AMOEBA–ABSOLUTENESS AND PROJECTIVE MEASURABILITY

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Abstract. We show that $\Sigma^1_4$–Amoeba–absoluteness implies that $\forall a \in \mathbb{R} (\omega_1^{L[a]} < \omega_1^V)$, and hence $\Sigma^1_3$–measurability. This answers a question of Haim Judah (private communication).

INTRODUCTION

We study the relationship between Amoeba forcing and projective measurability. Recall that the Amoeba partial order $\mathbb{A}$ is defined as follows.

\[ A \in \mathbb{A} \iff A \subseteq 2^{\omega} \land A \text{ open} \land \mu(A) < \frac{1}{2} \]

\[ A \leq B \iff B \subseteq A \]

Amoeba forcing generically adds a measure one set of random reals. Its importance in the investigation of measurability of projective sets stems from the classical result, due to Solovay, that

\[ (*): \quad \text{all } \Sigma^1_2 \text{-sets are measurable} \iff \forall a \in \mathbb{R} (\mu(Ra(L[a]))) = 1 \]

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(see, e.g., [JS 2, 0.1. and § 3]). Here $Ra(M)$ denotes the set of reals random over a model $M$ of set theory.

The connection between Amoeba forcing and projective measurability was made more explicit through Judah’s study of absoluteness between models $V \subseteq W$ of set theory such that $W$ is a forcing extension of $V$ [Ju].

**Definition** (Judah [Ju, § 2]). Let $V$ be a universe of set theory. Given a forcing notion $P \in V$ we say that $V$ is $\Sigma^1_n - P$–absolute iff for every $\Sigma^1_n$–sentence $\phi$ with parameters in $V$ we have $V \models \phi$ iff $V^P \models \phi$. (So this is equivalent to saying that $R^V \prec \Sigma^1_n R^{V^P}$.)

Note that Shoenfield’s Absoluteness Lemma [Je, Theorem 98] says that $V$ is always $\Sigma^1_2 - P$–absolute. Furthermore, using (*), Judah showed [Ju, § 2] (**) all $\Sigma^1_2$–sets are measurable in $V \iff V$ is $\Sigma^1_3 - A$–absolute.

Whereas there is no way of getting a characterization of $\Sigma^1_3$–measurability analogous to (*), (**) suggests the investigation of the relation between $\Sigma^1_3$–measurability and $\Sigma^1_4 - A$–absoluteness. The main goal of this note is to establish one implication, namely that $\Sigma^1_4 - A$–absoluteness implies $\Sigma^1_3$–measurability (Theorem 5 in § 2). Our tools for proving this theorem are a partial earlier result of Judah’s, who showed Theorem 5 under the additional assumption that $\forall a \in \mathbb{R} (\omega^L[a] < \omega^Y_1)$, and combinatorial ideas due to Cichoń and Pawlikowski [CP], which will eventually yield that Judah’s additional assumption is in fact a consequence of $\Sigma^1_4 - A$–absoluteness (§ 1 and Theorem 4 in § 2).

**Notation.** We shall mostly work with $2^\omega$ or $\omega^\omega$ instead of $\mathbb{R}$. $\mathcal{L}$ denotes the ideal of Lebesgue measure zero sets, and $\mathcal{B}$ is the ideal of meager sets. $\Sigma^1_n(\mathcal{L})$ stands for all $\Sigma^1_n$–sets are Lebesgue measurable; and $\Sigma^1_n(\mathcal{B})$ means all $\Sigma^1_n$–sets have the property of Baire. For a non–trivial $\sigma$–ideal $\mathcal{I} \subset P(2^\omega)$, let $add(\mathcal{I})$ be the size of the smallest family of members in $\mathcal{I}$ whose union is not in $\mathcal{I}$; $cov(\mathcal{I})$ denotes the least $\kappa$ such that $2^\omega$ can be covered by $\kappa$ sets from $\mathcal{I}$; $unif(\mathcal{I})$ is the cardinality of the smallest subset of the reals which does not lie in $\mathcal{I}$; and $cof(\mathcal{I})$ is the size of the smallest $\mathcal{F} \subset \mathcal{I}$ such that every member of $\mathcal{I}$ is included in a member of $\mathcal{F}$. We always have $add(\mathcal{I}) \leq cov(\mathcal{I}) \leq cof(\mathcal{I})$ and $add(\mathcal{I}) \leq unif(\mathcal{I}) \leq cof(\mathcal{I})$ (see, e.g., [CP] for details concerning these invariants in case $\mathcal{I} = \mathcal{L}$ or $\mathcal{B}$).

