Formal solutions and the first-order theory of acylindrically hyperbolic groups

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**Abstract**
We generalise Merzlyakov’s theorem about the first-order theory of non-abelian free groups to all acylindrically hyperbolic groups. As a corollary, we deduce that if \(G\) is an acylindrically hyperbolic group and \(E(G)\) denotes the unique maximal finite normal subgroup of \(G\), then \(G\) and the HNN extension \(G \ast_{E(G)}\), which is simply the free product \(G \ast \mathbb{Z}\) when \(E(G)\) is trivial, have the same \(\forall \exists\)-theory. As a consequence, we prove the following conjecture, formulated by Casals-Ruiz, Garreta and de la Nuez González: acylindrically hyperbolic groups have trivial positive theory. In particular, one recovers a result proved by Bestvina, Bromberg and Fujiwara, stating that, with only the obvious exceptions, verbal subgroups of acylindrically hyperbolic groups have infinite width.

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INTRODUCTION

Given a group $G$, a natural model-theoretic question is whether or not $G$ and $G \rtimes \mathbb{Z}$ have the same first-order theory. This problem was first considered by Tarski in the case of free groups. Around 1945, he posed the following question: are all non-abelian free groups elementarily equivalent? A positive answer to this question was given by Sela in [36] (see also [24] by Kharlampovich and Myasnikov). Then, Sela generalised this result in two directions: first, he proved in [37] that every torsion-free non-elementary hyperbolic group $G$ is elementarily equivalent to $G \rtimes \mathbb{Z}$. A few years later, he established the same result in the case where $G$ is a non-trivial free product, different from the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ (see [38]). More precisely, he proved the following stronger result: $G$ is elementarily embedded into $G \rtimes \mathbb{Z}$.

All these groups (namely non-elementary hyperbolic groups and non-elementary free products) have in common the property of being acylindrically hyperbolic, meaning that they admit a non-elementary acylindrical action on a hyperbolic space (for details, we refer the reader to Subsection 2.3). The main result of this paper is a partial generalisation of the above-mentioned theorems of Sela to all acylindrically hyperbolic groups (see Theorem 1.1). This wide class of groups, introduced by Osin in [27] in order to unify several classes of negatively curved groups considered by different authors (in particular, see [13]), has been intensively studied in the past few years. Examples of acylindrically hyperbolic groups include, notably, all non-elementary (relatively) hyperbolic groups, all but finitely many mapping class groups of surfaces of finite type, $\text{Out}(F_n)$ for $n \geq 2$, most 3-manifold groups, all non-cyclic and directly indecomposable right-angled Artin groups, and more generally any group acting geometrically on a CAT(0) space and containing a rank-one isometry, many fundamental groups of graphs of groups, and many other groups.

Despite an intense activity around acylindrically hyperbolic groups in geometric group theory, very little is known about the first-order theory of these groups. Dahmani, Guirardel and Osin proved that acylindrically hyperbolic groups are not superstable (see [13, Theorem 8.1]). Recently, Groves and Hull adapted some of Sela’s techniques to the context of acylindrically hyperbolic groups and initiated the study of solutions of systems of equations over such groups (see [18]). Last, building on Groves’ and Hull’s version of Sela’s shortening argument (for further details, see Subsection 3.4), the second-named author of the present paper proved a generalisation of Merzlyakov’s celebrated theorem [25] for torsion-free acylindrically hyperbolic groups (see [16]). An important part of our paper is devoted to an extension of Merzlyakov’s theorem to all acylindrically hyperbolic groups, possibly with torsion; this involves techniques used in [2] by the first-named author in the setting of hyperbolic groups.

An $\forall \exists$-sentence is a first-order sentence of the form $\forall x \exists y \psi(x, y)$, where $x$ and $y$ are two tuples of variables, and $\psi$ is a quantifier-free formula in these variables. The set of such sentences satisfied by a group $G$ is called the $\forall \exists$-theory of $G$. We also say that the inclusion $i$ of a group $G$
into an overgroup $G'$ is an $\exists \forall \exists$-elementary embedding if the following condition is satisfied: for every first-order formula of the form

$$\phi(t) : \exists x \forall y \exists z \psi(x, y, z, t),$$

where $\psi(x, y, z, t)$ is a quantifier-free formula, and for every tuple $g$ of elements of $G$ of the same arity as $t$, if the statement $\phi(g)$ holds in $G$, then $\phi(i(g))$ holds in $G'$. Before stating our main result, recall that every acylindrically hyperbolic group $G$ admits a unique maximal finite normal subgroup, denoted by $E(G)$ (see [13, Theorem 2.24]). In what follows, $G \ast_{E(G)}$ denotes the HNN extension where the stable letter acts trivially, that is the group

$$G \ast_{E(G)} (\mathbb{Z} \times E(G)) = \langle G, t \mid [t, g] = 1, \forall g \in E(G) \rangle.$$

**Theorem 1.1.** Let $G$ be an acylindrically hyperbolic group. The canonical inclusion of $G$ into $G \ast_{E(G)}$ is an $\exists \forall \exists$-elementary embedding. In particular, $G$ and $G \ast_{E(G)}$ have the same $\forall \exists$-theory.

**Remark 1.2.** This result was proved by the first author in [2] under the stronger assumption that the group $G$ is hyperbolic (possibly with torsion).

**Remark 1.3.** Note that if the finite group $E(G)$ is trivial, the group $G \ast_{E(G)}$ is simply the free product $G \ast \mathbb{Z}$. If $E(G)$ is non-trivial, one easily sees that $G \ast \mathbb{Z}$ cannot have the same $\forall \exists$-theory as $G$, since the existence of a non-trivial normal finite subgroup is expressible by means of a $\forall \exists$-sentence.

**Remark 1.4.** As an immediate consequence of Theorem 1.1, one recovers a result of Hull and Osin stating that an acylindrically hyperbolic group $G$ with $E(G) = 1$ is mixed identity free (see [23]).

For now, it is an open question whether Theorem 1.1 remains true if one considers the whole first-order theories of $G$ and $G \ast_{E(G)}$ instead of the $\forall \exists$ or $\exists \forall \exists$-fragments of these theories. This question can be viewed as a broad generalisation of Tarski’s problem about elementary equivalence of non-abelian free groups.

**Question 1.5.** Let $G$ be an acylindrically hyperbolic group.

1. Are $G$ and $G \ast_{E(G)}$ elementarily equivalent?
2. Is $G$ elementarily embedded into $G \ast_{E(G)}$?

As mentioned before, Sela proved that the answer to both of these questions is ‘Yes’ under the stronger assumption that $G$ is a torsion-free non-elementary hyperbolic group or a non-trivial and non-dihedral free product. In all other cases, the answer is not known.

Moreover, let us note that we do not know of any example of a finitely generated group $G$ that is not acylindrically hyperbolic and that has the same first-order theory, or even the same $\forall \exists$-theory, as $G \ast \mathbb{Z}$. The question of the existence of such a group is closely related to that of the preservation of acylindrical hyperbolicity under elementary equivalence among finitely generated groups (see Section 10 for further comments). It is worth mentioning the following corollary of Theorem 1.1 (see Proposition 10.5).
**Corollary 1.6.** Let $G$ be an acylindrically hyperbolic group, and let $H$ be a group that admits a non-trivial splitting over a virtually abelian group. Suppose that $G$ and $H$ are elementarily equivalent (or simply that they have the same $\exists \forall \exists$-theory). Then the group $H$ is acylindrically hyperbolic.

**Remark 1.7.** As a consequence, if there exists a group $G$ that is not acylindrically hyperbolic and such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent, then all non-trivial splittings of $G$ (if they exist) have sufficiently complicated edge groups. For instance, if $G$ is a generalised Baumslag–Solitar group, then $G$ and $G \ast \mathbb{Z}$ are not elementarily equivalent.

**Positive theory, verbal subgroups**

A first-order sentence is called **positive** if it does not involve inequalities. We say that a group $G$ has **trivial positive theory** if every positive sentence satisfied by $G$ is satisfied by all groups. In [25], Merzlyakov proved that non-abelian free groups have trivial positive theory. As a consequence, $G$ has trivial positive theory if and only if it has the same positive theory as $F_n$, for any $n \geq 2$. Recently, in [8] and [9], Casals-Ruiz, Garreta, Kazachkov and de la Nuez González proved that many groups acting non-trivially on trees have trivial positive theory. In particular, they showed that every acylindrically hyperbolic group that acts hyperbolically and irreducibly on a tree has trivial positive theory (see [8, Corollary 8.2]). They also established the following quantifier elimination result (see [8, Theorem 6.3]): a group has trivial positive theory if and only if it has trivial $\forall \exists$-theory. Using this fact and relying on Theorem 1.1, we prove the following result (see Section 9), which was conjectured in [8, Conjecture 9.1].

**Corollary 1.8.** Acylindrically hyperbolic groups have trivial positive theory.

Let $G$ be a group, and let $w$ be an element of the free group $F(x_1, \ldots, x_k)$. This element $w$ induces a map $g \in G^k \mapsto w(g) \in G$, also denoted by $w$. We say that the **verbal subgroup** $w(G) = \langle \{w(g), \ g \in G^k\} \rangle$ has **finite width** if there exists an integer $m \in \mathbb{N}$ such that any $g \in w(G)$ can be represented as a product of at most $m$ values of $w$ and their inverses, and the smallest such integer $m$ is called the **width** of $w(G)$. For instance, for $n \geq 3$, there exists a constant $C(n)$ such that every element of $SL_n(\mathbb{Z})$ is a product of $C(n)$ commutators (see [5]); in other words, the derived subgroup of $SL_n(\mathbb{Z})$ (that is $SL_n(\mathbb{Z})$ itself) has finite width. Otherwise, one says that $w(G)$ has **infinite width**.

Let $e_i$ be the sum of the exponent of $x_i$ in $w$, for every $1 \leq i \leq k$. If $e_1, \ldots, e_k$ are all equal to 0, define $d(w) = 0$. Otherwise, let $d(w)$ be their greatest common divisor. The following holds (see Section 9, Lemma 9.1): if $G$ has trivial positive theory, then $w(G)$ has infinite width, except if $w$ is trivial or $d(w) = 1$ (in which cases the width is equal to 1). As a consequence of Corollary 1.8, one recovers the main result of [6], due to Bestvina, Bromberg and Fujiwara.

**Corollary 1.9.** Let $G$ be an acylindrically hyperbolic, let $k \geq 1$ be an integer and let $w$ be a non-trivial element of $F_k$. If $d(w) \neq 1$, then $w(G)$ has infinite width.

It is worth noting that we do not know any group $G$ with non-trivial positive theory and such that $w(G)$ has infinite width for every non-trivial $w$ satisfying $d(w) \neq 1$ (see [8, Section 9.8] for further discussion).
**Merzlyakov’s theorem**

In Section 7, we deduce Theorem 1.1 from a generalisation of Merzlyakov’s theorem [25]. Assuming that a non-abelian free group $F$ satisfies the positive first-order sentence

$$\forall x \exists y \Sigma(x, y) = 1,$$

where $\Sigma(x, y) = 1$ denotes a finite system of equations, Merzlyakov’s theorem asserts that there exists a retraction from $\langle x, y \mid \Sigma(x, y) = 1 \rangle$ onto the free group $F(x)$ on $x$. Upon closer inspection, this result resembles the classical implicit function theorem in the sense that it enables one to convert the relations between the tuples $x$ and $y$ into a function. This is why Merzlyakov’s theorem is sometimes referred to as an implicit function theorem for groups. This fundamental result was one of the first steps in Sela’s positive answer to Tarski’s question about the elementary equivalence of non-abelian free groups.

Let us mention that previous generalisations of Merzlyakov’s theorem were proved for torsion-free hyperbolic groups, for hyperbolic groups with torsion and for $\pi$-groups (that is pairs of the form $(F, \pi)$ where $\pi : F \to G$ is a homomorphism), respectively, by Sela (see [37]), by Heil (see [22]) and by de la Nuez González (see [14]).

Given a group $G$ and an element $g \in G$, we denote by $\text{ad}(g)$ the inner automorphism $x \in G \mapsto gxg^{-1}$. Before stating our generalisation of Merzlyakov’s theorem to all acylindrically hyperbolic groups, let us introduce the following definition.

**Definition 1.10.** Let $G$ be a group, and let $H$ be a subgroup of $G$. We define the subgroup $\text{Aut}_G(H)$ of $\text{Aut}(H)$ as follows:

$$\text{Aut}_G(H) = \{ \sigma \in \text{Aut}(H) \mid \exists g \in G, \ \text{ad}(g)|_H = \sigma \}. $$

We prove the following version of Merzlyakov’s theorem (in Section 5, we give a more general statement allowing us to deal with finite disjunctions of finite systems of equations and inequalities). In the case where $G$ is torsion-free and the first-order sentence considered in the theorem is positive, this result was proved by the second author (see [16]).

**Theorem 1.11.** Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$ (called constants). Fix a presentation $\langle a \mid R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\Sigma(x, y, a) = 1 \land \Psi(x, y, a) \neq 1$$

be a finite system of equations and inequalities over $G$, where $x$ and $y$ are two tuples of variables. Let $G_\Sigma$ denote the following finitely generated group, finitely presented relative to $\langle a \mid R(a) = 1 \rangle$:

$$\langle x, y, a \mid R(a) = 1, \Sigma(x, y, a) = 1 \rangle.$$

Let $p = |x|$ be the arity of $x$, and let $x_i$ denote the $i$th component of $x$. Suppose that $G$ satisfies the following first-order sentence:

$$\forall x \exists y \Sigma(x, y, a) = 1 \land \Psi(x, y, a) \neq 1.$$
Then, for every $p$-tuple $\sigma = (\sigma_1, \ldots, \sigma_p) \in \text{Aut}_G(E(G))^p$, there exists a morphism
\[ \pi_\sigma : G_\Sigma \to G_\sigma = G \ast_{E(G)} \langle x, E(G) \mid \text{ad}(x_i)|_{E(G)} = \sigma_i, \forall i \in [1, p] \rangle, \]
called a formal solution, enjoying the following properties:

- $\pi_\sigma(x) = x$.
- $\pi_\sigma(a) = a$.
- $\Psi(x, \pi_\sigma(y), a) \neq 1$.

Moreover, the image of $\pi_\sigma$ is a subgroup of $G_\sigma$ of the form
\[ \langle g, a \rangle \ast_{E(G)} \langle x, E(G) \mid \text{ad}(x_i)|_{E(G)} = \sigma_i, \forall i \in [1, p] \rangle \]
for some tuple $g$ of elements of $G$.

Remark 1.12. Note that $G_\sigma$ is isomorphic to the group $G \ast_{E(G)} (F_p \times E(G))$ obtained from $G$ by adding $p$ stable letters commuting with $E(G)$. Indeed, by definition of $\text{Aut}_G(E(G))$, for every $1 \leq i \leq p$, there exists an element $g_i \in G$ such that $\text{ad}(x_i)|_{E(G)} = \text{ad}(g_i)|_{E(G)}$. It follows that $t_i = x_i g_i^{-1}$ commutes with $E(G)$.

Remark 1.13. This theorem captures the spirit of Merzlyakov’s original theorem, in the following sense: let $g = (g_1, \ldots, g_p)$ be a tuple of elements of $G$, of the same arity as $x$. Let $\sigma = (\text{ad}(g_1)|_{E(G)}, \ldots, \text{ad}(g_p)|_{E(G)})$, and let $\varphi : G_\sigma \to G$ be the retraction that maps $x_i$ to $g_i$ and coincides with the identity on $G$. The homomorphism $\varphi \circ \pi_\sigma$ from $G_\Sigma$ to $G$ maps $x$ to $g$. Denote by $h$ the image of $y$ under this homomorphism. The equalities $\Sigma(g, h, a) = 1$ hold in $G$. In other words, just as with Merzlyakov’s original theorem, the theorem above gives a mechanism for associating to every tuple $g \in G^p$ another tuple $h$ of the same arity as $y$ such that the equalities $\Sigma(g, h, a) = 1$ hold in $G$. However, note that the image of $\Psi(x, \pi_\sigma(y), a)$ by $\varphi$ may be trivial.

Example 1.14. Let $G = \langle x, y, a \mid a^3 = 1, [x, a] = [y, a] \rangle \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}_2$. Let $\sigma$ be the automorphism of $\langle a \rangle$ that maps $a$ to $a^2$, and let us consider the following first-order sentence, which is clearly satisfied by $G$: $\forall x \exists y \ ([x, a] = [y, a]) \wedge (x \neq y)$. By definition, one has:

- $G_\Sigma = \langle x, y, a \mid a^3 = 1, [x, a] = [y, a] \rangle$,
- $G_{id} = G \ast_{\langle a \rangle} \langle x, a \mid xax^{-1} = a \rangle$,
- $G_\sigma = G \ast_{\langle a \rangle} \langle x, a \mid xax^{-1} = a^2 \rangle$.

The morphism $\pi_{id}$ can be defined by $\pi_{id}(x) = x, \pi_{id}(a) = a$, and $\pi_{id}(y) = g_1$. The morphism $\pi_\sigma$ can be defined by $\pi_\sigma(x) = x, \pi_\sigma(a) = a$, and $\pi_\sigma(y) = g_2$. Note that $\pi_{id}(G_\Sigma)$ and $\pi_\sigma(G_\Sigma)$ are both isomorphic to $G$. 
The structure of the proof of Theorem 1.11 given in this paper, which is quite different from Merzlyakov’s original combinatorial proof, is inspired from Sela’s geometric proof of Merzlyakov’s theorem (we refer to [35]). Nevertheless, both proofs rely crucially on small cancellation theory (combinatorial in one case, geometric in the other case). We also took inspiration from Sacerdote’s paper [32].

An outline of the proof of Theorem 1.1

To illustrate the main ideas and to highlight the difficulties encountered, we sketch a proof of Theorem 1.1 in the particular case where the maximal normal finite subgroup $E(G)$ is trivial.

Suppose that $G$ satisfies a first-order sentence

$$\theta : \forall x \exists y \Sigma(x, y) = 1 \land \Psi(x, y) \neq 1.$$ 

Let $\Gamma = G \ast \langle t \rangle \simeq G \ast \mathbb{Z}$. Observe that the following two assertions are equivalent, where $p$ denotes the arity of $x$.

- $\Gamma$ satisfies the sentence $\theta$.
- For every $\gamma \in \Gamma^p$, there exists a retraction $r$ from $\Gamma_{\Sigma, \gamma} = \langle \Gamma, y | \Sigma(\gamma, y) = 1 \rangle$ onto $\Gamma$ such that no component of $\Psi(\gamma, y)$ is killed by $r$, that is, the inequations remain valid in the image of $r$.

To prove that $\Gamma$ satisfies the sentence $\theta$, we will construct such a retraction $r : \Gamma_{\Sigma, \gamma} \rightarrow \Gamma$, for any $\gamma \in \Gamma^p$. The very first step of the construction of this retraction relies on the existence of a quasi-convex free subgroup $F(a, b) \subset G$ (see [13, Theorem 6.14] combined with [4, Lemma 3.1]). From a sequence of elements $(w_n(a, b))_{n \in \mathbb{N}} \in F(a, b)^\mathbb{N}$ satisfying certain small cancellation conditions in the free group $F(a, b)$, one defines a test sequence $(\varphi_n : \Gamma \rightarrow G)_{n \in \mathbb{N}}$ by $\varphi_n | G = \text{id}_G$ and $\varphi_n(t) = w_n(a, b)$. Since, by assumption, the sentence $\theta$ is true in the group $G$, each morphism $\varphi_n$ extends to a morphism $\bar{\varphi}_n : \Gamma_{\Sigma, \gamma} \rightarrow G$ mapping $y$ to a tuple $g_n$ such that $\Sigma(\bar{\varphi}_n(\gamma), g_n) = 1$ and $\Psi(\bar{\varphi}_n(\gamma), g_n) \neq 1$.

The fact that $F(a, b)$ is quasi-isometrically embedded into $G$ enables us to prove that the sequence of elements $(\psi_n(t) = w_n(a, b))_{n \in \mathbb{N}}$ satisfies nice geometric conditions in $G$, which, in some sense, encapsulate the first-order sentence $\exists y \Sigma(\gamma, y) = 1 \land \Psi(\gamma, y) \neq 1$.

Then, using the non-elementary acylindrical action of $G$ on a hyperbolic space, one can show that the sequence $(\psi_n)_{n \in \mathbb{N}}$ converges to an action of $\Gamma_{\Sigma, \gamma}$ on a limiting real tree in the Gromov–Hausdorff topology, via the well-known Bestvina–Paulin method. This tree $T$ comes equipped with an isometric action of a quotient $L$ of $\Gamma$, called a divergent limit group. The action of $L$ on this tree can be analysed using the Rips machine, adapted by Groves and Hull in [18] to the setting of acylindrically hyperbolic groups, which converts the action $L \acts T$ into an action of $L$ on a simplicial tree, that is, a splitting of $L$. The properties of the test sequence $(\psi_n)_{n \in \mathbb{N}}$ are reflected in this splitting of $L$, and the rest of the proof consists in constructing a retraction from $L$ onto $\Gamma$, using this splitting.

One key ingredient in this construction is a generalisation of Sela’s shortening argument. Due to the lack of equationality Noetherianity of acylindrically hyperbolic groups (see below), the sequence $(\psi_n : \Gamma_{\Sigma, \gamma} \rightarrow G)_{n \in \mathbb{N}}$ does not factor through the quotient epimorphism $\psi_\infty : \Gamma_{\Sigma, \gamma} \rightarrow L$ in general, which is the source of difficulties. Note that our version of the shortening argument is slightly
different from the one proved by Groves and Hull, see [18, Theorem 5.29 and Lemma 6.5] and Section 3 (Remark 3.30) for further details.

Recall that a group is said to be equationally Noetherian if the set of solutions of any system of equations in finitely many variables coincides with the set of solutions of a certain finite subsystem of this system. As a consequence of the Hilbert Basis Theorem, linear hyperbolic groups are equationally Noetherian, and it was proved by Sela in [37] (torsion-free case) and by Reinfeldt and Weidmann in [29] (general case) that the linearity assumption can be dropped. Equational Noetherianity has proved extremely useful in the study of the first-order theory of hyperbolic groups, notably because limit groups over hyperbolic groups are not finitely presentable in general, which constrains us to deal with infinite systems of relations. Unfortunately, since equational Noetherianity is inherited by subgroups, and since, for instance, $H \ast \mathbb{Z}$ is acylindrically hyperbolic for any non-trivial group $H$, acylindrically hyperbolic groups are typically not equationally Noetherian. This is a major obstacle to constructing the desired retraction from $\Gamma_{\Sigma, \gamma}$ onto $\Gamma$.

We overcome this problem by introducing a method of approximating, in a precise sense, limit groups over acylindrically hyperbolic groups by finitely presented groups relative to a subgroup. The idea of approximating limit groups by finitely presented groups already appears, in a slightly different form, in [34, Theorem 3.2] (see also [17; 18, Lemma 6.3; 29, Lemma 6.1]). More precisely, in the present case, there exists a quotient $A$ of $\Gamma_{\Sigma, \gamma}$, called an approximation of $L$, which is finitely presented relative to $G$, maps onto $L$, and has a splitting that mimics the splitting of $L$ outputted by the Rips machine. Since $A$ is finitely presented relative to $G$, each morphism $\psi_n : \Gamma_{\Sigma, \gamma} \to G$ factors through the quotient epimorphism $\Gamma_{\Sigma, \gamma} \to A$, for $n$ sufficiently large, as shown in the commutative diagram below.

We prove that the shortening argument applies to the resulting sequence $(\rho_n : A \to G)_{n \in \mathbb{N}}$, together with the splitting of $A$ mimicking the splitting of $L$. We refer the reader to Section 3 for further details. Note, however, that the sequence $(\rho_n : A \to G)_{n \in \mathbb{N}}$ is not discriminating as soon as $L$ is a strict quotient of $A$; in other words, the stable kernel of this sequence is not trivial, and $A$ is not a $G$-limit group a priori, which leads to new technical difficulties.

A further bad consequence of the lack of equational Noetherianity is that there is no descending chain condition for limit groups over acylindrically hyperbolic groups in general. This is another obstacle to the construction of the retraction. Fortunately, it is proved in [18, Convention 4.6 and Lemma 4.7] that the size of the finite edge groups appearing in the splitting of $L$ outputted by the Rips machine is bounded from above by a constant that depends only on the acylindrical action of $G$ on a fixed hyperbolic space, and on the hyperbolicity constant of this space. This result is remarkable since the order of a finite subgroup of $G$ is not bounded in general, and allows us to appeal to accessibility results and to prove that our construction, which is iterative, eventually terminates.
2 | PRELIMINARIES

2.1 | Conventions

For a group $G$ generated by a (not necessarily finite) set $S$, the word length $|g|_S$ of an element $g \in G$ is the length of the shortest word in $S \cup S^{-1}$ representing $g$ in $G$. We usually denote the Cayley graph of $G$ (with respect to $S$) by $X$ and regard $X$ as a metric space by setting $d(g, h) = |g^{-1}h|_S$. Throughout this paper, all groups acting on metric spaces act by isometries, and all metric spaces are geodesic.

2.2 | Equations over groups

An equation in variables $x = (x_1, \ldots, x_p)$ is an equality $w(x) = 1$ for $w(x) \in F(x)$ (where $F(x)$ is the free group on $x$); an equation over a group $G$ in variables $x$ is an equality of the form $w(x, a) = 1$ where $w(x, a) \in F(x) \ast G$ and $a$ is a tuple of elements from $G$. A solution to the equation $w(x, a) = 1$ over a group $G$ consists of a tuple $g \in G^p$ for which the element $w(g, a)$, obtained by replacing every occurrence of $x_i^{\pm 1}$ with $g_i^{\pm 1}$, is trivial. Given a subset $\Sigma(x, a) = \{w_i(x, a)\}_{i \in I} \subset F(x) \ast G$, we refer to the conjunction $\wedge_{i \in I} w_i(x, a) = 1$ as a system of equations. We abbreviate and write $\Sigma(x, a) = 1$, and say that a tuple $g \in G^p$ is a solution to $\Sigma(x, a) = 1$ if for every $w_i(x, a) \in \Sigma(x, a)$, one has $w_i(g, a) = 1$.

Similarly, an inequation in variables $x = (x_1, \ldots, x_p)$ is an inequality $w(x) \neq 1$ for $w(x) \in F(x)$ (and an inequation over a group $G$ is an inequality $w(x, a) \neq 1$ where $w(x, a) \in F(x) \ast G$ and $a$ is a tuple of elements from $G$). Just like systems of equations, systems of inequations are conjunctions of inequations; we say that a tuple $g \in G^p$ satisfies the system of inequations $\Phi(x, a) \neq 1$ in $G$ if for every $w_i(x, a) \in \Phi(x, a)$, $w_i(g, a) \neq 1$ holds.

Note that there is a one-to-one correspondence between the set of solutions to the system of equations $\Sigma(x, a) = 1$ over a group $G$ and the set of homomorphisms

$$\varphi : G_\Sigma = \langle x, a \mid R(a) \cup \Sigma(x, a) \rangle \to G$$

(where $R(a)$ is a set of relations for which $\langle a \mid R(a) \rangle$ is a presentation of the subgroup of $G$ generated by $a$). If $g$ is a solution to $\Sigma(x, a) = 1$, there exists a homomorphism $\varphi : G_\Sigma \to G$ mapping $x$ to $g$ and $a$ to $a$; on the other hand, given such a homomorphism $\varphi$, the tuple $\varphi(x) \in G^p$ is a solution to $\Sigma(x, a) = 1$ over $G$. In addition, a solution $g$ to the system of equations $\Sigma(x, a) = 1$ satisfies the system of inequations $\Phi(x, a) \neq 1$ if and only if there exists a homomorphism $\varphi : G_\Sigma \to G$ which maps $x$ to $g$ and $a$ to $a$, and such that for every $w_i(x, a) \in \Phi(x, a)$, $\varphi(w_i(x, a)) \neq 1$. Thus, we regard the study of equations (and their solutions) over $G$, as the study of homomorphisms from the group $G_\Sigma$ to $G$.

2.3 | Acylindrically hyperbolic groups

The aim of this subsection is to familiarise the reader, in a rather shallow manner, with acylindrically hyperbolic groups.
Definition 2.1. A geodesic metric space \((X, d)\) is called \(\delta\)-hyperbolic if every geodesic triangle \(\Delta = (x, y, z)\) in \(X\) is \(\delta\)-slim: every side of \(\Delta\) is contained in the closed \(\delta\)-neighbourhood of the union of the two other edges. The space \((X, d)\) is called hyperbolic if it is \(\delta\)-hyperbolic for some \(\delta\).

