The double Ringel-Hall algebra on a hereditary abelian
finitary length category

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Abstract

In this paper, we study the category $\mathcal{H}(\rho)$ of semi-stable coherent sheaves of a
fixed slope $\rho$ over a weighted projective curve. This category has nice properties: it
is a hereditary abelian finitary length category. We will define the Ringel-Hall algebra
of $\mathcal{H}(\rho)$ and relate it to generalized Kac-Moody Lie algebras. Finally we obtain the
Kac type theorem to describe the indecomposable objects in this category, i.e. the
indecomposable semi-stable sheaves.

Key words: Ringel-Hall algebra, generalized Kac-Moody Lie algebra, coherent sheaf,
semi-stability.

1 Introduction

It is known that by Ringel [14] and Green [7], the composition subalgebra of the Ringel-
Hall algebra of a finite dimensional hereditary algebra $\Lambda$ over a finite field $k$ is isomorphic
to the positive part of the quantized enveloping algebra of the Kac-Moody Lie algebra
$\mathfrak{g}$ determined by the Euler form. By Sevenhant and Van den Bergh [17], and Deng and
Xiao [3], the double Ringel-Hall algebra with a new gradation is essentially isomorphic
to the quantized enveloping algebra of a generalized Kac-Moody algebra $\mathfrak{g}'$, which ad-
mits imaginary simple roots and contains $\mathfrak{g}$ as a Lie subalgebra. By Kac [9], the set of
dimension vectors of the indecomposable modules of $\Lambda$ coincides with the set of positive
roots of $\mathfrak{g}$. Moreover, if $\alpha$ is a real root, there is a unique, up to isomorphism, inde-
composable module with dimension vector $\alpha$. In [4], it was shown that the Ringel-Hall
algebra approach provides a new and self-contained proof of the above Kac theorem. The case for valued quivers with loops was considered in [18].

We generalize these results from the module category mod(Λ) to an arbitrary hereditary abelian finitary length k-category \( \mathcal{A} \). Recall that a category \( \mathcal{A} \) is called finitary if Hom-spaces and Ext-spaces are finite sets. It is called a length category if any object has a finite Jordan-Hölder series. To the category \( \mathcal{A} \) we associate the composition Ringel-Hall algebra \( \mathcal{C}(\mathcal{A}) \) and the double Ringel-Hall algebra \( \mathcal{D}(\mathcal{A}) \). By the Euler form on \( \mathcal{A} \) we define two Borcherds-Cartan matrices \( C_0 \) and \( C \), and associate to them two generalized Kac-Moody Lie algebras \( g(C_0) \) and \( g(C) \).

We prove that the double composition algebra \( \mathcal{C}(\mathcal{A}) \) of the Ringel-Hall algebra is isomorphic to the quantized enveloping algebra of \( g(C_0) \) (see Theorem 4.1); and the double Ringel-Hall algebra \( \mathcal{D}(\mathcal{A}) \) essentially gives a realization of the quantized enveloping algebra of \( g(C) \) (see Theorem 4.5). Roughly speaking, by enlarging the Cartan part of \( \mathcal{D}(\mathcal{A}) \) and giving a new gradation we obtain the quantized enveloping algebra of \( g(C) \). Furthermore, the set of dimension vectors of the indecomposable objects in \( \mathcal{A} \) is the union of the positive roots of \( g(C_0) \) and \( W_0(\cup_{s \geq 2}sI^{im}) \), where \( W_0 \) is the Weyl group of \( g(C_0) \) and \( I^{im} \) is the set of imaginary simple roots of \( g(C_0) \) (see Theorem 4.8). Moreover, if \( \alpha \) is a real root, there is a unique, up to isomorphism, indecomposable object in \( \mathcal{A} \) with dimension vector \( \alpha \).

As a special case, we consider the category \( \mathcal{H}^{(\rho)} \) of semi-stable coherent sheaves of a fixed slope \( \rho \) over a weighted projective curve. All the above results can apply to this category.

The paper is organized as follows. In Section 2, we recall the basic knowledge of generalized Kac-Moody Lie algebra and its quantized enveloping algebra. In Section 3 we define the composition and the double Ringel-Hall algebra of a hereditary abelian finitary length category \( \mathcal{A} \). The structure of the composition and the double Ringel-Hall algebra, and the relation with generalized Kac-Moody Lie algebras are studied in Section 4. In Section 5 we apply our results to the category \( \mathcal{H}^{(\rho)} \) and classify the dimension vectors of the indecomposable semi-stable sheaves of slope \( \rho \).

Throughout the paper \( k \) will be a fixed finite field \( \mathbb{F}_q \), and \( v = \sqrt{q} \) be a complex
number (and not a root of unity).

Finally we note that a deeper relation between the indecomposable coherent sheaves over weighted projective lines and the root system of the loop algebras of Kac-Moody algebras was found by Crawley-Boevey in [2].

2 Generalized Kac-Moody Lie algebras

In this section we recall the definition of generalized Kac-Moody Lie algebra and its quantized enveloping algebra. Generalized Kac-Moody Lie algebras were introduced by Borcherds [1], and their quantized version was defined by Kang [10]. For reference one sees also [17] and [4].

Let $I$ be an index set (possibly infinite or even uncountable).

Definition 2.1. A complex matrix $C = (c_{ij})_{i,j \in I}$ is called a Borcherds-Cartan matrix if the following holds:

(i) $c_{ii} = 2$ or $c_{ii} \leq 0$, for any $i \in I$;

(ii) $c_{ij} \leq 0$, for any $i, j \in I$ and $i \neq j$;

(iii) $c_{ij} \in \mathbb{Z}$, for any $i, j \in I$ and $c_{ii} = 2$;

(iv) $c_{ij} = 0$ if and only if $c_{ji} = 0$, for any $i, j \in I$.

Set $I^{re} = \{i \in I : c_{ii} = 2\}$ and $I^{im} = \{i \in I : c_{ii} \leq 0\}$. The index set $I$ is the disjoint union of $I^{re}$ and $I^{im}$.

Definition 2.2. A Borcherds-Cartan matrix $C$ is called symmetrizable, if there exists positive number $\varepsilon_i$ for $i \in I$ satisfying that $\varepsilon_i c_{ij} = \varepsilon_j c_{ji}$ for any $i, j \in I$.

Two symmetrizable Borcherds-Cartan matrices $C = (c_{ij})$ (with symmetrization $\varepsilon_i$) and $C' = (c'_{ij})$ (with symmetrization $\varepsilon'_i$) are identified, if they correspond to the same symmetrization, namely $\varepsilon_i c_{ij} = \varepsilon'_i c'_{ij}$ for any $i, j \in I$.

Remark 2.3. Under the above identification, a symmetrizable Borcherds-Cartan matrix is one-to-one correspondent to a symmetric bilinear form $(-, -) : \mathbb{C}I \times \mathbb{C}I \rightarrow \mathbb{C}$ satisfying that $(i, j) \leq 0$ for any $i \neq j$ in $I$, and that if $(i, i)$ is positive then $\frac{2(i, j)}{(i, i)} \in \mathbb{Z}$.

Such a bilinear form is called a generalized Kac-Moody bilinear form.
Indeed, given a symmetrizable Borcherds-Cartan matrix $C = (c_{ij})$ with symmetrization $\varepsilon_i$, the bilinear form defined by $(i, j) = \varepsilon_i c_{ij}$, for any $i, j \in I$, is a generalized Kac-Moody bilinear form. Conversely, one can associate to a generalized Kac-Moody bilinear form $(-,-) : \mathbb{C}I \times \mathbb{C}I \rightarrow \mathbb{C}$ a symmetrizable Borcherds-Cartan matrix $C$ defined by

$$
c_{ij} = \begin{cases} 2(i,j) & \text{if } (i,i) > 0 \\ (i,j) & \text{otherwise} \end{cases}
$$

with symmetrization

$$
\varepsilon_i = \begin{cases} (i,i) / 2 & \text{if } (i,i) > 0 \\ 1 & \text{otherwise} \end{cases}
$$

The pair $(I, (-, -))$ is called a Borcherds datum following the notion Cartan datum of Lustig [12].

Recall that a complex matrix $C = (c_{ij})_{i,j \in I}$ is called a generalized Cartan matrix provides that

(i) $I$ is a finite set;
(ii) $c_{ii} = 2$, for any $i \in I$;
(iii) $c_{ij} \in \mathbb{Z}_{\geq 0}$, for any $i, j \in I$ and $i \neq j$;
(iv) $c_{ij} = 0$ if and only if $c_{ji} = 0$, for any $i, j \in I$.

Clearly generalized Cartan matrices are Borcherds-Cartan matrices with $I = I^{re}$ being finite. In particular, symmetrizable generalized Cartan matrices are symmetrizable Borcherds-Cartan matrices.

**Definition 2.4.** To a symmetrizable Borcherds-Cartan matrix $C$, we associate a complex Lie algebra $g(C)$, called the generalized Kac-Moody Lie algebra, which is generated by $\{e_i, f_i, h_i : i \in I\}$ with relations

(i) $[h_i, h_j] = 0$, $\forall i, j \in I$;
(ii) $[h_i, e_j] = c_{ij} e_j$, $[h_i, f_j] = -c_{ij} f_j$, $\forall i, j \in I$;
(iii) $[e_i, f_j] = \delta_{ij} h_i$, $\forall i, j \in I$;
(iv) $(ade_i)^{1-c_{ij}} e_i = 0$, $(adf_i)^{1-c_{ij}} f_j = 0$, $\forall i \in I^{re}$ and $j \in I$ with $i \neq j$;
(v) $[e_i, e_j] = 0$, $[f_i, f_j] = 0$, $\forall i, j \in I$ with $c_{ij} = 0$. 


For \( i \in \mathcal{I}^e \), we define a linear transformation \( \hat{r}_i : \mathbb{C}\mathcal{I} \to \mathbb{C}\mathcal{I} \) sending \( j \in \mathcal{I} \) to \( j - c_{ij}i \). The Weyl Group \( W = W(C) \) of the generalized Kac-Moody algebra \( g(C) \) is the subgroup of \( GL(\mathbb{C}\mathcal{I}) \) generated by the reflections \( \{\hat{r}_i : i \in \mathcal{I}^e\} \). The root system \( \Delta = \Delta(C) \) of \( g(C) \) can be described as follows:

\[
\Delta = \Delta_+ \cup \Delta_-, \quad \Delta_+ = -\Delta_+ \quad \Delta = \Delta^e \cup \Delta^{im},
\]

\[
\Delta^e = W(\mathcal{I}^e) = \{w(i) : w \in W, \ i \in \mathcal{I}^e\},
\]

\[
\Delta^{im} = W(\mathcal{F} \cup -\mathcal{F}),
\]

where \( \mathcal{F} = \{0 \neq \mu \in \mathbb{N}\mathcal{I} : (\mu, i) \leq 0, \forall \ i \in \mathcal{I}^e, \ \text{supp}(\mu) \text{ is connected}\}\setminus \bigcup_{s \geq 2} s\mathcal{I}^{im} \), called the fundamental region. Note that \( i \in \mathcal{I}^e \) are real simple roots and \( i \in \mathcal{I}^{im} \) are imaginary simple roots.

