(R)$ be the set of zero-divisors of $R$. In [1], D. F. Anderson and Livingston associate a graph, $\Gamma(R)$, to $R$ with vertices $Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of $R$, and for distinct $x, y \in Z(R) \setminus \{0\}$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Recall that a graph is said to be connected if for each pair of distinct vertices $v$ and $w$, there is a finite sequence of distinct vertices $v = v_1, \cdots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $(v, w)$, between connected vertices $v$ and $w$ is the length of the shortest path connecting them. The diameter of a connected graph is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex, and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. In [1], the authors proved that $\Gamma(R)$ is always connected and its diameter, $\text{diam}(\Gamma(R))$, is always less than or equal to 3 [1, Theorem 2.3]. They also proved that $\Gamma(R)$ is a complete graph if and only if either $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$, cf. [1, Theorem 2.8]. More recently, Axtell, Coykendall and Stickles [5] and Lucas [10] have studied the diameter of the corresponding graphs of the polynomial ring $R[x]$ and the power series ring $R[[x]]$. Recall that the girth of $G$ is the length of a shortest cycle in $G$ and is denoted by girth $(G)$. If $G$ has no cycles, we define the girth of $G$ to be infinite. In [1], the authors proved that the girth of $\Gamma(R)$ is either infinite or less than or equal to four when $R$ is Artinian and conjectured that this would hold if $R$ was

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not Artinian, cf. \cite{5} Theorem 2.4. In \cite{9} Theorem 1.6] DeMeyer and Schneider and in \cite{11} Theorem 1.4] Mulay proved this conjecture independently (see also \cite{5} Theorem 2).

Let $M$ be an $R$-module, the idealization $R(+)M$ (also called trivial extension), introduced by Nagata in 1956, cf. \cite{12}, is a ring where the module $M$ can be viewed as an ideal such that its square is $(0)$. In \cite{6}, Axtell and Stickles considered zero divisor graphs of idealization of commutative rings. They characterize the diameter and the girth of the zero-divisor graph of an idealization and show when this graph is complete.

In this paper, we deal with some applications of a similar general construction, introduced recently in \cite{8}, called the amalgamated duplication of a ring $R$ along an ideal $I$, and denoted by $R \bowtie I$. When $I^2 = 0$, the new construction $R \bowtie I$ coincides with the Nagata’s idealization $R(+)I$. More precisely, the amalgamated duplication of $R$ along an ideal $I$ is a ring that is defined as the following subring of $R \times R$,

$$R \bowtie I = \{(r, r+i) | r \in R, i \in I\}.$$

More generally, this construction can be given starting with a ring $R$ and an ideal $I$ of an overring $S$ of $R$ (such that $S \subseteq Q(R)$, where $Q(R)$ is the total ring of fractions of $R$); this extension has been studied, in the general case, and form the different point of view of pullbacks, by D’Anna and Fontana \cite{8}. One main difference of this construction, with respect to the idealization is that the ring $R \bowtie I$ can be a reduced ring (and it is always reduced if $R$ is a domain). As it happens for the idealization, one interesting application of this construction is the fact that it allows to produce rings satisfying (or not satisfying) preassigned conditions. Moreover, in many cases, the amalgamated duplication of a ring preserves the property of being reduced (see \cite{7}, \cite{8}). Recently, D’Anna proved that, if $R$ is a Cohen-Macaulay local ring, then $R \bowtie I$ is Gorenstein if and only if $I$ is a canonical ideal, cf. \cite{7}, where was known for trivial extension, cf. \cite{13}. This was our motivation to study the zero-divisor graph of $R \bowtie I$.

In this paper we study the diameter and girth of the graph of $\Gamma(R \bowtie I)$. In section 2, we review some properties of the ring $R \bowtie I$ and classify the zero-divisors of this ring. In section 3, we completely characterize the girth of zero-divisor graph $\Gamma(R \bowtie I)$. More precisely, it is shown that $R$ is not integral domain if and only if girth $(\Gamma(R \bowtie I)) = 3$. Also if $R$ is integral domain then girth $(\Gamma(R \bowtie I)) = 4$ provided $|I| \geq 3$, and girth $(\Gamma(R \bowtie I))$ is infinite if $|I| = 2$. In section 4, it is shown that for any non-zero ideal $I$ the following are equivalent:

(a) The graph $\Gamma(R \bowtie I)$ is a complete graph;
(b) $(Z(R))^2 = 0$ and $I \subseteq Z(R)$;
(c) $(Z(R \bowtie I))^2 = 0$.

