DIRAC REDUCTIONS AND CLASSICAL W-ALGEBRAS

GAHNG SAHN LEE, ARIM SONG, UHI RINN SUH

Abstract. In the first part of this paper, we generalize Dirac reduction to the extent of non-local Poisson vertex superalgebra and non-local SUSY Poisson vertex algebra cases. Next, we modify this reduction so that we explain the structures of classical W-superalgebras and SUSY classical W-algebras in terms of the modified Dirac reduction.

1. Introduction

Dirac reductions of a Poisson algebra were first introduced in the paper [17] of Dirac. For a Poisson algebra \((P, \{, \})\) and a finite subset \(\{\theta_i | i \in I\} \subset P\), if the matrix \(C_{ij} = \{\theta_i, \theta_j\}\) is invertible, then

\[
\{a, b\}^D := \{a, b\} - \sum_{i \in I} \{a, \theta_i\}(C^{-1})_{ij}\{\theta_j, b\}, \quad a, b \in P
\]

(1.1)

defines a new Poisson bracket called Dirac reduced bracket. Since this new bracket satisfies \(\{\theta_i, P\}^D = 0\) for any \(i \in I\), it induces a Poisson bracket on \(P/I_P\), where \(I_P\) is the associative algebra ideal generated by \(\{\theta_i | i \in I\}\). Moreover, in [15], De Sole-Kac-Valeri generalized the notion to Poisson vertex algebras (PVAs) cases. The Dirac reduced bracket of a Poisson vertex algebra \((P, \{, \})\) associated with the finite subset \(\{\theta_i | i \in I\} \subset P\) is defined by

\[
\{a, b\}^D = \{a, b\} - \sum_{i,j \in I} \{\theta_j, \lambda \theta_i\} (C^{-1})_{ji}(\lambda + \partial)\{a, \lambda \theta_i\}, \quad a, b \in P,
\]

(1.2)

provided that the matrix \(C(\lambda) = ((\{\theta_j, \lambda \theta_i\}))_{i,j \in I}\) is invertible with respect to the product \(A(\lambda) \circ B(\lambda) = A(\lambda + \partial)B(\lambda)\) and \((C^{-1})(\lambda)\) is the inverse of \(C(\lambda)\). Similarly to the Poisson algebra case, the bracket \(\{, \}^D\) induces a Poisson vertex algebra bracket on \(P/I_P\), where \(I_P\) is the differential algebra ideal generated by \(\{\theta_i | i \in I\}\).

Integrable Hamiltonian systems have been studied in the framework of PVAs by the authors in [4, 12, 14] and Dirac reductions are useful to obtain new systems [10]. For example, the bi-Poisson structure of the Gelfand-Dickey hierarchy associated with the Lax operator \(L = \partial^n + u_1\partial^{n-1} + \cdots + u_n\) can be understood as a Dirac reduction of the Poisson structure induced from \(\tilde{L} = \partial^n + u_1\partial^{n-1} + u_2\partial^{n-2} + \cdots + u_n\) (see [1, 21, 20]). Moreover, if we admit non-local Poisson structures for Hamiltonian equations, more systems can be understood by Dirac reductions. See [10] for detailed explanations and examples.

The third author described in [34] integrable Hamiltonian systems associated with a certain family of Poisson vertex superalgebras (PVSA). Hence a natural question is whether we can get reduced systems via super-analogue of non-local PVAs and Dirac reductions. All the notions and properties for non-local PVAs in [10] can be generalized to the super-analogue by Koszul-Quillen sign rule. On the other hand, we introduce in Definition 2.6 the Dirac reduction for PVSA \(P\).
The main idea is that we consider $C(\lambda) = (\{\theta_j \lambda \theta_i\})_{i,j \in I}$ as an element in $\mathfrak{g} \{m|n\} \otimes \mathcal{P}((\lambda^{-1}))$ where $m$ and $n$ are the number of even and odd indices in $I$. If $C(\lambda)$ is invertible via the multiplication in (2.7), then the Dirac reduction of the Poisson $\lambda$-bracket \{ $\lambda$ \} on $\mathcal{P}$ associated with $\theta_I = \{ \theta_i | i \in I \}$ is given by
\[
\{a_\lambda b\}^D = \{a_\lambda b\} - \sum_{i,j \in I} (-1)^{p(a)+p(j)}(p(b)+p(i)+p(i)+p(j)) \{\theta_j \lambda \partial b\} \partial (C^{-1})_{ji}(\partial + D) \{a_\lambda \theta_i\},
\]
where $a, b$ are homogeneous elements in $\mathcal{P}$ and $p(i) = p(\theta_i)$. In Theorem 2.9 and Theorem 2.10, we show the following statements.

**Theorem 1.1.** Let $\{\theta_I\}$ be the differential superalgebra ideal of $\mathcal{P}$ generated by $\theta_I$.

1. The bracket $\{ \lambda \}^D$ in (1.3) gives another PVSA structure on $\mathcal{P}$. In other words, it satisfies the sesquilinearity, skewsymmetry, Jacobi identity and Leibniz rule for PVSA.
2. The Dirac reduced bracket induces a PVSA $\lambda$-bracket on $\mathcal{P}/\{\theta_I\}$.

Furthermore, we also consider supersymmetric (SUSY) PVAs, which are PVAS with an odd derivation $D$. SUSY vertex algebras and SUSY PVAs introduced in [23, 22] to provide algebraic frameworks for SUSY conformal field theory. Moreover, some superintegrable Hamiltonian systems can be understood in terms of SUSY PVAs. See [6], for instance. The simplest example of superintegrable systems described by a SUSY PVA is the super-KdV equation:
\[
\frac{du}{dt} = D_6^2 u + 3DD^2 u + 3uD^3 u = \{u u \lambda u\}|_{\lambda = \lambda = 0},
\]
where $\{u_\lambda u\} = (D^2 + \frac{1}{2}(\lambda + \frac{1}{2}D)) u - \lambda^2 u$. In spite of the fact that the second Poisson structure of the super-KdV has a non-local property, non-local super Hamiltonian operators have not been studied in the theory of SUSY PVAs. See [27, 29, 31] for the SUSY bi-Poisson structure of the super KdV.

As the first step to understand non-local super Hamiltonians, we introduce the notion of non-local SUSY PVAS in Definition 3.1. Simply speaking, a non-local SUSY PVA $\mathcal{P}$ is a $\mathbb{C}[D]$-module endowed with a non-local $\Lambda$-bracket
\[
\{\lambda\} : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}[\chi](\{\lambda^{-1}\}), \tag{1.4}
\]
which induces an admissible non-local Poisson $\Lambda$-bracket and satisfies the skewsymmetry, Jacobi identity and Leibniz rule. The SUSY Dirac reduced $\Lambda$-bracket on $\mathcal{P}$ associated with the finite subset $\theta_I$ of $\mathcal{P}$ is defined as follow. Assuming the invertibility of $C(\lambda) := (\{\theta_j \lambda \theta_i\})_{i,j \in I}$, the Dirac reduced bracket of $\mathcal{P}$ associated with $\theta_I$ is given by
\[
\{a_\lambda b\}^D = \{a_\lambda b\} - \sum_{i,j \in I} (-1)^{p(a)+p(j)}(p(b)+p(i)+p(i)+p(j)) \{\theta_j \lambda \partial b\} \partial (C^{-1})_{ji}(\partial + D) \{a_\lambda \theta_i\},
\]
for homogeneous $a, b \in \mathcal{P}$. Then the SUSY analogue of Theorem 1.1 holds. See Theorem 3.11 and Theorem 3.12.

**Theorem 1.2.** Let $\{\theta_I\}$ be the differential superalgebra ideal of $\mathcal{P}$ generated by $\theta_I$.

1. The bracket $\{ \lambda \}^D$ in (1.3) gives another SUSY PVA structure on $\mathcal{P}$. In other words, it satisfies the sesquilinearity, skewsymmetry, Jacobi identity and Leibniz rule for SUSY PVAs.
2. The Dirac reduced bracket on $\mathcal{P}$ induces a SUSY Poisson $\Lambda$-bracket on $\mathcal{P}/\{\theta_I\}$.
In the second part of this paper, we deal with classical W-algebras. Let \( \mathfrak{g} \) be a simple Lie algebra with a Dynkin grading \( \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \) associated with an \( sl_2 \)-triple \( \{ E, H, F \} \) and a nondegenerate symmetric invariant bilinear form \( (\cdot, \cdot) \). Let \( \mathfrak{m} = \mathfrak{L} \oplus \bigoplus_{i \geq 1} \mathfrak{g}(i) \), where \( \mathfrak{L} \) is a Lagrangian subspace with respect to the skew symmetric bilinear form \( \{ \cdot, \cdot \} := (F[\cdot], \cdot) \) on \( \mathfrak{g} \). The classical finite W-algebras \( \mathcal{W}^{\text{fin}}(\mathfrak{g}, F) \) is a Hamiltonian reduction of \( \mathbb{C}[\mathfrak{g}^*] \simeq S(\mathfrak{g}) \) via the the moment map \( \mu : \mathfrak{g}^* \to \mathfrak{m}^* \) at the regular value \( \chi|_\mathfrak{m} = (F[\cdot], \cdot) \in \mathfrak{g}^* \). It is known that, for any isotropic subspace \( \ell \), if we take \( n := \ell^\perp \oplus \bigoplus_{i \geq 1} \mathfrak{g}(i) \) and replace \( \mathfrak{m} \) by \( \ell \oplus \bigoplus_{i \geq 1} \mathfrak{g}(i) \) then

\[
\mathcal{W}^{\text{fin}}(\mathfrak{g}, F) = \{ w \in S(\mathfrak{p}) | \rho(\{n, w\}) = 0 \text{ for } n \in \mathfrak{n} \}.
\]

Here \( \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p} \) and \( \rho(\mathfrak{m}) = \mathfrak{m}_\mathfrak{p} + (F[\mathfrak{a}] \mathfrak{a}) \) for \( \mathfrak{a} \in \mathfrak{g}, \mathfrak{a}_\mathfrak{p} \in \mathfrak{p} \) and \( \mathfrak{a}_\mathfrak{m} \in \mathfrak{m} \) such that \( \mathfrak{a} = \mathfrak{a}_\mathfrak{m} + \mathfrak{a}_\mathfrak{p} \). As an affine analogue, Drinfeld-Sokolov [18] introduced the classical affine W-algebra \( W_{K}(\mathfrak{g}, F) \) as a Hamiltonian reduction of the affine PVA \( V^k(\mathfrak{g}) \). Note that, in this paper, we fix \( \ell = 0 \). Hence, \( \mathfrak{n} = \bigoplus_{i > 0} \mathfrak{g}(i) \) and \( \mathfrak{m} = \bigoplus_{i \geq 1} \mathfrak{g}(i) \).

Kac-Roan-Wakimoto [24] introduced W-algebras associated with Lie superalgebras and the corresponding classical finite W-algebras were established in [2], which we call W-superalgebras in this paper. For a finite simple Lie superalgebra \( \mathfrak{g} \) with a nondegenerate supersymmetric invariant even bilinear form \( (\cdot, \cdot) \), let us take an \( sl_2 \)-triple \( (E, H, F) \), which gives rise to the Dynkin grading \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \). The classical W-superalgebra

\[
\mathcal{W}^{k}(\mathfrak{g}, F) = \{ w \in \mathcal{P}(\mathfrak{p}) | \rho(\{n, w\}) = 0 \text{ for } n \in \mathfrak{n} \}
\]
is a PVSA endowed with the \( \lambda \)-bracket induced from \( V^k(\mathfrak{g}) \). Here \( \mathcal{P}(\mathfrak{p}) \) is the differential superalgebra generated by \( \mathfrak{p} \). In order to describe free generators of \( \mathcal{W}^{k}(\mathfrak{g}, F) \), consider a homogeneous basis \( \{ v_i | i \in J^F \} \) of \( \ker(\text{ad}F) =: \mathfrak{g}^F \). Then there is a unique element \( \omega_i \in \mathcal{W}^{k}(\mathfrak{g}, F) \) such that \( \omega_i - v_i \in \bigoplus_{i > 1} \mathcal{P}(\mathfrak{g}^F) \otimes (\mathbb{C}[\partial] \otimes [E, \mathfrak{g}_{> 0}]) \otimes \mathfrak{m}^n \) for the differential superalgebra \( \mathcal{P}(\mathfrak{g}^F) \) generated by \( \mathfrak{g}^F \).

Then the subset \( \{ \omega_i | i \in J^F \} \) freely generates \( \mathcal{W}^{k}(\mathfrak{g}, F) \) as a differential superalgebra. Moreover, in [35] [26], the third author computed the summand \( \gamma_i \) of \( \omega_i - v_i \in \mathcal{P}(\mathfrak{g}^F) \otimes (\mathbb{C}[\partial] \otimes [E, \mathfrak{g}_{> 0}]) \). Using the explicit formula of \( \gamma_i \), the \( \lambda \)-bracket between the generators can be expressed with elements in \( \mathbb{C}[\lambda] \otimes C[\partial^n \omega_i | n \in \mathbb{Z}, i \in I] \).

On the other hand, in various physical articles [16] [28] [33] [32] [19], classical W-algebras were introduced as Dirac reductions of affine PVAs. When \( \mathfrak{g} \) is a Lie algebra, De Sole-Kac-Valeri described a W-algebra \( \mathcal{W}^{k}(\mathfrak{g}, F) \) as a limit of Dirac reduction of \( V^k(\mathfrak{g}) \) (see [13]). In this paper, we aimed to explain the \( \lambda \)-bracket of a W-superalgebra \( \mathcal{W}^{k}(\mathfrak{g}, F) \), in terms of a Dirac reduced bracket of \( V^k(\mathfrak{g}) \).

Unfortunately, it was not possible to find \( \theta_1 \subset V^k(\mathfrak{g}, F) \) making \( C(\lambda) = \{(\theta_1, \lambda)\} \) invertible and the quotient algebra \( V^k(\mathfrak{g})/\theta_1 \) with the Dirac reduced bracket isomorphic to \( \mathcal{W}^{k}(\mathfrak{g}, F) \). In case \( \mathfrak{g} \) is a Lie algebra, De Sole-Kac-Valeri [13] resolved the problem by including an extra parameter \( t \) in the constraints \( \theta_1(t) \). In this paper, instead of considering an additional parameter, we define the modified Dirac reduced bracket on the quotient differential superalgebra \( \mathcal{P}/\mathbb{P} \) associated with a finite subset \( \theta_1 \) of PVSA \( \mathbb{P} \). To be precise, let \( \pi : \mathcal{P} \to \mathbb{P}/\theta_1 \) be the canonical projection map and \( \tilde{C}(\lambda) \) be the matrix whose \( ij \)-entry is \( \tilde{C}_{ij} = \pi(C_{ij}) \in \mathbb{P}/\theta_1[\lambda] \) for the \( ij \)-entry \( C_{ij} \) of \( C(\lambda) \) in [13]. Provided that \( \tilde{C}(\lambda) \) is invertible, we replace \( (C^{-1})_{ij}(\lambda) \) in [13] by \( \tilde{C}_{ij}^{-1}(\lambda) \) and obtain the modified Dirac reduced bracket. Consequently, we prove Theorem 1.12 which implies the following statement.

3
**Theorem 1.3.** Let $\theta_I = \{q^i_m - (F)q^i_m\}_{(i,m) \in I}$, where $\{q^i_m\}_{(i,m) \in I}$ is a basis of $[E, g]$ and $\pi : V^k(g) \to V^k(g)/\langle \theta_I \rangle$ is the canonical projection map. Then the differential superalgebra $V^k(g)/\langle \theta_I \rangle$ endowed with the modified Dirac reduced bracket is isomorphic to $W^k(g, F)$.

As a counterpart of W-algebras in the theory of SUSY vertex algebras, one can consider SUSY W-algebras. A SUSY W-algebra was introduced to understand the superfield formalism in super-Toda theory in [7, 8, 25]. The SUSY version of BRST formalism was introduced by Madsen-Ragoucy in [26] and it was interpreted via SUSY vertex algebras in [30]. On the other hand, the third author showed in [35] that SUSY classical W-algebras can be obtained from the quasi-classical limit of SUSY BRST complexes or the SUSY analogue of Drinfeld-Sokolov reductions.

A SUSY classical W-algebra $W^k(g, f)$ is governed by a Lie superalgebra $g$ and an odd nilpotent element $f$ in a subalgebra $a = \mathfrak{osp}(1|2)$. Let us consider a subspace $p = \oplus_{i=0}^g g(i)$ and the differential superalgebra $\mathcal{P}(p)$ generated by $p$. For $a \in g$, we denote by $a_- \in p$ and $a_+ \in n$ the elements satisfying $a = a_+ + a_-$ and define $\rho_-(a) := a_+ + (f|a)$. Then the SUSY W-algebra

$$W^k(g, f) = \{\hat{\omega} \in \mathcal{P}(p) | \rho_-(\{\hat{a}_A \hat{\omega}\}) = 0 \text{ for any } n \in n\}$$

is a SUSY PVA, whose $A$-bracket is induced from the affine SUSY PVA $\mathcal{V}^k(g)$. When a homogeneous basis $\{u_i \mid i \in J_f\}$ of $\ker(\text{ad}f) = g^0$ is given, one can find a unique element $\hat{v}_i \in W^k(g, f)$ for $i \in J_f$ such that $\hat{v}_i - \hat{u}_i \in \bigoplus_{n \geq 1} \mathcal{P}(g^0) \otimes (C[D] \otimes [e, g_{\leq 1}])^{\otimes n}$. The $A$-brackets between the generators $\hat{v}_i$ can be found in [35].

Similarly to the classical W-superalgebra case, we substitute $C(A)$ in (1.5) by $\tilde{C}(A)$ whose $ij$-entry is $\pi_i \left|_{\mathcal{P}(p) / \langle \theta_I \rangle} \right.$$ [A]$ to obtain a bracket called modified SUSY Dirac reduction on the quotient space $\mathcal{P}(p) / \langle \theta_I \rangle$. Then the following theorem holds. See Theorem 5.10 for the precise statement.

**Theorem 1.4.** Let $\theta_I = \{q^i_m - (F)q^i_m\}_{(i,m) \in I}$, where $\{q^i_m\}_{(i,m) \in I}$ is a basis of $[e, g]$ and $\pi : \mathcal{V}^k(g) \to \mathcal{V}^k(g)/\langle \theta_I \rangle$ be the canonical projection map. Then the differential superalgebra $\mathcal{V}^k(g)/\langle \theta_I \rangle$ endowed with the modified Dirac reduced bracket is isomorphic to the SUSY classical W-algebra $W^k(g, f)$.

This paper is organized as follows. In Section 2 we review non-local PVAs and introduce Dirac reduction for Poisson vertex superalgebras and Poisson superalgebras. Analogously, in Section 3 we introduce non-local SUSY PVAs and Dirac reduction for SUSY PVAs. In Section 4 we introduce modified Dirac reduced brackets for Poisson vertex superalgebras and describe the Poisson structures of W-superalgebras as modified Dirac reduced brackets of affine PVAs. In Section 5 we explain SUSY analogue of Section 4 and SUSY W-algebras via modified Dirac reduced brackets of SUSY affine PVAs.

Throughout this paper, the base field is $\mathbb{C}$. The set of integers (resp. nonnegative integers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Z}_+$).
is a linear operator $d$ on a superalgebra $A$ with satisfying

$$d(ab) = d(a)b + (-1)^a d(a)b$$

for $a, b \in A$, then $A$ is called a differential superalgebra with a derivation $d$. Here and further, for simplicity, we just call the tuple $(A, d)$ a differential algebra. In the rest of the paper, whenever the parity $\tilde{a}$ of an element $a$ in a vector superspace is considered, we assume that $a$ is homogeneous even though it is not mentioned.

2.1. Non-local Poisson vertex superalgebras.

A vector superspace $g$ endowed with a bilinear bracket $[\cdot, \cdot] : g \times g \to g$ is a Lie superalgebra if it satisfies

- (skewsymmetry) $[a, b] = -(-1)^{\tilde{a}\tilde{b}}[b, a]$,
- (Jacobi identity) $(-1)^{\tilde{a}\tilde{b}}[a, [b, c]] + (-1)^{\tilde{b}\tilde{c}}[b, [c, a]] + (-1)^{\tilde{c}\tilde{a}}[c, [a, b]] = 0$

for $a, b, c \in g$. If a Lie superalgebra $(P, \{\cdot, \cdot\})$ is a unital supercommutative associative algebra with the Leibniz rule:

$$\{a, bc\} = \{a, b\} c + (-1)^{\tilde{a}\tilde{b}} \{a, c\}$$

for $a, b, c \in P$, then $P$ is called a Poisson superalgebra.

To define a non-local Poisson vertex superalgebra in a similar manner, let us introduce an admissible non-local $\lambda$-bracket. See [10] for detailed properties of non-local Poisson vertex algebras. On a $C[\partial]$-module $R$, a non-local $\lambda$-bracket is a parity preserving bilinear map $[\cdot, \cdot, \cdot : R \times R \to R((\lambda^{-1}))$ satisfying the sesquilinearity:

$$[\partial a \lambda b] = -\lambda[a \lambda b], \ [a \lambda \partial b] = (\lambda + \partial)[a \lambda b]$$

for $a, b \in R$. We denote by $[a \lambda b] = \sum_{n \in \mathbb{Z}} \lambda^n a_{(n)} b$ for $a_{(n)} b \in R$.