**Definition 2.5.** Let \( v \) be a complex number (not a root of unity). The quantized enveloping algebra \( U_v(g(C)) \) of a generalized Kac-Moody Lie algebra \( g(C) \) is the algebra over \( \mathbb{C} \) generated by \( \{E_i, F_i : i \in \mathcal{I}\} \) and \( \{K_\mu : \mu \in \mathbb{Z}\mathcal{I}\} \) with relations

(i) \( K_0 = 1, \ K_\mu K_\nu = K_{\mu+\nu}, \ \forall \ \mu, \nu \in \mathbb{Z}\mathcal{I} \);

(ii) \( K_\mu E_i = v^{(\mu,i)} E_i K_\mu, \ K_\mu F_i = v^{-(\mu,i)} F_i K_\mu, \ \forall \ i \in \mathcal{I}, \ \mu \in \mathbb{Z}\mathcal{I} \);

(iii) \( E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v_i - v_{-i}}, \ \forall \ i, j \in \mathcal{I} \);

(iv) \( \sum_{p=0}^{1-c_{ij}} (-1)^p \left[ \frac{1 - c_{ij}}{p} \right]_{v_i} E_i^p F_j E_i^{1-c_{ij}} - p = 0, \ \forall \ i \in \mathcal{I}^e, \ i \neq j \in \mathcal{I} \);

(iv)' \( \sum_{p=0}^{1-c_{ij}} (-1)^p \left[ \frac{1 - c_{ij}}{p} \right]_{v_i} F_i^p F_j E_i^{-c_{ij}} - p = 0, \ \forall \ i \in \mathcal{I}^e, \ i \neq j \in \mathcal{I} \);

(v) \( E_i E_j - E_j E_i = 0, \ F_i F_j - F_j F_i = 0, \ \forall i, j \in \mathcal{I} \) with \( c_{ij} = 0 \);

where \( v_i = v^{\varepsilon_i} \) (\( \varepsilon_i \) are the symmetrization of \( C \)), and

\[
[n]_{v_i} = \frac{v_i^n - v_{-i}^{-n}}{v_i - v_{-i}^{-1}}, \quad [n]_{v_i}! = \prod_{k=1}^{n} [k]_{v_i},
\]

\[
\begin{pmatrix} m \\ n \end{pmatrix}_{v_i} = \frac{[m]_{v_i}!}{[m-n]_{v_i}![n]_{v_i}!}.
\]

The quantized enveloping algebra admits a natural triangle decomposition \( U_v(g(C)) = U_v^-(g(C)) \otimes U_v^0(g(C)) \otimes U_v^+(g(C)) \), where the negative part \( U_v^-(g(C)) \) is generated by
$F_i$ and $K_\mu$, the Cartan part by $K_\mu$, and the positive part by $E_i$ and $K_\mu$. Define the formal character of $U^-_v(\mathfrak{g}(C))$ by

$$\text{ch} U^-_v(\mathfrak{g}(C)) = \sum_{\mu \in \mathbb{N}^I} \dim_{\mathbb{C}} U^-_v(\mathfrak{g}(C))_{-\mu} e(-\mu).$$

By [1] (see also [17]), we have the following proposition.

**Proposition 2.6.** The formal character of $U^-_v(\mathfrak{g}(C))$ is

$$\text{ch} U^-_v(\mathfrak{g}(C)) = \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{-\text{mult}_{g(C)}(\alpha)}.$$

It is well-known that $U_\nu(\mathfrak{g}(C))$ is a Hopf algebra with comultiplication $\Delta$, counit $\epsilon$ and the antipode $S$ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1,$$

$$S(E_i) = -K_{-i}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_{-i}.$$

Write $U^-_\nu(\mathfrak{g}(C))$ (respectively $U^\leq_\nu(\mathfrak{g}(C))$) for the subalgebras of $U_\nu(\mathfrak{g})$ generated by $E_i, K_\mu$ (respectively by $F_i, K_\mu$). It is clear they are Hopf subalgebras of $U_\nu(\mathfrak{g}(C))$.

Define a bilinear form $\phi : U^-_\nu(\mathfrak{g}(C)) \times U^\leq_\nu(\mathfrak{g}(C)) \rightarrow \mathbb{C}$ by

$$\phi(E_i, F_j) = \delta_{ij} \frac{-1}{v_i - v_j}, \quad \phi(K_i, K_j) = v^{-(i,j)},$$

$$\phi(K_i, F_j) = 0 = \phi(E_i, K_j),$$

for any $i,j \in \mathcal{I}$, and extend it according to the following relations: for any $a, a'$ in $U^-_\nu(\mathfrak{g}(C))$ and $b, b'$ in $U^\leq_\nu(\mathfrak{g}(C))$,

(i) $\phi(a,1) = \epsilon(a)$, $\phi(1,b) = \epsilon(b)$;

(ii) $\phi(ab, b') = \phi(\Delta(a), b \otimes b')$;

(iii) $\phi(a, b') = \phi(a \otimes a', \Delta^\text{op}(b'))$;

(iv) $\phi(S(a), b) = \phi(a, S^{-1}(b))$,

where $\phi(a \otimes a', b \otimes b') = \phi(a,b)\phi(a', b')$, and $\Delta^\text{op}(b) = \sum b_2 \otimes b_1$, if $\Delta(b) = \sum b_1 \otimes b_2$.

Such a bilinear form $\phi$ satisfying (i) – (iv) is called a skew-Hopf pairing. Note that sometimes the triple $(U^-_\nu(\mathfrak{g}), U^\leq_\nu(\mathfrak{g}), \phi)$ is called a skew-Hopf pairing.
Proposition 2.7. (Proposition 2.4 [17]) The skew-Hopf pairing \((U^+_v(g), U^-_v(g), \phi)\) defined above is restricted non-degenerate, that means its restricted form \(\phi : U^+_v(g) \times U^-_v(g) \rightarrow \mathbb{C}\) is non-degenerate.

3 The double Ringel-Hall algebra

Let \(k = \mathbb{F}_q\) be a fixed finite field, and \(\mathcal{A}\) be an abelian category. Assume that \(\mathcal{A}\) is \(k\)-linear, Hom-finite and Ext-finite. That is, for all objects \(X, Y\) and \(Z\) in \(\mathcal{A}\), the sets \(\text{Hom}(X,Y)\) and \(\text{Ext}^1(X,Y)\) are finite dimensional \(k\)-vector spaces and the composition \(\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \rightarrow \text{Hom}(X,Z)\) is \(k\)-bilinear. Assume further that \(\mathcal{A}\) is hereditary, i.e. \(\text{Ext}^i(-,-)\) vanishes for all \(i \geq 2\), and that \(\mathcal{A}\) is a length category, i.e. all objects in \(\mathcal{A}\) have a composition series of finite length.

Following Ringel [13], Green [7] and Xiao [19], we associate to such a category \(\mathcal{A}\) a Hopf algebra, called the Ringel-Hall algebra, and a doubled version of the Ringel-Hall algebra, by using a skew-Hopf pairing.

3.1 Ringel-Hall algebras and skew-Hopf pairings

Let \(\mathcal{A}\) be an abelian category as above. Let \(\mathcal{P}\) be the set of isomorphism classes of objects in \(\mathcal{A}\), \(\mathcal{P}_1\) the complement set of \(\{0\}\) in \(\mathcal{P}\), and \(\mathcal{I}\) the set of isomorphism classes of simple objects in \(\mathcal{A}\). For \(\alpha \in \mathcal{P}\), write \(M_\alpha\) for a representative object of \(\alpha\) in \(\mathcal{A}\). In particular if \(i \in \mathcal{I}\), we write \(S_i\) for a simple object in \(\mathcal{A}\) corresponding to \(i\). The Grothendieck group of the category \(\mathcal{A}\) is the free abelian group \(\mathbb{Z}\mathcal{I}\) with basis \(\mathcal{I}\), as \(\mathcal{A}\) is a length category. For any object \(M\) in \(\mathcal{A}\), write \(\dim M\) for the image of \(M\) in \(\mathbb{Z}\mathcal{I}\), called the dimension vector of \(M\), which is given by the composition factors of \(M\), or equivalently, uniquely determined by the rule: \(\dim L = \dim M + \dim N\) for any exact sequence in \(\mathcal{A}\) : \(0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0\).

For \(\alpha, \beta\) and \(\gamma\) in \(\mathcal{P}\), the Ringel-Hall number \(g^\gamma_{\alpha\beta}\) counts the number of subobjects \(X\) of \(M_\gamma\) satisfying \(X \cong M_\beta\) and \(M_\gamma/X \cong M_\alpha\). The Euler form \(\langle \alpha, \beta \rangle = \dim_k \text{Hom}_{\mathcal{A}}(M_\alpha, M_\beta) - \dim_k \text{Ext}^1_{\mathcal{A}}(M_\alpha, M_\beta)\), and the symmetric Euler form \(\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle\). Denote by
$a_\alpha$ the cardinality of the automorphism group of $M_\alpha$.

Lemma 3.1. The symmetric Euler form $(-,-) : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$ is a generalized Kac-Moody bilinear form.

Proof. By definition, one needs to check: (i) For $i,j \in I$ and $i \neq j$, $(i,j) \leq 0$, and (ii) If $(i,i) > 0$, then $\frac{2(i,j)}{(i,i)} \in \mathbb{Z}$.

(i) is obvious. For (ii), note that for $i,j \in I$, $\text{End}_{\mathcal{A}}(S_i)$ is a finite skew field (hence a field), and $\text{Ext}_{\mathcal{A}}^1(S_i, S_j)$ has the natural structure as $\text{End}_{\mathcal{A}}(S_j) \cdot \text{End}_{\mathcal{A}}(S_i)$-bimodule. Suppose $(i,i) > 0$. Namely

$$2(\dim_k \text{End}_{\mathcal{A}}(S_i) - \dim_k \text{Ext}_{\mathcal{A}}^1(S_i, S_i)) =$$

$$\dim_k \text{End}_{\mathcal{A}}(S_i)(1 - \dim_k \text{Ext}_{\mathcal{A}}^1(S_i, S_i)) > 0$$

This implies $S_i$ has no self-extensions, and $(i,i) = 2\dim_k \text{End}_{\mathcal{A}}(S_i)$. Hence, when $i \neq j$,

$$\frac{2(i,j)}{(i,i)} = -\frac{\dim_k \text{Ext}_{\mathcal{A}}^1(S_i, S_j) + \dim_k \text{Ext}_{\mathcal{A}}^1(S_j, S_i)}{\dim_k \text{End}_{\mathcal{A}}(S_i)}$$

$$= -\dim_k \text{End}_{\mathcal{A}}(S_i) \text{Ext}_{\mathcal{A}}^1(S_i, S_j) - \dim_k \text{End}_{\mathcal{A}}(S_j) \text{Ext}_{\mathcal{A}}^1(S_j, S_i)$$

is an integer. \hfill \square

By Remark 2.3, the symmetric Euler form determines a symmetrizable Borcherds-Cartan matrix, denoted by $C_0 = (c_{ij})_{i,j \in I}$, where $c_{ij} = \frac{2(i,j)}{(i,i)}$ if $(i,i) > 0$ and $(i,j)$ otherwise, with symmetrization $\varepsilon_i = \frac{(i,i)}{2} = (i,i)$ if $(i,i) > 0$ and 1 otherwise.

Write $v$ for the complex number $\sqrt{q}$, and $v_i$ for $v^\varepsilon_i$ for $i \in I$.

