Spherical polytropic balls cannot mimic black holes

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The so-called black hole shadow is a dark region which is expected to appear in a fine image of optical observation of black holes. It is essentially an absorption cross section of the black hole, and the boundary of shadow is determined by unstable circular orbits of photons (UCOP). If there exists a compact object possessing UCOP but no black hole horizon, it can provide us with the same shadow image as black holes, and detection of a shadow image cannot be direct evidence of black hole existence. This paper examines whether or not such compact objects can exist under some suitable conditions. We investigate thoroughly the static spherical polytropic ball of perfect fluid with single polytrope index, and then investigate a representative example of a piecewise polytropic ball. Our result is that the spherical polytropic ball which we have investigated cannot possess UCOP, if the speed of sound at the center is subluminal (slower than light). This means that, if the polytrope treated in this paper is a good model of stellar matter in compact objects, the detection of a shadow image can be regarded as good evidence of black hole existence. As a by-product, we have found the upper bound of the mass-to-radius ratio of a polytropic ball with single index, \( M_\ast/R_\ast < 0.281 \), under the condition of subluminal sound speed.

Subject Index E01, E30, E31

1. Introduction and our aim

In recent years, the resolution of images from very long baseline interferometer (VLBI) radio observation is approaching the visible angular size of Sgr A\textsuperscript{\ast},\textsuperscript{1} about 10 microarcseconds, which is the largest visible angular size of known black hole candidates [1,2]. The so-called black hole shadow is expected to be resolved by such fine observation in the near future (see Refs. [3,4] and references therein). It seems to be a current common understanding that seeing the black hole shadow is proof of the existence of a black hole horizon.

However, this common understanding has not been confirmed in general relativity. Remember that the black hole shadow is a dark region appearing in an optical image of a black hole, on which some photons would be detected if the black hole did not exist. Therefore, the shadow is essentially an absorption cross section of the black hole. However, it should be emphasized that photons on the edge of the shadow have been circulating around the black hole before coming to the observer. The innermost circular orbit of those photons is not a great circle on the black hole horizon, but an unstable circular orbit of photons (UCOP). That is to say, the boundary of the shadow is determined

\textsuperscript{1}A black hole candidate of \( 4 \times 10^6 M_\odot \) at the center of our galaxy, 8 kpc from Earth.
not by the black hole horizon, but by the UCOP. This indicates that the direct origin of the shadow is the UCOP, not the black hole horizon. Hence, although we can conclude the existence of a UCOP once a shadow is observed, we cannot conclude immediately the existence of a black hole horizon even if a shadow is clearly observed.

Here, let us assign the term "black hole mimicker" to a compact object possessing a UCOP but no black hole horizon. If there exists a black hole mimicker, it can provide us with the same shadow image as a black hole in optical observation, and the detection of a shadow image cannot be direct evidence of black hole existence. Therefore, we are interested in an existence/non-existence condition for black hole mimickers.

Some exotic candidates for black hole mimickers have been proposed, such as gravastars and boson stars. Under the assumption that any black hole mimicker emits thermal radiation from its surface, the current observational data of Sgr A* (the mass accretion rate and the observed flux) exclude the possibility that Sgr A* may be one of these exotic black hole mimickers [5]. Further, the current observational data of some black hole candidates indicate that the gravastar cannot exist in nature, if a gravastar always emits thermal radiation from its surface [6]. For the case that the gravastar does not emit radiation from its surface, the general feature of gravastar shadow, which enables us to distinguish gravastars from black holes in the shadow image, has already been examined [7]. Investigation of these exotic models may be interesting. However, we focus on a rather more usual model in this paper.

Consider a static spherical ball of perfect fluid matter in the framework of general relativity, which connects to Schwarzschild geometry at its surface. A fluid ball that does not possess a black hole horizon becomes a black hole mimicker if it possesses one of following properties:

(A) The ball is so compact that there appears a UCOP in the outside Schwarzschild geometry.
(B) The ball is not so compact as case (A), but a UCOP appears inside the ball.

In case (A), if the surface of the fluid ball neither emits nor reflects any radiation, this ball can provide us with the same shadow image as a black hole. In case (B), if the fluid outside the UCOP is completely transparent and if the fluid inside the UCOP is not transparent, this ball can provide us with the same shadow image as a black hole.

For case (A), the mass-to-radius ratio \(3M_*/R_*\) of the fluid ball is the key quantity, where \(M_*\) and \(R_*\) are respectively the total mass and surface radius of the ball measured in the dimension of length. If this ratio is less than unity \((3M_*/R_* < 1)\), then no UCOP appears outside the fluid ball because the radius of the UCOP in the Schwarzschild geometry is \(3M_*\). Note that, in order to avoid the inequality \(3M_*/R_* < 3/2\) must hold. Further, by adding some reasonable conditions to the fluid ball, the upper bound of the ratio, \(3M_*/R_* \leq \mathcal{U}\), should decrease,

\[
\frac{3M_*}{R_*} \leq \mathcal{U} < \frac{3}{2}.
\]

Hence, the problem in case (A) is whether or not the upper bound \(\mathcal{U}\) becomes less than unity under some reasonable conditions of the fluid ball.

The upper bound \(\mathcal{U}\) has already been estimated for some situations. For a fluid ball with any equation of state satisfying the three conditions of non-increasing energy density in the outward direction, a barotropic form of the equation of state, and subluminal (slower than light) sound speed, an upper bound was obtained in our previous work [8] of \(\mathcal{U} \simeq 1.0909209\). Also, for a core–envelope model of a neutron star [9,10], another value of the upper bound has been obtained, \(\mathcal{U} \simeq 1.018\), where the equation of state in the envelope region is determined by nuclear matter physics (see section 2.
of [10] for details), while the equation of state in the core region is treated by the same method used in our previous work [8]. These estimations still give a bound greater than unity, \( \mathcal{U} > 1 \). Further investigation is necessary to consider what kind of equation of state results in \( \mathcal{U} < 1 \).

On the other hand, concerning case (B), there is no existing work analyzing UCOP inside a fluid ball as far as we know. We need to formulate the criterion for judging the existence/non-existence of a UCOP, and then apply the criterion to our model of a fluid ball.

