THE BLOWUP FORMULA FOR THE INSTANTON PART OF VAFA–WITTEN INVARIANTS ON PROJECTIVE SURFACES

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Abstract. We prove a blow-up formula for the generating series of virtual $\chi_y$-genera for moduli spaces of sheaves on projective surfaces, which is related to a conjectured formula for topological $\chi_y$-genera of Göttsche. Our formula is a refinement of one by Vafa–Witten relating to S-duality.

We prove the formula simultaneously in the setting of Gieseker stable sheaves on polarised surfaces and also in the setting of framed sheaves on $\mathbb{P}^2$. The proof is based on the blow-up algorithm of Nakajima–Yoshioka for framed sheaves on $\mathbb{P}^2$, which has recently been extend to the setting of Gieseker $H$-stable sheaves on $H$-polarised surfaces by Kuhn–Tanaka.

1. Introduction

Hilbert schemes of points on a projective surface have been extensively studied; and while the picture is not complete, their geometry is well understood. In particular, there are many deep results concerning their enumerative invariants (e.g. [Go90], [Che96], [EGL01], [MOP19]). For their higher rank analogues, the moduli spaces of (semi-)stable sheaves on a surface, the situation is more complicated. Results for topological invariants for moduli spaces in ranks 2 and 3 on some surfaces have been obtained e.g. in [LQ99], [Yos96], [MM18] and [Koo15], but, in general, one does not expect any reasonable formulas for the topological invariants.

More recently, there has been an interest in studying virtual analogues of the topological invariants, which are defined using the perfect obstruction theory coming from the moduli problem. This interest is partially motivated by the S-duality conjecture [VW94] concerning Vafa–Witten invariants and by the papers [TT20], [TT17], where Tanaka–Thomas proposed a mathematical definition of these invariants as the sum of two parts – one of which is given by the signed virtual Euler characteristics of moduli spaces of stable sheaves. The virtual invariants have the benefit of being invariant under deformations of the surface that preserve the polarization, and – strikingly – they are in many cases predicted to depend only on finitely many basic invariants of the underlying surface. For example, explicit formulas up to rank 5 have been conjectured for the virtual Euler characteristics in [GKL21] (see also [GK18], [GK20], [GK19a], [GKW21]; and [GK19b] for an excellent survey).

In this paper, we prove the first completely general result for virtual $\chi_y$-genus and Euler characteristic of the higher rank moduli spaces: Namely, we prove a blowup formula for the virtual $\chi_y$-genera of the moduli spaces, and thereby for virtual Euler characteristics. This is a first and important step towards establishing the existing conjectures, and provides further evidence for them. Similar formulas for the topological invariants had been obtained in [LQ99] for the virtual Hodge polynomial in rank 2, and in [Go99, Prop. 3.1] for virtual Poincaré polynomials of arbitrary rank moduli spaces. Our result also confirms part of a conjecture of

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Göttsche [G699, Rem. 3.2] (stated more generally for virtual Hodge polynomials) when the moduli spaces are unobstructed, so that their virtual and topological \( \chi_r \)-genera agree.

In the course of our proof, we prove a blowup formula for an equivariant analogue of the \( \chi_r \)-genus defined on moduli spaces of framed sheaves on \( \mathbb{P}^2 \) and on its blowup. These moduli spaces of framed sheaves and their invariants are of independent interest in physics through in Nekrasov’s conjecture and related topics (see for example [Nek03, NY05a, NO06, BE06, GNY08, GL10 and GNY11]).

For the rest of the paper, we fix a positive integer \( r > 0 \), which will denote the rank of the sheaves being considered. As mentioned above, we will prove blow-up formulas for two different kinds of moduli spaces. These two situations are:

**Situation A - Gieseker stable sheaves** [Moc09, KT21]. In this situation, \( X \) is a smooth projective (connected) surface with a fixed polarisation \( H \) and \( p : \tilde{X} \to X \) is the blow-up of \( X \) at a chosen reduced point \( pt \in X \) with exceptional divisor \( C \). Moreover, \( L_X \) is a fixed line bundle on \( X \) with \( c_1 := c_1(L_X) \), such that the intersection number \( (L_X, H) \) is coprime to \( r \).

We consider classes \( c := r + c_1 + ch_2 \in H^*(X, \mathbb{Q}) \) for various \( ch_2 \in H^4(X, \mathbb{Q}) \). On the blow-up \( \tilde{X} \) we will write \( \tilde{c}_1 := c_1 - k[C] \) for various \( k \in \mathbb{Z} \) and consider classes \( \tilde{c} := r + \tilde{c}_1 + ch_2 \) for various \( ch_2 \in H^*(X, \mathbb{Q}) \). In this situation, we consider the following moduli stacks:

(i) \( M(c) \), the moduli stack of oriented Gieseker \( H \)-stable sheaves on \( X \) with Chern character \( c \). Here an oriented sheaf is a pair \((E, \psi : det E \to L_X)\) where \( E \) is a Gieseker \( H \)-stable sheaf and \( \psi \) is an isomorphism.

(ii) \( \widetilde{M}(\tilde{c}) \) the moduli stack of oriented \( p^*H \) slope-stable sheaves on \( \tilde{X} \) with Chern character \( \tilde{c} \). Note that slope stability for \( p^*H \) is the same as Gieseker stability for the polarization \( p^*H - \varepsilon[C] \) for any small enough \( \varepsilon \in \mathbb{Q}_{>0} \) depending on \( \tilde{c} \). Here, we define an orientation to be an isomorphism \( det E \cong p^*L_X \otimes \mathcal{O}_{\tilde{X}}(-kC) \) if \( \tilde{c}_1 = p^*c_1 - k[C] \).

**Situation B - Framed sheaves on \( \mathbb{P}^2 \)** [NY05a, NY05b]. In this situation, we let \( X := \mathbb{P}^2 \) and consider the blow-up \( p : \tilde{X} = \mathbb{P}^2 \to X \) at \( pt = [1 : 0 : 0] \) with exceptional divisor \( C \). We set \( L_X = \mathcal{O}_{\mathbb{P}^2} \) and \( c_1 = 0 \). We consider classes on \( X \) of the form \( c := r + 0 + ch_2 \) for various \( ch_2 \in H^4(\mathbb{P}^2, \mathbb{Q}) \). For classes on the blow-up we consider \( \tilde{c}_1 := -k[C] \) for some \( k \in \mathbb{Z} \) and \( \tilde{c} := r + \tilde{c}_1 + ch_2 \) for various \( ch_2 \in H^4(\mathbb{P}^2, \mathbb{Q}) \) In this situation, we consider the following moduli spaces:

(i) \( M(c) \), the moduli space of **framed sheaves on \( \mathbb{P}^2 \)** with Chern character \( c \). That is, pairs \((E, \varphi)\) where \( E \) is a coherent sheaf on \( X \) which is locally free in a neighbourhood of \( \ell_\infty := \{[0 : z_1 : z_2] \} \) and where \( \varphi \) is an isomorphism \( E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r} \).

(ii) \( \widetilde{M}(\tilde{c}) \), the moduli space of **framed sheaves on \( \tilde{X} \)** with Chern character \( \tilde{c} \), where the framing is with respect to the preimage \( p^{-1}\ell_\infty \) of \( \ell_\infty \) under \( p \), which we denote again by \( \ell_\infty \).

**Remark 1.1.** In Situation A, we use moduli stacks of oriented sheaves instead of the usual moduli spaces, since the stacks always have universal sheaves, which simplifies some of the presentation. The moduli stacks are degree \( 1/r \) gerbes over the coarse moduli spaces and all the invariants that we consider differ only by a factor of \( 1/r \).

We will now address how one can define algebraic invariants in Situations A and B. In Situation A, this uses that the moduli stacks are proper and carry a perfect obstruction theory, while in Situation B, one uses the fact that the moduli spaces carry a natural torus action with isolated torus fixed points.
In what follows below, we let $M$ denote either $M(c)$ or $\tilde{M}(\tilde{c})$ in either situation, we let $\mathcal{E}$ denote the universal sheaf on $M \times X$ or $M \times \tilde{X}$ and we let $\pi$ denote the projection morphism $\pi : M \times X \rightarrow M$ or $\pi : M \times \tilde{X} \rightarrow M$.

(a) **Situation A**: $M$ is a proper Deligne–Mumford stack with a canonical perfect obstruction theory of virtual dimension $-2r \, ch_2 + c_1^2 - (r^2 - 1) \chi(X, \mathcal{O}_X)$. The $K$-theoretic class of the virtual tangent bundle is

$$T_M^{\text{vir}} = \mathcal{O}_M^{\mathbb{P}^\chi(X, \mathcal{O}_X)} - R\pi_* R\mathcal{Hom}(\mathcal{E}, \mathcal{E}).$$

(b) **Situation B**: $M$ is representable by a smooth quasi-projective variety with dimension $-2r \, ch_2 + c_1^2$. There is a canonical action of $\tilde{T} := (\mathbb{C}^*)^{2+r}$ on $M$ whose fixed locus is a set of isolated points. This action extends in a canonical way to $M \times X$ (respectively $M \times \tilde{X}$), such that $\mathcal{E}$ has a canonical structure of equivariant sheaf. The $\tilde{T}$-equivariant class of the tangent bundle in $K$-theory is

$$T_M^{\tilde{T}} = -R\pi_* R\mathcal{Hom}(\mathcal{E}, \mathcal{E}(-\ell_\infty)).$$

As references for these properties: For Situation A, see either [Moc09, §5.6] or [KT21, Cor. 3.26 & Prop. 5.12] and for Situation B see [NY05a, §2 & §3] and [NY11b, §5]. Using these properties, we now define a notion of “integration” for Chow cohomology classes on moduli spaces in each situation. We will let $F$ denote a field of characteristic zero, and write $A^*$ for Chow cohomology rings with coefficients in $F$.

