Compact convex structure of measurements and its applications to simulability, incompatibility, and convex resource theory of continuous-outcome measurements

Yui Kuramochi∗
Photon Science Center, Graduate School of Engineering,
The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

Abstract

We introduce the post-processing preorder and equivalence relations for general measurements on a possibly infinite-dimensional general probabilistic theory described by an order unit Banach space $E$ with a Banach predual. We define the measurement space $M(E)$ as the set of post-processing equivalence classes of continuous measurements on $E$. We define the weak topology on $M(E)$ as the weakest topology in which the state discrimination probabilities for any finite-label ensembles are continuous and show that $M(E)$ equipped with the convex operation corresponding to the probabilistic mixture of measurements can be regarded as a compact convex set regularly embedded in a locally convex Hausdorff space. We also prove that the measurement space $M(E)$ is infinite-dimensional except when the system is 1-dimensional and give a characterization of the post-processing monotone affine functional. We apply these general results to the problems of simulability and incompatibility of measurements. We show that the robustness measures of unsimulability and incompatibility coincide with the optimal ratio of the state discrimination probability of measurement(s) relative to that of simulable or compatible measurements, respectively. The latter result for incompatible measurements generalizes the recent result for finite-dimensional quantum measurements. Throughout the paper, the fact that any weakly∗ continuous measurement can be arbitrarily approximated in the weak topology by a post-processing increasing net of finite-outcome measurements is systematically used to reduce the discussions to finite-outcome cases.

Keywords: general probabilistic theory, weak topology of measurements, simulability, incompatibility, robustness measure, convex resource theory, comparison of statistical experiments

∗Email: kuramochi@qi.t.u-tokyo.ac.jp
1 Introduction

The measurement process is one of the indispensable constituents of the quantum theory, or more generally any kind of operational physical theory, since it connects the predictions by an abstract mathematical model to the observed experimental events, making the theory comparable with the real world. In spite of such a general importance, little is known for the property of the totality of measurements of a given system. One of the reasons for this might be its mathematical difficulty, especially that the class of measurements is a proper class, i.e. a class larger than any set, because we have no restrictions to the outcome space of a measurement.

A related important problem of the measurement we investigate in this paper is how we should consider continuous-outcome measurements. In quantum theory and technology, continuous-outcome measurements, like the homodyne detection of a photon field, play fundamental roles, for example in the continuous-variable quantum key distribution \[37\]. We cannot however naively think that the continuous measurement described by a positive-operator valued measure (POVM) is exactly realized in a real experiment because it is impossible for an experimental device to exactly record a continuous variable, e.g. a real number, which requires infinite bits of information. One way to reconcile such a contradiction is to think that the theoretical description of a continuous-outcome measurement approximates in some sense the real measurement process which has a finite outcome space. If we take this standpoint, then we have to answer in what sense this “approximation” is.

Another related mathematical problem is that the operation of the probabilistic mixture of two (or generally more than two) measurements that does not post-process the measurement outcome is not closed in a certain set, but is defined on the class of measurements. For instance, two general measurements on a quantum system have different outcome spaces \(X\) and \(Y\) and the outcome space of the probabilistic mixture of the two measurements is the disjoint union of \(X\) and \(Y\). Thus the outcome space becomes larger if we take probabilistic mixture and this operation cannot be closed within some set of measurements. Presumably because of this kind of difficulty, the probabilistic mixture operation has not been sufficiently studied, while in some works it is natural to consider this operation. For example, the class of measurements simulable \[19, 45, 14\] by a certain set of measurements and the class of pairs of compatible (i.e. jointly measurable) measurements \[24, 57, 34, 7\] are closed under this operation. Moreover the state discrimination probability recently considered in the context of convex resource theory of measurements (POVMs) \[51, 56, 44\] is affine with respect to this operation.

The purpose of this paper is to study the measurement space \(\mathcal{M}(E)\), which is the set of post-processing equivalence classes of measurements on a given (possibly infinite-dimensional) order unit Banach space \(E\) with a predual. Such an ordered Banach space \(E\) corresponds to the set of observables on the state space of a general probabilistic theory (GPT) \[17, 23, 26, 14, 48\]. We also apply this general formulation of measurements to the problems of the simulability and (in)compatibility of measurements.

This paper is organized as follows. In Section 2, we give preliminary results for order unit Banach spaces (GPTs). We introduce two kinds of formulations of GPT. The first formulation is based on compact state space and considers the continuous affine functionals as the observables, while the second one only requires the norm completeness of the state
space and considers the bounded affine functionals as the observables. In this paper the former one will appear as the measurement space in the main part, while we consider the state space of the second type as the physical system. This is because ordinary formulation of the quantum theory in infinite dimensions is described by the second one, but not by the first one since the set of density operators is not compact in the trace-norm in infinite dimensions.

In section 3, we give some basic facts on measurement, which is in this paper defined as an abstract GPT-to-classical channel, and post-processing relations among measurements. The results in Section 3 is essentially the same as those in restricted situations, for example when the system is quantum or more generally that described by a von Neumann algebra [34].

In Section 4, based on the Blackwell-Sherman-Stein theorem (BSS theorem) for measurements (Theorem 1), we introduce the measurement space and the weak topology on it. We show that the measurement space equipped with the weak topology and convex combination corresponding to the probabilistic mixture can be regarded as a compact convex set in a locally convex Hausdorff space (Theorems 3 and 5). We also prove that any w*-measurement can be approximated by finite-outcome ones (Theorem 4) and that the measurement space $\mathcal{M}(E)$ is an infinite-dimensional convex set except when $E$ is 1-dimensional (Theorem 6).

The weak topology is known in the area of theory of statistical experiments (statistical decision theory) [36, 54], a branch of mathematical statistics, and our formalism contain this theory as a special case. How the theory of statistical experiments is reduced to that of measurements is addressed in Appendix D.

In Section 5, we consider more general class of preorders on a compact convex set that is characterized by a set of continuous affine functionals. By the BSS theorem, the post-processing order on the measurement space, the main subject of this paper, is an example of such an order. We give characterizations of post-processing monotone affine functionals (Theorem 7 and Corollary 1). Moreover, by using the condition when the order is a partial or total order (Proposition 17) and the infinite-dimensionality of the measurement space, we prove that the post-processing order on the measurement space is not total (Corollary 2).

At last, we will see that the class of preorders in consideration is characterized by the independence and continuity axioms, which is a result analogous to the von Neumann-Morgenstern utility theorem [42, 10].

The following Sections 6, 7, and 8 are devoted to the applications of the general theory of the compact convex structure to the simulability and incompatibility of measurements. In Section 6, we introduce the notion of simulability based on the weak topology, which is a weaker notion than the previously known (strong) simulability [19, 45, 14]. We show that the simulability is characterized by the outperformance on the state discrimination probability (Theorem 9), which generalizes the finite-dimensional result [51]. We define the robustness of unsimulability of a measurement as the minimal noise needed to make the measurement simulable and prove in Theorem 10 that the robustness measure is the optimal ratio of the state discrimination probability of the measurement relative to that of simulable ones.

In Section 7, we consider related classes of extremal, maximal, and simulation irreducible measurements. Based on the characterization of the extremality (Theorem 11) and simulation irreducibility (Proposition 22), we show that any measurements is simulable by the simulation irreducible measurements (Theorem 12), which is known in the finite-dimensional quantum systems [22] and finite-dimensional GPTs [13].
In Section 8, we consider incompatibility of measurements and prove that any incompatible measurements outperform the compatible ones in the state discrimination task (Theorem 13). We also introduce the quantity called the robustness of incompatibility for a family of measurements as the minimal noise needed to make the measurements compatible and show that this quantity coincides with the optimal ratio of state discrimination probabilities with pre- and post-measurement information (Theorem 14). The results in Section 8 generalize the finite-dimensional results in [7, 51, 56].

Section 9 concludes the paper.

2 Preliminaries

In this preliminary section, we review basic properties of order unit Banach spaces (GPTs), (compact) convex structures, and classical spaces as well as fix the notation. For general references, we refer to [25, 1, 23, 49] for ordered topological linear spaces. For a more complete review of the GPT and ordered vector spaces, see [35] (Chapter 1).

2.1 Order unit Banach spaces (with preduals)

In this subsection we introduce the notions of the order unit Banach space and that with a predual. In this paper the former appears as the space of continuous affine functionals on the measurement space, while the latter as the space of observables on a physical state space.

Throughout the paper linear spaces are assumed to be over the reals \( \mathbb{R} \) unless otherwise stated. For a normed linear space \( E \), its Banach dual and double dual are denoted as \( E^* \) and \( E^{**} \), respectively. The scalar \( \psi(x) \) \( (x \in E, \psi \in E^*) \) is occasionally written in the bilinear form as \( \langle \psi, x \rangle \) or \( \langle x, \psi \rangle \). For a subset \( A \) of a normed linear space \( E \) and \( r \in [0, \infty) \), we write as \( (A)_r := \{ x \in A \mid \| x \| \leq r \} \).

Let \((E, F)\) be a pair of dual pair of linear spaces (e.g. a Banach space \( E \) and its dual \( E^* \)) separated with the bilinear form \( \langle \cdot, \cdot \rangle : E \times F \to \mathbb{R} \). For a subset \( A \subset E \), the polar \( A^o \) and the bipolar \( A^{oo} \) of \( A \) in the pair \((E, F)\) are defined by

\[
A^o := \{ y \in F \mid \langle x, y \rangle \geq -1 \ (\forall x \in A) \}
\]

and

\[
A^{oo} := \{ x \in E \mid \langle x, y \rangle \geq -1 \ (\forall y \in A^o) \},
\]

respectively. According to the bipolar theorem, \( A^{oo} \) is the \( \sigma(E, F) \)-closed convex hull of \( A \cup \{0\} \).

A subset \( K \) of a linear space \( E \) is called a cone if it satisfies

(i) \( K + K \subset K \),

(ii) \( \lambda K \subset K \ (\forall \lambda \in [0, \infty)) \).

A subset \( K \subset E \) is called a positive cone (or proper cone) if \( K \) is a cone satisfying

(iii) \( K \cap (-K) = \{0\} \).
A positive cone $K$ on $E$ induces a partial order $\leq$ by $x \leq y :\iff y - x \in K \; (x, y \in E)$. An order $\leq$ on a linear space induced by a positive cone is called a linear order. Conversely any partial order $\leq$ on $E$ induces the positive cone $K = \{ x \in E \mid x \geq 0 \}$ and the order induced by $K$ coincides with $\leq$ if

(a) $x \leq y \implies x + z \leq y + z \; (x, y, z \in E),$
(b) $x \leq y \implies \lambda x \leq \lambda y \; (x, y \in E; \lambda \in [0, \infty))$.

A linear space $E$ equipped with such a positive cone or a linear order is called an ordered linear space. The positive cone of an ordered linear space $E$ is denoted by $E_+$ and each element of $E_+$ is called positive.

An ordered linear space $E$ is called Archimedean if for any $x \in E$, if there exists $y \in E$ such that $nx \leq y$ for all positive integer $n$, then $x \leq 0$. A positive element $u \in E_+$ is called an order unit if for any $x \in E$ there exists $\lambda \in [0, \infty)$ such that $-\lambda u \leq x \leq \lambda u$. For an Archimedean ordered linear space $E$ with an order unit $u$, we define the order unit norm on $E$ by $\|x\| := \inf \{ \lambda \in [0, \infty) \mid -\lambda u \leq x \leq \lambda u \} \; (x \in E)$. This norm satisfies $-\|x\| u \leq x \leq \|x\| u$ for any $x \in E$. We call $(E, u_E)$ an order unit Banach space if $E$ is an Archimedean ordered linear space with the order unit $u_E$ and the order unit norm induced by $u_E$ is complete. Throughout this paper the order unit of an order unit Banach space $E$ is always written as $u_E$.

Let $E$ be an ordered linear space. A convex subset $B \subset E_+$ is called a base of the positive cone $E_+$ if for each positive element $x \in E_+$ there exists a unique $\lambda \in [0, \infty)$ and $b \in B$ such that $x = \lambda b$. For $x \in E_+ - E_+ = \text{lin}(E_+)$ (here $\text{lin}(\cdot)$ denotes the linear span), we define the base norm $\|x\|_B := \inf \{ \alpha + \beta \mid x = \alpha b_1 - \beta b_2; \; b_1, b_2 \in B; \alpha, \beta \in [0, \infty) \}$. The base norm $\|\cdot\|_B$ on $\text{lin}(E_+)$ coincides with Minkowski functional of $\text{conv}(B \cup (-B))$, where $\text{conv}(\cdot)$ denotes the convex hull. An ordered linear space $E$ is called base-normed if $E_+$ is generating, i.e. $E = \text{lin}(E_+)$, and $E_+$ has a base $B$. If the base $B$ of a base-normed space $E$ induces a complete norm, then $E$ is called a base-normed Banach space.

For a compact convex set $K$ on a locally convex Hausdorff space $V$, we denote by $A_c(K)$ the set of continuous real affine functionals on $K$. Then $(A_c(K), 1_K)$ is an order unit Banach space and the order unit norm coincides with the supremum norm $\|f\| = \sup_{x \in K} |f(x)|$, where $1_S(\cdot) \equiv 1$ denotes the unit constant function on a set $S$.

Conversely, any order unit Banach space $(E, u_E)$ can be regarded as $(A_c(K), 1_K)$ for some compact convex set $K$ in the following way. The dual space $E^*$ is an ordered linear space with the dual positive cone $E_+^* := \{ \psi \in E^* \mid \langle \psi, x \rangle \geq 0 \; (\forall x \in E_+) \}$ and a positive linear functional $\psi \in E_+^*$ is called a state (on $E$) if $\|\psi\| = 1$, or equivalently $\langle \psi, u_E \rangle = 1$. The set of states on $E$ is written as $S(E)$, which is a weakly*-compact convex subset of $E^*$ and $(E, u_E)$ is isomorphic to $(A_c(S(E)), 1_{S(E)})$ by the following correspondence:

$$E \ni x \mapsto f_x \in A_c(S(E)), \quad f_x(\psi) := \langle \psi, x \rangle \quad (x \in E, \psi \in S(E))$$

([1], Theorem II.1.8). The dual space $E^*$ is a base-normed Banach space with the base $S(E)$ and the base norm on $E^*$ coincides with the dual norm $\|\psi\| = \sup_{x \in (E)_+} |\langle \psi, x \rangle|$ ([1], Theorem II.1.15).
A similar base norm property also holds for a Banach predual of an order unit Banach space. A Banach space $E$ is said to have a predual $E_*$ if $E$ is isometrically isomorphic to the Banach dual $(E_*)^*$ of the normed linear space $E_*$. We can and do take a predual $E_*$ as a norm closed linear subspace of $E^*$ and such $E_*$ is called a Banach predual of $E$. Let $(E,u_E)$ be an order unit Banach space with a Banach predual $E_*$.

Then $E_*$ is ordered by the predual positive cone $E_{*+} := \{ \psi \in E_* \mid \langle \psi, x \rangle \geq 0 \ (\forall x \in E_{*+}) \}$. It is known that $E_{*+}$ is weakly* closed and hence by the bipolar theorem $E_{*+}$ is the dual cone of $E_{*+}$. The positive cone $E_{*+}$ generates $E_*$ and has the base $S_*(E) := S(E) \cap E_*$, which is the set of weakly* continuous states on $E$. Furthermore the base norm on $E_*$ induced by $S_*(E)$ coincides with the original norm $\| \psi \|$ i.e. for $\psi \in E_*$

$$\sup_{x \in (E)_1} | \langle \psi, x \rangle | := \| \psi \| = \inf \{ \alpha + \beta \mid \psi = \alpha \phi_1 - \beta \phi_2 ; \alpha, \beta \in [0, \infty) ; \phi_1, \phi_2 \in S_*(E) \}.$$ 

An order unit Banach space with a Banach predual can be represented as the set of bounded affine functionals on a convex set as follows. We denote by $A_b(C)$ by the set of bounded real affine functionals on a convex set $C$. Then $(A_b(C),1_C)$ is an order unit Banach space. Furthermore the pointwise convergence topology on $A_b(C)$ is defined, which is the weakest topology such that $A_b(C) \subseteq f : f(x) \in \mathbb{R}$ is continuous for any $x \in C$. Now let $(E,u_E)$ be an order unit Banach space with a Banach predual $E_*$. Then $E$ and $A_b(S_*(E))$ are isomorphic by the correspondence

$$E \ni x \mapsto g_x \in A_b(S_*(E)),$$

$$g_x(\psi) := \langle \psi, x \rangle \ (x \in E, \psi \in S_*(E)).$$

Moreover, by this identification the weak* topology $\sigma(E,E_*)$ on $E$ and the pointwise convergence topology on $A_b(S_*(E))$ coincide.

**Example 1** (Operator algebraic and quantum theories [53]). Let $\mathcal{A}$ be a $C^*$-algebra with a unit $1_\mathcal{A}$ and let $\mathcal{A}_{\text{sa}}$ denote the set of self-adjoint elements of $\mathcal{A}$. By taking the ordinary positive cone $\mathcal{A}_+ := \{ a^* a \mid a \in \mathcal{A} \} \subseteq \mathcal{A}_{\text{sa}}$ (i.e., the self-adjoint elements of $\mathcal{A}$), $(\mathcal{A}_{\text{sa}}, 1_\mathcal{A})$ is an order unit Banach space and the order unit norm on $\mathcal{A}_{\text{sa}}$ coincides with the $C^*$-norm restricted to $\mathcal{A}_{\text{sa}}$. If we further assume that $\mathcal{A}$ is a $W^*$-algebra, which is a $C^*$-algebra with a (unique) complex Banach predual $\mathcal{A}_*$, then the unique Banach predual of $\mathcal{A}_{\text{sa}}$ is given by the self-adjoint part $\mathcal{A}_{\text{sa}}$ of $\mathcal{A}_*$.

An important example of this is the ordinary quantum theory. Let $\mathcal{H}$ be a complex Hilbert space. Then the set $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is a special kind of $W^*$-algebra and the predual $\mathcal{L}(\mathcal{H})_*$ can be identified with the set $\mathcal{T}(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ by the bilinear form $\langle T, a \rangle := \text{tr}(Ta)$ $(T \in \mathcal{T}(\mathcal{H}), a \in \mathcal{L}(\mathcal{H}))$, where $\text{tr}(\cdot)$ denotes the trace. By this identification $S_*(\mathcal{L}(\mathcal{H})_{\text{sa}})$ corresponds to the set of density operators on $\mathcal{H}$. Note that if $\mathcal{H}$ is infinite-dimensional, the Banach dual $\mathcal{L}(\mathcal{H})^*$ and the state space $S(\mathcal{L}(\mathcal{H})_{\text{sa}})$ do not coincide with $\mathcal{L}(\mathcal{H})_*$ and $S_*(\mathcal{L}(\mathcal{H})_{\text{sa}})$, respectively.

Let $(E_i,u_{E_i}) \ (i \in I)$ be a (possibly infinite) family of order unit Banach spaces. Then we can define another order unit Banach space $(\bar{E},u_{\bar{E}})$, called the direct sum space, by

$$\bar{E} := \{ (x_i)_{i \in I} \in \prod_{i \in I} E_i \mid \sup_{i \in I} \| x_i \| < \infty \},$$

$$\bar{E}_{*+} := \{ (x_i)_{i \in I} \in \bar{E} \mid x_i \geq 0 \ (\forall i \in I) \},$$

$$u_{\bar{E}} := (u_{E_i})_{i \in I}.$$
Let \((S, \oplus, \circ, \emptyset, 1, \mu, \tau)\) be an order unit Banach space. The order unit norm on \(\widetilde{E}\) is then given by \(\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\| \quad ((x_i)_{i\in I} \in \widetilde{E})\). The Banach space \(\widetilde{E}\) is occasionally written as \(\bigoplus_{i\in I} E_i\).

Suppose further that each \(E_i\) has a Banach predual \(E_i^*\). Then \(\widetilde{E}\) has the predual \(\widetilde{E}_* := \{ (\psi_i)_{i\in I} \in \prod_{i\in I} E_i^* \mid \sum_{i\in I} \|\psi_i\| < \infty \}\) with the bilinear form
\[
((\psi_i)_{i\in I}, (x_i)_{i\in I}) := \sum_{i\in I} \langle \psi_i, x_i \rangle
\]

If \(I\) is a finite set, the dual space \(\widetilde{E}_*\) can be identified with \(\prod_{i\in I} E_i^*\) with the positive cone \(\prod_{i\in I} E_i^*_+\) by the bilinear form \((1)\). Note that this identification of \(\widetilde{E}_*\) is not true when \(I\) is infinite.

For a finite number of order unit Banach spaces \((E_1, u_{E_1}), (E_2, u_{E_2}), \ldots, (E_n, u_{E_n})\), the direct sum space \(\widetilde{E}\) and each element \((x_i)_{i=1}^n \in \widetilde{E}\) are occasionally written as \(E_1 \oplus E_2 \oplus \cdots \oplus E_n\) and \(x_1 \oplus x_2 \oplus \cdots \oplus x_n\), respectively.

### 2.2 Abstract convex structures

The order unit Banach space and that with a predual introduced in Section 2.1 can be regarded as the spaces of observables of physical systems. Here we conversely derive these notions from abstract state spaces based on the line of Gudder \([17, 18]\).

**Definition 1** (Convex structures). 1. A set \(S\) endowed with a map
\[
[0, 1] \times S \times S \ni (\lambda, s, t) \mapsto \langle \lambda; s, t \rangle \in S
\]

is called a **convex prestructure** \([17, 18]\) and \(\langle \cdot; \cdot, \cdot \rangle\) is called the convex combination on \(S\). We always assume that any convex subset \(C\) of a linear space is equipped with the usual convex combination \(\langle \lambda; s, t \rangle = \lambda s + (1 - \lambda)t \quad (\lambda \in [0, 1]; s, t \in C)\).

2. Let \((S_i, \langle \cdot; \cdot, \cdot \rangle_i) (i = 1, 2)\) be convex prestructures. A map \(\Psi: S_1 \rightarrow S_2\) is called **affine** if
\[
\Psi(\langle \lambda; s, t \rangle_i) = \langle \lambda; \Psi(s), \Psi(t) \rangle_2 \quad (\forall \lambda \in [0, 1]; \forall s, t \in S_i).
\]

An affine bijection \(\Psi: S_1 \rightarrow S_2\) is called an affine isomorphism. Note that if \(\Psi\) is an affine isomorphism, its inverse \(\Psi^{-1}\) is also affine.

3. Let \((S, \langle \cdot; \cdot, \cdot \rangle)\) be a convex prestructure. An affine map \(f: S \rightarrow \mathbb{R}\) is called an **affine functional** on \(S\). We denote by \(A_b(S)\) the set of bounded affine functionals on \(S\). \(A_b(S)\) endowed with the supremum norm \(\|f\| := \sup_{s \in S} |f(s)|\) is a Banach space. If \(S\) is a topological space, we denote by \(A_c(S)\) the set of continuous affine functionals on \(S\).
4. A convex prestructure \((S, \langle \cdot ; \cdot, \cdot \rangle)\) is called a **compact convex structure** if \(S\) is a compact Hausdorff topological space and \(A_c(S)\) separates points of \(S\), i.e. for any \(s, t \in S\), \(f(s) = f(t)\) \((\forall f \in A_c(S))\) implies \(s = t\).

5. A convex prestructure \((S, \langle \cdot ; \cdot, \cdot \rangle)\) is called a **norm-complete convex structure** if \(A_b(S)\) separates points of \(S\) and the metric \(d\) on \(S\) defined by

\[
d(s, t) := \sup \{ |f(s) - f(t)| \mid f \in A_b(S), \|f\| \leq 1 \} \quad (s, t \in S)
\]

is complete. \(\square\)

The notion of compact (norm-complete) convex structure corresponds to that of order unit Banach space (with a Banach predual) as in the following proposition

**Proposition 1.** 1. Let \((S, \langle \cdot ; \cdot, \cdot \rangle)\) be a compact convex structure. Then \((A_c(S), 1_S)\) endowed with the positive cone \(A_c(S)_+ := \{ f \in A_c(S) \mid f(s) \geq 0 (\forall s \in S) \}\) is an order unit Banach space. If we define \(\Psi: S \ni s \mapsto \Psi(s) \in A_c(S)^*\) by

\[
\langle \Psi(s), f \rangle := f(s) \quad (s \in S, f \in A_c(S)),
\]

then the map \(\Psi\) is a continuous affine isomorphism between \(S\) and \(S(A_c(S))\) so that we can identify \(S\) with \(S(A_c(S))\).

2. Let \((S, \langle \cdot ; \cdot, \cdot \rangle)\) be a norm-complete convex structure. Then \((A_b(S), 1_S)\) endowed with the positive cone \(A_b(S)_+ := \{ f \in A_b(S) \mid f(s) \geq 0 (\forall s \in S) \}\) is an order unit Banach space. The map \(\Phi: S \to A_b(S)^*\) defined by

\[
\langle \Phi(s), f \rangle := f(s) \quad (s \in S, f \in A_b(S))
\]

is an isometry so that \(S\) may be identified with the norm-closed convex subset \(\Phi(S)\) of \(A_b(S)^*\). The linear subspace \(E_* := \text{lin}(S) \subset A_b(S)^*\) is a Banach predual of \(A_b(S)\) and \(S\) coincides with the base \(S_*(A_b(S))\) of \(E_*\).

Proposition 1.2 is what is called in [35] Ludwig’s embedding theorem [40] (IV, Theorem 3.7). The claim 1 can be shown similarly as claim 2. For completeness short proofs are included in Appendix A.

We can rephrase Proposition 1.1 in terms of the regular embedding [H, Section II.2]. For a compact convex structure \((S, \langle \cdot ; \cdot, \cdot \rangle)\), a continuous affine injection \(\Psi: S \to V\) into a locally convex Hausdorff space \(E\) is called a regular embedding if \(E = \text{lin}(\Psi(S))\) and \(0 \notin \text{aff}(\Psi(S))\), where \(\text{aff}(\cdot)\) denotes the affine hull. If such \(\Psi\) exists, \(S\) is said to be regularly embedded into \(E\).

**Proposition 2.** Let \((S, \langle \cdot ; \cdot, \cdot \rangle)\) be a compact convex structure and let \(\Psi: S \to A_c(S)^*\) be the map in Proposition 1. Then \(\Psi\) is a regular embedding into \(A_c(S)\) equipped with the weak* topology. Furthermore, such a regular embedding is unique in the following sense: if \(\Phi: S \to E\) is another regular embedding into a locally convex Hausdorff space \(E\), there exists a continuous linear isomorphism \(J: E \to A_c(S)^*\) such that \(\Psi = J \circ \Phi\).
Proof. The first part of the claim is immediate from Proposition 1 and from that $S(A_c(S)) = \Psi(S)$ is a base of the positive cone $A_c(S)_+$. It also follows that $A_c(S)$ separates points of $S$, $1_S \in A_c(S)$, and $S(A_c(S)) = S$. Therefore $(S, A_c(S))$ is an abstract convex in the sense of [1] (Section II.2) and the rest of the claim follows from Theorem II.2.4 of [1].

As we have seen in Sections 2.1 and 2.2, the order unit Banach space and the convex state space are dual notions and we can always translate a general statement on the one side to the other. In physical terms, these notions correspond to the descriptions of the systems in the Heisenberg and Schrödinger pictures, respectively. In what follows in this paper we mainly consider order unit Banach spaces with preduals as the spaces of observables of physical systems, while the measurement space will be introduced in Section 4 as a special kind of compact convex structure.

2.3 Classical space

An order unit Banach space $E$ is called classical if it satisfies either (all) of the following equivalent conditions ([1], Theorem II.4.1):

(i) The state set $S(E)$ is a Bauer simplex, i.e. the set $\partial S(E)$ of extremal points (or pure states) of $S(E)$ is compact and any $\phi \in S(E)$ is a barycenter of a unique simplicial boundary measure [1].

(ii) $(E, u_E)$ is isomorphic to $(C(X), 1_X)$ for some compact Hausdorff space $X$ as an order unit Banach space, where $C(X)$ denotes the set of real continuous functions on $X$ equipped with the positive cone $C(X)_+ = \{f \in C(X) \mid f(x) \geq 0 \forall x \in X\}$.

(iii) The partially ordered set $(E, \leq)$ is a lattice.

Now, by generalizing the finite-dimensional result in [2] (Corollary 1), we give another characterization of a classical space in terms of a well-behaving product operation, or a universal broadcasting channel as in the following proposition. See also [54] (Corollary 5.7.9) for the uniqueness part.

Proposition 3. An order unit Banach space $E$ is classical if and only if there exists a bilinear map $B: E \times E \to E$ such that

(i) (broadcasting property) $B(a, u_E) = B(u_E, a) = a$ ($\forall a \in E$);

(ii) (bipositivity) $B(a, b) \geq 0$ ($\forall a, b \in E_+$).

Furthermore, such a bilinear map $B$ is, if exists, unique and satisfies the commutativity $B(a, b) = B(b, a)$ and the associativity $B(B(a, b), c) = B(a, B(b, c))$ ($\forall a, b, c \in E$).

The proof of Proposition 3 is analogous to the finite-dimensional case [2] and to the Gelfand’s representation theorem for abelian $C^*$-algebras [33]. See Appendix B for detail.

For a classical space $E$, the unique bilinear map $B(a, b)$ in Proposition 3 is written as $a \cdot b$ ($a, b \in E$) and called the product on $E$.

A possibly infinite direct sum of classical spaces is also classical. If $E$ is a classical space with a Banach predual $E_*$, then $E$ is isomorphic to the set of self-adjoint elements of an
abelian \( W^* \)-algebra. By the uniqueness of the complex Banach predual of a \( W^* \)-algebra, the Banach predual of a classical space is, if exists, unique. The double dual \( E^{**} \) of a classical space \( E \) is also a classical space.

An element \( P \) of a classical space \( E \) is called a projection if \( P \cdot P = P \). A projection \( P \) always satisfies \( 0 \leq P \leq u_E \).

The following proposition, which is immediate from the general properties from \( W^* \)-algebras (von Neumann algebras), will be used in the main part.

**Proposition 4.** Let \( E \) be a classical space with a Banach predual \( E_* \).

1. For each \( a \in E \) there exists a sequence \( (a_n)_{n \in \mathbb{N}} \) of finite sums of projections on \( E \) such that \( \|a - a_n\| \to 0 \).

2. For each positive weakly\(^*\) continuous linear functional \( \varphi \in E_{**} \) there exists the smallest projection \( P \in E \) such that \( \langle \varphi, P \rangle = \| \varphi \| \). Such \( P \) is called the support projection of \( \varphi \) and written as \( s(\varphi) \).

### 3 Measurements

In this section we introduce and prove basic facts on the channels, measurements, and the post-processing order and equivalence relations between them.

#### 3.1 Channels and post-processing relations

Before introducing measurements, we consider more general class of channels between order unit Banach spaces.

For simplicity, in what follows in this paper, if we say that \( E \) is an order unit Banach space, we understand that \( E \) is endowed with an order unit which is written as \( u_E \).

A linear map \( \Psi: E \to F \) between ordered linear spaces \( E \) and \( F \) is called positive if \( \Psi(E_+) \subset F_+ \). If \( E \) and \( F \) are order unit Banach spaces, a positive linear map \( \Psi: E \to F \) that is unital, i.e. \( \Psi(u_E) = u_F \), is called a channel (in the Heisenberg picture). The domain \( E \) and the codomain \( F \) of a channel \( \Psi: E \to F \) are called the outcome and input spaces of \( \Psi \), respectively. We write the set of channels from \( E \) to \( F \) as \( \text{Ch}(E \to F) \).

**Proposition 5.** Let \( E \) and \( F \) be order unit Banach spaces. Then any positive linear map \( \Psi: E \to F \) is bounded and the uniform norm is given by \( \| \Psi \| = \| \Psi(u_E) \| \). If \( \Psi \) is channel, then \( \| \Psi \| = 1 \).

**Proof.** For any \( a \in E \), we have \( -\|a\|u_E \leq a \leq \|a\|u_E \) and the positivity of \( \Psi \) implies \( -\|a\|\Psi(u_E) \leq \Psi(a) \leq \|a\|\Psi(u_E) \) and hence \( -\|a\|\|\Psi(u_E)\|u_F \leq \Psi(a) \leq \|a\|\|\Psi(u_E)\|u_F \). Thus \( \|\Psi(a)\| \leq \|\Psi(u_E)\|\|a\| \) and we obtain \( \|\Psi\| \leq \|\Psi(u_E)\| \). Since \( \|\Psi(u_E)\| \leq \|\Psi\| \) is obvious, the first part of the claim is proved. If \( \Psi \) is a channel, then \( \|\Psi\| = \|\Psi(u_E)\| = \|u_F\| = 1 \).

Let \( E \) be a Banach space and let \( F \) be a Banach space with a Banach predual \( F_\ast \). Then the set \( \mathcal{L}(E \to F) \) of bounded linear maps from \( E \) to \( F \) is endowed with a locally convex
Hausdorff topology called the BW-topology ([16], Chapter 7) in the following way. The BW-topology is the weakest topology such that

\[ \mathcal{L}(E \to F) \ni \Psi \mapsto \langle \psi, \Psi(a) \rangle \in \mathbb{R} \]

is continuous for any \( a \in E \) and any \( \psi \in F_\ast \). A net \((\Psi_i)_{i \in I}\) in \( \mathcal{L}(E \to F) \) is BW-convergent to \( \Psi \in \mathcal{L}(E \to F) \) if and only if \( \langle \psi, \Psi_i(a) \rangle \to \langle \psi, \Psi(a) \rangle \) for any \( a \in E \) and \( \psi \in F_\ast \), or equivalently \( \Psi_i(a) \xrightarrow{\text{weakly}^*} \Psi(a) \) for any \( a \in E \). It follows from Tychonoff’s theorem that the closed unit ball \((\mathcal{L}(E \to F))_1\) is BW-compact.

**Proposition 6.** Let \( E \) and \( F \) be order unit Banach spaces. Suppose that \( F \) has a Banach predual \( F_\ast \). Then \( \text{Ch}(E \to F) \) is a BW-compact convex subset of \( \mathcal{L}(E \to F) \).

**Proof.** It is easy to show the convexity of \( \text{Ch}(E \to F) \). Since \( \text{Ch}(E \to F) \) is a subset of the BW-compact set \((\mathcal{L}(E \to F))_1\) by Proposition 5, it suffices to show that \( \text{Ch}(E \to F) \) is BW-closed and this follows from the weak* closedness of the positive cone \( F_+ \).

Let \( E \) and \( F \) be order unit Banach spaces with preduals \( E_\ast \) and \( F_\ast \), respectively. Then a weakly* continuous channel \( \Psi: E \to F \) is briefly called a w*-channel. The set of w*-channels from \( E \) to \( F \) is denoted by \( \text{Ch}_{w^*}(E \to F) \). For a w*-channel \( \Psi: E \to F \), there exists a unique bounded linear map \( \Psi_\ast: F_\ast \to E_\ast \) such that

\[ \langle \psi, \Psi(a) \rangle = \langle \Psi_\ast(\psi), a \rangle \]  

\((a \in E, \psi \in F_\ast)\). This map satisfies \( \Psi_\ast(S_\ast(F)) \subset S_\ast(E) \). Conversely for each affine map \( \Psi_\ast: S_\ast(F) \to S_\ast(E) \) there exists a unique w*-channel \( \Psi: E \to F \) satisfying \((2)\) for any \( a \in E \) and \( \psi \in S_\ast(F) \). The above map \( \Psi_\ast \) is called the predual of \( \Psi \) corresponds to the channel in the Schrödinger picture.

We now introduce the post-processing relations for channels.

**Definition 2.** Let \( \Psi \in \text{Ch}(F \to E) \) and \( \Phi \in \text{Ch}(G \to E) \) be channels with the same input space \( E \).

1. \( \Psi \) is said to be a **post-processing** of \( \Phi \), written as \( \Psi \preceq_{\text{post}} \Phi \), if there exists \( \Lambda \in \text{Ch}(F \to G) \) such that \( \Psi = \Phi \circ \Lambda \).

2. \( \Psi \) is said to be **post-processing equivalent** to \( \Phi \), written as \( \Psi \sim_{\text{post}} \Phi \), if \( \Psi \preceq_{\text{post}} \Phi \) and \( \Phi \preceq_{\text{post}} \Psi \) hold.

By noting that any composition of channels is again a channel, we can easily see that the relations \( \preceq_{\text{post}} \) and \( \sim_{\text{post}} \) are respectively binary preorder and equivalence relations defined on the class of channels with a fixed input space.

We next introduce the w*-extension of a channel. For this we need the following characterization of the double dual Banach space. As usual, we regard every normed linear space \( E \) as a linear subspace of the double dual Banach space \( E^{**} \).

**Proposition 7.** Let \( E \) be a Banach space, let \( F \) be a Banach space with a Banach predual \( F_\ast \), and let \( \Psi: E \to F \) be a bounded linear map. Then \( \Psi \) is uniquely extended to a weakly* continuous (i.e. \( \sigma(E^{**}, E^*)/\sigma(F, F_\ast) \)-continuous) linear map \( \overline{\Psi}: E^{**} \to F \). The map \( \overline{\Psi} \) is called the w*-extension of \( \Psi \).
Proof. Let $\Psi^* : F^* \to E^*$ be the dual map of $\Psi$ and let $\Phi : F_+ \to E^*$ be the restriction of $\Psi^*$ to $F_+ (= (F_*)^*)$. We define $\overline{\Psi} : E^{**} \to F(= (F_*)^*)$ by the dual map of $\Phi$. Then $\overline{\Psi}$ is $\sigma(E^{**}, E^*)/\sigma(F, F_+)$-continuous by definition. Furthermore for any $a \in E$ and $\psi \in F_+$

$$\langle \psi, \Psi(a) \rangle = \langle \Phi(\psi), a \rangle = \langle \Psi^*(\psi), a \rangle = \langle \psi, \Psi(a) \rangle,$$

which implies $\overline{\Psi}(a) = \Psi(a)$ ($a \in E$). Therefore $\overline{\Psi}$ satisfies the required conditions of the claim. The uniqueness of $\overline{\Psi}$ follows from the weak$^*$ density of $E$ in $E^{**}$. □

If $E$ is an order unit Banach space, the double dual space $E^{**}$ with the order unit $u_{E^{**}} = u_E$ and the double dual positive cone $E^+_+ := \{ a'' \in E^{**} \mid \langle \psi, a'' \rangle \geq 0 \ (\forall \psi \in E^+_+) \}$ is an order unit Banach space with the Banach predual $E^*$. Then we have $E_+ = E^+_+ \cap E$, i.e. the orders on $E$ and $E^{**}$ are consistent. Moreover by the bipolar theorem $E_+$ is a weakly$^*$ dense subset of $E^+_+$. □

**Proposition 8.** Let $\Psi \in \text{Ch}(E \to F)$ be a channel. Suppose that the order unit Banach space $F$ has a Banach predual $F_+$. Then the w$^*$-extension $\overline{\Psi} : E^{**} \to F$ of $\Psi$ is a w$^*$-channel.