Recall that if a group \(G\) acts on a hyperbolic space \((X, d)\) by isometries, an isometry \(g \in G\) is called elliptic if some (equivalently, any) orbit of \(g\) is bounded. An isometry \(g \in G\) is called hyperbolic if for some (equivalently, any) \(x \in X\), the map \(Z \to X\) defined via \(m \mapsto g^m x\) is a quasi-isometric embedding; we call the image of such a quasi-isometric embedding a quasi-geodesic axis of \(g\) (or in other words, a quasi-geodesic axis of \(g\) is an orbit of \(g\) in \(X\)). The Gromov boundary of \(X\), denoted by \(\partial X\), is defined as the collection of equivalence classes of quasi-isometric embeddings \(\mathbb{N} \to X\) (where two embeddings are equivalent if their images lie at bounded Hausdorff distance from one another). A hyperbolic element \(g \in G\) has therefore exactly two limit points \(g^+\infty\) and \(g^-\infty\) on \(\partial X\), represented by the quasi-isometric embeddings \(n \mapsto g^n x\) and \(n \mapsto g^{-n} x\) (for some \(x \in X\), respectively). Two hyperbolicelements \(g\) and \(h\) are called independent if \(\{g^{\pm\infty}\} \cap \{h^{\pm\infty}\} = \emptyset\). We call the action of \(G\) on \(X\) non-elementary if there are two (or equivalently, infinitely many) independent hyperbolic elements in \(G\).

The notion of an acylindrical group action on a metric space was first introduced by Bowditch in [7], and was inspired by Sela’s notion of a \(k\)-acylindrical group action on a tree: a group action on a tree is called \(k\)-acylindrical if it contains no arcs of length greater than \(k\) which are fixed by a non-trivial element of the group (and hence, the tree contains no ‘cylinders’). This notion was later generalised by imposing a bound on the cardinality of a subgroup which fixes an arc of length greater than \(k\) in the tree, and coarsified in the following manner.

Definition 2.2. A group action on a metric space \(G \acts (X, d)\) is called acylindrical if for every \(\varepsilon \geq 0\) there exist \(N > 0\) and \(R > 0\) such that for every \(x, y \in X\) satisfying \(d(x, y) \geq R\),

\[|\{g \in G \mid d(x, gx) \leq \varepsilon \text{ and } d(y, gy) \leq \varepsilon\}| \leq N.\]

The following lemma which appears in [13] will play an important role throughout this paper. Note that in [13], this result is stated for hyperbolic (also called loxodromic) elements which satisfy the Bestvina-Fujiwara weak proper discontinuity condition (WPD in short; an element \(g\) of \(G\) satisfies the WPD condition if the action of \(G\) in a direction of a quasi-axis of \(g\) is acylindrical). In fact, it turns out that all hyperbolic elements are automatically WPD when the action of the group is acylindrical and the lemma below holds for all hyperbolic elements in \(G\).

Lemma 2.3 [13, Lemma 6.5 and Corollary 6.6]. Let \(G\) be a group acting acylindrically on a hyperbolic space \(X\) and let \(g \in G\) be a hyperbolic element. Then \(g\) is contained in the unique maximal and virtually cyclic subgroup \(\Lambda(g)\) which consists of all \(h \in H\) for which the Hausdorff distance between \(\ell'\) and \(h\ell'\) is finite, where \(\ell'\) is some quasi-geodesic axis of \(g\) in \(X\). In addition, the following are equivalent for any \(h \in G\):

1. \(h \in \Lambda(g)\);
2. \(h^{-1} g^m h = g^k\) for some \(0 \neq m, k \in \mathbb{Z}\);
3. \(h^{-1} g^n h = g^{\pm n}\) for some \(n \in \mathbb{N}^*\).

In addition, there exists \(r \in \mathbb{N}\) such that the centraliser of \(g^r\) is given by

\[C_G(g^r) = \{h \in G \mid \exists n \in \mathbb{N}, \ h^{-1} g^n h = g^n\} \subset \Lambda(g).\]
Suppose now that a group \( G \) acts acylindrically on a hyperbolic space \((X, d)\). The action of \( G \) on \( X \) falls into exactly one of three categories [27, Theorem 1.1].

1. The action of \( G \) is elliptic, that is every \( G \)-orbit is bounded.
2. \( G \) is virtually cyclic and contains a hyperbolic element.
3. \( G \) contains two (equivalently, infinitely many) pairwise independent hyperbolic elements.

If an action falls into category (1) or (2) above, it is termed elementary.

**Definition 2.4.** A group \( G \) is said to be **acylindrically hyperbolic** if it admits a non-elementary and acylindrical action on a hyperbolic space.

As a matter of fact, we can always choose the hyperbolic space on which \( G \) acts to be a simplicial graph, as shown by the following result.

**Theorem 2.5** [27, Theorem 1.2]. If \( G \) is acylindrically hyperbolic then there exists a (not necessarily finite) generating set \( S \) of \( G \) such that the Cayley graph \( X \) of \( G \) with respect to \( S \) is hyperbolic and such that the natural action of \( G \) on \( X \) is non-elementary and acylindrical.

**Remark 2.6.** If the group \( G \) is not hyperbolic, then the generating set \( S \) mentioned in Theorem 2.5 is necessarily infinite.

### 3 LIMIT GROUPS, APPROXIMATIONS AND THE SHORTENING ARGUMENT

In this section, we define **limit groups** over acylindrically hyperbolic groups and discuss some of their prominent properties following the recent work of Groves and Hull [18]. We focus our interest on **divergent** limit groups (see Definition 3.1) as these come armed with a limiting action on a real tree. Under some conditions, a divergent limit group splits as a **graph of actions** (see Definition 3.3); this splitting is famously known as the output of the **Rips machine**, which was introduced in unpublished work of Rips in the early 1990s.

The lack of equational Noetherianity in acylindrically hyperbolic groups imposes a great obstacle to exploiting Sela’s **shortening argument** (see Subsection 3.4) in our setting. Hence, we conclude this section by introducing a method of approximating limit groups by finitely presented groups (relative to a subgroup) in a sense that captures the structure of the aforementioned splitting, and which will allow us to establish a generalisation of Sela’s shortening argument to acylindrically hyperbolic groups. Note that a similar method of approximating limit groups appears in [18], however, the construction of the approximation is different, and the version of the shortening argument we prove is slightly different from the one proved in [18] (see Remark 3.30 for further details).

Throughout this section, assume that \((X, d)\) is a \(\delta\)-hyperbolic simplicial graph on which \( G \) acts acylindrically and non-elementarily (note that, by Theorem 2.5, this assumption is not restrictive). We also stick to the language of **ultra-filters** as in [18] and as is common in non-standard analysis; this enables us to phrase statements with relative ease, rather than often passing to subsequences. Recall that a **non-principal ultra-filter** is a finitely additive probability measure \(\omega : 2^\mathbb{N} \to \{0, 1\}\) satisfying \(\omega(F) = 0\) for every finite \(F \subset \mathbb{N}\). For a statement \(P\) depending on an index \(n \in \mathbb{N}\), we say
that $P$ holds $\omega$-almost surely if

$$\omega(\{ n \in \mathbb{N} \mid P \text{ holds for } n \}) = 1.$$ 

The $\omega$-limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ is $x \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\omega(\{ n \in \mathbb{N} \mid |x - x_n| < \varepsilon \}) = 1.$$

In this case, we denote $\lim_\omega(x_n) = x$. We say that $\lim_\omega(x_n) = \infty$ if $\omega(\{ n \in \mathbb{N} \mid x_n > N \}) = 1$ holds for every $N \in \mathbb{N}$. Every sequence of real numbers has a unique $\omega$-limit in $\mathbb{R} \cup \{\infty\}$.

### 3.1 Limit groups over acylindrically hyperbolic groups

We define limit groups over acylindrically hyperbolic groups as in the standard case over free groups; for a more detailed description of the construction, and for additional properties of such limit groups, we refer the reader to [18].

**Definition 3.1.** Let $H$ be a finitely generated group, and let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $\text{Hom}(H, G)^\mathbb{N}$. The stable kernel of $(\varphi_n)_{n \in \mathbb{N}}$ (with respect to $\omega$) is

$$\ker_\omega((\varphi_n)_{n \in \mathbb{N}}) = \{ g \in H \mid g \in \ker(\varphi_n) \text{ $\omega$-almost surely} \}.$$ 

Fixing a finite generating set $S$ of $H$, we associate a scaling factor to every morphism in the sequence $(\varphi_n)_{n \in \mathbb{N}}$, defined by

$$||\varphi_n|| = \min_{y \in X} \max_{s \in S} d(y, \varphi_n(s)y).$$

Note that $X$ is a simplicial graph, so the scaling factor above is well-defined.

Using the notion of a stable kernel of a sequence of homomorphisms, we can define limit groups over acylindrically hyperbolic groups.

**Definition 3.2.** Keeping the notation from Definition 3.1, a $G$-limit group is a group of the form $L = H/\ker_\omega((\varphi_n)_{n \in \mathbb{N}})$. We call the limit group $L$ divergent if $\lim_\omega(||\varphi_n||) = \infty$.

The sequence $(\varphi_n)_{n \in \mathbb{N}}$ is called the defining sequence of homomorphisms for $L$, and we denote by $\varphi_\infty : H \to L$ the natural quotient map and refer to $\varphi_\infty$ as the limit map associated with the sequence $(\varphi_n)_{n \in \mathbb{N}}$. As previously mentioned, every divergent limit group $L$ comes equipped with a non-trivial and minimal action on a real tree; the construction of this real tree is commonly referred to as the Bestvina–Paulin method. We briefly explain how the real tree on which a divergent limit group acts is constructed (a detailed proof appears in [18, Theorem 4.4]). Consider the sequence $(X_n, d_n, o_n)_{n \in \mathbb{N}}$ of pointed simplicial graphs, where $X_n = X$ for every $n$, $d_n = \frac{1}{||\varphi_n||} \cdot d$ and $o_n$ is a point in $X_n$ chosen to satisfy

$$\max_{s \in S} d(o_n, \varphi_n(s) o_n) = ||\varphi_n||.$$
We can always choose a point $o_n$ which satisfies this property since the metric on $X$ is discrete. The *ultra-limit* $(\prod_{n \in \mathbb{N}} X_n) / \omega$ of the sequence $(X_n, d_n, o_n)_{n \in \mathbb{N}}$ is given by

$$\left( \prod_{n \in \mathbb{N}} X_n \right) / \omega = \left\{ (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n \mid \lim_{\omega} (d_n(o_n, x_n)) < \infty \right\},$$

where the equivalence relation $\sim_\omega$ on $\prod_{n \in \mathbb{N}} X_n$ is defined by setting $(x_n)_{n \in \mathbb{N}} \sim_\omega (y_n)_{n \in \mathbb{N}}$ if and only if $\lim_{\omega} (d_n(x_n, y_n)) = 0$. The ultra-limit $(\prod_{n \in \mathbb{N}} X_n) / \omega$ is equipped with a complete metric $d_\omega$ defined by $d_\omega((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \lim_{\omega} d_n(x_n, y_n)$; furthermore, note that in this case every $X_n$ is $(\delta_n = 1 || \varphi_n || \cdot \delta)$-hyperbolic and the ultra-limit is $(\lim_\omega (\delta_n) = 0)$-hyperbolic or in other words a real tree. Note that every homomorphism $\varphi_n$ endows $H$ with an action on $X$ (and $X_n$) by setting $hx = \varphi_n(h)x$ for every $h \in H$. These actions enable us to define an action of $H$ on the ultra-limit $(\prod_{n \in \mathbb{N}} X_n) / \omega$ via $h(x_n)_{n \in \mathbb{N}} = (hx_n)_{n \in \mathbb{N}}$.

Finally, we choose a minimal and $H$-invariant subtree $T$ of $(\prod_{n \in \mathbb{N}} X_n) / \omega$. Since every element of $\ker((\varphi_n)_{n \in \mathbb{N}})$ acts trivially on $T$, the action of $H$ on $T$ induces the desired action of the divergent limit group $L$ on the real tree $T$. Furthermore, $L$ acts on $T$ without global fixed points, that is, the action is non-trivial.

### 3.2 Graphs of actions and the Rips machine

Under certain conditions, a group acting on a real tree splits as a *graph of actions*. This splitting endows the group with an action on a simplicial tree which is generally easier to understand than an action on a real tree. Groves and Hull proved in [18] that divergent limit groups over acylindrically hyperbolic groups and their canonical actions on real trees satisfy the desired conditions which are required to invoke Guirardel’s version of the Rips machine (see [21]). We present the relevant definitions and results from both works and assume that the reader is familiar with the standard terminology associated with the Rips machine, as in [21].

**Definition 3.3** [21, Definition 1.2]. A *graph of actions* $G$ consists of:

1. an underlying graph of groups $\mathbb{A} = (A, (A_v)_{v \in V(A)}, (A_e)_{e \in E(A)}, (i_e)_{e \in E(A)})$,
2. a collection of real trees $(T_v, d_v)_{v \in V(A)}$ such that $A_v$ acts on $T_v$,
3. a collection of points $(p_e \in T_{\ell(e)} \mid e \in E(A))$ such that every $p_e$ is fixed by $i_e(A_e)$, called *attaching points*,
4. a function $\ell : E(A) \rightarrow \mathbb{R}_{\geq 0}$ assigning *lengths* to the edges of $A$, and such that $\ell(e) = \ell(\bar{e})$ for every $e \in E(A)$.

We usually present the information above as a tuple and write

$$G = G(\mathbb{A}) = (\mathbb{A}, (T_v)_{v \in V(A)}, (p_e)_{e \in E(A)}, \ell).$$

A graph of actions $G(\mathbb{A})$ enables one to canonically construct a real tree $T_G$ on which $G = \pi_1(\mathbb{A})$ acts: replace each vertex $\bar{v}$ of the Bass–Serre tree $T_A$ corresponding to $\mathbb{A}$ by a copy of $T_v$ (where $v$ is the image of $\bar{v}$ under the quotient map $q : T_A \rightarrow G \setminus T_A = A$), and replace any edge $\bar{e}$ of $T_A$ by a segment of length $\ell(e)$ (where $e = q(\bar{e})$). We also ask that if $t(\bar{e}) = \bar{v}$ in $T_A$ then $t(\bar{e}) = \bar{p}_e$ in $T_G$, that is we attach the tree $T_v$ via the attaching point $p_e$. The action of $\pi_1(\mathbb{A})$ on $T_A$ extends naturally.
to an action $\pi_1(\mathbb{A}) \curvearrowright T_G$. We next define notions of stability concerning with group actions on real trees which will allow us to describe the output of the Rips machine.

**Definition 3.4.** Suppose that $L$ is a group acting on a real tree $T$.

1. A subtree $T' \subset T$ is called *stable* if for every non-degenerate subtree $T'' \subset T'$, \( \text{Stab}_L(T') = \text{Stab}_L(T'') \). Otherwise, $T'$ is called *unstable*. An action on a real tree is *stable* if any non-degenerate arc contains a non-degenerate stable subarc.

2. The action $L \curvearrowright T$ is said to satisfy the *ascending chain condition* if for any sequence of nested arcs $I_1 \supset I_2 \supset \cdots$ in $T$ whose lengths tend to 0, the corresponding sequence of stabilisers $\text{Stab}_L(I_1) \subset \text{Stab}_L(I_2) \subset \cdots$ eventually stabilises.

We are now ready to state a relative version of the Rips machine which appears in [21].

**Theorem 3.5** [21, Theorem 5.1]. Let $L$ be a group acting minimally and non-trivially on a real tree $T$ by isometries. Let $U$ be a subgroup of $L$ such that $L$ is finitely generated over $U$ and such that $U$ fixes a point in a real tree $T$ on which $L$ acts. Assume in addition that the action of $L$ on $T$ satisfies the ascending chain condition, and that for any unstable arc $I \subset T$,

1. $\text{Stab}_L(I)$ is finitely generated, and
2. $\text{Stab}_L(I)$ is not a proper subgroup of any conjugate of itself.

Then one of the following holds.

1. $L$ splits over the stabiliser of an unstable arc and $U$ is contained in one of the factors.
2. $L$ splits over the stabiliser $N$ of an infinite tripod and $U$ is contained in one of the factors, and the normaliser of $N$ contains a non-abelian free group generated by two hyperbolic elements whose axes do not intersect.
3. The action $L \curvearrowright T$ decomposes as a graph of actions $\mathbb{R}_L$ where each vertex action is
   a. either simplicial: a simplicial action on a simplicial tree,
   b. of Seifert-type: the action of $L_v$ has kernel $N_v$ and the faithful action of $L_v/N_v$ is dual to an arational measured foliation on a compact 2-orbifold with boundary,
   c. or axial: $T_v$ is a line, and the image of $L_v$ in $\text{Isom}(T_v)$ is a finitely generated group acting with dense orbits on $T_v$.

We are interested in decompositions of divergent limit groups over acylindrically hyperbolic groups; the following lemma that appears in [18], also known as the stability lemma, implies that such limit groups indeed satisfy the stability conditions required for applying Theorem 3.5. Recall that $L$ is a divergent limit group with defining sequence of homomorphisms $(\varphi_n)_{n \in \mathbb{N}} \in \text{Hom}(H, G)$. In the following lemma, $\delta$ denotes the hyperbolicity constant of a hyperbolic space on which $G$ acts acylindrically and non-elementarily, and $N$ and $R$ denote the acylindricity constants appearing in Definition 2.2.

**Lemma 3.6** [18, Lemma 4.7]. There is a constant $C$ depending only on $\delta$, $N$ and $R$ such that the action of $L$ on the real tree $T$ constructed in Subsection 3.1 satisfies the following conditions.

1. If $A \subset L$ stabilises a non-trivial arc of $T$, or if $A$ preserves a line in $T$ and fixes its ends, then $A$ is an extension of an abelian group by a finite group of order $\leq C$.
2. The stabiliser of a tripod in $T$ is of order $\leq C$. 

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(3) The stabiliser of an unstable arc $I \subset T$ is of order $\leq C$.

(4) If $K \subset L$ is locally stably elliptic, that is for every finitely generated subgroup $K' \subset K$, the action of $\varphi_n(\tilde{K}')$ (where $\tilde{K}'$ is a lift of $K'$ to $H$) on $X_n$ is elliptic $\omega$-almost surely, then the order of $K$ is $\leq C$.

**Corollary 3.7.** The fact that stabilisers of unstable arcs are finite implies that the action of $L$ on $T$ satisfies the ascending chain condition and the rest of the conditions required for Theorem 3.5. Hence, if $L$ does not split non-trivially over a finite subgroup of order $\leq C$, it must split as a graph of actions as in Theorem 3.5.

### 3.3 Approximations of limit groups

Recall that $G$ is a group that admits an acylindrical and non-elementary action on a $\delta$-hyperbolic simplicial graph $(X, d)$ and that $L$ is a limit group with defining sequence $(\varphi_n)_{n \in \mathbb{N}} \in \text{Hom}(H, G)^\mathbb{N}$, where $H$ is a finitely generated group. For the remainder of this section, we also assume that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is divergent.

**Standing Assumption 3.8.** In what follows, we assume that $H$ is finitely presented over an infinite finitely generated (but not necessarily finitely presented) subgroup $U \subset H$. We denote by $S$ a finite generating set of $H$. In addition, we suppose that $U$ acts elliptically on the limiting tree $T$, and that the restriction of the limit map $\varphi_\infty$ to $U$ is injective, which allows us to identify $U$ with its image under $\varphi_\infty$.

In this subsection, we aim to prepare the grounds for proving a version of the shortening argument for acylindrically hyperbolic groups (slightly different from the version proved in [18] by Groves and Hull). First, let us recall that the group $G$ is not equationally Noetherian in general. Therefore, the sequence $(\varphi_n : H \to G)_{n \in \mathbb{N}}$ defining the $G$-limit group $L$ does not factor through the quotient map $\varphi_\infty$ a priori. This is a major obstacle to generalising the standard proof of the shortening argument, since we cannot rely on the splitting of $L$ as a graph of actions outputted by the Rips machine for shortening the morphisms of the sequence. Before we explain our approach for overcoming this difficulty (which is very similar to the approach taken in [18, Lemma 6.3], coming from [34, Theorem 3.2]), we begin by defining approximations of limit groups.

**Definition 3.9.** Given a finite set of relations $R \subset \ker(\varphi_\infty)$, we define the $R$-approximation $A$ of $L$ as $A = H/\langle \langle R \rangle \rangle$. In general, we call a group $A$ obtained in this manner an approximation of $L$.

**Remark 3.10.** Since the set $R$ is a subset of $\ker(\varphi_\infty)$, the group $A$ acts on the limiting tree $T$. Both quotient maps $H \twoheadrightarrow A$ and $A \twoheadrightarrow L$ are equivariant with respect to the corresponding actions on $T$.

Our motivations for introducing approximations of limit groups are the following.

(1) Since $H$ is finitely presented over $U$ (see Standing Assumption 3.8), every $R$-approximation $A$ of $L$ is finitely presented over $U$. Therefore, the homomorphisms in the sequence $(\varphi_n : H \to G)_{n \in \mathbb{N}}$ factor $\omega$-almost surely through the quotient map $q : H \to A$. We will denote the maps arising from this factorisation by $\vartheta_n$, that is $\varphi_n = \vartheta_n \circ q$, and the corresponding limit
map by $\vartheta_{\infty} : A \to L$. This factorisation will be crucial in our proof of the general version of the shortening argument.

(2) Suppose that $L$ admits a nice splitting as a graph of groups (see below for more details). We will see that, provided that the finite set of relations $R \subset \ker(\varphi_{\infty})$ is carefully chosen, the approximation $A$ admits a splitting that mimics the splitting of $L$, in a precise sense.

In the proof of the shortening argument, as well as in the proof of Merzlyakov’s theorem, we will consider different splittings of $L$; for the definition of JSJ decompositions we refer the reader to 39.

- If $L$ splits non-trivially relative to $U$ over a finite subgroup of order $\leq C$ (the constant appearing in Lemma 3.6), a reduced JSJ splitting of $L$ relative to $U$ over finite subgroups of order $\leq C$, denoted by $J_L$ (see Proposition 3.13 and Corollary 3.15).
- If $L$ does not split non-trivially relative to $U$ over a finite subgroup of order $\leq C$, a splitting of $L$ as a graph of actions outputted by the Rips machine 3.5, denoted by $R_L$ (see Proposition 3.13 and Corollary 3.16). This is our main motivation for approximating limit groups.
- More generally, a splitting $R.J_L$ of $L$ obtained from $J_L$ by replacing the unique vertex $u$ fixed by $U$ with $R_{L_u}$ (see Proposition 3.13 and Corollary 3.17).

Before we construct approximations of $L$ equipped with splittings that mimic one of the aforementioned splittings, we define with more details the sense in which an approximation of $L$ mimics a certain splitting.

**Definition 3.11.** Let $A$ be an approximation of $L$ as in Definition 3.9. Let $\pi : A \to L$ be the natural epimorphism (obtained by quotienting out by the image of $\ker(\varphi_{\infty})$ in $A$). Suppose that $L$ splits as a graph of groups $S_L$. One says that $A$ is an $S_L$-approximation of $L$ if the following four conditions hold.

1. $A$ splits as a graph of groups $S_A$ with the same underlying graph as $S_L$, and in which all the edge groups are finitely presented and all the vertex groups are finitely presented (relative to $U$).
2. $\pi$ induces an isomorphism of graphs, denoted by $f$, between the underlying graph of $S_A$ and the underlying graph of $S_L$.
3. If $i : A_e \leftrightarrow A_v$ denotes the inclusion of an edge group of $S_A$ into an adjacent vertex group, and $j : L_{f(e)} \leftrightarrow L_{f(v)}$ denotes the corresponding inclusion in the graph of groups $S_L$, then the following diagram commutes:

$$
\begin{array}{ccc}
A_e & \xrightarrow{i} & A_v \\
\downarrow{\pi} & & \downarrow{\pi} \\
L_{f(e)} & \xrightarrow{j} & L_{f(v)}
\end{array}
$$

4. $\pi$ maps every edge group $A_e$ of $S_A$ into the corresponding edge group $L_{f(e)}$ of $S_L$.

For readability, we omit the isomorphism $f$ and denote the vertex $f(v)$ and the edge $f(e)$ by $v$ and $e$, respectively.

**Remark 3.12.** The second and third conditions in the definition above can be phrased, equivalently, as follows: there exists a $\pi$-equivariant isomorphism of graphs between the Bass–Serre trees of the splittings $S_A$ and $S_L$. 

A similar method of approximating limit groups appears in [34, Theorem 3.2] (see also [17; 18, Lemma 6.3; 29, Lemma 6.1]). Note that in these papers, in the process of approximating a limit group, one constructs countably many approximations; each of them approximates the limit group \( L \) to a greater extent than its predecessors. Below, we give an alternative construction, which approximates (only) the specific properties of the limit group \( L \) required for the proofs of the shortening argument and Merzlyakov’s theorem.

**Proposition 3.13.** Suppose that \( L \) splits as a graph of groups \( \mathbb{S}_L \) in which all the edge groups are virtually abelian. Then there exists an \( \mathbb{S}_L \)-approximation \( A \) of \( L \), which in addition satisfies the following two properties.

1. If \( L_e \) is a finitely generated edge group of \( \mathbb{S}_L \), then the quotient map \( \pi : A \to L \) maps \( A_e \) onto \( L_e \) (and thus \( \pi|_{A_e} : A_e \to L_e \) is an isomorphism).
2. Let \( L_v \) be a vertex group of \( \mathbb{S}_L \); if all the edge groups of \( \mathbb{S}_L \) adjacent to \( L_v \) are finitely generated, then the quotient map \( \pi : A \to L \) maps \( A_v \) onto \( L_v \). Moreover, if \( L_v \) is finitely presented (relative to \( U \)), then the map \( \pi|_{A_v} : A_v \to L_v \) is an isomorphism.

In addition, for any finite set of relations \( F \subset \ker\(\phi_\infty\) \), we can choose the approximation \( A \) such that the image of \( F \) in \( A \) is trivial.

**Proof.** We choose to construct the vertex and edge groups of \( \mathbb{S}_A \) before constructing the group \( A \) itself. By doing so, we hope to give the reader a better understanding of the structure of the approximation \( A \). Since the proof of this proposition is quite intricate, we divide it into four steps.

**Step 1.** We will fix explicit presentations of the edge and vertex groups of the graph of groups \( \mathbb{S}_L \) that will be used throughout the proof. Since \( L \) is finitely generated, every vertex group \( L_v \) of \( \mathbb{S}_L \) is finitely generated relative to its adjacent edge groups. Fix a presentation \( (S \mid P_U \cup P) \) of \( H \), where \( P_U \) consists of relations involving only elements from \( U \) and \( P \) is finite. We also fix a finite generating set \( X_U \) of \( U \).

For each edge \( e \in E(\mathbb{S}_L) \), fix a presentation \( (X_e \mid R_e) \) of the edge group \( L_e \). If \( L_e \) is finitely generated (and hence finitely presented, as a virtually abelian group), we choose this presentation to be finite. Note that otherwise, \( X_e \) and \( R_e \) are both infinite. Then, for each vertex \( v \in V(\mathbb{S}_L) \), fix a presentation of the vertex group \( L_v \) of the form

\[
\langle X_v = Y_v \cup X_{e_1}^v \cup \cdots \cup X_{e_n}^v \mid R_v = Q_v \cup R_{e_1}^v \cup \cdots \cup R_{e_n}^v \rangle,
\]

where

- \( e_1, \ldots, e_n \) are the edges adjacent to \( v \) in the underlying graph of \( \mathbb{S}_L \).
- Each \( \langle X_{e_i}^v \mid R_{e_i}^v \rangle \) is a copy of the corresponding edge group within \( L_v \).
- Recall that \( L \) is finitely generated; therefore, \( L_v \) is finitely generated relative to its adjacent edge groups. Hence, the set \( Y_v \) can be chosen finite (and if \( L_v \) contains \( U \) we choose \( Y_v \) to be the union of \( X_U \) and a finite set).
- \( Q_v \) is a (possibly infinite) set of relations.

In addition, we fix a presentation of \( L \) as the fundamental group of \( \mathbb{S}_L \), that is

\[
L = \left( \bigcup_{v \in V(\mathbb{S}_L)} X_v \cup \{t_e \mid e \in E\} \right) \bigcup_{v \in V(\mathbb{S}_L)} R_v \cup R,
\]
where

- $E = E(S_L) \setminus T_{S_L}$ for a spanning subtree $T_{S_L}$ of the underlying graph of $S_L$, and
- $R$ is a (possibly infinite) set of relations which includes relations of two types:
  - relations which identify the set of generators $X_e$ of the edge group $L_e$ with their images in the adjacent vertex groups whenever $e \in T_{S_L}$;
  - relations of the form $t_e^{-1}i_e(x_e)t_e = i_e(x_e)$ where $e \in E$, $x_e \in X_e$ and the maps $i_e$ and $i_e^\ast$ are the inclusion maps of the edge group $L_e$ into the adjacent vertex groups.