Our forcing notation is rather standard (see [Je] for any notion left undefined here). We confuse to some extent Boolean–valued models $V^P$ and forcing extensions $V[G]$, $G$
\(\mathbb{P}\)-generic over \(V\). For p.o.s \(\mathbb{P}, \mathbb{Q}\), \(\mathbb{P} <_c \mathbb{Q}\) means that \(\mathbb{P}\) can be completely embedded in \(\mathbb{Q}\). For a sentence of the \(\mathbb{P}\)-forcing language \(\phi\), \(\|\phi\|\) is the Boolean value of \(\phi\). \(\mathbb{P}\)-names for objects in the forcing extension are denoted by symbols like \(\check{r}\). Finally, \(\mathbb{B}\) will stand for the random algebra, \(\mathbb{C}\) for the Cohen algebra, and \(\mathbb{D}\) for the Hechler p.o. (see, e.g., [BJS]).

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§ 1. The combinatorial component

We start with a straightforward generalization of one version of the main result of [CP]. The proof is included for completeness’ sake.

Theorem 1 (Cichoń – Pawlikowski [CP, § 1]). Assume that \(\mathbb{C} <_c \mathbb{P}\), and that for any uncountable \(T \subseteq \mathbb{P}\) there is an \(s \in \mathbb{C}\) such that for all \(\ell \in \omega\) there exists \(F \subseteq T\) of size \(\ell\) such that any \(t\) extending \(s\) is compatible with \(\bigcap F \in \mathbb{P}\). Then there is a family \(\{A_x; x \in \omega^\omega \cap V\}\) of Lebesgue measure zero sets in \(V^\mathbb{C}\) such that for all \(z \in V^\mathbb{P}\), \(\{x \in \omega^\omega \cap V; z \notin A_x\}\) is at most countable.

Proof. Let \(\{\tau_n; n \in \omega\}\) be a one–to–one enumeration of \(\omega^{<\omega}\); set \(\text{code}(\tau) = n\) iff \(\tau = \tau_n\) for any \(\tau \in \omega^{<\omega}\). Let \(\{C^n(i); i \in \omega\}\) be an enumeration of all open intervals in the unit interval \(I = [0, 1]\) with rational endpoints of length \(2^{-n}\). For \(x, y \in \omega^\omega\) let

\[
B_{x,y}^n = \begin{cases} C^n(\tau_{y(n)}(\text{code}(x|y(n+1)))) & \text{if } \text{code}(x|y(n+1)) \in \text{dom}(\tau_{y(n)}) \\ \emptyset & \text{if not} \end{cases}
\]

Let \(B_{x,y} = \bigcap_n \bigcup_{m>n} B_{x,y}^m\). Clearly \(\mu(B_{x,y}) = 0\). We claim that if \(c\) is Cohen over \(V\), \(A_x = B_{x,c}\) for \(x \in \omega^\omega \cap V\), then \(\{A_x; x \in \omega^\omega \cap V\}\) is the required family.