Consider a superspace $R_{\lambda, \mu} := R[[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}]][[\lambda, \mu]]$ (2.3) and let $\iota_{\mu, \lambda} : R_{\lambda, \mu} \hookrightarrow R((\lambda^{-1}))((\mu^{-1}))$ be defined by the geometric expansion of $(\lambda + \mu)^m$ for an integer $m$ in the domain $|\mu| > |\lambda|$. Then we can identify $R_{\lambda, \mu}$ with the image $\iota_{\mu, \lambda}(R_{\lambda, \mu}) \subset R((\lambda^{-1}))((\mu^{-1}))$. If a non-local $\lambda$-bracket on $R$ satisfies

$$[a \lambda [b \mu c]] \in R_{\lambda, \mu}$$

for all $a, b, c \in R$, then it is said to be admissible. If a $C[\partial]$-module $R$ and an admissible $\lambda$-bracket $[\cdot, \cdot, \cdot]$ are given to satisfy the following axioms:

- (skewsymmetry) $[b_{\lambda a}] = -(-1)^{\tilde{a}\tilde{b}} [a_{\lambda-b} b]$ in $R((\lambda^{-1}))$,
- (Jacobi identity) $[a \lambda [b \mu c]] = [[a \lambda b] \lambda, \mu c] + (-1)^{\tilde{b}\tilde{c}} [b \mu [a \lambda c]]$ in $R_{\lambda, \mu}$

for $a, b, c \in R$. We call $R$ a non-local Lie conformal superalgebra (LCA). The skewsymmetry can be rewritten as

$$\sum_{n \in \mathbb{Z}} \lambda^n a_{(n)} b = -(-1)^{\tilde{a}} \sum_{n \in \mathbb{Z}} (-\lambda - \partial)^n a_{(n)} b = -(-1)^{\tilde{a}} \sum_{n \in \mathbb{Z}, \tau \in \mathbb{Z}_+} \binom{n}{\tau} (-1)^{n}(\partial^n a_{(n)} b) \lambda^{n-r}$$

and the three terms in the Jacobi identity are in $R((\lambda^{-1}))((\mu^{-1}))$, $R(((\lambda + \mu)^{-1}))((\lambda^{-1}))$ and $R(((\mu^{-1}))((\lambda^{-1}))$, respectively. By the admissibility, the LHS and RHS of Jacobi identity can be compared in $R_{\lambda, \mu}$ via the identification $\iota_{\mu, \lambda}$, $\iota_{\lambda, \mu}$, and $\iota_{\mu, \lambda}$.

**Definition 2.1.** A non-local Poisson vertex superalgebra (non-local PVSA or just PVSA) is a quadruple $(P, \partial, \{\cdot, \cdot, \cdot\})$ which satisfies the following axioms:
\begin{itemize}
\item $(\mathcal{P}, \partial, \{\cdot, \cdot\})$ is a non-local LCA,
\item $(\mathcal{P}, \partial, \cdot)$ is a unital supercommutative associative differential algebra,
\item (left Leibniz rule) $\{a \lambda bc\} = (a \lambda b) c + (-1)^{\delta b} (a \lambda c)$ for $a, b, c \in \mathcal{P}$.
\end{itemize}

The supersymmetric algebra generated by a superspace $V$ is

$$S(V) := S(V_0) \otimes \Lambda(V_1),$$

where $S(V_0)$ is the symmetric algebra generated by $V_0$ and $\Lambda(V_1)$ is the exterior algebra generated by $V_1$. Suppose $V$ is a superspace with a homogeneous basis $B = B_0 \cup B_1$, where $B_0 = \{u_i | i \in I_0\}$ and $B_1 = \{u_i | i \in I_1\}$ consist of even and odd elements, respectively. Then the supersymmetric algebra

$$\mathcal{P} = S(\mathbb{C}[\partial] \otimes V) = \mathbb{C}[u_i^{(m)} | i \in I_0 \cup I_1, m \in \mathbb{Z}_+]$$

is a differential algebra with even derivation $\partial$ on $\mathcal{P}$ defined by $\partial(u_i^{(m)}) = u_i^{(m+1)}$.

**Proposition 2.2.** (Master formula of non-local PVSA). For the differential algebra $\mathcal{P}$ in (2.4), let $\frac{\partial}{\partial u_i^{(m)}}$ be the derivation of parity $u_i$ on $\mathcal{P}$ such that $\frac{\partial}{\partial u_i^{(m)}} u_j^{(n)} = \delta_{m,n}\delta_{i,j}$. If $\mathcal{P}$ is a PVSA with non-local $\lambda$-bracket $\{\cdot, \cdot\}_\lambda$, then for $f, g \in \mathcal{P}$,

$$\{f, g\}_\lambda = \sum_{i,j, \{I_0,I_1\}, I_0 \cup I_1, m, n \in \mathbb{Z}_+} X_{ij}^{fg} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n \{u_{i\lambda + \partial} u_j\} - (\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}},$$

(2.5)

where $X_{ij}^{fg} = (-1)^{\tilde{f}\tilde{g}} (-1)^{\tilde{a}_i\tilde{a}_j} (\lambda + \partial)^n \{u_{i\lambda + \partial} u_j\} - (\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}$.

**Proof.** By the skewsymmetry and the left Leibniz rules of PVSA, one can deduce the right Leibniz rule:

$$\{ab\lambda\}_\lambda = (-1)^{\tilde{a}\tilde{c}} (a_{\lambda + \partial} c) \cdot b + (-1)^{\tilde{b}\tilde{a}} (b_{\lambda + \partial} c) \cdot a,$$

where $\{a_{\lambda + \partial} c\} \cdot b = \sum_{n \in \mathbb{Z}_0} a_{(n)} c (\lambda + \partial)^n b$. The formula (2.5) follows from the sesquilinearity and left and right Leibniz rule of $\lambda$-brackets. We refer to Theorem 4.8 in [10] and Proposition 4.4 in [34] for detailed proof. \qed

**Proposition 2.3.** Let $V$ be a vector superspace and $\mathcal{P} := S(\mathbb{C}[\partial] \otimes V)$. If a bracket $[\cdot, \cdot] : V \otimes V \to \mathcal{P}(\langle \lambda^{-1} \rangle)$ satisfies the admissibility, skewsymmetry and Jacobi identity, then it can be uniquely extended to a non-local PVSA $\lambda$-bracket on $\mathcal{P}$ using the sesquilinearity and Leibniz rule.

**Proof.** It can be proved by combining the results of Theorem 4.8 of [10] and Theorem 1.15 of [4]. \qed

**Example 2.4.** Let $\mathfrak{g}$ be a Lie superalgebra with a nondegenerate supersymmetric invariant even bilinear form $(\cdot | \cdot)$. For $k \in \mathbb{C}$, the differential algebra $\mathcal{V}^k(\mathfrak{g}) := S(\mathbb{C}[\partial] \otimes \mathfrak{g})$ with the $\lambda$-bracket

$$\{a\lambda b\} = [a, b] + k\lambda(a|b)$$

(2.6)

is called an affine PVSA. This $\lambda$-bracket satisfies all the conditions of Proposition 2.3. Hence the PVSA structure of $\mathcal{V}^k(\mathfrak{g})$ is fully determined by the master formula (2.5).

**Example 2.5.** The differential algebra $\mathcal{P} = \mathbb{C}[u^{(n)} | n \in \mathbb{Z}_+]$ endowed with the $\lambda$-bracket

$$\{u_{\lambda} u\} = \lambda^{-1}$$

is a non-local PVSA. By Proposition 2.3, the $\lambda$-bracket can be extended to the whole space $\mathcal{P}$. See [10] for more examples.
2.2. Dirac reduction of Poisson vertex superalgebras.

Let \( \mathcal{P} \) be a PVSA and denote by \( \text{Mat}_{(r|s)} = \text{End}(\mathbb{C}_{(r|s)}) \) for nonnegative integers \( r \) and \( s \). Consider the superspace \( \mathcal{M}_{(r|s)}(\lambda) = \text{Mat}_{(r|s)} \otimes \mathcal{P}((\lambda^{-1})) \), where the parity of \( a \otimes F(\lambda) \in \mathcal{M}_{(r|s)}(\lambda) \) is given by \( \tilde{a} + \tilde{F} \). It is an associative algebra with respect to the product

\[
(a \otimes F(\lambda)) \circ (b \otimes G(\lambda)) = (a \otimes F(\lambda + \partial)) (b \otimes G(\lambda))
\]

\[
= (-1)^{\tilde{b}E} ab \otimes F(\lambda + \partial)G(\lambda).
\]

Then \( \text{Id}_{(r|s)} \otimes 1 \) is the unity in \( \mathcal{M}_{(r|s)}(\lambda) \) for the identity matrix \( \text{Id}_{(r|s)} \) in \( \text{Mat}_{(r|s)} \). In addition, we say \( A(\lambda) \in \mathcal{M}_{(r|s)}(\lambda) \) is invertible if there is an element \( A^{-1}(\lambda) \in \mathcal{M}_{(r|s)}(\lambda) \) such that

\[
A(\lambda) \circ A^{-1}(\lambda) = A^{-1}(\lambda) \circ A(\lambda) = \text{Id}_{(r|s)} \otimes 1.
\]

For the simplicity of notation, we sometimes denote \( \mathcal{M}_{(r|s)}(\lambda) \) by \( \mathcal{M}(\lambda) \).

Let \( I = I_0 \cup I_1 \subset \mathbb{Z} \) be a finite index set, where \( I_i = I \cap (2\mathbb{Z} + i) \) and \( i \in \{0, 1\} \). Let \( \theta_I := \{ \theta_i | i \in I \} \) be a subset of \( \mathcal{P} \) consisting of homogeneous elements such that \((-1)^{\tilde{\theta}_i} = (-1)^i \). For \( r = |I_0| \) and \( s = |I_1| \), consider the element

\[
C(\lambda) := \sum_{i,j \in I} e_{ij} \otimes \{ \theta_j + \theta_i \} \in \mathcal{M}_{(r|s)}(\lambda).
\]

**Definition 2.6.** Assume that the element \( C(\lambda) \) in (2.8) is invertible and write its inverse as \((C^{-1})(\lambda) = \sum_{i,j} e_{ij} \otimes (C^{-1})_{ij}(\lambda) \). The Dirac reduced bracket of the Poisson \( \lambda \)-bracket \( \{ \cdot, \cdot \} \) on \( \mathcal{P} \) associated with \( \theta_I \) is the bilinear map

\[
\{ \cdot, \cdot \}^D : \mathcal{P} \times \mathcal{P} \to \mathcal{P}((\lambda^{-1}))
\]

given by

\[
\{ a, b \}^D = \{ a, b \} - \sum_{i,j \in I} (-1)^{(\tilde{a} + \tilde{b}) + i + j} \{ \theta_j + \theta_i \} \cdot (C^{-1})_{ji}(\lambda + \partial) \{ a, \theta_i \},
\]

where \( a, b \) are in \( \mathcal{P} \).

For the next proposition, we generalize the multiplication (2.7) as follows:

\[
(\text{Mat}_{(p_1|q_1)}(r_1|s_1) \otimes \mathcal{P}((\lambda^{-1}))) \times (\text{Mat}_{(p_2|q_2)}(r_2|s_2) \otimes \mathcal{P}((\lambda^{-1}))) \to (\text{Mat}_{(p_1|q_1)}(r_2|s_2) \otimes \mathcal{P}((\lambda^{-1})))
\]

\[
\left( (a \otimes F(\lambda)), (b \otimes G(\lambda)) \right) \mapsto (-1)^{\tilde{b}E} ab \otimes F(\lambda + \partial)G(\lambda).
\]

Here \( p_1, q_1, r_1 = p_2, q_2, r_2, s_2 \) are nonnegative integers and \( \text{Mat}_{(p_1|q_1)}(r_1|s_1) \) is the set of linear maps from \( \mathbb{C}_{(p_1|q_1)} \) to \( \mathbb{C}_{(r_1|s_1)} \).

**Proposition 2.7.** Let \( \mathcal{P} = \mathbb{C}[[\theta_k^{(n)}]| k \in I, n \in \mathbb{Z}_+] \) be a PVSA with the derivation \( \partial : \theta_k^{(n)} \to \theta_k^{(n+1)} \). If \( C(\lambda) \) in (2.8) is invertible, the corresponding Dirac reduced bracket is trivial.

**Proof.** Let us use the identification \( \mathcal{P}((\lambda^{-1})) = \text{Mat}_{(1|0)} \otimes \mathcal{P}((\lambda^{-1})) \). For \( a, b \in \mathcal{P} \), recall the master formula (2.10):

\[
\{ a, b \} = \sum_{i,j \in I, n \in \mathbb{Z}_+} (-1)^{\tilde{a} + \tilde{b} + i + j} \frac{\partial b}{\partial \theta_i^{(n)}} (\lambda + \partial)^n \{ \theta_i + \theta_j \} \cdot (-\lambda - \partial)^m \frac{\partial a}{\partial \theta_j^{(m)}}.
\]

\[\text{(2.10)}\]
For given $i, j \in I$, the RHS of (2.10) can be written as

$$\sum_{n \in \mathbb{Z}_+} (-1)^{b+l} \left( e_j^T \otimes \frac{\partial b}{\partial \theta_i^{(n)}} (\lambda + \partial)^n \right) \left( e_{ji} \otimes (C_{ij}(\lambda + \partial)) \right) \left( e_i \otimes (-\lambda - \partial)^m \frac{\partial a}{\partial \theta_i^{(m)}} \right) ,$$

(2.11)

where $e_i \in \text{Mat}_{\{r\}(1)}$ is the $i$-th standard column vector, $e_i^T \in \text{Mat}_{\{1\}(0)}$ is its transpose and $e_{ij}$ is the $ij$-th elementary matrix in $\text{Mat}_{\{r\}(0)}$. Note that

$$e_{ji} \otimes C_{ij}(\lambda) = \sum_{k,l \in I} (e_{jk} \otimes C_{jk}(\lambda + \partial))(e_{kl} \otimes (C^{-1})_{kl}(\lambda + \partial))(e_{li} \otimes C_{li}(\lambda))$$

(2.12)

and

$$\sum_{j \in \text{I}, n \in \mathbb{Z}_+} (-1)^{i}(e_{li} \otimes C_{li}(\lambda + \partial)) \left( e_i \otimes (-\lambda - \partial)^m \frac{\partial a}{\partial \theta_i^{(m)}} \right) = (-1)^{\bar{a}i} e_i \otimes \{a_\lambda \theta_i\},$$

(2.13)

$$\sum_{j \in \text{I}, n \in \mathbb{Z}_+} \left( e_j^T \otimes \frac{\partial b}{\partial \theta_j^{(n)}} (\lambda + \partial)^n \right) (e_{jk} \otimes C_{jk}(\lambda + \partial)) = e_j^T \otimes \{k \lambda \, b\}.$$  

If we substitute $(e_{ji} \otimes C_{ij}(\lambda + \partial))$ in (2.11) with the RHS of (2.12), then by (2.13) we get

$$\{a_\lambda b\} = \sum_{k,l \in I} (-1)^{\bar{a}(\bar{b}+l)} (e_k \otimes \theta_{k\lambda + \bar{a}b}) (e_{kl} \otimes (C^{-1})_{kl}(\lambda + \partial)) (e_l \otimes \{a_\lambda \theta_l\})$$

$$= \sum_{k,l \in I} (-1)^{\bar{a}(\bar{b}+l)+k+l} \{\theta_{k\lambda + \bar{a}b}\} (C^{-1})_{kl}(\lambda + \partial) \{a_\lambda \theta_l\}.$$  

(2.14)

Hence, $\{a_\lambda b\}^D = \{a_\lambda b\} - [\text{RHS of (2.11)}] = 0$. 

Let us introduce the adjoint of $F(\lambda) = \sum_{n \in \mathbb{Z}} F_n \lambda^n \in \mathcal{P}(\mathcal{P}(\lambda^{-1}))$ for $F_n \in \mathcal{P}$ which is defined by

$$F^*(\lambda) = \sum_{n \in \mathbb{Z}} (-\lambda - \partial)^n F_n.$$  

In addition, the adjoint of an even element $A(\lambda) = \sum_{i,j \in \mathbb{Z}} e_{ij} \otimes A_{ij}(\lambda) = \sum_{i,j \in \mathbb{Z}} e_{ij} \otimes A_{ij}\lambda^n \in \mathcal{M}(\{r\}(1))$ is given by

$$A^*(\lambda) := \sum_{i,j \in \mathbb{Z}} (-1)^{ij} e_{ji} \otimes A_{ij}(\lambda).$$

(2.15)

Then $(A(\lambda) \circ B(\lambda))^* = B^*(\lambda) \circ A^*(\lambda)$ for any even elements $A(\lambda), B(\lambda) \in \mathcal{M}(\lambda)$.

Now we show that Dirac reduced bracket (2.9) is indeed a PVSA bracket. The following lemma is needed to prove Theorem 2.9.

**Lemma 2.8.** Let $\{(\cdot, \lambda)\} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}(\mathcal{P}(\lambda^{-1}))$ be a non-local PVSA $\lambda$-bracket on $\mathcal{P}$. Suppose $C(\lambda) = \left(C_{ij}(\lambda)\right)_{i,j \in \mathcal{I}} \in \mathcal{M}(\lambda)$ in (2.8) is invertible and let $(C^{-1})(\lambda) = \left((C^{-1})_{ij}(\lambda)\right)_{i,j \in \mathcal{I}} \in \mathcal{M}(\lambda)$ be its inverse. For $a \in \mathcal{P}$, assume that

$$\{a_\lambda C_{ij}(\mu)\} = \mathcal{P}_{\lambda, \mu} \text{ for all } i,j \in \mathcal{I}.$$  

Then we have $\{a_\lambda (C^{-1})_{ij}(\mu)\}, \{(C^{-1})_{ij}(\lambda)\}_{\lambda + \mu} \in \mathcal{P}_{\lambda, \mu}$. If $a \in \mathcal{P}$ is homogenous, then

$$\{a_\lambda (C^{-1})_{ij}(\mu)\} = - \sum_{r, i \in \mathcal{I}} \alpha(i + r + (i + r)\beta + t) (C^{-1})_{ir}(\lambda + \mu + \partial) \{a_\lambda C_{rt}(\mu)\} (\mu + \partial)^n (C^{-1})_{ij}(\mu)$$

(2.16)
Using the left Leibniz rule, we obtain

\[ \{(C^{-1})_{ij}(\lambda) \lambda_{+\mu} a\} = - \sum_{r,t \in I} (-1)^{(i+j+r+t)+(r+t)} \{C_{rt} \lambda_{+\mu} \partial a\}(\lambda + \partial)^n (C^{-1})_{ij}(\lambda)(C^{-1})^*_{ir}(\mu), \]  

(2.17)

where \((C^{-1})^*_{ir}(\mu) := ((C^{-1})_{ir})^*(\mu)\).

**Proof.** For the first assertion, see Lemma 2.1, Lemma 2.3 and Lemma 2.4 in [10].

In order to show (2.16) and (2.17), observe that

\[ e_{ij} \otimes (C^{-1})_{ij}(\lambda) = \sum_{r,t \in I} (e_{ir} \otimes (C^{-1})_{ir}(\lambda) \circ (C_{rt} \otimes (C^{-1})_{ij}(\lambda)) \right. \]

\[ = \sum_{r,t \in I} (-1)^{(i+r)+(r+t)} \{e_{ij} \otimes (C^{-1})_{ir}(\lambda + \partial)C_{rt}(\lambda + \partial)(C^{-1})_{ij}(\lambda)\}. \]

Hence we have

\[ (C^{-1})_{ij}(\lambda) = \sum_{r,t \in I} (-1)^{(i+r)+(r+t)}(C^{-1})_{ir}(\lambda + \partial)C_{rt}(\lambda + \partial)(C^{-1})_{ij}(\lambda). \]

Using the left Leibniz rule, we obtain

\[ \{a_{\lambda}(C^{-1})_{ij}(\mu)\} = \sum_{r,t \in I} (-1)^{(i+j+r+t+j+r)} \{a_{\lambda}(C^{-1})_{ir}(\mu + \partial)C_{rt}(\mu + \partial)(C^{-1})_{ij}(\mu)\} \]

\[ = \sum_{t \in I} (-1)^{(i+j)} \{a_{\lambda} \sum_{r \in I} (-1)^{(i+r)+(r+t)} (C^{-1})_{ir}(\mu + \partial)C_{rt}(\mu + \partial)(C^{-1})_{ij}(\mu)\}(C^{-1})_{ij}(\mu) \]  

(2.18)

\[ + \sum_{r \in I} (-1)^{(i+j+r)+(r+t)}(C^{-1})_{ir}(\lambda + \mu + \partial) \{a_{\lambda} \sum_{t \in I} (-1)^{(r+t)+(t+j)} C_{rt}(\mu + \partial)(C^{-1})_{ij}(\mu)\} \]

\[- \sum_{r \in I} (-1)^{(j+i+r+t+j+i+r)}(C^{-1})_{ir}(\lambda + \mu + \partial) \{a_{\lambda} C_{rt}(\mu + \partial)\}(C^{-1})_{ij}(\mu). \]  

(2.19)

Since \( \sum_{r \in I} (-1)^{(i+r)+(r+t)}(C^{-1})_{ir}(\mu + \partial)C_{rt}(\mu + \partial) = \delta_{it} \), we have \( \text{2.19} = 0 \). Similarly, \( \text{2.18} = 0 \) and hence we showed \( \text{2.16} \).

For (2.17), using the right Leibniz rule, one can observe that

\[ \{((C^{-1})_{ij}(\lambda) \lambda_{+\mu} a\} = \sum_{t \in I} (-1)^{(i+j)} \{a_{\lambda} \sum_{r \in I} (-1)^{(i+r)+(r+t)} (C^{-1})_{ir}(\lambda + \partial)C_{rt}(\lambda + \partial)(C^{-1})_{ij}(\lambda) \lambda_{+\mu} a\} \]

\[ = \sum_{t \in I} (-1)^{(i+j)}(t+i+a) \{a_{\lambda} \sum_{r \in I} (-1)^{(i+r)+(r+t)} (C^{-1})_{ir}(\lambda + \partial)C_{rt}(\lambda + \partial)(C^{-1})_{ij}(\lambda) \lambda_{+\mu} a\} \]  

(2.21)

\[ + \sum_{r \in I} (-1)^{(i+r)} \{a_{\lambda} \sum_{t \in I} (-1)^{(r+t)+(t+j)} C_{rt}(\lambda + \partial)(C^{-1})_{ij}(\mu) \lambda_{+\mu} a\} \]  

(2.22)

\[ - \sum_{r \in I} (-1)^{(r+t)+(t+j+i+r)} \{C_{rt}(\lambda + \partial) \lambda_{+\mu} \partial a\} \]  

(2.23)

Since \( \text{2.21} = \text{2.22} = 0 \), (2.17) holds.

