SINGULAR SUBELLIPTIC EQUATIONS AND SOBOLEV INEQUALITIES ON NILPOTENT LIE GROUPS

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Abstract. In this article we study singular subelliptic $p$-Laplace equations and best constants in Sobolev inequalities on nilpotent Lie groups. We prove solvability of these subelliptic $p$-Laplace equations and existence of the minimizer of the corresponding variational problem. It leads to existence of the best constant in the corresponding $(q,p)$-Sobolev inequality, $0 < q < 1$, $1 < p < \nu$.

1. Introduction

In this article, we investigate the following singular Dirichlet boundary value problem for the subelliptic $p$-Laplace equation

$$-\text{div}_H[(\nabla_H u)^{p-2}\nabla_H u] = f(x)u^{-\delta} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Omega$ is a bounded domain on a stratified nilpotent Lie group $G$ (Carnot group).

We assume that $1 < p < \nu$, $0 < \delta < 1$, where $\nu$ is the homogeneous dimension of $G$ and $f \in L^m(\Omega) \setminus \{0\}$ is nonnegative, where $m \geq 1$ to be made precise below. More precisely, we establish existence, uniqueness and boundedness of weak solutions (Definition 3.1) for the problem (1.1). Further, we observe that such solutions are associated to the following minimizing problem given by

$$\mu(\Omega) := \inf_{u \in W^{1,p}_0(\Omega)} \left\{ \int_\Omega |\nabla_H u|^p \, dx : \int_\Omega |u|^{1-\delta} f \, dx = 1 \right\}.$$

As a consequence, we obtain the following $(q,p)$-Sobolev inequality, $0 < q = 1 - \delta < 1$, $1 < p < \nu$:

$$S \left( \int_\Omega |u|^{1-\delta} f \, dx \right)^{\frac{1}{1-\delta}} \leq \left( \int_\Omega |\nabla_H u|^p \, dx \right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}_0(\Omega),$$

for some constant $S > 0$.

By singularity, we mean the nonlinearity on the right hand side of (1.1) blow up near the origin. Singular problems has been thoroughly studied over the last three decades and there is a colossal amount of literature in this concern. Most of these results are investigated in the Euclidean case and recently, some works has been done in the Riemannian manifolds as well.

To motivate our present study, let us state some known results related to our question. In the Euclidean case, Crandall-Rabinowitz-Tar tar [10] proved existence of a unique positive classical solution for the singular Laplace equation $-\Delta u = u^{-\delta}$, $\delta > 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ subject to the Dirichlet boundary

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condition \( u = 0 \) on \( \partial \Omega \). Lazer-McKenna [23] established that such solutions are weak solution in \( W^{1,2}_0(\Omega) \) if and only if \( 0 < \delta < 3 \). This restriction on \( \delta \) has been removed by Boccardo-Orsina [4] to investigate the existence of weak solutions in \( W^{1,2}_{loc}(\Omega) \). Indeed, they proved existence of weak solutions in \( W^{1,2}_0(\Omega) \) when \( 0 < \delta \leq 1 \) and in \( W^{1,2}_{loc}(\Omega) \) such that \( u^{\frac{1}{\delta-1}} \in W^{1,2}_0(\Omega) \) when \( \delta > 1 \). We would like to point out that for \( \delta > 1 \), the fact that \( u^{\frac{1}{\delta-1}} \in W^{1,2}_0(\Omega) \) is referred to as the Dirichlet boundary condition \( u = 0 \) on \( \partial \Omega \). We also refer to [2, 3] and the references therein in the semilinear context. Such results has been further extended to the singular \( p \)-Laplace equations by Canino-Sciummo-Trombetta [7], De Cave [13], Haitao [20], Giacomoni-Schindler-Takáč [19], Bal-Garain [4] and the references therein.

As far as we are aware, singular problems are very less understood in the non-Euclidean setting. In the Riemannian setting, some regularity results for singular semilinear problems is studied in do Ó and Clemente [14]. Wang-Wang [27] obtained existence results for the purely singular problem and symmetry properties of solutions to the perturbed singular problem in the Heinsenberg group for the semilinear case. Recently, in the quasilinear case, Garain-Kinnunen [18] established non-existence result for the singular \( p \)-Laplace equation in a general metric measure space which satisfies a doubling property and a Poincaré inequality.

Our main purpose in this article is to investigate singular problems in the subelliptic setting on nilpotent Lie groups. To this end, as mentioned at the beginning, we obtain existence, uniqueness and boundedness of weak solutions (see Theorem 3.3) for the problem (1.1). Moreover, we prove that such solutions are associated to the best constant \( \mu(\Omega) \) defined in (1.2) which also gives rise to the Sobolev type inequality (1.3) (see Theorem 3.4). In the Euclidean case, such minimizing problems has been studied in Anello-Faraci-Iannizzotto [1] for the local case, in the nonlocal case in Ercole-Pereira [15], for the weighted and anisotropic case in Bal-Garain [4]. These are based on the approximation technique introduced in Boccardo-Orsina [6]. We adopt this technique in the subelliptic setting to prove our main results. To the best of our knowledge, our main results are new even for the linear case \( p = 2 \).

This article is organized as follows: In Section 2, we mention some preliminary results in the subelliptic setting. In Section 3 and 4, we state our main results and obtain some preliminary results respectively. Finally, in Section 5, we prove our main results.

**Notation:** We write \( c \) or \( C \) to denote a constant which may vary from line to line or even in the same line. If \( c \) or \( C \) depends on the parameters \( r_1, r_2, \ldots, r_k \), we write \( c = c(r_1, r_2, \ldots, r_k) \) or \( C = C(r_1, r_2, \ldots, r_k) \) respectively. For \( a \in \mathbb{R} \), we denote by \( a^+ = \max\{a, 0\} \), \( a^- = \min\{-a, 0\} \). For given constants \( c, d \), a set \( S \) and a function \( u \), by \( c \leq u \leq d \) in \( S \), we mean \( c \leq u \leq d \) almost everywhere in \( S \).

