Induced Riemannian Structures and Topology of Null Hypersurfaces in Lorentzian Manifold

Karimumuryango M*
Faculty of Science and Engineering, University of Burundi, Burundi

Abstract

Given a null hypersurface of a Lorentzian manifold, we induce a Riemannian metric on the null hypersurface using a normalizing field defined on some open set containing the null hypersurface. We establish links between the null geometry and basics invariants of the associated Riemannian metric. This allowed us, using some comparison theorems from Riemannian geometry, to get important results on the topology of the null hypersurface.

Keywords: Null hypersurface; Rigging; Closed normalization; Associated Riemannian metric

Introduction

In spite sharing common root, Lorentzian and Riemannian geometries diverge very fast. For example, in Lorentzian case, due to the causal character of three categories of vector fields (namely, spacelike, timelike and null), the induced metric on a hypersurface is a non-degenerate metric tensor field or degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On no-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) of the unit orthogonal vector field is known to lead to a canonical splitting of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations, along with the second fundamental form, shape operator, induced connection, etc. The case the normal vector field is null, the hypersurface is called null (or lightlike). Null hypersurfaces are then exclusive objects from Lorentzian manifolds, and have not Riemannian counterpart, making them interesting by their own from a geometric point of view, but also they are key objects for modern physics (quantum gravity effects). The geometry of null submanifolds is different and rather difficult since (contrary to the non-degenerate counterpart) the normal vector bundle intersects (non trivially) with the tangent bundle. Thus, one can not find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects as usual. This degeneracy of the induced metric makes impossible to study them as part of standard submanifold theory, forcing to develop specific techniques and tools. For the most part, these tools are specific to a given problem, or sometimes with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Indeed, Duggal and Bejancu introduced a non-degenerate screen distribution or equivalently a null transversal line vector bundle as we may see below so as to get a three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, shape operators, induced connection, curvature, etc. [1]. Unfortunately, the screen distribution is not unique and there is no preferred one in general. The least we can say is that for the above approach to be complete and consistent, we still need to build a distinguished normalization to accompany it. Most of the recent works of the first named author are indeed devoted to this normalization problem [2-4]. Given the collective expertise in Riemannian geometry, the ideal situation on could expect is that the developed tools could bring to a full reduction of problems in null geometry to purely Riemannian ones. In, the present first named author, after fixing a pair of normalization, constructed an associated Riemannian metric to the "normalized null structure" [5,6]. These ideas have been generalized and improved where authors used riggings defined on neighborhood of the null hypersurface. In the present paper, we first consider the associated Riemannian metric as [5] but arising from a null rigging defined on neighborhood of the null hypersurface, and establish links between the null geometry and basics invariants of the associated Riemannian metric. Also, note that one of the major issues of Riemannian geometry is how to obtain topological or differential properties of a manifold from some known properties of its curvatures. For example what can be said about a complete Riemannian manifold when some suitable estimates are know for the sectional or Ricci curvature? These considerations have been on much scrutiny with excellent results: Myers (compactness), Klingenberg (on the injectivity radius), Cheeger-Gromoll (splitting theorem), Shoen-Yau (3-manifolds that are diffeomorphic to the standard $R^3$), Gromov’s estimate of the number of generators of the fundamental group and the Betti numbers when lower curvature bounds are given. For further background on this problem we refer to the excellent texts [7-11]. Since in the present paper we have established links between the null geometry and basics invariants of the associated Riemannian metric, it is reasonable to expect that the geometry of the null hypersurface provides insight informations on its topology. This constitutes our second and main goal. The plan of the article is as follows. Section (2) sets notations and definitions on riggings (normalizations) and review basics properties on null hypersurfaces. The associated Riemannian distance structure on the rigged (or normalized) null hypersurface are introduced and discussed. The relationship between the null and the associated Riemannian geometry is considered in section (4) where we proceed to a connection of the main geometric objects (invariants) of both side involved in our analysis. In the last sections, thanks to some Riemannian comparison theorems we get some topological facts on the null hypersurfaces from its null geometry.

*Corresponding author: Karimumuryango M, Professor, Faculty of Science and Engineering, University of Burundi, Burundi, Tel: +25771058855; E-mail: kmizedore@gmail.com

Received September 21, 2017; Accepted October 31, 2017; Published November 03, 2017

Citation: Karimumuryango M (2017) Induced Riemannian Structures and Topology of Null Hypersurfaces in Lorentzian Manifold. J Phys Math 8: 250. doi: 10.4172/2090-0902.1000250

Copyright: © 2017 Karimumuryango M. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
Riggins and Preliminaries of Null Hypersurfaces

Let \( (M, g) \) be a \((n+2)\)-dimensional Lorentzian manifold and \( M \) a null hypersurface in \( M \). This means that at each \( p \in M \), the restriction \( g_{\nu \mu}(\cdot, \cdot) \) is degenerate, that is there exists a non-zero vector \( U \in T_pM \) such that \( g(U, \cdot) = 0 \) for \( X \in T_pM \). Hence, in null setting, the normal bundle \( TM^\perp \) of the null hypersurface \( M \) is a rank 1 vector subbundle of the tangent bundle \( TM \), contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection \([0] \) with the tangent one plays an important role in the introduction of the main induced geometric objects on \( M \). Let start with the usual tools involved in the study of such hypersurfaces according to Duggal [1]. They consist in fixing on the null hypersurface a geometric data formed by a lightlike section and screen distribution. By screen distribution on \( M \), we mean a complementary bundle of \( TM^\perp \) in \( TM \). In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface \( M \) be paracompact, but each of them is canonically isomorphic to the factor vector bundle \( TM/TM^\perp \). For reasons that will become obvious in few lines below, let denote such a distribution by \( S(N) \). We then have,

\[
TM = S(N) \oplus_{\text{orth}} TM^\perp. \tag{1}
\]

Where \( \ominus_{\text{orth}} \) denotes orthogonal direct sum. From ref. [1], it is know that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle \( \nu(TM) \) of \( TM \) over \( M \), such that for any non-zero section \( \xi \) of \( TM^\perp \) on coordinate neighborhood \( U \subset M \) there exists a unique section \( N \) of \( tr(TM) \) on \( U \) satisfying

\[
\overline{F}(N, \xi) = 1, \overline{F}(N, N) = \overline{F}(N, W) = 0 \vee W \in \Gamma(S(N)) \tag{2}
\]

Then \( TM \) is decomposed as following:

\[
TM = S(N) \ominus_{\text{orth}} \left( TM^\perp \ominus tr(TM) \right) = TM \ominus tr(TM). \tag{3}
\]

We call \( tr(TM) \) a (null) transversal vector bundle along \( M \). In fact, from eqns. (2) and (3) one shows that, conversely, a choice of a transversal bundle \( tr(TM) \) determines uniquely the distribution \( S(N) \). A vector field \( N \) as in eqn. (2) is called a null transversal vector field of \( M \). It is then noteworthy that the choice of null transversal vector field \( N \) along \( M \) determines both the null transversal vector bundle, the screen distribution \( S \) and a unique radical vector field, say satisfying eqn. (2). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface.