Proof. The unitality of $\overline{\Psi}$ follows from $\overline{\Psi}(u_E) = \Psi(u_E) = u_F$. To show the positivity, take an element $a'' \in E^{**}_+$. Then there exists a net $(a_i)_{i \in I}$ in $E_+$ weakly$^*$ converging to $a''$. Then since the positive cone $F_+$ is weakly$^*$ closed, we have $\overline{\Psi}(a'') = \lim_{i \in I} \Psi(a_i) \in F_+$, where the limit is with respect to $\sigma(F, F_+)$. Therefore $\Psi$ is a w$^*$-channel. □

The following proposition implies that the w$^*$-extension of a channel is the least channel in the post-processing order that upper bounds the original channel (cf. [34], Lemma 7).

**Proposition 9.** Let $\Psi \in \text{Ch}(E \to F)$ and $\overline{\Psi} \in \text{Ch}_{w^*}(E^{**} \to F)$ be the same as in Proposition 8. Then for any w$^*$-channel $\Phi \in \text{Ch}_{w^*}(G \to F)$, where $G$ has a Banach predual $G_+$, $\Psi \leq_{\text{post}} \Phi$ if and only if $\overline{\Psi} \leq_{\text{post}} \Phi$.

Proof. Assume $\Psi \leq_{\text{post}} \Phi$. Then there exists a channel $\Lambda \in \text{Ch}(E \to G)$ such that $\Psi = \Phi \circ \Lambda$. Let $\overline{\Lambda} \in \text{Ch}_{w^*}(E^{**} \to G)$ be the w$^*$-extension of $\Lambda$. Then $\Phi \circ \overline{\Lambda} \in \text{Ch}_{w^*}(E^{**} \to F)$ and for any $a \in E$ we have $\Phi \circ \overline{\Lambda}(a) = \Phi \circ \Lambda(a) = \Psi(a)$. Therefore the uniqueness of the w$^*$-extension implies $\overline{\Psi} = \Phi \circ \overline{\Lambda} \leq_{\text{post}} \Phi$. The converse implication follows from $\Psi \leq_{\text{post}} \overline{\Psi}$, which holds because $\Psi$ is the restriction of $\overline{\Psi}$ to $E$. □

Let $\Phi \in \text{Ch}(F \to E)$ and $\Psi \in \text{Ch}(G \to E)$ be channels. For $\lambda \in [0, 1]$ we define the direct convex combination channel $\lambda \Phi + (1 - \lambda) \Psi \in \text{Ch}(F \oplus G \to E)$ by

$$[\lambda \Phi + (1 - \lambda) \Psi](a \oplus b) := \lambda \Phi(a) + (1 - \lambda) \Psi(b) \quad (a \oplus b \in F \oplus G).$$

The channel $\lambda \Phi + (1 - \lambda) \Psi$ corresponds to performing $\Phi$ and $\Psi$ independently with probabilities $\lambda$ and $1 - \lambda$, respectively. If $\Phi$ and $\Psi$ are w$^*$-channels, so is $\lambda \Phi + (1 - \lambda) \Psi$. As mentioned in Section II this operation is not closed in a set, but in this case defined on the class of channels with a fixed input space.

The first claim of the next proposition indicates that the convex operation is consistent with the post-processing order.
Proposition 10. 1. Let $\Phi_i \in \text{Ch}(F_i \to E)$ and $\Psi_i \in \text{Ch}(G_i \to E)$ ($i = 1, 2$) be channels. Then $\Phi_1 \preceq_{\text{post}} \Phi_2$ and $\Psi_1 \preceq_{\text{post}} \Psi_2$ imply $\lambda \Phi_1 + (1 - \lambda) \Psi_1 \preceq_{\text{post}} \lambda \Phi_2 + (1 - \lambda) \Psi_2$ for any $\lambda \in [0, 1]$.

2. If $\Psi, \Phi \in \text{Ch}(F \to E)$ are channels with the common input and outcome spaces, then $\lambda \Psi + (1 - \lambda) \Phi \preceq_{\text{post}} \lambda \Psi \oplus (1 - \lambda) \Phi$ for any $\lambda \in [0, 1]$.

Proof. 1. By assumption there exist channels $\Theta \in \text{Ch}(F_1 \to F_2)$ and $\Xi \in \text{Ch}(G_1 \to G_2)$ such that $\Phi_1 = \Phi_2 \circ \Theta$ and $\Psi_1 = \Psi_2 \circ \Xi$. We define $\Omega \in \text{Ch}(F_1 \oplus F_2 \to G_1 \oplus G_2)$ by $\Omega(a \oplus b) := \Theta(a) \oplus \Xi(b)$ ($a \oplus b \in F_1 \oplus F_2$). Then it readily follows that

$$\lambda \Phi_1 + (1 - \lambda) \Psi_1 = [\lambda \Phi_2 \oplus (1 - \lambda) \Psi_2] \circ \Omega \preceq_{\text{post}} \lambda \Phi_2 \oplus (1 - \lambda) \Psi_2.$$ 

2. Define a channel $\Delta \in \text{Ch}(F \to F \oplus F)$ by $\Delta(a) := a \oplus a$ ($a \in F$). Then for each $a \in F$, $[\lambda \Psi \oplus (1 - \lambda) \Phi] \circ \Delta(a) = \lambda \Psi(a) + (1 - \lambda) \Phi(a)$, which implies $\lambda \Psi + (1 - \lambda) \Phi = [\lambda \Psi \oplus (1 - \lambda) \Phi] \circ \Delta \preceq_{\text{post}} \lambda \Psi \oplus (1 - \lambda) \Phi$. □

3.2 Measurements

In this paper we consider w*-measurement as abstract GPT-to-classical channels, generalizing the quantum-to-classical channels. This kind of formulation, rather than the ordinary way of considering POVMs or effect-valued measures (EVMs) (e.g. [14]), is useful for developing the general theory of measurements as in the succeeding sections.

In the rest of this paper, unless otherwise stated, we fix an input order unit Banach space $(E, u_E)$ and its Banach predual $E^*$.

A channel $\Psi \in \text{Ch}(F \to E)$ is said to be a measurement if the outcome space $F$ is classical. When $F$ is a classical space with a Banach predual, then a w*-channel $\Psi \in \text{Ch}_{w^*}(F \to E)$ is called a w*-measurement. If we say that $\Psi \in \text{Ch}_{w^*}(F \to E)$ is a w*-measurement, we understand that $F$ is a classical space with the Banach predual $F^*$.

Proposition 11. Let $\Psi \in \text{Ch}_{w^*}(F \to E)$ and $\Phi \in \text{Ch}_{w^*}(G \to E)$ be w*-measurements. Then $\Psi \preceq_{\text{post}} \Phi$ if and only if there exists a w*-channel $\Gamma \in \text{Ch}_{w^*}(F \to G)$ such that $\Psi = \Phi \circ \Gamma$.

Proof. "If" part of the claim is obvious. Assume $\Psi \preceq_{\text{post}} \Phi$. Then by the proof of Proposition 9 there exists a w*-channel $\Lambda \in \text{Ch}_{w^*}(F^{**} \to G)$ such that $\Psi = \Phi \circ \Lambda$, where $\Psi \in \text{Ch}_{w^*}(F^{**} \to E)$ is the w*-extension of $\Psi$. From [20] (in the proof of Lemma 3.12), there exists a w*-channel $\Xi \in \text{Ch}_{w^*}(F \to F^{**})$ such that $\langle \varphi, \Xi(a) \rangle = \langle \varphi, a \rangle$ ($\varphi \in F^*, a \in F$). Then for $a \in F$ and $\psi \in E^*$

$$\langle \psi, \Xi(a) \rangle = \langle \Psi^*(\psi), \Xi(a) \rangle = \langle \Psi^*(\psi), a \rangle = \langle \psi, \Psi(a) \rangle,$$

where we used $\Psi^*(\psi) \in F^*$ in the second equality. This implies $\Psi = \Psi^* \circ \Xi = \Phi \circ \Lambda \circ \Xi$. Since $\Lambda \circ \Xi$ is a w*-channel, this proves the "only if" part of the claim. □

As we can see from the proof, Proposition 11 still holds when the outcome spaces $F$ and $G$ are relaxed to the self-adjoint parts of arbitrary $W^*$-algebras.
The above definition of \( w^* \)-measurement is related to the more common notion of normalized EVM. A triple \((X, \Sigma, M)\) is said to be an EVM on \( E \) if \( \Sigma \) is a \( \sigma \)-algebra on a set \( X \) and \( M : \Sigma \to E^+_\mathbb{R} \) is a map such that

(i) \( M(X) = u_E, M(\emptyset) = 0 \),

(ii) for any disjoint and countable family \((A_k)_{k \in \mathbb{N}} (\mathbb{N} := \{1, 2, \ldots\})\) in \( \Sigma \), \( M(\bigcup_{k \in \mathbb{N}} A_k) = \sum_{k \in \mathbb{N}} M(A_k) \), where the RHS converges weakly*.

For \( \psi \in E^* \) (respectively, \( \psi \in S^*(E) \)) the function \( \mu^M_\psi : \Sigma \ni A \mapsto \langle \psi, M(A) \rangle \in \mathbb{R} \) is a signed (respectively, probability) measure. Conversely for any affine map

\[ S^*(E) \ni \psi \mapsto \nu_{\psi} \]

that maps each weakly* continuous state to a probability measure on a measurable space \((X, \Sigma)\), there exists a unique EVM \((X, \Sigma, M)\) such that \( \nu_{\psi} = \mu^M_\psi (\psi \in S^*(E)) \).

For a measurable space \((X, \Sigma)\), we denote by \( B(X, \Sigma) \) the set of real bounded \( \Sigma \)-measurable functions on \( X \). Then the order unit Banach space \((B(X, \Sigma), 1_X)\) equipped with the positive cone

\[ B(X, \Sigma)^+ = \{ f \in B(X, \Sigma) \mid f(x) \geq 0 (\forall x \in X) \} \]

is a classical space.

Let \((X, \Sigma, M)\) be an EVM on \( E \). For each function \( f \in B(X, \Sigma) \), the integral \( \int_X f(x) dM(x) \in E \) is well-defined by

\[ \left\langle \int_X f(x) dM(x), \psi \right\rangle := \int_X f(x) d\mu^M_\psi(x) \quad (\psi \in E^*). \]

Then the map

\[ \gamma^M : B(X, \Sigma) \ni f \mapsto \int_X f(x) dM(x) \in E \]

is a measurement and called the measurement associated with the EVM \( M \). The \( w^* \)-extension \( \Gamma^M \in \text{Ch}_{w^*}(B(X, \Sigma)^* \to E) \) of \( \gamma^M \) is called the \( w^* \)-measurement associated with \( M \). Thus for each EVM \( M \) there corresponds a natural \( w^* \)-measurement \( \Gamma^M \). If \( E = \mathcal{L}(\mathcal{H})_{sa} \) for a separable Hilbert space (or more generally \( E \) is the self-adjoint part of a \( \sigma \)-finite \( W^* \)-algebra), we can show that for EVMs \((X, \Sigma_1, M)\) and \((Y, \Sigma_2, N)\) on \( E \), \( \Gamma^M \preceq_{\text{post}} \Gamma^N \) holds if and only if there exists a weak Markov kernel \( p(\cdot | \cdot) \) such that \( M(A) = \int_X p(A|y) dN(y) \) [32, 34].

Conversely, the following proposition indicates that any \( w^* \)-measurement can be regarded as the associated \( w^* \)-measurement of an EVM up to post-processing equivalence.

**Proposition 12.** For any \( w^* \)-measurement \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) there exists an EVM \((X, \Sigma, M)\) on \( E \) such that \( \Gamma \sim_{\text{post}} \Gamma^M \).

Proposition [12] can be shown analogously as in [32] (Proposition 3). In Appendix C we give another proof using the Riesz-Markov-Kakutani-type representation theorem for EVMs.
3.3 Finite-outcome measurements

A special class of EVMs called finite-outcome EVMs plays a fundamental role in the later sections of this paper.

Let $F$ be an order unit Banach space. A family (map) $M = (M(x))_{x \in X} \in F^X$ is called a subnormalized finite-outcome EVM, or just a subnormalized EVM, on $F$ if $X$ is a finite set called the outcome set of $X$ with the outcome set sections of this paper.

A subnormalized EVM $(M(x))_{x \in X}$ is called a normalized finite-outcome EVM, or just an EVM, if $\sum_{x \in X} M(x) = u_F$. We write the sets normalized and subnormalized EVMs on $F$ with the outcome set $X$ by $\text{EVM}(X;E)$ and $\text{EVM}^{\text{sub}}(X;E)$, respectively. For each EVM $(M(x))_{x \in X}$ on $E$ there corresponds the associated w*-measurement $\Gamma_M \in \text{Ch}(\ell^\infty(X) \to E) =$ $\text{Ch}_{w^*}(\ell^\infty(X) \to E)$ defined by

$$\Gamma_M(f) = \sum_{x \in X} f(x)M(x) \ (f \in \ell^\infty(X)),$$

where $\ell^\infty(X)$ denotes the classical space of (bounded) real functions on $X$ equipped with the order unit $1_X$ and the positive cone

$$\ell^\infty(X)_+ = \{ f \in \ell^\infty(X) \mid f(x) \geq 0 \ (\forall x \in X) \}.$$

The classical space $\ell^\infty(X) = \ell^\infty(X)^{**}$ is finite-dimensional and conversely any finite-dimensional classical space $F$ is isomorphic to $\ell^\infty(\mathcal{P}_\text{atom}(F))$, where $\mathcal{P}_\text{atom}(F)$ denotes the set of atomic projections in $F$. The sets $\text{EVM}(X;E)$ and $\text{EVM}^{\text{sub}}(X;E)$ are compact convex subsets of $E^X$ equipped with the product topology $\sigma(\ell^\infty,\ell^\infty)$ of the weak* topology $\sigma(E,E^*)$. With respect to this topology on $\text{EVM}(X;E)$, the map

$$\text{EVM}(X;E) \ni M \mapsto \Gamma_M \in \text{Ch}(\ell^\infty(X) \to E)$$

is a continuous affine isomorphism, where the topology of $\text{Ch}(\ell^\infty(X) \to E)$ is the BW-topology.

A w*-measurement $\Gamma \in \text{Ch}(\ell^\infty(X) \to E)$ for some finite set $X$ is called finite-outcome. For finite-outcome EVMs, the post-processing relation is characterized as follows.

**Proposition 13.** 1. For any finite-outcome EVM $M \in \text{EVM}(X;E)$ and a channel $\Lambda \in \text{Ch}(F \to E)$, $\Gamma^M \preceq_{\text{post}} \Lambda$ if and only if there exists an EVM $\mathbf{N} \in \text{EVM}(X;F)$ such that $M(x) = \Lambda(\mathbf{N}(x)) \ (\forall x \in X)$.

2. For any finite-outcome EVMs $\mathbf{A} \in \text{EVM}(X;E)$ and $\mathbf{B} \in \text{EVM}(Y;E)$, $\Gamma^\mathbf{A} \preceq_{\text{post}} \Gamma^\mathbf{B}$ if and only if there exists a stochastic matrix

$$p(\cdot|x) \in \text{Stoch}(X,Y) := \{ q(\cdot|\cdot) \in \mathbb{R}^{X \times Y} \mid q(x|y) \geq 0, \sum_{x' \in X} q(x'|y) = 1 \ (x \in X, y \in Y) \}$$

such that

$$A(x) = \sum_{y \in Y} p(x|y)B(y) \ (x \in X). \quad (3)$$
Proof. The claim 1 is immediate from the isomorphism between $\text{EVM}(X; F)$ and $\text{Ch}(\ell^\infty(X) \to F)$. To show the claim 2, assume $\Gamma^A \preceq_{\text{post}} \Gamma^B$ and take a channel $\Psi \in \text{Ch}(\ell^\infty(X) \to \ell^\infty(Y))$ such that $\Gamma^A = \Gamma^M \circ \Psi$. Define a stochastic matrix $p(\cdot|\cdot) \in \text{Stoch}(X,Y)$ by

$$p(x|y) := \Psi(\delta_x)(y) \quad (x \in X, y \in Y),$$

where $\delta_x \in \ell^\infty(X)$ is given by

$$\delta_x(x') := \begin{cases} 1 & \text{if } x = x'; \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily check that $p(\cdot|\cdot)$ satisfies (3). Conversely, if (3) holds for some stochastic matrix $p(\cdot|\cdot)$, then the channel $\Psi$ defined by (4) satisfies $\Gamma^A = \Gamma^B \circ \Psi$. \qed

An EVM $M \in \text{EVM}(X; E)$ is called trivial if each element $M(x)$ ($x \in X$) is proportional to $u_E$. The associated w*-measurement $\Gamma^M$ is then minimal with respect to the post-processing order, i.e. $\Gamma^M \preceq_{\text{post}} \Lambda$ for any measurement (indeed, any channel) $\Lambda$.

4 Compact convex structure of measurements

In this section we define the measurement space and the weak topology on it, and prove some general properties of them. Among these results, the most important one is Theorem 4 which states that any measurement can be approximated by a net of finite-outcome ones and will be used in the later application parts to reduce the discussions to the finite-outcome cases.

The results in this section are generalizations of the known facts in the theory of statistical experiments [36, 54]. See Appendix D for how statistical experiments can be regarded as a special class of measurements.

4.1 Gain functional and the Blackwell-Sherman-Stein (BSS) theorem

We begin with the notion of gain functional, or state-discrimination probability functional, which will play a central role in this paper.

Definition 3 (Ensemble and gain functional). 1. For a finite set $X \neq \emptyset$, a family $\mathcal{E} = (\varphi_x)_{x \in X} \in E^X_*$ is called a w*-family. The set $X$ is then called the label set of $\mathcal{E}$. A w*-family $\mathcal{E} = (\varphi_x)_{x \in X}$ is called an ensemble if $\varphi_x \geq 0$ ($x \in X$) and the normalization condition $\sum_{x \in X} \langle \varphi_x, u_E \rangle = 1$ holds.

2. For a w*-family $\mathcal{E} = (\varphi_x)_{x \in X}$ and a measurement $\Gamma \in \text{Ch}(F \to E)$, we define the gain functional by

$$P_g(\mathcal{E}; \Gamma) := \sup_{M \in \text{EVM}(X; F)} \sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle.$$  \hspace{1cm} (5)

If $\mathcal{E}$ is an ensemble, the gain functional $P_g(\mathcal{E}; \Gamma)$ is occasionally called the state discrimination probability. \qed
In the operational language, an ensemble $\mathcal{E} = (\varphi_x)_{x \in X}$ corresponds to the situation where system's state is prepared to be $\langle \varphi_x, u_E \rangle^{-1} \varphi_x$ with the probability $\langle \varphi_x, u_E \rangle$. The value $P_g(\mathcal{E}; \Gamma)$ is then the optimal probability that we can properly guess the state label $x \in X$ when we have access to the outcome of the measurement $\Gamma$. Here each EVM $M \in \text{EVM}(X; F)$ in (5) corresponds to a randomized decision rule of $x \in X$ when the measurement outcome of $\Gamma$ is given (cf. [54, Section 4.5]).

If $\Gamma$ is a w*-measurement in Definition $3^3$ for each w*-family $\mathcal{E} = (\varphi_x)_{x \in X}$ we can always take an EVM $M \in \text{EVM}(X; F)$ that attains the optimal value for $P_g(\mathcal{E}; \Gamma)$, i.e.

$$P_g(\mathcal{E}; \Gamma) = \sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle.$$  

We remark that we can construct the theory developed in this section based instead on the loss functional defined by

$$L(\mathcal{E}; \Gamma) := \inf_{M \in \text{EVM}(X; F)} \sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle = -P_g((-\varphi_x)_{x \in X}; \Gamma)$$

(cf. [4, 28, 39]).

Now we prove some elementary properties of the gain functional.

**Proposition 14.** Let $\mathcal{E} = (\varphi_x)_{x \in X}$ be a w*-family and let $\Gamma \in \text{Ch}(F \to E)$ and $\Lambda \in \text{Ch}(G \to E)$ be measurements.

1. $P_g(\mathcal{E}; \lambda \Gamma \oplus (1 - \lambda)\Lambda) = \lambda P_g(\mathcal{E}; \Gamma) + (1 - \lambda)P_g(\mathcal{E}; \Lambda)$ for any $\lambda \in [0, 1]$.
2. $P_g(\alpha \mathcal{E}; \Gamma) = \alpha P_g(\mathcal{E}; \Gamma)$ for any $\alpha \in [0, \infty)$.
3. There exist a positive number $\alpha > 0$, a linear functional $\psi \in E_\ast$, and an ensemble $\mathcal{E}' = (\varphi'_x)_{x \in X}$ such that $\varphi_x = \alpha \varphi'_x + \psi$. Then it also holds that $P_g(\mathcal{E}; \Gamma) = \alpha P_g(\mathcal{E}'; \Gamma) + \langle \psi, u_E \rangle$.

**Proof.** By noting

$$\text{EVM}(X; F \oplus G) = \{ (M(x) \oplus N(x))_{x \in X} \in (F \oplus G)^X \mid M \in \text{EVM}(X; F), N \in \text{EVM}(X; G) \}$$

we obtain

$$P_g(\mathcal{E}; \lambda \Gamma \oplus (1 - \lambda)\Lambda) = \sup_{M \in \text{EVM}(X; F), N \in \text{EVM}(X; G)} \sum_{x \in X} \langle \varphi_x, \lambda \Gamma(M(x)) + (1 - \lambda)\Lambda(N(x)) \rangle$$

$$= \lambda \sup_{M \in \text{EVM}(X; F)} \sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle + (1 - \lambda) \sup_{N \in \text{EVM}(X; G)} \sum_{x \in X} \langle \varphi_x, \Lambda(N(x)) \rangle$$

$$= \lambda P_g(\mathcal{E}; \Gamma) + (1 - \lambda)P_g(\mathcal{E}; \Lambda),$$

which proves the claim $1$. The claim $2$ is evident from the definition.

We now show the claim $3$. Since $E_{\ast\ast}$ is generating, we have a decomposition $\varphi_x = \varphi_x^+ - \varphi_x^-$ ($\varphi_x^\pm \in E_{\ast\ast}$) for each $x \in X$. By adding a common non-zero functional $\varphi \in E_{\ast\ast}$ to $\varphi_x^\pm$ if
necessary, we may assume \( \varphi_x^+ \neq 0 \) for all \( x \in X \). Define \( \psi := -\sum_{x \in X} \varphi_x^- \). Then \( \varphi_x - \psi \) is positive and non-zero for all \( x \in X \). Therefore we may write \( \varphi_x = \alpha \varphi'_x + \psi \) (\( x \in X \)) and \( \mathcal{E}' = (\varphi'_x)_{x \in X} \) is an ensemble, where \( \alpha := \sum_{x \in X} \langle \varphi_x - \psi, u_E \rangle > 0 \) and \( \varphi'_x := \alpha^{-1}(\varphi_x - \psi) \). The rest of the claim follows from the definition.

By Proposition 29 any gain functional coincides with a state discrimination probability functional up to a positive factor and a constant functional.

The following BSS theorem states that the family of the gain functionals completely characterizes the post-processing order relation for \( w^* \)-measurements. While we can prove the following theorem using the corresponding result for statistical experiments \([30, 54]\) and Proposition 29 in Appendix D, here we give a direct proof based on the line of \([39]\). The finite division of classical space used in the proof are also of great importance in the later development of the theory.

**Theorem 1** (BSS theorem for \( w^* \)-measurements). Let \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \) be \( w^* \)-measurements. Then the following conditions are equivalent.

(i) \( \Gamma \preceq_{\text{post}} \Lambda \).

(ii) \( P_g(\mathcal{E}; \Gamma) \leq P_g(\mathcal{E}; \Lambda) \) for any \( w^* \)-family \( \mathcal{E} \).

(iii) \( \Lambda(\mathcal{E}; \Gamma) = \Lambda(\mathcal{E}; \Lambda) \) for any ensemble \( \mathcal{E} \).

(iv) For each EVM \( M \in \text{EVM}(X; F) \) there exists an EVM \( N \in \text{EVM}(X; G) \) such that \( \Gamma(M(x)) = \Lambda(N(x)) \) (\( x \in X \)).

For the proof of Theorem 1 we introduce here the concept of finite division. Let \( F \) be a classical space with a Banach predual \( F_* \). A finite subset \( \Delta \subset F \) is said to be a finite division of \( F \) if each element \( Q \in \Delta \) is a non-zero projection and \( \sum_{Q \in \Delta} Q = u_F \). The set of finite divisions on \( F \) is denoted by \( \mathcal{D}(F) \). For each \( \Delta \in \mathcal{D}(F) \) we write as \( F_\Delta := \text{lin}(\Delta) \), which is the finite-dimensional subalgebra of \( F \) generated by \( \Delta \). For finite divisions \( \Delta, \Delta' \in \mathcal{D}(F) \), \( \Delta' \) is said to be finer than \( \Delta \), written as \( \Delta \leq \Delta' \), if \( Q = \sum_{R \in \Delta'} R \leq Q \) for all \( Q \in \Delta \). Then \( \leq \) is a directed partial order on \( \mathcal{D}(F) \). An element of the subalgebra \( \bigcup_{\Delta \in \mathcal{D}(F)} F_\Delta \subset F \) is said to be a simple element. Note that the set of simple elements is norm dense in \( F \) by Proposition 4. If \( F \) is the real \( L^\infty \)-space of a some \( (\sigma-) \)finite measure, then the set of simple elements is exactly the set of measurable simple functions.

**Lemma 1** (cf. \([33]\), Lemma 5). Let \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \) be \( w^* \)-measurements and let \( \Gamma_\Delta \in \text{Ch}_{w^*}(F_\Delta \to E) \) denote the restriction of \( \Gamma \) to \( F_\Delta \). Then \( \Gamma_\Delta \preceq_{\text{post}} \Lambda \) (\( \forall \Delta \in \mathcal{D}(F) \)) implies \( \Gamma \preceq_{\text{post}} \Lambda \).

**Proof.** By assumption for each \( \Delta \in \mathcal{D}(F) \) there exists \( \Psi_\Delta \in \text{Ch}(F_\Delta \to G) \) such that \( \Gamma_\Delta = \Lambda \circ \Psi_\Delta \). Let \( F_0 := \bigcup_{\Delta \in \mathcal{D}(F)} F_\Delta \) and define a map \( \widetilde{\Psi}_\Delta : F_0 \to G \) by

\[
\widetilde{\Psi}_\Delta(a) := \begin{cases} 
\Psi_\Delta(a) & \text{if } a \in F_\Delta; \\
0 & \text{otherwise.}
\end{cases}
\]

18
Then since \( \|\tilde{\Psi}_\Delta(a)\| \leq \|a\| \) \((\Delta \in \mathcal{D}(F), a \in F_0)\), Tychoff’s theorem implies that there exist a subnet \((\tilde{\Psi}_{\Delta(i)})_{i \in I}\) and a map \(\Psi_0 : F_0 \to G\) such that \(\tilde{\Psi}_{\Delta(i)}(a) \xrightarrow{\text{weak*}} \Psi_0(a) \in (G)\|a\|\) for each \(a \in F_0\). Then \(\Psi_0\) is a unital bounded linear map that maps a positive element in \(F_0\) to a positive one in \(G\). Therefore, since \(F_0\) is norm dense in \(F\), \(\Psi_0\) is uniquely extended to a channel \(\Psi \in \text{Ch}(F \to G)\). Then for every \(a \in F_0\), we have

\[
\Lambda \circ \Psi(a) = \lim_{i \in I} \Lambda \circ \tilde{\Psi}_{\Delta(i)}(a) = \Gamma(a),
\]

where we used the \(\text{weak*}\) continuity of \(\Lambda\) in the first equality. By the norm density of \(F_0\) in \(F\), this implies \(\Gamma = \Lambda \circ \Psi \preceq_{\text{post}} \Lambda\), which proves the claim. 

**Proof of Theorem 1.**

(i) \(\Rightarrow\) (ii). Assume (i) and take a channel \(\Psi \in \text{Ch}(F \to G)\) such that \(\Gamma = \Lambda \circ \Psi\). Let \(\mathcal{E} = (\varphi_x)_{x \in X}\) be a \(\text{w*}\)-family. Then

\[
P (\mathcal{E}; \Gamma) = \sup_{\mathcal{M} \in \text{EVM}(X; F)} \sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle
\]

\[
= \sup_{\mathcal{M} \in \text{EVM}(X; F)} \sum_{x \in X} \langle \varphi_x, \Lambda \circ \Psi(M(x)) \rangle
\]

\[
\leq \sup_{\mathcal{N} \in \text{EVM}(X; G)} \sum_{x \in X} \langle \varphi_x, \Lambda(N(x)) \rangle
\]

\[
= P (\mathcal{E}; \Lambda),
\]

where the inequality follows from \(\{(\Psi(M(x)))_{x \in X} | M \in \text{EVM}(X; F)\} \subset \text{EVM}(X; G)\).

(ii) \(\Rightarrow\) (iii) is obvious.

(iii) \(\Rightarrow\) (iv) follows from Proposition 1.

(iii) \(\Rightarrow\) (iv) can be shown similarly as in [39] (Proposition 2) by applying the Hahn-Banach separation theorem to \(\{(\Lambda(N(x)))_{x \in X} | N \in \text{EVM}(X; G)\}\).

(iv) \(\Rightarrow\) (i). Assume (iv). Since \((\Lambda(N(x)))_{x \in X}\), \(\mathcal{E} \in \text{EVM}(X; F)\) for any \(\Delta \in \mathcal{D}(F)\), the assumption (iv) implies that there exists an EVM \(N_\Delta \in \text{EVM}(\Delta; F)\) such that \(\Gamma(Q) = \Lambda(N_\Delta(Q))\) \((Q \in \Delta)\). Define a channel \(\Phi_\Delta \in \text{Ch}(F_\Delta \to G)\) by \(\Phi_\Delta(Q) := N_\Delta(Q)\) \((Q \in \Delta)\). Then, for any \(\Delta \in \mathcal{D}(F)\), we have \(\Gamma_\Delta = \Lambda \circ \Phi_\Delta \preceq_{\text{post}} \Lambda\), where \(\Gamma_\Delta\) is the restriction of \(\Gamma\) to \(F_\Delta\). Therefore Lemma 1 implies \(\Gamma \preceq_{\text{post}} \Lambda\).

For a general pair of measurements which are not necessarily weakly* continuous, a theorem corresponding to Theorem 1 will be

**Theorem 2.** Let \(\Gamma \in \text{Ch}(F \to E)\) and \(\Lambda \in \text{Ch}(G \to E)\) be measurements and let \(\Gamma \in \text{Ch}_{\text{w*}}(F^{**} \to E)\) and \(\Lambda \in \text{Ch}_{\text{w*}}(G^{**} \to E)\) be the \(\text{w*}\)-extensions of \(\Gamma\) and \(\Lambda\), respectively. Then the following conditions are equivalent.

(i) \(\Gamma \preceq_{\text{post}} \Lambda\).

(ii) \(\Gamma \preceq_{\text{post}} \Lambda\).

(iii) \(P (\mathcal{E}; \Gamma) \leq P (\mathcal{E}; \Lambda)\) for any ensemble \(\mathcal{E}\).

For the proof of Theorem 2 we need the following lemmas.
Lemma 2. Let $E_1$ be an order unit Banach space and let $X \neq \emptyset$ be a finite set. Then $EVM(X; E_1)$ is dense in $EVM(X; E_1^{**})$ with respect to the weak* topology $\sigma((E_1^{**})^X, (E_1^*)^X)$.

Proof. The claim is trivial when $|X| = 1$, where $|\cdot|$ denotes the cardinality. If $|X| > 1$, fix an element $x_0 \in X$ and define $X' := X \setminus \{x_0\}$. Then we have a one-to-one affine correspondence

$$EVM(X; E_1) \ni (M(x))_{x \in X} \mapsto (M'(x))_{x \in X'} \in EVM_{\text{sub}}(X'; E_1).$$

We can similarly identify $EVM(X; E_1^{**})$ with $EVM_{\text{sub}}(X'; E_1^{**})$. Therefore the claim will follow if we can show the weak* density of $EVM_{\text{sub}}(X; E_1)$ in $EVM_{\text{sub}}(X; E_1^{**})$ for any finite set $X$. Define

$$K := \{ (\phi_x - \psi)_{x \in X} \in (E_1^*)^X \mid (\phi_x)_{x \in X} \in (E_1^+)^X, \psi \in S(E_1) \}.$$ 

Then $K$ is a convex subset of $(E_1^+)^X$ containing $0$. Moreover, for $M \in E_1^X$ we have

$$M \in EVM_{\text{sub}}(X; E_1)$$

$$\iff \sum_{x \in X} \langle \phi_x, M(x) \rangle + \langle \psi, u_E - \sum_{x \in X} M(x) \rangle \geq 0 \quad (\forall (\phi_x)_{x \in X} \in (E_1^+)^X, \forall \psi \in S(E_1))$$

$$\iff \sum_{x \in X} \langle \phi_x - \psi, M(x) \rangle \geq -1 \quad (\forall (\phi_x)_{x \in X} \in (E_1^+)^X, \forall \psi \in S(E_1)),$$

which implies that $EVM_{\text{sub}}(X; E_1)$ is the polar of $K$ in the pair $(E_1^X, (E_1^*)^X)$. A similar reasoning yields that $EVM_{\text{sub}}(X; E_1^{**})$ is the polar of $K$ in the pair $((E_1^*)^X, (E_1^*)^X)$. Therefore if we can show that $K$ is $\sigma((E_1^*)^X, (E_1^*)^X)$-closed, the claim follows from the bipolar theorem. By the Krein–Smulian theorem, it is sufficient to prove the $\sigma((E_1^*)^X, (E_1^*)^X)$-closedness of $(K)_r$ for any $r \in (0, \infty)$. Let $(\phi_x^i - \psi^i)_{x \in X} (i \in I)$ be a net in $(K)_r$ weakly* converging to $(\xi_x)_{x \in X} \in (E_1^*)^X$, where $\phi_x^i \in E_1^{i+}$ and $\psi^i \in S(E_1)$ $(x \in X, i \in I)$. Then $\|\psi^i\| = 1$ and hence

$$\|\phi_x^i\| \leq \|\psi^i\| + \|\phi_x^i - \psi^i\| \leq 1 + \sum_{x' \in X} \|\phi_x^i - \psi^i\| \leq 1 + r.$$

Therefore by the Banach-Alaoglu theorem there exist a subnet $((\phi_x^{(j)}(x))_{x \in X}, (\psi^j(x))_{x \in X}) (j \in J)$ and $((\phi_x)_{x \in X}, (\psi)_{x \in X}) \in (E_1^{i+})^X \times S(E_1)$ such that $\phi_x^{(j)} \xrightarrow{\text{weakly*}} \phi_x$ and $\psi^{(j)} \xrightarrow{\text{weakly*}} \psi$. Then $(\xi_x)_{x \in X} = (\phi_x - \psi)_{x \in X} \in K$ and hence $K$ is $\sigma((E_1^*)^X, (E_1^*)^X)$-closed. \hfill $\Box$

The following lemma states that a measurement is equivalent to its w*-extension if we concern only the state discrimination probabilities of ensembles.

Lemma 3. Let $\Gamma \in \text{Ch}(F \to E)$ be a measurement and let $\overline{\Gamma} \in \text{Ch}_{w^*}(F^{**} \to E)$ be the w*-extension of $\Gamma$. Then $P_g(\mathcal{E}; \Gamma) = P_g(\mathcal{E}; \overline{\Gamma})$ for any w*-family $\mathcal{E}$.

Proof. Let $\mathcal{E} = (\nu_x)_{x \in X}$ be an arbitrary w*-family. From $\Gamma \leq_{\text{post}} \overline{\Gamma}$, we can show $P_g(\mathcal{E}; \Gamma) \leq P_g(\mathcal{E}; \overline{\Gamma})$ similarly as in Theorem \[. Take an arbitrary $M'' \in EVM(X; F^{**})$. Then by Lemma \[ there exists a net $(M_i)_{i \in I}$ in $EVM(X; F)$ weakly* converging to $M''$. Then by the weak* continuity of $\overline{\Gamma}$,

$$\sum_{x \in X} \langle \phi_x, \overline{\Gamma}(M''(x)) \rangle = \lim_{i \in I} \sum_{x \in X} \langle \phi_x, \overline{\Gamma}(M_i(x)) \rangle = \lim_{i \in I} \sum_{x \in X} \langle \phi_x, \Gamma(M_i(x)) \rangle \leq P_g(\mathcal{E}; \Gamma).$$

By taking the supremum of $M''$, we obtain $P_g(\mathcal{E}; \overline{\Gamma}) \leq P_g(\mathcal{E}; \Gamma)$.

$\Box$
Proof of Theorem 2. The equivalence (i) \(\iff\) (ii) is immediate from Proposition 9. The equivalence (ii) \(\iff\) (iii) follows from Theorem 1 and Lemma 3.

4.2 Measurement space

Let us denote the class of \(w^*-\)measurements on \(E\) by \(\text{Meas}(E)\), which is a proper class since the class of classical spaces with preduals is proper. Here a proper class is a class that is not a set. Based on the BSS theorem, we can construct the set of post-processing equivalence classes of \(w^*-\)measurements as follows.

Proposition 15. There exist a set \(\mathcal{M}(E)\) and a class-to-set surjection

\[
\text{Meas}(E) \ni \Gamma \mapsto \llbracket \Gamma \rrbracket \in \mathcal{M}(E)
\]

such that \(\Gamma \sim_{\text{post}} \Lambda\) if and only if \(\llbracket \Gamma \rrbracket = \llbracket \Lambda \rrbracket\) for any \(\Gamma, \Lambda \in \text{Meas}(E)\).

