The modular class of a regular Poisson manifold and the Reeb invariant of its symplectic foliation

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Abstract. We show that, for any regular Poisson manifold, there is an injective natural linear map from the first leafwise cohomology space into the first Poisson cohomology space which maps the Reeb class of the symplectic foliation to the modular class of the Poisson manifold. The Riemannian interpretation of those classes will permit us to show that a regular Poisson manifold whose symplectic foliation is of codimension one is unimodular if and only if its symplectic foliation is Riemannian foliation. It permit us also to construct examples of unimodular Poisson manifolds and other which are not unimodular. Finally, we prove that the first leafwise cohomology space is an invariant of Morita equivalence.

Key words. Poisson manifold, Modular class, Riemannian foliation.

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1 Introduction

The Reeb invariant of a foliated manifold is an obstruction lying in the first leafwise cohomology to the existence of a volume normal form invariant by the vector fields tangent to the foliation [5]. The modular class of a Poisson manifold was introduced by Weinstein [11]. It is an obstruction lying in the first Poisson cohomology to the existence of a volume form invariant with

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respect to the Hamiltonian flows. For a regular Poisson manifold, Weinstein [11] pointed out that the two classes are closely related without giving an explicit relation between them. In fact, the two classes represent the same mathematical object. We will show that the first leafwise cohomology space is, in natural way, a subspace of the first Poisson cohomology space and the Reeb class of the symplectic foliation agrees with the modular class of the Poisson manifold.

On a Riemannian foliated manifold, we remark that the tangent mean curvature gives arise to a tangential 1-form whose leafwise cohomology class is the Reeb invariant. This remark and the fact that the Reeb invariant is the same object as the modular class permit us to have the following corollaries.

**Corollary 1.1** Let \((P, \pi)\) be a regular Poisson manifold. If the symplectic foliation is Riemannian then the modular class of \(P\) vanishes.

**Corollary 1.2** Let \((P, \pi)\) be a regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. The following assertions are equivalent:
1) The modular class of \(P\) vanishes.
2) The symplectic foliation is Riemannian.

**Corollary 1.3** Let \((P, \pi)\) a simply connected and compact regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. Then \(\text{mod}(P) \neq 0\).

With this corollary in mind, we construct many examples of regular Poisson manifold with vanishing modular class and many examples with non-vanishing modular class. The first Poisson cohomology spaces of Morita equivalent Poisson manifolds are isomorphic, according to Ginzburg an Lu [4], and Ginzburg [3] has shown that the modular classes are compatible with this isomorphism. We will show that the first leafwise cohomology spaces are also compatible with this isomorphism.
2 The Reeb class of a foliation and its Riemannian interpretation

Let $M$ be a differentiable manifold endowed with a transversally oriented foliation $\mathcal{F}$ of dimension $p$ and of codimension $q$. We denote $T \mathcal{F}$ the tangent bundle to the foliation and $\mathcal{X}(\mathcal{F})$ the space of vector fields tangent to $\mathcal{F}$. Let $\mathcal{A}_r^{\mathcal{F}}$ denote the space of sections of the bundle $\wedge^r T^* \mathcal{F} \rightarrow M$. The elements of $\mathcal{A}_r^{\mathcal{F}}$ are called tangential differential $r$-forms. The expression

$$d_{\mathcal{F}} \alpha(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{r+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1}),$$

(1)

where $\alpha \in \mathcal{A}_r^{\mathcal{F}}$ and $X_1, \ldots, X_{r+1} \in \mathcal{X}(\mathcal{F})$, defines a degree one differential operator $d_{\mathcal{F}}$ that satisfies $d_{\mathcal{F}}^2 = 0$. The induced cohomology $H^*_{\mathcal{F}}(M)$ is called the leafwise cohomology.

An orientation of the normal bundle to $\mathcal{F}$ is a differential $q$-form $\nu$ on $M$ such that $\nu_x \neq 0$ for any $x \in M$ and such that $i_X \nu = 0$ for any $X \in \mathcal{X}(\mathcal{F})$.

For any $X \in \mathcal{X}(\mathcal{F})$, $L_X \nu$ is proportional to $\nu$ and one can define a tangential 1-form $\alpha_{\mathcal{F}}$ by

$$L_X \nu = \alpha_{\mathcal{F}}(X) \nu.$$  

(2)

From $L_{[X,Y]} \nu = L_X \circ L_Y \nu - L_Y \circ L_X \nu$, we have

$$d_{\mathcal{F}} \alpha_{\mathcal{F}} = 0.$$  

(3)

The cohomology class of $\alpha_{\mathcal{F}}$ denoted by $\text{mod}(\mathcal{F})$ is the Reeb class of the foliation.