Definition 2.1

Let \( M \) be a null hypersurface of a Lorentzian manifold. A rigging for \( M \) is a vector field \( \xi \) defined on some open set containing \( M \) such that \( \xi \notin T_pM \) for each \( p \in M \).

We say that we have a null rigging in case the restriction of \( \xi \) to the null hypersurface is a null vector field. From now on we fix a null rigging \( \xi \) for \( M \). In particular this rigging fixes a unique null vector field \( \xi \in \Gamma(TM^\perp) \) called the rigging vector field, all of them defined in an open set containing \( M \) (hence globally on \( M \)) such that eqns. (1)-(3) hold. Whence, from now on, by a normalized (or rigged) null hypersurface we mean a triple \((M, g, N)\) where \( g = \xi g \) is the induced metric on \( M \) and \( N \) a null rigging for \( M \). In fact, in case the ambient manifold \( M \) has Lorentzian signature, at an arbitrary point \( p \) in \( M \), a real null cone \( C_p \) is invariantly defined in ambient tangent space \( T_pM \) and is tangent to \( M \) along a generator emanating from \( p \). This generator is exactly the radical fiber \( \xi = T_pM^\perp \) and for each null rigging \( N \) for \( M \) and each \( \xi \in M \) we have \( N_z = C_p \setminus \xi_z \). Actually, a lightlike hypersurface \( M \) of a Lorentzian manifold is a hypersurface which is tangent to the lightlike cone \( C_p \) at each point \( p \in M \). Recall that a spacetime \((M, g)\) is a connected Lorentzian manifold which is "time-oriented", i.e a causal cone at each \( p \in M \) \((\text{the future}'\text{causal cone})\) has been continuously chosen. Hence, null hypersurfaces in spacetimes can be naturally given an orientation by such a continuous distribution of causal cones \( C_p \). Let \( N \) be a null rigging of a null hypersurface of a Lorentzian manifold \((M, g, N)\) and \( \theta = \xi g(N, \xi) \) the 1-form metrically equivalent to \( N \). Then, take

\[
\eta = \iota \theta \tag{4}
\]

to be its restriction to \( M \), the map \( i: M \to M \) being the inclusion map. The normalized null hypersurface \((M, g, N)\) will be said to closed if \( \theta \) is closed on \( M \). It is easy to check that \( S(N) = ker \theta \) and that the screen distribution \( S(N) \) is integrable whenever is closed.

On normalized null hypersurface \((M, g, N)\), the local Gauss and Weingarten type formulas are given by:

\[
\nabla_{\xi} X = \nabla_{\xi} X + B^\xi(X, Y) N, \tag{5}
\]

\[
\nabla_{\xi} N = \nabla_{\xi} N = -A_{\xi} + \tau \xi (X, N), \tag{6}
\]

\[
\nabla_{Y} P Y = \nabla_{Y} P Y + C^\xi(X, PY)\xi, \tag{7}
\]

\[
\nabla_{\xi} \xi = -A_{\xi}^* X - \tau ^\xi(X)\xi, \tag{8}
\]

for any \( X, Y \in \Gamma(TM) \), where \( \nabla \) denotes the Levi-Civita connection on \((M, g)\) \(\nu \) denotes the connection on \( M \) induced from \( \nabla \) through the projection along the null rigging \( N \), \( \nu \) denotes the Levi-Civita connection on the screen distribution \( S(N) \) induced from \( \nu \) through the projection morphism \( \Gamma(TM) \) onto \( \Gamma(S(N)) \) with respect to the decomposition. Now the \((0,2)\) tensor \( B^\mu \) \(C^\mu \) are the second fundamental forms on \( TM \) and \( S(N) \) respectively, \( A_{\xi} \) \( A_{\xi}^* \) are the shape operators on \( TM \) and \( S(N) \) respectively and \( \tau ^\xi \) a 1-form on \( TM \) defined by \( \tau ^\xi = \xi g(N, \xi) \). For the second fundamental \( B^\mu \)\( C^\mu \) the following hold:

\[
B^\mu(X, Y) - g(A_{\xi} N, X) Y - g(A_{\xi} Y, X) = 0, \tag{9}
\]

\[
\forall X, Y \in \Gamma(TM) \quad \text{and} \quad \tau ^\xi = \xi g(N, \xi) = 0. \tag{10}
\]

It follows from eqn. (10) that integral curves of \( \xi \) are geodesics both \( M \) and \( M \) as consider these integral curves to be geodesics which means that

\[
\tau ^\xi = 0 \quad \text{for all } \xi \in \Gamma(TM). \tag{11}
\]

A null hypersurface \((M, g, N)\) is called totally umbilical (resp. geodesic) if there exists a smooth function \( \rho \) on \( M \) such that at each \( p \notin M \) and for all \( u, v \in T_pM \) \( BN(u, v) = (\rho g(u, v)\text{resp } B^\nu ) \text{vanishes identically on } M \). These are intrinsic notions on any null hypersurface in the following way. Note that \( N \) being a null rigging for \( M \), a vector field \( N \in \Gamma(TM) \) is a null rigging for \( M \) if and only if it is defined in an open set containing \( M \) and there exist a function \( \psi \) and section \( \zeta \) of \( TM \) with the properties that \( \psi \), \( i \) is nowhere vanishing, being \( i \) the inclusion map and \( \psi g(\zeta, \zeta)^{1/2} = 0 \). Then we have for details on changes in normalizations) \( B^\nu = \frac{1}{\psi'} B^\nu \), which shows that total umbilicity and totally geodesibility are intrinsic properties for \( M \). The total umbilicity and the total geodesibility conditions for \( M \) can also be written respectively as \( A_{\xi}^* = \rho P \) and \( A_{\xi}^* = 0 \). Also, the screen distribution \( S(N) \) is totally umbilical (resp. totally geodesic) if \( C^\xi = C^\xi = 0 \) for all \( X, Y \in \Gamma(TM) \) resp. \( C^\xi = 0 \) which is equivalent to \( A_{\xi} = \rho P \) (resp \( A_{\xi} = 0 \)). It is noteworthy
to mention that the shape operators $A^c_i$ and $A_x$ are $\mathfrak{S}(N)$-valued. The induced connection $\nabla$ is torsion-free, but not necessarily $g$-metric unless $M$ be totally geodesic. In fact we have for all tangent vector fields $X,Y$ and $Z$ of $M$:

$$\langle \nabla_X Y, Z \rangle = B^X(Y)\eta(Z) + B^Z(Y)\eta(X). \tag{12}$$

Let denote by $\nabla$ and $\Gamma$ the Weingarten formula, we get

$$\langle \nabla_{\xi} \eta, \eta \rangle = \sum_{i,j}^{\mathfrak{S}(N)} F^i_{ij} \eta^{,i} \eta^{,j}. \tag{13}$$

