Random Parametrization Double Tensors Integrals and Their Applications

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Abstract

In this work, we extend double tensor integrals (DTI) from our previous work to parametrization
double tensors integrals (PDTI) by applying integral kernel transform bounds to upper bound PDTI
norm and establishing a new perturbation formula. Besides, the convergence property of random PDTI
is investigated and this property is utilized to characterize the relation between the original derivative
tensor and the action result of PDTI to the original derivative tensor. These tools help us to derive new
tail bounds for random tensors according to more general operator inequalities, e.g., Heinz inequality
and Birman-Koplienko-Solomyak inequality. Moreover, new tail bounds about random tensors are also
obtained according to our new derived perturbation formula and integral kernel transform bounds.

Index terms— Einstein product, parametrization double tensor integrals (PDTI), random PDTI, tail
bound, perturbation formula, convergence in the random tensor mean, derivative of tensor-valued function

1 Introduction

In order to consider the random tensor mean problem, we defined the notion about double tensor inte-
grals (DTI) and discussed perturbation formula, Lipschitz estimation, and continuity issues for random DTI
in [1]. Motivated by works in [2–4] about applying double operator integration theory to noncommutative
geometry, we extend DTI definition discussed in [1] to parametrization double tensors integrals (PDTI).
The idea to apply double operator integration techniques in the general area of operator inequalities can be
traced back to the 1970s. For example, the original proof of Birman-Kopilenko-Solomyak inequality given
in [5] depends on profound facts from double operator integration theory. The works from [3, 4] provide a
framework by combining parametrization double operator integrals with Fourier transform bounds of pertur-
bation function to prove various operator inequalities, e.g., Heinz inequality, Birman-Koplenko-Solomyak
inequality, in a systematic approach.

In this work, we apply the framework from [3, 4] to random DTI. First, we extend operators from
matrices format to tensors format by defining PDTI and consider more general integral kernel transform
bounds, which will be used to upper bound PDTI norm. This will help us to associate the underlying
perturbation function properties with PDTI norm estimation. Only Fourier transform is considered in [3, 4].
Second, we derive a more general perturbation formula, compared to Lemma 4 in [4], in Theorem 2. Third,
the convergence of random PDTI is provided by Lemma 6, which is used with Theorem 2 to characterize
the relation between the original derivative tensor and the action result of PDTI to the original derivative
tensor, see Lemma 7. All these tools will help us to derive various new inequalities about random tensors.

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Originally, the Heinz inequality was proved in [6–8]. We extend the Heinz inequality by a tail bound format of random tensors in Theorem 3. Birman-Koplienko-Solomyak inequality was first proved in [5] with an alternative proof provided in [9]. Ando’s proof was later extended to semifinite von Neumann algebras in [10]. We extend this Birman-Koplienko-Solomyak inequality to a more general setting by tail bounds of random tensors in Theorem 5. Other new inequalities are also obtained based on our new derived perturbation formula and integral kernel transform bounds, for example, Theorem 6 and its corollary.

The rest of this paper is organized as follows. The terminologies related to tensors and fundamental facts about tensors are introduced in Section 2. The extension of double tensor integrals, Parametrization Double Tensor Integrals (PDTI), is presented in Section 3. A new perturbation formula for a more general divided difference form is derived in Section 4. In Section 5, we will establish continuity conditions for PDTI using the convergence in mean for random tensors. In Section 6, we will apply the proposed PDTI to build several new inequalities of random tensors. Finally, conclusions will be drawn in Section 7.

## 2 Fundamental of Tensors

Without loss of generality, one can partition the dimensions of a tensor into two groups, say \( M \) and \( N \) dimensions, separately. Thus, for two order-(\( M+N \)) tensors: \( \mathcal{X} \overset{\text{def}}{=} (x_{i_1\ldots i_M, j_1\ldots j_N}) \in \mathbb{C}^{I_1\times\cdots\times I_M \times J_1\times\cdots\times J_N} \) and \( \mathcal{Y} \overset{\text{def}}{=} (y_{i_1\ldots i_M, j_1\ldots j_N}) \in \mathbb{C}^{I_1\times\cdots\times I_M \times J_1\times\cdots\times J_N} \), according to \(1\), \(1\), the tensor addition \( \mathcal{X} + \mathcal{Y} \) is given by

\[
(\mathcal{X} + \mathcal{Y})_{i_1\ldots i_M, j_1\ldots j_N} \overset{\text{def}}{=} x_{i_1\ldots i_M, j_1\ldots j_N} + y_{i_1\ldots i_M, j_1\ldots j_N}.
\]

On the other hand, for tensors \( \mathcal{X} \overset{\text{def}}{=} (x_{i_1\ldots i_M, j_1\ldots j_N}) \in \mathbb{C}^{I_1\times\cdots\times I_M \times J_1\times\cdots\times J_N} \) and \( \mathcal{Y} \overset{\text{def}}{=} (y_{j_1\ldots j_N, k_1\ldots k_L}) \in \mathbb{C}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_L} \), the Einstein product (or simply referred to as tensor product in this work) \( \mathcal{X} \ast_N \mathcal{Y} \) is given by

\[
(\mathcal{X} \ast_N \mathcal{Y})_{i_1\ldots i_M, j_1\ldots j_N, k_1\ldots k_L} \overset{\text{def}}{=} \sum_{j_1\ldots j_N} x_{i_1\ldots i_M, j_1\ldots j_N} y_{j_1\ldots j_N, k_1\ldots k_L}.
\]

One can find more preliminary facts about tensors based on Einstein product in \(1\). In the remaining of this paper, we will represent the scalar value \( I_1 \times \cdots \times I_N \) by \( \mathbb{I}^N \).

We also list other crucial tensor operations here. The trace of a square tensor is equivalent to the summation of all diagonal entries such that

\[
\text{Tr}(\mathcal{X}) \overset{\text{def}}{=} \sum_{1 \leq i, j \leq [M]} \mathcal{X}_{i_1,\ldots,i_M,i_1,\ldots,i_M}.
\]

The inner product of two tensors \( \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M \times J_1\times\cdots\times J_N} \) is given by

\[
\langle \mathcal{X}, \mathcal{Y} \rangle \overset{\text{def}}{=} \text{Tr}(\mathcal{X}^H \ast_M \mathcal{Y}^\ast).
\]

From Theorem 3.2 in \(12\), every Hermitian tensor \( \mathcal{H} \in \mathbb{C}^{I_1\times\cdots\times I_N \times I_1\times\cdots\times I_N} \) has the following decomposition

\[
\mathcal{H} = \sum_{i=1}^{I_N^N} \lambda_i \mathcal{U}_i \ast_1 \mathcal{U}_i^H \quad \text{with} \quad \langle \mathcal{U}_i, \mathcal{U}_i \rangle = 1 \quad \text{and} \quad \langle \mathcal{U}_i, \mathcal{U}_j \rangle = 0 \quad \text{for} \quad i \neq j,
\]

\[
\overset{\text{def}}{=} \sum_{i=1}^{I_N^N} \lambda_i \mathcal{P}_i \mathcal{U}_i
\]

(5)
where \( \mathcal{U}_t \in \mathbb{C}^{I_1 \times \cdots \times I_N \times 1} \), and the tensor \( \mathcal{P}_{\mathcal{U}} \) is defined as \( \mathcal{U} \star_1 \mathcal{U}_t^H \). The values \( \lambda_i \) are named as eigenvalues. A Hermitian tensor with the decomposition shown by Eq. (5) is named as eigen-decomposition. A Hermitian tensor \( \mathcal{H} \) is a positive definite (or positive semi-definite) tensor if all its eigenvalues are positive (or nonnegative).

### 3 Parametrization Double Tensor Integrals

Let \( \psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) be a function with the following decomposition format in integrand as:

\[
\psi(\lambda_A, \lambda_B) = \int_{\Sigma} f_{A, \sigma}(\lambda_A)f_{B, \sigma}(\lambda_B) d\mu(\sigma),
\tag{6}
\]

where \( \mu(\sigma) \) is a measure on measurable space \( (\Sigma, \mu) \). Functions \( f_{A, \sigma} : \mathbb{R} \to \mathbb{C} \) and \( f_{B, \sigma} : \mathbb{R} \to \mathbb{C} \) are two bounded complex-valued functions satisfying

\[
\int_{\Sigma} \|f_{A, \sigma}(\lambda_A)\|_\infty \|f_{B, \sigma}(\lambda_B)\|_\infty d\mu(\sigma) < \infty.
\tag{7}
\]

Let us collect all \( \psi \) functions having the form as shown by Eq. (6) by a set \( \Psi \) such that, for any given two functions \( \psi_1, \psi_2 \in \Psi \) with

\[
\psi_1(\lambda_A, \lambda_B) = \int_{\Sigma_1} f_{A_1, \sigma_1}(\lambda_A)f_{B_1, \sigma_1}(\lambda_B) d\mu_1(\sigma_1);
\]

\[
\psi_2(\lambda_A, \lambda_B) = \int_{\Sigma_2} f_{A_2, \sigma_2}(\lambda_A)f_{B_2, \sigma_2}(\lambda_B) d\mu_2(\sigma_2),
\tag{8}
\]

we have new measure \( (\Sigma_3, \mu_3) \) and new functions \( f_{A_3, \sigma_3}, f_{B_3, \sigma_3} \) satisfying Eq. (7) such that the following relation is valid\(^1\):

\[
\int_{\Sigma_1} f_{A_1, \sigma_1}(\lambda_A)f_{B_1, \sigma_1}(\lambda_B) d\mu_1(\sigma_1) + \int_{\Sigma_2} f_{A_2, \sigma_2}(\lambda_A)f_{B_2, \sigma_2}(\lambda_B) d\mu_2(\sigma_2) = \int_{\Sigma_3} f_{A_3, \sigma_3}(\lambda_A)f_{B_3, \sigma_3}(\lambda_B) d\mu_3(\sigma_3).
\tag{9}
\]

