Factorization of the Universal $\mathcal{R}$-matrix for $U_q(\hat{sl}_2)$

Jintai Ding $^*$, Sergei Khoroshkin $^*$, Stanislav Pakuliak $^{**}$

$^*$ Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221-0025, USA
$^*$ Institute of Theoretical & Experimental Physics, 117259 Moscow, Russia
$^\ast$ Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia
$^{\circ}$ Bogoliubov Institute for Theoretical Physics, 003143 Kiev, Ukraine

Abstract

The factorization of the universal $\mathcal{R}$-matrix corresponding to so called Drinfeld Hopf structure is described on the example of quantum affine algebra $U_q(\hat{sl}_2)$. As a result of factorization procedure we deduce certain differential equations on the factors of the universal $\mathcal{R}$-matrix, which allow to construct uniquely these factors in the integral form.

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1E-mail: Jintai.Ding@math.uc.edu
2E-mail: khor@heron.itep.ru
3E-mail: pakuliak@thsun1.jinr.ru
1 Introduction

The theory of quantum groups is the origin of many group-theoretical methods for investigation of the quantum integrable models. The quantum group theory based on the quantum inverse scattering method \cite{FT} was described in the pioneering works \cite{D, J} as the Hopf algebra deformation of the universal enveloping algebras of contragredient Lie algebras. In most applications the quantum groups as the Hopf algebras appeared together with $R$-matrices, either in the form of numerical matrices or $L$-operators or universal $R$-matrices. The latter are the elements in the completed square of the corresponding Hopf algebras, which satisfy certain conditions. A direct consequence of these relations is the fact that the universal $R$-matrix satisfy Yang-Baxter relation and different $L$-operators and numerical $R$-matrices can be obtained from the universal one by specializing to certain representations of the original algebra.

Recently it became clear \cite{F, ABRR, JKOS} that the deformed algebras which are behind the integrability of the elliptic models \cite{ABF, Ba} and their counterparts from the quantum field theories \cite{ZZ} can be obtained using twisting procedure slightly relaxing the coassociativity axiom of the original Hopf algebras. The resulting algebras turn out to be the quasi-Hopf algebras \cite{D2}. The notion of the universal $R$-matrix survive during the twisting procedure although the algebra itself can loose some properties. The precise construction of the twisting element refers to the solution of certain difference equation. As a consequence, the universal $R$-matrix in twisted algebra appears in this approach as infinite product of shifted universal $R$-matrices for original quantum affine algebra, which is unobservable for practical use.

Another way to get the same result is to use so called 'new realization' of quantum affine algebras and their generalizations \cite{D1}. This realization was introduced by Drinfeld in order to show the deformation of standard loop basis of affine Lie algebras. It happened to be very useful in representation theory. 'New realization' possesses its own comultiplication structure (we call it 'Drinfeld comultiplication'), different from standard comultiplication structure for quantized Kac-Moody algebras. It was proved in \cite{KT} that this comultiplication structure can be obtained from the standard one as a twist by certain factor of universal $R$-matrix. Another advantage of 'new realizations' was noted in \cite{JKOS}: their elliptic analogs can be described with a help of very simple twist.

Enriquez, Felder and Rubtsov \cite{EF, ER} suggested to reverse the calculations in \cite{KT}: one can try to describe traditional Hopf structure of quantum affine algebras and their elliptic analogs starting from Drinfeld comultiplication and its elliptic analog. It follows from \cite{KT}, that this problem is equivalent to a Riemann type problem of factorization of essential part of the universal $R$-matrix for Drinfeld comultiplication. They managed to get in this way an $L$-operator description of elliptic face type algebras and generalized this approach to the curves of higher genus.

In this paper we develop the ideas of \cite{EF} and solve explicitly the factorization problem for quantum affine algebra $U_q(\hat{sl}_2)$. Our method is different from \cite{EF} and \cite{ER}. It is based on the use of the results of \cite{DK, DKP}, where an integral presentation of the universal $R$-matrix for Drinfeld comultiplication was studied. Applying the projectors which describe the factorization to this integral presentation we deduce differential equations for the factors of the universal $R$-matrix for $U_q(\hat{sl}_2)$. These differential equations have precise unique solution in noncommutative power series. In particular cases, for instance, for level one representations, they allow to describe an evaluation of the universal $R$-matrix in infinite-dimensional representations. The method is quite general, it can be applied for the elliptic and Yangian algebras as well, though we restrict ourselves to $U_q(\hat{sl}_2)$ in this paper.

The paper is organized as follows. First, we remind two descriptions of quantum affine algebra $U_q(\hat{sl}_2)$ and formulate the main results. Section 3 is devoted to 'new realization' of $U_q(\hat{sl}_2)$. We describe here its Hopf structure, the Hopf pairing between two Borel subalgebras in a current form and review all known description of the corresponding universal $R$-matrix and of the pairing tensor. In particular, we remind the results of \cite{DK, DKP}, where an integral presentation and the differential equation for the universal $R$-matrix was studied.

The main results are proved in Section 4. Here we first review the projection technique, elaborated
2. The ‘new realization’ of the quantum affine algebra $U_q(\widehat{sl}_2)$

1. $U_q(\widehat{sl}_2)$ as quantized Kac-Moody algebra.

Quantum affine algebra $U_q(\widehat{sl}_2)$ is an associative algebra generated by the elements, $e_{\pm \alpha_i}$, $k_{\alpha_i}^{\pm 1}$, $d$, $i = 0, 1$ subjected to the commutation relations:

$$[d, e_{\pm \alpha_i}] = \pm \delta_{ij} e_{\pm \alpha_i}, \quad k_{\alpha_i} e_{\pm \alpha_j} k_{\alpha_i}^{-1} = q_i^{\pm a_{ij}} e_{\pm \alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}}$$

(2.1)

and also the Serre relations

$$e_{\pm \alpha_i}^3 + [3] q_{\pm \alpha_i} e_{\pm \alpha_i} e_{\pm \alpha_i} + [3] q e_{\pm \alpha_i} e_{\pm \alpha_i} e_{\pm \alpha_i} + e_{\pm \alpha_i} e_{\pm \alpha_i}^3 = 0, \quad i \neq j,$$

(2.2)

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ is Gauss $q$-number and $a_{ij} = (\alpha_i, \alpha_j)$ is Cartan matrix of the affine algebra $\widehat{sl}_2$.

The element $c$ given by the relation

$$q^c \equiv k_{\alpha_0} k_{\alpha_1}$$

(2.3)

is central and its value on the particular representation is called ‘level’.

One of the possible Hopf structures is given by the formulas:

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \quad \Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1}$$

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i}, \quad \Delta(d) = d \otimes 1 + 1 \otimes d$$

$$\varepsilon(e_{\pm \alpha_i}) = 0, \quad \varepsilon(k_{\alpha_i}^{\pm 1}) = 1, \quad \varepsilon(d) = 0,$$

(2.4)

$$a(e_{\alpha_i}) = -k_{\alpha_i}^{-1} e_{\alpha_i}, \quad a(e_{-\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}, \quad a(k_{\alpha_i}^{\pm 1}) = k_{\alpha_i}^{\mp 1}, \quad a(d) = -d,$$

where $\Delta$, $\varepsilon$ and $a$ are comultiplication, counit and antipode maps respectively.

2. The ‘new realization’ of $U_q(\widehat{sl}_2)$.

The so called new realization of quantum affine algebras was introduced by V. Drinfeld in [D]. In this description the algebra $U_q(\widehat{sl}_2)$ is generated by the infinite set of generators $d$, $q^c$, $k^{\pm 1}$, $e_n$, $f_n$, $n \in \mathbb{Z}$, $a_{\pm m}$, $m \geq 0$ subjected to quadratic commutation relations, which are given as formal power series identities on their generating functions

$$e(z) = \sum_{n \in \mathbb{Z}} e_n z^{-n}, \quad f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n},$$

$$\psi^\pm(z) = \sum_{n \geq 0} \psi_{n}^{\pm} z^{\mp n} = k^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} a_{\pm n} z^{\mp n} \right),$$

(2.5)
as follows:

\[ [q^c, \text{everything}] = 0 \, , \]

\[ x^d e(z) x^{-d} = e(xz) , \quad x^d f(z) x^{-d} = f(xz) , \quad x^d \psi^\pm(z) x^{-d} = \psi^\pm(xz) \, , \quad (2.6a) \]

\[ (z - q^2 w) e(z) e(w) = (q^2 z - w) e(w) e(z) \, , \quad (2.6b) \]

\[ (z - q^{-2} w) f(z) f(w) = (q^{-2} z - w) f(w) f(z) \, , \quad (2.6c) \]

\[ \frac{(q^{c/2} z - q^{-2} w)}{(q^{2-c/2} z - w)} \psi^+(z) e(w) = e(w) \psi^+(z) \, , \quad (2.6d) \]

\[ \frac{(q^{c/2} z - q^{-2} w)}{(q^{-2-c/2} z - w)} \psi^+(z) f(w) = f(w) \psi^+(z) \, , \quad (2.6e) \]

\[ \psi^-(z) e(w) = \frac{(q^{-2-c/2} z - w)}{(q^{-2+c/2} z - q^{-2} w)} e(w) \psi^-(z) \, , \quad (2.6f) \]

\[ \psi^-(z) f(w) = \frac{(q^{-2+c/2} z - w)}{(q^{c/2} z - q^{-2} w)} f(w) \psi^-(z) \, , \quad (2.6g) \]

\[ \frac{(z - q^{-c} w) (z - q^{-2+c} w)}{(q^{2-c} z - q^{-2} w)} \psi^+(z) \psi^-(w) = \psi^-(w) \psi^+(z) \, , \quad (2.6h) \]

\[ \psi^+(z) \psi^+(w) = \psi^+(w) \psi^+(z) \, , \quad (2.6i) \]

\[ [e(z), f(w)] = \frac{1}{q - q^{-1}} \left( \delta(z/q^2 w) \psi^+(z q^{-c/2}) - \delta(z q^c / w) \psi^-(w q^{-c/2}) \right) \, , \quad (2.6j) \]

where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \).

The coalgebraic structure of the algebra \( U_q(\hat{sl}_2) \) which is equivalent to \( [2,4] \) cannot be formulated in total currents \( [2,3] \). Let

\[ e_+(z) = \oint \frac{dw}{2 \pi w} \frac{e(w)}{1 - w/z} = \sum_{k \geq 0} e_k z^{-k} , \quad e_-(z) = - \oint \frac{dw}{2 \pi w} \frac{e(w) z/w}{1 - z/w} = - \sum_{k < 0} e_k z^{-k} \quad (2.7) \]

\[ f_+(z) = \oint \frac{dw}{2 \pi w} \frac{f(w) w/z}{1 - w/z} = \sum_{k \geq 0} f_k z^{-k} , \quad f_-(z) = - \oint \frac{dw}{2 \pi w} \frac{e(w)}{1 - z/w} = - \sum_{k < 0} f_k z^{-k} \]

be the half-currents. Then the coalgebraic structure of \( U_q(\hat{sl}_2) \) in new realization have the form (see [KLP], where the formulas \( [2.8] \) have been proved in slightly different situation)

\[ \Delta(e_\pm(z)) = e_\pm(z) \otimes 1 + \sum_{k \geq 0} (-q)^k (q - q^{-1})^{2k} f_\pm^k (z q^{c_1}) \otimes e_\pm^{k+1} (z q^{c_1}) \, , \quad (2.8a) \]

\[ \Delta(f_\pm(z)) = 1 \otimes f_\pm(z) + \sum_{k \geq 0} (-q)^{-k} (q - q^{-1})^{2k} f_\pm^{k+1} (z q^{c_2}) \otimes \psi_\pm (z q^{c_2}) e_\pm^k (z q^{c_2}) \, , \quad (2.8b) \]

\[ \Delta(\psi_\pm(z)) = \sum_{k \geq 0} (-1)^k [k + 1] q (q - q^{-1})^{2k} f_\pm^k (z q^{2+c_2}) \otimes \psi_\pm (z q^{c_2}) e_\pm^k (z q^{2+c_2}) \, , \quad (2.8c) \]

where \( c_1 = c \otimes 1 \) and \( c_2 = 1 \otimes c \).

3. The connection between two descriptions.

In the original paper [D] the Chevalley generators were presented in terms of the current generators. In the case under consideration they have the form:

\[ e_{\alpha_1} = e_0 , \quad e_{-\alpha_1} = f_0 , \quad e_{\alpha_0} = q^c f_1 k^{-1} , \quad e_{-\alpha_0} = k e_{-1} q^{-c} , \quad k_{\alpha_1} = k , \quad k_{\alpha_0} = q^c k^{-1} . \quad (2.9) \]
The inverse formulas appeared lately in [K, KT] in general case of arbitrary quantum affine algebra $U_q(\hat{g})$ (see also [Da] for $U_q(sl_2)$). It turns out the the generators in ‘new realization’ coincide up to central elements with Cartan-Weyl generators of quantum affine algebra. We describe shortly this construction in case under consideration.

First of all, define the generators

$$e_\delta \overset{\text{def}}{=} e_{\alpha_1}e_{\alpha_0} - q^2 e_{\alpha_0}e_{\alpha_1}, \quad e_{-\delta} \overset{\text{def}}{=} e_{-\alpha_0}e_{-\alpha_1} - q^{-2} e_{-\alpha_1}e_{-\alpha_0}.$$  

They satisfy the commutation relation

$$[e_{\delta}, e_{-\delta}] = \frac{q^c - q^{-c}}{q - q^{-1}}.$$

Now we define the Cartan-Weyl basis of $U_q(\hat{g}_2)$. For $n \geq 0$ we define the generators which correspond to all real roots $\pm \alpha_1 \pm n\delta$,

$$E_{\alpha_1+n\delta} \overset{\text{def}}{=} [2]_q^{-n} \left( \text{ad} e_{\delta} \right)^n e_{\alpha_1}, \quad E_{\alpha_1-n\delta} \overset{\text{def}}{=} q^n [2]_q^{-n} \left( \text{ad} e_{-\delta} \right)^n e_{\alpha_1},$$

$$E_{-\alpha_1+n\delta} \overset{\text{def}}{=} q^{-cn}[-2]_q^{-n} \left( \text{ad} e_{\delta} \right)^n e_{-\alpha_1}, \quad E_{-\alpha_1-n\delta} \overset{\text{def}}{=} [-2]_q^{-n} \left( \text{ad} e_{-\delta} \right)^n e_{-\alpha_1}.$$ 

Define also auxiliary generators

$$E_{n\delta} = [E_{\alpha_1}, E_{-\alpha_1+n\delta}], \quad E_{-n\delta} = [E_{\alpha_1-n\delta}, E_{-\alpha_1}], \quad n > 0$$

related to imaginary roots $\pm n\delta$ generators $a_{\pm n}$

$$\pm (q - q^{-1}) \sum_{n=0}^\infty q^{\pm \frac{n^2}{2}} E_{\pm n\delta} z^{\mp n} = k^{\pm 1}_{\alpha_1} \exp \left( \pm (q - q^{-1}) \sum_{n>0} a_{\pm n} z^{\mp n} \right).$$

The identification with Drinfeld’s generators reads as follows

$$e_n = E_{\alpha_1+n\delta}, \quad f_n = E_{-\alpha_1+n\delta}, \quad \forall n \in \mathbb{Z}, \quad \psi_0^\pm = k^{\pm 1}_{\alpha_1},$$

$$E_{n\delta} \equiv \frac{q^{-\frac{n^2}{2}}}{q - q^{-1}} \psi_n^+, \quad n > 0, \quad E_{-n\delta} \equiv - \frac{q^{\frac{n^2}{2}}}{q - q^{-1}} \psi_n^-, \quad n < 0.$$  

### 2.2 The main results

Let us remind that a universal $\mathcal{R}$-matrix for the quantum affine algebra $U_q(\hat{g})$ is an element in some completion of $U_q(\hat{g}) \otimes U_q(\hat{g})$ which satisfies

$$\mathcal{R}\Delta(x) = \Delta'(x)\mathcal{R}, \quad \forall x \in U_q(\hat{g}), \quad (2.10a)$$

$$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}, \quad (\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}, \quad (2.10b)$$

where for $\Delta(x) = x(1) \otimes x(2), \Delta'(x) = x(2) \otimes x(1)$ means the opposite comultiplication and if $\mathcal{R} = \sum a_i \otimes b_i, \mathcal{R}^{13}$ means $\sum a_i \otimes 1 \otimes b_i,$ etc.