**Step 2.** After having fixed the relevant presentations, we seek to pick a finite set $X_L \subset \bigcup_{v \in V(S_L)} X_v \cup \{t_e, e \in E\}$ that generates $L$. The elements in $X_L$ will be used to define the vertex and edge groups of $S_A$. We choose $X_L$ to be extensive enough so that each of the relations in the finite set $\varphi_\infty(P \cup F)$ can be written as a product of conjugates of relations from the presentation of $L$ above as the fundamental group of $S_L$, involving only elements from $X_L$.

For every $s \in S$, write $\varphi_\infty(s)$ as a product of generators appearing in the presentation of $L$ above. Let $X_S$ be the finite subset of the generating set $\bigcup_{v \in V(S_L)} X_v \cup \{t_e, e \in E\}$ of $L$ composed of the generators appearing in these products.

Similarly, each relation $r$ in the finite set $\varphi_\infty(P \cup F)$ can be written as a product of conjugates of relations appearing in the presentation of $L$ above. Let $X_R$ be the finite subset of $\bigcup_{v \in V(S_L)} X_v \cup \{t_e, e \in E\}$ of $L$ composed of the generators appearing in these products.

Finally, let $X_L = X_S \cup X_R$.

**Step 3.** We finally construct the edge and vertex groups of the splitting $S_A$. For every $e \in E(S_L)$ we define $A_e$ as follows: if $L_e$ is finitely generated (and hence, finitely presented), let $A_e = L_e$. We fix an alternative notation for the finite presentation of $A_e$: $\langle X'_e \mid R'_e \rangle$. If $L_e$ is not finitely presented, let $A_e$ be the subgroup of $L_e$ generated by $L_e \cap X_L$; note that $A_e$ is finitely presented (as a finitely generated virtually abelian group) and fix a finite presentation $\langle X'_e \mid R'_e \rangle$ of $A_e$. Up to modifying the original presentation of $L_e$, we may assume that $X'_e$ is a subset of $X_L$.

For every $v \in V(S_L)$, we define $A_v$ as follows: if $L_v$ is finitely presented over $U$, we let $A_v = L_v$ (and fix an alternative notation for the presentation of $A_v$: $\langle X'_v \mid R'_v \rangle$); otherwise, we set $A_v$ to be the group admitting the following presentation:

$$\langle X'_v = Y_v \cup (X'_{e_1})^v \cup \ldots \cup (X'_{e_{n_v}})^v \mid R'_v = Q'_v \cup \ldots \cup (R'_{e_{n_v}})^v \rangle,$$

where

- $e_1, \ldots, e_{n_v}$ are the edges adjacent to $v$ in the underlying graph of $S_L$;
- $Y_v$ is as in the presentation of $L_v$;
- each $\langle (X'_{e_i})^v \mid (R'_{e_i})^v \rangle$ is a copy of $A_{e_i}$ within $A_v$;
- if $L_v$ does not contain $U$, one has $Q'_v = Q_v \cap R_L$ if $Q_v$ is infinite, and $Q'_v = Q_v$ otherwise. If $L_v$ contains $U$, we pick $Q'_v$ in the same manner, but include in $Q'_v$ the (possibly infinite) subset of $Q_v$ which consists of relations involving only elements from $U$.

Recall that $R$ is the set of relations from the presentation of $L$ above. Let $R'$ be the finite set of relations which involve only the generators $X'_e$ of $A_e$ for each $e \in E(S_L)$. Let $R = \bigcup_{v \in V(S_L)} R'_v \cup R'$, where all of the relations in the union are written with the letters of the generating set $S$ of $H$. Note that $R$ is finite. Now, let $A$ be the $R$-approximation of $L$, that is define $A = H/\langle \langle R \rangle \rangle$. 
Step 4. We now show that \( A \) satisfies the desired properties. First, note that \( A \) admits the presentation \( \langle S \mid P_U \cup P \cup R \rangle \) in the generators of \( H \). By expressing this presentation in terms of the generators of \( X_L \), we obtain the following presentation of \( A \):

\[
A = \left\langle \bigcup_{v \in \mathcal{V}(S_L)} X'_v \cup \{ t_e, e \in E \} \mid R_L \cup R \right\rangle.
\]

But since \( R_L \) is contained in \( R \) by definition of the \( R'_v \), one can omit \( R_L \) in the previous presentation of \( A \). Hence, \( A \) is simply the fundamental group of the graph of groups \( \mathbb{S}_A \) obtained from \( \mathbb{S}_L \) by replacing each vertex group \( L_v \) with the group \( A_v \), and each edge group \( L_e \) with the group \( A_e \).

In addition, all of the relations in \( \mathcal{R} \) hold in \( A \). This shows that condition (1) of Definition 3.11 holds.

Next, note that the map \( \pi' \) defined by mapping each generator in the presentation of \( A \) above to the corresponding generator in the presentation of \( L \) as the fundamental group of \( \mathbb{S}_L \) coincides with the natural epimorphism \( \pi : A \to L \) obtained by quotienting out the image of \( \ker(\varphi_{\infty}) \) in \( A \). Indeed, denote by \( q \) the quotient map \( H \to A \) and observe that for every \( s \in S \) one has \( \pi' \circ q(s) = \varphi_{\infty}(s) \); this implies that \( \pi' = \pi \). Last, properties (2), (3) and (4) appearing in Definition 3.11 are clearly satisfied.

To finish, let us check that properties (1) and (2) of Proposition 3.13 hold: for property (1), recall that whenever an edge group \( L_e \) of \( \mathbb{S}_L \) is finitely generated, we defined \( A_e \) to be \( L_e \). For property (2), recall that if all the edge groups adjacent to a vertex group \( L_v \) of \( \mathbb{S}_L \) are finitely generated, then the generators \( X'_v \) of \( A_v \) correspond to the generators \( X_v \) of \( L_v \). This implies that the restriction of the map \( \pi : A \to L \) to \( A_v \) is a surjection. If in addition \( L_v \) is finitely presented (over \( U \)), then \( A_v \) and \( L_v \) admit the same presentation and \( \pi \) maps \( A_v \) isomorphically to \( L_v \).

Remark 3.14. Suppose that \( L \) admits a splitting \( \mathbb{S}_L \), and let \( \{ h_1, \ldots, h_k \} \) be a finite set of elements of \( L \). Write each element \( h_i \) as a product \( s_{i,1} \cdots s_{i,m_i} \) of generators appearing in the presentation of \( L \) as the fundamental group of \( \mathbb{S}_L \). By choosing the finite set of relations \( \mathcal{F} \) in Proposition 3.13 wisely, we can make sure that \( L \) has a \( \mathbb{S}_L \)-approximation \( \mathbb{A} \) such that each \( h_i \) has a preimage \( a_i \) in \( \mathbb{A} \) that admits the same decomposition as \( h_i \) as a product of generators. More precisely, let \( \tilde{h}_i, \tilde{s}_{i,1}, \ldots, \tilde{s}_{i,m_i} \) be lifts of \( h_i, s_{i,1}, \ldots, s_{i,m_i} \) to \( H \), for \( 1 \leq i \leq k \). Let

\[
\mathcal{F} = \{ \tilde{h}_i^{-1} \tilde{s}_{i,1} \cdots \tilde{s}_{i,m_i}, 1 \leq i \leq k \} \subset \ker(\varphi_{\infty}),
\]

and let \( A \) be an \( \mathbb{S}_L \)-approximation of \( L \) in which the relations in \( \mathcal{F} \) hold. Then by Proposition 3.13, all of the generators \( s_{i,j} \) appear in the presentation of \( A \) as the fundamental group of \( \mathbb{S}_A \), and the image of \( h_i \) in \( A \) can be simply written as \( s_{i,1} \cdots s_{i,m_i} \). We will use this method in our proof of the general version of the shortening argument.

We now deduce from Proposition 3.13 a series of three corollaries.

Corollary 3.15. Suppose that \( L \) admits a splitting \( \mathbb{S}_L \) in which all the edge groups are finite (for instance, \( \mathbb{S}_L \) can be a reduced JSJ splitting of \( L \) relative to \( U \) over finite subgroups of order \( \leq C \), denoted by \( \mathbb{J}_L \)). In this case, all the vertex groups of \( \mathbb{S}_L \) are finitely generated. Then there exists an \( \mathbb{S}_L \)-approximation \( \mathbb{A} \) of \( L \), whose splitting is denoted by \( \mathbb{S}_A \), such that \( \mathbb{S}_A \) and \( \mathbb{S}_L \) share the same edge groups, and the vertex groups of \( \mathbb{S}_A \) surject onto those of \( \mathbb{S}_L \).
Proof. Since edge groups of $S_L$ are finite, they are virtually abelian and finitely generated. Thus, the existence of $A$ is an immediate consequence of Proposition 3.13.

As mentioned earlier, the main motivation for defining approximations of limit groups is to approximate in an accurate manner splittings of $L$ as a graph of actions outputted by the Rips machine 3.5. The following lemma proves that the approximations given by Proposition 3.13 capture many of the properties of such splittings.

**Corollary 3.16.** Suppose that $L$ does not split non-trivially over a finite subgroup of order $\leq C$, and let $R_L$ be a splitting of $L$ as a graph of actions outputted by the Rips machine. Let $A$ be an $R_L$-approximation of $L$ given by Proposition 3.13 and let $R_A$ denote its splitting. Then the following hold:

1. every edge group of $R_A$ is finitely presented and virtually abelian, and the quotient map $\pi : A \rightarrow L$ maps every edge group of $R_A$ into the corresponding edge group of $R_L$;
2. if $L_v$ is a simplicial vertex group of $R_L$ and $A_v$ is the corresponding vertex group of $R_A$, then $\pi$ maps $A_v$ onto a finitely generated (relative to $U$) subgroup of $L_v$;
3. if $L_v$ is a Seifert-type vertex group of $R_L$ and $A_v$ is the corresponding vertex group of $R_A$, then $\pi$ maps $A_v$ isomorphically to $L_v$;
4. if $L_v$ is an axial vertex group of $R_L$ and $A_v$ is the corresponding group of $R_A$, then $A_v$ is finitely presented (relative to $U$) and virtually abelian, and $\pi$ maps $A_v$ into $L_v$.

Proof. Recall that the edge groups of $R_L$ are virtually abelian, and that each edge group $A_e$ of $R_A$ is finitely presented (condition (1) in Definition 3.11), and that $\pi$ maps $A_e$ into the corresponding edge group $L_e$ (condition (4) in Definition 3.11). Hence, $A_e$ is virtually abelian and the first assertion above holds.

Next, note that the second assertion is an immediate consequence the construction of $A$ in Proposition 3.13.

For (3), let $L_v$ be a Seifert-type vertex group of $R_L$. Note that $L_v$ is finitely presented (relative to $U$). We will prove that $\pi$ maps the corresponding vertex group $A_v$ of $R_A$ isomorphically to $L_v$. By the second assertion of Proposition 3.13, it is enough to show that $L_v$ does not contain infinitely generated abelian subgroups, and thus that all the edge groups adjacent to $L_v$ are finitely generated. This follows from the following easy observation: since $L_v$ is a Seifert-type vertex group, it is hyperbolic, and therefore all of its abelian subgroups are virtually cyclic.

Finally, for (4), note that any subgroup of $L_v$ which is finitely generated (relative to $U$) is finitely presented (relative to $U$) since $L_v$ is virtually abelian. This implies that, as part of the construction of $A_v$ in the proof of Proposition 3.13, $A_v$ is in fact a subgroup of $L_v$ which is finitely presented (relative to $U$) and $\pi$ maps $A_v$ injectively to $L_v$.

The last corollary is a combination of Corollaries 3.15 and 3.16, and it can be proved in a similar way.

**Corollary 3.17.** Let $J_L$ be a reduced JSJ splitting of $L$ over finite groups of order $\leq C$ relative to $U$, and let $R_J$ be a splitting of $L$ obtained from $J_L$ by replacing the unique vertex $u$ fixed by $U$ with $R_{L_u}$. Note that $L_u$ can be viewed as a limit group whose defining sequence of homomorphisms is $(\varphi_n : H_u)_{n \in \mathbb{N}}$, where $H_u$ denotes a finitely generated subgroup of $H$ that contains $U$ and such that $\varphi_\infty(H_u) = L_u$. Then any $R_J$-approximation $A$ of $L$ outputted by Proposition 3.13 admits a splitting $R_J A$ satisfying the following conditions.
Denote by $\mathbb{R}_{A_u}$ the subgraph of $\mathbb{R}_J^A$ which corresponds to the subgraph $\mathbb{R}_{L_u}$ of $\mathbb{R}_J^L$. Denote by $A_u$ the fundamental group of $\mathbb{R}_{A_u}$ and note that $A_u$ is a lift of $L_u$ to $A$. Then $A_u$ is an $\mathbb{R}_{L_u}$-approximation of $L_u$. Furthermore, the splitting $\mathbb{R}_{A_u}$ of $A_u$ enjoys the properties described in Corollary 3.16.

Let $J_A$ be the splitting of $A$ obtained by collapsing to a point the subgraph $\mathbb{R}_{A_u}$ of $\mathbb{R}_J^A$. Then $A$, equipped with the splitting $J_A$, is a $J_L$-approximation of $L$.

Remark 3.18. Note that the splitting $\mathbb{R}_J^L$ is not unique in general since the finite edge groups adjacent to the vertex $u$ in $J_L$ may fix several vertex groups in the splitting $\mathbb{R}_{L_u}$ of $L_u$.

### 3.4 The shortening argument

The shortening argument encompasses a wide array of results, all of which share a similar nature: shortening homomorphisms. The classical result asserts that given a sequence of homomorphisms from a finitely generated group to another group from a certain class, either one can shorten the homomorphisms (in some sense), or the stable kernel of the sequence is non-trivial. For the class of acylindrically hyperbolic groups, a version of the shortening argument is proven in [18, Theorem 5.29]. In this section, we provide two additional versions of this result which will help us deal with inequalities in the proof of Merzlyakov’s theorem. We first define the notion of a short homomorphism.

**Definition 3.19.** Recall that the scaling factor (or length) of a homomorphism $\varphi : H \to G$ is defined by

$$||\varphi|| = \inf_{y \in X} \max_{s \in S} d(y, \varphi(s)y),$$

where $S$ is a finite generating set of $H$. We call $\varphi$ short relative to $U$ if for every homomorphism $\phi : H \to G$ whose restriction to $U$ coincides with $\varphi|_U$ up to conjugation (that is, there exists $g \in G$ such that $\phi(h) = g\varphi(h)g^{-1}$ for every $h \in U$), one has:

$$||\varphi|| \leq ||\phi||.$$

We would like to point out once again the main difficulty which stands in our way: unlike hyperbolic groups, acylindrically hyperbolic groups are not equationally Noetherian in general. Therefore, the sequence of homomorphisms $(\varphi_n)_{n \in \mathbb{N}}$ does not necessarily factor via the limit group $L$ ($\omega$-almost surely) and one cannot use automorphisms of $L$ in order to shorten the homomorphisms in the sequence $(\varphi_n)_{n \in \mathbb{N}}$. To combat this, we use approximations of $L$ which were defined in the previous subsection (see Definitions 3.9 and 3.11). Since the homomorphisms in the sequence $(\varphi_n)_{n \in \mathbb{N}}$ do factor via an approximation $A$ of $L$ $\omega$-almost surely, we can use automorphisms of $A$ in order to shorten the sequence $(\varphi_n)_{n \in \mathbb{N}}$. The automorphisms which we use are lifts of a certain type of modular automorphisms of $L$ (see Definitions 3.23 and 3.25) to $A$.

Before stating and proving our two versions of the shortening argument, we begin by collecting a few definitions and results.
Definition 3.20 [29, Definition 3.13]. Let $G$ be a group which splits as a graph of groups $\mathbb{S}$ and let $G_v$ be one of its vertex groups. Suppose that $\alpha_v \in \text{Aut}(G_v)$ satisfies the following property: for every edge group $G_{e_i}$ adjacent to $G_v$, there exists an element $c_{e_i} \in G_v$ such that $\alpha_v$ restricts to conjugation by $c_{e_i}$ on $G_{e_i}$. Recall that each element of $G$ can be realised as a loop in the graph of groups $\mathbb{S}$. The homomorphism $\alpha : G \to G$ defined by

$$[a_0,e_1,a_1,...,e_k,a_k] \mapsto [b_0,e_1,b_1,...,e_k,b_k],$$

where

$$b_i = \begin{cases} a_i & a_i \not\in A_v \\ c_{e_i}^{-1}\alpha_v(a_i)c_{e_{i+1}} & a_i \in A_v \end{cases}$$

is called a natural extension of $\alpha_v$.

Remark 3.21. Note that the elements $c_{e_i}$ are not unique in general, and hence the morphism $\alpha$ above is not uniquely defined by $\alpha_v$.

The following short lemma shows that such a natural extension $\alpha$ is an automorphism of $G$.

Lemma 3.22. Let $G$ be a group that splits as a graph of groups $\mathbb{S}$; let $G_v$ be one of its vertex groups. Let $\alpha_v \in \text{Aut}(G_v)$ satisfy the properties appearing in Definition 3.20 and let $\alpha : G \to G$ be a natural extension of $\alpha_v$. Then $\alpha$ is a well-defined automorphism of $G$ whose restriction to $G_v$ is $\alpha_v$, and whose restriction to every edge group of $\mathbb{S}$ is a conjugation (by some element, depending on the edge).

Proof. Since $G$ can be realised as a sequence of amalgamated products followed by a sequence of HNN extensions, it is enough to prove the lemma in the case where $\mathbb{S}$ has only one edge.

First case. Suppose that $G = A *_{C} B$, and assume that $\alpha_v$ is an automorphism of $A$ such that $\alpha_v|_{C} = \text{ad}(a)$ for some $a \in A$. Define $\alpha$ as in Definition 3.20, that is: $\alpha|_{A} = \alpha_v$ and $\alpha|_{B} = \text{ad}(a)$. This endomorphism is well-defined, and it is clearly surjective since its image contains $\alpha_v(A) = A$ and $aBa^{-1}$, which generate $G$. Let us prove that $\alpha$ is injective. Consider a non-trivial element $g = a_1b_1a_2b_2\cdots a_nb_n \in G$ written in normal form. The elements $a_i$ and $b_i$ do not belong to $C$, except maybe $a_1$ or $b_n$. One can write $\alpha(g) = a'_1b'_1a'_2b'_2\cdots a'_nb'_na'_{n+1}$ with $a'_i = \alpha_v(a_i)a$, $a'_i = a^{-1}\alpha_v(a_i)a$ for $1 < i < n$ and $a'_{n+1} = a^{-1}$. Observe that $a'_i$ does not belong to $C$ for $1 < i < n$, otherwise $a'_i = a^{-1}\alpha_v(a_i)a = c \in C$, thus $\alpha_v(a_i) = ac^{-1} = \alpha_v(c)$. It follows that $a_i = c$; this is a contradiction. Hence, the previous decomposition of $\alpha(g)$ is in normal form, which proves that $\alpha(g)$ is not trivial.

Second case. Suppose that $G = \langle A, t \mid tct^{-1} = \sigma(c), \forall c \in C_1 \rangle$, where $\sigma$ denotes an isomorphism between two subgroups $C_1$ and $C_2 = \sigma(C_1)$ of $A$. Suppose that $\alpha_v$ is an automorphism of $A$ such that $\alpha_v|_{C_1} = \text{ad}(a_i)$ for some $a_i \in A$, for $1 \leq i \leq 2$. Define $\alpha$ as follows: $\alpha|_{A} = \alpha_v$ and $\alpha(t) = a_2ta_1^{-1}$. As in the first case, one easily sees that $\alpha$ is well-defined and surjective. The injectivity follows from Britton’s lemma, by a similar argument as above. □

We next define the modular group of a limit group (see [18, Definition 5.22]).
Definition 3.23. Suppose that $L$ admits a splitting as a graph of actions $\mathbb{R}_L$ outputted by the Rips machine. The modular group $\text{Mod}_{\mathbb{R}_L}(L)$ associated with the splitting $\mathbb{R}_L$ is the subgroup of $\text{Aut}(L)$ generated by the following automorphisms.

1. Inner automorphisms.
2. Dehn twists over the virtually abelian edge groups of $\mathbb{R}_L$: if $L_e$ is an edge group of $\mathbb{R}_L$ and $c \in Z(L_e)$ then the Dehn twist by $c$ is the automorphism of $L$ given by
   
   \[ \begin{align*}
   \tau_c(a) &= a, \tau_c(b) = cbc^{-1} & \text{if } L = A_1 \ast_A e A_2, a \in A_1 \text{ and } b \in A_2 \\
   \tau_c(a) &= a, \tau_c(t) = tc & \text{if } L = A \ast_L e \text{ with stable letter } t \text{ and } a \in A.
   \end{align*} \]
3. Natural extensions of automorphisms of Seifert-type vertex groups which are induced by homeomorphisms of the underlying 2-orbifold and which fix the boundary and conical points.
4. Natural extensions of automorphisms of axial vertex groups, which satisfy the condition below. Denote by $L_v$ an axial vertex group of $\mathbb{R}_L$, then by [18, Lemma 5.1] every subgroup $B \leq L_v$ is virtually abelian, and has a unique maximal subgroup $B^+$ of index at most 2 which is finite-by-abelian. Denote by $E(L_v)$ the subgroup of $L_v$ generated by its adjacent edge groups.

We allow natural extensions of automorphisms $\alpha_v$ of $L_v$ for which:

(a) $\alpha_v$ fixes the subgroup $P^+_v$ of $L_v$ which consists of all $g \in L_v$ such that $g \in \ker(\phi)$ for every homomorphism $\phi : L_v \to \mathbb{Z}$ satisfying $E(L_v) \cap A^+_v \subset \ker(\phi)$, and
(b) $\alpha_v$ restricts to conjugation on every subgroup $B \leq L_v$ for which $B^+ = P^+_v$.

Remark 3.24. Note that every modular automorphism of one of the types (1)–(4) above restricts to conjugation on every finite subgroup of $L$, and hence modular automorphisms always restrict to conjugation on finite subgroups of $L$.

Our next goal is to show that given a modular automorphism $\alpha$ of $L$, under some restrictions, one can find an approximation $A$ of $L$ and a lift $\tilde{\beta} \in \text{Aut}(A)$ of $\alpha$. As Lemma 3.26 will show, every modular automorphism of types (1)–(3) above admits a lift to an approximation of $L$. This follows from the extent to which one can approximate the edge groups and the Seifert-type vertex groups of the splitting of $L$ as a graph of actions, as evident in Corollary 3.16. However, dealing with axial vertex groups is slightly more complicated. We therefore discuss further the structure of axial vertex groups of $L$ and describe a few properties of modular automorphisms of type (4) used in the proof of the shortening argument. We follow [29, Subsection 4.2.1] and refer the reader to [29] for further details.

Suppose that $L$ admits a splitting as a graph of actions $\mathbb{R}_L$ outputted by the Rips machine and that $L_v$ is an axial vertex group of $L$. Denote by $E \leq L_v$ the torsion subgroup of $L_v$ and let $\pi_E$ be the quotient map $L_v \to L_v/E$. Recall that $L_v$ has a subgroup $L^+_v$ of index at most 2 which is finite-by-abelian, and let $(L_v/E)^+$ be the image of $L^+_v$ in $L_v/E$. The group $(L_v/E)^+$ admits a decomposition $(L_v/E)^+ = A \oplus B$ where $A$ is a finitely generated free abelian group and $B$ is the torsion-free (and abelian) kernel of the action of $(L_v/E)^+$ on the line $T_v \subset T$. Let $\tilde{A} = \pi_E^{-1}(A)$ and $\tilde{B} = \pi_E^{-1}(B)$. As in [29, Subsection 4.2.1], there exists an element $s \in L_v$ such that $L_v = \langle A, B, s \rangle$ and every $g \in L_v$ can be written as a product of the form

\[ g = abs^\eta \]

where $a \in \tilde{A}, b \in \tilde{B}$ and $\eta \in \{0, 1\}$. 
We now define the subgroup \( \text{Aut}^*(L_v) \) of \( \text{Aut}(L_v) \) to be the subgroup which consists of all the automorphisms \( \alpha_v \in \text{Aut}(L_v) \) which satisfy the following three properties:

1. \( \alpha_v \) preserves \( \tilde{A} \);
2. \( \alpha_v \) restricts to the identity on \( \langle \tilde{B}, s \rangle \);
3. consider the action of \( L_v \) on the line \( T_v \subset T \). Then for every \( x \in T_v \), \( \alpha_v \) restricts to conjugation on the stabiliser \( (L_v)_x \) of \( x \).

This leads us to define the following subgroup of \( \text{Mod}_{\mathbb{R}L}(L) \).

**Definition 3.25.** The group \( \text{Mod}^*_{\mathbb{R}L}(L) \) is the subgroup of \( \text{Mod}_{\mathbb{R}L}(L) \) generated by:

1. modular automorphisms of types (1)–(3) (see Definition 3.23);
2. modular automorphisms \( \alpha \) of type (4) which satisfy the following: if \( \alpha \) is a natural extension of \( \alpha_v \in \text{Aut}(L_v) \) for an axial vertex group \( L_v \) of \( L \), then \( \alpha_v \in \text{Aut}^*(L_v) \).

The motivation behind Definition 3.25 comes from the fact that by [29, Subsection 4.2.1] it is enough to use modular automorphisms which lie in \( \text{Mod}^*_{\mathbb{R}L}(L) \) in the proof of the shortening argument. We finish this discussion with the following easy observation.

**Observation.** suppose that \( L_v \) is an axial vertex group of \( L \); since the torsion subgroup \( E \) of \( L_v \) is finite (by Lemma 3.6) and since the subgroup \( A \) of \( (L_v/E)^+ \) is finitely generated, there are finitely generated subgroups of \( L_v \) which contain \( \tilde{A} \). In addition, let \( \alpha_v \in \text{Aut}^*(L_v) \) and suppose that \( L'_v \) is any subgroup of \( L_v \) which contains \( \tilde{A} \), then the restriction of \( \alpha_v \) to \( L'_v \) is an automorphism.

**Proposition 3.26.** Suppose that \( L \) does not split non-trivially over a subgroup of order \( \leq C \) and let \( \mathbb{R}_L \) be the graph of actions decomposition of \( L \) outputted by the Rips machine. Let \( \alpha \in \text{Mod}^*_{\mathbb{R}L}(L) \). Then there exists an \( \mathbb{R}_L \)-approximation \( A \) of \( L \) and \( \beta \in \text{Aut}(A) \) such that the following diagram commutes \( \omega \)-almost surely.

\[
\begin{array}{ccc}
H & \xrightarrow{q} & A \\
\downarrow{\varphi_n} & & \downarrow{\theta_n} \\
A & \xrightarrow{\beta} & A \\
\downarrow{\theta_\infty} & & \downarrow{\theta_\infty} \\
L & \xleftarrow{\alpha} & L \\
\end{array}
\]

In particular, for every \( h \in H \) such that \( \varphi_n(h) \neq 1 \) \( \omega \)-almost surely, \( \varphi_n \circ \beta \circ q(h) \neq 1 \) \( \omega \)-almost surely.

**Proof.** Recall that the maps \( \varphi_n \) factor via \( A \) \( \omega \)-almost surely and that the maps arising from this factorisation are denoted by \( \theta_n \), that is \( \varphi_n = \theta_n \circ q \) \( \omega \)-almost surely. The limit map associated to the sequence \( (\theta_n)_{n \in \mathbb{N}} \) is denoted by \( \theta_\infty \). The fact that \( \varphi_n = \theta_n \circ q \) \( \omega \)-almost surely implies that the map \( \theta_\infty : A \rightarrow L \) can also be obtained by quotienting out the image of \( \ker(\varphi_\infty) \) in \( A \).

Write \( \alpha = \alpha_k \circ \cdots \circ \alpha_1 \) where for every \( 1 \leq i \leq k \), \( \alpha_i \in \text{Mod}_{\mathbb{R}L}(L) \) is a modular automorphism of \( L \) of one of the types (1)–(4) appearing in Definition 3.23. Furthermore, if \( \alpha_i \) is a modular automorphism of type (4), we assume that it satisfies the condition appearing in Definition 3.25. It is enough to find an \( \mathbb{R}_L \)-approximation \( A \) of \( L \) and an automorphism \( \beta \in \text{Aut}(A) \) for which

\[
\theta_\infty \circ \beta = \alpha \circ \theta_\infty.
\]
In fact, it is enough to show that there is an $\mathbb{R}_L$-approximation $A$ of $L$ and automorphisms $\beta_1, \ldots, \beta_k$ of $L$ such that the following holds for every $1 \leq j \leq k$:

$$\theta_\infty \circ \beta_j = \alpha_j \circ \theta_\infty.$$  

We begin with the construction of the approximation $A$ of $L$. We define a finite subset $C$ of $L$ as follows.

1. Whenever $\alpha_i$ is a Dehn twist for $1 \leq i \leq k$, we add to $C$ an element $c$ which lies in an edge group of $\mathbb{R}_L$ and such that $\alpha_i$ is a Dehn twist by $c$.

