COMPLEX ANALYTIC REALIZATIONS
FOR QUANTUM ALGEBRAS

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Abstract

A method for obtaining complex analytic realizations for a class of deformed algebras based on their respective deformation mappings and their ordinary coherent states is introduced. Explicit results of such realizations are provided for the cases of the $q$-oscillators ($q$-Weyl-Heisenberg algebra) and for the $su_q(2)$ and $su_q(1,1)$ algebras and their co-products. They are given in terms of a series in powers of ordinary derivative operators which act on the Bargmann-Hilbert space of functions endowed with the usual integration measure. In the $q \to 1$ limit these realizations reduce to the usual analytic Bargmann realizations for the three algebras.

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I.-Introduction

The representation theory of quantum algebras and groups constitutes an open field of research. Some of the features specific to quantum algebras, for example those appearing when the deformation parameter $q$ becomes a root of unity, have been studied already. It will be shown here for some of the simplest types of deformed algebras that also for $q$ real there are some interesting realizations which can elucidate the relation between deformation and non-linearity.

We will confine our scope in this paper to three of the simplest quantum algebras i.e., the so called q-oscillator and the $su_q(2)$ and $su_q(1,1)$ algebras. We will seek complex analytic realizations of the above algebras which, unlike the ones existing in the literature which are based on the so called $q$-coherent states and involve $q$-deformed (Jackson) derivatives, will be given instead in terms of a series of higher powers of ordinary derivatives. The representation spaces of these deformed realizations will be the ordinary Hilbert spaces of square-integrable analytic functions $L^2(G/H,d\mu(\zeta))$, built on the corresponding cosets of the non-deformed Lie groups, i.e $G/H = \frac{Weyl-Heisenberg}{U(1)}$, $\frac{SU(2)}{U(1)}$ and $\frac{SU(1,1)}{U(1)}$. The invariant measure of integration $d\mu(\zeta)$, the so-called Bargmann measure, will be explicitly given below for each case. One feature of the obtained realizations of the quantum algebra generators is that they constitute a deformation of the ordinary Lie algebra generators in the sense that they involve a series in powers of ordinary derivatives (the coefficients of which depend on the deformation parameter $q$) that reproduces in the ‘classical’ $q \to 1$ limit the Lie algebra vector field generators.

The outline the paper is as follows: in Section II, the required coherent states (CS) formulae for each of the non-deformed Weyl-Heisenberg $(wu)$, $su(2)$ and $su(1,1)$ algebra are given. Also the deformation mappings relating the quantum versions of the above algebras to the generators of the respective non-deformed ones will be provided, as they will be important in the next Section for the analytic realizations. Section IV will extend the method of obtaining the generator realizations, described in Section III, to the coproducts realized on complex functions depending on two arguments. Finally
some conclusions are given in Section V.

II.- Coherent states and deforming mappings

Let $G_+, G_-$ and $G_0$ be the generic expressions for the generators of the three-dimensional algebras $G = wh$, $su(2)$ and $su(1, 1)$. The (unnormalized, see below) coherent states (see [16, 17, 18, 19, 20] to which we refer for details on the general group definition of coherent states) can be defined generically as:

$$|\zeta\rangle = e^{\bar{\zeta}G_+}|\phi\rangle, \ (\zeta| = <\phi|e^{\zeta G_-}, \ (1)$$

where $|\phi\rangle$ is the corresponding lowest weight state of the different algebras, i.e. $|\phi\rangle \equiv |n = 0\rangle$ is the Fock vacuum state for the oscillator; $|\phi\rangle$ is given by $|j, m = -j\rangle$ for $su(2)$ with $j = 1/2, 1, 3/2, \ldots$ and $|\phi\rangle \equiv |k, l = 0\rangle$ for $su(1, 1)$ with $k = 1, 3/2, 2, 5/2, \ldots$. The round ket indicates that the CS are unnormalized; the normalized ones are given by $|\zeta\rangle = \frac{1}{\sqrt{\langle \zeta | \zeta \rangle}}|\zeta\rangle$.

The complex variables $\zeta$ ($\zeta = \alpha, z, \xi$) label the CS; $\zeta$ and $\bar{\zeta}$ are the projective coordinates of the respective coset spaces $G/H$ where $H$ is the isotropy group of the vacuum state $|\phi\rangle$, namely $G/H = WH/U(1) \approx R^2$, $SU(2)/U(1) \approx S^2$; $SU(1, 1)/U(1) \approx S^{1, 1}$; i.e. the two-dimensional plane, sphere and hyperboloid respectively. Moreover $G_+$ stands for the generic creation operator which together with the two other generators $G_-, G_0$, close into the respective Lie algebras $G$.

