Can spontaneous symmetry breaking order develop in a topological phase?

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Landau’s symmetry breaking theory is the basis of modern condensed matter physics, from which most phases can be understood. The recent discovery of topological insulators simulates the study on the topological phase which is beyond this theory. An interesting question is whether the topological phase and spontaneous symmetry breaking order can coexist. We study this problem based on the topological phase in the Haldane model. Firstly by including a staggered potential, we show that if the system is subjected to the symmetry needed by the charge density wave (CDW) order in advance, the topological phase can coexist with the CDW order, realizing the topological CDW phase. While when the NN interaction which preserves the symmetry of the origin Hamiltonian is included, the CDW order can not develop from spontaneous symmetry breaking in the topological phase. The results imply that a spontaneous symmetry breaking order may not develop in a topological phase.

Introduction. The discovery of topological insulators (TIs) has aroused great interests in the community of condensed matter physics. Many interesting properties are predicted or realized based on TIs, such as: Majorana fermions, quantum anomalous Hall effect et al. These studies not only have fundamental interests, but also suggest TIs’s potential applications in spintronics and quantum computing. Besides the studies on the weak-interacting TIs, the effect of electronic correlations has become an active field. There appear many studies discussing the issues related to interacting driven TIs, the interplay of spin-orbit coupling and interactions, the classification of the interacting topological phases, fractional Chern insulator, et al. The trend of topology also spreads to the field of cold atoms. Great efforts are devoted to the realization of the topological phases based on optical lattice.

TIs belong to a kind of topological phase, which has been studied for a long time. As has been known, most phases of matter can be understood through spontaneous symmetry-breaking. But a topological phase cannot be described by this theory. This phase does not break any symmetry, but are distinct phase of matter. An interesting question is that whether some spontaneous symmetry breaking order can develop in the topological phase which is beyond the description of spontaneous symmetry-breaking. In this paper, we study the problem based on the Haldane model using analytical and exact numerical methods. We firstly include a staggered potential and show that if the system is subjected to the symmetry needed by the CDW order in advance, the topological phase can coexist with the CDW order, realizing the topological CDW phase. Then a nearest-neighbor (NN) interaction is considered. Contrary to the staggered potential, this term preserves the symmetry of the origin Hamiltonian. We find no CDW order develops from spontaneous symmetry breaking in the topological phase. The results imply that a spontaneous symmetry breaking order may not develop in a topological phase.

The model. The lattice model under consideration is the Haldane model, which consists of two parts, a nearest-neighbor hopping term and a next-nearest-neighbor hopping spin-orbit term on the honeycomb lattice,

\[ H_0 = \sum_{(ij)} t c_{(ij)}^\dagger c_j + i\lambda \sum_{(ij)} \nu_{ij} c_{(ij)}^\dagger c_j. \]

Here \( c_{(ij)}^\dagger \) annihilates (creates) a fermion on site \( i \). \( t \) is the hopping amplitude, \( \lambda \) is the spin-orbit coupling, \( (ij) \) denotes nearest-neighbor and \( \langle\langle ij\rangle\rangle \) next-nearest-neighbor sites. \( \nu_{ij} = \text{sgn}(\hat{d}_1 \times \hat{d}_2)z \) with \( \hat{d}_1, \hat{d}_2 \) the unit vector connecting the sites \( j \) and \( i \), thus \( \nu_{ij} \) is \(-1\) for left turn and \(1\) for right turn.

In the momentum space, the Hamiltonian at \( \mathbf{k} = (k_x, k_y) \) writes as: \( H_0(\mathbf{k}) = h_x(\mathbf{k})\sigma_x + h_y(\mathbf{k})\sigma_y + h_z(\mathbf{k})\sigma_z \), with

\[ h_x(\mathbf{k}) = t[2\cos(\frac{\sqrt{3}}{2} k_x)\cos(\frac{1}{2} k_y) + \cos(k_y)] \]
\[ h_y(\mathbf{k}) = t[2\cos(\frac{\sqrt{3}}{2} k_x)\sin(\frac{1}{2} k_y) - \sin(k_y)] \]
\[ h_z(\mathbf{k}) = \lambda[-2\sin(\frac{\sqrt{3}}{2} k_x) + 4\sin(\frac{\sqrt{3}}{2} k_x)\cos(\frac{3}{2} k_y)] \]