2. Zero-divisors of the ring $R \bowtie I$

Let $R$ be a commutative ring with identity element $1$ and let $I$ be an ideal of $R$. We define $R \bowtie I = \{(r, s) | r, s \in R, s - r \in I\}$. It is easy to check that $R \bowtie I$ is a subring, with unit element $(1, 1)$, of $R \times R$ (with the usual componentwise operations) and that $R \bowtie I = \{(r, r+i) | r \in R, i \in I\}$.
We recall that the idealization $R(+)M$, introduced by Nagata [12] for every $R$-module $M$, is defined as the $R$-module $R \oplus M$ with multiplication defined by $(r,m)(s,n) = (rs, rn + sm)$.

It is easy to see that, if $\pi_i (i = 1, 2)$ are the projections of $R \times R$ on $R$, then $\pi_i (R \otimes I) = R$ and hence if $O_i = \ker(\pi_i|_{R \otimes I})$, then $(R \otimes I)/O_i \cong R$. Moreover $O_1 = \{(0,i)|i \in I\}$, $O_2 = \{(i,0)|i \in I\}$ and $O_1 \cap O_2 = (0)$. Now we state some properties of the ring $R \otimes I$ from [8], that will be considered numerous times.

**Proposition 2.1.** (see [8]) The following hold:

(a) The ring $R \otimes I$ is reduced if and only if $R$ is reduced. In particular, if $R$ is an integral domain, $R \otimes I$ is reduced and it has exactly two minimal primes which are $O_1$ and $O_2$.

(b) The ring $R \otimes I$ is isomorphic to the idealization $R(+)I$ if and only if $I$ is a nilpotent ideal of index 2 in $R$.

(c) If in the $R$-module direct sum $R \oplus I$ we consider a multiplicative structure by setting

$$(r,i)(s,j) = (rs, rj + si + ij),$$

then the map $f : R \oplus I \rightarrow R \otimes I$ defined by $f((r,i)) = (r, r + i)$ is a ring isomorphism. So if we consider the ring $R \otimes I$ as $R \oplus I$, and $(r,i)(s,j) = (rs, rj + si + ij)$, then $O_1 = \{(0,i)|i \in I\}$ and $O_2 = \{(-i,i)|i \in I\}$.

In the rest of this paper we will use freely Proposition 2.1 part c when we refer to the amalgamated duplication of $R$ along $I$.

To consider the zero-divisor graph of $R \otimes I$ we need the following result.

**Proposition 2.2.** Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Then

$$Z(R \otimes I) = \{(0,i)|i \in I\} \cup \{(i,-i)|i \in I\} \cup \{(x,i)|x \in Z(R) \setminus \{0\}, i \in I\} \cup \{(x,i)|x \in R \setminus Z(R), \text{there exists } j \in I, j(x+i) = 0\}.$$

**Proof.** It is easy to see that $(0,i)$ is adjacent to $(j,-j)$ for any $i, j \in I$. If $(a,i) \in R \otimes I$ with $a \in Z(R)$, then there exists $b \neq 0$ such that $ab = 0$. Consider the following two cases, namely the case where $b$ does not belong to $Ann(I)$ and the case where $b$ belongs to $Ann(I)$.

**Case 1.** Assume that $b \notin Ann(I)$. There exists $j \in I$ such that $bj \neq 0$. We obtain $(a,i)(b,j,-b,j) = 0$ and so $(a,i) \in Z(R \otimes I)$.

**Case 2.** Assume $b \in Ann(I)$. Then $bi = 0$ and so $(a,i)(b,0) = 0$. Therefore $(a,i) \in Z(R \otimes I)$.

On the other hand, if $(a,i) \in Z(R \otimes I)$, $a \neq 0$ and $a \in Z(R)$, then there exists a non-zero element $s \in R$ such that $as = 0$. Thus $(a,i) \in \{(x,i)|x \in Z(R) \setminus \{0\}, i \in I\}$. If $a \notin Z(R)$, then $(x,i)(y,j) = 0$, implies that $y = 0$ and $j(i + x) = 0$.

