Two level natural selection
under the light of Quasi-Stationary Distributions

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Abstract

In a view for a simple model where natural selection at the individual level is confronted to selection effects at the group level, we consider some individual-based models of some large population subdivided in a large number of groups. We then obtain the convergence to the law of a stochastic process with some Feynman-Kac penalization. To analyze the limiting behavior of this law, we use a recent approach, designed for the convergence to quasi-stationary distributions (QSD) in total variation, that generalize the principles of Harris recurrence. Thanks to this, we can deal with the fixation of the stochastic process and relate the convergence to equilibrium to the one where fixation implies extinction. We notably establish different regimes of convergence. Besides the case of an exponential rate (the rate being uniform over the initial condition), critical regimes with convergence in $1/t$ are also to notice.

Introduction

Each group contains $n \in \mathbb{N}$ individuals. There are two types of individuals: type I individuals are selectively advantageous at the individual (I) level and type G individuals are selectively advantageous at the group (G) level. Replication and selection occur concurrently at the individual and group level according to the Moran process and are illustrated in Fig 1. Type I individuals replicate at rate $w_I (1 + s), s \geq 0$ and type G individuals at rate $w_I$. When an individual gives birth, another individual in the same group is selected uniformly at random to die. To reflect the antagonism at the higher level of selection, groups replicate at a rate which increases with the number of type G individuals they contain.

As a simple case, we take this rate to be $w_G \times [1 + r(k/n)]$, where $k/n$ is the fraction of individuals in the group that are type G, $r(x), x \in [0, 1]$ is the selection coefficient at the group level. Like at the individual level, the population of groups is maintained at $m$ by selecting a group uniformly at random to die whenever a group replicates. The offspring of groups are assumed to be identical to their parent.

Let $X_i^t$ be the number of type G individuals in group $i$ at time $t$. Then

$$
\mu_{m:n}^{m:n} := \frac{1}{m} \sum_{i \leq m} \delta_{X_i^t/n}
$$
is the empirical measure at time $t$ of the proportion of type $G$ by group; for a given number of groups $m$ and individuals per group $n$. $\delta_x(y) = 1$ if $x = y$ and zero otherwise. The $X_i^t$ are divided by $n$ so that $\mu_t^{m,n}$ is a probability measure on $E_n := [0; 1/n; \ldots; 1]$. For fixed $T > 0$, $\mu_t^{m,n} \in D([0; T]; M_1(E_n))$, the set of càdlàg processes on $[0; T]$ taking values in $M_1(E_n)$, where $M_1(S)$ is the set of probability measures on a set $S$. With the particle process described above, $\mu_t^{m,n}$ has generator

$$(L^{m,n} \psi)(v) = \sum_{i,j}(w_l R_{ij}^l + w_G R_{ij}^G)(v) \times (\psi[v + 1/m (\delta_j/n - \delta_i/n)] - \psi[v])$$

where $\psi \in C_b(M_1([0; 1]))$ is a bounded continuous functions, and $v \in M_1(E_n) \subset M_1([0; 1])$. The transition rates $(w_l R_{ij}^l + w_G R_{ij}^G)$ are given by

$$R_{ij}^l(v) := \begin{cases} m v(i/n) i (1 - i/n) (1 + s) & \text{if } j = i - 1; i < n, \\ m v(i/n) i (1 - i/n) & \text{if } j = i + 1; i > 0 \\ 0 & \text{otherwise} \end{cases} \quad (1. R_l)$$

and

$$R_{ij}^G(v) := m v(i/n) v(j/n) (1 + r[j/n]). \quad (2. R_G)$$

$R_{ij}^l$ represents individual-level events while $R_{ij}^G$ represents group-level events.

1 Limiting behavior

Luo and Mattingly consider extensively the limit $n, m \to \infty$, where the limit $\mu_t^{s,r}$ satisfies:

$$\partial_t \langle \mu_t^{s,r} \mid f \rangle = -w_l s \langle \mu_t^{s,r} \mid x(1 - x) f' \rangle + w_G \langle \mu_t^{s,r} \mid r f \rangle - \langle \mu_t^{s,r} \mid f \rangle \langle \mu_t \mid r \rangle.$$ 

They also proved that, with the rates $w_l = n \omega_l, w_G = m \omega_G, n/m \to \theta, s = \sigma/n, r = \rho/m$, in the limit $n \to \infty$, the process converges weakly to $\nu_t^{s,r}$, where $\nu_t^{s,r}$ satisfies the following martingale problem:

with $L_{WF} f(x) := x (1 - x) \left[ \frac{\partial^2}{\partial x^2} f(x) - \sigma \partial_s f(x) \right]$,

$$N_t^f = \langle \nu_t^{s,r} \mid f \rangle - \langle \nu_0^{s,r} \mid f \rangle - \omega_l \int_0^t \langle \nu_s^{s,r} \mid L_{WF} f \rangle \, ds + \omega_G \int_0^t \langle \nu_s^{s,r} \mid r \rangle \times \langle \nu_s^{s,r} \mid f \rangle \, ds$$

is a martingale with conditional quadratic variation:

$$< N_t^f >_t = \omega_G \int_0^t \left\{ \langle \nu_s \mid f^2 \rangle - \langle \nu_s \mid f \rangle^2 \right\} \, ds.$$

There is an intermediate limit between these two, where the fluctuations between groups are still neglected (rather in order to simplify the following analysis than biologically relevant):
Theorem 1.1. Suppose $n, m \to \infty$ and the rates $w_1/n \to \omega_1$, $n s \to \sigma$, $w_G$ and $\{r(x)\}_{x \in [0,1]}$ constant (or converge to constant limiting values). Suppose the particles in the process $\mu_t^{m,m}$ are initially independently and identically distributed according to the measure $\mu_0^{m,m}$, where $\mu_0^{m,m} \to \mu_0$ as $m, n \to \infty$. Then, $\mu_t^{m,m}$ converges weakly to $\mu_t^{s,r} \in D([0;T];\mathcal{M}_1([0;1]))$.

$\mu_t^{s,r}$ is a unique solution of the differential equation:

$$
\partial_t \langle \mu_t^{s,r} \mid f \rangle = \langle \mu_t^{s,r} \mid \mathcal{L}_{WF} f \rangle + \langle \mu_t^{s,r} \mid r f \rangle - \langle \mu_t^{s,r} \mid f \rangle \times \langle \mu_t^{s,r} \mid r \rangle , \quad \mu_0^{s,r} = \mu_0 \quad (3)
$$

where $\mathcal{L}_{WF} f(x) = x (1-x) \left[ \partial_x^2 f(x) - \sigma \partial_x f(x) \right]$. (4. $\mathcal{L}_{WF}$)

with initial condition $\mu_0$.

2 Characterization of the solution of (3)

2.1 Definition as a conditional law

The solution of such equation shall then be described using the notion of QSD. Since subtracting a constant to $r$ does not change the value of $\langle \mu_t \mid r f \rangle - \langle \mu_t \mid f \rangle \langle \mu_t \mid r \rangle$, we assume in the following that $r \leq 0$, so that we can represent it as a death rate.

Consider $X_t$ the solution of the SDE, with IC $X_0 \sim \mu_0$:

$$
dX_t := -\sigma X_t (1-X_t) dt + \sqrt{2X_t (1-X_t)} dB_t. \quad (5. X)
$$

The existence and uniqueness of such process can be found e.g. in [4] ... We also consider a bias:

$$
Z_t := \exp \int_0^t r(X_s) \, ds \quad (6. Z)
$$

It can then be interpreted as the probability that the process has survived while confronted to a death rate of $r$, conditionally on $(X_t)_{t \geq 0}$. More formally, with $T_0 \sim \exp(1)$ (exponential r.v. with rate 1), we can define the extinction time as:

$$
\tau_0 := \inf \{ t \geq 0 ; -\ln(Z_t) \geq T_0 \}, \quad (7. \tau_0)
$$

since $\mathbb{P}(t < \tau_0) = \mathbb{P}(-\ln(Z_t) < T_0) = \mathbb{E}(Z_t)$ and $\partial_t \mathbb{P}(t < \tau_0) = \mathbb{E}(r(X_t) Z_t)$. (8. $\tau_{0,1}$)

Clearly, 0 and 1 are absorbing for the dynamics of $X$, so we first treat these fixations as another kind of extinction. The hitting times of 0 and 1 are denoted $\tau_0$ and $\tau_1$, and we consider any combination:

$$
\tau_{0,\theta} := \tau_0 \land \tau_\theta, \quad \tau_{1,\theta} := \tau_\theta \land \tau_1, \quad \tau_{0,1} := \tau_0 \land \tau_1, \quad \tau_{0,1,\theta} := \tau_\theta \land \tau_0 \land \tau_1. \quad (8. \tau_{0,1,\theta})
$$

We then define $\mu_t, \xi_t \in \mathcal{M}_1([0,1])$ by:

$$
\langle \mu_t \mid f \rangle := \mathbb{E} \left[ f(X_t) \, Z_t \right] / \mathbb{E} \left[ Z_t \right], \quad (9. \mu_t)
$$

$$
\langle \xi_t \mid f \rangle := \mathbb{E} \left[ f(X_t) \, Z_t \mid t < \tau_{0,1}/ \mathbb{E} \left[ Z_t \mid t < \tau_{0,1} \right] \right]. \quad (10. \xi_t)$$
By the Ito formula, for any \( f \in C_b^2 \):
\[
\mathbb{E}[f(X_t) Z_t] = \langle \mu_0 | f \rangle + \int_0^t \mathbb{E}[\mathcal{L}_W f(X_s) Z_s] \, ds + \int_0^t \mathbb{E}[f(X_s) r(X_s) Z_s] \, ds,
\]
\[
\mathbb{E}[Z_t] = 1 + \int_0^t \mathbb{E}[r(X_s) Z_s] \, ds,
\]
Thus \( \partial_t \langle \mu_t | f \rangle = \frac{\mathbb{E}[\mathcal{L}_W f(X_t) Z_t]}{\mathbb{E}[Z_t]} + \frac{\mathbb{E}[f(X_t) r(X_t) Z_t]}{\mathbb{E}[Z_t]} - \frac{\mathbb{E}[f(X_t) Z_t]}{\mathbb{E}[Z_t]} \times \frac{\mathbb{E}[r(X_t) Z_t]}{\mathbb{E}[Z_t]}
\]
so that the distribution we have defined indeed satisfies (3).