For suppose not. Then there are a \(\mathbb{P}\)-name \(\check{z}\), an uncountable set \(T \subseteq \omega^\omega \cap V\), \(T \in V\), conditions \(p_x \in \mathbb{P}\), and \(k_x \in \omega\) \((x \in T)\) such that
Choose $T' \subseteq T$ uncountable and $k \in \omega$ such that $\forall x \in T' (k_x = k)$. Fix $s \in \mathcal{C}$ according to $T'$. Let $\ell \geq k$, $\ell \geq lh(s)$, and choose $F \subseteq \omega^\omega$ of size $2^\ell$ such that $\{p_x; x \in F\}$ satisfies the requirements of the Theorem. Next let $n > \ell$ be such that $|\{x| n; x \in F\}| = 2^\ell$. Let $F = \{x_i; i \in 2^\ell\}$, and choose $i_0, ..., i_{2^\ell - 1}$ such that $C^\ell(i_0) \cup ... \cup C^\ell(i_{2^\ell - 1}) = 1$. Let $m \in \omega$ be such that $\tau_m(code(x_0|n)) = i_0, ..., \tau_m(code(x_{2^\ell - 1}|n)) = i_{2^\ell - 1}$. Let $t \leq s$ be such that $t(\ell) = m$, $t(\ell + 1) = n$. Then $\bigcup_{i \in 2^\ell} C^\ell(\tau_{t(\ell)}(code(x_i|t(l + 1)))) = 1$, i.e.

$$t \cap \bigcap\{p_x; x \in F\} \models \not\exists \tilde{\varepsilon} \in \bigcup_{i \in 2^\ell} C^\ell(\tau_{e(l)}(code(x_i|\tilde{\varepsilon}(l + 1)))) = \bigcup_{i \in 2^\ell} B_{x_i, \tilde{\varepsilon}},$$

contradicting (*). □

As each open set in $2^\omega$ can be written as a countable disjoint union of sets of the form $[\sigma] = \{f \in 2^\omega; \sigma \subseteq f\}$, where $\sigma \in 2^{< \omega}$, we can think of a condition $A$ in the Amoeba algebra $A$ as a function $\phi: \omega \rightarrow \bigcup_{i \in \omega} P(2^i)$ with $\phi(i) \in P(2^i)$ such that $\sigma \in \phi(i)$ iff $\sigma \in 2^i$ and $\sigma$ lies in the countable disjoint decomposition of $A$. We can furthermore assume that $\phi$ has the property:

$$(*) \quad \forall \sigma \in 2^i \setminus \phi(i) (\mu(\{[\tau]; \tau \supseteq \sigma \land \exists j > i (\tau \in \phi(j))\}) < 2^{-i}).$$

(Then $\phi$ is unique.) We define a p.o. $A'$ as follows.

$$(u, \phi) \in A' \iff \begin{cases} 1) dom(\phi) = \omega \land \forall i \in \omega (\phi(i) \in P(2^i)) \land \phi \text{ satisfies } (*) \\ 2) u \subseteq \phi \text{ (}u\text{ is an initial segment of } \phi) \\ 3) \mu(\{[\sigma]; \exists i \in \omega (\sigma \in \phi(i))\}) < \frac{1}{2} \end{cases}$$

$$(u, \phi) \leq (v, \psi) \iff u \supseteq v \land \forall i \forall \sigma \in \psi(i) \exists j \leq i \exists \tau \in \phi(j) (\sigma \supseteq \tau)$$

**Lemma 1.** $A$ and $A'$ are equivalent.

**Proof.** We define $\Phi: A \rightarrow A'$ as follows. $\Phi(\phi) = (u, \phi)$, where $u \subseteq \phi$ is such that $dom(u)$ is maximal with the following property: for any extension $\psi \supseteq \phi$ in $A$, $\psi|dom(u) = \phi|dom(u)$. We claim that $\Phi$ is a dense embedding.

Clearly $\psi \leq \phi$ implies $\Phi(\psi) \leq \Phi(\phi)$, and $\psi \perp \phi$ implies $\Phi(\psi) \perp \Phi(\phi)$. To check density, choose $(u, \phi) \in A'$. Let $i := dom(u) - 1$; and set $S_\phi := \{\sigma \in 2^i; \text{ for no } j \leq i \text{ does there exist } \tau \in u(j) \text{ such that } \sigma \supseteq \tau\}$. For $\sigma \in S_\phi$ we have $m_\sigma := \mu([\sigma] \setminus \{[\tau]; \tau \supseteq \sigma \land \exists i \geq dom(u) (\tau \in \phi(i))\}) > 0$. Let $a := \min\{m_\sigma; \sigma \in S_\phi\}$; and note that $\sum_{\sigma \in S_\phi} m_\sigma > \frac{1}{2}$. 4
Now define $\psi$ satisfying (*) such that

