Methods to compute determinant of a 3×3 matrix

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Abstract. Sarrus's method is a rule to compute the determinant of a 3×3 matrix, by write out six entries of matrix (two columns or rows of matrix). Recently in 2014, Hajrizaj gave three new methods to compute determinant of a 3×3 matrix, by write out four entries of a 3×3 matrix. By position of copied entries from Hajrizaj’s methods, we do combination for four duplicate entries to get fourteen methods of computing determinant of a 3×3 matrix.

1. Introduction

Seki Kowa and Gottfried W. Leibniz, separately, was concept of the determinant together with applications of array manipulation to solve systems of linear equations.

Definition 1. [1] For an n×n matrix $A = [a_{ij}]$, the determinant of $A$ is defined to be the scalar

$$\det A = \sum_p \sigma(p)a_{p_1}a_{p_2}...a_{p_n}$$

(1)

where the sum is taken over the $n!$ permutations $p = (p_1, p_2, ..., p_n)$ of $(1, 2, ..., n)$ and

$$\sigma(p) = \begin{cases} +1 & \text{if } p \text{ can be restored to natural order by an even number of interchanges} \\ +1 & \text{if } p \text{ can be restored to natural order by an odd number of interchanges} \end{cases}$$

Observe that each term $a_{p_1}a_{p_2}...a_{p_n}$ (1) contains exactly one entry from each row and each column of $A$. The determinant of $A$ can be denoted by $\det(A)$ or $|A|$ whichever is more convenient.

Note: The determinant of a nonsquare matrix is not defined.

By definition, determinant is a real-valued function of a matrix variable in the sense that it associates a real number with a square matrix. Determinant function was having important applications to the theory of systems of linear equations and will also lead us to an explicit formula for the inverse of an invertible matrix. Other methods to compute determinant are cofactor expansion, row reductions, Chio’s condensation, Dodgson’s condensation and Sarrus’s rule. Sarrus’s rule is method simplicity for Definition 1. Following section show the Sarrus’s rule for 3×3 matrix.

Corollary 2. [2] For an 3×3 matrix $A$, the determinant of $A$ is defined to be the scalar

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(2)

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1.1. Sarrus's rule
Sarrus's rule or Sarrus's scheme is a method and a memorization scheme to compute the determinant of a matrix. If matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\det(A)$ can be computed by the following schemes:

- **Scheme 1**: $+a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
- **Scheme 2**: $-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$
- **Scheme 3**: $+a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}$
- **Scheme 4**: $-a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} - a_{11}a_{22}a_{33}$

*Figure 1.* Schemes of Sarrus’s rule [3].

By those schemes, determinant is computed by summing the products on the leftward arrows and subtracting the products on the rightward arrows and each scheme is equal to equation (2). Then Sarrus’s rule is determinant function.

1.2. Hajrizaj's method
In 2009, Hajrizaj introduced modifications of Sarrus's rule for third order matrix. If matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then its determinant can be computed by the following schemes:

- **Scheme 1**: $+a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
- **Scheme 2**: $-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$
- **Scheme 3**: $+a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}$

*Figure 2.* Schemes of Hazrijaz’s method [4].

By schemes in Figure 2, determinant is computed by summing the products on the leftward arrows and subtracting the products on the rightward arrows and each scheme is equal to equation (2).
2. Method
In Sarrus’s rule [3] entries of matrix was copied into twenty four possible positions which on two rows of entries or two columns of entries. While in Hajrizaj’s method [4] entries of matrix was copied into twelve possible positions, such as following:

\[
\begin{bmatrix}
3 & 2 & 1 \\
4 & a_{11} & a_{12} & a_{13} \\
5 & a_{21} & a_{22} & a_{23} \\
6 & a_{31} & a_{32} & a_{33} \\
7 & 8 & 9 & 12 \\
11 & 10
\end{bmatrix}
\]

Figure 3. Duplicate entries position for Hajrizaj’s method.

From Hajrizaj’s method, we conclude that each entry of third order matrix can duplicate into one or two positions. Entry \(a_{11}\) can duplicate to position 7 or 12, entry \(a_{12}\) can duplicate to position 8, entry \(a_{13}\) can duplicate to position 4 or 9, entry \(a_{21}\) can duplicate to position 11, entry \(a_{22}\) does not duplicate to any position, entry \(a_{23}\) can duplicate to position 5, entry \(a_{31}\) can duplicate to position 3 or 10, entry \(a_{32}\) can duplicate to position 2, and entry \(a_{33}\) can duplicate to position 1 or 6.

We modify Hajrizaj’s method by combination of four duplicate entries from twelve possible positions (Figure 3). We observe, in such combination, that each entry only duplicate to one position.

3. Results and discussion
In 2014, Assen and Rao modified of Hajrizaj’s rule for third order matrix by duplication of five entries [3]. These are several schemes of modification of Hajrizaj’s rule by Assen and Rao.

![Scheme 1](image1.png) ![Scheme 2](image2.png) ![Scheme 3](image3.png)

Figure 4. Several schemes of Assen and Rao’s method.

In Figure 4, we see that entry \(a_{21}\) does not multiply to get equation (2). In fact, scheme 3 in Figure 4 is the same to scheme 3 of Hajrizaj’s method (Figure 3). While, scheme 1 and scheme 2 of Assen and Rao’s method are rewrite, without entry \(a_{21}\), as scheme 2 and scheme 13 in Figure 5.

These are results of our modification of Hajrizaj’s method.
Each scheme in Figure 5, value of determinant function is computed by summing the products on the leftward arrows and subtracting the products on the rightward arrows and it is equal to equation (2). Hazrijaz’s method modification are determinant function for third order matrix. By these researches, we have many variations to calculate determinant of third order matrix.

4. Conclusion
We have fourteen schemes to compute determinant of 3×3 matrix. Further, we going to modified Hajrizaj's methods for five entries duplication.

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