Proof. Let \(\text{Ens}_k(E)\) be the set of ensembles with the label set \(\mathbb{N}_k := \{1, 2, \ldots, k\} (k \in \mathbb{N})\) and let \(\text{Ens}(E) := \bigcup_{k \in \mathbb{N}} \text{Ens}_k(E)\). We define a map

\[
\text{Meas}(E) \ni \Gamma \mapsto \llbracket \Gamma \rrbracket := \left( P_g(\mathcal{E}; \Gamma) \right)_{\mathcal{E} \in \text{Ens}(E)} \in \mathbb{R}^{\text{Ens}(E)}
\]

and a set \(\mathcal{M}(E) \subset \mathbb{R}^{\text{Ens}(E)}\) by the image of the map (6). Then by Theorem 1, \(\mathcal{M}(E)\) and \(\llbracket \cdot \rrbracket\) satisfy the required condition of the statement.

In what follows in this paper, we fix a set \(\mathcal{M}(E)\) and a map \(\llbracket \cdot \rrbracket\) satisfying the conditions of Proposition 15. The set \(\mathcal{M}(E)\) is called the measurement space of \(E\). Each element of \(\mathcal{M}(E)\) is also called a measurement, or an equivalence class of measurements if the distinction is necessary. For each \(\omega \in \mathcal{M}(E)\), a \(w^*-\)measurement \(\Gamma \in \text{Meas}(E)\) with \(\llbracket \Gamma \rrbracket = \omega\) is called a representative of \(\omega\).

We define the post-processing partial order \(\preceq_{\text{post}}\) on \(\mathcal{M}(E)\) by \(\llbracket \Gamma \rrbracket \preceq_{\text{post}} \llbracket \Lambda \rrbracket\) if and only if \(\Gamma \preceq_{\text{post}} \Lambda\). We symbolically write as \(\llbracket u_E \rrbracket := \llbracket \Gamma_{M_0} \rrbracket\) and call it the trivial measurement.

By Theorem 1, for each \(w^*-\)family \(\mathcal{E}\) the gain functional \(P_g(\mathcal{E}; \cdot)\) on \(\mathcal{M}(E)\) is well-defined by

\[
P_g(\mathcal{E}; \llbracket \Gamma \rrbracket) := P_g(\mathcal{E}; \Gamma) \quad (\llbracket \Gamma \rrbracket \in \mathcal{M}(E)).
\]

We define the convex combination (probabilistic mixture) map

\[
\langle \cdot; \cdot; \cdot \rangle : [0, 1] \times \mathcal{M}(E) \times \mathcal{M}(E) \to \mathcal{M}(E)
\]

by

\[
(\lambda; \llbracket \Gamma \rrbracket, \llbracket \Lambda \rrbracket) := \llbracket (1 - \lambda)\Gamma \oplus \lambda \Lambda \rrbracket \quad (\lambda \in [0, 1]; \llbracket \Gamma \rrbracket, \llbracket \Lambda \rrbracket \in \mathcal{M}(E)),
\]

which is well-defined by Proposition 10. The gain functional \(P_g(\mathcal{E}; \cdot)\) for a \(w^*-\)family \(\mathcal{E}\) is an affine functional on the convex prestructure \((\mathcal{M}(E), \langle \cdot; \cdot; \cdot \rangle)\) by Proposition 14.

For each measurement \(\Gamma\), we also denote by \(\llbracket \Gamma \rrbracket\) the equivalence class of the \(w^*-\)extension of \(\Gamma\). Note that for any measurements \(\Gamma\) and \(\Lambda\), (7) is also well-defined by Lemma 3 as well as (8) is well-defined by Proposition 10.
4.3 Weak topology on the measurement space

Now we are in a position to define the weak topology on \( \mathcal{M}(E) \).

**Definition 4** (Weak topology). We define the weak topology on \( \mathcal{M}(E) \) as the weakest topology on \( \mathcal{M}(E) \) such that the gain functional \( P_g(\mathcal{E}; \cdot) \) on \( \mathcal{M}(E) \) is continuous for any \( \ast \)-family \( \mathcal{E} \).

In terms of net, the weak topology is characterized as follows: a net \( (\omega_i)_{i \in I} \) in \( \mathcal{M}(E) \) weakly converges to \( \omega \in \mathcal{M}(E) \) if and only if \( P_g(\mathcal{E}; \omega_i) \to P_g(\mathcal{E}; \omega) \) for any \( \ast \)-family, or ensemble, \( \mathcal{E} \).

The following theorem is a basic result for the weak topology.

**Theorem 3.** The weak topology on \( \mathcal{M}(E) \) is a compact Hausdorff topology.

**Proof.** The Hausdorff property of the weak topology follows from that the gain functionals separate points of \( \mathcal{M}(E) \) by Theorem [1].

To show the compactness, take an arbitrary net \( (\Gamma_i)_{i \in I} \) in \( \mathcal{M}(E) \) and let \( F_i \) be the classical outcome space of the representative \( \Gamma_i \). Let \( (\tilde{F}, u_{\tilde{F}}) \) be the direct sum of \( (F_i, u_{F_i})_{i \in I} \) and for each \( i \in I \) define \( \tilde{\Gamma}_i \in \text{Ch}(\tilde{F} \to E) \) by

\[
\tilde{\Gamma}_i((a_{i^j})_{j \in J}) := \Gamma_i(a_i), \quad ((a_{i^j})_{j \in J}) \in \tilde{F}.
\]

By the BW-compactness of \( \text{Ch}(\tilde{F} \to E) \) (Proposition [6]), there exists a subnet \( \Gamma_{i(j)} \) \( j \in J \) BW-convergent to a channel \( \tilde{\Gamma}_0 \in \text{Ch}(\tilde{F} \to E) \). We show \( [\Gamma_{i(j)}] \xrightarrow{\text{weakly}} [\tilde{\Gamma}_0] \), from which the compactness of \( \mathcal{M}(E) \) follows. Then it suffices to prove \( P_g(\mathcal{E}; \Gamma_{i(j)}) \to P_g(\mathcal{E}; \tilde{\Gamma}_0) \) for any \( \ast \)-family \( \mathcal{E} = (\varphi_x)_{x \in X} \). For each \( i \in I \) we take an EVM \( M_i \in \text{EVM}(X; F_i) \) such that

\[
\sum_{x \in X} \langle \varphi_x, \Gamma_i(M_i(x)) \rangle = P_g(\mathcal{E}; \Gamma_i).
\]

We define \( \tilde{M} \in \text{EVM}(X; \tilde{F}) \) by \( \tilde{M}(x) := (M_i(x))_{i \in I} \ (x \in X) \). Take an arbitrary EVM \( \tilde{N} \in \text{EVM}(X; \tilde{F}) \) with \( \tilde{N}(x) = (N_i(x))_{i \in I} \ (x \in X) \). Then we have

\[
\sum_{x \in X} \langle \varphi_x, \tilde{\Gamma}_0(\tilde{N}(x)) \rangle = \lim_{j \in J} \sum_{x \in X} \langle \varphi_x, \Gamma_{i(j)}(N_{i(j)}(x)) \rangle \\
\leq \liminf_{j \in J} P_g(\mathcal{E}; \Gamma_{i(j)}) \\
\leq \limsup_{j \in J} P_g(\mathcal{E}; \Gamma_{i(j)}) \\
= \limsup_{j \in J} \sum_{x \in X} \langle \varphi_x, \Gamma_{i(j)}(M_{i(j)}(x)) \rangle \\
= \limsup_{j \in J} \sum_{x \in X} \langle \varphi_x, \tilde{\Gamma}_{i(j)}(\tilde{M}(x)) \rangle \\
= \sum_{x \in X} \langle \varphi_x, \tilde{\Gamma}_0(\tilde{M}(x)) \rangle \\
\leq P_g(\mathcal{E}; \tilde{\Gamma}_0).
\]
By taking the supremum of $\tilde{N}$ in the above (in)equalities, we obtain

$$P_g(\mathcal{E}; \tilde{\Gamma}_0) = \sum_{x \in X} \langle \varphi_x, \tilde{\Gamma}_0(\tilde{M}(x)) \rangle = \lim_{j \to \infty} P_g(\mathcal{E}; \Gamma_{i(j)}),$$

which proves $[\Gamma_{i(j)}] \xrightarrow{\text{weakly}} [\tilde{\Gamma}_0]$.

We next show that the post-processing order $\leq_{\text{post}}$ on $\mathcal{M}(E)$ is compatible with the weak topology in the following sense.

**Definition 5** ([15], Chapter VI). Let $X$ be a topological space. A preorder $\leq$ on $X$ is said to be closed if the graph $\{(x, y) \in X \times X \mid x \leq y\}$ is closed in the product topology on $X \times X$. If $\leq$ is a closed partial order, then the poset $(X, \leq)$ is called a pospace.

In terms of net, the above condition says that the order and the limit commute in the following sense: for any nets $(x_i)_{i \in I}, (y_i)_{i \in I}$ in $X$, if $x_i \leq y_i \ (\forall i \in I)$, $x_i \to x \in X$, and $y_i \to y \in X$ hold, then $x \leq y$.

**Proposition 16.** The poset $(\mathcal{M}(E), \leq_{\text{post}})$ equipped with the weak topology is a pospace.

**Proof.** Let $(\omega_i)_{i \in I}$ and $(\nu_i)_{i \in I}$ be nets in $\mathcal{M}(E)$ satisfying $\omega_i \leq_{\text{post}} \nu_i \ (\forall i \in I)$, $\omega_i \xrightarrow{\text{weakly}} \omega \in \mathcal{M}(E)$, and $\nu_i \xrightarrow{\text{weakly}} \nu \in \mathcal{M}(E)$. Then for any w*-family $\mathcal{E}$ we have $P_g(\mathcal{E}; \omega_i) \leq P_g(\mathcal{E}; \nu_i) \ (\forall i \in I)$ by Theorem 1. By taking the limit we obtain $P_g(\mathcal{E}; \omega) \leq P_g(\mathcal{E}; \nu)$ Since $\mathcal{E}$ is arbitrary, this implies $\omega \leq_{\text{post}} \nu$ by Theorem 1.

A net $(x_i)_{i \in I}$ in a poset $(X, \leq)$ is called increasing if $i \leq j$ implies $x_i \leq x_j \ (i, j \in I)$. For a net $(x_i)_{i \in I}$ in a poset $(X, \leq)$, its supremum $\sup_{i \in I} x_i$ is the supremum (the least upper bound) of the image $\{x_i \mid i \in I\}$ in $X$, if it exists.

**Lemma 4** ([15], Proposition VI-1.3). Let $(X, \leq)$ be a compact pospace. Then any increasing net $(x_i)_{i \in I}$ in $X$ has a supremum $\sup_{i \in I} x_i$ in $X$ to which $(x_i)_{i \in I}$ converges topologically.

A measurement $\omega \in \mathcal{M}(E)$ is said to be finite-outcome if there exists a finite-outcome EVM $M$ such that $\omega = [\Gamma^M]$, or equivalently if $\omega$ has a representative with a finite-dimensional outcome space. We denote by $\mathcal{M}_{\text{fin}}(E)$ the set of finite-outcome measurements in $\mathcal{M}(E)$. The following theorem states that any measurement can be approximated by an increasing net of finite-outcome measurements and will be used in Sections 6 and 8 to reduce the discussions to the finite-outcome cases.

**Theorem 4** (Approximation by finite-outcome measurements). Let $\Gamma \in \text{Ch}_{w^*}(F \to E)$ be a w*-measurement and let $\Gamma_\Delta$ be the restriction of $\Gamma$ to $F_\Delta$ for each finite division $\Delta \in \mathcal{D}(F)$. Then the net $([\Gamma_\Delta])_{\Delta \in \mathcal{D}(F)}$ in $\mathcal{M}_{\text{fin}}(E)$ is an increasing net and weakly converges to $[\Gamma] = \sup_{\Delta \in \mathcal{D}(F)} [\Gamma_\Delta]$. Furthermore, there exists a net $(\Lambda_\Delta)_{\Delta \in \mathcal{D}(F)}$ in $\text{Ch}(F \to E)$ such that $\Lambda_\Delta \sim_{\text{post}} \Gamma_\Delta \ (\forall \Delta \in \mathcal{D}(F))$ and $\|\Lambda_\Delta(a) - \Gamma(a)\| \to 0 \ (\forall a \in F)$.

**Proof.** If $\Delta \leq \Delta' \ (\Delta, \Delta' \in \mathcal{D}(F))$, the w*-measurement $\Gamma_\Delta$ is the restriction of $\Gamma_{\Delta'}$ to the subalgebra $F_\Delta$ of $F_{\Delta'}$, and so $\Gamma_\Delta \leq_{\text{post}} \Gamma_{\Delta'}$. Thus the net $([\Gamma_\Delta])_{\Delta \in \mathcal{D}(F)}$ in $\mathcal{M}_{\text{fin}}(E)$ is increasing. Therefore by Theorem 3 Proposition 16 and Lemma 4 there exists a supremum
Theorem 1

A compact convex structure. Furthermore, $M$ convex structure.

Proof. By Proposition 2 the latter part of the claim follows from the first one. In Theorem 3 since any gain functional $P$ we have established that the weak topology is a compact Hausdorff topology. Moreover, $\phi$ state $E$.

The convex prestructure $\mathcal{E}$ in $M$.

Hausdorff space $E$.

To show the latter part, for each non-zero projection $P \in F$ take a weakly* continuous state $\varphi_P \in S_c(F)$ such that $s(\varphi_P) \leq P$. For each finite division $\Delta \in D(F)$ we define a linear map $E_\Delta: F \to F_\Delta$ by

$$E_\Delta(a) := \sum_{Q \in \Delta} \varphi_Q(a)Q.$$ 

By noting $\langle \varphi_Q, Q' \rangle = 0$ for $Q, Q' \in \Delta$ with $Q \neq Q'$, we can see that $E_\Delta$ is a conditional expectation onto $F_\Delta$, i.e. $E_\Delta$ satisfies

$$E_\Delta(a) = a \quad (a \in F_\Delta), \quad \|E_\Delta(b)\| \leq \|b\| \quad (b \in F),$$

from which it follows that $E_\Delta \in \text{Ch}(F \to F_\Delta)$. Now let $\Lambda_\Delta := \Gamma \circ E_\Delta (\Delta \in D(F))$, which is a channel in $\text{Ch}(F \to E)$. Then since $\Lambda_\Delta = \Gamma_\Delta \circ E_\Delta$ and $\Gamma_\Delta$ coincides with the restriction of $\Lambda_\Delta$ to the subalgebra $F_\Delta$ of $F$, we have $\Gamma_\Delta \sim_{\text{post}} \Lambda_\Delta$. Take an element $a \in F$. Then since $a$ is approximated in norm by a sequence of simple elements in $F$, for every $\epsilon > 0$ there exists a simple element $a_\epsilon \in F$ such that $\|a - a_\epsilon\| < \epsilon/2$. Let $\Delta_\epsilon \in D(F)$ be a finite division satisfying $a_\epsilon \in F_{\Delta_\epsilon}$. Then for any $\Delta \in D(F)$ with $\Delta_\epsilon \leq \Delta$, we have $E_\Delta(a_\epsilon) = a_\epsilon$ and hence

$$\|a - E_\Delta(a)\| \leq \|a - a_\epsilon\| + \|E_\Delta(a_\epsilon - a)\| \leq 2\|a - a_\epsilon\| < \epsilon.$$ 

Therefore for any $\Delta \geq \Delta_\epsilon$,

$$\|\Gamma(a) - \Lambda_\Delta(a)\| = \|\Gamma(a - E_\Delta(a))\| \leq \|a - E_\Delta(a)\| < \epsilon,$$

which proves the latter part of the claim. \qed

We now establish the compatibility of the weak topology and the convex structure on $\mathcal{M}(E)$.

Theorem 5. The convex prestructure $(\mathcal{M}(E), \langle \cdot; \cdot, \cdot \rangle)$ equipped with the weak topology is a compact convex structure. Furthermore, $\mathcal{M}(E)$ is regularly embedded into the locally convex Hausdorff space $A_c(\mathcal{M}(E))^*$, which is unique up to continuous linear isomorphism.

Proof. By Proposition 2 the latter part of the claim follows from the first one. In Theorem 3 we have established that the weak topology is a compact Hausdorff topology. Moreover, since any gain functional $P_\delta(\mathcal{E}; \cdot)$ is a weakly continuous affine functional on $\mathcal{M}(E)$, by Theorem 1 $A_c(\mathcal{M}(E))$ separates points of $\mathcal{M}(E)$. Therefore $(\mathcal{M}(E), \langle \cdot; \cdot, \cdot \rangle)$ is a compact convex structure. \qed

From now on we identify $\mathcal{M}(E)$ with the weakly* compact set $S(A_c(\mathcal{M}(E)))$ on $A_c(\mathcal{M}(E))^*$. In this identification the convex combination $\langle \lambda; [\Gamma], [\Lambda] \rangle = [\lambda \Gamma \oplus (1 - \lambda)\Lambda]$ becomes the ordinary convex combination $\lambda [\Gamma] + (1 - \lambda) [\Lambda] \ (\lambda \in [0, 1]; [\Gamma], [\Lambda] \in \mathcal{M}(E))$. 

24
4.4 Infinite-dimensionality of $\mathfrak{M}(E)$

We now prove that the measurement space is infinite-dimensional except in the trivial case $\dim E = 1$.

**Theorem 6** (Infinite-dimensionality of $\mathfrak{M}(E)$). If $\dim E > 1$, then the measurement space $\mathfrak{M}(E)$ is an infinite-dimensional convex set, i.e. for any convex set $K$ in a finite-dimensional Euclidean space there exists no affine bijection $\Psi : \mathfrak{M}(E) \to K$.

Theorem 6 indicates that the measurement space $\mathfrak{M}(E)$ has sufficiently many w*-measurements and also that considerations on a proper topology, as we have done in this section, is indeed necessary.

We first prove the following lemma.

**Lemma 5.** Let $M \in \text{EVM}(X; E)$ be a finite-outcome EVM and let $\mathcal{E} = (\varphi_y)_{y \in Y}$ be a w*-family. Then

$$P_{\gamma}(\mathcal{E}; \Gamma^M) = \sum_{x \in X} \max_{y \in Y} \langle \varphi_y, M(x) \rangle$$

holds.

**Proof.** The gain functional $P_{\gamma}(\mathcal{E}; \Gamma^M)$ is evaluated to be

$$P_{\gamma}(\mathcal{E}; \Gamma^M) = \sup_{\rho(\cdot|\cdot) \in \text{Stoch}(Y, X)} \sum_{x \in X} \sum_{y \in Y} p(y|x) \langle \varphi_y, M(x) \rangle$$

$$\leq \sup_{\rho(\cdot|\cdot) \in \text{Stoch}(Y, X)} \sum_{x \in X} \sum_{y \in Y} p(y|x) \max_{y' \in Y} \langle \varphi_{y'}, M(x) \rangle$$

$$= \sum_{x \in X} \max_{y \in Y} \langle \varphi_y, M(x) \rangle.$$  

The equality of the above inequality is attained by putting $p(y|x) = \delta_{y,y'(x)}$, where for each $x \in X$, we take $y'(x) \in Y$ such that $\langle \varphi_{y'(x)}, M(x) \rangle = \max_{y \in Y} \langle \varphi_y, M(x) \rangle$. \hfill \Box

**Proof of Theorem 6.** By the assumption $\dim E > 1$, there exists an element $a \in E$ such that $0 \leq a \leq u_E$ and $(a, u_E)$ is linearly independent. Therefore if we put $a' := u_E - a$, $(a, a')$ is also linearly independent. Thus by the Hahn-Banach separation theorem, there exist linear functionals $\psi_1, \psi_2 \in E_*$ such that

$$\langle \psi_1, a \rangle = \langle \psi_2, a' \rangle = 1 > 0 = \langle \psi_1, a' \rangle = \langle \psi_2, a \rangle.$$  

For each $p \in [0, 1]$ and each $q \in (0, 1)$, define a w*-family $\mathcal{E}_p$ and an EVM $M_p \in \text{EVM}(N_2; E)$ by

$$\mathcal{E}_p := (0, (1-p)\psi_2 - p\psi_1),$$

$$M_p(1) := (1-q)a, \quad M_p(2) := qa + a'.$$

Then by Lemma 5 we have

$$P_{\gamma}(\mathcal{E}_p; \Gamma^{M_p}) = \max(0, (1-p)\psi_2 - p\psi_1, M_p(1)) + \max(0, (1-p)\psi_2 - p\psi_1, M_p(2))$$

$$= \max(0, -p(1-q)) + \max(0, -pq + 1 - p)$$

$$= \max(0, 1 - (q + 1)p) := f_q(p). \quad (9)$$

25
Now suppose that $\mathcal{M}(E)$ is finite-dimensional. Then since the map
\[ \mathcal{M}(E) \ni \omega \mapsto (P_\omega (E_p; \omega))_{p \in [0,1]} \in \mathbb{R}^{[0,1]} \]  
\[ (10) \]
is affine, the image $A$ of the map (10) contains finite number of linearly independent elements in $\mathbb{R}^{[0,1]}$. On the other hand, by (9), $A$ contains the functions $\{f_q | q \in (0,1)\}$ and it is easy to see that any finite subset of $\{f_q | q \in (0,1)\}$ is linearly independent, which is a contradiction. Therefore $\mathcal{M}(E)$ is infinite-dimensional.

5 Order characterized by a set of continuous affine functionals

In Section 4 we have seen that the post-processing order on the measurement space is uniquely characterized by the set of gain functionals. In this section, as a generalization of the post-processing order, we consider a preorder on a compact convex structure that is characterized by a set of continuous affine functionals.

Throughout this section, if we call a set $S$ a compact convex structure, we understand that $S$ is identified with the state space $S(A_c(S))$ and the convex combination on $S$ is the ordinary one $\lambda \omega + (1 - \lambda) \nu$ on the linear space $A_c(S)^*$. The main subject of this section is the order of the following kind.

Definition 6. Let $S$ be a compact convex structure and let $A \subset A_c(S)$ be a set of continuous affine functionals. We define a preorder $\preceq_A$ on $S$ by
\[
\omega \preceq_A \nu \quad \text{def.} \quad [f(\omega) \leq f(\nu) \quad (\forall f \in A)]
\]
for any $\omega, \nu \in S$. \hspace{1cm} \Box

For example, when $S = \mathcal{M}(E)$, the post-processing order $\preceq_{\text{post}}$ can be written as $\preceq_A$ where $A$ is the set of gain functionals. Later in Theorem 8 we will give an axiomatic characterization of this kind of order.

We first consider monotonically increasing affine functionals for this kind of order. We remind the reader that for a set $X$ equipped with a preorder $\leq$, a function $f : X \to \mathbb{R}$ is monotonically increasing (in $\leq$) if
\[ x \leq y \implies f(x) \leq f(y) \quad (x, y \in X). \]
The following theorem characterizes the monotonically increasing affine functional in $\preceq_A$.

Theorem 7. Let $S$ be a compact convex structure, let $f : S \to \mathbb{R}$ be an affine functional, let $A \subset A_c(S)$ be a set of continuous affine functionals, and let $\mathcal{U}_A$ denote the set of affine functionals on $S$ that can be written as
\[
\alpha 1_S + \sum_{j=1}^{n} \beta_j g_j \quad (n \in \mathbb{N}; \alpha \in \mathbb{R}; \beta_1, \ldots, \beta_n \in \mathbb{R}_+; g_1, \ldots, g_n \in A),
\]
i.e. $\mathcal{U}_A := \text{cone}(A \cup \{\pm 1_S\})$ where $\text{cone}(\cdot)$ denotes the conic hull. Then the following assertions hold.

26
1. \( f \) is monotonically increasing in \( \preceq_A \) and continuous if and only if \( f \) is a uniform limit of a sequence in \( U_A \).

2. \( f \) is monotonically increasing in \( \preceq_A \) and bounded (i.e. \( \sup_{\omega \in S} |f(\omega)| < \infty \)) if and only if \( f \) is a pointwise limit of a uniformly bounded net in \( U_A \).

3. \( f \) is monotonically increasing in \( \preceq_A \) if and only if \( f \) is a pointwise limit of a net in \( U_A \).

If we apply Theorem 7 to the measurement space \( \mathcal{M}(E) \), we readily obtain

**Corollary 1.** Let \( f : \mathcal{M}(E) \to \mathbb{R} \) be an affine functional and let \( U \) denote the set of affine functionals on \( \mathcal{M}(E) \) that can be written as

\[
\alpha 1_{\mathcal{M}(E)} + \sum_{i=1}^{n} \beta_i P_{g}(\mathcal{E}_i; \cdot) \quad (n \in \mathbb{N}; \alpha \in \mathbb{R}; \beta_1, \ldots, \beta_n \in \mathbb{R}_+; \mathcal{E}_1, \ldots, \mathcal{E}_n \in \mathsf{Ens}(E)).
\]  

(11)

Then the following assertions hold.

1. \( f \) is monotonically increasing in \( \preceq_{\text{post}} \) and weakly continuous if and only if \( f \) is a uniform (i.e. norm) limit of a sequence in \( U \).

2. \( f \) is monotonically increasing in \( \preceq_{\text{post}} \) and bounded if and only if \( f \) is a pointwise limit of a uniformly bounded net in \( U \).

3. \( f \) is monotonically increasing in \( \preceq_{\text{post}} \) if and only if \( f \) is a pointwise limit of a net in \( U \).

We remark the affine functional (11) can be written as

\[
\alpha' 1_{\mathcal{M}(E)} + \beta' P_{g}(\overrightarrow{\mathcal{E}}; \cdot)
\]

for some \( \alpha' \in \mathbb{R} \), \( \beta' \in \mathbb{R}_+ \), and a partitioned ensemble \( \overrightarrow{\mathcal{E}} \), which will be defined later in Definition 10. This indicates that up to constant factors elements of \( U \) can be regarded as the state discrimination probability with some pre-measurement information.

Now we prove Theorem 7.

**Proof of Theorem 7.**

1. Let \( U_{\preceq A} \) denote the set of continuous affine functionals that are monotonically increasing in \( \preceq_A \). Then to show the claim, we have only to prove \( \overline{U_A} = U_{\preceq A} \), where the closure is with respect to the norm topology. Let \( U'_A := \{ \psi \in A_c(S)^* \mid \langle \psi, g \rangle \geq 0 \ (\forall g \in U_A) \} \) be the dual cone of \( U_A \). We show

\[
U'_A = \{ r(\nu - \omega) \mid r \in (0, \infty); \omega, \nu \in S; \omega \preceq_A \nu \}.
\]  

(12)

The inclusion (LHS) \( \supset \) (RHS) is immediate from the definition. To prove the converse inclusion, take arbitrary \( \psi \in U'_A \). If \( \psi = 0 \), then \( \psi = \omega - \omega \in (\text{RHS}) \) for any \( \omega \in S \). Assume \( \psi \neq 0 \). Since \( S \) generates \( A_c(S)^* \), we can write as \( \psi = r_1 \nu - r_2 \omega \) for some \( r_1, r_2 \in \mathbb{R}_+ \) and some \( \omega, \nu \in S \). Since \( \pm 1_S \in U_A \), we have \( 0 = \langle \psi, 1_S \rangle = r_1 - r_2 \). Hence \( \psi = r_1(\nu - \omega) \) and \( r_1 \neq 0 \) from \( \psi \neq 0 \). Then from \( \psi \in U_A \) we have

\[
g(\nu) - g(\omega) = r_1^{-1} \langle \psi, g \rangle \geq 0 \quad (\forall g \in A),
\]
which implies \( \omega \preceq_{A} \nu \). Therefore \( \psi \) is in the RHS of \( \mathbf{12} \) and we have proved \( \mathbf{12} \). Now let \( \mathcal{U}_{A}^{**} \) be the double dual cone of \( \mathcal{U}_{A} \) in the pair \( (A_{c}(S), A_{c}(S)^{*}) \). Then from \( \mathbf{12} \)

\[
\mathcal{U}_{A}^{**} = \{ f \in A_{c}(S) \mid f(\omega) \leq f(\nu) \text{ for any } \omega, \nu \in S \text{ with } \omega \preceq_{A} \nu \} = \mathcal{U}_{\leq_{A}}.
\]

On the other hand, by the bipolar theorem \( \mathcal{U}_{A}^{**} \) is the weak closure (i.e. \( \sigma(A_{c}(S), A_{c}(S)^{*}) \)-closure) of \( \mathcal{U}_{A} \). Since the weak and the norm closures coincide for a convex set on a Banach space (e.g. \[49\], Section 9.2), we have \( \mathcal{U}_{A}^{**} = \mathcal{U}_{A} \). Thus we obtain \( \mathcal{U}_{A} = \mathcal{U}_{\leq_{A}} \).

2. We first note that the Banach dual space \( A_{c}(S)^{**} \) can be identified with \( A_{b}(S) \) and the weak* topology \( \sigma(A_{c}(S)^{**}, A_{c}(S)^{*}) \) on \( A_{c}(S)^{**} \) is, by this identification, the pointwise convergence topology on \( A_{b}(S) \). Let us define

\[
\mathcal{K} := \mathcal{U}_{A}^{*} + (A_{c}(S)^{*})_{1} = \{ \psi + \phi \in A_{c}(S)^{*} \mid \psi \in \mathcal{U}_{A}^{*}; \phi \in (A_{c}(S)^{*})_{1} \},
\]

which is a convex set containing 0. Since \( \mathcal{U}_{A}^{*} \) is weakly* closed and \( (A_{c}(S)^{*})_{1} \) is weakly* compact, \( \mathcal{K} \) is weakly* closed. Then for any \( g \in A_{b}(S) \) we have

\[
\langle g, r(\nu - \omega) + \phi \rangle \geq -1 \quad (r \in \mathbb{R}_{+}; \omega, \nu \in S; \omega \preceq_{A} \nu; \phi \in (A_{c}(S)^{*})_{1})
\]

\[
\iff \langle g, \nu - \omega \rangle \geq 0 \quad (\omega, \nu \in S; \omega \preceq_{A} \nu) \text{ and } \langle g, \phi \rangle \geq -1 \quad (\phi \in (A_{c}(S)^{*})_{1})
\]

\[
\iff g \text{ is monotonically increasing in } \preceq_{A} \text{ and } \|g\| \leq 1,
\]

which implies that the polar \( \mathcal{K}^{\circ} \) of \( \mathcal{K} \) in the pair \( (A_{c}(S)^{*}, A_{b}(S)) \) is the set of bounded affine functionals in the unit ball \( (A_{b}(S))_{1} \) that are monotonically increasing in \( \preceq_{A} \). Similarly, the polar \( \mathcal{K}_{o} \) of \( \mathcal{K} \) in the pair \( (A_{c}(S)^{*}, A_{c}(S)) \) is the set of continuous affine functionals in the unit ball \( (A_{c}(S))_{1} \) that are monotonically increasing in \( \preceq_{A} \), i.e. \( \mathcal{K}_{o} = (\mathcal{U}_{\leq_{A}})_{1} \). Since \( \mathcal{K} \) is a weakly* closed convex set containing 0, the bipolar theorem implies that \( \mathcal{K} \) is the polar of \( \mathcal{K}_{o} \) in the pair \( (A_{c}(S)^{*}, A_{c}(S)) \). Thus again by the bipolar theorem, \( \mathcal{K}^{\circ} \) is the closure of \( \mathcal{K}_{o} \) in the pointwise convergence topology on \( A_{b}(S) \).

Now assume that \( f \) is monotonically increasing and bounded. Then \( f \in \|f\|\mathcal{K}^{\circ} \) and hence by the above result there exists a net \( (f_{i})_{i \in I} \) in \( (\mathcal{U}_{\leq_{A}})_{\|f\|} \) converging pointwise to \( f \). Thus, from the claim \( \text{[1]} \) for each \( i \in I \) there exists a sequence \( (f_{i,n})_{n \in \mathbb{N}} \) in \( \mathcal{U}_{A} \) uniformly converging to \( f_{i} \). We may assume that \( \|f_{i,n}\| \leq \|f\| \) for all \( i \in I \) and all \( n \in \mathbb{N} \). Then \( (f_{i,n})_{i \in I, n \in \mathbb{N}} \) is a uniformly bounded net in \( \mathcal{U}_{A} \) and converges to \( f \) for each point in \( S \), which proves the “only if” part of the claim. The “if” part of the claim is obvious.

3. Let \( A_{\text{alg}}(S) \) denote the set of affine functionals on \( S \). In a similar manner as the case of \( A_{b}(S) \) and \( A_{c}(S)^{*} \), we can prove that \( A_{\text{alg}}(S) \) can be identified with the algebraic dual \( (A_{c}(S)^{*})' \) of \( A_{c}(S)^{*} \) (i.e. \( (A_{c}(S)^{*})' \) is the set of real linear functionals on \( A_{c}(S)^{*} \)). Moreover the bilinear form \( \langle \cdot , \cdot \rangle \) on \( A_{\text{alg}}(S) \times A_{c}(S)^{*} \) such that

\[
\langle g, \omega \rangle = g(\omega) \quad (g \in A_{\text{alg}}(S); \omega \in S)
\]

is well-defined and separating. By this correspondence, the topology \( \sigma(A_{\text{alg}}(S), A_{c}(S)^{*}) \) is the pointwise convergence topology on \( A_{\text{alg}}(S) \). Then as in the proof of the claim \( \text{[1]} \) we can show that the set of affine functionals in \( A_{\text{alg}}(S) \) that are monotonically increasing in \( \preceq_{A} \) is the closure of \( \mathcal{U}_{A} \) in the pointwise convergence topology on \( A_{\text{alg}}(S) \), from which the claim immediately follows. \( \square \)
We next give conditions of a set $A$ of continuous affine functionals when the order $\preceq_A$ is a partial or a total order.

**Proposition 17.** Let $S$ be a compact convex structure and let $A \subset A_c(S)$ be a set of continuous affine functionals. Then the following assertions hold.

1. $\preceq_A$ is a partial order, i.e. $\omega \preceq_A \nu$ and $\nu \preceq_A \omega$ imply $\omega = \nu$ for any $\omega, \nu \in S$, if and only if the linear span $\text{lin}(A \cup \{1_S\})$ is norm dense in $A_c(S)$.

2. $\preceq_A$ is a total order, i.e. either $\omega \preceq_A \nu$ or $\nu \preceq_A \omega$ holds for any $\omega, \nu \in S$, if and only if there exists an element $f_0 \in A_c(S)$ such that $A \subset \text{cone}(\{ f_0, \pm 1_S \})$.

**Proof.** 1. Assume that $\text{lin}(A \cup \{1_S\})$ is norm dense in $A_c(S)$. Take arbitrary $\omega, \nu \in S$ such that $\omega \preceq_A \nu$ and $\nu \preceq_A \omega$. Then from the definition of $\preceq_A$ we have

$$\langle \nu - \omega, f \rangle = 0 \quad (\forall f \in A)$$

$$\implies \langle \nu - \omega, f \rangle = 0 \quad (\forall f \in \text{lin}(A \cup \{1_S\})).$$

Since $\text{lin}(A \cup \{1_S\})$ is norm dense in $A_c(S)$, this implies $\nu - \omega = 0$. Thus $\preceq_A$ is a partial order.

Conversely, assume that $\text{lin}(A \cup \{1_S\})$ is not norm dense in $A_c(S)$. Then by the Hahn-Banach theorem, there exists a non-zero linear functional $\psi \in A_c(S)^*$ such that

$$\langle \psi, f \rangle = \langle \psi, 1_S \rangle = 0 \quad (\forall f \in A).$$

(13)

Since $S$ generates $A_c(S)^*$, by noting $\langle \psi, 1_S \rangle = 0$ we may write as $\psi = r(\nu - \omega)$ for some $r \in (0, \infty)$ and some $\omega, \nu \in S$. Then from (13) we have

$$f(\omega) = f(\nu) \quad (\forall f \in A)$$

and hence $\omega \preceq_A \nu$ and $\nu \preceq_A \omega$ hold. Since $\omega \neq \nu$ by $\psi \neq 0$, this proves that $\preceq_A$ is not a partial order.

2. Assume that $A \subset \text{cone}(\{ f_0, \pm 1_S \})$ for some $f_0 \in A_c(S)$. Then we have

$$\omega \preceq_{\{ f_0 \}} \nu \implies \omega \preceq_A \nu \quad (\omega, \nu \in S).$$

Moreover we can easily see that $\preceq_{\{ f_0 \}}$ is total from the definition. Thus from the above implication it follows that $\preceq_A$ is also total, which proves the “only if” part of the claim.

Conversely assume that $A$ is not included in $\text{cone}(\{ f_0, \pm 1_S \})$ for any $f_0 \in A_c(S)$. Then we can take $f_1 \in A$ which is not a constant functional. Moreover since $A$ is not included in $\text{cone}(\{ f_1, \pm 1_S \})$, we can take an element $f_2 \in A \setminus \text{cone}(\{ f_1, \pm 1_S \})$, which implies

$$f_1 \notin \text{cone}(\{ f_2, \pm 1_S \}) \quad \text{and} \quad f_2 \notin \text{cone}(\{ f_1, \pm 1_S \}).$$

Since $\text{cone}(\{ f_2, \pm 1_S \})$ is a finitely generated cone and hence is closed in the norm topology ([3, Proposition 2.41]), the Hahn-Banach separation theorem implies that there exists a linear functional $\psi_1 \in A_c(S)^*$ such that

$$\langle \psi_1, f_1 \rangle > \langle \psi_1, 1_S \rangle = 0 \geq \langle \psi_1, f_2 \rangle.$$

(14)
Then from \( \langle \psi_1, 1_S \rangle = 0 \) and \( \psi_1 \neq 0 \) we may write as \( \psi_1 = r_1(\nu_1 - \omega_1) \) for some \( r_1 \in (0, \infty) \) and some \( \omega_1, \nu_1 \in S \). Then from (14) we obtain

\[
f_1(\omega_1) < f_1(\nu_1), \quad f_2(\omega_1) \geq f_2(\nu_1). \tag{15}
\]

By interchanging \( f_1 \) and \( f_2 \) in the above discussion, we can take elements \( \omega_2, \nu_2 \in S \) such that

\[
f_1(\omega_2) \geq f_1(\nu_2), \quad f_2(\omega_2) < f_2(\nu_2). \tag{16}
\]

Now if \( f_2(\omega_1) > f_2(\nu_1) \), (15) implies that \( \omega_1 \) and \( \nu_1 \) are incomparable in \( \preceq_A \). Similarly if \( f_1(\omega_2) > f_1(\nu_2) \), (16) implies that \( \omega_2 \) and \( \nu_2 \) are incomparable in \( \preceq_A \). Now assume \( f_2(\omega_1) = f_2(\nu_1) \) and \( f_1(\omega_2) = f_1(\nu_2) \). Then from (15) and (16) we have

\[
f_1\left(\frac{\omega_1 + \nu_2}{2}\right) < f_1\left(\frac{\omega_2 + \nu_1}{2}\right), \quad f_2\left(\frac{\omega_1 + \nu_2}{2}\right) > f_2\left(\frac{\omega_2 + \nu_1}{2}\right),
\]

which implies the incomparability of \( \frac{\omega_1 + \nu_2}{2} \) and \( \frac{\omega_2 + \nu_1}{2} \) in \( \preceq_A \). Therefore \( \preceq_A \) is not total. \( \Box \)

By using this result we prove the non-totality of the measurement space \( \mathcal{M}(E) \) except when \( \dim E = 1 \).