**Theorem 2.9.** Let \( \mathcal{P} \) be a PVSA and let \( \theta_1 \in \mathcal{P} \) be a finite set of homogeneous elements. If \( C(\lambda) \in M(\lambda) \) in \( \text{2.8} \) is invertible, then the Dirac reduced bracket \( \{\cdot, \cdot\}^D \) given by \( \text{2.9} \) is a Poisson \( \lambda \)-bracket on \( \mathcal{P} \).
Note that we used Lemma 2.8 to obtain (2.26) and (2.30).

The sesquilinearity, admissibility and Leibniz rule can be proved by direct computations (see Theorem 2.2(a) of [15]).

To show the skewsymmetry, we substitute $\lambda$ by $-\lambda - \partial$ in (2.27) and let $\partial$ act on the left. Then we have

$$\{a_{-\lambda - \partial} b\}^D = \{a_{-\lambda - \partial} b\} - \sum_{\alpha, \beta \in I} \underbrace{(-\{a_{-\lambda - \partial} \theta_\beta\})(|_{x=-\lambda - \partial} (C^{-1})_{\alpha \beta}(x))}_{= \{a_{-\lambda - \partial} b\}} \{\theta_\alpha - \lambda - \partial b\}$$

$$= \{a_{-\lambda - \partial} b\} - \sum_{\alpha, \beta \in I} \underbrace{(-1)^{\hat{\alpha} + \hat{\beta} + \alpha + \beta + 1}\{\theta_\beta \lambda + \partial b\}}_\sim (C^{-1})_{\beta \alpha}(\lambda + \partial)\{b_{\lambda \theta_\alpha}\}.$$ 

Here, for the last equality, we used the fact that $(C^{-1})_{\beta \alpha}(\lambda) = (-1)^{\alpha + \beta + 1}(|_{x=-\lambda - \partial} (C^{-1})_{\alpha \beta}(x))$, which is induced from the identity $C^{-1}(\lambda)\circ C(\lambda) = (C(\lambda) \circ (C^{-1})_\lambda)^*$. By multiplying $-(-1)^{\hat{a} \hat{b}}$ on the both sides, we get the sesquilinearity.

To prove the Jacobi identity, one can expand the following equation by Definition 2.6 of Dirac reduced bracket.

$$\{a_\lambda \{b_\mu c\}\}^D = \{a_\lambda \{b_\mu c\}\}$$

$$- \sum_{\gamma, \delta \in I} (-1)^{\hat{b} + \hat{\delta} + \gamma + \delta}\{a_\lambda \{b_\delta y c\}\} (|_{y = \mu + \partial} (C^{-1})_{\beta \gamma}(\mu + \partial)\{b_\mu \theta_\gamma\})$$

$$+ \sum_{\alpha, \beta, \gamma, \delta \in I} (-1)^{\hat{\beta} + \hat{\gamma} + \hat{\gamma} + \beta + \gamma + \delta}\{a_\lambda \{b_\gamma y c\}\} (|_{y = \mu + \partial} (C^{-1})_{\beta \gamma}(\mu + \partial)\{b_\mu \theta_\gamma\})$$

$$= \{a_\lambda \{b_\mu c\}\} - \sum_{\alpha, \beta \in I} (-1)^{\hat{a} + \hat{\beta} + \alpha + \beta + 1}\{\theta_\beta \lambda + \partial b\} \{\theta_\alpha \lambda + \partial c\} \sim (C^{-1})_{\beta \alpha}(\lambda + \partial)\{a_\lambda \theta_\alpha\}.$$ 

Note that we used Lemma 2.8 to obtain (2.26) and (2.30).

We also get the expansion of $\{b_\mu \{a_\lambda c\}\}^D$ by exchanging the roles of $a$ and $b$ and roles of $\lambda$ and $\mu$ in the above equation. We shall denote the corresponding terms to (2.24) to (2.31) by (2.24') to (2.31').

\[\tag{2.24'} \]
We claim that the sum of three terms in the following triples a re all trivial.

\[
\{\{a_\beta b\}^D_{\lambda+\mu c}\}^D = \{\{\alpha\beta\}\lambda+\mu c\}
\]

\[- \sum_{\alpha,\beta I} (-1)^{(\alpha+\beta)}(\alpha+\beta+\alpha+\beta) \{\theta_{\beta x} b\}_{\lambda+\mu \partial c} \to \left(\frac{\partial_{\lambda x+\partial}}{\partial_{\lambda x+\partial}}\right)_{\beta_\alpha}(\lambda + \partial) \{a_\lambda \theta_{\alpha}\}\]

\[- \sum_{\alpha,\beta I} (-1)^{(\alpha+\beta)}(\alpha+\beta+\alpha+\beta) \{\theta_{\beta x} c\}_{\lambda+\mu \partial c} \to \left(\frac{\partial_{\lambda x+\partial}}{\partial_{\lambda x+\partial}}\right)_{\beta_\alpha}(\lambda + \partial) \{a_\lambda \theta_{\alpha}\}\]

\[- \sum_{\alpha,\beta I} (-1)^{(\alpha+\beta)}(\alpha+\beta+\alpha+\beta) \{\theta_{\beta x} c\}_{\lambda+\mu \partial c} \to \left(\frac{\partial_{\lambda x+\partial}}{\partial_{\lambda x+\partial}}\right)_{\beta_\alpha}(\lambda + \partial) \{a_\lambda \theta_{\alpha}\}\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.23), (2.24), (2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]

We claim that the sum of three terms in the following triples are all trivial.

\[
\begin{align*}
(2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31), (2.32), (2.33),
\end{align*}
\]
To show the similar result for the second triple \( (2.26), (2.31)', (2.33) \), consider the definition of the adjoint and the skewsymmetry. Then we have

\[ \sum_{\alpha, \beta, \gamma, \delta \in I} (-1)\tilde{a}\tilde{b}+\tilde{a}\tilde{c}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial)\]

By changing the indices, we can show that the sum \( (2.26) + (2.31)' + (2.33) \) is

\[ \sum_{\gamma, \delta, \zeta, \eta \in I} (-1)\tilde{a}\tilde{b}+\tilde{a}\tilde{c}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial) \]

The sum of fourth triple \( (2.28), (2.25), (2.33) \) equals to 0 by the similar computations to the case of the first triple.

Consider the fifth triple \( (2.29), (2.29)', (2.31) \). Then

\[ \sum_{\alpha, \beta, \gamma, \delta \in I} (-1)\tilde{a}\tilde{c}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial) \]

and

\[ \sum_{\alpha, \beta, \gamma, \delta \in I} (-1)\tilde{a}\tilde{d}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial) \]

Hence the sum of the three terms of equals to

\[ \sum_{\alpha, \beta, \gamma, \delta \in I} (-1)\tilde{a}\tilde{c}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial) \]

Consider the sixth triple \( (2.30), (2.30)', (2.33) \). We have

\[ \sum_{\alpha, \beta, \gamma, \delta, \zeta, \eta \in I} (-1)\tilde{b}\tilde{d}+\tilde{b}\tilde{d}+\tilde{a}\tilde{c}+\tilde{a}\tilde{d}+\tilde{b}\tilde{c}+\tilde{b}\tilde{d}+\tilde{c}\tilde{d}(\tilde{C}^{-1})\tilde{\lambda}(\mu+\partial) \]

12
2.2. Consider a subset \( r \) since \( \lambda + \mu + 0 = 0 \) for \( \lambda, \mu, 0 \) \( \in \mathbb{R} \). On the other hand, the result in this section can be understood as a super-

Consider the PVSA \( C(\lambda) \in \mathcal{M}(\lambda) \) in Theorem 2.10. Then the following statements hold.

(a) For any \( i \in I \) and \( a \in \mathcal{P} \), we have \( \{a \lambda \theta_i\}^D = \{\theta_i \lambda a\}^D = 0 \).

(b) Consider the differential algebra ideal \( \langle \theta_1 \rangle \) of \( \mathcal{P} \) generated by \( \theta_1 \). Then the corresponding Dirac reduction induces a PVSA structure on \( \mathcal{P} / \langle \theta_1 \rangle \). In other words, the bracket \( \{\lambda\}^D : \mathcal{P} / \langle \theta_1 \rangle \times \mathcal{P} / \langle \theta_1 \rangle \to \mathcal{P} / \langle \theta_1 \rangle \) is a well-defined Poisson \( \lambda \)-bracket.

Proof. For \( i \in I \) and \( a \in \mathcal{P} \), we have

\[
\{a \lambda \theta_i\}^D = \{a \lambda \theta_i\} - \sum_{i, j \in I} (-1)^{\alpha + \beta + \gamma} \{\theta_i \lambda \theta_j\} \to (C^{-1})_{\beta \alpha} (\lambda + 0) (a \lambda \theta_i) = 0
\]

since

\[
\sum_{i, j \in I} (-1)^{i + \beta + \alpha + j} \{\theta_i \lambda \theta_j\} \to (C^{-1})_{\beta \alpha} (\lambda) = \delta_{i \alpha}.
\]

In addition, the equality \( \{\theta_i \lambda a\}^D = 0 \) holds by the skewsymmetry. The statement (b) directly follows from (a).

Theorem 2.10. Consider the PVSA \( \mathcal{P} \) and the invertible element \( C(\lambda) \in \mathcal{M}(\lambda) \) in Theorem 2.9. Then the following statements hold.

(a) For any \( i \in I \) and \( a \in \mathcal{P} \), we have \( \{a \lambda \theta_i\}^D = \{\theta_i \lambda a\}^D = 0 \).

(b) Consider the differential algebra ideal \( \langle \theta_1 \rangle \) of \( \mathcal{P} \) generated by \( \theta_1 \). Then the corresponding Dirac reduction induces a PVSA structure on \( \mathcal{P} / \langle \theta_1 \rangle \). In other words, the bracket \( \{\lambda\}^D : \mathcal{P} / \langle \theta_1 \rangle \times \mathcal{P} / \langle \theta_1 \rangle \to \mathcal{P} / \langle \theta_1 \rangle \) is a well-defined Poisson \( \lambda \)-bracket.

Proof. For \( i \in I \) and \( a \in \mathcal{P} \), we have

\[
\{a \lambda \theta_i\}^D = \{a \lambda \theta_i\} - \sum_{i, j \in I} (-1)^{\alpha + \beta + \gamma + \delta} \{\theta_i \lambda \theta_j\} \to (C^{-1})_{\beta \alpha} (\lambda + 0) (a \lambda \theta_i) = 0
\]

since

\[
\sum_{i, j \in I} (-1)^{i + \beta + \alpha + j} \{\theta_i \lambda \theta_j\} \to (C^{-1})_{\beta \alpha} (\lambda) = \delta_{i \alpha}.
\]

In addition, the equality \( \{\theta_i \lambda a\}^D = 0 \) holds by the skewsymmetry. The statement (b) directly follows from (a).

2.3. Dirac reduction of Poisson superalgebras.

As a finite analogue of Section 2.2, we briefly introduce the Dirac reduction for Poisson superalgebras. On the other hand, the result in this section can be understood as a super-

analogue of [17].

Let \( \{P, \{\cdot, \cdot\} \} \) be a Poisson superalgebra and \( I = I_0 \cup I_1 \) be the same index set as in Section 2.2. Consider a subset \( \theta_I = \{\theta_i | i \in I\} \subseteq P \) such that \( (-1)^{\theta_i} = (-1)^i \) and the matrix

\[
C := \sum_{i, j \in I} e_{ij} \otimes \{\theta_i, \theta_j\} \in \text{Mat}(r | s) \otimes P
\]

where \( r = |I_0| \) and \( s = |I_1| \).
Definition 2.11. Assuming the element $C$ in (2.40) is invertible, the Dirac reduced bracket on $P$ associated with $\theta_I$ is the bilinear map $\{\cdot,\cdot\}_D : P \times P \to P$ given by
\[
\{a, b\}_D = \{a, b\} - \sum_{i,j\in I} (-1)^{ij+1} \{a, \theta_i\} (C^{-1})_{ij} \{\theta_j, b\}.
\]
(2.41)

By the similar proof to Theorem 2.9 and 2.10 we get the following theorem.

Theorem 2.12. Suppose the matrix $C$ in (2.40) is invertible.
(a) The Dirac reduced bracket $\{\cdot,\cdot\}_D$ given in (2.41) is a Poisson bracket on $P$.
(b) All the elements in $\theta_I$ are central with respect to $\{\cdot,\cdot\}_D$.
(c) Let $\langle \theta_I \rangle$ be the associative algebra ideal of $P$ generated by $\theta_I$. Then the bracket $\{\cdot,\cdot\}_D$ induces a well-defined Poisson bracket on $P/\langle \theta_I \rangle$.

\[\square\]

3. Dirac reduction of supersymmetric Poisson vertex algebras

In this section, we introduce non-local $N_k = 1$ supersymmetric (SUSY) Poisson vertex algebras (PVAs) and their Dirac reduction. Since we only deal with the $N_k = 1$ case, we use the terms SUSY LCAs and SUSY PVAs instead of $N_k = 1$ SUSY LCAs and $N_k = 1$ SUSY PVAs, respectively. For detailed properties of local SUSY PVAs, we refer to [22].

3.1. Supersymmetric Poisson vertex algebras.

For an odd indeterminate $\chi$ and an even indeterminate $\lambda$, let us consider the noncommutative associative algebra $\mathbb{C}[\Lambda]$ generated by $\Lambda = (\lambda, \chi)$ subject to the relations:
\[
\chi^2 = -\lambda, \quad \chi\lambda = \lambda\chi.
\]
(3.1)

Then $\mathbb{C}[\Lambda] \cong \mathbb{C}[\lambda] \oplus \chi \mathbb{C}[\lambda]$ as vector superspaces. Similarly, we define another associative algebra $\mathbb{C}((\Lambda^{-1})) := \mathbb{C}((\lambda^{-1})) \oplus \chi \mathbb{C}((\lambda^{-1}))$ generated by $\chi, \lambda$ and $\lambda^{-1}$ satisfying $\lambda\lambda^{-1} = \lambda^{-1}\lambda = 1, \chi\lambda^{-1} = \lambda^{-1}\chi$ along with the relation (3.1).

Let $\mathcal{R}$ be a superspace endowed with an odd derivation $D : \mathcal{R} \to \mathcal{R}$. In other words, we consider a $\mathbb{C}[D]$-module $\mathcal{R}$. We extend the $\mathbb{C}[D]$-module structure to $\mathcal{R}((\lambda^{-1})) := \mathbb{C}((\lambda^{-1})) \otimes \mathcal{R}$ via the relations
\[
D\chi = -\chi D + 2\lambda, \quad D\lambda = \lambda D.
\]
(3.2)

A $(N_k = 1)$ SUSY non-local $\Lambda$-bracket is a parity reversing bilinear map
\[
\langle \cdot , \cdot \rangle : \mathcal{R} \times \mathcal{R} \to \mathcal{R}((\lambda^{-1}))
\]
satisfying the sesquilinearity:
\[
[D a_\Lambda b] = \chi[a_\Lambda b], \quad [a_\Lambda D b] = -((-1)^\tilde{a}(D + \chi))[a_\Lambda b]
\]
(3.3)
for $a, b \in \mathcal{R}$. We denote the coefficient of $\lambda^n\chi^i$ of $[a_\Lambda b]$ for $n \in \mathbb{Z}$, $i \in \{0, 1\}$ by $a_{(n,i)} b \in \mathcal{R}$ so that $[a_\Lambda b] = \sum_{n \in \mathbb{Z}, i \in \{0, 1\}} \lambda^n \chi^i a_{(n,i)} b$. In addition, we let $\partial := D^2$.

Consider a pair $\Gamma = (\mu, \gamma)$ of an even indeterminate $\mu$ and an odd indeterminate $\gamma$ and the associative algebra $\mathbb{C}((\Gamma^{-1}))$ which is isomorphic to $\mathbb{C}((\Lambda^{-1}))$ via the map defined by $\mu \mapsto \lambda$ and $\gamma \mapsto \chi$. Then
\[
[a_\Lambda [b_{\Gamma c}]] = \sum_{m, n \in \mathbb{Z}, i, j \in \{0, 1\}} (-1)^{(\tilde{a}+1)\mu n}\gamma^j \lambda^m \chi^i a_{(m,i)} (b_{(n,j)c}) \in \mathcal{R}((\lambda^{-1}))((\Gamma^{-1})).
\]
(3.4)
Assuming $\Lambda$ and $\Gamma$ supercommute, one can check that $\mathcal{R}((\Lambda^{-1}))((\Gamma^{-1})) \simeq \mathbb{C}_\chi \otimes \mathcal{R}((\Lambda^{-1}))((\mu^{-1}))$ where $\mathbb{C}_\chi := \mathbb{C} \oplus \mathbb{C}_\chi \oplus \mathbb{C}_\gamma \oplus \mathbb{C}_\chi \gamma$. Recall the notation (2.3) and denote $\mathcal{R}_\Lambda, \mathcal{R}_\Gamma := \mathbb{C}_\chi \otimes \mathcal{R}_{\Lambda, \mu}$. The superspace $\mathcal{R}_{\Lambda, \Gamma}$ can be embedded in $\mathcal{R}((\Lambda^{-1}))((\Gamma^{-1}))$ and if

$$[a_\Lambda[b_\Gamma c]] \in \mathcal{R}_{\Lambda, \Gamma}$$

for any $a, b, c \in \mathcal{R}$, then the $\Lambda$-bracket is called \textit{admissible}.

\textbf{Definition 3.1.} A \textit{non-local SUSY LCA} is a $\mathbb{C}[D]$-module $\mathcal{R}$ endowed with an admissible non-local $\Lambda$-bracket which satisfies the following axioms for $a, b, c \in \mathcal{R}$:

- (skewsymmetry) $[a_\Lambda b] = (-1)^{\hat{a}\hat{b}} [b_{-\Lambda - \nabla} a]$, for $-\Lambda - \nabla = (-\lambda - \partial, -\chi - D)$,
- (Jacobi identity) $[a_\Lambda[b_\Gamma c]] + (-1)^{\hat{a}}[[a_\Lambda b]_{\Lambda + \Gamma} c] = (-1)^{\hat{a} + 1} \hat{b} [b_\Gamma [a_\Lambda c]]$ in $\mathcal{R}_{\Lambda, \Gamma}$.

The RHS of the skewsymmetry can be written as

$$[-[b_{-\Lambda - \nabla} a] = \sum_{n \in \mathbb{Z}, i = 0, 1} (-\lambda - \partial)^n (-\chi - D)^i b_{(n|i)} a$$

and the Jacobi identity can be understood via (3.4) and

$$\left[\chi^i a_{(n|i)} b_{\Lambda + \Gamma} c\right] = (-\chi)^i [a_{(n|i)} b_{\Lambda + \Gamma} c].$$

\textbf{Definition 3.2.} A \textit{non-local SUSY PVA} is a tuple $(\mathcal{P}, \mathcal{D}, \{\Lambda\cdot\})$ which satisfies the following axioms:

- $(\mathcal{P}, \mathcal{D}, \cdot)$ is a non-local SUSY LCA.
- $(\mathcal{P}, \mathcal{D}, \cdot)$ is an unital supercommutative associative differential algebra.
- \{a_\Lambda bc\} = \{a_\Lambda b\} c + (-1)^{\hat{a} + \hat{b} + \hat{c}} b\{a_\Lambda c\}$ for $a, b, c \in \mathcal{P}$.

Note that the last axiom is called the \textit{left Leibniz rule}.

\textbf{Proposition 3.3.} A non-local SUSY PVA is a non-local PVSA. More precisely, if $\mathcal{P}$ is a non-local SUSY PVA, then the $\lambda$-bracket defined by

$$\{a_\Lambda b\} := \sum_{n \in \mathbb{Z}} \lambda^n a_{(n|1)} b, \quad a, b \in \mathcal{P}$$

induces a non-local PVSA structure on $\mathcal{P}$.

\textbf{Proof.} The sesquilinearity, skewsymmetry, Jacobi identity and Leibniz rule of PVAs can be checked similarly to Proposition 4.3 in \cite{35}. The only thing to check is the admissibility. By (3.4), we have

$$\{a_\Lambda \{b_\mu c\}\} = \sum_{m, n \in \mathbb{Z}} \mu^n \lambda^m a_{(n|1)} (b_{(n|1)} c)$$

and the admissibility of the $\Lambda$-bracket implies $\{a_\Lambda \{b_\mu c\}\} \in \mathcal{R}_{\Lambda, \mu}$. \hfill \hspace{1cm} \Box

We note that non-local SUSY $\Lambda$-bracket on a $\mathbb{C}[D]$-module $\mathcal{R}$ is admissible if and only if the induced $\lambda$-bracket on $\mathcal{R}$ is admissible. The only if part can be shown as in the proof of Proposition 3.3. In order to see the converse, observe that if the $\lambda$-bracket is admissible then $[a_\Lambda[b_\mu c]], \{D\alpha_\Lambda[b_\mu c], \{a_\Lambda[D\beta_\mu c]\}, \{D\alpha_\Lambda[D\beta_\mu c]\} \in \mathcal{R}_{\Lambda, \mu}$ for $a, b, c \in \mathcal{R}$. In other words, $\sum_{n, m \in \mathbb{Z}} \lambda^n \mu^m a_{(n|1)} (b_{(n|1)} c) \in \mathcal{R}_{\Lambda, \mu}$ for $i, j = 0, 1$. Hence, by (3.4), we have $[a_\Lambda[b_\mu c]] \in \mathcal{R}_{\Lambda, \Gamma}$.