Moreover, since 0 and 1 are absorbing:
\[
\mu_t = x_t^0 \delta_0 + x_t^1 \delta_1 + x_t^2 \xi_t, \quad \text{where } x_t^e := \mathbb{E}[Z_t ; t < \tau_{0,1}] / \mathbb{E}[Z_t]
\]
\[
x_t^0 := \mathbb{E}[Z_{\tau_0} \exp[-r_0(t - \tau_0)] ; \tau_0 < t] / \mathbb{E}[Z_t], \quad x_t^1 := \mathbb{E}[Z_{\tau_1} \exp[-r_1(t - \tau_1)] ; \tau_1 < t] / \mathbb{E}[Z_t].
\]

2.2 Uniqueness of the solution to equation (3)

Let \( \tilde{\mu} \) be a solution to equation (3), \( P_t \) the semi-group associated to \( X_t \), the Wright-Fisher diffusion defined by (5, X), \( f_0 \in C_b^2([0,1]) \), and for \( 0 \leq s \leq t \):
\[
\tilde{n}_t := \exp \left[ \int_0^t \langle \tilde{\mu}_s | r \rangle \, ds \right], \quad f_t^s(x) = \tilde{n}_s \times P_{t-s} f_0(x),
\]
so that:
\[
\partial_s f_t^s(x) := \langle \tilde{\mu}_s | r \rangle f_t^s(x) - \mathcal{L}_W f_t^s(x),
\]
\[
\langle \tilde{\mu}_t | \tilde{n}_t f_0 \rangle = \langle \tilde{\mu}_t | f_t^0 \rangle := \langle \mu_0 | P_t f_0 \rangle + \int_0^t [\langle \tilde{\mu}_s | \mathcal{L}_W f_t^s \rangle + \langle \tilde{\mu}_s | r f_t^s \rangle - \langle \tilde{\mu}_s | f_t^s \rangle \times \langle \tilde{\mu}_s | r \rangle \, ds,
\]
so that \( \tilde{\nu}_t(dx) := \tilde{n}_t \tilde{\mu}_t(dx) \) solves
\[
\langle \tilde{\nu}_t | f_0 \rangle = \langle \tilde{\nu}_0 | P_t f_0 \rangle + \int_0^t \langle \tilde{\nu}_s | r \times P_{t-s} f_0 \rangle \, ds. \tag{12}
\]

Recall that \( \mu_t \) defined in (9, \( \mu_t \)) also satisfies (3) – cf (11). Define similarly:
\[
n_t := \exp \left[ \int_0^t \langle \mu_s | r \rangle \, ds \right], \quad \nu_t(dx) := n_t \mu_t(dx). \tag{13, \nu}
\]
As previously, \( \nu \) is also solution to (12), and we deduce:
\[
|\langle \nu_t - \tilde{\nu}_t | f_0 \rangle| \leq \int_0^t |\langle \nu_s - \tilde{\nu}_s | r \times P_{t-s} f_0 \rangle| \, ds
\]
\[
\leq \|f_0\|_\infty \times \|r\|_\infty \int_0^t \|\nu_s - \tilde{\nu}_s\|_{TV} \, ds.
\]
Since this is true for any \( f_0 \in C_b^2([0,1]) \), with an upper-bound proportional to \( \|f_0\|_\infty \):
\[
\|\nu_t - \tilde{\nu}_t\|_{TV} \leq \|r\|_\infty \int_0^t \|\nu_s - \tilde{\nu}_s\|_{TV} \, ds.
\]
By Gronwall’s Lemma (with the total variation uniformly bounded), this proves that \( \nu_t = \tilde{\nu}_t \) for any \( t > 0 \). Since \( \mu_t \) is deduced from \( \nu_t \) by renormalization, \( \mu_t = \mu_t \) for any \( t > 0 \).
2.3 QSDs and exponential convergence

In any case, $\delta_0$ and $\delta_1$ are QSDs for the extinction $\tau_0$, i.e. stable distributions for the dynamics given by (3).

We define the semi-groups associated to our different extinctions:

\[
\begin{align*}
\mu P_t(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_0), & \mu A_t(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_0) \\
\mu P_t^{01}(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_{0,1}), & \mu A_t^{01}(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_{0,1}) \\
\mu P_t^1(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_1), & \mu A_t^1(dx) &:= \mathbb{P}_\mu(X_t \in dx \mid t < \tau_1).
\end{align*}
\]

**Proposition 2.3.1.** There exists a unique QSD $\alpha \in \mathcal{M}_1([0,1])$ and a unique capacity of survival $\eta$ associated to the extinction $\tau_{0,1}$. With the associated extinction rate $\rho_\alpha$, it means:

\[
\forall t > 0, \quad \alpha P_t^{01}(dx) = \exp[-\rho_\alpha t] \alpha(dx) \\
\eta(x) := \lim_{t \to \infty} \exp[\rho_\alpha t] \mathbb{P}_x(t < \tau_{0,1})
\]

Moreover, we have the following exponential convergences at rate $\xi$:

\[
\exists C, \xi > 0, \; \forall \mu \in \mathcal{M}_1([0,1]), \; \|\mu A_t^{01} - \alpha\|_{TV} \leq C \exp[-\xi t]. \quad (14. \; \alpha)
\]

\[
\exists C' > 0, \; \forall \mu \in \mathcal{M}_1([0,1]), \; \|\exp[\rho_\alpha t] \mathbb{P}_\mu(t < \tau_{0,1}) - \langle \mu \mid \eta \rangle \| \leq C' \exp[-\xi t] \quad (15)\]

**a fortiori**

\[
\|\eta_*\| := \sup_{x \in (0,1), t > 0} \exp[\rho_\alpha t] \mathbb{P}_x(t < \tau_{0,1}) < \infty \quad (16. \; \|\eta_*\|)
\]

Let $\rho_0 = -\tau_0$ (resp. $\rho_1$), the extinction rate of $\delta_0$ (resp. $\delta_1$). We show in the following that the long-time behavior of the process with only the local extinction rate depends mainly on $\rho_0$, $\rho_0$ and $\rho_1$.

In the following convergences, we will often have uniform bounds for probability measures belonging for some $n \geq 1$ and $\xi > 0$ to:

\[
\mathcal{M}_{n,\xi} := \{\mu \in \mathcal{M}_1([0,1]) \mid \mu[1/n, 1] \geq \xi\}, \quad \bigcup_{n,\xi} \mathcal{M}_{n,\xi} = \mathcal{M}_1([0,1]) \setminus \{\delta_0\}. \quad (17. \; \mathcal{M}_{n,\xi})
\]

or in

\[
\mathcal{M}_{n,\xi}^{0,1} := \{\mu \in \mathcal{M}_1([0,1]) \mid \mu[1/n, 1 - 1/n] \geq \xi\}, \quad \bigcup_{n,\xi} \mathcal{M}_{n,\xi}^{0,1} = \mathcal{M}_1([0,1]) \setminus \{x \delta_0 + (1 - x) \delta_1 \mid x \in [0,1]\}. \quad (18. \; \mathcal{M}_{n,\xi}^{0,1})
\]

2.3.1 $\rho_1 < \rho_0 < \rho_\alpha$

**Proposition 2.3.2.** Assume that $\rho_1 < \rho_0 < \rho_\alpha$. Then, $\delta_1$ is the only stable QSD, with convergence rate $\rho_0 - \rho_1$, i.e.:

\[
\forall n \geq 1, \forall \xi > 0, \exists C_{n,\xi} > 0, \; \forall \mu \in \mathcal{M}_{n,\xi}, \; \|\mu A_t - \delta_1\|_{TV} \leq C_{n,\xi} \exp[-(\rho_0 - \rho_1) t].
\]

We also have an additional level of convergence:

**Proposition 2.3.3.** Assume that $\rho_0 < \rho_\alpha$. Then, there exists $C > 0$ s.t.:

\[
\forall \mu \in \mathcal{M}_1([0,1]) \setminus \{\delta_1\}, \quad \|\mu A_t^1 - \delta_0\|_{TV} \leq C \exp[-(\rho_\alpha - \rho_0) t].
\]
2.3.2 \( \rho_1 < \rho_0 < \rho_0 \)

**Proposition 2.3.4.** Assume that \( \rho_1 < \rho_0 < \rho_0 \). Then, \( \delta_1 \) is again the only stable QSD, with convergence rate \( \rho_0 - \rho_1 \), i.e. :

\[
\forall n \geq 1, \forall \xi > 0, \exists C_{n,\xi} > 0, \forall \mu \in \mathcal{M}_{n,\xi}, \|\mu A_t - \delta_1\|_{TV} \leq C_{n,\xi} \exp[-(\rho_0 - \rho_1) t].
\]

Again, we have an additional level of convergence :

**Proposition 2.3.5.** Assume that \( \rho_0 < \rho_0 \). Then :

\[
\exists \zeta^1 > 0, \forall n \geq 1, \forall \xi > 0, \exists C_{n,\xi} > 0, \forall \mu \in \mathcal{M}_{n,\xi} \setminus \{\delta_1\}, \|\mu A_t^1 - \alpha_1\|_{TV} \leq C_{n,\xi} \exp[-\zeta^1 t],
\]

where the QSD \( \alpha_1 \) has extinction rate \( \rho_0 \) and is given as \( \alpha_1 = y_0 \delta_0 + y_\alpha \alpha \) with the relations :

\[
y_0 = \frac{\rho_0 \times \mathbb{P}_\alpha(\tau_0 = \tau_{1,\alpha})}{(\rho_0 - \rho_\alpha)}, \quad y_0 + y_\alpha = 1
\]

and thus \( y_\alpha := \frac{(\rho_0 - \rho_\alpha)}{\rho_0 - \rho_\alpha \times \mathbb{P}_\alpha(\tau_{1,\alpha} = \tau_{0,1,\alpha})} \), \( y_0 := \frac{\rho_\alpha \times \mathbb{P}_\alpha(\tau_0 = \tau_{1,\alpha})}{\rho_0 - \rho_\alpha \times \mathbb{P}_\alpha(\tau_{1,\alpha} = \tau_{0,1,\alpha})} \).

Moreover, we know the associated capacity of survival \( \eta^1 := \eta/y_\alpha (\eta^1(0) = 0) \) and :

\[
\forall n, \xi, \exists C_{n,\xi} > 0, \forall \mu \in \mathcal{M}_{n,\xi},
|\exp[\rho_\alpha t] \mathbb{P}_\mu(t < \tau_{1,\alpha}) - \langle \mu | \eta^1 \rangle| \leq C_{n,\xi} \exp[-\zeta^1 t]
\]

and \( \|\eta^1\| := \sup_{x \in [0,1), t>0} \exp[\rho_\alpha t] \mathbb{P}_x(t < \tau_{1,\alpha}) < \infty \) \( (21) \|\eta^1\| \)

2.3.3 \( \rho_1 < \rho_0 = \rho_0 \)

**Proposition 2.3.6.** Assume that \( \rho_1 < \rho_0 = \rho_0 \). Then, \( \delta_1 \) is again the only stable QSD, with convergence rate \( \rho_0 - \rho_1 \), i.e. :

\[
\forall n \geq 1, \forall \xi > 0, \exists C_{n,\xi} > 0, \forall \mu \in \mathcal{M}_{n,\xi}, \|\mu A_t - \delta_1\|_{TV} \leq C_{n,\xi} \times (1 + t) \exp[-(\rho_0 - \rho_1) t].
\]

For the next level of convergence :

**Proposition 2.3.7.** Assume that \( \rho_0 = \rho_0 \). Then, \( \delta_0 \) is the only QSD with extinction \( \tau_{1,\alpha} \).