It is known [KT] that the universal $\mathcal{R}$-matrix for quantum affine algebra with Hopf structure given by (2.4) have the following form

$$\mathcal{R}_{\text{can}} = \mathcal{R}_+^{21} \cdot \mathcal{K} \cdot \mathcal{R}_+^{1-}, \quad (2.11)$$

where $A^{21}$ means the transposition of the left and right tensor space and the element $\mathcal{K}$

$$\mathcal{K} = q^{-\frac{h(\hat{h})}{2}} q^{-\frac{\epsilon d + d \epsilon}{2}} \exp \left( (q^{-1} - q) \sum_{n>0} \frac{n}{[2n]_q} a_n \otimes a_{-n} \right) q^{-\frac{\epsilon d + d \epsilon}{2}} \quad (2.12)$$
depends only on the Cartan and imaginary root generators. The $R$-matrix (2.11) belongs to the tensor product $U_q(b_+) \otimes U_q(b_-)$, where $U_q(b_+)$ is generated by $e_m$, $m \geq 0$, $k^{\pm 1}$, $a_n$, $f_n$, $n > 0$ and $U_q(b_-)$ is generated by $f_n$, $n \leq 0$, $k^{\pm 1}$, $a_m$, $e_m$, $m < 0$. The multiplicative expressions for the elements $R_{\pm, \mp}$ are known (see [KT] or formulas (4.1) below).

The main subject of the paper is an integral presentation for the factors $R_{\pm, \mp}$ of the universal $R$-matrix (2.11). Let $d_\alpha$ be the following gradation operator:

\[ [d_\alpha, e(z)] = e(z), \quad [d_\alpha, f(z)] = -f(z), \quad [d_\alpha, \psi_\pm(z)] = 0 \]

and (do not confuse $\tau$ with spectral parameter)

\[ R_{\pm, \mp}(\tau) = \tau^{-d_\alpha \otimes 1} R_{\pm, \mp} \tau^{d_\alpha \otimes 1}, \quad \overline{R}_{\pm, \mp}(\tau) = 1 \otimes 1 + \sum_{n>0} R_{\pm, \mp}^{(n)} \tau^n. \]  

The following differential equations can be regarded as a main result of the paper.

**Theorem 1** Let $q^N \neq 1$ for $N \in \mathbb{Z} \setminus \{0\}$. Then

\[ \frac{dR_{+, -}(\tau)}{d\tau} = R_{+, -}(\tau) \cdot I_{+, -}(\tau), \]  

\[ \frac{dR_{-, +}(\tau)}{d\tau} = I_{-, +}(\tau) \cdot R_{-, +}(\tau), \]  

where

\[ I_{\pm, \mp}(\tau) = \sum_{n>0} I_{\pm, \mp}^{(n)} \tau^n \]

and

\[ I_{+, -}^{(n)} = \frac{(-1)^n(q^{-1} - q)}{[n]_q!([n-1]_q)!} \oint \frac{dz}{2\pi iz} S_{j_0}^{n-1}(f_+(z)) \otimes S_{e_0}^{n-1}(e_-(z)) \]  

\[ I_{-, +}^{(n)} = \frac{(-1)^n(q^{-1} - q)}{[n]_q!([n-1]_q)!} \oint \frac{dz}{2\pi iz} S_{j_0}^{n-1}(f_-(z)) \otimes S_{e_0}^{n-1}(e_+(z)) \].

The screening operators $S_{e_0}$ and $S_{j_0}$ are defined through left/right adjoint actions (2.4) which use the standard Hopf structure (2.4):

\[ S_{e_0}(x) = e_0 x - kxk^{-1}e_0, \quad S_{j_0}(x) = j_0 x - j_0 k^{-1} j_0 k. \]

It is possible also to express the action of these screening operators on the fields $e_\pm(z)$ and $f_\pm(z)$ via the powers of the fields:

\[ S_{e_0}^{n-1}(e_\pm(z)) = \prod_{k=2}^n (1 - q^{2(k-1)}) e_\pm(z), \quad S_{j_0}^{n-1}(f_\pm(z)) = \prod_{k=2}^n (q^{-2(k-1)} - 1) f_\pm(z). \]

The differential equations (2.15a) and (2.15b) define the recurrence relations between homogeneous components $R_{\pm, \mp}^{(n)}$ of $R_{\pm, \mp}(\tau)$. Moreover, these equations have unique solutions in power series over $\tau$ with initial conditions $R_{\pm, \mp}(0) = 1 \otimes 1$.

**Theorem 2** The elements $R_{\pm, \mp}$ can be presented as series of multiple formal integrals

\[ R_{\pm, \mp} = 1 \otimes 1 + \sum_{n>0} \overline{R}_{\pm, \mp}^{(n)} \]

\[ \times \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_m}{z_m} S_{j_0}^{j_0-1}(f_+(z_1)) \cdots S_{j_0}^{j_0-1}(f_+(z_m)) \otimes S_{e_0}^{j_0-1}(e_+(z_1)) \cdots S_{e_0}^{j_0-1}(e_+(z_m)) \]
and
\[ C_+(j_1, j_2, \ldots, j_m) = \frac{(q^{-1} - q)^m}{j_1(j_1 + j_2)(j_1 + j_2 + j_3) \cdots (j_1 + j_2 + \cdots + j_m)} \prod_{i=1}^m \frac{1}{[j_i]_q ![j_i - 1]_q !}, \]
\[ C_-(j_1, j_2, \ldots, j_m) = \frac{(q^{-1} - q)^m}{j_m(j_m + j_{m-1})(j_m + j_{m-1} + j_{m-2}) \cdots (j_m + j_{m-1} + \cdots + j_1)} \prod_{i=1}^m \frac{1}{[j_i]_q ![j_i - 1]_q !}. \]

(2.20)

Applying the results of \([DM]\), we can prove that in integral representations of level \( k > 0 \) the fields \( e_\pm(z) \) and \( f_\pm(z) \) are annihilated by \( k + 1 \)-th degree of screenings. So we have under the same condition on \( q \)

**Corollary 1** Let the \( \mathcal{R} \)-matrix \([2.11]\) acts in tensor product of integrable representations, one of which has level \( k > 0 \). Then the summation indices \( j_i, \ i = 1, \ldots, m \) in \([2.19]\) satisfy the inequalities \( 1 \leq j_i \leq k \). In particular, if one of the representations has level 1, then the \( \mathcal{R} \)-matrix \([2.11]\) has a form

\[ \mathcal{R}_{\text{can}} = \exp \left( \frac{q^{-1} - q}{2\pi i} \oint \frac{dz}{z} e_+ (z) \otimes f_- (z) \right) \cdot \mathcal{K} \cdot \exp \left( \frac{q^{-1} - q}{2\pi i} \oint \frac{dz}{z} f_+ (z) \otimes e_- (z) \right), \]

(2.21)

where the factor \( \mathcal{K} \) is given by \([2.12]\).

### 3.1 The universal \( \mathcal{R} \)-matrix in a multiplicative form

In \([1]\) the quantum affine algebras have been constructed by means of the quantum double construction with comultiplication different from those given in formulas \((2.4)\). This is so called Drinfeld Hopf structure written in terms of generating functions as follows:

\[ \Delta^{(1)} e(z) = e(z) \otimes 1 + \psi^-(zq^{q^2}) \otimes e(zq^{-1}), \]

(3.1a)

\[ \Delta^{(1)} f(z) = 1 \otimes f(z) + f(zq^2) \otimes \psi^+(zq^{q^2}), \]

(3.1b)
\[ \Delta^{(1)} \psi^\pm (z) = \psi^\pm (zq^{\frac{\sigma}{2}}) \otimes \psi^\pm (zq^{-\frac{\sigma}{2}}), \quad (3.1c) \]
\[ a^{(1)}(e(z)) = - \left( \psi^-(zq^{-\frac{\sigma}{2}}) \right)^{-1} e(zq^{-c}), \quad a^{(1)}(f(z)) = -f(zq^{-c}) \left( \psi^+(zq^{\frac{\sigma}{2}}) \right)^{-1}, \quad (3.1d) \]
\[ a^{(1)}(\psi^\pm(z)) = (\psi^\pm(z))^{-1}, \quad a^{(1)}(c) = -c, \quad a^{(1)}(d) = -d, \quad (3.1e) \]
\[ \varepsilon(c) = \varepsilon(d) = \varepsilon(e(z)) = 0, \quad \varepsilon(\psi^\pm(z)) = 1. \quad (3.1f) \]

There exists two Drinfeld's Hopf structures. That is why we denoted coproduct and antipode maps in (3.1) as \( \Delta^{(1)} \) and \( a^{(1)} \). The counit maps are the same in the all Hopf structures considered in this paper, so we do not use for them different notations. The second structure is given by the formulas

\[ \Delta^{(2)}e(z) = e(z) \otimes 1 + \psi^+(zq^{-\frac{\sigma}{2}}) \otimes e(zq^{-c_1}), \quad (3.2a) \]
\[ \Delta^{(2)}f(z) = 1 \otimes f(z) + f(zq^{-c_2}) \otimes \psi^-(zq^{-\frac{\sigma}{2}}), \quad (3.2b) \]
\[ a^{(2)}(e(z)) = - \left( \psi^+(zq^{\frac{\sigma}{2}}) \right)^{-1} e(zq^{\sigma}), \quad a^{(2)}(f(z)) = -f(zq^{\sigma}) \left( \psi^-(zq^{\frac{\sigma}{2}}) \right)^{-1}, \quad (3.2c) \]
and the rest maps are the same as in (3.1).

To exploit the method of quantum double construction we should decompose the algebra under consideration into two Borel subalgebras corresponding to the chosen Hopf structure. This decomposition for (3.1) is \( U_F = \{ f(z), \psi^+(z), c, d \} \) and \( U_E = \{ e(z), \psi^-(z), c, d \} \). The pairing between these dual subalgebras can be reconstructed from comparing the general multiplication in quantum double with the commutation relations (2.6a) and (2.6b). The only nontrivial pairings between generators are \( \langle c, d \rangle = \langle d, c \rangle = 1 \) and (in terms of the formal series)

\[ \langle f(w), e(z) \rangle = \frac{1}{q^{-1} - q} \delta(z/w) \quad (3.3) \]

and

\[ \langle \psi^+(w), \psi^-(z) \rangle = g \left( \frac{z}{w} \right) = \frac{q^2 - \frac{\sigma}{w}}{1 - q^2 \frac{\sigma}{w}} = q^2 + (q^2 - q^{-2}) \sum_{k>0} q^{2k} \frac{z^k}{w^k}. \quad (3.4) \]

The pairing (3.3) and (3.4) \( U_F \otimes U_E \to \mathbb{C} \) is the Hopf pairing, satisfying the property

\[ \langle k_1 f, k_2 e \rangle = \langle k_1, k_2 \rangle \langle f, e \rangle \quad (3.5) \]

for any \( k_1 \in K_+, \ k_2 \in K_-, \ f \in U_f, \ e \in U_e \). Here \( K_{\pm} \) are the algebras, generated by \( q^{\pm h}, \ a_n \) for \( n \geq 0 \) and \( n \leq 0 \) and \( U_f \) and \( U_e \) are nilpotent subalgebras of \( U_F \) and \( U_E \) which are generated only by modes \( f_n \) and \( e_n \), \( n \in \mathbb{Z} \) respectively. The property (3.5) signifies that the universal \( \mathcal{R} \)-matrix corresponding to the coproduct \( \Delta^{(1)} \) will have factorized form

\[ \mathcal{R} = \mathcal{K} \cdot \overline{\mathcal{R}}, \quad (3.6) \]

where \( \mathcal{K} \) is given by (2.12).

To describe the element \( \overline{\mathcal{R}} \) one should consider the restriction of the pairing (3.3) and (3.4) on the dual subalgebras \( U_f \) and \( U_e \). It can be obtained inductively and have the form

\[ \langle f(z_1) \cdots f(z_n), e(w_1) \cdots e(w_n) \rangle = (q^{-1} - q)^{-n} \sum_{\sigma \in S_n} \prod_{k} \delta \left( \frac{z_k}{w_{\sigma(k)}} \right) \prod_{k<l} g \left( \frac{z_k}{z_l} \right). \quad (3.7) \]

As usual, one should start with different Cartan elements \( c, d \) and \( \psi_0^\pm \) in these dual subalgebras and factorizing over simple central elements identify these Cartan elements. In this way one can prove the relation \( \psi_0^+ = (\psi_0^-)^{-1} \).
where the series $g(z)$ is given by the formula (3.4) and $S_n$ is symmetric group of permutations of $n$ elements. The Hopf pairing between $U_e$ and $U_f$, corresponding to comultiplication $\Delta^{(2)}$ has analogous form (see (3.10) below).

The element $\overline{R}$ in the multiplicative form reads as follows \cite{KT}

$$\overline{R} = \prod_{n \in \mathbb{Z}} \exp_{q^2} \left( (q^{-1} - q) f_{-n} \otimes e_n \right),$$

where

$$\exp_{q^2}(x) = 1 + x + \frac{x^2}{1 + q^2} + \frac{x^3}{(1 + q^2)(1 + q^2 + q^4)} + \cdots + \frac{x^n}{(n)q^2!} + \cdots,$$

where $(a)_{q^2} = \frac{a^{\omega_{q^2}-1}}{q^2-1}$. In terms of the pairing the formula (3.8) is equivalent to the relation

$$\langle f_{-n_1}^{k_1} \cdots f_{-n_s}^{k_s}, e_{m_1}^{l_1} \cdots e_{m_s}^{l_s} \rangle = \prod_i \delta_{n_i, m_i} \cdot \delta_{k_i, l_i} \cdot (q^{-1} - q)^{k_i}(k_i)q^2!$$

for all $n_1 < \ldots < n_s$, $m_1 < \ldots < m_s$, which can be deduced from (3.7) by induction.