$\square$

**Remark 2.3.** Consider the following subsets of $\Gamma(R \otimes I)$:

1. $T_1 = \{(o,i)|i \in I\}$;
2. $T_2 = \{(i,-i)|i \in I\}$;
3. $T_3 = \{(x,i)|x \in Z(R) \setminus \{0\}, i \in I\}$;
(4) \( T_4 = \{ (x,i) | x \in R \setminus Z(R), j(x+i) = 0 \text{ for some } j \in I \} \).

Then the following hold:

(a) Each element of \( T_1 \setminus \{ (0,0) \} \) is adjacent to any element of \( T_2 \setminus \{ (0,0) \} \).
   This implies that there exists a complete bipartite graph \( K_{|I|-1,|I|-1} \) in the structure of \( \Gamma(R \otimes I) \).

(b) If \( i \in I \setminus Z(R) \), then the vertices \( (0,i) \) is adjacent only to vertices of \( T_2 \setminus \{ (0,0) \} \), and \( (i,-i) \) is adjacent only to vertices \( T_1 \setminus \{ (0,0) \} \).

(c) There exists a subgraph of \( \Gamma(R \otimes I) \) isomorphic to \( \Gamma(R) \).

3. Girth of \( \Gamma(R \otimes I) \)

In this section we study the girth of \( \Gamma(R \otimes I) \). If \( |I| = 1 \), then \( \Gamma(R) = \Gamma(R \otimes I) \) and so girth \( (\Gamma(R)) = \text{girth} \( (\Gamma(R \otimes I)) \). Thus we are interested in girth \( (\Gamma(R \otimes I)) \) for \( |I| \geq 2 \). The first result gives complete answer for the rings that are not integral domain.

**Proposition 3.1.** Let \( I \) be an ideal of \( R \). Then girth \( (\Gamma(R \otimes I)) = 3 \) if \( R \) is not integral domain.

**Proof.** Clearly \( Z(R) \cap I \neq \{ 0 \} \). Consider \( 0 \neq x \in Z(R) \cap I \). Then there exists \( 0 \neq y \in R \) such that \( xy = 0 \). Thus \( (0,x) \rightarrow (y,0) \rightarrow (x,-x) \rightarrow (0,x) \) is a cycle of length 3 in the graph \( \Gamma(R \otimes I) \). Therefore girth \( (\Gamma(R \otimes I)) = 3 \).

**Proposition 3.2.** Let \( R \) be an integral domain and let \( I \) be an ideal of \( R \). Then girth \( (\Gamma(R \otimes I)) = 4 \) provided \( |I| \geq 3 \). In addition, if \( |I| = 2 \), then \( I = R \cong \mathbb{Z}_2 \), and girth \( (\Gamma(R \otimes I)) = \infty \).

**Proof.** By assumption the only vertices of \( \Gamma(R \otimes I) \) are

\( \{ (0,i) | i \in I \setminus \{ 0 \} \} \cup \{ (i,-i) | i \in I \setminus \{ 0 \} \} \).

Thus \( \Gamma(R \otimes I) \) is a complete bipartite graph. If \( |I| \geq 3 \), then for two distinct non-zero elements \( i, j \in I \), we have a cycle \( (0,i) \rightarrow (i,-i) \rightarrow (0,j) \rightarrow (j,-j) \rightarrow (0,i) \), in \( \Gamma(R \otimes I) \) and hence girth \( (\Gamma(R \otimes I)) = 4 \).

Assume \( |I| = 2 \). Then the graph \( \Gamma(R \otimes I) \) is isomorphic to \( (0,i) \rightarrow (i,-i) \), and so girth \( (\Gamma(R \otimes I)) = \infty \).

We obtain the following result by considering Propositions 3.1 and 3.2.

**Corollary 3.3.** Let \( I \) be a non-zero ideal of \( R \). Then the following hold:

(a) girth \( (\Gamma(R \otimes I)) = 3 \) if and only if \( R \) is not an integral domain.

(b) girth \( (\Gamma(R \otimes I)) = 4 \) if and only if \( R \) is an integral domain and \( |I| \geq 3 \).

(c) girth \( (\Gamma(R \otimes I)) = \infty \) if and only if \( I = R \cong \mathbb{Z}_2 \).

**Corollary 3.4.** The following statements are equivalent:

(a) \( R \) is integral domain.