In this paper, we regard the polytrope as a representative example of the barotropic equation of state. The model investigated here is a static spherical ball of perfect fluid with the polytropic equation of state. Our investigation focuses mainly on the simple polytrope model whose polytrope index is fixed at one value for the whole region inside the fluid ball. After a thorough investigation of the simple model, an example of a piecewise polytropic fluid ball [11] is investigated in the framework of a core–envelope model, where the value of the polytrope index in the core region differs from that in the envelope region. Although a thorough study of the piecewise polytrope model is left for the future, we can find a good insight into the core–envelope model of polytropic fluid balls.

For both polytrope models, we examine whether or not the cases (A) and (B) are possible for the polytropic fluid ball. Our result is that there cannot exist a UCOP either outside or inside a polytropic fluid ball if the sound speed at the center is subluminal. This implies that, if the polytrope investigated in this paper is a good model of stellar matter in compact objects, detection of a shadow image can be regarded as good evidence of the existence of a black hole.

In Sect. 2, the setup of our analysis is described. Sections 3 and 4 are devoted to analyses of, respectively, cases (A) and (B) for the simple polytrope model. In Sect. 5, the analyses in the previous sections are extended to an example of the core–envelope piecewise polytrope model. Section 6 presents a summary and discussion.

### 2. Static spherical polytropic ball

We consider a static spherical ball made of polytropic perfect fluid. The metric of this spacetime is given by a line element,

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi(r)} c^2 dt^2 + \frac{dr^2}{1 - 2Gm(r)/c^2 r} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]

where \((t, r, \theta, \varphi)\) are spherical polar coordinates, \(\Phi(r)\) gives the lapse function, and \(m(r)\) is the mass of fluid contained in the spherical region of radius \(r\). The energy–momentum tensor of perfect fluid is \(T_{\mu\nu} = [\sigma(r)c^2 + p(r)]u_\mu u_\nu + p(r)g_{\mu\nu}\), where \(u = e^{-\Phi}\partial_c t\) is the four-velocity of static fluid, and \(\sigma(r)\) and \(p(r)\) are respectively the mass density and pressure of the fluid.

The condition \(m(0) = 0\) should hold due to the regularity of spacetime at the center. This implies a finite mass density at the center, \(\sigma_c = \sigma(0) \neq \infty\), where the suffix \(c\) denotes the value at the center. We normalize all quantities by \(\sigma_c\):

\[
R := \frac{\sqrt{G \sigma_c}}{c} r, \quad \Sigma(R) := \frac{\sigma(r)}{\sigma_c}, \quad M(R) := \frac{\sqrt{G^3 \sigma_c}}{c^3} m(r), \quad P(R) := \frac{p(r)}{\sigma_c c^2},
\]

There have been some attempts to study a nonlinear instability of black hole mimickers (see [12] and references therein). This is an interesting approach. However, in those works, while the argument for nonlinear instability of a black hole mimic is conjectured by a combination of linear analyses, no definite proof of nonlinear instability has been obtained.
These are dimensionless. The lapse function, \( \Phi(r) := \Phi(R) \), does not need normalization because \( \Phi \) is already dimensionless by definition (2).

The barotropic equation of state is generally expressed as \( P = P(\Sigma) \). We adopt the polytrope as a representative form of the barotropic matter,

\[
P(\Sigma) = K \Sigma^{1+1/n},
\]

where \( K \) and \( n \) are positive constants, and \( n \) is called the polytrope index. By normalization (3), the mass density at the center is unity, \( \Sigma_c = 1 \). Therefore, the coefficient \( K \) is equal to the pressure at the center in our normalization (3),

\[
P_c = K (= P(\Sigma = 1)).
\]

The form (4) is the simple polytrope whose index, \( n \), is fixed at one value for the whole region inside the fluid ball. We are going to extend the simple form (4) to the piecewise polytrope in Sect. 5, but in this section we focus on the simple form (4).

The surface of the fluid ball is defined by zero pressure, \( P^* = 0 \), where the suffix \( * \) denotes the value at the surface. Therefore, the mass density at the surface is zero due to the polytropic equation of state (4), \( \Sigma^* = 0 \). The normalized mass density takes values in the interval

\[
(\Sigma_* = 0) \leq \Sigma \leq 1 (= \Sigma_c).
\]

Regarding \( \Sigma \) as an independent variable, this finite interval of \( \Sigma \) seems to be useful for numerical analysis. Hence, we regard all variables as functions of \( \Sigma \),

\[
R = R(\Sigma), \quad M = M(\Sigma), \quad \Phi = \Phi(\Sigma),
\]

and \( P(\Sigma) \) is already expressed as a function of \( \Sigma \) in (4). The surface radius \( R_* \) and total mass \( M_* \) of the polytropic ball are determined by

\[
R_* = R(\Sigma = 0), \quad M_* = M(\Sigma = 0).
\]

The speed of sound \( V \) in the polytropic ball is given by

\[
V^2 = \frac{dP(\Sigma)}{d\Sigma} = P_c \left(1 + \frac{1}{n}\right) \Sigma^{1/n}.
\]

This speed of sound is normalized by the speed of light. Obviously, this \( V \) decreases from the center \( (\Sigma_c = 1) \) to the surface \( (\Sigma_* = 0) \). The highest speed of sound is given at the center,

\[
V_c^2 = P_c \left(1 + \frac{1}{n}\right).
\]

In this paper, we assume a subluminal condition for the speed of sound,

\[
V_c \leq 1.
\]

Here it may be fair to note that the possibility of omitting the subluminal condition (10) has been discussed for the nuclear matter of neutron stars [9]. The omission of condition (10) may be possible if the nuclear matter possesses the properties that the matter temperature is zero and the sound wave is dispersed and dumped (absorbed) by the matter instantaneously. Even though such properties are good approximate ones for nuclear matter in neutron stars, they may not be exact and correct for all regions inside neutron stars. Furthermore, our analysis is not restricted to neutron stars but is designed to include any matter described by a polytropic equation of state under conditions which are...
reasonable from the viewpoint of general relativity. Hence, let us require the condition of a subluminal speed of sound \( c_{\text{sound}} \), which we regard as a rigorous general relativistic property of matter.