2. Nilpotent Lie groups and Sobolev spaces

Recall that a stratified homogeneous group [17], or, in another terminology, a Carnot group [22] is a connected simply connected nilpotent Lie group \( G \) whose Lie algebra \( V \) is decomposed into the direct sum \( V_1 \oplus \cdots \oplus V_m \) of vector spaces such that \( \dim V_i \geq 2 \), \( [V_i, V_i] \subseteq V_{i+1} \) for \( 1 \leq i \leq m - 1 \) and \( [V_1, V_m] = \{0\} \). Let \( X_{11}, \ldots, X_{1n_1} \) be left-invariant basis vector fields of \( V_1 \). Since they generate \( V \), for each \( i, 1 < i \leq m \), one can choose a basis \( X_{ik} \) in \( V_i \), \( 1 \leq k \leq n_i = \dim V_i \), consisting of commutators of order \( i - 1 \) of fields \( X_{1k} \in V_1 \). We identify elements \( g \) of \( G \) with vectors \( x \in \mathbb{R}^N \), \( N = \sum_{i=1}^{m} n_i \), \( x = (x_{ik}), 1 \leq i \leq m, 1 \leq k \leq n_i \) by means of
calculated by the formula

\[ \delta_t x = (t^i x_{ik} 1 \leq i \leq m, 1 \leq k \leq n, \]

\[ = (tx_{11}, \ldots, tx_{1n}, t^2 x_{21}, \ldots, t^2 x_{2n}, \ldots, t^n x_{mn1}, \ldots, t^n x_{mn}) \]

are automorphisms of \( G \) for each \( t > 0 \). Lebesgue measure \( dx \) on \( \mathbb{R}^N \) is the bi-invariant Haar measure on \( G \) (which is generated by the Lebesgue measure by means of the exponential map), and \( d(\delta_t x) = t^n \ dx \), where the number \( \nu = \sum_{i=1}^m \ i n_i \) is called the homogeneous dimension of the group \( G \). The measure \( |E| \) of a measurable subset \( E \) of \( G \) is defined by \( |E| = \int_E \ dx \).

Recall that a continuous map \( \gamma : [a, b] \to G \) is called a continuous curve on \( G \). This continuous curve is rectifiable if

\[ \sup \left\{ \sum_{k=1}^m |\gamma(t_k)\gamma(t_k+1)|^{-1} \right\} < \infty, \]

where the supremum is taken over all partitions \( a = t_1 < t_2 < \ldots < t_m = b \) of the segment \([a, b]\). In [24] it was proved that any rectifiable curve is differentiable almost everywhere and \( \dot{\gamma}(t) \in V_1 \): there exists measurable functions \( a_i(t), t \in (a, b) \) such that

\[ \dot{\gamma}(t) = \sum_{i=1}^n a_i(t) X_i(\gamma(t)) \]

for almost all \( t \in (a, b) \). The length \( l(\gamma) \) of a rectifiable curve \( \gamma : [a, b] \to G \) can be calculated by the formula

\[ l(\gamma) = \int_a^b \left( \langle \gamma(t), \dot{\gamma}(t) \rangle_0^n \right)^{\frac{1}{2}} \ dt = \int_a^b \left( \sum_{i=1}^n |a_i(t)|^2 \right)^{\frac{1}{2}} \ dt, \]

where \( \langle \cdot, \cdot \rangle_0 \) is the inner product on \( V_1 \). The result of [24] implies that one can connect two arbitrary points \( x, y \in G \) by a rectifiable curve. The Carnot-Carathéodory distance \( d(x, y) \) is the infimum of the lengths over all rectifiable curves with endpoints \( x \) and \( y \) in \( G \). The Hausdorff dimension of the metric space \((G, d)\) coincides with the homogeneous dimension \( \nu \) of the group \( G \).

2.1. Sobolev spaces on Carnot groups. Let \( G \) be a Carnot group with one-parameter dilatation group \( \delta_t \), \( t > 0 \), and a homogeneous norm \( \rho \), and let \( E \) be a measurable subset of \( G \). The Lebesgue space \( L^p(E), \ p \in [1, \infty] \), is the space of \( p \)-th power integrable functions \( f : E \to \mathbb{R} \) with the standard norm:

\[ \|f\|_{L^p(E)} = \left( \int_E |f(x)|^p \ dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \]

and \( \|f\|_{L^\infty(E)} = \text{esssup}_E |f(x)| \) for \( p = \infty \). We denote by \( L^p_{\text{loc}}(E) \) the space of functions \( f : E \to \mathbb{R} \) such that \( f \in L^p(F) \) for each compact subset \( F \) of \( E \).

Let \( \Omega \) be an open set in \( G \). The (horizontal) Sobolev space \( W^{1,p}(\Omega), 1 \leq p \leq \infty \), consists of the functions \( f : \Omega \to \mathbb{R} \) which are locally integrable in \( \Omega \), having the weak derivatives \( X_{1i} f \) along the horizontal vector fields \( X_{1i}, i = 1, \ldots, n_1 \), and the finite norm

\[ \|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla_H f\|_{L^p(\Omega)}, \]
where $\nabla_{H} f = (X_{11} f, \ldots, X_{1n} f)$ is the horizontal subgradient of $f$. If $f \in W^{1,p}(U)$ for each bounded open set $U$ such that $U \subset \Omega$ then we say that $f$ belongs to the class $W^{1,p}_{bc}(\Omega)$.

For the rest of the article, we assume that $\Omega \subset \mathbb{R}$ is a bounded domain. The Sobolev space $W^{1,p}_{0}(\Omega)$ is defined to be the closure of $C^{\infty}_{c}(\Omega)$ under the norm

$$
\|f\|_{W^{1,p}_{0}(\Omega)} = \|f\|_{L^{p}(\Omega)} + \|\nabla_{H} f\|_{L^{p}(\Omega)}.
$$

For the following result, refer to [16, 25, 26, 28].

**Lemma 2.1.** The space $W^{1,p}_{0}(\Omega)$ is a real separable and uniformly convex Banach space.

The following embedding result follows from [12] (2.8) and [16, 21] Theorem 8.1, see also [8] Theorem 2.3.