The normal bundle to $\mathcal{F}$ carries an orientation $\nu$ such that $L_X \nu = 0$ for any $X \in \mathcal{X}(\mathcal{F})$ if and only if $\text{mod}(\mathcal{F}) = 0$ (see [5]).

Now, we give the Riemannian interpretation of the Reeb class.

Let $g$ be a Riemann metric on $M$ and let $\nabla$ be the associated Levi-Civita connection. We denote $T^\perp \mathcal{F}$ the orthogonal distribution to $T \mathcal{F}$ and $\mathcal{X}(\mathcal{F}^\perp)$ the space of vector fields tangent to $T^\perp \mathcal{F}$. For any vector field $X$, denote $X^\mathcal{F}$ its component in $\mathcal{X}(\mathcal{F})$ and $X^{\mathcal{F}^\perp}$ its component in $\mathcal{X}(\mathcal{F}^\perp)$. The orthogonal volume form to the foliation is the differential $q$-form $\eta$ defined by
\[ \eta(Y_1, \ldots, Y_q) = 1 \] for any orthonormal oriented frame \((Y_1, \ldots, Y_q)\) in \(\mathcal{X}(\mathcal{F}^\perp)\) and \(i_X \eta = 0\) for any \(X \in \mathcal{X}(\mathcal{F})\).

A straightforward calculation gives
\[ L_X \eta(Y_1, \ldots, Y_q) = \sum_{i=1}^{q} g(\nabla_{Y_i} X, Y_i) = -\sum_{i=1}^{q} g(\nabla_{Y_i} Y_i, X). \quad (4) \]

The second fundamental form of \(T^\perp \mathcal{F}\) is the tensor field \(B^\perp : \mathcal{X}(\mathcal{F}^\perp) \times \mathcal{X}(\mathcal{F}^\perp) \rightarrow \mathcal{X}(\mathcal{F})\) given by
\[ B^\perp(Y_1, Y_2) = \frac{1}{2} [\nabla_{Y_1} Y_2 + \nabla_{Y_2} Y_1]^F. \]

Its trace with respect to \(g\), called the tangent mean curvature, is a vector field \(H^\perp\) tangent to \(\mathcal{F}\). We define a tangential 1-form \(K^\perp \in \mathcal{A}^1_F\) by
\[ K^\perp(X) = g(X, H^\perp), \quad X \in \mathcal{X}(\mathcal{F}). \quad (5) \]

(3) can be written
\[ L_X \eta(Y_1, \ldots, Y_q) = -K^\perp(X) \quad (6) \]
and so
\[ mod(\mathcal{F}) = -[K^\perp]. \quad (7) \]

**Remark.** This formula can be compared to the metric formula for the Godbillon-Vey invariant (see [8]).

**Proposition 2.1** Let \(M\) be a differentiable manifold endowed with a transversally oriented foliation \(\mathcal{F}\) of dimension \(p\) and of codimension \(q\). The following assertions are equivalent:
1) The normal bundle of \(\mathcal{F}\) carries an orientation invariant by the vector fields tangent to the foliation.
2) \(mod(\mathcal{F}) = 0\).
3) There is a Riemann metric on \(M\) with vanishing tangent mean curvature.

**Proof:** We have shown that 1) \(\iff\) 2) and that 3) \(\implies\) 2). We show now 2) \(\implies\) 3). Let \(g\) be a Riemann metric on \(M\). If \(mod(\mathcal{F}) = 0\) then there is a smooth function \(h \in C^\infty(M)\) such that \(dFh = K^\perp\). It’s easy to verify that the tangent mean curvature of the Riemann metric \(g_1 = e^{\frac{-2h}{q}} g\) vanishes. \(\square\)

It’s known that the second fundamental form \(B\) vanishes if and only if \(g\) is bundle-like [9] so we get the following proposition.
Proposition 2.2 Let $M$ be a differentiable manifold endowed with a transversally oriented foliation $F$ of codimension 1. The following assertions are equivalent:
1) $\text{mod}(F) = 0$.
2) $F$ is a Riemannian foliation.
3) $F$ is defined by a closed 1-form.

3 The modular class of a Poisson manifold

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman’s monograph [10].