Definition 3.2. The ‘positive’ Ringel-Hall algebra, denoted by $\mathcal{H}^+(\mathcal{A})$, is defined to be the Hopf algebra over $\mathbb{C}$ with basis $\{K_\mu u_\alpha^+ : \mu \in \mathbb{Z}I, \alpha \in \mathcal{P}\}$ whose Hopf structure is given by the following:

(i) (multiplication and unit)

$$u_\alpha^+ u_\beta^+ = v^{(\alpha,\beta)} \sum_{\gamma \in \mathcal{P}} g_{\alpha\beta}^\gamma u_\gamma^+$$

$$K_\mu K_\nu = K_{\mu+\nu},$$

$$K_\mu u_\alpha^+ = v^{(\mu,\alpha)} u_\alpha^+ K_\mu, \quad 1 = u_0^+ = K_0;$$
(ii) (comultiplication and counit)

\[ \Delta(u^+_\gamma) = \sum_{\alpha, \beta \in P} v^{(\alpha, \beta)} a_\alpha a_\beta g^\gamma_{\alpha \beta} u^+_\alpha K^\beta \otimes u^+_\beta, \]

\[ \Delta(K_\mu) = K_\mu \otimes K_\mu, \quad \epsilon(u^+_\alpha) = \delta_{\alpha,0}, \quad \epsilon(K_\mu) = 1; \]

(iii) (antipode)

\[ S(K_\mu) = K_{-\mu}, \]

\[ S(u^+_\alpha) = \delta_{\gamma,0} + \sum_{m \geq 1} (-1)^m \sum_{\pi \in P, \gamma_1, \ldots, \gamma_m \in P_1} v^2 \sum_{\pi < j} g_{\gamma_j \gamma_2} \ldots \gamma_m a_{\gamma_1} \ldots a_{\gamma_m} g^\gamma_{\gamma_j \gamma_2} \ldots g^\pi_{\gamma_m \gamma_1} u^+_\pi K_{-\gamma_1} \ldots K_{-\gamma_m} K_{-\gamma_1} \ldots K_{-\gamma_m}. \]

**Definition 3.3.** The ‘negative’ Ringel-Hall algebra, denoted by \( \mathcal{H}^- (\mathcal{A}) \), is defined to be the Hopf algebra over \( \mathbb{C} \) with basis \( \{ K_\mu u^-_\alpha : \mu \in \mathbb{Z}, \alpha \in P \} \) whose Hopf structure is given by the following:

(i) (multiplication and unit)

\[ u^-_\alpha u^-_\beta = v^{(\alpha, \beta)} \sum_{\gamma \in P} g^\gamma_{\alpha \beta} u^-_\gamma, \quad K_\mu K_\nu = K_{\mu + \nu}, \]

\[ K_\mu u^-_\alpha = v^{- (\mu, \alpha)} u^-_\alpha K_\mu, \quad 1 = u^+:_0 = K_0; \]

(ii) (comultiplication and counit)

\[ \Delta(u^-_\gamma) = \sum_{\alpha, \beta \in P} v^{(\beta, \alpha)} a_\alpha a_\beta g^\gamma_{\beta \alpha} u^-_\alpha \otimes u^-_\beta K^-_{-\alpha}, \]

\[ \Delta(K_\mu) = K_\mu \otimes K_\mu, \quad \epsilon(u^-_\alpha) = \delta_{\alpha,0}, \quad \epsilon(K_\mu) = 1; \]

(iii) (antipode)

\[ S(K_\mu) = K_{-\mu}, \]

\[ S(u^-_\alpha) = \delta_{\gamma,0} + \sum_{m \geq 1} (-1)^m \sum_{\pi \in P, \gamma_1, \ldots, \gamma_m \in P_1} v^2 \sum_{\pi < j} g_{\gamma_j \gamma_2} \ldots \gamma_m a_{\gamma_1} \ldots a_{\gamma_m} g^\gamma_{\gamma_j \gamma_2} \ldots g^\pi_{\gamma_m \gamma_1} u^-_\pi K_{-\gamma_1} \ldots K_{-\gamma_m} K_{-\gamma_1} \ldots K_{-\gamma_m}. \]

**Remark 3.4.** See Schiffmann’s lecture note [16], in a more general setting, for the proof of the Hopf structure defined as above. The multiplication of Ringel-Hall algebras was defined by Ringel [13], the bialgebra structure was defined by Green [7], and the antipode was found by Xiao [19].
Following Ringel [14], we define a bilinear form \( \varphi : \mathcal{H}^{+}(\mathcal{A}) \times \mathcal{H}^{-}(\mathcal{A}) \to \mathbb{C} \) by

\[
\varphi(K_{\mu}u_{\alpha}^{+}, K_{\nu}u_{\beta}^{-}) = v^{-(\mu, \nu) - (\alpha, \nu) + (\mu, \beta)} \frac{1}{a_{\alpha}} \delta_{\alpha \beta},
\]

for any \( \mu, \nu \in \mathbb{Z}I \) and \( \alpha, \beta \in \mathcal{P} \). In particular for \( i \in I \),

\[
\varphi(u_{i}^{+}, u_{i}^{-}) = \frac{1}{a_{i}} = \begin{cases} 
\frac{1}{q^{i-1}} = \frac{1}{v_{i}^{i-1}} & \text{if } (i, i) > 0 \\
\frac{1}{q^{\dim_{k}(\text{End}(S_{i}))}} = \frac{1}{v_{i}^{i}} & \text{otherwise.}
\end{cases}
\]

Similar to Xiao [19] Proposition 5.3, we have the following lemma.

**Lemma 3.5.** With the bilinear form \( \varphi \) defined as above, \((\mathcal{H}^{+}(\mathcal{A}), \mathcal{H}^{-}(\mathcal{A}), \varphi)\) is a skew-Hopf pairing.

One needs to check \( \varphi \) satisfies similar relations as (i) – (iv) stated before Proposition 2.7. The proof is straightforward and hence omitted.

### 3.2 The double Ringel-Hall algebra

The **double Ringel-Hall algebra** of the category \( \mathcal{A} \) is defined to be the reduced Drinfeld double of the skew-Hopf pairing \((\mathcal{H}^{+}(\mathcal{A}), \mathcal{H}^{-}(\mathcal{A}), \varphi)\), denoted by \( \mathcal{D}(\mathcal{A}) \). It is the quotient of the Hopf algebra \( \mathcal{H}^{+}(\mathcal{A}) \otimes \mathcal{H}^{-}(\mathcal{A}) \) factoring out the Hopf ideal generated by \( \{K_{\mu} \otimes K_{-\mu} - 1 \otimes 1 : \mu \in \mathbb{Z}I\} \), with the Hopf structure inherited from \( \mathcal{H}^{+}(\mathcal{A}) \otimes \mathcal{H}^{-}(\mathcal{A}) \).

It has a triangle decomposition of the form

\[
\mathcal{D}(\mathcal{A}) = \mathfrak{h}^{-}(\mathcal{A}) \otimes \mathcal{T} \otimes \mathfrak{h}^{+}(\mathcal{A}),
\]

where \( \mathcal{T} \) is the subalgebra of \( \mathcal{D}(\mathcal{A}) \) generated by \( \{K_{\mu} : \mu \in \mathbb{Z}I\} \), and \( \mathfrak{h}^{+}(\mathcal{A}) \) (\( \mathfrak{h}^{-}(\mathcal{A}) \)) is the subalgebra of \( \mathcal{H}^{+}(\mathcal{A}) \) (\( \mathcal{H}^{-}(\mathcal{A}) \)) generated by \( \{u_{\alpha}^{+} : \alpha \in \mathcal{P}\} \) (\( \{u_{\beta}^{-} : \beta \in \mathcal{P}\} \), respectively).

**Lemma 3.6.** In \( \mathcal{D}(\mathcal{A}) \) we have for \( i, j \in I \) that

\[
u_{i}^{+} u_{j}^{-} - u_{j}^{-} u_{i}^{+} = -\varphi(u_{i}^{+}, u_{j}^{-})(K_{i} - K_{-i}).
\]

**Proof.**

\[
\Delta^{2}(u_{i}^{+}) = u_{i}^{+} \otimes 1 \otimes 1 + K_{i} \otimes u_{i}^{+} \otimes 1 + K_{i} \otimes K_{i} \otimes u_{i}^{+},
\]

\[
\Delta^{2}(u_{j}^{-}) = 1 \otimes 1 \otimes u_{j}^{-} + 1 \otimes u_{j}^{-} \otimes K_{-j} + u_{j}^{-} \otimes K_{-j} \otimes K_{-j}.
\]

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Hence

\[
\begin{align*}
    u_j^+ u_i^- &= (1 \otimes u_j^-)(u_i^+ \otimes 1) \\
    &= \varphi(u_i^+, S(u_j^-)) \cdot 1 \otimes K_{-j} \cdot \varphi(1, K_{-j}) + \varphi(K_i, S(1)) \cdot u_i^+ \otimes u_j^- \cdot \varphi(1, K_{-j}) + \\
    &\quad \varphi(K_i, S(1)) \cdot K_i \otimes 1 \cdot \varphi(u_i^+, u_j^-) \\
    &= -1 \otimes K_{-j} \cdot \varphi(u_i^+, u_j^-) + K_i \otimes 1 \cdot \varphi(u_i^+, u_j^-) + u_i^+ \otimes u_j^-,
\end{align*}
\]

where \( \varphi(u_i^+, S(u_j^-)) = \varphi(u_i^+, -u_j^- K_j) = -\varphi(\Delta(u_i^+), u_j^- \otimes K_j) = -\varphi(u_i^+, u_j^-) \). On the other hand, \( u_i^+ u_j^- = (u_i^+ \otimes 1)(1 \otimes u_j^-) = u_i^+ \otimes u_j^- \). Hence in the double Ringel-Hall algebra \( \mathcal{D}(\mathcal{A}) \), \( u_i^+ u_j^- - u_j^- u_i^+ = -\varphi(u_i^+, u_j^-)(K_i - K_{-i}) \). \( \square \)

By setting \( \deg(u_{\alpha}^+) = \dim(M_\alpha) \), \( \deg(u_{\alpha}^-) = -\dim(M_\alpha) \) and \( \deg(K_\mu) = 0 \), for \( \alpha \in \mathcal{P} \) and \( \mu \in \mathbb{Z} \mathcal{I} \), the Hopf algebras \( \mathcal{H}^+(\mathcal{A}) \) and \( \mathcal{H}^-(\mathcal{A}) \) become \( \mathbb{N} \mathcal{I} \)-graded and \( -\mathbb{N} \mathcal{I} \)-graded, respectively. Hence the double Ringel-Hall algebra \( \mathcal{D}(\mathcal{A}) \) is \( \mathbb{Z} \mathcal{I} \)-graded. For any \( \mu \in \mathbb{N} \mathcal{I} \), the homogeneous space \( \mathfrak{h}^\pm(\mathcal{A})_{\pm\mu} \) is a finite dimensional \( \mathbb{C} \)-vector space with basis \( \{ u_{\alpha}^\pm : \alpha \in \mathcal{P}_\mu \} \), where \( \mathcal{P}_\mu = \{ \alpha \in \mathcal{P} : \dim(M_\alpha) = \mu \} \).

The double Ringel-Hall algebra \( \mathcal{D}(\mathcal{A}) \) has an important algebra automorphism \( \omega : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A}) \) defined on generators by

\[
\omega(u_{\alpha}^+) = u_{\alpha}^-, \quad \omega(u_{\alpha}^-) = u_{\alpha}^+, \quad \omega(K_\mu) = K_{-\mu},
\]

for all \( \alpha \in \mathcal{P} \) and \( \mu \in \mathbb{Z} \mathcal{I} \). It is easy to see that the operator \( \omega \) is an involution, i.e. \( \omega^2 = id \), and that \( \omega \) induces algebra isomorphisms \( \mathfrak{h}^+(\mathcal{A}) \xrightarrow{\simeq} \mathfrak{h}^-(\mathcal{A}) \) and \( \mathcal{H}^+(\mathcal{A}) \xrightarrow{\simeq} \mathcal{H}^-(\mathcal{A}) \).