**Notation 1.2.** Let $M$ be a moduli space from either Situation A or B. Define the following integral notation:

(a) **Situation A**: Using the perfect obstruction theory, we obtain a virtual fundamental class $[M]^{\text{vir}}$. Using this, we define for $\alpha \in A^*(M)$:

$$\int_M \alpha := \int_{[M]^{\text{vir}}} \alpha := \deg (\alpha \cap [M]^{\text{vir}}),$$

where for a Chow class $\gamma \in A_*(M)$ we let $\deg \gamma$ be the number obtained by pushforward along the structure morphism $M \rightarrow \text{Spec} \mathbb{C}$.

(b) **Situation B**: In this situation the moduli space is non-proper, but has a proper $\tilde{T}$-fixed locus. We define integration via the Atiyah–Bott localisation formula. Hence for a $\tilde{T}$-equivariant class $\alpha \in A^*_{\tilde{T}}(M)$:

$$\int_M \alpha := \sum_{x \in M^{\tilde{T}}} \alpha|_x \in \left( T_M^{\tilde{T}}|_x \right).$$

Moreover, in light of the unified notations of Situations A and B we suppress the equivariant notation when considering Situation B unless it is necessary. Hence, from now on, in Situation B we write $A^*(M)$ instead of $A^*_{\tilde{T}}(M)$, $T_M$ instead of $T_M^{\tilde{T}}$ and always consider $\mathcal{E}$ as an equivariant sheaf.

**Remark 1.3.** If we consider the moduli spaces in Situation B as having the trivial perfect obstruction theory, then we have $T_M^{\text{vir}} = T_M$ and it makes sense to use the word *virtual* when discussing the enumerative invariants of both situations.

Using the above notation, in this article, we will consider invariants of the form

$$\int_M \Theta(T^{\text{vir}}),$$

where $\Theta : K(M) \rightarrow A^*(M)$ is some characteristic class. In particular, we are interested in the case where $\Theta = \Theta_{\chi}$ is the multiplicative class defined on line
bundles by \( \Theta_y(L) = (1 - y \text{ch}(L)^{-1}) \cdot \text{td}(L) \), where \( y \) is a formal variable. In this case, we set
\[
\chi^\text{vir}(M) := \int_M \Theta_y(T^\text{vir}) \in \mathbb{Q}[y].
\]
This is the virtual \( \chi_y \)-genus of \( M \) as defined by Fantechi–Göttsche in [FG10, §4]. The main result of the article is the following theorem:

**Theorem 1.4 (Main Theorem).** Consider Situation \( A \) or \( B \) such that \( 0 \leq k < r \). Consider the generating series of virtual \( \chi_y \)-genera
\[
Z(q, y) := \sum_{c_1 + \cdots + c_2 = k} \chi^\text{vir}_y(M + c_1 + c_2) q^{\vdim M(r+c_1+c_2)}, \quad \text{and}
\]
\[
\hat{Z}(q, y) := \sum_{c_1 + \cdots + c_2 = k} \chi^\text{vir}_y(M + \hat{c}_1 + c_2) q^{\vdim M(r+\hat{c}_1+c_2)}.
\]
Then, we have the identity
\[
\hat{Z} = \mathcal{Y}_k \cdot Z,
\]
where
\[
\mathcal{Y}_k = \mathcal{Y}_k(q, y) := \prod_{n > 0} (1 - (q^2 y^n)^r)^{-r} \sum_{k_1 + \cdots + k_r = k} (q^2 y)^{\sum_{i<j} (k_i - k_j) / 2} y^{\sum_{i<j} (k_i - k_j) / 2}.
\]

**Remark 1.5.** The factor \( \mathcal{Y}_k \) appearing in Theorem 1.4 is independent of all input data except \( k \) and the rank \( r \). This is the same blow-up behaviour observed for the topological invariants studied in [Gö90], [Che96], [LQ98, LQ99] and [Gö99].

**Remark 1.6.** The result in Theorem 1.4 was originally conjectured in Situation \( A \) for the topological \( \chi_y \)-genus by Görtzche in [Gö99, Rem. 3.2] (stated more generally for virtual Hodge polynomials). The factor \( \mathcal{Y}_k \) can be rewritten in Görtzche’s notation as follows. Let \( A = (a_{ij})_{i,j} \) be the \((r - 1) \times (r - 1)\)-matrix with entries \( a_{ij} = 1 \) for \( i \leq j \) and \( a_{ij} = 0 \) and \( I \) be the column vector of length \( r - 1 \) with all entries equal to one. Let \( \eta \) denote the Dedekind eta function. Then we have
\[
\mathcal{Y}_k = \frac{(q^{2r} y^r)^{r/24}}{\eta(q^{2r} y^r)^r} \sum_{v \in \mathbb{Z}^{r-1} + \frac{k}{r} I} (q^{2r} y^r)^v \text{exp}(\text{exp}(v + A I)).
\]
In the cases where the moduli problem is unobstructed (i.e. if every stable sheaf \( E \) on \( X \) and \( \hat{X} \) has \( \text{Ext}^2(E, E) = 0 \) topological and virtual \( \chi_y \)-genus coincide. Hence, in these cases, Theorem 1.4 proves Görtzche’s topological conjecture.

One can also consider virtual Euler characteristic and virtual holomorphic Euler characteristic. These are defined respectively as
\[
e^\text{vir}(M) := \int_M e(T^\text{vir}_M) \quad \text{and} \quad \chi^\text{vir}(M) := \int_M \text{td}(T^\text{vir}_M),
\]
which are the respective specialisations \( \chi^\text{vir}_{-1} \) and \( \chi^\text{vir}_0 \) of \( \chi^\text{vir}_y \). Using these characterisations, Theorem 1.4 has the following corollary.

**Corollary 1.7.** In the setting of Theorem 1.4, using appropriately defined generating series, we have
\[
\hat{Z}_e^\text{vir} = \mathcal{Y}_{k,e^\text{vir}} \cdot Z_e^\text{vir} \quad \text{and} \quad \hat{Z}_\chi^\text{vir} = \mathcal{Y}_{k,\chi^\text{vir}} \cdot Z_\chi^\text{vir},
\]
where
\[
\mathcal{Y}_{k,e^\text{vir}} = \prod_{n > 0} (1 - q^{2r n})^{-r} \sum_{\sum_i k_i = k} q^{\sum_{i<j} (k_i - k_j)^2} \quad \text{and} \quad \mathcal{Y}_{k,\chi^\text{vir}} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{else}. \end{cases}
\]
Remark 1.8. The blow-up formula for virtual holomorphic Euler characteristics of Corollary 1.7 was previously proved by Nakajima–Yoshioka in [NY11c, Thm. 2.11] for Situation B.

Outline of the Article. In order to prove Theorem 1.4, we use the blow-up algorithm of Nakajima–Yoshioka [NY11c] (extended to Situation A by [KT21]) to obtain a weak universal blowup formula for invariants of the form being studied. A key aspect of this formula is its universality, namely it shows that the desired blow-up formula exists and is independent of much of the input data. In particular, the formula will be the same in Situation A and Situation B. We then analyse the equivariant geometry arising in Situation B to determine the exact form of the desired formula. This proceeds via a reduction to rank 1, where we can use the $(q,t)$-Nekrasov–Okounkov formula [RW18, Thm. 1.3] to conclude.

The article is structured as follows:

- **Section 2**: We recall the blow-up algorithm of Nakajima–Yoshioka [NY11c] and show how it is extended to both situations by [KT21].
- **Section 3**: We state and prove a weak universal blowup formula for invariants defined by classes which are multiplicative in the tangent bundle.
- **Section 4**: We prove the Main Theorem by determining the coefficients from the weak universal blowup formula. We do this by examining the equivariant structure arising in Situation B.

Basic Notation. We fix a positive integer $r > 0$ and will always assume the notation described in Situations A and B. In particular, we consider classes of the form $c := r + c_1 + c_2 \in A^*(X)$ and $\tilde{c} := r + \tilde{c}_1 + c_2 \in A^*(\tilde{X})$ described there. We will also use the integral notation described in Notation 1.2.

Other basic notation includes:

- $C_m := \mathcal{O}_C(-m - 1) := \mathcal{O}_C \oplus \mathcal{O}_X((m + 1)C)$ denotes the unique degree $-m - 1$ line bundle on the exceptional divisor $C$.
- $e_m := \text{ch}(j_*C_m) = [C] - (m + 1/2)[\text{pt}]$ is the Chern character of $C_m$ taken as a coherent sheaf on $\tilde{X}$.
- For a moduli space $M$ of sheaves on $X$ (or $\tilde{X}$) denote the universal sheaf on $M \times X$ (or $M \times \tilde{X}$) by $E$ and let $\pi : M \times X \to M$ (resp. $\pi : M \times \tilde{X} \to M$) be the projection. We will denote $T^\text{vir}_M$ simply by $T^\text{vir}$.
- $R\text{Hom}_\mathbb{Z}(F, G) := R\pi_*R\text{Hom}(F, G)$.
- For a $\mathbb{C}^*$-representation $h$ we write $e^h$ to the the associated equivariant line bundle.
- $\mathbb{F}$ denotes the coefficient field for Chow cohomology groups.
- $t_1, t_2, e_1, \ldots, e_r$ are the fundamental one dimensional $\mathbb{T}$-representations. See Notation 4.3 for more details.

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2. Nakajima–Yoshioka blow-up algorithm

In [NY11c], Nakajima and Yoshioka introduced an algorithm to express certain tautological integrals on $\widehat{M}(\tilde{c})$ in terms of similar integrals on $M(c + n[pt])$ for various $n$, which applies in Situation B. The results necessary to apply the algorithm in Situation A were established in [KT21]. In this section we will review the Nakajima–Yoshioka blow-up algorithm in both situations simultaneously and draw some conclusions. First, we recall the concept of $m$-stable sheaves.

