Dimension Drop for Transient Random Walks on Galton-Watson Trees in Random Environments

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August 2, 2018

We prove that the dimension drop phenomenon holds for the harmonic measure associated to a transient random walk in a random environment (as defined in [7, 14]) on an infinite Galton-Watson tree without leaves. We use regeneration times and ergodic theory techniques from [16] to give an explicit construction of the invariant measure for the forward environment seen by the particle at exit times which is absolutely continuous with respect to the joint law of the tree and the path of the random walk.

Keywords: Galton-Watson tree, random walk, harmonic measure, Hausdorff dimension, invariant measure, dimension drop, random walk in random environments.

AMS 2010 subject classifications:
Primary 60J50, 60J80, 37A50; secondary 60J05, 60J10.

Version Notes: This is version 1.0. Any feedback is highly appreciated.

Introduction

Consider an infinite rooted tree $t$. A ray in $t$ is an infinite path starting from its root, that never backtracks. The set of all rays in $t$ is called its boundary and denoted $\partial t$. We put a natural topology on $\partial t$ by saying that two rays are close to each other when they coincide in a large ball centered at the root of $t$.

We are interested in natural Borel probability measures on $\partial t$. In many cases, such a measure $\theta$ happens to have full support. However, it may turn out that $\theta$-almost every ray $\xi$ belongs to a Borel subset $C_\theta$ of $\partial t$ that is “small”, compared to the whole boundary. To make this rigorous and quantitative, we put a suitable metric on the boundary $\partial t$ and use Hausdorff dimensions. The Hausdorff dimension of the measure $\theta$ is the Hausdorff dimension of a minimal Borel subset $C_\theta$ such that $\theta(C_\theta) = 1$. One

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says that the *dimension drop phenomenon* occurs for \( \theta \) when this dimension is strictly less than the dimension of the whole boundary.

Now, let us suppose that we have a transition matrix \( P^t \) on the vertices of \( t \) that defines a nearest-neighbour random walk on \( t \), which is *transient*. By transience, almost every trajectory of the walk “goes to infinity” and shares infinitely many vertices with a unique random ray \( \Xi \). The law of this random ray \( \Xi \) is called the *harmonic measure* on \( \partial t \), with respect to the transition matrix \( P^t \).

When the tree \( T \) is a random, Galton-Watson tree, the dimension drop phenomenon for the harmonic measure is already known to occur in some cases: see [15] for the simple random walk, [16] for transient \( \lambda \)-biased random walks and [4, 10, 19] for other models depending on random lengths on the Galton-Watson tree.

In this work, we prove that the dimension drop phenomenon occurs for the harmonic measure with respect to a transition matrix defined by a *random environment* on the Galton-Watson tree. This model was introduced in [14] and has been extensively studied (see for instance [1, 9, 3, 2]). We will use the definition from [7], which is a generalization and can be described as follows. Let \( p := (p_k)_{k \geq 1} \) a sequence of non-negative real numbers such that \( p_1 < 1 \) and \( \sum_{k \geq 1} p_k = 1 \). Assume that \( m := \sum_{k \geq 1} kp_k < \infty \). Let \( (N, A) \) a random element in \( \mathbb{N}^* \times \bigcup_{k \geq 1} (0, \infty)^k \) such that the marginal law of \( N \) is \( p \) and, for any \( k \geq 1 \), if \( N = k \), then \( A \) lies in \( (0, \infty)^k \). Build a random weighted Galton-Watson tree \( T \) in the following way: we start from a root \( \emptyset \) and pick a random element \( (N, A) \). The number of children of the root is then \( N \) and the children 1, 2, \ldots, \( N \) of the root have respective weights \( A(1), A(2), \ldots, A(N) \), where \( (A(1), \ldots, A(N)) := A \). Then, continue recursively in an independent manner on the subtrees starting from the vertices 1, \ldots, \( N \).

Now, conditionally on \( T \), we start a nearest-neighbour random walk \( X \) from \( \emptyset \) such that, if the walk is at a vertex \( x \) of \( T \), the walk may jump to one of the children of \( x \) with probability equal to the weight of this child divided by the sum of the weights of the children of \( x \) plus 1 while it goes back to the parent of \( x \) with probability equal to 1 divided by the same sum. We know, from [14] and [7] a transience criterion for this model, and assume throughout this work that we are in this regime. Our main result is the following theorem:

**Theorem 0.1** (Dimension drop for HARM). Let \( T \) a random weighted Galton-Watson tree. The harmonic measure \( \text{HARM}_T \) is almost surely exact-dimensional and its Hausdorff dimension is almost surely a constant that equals

\[
\dim_{\text{H}} \text{HARM}_T = \frac{1}{\mathbb{E}[\kappa(T)]} \mathbb{E}[-\log(\text{HARM}_T(\Xi_1)) \kappa(T)],
\]

with \( \kappa \) defined by (3.2). It is almost surely strictly less than the Hausdorff dimension of the whole boundary \( \partial T \) (which is almost surely \( \log m \)), unless the model reduces to a transient \( \lambda \)-biased random walk (with a deterministic and constant \( \lambda < m \)) on an \( m \)-regular tree.

Our results are inspired by the work of Lyons, Peres and Pemantle on transient \( \lambda \)-biased random walks on Galton-Watson trees ([16]). We use in the same way the notions
of exit times and regeneration times to build an invariant measure for the forward environment seen by the particle at exit times. The construction of this measure, via a Rokhlin tower, was already suggested in [16].

The paper is organized as follows. In Section 1, we introduce our notations and define the space of weighted trees, the transient paths on such trees, the exit times and regeneration times of such paths. We also recall a transience criterion by Lyons and Pemantle ([14]), generalized by Faraut ([7]). In Section 2, we show that there are almost surely infinitely many regeneration times and find an invariant measure for the forward environment seen by the particle at such times. Again, this follows the ideas of [16], but we give detailed proofs in our setting of weighted trees for completeness. The heart of this work is Section 3, where we give a detailed “tower construction” over the preceding dynamical system to build an invariant measure for the forward environment seen by the particle at exit times. We then show that this measure has a density with respect to the joint law of the tree and the random path on it and give an expression of this density. We conclude in Section 4 by projecting this measure on the space of trees with a random ray on it and using the general theory of flow rules on Galton-Watson trees developed in [15].

Acknowledgements. The author is very grateful to his Ph.D. supervisors Julien Barral and Yueyun Hu for many interesting discussions and constant help and support. He also thanks Élie Aïdekon for encouraging him to write this work.

1 Notations and preliminaries

1.1 Words and paths

Let $\mathbb{N}^* := \{1, 2, \ldots \}$ be our alphabet. The set of all finite words on $\mathbb{N}^*$ is

$$\mathcal{U} := \bigcup_{k=0}^{\infty} (\mathbb{N}^*)^k,$$

with the convention $(\mathbb{N}^*)^0 := \{\varnothing\}$, $\varnothing$ being the empty word. The length $|x|$ of a word $x$ is the unique integer such that $x$ belongs to $(\mathbb{N}^*)^{|x|}$. The concatenation of the words $x$ and $y$ is denoted by $xy$. A word $x = (i_1, i_2, \ldots, i_{|x|})$ is called a prefix of a word $y = (j_1, j_2, \ldots, j_{|y|})$ when either $x = \varnothing$ or $|x| \leq |y|$ and $i_\ell = j_\ell$ for any $\ell \leq |x|$. We denote $\leq$ this partial order and $x \wedge y$ the greatest common prefix of $x$ and $y$. The parent of a non-empty word $x = (i_1, i_2, \ldots, i_{|x|})$ is $x_* := (i_1, i_2, \ldots, i_{|x|-1})$ if $|x| \geq 2$; otherwise it is the empty word $\varnothing$. We also say that $x$ is a child of $x_*$. When a finite word $x$ is a prefix of a word $y$, we define $x^{-1}y$ as the unique word $z$ such that $y = xz$.

We add an artificial parent of the empty word. Let $\varnothing_*$ an arbitrary element that does not belong to $\mathcal{U}$ and $\mathcal{U}_* := \mathcal{U} \cup \{\varnothing_*\}$. We let $|\varnothing_*| := -1$, $(\varnothing)_* := \varnothing_*$ and, for all $x$ in $\mathcal{U}$, $x\varnothing_* = \varnothing_*x = x_*$. 

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An infinite path in $\mathcal{U}$ is a sequence $x = (x_0, x_1, \ldots)$ such that for any $k \geq 0$, $x_{k+1}$ is either a child of $x_k$ or its parent. A transient path is an infinite path $x$ such that $\lim_{k \to \infty} |x_k| = \infty$.