**Corollary 2.** Assume \( \dim E > 1 \). Then the post-processing order \( \preceq_{\text{post}} \) on \( \mathcal{M}(E) \) is not total.

**Proof.** Assume that \( \preceq_{\text{post}} \) is a total order and let \( A \) denote the set of gain functionals on \( \mathcal{M}(E) \). Since \( \preceq_{\text{post}} \) coincides with \( \preceq_A \), Proposition 17 implies that \( A \) is included in cone\( (\{f_0, \pm 1_{\mathcal{M}(E)}\}) \) for some \( f_0 \in A_c(\mathcal{M}(E)) \) and hence \( \text{lin}(A \cup \{1_{\mathcal{M}(E)}\}) \) is finite-dimensional. Furthermore, since \( \preceq_{\text{post}} \) is a partial order on \( \mathcal{M}(E) \), again by Proposition 17 the linear span \( \text{lin}(A \cup \{1_{\mathcal{M}(E)}\}) \), which is norm closed by the finite dimensionality, coincides with \( A_c(\mathcal{M}(E)) \). Hence \( A_c(\mathcal{M}(E)) \) and \( A_c(\mathcal{M}(E))^\ast \) are finite-dimensional, which contradicts the infinite-dimensionality of \( \mathcal{M}(E) \subset A_c(\mathcal{M}(E))^\ast \) established in Theorem 6. \( \Box \)

Throughout this section, we have considered the class of orders of the form \( \preceq_A \), which contains the post-processing order as a special case, and proved general statements under this general setup. As finishing this section we give an axiomatization of this kind of order analogous to that of the preference relation characterized by the utility [12, 10] or of the adiabatic accessibility relation characterized by the thermodynamic entropy [16, 38].

**Theorem 8** (Von Neumann-Morgenstern utility theorem without the totality (completeness) axiom). Let \( S \) be a compact convex structure and let \( \preceq \) be a preorder on \( S \). Then the following conditions are equivalent.

(i) There exists a subset \( A \subset A_c(S) \) such that \( \preceq \) coincides with \( \preceq_A \).

(ii) The order \( \preceq \) satisfies both of the following conditions.

(a) (Independence axiom). For any \( \omega, \nu, \mu \in S \) and any \( \lambda \in (0, 1) \),

\[
\omega \preceq \nu \implies \lambda \omega + (1 - \lambda) \mu \preceq \lambda \nu + (1 - \lambda) \mu.
\]
(b) (Continuity axiom). The preorder \( \preceq \) is closed in the sense of Definition 5. Moreover, for any non-empty subsets \( A, B \subseteq A_c(S) \), the orders \( \preceq_A \) and \( \preceq_B \) coincide if and only if \( \text{cone}(A \cup \{ \pm 1_S \}) = \text{cone}(B \cup \{ \pm 1_S \}) \), where \( \text{cone}(\cdot) \) denotes the closed conic hull with respect to the norm topology.

Theorem 8 was proved in [10] when \( S \) is a Bauer simplex \( S(C(X)) \) for a compact metric space \( X \). The proof in [10] can be straightforwardly generalized to the more general case of Theorem 8 with slight modifications. See Appendix E for the proof.

6 Simulability and robustness of unsimulability

In this section, we consider simulability \([19, 45, 14]\) of a measurement relative to a given set of measurements. The main result in this section is Theorem 10 that characterizes the operational meaning of the robustness measure of unsimulability.

6.1 Simulability: definition and basic properties

**Definition 7** (Simulability). Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) be a set of measurements. A measurement \( \omega \in \mathcal{M}(E) \) is said to be simulable (respectively, strongly simulable) by \( \mathcal{L} \) if there exists \( \nu \in \text{conv}(\mathcal{L}) \) (respectively, \( \nu \in \text{conv}(\mathcal{L}) \)) such that \( \omega \preceq_{\text{post}} \nu \), where \( \text{conv}(\cdot) \) denotes the closed convex hull in the weak topology. We also say that a measurement \( \Gamma \) is (strongly) simulable by \( \mathcal{L} \) if the equivalence class \([\Gamma]\) is (strongly) simulable by \( \mathcal{L} \). The sets of measurements in \( \mathcal{M}(E) \) (strongly) simulable by \( \mathcal{L} \) is written as \( \text{sim}(\mathcal{L}) \) (\( \text{sim}_{\text{str}}(\mathcal{L}) \)). If \( \mathcal{L} \) is finite, \( \text{sim}(\mathcal{L}) \) and \( \text{sim}_{\text{str}}(\mathcal{L}) \) coincide since \( \text{conv}(\mathcal{L}) = \overline{\text{conv}(\mathcal{L})} \).

The operational meaning of the strong simulability is as follows. Suppose that an experimenter is able to perform only restricted measurements belonging to \( \mathcal{L} \subset \mathcal{M}(E) \). Then a measurement strongly simulable by \( \mathcal{L} \) is also realized by the experimenter by classical pre- and post-processing a finite measurements belonging to \( \mathcal{L} \). Here each element of \( \text{conv}(\mathcal{L}) \) corresponds to take a classical pre-processing.

The following order theoretic terminology is useful in representing the set of simulable measurements.

**Definition 8.** Let \((X, \leq)\) be a poset and let \( Y \subset X \). We define the lower closure of \( Y \) by

\[
\downarrow Y := \{ x \in X \mid \exists y \in Y, x \leq y \}.
\]

If \( Y = \downarrow Y \), \( Y \) is said to be a lower set. The lower closure \( \downarrow Y \) is the smallest lower set containing \( Y \).

By using this notation, the sets \( \text{sim}_{\text{str}}(\mathcal{L}) \) and \( \text{sim}(\mathcal{L}) \) in Definition 7 can be written as \( \text{sim}_{\text{str}}(\mathcal{L}) = \downarrow \text{conv}(\mathcal{L}) \) and \( \text{sim}(\mathcal{L}) = \downarrow \overline{\text{conv}(\mathcal{L})} \), respectively.

As for the operational meaning of the simulability, the following proposition suggests that the simulable measurements are exactly the measurements that are arbitrary approximated by strongly simulable ones in the weak topology.
**Proposition 18.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \). Then \( \text{sim}(\mathcal{L}) = \overline{\text{sim}_{str}(\mathcal{L})} \).

For the proof we need some lemmas.

**Lemma 6.** Let \( (X, \leq) \) be a compact pospace and let \( Y \) be a compact subset of \( X \). Then the lower closure \( \downarrow Y \) is also compact.

**Proof.** Let \( (x_i)_{i \in I} \) be a net in \( \downarrow Y \). Then for each \( i \in I \) we take \( y_i \in Y \) satisfying \( x_i \leq y_i \). By the compactness of \( X \) and \( Y \), there exist subnets \( (x_{i(j)})_{j \in J} \) and \( (y_{i(j)})_{j \in J} \) satisfying \( x_{i(j)} \to x \in X \) and \( y_{i(j)} \to y \in Y \). Then by the pospace condition of \( X \) we have \( x \leq y \), which implies \( x \in \downarrow Y \). Therefore \( \downarrow Y \) is compact.

**Lemma 7.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \). Then \( \text{sim}(\mathcal{L}) \) is a compact convex subset of \( \mathcal{M}(E) \).

**Proof.** Since \( \text{conv}(\mathcal{L}) \) is weakly compact, the compactness of \( \text{sim}(\mathcal{L}) = \downarrow \text{conv}(\mathcal{L}) \) follows from Lemma 6. To prove the convexity, take measurements \( \omega_1, \omega_2 \in \text{sim}(\mathcal{L}) \) and \( \nu_1, \nu_2 \in \text{conv}(\mathcal{L}) \) satisfying \( \omega_j \preceq_{\text{post}} \nu_j \) \((j = 1, 2)\). Then by Proposition 10 for each \( \lambda \in [0, 1] \) we have

\[
\lambda \omega_1 + (1 - \lambda) \omega_2 \preceq_{\text{post}} \lambda \nu_1 + (1 - \lambda) \nu_2 \in \text{conv}(\mathcal{L}),
\]

which implies \( \lambda \omega_1 + (1 - \lambda) \omega_2 \in \text{sim}(\mathcal{L}) \). Therefore \( \text{sim}(\mathcal{L}) \) is convex.

For a subset \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) and a finite set \( X \) we define

\[
\text{EVM}_{\text{sim}(\mathcal{L})}(X; E) := \{ M \in \text{EVM}(X; E) \mid [\Gamma^M] \in \text{sim}(\mathcal{L}) \},
\]

which is the set of EVMs simulable by \( \mathcal{L} \) with the outcome set \( X \).

**Lemma 8.** Let \( X \) be a finite set.

1. The map

\[
\text{EVM}(X; E) \ni M \mapsto [\Gamma^M] \in \mathcal{M}(E)
\]

is continuous with respect to the weak* topology on \( \text{EVM}(X; E) \) and the weak topology on \( \mathcal{M}(E) \).

2. For any subset \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \), \( \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \) is a weakly* compact convex subset of \( \text{EVM}(X; E) \).

**Proof.** 1. If a net \( (M_i)_{i \in I} \) in \( \text{EVM}(X; E) \) weakly* converges to \( M \in \text{EVM}(X; E) \), then by Lemma 5 for any w* family \( \mathcal{E} = (\varphi_y)_{y \in Y} \) we have

\[
P_\mathcal{E}(\mathcal{E}; \Gamma^M_i) = \sum_{x \in X} \max_{y \in Y} \langle \varphi_y, M_i(x) \rangle \to \sum_{x \in X} \max_{y \in Y} \langle \varphi_y, M(x) \rangle = P_\mathcal{E}(\mathcal{E}; \Gamma^M),
\]

which proves the continuity of (17).

2. Since \( \text{sim}(\mathcal{L}) \) is weakly closed by Lemma 4 the compactness of \( \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \) follows from the claim 1. To show the convexity, take EVMs \( M_1, M_2 \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \) and \( \lambda \in [0, 1] \). Then by Proposition 10 and Lemma 7 we have

\[
[\Gamma^{\lambda M_1 + (1 - \lambda) M_2}] = [\lambda \Gamma^{M_1} + (1 - \lambda) \Gamma^{M_2}] \preceq_{\text{post}} \lambda [\Gamma^{M_1}] + (1 - \lambda) [\Gamma^{M_2}] \in \text{sim}(\mathcal{L}).
\]

Since \( \text{sim}(\mathcal{L}) \) is a lower set, this implies \( \lambda M_1 + (1 - \lambda) M_2 \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \).
Proof of Proposition 18. The inclusion $\overline{\text{sim}}_{\text{sfr}}(\mathcal{L}) \subset \text{sim}(\mathcal{L})$ is obvious. Since $\text{sim}(\mathcal{L})$ is weakly compact by Lemma 7, this implies $\overline{\text{sim}}_{\text{sfr}}(\mathcal{L}) \subset \text{sim}(\mathcal{L})$. To show the converse inclusion, we take $\omega \in \text{sim}(\mathcal{L})$ and prove $\omega \in \overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$.

We first assume that $\omega$ is finite-outcome. Then $\omega = [\Gamma^\mathcal{M}]$ for some finite-outcome EVM $\mathcal{M} \in \text{EVM}(X; E)$. By the definition of $\text{sim}(\mathcal{L})$, there exist a measurement $\nu \in \overline{\text{conv}}(\mathcal{L})$ and a net $(\nu_i)_{i \in I}$ in $\text{conv}(\mathcal{L})$ such that $\omega \leq_{\text{post}} \nu$ and $\nu_i \xrightarrow{\text{weakly}} \nu$. Let $\Gamma_i \in \text{Ch}_{\text{w*}}(F_i \to E)$ ($i \in I$) be a representative of $\nu_i$ and let $\tilde{F} = \bigoplus_{i \in I} F_i$, $\tilde{\Gamma}_i \in \text{Ch}_{\text{w*}}(\tilde{F} \to E)$, $(\tilde{\Gamma}_{i(j)})_{j \in J}$, and $\tilde{\Gamma}_0 \in \text{Ch}(\tilde{F} \to E)$ be the same as in the proof of Theorem 3. Then from the proof of Theorem 3 we have $\tilde{\Gamma}_0 = \lim_{j \in J} \tilde{\Gamma}_{i(j)} = \lim_{j \in J} \nu_{i(j)} = \nu$.

Let $\tilde{\Gamma} \in \text{Ch}_{\text{w*}}(\tilde{F}^* \to E)$ be the $w^*$-extension of $\tilde{\Gamma}_0$. Since $[\Gamma^\mathcal{M}] \leq_{\text{post}} \nu = [\tilde{\Gamma}_0] = [\tilde{\Gamma}]$, there exists an EVM $\mathcal{N}'' \in \text{EVM}(X; \tilde{F}^*)$ such that $\mathcal{M}(x) = \tilde{\Gamma}(\mathcal{N}''(x))$ ($x \in X$). By Lemma 2 there exists a net $(\mathcal{N}_k)_{k \in K}$ in $\text{EVM}(X; \tilde{F})$ such that $\mathcal{N}_k \xrightarrow{\text{weakly}*} \mathcal{N}''$. Let $\mathcal{N}_k(x) = (\mathcal{N}_{k,i}(x))_{i \in I}$ ($k \in K$, $x \in X$) and define $\mathcal{M}_{k,j}(x) := \tilde{\Gamma}_{i(j)}(\mathcal{N}_k(x)) = \Gamma_{i(j)}(\mathcal{N}_{k,i(j)}(x))$, $\mathcal{M}_k(x) := \tilde{\Gamma}(\mathcal{N}_k(x)) = \tilde{\Gamma}_0(\mathcal{N}_k(x)) = \lim_{j \in J} \mathcal{M}_{k,j}(x)$.

Then by the weak* continuity of $\tilde{\Gamma}$, we have $\mathcal{M}_k \xrightarrow{\text{weakly}*} \mathcal{M}$. Since $[\Gamma^\mathcal{M}_{k,j}] \in \overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$ by (18), Lemma 8.2 implies $[\Gamma^\mathcal{M}_k] \in \overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$ and hence again by Lemma 8.2 we have $\omega = [\Gamma^\mathcal{M}] = \lim_{k \in K} [\Gamma^\mathcal{M}_k] \in \overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$.

For general $\omega \in \text{sim}(\mathcal{L})$, Theorem 3 implies that there exists a post-processing increasing net $(\omega_{\alpha})_{\alpha \in A}$ in $\mathcal{M}_{\text{fin}}(E)$ weakly converging to $\omega = \sup_{\alpha \in A} \omega_{\alpha}$. Since $\text{sim}(\mathcal{L})$ is a lower set, we have $\omega_{\alpha} \in \text{sim}(\mathcal{L})$ ($\alpha \in A$) and hence $\omega_{\alpha} \in \overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$ from what we have shown above. Therefore $\omega = \lim_{\alpha \in A} \omega_{\alpha}$ is also in $\overline{\text{sim}}_{\text{sfr}}(\mathcal{L})$, which completes the proof.

One might expect from Lemma 8.11 that for an infinite-dimensional classical space $F$ the map

$$\text{Ch}(F \to E) \ni \Gamma \rightarrow [\Gamma] \in \mathcal{M}(E)$$

is also continuous with respect to the BW-topology on $\text{Ch}(F \to E)$ and the weak topology on $\mathcal{M}(E)$. This is in fact not true. We have still a result analogous to Lemma 8.2. Let us show a slightly more general result.

For a subset $\mathcal{L} \subset \mathcal{M}(E)$ and a classical space $F$ we define

$$\text{Ch}^\mathcal{L}(F \to E) := \{ \Gamma \in \text{Ch}(F \to E) \mid [\Gamma] \in \mathcal{L} \},$$

which is the inverse image of $\mathcal{L}$ under the map (19).

Proposition 19. Let $\mathcal{L} \subset \mathcal{M}(E)$ be a lower subset with respect to the post-processing order.

1. $\mathcal{L}$ is weakly compact if and only if $\text{Ch}^\mathcal{L}(F \to E)$ is BW-compact for any classical space $F$.

2. $\mathcal{L}$ is convex if and only if $\text{Ch}^\mathcal{L}(F \to E)$ is convex for any classical space $F$. 

33
Proof. 1. Suppose that $\mathcal{L}$ is weakly compact. Let $F$ be a classical space and take a net $(\Gamma_i)_{i \in I}$ in $\text{Ch}^\omega(F \to E)$. By the compactness of $\mathcal{L}$ and $\text{Ch}(F \to E)$, there exist a subnet $((\Gamma_{i(j)})_{j \in J})_{j \in J}$, a measurement $\omega \in \mathcal{L}$, and a measurement $\Gamma \in \text{Ch}(F \to E)$ such that $\Gamma_{i(j)} \xrightarrow{\text{weakly}} \omega$ and $\Gamma_{i(j)} \xrightarrow{\text{BW}} \Gamma$. Then for any ensemble $\mathcal{E} = (\varphi_x)_{x \in X}$ we have $P_\omega(\mathcal{E}; \Gamma_{i(j)}) \to P_\omega(\mathcal{E}; \omega)$. Thus for any $M \in \text{EVM}(X; F)$,

$$\sum_{x \in X} \langle \varphi_x, \Gamma(M(x)) \rangle = \lim_{j \in J} \sum_{x \in X} \langle \varphi_x, \Gamma_{i(j)}(M(x)) \rangle \leq \lim_{j \in J} P_\omega(\mathcal{E}; \Gamma_{i(j)}) = P_\omega(\mathcal{E}; \omega),$$

which implies $P_\omega(\mathcal{E}; \Gamma) \leq P_\omega(\mathcal{E}; \omega)$. Therefore by Theorem 2 we obtain $[\Gamma] \preceq_{\text{post}} \omega$. Since $\mathcal{L}$ is a lower set, this implies $[\Gamma] \in \mathcal{L}$ and hence $\Gamma \in \text{Ch}^\omega(F \to E)$, which proves the compactness of $\text{Ch}^\omega(F \to E)$.

Conversely suppose that $\text{Ch}^\omega(F \to E)$ is BW-compact for any classical space $F$. Let $(\Gamma_i)_{i \in I}$ be a net in $\mathcal{L}$ with the representatives $\Gamma_i \in \text{Ch}_{w^*}(F_i \to E_i)$ $(i \in I)$. We take $\tilde{F} = \bigoplus_{i \in I} F_i$, $\tilde{\Gamma}_i \in \text{Ch}(\tilde{F} \to E)$, $(\tilde{\Gamma}_{i(j)})_{j \in J}$, and $\tilde{\Gamma}_0 \in \text{Ch}(\tilde{F} \to E)$ in the same way as in Theorem 3. Then $\tilde{\Gamma}_{i(j)} \xrightarrow{\text{weakly}} \tilde{\Gamma}_0$. We can also easily see that $\Gamma_i \sim_{\text{post}} \tilde{\Gamma}_i$. Thus by assumption the BW-limit $\tilde{\Gamma}_0$ of $(\tilde{\Gamma}_{i(j)})_{j \in J}$ is in $\text{Ch}^\omega(\tilde{F} \to E)$, which implies $[\tilde{\Gamma}_0] \in \mathcal{L}$. Therefore $\mathcal{L}$ is compact.

2. Suppose that $\mathcal{L}$ is convex and take a classical space $F$, measurements $\Gamma_1, \Gamma_2 \in \text{Ch}^\omega(F \to E)$, and $\lambda \in [0, 1]$. Then by Proposition 10 and the convexity of $\mathcal{L}$,

$$[\lambda \Gamma_1 + (1 - \lambda) \Gamma_2] \preceq_{\text{post}} [\lambda \Gamma_1 \oplus (1 - \lambda) \Gamma_2] = \lambda \Gamma_1 + (1 - \lambda) \Gamma_2 \in \mathcal{L}.$$

Since $\mathcal{L}$ is a lower set, this implies $\lambda \Gamma_1 + (1 - \lambda) \Gamma_2 \in \text{Ch}^\omega(F \to E)$, which proves the convexity of $\text{Ch}^\omega(F \to E)$.

Conversely assume that $\text{Ch}^\omega(F \to E)$ is convex for any classical space $F$. Let $\omega_1, \omega_2 \in \mathcal{L}$ and let $\Lambda_j \in \text{Ch}_{w^*}(F_j \to E)$ be a representative of $\omega_j$ $(j = 1, 2)$. Define $w^*$-measurements $\tilde{\Lambda}_j \in \text{Ch}_{w^*}(F_1 \oplus F_2 \to E)$ $(j = 1, 2)$ by

$$\tilde{\Lambda}_1(a \oplus b) := \Lambda_1(a), \quad \tilde{\Lambda}_2(a \oplus b) := \Lambda_2(b) \quad (a \in F_1, b \in F_2).$$

Then it is easy to show $\Lambda_j \sim_{\text{post}} \tilde{\Lambda}_j$ $(j = 1, 2)$. Hence $\tilde{\Lambda}_j \in \text{Ch}^\omega(F_1 \oplus F_2 \to E)$ and the assumption implies

$$\lambda \Lambda_1 \oplus (1 - \lambda) \Lambda_2 = \lambda \tilde{\Lambda}_1 + (1 - \lambda) \tilde{\Lambda}_2 \in \text{Ch}^\omega(F_1 \oplus F_2 \to E).$$

Thus $\lambda \omega_1 + (1 - \lambda) \omega_2 = [\lambda \Lambda_1 \oplus (1 - \lambda) \Lambda_2] \in \mathcal{L}$, which proves the convexity of $\mathcal{L}$. \qed

Corollary 3. Let $\mathcal{L} \subset \mathfrak{M}(E)$. Then for any classical space $F$, $\text{Ch}^{\text{sim}}(\mathcal{L})(F \to E)$ is a BW-compact convex subset of $\text{Ch}(F \to E)$. 

34
6.2 Simulability and outperformance in the state discrimination task

We introduce the gain functional relative to a set of measurements based on the following proposition.

**Proposition 20.** Let \( \mathcal{E} = (\varphi_x)_{x \in X} \) be a \( \mathcal{L}^* \)-family and let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \). Then the following equalities hold:

\[
\sup_{\omega \in \text{sim}(\mathcal{L})} P_g(\mathcal{E}; \omega) \leq \sup_{\omega \in \text{conv}(\mathcal{L})} P_g(\mathcal{E}; \omega) \leq \sup_{\omega \in \mathcal{L}} P_g(\mathcal{E}; \omega) = \sup_{M \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E)} \sum_{x \in X} \langle \varphi_x, M(x) \rangle.
\]

**Proof.** The first two equalities follow from the monotonicity in \( \preceq_{\text{post}} \), the affinity, and the weak continuity of \( P_g(\mathcal{E}; \cdot) \). By the compactness of \( \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \) (Lemma 8), the maximal value of the RHS of (20) is attained by some \( M_0 \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \). Then from \( [\Gamma_{M_0}] \in \text{sim}(\mathcal{L}) \) we have

\[
(\text{RHS of (20)}) = \sum_{x \in X} \langle \varphi_x, M_0(x) \rangle \leq P_g(\mathcal{E}; [\Gamma_{M_0}]) \leq \sup_{\omega \in \text{sim}(\mathcal{L})} P_g(\mathcal{E}; \omega).
\]

On the other hand, by the compactness of \( \text{sim}(\mathcal{L}) \), we can take \( \omega_0 \in \text{sim}(\mathcal{L}) \) such that \( \sup_{\omega \in \text{sim}(\mathcal{L})} P_g(\mathcal{E}; \omega) = P_g(\mathcal{E}; \omega_0) \). Let \( \Lambda_0 \in \text{Ch}_{\mathcal{L}^*}(F \to E) \) be a representative of \( \omega_0 \). Then we can take \( N_0 \in \text{EVM}(X; F) \) such that

\[
P_g(\mathcal{E}; \omega_0) = \sum_{x \in X} \langle \varphi_x, \Lambda_0(N_0(x)) \rangle.
\]

Since \( (\Lambda_0(N_0(x)))_{x \in X} \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \), this implies

\[
\sup_{\omega \in \text{sim}(\mathcal{L})} P_g(\mathcal{E}; \omega) \leq (\text{RHS of (20)}),
\]

which completes the proof. \( \square \)

We write the quantity (20) as \( P_g(\mathcal{E}; \mathcal{L}) \). If \( \mathcal{E} \) is an ensemble, \( P_g(\mathcal{E}; \mathcal{L}) \) is the optimal probability that we correctly guess the original state when we can perform measurements in \( \text{sim}(\mathcal{L}) \) (or \( \mathcal{L} \)).

Now, as a generalization of the finite-dimensional result [51] (Eq. (14)), we prove that the outperformance in the state discrimination task characterizes the simulability.

**Theorem 9.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) and let \( \omega \in \mathcal{M}(E) \) be a measurement. Then \( \omega \) is simulable by \( \mathcal{L} \) if and only if

\[
P_g(\mathcal{E}; \omega) \leq P_g(\mathcal{E}; \mathcal{L})
\]

holds for any ensemble \( \mathcal{E} \).
Proof. Assume that $\omega$ is simulable by $\mathcal{L}$. Then by the definition of simulability and Proposition 20 we can readily see that (21) holds. To show the converse implication, we assume $\omega \notin \text{sim}(\mathcal{L})$ and find an ensemble $\mathcal{E}$ that does not satisfy (21).

We first consider the case when $\omega$ is finite-outcome. Take an EVM $M \in \text{EVM}(X; \mathcal{E})$ such that $\omega = \Gamma_M$. Then $M \notin \text{EVM}_\text{sim}(\mathcal{L})(X; \mathcal{E})$. Since $\text{EVM}_\text{sim}(\mathcal{L})(X; \mathcal{E})$ is a weakly compact convex set by Lemma 8, the Hahn-Banach separation theorem implies that there exists a non-zero $w^*$-family $\mathcal{E} = (\phi_x)_{x \in X} \in E^*_X$ such that $\sum_{x \in X} \langle \phi_x, M(x) \rangle > \sup_{N \in \text{EVM}_\text{sim}(\mathcal{L})(X; \mathcal{E})} \sum_{x \in X} \langle \phi_x, N(x) \rangle$.

By Proposition 14.3 we can take $\mathcal{E}$ as an ensemble. Then (22) implies $P_g(\mathcal{E}; \omega) > P_g(\mathcal{E}; \mathcal{L})$. Therefore $\mathcal{E}$ violates (21).

For general $\omega$, by Theorem 4 there exists an increasing net $(\omega_i)_{i \in I}$ of finite-outcome measurements weakly converging to $\omega = \sup_{i \in I} \omega_i$. Since $\mathcal{M}(\mathcal{E}) \setminus \text{sim}(\mathcal{L})$ is weakly open by Lemma 7, there exists some $i \in I$ satisfying $\omega_i \notin \text{sim}(\mathcal{L})$. Then from what we have shown in the last paragraph, there exists an ensemble $\mathcal{E}$ satisfying $P_g(\mathcal{E}; \omega_i) > P_g(\mathcal{E}; \mathcal{L})$. Therefore by the monotonicity of $P_g(\mathcal{E}; \cdot)$ and $\omega_i \leq_{\text{post}} \omega$, we obtain $P_g(\mathcal{E}; \omega) > P_g(\mathcal{E}; \mathcal{L})$, which completes the proof.

From Theorem 9 and Lemma 3 we immediately obtain

**Corollary 4.** Let $\emptyset \neq \mathcal{L} \subset \mathcal{M}(\mathcal{E})$ and let $\Gamma$ be a measurement. Then $\Gamma$ is simulable by $\mathcal{L}$ if and only if $P_g(\mathcal{E}; \Gamma) \leq P_g(\mathcal{E}; \mathcal{L})$ holds for any ensemble $\mathcal{E}$.

### 6.3 Robustness of unsimulability

Now we introduce the robustness measure of unsimulability relative to a set of measurements.

**Definition 9.** Let $\emptyset \neq \mathcal{L} \subset \mathcal{M}(\mathcal{E})$ and let $\Gamma \in \text{Ch}(F \to E)$ be a measurement. We define the *robustness of unsimulability* of $\Gamma$ relative to $\mathcal{L}$ by

$$R_{\text{uns}}(\Gamma; \mathcal{L}) := \inf_{r, \Lambda} r$$

subject to $r \in [0, \infty)$,

$$\Lambda \in \text{Ch}(F \to E)$$

$$\frac{\Gamma + r\Lambda}{1 + r} \in \text{Ch}_{\text{sim}(\mathcal{L})}(F \to E)$$

(23)
where \( R_{\text{uns}}(\Gamma; \mathcal{L}) := \infty \) when the feasible region of (23) is empty. The optimization problem (23) can be written as

\[
R_{\text{uns}}(\Gamma; \mathcal{L}) = \inf_{r, \Psi} r \\
\text{subject to } r \in [0, \infty) \\
\Psi \in \text{Ch}^{\text{sim}}(F \to E) \\
\Gamma \leq (1 + r) \Psi,
\]

where the order \( \leq \) on the set of linear operators between the ordered linear spaces \( G, H \) is defined by

\[ \Phi \leq \Xi : \text{def. } \iff [\Phi(a) \leq \Xi(a) \ (\forall a \in G_+)] \]

for linear maps \( \Phi, \Xi : G \to H \).

The meaning of \( R_{\text{uns}}(\Gamma; \mathcal{L}) \) is the minimal amount of noise that should be added to make the measurement \( \Gamma \) simulable by \( \mathcal{L} \). In the resource theoretic perspective, measurements in \( \text{sim}(\mathcal{L}) \) are considered to be free and the ability to perform an unsimulable measurement is considered to be resourceful. In this viewpoint \( R_{\text{uns}}(\Gamma; \mathcal{L}) \) quantifies how resourceful \( \Gamma \) is relative to the free measurements in \( \mathcal{L} \) or \( \text{sim}(\mathcal{L}) \).

Motivated by recent results on robustness measures, we prove the following theorem, the main result of this section.

**Theorem 10.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) and let \( \Gamma \in \text{Ch}(F \to E) \) be a measurement. Then the equality

\[
1 + R_{\text{uns}}(\Gamma; \mathcal{L}) = \sup_{\mathcal{E} : \text{ensemble}} \frac{P_{\mathcal{E}}(\mathcal{E}; \Gamma)}{P_{\mathcal{E}}(\mathcal{E}; \mathcal{L})}
\]

holds, where the supremum is taken over all the ensembles.

We remark that if we put \( \mathcal{L} = \{ [u_E] \} \), the singleton consisting of the trivial measurement, then the robustness measure \( R_{\text{uns}}(\Gamma; \mathcal{L}) \) is the one called the “robustness of measurement” in [52] and Theorem 10 in this case is the infinite-dimensional version of Eq. (13) in [52].

For the first step of the proof, we show some elementary properties of \( R_{\text{uns}}(\cdot; \cdot) \).

**Lemma 9.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \). Then for each measurement \( \Gamma \in \text{Ch}(F \to E) \) with \( r := R_{\text{uns}}(\Gamma; \mathcal{L}) < \infty \), there exists a measurement \( \Psi \in \text{Ch}^{\text{sim}}(F \to E) \) such that \( \Gamma \leq (1 + r) \Psi \).

**Proof.** By the definition of \( R_{\text{uns}}(\Gamma; \mathcal{L}) \), there exists a sequence \((r_n, \Psi_n)\) \((n \in \mathbb{N})\) in \([r, \infty) \times \text{Ch}^{\text{sim}}(F \to E)\) such that

\[
\Gamma \leq (1 + r_n) \Psi_n, \quad r_n \downarrow r.
\]

Then by the BW-compactness of \( \text{Ch}^{\text{sim}}(F \to E) \) there exists a subnet \((\Psi_{n(i)})_{i \in I}\) BW-convergent to a simulable measurement \( \Psi \in \text{Ch}(F \to E) \). By the weak* closedness of \( E_+ \), this implies \( \Gamma \leq (1 + r) \Psi \).

**Lemma 10.** Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) and let \( \Gamma_j \in \text{Ch}(F_j \to E) \) \((j = 1, 2)\) be measurements. Then \( \Gamma_1 \preceq_{\text{post}} \Gamma_2 \) implies \( R_{\text{uns}}(\Gamma_1; \mathcal{L}) \leq R_{\text{uns}}(\Gamma_2; \mathcal{L}) \), i.e. the robustness of unsimulability is monotonically increasing in the post-processing order.
Proof. We may assume \( r_2 := R_{\text{uns}}(\Gamma_2; \mathcal{L}) < \infty \). Then by Lemma 9 there exists \( \Psi_2 \in \text{Ch}^{\text{sim}(\mathcal{L})}(F_2 \to E) \) satisfying \( \Gamma_2 \leq (1 + r_2)\Psi_2 \). By assumption there exists \( \Psi \in \text{Ch}(F_1 \to F_2) \) such that \( \Gamma_1 = \Gamma_2 \circ \Psi \). Then we have \( \Gamma_1 \leq (1 + r_2)\Psi_2 \circ \Psi \). Since \( \Psi_2 \circ \Psi \) is simulable by \( \mathcal{L} \), this implies \( R_{\text{uns}}(\Gamma_1; \mathcal{L}) \leq r_2 = R_{\text{uns}}(\Gamma_2; \mathcal{L}) \). \( \square \)

Lemma 11. Let \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \), let \( \Gamma \in \text{Ch}(F \to E) \) be a measurement, and let \( \Gamma \in \text{Ch}_{w^*}(F^* \to E) \) be the \( w^* \)-extension of \( \Gamma \). Then \( R_{\text{uns}}(\Gamma; \mathcal{L}) = R_{\text{uns}}(\Gamma; \mathcal{L}) \).

Proof. From \( \Gamma \leq_{\text{post}} \Gamma \), we have \( R_{\text{uns}}(\Gamma; \mathcal{L}) \leq R_{\text{uns}}(\Gamma; \mathcal{L}) \) by Lemma 10. Thus without loss of generality we may assume \( r := R_{\text{uns}}(\Gamma; \mathcal{L}) < \infty \). Then by Lemma 9 there exists a measurement \( \Psi \in \text{Ch}^{\text{sim}(\mathcal{L})}(F \to E) \) such that \( \Gamma \leq (1 + r)\Psi \). Let \( \overline{\Psi} \in \text{Ch}_{w^*}(F^* \to E) \) be the \( w^* \)-extension of \( \Psi \). Since \( [\overline{\Psi}] = [\Psi], \overline{\Psi} \) is simulable by \( \mathcal{L} \). Moreover, by the weak* density of \( F_+ \) in \( F^* \), we have \( \Gamma \leq (1 + r)\overline{\Psi} \). This implies \( R_{\text{uns}}(\Gamma; \mathcal{L}) \leq r \), which proves the claim. \( \square \)

We now prove

Lemma 12. In the setting of Theorem 10, the inequality

\[
1 + R_{\text{uns}}(\Gamma; \mathcal{L}) \geq \sup_{\mathcal{E} : \text{ensemble}} \frac{P_{\mathcal{L}}(\mathcal{E}; \Gamma)}{P_{\mathcal{L}}(\mathcal{E}; \mathcal{L})} \tag{25}
\]

holds.

Proof. We may assume \( r := R_{\text{uns}}(\Gamma; \mathcal{L}) < \infty \). Then by Lemma 9 there exists \( \Psi \in \text{Ch}^{\text{sim}(\mathcal{L})}(F \to E) \) such that \( \Gamma \leq (1 + r)\Psi \). Then for any ensemble \( \mathcal{E} = (\varphi_x)_{x \in X} \) and \( \mathcal{M} \in \text{EVM}(X; F) \) we have

\[
\sum_{x \in X} \langle \varphi_x, \Gamma(\mathcal{M}(x)) \rangle \leq (1 + r) \sum_{x \in X} \langle \varphi_x, \Psi(\mathcal{M}(x)) \rangle \\
\leq (1 + r)P_{\mathcal{L}}(\mathcal{E}; \Psi) \\
\leq (1 + r)P_{\mathcal{L}}(\mathcal{E}; \mathcal{L}).
\]

By taking the supremum of \( \mathcal{M} \), we obtain \( P_{\mathcal{L}}(\mathcal{E}; \Gamma) \leq (1 + r)P_{\mathcal{L}}(\mathcal{E}; \mathcal{L}) \), which implies (25). \( \square \)

To establish the converse inequality, we first consider the case when \( \Gamma \) is finite-outcome.

Lemma 13. The statement of Theorem 10 is true when \( \Gamma = \Gamma^\mathcal{M} \) for some finite-outcome EVM \( \mathcal{M} \in \text{EVM}(X; E) \).