(b) girth \( (\Gamma(R)) = 4 \) or \( \infty \).
(c) \( R \otimes I \) has exactly two minimal prime ideals \( Q_1 \) and \( Q_2 \) such that \( Q_1 \cap Q_2 = (0) \) (i.e \( Q_1 = \{(0,i) | i \in I\} \) and \( Q_2 = \{(i,-i) | i \in I\} \)).

(d) \( \Gamma(R \otimes I) \) is a complete bipartite graph.

**Proof.** (a)\(\Rightarrow\)(b). This follows from Corollary 3.3.

(b)\(\Rightarrow\)(c). This follows from [7, Proposition 2].

(c)\(\Rightarrow\)(d). This follows from [2, Theorem 2.4].

(d)\(\Rightarrow\)(a). Since \( \Gamma(R \otimes I) \) is complete bipartite graph, we have that girth \( \Gamma(R \otimes I) \) = 4 or \( \infty \), so \( R \) is integral domain.

\[ \square \]

### 4. Diameter of \( \Gamma(R \otimes I) \)

In this section we study the diameter of \( \Gamma(R \otimes I) \). It is clear that if \( \text{diam} (\Gamma(R)) > 1 \), then \( \text{diam} (\Gamma(R \otimes I)) > 1 \). However, it is possible to have a ring such that \( \text{diam} (\Gamma(R)) = 1 \) but \( \text{diam} (\Gamma(R \otimes I)) \neq 1 \), as the following examples show.

**Example 4.1.** Let \( R = \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( I = \mathbb{Z}_2 \times \{0\} \). Set \( x = (1,0), y = (0,1) \) and \( z = (1,1) \). In the graph \( \Gamma(R \otimes I) \), \((z,x)\) is adjacent exactly to \((0,x)\). On the other hand, \((0,x)\) and \((x,0)\) are not adjacent. Thus \( \text{diam} (\Gamma(R \otimes I)) = 3 \) but it is easy to see that \( \text{diam} (\Gamma(R)) = 1 \).

**Example 4.2.** Let \( R = \mathbb{Z}_6 \) and \( I = \{0,3\} \). Then \( \text{diam} (\Gamma(R)) = 2 \). On the other hand, \((1,3)\) is in \( \Gamma(R \otimes I) \) is adjacent exactly to the vertex \((0,3)\). Since \((0,3),(3,0)\) = 2, so \((1,3),(3,0)\) = 3 and hence \( \text{diam} (\Gamma(R \otimes I)) = 3 \).

**Example 4.3.** Let \( R = \mathbb{Z}_8 \) and \( I = \{0,4\} \). Then

\[ Z(R \otimes I) = \{(0,4),(4,4),(6,0),(2,0),(4,0),(2,4),(6,4)\}. \]

It is clear that \( \text{diam} (R) = 2 = \text{diam} (R \otimes I) \).

**Example 4.4.** Let \( R = \mathbb{Z}_2 \times F \), where \( F \) is a field. Let \( I = \{(0,0),(1,0)\} \). Then \( \Gamma(R) \) is a star graph by [11, Theorem 2.13], and so \( \text{diam} (\Gamma(R)) = 2 \). Consider the element \((0,1),(1,0)\) in \( \Gamma(R \otimes I) \). It is clear that \((0,1),(1,0)\) is adjacent to \((1,0),(1,0)\), and \((0,1),(1,0)\) is not adjacent to \((0,0),(1,0)\). Thus \((0,1),(1,0)\) is just adjacent to \((1,0),(1,0)\). Now since \((1,0),(1,0)\) is not adjacent to \((0,1),(0,0)\), so the distance of \((0,1),(1,0)\) to \((1,0),(0,0)\) is equal 3. Therefore \( \text{diam} (\Gamma(R \otimes I)) = 3 \).

**Lemma 4.5.** Let \( R \) be a commutative ring. Then \( (Z(R \otimes I))^2 = 0 \) if and only if \( (Z(R))^2 = 0 \) and \( \text{I} \subseteq Z(R) \).

**Proof.** “only if” Let \( x,y \in Z(R) \). Then \((x,0),(y,0)\) in \( Z(R \otimes I) \). Thus \((x,0)(y,0) = (0,0)\) which implies \( xy = 0 \). Therefore \( (Z(R))^2 = 0 \). Now assume \( i \in I \). Then for any \( z \in Z(R) \), we have \((0,i) = (0,i) = (0,0)\). Thus \( i^2 = 0 \) and so \( i \in Z(R) \).