Let $P$ be a Poisson manifold with Poisson tensor $\pi$. We have a bundle map $\pi : T^*P \to TP$ defined by

$$\beta(\pi(\alpha)) = \pi(\alpha, \beta), \quad \alpha, \beta \in T^*P.$$  \hfill (8)

On the space of differential 1-forms $\Omega^1(P)$, the Poisson tensor induces a Lie bracket

$$[\alpha, \beta]_\pi = L_{\pi(\alpha)}\beta - L_{\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) = i_{\pi(\alpha)}d\beta - i_{\pi(\beta)}d\alpha + d(\pi(\alpha, \beta)).$$  \hfill (9)

For this Lie bracket and the usual Lie bracket on vector fields, the bundle map $\pi$ induces a Lie algebra homomorphism $\pi : \Omega^1(P) \to \mathcal{X}(P)$:

$$\pi([\alpha, \beta]_\pi) = [\pi(\alpha), \pi(\beta)].$$  \hfill (10)

The Poisson cohomology of a Poisson manifold $(P, \pi)$ is the cohomology of the chain complex $(\mathcal{X}^*(P), d_\pi)$ where, for $0 \leq p \leq \text{dim}P$, $\mathcal{X}^p(P)$ is the $C^\infty(P, \mathbb{R})$-module of $p$-multi-vector fields and $d_\pi$ is given by

$$d_\pi Q(\alpha_0, \ldots, \alpha_p) = \sum_{j=0}^p (-1)^j \pi(\alpha_j) Q(\alpha_0, \ldots, \hat{\alpha_j}, \ldots, \alpha_p)$$

$$+ \sum_{i<j} (-1)^{i+j} Q([\alpha_i, \alpha_j]_\pi, \alpha_0, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_p).$$  \hfill (11)

We denote $H^*_\pi(P)$ the spaces of cohomology.
The modular class of \((P, \pi)\) is the obstruction to the existence of a volume form on \(P\) which is invariant with respect to Hamiltonian flows. More explicitly, let \(\mu\) be a volume form on \(P\). As shown in \([11]\), the operator \(\phi_\mu : f \mapsto \text{div}_\pi(df)\) is a derivation and hence a vector field called the modular vector field of \((P, \pi)\) with respect to the volume form \(\mu\). Also \(L_{\phi_\mu} \pi = 0\) and \(L_{\phi_\mu} \mu = 0\). \hfill (12)

If we replace \(\mu\) by \(a\mu\), where \(a\) is a positive function, the modular vector fields becomes \(\phi_{a\mu} = \phi_\mu + \pi(d(\log a))\). \hfill (13)

Thus the first Poisson cohomology class of \(\phi_\mu\) is independent of \(\mu\), we call it the modular class of \((P, \pi)\) and we denote it \(\text{mod}(P)\). The Poisson manifold is unimodular if its modular class vanishes.

4 Link between the Reeb class and the modular class of a regular Poisson manifold

Let \((P, \pi)\) be a Poisson manifold whose symplectic foliation, denoted by \(\mathcal{F}\), is a regular foliation transversally oriented of dimension \(2p\) and of codimension \(q\). As shown in \([11, \text{pp. 385}]\), the modular vector field of \(P\) is closely related to the Reeb tangential 1-form. More explicitly, let \(\omega \in \Gamma(\wedge^2 T^*\mathcal{F})\) be the leafwise symplectic form given by

\[
\omega(u, v) = \pi(\pi^{-1}(u), \pi^{-1}(v)), \quad u, v \in T\mathcal{F}.
\]

Let \(\nu\) be a transverse orientation of \(\mathcal{F}\) i.e a differential \(q\)-form on \(P\) such that \(i_{\pi(df)}\nu = 0\) for any smooth function \(f\) and \(\nu_x \neq 0\) for any \(x \in P\). Choose \(F\) a supplement distribution to \(T\mathcal{F}\) and extend \(\omega\) to a differential 2-form on \(P\) by setting \(i_X\omega = 0\) for any \(X\) tangent to \(F\). The form \(\mu = \wedge^p \omega \wedge \nu\) is a volume form on \(P\).

For any \(f \in C^\infty(P)\), since \([L_{\pi(df)}(\wedge^p \omega)] \wedge \nu = 0\), we have

\[
L_{\pi(df)} \mu = (\wedge^p \omega) \wedge L_{\pi(df)} \nu = \alpha_F(\pi(df))\mu = -\pi(\alpha)(f)\mu
\]

where \(\alpha\) is any differential 1-form on \(P\) whose restriction to \(\mathcal{F}\) is \(\alpha_F\).
\( \pi(\alpha) \) depends only on \( \alpha_F \) and we will denote it by \( \pi(\alpha_F) \). We get
\[
\phi(\wedge p \omega) = - \pi(\alpha_F).
\] (14)

In the following, we will give a more precise interpretation of this relation.