**Lemma 2.2.** Let $1 < p < \nu$, then the space $W^{1,p}_{0}(\Omega)$ is continuously embedded in $L^{q}(\Omega)$ for every $1 \leq q \leq p^{*}$ where $p^{*} = \nu p/(\nu - p)$. Moreover, the embedding is compact for every $1 \leq q < p^{*}$.

Hence, for $1 < p < \nu$ we can consider the Sobolev space $W^{1,p}_{0}(\Omega)$ with the norm

$$
\|f\| := \|f\|_{W^{1,p}_{0}(\Omega)} = \|\nabla_{H} f\|_{L^{p}(\Omega)}.
$$

The following algebraic inequality from [11] Lemma 2.1 will be useful for us.

**Lemma 2.3.** Let $1 < p < \infty$. Then for any $a, b \in \mathbb{R}^{N}$, there exists a positive constant $C = C(p)$ such that

$$
\langle |a|^{p-2} a - |b|^{p-2} b, a - b \rangle \geq C(|a| + |b|)^{p-2}|a - b|^{2}.
$$

3. **Singular subelliptic $p$-Laplace equations**

The system of basis vectors $X_{1}, X_{2}, \ldots, X_{n}$ of the space $V_{1}$ (here and throughout we set $n_{1} = n$ and $X_{1i} = X_{i}$, where $i = 1, \ldots, n$) satisfies the Hörmander’s hypoellipticity condition. We study the singular problem in the geometry of the vector fields satisfying the Hörmander’s hypoellipticity condition.

**Definition 3.1.** (Weak solution) We say that $u \in W^{1,p}_{0}(\Omega)$ is a weak solution of (1.1) if $u > 0$ in $\Omega$ and for every $\omega \in \Omega$ there exists a positive constant $c(\omega) > 0$ such that $u \geq c(\omega) > 0$ in $\omega$ and for every $\phi \in C^{1}_{c}(\Omega)$, we have

$$
\int_{\Omega} |\nabla_{H} u|^{p-2} \nabla_{H} u \nabla_{H} \phi \, dx = \int_{\Omega} f(x) u^{-\delta} \phi \, dx.
$$

**Remark 3.2.** First, we claim that if $u \in W^{1,p}_{0}(\Omega)$ is a weak solution of (1.1), then (3.1) holds for every $\phi \in W^{1,p}_{0}(\Omega)$. Let $\psi \in W^{1,p}_{0}(\Omega)$, then there exists a sequence of nonnegative functions $\{\psi_{n}\}_{n \in \mathbb{N}} \subset C^{1}_{c}(\Omega)$ such that $0 \leq \psi_{n} \rightarrow |\psi|$ strongly in $W^{1,p}_{0}(\Omega)$ as $n \rightarrow \infty$ and pointwise almost everywhere in $\Omega$. We observe that

$$
\left| \int_{\Omega} f(x) u^{-\delta} \psi \, dx \right| \leq \int_{\Omega} f(x) u^{-\delta} |\psi| \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f(x) u^{-\delta} \psi_{n} \, dx
$$

$$
= \liminf_{n \rightarrow \infty} -\text{div}_{H}([\nabla_{H} u]^{p-2} \nabla_{H} u), \psi_{n})
$$

$$
\leq \|u\|^{p-1} \lim_{n \rightarrow \infty} \|\psi_{n}\| \leq \|u\|^{p-1} \|\psi\|.
$$
Let $\phi \in W^{1,p}_0(\Omega)$, then there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset C^1_0(\Omega)$, which converges to $\phi$ strongly in $W^{1,p}_0(\Omega)$. Choosing $\psi = \phi_n - \phi$ in (3.2), we obtain

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} f(x)u^{-\delta}(\phi_n - \phi) \, dx \leq \|u\|^{p-1} \lim_{n \to \infty} \|\phi_n - \phi\| = 0.
\end{equation}

Again, since $\phi_n \to \phi$ strongly in $W^{1,p}_0(\Omega)$ as $n \to \infty$, we have

\begin{equation}
\lim_{n \to \infty} \int_{\Omega} |\nabla_H u|^{p-2} \nabla_H u \nabla_H (\phi_n - \phi) \, dx = 0.
\end{equation}

Hence, using (3.3) and (3.4) in (3.1) the claim follows.

**Statement of the main results:** Our main results in this article are stated below. The first one is the following existence, uniqueness and regularity result.

**Theorem 3.3.** Let $0 < \delta < 1 < p < \nu$ and $f \in L^m(\Omega) \setminus \{0\}$ be nonnegative, where $m = \left(\frac{p-\nu}{\nu}\right)'$. Then the problem (1.1) admits a unique positive weak solution $u_\delta$ in $W^{1,p}_0(\Omega)$. Moreover, if $m > \frac{p^*}{p^* - n}$, then $u_\delta \in L^\infty(\Omega)$.

The second main result asserts that the Sobolev inequality (1.3) holds and its best constant $\mu(\Omega)$ defined above in (1.2) is associated to the problem (1.1).

**Theorem 3.4.** Let $0 < \delta < 1 < p < \nu$ and $f \in L^m(\Omega) \setminus \{0\}$ be nonnegative, where $m = \left(\frac{p-\nu}{\nu}\right)'$. Assume that $u_\delta \in W^{1,p}_0(\Omega)$ is given by Theorem 3.3. Then, we have

\begin{equation}
\mu(\Omega) := \inf_{u \in W^{1,p}_0(\Omega)} \left\{ \int_{\Omega} |\nabla_H u|^p \, dx : \int_{\Omega} |u|^{p-\delta} f \, dx = 1 \right\} = \left( \int_{\Omega} |\nabla_H u_\delta|^p \, dx \right)^{\frac{\nu-\delta}{\nu-p}}.
\end{equation}

Further, the following Sobolev inequality

\begin{equation}
S \left( \int_{\Omega} |v|^{p-\delta} f \, dx \right)^{\frac{\nu}{\nu-p}} \leq \int_{\Omega} |\nabla_H u|^p \, dx,
\end{equation}

holds for every $v \in W^{1,p}_0(\Omega)$, if and only if $S \leq \mu(\Omega)$.