**Lemma 3.7.** (i) The operator \( \omega \) is a coalgebra anti-morphism of \( \mathcal{D}(\mathcal{A}) \), i.e. \( \Delta \circ \omega = \omega \circ \Delta^\text{op} \).

(ii) For any \( x \in \mathcal{H}^+(\mathcal{A}) \) and \( y \in \mathcal{H}^-(\mathcal{A}) \), we have \( \varphi(x, y) = \varphi(\omega(y), \omega(x)) \).

(iii) The relation between \( \omega \) and the antipode \( S \) is \( S \circ \omega = \omega \circ S^{-1} \).

**Proof.** (i) Since \( \omega \) and \( \Delta \) are algebra morphisms, it suffices to check for the algebra
generators \( K_\mu \ (\mu \in \mathbb{Z}I) \) and \( u_\gamma^\pm \ (\gamma \in \mathcal{P}) \). We have that

\[
\Delta \circ \omega(K_\mu) = \Delta(K_{-\mu}) = K_{-\mu} \otimes K_{-\mu} = \omega \circ \Delta^{op}(K_\mu),
\]

\[
\Delta \circ \omega(u_\gamma^+) = \Delta(u_\gamma^-) = \sum_{\alpha, \beta \in \mathcal{P}} v^{(\beta, \alpha)} a_{\alpha} a_{\beta} g_{\beta \alpha} u_\alpha^- \otimes u_\beta^- K_{-\alpha},
\]

\[
\omega \circ \Delta^{op}(u_\gamma^+) = \omega(\sum_{\alpha, \beta \in \mathcal{P}} v^{(\alpha, \beta)} a_{\alpha} a_{\beta} g_{\alpha \beta} u_\alpha^+ \otimes u_\beta^- K_{\alpha})
\]

\[
= \sum_{\alpha, \beta \in \mathcal{P}} v^{(\alpha, \beta)} a_{\alpha} a_{\beta} g_{\alpha \beta} u_\alpha^+ \otimes u_\beta^- K_{\beta}.
\]

So \( \Delta \circ \omega(u_\gamma^+) = \omega \circ \Delta^{op}(u_\gamma^+) \). Similarly \( \Delta \circ \omega(u_\gamma^-) = \omega \circ \Delta^{op}(u_\gamma^-) \) holds.

\( (ii) \) By the linearity of \( \omega \) and the bilinearity of \( \varphi \), it is sufficient to check for basis elements. Take \( x = K_\mu u_\alpha^+ \) and \( y = K_\nu u_\beta^- \). Then

\[
\varphi(\omega(y), \omega(x)) = \varphi(K_{-\nu} u_\beta^+, K_{-\mu} u_\alpha^-) = v^{(\nu, \mu) + (\beta, \alpha) - (\alpha, \beta)} \frac{1}{a_{\alpha} a_{\beta}} \delta_{\alpha \beta}
\]

\[
= \varphi(x, y).
\]

\( (iii) \) It suffices to check for \( K_\mu \ (\mu \in \mathbb{Z}I) \) and \( u_\gamma^\pm \ (\gamma \in \mathcal{P}) \). We have

\[
S \circ \omega(K_\mu) = S(K_{-\mu}) = K_\mu = \omega \circ S^{-1}(K_\mu),
\]

\[
S \circ \omega(u_\lambda^+) = S(u_\lambda^-) = \delta_{\gamma 0} + \sum_{m \geq 1} (-1)^m \sum_{\pi \in \mathcal{P}, \gamma_1, \ldots, \gamma_m \in \mathcal{P}} \frac{a_{\gamma_1} \cdots a_{\gamma_m}}{a_{\gamma}} g_{\gamma_1 \cdots \gamma_m} g_{\gamma_m \cdots \gamma_1} u_\pi^- K_\gamma
\]

\[
= \omega \circ S^{-1}(u_\lambda^+),
\]

and similar for \( u_\lambda^- \). \( \square \)

Consequently, the skew-Hopf pairing \( \varphi : \mathcal{H}^+(\mathcal{A}) \times \mathcal{H}^-(\mathcal{A}) \rightarrow \mathbb{C} \) defined in the last subsection gives rise to a Hopf pairing \( \psi : \mathcal{H}^+(\mathcal{A}) \times \mathcal{H}^+(\mathcal{A}) \rightarrow \mathbb{C} \) defined by \( \psi(a, b) = \varphi(a, \omega(b)) \). That is, for any \( a, a' \) and \( b, b' \) in \( \mathcal{H}^+(\mathcal{A}) \), the following holds:

\( (i)' \) \( \psi(a, 1) = \epsilon(a), \ \psi(1, b) = \epsilon(b); \)

\( (ii)' \) \( \psi(a, bb') = \psi(\Delta(a), b \otimes b'); \)

\( (iii)' \) \( \psi(aa', b) = \psi(a \otimes a', \Delta(b)); \)

\( (iv)' \) \( \psi(S(a), b) = \psi(a, S(b)). \)

For any \( \mu \in \mathcal{N}I \) and \( \alpha, \beta \in \mathcal{P}_\mu \), we have

\[
\psi(u_\alpha^+, u_\beta^+) = \varphi(u_\alpha^+, u_\beta^-) = \frac{1}{a_{\alpha}} \delta_{\alpha \beta}.
\]
So the restriction of $\psi$ to $h^+(\mathcal{A})_\mu$, and hence to $h^+(\mathcal{A}) = \bigoplus_{\mu \in \mathbb{Z}^I} h^+(\mathcal{A})_\mu$, is a definite positive symmetric bilinear form.

**Remark 3.8.** Note that the Ringel-Hall algebra and the bilinear form $\psi$ can actually be defined over the rational field $\mathbb{Q}$. So it makes sense to talk about the definite positivity.

### 4 Main results

In Section 4.1 we clarify the relation of the double Ringel-Hall algebra and its composition subalgebra with generalized Kac-Moody Lie algebras. In Section 4.2 we classify the dimension vectors of indecomposable objects in the category $\mathcal{A}$, via the root system of the generalized Kac-Moody Lie algebra corresponding to the double composition algebra.

#### 4.1 The double composition algebras

Recall that $\mathcal{A}$ is a hereditary abelian finitary length category, and $\mathcal{D}(\mathcal{A})$ the double Ringel-Hall algebra with a triangle decomposition $\mathcal{D}(\mathcal{A}) = h^-(\mathcal{A}) \otimes T \otimes h^+(\mathcal{A})$.

Let $C(\mathcal{A})$ be the subalgebra of $\mathcal{D}(\mathcal{A})$ generated by $\{u_i^\pm : i \in I\}$ and $T$, called the *double composition algebra*. It is a Hopf subalgebra and $\mathbb{Z}I$-graded as well as $\mathcal{D}(\mathcal{A})$. It also admits a triangle decomposition $C(\mathcal{A}) = c^-(\mathcal{A}) \otimes T \otimes c^+(\mathcal{A})$, where $c^+(\mathcal{A})$ (and $c^-(\mathcal{A})$) are the subalgebra of $C(\mathcal{A})$ generated by $u_i^+ (i \in I)$ (and $u_i^- (i \in I)$, respectively). Let $C^+(\mathcal{A})$ and $C^-(\mathcal{A})$ be the intersection of $C(\mathcal{A})$ with $H^+(\mathcal{A})$ and $H^-(\mathcal{A})$ respectively. So the involution $\omega$ of $\mathcal{D}(\mathcal{A})$ defined before Lemma 3.7 restricts to an involution of $C(\mathcal{A})$, switching $u_i^+$ and $u_i^-$. It also induces algebra isomorphisms $c^+(\mathcal{A}) \xrightarrow{\sim} c^-(\mathcal{A})$ and $C^+(\mathcal{A}) \xrightarrow{\sim} C^-(\mathcal{A})$. Therefore it is clear that the restriction to $C^+(\mathcal{A}) \times C^-(\mathcal{A})$ of the bilinear form $\varphi : H^+(\mathcal{A}) \times H^-(\mathcal{A}) \rightarrow \mathbb{C}$, defined before Lemma 3.5, gives rise to another skew-Hopf pairing $(C^+(\mathcal{A}), C^-(\mathcal{A}), \varphi)$.

Recall that in Section 3.1 we defined the symmetric Euler form $(-,-) : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$, which is a generalized Kac-Moody bilinear form by Lemma 3.1. We write $C_0$ for the
corresponding Borcherds-Cartan matrix. Let \( g(C_0) \) be the associated generalized Kac-Moody Lie algebra and \( U_v(g(C_0)) \) the quantized enveloping algebra with generators \( E_i, F_i \ (i \in \mathcal{I}) \) and \( K_\mu \ (\mu \in \mathbb{Z}\mathcal{I}) \) (see Definition 2.5).

**Theorem 4.1.** The map \( \Phi : U_v(g(C_0)) \rightarrow \mathcal{C}(\mathcal{A}) \) from the quantized enveloping algebra to the double composition algebra, defined by

\[
\Phi(K_i) = K_i, \quad \Phi(E_i) = u_i^+, \quad \Phi(F_i) = \begin{cases} -v_i u_i^- & i \in \mathcal{I}^{re} \\ \frac{v^{2\dim_k \text{End}_{\mathcal{A}}(S_i)} - 1}{v_i - v_i^{-1}} u_i^- & i \in \mathcal{I}^{im} \end{cases}
\]

is a Hopf algebra isomorphism.

**Proof.** We proceed by two steps. The aim of step one is to show that \( \Phi \) is a well-defined surjective Hopf algebra homomorphism. We claim that \( \Phi(E_i), \Phi(F_i) \ (i \in \mathcal{I}) \) and \( \Phi(K_\mu) \ (\mu \in \mathbb{Z}\mathcal{I}) \) satisfy the relations (i) – (v) in Definition 2.5. Indeed (i) and (ii) follow from Definition 3.2 and 3.3 directly, and (iii) follows from Lemma 3.6. The remaining relations (iv), (iv)' and (v) follow from the same calculation as in Proposition 6.2, 6.3 in [3], using the fact that \( i \in \mathcal{I}^{re} \) iff \( i \) has no self-extensions. Hence \( \Phi \) is a well-defined surjective algebra homomorphism. Since it is clear that \( \Phi \) commutes with the comultiplications and the antipodes on the generators \( E_i, F_i \) and \( K_i \), step one is done.

It remains to show that \( \Phi \) is injective. Both \( U_v(g(C_0)) \) and \( \mathcal{C}(\mathcal{A}) \) admit a triangle decomposition and they are compatible with \( \Phi \). Also the Cartan parts are preserved by \( \Phi \). Recall that we have skew-Hopf pairings \( \phi : U_v^+ (g(C_0)) \times U_v^- (g(C_0)) \rightarrow \mathbb{C} \) and \( \varphi : \mathcal{C}^+(\mathcal{A}) \times \mathcal{C}^- (\mathcal{A}) \rightarrow \mathbb{C} \). It is straightforward to check that they are compatible with \( \Phi \), namely \( \phi(a, b) = \varphi(\Phi(a), \Phi(b)) \), for all \( a \in U_v^+ (g(C_0)) \) and \( b \in U_v^- (g(C_0)) \). For example we check now \( \phi(E_i, F_j) = \varphi(\Phi(E_i), \Phi(F_j)) \). Indeed if \( j \) is real, then \( \varphi(\Phi(E_i), \Phi(F_j)) = \varphi(u_i^+, -v_j u_j^-) = -v_i \delta_{ij} \frac{1}{v_i - v_j} = \delta_{ij} \frac{1}{v_i - v_j} = \phi(E_i, F_j) \). If \( j \) is imaginary, then

\[
\varphi(\Phi(E_i), \Phi(F_j)) = \varphi(u_i^+, \frac{v^{2\dim_k \text{End}_{\mathcal{A}}(S_j)} - 1}{v_j^{-1} - v_j} u_j^-) = \frac{v^{2\dim_k \text{End}_{\mathcal{A}}(S_j)} - 1}{v_j^{-1} - v_j} \delta_{ij} = \delta_{ij} \frac{1}{v_i - v_i^{-1}} = \phi(E_i, F_j).
\]
Suppose now $u \in U^+_v (g(C_0))$ lies in the kernel of $\Phi$. For any $v \in U^-_v (g(C_0))$, we have that $\phi(u, v) = \varphi(\Phi(u), \Phi(v)) = 0$. But $\phi : U^+_v (g(C_0)) \times U^-_v (g(C_0)) \rightarrow \mathbb{C}$ is non-degenerate by Proposition 2.7. Hence $u$ must be zero. This completes the proof.

