On the Adler-Bell-Jackiw anomaly in a Horava-Lifshitz–like QED

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received 20 October 2015; accepted in final form 22 December 2015

PACS 11.10.Wx - Finite-temperature field theory
PACS 11.10.Er - Charge conjugation, parity, time reversal, and other discrete symmetries
PACS 11.30.Ly - Other internal and higher symmetries

Abstract – We show the absence of the Adler-Bell-Jackiw (ABJ) anomaly for a Horava-Lifshitz–like QED with any even $z$. Besides this, we study the graph contributing to the ABJ anomaly at non-zero temperature and extend Fujikawa’s methodology for studying the integral measure to our model.

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Introduction. – over the last years, the studies of field theory models with a strong space-time asymmetry, also known as Horava-Lifshitz–like (HL-like) models, have attracted great attention. Originally such models were inspired by studies of critical phenomena many years ago [1,2]. The first studies of such models within the context of the quantum field theory were performed in [3–6] where their renormalizability has been discussed. Further, formulation of the Horava-Lifshitz gravity [7] crucially increased the interest in this class of models. Certainly, the HL-like gauge theories play a very important role within these models. The most interesting results for these studies are, first, obtaining the two- and three-point functions for five-dimensional [8] and four-dimensional [9] cases, second, the explicit calculation of the effective potential in the HL-like scalar QED [10–13].

All this certainly calls the interest in the study of more sophisticated aspects of the HL-like theories. One of them is just the problem of anomalies, especially the famous Adler-Bell-Jackiw anomaly (triangle anomaly) [14,15] implying breaking of the chiral symmetry. It is known that just this anomaly causes ambiguities in the theories with “small” Lorentz symmetry breaking [16]. Therefore, it is natural to verify the presence of such anomaly in the HL-like extension of the QED. Namely this problem is considered in this paper.

An HL-like Abelian gauge model. – Let us formulate the HL-like extension of QED involving coupling with the extra pseudovector field $B^\mu$. For the sake of concreteness and simplicity, we restrict ourselves to the case of even $z = 2n$. The motivation for this study consists in the interest in studying whether any Horava-Lifshitz–like analogue of the Carroll-Field-Jackiw term $\epsilon^{\mu\nu\rho\sigma} B_\mu A_\nu \partial_\rho A_\sigma$ known to be related with the ABJ anomaly in usual Lorentz-breaking theories [16] can arise, and such a term involves a pseudovector $B_\mu$ as a necessary ingredient.

The results we obtained can be straightforwardly generated for the case of an arbitrary even critical exponent. In this case, the Lagrangian of the spinor sector of the theory is

$$L = \bar{\psi} (i\gamma^0 D_0 + (i\gamma^i D_i))^{2n} + m^{2n} + \mathcal{B}_{\gamma\delta}) \psi, \quad (1)$$

where $z = 2n$ is a critical exponent. Here, $D_\mu = \partial_\mu - i e A_\mu$, with $\mu = 0, i$ and $i = 1, 2, 3$, is a gauge covariant derivative, with the corresponding gauge transformations being $\psi \rightarrow e^{ic\xi} \psi$, $\psi \rightarrow \psi e^{-ic\xi}$, and $A_{0,i} \rightarrow A_{0,i} + \partial_0 \xi$, and $B_\mu$ is an extra pseudovector field. The dimensions of our objects look as follows: the dimension of $\partial_0$ and $eA_i$ is 1, as in the usual case, of $\partial_0$ and $eA_0$ is $z$, as the Horava-Lifshitz formalism requires, of $\psi$ is $d/2$, and of $B_\mu$ is $2n$. We note that in the absence of a kinetic term for the vector field, the dimension of the $A_0$ and $A_i$ fields cannot be fixed unambiguously. Also, we introduced $m^{2n}$ into the mass term for the spinor field in order to have the constant $m$ with a mass dimension equal to one.

The free propagator for the fermionic fields is

$$\langle \psi(k)\bar{\psi}(-k) \rangle = S(k) = i \frac{\gamma^0 k_0 - (\vec{k}^{2n} + m^{2n})}{k_0^2 - (\vec{k}^{2n} + m^{2n})^2}. \quad (2)$$

In principle, the number of different vertices for an arbitrary $n$ is very large. However, the ABJ anomaly involves

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as usual two vector fields and one axial field. If we consider the vertices with no more than two vector fields and no more than one derivative applying within any vertex, the number of vertices is drastically restricted, so, denoting \( \nabla^2 = \partial^2 \partial \), we have only
\[
V_1 = ie\tilde{\psi}^0 A_0 \psi, \quad V_2 = ieC_2 \tilde{\psi}^\gamma \gamma^\gamma (\partial A_j) \nabla^{2n-2} \psi, \\
V_3 = 2ieC_3 \tilde{\psi} A^i \partial \nabla^{2n-2} \psi, \quad V_4 = e^2 C_4 \tilde{\psi} A_i A^i \nabla^{2n-2} \psi.
\]
(3)

