GENERATORS OF MATRIX MONOIDS OVER COMMUTATIVE SEMIRINGS

THOMAS AIRD

Department of Mathematics, University of Manchester, Manchester M13 9PL, UK.

Abstract. We determine minimal generating sets for matrix monoids over semirings and semifields. In particular, we produce minimal generating sets for the monoids of upper triangular and unitriangular matrices over a commutative semiring. We construct minimal and irredundant generating sets for the monoids of $2 \times 2$ and $3 \times 3$ matrices over the tropical integers, showing that $M_3(\mathbb{Z}_{\text{max}})$ is not finitely generated.

Constructing minimal and irredundant generating sets for semigroups is a widely studied area of research, for example [1, 8]. This is related to the classical problem of calculating the rank of a semigroup, that is, the minimum cardinality of a generating set of a semigroup. This important invariant of a semigroup has again been widely researched, for example [3, 10].

Recently, there has been interest in semigroups of matrices over tropical semirings. These matrix monoids have many useful properties, including been used to admit faithful representations of semigroups which cannot be faithfully represented by matrices over fields [11]. Hence, there has been research into constructing minimal generating sets for matrix monoids, particularly semigroups of matrices over tropical semirings. East, Jonušas and Mitchell [6] found generating sets for $2 \times 2$ full matrix monoids over the min-plus natural number semiring, max-plus natural number semiring, and their finite quotients. Moreover, Hivert, Mitchell, Smith, and Wilson [9] found minimal generating sets for a number of submonoids of the monoid of boolean matrices and showed that the generating sets given in [6] are minimal generating sets.

In this paper, we focus on matrix monoids, constructing minimal and irredundant generating sets for monoids of upper triangular and unitriangular matrices over commutative semirings and for full matrix monoids over semifields. We have an explicit focus on the tropical integer semiring, showing that the monoid of $3 \times 3$ matrices over the tropical integers is not finitely generated.

In addition to this introduction, this paper comprises 5 sections. In Section 1, we introduce some notation and definitions that we use throughout the rest of the paper.
In Section 2, we describe the minimal and irredundant generating sets of the monoid of upper triangular matrices over a commutative semiring, showing that the monoid $UT_n(\mathbb{Z}_{\text{max}})$ is finitely generated for all $n \in \mathbb{N}$.

In Section 3, we turn our attention to unitriangular matrices. Again describing the minimal and irredundant generating sets for the monoid of unitriangular matrices over a commutative semiring, and showing that $U_n(\mathbb{Z}_{\text{max}})$ is not finitely generated for $n \geq 2$.

In Section 4, we look at the $2 \times 2$ full matrix monoid $M_2(\mathbb{Z}_{\text{max}})$, showing that $M_2(\mathbb{Z}_{\text{max}})$ is finitely generated, and constructing a minimal and irredundant generating set for it.

In Section 5, we look at the $3 \times 3$ full matrix monoid $M_3(S)$ over a semiring $S$, and show that if $S$ is anti-negative semifield then $M_3(S)$ is finitely generated if and only if $S$ is finite. We use this to show that $M_3(\mathbb{Z}_{\text{max}})$ is not finitely generated. Then, we explicitly construct a minimal and irredundant generating set for $M_3(\mathbb{Z}_{\text{max}})$ and show that the subsemigroup of $M_3(\mathbb{Z}_{\text{max}})$ consisting of the matrices that can be expressed as products of regular matrices, is 4-generated.

1. Preliminaries

For a semigroup $S$, $X \subseteq S$ is a (semigroup) generating set for $S$, if for all $s \in S$, there exists $x_1, \ldots, x_m \in X$ such that $s = x_1 \cdots x_m$. For a group $G$, $X$ is a group generating set for $G$ if $X \cup X^{-1} \cup \{1_G\}$ is a (semigroup) generating set for $G$. We say a generating set $X$ for a semigroup $S$ is minimal if $|X| \leq |Y|$ for any other generating set $Y$ for $S$ and say an element $x \in X$ is irredundant if $X \setminus \{x\}$ is not a generating set for $S$. If every $x \in X$ is irredundant then we say $X$ is irredundant. More generally, we say a set $X$ is minimal with a given property if it has the property and $|X| \leq |Y|$ for any other set $Y$ that has the property, and say a set is irredundant with a given property if it has the property and no proper subset of it has the property.