4.2 The double Ringel-Hall algebra as the quantized enveloping algebra of a generalized Kac-Moody algebra

Our next step is to measure the difference between the double composition algebra $C(\mathcal{A})$ and the double Ringel-Hall algebra $D(\mathcal{A})$, and to approximate $D(\mathcal{A})$ with the quantized enveloping algebra of a larger generalized Kac-Moody Lie algebra, following the method of Sevenhant and Van den Bergh [17] (used also by Deng and Xiao [3]).

We define $\Xi^\pm_0$ to be $h^\pm_0 = \mathbb{C}$, and define $\Xi^\pm_i$ to be $h^\pm_i (\Lambda)^\pm_i$ with basis $\{u^\pm_i\}$ (for $i \in \mathcal{I}$). For $\theta \in \mathbb{N} \mathcal{I}$, $\theta \neq 0$ and $\theta \notin \mathcal{I}$, define $\Xi^\pm_\theta$ to be the subalgebra of $h^\pm_i (\Lambda)^\pm_\theta$ generated by $\sum_{\mu + \nu = \theta, \mu, \nu \neq \theta} h^\pm_i (\Lambda)^\pm_\mu h^\pm_i (\Lambda)^\pm_\nu$.

Set

$L^+_\theta = \{ x \in h^+_\theta (\Lambda) : \varphi(x, \Xi^-_\theta) = 0 \} = \{ x \in h^+_\theta (\Lambda) : \psi(x, \Xi^+_\theta) = 0 \}$,

$L^-_\theta = \{ y \in h^-_\theta (\Lambda) : \varphi(\Xi^+_\theta, y) = 0 \} = \{ y \in h^-_\theta (\Lambda) : \psi(y, \Xi^-_\theta) = 0 \}$.

It is easy to see that $h^\pm (\mathcal{A})^\pm_\theta = \Xi^\pm_\theta \oplus L^\pm_\theta$ as a $\mathbb{C}$-vector space, and that $\omega : \Xi^+_\theta \rightarrow \Xi^-_\theta$, $\omega : L^+_\theta \rightarrow L^-_\theta$. Similar to Lemma 3.1 in [17], we show that elements in $L^\pm_\theta$ are primitive, as well as $u^\pm_i$.

Lemma 4.2. (i) For any $x \in L^+_\theta$ and $y \in L^-_\theta$, we have that

$\Delta(x) = x \otimes 1 + K_\theta \otimes x$, $S(x) = -K^-_\theta x$,

$\Delta(y) = 1 \otimes y + y \otimes K^-_\theta$, $S(y) = -y K_\theta$.

(ii) For any $x \in L^+_\theta$ and $y \in L^-_\theta$, we have that

$xy - yx = -\varphi(x, y) (K_\theta - K^-_\theta)$.  

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Proof. (i) We only need to prove the formula for \( x \in L^+_\theta \), and then apply the involution \( \omega \) to obtain the formula for \( y \in L^-_\theta \).

For \( \theta \in \mathbb{N}I \), let us take a normal orthogonal basis \( \{ x(\theta, p) : 1 \leq p \leq \dim L^+_\theta \} \) of \( L^+_\theta \), with respect to the definite positive bilinear form \( \psi : \mathfrak{h}^+(\mathcal{A})_\theta \times \mathfrak{h}^+(\mathcal{A})_\theta \to \mathbb{C} \)
defined in the end of the last subsection. Write \( J \) for the index set \( \{ (\theta, p) : \theta \in \mathbb{N}I, 1 \leq p \leq \dim L^+_\theta \} \). Then \( \{ x(\theta, p) : (\theta, p) \in J \} \) is a normal orthogonal basis of \( \oplus_{\theta \in \mathbb{N}I} L^+_\theta \).

We extend it to a normal orthogonal basis \( \{ x(\theta, p) : (\theta, p) \in J' \} \) of \( \mathfrak{h}^+(\mathcal{A}) \), where \( J' \) stands for the index set \( \{ (\theta, p) : \theta \in \mathbb{N}I, 1 \leq p \leq \dim \mathfrak{h}^+(\mathcal{A})_\theta \} \). In particular
\[
\psi(x(\theta, p), x(\theta', p')) = \delta_{\theta, \theta'} \delta_{p, p'}.
\]

Note that \( \{ x(\theta, p) : (\theta, p) \in J' \setminus J \} \) forms a basis of \( \oplus_{\theta \in \mathbb{N}I} \mathbb{Z}^+_\theta \), and each element \( x(\theta, p) \) is homogeneous of degree \( \theta \). For example when \( \theta = 0 \), \( \mathfrak{h}^+(\mathcal{A})_0 \) is one dimensional and \( x(0, 1) = 1 \). When \( \theta = i \in I \), \( \mathfrak{h}^+(\mathcal{A})_i = \mathbb{Z}^+_i \) is one dimensional and \( x(i, 1) = \sqrt{a_i} \cdot u_i^+ \), where \( a_i \) is the cardinality of the automorphism group of the simple object \( S_i \) in \( \mathcal{A} \).

Suppose the comultiplication \( \Delta \) sends a basis element \( x(\theta, p) \) to a linear combination of the form
\[
\sum_{(\theta_1, p_1), (\theta_2, p_2) \in J'} c(\theta_1, p_1), (\theta_2, p_2) x(\theta_1, p_1) K_{\theta_2} \otimes x(\theta_2, p_2) \text{ with complex coefficients.}
\]

Note that \( c(\theta, p), (0, 1) = 1 = c(0, 1), (\theta, p) \). For any \( (\tau_1, q_1), (\tau_2, q_2) \in J' \), we have
\[
\psi(x(\theta, p), x(\tau_1, q_1) x(\tau_2, q_2)) = \psi(\Delta(x(\theta, p)), x(\tau_1, q_1) \otimes x(\tau_2, q_2))
\]
\[
= \sum c(\theta_1, p_1), (\theta_2, p_2) \psi(x(\theta_1, p_1) K_{\theta_2}, x(\tau_1, q_1)) \psi(x(\theta_2, p_2), x(\tau_2, q_2))
\]
\[
= c(\tau_1, q_1), (\tau_2, q_2),
\]

because \( \psi(x(\theta_2, p_2), x(\tau_2, q_2)) = \delta_{\theta_2, \tau_2} \delta_{p_2, q_2} \) and \( \psi(x(\theta_1, p_1) K_{\theta_2}, x(\tau_1, q_1)) = \psi(x(\theta_1, p_1) \otimes K_{\theta_2}, \Delta(x(\tau_1, q_1))) = \psi(x(\theta_1, p_1) \otimes K_{\theta_2}, x(\tau_1, q_1) \otimes 1) = \delta_{\theta_1, \tau_1} \delta_{p_1, q_1}.\)

Now we take \( (\theta, p) \) from \( J \) so that \( x(\theta, p) \) belongs to \( L^+_\theta \). Then the bilinear form \( \psi(x(\theta, p), x(\tau_1, q_1) x(\tau_2, q_2)) \) is nonzero only when either \( \tau_1 = \theta \) and \( \tau_2 = 0 \), or \( \tau_1 = 0 \) and \( \tau_2 = \theta \). In the first case \( x(\tau_1, p_1) = x(\theta, p) \) and \( x(\tau_2, q_2) = 1 \), and in the other case \( x(\tau_1, p_1) = 1 \) and \( x(\tau_2, q_2) x(\theta, p) \). Hence \( \Delta(x(\theta, p)) = x(\theta, p) \otimes 1 + K_{\theta} \otimes x(\theta, p) \). The first formula follows since the comultiplication \( \Delta \) is linear.

By definition of the comultiplication and counit we have for \( x \in L^+_\theta \) that \( x = (\epsilon \otimes 1) \circ \Delta(x) = \epsilon(x) + \epsilon(K_{\theta})x = \epsilon(x) + x \). Hence \( \epsilon(x) = 0 \). By definition of the antipode we have that \( 0 = (S \otimes id) \circ \Delta(x) = S(x) + S(K_{\theta})x = S(x) + K_{-\theta}x \). Hence \( S(x) = -K_{-\theta}x \). 

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(ii) follows from (i) and similar calculation with Lemma \cite{3.6}.

We enlarge the index set \( \mathcal{I} \) to \( \mathcal{I}' = \mathcal{I} \cup \mathcal{J} \), where \( \mathcal{J} = \{(\theta, p) : \theta \in \mathfrak{N}L, 1 \leq p \leq \dim_{\mathbb{C}}L_{\theta} \} \). There exists a natural linear map \( \varpi : \mathbb{Z}\mathcal{I}' \to \mathbb{Z}\mathcal{I} \) defined by \( \varpi(i) = i \) for \( i \in \mathcal{I} \), and \( \varpi(j) = \theta \) for \( j = (\theta, p) \in \mathcal{J} \). The generalized Kac-Moody bilinear form on \( \mathbb{Z}\mathcal{I} \) can be extended to a bilinear form on \( \mathbb{Z}\mathcal{I}' \) by \( (i, j)' = (\varpi(i), \varpi(j)) \) for \( i, j \in \mathcal{I}' \).

We are going to show that it is again a generalized Kac-Moody bilinear form (with similar argument as Proposition 3.2 in \cite{17}), and hence determines a Borcherds datum \((\mathcal{I}', (-,-))\).

Note that \( \mathcal{I}' \) is a subset of \( \mathcal{J}' \) appeared in the proof above. To each \( i \in \mathcal{I}' \), we have also associated a primitive homogeneous element \( x_i \in \mathfrak{h}^+(\mathcal{A}') \) with degree \( \varpi(i) \).

Moreover, \( \psi(x_i, x_j) = \delta_{ij} \).

**Proposition 4.3.** The following holds for any \( i, j \in \mathcal{I}' \),

(i) if \( i \neq j \), \( (i, j)' \leq 0 \);

(ii) if \( j \in \mathcal{J} \), \( (j, j)' \leq 0 \);

(iii) if \( (i, i)' > 0 \), \( \frac{2(i, j)'}{(i, i)'} \in \mathbb{Z} \).

In particular, \((-,-)' : \mathbb{Z}\mathcal{I}' \times \mathbb{Z}\mathcal{I}' \to \mathbb{Z} \) is a generalized Kac-Moody bilinear form.