One of the generic relations that will be extensively used below reads

$$[G_\pm, f(G_0)] = (f(G_0 \mp 1) - f(G_0))G_\pm \ \ (2)$$

for any analytic function $f$ of $G_0$; this relation is common to the three algebras considered and is a consequence of the first commutator in eqs. (4),(5) and (6) below. It also follows from the definition (1) that $G_+|\zeta\rangle = \partial_{\bar{\zeta}}|\zeta\rangle, \ (\zeta|G_- = \partial_{\zeta}|\zeta\rangle$.

Let us now turn to the deforming mapping by which the generators of the quantum $q$-oscillator, $su_q(2)$ and $su_q(1, 1)$ are written uniquely in terms of their non-deformed counterparts. Generically (see eg. [21, 22, 23, 24, 25]),
\[ G^q_\pm = G_\pm F^\pm(G_0) \] (3a)

\[ G_0^q = G_0 \] (3b)

where \( G_0^q \equiv N, J_3, K_3 \) and \( G^q_\pm \equiv a^q_\pm, J^q_\pm, K^q_\pm \) and \( F^\pm(G_0) \) is given for each algebra by

\[
\begin{array}{c|c|c|c}
F^+(G_0) & \sqrt{\frac{N+1}{N+1}} & \sqrt{\frac{[J_3+j+1][J_3-j]}{(J_3+j+1)(J_3-j)}} & \sqrt{\frac{[K_3-k+1][K_3+k]}{(K_3-k+1)(K_3+k)}} \\
F^-(G_0) & \sqrt{\frac{N-1}{N}} & \sqrt{\frac{[J_3-j-1][J_3+j]}{(J_3-j-1)(J_3+j)}} & \sqrt{\frac{[K_3-k][K_3+k-1]}{(K_3-k)(K_3+k-1)}} \\
\end{array}
\]

Table 1

The deformed generators have the respective commutation relations:

\[[N, a^+_q] = \pm a^+_q, \quad [a^-_q, a^+_q] = [N + 1] - [N] \quad (q - \text{oscillator}); \quad (4)\]

\[[J^q_3, J^q_\pm] = \pm J^q_\pm, \quad [J^q_\pm, J^q_\pm] = [2J^q_3] \quad (su_q(2)) \; ; \quad (5)\]

and

\[[K^q_3, K^q_\pm] = \pm K^q_\pm, \quad [K^q_\pm, K^q_\pm] = -[2K^q_3] \quad (su_q(1, 1)) \; , \quad (6)\]

where the square bracket is defined by \([x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}\).

III.- Realizations of the \( q \)-algebra generators

We shall now develop a method which will enable us to specify complex analytic realizations of the generators of the above quantum algebras. This method will utilize the factorization provided by the deforming mappings (3)
by which a deformed generator is given by a non-deformed one times a deformation operator factor $F$ (Table 1).

Let us consider the action of $G^q_i$ on a generic state $|\Psi> >$ leading to another state $|\Phi_i> >$,

$$G^q_i|\Psi> = |\Phi_i> , \quad (i = +, -, 0) .$$  \hfill (7)

Then,

$$(\zeta|G^q_i|\Psi> = (\zeta|\Phi_i>$$  \hfill (8)

may be used to define the representative $\pi_\zeta(G^q_i)$ of the generators $G^q_i$ acting on functions $\Psi(\zeta)$ defined on the general $\zeta$-Bargmann space,

$$\pi_\zeta(G^q_i)\Psi(\zeta) \equiv \Phi_i(\zeta)$$  \hfill (9)

with $\Psi(\zeta) = (\zeta|\Psi>$ and $\Phi_i(\zeta) = (\zeta|\Phi_i> .$

Since $G^q_0 = G_0$ (eq. (3b)), in what follows we shall concentrate on $G^q_\pm$ only and give the result for $\pi_\zeta(G^q_0)$ at the end. Using eq.(3) the l.h.s. of eq.(8) is written

$$(\zeta|G^q_\pm|\Psi> = (\zeta|G_\pm F^\pm(G_0)|\Psi> = \tau_\pm(\zeta|f^\pm(G_0)|\Psi> ,$$  \hfill (10)

where the actions $(\zeta|G_\pm$ have been evaluated using the following formulae for the $wh$, $su(2)$ and $su(1, 1)$ coherent states:

$$\alpha|a^+ = \alpha(\alpha| and \quad (\alpha|a = \partial_\alpha(\alpha| \quad (\text{ordinary oscillator}) ;$$  \hfill (11)

$$ (z|J_+ = (z|(J - J_3)z \quad and \quad (z|J_- = (J + J_3)z^{-1} \quad (su(2)) ;$$  \hfill (12)

and

$$ (\xi|K_+ = (\xi|(K_3 + k)\xi \quad and \quad (\xi|K_- = (\xi|(K_3 - k)\xi^{-1} \quad (su(1, 1)) .$$  \hfill (13)

and $\tau_+ (\tau_-)$ are given by $\alpha, z, \xi (\partial_\alpha, z^{-1}, \xi^{-1})$. Then, by virtue of the above relations, the factorizations (3a) and Table 1, we obtain for the $f^\pm(G_0)$ defined by (10) the following values:
where we have also introduced the definitions of $\tau_{\pm}$.