The outside region of the polytropic ball, \( R > R_s \), is described by the Schwarzschild geometry of mass \( M_\ast \). The inside region \( R \leq R_s \) is determined by the Einstein equation and conservation law \( T^{\mu\nu} : v = 0 \), which reduce to the Tolman–Oppenheimer–Volkoff (TOV) equations,

\[
\frac{dM}{d\Sigma} = 4\pi R^2 \Sigma \frac{dR}{d\Sigma}, \quad (11a)
\]

\[
\frac{dP}{d\Sigma} = -\frac{(\Sigma + P)(M + 4\pi R^3 P)}{R (R - 2M)} \frac{dR}{d\Sigma}, \quad (11b)
\]

\[
\frac{d\Phi}{d\Sigma} = -\frac{1}{\Sigma + P} \frac{dP}{d\Sigma}. \quad (11c)
\]

The two functions \( R(\Sigma) \) and \( M(\Sigma) \) are obtained by solving (11a) and (11b) under the equation of state (4). Substituting those solutions into (11c), \( \Phi(\Sigma) \) is obtained. The solutions of the TOV equations depend on two parameters, \( V_c \) and \( n \), for the simple polytrope (4).

Under the setup given above, our aim is to analyze the problem of whether or not the properties (A) and (B) described in Sect. 1 hold for a polytropic ball under the condition of subluminal speed of sound \( c_{\text{sound}} \). In the following analyses, the TOV equations (11a) and (11b) are solved numerically. A technical remark on the numerical calculation is summarized in Appendix A, which applies to Sects. 3, 4, and 5. All our numerical analyses are performed with Mathematica ver. 10.

3. Problem A: can a UCOP appear outside the simple polytropic ball?

The problem in this section is whether or not the inequality \( R_s < 3M_\ast \) holds for the simple polytropic ball with equation of state (4) under the condition (10). If \( R_s < 3M_\ast \), then a UCOP appears outside the polytropic ball. Otherwise, if \( 3M_\ast < R_s \), then a UCOP does not appear outside the ball. Our strategy is as follows:

A1: Numerically solve the TOV equations (11a) and (11b) for given values of parameters \((V_c, n)\), and calculate the mass-to-radius ratio, \( 3M_\ast / R_s \).

A2: Iterate step A1 with varying parameters \((V_c, n)\), so as to obtain the ratio \( 3M_\ast / R_s \) as a function of \((V_c, n)\).

A3: Find the maximum value of \( 3M_\ast / R_s \) as a function of \((V_c, n)\). If the maximum is less than unity, we conclude that the inequality \( 3M_\ast < R_s \) holds for all values of \((V_c, n)\), and no UCOP appears outside the simple polytropic ball of equation of state (4) under the condition of subluminal speed of sound (10).

In Newtonian gravity, the total mass and surface radius of the simple polytropic ball are finite in the index interval \( 0 < n < 5 \), but diverge in the interval \( 5 \leq n \), for any value of \( V_c > 0 \). However, in Einstein gravity, the thorough numerical analysis of the simple polytropic ball by Nilsson and Uggla [13] has revealed a complicated behavior of \( M_\ast \) and \( R_s \) in the half-infinite interval of central sound speed, \( 0 < V_c \):

- In the polytropic index interval \( 0 < n < 3.339 \), both \( M_\ast \) and \( R_s \) are finite.
- In the interval \( 3.339 \leq n < 5 \), a complicated behavior is found.
  - Both \( M_\ast \) and \( R_s \) are finite for almost all values of \((V_c, n)\) in the present parameter region.
  - However, both \( M_\ast \) and \( R_s \) diverge at some discrete points \( (V_c^{(i)}, n^{(i)}) \) in the \( V_c-n \) plane, where \( i = 1, 2, \ldots, N_{\text{div}} \). The number of such divergence points, \( N_{\text{div}} \) (=finite or countable infinity), cannot be read from Ref. [13].
Note that, although the existence of some divergence points \((V_c^{(i)}, n^{(i)})\) in the \(V_c-n\) plane has been definitely confirmed, their accurate positions have not been specified.\(^3\)

- In the interval \(5 \leq n\), both \(M_\ast\) and \(R_\ast\) are infinite.

Note that the mass-to-radius ratio has not been analyzed in Ref. [13]. Analysis of this ratio is our task.

From the above behavior of \(M_\ast\) and \(R_\ast\) found by Nilsson and Ugglä, the physically interesting region of the parameters is

\[
0 < V_c \leq 1, \quad 0 < n < 5,
\]

where the interval of \(V_c\) denotes the condition of subluminal speed of sound (10). It is enough for our aim to calculate the ratio \(3M_\ast/R_\ast\) in this parameter region. However, the parameter points \((V_c^{(i)}, n^{(i)})\), where \(M_\ast\) and \(R_\ast\) diverge in the region (12), may not be included in the grid points of the numerical analysis (see step A2 of our strategy). In order to guess the behavior of \(3M_\ast/R_\ast\) at the points \((V_c^{(i)}, n^{(i)})\), we observe the solutions of the TOV equations in the interval \(5 \leq n\), where \(M_\ast\) and \(R_\ast\) also diverge. Figure 1 is an example with \(V_c = 0.6\) and \(n = 6\). This figure shows that, although the mass \(M(\Sigma)\) and radius \(R(\Sigma)\) diverge as the surface \((\Sigma = 0)\) is approached, the mass-to-radius ratio \(M(\Sigma)/R(\Sigma)\) converges to zero. The same behavior is observed for the other values of \((V_c, n)\) in the interval \(5 \leq n\). Hence, it is expected that, even at the parameter points \((V_c^{(i)}, n^{(i)})\) where \(M_\ast\) and \(R_\ast\) diverge in the region (12), the mass-to-radius ratio converges to zero.

With the help of the above discussion, we can safely carry out our strategy of numerical analysis, composed of steps A1, A2, and A3. The result is shown in Fig. 2, in which the contours of \(3M_\ast/R_\ast\) are plotted. Although our main interest is in the parameter region (12), we have calculated a slightly larger

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\(^3\)Two examples of such divergence points are \((V_c, n) = (V_c^{(1)}, 3.357), (V_c^{(2)}, 4.414)\), where the values \(V_c^{(1)}\) and \(V_c^{(2)}\) cannot be read from Ref. [13].