“if” Suppose that \( (Z(R))^2 = 0 \), and \( \text{I} \subseteq Z(R) \). For any \( x \in Z(R) \), \( i \in I \), the elements \((x,i)\) are adjacent to one another. The only elements that we should study, are \((x,i)\) where \( x \in R \setminus Z(R) \). Let \((x,i) \in Z(R \otimes I) \) and \( x \in R \setminus Z(R) \). Then there exists \( 0 \neq k \in I \) such that \( k(x+i) = 0 \). Since \( I \subseteq Z(R) \) and \( (Z(R))^2 = 0 \), we have that \( kx = 0 \). Thus \( x \in Z(R) \), which is a contradiction. Therefore the assertion holds.
Remark 4.6. Note that in example 4.1, \( \Gamma(R) \) is complete graph and \( I \subseteq Z(R) \), but \( \Gamma(R \otimes I) \) is not complete.

In the following Example, it is shown that the condition \( I \subseteq Z(R) \) in Lemma 4.5 can not omitted.

Example 4.7. Let \( R = \mathbb{Z}_{p^2} \) where \( p \) be a prime integer. It is easy to see that \( Z(R) = \{0, p, 2p, \cdots, p(p-1)\} \) and \( (Z(R))^2 = 0 \). Let \( I = \mathbb{Z}_{p^2} \). then \( (1, p-1) \in Z(R \otimes I) \) which is not adjacent to \( (p, 0) \). Thus \( (Z(R \otimes I))^2 \neq 0 \).

Theorem 4.8. Let \( I \) be a non-zero ideal of \( R \). Then the following are equivalent:

(a) The graph \( \Gamma(R \otimes I) \) is a complete graph

(b) \( (Z(R))^2 = 0 \) and \( I \subseteq Z(R) \).

(c) \( (Z(R \otimes I))^2 = 0 \).

Proof: (a)⇒(b). Assume \( \Gamma(R \otimes I) \) is a complete graph. Then \( \Gamma(R) \) is a complete graph and so by \[1\] Theorem 2.8] \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( xy = 0 \) for all \( x, y \in Z(R) \). If \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) then for any non-zero ideal \( I \) of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), two vertices \((0, 1), (0, 0)\) and \((0, 0), (0, 1)\) are not adjacent and hence \( \Gamma(R \otimes I) \) is not complete graph. Thus \( xy = 0 \) for all \( x, y \in Z(R) \), that means \( (Z(R))^2 = 0 \). Observing that for any \( i, j \in I \) and \( i = 0, j = 0 \) implies \( I \subseteq Z(R) \).

(b)⇒(c). The assertion follows from Lemma 4.5.

(c)⇒(a). This is clear.

Lemma 4.9. Assume \( R \) is not integral domain and \( I \nsubseteq Z(R) \). If \( Z(R) \) is an ideal then \( \text{diam}(\Gamma(R \otimes I)) = 3 \).

Proof: Choose \( i \in I \setminus Z(R) \). Since \( I \cap Z(R) \neq \{0\} \), there exists \( k \in I \), such that \( \text{Ann}_R(k) \neq \{0\} \). Let \( 0 \neq x \in \text{Ann}_R(k) \). Then \( k(x - i + i) = 0 \). Set \( y = x - i \). The vertex \((y, i)\) is adjacent to \((0, k)\). If \( y \notin Z(R) \), then \((y, i)\) is adjacent exactly to vertices \((0, l)\) where \( y + i \in \text{Ann}_R(l) \). Since the distance of such vertices \((0, l)\) to \((0, i)\) is 2, we have \( \text{diam}(\Gamma(R \otimes I)) \geq 3 \). If \( x - i = y \in Z(R) \), then we have \( i \in Z(R) \), since \( x \in Z(R) \) - a contradiction.

Corollary 4.10. If \( I \nsubseteq Z(R) \) and there is a vertex of \( \Gamma(R) \) which is adjacent to every other vertex of \( \Gamma(R) \), then \( \text{diam}(\Gamma(R \otimes I)) = 3 \).