Moreover :

\[
\exists t, C > 0, \forall t \geq t, \forall \mu \in \mathcal{M}_1([0,1]), \|\mu A_t^1 - \delta_0\|_{TV} \leq C/(1 + t),
\]

\[
\forall n, \xi, \exists t, C_{n,\xi} > 0, \forall t \geq t, \forall \mu \in \mathcal{M}_{n,\xi}^{0,1}, \|\mu A_t^1 - \delta_0\|_{TV} \geq C_{n,\xi}/t.
\]

2.3.4 \( \rho_0 = \rho_1 < \rho_0 \)

**Proposition 2.3.8.** Assume that \( \rho_0 = \rho_1 < \rho_0 \). Then, any convex combination of \( \delta_0 \) and \( \delta_1 \) is a QSD, with extinction rate \( \rho_1 \). The convergence still happens with convergence rate \( \rho_0 - \rho_1 \), i.e. :

\[
\exists C > 0, \forall \mu \in \mathcal{M}_1([0,1]), \exists x \in [0,1], \|\mu A_t^1 - (x \delta_0 + (1 - x) \delta_1)\|_{TV} \leq C \exp[-(\rho_0 - \rho_1) t].
\]

Moreover, the proportion \( x \) for the limiting QSD is :

\[
x(\mu) := E_\mu[\exp(\rho_1 \tau_{0,1,\alpha})] / E_\mu[\exp(\rho_1 \tau_{0,1,\alpha})] ; \tau_{0,1,\alpha} = \tau_0 \] \( (23) \ x(\mu) \)
The next level of convergence (with extinction $\tau_{0,1,\delta}$) is the already known convergence to $\alpha$ at exponential rate.

2.3.5 $\rho_\alpha < \rho_0 \wedge \rho_1$

**Proposition 2.3.9.** Assume that $\rho_\alpha < \rho_0 \wedge \rho_1 := \rho$. Then, there is only one stable QSD $\alpha_{0,1}$, with convergence rate $\rho - \rho_\alpha$, i.e.:

$$\forall n \geq 1, \forall \xi > 0, \exists C_{n,\xi} > 0, \forall \mu \in M_{n,\xi}^{0,1}, \|\mu A_t - \alpha_{0,1}\|_{TV} \leq C_{n,\xi} \exp[-(\rho_\alpha - \rho) t],$$

where $\alpha_{0,1}$ has extinction rate $\rho_\alpha$ and is given as $\alpha_{0,1} = y_0 \delta_0 + y_1 \delta_1 + y_\alpha \alpha$ with:

$$\frac{y_0}{y_\alpha} = \frac{\rho_\alpha \times P_\alpha(\tau_0 = \tau_{0,1,\delta})}{(\rho_0 - \rho_\alpha)}, \quad \frac{y_1}{y_\alpha} = \frac{\rho_\alpha \times P_\alpha(\tau_1 = \tau_{0,1,\delta})}{(\rho_1 - \rho_\alpha)},$$

and of course $y_0 + y_1 + y_\alpha = 1$.

If $\rho_1 < \rho_0$, any IC $x \delta_0 + (1 - x) \delta_1$ with $x \in (0,1)$ converges at rate $\rho_0 - \rho_1$ to $\delta_1$.

If $\rho_1 = \rho_0$, any such distribution is a QSD with the extinction rate $\rho_0$.

2.3.6 $\rho_1 = \rho_\alpha < \rho_0$

**Proposition 2.3.10.** Assume that $\rho_1 = \rho_\alpha < \rho_0$. Then, $\delta_1$ is again the only stable QSD, yet the convergence is not exponential, and more precisely:

$$\exists C > 0, \forall \mu \in M_1([0,1]), \|\mu A_t - \delta_1\|_{TV} \leq C/(1 + t),$$

$$\forall n \geq 2, \forall \xi > 0, \exists c_{n,\xi} > 0, \forall \mu \in M_{n,\xi}^{0,1}, \|\mu A_t - \delta_1\|_{TV} \geq c_{n,\xi}/(1 + t).$$

For the next level of convergence, we refer to Proposition 2.3.5.

2.3.7 $\rho_0 = \rho_1 = \rho_\alpha$

**Proposition 2.3.11.** Assume that $\rho_0 = \rho_1 = \rho_\alpha$. Then, any convex combination of $\delta_0$ and $\delta_1$ is a QSD, with extinction rate $\rho_1$. They are the only ones, and among them, only one is stable:

$$\forall n, \xi, \exists C_{n,\xi} > 0, \forall \mu \in M_n^{0,1}, \|\mu A_t - (x \delta_0 + (1 - x) \delta_1)\|_{TV} \leq C_{n,\xi}/(1 + t),$$

where the proportion $x$ for the limiting QSD is:

$$x := P_\alpha(\tau_0 = \tau_{0,1,\delta})/P_\alpha(\tau_{0,1,\delta} = \tau_{0,1,\delta}).$$

**Remark:** The distribution inside the interval vanishes so slowly that its flux to 0 and 1 governs the limiting distribution (with a much quicker stabilization to $\alpha$).
2.3.8 Limits of the parameters

**Proposition 2.3.12.** Given any \( s > 0 \) and any bounded function \( r \), \( \lim_{\sigma \to \infty} \rho_\alpha(\sigma) = +\infty \).

**Proposition 2.3.13.** Given any \( \sigma > 0 \), \( s \geq 0 \), and a continuous (and negative) function \( r^0 \) with its maximum only in the interior of \((0,1)\), there exists a critical value \( R_\lor > 0 \) such that for any \( R > R_\lor \) and considering the system with \( r = R r^0 \), we indeed have \( \rho_\alpha < \rho_0 \wedge \rho_1 \).

Polymorphism is maintained by any sufficiently large group selection favoring it.

**Proposition 2.3.14.** Conversely, given any \( \sigma > 0 \), \( s \geq 0 \), and a bounded function \( r^0 \), there exists a critical value \( R_\land > 0 \) such that for any \( R < R_\land \) and considering the system with \( r = R r^0 \), we indeed have \( \rho_0 \wedge \rho_1 < \rho_\alpha \).

When the group selection is too small, polymorphism cannot maintain itself.

One could expect \( \rho_\alpha(\sigma) \) to be first a decreasing function of \( \sigma \) and then increasing. Yet, it seems not to hold true for any general \( r \). Think for instance of two types of equilibria that compete inside \((0,1)\), i.e. \( r \) with two localized modes, with a specific optimal value \( \sigma_1 < \sigma_2 \) for each. It can happen if there is a very strong mode of \( r \) close to a border, that is responsible for the first equilibrium. Then, one may find some \( \sigma \in (\sigma_1, \sigma_2) \) such that \( \rho_\alpha(\sigma) > \rho_\alpha(\sigma_1) \lor \rho_\alpha(\sigma_2) \), which contradicts the predicted profile of \( \rho_\alpha \).

We conjecture that \( \lim_{\sigma \to 0} \rho_\alpha(\sigma) = \infty \) also holds for any \( s > 0 \) and any bounded function \( r \). To ensure this, one should study the behavior of \( \mu_t \) around the boundary \( x = 1 \) for very small \( \sigma \). Since the amplification through the Feynman-Kac penalization when the process \( X \) stays close to 1 compete with the fixation rate at 1, this analysis is beyond the reach of this work. Yet, it is not difficult to obtain that if our conjecture were false, then the survival of the QSD would mainly rely on a vicinity of 1, because of:

**Proposition 2.3.15.** For any \( s, \rho, \epsilon > 0 \), there exists \( t \geq 1 \) such that : \( \lim_{\sigma \to 0} \mathbb{P}_{1-\epsilon}(t < \tau_0) \leq \exp(-\rho t) \).

The previous conjecture would imply the following result :

"Given any bounded function \( r \), there is a critical value \( s_\lor \) such that for any \( s \geq s_\lor \) and \( \sigma \in \mathbb{R}_+ \), \( \rho_0 \wedge \rho_1 < \rho_\alpha \)."

Such result would imply that polymorphism cannot subsist when the selection at the individual level is too large.

2.4 Proofs

2.4.1 Proof of Proposition 2.3.1

We rely on the method used in [7] and more precisely on the proof of the third illustration presented in Subs. 3.4. to ensure that :

\[
\exists \zeta > 0, \quad \forall n \geq 1, \xi > 0, \exists C_{n,\xi} > 0, \quad \forall \mu \in \mathcal{M}_{n,\xi}^{0,1}, \quad \|\mu A_t^{0,1} - \alpha\|_{TV} \leq C_{n,\xi} \exp[-\zeta t].
\] (25)
The diffusion is indeed regular on any \( D_n := [1/n, 1 - 1/n] \) (for \( n \geq 3 \)) so that applying Harnack inequality, we prove similarly as in [7] :

\[
\text{with } \tau_0^n := \inf \{ t > 0 \mid X_t \notin D_n \}, \quad \forall 0 < t_c < t_{mx}, \quad \exists c_{mx} > 0, \forall x \in D_n, \quad \Pr_x (X_{t_{mx}} \in dx ; t_{mx} < \tau_0^{n+1}) \geq c_{mx} \Pr_{1/2} (X_{t_c} \in dx \mid t_c < \tau_0^3) := c_{mx} \alpha_c(dx).
\] (26)

We refer to the step 4 of the proof given in Sect. 4 of [2] to ensure that for any \( n \geq 3 \) and \( t > 0 \), there exists \( c_n > 0 \) s.t. :

\[
\forall x, y \in D_n, \quad \Pr_x (X_t \in dx ; t < \tau_{0,1}) \leq c_n \Pr_y (X_t \in dx ; t < \tau_{0,1}).
\] (27)

It is well-known (?) that for any \( t > 0 \),

\[
\Pr_x (t < \tau_{0,1}) \xrightarrow{x \to 0} 0, \quad \Pr_x (t < \tau_{0,1}) \xrightarrow{x \to 1} 0.
\]

Like in [7] (cf Lemma 3.4.5. and hereafter), we deduce that for any \( \rho > 0 \), there exists \( \Delta_c = D_{nc} \) s.t. :

\[
\sup_{x \in (0,1)} \mathbb{E}_x \exp[\rho V_{\Delta_c}] < \infty \quad \text{where } V_{\Delta_c} := \tau_{0,1,0} \land \inf \{ t > 0 \mid X_t \in \Delta_c \}.
\] (28)

Applying Theorem 2.5 in [7] with (26), (27) and (28) (where conditions (A0) – on \( \{ D_n \} \) – and (A1) – on the jumps – hold immediately) concludes the proof of (25).

To end the proof of Proposition 2.3.1 we only need to ensure that there exists \( n_e \geq 3, \xi_e, \tau_e > 0 \) s.t. :

\[
\forall \mu \in \mathcal{M}_1((0,1)], \quad \mu A_{t_e} \in \mathcal{M}_{n_e, \xi_e}.
\]

This can be done exactly as in step 1, Subs. 5.1 of [1] both for a vicinity of 0 and a vicinity of 1.