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Fig. 1. Numerical solution of TOV equations with the simple polytropic equation of state (4) at \((V_c, n) = (0.6, 6)\). Although \(M_\ast\) and \(R_\ast\) are infinite, the mass-to-radius ratio \(M_\ast/R_\ast\) becomes zero.
Fig. 2. Contours of $3M_*/R_*$ on the $V_c$–$n$ plane for the simple polytropic ball. Numerical calculation is performed in the region $0.05 \leq V_c \leq 1.5$ and $0.01 \leq n \leq 5.5$. Obviously, the inequality $3M_*/R_* < 1$ holds under the condition of subluminal speed of sound, $V_c \leq 1$. For small values of $V_c$ and $n$, the numerical errors become manifest.

It is obvious that the maximum value of $3M_*/R_*$ in the region (12) appears on the vertical line at $V_c = 1$ in Fig. 2. This maximum takes a value between $3M_*/R_* = 0.8$ and $0.85$. (A more precise value is calculated in Sect. 6.) Hence, we can conclude that the inequality $3M_*/R_* < 1$ holds in the physically interesting region (12). No UCOP appears in the outside Schwarzschild geometry under the condition of subluminal speed of sound (10) for the simple polytropic ball of equation of state (4).

4. Problem B: can a UCOP appear inside the simple polytropic ball?

The problem in this section is whether a UCOP can exist inside the simple polytropic ball of equation of state (4) under the condition of subluminal speed of sound (10). First, the conditions for the existence of a UCOP in static spherical spacetimes are summarized. Next, these conditions are applied to the simple polytropic ball, and we show that no UCOP can exist inside the simple polytropic ball.

4.1. UCOP in static spherical spacetimes

The feature of a UCOP in the static spherical spacetime of metric (2) is determined by the null geodesic equation. Denoting the affine parameter and radial coordinate of the null geodesic by, respectively, $\lambda$ and $R_{null}(\lambda)$ under the normalization (3), the radial component of the null geodesic

\[ \text{In Fig. 2, the ratio } 3M_*/R_* \text{ is small enough in the interval } 5 \leq n \leq 5.5. \text{ This is consistent with Fig. 1 and the discussion following (12). Further, we have found numerical implications, although the details are not shown here, that the points } (V_c(i), n(i)) \text{, where } M_* \text{ and } R_* \text{ diverge in the region (12), form a line along the valley shown in Fig. 2, and this line seems to approach } (V_c, n) = (0, 5). \]
equation on spacetime (2) is

\[ \left( \frac{dR_{\text{null}}}{d\lambda} \right)^2 + \omega^2 U_{\text{eff}}(R_{\text{null}}) = 0, \]  

(13)

where \( U_{\text{eff}} \) is the effective potential given by

\[ U_{\text{eff}}(R) = \left[ b^2/R^2 - \exp(-2\Phi(R)) \right] \left( 1 - \frac{2M(R)}{R} \right), \quad b = l/\omega, \]  

(14)

where \( M(R) \) and \( \Phi(R) \) are regarded as functions of \( R \) given by solving the TOV equations, \( b \) is an impact parameter, and \( l \) and \( \omega \) are respectively the orbital angular momentum and the frequency of photons measured at infinity.

A photon propagating on a UCOP remains at a constant radius \( (dR_{\text{null}}/d\lambda = 0) \), but it is unstable. Hence, the radius of the UCOP, \( R_u \), is determined by the conditions

\[ U_{\text{eff}}(R_u) = 0, \quad \frac{dU_{\text{eff}}}{dR}(R_u) = 0, \quad \frac{d^2U_{\text{eff}}}{dR^2}(R_u) \leq 0. \]  

(15)

This implies that, if a UCOP exists inside the polytropic ball, the top of the potential barrier touches below the zero level at \( R_u \), as shown in Fig. 3.

By substituting (14) into (15), we obtain

\[ \frac{2}{R_u} F(R_u) - \frac{dF}{dR}(R_u) = 0, \]  

(16a)

\[ \frac{1}{R_u} \frac{dF}{dR}(R_u) - \frac{d^2F}{dR^2}(R_u) \geq 0, \]  

(16b)

\[ b^2 = \frac{R_u^2}{F(R_u)}, \]  

(16c)

where

\[ F(R) := \exp(2\Phi(R)) \quad (= g_{00}). \]  

(17)

The radius of the UCOP, \( R_u \), is determined by (16a) and (16b), and then the impact parameter of the null geodesic circulating forever on the UCOP is obtained by (16c). Therefore, the existence condition for a UCOP consists of two parts, an algebraic equation (16a) and an inequality (16b), which do not include the impact parameter.

### 4.2. Non-existence of a UCOP inside the simple polytropic ball

In order to apply the existence conditions for a UCOP (16a) and (16b) to simple polytropic balls, we need a concrete functional form of \( \Phi(\Sigma) = (1/2) \ln F(\Sigma) \). Substituting the equation of state (4)
The integration constant for this equation is determined by the junction condition of the metric at the surface of the polytropic ball, \( \exp(2\Phi_*) = 1 - 2M_*/R_* \). We obtain

\[
F(\Sigma) = e^{2\Phi(\Sigma)} = \frac{F_*}{(1 + P_\Sigma \Sigma^{1/n})^{2(n+1)}} = F_* \left(1 + \frac{P(\Sigma)}{\Sigma}\right)^{-2(n+1)},
\]

where \( F_* = 1 - 2M_*/R_* \). Regarding \( \Sigma \) as a function of \( R \), which is given by solving the TOV equations (11a) and (11b), we obtain \( F(R) \) as a function of \( R \).