Proof. By the identification between \( \text{Ch}(\ell^\infty(X) \to E) \) and \( \text{EVM}(X; E) \) (cf. Section 3.3), the robustness of unsimulability can be written as

\[
R_{\text{uns}}(\Gamma^\mathcal{M}; \mathcal{L}) = \inf_{r, K} r \\
\text{subject to} \quad r \in [0, \infty) \\
K \in \text{EVM}_{\text{sim}(\mathcal{L})}(X; E) \\
(1 + r)K(x) \geq \mathcal{M}(x) (x \in X).
\]

38
Thus if we define the convex cone
\[
\mathcal{K}_\lambda(X; E) := \{ \lambda M \in E^X \mid \lambda \in [0, \infty), \; M \in \mathbf{EVM}_{\text{sim}(\mathcal{U})}(X; E) \}
\]
generated by \( \mathbf{EVM}_{\text{sim}(\mathcal{U})}(X; E) \) we have
\[
1 + R_{\text{uns}}(\Gamma^M, \mathcal{U}) = \inf_{s, K} s \\
\text{subject to} \quad s \in \mathbb{R}, \quad K \in \mathcal{K}_\lambda(X; E) \\
\sum_{x \in X} K(x) = s u_E \\
K(x) \geq M(x) (x \in X),
\]
which is equal to
\[
1 + R_{\text{uns}}(\Gamma^M, \mathcal{U}) = \inf_{s, K} s \\
\text{subject to} \quad s \in \mathbb{R}, \quad K \in \mathcal{K}_\lambda(X; E) \\
\sum_{x \in X} K(x) \leq s u_E \\
K(x) \geq M(x) (x \in X),
\]
which is equal to
\[
1 + R_{\text{uns}}(\Gamma^M, \mathcal{U}) = \inf_{s, K} s \\
\text{subject to} \quad s \in \mathbb{R}, \quad K \in \mathcal{K}_\lambda(X; E) \\
\sum_{x \in X} K(x) \leq s u_E \\
K(x) \geq M(x) (x \in X).
\]
The optimization problem (26) can be written in the standard form of the conic programming [50, 3]
\[
\inf_v \langle c^*, v \rangle \quad \text{subject to} \quad v \in C, \; A(v) + b \in K.
\]
where \( C \) and \( K \) are respectively closed convex cones on Banach spaces \( V \) and \( U \), \( A : V \to U \) is a bounded linear map, \( c^* \in V^* \), and \( b \in U \). Indeed (26) coincides with (27) if we put
\[
V := E^X \times \mathbb{R}, \quad C := \mathcal{K}_\lambda(X; E) \times \mathbb{R} \\
U := E^X \times E, \quad K := E^X_+ \times E_+, \\
\langle c^*, (w, s) \rangle := s \quad ((w, s) \in V), \quad b := ((-M(x))_{x \in X}, 0), \\
A((w, s)) := \left( w, su_E - \sum_{x \in X} w(x) \right) \quad ((w, s) \in V),
\]
provided that \( \mathcal{K}_\lambda(X; E) \) is closed. We prove a stronger fact that \( \mathcal{K}_\lambda(X; E) \) is weakly* closed. Take \( r \in (0, \infty) \) and a net \( (K_i)_{i \in I} \) in \( (\mathcal{K}_\lambda(X; E))_r \) weakly* converging to some \( K \in E^X \). By the definition of \( \mathcal{K}_\lambda(X; E) \) we may write as \( K_i = \lambda_i N_i \) for some \( \lambda_i \in [0, \infty) \) and \( N_i \in \mathbf{EVM}_{\text{sim}(\mathcal{U})}(X; E) \). Then from \( \|K_i(x)\| \leq r \) we have
\[
\lambda_i = \|\lambda_i u_E\| = \left\| \sum_{x \in X} K_i(x) \right\| \leq \sum_{x \in X} \|K_i(x)\| \leq |X| r.
\]
Hence we can take subnets \( (\lambda_{i(j)})_{j \in J} \) and \( (N_{i(j)})_{j \in J} \) converging respectively to some \( \lambda \in [0, |X| r] \) and \( N \in \mathbf{EVM}_{\text{sim}(\mathcal{U})}(X; E) \). Then we have \( K = \lambda N \in \mathcal{K}_\lambda(X; E) \). Thus by the Krein–Smulian theorem \( \mathcal{K}_\lambda(X; E) \) is weakly* closed.
Now let $v_0 := ((u_E)_{x \in X}, |X| + 1) \in V$. Since the trivial EVM $(|X|^{-1} u_E)_{x \in X}$ is in $\text{EVM}_{\text{sim}(\omega)}(X; E)$, we have $v_0 \in C$ and hence

$$U \ni A(C) - K \ni \mathbb{R}_+ A(v_0) - K$$

$$= \{ (\lambda u_E - w(x))_{x \in X}, \lambda u_E - v) \mid \lambda \in \mathbb{R}_+, w \in E_+^X, v \in E_+ \}$$

$$= U.$$

This implies $-b \in \text{int}(A(C) - K)(= U)$, where $\text{int}(\cdot)$ denotes the interior. Therefore the optimization problem (26) has no duality gap ([50], Proposition 2.9; [3], Theorem 2.187) and hence (27) coincides with

$$\sup_{u^*} \langle u^*, b \rangle \text{ subject to } u^* \in -K^*, A^*(u^*) + c^* \in C^*, \quad (28)$$

where

$$K^* := \{ u^* \in U^* \mid \langle u^*, u \rangle \geq 0 (\forall u \in K) \}$$

$$C^* := \{ v^* \in V^* \mid \langle v^*, v \rangle \geq 0 (\forall v \in C) \}$$

are dual cones. Since we have

$$K^* = (E_+^*)^X \times E_+^*, \quad C^* = \mathcal{K}_\mathcal{L}(X; E)^* \times \{0\},$$

$$\mathcal{K}_\mathcal{L}(X; E)^* := \{ (\psi_x)_{x \in X} \in (E^*)^X \mid \sum_{x \in X} \langle \psi_x, K(x) \rangle \geq 0 (\forall K \in \mathcal{K}_\mathcal{L}(X; E)) \},$$

$$A^*((\psi_x)_{x \in X}, \chi) = ((\psi_x - \chi)_{x \in X}, (\chi, u_E)) \quad (((\psi_x)_{x \in X}, \chi) \in U^* = (E^*)^X \times E^*),$$

the dual problem (28) is explicitly written as

$$\sup_{\chi, (\psi_x)_{x \in X}} \sum_{x \in X} \langle \psi_x, M(x) \rangle$$

subject to $\chi \in E_+^*, \quad (\psi_x)_{x \in X} \in (E_+^*)^X$

$$\quad (x - \psi_x)_{x \in X} \in \mathcal{K}_\mathcal{L}(X; E)^*, \quad \langle \chi, u_E \rangle = 1$$

and hence

$$1 + R_{\text{uns}}(\Gamma^M, \Omega) = \sup_{\chi, (\psi_x)_{x \in X}} \sum_{x \in X} \langle \psi_x, M(x) \rangle$$

subject to $\chi \in E_+^*, \quad (\psi_x)_{x \in X} \in (E_+^*)^X$

$$\quad (x - \psi_x)_{x \in X} \in \mathcal{K}_\mathcal{L}(X; E)^*, \quad \langle \chi, u_E \rangle \leq 1. \quad (29)$$

We now show

$$1 + R_{\text{uns}}(\Gamma^M, \Omega) = \sup_{\chi, (\psi_x)_{x \in X}} \sum_{x \in X} \langle \psi_x, M(x) \rangle$$

subject to $\chi \in E_+, \quad (\psi_x)_{x \in X} \in (E_+^*)^X$

$$\quad (x - \psi_x)_{x \in X} \in \mathcal{K}_\mathcal{L}(X; E)^*, \quad \langle \chi, u_E \rangle \leq 1. \quad (30)$$
Since the common objective function of (29) and (30) is weakly* (i.e. in \(\sigma((E^*)_X \times E^*, E^X \times E)\)) continuous, we have only to prove that the feasible region of (30) is weakly* dense in that of (29). Define

\[
C := \{ ((a_x - b_x)_{x \in X}, a + \sum_{x \in X} b_x - u_E) \mid a \in E_+, (a_x)_{x \in X} \in E^X_+, (b_x)_{x \in X} \in K_L(X; E) \},
\]

which is a convex set in \(E^X \times E\) containing 0. Then for \(((\psi_x)_{x \in X}, \chi) \in (E^*)_X \times E^*\) we have

\[
\sum_{x \in X} \langle \psi_x, c_x \rangle + \langle \chi, c_0 \rangle \geq -1 \quad (\forall ((c_x)_{x \in X}, c_0) \in C)
\]

\[
\iff \langle \chi, a \rangle + \sum_{x \in X} \langle \psi_x, a_x \rangle + \sum_{x \in X} \langle \chi - \psi_x, b_x \rangle - \langle \chi, u_E \rangle \geq -1
\]

\[
(\forall a \in E_+, \forall (a_x)_{x \in X} \in E^X_+, \forall (b_x)_{x \in X} \in K_L(X; E))
\]

\[
\iff \chi \in E_+, (\psi_x)_{x \in X} \in (E^*)_X, (\chi - \psi_x)_{x \in X} \in K_L(X; E)^*, \langle \chi, u_E \rangle \leq 1.
\]

Therefore the polar of \(C\) in the pair \((E^X \times E, (E^*)_X \times E^*)\) coincides with the feasible region of (29). Similarly the polar of \(C\) in the pair \((E^X \times E, (E^*)_X \times E_*)\) is the feasible region of (30). Thus, by the bipolar theorem, we have only to prove that \(C\) is closed in the weak* topology \(\sigma(E^X \times E, (E^*)_X \times E_*)\). By the Krein-Šmulian theorem, this reduces to show that \((C)_r\) is weakly* closed for any \(r \in (0, \infty)\). Now suppose that

\[
((a_x - b_x)_{x \in X}, a + \sum_{x \in X} b_x - u_E) \in (C)_r
\]

with \(a \in E_+, (a_x)_{x \in X} \in E^X_+,\) and \((b_x)_{x \in X} \in K_L(X; E).\) Then

\[
\|a\|, \|b_x\| \leq \left\| a + \sum_{x' \in X} b_{x'} \right\| \leq \left\| a + \sum_{x' \in X} b_{x'} - u_E \right\| + 1 \leq r + 1 \quad (x \in X),
\]

\[
\|a_x\| \leq \|a_x - b_x\| + \|b_x\| \leq 2r + 1 \quad (x \in X).
\]

Therefore, by noting the weak* closedness of \(K_L(X; E)\), the weak* closedness of \((C)_r\) follows from the Banach-Alaoglu theorem as in the proof of Lemma 2. Thus we have proved (30).

Now by (30) there exists a sequence \(((\psi_x^k)_{x \in X}, \chi^k) \mid (k \in \mathbb{N})\) in the feasible region of (30) satisfying

\[
\sum_{x \in X} \langle \psi_x^k, M(x) \rangle > 1 + R_{\text{uns}}(\Gamma^M; \mathcal{L}) - \frac{1}{k},
\]

Then

\[
N_k := \sum_{x \in X} \langle \psi_x^k, u_E \rangle \geq \sum_{x \in X} \langle \psi_x^k, M(x) \rangle > 0
\]

for all \(k \in \mathbb{N}\). Let \(\varphi_x^k := N_k^{-1} \psi_x^k\) and define \(\mathcal{E}_k := (\varphi_x^k)_{x \in X}\), which is an ensemble. Then we
have
\[ P_{\mathcal{E}}(\mathcal{E}_k; \Gamma^M) \geq \sum_{x \in X} \langle \varphi^k_x, M(x) \rangle \]
\[ = \frac{1}{N_k} \sum_{x \in X} \langle \psi^k_x, M(x) \rangle \]
\[ > \frac{1}{N_k} \left( 1 + R_{\text{uns}}(\Gamma^M; \mathcal{L}) - \frac{1}{k} \right). \quad (31) \]

From \((\chi^k - \psi^k_x)_{x \in X} \in \mathcal{K}_E(X; E)^*\), we have
\[ \sum_{x \in X} \langle \chi^k - \psi^k_x, N(x) \rangle \geq 0 \quad (\forall N \in \text{EVM}_{\text{sim}}(X; E)) \]
and hence for any \(N \in \text{EVM}_{\text{sim}}(X; E)\)
\[ 1 \geq \langle \chi^k, u \rangle = \sum_{x \in X} \langle \chi^k, N(x) \rangle \geq \sum_{x \in X} \langle \psi^k_x, N(x) \rangle = N_k \sum_{x \in X} \langle \varphi^k_x, N(x) \rangle. \]

Therefore we have
\[ P_{\mathcal{E}}(\mathcal{E}_k; \mathcal{L}) = \sup_{N \in \text{EVM}_{\text{sim}}(X; E)} \sum_{x \in X} \langle \varphi^k_x, N(x) \rangle \leq \frac{1}{N_k} \]

By combining this with \((31)\) we obtain
\[ P_{\mathcal{E}}(\mathcal{E}_k; \Gamma^M) > P_{\mathcal{E}}(\mathcal{E}_k; \mathcal{L}) \left( 1 + R_{\text{uns}}(\Gamma^M; \mathcal{L}) - \frac{1}{k} \right), \]
which implies
\[ 1 + R_{\text{uns}}(\Gamma^M; \mathcal{L}) \leq \sup_{N \in \mathcal{E}} \frac{P_{\mathcal{E}}(\mathcal{E}_k; \Gamma^M)}{P_{\mathcal{E}}(\mathcal{E}_k; \mathcal{L})} \leq \sup_{\mathcal{E}: \text{ensemble}} \frac{P_{\mathcal{E}}(\mathcal{E}; \Gamma^M)}{P_{\mathcal{E}}(\mathcal{E}; \mathcal{L})}. \]

Thus by Lemma \([12]\) the statement of Theorem \([10]\) is true in this case. \(\square\)

To reduce the proof to the finite-outcome case, we need the following lemma.

**Lemma 14.** Let \(F\) be a classical space and let \(\emptyset \neq \mathcal{L} \subset \mathcal{M}(E)\). Then the extended real-valued function
\[ \text{Ch}(F \to E) \ni \Gamma \mapsto R_{\text{uns}}(\Gamma; \mathcal{L}) \in [0, \infty] \quad (32) \]
is lower semicontinuous with respect to the BW-topology, i.e., for any net \((\Gamma_i)_{i \in I}\) in \(\text{Ch}(F \to E)\) BW-convergent to \(\Gamma \in \text{Ch}(F \to E)\) and any \(r < R_{\text{uns}}(\Gamma; \mathcal{L})\), \(r < R_{\text{uns}}(\Gamma_i; \mathcal{L})\) holds eventually.

**Proof.** Suppose that there exist a net \((\Gamma_i)_{i \in I}\) in \(\text{Ch}(F \to E)\) BW-convergent to \(\Gamma \in \text{Ch}(F \to E)\) and \(r < R_{\text{uns}}(\Gamma; \mathcal{L})\) such that \(R_{\text{uns}}(\Gamma_i; \mathcal{L}) \leq r\) frequently. By taking a subnet we may assume \(R_{\text{uns}}(\Gamma_i; \mathcal{L}) \leq r\) for all \(i \in I\). Then for every \(i \in I\) there exists a simulable measurement \(\Psi_i \in \text{Ch}_{\text{sim}}(\mathcal{L})(F \to E)\) such that \((1 + r)\Psi_i \geq \Gamma_i\). Since \(\text{Ch}_{\text{sim}}(\mathcal{L})(F \to E)\) is BW-compact by Corollary \([3]\) there exists a subnet \((\Psi_{i(j)})_{j \in J}\) BW-converging to \(\Psi \in \text{Ch}_{\text{sim}}(\mathcal{L})(F \to E)\). Then by the weak* closedness of \(E_+\) we have \((1 + r)\Psi \geq \Gamma\) and hence \(R_{\text{uns}}(\Gamma; \mathcal{L}) \leq r\), which contradicts the assumption. Therefore \((32)\) is lower semicontinuous. \(\square\)
Proof of Theorem 10. By Lemmas 3 and 11 we have only to prove (24) when \( F \) has the Banach predual \( F_\ast \) and \( \Gamma \) is a \( w_\ast \)-measurement. Then by Theorem 4 there exists a net \( (\Lambda_\Delta)_{\Delta \in D(F)} \) in \( \text{Ch}(F \to E) \) such that \( (\Lambda_\Delta)_{\Delta \in D(F)} \) is an increasing net in \( \mathcal{M}\_\text{fin}(E) \), \( \lim_{\text{weakly}} \sup_{\Delta \in D(F)} [\Lambda_\Delta] = [\Gamma] \), and \( \Lambda_\Delta \xrightarrow{\text{BW}} \Gamma \). Thus from Lemmas 10 and 14 we have

\[
R_{\text{uns}}(\Lambda_\Delta; \mathcal{L}) \uparrow R_{\text{uns}}(\Gamma; \mathcal{L}).
\]

(33)

Since \( \Lambda_\Delta \) is post-processing equivalent to a finite-outcome measurement, Lemma 13 implies

\[
1 + R_{\text{uns}}(\Lambda_\Delta; \mathcal{L}) = \sup_{\mathcal{E} : \text{ensemble}} \frac{P_{\text{g}}(\mathcal{E}; \Lambda_\Delta)}{P_{\text{g}}(\mathcal{E}; \mathcal{L})}.
\]

(34)

From (33) and (34) we obtain

\[
1 + R_{\text{uns}}(\Gamma; \mathcal{L}) = \sup_{\Delta \in D(F)} (1 + R_{\text{uns}}(\Lambda_\Delta; \mathcal{L}))
\]

\[
= \sup_{\Delta \in D(F)} \sup_{\mathcal{E} : \text{ensemble}} \frac{P_{\text{g}}(\mathcal{E}; \Lambda_\Delta)}{P_{\text{g}}(\mathcal{E}; \mathcal{L})}
\]

\[
= \sup_{\mathcal{E} : \text{ensemble}} \sup_{\Delta \in D(F)} \frac{P_{\text{g}}(\mathcal{E}; \Lambda_\Delta)}{P_{\text{g}}(\mathcal{E}; \mathcal{L})}
\]

\[
= \sup_{\mathcal{E} : \text{ensemble}} \frac{P_{\text{g}}(\mathcal{E}; \Gamma)}{P_{\text{g}}(\mathcal{E}; \mathcal{L})},
\]

where the last equality follows from the post-processing monotonicity and the weak continuity of the gain functional \( P_{\text{g}}(\mathcal{E}; \cdot) \).

As finishing this section, we slightly generalize Theorem 10 to the partitioned ensembles, which we will consider again in Section 8.

Definition 10 (Partitioned ensemble). 1. For a finite set \( X \neq \emptyset \), a family \( \mathcal{E} = (\mathcal{E}_x)_{x \in X} \) of \( w_\ast \)-families \( \mathcal{E}_x = (\varphi_{x,y})_{y \in Y_x} \) \( \Gamma \) in \( \chi (F \to E) \) such that \( (\mathcal{E}_x)_{x \in X} \) is called a partitioned ensemble if \( \varphi_{x,y} \geq 0 \) \( (x \in X, y \in Y_x) \) and \( \mathcal{E} \) satisfies the normalization condition

\[
\sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, u_E \rangle = 1.
\]

Each component \( \mathcal{E}_x \) is called a subensemble of \( \mathcal{E} \).

2. Let \( \mathcal{E} \) be a partitioned ensemble. For each measurement \( \Gamma \in \chi (F \to E) \), each measurement \( \omega \in \mathcal{M}(E) \), and each subset \( \emptyset \neq \mathcal{L} \subset \mathcal{M}(E) \) we define

\[
P_{\text{g}}(\mathcal{E}; \Gamma) := \sum_{x \in X} P_{\text{g}}(\mathcal{E}_x; \Gamma),
\]

\[
P_{\text{g}}(\mathcal{E}; \omega) := \sum_{x \in X} P_{\text{g}}(\mathcal{E}_x; \omega),
\]

\[
P_{\text{g}}(\mathcal{E}; \mathcal{L}) := \sum_{x \in X} P_{\text{g}}(\mathcal{E}_x; \mathcal{L}),
\]

where \( \mathcal{E} = (\mathcal{E}_x)_{x \in X} \).
The operational meaning of the quantity $P_g(\vec{E}; \mathcal{L})$ in Definition 10 is as follows. Consider the situation where Alice prepares the system’s state according to the ensemble $(\Phi_{x,y})_{x \in X; y \in Y_x}$ and Bob can perform only the measurements belonging to $\mathcal{L}$. We also assume that before Bob perform a measurement, Alice announces the value of $x \in X$ to Bob so that to Bob the system’s state corresponds to (up to the normalization factor) the subensemble $\mathcal{E}_x$. Then Bob chooses an appropriate measurement from $\mathcal{L}$, perform it, and guesses the original label $y \in Y_x$ based on the measurement outcome. The quantity $P_g(\vec{E}; \mathcal{L})$ is then the optimal average probability that Bob can correctly guess the label $y \in Y_x$. A similar interpretation also applies to the quantity $P_g(\vec{E}; \Gamma)$.

Now Theorem 10 is generalized to

**Corollary 5.** Let $\mathcal{L}$ and $\Gamma$ be the same as in Theorem 10. Then

$$1 + R_{\text{uns}}(\Gamma; \mathcal{L}) = \sup_{\vec{E}: \text{partitioned ensemble}} \frac{P_g(\vec{E}; \Gamma)}{P_g(\vec{E}; \mathcal{L})}.$$ (35)

**Proof.** Let $\vec{E} = (\mathcal{E}_x)_{x \in X}$ be a partitioned ensemble. Then by Theorem 10 we have

$$P_g(\mathcal{E}_x; \Gamma) \leq (1 + R_{\text{uns}}(\Gamma; \mathcal{L})) P_g(\mathcal{E}_x; \mathcal{L}) \quad (x \in X).$$

Hence

$$P_g(\vec{E}; \Gamma) = \sum_{x \in X} P_g(\mathcal{E}_x; \Gamma)$$
$$\leq (1 + R_{\text{uns}}(\Gamma; \mathcal{L})) \sum_{x \in X} P_g(\mathcal{E}_x; \mathcal{L})$$
$$= (1 + R_{\text{uns}}(\Gamma; \mathcal{L})) P_g(\vec{E}; \mathcal{L}).$$

Therefore by Theorem 10 we obtain

$$1 + R_{\text{uns}}(\Gamma; \mathcal{L}) \geq \sup_{\vec{E}: \text{partitioned ensemble}} \frac{P_g(\vec{E}; \Gamma)}{P_g(\vec{E}; \mathcal{L})}$$
$$\geq \sup_{\mathcal{E}: \text{ensemble}} \frac{P_g(\mathcal{E}; \Gamma)}{P_g(\mathcal{E}; \mathcal{L})}$$
$$= 1 + R_{\text{uns}}(\Gamma; \mathcal{L}),$$

where the second inequality follows by restricting $\vec{E}$ to single-element families. $\square$

### 7 Extremal, maximal, and simulation irreducible measurements

In this section, we show some basic properties of extremal, maximal, and simulation irreducible measurements and prove that any measurement is simulable by the set of simulation irreducible measurements (Theorem 12).
7.1 Extremal measurement

As we have seen in Theorem \[1\] the measurement space \(\mathcal{M}(E)\) can be regarded as compact convex set and hence has sufficiently many extremal points by the Krein-Milman theorem. Here one should not confuse the extremality in \(\mathcal{M}(E)\) and the “extremal POVM,” \((\text{e.g.} \ [6], \text{Section 9.3}),\) which in our terminology corresponds to the extremal point of \(\text{Ch}_{w^*}(F \to E)\) for some fixed classical space \(F\).

In this subsection we prove the following theorem that characterizes the set \(\partial_c \mathcal{M}(E)\) of extremal points of \(\mathcal{M}(E)\). This is a generalization of the corresponding result for classical statistical experiments ([54], Theorem 7.3.15, (i) \(\iff\) (vi)). See also [20] (Corollary 3.8) for the related result on the extremality of quantum statistical experiments.

**Theorem 11** (Characterization of extremal measurement). A measurement \(\omega \in \mathcal{M}(E)\) is an extremal point of \(\mathcal{M}(E)\) if and only if \(\omega\) has a representative \(\Gamma \in \text{Ch}_{w^*}(F \to E)\) that is an injection.

For the proof we need the following lemma.

**Lemma 15.** Let \(F\) be an order unit Banach space. Then the identity channel \(\text{id}_F\) is an extremal point of \(\text{Ch}(F \to F)\).

**Proof.** Take channels \(\Psi_1, \Psi_2 \in \text{Ch}(F \to F)\) and \(\lambda \in (0, 1)\) such that \(\text{id}_F = \lambda \Psi_1 + (1 - \lambda) \Psi_2\). Then for any pure state \(\psi \in \partial_c S(F)\) we have

\[
\psi = \text{id}_{F^*}(\psi) = \lambda \Psi_1^*(\psi) + (1 - \lambda) \Psi_2^*(\psi),
\]

where the star denotes the Banach dual map. Since \(\Psi_j^*(\psi) \in S(F)\) \((j = 1, 2)\), the extremality of \(\psi\) implies \(\Psi_1^*(\psi) = \Psi_2^*(\psi) = \psi\). Since \(\text{lin}(S(F)) = F^*\), the Krein-Milman theorem and the weak* continuity of the Banach dual maps imply \(\Psi_1^* = \Psi_2^* = \text{id}_{F^*}\). Therefore \(\Psi_1 = \Psi_2 = \text{id}_F\), which proves the claim.

We remark that a proof similar to the above one applies to the more general result that the identity map on an arbitrary Banach space \(X\) is an extremal point of the unit ball of the set bounded operators on \(X\) [29].

**Proof of Theorem 11.** Assume \(\omega \in \partial_c \mathcal{M}(E)\) and take a minimally sufficient (see Appendix \[E\]) representative \(\Gamma \in \text{Ch}_{w^*}(F \to E)\) of \(\omega\). We show the injectivity of \(\Gamma\). Take an element \(a \in F\) such that \(\Gamma(a) = 0\). Without loss of generality we may assume \(\|a\| \leq 1\). Let \(e_\pm := \frac{1}{2}(u_F \pm a)\) and define \(\Gamma_{\pm} : F \to E\) by \(\Gamma_{\pm}(b) := 2\Gamma(b \cdot e_\pm)\ (b \in F)\). Then \(0 \leq e_\pm \leq u_F\) and \(\Gamma_{\pm} \in \text{Ch}_{w^*}(F \to E)\). Define channels \(\Psi_1 \in \text{Ch}(F \to F \oplus F)\) and \(\Psi_2 \in \text{Ch}(F \oplus F \to F)\) by

\[
\Psi_1(b) := b \oplus b, \quad \Psi_2(b \oplus c) := b \cdot e_+ + c \cdot e_- \quad (b, c \in F).
\]

Then we have

\[
\left(\frac{1}{2} \Gamma_+ \oplus \frac{1}{2} \Gamma_-\right) \circ \Psi_1(b) = \Gamma(b \cdot e_+) + \Gamma(b \cdot e_-) = \Gamma(b) \quad (b \in F),
\]

\[
\Gamma \circ \Psi_2(b \oplus c) = \Gamma(b \cdot e_+) + \Gamma(c \cdot e_-) = \left(\frac{1}{2} \Gamma_+ \oplus \frac{1}{2} \Gamma_-\right) (b \oplus c) \quad (b, c \in F),
\]

45
which implies $\Gamma \sim_{\text{post}} \frac{1}{2}\Gamma_+ \oplus \frac{1}{2}\Gamma_-$. Thus by the extremality of $\omega = [\Gamma]$ it follows that $\Gamma \sim_{\text{post}} \Gamma_{\pm}$. Therefore by Proposition \[\text{there exist channels } \Phi_{\pm} \in \text{Ch}_{ws}(F \to F) \text{ such that } \Gamma_{\pm} = \Gamma \circ \Phi_{\pm}. \text{ Then we have}
\]
$$
\Gamma = \frac{1}{2}\Gamma_+ + \frac{1}{2}\Gamma_- = \Gamma \circ \left( \frac{1}{2}\Phi_+ + \frac{1}{2}\Phi_- \right).
$$

By the minimal sufficiency of $\Gamma$, this implies $\frac{1}{2}\Phi_+ + \frac{1}{2}\Phi_- = \text{id}_F$ and hence by Lemma \[\text{we obtain } \Phi_{\pm} = \text{id}_F. \text{ Therefore we have}
\]
$$
\Gamma(b) = \Gamma \circ \Phi_+(b) = \Gamma_+(b) = 2\Gamma(b \cdot e_+) \quad (b \in F).
$$

Now suppose that $\|2e_+\| > 1$. Then there exists a non-zero projection $Q \in F$ and $\delta > 0$ such that $2Q \cdot e_+ \geq (1 + \delta)Q$. Thus
$$
\Gamma(Q) = \Gamma(2Q \cdot e_+) \geq (1 + \delta)\Gamma(Q),
$$
which implies $\Gamma(Q) = 0$. Hence by the faithfulness (cf. Appendix \[\text{of } \Gamma \text{ we obtain } Q = 0, \text{ which is a contradiction. Therefore } \|2e_+\| \leq 1. \text{ From } \Gamma(u_F - 2e_+) = 0 \text{ and } 2e_+ \leq u_F, \text{ the faithfulness of } \Gamma \text{ again yields } 2e_+ = u_F. \text{ Therefore we obtain } a = 0, \text{ proving the injectivity of } \Gamma.
$$

Conversely suppose that $\Gamma \in \text{Ch}_{ws}(F \to E)$ is an injective representative of $\omega$. To show the extremality of $\omega$, take w*-measurements $\Lambda_j \in \text{Ch}_{ws}(G_j \to F)$ ($j = 1, 2$) and $\lambda \in (0, 1)$ satisfying $\Gamma \sim_{\text{post}} \lambda \Lambda_1 \oplus (1 - \lambda)\Lambda_2$. Then there exists channels $\Xi \in \text{Ch}(F \to G_1 \oplus G_2)$ and $\Theta \in \text{Ch}(G_1 \oplus G_2 \to F)$ such that
$$
\Gamma = (\lambda \Lambda_1 \oplus (1 - \lambda)\Lambda_2) \circ \Xi, \quad \lambda \Lambda_1 \oplus (1 - \lambda)\Lambda_2 = \Gamma \circ \Theta.
$$

Then we have $\lambda \Lambda_1(b) = \Gamma \circ \Theta(b \oplus 0)$ ($b \in G_1$). Thus by putting $b = u_{G_1}$, we obtain
$$
\Gamma(\lambda^{-1}\Theta(u_{G_1} \oplus 0)) = \Lambda_1(u_{G_1}) = u_E = \Gamma(u_F).
$$

By the injectivity of $\Gamma$ this implies $\lambda^{-1}\Theta(u_{G_1} \oplus 0) = u_F$. Hence we may define $\Phi_1 \in \text{Ch}(G_1 \to F)$ by
$$
\Phi_1(b) := \lambda^{-1}\Theta(b \oplus 0) \quad (b \in G_1).
$$

Similarly the linear map $\Phi_2$ defined by
$$
\Phi_2(c) := (1 - \lambda)^{-1}\Theta(0 \oplus c) \quad (c \in G_2)
$$
is a channel in $\text{Ch}(G_2 \to F)$. Then by the definition of $\Phi_j$ we have $\Lambda_j = \Gamma \circ \Phi_j \leq_{\text{post}} \Gamma$ ($j = 1, 2$). If we write as
$$
\Xi(a) = \Xi_1(a) \oplus \Xi_2(a) \quad (a \in F),
$$
where $\Xi_j \in \text{Ch}(F \to G_j)$, then
$$
\Gamma = \lambda \Lambda_1 \circ \Xi_1 + (1 - \lambda)\Lambda_2 \circ \Xi_2 = \Gamma \circ (\lambda \Phi_1 \circ \Xi_1 + (1 - \lambda)\Phi_2 \circ \Xi_2).
$$

By the injectivity of $\Gamma$ this implies $\lambda \Phi_1 \circ \Xi_1 + (1 - \lambda)\Phi_2 \circ \Xi_2 = \text{id}_F$. Hence by Lemma \[\text{we have } \Phi_1 \circ \Xi_1 = \Phi_2 \circ \Xi_2 = \text{id}_F. \text{ Therefore we obtain}
\]
$$
\Gamma = \Gamma \circ \Phi_j \circ \Xi_j = \Lambda_j \circ \Xi_j \leq_{\text{post}} \Lambda_j \quad (j = 1, 2).
$$

This proves $\Gamma \sim_{\text{post}} \Lambda_1 \sim_{\text{post}} \Lambda_2$ and hence $\omega = [\Gamma]$ is extremal.
7.2 Maximal measurement

We next study post-processing maximal measurements [11 9 5].

Definition 11. A measurement \( \omega \in \mathcal{M}(E) \) is said to be post-processing maximal, or just maximal, if \( \omega \) is a maximal element of \( \mathcal{M}(E) \) in the post-processing order, i.e. for any \( \nu \in \mathcal{M}(E) \), \( \omega \succeq_{\text{post}} \nu \) implies \( \nu \succeq_{\text{post}} \omega \). The set of maximal measurements in \( \mathcal{M}(E) \) is denoted by \( \mathcal{M}_{\text{max}}(E) \).

From Theorem 3, Proposition 16, and Lemma 4, application of Zorn’s lemma immediately gives

Corollary 6. \( \mathcal{M}(E) = \downarrow \mathcal{M}_{\text{max}}(E) \), i.e. every measurement in \( \mathcal{M}(E) \) is a post-processing of a maximal measurement.

For a finite-outcome POVM \( M \) on a quantum system, \( \Gamma^M \) is maximal if and only if each element of \( M \) is rank-1 [11]. A similar characterization can be shown for continuous-outcome quantum POVMs [47 34].

We now prove

Proposition 21. \( \mathcal{M}_{\text{max}}(E) \) is a face of \( \mathcal{M}(E) \).

For the proof of Proposition 21 we need the following lemma, which can be shown in the same way as in [33] (Lemma 6).

Lemma 16. Let \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \) be w*-measurements. Suppose \( \Gamma \succeq_{\text{post}} \Lambda \). Then there exists a w*-measurement \( \tilde{\Lambda} \in \text{Ch}_{w^*}(\tilde{G} \to E) \) such that \( \tilde{\Lambda} \sim_{\text{post}} \Lambda, F \) is included in \( \tilde{G} \) as a unital subalgebra of \( \tilde{G} \), and \( \Gamma \) is the restriction of \( \tilde{\Lambda} \) to the subalgebra \( F \).

Here for a classical space \( G \), a subset \( F \subset G \) is called a (unital) subalgebra of \( G \) if \( F \) is linear subspace of \( G \) (containing the unit \( u_G \)) and \( F \) is closed under the multiplication on \( G \).

Proof of Proposition 21 (Convexity). Take \( \omega_1, \omega_2 \in \mathcal{M}_{\text{max}}(E) \) and \( \lambda \in (0, 1) \). We prove \( \lambda \omega_1 + (1 - \lambda) \omega_2 \in \mathcal{M}_{\text{max}}(E) \). Assume \( \lambda \omega_1 + (1 - \lambda) \omega_2 \succeq_{\text{post}} \nu \in \mathcal{M}(E) \) and let \( \Gamma_j \in \text{Ch}_{w^*}(F_j \to E) \) \( (j = 1, 2) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \) be representatives of \( \omega_j \) and \( \nu \), respectively. By Lemma 16 we can take \( \Lambda \) so that \( F_1 \oplus F_2 \) is a unital subalgebra of \( G \) and \( \lambda \Gamma_1 \oplus (1 - \lambda) \Gamma_2 \) is the restriction of \( \Lambda \) to \( F_1 \oplus F_2 \). We define projections \( s_1, s_2 \in G \) by

\[
s_1 := u_{F_1} \oplus 0, \quad s_2 := 0 \oplus u_{F_2}.
\]

Then w*-measurements \( \Lambda_j \in \text{Ch}_{w^*}(s_j \cdot G \to E) \) \( (j = 1, 2) \) are well-defined by

\[
\Lambda_1(a) := \lambda^{-1} \Lambda(a) \quad (a \in s_1 \cdot G),
\]

\[
\Lambda_2(b) := (1 - \lambda)^{-1} \Lambda(b) \quad (b \in s_2 \cdot G),
\]

where \( s_j \cdot G := \{ s_j \cdot c \mid c \in G \} \) which has the order unit \( s_j \). Indeed we have

\[
\Lambda_1(s_1) = \lambda^{-1} \Gamma_1 \oplus (1 - \lambda) \Gamma_2 (u_{F_1} \oplus 0) = \Gamma_1 (u_{F_1}) = u_E,
\]

\[
\Lambda_2(s_2) = (1 - \lambda)^{-1} \Gamma_1 \oplus (1 - \lambda) \Gamma_2 (0 \oplus u_{F_2}) = \Gamma_2 (u_{F_2}) = u_E.
\]
Define channels $\Psi_j \in \text{Ch}(F_j \to s_j \cdot G)$ by

$$\Psi_1(a) := a + 0 \quad (a \in F_1), \quad \Psi_2(b) := 0 + b \quad (b \in F_2).$$

Then we have

$$\Lambda_1 \circ \Psi_1(a) = \lambda^{-1} \Lambda(a + 0) = \Gamma_1(a) \quad (a \in F_1)$$

and similarly $\Lambda_2 \circ \Psi_2 = \Gamma_2$. Thus by the maximality of $[\Gamma_1] = \omega_1$ and $[\Gamma_2] = \omega_2$ there exist channels $\Phi_j \in \text{Ch}_{\text{w*}}(s_j \cdot G \to F_j) \quad (j = 1, 2)$ such that $\Lambda_j = \Gamma_j \circ \Phi_j$. Then for any $c \in G$,

$$\Lambda(c) = \Lambda(s_1 \cdot c) + \Lambda(s_2 \cdot c)$$

$$= \lambda \Lambda_1(s_1 \cdot c) + (1 - \lambda) \Lambda_2(s_2 \cdot c)$$

$$= \lambda \Gamma_1 \circ \Phi_1(s_1 \cdot c) + (1 - \lambda) \Gamma_2 \circ \Phi_2(s_2 \cdot c)$$

$$= (\lambda \Gamma_1 + (1 - \lambda) \Gamma_2) \circ \tilde{\Phi}(c),$$

where we defined $\tilde{\Phi} \in \text{Ch}(G \to F_1 \oplus F_2)$ by $\tilde{\Phi}(c) := \Phi_1(s_1 \cdot c) \oplus \Phi_2(s_2 \cdot c) \quad (c \in G)$. Therefore this shows $\nu = [\Lambda] \leq_{\text{post}} \lambda \omega_1 + (1 - \lambda) \omega_2$, which proves the convexity of $\mathcal{M}_{\text{max}}(E)$.