Note that the ordering of the $q$-exponents in (3.8) dictates that modes $f_n$ in the first tensor space are ordered by nonincreasing indices while $e_n$ in the second tensor space by nondecreasing order, so the element $\overline{R}$ belongs to the tensor product of certain completion of the nilpotent subalgebras $U_f$ and $U_e$, such that the element (3.8) is well defined operator when it acts in the tensor product of lowest weight representation with arbitrary one or in tensor product of arbitrary representation with highest weight representation.

A distinguished feature of the pairing (3.7) which follows from the form (3.8) is that it provides a way to order any monomial of generators $e_n$ as a sum of monomials $e_{k_1} \cdots e_{k_n}$ with nondecreasing indices $k_i$, $k_1 \leq \ldots \leq k_n$ and to order any monomial of generators $f_n$ as a sum of monomials $f_{l_1} \cdots f_{l_n}$ with nonincreasing indices $l_i$, $l_1 \geq \ldots \geq l_n$. For the orderings in opposite directions one should use the pairing corresponding to comultiplication $\Delta^{(2)}$:

$$\langle e(w_1) \cdots e(w_n), f(z_1) \cdots f(z_n) \rangle^{(2)} = (q - q^{-1})^{-n} \sum_{\sigma \in S_n} \prod_k \delta_{\frac{z_k}{w_{\sigma(k)}}} \prod_{k < l} g'(\frac{z_l}{z_k}),$$

where

$$g'(z) = \frac{q^2 - z}{1 - q^{-2}z} = q^{-2} + (q^{-2} - q^2) \sum_{k>0} q^{-2k}z^k.$$

An application of this technique is presented in the Appendix.

### 3.2 Integral presentation for the element $\overline{R}'$ and the fundamental differential equation

It is a common place to extend highest (lowest) weight representations of a Kac-Moody algebra to a representations of bigger algebra. This algebra can be defined as a completion with respect to a minimal topology, compatible with action on highest weight representations \cite{DK}. Since we are interested only by separate action of the algebras $U_e$ and $U_f$, these completions can be defined quite explicitly. Namely, the completed algebras $\overline{U}_e$ and $\overline{U}_f$, which act in highest weight representations, are generated as linear spaces by the series over monomials $e_{k_1} \cdots e_{k_i} (f_{k_1} \cdots f_{k_i})$ where $k_1 \leq \ldots \leq k_i$ and $\sum_i k_i$ is fixed. Analogously, for the completed algebras $\overline{U}_e$ and $\overline{U}_f$, which act in lowest weight representations, we use the series over monomials, ordered in opposite direction.

The following presentation of the universal $\mathcal{R}$-matrix was constructed in \cite{DK} in a completed tensor product of $\overline{U}_f \otimes \overline{U}_e$:

$$\overline{R}' = \mathcal{K} \cdot \overline{R},$$

\begin{equation}
(3.12)
\end{equation}
where $\mathcal{K}$ coincides with $R_{(2.12)}$ and

$$\mathcal{K} = 1 + \sum_{n>0} \frac{1}{n!(2\pi i)^n} \oint_{D_n} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} t(z_1) \cdots t(z_n).$$ (3.13)

Here

$$t(z) = (q^{-1} - q)f(z) \otimes e(z)$$ (3.14)

and $D_n$ is $n$-dimensional torus $|z_i| = 1$ for $|q| > 1$ and is the $n$-cycle $|z_i \prod_{j=1,\ldots,n,j \neq i} (z_i - q^2 z_j) | = 1$,

$i = 1, \ldots, n$ for any $q$, such that $q^N \neq 1$ , $N \in \mathbb{Z} \setminus \{0\}$ and the integrand is understood as analytical continuation of the product $t(z_1) \cdots t(z_n)$ from the region $|z_1| \gg |z_2| \gg \ldots \gg |z_n|$.

The results of the paper are strongly based on the following

**Proposition 3.1** (DKP) (i) The action of $\mathcal{R}$-matrix $R_{(2.12)}$ in tensor product of highest weight modules is well defined and coincides with the action of the $\mathcal{R}$-matrix $R_{(3.4)}$:

(ii) The element $\mathcal{R}'(\tau) = \tau^{-d_a \otimes 1} \mathcal{K} \tau^{d_a \otimes 1}$ satisfy the following differential equation:

$$\tau \frac{d\mathcal{R}'(\tau)}{d\tau} = \mathcal{R}'(\tau) \cdot I(\tau) = I(\tau) \cdot \mathcal{R}'(\tau).$$ (3.15)

Here the generating series $I(\tau)$ also belongs to the tensor product of the same completions $U^\leq_f \otimes U^\leq_e$ and so is a well defined operator acting in the tensor product of h.w.r. The coefficients of the formal series $I(\tau)$

$$I(\tau) = \sum_{n=1}^{\infty} I^{(n)} \tau^n$$ (3.16)

are the commuting quantities

$$I^{(n)} = \frac{(q - q^{-1})(-1)^n}{[n - 1]_q! [n]_q!} \oint \frac{dz}{2\pi i z} f^{(n)}(z) \otimes e^{(n)}(z)$$ (3.17)

constructed from the composed currents $f^{(n)}(z)$ and $e^{(n)}(z)$ (their definitions of the multiple residues are given by $I_{(1.23)}$

$$f^{(n)}(z) = (q^{-1} - q)^{n-1} [n]_q! [n - 1]_q! f(q^{2(n-1)} z)f(q^{2(n-2)} z) \cdots f(z),$$

$$e^{(n)}(z) = (q - q^{-1})^{n-1} [n]_q! [n - 1]_q! e(z) e(q^2 z) \cdots e(q^{2(n-1)} z).$$ (3.18)

The composed currents can be obtained inductively as the residues of the products of the preceding currents and by the constructions the composed currents are elements from the completions $U^\leq_f \otimes U^\leq_e$. The commutativity of the zero modes of two-tensors

$$\ell^{(n)}(z) = \frac{(q - q^{-1})(-1)^n}{[n - 1]_q! [n]_q!} f^{(n)}(z) \otimes e^{(n)}(z)$$ (3.19)

follows from the commutation relations

$$[\ell^{(n)}(z_1), \ell^{(m)}(z_2)] = \delta(q^{2n} z_1/z_2) \ell^{(n+m)}(z_1) - \delta(q^{-2m} z_1/z_2) \ell^{(n+m)}(z_2).$$ (3.20)

Due to this commutativity the differential equation can be easily solved

$$\mathcal{K}' = \exp \left(\sum_{n>0} \frac{I^{(n)}}{n}\right) = \exp \left((q - q^{-1}) \sum_{n>0} \frac{(-1)^n}{[n - 1]_q! [n]_q!} \oint \frac{dz}{2\pi i z} f^{(n)}(z) \otimes e^{(n)}(z)\right).$$ (3.21)
3.3 The pairing tensor as a formal integral \[\mathcal{E}_f, \mathcal{E}_e\]

In the completions \(\mathcal{U}_e^\prec, \mathcal{U}_e^\succ, \mathcal{U}_f^\prec\) and \(\mathcal{U}_f^\succ\) the defining relations for the total currents can be strengthened.

Lemma 3.1 (i) In the algebras \(\mathcal{U}_e^\prec\) and \(\mathcal{U}_f^\prec\) the following identity of formal power series take place:

\[
e(z_1) \cdots e(z_n) = \prod_{k<l} g\left(\frac{z_k}{z_l}\right) e(z_n) \cdots e(z_1), \quad f(z_1) \cdots f(z_n) = \prod_{k<l} g'\left(\frac{z_k}{z_l}\right) f(z_n) \cdots f(z_1)
\]  

(3.22)

(ii) In the algebras \(\mathcal{U}_f^\succ\) and \(\mathcal{U}_e^\succ\) the following identity of formal power series take place:

\[
e(z_1) \cdots e(z_n) = \prod_{k<l} g'\left(\frac{z_k}{z_l}\right) e(z_n) \cdots e(z_1), \quad f(z_1) \cdots f(z_n) = \prod_{k<l} g\left(\frac{z_k}{z_l}\right) f(z_n) \cdots f(z_1),
\]

where \(g(z)\) and \(g'(z)\) are given by the series \((3.4)\) and \((3.11)\) respectively.

Proof. One can prove the statement of this Lemma using pairing arguments and explicit formula \((3.7)\) for the pairing. For instance, the pairing \((3.7)\) can be extended by continuity to the pairing of \(\mathcal{U}_e^\prec\) with \(U_f\). Then

\[
(q^{-1} - q^2) \left\langle f(w_1) f(w_2), e(z_1) e(z_2) - g\left(\frac{z_2}{z_1}\right) e(z_2) e(z_1)\right\rangle =
\]

\[
= \delta\left(\frac{z_1}{w_1}\right) \delta\left(\frac{z_2}{w_2}\right) g\left(\frac{z_2}{w_1}\right) \delta\left(\frac{z_1}{w_2}\right) -
\]

\[
- g\left(\frac{z_1}{w_1}\right) \left[ \delta\left(\frac{z_2}{w_1}\right) \delta\left(\frac{z_1}{w_2}\right) + g\left(\frac{z_1}{z_2}\right) \delta\left(\frac{z_2}{w_1}\right) \delta\left(\frac{z_1}{w_2}\right) \right] \equiv 0,
\]

where the vanishing is due to the functional relation \(g(z)g(z^{-1}) = 1\). Note that for two of four equalities in Lemma 3.1 we should use the Hopf pairing \((3.10)\) attached to the coproduct \(\Delta^{(2)}\).

Let us introduce the following formal integral

\[
\mathcal{R}' = \exp\left(\oint \frac{dz}{2\pi i} t(z)\right) : = \sum_{n \geq 0} \frac{(q^{-1} - q)^n}{n!(2\pi i)^n} : \oint \cdots \oint \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} f(z_1) \cdots f(z_n) \otimes e(z_1) \cdots e(z_n) :,
\]

(3.24)

The dots \(:\cdots:\) mean the following ordering of the result of the formal integration: we present the monomial of generators \(e_n\) as a sum of monomials \(e_{k_1} \cdots e_{k_n}\) with increasing indices \(k_i, k_1 \leq \cdots \leq k_n\) and to order any monomial of generators \(f_n\) as a sum of monomials \(f_{l_1} \cdots f_{l_n}\) with decreasing indices \(l_i, l_1 \geq \cdots \geq l_n\). The procedure is correctly defined only for \(|q| < 1\) since the resulting coefficients at ordered monomials include the sums of geometric progressions. The element \(\mathcal{R}'\) belongs, by a construction, to a completed tensor product of \(\mathcal{U}_f^\succ\) and \(\mathcal{U}_e^\prec\).

Lemma 3.2

\[
\mathcal{R}', x \otimes 1) = x, \quad (\mathcal{R}', 1 \otimes y) = y
\]

(3.25)

for any \(x \in \mathcal{U}_f^\prec\) and \(y \in \mathcal{U}_e^\prec\).

Let us calculate, for instance, \((\mathcal{R}', e(z_1) e(z_2) \otimes 1)\). Due to the formulas of the pairing, we have after integrations of delta-functions:

\[
(\mathcal{R}', e(z_1) e(z_2) \otimes 1) = \frac{1}{2} (e(z_1) e(z_2) + g(z_2/z_1) e(z_2) e(z_1))
\]

which is equal to \(e(z_1) e(z_2)\) due to Lemma 3.1. Due to the Lemma 3.2, we can use \(\mathcal{R}'\) as a tensor of pairing between \(\mathcal{U}_f^\succ\) and \(\mathcal{U}_e^\prec\). Moreover, we will see further, that it actually coincides with \((3.8)\) and thus use further for \(\mathcal{R}'\) the same notation \(\mathcal{R}\).
Nevertheless, from the definition we can define an action of (3.24) only on tensor product of lowest weight and highest weight representations; also the quantum double, including \( U_f \) and \( U_e \) as dual Hopf subalgebras is not correctly defined due to divergences. Thus we use the expression (3.24) not as the universal \( \mathcal{R} \)-matrix, but as a tensor of pairing.

4 Factorization of the universal \( \mathcal{R} \)-matrix

As it was mentioned in Introduction, we want, following [EF, ER], to decompose the factor \( \mathcal{R} \) of the universal \( \mathcal{R} \)-matrix (3.6), corresponding to Drinfeld comultiplication, into a product of two cocycles, which can be used for restoring of the canonical comultiplication structure (2.4) and corresponding \( \mathcal{R} \)-matrix. For multiplicative presentation of \( \mathcal{R} \) (3.8) such a factorization is clear: 

\[
\mathcal{R} = \mathcal{R}^{+,-} \cdot \mathcal{R}^{-,+}
\]

where \( \mathcal{R}^{+,-} \) consists of the product of \( q \)-exponents \( \exp_q^2 \left( (q^{-1} - q)f_n \otimes e_n \right) \) for \( n < 0 \) and \( \mathcal{R}^{-,+} \) consists of the product of \( q \)-exponents for \( n \geq 0 \), such that the universal \( \mathcal{R} \)-matrix for canonical comultiplication has a form

\[
\mathcal{R}_{\text{can}} = \prod_{n \geq 0} \exp_q^2 \left( (q^{-1} - q)f_n \otimes e_n \right) \cdot \mathcal{K} \cdot \prod_{n > 0} \exp_q^2 \left( (q^{-1} - q)f_n \otimes e_n \right).
\]

(4.1)

Our goal is to develop general technique of factorization applicable to a situation when multiplicative expression for \( \mathcal{R} \)-matrix is missing. Algebraical background for such a factorization consists of the use of projection operators from (current) Borel subalgebras to their orthogonal subalgebras, developed in [EF, ER]. Its survey is given in the first subsection and is applied to the tensors from previous subsection in the second one.

The projections of the contour integrals \( \mathcal{R}' \) are not well defined. Still, we notice in the next subsection, that the projections of the logarithmic derivatives of \( \mathcal{R}' \) make sense and we use this to deduce differential equations for the factors \( \mathcal{R}^{\pm,\mp} \), which determine them uniquely. The main technical problem reduces to the calculation of the projections of the composed currents, entering into differential equations for \( \mathcal{R}' \). This is done is the last subsection with a help of screening operators. These calculations finish the proof of main theorems.