For such a path $x$, we define:

- the set of fresh times:
  $$
  ft(x) := \{ s \geq 0 : \forall k < s, x_k \neq x_s \} =: \{ ft_0(x), ft_1(x), \ldots \},
  $$
  where $ft_0(x) < ft_1(x) < \cdots$;

- the set of exit times:
  $$
  et(x) := \{ s \geq 0 : \forall k > s, x_k \neq (x_s)_s \} =: \{ et_0(x), et_1(x), \ldots \},
  $$
  where $et_0(x) < et_1(x) < \cdots$;

- the exit points, $ep_k(x) := x_{et_k(x)}$, for $k = 0, 1, \ldots$;

- the set of regeneration times:
  $$
  rt(x) := ft(x) \cap et(x).
  $$

Likewise, the regeneration times (if there are any) are ordered $rt_0(x) < rt_1(x) < \cdots$, and if there are at least $k$ regeneration times, $rp_k(x) := x_{rt_k(x)}$ is the $k$-th regeneration point and $rh_k(x) := |rp_k(x)|$ is the $k$-th regeneration height.

- for $u \in \mathcal{U}$, the first hitting time of the path $x$ to $u$ is
  $$
  \tau_u(x) := \inf \{ s \geq 0 : x_s = u \},
  $$
  with the convention $\inf \emptyset = +\infty$.

A ray is an infinite path $\rho$ such that $\rho_0 = \emptyset$ and for each $k \geq 0$, $\rho_{k+1}$ is a child of $\rho_k$. In particular, for each $k \geq 0$, $|\rho_k| = k$. Any transient path $x$ starting from $x_0 = \emptyset$ defines a ray
  $$
  \text{ray}(x) := (ep_0(x), ep_1(x), \ldots).
  $$

Let $\mathcal{U}_\infty := (\mathbb{N}^*)^{\mathbb{N}^*}$ the set of all infinite words. For $k \geq 0$, the $k$-th truncation of an infinite word $\xi$ is the finite word composed of its $k$ first letters and is denoted $\xi_k$, with $\xi_0 := \emptyset$. The mapping $\xi \mapsto (\xi_0, \xi_1, \xi_2, \ldots)$ is a bijection between infinite words and rays, therefore we will often abuse notation and write $\xi$ for both the infinite word and the ray associated to it. When a finite word $x$ is a truncation of an infinite word $\xi$, we still say that $x$ is a prefix of $\xi$. For two distinct infinite words $\xi$ and $\eta$, we may again consider their greatest common prefix $\xi \land \eta \in \mathcal{U}$ and define the natural distance between each other by
  $$
  d_{\mathcal{U}_\infty}(\xi, \eta) = e^{-|\xi \land \eta|}.
  $$

This makes $\mathcal{U}_\infty$ into a complete, separable, ultrametric space.
1.2 Trees and flows

We represent our trees as subsets of the finite words on the alphabet \(N^*\). A (rooted, planar, locally finite) tree \(t\), without leaves, is a subset of \(U^*\) such that:

- \(\emptyset\) and \(\emptyset^*\) are in \(t\);
- for any \(x \neq \emptyset^*\) in \(t\), there exists a unique positive integer, denoted by \(\nu_t(x)\) and called the number of children of \(x\) in \(t\), such that for any \(i \in N^*\), \(x_i\) is in \(t\) if and only if \(i \leq \nu_t(x)\).

In this context, we call \(\emptyset\) the root of \(t\). The tree \(t\) is endowed with the undirected graph structure obtained by drawing an edge between each vertex and its children.

We say that an infinite path \(x\) in \(U_\ast\) belongs in \(t\) if for any \(k \geq 0\), \(x_k\) is in \(t\). The boundary of \(t\) is the set \(\partial t\) of all rays that belong in it. It is a compact subspace of \(U_\infty\).

A flow on \(t\) is a function \(\theta\) from \(t\) to \([0, 1]\), such that \(\theta(\emptyset) = 1\) and for any \(x \in t\),

\[\theta(x) = \sum_{i=1}^{\nu(x)} \theta(x_i)\]

Let \(M\) a Borel probability on \(\partial t\). We may define a flow \(\theta_M\) on \(t\) by setting, for all \(x \neq \emptyset^*\) in \(t\), \(\theta_M(x) = M\) \(\{\xi \in \partial t : x \prec \xi\}\). By a monotone class argument, the mapping \(M \mapsto \theta_M\) is a bijection and we will write \(\theta\) for both the flow on \(t\) and the associated Borel probability measure on \(\partial t\).

The (upper) Hausdorff dimension of a flow \(\theta\) on the tree \(t\) is defined by

\[\dim_H(\theta) := \inf \{\dim_H(B) : B\text{ Borel subset of }\partial t, \theta(B) = 1\}\]

When, for some \(\alpha \geq 0\), \(\theta\) satisfies

\[-\frac{1}{n} \log (\theta(\xi_n)) \rightarrow \alpha\quad \text{for } \theta\text{-almost every } \xi,\]

one says that \(\theta\) is exact-dimensional and many alternative definitions of its dimension coincide (see [6, chapter 10] and [18]). In particular, \(\dim_H \theta = \alpha\). We say that the dimension drop phenomenon occurs for \(\theta\) when \(\dim_H \theta < \dim_H \partial t\).

1.3 Weighted trees

A weighted tree is a tree \(t \in \mathcal{T}\) together with a function \(A_t\) from \(t \setminus \{\emptyset, \emptyset^*\}\) to \((0, \infty)\).

For \(x\) in \(t \setminus \{\emptyset^*, \emptyset\}\), we call \(A_t(x)\) the weight of \(x\) in \(t\).

We will only work with weighted trees but to lighten notations, we will write \(t\) when we should write \((t, A_t)\). We still, however, write \(x \in t\) when we mean that a word \(x\) is a vertex of the weighted tree \(t\).

We define the (local) distance between two weighted trees \(t\) and \(t'\) by

\[d_w(t, t') = \sum_{r \geq 0} 2^{-r-1} \delta_m^{(r)}(t, t')\]
where $\delta_w^{(r)}$ is defined by
\[
\delta_w^{(r)}(t, t') = \begin{cases} 
1 & \text{if } t \text{ and } t' \text{ (without their weights) do not agree up to height } r; \\
\min(1, \sup \{|A_t(x) - A_{t'}(x)| : x \in t, 1 \leq |x| \leq r\}) & \text{otherwise.}
\end{cases}
\]

We denote $\mathcal{T}_w$ the metric space of all weighted trees. It is a Polish space. For a weighted tree $t$ and a vertex $x \in t$, we denote
\[
t[x] := \{u \in U : xu \in t\},
\]
the reindexed subtree starting from $x$ with weights
\[
A_t[x](y) := A_t(xy), \quad \forall y \in t[x] \setminus \{o, o_s\}.
\]
The tree $t$ pruned at a vertex $x \neq o_s$ in $t$ is the subset of $U$ defined by
\[
t^{\leq x} := \{y \in t : x \neq y\}.
\]
Notice that $x$ is in $t^{\leq x}$. We will write $t^{\leq x}$ when we mean $t^{\leq x}$ together with the restriction of $A_t$ to $t^{\leq x}$. The set of all pruned (weighted) trees is
\[
\mathcal{T}_w^{\leq} := \{t^{\leq x} : t \in \mathcal{T}_w, x \in t\}.
\]
We equip it with a local distance similar to $d_w$.

For two weighted trees $t$ and $t'$, and $x \neq o_s$ in $t$, we define the glued weighted tree $t^{\leq x} \triangleleft t'$ as the tree
\[
t^{\leq x} \triangleleft t' := t^{\leq x} \cup \{xy : y \in t' \setminus \{o_s, o\}\}
\]

We equip it with the weights :
\[
A_{t^{\leq x} \triangleleft t'}(z) = \begin{cases} 
A_t(z) & \text{if } x \neq z; \\
A_{t'}(x^{-1}z) & \text{otherwise.}
\end{cases}
\]
Notice that in particular the weight of $x$ in $t^{\leq x} \triangleleft t'$ is still $A_t(x)$.

1.4 Flow rules on weighted trees

A (positive and consistent) flow rule on weighted trees is a measurable mapping $t \mapsto \Theta_t$ from a Borel subset $\mathcal{I}$ of $\mathcal{T}_w$ to the set of functions $[0, 1]^U$ such that:

- for any weighted tree $t$ in $\mathcal{I}$, $\Theta_t$ is a flow on $t$;
- for any $x$ in $t$, $t[x] \in \mathcal{I}$ and $\Theta_t(x) > 0$;
- for any $xy$ in $t$,
  \[\Theta_t(xy) = \Theta_t(x)\Theta_{t[x]}(y).\]
Notice that in our context, $\Theta_t$ might depend on the weights of the weighted tree $t$. As a simple first example, let us define the flow rule $\overline{\text{VIS}}$ as in [11] (it is denoted there $\nu$; when the weights are all equal, we recover the flow rule $\text{VIS}$ of [15]). For a weighted tree $t$ and $x$ in $t \setminus \{o_\ast\}$, we let

$$\overline{\text{VIS}}_t(x) := \prod_{o < u \leq x} \frac{A_t(y)}{\sum_{i=1}^{\nu_t(y_\ast)} A_t(y_\ast)},$$

In other words, this is the law of the trajectory of a non-backtracking random walk starting from $o_\ast$, which randomly chooses a child of its current position with probability proportional to its weight. See [11, Theorem 7] for the exact dimensionality and the Hausdorff dimension of $\overline{\text{VIS}}_T$, when $T$ is a random weighted Galton-Watson tree.