We define the following norm function over the set \( \Psi \) as

\[
\|\psi\|_\Psi \overset{\text{def}}{=} \min_{\lambda_A} \int_{\Sigma} \|f_{A, \sigma}(\lambda_A)\|_\infty \|f_{B, \sigma}(\lambda_B)\|_\infty d\mu(\sigma),
\tag{10}
\]

where the minimum is taken over all possible representations of Eq. (6). With the condition provided by Eq. (9), it is easy to verify that the norm defined by Eq. (10) over the space \( \Phi \) has the triangle inequality:

\[
\|\psi_1 + \psi_2\|_\Psi \leq \|\psi_1\|_\Psi + \|\psi_2\|_\Psi.
\tag{11}
\]

Let \( \mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) be Hermitian tensors with the following eigen-decompositions:

\[
\mathcal{A} = \sum_{i=1}^{I_N^2} \lambda_{A,i} \mathcal{U}_{A,i} \star_1 \mathcal{U}_{A,i}^H \overset{\text{def}}{=} \sum_{i=1}^{I_N^2} \lambda_{A,i} \mathcal{P}_{A,i},
\tag{12}
\]

\(^1\)In [3], this condition should be added to prove the Banach space of \( \|\psi\|_\Psi \).
and

\[ B = \sum_{j=1}^{I^N} \lambda_{B,j} \mathcal{U}_{B,j} \ast \mathcal{U}_{B,j}^H \overset{\text{def}}{=} \sum_{j=1}^{I^N} \lambda_{B,j} \mathcal{P}_{B,j}, \]  

(13)

where \( \mathcal{P}_{A,i} \) and \( \mathcal{P}_{B,j} \) are projection tensors of tensors \( \mathcal{A} \) and \( \mathcal{B} \), respectively. We also have the function \( \psi(\lambda_A, \lambda_B) \) associated to eigenvalues of \( \lambda_A \) and \( \lambda_B \) defined by Eq. (6). Then, we can define a parametrize double tensor integrals (PDTI) over the measurable space \((\Sigma, \mu)\), represented by \( T_\psi(\mathcal{X}) \), as:

\[
T_\psi(\mathcal{X}) = \int_\Sigma \left( \sum_{i=1}^{I^N} f_{A,i} (\lambda_{A,i}) \mathcal{P}_{A,i} \right) \ast_N \mathcal{X} \ast_N \left( \sum_{j=1}^{I^N} f_{B,j} (\lambda_{B,j}) \mathcal{P}_{B,j} \right) d\mu(\sigma)
\]

(14)

where \( \mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \). \( T_\psi(\mathcal{X}) \) is called a random PDTI if \( \lambda_{A,i}, \lambda_{B,j} \) are random variables and \( \mathcal{P}_{A,i}, \mathcal{P}_{B,j} \) are random tensors.

From the definition provided by Eq. (14), we have the following Lemma about \( T_\psi(\mathcal{X}) \).

**Lemma 1 (Kernel of the mapping \( \psi \rightarrow T_\psi \) is zero)** Given the function \( \psi(\lambda_A, \lambda_B) \) defined by Eq. (6), the Kernel space of the mapping \( \psi \rightarrow T_\psi \) is zero.

**Proof:** It is enough to prove that if functions \( f_{A,i}, \sigma \) and \( f_{B,j}, \sigma \) have the following property:

\[
\int_\Sigma f_{A,i} (\lambda_A) f_{B,j} (\lambda_B) d\mu(\sigma) = 0,
\]

we have

\[
\text{Tr} (T_\psi(\mathcal{X}) \ast_N \mathcal{Y}) = 0,
\]

(16)

where \( \mathcal{X} \) and \( \mathcal{Y} \) are any tensors with dimensions \( \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \).

Suppose we have the following expression for tensors \( \mathcal{X} \) and \( \mathcal{Y} \):

\[
\mathcal{X} = \mathcal{U}_A \ast \mathcal{V}_B,
\]

(17)

where \( \mathcal{U}_A, \mathcal{V}_B \in \mathbb{C}^{I_1 \times \cdots \times I_N} \); and

\[
\mathcal{Y} = \mathcal{U}_B \ast \mathcal{V}_A,
\]

(18)

where \( \mathcal{U}_B, \mathcal{V}_A \in \mathbb{C}^{I_1 \times \cdots \times I_N} \). For any \( \sigma \in \Sigma \), we have

\[
\text{Tr} \left( \sum_{i=1}^{I^N} f_{A,i} (\lambda_{A,i}) \mathcal{P}_{A,i} \right) \ast_N \mathcal{X} \ast_N \left( \sum_{j=1}^{I^N} f_{B,j} (\lambda_{B,j}) \mathcal{P}_{B,j} \right) \ast_N \mathcal{Y}
\]

\[
= \sum_{i=1}^{I^N} f_{A,i} (\lambda_{A,i}) \mathcal{P}_{A,i} \ast_N \mathcal{U}_A, \mathcal{V}_A \left( \sum_{j=1}^{I^N} f_{B,j} (\lambda_{B,j}) \mathcal{P}_{B,j} \ast_N \mathcal{U}_B, \mathcal{V}_B \right)
\]

\[
= \sum_{i=1}^{I^N} \sum_{j=1}^{I^N} (f_{A,i} (\lambda_{A,i}) f_{B,j} (\lambda_{B,j})) \langle \mathcal{P}_{A,i}, \mathcal{U}_A, \mathcal{V}_A \rangle \langle \mathcal{P}_{B,j}, \mathcal{U}_B, \mathcal{V}_B \rangle.
\]

(19)
If we integrate both sides at Eq. (19) with respect to \( \sigma \), we have

\[
\text{Tr} \left( T_\psi (X) *_N Y \right) = \sum_{i=1}^{N} \sum_{j=1}^{N} (f_{A,i} (\lambda_{A,i}) f_{B,j} (\lambda_{B,j})) \left[ \int \frac{f_{A,i} (\lambda_{A,i}) f_{B,j} (\lambda_{B,j})}{\lambda_{A,i}} d\mu(\sigma) \right]
\]

Then, if the function \( \psi \) becomes 0, we have \( \text{Tr} \left( T_\psi (X) *_N Y \right) = 0 \). This indicates that \( T_\psi \) will be zero. \( \square \)

Our next lemma is about the norm estimate of \( T_\psi \). The spectral norm of a tensor is assumed here, i.e., \( \| X \| = s_{\text{max}}(A) \), where \( s_{\text{max}} \) represents the largest singular value of the tensor \( A \), see Theorem 3.2 in [12] about the singular values definition of a tensor.

**Lemma 2 (Norm estimate of \( T_\psi \) by \( \psi \) norm)** Let \( T_\psi (X) \) defined by Eq. (14), we have the following spectral norm estimate

\[
\| T_\psi (X) \| \leq (I_N)^2 \| \psi \|_\psi \| X \|.
\]

**Proof:** Suppose we select a \( \psi \in \Psi \) and \( \epsilon > 0 \) such that

\[
\int_{\Sigma} \| f_{A,i} (\lambda_{A,i}) \|_\infty \| f_{B,j} (\lambda_{B,j}) \|_\infty d\mu(\sigma) < (\| \psi \|_\psi + \epsilon).
\]

We also have

\[
\left\| \left( \sum_{i=1}^{N} f_{A,i} (\lambda_{A,i}) P_{A,i} \right) *_N X *_N \left( \sum_{j=1}^{N} f_{B,j} (\lambda_{B,j}) P_{B,j} \right) \right\|
\leq 1 \left\| \sum_{i=1}^{N} f_{A,i} (\lambda_{A,i}) P_{A,i} \right\| \| X \| \left\| \sum_{j=1}^{N} f_{B,j} (\lambda_{B,j}) P_{B,j} \right\|
\leq 2 \left( I_N \right)^2 \| f_{A,i} (\lambda_{A,i}) \|_\infty \| f_{B,j} (\lambda_{B,j}) \|_\infty \| X \|,
\]

where the inequality \( \leq_1 \) is based on the submultiplicative of spectral norm and the inequality \( \leq_2 \) is based on the triangle inequality and the fact that the spectral norm of \( P_{A,i} \) and \( P_{B,j} \) are one.