(Extremality). Take $\nu_1, \nu_2 \in \mathcal{M}(E)$ and $\lambda \in (0, 1)$ such that $\lambda \nu_1 + (1 - \lambda) \nu_2 \in \mathcal{M}_{\text{max}}(E)$. If $\nu_1 \leq_{\text{post}} \nu_1' \in \mathcal{M}(E)$, we have

$$\lambda \nu_1 + (1 - \lambda) \nu_2 \leq_{\text{post}} \lambda \nu_1' + (1 - \lambda) \nu_2$$

by Proposition [10]. Therefore the maximality of $\lambda \nu_1 + (1 - \lambda) \nu_2$ implies

$$\lambda \nu_1 + (1 - \lambda) \nu_2 = \lambda \nu_1' + (1 - \lambda) \nu_2$$

and hence $\nu_1 = \nu_1'$, which proves $\nu_1 \in \mathcal{M}_{\text{max}}(E)$. We can show $\nu_2 \in \mathcal{M}_{\text{max}}(E)$ similarly. Thus $\mathcal{M}_{\text{max}}(E)$ is a face.

### 7.3 Simulation irreducible measurement

We now introduce the simulation irreducibility of measurements, generalizing the finite-dimensional concept in [14].

**Definition 12** (Simulation irreducible measurement). A measurement $\omega \in \mathcal{M}(E)$ is said to be **simulation irreducible** if $\omega \in \text{sim}_{\text{str}}(\mathcal{L})$ implies $\omega \in \mathcal{L}$ for any subset $\mathcal{L} \subset \mathcal{M}(E)$. The set of simulation irreducible measurements in $\mathcal{M}(E)$ is denoted by $\mathcal{M}_{\text{irr}}(E)$.

A simulation irreducible measurement is a measurement that can be simulated only by itself.

Now we give equivalent characterizations of the simulation irreducibility.

**Proposition 22** (cf. [14], Proposition 5). For a measurement $\omega \in \mathcal{M}(E)$, the following conditions are equivalent.

(i) $\omega \in \mathcal{M}_{\text{max}}(E) \cap \partial_e \mathcal{M}(E)$.

(ii) For any subset $\mathcal{L} \subset \mathcal{M}(E)$, $\omega \in \text{sim}(\mathcal{L})$ implies $\omega \in \overline{\mathcal{L}}$, where the closure is with respect to the weak topology.
\((iii)\) \(\omega\) is simulation irreducible.

Specifically, \(M_{\text{str}}(E) = M_{\text{max}}(E) \cap \partial_{e} M(E)\) holds.

**Proof.** \((i) \implies (iii)\). Assume \((i)\) and suppose \(\omega \in \text{sim}(\mathcal{L})\) for some subset \(\mathcal{L} \subset M(E)\). Then there exists a measurement \(\nu \in \text{conv}(\mathcal{L})\) such that \(\omega \leq_{\text{post}} \nu\). Since \(\omega\) is maximal, this implies \(\omega = \nu \subset \text{conv}(\mathcal{L})\). By the extremality of \(\omega\), this implies that \(\omega\) is also an extremal point of \(\text{conv}(\mathcal{L})\). Hence by Milman’s partial converse to the Krein-Milman theorem ([19], II.10.5) we have \(\omega \in \overline{\mathcal{L}}\).

\((iii) \implies (i)\). Assume \((iii)\) and let \(\omega \in \text{sim}_{\text{str}}(\mathcal{L})\) for some subset \(\mathcal{L} \subset M(E)\). Then there exists a finite subset \(\mathcal{F} \subset \mathcal{L}\) such that \(\omega\) is a post-processing of a convex combination of elements of \(\mathcal{F}\). Thus \(\omega \in \text{sim}_{\text{str}}(\mathcal{F}) = \text{sim}(\mathcal{F})\). Then by assumption we have \(\omega \in \overline{\mathcal{F}} = \mathcal{F} \subset \mathcal{L}\), which proves the simulation irreducibility of \(\omega\).

\((iii) \implies (i)\). Assume \((iii)\). Let \(\omega \leq_{\text{post}} \omega' \in M(E)\). Then \(\omega \in \text{sim}_{\text{str}}(\{\omega'\})\) and the assumption implies \(\omega = \omega'\). Thus \(\omega \in M_{\text{max}}(E)\). To prove the extremality, take \(\omega_1, \omega_2 \in M(E)\) and \(\lambda \in (0, 1)\) such that

\[
\omega = \lambda \omega_1 + (1 - \lambda) \omega_2. \tag{36}
\]

Then \(\omega \in \text{sim}_{\text{str}}(\{\omega_1, \omega_2\})\) and the simulation irreducibility of \(\omega\) implies either \(\omega = \omega_1\) or \(\omega = \omega_2\). In both cases, from (36) we obtain \(\omega = \omega_1 = \omega_2\). Therefore \(\omega \in \partial_{e} M(E)\). \(\square\)

In [14], it is shown that a finite-outcome measurement \(\Gamma^M\) with \(M \in \text{EVM}(X; E)\) is simulation irreducible if and only if it is maximal and \(M\) is extremal in \(\text{EVM}(X; E)\), which is a condition different from our extremality in \(M(E)\). Indeed, for maximal measurements, we can show these two notions of extremality coincide as in the following proposition.

**Proposition 23.** Let \(\omega \in M_{\text{max}}(E)\) be a maximal measurement and let \(\Gamma \in \text{Ch}_{\omega^*}(F \to E)\) be a minimally sufficient representative of \(\omega\). Then \(\omega \in \partial_{e} M(E)\) if and only if \(\Gamma \in \partial_{e} \text{Ch}_{\omega^*}(F \to E)\).

**Proof.** Suppose \(\omega \in \partial_{e} M(E)\). Then by Theorem [11] and the uniqueness of the minimally sufficient measurement (Proposition [31]), \(\Gamma\) is injective. To show \(\Gamma \in \partial_{e} \text{Ch}_{\omega^*}(F \to E)\), take \(\lambda \in (0, 1)\) and \(\Gamma_1, \Gamma_2 \in \text{Ch}_{\omega^*}(F \to E)\) such that \(\Gamma = \lambda \Gamma_1 + (1 - \lambda) \Gamma_2\). Then from Proposition [11] we have

\[
\Gamma \preceq_{\text{post}} \lambda \Gamma_1 + (1 - \lambda) \Gamma_2
\]

and hence \(\omega = [\Gamma] \in \text{sim}_{\str}([\Gamma_1], [\Gamma_2])\). Therefore by Proposition [22] this implies \(\Gamma = [\Gamma_1] = [\Gamma_2]\). Thus there exist channels \(\Psi_j \in \text{Ch}(F \to F)\) \((j = 1, 2)\) such that \(\Gamma_j = \Gamma \circ \Psi_j\). Thus

\[
\Gamma = \lambda \Gamma_1 + (1 - \lambda) \Gamma_2 = \Gamma \circ (\lambda \Psi_1 + (1 - \lambda) \Psi_2).
\]

and the injectivity of \(\Gamma\) implies \(\lambda \Psi_1 + (1 - \lambda) \Psi_2 = \text{id}_F\). Hence by Lemma [15] we obtain \(\Psi_1 = \Psi_2 = \text{id}_F\). Therefore \(\Gamma_1 = \Gamma_2 = \Gamma\), which proves \(\Gamma \in \partial_{e} \text{Ch}_{\omega^*}(F \to E)\).

Conversely assume \(\Gamma \in \partial_{e} \text{Ch}_{\omega^*}(F \to E)\) and take \(\lambda \in (0, 1)\) and \(\omega_1, \omega_2 \in M(E)\) such that \(\omega = \lambda \omega_1 + (1 - \lambda) \omega_2\). Let \(\Lambda_j \in \text{Ch}_{\omega^*}(G_j \to E)\) be a representative of \(\omega_j\) \((j = 1, 2)\). Since \(\Gamma \sim_{\text{post}} \lambda \Lambda_1 + (1 - \lambda) \Lambda_2\), there exists a channel \(\Theta \in \text{Ch}_{\omega^*}(F \to G_1 \oplus G_2)\) such that

\[
\Gamma = (\lambda \Lambda_1 + (1 - \lambda) \Lambda_2) \circ \Theta.
\]

49
If we write as
\[ \Theta(a) = \Theta_1(a) \oplus \Theta_2(a) \quad (a \in F), \]
then \( \Theta_j \in \text{Ch}_{w*}(F \to G_j) \) \((j = 1, 2)\) and we have
\[ \Gamma = \lambda \Lambda_1 \circ \Theta_1 + (1 - \lambda) \Lambda_2 \circ \Theta_2. \]

Therefore the extremality of \( \Gamma \) in \( \text{Ch}_{w*}(F \to E) \) implies \( \Gamma = \Lambda_1 \circ \Theta_1 = \Lambda_2 \circ \Theta_2 \) and hence \( \omega \preceq_{\text{post}} \omega_1, \omega_2 \). Thus the maximality of \( \omega \) implies \( \omega = \omega_1 = \omega_2 \), which proves \( \omega \in \partial \mathcal{M}(E) \). \( \square \)

### 7.4 Simulability by simulation irreducible measurements

We now show that every measurement is simulable by the set of simulation irreducible measurements, generalizing the finite-dimensional results in [22, 14]. While the proof for the corresponding finite-dimensional result [22, 14] is constructive, the proof of the following theorem, a part of which is analogous to the common proof of the Krein-Milman theorem, is non-constructive and based on the well-ordering theorem and Theorem 9.

**Theorem 12.** \( \mathcal{M}(E) = \text{sim}(\mathcal{M}_{\text{irr}}(E)) \), i.e. every measurement in \( \mathcal{M}(E) \) is simulable by the set of simulation irreducible measurements.

**Proof.** By Theorem 9 we have only to prove that for any measurement \( \omega_0 \in \mathcal{M}(E) \) and an ensemble \( \mathcal{E} \in \text{Ens}(E) \) the inequality
\[ P_g(\mathcal{E}; \omega_0) \leq \sup_{\nu \in \mathcal{M}_{\text{irr}}(E)} P_g(\mathcal{E}; \nu) \]
holds, where \( \text{Ens}(E) \) is the set of ensembles defined in Proposition 15. Well-order \( \text{Ens}(E) \) so that \( \text{Ens}(E) = \{ \mathcal{E}_\alpha \mid 0 \leq \alpha < \gamma \} \) and \( \mathcal{E}_0 = \mathcal{E} \), where the index \( \alpha \) runs over all the ordinals smaller than the ordinal \( \gamma \). Define
\[ F_0 := \{ \omega \in \mathcal{M}(E) \mid P_g(\mathcal{E}_0; \omega) = \sup_{\nu \in \mathcal{M}(E)} P_g(\mathcal{E}_0; \nu) \}, \]
which is a non-empty compact face of \( \mathcal{M}(E) \). We then inductively define \((F_\alpha)_{0 \leq \alpha < \gamma}\) by
\[ F_\alpha := \{ \omega \in \bigcap_{0 \leq \beta < \alpha} F_\beta \mid P_g(\mathcal{E}_\alpha; \omega) = \sup_{\nu \in \mathcal{M}(E)} P_g(\mathcal{E}_\alpha; \nu) \} \quad (0 < \alpha < \gamma). \]
Then \((F_\alpha)_{0 \leq \alpha < \gamma}\) is a decreasing transfinite sequence of compact faces. Moreover, if \( F_\beta \neq \emptyset \) for all \( 0 \leq \beta < \alpha \), then, being the intersection of compact sets satisfying the finite-intersection property, \( \bigcap_{0 \leq \beta < \alpha} F_\beta \) is non-empty, and hence so is \( F_\alpha \). Thus by induction \( F_\alpha \neq \emptyset \) for all \( 0 \leq \alpha < \gamma \). Therefore \( F := \bigcap_{0 \leq \alpha < \gamma} F_\alpha \) is a non-empty compact face. If \( \omega_1, \omega_2 \in F \), then
\[ P_g(\mathcal{E}_\alpha; \omega_1) = \sup_{\nu \in \bigcap_{0 \leq \beta < \alpha} F_\beta} P_g(\mathcal{E}_\alpha; \nu) = P_g(\mathcal{E}_\alpha; \omega_2) \]
for all \( 0 \leq \alpha < \gamma \) and hence Theorem 11 implies \( \omega_1 = \omega_2 \). Therefore \( F \) is a singleton \( \{ \nu \} \).

Since \( F \) is a face in \( \mathcal{M}(E) \), \( \nu \) is an extremal point of \( \mathcal{M}(E) \). To show the maximality of \( \nu \), take a measurement \( \nu' \in \mathcal{M}(E) \) satisfying \( \nu \preceq_{\text{post}} \nu' \). Then we have
\[ P_g(\mathcal{E}_0; \nu') \leq \sup_{\nu \in \mathcal{M}(E)} P_g(\mathcal{E}_0; \nu) = P_g(\mathcal{E}_0; \nu) \leq P_g(\mathcal{E}_0; \nu'), \]
and hence \( \nu \in \partial \mathcal{M}(E) \). Therefore, \( \mathcal{M}(E) = \text{sim}(\mathcal{M}_{\text{irr}}(E)) \), i.e., every measurement in \( \mathcal{M}(E) \) is simulable by the set of simulation irreducible measurements.
Then the convex prestructure \(\mathcal{M}(E)\) equipped with the product topology of the weak topology on \(\mathcal{M}(E)\) is a compact convex structure.

8 Incompatibility and robustness of incompatibility

In this section, we consider incompatibility of measurements and generalizes some known results in finite dimensions [7, 51, 56]. The main result in this section is Theorem 14 that characterizes the operational meaning of the robustness of incompatibility.

8.1 Basic properties of (in)compatible measurements

Definition 13 (Compatibility and incompatibility of measurements). Let \(X \neq \emptyset\) be a set which may be finite or infinite. A family \((\Gamma_x)_{x \in X}\) of measurements is called compatible, or jointly measurable, if there exists a measurement \(\Lambda\) such that \(\Gamma_x \leq_{\text{post}} \Lambda\) for all \(x \in X\) and incompatible if not. Such a measurement \(\Lambda\), if exists, is called a mother measurement. We can always take \(\Lambda\) to be a w*-measurement by replacing \(\Lambda\) with the w*-extension \(\overline{\Lambda}\) if necessary.

A family of measurements \(\{\Gamma_x\}_{x \in X}\) in \(\mathcal{M}(E)\) is called (in)compatible if the family \((\Gamma_x)_{x \in X}\) of representatives is (in)compatible. Note that this definition does not depend on the choices of \(\Gamma_x\). We define the sets of compatible and incompatible measurements in \(\mathcal{M}(E)\) by

\[
\mathcal{M}_{\text{comp}}^X(E) := \{ (\omega_x)_{x \in X} \in \mathcal{M}(E)^X \mid \exists \nu \in \mathcal{M}(E), \left[ \omega_x \leq_{\text{post}} \nu \, (\forall x \in X) \right] \},
\]

\[
\mathcal{M}_{\text{incomp}}^X(E) := \mathcal{M}(E)^X \setminus \mathcal{M}_{\text{comp}}^X(E),
\]

respectively.

Before investigating the properties of the sets \(\mathcal{M}_{\text{comp}}^X(E)\) and \(\mathcal{M}_{\text{incomp}}^X(E)\), let us show that we can introduce a natural compact convex structure on the Cartesian power \(\mathcal{M}(E)^X\).

Proposition 24. Let \(X \neq \emptyset\) and define the convex combination map on \(\mathcal{M}(E)^X\) by

\[
\langle \lambda; (\omega_x)_{x \in X}, (\nu_x)_{x \in X} \rangle := (\lambda \omega_x + (1 - \lambda) \nu_x)_{x \in X} \quad (\lambda \in [0, 1]; \, \omega_x, \nu_x \in \mathcal{M}(E) \, (x \in X)).
\]

Then the convex prestructure \((\mathcal{M}(E)^X, \langle \cdot, \cdot, \cdot \rangle)\) equipped with the product topology of the weak topology on \(\mathcal{M}(E)\) is a compact convex structure.
Proof. By Tychonoff’s theorem, the product topology of the weak topology is a compact Hausdorff topology. Moreover the family of continuous affine functionals

$$\mathcal{M}(E)^X \ni (\omega_x')_{x' \in X} \mapsto P_\xi(\mathcal{E}; \omega_x) \in \mathbb{R} \quad (x \in X, \xi; \text{ ensemble})$$

separates points of \(\mathcal{M}(E)^X\). Therefore \((\mathcal{M}(E)^X, \langle \cdot, \cdot \rangle)\) is a compact convex structure. \(\square\)

Thus we identify \(\mathcal{M}(E)^X\) with the state space \(S(A_c(\mathcal{M}(E)^X))\) regularly embedded into \(A_c(\mathcal{M}(E)^X)^\ast\). We also define the post-processing partial order \(\preceq_{\text{post}}\) on \(\mathcal{M}(E)^X\) by the product order of \(\preceq_{\text{post}}\) on \(\mathcal{M}(E)\), i.e.

$$\omega_x \preceq_{\text{post}} \nu_x : \Longleftrightarrow [\omega_x \preceq_{\text{post}} \nu_x \quad (\forall x \in X)].$$

**Proposition 25.** Let \(X \neq \emptyset\). Then \(\mathcal{M}^X_{\text{comp}}(E)\) is a weakly compact, convex, lower subset of \(\mathcal{M}(E)^X\).

Proof. (Compactness). Let \((\omega^i_x)_{x \in X}\) (\(i \in I\)) be a net in \(\mathcal{M}^X_{\text{comp}}(E)\). Then for each \(i \in I\) we take \(\nu_i \in \mathcal{M}(E)\) such that \(\omega^i_x \preceq_{\text{post}} \nu_i \quad (x \in X)\). Since \(\mathcal{M}(E)^X \times \mathcal{M}(E)\) is compact in the product topology of the weak topology, there exist subnets \((\omega^i(j))_{x \in X}\) and \(\nu_i(j)\) (\(j \in J\)) and elements \((\omega_x)_{x \in X} \in \mathcal{M}(E)^X\) and \(\nu \in \mathcal{M}(E)\) such that \(\omega^i(j) \xrightarrow{\text{weakly}} \omega_x \quad (x \in X)\) and \(\nu_i(j) \xrightarrow{\text{weakly}} \nu\). Moreover, since \(\mathcal{M}(E)\) is a pospace, this implies \(\omega_x \preceq_{\text{post}} \nu \quad (x \in X)\) and hence \((\omega_x)_{x \in X} \in \mathcal{M}^X_{\text{comp}}(E)\), which proves the compactness of \(\mathcal{M}^X_{\text{comp}}(E)\).

(Convexity). Let \((\omega^1_x)_{x \in X}, (\omega^2_x)_{x \in X} \in \mathcal{M}^X_{\text{comp}}(E)\) and take measurements \(\nu_1, \nu_2 \in \mathcal{M}(E)\) such that \(\omega^j_x \preceq_{\text{post}} \nu_j \quad (j = 1, 2; x \in X)\). Then for each \(\lambda \in [0, 1]\) we have

$$\lambda \omega^1_x + (1 - \lambda) \omega^2_x \preceq_{\text{post}} \lambda \nu_1 + (1 - \lambda) \nu_2 \quad (x \in X),$$

which implies \((\lambda \omega^1_x + (1 - \lambda) \omega^2_x)_{x \in X} \in \mathcal{M}^X_{\text{comp}}(E)\).

(Lower set condition). Suppose \(\mathcal{M}(E)^X \ni (\nu_x)_{x \in X} \preceq_{\text{post}} (\omega_x)_{x \in X} \in \mathcal{M}^X_{\text{comp}}(E)\) and take a measurement \(\nu \in \mathcal{M}(E)\) satisfying \(\omega_x \preceq_{\text{post}} \nu \quad (x \in X)\). Then \(\nu_x \preceq_{\text{post}} \nu \quad (x \in X)\) and hence \((\nu_x)_{x \in X}\) is also compatible. \(\square\)

It is common to consider the (in)compatibility of finite family of measurements (e.g. [7]). The following proposition ensures that the (in)compatibility of arbitrary finite subfamilies sufficiently characterizes that of an infinite family of measurements.

**Proposition 26.** Let \(X \neq \emptyset\). Then \((\omega_x)_{x \in X} \in \mathcal{M}(E)^X\) is compatible if and only if \((\omega_x)_{x \in A}\) is compatible for any finite subset \(\emptyset \neq A \subset X\).

Proof. Let us denote the set of non-empty finite subsets of \(X\) by \(\mathcal{F}(X)\), which is directed by the set inclusion \(\subset\). Assume that \((\omega_x)_{x \in A}\) is compatible for any \(A \in \mathcal{F}(X)\). Then for each \(A \in \mathcal{F}(X)\) there exists \(\nu_A \in \mathcal{M}(E)\) such that \(\omega_x \preceq_{\text{post}} \nu_A \quad (x \in A)\). By the compactness of \(\mathcal{M}(E)\), there exists a subnet \((\nu_{A(i)})_{i \in I}\) of \((\nu_A)_{A \in \mathcal{F}(X)}\) weakly converging to some \(\nu \in \mathcal{M}(E)\). Then since \(\omega_x \preceq_{\text{post}} \nu_{A(i)}\) eventually for each \(x \in X\), the pospace property of \(\mathcal{M}(E)\) implies \(\omega_x \preceq_{\text{post}} \nu\) for each \(x \in X\), which proves the “if” part of the claim. The “only if” part is obvious. \(\square\)
8.2 Outperformance in the state discrimination task

Let \((\omega_x)_{x \in X} \in \mathcal{M}(E)^X\) and let \(\mathcal{L} := \{\omega_x \in \mathcal{M}(E) \mid x \in X\}\). For each w*-family \(\mathcal{E}\) and partitioned ensemble \(\overrightarrow{\mathcal{E}} = (\mathcal{E}_z)_{z \in Z}\) with \(\mathcal{E}_z = (\varphi_{z,y})_{y \in Y_z}\), we define

\[
\begin{align*}
P_g(\mathcal{E}; (\omega_x)_{x \in X}) &:= P_g(\mathcal{E}; \mathcal{L}), \\
P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X}) &:= P_g(\overrightarrow{\mathcal{E}}; \mathcal{L}).
\end{align*}
\]

For a family \((\Gamma_x)_{x \in X}\) of measurements, we define

\[
\begin{align*}
P_g(\mathcal{E}; (\Gamma_x)_{x \in X}) &:= P_g(\mathcal{E}; ([\Gamma_x])_{x \in X}) = P_g(\mathcal{E}; \mathcal{L}'), \\
P_g(\overrightarrow{\mathcal{E}}; (\Gamma_x)_{x \in X}) &:= P_g(\overrightarrow{\mathcal{E}}; ([\Gamma_x])_{x \in X}) = P_g(\overrightarrow{\mathcal{E}}; \mathcal{L}'),
\end{align*}
\]

where \(\mathcal{L}' := \{[\Gamma_x] \in \mathcal{M}(E) \mid x \in X\}\). We also define

\[
P_g^{\text{comp}}(\overrightarrow{\mathcal{E}}) := \sup_{Y: \text{set}; (\nu_y)_{y \in Y} \in \mathcal{M}(E)_{\text{comp}}} P_g(\overrightarrow{\mathcal{E}}; (\nu_y)_{y \in Y}) = \sup_{\nu \in \mathcal{M}(E)} \sum_{z \in Z} P_g(\mathcal{E}_z; \nu),
\]

where the second equality follows from the post-processing monotonicity of the gain functional.

The operational meaning of the quantity \(P_g^{\text{comp}}(\overrightarrow{\mathcal{E}})\) is as follows \cite{7, 51}. Suppose that Alice prepares system’s state according to the ensemble \((\varphi_{z,y})_{z \in Z, y \in Y_z}\). Then Bob performs the measurement on the system and, after the measurement, Alice informs Bob of the value of \(z \in Z\). This is in contrast to the operational setting for \(P_g(\mathcal{E}; \mathcal{L})\) in which Bob is informed of the label \(x \in X\) before he performs the measurement and can choose a proper measurement from \(\mathcal{L}\) to which incompatible measurements may belong. We now show that an incompatible family of measurements outperforms in this state discrimination task for some partitioned ensemble, generalizing the result for finite-dimensional quantum systems in \cite{12} (Theorem 2).

**Theorem 13.** Let \(X \neq \emptyset\) and let \((\omega_x)_{x \in X} \in \mathcal{M}(E)^X\). Then \((\omega_x)_{x \in X} \in \mathcal{M}_{\text{comp}}(E)\) if and only if

\[
P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X}) \leq P_g^{\text{comp}}(\overrightarrow{\mathcal{E}})
\]

holds for all partitioned ensemble \(\overrightarrow{\mathcal{E}}\).

For the proof of the theorem, we first consider the set of compatible EVMs. For a family \((Y_x)_{x \in X}\) of finite sets indexed by a finite set \(X\), we define

\[
\mathbf{EVM}((Y_x)_{x \in X}; E) := \prod_{x \in X} \mathbf{EVM}(Y_x; E) \subset \prod_{x \in X} E^{Y_x},
\]

\[
\mathbf{EVM}_{\text{comp}}((Y_x)_{x \in X}; E) := \{ (M_x)_{x \in X} \in \mathbf{EVM}((Y_x)_{x \in X}; E) \mid (\Gamma^{M_x})_{x \in X} \text{ is compatible} \}.
\]

A family \((M_x)_{x \in X}\) EVMs is called (in)compatible if the family \((\Gamma^{M_x})_{x \in X}\) of the associated measurements is (in)compatible.
Lemma 17. Let $X$ and $Y_x$ ($x \in X$) be non-empty finite sets. Then $\operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$ is a weakly* compact (i.e. $\sigma(\prod_{x \in X} E^Y_x, \prod_{x \in X} E^*_x)$-compact) convex subset of $\operatorname{EVM}((Y_x)_{x \in X}; E)$.

Proof. (Compactness). By Lemma 8.11 the map

$$\operatorname{EVM}((Y_x)_{x \in X}; E) \ni (M_x)_{x \in X} \mapsto ([\Gamma^{M_x}])_{x \in X} \in \mathcal{M}(E)^X$$

(38)

is continuous with respect to $\sigma(\prod_{x \in X} E^Y_x, \prod_{x \in X} E^*_x)$ and the product topology of the weak topology on $\mathcal{M}(E)$. Since $\operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$ is the inverse image of the compact set $\mathcal{M}^X_{\text{comp}}(E)$ under the map (38) and $\operatorname{EVM}((Y_x)_{x \in X}; E)$ is compact, $\operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$ is also compact.

(Convexity). Let $\tilde{M}^j = (M^j_x)_{x \in X} \in \operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$ ($j = 1, 2$). Then by Proposition 10 for each $\lambda \in [0, 1]$,

$$([\lambda M^1_x + (1-\lambda)M^2_x])_{x \in X} \leq_{\text{post}} ([\lambda M^1_x] + (1-\lambda)[M^2_x])_{x \in X} \in \mathcal{M}^X_{\text{comp}}(E).$$

Since $\mathcal{M}^X_{\text{comp}}(E)$ is a lower set by Proposition 26, this implies

$$(\lambda M^1_x + (1-\lambda)M^2_x)_{x \in X} \in \operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E),$$

which proves the convexity. \qed

Lemma 18. Let $\mathcal{E} = (\mathcal{E}_x)_{x \in X}$ be a partitioned ensemble with $\mathcal{E}_x = (\varphi_{x,y})_{y \in Y_x}$. Then

$$P^\text{comp}_g(\mathcal{E}) = \sup_{(N_x)_{x \in X} \in \operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)} \sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, N_x(y) \rangle.$$ (39)

Proof. For any family $(N_x)_{x \in X} \in \operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$ of compatible EVMs, we have

$$\sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, N_x(y) \rangle \leq \sum_{x \in X} P_g(\mathcal{E}_x; [\Gamma^{N_x}]) \leq P^\text{comp}_g(\mathcal{E}),$$

which implies (LHS) $\leq$ (RHS) of (39). Conversely for any $\omega \in \mathcal{M}(E)$ with the representative $\Gamma \in \mathcal{C}h_{w^*}(F \rightarrow E)$, for each $x \in X$ we can take $M_x \in \operatorname{EVM}(Y_x; F)$ such that

$$P_g(\mathcal{E}_x; \omega) = \sum_{y \in Y_x} \langle \varphi_{x,y}, \Gamma(M_x(y)) \rangle.$$

Therefore we have

$$P_g(\mathcal{E}; \omega) = \sum_{x \in X} P_g(\mathcal{E}_x; \omega) = \sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, \Gamma(M_x(y)) \rangle.$$

Since $(\Gamma \circ M_x)_{x \in X} \in \operatorname{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$, this implies (LHS) $\leq$ (RHS) of (39). \qed

Proof of Theorem 13. The “only if” part of the claim is obvious from the definition of $P^\text{comp}_g(\mathcal{E})$. To show the “if” part, we assume $(\omega_x)_{x \in X} \in \mathcal{M}^X_{\text{incomp}}(E)$ and find a partitioned ensemble $\mathcal{E}$ violating (37). By Proposition 26 there exists a finite subset $X_0 \subset X$ such that
\((\omega_x)_{x \in X_0} \in \mathfrak{M}_{\text{incomp}}^X(E)\). If we can find a partitioned ensemble \(\overrightarrow{\mathcal{E}}\) such that \(P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X_0}) > P_g^{\text{comp}}(\overrightarrow{\mathcal{E}})\), then
\[
P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X}) \geq P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X_0}) > P_g^{\text{comp}}(\overrightarrow{\mathcal{E}}).
\]
Therefore we may assume that \(X\) is finite.

We first assume that every \(\omega_x (x \in X)\) is finite-outcome and \(\omega_x = [\Gamma^{M_x}]\) for some finite-outcome EVM \(M_x \in EVM(Y_x; E)\). Then since \((M_x)_{x \in X} \notin EVM_{\text{comp}}((Y_x)_{x \in X}; E)\) and \(EVM_{\text{comp}}((Y_x)_{x \in X}; E)\) is convex and weakly compact, the Hahn-Banach separation theorem implies that there exist weakly* continuous linear functionals \(\varphi_{x,y} \in E^*_s (x \in X, y \in Y_x)\) such that
\[
\sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, M_x(y) \rangle > \sup_{(N_x)_{x \in X} \in EVM_{\text{comp}}((Y_x)_{x \in X}; E)} \sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, N_x(y) \rangle.
\]
Similarly as in Proposition \[25\] we can take \((\varphi_{x,y})_{x \in X, y \in Y_x}\) so that \(\overrightarrow{\mathcal{E}} := (\mathcal{E}_x)_{x \in X}\) with \(\mathcal{E}_x := (\varphi_{x,y})_{y \in Y_x}\) is a partitioned ensemble. Then
\[
P_g(\overrightarrow{\mathcal{E}}; (\Gamma^{M_x})_{x \in X}) \geq \sum_{x \in X} P_g(\mathcal{E}_x, \Gamma^{M_x})
\]
\[
\geq \sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, M_x(y) \rangle
\]
\[
> \sup_{(N_x)_{x \in X} \in EVM_{\text{comp}}((Y_x)_{x \in X}; E)} \sum_{x \in X} \sum_{y \in Y_x} \langle \varphi_{x,y}, N_x(y) \rangle
\]
\[
= P_g^{\text{comp}}(\overrightarrow{\mathcal{E}}),
\]
where the last equality follows from Lemma \[18\]. Therefore \(\overrightarrow{\mathcal{E}}\) violates \[37\].

We now consider general \((\omega_x)_{x \in X} \in \mathfrak{M}_{\text{incomp}}^X(E)\). Then by Theorem \[1\] there exists a post-processing increasing net \((\omega^{i}_x)_{x \in X} (i \in I)\) of finite-outcome measurements such that \(\omega^{i}_x \overset{\text{weakly}}{\longrightarrow} \sup_{i \in I} \omega^{i}_x = \omega_x\) for each \(x \in X\). Since \(\mathfrak{M}_{\text{incomp}}^X(E)\) is an open subset of \(\mathfrak{M}(E)^X\) by Proposition \[24\] there is some \(i \in I\) such that \((\omega^i_x)_{x \in X} \in \mathfrak{M}_{\text{incomp}}^X(E)\). Then from what we have shown in the last paragraph, there exists a partitioned ensemble \(\overrightarrow{\mathcal{E}}\) such that \(P_g(\overrightarrow{\mathcal{E}}; (\omega^i_x)_{x \in X}) > P_g^{\text{comp}}(\overrightarrow{\mathcal{E}})\). Since we have \(P_g(\overrightarrow{\mathcal{E}}; (\omega_x)_{x \in X}) \geq P_g(\overrightarrow{\mathcal{E}}; (\omega^i_x)_{x \in X})\) by the monotonicity of \(P_g(\overrightarrow{\mathcal{E}}; \cdot)\), we can readily see that \(\overrightarrow{\mathcal{E}}\) violates \[37\].

**Corollary 7.** Let \(X \neq \emptyset\) and let \((\Gamma_x)_{x \in X}\) be a family of measurements. Then \((\Gamma_x)_{x \in X}\) is compatible if and only if \(P_g(\overrightarrow{\mathcal{E}}; (\Gamma_x)_{x \in X}) \leq P_g^{\text{comp}}(\overrightarrow{\mathcal{E}})\) for any partitioned ensemble \(\overrightarrow{\mathcal{E}}\).

**Proof.** Let \(\overline{\Gamma}_x\) denote the w*-extension of \(\Gamma_x\). Then by Proposition \[9\] it holds that \((\overline{\Gamma}_x)_{x \in X}\) is compatible if and only if \((\overline{\Gamma}_x)_{x \in X}\) is compatible. Moreover we have \(P_g(\overrightarrow{\mathcal{E}}; (\Gamma_x)_{x \in X}) = P_g(\overrightarrow{\mathcal{E}}; (\overline{\Gamma}_x)_{x \in X})\) by Lemma \[3\] Thus the claim immediately follows from Theorem \[13\].

### 8.3 Robustness of incompatibility

We now define the robustness of incompatibility \[21\] \[51\] \[56\] \[8\].
**Definition 14 (Robustness of incompatibility).** Let $X \neq \emptyset$ and let $\Gamma_x \in \text{Ch}(F_x \to E)$ ($x \in X$) be measurements. Then we define the robustness of incompatibility by

$$R_{\text{inc}}((\Gamma_x)_{x \in X}) := \inf_{r, (\Lambda_x)_{x \in X}} r$$

subject to $r \in [0, \infty)$

$$(\Lambda_x)_{x \in X} \in \prod_{x \in X} \text{Ch}(F_x \to E)$$

$$\left(\frac{\Gamma_x + r \Lambda_x}{1 + r}\right)_{x \in X} \text{ is compatible},$$

which coincides with

$$\inf_{r, (\Psi_x)_{x \in X}} r$$

subject to $r \in [0, \infty)$

$$(\Psi_x)_{x \in X} \in \prod_{x \in X} \text{Ch}(F_x \to E)$$

$$(\Psi_x)_{x \in X} \text{ is compatible}$$

$$\Gamma_x \leq (1 + r)\Psi_x \quad (\forall x \in X).$$

Here $R_{\text{inc}}((\Gamma_x)_{x \in X}) := \infty$ if the feasible region is empty.

The robustness $R_{\text{inc}}((\Gamma_x)_{x \in X})$ quantifies the minimal amount of noise which should be added to the family $(\Gamma_x)_{x \in X}$ of measurements to make it compatible.

We now prove the main result of this section that the robustness of incompatibility coincides with the maximal relative increase in the state discrimination probability of a partitioned ensemble compared to compatible measurements, generalizing the result in [51, 56] for finite-dimensional quantum systems.

**Theorem 14.** In the setting of Definition 14, the equality

$$1 + R_{\text{inc}}((\Gamma_x)_{x \in X}) = \sup_{\vec{\mathcal{E}}: \text{partitioned ensemble}} \frac{P_g(\vec{\mathcal{E}}; (\Gamma_x)_{x \in X})}{P_g^{\text{comp}}(\vec{\mathcal{E}})}$$

holds, where the supremum is taken over all the partitioned ensembles.

While the following proof of Theorem 14 is almost parallel to those of Theorem 10 and the previous work [51], we give it here for completeness.

We first establish some elementary properties of the set of compatible measurements with fixed outcome spaces and those of the robustness measure.

Let $(F_x)_{x \in X}$ be a family of classical spaces. We regard the product set $\prod_{x \in X} \text{Ch}(F_x \to E)$ as a compact convex set by considering the direct product topology of the BW-topologies on $\text{Ch}(F_x \to E)$ and the convex operation

$$\lambda(\Gamma_x)_{x \in X} + (1 - \lambda)(\Lambda_x)_{x \in X} = (\lambda\Gamma_x + (1 - \lambda)\Lambda_x)_{x \in X}$$

($\lambda \in [0, 1]; \Gamma_x, \Lambda_x \in \text{Ch}(F_x \to E)$ ($x \in X$)). We also denote by $\text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)$ the set of compatible measurements in $\prod_{x \in X} \text{Ch}(F_x \to E)$.
Lemma 19. Let $X \neq \emptyset$, let $(F_x)_{x \in X}$ be a family of classical spaces, and let $(\Gamma_x)_{x \in X} \in \prod_{x \in X} \text{Ch}(F_x \to E)$.

1. The set $\text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)$ is a compact convex subset of $\prod_{x \in X} \text{Ch}(F_x \to E)$.

2. If $X$ is finite, then $r := R_{\text{inc}}((\Gamma_x)_{x \in X}) < \infty$.

3. If $R_{\text{inc}}((\Gamma_x)_{x \in X}) < \infty$, then there exists a compatible family $(\Psi_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)$ such that $\Gamma_x \leq (1 + r)\Psi_x \quad (\forall x \in X)$.

4. $R_{\text{inc}}((\Gamma_x)_{x \in X}) \leq R_{\text{inc}}((\Gamma_x)_{x \in X})$ for any subset $\emptyset \neq Y \subset X$.

5. $R_{\text{inc}}((\Gamma_x)_{x \in X}) = \sup_{A \in F(X)} R_{\text{inc}}((\Gamma_x)_{x \in A})$, where $F(X)$ denotes the set of non-empty finite subsets of $X$.