For other examples of flow rule, see $\text{HARM}$ (the harmonic flow rule) in the next subsection and $\text{UNIF}$ (the limit uniform measure) in [15, Section 6] and in Section 4.

### 1.5 Random walks on weighted trees

Let $t$ a weighted tree. We associate to $t$ a transition matrix : for all $x \neq o_\ast$ in $t$ and all $y$ in $t$

$$P^t(x, y) = \begin{cases} 1 / \left(1 + \sum_{j=1}^{\nu_t(x)} A_t(xj)\right) & \text{if } y = x_\ast; \\ A_t(xi) / \left(1 + \sum_{i=1}^{\nu_t(x)} A_t(xj)\right) & \text{if } y = xi, \text{ for } 1 \leq i \leq \nu_t(x); \\ 0 & \text{otherwise.} \end{cases} \hspace{1cm} (1.2)$$

The walk is reflected at $o_\ast$, that is, $P^t(o_\ast, o) = 1$. For $x$ in $t$, we denote $P^t_x$ the probability measure under which the random path $X = (X_0, X_1, \ldots)$ in $t$ is a Markov chain starting from $x$ with transition matrix $P^t$. The associated expectation is denoted $E^t_x$. Since we will later consider random weighted tree, $P^t_x$ and $E^t_x$ will often be referred to as the “quenched” probabilities and expectations.

We say that a weighted tree $t$ is everywhere transient if, for all $x \neq o_\ast$, the random path $X$ in $t[x]$ is $P^t_x$-almost surely transient. When a weighted tree $t$ is everywhere transient, the harmonic measure $\text{HARM}_t$ on its boundary $\partial t$ is the law of $\Xi := \text{ray}(X)$. The mapping $t \mapsto \text{HARM}_t$ on the set of everywhere transient weighted trees is then a (positive and consistent) flow rule.

### 1.6 Weighted Galton-Watson trees

We consider a reproduction law $p = (p_k)_{k \geq 0}$, that is a sequence of non-negative real numbers such that $\sum_{k=0}^{\infty} p_k = 1$. We assume throughout this work that $p_0 = 0$ and $m := \sum_{k \geq 1} p_k k < \infty$. Under some probability $P$, let $(N, A)$ a random element in $\mathbb{N}^* \times \bigcup_{k \geq 1} (0, \infty)^k$ such that the marginal law of $N$ is $p$ and, for any $k \geq 1$, if $N = k$, then $A$ lies in $(0, \infty)^k$.

The law of the random weighted Galton-Watson tree $T$ under the probability $P$ is defined recursively in the following way:
• The joint law of \((\nu_T(\phi), A_T(1), A_T(2), \ldots, A_T(\nu_T(\phi)))\) is the law of \((N, A)\);
• the reindexed subtrees \(T[1], \ldots, T[\nu_T(\phi)]\) are independent and have the same law as \(T\).

We call the resulting tree a weighted Galton-Watson tree and denote its law \(\text{GW}\). Note that, if we forget the weights, we recover a classical Galton-Watson tree of reproduction law \(p\). The branching property is still valid in this setting of weighted trees. More precisely, let \(k \in \mathbb{N}^*, B_1, B_2, \ldots, B_k\) Borel sets of \((0, \infty)\) and \(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\) Borel sets of \(\mathcal{F}_w\).

\[
P(\nu_T(\phi) = k, A_T(1) \in B_1, \ldots, A_T(k) \in B_k, T[1] \in \mathcal{B}_1, \ldots, T[k] \in \mathcal{B}_k) = p_k \mathbb{P}(A \in B_1 \times \cdots \times B_k \mid N = k) \prod_{i=1}^k \text{GW}(B_i). \tag{1.3}
\]

Let \(f\) a Borel, positive or bounded, function on \(\mathcal{F}_w\) and \(g\) a Borel, positive or bounded, function on the space \(\mathcal{F}_w\). A consequence of the previous identity is the following version of the branching property that we will use constantly throughout this work.

\[
\mathbb{E}\left[1_{\{x \in T\}} f(T \leq x) g(T[x]) \right] = \mathbb{E}\left[1_{\{x \in T\}} f(T \leq x)\right] \mathbb{E}[g(T)]. \tag{1.4}
\]

We associate to \(T\) a transition matrix \(P^T\) as in (1.2) and the associated random walk \(X\).

**Example 1.1.** Let \(\lambda > 0\). If the weights are all constant and equal to \(\lambda^{-1}\), the model is called the \(\lambda\)-biased random walk on a Galton-Watson tree. Given the Galton-Watson tree \(T\), the transition matrix \(P^T\) is defined by:

\[
P^T(x, y) = \begin{cases} 
\lambda/ (\lambda + \nu_T(x)) & \text{if } y = x; \\
1/(\lambda + \nu_T(x)) & \text{if } y = x_i, \text{ for } 1 \leq i \leq \nu_T(x); \\
0 & \text{otherwise.}
\end{cases}
\]

The associated random walk is almost surely transient if and only if \(\lambda < m\) (see [13]). The dimension drop for the harmonic measure is established in [15] for \(\lambda = 1\) (simple random walk) and in [16] for \(0 < \lambda < m\).

In [14] we have a transience criterion for the random walk \(X\) on \(T\) with (quenched) transition matrix \(P^T\), when the weights are i.i.d. It is generalized for our setting in [7, Theorem 1.1]. One can see that the integrability assumptions are not needed for the proof of the transient case.

**Fact 1.1** ([7, theorem 1.1]). If \(\min_{\alpha \in [0,1]} \mathbb{E}\left[\sum_{i=1}^{\nu_T(\phi)} A_T(i)^\alpha\right] > 1\), then for \(\text{GW}\)-almost every weighted tree \(t\), the random walk defined by \(P^t\) is transient.

We will assume throughout this work that we are in this regime.
1.7 Basic facts of ergodic theory

We recall here some definitions and basic properties that are used in this paper. The notations of this section are local to this section.

Definition 1.1. Let \((X, \mathcal{F}_X)\) and \((Y, \mathcal{F}_Y)\) two measurable spaces and \(S_X : X \to X\), \(S_Y : Y \to Y\) two measurable transformations. A semi-conjugacy between \((X, \mathcal{F}_X, S_X)\) and \((Y, \mathcal{F}_Y, S_Y)\) is a surjective measurable mapping \(h : X \to Y\) such that \(h \circ S_X = S_Y \circ h\).

One says that \(h\) is a conjugacy between \((X, \mathcal{F}_X, S_X)\) and \((Y, \mathcal{F}_Y, S_Y)\) if, in addition, the semi-conjugacy \(h\) is also injective.

The following well-known fact can be checked very directly, so we omit the proof.

Fact 1.2. Let \((X, \mathcal{F}_X, S_X)\) and \((Y, \mathcal{F}_Y, S_Y)\) two measurable spaces endowed with a measurable transformation. Let \(h : X \to Y\) a semi-conjugacy and \(\mu_X\) a probability measure on \(\mathcal{F}_X\). Then, if the system \((X, \mathcal{F}_X, S_X, \mu_X)\) is measure-preserving (resp. ergodic, mixing), so is \((Y, \mathcal{F}_Y, S_Y, \mu_Y \circ h^{-1})\).

Definition 1.2. Let \((X, \mathcal{F}, S, \mu)\) a measure-preserving system (with \(\mu(X) = 1\)) and \(A\) in \(\mathcal{F}\) such that \(\mu(A) > 0\). For \(x\) in \(X\), let

\[
n_A(x) = \inf \left\{ k \geq 1 : S^k(x) \in A \right\},
\]

with the convention \(\inf \emptyset := +\infty\). For \(B\) in \(\mathcal{F}\), let \(\mu_A(B) = \mu(A \cap B)/\mu(A)\) and for \(x\) in \(X\), let \(S_A(x) := S^{n_A(x)}(x)\) if \(n_A(x)\) is finite and (say) \(S_A(x) := x\) if \(n_A(x) = \infty\). The induced system on \(A\) is defined as \((A, \mathcal{F} \cap A, S_A, \mu_A)\).