4.1 The biorthogonal decompositions of Hopf algebras

Let \( \mathcal{A} \) be a bialgebra with unit and counit, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two subalgebras of \( \mathcal{A} \) satisfying the following conditions:

(i) Algebra \( \mathcal{A} \) admits a decomposition \( \mathcal{A} = \mathcal{A}_1 \mathcal{A}_2 \), that is the multiplication map

\[
\mu: \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}
\]

is an isomorphism of linear spaces;

(ii) \( \mathcal{A}_1 \) is left coideal, \( \mathcal{A}_2 \) is right coideal:

\[
\Delta(\mathcal{A}_1) \subset \mathcal{A} \otimes \mathcal{A}_1, \quad \Delta(\mathcal{A}_2) \subset \mathcal{A}_2 \otimes \mathcal{A}.
\]

(4.3)

Then the operators

\[
P^1_A: P^1_A(a_1a_2) = a_1 \varepsilon(a_2), \quad P^2_A: P^2_A(a_1a_2) = \varepsilon(a_1)a_2, \quad a_1 \in \mathcal{A}_1, \quad a_2 \in \mathcal{A}_2
\]

are well defined projection operators from \( \mathcal{A} \) to \( \mathcal{A}_i \), satisfying the following property:

\[
\mu(P^1_A \otimes P^2_A)\Delta(a) = a
\]

(4.4)

4.1 The biorthogonal decompositions of Hopf algebras

Let \( \mathcal{A} \) be a bialgebra with unit and counit, \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two subalgebras of \( \mathcal{A} \) satisfying the following conditions:

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\[
\mu: \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}
\]

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(ii) \( \mathcal{A}_1 \) is left coideal, \( \mathcal{A}_2 \) is right coideal:

\[
\Delta(\mathcal{A}_1) \subset \mathcal{A} \otimes \mathcal{A}_1, \quad \Delta(\mathcal{A}_2) \subset \mathcal{A}_2 \otimes \mathcal{A}.
\]

(4.3)

Then the operators

\[
P^1_A: P^1_A(a_1a_2) = a_1 \varepsilon(a_2), \quad P^2_A: P^2_A(a_1a_2) = \varepsilon(a_1)a_2, \quad a_1 \in \mathcal{A}_1, \quad a_2 \in \mathcal{A}_2
\]

are well defined projection operators from \( \mathcal{A} \) to \( \mathcal{A}_i \), satisfying the following property:

\[
\mu(P^1_A \otimes P^2_A)\Delta(a) = a
\]

(4.4)
for any \( a \in A \). Here \( \varepsilon \) is counit. In Sweedler notation (4.1) means
\[
P_A^1(a')P_A^2(a'') = a.
\]

The correctness of definition of \( P_A^i \) follows from the condition (i). Denote by \( \phi : A \to A \) the linear map \( \phi(a) = P_A^1(a')P_A^2(a'') \). We claim that \( \phi \) is the map of left \( A_1 \)-modules and of right \( A_2 \)-modules, that is, \( \phi(a_1 a) = a_1 \phi(a) \), \( \phi(aa_2) = \phi(a)a_2 \). Indeed, for any \( a_1 \in A_1 \), \( a \in A \) we have
\[
\phi(a_1 a) = P_A^1(a_1 a')P_A^2(a'') = a_1 P_A^1(a')P_A^2(a'').
\]
From (4.3) we know that \( a_i'' \in A_1 \), so \( P_A^2(a_i'' a'') = \varepsilon(a_i'')P_A^2(a'') \) and
\[
\phi(a_1 a) = P_A^1(a_1 a')P_A^2(a'') = P_A^1(a_1 a')P_A^2(a'') = a_1 P_A^1(a')P_A^2(a'') = a_1 \phi(a)
\]
In analogous manner we prove, that \( \phi(aa_2) = \phi(a)a_2 \). Noting that \( \phi(1) = 1 \), we conclude that \( \phi(a) = a \) for any \( a \in A \).

Let now \( B \) be bialgebra dual to \( A \) with opposite comultiplication, that is there exists nondegenerate Hopf pairing \( \langle \cdot, \cdot \rangle : A \otimes B \to \mathbb{C} \), satisfying the conditions
\[
\langle a, b_1 b_2 \rangle = \langle \Delta(a), b_1 \otimes b_2 \rangle, \quad \langle a_1 a_2, b \rangle = \langle a_2 \otimes a_1, \Delta(b) \rangle,
\]
and \( \mathcal{R} = \sum a^\alpha \otimes b_\alpha \) be the tensor of the pairing. Let \( \mathcal{R}_i = (P_A^i \otimes \text{id})\mathcal{R} \). The addition identity (4.4) yields the factorization
\[
\mathcal{R} = \mathcal{R}_1 \cdot \mathcal{R}_2.
\]
Indeed, the tensor \( \mathcal{R} \) is uniquely characterized by one of the properties
\[
\langle \mathcal{R}, b \otimes 1 \rangle = b, \quad \text{for any } b \in B, \quad \langle \mathcal{R}, 1 \otimes a \rangle = a, \quad \text{for any } a \in A.
\]
Let us calculate \( \langle \mathcal{R}_1 \mathcal{R}_2, 1 \otimes a \rangle \). We have
\[
\langle \mathcal{R}_i, 1 \otimes a \rangle = \langle (P_A^i \otimes \text{id})\mathcal{R}, 1 \otimes a \rangle = P_A^i(\mathcal{R}, 1 \otimes a) = P_A^i(a).
\]
Then
\[
\langle \mathcal{R}_1 \mathcal{R}_2, 1 \otimes a \rangle = \langle \mathcal{R}_1, 1 \otimes a' \rangle \langle \mathcal{R}_2, 1 \otimes a'' \rangle = P_A^1(a')P_A^2(a'') = a
\]
due to (4.4). It proves (4.5).

We can get a factorization of \( \mathcal{R} \), \( \mathcal{R} = \mathcal{R}_2 \mathcal{R}_1 \) if we start from the decomposition \( B = B_1 B_2 \) into a product of two subalgebras, being right and left (since the opposite comultiplication is used in the pairing) coideals of \( B \): \( \Delta(B_1) \subset B_1 \otimes B \), \( \Delta(B_2) \subset B \otimes B_2 \) and use \( \mathcal{R}_i = (1 \otimes P_B^i)\mathcal{R} \), where \( P_B^1(b_1 b_2) = b_1 \varepsilon(b_2) \), \( P_B^2(b_1 b_2) = b_1 \varepsilon(b_2) \). The natural question arise: when these two decomposition coincide, that is \( \mathcal{R}_1 = \mathcal{R}_1 \), \( \mathcal{R}_2 = \mathcal{R}_2 \)? We claim the following

**Proposition 4.1** The decomposition of pairing tensor induced by the decomposition of the algebra \( A \) coincides with the decomposition of this tensor induced by the decomposition of the algebra \( B \) if \( A_i \) and \( B_j \) are mutually orthogonal, that is
\[
\langle a_i, b_j \rangle = \varepsilon(a_i)\varepsilon(b_j), \quad \text{for any } a_i \in A_i, b_j \in B_j, \quad i \neq j.
\]
Indeed, let us compute \( \langle \mathcal{R}_1, b \otimes a \rangle \) and \( \langle \mathcal{R}_1, b \otimes a \rangle \) for any \( a \in A \), \( b \in B \). We have
\[
\langle \mathcal{R}_1, b_1 b_2 \otimes a_1 a_2 \rangle = \langle a_1 \varepsilon(a_2), b_1 b_2 \rangle = \varepsilon(a_2)\langle a_1', b_1 \rangle \langle a_1'', b_2 \rangle.
\]
We know that \( a_i'' \in A_1 \), so
\[
\langle \mathcal{R}_1, b_1 b_2 \otimes a_1 a_2 \rangle = \varepsilon(a_2)\varepsilon(b_2)\langle a_1', \varepsilon(a_1''), b_1 \rangle = \varepsilon(a_2)\varepsilon(b_2)\langle a_1', b_1 \rangle.
\]
Analogously,
\[
\langle \mathcal{R}_1, b_1 b_2 \otimes a_1 a_2 \rangle = \langle a_1 a_2, b_1 \varepsilon(b_2) \rangle = \varepsilon(b_2) \langle a_1, b'_1 \rangle \langle a_2, b'_1 \rangle = \\
= \varepsilon(a_2) \varepsilon(b_2) \langle a_1, b'_1 \rangle = \varepsilon(a_2) \langle a_1, b_1 \rangle
\]
since \( b'_1 \in \mathcal{B}_1 \). We see that \( \mathcal{R}_1 = \bar{\mathcal{R}}_1 \). The same story takes place for other pair. We call further the decompositions \( \mathcal{A} = \mathcal{A}_1 \mathcal{A}_2 \), \( \mathcal{B} = \mathcal{B}_1 \mathcal{B}_2 \), satisfying the condition describe above, as biorthogonal decompositions of \( \mathcal{A} \) and \( \mathcal{B} \equiv (\mathcal{A}^{*})^{op} \).

Let now \( \mathcal{A} \) be a Hopf algebra and element \( \mathcal{R} \) is considered as an element from square tensor of its quantum double \( D(\mathcal{A}) \). For any biorthogonal decomposition the tensors \( \mathcal{R}_2 \) and \((\mathcal{R}_1^2)^{-1}\), where \( \mathcal{R}_i = (P^i_\mathcal{A} \otimes 1)\mathcal{R} = (1 \otimes P^i_\mathcal{B})\mathcal{R} \in \mathcal{A}_i \otimes \mathcal{B}_i = (P^i_\mathcal{A} \otimes P^i_\mathcal{B})\mathcal{R} \) are two cocycles in the double \( D(\mathcal{A}) \), that is,
\[
\mathcal{R}_2^{12} \cdot (\Delta \otimes 1) \mathcal{R}_2 = \mathcal{R}_2^{23} \cdot (1 \otimes \Delta) \mathcal{R}_2 , \\
(\Delta' \otimes 1) \mathcal{R}_1 \cdot \mathcal{R}_1^{12} = (1 \otimes \Delta') \mathcal{R}_1 \cdot \mathcal{R}_1^{23} .
\]
Indeed, both sides belong to \( \mathcal{A}_2 \otimes D(\mathcal{A}) \otimes \mathcal{B}_2 \). From the other hand, we have
\[
(\Delta' \otimes 1) \mathcal{R}_1 \cdot \mathcal{R}_1^{12} \cdot (\Delta \otimes 1) \mathcal{R}_2 = (1 \otimes \Delta') \mathcal{R}_1 \cdot \mathcal{R}_1^{23} \cdot \mathcal{R}_2^{23} \cdot (1 \otimes \Delta) \mathcal{R}_2
\]
due to the properties of universal \( \mathcal{R} \)-matrix, so the coassociator
\[
\Phi = \mathcal{R}_2^{12} \cdot (\Delta \otimes 1) \mathcal{R}_2 \cdot \left( \mathcal{R}_2^{23} \cdot (1 \otimes \Delta) \mathcal{R}_2 \right)^{-1}
\]
can be presented also as
\[
\Phi = \left( (\Delta' \otimes 1) \mathcal{R}_1 \cdot \mathcal{R}_1^{12} \right)^{-1} \cdot (1 \otimes \Delta') \mathcal{R}_1 \cdot \mathcal{R}_1^{23}
\]
and thus belongs to the intersection of \( \mathcal{A}_2 \otimes D(\mathcal{A}) \otimes \mathcal{B}_2 \) and \( \mathcal{A}_1 \otimes D(\mathcal{A}) \otimes \mathcal{B}_1 \), which means that it has a form \( 1 \otimes d \otimes 1 \) for some \( d \in D(\mathcal{A}) \). Then the pentagon identity on \( \Phi \) says that there is no nontrivial coassociator of such a form.

### 4.2 Application to the algebra \( U_q(\hat{sl}_2) \)

Let \( U^+_F \) be a subalgebra of \( U_F \), generated by all \( a_n, f_n, n > 0 \) and \( q^h; U^-_F \) be a subalgebra of \( U_F \), generated by all \( f_n, n \leq 0 \). We choose them as \( A_1 \) and \( A_2 \). The corresponding projectors will be denoted as \( P^{\pm*}_f \) and \( P^{\pm*}_e \). Let also \( U^-_E \) be a subalgebra of \( U_E \), generated by all \( a_n, e_n, n < 0; U^+_e \) be a subalgebra of \( U_e \subset U_E \), generated by all \( e_n, n \geq 0 \). We choose them as \( B_1 \) and \( B_2 \). The corresponding projectors will be denoted as \( P^-_e \) and \( P^+_e \). That is,
\[
P^{\pm*}_f(a_1 a_2) = a_1 \varepsilon(a_2), \quad P^{\pm*}_f(a_1 a_2) = \varepsilon(a_1) a_2, \quad a_1 \in U^+_F, \quad a_2 \in U^-_F, \quad (4.6)
\]
\[
P^-_e(a_1 a_2) = a_1 \varepsilon(a_2), \quad P^+_e(a_1 a_2) = \varepsilon(a_1) a_2, \quad a_1 \in U^-_E, \quad a_2 \in U^+_e. \quad (4.7)
\]
It follows from the definition of the open sets in \( U^+_F \) and \( U^-_E \) \[DKP\], that the projections of small enough open neighborhoods of zero are zero, which means that the projectors can be defined also on the completed spaces:
\[
P^{\pm*}_f : U^+_F \to U^+_F, \quad P^{\pm*}_e : U^-_F \to U^-_F, \quad P^+_e : U^-_E \to U^+_e, \quad P^-_e : U^+_E \to U^-_E. \quad (4.8)
\]

The subalgebras \( U^+_F, U^-_F, U^-_E, U^+_e \) and the corresponding projectors (4.6) and (4.7) satisfy all the conditions of biorthogonal decomposition and can be applied to decomposition of the pairing tensor \( \mathcal{K} \cdot \bar{\mathcal{R}} \). Due to the property (3.5) of the Hopf pairing, the factorization (4.7) in this case has the form
\[
\mathcal{K} \cdot \bar{\mathcal{R}} = \mathcal{K} \cdot \mathcal{R}_{+-} \cdot \mathcal{R}_{-+}, \quad (4.9)
\]
Here generated by all elements of dual subalgebras. So they coincide, as well as in the notations.

Since the projections admit prolongation to completed (in opposite directions) subalgebras, the application of \((P_\pm^* \otimes P_\mp^*)\) to the tensor is well defined, and we can repeat the arguments of the previous subsection to the factorization of this tensor. So we have an equality

\[
\mathcal{R}'' = \mathcal{R}'_+ \cdot \mathcal{R}'_- \cdot \mathcal{R}'_{++} \cdot \mathcal{R}'_{--},
\]

where

\[
\mathcal{R}_{\pm,\mp} = (P_\pm^* \otimes 1)\mathcal{R} = (1 \otimes P_\mp^*)\mathcal{R} = (P_\pm^* \otimes P_\mp^*)\mathcal{R}.
\]

It coincides with the natural factorization for multiplicative expression of the universal \(\mathcal{R}\)-matrix, mentioned in the beginning of this section.