It is easy to check such a hypersurface is null if and only if $F$ is a solution of the partial differential equation

$$1 = \sum_{i,j}^{\mathfrak{S}(N)} (F^i_{ij})^2. \tag{20}$$

(Which we assume from now on) and that the rank one normal bundle $TM^\perp$ is spanned by the global vector field

$$\eta = \sum_{i,j}^{\mathfrak{S}(N)} F^i_{ij} \eta^{,i} \eta^{,j}. \tag{21}$$

Then, the vector field defined by

$$N = \nabla + \frac{1}{2} = \frac{1}{2} \left( - \frac{\partial}{\partial x^i} \sum_{i,j}^{\mathfrak{S}(N)} F^i_{ij} \eta^{,i} \eta^{,j} \right) \tag{22}$$

is null rigging for the null Monge hypersurface $M$. For this null rigging, we get $\tau^0 = 0$ and let $\nabla$ denote the Levi-Civita connection of $R^{n+2}$ and $\mathcal{X}(TM)$ a tangent vector field. We have:

$$\tau^X(X) = \langle \nabla_X N, \xi \rangle = \langle \nabla_X \left( \nabla^2(x, \frac{1}{2} \xi) \right), \xi \rangle = \frac{1}{2} \nabla^2(x, \xi, \xi) 0. \tag{23}$$

Now, we show that the normalization given by eqn. (22) is closed (and in particular, the distribution $\mathcal{S}(N)$ is integrable). The 1-form $\theta$ metrically equivalent to $N$ is given by $\theta = -\langle \nabla^X, \xi \rangle$ and $\eta = \tau^X$ (i.e. the inclusion map). Let $X$ be two vector fields on $M$ smoothly extended in two vector fields on $R^{n+2}$ (we also denote these extensions by $X$ and $Y$). We have

$$2d\eta(X,Y) = X\langle N, Y \rangle - Y\langle N, X \rangle - \langle N, [X,Y] \rangle = \nabla^X(Y, N) - \nabla^Y(X, N) = \frac{1}{2} \nabla^X (\nabla^Y, X) - \frac{1}{2} \nabla^Y (\nabla^X, X). \tag{24}$$

Hence

$$\langle A_x Y, X \rangle - \langle A_y X, Y \rangle = \tau^X(X)\eta(Y) - \tau^Y(Y)\eta(X). \tag{17}$$

as announced. In case the normalization is closed the (connection)1-form $\tau$ is related to the shape operator $A_x$ as shown in the following lemma.

**Lemma 2.1:** For all $X,Y \in TM$

$$\langle A_x Y, X \rangle - \langle A_y X, Y \rangle = \tau^X(X)\eta(Y) - \tau^Y(Y)\eta(X) - 2d\eta(X,Y) \tag{16}$$

where (throughout) $\langle \rangle = g$ stands for the Lorentzian metric.

**Proof:** Recall that $\eta = \tau^X \theta$ where $\theta = \langle N, \gamma \rangle$ the taking the differential of $\theta$ and using the Weingarten formula, we have for all $X,Y \in TM$

$$2d\eta(X,Y) = 2d\theta(X,Y) = \langle \nabla_X N, Y \rangle - \langle \nabla_Y N, X \rangle = -\langle A_x Y, X \rangle + \tau^X(X)\eta(Y) + \langle A_y X, Y \rangle + \tau^Y(Y)\eta(X).$$

Hence

$$\langle A_x Y, X \rangle - \langle A_y X, Y \rangle = \tau^X(X)\eta(Y) - \tau^Y(Y)\eta(X). \tag{17}$$

as announced. In case the normalization is closed the (connection)1-form $\tau$ is related to the shape operator $A_x$ as follows.

**Lemma 2.2:** Let $(M,g,N)$ be a closed normalized of null hypersurface $M$ in a Lorentzian manifold such that $\tau^X(0) = 0$. Then

$$\tau(X) = -\langle A_x \xi, \eta \rangle. \tag{18}$$

**Proof:** Assume $\eta = \tau^X \theta$ and let $X,Y$ be tangent vector fields to $M$. The condition $\langle X, \eta \rangle - Y, \eta \rangle - \langle X, [N, Y] \rangle = 0$ is equivalent to

$$\langle \nabla_X N, Y \rangle - \langle \nabla_Y N, X \rangle = 0$$

Then by the Weingarten formula, we get

$$\langle A_x Y, X \rangle + \tau^X(X)\eta(Y) = \langle A_y X, Y \rangle + \tau^Y(Y)\eta(X). \tag{17}$$

In this relation, take $Y = \xi$ to get

$$\tau^X(X)\eta(Y) = -\langle A_x \xi, X \rangle + \tau^X(\xi)\eta(Y) \tag{17}$$

which gives the desired formula as $\tau^X(\xi) = 0$.

**Example 2.1**

In the pseudo-Euclidean space $R^{n+2}_+$, the pseudo-Euclidean space $R^{n+2}_-$ (as $\mathfrak{S}(N)$) $\langle \rangle = g = -\sum_{i,j}^{\mathfrak{S}(N)} dx^i \otimes dx^j + \sum_{i=1}^{\mathfrak{S}(N)} dx^i \otimes dx^i$ where $(x^1, ..., x^{n+1})$ stands for the usual rectangular coordinates of $R^{n+2}$. Consider the Monge hypersurface

$$M = \{x^1, ..., x^{n+1} \in R^{n+2}, x^0 = F(x^1, ..., x^{n+1}) \tag{19}$$

Where $F: \Omega \rightarrow R$ is a smooth function defined on $\Omega$ an open set of $R^{n+2}$. Throughout, $V$ denote a the constant vector field $\frac{\partial}{\partial x^0}$. It is easy to

**Riemannian Distance Associated**

Let $(M,g,N)$ be a normalized null hypersurface, of a Lorentzian manifold. For $p \in M$ let $\Omega_p$ denote the space of all piecewise smooth curves $\gamma: [0,1] \rightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$. The ambient manifold being Lorentzian, the induced metric on $M$ has signature $(0,n)$. In particular, for each $\epsilon \in [0,1]$, we have $\|\gamma(\epsilon)\|^2 + \eta(\gamma(\epsilon)) = 0$ if and only if $\gamma' = 0$. We define the $\eta$-arc length of $y \in \Omega_p$ by

$$L_\eta(y) = \int_0^1 \|\gamma'(t)\|^2 + \eta(\gamma(t)) dt. \tag{26}$$

From the above facts and using standard techniques as in Riemannian setting, and noting that a tangent vector $X$ belongs to $\mathcal{S}(N)$ if and only if $\eta(X) = 0$, one gets the following lemma.

**Lemma 3.1:** The map $d^\eta: M \times [0, \epsilon)$ given by

$$d^\eta(p,q) = \inf_{y \in \Omega_p} L_\eta(y) \tag{27}$$

is a distance function on $M$. 