4. **Preliminary results**

For $n \in \mathbb{N}$, we investigate the following approximated problem

\begin{equation}
- \text{div}_H(|\nabla_H u|^{p-2} \nabla_H u) = \frac{f_n}{(u^+ + \frac{1}{n})^\delta} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,
\end{equation}

where $f_n(x) = \min\{f(x), n\}$, for $f \in L^m(\Omega) \setminus \{0\}$ is nonnegative, where $m = \left(\frac{p-\nu}{\nu}\right)'$, provided $0 < \delta < 1 < p < \nu$ and $p^* = \frac{np}{n-p}$.

**Lemma 4.1.** Let $1 < p < \nu$ and $g \in L^\infty(\Omega) \setminus \{0\}$ be nonnegative in $\Omega$. Then there exists a unique solution $u \in W^{1,p}_0(\Omega)$ of the problem

\begin{equation}
- \text{div}_H(|\nabla_H u|^{p-2} \nabla_H u) = g \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega.
\end{equation}

Moreover, for every $\omega \subset \Omega$, there exists a constant $c(\omega)$, satisfying $u \geq c(\omega) > 0$ in $\omega$. 

Proof. Existence: We define the energy functional \( I : W^{1,p}_0(\Omega) \to \mathbb{R} \) by
\[
I(u) := \frac{1}{p} \int_{\Omega} |\nabla_H u|^p \, dx - \int_{\Omega} gu \, dx.
\]
By using that \( g \in L^\infty(\Omega) \) and Lemma 2.2 we have
\[
I(u) \geq \frac{\|u\|^p}{p} - C|\Omega|^{\frac{1}{p}} \|g\|_{L^\infty(\Omega)} \|u\|,
\]
where \( C > 0 \) is the Sobolev constant. Therefore, since \( p > 1 \), we have \( I \) is coercive. Moreover, \( I \) is weakly lower semicontinuous. As a consequence \( I \) has a minimizer, say \( u \in W^{1,p}_0(\Omega) \) which solves the equation
\[
- \text{div}_H(|\nabla_H u|^{p-2}\nabla_H u) = g \quad \text{in} \ \Omega.
\]
Positivity: Choosing \( u_- := \min\{u, 0\} \) as a test function in (4.4) and since \( g \geq 0 \), we obtain
\[
\int_{\Omega} |\nabla_H u_-|^p \, dx = \int_{\Omega} gu_- \, dx \leq 0,
\]
which gives, \( u \geq 0 \) in \( \Omega \). Further \( g \neq 0 \) gives \( u \neq 0 \) in \( \Omega \). Applying [25] Theorem 5] for every \( \omega \in \Omega \), there exists a constant \( c(\omega) \) such that \( u \geq c(\omega) > 0 \) in \( \Omega \). Thus \( u > 0 \) in \( \Omega \).

Uniqueness: Let \( u, v \in W^{1,p}_0(\Omega) \) solves the problem (4.4). Therefore,
\[
\int_{\Omega} |\nabla_H u|^{p-2}\nabla_H u \nabla_H \phi \, dx = \int_{\Omega} g\phi \, dx,
\]
and
\[
\int_{\Omega} |\nabla_H v|^{p-2}\nabla_H v \nabla_H \phi \, dx = \int_{\Omega} g\phi \, dx
\]
holds for every \( \phi \in W^{1,p}_0(\Omega) \). We choose \( \phi = (u-v)^+ \) and then subtracting (4.5) with (4.6), we obtain
\[
\int_{\Omega} \langle |\nabla_H u|^{p-2}\nabla_H u - |\nabla_H v|^{p-2}\nabla_H v, \nabla_H (u-v)^+ \rangle \, dx = 0.
\]
By Lemma 2.3 we get \( u \leq v \) in \( \Omega \). Similarly, we get \( v \leq u \) in \( \Omega \). Hence the uniqueness follows. \( \square \)

Lemma 4.2. For every \( n \in \mathbb{N} \), the problem (1.1) admits a solution \( u_n \in W^{1,p}_0(\Omega) \) which is positive in \( \Omega \). Moreover, \( u_n \) is unique and \( u_{n+1} \geq u_n \) in \( \Omega \) for every \( n \), and for every \( \omega \in \Omega \), there exists a constant \( c(\omega) \) (independent of \( n \)) such that \( u_n \geq c(\omega) > 0 \) in \( \omega \). Further, \( \|u_n\| \leq c \), for some positive constant \( c \), which is independent of \( n \).
Proof. **Existence:** Let \( n \in \mathbb{N} \), then for every \( h \in L^p(\Omega) \), there exists a unique \( u \in W^{1,p}_0(\Omega) \) such that

\[
- \text{div}(|\nabla u|^{p-2}\nabla u) = \frac{f_n}{(h^+ + \frac{1}{n})^\delta} \quad \text{in} \ \Omega.
\]

(4.8)

Therefore, we define \( T : L^p(\Omega) \to L^p(\Omega) \) by \( T(h) = I(u) = u \), where \( u \) solves (4.8) and \( I : W^{1,p}_0(\Omega) \to L^p(\Omega) \) is the continuous and compact inclusion mapping from Lemma 2.2. First we observe that \( T \) is continuous. Indeed, let \( \{h_k\}_{k \in \mathbb{N}} \subset L^p(\Omega) \) and \( h \in L^p(\Omega) \) be such that \( h_k \to h \) in \( L^p(\Omega) \). Suppose \( T(h_k) = u_k \) and \( T(h) = u \). Then we claim that \( u_k \to u \) in \( L^p(\Omega) \). By the definition of the mapping \( T \), for every \( \phi \in W^{1,p}_0(\Omega) \), we have

\[
\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla \phi \, dx = \int_\Omega \frac{f_n}{(h_k^+ + \frac{1}{n})^\delta} \phi \, dx
\]

(4.9)

and

\[
\int_\Omega |\nabla u|^{p-2}\nabla u \phi \, dx = \int_\Omega \frac{f_n}{(h^+ + \frac{1}{n})^\delta} \phi \, dx.
\]