Using the CS defined by (1), the r.h.s of eq.(10) can be cast in the following form:

$$
\tau_{\pm}(\zeta|f^\pm(G_0)|\Psi) = \tau_{\pm} < \phi|e^{\zeta G_-} f^\pm(G_0)e^{-\zeta G_-} . e^{\zeta G_-} |\Psi >
$$

$$
= \tau_{\pm} < \phi |\{f^\pm(G_0) + \zeta [G_-, f^\pm(G_0)] + \frac{\zeta^2}{2!} [G_-, [G_-, f^\pm(G_0)]] + ... \}e^{\zeta G_-}|\Psi > .
$$

(14)

Using now relation (2) we may arrange the nested commutators in the last equation in such a way that the functions of the generators $G_0$ always appear at the left of the monomials of $G_-$. This yields

$$
\tau_{\pm}(\zeta|f^\pm(G_0)|\Psi) = \tau_{\pm} < \phi |\sum_m \zeta^m m! B^\pm_m G^m_- \}e^{\zeta G_-}|\Psi > ,
$$

(15)

where

$$
B^\pm_m \equiv \binom{m}{m} f^\pm(G_0 + m) - \binom{m}{m-1} f^\pm(G_0 + m - 1) + ...(-1)^m \binom{m}{0} f^\pm(G_0)
$$

$$
= \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} f^\pm(G_0 + p)
$$

(16)

and $m = 0, 1, 2, ...$

As the $G_0$ generators are all diagonal in the basis of the states constructed from their lowest weight $|\phi >$, we will have generically that

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Table 2

| $\tau_+$ | $\tau_-$ | $f^+(G_0)$ | $f^-(G_0)$ |
|----------|----------|-------------|-------------|
| $\alpha$ | $\partial_\alpha$ | $\sqrt{\frac{N+1}{N+1}} (j - J_3)$ | $\sqrt{\frac{N}{N}} (j + J_3)$ |
| $\xi$ | $\xi^{-1}$ | $\sqrt{\left[\frac{(J_3 + j + 1)(J_3 - j)}{(K_3 + k)(K_3 + k + 1)}\right]}$ | $\sqrt{\left[\frac{(K_3 - k)(K_3 + k + 1)}{(K_3 - k + 1)(K_3 + k)}\right]}$ |

In the above table, $wh_q$, $su_q(2)$, and $su_q(1, 1)$ refer to the weight, spin, and quantum groups, respectively.
\[ \langle \phi | B_m^\pm = \langle \phi | b_m^\pm, \quad (17) \]

where the numerical eigenvalues \( b_m^\pm \) will be evaluated explicitly below for each of the three algebras considered. Using (17), (1) and that \( (\zeta|G^- = \partial_\zeta(\zeta), \text{eq. (15)} \) now leads to

\[ \tau_\pm(\zeta|f^\pm(G_0)|\Psi = \tau_\pm \{ b_0^\pm + b_1^\pm \zeta \partial_\zeta + \frac{b_2^\pm}{2!} \zeta^2 \partial_\zeta^2 + \ldots \} (\zeta|\Psi \equiv \pi_\zeta(G_\pm)\Psi(\zeta), \quad (18) \]

where \( \pi_\zeta(G_\pm) \) defines the realization of \( G_\pm \) on the functions \( \Psi(\zeta) \). Let us notice that in the \( su_q(2) \) case, the expansions (15) and (18) terminate since then \( G_{2j}^{2j+1} \equiv J_{2j+1} \) is zero on any vector of the representation space.

Collecting now all the above results and replacing \( \tau_\pm \) by their values (Table 2) the following realizations for the deformed generators are obtained:

\(a\) Quantum deformed oscillator

\[ \pi_\alpha(a_+^\alpha) = \partial_\alpha \sum_{m=0}^{\infty} \frac{b_0^\pm}{m!} \alpha^m \partial_\alpha^m \quad (19a) \]

\[ \pi_\alpha(a_-^\alpha) = \alpha \sum_{m=0}^{\infty} \frac{b_-^\pm}{m!} \alpha^m \partial_\alpha^m \quad (19b) \]

\[ \pi_\alpha(N) = \alpha \partial_\alpha, \quad (19c) \]

where

\[ b_m^\pm = (m)_m f_m^\pm - (m)_{m-1} f_{m-1}^\pm + \ldots (-1)^m (m)_0 f_0^\pm = \sum_{p=0}^{m} (-1)^m (m)_p f_p^\pm \quad (20a) \]

with

\[ f_p^+ = \sqrt{\frac{p+1}{p+1}} \quad \text{and} \quad f_p^- = \sqrt{\frac{|p|}{p}} \quad (20b) \]

since for instance \( \langle 0|f^+(N+p) = \langle 0|\sqrt{\frac{|p|+1}{p+1}} \equiv \langle 0|f_p^+ \).
The carrier space for this realization $L^2(C, \frac{d^2\alpha}{(\alpha|\alpha)}$) possesses an orthonormal basis formed by the monomials \( \{ \gamma_n(\alpha) \equiv (\alpha|n >= \frac{\alpha^n}{\sqrt{n!}}; n = 0, 1, 2, ... \} \), where

\[
< \gamma_n(\alpha), \gamma_{n'}(\alpha) > = \delta_{nn'} = \frac{1}{\pi} \int d^2\alpha e^{-\alpha\bar{\alpha}} \gamma_n(\alpha) \gamma_{n'}(\alpha)
\]  

(21)

since

\[
1 = \frac{1}{\pi} \int d^2\alpha e^{\alpha\bar{\alpha}} |\alpha| (\alpha| .
\]  

(22)

The action \( \pi_\alpha \) of the generators \( a^\pm_q, N \) on the basis vectors is given by the usual expressions

\[
\pi_\alpha(a^+_q) \frac{\alpha^n}{\sqrt{n!}} = \sqrt{[n + 1]} \frac{\alpha^{n+1}}{\sqrt{(n + 1)!}}
\]  

(23a)

\[
\pi_\alpha(a^-_q) \frac{\alpha^n}{\sqrt{n!}} = \sqrt{[n]} \frac{\alpha^{n-1}}{\sqrt{(n - 1)!}}
\]  

(23b)

\[
\pi_\alpha(N) \frac{\alpha^n}{\sqrt{n!}} = n \frac{\alpha^n}{\sqrt{n!}}
\]  

(23c)

In the \( q \rightarrow 1 \) limit eq. (20b) shows that \( f^\pm_p \rightarrow 1 \) and it is simple to check in eq. (20a) that \( b^+_q \rightarrow 1 \) and \( b^-_{q\neq 0} \rightarrow 0 \). As a result, the previous expressions yield the expected 'classical' limit relations \( \pi_\alpha(a^+_q) \rightarrow \alpha, \pi_\alpha(a^-_q) \rightarrow \partial_\alpha \) (and of course \( \pi_\alpha(N) = \alpha\partial_\alpha \)) of the oscillator Bargmann algebra.

b) \( su_q(2) \) algebra

Proceeding similarly, we find

\[
\pi_z(J^+_q) = z \sum_{m=0}^{2j} \frac{b^+_m}{m!} z^m \partial_z^m
\]  

(24a)

\[
\pi_z(J^-_q) = z^{-1} \sum_{m=0}^{2j} \frac{b^-_m}{m!} z^m \partial_z^m
\]  

(24b)

\[
\pi_z(J^3_q) = -j + z\partial_z,
\]  

(24c)

where now
\[ b_m^\pm = \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} f_p^\pm \] (25a)

and

\[ f_p^+ = (2j - p) \sqrt{\frac{[2j - p][p + 1]}{(2j - p)(p + 1)}} , \quad f_p^- = p \sqrt{\frac{[p][p - 2j - 1]}{p(p - 2j - 1)}} \] (25b)

which follow from computing \( \langle j, -j | f^\pm (J_3 + p) \rangle \) using Table 2.

The Hilbert representation space \( L^2(C - \{ \infty \}, \frac{(2j + 1)}{\pi} \frac{d^2z}{(1 + |z|^2)^{2j+1}}) \) admits the following basis \( \{ \gamma_n(z) \equiv (\frac{2j}{j+n})^{1/2} z^{j+n}, n = -j, ..., j \} \) with orthonormality conditions,

\[ < \gamma_n(z), \gamma_{n'}(z) > = \delta_{nn'} = \frac{(2j + 1)}{\pi} \int \frac{d^2z}{(1 + |z|^2)^{2j+2}} \bar{\gamma}_n(z) \gamma_{n'}(z) \] , (26)

where the new measure is designed so that the \( \gamma_n(z) \) states are orthonormal since

\[ 1 = \frac{(2j + 1)}{\pi} \int \frac{d^2z}{(1 + |z|^2)^{2j+2}} |z| \] . (27)