Lemma 1.3. With the notations and assumptions of the previous definition, the system \((A, \mathcal{F} \cap A, S_A, \mu_A)\) is measure-preserving. Moreover, we have that the whole system \((X, \mathcal{F}, S, \mu)\) is ergodic if and only if \(\mu\left( \bigcup_{k \geq 1} S^{-k}(A) \right) = 1\) and \((A, \mathcal{F} \cap A, S_A, \mu_A)\) is ergodic.

We provide a short proof of the “if” part, since we did not find it in the litterature. For the other assertions, see for instance [5, Lemma 2.43].

Proof. For \(k\) in \(\mathbb{N}^* \cup \{\infty\}\), let \(A_k := \{x \in A : n_A(x) = k\}\). Assume that

\[
\mu\left( \bigcup_{k \geq 1} S^{-k}(A) \right) = 1
\]

and \((A, \mathcal{F} \cap A, S_A, \mu_A)\) is ergodic. Let \(C\) in \(\mathcal{F}\) such that \(S^{-1}(C) = C\). We prove that \(C \cap A\) is \(S_A\)-invariant. Indeed,

\[
S_A^{-1}(C \cap A) = S_A^{-1}(C \cap A) \cap A_\infty \cup \bigcup_{k \geq 1} S^{-k}(C \cap A) \cap A_k
\]

\[
= C \cap A_\infty \cup \bigcup_{k \geq 1} C \cap S^{-k}(A) \cap A_k
\]

\[
= C \cap A_\infty \cup \bigcup_{k \geq 1} C \cap A_k = C \cap A.
\]
Thus, \( \mu(C \cap A) \) equals 0 or \( \mu(A) \). If it is 0, then
\[
\mu(C) = \mu \left( C \cap \bigcup_{k \geq 1} S^{-k}(A) \right) \leq \sum_{k \geq 1} \mu \left( C \cap S^{-k}(A) \right) = 0,
\]
since, for any \( k \geq 1 \),
\[
\mu \left( C \cap S^{-k}(A) \right) = \mu \left( S^{-k}(C) \cap S^{-k}(A) \right) = \mu(C \cap A).
\]
If \( \mu(C \cap A) = \mu(A) \), we reason on the complement \( C^c \) of \( C \), which is still invariant by \( S \) and satisfies \( \mu(C^c \cap A) = 0 \).

## 2 Regeneration Times

Let \( \mathcal{T}_{w,p} \) be the space of all trees \( t \) in \( \mathcal{T}_w \) with a distinguished transient path \( x \) starting from the root. On \( \mathcal{T}_{w,p} \), we define the distance \( d_{w,p} \) by
\[
d_{w,p} \left( (t, x), (t', x') \right) = \sum_{r \geq 0} 2^{-r-1} \delta_{w,p}^{(r)} \left( ((t, x), (t', x')) \right),
\]
where \( \delta_{w,p}^{(r)} \left( ((t, x), (t', x')) \right) = 1 \) if the vertices of \( t \) and of \( t' \) do not agree up to height \( r \) or if the paths \( x \) and \( x' \) do not coincide before the first time they reach height \( r + 1 \). Otherwise, \( \delta_{w,p}^{(r)} \left( ((t, x), (t', x')) \right) = \delta_{w}^{(r)} (t, t') \). The metric space \( \mathcal{T}_{w,p} \) is again Polish.

Following [16, proof of Proposition 3.4], for any \( s \) in \( ft(x) \), we consider the tree and the path before time \( s \):
\[
\phi_s (t, x) := \left( t_{ \leq x_s}, (x_i)_{0 \leq i \leq s} \right).
\]
Likewise if \( s \) is in \( et(x) \), the reindexed tree and path after time \( s \) is
\[
\psi_s (t, x) = (t \left[ x_s \right], x \left[ s \right]),
\]
where
\[
x \left[ s \right] := \left( x_s^{-1} x_{s+k} \right)_{k \geq 0}.
\]
From the definition of fresh times and exit times, each path is in the corresponding tree or pruned tree.

Now, let \( T \) a random weighted tree of law \( GW \) and, conditionally on \( T \), let \( X \) a trajectory of the random walk with transition matrix \( P_T \), starting from \( \phi \). We denote \( P \) the “annealed” probability, that is, the probability associated to the expectation \( E \) defined by
\[
E \left[ f(T, X) \right] := E \left[ E^T_{\phi} [ f(T, X) ] \right],
\]
for all suitable measurable functions \( f \) on \( \mathcal{T}_{w,p} \).

For short, we write \( ft \) for \( ft(X) \), \( fp \) for \( fp(X) \), etc, and \( \psi_s \), \( \phi_s \) for \( \psi_s(T, X) \) and \( \phi_s(T, X) \).
Lemma 2.1. Let $s \in \mathbb{N}^*$, and $f$ and $g$ measurable and non-negative (or bounded) functions, respectively on $\mathcal{T}_{w,p}^\leq$ and $\mathcal{T}_{w,p}$. Then

$$E \left[ 1_{\{s \in rt\}} f(\phi_s) g(\psi_s) \right] = E \left[ 1_{\{s \in rt\}} f(\phi_s) \right] E \left[ g(T, X) 1_{\{\tau_x = \infty\}} \right].$$

Proof. We first decompose the expectation according to the value of $X_s$.

$$E \left[ 1_{\{s \in rt\}} f(\phi_s) g(\psi_s) \right] = \sum_{x \in \mathcal{U}} E \left[ 1_{\{x \in T\}} E^T_\circ \left[ 1_{\{X_s = x, s \in rt\}} f \left( T^{\leq x}, (X_i)_{0 \leq i \leq s} \right) 1_{\{s \in rt\}} g \left( T[x], (x^{-1}X_{s+k})_{k \geq 0} \right) \right] \right].$$

By the Markov property at time $s$, for any fixed $x$ in $T$, the quenched expectation can be rewritten

$$E^T_\circ \left[ 1_{\{X_s = x, s \in rt\}} f \left( T^{\leq x}, (X_i)_{0 \leq i \leq s} \right) 1_{\{s \in rt\}} g \left( T[x], (x^{-1}X_{s+k})_{k \geq 0} \right) \right] = E^T_\circ \left[ 1_{\{X_s = x, s \in rt\}} f \left( T^{\leq x}, (X_i)_{0 \leq i \leq s} \right) \right] E^T_x \left[ 1_{\{\tau_s = \infty\}} g \left( T[x], (x^{-1}X_{k})_{k \geq 0} \right) \right].$$

Now, the first quenched expectation is only a function of the weighted tree $T^{\leq x}$ while the second is only a function of $T[x]$. So we can use the branching property (1.4) and sum over $x$ in $\mathcal{U}$ to get the result. \hfill \Box

Definition 2.1. The conductance of a weighted tree $t$ is

$$\beta(t) := P^t_\circ (\tau_{o_\circ} = \infty).$$

From the theory of discrete-time Markov chains, one can check that the tree $t$ is transient if and only if $\beta(t) > 0$.

Lemma 2.2. For $GW$-almost every weighted tree $t$, for $P^t_\circ$-almost every path $x$, the set $rt(x)$ is infinite.

Proof. This proof is very similar to [16, Lemma 3.3]. For $k \geq 1$, let $\mathcal{F}_k$ the $\sigma$-algebra on $\mathcal{T}_{w,p}$ generated by $X_0, X_1, \ldots, X_k$ and $\mathcal{F}_\infty$ the $\sigma$-algebra generated by the whole path $X$. For $N \in \mathbb{N}$, let $ft(N)$ the first fresh time after (or at) time $N$. Then,

$$P \left[ \bigcup_{s \geq N} \{ s \in rt \} \mid \mathcal{F}_N \right] \geq P \left[ ft(N) \in rt \mid \mathcal{F}_N \right] = \sum_{s \geq N} E \left[ 1_{\{ft(N) = s\}} 1_{\{s \in rt\}} \mid \mathcal{F}_N \right].$$

Thus we can use Lemma 2.1 to get

$$P \left[ \bigcup_{s \geq N} \{ s \in rt \} \mid \mathcal{F}_N \right] \geq \sum_{s \geq N} E \left[ 1_{\{ft(N) = s\}} \mid \mathcal{F}_N \right] E \left[ 1_{\{\tau_s = \infty\}} \mid \mathcal{F}_N \right] = E \left[ \beta(T) \right] > 0,$$
By regular martingale convergence theorem and the fact that for any $N$ in $\mathbb{N}$, the event $\bigcup_{s \geq N} \{ s \in rt \}$ is in $\mathcal{F}_\infty$, we have almost surely,

\[
1 \bigcup_{s \geq N} \{ s \in rt \} = \lim_{k \to \infty} \mathbb{P} \left[ \bigcup_{s \geq N} \{ s \in rt \} \, \bigg| \, \mathcal{F}_{N+k} \right] \geq \mathbb{P} \left[ \bigcup_{s \geq N+k} \{ s \in rt \} \, \bigg| \, \mathcal{F}_{N+k} \right] \geq \mathbb{E} [\beta(T)] > 0.
\]

Hence, $1 \bigcup_{s \geq N} \{ s \in rt \} = 1$, almost surely. \qed

We will now work on the space of weighted trees with transient paths that have infinitely many regeneration times. We still denote it $\mathcal{T}_{w,p}$ in order not to add another notation.