\textbf{Proposition 3.4.}
\( \text{(1)} \) Let \( V \) be a vector superspace and \( \mathcal{R}(V) := \mathbb{C}[D] \otimes V \). Suppose 
\[
\{ \cdot, \cdot \} : V \times V \to \mathcal{R}(V)((\Lambda^{-1}))
\]
is an odd bilinear map satisfying the admissibility, skewsymmetry and Jacobi identity of SUSY LCAs. Then \( \Lambda \)-bracket can be extended to \( \mathcal{R}(V) \) by the sesquilinearity, which gives a non-local SUSY LCA structure on the space \( \mathcal{R}(V) \).

\( \text{(2)} \) Let \( \mathcal{R} \) be a SUSY LCA. Then the supersymmetric algebra \( \mathcal{P} := S(\mathcal{R}) \) endowed with the \( \Lambda \)-bracket induced by that of \( \mathcal{R} \) by Leibniz rule is a SUSY PVA.

\textbf{Proof.} Since \( \chi \mathcal{R}_{\Lambda, \Gamma}, \gamma \mathcal{R}_{\Lambda, \Gamma} \), and \( (D + \chi + \gamma) \mathcal{R}_{\Lambda, \Gamma} \) are subsets of \( \mathcal{R}_{\Lambda, \Gamma} \), we have
\[
[D\mathcal{V}[V_\Gamma V]], [V_\Lambda[D\mathcal{V}V]], [D\mathcal{V}[V_\Gamma D\mathcal{V}]] \subset \mathcal{R}_{\Lambda, \Gamma}.
\]
One can inductively show that the \( \Lambda \)-bracket on \( \mathcal{R}(V) \) is admissible. For the skewsymmetry and Jacobi identity, we refer to \([6]\). Hence we showed (1). Similarly, the skewsymmetry and Jacobi identity for (2) can be checked as Theorem 2.15 in \([6]\). The admissibility of the \( \Lambda \)-bracket on \( S(\mathcal{R}) \) is equivalent to the admissibility of the corresponding \( \Lambda \)-bracket \([5,6]\) and it already has been shown in Proposition 2.3.

Let us consider the differential algebra
\[
\mathcal{P} = \mathbb{C}[u_i^{[n]} | i \in I, n \in \mathbb{Z}_+]
\]  

of polynomials generated by homogeneous variables \( u_i \) endowed with the odd derivation \( D \) such that \( D : u_i^{[n]} \mapsto u_i^{[n+1]} \). If we denote \( \tilde{i} := \hat{i} \), then \( \tilde{u}_i^{[n]} \equiv \hat{i} + n \mod 2 \). Denote by \( \frac{\partial}{\partial u_i^{[m]}} : \mathcal{P} \to \mathcal{P} \) the derivation of parity \( \tilde{u}_i^{[m]} \), which satisfies the property \( \frac{\partial}{\partial u_i^{[m]}}(u_j^{[n]}) = \delta_{i,j}\delta_{m,n} \). If \( \mathcal{P} \) is a non-local SUSY PVA, then the \( \Lambda \)-brackets between two elements \( a, b \in \mathcal{P} \) can be obtained by the following theorem.

\textbf{Theorem 3.5} \([6]\), Master formula. Suppose \( \mathcal{P} \) in (3.9) is a non-local SUSY PVA. For \( a, b \in \mathcal{P} \), we have
\[
\{ a, b \} = \sum_{i,j \in I, m,n \in \mathbb{Z}_+} S(a_{i,m}, b_{j,n}) \{ a_{i,m+n} \} \to a_{i,m} \to a_{i,m+n} \to a_{i,m}
\]
and
\[
\{ a, b \} = \sum_{i,j \in I, m,n \in \mathbb{Z}_+} S(a_{i,m}, b_{j,n}) (-1)^{n(i+m+1)+m(i+j+1)+m(m-1)/2} \to b_{j,m} \to (\chi + D)^n \{ u_{i+m+n} u_j \} \to a_{i,m+n} \to a_{i,m},
\]
where
- \( \Lambda + \nabla = (\lambda + \partial, \chi + D) \),
- \( a_{i,m} := \frac{\partial}{\partial u_i^{[m]}} a \) and \( b_{j,n} := \frac{\partial}{\partial u_j^{[n]}} b \),
- \( S(a_{i,m}, b_{j,n}) := (-1)^{b_{j,n}} (-1)^{b_{j,n+1}} (-1)^{\tilde{u}_j^{[n+1]}} (-1)^{\tilde{u}_j^{[n+1]}} (-1)^{\tilde{a}_{i,m}} \tilde{u}_j^{[n]}. \)

\textbf{Example 3.6} (affine SUSY PVA). Let \( \mathfrak{g} \) be a simple finite Lie superalgebra with a nondegenerate supersymmetric invariant even bilinear form \( (\cdot, \cdot) \). For the parity reversed space \( \bar{\mathfrak{g}} \) of \( \mathfrak{g} \), the affine SUSY PVA is \( \mathcal{P}(\mathfrak{g}) = S(\mathbb{C}[D] \otimes \bar{\mathfrak{g}}) \) endowed with the \( \Lambda \)-bracket
\[
\{ a \bar{\Lambda} b \} = (-1)^{\tilde{a}} (\overline{[a,b]} + k\chi(a,b))
\]
for \( a, b \in \mathfrak{g} \). On the other hand, since
\[ \{D\bar{a}, D\bar{b}\} = \chi \{\bar{a}, \bar{b}\} = -\lambda[a, b] + \chi(D[a, b] + k\lambda(a[b))]. \]

the affine PVSA of level \( k \) can be embedded in \( \mathcal{V}^k(\mathfrak{g}) \) via the injective PVSA homomorphism \( a \mapsto D\bar{a} \) for \( a \in \mathfrak{g} \).

**Example 3.7.** Let \( \mathcal{P} = \mathbb{C}[u[n]|n \in \mathbb{Z}_+ \) be the differential algebra generated by an even variable \( u \). The bracket

\[ \{u_A u\} = \lambda^{-1} \chi \]

defines a non-local SUSY PV structure on \( \mathcal{P} \). The map from the non-local PVSA in Example 2.9 to \( \mathcal{P} \) defined by \( u \mapsto u \) is an injective PVSA homomorphism.

3.2. **Dirac reduction of supersymmetric PVAs.**

Let \( \mathcal{P} \) be a SUSY PV and let \( \text{Mat}_{(r,s)} = \text{End}(C_{(r,s)}) \) for nonnegative integers \( r \) and \( s \). Consider \( F(\lambda) = F_0(\lambda) + F_1(\lambda)\chi, G(\lambda) = G_0(\lambda) + G_1(\lambda)\chi \in \mathcal{P}(\Lambda^{-1}) \) for \( F_i(\lambda), G_i(\lambda) \in \mathcal{P}(\Lambda^{-1}) \) where \( i \in \{0, 1\} \). Then the product \( \circ \) on \( \mathcal{P} \) defined by

\[ F(A) \circ G(\lambda) := F(A + \vartheta)G(\lambda) = (F_0(\lambda + \vartheta) + F_1(\lambda + \vartheta)(\chi + D))(G_0(\lambda) + G_1(\lambda)\chi) \]

gives an associative algebra structure on \( \mathcal{P}(\Lambda^{-1}) \). The adjoint \( F^*(\lambda) \in \mathcal{P}(\Lambda^{-1}) \) of \( F(\lambda) \in \mathcal{P}(\Lambda^{-1}) \) is defined by

\[ F^*(\lambda) = \sum_{n \in \mathbb{Z}_+} \left( (-\lambda - \vartheta)^n F(n[0]) + (-1)^{k+1}(-\chi - D)((-\lambda - \vartheta)^n F(n[1]) \right), \]

where \( F(\lambda) = \sum_{n \in \mathbb{Z}} (F(n[0])\lambda^n + F(n[1])\lambda^n) \).

Similarly, the superspace \( \mathcal{M}_{(r,s)}(\Lambda) := \text{Mat}_{(r,s)} \otimes \mathcal{P}(\Lambda^{-1}) \) is an associative algebra with respect to the product

\[ (a \otimes F(\lambda)) \circ (b \otimes G(\lambda)) = (-1)^{k\lambda} ab \otimes F(\lambda) \circ G(\lambda). \quad (3.10) \]

The multiplication \( (3.10) \) can be generalized to the map:

\[ \circ : \left( \text{Mat}_{(p_1,q_1)}(r_1,s_1) \otimes \mathcal{P}(\Lambda^{-1}) \right) \times \left( \text{Mat}_{(p_2,q_2)}(r_2,s_2) \otimes \mathcal{P}(\Lambda^{-1}) \right) \rightarrow \text{Mat}_{(p_1,q_1)}(r_2,s_2) \otimes \mathcal{P}(\Lambda^{-1}) \]

\[ \left((a \otimes F(\lambda)), (b \otimes G(\lambda)) \right) \mapsto \left((-1)^{k\lambda} ab \otimes F(\lambda + \vartheta)G(\lambda) \right) \]

for \( p_1, q_1, r_1 = p_2, q_2, r_2, s_2 \in \mathbb{Z}_+ \). Likewise Section 2.9 we also denote \( \mathcal{M}_{(r,s)}(\Lambda) \) by \( \mathcal{M}(\Lambda) \).

Since the identity element in \( \mathcal{M}(\Lambda) \) is \( \text{Id}_{(r,s)} \otimes 1 \), an element \( A(\lambda) \in \mathcal{M}(\Lambda) \) is called **invertible** if there is an element \( A^{-1}(\lambda) \in \mathcal{M}(\Lambda) \) such that

\[ A(\lambda) \circ A^{-1}(\Lambda) = A^{-1}(\Lambda) \circ A(\lambda) = \text{Id}_{(r,s)} \otimes 1. \quad (3.11) \]

Let \( I = I_0 \sqcup I_1 \subset \mathbb{Z} \) be a finite index set, where \( I_i = I \cap (2\mathbb{Z} + i) \) and \( i \in \{0, 1\} \). Let \( \theta_I := \{ \theta_i | i \in I \} \) be a subset of \( \mathcal{P} \) consisting of homogeneous elements such that \( (-1)^{\theta_i} = (-1)^i \). For \( r = |I_0| \) and \( s = |I_1| \) consider the element

\[ C(\Lambda) := \sum_{i \in I} e_{ij} \otimes \{ \theta_{j, \Lambda} \theta_i \} \in \mathcal{M}(\Lambda). \quad (3.12) \]

**Definition 3.8.** Suppose \( C(\Lambda) \) in \( (3.12) \) is invertible. The **Dirac reduced bracket** of the SUSY \( \Lambda \)-bracket \( \{\cdot, \cdot\} \) on \( \mathcal{P} \) associated with \( \theta_{I} \) is the bilinear map

\[ \{\cdot, \cdot\}^D : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}(\Lambda^{-1}) \]

\[ (3.13) \]
given by
\[
\{a_\Lambda b\}^D = \{a_\Lambda b\} - \sum_{i,j \ell} (-1)^{(\bar{a}+j)(\bar{b}+i)} \{\theta_{j_\Lambda \Lambda + \nabla} b\}_{\ell} (C^{-1})_{ji} (\Lambda + \nabla) \{a_\Lambda \theta_i\} \tag{3.14}
\]
for \(a, b \in \mathcal{P}\).

**Proposition 3.9.** Let \(\mathcal{P} = \mathbb{C}[\theta_k^n | k \in I_0 \cup I_1, n \in \mathbb{Z}_+\) be a SUSY PVA. If \(C(\Lambda)\) in \(\mathcal{M}(\Lambda)\) is invertible, the corresponding Dirac reduced bracket is the trivial bracket.

**Proof.** We use the SUSY analogue of the argument proving Proposition 2.7.

By the master formula in Theorem 3.5, we have
\[
\{a_\Lambda b\} = \sum_{i,j \ell, m,n \in \mathbb{Z}_+} (-1)^{\bar{a}_b \bar{b}_m + bn + im + j + n + m} \frac{\partial b}{\partial \theta_{j}^m} (\chi + D)^n \{\theta_{i \Lambda \Lambda + \nabla} \theta_{j}\} (\Lambda + \nabla) \frac{\partial a}{\partial \theta_{i}^m} \tag{3.15}
\]
for \(a, b \in \mathcal{P}\). If we identify \(\mathcal{P}(\Lambda^{-1}) = \text{Mat}_{(10)}(\mathbb{C}) \otimes \mathcal{P}(\Lambda^{-1})\), then the RHS of \(\text{(3.14)}\) can be written as
\[
\sum_{m,n \in \mathbb{Z}_+} (-1)^{\bar{a}_b \bar{b}_m + bn + im + j + n + m} \frac{\partial b}{\partial \theta_{j}^m} (\chi + D)^n \left( e_{ji} \otimes C_{ji}(\Lambda + \nabla) \right) (e_i \otimes (\Lambda + \nabla) \frac{\partial a}{\partial \theta_{i}^m} \right) \tag{3.16}
\]
for given \(i,j \in I\). Note that
\[
e_{ji} \otimes C_{ji}(\Lambda) = \sum_{k,l \ell} (e_{jk} \otimes C_{jk}(\Lambda + \nabla))(e_{kl} \otimes (C^{-1})_{kl}(\Lambda + \nabla))(e_{li} \otimes C_{li}(\Lambda)) \tag{3.17}
\]
and
\[
\{a_\Lambda \theta_j\} = \sum_{i \ell, m \in \mathbb{Z}_+} (-1)^{(\bar{a}+i+m)(\bar{b}+i+j+1) + \frac{m(m-1)}{2}} \{\theta_{i \Lambda \Lambda + \nabla} \theta_{j}\} (\Lambda + \nabla) \frac{\partial a}{\partial \theta_{i}^m}, \tag{3.18}
\]
\[
\{\theta_{i \Lambda} b\} = \sum_{j \ell, m \in \mathbb{Z}_+} (-1)^{(\bar{b}+j+n)(\bar{b}+j+n+1) + n(1)} \frac{\partial b}{\partial \theta_{j}^m} (\chi + D)^n \{\theta_{i \Lambda} \theta_{j}\} \tag{3.19}
\]
if we use the fact that \([m \quad 2] = \frac{m(m-1)}{2} \mod 2\). Now substitute \((e_{ji} \otimes C_{ji}(\Lambda + \nabla))\) in \(\text{(3.16)}\) with the RHS of \(\text{(3.17)}\). Then by \(\text{(3.18)}\) and \(\text{(3.19)}\), we get
\[
\{a_\Lambda b\} = \sum_{k,l \ell} (-1)^{(\bar{a}+k)(\bar{b}+l)} (e_{kl} \otimes \{\theta_{k \Lambda \Lambda + \nabla} \theta_{l}\}) (C^{-1})_{kl}(\Lambda + \nabla) (e_{li} \otimes \{a_\Lambda \theta_{l}\})
\]
\[
= \sum_{k,l \ell} (-1)^{(\bar{a}+k)(\bar{b}+l)} \{\theta_{k \Lambda \Lambda + \nabla} \theta_{l}\} (C^{-1})_{kl}(\Lambda + \nabla) \{a_\Lambda \theta_{l}\} \tag{3.20}
\]
Hence, \(\{a_\Lambda b\}^D = \{a_\Lambda b\} - \text{RHS of (3.20)} = 0. \square

Let us introduce the following lemma which is needed to prove Theorem 3.11.

**Lemma 3.10.** Let \(\{\cdot, \cdot\} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}(\Lambda^{-1})\) be a non-local SUSY \(\Lambda\)-bracket on \(\mathcal{P}\). Suppose \(C(\Lambda) = (C_{ij}(\Lambda))_{i,j \in I} \in \mathcal{M}(\Lambda)\) in \(\mathcal{M}(\Lambda)\) is invertible and let \((C^{-1})(\Lambda) = ((C^{-1})_{ij}(\Lambda))_{i,j \in I}\) be its inverse. For \(a \in \mathcal{P}\), assume that
\{\alpha \mathcal{C}_{ij}(\Gamma)\} \in \mathcal{P}_{\Lambda, \Gamma} \text{ for all } i, j \in I.

Then we have \{\alpha (C^{-1})_{ij}(\Gamma)\}, \{(C^{-1})_{ij}(\Gamma) \Lambda^+ a\} \in \mathcal{P}_{\Lambda, \Gamma}. \text{ If } a \in \mathcal{P} \text{ is homogenous, then}

\begin{align*}
\{\alpha (C^{-1})_{ij}(\Gamma)\} &= - \sum_{r,t,I,n \in \mathbb{Z}} (-1)^{(i+t)(j+r)+(\tilde{a}+1)(i+r+1)} (C^{-1})_{ir}(\Lambda + \Gamma + \nabla) \{\alpha C_{rt}(\Gamma + \nabla)\} (C^{-1})_{ij}(\gamma) \tag{3.21} \\
\{(C^{-1})_{ij}(\Lambda) \Lambda^+ a\} &= - \sum_{r,t,I,n \in \mathbb{Z}} (-1)^{(i+j+r+t)+(r+j)(t+j)} \{C_{rt}(\Lambda + \nabla) \Lambda^+ \nabla a\}_{\cdot} (C^{-1})_{ij}(\Lambda)(C^{-1})_{ir}^*(\Gamma). \tag{3.22}
\end{align*}

\textbf{Proof.} It is same argument as in the proof of Lemma 2.8. For the admissibility, we can apply the similar proof to Lemma 2.4 in [10].

Let us show (3.21) and (3.22). Since \((C^{-1})(\Lambda) \circ C(\Lambda) = C(\Lambda) \circ (C^{-1})(\Lambda) = \text{Id}_{(r,t)} \otimes 1\), we have

\[(C^{-1})_{ij}(\Lambda) = \sum_{r,t,I} (-1)^{(i+t)(j+r)} (C^{-1})_{ir}(\Lambda) \circ C_{rt}(\Lambda) \circ (C^{-1})_{ij}(\Lambda) \tag{3.23}
\]

and

\[\sum_{r,t,I} (-1)^{(i+r+1)(r+t)} C_{ir}^{-1}(\Gamma) \circ C_{rt}(\Gamma) = \sum_{r,t,I} (-1)^{(r+t+1)(r+i)} C_{ir}^{-1}(\Gamma) \circ C_{rt}^{-1}(\Gamma) = \delta_{it}. \tag{3.24}\]

Now, using the left and right Leibniz rules, we obtain

\begin{align*}
\{\alpha (C^{-1})_{ij}(\Gamma)\} &= \sum_{r,t,I} (-1)^{(i+t)(j+r)} \{\alpha C_{ir}^{-1}(\Gamma) \circ C_{rt}(\Gamma) \circ C_{ij}^{-1}(\Gamma)\} \\
&= \sum_{t,I} (-1)^{(i+t)(j+r)} \{\alpha \sum_{r,I} (-1)^{(i+r+1)(r+t)} C_{ir}^{-1}(\Gamma) \circ C_{rt}(\Gamma)\} C_{ij}^{-1}(\Gamma) \tag{3.25} \\
+ \sum_{r,I} (-1)^{(i+j)(r+t)+(\tilde{a}+1)(i+r+1)} C_{ir}^{-1}(\Lambda + \Gamma + \nabla) \{\alpha \sum_{t,I} (-1)^{(r+t+1)(t+j)} C_{rt}(\Gamma) \circ C_{ij}^{-1}(\Gamma)\} \tag{3.26} \\
- \sum_{r,t,I} (-1)^{(i+j)(i+r)+(\tilde{a}+1)(i+r+1)} C_{ir}^{-1}(\Lambda + \Gamma + \nabla) \{\alpha C_{rt}(\Gamma + \nabla)\} C_{ij}^{-1}(\Gamma).
\end{align*}

By (3.24), we have (3.25) = (3.26) = 0 and hence (3.21) holds. The last assertion (3.22) can be proved similarly. \qed

\textbf{Theorem 3.11.} Let \(\mathcal{P}\) be a non-local SUSY Poisson vertex algebra with the SUSY \(\Lambda\)-bracket \(\cdot_{\Lambda}\). Let \(\theta_i \in \mathcal{P}\) for each \(i \in I\) be homogeneous element. If the \(C(\Lambda) \in \mathcal{M}(\Lambda)\) defined in (3.12) is invertible, then the Dirac reduced bracket \(\cdot_{\Lambda}^D\) given by (3.14) is a SUSY \(\Lambda\)-bracket on \(\mathcal{P}\).

\textbf{Proof.} The sesquilinearity, admissibility and Leibniz rule can be proved by direct computations.
To show the skewsymmetry, we substitute $\Lambda$ by $-\Lambda - \nabla$ in (3.14) and let $\nabla$ act on the left. Then we have
\[
\{a - \Lambda - \nabla \ b\}^D = \{a - \Lambda - \nabla \ b\} - \sum_{i,j \in I} (-1)^{\tilde{a}_i + \tilde{a}_j + i + j} (\tilde{a}_i + 1)(\tilde{b}_i + 1)(\tilde{b}_j + 1)(i + j + 1)
\]
\[
\left(\{a - \Lambda - \nabla \theta_i\}\right)\left(\{X = -\Lambda - \nabla (C^{-1}) j_i (X)\}\right) \left(\{\theta_j - \Lambda - \nabla \ b\}\right)
\]
\[
= \{a - \Lambda - \nabla \ b\} - \sum_{i,j \in I} (-1)^{\tilde{a}_i + \tilde{b}_j + i + j} \{\theta_i \Lambda + \nabla a\} \{C^{-1}\} j_i (\Lambda + \nabla) \{b \Lambda \theta_j\}.
\]
Here, for the last equality, we used the fact that $(C^{-1}) j_i (\Lambda) = (-1)^{i j + i j + 1} (C^{-1})^* j_i (\Lambda)$. Then, by multiplying $(-1)^{\tilde{a}_i}$ on the both sides, we get the skewsymmetry.