The action of the deformed generator on the unit vector of the representation space reproduces the familiar \( q \)-Fock expressions

\[ \pi_z(J_3^q) \gamma_m(z) = \sqrt{[j - m][j + m + 1]} \gamma_{m+1}(z) \] (28a)

\[ \pi_z(J_3^q) \gamma_m(z) = \sqrt{[j + m][j - m + 1]} \gamma_{m-1}(z) \] (28b)

and

\[ \pi_z(J_3^q) \gamma_m(z) = m \gamma_m(z) \] . (28c)

In the zero deformation limit, \( q \rightarrow 1 \), \( f_p^+ \rightarrow (2j - p) \) and \( f_p^- \rightarrow p \) (eq. (25b)). Then, \( b_0^+ \rightarrow 2j \), \( b_1^+ \rightarrow -1 \) and all the other \( b^+ \)'s become zero. Thus, \( \pi_z(J_3^q) \rightarrow 2j z - z^2 \partial_z \). Similarly \( b_1^- \rightarrow 1 \) and the other \( b^- \)'s are zero in the same limit and we obtain \( \pi_z(J_3^q) \rightarrow \partial_z \), the realization \( \pi_z(J_3) = -j + z \partial_z \) remaining
unaffected. Thus, the reduction to the ordinary coset representation (see e.g. [17, 18, 19]) of the $su(2)$ algebra is provided by the classical $q \to 1$ limit.

c) $su_q(1, 1)$ algebra

Finally, for the $su_q(1, 1)$ generators we find the realization

\[
\pi_\xi(K^+_q) = \xi \sum_{m=0}^{\infty} \frac{b^+_m}{m!} \xi^m \partial_\xi^m
\]

(29a)

\[
\pi_\xi(K^-_q) = \xi^{-1} \sum_{m=0}^{\infty} \frac{b^-_m}{m!} \xi^m \partial_\xi^m
\]

(29b)

\[
\pi_\xi(K^3_q) = k + \xi \partial_\xi
\]

(29c)

with

\[
b^\pm_m = \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} f^\pm_p
\]

(30a)

where now Table 2 gives for $< k, 0|f^\pm_p(K^3 + p)$ the expressions

\[
f^+_p = (2k + p) \sqrt{\frac{[p + 1][2k + p]}{(p + 1)(2k + p)}} \quad \text{and} \quad f^-_p = p \sqrt{\frac{[p][2k + p - 1]}{p(2k + p - 1)}}.
\]

(30b)

The representation space is now the Hilbert space of square integrable functions with support on the open unit disk on the complex plane, $D = \{\xi \in \mathbb{C} : |\xi|^2 < 1\}$, denoted by $L^2(D, \frac{d^2\xi}{(1-|\xi|^2)^{(2k-1)}})$ which has the basis of vectors $\{\gamma_n(\xi) \equiv (\xi|n > = \binom{\Gamma(2k+n)}{n!\Gamma(2k)} \frac{1}{\xi^n}$, $n = 0, 1, 2, \ldots\}$, satisfying the orthonormality condition

\[
< \gamma_n(\xi), \gamma_{n'}(\xi) > = \delta_{nn'} \int_{|\xi|<1} \frac{d^2\xi}{(1-|\xi|^2)^{(2k-2)}} \gamma_n(\xi)\gamma_{n'}(\xi)
\]

(31a)

corresponding to the completeness relation

\[
1 = \frac{(2k - 1)}{\pi} \int_{|\xi|<1} \frac{d^2\xi}{(1-|\xi|^2)^{(2k-2)}} \gamma_n(\xi)|\xi|.
\]

(31b)

The action on the above basis monomials is given by

\[
\pi_\xi(K^q_{+) \gamma_n(\xi) = \sqrt{[2k + n][n + 1]} \gamma_{n+1}(\xi),
\]

(32a)
\[ \pi_\xi(K^q_3) \gamma_n(\xi) = (k + n)\gamma_n(\xi) \quad . \] (32c)

If there is no deformation \( f^p_+ \rightarrow 2k + p \) and \( f^p_- \rightarrow p \) (eq. (30b)). Then, eq. (30a) gives \( b_{n=0} = b_{n \geq 2} = 0 \) and \( b_1^- = 1 \) so \( \pi_\xi(K^q_\+ \rightarrow \pi_\xi(K^q_\-) = \partial_\xi \), and \( \pi_\xi(K^q_\+) \rightarrow \pi_\xi(K_\+) = 2k\xi + \xi^2\partial_\xi \); \( \pi_\xi(K^q_3) \) is again given by (32c).

Having derived a complex analytic realization of the three algebra generators we shall give in the next Section the realizations of the corresponding co-products.

**IV.- Analytic realizations for the co-products**

In order to construct a realization of the co-products we shall write them generically in the form

\[ \Delta G^q_\pm = G^q_\pm \otimes g(G^q_0) + g'(G^q_0) \otimes G^q_\pm \] (33a)

\[ \Delta G^q_0 = G^q_0 \otimes 1 + 1 \otimes G^q_0 \quad , \] (33b)

where \( g(G^q_0) \) and \( g'(G^q_0) \) will be explicitly found for each algebra separately.

