Abstract. Let $G = (V, E)$ be a multigraph without loops and for any $x \in V$ let $E(x)$ be the set of edges of $G$ incident to $x$. A homogeneous edge-coloring of $G$ is an assignment of an integer $m \geq 2$ and a coloring $c : E \rightarrow S$ of the edges of $G$ such that $|S| = m$ and for any $x \in V$, if $|E(x)| = mq_x + r_x$ with $0 \leq r_x < m$, there exists a partition of $E(x)$ in $r_x$ color classes of cardinality $q_x + 1$ and other $m-r_x$ color classes of cardinality $q_x$. The homogeneous chromatic index $\chi_h(G)$ is the least $m$ for which there exists such a coloring. We determine $\chi_h(G)$ in the case that $G$ is a complete multigraph, a tree or a complete bipartite multigraph.

1. Introduction

Let $G = (V, E)$ be a multigraph without loops (see [2, 9] as a reference). The usual definition of coloring of the edges of a multigraph is a mapping from the set of edges $E$ into a finite set of colors such that two adjacent edges have different colors. The chromatic index $\chi'(G)$ is the minimum number of colors for which there exists such a coloring for $G$ (see [5, 6, 7]).

In [4] Gionfriddo, Milazzo and Voloshin define a coloring of the edges of a multigraph by a mapping between $E$ and a set of colors for which each non-pendant vertex, i.e. of degree at least 2, is incident to at least two edges of the same color. In their paper they give the definition of upper chromatic index, $\chi^*(G)$, which is the maximum $k$ for which there exists an edge coloring with $k$ colors, and they determine $\chi^*(G)$ that for an arbitrary multigraph $G = (V, E)$. The study of such a coloring is related to the coloring theory of mixed hypergraphs (see [8, Problem 13]).

In [3] Gionfriddo, Amato and Ragusa proceed in the way of studying edge colorings of a multigraph in which each non-pendant vertex is incident to at least two edges of the same color. In particular, they give the definition of equipartite edge coloring of a multigraph $G$, where, fixed an integer $h$, they search for the maximum number of colors for which for any $x \in V$ there exists a partition of $E(x)$ in color classes of the same cardinality $h$ with the exception of one of smaller cardinality.

In this paper, proceeding in this direction, we give the definition of homogeneous edge-coloring of a multigraph as an assignment of an integer $m \geq 2$ and a coloring such that for any $x \in V$ $E(x)$ has a partition in $m$ classes of colors whose cardinality differs for at most 1, in the case that $|E(x)| \geq m$, and all the edges of $E(x)$ has different colors, in the case that $|E(x)| < m$. In particular, we search for the homogeneous chromatic index, which is the minimum number of colors $\chi_h(G)$ for which there exists such a coloring and we prove that, if $G$ is either a complete multigraph, a tree or a complete bipartite multigraph, then either $\chi_h(G) = 2$ or $\chi_h(G) = 3$. 

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2. Homogeneous edge-coloring

Definition 2.1. A homogeneous edge-coloring of $G$ (or $m$-homogeneous edge-coloring of $G$) is an assignment of an integer $m \geq 2$ and a coloring $c : E \to S$ of the edges of $G$ such that $|S| = m$ and for any $x \in V$, if $|E(x)| = mq_x + r_x$ with $0 \leq r_x < m$, there exists a partition of $E(x)$ in $r_x$ color classes of cardinality $q_x + 1$ and other $m - r_x$ color classes of cardinality $q_x$.

Remark 2.2. If $|E(x)| < m$, then $q_x = 0$ and $r_x = |E(x)|$: in this case, the previous definition implies that any two edges of $E(x)$ must be colored with different colors.

Given $x \in V$ and $k \in \{1, \ldots, m\}$ it may happen that $c(\sigma) \neq k$ for any $\sigma \in E(x)$. However, if $i, j \in c(E(x))$, with $i \neq j$, then the number of edges of $E(x)$ colored with $i$ and the number of edges of $E(x)$ colored with $j$ either are equal or differ by 1.

Definition 2.3. Let $G = (V, E)$ be a graph. The homogeneous chromatic index $\chi'(G)$ is the minimum integer $m$ such that $G$ admits a $m$-homogeneous edge-coloring.

Remark 2.4. It is useful to underline the following facts.