In the momentum space they look like
\[
V_1 = ie\tilde{\psi}(k)\gamma^\gamma A_0(p)\psi(-p - k), \\
V_2 = eC_2 np \tilde{\psi}(k)\gamma^\gamma A_j(p)\psi(-k - \vec{k})2n-2\psi(-p - k), \\
V_3 = -2C_3 n(p_i + k_i) \tilde{\psi}(k)A^i(p)\psi(-k - \vec{k})2n-2\psi(-p - k), \\
V_4 = e^2 C_4 n(\tilde{k}A_i(p_i)A^i(p_2))\psi(-p - k). \\
\]
(4)

Here \( C_{2,n}, C_{3,n}, C_{4,n} \) are the numbers generated by permutations of the Dirac matrices. Their explicit form is not important for us; however, it can be found, that is, \( C_{2,n} = n(-1)^{n-1}, \), \( C_{3,2k} = 2k, \) \( C_{3,2k+1} = -(2k + 1), \) \( C_{4,2k} = 2k, \) \( C_{4,2k+1} = -(2k + 1). \) There is also the extra vertex \( \tilde{\psi} \).  

**ABJ anomaly.** – The ABJ anomaly is given by the Feynman diagram depicted in fig. 1.

Here the thick wavy line is for the external \( B_\mu \) field.

We note that the quartic vertex does not contribute to the ABJ anomaly yielding a total derivative.  

There are six possible contributions to it characterized by different positions of the vertices \( V_1, V_2, V_3 \) (beside the axial vertex). It is clear that the diagrams with two \( V_1 \) vertices and with two \( V_2 \) vertices do not contribute to the ABJ anomaly (indeed, in the first of these cases the indices of the external \( A_0 \) fields are the same, and in the second one, the contribution is of second order in derivatives). The remaining four graphs can be described as follows. The first one involves \( V_1 \) and \( V_2 \) vertices, the second one – \( V_1 \) and \( V_3 \) ones, for the third one – \( V_2 \) and \( V_3 \) ones, finally, the fourth one is formed with the use of two \( V_3 \) vertices. They are depicted in fig. 2.

The contributions of these graphs are
\[
S_1 = e^2 C_{2,n} \text{tr} \int \frac{dk_0 dk^2}{(2\pi)^4} A_0(-p)\gamma^0 S(k)\gamma^m B_m \\
\times \gamma_5 S(k)\gamma^\gamma \gamma^\gamma (\partial A_j)(p)S(k), \\
S_2 = ie^2 C_{2,n} \text{tr} \int \frac{dk_0 dk^2}{(2\pi)^4} A_0(-p)\gamma^0 S(k)\gamma^m B_m \\
\times \gamma_5 S(k)A_i(p)(k + p)S(k + p), \\
S_3 = 4ie^2 C_{2,n} C_{3,n} \text{tr} \int \frac{dk_0 dk^2}{(2\pi)^4} A_0(-p)k_0 S(k) \\
\times \gamma^m B_m \gamma_5 S(k)\gamma^\gamma \gamma^\gamma (\partial A_j)(p)S(k), \\
S_4 = 4e^2 C_{2,n} \text{tr} \int \frac{dk_0 dk^2}{(2\pi)^4} A_i(-p)(k)S(k)\gamma^m B_m \\
\times \gamma_5 S(k)A_i(p)(k + p)S(k + p). 
\]
(5)

Here we omitted the dependence of propagators on the external momentum \( p \) within \( S_1 \) and \( S_3 \) since taking it into account will yield only the second- and higher-order contributions in derivatives. It is clear that the contributions \( S_2, S_3, S_4 \) vanish. Indeed, in \( S_2 \) and \( S_4 \) the structure of products of Dirac matrices is insufficient to give a non-zero trace, and the integrand of \( S_3 \) is odd with respect to the internal moments.

So, it remains to consider \( S_1 \). Its explicit form is given by
\[
S_1 = e^2 C_{2,n} \text{tr} \int \frac{dk_0 dk^2}{(2\pi)^4} A_0(-p)\gamma^0 S(k)\gamma^m B_m \gamma_5 S(k) \\
\times \gamma^\gamma \gamma^\gamma (\partial A_j)(p)\vec{k}^{2n-2} S(k). 
\]
(6)

By calculating the trace over the Dirac matrices, we obtain
\[
S_1 = -4ie^2 C_{2,n} \epsilon^{ijk} A_0(-p)B_j A_i(p) p_k \\
\times \int \frac{dk_0 dk^2}{(2\pi)^4} \left[ 3k_0^2 + (\vec{k}^{2n} + m^2 n^2)(\vec{k}^{2n} + m^2 n^2)\vec{k}^{2n-2} \right] \\
\left[ k_0^2 - (\vec{k}^{2n} + m^2 n^2) \right]^3. 
\]
(7)

A straightforward integration over \( k_0 \) shows that this integral vanishes for any \( n \). We note that it is rather natural since the corresponding contribution is not gauge invariant.