2. We keep the notations from the discussion appearing before Definition 3.25. Suppose now that $\alpha_i$ is a modular automorphism of type (4); denote by $\alpha_v \in \text{Aut}^*(L_v)$ an automorphism of an axial vertex group $L_v$ of $L$ such that $\alpha_i$ is a natural extension of $\alpha_v$. Let $L_{e_1}, \ldots, L_{e_k}$ be the edge groups adjacent to $L_v$ and recall that $\alpha_v$ restricts to conjugation by elements $c_1, \ldots, c_k \in L_v$ on $L_{e_1}, \ldots, L_{e_k}$, respectively. Let $L'_v$ be a finitely generated subgroup of $L_v$ (over $U$) which contains $\tilde{A}$ and $c_1, \ldots, c_k$; such a group exists by the observation following Definition 3.25. Denote by $S'_v$ a finite set of generators of $L'_v$. We add the elements in $S'_v$ to $C$.

Let $A$ be an $\mathbb{R}_L$-approximation of $L$ which satisfies the following condition: for every $c \in C$ belonging to an edge group $L_e$ of $\mathbb{R}_L$, there is an element $c' \in A_e$ such that $\theta_\infty(c') = c$. Such an approximation $A$ of $L$ exists by Remark 3.14. Recall that by Lemma 3.11, $A$ admits a splitting $\mathbb{R}_A$ which satisfies the following: $\theta_\infty$ maps every vertex group or edge group of $A_v$ to the corresponding vertex or edge group of $L_v$. In addition, $\theta_\infty$ maps every stable letter in the presentation of $A$ as the fundamental group of $\mathbb{R}_A$ to the corresponding stable letter in the presentation of $L$ as the fundamental group of $\mathbb{R}_L$.

We next construct the automorphisms $\beta_1, \ldots, \beta_k$ of $A$. Let $1 \leq j \leq k$. We divide the construction of $\beta_j$ into cases, depending on the type of the modular automorphism $\alpha_j \in \text{Mod}_{\mathbb{R}_L}(L)$.

1. $\alpha_j$ is a modular automorphism of $L$ of type (1), that is conjugation by some element $g \in L$. Let $g' \in A$ be such that $\theta_\infty(g') = g$ and set $\beta_j$ to be conjugation by $g'$. It is clear that the desired equality holds.

2. $\alpha_j$ is a modular automorphism of $L$ of type (2), that is a Dehn twist by some element $c \in C$ which lies in an edge group $L_e$ of $\mathbb{R}_L$. By the manner in which the approximation $A$ was chosen, there is an element $c' \in A_e$ such that $\theta_\infty(c') = c$. Let $\beta_j$ be a Dehn twist by $c'$. Assume that by collapsing every edge except for $e$, $A$ splits as an amalgamated product over $A_e$, that is $A = A_1 \ast_{A_e} A_2$; the case where $A$ splits as an HNN extension is similar. It follows that $L$ splits as an amalgamated product over $L_e$; write $L = L_1 \ast_{L_e} L_2$. In addition, by the properties of the approximation $A$, one has $\theta_\infty(A_i) \subset L_i$ for $i \in \{1, 2\}$.

Let $g \in A$ and write $g$ as an alternating product of elements from $A_1$ and $A_2$, that is $g = a_1 b_1 a_2 \cdots a_m b_m$ with $a_i \in A_1$ and $b_i \in A_2$ for $1 \leq i \leq m$. We have

$$\theta_\infty \circ \beta_j(g) = \theta_\infty \circ \beta_j(a_1 b_1 a_2 \cdots a_m b_m)$$

$$= \theta_\infty(a_1 (c' b_1 (c')^{-1}) a_2 \cdots a_m (c' b_m (c')^{-1}))$$

$$= \theta_\infty(a_1 (c \theta_\infty(b_1) c^{-1}) \theta_\infty(a_2) \cdots \theta_\infty(a_m) (c \theta_\infty(b_m) c^{-1}))$$

$$= \alpha_j(\theta_\infty(a_1) \theta_\infty(b_1) \theta_\infty(a_2) \cdots \theta_\infty(a_m) \theta_\infty(b_m))$$

$$= \alpha_j \circ \theta_\infty(g).$$
(3) $\alpha$ is a modular automorphism of type (3), that is a natural extension of an automorphism $\alpha_v$ of a vertex group $L_v$ of $\mathbb{R}_L$ of Seifert-type, as described in Definition 3.23. Let $e^v_1, \ldots, e^v_\ell \in E(\mathbb{R}_L)$ be an enumeration of the edges of $\mathbb{R}_L$ which are adjacent to $v$ and recall that $\alpha$ restricts to conjugation by some $c^v_i \in L_v$ on $L^v_i$ for every $1 \leq i \leq \ell$. In addition, by Corollary 3.16, $\theta_\infty$ maps $A_v$ isomorphically to $L_v$. This implies that $\alpha_v$ is an isomorphism of $A_v$, and that there are elements $c^v_i, \ldots, c^v_\ell \in A_v$ such that $\theta_\infty(c^v_i) = c^v_i$ and $\alpha_v$ restricts to conjugation by $c^v_i$ on $A^v_i$ for every $1 \leq i \leq \ell$. Let $\beta_j$ be the natural extension of $\alpha_v$ to $A$, with respect to the elements $c^v_1, \ldots, c^v_\ell$. Now let $g \in A$ and write $g$ as a loop in the graph of groups $\mathbb{R}_A$, that is $g = [a_0, e_1, a_1, \ldots, e_k, a_k]$. Then $\beta_j(g) = [b_0, e_1, b_1, \ldots, e_k, b_k]$ where

$$b_i = \begin{cases} a_i & a_i \in A_v \\ c^v_i -1 \alpha_v(a_i)c^v_{i+1} & a_i \in A_v. \end{cases}$$

To finish, note that $\theta_\infty(g)$ can be written as a loop $[\theta_\infty(a_0), e_1, \theta_\infty(a_1), \ldots, e_k, \theta_\infty(a_k)]$ in $\mathbb{R}_L$, which implies that $\alpha \circ \theta_\infty(g) = [c_0, e_1, c_1, \ldots, e_k, c_k]$ where

$$c_i = \begin{cases} \theta_\infty(a_i) & \theta_\infty(a_i) \notin A_v \\ c^v_i -1 \alpha_v(\theta_\infty(a_i))c^v_{i+1} & a_i \in A_v. \end{cases}$$

One easily sees that $\theta_\infty(b_i) = c_i$ for $1 \leq i \leq k$, which implies that

$$\theta_\infty \circ \beta_j \circ \cdots \circ \beta_1 = \alpha \circ \cdots \circ \alpha_1 \circ \theta_\infty.$$

(4) $\alpha$ is a modular automorphism of $L$ of type (4), and which satisfies the condition appearing in Definition 3.25. In particular, $\alpha$ is a natural extension of an automorphism $\alpha_v \in \text{Aut}^*(L_v)$ of an axial vertex group $L_v$ of $L$. Note that $\theta_\infty$ maps $A_v$ into $L_v$; we identify $A_v$ with its image in $L_v$. By the manner in which the set $C$ was defined, and by the observation following Definition 3.25, we have that the restriction of $\alpha_v$ to $A_v$ is an automorphism. One can continue as in (3) above, by taking a natural extension of $\alpha_v|_{A_v}$ to $A$.

Finally, let $\beta = \beta_k \circ \cdots \circ \beta_1$. The construction of the automorphisms $\beta_1, \ldots, \beta_k$ implies that the diagram appearing in the statement of this proposition does commute $\omega$-almost surely. Finally, let $h \in H$ be such that $\varphi_n(h) \neq 1$ $\omega$-almost surely. It follows that $\theta_\infty \circ q(h) \neq 1$. Therefore, $\alpha \circ \theta_\infty \circ q(h) \neq 1$ which implies that $\theta_\infty \circ \beta \circ q(h) \neq 1$. Hence, $\theta_\infty \circ \beta \circ q(h) \neq 1$ $\omega$-almost surely. \hfill $\square$

**Remark 3.27.** Note that since the action of $U$ on $T$ is elliptic, the modular automorphism $\alpha$ of $L$ restricts to conjugation on $U$. The proof of Proposition 3.26 implies that $\beta$ also restricts to conjugation on $U$.

**Theorem 3.28** (The shortening argument). Suppose that $L$ does not split non-trivially over a finite subgroup of order $\leq C$, then $\omega$-almost surely the homomorphisms $\varphi_n$ are not short relative to $U$. More explicitly, denote by $\mathbb{R}_L$ the splitting of $L$ as a graph of actions outputted by the Rips machine. Then
there is an $\mathbb{R}_L$-approximation $A$ of $L$ admitting a splitting $\mathbb{R}_A$, and an automorphism $\beta \in \text{Aut}(A)$, for which the following holds: denote by $q$ the quotient map $H \to A$ and let $(\gamma_n : A \to G)_{n \in \mathbb{N}}$ be such that $\varphi_n = \gamma_n \circ q \omega$-almost surely. Then the sequence $(\phi_n = \gamma_n \circ \beta \circ q : H \to G)_{n \in \mathbb{N}}$ satisfies the following:

1. $\phi_n|_U$ coincides with $\varphi_n|_U$ up to conjugation $\omega$-almost surely;
2. $||\phi_n|| < ||\varphi_n|| \omega$-almost surely;
3. for every $h \in H$ such that $\varphi_n(h) \neq 1$ $\omega$-almost surely, $\phi_n(h) \neq 1$ $\omega$-almost surely.

**Remark 3.29.** Note that conditions (1) and (2) above imply that $\varphi_n$ is not short relative to $U$ $\omega$-almost surely. Furthermore, condition (2) can be equivalently phrased as follows: $||\gamma_n \circ \beta|| < ||\gamma_n|| \omega$-almost surely, where the lengths are taken with respect to the set $q(S)$. Condition (3) is equivalent to each of the following two conditions:

3’. $\ker_{\omega}(\{\phi_n\}_{n \in \mathbb{N}}) \subset \ker_{\omega}(\{\varphi_n\}_{n \in \mathbb{N}})$;
3’’. $\beta^{-1}(\ker_{\omega}(\{\gamma_n\}_{n \in \mathbb{N}})) \subset \ker_{\omega}(\{\gamma_n\}_{n \in \mathbb{N}})$.

**Remark 3.30.** The version of the shortening argument appearing in [18, Theorem 5.29] satisfies conditions (1) and (2) above, but does not necessarily satisfy condition (3). To obtain condition (3) we approximate a single modular automorphism of $L$, rather than a sequence of modular automorphisms, and the result follows from Lemma 3.26.

The second version of the shortening argument that will be proved is a strengthened version of Theorem 3.28 which accommodates the use of JSJ decompositions of limit groups over finite groups of order less than $C$. Note that in this version of the shortening argument, we assume that a single vertex group of a JSJ decomposition of $L$ over $C$ admits a splitting outputted by the Rips machine and we shorten the homomorphisms with respect to the generators of this vertex group.

**Theorem 3.31.** Let $J_L$ be a reduced JSJ splitting of $L$ over finite groups of order $\leq C$ relative to $U$ and let $u$ be the vertex fixed by $U$. Let $R_{L_u}$ be the splitting of $L_u$ as a graph of actions outputted by the Rips machine, and let $R_{J_L}$ be the splitting of $L$ obtained from $J_L$ by replacing $u$ with $R_{L_u}$. Let $H_u$ be a finitely generated subgroup of $H$ containing $U$ and such that $\varphi_{\infty}(H_u) = L_u$. Let $S_u$ be a finite generating of $H_u$. Then the following hold.

1. There exists an $R_{J_L}$-approximation $A$ of $L$ admitting a splitting $R_{J_A}$ as in Corollary 3.17 and a sequence $(\delta_n : A \to G)_{n \in \mathbb{N}}$ that satisfies $\varphi_n = \delta_n \circ q \omega$-almost surely (where $q$ is the quotient map $H \to A$).
2. Denote by $R_{A_u}$ the subgraph of $R_{J_A}$ corresponding to the subgraph $R_{L_u}$ of $R_{J_L}$. Denote by $A_u$ the fundamental group of $R_{A_u}$. There exists an automorphism $\beta_u$ of $A_u$ that admits a natural extension $\beta \in \text{Aut}(A)$, and such that the sequence $(\phi_n = \delta_n \circ \beta \circ q : H \to G)_{n \in \mathbb{N}}$ satisfies the following properties:
   a. $\phi_n|_U$ coincides with $\varphi_n|_U$ up to conjugation $\omega$-almost surely;
   b. $||\phi_n||_H \leq ||\varphi_n||_{H_u} \omega$-almost surely (where the lengths are taken with respect to the $S_u$);
   c. for every $h \in H$ such that $\varphi_n(h) \neq 1$ $\omega$-almost surely, $\phi_n(h) \neq 1$ $\omega$-almost surely.

**Remark 3.32.** As in Theorem 3.28, the condition (2) above is equivalent to

$$||\delta_n \circ \beta||_{q(H_u)} < ||\delta_n||_{q(H_u)}.$$. 
We need the following lemma in order to prove Theorems 3.28 and 3.31.

**Lemma 3.33.** Suppose that $L$ does not split non-trivially over a finite subgroup of order at most $C$, and that $e \subset T$ (where $T$ is the limiting tree on which $L$ acts) is an edge of a simplicial subtree of $T$; denote its stabiliser by $L_e$. Then there is an element $\tilde{e}_c \in H$ whose image $c_e$ in $L$ is contained in $Z(L_e)$ and such that $\varphi_n(\tilde{e}_c)$ is hyperbolic $\omega$-almost surely.

To prove this lemma, we use the following result.

**Lemma 3.34** [28, Lemma 3.5]. Let $(X,d)$ be a $\delta$-hyperbolic space and let $g : X \to X$ be an isometry. Suppose that for $x, y \in X$

$$d(x, g(x)) + d(y, g(y)) < 2d(x, y) - 4\delta.$$  

Then for some $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \max\{d(x, g(x)), d(y, g(y))\}$, the isometry $g$ acts by $(\lambda, 2\delta)$-quasi-translation on a subgeodesic of $[x, y]$: for every $p \in [x, y]$ at distance greater than $\max\{d(x, g(x)), d(y, g(y))\}$ from both $x$ and $y$ (if such a point $p$ exists), one has

$$d(g(p), p_\lambda) < 2\delta$$  

and

$$d(g^{-1}(p), p_{-\lambda}) < 2\delta,$$

where $p_\lambda$ and $p_{-\lambda}$ are the points on $[x, y]$ which lie at distance $\lambda$ from $p$.

**Proof of Lemma 3.33.** Since $L$ does not split non-trivially over a finite subgroup of order $\leq C$, Lemma 3.6 implies that there exists $c \in L_e$ of infinite order and such that $\varphi_n(\tilde{e}_c)$ is hyperbolic $\omega$-almost surely. Fix $\varepsilon = 8\delta$ and let $N$ and $R$ be the corresponding acylindricity constants. We claim that $c_e = c^{N+1}$ lies in $Z(L_e)$. Let $g \in L_e$, write $e = [x, y]$ and let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be approximating sequences for $x$ and $y$, respectively; let $\tilde{g}$ be a lift of $g$ to $H$. Since $g$ fixes both $x$ and $y$ in the limiting tree, $\varphi_n(\tilde{g})$ must displace both $x_n$ and $y_n$ by a distance which is significantly smaller than $d(x_n, y_n)$ $\omega$-almost surely. More precisely, we have

$$d(x_n, y_n) > 100 \cdot R$$  

$$d(x_n, y_n) > 100 \cdot \max\{d(x_n, \varphi_n(\tilde{g}) x_n), d(y_n, \varphi_n(\tilde{g}) y_n)\}$$  

$$d(x_n, y_n) > 100 \cdot \max\{d(x_n, \varphi_n(\tilde{e})^j x_n), d(y_n, \varphi_n(\tilde{e})^j y_n)\}$$

holds for all $j \in [1, N+1]$ $\omega$-almost surely. Choose two points $p_n, q_n \in e$ satisfying $d(x_n, p_n) < d(x_n, q_n), (p_n, q_n) > R$ and

$$\min\{d(x_n, p_n), d(q_n, y_n)\} > 10 \cdot \left( \max_{h \in [g, c, c^2, \ldots, c^{N+1}]} \{d(x_n, \varphi_n(\tilde{h}) x_n), d(y_n, \varphi_n(\tilde{h}) y_n)\} \right)$$

$\omega$-almost surely. It follows from Lemma 3.34 that each of $g, c, c^2, \ldots, c^{N+1}$ acts on a subsegment of $[x, y]$ which contains $p_n$ and $q_n$ by $2\delta$-quasi-translation, and therefore both $d(p_n, [\varphi_n(\tilde{g}), \varphi_n(\tilde{e})^j](p_n) \leq 8\delta$ and $d(q_n, [\varphi_n(\tilde{g}), \varphi_n(\tilde{e})^j](q_n) \leq 8\delta$ hold for every $j \in [1, N+1]$ $\omega$-almost surely. The acylindricity condition implies that not all of the $N + 1$ commutators can be
distinct and there are $i, j \in [1, N + 1]$ for which
\[
\begin{align*}
[\varphi_n(\tilde{g}), \varphi_n(\tilde{c})^j] &= \varphi_n(\tilde{g})\varphi_n(\tilde{c})^j\varphi_n(\tilde{g})^{-1}\varphi_n(\tilde{c})^{-j} \\
&= \varphi_n(\tilde{g})\varphi_n(\tilde{c})^j\varphi_n(\tilde{g})^{-1}\varphi_n(\tilde{c})^{-i} \\
&= [\varphi_n(\tilde{g}), \varphi_n(\tilde{c})^i].
\end{align*}
\]
In particular, $\varphi_n(\tilde{g})$ commutes with $\varphi_n(\tilde{c})^{j-i} \omega$-almost surely. Since $|j - i| \leq N$, the element $c_e = c^{N!}$ is a power of $c^{j-i}$ and $\varphi_n(\tilde{g})$ commutes with $\varphi_n(\tilde{c})^j$ $\omega$-almost surely; hence $g$ commutes with $c_e$. \hfill \Box

**Proof of Theorem 3.28.** The proof of this theorem is divided in two: using results from [29] and [31] and explaining briefly how they adapt to our setting, we first find a suitable modular automorphism $\alpha \in \text{Mod}^*_{\mathbb{R}_L}(L)$. Then, by means of Proposition 3.26, we find an $\mathbb{R}_L$-approximation $A$ of $L$ and an automorphism $\beta$ of $A$ which satisfy the desired properties.

The idea behind the proof amounts to finding a finite sequence of modular automorphisms of $L$, each of which shortens the actions of the generators $S$ of $H$ over $U$ with respect to the different vertex actions in the graph of actions decomposition $\mathbb{R}_L$ of $L$. The construction of the modular automorphism $\alpha \in \text{Mod}^*_{\mathbb{R}_L}(L)$ relies on the proofs appearing in [29] for the axial and Seifert-type cases, and on the proof appearing in [31] for the simplicial case. We begin with vertex actions $L_v \sim T_v$ which admit dense orbits, namely axial and Seifert-type vertex actions. Recall that if $L_v$ is of Seifert-type, then the index of the 2-orbifold subgroup of $L_v$ is at most $C$. Therefore, by [29, Subsections 4.2.1 and 4.2.2], for every such vertex action and every finite subset $F \subset H$, there exists a modular automorphism $\alpha^F_v \in \text{Mod}^*_{\mathbb{R}_L}(L)$ of type (3) or (4) which satisfies the following: denote by $o = (o_n)_{n \in \mathbb{N}}$ the basepoint of $T$, then for every $f \in F$,
\[
d_T(o, \alpha^F_v(\varphi_\infty(f))o) < d_T(o, \varphi_\infty(f)o)
\]
whenever $[o, \varphi_\infty(f)o]$ has a non-degenerate intersection with a translate of $T_v$ in $T$, and $d_T(o, \alpha^F_v(\varphi_\infty(f))o) = d_T(o, \varphi_\infty(f)o)$ otherwise.

This allows us to shorten the actions (on the real tree $T$) of all the generators which intersect (a translate of) an axial or a Seifert-type component of $T$ non-degenerately: let $v_1, ..., v_m$ be an enumeration of the axial and Seifert-type vertices of $\mathbb{R}_L$; we define a sequence $(\alpha_1, ..., \alpha_m) \in \text{Mod}^*_{\mathbb{R}_L}(L)^m$ iteratively. Let $\alpha_1 = \alpha^F_{v_1}$, and after $\alpha_1, ..., \alpha_i$ were defined let
\[
\alpha_{i+1} = \alpha^F_{v_{i+1}} \circ \alpha_1 \circ \cdots \circ \alpha_i.
\]
Now note that for every $s \in S$ such that $[o, \varphi_\infty(f)o]$ has a non-degenerate intersection with (a translate of) an axial or a Seifert-type component of $T$, we have
\[
d_T(o, \alpha_1 \circ \cdots \circ \alpha_i \circ \varphi_\infty(s)o) < d_T(o, \varphi_\infty(s)o).
\]

Bring to mind that the modular automorphism $\alpha_1 \circ \cdots \circ \alpha_i$ of $L$ does not necessarily shorten the actions of $S$ on $T$: it could be that for every $s \in S$, $[o, \varphi_\infty(s)o]$ is contained entirely in the discrete part of $T$, that is it does not intersect non-degenerately (translates of) axial and Seifert-type components of $T$. We therefore adapt [31, Theorem 6.1] to our settings. This theorem states...
that for every finite set $F \subset H$ there is a modular automorphism $\alpha^F_{\text{sim}} \in \Mod_+^\ast (L)$ of $L$, which can be written as a composition of Dehn twists about elements which lie in the edge groups of $\mathbb{R}_L$, and which satisfies the following: for every $f \in F$ which does not fix $o$, and such that $[o, \varphi_\infty (f) o]$ lies entirely in the discrete part of $T$, let $f^F_{\alpha^F_{\text{sim}}}$ be a lift of $\alpha^F_{\text{sim}} (\varphi_\infty (f))$ to $H$; then

$$d(o_n, \varphi_n (f^F_{\alpha^F_{\text{sim}}}) o_n) < d(o_n, \varphi_n (f) o_n),$$

$\omega$-almost surely. In addition, for every $f \in F$,

$$d_T (o, \alpha^F_{\text{sim}} (\varphi_\infty (f)) o) = d_T (o, \varphi_\infty (f) o).$$

Note that in this case the modular automorphism $\alpha^F_{\text{sim}}$ does not shorten the actions of the elements in $F$ on $T$, but rather shortens the actions of the elements in $F$ directly on the spaces $X_n$.

In the proof of [31, Theorem 6.1], one finds Dehn twists over the edge groups of $\mathbb{R}_L$, where each Dehn twist does not affect the displacement of $o$ by the elements of $F$ in $T$, but does affect the displacement of $o_n$ by $\varphi_n (F)$ in $X_n$. The construction of a Dehn twist over the edge group corresponding to an edge $e$ of $\mathbb{R}_L$ is divided into three cases (below, $\tilde{e}$ denotes an edge of a simplicial subtree of $T$ which corresponds to $e$ as in the paragraph following Definition 3.3):

1. $o$ lies in the interior of $\tilde{e}$ and $L$ splits as an amalgamated product over the stabiliser of $\tilde{e}$;
2. $o$ lies in the interior of $\tilde{e}$ and $L$ splits as an HNN extension over the stabiliser of $\tilde{e}$;
3. $o$ does not lie in the interior of $\tilde{e}$, and is one of its vertices.

In the last case, one considers all such edges $\tilde{e}$ in (simplicial subtrees of) $T$ which are adjacent to $o$ and shortens the action of the generators with respect to all of these edges simultaneously. The proof appearing in [31] can be transitioned almost seamlessly to our setting. In [31], the stabilisers of edges in the limiting tree are cyclic, whereas in our case they are virtually abelian. Therefore, in our case we cannot take Dehn twists by any element in an edge group $L_e$ of $\mathbb{R}_L$, and take Dehn twists by a power of an element $c_e \in Z(L_e)$, of infinite order, which exists by Lemma 3.33. The only other parts which do not carry over to our settings are [31, Lemmas 6.2, 6.5, 6.8 and 6.11] which assert that there are elements of infinite order in the edge groups of $\mathbb{R}_L$ which satisfy the following: let $\tilde{c} \in H$ be a lift of such an element, then $\varphi_n (\tilde{c})$ displaces certain points in $X_n$ by a distance bounded from below by $10 \delta$ or $20 \delta$ $\omega$-almost surely. These are the elements by which one takes the Dehn twists. We can easily overcome this: by [7], there is $\eta > 0$ such that

$$\eta < \ell (g) = \lim_{n \to \infty} \frac{1}{n} d(g^n o, o)$$

for every hyperbolic $g \in G$. In addition, denote $||g|| = \inf_{x \in X} \{d(x, g x)\}$ and by [10, Lemma 10.6.4] we have that $\ell (g) \leq ||g||$ and clearly $\ell (g^n) = n \ell (g)$. Therefore, given $D > 0$ there exists $N$ such that

$$D < N \eta < N \ell (g) = \ell (g^N) \leq ||g^N|| = \inf_{x \in X} \{d(x, g^N x)\}$$

for every hyperbolic $g \in G$. We also remark that one might have to enlarge the constant $C_0$ appearing in [31] to accommodate with the choice of $N$ above. This implies that the proof of [31, Theorem 6.1] can be carried out in our setting.
Now let
\[ \alpha = \alpha_{\sim_{\text{sim}}}^{\cdots} \circ \alpha_1 \circ \varphi_\infty(S) \circ \alpha_m \circ \cdots \circ \alpha_1 \in \text{Mod}^\sim_{\text{sim}}(L). \]

For every \( s \in S \), denote by \( s_\alpha \in H \) a lift of \( \alpha(\varphi_\infty(s)) \) to \( H \). Since \( d_T(o, \alpha(\varphi_\infty(s)) o) < d_T(o, \varphi_\infty(s) o) \) whenever \([o, \varphi_\infty(s)] \) intersects a translate of an axial or a Seifert type component of \( T \) non-degenerately, the following holds \( \omega \)-almost surely:
\[ d(o_n, \varphi_n(s_\alpha) o_n) < d(o_n, \varphi_n(s) o_n). \]

By Proposition 3.26, there exists an \( \mathbb{R}_L \)-approximation \( A \) of \( L \) and \( \beta \in \text{Aut}(A) \) such that the following diagram commutes \( \omega \)-almost surely.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (H) at (-3,2) {$H$};
  \node (G) at (3,2) {$G$};
  \node (L) at (0,-2) {$L$};

  \draw[->] (A) -- (H) node [midway, above] {$\varphi_n$};
  \draw[->] (A) -- (G) node [pos=.6, above] {$\varphi_\infty$};
  \draw[->] (A) -- (L) node [midway, below] {$\theta_\omega$};
  \draw[->] (H) -- (L) node [midway, below] {$\theta_\omega$};
  \draw[->] (G) -- (L) node [midway, above] {$\theta_\omega$};
\end{tikzpicture}
\end{center}

We claim that the approximation \( A \) and its automorphism \( \beta \) satisfy the properties described in the theorem.

By Remark 3.27, since the action of \( U \) on \( T \) is elliptic, \( \beta \) restricts to conjugation on \( U \). Hence, condition (1) holds. For (2), note that for every \( s \in S \), \( \alpha(\varphi_\infty(s)) \) and \( \beta(q(s)) \) share the same lift in \( H \); setting \( \varphi_n = \theta_n \circ \beta \circ q \), it follows that
\[ d(o_n, \theta_n \circ \beta \circ q(s) o_n) = d(o_n, \varphi_n(s) o_n) < d(o_n, \varphi_n(s) o_n), \]
and in particular, since \( o_n \) realises the infimum \( \inf_{x \in X} \max_{s \in S} d(x, \varphi_n(s) x) \),
\[ ||\varphi_n|| = \inf_{x \in X} \max_{s \in S} d(x, \varphi_n(s) x) \]
\[ \leq \max_{s \in S} d(o_n, \varphi_n(s) o_n) \]
\[ < \max_{s \in S} d(o_n, \varphi_n(s) o_n) \]
\[ = ||\varphi_n|| \]
\( \omega \)-almost surely. Last, Property (3) follows directly from Proposition 3.26.

The proof of Theorem 3.31 is very similar to the proof of Theorem 3.28.

**Proof of Theorem 3.31.** The proof is identical to that of Theorem 3.28, with one change (applied to both Theorem 3.28 and Proposition 3.26): one has to take natural extensions of automorphisms of axial and Seifert-type vertex group with respect to the entire graph of groups decomposition \( \mathbb{R}_{L_J} \) of \( L \) (and not with respect to a graph of actions outputted by the Rips machine) instead of modular automorphisms of \( L \) with respect to \( \mathbb{R}_L \).
4 | TEST SEQUENCES

The goal of this section is to define test sequences and prove important preliminary results about these sequences that will play a crucial role in the proof of Merzlyakov’s Theorem 1.11.