1. An edge-coloring of a graph $G$ is a homogeneous edge-coloring. In particular, $\chi'(G) \leq \chi'(G)$, where $\chi'(G)$ is the chromatic index of $G$.

2. A path $P_n$, with $n \geq 2$, is the graph with vertices $\{x_1, \ldots, x_n\}$ and edges $\{x_i, x_{i+1}\}$ for $i = 1, \ldots, n - 1$. It is easy to see that $\chi'(P_n) = 2 = \chi'(P_n)$.

3. A cycle $C_n$, with $n \geq 3$, is the graph with vertices $\{x_1, \ldots, x_n\}$ and edges $\{x_i, x_{i+1}\}$ for $i = 1, \ldots, n - 1$ and $\{x_n, x_1\}$. If $n$ is even, then $\chi'(C_n) = 2 = \chi'(C_n)$; if $n$ is odd, then $\chi'(C_n) = 3 = \chi'(C_n)$.

4. A star $S_n$ is the graph with vertices $\{x_0, \ldots, x_n\}$ and edges $\{x_0, x_i\}$ for $i = 1, \ldots, n$. It is easy to see that $\chi'(S_n) = 2$. However, in this case $\chi'(S_n) < \chi'(S_n) = n$.

5. A wheel $W_n$ is the graph with vertices $\{x_1, \ldots, x_n\}$, with $n \geq 4$, such that the subgraph induced by $\{x_2, \ldots, x_n\}$ is the cycle $C_{n-1}$ and $x_1$ is adjacent to the other vertices $\{x_2, \ldots, x_n\}$. If $S = \{1, 2\}$, then the following mapping $c : E \to S$ is a 2-homogeneous coloring of $W_n$:  

- if $i \in \{2, \ldots, n\}$, we define:

$$c(\{x_1, x_i\}) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 2 & \text{if } i \text{ is odd} \end{cases}$$

- if $i \in \{1, \ldots, n-1\}$, we define:

$$c(\{x_i, x_{i+1}\}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$$
HOMOGENEOUS EDGE-COLORINGS OF GRAPHS

- and

\[ c(\{x_n, x_2\}) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases} \]

This means that \( \bar{\chi}(W_n) = 2 \). In this case \( \bar{\chi}(W_n) < \chi'(W_n) = n - 1 \) for \( n \geq 4 \).

**Example 2.5.** As we have just seen the wheel \( W_5 \) admits this 2-homogeneous edge-coloring:

\[
\begin{array}{c}
1 & 2 \\
1 & 2 \\
2 & 1 \\
2 & 1 \\
1 & 2
\end{array}
\]

So \( \bar{\chi}(W_5) = 2 \), but \( W_5 \) has no 3-homogeneous edge-coloring. In fact, the following are, up to permutation, the only possible colorings of \( x_1 \).

The other vertices \( x_2, x_3, x_4, x_5 \) have degree 3 and so in a 3-homogeneous edge-coloring the three corresponding edges have three different colors. However this does not happen in any of the previous cases.

**Theorem 2.6.** If \( G \) is an eulerian graph, then \( G \) admits a \( \frac{\Delta(G)}{2} \)-homogeneous edge-coloring.

**Proof.** This follows immediately by [3, Theorem 2.1].

\[ \square \]

3. COMPLETE GRAPHS

**Theorem 3.1.** Let \( n \geq 4 \) be an even integer. Then \( \bar{\chi}(K_n) = 2 \).

**Proof.** Let \( S = \{1, 2\} \) be a set of colors and \( n = 2k \). Any vertex \( x \in K_n \) has degree \( n - 1 = 2k - 1 \). So we will show that, for any \( x \in V \), we color \( k \) edges of \( E(x) \) with 1 and the remaining \( k - 1 \) edges of \( E(x) \) with 2. Indeed, let \( V = \{x_1, \ldots, x_n\} \) and let us consider a mapping \( c : E \rightarrow S \) defined in the following way:

\[ c(\{x_i, x_j\}) = \begin{cases} 1 & \text{if } i + j \text{ is odd} \\ 2 & \text{if } i + j \text{ is even} \end{cases} \]
for any \( i, j \in \{1, \ldots, n\} \), with \( i \neq j \). So, for any fixed \( i \), we see that \( k \) edges in \( E(x_i) \) are colored with 1 and \( k - 1 \) edges of \( E(x_j) \) are colored with 2. This proves the statement. \( \square \)

Theorem 3.2. Let \( n \geq 5 \) be an odd integer such that \( n \equiv 1 \mod 4 \). Then \( \chi(K_n) = 2 \).