Citation: Karimumuryango M (2017) Induced Riemannian Structures and Topology of Null Hypersurfaces in Lorentzian Manifold. J Phys Math 8: 250. doi: 10.4172/2090-0902.1000250
**Definition 3.1**

A normalized null hypersurface \((M, g, N)\) is said to be \(\eta\)-complete if the metric space \((M, d^\eta)\) is a complete space.

**Notations 3.1:** For \(x \in M\) we set

\[
S^\eta(1) = \{ x \in TM \mid \langle X, X \rangle = -\eta(X)^2 \},
\]

\[
S^\xi(1) = \bigcup_{x \in M} S^\eta(1),
\]

and for all \(X \in \Gamma(TM)\), we set

\[
O^{\eta}(X) = \{ Y \in TM \mid \langle X, Y \rangle = -\eta(X)\eta(Y) \},
\]

where \(\langle \cdot, \cdot \rangle\) stands for both \(g\) or \(\bar{g}\). Observe that, \(Y \in O^{\eta}(X)\) iff \(X \in O^{\eta}(Y)\).

A remarkable fact is that due to the degeneracy of the induced metric \(g\) on the null hypersurface \(M\), it is not possible to define the natural dual (musical) isomorphisms \(\#\) and \(\#^*\) between the tangent and dual tangent spaces, respectively. However, this construction is made possible by setting a rigging (normalization) \(N\) (we refer to ref. [5] for further details). Consider a normalized null hypersurface \((M, g, N)\) and we define one form define by:

\[
\iota_x : \Gamma(TM) \to \Gamma(TM'),
\]

\[
x \mapsto X^\eta = g(X, X) + \eta(X)\eta(Y) \forall Y \in \Gamma(TM), X^\eta(Y) = g(X, Y) + \eta(X)\eta(Y),
\]

(24)

Cleary, such a \(#\) is an isomorphism of \(\Gamma(TM)\) to \(\Gamma(TM')\) and can be used to generalize the usual non-degenerate theory. In the latter case, \(\Gamma(S(N))\) coincides with \(\Gamma(TM)\), and as a consequence the 1-form \(\eta\) vanishes identically and the projection morphism \(P\) becomes the identity map on \(\Gamma(TM)\).

**Definition 3.2**

\[
S^\eta(1) = \{ x \in TM \mid \langle X, X \rangle = -\eta(X)^2 \} = \{ x \in TM \mid g(x, x) = 1 \}
\]

(3)

(27)

Observe that in ambient Lorentzian case, the Riemannian distance \(d^R\) associated to \((M, g)\) coincides with the metric \(d^\eta\) as given in section (3), i.e \(d = d^\eta\). It follows the famous Hopf-Rinow theorem that the null hypersurface \((M, g, N)\) is \(\eta\)-complete if and only if the Riemannian manifold \((M, g, N)\) is complete. Also, for all \(x \in M\),

\[
S^\xi(1) = \{ x \in TM \mid \langle X, X \rangle = 1 - \eta(X)^2 \} = \{ x \in TM \mid g(x, x) = 1 \}
\]

(26)

(25)

\[
g_{\#}(\xi, \xi) = 1
\]

and the last equality in eqn. (26) is telling us that the restrict to \(S(N)\) the metrics \(g\) and \(\bar{g}\) coincide. We know from eqn. (12) that the induced metric \(g\) is not compatible with the induced connection in general and this compatibility arises if and only the null hypersurface \(M\) is totally geodesic in \(\bar{M}\). Let \(\nabla^v\) denote the Levi-Civita connection of the non-degenerate associate metric \(g_{\#}\) on \((M, g, N)\). We are now interested in characterizing the normalizations for which the Levi-Civita connection \(\nabla^v\) of \(g\) agrees with the induced connection \(\nabla\) due to \(N\), i.e \(\nabla = \nabla^v\). For this we recall the following.

**Lemma 3.2:** For all \(X, Y, Z \in \Gamma(TM)\) we have,

\[
\nabla^\eta(X,Y)Z = \eta(Y)B^\eta(X, PZ) - C^\eta(X, PZ)
\]

\[
+ \eta(Z)B^\eta(X, PY) - C^\eta(X, PY)
\]

\[
+ 2\tau^\eta(X)\eta(Y)\eta(Z)
\]

We derive the following result on the compatibility condition.

**Theorem 3.1**

Let \((M, g, N)\) be a normalized null hypersurface of a pseudo-Riemannian manifold \((\bar{M}, \bar{g})\). The induced connections \(\nabla\) and \(\nabla^v\) of the associated metric \(g_{\#}\) on \((M, g, N)\) agree if and only if for all \(X, Y, Z \in \Gamma(TM)\),

\[
\begin{cases}
B^\eta(X, PY) = C^\eta(X, PY) & \tau^\eta(X) = 0
\end{cases}
\]

In particular for a closed normalization,
where 
\[ \nabla_s^c Y = \nabla_s Y - \frac{1}{2} [g(A_s', Y, Y) - g(A_s, X, Y) - g(A, Y, X)] \]

**Proof.** Both connections V₀ and V are torsion free. We can write
\[ \nabla_s Y = \nabla_s Y + D(X, Y), VX, Y = \Gamma(TM), \]
with D a symmetric tensor. As V₀ is gₘetric, we have
\[ g_s(D(X, Y), Z) + g_s(Y, D(X, Z)) = Xg_s(Y, Z) - g_s(Vs, Y, Z) = (Vs, g_s(Y, Z). \]

Then using this and Lemma (3.2), we have
\[ g_s(D(X, Y), Z) + g_s(Y, D(X, Z)) = \eta(Y)[B^s(X, PZ) - C^s(X, PZ)] + \eta(Z)[B^s(X, PY) - C^s(X, PY)] \]
\[ + 2\eta(X)[\nabla_s(Y, \eta)(Z)]. \]

By circumferent permutation we get similar expressions for \( g_s(D(Y, X), Z) + g_s(Z, D(Y, X)) \) and \( g_s(D(Z, X), Y) \).