**Proof.** (i) Take any \( i, j \in \mathcal{I}' \) and \( i \neq j \). We have

\[
\Delta(x_i x_j) = \Delta(x_i) \Delta(x_j) = (x_i \otimes 1 + K_{\deg x_j} \otimes x_i)(x_j \otimes 1 + K_{\deg x_j} \otimes x_j) = x_i x_j \otimes 1 + x_i K_{\deg x_j} \otimes x_j + K_{\deg x_i} x_j \otimes x_i + K_{\deg x_i} K_{\deg x_j} \otimes x_i x_j.
\]

Hence

\[
\psi(x_i x_j, x_i x_j) = \psi(\Delta(x_i x_j), x_i \otimes x_j) = \psi(x_i K_{\deg x_j}, x_i) \psi(x_j, x_j) = \psi(x_i \otimes K_{\deg x_j}, \Delta(x_i)) = \psi(x_i \otimes K_{\deg x_j}, x_i \otimes 1 + K_{\deg x_i} \otimes x_i) = 1,
\]

and similarly

\[
\psi(x_i x_j, x_j x_i) = \psi(\Delta(x_i x_j), x_j \otimes x_i) = \psi(K_{\deg x_i} x_j, x_j) \psi(x_i, x_i) = \psi(K_{\deg x_i} \otimes x_j, \Delta(x_j)) = \psi(K_{\deg x_i} \otimes x_j, x_j \otimes 1 + K_{\deg x_j} \otimes x_j) = \psi(K_{\deg x_i}, K_{\deg x_j}) = \psi^{(i, j)'}.
\]

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Now for any \( a, b \in \mathbb{R} \), because \( \psi \) is positive definitive,

\[
0 \leq \psi(ax_ix_j + bx_jx_i, ax_ix_j + bx_jx_i) = a^2 + 2v(i,j)ab + b^2.
\]

It follows that \( v(i,j)' \leq 1 \), and therefore \( (i, j)' \leq 0 \), as \( v = \sqrt{q} > 1 \).

(ii) Suppose \( j \in \mathcal{J} \). Then \( (i, j)' \leq 0 \) for any \( i \in \mathcal{I} \) by (i). It follows that \( (i, j)' \leq 0 \) for any \( i \in \mathcal{I}' \). In particular \((i,i)' \leq 0\).

(iii) Suppose \( (i,i)' > 0 \). Then \( i \in \mathcal{I} \) by (ii). Write \( \varpi(j) = \sum_{k \in \mathcal{I}} a_kk \) where \( a_k \in \mathbb{Z} \).

Then \( \frac{2(i,j)'}{(i,i)'} = \sum_{k \in \mathcal{I}} a_k \frac{2(i,k)}{(i,i)} \in \mathbb{Z} \), because \( \frac{2(i,k)}{(i,i)} \in \mathbb{Z} \) in an integer by Proposition 3.1.

It follows from the definition of \( L_g^+ \) that \( x_i \) \((i \in \mathcal{I}')\) generates \( \mathfrak{h}^+(\mathcal{A}) \). Dually \( y_i = \omega(x_i) \) \((i \in \mathcal{I}')\) generates \( \mathfrak{h}^-(\mathcal{A}) \). Let us denoted by \( C = (c'_{ij})_{i,j \in \mathcal{I}'} \) the symmetrizable Borcherds-Cartan matrix corresponding to the Borcherds datum \((\mathcal{I}', (-,-)')\). Namely

\[
c'_{ij} = \begin{cases} 
\frac{2(i,j)'}{(i,i)'} & \text{if } (i,i)' > 0 \\
(i,j)' & \text{otherwise}
\end{cases}
\]

with symmetrization

\[
\varepsilon_i = \begin{cases} 
\frac{(i,i)'}{2} & \text{if } (i,i)' > 0 \\
1 & \text{otherwise.}
\end{cases}
\]

It is clear that the Borcherds-Cartan matrix \( C_0 = (c_{ij}) \) with index set \( \mathcal{I} \), is a submatrix of \( C \).

**Lemma 4.4.** The following relations hold in \( \mathcal{D}(\mathcal{A}) \):

\[
\sum_{p=0}^{1-c'_{ij}} (-1)^p \left[ 1 - c'_{ij} \right] v^p_{x_i x_j x_i} x_i^p x_j x_i^{1-c'_{ij}-p} = 0, \quad \forall \ i \in \mathcal{I}' = \mathcal{I}'^e, \ j \in \mathcal{I}' \text{ with } i \neq j,
\]

\[
\sum_{p=0}^{1-c'_{ij}} (-1)^p \left[ 1 - c'_{ij} \right] v^p_{y_i y_j y_i} y_i^p y_j y_i^{1-c'_{ij}-p} = 0, \quad \forall \ i \in \mathcal{I}' = \mathcal{I}'^e, \ j \in \mathcal{I}' \text{ with } i \neq j,
\]

\[
x_i x_j - x_j x_i = 0 = y_i y_j - y_j y_i, \quad \forall i, j \in \mathcal{I}' \text{ with } c'_{ij} = 0,
\]

where \( v = v^{\varepsilon_i} \) for \( i \in \mathcal{I}' \).
Proof. We have shown in Proposition 4.3 that \((-,-)'\) is a generalized Kac-Moody bilinear form. Now the Lemma follows from the same calculation as Proposition 6.2 and 6.3 in [3].

Let \(\mathfrak{g}(C)\) be the generalized Kac-Moody algebra associated to \(C\) and \(U_v(\mathfrak{g}(C))\) the quantized enveloping algebra. Recall that \(U_v(\mathfrak{g}(C))\) is generated by \(E_i, F_i (i \in I')\) and \(K_\mu (\mu \in \mathbb{Z}I')\) with respect to the relations given in Definition 2.5.

**Theorem 4.5.** The map \(\Psi : U_v(\mathfrak{g}(C)) \longrightarrow \mathcal{D}(\mathcal{A})\) from the quantized enveloping algebra to the double Ringel-Hall algebra, defined by

\[
\Psi(E_i) = x_i, \quad \Psi(F_i) = \frac{1}{v_i - v_i} y_i, \quad \forall i \in I',
\]

\[
\Psi(K_\mu) = K_{\varpi(\mu)}, \quad \forall \mu \in \mathbb{Z}I',
\]

is a Hopf algebra epimorphism. Moreover, the restriction of \(\Psi\) to \(\mathfrak{h}^+(\mathcal{A})\) and \(\mathfrak{h}^-(\mathcal{A})\) gives rise to algebra isomorphisms to \(U_v^+(\mathfrak{g}(C))\) and \(U_v^-(\mathfrak{g}(C))\) respectively.

Proof. We use the same strategy as in Theorem 4.1. Firstly we show that \(\Psi(E_i), \Psi(F_i) (i \in I')\) and \(\Psi(K_\mu) (\mu \in \mathbb{Z}I')\) satisfy the relations in Definition 2.5. Indeed the relations (i) and (ii) follow from Definition 3.2 and 3.3 and the fact that \(x_i\) and \(y_i\) are homogeneous. The relation (iii) follows from Lemma 4.2 (ii): for \(i = (\theta, p) \in I'\) (hence \(\theta = \varpi(i)\)),

\[
\Psi(E_i)\Psi(F_j) - \Psi(F_j)\Psi(E_i) = \frac{1}{v_j - v_j} (x_iy_j - y_jx_i)
= \frac{1}{v_j - v_j} (x_iy_j)(K_\theta - K_{-\theta})
= \frac{1}{v_j - v_j} (y_jx_i)(K_\theta - K_{-\theta})
= \delta_{ij} \frac{K_\theta - K_{-\theta}}{v_i - v_i}.
\]

The remaining relations follow from Lemma 4.4. Note that \(\Psi\) is compatible with the comultiplications and antipodes by Lemma 4.2 (i). Hence \(\Psi\) is a well-defined surjective Hopf algebra homomorphism.

For injectivity of \(\Psi\) on \(U_v^\pm(\mathfrak{g}(C))\), as we argued in the proof of Theorem 4.1 it suffices to show that \(\Psi\) is compatible with the skew-Hopf pairings \(\phi\) and \(\varphi\) on \(U_v(\mathfrak{g}(C))\).
and $\mathcal{D}(\mathcal{A})$ respectively. Namely $\phi(a, b) = \varphi(\Psi(a), \Psi(b))$, for all $a \in U^+_v(\mathfrak{g}(C)), b \in U^-_v(\mathfrak{g}(C))$. Indeed $\varphi(\Psi(E_i), \Psi(F_j)) = \varphi(v_i^{-1}v_j y_j, v_i^{-1}v_i) = \phi(E_i, F_j)$. Actually the kernel of $\Psi$ is contained in the Cartan part. It is the ideal generated by $\{K_j - K_\theta : j = (\theta, p) \in \mathcal{J}\}$.

\[\square\]

### 4.3 Positive roots and indecomposable objects

Recall that the fundamental region of a generalized Kac-Moody Lie algebra $\mathfrak{g}$ is given by $\mathcal{F} = \{0 \neq \mu \in \mathbb{N} \mathcal{I} : (\mu, i) \leq 0, \forall i \in \mathcal{I}^{re}, \text{supp}(\mu) \text{ is connected}\} \cup \cup_{s \geq 2} s \mathcal{I}^{im}$.

**Lemma 4.6.** For nonzero $\theta \in \mathbb{N} \mathcal{I}$, the space $L^\pm_\theta$ is zero unless $\theta$ lies in the union of the fundamental region $\mathcal{F}_0$ of the generalized Kac-Moody Lie algebra $\mathfrak{g}(C_0)$ and $\cup_{s \geq 2} s \mathcal{I}^{im}$.

**Proof.** Take any nonzero $\theta \in \mathbb{N} \mathcal{I}$ such that the space $L^+_\theta$ is nonzero. Then $\theta$ does not belong to $\mathcal{I}$, since $\Xi^+_i = \mathfrak{h}^+(\mathcal{A})_i$ for $i \in \mathcal{I}$ and hence $L_i = 0$. Now by Lemma 4.3 (i) for any $i \in \mathcal{I}$ the bilinear form $(\theta, i) \leq 0$.

We assume the support of such a $\theta$ is not connected and write $\text{supp}(\theta) = X_1 \cup X_2$, where $X_1$ and $X_2$ are disjoint nonempty subsets of $\mathcal{I}$ and the bilinear form $(i, j) = 0$ for all $i \in X_1$ and $j \in X_2$. This means that objects of $\mathcal{A}$ with dimension vector supported in $X_1$ and in $X_2$ have no non-trivial extensions with each other. Therefore any object $M$ with dimension vectors $\theta$ can be decomposed into $M = M_1 \oplus M_2$, such that the dimension vector of $M_i$ is supported in $X_i$, and $M_1$ and $M_2$ has no nontrivial extensions. This implies that the multiplication of the elements corresponding to $M_1$ and $M_2$ in the Ringel-Hall algebra $\mathfrak{h}^+(\mathcal{A})$ gives rise to a unique term corresponding to $M$. We have thus deduced $\Xi^+_\theta = \mathfrak{h}^+(\mathcal{A})_\theta$, and hence $L^+_\theta = 0$, which is a contradiction to the choice of $\theta$.

\[\square\]

To the Borcherds-Cartan matrix $C$ indexed by $\mathcal{I}'$ obtained in the last subsection, we associate the root system $\Delta = \Delta(C)$, the simple reflections $\tilde{r}_i : \mathbb{C} \mathcal{I}' \longrightarrow \mathbb{C} \mathcal{I}'$ for $i \in \mathcal{I}'^{re}$, and the Weyl group $W = W(C)$ of the generalized Kac-Moody Lie algebra $\mathfrak{g}(C)$ as defined in Section 2. To the Borcherds-Cartan submatrix $C_0$ indexed by $\mathcal{I}$, we associate the root system $\Delta_0 = \Delta(C_0)$, the simple reflection $r_i : \mathbb{C} \mathcal{I} \longrightarrow \mathbb{C} \mathcal{I}$ for $i \in \mathcal{I}$ and the Weyl group $W_0 = W(C_0)$. It is clear that $\Delta_0$ is a subsystem of $\Delta$, and $\mathcal{I}'^{re} = \mathcal{I}^{re}$.
Lemma 4.7. (i) For any \( i \in \mathcal{I}^r \), \( \varpi \circ \tilde{r}_i = r_i \circ \varpi : C\mathcal{I}' \rightarrow C\mathcal{I} \).