(4.10)

Let

\[
h_{k,n} = ((h_k^+ + \frac{1}{n})^{-\delta} - (h^+ + \frac{1}{n})^{-\delta}).
\]

Then, choosing \( \phi = u_k - u \) in (4.9) and (4.10) and then subtracting the resulting equations and using Lemma 2.3 we get

\[
\int_\Omega (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u_k - \nabla u_k, \nabla (u_k - u)) \, dx
\]

(4.11)

\[
eq n \int_\Omega f_n (h_k^+ + \frac{1}{n})^{-\delta} - (h^+ + \frac{1}{n})^{-\delta}) (u_k - u) \, dx \leq n \int_\Omega |h_{k,n}| u_k - u | \, dx \leq n \int_\Omega |h_{k,n}| u_k - u | \, dx \leq C n \int_\Omega |h_{k,n}| u_k - u | \, dx,
\]

where \( C > 0 \) is the Sobolev constant. Thus, using Lemma 2.2 in (4.11) we have

\[
\|u_k - u\| \leq C n \|h_{k,n}\| \|u_k - u\|_{L^{p^*}(\Omega)},
\]

(4.12)

where \( t = p \) if \( p \geq 2 \) and \( t = 2 \) if \( p < 2 \). Note that up to a subsequence, \( h_{k,n} \to 0 \) as \( k \to \infty \) pointwise almost everywhere in \( \Omega \) and \( |h_{k,n}| \leq 2n^{\delta+1} \) and thus, by the Lebesgue dominated convergence theorem, from (1.12) up to a subsequence \( \|u_k - u\| \to 0 \) as \( k \to \infty \). Since the limit is independent of the choice of the subsequence, by Lemma 2.2 \( u_k \to u \) strongly in \( L^p(\Omega) \). Therefore, the continuity of \( T \) follows.

To prove that \( T \) is compact, choosing \( u \) as a test function in (4.8) and by Lemma 2.2 we get

\[
\|u\|^p \leq \int_\Omega n^{\delta+1} u \, dx \leq n^{\delta+1} |\Omega|^{\frac{p-1}{p}} C \|u\|,
\]

where \( C > 0 \) is the Sobolev constant. Thus,

\[
\|u\| \leq c,
\]

(4.13)

where \( c > 0 \) is a constant independent of \( h \). Let \( \{h_k\}_{k \in \mathbb{N}} \subset L^p(\Omega) \) be a bounded sequence, then by (4.13) we have

\[
\|T(h_k)\| \leq c,
\]
Lemma 4.4. Suppose that for every \( n \geq 2.2 \), we get \( u \). Uniform boundedness: Choosing \( u \). Uniqueness follows similarly. From (4.14), we know that \( \omega \) for some constant \( c \) positive constant \( n \). Noting this fact and using Lemma 2.3 in (4.15), it follows that for some positive constant \( c \). As a consequence of Lemma 4.2 we define the pointwise limit \( \omega \). Remark 4.3. Using the inequalities \( h \) of the embedding \( h \) compact. Let us set \( S := \{ h \in L^p(\Omega) : \lambda^2T(h) = h, \quad 0 \leq \lambda \leq 1 \} \). Then for any \( h_1, h_2 \in S \), by (4.13) we have
\[
\| h_1 - h_2 \|_{L^p(\Omega)} = \lambda \| T(h_1) - T(h_2) \|_{L^p(\Omega)} \leq c,
\]
for some constant \( c > 0 \), which is independent of \( h_1, h_2 \). Thus, by Schauder’s fixed point theorem, there exists a fixed point \( u_n \in W^{1,p}(\Omega) \) such that \( T(u_n) = u_n \). As a consequence, \( u_n \) solves the problem (4.1). Moreover, by Lemma 4.1, we have \( u_n > 0 \) in \( \Omega \) and for every \( \omega \in \Omega \), there exists a constant \( c(\omega) \) such that \( (4.14) \)
\[
u_1 \geq c(\omega) > 0 \text{ in } \omega.
\]
Monotonicity and uniqueness: Choosing \( \phi = (u_n - u_{n+1})^+ \) as a test function in (4.11) we have
\[
J = \langle \nabla_H u_n \nabla^2 \nabla_H u_n - |\nabla_H u_{n+1}|^{p-2} \nabla_H u_{n+1}, \nabla (u_n - u_{n+1})^+ \rangle
\]
(4.15)
\[
= \int_{\Omega} \left\{ \frac{f_n}{(u_n + \frac{1}{n})^\delta} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} (u_n - u_{n+1})^+ \, dx.
\]
Using the inequalities \( f_n(x) \leq f_{n+1}(x) \), we obtain
\[
J \leq \int_{\Omega} f_{n+1} \left\{ \frac{1}{(u_n + \frac{1}{n})^\delta} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^\delta} \right\} (u_n - u_{n+1})^+ \, dx \leq 0.
\]
Noting this fact and using Lemma 2.3 in (4.14), it follows that \( u_{n+1} \geq u_n \) in \( \Omega \). Uniqueness follows similarly. From (4.14), we know that \( u_1 \geq c(\omega) > 0 \) for every \( \omega \in \Omega \). Hence using the monotonicity, for every \( \omega \in \Omega \), we get \( u_n \geq c(\omega) > 0 \) in \( \omega \), for some positive constant \( c(\omega) \) (independent of \( n \)).
Uniform boundedness: Choosing \( u_n \) as a test function in (4.11) and by Lemma 2.3 we get
\[
(4.16) \quad \| u_n \|_p \leq \int_{\Omega} f_n^{1-\delta} u_n \, dx \leq \| f \|_{L^m(\Omega)} \left( \int_{\Omega} u_n^{(1-\delta)\eta} \, dx \right)^{\frac{1}{\eta}} = \| f \|_{L^m(\Omega)} \left( \int_{\Omega} u_n^{p \eta} \, dx \right)^{\frac{1}{\eta}} \leq c \| f \|_{L^m(\Omega)} \| u_n \|^{1-\delta},
\]
for some constant \( c > 0 \), independent of \( n \). Therefore, we have \( \| u_n \| \leq c \), for some positive constant \( c \) (independent of \( n \)).