**Definition 2.1** ($m$-stable sheaves). [KT21, Lem. 3.29] & [NY11c, §1.1] We denote by $\hat{M}_m(\tilde{c})$ the moduli stack of $m$-stable sheaves which parameterises:

(i) In Situation A: Sheaves $E$ on $\hat{X}$ with Chern character $\tilde{c}$ such that the torsion-free quotient of the coherent sheaf $p_*E$ is $H$-stable on $X$, and such that $\text{Hom}(O_C(-m), E) = 0$ and $\text{Hom}(E, O_C(-m - 1)) = 0$.

(ii) In Situation B: Framed sheaves $(E, \Phi)$ on $\mathbb{P}^2$ such that $\text{ch}(E) = \hat{c}$ and $E$ is torsion-free away from the exceptional divisor $C$, and such that $\text{Hom}(O_C(-m), E) = 0$ and $\text{Hom}(E, O_C(-m - 1)) = 0$.

**Remark 2.2.** The discussion preceding Notation 1.2 also holds for $M = \hat{M}_m(\tilde{c})$. Hence, we can extend the integral notation from Notation 1.2 to $M = \hat{M}_m(\tilde{c})$.

In order to simplify the exposition, we will only state the results of this section for spaces of $m$-stable sheaves. This is justified by the following result.

**Proposition 2.3.** Suppose we are in either Situation A or B.

(i) [KT21, Prop. 3.25 & Cor. 3.27] & [NY11a, Prop. 7.1] If $\tilde{c}$ is of the form $p^*c$ then there is a natural isomorphism $p_*(-) : \hat{M}^0(\tilde{c}) \simto M(c)$ whose inverse is $p^*(-) : M(c) \simto \hat{M}^0(\tilde{c})$.

(ii) [NY11b, Prop. 3.37] & [NY11a, Prop. 7.1] For each $c$ there is an $m' \geq 0$ such that $\hat{M}_m(\tilde{c}) = M(\tilde{c})$ for all $m \geq m'$.

(iii) In Situation A, the isomorphisms in (i) and (ii) preserve the perfect obstruction theories. In Situation B, the isomorphisms are compatible with the $\tilde{T}$-equivariant structure.

The Nakajima–Yoshioka blow-up algorithm will be applied to integrals with the following type of integrand classes.

**Notation 2.4.** Suppose we are in either Situation A or B and let $M$ be $\hat{M}_m(\tilde{c})$ with $\pi : \hat{X} \times M \to M$ as the projection. We use the symbol $\Phi$ to denote a rule that associates a class $\Phi(\mathcal{F}) \in A^*(M)$ to each coherent sheaf $\mathcal{F}$ on $\hat{X} \times M$. More precisely, we require that $\Phi(\mathcal{F})$ is given as a power series in classes of the form $\pi_*(c_{i_1}(\mathcal{F}) \cdots c_{i_N}(\mathcal{F}) \cdot \alpha)$ for some list of integers $(i_1, \ldots, i_N)$ and some Chow class $\alpha \in A^*(\hat{X})$.

**Example 2.5.** In both Situation A and B, any polynomial in Chern classes of $\text{vir}$, $R\text{Hom}_\mathbb{C}(E, C_m)$ or $R\text{Hom}_\mathbb{C}(C_m, E)$ will satisfy the conditions of Notation 2.4 for $\Phi(\mathcal{E})$. This follows from the virtual and equivariant versions of the Grothendieck–Riemann–Roch Theorem, and the characterisation of the respective tangent bundles given by Remark 2.2 and the discussion preceding Notation 1.2.

In what follows, it will be useful to better understand the $K$-theory classes mentioned in Example 2.5.

**Lemma 2.6.** [NY11c, §4] Suppose we are in Situation A or B and consider the moduli space $M = \hat{M}_m(\tilde{c})$ with universal sheaf $\mathcal{E}$ and projection $\pi : \hat{X} \times M \to M$. Then we have:
(i) For any \( m \in \mathbb{Z} \), the respective \( K \)-theoretic ranks of \( R\text{Hom}_x(\mathcal{E}, C_m)[1] \) and \( R\text{Hom}_x(C_m, \mathcal{E})[1] \) are given by
\[ rm + (\bar{c}_1, [C]) \quad \text{and} \quad r(m + 1) + (\bar{c}_1, [C]). \]
(ii) For \( m = m_0 \), we have the vector bundles:
\[ R\text{Hom}_x(C_m)[1] = \text{Ext}^1_x(\mathcal{E}, C_m) \quad \text{and} \quad R\text{Hom}_x(C_{m-1}, \mathcal{E})[1] = \text{Ext}^1_x(C_{m-1}, \mathcal{E}). \]
(iii) Suppose \( m \geq 0 \), that \( \bar{c} \) satisfies \( (\bar{c}_1, [C]) = 0 \), and consider the space \( \bar{M}^0(\bar{c}) \).
Then, in Situation \( B \), we have the following identities of \( K \)-theory classes:
\[ R\text{Hom}_x(C_m, \mathcal{E}) = \left( \sum_{i=0}^m t_1^{-1} t_2^{i-m} \right) R\text{Hom}_x(C_0, \mathcal{E}), \]
\[ R\text{Hom}_x(\mathcal{E}, C_m) = \left( \sum_{i=1}^m t_1^{m+1} t_2^{i} \right) R\text{Hom}_x(C_0, \mathcal{E})^\vee. \]
In Situation \( A \), the identity holds if one replaces both \( t_1 \) and \( t_2 \) with the trivial line bundle.\(^1\)
(iv) Let \( m_1, m_2 \) be integers. The class \( \text{Hom}_{\bar{X}}(C_{m_1}, C_{m_2}) \) has \( K \)-theoretic rank equal to \(-1\). In Situation \( B \), the \( K \)-theory class of \( \text{Hom}_{\bar{X}}(C_{n_1}, C_{n_2}) \) equals a sum of terms of the form \( \pm t_1 t_2 \).

Proof. Point (i) follows from a Grothendieck–Riemann–Roch computation, while (ii) follows from the definition of \( m \)-stability, see for example [NY11c, §4]. For point (iii), using Bott’s formula for pushforwards in equivariant \( K \)-theory, we have for any \( E \in K^T(\mathbb{P}^2) \):
\[ R\Gamma(E) = \sum_F \frac{t_F(E)}{A_1 - T_1 F} \]
where the sum ranges over the torus fixed points of \( \mathbb{P}^2 \). Using this, one can compute
\[ R\Gamma(\mathcal{O}_C(m)) = \frac{t_1^{-m} - t_1^{-m-1}}{1 - t_1^{-1} - t_1 t_2^{-1} + t_2^{-1}} + \frac{t_2^m - t_2^{-m-1}}{1 - t_2^{-1} - t_2 t_1^{-1} + t_1^{-1}} \]
\[ = \begin{cases} \left( t_1^{-m} \sum_{i=0}^{m-2} t_1^{i} t_2^{-m-2-i} \right), & \text{if } m < 0; \text{ and} \\ \left( t_1 t_2 \right)^{-m} \sum_{i=0}^{m} t_1^{i} t_2^{-i}, & \text{if } m \geq 0. \end{cases} \]
By the equivariant localization formula in \( K \)-theory, we also have \( R\pi_* (\mathcal{O}_C(m)) = R\Gamma(\mathcal{O}_C(m)) \mathcal{O}_{\bar{p}_2} \). These facts also hold non-equivariantly on any surface if one imposes \( t_1 = t_2 = 1 \). Now, (iii) follows from a calculation using equivariant Serre duality and the observation that, on \( \bar{M}^0(p^c) \), we have the adjunction formula
\[ R\text{Hom}_x(p^* F, C_m) = R\text{Hom}_x(F, p_* C_m) \]
where \( F \) is the universal sheaf over \( \mathbb{P}^2 \times M_{\mathbb{P}2}(c) \). Point (iv) follows similarly, using that \( K \)-theoretically, we have \( C_m = \mathcal{O}_{\bar{X}}((m+1) C) - \mathcal{O}_{\bar{X}}(m C) \).

The Nakajima–Yoshioka blow-up algorithm allows us to express intersections of \( \int_{\bar{M}^0(\bar{c})} \Phi(\mathcal{E}) \) in terms of integrals over \( M(c') \) for various \( c' \). Indeed, it gives a consistent way to compute classes \( \Phi_n(\mathcal{E}) \) such that
\[ \int_{\bar{M}^0(\bar{c})} \Phi(\mathcal{E}) = \sum_{n \geq 0} \int_{M(c+npt)} \Phi_n(\mathcal{E}). \]
\(^1\)Here \( t_i \) is a one dimensional \( T \)-representation. See Notation 4.3 for more details.
For special choices of $\Phi$, we can say much more about the classes $\Phi_n$ (see Theorem 3.2). We now state the key components of the algorithm.

**Theorem 2.7 (Wall-Crossing Formula).** [KT21, Thm. 1.1] \& [NY11c, Thm. 1.5] Suppose we are in Situation $A$ or $B$, and let $\Phi(\mathcal{E})$ be as in Notation 2.4. Then we have the following identity:

$$
\int_{\tilde{M}^{m+1}(\tilde{c})} \Phi(\mathcal{E}) - \int_{\tilde{M}^m(\tilde{c})} \Phi(\mathcal{E}) = \sum_{j>0} \frac{1}{j!} \int_{\tilde{M}^{m}(\tilde{c}-j\epsilon_m)} \lim_{h_j \to 0} \lim_{h_{i=1}^j} \Phi \left( \mathcal{E} \oplus \bigoplus_{i=1}^j C_{m} \otimes e^{-h_i} \right) \Psi_m^j(\mathcal{E}),
$$

where $C_m := \mathcal{O}_C(-m-1)$, $\epsilon_m := \text{ch}(C_m)$, and where

$$
\Psi_m^j(\mathcal{E}) := \prod_{1 \leq i_1 \neq \cdots \neq j \leq (h_{i_1} - h_{i_2})} \prod_{i=1}^{j-1} e(-R\text{Hom}_\pi(\mathcal{E}, C_m) e^{-h_i}) e(-R\text{Hom}_\pi(C_m, \mathcal{E}) e^{h_i}).
$$

**Remark 2.8.** In regards to Theorem 2.7:

(i) The notation for taking the residue at $h_i = 0$ means: Taking the coefficient of $h_i^{-1}$ after expanding at $h_i = 0$.