and \( \sigma_{x,y,z} \) the Pauli matrices. The energy spectrum is: \( E(\mathbf{k}) = \pm \sqrt{h_x(\mathbf{k})^2 + h_y(\mathbf{k})^2 + h_z(\mathbf{k})^2} \). At the two inequivalent Dirac points \( K_1 = \frac{4\pi}{3\sqrt{3}}(1,0), K_2 = \frac{4\pi}{3\sqrt{3}}(\frac{1}{2}, \frac{\sqrt{3}}{2}) \), the spin-orbit term opens a gap with the size \( 6\sqrt{3}\lambda \). Since the gap has different sigh at the two Dirac points, the system exhibits nontrivial topological properties characterized by a nonzero Chern number.

Topological CDW induced by a staggered potential. To create a topological phase with CDW order, we include a staggered potential to Eq.\( H_0 \),

\[ H_s = \nu_0 \sum_i (-1)^i n_i, \]

with \( \nu_0 \) the strength of the potential. The Hamiltonian in the momentum space then changes to \( H'_0(\mathbf{k}) = h_x(\mathbf{k})\sigma_x + h_y(\mathbf{k})\sigma_y + (h_z(\mathbf{k}) + \Delta)\sigma_z \). The energy spectrum is \( E'(\mathbf{k}) = \pm \sqrt{h_x(\mathbf{k})^2 + h_y(\mathbf{k})^2 + (h_z(\mathbf{k}) + \nu_0)^2} \) and the gap becomes \( 2|3\sqrt{3}\lambda - \nu_0| \). At \( \nu_0 = 3\sqrt{3}\lambda \) the gap closes and the system experiences a TPT.
In the topological phase, due to the presence of the staggered potential, the average numbers of the electrons on the two sublattices differ and the CDW order appears. The staggered potential is increased, \( \langle n_i \rangle \) increases. At the critical value \( \nu_0 \), \( \rho \) shows a jump. However this jump varies with the sizes of the lattice and tends to vanish for large sizes. We also calculate the averages of NN and NNN hopping amplitudes and the results show that the NN hopping is continuous, but the NNN hopping has a jump, as shown in Fig.1.

Using the Green's function method, the above averages can be expressed as,

\[
\langle c_i^{\dagger} c_j \rangle = \frac{1}{N} \sum_k \left( \frac{1}{2} - \frac{h_z(k) + \nu_0}{2E'_k} \right) e^{ik(j-i)},
\]

\[
\langle c_i^{\dagger} c_j^{\dagger} c_j c_i \rangle = \frac{1}{N} \sum_k \left( \frac{1}{2} + \frac{h_z(k) + \nu_0}{2E'_k} \right) e^{ik(j-i)},
\]

\[
\langle c_i^{\dagger} c_j c_j^{\dagger} c_i \rangle = -\frac{1}{N} \sum_k \frac{a - ib}{2E'_k} e^{ik(j-i)},
\]

with \( A,B \) denoting the two sublattices. So \( \rho = \frac{1}{N} \sum_k \frac{h_z(k) + \nu_0}{E'_k} \). For the case of \( \nu_0 = 0 \), \( \rho = 0 \) since \( h_z(k) \) is asymmetric with respect to \( k \). The NN hopping amplitude \( \chi_1 = \langle c_i^{\dagger} c_j \rangle_{ij \in NN} \) is real, while the NNN one \( \chi_2 = \langle c_i^{\dagger} c_j \rangle_{ij \in NNN} \) is complex. The jump near the transition point can be understood from the above formulae. The jump only occurs for the average of the oper-

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**FIG. 1:** (Color online) (a) \( \rho = |\frac{1}{2} - \langle n_i \rangle| \) with \( \langle n_i \rangle \) the average density of the electrons on the \( i-th \) site. (b) the difference \( \delta \rho = \rho(n_0 + \varepsilon) - \rho(n_0) \) with \( \varepsilon \) small finite value for \( N_x = 48 \). (c) the average of the NN hopping amplitude. The real part (d) and the imaginary part (e) of the NNN hopping amplitude. (f) the difference \( \Delta \chi_2 = \chi_2(n_0 + \varepsilon) - \chi_2(n_0) \) for \( N_x = 48 \). Here the number of the unit cells in \( y \) direction \( N_y \), \( N_y = 48 \), and the total sites \( N = 2N_x \times N_y \), \( t = 1 \) and \( \lambda = 0.1 \) are fixed for all calculations if not specialized.