Remark 4.3. As a consequence of Lemma 4.2 we define the pointwise limit \( u_\delta \) of \( u_n \) in \( \Omega \). Note that by the monotonicity of \( u_n \) in Lemma 4.3, we have \( u_\delta \geq u_n \) in \( \Omega \) for every \( n \in \mathbb{N} \).

Lemma 4.4. Suppose that \( f \in L^q(\Omega) \setminus \{ 0 \} \) is nonnegative, where \( q > \frac{p-2}{p-2-q} \). Then \( \| u_n \|_{L^\infty(\Omega)} \leq C \), for some positive constant \( C \) independent of \( n \), where \( \{ u_n \}_{n \in \mathbb{N}} \) is the sequence of solutions of the problem (4.1) given by Lemma 4.2.
Proof. For \( k \geq 1 \), we define \( A(k) = \{ x \in \Omega : u_n(x) \geq k \} \). Choosing \( \phi_k(x) = (u_n - k)^+ \) as a test function in \( \mathbb{L}^1(\Omega) \), first using Hölder’s inequality with the exponents \( p^*, p' \) and then, by Young’s inequality with exponents \( p \) and \( p' \), we obtain

\[
\|\phi_k\|^p = \int_{\Omega} \frac{f_n}{\left(u_n + \frac{1}{k}\right)^q} \phi_k \, dx \leq \int_{A(k)} f(x) \phi_k \, dx \leq \left( \int_{A(k)} f^{p^*} \, dx \right)^{\frac{1}{p^*}} \left( \int \phi_k^{p'} \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{A(k)} f^{p^*} \, dx \right)^{\frac{1}{p^*}} \|\phi_k\| \leq C \left( \int_{A(k)} f^q \, dx \right)^{\frac{1}{q}} \|\phi_k\| \leq \epsilon\|\phi_k\|^p + C(\epsilon) \left( \int_{A(k)} f^{p^*} \, dx \right)^{\frac{1}{p^*}},
\]

where \( C \) is the Sobolev constant and \( C(\epsilon) > 0 \) is some constant depending on \( \epsilon \in (0, 1) \) but independent of \( n \). Note that \( q > \frac{p^*}{p - p^*} \) gives \( q > p^* \). Therefore, fixing \( \epsilon \in (0, 1) \) and again using Hölder’s inequality with exponents \( \frac{1}{r} \) and \( \left( \frac{p}{p^*} \right)' \), for some constant \( C > 0 \) which is independent of \( n \), we obtain

\[
\|\phi_k\|^p \leq C \left( \int_{A(k)} f^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C \left( \int_{\Omega} f^q \, dx \right)^{\frac{1}{q}} \|A(k)\| \leq C|A(k)| \left( \frac{1}{p^*} \right),
\]

(4.18)

Let \( h > 0 \) be such that \( 1 \leq k < h \). Then, \( A(h) \subset A(k) \) and for any \( x \in A(h) \), we have \( u_n(x) \geq h \). So, \( u_n(x) - k \geq h - k \) in \( A(h) \). Combining these facts along with \( (4.15) \) and again using Lemma \( 2.2 \) for some constant \( C > 0 \) (independent of \( n \)), we arrive at

\[
(h - k)^p|A(h)|^{\frac{1}{p^*}} \leq \left( \int_{A(h)} (u_n - k)^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq \left( \int_{A(h)} (u_n - k)^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C\|\phi_k\|^p \leq C|A(k)|^{\frac{1}{p^*}} \left( \frac{1}{p^*} \right).
\]

Thus, for some constant \( C > 0 \) (independent of \( n \)), we have

\[
|A(h)| \leq \frac{C}{(h - k)^p} |A(k)|^\alpha,
\]

where

\[
\alpha = \frac{p^* p'}{p p} \left( \frac{1}{p^*} \right)'.
\]

Due to the assumption, \( q > \frac{p^*}{p - p^*} \), we have \( \alpha > 1 \). Hence, by \( 22 \) Lemma B.1], we have \( \|u_n\|_{L^\infty(\Omega)} \leq C \), for some positive constant \( C > 0 \) independent of \( n \). \( \square \)

**Lemma 4.5.** Let \( \{u_n\}_{n \in \mathbb{N}} \) be the sequence of solutions of the problem \( 1.1 \) given by Lemma \( 4.2 \). Then, for every \( n \in \mathbb{N} \), we have

\[
\|u_n\|^p \leq \|\phi\|^p + p \int_{\Omega} \frac{(u_n - \phi)}{(u_n + \frac{1}{n})^q} \, dx, \quad \forall \phi \in W_0^1 p(\Omega).
\]

(4.19)
Moreover, for every $n \in \mathbb{N}$, we have
\begin{equation}
\|u_n\| \leq \|u_{n+1}\|.
\end{equation}

**Proof.** By Lemma 4.1, for any $h \in W_0^{1,p}(\Omega)$, there exists a unique solution $v \in W_0^{1,p}(\Omega, w)$ of the problem
\begin{equation}
-\text{div}(|\nabla u|^p - 2\nabla v) = \frac{f_n(x)}{(h^+ + \frac{1}{n}t)^p}, \quad v > 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega.
\end{equation}

Note that $v$ is also a minimizer of the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
\[ J(\phi) := \frac{1}{p} \|\phi\|^p - \int_{\Omega} \frac{f_n}{(h^+ + \frac{1}{n}t)^p} \phi \, dx. \]

Thus, $J(v) \leq J(\phi)$, for every $\phi \in W_0^{1,p}(\Omega)$ and we obtain
\begin{equation}
\frac{1}{p} \|v\|^p - \int_{\Omega} \frac{f_n}{(h^+ + \frac{1}{n}t)^p} v \, dx \leq \frac{1}{p} \|\phi\|^p - \int_{\Omega} \frac{f_n}{(h^+ + \frac{1}{n}t)^p} \phi \, dx.
\end{equation}

Setting $v = h = u_n$ (which is positive in $\Omega$) in (4.22), the estimate (4.19) follows.