We need also further another pair of projections operators, connected to the comultiplication \(\Delta^{(2)}\). Their restrictions to the algebras \(U_f\) and \(U_e\) can be defined as follows. Let \(U_f^+\) be a subalgebra of \(U_f\), generated by all \(f_n, n > 0\) and \(U_e^-\) be a subalgebra of \(U_e\), generated by all \(e_n, n < 0\). We put

\[
P_f^-(a_1a_2) = a_1\varepsilon(a_2), \quad P_f^+(a_1a_2) = \varepsilon(a_1)a_2, \quad a_1 \in U_f^-, \quad a_2 \in U_f^+,
\]

\[
P_e^{**}(a_1a_2) = a_1\varepsilon(a_2), \quad P_e^{*+}(a_1a_2) = \varepsilon(a_1)a_2, \quad a_1 \in U_e^{**}, \quad a_2 \in U_e^{*+}.
\]

As before, they can be prolonged to corresponding completed algebras.

The computation of the projections for the products of the currents can be carried out by means of the commutation relations between half-currents \((2.7)\) \(e_\pm(z)\) and \(f_\pm(z)\), where we put everywhere \(z = z_1/z_2\):

\[
e_{\pm}(z_1)e_{\pm}(z_2) = g(z^{-1})e_{\pm}(z_2)e_{\pm}(z_1) + \psi(z^{-1}) \left( z^{-1}e^2_{\pm}(z_1) + e^2_{\pm}(z_2) \right),
\]

\[
e_{\pm}(z_1)e_{\pm}(z_2) = g'(z)e_{\pm}(z_2)e_{\pm}(z_1) + \psi'(z) \left( e^2_{\pm}(z_1) + ze^2_{\pm}(z_2) \right),
\]

\[
e_{+}(z_1)e_{-}(z_2) = g(z^{-1})e_{-}(z_2)e_{+}(z_1) + \psi(z^{-1}) \left( z^{-1}e^2_{+}(z_1) + e^2_{-}(z_2) \right),
\]

\[
e_{-}(z_1)e_{+}(z_2) = g'(z)e_{+}(z_2)e_{-}(z_1) + \psi'(z) \left( e^2_{-}(z_1) + ze^2_{+}(z_2) \right),
\]

\[
f_{\pm}(z_1)f_{\pm}(z_2) = g'(z^{-1})f_{\pm}(z_2)f_{\pm}(z_1) + \psi'(z^{-1}) \left( f^2_{\pm}(z_1) + z^{-1}f^2_{\pm}(z_2) \right),
\]

\[
f_{\pm}(z_1)f_{\pm}(z_2) = g(z)f_{\pm}(z_2)f_{\pm}(z_1) + \psi(z) \left( zf^2_{\pm}(z_1) + f^2_{\pm}(z_2) \right),
\]

\[
f_{+}(z_1)f_{-}(z_2) = g'(z^{-1})f_{-}(z_2)f_{+}(z_1) + \psi'(z^{-1}) \left( f^2_{+}(z_1) + z^{-1}f^2_{-}(z_2) \right),
\]

\[
f_{-}(z_1)f_{+}(z_2) = g(z)f_{+}(z_2)f_{-}(z_1) + \psi(z) \left( zf^2_{-}(z_1) + f^2_{+}(z_2) \right).
\]

Here

\[
g(z) = \frac{q^2 - z}{1 - q^2 z}, \quad g'(z) = \frac{q^2 - z}{1 - q^2 z}, \quad \psi(z) = \frac{1 - q^2}{1 - q^2 z}, \quad \psi'(z) = \frac{1 - q^2}{1 - q^2 z}.
\]

For the calculation of the projection of the product of the currents, say \(P_f^{**}(f(z_1) \cdots f(z_n))\) we first replace the product of the currents by the sum of the products of half-currents \(f_\pm(z_i)\), using the relation \(f(z) = f_+(z) - f_-(z)\), then move successively all the \(f_-(z_i)\) to the right and all \(f_+(z_i)\) to the left. The projector \(P_f^{**}\) kills all the factors \(f_-(z_i)\) which stand from the right leaving at the end the products of some \(f_+(z_i)\).
4.3 Differential equations for the elements $\mathcal{R}_{\pm,\mp}(\tau)$

Let us rewrite the differential equation (3.15) for the element $\mathcal{R} \in U_f \otimes U_e$ in the form

$$I(\tau) = (\mathcal{R}(\tau))^{-1} \cdot \frac{d\mathcal{R}(\tau)}{d\tau} = \frac{d\mathcal{R}(\tau)}{d\tau} \cdot (\mathcal{R}(\tau))^{-1}$$

and act by left and right hand sides of this equality onto tensor product of highest weight modules over $U_f(\mathfrak{sl}_2)$. According to the Proposition 3.3 we can replace the element $\mathcal{R}$ by its multiplicative counterpart $\mathcal{E}$ which possesses the factorization (4.9). Equations (4.15) will have the form

$$(\mathcal{R}_{+,+}(\tau))^{-1} \cdot \frac{d\mathcal{R}_{+,+}(\tau)}{d\tau} + \frac{d\mathcal{R}_{+-}(\tau)}{d\tau} \cdot (\mathcal{R}_{--}(\tau))^{-1} = \mathcal{R}_{--,}(\tau) \cdot I(\tau) \cdot (\mathcal{R}_{--,}(\tau))^{-1},$$

$$(\mathcal{R}_{+,+}(\tau))^{-1} \cdot \frac{d\mathcal{R}_{+,+}(\tau)}{d\tau} + \frac{d\mathcal{R}_{+-}(\tau)}{d\tau} \cdot (\mathcal{R}_{--}(\tau))^{-1} = (\mathcal{R}_{+,+}(\tau))^{-1} \cdot I(\tau) \cdot \mathcal{R}_{+,+}(\tau).$$

Let us apply the projections $P_f^+ \otimes P_e^-$ to the equality (4.16a) and $P_f^- \otimes P_e^+$ to the equation (4.16b). Let us consider (4.16a). It is clear that the projection $P_f^+ \otimes P_e^-$ kills the second term in the l.h.s. of this equality and the first term is stable under this projection. Let us apply the same projections to the r.h.s. of (4.16a). As we already mentioned the generating series $I(\tau)$ contain the elements which belong to the tensor product $U_f \otimes U_e$ (see details in [DKP]). According to this completions application of the projection $P_f^+$ and $P_e^-$ means the following. One should move all generators which belong to subalgebra $U_f^-$ to the left in the first tensor space of the r.h.s. of (4.16a) and kill all the terms which contain these sort of generators on the left. Analogously, for the projection $P_e^-$ in the second tensor space of the r.h.s. of (4.16a) we will move all generators which belong to $U_e^+$ to the right and kill all the terms where such generators survive on the right hand side. We conclude that

$$(P_f^+ \otimes P_e^-)\mathcal{R}_{+,+}(\tau) = 1 \otimes 1$$

and

$$(P_f^+ \otimes P_e^-) \left( \mathcal{R}_{+,+}(\tau) \cdot I(\tau) \cdot (\mathcal{R}_{+,+}(\tau))^{-1} \right) = (P_f^+ \otimes P_e^-) I(\tau) \equiv I_{\pm,\mp}(\tau).$$

The substitution of (4.17) to (4.16a) and (4.16b) proves the following

**Proposition 4.2** The projections $\mathcal{R}_{+,+}(\tau) = (P_f^+ \otimes P_e^-)\mathcal{R}(\tau)$ of the universal $\mathcal{R}$ matrix $\mathcal{R}(\tau)$ (see (2.14)) satisfy the following differential equations:

$$\tau (\mathcal{R}_{--,}(\tau))^{-1} \cdot \frac{d\mathcal{R}_{--,}(\tau)}{d\tau} = (P_f^- \otimes P_e^+) I(\tau),$$

$$\tau \frac{d\mathcal{R}_{--}(\tau)}{d\tau} (\mathcal{R}_{--,}(\tau))^{-1} = (P_f^+ \otimes P_e^-) I(\tau),$$

where $I(\tau)$ is given by (3.10) and (3.11).

One can see that the differential equations (4.18) and (4.19) are equivalent to the following recurrence relations:

$$n\mathcal{R}_{+,+}^{(n)} = \mathcal{R}_{+,+}^{(n-1)} + \mathcal{R}_{+,+}^{(n-2)} I_1^{(1)} + \cdots + \mathcal{I}_{+,+}^{(1)} I_{+,+}^{(n-1)} + I_{+,+}^{(n)},$$

$$n\mathcal{R}_{--,}^{(n)} = I_1^{(1)} \mathcal{R}_{--,}^{(n-1)} + I_1^{(2)} \mathcal{R}_{--,}^{(n-2)} + \cdots + I_{--,}^{(n-1)} \mathcal{R}_{--,}^{(1)} + I_{--,}^{(n)}.$$

The system of the recurrent relations (4.20) with noncommutative coefficients $I_{\pm,\pm}^{(n)}$ have formal solution

$$\mathcal{R}_{\pm,\mp}^{(n)} = \sum_{m=1}^{n} \sum_{j_1+j_2+\cdots+j_m=n} C_{\pm}(j_1,j_2,\ldots,j_m) I_{\pm,\pm}^{(1)} I_{\pm,\pm}^{(2)} \cdots I_{\pm,\pm}^{(j_m)},$$
The application of the screening operators to a factorization problem is based on their compatibility with currents (4.17) onto subalgebras $U_{e,f}^\pm$. We solve this problem in the next subsection.

### 4.4 Projections of composed currents and screening operators

To calculate the projections to the subalgebras $U_{e,f}^\pm$ from the composed currents we need the following recurrent definitions of these currents

$$e^{(n)}(z) = - \lim_{w \to z} e^{(n-1)}(z q^2) e(w) \frac{dw}{w} = - \oint_{w \text{ around } z} \frac{dw}{2\pi i w} e^{(n-1)}(z q^2) e(w),$$

$$f^{(n)}(z) = \lim_{w \to z} f^{(n-1)}(z) f(w) \frac{dw}{w} = \oint_{w \text{ around } z} \frac{dw}{2\pi i w} f^{(n-1)}(z) f(w),$$

Let us define the screening operators $S_{e_0}, \tilde{S}_{e_0}, S_{f_0}, \tilde{S}_{f_0}$, which act as the following $q$-commutators in the algebra $U_q(\mathfrak{sl}_2)$:

$$S_{e_0}(x) = e_0 x - k x k^{-1} e_0, \quad S_{f_0}(x) = x f_0 - f_0 k x k^{-1},$$

$$\tilde{S}_{e_0}(x) = x e_0 - e_0 k^{-1} x k, \quad \tilde{S}_{f_0}(x) = f_0 x - k^{-1} x k f_0.$$

The screening operators coincide with adjoint action of the elements $e_0$ and $f_0$ with respect to comultiplication (2.4):

$$S_{e_0}(x) = e'_0 \cdot x \cdot a (e''_0), \quad S_{f_0}(x) = a (f''_0) \cdot x \cdot f'_0,$$

and are connected via the conjugation by $k$:

$$\tilde{S}_{e_0}(x) = - q^2 k^{-1} S_{e_0}(x) k, \quad \tilde{S}_{f_0}(x) = - q^{-2} k^{-1} S_{f_0}(x) k.$$

The application of the screening operators to a factorization problem is based on their compatibility with projection operators.

**Lemma 4.1**

(i) Subalgebras $U_{e,f}^\pm$ are invariant with respect to the screening operators $S_{e_0}$ and $\tilde{S}_{e_0}$: subalgebras $U_{e,f}^\mp$ are invariant with respect to the screening operators $S_{f_0}$ and $\tilde{S}_{f_0}$;

(ii) The screening operators $S_{e_0}$ and $\tilde{S}_{e_0}$ commute with the projectors $P_{e}^\pm$; the screening operators $S_{f_0}$ and $\tilde{S}_{f_0}$ commute with the projectors $P_{f}^\pm$:

$$P_{e}^\pm S_{e_0}(x) = S_{e_0} P_{e}^\pm(x), \quad P_{e}^\pm \tilde{S}_{e_0}(x) = \tilde{S}_{e_0} P_{e}^\pm(x), \quad x \in U_e,$$

$$P_{f}^\pm S_{f_0}(x) = S_{f_0} P_{f}^\pm(x), \quad P_{f}^\pm \tilde{S}_{f_0}(x) = \tilde{S}_{f_0} P_{f}^\pm(x), \quad x \in U_f.$$
The proof of statement (i) of the Lemma consists of a short calculation based on the use of (2.6c) and (2.6d). The statement (ii) follows from (i) together with a remark, that \( \varepsilon S(x) = 0 \) for any screening \( S \) in consideration.

The main result of this subsection is the following

**Proposition 4.3** The projections of the currents \( e^{(n)}(z) \) and \( f^{(n)}(z) \) onto subalgebras \( U^\pm_{e,f} \) are given by the formulas

\[
P_e^+ \left( e^{(n)}(z) \right) = S_{e_0}^{n-1} \left( e_+ (zq^{2(n-1)}) \right), \quad P_e^- \left( e^{(n)}(z) \right) = -S_{e_0}^{n-1} \left( e_-(z) \right), \quad (4.28a)
\]

\[
P_f^+ \left( f^{(n)}(z) \right) = S_{f_0}^{n-1} \left( f_+ (z) \right), \quad P_f^- \left( f^{(n)}(z) \right) = -S_{f_0}^{n-1} \left( f_-(zq^{2(n-1)}) \right). \quad (4.28b)
\]

**Proof.** One can prove by induction (see [DKP] for details) that the currents \( e^{(n)}(z) \) and \( f^{(n)}(z) \) satisfy the following quadratic relations:

\[
(w - zq^{2(n-2)})(w - zq^{2(n-1)})e(w)e^{(n-1)}(z) = q^{2(n-1)}(w - zq^{-2})(w - z)e^{(n-1)}(z)e(w), \quad (4.29a)
\]

\[
(w - zq^{2(n-2)})(w - zq^{2(n-1)})f^{(n-1)}(z)f(w) = q^{2(n-1)}(w - zq^{-2})(w - z)f(w)f^{(n-1)}(z), \quad (4.29b)
\]

Moreover, the product \( e(w)e^{(n-1)}(z) \) has unique simple pole at \( w = q^{2(n-1)}z \); the product \( e(w)e^{(n-1)}(z) \) has unique simple zero at \( w = q^{-2}z \); the product \( f(w)f^{(n-1)}(z) \) has unique simple pole at \( w = q^{-2}z \); and the product \( f(w)f^{(n-1)}(z) \) has unique simple zero at \( w = q^{2(n-1)}z \).