1) $\forall i \in \text{dom}(u) \ (\psi(i) = \phi(i))$

2) $\forall i \geq \text{dom}(u) \ \forall \tau_1 \in \phi(i) \ \exists j \leq i \ \exists \tau_2 \in \psi(j) \ (\tau_2 \subseteq \tau_1)$

3) $\frac{1}{2^i} \geq \mu(\bigcup\{\tau; \ \exists i \in \omega \ (\tau \in \psi(i))\}) > \frac{1}{2} - \frac{1}{2^n}$

4) for each $\sigma \in S_\phi$, $\mu(\{\sigma\} \setminus \bigcup\{\tau; \ \tau \supseteq \sigma \land \exists i \geq n \ (\tau \in \psi(i))\}) > \frac{1}{2}$

This is clearly possible. By construction we have $\Phi(\psi) = (u, \psi) \leq (u, \phi)$. □

Next define $A'' \subseteq A'$ by

$$(u, \phi) \in A'' \iff \left\{ \begin{array}{ll}
\text{for some } n \in \omega \text{ we have } & \mu(\bigcup\{\sigma; \ \exists i \in \text{dom}(u) \ (\sigma \in u(i))\}) > \frac{1}{2} - \frac{1}{2^n}, \\
\mu(\bigcup\{\sigma; \ \exists i \in \text{dom}(u) - 1 \ (\sigma \in u(i))\}) & < \frac{1}{2} - \frac{1}{2^n}, \\
\text{and } & \mu(\bigcup\{\sigma; \ \exists i \geq \text{dom}(u) \ (\sigma \in \phi(i))\}) < \frac{1}{2^n}.
\end{array} \right.$$ 

Clearly $A''$ is dense in $A'$. Finally we want to define $h : A'' \rightarrow \mathbb{C}$ giving rise to a complete embedding of $\mathbb{C}$ into $A$. To this end, let $f : \omega \rightarrow \omega$ be such that $\forall n \exists \infty i \ (f(i) = n)$. For $(u, \phi) \in A''$ and $n \in \omega$ such that $\frac{1}{2} - \frac{1}{2^n} \geq \mu(\bigcup\{\sigma; \ \exists i \in \text{dom}(u) \ (\sigma \in u(i))\}) > \frac{1}{2} - \frac{1}{2^n}$ and each $j \leq n$ choose $i_j$ minimal such that $\mu(\bigcup\{\sigma; \ \exists i \in j \ (\sigma \in u(i))\}) > \frac{1}{2} - \frac{1}{2^n}$, and let $h((u, \phi)) = \langle f(i_0) \rangle \ldots \langle f(i_n) \rangle$. We leave it to the reader to verify that $h : A'' \rightarrow \mathbb{C}$ is indeed a projection (in the forcing theoretic sense). Furthermore, given $T \subseteq A''$ uncountable we can find $T' \subseteq T$ uncountable and $u$ such that all elements of $T'$ are of the form $(u, \phi)$ for some $\phi$. Then there is an $s \in \mathbb{C}$ such that $\forall (u, \phi) \in T' \ (h((u, \phi)) = s)$. Next, given $\ell \in \omega$, we can find $F \subseteq T'$ of size $\ell$ such that $\cap F \in A''$. Clearly $h(\cap F) = s$ and so any extension of $s$ in $\mathbb{C}$ will be compatible with $\cap F$. Hence we have proved that $A''$ satisfies the requirements of Theorem 1. Using Lemma 1 we get

**Theorem 2.** There is a family $\{A_x; \ x \in \omega^\omega \cap V\}$ of Lebesgue measure zero sets in $V^A$ such that for all $z \in V^A$, $\{x \in \omega^\omega \cap V; \ z \notin A_x\}$ is at most countable. □

**Corollary 1.** Let $V \subseteq W$ be models of ZFC such that $\omega_1^V = \omega_1^W$. Then there is no real random in $W^A$ over $V^A$.