For \( 1 \leq p \leq \dim P \), we consider the subspace \( X_0^p(P) \subset X^p(P) \) of \( p \)-multivector fields \( Q \) such that \( i_\alpha Q = 0 \) for any \( \alpha \in \ker \pi \). It’s easy to verify that \( d_\pi(X_0^p) \subset X_0^{p+1} \). The natural injection \( X_0^p \hookrightarrow X^p \) induces a linear map \( H^*(X_0^p) \longrightarrow H^p(P) \) which is injective for \( * = 1 \).

Let \( \pi : A^p(F)(P) \longrightarrow X^p_0(P) \) be the map given by
\[
\pi(\omega)(\alpha_1, \ldots, \alpha_p) = \omega(\pi(\alpha_1), \ldots, \pi(\alpha_p)).
\]

It is easy to verify that \( \pi \) is an isomorphism and \( \pi(d_F \omega) = d_\pi(\omega) \) and hence \( \pi \) induces an isomorphism
\[
\pi^* : H^p_\pi(P) \longrightarrow H^p(X^p_0(P)).
\] (15)

So, we have shown the following theorem.

**Theorem 4.1** Let \((P, \pi)\) be a regular Poisson manifold for which the symplectic foliation is transversally oriented. \( \pi \) induces a linear injection
\[
\pi^* : H^1_\pi(P) \hookrightarrow H^1_\pi(P)
\] (16)

and we have
\[
\pi^*(\text{mod}(\mathcal{F})) = \text{mod}(P).
\] (17)

**Remark.** The fact that the leafwise cohomology spaces embed in the Poisson cohomology is known (see [10]).

The following corollaries are a consequence of this theorem, Proposition 2.1 and Proposition 2.2.

**Corollary 4.1** Let \((P, \pi)\) be a regular Poisson manifold. If the symplectic foliation is Riemannian then the modular class of \( P \) vanishes.

**Corollary 4.2** Let \((P, \pi)\) be a regular Poisson manifold for which the symplectic foliation is transversally oriented. The following assertions are equivalent:

1) \( P \) is unimodular.

2) The symplectic foliation carries an invariant normal volume form.

3) There is a Riemannian metric \( g \) on \( P \) such that \( \pi(K^\perp) \) is a hamiltonian vector fields.
Corollary 4.3 Let \((P, \pi)\) be a regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. The following assertions are equivalent:
1) \(P\) is unimodular.
2) The symplectic foliation is Riemannian.

Corollary 4.4 Let \((P, \pi)\) a simply connected and compact regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. Then \(\text{mod}(P) \neq 0\).

5 Examples

According to the above sections, we will give some illustration examples of regular Poisson manifolds with vanishing modular class (Reeb class) and other with non-vanishing modular class (Reeb class).

1. Any oriented foliation of codimension 1 on the sphere \(S^3\) is the symplectic foliation of a Poisson structure on \(S^3\) with non-vanishing modular class.

2. Let \((G, \omega)\) be a symplectic Lie group (for example the affine group \(GA(n)\)) (see [6]). Let \(P \times G \rightarrow P\) a locally free action of \(G\) on a differentiable manifold \(P\) whose associated foliation will be denoted by \(\mathcal{F}\). The symplectic form \(\omega\) gives rise to a tangential 2-form on \(\mathcal{F}\) which is symplectic on the restriction to any leaf of \(\mathcal{F}\). This gives canonically a Poisson structure \(\pi\) on \(P\) whose symplectic foliation is \(\mathcal{F}\).

If the action of \(G\) leaves invariant a Riemannian metric on \(P\), the foliation is Riemannian and the modular class of \((P, \pi)\) vanishes. This is the case if \(P\) a Lie group and \(G\) a Lie subgroup which acts by left translations.

Another interesting case is the case where \(H\) is a Lie group with \(G\) as subgroup and \(\Gamma\) is a discrete subgroup of \(H\) such that there is a Riemann metric on \(H\) which is right \(G\)-invariant and left \(\Gamma\)-invariant. Then the natural homogenous action of \(G\) on \(P = \Gamma \setminus H\) is locally free and the associated foliation is Riemannian. For example, the natural action of \(\mathbb{R}^{2p}\) on the torus \(T^n = \mathbb{Z}^n \setminus \mathbb{R}^n\).