As the co-product of each algebra generator acts in the tensor product of the representation space of the algebra, we shall look for a analytic functional realization carried by functions of two variables. To this aim consider

\[ (\zeta_1| \otimes (\zeta_2| \Delta G^q_\pm |\Psi > = (\zeta_1| \otimes (\zeta_2| (G^q_\pm \otimes g(G^q_0))|\Psi > + \\
+ (\zeta_1| \otimes (\zeta_2| g'(G^q_0) \otimes G^q_\pm |\Psi > \quad . \] (34)

Using the deformation mapping (3) and eq. (10) this equation gives

\[ (\zeta_1| \otimes (\zeta_2| \Delta G^q_\pm |\Psi > = \tau^1_\pm (\zeta_1| \otimes (\zeta_2| f^\pm (G^q_0) \otimes g(G^q_0)|\Psi > + \]

\[ = \tau^1_\pm (\zeta_1| \otimes (\zeta_2| f^\pm (G^q_0) \otimes g(G^q_0)|\Psi > + \]
\( + \tau^2_\pm (\zeta_1 | \otimes (\zeta_2 | g'(G_0) \otimes f^\pm (G_0))|\Psi > \equiv I_1 + I_2 , \) (35)

where \( \tau^1_\pm \) and \( \tau^2_\pm \), where the indices refer to the first or second term in the tensor product, are the same as in Table 2 for the different algebras. We shall now compute \( I_1 \) and limit ourselves to give the formulae for \( I_2 \), since they are derived in exactly the same fashion. Using again definition (1) we find

\[
I_1 = \tau^1_\pm < \psi | \otimes < \phi | (e^{\zeta_1 G_{-}} \otimes e^{\zeta_2 G_{-}})(f^\pm (G_0) \otimes g(G_0) - e^{-\zeta_1 G_{-}} \otimes e^{-\zeta_2 G_{-}}) \\
(\psi | \otimes \psi )|\Psi > = \tau^1_\pm < \psi | \otimes < \phi | (e^{\zeta_1 G_{-}} f^\pm (G_0) e^{-\zeta_1 G_{-}} \otimes e^{\zeta_2 G_{-}} g(G_0) e^{-\zeta_2 G_{-}}) \\
(\psi | \otimes \psi )|\Psi > . \quad (36)
\]

Expanding the exponentials in the above formula and rearranging the ensuing nested commutators for \( f^\pm (G_0) \) and \( g(G_0) \) as we did for eqs. (14,15) we obtain

\[
I_1 = \tau^1_\pm < \psi | \otimes < \phi | \left( \sum_{n=0}^{l} \frac{\zeta_1^n}{n!} B^n \pm G^n_{-} \otimes \sum_{m=0}^{l} \frac{\zeta_2^m}{m!} C_m G^m_{-} \right) (\psi | \otimes \psi )|\Psi > , \quad (37)
\]

where \( l \) is the appropriate limit for each algebra, \( B^n \pm \) are the functions of \( G_0 \) introduced in (15),(16) and (17), and the operators \( C_m \) are defined (cf. (16)) by the expansion

\[
C_m = \binom{m}{m} g(G_0 + m) - \binom{m}{m-1} g(G_0 + m - 1) + ... + (-1)^{m} \binom{m}{0} g(G_0) \]
\[
= \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} g(G_0 + p) . \quad (38)
\]

On the vacuum, \( < \psi | C_m = < \psi | c_m \) (the numerical eigenvalues will be specified later for each algebra separately). Then, using that \((\zeta | G_{-} = \partial_{\zeta} (\zeta | \), eq.(37) may be written as

\[
I_1 = \tau^1_\pm (\zeta_1 | \otimes (\zeta_2 | \left[ \sum_{n,m=0}^{l} \frac{b^n \pm c^n}{n! m!} \zeta_1^n \zeta_2^m C^n_{-} \otimes G^m_{-} \right] |\Psi > 
\]