Proof. Let $\{A_x; \ x \in \omega^\omega \cap W\}$ be as in Theorem 2 and note that $\forall z \in \omega^\omega \cap W^A \ \exists x \in \omega^\omega \cap V \ (z \in A_x)$. Hence any real in $W^A$ lies in a measure zero set coded in $V^A$. □

Using a similar argument as in [CP, § 3] we can prove

**Corollary 2.** After adding one Amoeba real, $\text{cov}(\mathcal{L}) = \text{add}(\mathcal{L}) = \omega_1$ and $\text{unif}(\mathcal{L}) = \text{cof}(\mathcal{L}) = 2^\omega$. □
We note that in [BJS, § 2] results much stronger than Theorem 2 and the Corollaries were proved for the Hechler p.o. $\mathbb{D}$; e.g. it was shown that after adding a Hechler real, $\text{add}(\mathcal{B}) = \text{unif}(\mathcal{B}) = \omega_1$ and $\text{cof}(\mathcal{B}) = \text{cov}(\mathcal{B}) = 2^\omega$ [BJS, 2.5]. Accordingly we ask:

**Question** [BJS, 2.7.]. Is $\text{unif}(\mathcal{B}) = \omega_1$ and $\text{cov}(\mathcal{B}) = 2^\omega$ after adding an Amoeba real?

Before ending this section I wish to include a few comments, some of which are due to Andrzej Rosłanowski.

**Definition** (implicit in [Tr 2]). A p.o. $\mathbb{P}$ is said to have $(\omega_1, \omega)$–caliber iff for any uncountable $T \subseteq \mathbb{P}$ of size $\omega_1$ there is a countable $F \subseteq T$ such that $\cap F \in \mathbb{P}$.

This is equivalent to: any set of ordinals $A$ in $V^\mathbb{P}$ of size $\geq \omega_1$ has a countable subset $B$ in $V$ [Tr 2]. It is easy to see that if $\mathbb{C} \leq_c \mathbb{P}$ and $\mathbb{P}$ has $(\omega_1, \omega)$–caliber, then the assumptions of Theorem 1 are satisfied. Furthermore the Amoeba algebra $\mathbb{A}$ has $(\omega_1, \omega)$–caliber (the proof for this is similar to the corresponding proof for the random algebra $\mathbb{B}$, given in [Tr 2]). This gives an alternative argument to prove Theorem 2. — Our reason for giving the (slightly more difficult) above argument involving $\mathbb{A}'$ and $\mathbb{A}''$ is that along the same lines results corresponding to Theorem 2 and the Corollary can be proved for p.o.s not having $(\omega_1, \omega)$–caliber. We include two examples for such p.o.s:

— the eventually different reals p.o. $\mathbb{E}$, due to Miller [Mi]:

$$(s, G) \in \mathbb{E} \iff s \in \omega^{<\omega} \land G \in [\omega^\omega]^{<\omega}$$

$$(s, G) \leq (t, H) \iff s \supseteq t \land G \supseteq H \land \forall g \in H \forall i (\text{dom}(t) \leq i < \text{dom}(s) \rightarrow s(i) \neq g(i))$$

— the localization p.o. $\mathbb{L}$ (see, e.g., [Tr 3, § 2]):

$$(\sigma, G) \in \mathbb{L} \iff \sigma \in ([\omega]^{<\omega})^{<\omega} \land \forall i \in \text{dom}(\sigma) (|\sigma(i)| = i + 1) \land G \in [\omega^\omega]^{\leq \text{dom}(\sigma) + 1}$$

$$(\sigma, G) \leq (\tau, H) \iff \sigma \supseteq \tau \land G \supseteq H \land \forall g \in H \forall i (\text{dom}(\tau) \leq i < \text{dom}(\sigma) \rightarrow g(i) \in \sigma(i))$$

Let $\{f_\alpha; \alpha < \omega_1\} \subseteq \omega^{\omega}$ be a family of pairwise eventually different reals (i.e. $\alpha \neq \beta \rightarrow \exists n \forall k \geq n (f_\alpha(k) \neq f_\beta(k))$). Then $\{(\langle \rangle, \{f_\alpha\}); \alpha < \omega_1\}$ is an uncountable set of conditions in $\mathbb{E}$ (and $\mathbb{L}$) such that no countable subset has nontrivial intersection, thus witnessing that $\mathbb{E}$ and $\mathbb{L}$ do not have $(\omega_1, \omega)$–caliber. We leave it to the reader to verify that both still satisfy the assumptions of Theorem 1, however (note that both have a definition similar to, but easier than, $\mathbb{A}''$).