Proof: Since there exists a vertex of \( \Gamma(R) \) which is adjacent to every other vertex, by \[1\] Theorem 2.5], \( Z(R) \) is an annihilator ideal or \( R \cong \mathbb{Z}_2 \times A \), where \( A \) is an integral domain. If \( Z(R) \) is an ideal, then \( \text{diam}(\Gamma(R \otimes I)) \geq 3 \) by Lemma 4.9. If \( R \cong \mathbb{Z}_2 \times A \), then \( I = \{0\} \times J \) or \( I = Z_2 \times J \) where \( J \) is a non-zero ideal of \( A \). Since \( \{0\} \times J \subseteq Z(R) \) and \( J \nsubseteq Z(R) \), we have \( I = Z_2 \times J \). If \( A = \mathbb{Z}_2 \), then \( I = R = \mathbb{Z}_2 \times \mathbb{Z}_2 \), and hence \( \text{diam}(\Gamma(R \otimes I)) \geq 3 \).

Now assume that \( A \neq \mathbb{Z}_2 \). Let \( A \neq J \) and \( a \in A \setminus J \). For arbitrary non-zero element \( b \in J \), consider \( X = (1, a) \), \( Y = (1, b) \), \( W = (b, b) \), and \( Z = (1, 0) \). Then
Theorem 4.12. If \( \text{dim} (\Gamma(R)) = 3 \), then \( \text{dim} (\Gamma(R \otimes I)) = 3 \).

Proof. First suppose that \( R \) is reduced ring. We follow the following two steps.

Step 1. We show that \( \text{diam} (\Gamma(R \otimes I)) \neq 2 \). Let \( \text{diam} (\Gamma(R \otimes I)) = 2 \). Since \( R \) is reduced, \( R \otimes I \) is reduced, cf. [7, Proposition 2]. On the other hand, since \( Z(R) \) is not an ideal, there exists \( x, y \in Z(R) \) such that \( x - y \notin Z(R) \), and so \( (x, 0) - (y, 0) \notin Z(R \otimes I) \). Hence \( Z(R \otimes I) \) is not an ideal of the ring \( R \otimes I \). Now by [10, Theorem 2.2], \( R \otimes I \) has exactly two two minimal prime ideals \( P_1, P_2 \). Therefore, \( P_1 \cap P_2 = \{0\} \), and \( \Gamma(R \otimes I) \) is complete bipartite graph. Thus by Corollary 3.4, \( R \) is an integral domain and so \( Z(R) \) is an ideal, which is a contradiction.

Step 2. We show that \( \text{diam} (\Gamma(R \otimes I)) \neq 1 \). Let \( \text{diam} (\Gamma(R \otimes I)) = 1 \). Then \( R \otimes I \) has exactly two minimal ideals by [10, Theorem 2.2], and so \( R \) is integral domain by Corollary 3.4. This is a contradiction.

Now suppose that \( R \) is not reduced ring. Then \( \text{diam} (\Gamma(R)) = 3 \) by [10, Corollary 2.5] and so \( \text{diam} (\Gamma(R \otimes I)) = 3 \) by Proposition 4.11.

Proposition 4.13. Let \( Z(R) \) be an ideal of \( R \) and \( I \subseteq Z(R) \). For all adjacent vertices \( a, b \) of \( \Gamma(R) \), let \( \text{Ann}(a, b) \neq 0 \). Then \( \text{diam} (\Gamma(R \otimes I)) = 2 \) provided \( \text{diam} (\Gamma(R)) = 2 \).

Proof. Since \( \text{diam} (\Gamma(R)) = 2 \), so \( \text{diam} (\Gamma(R \otimes I)) \geq 2 \). Let \( (x, i) \) and \( (y, j) \) be two vertices of \( \Gamma(R \otimes I) \). Consider the following cases:

Case 1. Let \( x = y = 0 \). Then \( (0, i) \) and \( (0, j) \) are adjacent to all vertices \( (k, -k) \) where \( k \in I \). Therefore \( ((0, i), (0, j)) \leq 2 \).

Case 2. Let \( x = 0 \) and \( y \neq 0 \). Since \( (y, j) \in Z(R \otimes I) \), we claim that \( y \in Z(R) \). If not, \( (y, j) \) is adjacent to vertices \( (0, k) \) where \( k \neq j \) and \( k \in Z(R) \). Since \( I \subseteq Z(R) \), we have that \( y \in Z(R) \) which is a contradiction. Therefore \( y \in Z(R) \). If there exists a non-zero element \( z \in \text{Ann}(y) \), and a non-zero element \( k \in I \) such that...
If \( zk \neq 0 \), then we have path \((0, i)\)\(\rightarrow\)(\(zk, \ -zk\))\(\rightarrow\)(\(y, j\)). If for any \(z \in \text{Ann}(y)\) and \(k \in I\) we have \(zk = 0\), then for an element \(0 \neq z \in \text{Ann}(y)\) we have path \((y, j)\)\(\rightarrow\)(\(z, 0\))\(\rightarrow\)(\(0, i\)). Therefore \(((y, j), (0, i)) \leq 2\).