### 2.4.2 Proof of Proposition 2.3.2

For \( \mu \in \mathcal{M}_{n, \xi} \), with the lower-bound of the mass absorbed at 1 before time 1 :

\[
\Pr_\mu (\tau_1 \leq \tau_{0,\theta}) \geq \xi \Pr_{1/n} (\tau_1 \leq 1) \exp[-\|r\|_\infty]
\]

\[
\mu P_1 \{ 1 \} \geq C_{n,\xi} \exp[-\rho_1 t], \quad \text{with } C_{n,\xi} := \frac{\exp[\|r\|_\infty - \rho_1]}{\xi \Pr_{1/n} (\tau_1 \leq 1)}.
\] (29)

Since \( \tau_0 \leq \tau_{0,1,0} \) and the extinction rate is \( \rho_0 \) once 0 is fixed, then using (16, \( \|\eta\|_\star \)) :

\[
\Pr_\mu (\tau_0 \leq t < \tau_\theta) \leq \Pr_\mu [\exp[-\rho_0 (t - \tau_{0,1,0})] ; \tau_{0,1,0} \leq t]
\]

\[
\leq \exp[-\rho_0 t] \left[ 1 + \rho_0 \int_0^t ds \exp[\rho_0 s]\Pr_\mu (s < \tau_{0,1,0}) \right]
\]

\[
\leq \exp[-\rho_0 t] \left[ 1 + \|\eta\|_\star \times \frac{\rho_0}{\rho_\alpha - \rho_0} \right].
\] (30)
With again \([16, \|\eta_s\|]\), and \([29]\) :

\[
\|\mu A_t - \delta_1\|_{TV} = \mu P_t[0,1)/(\mu P_t[0,1] + \mu P_t\{1\})
\]

\[
\leq \frac{\mathbb{P}_\mu(\tau_0 \leq t < \tau_0) + \mathbb{P}_\mu(t < \tau_0, 1, \cdot)}{\mu P_t\{1\}} \leq C'_{n, \xi} \exp[-(\rho_0 - \rho_1) t]
\]

where \(C'_{n, \xi} := [2 + \|\eta_s\| \times \rho_0/(\rho_0 - \rho_0)] / C_{n, \xi}\). \(\square\)

### 2.4.3 Proof of Proposition 2.3.3

Let \(t \geq 1\) and assume first that \(\mu([0,x]) \geq \xi\) for \(x \in (0,1)\) and \(\xi > 0\).

\[
\mathbb{P}_\mu(t < \tau_0, 1, \cdot) \leq \|\eta_s\| \exp[-\rho_0 t]
\]

With the rough lower-bound \(\mu P_t^1\{0\} \geq \exp[-\|r\|_\infty] \mathbb{P}_\mu(\tau_0 \leq 1)\) :

\[
\begin{align*}
\mathbb{P}_\mu(\tau_0 \leq t < 1, \cdot) & \geq \exp[-\|r\|_\infty] \mathbb{P}_\mu(\tau_0 \leq 1) \times \exp[-\rho_0 (t-1)] \\
& \quad \text{with } C := \frac{\|\eta_s\| \exp[\|r\|_\infty - \rho_0]}{\mathbb{P}_x(\tau_0 \leq 1)} > 0, \\
\|
\mu A_t^1 - \delta_0\|_{TV} & = \frac{\mathbb{P}_\mu(t < \tau_0, 1, \cdot)}{\mathbb{P}_\mu(\tau_0 \leq 1)} \\
& \leq C \exp[-(\rho_0 - \rho_0) t]. \quad (31)
\end{align*}
\]

The case where \(\mu\) has support on \([0,1]\) is trivial, since then \(\mu A_t^1 = \delta_0\).

Finally, for the general case of \(\mu \in \mathcal{M}_1([0,1]) \setminus \{\delta_1\}\), where \(\mu(0,1) > 0\), remark that, for any \(s > 0\), there exists \(m_s \in (0,1)\) s.t. :

\[
\mu A_s^1 = m_s \mu A_0^{11} + (1 - m_s) \delta_0
\]

where for any \(x > 0\) , \(\mu A_s^{01} \underset{s \to \infty}{\rightarrow} \alpha([0,x]) > 0\).

by Proposition 2.3.1 with the rate of convergence uniform over \(\mu\). Thus, we deduce some \(t_\vee > 0\) s.t. :

\[
\forall \mu \in \mathcal{M}_1([0,1]) \setminus \{\delta_1\}, \quad \mu A^1_{t_\vee}([0,x]) \geq \mu A_{t_\vee}^{10} \geq \alpha([0,x])/2 := \xi.
\]

Thus, for any \(t \geq t_\vee\), by \((31)\) :

\[
\|\mu A_t^1 - \delta_0\|_{TV} = \|\mu A_{t_\vee}^1, A_{t-t_\vee}^{11} - \delta_0\|_{TV} \leq C \exp[\rho_0 - \rho_0 t_\vee] \exp[-(\rho_0 - \rho_0) t]. \quad \square
\]

### 2.4.4 Proof of Proposition 2.3.4 and 2.3.5

For this Proposition, we need to adapt the proof of \([7]\). The main step is to prove that the mass on the interval \((0,1)\) does not vanish :

**Lemma 2.4.1.** Assume that \(\rho_0 < \rho_0\). Then, there exists \(n_{xt} \geq 2, \xi_{xt} > 0\) s.t. :

\[
\forall n \in \mathbb{N}, \forall \xi > 0, \exists t_{xt} > 0, \\
\forall \mu \in \mathcal{M}_n, \xi, \forall t \geq t_{xt}, \quad \mu A_t^1(1/n_{xt}, 1 - 1/n_{xt}) \geq \xi_{xt}.
\]
Lemma 2.4.2. Assume that $\rho_\alpha < \rho_0$ and $\alpha^1_\epsilon \in \mathcal{M}_1([0,1])$. Then, there exists $t_{ps}, c_{ps} > 0$ s.t.:

$$\forall x \in [0,1), \forall t \geq t_{ps}, \quad P_x(t < \tau_{1,\theta}) \leq c_{ps} P_{\alpha^1_\epsilon}(t < \tau_{1,\theta}).$$

The measure $\alpha^1_\epsilon$ comes from a mixing estimate that we recall –cf [26] –:

Lemma 2.4.3. Let $n \geq 2$ and $\xi > 0$. Then, there exists $\alpha^1_\epsilon \in \mathcal{M}_1([0,1])$ (possibly independent of $n$ and $\xi$), $t_{mx}, c_{mx} > 0$ s.t.:

$$\forall \mu \in \mathcal{M}_1([0,1]) \text{ s.t. } \mu(1/n, 1 - 1/n) \geq \xi, \quad \mu A^1_{t_{mx}}(dx) \geq c_{mx} \alpha^1_\epsilon(dx).$$

Proof of Proposition 2.3.4: Proposition 2.3.4 is deduced in the same way as Proposition 2.3.3 with \((22, \|\eta^1\|)\) obtained from Proposition 2.3.5.

Proof of Proposition 2.3.5 with Lemmas 2.4.1-3:

Combining these three lemmas and applying exactly the same reasoning as in Subsection 4.3 in [4] proves that there exists a unique QSD $\alpha_1$ associated to $\tau_{1,\theta}$, with the convergence stated in Proposition 2.3.5. Moreover, as stated in Proposition 2.3.5, we can identify $\alpha_1$ and $\eta^1$.

Let $\alpha^y_1 := y \alpha + (1 - y) \delta_0$. For any $t \geq 0$:

$$\alpha^y_1 P^1_t(dx) = y \exp[-\rho_\alpha t] \alpha(dx) + [(1 - y) \exp[-\rho_0 t] + y \tilde{P}_\alpha(\tau_0 \leq t < \tau_{1,\theta})] \delta_0(dx).$$

As proved in Theorem 2.6 in [3], the exit state is independent from the exit time when the initial condition is a QSD, with an exponential law for the exit time. Thus:

$$\tilde{P}_\alpha(\tau_0 \leq t < \tau_{1,\theta}) = E_\alpha[\exp[-\rho_0 (t - \tau_0)]; \tau_0 = \tau_{0,1,\theta} \leq t] = P_\alpha(\tau_0 = \tau_{0,1,\theta}) E_\alpha[\exp[-\rho_0 (t - \tau_{0,1,\theta})]; \tau_{0,1,\theta} \leq t] = P_\alpha(\tau_0 = \tau_{0,1,\theta}) \int_0^t \exp[-\rho_0 (t - s)] \rho_\alpha \exp[-\rho_\alpha s] ds = (\exp[-\rho_\alpha t] - \exp[-\rho_0 t]) \frac{\rho_\alpha P_\alpha(\tau_0 = \tau_{0,1,\theta})}{\rho_0 - \rho_\alpha}. \quad (34)$$

With our choice (20) i.e. $\frac{1 - y_0}{y_0} = \frac{\rho_0 P_\alpha(\tau_0 = \tau_{0,1,\theta})}{\rho_0 - \rho_\alpha}$, we see that we obtain indeed:

$$\alpha^{y_0}_1 P^1_t = \exp[-\rho_\alpha t] \alpha^{y_0}_1.$$

The proof that $\eta^1$ is uniquely defined, the convergences in (19) and (21) and the upper-bound in (22, $\|\eta^1\|$) are exactly the same as in [7]. It remains to identify $\eta^1$. Clearly:

$$\eta^1(0) := \lim_{t \to \infty} \exp[\rho_\alpha t] P_0(t < \tau_{1,\theta}) = \lim_{t \to \infty} \exp[-(\rho_0 - \rho_\alpha) t] = 0. \quad (35)$$

Let

$$\eta^1_t(x) := \exp[\rho_\alpha t] P_x(t < \tau_{1,\theta}) \quad (36, \eta^1_t)$$

$$\eta^1_{2t}(x) = \eta_t(x) (\delta_x A^0_t \mid \eta^1_t) + \eta^t_1(x) \mu A^1_t(0) \eta^1_t(0) \quad (37)$$

11
From (35) and (22. $\|\eta^1\|$), the second term in the right-hand side is clearly negligible. From (34) and (20), we see that:

$$\langle \alpha \mid \eta^1 \rangle = \exp[\rho_\alpha t] \mathbb{P}_\alpha(t < \tau_{0,1,\theta}) + \exp[\rho_\alpha t] \mathbb{P}_x(\tau_0 \leq t < \tau_{1,\theta})$$

$$= 1 + (1 - \exp[-(\rho_0 - \rho_\alpha) t]) \frac{\rho_\alpha \mathbb{P}_\alpha(\tau_0 = \tau_{0,1,\theta})}{\rho_0 - \rho_\alpha}$$

$$\xrightarrow{t \to \infty} 1 + (1 - \eta_\alpha) = 1/\eta_\alpha.$$

(38)

From (37), (38), (15), (14), and (22. $\|\eta^1\|$), we conclude:

$$\eta^1_t(x) \xrightarrow{t \to \infty} \eta(x)/\eta_\alpha = \eta^1(x).$$

□

Proof of Lemma 2.4.2 : From (21), with the notation (36. $\eta^1$):

$$\mathbb{P}_{\alpha t^1}(t < \tau_{1,\theta}) = \langle \alpha^1 \mid \eta^1 \rangle \exp[-\rho_\alpha t]$$

where $\langle \alpha^1 \mid \eta^1 \rangle \xrightarrow{t \to \infty} \langle \alpha^1 \mid \eta \rangle / \eta_\alpha$.