It means that the residues in (4.23a)-(4.23d) can be presented as the following formal integrals:

\[
e^{(n)}(z) = \oint \frac{dw}{2\pi iw} \left( e^{(n-1)}(zq^2)e(w) - q^{-2(n-1)}e(w)e^{(n-1)}(zq^2)\alpha_n \left( \frac{z}{w}; q \right) \right), \quad (4.30a)
\]

\[
= \oint \frac{dw}{2\pi iw} \left( e(w)e^{(n-1)}(z) - q^{-2(n-1)}e^{(n-1)}(z)e(w)\beta_n \left( \frac{w}{z}; q \right) \right), \quad (4.30b)
\]

\[
f^{(n)}(z) = \oint \frac{dw}{2\pi iw} \left( f^{(n-1)}(z)f(w) - q^{2(n-1)}f(w)f^{(n-1)}(z)\beta_n \left( \frac{z}{w}; q^{-1} \right) \right), \quad (4.30c)
\]

\[
= \oint \frac{dw}{2\pi iw} \left( f(w)f^{(n-1)}(zq^2) - q^{2(n-1)}f^{(n-1)}(zq^2)f(w)\alpha_n \left( \frac{w}{z}; q^{-1} \right) \right), \quad (4.30d)
\]

where

\[
\alpha_n(x; q) = \frac{1 - q^{2(n-1)}x}{(1 - x)(1 - q^2x)}, \quad \beta_n(x; q) = \frac{1 - q^2x}{(1 - q^{-2(n-1)}x)(1 - q^{-2(n-1)}x)}. \quad (4.31)
\]

The r.h.s. of (4.30a)-(4.30d) can be presented as total integrals of left/right adjoint actions of the currents \( e(w) \) and \( f(w) \) with respect to coproduct \( \Delta^{(1)} \). For instance, the relation (4.30a) we can read as

\[
e^{(n)}(z) = \int \frac{dw}{2\pi iw} a_+ \left( e(w) \right) X e'(w),
\]

where \( X = e^{(n-1)}(zq^2) \).

We can rewrite (4.30a)-(4.30d) as

\[
e^{(n)}(z) = e^{(n-1)}(zq^2)e_0 - q^{-2(n-1)}e_0 e^{(n-1)}(zq^2) + \sum_{k<0} \alpha_{n,k}(q)e_k e^{(n-1)}(zq^2)z^{-k}, \quad (4.32a)
\]

\[
e^{(n)}(z) = e_0 e^{(n-1)}(z) - q^{-2(n-1)}e^{(n-1)}(z)e_0 + \sum_{k>0} \beta_{n,k}(q) e^{(n-1)}(z)e_k z^{-k}, \quad (4.32b)
\]

\[
f^{(n)}(z) = f^{(n-1)}(z)f_0 - q^{2(n-1)}f_0 f^{(n-1)}(z) + \sum_{k<0} \beta_{n,k}(q^{-1}) f_k f^{(n-1)}(z)z^{-k}, \quad (4.32c)
\]
\[ f^{(n)}(z) = f_0 f^{(n-1)}(z q^2) - q^{2(n-1)} f^{(n-1)}(z q^2) f_0 + \sum_{k>0} \alpha_{n,k}(q^{-1}) f^{(n-1)}(z q^2) f_k z^{-k}, \]  

(4.32d)

where \( \alpha_{n,k}(q) \) and \( \beta_{n,k}(q) \) are coefficients of the expansion of the rational functions \( \alpha_n(x; q) \) and \( \beta_n(x; q) \) into series with respect to \( x \).

Due to the definitions (4.7) and (4.11) of the projection operators, the sums in the right hand sides of (4.32) disappear under corresponding projections and we obtain

\[
P_e^+ \left( e^{(n)}(z) \right) = P_e^+ \left( e^{(n-1)}(z q^2) e_0 - q^{-2(n-1)} e_0 e^{(n-1)}(z q^2) \right) = P_e^+ \hat{S}_{e_0} e^{(n-1)}(z q^2),
\]

(4.33a)

\[
P_e^- \left( e^{(n)}(z) \right) = P_e^- \left( e_0 e^{(n-1)}(z) - q^{2(n-1)} e^{(n-1)}(z) e_0 \right) = P_e^- \hat{S}_{e_0} e^{(n-1)}(z),
\]

(4.33b)

\[
P_f^+ \left( f^{(n)}(z) \right) = P_f^+ \left( f^{(n-1)}(z) f_0 - q^{-2(n-1)} f_0 f^{(n-1)}(z) \right) = P_f^+ \hat{S}_{f_0} f^{(n-1)}(z),
\]

(4.33c)

\[
P_f^- \left( f^{(n)}(z) \right) = P_f^- \left( f_0 f^{(n-1)}(z q^2) - q^{2(n-1)} f^{(n-1)}(z q^2) f_0 \right) = P_f^- \hat{S}_{f_0} f^{(n-1)}(z q^2).
\]

(4.33d)

Iteration of the formulas (4.33) together with (4.26), (4.27), proves the Proposition 4.3. The additional minus in (4.28) appear due to the definitions (2.7) \( P_f^- (e(z)) = -e_-(z) \) and \( P_f^- (f(z)) = -f_-(z) \). Note that the coefficients in front of \( e^{(n-1)}(z) e_0 \) and \( f_0 f^{(n-1)}(z) \) in (4.33b) and (4.33c) are changed in contrast to (4.32b) and (4.32c) in order to have possibility to apply the statement of the Lemma 4.2.

The combination of Propositions 4.2 and 4.3 complete the proof of Theorems 1 and 2. Note that in its formulations the screenings \( \hat{S}_{e_0} \) and \( \hat{S}_{f_0} \) do not appear. We can avoid their use since

\[
\hat{S}_{f_0}^k (f_+(z)) \otimes \hat{S}_{e_0}^k (e_-(z)) = \hat{S}_{f_0}^k (f_+(z)) \otimes \hat{S}_{e_0}^k (e_-(z))
\]

because of (4.25). The Corollary 1 to Theorem 2 is also a direct consequence of the Proposition 4.3 due to [DM].

The adjoint action of the screening operators onto half-currents can be expressed through the powers of these half-currents. We have the following

**Lemma 4.2**

\[
S_{e_0}^{n-1} (e_+(z)) = \prod_{k=2}^{n} (1 - q^{-2(k-1)}) e_+^n(z), \quad S_{e_0}^{n-1} (e_-(z)) = \prod_{k=2}^{n} (1 - q^{2(k-1)}) e_-^n(z),
\]

(4.34)

\[
S_{f_0}^{n-1} (f_+(z)) = \prod_{k=2}^{n} (q^{2(k-1)} - 1) f_+^n(z), \quad S_{f_0}^{n-1} (f_-(z)) = \prod_{k=2}^{n} (q^{-2(k-1)} - 1) f_-^n(z).
\]

**Proof.** Let us first equality in (4.34) since the rest are analogous. The proof is by induction over \( n \). For \( n = 2 \) the identity

\[
\hat{S}_{e_0} (e_+(z)) = [e_0, e_+(z)]_{q^{-2}} = e_0 e_+(z) - q^{-2} e_+(z) e_0 = (1 - q^{-2}) e_+^2(z).
\]

follows from the commutation relations (2.6c). Suppose that the identity \( \hat{S}_{e_0}^{m-1} (e_+(z)) = \prod_{k=2}^{m} (1 - q^{-2(k-1)}) e_+^m(z) \) is valid for \( m = 2, \ldots, n - 1 \). Then we calculate

\[
\hat{S}_{e_0}^{n} (e_+(z)) = \prod_{k=2}^{n} (1 - q^{-2(k-1)}) \hat{S}_{e_0} (e_+^{n-1}(z)) = \prod_{k=2}^{n} (1 - q^{-2(k-1)}) e_+^n(z)
\]

which prove the Lemma.

This Lemma allows one to write down the integral formulas for the elements \( R_{\pm, \pm}^{(n)} \) in the form of the formal integrals from two-tensors constructed from powers of half-currents:

\[
R^{(n)}_{\pm, \pm} = (-2\pi i)^{-n} \sum_{m=1}^{n} \sum_{j_1 + j_2 + \ldots + j_m = n} \hat{C}_{\pm} (j_1, j_2, \ldots, j_m) \times
\]

\[
\int \cdots \int \frac{dz_1}{z_1} \cdots \frac{dz_m}{z_m} f^{j_1}_\pm (z_1) \cdots f^{j_m}_\pm (z_m) \otimes e^{j_1}_\pm (z_1) \cdots e^{j_m}_\pm (z_m)
\]

(4.35)
We would like to factorize the element \((3.24)^{5.1}\) Another form of the differential equations and combinatorial identity. We will give the proof of this identity, proposed by A. Okounkov.

Let

\[
C_+(j_1, j_2, \ldots, j_m) = \frac{(q^{-1} - q)^{2n-m}}{j_1(j_1 + j_2)(j_1 + j_2 + j_3) \cdots (j_1 + j_2 + \cdots + j_m)} \prod_{i=1}^{m} \frac{1}{[j_i]},
\]

\[
C_-(j_1, j_2, \ldots, j_m) = \frac{(q^{-1} - q)^{2n-m}}{j_m(j_m + j_{m-1})(j_m + j_{m-1} + j_{m-2}) \cdots (j_m + j_{m-1} + \cdots + j_1)} \prod_{i=1}^{m} \frac{1}{[j_i]}.
\]

Note, that the presentation \((4.35)\) is specific for the case under consideration and in general situation of \(U_q(\hat{g})\) the only possibility is to use the screening operators.

5 Factorization of the formal pairing tensor

The recurrence relations and corresponding differential equations for the factorized part of \(\mathcal{R}\)-matrix can be also deduced from the factorization of the pairing tensor in a form of formal integral. We will see that in this approach they appear in a different form. Its equivalence to the results above reduces to certain combinatorial identity. We will give the proof of this identity, proposed by A. Okounkov.

5.1 Another form of the differential equations and combinatorial identity

We would like to factorize the element \((3.24)\)

\[
\overline{\mathcal{R}} = \exp \int t(z) \frac{dz}{z} := \sum_{n \geq 0} \frac{1}{n!} \int \cdots \int \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} t(z_1) \cdots t(z_n): (5.1)
\]

into a product

\[
\overline{\mathcal{R}} = \mathcal{R}_{+-} \cdot \mathcal{R}_{-+}. (5.2)
\]

Here, as before, \(t(z) = (q^{-1} - q)f(z) \otimes e(z)\). If we use the notation \((a \otimes b)_{\pm,\mp}\) for \(P_f^\pm(a) \otimes P_e^\pm(b)\), then the relation \((5.2)\) means that

\[
\mathcal{R}_{\pm,\mp} = \sum_{n \geq 0} \frac{1}{n!} \int \cdots \int \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} (t(z_1) \cdots t(z_n))_{\pm,\mp}.
\]

Let

\[
\overline{\mathcal{R}}^{(n)} = \frac{1}{n!} \int \cdots \int \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} t(z_1) \cdots t(z_n):
\]

and

\[
\mathcal{R}_{\pm,\mp}^{(n)} = \frac{1}{n!} \int \cdots \int \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} (t(z_1) \cdots t(z_n))_{\pm,\mp}.
\]

Then the factorization \((5.2)\) is equivalent to

\[
\overline{\mathcal{R}}^{(n)} = \sum_{0 \leq l \leq n} \mathcal{R}_{\pm,\mp}^{(l)} \mathcal{R}_{\mp,\pm}^{(n-l)}. (5.3)
\]

We can calculate, for example, \(\mathcal{R}_{\pm,\mp}^{(n)}\) using \((5.3)\) by induction over \(n:\)

\[
\mathcal{R}_{\pm,\mp}^{(n)} = \frac{1}{n} \left( : \overline{\mathcal{R}}^{(n-1)} \int t(z) \frac{dz}{z} : \right)_{\pm,\mp} = \frac{1}{n} \sum_{0 \leq l \leq n-1} \mathcal{R}_{\pm,\mp}^{(n-1-l)} \left( : \mathcal{R}_{\mp,\pm}^{(l)} \int t(z) \frac{dz}{z} : \right)_{\pm,\mp}, (5.4)
\]

Denote by

\[
\tilde{j}_{\mp,\pm}^{(l)} = \left( : \mathcal{R}_{\mp,\pm}^{(l)} \int t(z) \frac{dz}{z} : \right)_{\mp,\pm}, \quad \tilde{j}_{\pm,\mp}^{(l)} = \left( : \int t(z) \frac{dz}{z} \mathcal{R}_{\pm,\mp}^{(l)} : \right)_{\pm,\mp}, (5.5)
\]
\[ \mathcal{I}_{\pm,\mp}(\tau) = \sum_{n>0} i^{(n)}_{\pm,\mp} \tau^n \]

With these notations we can organize the recurrence relations (5.4) as the following differential equations.

**Proposition 5.1** The following differential equations follow from the factorization (5.2):

\[ \tau \frac{\partial R_{+,+}(\tau)}{\partial \tau} = R_{+,+}(\tau) \cdot \mathcal{I}_{+,+}(\tau), \quad (5.6) \]

\[ \tau \frac{\partial R_{-,+}(\tau)}{\partial \tau} = \mathcal{I}_{-,+}(\tau) \cdot R_{-,+}(\tau), \quad (5.7) \]

Due to the uniqueness of logarithmic derivative, we should have the identifications:

\[ \mathcal{I}_{\pm,\mp}(\tau) = I_{\pm,\mp}(\tau). \quad (5.8) \]

The calculation of the integrals (5.5) is very nontrivial technical problem. Nevertheless, we can get some profit comparing (5.6), (5.7) with (2.15a), (2.15b). Comparing the expressions (5.5) with (2.17) and (4.34) we see that the equality (5.8) can be considered as an effective way to calculate the integrals (5.5).

Using pairing arguments we will demonstrate in the Appendix that more general then (5.8) identities

\[ \oint \cdots \oint \frac{dz_1}{z_1} \cdots \frac{dz_k}{z_k} \left( (f(z_1) \cdots f(z_k))_{+} f_+(z_{k+1}) \right) \otimes \left( (e(z_1) \cdots e(z_k))_{+} e_-(z_{k+1}) \right) = k! (1 - q^2)^k \frac{1}{(k+1)q^2} f^{k+1}(z_{k+1}) \otimes e^{-1}(z_{k+1}), \quad (5.9a) \]

\[ \oint \cdots \oint \frac{dz_1}{z_1} \cdots \frac{dz_k}{z_k} \left( f_-(z_{k+1}) (f(z_k) \cdots f(z_1))_{+} \right) \otimes \left( e_+(z_{k+1})(e(z_k) \cdots e(z_1))_{+} \right) = k! (1 - q^2)^k \frac{1}{(k+1)q^2} f^{-1}(z_{k+1}) \otimes e^{k+1}(z_{k+1}). \quad (5.9b) \]

(as well as (5.8)) are equivalent to certain combinatorial identities. In particular, the equality (5.9a) is equivalent for \( |q| < 1 \) to

\[ \frac{1}{(n)_q!(n+1)_q!} = (1 - q^2)^n \sum_{0 \leq \lambda_n \leq \cdots \leq \lambda_1} C_{(\lambda)}(q) q^{4(\lambda_1 + 2\lambda_2 + \cdots + n \lambda_n)}, \quad (5.10) \]

where the constants \( C_{(\lambda)}(q) \) are parameterized by the partition \( \{\lambda_m\} \) of natural number \( n \)

\[ C_{(\lambda)}(q) = \prod_{i=1}^{n} \frac{1}{(\lambda_i - \lambda_{i+1})q^2}, \quad (5.11) \]

and \( \{\lambda'_j\} \) is dual to \( \{\lambda_k\} \) partition: \( \lambda'_j = \#k \), such that \( \lambda_k \geq j \). In (5.11) we assume that \( (0)_q! \equiv 1 \). Alternatively, these constants \( C_{(\lambda)}(q) \) can be defined as follows. Let \( m : \{\lambda_1, \ldots, \lambda_n\} \to \{m_1, \ldots, m_k\} \), \( 1 \leq k \leq n \) be a map described by the following rule

\[ \lambda_1, \ldots, \lambda_n \to \lambda_1, \ldots, \lambda_{m_1}, \lambda_{m_1+1}, \ldots, \lambda_{m_1+m_2}, \ldots, \lambda_{m_1+\cdots+m_{k-1}+1}, \ldots, \lambda_{m_1+\cdots+m_k} \]

such that \( \lambda_1 = \ldots = \lambda_{m_1} > \lambda_{m_1+1} = \ldots = \lambda_{m_1+m_2} > \ldots > \lambda_{m_1+\cdots+m_{k-1}+1} = \ldots = \lambda_{m_1+\cdots+m_k} \). Then the coefficients \( C_{(\lambda)}(q) \) are given by \( \prod_{i=1}^{k} \frac{1}{(m_i)q^2} \).