Proof. Let \( S = \{1, 2\} \) be a set of colors. Any vertex \( x \) of the graph \( K_n \) has even degree \( n - 1 \). We want to show that we can color \( \frac{n-1}{2} \) of these edges with 1 and \( \frac{n-1}{2} \) of these edges with 2. By [1, Theorem 1.2] we see that there exist \( \frac{n-1}{4} \) cycles of length \( n \) that decompose \( K_n \). Since \( \frac{n-1}{2} \) is even, we can color all the edges of \( \frac{n-1}{4} \) cycles with 1 and all the edges of \( \frac{n-1}{4} \) cycles with 2. This proves the statement.

Let \( n = 4h + 1 \) for some \( h \in \mathbb{N} \). Another possible coloring is the following mapping \( c : E \to S \):

\[
c((x_i, x_j)) = \begin{cases} 1 & \text{if } j \equiv i - 1, \ldots, i - h, i + 1, \ldots, i + h \mod n \\ 2 & \text{otherwise,} \end{cases}
\]

for any \( i, j \in \{1, \ldots, n\} \), with \( i \neq j \). \( \square \)

Proposition 3.3. Let \( n \in \mathbb{N} \) be an odd integer and let \( S = \{1, 2\} \) be a set of colors. Then in any coloring \( c \) of the edges of a cycle \( C_n \) with \( S \) there are precisely an odd number of vertices whose adjacent edges have the same color.

Proof. Let \( n = 2k + 1 \). The proof works by induction on \( k \). If \( k = 1 \), the statement is clear. Let \( k \geq 2 \) be an odd integer and suppose that the statement holds for \( k - 1 \). First note that, since \( n \) is odd, there exists at least a vertex \( v \) in \( C_n \) whose adjacent edges have the same color. Let \( V = \{x_1, \ldots, x_n\} \) be the set of vertices of \( C_n \). We can suppose that \( v = x_1 \) and that 1 is the color of its adjacent edges. Consider the cycle \( C_{n-2} \) of vertices \( \{x_3, \ldots, x_n\} \) obtained by \( C_n \) adding the edge \( \{x_n, x_3\} \) and color this cycle with 1. By hypothesis on \( x_1 \) we know that \( \{x_1, x_n\} \) and \( \{x_1, x_2\} \) are both colored with 1. So, if \( r \) is the number of vertices of \( C_n \) whose adjacent edges have the same color and \( s \) is the number of vertices of \( C_{n-2} \) whose adjacent edges have the same color, we see that either \( r = s + 2 \) or \( r = s \). Indeed:

- if \( c(\{x_2, x_3\}) = 2 \) and \( c(\{x_3, x_4\}) = 2 \), then \( r = s + 2 \);
- if \( c(\{x_2, x_3\}) = 2 \) and \( c(\{x_3, x_4\}) = 1 \), then \( r = s \);
- if \( c(\{x_2, x_3\}) = 1 \), then \( r = s + 2 \).

By applying the inductive hypothesis on \( C_{n-2} \) we see that \( s \) is odd and so \( r \) is odd too. \( \square \)

Theorem 3.4. Let \( n \geq 3 \) be an odd integer such that \( n \equiv 3 \mod 4 \). Then \( \chi(K_n) = 3 \).

Proof. Let \( n = 4k + 3 \) and suppose that \( \chi(K_n) = 2 \). Take \( S = \{1, 2\} \). Then each vertex has \( 2k + 1 \) edges colored with 1 and \( 2k + 1 \) colored with 2. By [1, Theorem 1.2] we see that there exist \( 2k + 1 \) cycles of length \( n \) that decompose \( K_n \). Moreover by Proposition 3.3 in any of these cycles there are precisely an odd number of vertices whose adjacent edges have the same color. This, together with the fact that the cycles of length \( n \) decomposing \( K_n \) are in odd number, gives a contradiction with the hypothesis that \( \chi(K_n) = 2 \), because there will be a vertex with at least \( 2k + 3 \) edges with the same color.

Now we show that \( \chi(K_n) = 3 \). Let \( S = \{1, 2, 3\} \) be a set of colors.