**Finite-temperature analysis of the ABJ anomaly.** – Let us consider now the finite-temperature study of the ABJ anomaly. To do it, we use the finite-temperature formalism described in [17] and generalized for the HL-like theories in [18]. A general approach to chiral anomaly at finite temperature can be found in [19]. Within our study we follow the lines of [20] where the triangle graph corresponding to the ABJ anomaly has been discussed at finite temperature in the usual Lorentz-breaking QED. First, we carry out the Wick rotation:
\[
S_1 = 4e^2 C_{2,n} \epsilon^{ijk} A_0(-p)B_j A_i(p) p_k \\
\times \int \frac{dk_0 dk^2}{(2\pi)^4} \left[ 3k_0^2 - (\vec{k}^{2n} + m^2 n^2)(\vec{k}^{2n} + m^2 n^2)\vec{k}^{2n-2} \right] \\
\left[ k_0^2 - (\vec{k}^{2n} + m^2 n^2) \right]^3. 
\]
(8)

Let us now use the Matsubara formalism, which consists in taking \( k_0 = (n + \frac{1}{2})2\pi/\beta \) and changing

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or in the massless limit (\(m \to 0\)), \(F\) at zero temperature in the function \(F\) vanishes for any \(n\). We note that the disappearance of \(\rho\) that in the case of the usual QED \(\|\) the chiral transformation is linear in \(\beta\). To verify this, one can expand \(F(M)\) in power series in \(M\) as

\[
F(M) \approx F(M = 0) + \frac{dF}{dM} \bigg|_{M=0} M. \tag{12}
\]

It is clear that \(F(M = 0) = 0\). And

\[
\frac{dF}{dM} \bigg|_{M=0} = \int_{0}^{\infty} dKK^{2n} \tanh[\pi K^{2n}] \text{sech}^2[\pi K^{2n}], \tag{13}
\]

that is, a constant which we denote as \(\rho\). Taking into account the explicit form of \(M\), we find \(F(M) \approx \rho m^n (\frac{2}{\pi^2})^{1/2}\).

We conclude that the potential anomaly differs from zero only if temperature is neither zero nor infinite.

**Transformations of the measure.** – It is well known that in the case of the usual QED \(\|\) the chiral transformations of the spinor fields

\[
\psi \to e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha \gamma_5} \tag{14}
\]

yield the additional contribution to the anomaly caused by the measure transformation and equal to

\[
\delta S_{\text{measure}} = \frac{1}{16\pi^2} \int d^4x \alpha(x)e^{\alpha\mu\nu} F_{\mu\lambda} F_{\nu\lambda}. \tag{15}
\]

Let us consider the possibility of the analogous contribution in our case.

Instead of the base of fields \(\phi_i\) satisfying the equation \(\mathcal{D}\phi_i = \lambda_i \phi_i\) (recall that the usual spinor action is \(\int d^4x \bar{\psi}(i\mathcal{D} - m)\psi\)), in our case, we will have the base \(\phi_i\) satisfying the equation \((i\gamma^0 D_0 - \mathcal{D}^{2n})\phi_i = \hat{\lambda}_i \phi_i\), cf. eq. (1). Since both the operator \(i\gamma^0 D_0\) and the operator \(\mathcal{D}^2\) are Hermitian, the operator \(i\gamma^0 D_0 - \mathcal{D}^{2n}\) will also be Hermitian possessing thus only real eigenvalues. Therefore, all \(\lambda_i\) are real.

The corresponding Jacobian will look like (cf. \(\|\))

\[
\exp(-2iJ) = \exp \left( -2i \lim_{N \to \infty} \sum_{l=1}^{N} \int dt d^3x \alpha(t, x) \bar{\phi}_i(t, x) \gamma_5 \phi_i(x) \right). \tag{16}
\]