A. Okounkov proposed an independent proof of this identity, based on the specialization formula for Macdonald polynomials \( \mathbb{M} \), proved in full generality by I. Cherednik \( \mathbb{C} \).
Consider the Cauchy identity for Macdonald polynomials:

\[
\prod_{i,j} \frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty} = \sum_{\lambda} P_\lambda(x; q, t) Q_\lambda(x; q, t),
\]

where \((a)_\infty = (1 - a)(1 - qa)(1 - q^2a) \cdots\), where \((a)_\infty = (1 - a)(1 - qa)(1 - q^2a) \cdots\), \(P_\lambda\) are Macdonald polynomials, and \(Q_\lambda\) are dual Macdonald polynomials, corresponding to Young diagram \(\lambda\). Set

\[
x = (t, t^2, t^3, \ldots, t^n, 0, 0, \ldots), \quad y = (t, t^2, \ldots, t^n, t^{n+1}, t^{n+2}, \ldots)
\]

and

\[
q = 0,
\]

which specializes Macdonald polynomials to Hall-Littlewood polynomials. Then the left-hand side of (5.12) becomes

\[
\frac{1}{(1 - t)^n(n + 1)!},
\]

and due to the formula (VI.6.11') from the book [M] for evaluating a Macdonald polynomial at a point \((t, t^2, \ldots, t^n)\)

\[
P_\lambda(t, \ldots, t^n; 0, t) = \ell(\lambda) \prod_{i=1}^{\ell(\lambda)} \frac{1 - t^{n-i+1}}{1 - t^{\lambda_i - 1}}.
\]

and also

\[
Q_\lambda(t, \ldots, t^{n+1}, t^{n+2}, \ldots; 0, t) = t^{n(\lambda) + |\lambda|},
\]

the right hand side of (5.12) becomes

\[
(n)_! \sum_{\ell(\lambda) \leq n} \frac{t^2 \sum i\lambda_i}{\prod_{k \geq 0} [\lambda_k - \lambda_{k+1}]}.
\]

In the formulas (5.14) and (5.15), \(n(\lambda) = \sum_i (i - 1) \lambda_i\), \(|\lambda| = \sum_i \lambda_i\) and \(\ell(\lambda) = \lambda'_1\) is the number of rows of the diagram \(\lambda\). Comparing (5.13) and (5.16) at \(t = q^2\) we obtain (5.10).

### 5.2 Some examples of calculations and vanishing of the cross terms in the integrals

Let us demonstrate how the factorization works in the simplest term \(\mathcal{R}^{(2)}\). It has the form

\[
\frac{(q^{-1} - q)^2}{(2\pi i)^2} \frac{1}{2} \oint d(z_1 d(z_2) (f_+(z_1) - f_-(z_1))(f_+(z_2) - f_-(z_2)) \otimes (e_+(z_1) - e_-(z_1))(e_+(z_2) - e_-(z_2))
\]

The action of projection operators \(P_f^+ \otimes P_e^-\) described above results in the following formula:

\[
\frac{(q^{-1} - q)^2}{(2\pi i)^2} \frac{1}{2} \oint d(z_1 \frac{dz_2}{z_2}) \left(f_+(z_1)f_+(z_2) - \psi(\frac{z_1}{z_2}, f_+(z_2)) \otimes \left(e_+(z_1)e_+(z_2) - \psi(\frac{z_2}{z_1}, e_+(z_2)) \right)\right).
\]

We can open the brackets and look to four summands. First, we leave the regular term as it is. The integrals

\[
\frac{1}{2\pi i} \oint \frac{dz_1}{z_1} e_-(z_1) \psi \left(\frac{z_1}{z_2}\right) \quad \text{and} \quad \frac{1}{2\pi i} \oint \frac{dz_1}{z_1} f_+(z_1) \psi \left(\frac{z_2}{z_1}\right)
\]

vanish due to definitions (or analytical properties of \(f_+(z)\) and \(e_-(z)\)). For the last term we need to calculate the integral

\[
\frac{1}{2\pi i} \oint \frac{dz_1}{z_1} \psi \left(\frac{z_1}{z_2}\right) \psi \left(\frac{z_2}{z_1}\right) \frac{1 - q^2}{1 + q^2}
\]
to obtain

\[ \mathcal{R}_{\pm,=}^{(2)} = \frac{(q^{-1} - q)^2}{2} \left( \oint \oint \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} f_+ (z_1) f_+ (z_2) \otimes e_- (z_1) e_- (z_2) + \frac{1 - q^2}{1 + q^2} \oint \frac{dz}{2\pi i z} f_+^2 (z) \otimes e_-^2 (z) \right) \]

which obviously coincide with (2.19) for \( n = 2 \).

As we can see from this exercise the most subtle point of these procedure is the vanishing of the cross terms (5.19). We can observe these vanishing property in general case. Indeed, the relations

\[ \mathcal{R}_{\pm,=} = (P_{f}^{\pm*} \otimes 1) \mathcal{R} = (1 \otimes P_{e}^{\mp}) \mathcal{R} = (P_{f}^{\pm*} \otimes P_{e}^{\mp}) \mathcal{R} \]

are equivalent to the equalities

\[
\begin{align*}
\oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} (f(z_1) \cdots f(z_n))_{\pm} \otimes (e(z_1) \cdots e(z_n))_{=} &= 0 \\
\oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} (f(z_1) \cdots f(z_n))_{\pm} \otimes e(z_1) \cdots e(z_n) &= (5.20) \\
\oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} f(z_1) \cdots f(z_n) \otimes (e(z_1) \cdots e(z_n))_{=} &= (5.21)
\end{align*}
\]

To proceed further we need the following Proposition 5.2. Let \( I \subset \{1, ..., n\} \), \( I = \{i_1 < ... < i_k\} \), \( J = \{1, ..., n\} \setminus I \), \( J = \{j_1 < ... < j_{n-k}\} \) be some subsets from the set \( \{1, ..., n\} \). For these sets denote by \( g_{I,J}(z_1, ..., z_n) \) and \( g'_{I,J}(z_1, ..., z_n) \) the following formal power series:

\[ g_{I,J}(z_1, ..., z_n) = \prod_{i \in I \setminus j \in J} g \left( \frac{z_i}{z_j} \right), \quad g'_{I,J}(z_1, ..., z_n) = \prod_{i \in I \setminus j \in J} g' \left( \frac{z_i}{z_j} \right). \]

**Proposition 5.2** The following equalities of formal power series take place in \( \mathcal{U}_f^> \otimes \mathcal{U}_e^< \):

\[
\begin{align*}
f(z_1) \cdots f(z_n) &= \sum_{I, J} g_{I,J}(z_1, ..., z_n) (f(z_{i_1}) \cdots f(z_{i_k}))_{+} (f(z_{j_1}) \cdots f(z_{j_{n-k}}))_{-} + (5.22a) \\
e(z_1) \cdots e(z_n) &= \sum_{I, J} g'_{I,J}(z_1, ..., z_n) (e(z_{i_1}) \cdots e(z_{i_k}))_{-} (e(z_{j_1}) \cdots e(z_{j_{n-k}}))_{+} + (5.22b)
\end{align*}
\]

**Proof.** For the proof let us apply (1.23) to the product \( f(z_1) \cdots f(z_n) \). We obtain

\[
f(z_1) \cdots f(z_n) = \mu (P_{f}^{+*} \otimes P_{f}^{-}) (1 \otimes f(z_1) + f(z_1) \otimes \psi^+ (z_1)) \cdots (1 \otimes f(z_n) + f(z_n) \otimes \psi^+ (z_n)) = \sum_{I \subset \{1, ..., n\}, J = \{i_1 < ... < i_k\}} (f(z_{i_1}) \cdots f(z_{i_k}))_{+} (a(z_1) \cdots a(z_n))_{-},
\]

where \( a(z_k) = \psi^+ (z_k) \) if \( k \in I \) and \( a(z_k) = f(z_k) \) otherwise. Moving all the \( \psi^+ (z_k) \) to the left in the product \( a(z_1) \cdots a(z_n) \) and noting that \( \varepsilon (\psi^+ (z_k)) = 1 \), we get the statement of the Proposition.

Let us consider the sum of the integrals

\[
\sum_{I, J} \oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} g_{I,J}(z_1, ..., z_n) (f(z_{i_1}) \cdots f(z_{i_k}))_{+} (f(z_{j_1}) \cdots f(z_{j_{n-k}}))_{-} \otimes e(z_1) \cdots e(z_n)
\]

which means that we replaced the product \( f(z_1) \cdots f(z_n) \) in the first tensor space using the statement of the proposition 5.2.
We can reorder the product $e(z_1) \ldots e(z_n)$ by means of (3.22) and rewrite this integral in a form
\[
\sum_{l, j} \oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} (f(z_{i_1}) \cdots f(z_{i_k}))_+ (f(z_{j_1}) \cdots f(z_{j_{n-k}}))_- \otimes e(z_1) \cdots e(z_k) e(z_{j_1}) \cdots e(z_{j_{n-k}})
\]
Now we separate integrations over $z_{i_1}, \ldots, z_{i_k}$ and over $z_{j_1}, \ldots, z_{j_{n-k}}$ and for the first integration use (5.20) (+) and for the second – (5.20) (−). As a result, we have
\[
\sum_{l, j} \oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} g_{l, j}(z_1, \ldots, z_n) (f(z_{i_1}) \cdots f(z_{i_k}))_+ (f(z_{j_1}) \cdots f(z_{j_{n-k}}))_- \otimes e(z_1) \cdots e(z_n) =
\]
\[
= \sum_{l, j} \oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} (f(z_{i_1}) \cdots f(z_{i_k}))_+ (f(z_{j_1}) \cdots f(z_{j_{n-k}}))_- \otimes (e(z_{i_1}) \cdots e(z_{i_k}))_- (e(z_{j_1}) \cdots e(z_{j_{n-k}}))_+ .
\]
This proves that the cross terms in the integral
\[
\oint \cdots \oint \prod_{i} \frac{dz_i}{z_i} (f(z_1) \cdots f(z_n))_+ \otimes (e(z_1) \cdots e(z_n))_-
\]
vanish.

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Appendix A. Pairing calculations

The aim of this Appendix is two-fold. First, we will prove the equivalence between equalities (5.9) and combinatorial identity (5.10). Second, we explain several hints how to work with orthogonal dual bases in nilpotent subalgebras of $U_q(\hat{sl}_2)$ during this proof. At the end, we exploit the evaluation homomorphism in order to verify the factorization of the element $\mathcal{R}$.

1. Equivalence between (5.9) and (5.10).

Using multiplicative presentation of the element $\mathcal{R}$ (3.8) we can define the dual bases in nilpotent subalgebras $\mathcal{U}_f^+$ and $\mathcal{U}_e^-$ corresponding to decomposition of the algebra $U_q(\hat{sl}_2)$ into two Borel subalgebras associated with first Drinfeld Hopf structure:
\[
E^{(n)} = \{ e_{p_1} e_{p_2} \cdots e_{p_n} \mid p_1 \leq p_2 \leq \cdots \leq p_n \},
\]
\[
F^{(p)} = \{ C_{\{p\}}(\mathbf{q}) f_{-p_1} f_{-p_2} \cdots f_{-p_n} \mid p_1 \leq p_2 \leq \cdots \leq p_n \},
\]
\[
\langle F^{(p)}, E^{(m)} \rangle = \delta^{(m)}_{\{p\}} = \delta^m_{p_1} \delta^m_{p_2} \cdots \delta^m_{p_n}, \quad p_k, m_k \in \mathbb{Z},
\]
where $C_{\{p\}}$ are given by (5.11). We see from (A.1) that ordered monomials of the generators $e_n$ in the completion $\mathcal{U}_e^-$ are dual to the non-ordered monomials constructed from modes $f_n$ in the completion $\mathcal{U}_f^+$. 