Let thus $t_{ps} > 0$ s.t. $\forall t \geq t_{ps}$, $\langle \alpha^1 \mid \eta^1 \rangle \geq \langle \alpha^1 \mid \eta \rangle / 2 > 0$. (39. $t_{ps}$)

(35) is clearly true and implies with (16. $\|\eta^1\|$) that (32) holds for $x = 0$.

For $x \in (0,1)$ and any $t > 0$:

$$\mathbb{P}_x(\tau_0 \leq t < \tau_{1,\theta}) = \mathbb{E}_x[\exp[-\rho_0(t - \tau_0)] ; \tau_0 = \tau_{0,1,\theta} \leq t]$$

$$\leq \mathbb{E}_x[\exp[-\rho_0(t - \tau_{0,1,\theta})] ; \tau_{0,1,\theta} \leq t]$$

$$= \exp[-\rho_0 t] \left(1 + \rho_0 \int_0^t \exp[\rho_0 s] \times \mathbb{P}_x(s \leq \tau_{0,1,\theta} \leq t) ds\right)$$

$$\leq \exp[-\rho_0 t] \left(1 + \rho_0 \|\eta\| \int_0^t \exp[(\rho_0 - \rho_\alpha) s] ds\right)$$

$$\leq \exp[-\rho_0 t] + \frac{\rho_0 \|\eta\|}{\rho_0 - \rho_\alpha} \exp[-\rho_\alpha t].$$

(40)

Combining (39. $t_{ps}$), (40) and (16. $\|\eta^1\|$) ends the proof of Lemma 2.4.2

□

Proof of Lemma 2.4.1 : Let $n_{xt} \geq 3$ s.t.:

$$\alpha(1/n_{xt} , 1 - 1/n_{xt}) \geq 1/2$$

(41. $n_{xt}$)

From (14. $\alpha$) and (15), we can find $t_{sb} > 0$ s.t. for any $\mu$ with $\mu(0,1) > 0$:

$$\forall t \geq t_{sb}, \quad \mathbb{P}_\mu(t < \tau_{0,1,\theta}) \geq \langle \mu \mid \eta \rangle / 2 \times \exp[-\rho_\alpha t],$$

$$\|\mu A^0_{1} - \alpha\|_{TV} \leq 1/4$$

Thus $\mu A^0_{1}(1/n_{xt} , 1 - 1/n_{xt}) \geq 1/4$. (42. $t_{sb}$)

Since 0 is absorbing and by (42. $t_{sb}$):

$$\forall t \geq 0, \quad \mu A^1_{1}(dx) = \mu A^1_{1}(0,1) \times \mu A^0_{1}(dx) + [1 - \mu A^1_{1}(0,1)] \delta_0(dx),$$

$$\forall t \geq t_{sb}, \quad \mu A^1_{1}(1/n_{xt} , 1 - 1/n_{xt}) \geq \mu A^1_{1}(0,1)/4$$

where:

$$\mu A^1_{1}(0,1) = \left(1 + \frac{\mathbb{P}_\mu(\tau_0 \leq t < \tau_{\theta}) t < \tau_{1,\theta})}{\mathbb{P}_\mu(t < \tau_{0,1,\theta}) t < \tau_{1,\theta})}\right)^{-1} \left(1 + \frac{\mathbb{P}_\mu(\tau_0 \leq t < \tau_{\theta})}{\mathbb{P}_\mu(t < \tau_{0,1,\theta})}\right)^{-1}.$$

(44)
Assume first that $\mu[1/n, 1 - 1/n] \geq \xi$ for some $n \geq 3$ and $\xi > 0$. Since $\eta$ is positive on $(0, 1)$, this implies, with (15), (40), (43) and (44), a lower-bound $\xi_{xt}$ that only depends on $n$ and $\xi$ s.t.:

$$\forall t \geq t_{sb}, \quad \mu A_t^1(1/n_{xt}, 1 - 1/n_{xt}) \geq \xi_{xt}. \quad (45)$$

The following lemma (already needed for the uniform bound in (14)) completes the proof:

**Lemma 2.4.4.**

$$\exists x_\vee \in (0, 1), \exists n' \geq 3, \exists t_e, \xi_e > 0, \forall x \in [x_\vee, 1), \quad \delta_x A_{t_e}^1(1/n', 1 - 1/n') \geq \xi_e.$$  

Indeed, if $\mu \in \mathcal{M}_{n, \xi}$ (w.l.o.g. $\mu\{1\} = 0$ since it vanishes immediately), either $\mu(1/n, x_\vee) \geq \xi/2$ and we deduce the result from (45), or $\mu(x_\vee, 1) \geq \xi/2$ and we deduce from Lemma 2.4.4 and (45):

$$\forall t \geq t_{sb} + t_e, \quad \mu A_t^1(1/n_{xt}, 1 - 1/n_{xt}) = [\mu A_{t_e}^1] A_{t - t_e}^1(1/n_{xt}, 1 - 1/n_{xt}) \geq \xi'_{xt}. \quad \square$$

2.4.5 Proofs of Proposition 2.3.6:

The calculations leading to (40) gives for the case $\rho_0 = \rho_\alpha$:

$$\forall \mu \in \mathcal{M}_1([0, 1]), \quad \mathbb{P}_\mu(\tau_0 \leq t < \tau_{1, \partial}) \leq \exp[-\rho_0 t] \left(1 + \rho_0 \|\eta\|_1 t\right). \quad (46)$$

With (46) instead of (30), like in the proof of Proposition 2.3.2 (i.e. with (16) $\|\eta\|_1$ and (29)), we deduce Proposition 2.3.6 \square

2.4.6 Proof of Proposition 2.3.7:

Now, since $\langle \mu \mid \eta \rangle$ is uniformly lower-bounded for $\mu \in \mathcal{M}_{n, \xi}^{0, 1}$ (for any $n \geq 3, \xi > 0$), by (15), for $t$ sufficiently large and any $\mu \in \mathcal{M}_{n, \xi}^{0, 1}$:

$$\mathbb{P}_\mu(t < \tau_{0, 1, \partial}) \geq c_{n, \xi} \exp[-\rho_0 t].$$

Combining this with (44) and (46) concludes the proof that for $t$ sufficiently large:

$$\|\mu A_t^1 - \delta_0\|_{TV} \geq C_{n, \xi}/t. \quad \square$$

**Remark:** To deal more precisely with the mass around 1, one may exploit Lemma 2.4.4 to conclude that the convergence is uniform for any $\mu$ s.t. $\mu A^1_{t_e}(1/n, 1) \geq \xi$ (?).

For the reverse inequality, assume first that $\mu \in \mathcal{M}_{n, \xi}^{0, 1}$. Using once more (15):

$$\mathbb{P}_\mu(\tau_0 \leq t < \tau_{1, \partial}) = \mathbb{E}_\mu \left[\exp[-\rho_0 t] \left(1 + \rho_0 \int_0^{\tau_{0, 1, \partial}} \exp[\rho_0 s] \, ds\right) ; \tau_0 = \tau_{0, 1, \partial} \leq t\right]$$

$$\geq \rho_0 \exp[-\rho_0 t] \int_0^t \exp[\rho_0 s] \mathbb{P}_\mu(\tau_0 = \tau_{0, 1, \partial} \in [s, t]) \, ds,$$

$$\geq c_{n, \xi} \exp[-\rho_0 t] \int_0^t \mathbb{P}_{\mu A^0_1}(\tau_0 = \tau_{0, 1, \partial} \leq t - s) \, ds. \quad (47)$$
Since \( \mathbb{P}_\alpha(\tau_0 = \tau_{0,1,0}) > 0 \) and by monotone convergence, there exists \( t_V > 0 \) s.t.:
\[
\forall t \geq t_V, \quad \mathbb{P}_\alpha(\tau_0 = \tau_{0,1,0} \leq t) \geq \mathbb{P}_\alpha(\tau_0 = \tau_{0,1,0} \leq t_V) := m_0 > 0. \tag{48. \ t_V}
\]
Now, according to (14, \( \alpha \)), we choose \( t_{sb} > 0 \) s.t.:
\[
\forall \mu \in \mathcal{M}_1((0, 1)), \forall s \geq t_{sb}, \quad \|\mu A^0_\alpha - \alpha\|_{TV} \leq m_0/2
\]
\[
\Rightarrow \quad \forall t - s \geq t_V, \quad \mathbb{P}_{\mu A^0_\alpha}(\tau_0 = \tau_{0,1,0} \leq t - s) \geq m_0/2. \tag{49. \ t_{sb}}
\]
Thus, (49. \( t_{sb} \)) and (47) imply that for any \( t \geq t_{sb} + t_V \):
\[
\mathbb{P}_\mu(\tau_0 \leq t < \tau_{1,0}) \geq \mathcal{C}_{n,\xi} \exp(-\rho_0 t) \times (t - t_{sb} - t_V).
\]
With (16. \( \|\eta^\bullet\| \)) and (44), this concludes the proof that:
\[
\mu[1/n, 1 - 1/n] \geq \xi \Rightarrow \quad \forall t \geq t_{sb} + t_V, \quad \|\mu A^1_t - \delta_0\|_{TV} \leq \mathcal{C}_{n,\xi}/t. \tag{50}
\]
Now, we prove that such upper-bound is in fact uniform thanks to Lemma 2.4.4 and its symmetrical counterpart (where 0 replaces 1). Indeed,
\[
\mu A^1_{te}(dx) = \mu A^1_{te}(0, 1) \times \mu A^0_{t_{sb}}(dx) + [1 - \mu A^1_{te}(0, 1)] \delta_0(dx),
\]
where \( \exists \xi_e > 0, \exists n_e \geq 2, \forall \mu \in \mathcal{M}_1((0, 1)), \quad \mu A^0_{t_{sb}}(1/n_e, 1 - 1/n_e) \geq \xi_e \)

Thus by (50):
\[
\forall t \geq t_{sb} + t_V, \quad \|\mu A^0_{t_{sb}} A^1_t - \delta_0\|_{TV} \leq \mathcal{C}_{n_e,\xi_e}/t. \tag{51}
\]
Since there exists \( y_t \in (0, 1) \) s.t.:
\[
\mu A^1_{te+t}(dx) = y_t [\mu A^0_{t_{sb}}] A^1_t + (1 - y_t) \delta_0,
\]
(51) concludes the proof of Proposition 2.3.7 (where \( t_{sb} + t_V \) replaces \( t_V \)). \( \square \)

**Remark**: In fact, our comparison of the survival from 0 and from \( \mu \) gives us a uniform upper-bound \( C > 0 \) s.t.:
\[
y_t = \mu A^1_t(0, 1) \times \left\langle \mu A^0_{t_{sb}} \big| \mathbb{P}(t < \tau_{1,0}) \right\rangle \leq C \mu A^1_t(0, 1)
\]
\[
\Rightarrow \quad \|\mu A^1_{te+t} - \delta_0\|_{TV} \leq \mu A^1_{te}(0, 1) \times C/t.
\]

### 2.4.7 Proof of Proposition 2.3.8:

Since \( \rho_0 = \rho_1 \), it is straightforward that any convex combination of \( \delta_0 \) and \( \delta_1 \) is a QSD, with extinction rate \( \rho_1 \).