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\[
\tau_1^\pm \left[ \sum_{n,m=0}^l \frac{b_n^\pm c_m}{n! m!} \zeta_1^n \zeta_2^m \partial_\zeta_1^n \partial_\zeta_2^m \right] (\zeta_1 \otimes (\zeta_2)) |\Psi> \\
= \pi_{\zeta_1}(G^q_{\pm}) \pi_{\zeta_2}(g(G_0)) \Psi(\zeta_1, \zeta_2), \tag{39}
\]
where we have used (18) for the identification

\[
\pi_{\zeta_1}(G^q_{\pm}) = \tau_1^\pm \sum_{n=0}^l \frac{b_n^\pm}{n!} \zeta_1^n \partial_\zeta_1^n \tag{40a}
\]
and written

\[
\pi_{\zeta_1}(g(G_0)) = \sum_{m=0}^l \frac{c_m}{m!} \zeta_2^m \partial_\zeta_2^m. \tag{40b}
\]
Collecting all results for eq.(33) the realization of the co-product is obtained,

\[
\pi_{\zeta_1 \zeta_2}(\Delta G^q_{\pm}) \Psi(\zeta_1, \zeta_2) \equiv (\zeta_1 \otimes (\zeta_2 | \Delta G^q_{\pm} | \Psi > =
\]

\[
[\pi_{\zeta_1}(G^q_{\pm}) \pi_{\zeta_2}(g(G_0)) + \pi_{\zeta_1}(g'(G_0)) \pi_{\zeta_2}(G^q_{\pm})] \Psi(\zeta_1, \zeta_2), \tag{41}
\]
and, similarly,

\[
\pi_{\zeta_1 \zeta_2}(\Delta G^q_{0}) = [\pi_{\zeta_1}(G_0) + \pi_{\zeta_2}(G_0)] \Psi(\zeta_1, \zeta_2). \tag{42}
\]

As there is no satisfactory co-product and bialgebra structure for the \(q\)-oscillator, we shall now specialize to the \(su_q(2)\) and \(su_q(1,1)\) cases for which we have, respectively,

\[
\Delta J^q_{\pm} = J^q_{\pm} \otimes q^J^q_{\pm} + q^{-J^q_{\pm}} \otimes J^q_{\pm}, \tag{43a}
\]

\[
\Delta J^q_3 = J^q_3 \otimes 1 \otimes 1 \otimes J^q_3, \tag{43b}
\]
and

\[
\Delta K^q_{\pm} = K^q_{\pm} \otimes q^K^q_{\pm} + q^{-K^q_{\pm}} \otimes K^q_{\pm}, \tag{44a}
\]

\[
\Delta K^q_3 = K^q_3 \otimes 1 \otimes 1 \otimes K^q_3. \tag{44b}
\]
so that \( g(G_0) = q^{G_0} \) and \( g'(G_0) = q^{-G_0} \) with \( G_0 = J_3, K_3 \) respectively.

a) \( su_q(2) \)

For the \( su_q(2) \) case eq. (41) and (42) give

\[
\pi_{z_1 z_2}(\Delta J^q_{\pm}) = \pi_{z_1}(J^q_{\pm})\pi_{z_2}(q^q J^q_{\pm}) + \pi_{z_1}(q^{-J^q_{\pm}})\pi_{z_2}(J^q_{\mp})
\]  

(45a)

\[
\pi_{z_1 z_2}(\Delta J^q_3) = \pi_{z_1}(J^q_3) + \pi_{z_2}(J^q_3)
\]  

(45b)

where (cf. eqs. (40))

\[
\pi_{z_1,2}(J^q_{\pm}) = z^\pm_{1,2} \sum_{n=0}^{2j} \frac{b^\pm_n}{n!} z^n_{1,2}\delta^m_{1,2}
\]  

(46a)

\[
\pi_{z_1,2}(J^q_3) = -j + z_{1,2}\delta_{1,2}
\]  

(46b)

\[
\pi_{z_1,2}(q^q J^q_3) = \sum_{m=0}^{2j} \frac{c^\pm_m}{m!} z^n_{1,2}\delta^m_{1,2}
\]  

(46c)

where the \( b \)'s are given by eq.(25) and (cf. eq. (38))

\[
c^\pm_m = (m_j q^{(-j+m)} - (m_{j-1}) q^{(-j+m-1)} + ...(-1)^m q^{j} = \sum_{p=0}^{m} (-1)^{m-p} \frac{m_p}{p!} q^{(-j+p)} .
\]  

(47)

The realization of the co-product acts on the tensor product of two representation spaces which is spanned by the basis

\[
\gamma_{nm}(z_1, z_2) \equiv (z_1 | \otimes (z_2 | n > \otimes | m > = (2j_{j+n}) \frac{1}{2j} (2j_{j+m}) \frac{1}{2j} z_1^{j+n} z_2^{j+m},
\]  

(48)

and which satisfies an orthonormality relation induced by that in each of the factors (eq. (26)). On the basis elements \( \gamma_{nm}(z_1, z_2) \) the action of \( \pi_{z_1 z_2} \) is given by (cf. eq.(28)),

\[
\pi_{z_1 z_2}(\Delta J^q_3) \gamma_{n,m}(z_1, z_2) =
\]

\[
= \sqrt{[j \mp n][j \pm n + 1]} q^n \gamma_{n+1,m} + q^{-n} \sqrt{[j \mp m][j \pm m + 1]} \gamma_{n,m+1} ,
\]  