Then, we can have the following relation

\[
\| T_\psi (X) \| \leq \int_{\Sigma} \left\| \left( \sum_{i=1}^{N} f_{A,i} (\lambda_{A,i}) P_{A,i} \right) *_N X *_N \left( \sum_{j=1}^{N} f_{B,j} (\lambda_{B,j}) P_{B,j} \right) \right\| d\mu(\sigma)
\leq \left( I_N \right)^2 \left[ \int_{\Sigma} \| f_{A,i} (\lambda_{A,i}) \|_\infty \| f_{B,j} (\lambda_{B,j}) \|_\infty \| X \| \right]
\leq \left( I_N \right)^2 (\| \psi \|_\psi + \epsilon) \| X \|.
\]

This Lemma is proved by taking \( \epsilon \to 0 \). \( \square \)

From Lemma 2 we only bound the PDTI in terms of \( \| \psi \|_\psi \). Following theorem will give the bound for \( \| \psi \|_\psi \) by the property of \( \psi \) function.

**Theorem 1** Suppose we are given an integral transform as:

\[
g(t) = \int_{\mathbb{R}} K(s,t) \hat{g}(s) ds.
\]
If the variable $t$ is associated to eigenvalues of $\lambda_A$ and $\lambda_B$ by the following bivariable function as

$$t = \beta(\lambda_A, \lambda_B),$$

(26)

and $\psi(\lambda_A, \lambda_B)$ is assumed to be expressed as

$$\psi(\lambda_A, \lambda_B) = g(\beta(\lambda_A, \lambda_B))$$

$$= \int_{\mathbb{R}} K(s, \beta(\lambda_A, \lambda_B))\tilde{g}(s)ds$$

$$= \int_{\mathbb{R}} f_{A,s}(\lambda_A)f_{B,s}(\lambda_B)\tilde{g}(s)ds,$$

(27)

where $f_{A,s}(\lambda_A) = f_{A,s}(\lambda_A)$ and $f_{B,s}(\lambda_B) = f_{B,s}(\lambda_B)$. For all $\sigma \in \Sigma$, we assume that $\|f_{A,\sigma}\|_\infty \leq c_A$ and $\|f_{B,\sigma}\|_\infty \leq c_B$, where both $c_A$ and $c_B$ are two positive real numbers.

Then, we have

$$\|\psi\|_\Psi \leq c_A c_B \left( \int_{\mathbb{R}} \left( \max_{t} |K(s,t)| \right) ds \right) \|g(t)\|_\infty.$$

(28)

**Proof:** From the definition of $\|\psi\|_\Psi$, we have

$$\|\psi\|_\Psi \leq \int_{\Sigma} \|f_{A,\sigma}(\lambda_A)\|_\infty \|f_{B,\sigma}(\lambda_B)\|_\infty d\mu(\sigma)$$

$$\leq 1 \quad c_A c_B \int_{\Sigma} d\mu(\sigma)$$

$$= 2 \quad c_A c_B \int_{\mathbb{R}} |\tilde{g}(s)| ds$$

$$\leq 3 \quad c_A c_B \left( \int_{\mathbb{R}} \left( \max_{t} |K(s,t)| \right) ds \right) \|g(t)\|_\infty,$$

(29)

where the inequality $\leq 1$ comes from assumptions about $\|f_{A,\sigma}(\lambda_A)\|_\infty$ and $\|f_{B,\sigma}(\lambda_B)\|_\infty$, the equality $= 2$ is obtained by setting $\Sigma = \mathbb{R}$ and $d\mu(\sigma) = |\tilde{g}(s)| ds$, and the inequality $\leq 3$ comes from Hölder’s inequality with $p = \infty$, $q = 1$. This theorem is proved. □

We will have following corollaries according to Theorem II by choosing different transform functions $K(s, t)$. But, we need the following Lemma about the $L^1$ estimate of Fourier transform.

**Lemma 3** If $g(t) : \mathbb{R} \to \mathbb{C}$ is an an absolutely continuous function with $g, g'$ are $L^2$ function, we have

$$\|\tilde{g}(s)\|_1 \leq \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right),$$

(30)

where $c$ is any positive real number and $\tilde{g}(s)$ is the Fourier transform of $g(t)$.

**Proof:** Since we have

$$\int_{\mathbb{R}} |\tilde{g}(s)| ds = \int_{s \in [-c, c]} |\tilde{g}(s)| ds + \int_{s \notin [-c, c]} |s|^{-1} |s\tilde{g}(s)| ds$$

$$\leq 1 \quad \sqrt{2c} \left( \int_{s \in [-c, c]} |\tilde{g}(s)|^2 ds \right)^{1/2}$$

$$+ \left( \int_{s \not\in [-c, c]} |s|^{-2} ds \right)^{1/2} \left( \int_{s \not\in [-c, c]} |s\tilde{g}(s)|^2 ds \right)^{1/2}$$

$$\leq 2 \quad \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2$$

(31)
where \( \leq_1 \) comes from the Cauchy–Schwarz inequality, and \( \leq_2 \) uses Plancherel identity and the \( L^2 \) norm has larger support \( \mathbb{R} \) than \( s \notin [-c, c] \). This Lemma is proved by taking the minimization over the positive variable \( c \). \( \square \)

**Corollary 1** Suppose we are given a Fourier transform

\[
g(t) = \int_{\mathbb{R}} \tilde{g}(s)e^{its}ds,
\]

where \( t = \sqrt{-1} \). If the variable \( t \) is associated to eigenvalues of \( \lambda_A \) and \( \lambda_B \) by the following bivariable function as

\[
t = \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right),
\]

where \( \gamma : \mathbb{R} \to \mathbb{R}^+ \) and \( \kappa : \mathbb{R} \to \mathbb{R}^+ \).

If \( \psi(\lambda_A, \lambda_B) = g \left( \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right) \right) \), then, we have

\[
\|\psi\|_\Psi \leq \min_{c>0} \left( \sqrt{2c}\|g(t)\|_2 + \sqrt{2/c}\|g'(t)\|_2 \right),
\]

where \( \| \|_2 \) is \( L^2 \) function norm.

**Proof:** Since \( t = \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right) \), we have

\[
\psi(\lambda_A, \lambda_B) = g \left( \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right) \right) = \int_{\mathbb{R}} \tilde{g}(s)(\gamma(\lambda_A))^{ias}(\kappa(\lambda_B))^{-ias}ds.
\]

If we set the following parameters: \( \Sigma = \mathbb{R}, d\mu(\sigma) = |\tilde{g}(s)|ds, f_{A,\sigma}(\lambda_A) = (\gamma(\lambda_A))^{ias} \) and \( f_{B,\sigma}(\lambda_B) = (\kappa(\lambda_B))^{-ias} \), we obtain

\[
\|\psi\|_\Psi \leq \int_{\Sigma} \|f_{A,\sigma}(\lambda_A)\|_\infty \|f_{B,\sigma}(\lambda_B)\|_\infty d\mu(\sigma)
\]

\[
\leq_1 \quad 1 \times 1 \times \int_{\Sigma} d\mu(\sigma)
\]

\[
= \quad 1 \times 1 \times \int_{\mathbb{R}} |\tilde{g}(s)| ds
\]

\[
\leq_2 \quad \min_{c>0} \left( \sqrt{2c}\|g(t)\|_2 + \sqrt{2/c}\|g'(t)\|_2 \right),
\]

where the inequality \( \leq_1 \) comes from assumptions about \( \|f_{A,\sigma}(\lambda_A)\|_\infty = 1 \) and \( \leq_2 \) comes from Lemma 3.

This Corollary is proved. \( \square \)

**Corollary 2** Suppose we are given a transform

\[
g(t) = \int_{\mathbb{R}} \tilde{g}(s)e^{o+its}ds.
\]

If the variable \( t \) is associated to eigenvalues of \( \lambda_A \) and \( \lambda_B \) by the following bivariable function as

\[
t = \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right),
\]

where \( \leq_1 \) comes from the Cauchy–Schwarz inequality, and \( \leq_2 \) uses Plancherel identity and the \( L^2 \) norm has larger support \( \mathbb{R} \) than \( s \notin [-c, c] \). This Lemma is proved by taking the minimization over the positive variable \( c \). \( \square \)
where $\gamma : \mathbb{R} \to \mathbb{R}^+$ and $\kappa : \mathbb{R} \to \mathbb{R}^+$.

If $\psi(\lambda_A, \lambda_B) = g \left( \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right) \right)$, then, we have

$$\|\psi\|_{\Psi} \leq \gamma^\alpha(\lambda_A^*) \kappa^{-\alpha}(\lambda_B^*) \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right);$$  \hspace{1cm} (39)

where $\gamma^\alpha(\lambda_A^*)$ and $\kappa^{-\alpha}(\lambda_B^*)$ are the maximum values of the functions $\gamma^\alpha(\lambda_A)$ and $\kappa^{-\alpha}(\lambda_B)$, respectively.