4.1 | Relative group presentations

A relative presentation of a group $G$ with respect to a subgroup $H \subset G$ and a subset $X \subset G$ is a presentation of the form

$$G = \langle H, X \mid R \rangle,$$

which is obtained from $H$ by adding the set of generators $X$ and the set of relations $R$. Thus $G = (H \ast F(X)) / \langle \langle R \rangle \rangle$, where $F(X)$ is the free group with basis $X$ and $\langle \langle R \rangle \rangle$ is the normal closure of $R$ in $H \ast F(X)$. In the case where the set $R$ is finite, one says that $G$ is finitely presented relative to $H$. We will frequently use relative group presentations throughout the following sections.

4.2 | Transverse covering

We will use the following definitions (see [20, Definitions 4.6 and 4.8]).

**Definition 4.1.** Let $T$ be a real tree endowed with an action of a group $G$, and let $(Y_j)_{j \in J}$ be a $G$-invariant family of non-degenerate closed subtrees of $T$. We say that $(Y_j)_{j \in J}$ is a transverse covering of $T$ if the following two conditions hold.

- **Transverse intersection:** if $Y_i \cap Y_j$ contains more than one point, then $Y_i = Y_j$.
- **Finiteness condition:** every arc of $T$ is covered by finitely many $Y_j$.

**Definition 4.2.** Let $T$ be a real tree, and let $(Y_j)_{j \in J}$ be a transverse covering of $T$. The **skeleton** of this transverse covering is the bipartite simplicial tree $S$ defined as follows.

1. $V(S) = V_0(S) \sqcup V_1(S)$ where $V_1(S) = \{ Y_j \mid j \in J \}$ and $V_0(S)$ is the set of points $x \in T$ that belong to at least two distinct subtrees $Y_i$ and $Y_j$.
2. There is an edge $\varepsilon = (Y_j, x)$ between $Y_j \in V_1(S)$ and $x \in V_0(S)$ if and only if $x$, viewed as a point of $T$, belongs to $Y_j$, viewed as a subtree of $T$.

**Remark 4.3.** The stabiliser of a vertex of $S$ is the stabiliser $G_{Y_j}$ or $G_x$ of the corresponding subtree or point of $T$. The stabiliser of an edge $\varepsilon = (Y_j, x)$ is $G_{Y_j} \cap G_x$. Moreover, the action of $G$ on $S$ is minimal provided that the action of $G$ on $T$ is minimal (see [20, Lemma 4.9]).

4.3 | Bounding the number of branch points

Let $T$ be a tree, and let $G$ be a group acting on $T$. Recall (see Definition 3.4) that an arc $I \subset T$ is called stable if for every non-degenerate subarc $J \subset I$, the stabiliser of $J$ is equal to the stabiliser of $I$. Otherwise, $I$ is called unstable.
Definition 4.4. An action on a real tree is $K$-superstable if every arc whose pointwise stabiliser has order greater than $K$ is stable.

Let $T$ be a real tree, and let $x$ be a point of $T$. A direction at $x$ is a connected component of $T \setminus \{x\}$. We say that $x$ is a branch point if there are at least three directions at $x$. The following result is a work in preparation by Guirardel and Levitt (improving [19]).

Theorem 4.5. Let $L$ be a group acting on a real tree $T_L$. Suppose that $L$ is finitely generated relative to a countable subgroup $\Gamma$ that is elliptic in $T_L$. Suppose that the following two conditions are satisfied:

1. the action is $K$-superstable for some constant $K$;
2. arc stabilisers are finitely generated.

Then every point stabiliser is finitely generated relative to $\Gamma$, the number of orbits of branch points in $T_L$ is finite and the number of orbit of directions at branch points in $T_L$ is finite.

4.4 Small cancellation condition

Let $(X, d)$ be a $\delta$-hyperbolic simplicial graph, let $G$ be a group acting on $(X, d)$ by isometries, and let $g$ be an element of $G$. We define the translation length of $g$ by $||g|| = \inf_{x \in X} d(x, gx)$. If $g$ is hyperbolic, the quasi-axis of $g$, denoted by $A(g)$, is the union of all geodesics joining $g^-$ and $g^+$. By [11, Lemma 2.26], the quasi-axis $A(g)$ is $11\delta$-quasi-convex. If $g'$ is another hyperbolic element of $G$, one defines the fellow travelling constant $\Delta(g, g')$ as follows:

$$\Delta(g, g') = \text{diam} \left( A(g)^{+100\delta} \cap A(g')^{+100\delta} \right) \in \mathbb{N} \cup \{\infty\},$$

where $A(g)^{+100\delta}$ is the $100\delta$-neighbourhood of $A(g)$ in $(X, d)$, and $A(g')^{+100\delta}$ is defined similarly. Recall that if $G$ acts acylindrically on $(X, d)$, then every hyperbolic element $g$ is contained in a unique maximal infinite virtually cyclic subgroup $\Lambda(g)$ of $G$ (see [13, Lemma 6.5]). Moreover, there exists a constant $N(g) \geq 0$ such that every element $h \in G$ satisfying $\Delta(g, hgh^{-1}) \geq N(g)$ belongs to $\Lambda(g)$ (see, for example, [12]). In addition, if $g$ and $h$ are hyperbolic, then either $\Lambda(g) = \Lambda(h)$ or $\Lambda(g) \cap \Lambda(h)$ is finite.

Definition 4.6. Let $\varepsilon > 0$. We say that a hyperbolic element $g \in G$ satisfies the $\varepsilon$-small cancellation condition if the following holds: for every $h \in G$, if

$$\Delta(g, hgh^{-1}) > \varepsilon ||g||,$$

then $h$ and $g$ commute (so $h$ belongs to $\Lambda(g)$). In particular, $g$ is central in $\Lambda(g)$.

Definition 4.7. Let $\varepsilon > 0$. We say that a tuple $(g_1, \ldots, g_p) \in G^p$ of hyperbolic elements satisfies the $\varepsilon$-small cancellation condition if the following condition holds: for every $h \in G$, for every $(i, j) \in [1, p]^2$, if

$$\Delta(g_i, hgh^{-1}) > \varepsilon \min(||g_i||, ||g_j||),$$
then $i = j$, and the elements $h$ and $g_i$ commute. In particular, $h$ belongs to $\Lambda(g_i)$. As a consequence, $g_i$ is central in $\Lambda(g_i)$ and for every $(i, j) \in [1, p]^2$, we have $\Lambda(g_i) \neq \Lambda(g_j)$, that is, $\Lambda(g_i) \cap \Lambda(g_j) = E(G)$.

4.5 Test sequences and preliminary lemmas

Let $G$ be an acylindrically hyperbolic group. By [27, Theorem 1.2], there exists a generating set $S$ of $G$ such that the Cayley graph $X$ of $G$ with respect to $S$ is $\delta$-hyperbolic, for some $\delta \geq 0$, and the natural action of $G$ on $X$ is acylindrical and non-elementary. We denote by $d$ the word metric on $X$ associated with $S$, and by $1$ the neutral of $G$, viewed as a point of $X$. Given a hyperbolic element $g \in G$, recall that $||g||$ denotes the translation length of $g$, and that $A(g)$ denotes the quasi-axis of $g$.

**Definition 4.8.** Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$. Fix a presentation $\langle a \mid R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let $\Sigma(x, y, a) = 1$ be a finite system of equations over $G$, where $x$ and $y$ are tuples of variables. Denote $G_{\Sigma} = \langle x, y, a \mid R(a) = 1, \Sigma(x, y, a) = 1 \rangle$. Let $p = |x|$ be the arity of $x$, and let $x_i$ denote the $i$th component of $x$. Let $(\sigma_1, \ldots, \sigma_p)$ be a $p$-tuple of elements of $\text{Aut}_c(E(G))$. A sequence of homomorphisms $(\varphi_n : G_{\Sigma} \to G)_{n \in \mathbb{N}}$ is called a $(\sigma_1, \ldots, \sigma_p)$-test sequence if the following four conditions hold.

1. For every integer $n$, the morphism $\varphi_n$ coincides with a conjugation on $a$.
2. For every integer $i \in [1, p]$, the translation length $||\varphi_n(x_i)||$ of $\varphi_n(x_i)$ tends to infinity as $n$ tends to infinity.
3. For every $(i, j) \in [1, p]^2$, there exists a real number $r_{i,j} \in (0, +\infty)$ such that the ratio

$$\frac{||\varphi_n(x_i)||}{||\varphi_n(x_j)||}$$

 tends to $r_{i,j}$ as $n$ tends to infinity.
4. There exists a sequence of positive real numbers $(\epsilon_n)_{n \in \mathbb{N}}$ converging to 0 such that, for every integer $n$, the tuple $\varphi_n(x)$ satisfies the $\epsilon_n$-small cancellation condition (see Definition 4.7), and the following equality holds, for every integer $1 \leq i \leq p$:

$$\Lambda(\varphi_n(x_i)) = \langle \varphi_n(x_i), E(G) \mid \text{ad(}\varphi_n(x_i))|_{\text{E(G)}} = \sigma_i \rangle.$$

In particular, the image of $\varphi_n(x_i)$ in $\Lambda(\varphi_n(x_i))/E(G)$ has no roots.

In the particular case where $(\sigma_1, \ldots, \sigma_p) = (\text{id}_{E(G)}, \ldots, \text{id}_{E(G)})$, one simply says that $(\varphi_n)_{n \in \mathbb{N}}$ is a test sequence.

**Remark 4.9.** Note that a test sequence is always divergent since the translation length $||\varphi_n(x_i)||$, which goes to infinity as $n$ goes to infinity by item (2) above, is smaller than the scaling factor $||\varphi_n||$ (see Definition 3.1) since $x_i$ belongs to the generating set $\{x, y, a\}$ of $G_{\Sigma}$.

**Remark 4.10.** Let $(\varphi_n : G_{\Sigma} \to G)_{n \in \mathbb{N}}$ be a $(\sigma_1, \ldots, \sigma_p)$-test sequence. Let $U$ be the subgroup of $G_{\Sigma}$ generated by $x$ and $a$. Since the translation length $|| \cdot ||$ is constant on conjugacy classes, one easily
sees that any sequence \((\theta_n : G_\Sigma \to G)_{n \in \mathbb{N}}\) such that \(\theta_n\) coincides on \(U\) with \(\varphi_n\) up to conjugation is also a \((\sigma'_1, ..., \sigma'_p)\)-test sequence for some \((\sigma'_1, ..., \sigma'_p) \in \text{Aut}_G(E(G))^p\).

**Remark 4.11.** Note that any subsequence of a \((\sigma_1, ..., \sigma_p)\)-test sequence is a \((\sigma_1, ..., \sigma_p)\)-test sequence as well.

The following easy lemma will be useful in the sequel.

**Lemma 4.12.** Let \((\varphi_n)_{n \in \mathbb{N}}\) be a \((\sigma_1, ..., \sigma_p)\)-test sequence. For every infinite subset \(A \subset \mathbb{N}\) and every integer \(1 \leq i \leq p\), we have

\[
\bigcap_{n \in A} \Lambda(\varphi_n(x_i)) = E(G).
\]

**Proof.** Suppose that \(g\) belongs to \(\Lambda(\varphi_n(x_i))\) for every \(n \in A\). Then, there exists an integer \(k_n\) and an element \(g_n \in E(G)\) such that \(g = \varphi_n(x_i)^{k_n} g_n\), for every \(n \in A\).

Now, observe that \(k_n\) must be equal to 0 for every \(n\) large enough, otherwise (up to extracting a subsequence) \(||\varphi_n(x_i)^{k_n}||\) goes to infinity, and so does the constant \(||g||\), which is a contradiction. It follows that \(g\) belongs to \(E(G)\). \(\square\)

Let \(G\) be an acylindrically hyperbolic group. We keep the same notations as above. Let \((\varphi_n : G_\Sigma \to G)_{n \in \mathbb{N}}\) be a \((\sigma_1, ..., \sigma_p)\)-test sequence. Let \(L\) be the quotient of \(G_\Sigma\) by the stable kernel of the sequence \((\varphi_n)_{n \in \mathbb{N}}\), and let \(\varphi_\infty : G_\Sigma \to L\) be the corresponding epimorphism. Let \(S\) be a finite generating set of \(L\) containing the images of \(x\) and \(a\) in \(L\) (still denoted by \(x\) and \(a\)) and let \((X, d)\) be a hyperbolic Cayley graph of \(G\) on which \(G\) acts acylindrically and non-elementarily. By Remark 4.9, the sequence \((\varphi_n)_{n \in \mathbb{N}}\) is divergent, and hence the construction of the real tree described in Subsection 3.1 is applicable. From now on, we denote by \(T_L\) this real tree, and we denote by \(o\) its base point.

**Lemma 4.13.** Define \(K = \{g \in G_\Sigma \mid \varphi_n(g) \in E(G) \omega\text{-almost surely}\}\) and \(F = \varphi_\infty(K)\). This group \(F\) is finite. Moreover, it is the unique maximal finite subgroup of \(L\) normalised by \(\varphi_\infty(x_i)\), for every component \(x_i\) of \(x\), \(1 \leq i \leq p\).

**Proof.** Note that \(F\) is clearly finite. Indeed for every finite subset \(K' \subset K\) one has \(|\varphi_\infty(K')| \leq |E(G)|\) because \(\varphi_n(K')\) is contained in \(E(G)\) \(\omega\)-almost surely, and hence \(|\varphi_\infty(K)| \leq |E(G)|\). We will prove that \(F\) is the unique maximal finite subgroup of \(L\) normalised by \(\varphi_\infty(x_i)\) for every \(1 \leq i \leq p\).

First, we will prove the existence of a unique maximal finite subgroup of \(L\) normalised by \(\varphi_\infty(x_i)\), denoted by \(F_i\). Then we will prove that \(F_i = F\). Let \(\{F_{i,j}\}_{j \in J}\) be the collection of all the finite subgroups of \(L\) that are normalised by \(\varphi_\infty(x_i)\), let \(A_i = \bigcup_{j \in J} F_{i,j}\) and let \(F_i\) be the subgroup of \(L\) generated by \(A_i\). Let \(K_i = \varphi_\infty^{-1}(F_i)\). We claim that \(K_i\) is contained in \(K\), that is, that \(\varphi_n(k)\) belongs to \(E(G)\) \(\omega\)-almost surely for every \(k \in K_i\). Suppose that \(k \in K_i\) is a preimage of an element of \(A_i\). By definition of \(A_i\), \(\varphi_\infty(k)\) is contained in a finite subgroup of \(L\) normalised by \(\varphi_\infty(x_i)\). As a consequence, there exists an integer \(m \geq 1\) such that \(\varphi_\infty([k, x_i^m]) = 1\). It follows that \(\varphi_n([k, x_i^m])\) is trivial \(\omega\)-almost surely. By Lemma 2.3, this implies that \(\varphi_n(k)\) belongs to \(\Lambda(\varphi_n(x_i))\) \(\omega\)-almost surely. Now, recall that \(\Lambda(\varphi_n(x_i))\) is generated by \(\varphi_n(x_i)\) and \(E(G)\) by definition of a test sequence. In particular, \(\Lambda(\varphi_n(x_i))\) is \(E(G)\)-by-\(Z\), which proves that \(\varphi_n(k)\) is contained in \(E(G)\) since \(\varphi_n(k)\) has finite order by definition of \(A_i\). This fact remains true if \(k\) is any element of \(K_i\) (not necessarily
a preimage of an element of \( A_i \) because \( F_i \) is generated by \( A_i \), and hence \( K_i \) is contained in \( K \). Since \( F \) is finite (by the previous paragraph), \( F_i \) is finite as well. Moreover, by construction, \( F_i \) is the unique maximal finite subgroup of \( L \) normalised by \( \varphi_\infty(x_i) \).

We will prove that \( K_i \) and \( K \) coincide (in particular, \( K_i \) does not depend on \( i \)). Recall that the inclusion \( K_i \subset K \) was proved in the previous paragraph. Now, let \( k \) be an element of \( K \), and let us prove that \( k \) belongs to \( K_i \). Let us define a subgroup \( K_i' \) of \( G_\Sigma \) as follows: \( K_i' = \langle \{ x_i^k x_i^{-1}\ell, \ell \in \mathbb{N} \} \rangle \). By definition of \( K \) the element \( \varphi_n(k) \) belongs to \( E(G) \) \( \omega \)-almost surely, and hence \( \varphi_n(x_i^k x_i^{-1}\ell) \) belongs to \( E(G) \) \( \omega \)-almost surely since \( E(G) \) is normal in \( G \). It follows that \( \varphi_n(K_i') \) is a subgroup of \( E(G) \) \( \omega \)-almost surely. In addition, this subgroup is normalised by \( \varphi_n(x_i) \) by construction. As a consequence, \( \varphi_\infty(K_i') \) is contained in \( F_i \). In particular, \( \varphi_\infty(k) \) belongs to \( F_i \), which implies that \( k \) belongs to \( K_i \). Hence, one has \( K_i = K \) for any integer \( 1 \leq i \leq p \) and so the finite groups \( F_1, \ldots, F_p \) are all equal to \( F \).

In the sequel, we abuse notation and denote by \( x_i \) both the element of \( G_\Sigma \) and its image in \( L \) under \( \varphi_\infty \).

**Lemma 4.14.** We keep the same notations and assumptions as above. Let \( Y \subset L \) be the stabiliser of the base point \( o \) in \( T_L \). Note that \( Y \) contains each element of the tuple \( a \). Indeed, each morphism \( \varphi_n \) restricts to a conjugation on \( a \). Suppose that the subgroup \( \Gamma = \langle x \cup Y \rangle \) of \( L \) does not fix a point in \( T_L \). Then the minimal subtree \( T_\Gamma \subset T_L \) of \( \Gamma \) is simplicial. Moreover, one has a precise description of the graph of groups \( T_\Gamma/\Gamma \): its underlying graph is a rose, its unique vertex group is \( Y \), every edge group coincides with \( F \) (which is the finite subgroup of \( L \) defined in the previous lemma), and the stable letters are the components \( x_1, \ldots, x_p \) of \( x \). It follows that \( \Gamma \) admits a splitting of the form

\[
\Gamma = Y *_F \langle x, F | \text{ad}(x_i)|_F = \alpha_i, \forall i \in [1, p] \rangle,
\]

where \( \alpha_i \) denotes the automorphism of \( F \) induced by the action of the stable letter \( x_i \) on \( F \).

**Proof.** Suppose that \( \Gamma \) does not fix a point of \( T_L \) and let \( T_\Gamma \subset T_L \) be the minimal subtree of \( \Gamma \). We first prove that each \( x_i \) acts hyperbolically on \( T_\Gamma \). Note that there exists an integer \( 1 \leq i \leq p \) such that \( ||\varphi_n(x_i)||/\lambda_n \) does not approach 0 as \( n \) goes to infinity. Otherwise \( \Gamma \), which is generated by \( Y \) and \( x \), would be elliptic in \( T_L \). Moreover, by the third condition of Definition 4.8, \( ||\varphi_n(x_i)||/||\varphi_n(x_j)|| \) tends to a real number \( r_{i,j} \in (0, +\infty) \), for every \( 1 \leq i, j \leq p \). Consequently every \( x_i \) is hyperbolic. We denote by \( \ell_i \) the limit of the sequence \( (||\varphi_n(x_i)||/\lambda_n)_{n\in\mathbb{N}} \), for every \( 1 \leq i \leq p \). Note that one has \( 0 < \ell_i < +\infty \).

Now, consider the subset \( T_\Gamma \) defined as the union of the axes of all the hyperbolic elements of \( \Gamma \). Let us prove that \( T \) is connected, that is, that \( T \) is a subtree of \( T_\Gamma \). Let \( p_1 \) and \( p_2 \) be two points of \( T \). By definition of \( T \), there exist two hyperbolic elements \( \gamma_1 \) and \( \gamma_2 \) of \( \Gamma \) such that \( p_1 \) belongs to the axis of \( \gamma_1 \) and \( p_2 \) belongs to the axis of \( \gamma_2 \). If these two axes are not disjoint, there is nothing to do. If they are disjoint, then it is well-known that \( \gamma_1 \gamma_2 \) is hyperbolic and that its axis \( A(\gamma_1 \gamma_2) \) intersects \( A(\gamma_1) \) and \( A(\gamma_2) \). Hence, \( T \) is a subtree of \( T_\Gamma \). In addition, \( T \) is \( \Gamma \)-invariant since \( \gamma_1 A(\gamma_2) = A(\gamma_1 \gamma_2 \gamma_1^{-1}) \) for every \( \gamma_1 \in \Gamma \) and every hyperbolic element \( \gamma_2 \in \Gamma \). By minimality of \( T_\Gamma \), one has \( T = T_\Gamma \). We will prove that \( T_\Gamma \) is a simplicial tree. \( \square \)

**Claim 1.** For every integers \( 1 \leq i, j \leq p \) and for every \( s \in Y \), if the intersection of the axes of \( x_j \) and \( sx_j s^{-1} \) contains two distinct points \( u \) and \( w \), then \( i = j \) and \( s \) belongs to \( F \).
Let us prove this claim. By assumption, the arc \([v, w]\) is contained in the intersection of the axes of \(sx_i s^{-1}\) and \(x_j\). Let \(\eta\) be the length of \([v, w]\) in the limiting tree \(T_\Gamma\). Let \(\bar{s}\) be a preimage of \(s\) in \(G_\Sigma\). The overlap \(\Delta(\varphi_n(\bar{s} x_i s^{-1}), \varphi_n(x_j))\) is comparable to \(\eta \lambda_n\) when \(n\) is large. As a consequence, \(\omega\)-almost surely, the following inequality holds:

\[
\Delta(\varphi_n(\bar{s} x_i s^{-1}), \varphi_n(x_j)) \geq \eta \lambda_n / 2.
\]

Moreover, the translation length \(||\varphi_n(x_i)||\) is equivalent to \(\ell_j \lambda_n\). Thus, \(\omega\)-almost surely, one has

\[
\Delta(\varphi_n(\bar{s} x_i s^{-1}), \varphi_n(x_j)) \geq \frac{\eta}{4 \ell_i} ||\varphi_n(x_i)||.
\]

According to the fourth condition of Definition 4.8, the tuple \((\varphi_n(x_1), ..., \varphi_n(x_p))\) satisfies the \(\varepsilon_n\)-small cancellation condition for some sequence of positive real numbers \((\varepsilon_n)_{n \in \mathbb{N}}\) converging to 0; \(\omega\)-almost surely, \(\varepsilon_n\) is smaller than \(\eta / (4 \ell_i)\). As a consequence, the integers \(i\) and \(j\) are equal, \(\omega\)-almost surely, so \(s = \varphi_\infty(\bar{s})\) belongs to \(F = \varphi_\infty(H)\), where \(H\) is the subgroup from Lemma 4.13.

**Claim 2.** For every \(1 \leq i \leq p\), the pointwise stabiliser of the axis \(A(x_i)\) of \(x_i\) is the finite group \(F\), and its setwise stabiliser \(Z_i\) is equal to \(\langle x_i, F \rangle\). Moreover, \(A(x_i)\) is transverse to its translates, which means that for every \(\gamma \in \Gamma \setminus Z_i\), the intersection of \(A(x_i)\) with \(A(\gamma x_i \gamma^{-1})\) is empty or reduced to a point.

Let us prove the second claim. Consider an element \(\gamma \in \Gamma\) such that the intersection of \(A(x_i)\) with \(A(\gamma x_i \gamma^{-1})\) contains two distinct points \(v\) and \(w\). Let \(\bar{\gamma}\) be a preimage of \(\gamma\) in \(G_\Sigma\). As in the proof of Claim 1, \(\varphi_n(\bar{\gamma})\) is contained in \(\Lambda(\varphi_n(x_i)) = E(G) \rtimes (\varphi_n(x_i))\). Hence, for every \(n\), there exists an integer \(p_n\) such that \(\varphi_n(\bar{\gamma} x_i^{p_n})\) belongs to \(E(G)\) \(\omega\)-almost surely. Note that the sequence \((p_n)_{n \in \mathbb{N}}\) is bounded \(\omega\)-almost surely, since \(||\varphi_n(x_i)||/\lambda_n\) stays away from 0 \(\omega\)-almost surely. In particular, \(p_n\) is constant \(\omega\)-almost surely, equal to a certain integer \(p\). It follows that \(\gamma x_i^p\) fixes the base point \(o\). In other words, \(\gamma\) is equal to \(sx_i^{-p}\) for some element \(s \in Y\). Now, observe that one has \(\gamma x_i^p = sx_i s^{-1}\). Since the intersection of \(A(x_i)\) with \(A(\gamma x_i \gamma^{-1})\) is neither empty nor reduced to a point, Claim 1 implies that \(s\) belongs to \(F\). This shows that \(Z_i = \langle x_i, F \rangle\) and that \(A(x_i)\) is transverse to its translates, which completes the proof of Claim 2.

We are now ready to prove that \(T_\Gamma\) is a simplicial tree. First, prove that the number of orbits of directions at branch points in \(T_\Gamma\) is finite by using Theorem 4.5. Note that \(\Gamma\) is countable since it is a subgroup of the finitely generated group \(L\) (which is a quotient of \(G_\Sigma\)). We need to check that the two assumptions of Theorem 4.5 are satisfied, namely that the action of \(\Gamma\) on \(T_\Gamma\) is \(K\)-superstable for some constant \(K\) (that is, that every arc whose stabiliser has order greater than \(K\) is stable) and that arc stabilisers are finitely generated. In fact, we will prove the following stronger fact: arc stabilisers are contained in the finite group \(F\). Let \(I \subset T_\Gamma\) be an arc. Observe that there exists a subarc \(J \subset I\) that is contained in the axis of a hyperbolic element of \(\Gamma\). Indeed, let \(p_1, p_2\) be two distinct points of \(I\), and let \(\gamma_1, \gamma_2 \in \Gamma\) be two hyperbolic elements such that \(p_1 \in A(\gamma_1)\) and \(p_2 \in A(\gamma_2)\) (which exist since \(T_\Gamma = T\)). Since \(T_\Gamma\) is a tree, if \(A(\gamma_1)\) and \(A(\gamma_2)\) are not disjoint then the arc \([p_1, p_2]\) is contained in the union \(A(\gamma_1) \cup A(\gamma_2)\), and thus there exists a subarc \(J \subset [p_1, p_2]\) that is contained in \(A(\gamma_1)\) or in \(A(\gamma_2)\). If \(A(\gamma_1)\) and \(A(\gamma_2)\) are disjoint, then \(\gamma_1 \gamma_2\) is hyperbolic and \([p_1, p_2]\) is contained in \(A(\gamma_1 \gamma_2)\), in which case one can take \(J = [p_1, p_2]\). In any case, there exists
a subarc $J \subset I$ that is contained in the axis of a hyperbolic element $\gamma \in \Gamma$. Then, observe that the axis of $\gamma$ is obtained by concatenating subarcs of the axes of some conjugates of $x_1, \ldots, x_p$, and so $J$ contains a subarc $J'$ that is contained in the axis of a conjugate $hx_ih^{-1}$ of $x_i$ for some $1 \leq i \leq p$ and $h \in \Gamma$. Now, let $g$ be an element that fixes $I$ pointwise. In particular $g$ fixes the subarc $J'$ and it follows that $J'$ is contained in $A(hx_ih^{-1})$ and in $gA(hx_ih^{-1})$. Since $A(x_i)$ is transverse to its translates, $g$ belongs to $F$. Hence, arc stabilisers are contained in the finite group $F$, and so the assumptions of Theorem 4.5 are satisfied. Thus, the number of orbits of directions at branch points in $T_G$ is finite.

Now, let $I$ be an arc of $T_G$. We aim to prove that there are only finitely many branch points on $I$ in $T_G$. Without loss of generality, one can assume that the length of $I$ is smaller than the translation length $\ell_i$ of $x_i$ for every $1 \leq i \leq p$. Assume towards a contradiction that there are infinitely many branch points on $I$. Then there exist necessarily two non-degenerate subsegments $J_1$ and $J_2$ in $I$, with $J_1 \cap J_2 = \emptyset$, and an element $\gamma \in \Gamma$ such that $\gamma J_1 = J_2$. Up to taking subarcs of $J_1$ and $J_2$, one can assume that $J_1$ and $J_2$ are contained in the axes of two hyperbolic elements $\gamma_1$ and $\gamma_2$, respectively (same proof as above). Since the axes of $\gamma_1$ and $\gamma_2$ are obtained by concatenating subarcs of the axes of some conjugates of $x_1, \ldots, x_p$, one can assume (up to taking subarcs of $J_1$ and $J_2$) that $J_1$ and $J_2$ are contained in the axes of two conjugates of $x_{i_1}$ and $x_{i_2}$ for some $1 \leq i_1, i_2 \leq p$. By Claim 1, one has $i_1 = i_2$ and $\gamma$ belongs to $\langle F, x_{i_1} \rangle$ (up to conjugacy). This is a contradiction since $F$ fixes the axis of $x_{i_1}$ pointwise, and since the intersection of $J$ with $x_{i_1}I$ is empty of reduced to a point (because the length of $I$ is smaller the translation length $\ell_i$ of $x_i$ for every $1 \leq i \leq p$). Therefore, there are only finitely many branch points on $I$, which proves that the tree $T_G$ is simplicial.