Proof. 1. (Compactness). Let $(\Gamma^i_x)_{x \in X}$ $(i \in I)$ be a net in $\text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)$ such that $\Gamma^i_x \xrightarrow{\text{BW}} \Gamma_x \in \text{Ch}(F_x \to E)$ $(x \in X)$. Then by the compatibility and Theorem 11 for each $i \in I$ there exists a measurement $\nu_i \in \mathcal{M}(E)$ such that

$$P_g(\mathcal{E}; \Gamma^i_x) \leq P_g(\mathcal{E}; \nu_i) \quad (x \in X)$$

for any ensemble $\mathcal{E}$. We take a subnet $(\nu_i(j))_{j \in J}$ weakly converging to some $\nu \in \mathcal{M}(E)$. Then for any $x \in X$, any ensemble $\mathcal{E} = (\varphi_z)_{z \in Z}$, and any EVM $M \in \text{EVM}(Z; F_x)$, we have

$$\sum_{z \in Z} \langle \varphi_z, \Gamma_x(M(z)) \rangle = \lim_{j \in J} \sum_{z \in Z} \langle \varphi_z, \Gamma^i_x(j)(M(z)) \rangle \leq \limsup_{j \in J} P_g(\mathcal{E}; \Gamma^i_x(j)) \leq \limsup_{j \in J} P_g(\mathcal{E}; \nu_i(j)) = P_g(\mathcal{E}; \nu).$$

By taking the supremum of $M$, we obtain $P_g(\mathcal{E}; \Gamma_x) \leq P_g(\mathcal{E}; \nu)$. Since the ensemble $\mathcal{E}$ is arbitrary, Theorem 2 implies $[\Gamma_x] \preceq_{\text{post}} \nu$. Therefore $(\Gamma_x)_{x \in X}$ is compatible, which proves the compactness of $\text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)$.

( Convexity). The convexity can be shown analogously as in Lemma 17 by using Propositions 10 and 25.

2. Define $\Psi'_x \in \text{Ch}(F_x \to E)$ $(x \in X)$ by

$$\Psi'_x(a) := \frac{\Gamma_x(a) + (|X| - 1)\phi_x(a)u_E}{|X|} \quad (a \in F_x),$$

where $\phi_x \in S(F_x)$ is a fixed state. We show that $(\Psi'_x)_{x \in X}$ is compatible, from which $R_{\text{inc}}((\Gamma_x)_{x \in X}) \leq |X| - 1 < \infty$ follows. Define $\Phi \in \text{Ch}(\bigoplus_{x \in X} F_x \to E)$ and $\Theta_x \in$
By assumption there are channels \( \Phi \) such that \( \Phi \circ \Theta \) is compatible, there exists a compatible family \( (\Psi_{x, x' \in X})_{x' \in X} \). Then by Lemma 19, there exists a compatible family \( (\Psi_{x, x' \in X})_{x' \in X} \). Since \( (\Psi_{x, x' \in X})_{x' \in X} \) is compatible, this implies \( \Phi \circ \Theta \) is post-

Lemma 20. Let \( X \neq \emptyset \) and let \( F_j^i (j = 1, 2; x \in X) \) be classical spaces. Then for any families \( (\Gamma_j^i)_{x \in X} \) such that \( \Gamma_j^1 \leq_{\text{post}} \Gamma_j^2 \) for \( x \in X \), it follows that there exists a compatible family \( (\Psi_{x, x' \in X})_{x' \in X} \) such that \( \Phi \circ \Theta \) is compatible, this implies \( \Phi \circ \Theta \) is post-

3. Since \( r = R_{\text{inc}}((\Gamma_{x \in X})_{x \in X}) < \infty \), there exists a sequence \( (r_n, (\Psi_{x, x' \in X})_{x \in X})_{n \in \mathbb{N}} \) such that \( r_n \downarrow r \), \( \Gamma_x \leq (1 + r_n)\Psi_{x, x' \in X} \), and \( (\Psi_{x, x' \in X})_{x \in X} \in \mathbf{Ch}^{\text{comp}}((F_{x \in X}; E) (x \in X; n \in \mathbb{N})). By the compactness of \( \mathbf{Ch}^{\text{comp}}((F_{x \in X}; E) \) there exists a subnet \( (\Psi_{x, x' \in X})_{x' \in X} (i \in I) \) converging to some \( (\Psi_{x, x' \in X})_{x \in X} \in \mathbf{Ch}^{\text{comp}}((F_{x \in X}; E) \). Then we have \( \Gamma_x \leq (1 + r)\Psi_{x} \) for \( x \in X \), which proves the result.

4. For simplicity we write as \( r_A := R_{\text{inc}}((\Gamma_{x \in X})_{x \in A}) \) for each subset \( A \subset X \). Without loss of generality we may assume \( r_A < \infty \). Then by the claim \( \Phi \circ \Theta \) is compatible, this implies \( \Phi \circ \Theta \) is post-

5. From the claim \( \Phi \circ \Theta \) is compatible, this implies \( \Phi \circ \Theta \) is post-

Lemma 20. Let \( X \neq \emptyset \) and let \( F_j^i (j = 1, 2; x \in X) \) be classical spaces. Then for any families \( (\Gamma_j^i)_{x \in X} \) such that \( \Gamma_j^1 \leq_{\text{post}} \Gamma_j^2 \) for \( x \in X \), it follows that there exists a compatible family \( (\Psi_{x, x' \in X})_{x \in X} \) such that \( \Phi \circ \Theta \) is compatible, this implies \( \Phi \circ \Theta \) is post-

Proof. We write as \( r_j := R_{\text{inc}}((\Gamma_j^i)_{x \in X}) \). Without loss of generality we may assume \( r_2 < \infty \). Then by Lemma 19, there exists a compatible family \( (\Psi_{x, x' \in X})_{x \in X} \) such that \( \Gamma_x \leq (1 + r_2)\Psi_{x} \). Then \( (\Phi \circ \Theta) \) implies \( \Gamma_x \leq (1 + r_2)\Psi_{x} \). Since \( (\Psi_{x, x' \in X})_{x \in X} \) is compatible, this implies \( r_1 \leq r_2 \).

58
Lemma 21. Let \( X \neq \emptyset \), let \( \Gamma_x \in \text{Ch}(F_x \to E) \) (\( x \in X \)) be measurements, and let \( \Gamma_x \in \text{Ch}_{w^*}(F_{x}^{**} \to E) \) be the \( w^* \)-extension of \( \Gamma_x \). Then

\[
R_{\text{inc}}((\Gamma_x)_{x \in X}) = R_{\text{inc}}((\Gamma_x)_{x \in X}).
\]

Proof. By Lemma 20 we have \( R_{\text{inc}}((\Gamma_x)_{x \in X}) \leq R_{\text{inc}}((\Gamma_x)_{x \in X}) \). We prove the converse inequality. Without loss of generality we may assume \( R_{\text{inc}}((\Gamma_x)_{x \in X}) < \infty \). Then by Lemma 19 we can take a compatible family \( (\Psi_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_{x}^{**})_{x \in X}; E) \) such that

\[
\Gamma_x \leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X}))\Psi_x \quad (x \in X).
\]

Let \( \Psi_x \in \text{Ch}_{w^*}(F_{x}^{**} \to E) \) be the \( w^* \)-extension of \( \Psi_x \) (\( x \in X \)). Then

\[
(\Psi_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_{x}^{**})_{x \in X}; E).
\]

Furthermore, since \( E_{+}^{**} \) is closed in the weak* topology \( \sigma(E^{**}, E^*) \), we have

\[
\Gamma_x \leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X}))\Psi_x,
\]

which implies \( R_{\text{inc}}((\Gamma_x)_{x \in X}) \geq R_{\text{inc}}((\Gamma_x)_{x \in X}) \).

We now show \( \text{(LHS)} \geq \text{(RHS)} \) in (40).

Lemma 22. In the setting of Theorem 14, the inequality

\[
1 + R_{\text{inc}}((\Gamma_x)_{x \in X}) \geq \sup_{\mathcal{E}^*: \text{partitioned ensemble}} \frac{P_g(\mathcal{E}^*: (\Gamma_x)_{x \in X})}{P_g^{\text{comp}}(\mathcal{E}^*)}
\]

holds.

Proof. Without loss of generality, we may assume \( R_{\text{inc}}((\Gamma_x)_{x \in X}) < \infty \). Then by Lemma 19 there exists a compatible family \( (\Psi_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_{x}^{**})_{x \in X}; E) \) such that

\[
\Gamma_x \leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X}))\Psi_x \quad (x \in X).
\]

Take an arbitrary partitioned ensemble \( \hat{\mathcal{E}}^* = (\mathcal{E}_y)_{y \in Y} \) with \( \mathcal{E}_y = (\varphi_{y, z})_{z \in Z_y} \). Then for any EVMs \( M_{x, y} \in \text{EVM}(Z_y; F_x) \) (\( x \in X, y \in Y \)) we have

\[
\begin{align*}
&\sup_{x \in X} \sum_{y \in Y} \sum_{z \in Z_y} \langle \varphi_{y, z}, \Gamma_x(M_{x, y}(z)) \rangle \\
&\leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X})) \sup_{x \in X} \sum_{y \in Y} \sum_{z \in Z_y} \langle \varphi_{y, z}, \Psi_x(M_{x, y}(z)) \rangle \\
&\leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X})) \sup_{x \in X} P_g(\mathcal{E}^*: \Psi_x) \\
&\leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X})) P_g^{\text{comp}}(\hat{\mathcal{E}}^*).
\end{align*}
\]

By taking the supremum of \( M_{x, y} \), we obtain

\[
P_g(\mathcal{E}^*: (\Gamma_x)_{x \in X}) \leq (1 + R_{\text{inc}}((\Gamma_x)_{x \in X})) P_g^{\text{comp}}(\hat{\mathcal{E}}^*),
\]

from which (42) follows. \hfill \Box
We now prove the theorem when $X$ is finite and each $\Gamma_x$ is finite-outcome.

**Lemma 23** (cf. [51]). The statement of Theorem 14 is true when $|X| < \infty$ and $\Gamma_x = \Gamma_{M_x}$ for some finite-outcome $EVM$ $M_x \in EVM(Y_x; E)$ ($x \in X$).

**Proof.** By the one-to-one correspondence between $\text{Ch}(\ell_\infty(Y_x) \to E)$ and $EVM(Y_x; E)$, the robustness measure $R_{\text{inc}}((\Gamma_{M_x})_{x \in X})$ can be written as

$$R_{\text{inc}}((\Gamma_{M_x})_{x \in X}) = \inf_{r,(N_x)_{x \in X}} r$$

subject to $r \in [0, \infty)$, $(N_x)_{x \in X} \in \text{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)$

$$M_x(y) \leq (1 + r)N_x(y) \quad (x \in X, y \in Y_x).$$

Define

$$K := \{ (\lambda N_x)_{x \in X} \mid \lambda \in [0, \infty), (N_x)_{x \in X} \in \text{EVM}_{\text{comp}}((Y_x)_{x \in X}; E) \}.$$ 

It can be shown similarly as in Lemma 13 that $K$ is a weakly* closed convex cone in $\prod_{x \in X} E^{Y_x}$. Then we have

$$1 + R_{\text{inc}}((\Gamma_{M_x})_{x \in X}) = \inf_{s,(N_x)_{x \in X}} s$$

subject to $s \in \mathbb{R}$, $(N_x)_{x \in X} \in K$

$$\sum_{y \in Y_x} N_x(y) \leq su_E \quad (x \in X)$$

$$M_x(y) \leq N_x(y) \quad (x \in X, y \in Y_x).$$

The optimization problem (43) can be written in the standard form (27) of the conic programming by putting

$$V := \left( \prod_{x \in X} E^{Y_x} \right) \times \mathbb{R}, \quad U := \left( \prod_{x \in X} E^{Y_x} \right) \times E^X,$$

$$C := K \times \mathbb{R}, \quad K := \left( \prod_{x \in X} (E_+^{Y_x}) \right) \times (E_+^X),$$

$$\langle c^*, (w, s) \rangle := s \quad ((w, s) \in V),$$

$$b := ((-M_x)_{x \in X}, (0)_{x \in X}) \in U,$$

$$A: V \ni ((w_{x,y})_{x \in X, y \in Y_x}, s) \mapsto \left( (w_{x,y})_{x \in X, y \in Y_x}, \left( su_E - \sum_{y \in Y_x} w_{x,y} \right)_{x \in X} \right) \in U,$$

where

$$((w_{x,y})_{x \in X, y \in Y_x})_{x \in X} \in V.$$ 

The convex cones $C$ and $K$ are weakly* closed in $V$ and $U$, respectively. Let $v_0 := \left( (|Y_x|^{-1} u_X)_{x \in X, y \in Y_x}, 2 \right) \in V$. Since the family $\left( (|Y_x|^{-1} u_E)_{y \in Y_x} \right)_{x \in X}$ of trivial observables is
compatible, we have $v_0 \in C$. Moreover

$$U \supset A(C) - K \supset \mathbb{R}_+ A(v_0) - K = \{ (\lambda |Y_x|^{-1} u_E - w_{x,y})_{x \in X, y \in Y}, (\lambda u_E - w'_{x,y})_{x \in X} \mid 
\lambda \in \mathbb{R}_+, w_{x,y}, w'_x \in E_+ \ (x \in X, y \in Y) \} = U,$$

which implies $-b \in \text{int}(A(C) - K)(= U)$. Therefore the optimal value of \((43)\) coincides with its dual problem \((28)\) with

$$K^* = \left( \prod_{x \in X} (E^*_x)_{Y_x} \right) \times (E^*_x)^X, \quad C^* = K^* \times \{0\},$$

$$\mathcal{K}^* = \{ (\omega_{x,y})_{x \in X, y \in Y} \in \prod_{x \in X} (E^*_x)_{Y_x} \mid \sum_{x \in X, y \in Y_x} \langle \omega_{x,y}, G_{x,y} \rangle \geq 0 (\forall (G_{x,y})_{x \in X, y \in Y} \in \mathcal{K}) \},$$

$$A^*((\psi_{x,y})_{x \in X, y \in Y}, (\chi_x)_{x \in X}) = \left( \psi_{x,y} - \chi_x \right)_{x \in X \times Y}, \sum_{x \in X} \langle \chi_x, u_E \rangle > \psi_{x,y}, \chi_x \in E^*.\right)$$

Therefore the dual problem can be written as

$$1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) = \sup_{(\psi_{x,y})_{x \in X, y \in Y}, (\chi_x)_{x \in X}} \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}, M_x(y) \rangle$$

subject to $\psi_{x,y}, \chi_x \in E^+_x \ (x \in X, y \in Y)$

$$(\chi_x - \psi_{x,y})_{x \in X, y \in Y} \in \mathcal{K}^*,$$

$$\sum_{x \in X} \langle \chi_x, u_E \rangle = 1,$$

which coincides with

$$1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) = \sup_{(\psi_{x,y})_{x \in X, y \in Y}, (\chi_x)_{x \in X}} \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}, M_x(y) \rangle$$

subject to $\psi_{x,y}, \chi_x \in E^*_x \ (x \in X, y \in Y)$

$$(\chi_x - \psi_{x,y})_{x \in X, y \in Y} \in \mathcal{K}^*,$$

$$\sum_{x \in X} \langle \chi_x, u_E \rangle \leq 1.\) \quad (44)$$

We next show that the feasible region of \((44)\) can be restricted to the weakly* functionals, i.e.

$$1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) = \sup_{(\psi_{x,y})_{x \in X, y \in Y}, (\chi_x)_{x \in X}} \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}, M_x(y) \rangle$$

subject to $\psi_{x,y}, \chi_x \in E^*_x \ (x \in X, y \in Y)$

$$(\chi_x - \psi_{x,y})_{x \in X, y \in Y} \in \mathcal{K}^*,$$

$$\sum_{x \in X} \langle \chi_x, u_E \rangle \leq 1.\) \quad (45)$$
For this we have only to show that the feasible region of (45) is weakly* dense in that of (44). An element \((\psi_{x,y})_{x \in X, y \in Y_x}, (x)_{x \in X}) \in \left(\prod_{x \in X} (E^*)^{Y_x}\right) \times (E^*)^X\) is in the feasible region of (44) if and only if
\[
-1 \leq \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}, a_{x,y} \rangle + \sum_{x \in X} \langle x, b_x \rangle + \sum_{x \in X, y \in Y_x} \langle x - \psi_{x,y}, G_{x,y} \rangle - \sum_{x \in X} \langle x, u_E \rangle
\]
\[
= \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}, a_{x,y} - G_{x,y} \rangle + \sum_{x \in X} \langle x, b_x + \sum_{y \in Y_x} G_{x,y} - u_E \rangle
\]
\[(\forall a_{x,y}, b_x \in E_+; \forall (G_{x,y})_{x \in X, y \in Y_x} \in \mathcal{K}).\]

Therefore if we define
\[
\mathcal{L} := \{ (a_{x,y} - G_{x,y})_{x \in X, y \in Y_x}, (b_x + \sum_{y \in Y_x} G_{x,y} - u_E)_{x \in X} \} | a_{x,y}, b_x \in E_+ (x \in X, y \in Y_x); (G_{x,y})_{x \in X, y \in Y_x} \in \mathcal{K} \},
\]
then \(\mathcal{L}\) is a convex subset of \(\left(\prod_{x \in X} E^{Y_x}\right) \times E^X\) containing the origin and the polar of \(\mathcal{L}\) in the pair \((\prod_{x \in X} E^{Y_x}) \times E^X, \left(\prod_{x \in X} (E^*)^{Y_x}\right) \times (E^*)^X\) coincides with the feasible region of (44). Similarly the polar of \(\mathcal{L}\) in the pair \((\prod_{x \in X} E^{Y_x}) \times E^X, \left(\prod_{x \in X} (E^*)^{Y_x}\right) \times (E^*)^X\) coincides with the feasible region of (45). Thus by the bipolar theorem and Krein-Šmulian theorem, it suffices to show that \((\mathcal{L})_r\) is weakly* closed for any \(r \in (0, \infty)\). Suppose that the element
\[
\left( (a_{x,y} - G_{x,y})_{x \in X, y \in Y_x}, (b_x + \sum_{y \in Y_x} G_{x,y} - u_E)_{x \in X} \right)
\]
with
\[
a_{x,y}, b_x \in E_+ (x \in X, y \in Y_x); (G_{x,y})_{x \in X, y \in Y_x} \in \mathcal{K}
\]
is in \((\mathcal{L})_r\). Then from \(b_x, G_{x,y} \geq 0\) we obtain
\[
\|b_x\|, \|G_{x,y}\| \leq \left| b_x + \sum_{y \in Y_x} G_{x,y} \right| \leq \left| b_x + \sum_{y \in Y_x} G_{x,y} - u_E \right| + 1 \leq r + 1,
\]
\[
\|a_{x,y}\| \leq \|a_{x,y} - G_{x,y}\| + \|G_{x,y}\| \leq 2r + 1
\]
\((x \in X, y \in Y_x)\). Thus by using the Banach-Alaoglu theorem, the weak* closedness of \((\mathcal{L})_r\) follows similarly as in Lemma 2. Therefore we have shown (45).

Now from (45) there exists a sequence \((\psi_{x,y}^k)_{x \in X, y \in Y_x}, (x)_{x \in X}) (k \in \mathbb{N})\) in the feasible region of (45) such that
\[
\sum_{x \in X, y \in Y_x} \langle \psi_{x,y}^k, M_x(y) \rangle > 1 + R_{\text{inc}}((\Gamma^{M_x})_{x \in X}) - \frac{1}{k}.
\]
Let \(N_k := \sum_{x \in X, y \in Y_x} \langle \psi_{x,y}^k, u_E \rangle\), which is \(> 0\) by the above inequality, and define a partitioned ensemble \(\mathcal{E}_k^x := \mathcal{E}_x^k\) by
\[
\mathcal{E}_x^k := (\varphi_{x,y}^k)_{y \in Y_x}, \quad \varphi_{x,y}^k := N_k^{-1} \psi_{x,y}^k.
\]

62
Then since \( (\chi^k_x - \psi^k_{x,y})_{x \in X, y \in Y_x} \in K^* \), for any \((N_x)_{x \in X} \in \text{EVM}_{\text{comp}}((Y_x)_{x \in X}; E)\) we have

\[
0 \leq \sum_{x \in X, y \in Y_x} \langle \chi^k_x - \psi^k_{x,y}, N_x(y) \rangle \\
= \sum_{x \in X} \langle \chi^k_x, u_E \rangle - \sum_{x \in X, y \in Y_x} \langle \psi^k_{x,y}, N_x(y) \rangle \\
\leq 1 - \sum_{x \in X, y \in Y_x} \langle \psi^k_{x,y}, N_x(y) \rangle
\]

and therefore

\[
\sum_{x \in X, y \in Y_x} \langle \varphi^k_{x,y}, N_x(y) \rangle \leq N_k^{-1}.
\]

By taking the supremum of \( N_x \) we obtain

\[
P^\text{comp}_g(\overrightarrow{E}^k) \leq N_k^{-1}.
\]

Thus

\[
P_g(\overrightarrow{E}^k, (\Gamma^M_x)_{x \in X}) = \sum_{x \in X} \max \{P_g(\mathcal{E}^k_x, \Gamma^M_{x', x}) \}
\]

\[
\geq \sum_{x \in X} P_g(\mathcal{E}^k_x, \Gamma^M_x)
\]

\[
\geq N_k^{-1} \sum_{x \in X, y \in Y_x} \langle \psi^k_{x,y}, M_x(y) \rangle
\]

\[
> N_k^{-1} \left( 1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) - \frac{1}{k} \right)
\]

\[
\geq P^\text{comp}_g(\overrightarrow{E}^k) \left( 1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) - \frac{1}{k} \right)
\]

and hence

\[
1 + R_{\text{inc}}((\Gamma^M_x)_{x \in X}) \leq \sup_{k \in \mathbb{N}} \frac{P_g(\mathcal{E}^k_x, (\Gamma^M_x)_{x \in X})}{P^\text{comp}_g(\overrightarrow{E}^k)} \leq \sup_{\overrightarrow{E} : \text{partitioned ensemble}} \frac{P_g(\mathcal{E}, (\Gamma^M_x)_{x \in X})}{P^\text{comp}_g(\overrightarrow{E})}.
\]

By combining this with Lemma 22 we obtain \(40\). \(\square\)

We next consider general measurements.

**Lemma 24.** Let \( X \neq \emptyset \) and let \( F_x (x \in X) \) be classical spaces. Then the extended real valued function

\[
\prod_{x \in X} \text{Ch}(F_x \to E) \ni (\Gamma_x)_{x \in X} \mapsto R_{\text{inc}}((\Gamma_x)_{x \in X}) \in [0, \infty]
\]

is lower semicontinuous with respect to the product topology of the BW topologies on \( \text{Ch}(F_x \to E) \).

63
Proof. Suppose that (46) is not lower semicontinuous. Then there exist a net \((\Gamma^i_x)_{x \in X} \ (i \in I)\) in \(\prod_{x \in X} \text{Ch}(F_x \to E)\) BW-convergent to some \((\Gamma_x)_{x \in X} \in \prod_{x \in X} \text{Ch}(F_x \to E)\) and \(r \in [0, R_{\text{inc}}((\Gamma^i_x)_{x \in X})]\) such that \(R_{\text{inc}}((\Gamma^i_x)_{x \in X}) < r\) for all \(i \in I\). Then for each \(i \in I\) there exists a compatible family \((\Psi^i_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)\) such that
\[
\Gamma^i_x \leq (1 + r)\Psi^i_x \quad (x \in X).
\]
By the compactness of \(\text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)\), there exists a subnet of \((\Psi^j_x(i))_{x \in X} \ (j \in J)\) BW-converging to some \((\Psi_x)_{x \in X} \in \text{Ch}^{\text{comp}}((F_x)_{x \in X}; E)\). Then we have \(\Gamma_x \leq (1 + r)\Psi_x \ (x \in X)\), which contradicts \(r < R_{\text{inc}}((\Gamma_x)_{x \in X})\). Therefore (46) is lower semicontinuous. \(\square\)

Lemma 25. Let \((\Gamma_x)_{x \in X}\) be a non-empty family of measurements on \(E\) and let \(\hat{\mathcal{F}} = (\mathcal{E}_y)_{y \in Y}\) be a partitioned ensemble. Then
\[
P_g(\hat{\mathcal{F}}; (\Gamma_x)_{x \in X}) = \sup_{A \in \mathcal{F}(X)} P_g(\hat{\mathcal{F}}; (\Gamma_x)_{x \in A}),
\]
where \(\mathcal{F}(X)\) denotes the set of non-empty finite subsets of \(X\).

Proof. The claim is immediate from
\[
P_g(\hat{\mathcal{F}}; (\Gamma_x)_{x \in X}) = \sum_{y \in Y} \sup_{x \in X} P_g(\mathcal{E}_y; \Gamma_x)
\]
and a similar expression for \(P_g(\hat{\mathcal{F}}; (\Gamma_x)_{x \in A}) \ (A \in \mathcal{F}(X))\). \(\square\)

Proof of Theorem 14. By Lemmas 3 and 21 we have only to prove (10) when for each \(x \in X\) \(\Gamma_x\) is a w*-measurement. Then by Theorem 4 there exists a net \((\Gamma^i_x)_{x \in X} \ (i \in I = \prod_{x \in X} \mathcal{D}(F_x))\) in \(\text{Ch}((F_x)_{x \in X}; E)\) satisfying the following conditions:

(i) \(\Gamma^i_x \stackrel{\text{BW}}{\longrightarrow} \Gamma_x \ (x \in X)\).

(ii) Each \([\Gamma^i_x]\) is finite-outcome \((i \in I, x \in X)\).

(iii) \(([\Gamma^i_x])_{i \in I}\) is an increasing net in \(\mathfrak{M}(E)\) weakly converging to \(\sup_{i \in I} [\Gamma^i_x] = [\Gamma_x]\) for each \(x \in X\).

From Lemmas 19, 20 and 24 we have
\[
1 + R_{\text{inc}}((\Gamma^i_x)_{x \in X}) = \sup_{A \in \mathcal{F}(X)} \sup_{i \in I} \left(1 + R_{\text{inc}}((\Gamma^i_x)_{x \in A})\right),
\]
where \(\mathcal{F}(X)\) denotes the set of non-empty finite subsets of \(X\). From Lemma 23 we also have
\[
1 + R_{\text{inc}}((\Gamma^i_x)_{x \in A}) = \sup_{\hat{\mathcal{F}}: \text{partitioned ensemble}} \frac{P_g(\hat{\mathcal{F}}; (\Gamma^i_x)_{x \in A})}{P_{g, \text{comp}}(\hat{\mathcal{F}})} \quad (i \in I, A \in \mathcal{F}(X)).
\]
Then from (47) and (48) we have
\[ 1 + R_{\text{inc}}((\Gamma^i_x)_{x \in X}) = \sup_{A \in \mathcal{F}(X)} \sup_{i \in I} \sup_{\mathcal{E} : \text{partitioned ensemble}} \frac{P_g(\mathcal{E}; (\Gamma^i_x)_{x \in A})}{P_g^{\text{comp}}(\mathcal{E})} \]
\[ = \sup_{\mathcal{E} : \text{partitioned ensemble}} \sup_{A \in \mathcal{F}(X)} \sup_{i \in I} \frac{P_g(\mathcal{E}; (\Gamma^i_x)_{x \in A})}{P_g^{\text{comp}}(\mathcal{E})} \]
\[ = \sup_{\mathcal{E} : \text{partitioned ensemble}} \frac{P_g(\mathcal{E}; (\Gamma^i_x)_{x \in X})}{P_g^{\text{comp}}(\mathcal{E})}, \]

where in the third equality we used the fact that for each partitioned ensemble \( \mathcal{E} = (\mathcal{E}_y)_{y \in Y} \) and \( A \in \mathcal{F}(X) \), the map
\[ \mathcal{M}(E)^A \ni (\omega_x)_{x \in A} \mapsto P_g(\mathcal{E}; (\omega_x)_{x \in A}) = \sum_{y \in Y} \max_{x \in A} P_g(\mathcal{E}_y; \omega_x) \]
is weakly continuous and monotonically increasing in the post-processing order. The fourth equality follows from Lemma 25.

9 Concluding remarks

In this paper, we have investigated general properties of the measurement space \( \mathcal{M}(E) \) for a given order unit Banach space \( E \) with a predual corresponding to a GPT. Among these general facts, the compactness of \( \mathcal{M}(E) \) (Theorem 3) and the density of finite-outcome measurements (Theorem 4) are proved to be essential in the applications to simulability and incompatibility of measurements with general outcome spaces in Sections 6, 7, and 8. Our study revealed that the compact convex structure naturally arises in the measurement space \( \mathcal{M}(E) \), whose physical meaning is fundamentally different from the state space of a general probabilistic theory. The compact convex structure of the measurement space \( \mathcal{M}(E) \) is based on the state discrimination probabilities of a finite-label ensembles. The general theory developed in this paper applies whenever such quantities are involved and not restricted to the specific examples considered in this paper.

9.1. The measurement space \( \mathcal{M}(E) \) has not only topological and convex structures but also the post-processing order structure. As we have shown in Theorem 8, the post-processing order is a special example of the orders characterized by the independence and the continuity axioms. From the mathematical point of view, this motivates us to ask when such an ordered compact convex set can be regarded as a measurement space \( \mathcal{M}(E) \), especially for \( E \) corresponding to a quantum or a classical system. From Theorem 6, we can see that such compact convex set has an infinite dimension except when it is a singleton.
2. We can also ask whether the measurement space $\mathcal{M}(E)$ characterizes the space $E$ up to weakly* isomorphism. To be specific, the question is formalized as follows: consider order unit Banach spaces $E_1$ and $E_2$ which respectively have the Banach preduals $E_{1*}$ and $E_{2*}$ and suppose that there exists a continuous, affine, and order isomorphism $\Psi: \mathcal{M}(E_1) \to \mathcal{M}(E_2)$ between the measurement spaces. Then is there a weakly* continuous, order bi-preservation, linear isomorphism between $E_1$ and $E_2$? Note that we can easily see that the converse implication holds, namely an isomorphism between $E_1$ and $E_2$ induces an isomorphism between the measurement spaces $\mathcal{M}(E_1)$ and $\mathcal{M}(E_2)$.

3. Recently in [12, 11, 55] it is shown that the weight of resource is related to the ratio of state exclusion probability. Specifically, in [12] the resource theory of measurements based on the state exclusion probability is studied. For an ensemble $\mathcal{E} = (\varphi_x)_{x \in X}$ and a measurement $\Gamma \in \text{Ch}(F \to E)$ the state exclusion probability is given by

$$P_{\text{ex}}(\mathcal{E}; \Gamma) := \sup_{M \in \text{EVM}(X;F)} \sum_{x, x' \in X : x \neq x'} \langle \varphi_x, \Gamma(M(x')) \rangle = P_g(\tilde{\mathcal{E}}; \Gamma),$$

(49)

where $\tilde{\mathcal{E}} := (\sum_{x' \in X : x' \neq x} \varphi_{x'})_{x \in X}$. Since (49) is apparently weakly continuous, the methods developed in Sections 6 and 8 will be straightforwardly generalized to this case.

4. We may also ask whether we can generalize our results for measurements to more general class of channels with non-classical outcome spaces. If we consider the order induced by the state discrimination probability, this order is the one induced by statistical morphisms, much weaker notion than that of channels, and does not coincide in general with the order induces by the post-processing channels [39]. For any of these orders, the compactness result (Theorem 3) seems to still hold because the classicality of the outcome spaces in the proof is used only to guarantee the limit channel has also classical outcome space.

Acknowledgement.
The author would like to thank Hayata Yamasaki for helpful discussions and comments on the paper and Masato Koashi for helpful discussions. This work was supported by Cross-Ministerial Strategic Innovation Promotion Program (SIP) (Council for Science, Technology and Innovation (CSTI)).

A Proof of Proposition 1

In this appendix we prove Proposition 1.

1. The first part of the claim is easy to verify. The affinity of $\Psi$ can be shown in the same way as [17] (Theorem 2.2). The continuity of $\Psi$ is immediate from the definition. The injectivity of $\Psi$ follows from that $A_c(S)$ separates points of $S$. Then since $S$ is a compact Hausdorff space, we have only to establish the surjectivity of $\Psi$. Suppose that there exists a state $\phi \in S(A_c(S)) \setminus \Psi(S)$. Since $\Psi(S)$ is a weakly* compact convex subset of $A_c(S)^*$, by the Hahn-Banach separation theorem we can take $f \in A_c(S)$ such that
sup_{s \in S} f(s) < \langle \phi, f \rangle. By replacing \( f \) with \( f + \|f\|_S \) if necessary, we may assume \( f \geq 0 \). Then we have
\[
\|f\| = \sup_{s \in S} |f(s)| = \sup_{s \in S} f(s) < \langle \phi, f \rangle \leq \|\phi\|\|f\| = \|f\|, 
\]
which is a contradiction. Therefore \( \Psi \) is a continuous affine isomorphism.

2. The first part of the claim is again easy to show. By the definition of the metric on \( S \), we can easily see that \( \Phi \) is an isometry. The affinity of \( \Phi \) can be again shown in the same way as in \[17\]. Consider the locally convex Hausdorff topology \( \sigma(A_b(S), S) \) on \( A_b(S) \), which is the pointwise convergence topology on \( A_b(S) \). Then the unit ball \( (A_b(S))_1 \) is compact in this topology and by \[27\] this implies that \( A_b(S) \) has the Banach predual \( \overline{\text{lin}(S)} = E_* \), where the closure is with respect to the norm topology. Let \( B(\supset S) \) be the base of the positive cone of \( E_* \). Assume \( S \not\subseteq B \) and take \( \psi \in B \setminus S \). Since \( S \) is norm-complete and convex, the Hahn-Banach separation theorem implies that there exists \( g \in A_b(S) = (E_*)_* \) such that \( \sup_{s \in S} g(s) < \langle g, \psi \rangle \). As in the proof of the claim 1, this yields a contradiction. Therefore \( S = B \) and hence \( E_* = \text{lin}(S) = \text{lin}(B) = \overline{E_*} \) is a Banach predual of \( A_b(S) \) with the base \( S \) of the predual positive cone \( E_{*+} \).

B Proof of Proposition 3

In this section we prove Proposition 3.

If \( E \) is classical, we may assume \( (E, u_E) = (C(X), 1_X) \) for some compact Hausdorff space \( X \). If we define \( B_0 : E \times E \to E \) by the pointwise multiplication \( B_0(f, g)(x) := f(x)g(x) \), then we can easily see that the conditions (i) and (ii) hold.

Conversely assume that there exists a bilinear map \( B : E \times E \to E \) satisfying (i) and (ii). Then as in \[2\] (Lemma 3), we can show
\[
\phi \circ B(a, b) = \phi(a)\phi(b) \tag{50}
\]
for any pure state \( \phi \in \partial_c S(E) \). We show that
\[
X := \{ \phi \in S(E) \mid \phi \circ B(a, b) = \phi(a)\phi(b) \ (\forall a, b \in E) \}
\]
is a Hausdorff topological space in the relative topology of the weak* topology on \( S(E) \). Take a net \( (\phi_i)_{i \in I} \) in \( X \) weakly* converging to \( \phi \in S(E) \). Then \( \phi \circ B(a, b) = \lim_{i \in I} \phi_i \circ B(a, b) = \lim_{i \in I} \phi_i(a)\phi_i(b) = \phi(a)\phi(b) \) for any \( a, b \in E \). Therefore, being a closed subset of \( S(E), X \) is weakly* compact. We define a linear map \( \Psi : E \to C(X) \) by \( \Psi(a)(\phi) := \langle \phi, a \rangle \ (a \in E, \phi \in X) \). Then \( \Psi \) is unital and positive. Furthermore, by the Krein-Milman theorem, for any \( a \in E \)
\[
\|a\| = \sup_{\phi \in S(E)} |\langle \phi, a \rangle| = \sup_{\phi \in \partial_c S(E)} |\langle \phi, a \rangle| = \sup_{\phi \in X} |\langle \phi, a \rangle| = \|\Psi(a)\| 
\]
\[ a \geq 0 \iff \langle \phi, a \rangle \geq 0 \quad (\forall \phi \in S(E)) \]
\[ \iff \langle \phi, a \rangle \geq 0 \quad (\forall \phi \in \partial_v S(E)) \]
\[ \iff \langle \phi, a \rangle \geq 0 \quad (\forall \phi \in X) \]
\[ \iff \Psi(a) \geq 0. \]

where we used \( \partial_v S(E) \subset X \). Thus to show that \( \Psi \) is an isomorphism between the order unit Banach spaces \( E \) and \( C(X) \), it suffices to prove that \( \Psi \) is a surjection. Since \( \Psi(a)(\phi)\Psi(b)(\phi) = \langle \phi, B(a, b) \rangle = \Psi(B(a, b))(\phi) (a, b \in E; \phi \in X) \) the image \( \Psi(E) \) is a norm-complete subalgebra of \( C(X) \) containing the unit \( 1_X = \Psi(u_E) \). Moreover \( \Psi(E) \) separates points of \( X \) since \( E \) separates \( S(E) \), a fortiori \( X(\subset S(E)) \). Therefore the Stone-Weierstrass theorem implies \( \Psi(E) = C(X) \), which proves the classicality of \( E \).

Let \( B' : E \times E \to E \) be another bilinear map satisfying (i) and (ii). Then we can similarly show \( \langle \phi, B'(a, b) \rangle = \langle \phi, a \rangle b(b) (\phi \in \partial_v S(E); a, b \in E) \). This implies \( \langle \phi, B(a, b) \rangle = \langle \phi, B'(a, b) \rangle (\phi \in \partial_v S(E); a, b \in E) \) and hence the Krein-Milman theorem implies \( B(a, b) = B'(a, b) \), which proves the uniqueness. The commutativity and the associativity of \( B \) follows again from \( \Psi \) and the Krein-Milman theorem.

\[ \square \]

\section{Proof of Proposition \[12\]}

In this appendix, we prove Proposition \[12\]. Throughout this appendix, we fix the system order unit Banach space \( E \) and its Banach predual \( E_\ast \) corresponding to the system.