If \( 3 \mid n - 1 \), then \( K_n \) is decomposed by \( \frac{n-1}{2} \) cycles of length \( n \) and we can color all the edges in \( \frac{n-1}{6} \) cycles with 1, all the edges in \( \frac{n-1}{6} \) cycles with 2 and all the edges
χ(n) is the following mapping in a way that all the edges in the same subset are colored either with 1 or 2 or 3. An easy computation shows that this gives the statement in the case n = 12h + 7 for some h ∈ N, another possible coloring is the following mapping c : E → S:

\[
\begin{cases}
1 & \text{if } j \equiv i - 1, \ldots, i - 2h - 1, i + 1, \ldots, i + 2h + 1 \mod n \\
2 & \text{if } j \equiv i - 2h - 2, \ldots, i - 4h - 2, i + 2h + 2, \ldots, i + 4h + 2 \mod n \\
3 & \text{if } j \equiv i - 4h - 3, \ldots, i - 6h - 3, i + 4h + 3, \ldots, i + 6h + 3 \mod n.
\end{cases}
\]

for any i, j ∈ {1, . . . , n}, with i ≠ j.

Let n ≡ 2 mod 3 and let V = \{x_1, . . . , x_n\}. We define a mapping c : E → S in the following way:

- if i, j < n
  \[
  c(x_i, x_j) = \begin{cases}
1 & \text{if } i + j \equiv 2 \mod 3 \\
2 & \text{if } i + j \equiv 1 \mod 3 \\
3 & \text{if } i + j \equiv 0 \mod 3
\end{cases}
  \]

- if i = n
  \[
  c(x_i, x_j) = \begin{cases}
1 & \text{if } j \equiv 1 \mod 3 \\
2 & \text{if } j \equiv 2 \mod 3 \\
3 & \text{if } j \equiv 0 \mod 3
\end{cases}
  \]

- if j = n
  \[
  c(x_i, x_j) = \begin{cases}
1 & \text{if } i \equiv 1 \mod 3 \\
2 & \text{if } i \equiv 2 \mod 3 \\
3 & \text{if } i \equiv 0 \mod 3.
\end{cases}
  \]

An easy computation shows that this gives the statement in the case n ≡ 2 mod 3, i.e. n = 12h + 11 for some h ∈ N. So given any vertex x ∈ V the edges in E(x) can be divided in three subsets, one with 4h + 4 and two with 4h + 3 elements, in such a way that all the edges in the same subset are colored either with 1 or 2 or 3.

Let n ≡ 0 mod 3, so that n = 12h + 3 for some h ∈ N. By [1] Theorem 1.2] we see that there are 6h + 1 cycles of length n that decompose Kn. To prove the statement it is sufficient to color all the edges in 2h cycles with 1, in other 2h cycles with 2 and other 2h cycles with 3. The edges in the last cycle can be colored alternatively with 1 and 2 with the exception of one edge colored with 3, accordingly to the sequence 1, 2, 1, 2, . . . , 1, 2, 3. So given any vertex x ∈ V the edges in E(x) can be divided in three subsets, one with 4h and two with 4h + 1 elements, in such a way that all the edges in the same subset are colored either with 1 or 2 or 3.

In this case another possible coloring is the following mapping c : E → S:

\[
\begin{cases}
1 & \text{if } i + j \equiv 2 \mod 3 \\
2 & \text{if } i + j \equiv 1 \mod 3 \\
3 & \text{if } i + j \equiv 0 \mod 3
\end{cases}
\]

for any i, j ∈ {1, . . . , n}, with i ≠ j.

\[
\begin{align*}
\overline{\chi}(\lambda K_n) &= \begin{cases}
3 & \text{if } \lambda \text{ is odd and } n \equiv 3 \mod 4 \\
2 & \text{otherwise}.
\end{cases}
\end{align*}
\]

**Theorem 3.5.** Let λ, n ∈ N. Then:
Proof. We will prove the following:

\[ \chi(\lambda K_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 2 & \text{if } n \equiv 1 \mod 4 \\ 2 & \text{if } \lambda \text{ is even and } n \equiv 3 \mod 4 \\ 3 & \text{if } \lambda \text{ is odd and } n \equiv 3 \mod 4. \end{cases} \]

First case: \( n \text{ even} \). Let \( S = \{1, 2\} \) a set of colors. Then by Theorem 3.1 each copy of \( K_n \) has a 2-homogeneous edge-coloring. We can use this coloring for \( \left\lfloor \frac{n}{2} \right\rfloor \) copies of \( K_n \) and for the remaining \( \frac{3}{2} \) copies of \( K_n \) the coloring obtained by permuting 1 and 2 in the previous one.