(The localization p.o. $\mathbb{L}$ arose from Bartoszyński’s characterization of the cardinal $\text{add}(\mathcal{L})$ [Ba], and is closely related to the Amoeba algebra $\mathbb{A}$. Truss [Tr 3, § 4] showed that
A \ll L. By the above discussion the converse cannot hold.)

§ 2. The projective part

We first introduce a notion closely related to absoluteness, and discuss the relationship between the two notions.

**Definition** (Judah [Ju, § 2]). Let $V$ be a universe of set theory. Given a forcing notion $\mathbb{P} \in V$ we say that $V$ is $\Sigma^1_n - \mathbb{P}$-correct iff for every $\Sigma^1_n$-formula $\phi(x)$ with parameters in $V$ and for every $\mathbb{P}$-name $\tau$ for a real we have $V[\tau] \models \phi(\tau)$ iff $V^\mathbb{P} \models \phi(\tau)$.

**Lemma 2.** Suppose $\mathbb{P} \ll \mathbb{Q}$. Then:

(i) $\Sigma^1_n - \mathbb{Q}$-correctness implies $\Sigma^1_n - \mathbb{P}$-correctness.

(ii) $\Sigma^1_{n+1} - \mathbb{Q}$-absoluteness + $\Sigma^1_n - \mathbb{P}$-correctness implies $\Sigma^1_{n+1} - \mathbb{P}$-absoluteness.

*Proof.* We prove both (i) and (ii) by induction on $n$.

(i) $n = 2$ follows from Shoenfield’s Absoluteness Lemma. Suppose it is true for $n \geq 2$ and assume $V$ is $\Sigma^1_{n+1} - \mathbb{Q}$-correct. Let $\phi(x)$ be a $\Sigma^1_{n+1}$-formula, $\phi(x) = \exists y \psi(y, x)$ where $\psi$ is $\Pi^1_n$. Suppose first that $V[\tau] \models \phi(\tau)$. Then $V[\tau] \models \exists x \psi(x, \tau)$. So there is a $\mathbb{P}$-name $\sigma$ such that $V[\tau] = V[\sigma, \tau] \models \psi(\sigma, \tau)$. By induction $V^\mathbb{P} \models \psi(\sigma, \tau)$; thus $V^\mathbb{P} \models \phi(\tau)$.

Assume now that $V^\mathbb{P} \models \phi(\tau)$. Hence $V^\mathbb{P} \models \exists x \psi(x, \tau)$; and we can again find a $\mathbb{P}$-name $\sigma$ such that $V^\mathbb{P} \models \psi(\sigma, \tau)$. By induction $V[\sigma, \tau] \models \psi(\sigma, \tau)$. So $\Sigma^1_n - \mathbb{Q}$-correctness implies $V^\mathbb{Q} \models \psi(\sigma, \tau)$; thus $V^\mathbb{Q} \models \phi(\tau)$. Hence by $\Sigma^1_{n+1} - \mathbb{Q}$-correctness $V[\tau] \models \phi(\tau)$.

(ii) $n = 1$ follows from Shoenfield’s Absoluteness Lemma. Suppose (ii) is true for $n \geq 1$ and assume $V$ is $\Sigma^1_{n+2} - \mathbb{Q}$-absolute and $\Sigma^1_{n+1} - \mathbb{Q}$-correct. By (i) $V$ is also $\Sigma^1_{n+1} - \mathbb{P}$-correct. Let $\phi$ be a $\Sigma^1_{n+2}$-sentence, $\phi = \exists x \psi(x)$, where $\psi$ is $\Pi^1_{n+1}$. Suppose first that $V \models \phi$; i.e. $V \models \psi(a)$ for some $a \in V$. By induction $V^\mathbb{P} \models \psi(a)$; thus $V^\mathbb{P} \models \phi$.