From (19), the first and second differentials of \( F(R) \) are calculated,

\[
\frac{dF(R)}{dR} = \frac{2M + 4\pi R^3P}{R(R - 2M)} F(R) \quad (>0),
\]

\[
\frac{d^2F(R)}{dR^2} = \frac{1}{R} \left[ \frac{4M - R}{R - 2M} + 4\pi R^3 \left( \frac{\Sigma + P}{R - 2M} + \frac{\Sigma + 3P}{M + 4\pi R^3P} \right) \right] \frac{dF(R)}{dR},
\]

where the TOV equations (11a), (11b), and (11c) are used. Substituting these differentials into the existence conditions for a UCOP (16a) and (16b), we obtain

\[
C_1|_{R = R_u} = 0, \quad C_2|_{R = R_u} \geq 0,
\]

where

\[
C_1 = R - 4\pi R^3P - 3M,
\]

\[
C_2 = 1 - \left[ \frac{4M - R}{R - 2M} + 4\pi R^3 \left( \frac{\Sigma + P}{R - 2M} + \frac{\Sigma + 3P}{M + 4\pi R^3P} \right) \right].
\]

If there does not exist an \( R_u \) that satisfies the conditions (21) for any value of parameters \( (V_c, n) \) in the physically interesting region (12), then it is concluded that no UCOP can appear inside the polytropic balls.

Note that the values of quantities \( C_1 \) and \( C_2 \) are calculated by substituting the solutions of the TOV equations (11a) and (11b). The solutions of the TOV equations are functions of \( \Sigma \) and depend on parameters \( (V_c, n) \). Hence, in our analysis, \( C_1 \) and \( C_2 \) can be obtained numerically as functions of three arguments, \( C_1(\Sigma, V_c, n) \) and \( C_2(\Sigma, V_c, n) \). Then, in order to check whether or not there exists \( R_u \) satisfying (21), our strategy is as follows:

**B1:** Numerically solve the TOV equations (11a) and (11b) for given values of parameters \( (V_c, n) \), and iterate this numerical calculation, varying \( V_c \) and fixing \( n \) at a given value. This iteration produces \( C_1(\Sigma, V_c, n) \) and \( C_2(\Sigma, V_c, n) \) as functions of \( (\Sigma, V_c) \) for the given value of \( n \).

**B2:** Plot two curves, \( C_1 = 0 \) and \( C_2 = 0 \), and identify two regions, \( C_2 > 0 \) and \( C_2 < 0 \), in the \( \Sigma-V_c \) plane for the given \( n \). If the curve \( C_1 = 0 \) does not intersect with the region \( C_2 \geq 0 \), then it is concluded that no UCOP exists inside the simple polytropic ball at the given value of \( n \).

**B3:** Iterate steps B1 and B2 with varying \( n \), and check whether or not the intersection of \( C_1 = 0 \) with \( C_2 \geq 0 \) exists at each value of \( n \). If the intersection does not appear for any value of \( n \), then we conclude that a UCOP can never appear inside the simple polytropic ball of equation of state (4).
Fig. 4. Plots of curves $C_1(\Sigma, V_c, n) = 0$ and $C_2(\Sigma, V_c, n) = 0$ in the $\Sigma-V_c$ plane for some fixed $n$. The regions $C_2 > 0$ and $C_2 < 0$ are also shown. It can be seen that the existence conditions of a UCOP (21) are not satisfied for single polytropic balls.

The numerical results of this strategy are shown in Fig. 4. We find that the curve $C_1 = 0$ remains in the region $C_2 < 0$ for all values of $n = 0.7, 1, 3, 4, 5, 6$. The same feature is found for the other values of $n$ as far as we have calculated. Hence, no UCOP appears inside the polytropic ball. Furthermore, the non-existence of a UCOP inside polytropic balls seems to hold for not only the physically interesting parameter region (12) but also for the whole parameter region $0 < n$ and $0 < V_c$.

5. Extension to the core–envelope piecewise polytrope models

In this section, the analysis of the simple polytrope model performed so far is extended to a representative model of core–envelope piecewise polytropic fluid balls. First, the formulation of our model is described in Sect. 5.1. Then, in Sects. 5.2 and 5.3, we investigate whether or not the properties (A) and (B) described in Sect. 1 hold for our model of piecewise polytropic balls. Our analysis will indicate the same result as for the simple polytropic ball: that no UCOP appears inside or outside the representative model of core–envelope piecewise polytropic balls.

5.1. Our model of the core–envelope piecewise polytropic perfect fluid ball

Dividing the interval of normalized mass density ($0 \leq \Sigma \leq 1$) into sub-intervals, the piecewise polytropic equation of state is defined as one whose polytrope index takes a different value at each sub-interval [11]. The core–envelope type is a polytrope composed of two sub-intervals (see Fig. 5),

$$ P(\Sigma) = \begin{cases} P_c \Sigma^{1+1/n_{cor}} & \text{in the core region, } \Sigma_b \leq \Sigma \leq 1 \\ K_{env} \Sigma^{1+1/n_{env}} & \text{in the envelope region, } 0 \leq \Sigma < \Sigma_b, \end{cases} \quad (23) $$
where $\Sigma_b$ is the mass density at the boundary between the core and envelope regions, the parameters $n_{\text{cor}}$ and $n_{\text{env}}$ are the polytrope indices of the core and envelope regions, respectively, and $P_c$ is the pressure at the center. We require that the pressure is continuous at the boundary between the two regions, $P(\Sigma_b - 0) = P(\Sigma_b + 0)$, in order to maintain the mechanical balance at the boundary. From this junction condition, the coefficient $K_{\text{env}}$ is given by the other parameters,

$$K_{\text{env}} = \frac{P_c \Sigma_b^{1/n_{\text{cor}} - 1/n_{\text{env}}}}{\Sigma_1^{1/n_{\text{cor}}}}. \tag{24}$$

This implies that $dP/d\Sigma$ becomes discontinuous at $\Sigma_b$, as shown in Fig. 5. And, through the TOV equation (11b), $dR/d\Sigma$ becomes discontinuous at $\Sigma_b$.

If we restricted our discussion to neutron stars, the nuclear matter near the surface might be in a solid state, and the matter around the core might be in a fluid state [9]. Such an inner structure of a neutron star may be approximately modeled by applying the above equation of state (23), and the phase transition density $\Sigma_b$ is determined by some reasonable nuclear matter physics. However, our discussion is not only for neutron stars but also for general spherical static balls of equation of state (23) under the condition of subluminal speed of sound. Therefore, we regard $\Sigma_b$ as a free parameter in our analysis.