(ii) We have written the right hand side of Theorem 2.7 as an infinite sum. However, from Remark 2.2 and the discussion preceding Notation 1.2 we know that the (virtual) dimension of $\tilde{M}^{m}(\tilde{c}-j\epsilon_m)$ decreases strictly as $j$ increases. Hence, for any choice of $\tilde{c}$, the right hand side is a finite sum.

**Remark 2.9.** In the version of Theorem 2.7 stated in [NY11c], only the action of a certain sub-torus of $\tilde{T} = (C^*)^{r+2}$ is considered. In correspondence with H. Nakajima, it was explained to us that this restriction is inessential, and that the results still hold when one considers the whole $\tilde{T}$-action.

**Lemma 2.10 (Twisting by $\mathcal{O}(C)$).** [KT21, Rem. 3.31] \& [NY11c, §1.5.4.] Suppose we are in Situation $A$ or $B$. For each $n \in \mathbb{Z}$, there is a natural isomorphism

$$
\tilde{M}^{m}(\tilde{c}) \cong \tilde{M}^{m+n}(\tilde{c} \cdot \text{ch}(\mathcal{O}_X(nC)))
$$

defined by sending a family of $m$-stable sheaves $F$ to its twist $F(nC)$. In Situation $A$, this isomorphism preserves the perfect obstruction theory and in Situation $B$, it is compatible with the $T$-equivariant structure.

**Remark 2.11.** Using the notation of Lemma 2.10, suppose that $\tilde{c} = \tilde{c} \cdot \text{ch}(\mathcal{O}_X(C))$. Then we have $(\tilde{c}_1, [C]) = (\tilde{c}_1, [C]) - rn$.

The final ingredient is the following.

**Proposition 2.12 (Grassmann-Bundle Formula).** Suppose that we are in Situation $A$ or $B$, and that $\tilde{c}$ satisfies $0 < \ell := (\tilde{c}_1, [C]) < r$ and let $\Phi(\mathcal{E})$ be as in Notation 2.4. Then

$$
\int_{\tilde{M}^1(\tilde{c})} \Phi(\mathcal{E}) = \frac{1}{\ell!} \int_{\tilde{M}^{m}(\tilde{c}-\ell \epsilon_0)} \lim_{h_0 \to 0} \lim_{h_{i=1}^\ell} \Phi \left( \mathcal{E} \oplus \bigoplus_{i=1}^\ell C_{m} \otimes e^{-h_i} \right) \Psi^{\ell}(\mathcal{E}),
$$

where $C_m := \mathcal{O}_C(-m-1)$, $\epsilon_m := \text{ch}(C_m)$, and where

$$
\Psi^{\ell}(\mathcal{E}) := \prod_{1 \leq i_1 \neq \cdots \neq \ell \leq (h_{i_1} - h_{i_2})} \prod_{i=1}^{\ell-1} e(-R\text{Hom}_\pi(C_m, \mathcal{E}) e^{h_i}).
$$

**Proof.** This is a reformulation of the results from [KT21, Thm. 1.2] and [NY11c, Thm. 1.2]. We used [Pra88, Prop. 2.2] to write it in the given form. □
Algorithm 2.13 (Blow-Up Algorithm For Single Terms). Let $0 \leq \ell < r$ and $n \geq 0$ be integers and set $\hat{c} := p^*c - \ell\epsilon_0 + n[pt]$.

- **INPUT:** The input for the algorithm is an integral of the form
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}). \]

- **OUTPUT:** The output is an identity of the form
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}) = \int_{M(c+n[pt])} \Phi(\mathcal{E})' + \sum_{i \in I} \int_{M_i} \Phi_i(\mathcal{E}), \]

  where
  
  (i) $\Phi'$ is as in Notation 2.4, and is zero if $0 < \ell < r$.

  (ii) $I$ is a finite collection indexing pairs $(M_i, \Phi_i)$ where $\Phi_i$ is as in Notation 2.4 and $M_i$ is of the form $\bar{M}^{m_i}(b_i)$ with $b_i = p^*c - \ell_i\epsilon_0 + n_i[pt]$ for integers $0 \leq \ell_i < r$ and $n_i \geq n$ such that $vdim \bar{M}^{m_i}(b_i) < vdim \bar{M}^{m}(\hat{c})$.

- **PROCEDURE:** The algorithm proceeds as follows:
  
  A Use Theorem 2.7 a total of $m - 1$ times to obtain the expression:
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}) = \int_{\bar{M}^{p(\hat{c})}} \Phi(\mathcal{E}) + \sum_{n=0}^{m-1} \sum_{j>0} \int_{\bar{M}^{m}(\hat{c} - j\epsilon_0 + n[pt])} \Phi_{n,j}(\mathcal{E}). \]

  B If $\ell = 0$, then we apply Proposition 2.3 to the first term on the right-hand side of step A. The second term on the right-hand side of step A becomes the collection indexed by $I$. We apply Lemma 2.10 for various values of $n \geq 0$ to obtain moduli spaces of the desired form. This completes the algorithm in the case $\ell = 0$.

  C If $0 < \ell < r$, then we twist the first term on the right-hand side of step A by $O(C)$ and apply Lemma 2.10. To the resulting term, apply Proposition 2.12 followed by Theorem 2.7. The term $\int_{\bar{M}^{p(\hat{c})}} \Phi(\mathcal{E})$ from step A becomes
  \[ \int_{\bar{M}^{1}(p^*c + (n+1+r)[pt])} \Phi''(\mathcal{E}). \]

  This, and the remaining terms from step A, become the collection indexed by $I$. We apply Lemma 2.10 for various values of $n \geq 0$ to obtain moduli spaces of the desired form. This completes the algorithm.

Algorithm 2.14 (Nakajima–Yoshioka Blow-Up Algorithm). Let $\ell \geq 0$ be an integer and suppose $\hat{c} := p^*c - \ell\epsilon_0$.

- **INPUT:** The input for the algorithm is an integral of the form
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}). \]

- **OUTPUT:** The output is an identity of the form
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}) = \sum_{n \geq 0} \int_{M(c+n[pt])} \Phi_n'(\mathcal{E}). \]

- **PROCEDURE:** The algorithm proceeds as follows:
  
  A We can assume $0 \leq \ell < r$ by applying Lemma 2.10 an appropriate number of times. Apply Algorithm 2.13 to obtain
  \[ \int_{\bar{M}^{m}(\hat{c})} \Phi(\mathcal{E}) = \int_{M(c)} \Phi_0'(\mathcal{E}) + \sum_{i \in I} \int_{M_i} \Phi_i(\mathcal{E}). \]
B Apply Algorithm 2.13 to those terms indexed by \( i \in I \) for which \( M_i \) has maximal virtual dimension, and remove those terms from \( I \). Collect the resulting output terms of type (i) to form \( \Phi_n' \) for the appropriate \( n \). Collect the resulting output terms of type (ii) of lower virtual dimension and add those terms to \( I \). We now use \( I \) to denote the indexing set for this new collection.

C Iterate Step B until \( I \) indexes the empty set. This will terminate, because after each step, \( I \) is finite and the maximal virtual dimension of the moduli spaces indexed by \( I \) decreases at each step.

3. Weak structure theorem

Here we explain a refined version of Nakajima–Yoshioka’s blowup formalism which holds when integrating a multiplicative class of the tangent bundle. This gives a unified version of Nakajima–Yoshioka’s [NY11c, Thm. 2.6] and Kuhn–Tanaka’s [KT21, Thm. 1.5]. In particular, we give a concise presentation of how the two situations are related, which is only implicit in their work. We begin this section by defining some useful notation.

**Notation 3.1.** Denote by \( \mathcal{S} \) the collection of symbols

\[
\left\{ \sum_{i \in \mathbb{Z}_{>0}, m \in \mathbb{Z}_{>0}} c_i \left( RHom_m (C_m, \_ ) \right), \ c_i \left( RHom_m (\_ , C_m) \right) \right\}_{i \in \mathbb{Z}_{>0}, m \in \mathbb{Z}_{>0}}
\]

and \( \mathcal{R} := F [\mathcal{S}] [\varepsilon_1, \varepsilon_2] \) the ring of power series in the symbols from \( \mathcal{S} \) and in variables \( \varepsilon_1, \varepsilon_2 \). For an element \( P \in \mathcal{R} \) and for \( \mathcal{F} \) a sheaf on \( M \times \hat{X} \), we denote by \( P(\mathcal{F}) \) the Chow class in \( A^* (M) \) obtained by evaluating the symbols of \( \mathcal{S} \) at \( \mathcal{F} \) and by setting \( \varepsilon_i = c_1 (t_i) \) in Situation \( B \), respectively \( \varepsilon_i = 0 \) in Situation \( A \).

We now state the main result of this section.