**Case 3.** Let \( x \neq 0 \) and \( y \neq 0 \). By the same argument as Case 2, \(x, y \in Z(R)\).

If \((x, y) = 2\), then there exists \(0 \neq z \in Z(R)\) such that \(yz = xz = 0\). If there exists \(k \in I\) such that \(zk \neq 0\), then \((x, i)\) and \((y, j)\) are adjacent to \((zk, -zk)\), and hence \(((x, i), (y, j)) \leq 2\). If \(zk = 0\) for each \(k \in I\), then we have path \((x, i)\)\(\rightarrow\)(\(z, 0\))\(\rightarrow\)(\(y, j\)), and the assertion holds.

If \((x, y) = 1\), then \(x\) and \(y\) are adjacent in \(\Gamma(R)\). Thus there exists \(0 \neq z \in Z(R)\) such that \(xz = yz = 0\), since \(\text{Ann}(x, y) \neq 0\). So by a same argument as above, \(((x, i), (y, j)) \leq 2\). Therefore \((a, b) \leq 2\) for any \(a, b \in \Gamma(R \otimes I)\) and hence \(\text{diam}(\Gamma(R \otimes I)) = 2\).

**Corollary 4.14.** Let \(R\) be a non-reduced ring, \(Z(R)\) be an ideal of \(R\), and \(I \subseteq Z(R)\). Then \(\text{diam}(\Gamma(R \otimes I)) = 2\) provided \(\text{diam}(\Gamma(R)) = 2\).

**Proof:** Let \(a, b \in Z(R)\). If \(a\) and \(b\) are adjacent and \(\text{Ann}(a, b) = 0\), then by [10, Theorem 2.4] \(\text{diam}(\Gamma(R)) = 3\). This is a contradiction. Therefore for any two adjacent vertices \(a\) and \(b\) of \(\Gamma(R)\) we have that \(\text{Ann}(a, b) \neq 0\). Now use Proposition 4.13.

**Lemma 4.15.** If \(I \nsubseteq Z(R)\) and \(\text{diam}(\Gamma(R \otimes I)) = 2\), then for any \(y \in Z(R) \setminus \{0\}\), \(\text{Ann}_R(y) \cap I \neq \{0\}\).

**Proof:** Let \(i \in I \setminus Z(R)\) and \(y \in Z(R) \setminus \{0\}\). Then the vertices \((0, i)\) and \((y, 0)\) are not adjacent. Thus there exists \((s, j) \in \Gamma(R \otimes I) \setminus \{(0, 0)\}\) such that \((s, j)\) adjacent to both vertices. So \(i(j + s) = 0\). Since \(i \notin Z(R)\), we have that \(0 \neq s = -j \in I\). In addition, \(sy = 0\) implies that \(s \in \text{Ann}_R(y)\). Therefore \(0 \neq s \in \text{Ann}_R(y) \cap I\).

Our last result provides a condition which is sufficient for \(Z(R)\) to be a prime ideal (that means \(R\) has exactly one associated prime).

**Proposition 4.16.** If there exists an element \((r, i) \in R \otimes I\) which is adjacent to every vertices of \(\Gamma(R \otimes I)\), then \(Z(R)\) is a prime ideal.

**Proof:** For any \(x \in Z(R)\), we have \((r, i)(x, 0) = 0\). Thus \(ix = 0\) for any \(x \in Z(R)\). If \(i \neq 0\) then \(Z(R) = \text{Ann}_R(i)\). If \(i = 0\), then \(r \neq 0\) and \(rx = 0\) for any \(x \in Z(R)\). This means that \(Z(R) = \text{Ann}_R(r)\). Therefore \(Z(R)\) is an ideal. On the other hand \(R \setminus Z(R)\) is a multiplicative closed subset of \(R\) and so \(Z(R)\) is prime ideal.

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