Last, let us give a precise description of the graph of groups $T_G/\Gamma$. Since the pointwise stabiliser of the axis of $x_i$ is $F$ for every $1 \leq i \leq p$, and since $x$ and $Y$ generate $\Gamma$ by definition, it follows from the paragraphs above that the underlying graph of $T_G/\Gamma$ is a rose such that the unique vertex group is $Y$, every edge group is equal to $F$ and the stable letters are $x_1, \ldots, x_p$. Therefore, $\Gamma$ can be decomposed as an amalgamated product as follows:

$$\Gamma = Y \star_F \langle x, F \mid \text{ad}(x_i)_{|F} = \alpha_i, \forall i \in [1, p] \rangle,$$

where $\alpha_i$ denotes the automorphism of $F$ induced by the action of $x_i$ on $F$ by conjugation.

**Corollary 4.15.** With the same notations and the same hypotheses as in Lemma 4.14, the tree $T_G$ is transverse to its translates, that is, for every element $h \in L \setminus \Gamma$, the intersection $hT_G \cap T_G$ is at most one point. In addition, if $e$ is an edge of $T_G$, there are only finitely many branch points on $e$ in $T_L$.

**Proof.** Let $h$ be an element of $L$ such that $hT_G \cap T_G$ is non-degenerate. As a consequence of the description of $T_G$ above, we can find two elements $u, v \in \Gamma$ such that the axes of $ux_iu^{-1}$ and $h(vx_jv^{-1})h^{-1}$ have a non-trivial overlap in the limiting tree $T_L$, for some $1 \leq i, j \leq p$, possibly equal. We denote by $\tilde{u}, \tilde{v}, \tilde{h}$ three preimages of $u, v, h$ in $G_2$. As in the previous proof, we have

$$\Delta(\varphi_n(x_i), \varphi_n(\tilde{u}^{-1}\tilde{h}\tilde{v})\varphi_n(x_j)\varphi_n(\tilde{u}^{-1}\tilde{h}\tilde{v})^{-1}) \geq \varepsilon_n \min(||\varphi_n(x_i)||, ||\varphi_n(x_j)||).$$

$\omega$-almost surely. Hence, the integers $i$ and $j$ are equal, and $\varphi_n(\tilde{u}^{-1}\tilde{h}\tilde{v})$ belongs to the group $\Lambda(\varphi_n(x_i)) = E(G) \times \langle \varphi_n(x_i) \rangle$. So, for every $n$, there is an integer $p_n$ such that $\varphi_n(\tilde{u}^{-1}\tilde{h}\tilde{v}x_i^{p_n})$ belongs to $E(G)$. On the other hand, since $x_i$ acts hyperbolically on $T_G$, the integer $p_n$ is bounded by a constant that does not depend on $n$. Otherwise, $||\varphi_n(x_i)||/||\varphi_n(\tilde{u}^{-1}\tilde{h}\tilde{v})|| \omega$-tends to 0. Hence,
since $\varphi_n(\overline{u^{-1}hv})/\lambda_n$ is bounded, $||\varphi_n(x_i)||/\lambda_n$ tends to 0, contradicting that $x_i$ is hyperbolic. As a consequence, one can assume that $p_n = p$ for all $n$. Thus, the image $\varphi_n(\overline{u^{-1}hv})$ belongs to $E(G) - \omega$-almost surely, that is, $\overline{u^{-1}hv}x^p_i$ belongs to $H$ and $u^{-1}hv_p$ belongs to $\varphi_\infty(H) = F$. Hence, there is an element $f \in F$ such that $u^{-1}hv_p^i = f$, that is $h = uf x^{-p}_i v^{-1}$. This element belongs to $\Gamma$ since $u, v, f$ and $x_i$ belong to $\Gamma$.

Last, let $e$ be an edge of $T_\Gamma$. One can assume without loss of generality that $e$ is adjacent to the base point $o$ (since $T_\Gamma$ is simplicial). Let us prove that there are only finitely many branch points on $e$ in $T_L$. First, note that the stabiliser of $e$ in $L$ is contained in the finite group $F$. Indeed, if an element $g \in L$ fixes the edge $e$ then it fixes the base point $o$, whose stabiliser is $Y \subset \Gamma$. Hence, $g$ belongs to the stabiliser of $e$ in $\Gamma$, and we proved in the proof of the previous lemma that this stabiliser is equal to $F$. This shows that the hypotheses of Theorem 4.5 are satisfied. It follows from this theorem that the number of orbits of directions at branch points in $T_L$ is finite. Now, assume towards a contradiction that there are infinitely many branch points on $e$. Then there exist necessarily two non-degenerate subsegments $I$ and $J$ in $e$, with $I \cap J = \emptyset$, and an element $g \in L$ such that $g I = J$. But we just proved that $T_\Gamma$ is transverse to its translates, so $g$ belongs to $\Gamma$. This is a contradiction since $T_\Gamma$ is a simplicial tree (by the previous lemma).

## 5 MERZLYAKOV’S THEOREM

The main theorem proved in this paper is the following generalisation of Merzlyakov’s theorem. Note that Theorem 1.11 stated in the introduction corresponds to the case where $\ell' = 1$ in the result below.

**Theorem 5.1.** Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$ (called constants). Fix a presentation $\langle a | R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\vartheta(x, y, a) : \bigvee_{k=1}^{\ell'} (\Sigma_k(x, y, a) = 1 \wedge \Psi_k(x, y, a) \neq 1)$$

be a finite disjunction of finite systems of equations and inequalities over $G$, where $x$ and $y$ are two tuples of variables. For every $1 \leq k \leq \ell'$, let $G_{2k}$ denote the following group, finitely presented relative to $\langle a | R(a) = 1 \rangle$:

$$\langle x, y, a | R(a) = 1, \Sigma_k(x, y, a) = 1 \rangle.$$

Let $p = |x|$ be the arity of $x$, and let $x_i$ denote the $i$th component of $x$. Suppose that $G$ satisfies the following first-order sentence:

$$\forall x \exists y \bigvee_{k=1}^{\ell'} (\Sigma_k(x, y, a) = 1 \wedge \Psi_k(x, y, a) \neq 1).$$

Then, for every $p$-tuple $\sigma = (\sigma_1, \ldots, \sigma_p) \in \text{Aut}_G(E(G))^p$, there exists an integer $1 \leq k \leq \ell'$ and a morphism

$$\pi_\sigma : G_{2k} \to G_\sigma = G *_{E(G)} \langle x, E(G) | \text{ad}(x_i)_{|E(G)} = \sigma_i, \forall i \in [1, p] \rangle.$$
such that the following hold:

1. $\pi_{\sigma}(x) = x$,
2. $\pi_{\sigma}(a) = a$,
3. $\Psi(x, \pi_{\sigma}(y), a) \neq 1$.

Moreover, the image of $\pi_{\sigma}$ is a subgroup of $G_{\sigma}$ of the form

$$\langle g, a \rangle \ast_{E(G)} \langle x, E(G) \mid \text{ad}(x_i)_{E(G)} = \sigma_i, \forall i \in [1, p] \rangle$$

for some tuple $g$ of elements of $G$.

In fact, as shown by Lemma 5.3, it is enough to prove the following result, which is \textit{a priori} weaker than Theorem 5.1 since the group $E(G)$ is replaced with a subgroup $E \subset E(G)$ that may be proper.

\textbf{Theorem 5.2.} Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$ (called constants). Fix a presentation $\langle a \mid R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\theta(x, y, a) : \bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1)$$

be a finite disjunction of finite systems of equations and inequations over $G$, where $x$ and $y$ are two tuples of variables. For every $1 \leq k \leq \ell$, let $G_{\Sigma_k}$ denote the following group, finitely presented relative to $\langle a \mid R(a) = 1 \rangle$:

$$\langle x, y, a \mid R(a) = 1, \Sigma_k(x, y, a) = 1 \rangle.$$

Let $p = |x|$ be the arity of $x$, and let $x_i$ denote the $i$th component of $x$. Suppose that $G$ satisfies the following first-order sentence:

$$\forall x \exists y \bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1).$$

Then, for every $p$-tuple $\sigma = (\sigma_1, \ldots, \sigma_p) \in \text{Aut}_G(E(G))^p$, there exists an integer $1 \leq k \leq \ell$, a finite subgroup $E$ of $E(G)$, and a morphism

$$\pi_{\sigma} : G_{\Sigma_k} \to G_{\sigma} = \langle G, x \mid \text{ad}(x_i)_{E} = \sigma_i_{\mid E}, \forall i \in [1, p] \rangle$$

such that the following hold:

1. $\pi_{\sigma}(x) = x$,
2. $\pi_{\sigma}(a) = a$,
3. $\Psi(x, \pi_{\sigma}(y), a) \neq 1$.

Moreover, the image of $\pi_{\sigma}$ is a subgroup of $G_{\sigma}$ of the form

$$\langle g, a \rangle \ast_{E} \langle x, E \mid \text{ad}(x_i)_{E} = \sigma_i_{\mid E}, \forall i \in [1, p] \rangle$$

for some tuple $g$ of elements of $G$. 
Lemma 5.3. Theorems 5.1 and 5.2 are equivalent.

Proof. Theorem 5.2 follows immediately from Theorem 5.1. Let us prove the converse. The proof consists in slightly modifying the first-order formula $\forall x \exists y \theta(x, y, a)$. Let $b$ be the tuple of elements of $G$ composed of $a$ and $E(G)$, and let $\sigma_1, \ldots, \sigma_N$ be an enumeration of the elements of $\text{Aut}_{E(G)}$. For $1 \leq i \leq N$, let $\mu_i(x, y, b)$ be the quantifier-free formula saying '$\theta(x, y, a)$ is true and $x$ acts on $E(G)$ as $\sigma_i$'. Since $\forall x \exists y \theta(x, y, a)$ holds in $G$, the following first-order sentence holds in $G$ as well:

$$\forall x \exists y \bigvee_{i=1}^{N} \mu_i(x, y, b).$$

Theorem 5.1 follows from Theorem 5.2 applied to this new first-order sentence. □

6 | PROOF OF MERZLYAKOV’S THEOREM 5.2 IN A PARTICULAR CASE

In this section, we deal with the case where $(\sigma_1, \ldots, \sigma_p) = (\text{id}_{E(G)}, \ldots, \text{id}_{E(G)})$. More precisely, we prove the following result, which is a partial version of Theorem 5.2.

Theorem 6.1. Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$ (called constants). Fix a presentation $\langle a | R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1)$$

be a finite disjunction of finite system of equations and inequalities over $G$, where $x$ and $y$ are two tuples of variables. For every $1 \leq k \leq \ell$, let $G_{\Sigma_k}$ denote the following group, finitely presented relative to $\langle a | R(a) = 1 \rangle$:

$$\langle x, y, a | R(a) = 1, \Sigma_k(x, y, a) = 1 \rangle.$$ 

Let $p = |x|$ be the arity of $x$, and let $x_i$ denote the $i$th component of $x$. Suppose that $G$ satisfies the following first-order sentence:

$$\forall x \exists y \bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1).$$

Then there exists an integer $1 \leq k \leq \ell$, a finite subgroup $E$ of $E(G)$, and a morphism

$$\pi_\sigma : G_{\Sigma_k} \rightarrow G_\sigma = \langle G, x | \text{ad}(x_i)_{E} = \text{id}_E, \forall i \in [1, p] \rangle$$

such that the following hold:

- $\pi_\sigma(x) = x$, 

- $\pi_\sigma(y) = y$, 

- $\pi_\sigma(a) = a$, 

- $\pi_\sigma(E) = E$. 

- $\pi_\sigma(R(a)) = R(a)$.
\[ \pi_\varphi(a) = a, \]
\[ \Psi(x, \pi_\varphi(y), a) \neq 1. \]

Moreover, the image of \( \pi_\varphi \) is a subgroup of \( G_\varphi \) of the form

\[ \langle g, a \rangle *_{E} \langle x, E \mid \text{ad}(x_i)|_E = \text{id}_E, \forall i \in [1, p] \rangle \]

for some tuple \( g \) of elements of \( G \).

Recall that a \((\text{id}_E(G), \ldots, \text{id}_E(G))\)-test sequence is simply called a test sequence. First, we build a test sequence enjoying two special properties.

### 6.1 Construction of a test sequence

The construction relies crucially on the existence of a quasi-isometrically embedded subgroup of \( G \) of the form \( F(a, b) \times E(G) \) (provided by [13, Theorem 6.14] together with [4, Lemma 3.1]), which will enable us to use small cancellation within \( F(a, b) \).

**Proposition 6.2.** Let \( G \) be an acylindrically hyperbolic group, and let \( a \) be a tuple of elements of \( G \). Fix a presentation \( \langle a \mid R(a) = 1 \rangle \) for the subgroup of \( G \) generated by \( a \). Let

\[ \bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1) \]

be a finite disjunction of finite system of equations and inequations over \( G \), where \( x \) and \( y \) are two tuples of variables. For every \( 1 \leq k \leq \ell \), denote

\[ G_{\Sigma_k} = \langle x, y, a \mid R(a) = 1, \Sigma_k(x, y, a) = 1 \rangle. \]

Suppose that \( G \) satisfies the following first-order sentence:

\[ \forall x \exists y \bigvee_{k=1}^{\ell} (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1). \]

Then, there exists an integer \( 1 \leq k \leq \ell \) and a test sequence \( (\varphi_n : G_{\Sigma_k} \to G)_{n \in \mathbb{N}} \) satisfying the following two conditions \( \omega \)-almost surely:

1. no component of the system of inequations \( \Psi(x, y, a) \) is killed by \( \varphi_n \),
2. and the morphism \( \varphi_n \) maps \( a \) to \( a \) (not only to a conjugate).

**Proof.** By [13, Theorem 6.14], there exists a hyperbolically embedded subgroup \( H \hookrightarrow_h G \) (see [13, Definition 2.1]) such that \( H = F(a, b) \times E(G) \), where \( F(a, b) \) denotes the free group on two generators \( a \) and \( b \), and the elements \( a \) and \( b \) are hyperbolic in \( G \).

By Theorem 2.5, there exists a (possibly infinite) generating set \( S \) of \( G \) such that the Cayley graph of \( G \) with respect to \( S \) is hyperbolic and such that the natural action of \( G \) on this Cayley graph is non-elementary and acylindrical. By [27, Lemma 5.1], the conclusion of Theorem 2.5 is
still satisfied for \( S \cup \{a, b\} \) instead of \( S \). Up to replacing \( S \cup \{a, b\} \), one can assume without loss of generality that \( a \) and \( b \) belong to \( S \). Let \( (X, d) \) be the Cayley graph of \( G \) with respect to this enlarged set \( S \).

Let \( d' \) denote the metric in the free group \( \langle a, b \rangle \) for the generating set \( \{a, b\} \). By [4, Lemma 3.1], there exist two constants \( q \) and \( r \) such that

\[
d'(1, h) \leq q d(1, h) + r \tag{1}
\]

for all \( h \in \langle a, b \rangle \).

Let \( p \) denote the arity of \( x \). For any integers \( 1 \leq i \leq p \) and \( n \geq 0 \), we define \( g_{i,n} = a^{(i-1)n+1} b a^{(i-1)n+2} b \cdots a^{in} b \). Let \( g_n \) be the \( p \)-tuple \( (g_{1,n}, \ldots, g_{p,n}) \).

There exists an integer \( 1 \leq k \leq \ell \) such that, for infinitely many integers \( n \), there exists a tuple \( h_n \) of elements of \( G \) such that \( \Sigma_k(g_{n, h_n}, a) = 1 \) and \( \Psi_k(g_{n, h_n}, a) \neq 1 \). By passing to a subsequence and relabelling, one can assume without loss of generality that this system of equalities and inequalities holds for all integers \( n \).

Let \( \varphi_n : \Gamma \rightarrow G \) be the morphism defined by \( \varphi_n(x) = g_n, \varphi_n(y) = h_n \) and \( \varphi_n(a) = a \). We will prove that \( (\varphi_n)_{n \in \mathbb{N}} \) is a test sequence.

For \( 1 \leq i \leq p \) and \( n \geq 0 \), let \( \tau_{i,n} \) be the path of \( X \) that links \( 1 \) to \( g_{i,n} \) and is labeled with the word \( g_{i,n} \) in \( a \) and \( b \), and consider the bi-infinite path \( \tau_{i,n} = \bigcup_{k \in \mathbb{Z}} g_{k,i,n} \tau_{i,n} \). This path \( \tau_{i,n} \) is a geodesic in the Cayley graph of the free group \( \langle a, b \rangle \) equipped with the metric \( d' \), and by the inequality 1 this graph is quasi-isometrically embedded into \( (X, d) \), thus \( \tau_{i,n} \) is a quasi-geodesic in \( (X, d) \), for some constants that do not depend on \( n \). Consequently, \( \tau_{i,n} \) lies in the \( \lambda \)-neighbourhood of the quasi-axis \( A(g_{i,n}) \) of \( g_{i,n} \) for some constant \( \lambda \geq 0 \) independent from \( n \). Similarly, let \( \alpha \) be the edge of \( X \) linking \( 1 \) to \( a \), let \( \alpha = \bigcup_{k \in \mathbb{Z}} a^k \alpha \) and let \( \mu \) be a constant such that \( \alpha \) lies in the \( \mu \)-neighbourhood of \( A(a) \).

Since \( g_{i,n} \) is cyclically reduced in \( \langle a, b \rangle \), an easy calculation shows that \( d'(1, g_{i,n}) \sim (i - 1/2)n^2 \). Thus, the second and third conditions of Definition 4.8 hold. In addition, note that it follows from the inequality (1) that there exists a constant \( R > 0 \) such that \( ||g_{i,n}|| \geq Rn^2 \) for all \( n \) large enough, and all \( i \in [1, p] \).

It remains to prove the fourth condition of Definition 4.8. Since \( a \) is hyperbolic, there exists a constant \( N \geq 0 \) such that, for every element \( g \in G \), if \( \Delta(a, ga g^{-1}) \geq N \), then \( g \) belongs to \( \Lambda(a) = \langle a \rangle \times E(G) \) (see Subsection 4.4). Let \( n_0 \) be an integer such that \( 16qRn_0 \) is large compared to \( N' = N + 204\delta + 2\lambda + 2\mu \), where \( q \) is the constant involved in the inequality (1). We will show that for every \( n \geq n_0 \), the tuple \( \varphi_n(x) = g_n \) satisfies the \((16q/n)\)-small cancellation condition (see Definition 4.7). Let \( n \) be an integer greater than \( n_0 \). Consider an element \( g \in G \) such that

\[
\Delta(g_{i,n}, gg_{j,n} g^{-1}) \geq 16q \min(||g_{i,n}||, ||g_{j,n}||)/n \tag{2}
\]

for some \( (i, j) \in [1, p]^2 \). We will show that \( i = j \) and that \( g \) belongs to the subgroup \( \langle g_{i,n} \rangle \times E(G) \). One can suppose without loss of generality that \( j \) is larger than \( i \). Thus, \( \omega \)-almost surely, one has \( \min(||g_{i,n}||, ||g_{j,n}||) = ||g_{j,n}|| \).

We first show that \( g \) belongs to the subgroup \( \langle a, b \rangle \times E(G) \). Since

\[
\Delta(g_{i,n}, gg_{j,n} g^{-1}) \geq 16q ||g_{i,n}||/n \geq 16qRn \geq 16qRn_0 \gg N',
\]
we can choose two subpaths \( \mu_{i,n} \) and \( \mu_{j,n} \) of \( \tau_{i,n} \) and \( g\tau_{j,n} \), respectively, of length \( N' \) and labelled by \( a^{N'} \), such that \( \text{diam}( (\mu_{i,n})^{+(100\delta+\lambda)} \cap (\mu_{j,n})^{+(100\delta+\lambda)}) \geq N' \). Denoting by \( o_{i,n} \) and \( o_{j,n} \) the initial points of \( \mu_{i,n} \) and \( \mu_{j,n} \), respectively, we have
\[
\text{diam}(o_{i,n} \alpha^{+(100\delta+\lambda)} \cap o_{j,n} \alpha^{+(100\delta+\lambda)}) \geq N'.
\]
It follows that
\[
\text{diam}(A(a)^{+(100\delta+\lambda+\mu)} \cap o_{i,n}^{-1} o_{j,n} A(a)^{+(100\delta+\lambda+\mu)}) \geq N'.
\]
By \cite[Lemma 2.13]{11}, we have
\[
\Delta(a, (o_{i,n}^{-1} o_{j,n})a(o_{i,n}^{-1} o_{j,n})^{-1}) \geq \text{diam}(A(a)^{+(100\delta+\lambda+\mu)} \cap o_{i,n}^{-1} o_{j,n} A(a)^{+(100\delta+\lambda+\mu)}) - (204\delta + 2\lambda + 2\mu) \\
\geq N' - (204\delta + 2\lambda + 2\mu) = N.
\]
It follows from this inequality that the element \( o_{i,n}^{-1} o_{j,n} \) belongs to \( \Lambda(a) = \langle a \rangle \times E(G) \). Now, observe that as \( o_{i,n} \) lies in \( \langle a, b \rangle \), it is on the quasi-geodesic \( \tau_{i,n} \). Similarly, \( o_{j,n} \) can be written as \( o_{j,n} = gw_{j,n} \) with \( w_{j,n} \) a word in \( a \) and \( b \). It follows that \( g \) belongs to the subgroup \( \langle a, b \rangle \times E(G) \).

Up to replacing \( g \) with \( gc \) for some \( c \in E(G) \), we can now assume that \( g \) belongs to the free group \( \langle a, b \rangle \). This does not affect the inequality \( \Delta(g_{i,n}, gg_{j,n}^{-1}) \geq 16q ||g_{i,n}||/n \); indeed, \( gcg_{j,n}(gc)^{-1} \) is equal to \( g_{j,n}g^{-1} \) since \( g_{j,n} \) centralises \( E(G) \), as an element of \( \langle a, b \rangle \).

Let \( Y \) be the Cayley graph of the free group \( \langle a, b \rangle \) equipped with the distance \( d' \). The following inequality can be easily deduced from the inequalities (1) and (2):
\[
\text{diam}( (\tau_{i,n})^{+(q(100\delta+r)+1)} \cap (g\tau_{j,n})^{+(q(100\delta+r)+1)}) \geq 16qd'(1, g_{i,n})/(2qn) = 8d'(1, g_{i,n})/n.
\]
Since \( Y \) is a tree, this inequality tells us that the axes of \( g_{i,n} \) and \( gg_{j,n}g^{-1} \) have an overlap of length larger than \( 8d'(1, g_{i,n})/n \) in this tree. Then, recall that \( d'(1, g_{i,n}) \) is asymptotically equivalent to \((i - 1/2)n^2\). Thus, \( 8d'(1, g_{i,n})/n \) is asymptotically equivalent to \( 8(i - 1/2)n \). Therefore, \( \omega \)-almost surely, the axes of \( g_{i,n} \) and \( gg_{j,n}g^{-1} \) have an overlap of length larger than \( 4(i - 1/2)n \) in the tree \( Y \).

To conclude, let us observe that \( 4(i - 1/2)n > 2in - 2 \), and that two distinct cyclic conjugates of \( g_{i,n} \) and \( g_{j,n} \) have at most their first \( 2in - 2 \) letters in common (recall that \( j \) is larger than \( i \) by assumption). Thus, if the axes of \( g_{i,n} \) and \( gg_{j,n}g^{-1} \) have a common subsegment in \( Y \) of length strictly larger than \( 2in - 2 \), then \( g_{i,n} \) and \( gg_{j,n}g^{-1} \) have the same axis. It follows that \( i = j \) and that \( g_{i,n} \) and \( g \) have a common root. Last, note that \( g_{i,n} \) has no root. It follows that \( g \) is a power of \( g_{i,n} \), which concludes the proof.

\[ \square \]

### 6.2 Proof of Theorem 6.1

Theorem 6.1 is an immediate consequence of Propositions 6.2 and 6.3 (applied with \((\sigma_1, \ldots, \sigma_p) = (id_{E(G)}, \ldots, id_{E(G)})\)).
Proposition 6.3. Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$. Fix a presentation $\langle a \mid R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\Sigma(x, y, a) = 1 \land \Psi(x, y, a) \neq 1$$

be a finite system of equations and inequations over $G$, where $x$ and $y$ are tuples of variables. Suppose that there exists a $(\sigma_1, \ldots, \sigma_p)$-test sequence $(\varphi_n : G_\Sigma \to G)_{n \in \mathbb{N}}$ satisfying the following two conditions $\omega$-almost surely:

1. no component of the system of inequations $\Psi(x, y, a)$ is killed by $\varphi_n$,
2. and the morphism $\varphi_n$ maps $a$ to $a$ (not only to a conjugate).

Then there exist a finite subgroup $E$ of $E(G)$ and a morphism

$$\pi_\sigma : G_\Sigma \to G_\sigma = \left\{ G, x \mid \text{ad}(x_i)_{|E} = \sigma_i_{|E}, \forall i \in [1, p] \right\}$$

such that the following hold:

- $\pi_\sigma(x) = x$,
- $\pi_\sigma(a) = a$,
- no component of the tuple $\Psi(x, y, a)$ is killed by $\pi_\sigma$.

Moreover, the image of $\pi_\sigma$ is a subgroup of $G_\sigma$ of the form

$$\langle g, a \rangle \ast_E \left\{ x, E \mid \text{ad}(x_i)_{|E} = \sigma_i_{|E}, \forall i \in [1, p] \right\}$$

for some tuple $g$ of elements of $G$.

Proof. By assumption, there exists a $(\sigma_1, \ldots, \sigma_p)$-test sequence $(\varphi_n : G_\Sigma \to G)_{n \in \mathbb{N}}$ that satisfies the following two conditions $\omega$-almost surely:

1. each component of $\varphi_n(\Psi(x, y, a))$ is non-trivial,
2. and the morphism $\varphi_n$ maps $a$ to $a$ (not only to a conjugate).

Let $U$ be the subgroup of $G_\Sigma$ generated by $x$ and $a$. Since the Cayley graph of $G$ with respect to $S$ (on which $G$ acts acylindrically and non-elementarily) is discrete, the length of any morphism $G_\Sigma \to G$ belongs to $\mathbb{N}$. As a consequence, there exists a sequence of morphisms $(\vartheta_n : G_\Sigma \to G)_{n \in \mathbb{N}}$ that satisfies simultaneously the following three conditions $\omega$-almost surely:

1. $\vartheta_n$ coincides with $\varphi_n$ on $U$ up to conjugation,
2. each component of $\vartheta_n(\Psi(x, y, a))$ is non-trivial,
3. and there is no morphism that satisfies simultaneously the conditions (1) and (2) above and that is strictly shorter than $\vartheta_n$.

Note that, by Remark 4.10, the sequence $(\vartheta_n)_{n \in \mathbb{N}}$ is a $(\sigma'_1, \ldots, \sigma'_p)$-test sequence for some $(\sigma'_1, \ldots, \sigma'_p) \in \text{Aut}_G(E(G))^p$, and hence it is divergent (by Remark 4.9). However, one cannot guarantee that $\sigma'_i$ coincides with $\sigma_i$. Moreover, $\vartheta_n$ maps $a$ to a conjugate of $a$, not necessarily to $a$ itself.

Let $L = G_\Sigma / \ker_\omega((\vartheta_n)_{n \in \mathbb{N}})$, and let $\theta_\infty : G_\Sigma \to L$ be the corresponding epimorphism. Observe that $\theta_\infty$ is injective on $U = \langle a, x_1, \ldots, x_p \rangle$: indeed, by Lemma 4.14, the limiting tree associated with the sequence $(\vartheta_n|U)_{n \in \mathbb{N}}$ is isometric to the Bass–Serre tree of the decomposition of $U$ as a
graph of groups whose underlying graph is a rose, such that the vertex group is \( \langle a \rangle \) and the edges groups are all equal to \( F \), with stable letters \( x_1, \ldots, x_p \). It follows that the sequence \( (\theta_n : G \to G)_{n \in \mathbb{N}} \) is discriminating since \( \theta_n \) is injective on the vertex group \( \langle a \rangle \) and since every element that does not belong to a conjugate of \( \langle a \rangle \) is hyperbolic in the limiting tree, and hence \( \theta_\infty \) is injective on \( U \). In the proof below, we abuse notation and denote by \( U \) the isomorphic image of \( U \) in the successive quotients of \( G_\Sigma \) involved in the construction of the formal solution \( \pi_x \).

In the rest of the proof, \( C \) denotes the constant defined in the Stability Lemma 3.6.