Assume now that $V^\mathbb{P} \models \phi$; i.e. $V^\mathbb{P} \models \psi(\tau)$ for some $\mathbb{P}$-name $\tau$. By $\Sigma^1_{n+1} - \mathbb{P}$-correctness $V[\tau] \models \psi(\tau)$. Hence $\Sigma^1_{n+1} - \mathbb{Q}$-correctness implies $V^\mathbb{Q} \models \phi$. Thus $V \models \phi$ by $\Sigma^1_{n+2} - \mathbb{Q}$-absoluteness.

**Lemma 3** (Truss [Tr 1, 6.5]). $\mathbb{D} \ll \mathbb{A}$.  □
**Definition** (Judah – Shelah [JS 1, §0]). A ccc notion of forcing \((\mathbb{P}, \leq)\) is called *Souslin* iff it can be thought of as a \(\Sigma^1_1\)–subset of the reals \(\mathbb{R}\) with both \(\leq\) and \(\perp\) (incompatibility) being \(\Sigma^1_1\)–relations (in the plane \(\mathbb{R}^2\)).

Note that all p.o.s discussed in this paper are Souslin.

**Theorem 3** (Judah [Ju, §2]). Assume that \(\forall a \in \mathbb{R} (\omega_1^{L[a]} < \omega_1^V)\), and \(\mathbb{P} \in V\) is a Souslin forcing. Then \(V\) is \(\Sigma^1_3 - \mathbb{P}\)-correct.

**Theorem 4.** \(\Sigma^1_4 - A\)-absoluteness implies that \(\forall a \in \mathbb{R} (\omega_1^{L[a]} < \omega_1^V)\).

**Corollary 3.** \(\Sigma^1_4 - A\)-absoluteness implies \(\Sigma^1_3 - A\)-correctness, \(\Sigma^1_4 - D\)-absoluteness, and \(\Sigma^1_3 - D\)-correctness.

**Theorem 5.** \(\Sigma^1_4 - A\)-absoluteness implies \(\Sigma^1_3 (L)\) and \(\Sigma^1_3 (B)\).

The proof of Theorem 4 follows the lines of the proof of 2.6 in [BJS]. Theorem 5 is a consequence of Theorem 4 and a result in [Ju, §2]. We give the proof here for completeness’ sake. — Note that \(\Sigma^1_3 - D\)-absoluteness is equivalent to \(\Sigma^1_2 (B)\) [Ju, §2]. Thus the implication \(\Sigma^1_3 - A\)-absoluteness \(\implies\) \(\Sigma^1_3 - D\)-absoluteness (immediate from Lemmata 2 and 3) is just another version of the Raisonnier–Stern Theorem; and Corollary 3 may be thought of as the corresponding result for \(\Sigma^1_4\).

**Proof of Theorem 4.** Suppose there is an \(a \in \mathbb{R}\) such that \(\omega_1^{L[a]} = \omega_1^V\). By \(\Sigma^1_3 - A\)-absoluteness we have \(\Sigma^1_2 (L)\); i.e. \(\forall b \in \mathbb{R} (\mu(Ra(L[b])) = 1)\) (see the beginning of this section). Note that \(x \in Ra(L[b])\) is equivalent to

\[
\forall c (c \notin L[b] \cap BC \vee \hat{c} \text{ is not null } \vee x \notin \hat{c}),
\]

where \(BC\) is the set of Borel codes which is \(\Pi^1_1\) [Je, Lemma 42.1], and for \(c \in BC, \hat{c}\) is the set coded by \(c\). As \(L[b]\) is \(\Sigma^1_2 [Je, Lemma 41.1], Ra(L[b])\) is a \(\Pi^1_2\)-set. Hence \(\forall b \in \mathbb{R} (\mu(Ra(L[b]))) = 1\) which is equivalent to

\[
\forall b \exists c (c \in BC \wedge \hat{c} \text{ is null } \wedge \forall x (x \in \hat{c} \vee x \in Ra(L[b])))
\]

is a \(\Pi^1_4\)-sentence. So it is true in \(V^A\) by \(\Sigma^1_4\)-absoluteness; in particular \(Ra(L[a][r])\) (where \(r\) is Amoeba over \(V\)) has measure one in \(V[r]\) which implies that there is a random real in \(V[r]\) over \(L[a][r]\), contradicting Corollary 1 in §1.