Now, we need to show the following equality to see the Jacobi identity:
\[
\{a \Lambda \{b \Gamma c\}^D\}^D + (-1)^{\tilde{b}_i + \tilde{a}_j} \{b \Gamma \{a \Lambda c\}^D\}^D + (-1)^{\tilde{a}_i} \{\{a \Lambda b\}^D \Lambda + \nabla c\}^D = 0. \tag{3.27}
\]
Each of term in the LHS of (3.27) can be expanded as follows. The first term in (3.27) is
\[
\{a \Lambda \{b \Gamma c\}^D\}^D
\]
\[
= \{a \Lambda \{b \Gamma c\}\}^D - \sum_{a, \beta \in I} \{a \Lambda (-1)^{\tilde{b} + b \alpha + \tilde{c} + c \beta + \alpha \beta} \{\theta_{\beta \Gamma + \nabla c}\} (C^{-1})_{\beta \alpha} (\Gamma + \nabla) \{b \Gamma \theta_{\alpha}\}\}^D
\]
\[
= \{a \Lambda \{b \Gamma c\}\}^D - \sum_{a, \beta \in I} (-1)^{\tilde{b} + b \alpha + \tilde{c} + c \beta + \alpha \beta} \{a \Lambda \{\theta_{\beta \Gamma + \nabla c}\}\}^D (C^{-1})_{\beta \alpha} (\Gamma + \nabla) \{b \Gamma \theta_{\alpha}\}\ (3.28)
\]
\[
- \sum_{a, \beta \in I} (-1)^{\tilde{b} + b \alpha + \tilde{c} + c \beta + (\tilde{a} + 1)(\tilde{c} + \beta + 1)} \{\theta_{\beta \Lambda + \nabla + \nabla c}\} \{a \Lambda \{C^{-1}\}_{\beta \alpha} (\Gamma + \nabla)\}^D \{b \Gamma \theta_{\alpha}\}\ (3.29)
\]
\[
- \sum_{a, \beta \in I} (-1)^{\tilde{b} + b \alpha + \tilde{c} + c \beta + (\tilde{a} + 1)(\tilde{c} + \alpha)} \{\theta_{\beta \Lambda + \nabla + \nabla c}\} \{C^{-1}\}_{\beta \alpha} (\Lambda + \Gamma + \nabla) \{a \Lambda \{b \Gamma \theta_{\alpha}\}\}^D. \ (3.30)
\]
The second term in (3.27) is
\[
\{b \Gamma \{a \Lambda c\}^D\}^D
\]
\[
= \{b \Gamma \{a \Lambda c\}\}^D - \sum_{a, \beta \in I} \{b \Gamma (-1)^{\tilde{a} + a \alpha + \tilde{c} + c \beta + \alpha \beta} \{\theta_{\beta \Lambda + \nabla c}\} (C^{-1})_{\beta \alpha} (\Lambda + \nabla) \{a \Lambda \theta_{\alpha}\}\}^D
\]
\[
= \{b \Gamma \{a \Lambda c\}\}^D - \sum_{a, \beta \in I} (-1)^{\tilde{a} + a \alpha + \tilde{c} + c \beta + \alpha \beta} \{b \Gamma \{\theta_{\beta \Lambda + \nabla c}\}\}^D (C^{-1})_{\beta \alpha} (\Lambda + \nabla) \{a \Lambda \theta_{\alpha}\}\ (3.32)
\]
\[
- \sum_{a, \beta \in I} (-1)^{\tilde{a} + \bar{a} \alpha + \tilde{c} + c \beta + (\tilde{b} + 1)(\tilde{c} + \beta + 1)} \{\theta_{\beta \Lambda + \nabla + \nabla c}\} \{b \Gamma \{C^{-1}\}_{\beta \alpha} (\Lambda + \nabla)\}^D \{a \Lambda \theta_{\alpha}\}\ (3.33)
\]
\[
- \sum_{a, \beta \in I} (-1)^{\tilde{a} + \bar{a} \alpha + \tilde{c} + c \beta + (\tilde{b} + 1)(\tilde{c} + \beta + 1)} \{\theta_{\beta \Lambda + \nabla + \nabla c}\} \{C^{-1}\}_{\beta \alpha} (\Lambda + \Gamma + \nabla) \{b \Gamma \{a \Lambda \theta_{\alpha}\}\}^D. \ (3.34)
\]
and the third term in \((3.27)\) is
\[
\{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\}^D \\
= \{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\}^D - \sum_{a,\beta \in I} \{ (-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} D_{\Lambda+\Gamma} c\}^D \\
= \{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\}^D - \sum_{a,\beta \in I} (-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma\tau} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} D_{\Lambda+\Gamma} c \\
- \sum_{a,\beta \in I} A(-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} D_{\Lambda+\Gamma} c \\
- \sum_{a,\beta \in I} B(-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma\tau} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Gamma)\{a_{\Lambda} \theta_{\alpha}\} D_{\Lambda+\Gamma} c ,
\]
where \(A = (-1)^{\hat{\tau}(\hat{a}+\hat{b}+1)(\hat{a}+\hat{c}+1)}\) and \(B = (-1)^{\hat{\tau}(\hat{a}+\hat{b}+1)(\hat{a}+\hat{c}+1)}\) Moreover, each Dirac reduced bracket in \((3.28)\), \((3.39)\) can be expanded into two terms via \((3.14)\). When \((\hat{\tau})\) indicates one of \((3.28)\), \((3.39)\), let us denote \((\hat{\tau}) = (\hat{\tau}.1) + (\hat{\tau}.2)\), where \((\hat{\tau}.1)\) is induced from the first term in the RHS of \((3.14)\) and \((\hat{\tau}.2)\) is induced from the second term in the RHS of \((3.14)\). For example, \((3.28)\) is
\[
\{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\}^D \\
= \{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\} \\
- \sum_{a,\beta \in I} (-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma\tau} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} .
\]
Now, we aim to show the sum of three terms in each triple below is trivial.
\[
\begin{align*}
(3.28.1), (3.32.1), (3.36.1), & (3.28.2), (3.33.1), (3.37.1), (3.29.1), (3.32.2), (3.39.1), \\
(3.31.1), (3.35.1), (3.36.2), & (3.29.2), (3.33.2), (3.38.1), (3.30.1), (3.35.2), (3.39.2), \\
(3.30.2), (3.34.2), (3.38.2), & (3.31.2), (3.34.1), (3.37.2).
\end{align*}
\]
First, since
\[
(3.28.1) = \{a_{\Lambda} \theta_{\alpha}\}, \\
(3.32.1) = \{b_{\Gamma} \{a_{\Lambda} c\}\}, \\
(3.36.1) = \{\{a_{\Lambda} b\} D_{\Lambda+\Gamma} c\},
\]
we have
\[
(3.28.1) + (-1)^{\hat{\tau}} \hat{\tau} (3.28.1) + (-1)^{\hat{\tau}} \hat{\tau} (3.36.1) = 0
\]
which directly follows from the Jacobi identity of \(\{\cdot\}^D\).

The terms in the second triple are
\[
\begin{align*}
(3.28.2) &= - \sum_{a,\beta \in I} (-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma\tau} \{\theta_{\beta A+\nabla} b\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} , \\
(3.33.1) &= - \sum_{a,\beta \in I} (-1)^{\hat{a}\hat{b} + \hat{a}\hat{c} + \alpha\beta\gamma\tau} \{b_{\Gamma} \{\theta_{\beta A+\nabla} c\} (C^{-1})_{\beta\alpha}(\Lambda + \nabla)\{a_{\Lambda} \theta_{\alpha}\} , \\
\end{align*}
\]
\begin{equation}(3.37)\end{equation}

Hence,

\begin{equation}(3.38)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{a} + \tilde{b} + a + b + \theta \alpha + \alpha \beta + \alpha (\tilde{a} + \beta)} \{\{\theta_{\beta \lambda + \Gamma} b \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\beta a} (\Lambda + \nabla) \{a \Lambda \theta \alpha \}.$$ 

In (3.39), which is the third term in the triple, we switch \(\alpha\) and \(\beta\) and get

\begin{equation}(3.39)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{b} + \tilde{c} + a + b + \alpha \beta + 1} \{\{a \Lambda \theta \beta \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\alpha \beta} (\Gamma + \nabla) \{b \Gamma \theta \alpha \}.$$ 

Then

\begin{equation}(3.32)\end{equation}

\begin{equation}(3.39)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{b} + \tilde{c} + a + b + \alpha \beta + 1} \{\{a \Lambda \theta \beta \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\alpha \beta} (\Gamma + \nabla) \{b \Gamma \theta \alpha \}.$$ 

The three terms in the fourth triple are

\begin{equation}(3.31)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{a} + \tilde{b} + \tilde{c} + a + b + \alpha \omega + \alpha \beta + \alpha (\tilde{a} + \beta)} \{\theta_{\beta \lambda + \Gamma} b \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\beta a} (\Lambda + \nabla) \{a \Lambda \theta \alpha \}.$$ 

\begin{equation}(3.35)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{a} + \tilde{b} + \tilde{c} + a + b + \alpha \omega + \alpha \beta + \alpha (\tilde{a} + \beta)} \{\theta_{\beta \lambda + \Gamma} b \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\beta a} (\Lambda + \nabla) \{a \Lambda \theta \alpha \}.$$ 

\begin{equation}(3.36)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{a} + \tilde{b} + \tilde{c} + a + b + \alpha \omega + \alpha \beta + \alpha (\tilde{a} + \beta)} \{\theta_{\beta \lambda + \Gamma} b \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\beta a} (\Lambda + \nabla) \{a \Lambda \theta \alpha \}.$$ 

Hence,

\begin{equation}(3.31)\end{equation}

$$\sum_{a,\beta \in I} (-1)^{\tilde{a} + \tilde{b} + \tilde{c} + a + b + \alpha \omega + \alpha \beta + \alpha (\tilde{a} + \beta)} \{\theta_{\beta \lambda + \Gamma} b \}_{\lambda + \Gamma + \theta c} (C^{-1})_{\beta a} (\Lambda + \nabla) \{a \Lambda \theta \alpha \}.$$
In the fifth triple, the first term is

\[
(3.29) 2 = \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{b} + b a + \bar{c} \beta + \alpha \beta} (-1)^{\bar{a} \bar{c} + \bar{b} \alpha + a \bar{b} + \bar{d} \delta + \bar{e} \epsilon + \bar{a} \beta} \sum_{\alpha} \{ \theta_{\epsilon} X \{ \theta_{\beta} Y \ c \} \}
\]

\[
\left( \begin{array}{l}
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right) \left( \begin{array}{l}
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right)
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{b} + \bar{c} \beta + \alpha \beta} \sum_{\alpha} \{ \theta_{\epsilon} X \{ \theta_{\beta} Y \ c \} \}
\]

\[
\left( \begin{array}{l}
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right) \left( \begin{array}{l}
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right)
\]

The second term in the fifth triple is

\[
(3.33) 2 = \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{b} + \bar{c} \beta + \alpha \beta} (-1)^{\bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} (-1)^{\bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \sum_{\alpha} \{ \theta_{\epsilon} Y \{ \theta_{\beta} X \ c \} \}
\]

\[
\left( \begin{array}{l}
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right) \left( \begin{array}{l}
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right)
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{b} + \bar{c} \beta + \alpha \beta} \sum_{\alpha} \{ \theta_{\epsilon} Y \{ \theta_{\beta} X \ c \} \}
\]

\[
\left( \begin{array}{l}
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right) \left( \begin{array}{l}
| Y = \Gamma + \nabla \{ b_{\Gamma} \theta_{\alpha} \} \\
| X = \Lambda + \nabla \{ a_{\Lambda} \theta_{\alpha} \} \\
| 0 = \nabla \{ a_{\Lambda} \theta_{\alpha} \}
\end{array} \right)
\]

The last equality of the above is obtained by switching \( \alpha \) and \( \delta \) (resp. \( \beta \) and \( \epsilon \)). The third term in the fifth triple is

\[
(3.33) 1
\]

\[
= - \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \{(C^{-1})_{\beta, \alpha} \{ A + \nabla \} \{ A + \nabla \} \} - \{ a_{\Lambda} \theta_{\alpha} \} \{ \theta_{\beta} \Gamma - \nabla \}
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \{(C^{-1})_{\beta, \alpha} \{ A + \nabla \} \{ A + \nabla \} \} - \{ a_{\Lambda} \theta_{\alpha} \} \{ \theta_{\beta} \Gamma - \nabla \}
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \{(C^{-1})_{\beta, \alpha} \{ A + \nabla \} \{ A + \nabla \} \} - \{ a_{\Lambda} \theta_{\alpha} \} \{ \theta_{\beta} \Gamma - \nabla \}
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \{(C^{-1})_{\beta, \alpha} \{ A + \nabla \} \{ A + \nabla \} \} - \{ a_{\Lambda} \theta_{\alpha} \} \{ \theta_{\beta} \Gamma - \nabla \}
\]

Here, the second equality follows from Lemma 5.10 and the last equality is obtained by substituting \( \alpha \) with \( \delta \), \( \beta \) with \( \alpha \), \( \delta \) with \( \epsilon \) and \( \epsilon \) with \( \beta \). Hence,

\[
(3.29) 2 + (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b}} (3.33) 2 + (-1)^{\bar{a} (3.33) 1}
\]

\[
= \sum_{\alpha, \beta, \delta, \epsilon, \in I} (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b} + \bar{c} \beta + \alpha \beta} \{(\theta_{\epsilon} A + \nabla \{ \theta_{\beta} \Gamma + \nabla \} c \} + (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b}} \{(\theta_{\beta} \Gamma + \nabla \{ \theta_{\epsilon} A + \nabla \} c \} + (-1)^{\bar{a} \bar{a} + \bar{b} \bar{b}} \{(\theta_{\epsilon} A + \nabla \{ \theta_{\beta} \Gamma + \nabla \} c \}
\]
\[
\left((C^{-1})_{\beta\alpha} (\Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\}\right) \left((C^{-1})_{\epsilon\delta} (\Lambda + \nabla) \{ a\Lambda \theta_{\delta}\}\right) = 0.
\]

The first term in the sixth triple can be expanded by Lemma 3.10 as follows:

\[
\begin{align*}
(3.30) & = \sum_{\alpha,\beta,\delta,\epsilon \in I} (-1)^{\bar{\alpha} + \bar{\epsilon} + \bar{\delta} + \bar{\beta}} \{\theta_{\beta \Lambda + \Gamma + \nabla} c\} \left((C^{-1})_{\beta\epsilon}\right) \left((C^{-1})_{\delta\alpha}\right) (\Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\}.
\end{align*}
\]

For the other two terms, by changing indices properly, we get

\[
\begin{align*}
(3.35) & = \sum_{\alpha,\beta,\delta,\epsilon \in I} (-1)^{\bar{\beta} + \bar{\delta} + \bar{\epsilon}} \{\theta_{\beta \Lambda + \Gamma + \nabla} c\} \left(C^{-1}\right)_{\beta\epsilon} (\Lambda + \Gamma + \nabla) \{ a\Lambda \{ \theta_{\delta\Gamma + \nabla} \theta_{\epsilon}\}\} \left(C^{-1}\right)_{\delta\alpha} (\Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\}.
\end{align*}
\]

Then

\[
\begin{align*}
(3.30) & = \sum_{\alpha,\beta,\delta,\epsilon \in I} (-1)^{\bar{\alpha} + \bar{\epsilon} + \bar{\beta}} \{\theta_{\beta \Lambda + \Gamma + \nabla} \epsilon\} + \left(-1\right)^{\bar{\beta}} \{\theta_{\beta \Lambda + \Gamma + \nabla} \epsilon\} (\Lambda + \Gamma + \nabla) \{ a\Lambda \{ \theta_{\delta\Gamma + \nabla} \theta_{\epsilon}\}\} + (-1)^{\bar{\alpha}} \{ a\Lambda \theta_{\delta}\} \left(C^{-1}\right)_{\beta\epsilon} (\Lambda + \Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\} = 0.
\end{align*}
\]

Let us observe the seventh triple. We have

\[
(3.30) = \sum_{\alpha,\beta,\delta,\epsilon,\zeta,\eta \in I} (-1)^{\bar{\beta} + \bar{\delta} + \bar{\epsilon} + \bar{\alpha} + \bar{\zeta} + \bar{\eta}} \{\theta_{\beta \Lambda + \Gamma + \nabla} \epsilon\} \left(C^{-1}\right)_{\beta\epsilon} (\Lambda + \Gamma + \nabla) \{ a\Lambda \theta_{\delta}\} \left(C^{-1}\right)_{\delta\zeta} (\Lambda + \Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\} = \sum_{\alpha,\beta,\delta,\epsilon,\zeta,\eta \in I} (-1)^{\bar{\alpha} + \bar{\epsilon} + \bar{\delta} + \bar{\zeta} + \bar{\eta}} \{\theta_{\beta \Lambda + \Gamma + \nabla} \epsilon\} \left(C^{-1}\right)_{\beta\epsilon} (\Lambda + \Gamma + \nabla) \{ a\Lambda \theta_{\delta}\} \left(C^{-1}\right)_{\delta\zeta} (\Lambda + \Gamma + \nabla) \{ b\Gamma \theta_{\alpha}\}.
\]

Again, by Lemma 3.10 the other two terms in the triple can be expressed as follows:

\[
(3.31)
\]
(3.35) 2

\[ \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \delta \epsilon} \{ \theta_{\beta \Lambda+\Gamma+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \]

\[ \{ \theta_{\eta \Gamma} \{ \theta_{\epsilon X} \theta_{\zeta} \} \} \left( \left| X_{\Lambda+\nabla} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \} \right| \right) \left( \left| Y_{\Gamma+\nabla} (C^{-1})_{\eta \alpha} (\Gamma + \nabla) \{ b_{\Gamma} \theta_{\alpha} \} \right| \right) = 0. \]

Hence,

(3.30) 2 + (-1)^{\alpha \beta \epsilon \delta} (3.35) 2 + (-1)^{\alpha} (3.35) 2

= \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \delta \epsilon} \{ \theta_{\beta \Lambda+\Gamma+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \]

\[ \{ \theta_{\epsilon \Lambda+\nabla} \{ \theta_{\epsilon \Gamma+\nabla} \theta_{\delta} \} \} \left( \left| X_{\Lambda+\nabla} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \} \right| \right) \left( \left| Y_{\Gamma+\nabla} (C^{-1})_{\eta \alpha} (\Gamma + \nabla) \{ b_{\Gamma} \theta_{\alpha} \} \right| \right) = 0. \]

Finally, the three terms in the last triple are

(3.31) 2 = \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \delta \epsilon} \{ \theta_{\beta \Lambda+\Gamma+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \{ \theta_{\epsilon \Lambda+\nabla} \{ b_{\Gamma} \theta_{\alpha} \} \} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \},

(3.34) 1 = \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \delta \epsilon} \{ \theta_{\beta \Lambda+\Gamma+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \{ b_{\Gamma} \{ \theta_{\epsilon \Lambda+\nabla} \theta_{\alpha} \} \} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \},

(3.37) 2 = \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \delta \epsilon} \{ \theta_{\beta \Lambda+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \{ \theta_{\epsilon \Lambda+\nabla} b \} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \}.

Hence,

(3.31) 2 + (-1)^{\alpha \beta \epsilon \delta} (3.34) 1 + (-1)^{\alpha} (3.37) 2

= \sum_{\alpha, \beta, \delta, \epsilon, \eta \in I} (-1)^{\alpha \beta \epsilon \delta} \{ \theta_{\beta \Lambda+\nabla} C \} (C^{-1})_{\beta \epsilon} (\Lambda + \Gamma + \nabla) \]

\[ \{ \theta_{\epsilon \Lambda+\nabla} \{ b_{\Gamma} \theta_{\alpha} \} \} + (-1)^{\beta \epsilon \delta} \{ b_{\Gamma} \{ \theta_{\epsilon \Lambda+\nabla} \theta_{\alpha} \} \} + \{ \theta_{\epsilon \Lambda+\nabla} b \} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \} \left( \left| X_{\Lambda+\nabla} (C^{-1})_{\epsilon \delta} (\Lambda + \nabla) \{ a_{\Lambda} \theta_{\delta} \} \right| \right) = 0. \]

Therefore, we showed the sum of each triple is zero so that the Jacobi identity holds. \hfill \square

**Theorem 3.12.** Let \( \mathcal{P} \) be a SUSY PVA and \( C(\Lambda) \) be the matrix in (3.12), which is assumed to be invertible.

(a) All the elements in \( \theta_I \) are central with respect to \( \{ \Lambda \}^D \) in (3.14).
(b) Let \((\theta_1)\) be the differential algebra ideal of \(\mathcal{P}\) generated by \(\theta_1\). Then the bracket \(\{\cdot, \cdot\}^D\) induces a well-defined SUSY PVA bracket on \(\mathcal{P}/(\theta_1)\).

Proof. For \(i \in I\) and \(a \in \mathcal{P}\), we have

\[
\{a_{\lambda} \theta_i\}^D = \{a_{\lambda} \theta_i\} - \sum_{\alpha, \beta \in I} (-1)^{(\alpha+\beta)(i+\alpha)} \{\theta_{\beta \lambda + \nabla} \theta_i\} - (C^{-1})_{\beta \alpha} (\Lambda + \nabla) \{a_{\lambda} \theta_\alpha\}.
\]

Since

\[
\sum_{\beta \in I} (-1)^{(i+\beta)(\beta+\alpha)} \{\theta_{\beta \lambda + \nabla} \theta_i\} \rightarrow (C^{-1})_{\beta \alpha} (\Lambda) = \delta_{\alpha},
\]

the equality \(\{a_{\lambda} \theta_i\}^D = 0\) holds, and \(\{\theta_i a\}^D = 0\) also holds by the skewsymmetry of \(\{\cdot, \cdot\}^D\). Hence (a) is proved and (b) is a direct consequence of (a).

4. W-superalgebras and modified Dirac reduction of PVSA

In this section, inspired by the definition of Dirac reduced bracket, we define a modified Dirac reduced bracket on a quotient space of PVSA. We also describe Poisson structures of W-superalgebras and modified Dirac reductions of affine PVSA. For the further properties of W-superalgebras, we refer to [34, 35].