**Theorem 3.2** (Weak Structure Theorem). Let \( k \geq 0 \) be an integer, \( \Theta \) a multiplicative class and \( \nu_1, \ldots, \nu_r \) be formal variables. There exists a unique collection of power series \( \{ \Omega_n \}_{n \geq 0} \) in \( F [\varepsilon_1, \varepsilon_2, \nu_1, \ldots, \nu_r] \) which satisfies the following condition:

Whenever we are in Situation \( A \) or \( B \) with \( \hat{c} \) satisfying \( \langle \hat{c}_1, [\mathcal{C}] \rangle = k \), then we have

\[
\int_{\hat{M}(\hat{c})} \Theta (T^{vir}) = \sum_{n \geq 0} \int_{\hat{M}(\nu, c + n[pt])} \Theta (T^{vir}) \Omega_n (\mathcal{E}),
\]

where \( \Omega_n (\mathcal{E}) \) denotes the Chow-cohomology class obtained by replacing the variable \( \nu_i \) by the class \( c_1 (RHom_\mathcal{E} (\mathcal{O}_{pt}, \mathcal{E})) \).

In fact, the collection \( \{ \Omega_n \}_{n \geq 0} \) is uniquely determined if one only assumes this condition to hold in Situation \( B \).

**Remark 3.3.** More generally, the existence of Theorem 3.2 holds (with the same proof) if instead of just a multiplicative class of the tangent bundle, one considers a product \( \Theta (T^{vir}) \Phi (\mathcal{E}) \), where \( \Theta \) is multiplicative, and \( \Phi \) is as in Notation 2.4, and is also multiplicative in its input, i.e. \( \Phi (\mathcal{F}_1 \oplus \mathcal{F}_2) = \Phi (\mathcal{F}_1) \cdot \Phi (\mathcal{F}_2) \). Here, \( \Phi \) may only be defined for a particular choice of \( X \) or for a particular situation. The uniqueness statement holds (with the same proof) whenever \( \Phi \) is defined in a way that makes sense in Situation \( B \).

The idea of the proof is to follow a refined version of the Nakajima–Yoshioka Blow-Up Algorithm (Algorithm 2.14) which keeps track of extra structure related to the coefficients \( \Phi_i \). We first collect some consequences from Theorem 2.7 and Proposition 2.12 when applied to multiplicative classes.

\[\text{Remark 3.3.} \text{ Recall that a characteristic class } \Theta : K(\_ ) \to A^* (\_ ) \text{ is multiplicative if it satisfies the identity } \Theta (F + E) = \Theta (F) \cdot \Theta (E) \text{ for all choices of } F \text{ and } E.\]
Lemma 3.4 (Series version of Nakajima–Yoshioka-Algorithm components). Suppose that $P \in \mathcal{R}$, and let $\Psi(\mathcal{E}) := \Theta(T^{\text{vir}})P(\mathcal{E})$. The components of Algorithm 2.14 (the Nakajima–Yoshioka Blow-Up Algorithm) define canonical series in $\mathcal{R}$ in the following ways:

(i) Wall-Crossing Formula (Theorem 2.7): For integers $k, m \geq 0$, there exists a canonical collection of power series $\{Q_j\}_{j \geq 1}$ in $\mathcal{R}$ which satisfies the following condition:

Whenever we are in Situation $A$ or $B$ with $\hat{c}$ satisfying $(\hat{c}_1, [C]) = k$, then we have

$$\int_{\tilde{M}^{m+1}(\hat{c})} \Theta(T^{\text{vir}})P(\mathcal{E}) - \int_{\tilde{M}^{m}(\hat{c})} \Theta(T^{\text{vir}})P(\mathcal{E}) = \sum_{j \geq 1} \int_{\tilde{M}^{m}(\hat{c} - j\epsilon_m)} \Theta(T^{\text{vir}})Q_j(\mathcal{E}).$$

(ii) Twisting by $\mathcal{O}(C)$ (Lemma 2.10): There exists a canonical power series $P' \in \mathcal{R}$ satisfying $P'(\mathcal{E}) = P(\mathcal{E}(C))$ whenever we are in Situation $A$ or $B$ with arbitrary $\hat{c}$.

(iii) Gr-Bundle Formula (Proposition 2.12): For each integer $0 < k < r$ there exists a canonical power series $Q$ in $\mathcal{R}$, which satisfies the condition:

Whenever we are in Situation $A$ or $B$ with $\hat{c}$ satisfying $(\hat{c}_1, [C]) = -k$, then we have

$$\int_{\tilde{M}^{1}(\hat{c})} \Theta(T^{\text{vir}})P(\mathcal{E}) = \int_{\tilde{M}^{1}(\hat{c} - k\epsilon_0)} \Theta(T^{\text{vir}})Q(\mathcal{E}).$$

Proof. For part (ii), one obtains $P'$ by replacing every occurrence of $R\text{Hom}_\pi(C_m, -)$ and $R\text{Hom}_\pi(-, C_m)$ in $P$ by $R\text{Hom}_\pi(C_{m+1}, -)$ and $R\text{Hom}_\pi(-, C_{m+1})$ respectively. For parts (i) and (iii), the description of the virtual tangent bundle from Remark 2.2 and the discussion preceding Notation 1.2 shows that $\Psi(\mathcal{E} \oplus \bigoplus_{i=1}^{j} C_m e^{-h_i})$ can be written as:

$$\Theta(T^{\text{vir}})P\left(\mathcal{E} \oplus \bigoplus_{i=1}^{j} C_m e^{-h_i}\right) = \prod_{1 \leq i_1 \leq i_2 \leq j} \Theta(R\text{Hom}_\pi(C_m, C_m) e^{h_{i_2} - h_{i_1}}) \cdot \prod_{i=1}^{j} \Theta(R\text{Hom}_\pi(\mathcal{E}, C_m) e^{-h_i}) \Theta(R\text{Hom}_\pi(C_m, \mathcal{E}) e^{h_i}).$$

We can make use of Lemma 2.6 (iv) and the basic properties of Chern classes to rewrite this expression canonically as an expression in variables $h_i$, elements of $\mathcal{S}$ and in $\epsilon_1, \epsilon_2$. It follows that the series $\Psi_m(\mathcal{E})$ and $\Psi(\mathcal{E})$ appearing in Algorithm 2.14 are of the desired form. This implies parts (i) and (iii).

We can now state a refined version of Algorithm 2.13.

Algorithm 3.5 (Refined Blow-Up Algorithm For Single Terms). Let $n, \ell \geq 0$ be integers and set $\hat{c} := \frac{r^*c}{\ell} - \ell_0 + n[pt]$.

**INPUT:** The input for the refined algorithm is an expression of the form

$$\int_{\tilde{M}^{m}(\hat{c})} \Theta(T^{\text{vir}})P(\mathcal{E}),$$

where $P \in \mathcal{R}$.

**OUTPUT:** The output is an identity of the form

$$\int_{\tilde{M}^{m}(\hat{c})} \Theta(T^{\text{vir}})P(\mathcal{E}) = \int_{\tilde{M}(c + n[pt])} \Theta(T^{\text{vir}})\Omega(\mathcal{E}) + \sum_{i \in I} \int_{\tilde{M}_i} \Theta(T^{\text{vir}})P_i(\mathcal{E}),$$
where
(i) $\Omega(\mathcal{E})$ is a power series in the symbols $c_i(R\text{Hom}_\pi(\mathcal{O}_\text{pt}, \mathcal{E}))$, and is zero if $0 < \ell < r$.
(ii) $I$ is a finite collection indexing tuples $(m_i, \ell_i, d_i, P_i)$, where $m_i, d_i, \ell_i \geq 0$, and where $P_i \in \mathcal{R}$. Moreover, for $M_i := \tilde{M}^{m_i}(b_i)$ with $b_i = p^*c - \ell_i \epsilon_0 + (n + d_i)[pt]$ they satisfy $vdim \tilde{M}^{m_i}(b_i) < vdim \tilde{M}^{m}$. 

\textbf{PROCEDURE:} The algorithm proceeds as follows:

A. Follow Algorithm 2.13 with the Wall-Crossing Component (Theorem 2.7), the Twisting Component (Lemma 2.10) and the Gr-Bundle Component (Proposition 2.12) replaced by their counterparts from Lemma 3.4.

B. If $0 < \ell < r$, then the algorithm is complete. If $\ell = 0$, then consider the term
$$\int_{M(c+n[pt])} \Phi(\mathcal{E}) = \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})P(\mathcal{E})$$
arising in step (i) of Algorithm 2.13 and apply (to this term) the following procedure:

(a) Use the formulas from Lemma 2.6 (iii) to rewrite all Chern classes $c_i(R\text{Hom}_\pi(\mathcal{E}, C_m))$ and $c_i(R\text{Hom}_\pi(C_m, \mathcal{E}))$ appearing in $P$ in terms of only the Chern classes $c_i(-R\text{Hom}_\pi(C_0, \mathcal{E}))$ for $i \geq 1$ and of $\epsilon_a = c_1(t_a)$ for $a = 1, 2$. Denote the resulting power series by $P'$. 

(b) Use the Wall-Crossing Formula from Lemma 3.4 to get
$$\int_{M(c+n[pt])} \Phi(\mathcal{E}) = \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})P'(\mathcal{E}) - \sum_{j \geq 1} \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})Q_j(\mathcal{E}).$$
for some $Q_j \in \mathcal{R}$.

(c) By Lemma 2.6 (ii), we have that $-R\text{Hom}_\pi(C_0, \mathcal{E})$ is the K-theory class of a rank $r$ vector bundle on $\tilde{M}^1$. Therefore, set all Chern classes of degree higher than $r$ equal to zero in $P'$ to get a power series $\Omega \in \mathbb{Z}[\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_r]$ such that
$$\int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})P'(\mathcal{E}) = \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})\Omega(\mathcal{E}).$$

(d) Use the Wall-Crossing Formula from Lemma 3.4 on the term from Step (c) to get the identity
$$\int_{M(c+n[pt])} \Phi(\mathcal{E}) = \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})\Omega(\mathcal{E}) + \sum_{j \geq 1} \int_{\tilde{M}^{c}(\mathcal{E})} \Theta(T^\text{vir})\left(Q'_j(\mathcal{E}) - Q_j(\mathcal{E})\right)$$
for canonical power series $Q'_j$.