**A particular case.** For presentation purposes, we first present a proof of Proposition 6.3 in the particular case where \( L \) does not split non-trivially over a finite group of order less than \( C \). Under this assumption, if one assumes (towards a contradiction) that the group \( U \) is elliptic in the limiting tree of the test sequence \( (\theta_n : G \to G)_{n \in \mathbb{N}} \), then by Theorem 3.28 there exists a sequence of homomorphisms \( (\rho_n : G \to G)_{n \in \mathbb{N}} \) satisfying the following three conditions \( \omega \)-almost surely:

1. \( \rho_n \) coincides with \( \theta_n \) (and therefore with \( \varphi_n \)) on \( U \) up to conjugation,
2. \( \rho_n \) kills no component of the tuple \( \Psi(x, y, a) \),
3. \( \rho_n \) is strictly shorter than \( \theta_n \) relative to \( H \).

This contradicts the definition of \( \theta_n \) as the shortest morphism satisfying both conditions (1) and (2) \( \omega \)-almost surely. Hence, \( U \) is not elliptic in the limiting tree of the test sequence \( (\theta_n : G \to G)_{n \in \mathbb{N}} \). The conclusion now follows from the following technical lemma, whose proof is postponed (see Lemma 6.5 for a more general version).

**Lemma 6.4.** Let \( F \) be the finite subgroup of \( L \) defined in Lemma 4.13. If \( U \) is not elliptic in the limiting tree, then the group \( L \) admits a splitting \( S_L \) with exactly two vertex groups \( \langle e, a \rangle \) (for some tuple \( e \) of elements of \( L \)) and \( \langle x, F \rangle \), and one edge group \( F \). Let \( A \) be an \( S_L \)-approximation of \( L \) as in Proposition 3.13. There exist a finite subgroup \( E \) of \( E(G) \) and an epimorphism \( r \) from \( A \) onto a group of the form

\[
\langle g, a \rangle \ast_E \langle x, E \mid \ad(x_i)\rangle_{|E} = \sigma_i|_{E}, \forall i \in \llbracket 1, p \rrbracket,
\]

where \( g \) denotes a tuple of elements of \( G \), such that \( r(x) = x \), \( r(a) = a \) and \( r \) kills no component of the image in \( A \) of the tuple \( \Psi(x, y, a) \).

Last, one defines the formal solution \( \pi_x : G_\Sigma \to G \) by \( \pi_x = r \circ q \) where \( q \) denotes the natural epimorphism from \( G_\Sigma \) onto \( A \). This concludes the proof of Proposition 6.3 in the particular case where \( L \) does not split non-trivially over a finite group of order less than \( C \). In general, however, this hypothesis is not satisfied and one has to deal with complications arising from splittings over finite subgroups. In particular, one needs a strengthened version of the relative shortening argument Theorem 3.28, namely Theorem 3.31.

**General case.** Since we are going to describe an iterative process, let us rename \( \theta_n \) to \( \theta_n^0 \), and \( L \) to \( L_0 \). For any \( G \)-limit group \( L_i \) that appears in the proof, we denote by \( L_i^U \) the vertex group containing \( U \) in a reduced JSJ decomposition \( J_i \) of \( L_i \), relative to \( U \), over finite groups of order less than \( C \).

The proof of Proposition 6.3 consists in constructing the following diagrams \( \omega \)-almost surely (the objects appearing in this diagram are defined below). Note that not all parts of this diagram commute as it appears. That is, \( \theta_n^i \) is not equal to \( \theta_n^{i+1} \circ q_{i+1} \), but there are other maps \( \rho_n^i : A_i \to G \).
defined below such that $\theta_n^i = \rho_n^{i+1} \circ q_{i+1}$.

This diagram is built iteratively, as follows: given the divergent sequence $(\theta_n^i : A_i \to G)_{n \in \mathbb{N}}$, one defines $L_i$ by $L_i = A_i / \ker(\omega(\theta_n^i)_{n \in \mathbb{N}})$. Let $J_i$ be a reduced JSJ splitting of $L_i$ over finite groups of order less than $C$, let $R_i$ be the splitting of the vertex group $L_i^U$ as a graph of actions outputted by the Rips machine and let $R_Ji$ be the splitting of $L$ obtained from $J_i$ by replacing the vertex fixed by $L_i^U$ with the graph of groups $R_i$. Let $A_{i+1}$ be an $R_iJ$-approximation of $L_i$ given by Proposition 3.13 and Corollary 3.17, and let $\rho_n^{i+1} : A_{i+1} \to G$ be the factorisation of $\theta_n^i : A_i \to G$ through the natural epimorphism $q_{i+1} : A_i \to A_{i+1}$.

Since $A_{i+1}$ is an $R_iJ$-approximation of $L_i$, it is also a $J_i$-approximation of $L_i$ (indeed, one can collapse to a point the subgraph corresponding to $R_i$). We denote by $A_{i+1}^U$ the vertex group of the splitting of $A_{i+1}$ corresponding to $L_i^U$. Note that $A_{i+1}^U$, unlike $L_i^U$, may split non-trivially relative to $U$ over finite subgroups of order less than $C$. It remains to define the divergent sequence $(\theta_n^{i+1} : A_{i+1} \to G)_{n \in \mathbb{N}}$.

If $U$ is elliptic in the limiting tree of the sequence $(\rho_n^{i+1})_{n \in \mathbb{N}}$, then by Theorem 3.31 there exists a sequence of homomorphisms $(\theta_n^{i+1} : A_{i+1} \to G)_{n \in \mathbb{N}}$ satisfying the following three conditions $\omega$-almost surely:

1. $\theta_n^{i+1}$ coincides with $\rho_n^{i+1}$ (and therefore with $\varphi_n$) on $U$ up to conjugation,
2. $\theta_n^{i+1}$ kills no component of the image of the tuple $\Psi(x, y, a)$ in $A_{i+1}$,
3. and the restriction $\theta_n^{i+1} |_{A_{i+1}^U}$ is strictly shorter than the restriction $\rho_n^{i+1} |_{A_{i+1}^U}$, relative to $U$.

In addition, since the length of $\theta_n^{i+1}$ belongs to $\mathbb{N}$, one can assume without loss of generality that $\theta_n^{i+1}$ is the shortest morphism from $A_{i+1}$ to $G$ that satisfies the first two conditions above $\omega$-almost surely. Note that the sequence $(\theta_n^{i+1})_{n \in \mathbb{N}}$ is divergent since it is a test sequence (see Remarks 4.9 and 4.10).
Why does the iteration eventually terminate? We have to prove that there exists an integer \(i\) such that \(U\) is not elliptic in the limiting tree of the sequence \((\theta_{i+1}^{(n)})_{n \in \mathbb{N}}\).

Claim. There exists an integer \(i\) such that \(q_{i+1}(A_i^U) = A_{i+1}^U\).

Before proving this claim, let us explain how to complete the proof of Proposition 6.3. First, note that if \(q_{i+1}(A_i^U) = A_{i+1}^U\), then if one shortens the restriction of \(\theta_{i+1}^{(n)}\) to \(A_{i+1}^U\), one automatically shortens the restriction of \(\rho_i^{(n)}\) to \(A_i^U\), which is not possible by definition of \(\rho_i^{(n)}\). As a consequence, if \(q_{i+1}(A_i^U) = A_{i+1}^U\), then \(U\) cannot be elliptic in the limiting tree of the sequence \((\theta_{i+1}^{(n)})_{n \in \mathbb{N}}\), otherwise one could get a contradiction by means of Theorem 3.31. To construct the formal solution, we will use the following lemma, whose proof is postponed.

Lemma 6.5. Let \(F\) be the finite subgroup of \(L_{i+1}\) defined in Lemma 4.13. If \(U\) is non-elliptic in the limiting tree of the sequence \((\theta_{i+1}^{(n)})_{n \in \mathbb{N}}\), then the group \(L_{i+1}\) admits a splitting \(\mathbb{S}\) with exactly two vertex groups \((\ell, a)\) (for some tuple \(\ell\) of elements of \(L_{i+1}\)) and \((x, F)\), and one edge group \(F\). Let \(A\) be an \(S\)-approximation of \(L_{i+1}\) given by Proposition 3.13. There exists a subgroup \(E\) of \(E(G)\) and an epimorphism \(r\) from \(A\) onto a group of the form

\[
\langle g, a \rangle \ast_E \langle x, E \mid \text{ad}(x_i) \rangle = \sigma_i | E, \forall i \in [1, p]),
\]

where \(g\) denotes a tuple of elements of \(G\), such that \(r(x) = x, r(a) = a\) and \(r\) kills no component of the image in \(A\) of the tuple \(\Psi(x, y, a)\).

As a consequence of this lemma, if \(U\) is not elliptic in the limiting tree of the sequence \((\theta_{i+1}^{(n)})_{n \in \mathbb{N}}\), one can define the formal solution \(\pi_\sigma : G_\Sigma \to G\) by \(\pi_\sigma = r \circ q_{i+1} \circ q_i \circ \cdots \circ q_0\).

Therefore, in order to conclude the proof of Proposition 6.3, we just have to prove the claim according to which there exists an integer \(i\) such that \(q_{i+1}(A_i^U) = A_{i+1}^U\). Let us denote by \(\eta_i\) the number of edges in a reduced JSJ splitting \(J_i\) of \(L_i\) over finite groups of order less than \(C\), relative to \(U\). Let \(E(J_i)\) be the set of edges of \(J_i\). We make the following two observations.

**First observation:** Note that \(L_i\) is finitely generated as a quotient of the finitely generated group \(G_\Sigma\). Using a folding sequence argument, Dunwoody proved in [15] that the sum \(\sum_{e \in E(J_i)} 1/|L_i e|\), where \(L_i e\) denotes the edge group of \(e\), is smaller than the rank \(\text{rank}(L_i)\) of \(L_i\) (that is, the minimal number of generators of \(L_i\)). Therefore, for every integer \(i\), one has \(\eta_i \leq C \text{rank}(L_i)\). In addition, one has \(\text{rank}(L_i) \leq \text{rank}(G_\Sigma)\) since \(L_i\) is a quotient of \(G_\Sigma\). Thus, \(\eta_i\) is bounded from above by \(C \text{rank}(G_\Sigma)\).

**Second observation:** We claim that \(\eta_{i+1}\) is greater than \(\eta_i\), with equality if and only if \(q_{i+1}(A_i^U) = A_{i+1}^U\).

Let us prove this claim. Since \(A_{i+1}\) is a \(J_i\)-approximation of \(L_i\), there exists by definition a splitting \(J'_i\) of \(A_{i+1}\) with the same underlying graph as \(J_i\), and whose edge groups have the same order as the corresponding edge groups in \(J_i\). In particular, \(J'_i\) is a splitting over finite groups of order less than \(C\), with \(\eta_i\) edges. Moreover, by Proposition 3.13 and Lemma 6.6, the splitting \(J'_i\) is reduced since \(J_i\) is reduced.

To establish the inequality \(\eta_{i+1} \geq \eta_i\), let us have a closer look at the defining sequence \((\theta_{i+1}^{(n)} : A_{i+1}^{(n)} \to G)_{n \in \mathbb{N}}\) of \(L_{i+1} = A_{i+1}^{(n)} / \ker((\rho_{i+1}^{(n)})_{n \in \mathbb{N}})\). In the proof of Theorem 3.31, each morphism \(\theta_{i+1}^{(n)}\) is obtained by precomposing \(\rho_{i+1}^{(n)}\) by an automorphism \(\alpha\) of \(A_{i+1}\) (independent from \(n\)) whose restriction to \(A_{i+1}^U\) is a modular automorphism (or, to be more precise, a lift of a modular automorphism of \(L_i^U\)), and whose restriction to any other vertex group of \(J'_i\) is a conjugation. This automorphism \(\alpha\) is obtained by means of Lemma 3.22, using the fact that modular automorphisms
coincide with the identity up to conjugation on finite subgroups of \( A_{i+1}^U \). As a consequence, \( L_{i+1} \) admits a splitting \( \mathcal{J}'_i \) with \( \eta_i \) edge groups, over finite groups of order less than \( C \), obtained from the splitting \( \mathcal{J}'_i \) of \( A_{i+1}^U \) by replacing each vertex group by its image by the quotient map \( \theta^{i+1}_\infty \). This shows that a reduced JSJ splitting of \( L_{i+1} \) has at least \( \eta_i \) edges; in other words, one has \( \eta_{i+1} \geq \eta_i \).

Now, suppose that \( \eta_i = \eta_{i+1} \), and let us prove that \( \mathcal{J}''_i \) is a reduced JSJ splitting of \( L_{i+1} \) over finite groups of order less than \( C \). Since we already know that \( \mathcal{J}''_i \) is a splitting of \( L_{i+1} \) over finite groups of order less than \( C \) with \( \eta_{i+1} \) edges, we just have to prove that \( \mathcal{J}''_i \) is reduced. To this end, let us verify that the conditions of Lemma 6.6 are satisfied. By definition of \( \mathcal{J}''_i \), the natural epimorphism \( \theta^{i+1}_\infty \) from \( A_{i+1} \) onto \( L_{i+1} \) maps each vertex group of \( \mathcal{J}'_i \) onto the corresponding vertex group of \( \mathcal{J}''_i \).

We will prove the following two facts:

1. \( \theta^{i+1}_\infty \) is injective on finite vertex groups,
2. and \( \theta^{i+1}_\infty \) maps infinite vertex groups onto infinite vertex groups.

Let us consider the following diagram, where \( \pi_i \) denotes the natural epimorphism from \( A_{i+1} \) onto \( L_i \) (in other words, \( \pi_i = \rho^{i+1}_\infty \)):

Let us make the following observation: for each vertex group \( V \subset A_{i+1} \) of \( \mathcal{J}'_i \) that does not contain \( U \), the kernel of the restriction of \( \pi_i \) to \( V \) coincides with the kernel of the restriction of \( \theta^{i+1}_\infty \) to \( V \). Indeed, recall that \( \theta^{i+1}_\infty \) is obtained by precomposing \( \rho^{i+1}_\infty \) by an automorphism \( \alpha \) of \( A_{i+1} \) whose restriction to \( V \) is a conjugation. As a consequence, the vertex groups \( \pi_i(V) \) and \( \theta^{i+1}_\infty(V) \) are isomorphic. But we know that \( \pi_i(V) \) is infinite if and only if \( V \) is infinite, and that in addition \( \pi_i(V) \) and \( V \) are isomorphic if they are finite, by construction of \( \mathcal{J}'_i \) and \( A_{i+1} \) (see Proposition 3.13 and Corollary 3.16). Therefore, \( \theta^{i+1}_\infty(V) \) is infinite if and only if \( V \) is infinite. In addition, \( \theta^{i+1}_\infty(V) \) and \( V \) are isomorphic if they are finite. Last, note that the image by \( \theta^{i+1}_\infty \) of the vertex group of \( \mathcal{J}'_i \) containing \( U \) is infinite since \( U \) is infinite and since \( \theta^{i+1}_\infty \) is injective on \( U \). Hence, the conditions (1) and (2) above are satisfied. Thus, Lemma 6.6 applies and tells us that \( \mathcal{J}''_i \) is reduced.

Hence, if \( \eta_i = \eta_{i+1} \), then \( \mathcal{J}''_i \) is a reduced JSJ splitting of \( L_{i+1} \) over finite groups of order less than \( C \). It follows that the image of the vertex group \( A_{i+1}^U \) in \( L_{i+1} \) coincides with \( L_{i+1}^U \). Therefore, one has \( A_{i+1}^U = q_{i+1}(A_{i+1}^U) \).

**Lemma 6.6.** Let \( G \) and \( H \) be two groups, with two splittings \( \mathcal{S}_G \) and \( \mathcal{S}_H \) over finite groups. Let \( T_G \) and \( T_H \) denote the Bass–Serre trees of these splittings. Suppose that there exists an epimorphism \( \theta : G \to H \) and a \( \theta \)-equivariant bijection \( f : T_G \to T_H \) such that \( \theta \) is injective on finite vertex groups and maps infinite vertex groups onto infinite vertex groups. Then, the following implication holds: if \( T_G \) is reduced, then \( T_H \) is reduced.

**Proof.** Suppose that \( T_G \) is reduced. Let \( \varepsilon = [v, w] \) be an edge of \( T_H \) such that \( H_v = H_e = H_w \). We have to prove that \( w \) is a translate of \( v \), that is, there exists an element \( h \in H \) such that \( w = hv \). Let \( e = [x, y] \) be a preimage of \( \varepsilon \) by \( f \). Since \( H_e \) is finite, \( H_v \) and \( H_w \) are finite, thus \( G_x \) and \( G_y \) are finite (indeed, by assumption, \( \theta \) maps infinite vertex groups onto infinite vertex groups). Moreover, \( \theta \) being injective on finite vertex groups, one has \( G_x = G_e = G_y \). It follows that \( y = gx \) for some
\[ g \in G. \text{ One has } f(y) = w \text{ and, since } f \text{ is } \vartheta\text{-equivariant, } f(gx) = \vartheta(g)f(x) = \vartheta(g)v. \text{ Hence, } w = \vartheta(g)v. \]

It remains to prove Lemma 6.5, whose statement is recalled below (for the sake of readability, the index \( i + 1 \) is replaced with \( i \)).

**Lemma.** Let \( F \) be the finite subgroup of \( L_i \) defined in Lemma 4.13. If \( U \) is non-elliptic in the limiting tree of the sequence \((\vartheta^i_n)_{n \in \mathbb{N}}\), then the group \( L_i \) admits a splitting \( \mathbb{S} \) with exactly two vertex groups \( \langle x, F \rangle \) (for some tuple \( x \) of elements of \( L_i \)) and \( \langle a, \mathbb{F} \rangle \), and one edge group \( F \). Let \( A \) be an \( \mathbb{S} \)-approximation of \( L_i \) given by Proposition 3.13. There exists a subgroup \( E \) of \( E(G) \) and an epimorphism \( r \) from \( A \) onto a group of the form

\[
\langle g, a \rangle \ast_E \langle x, F | \text{ad}(x_\iota) | F = \sigma_\iota | E, \forall \iota \in [1, p] \rangle,
\]

where \( g \) denotes a tuple of elements of \( G \), such that \( r(x) = x, r(a) = a \) and \( r \) kills no component of the image in \( A \) of the tuple \( \Psi(x, y, a) \).

**Proof.** By assumption, the group \( U \) is non-elliptic in the limiting tree \( T := T_{L_i^U} \) associated with the divergent sequence \((\vartheta^i_n)_{n \in \mathbb{N}}\). First, we aim to construct a splitting \( \mathbb{S} \) of \( L_i \) with exactly two vertex groups \( \langle x, F \rangle \) and \( \langle a, F \rangle \) (for some tuple \( x \) of elements of \( L_i \)) and one edge group \( F \). Let \( \Gamma \) be the stabiliser of the base point \( o \) in \( T \). Let \( \Gamma \) be the subgroup \( \langle U, Y \rangle \) of \( L_i^U \). Since \( U \) is contained in \( \Gamma \), this group is non-elliptic in the limiting tree \( T \) (otherwise \( U \) should be elliptic as well). Let \( T_{\Gamma} \subset T \) be the minimal invariant subtree of \( \Gamma \). By Lemma 4.14, the tree \( T_{\Gamma} \) is simplicial and \( \Gamma \) admits the following splitting:

\[
\Gamma = Y \ast_F \langle x, F | \text{ad}(x_\iota) | F = \alpha_\iota, \forall \iota \in [1, p] \rangle,
\]

where \( F \) denotes the finite subgroup of \( L \) defined in Lemma 4.13, and \( \alpha_\iota \) denotes the automorphism of \( F \) induced by the action of \( x_\iota \).

Let \( \sim \) be the relation on \( T \) defined by \( x \sim y \) if \( [x, y] \cap uT_{\Gamma} \) contains at most one point, for every \( u \in L_i^U \). Note that \( \sim \) is an equivalence relation. Let \((Y_j)_{j \in J}\) denote the equivalence classes that are not reduced to a point. Each \( Y_j \) is a subtree of \( T \). Let us prove that \((Y_j)_{j \in J} \cup \{uT_{\Gamma} | u \in L_i^U / \Gamma}\) is a transverse covering of \( T \), in the sense of Definition 4.1.

- **Transverse intersection.** For every \( i \neq j \), the intersection \( Y_i \cap Y_j \) is clearly empty. For every \( i \) and \( u \in L_i^U \), \( Y_i \cap uT_{\Gamma} \) contains at most one point by definition. For every \( u, u' \in L_i^U \) such that \( u'^{-1} \notin \Gamma, |uT_{\Gamma} \cap u'T_{\Gamma}| \leq 1 \) thanks to Lemma 4.15.

- **Finiteness condition.** Let \( x \) and \( y \) be two points of \( T \). By Lemma 4.15, there exists a constant \( \varepsilon > 0 \) such that, for every \( u \in U \), if the intersection \( [x, y] \cap uT_{\Gamma} \) is non-degenerate, the length of \( [x, y] \cap uT_{\Gamma} \) is bounded from below by \( \varepsilon \). Consequently, the arc \( [x, y] \) is covered by at most \( [d(x, y)/\varepsilon] + 1 \) distinct subtrees \( Y_j \).

Hence, the collection \((Y_j)_{j \in J} \cup \{uT_{\Gamma} | u \in L_i^U \}\) is a transverse covering of \( T \). One can construct what Guirardel calls the skeleton of this transverse covering (see Definition 4.2), denoted by \( T_c \). Since the action of \( L_i^U \) on \( T \) is minimal (by definition of \( T \)), the same holds for the action of \( L_i^U \) on \( T_c \). According to Lemma 4.9 of [20]. The question is now to understand the decomposition \( \Delta_c = T_c / L_i^U \) of \( L_i^U \) as a graph of groups.
We begin with a description of the stabiliser in $L_i^U$ of an edge $e$ of $T_Γ$. Let $u$ be an element of $L_i^U$ that fixes $e$. Then $e$ is contained in $T_Γ \cap uT_Γ$, so $u$ belongs to $Γ$, thanks to Lemma 4.15. It follows that $u$ belongs to $F$, because the stabiliser of $e$ in $Γ$ is contained in $F$ (indeed, recall that $T_Γ$ is isometric to the Bass–Serre tree of the splitting of $Γ$ as a graph of groups whose underlying graph is a rose, such that every edge group is $F$, by Lemma 4.14). Thus, the stabiliser of $e$ in $L_i^U$ is equal to $F$.

We now prove that if one of the subtrees of the covering other than $T_Γ$ intersects $T_Γ$ in a point, then this point is necessarily one of the extremities of a translate of the edge $e \in T_Γ$. Assume towards a contradiction that $Y_j \cup uT_Γ$ with $u \notin Γ$ intersects $T_Γ$ in a point $x$ that is not one of the extremities of $e$. Then, $T_c$ contains an edge $ε = (x, T_Γ)$ whose stabiliser is $Stab(x) \cap Γ$ (where $Stab(x)$ denotes the stabiliser of $x$ in $L_i^U$), which is contained in $F$ by the previous paragraph. So the splitting $Δ_c$ of $L_i^U$ is a non-trivial splitting over the finite subgroup $F$, relative to $Γ$. This is impossible since $|F| \leq C$ (because $φ_n$ maps $F$ into $E(G)$ \(ω\)-almost surely) and $L_i^U$ does not split relative to $Γ$ over a finite subgroup of order $\leq C$ non-trivially, by definition of $L_i^U$. Hence, if $Y_j \cap T_Γ = \{x\}$ or $uT_Γ \cap T_Γ = \{x\}$ with $u \notin Γ$, then the point $x$ is one of the extremities of $e$ in $T_Γ$. As a consequence, $Stab(x)$ is a conjugate of $Y$ (the stabiliser of the base point $o$) in $Γ$, and every edge adjacent to $T_Γ$ in $T_c$ is of the form $(γx, T_Γ) = γ ε$ with $ε = (x, T_Γ)$.

Therefore, $ε$ is the only edge adjacent to $T_Γ$ in the quotient graph $Δ_c$. Its stabiliser is $Y$. By collapsing all edges of $Δ_c$ except $ε$, one gets a splitting of $L_i^U$ of the following form: $L_i^U = Γ \ast_Y H$ for some subgroup $H \subset L_i^U$. Recall that

$$Γ = Y \ast_F \langle x, F \mid \text{ad}(x_i)|_F = α_i, \forall i \in [1, p] \rangle,$$

and hence the previous splitting of $L_i^U$ can be written as

$$L_i^U = H \ast_F \langle x, F \mid \text{ad}(x_i)|_F = α_i, \forall i \in [1, p] \rangle.$$

Since every finite subgroup of $L_i^U$ is conjugate to a finite subgroup of $H$, the group $L_i$ splits as

$$L_i = K \ast_F \langle x, F \mid \text{ad}(x_i)|_F = α_i, \forall i \in [1, p] \rangle,$$

for some subgroup $K$ of $L_i$ such that $\langle a \rangle \subset Y \subset H \subset K$. Denote this splitting of $L_i$ by $Σ$.

If $a$ were empty, one could just retract $L_i$ onto the free group $F(x)$ on $x$. But $a$ is not empty in general, which makes the construction of the retraction a little bit more involved.

Let $A$ be an $S$-approximation of $L_i$ given by Proposition 3.13, and let $S_A$ be the corresponding splitting of $A$. By Remark 3.14, one can assume that the components of $Ψ_k(x, y, a)$ in $L_i$ and $A$ have exactly the same normal forms when written in $S$ and $S_A$. The splitting $S_A$ is of the form

$$A = K' \ast_F \langle x, F \rangle = \langle K', x \mid \text{ad}(x_i)|_F = α_i, \forall i \in [1, p] \rangle.$$

Note that we abuse notation and still denote by $F$ a preimage of $F \subset L_i$ in $A$. Same comment about $x$, and $a$ (which is contained in $K'$).

We claim that there exists a subgroup $E$ of $E(G)$ and an epimorphism $r$ from $A$ onto a group of the form

$$\langle g, a \rangle \ast_E \langle x, E \mid \text{ad}(x_i)|_E = \sigma_i|_E, \forall i \in [1, p] \rangle,$$
where \( g \) denotes a tuple of elements of \( G \), such that \( r(\mathbf{x}) = \mathbf{x} \), \( r(\mathbf{a}) = \mathbf{a} \) and \( r \) kills no component of the image in \( A \) of the tuple \( \Psi(\mathbf{x}, \mathbf{y}, \mathbf{a}) \).

For every integer \( n \), denote by \( \psi_n : A \to G \) the factorisation of the homomorphism \( \theta^n_i : A_i \to G \) through the natural epimorphism from \( A_i \) onto \( A \).

This homomorphism \( \psi_n \) restricts to a conjugation on \( \langle \mathbf{x}, \mathbf{a} \rangle \). Up to postcomposing \( \psi_n \) with an inner automorphism of \( G \), one can now assume without loss of generality that \( \psi_n \) coincides with the identity on \( \langle \mathbf{x}, \mathbf{a} \rangle \). In particular, the inner automorphism \( \text{ad}(\psi_n(x_i)) \) induces the same automorphism \( \sigma_i \) of \( E(G) \) as \( \text{ad}(\varphi_n(x_i)) \), where \( (\varphi_n : G_{\Sigma} \to G)_{n \in \mathbb{N}} \) denotes the initial \((\sigma_1, \ldots, \sigma_p)\)-test sequence.

For every integer \( n \), since \( \psi_n \) is the identity on \( \mathbf{a} \), the group \( \psi_n(K') \) contains \( \mathbf{a} \). Since \( A \) is finitely presented relative to \( U = \langle \mathbf{x}, \mathbf{a} \rangle \), and since \( F \) is a finite group, \( K' \) and \( \psi_n(K') \) are finitely generated relative to \( \mathbf{a} \). Therefore, there exists a tuple \( g_n \) of elements of \( G \) such that \( \psi_n(K') = \langle g_n, \mathbf{a} \rangle \).

Let \( E_0 = \psi_n(F) \subset E(G) \). Let us consider the following amalgamated product:

\[
Q_n = \langle g_n, \mathbf{a} \rangle \ast_E \langle \mathbf{x}, E | \text{ad}(x_i)|_E = \sigma_i, \forall i \in [1, p] \rangle.
\]

For every integer \( n \), one can define a morphism \( \pi_n \) from \( A \) onto \( Q_n \) by \( \pi_n(x_i) = x_i \) and \( \pi_n = \psi_n \) on \( K' \). This morphism is well-defined. Indeed, for every integer \( 1 \leq i \leq p \), as \( x_i \) normalises \( F \), there exists an automorphism \( \alpha_i \) of \( F \) such that \( x_i f x_i^{-1} = \alpha_i(f) \) for every \( f \in F \). The following relation holds \( \omega \)-almost surely:

\[
\psi_n \circ \alpha_i = \sigma_i \circ \psi_n
\]

for every integer \( n \). This relation shows that \( \pi_n \) is well-defined. In addition, this morphism is surjective because its image contains \( \mathbf{x} \) and \( \psi_n(K') = \langle g_n, \mathbf{a} \rangle \), which generate the group \( Q_n \). It remains to prove that \( \pi_n \) kills no component of the image of \( \Psi_k(\mathbf{x}, \mathbf{y}, \mathbf{a}) \) in \( A \) \( \omega \)-almost surely.