Summing the first two expressions minus the last one leads to
\[ 2g_s(D(X, Y), Z) + \eta(X)[g_s(A_s, Z) - g_s(A_s, Y)] \]
\[ + \eta(Z)[2g_s(A_s, Y) - g_s(A_s, X) - g_s(A_s, Y, X)] \]
\[ + 2\eta(X)[\nabla_s(Y, \eta)(Z) + r^s(X)\eta(Y, \eta)(Z) - r^s(Z)\eta(X, \eta)(Y)]. \]

(29)

Now using eqns. (26) and (29), we get
\[ 2g_s(D(X, Y), Z) = 2g_s[\eta(X)(i_s, d\eta)^s, Y] + \eta(Y)(i_s, d\eta)^s, Z], \]
\[ + g_s(2g_s(A_s, Y) + \eta(X)r^s(Y) + \eta(Y)r^s(X) - g_s(A_s, Y) - g_s(A_s, Y, X)]. \]

(30)

It follows the non-degeneracy of g that
\[ \eta(X)(i_s, d\eta)^s + \eta(Y)(i_s, d\eta)^s, \]
\[ + \eta(X)[i_s, d\eta]^s, + \eta(Y)[i_s, d\eta]^s . \]

(31)

which, using the fact that the operators \( A_s' \) and \( A_s \) are \( S(N) \) valued and \( g_{s(x)} = g_{s(n)} \) gives the desired formulas. with
\[ d\eta(X, Y) = \frac{1}{2} [\eta(X)Y - \eta(Y)X] \cdot \]

**Remark 4.1**

From above proposition (4.1), it follows that if the screen distribution is integrable (which is equivalent to the symmetry of \( C^s \) on \( S(N) \)) we have for all \( S(N) \)
\[ \nabla_s^c Y = \nabla_s Y + B^s(X, Y) - C^s(X, Y)]. \]

(32)

Throughout, the normalization will be assumed closed. Beyond its technical aspect this assumption guarantees integrability of the screen distribution \( S(N) \).

**Some curvature relations**

In this section we relate various curvature tensors of the null geometry of \((M, g, N)\) to those of the associated non-degenerate metric \( g_s \) on \( M \). Let \( R \) and \( R \) denote the Riemann curvature tensors of \( V^0 \) and \( V \) respectively. Using proposition (4.1) we prove the following.

**Proposition 4.2:** Let \((M, g, N)\) be a closed normalized null hypersurface with rigged vector field \( \xi \). Then, for all \( X, Y, \xi, \zeta, (TM) \) the following hold.

\[ R_s^{(X, Y, Z)} = R_{s(X, Y, Z)} + \frac{1}{2} [\phi_{\alpha, \beta} \xi, Y - \phi_{\alpha, \beta} \xi, X] + \frac{1}{2} [\eta(Y, \eta(Z), \eta(X) + \eta(X, \eta(Z), \eta(Y) \eta(Y) + \eta(Y) \eta(Y)]. \]

(33)

Then eqns. (37) and (38) consist on repeated applications of (30) in Proposition (4.1).

**Remark 4.2**

Note that by lemma (2.2), for a closed and conformal normalization (with factor \( \varphi \) we have \( \tau^s = 0 \) and \( \phi(X, Z) = 2(1 - \varphi)B_s(X, Z) \). Then eqns. (37) and (38) take the forms
\[ g_s(R_s^{(X, Y, Z), PW}) = g(R_{s(X, Y, Z), PW}) \]
\[ + \frac{1}{2} [\phi_{\alpha, \beta} \xi, Y, W - \phi_{\alpha, \beta} \xi, X, W] \]
\[ = -g(R_{s(X, Y, Z), U}, PW) \]
\[ - \frac{1}{2} [\eta(Y, \eta(Z), \eta(X) + \eta(Y) \eta(Y) + \eta(Y) \eta(Y)]. \]

(34)

**Proof:** The Riemann curvature tensor field \( R^n \) of type (1.3) is defined by
\[ R^n(X, Y, Z) = [V^n, V^n Y]Z - V^n[i_{X, Y}]. \]

(36)

Then eqns. (37) and (38) consist on repeated applications of (30) in Proposition (4.1).

In the following we let \( Ric \) and \( Ric \) denote the Ricci curvature of \( V^0 \) and \( V \) respectively. Recall that \( Ric^n(X, Y) = trace Z \rightarrow R^n(X, Y, Z) \in TM \). This is a symmetric (0,2)-tensor on TM. Unfortunately, the corresponding quantity \( Ric^n(X, Y) \) obtained from \( V \) is no longer symmetric in general, due to the fact that the induced Riemann curvature \( R \) on the normalized null hypersurface \((M, g, N)\) fails to have the usual algebraic curvature symmetries in general. Precisely, the induced Ricci tensor \( Ric \) is given by
\[ Ric_s^{(X, Y, Z)} = Ric_{s(X, Y, Z)} + B_s^{(X, Y), \alpha} A_s - \alpha B_{\alpha}(\xi, Y)X - g(A_s, X, A_s)]. \]

(37)

where \( Ric_{s(X, Y, Z)} \) denotes the Ricci curvature of the ambient manifold. We define the (0,2)-symmetrized Ricci tensor \( Ric \) on the null hypersurface by
\[ Ric_{s(X, Y, Z)} = \frac{1}{2} [Ric_{s(X, Y, Z)} + Ric_{s(Y, X, Z)}]. \]
for all $X, Y, T Y M$.

**Theorem 4.1**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a $(n + 2)$-pseudo-Riemannian manifold. Then,

$$\text{Ric}^0(X, Y) = \text{Ric}(X, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i.$$  

Thus from eqn. (37) and (38), we get,

$$R^0t^r(\xi, Y) = g_0(R^0(\xi, Y), \xi) + \sum_{i=1}^{n} g_i R^0(E_i, X)Y_E_i = \text{Ric}(X, Y) + \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$$

and $R^0t^r(\xi, Y) = \text{Ric}(X, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$.

**Proof.** Let $p \in M$ and $(E_i) = (\xi, E_1, \ldots, E_n)$ be a quasiorthonormal basis for $(T_p M, g)$ with $\text{Span}(E_1, \ldots, E_n) = S(N)$. Then, we have:

$$\text{Ric}^0(X, Y) = \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i.$$  

(40)

From eqn. (37) and (38), we get,

$$R^0t^r(\xi, Y) = g_0(R^0(\xi, Y), \xi) + \sum_{i=1}^{n} g_i R^0(E_i, X)Y_E_i = \text{Ric}(X, Y) + \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i,$$

and $R^0t^r(\xi, Y) = \text{Ric}(X, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$.

Where

$$\text{Ric}(X, Y) = \sum_{i=1}^{n} g_i R(E_i, X)Y_E_i + \sum_{i=1}^{n} g^r_i R(\xi, X)Y_E_i,$$

(41)

is induced Ricci tensor curvature on a null hypersurface. But

$$(\langle X, (\text{Ric}(X, Y) \rangle = \langle R(\xi, X)Y_E_i + \sum_{i=1}^{n} g^r_i R(\xi, X)Y_E_i, Y \rangle \neq 0$$

and \((\text{Ric}(X, Y) \rangle = \langle R(\xi, X)Y_E_i + \sum_{i=1}^{n} g^r_i R(\xi, X)Y_E_i, Y \rangle \neq 0$$

and $\text{Ric}(X, Y) = \sum_{i=1}^{n} g_i R(E_i, X)Y_E_i + \sum_{i=1}^{n} g^r_i R(\xi, X)Y_E_i$.

By substituting previous terms in the above expression of $\text{Ric}^0(X, Y)$ we get the desired formula.