4.1. Modified Dirac reduction of PVSA

Let \(\mathcal{P}\) be a differential algebra freely generated by a finite homogeneous set \(\theta_j := \{\theta_j| j \in J\}\) with an even derivation \(\partial\), i.e., \(\mathcal{P} \cong \mathbb{C}[\partial^n \theta_j| j \in J, n \in \mathbb{Z}_{\geq 0}]\). To simplify notations, we assume that the index set \(J\) is a subset of \(\mathbb{Z}\) and \(j \in J\) is even (resp. odd) if \(\theta_j\) is even (resp. odd). Consider the differential algebra ideal \(I\) of \(\mathcal{P}\) generated by \(\theta_j := \{\theta_j| j \in J \subset J\}\) and let \(\tilde{\mathcal{P}} := \mathcal{P}/I\) be the quotient algebra. Then we can identify

\[
\tilde{\mathcal{P}} \cong \mathbb{C}[\partial^n \theta_j| j \in J \setminus I, n \in \mathbb{Z}_{\geq 0}]
\]

as differential algebras.

From now on we simply write as Mat \(_I\) = Mat \(_{(ij)\in I}\) and \(I_{d}\) = Id \(_{(ij)\in I}\), where \(r\) (resp. \(s\)) denotes the number of even (resp. odd) elements in \(\theta_j\). Note that \(\tilde{M}(\lambda) := \text{Mat}_{\lambda} \oplus \tilde{\mathcal{P}}((\lambda)^{-1})\) is a unital associative algebra with respect to the product \(\circ\) in (2.7). Then \(I_{d} \oplus 1 \in \tilde{M}(\lambda)\) for the identity matrix \(Id_{d}\) \(\in\) Mat \(_I\) is the unity and an element \(\tilde{A}(\lambda) \in \tilde{M}(\lambda)\) is called invertible if there exists \(\tilde{A}^{-1}(\lambda) \in \tilde{M}(\lambda)\) such that

\[
\tilde{A}(\lambda) \circ \tilde{A}^{-1}(\lambda) = \tilde{A}^{-1}(\lambda) \circ \tilde{A}(\lambda) = \text{Id}_{d} \oplus 1.
\]

Now we suppose \(\mathcal{P}\) is a PVSA endowed with the Poisson bracket \(\{\cdot, \cdot\}\) and consider

\[
\tilde{C}(\lambda) := \sum_{i,j \in I} \epsilon_{ij} \otimes \pi \{\theta_{\lambda \theta_i}\} \in \tilde{M}(\lambda),
\]

where \(\pi : \mathcal{P}((\lambda)^{-1}) \rightarrow \tilde{\mathcal{P}}((\lambda)^{-1})\) is the canonical quotient map.

**Definition 4.1.** Assume that \(\tilde{C}(\lambda)\) in (4.3) is invertible and \(\tilde{C}^{-1}(\lambda) = (\tilde{C}_{ij}^{-1}(\lambda))_{i,j \in I}\) is its inverse. Then the modified Dirac reduced bracket \(\pi \{\cdot, \cdot\}^D : \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}((\lambda)^{-1})\) on \(\tilde{\mathcal{P}}\) associated with \(\theta_1\) is the map defined by

\[
\pi \{a_{\lambda} b\}^D = \pi \{a_{\lambda} b\} - \sum_{i,j \in I} (-1)^{(i+j)(i+j)+j} \pi \{\theta_{i \lambda + \theta j}\} \rightarrow (\tilde{C}^{-1})_{ij} (\lambda + \partial) \pi \{a_{\lambda} \theta_j\}
\]

for \(a, b \in \tilde{\mathcal{P}}\). By (4.1), the RHS of (4.4) is computed regarding \(a, b \in \mathbb{C}[\partial^n \theta_j| j \in J \setminus I, n \in \mathbb{Z}_{\geq 0}]\).
Note that the bracket $\pi \{ \cdot, \cdot \}^D$ in (4.4) is not necessarily a PVSA $\lambda$-bracket. Especially, the Jacobi identity is not guaranteed since $\pi$ is not a LCA homomorphism. In other words, $\pi \{ a_\lambda \pi \{ b_\mu c \} \}$ may not be same as $\pi \{ a_\lambda \pi \{ b_\mu c \} \}$ for $a, b, c \in \overline{P}$ so that the Jacobi identity of $\{ \cdot, \cdot \}$ does not imply that of $\pi \{ \cdot, \cdot \}^D$. However, other axioms of PVSA hold for $\pi \{ \cdot, \cdot \}^D$ by the corresponding axioms for $\{ \cdot, \cdot \}$.

**Proposition 4.2.** If the modified Dirac reduced bracket in (4.4) satisfies the Jacobi identity then $\overline{\mathcal{P}}$ is a PVSA.

**Proof.** We need to show that (4.4) satisfies the sesquilinearity, skew-symmetry and Leibniz rule. Since $\pi$ is differential algebra homomorphism, these properties can be proved by the similar proof to Theorem 2.9. □

Now we close this section with the following lemma, which is used to describe the PVSA structures of W-superalgebras using modified Dirac reductions.

**Lemma 4.3.** Let $\theta'_j := \{ \theta'_j \} j \in I$ be another basis of a vector superspace $V$ spanned by $\theta_1$. Suppose $(-1)^{\theta'_j} = (-1)^{\theta_1}$. If $\overline{C}(\lambda)$ in (4.3) is invertible, then $\overline{\mathcal{C}}(\lambda) = (\pi \{ \theta_j, \theta'_j \} )$ is invertible and the bracket (4.4) can be written as

$$\pi \{ a_\lambda b \}^D = \pi \{ a_\lambda b \} - \sum_{i,j} (-1)^{(a+i)(b+j)} \pi \{ \theta_i \lambda \theta_j b \} \overline{C}_{ij}^1 (\lambda + \partial) \pi \{ a_\lambda \theta'_j \}. \tag{4.5}$$

**Proof.** Take the invertible matrix $K = \sum e_i \otimes K_{ij} \in \overline{M}(\lambda)$ with $K_{ij} \in \mathbb{C}$ satisfying

$$\sum_{j \in I} e_i \otimes \theta'_j = K \left( \sum_{j \in I} e_j \otimes \theta_j \right). \tag{4.6}$$

Observe that $K \overline{\mathcal{C}}(\lambda) = \overline{\mathcal{C}}(\lambda)$ and $\overline{\mathcal{C}}(\lambda)$ is invertible. Hence the matrix $\overline{\mathcal{C}}(\lambda)$ is also invertible and

$$\overline{\mathcal{C}}^{-1}(\lambda + \partial) = \overline{\mathcal{C}}^{-1}(\lambda + \partial) K^{-1}. \tag{4.7}$$

By (4.6) and (4.7), we have

$$\sum_{j \in I} (\overline{\mathcal{C}}^{-1}(\lambda + \partial) K^{-1})(K(e_j \otimes \pi \{ a_\lambda \theta_j \})) = \sum_{j \in I} \overline{\mathcal{C}}^{-1}(\lambda + \partial)(e_j \otimes \pi \{ a_\lambda \theta'_j \}). \tag{4.8}$$

By (4.8), we obtain

$$\sum_{i,j \in I} (-1)^{\tilde{a}b + \tilde{a}j} \left( e_i^T \otimes \pi \{ \theta_i \lambda \theta_j b \} \right) \overline{\mathcal{C}}^{-1}(\lambda + \partial) (e_j \otimes \pi \{ a_\lambda \theta_j \})$$

$$= \sum_{i,j \in I} (-1)^{\tilde{a}b + \tilde{a}j} \left( e_i^T \otimes \pi \{ \theta_i \lambda \theta_j b \} \right) \overline{\mathcal{C}}^{-1}(\lambda + \partial) (e_j \otimes \pi \{ a_\lambda \theta'_j \}). \tag{4.9}$$

The last terms of (4.4) and (4.5) equal to the LHS and RHS of (4.9), respectively. Hence we proved the lemma. □

### 4.2. Structures of $W$-superalgebras.

Let $\mathfrak{g}$ be a finite simple Lie superalgebra with a nondegenerate supersymmetric invariant even bilinear form $\langle \cdot, \cdot \rangle$. Suppose $F \in \mathfrak{g}$ is an nilpotent element in an $\mathfrak{s} \mathfrak{l}_2$-triple $\{ E, H, F \}$. Assume that the bilinear form is normalized by $\langle E, F \rangle = 1$. Consider the eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \tag{4.10}$$
where \( g(i) = \{ g \in g | [H, g] = ig \} \). Then \( F \in g(-1) \) and \( E \in g(1) \). Let us write
\[
\begin{align*}
g_{s2j} &= g(i), & g_{s3j} &= g(i), & g_{s4j} &= g(i), & g_{s5j} &= g(i) \\
\end{align*}
\] (4.11)
for each \( j \in \mathbb{Z}/2 \). In particular, we denote
\[
\begin{align*}
m &:= g_{s21}, & n &:= g_{s20}, & p &:= g_{s21}. \\
\end{align*}
\] (4.12)
On the other hand, by the \( sl_2 \)-representation theory, the Lie algebra \( g \) can be decomposed into
\[
g = g^F \oplus [E, g] \\
\] (4.13)
for \( g^F := \{ x \in g | [F, x] = 0 \} \subset g_{s20} \). We denote by \( a^k \) the projection of \( a \in g \) to \( g^F \).

Recall that \( W^k(g) \) is the affine PVSA of level \( k \) as defined in Example 2.4. For a subspace \( a \) of \( g \), the differential algebra
\[
\mathcal{P}(a) := S(C[\partial] \otimes a) \\
\] (4.14)
can be viewed as a subalgebra of \( W^k(g) \) as differential algebras.

**Definition 4.4.** Let \( \mathcal{F} \) be the differential algebra ideal of \( V^k(g) \) generated by \( \{ m - (F|m) | m \in m \} \) and let \( \rho : V^k(g)[\lambda] \to V^k(g)/\mathcal{F}[\lambda] \) be the canonical projection map. The \( W \)-superalgebra of level \( k \) associated with \( g \) and \( F \in g \) is defined by
\[
\mathcal{W}^k(g,F) := \{ a \in \mathcal{P}(a) | \rho(n_a) = 0 \text{ for any } n \in n \}, \\
\] (4.15)
where \( \{ \cdot , \cdot \} \) is the \( \lambda \)-bracket of \( V^k(g) \).

It is well known that \( \mathcal{W}^k(g,F) \) is a PVSA whose \( \lambda \)-bracket is induced from that of \( V^k(g) \). More precisely, since there is a canonical differential algebra isomorphism \( \iota : \mathcal{W}^k(g)/\mathcal{F}[\lambda] \to \mathcal{P}(p)[\lambda] \), the bracket \( \iota(\rho(\cdot , \cdot )) \) defined on \( \mathcal{P}(p) \) induces a PVSA \( \lambda \)-bracket on \( \mathcal{W}^k(g,F) \). In the rest of this section, we discuss \( \lambda \)-brackets between generators of the \( W \)-superalgebra described in Proposition 4.5.

Let us introduce a \( \mathbb{Z}/2 \)-grading \( \Delta \) on \( V^k(g) \), called the conformal weight. For \( a \in g(i) \), the conformal weight \( \Delta_a \) of \( a \) is defined by
\[
\Delta_a := 1 - i. \\
\] (4.16)
Let
\[
\Delta_{BA} = \Delta_A + 1, \quad \Delta_{AB} = \Delta_A + \Delta_B, \\
\] (4.17)
where \( \Delta_A \) and \( \Delta_B \) are conformal weights of homogeneous elements \( A \) and \( B \). The conformal weight \( \Delta \) naturally induces the \( \mathbb{Z}/2 \)-grading on the \( W \)-algebra \( \mathcal{W}^k(g,F) \), which we also denote by \( \Delta \). Furthermore, we can find a homogeneous generating set of \( \mathcal{W}^k(g,F) \) which satisfy the properties in the following proposition.

**Proposition 4.5.** [24] Let \( \{ q_i | i \in J^F \} \) be a basis of \( g^F \), homogeneous with respect to both conformal weight and parity. Then there exists a unique subset \( \{ \omega_i | i \in J^F \} \) of homogeneous elements in \( \mathcal{W}^k(g,F) \) satisfying the following properties:
\[
\begin{align*}
\text{(i)} \quad & \mathcal{W}^k(g,F) \simeq \mathbb{C}[\partial^n \omega | i \in J^F, n \in \mathbb{Z}_+] \text{ as differential algebras}, \\
\text{(ii)} \quad & \Delta_{\omega_i} = \Delta_{q_i} \text{ for } i \in J^F, \\
\text{(iii)} \quad & \omega_i \text{ for } i \in J^F \text{ is decomposed into} \omega_i = q_i + \sum_{n \geq 1} \gamma_i^n \\
\text{for } & \gamma_i^n \in \mathcal{P}(g^F) \otimes (\mathbb{C}[\partial] \otimes [E, g_{s20}])^{\otimes n}. \\
\end{align*}
\] (4.18)
Remark 4.6. A generating set \( \{ \omega'_i \mid i \in J^F \} \) satisfying (i) and (ii) can be found by Drinfeld-Sokolov reduction (see \cite{34}). Starting from \( \omega'_i \), we can find \( \omega_i \) by induction on the conformal weights as follows. For \( i \in J^F \), denote by \( p[\Delta_i] \) the subspace of \( p \) spanned by homogeneous elements whose conformal weights are strictly less than \( \Delta_{\alpha} \). Then
\[
\omega'_i - q_i = \sum_{n \in \mathbb{Z}} \gamma''_i^n \in \mathcal{P}(p[\Delta_i])
\]
for some \( \gamma''_i^n \in \mathcal{P}(g^F) \otimes (\mathbb{C}[\partial] \otimes [E, g_{\alpha}])^\otimes n \). The element \( \gamma''_i^0 \in \mathcal{P}(g^F) \) can be written as a differential polynomial in \( q_{\alpha} \)'s where \( \alpha \in J^F \) and \( \Delta_{\alpha} < \Delta_{\alpha} \). Consider the element \( \omega''_i^0 \) which is obtained from \( \gamma''_i^0 \) by substituting \( q_{\alpha} \)'s with \( \omega_{\alpha} \)'s. Then
\[
\omega_i := \omega'_i - \omega''_i^0
\]
is the generator satisfying (i), (ii), (iii) in Proposition 4.3.

In addition, the uniqueness of \( \{ \omega_i \mid i \in J^F \} \) in Proposition 4.3 follows from the fact that
\[
\mathcal{W}^k(g, F) \cap \sum_{n \geq 1} \mathcal{P}(g^F) \otimes (\mathbb{C}[\partial] \otimes [E, g_{\alpha}])^\otimes n = 0. \tag{4.19}
\]
It is a direct consequence of the property (i) in Proposition 4.3. By the uniqueness, the map \( \omega \) in Definition 4.7 is completely determined.

Definition 4.7. Recall the basis \( \{ q_i \mid i \in J^F \} \) of \( g^F \) and the generating set \( \{ \omega_i \mid i \in J^F \} \) of \( \mathcal{W}^k(g, F) \) in Proposition 4.3. The injective linear map \( \omega \) is defined by
\[
\omega : g^F \rightarrow \mathcal{W}^k(g, F), \quad q_i \mapsto \omega_i. \tag{4.20}
\]
In order to describe the \( \lambda \)-bracket of \( \mathcal{W}^k(g, F) \), let us introduce some notations. Let \( \{ q^1_i \mid i \in J^F \} \) be the basis of \( g^E := \{ a \in g[E, a] = 0 \} \) which is dual to \( \{ q_i \mid i \in J^F \} \) in Proposition 4.3, i.e.,\( (q^1_i | q_j) = \delta_{ij} \). For \( \alpha_i \in \mathbb{Z}/2 \) such that \( q^1 \in g(\alpha_i) \), we denote
\[
q^1_{i,m} := (adF)^m q^1_i, \quad q^m_{i,m} := k_{i,m}(adE)^m q_i. \tag{4.21}
\]
Here, we have \( 0 \leq m \leq 2\alpha_i \) and \( k_{i,m} \)'s are nonzero constants chosen to satisfy \( (q^1_{m} | q^m_{j}) = \delta_{m,m} \delta_{ij} \). For the explicit choices of \( k_{i,m} \), see Section 3 of \cite{35}.

Theorem 4.8. \cite{35} Recall the map \( \omega \) in Definition 4.7 and take \( a \in g(-t_1) \cap g^F \) and \( b \in g(-t_2) \cap g^F \). Identify the index set \( J^F \) with a subset of \( \mathbb{Z} \) so that \( (-1)^t = (-1)^i \) for any \( i \in J^F \). Then
\[
\{ \omega(a) \omega(b) \} = \omega(\{ a, b \}) + k\lambda(a|b)
\]
\[
- \sum_{p \in \mathbb{Z}, t_2 \geq t_1 < n_0 \geq 0} \cdots \sum_{j_p 
eq n_p \neq t_2} \sum_{j_t = 1}^p (-1)^{p+\tilde{a_b}+\tilde{a_j}+\tilde{p}+j_{t_0}} (\omega([q^0_n, b]^{1}) + (q^0_n | q^0_n) k(\lambda + \partial))
\]
\[
\prod_{t=1}^p (-1)^{1+t_{1}^{t_{1}}} \left( \omega([q^1_{n_1}, q^{n_{1-1}}_{j_{t_{1-1}}}]) + (q^1_{n_1} | q^{n_{1-1}}_{j_{t_{1-1}}}) k(\lambda + \partial) \right) (\omega([a, q^{n_{p+1}-1}_j]) + (a | q^{n_{p+1}}_j) k\lambda)
\]
where
- the pairs \( (j_t, n_t) \) for \( t = 0, 1, \ldots, p \) are elements of \( I = \{ (j, n) \mid j \in J^F, n = 0, 1, \ldots, 2\alpha_j - 1 \} \),
- the partial order \( \prec \) on \( \frac{\mathbb{Z}}{2} \cup I \) is given by
\[
(j, n) < t \iff n - \alpha_j + 1 \leq t, \quad t < (j, n) \iff t + 1 \leq n - \alpha_j,
\]
\[
(j, n) < (i, m) \iff n - \alpha_j + 1 \leq m - \alpha_i.
\]
4.3. \textbf{W-superalgebras via modified Dirac reduction.}

In this section, we continue using the notations defined in Section 4.2. Consider the subset
\[ \theta_I = \{ q^i_m - (F)q^i_m | (i, m) \in I \} \subset \mathcal{V}^k(\mathfrak{g}) \]  
(4.22)
for \( I = \{(i, m) | i \in J^F, 0 \leq m < 2\alpha_i \} \). The set (4.22) would play the role of \( \theta_I \) in Section 4.1. Recall in Theorem 4.8 that the index set \( J^F \) was considered as a subset of \( \mathbb{Z} \) and assume \((-1)^{\check{m}} = (-1)^i \) for \( i \in J^F \). Hence, \((-1)^{\check{m}'} = (-1)^i \) for any \((i, m) \in I \).

Let \( \mathcal{I} \) be the differential algebra ideal of \( \mathcal{V}^k(\mathfrak{g}) \) generated by \( \theta_I \) and denote the canonical projection map by
\[ \pi : \mathcal{V}^k(\mathfrak{g}) \to \mathcal{V}^k(\mathfrak{g}) / \mathcal{I} =: \hat{\mathcal{V}}^k(\mathfrak{g}) . \]

Note that both \( \{ q^i_m | (i, m) \in I \} \) and \( \{ q^j_{\check{m}+1} | (j, n) \in I \} \) are the bases of \([E, \mathfrak{g}] \). Therefore, by Lemma 4.3 one can express the modified Dirac reduced bracket on \( \hat{\mathcal{V}}^k(\mathfrak{g}) \) associated with \( \theta_I \) using
\[ C(\lambda) = \sum_{(i, m), (j, n) \in I} e_{(j, n), (i, m)} \otimes \pi \{ q^i_m \lambda q^{i+1}_{\check{m}+1} \} \in \text{Mat}_I \otimes \hat{\mathcal{V}}^k(\mathfrak{g})((\lambda^{-1})). \]  
(4.23)

Since the quotient space \( \hat{\mathcal{V}}^k(\mathfrak{g}) \) is isomorphic to \( S(\mathbb{C}[\partial] \otimes \mathfrak{g}^F) \) as superalgebras, we would identify the elements of \( \hat{\mathcal{V}}^k(\mathfrak{g}) \) with the corresponding elements of \( S(\mathbb{C}[\partial] \otimes \mathfrak{g}^F) \). Under this identification, the projection map \( \pi : \mathcal{V}^k(\mathfrak{g}) \to \hat{\mathcal{V}}^k(\mathfrak{g}) \cong S(\mathbb{C}[\partial] \otimes \mathfrak{g}^F) \) is written as
\[ \pi(a) = a^I + (F)a \]  
(4.24)
for all \( a \in \mathfrak{g} \).

**Lemma 4.9.** Recall the elements in (4.21) and let \( a \in \mathfrak{g}(-t) \cap \mathfrak{g}^F \). For the map \( \pi \) in (4.24), we have

1. \( \pi \{ q^i_m \lambda q^{i+1}_{\check{m}+1} \} = 1 \) for any \((i, m) \in I \),
2. \( \pi \{ q^i_m \lambda q^{i+1}_{\check{m}+1} \} = 0 \) if \( n - \alpha_j + 1 > m - \alpha_i \),
3. \( \pi \{ a \lambda q^{i+1}_{\check{m}+1} \} = 0 \) if \( n - \alpha_j + 1 > t \),
4. \( \pi \{ q^i_m \lambda a \} = 0 \) if \( n - \alpha_j < -t \).