The speed of sound $V$ is given by $V^2 = dP/d\Sigma$. There are thus two local maxima of $V$, at the center ($\Sigma = 1$) and at the boundary approached from the envelope ($\Sigma \to \Sigma_b - 0$). Hence, the subluminal speed of sound condition is given by

$$V_c^2 := P_c \left(1 + \frac{1}{n_{\text{cor}}}\right) \leq 1,$$

$$V_b^2 := K_{\text{env}} \left(1 + \frac{1}{n_{\text{env}}}\right) \Sigma_b^{1/n_{\text{env}}} = P_c \left(1 + \frac{1}{n_{\text{env}}}\right) \Sigma_b^{1/n_{\text{cor}}} \leq 1, \tag{25}$$

where the relation (24) is used in the second equality for $V_b$.

Here let us note that, according to our previous paper [8] which analysed the mass-to-radius ratio $3M_*/R_*$ of generic perfect fluid balls of any equation of state under the subluminal speed of sound condition, the ratio $3M_*/R_*$ tends to decrease as the speed of sound increases inside the ball. Therefore, in search of the upper bound of $3M_*/R_*$ for core–envelope piecewise polytropic balls, it seems

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5 When going down from the surface to the center of a neutron star, there may occur a phase transition from a solid state to a fluid state at some radius $R_b = R(\Sigma_b)$ due to the increase in matter density in the inward direction.
to be reasonable to set the speed of sound at $\Sigma = 1$ and $\Sigma_b$ to the speed of light,

$$V_c = 1, \quad V_b = 1.$$  \hfill (26)

The polytropic ball under this extreme speed of sound condition is the model we focus on in the following analyses.

Condition (26) and relation (24) give the relations

$$P_c(n_{cor}) = \frac{n_{cor}}{n_{cor} + 1},$$  \hfill (27a)

$$n_{env}(\Sigma_b, n_{cor}) = \left[ \frac{n_{cor} + 1}{n_{cor}} \Sigma_b^{1/n_{cor}} - 1 \right]^{-1},$$  \hfill (27b)

$$K_{env}(\Sigma_b, n_{cor}) = \frac{n_{cor}}{n_{cor} + 1} \Sigma_b^{1/n_{cor} - 1/n_{env}}.$$  \hfill (27c)

Through these relations, we find that the core–envelope piecewise polytrope (23) depends on a variable $\Sigma$ and two free parameters $(\Sigma_b, n_{cor})$. Actually, our equation of state (23) is a function of three arguments, $P(\Sigma, \Sigma_b, n_{cor})$, under the extreme speed of sound condition (26).

The parameter $\Sigma_b$ obviously takes a value in the interval

$$0 < \Sigma_b < 1.$$  \hfill (28a)

On the other hand, the physically interesting interval of the piecewise polytrope index, $n_{cor}$, is not obvious. Remember that, for the simple polytrope (4), the physically interesting interval of the polytrope index $(0 < n < 5)$ was determined by Nilsson and Uggla’s thorough numerical analysis of the simple polytrope [13]. Therefore, their analysis should be extended to piecewise polytropic balls when we need an accurate interval of the index $n_{cor}$. However, in the following analyses, we simply assume that a physically reasonable interval for the piecewise polytrope index $n_{cor}$ is the same as the simple polytrope index,

$$0 < n_{cor} < 5.$$  \hfill (28b)

In this paper, we do not aim for a thorough analysis of core–envelope piecewise polytropic balls, but we perform a test analysis with a representative model under the extreme speed of sound condition (26) in the parameter region (28). We expect that our model can represent the typical behavior of core–envelope piecewise polytropic balls.

Furthermore, here we introduce one more expectation: Remember that, for the simple polytrope (4) as mentioned in Sect. 3, Ref. [13] revealed the existence of divergence of the mass $M_*$ and radius $R_*$ at some isolated parameter points in the physically interesting parameter region (12). Therefore, for the core–envelope piecewise polytrope (23), the same divergence of $M_*$ and $R_*$ may occur. However, let us expect that the ratio $3M_*/R_*$ converges to zero even if $M_*$ and $R_*$ of the piecewise polytropic ball diverge, since $3M_*/R_*$ for the simple polytropic ball is expected to converge to zero as shown in Fig. 1.

5.2. Problem A: can a UCOP appear outside the core–envelope piecewise polytropic ball?

The problem that we numerically analyze in this section is whether or not the inequality $R_* < 3M_*$ holds for the fluid ball with equation of state (23) under the extreme speed of sound condition (26).
Fig. 6. Contours of $3M_*/R_*$ on the $\Sigma_b$–$n_{\text{cor}}$ plane for the core–envelope piecewise polytropic ball. The numerical calculation is performed in the region $0.05 \leq \Sigma_b \leq 1.0$ and $0.2 \leq n \leq 6$. Contours of $3M_*/R_*$ are plotted at intervals of 0.05 in the left panel and at intervals of 0.002 in the right panel. It is found that the inequality $3M_*/R_* < 1$ holds.

Our strategy to calculate the ratio $3M_*/R_*$ is as follows:

A1′: Numerically solve the TOV equations (11a) and (11b) for given values of parameters $(\Sigma_b, n_{\text{cor}})$, and calculate the ratio $3M_*/R_*$. 

A2′: Iterate step A1′ with varying parameters $(\Sigma_b, n_{\text{cor}})$, so as to obtain the ratio $3M_*/R_*$ as a function of parameters $(\Sigma_b, n_{\text{cor}})$. 

A3′: Find the maximum value of $3M_*/R_*$ as a function of $(\Sigma_b, n_{\text{cor}})$. If the maximum is less than unity, we conclude that the inequality $3M_*/R_* < 1$ holds for all values of $(\Sigma_b, n_{\text{cor}})$ in the parameter region (28), and no UCOP appears outside the core–envelope piecewise polytropic ball under the extreme speed of sound condition (26).

Our numerical result is shown in Fig. 6, in which the contours of $3M_*/R_*$ are plotted on the $\Sigma_b$–$n_{\text{cor}}$ plane. Because the contour of $3M_*/R_* = 1$ does not appear, the inequality $3M_*/R_* < 1$ holds for our model of a core–envelope piecewise polytropic ball under the extreme speed of sound condition (26) in our parameter region of interest (28).