(e) Apply Proposition 2.3 to the first term on the right-hand side of Step (d). The second term on the right-hand side of step (d) is absorbed into the collection indexed by $I$. We can apply the Twisting Component from Lemma 3.4 for various values of $n$ to obtain moduli spaces of the desired form. This completes the algorithm in the case $\ell = 0$.

\textbf{Remark 3.6.} It follows from the algorithm that the output depends only on $\Theta, P, \ell$ and $m$, but is independent of $c$ and $n$, and on whether we are in Situation $A$ or $B$, in the following sense: The series $\Omega$ depends only on $\Theta, P, \ell$ and $m$. Moreover,
there is an infinite collection \( \mathcal{T} \) of tuples \((m_i, \ell_i, d_i, P_i)\) which depends only on \( \Theta, P, \ell \)
and \(n\), such that: \( I \) is exactly the subset of \( \mathcal{T} \) consisting of those tuples for which \( \text{vdim} M^{m_i}(p^c - \ell_i e_0 + (n + d_i)[pt]) \) is non-negative.

**Proof of Theorem 3.2. Existence of \( \{\Omega_n\}_{n \geq 0} \):** We have a refined version of Algorithm 2.14 (the Nakajima–Yoshioka Blow-Up Algorithm), obtained by replacing Algorithm 2.13 by the refined version, Algorithm 3.5, and by replacing Lemma 2.10 by its counterpart in Lemma 3.4. Then, this modified algorithm applies to any \( \Phi(\mathcal{E}) = \Theta(T^{\text{vir}}) P(\mathcal{E}) \) with \( P \in \mathbb{R} \), whenever we are in Situation A or B with \( \hat{c} \) such that \( (\hat{c}_1, [C]) = k \geq 0 \). It shows that there are classes \( \Omega_n(\mathcal{E}) \), which are power series in \( c_i(R\text{Hom}_x(\mathcal{O}_{pt}, \mathcal{E})) \) and that satisfy the equation

\[
\left(3.1\right) \quad \int_{M(\mathcal{E})} \Theta(T^{\text{vir}}) = \sum_{n \geq 0} \int_{M(p_c+n[pt])} \Theta(T^{\text{vir}}) \Omega_n(\mathcal{E}).
\]

We observe that the algorithm creates the same collection of formal power series \( \{\Omega_n\}_{n \geq 0} \subset F[\xi_1, \xi_2, \nu_1, \ldots, \nu_N] \) for any choice of data from Situation A or B with \( \hat{c} \) such that \( (\hat{c}_1, [C]) = k \geq 0 \). This is due to the independence in the single step asserted in Remark 3.6.

**Uniqueness of \( \{\Omega_n\}_{n \geq 0} \):** We prove that any collection of power series \( \{\Omega_n\} \) satisfying the condition in the statement of Theorem 3.2 must be unique. As mentioned there, the following stronger statement is true:

The collection \( \{\Omega_n\} \) is the unique collection which satisfies the requirement that \( (3.1) \) holds in Situation B for all \( \hat{c} = r - k \epsilon_0 + n_0[pt] \) with \( n_0 \in \mathbb{Z}_{\geq 0} \).

Hence, for the rest of the proof we will assume we are in Situation B and that we have some collection \( \{\Omega_n\} \) for which \( (3.1) \) holds for all \( \hat{c} = r - k \epsilon_0 + n_0[pt] \) with \( n_0 \in \mathbb{Z}_{\geq 0} \). Now, for any \( n_0 \in \mathbb{Z}_{\geq 0} \), equation \( (3.1) \) can be rearranged as

\[
\left(3.2\right) \quad \int_{M(r+0[pt])} \Theta(T^{\text{vir}}) \Omega_n(\mathcal{E})
\]

\[
= \int_{M(r-k \epsilon_0 - n_0[pt])} \Theta(T^{\text{vir}}) - \sum_{n=0}^{n_0-1} \int_{M(r-(n_0-n)[pt])} \Theta(T^{\text{vir}}) \Omega_n(\mathcal{E}).
\]

As shown by Nakajima–Yoshioka in [NY05a, Prop. 2.9 & Thm 2.11], the moduli space \( M(r+0[pt]) \) is a point with tangent bundle \( T^{\text{vir}}_{M(r+0[pt])} = 0 \). Now, letting \( t_1, t_2, e_1, \ldots, e_r \) be the characters of the (trivial) \( \hat{T} \)-action on \( M(r+0[pt]) \), the universal sheaf of \( M(r+0[pt]) \) is \( \mathcal{E}_0 = \bigoplus_{i=1}^r \mathcal{O}_{pt} e_i \). In particular, one can compute that \( R\text{Hom}_x(\mathcal{O}_{pt}, \mathcal{E}_0) = \bigoplus_{i=1}^r \mathcal{E}_i \) and we have the following equation in \( A^*_c(pt) \):

\[
\Omega_n \left( \bigoplus_{i=1}^r \mathcal{O}_{pt} e_i \right) = \int_{M(r-k \epsilon_0 - n_0[pt])} \Theta(T^{\text{vir}}) - \sum_{n=0}^{n_0-1} \int_{M(r-(n_0-n)[pt])} \Theta(T^{\text{vir}}) \Omega_n(\mathcal{E}).
\]

We observe that the left hand side is by definition a power series in the terms \( c_i(R\text{Hom}_x(\mathcal{O}_{pt}, \mathcal{E}_i)) = \sigma_i(c_1(e_i), \ldots, c_r(e_i)) \) for \( 1 \leq i \leq r \), where \( \sigma_i \) denotes the \( i \)-th elementary symmetric polynomial. These are analytically independent (since the terms \( c_1(e_i) \) are), so the equality \( (3.2) \) determines \( \Omega_n \) inductively from the \( \Omega_n \) for \( n = 0, \ldots, n_0 - 1 \). (Note that this includes the base case of the induction.) Thus the collection of all the \( \Omega_n \) is uniquely determined.

\[\square\]

\[\text{3The } \hat{T} \text{-action on the moduli spaces is described further in Definition 4.2.}\]
4. Proof of the blowup formula for the $\chi_y$-genus

In this section, we prove our main theorem, Theorem 1.4. For this, we apply Theorem 3.2 to the class $\Theta_y$ giving rise to the $\chi_y$-genus. Then, it is clearly enough to show that each power series $\Omega_n$ is constant and equal to the coefficient of $q^n$ in the power series $Y_k$ appearing in Theorem 1.4. To do this, we use the fact that by (3.2), the $\Omega_n$ can be uniquely determined from the equivariant theory of framed moduli spaces in Situation B.

**Remark 4.1.** An easier version of the proof using the total Chern class $\Theta(-) = c(-)$ in place of $\Theta_y$ gives a direct proof of the blowup formula for virtual Euler characteristics.

In the rest of this section, we work in Situation $B$ and we let $\{\Omega_n\}$ denote the collection of power series obtained from Theorem 3.2 by the choice $\Theta = \Theta_n$. Note that, by definition, these are power series with coefficients in $F = \mathbb{Q}(y)$. We recall the $\hat{T}$-action on the moduli spaces $M(c)$ and $\hat{M}(\hat{c})$ from [NY05a].

**Definition 4.2 (\(\hat{T}\)-Action).** [NY05a, §2 – §3] Let $M$ be one of $M(c)$ or $\hat{M}(\hat{c})$ and let $(E, \varphi) \in M$ be a framed sheaf. For $(t_1, t_1, e_1, \ldots, e_r) \in \hat{T} \cong (\mathbb{C}^*)^{2+r}$ we define the following:

(i) In the case of $M = M(c)$: Let $F_{t_1, t_2}$ denote the automorphism of $\mathbb{P}^2$ defined via $[z_0 : z_1 : z_2] \mapsto [z_0, t_1 z_1 : t_2 z_2]$. In the case of $M = \hat{M}(\hat{c})$: We denote again by $F_{t_1, t_2}$ the unique lift of this automorphism to $\mathbb{P}^2$.

(ii) In either case, we let $\varphi_{(t_1, t_1, e_1, \ldots, e_r)}$ denote the composition

$$(F_{t_1, t_2}^{-1})^*E \xrightarrow{\varphi} (F_{t_1, t_2}^{-1})^*O_{\ell_\infty}^{\oplus r} \xrightarrow{\sim} O_{\ell_\infty}^{\oplus r} \xrightarrow{\varphi} O_{\ell_\infty}^{\oplus r},$$

where the middle arrow comes from the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2$ (respectively $\mathbb{P}^2$), and the last arrow is $(x_1, \ldots, x_r) \mapsto (e_1 x_1, \ldots, e_r x_r)$.

We now recall the description of the torus fixed points of the moduli spaces of framed sheaves on $\mathbb{P}^2$ and $\mathbb{P}^2$ given in [NY05a]. We also recall their description of the equivariant $K$-theory class given by the restriction of the tangent bundle to each fixed point. The results are as follows:

**Theorem 4.4.** [NY05a, Prop. 2.9 & Thm. 2.11] Let $c = r - n[pt]$ for $n \geq 0$ an integer. Then we have:

(i) The fixed points of $M(c)$ are in bijection to the set of $r$-tuples of Young diagrams $Y = (Y_1, \ldots, Y_r)$ satisfying $\sum_{i=1}^r |Y_i| = n$.