It is then not difficult to adapt the proof of Proposition 2.3.3 and since \( \mathbb{P}_\mu(\tau_{0,1} \leq 1) \) is lower-bounded uniformly over any \( \mu \in \mathcal{M}_1([0, 1]) \), we obtain
\[
\forall \mu \in \mathcal{M}_1([0, 1]), \quad \mu A_t(0, 1) \leq C \exp[-(\rho_0 - \rho_1) t].
\]
\[ \mu A_t \{ 0 \} = \frac{E_{\mu} \left[ \exp \left[ -\rho_1 \left( t - \tau_{0,1,0} \right) \right] \right]}{P_{\mu} \left( t < \tau_{0,1,0} \right) + E_{\mu} \left[ \exp \left( -\rho_1 \left( t - \tau_{0,1,0} \right) \right) \right]} \quad \tau_{0,1,0} = \tau_0 \leq t \]

\[ = \frac{E_{\mu} \left[ \exp \left[ \rho_1 \tau_{0,1,0} \right] ; \tau_{0,1,0} = \tau_0 \leq t \right]}{E_{\mu} \left[ \exp \left[ \rho_1 \tau_{0,1,0} \right] ; \tau_{0,1,0} = \tau_0 \leq t \right]} \times \left( 1 + \frac{\exp[\rho_1 t] P_{\mu}(t < \tau_{0,1,0})}{E_{\mu} \left[ \exp \left( \rho_1 \tau_{0,1,0} \right) \right]} \right)^{-1} \quad \text{for } \tau_{0,1,0} = \tau_0 \leq t \]

The limit as \( t \to \infty \) is well-defined and the convergence occurs at exponential rate since:

\[ 0 \leq E_{\mu} \left[ \exp \left[ \rho_1 \tau_{0,1,0} \right] \right] - E_{\mu} \left[ \exp \left[ \rho_1 \tau_{0,1,0} \right] \right] \leq \rho_1 \int_{\mathbb{R}} \exp[\rho_1 s] P_{\mu A^t}(s < \tau_{0,1,0}) \, ds \]

\[ \leq \left\| \eta_\star \right\| \exp \left[ \rho_0 t \right] \left[ 1 + \rho_1 \rho_0 \right] \exp[\rho_0 t] := C \exp \left[ -\rho_0 t \right] \]

The same holds of course for the case \( \{ \tau_{0,1,0} = \tau_{0,1} \} \) and \( E_{\mu} \left[ \exp \left( \rho_1 \tau_{0,1,0} \right) \right] \leq \rho_1, \tau_{0,1,0} = \tau_{0,1} \leq t \) converges with exponential rate. Therefore with (52) and the well-defined notation (23, \( x(\mu) \)) we can define some \( C > 0 \) s.t. \( \forall \mu \in \mathcal{M}_1([0,1]) : \)

\[ \left| \mu A_t \{ 1 \} - (1 - x(\mu)) \right| \vee \left| \mu A_t \{ 0 \} - x(\mu) \right| \vee \left| \mu A_t(0,1) \right| \leq C \exp \left[ -\rho_0 \, t \right] \]

which concludes the proof of Proposition 2.3.8. \( \square \)

### 2.4.8 Proof of Proposition 2.3.9 :

This proof is very similar to the one of Proposition 2.3.5 so we won’t go into much detail. Lemmas 2.4.3 and 2.4.2 are of course replaced by:

**Lemma 2.4.5.** Assume that \( \rho_0 < \rho := \rho_0 \wedge \rho_1 \). Then, there exists \( n_{xt} \geq 3, \xi_{xt} > 0 \) s.t.:

\[ \forall n \in \mathbb{N}, \forall \xi > 0, \exists t_{xt} > 0, \quad \forall \mu \in \mathcal{M}^{01}_{n,\xi}, \forall t \geq t_{xt}, \quad \mu A_t(1/n_{xt}, 1 - 1/n_{xt}) \geq \xi_{xt}. \]

**Lemma 2.4.6.** Assume that \( \rho_0 < \rho := \rho_0 \wedge \rho_1 \) and \( \alpha^1 \in \mathcal{M}_1([0,1]) \). Then, there exists \( t_{ps}, c_{ps} > 0 \) s.t.:

\[ \forall x \in [0,1], \forall t \geq t_{ps}, \quad P_x(t < \tau_0) \leq c_{ps} P_{\alpha^1}(t < \tau_0). \]

We leave the proofs to the reader, and just mention that we can take as an upper-bound for \( P_x(\tau_1 \leq t < \tau_0) \) the same formula as for \( P_x(\tau_0 \leq t < \tau_1) = P_x(\tau_0 \leq t < \tau_{1,0}) \), with \( \rho_1 \) instead of \( \rho_0 \) (cf 40).

For the rest of the proof, we remark that, for \( \alpha^1 := y_a \alpha + y_0 \delta_0 + y_1 \delta_1 \) with \( y_a + y_0 + y_1 = 1 \), (53) has to be changed by:

\[ \alpha^1 P_t(dx) = y_a \exp[-\rho_0 t] \alpha(dx) + [y_0 \exp[-\rho_0 t] + y_a \exp[-\rho_0 \tau_0 \leq t < \tau_{1,0}] \delta_0(dx) + [y_1 \exp[-\rho_1 t] + y_a \exp[-\rho_1 \tau_1 \leq t < \tau_{1,0}] \delta_1(dx). \]

Again: \( \alpha^1 P_t(dx) = \exp[-\rho_0 t] \alpha^1(dx) \) iff the conditions in (24, \( \alpha_{01} \)) are satisfied. \( \square \)
2.4.9 Proof of Proposition 2.3.10:

Let us first prove that we only need to control \( \| \mu A_t^0 - \delta_1 \|_{TV} \) like it is done in Proposition 2.3.7. From Proposition 2.3.5, we know that for some \( \alpha_1 := y_0 \alpha + y_0 \delta_0 \), with \( y_0, y_0 \in (0, 1) \), there exists \( C^1, \zeta^1 > 0 \) s.t.:

\[
\| \mu A_t^1 - \alpha_1 \|_{TV} \leq C^1 \exp[-\zeta^1 t].
\]  \hspace{1cm} (54. \( \zeta_1 \))

Consequently, for \( t \) sufficiently large:

\[
\frac{y_0}{2 y_\alpha} \leq \frac{\mu A_t \{0\}}{\mu A_t \{0, 1\}} \leq \frac{2 y_0}{y_\alpha}.
\]  \hspace{1cm} (55)

On the other hand:

\[
\| \mu A_t - \delta_1 \|_{TV} = \left[ 1 + \frac{\mu A_t \{1\}}{\mu A_t \{0, 1\} + \mu A_t \{0\}} \right]^{-1}, \quad \| \mu A_t^0 - \delta_1 \|_{TV} = \left[ 1 + \frac{\mu A_t \{1\}}{\mu A_t \{0, 1\}} \right]^{-1}.
\]

Consequently, (55) implies that \( \| \mu A_t - \delta_1 \|_{TV} \) has the same rate of convergence as \( \| \mu A_t^0 - \delta_1 \|_{TV} \) (as long as it indeed converges to 0).

Now, from the proof of Proposition 2.3.7, we deduce quite immediately:

- with the notation \( \mu A_t^0 (dx) := \mathbb{P}_\mu (X_t \in dx \mid t < \tau_{0, \theta}) \)
  \( \exists t_\nu, C > 0, \forall t \geq t_\nu, \forall \mu \in \mathcal{M}_1([0, 1]), \| \mu A_t^0 - \delta_1 \|_{TV} \leq C/t, \)
- \( \forall n \geq 3, \forall \xi > 0, \exists t_{n, \xi}, c_{n, \xi} > 0, \forall t \geq t_{n, \xi}, \forall \mu \in \mathcal{M}_1^{n, \xi}, \| \mu A_t^0 - \delta_1 \|_{TV} \geq c_{n, \xi}/t. \) \hspace{1cm} (56. \( \eta_t \))

2.4.10 Proof of Proposition 2.3.11:

Any convex combination of \( \delta_0 \) and \( \delta_1 \) is clearly a QSD with extinction rate \( \rho := \rho_0 = \rho_1 = \rho_\alpha \).

For \( t \geq 0 \) and \( x \in [0, 1] \), let:

\[
\eta_t(x) := \exp[\rho t] \mathbb{P}_x (t < \tau_{0, 1, \theta}) \quad \text{and} \quad E_t^0(x) := \mathbb{E}_\mu \left[ \exp[\rho \tau_{0, 1, \theta}] \mid \tau_{0, 1, \theta} = \tau_0 \leq t \right] \quad \text{and} \quad E_t^1(x) := \mathbb{E}_\mu \left[ \exp[\rho \tau_{0, 1, \theta}] \mid \tau_{0, 1, \theta} = \tau_1 \leq t \right].
\]  \hspace{1cm} (56. \( \eta_t \))

Let then \( k \geq 1 \) and \( \mu \in \mathcal{M}_1([0, 1]) \) with \( \mu(0, 1) > 0 \), so that \( \langle \mu \mid \eta \rangle > 0 \). Then:

\[
\langle \mu \mid E_t^k \rangle = \sum_{j=0}^{k-1} \langle \mu \mid \eta_j \rangle \langle \mu A_j^0 \mid E_0^1 \rangle,
\]

where by (14. \( \alpha \)) and (15. \( C \)), with the upper-bound \( e^\rho \) of \( E_t^1 \), there exists \( C > 0 \) s.t.:

\[
| \langle \mu \mid \eta_j - \eta \rangle | \leq C \exp[-j \zeta], \quad | \langle \mu A_j^0 - \alpha \mid E_0^1 \rangle | \leq C \exp[-j \zeta].
\]

Consequently:

\[
| \langle \mu \mid E_t^k \rangle - k \langle \mu \mid \eta \rangle \langle \alpha \mid E_0^1 \rangle | \leq 2 C/(1 + \exp[-\zeta]) < \infty.
\]  \hspace{1cm} (58)