(ii) There exists a group isomorphism \( W \rightarrow W_0 \) by sending \( \tilde{r}_i \) to \( r_i \), for \( i \in \mathcal{I}^r \).

(iii) The linear map \( \varpi : \mathbb{Z}\mathcal{I}' \rightarrow \mathbb{Z}\mathcal{I} \) sends the root system \( \Delta \) to the union of the root system \( \Delta_0 \) and \( W_0(\cup_{s \geq 2}s\mathcal{I}^m) \).

Proof. (i) It suffices to check that \( \varpi \circ \tilde{r}_i(j) = r_i \circ \varpi(j) \) for any \( i \in \mathcal{I}^r \) and \( j \in \mathcal{I}' \) by the linearity of \( \varpi \) and the reflections. Recall that if we write \( C = (c_{ij}) \) then \( c_{ij} = 2(\varpi(i), \varpi(j))_{(i,i)} = 2(\varpi(i), \varpi(j))_{(i,i)} \) for all \( i, j \in \mathcal{I}' \). Hence,

\[
\varpi \circ \tilde{r}_i(j) = \varpi(j - c_{ij}i) = \varpi(j) - c_{ij}i, \\
r_i \circ \varpi(j) = \varpi(j) - 2(i, \deg f_j)_{(i,i)}i = \varpi(j) - c_{ij}i.
\]

(ii) Because the matrix \( C_0 \) is a submatrix of \( C \) and \( \mathcal{I}^r = \mathcal{I}^e \), the simple reflections \( r_i \) satisfy the same relations as \( \tilde{r}_i \). Indeed, the relations are \( r_i^2 = 1 \) and \( (r_i r_j)^{m_{ij}} = 1 \) where \( m_{ij} = 2, 3, 4, 6 \) or \( \infty \) depending on \( 4(i,j)^2 = 0, 1, 2, 3 \) or \( > 3 \) respectively (see [1]).

(iii) It is clear that \( \varpi \) identifies \( \mathcal{I}^r \) with \( \mathcal{I}^e \). By Lemma 4.6 the map \( \varpi \) sends the fundamental region of the Lie algebra \( g(C) \) to the fundamental region of \( g(C_0) \) and \( \cup_{s \geq 2}s\mathcal{I}^m \). Since the root system is generated by the simple real roots and the fundamental region under the Weyl group reflections, the statement follows from (i) and (ii).

Although we cannot classify the indecomposable objects in the category \( \mathcal{A} \), we give the following correspondence between their dimension vectors and the positive roots of the generalized Kac-Moody Lie algebra \( g(C_0) \).

Theorem 4.8. Let \( \Phi^+ \) be the set of dimension vectors of indecomposable objects in \( \mathcal{A} \).

Then \( \Phi^+ = \Delta_0^+ \cup W_0(\cup_{s \geq 2}s\mathcal{I}^m) \) as subsets of \( \mathbb{N}\mathcal{I} \), and for any real root \( \alpha \in \Delta_0^+ \) there exists a unique, up to isomorphism, indecomposable object with dimension \( \alpha \).

Proof. For \( \alpha \in \Phi^+ \), write \( I(\alpha, q) \) for the number of indecomposable objects in \( \mathcal{A} \) over \( \mathbb{F}_q \) with dimension vector \( \alpha \). The formal character of \( \mathfrak{h}^-(\mathcal{A}) \) equals

\[
\text{ch}\mathfrak{h}^-(\mathcal{A}) = \sum_{\mu \in \mathbb{N}\mathcal{I}} \dim \mathfrak{h}^-(\mathcal{A})_{-\mu} e(-\mu) = \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-I(\alpha, q)}. \]
By Corollary 4.4 and Proposition 2.6, we have that
\[
\text{ch}^{-}(\mathcal{A}) = \varpi(\text{ch}U^{-}(g(C))) = \varpi(\Pi_{\beta \in \Delta^{+}}(1 - e(-\beta))^{-\text{mult}_{g(C)}\beta}) = \Pi_{\beta \in \Delta^{+}}(1 - e(-\varpi(\beta)))^{-\text{mult}_{g(C)}\beta}.
\]

Hence
\[
\Phi^{+} = \{ \varpi(\beta) : \beta \in \Delta^{+} \}
\]
and for any \( \alpha \) in \( \Phi^{+} \),
\[
I(\alpha, q) = \sum_{\beta \in \Delta^{+}, \varpi(\beta) = \alpha} \text{mult}_{g(C)}\beta.
\]
If \( \alpha \) is a real root, then \( I(\alpha, q) = \text{mult}_{g(C)}\alpha = 1 \).

By Lemma 4.7(iii), it remains to prove that for any \( s \geq 2 \) and \( i \in \mathcal{I}^{im}, si \in \Phi^{+} \). In the case \((i, i) < 0\), it is clear that \( L_{2i}^{\pm} \neq 0 \), so there exists \((2i, 1) \in \mathcal{J} \). It follows that \( ni + (2i, 1) \) is in the fundamental region of the Lie algebra \( g(C) \). Therefore its image under \( \varpi \), i.e. \((n + 2)i\), lies in \( \Phi^{+} \) (for any \( n \geq 0 \)). In the case \((i, i) = 0\), let \( k' \) denotes the field \( \text{End}(S, S) \). We claim that there exists an indecomposable object \( L_{n} \) which has both the length and the Loewy length \( n \), each composition factor is \( S \), and \( \text{Hom}(S, L_{n}) = k' = \text{Ext}^{1}(S, L_{n}) \). This would finish the proof of the Theorem.

We prove the claim by induction. Without lose the generality, we can assume \( k' = k \). When \( n = 1 \), \( L_{1} = S \) is the only choice. When \( n = 2 \), . Because \( \text{Ext}^{1}(S, S) = k' \), there is uniquely a non-split short exact sequence \( 0 \rightarrow S \rightarrow L_{2} \rightarrow S \rightarrow 0 \) with \( L_{2} \) indecomposable. Moreover we apply \( \text{Hom}(S, -) \) to the sequence and find that \( \text{Hom}(S, L_{2}) = k' = \text{Ext}^{1}(S, L_{2}) \). Assume now there exists such a unique \( L_{n-1} \) satisfying that \( \text{Hom}(S, L_{n-1}) = k' = \text{Ext}^{1}(S, L_{n-1}) \). Then there exists uniquely a non-split sequence \( 0 \rightarrow L_{n-1} \rightarrow E \rightarrow S \rightarrow 0 \). Note that the Loewy length of \( E \) must be \( n-1 \leq l.length(E) \leq n \). If \( l.length(E) = n-1 \), then \( E = L_{n-1} \oplus S \), namely the sequence \( 0 \rightarrow L_{n-1} \rightarrow E \rightarrow S \rightarrow 0 \) splits, that is a contradiction. Hence \( l.length(E) = n \). Since the length of \( E \) is also \( n \), \( E \) must be indecomposable and we define \( L_{n} = E \). Apply \( \text{Hom}(S, -) \) to the sequence and find that \( \text{Hom}(S, L_{n}) = k' = \text{Ext}^{1}(S, L_{n}) \). \( \square \)
5 Coherent sheaves on a weighted projective curve

In this section we apply our main results to a special case, namely, the category of coherent sheaves over a weighted projective curve. To be more precise, we fix a slope consider semistable sheaves of the fixed slope so that we get a hereditary abelian finitary length category. Our method will give the classification of the dimension vectors of the indecomposable objects in this category.

5.1 General features

We introduce the category of coherent sheaves on a weighted projective curve axiomatically, following [11]. All results in this subsection can be found in [6] and [11].

Recall that our base field $k$ is a finite field. By a category of coherent sheaves on a weighted projective curve, we mean a category $\mathcal{H}$ satisfying the following axioms $(H1) - (H6)$ (and objects in $\mathcal{H}$ are coherent sheaves):

$(H1)$ $\mathcal{H}$ is an abelian, $k$-linear category.

$(H2)$ $\mathcal{H}$ is small and Hom-finite.

$(H3)$ $\mathcal{H}$ admits a self-equivalence $\tau$ satisfying Serre duality, i.e. $\text{DExt}^1(X,Y) = \text{Hom}(Y,\tau X)$.

$(H4)$ $\mathcal{H}$ is Noetherian, and not each object having finite length.

It follows from $(H3)$ that the category $\mathcal{H}$ is hereditary. Assume $\mathcal{H}$ satisfies $(H1)-(H4)$, then each $X \in \mathcal{H}$ has the form $X = X_0 \oplus X_+$, where $X_0 \in \mathcal{H}_0 = \{X \in \mathcal{H} \mid X \text{ has finite length}\}$ and $X_+ \in \mathcal{H}_+ = \{X \in \mathcal{H} \mid X \text{ has no simple subobjects}\}$. Sheaves in $\mathcal{H}_+$ are called bundles. The full subcategory $\mathcal{H}_0$ splits into a direct sum of tubes $\mathcal{H}_0 = \bigsqcup_{X \in C} \mathcal{I}_X$, where $\mathcal{I}_X = ZA_\infty/\tau^n$ for some $n \geq 0$ and $C$ is the index set. The simple objects $S$ in $\mathcal{H}_0$ are of two types: (i) $\tau S \cong S$ (called ordinary); (ii) $\tau S \ncong S$ (called exceptional).

$(H5)$ There exists an additive function $\text{rk} : \mathcal{H} \to \mathbb{Z}_{\geq 0}$, called the rank, such that for any object $X$ the following holds,

(i) $\text{rk}(X) = 0 \iff X \in \mathcal{H}_0$

(ii) $\text{rk}(\tau X) = \text{rk}(X)$
(iii) $\exists X \in \mathcal{H}_+$, such that $\text{rk}(X) = 1$.

Bundles of rank one are called line bundles.

(H6) $\mathcal{H}_0$ has only finitely many exceptional simples. Moreover, for each tube $\mathcal{S}_X$ and any line bundle $L$, we have $\sum_{S: \text{simples in } \mathcal{S}_X} \dim_k \text{Hom}(L,S) = 1$.

**Remark 5.1.** The index set $C$ has the structure of a smooth projective curve. Let $p : C \to \mathbb{N}$ be defined by $p(X) = \#\{\text{simples in } \mathcal{S}_X\}$. Then $(C,p)$ is the weighted projective curve associated to $\mathcal{H}$.

By definition the function $p$ takes value 1 at ordinary points. We collect the values at those exceptional points into a set $(p_1,p_2,\cdots,p_t)$, called the weight sequence. The average Euler form is the bilinear form on the Grothendieck group $K_0(\mathcal{H})$ defined by

$$\langle\langle [X],[Y]\rangle\rangle = \frac{1}{p}\sum_{j=0}^{p-1} \langle\langle \tau^jX,[Y]\rangle\rangle$$

where $p = \text{l.c.m}(p_1,p_2,\cdots,p_t)$ and $\langle\langle [X],[Y]\rangle\rangle = \dim_k \text{Hom}(X,Y) - \dim_k \text{Ext}^1(X,Y)$ is the ordinary Euler form.