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**FIG. 2:** (Color online) (a) the three lowest eigenenergies. (b) the energy difference of the eigenstates: \( \Delta_1 = E_1 - E_0 \) and \( \Delta_2 = E_2 - E_0 \). (c) \( \Delta_1 \) and \( \Delta_2 \) near the transition point. (d) the average of the NN hopping amplitude. (e) the average of the NNN hopping amplitude. (f) the average of the NN particle correlation. The inset shows the 1st Brillouin zone and the \( k \) points for a lattice with \( N_x = N_y = 3 \), on which all ED calculations are performed.

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**FIG. 3:** (a) \( |\frac{1}{2} - \langle n_i \rangle| \) with \( \langle n_i \rangle \) the average density of the electrons on the \( i \) - \( th \) site. (b) the static structure factor at \( k = 0 \). The inset shows the two configurations of spontaneous symmetry breaking CDW order.
atons on the same sublattice since the expression for such averages contains the coefficient $h_i(k)$ which changes its sign at the critical point. Thus near the Dirac point, 
$$|\frac{h_i(k)+\epsilon_0}{E_k}| \simeq 1$$
and has different signs at the two sides of the transition point. So as the parameter $\nu_0$ passes the transition point, the value of $\frac{h_i(k)+\epsilon_0}{E_k}$ has a jump. Though the jump exists, its value is scaled by the number of the sites $N$. This is the reason that the jump is invisible for large sizes. However if we do a difference between $\nu_0+\epsilon$ and $\nu_0$ with $\epsilon$ small finite value, this jump is still visible, as shown in Fig.

Till now by including a staggered potential, we realize the topological CDW phase, in which the nontrivial topology and the CDW order coexist. We also show that besides the closing of the gap, the two-operator averages also show unusual properties near the TPT. Specially we notice that the TPT can be characterized by this discontinuity even at small sizes although the property is affected by the system’s size.

**TPT driven by the NN interaction.** Although the CDW order is realized in a topological phase by simply including a staggered potential, it is not due to the spontaneous symmetry breaking. It is interesting to study whether the CDW order can be generated from a term which preserves the symmetry of the original system. The NN interaction is such a term and favors a CDW order. Next we add the NN interaction to Eq. (1),

$$H_I = V_1 \sum_{\langle ij \rangle} n_in_j, \quad (3)$$

and study the topological phase and the induced TPT.

Using the ED method, the eigenenergy of the interacting system can be directly obtained. In the non-interacting limit, the ground-state is gapped from the excited states. As $V_1$ is increased, the gap decreases and closes at a critical interaction $V_1^c = 1.38666 \pm 0.00001$, which marks the TPT. After that the ground-state becomes two-fold degenerate and is separated from the excited states by a gap which increases with $V_1$. In Fig.2 we also plot the energy difference between the eigenstates: $\Delta_{10} = E_1 - E_0$ and $\Delta_{20} = E_2 - E_0$ with $E_0, E_1, E_2$ the eigenenergies of the ground-state, the first and second excited states, in which the TPT is shown more clearly. To further study the transition, with the ground-state wave function $|\psi_0\rangle$, we calculate the averages of the NN and NNN hopping amplitudes, $\chi_1, \chi_2$; the NN particle correlation; $\langle n_{i,n_j}\rangle_{NN} = \langle \psi_0|n_{i,n_j}|\psi_0\rangle_{\langle ij \rangle \in NN}$. These quantities show discontinuous at the same critical energy $V_1^c$.

To characterize the topological nature of the phase transition, using generalized boundary condition, $\psi(x_i + N_x, y_1 + N_y) = e^{i(\phi_x+\phi_y)}\psi(x_i,y_1)$, where $(x_i, y_1)$ are the coordinates for the $i-th$ site and $(\phi_x, \phi_y)$ the twisted phases, the Chern number can be defined,

$$C = \frac{1}{2\pi i} \int_0^{2\pi} \int_0^{2\pi} d\phi_x d\phi_y F_{12}(\phi_x, \phi_y). \quad (4)$$

Here the field strength $F_{12}(\phi_x, \phi_y) = \partial_{\phi_x} A_2(\phi_x, \phi_y) - \partial_{\phi_y} A_1(\phi_x, \phi_y)$ with the Berry connection $A_{1,2}(\phi_x, \phi_y) = \langle \Psi|\partial_{\phi_{\phi_x(y)}}|\Psi\rangle$ and $|\Psi\rangle$ the many-body wave function. The calculations show that $C$ changes its value from one to zero at the transition point, confirming the topological nature of the phase transition.