**Proposition 2.3.** Let $f$ and $g$ measurable (or bounded) functions. For any $n \geq 1$,

\[
\mathbb{E} [f(\Phi_{rt_n}) g(\Psi_{rt_n})] = \mathbb{E} [f(\Phi_{rt_n})] \mathbb{E} [g(T, X) \mid \tau_{o_s} = \infty] = \mathbb{E} [f(\Phi_{rt_n})] \mathbb{E} [g(\Psi_{rt_n})].
\]  

(2.1)

**Proof.** Again, this proof is similar to [16, p. 255]. For $1 \leq n \leq s$, let $C^n_s$ the event that exactly $n$ edges have been crossed exactly one time before time $s$. Reasoning on the value of the $n$-th regeneration time, we first get

\[
\mathbb{E} [f(\Phi_{rt_n}) g(\Psi_{rt_n})] = \sum_{s \geq n} \mathbb{E} \left[ 1_{\{rt_n=s\}} f(\Phi_s) g(\Psi_s) \right] = \sum_{s \geq n} \mathbb{E} [1_{\{s \in rt\}} 1_{C_{n-1}^{s-1}} f(\Phi_s) g(\Psi_s)].
\]

On the event $\{s \in rt\}$, the indicator $1_{C_{n-1}^{s-1}}$ is a function of $\Phi_s$, thus, using Lemma 2.1, we obtain

\[
\mathbb{E} [f(\Phi_{rt_n}) g(\Psi_{rt_n})] = \mathbb{E} \left[ 1_{\{\tau_{o_s} = \infty\}} g(T, X) \sum_{s \geq n} \mathbb{E} \left[ 1_{\{s \in rt\}} 1_{C_{n-1}^{s-1}} f(\Phi_s) \right] \right] = \mathbb{E} [g(T, X) \mid \tau_{o_s} = \infty] \sum_{s \geq n} \mathbb{E} \left[ 1_{\{\tau_{o_s} = \infty\}} \right] \mathbb{E} \left[ 1_{\{s \in rt\}} 1_{C_{n-1}^{s-1}} f(\Phi_s) \right].
\]

Using Lemma 2.1 the other way around,

\[
\mathbb{E} [f(\Phi_{rt_n}) g(\Psi_{rt_n})] = \mathbb{E} [g(T, X) \mid \tau_{o_s} = \infty] \sum_{s \geq n} \mathbb{E} \left[ 1_{\{s \in et\}} 1_{\{s \in rt\}} 1_{C_{n-1}^{s-1}} f(\Phi_s) \right] = \mathbb{E} [g(T, X) \mid \tau_{o_s} = \infty] \mathbb{E} [f(\Phi_{rt_n})].
\]

Finally, taking $f$ constant equal to one yields the last equality. \qed

We define on $\mathcal{T}_{w,p}$ the shift at exit times

\[
S_e : (t, x) \mapsto \Psi_{et_1(x)} (t, x) = (t [ep_1], x [et_1]),
\]
and the shift at regeneration times
\[ S_r : (t, x) \mapsto \Psi_{rt_1(x)} (t, x) = (t [rp_1], x [rt_1]). \]

For \( k \geq 1 \), let
\[ S^k_e := S_e \circ \cdots \circ S_e. \]

Since regeneration times are exit times, for any \( (t, x) \) in \( \mathcal{T}_{w,p} \),
\[ S_r (t, x) = S^1_e (t, x). \]

**Corollary 2.4.** The law \( \mu_r \) of \( (T, X) \) on \( \mathcal{T}_{w,p} \), under the probability measure \( \mathbb{P}^* := \mathbb{P} [\cdot \mid \tau_\omega = \infty] \) is invariant and mixing with respect to the shift \( S_r \).

**Proof.** For the invariance, take \( f \) constant equal to one in (2.1).

Now let \( f \) and \( g \) non-negative measurable functions on \( \mathcal{T}_{w,p} \). By a monotone class argument, we may assume that \( g \) only depends on the \( N \) first generations of the weighted tree and on the path until it escapes these generations for the first time.

Since the \( N \)-th regeneration point is at least of height \( N \), we get, using (2.1), for all \( k \geq N \),
\[ E^* [f \circ S^k_e (T, X) g (T, X)] = E^* [f (T [rp_k], X [rt_k]) g (T, X)] = E^* [f (T, X)] E^* [g (T, X)], \]

thus the system is mixing.

**3 Tower Construction of an Invariant Measure for the Shift at Exit Times**

We now build a Rokhlin tower over the system \((\mathcal{T}_{w,p}, S_r, \mu_r)\) in order to obtain a probability measure which is invariant with respect to the shift \( S_e \). This is a classical and general construction but we provide details in our specific case for the reader’s convenience.

For any \( i \geq 1 \), let
\[ E_i := \{ (t, x) \in \mathcal{T}_{w,p} : rh_1 (x) \geq i \}. \]

We then have
\[ \mathcal{T}_{w,p} = E_1 \supset E_2 \supset \cdots. \]

For \( i \geq 1 \), let \( \tilde{E}_i := E_i \times \{ i \} \) and \( \tilde{E} := \bigsqcup_{i \geq 1} \tilde{E}_i \). Let \( \phi_i : E_i \to \tilde{E}_i \) the natural bijection.

We define the measure \( \tilde{\mu}_r^0 \) by : for any measurable \( A \) in \( \tilde{E} \),
\[ \tilde{\mu}_r^0 (A) := \sum_{i \geq 1} \mu_r (\phi_i^{-1} (A \cap \tilde{E}_i)). \]

The total mass of \( \tilde{\mu}_r^0 \) is
\[ \tilde{\mu}_r^0 (\tilde{E}) = \sum_{i \geq 1} \mathbb{P}^* (rh_1 \geq i) = E^* [rh_1]. \]
Lemma 3.1. The expectation $\mathbb{E}^* [\rho_1]$ is finite and equals $\mathbb{E} [\beta (T)]^{-1}$.

We will prove this lemma later. We now write $\tilde{\mu}_r := \tilde{\mu}_r^0 / \tilde{\mu}_r^0 \left( \tilde{E} \right)$. We define the shift $\tilde{S}$ on $\tilde{E}$ by:

$$\tilde{S} (t, x, i) := \begin{cases} (t, x, i + 1) & \text{if } \rho_1(x) \geq i + 1; \\ (S_r (t, x), 1) & \text{if } \rho_1(x) = i. \end{cases}$$ (3.1)

Lemma 3.2. The measure $\tilde{\mu}_r$ is invariant and ergodic with respect to the shift $\tilde{S}$.

Proof. Let $f : \tilde{E} \to \mathbb{R}_+$ a measurable function.

$$\int f \circ \tilde{S} (t, x, i) \, d\tilde{\mu}_r (t, x, i) = \sum_{j \geq 1} \int_{E_j} f \circ \tilde{S} (t, x, j) \, d\mu_r (t, x)$$

$$= \mathbb{E}^* [\rho_1]^{-1} \sum_{j \geq 1} \int_{E_j} f (S_r (t, x), 1) \, d\mu_r (t, x) + \int_{E_{j+1}} f (t, x, j + 1) \, d\mu_r (t, x)$$

$$= \mathbb{E}^* [\rho_1]^{-1} \left( \int_{E_1} f (S_r (t, x), 1) \, d\mu_r (t, x) + \int_{E \setminus E_1} f (t, x, i) \, d\tilde{\mu}_r (t, x, i) \right).$$

The fact that $S_r$ is invariant with respect to $\mu_r$ concludes the proof of the invariance.