3. Let \(G\) be an unimodular Lie group and \(H\) an unimodular subgroup of \(G\). Let \(\Gamma\) a discrete subgroup of \(G\). The homogenous left action of \(\Gamma\) on \(G/H\) lives invariant a volume form on \(G/H\). Let \((M, \omega)\) a symplectic manifold with a free proper symplectic action \(M \times \Gamma \rightarrow M\). The manifold \(P = \)
$M \times \Gamma G/H$ carries a foliation $\mathcal{F}$ whose leaves are of the form $M \times \{.\}$. The symplectic form $\omega$ gives arise to a tangential symplectic form and then a Poisson structure $\pi$. Although the foliation is not Riemannian, the modular class of $\pi$ vanishes.

4. The affine group $GA(2)$ can be considered as a subgroup of two simply connected Lie groups of dimension 3 both having a cocompact discrete subgroup $\Gamma$. The first one is $\tilde{SL}(2, \mathbb{R})$ the universal covering of $SL(2, \mathbb{R})$ and the second is $G_3$ whose Lie algebra is given by the relations

$$[e_1, e_2] = -e_1, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = -e_3.$$ 

There is Poisson structure (constructed as above) on the compact 3-manifold $M = \Gamma \backslash H$ (where $H = \tilde{SL}(2, \mathbb{R})$ or $G_3$) whose the symplectic foliation is the foliation given by the homogenous action of $GA(2)$. This foliation (of codimension one) is not Riemannian and so the modular class of the Poisson structure don’t vanish.

6 The first leafwise cohomology space is an invariant of Morita equivalence

Following [17], recall that a full dual pair $P_1 \xleftarrow{\rho_1} W \xrightarrow{\rho_2} P_2$ consists of two Poisson manifolds $(P_1, \pi_1)$ and $(P_2, \pi_2)$, a symplectic manifold $W$, and two submersions $\rho_1 : W \rightarrow P_1$ and $\rho_2 : W \rightarrow P_2$ such that $\rho_1$ is Poisson, $\rho_2$ is anti-Poisson, and the fibers of $\rho_1$ and $\rho_2$ are symplectic orthogonal to each other. A Poisson (or anti-Poisson) mapping is said to be complete if the pull-back of a complete Hamiltonian flow under this mapping is complete. A full dual pair is called complete if both $\rho_1$ and $\rho_2$ are complete. The Poisson manifolds $P_1$ and $P_2$ are Morita equivalent if there exists a complete full dual pair $P_1 \xleftarrow{\rho_1} W \xrightarrow{\rho_2} P_2$ such that $\rho_1$ and $\rho_2$ both have connected and simply connected fibers. Morita equivalent Poisson manifolds $P_1$ and $P_2$ have isomorphic first Poisson cohomology spaces. More explicitly, there is a natural isomorphism

$$E : H^1_{\pi}(P_1) \xrightarrow{\cong} H^1_{\pi}(P_2)$$

which is defined by (see [4] Lemma 5.2))

$$E([\xi_1]) = [\xi_2] \iff \exists F \in C^\infty(W) \quad \xi_1 = (\rho_1)_* X_F, \xi_2 = -(\rho_2)_* X_F.$$
Let $\xi_1$ be a representant of a class in $\pi_1^\ast(H_1^F(P_1))$. Then there exists a differential 1-form $\alpha$ on $P$ such that $\pi(\alpha) = \xi_1$. Let $\xi_2$ a representant of $E([\xi_1])$. $[\xi_2] \in \pi_2^\ast(H_1^F(P_2))$ if and only if $\xi_2$ is tangent to the symplectic foliation which is true if, for any local Casimir function $f$, $df(\xi_2) = 0$.

Let $f$ be a Casimir function. Remark that the relation $(\rho_1)_\ast X_F = \xi_1$ is equivalent to the existence of $\rho_1$-vertical vector field $V$ such that

$$dF = \rho_1^\ast(\alpha) + \omega^{-1}(V)$$

where $\omega : T^*W \to TW$ is the identification associated to the symplectic form on $W$.

We have

$$df(\xi_2) = -df((\rho_2)_\ast X_F)$$

$$= -d(f \circ \rho_2)(X_F)$$

$$= dF(X_{f_{\rho_2}})$$

$$= \rho_1^\ast(\alpha)(X_{f_{\rho_2}}) + \omega(V, X_{f_{\rho_2}}).$$

$X_{f_{\rho_2}}$ is $\rho_2$-vertical since $f$ is a Casimir function so $\omega(V, X_{f_{\rho_2}}) = 0$ and $X_{f_{\rho_2}}$ is $\rho_1$-vertical so $\rho_1^\ast(\alpha)(X_{f_{\rho_2}}) = 0$. So we can conclude.

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