By Lemma 4.2, we know $u_n \leq u_{n+1}$ for every $n \in \mathbb{N}$. Thus, choosing $\phi = u_{n+1}$ in (4.19) we obtain $\|u_n\| \leq \|u_{n+1}\|$. Hence the result follows.

**Lemma 4.6.** Let \( \{u_n\}_{n \in \mathbb{N}} \) be the sequence of solutions of the problem (4.1) given by Lemma 4.4 and suppose \( u_\delta \) is the pointwise limit of \( u_n \) given by Remark 4.3.

Then, up to a subsequence
\begin{equation}
\lim_{n \to \infty} u_n \to u_\delta \text{ strongly in } W_0^{1,p}(\Omega).
\end{equation}

Moreover, $u_\delta$ is a minimizer of the energy functional $J_\delta : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
\begin{equation}
J_\delta(v) := \frac{1}{p} \|v\|^p - \frac{1}{1-\delta} \int_{\Omega} (v^+)^{1-\delta} f \, dx.
\end{equation}

**Proof.** By Remark 4.3 we know that $u_n \leq u_\delta$ in $\Omega$ for every $n \in \mathbb{N}$. Thus, by choosing $\phi = u_\delta$ in (4.19) we get $\|u_n\| \leq \|u_\delta\|$ for every $n \in \mathbb{N}$. Hence, using the norm monotonicity $\|u_n\| \leq \|u_{n+1}\|$ from (4.20), we have
\begin{equation}
\lim_{n \to \infty} \|u_n\| \leq \|u_\delta\|.
\end{equation}

Moreover, since $u_n \rightharpoonup u_\delta$ weakly in $W_0^{1,p}(\Omega)$, we get
\begin{equation}
\|u_\delta\| \leq \liminf_{n \to \infty} \|u_n\|.
\end{equation}

Thus from (4.25) and (4.26) along with the uniform convexity of $W_0^{1,p}(\Omega)$ from Lemma 2.1 the convergence in (4.23) follows.

To prove the second result, it is enough to show that, for all $v \in W_0^{1,p}(\Omega)$, we have
\begin{equation}
J_\delta(u_\delta) \leq J_\delta(v).
\end{equation}

To this end, for any $n \in \mathbb{N}$ and recalling that $f_n(x) = \min\{f(x), n\}$, we define $J_n : W_0^{1,p}(\Omega) \to \mathbb{R}$ by
\[ J_n(v) := \frac{1}{p} \|v\|^p - \int_{\Omega} H_n(v) f_n \, dx, \]

where
\[ H_n(t) := \frac{1}{1-\delta} \left( t^+ + \frac{1}{n} \right)^{1-\delta} - \left( \frac{1}{n} \right)^{-\delta} t^- . \]
Then we observe that \( J_n \) is \( C^1 \), bounded below and coercive, and therefore, \( J_n \) has a minimizer, say \( v_n \in W^{1,p}_0(\Omega) \). This gives, \( J_n(v_n) \leq J_n(v_n^+ \phi) \), from which it follows that \( v_n \geq 0 \) in \( \Omega \). Noting that \( \langle J_n(v_n), \phi \rangle = 0 \) for all \( \phi \in W^{1,p}_0(\Omega) \), we conclude that \( v_n \) solves (4.1). By the uniqueness result from Lemma 4.2 we get \( u_n = v_n \) in \( \Omega \). Hence \( u_n \) is a minimizer of \( J_n \) and we have
\[
J_n(u_n) \leq J_n(v^+), \quad \forall v \in W^{1,p}_0(\Omega).
\]
Below we pass to the limit as \( n \to \infty \) in (4.28) to prove the claim (4.27). Indeed, by Remark 4.3 we know that \( u_n \leq u_\delta \) in \( \Omega \), which along with the Lebesgue dominated convergence theorem gives us
\[
\lim_{n \to \infty} \int_\Omega H_n(u_n) f_n dx = \frac{1}{1 - \delta} \int_\Omega (u_\delta)^{1-\delta} f dx.
\]
Also, by the strong convergence from (4.23), we have
\[
\lim_{n \to \infty} \|u_n\| = \|u_\delta\|.
\]
Hence, by (4.29) and (4.30), we have
\[
\lim_{n \to \infty} J_n(u_n) = J_\delta(u_\delta).
\]
Again,
\[
\lim_{n \to \infty} \int_\Omega H_n(v^+) f_n dx = \frac{1}{1 - \delta} \int_\Omega (v^+)^{1-\delta} f dx,
\]
for any \( v \in W^{1,p}_0(\Omega) \). Now, letting \( n \to \infty \) in (4.28) and then using the estimates (4.31), (4.32) along with \( \|v^+\| \leq \|v\| \), the inequality (4.27) holds. Hence the result follows.

5. Proof of the main results

Proof of Theorem 3.3

**Uniqueness:** Suppose \( u, v \in W^{1,p}_0(\Omega) \) are weak solutions of (1.1). Then by Remark 3.2 choosing \( \phi = (u - v)^+ \in W^{1,p}_0(\Omega) \) as a test function in (3.1) we have
\[
\int_\Omega |\nabla H u|^{p-2} \nabla H u \nabla H (u - v)^+ dx = \int_\Omega f u^{-\delta} (u - v)^+ dx,
\]
\[
\int_\Omega |\nabla H u|^{p-2} \nabla H u \nabla H (u - v)^+ dx = \int_\Omega f v^{-\delta} (u - v)^+ dx.
\]
Subtracting (5.1) and (5.2), we have
\[
\int_\Omega (|\nabla H u|^{p-2} \nabla H u - |\nabla H v|^{p-2} \nabla H v, \nabla H (u - v)^+) dx = \int_\Omega f (u^{-\delta} - v^{-\delta}) (u - v)^+ dx \leq 0.
\]
Therefore, by Lemma 2.3 we obtain \( u \leq v \) in \( \Omega \). In a similar way, we have \( v \leq u \) in \( \Omega \). Hence the result follows.