So the NN interaction drives a TPT at a finite value $V_1^c$. Next we study the CDW order in the system. We calculate the distribution of the electrons $(n_{i}) = 1/2 + \rho \langle \psi_{1/2} | \tilde{V} | \psi_{1/2} \rangle$ and the static structure factor (SSF), which is defined as:

$$S(k) = \frac{1}{N} \sum_{i,j} \langle (n_{i,A} - n_{i,B})(n_{j,A} - n_{j,B}) \rangle e^{ik(i-j)}. \quad (5)$$

The CDW order is characterized by $S_{CDW} = S(k = 0)$. As shown in Fig.3 in the topological phase the electrons distribute uniformly and $S_{CDW}$ is very small, implying that no symmetry breaking CDW order coexists with the topological one. After the TPT, the SSF obtains a finite value. By a proper linear combination of the two nearly degenerate ground-state, two new states with the CDW order can be constructed. Thus the CDW order appears only after the topological phase is broken. These results give clear evidence that the topological and the symmetry breaking phases do not coexist.

To understand the above results further, we present a mean-field treatment. The NN interaction can be decoupled as,

$$n_in_j = n_i\langle n_j \rangle + \langle n_i \rangle n_j - \langle n_i \rangle \langle n_j \rangle$$

This procedure yields a mean-field Hamiltonian which writes,

$$H_{mf} = \sum_k [h_x(k)\sigma_x + h_y(k)\sigma_y + (h_z(k) - 3\rho V_1)\sigma_z] + C_0 \quad (6)$$

with the constant $C_0 = \frac{1}{2} NV_1 \rho^2 + \frac{3}{2} NV_1$. The ground-state is obtained by minimizing the free energy $F = -\frac{1}{N} \sum_k \ln(1 + e^{\beta E_{mf}(k)})$ with $\beta = 1/k_BT$ and $E_{mf}(k)$ the new energy spectrum. This yield a self-consistent equation,

$$\rho + \frac{1}{N} \sum_k \frac{h_z(k) - 3\rho V_1}{E_{mf}(k)} = 0,$$

in which $\rho$ can be determined self-consistently.

As shown in Fig.4 (a), $\rho$ is zero at small $V_1$ in the topological phase. Then at a critical point, $\rho$ jumps to a finite value $\rho_c$. Specifically the value approximates $3\rho_c V_1 \simeq 3\sqrt{3}\lambda$, implying the appearance of the CDW order and the disappearance of the topological phase is simultaneous. So even the mean-field results support the sudden appearance of the CDW order, implying that the topological phase and the symmetry breaking CDW order do not coexist. To make a comparison, we also show the mean field result for the case of $\lambda = 0$, when the system is a semimetal and has two inequivalent Dirac
CDW order in advance. As the strength of the staggered potential, which subjects the system to the symmetry needed by the Haldane model, we firstly include a staggered potential, and we show that a TPT is driven by the NN interaction. However no CDW order develop from spontaneous symmetry breaking order. The results imply that a topological phase can not coexist with a order from spontaneous symmetry breaking.

Conclusions. In conclusion, we study the problem whether the topological phase and spontaneous symmetry breaking order can coexist. Starting from the Haldane model, we firstly include a staggered potential, which subjects the system to the symmetry needed by the CDW order in advance. As the strength of the staggered potential is below a critical value, the topological phase can coexist with the CDW order, realizing the topological CDW phase. Then a TPT is driven by the staggered potential, which can be characterized by the closing of the gap and the vanishing of the Chern number. At the transition point, the averages of the NNN hopping amplitude shows discontinuity, which can also be viewed as the signature of the TPT.

Next we add the NN interaction which preserves the symmetry of the origin Hamiltonian. By calculating the energy spectrum, the Chern number, the averages of the NN and NNN hopping amplitudes, the NN particle correlation, we show that a TPT is driven by the NN interaction. However no CDW order develop from spontaneous symmetry breaking in the topological phase. These results imply that a topological phase can not coexist with a order from spontaneous symmetry breaking.

Finally we want to emphasize that although the present work focuses on a concrete example, the question put forward in the paper is quite general and needs more further studies. The model under consideration may be realized in optical lattice and the related physics can be also studied experimentally.

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