**Corollary 4.1**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a $(n + 2)$-pseudo-Riemannian manifold with constant curvature $k$. Then

$$\text{Ric}^0(X, Y) = nk(\xi, Y) + \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$$

and

$$\text{Ric}^0(X, Y) = nk(\xi, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i.$$  

(42)

**Theorem 4.2**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a $(n + 2)$-pseudo-Riemannian manifold. Then

$$\text{Ric}^0(X, Y) = \text{Ric}(X, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$$

and

$$\text{Ric}^0(X, Y) = \text{Ric}(X, Y) + \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i.$$  

(43)

**Corollary 4.2**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a $(n + 2)$-pseudo-Riemannian manifold with constant curvature $k$. Then

$$\text{Ric}^0(X, Y) = nk(\xi, Y) + \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i$$

and

$$\text{Ric}^0(X, Y) = nk(\xi, Y) - \left(\langle A^r \xi, Y \rangle - \langle A^r, X \rangle \right) + \sum_{i=1}^{n} g^r_i R^0(E_i, X)Y_E_i.$$  

(44)

**Theorem 4.3**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a $(n + 2)$-pseudo-Riemannian manifold. Then

$$\text{Ric}(X, Y) = \text{Ric}(X, Y) - \langle A^r, X \rangle + 2d\tau^r(\xi, X).$$  

(45)

**Proof.** To get eqn. (45), take $Y = \xi$ in the (4.1). Recall that from a geometric point of view and in practice, one gets the scalar curvature by contracting with a (non-degenerate) metric (the symmetric) Ricci curvature. It turns out that in the null geometry setting, such a scalar quantity cannot be calculated by the usual way (degeneracy of the induced metric and the failure of symmetry in the induced Ricci curvature) [12]. This justified introduction of a symmetrized Ricci curvature tensor $\text{Ric}^s$ and our use of the associated non-degenerate metric $g_0$ in calculating this scalar quantity. More precisely, the extrinsic scalar curvature $r^s$ on the null hypersurface $(M, g, N)$ is given by $g_0^{-1}$ trace of the symmetrized Ricci curvature $\text{Ric}^s$. With respect to a local quasiorthonormal frame $(\xi, e_1, \ldots, e_n)$ for $(M, g)$ we have

$$r^s = \text{Ric}^s_{\text{null}}.$$  

(46)

Now let $r^s$ denote the scalar curvature of the non-degenerate metric $g_0$ on $M$ that is the contraction of $\text{Ric}^s$ with respect to $g$. In the following, we state a formula relating the extrinsic scalar curvature $r^s$ to the associated scalar curvature $r^s$.

**Theorem 4.4**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^r(\xi) \neq 0$ in a pseudo-Riemannian manifold. Then

$$r^s = r^0 - \langle A^r, \xi \rangle + \tau^r(\xi, \xi) + \tau^r(\xi, \xi).$$  

(47)

**Proof.** We have

$$r^s = \text{Ric}^s_{\text{null}}$$

in a local quasiorthonormal frame $(\xi, e_1, \ldots, e_n)$ for $(M, g_0)$ with span $(\xi, e_1, \ldots, e_n) = S(N)$. But

$$\text{Ric}^s_{\text{null}} = \text{Ric}^s - \langle A^r, \xi \rangle + \tau^r(\xi, \xi) + \tau^r(\xi, \xi).$$

Moreover, by contracting each side with $g_0$ and taking into account Proposition (4.1) along with the following facts:

$$\langle V, \tau^r(\xi, e_i) \rangle = \langle V, \tau^r(\xi, e_i) \rangle + \langle V, \tau^r(\xi, e_i) \rangle,$$

$$\text{Ric}^s((\xi, e_1, \ldots, e_n)) = \langle V, \tau^r(\xi, e_i) \rangle,$$

and

$$\text{Ric}^s((\xi, e_1, \ldots, e_n)) = \langle V, \tau^r(\xi, e_i) \rangle.$$  

(48)

Hence, by contracting each side with $g_0$ and taking into account Proposition (4.1) along with the following facts:
Corollary 4.3
Let $(M,g,N)$ be a closed normalized null hypersurface of a $(n+2)$-dimensional Lorentzian space and
\[ r^2 = n^2k + 2tr(A'_iA_i) - tr(A'_iA_i)_0. \]
Then, we have the following [1].

\[ \eta \in \Sigma, \text{ we get the announced relation} \]
\[ = g_\pi(R(X,Y)^\tau Y',X') - \frac{1}{2}g(A_\pi X,Y')\eta(A'X',Y') \]
\[ + \tau^\pi(Y')g(A'_X,Y')\tau^\pi(Y') \]
\[ = -g(R(Y',X')X',Y') \]
\[ = \eta(Y')g(R(Y',\xi')X,Y') \]
\[ = [1 - \eta(Y')]p_{\pi}(Y)^2 K_{\pi}(\pi(\xi',Y')). \]

Similarly, we have
\[ g_\pi(R^\tau(X',Y')X',Y') = \eta(-X')g(R(X',\xi')\pi(X,Y') - \tau^\pi(X')B^\pi(X,Y') \]
\[ + \tau^\pi(X')B^\pi(X,Y'). \]

Proof. From $X,Y \in \mathcal{S}_0^1(1)$, $Y \subset \mathcal{O}_1(X)$, we infer that $g_\pi(X,Y) = g_\pi(Y,X) = 1$ and $g_\pi(X,Y) = 0$. It follows that $g_\pi(X,Y)g_\pi(Y,X) - g_\pi(X,Y)_0 = 1$ and $K_\pi(X,Y) - g_\pi(R(X,Y)_0) = 0$. Here, for a vector field $X$, we brief $X'$ and $X''$ for $PX$ and $\eta(X)\xi'$ respectively, where $P$ is the morphism projection of $TM$ onto $\Sigma(N)$ and then, $X = X' + X''$.
in eqn. (56). For a totally geodesic null hypersurface, the second fundamental form $B^N$ vanishes identically and we get the following.

**Corollary 4.4**

Let $(M,g,N)$ be a closed normalized null hypersurface with null field $\xi$ and $\tau^N(\xi)=0$ in a Lorentzian space form $\overline{M}(k)$ Then

$$K_\xi(X,Y) = k\left[1-\eta^2(X)-\eta^2(Y)\right].$$

**Corollary 4.5**

Let $(M,g,N)$ be a closed normalized null hypersurface with null field $\xi$ and $\tau^N(\xi)=0$ in a Lorentzian space form $\overline{M}(k)$ Then of a Lorentzian manifold.

Then, the normalizing one-form $\tau^N$ and $\xi$. Moreover, $\rho$ being the nowhere vanishing conformal factor of $N$, we have for all $X,Y \in \mathcal{S}^N_1, Y \in O_k(X)$

$$K_\xi(X,Y) = \left(k-2\rho^2\eta^2(X) - \eta^2(Y)\right) \frac{(\xi_\rho - k\eta_\xi)}{\rho^2}. \quad (57)$$

**Proof.** obtain relation eqn. (57) by setting $\tau^N=0$ in eqn. (56). Now from Remark (4.2) and definition (3.2) the last claim follows.