Although the numerical analysis shown in Fig. 6 has been performed under the extreme speed of sound condition (26), our original interest is the case under the subluminal speed of sound condition (25). In order to consider the subluminal case, let us remember the discussion for introducing the extreme speed of sound condition (26). Then, according to our previous paper [8], we can expect that no UCOP appears outside core–envelope piecewise polytropic balls under the subluminal speed of sound condition (25) in our parameter region of interest (28).

5.3. Problem B: can a UCOP appear inside the core–envelope piecewise polytropic ball? The problem that we numerically analyze in this section is whether or not a UCOP can exist inside the fluid ball of equation of state (23) under the extreme speed of sound condition (26). The calculation and discussion up to Eq. (22) in Sect. 4 is applicable to the core–envelope piecewise polytropic ball. Therefore, our task is to check whether or not the existence conditions (21) hold.
Note that, for the core–envelope piecewise polytropic ball under the extreme speed of sound condition (26), the quantities \( C_1 \) and \( C_2 \) in Eq. (22) can be regarded as functions of three arguments, \( C_1(\Sigma, \Sigma_b, n_{\text{cor}}) \) and \( C_2(\Sigma, \Sigma_b, n_{\text{cor}}) \), since the values of \( C_1 \) and \( C_2 \) are calculated by substituting the solution of the TOV equations (11a) and (11b). Then, our strategy is as follows:

B1': Numerically solve the TOV equations (11a) and (11b) for given values of parameters \((\Sigma_b, n_{\text{cor}})\), and iterate this numerical calculation with varying \( \Sigma_b \) and \( n_{\text{cor}} \) fixed at a given value. This iteration produces \( C_1(\Sigma, \Sigma_b, n_{\text{cor}}) \) and \( C_2(\Sigma, \Sigma_b, n_{\text{cor}}) \) as functions of \((\Sigma, \Sigma_b)\) for the given value of \( n_{\text{cor}} \).

B2': Plot two curves, \( C_1 = 0 \) and \( C_2 = 0 \), and identify the two regions \( C_2 > 0 \) and \( C_2 < 0 \) in the \( \Sigma – \Sigma_b \) plane for the given \( n_{\text{cor}} \). If the curve \( C_1 = 0 \) does not intersect with the region \( C_2 \geq 0 \), it is concluded that, at the given value of \( n_{\text{cor}} \), no UCOP exists inside the core–envelope piecewise polytropic ball under the extreme speed of sound condition (26).

B3': Iterate steps B1' and B2' with varying \( n_{\text{cor}} \), and check whether or not the intersection of \( C_1 = 0 \) with \( C_2 \geq 0 \) exists at each value of \( n_{\text{cor}} \). If the intersection does not appear for any value of \( n_{\text{cor}} \) in our parameter region of interest (28), then we conclude that a UCOP can never appear inside the core–envelope piecewise polytropic ball under the extreme speed of sound condition (26).

By the above strategy, it is found that the quantity \( C_1(\Sigma, \Sigma_b, n_{\text{cor}}) \) remains positive and does not become zero \((C_1 > 0)\) in the \( \Sigma – \Sigma_b \) plane for low values of the polytrope index, \( 0 < n_{\text{cor}} < n_{\text{cor-low}} \), where \( n_{\text{cor-low}} \sim 1.5 \). Also, for higher values of the polytrope index, \( n_{\text{cor-low}} < n_{\text{cor}} < 5 \), although the quantity \( C_1 \) becomes zero at some values of \( \Sigma \) and \( \Sigma_b \) in the \( \Sigma – \Sigma_b \) plane at every value of \( n_{\text{cor}} \), the curve \( C_1 = 0 \) remains in the region \( C_2 < 0 \). An example for \( n_{\text{cor}} = 3 \) is shown in Fig. 7. Hence, we can conclude numerically that no UCOP appears inside core–envelope piecewise polytropic balls under the extreme speed of sound condition (26) in our parameter region of interest (28).

Although the above numerical analysis has been performed under the extreme speed of sound condition (26), our original interest is the case under the subluminal speed of sound condition (25). In order to consider the subluminal case, let us refer to the simple polytropic balls analyzed in Sect. 4, and note that the curve \( C_1 = 0 \) in the \( \Sigma – V_c \) plane for the simple polytrope, shown in Fig. 4, appears in the region of high values of \( V_c \). Hence, if similar behavior is expected for core–envelope piecewise polytropic balls, the curve \( C_1 = 0 \) in the \( \Sigma – \Sigma_b \) plane tends to disappear as the sound speeds \( V_b \) and
$V_c$ decrease. That is, the intersection of $C_1 = 0$ with $C_2 < 0$ does not appear under both the extreme speed of sound condition (26) and the subluminal speed of sound condition (25). This discussion makes us expect that no UCOP appears inside core–envelope piecewise polytropic balls under the subluminal speed of sound condition (25) in our parameter region of interest (28).

6. Summary and discussions

We have investigated whether or not a UCOP can exist in the spacetime of a static spherical ball of perfect fluid. The equations of state we have considered are the simple polytrope (4) and the core–envelope piecewise polytrope (23). By numerical analyses of the TOV and null geodesic equations, our result is as follows: For the simple polytropic balls, we have performed a thorough numerical investigation and concluded that no UCOP can exist inside or outside any simple polytropic ball under the subluminal speed of sound condition. For the core–envelope piecewise polytropic balls, we have numerically investigated a representative model under the extreme speed of sound condition (26), and concluded again that no UCOP can exist inside or outside a core–envelope piecewise polytropic ball under the extreme speed of sound condition. Further, according to our previous paper [8] and Sect. 4 of this paper, it is expected that the conclusion under the extreme speed of sound condition is also true of the case under the subluminal speed of sound condition.

The above conclusions mean that the polytropic balls investigated in this paper cannot be black hole mimickers which possess UCOPs but no black hole horizon. This implies that, if the polytrope treated in this paper is a good model of the stellar matter in compact objects, the detection of shadow image by optical observation can be regarded as good evidence of the existence of a black hole. Note that, to obtain a more definite conclusion for the core–envelope piecewise polytropic ball under the subluminal speed of sound condition, more detailed numerical research is necessary as discussed in the last two paragraphs of Sect. 5.1.