To provide the convergence of the integral, we introduce the regularization by inserting the function \(f(\frac{1}{N})\),

\[
1/(2\pi) \int dK_0 \to 1/\beta \sum. \quad \text{Thus, we obtain}
\]

\[
S_1 = \frac{e^2}{m} \epsilon^{0ijk} A_0(-p) B_i A_j(p) p_k \int_{0}^{\infty} dK K^2 \frac{M}{\pi^4} \times \frac{3 (n + \frac{1}{2})^2 - (K^{2n} + M^{2n})^2}{(n + \frac{1}{2})^2 + (K^{2n} + M^{2n})^2} \tag{9}
\]

with \(K^{2n} = \overline{K}^{2n} \frac{\beta}{\pi}\) and \(M^{2n} = m^{2n} \frac{\beta}{\pi}\), where we have used spherical coordinates, i.e., \(\int d^3k = (2\pi/\beta)^2 \int dK 4\pi K^2\). Finally, one finds that eq. (7) takes the form

\[
S_1 = e^2 C_{2, n} m^{1-2n} \epsilon^{0ijk} A_0(-p) B_i A_j(p) p_k F(M) \tag{10}
\]

with

\[
F(M) = \int_{0}^{\infty} dK K^{2n} M^{2n-1} \tan[\pi (K^{2n} + M^{2n})] \times \text{sech}^2[\pi (K^{2n} + M^{2n})]. \tag{11}
\]
where $M_0$ is a constant with a dimension of mass, so that $f(x)_{|x \to \infty} = 0$. After proceeding as in [21], the factor $J$ determining the Jacobian takes the form

$$J = \text{Tr} \int d^3x \alpha(t, x) \gamma_5 f \left( \frac{(i \gamma^0 D_0 - D^{2n})^2}{M_0^4} \right),$$

(17)

where, unlike [21], $D = \gamma^i D_i$ is a purely spatial contraction. The explicit form of the trace is

$$J = \text{Tr} \int d^3x \alpha(t, x) e^{-ikx} \gamma_5$$

$$\times f \left( \frac{(i \gamma^0 D_0 - D^{2n})^2}{M_0^4} \right) e^{ikx}.$$  

(18)

One can argue as in [21] that only the second order in $(i \gamma^0 D_0 - D^{2n})^2$ contributes to the measure (indeed, it yields only the quadratic contribution to the effective action), so, expanding the function $f$ up to the second order, we have

$$J = \frac{1}{2} \text{Tr} \int d^3x \alpha(t, x) e^{-ikx} f''(0)$$

$$\times \gamma_5 \left( \frac{i \gamma^0 D_0 - D^{2n}}{M_0^2} \right)^2 e^{ikx}.$$  

(19)

Then, the simple transformation shows that $(i \gamma^0 D_0 - D^{2n})^2 = (i \gamma^0 D_0 - (D^2 + \frac{1}{4}[\gamma^i, \gamma^j]F_{ij})^n)^2$, thus,

$$J = \frac{1}{2} \text{Tr} \int d^3x \alpha(t, x)$$

$$\times e^{-ikx} f''(0) \gamma_5 \left( \frac{i \gamma^0 D_0 - D^2 + \frac{1}{4}[\gamma^i, \gamma^j]F_{ij}}{M_0^2} \right)^2 e^{ikx}.$$  

(20)

However, unlike the usual case where the trace yields the Levi-Civita symbol, this expression involves either the trace of the product $\gamma_5 [\gamma^i, \gamma^j] [\gamma^k, \gamma^l]$ which is zero, or the trace of $\gamma_5 [\gamma^i, \gamma^j] [\gamma^k, \gamma^l]$ which is also zero. Hence, the factor $J$ is zero, and the Jacobian is consequently trivial being equal to 1. We conclude that in our case both the variation of the measure and the triangle contribution vanish.

Summary. – Let us discuss our results. We find that the only nontrivial contribution vanishes at zero temperature and tends to zero at a very high temperature. In principle, the vanishing of the CFJ-like contribution at the zero temperature has a natural reason – actually, since the action of the theory does not involve any terms linear in zero temperature has a natural reason – actually, since the chiral principle, the vanishing of the CFJ-like contribution at the classical level. We explicitly demonstrated that the similar situation will occur for other even $\gamma$ (indeed, in these cases the propagators and the vertices will involve even numbers of the spatial $\gamma^i$ Dirac matrices). Also, we have showed that the contribution to the effective action generated by the transformation of the measure is trivial. As a by-product, we can see that the one-loop contribution to the pion decay (which can be obtained by replacement of $D_{\gamma_5}$ by $\gamma_5$ in the corresponding vertex) is zero since the number of space-like $\gamma_5$ matrices in its contribution is insufficient.

All our discussions were carried out for even $\gamma$. At the same time, odd $\gamma$ (we note that the contribution of the integral measure to the possible chiral anomaly has been discussed in [22,23], for the case of an essentially odd critical exponent $\gamma$) will essentially differ. We are going to consider this situation in our next paper.

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This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). The work by AYuP has been partially supported by the CNPq project No. 303438/2012-6.

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