For the ergodicity, we remark that, by construction,

$$\bigcup_{k=1}^{\infty} \tilde{S}^{-k} \left( E_1 \right) = \tilde{E}.$$

and the induction of the system on $\tilde{E}_1$ is canonically conjugated to $(\mathcal{S}_{w,p}, S_r, \mu_r)$, thus is ergodic and (see Subsection 1.7) so is the whole system. \( \square \)

Proof of Lemma 3.1. This proof can be found in [1, Subsection 3.1]. We reproduce it with our notations for the reader’s convenience. From Proposition 2.3, we know that under $\mathbb{P}^*$, we have $\rho_0 = 0$ and the increments $\rho_1, \rho_2 - \rho_1, \ldots, \rho_{k+1} - \rho_k, \ldots$ are i.i.d. For $n \geq 1$,

$$\mathbb{P}^* (n \in \rho) = \mathbb{E}^* [\# \rho \cap \{0, 1, \ldots, n\}] - \mathbb{E}^* [\# \rho \cap \{0, 1, \ldots, n - 1\}].$$

So, by the renewal theorem ([8, p 360]),

$$\mathbb{P}^* (n \in \rho) \xrightarrow{n \to \infty} 1 / \mathbb{E}^* [\rho_1].$$

On the other hand,

$$\mathbb{P}^* (n \in \rho) = \mathbb{E} [\beta (T)]^{-1} \sum_{|x| = n} \mathbb{E} \left[ 1_{\{x \in T\}} 1_{\{\tau_x < \infty\}} 1_{\{\tau_x > \tau_x\}} 1_{\{\forall k \geq \tau_x, X_k \neq x_x\}} \right]$$

$$= \sum_{|x| = n} \mathbb{E} \left[ 1_{\{x \in T\}} 1_{\{\tau_x < \infty\}} 1_{\{\tau_x > \tau_x\}} \right] = \mathbb{P} \left[ \tau_x > \tau^{(n)} \right],$$

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where \( \tau^{(n)} := \inf \{ k \geq 0 : |X_k| = n \} \). By dominated convergence,

\[
P_\tau \left[ \tau_0 > \tau^{(n)} \right] \xrightarrow{n \to \infty} P[\tau_0 = \infty] = E[\beta(T)].
\]

In order to construct a \( S_e \)-invariant measure on \( \mathcal{F}_{w,p} \), all we need now is the right semi-conjugacy. Let \( h_e : E \to \mathcal{F}_{w,p} \) defined by

\[
h_e(t, x, i) := (t [ep_{i-1}], x [et_{i-1}]).
\]

By construction,

\[h_e \circ S = S_e \circ h_e,\]

that is \( h_e \) is a semi-conjugacy on its image, so we get the desired ergodic system.

**Corollary 3.3.** The probability measure \( \mu_e := \bar{\mu} \circ h_e^{-1} \) on \( \mathcal{F}_{w,p} \) is invariant and ergodic with respect to the shift \( S_e \).

We now investigate further the law \( \mu_e \). Let \( f : \mathcal{F}_{w,p} \to \mathbb{R}_+ \) a measurable function. By definition,

\[
\int f(t, x) \, d\mu_e(t, x) = \int f(t [ep_{i-1}], x [et_{i-1}]) \, d\bar{\mu}_T(t, x, i)
\]

\[
= E[\beta(T)] \sum_{i \geq 1} \int 1_{\{rh_i \geq i\}} f(t [ep_{i-1}], x [et_{i-1}]) \, d\mu_T(t, x)
\]

\[
= E[\beta(T)] \sum_{j \geq 1} \sum_{i=0}^{j-1} E \left[ 1_{\{rh_1 = j, \tau_{rh_1} = \infty\}} f(T[ep_1], X[et_1]) \right].
\]

Reasoning on the value of \( rp_1 \) and its strict ancestors, we get for all \( j \geq 1 \),

\[
\sum_{i=0}^{j-1} E \left[ 1_{\{rh_1 = j, \tau_{rh_1} = \infty\}} f(T[ep_1], X[et_1]) \right]
\]

\[
= \sum_{x \in \mathcal{U}, x \succeq y \succ x} \sum_{s \geq 1} E \left[ 1_{\{x \in T, rp_1 = x, \tau_{rh_1} = \infty, ep_{|y|} = y, et_{|y|} = s\}} f(T[y], X[s]) \right].
\]

Summing over \( j \), we obtain

\[
\sum_{j \geq 1} \sum_{i=0}^{j-1} E \left[ 1_{\{rh_1 = j, \tau_{rh_1} = \infty\}} f(T[ep_1], X[et_1]) \right]
\]

\[
= \sum_{y \in \mathcal{T}} \sum_{s \geq 1} E \left[ 1_{\{y \in T, rp_1 = x, \tau_{bh_1} = \infty, ep_{|y|} = y, et_{|y|} = s\}} f(T[y], X[s]) \right] \sum_{x \succeq y} 1_{\{x \in T, rp_1 = x\}}
\]

\[
= \sum_{y \in \mathcal{T}} \sum_{s \geq 1} E \left[ 1_{\{y \in T, rp_1 = x, \tau_{rh_1} = \infty, et_{|y|} = s\}} f(T[y], X[s]) \right] 1_{\{rp_1 = y\}}.
\]

We want to use the Markov property at time \( s \). For \( s \geq 1 \), and \( y \in \mathcal{T} \), let \( D_s(y) \) the event that:
• the walk has not hit \( \emptyset \) before time \( s \);
• the walk hits \( y \) at time \( s \) and \( y_\ast \) at time \( s-1 \);
• for all \( \emptyset < z \leq y \), there exist \( 1 \leq i < j \leq s \) such that \( X_i = z \) and \( X_j = z_\ast \).

For fixed \( y \in \mathcal{U} \) and \( s \geq 1 \), on the event \( \{y \in T\} \), we denote \( X' \) a random walk in \( T \)
starting from \( y \) independant of \( X_0, X_1, \ldots, X_s \) and \( y^{-1}X' := (y^{-1}X'_0, y^{-1}X'_1, \ldots) \). We obtain

\[
1_{\{y \in T\}} E^T_\emptyset \left[ f \left( T[y], X[s] \right) 1_{\{\forall k \geq s, X_k \neq y_\ast \}} 1_{D_s(y)} \right] \\
= 1_{\{y \in T\}} E^T_y \left[ f \left( T[y], y^{-1}X' \right) 1_{\{\forall k \geq 0, X_k' \neq y_\ast \}} \right] P^T_\emptyset [D_s(y)].
\]

We remark that the law of a random walk \( Y \) in the weighted tree \( T[y] \), starting from \( \emptyset \), on the event \( \{\forall k \geq 0, Y_k \neq \emptyset \} \), thus

\[
1_{\{y \in T\}} E^T_y \left[ f \left( T[y], y^{-1}X' \right) 1_{\{\forall k \geq 0, X_k' \neq y_\ast \}} \right] P^T_\emptyset [D_s(y)] \\
= 1_{\{y \in T\}} E^T_\emptyset \left[ f \left( T[y], Y \right) 1_{\{\forall k \geq 0, Y_k \neq y_\ast \}} \right] P^T_\emptyset [D_s(y)] \\
= 1_{\{y \in T\}} E^T_\emptyset \left[ f \left( T[y], Y \right) | \tau_{y_\ast} (Y) = \infty \right] P^T_\emptyset \left[ \tau_{y_\ast} (Y) = \infty \right] P^T_\emptyset [\tau_1 \succ y, \tau_0 = \infty].
\]

Summing over \( s \), we obtain

\[
E \left[ \beta(T) \right]^{-1} \int f(t, x) d\mu_\emptyset (t, x) \\
= \sum_{y \in \mathcal{U}} E \left[ 1_{\{y \in T\}} E^T_\emptyset \left[ f \left( T[y], Y \right) | \tau_{y_\ast} (Y) = \infty \right] P^T_\emptyset \left[ \tau_1 \succ y, \tau_0 = \infty \right] \right].
\]

We may write

\[
P^T_\emptyset [\tau_1 \succ y, \tau_0 = \infty] = P^{T \leq y \cap T[y]}_\emptyset [\tau_1 \succ y, \tau_0 = \infty] =: h \left( T^{\leq y}, T[y] \right).
\]

By the branching property, for any \( y \) in \( \mathcal{U} \),

\[
E \left[ 1_{\{y \in T\}} E^T_\emptyset \left[ f \left( T[y], Y \right) | \tau_{y_\ast} (Y) = \infty \right] P^T_\emptyset [\tau_1 \succ y, \tau_0 = \infty] \right] \\
= E \left[ 1_{\{y \in T\}} E^T_\emptyset \left[ f \left( \tilde{T}, Y \right) | \tau_{y_\ast} (Y) = \infty \right] h \left( T^{\leq y}, \tilde{T} \right) \right],
\]

where \( \tilde{T} \) is a weighted tree whose law is \( GW \), independant of \( T^{\leq y} \) and \( 1_{\{y \in T\}} \). As a consequence, the conditional expectation of \( 1_{\{y \in T\}} h \left( T^{\leq y}, \tilde{T} \right) \) given \( \tilde{T} = t \) equals

\[
E \left[ 1_{\{y \in T\}} h \left( T^{\leq y}, \tilde{T} \right) \right] =: \kappa_y(t).
\]