Fix a line bundle $L_0$. We define the degree to be the additive function $\text{deg} : K_0(\mathcal{H}) \to \frac{1}{p}\mathbb{Z}$ via

$$\text{deg}([X]) = \langle\langle [L_0],[X]\rangle\rangle - \langle\langle [L_0],[L_0]\rangle\rangle \text{rk}(X).$$

For a non-zero object $X \in \mathcal{H}$, define the slope of $X$ to be $\mu_X := \frac{\text{deg}(X)}{\text{rk}(X)}$. We call the rational number $g_\mathcal{H} = 1 - \langle\langle [L_0],[L_0]\rangle\rangle$ the orbifold genus of $\mathcal{H}$, and $\chi_\mathcal{H} = 2(1 - g_\mathcal{H})$ the orbifold Euler characteristic of $\mathcal{H}$.

Write $g_0$ for the ordinary genus of the underlying smooth projective curve. So $g_0 = 0$ means the projective line, and $g_0 = 1$ means the elliptic curve.

**Remark 5.2.** The orbifold Euler characteristic $\chi_\mathcal{H}$ can be used to classify the category $\mathcal{H}$. If $\chi_\mathcal{H} > 0$ (domestic case), all possible cases are as follows:

(i) $g_0 = 0$, no weights
(ii) $g_0 = 0$, weights $(p)$
(iii) $g_0 = 0$, weights $(p,q)$
(iv) $g_0 = 0$, weights $(2,2,n)$
(v) $g_0 = 0$, weights $(2,3,3)$
(vi) $g_0 = 0$, weights $(2,3,4)$
(vii) $g_0 = 0$, weights $(2,3,5)$. 

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If $X_{\mathcal{H}} = 0$ (tubular case), all the possible cases are:

(i) $g_0 = 0$, weights (3,3,3)  
(ii) $g_0 = 0$, weights (2,4,4)  
(iii) $g_0 = 0$, weights (2,3,6)  
(iv) $g_0 = 0$, weights (2,2,2,2)  
(v) $g_0 = 1$, no weights.

All the other cases are wild and $X_{\mathcal{H}} < 0$.

A non-zero bundle $X \in \mathcal{H}$ is called stable (respectively, semi-stable) of slope $\rho \in \mathbb{Q}$ if:

(i) $\mu X = \rho$

(ii) if $X' \subset X$, $\mu X' < \mu X$ (respectively, $\mu X' \leq \mu X$). By [6] and [11], if the orbifold Euler characteristic $X_{\mathcal{H}} \geq 0$, each indecomposable bundle is semi-stable. Moreover if $X_{\mathcal{H}} > 0$, each indecomposable bundle is stable.

**Proposition 5.3.** (Riemann-Roch) For each $X,Y \in K_0(\mathcal{H})$, we have

$$\langle \langle X,Y \rangle \rangle = (1 - g_{\mathcal{H}}) \text{rk}(X) \text{rk}(Y) + \begin{vmatrix} \text{rk}(X) & \text{rk}(Y) \\ \text{deg}(X) & \text{deg}(Y) \end{vmatrix}.$$ 

**Lemma 5.4.** Let $\mathcal{A}$ be a hereditary abelian category. For each $\mathbb{Z}$-linear form $\lambda : K_0(\mathcal{A}) \to \mathbb{Z}$, the full subcategory $\mathcal{A}(\lambda)$ of $\mathcal{A}$ controlled by $\lambda$, consisting of all objects $X$ from $\mathcal{A}$ with (i) $\lambda(X) = 0$ (ii) $\forall X' \subset X$, $\lambda(X') \leq \lambda(X)$, is an exact extension-closed subcategory of $\mathcal{A}$. In particular $\mathcal{A}(\lambda)$ is again a hereditary abelian category.

**Proof.** We have to show $\mathcal{A}(\lambda)$ is closed under taking kernels, cokernels and extensions. Let us only prove for kernels. Take any $X,Y \in \mathcal{A}(\lambda)$, $f : X \to Y$, the short exact sequence $0 \to \text{Ker}(f) \to X \to \text{Im}(f) \to 0$ implies that $\lambda(X) = \lambda(\text{Ker}(f)) + \lambda(\text{Im}(f))$. On the other hand, $\text{Ker}(f) \subset X$ and $\text{Im}(f) \subset Y$, so $\lambda(\text{Ker}(f)) \leq 0$ and $\lambda(\text{Im}(f)) \leq 0$. We have $\lambda(\text{Ker}(f)) = 0$. Subobjects of $\text{Ker}(f)$ are subobjects of $X$, therefore $\text{Ker}(f) \in \mathcal{A}(\lambda)$. 

For each $\rho \in \mathbb{Q}$, let $\mathcal{H}^{(\rho)}$ be the full subcategory of $\mathcal{H}$ consisting of all semi-stable bundles of slope $\rho$ (including the 0 object).

**Proposition 5.5.** $\mathcal{H}^{(\rho)}$ is an exact subcategory of $\mathcal{H}$, closed under extensions. Furthermore, each $X \in \mathcal{H}^{(\rho)}$ has finite length in $\mathcal{H}^{(\rho)}$, and the simple objects in $\mathcal{H}^{(\rho)}$ are the stable ones.
Proof. Write \( \rho = \frac{d}{r} \) with \( d \) and \( r \) coprime to each other. Consider the linear form \( \lambda = r \text{deg} - drk : K_0(\mathcal{H}) \to \mathbb{Z} \). By definition, the subcategory \( \mathcal{H}(\lambda) \) of \( \mathcal{H} \) controlled by \( \lambda \) is just \( \mathcal{H}(\rho) \). Hence it follows from Lemma 5.4 that \( \mathcal{H}(\rho) \) is an exact subcategory of \( \mathcal{H} \), closed under extensions.

It is clear that the stable objects are exactly the simple ones in the \( \mathcal{H}(\rho) \). For each semi-stable \( X \), if it is not stable, there exists a proper subobject \( X' \) of \( X \), which is semi-stable and of the same slope \( \rho \). We can continue this process on the \( X' \), which has smaller rank. Finally we get a stable object \( Y \), and a short exact sequence \( 0 \to Y \to X \to X/Y \to 0 \). Therefore, \( X \) has finite length in \( \mathcal{H}(\rho) \), and the length is bounded by the rank \( \text{rk}(X) \). \( \square \)

Remark 5.6. If \( \chi_{\mathcal{H}} > 0 \), \( \mathcal{H}(\rho) \) is a semi-simple category. If \( \chi_{\mathcal{H}} = 0 \), we have the category equivalences \( \mathcal{H}(\rho) \cong \mathcal{H} = \mathcal{H}_0 \).

5.2 The category \( \mathcal{H}(\rho) \)

From now on, we fix a rational number \( \rho \in \mathbb{Q} \) and consider the full subcategory \( \mathcal{H}(\rho) \) of \( \mathcal{H} \) which consists of all semi-stable bundles of slope \( \rho \). By Proposition 5.5, \( \mathcal{H}(\rho) \) is a hereditary abelian finitary length category and the simples in \( \mathcal{H}(\rho) \) are the stable sheaves of slope \( \rho \). Namely each semi-stable sheaf in \( \mathcal{H}(\rho) \) has a composition series of finite length, and the composition factors are stable sheaves of the same slope \( \rho \). Let \( I \) be the set of isomorphism classes of stable objects in \( \mathcal{H}(\rho) \). Then the Grothendieck group \( K_0(\mathcal{H}(\rho)) \) is isomorphic to the free abelian group \( \mathbb{Z}I \) generated by \( I \). As in Section 3.1, we call the image in \( \mathbb{Z}I \) of a semi-stable sheaf its dimension vector.

We aim to understand the indecomposable objects in \( \mathcal{H}(\rho) \). The wild cases \( \chi_{\mathcal{H}} < 0 \) are difficult and very little is known. However, we obtain the classification of the dimension vectors of the indecomposable objects with the help of Ringel-Hall algebras and generalized Kac-Moody Lie algebras.

On the Grothendieck group \( \mathbb{Z}I \) are defined the Euler form \( \langle [X], [Y] \rangle = \dim_k \text{Hom}_{\mathcal{H}(\rho)}(X, Y) - \dim_k \text{Ext}^1_{\mathcal{H}(\rho)}(X, Y) \) and the symmetric Euler form \( \langle [X], [Y] \rangle = \langle [X], [Y] \rangle + \langle [Y], [X] \rangle \), where \( X \) and \( Y \) are objects in \( \mathcal{H}(\rho) \). By Proposition 3.1, the later is a generalized Kac-Moody bilinear form. By Remark 2.3, it uniquely determines a symmetrizable
Borcherds-Cartan matrix, say \( C_0 \). Let \( \mathfrak{g}(C_0) \) be the corresponding generalized Kac-Moody Lie algebra (see Definition 2.3) and \( U_v(\mathfrak{g}(C_0)) \) the corresponding quantized enveloping algebra (see Definition 2.5).

On the other hand, we can associate a Hopf algebra, the double Ringel-Hall algebra \( D(\mathcal{H}) \), to the category \( \mathcal{H}^{(\rho)} \) (see Section 3). As studied in Sections 5, the composition subalgebra of \( D(\mathcal{H}) \) is isomorphic to the quantized enveloping algebra \( U_v(\mathfrak{g}(C_0)) \) (see Theorem 4.1). Moreover, the double Ringel-Hall algebra \( D(\mathcal{H}) \) is a quotient of the quantized enveloping algebra \( U_v(\mathfrak{g}(C)) \) of some larger generalized Kac-Moody algebra \( \mathfrak{g}(C) \) which is obtained by extending the torus of \( \mathfrak{g}(C_0) \) (see Theorem 4.5).

We write \( \Phi^+ \) for the set of dimension vectors of indecomposable objects in \( \mathcal{H}^{(\rho)} \). Let \( \Delta^+_0 \) and \( W_0 \) be respectively the set of positive roots and the Weyl group of \( \mathfrak{g}(C_0) \) (see Section 2 for definition). Let \( I^{im} \) be the subset of \( I \) containing of isomorphism classes of the simple objects in \( \mathcal{H}^{(\rho)} \) which have nontrivial self-extensions. Indeed \( I^{im} \) is the set of imaginary simple roots of \( \mathfrak{g}(C_0) \).

Finally, we have the following Corollary of Theorem 4.8 which relates the set \( \Phi^+ \) to the positive roots of \( \mathfrak{g}(C_0) \).

**Corollary 5.7.** As subsets of \( \mathbb{N}I \) we have \( \Phi^+ = \Delta^+_0 \cup W_0(\cup s \geq 2 sI^{im}) \). Moreover for each real root \( \alpha \in \Delta^+_0 \), there exits a unique (up to isomorphism) indecomposable object with dimension vector \( \alpha \).

Hence, for a fixed slope \( \rho \), the dimension vector of an indecomposable semi-stable sheaf of slope \( \rho \) is either a positive root of the generalized Kac-Moody algebra \( \mathfrak{g}(C_0) \), or a imaginary root lying in \( W_0(\cup s \geq 2 sI^{im}) \). By definition the positive roots of \( \mathfrak{g}(C_0) \) is obtained by applying the Weyl group \( W_0 \) to the real simple roots \( I^{re} = I \setminus I^{im} \). Hence the set \( \Phi^+ = \Delta^+_0 \cup W_0(\cup s \geq 2 sI^{im}) \) can be computed if the Euler form is given.

**Remark 5.8.** Suppose \( \mathcal{H} \) is not weighted. Let \( X, Y \) be the stable bundles in \( \mathcal{H}^{(\rho)} \), then the Euler form \( \langle X, Y \rangle = (1 - g_0)rk(X)rk(Y) \).
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