**Proof:** Since $t = \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right)$, we have

$$\psi(\lambda_A, \lambda_B) = g \left( \log \left( \frac{\gamma(\lambda_A)}{\kappa(\lambda_B)} \right) \right) = \int_{\mathbb{R}} \tilde{g}(s) \gamma^\alpha(\lambda_A)(\gamma(\lambda_A))^{i\kappa} \kappa^{-\alpha}(\lambda_B)(\kappa(\lambda_B))^{-is} ds.$$ \hspace{1cm} (40)

If we set the following parameters: $\Sigma = \mathbb{R}$, $d\mu(\sigma) = |\tilde{g}(s)| ds$, $f_{A,\sigma}(\lambda_A) = \gamma^\alpha(\lambda_A)(\gamma(\lambda_A))^{i\kappa}$ and $f_{B,\sigma}(\lambda_B) = \kappa^{-\alpha}(\lambda_B)(\kappa(\lambda_B))^{-is}$, we obtain

$$\|\psi\|_{\Psi} \leq \int_{\Sigma} \|f_{A,\sigma}(\lambda_A)\|_\infty \|f_{B,\sigma}(\lambda_B)\|_\infty d\mu(\sigma) \leq_1 \gamma^\alpha(\lambda_A^*) \kappa^{-\alpha}(\lambda_B^*) \int_{\Sigma} d\mu(\sigma) = \gamma^\alpha(\lambda_A^*) \kappa^{-\alpha}(\lambda_B^*) \int_{\mathbb{R}} |\tilde{g}(s)| ds \leq_2 \gamma^\alpha(\lambda_A^*) \kappa^{-\alpha}(\lambda_B^*) \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right),$$ \hspace{1cm} (41)

where the inequality $\leq_1$ comes from the definition of $\gamma^\alpha(\lambda_A^*)$ and $\kappa^{-\alpha}(\lambda_B^*)$, and $\leq_2$ comes from Lemma 3. This Corollary is also proved.

**4 Perturbation Formula**

The main purpose of this section is to prepare a perturbation formula for the tensor operator $T_\psi$ with respect to a more general divided difference form. We begin with some preparation lemmas.

**Lemma 4** The mapping $\psi \to T_\psi$ is a homomorphism.

**Proof:** We note that $\Psi$ is a Banach algebra since it is closed under the multiplication and it is also continuous with respect to the norm of $\Phi$ defined by Eq. (10).

We define $\psi_1$ and $\psi_2$ as follows

$$\psi_1(\lambda_A, \lambda_B) = \int_{\Sigma_1} f_{A,\sigma_1}(\lambda_A) f_{B,\sigma_1}(\lambda_B) d\mu_1(\sigma_1);$$

$$\psi_2(\lambda_A, \lambda_B) = \int_{\Sigma_2} f_{A,\sigma_2}(\lambda_A) f_{B,\sigma_2}(\lambda_B) d\mu_2(\sigma_2),$$ \hspace{1cm} (42)

and assume that $\psi_3$ is the product of $\psi_1$ and $\psi_2$. Then, we can further define the following terms:

$$F_{A,\sigma_1} \overset{\text{def}}{=} \sum_{i=1}^{N} f_{A,\sigma_1}(\lambda_{A,i}) \mathcal{P}_{A,i}, \quad F_{B,\sigma_1} \overset{\text{def}}{=} \sum_{i=1}^{N} f_{B,\sigma_1}(\lambda_{B,i}) \mathcal{P}_{B,i},$$

$$F_{A,\sigma_2} \overset{\text{def}}{=} \sum_{i=1}^{N} f_{A,\sigma_2}(\lambda_{A,i}) \mathcal{P}_{A,i}, \quad F_{B,\sigma_2} \overset{\text{def}}{=} \sum_{i=1}^{N} f_{B,\sigma_2}(\lambda_{B,i}) \mathcal{P}_{B,i}.$$ \hspace{1cm} (43)
From the spectral mapping theorem, we have

\[ F_{A,\sigma_1} \ast_N F_{A,\sigma_2} = \sum_{i=1}^{I^N} f_{A,\sigma_1}(\lambda_{A,i}) f_{A,\sigma_2}(\lambda_{A,i}) \mathcal{P}_{A,i}, \]

\[ F_{B,\sigma_1} \ast_N F_{B,\sigma_2} = \sum_{i=1}^{I^N} f_{B,\sigma_1}(\lambda_{B,i}) f_{B,\sigma_2}(\lambda_{B,i}) \mathcal{P}_{B,i} \]

\[ = F_{B,\sigma_2} \ast_N F_{B,\sigma_1}. \]  

(44)

From the definition of \( T_\psi \) provided by Eq. (14) and \( \psi_3 = \psi_1 \psi_2 \), we have

\[ T_{\psi_3} = T_{\psi_1 \psi_2} = \int_{\Sigma_1 \times \Sigma_2} F_{A,\sigma_1} \ast_N F_{A,\sigma_2} \ast_N \mathcal{X} \ast_N F_{B,\sigma_2} \ast_N F_{B,\sigma_1} \mu_1(\Sigma_1) \times \mu_2(\Sigma_2) \]

\[ = \int_{\Sigma_1} \int_{\Sigma_2} F_{A,\sigma_1} \ast_N \left[ \int_{\Sigma_2} F_{A,\sigma_2} \ast_N \mathcal{X} \ast_N F_{B,\sigma_2} \mu_2(\Sigma_2) \right] \ast_N F_{B,\sigma_1} \mu_1(\Sigma_1) \]

\[ = T_{\psi_1}(T_{\psi_2}(\mathcal{X})). \]  

(45)

Therefore, the mapping \( \psi \to T_\psi \) is a homomorphism. \( \square \)

**Lemma 5** Let \( f \) be a bounded real-valued function with the following properties for any given positive integer \( m \):

\[ \phi_1(\lambda_A, \lambda_B) = f(\lambda^m_A), \text{ and } \phi_2(\lambda_A, \lambda_B) = f(\lambda^m_B), \]

then

\[ T_{\phi_1}(\mathcal{X}) = F_{A}^m \ast_N \mathcal{X}, \text{ and } T_{\phi_2}(\mathcal{X}) = \mathcal{X} \ast_N F_{B}^m, \]

(47)

where

\[ F_{A}^m = \sum_{i=1}^{I^N} f(\lambda^m_A) \mathcal{P}_{A,i}, \text{ and } F_{B}^m = \sum_{i=1}^{I^N} f(\lambda^m_B) \mathcal{P}_{B,i}, \]

(48)

where \( \mathcal{P}_{A,i} \) and \( \mathcal{P}_{B,i} \) are the projection tensors for the underlying mappings \( \phi_1 \to T_{\phi_1} \) and \( \phi_2 \to T_{\phi_2} \), respectively. We assume that \( \sum_{i=1}^{I^N} f(\lambda^m_A) \mathcal{P}_{A,i} \) and \( \sum_{i=1}^{I^N} f(\lambda^m_B) \mathcal{P}_{B,i} \) are positive definite tensors.

**Proof:** Since both functions \( \phi_1 \) and \( \phi_2 \) are belong to \( \Phi \), this Lemma is proved by the definition of \( T_\psi \) provided by Eq. (14) and Lemma [1] \( \square \)

We are ready to present the main theorem of this section.

**Theorem 2** Let \( f, g_A, g_B, h_A, h_B \) be bounded real-valued functions, and \( \mathcal{E}_A \) and \( \mathcal{E}_B \) be Hermitian tensors. We use \( \text{Sp}(\mathcal{E}_A) \) and \( \text{Sp}(\mathcal{E}_B) \) to represent the sets of eigenvalues of \( \lambda_A \) and \( \lambda_B \) for Hermitian tensors \( \mathcal{E}_A \) and \( \mathcal{E}_B \), respectively. We also assume that \( m_A, n_A, k_A \) and \( m_B, n_B, k_B \) are natural numbers. Let the function

\[ \psi(\lambda_A, \lambda_B) = \begin{cases} 
\frac{h_A(\lambda^m_A) f(\lambda^m_A) - f(\lambda^m_B)}{\lambda^m_A - \lambda^m_B} & \text{if } (\lambda_A, \lambda_B) \in \text{Sp}(\mathcal{E}_A) \times \text{Sp}(\mathcal{E}_B); \\
0 & \text{otherwise}.
\end{cases} \]  

(49)
Moreover, if we have
\[
G^m_A = g_A(\mathcal{E}^m_A), \quad G^m_B = g_B(\mathcal{E}^m_B)
\]
\[
H^r_A = h_A(\mathcal{E}^r_A), \quad H^r_B = h_B(\mathcal{E}^r_B)
\]
\[
F^k_A = f(\mathcal{E}^k_A), \quad F^k_B = f(\mathcal{E}^k_B),
\]
then,
\[
\mathcal{H}^{n_A}_A \star_N \left( F^k_A \star_N X - X \star_N F^k_B \right) \star_N \mathcal{H}^{n_B}_B = T_\psi \left( G^m_A \star_N \left( E^k_A \star_N X - X \star_N E^k_B \right) \star_N G^m_B \right).
\]

In addition, we also have
\[
\left\| \mathcal{H}^{n_A}_A \star_N \left( F^k_A \star_N X - X \star_N F^k_B \right) \star_N \mathcal{H}^{n_B}_B \right\|
\leq \left( \epsilon_1^N \right)^2 \left\| G^m_A \star_N \left( E^k_A \star_N X - X \star_N E^k_B \right) \star_N G^m_B \right\|
\]

Proof:

We define the following functions with respect to \( f, g_A, g_B, h_A, h_B \).
\[
\rho_A(\lambda_A, \lambda_B) \overset{\text{def}}{=} \lambda_A^k g_A(\lambda_A^m_A) g_B(\lambda_B^m_B), \quad \rho_B(\lambda_A, \lambda_B) \overset{\text{def}}{=} \lambda_B^k g_A(\lambda_A^m_A) g_B(\lambda_B^m_B)
\]
\[
\varsigma_A(\lambda_A, \lambda_B) \overset{\text{def}}{=} f(\lambda_A^k) h_A(\lambda_A^m_A) h_B(\lambda_B^m_B), \quad \varsigma_B(\lambda_A, \lambda_B) \overset{\text{def}}{=} f(\lambda_B^k) h_A(\lambda_A^m_A) h_B(\lambda_B^m_B),
\]
where \((\lambda_A, \lambda_B) \in \text{Sp}(\mathcal{E}_A) \times \text{Sp}(\mathcal{E}_B)\). From Lemma 5, we have
\[
T_{\rho_A}(X) = G^m_A \star_N \mathcal{E}^{k_A} \star_N X \star_N G^m_B, \quad T_{\rho_B}(X) = G^m_A \star_N X \star_N \mathcal{E}^{k_B} \star_N G^m_B,
\]
\[
T_{\varsigma_A}(X) = H^{n_A}_A \star_N X^{k_A} \star_N X \star_N H^{n_B}_B, \quad T_{\varsigma_B}(X) = H^{n_A}_A \star_N X \star_N F^{k_B} \star_N H^{n_B}_B.
\]

By applying homomorphism of the mapping \( \psi \to \Psi \) from Lemma 4, we have
\[
T_\psi \left( G^m_A \star_N \left( E^k_A \star_N X - X \star_N E^k_B \right) \star_N G^m_B \right)
= T_\psi \left( T_{\rho_A}(X) - T_{\rho_B}(X) \right)
= T_\psi(\rho_A - \rho_B)(X)
= \mathcal{H}^{n_A}_A \star_N \left( F^{k_A} \star_N X - X \star_N F^{k_B} \right) \star_N \mathcal{H}^{n_B}_B.
\]

Therefore, Eq. (51) is established.

Eq. (52) is true from Eq. (51) and Lemma 2.

Following corollary is the variation of Theorem 2 by changing the negative sign in Eq. (49) to be the positive sign. The proof will be almost identical so we skip it.

Corollary 3 Let \( f, g_A, g_B, h_A, h_B \) be bounded real-valued functions, and \( \mathcal{E}_A \) and \( \mathcal{E}_B \) be Hermitian tensors. We use \( \text{Sp}(\mathcal{E}_A) \) and \( \text{Sp}(\mathcal{E}_B) \) to represent the sets of eigenvalues of \( \lambda_A \) and \( \lambda_B \) for Hermitian tensors \( \mathcal{E}_A \) and \( \mathcal{E}_B \), respectively. We also assume that \( m_A, n_A, k_A \) and \( m_B, n_B, k_B \) are natural numbers. Let the function
\[
\psi(\lambda_A, \lambda_B) = \begin{cases} \frac{h_A(\lambda_A^m_A) f(\lambda_A^k)}{g_A(\lambda_A^m_A)} - \frac{f(\lambda_B^k)}{h_B(\lambda_B^m_B)}, & \text{if } (\lambda_A, \lambda_B) \in \text{Sp}(\mathcal{E}_A) \times \text{Sp}(\mathcal{E}_B); \\ 0, & \text{otherwise}. \end{cases}
\]
Moreover, if we have
\[
G^m_A = g_A(E^m_A), \quad G^m_B = g_B(E^m_B)
\]
\[
H^m_A = h_A(E^m_A), \quad H^m_B = h_B(E^m_B)
\]
\[
F^{k_A}_A = f(E^{k_A}_A), \quad F^{k_B}_B = f(E^{k_B}_B),
\]
then,
\[
\mathcal{H}^n_A \ast_N \left( F^{k_A}_A \ast_N \mathcal{X} + \mathcal{X} \ast_N F^{k_B}_B \right) \ast_N \mathcal{H}^n_B = T_\psi \left( G^m_A \ast_N \left( E^{k_A}_A \ast_N \mathcal{X} + \mathcal{X} \ast_N E^{k_B}_B \right) \ast_N G^m_B \right).
\]

In addition, we also have
\[
\left\| \mathcal{H}^n_A \ast_N \left( F^{k_A}_A \ast_N \mathcal{X} + \mathcal{X} \ast_N F^{k_B}_B \right) \ast_N \mathcal{H}^n_B \right\| \leq (\varepsilon_1^N)^2 \| \psi \| \left\| G^m_A \ast_N \left( E^{k_A}_A \ast_N \mathcal{X} + \mathcal{X} \ast_N E^{k_B}_B \right) \ast_N G^m_B \right\|.
\]

5 Limiting Behavior of Random Parametrization Double Tensor Integrals

In this section, we will establish continuity of random PDTI. We need the following definition to define the convergence in mean for random tensors.

**Definition 1** We say that a sequence of random tensor \( X_n \) converges in the \( r \)-th mean towards the random tensor \( X \) with respect to the tensor norm \( \| \cdot \| \), if we have
\[
\mathbb{E} (\| X_n \|) \text{ exists},
\]
and
\[
\mathbb{E} (\| X \|) \text{ exists},
\]
and
\[
\lim_{n \to \infty} \mathbb{E} (\| X_n - X \|) = 0.
\]

We adopt the notation \( X_n \xrightarrow{r} X \) to represent that random tensors \( X_n \) converges in the \( r \)-th mean to the random tensor \( X \) with respect to the tensor norm \( \| \cdot \| \).

All limiting behaviors involving randomness discussed in this paper are based on convergence converges in the 1-th mean.

We define a special subset \( \Psi_U \) within \( \Psi \) that satisfies the following condition. If \( \psi \in \Psi_U \), we have \( \Sigma, f_{A,\sigma} \) and \( f_{B,\sigma} \) in Eq. (14) with the requirement that there is a increasing sequence of measurable subsets \( S_k \in \Sigma \) for \( i = 1, 2, \cdots \) such that
\[
\Sigma = \bigcup_{i=1}^{\infty} S_i,
\]
and the family of functions \( \{ f_{A,\sigma}, f_{B,\sigma} \} \) is uniformly continuous for every \( i = 1, 2, \cdots \).

According to the \( T_\psi(\mathcal{X}) \) definition shown below,
\[
T_\psi(\mathcal{X}) = \int_\Sigma \left( \sum_{i=1}^{N} \lambda_{A,i} \mathcal{P}_{A,i} \right) \ast_N \mathcal{X} \ast_N \left( \sum_{j=1}^{N} f_{B,\sigma} (\lambda_{B,j}) \mathcal{P}_{B,j} \right) d\mu(\sigma),
\]
the randomness of \( T_\psi(\mathcal{X}) \) comes from random variables \( \lambda_{A,i}, \lambda_{B,i} \) and random tensors \( \mathcal{P}_{A,i}, \mathcal{P}_{B,i} \).
Lemma 6 Let \( \psi(\lambda_A, \lambda_B) = \int_{\Sigma} f_{A,\sigma}(\lambda_A) f_{B,\sigma}(\lambda_B) d\mu(\sigma) \in \Psi_U \) such that functions \( \|f^{(k)}_{A,\sigma}\|_{\infty} \) and \( \|f^{(k)}_{B,\sigma}\|_{\infty} \) are bounded for \( k = 0, 1, 2 \), where superscript \( (k) \) represents the \( k \)-th derivative. The measure space \( (\Sigma, \mu) \) follows Eq. 63. Also let \( \{E_{A,t}\}, \{E_{B,t}\} \) be two indexed families of independent random Hermitian tensors for \( t \in \mathbb{R} \) with formats 2:

\[
E_{A,t} = \sum_{i=1}^{\mathbb{N}} \lambda_{A,i} P_{A,t,i}, \quad E_{B,t} = \sum_{i=1}^{\mathbb{N}} \lambda_{B,i} P_{B,t,i}
\]

such that

\[
\lim_{t \to 0} E(\|E_{A,t} - E_{A,0}\|) = 0, \quad \lim_{t \to 0} E(\|E_{B,t} - E_{B,0}\|) = 0.
\]

If \( T_{\psi,t} \) is the random PDTI associated with \( \psi \) and random tensors \( P_{A,t,i}, P_{B,t,i} \), then we have

\[
\lim_{t \to 0} E(\|T_{\psi,t} - T_{\psi,0}\|) = 0.
\]

Proof:

Given \( \epsilon > 0 \), we wish to show that there is a function \( \psi_\epsilon \in \Psi_U \) such that

\[
\|\psi - \psi_\epsilon\| < \frac{\epsilon}{(1^N)^2}.
\]

The standard smoothing technique will be adopted here. We begin with the selecting the integer \( i_\epsilon \in \mathbb{N} \) such that

\[
\int_{\Sigma \setminus S_{i_\epsilon}} \|f_{A,\sigma}(\lambda_A)\|_{\infty} \|f_{B,\sigma}(\lambda_B)\|_{\infty} d\mu(\sigma) < \frac{\epsilon}{3 (1^N)^2}.
\]

Then, given \( y_\epsilon > 0 \), we set

\[
f_{A,\sigma,\epsilon}(\lambda_A) = \begin{cases} f_{A,\sigma}(\lambda_A) \otimes \frac{y_\epsilon}{\pi(\lambda_A^2 + y_\epsilon^2)}, & \text{if } \sigma \in S_{i_\epsilon}; \\ 0, & \text{otherwise}. \end{cases}
\]

where \( \otimes \) is the convolution operator. Similarly, we also set \( f_{B,\sigma,\epsilon}(\lambda_B) \) as

\[
f_{B,\sigma,\epsilon}(\lambda_B) = \begin{cases} f_{B,\sigma}(\lambda_B) \otimes \frac{y_\epsilon}{\pi(\lambda_B^2 + y_\epsilon^2)}, & \text{if } \sigma \in S_{i_\epsilon}; \\ 0, & \text{otherwise}. \end{cases}
\]