Summing over \( y \in \mathcal{U} \), we finally obtain the following theorem which summarizes the results of this section.
Theorem 3.4. The system \((\mathcal{T}_{w,p}, S_e, \mu_e)\) is measure-preserving and ergodic. The probability measure \(\mu_e\) has the following expression: for all non-negative measurable functions \(f\),

\[
\int f(t,x) \, d\mu_e(t,x) = \frac{1}{E[\kappa(T)]} \, E \left[ f(T,X) \, \kappa(T) \beta(T)^{-1} 1_{\{\tau_{\rho_*} = \infty\}} \right]
\]

where, for all weighted trees \(t\),

\[
\kappa(t) := E \left[ \sum_{y \in t} P^{T \leq y \, \rho_1 \succ y} (\tau_{\rho_*} = \infty) \right] \quad \text{and} \quad \beta(t) := P^{\rho_1} (\tau_{\rho_*} = \infty).
\] (3.2)

4 Invariant Measure for the Harmonic Flow Rule

We now slightly change our point of view. We will forget everything about the random path \(X\), except the ray it defines. Let \(\mu_{HARM}\) the projection on \(\mathcal{T}_w\) of the probability measure \(\mu_e\), that is, the probability defined by:

\[
\int f(t) \, d\mu_{HARM}(t) = \frac{1}{E[\kappa(T)]} \, E [f(T)\kappa(T)],
\] (4.1)

for all non-negative measurable functions \(f\) on \(\mathcal{T}_w\). We denote \(\mathcal{T}_{w,r}\) the space of all weighted trees with a distinguished ray, that is:

\[
\mathcal{T}_{w,r} := \{(t,\xi) : t \in \mathcal{T}_w, \xi \in \partial t\}.
\]

We view it as a metric subspace of \(\mathcal{T}_{w,p}\).

We build a Borel probability measure \(\mu_{HARM} \bowtie HARM\) on \(\mathcal{T}_{w,r}\) by:

\[
\int_{\mathcal{T}_{w,r}} f(t,\xi) \, d(\mu_{HARM} \bowtie HARM)(t,\xi) = \int_{\mathcal{T}_w} \left( \int_{\partial t} f(t,\xi) \, d\mu_{HARM}(\xi) \right) \, d\mu_{HARM}(t),
\]

for all positive measurable functions \(f : \mathcal{T}_{w,r} \to \mathbb{R}_+\). The shift \(S\) on \(\mathcal{T}_{w,r}\) is

\[
S(t,\xi) = \left( t [\xi_1], \xi_1^{-1} \xi \right).
\]

To check that this new system is (canonically) semi-conjugated to the one of Theorem 3.4 we need the following lemma.

Lemma 4.1. Let \(t\) in \(\mathcal{T}_w\). Under the probability \(E^t_{\rho_1}\), ray \((X)\) is independent of the event \(\{\tau_{\rho_*} = \infty\}\).

Proof. Let \(x\) in \(t\). By the Markov property, first at time \(\tau_{\rho_*}\) and then at time 1, we have

\[
P^t_{\rho_1} (x \in \text{ray} (X), \tau_{\rho_*} < \infty) = P^t_{\rho_1} (x \in \text{ray} (X)) \, P^t_{\rho_1} (\tau_{\rho_*} < \infty).
\]

Since the cylinders \(\{\xi \in \partial t : x < \xi\}\), for \(x\) in \(t\), generate the Borel \(\sigma\)-algebra of \(\partial t\), we conclude by a monotone class argument. □
Proposition 4.2. The system \((\mathcal{T}_{w,r}, S, \mu_{\text{HARM}} \ltimes \text{HARM})\) is measure-preserving and ergodic. Furthermore, the probability measure \(\mu_{\text{HARM}}\) and \(GW\) are mutually absolutely continuous.

Proof. Let \(h_{p,r} : \mathcal{T}_{w,p} \to \mathcal{T}_{w,r}\) defined by \(h_{p,r}(t,x) = (t, \text{ray}(x))\). The mapping \(h_{p,r}\) is surjective and satisfies \(h_{p,r} \circ S = S \circ h_{p,r}\), so is a semi-conjugacy. By the previous lemma, the probability measure \(\mu_{\text{HARM}} \ltimes \text{HARM}\) equals \(\mu_e \circ h_{p,r}^{-1}\).

We already know that \(\mu_{\text{HARM}}\) is absolutely continuous with respect to \(GW\). We only need to show that, for \(GW\)-almost every tree \(t\), the density \(\kappa(t)\) is positive. This is the case, because

\[
\kappa(t) = E \left[ \sum_{y \in T} P_{\phi}^{T \leq y, \phi} (y > y, \tau_0 = \infty) \right] \\
\geq E \left[ P_{\phi}^{T \leq y, \phi} (y > \emptyset, \tau_0 = \infty) \right] = P_{\phi} (\tau_0 = \infty) = \beta(t),
\]

and \(GW\)-almost every tree \(t\) is transient, thus is such that \(\beta(t) > 0\).

We could also have used [15, Proposition 5.2] to prove the ergodicity and the absolute continuity of \(GW\) with respect to \(\mu_{\text{HARM}}\) since our measure \(\mu_{\text{HARM}}\) was already known to be absolutely continuous with respect to \(GW\). To conclude that there indeed is a dimension drop phenomenon, we proceed as in [15, Theorem 7.1] and compare our flow rule \(\text{HARM}\) to an other flow rule called \(\text{UNIF}\). Before that, let us collect two equations about the flow rule \(\text{HARM}\).

Lemma 4.3. Let \(t\) in \(\mathcal{T}_w\), and \(\Xi\) a random ray in \(\partial T\) whose law is \(\text{HARM}_{t}\). Then,

\[
\text{HARM}_{T}(i) = P_{\emptyset}^{T}(i \prec \Xi) = \frac{A_t(i) \beta(t|i)}{\sum_{j \neq i} A_t(j) \beta(t|j))},
\]

\[
\beta(t) = \frac{\sum_{j=1}^{n_{\emptyset}} A_t(j) \beta(t|j))}{1 + \sum_{j=1}^{n_{\emptyset}} A_t(j) \beta(t|j)}.
\]

Proof. We first apply the Markov property at time 1 to get

\[
P_{\emptyset}^{t}(i \prec \Xi) = \frac{1}{1 + \sum_{j=1}^{n_{\emptyset}} A_t(j)} \left[ A_t(i) \left( P_{\emptyset}^{t}(\tau_0 = \infty) + P_{i}^{t}(\tau_0 < \infty, i \prec \Xi) \right) \right] \\
+ \sum_{j \neq i} A_t(j) P_{j}^{t}(\tau_0 < \infty, i \prec \Xi) + 1 \times P_{\emptyset}^{t}(i \prec \Xi).
\]

Using again the Markov property at time \(\tau_0\) yields the first formula. The proof of the second formula is similar.

The uniform flow rule \(\text{UNIF}\) is defined as in [15]. Let \(W(T)\) the almost sure limit in \((0, \infty)\) of \(Z_n(T)/c_n\) as \(n\) goes to infinity ; \(Z_n(T)\) being the number of vertices of
height $n$ in $T$, and the sequence $(c_n)_{n \geq 1}$ being the (deterministic) Seneta-Heyde norming sequence of the reproduction law $p$ (see [17, chapter 5, Section 1]). The rule $\text{UNIF}_T$ is defined on the first generation by

$$\text{UNIF}_T(i) = \frac{W(T[i])}{\sum_{\ell \geq 1} W(T[\ell])}, \quad \forall 1 \leq i \leq \nu_T(\emptyset).$$

(4.4)

For other generations, we use the rule

$$\text{UNIF}_T(xy) = \text{UNIF}_T(x)\text{UNIF}_T(y).$$

From the fact that $\lim_{n \to \infty} c_n/n = m$, we deduce the recursive equation

$$W(T) = \frac{1}{m} \sum_{\ell \geq 1} W(T[\ell]).$$

(4.5)

Notice that the equations (4.4) and (4.5) are the counterparts for $\text{UNIF}_T$ of (4.2) and (4.3). It is known (see [15, Section 6]) that, under the additional assumption that $E[N \log N] < \infty$, we have that almost surely

$$\dim_H \text{UNIF}_T = \log m = \dim_H \partial T,$$

that is, $\text{UNIF}_T$ is maximal in some sense. However, we will not need this fact (except the last equality, which is true without additional assumptions, see [12, Proposition 6.4]).

**Lemma 4.4.** For $\text{GW}$-almost any weighted tree $t$, $\text{HARM}_t \neq \text{UNIF}_t$, unless $p_m = 1$ for some integer $m \geq 2$ and the weights are all deterministic and equal.