(ii) Let $Y = (Y_1, \ldots, Y_r)$ correspond to a fixed point of $M(c)$. The equivariant $K$-theory class of the restricted tangent bundle $T_{M(c)}|_{\mathfrak{Y}}$ is given by

$$\sum_{a, b = 1}^r N^Y_{a, b}(t_1, t_2),$$

where $N^Y_{a, b}(t_1, t_2)$ is given by

$$e_a e^{-1}_b \times \left( \sum_{s \in Y_a} \left( t_1^{-y_{a}(s)} t_2^{-y_{a}(s)+1} \right) + \sum_{t \in Y_b} \left( t_1^{y_{b}(t)+1} t_2^{-y_{b}(t)} \right) \right).$$
Here, \( a_Y(i,j) := \lambda_i - j \) and \( b_Y(i,j) := \lambda_j^i - i \) are the arm length and leg length of the box \((i,j)\) in \(Y\), where \(\lambda\) is the partition associated to \(Y\) with transpose \(\lambda^t\).

**Theorem 4.5.** [NY05a, Prop. 3.2 & Thm. 3.4] Let \(k, n \in \mathbb{Z}\) be integers and \(e_0 = [C] - \frac{1}{2}[pt]\). If \(\hat{c} = r + k\epsilon_0 - n[pt]\), then we have:

(i) The fixed points of \(\hat{M}(\hat{c})\) are in natural bijection to the set of triples \((Y, Z, k)\), where \(Y = (Y_1, \ldots, Y_r)\) and \(Z = (Z_1, \ldots, Z_r)\) are \(r\)-tuples of Young diagrams and \(k = (k_1, \ldots, k_r) \in \mathbb{Z}^r\) such that

\[
\sum_{j=1}^r k_i = k, \quad \text{and} \quad \sum_{j=1}^r (|Y_j| + |Z_j|) + \frac{1}{2r} \sum_{1 \leq i < j \leq r} (k_i - k_j)^2 = n + \frac{k(r - k)}{2r}.
\]

(ii) Let \(F\) be the fixed point of \(\hat{M}(\hat{c})\) given by \((Y, Z, k)\). The equivariant \(K\)-theory class of the restricted tangent bundle \(T_{\hat{M}(\hat{c})|F}\) is given by

\[
\sum_{a,b=1}^r L_{a,b}(t_1, t_2) + t_1^{k_a - k_b} N_{a,b}(t_1, t_2) + t_1 t_2^{k_a - k_b} N_{a,b}(t_1, t_2),
\]

where the expressions \(N_{a,b}\) are defined as in Theorem 4.4 and

\[
L_{a,b}(t_1, t_2) = e_b e_a^{-1} \times \begin{cases} \sum_{i+j \geq 0 \atop i+j \leq k_a - k_b} t_1^i t_2^j, & \text{if } k_a > k_b; \\ \sum_{i+j \geq 0 \atop i+j \leq k_a - k_b} t_1^{i+1} t_2^{j+1}, & \text{if } k_a + 1 < k_b; \\ 0, & \text{otherwise.} \end{cases}
\]

**Example 4.6.** From the dimension formula \(c_Y^2 - 2r \text{ch}_2\), one can see that the spaces \(M(r + 0)\) and \(\hat{M}(r + 0)\) consist of a collection of reduced points, while the spaces \(\hat{M}(r - k\epsilon_0)\) are empty for \(k > 0\). From Theorems 4.4 and 4.5, it follows that in fact \(M(r + 0)\) and \(\hat{M}(r + 0)\) consist of a single reduced point. Moreover, the universal sheaf \(\mathcal{E}_0\) on \(\mathbb{P}^2 \times M(r) \simeq \mathbb{P}^2\) is isomorphic to a direct sum \(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2} e_i\). It follows that \(R\text{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{E}_0) \simeq \bigoplus_{i=1}^r e_i\).

**Proposition 4.7.** The collection \(\{\Omega_n\}\) obtained from applying Theorem 3.2 (the Weak Structure Theorem) to \(\Theta = \Theta_y\) is a collection of elements from \(F[[\epsilon_1, \epsilon_2]]\), where \(F = \mathbb{Q}(y)\).

**Proof.** We prove by induction on \(n\) that the \(\Omega_n\) are constant in \(\nu_1, \ldots, \nu_r\). For this, we use the inductive formula (3.2). For \(n_0 = 0\), we get that

\[
\Omega_0 = \int_{\hat{M}(r - k\epsilon_0)} \Theta(T^{vir}).
\]

By Example 4.6, the right hand side is equal to zero if \(k \neq 0\), and equal to one if \(k = 0\), therefore \(\Omega_0\) is constant in either case. Now let \(n_0 > 0\) be fixed, and suppose we know that \(\Omega_n\) lies in \(F[[\epsilon_1, \epsilon_2]]\) for each \(n < n_0\). Then (3.2) becomes

\[
\Omega_{n_0} \left( \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^2} e_i \right) = \int_{\hat{M}(-n_0[pt] - k\epsilon_0)} \Theta(T) - \sum_{n=0}^{n_0-1} \Omega_n \int_{\hat{M}((n-n_0)[pt])} \Theta(T).
\]

By Example 4.6, we know that the left hand side is a power series in the variables \(\{c_1(e_1), \ldots, c_1(e_r), c_1(t_1), c_1(t_2)\}\) which is symmetric in the \(c_i(e_i)\) variables. We may equivalently write it as a power series in the variables \(d_i := \text{ch}(e_i) - 1\) and \(s_i := \text{ch}(t_i) - 1\) via the substitutions \(c_1(e_i) = \log(1 + d_i)\) and \(c_1(t_i) = \log(1 + s_i)\), where we use the formal power series expansion of \(\log(1 + x)\).
We more closely examine the right hand side of (4.1). By definition of integration as a sum over fixed points, we can rewrite each integral appearing on the right hand side as a sum

$$\sum_{G} \frac{\Theta(T|G)}{e(T|G)},$$

where $G$ ranges over the fixed points of a given moduli space. Note that for a nontrivial $T$-equivariant line bundle $L$ with $\alpha = c_1(L)$, we have

$$\frac{\Theta_j(L)}{e(L)} = \frac{1 - ye^{-\alpha}}{1 - e^{-\alpha}} = \frac{1}{\alpha} \text{td}(\alpha)(1 - ye^{-\alpha}).$$

In particular, this fraction is well-defined. Write $e_i := ch(e_i)$ and $t_i := ch(t_i)$. Using Theorems 4.4 and 4.5, and examining the formulas given there, we find that each integral is a sum of products over terms of the form

$$\frac{1 - y(e_a/e_b)t_i^it_j^j}{1 - (e_a/e_b)t_i^it_j^j},$$

where $a, b \in \{1, \ldots, r\}$, and where $(i, j) \neq (0, 0)$ if $a = b$. We may rewrite each such term as a fraction of two coprime elements in $F[e_1, \ldots, e_r, t_1, t_2]$ by cancelling the potential negative powers of $e_a$, $t_1$ and $t_2$ appearing in numerator and denominator. After doing this, for any term with $a \neq b$, the denominator is now of one of the forms

$$(4.2) \quad e_b - e_a t_1^i t_2^j, \quad e_b t_1^{-i} - e_a t_2^j, \quad e_b t_2^{-j} - e_a t_1^i, \quad \text{or} \quad e_b t_1^{-i} t_2^{-j} - e_a,$$

depending on which of the indices $i, j$ are positive. By the Gauss Lemma, each of these terms is irreducible in $F[e_1, \ldots, e_r, t_1, t_2]$. Moreover, note that each term vanishes at $e_1 = \cdots = e_r = t_1 = t_2 = 1$. Changing variables to $d_i := e_i - 1$ and $s_i := t_i - 1$, we find that each expression in (4.2) gives rise to an irreducible polynomial that vanishes at the origin in $F[d_1, \ldots, d_r, s_1, s_2]$.

Multiplying both sides of (4.1) with all the denominators of the form (4.2) appearing in the right hand side, we obtain an equality of the form

$$P(d_1, \ldots, d_r, s_1, s_2) \Omega_{n_0}(d_1, \ldots, d_r, s_1, s_2) = Q(d_1, \ldots, d_r, s_1, s_2)$$

in the power series ring $F[s_1, s_2, d_1, \ldots, d_r]$, where $P$ and $Q$ are polynomials in the variables $d_1, \ldots, d_r$ and where the total degree of $Q$ (in the $d_i$) is at most the total degree of $P$. We claim that this implies that $P$ divides $Q$ in $F[s_1, s_2][d_1, \ldots, d_r]$. Since $F[s_1, s_2]$ is a unique factorization domain, it is enough to show the claimed divisibility over its field of fractions. Since $P$ is a product of irreducible polynomials that vanish at $d_1 = \cdots = d_r = 0$, the statement follows from Lemma 4.8 below. Thus, we find that $Q_{n_0} = Q/P$ lies in $F[s_1, s_2][d_1, \ldots, d_r]$, and for degree reasons it is constant as a polynomial in the $d_i$. This is what we wanted to show. \hfill \Box

**Lemma 4.8.** Let $P, Q \in k[x_1, \ldots, x_n]$ be elements of a polynomial ring over the field $k$. Suppose that $P$ is irreducible and vanishes at the origin. Then if $P$ divides $Q$ in the power series ring $k[[x_1, \ldots, x_n]]$, it already divides $Q$ in $k[x_1, \ldots, x_n]$.

**Proof.** Let $I = (x_1, \ldots, x_n)$ be the ideal of the origin and consider the localization $k[x_1, \ldots, x_n]_I$. Since the completion $k[x_1, \ldots, x_n]_I \to k[[x_1, \ldots, x_n]]$ is fully faithful, it follows that $P$ divides $Q$ in the power series ring if and only if it does so in the localization. In particular, there is some equality of the form $P R_1 = Q R_2$ in $k[x_1, \ldots, x_n]$, where $R_1, R_2 \notin I$. Since $P \in I$ by assumption, it cannot divide $R_2$, therefore $P$ divides $Q$ by unique factorization. \hfill \Box

Using Proposition 4.7, one can already deduce Theorem 1.4 for the Gieseker moduli spaces by taking appropriate limits in the $\varepsilon_i$-variables. We want to go further and prove the desired formula in Situation B. This will use the following
result, which is nothing but a restatement of the rank 1 case of Theorem 1.4 for
the framed moduli spaces.