**Proof of Corollary 3.** Follows from Theorems 3 and 4 and Lemmata 2 and 3.

**Proof of Theorem 5** (Judah). Let \(\phi(x)\) be a \(\Sigma^1_3\)-formula and \(A = \{x; \phi(x)\}\). We shall show that \(A\) is measurable in \(V\). First note that the sentence \(A\) has measure zero is
equivalent to
\[ \exists c (c \in BC \land \mu(c) = 0 \land \forall x (\neg \phi(x) \lor x \in c)), \]
which is \(\Sigma^1_4\). So by \(\Sigma^1_4 - \Delta\)-absoluteness, if \(A\) is null in \(V^A\), it is also null in \(V\).

Hence assume that \(A\) is not null in \(V^A\). As \(\mu(Ra(V)) = 1\) in \(V^A\), there is \(r \in Ra(V) \cap A\) in \(V^A\); i.e. \(V^A \models \phi(r)\). By \(\Sigma^1_3 - \Delta\)-correctness \(V[r] \models \phi(r)\). Now let \(\phi(x) = \exists y \psi(x, y)\), where \(\psi\) is \(\Pi^1_2\). Then there is an \(s \in V[r]\) such that \(V[r] \models \psi(r, s)\). If \(a \in V\) codes the parameters of \(\psi\) and of the name of \(s\), we have by Shoenfield’s Absoluteness Lemma \(L[a][r] \models \psi(r, s)\). Let \(\check{r}\) be the \(\mathbb{B}\)-name for the random real \(r\) and \(s(\check{r})\) a \(\mathbb{B}\)-name for \(s\).

Then the Boolean value \(\|\psi(\check{r}, s(\check{r}))\|\) is non–zero. Furthermore, if \(r' \in \|\psi(\check{r}, s(\check{r}))\| \cap V\) is random over \(L[a]\), then \(L[a][r'] \models \psi(r', s(r'))\) and — by absoluteness — \(V \models \psi(r', s(r'))\); in particular \(V \models \phi(r')\).

By \(\Sigma^1_3 - \Delta\)-absoluteness we have that \(\mu(Ra(L[a])) = 1\) in \(V\) (cf Introduction). And the previous discussion gives us that \(Ra(L[a]) \cap \|\psi(\check{r}, s(\check{r}))\| \subseteq A\). This shows that any non–null \(\Sigma^1_3\)-set has positive inner measure; and it is easy to conclude from this that any \(\Sigma^1_3\)-set is indeed measurable.

Finally, \(\Sigma^1_3(B)\) follows along the same lines because \(\Delta\) adds a comeager set of Cohen reals. \(\square\)

**QUESTIONS.**

1) Does \(\Sigma^1_3(L)\) imply \(\Sigma^1_4 - \Delta\)-absoluteness?

2) Does \(\Sigma^1_4 - \text{Amoeba–meager–absoluteness (or } \Sigma^1_4 - \mathbb{D}\text{-absoluteness)}\) imply \(\Sigma^1_3(B)\)? (cf [Tr 1, § 5] for Amoeba–meager forcing — the problem here is whether \(\Sigma^1_4 - \text{Amoeba–meager–absoluteness}\) implies \(\forall a \in \mathbb{R} (\omega^L[a] < \omega^V_1)\); cf [BJS, § 2] for \(\mathbb{D}\) — the problem here is that \(\mathbb{D}\) does not add a comeager set of Cohen reals)

3) Does \(\forall n (V \text{ is } \Sigma^1_n - \Delta\text{-absolute})\) imply projective measurability?

4) (Judah) Does \(\Sigma^1_3(L)\) imply \(\Sigma^1_3(B)\)? (cf Corollary 3)

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