By selecting the value \( y_\epsilon \) larger enough, we have

\[
\|f_{A,\sigma,\epsilon}(\lambda_A) - f_{A,\sigma}(\lambda_A)\|_{\infty} < \frac{\epsilon}{C_\epsilon (1^N)^2}, \quad \text{and} \quad \|f_{B,\sigma,\epsilon}(\lambda_B) - f_{B,\sigma}(\lambda_B)\|_{\infty} < \frac{\epsilon}{C_\epsilon (1^N)^2},
\]

where \( C_\epsilon \) is defined as

\[
C_\epsilon = 3 \mu(S_{i_\epsilon}) \max_{\sigma \in S_{i_\epsilon}} (\|f_{A,\sigma}\|_{\infty}, \|f_{B,\sigma}\|_{\infty}).
\]
The term $C_\epsilon$ is finite since the family of functions $\{f_{A,\sigma}, f_{B,\sigma}\}$ for $\sigma \in S_i$ is uniformly continuous for every $i = 1, 2, \cdots$. The function $\psi_\epsilon$ can be defined as

$$
\psi_\epsilon(\lambda_A, \lambda_B) \overset{\text{def}}{=} \int_{S_i} f_{A,\sigma,\epsilon}(\lambda_A)f_{B,\sigma,\epsilon}(\lambda_B)d\mu(\sigma),
$$

(74)

then, we have

$$
\psi(\lambda_A, \lambda_B) - \psi_\epsilon(\lambda_A, \lambda_B) = \int_{S_i} f_{A,\sigma}(\lambda_A)\left[ f_{B,\sigma}(\lambda_B) - f_{B,\sigma,\epsilon}(\lambda_B) \right]d\mu(\sigma) + \int_{S_i} \left[ f_{A,\sigma}(\lambda_A) - f_{A,\sigma,\epsilon}(\lambda_A) \right] f_{B,\sigma,\epsilon}(\lambda_B)d\mu(\sigma) + \int_{\Sigma \setminus S_i} f_{A,\sigma}(\lambda_A)f_{B,\sigma,\epsilon}(\lambda_B)d\mu(\sigma)
$$

(75)

By applying Eqs. (69) and (72) to Eq. (75), we can have $\|\psi - \psi_\epsilon\|_\Psi < \frac{\epsilon}{(1_N)^2}$. Our next goal is to show

$$
\|T_{\psi,\epsilon}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| < \|\mathcal{X}\| \epsilon, \text{ if } t < \delta.
$$

(76)

If we set

$$
F_{A,\sigma,t,\epsilon} \overset{\text{def}}{=} f_{A,\sigma,\epsilon}(\mathcal{E}_{A,t}) = \sum_{i=1}^{1_N} f_{A,\sigma,\epsilon}(\epsilon_{A,i}) P_{A,t,i},
$$

(77)

and

$$
F_{B,\sigma,t,\epsilon} \overset{\text{def}}{=} f_{B,\sigma,\epsilon}(\mathcal{E}_{B,t}) = \sum_{i=1}^{1_N} f_{B,\sigma,\epsilon}(\epsilon_{B,i}) P_{B,t,i};
$$

(78)

then, from Theorem 2 and Theorem 4 of [3], we have

$$
\|F_{A,\sigma,t,\epsilon} - F_{A,\sigma,0,\epsilon}\| \leq C_\epsilon \|\mathcal{E}_{A,t} - \mathcal{E}_{A,0}\|,
$$

(79)

and

$$
\|F_{B,\sigma,t,\epsilon} - F_{B,\sigma,0,\epsilon}\| \leq C_\epsilon \|\mathcal{E}_{B,t} - \mathcal{E}_{B,0}\|,
$$

(80)

where the constant $C_\epsilon$ can be expressed as

$$
C_\epsilon = \max_{\sigma \in S_i} \max_{i=0,1,2} \left( \max_{i=0,1,2} \|f_{A,\sigma}^{(i)}\|_\infty, \max_{i=0,1,2} \|f_{B,\sigma}^{(i)}\|_\infty \right),
$$

(81)

where the superscript $(i)$ is the $i$-th derivative.

By taking expectations for the both sides of Eqs. (79) and (80), and from the assumptions provided by Eq. (66), we have

$$
\mathbb{E}(\|F_{A,\sigma,t,\epsilon} - F_{A,\sigma,0,\epsilon}\|) \leq \left( \frac{\epsilon}{\mu(S_i)} \right)^{1/2} \mathbb{E}(\|\mathcal{E}_{A,t} - \mathcal{E}_{A,0}\|) < \frac{\epsilon}{\sqrt{\mu(S_i)C_\epsilon}},
$$

(82)
where $\mu(S_{i_\epsilon})$ is the measure for the $S_{i_\epsilon}$. Similarly, we also have

$$
\mathbb{E} \left( \|F_{B,\sigma,t,\epsilon} - F_{B,\sigma,0,\epsilon}\| \right) \leq \left( \frac{\epsilon}{\mu(S_{i_\epsilon})} \right)^{1/2}, \text{ if } t < \delta \text{ such that } \mathbb{E} \left( \|\mathcal{E}_{A,t} - \mathcal{E}_{A,0}\| \right) < \frac{\sqrt{\epsilon}}{\sqrt{\mu(S_{i_\epsilon})}}. 
$$

Then, we have

$$
\mathbb{E} \left( \|T_{\psi,t}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| \right) = \mathbb{E} \left( \left\{ \int_{S_{i_\epsilon}} [F_{A,\sigma,t,\epsilon} - F_{A,\sigma,0,\epsilon}] * N \mathcal{X} * N [F_{B,\sigma,t,\epsilon} - F_{B,\sigma,0,\epsilon}] \, d\mu(\sigma) \right\} \right) 
\leq \left\{ \int_{S_{i_\epsilon}} \mathbb{E} (F_{A,\sigma,t,\epsilon} - F_{A,\sigma,0,\epsilon}) \mathbb{E} (F_{B,\sigma,t,\epsilon} - F_{B,\sigma,0,\epsilon}) \, d\mu(\sigma) \right\} \|\mathcal{X}\| 
$$

(84)

By taking expectation of the both sides of Eq. (84) and applying Eqs. (82) and (83), we obtain

$$
\mathbb{E} \left( \|T_{\psi,t}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| \right) \leq \|\mathcal{X}\| \epsilon. 
$$

(85)

Finally, given $t < \delta$, we have

$$
\mathbb{E} \left( \|T_{\psi,t}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| \right) \leq \mathbb{E} \left( \|T_{\psi,t}(\mathcal{X}) - T_{\psi,t}(\mathcal{X})\| \right) + \mathbb{E} \left( \|T_{\psi,t}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| \right) 
+ \mathbb{E} \left( \|T_{\psi,0}(\mathcal{X}) - T_{\psi,0}(\mathcal{X})\| \right) 
\leq 3 \|\mathcal{X}\| \epsilon, 
$$

(86)

where the first and third terms are obtained from Eq. (63) and Lemma 2 and the second term comes from Eq. (83).

Following Lemma is the derivative tensor relation after the action of $T_\psi$.

**Lemma 7** Let $\mathcal{E}_t$ for $t \in \mathbb{R}$ be a family of Hermitian tensors such that

$$
\lim_{t \to 0} \|\mathcal{E}_t - \mathcal{E}_0\| = 0. 
$$

(87)

Moreover, we also have

$$
G^m_0 = g(\mathcal{E}^m_0), \quad G^m_t = g(\mathcal{E}^m_t); \\
H^m_0 = h(\mathcal{E}^m_0), \quad H^m_t = h(\mathcal{E}^m_t); \\
F^k_0 = f(\mathcal{E}^k_0), \quad F^k_t = f(\mathcal{E}^k_t). 
$$

(88)

If $\psi \in \Psi_U$ and if

$$
\mathcal{A}_0 = \lim_{t \to 0} G^m_t \mathcal{E}^k_t - \mathcal{E}^k_0 \mathcal{G}^m_0 
$$

(89)

exists, then the limit

$$
\mathcal{B}_0 = \lim_{t \to 0} H^m_t \frac{F^k_t - F^k_0}{t} \mathcal{H}^m_0 
$$

(90)

exist. Moreover, we have

$$
\mathcal{B}_0 = T_\psi (\mathcal{A}_0), 
$$

(91)

where $\psi$ can be expressed as

$$
\psi(\lambda_A, \lambda_B) = \begin{cases} 
\frac{h(\lambda_B^n)}{g(\lambda_A^n)} f(\lambda_B^k) - f(\lambda_A^k), & \text{if } (\lambda_A, \lambda_B) \in \text{Sp}(\mathcal{E}_0) \times \text{Sp}(\mathcal{E}_0); \\
0, & \text{otherwise}.
\end{cases}
$$

(92)
Proof:
By setting
\[ A_t = G_t m \frac{\xi^k_t - \xi^k_0}{t^k} G^m_t, \]
and
\[ B_t = H_t^n \frac{\xi^k_t - \xi^k_0}{t^k} H^m_t, \]
we have \( B_t = T \psi, t (A_t) \) from Theorem 2. Then, we have
\[
\lim_{t \to 0} \|B_t - B_0\| \leq \lim_{t \to 0} (\|T \psi, t (A_t - A_0)\| + \|(T \psi, t - T \psi)(A_0)\|)
\]
\[
\leq \epsilon,
\]
(95)
where \( \lim_{t \to 0} A_t = A_0 \) comes from the assumption provided by Eq. (87), and \( T \psi, t - T \psi \) comes from Lemma 6.