**Existence:** For every \( n \in \mathbb{N} \), using Lemma 1.2 there exists \( u_n \in W^{1,p}_0(\Omega) \) such that
\[
\int_\Omega |\nabla H u_n|^{p-2} \nabla H u_n \nabla H \phi dx = \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\delta} \phi dx, \quad \forall \phi \in C_c^1(\Omega).
\]
Limit pass: By the strong convergence (4.23) in Lemma 4.1 up to a subsequence, we have \( \nabla_H u_n \to \nabla_H u_\delta \) pointwise almost everywhere in \( \Omega \) as \( n \to \infty \). Hence, for every \( \phi \in C^1_c(\Omega) \), we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla_H u_n|^{p-2} \nabla_H u_n \nabla_H \phi \, dx = \int_{\Omega} |\nabla_H u_\delta|^{p-2} \nabla_H u_\delta \nabla_H \phi \, dx.
\]

Let \( \text{supp} \, \phi = \omega \subseteq \Omega \) and so by Lemma 1.2 there exists a constant \( c(\omega) > 0 \), which is independent of \( n \) such that \( u_n \geq c(\omega) > 0 \) in \( \omega \). Thus, \( u_\delta \geq c(\omega) > 0 \) in \( \omega \) and also we have

\[
\frac{f_n}{u_n^p} \phi \leq \frac{f}{c(\omega)^p} ||\phi||_{L^\infty(\Omega)} \in L^1(\Omega).
\]

By Remark 1.3 using the pointwise convergence \( u_n \to u_\delta \) almost everywhere in \( \Omega \) and the Lebesgue dominated convergence theorem, we have

\[
\lim_{n \to \infty} \int_{\Omega} \frac{f_n}{(u_n + \delta)^p} \phi \, dx = \int_{\Omega} \frac{f}{u_\delta^p} \phi \, dx.
\]

Combining the estimates (5.4) and (5.5), we obtain \( u_\delta \) is a weak solution of (1.1).

Boundedness: Using Lemma 1.3 we have \( u_\delta \in L^\infty(\Omega) \). □

Proof of Theorem 3.4: First, we prove (3.5). Let us set

\[
J(\delta) := \min_{u \in S_\delta} \{ \int_{\Omega} |u|^{1-\delta} f \, dx = 1 \}.
\]

Thus it is enough to obtain

\[
\mu(\Omega) := \inf_{v \in S_\delta} ||v||^p = ||u_\delta||^{\frac{p(1-\delta)}{1-\frac{p}{\delta}}}
\]

We observe that \( U_\delta = \theta_\delta u_\delta \in S_\delta \), where

\[
\theta_\delta = \left( \int_{\Omega} u_\delta^{1-\delta} f \, dx \right)^{-\frac{1}{1-\frac{p}{\delta}}}.
\]

By Remark 3.2 choosing \( \phi = u_\delta \in W_0^{1,p}(\Omega) \) as a test function in (3.1), we have

\[
\int_{\Omega} |\nabla_H u_\delta|^p \, dx = ||u_\delta||^p = \int_{\Omega} u_\delta^{1-\delta} f \, dx.
\]

Therefore, we have

\[
||U_\delta||^p = \int_{\Omega} |\nabla_H U_\delta|^p \, dx = \theta_\delta^p \int_{\Omega} |\nabla_H u_\delta|^p \, dx = ||u_\delta||^{\frac{p(1-\delta)}{1-\frac{p}{\delta}}}
\]

Let \( v \in S_\delta \) and define by \( \mu = ||v||^{-\frac{p}{1-\frac{p}{\delta}}} \). Then by Lemma 4.5 since \( u_\delta \) is a minimizer of the functional \( J_\delta \) given by (4.24), we have

\[
J_\delta(u_\delta) \leq J_\delta(\mu |v|).
\]

Using (5.8), we have

\[
J_\delta(u_\delta) = \frac{1}{p} ||u_\delta||^p - \frac{1}{1-\delta} \int_{\Omega} u_\delta^{1-\delta} f \, dx = \left( \frac{1}{p} - \frac{1}{1-\delta} \right) ||u_\delta||^p.
\]

Again, since \( v \in S_\delta \), we have

\[
J_\delta(\mu |v|) = \frac{\mu^p}{p} |||v|||^p - \frac{\mu^{1-\delta}}{1-\delta} \leq \frac{\mu^p}{p} ||v||^p - \frac{\mu^{1-\delta}}{1-\delta} = \left( \frac{1}{p} - \frac{1}{1-\delta} \right) ||v||^{\frac{p(p-1)}{p+1}}.
\]
Since $v \in S_\delta$ is arbitrary, using (5.9) and (5.10) in (5.8), we obtain
\begin{equation}
\|u_\delta\|^{p(1-\delta-p)/(1-\delta)} \leq \inf_{v \in S_\delta} \|v\|^p.
\end{equation}
Using (5.7) and (5.11), we obtain
\begin{equation}
\|U_\delta\|^p = \|u_\delta\|^{p(1-\delta-p)/(1-\delta)} \leq \inf_{v \in S_\delta} \|v\|^p.
\end{equation}
Since $U_\delta \in S_\delta$, from (5.12), we obtain (3.5).

To prove the second part, let (3.6) holds. If $S > \mu(\Omega)$, then from (3.5) and (5.7) above, we obtain
\begin{equation}
S \left(\int_\Omega |U_\delta|^{1-\delta} f \, dx \right)^{\frac{1}{1-\delta}} > \int_\Omega |\nabla H U_\delta|^p \, dx.
\end{equation}
Since $U_\delta \in W^{1,p}_0(\Omega)$, (5.13) violates the hypothesis (3.6). Conversely, let
\[ S \leq \mu(\Omega) = \inf_{v \in S_\delta} \|v\|^p \leq \|U\|^p, \]
for all $U \in S_\delta$. We observe that the claim directly follows if $v = 0$. So we consider the case of $v \in W^{1,p}_0(\Omega, w) \setminus \{0\}$. This gives
\[ U = \left(\int_\Omega |v|^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}} v \in S_\delta. \]
Therefore, we have
\[ S \leq \left(\int_\Omega |v|^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}} \|v\|^p. \]
Hence, the result follows. \qed

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