Likewise \( | \langle \mu \mid E_t^k + E_t^1 \rangle - k \langle \mu \mid \eta \rangle \langle \alpha \mid E_0^1 + E_1^1 \rangle | \leq 4 C/(1 + \exp[-\zeta]) < \infty. \)
From (52) and (16, ||η||), we deduce that there exists $C' > 0$ s.t. :

$$\left| \mu A_k \{0\} - \frac{\langle \alpha \cdot E_0 \rangle}{\langle \alpha \cdot E_0 + E_1 \rangle} \right| \leq \frac{C'}{k \langle \mu \cdot \eta \rangle}$$

(59)

The symmetrical result for $\mu A_k \{1\}$ holds of course true, and since the sum of the limits equals 1, we deduce also

$$|\mu A_k(0,1)| \leq \frac{C'}{k \langle \mu \cdot \eta \rangle}.$$  

(60)

Again, from Theorem 2.6 in [5], the exit state is independent from the exit time when the initial condition is a QSD, with an exponential law for the exit time. Thus :

$$\langle \alpha \cdot E_0 \rangle = \mathbb{P}_\alpha (\tau_0 = \tau_{0,1,0}, \int_0^1 \exp[\rho s] \rho \exp[-\rho s] ds = \mathbb{P}_\alpha (\tau_0 = \tau_{0,1,0}).$$

To end the proof, just remark that $\langle \mu \cdot \eta \rangle$ is lower-bounded for any $\mu \in \mathcal{M}_{n, \xi}^0$. □

### 2.4.11 Proof of Proposition 2.3.12:

We choose arbitrary some $t_0$, for instance $t_0 := 1$. We show then that there exists $c > 0$ such that for any $t > 0$, with $\sigma$ sufficiently large :

$$\sup_{x \in (0,1)} \mathbb{P}_x (t_0 < \tau_{0,1,0}) \leq c.$$

By the Markov property, choosing $t$ sufficiently small (to ensure $c \leq \exp[-\rho t]$ with large value of $\rho$), it implies : $(-1/t) \cdot \log[\mathbb{P}_x(t < \tau_{0,1,0})] \to +\infty$

and a fortiori the result on the QSD.

Since $r$ is bounded, the result (for $t \leq t_0$) is equivalent to the case where $r = 0$.

We can notice that

$$X(t_0) = x_0 + s \cdot T(t_0) + \sigma \tilde{B}[T(t_0)] \text{ with } T(t_0) := \int_0^{t_0 / \tau_0,1} \sigma^2 X_u (1 - X_u) du,$$

and $\tilde{B}$ has the law of a Brownian Motion.

For initial conditions in $[\delta, 1 - \delta]$, consider $\tau_\delta := \inf\{t \geq 0 ; X_t \notin [\delta, 1 - \delta]$. The previous inequalities ensure : $\mathbb{P}_x(\tau_\delta \leq t/2, t < \tau_{0,1}) \leq \epsilon$. On the other hand, by Itô's formula :

$$\mathbb{E}_x \left( X_t - x - \int_0^{t/2 \wedge \tau_\delta} s X_u (1 - X_u) du \right)^2 = \mathbb{E}_x \left( \int_0^{t/2 \wedge \tau_\delta} \sigma \sqrt{X_u (1 - X_u)} dB_u \right)^2$$

$$= \mathbb{E}_x \left[ \int_0^{t/2 \wedge \tau_\delta} \sigma^2 X_u (1 - X_u) du \right]$$

thus $\mathbb{P}_x(t/2 < \tau_\delta) \times (\sigma \delta)^2 t/2 \leq C = (2 + s t/4)^2$, independent from $x$.

For $\sigma$ sufficiently large, this term is indeed lower than $\epsilon$. Since $\mathbb{P}_x(t < \tau_{0,1}) \leq \mathbb{P}_x(\tau_\delta \leq t/2, t < \tau_{0,1}) + \mathbb{P}_x(t/2 < \tau_\delta)$, this concludes the proof. □
2.4.12 Proof of Proposition 2.3.13:

Define \( r_2, r_3 \) such that \( \max r(x) < r_3 < r_2 < r(1) \wedge r(0) \) and the open sets \( A := r^{-1}([0, r_3)) \subset B := r^{-1}([0, r_2)) \subset (0, 1) \) (recall that \( r \) is assumed to be continuous). We choose arbitrary \( t_0 \). A classical result on diffusion ensures that there exists \( \rho > 0 \) such that:

\[
\inf_{x \in A} \mathbb{P}_x (X_{t_0} \in A, \forall s \leq t, X_s \in B) \geq \exp[-\rho t_0].
\]

Then, it implies by the Markov property:

\[
\inf_{x \in A} \mathbb{P}_x (\forall s \leq t, X_s \in B; t < \tau_{0,1}) \geq C \exp[-(\rho + R r_2) t].
\]

From the Harnack inequality, we know that \( \alpha^{(R)} \) has a lower-bounded density on any open set of \((0, 1)\) so that \( \alpha^{(R)}(A) > 0 \) and

\[
\alpha^{(R)}(B) \geq \exp[\rho^{(R)}_\alpha t] \mathbb{P}_{\alpha^{(R)}} (\forall s \leq t, X_s \in B; t < \tau_{0,1}) \geq C \alpha^{(R)}(A) \exp[-(\rho + R r_2 - \rho^{(R)}_\alpha) t].
\]

This proves \( \rho^{(R)}_\alpha \leq \rho + R r_2 < R (r(0) \wedge r(1)) = \rho^{(R)}_0 \wedge \rho^{(R)}_1 \) for \( R \) sufficiently large. \( \square \)

2.4.13 Proof of Proposition 2.3.14:

In this case:

\[
\rho^{(R)}_0 \wedge \rho^{(R)}_1 = R \times r^0(0) \lor r^0(1) \xrightarrow{R \to 0} 0.
\]

In fact, \( \rho^{(R)}_\alpha \xrightarrow{R \to 0} \rho^{(0)}_\alpha > 0 \), where \( \rho^{(0)}_\alpha \) is thus the death rate of the QSD for the Wright-Fisher diffusion conditioned not to touch the boundary, with \( r = 0 \).

By Proposition 2.3.1, for any \( x \in (0, 1) \) and \( t \geq 0 \):

\[
\mathbb{P}_x (t < \tau_{0,1}) \leq \|\eta\| \exp[-\rho^{(0)}_\alpha t] \mathbb{P}_x (t < \tau_{0,1}) \leq \|\eta\| \exp[-(\rho^{(0)}_\alpha - R \|r\|_\infty) t]
\]

and in particular, with the QSD \( \alpha(R) \) as initial condition, with deduce \( \rho^{(R)}_\alpha \leq \rho^{(0)}_\alpha - R \|r\|_\infty \).

Conversely:

\[
\exp[-(\rho^{(0)}_\alpha + R \|r\|_\infty - \rho^{(R)}_\alpha) t] \leq \exp[+(\rho^{(R)}_\alpha) t], \mathbb{P}_{\alpha^{(R)}} (t < \tau_{0,1}) \xrightarrow{t \to \infty} \langle \alpha^{(0)} \mid \eta^{(R)} \rangle,
\]

which implies \( \rho^{(R)}_\alpha \leq \rho^{(0)}_\alpha + R \|r\|_\infty \). \( \square \)

2.4.14 Proof of Proposition 2.3.15:

3 Tightness

(largely taken from [6])

The proof of Theorem 1.1 follows a standard procedure [11, 10, 3] in which we prove:

(i) the tightness of the sequence of stochastic processes – which implies a subsequential limit, and (ii) the uniqueness of this limit. For the tightness of \( \{\mu^{m,n}_t\}_{m,n} \) on \( D([0, T], \mathcal{P}([0, 1])) \), it is sufficient, by Theorem 14.26 in Kallenberg [12] to show that \( \{\mu^{m,n}_t \mid f\} \) is tight on \( D([0, T], \mathbb{R}) \) for any test function \( f \) from a countably dense subset of continuous, positive functions on \([0, 1]\).
3.1 Semimartingale property of multilevel selection process

It will be useful for what follows to treat $\langle \mu_{t}^{m,n} | f \rangle$ as a semimartingale. Below, $D_{+}^{f}$ is the first order difference quotient of $f$ taken from the right, $D_{-}^{f}$ is the first order difference quotient of $f$ taken from the left, and $D_{xx}^{f}$ is the second order difference quotient.

We recall that in our limit, $n, m \to \infty$, $wI/n \to \omega_I$, $n \times s \to \sigma$, $wG$ and $\{r(x)\}$ are fixed. It is easy to adapt the proof of [6] in order to state:

**Lemma 3.1.1.** For $f \in C^2([0, 1])$ and $\mu_{t}^{m,n}$ with generator $L_{m,n}$ defined in (1),

$$\langle \mu_{t}^{m,n} | f \rangle - \langle \mu_{0}^{m,n} | f \rangle = A_{t}^{m,n}(f) + M_{t}^{m,n}(f)$$

where $A_{t}^{m,n}(f)$ is a process of finite variation, $A_{t}^{m,n}(f) := \int_{0}^{t} a_{s}^{m,n}(f) ds$, with:

$$a_{t}^{m,n}(f) = \frac{wI}{n} \sum_{i} \mu_{t}^{m,n}(j/n) \frac{i}{n} (1 - i/n) \left[ D_{xx} f (j/n) - n \times s D_{-} f (j/n) \right]$$

$$+ wG \left\{ \sum_{j} \mu_{t}^{m,n}(j/n) r(j/n) f(j/n) - \sum_{i} \mu_{t}^{m,n}(i/n) f(i/n) \sum_{j} \mu_{t}^{m,n}(j/n) r(j/n) \right\}$$

and $M_{t}^{m,n}(f)$ is a càdlàg martingale with (conditional) quadratic variation:

$$\langle M^{m,n}(f) \rangle_{t} = \frac{1}{m} \int_{0}^{t} \left\{ \frac{wI}{n} \sum_{i} \mu_{s}^{m,n}(i/n) \frac{i}{n} (1 - i/n) \left[ (D_{+} f (i/n))^2 + (1 + \frac{n}{s/n}) (D_{-} f (i/n))^2 \right]$$

$$+ wG \sum_{i,j} \mu_{s}^{m,n}(i/n) \mu_{s}^{m,n}(j/n) \left( 1 + r(j/n) \right) \left[ f(i/n) - f(j/n) \right] \right\} ds$$

3.2 Proof of the convergence to our limit

We prove here that the drift term is tight while the martingale converges to zero.

For the finite variation term $A_{t}^{m,n}(f)$, assuming w.l.o.g. $wI/n \leq 2 \omega_I$, $n/s \leq 2 \sigma$:

$$|a_{t}^{m,n}(f)| \leq \frac{\omega_I}{2} \left[ \|f''\|_{\infty} + 2 \sigma \|f'\|_{\infty} \right] + 2 wG \|r\|_{\infty} \|f\|_{\infty} := G_{f}$$

therefore: $\sup_{t \in [0,T]} |A_{t}^{m,n}(f)| \leq G_{f}T$

where $G_{f}$ is a constant that depends on $f$. Moreover, for any prescribed $\epsilon$, we can always choose $\delta_{\epsilon}$ to be sufficiently small so that, for any $0 \leq t \leq t + \delta$ with $\delta \leq \delta_{\epsilon}$, for any $n, m$:

$$|A_{t+\delta}^{m,n} - A_{t}^{m,n}| \leq \delta G_{f} \leq \epsilon.$$ By Prop. 3.26 chp. 3 in [5], this proves immediately that the sequence $(A_{t}^{m,n})$ is tight, and any limit is continuous.