If we have the following condition in Lemma 7, \( E_t \) for \( t \in \mathbb{R} \) be a family of random Hermitian tensors such that
\[
\lim_{t \to 0} E_t (\|E_t - E_0\|) = 0
\]
(96)
then, we have
\[
\lim_{t \to 0} E (\|T \psi, t (A_t) - B_0\|) = 0.
\]
(97)
The proof is similar to Lemma 7.

6 New Inequalities By PDTI

In this section, we will apply the proposed PDTI to derive several new inequalities.

Theorem 3 Let \( A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) be random Hermitian tensors and \( \mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N} \) be a Hermitian tensor. For every \( 0 \leq \omega \leq m \), we have
\[
\Pr (\|A^m * N \mathcal{X} * N B^\omega - A^\omega * N \mathcal{X} * N B^m\| \geq \theta) \leq \frac{(I_1^N)^2}{\theta} \left[ \min_{c > 0} \left( \sqrt{2c \|g(t)\|_2 + \sqrt{2/c \|g'(t)\|_2}} \right) \right]
\times E (\|A^m * N \mathcal{X} - \mathcal{X} * N B^m\|),
\]
(98)
where \( g(t) \) is
\[
g(t) = \frac{\exp \left( \frac{(m - 2\omega)t}{2} \right) - \exp \left( \frac{(2\omega - m)t}{2} \right)}{\exp \left( \frac{mt}{2} \right) - \exp \left( \frac{-mt}{2} \right)}.
\]
(99)
Proof:
From Theorem 2, we have
\[
A^m * N \mathcal{X} * N B^\omega - A^\omega * N \mathcal{X} * N B^m = T \psi (A^m * N \mathcal{X} - \mathcal{X} * N B^m),
\]
(100)
This theorem is proved by applying Markov inequality to Eq. (103).

Then, Eq. (101) will be obtained by setting \( t = \log \frac{\lambda_{A}}{\lambda_{B}} \) in Eq. (99).

By applying Lemma 2 and Corollary 1 to the function \( g(t) \) provided by Eq. (99), we have

\[
\|A^{m} \ast_{N} X \ast_{N} B^{\omega} - A^{\omega} \ast_{N} X \ast_{N} B^{m}\| \leq \left( \mathbb{I}_{1}^{N} \right)^{2} \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_{2} + \sqrt{2/c} \|g'(t)\|_{2} \right) \right] \times \|A^{m} \ast_{N} X - X \ast_{N} B^{m}\|.
\]

Therefore, we have

\[
\Pr (\|A^{m} \ast_{N} X \ast_{N} B^{\omega} - A^{\omega} \ast_{N} X \ast_{N} B^{m}\| \geq \theta) \\
\leq \Pr \left( \left\{ \left( \mathbb{I}_{1}^{N} \right)^{2} \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_{2} + \sqrt{2/c} \|g'(t)\|_{2} \right) \right] \|A^{m} \ast_{N} X - X \ast_{N} B^{m}\| \geq \theta \right\} \right) \\
= \Pr \left( \|A^{m} \ast_{N} X - X \ast_{N} B^{m}\| \geq \frac{\theta}{\left( \mathbb{I}_{1}^{N} \right)^{2} \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_{2} + \sqrt{2/c} \|g'(t)\|_{2} \right) \right]} \right). \quad (103)
\]

This theorem is proved by applying Markov inequality to Eq. (103).

If \( m = 1 \), Theorem 3 becomes the tail bound for Heinz inequality [8].

Following corollary is obtained by applying Corollary 3 to the same conditions of Theorem 3 for the tensor \( A^{m} \ast_{N} X \ast_{N} B^{\omega} + A^{\omega} \ast_{N} X \ast_{N} B^{m} \). We will skip the proof here due to the similarity of the proof provided by Theorem 3.

**Corollary 4** Let \( A, B \in \mathbb{C}^{I_{1} \times \cdots \times I_{N} \times I_{1} \times \cdots \times I_{N}} \) be random Hermitian tensors and \( X \in \mathbb{C}^{I_{1} \times \cdots \times I_{N} \times I_{1} \times \cdots \times I_{N}} \) be a Hermitian tensor. For every \( 0 \leq \omega \leq m \), we have

\[
\Pr (\|A^{m} \ast_{N} X \ast_{N} B^{\omega} + A^{\omega} \ast_{N} X \ast_{N} B^{m}\| \geq \theta) \leq \frac{\left( \mathbb{I}_{1}^{N} \right)^{2} \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_{2} + \sqrt{2/c} \|g'(t)\|_{2} \right) \right]}{\theta} \times \mathbb{E} (\|A^{m} \ast_{N} X - X \ast_{N} B^{m}\|) \quad (104)
\]

where \( g(t) \) is

\[
g(t) \overset{\text{def}}{=} \exp \left( \frac{(m-2\theta)t}{2} \right) + \exp \left( \frac{(2\theta-m)t}{2} \right) \overset{\text{def}}{=} \exp \left( \frac{mt}{2} \right) + \exp \left( \frac{-mt}{2} \right) \quad (105)
\]

Before presenting the following theorem, we have to introduce some notations. Given the tensor \( A \in \mathbb{C}^{I_{1} \times \cdots \times I_{N} \times I_{1} \times \cdots \times I_{N}} \), we use the absolute symbol \( |A| \) to represent the following:

\[
|A| \overset{\text{def}}{=} \sqrt{A^{\ast_{N}} A} \quad (106)
\]

Also, we use the symbol \([A, B] \), where \( A, B \in \mathbb{C}^{I_{1} \times \cdots \times I_{N} \times I_{1} \times \cdots \times I_{N}} \), to represent the commutator between two tensors, it is defined as:

\[
[A, B] \overset{\text{def}}{=} A \ast_{N} B - B \ast_{N} A. \quad (107)
\]
Theorem 4 Let $A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be random positive definite tensors and $X \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be a Hermitian tensor. For every $0 \leq \nu \leq 1$ and two nonnegative real numbers $r_0, r_1$ satisfying $r_0 + r_1 = 1$, we have

$$\Pr \left( \| [A |A|^{-\nu}, B] \| \geq \theta \right) \leq \frac{(\|I_N\|_2^2) \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right]}{\theta} \times \mathbb{E} \left( \| [A |A|^{-r_0\nu} \ast_N [A, B] \ast_N |A|^{-r_1\nu}] \| \right)$$

(108)

where $g(t)$ is

$$g(t) \equiv \frac{\exp \left( \frac{(1-2r_1\nu)t}{2} \right) - \exp \left( \frac{(2r_0\nu-1)t}{2} \right)}{\exp \left( \frac{t}{2} \right) - \exp \left( -\frac{t}{2} \right)}.$$  

(109)

Proof:

From Theorem 2, we have

$$[A |A|^{-\nu}, B] = T_{\psi} \left( [A |A|^{-r_0\nu} \ast_N [A, B] \ast_N |A|^{-r_1\nu}] \right),$$

(110)

where $\psi$ is

$$\psi(\lambda_A, \lambda_B) = \lambda_A^{r_0\nu} A^{1-\nu} - \lambda_B^{1-\nu} A^{r_1\nu}.$$  

(111)

Then, Eq. (109) will be obtained by setting $t = \log \frac{\lambda_A}{\lambda_B}$ in Eq. (111).

By applying Lemma 2 and Corollary 1 to the function $g(t)$ provided by Eq. (109), we have

$$\| [A |A|^{-\nu}, B] \| \leq \left( \|I_N\|_2^2 \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \right) \times \| [A |A|^{-r_0\nu} \ast_N [A, B] \ast_N |A|^{-r_1\nu}] \|.$$  

(112)

Therefore, we have

$$\Pr \left( \| [A |A|^{-\nu}, B] \| \geq \theta \right) \leq \Pr \left( \left( \|I_N\|_2^2 \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \right) \| [A |A|^{-r_0\nu} \ast_N [A, B] \ast_N |A|^{-r_1\nu}] \| \geq \theta \right)$$

$$= \Pr \left( \| [A |A|^{-r_0\nu} \ast_N [A, B] \ast_N |A|^{-r_1\nu}] \| \geq \frac{\theta}{\left( \|I_N\|_2^2 \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \right)} \right).$$  

(113)

This theorem is proved by applying Markov inequality to Eq. (113). \qed

Theorem 5 Let $A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be random positive definite tensors. For every $0 \leq \omega \leq 1$ and positive integers $m, n$, we have

$$\Pr \left( \| A^{\omega m} - B^{m \omega} \| \geq \theta \right) \leq \frac{(\|I_N\|_2^2 \omega \theta}{\omega} \left[ \min_{c > 0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \times \mathbb{E} \left( \| A^{\omega m} - B^{m \omega} \| \right)$$

(114)

where $g(t)$ is

$$g(t) \equiv \frac{\exp \left( \frac{\omega t}{2} \right) - \exp \left( -\frac{\omega t}{2} \right)}{\exp \left( \frac{t}{2} \right) - \exp \left( -\frac{t}{2} \right)}.$$  

(115)
Proof: Since the spectral norm is the same for taking a negative sign for any tensor, it is enough to consider the situation that $A^n - B^m$ is a positive definite tensor.