**Proof.** Let us show (2). By (4.21), we have
\[ \pi \{ q^i_m \lambda q^{i+1}_{\check{m}+1} \} = \left[ q^i_m, q^{i+1}_{\check{m}+1} \right] + (q^i_m q^{i+1}_{\check{m}+1} + (q^i_m q^{i+1}_{\check{m}+1}) k \lambda \]
\[ = [q^i_m, q^{i+1}_{\check{m}+1}] + \delta_{i,j} \delta_{m,n} + \delta_{i,j} \delta_{m,n+1} k \lambda . \]  
(4.25)

Note that if the condition of (2) holds, then neither \((i, m) = (j, n)\) nor \((i, m) = (j, n + 1)\). Also, \([q^i_m, q^{i+1}_{\check{m}+1}] \in \mathfrak{g}((n - \alpha_j + 1) - (m - \alpha_i)) \subset \mathfrak{g}^F \) under the condition. Therefore, its projection to \( \mathfrak{g}^F \) vanishes. Hence we proved (2). Similarly, the others can be proved by direct computations. \( \square \)

Then, by Lemma 4.9 we have
\[ C(\lambda) = \sum_{(i, m) \in I} e_{(j, n), (i, m)} \otimes 1 + \sum_{(j, n) \in (i, m)} e_{(j, n), (i, m)} \otimes \pi \{ q^i_m \lambda q^{i+1}_{\check{m}+1} \}, \]  
(4.26)
where \( < \) is the partial order defined in Theorem 4.8.
Proposition 4.10. The element \( C(\lambda) \) in (4.23) is invertible and its inverse is
\[
(C^{-1})(\lambda) = \text{Id}_I + \sum_{p \in \mathbb{Z}_+} \sum_{(j_0, n_0) < (j_1, n_1) < \cdots < (j_p, n_p)} (-1)^{p+j_0+j_0j_p} (-1)^{j_p-1-j_{p-1}j_p} e_{(j_0, n_0), (j_p, n_p)} \otimes \prod_{t=1}^{p-1} (-1)^{j_{t-1}+j_{t-1}j_t} \pi \{ q_{n_t}^{j_t} \lambda, \partial q_{j_t-1}^{n_{t-1}+1} \} \rightarrow \pi \{ q_{n_p}^{j_p} \lambda, q_{j_{p-1}}^{n_{p-1}+1} \},
\]
where \((j_0, n_0), \cdots, (j_p, n_p) \in I\).

Proof. By the equation (4.26), we can write
\[
C(\lambda) = \text{Id}_I + T(\lambda)
\]
for the identity \( \text{Id}_I \) in \( \hat{M}(\lambda) \) and \( T(\lambda) = \sum_{(j,n) \in (i,m)} e_{(j,n), (i,m)} \otimes \pi \{ q_{m}^{j} q_{p}^{n+1} \}. \) Then \( C(\lambda) \) is invertible and its inverse is
\[
(C^{-1})(\lambda) = \text{Id} - T(\lambda) + T(\lambda) \circ T(\lambda) - T(\lambda) \circ T(\lambda) \circ T(\lambda) + \cdots
= \text{Id} - \sum_{p \in \mathbb{Z}_+} (-1)^{p} (T(\lambda + \partial))^p T(\lambda).
\]
Equivalently,
\[
(C^{-1})(\lambda + \partial) = \sum_{p \in \mathbb{Z}_+} (-1)^{p} (T(\lambda + \partial))^p.
\]
Let us use the induction on \( p \) to compute each term of \((C^{-1})(\lambda + \partial)\). By induction hypothesis,
\[
T(\lambda + \partial)^{p+1} = T(\lambda + \partial)^p (T(\lambda + \partial))
\]
\[
= T(\lambda + \partial)^p \left( \sum_{(j_p, n_p) \in (j_{p+1}, n_{p+1})} e_{(j_p, n_p), (j_{p+1}, n_{p+1})} \otimes \pi \{ q_{n_{p+1}}^{j_{p+1}} \lambda, q_{j_p}^{n_p+1} \} \rightarrow \right)
\]
\[
= \sum_{(j_0, n_0) < \cdots < (j_p, n_p)} (-1)^{j_0+j_0j_p} e_{(j_0, n_0), (j_p, n_p)} \otimes \prod_{t=1}^{p+1} (-1)^{j_{t-1}+j_{t-1}j_t} \pi \{ q_{n_t}^{j_t} \lambda, \partial q_{j_t-1}^{n_{t-1}+1} \} \rightarrow \pi \{ q_{n_p}^{j_p} \lambda, q_{j_{p-1}}^{n_{p-1}+1} \}.
\]
Thus, we have
\[
(T(\lambda + \partial))^p = \sum_{(j_0, n_0) < \cdots < (j_p, n_p)} (-1)^{j_0+j_0j_p} e_{(j_0, n_0), (j_p, n_p)} \otimes \prod_{t=1}^{p} (-1)^{j_{t-1}+j_{t-1}j_t} \pi \{ q_{n_t}^{j_t} \lambda, \partial q_{j_t-1}^{n_{t-1}+1} \} \rightarrow \pi \{ q_{n_p}^{j_p} \lambda, q_{j_{p-1}}^{n_{p-1}+1} \}
\]
which implies (4.27). \( \square \)

The modified Dirac reduced bracket on \( \tilde{\mathfrak{g}}^k(g) \) immediately follows from Lemma 1.9 and Proposition 4.10.
Theorem 4.11. Recall the subset $\theta_1$ of $\mathcal{V}^k(g)$ in (4.22). If we denote the differential algebra ideal of $\mathcal{V}^k(g)$ generated by $\theta_1$ by $I$ and the canonical projection map by $\pi: \mathcal{V}^k(g) \to \tilde{\mathcal{V}}^k(g) := \mathcal{V}^k(g)/I$, then the modified Dirac reduced bracket $\pi\{\cdot, \cdot\}^D$ on $\tilde{\mathcal{V}}^k(g)$ associated with $\theta_1$ can be written as

$$\pi\{a, b\}^D = \pi\{a, b\} - \sum_{p, t_2, \ldots, t_{2k-1} \in \mathbb{Z}, t_0, \ldots, t_{2k-1}} (-1)^{p+t_2+t_1+\ldots+t_{2k-1}} \pi\{q^{\mu_0}_{t_0, \ldots, t_{2k-1}} e^{\mu}_p\} \rightarrow \prod_{l=1}^p (-1)^{j_{l-1}+j_l} \pi\{q^{\mu_l}_{t_l, \ldots, t_{2k-1}} e^{\mu_{l-1}}\} \pi\{a, b\}$$

(4.31)

for any $a, b \in g^F$.

Note that the bracket $\{\cdot, \cdot\}$ in Theorem 4.11 is the $\lambda$-bracket for $\mathcal{V}^k(g)$ given in Example 2.4. Now, one can deduce the following theorem.

Theorem 4.12. The map $\omega: g^F \to \mathcal{V}^k(g, F)$ in (4.20) can be uniquely extended to the differential algebra isomorphism

$$\omega: \tilde{\mathcal{V}}^k(g) \to \mathcal{V}^k(g, F).$$

Moreover, the map $\omega$ naturally induces the PVSA $\lambda$-bracket of $\tilde{\mathcal{V}}^k(g)$, which is identical to the modified Dirac reduced bracket $\pi\{\cdot, \cdot\}^D$ in Theorem 4.11. In other words, $\omega$ is a PVSA isomorphism between $(\tilde{\mathcal{V}}^k(g), \pi\{\cdot, \cdot\}^D)$ and $(\mathcal{V}^k(g, F), \{\cdot, \cdot\})$.

Proof. The unique extension of $\omega$ and its bijectivity are ensured by Proposition 4.5. Therefore, it is enough to check that $\omega$ is a PVSA homomorphism. Observe that

$$\pi\{q^{\mu_l}_{t_l, \ldots, t_{2k-1}} e^{\mu_{l-1}}\} \rightarrow \prod_{l=1}^p (-1)^{j_{l-1}+j_l} \pi\{q^{\mu_l}_{t_l, \ldots, t_{2k-1}} e^{\mu_{l-1}}\} (k + \partial)$$

(4.32)

for $(j_{l-1}, n_{l-1}) < (j_l, n_l)$. Thus, the desired statement immediately follows from Theorem 4.8 and Theorem 4.11.

Remark 4.13. For the subset $\theta_1 \subset \mathcal{V}^k(g)$ in (4.22), we cannot get invertible matrix if we consider

$$C(\lambda) = \sum_{(i, j), (i, j) \in I} e_{(i, j), (i, j)} \otimes \{q^{\mu}_{i, j} e^{\mu_{j+1}}\} \in \text{Mat}_I \otimes \mathcal{V}^k(g) (\{\lambda^{-1}\})$$

(4.33)

instead of (4.23). For example, let $g = sl_2(\mathbb{C})$ and denote $F = e_{21}, E = e_{12}, H = e_{11} - e_{22}$. Then $\{F\}$ and $\{E\}$ are dual bases of $g^F$ and $g^E$, respectively, and $C(\lambda)$ in (4.33) is

$$C(\lambda) = \begin{pmatrix} [E, \lambda - \frac{1}{2} H] & \{E, \lambda - \frac{1}{2} E\} \\ [-H, \lambda - \frac{1}{2} H] & \{-H, \lambda - \frac{1}{2} E\} \end{pmatrix} = \begin{pmatrix} E & 0 \\ k \lambda & E \end{pmatrix},$$

which is not invertible. Therefore, we cannot use the Dirac reduction in Section 2.2.

We close this section by computing the modified Dirac reduced bracket in Theorem 4.11 when $g = \mathfrak{osp}(1|2)$.

Example 4.14. Consider the Lie superalgebra $g := \mathfrak{osp}(1|2) \subset \mathfrak{gl}(2|1)$ equipped with the supertrace form $(\cdot | \cdot)$ given by

$$(A | B) = \text{str}(AB).$$

Denote the index set by $J = \{1, 2, \bar{1}\}$, where 1, 2 are even indices and $\bar{1}$ is an odd index. Let

$$E = e_{12}, \quad e = e_{1\bar{1}} + e_{\bar{1}2}, \quad H = e_{11} - e_{22}, \quad f = e_{\bar{1}1} - e_{2\bar{1}}, \quad F = e_{21}.$$


Take the basis \( \{ q_a := F, q_b := f \} \) of \( g^F \) and its dual basis \( \{ q^a := E, q^b := \frac{1}{2} c \} \) of \( g^E \). Order the index set \( I \) for the matrix \( C(\lambda) \) as
\[
I := \{(a, 0), (a, 1), (b, 0)\}.
\]
Then \( C(\lambda) \) is
\[
C(\lambda) = \begin{pmatrix}
1 & \pi\{q_1^a q_1^a\} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & k\lambda & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
and its inverse is
\[
C^{-1}(\lambda) = \begin{pmatrix}
1 & -k\lambda & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Therefore, for any two elements of \( \mathbb{C}[F, f] \), we can compute the bracket \( \pi\{\cdot, \cdot\}^D \) between them. In particular,
\[
\pi\{F_\lambda f\}^D = -\left( f \left( -\frac{1}{2} k\lambda \right) + k(\lambda + \partial)(-f) \right) = \frac{3}{2} k\lambda f + k\partial f.
\]

5. SUSY W-algebras and modified Dirac reduction of SUSY PVAs

In this section, we define a modified Dirac reduced bracket on a quotient space of SUSY PVA. We also show that SUSY W-algebras can be described by modified Dirac reductions of SUSY affine PVAs. For further properties of SUSY W-algebras, we refer to [30, 35].

5.1. Modified Dirac reduction of SUSY PVAs.

Let \( \mathcal{P} \) be a differential algebra freely generated by a finite homogeneous set \( \theta_J := \{ \theta_j \mid j \in J \} \) with an odd derivation \( D \), i.e., \( \mathcal{P} \cong \mathbb{C}[D^n \theta_j \mid j \in J, n \in \mathbb{Z}_+] \). Assume that the index set \( J \) is a subset of \( \mathbb{Z} \) and \( j \in J \) is even (resp. odd) if \( \theta_j \) is even (resp. odd). Consider the differential algebra ideal \( \mathcal{I} \) of \( \mathcal{P} \) generated by \( \theta_I := \{ \theta_j \mid j \in I \} \) and let \( \mathcal{P}/\mathcal{I} \) be the quotient algebra. Then we can identify
\[
\mathcal{P} \cong \mathbb{C}[D^n \theta_j \mid j \in J \setminus I, n \in \mathbb{Z}_+]
\]
as differential algebras. The space \( \mathcal{M}(\Lambda) := \text{Mat}_I \otimes \mathcal{P}(\Lambda^{-1}) \) is an associative algebra with unity \( 1 \) for the product defined in (3.10). An invertible element in \( \mathcal{M}(\Lambda) \) is also defined as in (3.11). Now we suppose that \( \mathcal{P} \) is a SUSY PVA with the \( \Lambda \)-bracket \( \{\cdot, \cdot\} \) and consider the following odd element
\[
\tilde{C}(\Lambda) := \sum_{i,j \in I} e_{ij} \otimes \pi\{\theta_j \Lambda \theta_i\} \in \mathcal{M}(\Lambda)
\]
where \( \pi : \mathcal{P}(\Lambda^{-1}) \to \mathcal{P}(\Lambda^{-1}) \) is the canonical quotient map of differential algebras.

**Definition 5.1.** Assume that \( \tilde{C}(\Lambda) \) in (5.2) is invertible and let \( \tilde{C}^{-1}(\Lambda) \in \mathcal{M}(\Lambda) \) be its inverse. Then the modified Dirac reduced bracket on \( \mathcal{P} \) associated with \( \theta_I \) is a bilinear map
\[
\pi\{\cdot, \cdot\}^D : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}(\Lambda^{-1})
\]
given by
\[
\pi\{a \Lambda b\}^D = \pi\{a \Lambda b\} - \sum_{i,j \in I} (-1)^{(a+j)(b+i)} \pi\{\theta_j \Lambda + \nabla \} \pi\{a \Lambda \theta_i\},
\]
where \( \tilde{C}^{-1}_{ij}(\Lambda) \) denotes the \( ij \)-entry of \( \tilde{C}^{-1}(\Lambda) \) and \( a, b \) are elements in \( \mathcal{P} \). Regard \( a, b \in \mathbb{C}[D^n \theta_j \mid j \in J \setminus I, n \in \mathbb{Z}_+] \), when we compute the RHS.
In general, the modified bracket \((5.4)\) does not give a SUSY PVA structure on \(\mathcal{F}\). More precisely, the modified \(\Lambda\)-bracket satisfies the sesquilinearity, skew-symmetry and Leibniz rule, but Jacobi identity for SUSY PVAs. That is because the quotient map \(\pi\) is only a differential algebra homomorphism, not a SUSY LCA algebra homomorphism.

Now we introduce the SUSY analogue of Lemma 4.3.

**Lemma 5.2.** Let \(\theta'_j := \{\theta'_j \mid j \in I\}\) be another basis of a vector superspace \(V\) spanned by \(\theta_I\) and suppose \((-1)^{\bar{\theta}'_j} = (-1)^i = (-1)^{\bar{\theta}_i}\). If \(\overline{C}(\Lambda)\) in \((5.2)\) is invertible, then \(\overline{C}(\Lambda) = (\pi\{\theta_j \Lambda \theta'_j\})_{i,j \in I}\) is invertible and the bracket \((5.4)\) can be written as

\[
\pi\{a_{\Lambda} b\}_D = \pi\{a_{\Lambda} b\} - \sum_{i,j \in I} (-1)^{(\bar{a} + j)(\bar{b} + i)} \pi\{\theta_j \Lambda \theta'_j\} \pi\{a_{\Lambda} \theta'_j\}. \tag{5.5}
\]

**Proof.** A similar proof to Lemma 4.3 works. \(\Box\)

### 5.2. Structures of SUSY W-algebras

Let \(\mathfrak{g}\) be a finite simple Lie superalgebra and \(\mathfrak{s} = \text{Span}_\mathbb{C}\{E, e, H = 2x, f, F\}\) be a subalgebra isomorphic to \(\mathfrak{osp}(1|2)\). Here, \(\{E, H, F\}\) is an \(sl_2\)-triple and \(e, f\) are odd elements of \(\mathfrak{g}\) satisfying \([e, e] = 2E, [f, f] = -2F, [H, f] = -F\) and \([H, e] = e\). Assume that \(\mathfrak{g}\) is equipped with a non-degenerate supersymmetric invariant even bilinear form \((\cdot | \cdot)\) such that \((E|F) = 2(x|x) = 1\). We can consider the eigenspace decomposition with respect to \(ad_x\):

\[
\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}(i). \tag{5.6}
\]

In this section, we use the notation \((4.11)\) and denote

\[
\mathfrak{n} := \mathfrak{g}_{\geq 0}, \quad \mathfrak{p} := \mathfrak{g}_{\leq 0}. \tag{5.7}
\]

Beware of that \(\mathfrak{p}\) is different from the \(\mathfrak{p}\) in Section 1.2. Here, \(\mathfrak{p}\) is the complement of \(\mathfrak{n}\) with respect to \((5.5)\). On the other hand, by the \(\mathfrak{osp}(1|2)\)-representation theory, \(\mathfrak{g}\) can be decomposed into

\[
\mathfrak{g} = \mathfrak{g}^f \oplus [e, \mathfrak{g}], \tag{5.8}
\]

where \(\mathfrak{g}^f = \{a \in \mathfrak{g} \mid [f, a] = 0\}\). For each \(a \in \mathfrak{g}\), denote by \(a^f\) its projection to \(\mathfrak{g}^f\) with respect to \((5.8)\).

Recall that \(\mathcal{V}^k (\mathfrak{g})\) is the affine SUSY PVA (see Example 3.6). For a subspace \(\mathfrak{a}\) of \(\mathfrak{g}\), the differential algebra \(\mathcal{P}(\tilde{\mathfrak{a}}) := S(\mathbb{C}[D] \otimes \tilde{\mathfrak{a}})\) can be viewed as a subalgebra of \(\mathcal{V}^k (\mathfrak{g})\) as differential algebras.

**Definition 5.3.** Let \(\mathcal{J}_f\) be the differential algebra ideal of \(\mathcal{V}^k (\mathfrak{g})\) generated by \(\{\tilde{n} - (f|n) \mid n \in \mathfrak{n}\}\) and let \(\rho : \mathcal{V}^k (\mathfrak{g})[\Lambda] \to \mathcal{V}^k (\mathfrak{g})/\mathcal{J}_f[\Lambda]\) be the canonical projection. Then for \(k \in \mathbb{C}\), the SUSY \(W\)-algebra of level \(k\) associated with \(\mathfrak{g}\) and \(f\) is defined by

\[
\mathcal{W}^k (\mathfrak{g}, f) := \{a \in \mathcal{P}(\tilde{\mathfrak{p}}) \mid \rho\{\tilde{n}_{\Lambda} a\} = 0 \text{ for any } n \in \mathfrak{n}\} \tag{5.9}
\]

where \(\{\tilde{n}_{\Lambda}\}\) is the \(\Lambda\)-bracket of \(\mathcal{V}^k (\mathfrak{g})\).

It is well-known that the SUSY \(W\)-algebra \(\mathcal{W}^k (\mathfrak{g}, f)\) is a SUSY PVA with \(\Lambda\)-bracket naturally induced from that of \(\mathcal{V}^k (\mathfrak{g})\). More precisely, for the canonical differential algebra isomorphism \(\iota : \mathcal{V}^k (\mathfrak{g})/\mathcal{J}_f[\Lambda] \to \mathcal{P}(\tilde{\mathfrak{p}})[\Lambda]\), the bracket \(\iota(\rho\{\tilde{n}_{\Lambda}\})\) defined on \(\mathcal{P}(\tilde{\mathfrak{p}})\) induces a SUSY PVA \(\Lambda\)-bracket on \(\mathcal{W}^k (\mathfrak{g}, f)\).
Let us introduce a $\frac{\omega}{2}$-grading $\Delta$ on $\mathcal{V}^k(\mathfrak{g})$ called conformal weight. The conformal weight $\Delta_a$ of $\bar{a}$ for $a \in \mathfrak{g}(i)$ is defined by
\[
\Delta_a := \frac{1}{2} - i.
\] (5.10)
and for homogeneous elements $A, B \in \mathcal{V}^k(\mathfrak{g})$, the conformal weights of $DA$ and $AB$ are
\[
\Delta_{DA} = \Delta_A + \frac{1}{2}, \quad \Delta_{AB} = \Delta_A + \Delta_B.
\] (5.11)
The conformal weight $\Delta$ naturally induces the $\frac{\omega}{2}$-grading on the SUSY W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. Furthermore, there is a set of free generators of $\mathcal{W}^k(\mathfrak{g}, f)$, which satisfies the properties in the following proposition.

**Proposition 5.4.** [35] Let $\{r_i | i \in J^f\}$ be a basis of $\mathfrak{g}^f$, homogeneous with respect to both conformal weight and parity. Then there exists a unique subset $\{\omega_i | i \in J^f\}$ of homogeneous elements in $\mathcal{W}^k(\mathfrak{g}, f)$ satisfying the follows properties:

(i) $\mathcal{W}^k(\mathfrak{g}, f) = \mathbb{C}[D^{m_i} | i \in J^f, n \in \mathbb{Z}_+]$ as differential algebras,

(ii) $\Delta_{\omega_i} = \Delta_{r_i}$ for each $i \in J^f$,

(iii) $\omega_i$ for $i \in J^f$ is decomposed into $\omega_i = \gamma_i + \sum_{n \geq 1} \gamma_i^n$ for $\gamma_i^n \in \mathcal{P}(\mathfrak{g}^f) \otimes (\mathbb{C}[D] \otimes [e, \mathfrak{g}_{\frac{\omega}{2}}])^\otimes n$.

The set $\{\omega_i | i \in J^f\}$ in Proposition 5.4 can be found by the SUSY analogue of Remark 4.6 and the uniqueness of such set follows from the fact that
\[
\mathcal{W}^k(\mathfrak{g}, f) \cap \sum_{n \geq 1} \mathcal{P}(\mathfrak{g}^f) \otimes (\mathbb{C}[D] \otimes [e, \mathfrak{g}_{\frac{\omega}{2}}])^\otimes n = 0.
\] (5.12)

By the uniqueness, the linear map $\omega$ in Definition 5.3 is determined completely.