An EVM \((X, \Sigma, M)\) on \( E \) is called regular \([6] \), Section 4.10) if \( X \) is a compact Hausdorff space, \( \Sigma \) is the Borel \( \sigma \)-algebra \( B(X) \) of \( X \), and \( \mu^M_\psi \) is a regular signed measure for any \( \psi \in E_\ast \). The following Riesz-Markov-Kakutani-type representation theorem can be shown similarly as in \([6] \) (Theorem 4.4).

\begin{proposition}
Let \( X \) be a compact Hausdorff space. Then for each channel \( \Psi \in \text{Ch}(C(X) \to E) \) there exists a unique regular EVM \((X, B(X), M)\) such that
\[
\Psi(f) = \int_X f(x)dM(x) \quad (f \in C(X)).
\]
\end{proposition}

\textbf{Proof of Proposition \[12\].} Since \( F \) is classical, we may assume \( F = C(X) \) for some compact Hausdorff space \( X \). Then by Proposition \[27\] there exists a unique regular EVM \((X, B(X), M)\) on \( E \) such that
\[
\Gamma(f) = \int_X f(x)dM(x) \quad (f \in F = C(X)).
\]

We show \( \Gamma \simeq_{\text{post}} \Gamma^M. \) Let \( \gamma^M \in \text{Ch}(B(X, B(X)) \to E) \) be the measurement associated with \( M \). Then \( \Gamma \) is the restriction of \( \gamma^M \) to the subalgebra \( C(X) \subset B(X, B(X)) \) and hence \( \Gamma \preceq_{\text{post}} \gamma^M \simeq_{\text{post}} \Gamma^M. \)

We now prove \( \gamma^M \simeq_{\text{post}} \Gamma \), where \( \Gamma \in \text{Ch}_{w^*}(C(X)^{**} \to E) \) is the \( w^* \)-extension of \( \Gamma \). By the ordinary Riesz-Markov-Kakutani representation theorem, the Banach dual space \( C(X)^{**} \)
is identified with the set $M(X)$ of signed regular measures on $X$ with the bilinear form
\[
\langle \nu, f \rangle = \int_X f(x) d\nu(x) \quad (f \in C(X), \nu \in M(X)).
\]
We define a linear map $\Phi: M(X) \to B(X, B(X))^*$ by
\[
\langle \Phi(\nu), f \rangle := \int_X f(x) d\nu(x) \quad (f \in B(X, B(X)), \nu \in M(X)).
\]
Then $\Phi$ is positive and sends a state (i.e. a probability measure) in $M(X)$ to a state in $B(X, B(X))^*$. Therefore the dual map $\Phi^*: B(X, B(X))^* \to C(X)^*(= M(X)^*)$ is a $w^*$-channel. Define a channel $\Psi \in \text{Ch}(B(X, B(X)) \to C(X)^*)$ by the restriction of $\Phi^*$ to $B(X, B(X))$. Then for any $f \in B(X, B(X))$ and $\psi \in E_*$
\[
\langle \psi, \Gamma \circ \Psi(f) \rangle = \langle \Gamma^*(\psi), \Psi(f) \rangle = \langle M^M, \Psi(f) \rangle = \int_X f(x) d\mu^M(x) = \langle \psi, M^M(f) \rangle,
\]
which implies $\gamma^M = \Gamma \circ \Psi \preceq_{\text{post}} \Gamma$.

Since $\Gamma \sim_{\text{post}} \Gamma$ by Proposition 9 this implies $\gamma^M \preceq_{\text{post}} \Gamma$ and hence again by Proposition 9 we obtain $\Gamma^M \preceq_{\text{post}} \Gamma$, which completes the proof.

\section{D Measurement space and types of statistical experiments}

In this appendix, we discuss the relation between the general theory of $w^*$-measurements developed in this paper and the theory of (classical) statistical experiments [36, 54]. It will be shown that there is a one-to-one correspondence between the statistical experiments with a given parameter set and the $w^*$-measurements with the discrete classical space corresponding to the parameter set. Conversely the class $\text{Meas}(E)$ of $w^*$-measurements for a given input space $E$ is shown to be regarded as a face-like subclass of the “larger” class of statistical experiments with the parameter set $S_*(E)$. The former statement indicates that our results on general $w^*$-measurements are more general than the corresponding results for statistical experiments [36, 54], while, according to the latter one, we can define a quantity or relation known in the general statistical experiments to $w^*$-measurements by restricting the quantity or relation defined in the “large” class of statistical experiments to the class of $w^*$-measurements. We remark that these correspondences are also valid in the setup of quantum statistical experiments and post-processing completely positive channels ([33], Section 2.2).

A (classical) statistical experiment is a parameterized family of probability measures. Formal definition is as follows.

\textbf{Definition 15} (Statistical experiment). A triple $E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta})$ is called a (classical) statistical experiment if $E$ is a classical space with a Banach predual, $\Theta \neq \emptyset$ is a set, and $(\varphi_\theta)_{\theta \in \Theta} \in S_*(E)^\Theta$ is a family of weakly* continuous states indexed by $\Theta$. $E$ and $\Theta$ are called the outcome (or sample) space and the parameter set of $E$, respectively. For each set $\Theta \neq \emptyset$ the class of statistical experiments with the parameter set $\Theta$ is denoted by $\text{Exper}(\Theta)$, which is a proper class.
As in the case of w*-measurements or channels, we can define the post-processing (or randomization) order and equivalence relations for statistical experiments:

**Definition 16** (Post-processing relation for statistical experiments). For any statistical experiments \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}) \) and \( F = (F, \Theta, (\psi_\theta)_{\theta \in \Theta}) \) with the same parameter set \( \Theta \neq \emptyset \), we define the following binary relations \( \preceq_{\text{post}} \) and \( \sim_{\text{post}} \) on \( E \) and \( F \):

(i) \( E \preceq_{\text{post}} F \) (\( E \) is a post-processing of \( F \)) if there exists a channel \( \Psi \in \text{Ch}(E \to F) \) such that \( \varphi_\theta = \psi_\theta \circ \Psi \) for all \( \theta \in \Theta \).

(ii) \( E \sim_{\text{post}} F \) (\( E \) is post-processing equivalent to \( F \)) if \( \Psi = \Id \) and suppose \( \Gamma(\varphi_\theta) = \Gamma(\psi_\theta) \) for all \( \theta \in \Theta \).

The relations \( \preceq_{\text{post}} \) and \( \sim_{\text{post}} \) are binary preorder and equivalence relations on \( \text{Exper}(\Theta) \), respectively.

An operational meaning of a statistical experiment \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}) \) is that the system is governed by the parameter \( \theta \in \Theta \) and the system’s state is prepared to \( \varphi_\theta \) when \( \theta \) prevails. If \( E \preceq_{\text{post}} F \) (respectively, \( E \sim_{\text{post}} F \)), then the information about \( \theta \) when we can access \( E \) is at least as much as (respectively, the same as) the information when we can access to \( F \).

The class of statistical experiments \( \text{Exper}(\Theta) \) equipped with the post-processing relations can be identified with a class of w*-measurements in the following way.

**Proposition 28.** Let \( \Theta \neq \emptyset \) be a set. For each statistical experiment \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}) \) we define a channel \( \Gamma_E \in \text{Ch}_{\text{w*}}(E \to \ell^\infty(\Theta)) \) by

\[
\Gamma_E(a) := \sum_{\theta \in \Theta} \varphi_\theta(a) \delta_\theta \quad (a \in E),
\]

where \( \ell^\infty(\Theta) \) denotes the classical space of bounded real-valued functions on \( \Theta \), \( \delta_\theta := 1_{\{\theta\}} \), and the summation is convergent in the weak* topology, or equivalently the pointwise convergence topology, on \( \ell^\infty(\Theta) \). Then following assertions hold.

1. **The class-to-class map**

\[
\text{Exper}(\Theta) \ni E \mapsto \Gamma_E \in \text{Meas}(\ell^\infty(\Theta)) \quad (51)
\]

is bijective.

2. For any statistical experiments \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}), F = (F, \Theta, (\psi_\theta)_{\theta \in \Theta}) \in \text{Exper}(\Theta) \), \( E \preceq_{\text{post}} F \) if and only if \( \Gamma_E \preceq_{\text{post}} \Gamma_F \).

**Proof.** 1. Take statistical experiments \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}), F = (F, \Theta, (\psi_\theta)_{\theta \in \Theta}) \in \text{Exper}(\Theta) \) and suppose \( \Gamma_E = \Gamma_F \). Then \( E = F \) and

\[
\varphi_\theta(a) = \Gamma_E(a)(\theta) = \Gamma_F(a)(\theta) = \psi_\theta(a) \quad (a \in E; \theta \in \Theta),
\]

which implies \( E = F \). Thus \( (51) \) is injective. If \( \Gamma \in \text{Ch}_{\text{w*}}(E \to \ell^\infty(\Theta)) \) is a w*-measurement, then for each \( \theta \in \Theta \), \( \varphi_\theta(a) := \Gamma(a)(\theta) \) (\( a \in E \)) is a weakly* continuous state and

\[
\Gamma(a) = \sum_{\theta \in \Theta} \varphi_\theta(a) \delta_\theta = \Gamma_E(a) \quad (a \in E),
\]

where \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}) \in \text{Exper}(\Theta) \). Therefore \( (51) \) is surjective.
2. To establish the "only if" part of the claim, suppose \( E \preceq_{\text{post}} F \) and take a channel \( \Psi \in \text{Ch}(E \to F) \) such that \( \varphi_\theta = \psi_\theta \circ \Psi (\theta \in \Theta) \). Then

\[
\Gamma_E(a) = \sum_{\theta \in \Theta} \varphi_\theta(a)\delta_\theta = \sum_{\theta \in \Theta} \psi_\theta \circ \Psi(a)\delta_\theta = \Gamma_F \circ \Psi(a) \quad (a \in E)
\]

which implies \( \Gamma_E \preceq_{\text{post}} \Gamma_F \). Conversely if \( \Gamma_E = \Gamma_F \circ \Phi \) for some \( \Phi \in \text{Ch}(E \to F) \), then by using the injectivity of \( (51) \), we have \( \varphi_\theta = \psi_\theta \circ \Phi (\theta \in \Theta) \), which proves the "if" part of the claim.

By Proposition 28 we can define the convex combination \( \lambda E \oplus (1 - \lambda)F \) (\( \lambda \in [0, 1] \)) of two statistical experiments \( E, F \in \text{Exper}(\Theta) \) by \( \Gamma_{\lambda E \oplus (1 - \lambda)F} := \lambda \Gamma_E \oplus (1 - \lambda)\Gamma_F \). If \( E = (E, \Theta, (\varphi_\theta)_{\theta \in \Theta}) \) and \( F = (F, \Theta, (\psi_\theta)_{\theta \in \Theta}) \), the convex combination is given by

\[
\lambda E \oplus (1 - \lambda)F = (E \oplus F, \Theta, (\lambda \varphi_\theta \oplus (1 - \lambda)\psi_\theta)_{\theta \in \Theta})
\]

Furthermore, we can define the set \( E(\Theta) \) of post-processing equivalence classes of statistical experiments by \( E(\Theta) := \mathcal{M}(\ell^\infty(\Theta)) \), where for each statistical experiment \( E \in \text{Exper}(\Theta) \) the corresponding equivalence class is defined by \( [E] := [\Gamma_E] \in \mathcal{M}(\ell^\infty(\Theta)) \). In [36, 54] the equivalence class \([E] \in E(\Theta)\) is called the type of \( E \).

The weak topology on \( E(\Theta) = \mathcal{M}(\ell^\infty(\Theta)) \) in our sense coincides with the weak topology on \( E(\Theta) \) in the sense of [36, 54], which can be seen from Theorem 7.4.15 of [54].

We next show that the class of \( w^* \)-measurements can be regarded as a special class of statistical experiments.

**Proposition 29.** Let \( E \) be an order unit Banach space with a Banach predual \( E_* \). Define

\[
\text{Meas}(E) \ni \Gamma \mapsto E_\Gamma \in \text{Exper}(S_*(E))
\]

by \( E_\Gamma := (F, S_*(E), (\phi \circ \Gamma)_{\phi \in S_*(E)}) \) for \( \Gamma \in \text{Ch}_{w^*}(F \to E) \). Then the following assertions hold.

1. The map \( (52) \) is injective and affine in the following sense:

\[
E_{\lambda \Gamma \oplus (1 - \lambda) \Lambda} = \lambda E_\Gamma \oplus (1 - \lambda)E_\Lambda \quad (\lambda \in [0, 1]; \Gamma, \Lambda \in \text{Meas}(E))
\]  

2. A statistical experiment \( E = (F, S_*(E), (\xi_\phi)_{\phi \in S_*(E)}) \in \text{Exper}(S_*(E)) \) is in the image of \( (52) \) if and only if the map

\[
S_*(E) \ni \phi \mapsto \xi_\phi \in S_*(F)
\]

is affine. \( E \) is called affine if the map \( (52) \) is affine.

3. The image of \( (52) \) is a face of \( \text{Exper}(S_*(E)) \) in the following sense: for any \( \lambda \in (0, 1) \), and any statistical experiments \( E = (F, S_*(E), (\xi_\phi)_{\phi \in S_*(E)}) \) and \( F = (G, S_*(E), (\eta_\phi)_{\phi \in S_*(E)}) \), if \( \lambda E \oplus (1 - \lambda)F \) is in the image of \( (52) \), then so are \( E \) and \( F \).

4. For any \( w^* \)-measurements \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \), \( \Gamma \preceq_{\text{post}} \Lambda \) if and only if \( E_\Gamma \preceq_{\text{post}} E_\Lambda \).
Proof. 1. For w*-measurements \( \Gamma, \Lambda \in \text{Meas}(E) \), suppose \( E_\Gamma = E_\Lambda \). Then \( \Gamma \) and \( \Lambda \) have the same outcome classical space \( F \) with a Banach predual and \( \phi \circ \Gamma = \phi \circ \Lambda \) for all \( \phi \in S_*(E) \). Since \( S_*(E) \) generates \( E_* \), this implies \( \Gamma = \Lambda \).

For any \( \lambda \in [0, 1] \) and any w*-measurements \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) and \( \Lambda \in \text{Ch}_{w^*}(G \to E) \) we have

\[
E_{\lambda \Gamma \oplus (1-\lambda)\Lambda} = (F \oplus G, S_*(E), (\phi \circ (\lambda \Gamma \oplus (1-\lambda)\Lambda))_{\phi \in S_*(E)})
\]

\[
= (F \oplus G, S_*(E), (\lambda \phi \circ \Gamma \oplus (1-\lambda) \phi \circ \Lambda)_{\phi \in S_*(E)})
\]

\[
= \lambda E_\Gamma \oplus (1-\lambda) E_\Lambda,
\]

which proves (53).

2. If \( E = E_\Gamma \) for some w*-measurement \( \Gamma \in \text{Meas}(E) \), we can readily see that \( E \) is affine. Conversely suppose that \( E \) is affine. Then the map (54) is uniquely extended to a bounded linear map \( \Gamma_*: E_* \to F_* \). If we define \( \Gamma: F \to E \) by the dual map of \( \Gamma_* \), it is easy to show that \( \Gamma \) is a w*-channel and \( E = E_\Gamma \).

3. By the claim 2, the assumption implies that the map

\[
S_*(E) \ni \phi \mapsto \lambda \xi_\phi \oplus (1-\lambda) \eta_\phi \in S_*(F \oplus G)
\]

is affine. Then we can easily see that the maps

\[
S_*(E) \ni \phi \mapsto \xi_\phi \in S_*(F), \quad S_*(E) \ni \phi \mapsto \eta_\phi \in S_*(G)
\]

are also affine, and therefore, again by the claim 2, \( E \) and \( F \) are in the image of (52).

4. The claim readily follows from the definitions of the post-processing orders on \( \text{Meas}(E) \) and \( \text{Exper}(S_*(E)) \) and from the injectivity of (52). \( \square \)

The affine injection (52) induces the following affine injection for the sets of equivalence classes:

\[
\mathcal{M}(E) \ni [\Gamma] \mapsto [E_\Gamma] \in \overline{E}(S_*(E)).
\]

The image of (55) is a face of \( \overline{E}(S_*(E)) \). As for the weak topology, we have

**Proposition 30.** The map (55) is continuous with respect to the weak topologies on \( \mathcal{M}(E) \) and on \( \overline{E}(S_*(E)) \), respectively, and hence the image of (55) is a compact face of \( \overline{E}(S_*(E)) \).

**Proof.** Let \( (\varphi_x)_{x \in X} \in (\ell^\infty(S_*(E)))^X \) be an ensemble. Then each \( \varphi_x \) corresponds to \( q_x \in \ell^1(S_*(E)) \) such that

\[
\langle \varphi_x, f \rangle = \sum_{\psi \in S_*(E)} f(\psi) q_x(\psi) \quad (f \in \ell^\infty(S_*(E)))
\]

and \( q_x(\psi) \geq 0 \ (\psi \in S_*(E)) \), where \( \ell^1(\Omega) \) denotes the set of summable real functions on a set \( \Omega \) equipped with the \( \ell^1 \)-norm \( \|q\|_1 := \sum_{\omega \in \Omega} |q(\omega)| \). Then for any w*-measurement
For any ensemble \( \mathbf{M} \in \mathbf{EVM}(X; F) \), we have

\[
\sum_{x \in X} \langle \varphi_x, \Gamma_{E_{T}}(\mathbf{M}(x)) \rangle = \sum_{x \in X} \sum_{\psi \in S_{*}(E)} q_x(\psi) \langle \psi, \Gamma(\mathbf{M}(x)) \rangle
\]

\[
= \sum_{x \in X} \left( \sum_{\psi \in S_{*}(E)} q_x(\psi) \psi, \Gamma(\mathbf{M}(x)) \right).
\]

Note that \( \sum_{\psi \in S_{*}(E)} q_x(\psi) \psi \) makes sense since the summation is at most countable and absolutely convergent with respect to the norm on \( E_{*} \). This implies

\[
P_g((\varphi_x)_{x \in X}; \Gamma_{E_{T}}) = P_g(\mathbf{E}; \Gamma),
\]

where

\[
\mathbf{E} := \left( \sum_{\psi \in S_{*}(E)} q_x(\psi) \psi \right)_{x \in X}
\]

is an ensemble on \( E \). Therefore

\[
\mathfrak{M}(E) \ni [\Gamma] \mapsto P_g((\varphi_x)_{x \in X}; \Gamma_{E_{T}}) \in \mathbb{R}
\]

is weakly continuous on \( \mathfrak{M}(E) \) for any ensemble \( (\varphi_x)_{x \in X} \), which implies the continuity of \( (55) \).

By Proposition 30, the measurement space \( \mathfrak{M}(E) \) can be regarded as a compact face of the set \( \mathfrak{E}(S_{*}(E)) \). The above proof also shows that the restriction of any gain functional on \( \mathfrak{E}(S_{*}(E)) \) restricted to (the image of) \( \mathfrak{M}(E) \) is a gain functional on \( \mathfrak{M}(E) \). We note that this does not imply that the theory of measurements reduces to that of statistical experiments since in general we cannot obtain all the information about a mathematical structure from another larger structure into which the structure in consideration is embedded.

Conversely, as we have seen in Proposition 28, the statistical experiment is a special kind of w*-measurements. Moreover, in the case of statistical experiments, the notions of maximal and simulation irreducible measurements are trivial. Indeed for \( \mathfrak{M}(\ell^\infty(\Theta)) = \mathfrak{E}(\Theta) \), the maximum element \([\text{id}_{\ell^\infty(\Theta)}]\) is the unique maximal, and hence simulation irreducible, measurement and the results in Sections 7.2, 7.3, and 7.4 are trivial and not interesting in this case.

**E Proof of Theorem 8**

In this section we prove Theorem 8 in the line of [10].

The proof of the following lemma is the same as in [10] and omitted.

**Lemma 26 ([10], Lemmas 1 and 2).** Let \( S \) be a compact convex structure and let \( \preceq \) be a preorder on \( S \) satisfying the independence and continuity axioms of Theorem 8. Then the following assertions hold.
1. For any $\omega, \nu, \mu \in S$ and any $\lambda \in (0, 1]$, the cancellation law

$$\lambda \omega + (1 - \lambda)\mu \preceq \lambda \nu + (1 - \lambda)\mu \implies \omega \preceq \nu$$

holds.

2. Define

$$C_{\preceq} := \{ \lambda (\nu - \omega) \in A_c(S) | \lambda \in (0, \infty); \omega, \nu \in S; \omega \preceq \nu \}.$$  \hspace{1cm} (56)

Then $C_{\preceq}$ is a convex cone in $A_c(S)^*$. Moreover for any $\omega, \nu \in S$,

$$\omega \preceq \nu \iff \nu - \omega \in C_{\preceq}$$

holds.

Lemma 27 ([10], Claim 1). Let $S$ be a compact convex structure and let $\preceq$ be a preorder on $S$ satisfying the independence and continuity axioms of Theorem 8. Then $C_{\preceq}$ defined by (56) is weakly* closed.

Proof. Since $C_{\preceq}$ is a convex set from Lemma 26, by the Krein-Šmulian theorem it suffices to show that $(C_{\preceq})_r$ is weakly* closed for any $r \in (0, \infty)$. We take an arbitrary net $(\psi_i)_{i \in I}$ in $(C_{\preceq})_r$ weakly* convergent to $\psi \in A_c(S)^*$ and prove $\psi \in C_{\preceq}$. From Proposition II.1.14 of [1], by noting that $\langle \psi_i, 1_S \rangle = 0$, for each $i \in I$ we can write as

$$\psi_i = \frac{\|\psi_i\|}{2} (\nu_i - \omega_i)$$

for some $\omega_i, \nu_i \in S$, where we take as $\omega_i = \nu_i$ when $\psi_i = 0$. Then by Lemma 26, $\omega_i \preceq \nu_i$ holds for all $i \in I$. Since $\|\psi_i\| \leq r$ for all $i \in I$, we can take a subnet $(\psi_{i(j)})_{j \in J}$, a real number $\lambda \in [0, r]$, and states $\omega, \nu \in S$ such that

$$\|\psi_{i(j)}\| \to \lambda, \quad \omega_{i(j)} \to \omega, \quad \nu_{i(j)} \to \nu.$$

Then

$$\psi = \frac{\lambda}{2} (\nu - \omega).$$

Furthermore from the continuity axiom we have $\omega \preceq \nu$. Therefore $\psi \in C_{\preceq}$, which completes the proof.

Proof of Theorem 8. The implication (i) $\implies$ (ii) is trivial. We assume (ii) and prove (i). Define

$$U := \{ f \in A_c(S) | \langle \psi, f \rangle \geq 0 (\forall \psi \in C_{\preceq}) \},$$

where $C_{\preceq}$ is defined by (56). Then $U$ is the dual cone of $C_{\preceq}$ in the pair $(A_c(S), A_c(S)^*)$. Since $C_{\preceq}$ is a weakly* closed convex cone by Lemmas 26 and 27, the bipolar theorem implies that

$$C_{\preceq} = \{ \psi \in A_c(S)^* | \langle \psi, f \rangle \geq 0 (\forall f \in U) \}.$$
Therefore from Lemma [26], for any \( \omega, \nu \in S \) we have
\[
\omega \preceq \nu \iff \nu - \omega \in C_{\leq} \iff \langle \nu - \omega, f \rangle \geq 0 \quad (\forall f \in U) \iff \omega \preceq_{U} \nu.
\]

Hence \( \preceq \) coincides with \( \preceq_{U} \), which proves [I].

Now we establish the remaining uniqueness part of the claim. Take subsets \( A, B \subset A_c(S) \) such that \( \preceq_A \) and \( \preceq_B \) coincide. Then by definition any \( f \in A \) is monotonically increasing in \( \preceq_A \) and hence in \( \preceq_B \). Thus by Theorem [7] we have \( f \in \overline{\text{cone}}(B \cup \{ \pm 1_S \}) \) and therefore \( \overline{\text{cone}}(A \cup \{ \pm 1_S \}) \subset \overline{\text{cone}}(B \cup \{ \pm 1_S \}) \) holds. The converse inclusion can be shown similarly and hence we obtain \( \overline{\text{cone}}(A \cup \{ \pm 1_S \}) = \overline{\text{cone}}(B \cup \{ \pm 1_S \}) \). Conversely suppose that \( \overline{\text{cone}}(A \cup \{ \pm 1_S \}) = \overline{\text{cone}}(B \cup \{ \pm 1_S \}) \). Then since we can easily see that the orders \( \preceq_{\text{cone}(A \cup \{ \pm 1_S \})} \) and \( \preceq_{\text{cone}(B \cup \{ \pm 1_S \})} \) respectively coincide with \( \preceq_A \) and \( \preceq_B \), the orders \( \preceq_A \) and \( \preceq_B \) coincide. \( \square \)

## F Minimal sufficiency

In this appendix, we summarize the facts on minimally sufficient w*-measurements ([32], Section 7.3) needed in Section [7].

A w*-measurement \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) is called **minimally sufficient** if for any \( \Psi \in \text{Ch}_{w^*}(F \to F) \), \( \Gamma \circ \Psi = \Gamma \) implies \( \Psi = \text{id}_F \), where \( \text{id}_S \) denotes the identity map on a set \( S \). It can be shown that every minimally sufficient w*-measurement \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) is faithful, i.e. \( \Gamma(a) = 0 \) implies \( a = 0 \) for \( a \in F_+ \). Since a classical space \( F \) with a predual is isomorphic to the self-adjoint part of an abelian \( W^* \)-algebra, the following proposition readily follows from [32] (Corollary 2).

**Proposition 31.** Let \( \Gamma \in \text{Ch}_{w^*}(F \to E) \) be a w*-measurement. Then there exists a minimally sufficient w*-measurement \( \Gamma_0 \in \text{Ch}_{w^*}(F_0 \to E) \) post-processing equivalent to \( \Gamma \). Furthermore such a minimally sufficient w*-measurement is unique up to isomorphism of the outcome space, i.e. if \( \Gamma_1 \in \text{Ch}_{w^*}(F_1 \to E) \) is a minimally sufficient w*-measurement and \( \Gamma \sim_{\text{post}} \Gamma_1 \), then there exists a weakly* continuous isomorphism \( \Phi : F_0 \to F_1 \) such that \( \Gamma_0 = \Gamma_1 \circ \Phi \).

Let us see how to construct such a minimally sufficient w*-measurement \( \Gamma_0 \) when \( \Gamma \) is faithful. Define \( F \subset \text{Ch}_{w^*}(F \to F) \) and \( F_0 \subset F \) by
\[
F := \{ \Psi \in \text{Ch}_{w^*}(F \to F) \mid \Gamma \circ \Psi = \Gamma \},
\]
\[
F_0 := \{ a \in F \mid \Psi(a) = a \ (\forall \Psi \in F) \}.
\]

Then \( F_0 \) is a weakly* closed unital subalgebra of \( F \) and by the mean ergodic theorem [30] there exists weakly* continuous conditional expectation (norm-1 projection) \( E \) from \( F \) onto \( F_0 \) such that \( E \circ \Psi = \Psi \circ E = E \ (\Psi \in F) \) and \( \Gamma \circ E = \Gamma \). Then it can be shown that the restriction \( \Gamma_0 \) of \( \Gamma \) to the subalgebra \( F_0 \) is a minimally sufficient w*-measurement and post-processing equivalent to \( \Gamma \).
The following statements can also be shown similarly as in the case of the quantum theory \[11, 31, 32\]. For a finite-outcome EVM \( M \in \text{EVM}(X; E) \), the associated \( \ast \)-measurement \( \Gamma^M \in \text{Ch}(\ell^\infty(X) \to E) \) is minimally sufficient if and only if \( M \) is pairwise linearly independent, i.e. \( (M(x), M(x')) \) is linearly independent for any \( x, x' \in X \) with \( x \neq x' \). Every finite-outcome EVM \( M \) is post-processing equivalent to a pairwise linearly independent EVM \( M_0 \) and such \( M_0 \) is unique up to the bijective permutation of outcome sets.

References

[1] Alfsen, E.M.: Compact Convex Sets and Boundary Integrals. Springer (1971)

[2] Barnum, H., Barrett, J., Leifer, M., Wilce, A.: Cloning and broadcasting in generic probabilistic theories. arXiv preprint quant-ph/0611295 (2006)

[3] Bonnans, J.F., Shapiro, A.: Perturbation analysis of optimization problems. Springer (2000)

[4] Buscemi, F.: Comparison of quantum statistical models: Equivalent conditions for sufficiency. Comm. Math. Phys. 310(3), 625–647 (2012). DOI 10.1007/s00220-012-1421-3. URL https://doi.org/10.1007/s00220-012-1421-3

[5] Buscemi, F., Keyl, M., DAriano, G.M., Perinotti, P., Werner, R.F.: Clean positive operator valued measures. J. Math. Phys. 46(8), 082109 (2005). DOI 10.1063/1.2008996. URL https://doi.org/10.1063/1.2008996

[6] Busch, P., Lahti, P.J., Pellonpää, J.P., Ylinen, K.: Quantum Measurement. Springer (2016)

[7] Carmeli, C., Heinosaari, T., Toigo, A.: Quantum incompatibility witnesses. Phys. Rev. Lett. 122, 130402 (2019). DOI 10.1103/PhysRevLett.122.130402. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.130402

[8] Designolle, S., Farkas, M., Kaniewski, J.: Incompatibility robustness of quantum measurements: a unified framework. New J. Phys. 21(11), 113053 (2019)

[9] Dorofeev, S., de Graaf, J.: Some maximality results for effect-valued measures. Indag. Math. (N.S.) 8(3), 349 – 369 (1997). URL http://dx.doi.org/10.1016/S0019-3577(97)81815-0

[10] Dubra, J., Maccheroni, F., Ok, E.A.: Expected utility theory without the completeness axiom. J. Econ. Theory 115(1), 118 – 133 (2004)

[11] Ducuara, A.F., Skrzypczyk, P.: Operational interpretation of weight-based resource quantifiers in convex quantum resource theories of states. arXiv preprint arXiv:1909.10486 (2019)

[12] Ducuara, A.F., Skrzypczyk, P.: Weight of informativeness, state exclusion games and excludible information. arXiv preprint arXiv:1908.10347 (2019)
[13] Ellis, A.J.: The duality of partially ordered normed linear spaces. J. London Math. Soc. 39(1), 730–744 (1964). DOI 10.1112/jlms/s1-39.1.730

[14] Filippov, S.N., Heinosaari, T., Leppäjärvi, L.: Simulability of observables in general probabilistic theories. Phys. Rev. A 97, 062102 (2018). DOI 10.1103/PhysRevA.97.062102. URL https://link.aps.org/doi/10.1103/PhysRevA.97.062102

[15] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: Continuous lattices and domains, vol. 93. Cambridge university press (2003)

[16] Giles, R.: Mathematical Foundations of Thermodynamics. Pergamon Press (1964)

[17] Gudder, S.: Convex structures and operational quantum mechanics. Commun. Math. Phys. 29(3), 249–264 (1973). DOI 10.1007/BF01645250. URL https://doi.org/10.1007/BF01645250

[18] Gudder, S.P.: Stochastic Methods in Quantum Mechanics. North Holland (1979)

[19] Guerini, L., Bavaresco, J., Terra Cunha, M., Acín, A.: Operational framework for quantum measurement simulability. J. Math. Phys. 58(9), 092102 (2017). DOI 10.1063/1.4994303

[20] Guţă, M., Jenčová, A.: Local asymptotic normality in quantum statistics. Commun. Math. Phys. 276(2), 341–379 (2007). DOI 10.1007/s00220-007-0340-1. URL http://dx.doi.org/10.1007/s00220-007-0340-1

[21] Haapasalo, E.: Robustness of incompatibility for quantum devices. J. Phys. A: Math. Theor. 48(25), 255303 (2015). URL https://doi.org/10.1088%2F1751-8121%2F48%2F25%2F255303

[22] Haapasalo, E., Heinosaari, T., Pellonpää, J.P.: Quantum measurements on finite dimensional systems: relabeling and mixing. Quantum Inf. Process. 11(6), 1751–1763 (2012). DOI 10.1007/s11128-011-0330-2. URL https://doi.org/10.1007/s11128-011-0330-2

[23] Hartkämper, A., Neumann, H.: Foundations of Quantum Mechanics and Ordered Linear Spaces, vol. 29 (1974)

[24] Heinosaari, T., Miyadera, T., Ziman, M.: An invitation to quantum incompatibility. J. Phys. A: Math. Theor. 49(12), 123001 (2016). URL http://stacks.iop.org/1751-8121/49/i=12/a=123001

[25] Jameson, G.: Ordered Linear Spaces. Springer (1970)

[26] Janotta, P., Hinrichsen, H.: Generalized probability theories: what determines the structure of quantum theory? J. Phys. A: Math. Theor. 47(32), 323001 (2014). URL http://stacks.iop.org/1751-8121/47/i=32/a=323001

[27] Kaijser, S.: A note on dual Banach spaces. Math. Scand. 41(2), 325–330 (1978). URL http://www.jstor.org/stable/24492477
[28] Kaniowski, K., Lubnauer, K., Luczak, A.: Quantum Blackwell–Sherman–Stein Theorem and Related Results. Open Systems & Information Dynamics 20(04), 1350017 (2013)

[29] Kato, T.: On a Theorem by Kakutani. Sugaku 12(4), 234–235 (1961). URL https://doi.org/10.11429/sugaku1947.12.234 (in Japanese)

[30] Kümmerer, B., Nagel, R.: Mean ergodic semigroups on W*-algebras. Acta Sci. Math. 41(1-2), 151–159 (1979)

[31] Kuramochi, Y.: Minimal sufficient positive-operator valued measure on a separable Hilbert space. J. Math. Phys. 56(10), 102205 (2015). DOI http://dx.doi.org/10.1063/1.4934235. URL http://scitation.aip.org/content/aip/journal/jmp/56/10/10.1063/1.4934235

[32] Kuramochi, Y.: Minimal sufficient statistical experiments on von Neumann algebras. J. Math. Phys. 58(6), 062203 (2017). DOI 10.1063/1.4986247. URL http://dx.doi.org/10.1063/1.4986247

[33] Kuramochi, Y.: Directed-completeness of quantum statistical experiments in the randomization order. arXiv preprint arXiv:1805.04357v1 (2018)

[34] Kuramochi, Y.: Quantum incompatibility of channels with general outcome operator algebras. J. Math. Phys. 59(4), 042203 (2018). DOI 10.1063/1.5008300. URL https://doi.org/10.1063/1.5008300

[35] Lami, L.: Non-classical correlations in quantum mechanics and beyond. Ph.D. thesis. URL https://ddd.uab.cat/record/187745

[36] Le Cam, L.: Asymptotic methods in statistical decision theory. Springer (1986)

[37] Leverrier, A., Grangier, P.: Unconditional security proof of long-distance continuous-variable quantum key distribution with discrete modulation. Phys. Rev. Lett. 102, 180504 (2009). DOI 10.1103/PhysRevLett.102.180504. URL https://link.aps.org/doi/10.1103/PhysRevLett.102.180504

[38] Lieb, E.H., Yngvason, J.: The physics and mathematics of the second law of thermodynamics. Phys. Rep. 310(1), 1 – 96 (1999)

[39] Luczak, A.: Comparison of channels in operator algebras. J. Math. Phys. 60(2), 022203 (2019). DOI 10.1063/1.5074187. URL https://doi.org/10.1063/1.5074187

[40] Ludwig, G.: An Axiomatic Basis for Quantum Mechanics: Derivation of Hilbert Space Structure. Springer (1985)

[41] Martens, H., de Muynck, W.M.: Nonideal quantum measurements. Found. Phys. 20(3), 255–281 (1990). DOI 10.1007/BF00731693. URL https://doi.org/10.1007/BF00731693

[42] von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior, 3rd ed. Princeton University Press (1953)
[43] Olubummo, Y., Cook, T.A.: The Predual of an Order-Unit Banach Space. Int. J. Theor. Phys. 38(12), 3301–3303 (1999). DOI 10.1023/A:1026646602561

[44] Oszmaniec, M., Biswas, T.: Operational relevance of resource theories of quantum measurements. Quantum 3, 133 (2019). DOI 10.22331/q-2019-04-26-133. URL https://doi.org/10.22331/q-2019-04-26-133

[45] Oszmaniec, M., Guerini, L., Wittek, P., Acín, A.: Simulating positive-operator-valued measures with projective measurements. Phys. Rev. Lett. 119, 190501 (2017). DOI 10.1103/PhysRevLett.119.190501. URL https://link.aps.org/doi/10.1103/PhysRevLett.119.190501

[46] Paulsen, V.: Completely Bounded Maps and Operator Algebras. Cambridge Studies in Advanced Mathematics. Cambridge University Press (2003). DOI 10.1017/CBO9780511546631

[47] Pellonpää, J.P.: Quantum instruments: II. Measurement theory. J. Phys. A: Math. Theor. 46(2), 025303 (2013). URL http://stacks.iop.org/1751-8121/46/i=2/a=025303

[48] Plávala, M.: All measurements in a probabilistic theory are compatible if and only if the state space is a simplex. Phys. Rev. A 94, 042108 (2016). DOI 10.1103/PhysRevA.94.042108. URL https://link.aps.org/doi/10.1103/PhysRevA.94.042108

[49] Schaefer, H.H.: Topological Vector Spaces (2nd ed.). Springer (1999)

[50] Shapiro, A.: On Duality Theory of Conic Linear Problems, pp. 135–165. Springer, Boston, MA (2001)

[51] Skrzypczyk, P., Šupič, I., Cavalcanti, D.: All sets of incompatible measurements give an advantage in quantum state discrimination. Phys. Rev. Lett. 122, 130403 (2019). DOI 10.1103/PhysRevLett.122.130403. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.130403

[52] Skrzypczyk, P., Linden, N.: Robustness of measurement, discrimination games, and accessible information. Phys. Rev. Lett. 122, 140403 (2019). DOI 10.1103/PhysRevLett.122.140403. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.140403

[53] Takesaki, M.: Theory of Operator Algebras I. Springer (1979)

[54] Torgersen, E.: Comparison of Statistical Experiments. Cambridge University Press (1991)

[55] Uola, R., Bullock, T., Kraft, T., Pellonpää, J.P., Brunner, N.: All quantum resources provide an advantage in exclusion tasks. arXiv preprint 1909.10484 (2019)

[56] Uola, R., Kraft, T., Shang, J., Yu, X.D., Gühne, O.: Quantifying quantum resources with conic programming. Phys. Rev. Lett. 122, 130404 (2019). DOI 10.1103/PhysRevLett.122.130404. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.130404
[57] Jenčová, A.: Incompatible measurements in a class of general probabilistic theories. Phys. Rev. A 98, 012133 (2018). DOI 10.1103/PhysRevA.98.012133. URL https://link.aps.org/doi/10.1103/PhysRevA.98.012133