Second case: \( n \equiv 1 \mod 4 \). This follows immediately by Theorem 3.2 each copy of \( K_n \) has a 2-homogeneous edge-coloring. We can use this coloring for all the copies of \( K_n \).

Third case: \( \lambda \text{ is even and } n \equiv 3 \mod 4 \). Let \( S = \{1, 2\} \) a set of colors. We can color \( \frac{2}{3} \) copies of \( K_n \) with 1 and \( \frac{1}{3} \) copies of \( K_n \) with 2.

Fourth case: \( \lambda \text{ is odd and } n \equiv 3 \mod 4 \). Proceeding as in Theorem 3.4 we see that \( \chi(\lambda K_n) > 2 \). We need to prove that \( \chi(\lambda K_n) = 3 \). We denote by \( S = \{1, 2, 3\} \) a set of colors and by \( c \) be the coloring given in Theorem 3.4.

If \( 3 \mid n - 1 \), then we can use \( c \) for each copy of \( K_n \).

Let either \( n \equiv 2 \mod 3 \) or \( n \equiv 0 \mod 3 \). We consider \( c_{132} \) the coloring obtained by \( c \) and by the permutation of the colors (132) and \( c_{123} \) the coloring obtained by \( c \) and by the permutation of the colors (123).

- If \( \lambda = 3m \), for some \( m \in \mathbb{N} \), we color \( m \) copies of \( K_n \) with \( c \), \( m \) with \( c_{132} \) and \( m \) with \( c_{123} \).
- If \( \lambda = 3m + 1 \), for some \( m \in \mathbb{N} \), we color \( m + 1 \) copies of \( K_n \) with \( c \), \( m \) with \( c_{132} \) and \( m \) with \( c_{123} \).
- If \( \lambda = 3m + 2 \), for some \( m \in \mathbb{N} \), we color \( m + 1 \) copies of \( K_n \) with \( c \), \( m + 1 \) with \( c_{132} \) and \( m \) with \( c_{123} \).

This proves the statement. \( \square \)

4. Trees and complete bipartite graphs

Theorem 4.1. If \( G = (V, E) \) is a tree and \( |V| \geq 3 \), then \( \chi(G) = 2 \).

Proof. Let \( |V| = n \). We proceed by induction on \( n \). If \( n = 3 \), then \( G \) is an open path and so \( \chi(G) = 2 \). Now let the statement hold for a tree with \( n - 1 \) vertices. Let \( x \in V \) be a pendant vertex and let \( G' = G - x \). Then \( G' \) is a tree with \( n - 1 \) vertices and by induction \( \chi(G') = 2 \). Considered a 2-homogeneous edge-coloring of \( G' \), it is easy to get a 2-homogeneous edge-coloring of \( G \), because \( d(x) = 1 \). \( \square \)

Theorem 4.2. Given \( m, n \in \mathbb{N} \), \( \chi(K_{n,m}) = 2 \).

Proof. Given \( S = \{1, 2\} \), it is sufficient to consider the following \( c : E \to S \):

\[ c(\{x_i, x_j\}) = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 2 & \text{if } i + j \text{ is odd}, \end{cases} \]

for any \( i, j \in \{1, \ldots, n\} \), with \( i \neq j \). \( \square \)

Theorem 4.3. Given \( \lambda, m, n \in \mathbb{N} \), \( \chi(\lambda K_{n,m}) = 2 \).
Proof. Let \( c \) be the coloring of \( K_{m,n} \) given in Theorem 4.2 and \( c' \) the coloring obtained by \( c \) permuting 1 and 2. Then the statement follows by considering the following coloring of \( \lambda K_{m,n} \): we use the coloring \( c \) for \( \lceil \frac{\lambda}{2} \rceil \) copies of \( K_{m,n} \) and \( c' \) for the remaining \( \lfloor \frac{\lambda}{2} \rfloor \) copies of \( K_{m,n} \).

□

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