Let us calculate left hand side of identity (5.9a). To do this we rewrite the products of the currents \( f(z_1) \cdots f(z_k) \) and \( e(z_1) \cdots e(z_k) \) in ordered form using the dual basis in the subalgebras \( U_f^\geq \) and \( U_e^\leq \):

\[
f(z_1) \cdots f(z_k) = \sum_{p_1 \geq \cdots \geq p_k} C_{\{p\}}(q) \langle f(z_1) \cdots f(z_k), e_{-p_1} \cdots e_{-p_k} \rangle f_{p_1} \cdots f_{p_k}, \quad \text{(A.2a)}
\]

\[
e(z_1) \cdots e(z_k) = \sum_{s_1 \leq \cdots \leq s_k} C_{\{s\}}(q) \langle f_{-s_1} \cdots f_{-s_k}, e(z_1) \cdots e(z_k) \rangle f_{p_1} \cdots f_{p_k}. \tag{A.2b}
\]

Then we have

\[
\oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_k}{z_k} \left( (f(z_1) \cdot f(z_k))_+ f_+(z) \right)_{-} \otimes \left( (e(z_1) \cdot e(z_k))_- + e_-(z) \right)_{-} = \quad \text{(A.3)}
\]

\[
\times \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_k}{z_k} (f_{-s_1} \cdots f_{-s_k}, e(z_1) \cdots e(z_k)) \langle f(z_1) \cdots f(z_k), e_{-p_1} \cdots e_{-p_k} \rangle = \quad \text{(A.7)}
\]

\[
\Delta \quad k! \sum_{0 \leq p_1 \leq \cdots \leq p_k} C_{\{p\}}(q) (f_{-p_1} \cdots f_{-p_k} f_+(z))_+ \otimes (e_{p_1} \cdots e_{p_k} e_-(z))_. \tag{A.4}
\]

The last step to prove the equivalence of (5.9a) to the combinatorial formula (5.10) is to use the formulas

\[
(f_{-p_1} \cdots f_{-p_k} f_+(z))_+ = \prod_{m=1}^{k} (1 - q^{2m}) q^{2(p_k+2p_{k-1}+\cdots+k_p)} f_+^{k+1}(z)_{-p_1-\cdots-p_k}, \tag{A.5a}
\]

\[
(e_{p_1} \cdots e_{p_k} e_-(z))_+ = \prod_{m=1}^{k} (1 - q^{2m}) q^{2(p_k+2p_{k-1}+\cdots+k_p)} e_-^{k+1}(z)_{p_1+\cdots+p_k}.
\tag{A.5b}
\]

which are particular cases of more general formulas

\[
\left( f_{-}^{n-k+1}(z_1) f_{+}^{n-k+1}(z_2) \right)_{+} = \prod_{m=n-k-1}^{n} \frac{1 - q^{2m}}{1 - q^{2m} z_1/z_2} f_{+}^{n+1}(z_2), \tag{A.5a}
\]

\[
\left( e_{+}^{n-k+1}(z_1) e_{-}^{n-k+1}(z_2) \right)_{-} = \prod_{m=n-k-1}^{n} \frac{1 - q^{2m}}{1 - q^{2m} z_2/z_1} e_{-}^{n+1}(z), \tag{A.5b}
\]

\[
\left( f_{+}^{n-k+1}(z_1) f_{+}^{k}(z_2) \right)_{+} = \prod_{m=n-k-1}^{n} \frac{(1 - q^{2m}) z_1/z_2}{1 - q^{2m} z_1/z_2} f_{+}^{n+1}(z_1), \tag{A.5c}
\]

\[
\left( e_{+}^{n-k+1}(z_1) e_{+}^{k}(z_2) \right)_{+} = \prod_{m=n-k-1}^{n} \frac{(1 - q^{2m}) z_2/z_1}{1 - q^{2m} z_2/z_1} e_{+}^{n+1}(z_1). \tag{A.5d}
\]

2. Equivalence between multiplicative and integral forms of the pairing tensor.

In \( U_f^\geq \otimes U_e^\leq \) there is an identity

\[
\prod_{n \in \mathbb{Z}} \exp q^2 \left( (q^{-1} - q) f_{-n} \otimes e_n \right) =: \exp \left( (q^{-1} - q) \oint f(z) \otimes e(z) \right), \tag{A.6}
\]
where, as usual, the dots in r.h.s. means the antiordering in left tensor space and ordering in right tensor space.

In order to prove (A.6) we need two formulas

\[
\oint dz_1 \ldots \oint dz_n \langle f(w_1) \ldots f(w_n), e(z_1) \ldots e(z_n) \rangle \langle f(z_1) \ldots f(z_n), e(u_1) \ldots e(u_n) \rangle = n!(q^{-1} - q)^{-n} \langle f(w_1) \ldots f(w_n), e(u_1) \ldots e(u_n) \rangle
\]  

(A.7)

and

\[
\langle f_{-p_1} f_{-p_2} \ldots f_{-p_n}, e_{s_1} e_{s_2} \ldots e_{s_n} \rangle = \delta_{p_1, s_1} \delta_{p_2, s_2} \ldots \delta_{p_n, s_n} C_{[p]}(q)(q^{-1} - q)^{-n}
\]  

(A.8)

for \( p_1 \leq p_2 \leq \ldots \leq p_n \) and \( s_1 \leq s_2 \leq \ldots \leq s_n \). The first formula can be proved using simple combinatorics and explicit formulas for the pairing (3.7) and the second one is the direct consequence of the factorization property of this pairing.

Now the statement of (A.6) follows from the following calculation

\[
\frac{1}{n!} \oint dz_1 \ldots \oint dz_n f(z_1) \ldots f(z_n) \otimes e(z_1) \ldots e(z_n) = \sum_{p_1 \leq \ldots \leq p_n} C_{[p]}(q) f_{-s_1} \ldots f_{-s_n} \otimes e_{p_1} \ldots e_{p_n} \times
\]

\[
\times \oint dz_1 \ldots \oint dz_n \langle f_{-p_1} \ldots f_{-p_n} \otimes e(z_1) \ldots e(z_n) \rangle \langle f(z_1) \ldots f(z_n) \otimes e_{s_1} \ldots e_{s_n} \rangle = \sum_{p_1 \leq p_2 \leq \ldots \leq p_n} C_{[p]}(q) f_{-p_1} \ldots f_{-p_n} \otimes e_{p_1} \ldots e_{p_n}
\]  

(A.9)

3. Evaluation map and universal \( R \)-matrix.

Let \( E, F \) and \( H \) be generators of \( U_q(s \lambda_2) \) with standard commutation relations

\[
q^H E q^{-H} = q^2 E, \quad q^H F q^{-H} = q^{-2} F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}
\]  

(A.10)

An evaluation map \( \mathcal{E}_{v_a} : U_q(s \lambda_2) \to U_q(s \lambda_2) \otimes \mathbb{C}[[a, a^{-1}]] \) is defined as follows:

\[
\mathcal{E}_{v_a}(e_n) = a^n q^{nH} E, \quad \mathcal{E}_{v_a}(f_n) = a^n F q^{-nH}, \quad \mathcal{E}_{v_a}(\psi_n^\pm) = \pm(q - q^{-1}) a^n q^{nH} \left( EF - q^n FE \right).
\]  

(A.11)

Under this map the half currents will take the form

\[
\mathcal{E}_{v_a}(f_-(z)) = -F \frac{1}{1 - \frac{a}{q} q^{-H}} E, \quad \mathcal{E}_{v_a}(e_+(z)) = \frac{1}{1 - \frac{a}{q} q^{-H}} E,
\]

\[
\mathcal{E}_{v_a}(f_+(z)) = F \frac{\frac{a}{q} q^{H}}{1 - \frac{a}{q} q^{-H}} E, \quad \mathcal{E}_{v_a}(e_-(z)) = -\frac{\frac{a}{q} q^{H}}{1 - \frac{a}{q} q^{-H}} E.
\]  

(A.12)

Using formulas (4.34) we can write down the formulas for the projections of two-tensors \( I_{\pm, \mp}^{(n)} \) as the contour integrals over unit circles (\( |a| < 1, |a| < 1 \))

\[
\mathcal{E}_{v_a} \otimes \mathcal{E}_{v_b} \left( I_{\pm, \mp}^{(n)} \right) = \frac{(q^{-1} - q)^{2n-1}}{(2\pi i)^n [n]_q} \oint_{|z|=1} \frac{dz}{z} F \frac{\frac{a}{q} q^{H}}{1 - \frac{a}{q} q^{-H}} \ldots F \frac{\frac{a}{q} q^{H}}{1 - \frac{a}{q} q^{-H}} \otimes \frac{\frac{\pm}{q} q^{H}}{1 - \frac{\pm}{q} q^{-H}} E \ldots \frac{\frac{\pm}{q} q^{H}}{1 - \frac{\pm}{q} q^{-H}} E,
\]

(A.13a)

\[
\mathcal{E}_{v_a} \otimes \mathcal{E}_{v_b} \left( I_{\pm, \mp}^{(n)} \right) = \frac{(q^{-1} - q)^{2n-1}}{(2\pi i)^n [n]_q} \oint_{|z|=1} \frac{dz}{z} F \frac{1}{1 - \frac{a}{q} q^{-H}} \ldots F \frac{1}{1 - \frac{a}{q} q^{-H}} \otimes \frac{1}{1 - \frac{\pm}{q} q^{-H}} E \ldots \frac{1}{1 - \frac{\pm}{q} q^{-H}} E.
\]  

(A.13b)
Commutation relations (A.10) and Cauchy theorem allow one to calculate these integrals to obtain:

\[ E_{v_a} \otimes E_{v_b} \left( f_{+,+-}^{(n)} \right) = \frac{(q^{-1} - q)^{2n-1}}{[n]_q} \sum_{j=1}^{n} \prod_{i 
eq j}^{n} \frac{q^{2(i-j)}}{1 - q^{2(i-j)}} \prod_{\ell=1}^{n} \frac{Z q^{2(\ell+j-1)}}{1 - Z q^{2(\ell+j-1)}} E^n \otimes E^n, \]  
(A.14a)

\[ E_{v_a} \otimes E_{v_b} \left( I_{-,+}^{(n)} \right) = \frac{(q^{-1} - q)^{2n-1}}{[n]_q} F^n \otimes E^n \sum_{j=1}^{n} \prod_{i 
eq j}^{n} \frac{1}{1 - q^{2(i-j)}} \prod_{\ell=1}^{n} \frac{1}{1 - Z^{-1} q^{2(\ell+j-1)}} , \]  
(A.14b)

where

\[ Z = \frac{a}{b} q^{H \otimes 1 - 1 \otimes H}. \]

For the other hand the expressions for \( E_{v_a} \otimes E_{v_b} \left( R_{+,\mp}^{(n)} \right) \) can be found in [KTS]:

\[ E_{v_a} \otimes E_{v_b} \left( R_{+,\mp}^{(n)} \right) = \frac{\frac{q^{-1} - q}{(n)_{q^{-1}}}}{Z} \left( \prod_{i=1}^{n} \frac{q^{2i} Z}{1 - q^{2i} Z} \right) F^n \otimes E^n, \]  
(A.15a)

\[ E_{v_a} \otimes E_{v_b} \left( R_{-,+}^{(n)} \right) = \frac{\frac{q^{-1} - q}{(n)_{q^{2}}} Z^{-1}}{Z} F^n \otimes E^n \left( \prod_{i=1}^{n} \frac{1}{1 - q^{2i} Z^{-1}} \right) . \]  
(A.15b)

We checked for \( n = 2, 3 \) that the elements (A.15) satisfy recurrence relations (4.20a) and (4.20b) with \( I_{\pm,\mp}^{(n)} \) given by (A.14). Unfortunately, we did not manage to verify directly the corresponding differential equations for \( E_{v_a} \otimes E_{v_b} (R_{\pm,\mp}(\tau)) \). It would be interesting to find such a proof.

References

[ABF] G. Andrews, R. Baxter, J. Forrester. Eight-vertex SOS model and generalized Rogers-Ramanujan identities. J. Stat. Phys. 35 (1984) 193–266.

[ABRR] D. Arnaudon, E. Buffenoir, E. Ragoucy, Ph. Roche. Universal solution of quantum dynamical Yang-Baxter equations. Lett. Math. Phys. 44 (1998) 201.

[Ba] R.J. Baxter. Exactly Solved Models in Statistical Mechanics. Academic, London, 1982.

[B] J. Beck. Braided group action and quantum affine algebras. Comm. Math. Phys. 165 (1994) 555–568.

[C] I. Cherednik. Macdonald’s evaluation conjectures and difference Fourier transform. Inv. Math. 122 (1995) 119–145.

[Da] I. Damiani. A basis of type Poincaré-Birkhoff-Witt for the quantum algebra of \( \hat{sl}_2 \). J. of Algebra 161 (1993) 291–310.

[DK] J. Ding, S. Khoroshkin. Weyl group extension of quantized current algebras. Preprint math.QA/9804139.

[DM] J. Ding, T.Miwa Zeros and poles of quantum current operators and the condition of quantum integrability. Preprint q-alg/9608001.

[DKP] J. Ding, S. Khoroshkin, S. Pakuliak. Integral representations for universal \( \mathcal{R} \)-matrix. Preprint ITEP-TH-67/99, math.QA/0008226.

[D] V. G. Drinfeld. A new realization of Yangians and quantized affine algebras. Soviet Math. Dokl. 36 (1988) 212–216.

[D1] V. G. Drinfeld. Quantum groups. Proc. ICM, AMS, Berkley, Ca (1986) 798–820.

[^1]: math.QA/9804139
[D2] V. G. Drinfeld. Quasi-Hopf algebras. *Leningrad Math. J.* **1** (1990) 1419–1457.

[J] M. Jimbo. A $q$-difference analogue of $U(g)$ and the Yang-Baxter equation. *Lett. Math. Phys.* **10** (1985) 63–69; Quantum $R$-matrix for the generalized Toda system. *Comm. Math. Phys.* **102** (1986) 537–547.

[JKOS] M. Jimbo, H. Konno, S. Odake, J. Shiraishi. Elliptic algebra $U_{q,p}(\hat{sl}_2)$: Drinfeld currents and vertex operators. Preprint [math.QA/9802002](http://arxiv.org/abs/math.QA/9802002). Quasi-Hopf twistors for elliptic quantum groups. Preprint [q-alg/9712023](http://arxiv.org/abs/q-alg/9712023).

[EF] B. Enriquez, G. Felder. Elliptic quantum groups $E_{\tau,\eta}(\hat{sl}_2)$ and quasi-Hopf algebras. Preprint [q-alg/9703018](http://arxiv.org/abs/q-alg/9703018).

[ER] B. Enriquez, V. Rubtsov. Quasi-Hopf algebras associated with $\mathfrak{sl}(2)$ and complex curves. Preprint [q-alg/9608005](http://arxiv.org/abs/q-alg/9608005).

[F] C. Frønsdal. Quasi-Hopf deformation of quantum groups. *Lett. Math. Phys.* **40** (1997) 117–134.

[FT] L. Faddeev, L. Takhtadjan. Quantum inverse scattering method. *Usp. Mat. Nauk.* **34** no. 5 (1979) 13–63.

[KLP] S. Khoroshkin, D. Lebedev, S. Pakuliak. Elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}_2)$ in the scaling limit. *Commun. Math. Phys.* **190** (1998) 597–627.

[KT] S. Khoroshkin, V. Tolstoy. Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan-Weyl realizations for quantum affine algebras. Max Plank Institut Preprint MPI/94–23, [hep-th/9404036](http://arxiv.org/abs/hep-th/9404036).

[KT1] S. Khoroshkin, V. Tolstoy. On Drinfeld realization of quantum affine algebras, *Journal of Geometry and Physics*, **11**, 1993, 101–108.

[KTS] S. Khoroshkin, V. Tolstoy, A. Stolin. Generalized Gauss decomposition of trigonometric $R$-matrices. *Modern Phys. Lett. A* **10** (1995) no. 19, 1375–1392.

[M] I. G. Macdonald. Symmetric functions and Hall polynomials. 2nd edition, Oxford University Press (1995).

[RS] N. Reshetikhin, M. Semenov-Tyan-Shansky. Central extensions of quantum current groups. *Lett. Math. Phys.* **19** (1990), no. 2, 133-142.

[TK] V. Tolstoy, S. Khoroshkin, The universal $R$-matrix for quantum nontwisted affine Lie algebras. *Funk. Analiz i ego pril.* **26** (1992) 85–88.

[ZZ] A. Zamolodchikov, Al. Zamolodchikov. Factorized $S$-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. *Ann. Phys. (N.Y.)* **120** (1979) 253–291.