Let \( v \) be component of the image of \( \Psi_k(\mathbf{x}, \mathbf{y}, \mathbf{a}) \) in \( A \). This element can be written in normal form in the splitting \( S_A \) as \( v = k'_0 \varepsilon_1 k'_1 \varepsilon_2 k'_2 \cdots t_{q+1}^{q-1} \), with \( k'_i \in K' \) and \( t_j \in \{ x_1, \ldots, x_p \} \) for every \( 1 \leq j \leq q \). For every \( j \), if \( t_j = t_{j+1} = x_i \) and \( \varepsilon_j = -\varepsilon_{j+1} \), then \( k'_j \notin F \). By Remark 3.14, the image of \( v \) in \( L_i \) can be written in normal form in a similar way, by replacing each \( k'_i \) by an element \( k_i \) that belongs to the subgroup \( K \) of \( L \). Therefore, for every \( j \), if \( t_j = t_{j+1} = x_i \) and \( \varepsilon_j = -\varepsilon_{j+1} \), then \( k_j \) does not belong to \( F \). It follows that \( \pi_n(k'_j) = \psi_n(k'_j) \) does not lie in \( E \) \( \omega \)-almost surely. Otherwise, if \( \pi_n(k'_j) \) belonged to \( E \) \( \omega \)-almost surely, then \( k_j \) would belong to \( F \) \( \omega \)-almost surely, contradicting the previous condition. Hence, for every \( n \) large enough, the element \( \pi_n(v) = \psi_n(k_0) t_{q+1}^{q} \psi_n(k_1) t_{q+1}^{q} \psi_n(k_2) \cdots t_{q+1}^{q} \psi_n(k_q+1) \) is non-trivial. Last, take \( r = \pi_N \) for \( N \) such that \( \pi_N \) kills no component of \( \Psi_k(\mathbf{x}, \mathbf{y}, \mathbf{a}) \) in \( A \). \( \square \)
7 | PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. First, recall that this theorem says that every acylindrically hyperbolic group $G$ is $\forall\exists$-embedded into the HNN extensions $G_{E(G)}^* = \langle G, t \mid [t, g] = 1, \forall g \in E(G) \rangle$. In fact, we just have to prove that $G$ is $\forall\exists$-embedded into $G_{E(G)}^*$, in virtue of the following easy and general lemma, which has nothing to do with acylindrical hyperbolicity.

**Lemma 7.1.** Let $G'$ be a group, and let $G$ be a subgroup of $G$. If $G$ is $\forall\exists$-embedded into $G'$, then $G$ is $\exists\forall\exists$-embedded into $G'$.

**Proof.** Suppose that $G$ is $\forall\exists$-embedded into $G'$. Let $\vartheta(t)$ be an $\exists\forall\exists$-formula with $m$ free variables. Suppose that there exists a tuple $g \in G^m$ such that $\vartheta(g)$ holds in $G$, and prove that $\vartheta(g)$ holds in $G'$.

The formula $\vartheta(t)$ can be written as $\exists x \mu(t, x)$, where $\mu(t, x)$ denotes a $\forall\exists$-formula with $m + n$ free variables, where $n$ is the arity of $x$. Since $\vartheta(g)$ holds in $G$, there exists a tuple $h \in G^n$ such that $\mu(g, h)$ holds in $G$. But the formula $\mu(t, x)$ is $\forall\exists$, thus $\mu(g, h)$ holds in $G'$. This concludes the proof of the lemma.

To prove Theorem 1.1, it remains to prove that every acylindrically hyperbolic group $G$ is $\forall\exists$-embedded into $G_{E(G)}^*$. The proof of this result relies on Theorem 1.11.

**Theorem 7.2.** Every acylindrically hyperbolic group $G$ is $\forall\exists$-embedded into $G_{E(G)}^*$.

**Proof.** Let

$$\bigvee_{k=1}^{\ell}(\Sigma_k(x, y, g) = 1 \land \Psi_k(x, y, g) \neq 1)$$

be a finite disjunction of systems of equations and inequations in $x$ and $y$. Suppose that $G$ satisfies the following first-order sentence $\mu(g)$:

$$\forall x \exists y \bigvee_{k=1}^{\ell}(\Sigma_k(x, y, g) = 1 \land \Psi_k(x, y, g) \neq 1).$$

Let $y$ be a tuple of elements of $\Gamma$ of the same arity as $x$. We will prove that there exists a tuple $y'$ of elements of $\Gamma$ of the same arity as $y$ such that the following holds in $\Gamma$:

$$\bigvee_{k=1}^{\ell}(\Sigma_k(y, y', g) = 1 \land \Psi_k(y, y', g) \neq 1).$$

To this end, we would like to construct a retraction $\pi$ from the group $\langle \Gamma, y \mid \Sigma_k(y, y, g) = 1 \rangle$ onto $\Gamma$, for some $1 \leq k \leq \ell'$, such that $\pi$ kills no component of the system of inequations $\Psi_k(y, y, g) \neq 1$. Indeed, given such a retraction $\pi$, one can simply take $y' = \pi(y)$. We could construct this retraction by mimicking the proof of Theorem 1.11, as sketched in the introduction, but in order to avoid unnecessary repetitions, we will appeal to Theorem 1.11. However, before applying this result, one has to fix the following problem: Theorem 1.11 does not apply directly in the present situation since
it only allows us to deal with constants from $G$, and $\gamma$ is not a tuple of elements of $G$ in general. To be able to use Theorem 1.11, we have first to slightly reformulate the problem.

Let $s$ be a generating tuple of $G$, possibly infinite. For every integer $n \geq 1$, let $s_n$ be the $n$-tuple composed of the first $n$ components of $s$ and let $G_n$ be subgroup of $G$ generated by $s_n$. For $n$ sufficiently large, the following two conditions are satisfied.

- The subgroup $G_n$ of $G$ contains the finite subgroup $E(G)$. Therefore, there is a finite system of equations $\vartheta(s_n, t) = 1$ expressing the fact that the stable letter $t$ centralises $E(G)$.
- The subgroup $\langle G_n, t \rangle$ of $\Gamma$ contains each component $\gamma_i$ of $\gamma$. As a consequence, each $\gamma_i$ can be written as a word $w_i(s_n, t)$.

Let $a$ be the tuple of elements of $G$ obtained by concatenating $g$ and $s_n$. Let $\Sigma'_k(t, y, a) = 1$ denote the finite system of equations

$$\left( \Sigma_k((w_1(s_n, t), \ldots, w_p(s_n, t)), y, g) = 1 \right) \land (\vartheta(s_n, t) = 1),$$

and let $\Psi'_k(t, y, a) \neq 1$ denote the finite system of inequations

$$\Psi_k((w_1(s_n, t), \ldots, w_p(s_n, t)), y, g) \neq 1.$$

By assumption, the group $G$ satisfies $\mu(g)$. Therefore, $G$ satisfies the following first-order sentence $\vartheta(a)$:

$$\forall t \exists y \bigvee_{k=1}^\ell (\Sigma'_k(t, y, a) = 1 \land \Psi'_k(t, y, a) \neq 1).$$

By Theorem 1.11, there exists an integer $1 \leq k \leq \ell$, a subgroup $G'$ of $G$ containing $\langle a \rangle$ and an epimorphism

$$\pi : G' \twoheadrightarrow \Gamma' := (\langle t \rangle \times E(G)) \ast_{E(G)} G'$$

such that

1. $\pi(t) = t$,
2. $\pi(a) = a$ (in particular $\pi(g) = g$ and $\pi(s_n) = s_n$, and therefore $\pi(\gamma) = \gamma$),
3. and such that no component of the system of inequations $\Psi'_k(t, y, a) \neq 1$ is killed by $\pi$.

As a consequence, the following system of equations and inequations holds in $\Gamma'$:

$$\bigvee_{k=1}^\ell (\Sigma'_k(t, \pi(y), a) = 1 \land \Psi'_k(t, \pi(y), a) \neq 1).$$

It follows that the following system of equations and inequations holds in $\Gamma'$:

$$\bigvee_{k=1}^\ell (\Sigma_k(y, \pi(y), g) = 1 \land \Psi_k(y, \pi(y), g) \neq 1).$$

Since $\Gamma'$ is a subgroup of $\Gamma$, this system holds in $\Gamma$ as well. One can take $\gamma' = \pi(y)$. \qed
8 PROOF OF MERZLYAKOV’S THEOREM 5.2 IN THE GENERAL CASE

8.1 Reduction to an overgroup $G_{2p}$ of $G$

As above, $G$ denotes an acylindrically hyperbolic group, and $p$ denotes the arity of $x$ in the considered first-order sentence. In the proof of Proposition 6.2, for defining a test sequence (that is, a $(\sigma_1, \ldots, \sigma_p)$-test sequence with $\sigma_i = \text{id}_{E(G)}$ for every $1 \leq i \leq p$), we used the fact that $G$ contains a quasi-convex non-abelian free subgroup $F_2$ that centralises $E(G)$. It seems quite involved to adapt this construction in order to get a non-central prescribed action of $F_2$ by conjugation on $E(G)$.

We shall circumvent this difficulty by means of Theorem 1.1 proved in the previous section. According to this result, the inclusion of $G$ into $G \ast_{E(G)} (E(G) \times Z)$ is an $\exists \forall \exists$-embedding. More generally, the inclusion of $G$ into $G_m := G \ast_{E(G)} (E(G) \times F_m)$ is an $\exists \forall \exists$-embedding, for any integer $m$. Take $m = 2p$, and let $t_1, \ldots, t_{2p}$ be a basis of $F_{2p}$. Let $(\sigma_1, \ldots, \sigma_p)$ be a $p$-tuple of elements of $\text{Aut}_E(G)$. For every $1 \leq i \leq p$, there exists an element $g_i \in G$ such that $\sigma_i = \text{ad}(g_i)_{|E(G)}$, by definition of $\text{Aut}_E(G)$. Note that $\sigma_i = \text{ad}(g_i t_i)_{|E(G)}$, since $t_i$ centralises $E(G)$. Let $\alpha_i$ be the automorphism of $G_{2p}$ that coincides with $\text{ad}(g_i)$ on $G$ and that maps $t_i$ to $g_i t_i$ and $t_j$ to $t_j$ for $j \neq i$. The composition $\alpha_1 \circ \cdots \circ \alpha_p$ is an automorphism of $G_{2p}$ that coincides with the conjugacy by $g_1 \cdots g_p$ on $G$ and maps $t_i$ to $g_i t_i$ for $1 \leq i \leq p$, and fixes $t_j$ for $p+1 \leq i \leq 2p$. As a consequence, up to replacing $t_i$ by $g_i t_i$, one can assume without loss of generality that $\text{ad}(t_i)_{|E(G)} = \sigma_i$ for every $1 \leq i \leq p$, and $G_{2p}$ has the following presentation:

$$G_{2p} = G \ast_{E(G)} \left< E(G), t_1, \ldots, t_{2p} \mid \begin{array}{l}
\text{ad}(t_i)_{|E(G)} = \sigma_i \text{ for } 1 \leq i \leq p \\
\text{ad}(t_i)_{|E(G)} = \text{id} \text{ for } p+1 \leq i \leq 2p
\end{array} \right>.$$

8.2 Construction of a $(\sigma_1, \ldots, \sigma_p)$-test sequence

We now build a $(\sigma_1, \ldots, \sigma_p)$-test sequence from $G_{\Sigma_k}$ to $G_{2p}$, for any $(\sigma_1, \ldots, \sigma_p)$ in $\text{Aut}_G(E(G))^p$.

Proposition 8.1. Let $G$ be an acylindrically hyperbolic group, and let $a$ be a tuple of elements of $G$. Fix a presentation $\langle a \mid R(a) = 1 \rangle$ for the subgroup of $G$ generated by $a$. Let

$$\ell \bigvee_{k=1}^\ell (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1)$$

be a finite disjunction of finite system of equations and inequations over $G$, where $x$ and $y$ are tuples of variables. For every $1 \leq k \leq \ell$, denote

$$G_{\Sigma_k} = \langle x, y, a \mid R(a) = 1, \Sigma_k(x, y, a) = 1 \rangle.$$

Let $p = |x|$ be the arity of $x$, and let $(\sigma_1, \ldots, \sigma_p)$ be a $p$-tuple of elements of $\text{Aut}_G(E(G))$. Suppose that $G$ satisfies the following first-order sentence:

$$\emptyset : \forall x \exists y \bigvee_{k=1}^\ell (\Sigma_k(x, y, a) = 1 \land \Psi_k(x, y, a) \neq 1).$$
Then there exists an integer $1 \leq k \leq \ell$ and a $(\sigma_1, \ldots, \sigma_p)$-test sequence $(\varphi_n : G_{\Sigma_k} \rightarrow G_{2p})_{n \in \mathbb{N}}$ such that $\varphi_n(\Psi_k(x, y, a))$ is non-trivial for every $n$ sufficiently large and such that $\varphi_n(a) = a$.

**Proof.** Recall that $G_{2p}$ has the following presentation:

$$G_{2p} = G \ast_{E(G)} \left\langle E(G), t_1, \ldots, t_{2p} \mid \text{ad}(t_i) \mid E(G) = \sigma_i \text{ for } 1 \leq i \leq p \right. \mid \text{ad}(t_i) \mid E(G) = \text{id} \text{ for } p + 1 \leq i \leq 2p \right\rangle.$$

For every $1 \leq i \leq p$ and for every integer $n \geq 1$, let $o_i$ be the order of $\sigma_i$, let $r_n$ be the remainder of the division of $n$ by $o_i$, and let $q_n = o_i + 1 - r_n$. Note that one has $2 \leq q_n \leq o_i + 1$ and that $n + q_n = 1 \mod o_i$. Let us define an element $g_{i,n}$ of $G$ by

$$g_{i,n} = t_i^n t_{i+p}^{n+1} t_i \cdots t_{i+1}^{2n} t_i^{q_n}.$$

Observe that $\text{ad}(g_{i,n}) \mid E(G) = \sigma_i$, thanks to our choice of $q_n$.

Since the inclusion of $G$ into $G_{2p}$ is an $\exists \forall \exists$-embedding, the group $G_{2p}$ also satisfies the first-order sentence $\theta$. By the pigeonhole principle, there exists an integer $1 \leq k \leq \ell$ and an infinite set $A \subset \mathbb{N}$ such that for every integer $n \in A$, the group $G_{2p}$ satisfies the following existential sentence:

$$\exists y_n \Sigma_k((g_{1,n}, \ldots, g_{p,n}), y_n, a) = 1 \land \Psi_k((g_{1,n}, \ldots, g_{p,n}), y_n, a) \neq 1.$$

Let us define $\varphi_n : G_{\Sigma} \rightarrow G_{2p}$ by $\varphi_n(x_1) = g_{i,n}$, $\varphi_n(a) = a$ and $\varphi_n(y) = y_n$. One can check that the sequence $(\varphi_n : G_{\Sigma_k} \rightarrow G_{2p})_{n \in \mathbb{N}}$ is a $(\sigma_1, \ldots, \sigma_p)$-test sequence. ◻

### 8.3 Proof of Merzlyakov's Theorem 1.11

By Proposition 8.1, there exists an integer $1 \leq k \leq \ell$ and a $(\sigma_1, \ldots, \sigma_p)$-test sequence $(\varphi_n : G_{\Sigma_k} \rightarrow G_{2p})_{n \in \mathbb{N}}$ such that $\varphi_n(\Psi_k(x, y, a))$ is non-trivial for every $n$ sufficiently large, where $G_{2p}$ is the group defined in the previous section. Then, Proposition 6.3 applied to $G_{2p}$ instead of $G$ provides a finite subgroup $E$ of $E(G_{2p}) = E(G)$ and a morphism

$$\pi_{\sigma} : G_{\Sigma_k} \rightarrow (G_{2p})_{\sigma} = \langle G_{2p}, x \mid \text{ad}(x) \mid E = \sigma_i, \forall i \in [1, p] \rangle$$

such that the following three conditions hold:

- $\pi_{\sigma}(x) = x$,
- $\pi_{\sigma}(a) = a$,
- no component of the tuple $\Psi(x, y, a)$ is killed by $\pi_{\sigma}$.

Moreover, the image of $\pi_{\sigma}$ is a subgroup of $(G_{2p})_{\sigma}$ of the form

$$\langle g, a \rangle \ast_E \langle x, E \mid \text{ad}(x) \mid E = \sigma_i, \forall i \in [1, p] \rangle$$

for some tuple $g$ of elements of $G_{2p}$. We conclude the proof of Theorem 1.11 by composing the morphism $\pi_{\sigma}$ with the retraction $r$ from $G_{2p}$ onto $G$ defined by $r(t_i) = x_i$ for every $1 \leq i \leq p$ and $r(t_i) = 1$ for every $p + 1 \leq i \leq 2p$. 


9 | TRIVIAL POSITIVE THEORY AND VERBAL SUBGROUPS

In this section, we prove Corollary 1.8, which claims that acylindrically hyperbolic groups have trivial positive theory. We also deduce Corollary 1.9 about verbal subgroups of acylindrically hyperbolic groups.

The proof of Corollary 1.8 relies on Theorem 1.1. According to this theorem, the canonical inclusion of an acylindrically hyperbolic group $G$ into $G^{*E(G)}$ is an $\exists\forall\exists$-elementary embedding (as defined in the introduction). In particular, $G$ and $G^{*E(G)}$ have the same $\forall\exists$-theory.

Proof of Corollary 1.8. Let $G$ be an acylindrically hyperbolic group. By [8, Theorem 6.3], if a group satisfies a non-trivial positive sentence, then it also satisfies a non-trivial positive $\forall\exists$-sentence. As a consequence, in order to prove that $G$ has trivial positive theory, it suffices to prove that $G$ has trivial positive $\forall\exists$-theory. Let $\exists\forall\exists$-sentence satisfied by $G$. Let $E(G)$ denote the maximal finite normal subgroup of $G$. It follows from Theorem 1.1 that the groups $G$ and $\Gamma = \langle G, x, y | [x, g] = [y, g] = 1, \forall g \in E(G) \rangle$ have the same $\forall\exists$-theory. As a consequence, $\exists\forall\exists$-sentence is satisfied by $\Gamma$. Now, observe that $\Gamma$ maps onto the free group $(x, y) \simeq F_2$. Since positive sentences are preserved under epimorphisms, $\exists\forall\exists$-sentence is satisfied by $F_2$. It follows that $\exists\forall\exists$-sentence is satisfied by all free groups. Therefore, $\exists\forall\exists$-sentence holds in all groups.

Before proving Corollary 1.9, let us prove a lemma. Given a word $w \in F_k$, recall that the verbal subgroup $w(G) = \langle \{w(g), g \in G^k\} \rangle$ is said to have finite width if there exists an integer $m \in \mathbb{N}$ such that any $g \in w(G)$ can be represented as a product of at most $m$ values of $w$ and their inverses. Otherwise, one says that $w(G)$ has infinite width.

Lemma 9.1. Let $G$ be a group, let $k \geq 1$ be an integer and let $w$ be an element of the free group $F(x_1, \ldots, x_k)$. Recall that we denote by $e_i$ the sum of the exponents of $x_i$ in $w$. If $e_1, \ldots, e_k$ are all equal to 0, define $d(w) = 0$. Otherwise, let $d(w)$ be their greatest common divisor. Suppose that $G$ has trivial positive theory, then $w(G)$ has infinite width, except if $w$ is trivial or $d(w) = 1$ (in which cases the width is equal to 1).

Proof. First, suppose that $w$ is trivial or $d(w) = 1$, and let us prove that the width of $w(G)$ is 1. If $w$ is trivial, this is obvious. Now, suppose that $w$ is non-trivial and that $d(w) = 1$. Then there exist $k$ integers $a_1, \ldots, a_k$ such that $a_1e_1 + \cdots + a_ke_k = 1$, and one has $w(g^{a_1}, \ldots, g^{a_k}) = g$ for every element $g \in G$. Hence, $w(G)$ is equal to $G$, and its width is equal to 1.

Now, suppose that $G$ has trivial positive theory, that $w$ is non-trivial and that $d(w) \neq 1$. Let us prove that $w(G)$ has infinite width. Assume towards a contradiction that $w(G)$ has finite width $\ell \geq 1$. Then $G$ satisfies the following positive first-order ($\forall\exists$)-sentence $\phi_n$ for every integer $n \geq 1$: every element of $g$ that can be represented as a product of $n$ elements of $\{w(g)^{\pm1}, g \in G^k\}$ can be represented as a product of $\ell$ elements of $\{w(g)^{\pm1}, g \in G^k\}$. Since $G$ has trivial positive theory, this sentence $\phi_n$ is satisfied by all groups. In particular, $\phi_n$ is true in the free group $F_2$, for every $n$. Thus, $w(F_2)$ has finite width (equal to $\ell$). It follows from [33, Lemma 3.1.1 and Theorem 3.1.2] (inspired from [30]) that either $w$ is trivial or $d(w) = 1$, contradicting our assumption. Hence, $w(G)$ has infinite width.

Last, prove Corollary 1.9.

Proof of Corollary 1.9. Since acylindrically hyperbolic groups have trivial positive theory, Corollary 1.9 is an immediate consequence of the previous lemma.
10 | QUESTIONS AND COMMENTS

In [38], Sela asked the following intriguing question.

**Question 10.1.** Which (algebraic, first-order) properties are satisfied by groups $G$ such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent?

If the answer to the generalised Tarski’s problem 1.5 is ‘Yes’, then every acylindrically hyperbolic group $G$ with trivial finite radical $E(G)$ is elementarily equivalent to $G \ast \mathbb{Z}$. As far as we are aware, no examples are known of finitely generated groups that have this property but are not acylindrically hyperbolic. This raises the following question.

**Question 10.2.** Is there a finitely generated group $G$ that is not acylindrically hyperbolic but is such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent (or at least have the same $\forall \exists$-theory)?

This question is closely related to the following one (see Proposition 10.4).

**Question 10.3.** Is acylindrical hyperbolicity preserved under elementary equivalence among finitely generated groups?

In [1], the first author proved that the property of being a hyperbolic group is preserved under elementary equivalence among finitely generated groups (this result was proved by Sela in [37] for torsion-free groups). Since acylindrically hyperbolic groups are not supposed to be finitely generated, Question 10.3 makes sense without assuming finite generation; however, the answer to this question is negative in general, even among countable groups. We refer the reader to [3] for further details.

The following result shows that a positive answer to Question 10.2 implies a negative answer to Question 10.3, and that the converse is true under the assumption that the answer to the generalised Tarski’s problem 1.5 is ‘Yes’.

**Proposition 10.4.** If there exists a finitely generated non-acylindrically hyperbolic group $G$ such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent, then acylindrical hyperbolicity is not preserved under elementary equivalence among finitely generated groups. Conversely, under the assumption that Question 1.5 admits a positive answer, the following implication holds: if acylindrical hyperbolicity is not preserved under elementary equivalence among finitely generated groups, then there exists a finitely generated non-acylindrically hyperbolic group $G$ such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent.

**Proof.** If there exists a finitely generated group $G$ such that $G$ and $G \ast \mathbb{Z}$ are elementarily equivalent, and $G$ is not acylindrically hyperbolic, then acylindrical hyperbolicity is not preserved under elementary equivalence among finitely generated groups since $G \ast \mathbb{Z}$ is acylindrically hyperbolic.

Now, assume that the answer to Question 10.3 is negative, namely, that there exist two elementarily equivalent finitely generated groups $G$ and $H$ such that $G$ is not acylindrically hyperbolic and $H$ is acylindrically hyperbolic. Observe that the maximal normal finite subgroup $E(H)$ coincides with the definable set $D_N(H) = \{ h' \in H \mid [h^N, h'] = 1, \forall h \in H \}$ for $N = |\text{Aut}(E(H))|$: indeed, the fact that $E(H)$ is contained in $D_N(H)$ is obvious since any element of $H$ induces an automorphism of $E(H)$ by conjugacy; conversely, by [13, Theorem 6.14], $E(H)$ is the intersection of all maximal
virtually cyclic subgroups $\Lambda(h)$, where $h$ runs through all hyperbolic elements of $H$, and thus it follows from Lemma 2.3 that the set $D_N(H)$ is contained in $E(H)$. Since this set is definable, $D_N(G)$ is isomorphic to $D_N(H) = E(H)$, and the quotients $G' = G/D_N(G)$ and $H' = H/D_N(H)$ are elementarily equivalent. Note that $H'$ is acylindrically hyperbolic since $H$ is acylindrically hyperbolic (see [26, Lemma 3.9]). In addition, by [38], $G' \ast \mathbb{Z}$ and $H' \ast \mathbb{Z}$ are elementarily equivalent. Now, if the generalised Tarski’s problem 1.5 admits a positive answer, then $H' \ast \mathbb{Z}$ is elementarily equivalent to $H'$, which is elementarily equivalent to $G'$. Hence, $G'$ and $G' \ast \mathbb{Z}$ are elementarily equivalent. But $G'$ is not acylindrically hyperbolic, otherwise $G$ would be acylindrically hyperbolic as well, as a finite extension of $G$. Thus, the answer to Question 10.2 is ‘Yes’. □

Last, it is worth mentioning the following partial answer to Question 10.3, following from Theorem 1.1 together with a theorem of Minasyan and Osin that gives a sufficient condition under which a group $H = A \ast_C B$ or $H = A \ast_C$ is acylindrically hyperbolic ([26, Corollaries 2.2 and 2.3]).

**Proposition 10.5.** Let $G$ be an acylindrically hyperbolic group, and let $H$ be a group that admits a non-trivial splitting over a virtually abelian group. Suppose that $G$ and $H$ are elementarily equivalent (or simply that they have the same $\exists \forall \exists$-theory). Then, $H$ is acylindrically hyperbolic.

A subgroup $C$ of $H$ is said to be weakly malnormal in $H$ if there exists an element $h \in H$ such that $hCh^{-1} \cap C$ is finite.

**Proof.** First, note that the group $H$ is not virtually cyclic since it has the same first-order theory as $G$, which contains a non-abelian free subgroup.

As a first step, let us assume that the radical $E(G)$ is trivial and that $C$ is abelian. Let us fix a non-trivial element $c$ of $C$. Assume towards a contradiction that $H$ is not acylindrically hyperbolic. Then, by [26, Corollaries 2.2 and 2.3], the subgroup $C$ is not weakly malnormal. Hence, for all $h \in H$, the intersection of $hCh^{-1}$ and $C$ is infinite. In particular, this intersection contains a non-trivial element $z$. Since $C$ is abelian, this element $z$ commutes both with $c$ and $hch^{-1}$. Therefore, the following $\exists \forall \exists$-sentence is satisfied by $H$:

$$\vartheta : \exists c \neq 1 \forall h \exists z \neq 1 ([c, z] = 1 \land [hch^{-1}, z] = 1).$$

Since $G$ and $H$ have the same $\exists \forall \exists$-theory, the sentence $\vartheta$ is satisfied by $G$ as well. By Theorem 1.1, the sentence $\vartheta$ is satisfied by $G \ast \langle t \rangle$, with $t$ of infinite order. This is a contradiction since no non-trivial element of $G \ast \langle t \rangle$ commutes both with $c$ and $tct^{-1}$; indeed, by writing the elements of $G \ast \langle t \rangle$ in normal form, one easily sees that the centraliser of $c$ in $G \ast \langle t \rangle$ is contained in $G$, and that the only element of $G$ that commutes with $tct^{-1}$ is the neutral element.

If $E(G)$ is non-trivial of order $N \geq 2$ and $C$ contains an abelian subgroup of index $d$, one has to modify the sentence $\vartheta$ a little bit in order to ensure that the elements $c$ and $z$ do not belong to $E(G)$ and belong to the abelian subgroup of $C$. For that we just replace the conditions ‘$\exists c \neq 1$’ and ‘$\exists z \neq 1$’ with the conditions ‘there exist $N + 1$ pairwise distinct elements $c^d_1, ..., c^d_{N+1}$’ and ‘there exist $N + 1$ pairwise distinct elements $z^d_1, ..., z^d_{N+1}$’. □

**Remark 10.6.** More generally, if one assumes that $C$ virtually satisfies a law, the same proof works modulo some adjustments.
Remark 10.7. Note that the sentence $\theta$ given in the previous proof shows in particular that Baumslag–SolitargroupsdonotsatisfytheconclusionofTheorem 1.1: $BS(m, n) = \langle a, t \mid ta^m t^{-1} = a^m \rangle$ is not $\exists\forall\exists$-embedded into $BS(m, n) \ast \mathbb{Z}$. This observation is interesting because the main result of [8] applies to non-solvable Baumslag–Solitar groups (and shows that these groups have trivial positive theory); hence, the weak small cancellation conditions used in [8] for dealing with positive theory are not sufficient if one wants to deal with inequations.

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