Next, let us discuss a by-product of our analysis. In Sects. 3 and 5.2, the ratio of total mass to surface radius of the polytropic ball, $3M_s/R_s$, has been the central issue. As mentioned in Sect. 1, the mass-to-radius ratio must be bounded above, $3M_s/R_s < 3/2$, in order to avoid gravitational collapse. Buchdahl [14] decreased the upper bound to $3M_s/R_s < 4/3$ by assuming non-increasing mass density in the outward direction and a barotropic equation of state. Next, Barraco and Hamity [15] decreased Buchdahl’s upper bound to $3M_s/R_s < 9/8$ by adding the dominant energy condition to Buchdahl’s assumptions. Furthermore, in our previous paper [8], we decreased Barraco and Hamity’s upper bound to $3M_s/R_s < 1.0909209$ by replacing the dominant energy condition with the subluminal speed of sound condition. All these upper bounds remained greater than unity, which permits the existence of some black hole mimickers. However, as shown in Fig. 2 of this paper, the upper bound of $3M_s/R_s$ is decreased to a value lower than unity by restricting the equation of state to the simple polytrope (4) and assuming the subluminal speed of sound condition (10). Since the upper bound is found on the vertical line at $V_c = 1$ in Fig. 2, a sectioned diagram of Fig. 2 at $V_c = 1$ is useful to read a precise value of the upper bound. This is shown in Fig. 8, where we define $f(n)$ by $3M_s/R_s$ as a function of $n$ at $V_c = 1$. From Fig. 8, it is concluded that the following inequality holds in the physically interesting parameter region (12) for simple polytropic balls under the subluminal speed of sound condition:

$$\frac{3M_s}{R_s} < 0.844,$$

where the upper bound is given by parameters $V_c = 1$ and $n \simeq 0.78$, and the value of the upper bound is numerically calculated, $f(0.78) \simeq 0.844$. On the other hand, when we restrict the equation of state
to the core–envelope piecewise polytrope (23) and assume the extreme speed of sound condition (26), the upper bound of the ratio \( \frac{3M_*}{R_*} \) can be roughly read from Fig. 6 as

\[
\frac{3M_*}{R_*} < 0.976. \tag{30}
\]

Finally, let us make a comment on a related topic. The possibility of trapping gravitational waves inside stellar objects has been discussed [16,17]. The potential of gravitational perturbation is analyzed in those discussions, while the potential of light propagation such as shown in Fig. 3 is discussed in our analysis. So, when one is interested in a combination of optical observation and gravitational wave observation in the search for black holes and gravitational waves, the existence/non-existence conditions of a UCOP and gravitational wave trapping may become an interesting issue.

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Appendix A. On the numerical treatment of the TOV equations

The right-hand sides of the TOV equations (11a) and (11b) are indeterminate form at the center because of the conditions \( M \to 0 \) and \( R \to 0 \) as \( \Sigma \to 1 \). Therefore, in solving the TOV equations numerically, we have made use of perturbative solutions near the center.

In order to consider a perturbation near the center, we regard the radius \( R \) as an independent variable, and the mass density as a function of radius, \( \Sigma(R) \). The TOV equations (11a) and (11b) are rearranged to

\[
\frac{dM(R)}{dR} = 4\pi R^2 \Sigma(R),
\]

\[
\frac{dP(R)}{dR} = -\left[ (\Sigma(R) + P(R)) \left[ M(R) + 4\pi R^3 P(R) \right] \right] R \left[ R - 2M(R) \right]. \tag{A1}
\]

For a sufficiently small radius \( R \ll 1 \), we introduce the perturbations

\[
M(R) = M_{(1)} R + M_{(2)} R^2 + M_{(3)} R^3 + \cdots, \\
P(R) = P_c + P_{(1)} R + P_{(2)} R^2 + P_{(3)} R^3 + \cdots, \\
\Sigma(R) = 1 + \Sigma_{(1)} R + \Sigma_{(2)} R^2 + \Sigma_{(3)} R^3 + \cdots. \tag{A2}
\]
where the conditions $M(R = 0) = 0$, $\Sigma(R = 0) = 1$, and $P(R = 0) = P_c$ are included. Substituting (A2) into (A1), we obtain $M(1) = 0$, $M_2 = 0$, $P(1) = 0$, and, for the remaining parts,

\[
M(R) = \frac{4}{3} \pi R^3 + \pi \Sigma(1) R^4 + \cdots ,
\]

\[
P(R) = P_c - \frac{2}{3} \pi (1 + 3 P_c) (1 + P_c) R^2 - \frac{\pi}{9} (7 + 15 P_c) \Sigma(1) R^3 + \cdots ,
\]

\[
\Sigma(R) = 1 + \Sigma(1) R + \Sigma(2) R^2 + \Sigma(3) R^3 + \cdots ,
\]

where the central pressure $P_c$ and coefficients $\Sigma(n)$ ($n = 1, 2, 3, \ldots$) are determined by a concrete form of the equation of state.

Substituting these perturbative expansions into the equation of state (4), we obtain

\[
\Sigma(1) = 0, \quad \Sigma(2) = -\frac{n}{n + 1} \frac{2\pi (1 + P_c) (1 + 3 P_c)}{3 P_c} .
\]

Hence, denoting a small radius by $R_\delta \ll 1$, the mass density $\Sigma_\delta$ and mass $M_\delta$ at $R = R_\delta$ are approximately given by $\Sigma_\delta = 1 + \Sigma(2) R_\delta^2$ and $M_\delta = (4\pi/3) R_\delta^3$. If the mass density near the center $\Sigma_\delta$ is given, then the others are determined by

\[
R_\delta = \sqrt{1 - \frac{\Sigma_\delta}{|\Sigma(2)|}}, \quad M_\delta = \frac{4}{3} \pi \left( \frac{1 - \Sigma_\delta}{|\Sigma(2)|} \right)^{3/2} .
\]

In the numerical calculation, we have solved the TOV equations (11a) and (11b) for the interval $0 < \Sigma \leq \Sigma_\delta$ with the initial condition (A5). Also, we have checked the convergence of numerical solutions with varying $\Sigma_\delta$. All the results in this paper are obtained using $\Sigma_\delta = 1 - 10^{-4}$.

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