**Definition 5.5.** Recall the basis $\{r_i | i \in J^f\}$ of $\mathfrak{g}^f$ and the set of generators $\{\omega_i | i \in J^f\}$ in Proposition 5.4. The injective map $\omega$ is defined by
\[
\omega : \mathfrak{g}^f \rightarrow \mathcal{W}^k(\mathfrak{g}, f), \quad r_i \mapsto \omega_i.
\] (5.13)

In order to see the $\Lambda$-bracket of $\mathcal{W}^k(\mathfrak{g}, f)$, we introduce the basis $\{r^j | j \in J^f\}$ of $\mathfrak{g}^c := \{a \in \mathfrak{g}| [e, a] = 0\}$ which is dual to $\{r_j | j \in J^f\}$ in Proposition 5.4 i.e., $(r^i|r_j) = \delta_{ij}$. For $\beta_i \in \mathbb{Z}/2$ such that $r^i \in \mathfrak{g}(\beta_i)$, we denote
\[
r^j_m := (\text{ad}f)^m r^j, \quad r^m_i := k_{i,m}(\text{ade})^m r_i.
\] (5.14)
Here, we have $0 \leq m \leq 4\beta_i$ and $k_{i,m}$’s are nonzero constants chosen to satisfy $(r^j_m|r^m_j) = \delta_{mn}\delta_{ij}$. For the explicit choice of $k_{i,m}$, we refer to Section 4 of [35].
Theorem 5.6 (BM). Recall the map \( \varpi : g^J \to \mathcal{W}^k(\bar{g}, f) \) in (5.13). Then for any \( a \in g(-t_1) \cap g^J \) and \( b \in g(-t_2) \cap g^J \), we have
\[
\langle \varpi(a) \varpi(b) \rangle = (-1)^{\tilde{a}} (\varpi([a, b]) + k\chi(a|b))
- \sum_{p \in Z_{20} - t_2} \sum_{\frac{k}{2} \leq (j_0, n_0) \leq \cdots \leq (j_p, n_p) \leq t_1} (-1)^{\tilde{a} + (\tilde{a} + 1)(b + 1) + (\tilde{a} + 1)(j_p + n_p) + j_p + n_p + (b + 1)(j_0 + n_0 + 1)}
\]
\[
(\tilde{a})^{j_0 + n_0} (\varpi([r_{j_0}^{n_0}, b]^J)) + (r_{n_0}^{j_0} b)(k\chi + D))
\sum_{t=1}^p (-1)^{(j_{t-1} + n_{t-1})(j_t + n_t)} (\varpi([r_{j_t}^{n_t}, r_{n_t}^{j_t-1}]^J)) + (r_{n_t}^{j_t} r_{n_t-1}^{j_t-1} k\chi + D))
\]
\[
(\tilde{a})^{j_1 + n_1} (\varpi([a, r_{n_1}^{j_1}]^J)) + (a|n_1^{j_1} k\chi),
\]
where
- the pairs \((j_t, n_t)\) for \( t = 0, 1, \ldots, p \) are elements of \( I = \{(j, n) \mid j \in J^J, j = 0, 1, \ldots, 4\beta_j - 1\} \),
- the partial order \( \prec \) on \( \frac{2}{\tilde{a}} \cup I \) is given by
\[
(j, n) \prec t \iff \frac{n}{2} - \beta_j + \frac{1}{2} \leq t, \quad t \prec (j, n) \iff t + \frac{1}{2} \leq \frac{n}{2} - \beta_j,
\]
\[
(j, n) \prec (i, m) \iff \frac{n}{2} - \beta_j + \frac{1}{2} \leq \frac{m}{2} - \beta_i.
\]

\[
\square
\]

5.3. SUSY W-algebras via modified Dirac reduction.

In this section, we use notations defined in Section 5.2. Recall the basis \( \{r_i \mid i \in J^J\} \) of \( g^J \) in Proposition 5.4 and bases of \( \bar{g} \) in (5.14). To simplify notations, we assume that \( J^J \) is a subset of \( \mathbb{Z} \) such that \((-1)^{r_i} = (-1)^i\) for any \( i \in J^J \).

Let us consider the index set
\[
I = \{(i, m) \mid i \in J^J, m = 0, 1, \ldots, 4\beta_i - 1\},
\]
where \( \beta_i \in \mathbb{Z} \) satisfies \( r_i \in \mathfrak{g}(\beta_i) \). Then \( \{r_i^m \mid (i, m) \in I\} \) is a basis of \([e, \mathfrak{g}]\). Let \( p(i, m) := i + m + 1 \) so that \( p(i, m) = p(r_i^m) \). In addition, if we denote
\[
(i, m)' := (i, 4\beta_i - m)
\]
for \( i \in J^J \) and \( m \in \mathbb{Z} \), then \( \{r_i^m \mid (i, m) \in I\} \) is another basis of \([e, \mathfrak{g}]\). Take a subset
\[
\theta_I = \{r_i^m - (f|r_i^m) \mid (i, m) \in I\}
\]
of \( \mathcal{V}^k(\bar{g}) \) and denote the differential algebra ideal generated by \( \theta_I \) in \( \mathcal{V}^k(\bar{g}) \) by \( \mathcal{I} \). Then the quotient space \( \bar{\mathcal{V}}^k(\bar{g}) := \mathcal{V}^k(\bar{g})/\mathcal{I} \) is isomorphic to \( \mathcal{S}(\mathbb{C}[D] \otimes \bar{g}^J) \) as differential algebras. The canonical projection map is defined by
\[
\pi : \mathcal{V}^k(\bar{g}) \longrightarrow \bar{\mathcal{V}}^k(\bar{g}) \cong \mathcal{S}(\mathbb{C}[D] \otimes \bar{g}^J), \quad \bar{a} \mapsto \bar{a} + (f|a) \text{ for } a \in \bar{g}.
\]
Consider another basis of the vector superspace spanned by \( \theta_I \):
\[
\theta_I' = \{r_i^m - (f|r_i^m) \mid (i, m)' \in I\}.
\]
By Lemma 5.2, the modified Dirac reduced bracket associated with $\theta_I$ can be obtained by the odd matrix

$$\tilde{C}(\Lambda) = \sum_{(i,m),(j,n+1) \in I} e_{(j,n+1), (i,m)} \otimes \pi \{ \bar{r}_{m \Lambda}^i \bar{p}_{j}^{n+1} \} \in \text{Mat}_I \otimes \mathfrak{F}_g((\Lambda^{-1})).$$

(5.20)

By direct computations, we get the following lemma.

**Lemma 5.7.** For $\{ r_{i}^1 \}, \{ r_{i}^{n+1} \}$ in (5.14) and $a \in \mathfrak{g}(-t) \cap \mathfrak{g}^I$, the following identities hold:

1. $\pi \{ \bar{r}_{m \Lambda}^i r_{i}^{m+1} \} = (-1)^{i+m}$,
2. $\pi \{ \bar{r}_{m \Lambda}^i p_{j}^{n+1} \} = 0$ if $\frac{\beta_j - \beta_i}{2} + \frac{1}{2} > \frac{\beta_j}{2}$,
3. $\pi \{ \bar{a}_{\Lambda} r_{i}^{n+1} \} = 0$ if $\frac{\beta_j - \beta_i}{2} + \frac{1}{2} > t$,
4. $\pi \{ \bar{p}_{i}^j a \} = - if t > \frac{\beta_j}{2} - \beta_i$.

By Lemma 5.7, we can write (5.20) as

$$\tilde{C}(\Lambda) = \sum_{(i,m) \in I} e_{(i,m), (i,m+1)} \otimes (-1)^{i+m} + \sum_{(j,n) \prec (i,m)} e_{(j,n+1), (i,m)} \otimes \pi \{ \bar{r}_{m \Lambda}^i p_{j}^{n+1} \}$$

(5.21)

for the partial order $\prec$ defined in Theorem 5.6.

**Proposition 5.8.** The matrix $\tilde{C}(\Lambda)$ in (5.20) is invertible. Moreover, its inverse $\tilde{C}^{-1}(\Lambda)$ is given by

$$\tilde{C}^{-1}(\Lambda) = \sum_{(i,m) \in I} e_{(i,m), (i,m+1)} \otimes (-1)^{i+m}$$

$$+ \sum_{p \geq 2} \sum_{(j_0,n_0) \prec \cdots \prec (j_p,n_p)} (-1)^{p+\sum_{k=0}^{p} (j_k,n_k) - \sum_{k=0}^{p} (j_k,n_k+1)} \otimes \tilde{C}^{-1}_{(j_0,\cdots,j_p),(n_0,\cdots,n_p)}$$

(5.22)

where

$$\tilde{C}^{-1}_{(j_0,\cdots,j_p),(n_0,\cdots,n_p)} = \left( \prod_{t=1}^{p} R_t(\Lambda + \nabla) \right) R_p(\Lambda)$$

and

$$R_s(\Lambda) = (-1)^{(j_s + n_s + 1) + \sum_{k=1}^{s} (j_k,n_k)} \pi \{ \bar{p}_{i}^{j_s} \bar{r}_{n_k}^{j_s} \bar{p}_{j_s+1}^{n+1} \}$$

for $s = 1, 2, \cdots, p$.

**Proof.** Multiply the constant matrix $K := \sum_{(i,m) \in I} e_{(i,m), (i,m+1)} \otimes (-1)^{i+m}$ on the right of $\tilde{C}(\Lambda)$, so that one can obtain

$$\tilde{C}(\Lambda) \cdot K = \sum_{(i,m) \in I} e_{(i,m+1), (i,m+1)} \otimes 1 + \sum_{(j,n) \prec (i,m)} e_{(j,n+1), (i,m)} \otimes (-1)^{i+m} \pi \{ \bar{r}_{m \Lambda}^i \bar{p}_{j}^{n+1} \}.$$  

(5.23)

Hence, if we write

$$\tilde{T}(\Lambda) = \sum_{(j,n) \prec (i,m)} e_{(j,n+1), (i,m+1)} \otimes (-1)^{i+m} \pi \{ \bar{r}_{m \Lambda}^i \bar{p}_{j}^{n+1} \}$$

(5.24)

then $\tilde{C}(\Lambda) \cdot K$ can be expressed as

$$\tilde{C}(\Lambda) \cdot K = \text{Id} + \tilde{T}(\Lambda)$$

(5.25)

for identity matrix $\text{Id}$. Since it is of the same form with (4.28), we can find its inverse with analogous calculation to Proposition 4.10. Denote the inverse by $\tilde{C}^{-1}$, i.e.,

$$\tilde{C}^{-1}(\Lambda + \nabla) \tilde{C}(\Lambda) \cdot K = \text{Id}.$$  

(5.26)

Then by multiplying $K$ on the left side of $\tilde{C}^{-1}$, one get the inverse of $\tilde{C}(\Lambda)$.
Theorem 5.9. Consider the subset $\theta_1$ in (5.17) and the differential algebra ideal $\mathcal{I}$ of $\mathcal{V}^k(\bar{g})$ generated by $\theta_1$. Denote the canonical projection map by $\pi : \mathcal{V}^k(\bar{g}) \rightarrow \tilde{\mathcal{V}}^k(\bar{g}) := \mathcal{V}^k(\bar{g})/\mathcal{I}$. Then the modified Dirac reduced bracket $\pi \{ \cdot, \cdot \}^D$ on $\tilde{\mathcal{V}}^k(\bar{g})$ associated with $\theta_1$ can be written as

$$
\pi \{ \bar{a}, \bar{b} \}^D = \pi \{ \bar{a}, \bar{b} \} - \sum_{p \in \mathbb{Z}_{\geq 0}} \sum_{r = 0}^{\infty} (-1)^p \pi (\bar{a}^p \bar{b}^{p+1}) \theta^{p+1} \quad (5.27)
$$

for any $a, b \in g^I$, where the bracket $\{ \cdot, \cdot \}$ is the $\Lambda$-bracket for $\mathcal{V}^k(\bar{g})$.

Comparing Theorem 5.9 and Theorem 5.10, we obtain the following theorem.

Theorem 5.10. The map $\mathcal{W} : g^I \rightarrow \mathcal{W}^k(\bar{g}, f)$ in (5.13) can be uniquely extended to the differential algebra isomorphism

$$
\mathcal{W} : \tilde{\mathcal{V}}^k(\bar{g}) \rightarrow \mathcal{W}^k(\bar{g}, f).
$$

Then $\mathcal{W}$ naturally induces the SUSY PVA $\Lambda$-bracket on $\tilde{\mathcal{V}}^k(\bar{g})$ which coincides with the modified Dirac reduced bracket $\pi \{ \cdot, \cdot \}^D$ in Theorem 5.9. In other words, $\mathcal{W}$ is a SUSY PVA isomorphism between $\tilde{\mathcal{V}}^k(\bar{g}), \pi \{ \cdot, \cdot \}^D$ and $(\mathcal{W}^k(\bar{g}, f), \{ \cdot, \cdot \})$.

Proof. The proof is similar to that of Theorem 4.12. \qed

In the end, we see how the bracket $\pi \{ \cdot, \cdot \}^D$ in Theorem 5.9 is computed in the example.

Example 5.11. Let $g = \mathfrak{osp}(1|2)$ be equipped with a supertrace form $(\cdot | \cdot)$. As in Example 4.14, denote the elements of $g$ by $E, e, H, f$ and $F$. Take the basis $\{ r_a := F \}$ of $g^I$ and its dual basis $\{ r^a := E \}$ of $g^\ast$. Then

$$
\begin{align*}
    r_0^a &= r^a = E, & r_1^a &= e, & r_2^a &= H, & r_3^a &= f, & r_4^a &= -2F, \\
    r_5^4 &= \frac{1}{2} E, & r_6^3 &= \frac{1}{2} e, & r_7^2 &= \frac{1}{2} H, & r_8^1 &= \frac{1}{2} f, & r_9^0 &= F.
\end{align*}
$$

The index set $I$ consists of the four elements:

$$
    (a, 4)' = (a, 0), \quad (a, 3)' = (a, 1), \quad (a, 2)' = (a, 2), \quad (a, 1)' = (a, 3),
$$

and the partial order in the index set $I$ is given by

$$
    (a, 0) \ll (a, 1) \ll (a, 2) \ll (a, 3).
$$

Then the odd matrix $\tilde{C}(\Lambda)$ in (5.28) is

$$
\tilde{C}(\Lambda) = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & \pi (\bar{r}_2^a \bar{r}_3^a) \\
0 & -1 & \pi (\bar{r}_1^a \bar{r}_4^a) & \pi (\bar{r}_2^a \bar{r}_4^a) \\
1 & \pi (\bar{r}_1^a \bar{r}_4^a) & \pi (\bar{r}_2^a \bar{r}_4^a) & \pi (\bar{r}_3^a \bar{r}_4^a)
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & -k \lambda \\
0 & -1 & k \chi & 0 \\
1 & -k \chi & 0 & F
\end{pmatrix}
$$

and its inverse is

$$
\tilde{C}^{-1}(\Lambda) = \begin{pmatrix}
-k \lambda \chi - F & -k^2 \lambda & k \chi & 1 \\
k^2 \lambda & -k \chi & -1 & 0 \\
k \chi & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
$$
Using this, we obtain the bracket $\pi \{ \cdot, \cdot \}^D$ on $\mathbb{C}[D^n F | n \in \mathbb{Z}_+]$. In particular,

\[
\pi \{ \mathcal{F}_\Lambda \mathcal{F} \}^D = -k(\chi + D)k(\chi + D)\mathcal{F} - k(\chi + D)\left( -k^2(\chi + D)(\lambda + \partial) - \mathcal{F} \right) \left( \frac{1}{2} k\chi \right) + 2\mathcal{F}k(\chi + D)\left( -\frac{1}{2} k\chi \right) = -\frac{1}{2} k^5 \lambda^2 \chi - \frac{3}{2} k^2 \lambda \mathcal{F} - \frac{1}{2} k^2 \chi D\mathcal{F} - k^2 \partial \mathcal{F}.
\]

REFERENCES

[1] Adler M.: On a trace functional for formal pseudodifferential operators and the symplectic structure of the Korteweg-de Vries equation. Invent. Math. 50 (1979), 219-248.
[2] Bais, F. A., Tjin, T., van Driel, P.: Covariantly coupled chiral algebras. Nuclear Phys. B 357 (1991), no. 2-3, 632–654.
[3] Balog, J., Féher, L., O’Raifeartaigh, L., Forgács, P., Wipf, A.: Toda theory and W-algebra from a gauged W2N2 point of view. Ann. Physics 203 (1990), no. 1, 76–136.
[4] Barakat, A., De Sole, A., Kac, V.: Poisson vertex algebras in the theory of Hamiltonian equations. Jpn. J. Math. 4 (2009), no. 2, 141–252.
[5] de Boer, J., Tjin, T.: Quantization and representation theory of finite W algebras. Comm. Math. Phys. 158 (1993), no. 3, 485–516.
[6] Carpentier, S., Suh, U.R.: Supersymmetric bi-Hamiltonian systems. Comm. Math. Phys. 382 (2021), no. 1, 317–350.
[7] Delduc, F., Ragoucy, E., Sorba, P.: Super-Toda theories and W-algebras from superspace Wess-Zumino-Witten models. Comm. Math. Phys. 146 (1992), no. 2, 403–426.
[8] Evans, J., Hollowood, T.: Supersymmetric Toda field theories. Nuclear Phys. B 352 (1991), no. 3, 723–768.
[9] De Sole, A., Kac, V.: Finite vs affine W-algebras. Jpn. J. Math. 1 (2006), no. 1, 137–261.
[10] De Sole, A., Kac, V.: Non-local Poisson structures and applications to the theory of integrable systems. Jpn. J. Math. 8 (2013), no. 2, 233–347.
[11] De Sole, A., Kac, V., Valeri, D.: Adler-Gelfand-Dickey approach to classical W-algebras within the theory of Poisson vertex algebras. Int. Math. Res. Not. IMRN 2015, no. 21, 11186–11235.
[12] De Sole, A., Kac, V., Valeri, D.: Classical affine W-algebras and the associated integrable Hamiltonian hierarchies for classical Lie algebras. Commun. Math. Phys. 360(3), 851–918 (2018).
[13] De Sole, A., Kac, V., Valeri, D.: Classical affine W-algebras for $\mathfrak{gl}_N$ and associated integrable Hamiltonian hierarchies. Commun. Math. Phys. 348 (2016), no. 1, 265–319.
[14] De Sole, A., Kac, V., Valeri, D.: Classical $W$-algebras and generalized Drinfeld-Sokolov bi-Hamiltonian systems within the theory of Poisson vertex algebras. Commun. Math. Phys. 323(2), 663–711 (2013).
[15] De Sole, A., Kac, V., Valeri, D.: Dirac reduction for Poisson vertex algebras. Comm. Math. Phys. 331 (2014), no. 3, 1155–1190.
[16] Dinar, Y. I.: W-algebras and the equivalence of bihamiltonian, Drinfeld-Sokolov and Dirac reductions. J. Geom. Phys. 84 (2014), 30–42.
[17] Dirac, P. A. M.: Generalized Hamiltonian dynamics. Canad. J. Math. 2 (1950), 129–148.
[18] Drinfeld, V.G., Sokolov, V.V.: Lie algebras and equations of Korteweg-de Vries type. J Math Sci 30, 1975–2036 (1985).
[19] Féher, L., O’Raifeartaigh, L., Ruelle, P., Tsutsui, I.: On the completeness of the set of classical W-algebras obtained from DS reductions. Comm. Math. Phys. 162 (1994), no. 2, 399–431.
[20] Gelfand I. M., Dickey L. A.: Family of Hamiltonian structures connected with integrable non-linear equations. Preprint, IPM, Moscow (in Russian), 1978. English version in Collected papers of I.M. Gelfand, vol. 1, Springer-Verlag (1987), 625–646.
[21] Gelfand I. M., Dickey L. A.: Fractional powers of operators and Hamiltonian systems. Funct. Anal. Appl. 10 (1976), 259–273.
[22] Heluani, R., Kac, V.: Supersymmetric vertex algebras. Comm. Math. Phys. 271 (2007), no. 1, 103–178.
[23] Kac, V.: Vertex algebras for beginners. Second edition. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1998.
[24] Kac, V., Roan, S.-S., Wakimoto, M.: Quantum reduction for affine superalgebras. Comm. Math. Phys. 241 (2003), no. 2-3, 307–342.
[25] Komata, S., Mohri, K., Nohara, H.: Classical and quantum extended superconformal algebra. Nuclear Phys. B 359 (1991), no. 1, 168–200.
[26] Madsen, J. O., Ragoucy, E.: Quantum Hamiltonian reduction in superspace formalism. Nuclear Phys. B 429 (1994), no. 2, 277–290.
[27] Manin, Yu. I., Radul, A. O.: A supersymmetric extension of the Kadomtsev-Petviashvili hierarchy. Comm. Math. Phys. 98 (1985), no. 1, 65–77.
[28] Marsden, J. E., Ratiu, T.: Reduction of Poisson manifolds. Lett. Math. Phys. 11 (1986), no. 2, 161–169.
[29] McArthur, I. N.: On the integrability of the super-KdV equation. Comm. Math. Phys. 148 (1992), no. 1, 177–188.
[30] Molev, A., Ragoucy, E., Suh, U. R. Supersymmetric W-algebras. Lett. Math. Phys. 111 (2021), no. 1, Paper No. 6, 25 pp.
[31] Oevel, W., Popowicz, Z.: The bi-Hamiltonian structure of fully supersymmetric Korteweg-de Vries systems. Comm. Math. Phys. 139 (1991), no. 3, 441–460.
[32] Ragoucy, E.: Twisted Yangians and folded W-algebras. Internat. J. Modern Phys. A 16 (2001), no. 13, 2411–2433.
[33] Ragoucy, E., Sorba, P.: Yangian realisations from finite W-algebras. Comm. Math. Phys. 203 (1999), no. 3, 551–572.
[34] Suh, U. R.: Classical affine W-superalgebras via generalized Drinfeld-Sokolov reductions and related integrable systems. Comm. Math. Phys. 358 (2018), no. 1, 199–236.
[35] Suh, U. R.: Structures of (supersymmetric) classical W-algebras. J. Math. Phys. 61 (2020), no. 11, 111701, 27 pp.
[36] Suh, U. R.: Classical affine W-algebras associated to Lie superalgebras. J. Math. Phys. 57 (2016), no. 2, 021703, 34 pp.