**Proof.** We prove it by contradiction. By [15, proposition 5.1], we may assume that almost surely, $\text{HARM}_T = \text{UNIF}_T$. Let $k \geq 2$ and $i$ and $j$ distinct integers in $[1, k]$. We reason on the event $\{\nu_T(\emptyset) = k\}$, assuming it has positive probability (we recall that, by assumption, $p(\nu_T(\emptyset) = 1) < 1$). Since $\text{HARM}_T(i) = \text{UNIF}_T(i)$ and $\text{HARM}_T(j) = \text{UNIF}_T(j)$, we have

$$\frac{A_T(i)\beta(T[i])}{W(T[i])} = \frac{\sum_{\ell=1}^k A_T[\ell] \beta(T[\ell])}{\sum_{\ell=1}^k W(T[\ell])} = \frac{A_T(j)\beta(T[j])}{W(T[j])}.$$

In particular,

$$A_T(i)\beta(T[i]) W(T[j]) = A_T(j)\beta(T[j]) W(T[i]).$$

(4.6)

We first take the conditional expectation with respect to the $\sigma$-algebra generated by $A_T(i)$, $A_T(j)$ and the tree $T[i]$ to get that $E[W(T)] < \infty$, so it is 1. Then, conditioning only with respect to $A_T(i)$ and $A_T(j)$, we get $A_T(i) = A_T(j)$. Let us denote $A_k := A_T(1) = \cdots = A_T(k)$. Simplifying in (4.6) and taking the conditional expectation with respect to the subtree $T[i]$ gives $\beta(T[i]) = \alpha W(T[i])$, for $\alpha = E[\beta(T)] \in (0, 1)$. Since the law of $T[i]$ is itself $\text{GW}$, we have, for $\text{GW}$-almost every tree $t$,

$$\beta(t) = \alpha W(t).$$
We reason again on the event \{\nu_T(s) = k\}. Using the recursive equations (4.3) and (4.5), we get
\[
\alpha W(T) = \beta(T) = \frac{A_k \alpha \sum_{j=1}^{k} W(T[j])}{1 + A_k \alpha \sum_{j=1}^{k} W(T[j])} = \frac{A_k \alpha m W(T)}{1 + A_k \alpha m W(T)}.
\]
We then obtain
\[
m A_k (\alpha W(T) - 1) = 1,
\]
which, by independence, can only happen if \(W\) and \(A_k\) are almost surely constant. This is possible only if the law \(p\) is degenerated and \(k = m\). In this case, we have \(A_k = \frac{1}{k^{(\alpha - 1)} - 1} > \frac{1}{k}\), that is, our random walk model reduces to transient \(\lambda\)-biased random walk on a regular tree, with deterministic \(\lambda < m\).

We now have all the ingredients to prove Theorem 0.1.

Proof of Theorem 0.1. This is very similar to [15, Theorem 7.1]. We detail the proof for completeness and because of the fact that we work on weighted trees (although, as we will see, this does not make any difference here). For \((t, \xi)\) in \(\mathcal{T}_{w,r}\), let
\[
f(t, \xi) := -\log (\text{HARM}_t(\xi_1)).
\]
For any \(k \geq 1\), we have
\[
f \circ S^k(t, \xi) = -\log \left(\text{HARM}_{t[\xi_k]}(\xi_k^{-1} \xi_{k+1})\right).
\]
On the other hand, using the flow-rule property of HARM, for any \(n \geq 1\),
\[
\text{HARM}_t(\xi_n) = \prod_{k=0}^{n-1} \text{HARM}_{t[\xi_k]}(\xi_k^{-1} \xi_{k+1}).
\]
Thus, by the ergodic theorem, and the fact that \(GW\) is absolutely continuous with respect to \(\mu_{\text{HARM}}\), we have, for \(GW\)-almost every weighted tree \(t\), for \(\text{HARM}_t\)-almost every \(\xi\),
\[
-\frac{1}{n} \log (\text{HARM}_t(\xi_n)) \xrightarrow{n \to \infty} \int \left( \int f(t, \xi) \, d\text{HARM}_t(\xi) \right) \, d\mu_{\text{HARM}}(t) = \frac{1}{E[\kappa(T)]} \mathbb{E} \left[ -\log (\text{HARM}_T(\Xi_1)) \kappa(T) \right].
\]
Now, assume that our model does not reduce to \(\lambda\)-biased random walk on an \(m\)-regular tree. By Shannon’s inequality (strict concavity of log and Jensen’s inequality), for \(GW\)-every \(t\),
\[
\sum_{i=1}^{\nu_T(s)} \text{HARM}_t(i)(-\log (\text{HARM}_t(i))) \leq \sum_{i=1}^{\nu_T(s)} \text{HARM}_t(i)(-\log (\text{UNIF}_t(i))), \quad (4.7)
\]

with equality if and only if, for all $1 \leq i \leq \nu(t)$, $H_{\text{HARM}}(i) = \text{UNIF}_T(i)$. By Lemma 4.4, this happens with $\mathbf{GW}$-probability strictly less than 1. Thus, integrating (4.7) with respect to $\mu_{\text{HARM}}$, which is equivalent to $\mathbf{GW}$, we get

$$\frac{1}{\mathbb{E}[\kappa(T)]} \mathbb{E} [- \log (H_{\text{HARM}}(\Xi_1)) \kappa(T)] < \frac{1}{\mathbb{E}[\kappa(T)]} \mathbb{E} [- \log (\text{UNIF}_T(\Xi_1)) \kappa(T)].$$

This last term is the integral with respect to $\mu_{\text{HARM}} \Join \text{HARM}$ of

$$- \log \left( \frac{W(t[\xi_1])}{\sum_{i=1}^{\nu(t)} W(t[i])} \right) = - \log \left( \frac{W(t[\xi_1])}{mW(t)} \right) = \log m + \log(W(t)) - \log(W(t[\xi_1])).$$

For $(t, \xi)$ in $\mathcal{T}_{w,p}$, let $g(t, \xi) = W(t)$. We want to prove that $g - g \circ S$ is integrable with integral 0. By the fact that our system is measure preserving and the ergodic theory (Lemma 6.2 in [15] (see also [17, Lemma 17.20]) all we need to show is that $g - g \circ S$ is bounded from below by an integrable function. This is indeed the case because

$$\log W(t) - \log W(t[\xi_1]) = \log \frac{\sum_{i=1}^{\nu(t)} W(t[i])}{mW(t[\xi_1])} \geq - \log m,$$

which concludes our final proof.

References

[1] Elie Aidékon. Transient random walks in random environment on a Galton-Watson tree. *Probab. Theory Related Fields*, 142(3-4):525–559, 2008.

[2] P. Andreoletti and X. Chen. Range and critical generations of a random walk on Galton-Watson trees. *ArXiv e-prints*, October 2015.

[3] P. Andreoletti and P. Debs. The number of generations entirely visited for recurrent random walks in a random environment. *J. Theoret. Probab.*, 27(2):518–538, 2014.

[4] Nicolas Curien and Jean-François Le Gall. The harmonic measure of balls in random trees. *Ann. Probab.*, 45(1):147–209, 2017.

[5] Manfred Einsiedler and Thomas Ward. *Ergodic theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[6] Kenneth J Falconer and KJ Falconer. *Techniques in fractal geometry*, volume 3. Wiley Chichester (W. Sx.), 1997.

[7] Gabriel Faraud. A central limit theorem for random walk in a random environment on marked Galton-Watson trees. *Electron. J. Probab.*, 16:no. 6, 174–215, 2011.

[8] William Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

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[9] Yueyun Hu, Zhan Shi, et al. The slow regime of randomly biased walks on trees. *The Annals of Probability*, 44(6):3893–3933, 2016.

[10] Shen Lin. The harmonic measure of balls in critical Galton-Watson trees with infinite variance offspring distribution. *Electron. J. Probab.*, 19:no. 98, 35, 2014.

[11] Quansheng Liu and Alain Rouault. On two measures defined on the boundary of a branching tree. In *Classical and Modern Branching Processes*, pages 187–201. Springer, 1997.

[12] Russell Lyons. Random walks and percolation on trees. *Ann. Probab.*, 18(3):931–958, 1990.

[13] Russell Lyons. Random walks, capacity and percolation on trees. *Ann. Probab.*, 20(4):2043–2088, 1992.

[14] Russell Lyons and Robin Pemantle. Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.*, 20(1):125–136, 1992.

[15] Russell Lyons, Robin Pemantle, and Yuval Peres. Ergodic theory on Galton-Watson trees: speed of random walk and dimension of harmonic measure. *Ergodic Theory Dynam. Systems*, 15(3):593–619, 1995.

[16] Russell Lyons, Robin Pemantle, and Yuval Peres. Biased random walks on Galton-Watson trees. *Probab. Theory Related Fields*, 106(2):249–264, 1996.

[17] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, New York, 2016.

[18] Pertti Mattila, Manuel Morán, and José-Manuel Rey. Dimension of a measure. *Studia Math*, 142(3):219–233, 2000.

[19] P. Rousselin. Invariant Measures, Hausdorff Dimension and Dimension Drop of some Harmonic Measures on Galton-Watson Trees. *ArXiv e-prints*, August 2017.