**Proposition 4.9.** Let \( W(t_1, t_2, y, q) \) be the rank one generating series for
equivariant \( \chi_y \)-genera for \( r = 1 \) on the moduli spaces of framed sheaves on \( \mathbb{P}^2 \).
More precisely,
\[
W(t_1, t_2, y, q) := \sum_{Y_s \in Y} \left( \Theta_y(t_1^{-1} l_1^{y(s)} t_2^{a_Y(s)+1}) \Theta_y(t_1^{l_1^{y(s)}+1} t_2^{-a_Y(s)}) \right)^{|Y|},
\]
where the sum ranges over all Young-diagrams \( Y \).

Then the contribution for \( r = 1 \) is computed by summing over the fixed
points of the associated moduli spaces. By Theorems 4.5 and 4.7
result, which is nothing but a restatement of the rank 1 case of Theorem 1.4 for
the framed moduli spaces.

We define a function \( \tilde{W} := \tilde{W}(t, q, u, T) \) by renaming variables as follows:
\[
t_1 \mapsto t, \quad t_2 \mapsto q^{-1}, \quad y \mapsto u, \quad q \mapsto T/u.
\]
Then
\[
\tilde{W}(t, q, u, T) = \sum_{Y_s \in Y} \left( \frac{1 - y t_1^{l_1^{y(s)}+1} t_2^{-a_Y(s)}}{1 - t_1^{l_1^{y(s)}+1} t_2^{-a_Y(s)}} \right)^{|Y|},
\]
where the sum ranges over all Young-diagrams \( Y \).

We define a function \( W(t_1, t_2, y, q) \) by renaming variables as follows:
\[
t_1 \mapsto t, \quad t_2 \mapsto q^{-1}, \quad y \mapsto u, \quad q \mapsto T/u.
\]

The desired formula now becomes
\[
\tilde{W}(t, q, u, T) = \sum_{Y_s \in Y} \left( \frac{1 - y t_1^{l_1^{y(s)}+1} t_2^{-a_Y(s)}}{1 - t_1^{l_1^{y(s)}+1} t_2^{-a_Y(s)}} \right)^{|Y|},
\]
where the sum ranges over all Young-diagrams \( Y \).

By the \((q,t)\)-Nekrasov–Okounkov formula [RW18, Thm. 1.3], we have the product
expansion
\[
\tilde{W}(t, q, u, T) = \prod_{n=1}^{\infty} \left( 1 - T^n \right)^{-1}.
\]

Expanding each occurence of \( \tilde{W} \) according to this formula and keeping track of all
the factors proves (4.3).

**Proof of Theorem 1.4 (the Main Theorem).** From Proposition 4.7, we have that
each element of the collection \( \{ \Omega_n \} \) is a power series in \( c_1(t_1) \) and \( c_1(t_2) \), hence
pulled back from \( A^*(pt) \). Consequently, the main equation from Theorem 3.2 (the
Weak Structure Theorem) implies that we have a relation of generating series
\[
\sum_n \int_{\overline{M}(r-ke_a-n[pt])} \Theta(T) q^{2rn-k(r+k)} = \left( \sum_n \Omega_n q^{2rn} \right) \cdot \sum_n \int_{\overline{M}(r-n[pt])} \Theta(T) q^{2rn}.
\]

We wish to show that \( \sum_n \Omega_n q^{2rn} \) is equal to \( Y_k \) as defined in the statement of the
Main Theorem. To do this, we will evaluate the equivariant integrals on both sides of
equation (4.4) and specialize the equivariant parameters corresponding to the \( e_i \).

The integrals arising in equation (4.4) are computed by summing over the fixed
points of the associated moduli spaces. By Theorems 4.4 and 4.5 the contribution
from each fixed point is a product of terms of the form \( \theta(e_i/e_a t_1^{l_1^{e_i}} t_2^{l_2^{e_i}}) \), where
\[
\theta(x) := \frac{1 - y x^{-1}}{1 - x^{-1}},
\]
with \( e_i = \text{ch}(e_i) \) and \( t_i = \text{ch}(t_i) \). Now, taking limits in a fixed order, we observe

\[
(4.5) \quad \lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \lim_{t_i \to a} \lim_{t_2 \to b} \theta(e_6 / e_a t_1 t_2) = \begin{cases} 
1, & \text{if } a < b \\
y, & \text{if } a > b \\
1 - y t_1^{-1} t_2^{-1} = \theta(t_1 t_2), & \text{if } a = b.
\end{cases}
\]

We now consider the generating series associated to the moduli spaces \( \mathcal{M}(c) \). By the definition of the integral notation from Notation 1.2 combined with Theorem 4.4 (i) we have:

\[
\sum_n \int_{\mathcal{M}(r-n[pt])} \Theta(T) q^{2rn} = \sum_Y \Theta(T|Y) q^{2r \sum_{i=1}^{\ell} Y_i},
\]

where the sum ranges over \( r \)-tuples of Young-diagrams \( Y = (Y_1, \ldots, Y_r) \). Now, by Theorem 4.4 (ii), we can compute \( \Theta(T|Y) \) to be:

\[
\prod_{a,b=1}^r \left( \prod_{s \in Y_a} \theta(e_{b}/e_{a} t_1^{a Y_a(s)} t_2^{(a Y_a(s) + 1)} \prod_{t \in Y_b} \theta(e_{b}/e_{a} t_1^{b Y_b(s)} + t_2^{a Y_a(s)}) \right).
\]

Using equation (4.5) gives that \( \lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \Theta(T|Y) \) is equal to:

\[
\prod_{1 \leq b < c \leq r} g^{Y_b|Y_c|} \prod_{a=1}^r \prod_{s \in Y_a} \theta(t_1^{a Y_a(s)} t_2^{a Y_a(s) + 1}) \theta(t_1^{b Y_b(s)} + t_2^{a Y_a(s)}).
\]

We conclude that:

\[
(4.6) \quad \lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \sum_n \int_{\mathcal{M}(r-n[pt])} \Theta(T) q^{2rn} = \left( \sum_Y \left( \prod_{s \in Y} \theta(t_1^{a Y_a(s)} t_2^{a Y_a(s) + 1}) \theta(t_1^{b Y_b(s)} + t_2^{a Y_a(s)}) \right) (q^{2r} y^{r-1})^{Y_r} \right)^r,
\]

where \( \mathcal{W} \) is as in Proposition 4.9.

We now consider the generating series associated to the moduli spaces \( \widetilde{\mathcal{M}}(\mathcal{C}) \). By the definition of the integral notation from Notation 1.2 combined with Theorem 4.5 (i) we have:

\[
\sum_n \int_{\mathcal{M}(r-n[pt]-k[pt])} \Theta(T) q^{2rn-k(2r+k)} = \sum_{(Y,Z,k)} \Theta(T|Y,Z,k) q^{2r \sum_{i=1}^{\ell} (Y_i + |Z_i|) + \sum_{1 \leq i < j < r} (k_i - k_j)^2}.
\]

Consider a fixed triple \( (Y,Z,k) \). In the notation of Theorem 4.5 (ii), we have:

\[
\lim_{c_i \to 0} \cdots \lim_{c_1 \to 0} \Theta(T(a,b),t_1,t_2) = \begin{cases} 
y^{(k_b - k_a + 1)/2}, & \text{if } a > b \text{ and } k_a > k_b \\
y^{(k_b - k_a)/2}, & \text{if } a > b \text{ and } k_a + 1 < k_b \\
1, & \text{else}
\end{cases}
\]

\[
= \begin{cases} 
y^{(k_a - k_b)^2/2}, & \text{if } a > b; \\
y^{(k_a - k_b)/2}, & \text{else}.
\end{cases}
\]
In particular, we have the identity
\[ \lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \prod_{a, b = 1}^r \Theta(L_{a,b}(t_1, t_2)) = y^{\sum_{a > b} (k_a - k_b)^2 / 2} y^{\sum_{a > b} (k_a - k_b)/2}. \]

Proceeding as before for \( N_{a,b}^r(-,-) \) gives that \( \lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \Theta(T'_{1, Y, Z, k}) \) is equal to
\[
\prod_{a = 1}^r \left( \prod_{s \in Y_a} \theta \left( (t_1/t_2)^{aZ_a(s)} (t_2/t_1)^{aY_a(s)+1} \right) \right) \\
\cdot \prod_{a = 1}^r \left( \prod_{s \in Z_a} \theta \left( (t_1/t_2)^{-aZ_a(s)} (t_2/t_1)^{-aY_a(s)+1} \right) \right) \\
\cdot y^{\sum_{a > b} (k_a - k_b)^2 / 2} y^{\sum_{a > b} (k_a - k_b)/2}.
\]

Now, considering the generating series on the left hand side of (4.7), we see that we can separate the sums over pairs of \( r \)-tuples of Young diagrams into \( 2r \) sums over Young diagrams after taking limits in the \( e_i \). In conclusion, we find that the limiting series is equal to
\[
\lim_{e_i \to 0} \cdots \lim_{e_1 \to 0} \sum_{n} \int_{\overline{M}(-n[pt]-k\tau)} \Theta(T) \ q^{2rn-k(r+k)} \\
= \left( W(t_1/t_2, t_2, y, y^{r-1}y^{2r}) \cdot W(t_1/t_2, t_1, y, y^{r-1}y^{2r}) \right)^r \\
\cdot \sum_{k_1 + \cdots + k_r = k} (q^2 y)^{\sum_{i<j} (k_i - k_j)^2 / 2} y^{\sum_{i<j} (k_i - k_j)/2}.
\]

Comparing this to equation (4.6) and using Proposition 4.9 gives the desired result.

\[ \square \]

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