If we apply $t = \log \frac{\lambda_A}{\lambda_B}$ to Eq. (115), we have $\psi(\lambda_A, \lambda_B)$ as

$$\psi(\lambda_A, \lambda_B) = \begin{cases} \frac{\lambda_A}{\lambda_A - \lambda_B}, & \text{if } \lambda_A \neq \lambda_B; \\ 0, & \text{otherwise}. \end{cases}$$

(116)

If we set the tensor $\mathcal{H}_t$ as

$$\mathcal{H}_t = B^m + t(A^n - B^m),$$

(117)

then, we have

$$\lim_{\delta t \to 0} \frac{\mathcal{H}_{t+\delta t}^{\omega} - \mathcal{H}_t^{\omega}}{\delta t} = \mathcal{H}_t^{\omega} (\mathcal{H}_1 - \mathcal{H}_0) \mathcal{H}_t^{\omega-1}.$$  

(118)

From Lemma 7 and Eqs. (116) and (118), we also have

$$\frac{d}{dt} (\mathcal{H}_t^{\omega}) = \lim_{\delta t \to 0} \frac{\mathcal{H}_{t+\delta t}^{\omega} - \mathcal{H}_t^{\omega}}{\delta t} = T_{\psi} \left( \mathcal{H}_t^{\omega} (\mathcal{H}_1 - \mathcal{H}_0) \mathcal{H}_t^{\omega-1} \right).$$  

(119)

By applying Lemma 2 and Corollary 1 to the function $g(t)$ provided by Eq. (115), we have

$$\left\| \frac{d}{dt} (\mathcal{H}_t^{\omega}) \right\| \leq (\mathbb{I}_1^N)^2 \left[ \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \left\| \mathcal{H}_t^{\omega} (\mathcal{H}_1 - \mathcal{H}_0) \mathcal{H}_t^{\omega-1} \right\|.$$  

(120)

Because $\mathcal{H}_t - t (\mathcal{H}_1 - \mathcal{H}_0)$ is a positive definite tensor and monotonicity of the function $t^{(1-\omega)}$, we have

$$\left\| \mathcal{H}_t^{(\omega-1)/2} \ast_N (\mathcal{H}_1 - \mathcal{H}_0)^{(1-\omega)} \ast_N \mathcal{H}_t^{(\omega-1)/2} \right\| \leq t^{(1-\omega)}.$$  

(121)

From Eq. (120) and Eq. (121), we have

$$\left\| \frac{d}{dt} (\mathcal{H}_t^{\omega}) \right\| \leq (\mathbb{I}_1^N)^2 \left[ \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \left\| \mathcal{H}_1 - \mathcal{H}_0 \right\|^{\omega}.$$

(122)

Therefore, we have

$$\|A^{n\omega} - B^{m\omega}\| = \left\| \int_0^1 \frac{d}{dt} (\mathcal{H}_t^{\omega}) \right\| \leq (\mathbb{I}_1^N)^2 \left[ \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \left\| \mathcal{H}_1 - \mathcal{H}_0 \right\|^{\omega} \int_0^1 t^{\omega-1} dt$$

$$= \frac{1}{\omega} \left[ \min_{c>0} \left( \sqrt{2c} \|g(t)\|_2 + \sqrt{2/c} \|g'(t)\|_2 \right) \right] \|A^n - B^m\|^\omega.$$  

(123)

where $=1$ is obtained by using $\mathcal{H}_1 = A^n$ and $\mathcal{H}_0 = B^m$ from Eq. (117). This theorem is proved by applying Markov inequality to Eq. (123). □

If $m = n = 1$, Theorem 5 becomes the tail bound for Birman-Koplenko-Solomyak inequality [5].

Following Theorem 6 will be another tail bound for new random tensors inequality based on PDTI.
Theorem 6  Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be random Hermitian tensors and $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be a Hermitian tensor. For two real numbers $\alpha, \beta$ such that $0 \leq \alpha, \beta \leq 1$, and two positive integers $m,n$, we have

$$\Pr \left( \left\| A^{m(1+\alpha)} \mathcal{X} \otimes N \mathcal{B}^{n(1-\alpha)} - A^{m(1-\beta)} \mathcal{X} \otimes N \mathcal{B}^{n(1+\beta)} \right\| \geq \theta \right) \leq \frac{\left( I_1 \right)^2 \alpha}{\theta} \left[ \min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right) \right] \mathbb{E} \left( \left\| A^m \mathcal{X} \otimes N \mathcal{B}^n \right\| \right)$$

(124)

where $g(t)$ is

$$g(t) = \frac{\exp \left( \frac{c(t)}{2} \right) - \exp \left( \frac{\beta t}{2} \right)}{\exp \left( \frac{\alpha t}{2} \right) - \exp \left( \frac{\beta t}{2} \right)}$$

(125)

Proof:

From Theorem 6, we have

$$A^{m(1+\alpha)} \mathcal{X} \otimes N \mathcal{B}^{n(1-\alpha)} - A^{m(1-\beta)} \mathcal{X} \otimes N \mathcal{B}^{n(1+\beta)} = T_{\psi} \left( A^m \mathcal{X} \otimes N \mathcal{B}^n \right),$$

(126)

where $\psi$ is

$$\psi(\lambda_A, \lambda_B) = \frac{m(1+\alpha)}{2} \lambda_A^{m(1+\alpha)} - \frac{m(1-\beta)}{2} \lambda_B^{m(1-\beta)} - \frac{n(1+\beta)}{2} \lambda_A^n - \frac{n(1-\beta)}{2} \lambda_B^n.$$

(127)

Then, Eq. (127) will be obtained by setting $t = \log \frac{\lambda_B^n}{\lambda_A^n}$ in Eq. (125).

By applying Lemma 2 and Corollary 1 to the function $g(t)$ provided by Eq. (125), we have

$$\left\| A^{m(1+\alpha)} \mathcal{X} \otimes N \mathcal{B}^{n(1-\alpha)} - A^{m(1-\beta)} \mathcal{X} \otimes N \mathcal{B}^{n(1+\beta)} \right\| \leq \frac{\left( I_1 \right)^2 \alpha}{\theta} \left[ \min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right) \right] \left\| A^m \mathcal{X} \otimes N \mathcal{B}^n \right\|.$$ 

(128)

Therefore, we have

$$\Pr \left( \left\| A^{m(1+\alpha)} \mathcal{X} \otimes N \mathcal{B}^{n(1-\alpha)} - A^{m(1-\beta)} \mathcal{X} \otimes N \mathcal{B}^{n(1+\beta)} \right\| \geq \theta \right) \leq \Pr \left( \left\{ \left( I_1 \right)^2 \alpha \left[ \min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right) \right] \left\| A^m \mathcal{X} \otimes N \mathcal{B}^n \right\| \right\} \geq \theta \right)$$

$$= \Pr \left( \left\| A^m \mathcal{X} \otimes N \mathcal{B}^n \right\| \geq \frac{\theta}{\left( I_1 \right)^2 \alpha \left[ \min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right) \right]} \right).$$

(129)

This theorem is proved by applying Markov inequality to Eq. (129).

Following corollary is obtained by applying Corollary 3 to the same conditions of Theorem 5 for the tensor $A^{m(1+\alpha)} \mathcal{X} \otimes N \mathcal{B}^{n(1-\alpha)} + A^{m(1-\beta)} \mathcal{X} \otimes N \mathcal{B}^{n(1+\beta)}$. We will skip the proof here due to the similarity of the proof provided by Theorem 6.
Corollary 5  Let $A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be random Hermitian tensors and $X \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be a Hermitian tensor. For two real numbers $\alpha, \beta$ such that $0 \leq \alpha, \beta \leq 1$, and two positive integers $m, n$, we have
\[
\Pr \left( \left\| A^{m(1+\alpha)} N \star X * N B^{n(1-\alpha)} N \right\| \geq \theta \right) \leq \frac{\left( \frac{2}{\theta} \right)^{2^N} \left( \min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right) \right)}{\left( \frac{\sqrt{2}}{\theta} \right)^{\min_{c>0} \left( \sqrt{2c} \left\| g(t) \right\|_2 + \sqrt{2/c} \left\| g'(t) \right\|_2 \right)}} \right) \right)
\]
where $g(t)$ is
\[
g(t) \overset{\text{def}}{=} \frac{\exp \left( \frac{\alpha t}{2} \right) + \exp \left( \frac{-\beta t}{2} \right)}{\exp \left( \frac{\alpha t}{2} \right) + \exp \left( \frac{\beta t}{2} \right)}. \tag{131}
\]

7  Conclusions

In this work, we extended our previous work about DTI to PDTI by deriving the upper bound for PDTI norm and new perturbation formula for a PDTI tensor. We also studied the convergence property of random PDTI and applied this property to characterize the tensor variation after the action of PDTI. With these new instruments, we are able to build new tail bounds for random tensors. We believe the proposed random PDTI and related tools can be applied to other fields of mathematics, e.g., noncommutative geometry.

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