A section $N$ is called conformal Killing (CKV in short) or conformal collineation on the underline manifold topology.

**Fact 4.1:** Assume $N$ is a closed conformal collineation rigging of $(M,g)$. Then, the normalizing one-form $\tau^N$ vanishes identically on $M$. Moreover, the screen distribution $\mathcal{S}(N)$ is integrable and totally umbilical.

**Proof.** Let $X,Y$ be tangent to $M$: We have:

$$2\rho g(X,Y) = 2\rho g(X,Y) = (L_{\xi_\rho}g)(X,Y) = \langle X, \xi_\rho \rangle = \langle X, \tau^N \rangle. \quad (55)$$

that is

$$-\langle A_\xi, Y \rangle + \tau^N(\xi)(Y)\eta(X) + \tau^N(Y)\eta(X) = 2\rho g(X,Y). \quad (56)$$

Hence, set $Y=\xi\xi$ to get:

$$2\tau^N(X) = 0$$

i.e $\tau^N(X) = \xi\xi$ vanishes on $M$. Then we get by the closed assumption

$$0 = \langle X, \tau^N \rangle = -C^N(X,PX) + C^N(Y,PY) \quad (as \tau^N = 0).$$

So, for $X,Y \in \mathcal{S}(N)$ we have

$$C^N(X,Y) = C^N(Y,X),$$

that is the screen distribution is integrable. Now, return to eqn. (56) to get

$$-C^N(X,PY) - C^N(Y,PX) = 2\rho g(X,Y), X,Y \in \mathcal{S}(TM).$$

Hence, for $X,Y \in \mathcal{S}(N)$,

$$C^N(X,Y) = -\rho g(X,Y).$$

But, as

$$C^N(\xi, PY) = \langle A_\xi, PY \rangle = -\tau^N(PY) = 0,$$

we deduce that

$$C^N(X,PY) = -\rho g(X,Y)$$

for all $X,Y \in \mathcal{S}(TM)$ which shows that $\mathcal{S}(N)$ is totally umbilical in null hypersurface $M$.

**Theorem 4.6**

Let $(M,g,N)$ be a null hypersurface of a Lorentzian space form $\overline{M}(k)$ with a closed and conformally Killing (but not Killing) normalization. Then $(M,g)$ is totally umbilical in $\overline{M}(k)$. Moreover, $\rho$ being the nowhere vanishing conformal factor of $N$, we have for all $X,Y \in \mathcal{S}^N_1, Y \in O_k(X)$

$$K_\xi(X,Y) = \left(k-2\rho^2\eta^2(X) - \eta^2(Y)\right) \frac{(\xi_\rho - k\eta_\xi)}{\rho^2}. \quad (57)$$

**Proof.** From the previous fact, $\mathcal{S}(N)$ is integrable and totally umbilical with umbilicity factor $\rho$. So, in this case, it is a well know fact that the following holds (MEN14, p.110)

$$\{\xi(\rho_0) + \rho_0^2(\xi) + k\} g(X,Y) = -\rho B^N(X,Y).$$

Hence, as $\tau^N(0) = 0$ and $\rho$ is everywhere non zero, we get

$$B^N(X,Y) = \left(\frac{\xi(\rho_0) - k}{\rho}\right) g(X,Y), X,Y \in \mathcal{S}(TM),$$

that is $(M,g)$ is totally umbilical. Now, using previous expressions of $B^N$ and $C^N$ eqn. (57), we get for all $X,Y \in \mathcal{S}^N_1, Y \in O(X)$

$$K_\xi(X,Y) = \left(k-2\rho^2\eta^2(X) - \eta^2(Y)\right) \frac{(\xi_\rho - k\eta_\xi)}{\rho^2}.$$ 

But for all $X,Y \in \mathcal{S}^N_1, Y \in O(X)$

$$g(X,Y) = 1 - \eta(X)^2, g(Y,Y) = 1 - \eta(Y)^2$$

and

$$g(X,Y) = -\eta(X)\eta(Y),$$

which after substitution leads to the desired expression of $K_\xi(X,Y)$.

**Relationship between Curvature and Topology of Null Hypersurface**

In this section, we study the null geometry of manifolds and link their invariants to those of the induced associated Riemannian metric on them through the normalization. Thereafter, we use some comparison theorems from Riemannian geometry to get informations on the underline manifold topology.

**Theorem 5.1**

Let $(M,g,N)$ be a closed normalized compact null hypersurface of a Lorentzian manifold $\overline{M}(k)$ and $\tau^N=0$

If

$$\text{Ric}(X,X) \geq \langle (A_\xi X, X) - \langle A_\xi X, X \rangle \rangle \text{pr}_{A_\xi}^2 \xi \xi - \langle (V, A_\xi)(X), X \rangle + \langle (V, A_\xi^2)(X), X \rangle$$
holds for all \(XTM\), then the universal Riemannian covering of the associated Riemannian manifold \((M, g)\) is isometric to a product \((M \times \mathbb{R}^c, g \times \delta)\) where \(\delta\) stands for the usual euclidean metric on \(\mathbb{R}^c\) and \((M, g)\) is a compact simply connected Riemannian manifold with non negative Ricci curvature.

**Proof.** This is a direct application of a splitting result by Cheeger and Gromoll on compact Riemannian manifolds with non negative Ricci curvature [9,10]. Indeed, the ambient manifold being Lorentzian, we know that \(M\) endowed with the associated metric \(g\) is Riemannian. Also, conditions \(\tau=0\) eqns. (64) and the expression (47) show that the Ricci curvature \(\text{Ric}^g\) of the Riemannian metric \(g\) satisfies \(\text{Ric}^g = 0\) that is non negative. as the hypersurface is compact, the claim follows [11,12].

**Theorem 5.2**

Let \((M^{n+1}, g, N)\) be a closed normalized compact null hypersurface of dimension \(n\) of Lorentzian manifold \((M, \eta)\) and \(\tau=0\), and

\[
\text{Ric}(X, X) < [(\xi', \eta, X) - (A, X, X)]pr A'_{i}
- \langle(V, A'_{i}(X), X) + \langle(V, A_{i}(X), X).
\]

Then every Killing vector field on \((M, g)\) is identically zero and group of isometry is finite.

**Proof.** Using Theorem (4.6), the assumption eqn. (64) is equivalent to saying that the hypersurface \(M\) endowed with the associated metric \(g\) has a negative sectional curvature. As it is compact, we conclude by theorem of Bochner [11] that every Killing vector field on null hypersurface is identically zero.

In dimension 3, Schoen and Yau proved in that a complete non-compact manifold with positive Ricci curvature is diffeomorphic to the standard euclidean space \(\mathbb{R}^3\). Using this and eqn. (47) lead to the following [8].