For the martingale part, assuming w.l.o.g. $n/s \leq 1$:

$$\langle M_{t}^{m,n}(f) \rangle_{t} \leq \frac{T}{m} \left\{ \frac{3 \omega I}{2} \|f''\|_{\infty} + wG (1 + \|r\|_{\infty}) \|f\|_{\infty}^2 \right\} := J_{f}/m \xrightarrow{m \to \infty} 0,$$
where \( J_f \) is a constant only depending on \( T > 0 \) and \( f \in C^2([0,1]) \). From Burkholder-Davis-Gundy’s inequality, since the jumps of \( M_t^{m,n}(f) \) are bounded by \( \|f\|_\infty/m : \\
\mathbb{E} \left[ \sup_{t \leq T} (M_t^{m,n}(f))^2 \right] \leq C J_f/m + \|f\|_\infty^2/m^2 \overset{n,m \to \infty}{\longrightarrow} 0.
This proves that \( M_t^{m,n}(f) \) converges to 0 in such a way that \( \langle \mu_t^{m,n} \mid f \rangle - \langle \mu_0^{m,n} \mid f \rangle \) is tight and any associated limit is continuous.

4 Decomposing the state space: special behavior at 0 and 1

Extension (useful?) Originally, I intended to prove in this part the stabilization of the transition rates towards 0 and 1. It appears however that my method (based on total variation) is clearly unsuited for such local properties (depend on the values in the vicinities of resp. 0 and 1). These results may however have an interest by themselves (?).

Since \( \forall x \), \( \mathbb{P}_x(\tau_0 \wedge t_0 \wedge t_1 < \infty) = 1 \), we can decompose \( \xi \in \mathcal{M}_1([0,1]) \) according to these different events, i.e. :

\[
\xi_t = y_t^0 \xi_t^0 + y_t^\alpha \xi_t^\alpha + y_t^r \xi_t^r
\]

where
\[
\langle \xi_t^r , f \rangle := \frac{\mathbb{E} \left[ f(X_t) Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_0,1) ; t < \tau_0,1 \right]}{\mathbb{E} \left[ Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_0,1) ; t < \tau_0,1 \right]}.
\]

\[
y_t^r := \frac{\mathbb{E} \left[ Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_0,1) ; t < \tau_0,1 \right]}{\mathbb{E} \left[ Z_t ; t < \tau_0,1 \right]},
\]

\[
\langle \xi_t^0 , f \rangle := \frac{\mathbb{E} \left[ f(X_t) Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_1 \wedge \tilde{\tau}_0) ; t < \tau_0,1 \right]}{\mathbb{E} \left[ Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_1 \wedge \tilde{\tau}_0) ; t < \tau_0,1 \right]},
\]

\[
y_t^0 := \frac{\mathbb{E} \left[ Z_t \mathbb{P}_{X_t} (\tilde{\tau}_0 < \tilde{t}_1 \wedge \tilde{\tau}_0) ; t < \tau_0,1 \right]}{\mathbb{E} \left[ Z_t ; t < \tau_0,1 \right]},
\]

and likewise for \( y_t^1 \) and \( \xi_t^1 \). We refer to \([3]\), Sect. 2.5, for the proof that, the law of \( X \) conditionally on \( \{\tau_0 < \tau_1,0\} \) (resp. \( \{\tau_0 < \tau_1,1\} \), \( \{\tau_1 < \tau_0,0\} \) is the one of a Markov process. For simplicity, we denote the associated law by \( \mathbb{P}^r \) (resp. \( \mathbb{P}^0 \), \( \mathbb{P}^1 \)). Moreover, if \( \alpha \) is a QSD of \( X \) associated to the extinction time \( \tau_{0,1,\alpha} := \tau_{0,1,\alpha} \) under \( \mathbb{P} \), then, there exists a QSD \( \alpha^r \) under \( \mathbb{P}^r \) (idem for \( \alpha^0 \) under \( \mathbb{P}^0 \), \( \alpha^1 \) under \( \mathbb{P}^1 \)), defined as :

\[
\alpha^r(dx) := \frac{\mathbb{P}_x(\tau_0 < \tau_{0,1})}{\mathbb{P}_x(\tau_0 < \tau_{0,1})} \alpha(dx).
\]

It is then quite straightforward that the extinction rate \( \lambda_\alpha \) is the same for \( \alpha \) and for \( \alpha^r \) (for \( \alpha^0 \), \( \alpha^1 \) also). Such results can be applied since :

**Theorem 4.1.** Under \( \mathbb{P} \), there exists a unique QSD \( \alpha \). Moreover :

\[
\exists C, \zeta > 0, \quad \forall \mu \in \mathcal{M}_1([0,1]), \forall t > 0,
\]

\[
\|\mathbb{P}_\mu \left[ X_t \in dx \mid t < \tau_{0,1,\alpha}, \tau_0 < \tau_{0,1} \right] - \alpha(dx) \|_{TV} \leq C e^{-\zeta t} \tag{67}
\]

\[
| \exp[\lambda_\alpha t] \mathbb{P}_\mu \left[ t < \tau_{0,1,\alpha} \mid \tau_0 < \tau_{0,1} \right] - \langle \mu \mid \eta \rangle | \leq C e^{-\zeta t} \tag{68}
\]

—and the other results for the Q-process.
Yet, to prove the uniqueness of and the rate of convergence to such QSDs, we need to consider more thoroughly the process $X$ under $\mathbb{P}^r$ (resp. $\mathbb{P}^0$, $\mathbb{P}^1$).

**Theorem 4.2.** Under $\mathbb{P}^r$, there exists a unique QSD $\alpha^r$. Moreover:

$$\exists C^r, \zeta^r > 0, \quad \forall \mu \in \mathcal{M}_1[(0,1)], \forall t > 0,$$

$$\| \mathbb{P}_\mu \left[ X_t \in dx \right] t < \tau_{0,1,0}, \tau_0 < \tau_{0,1} \|_{TV} - \alpha^r(dx) \leq C^r e^{-\zeta^r t} \quad (69)$$

$$\| \exp[\lambda_0 t] \mathbb{P}_\mu \left[ t < \tau_{0,1,0}, \tau_0 < \tau_{0,1} \right] - \langle \mu, \eta^r \rangle \|_\infty \leq C^r e^{-\zeta^r t} \quad (70)$$

—and the other results for the Q-process.

**Theorem 4.3.** Under $\mathbb{P}^0$ (the same with 1 instead of 0), there exists a unique QSD $\alpha^0$. Moreover:

$$\exists C^0, \zeta^0 > 0, \quad \forall \mu \in \mathcal{M}_1[(0,1)], \forall t > 0,$$

$$\| \mathbb{P}_\mu \left[ X_t \in dx \right] t < \tau_{0,1,0}, \tau_0 < \tau_{0,1} \|_{TV} - \alpha^0(dx) \leq C^0 e^{-\zeta^0 t} \quad (71)$$

$$\| \exp[\lambda_0 t] \mathbb{P}_\mu \left[ t < \tau_{0,1,0}, \tau_0 < \tau_{0,1} \right] - \langle \mu, \eta^0 \rangle \|_\infty \leq C^0 e^{-\zeta^0 t} \quad (72)$$

—and the other results for the Q-process.

**Lemma 4.0.1.** Assume that (67), (68) and (70) hold. Then we can relate $\eta^r$ to $\eta$ through:

$$\forall x \in (0,1), \quad \eta^r(x) := \frac{\mathbb{P}_\alpha (\tau_0 < \tau_{0,1})}{\mathbb{P}_x (\tau_0 < \tau_{0,1})} \eta(x). \quad (73)$$

Similar results hold of course for $\eta^0$ and $\eta^1$.

**Lemma 4.0.2.** Assume that (67), (68) hold and that the Q-process, with law $\mathbb{Q}^r$, is well-defined under $\mathbb{P}^r$. Then $\mathbb{Q}^r = \mathbb{Q}$, i.e. this Q-process is the same as the one in Theorem 4.1.

### 4.0.1 Proof of Lemma 4.0.1

Let $x \in (0,1)$ and for $\mu \in \mathcal{M}_1[(0,1)]$,

$$\mu A_t(dx) := \mathbb{P}_\mu \left[ X_t \in dx \right] t < \tau_{0,1,0}. \quad (74. A_t)$$

By (20): $\eta^r(x) := \lim_{t \to \infty} \exp[\lambda_0 t] \mathbb{P}_x \left[ t < \tau_{0,1,0}; \tau_0 < \tau_{0,1} \right] / \mathbb{P}_x (\tau_0 < \tau_{0,1})$,

$$| \exp[\lambda_0 t] \mathbb{P}_x \left[ t < \tau_{0,1,0}; \tau_0 < \tau_{0,1} \right] - \eta(x) \mathbb{P}_\alpha (\tau_0 < \tau_{0,1}) | \leq \eta(x) | (\delta_x A_t - \alpha, \mathbb{P}_x (\tau_0 < \tau_{0,1}) | + \mathbb{P}_\delta_x A_t (\tau_0 < \tau_{0,1}) | \exp[\lambda_0 t] \mathbb{P}_x \left[ t < \tau_{0,1,0} \right] - \eta(x) | \leq \eta(x) \| \delta_x A_t - \alpha \|_{TV} + | \exp[\lambda_0 t] \mathbb{P}_x \left[ t < \tau_{0,1,0} \right] - \eta(x) | \quad \to 0 \quad \text{by (67), (68)} \quad \Box$$
4.0.2 Proof of Lemma 4.0.2

The most straightforward is to compute the associated semi-group.

For $\varphi \in B_b(0,1)$, $x \in (0,1)$,

$$\langle \delta_x Q^r_t \mid \varphi \rangle = \frac{e^{\lambda \alpha t}}{\eta^r(x)} \langle \delta_x P^r_t \mid \eta \times \varphi \rangle$$

Then, by Lemma 4.0.1 and since the semi-group $P^r$ is defined by :

$$\langle \delta_x P^r_t \mid \varphi \rangle := \langle \delta_x P_t \mid \varphi \times \mathbb{P}. (\tau_0 < \tau_{0,1}) / \mathbb{P}_x (\tau_0 < \tau_{0,1}) ,$$

$$\langle \delta_x Q^r_t \mid \varphi \rangle = \frac{e^{\lambda \alpha t}}{\eta(x)} \langle \delta_x P_t \mid \eta \times \varphi \rangle = \langle \delta_x Q_t \mid \varphi \rangle.$$  

□

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