(49a)
\[ \pi_{z_1 z_2} (\Delta J^q_3) \gamma_{n,m}(z_1, z_2) = (n + m) \gamma_{n,m}(z_1, z_2) \] . \tag{49b}

b) \( su_q(1, 1) \)

Finally, for the \( su_q(1, 1) \) case, eq. (41), we find

\[ \pi_{\xi_1 \xi_2} (\Delta K^q_\pm) = \pi_{\xi_1} (K^q_\pm) \pi_{\xi_2} (q^K_3) + \pi_{\xi_1} (q^{-K^q_3}) \pi_{\xi_2} (K^q_\pm) \] \tag{50a}

\[ \pi_{\xi_1 \xi_2} (\Delta K^q_3) = \pi_{\xi_1} (K^q_3) + \pi_{\xi_2} (K^q_3) \] , \tag{50b}

where

\[ \pi_{\xi_1,2} (K^q_\pm) = \xi_{1,2}^\pm \frac{b^m_{\pm}}{m!} \xi_{1,2}^m \partial_{\xi_{1,2}}^m \] \tag{51a}

\[ \pi_{\xi_1,2} (K^q_3) = k + \xi_{1,2} \partial_{\xi_{1,2}} \] \tag{51b}

\[ \pi_{\xi_1,2} (q^K_3) = \sum_{m=0}^\infty c^m_{\pm} \xi_{1,2}^m \partial_{\xi_{1,2}}^m \] \tag{51c}

where the coefficients \( b^m_{\pm} \) are given in eq.(28) and the \( c^m_{\pm} \) in this case read (cf. eq. (47))

\[ c^m_{\pm} = \binom{m}{k} q^{\pm(k+m)} - \binom{m}{k+1} q^{\pm(k+m-1)} + \ldots + (-1)^m q^{\pm k} = \sum_{p=0}^m (-1)^{m-p} q^{\pm(k+p)} \] . \tag{52}

As for \( su_q(2) \), the orthonormal basis which spans the representation space of the above co-product realization inherits its orthonormality properties from eq.(31) and reads

\[ \gamma_{nm}(\xi_1, \xi_2) \equiv (\xi_1 | \otimes (\xi_2 | n > \otimes m > =

\[ = \left( \frac{\Gamma(2k+n)}{n! \Gamma(2k)} \right)^{\frac{1}{2}} \left( \frac{\Gamma(2k+m)}{m! \Gamma(2k)} \right)^{\frac{1}{2}} \xi_1^n \xi_2^m \] . \tag{53}

Extending the action expressed by eqs. (32) to the co-product realization of the \( su_q(1, 1) \) generators we obtain
\[ \pi_{\xi_1, \xi_2} (\Delta K_4^q) \gamma_{n,n}(\xi_1, \xi_2) = \]
\[ = \sqrt{[n + 1][n + 2k]} q^{k+n} \gamma_{n+1,m}(\xi_1, \xi_2) + q^{-(k+m)} \sqrt{[m + 1][m + 2k]} \gamma_{n,m+1}(\xi_1, \xi_2) \]
\[(51a)\]
\[ \pi_{\xi_1, \xi_2} (\Delta K_3^q) \gamma_{n,m}(\xi_1, \xi_2) = \sqrt{[n][n - 1 + 2k]} q^{k+n} \gamma_{n-1,m}(\xi_1, \xi_2) + q^{-(k+m)} \sqrt{[m][m - 1 + 2k]} \gamma_{n,m-1}(\xi_1, \xi_2) \]
\[(54b)\]

and

\[ \pi_{\xi_1, \xi_2} (\Delta K_3^q) \gamma_{n,m}(\xi_1, \xi_2) = (2k + n + m) \gamma_{n,m}(\xi_1, \xi_2) . \]
\[(54c)\]

\textbf{V. - Conclusions}

In this paper we have looked for realizations of quantum algebras in terms of ordinary differential operators. We have found their expression for certain \( q \)-algebras for which there exist functional deforming mappings relating the deformed and non-deformed generators which allow us to write explicitly the deformed generators as elements of the original enveloping algebra. The \( q \)-oscillator, \( su_q(2) \) and \( su_q(1, 1) \) quantum algebras are particular cases of this class, and the method of constructing realizations relies on the ordinary coherent states for the undeformed Lie algebras.

Finally, we recall that the realization for the deformed algebra generators obtained here is given in terms of a series of powers of derivatives with \( q \)-dependent coefficients which in the classical \( q \to 1 \) limit reproduces the vector field generators of the Lie algebra. Thus, the appearance of ordinary but higher order derivatives provides an alternative way of describing the deformation process.
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