**Theorem 5.3**

Let \((M, g, N)\) be a closed normalized \(d^-\)-complete non-compact null hypersurface of Lorentzian manifold \((M, \eta)\) and \(\tau=0\) and

\[
\text{Ric}(X, X) > [(\xi', \eta, X) - (A, X, X)]pr A'_{i}
- \langle(V, A'_{i}(X), X) + \langle(V, A_{i}(X), X).
\]

Then the manifold structure of the null hypersurface \((M, g, N)\) is diffeomorphic to \(\mathbb{R}^3\).

**Proof.** The ambient lorentzian manifold has dimension 4, so the null hypersurface is 3-dimensional. The inequality ensures that the associated metric \(g\) has a positive Ricci curvature. By the Shoen-Yau above quoted theorem [8], the claim follows.

**Theorem 5.4**

Let \((M, g, N)\) be a closed \(d^-\)-complete null hypersurface of a Lorentzian space form \((\tilde{M}(k), \tilde{\eta})\) with conformally Killing (but not Killing) normalization. Assume that for all \(X, Y \in S^1(\tau), Y \in O_\tau(X)\) we have:

\[
\frac{[\xi(\rho) - k]^{2}}{\rho^{2}} \eta(X) \eta(Y)^{2} \leq (3k - 2\xi(\rho))|\eta(X)|^{2} + |\eta(Y)|^{2} - 1.
\]

Then the universal covering of the null hypersurface is diffeomorphic to \(\mathbb{R}^3\).

**Proof.** Now, \((M^{n+1}, g, N)\) being dcomplete, it follows Remark 3.1 that the Riemannian manifold \((M, g)\) is complete. Also, using Theorem (4.6), the assumption eqn. (64) is equivalent to saying that the hypersurface \(M\) endowed with the associated metric \(g\) has a nonpositive sectional curvature. we conclude by Hadmar theorem that the universal of null hypersurface is diffeomorphic to \(\mathbb{R}^3\).

The authors proved, using a correspondence for isometric immersions into product spaces that, on a complete Riemannian manifold \(M\) with negative Ricci curvature, and whose scalar curvature is bounded above by a negative constant, the standard Euclidean space \((R^n, Euc)\) cannot be isometrically immersed into the Lorentzian product space \(M \times Euc\). A direct consequence of this fact is the following [14].

**Theorem 5.5**

Let \((M, g, N)\) be a \(d^-\)-complete null hypersurface of \((n+1)\)-dimensional Lorentzian manifold \((M, \eta)\) with a closed normalization and \(\tau=0\). Assume that

\[
\text{Ric}(X, X) < [(\xi', \eta, X) - (A, X, X)]pr A'_{i}
- \langle(V, A'_{i}(X), X) + \langle(V, A_{i}(X), X),
\]

and

\[
r \leq |tr A_{i} - tr A_{i}'| pr A_{i}' - tr(V, A_{i}') + tr(V, A_{i}) < c,
\]

for some positive constant \(c\). Then the standard Euclidean space \((\mathbb{R}^n, Euc)\) cannot be isometrically immerssed into the Lorentzian product space \(M \times Euc\) where the underline manifold \(M\) is endowed with the Riemannian metric \(g\). Let \(R^n\) be the standard Lorentzian 4-dimensional manifold with negative sectional curvature. By the Shoen-Yau above quoted theorem [8], we conclude by Hadmar theorem that the universal of null hypersurface is diffeomorphic to \(\mathbb{R}^3\).

**Proof.** The null hypersurface \(M\) being \(d^-\)-complete, it follows that \(M\) endowed with the associated metric \(g\) is a complete Riemannian manifold. Also, as \(\tau=0\) from eqns. (62) and (64) we infer that the Ricci curvature of the associated Riemannian metric \(g\) is negative, and by eqns. (51) and (63) its scalar curvature is bounded above by the negative constant \(-c\). This completes the proof.

**Theorem 5.6**

Let \((\tilde{M}, \tilde{\eta})\) a Lorentzian manifold and \((M, g, N)\) be a closed normalized complete null hypersurface in \(\tilde{M}, \tilde{\eta}\) with

\[
[\xi(\rho) - k]^{2} \eta(X) \eta(Y)^{2} \leq (3k - 2\xi(\rho))|\eta(X)|^{2} + |\eta(Y)|^{2} - 1.
\]

Then no complete Riemannian surface \(\sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j}\) of constant curvature \(c\) is isometrically immersed into \(M \times Euc\).

**Proof.** The null hypersurface \(M\) being \(d^-\)-complete, it follows that \(M\) endowed with the associated metric \(g\) is a complete Riemannian manifold. Also, as \(\tau=0\), from eqns. (62) and (64).

**Conclusion**

We infer that the sectional curvature of the associated Riemannian metric \(g\) is negative, by using theorem José [14,15] on correspondence for isometric immersions into product spaces, we complete the proof with the research result of Through the research establishment of links between the null geometry and basics invariants of the associated Riemannian metric is explained successfully.

**References**

1. Duggal KL, Bejancu A (1996) Lightlike submanifolds of semi-riemannian Manifolds and applications. Kluwer Acad Publishers, USA.
2. Atindogbe C (2010) Normalisation and prescribed extrinsic scalar curvature on light like hypersurfaces. Journal of Geometry and Physics 60: 1762-1770.
3. Atindogbe C, Berard-Bergery L (2013) Distinguished Normalization on Non-
Minimal Null Hypersurfaces. Mathematical Sciences and Applications E-notes 1: 18-35.

4. Atindogbe C, Duggal KL (2004) Conformal screen on lightlike hypersurfaces. Int J Pure and Applied Math 11: 421-442.

5. Atindogbe C, Ezin JP, Tossa J (2003) Pseudo-inversion of degenerate metrics. Int J Mathematics and Mathematical Sciences 55: 3479-3501.

6. Gutiérrez M, Olea B (2012) Lightlike hypersurfaces in Lorentzian manifolds. Cornell University Library.

7. Hebey H (1997) Introduction to Nonlinear Variety Analysis. Diderot Publisher, France.

8. Schoen R, Yau ST (1982) Complete three dimensional manifold with positive curvature, in Seminar on Differential Geometry, Princeton University Press, USA.

9. Cheeger J, Gromoll D (1971) The splitting theorem for manifold of nonnegative Ricci curvature. Journal of Differential Geometry 6: 119-128.

10. Cheeger J, Gromoll D (1972) On the structure of complete manifolds of nonnegative Ricci curvature. Annals of Mathematics 96: 413-443.

11. Eschenburg JH (1994) Comparison theorems in Riemannian geometry. Augsburg.

12. Atindogbe C (2009) Scalar curvature on lightlike hypersurfaces. Applied Sciences 11: 9-18.

13. Bochner S (1946) Vector fields and Ricci curvature. Bull Amer Maths Sco 52: 776-797.

14. José AG, Jiménez A, Mira P (2010) A correspondence for isometric immersions into product spaces and its application. Journal of Geometry and Physics 60: 1819-1832.

15. Katsuno K (1980) Null Hypersurfaces in Lorentzian Manifolds. Math Proc Cab Phil Soc 88: 175-182.