We call $x \in S$ a unit of a monoid $S$ if there exists $x^{-1} \in S$ such that $xx^{-1} = x^{-1}x = 1_S$. Let $U(S)$ be the group of units of $S$. We say a non-unit $x \in S$ is prime if for every product $x = uv$, exactly one of $u$ or $v$ is a unit. For a monoid $S$, let $J$ be Green’s $J$-relation, that is, the equivalence relation on $S$ defined by $xJy$ if and only if $SxS = S y S$. For $a \in S$, denote the $J$-class containing $a$ by $J_a$. We call $J$ a prime $J$-class if every element of $J$ is prime. It is easy to see that every generating set of $S$ must contain a representative from each prime $J$-class of $S$.

Let $S$ be a semiring, that is, a set $S$ with two binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity $0_S$, $(S, \cdot)$ is a monoid with identity $1_S$ such that $x0_S = 0_Sx = 0_S$ for all $x \in S$, and multiplication distributes over addition. We say $S$ is commutative if $(S, \cdot)$ is commutative. Define $S^* = S \setminus \{0_S\}$, and say $S$ is a semifield if $(S^*, \cdot)$ is an abelian group.

For a semiring $S$, let $U(S)$ be the group of units of $(S, \cdot)$. We say $x \in S$ is additively invertible if there exists $y \in S$ such that $x + y = 0_S$. Let $V(S)$ be the subset of additively invertible elements of $S$, i.e. the group of units of $(S, +)$. It is easy to see that for all $x, y \in V(S)$ and $z \in S$, $x + y \in V(S)$ and $zx, xz \in V(S)$. Note that $V(S)$ is a (possibly non-unital) ring and $V(S) = S$.
if and only if $1_S \in V(S)$. We say $S$ is anti-negative if for $x, y \in S, x + y = 0_S$ if and only if $x, y = 0_S$. Thus, $S$ is anti-negative if and only if $V(S) = \{0_S\}$.

We denote by $M_n(S)$ the monoid of all $n \times n$ matrices with entries in $S$ under matrix multiplication. Similarly, we let $UT_n(S)$ be the submonoid of $M_n(S)$ of all $n \times n$ upper triangular matrices over $S$ with $0_S$ entries below the diagonal, and $U_n(S)$ be the submonoid of $UT_n(S)$ of all $n \times n$ unitriangular matrices over $S$ with $0_S$ entries below the diagonal and $1_S$ entries on the diagonal.

Finally, for $1 \leq i \leq n$, let $A_i(\lambda) \in UT_n(S)$ be the diagonal matrix with $1_S$ on the diagonal apart from $\lambda$ as the $(i, i)$th entry, and for $1 \leq i, j \leq n$ with $i \neq j$, let $E_{ij}(\lambda) \in UT_n(S)$ be the matrix where all diagonal entries are $1_S$, $(E_{ij})_{ij} = \lambda$, and all other entries are $0_S$. We write $E_{ij}$ to denote $E_{ij}(1_S)$.

Throughout, let $\mathbb{B} = \{0, 1\}$ denote the Boolean Semifield, that is the semifield with the operations $x + y = \max\{x, y\}$ and $x \cdot y = \min\{x, y\}$, and let $\mathbb{Z}_{\text{max}} = \mathbb{Z} \cup \{-\infty\}$ denote the tropical integers, that is, the semifield with semiring addition given by maximum and semiring multiplication given by addition where for all $a \in \mathbb{Z}, -\infty < a$ and $-\infty + a = -\infty = a + -\infty$.

Now, we require two lemmas which we will use throughout the paper.

**Lemma 1.1.** Let $S$ be a commutative semiring. Then, $xy$ is a unit if and only if $x$ and $y$ are units.

**Lemma 1.2.** Let $S$ be a commutative semiring and $X \in S$ where $S = M_n(S), UT_n(S), \text{ or } U_n(S)$. If $XJI_n$ in $S$, then $X$ is a unit in $S$.

**Proof.** If $XJI_n$, then there exists $A, B \in S$ such that $AXB = I_n$. Hence, by the main theorem in [13], $XBA = BAX = I_n$, and thus $X \in U(S)$. \hfill $\Box$

2. **Upper Triangular Matrix Monoids**

In this section, we produce minimal generating sets for the monoid of upper triangular matrices over semirings, anti-negative semirings, and anti-negative semifields. First, we need the following lemma, which tells us when an upper triangular matrix over a commutative semiring is invertible. This can be deduced from [13, Theorem 4.2].

**Lemma 2.1.** Let $S$ be a commutative semiring and $n \in \mathbb{N}$. $X \in UT_n(S)$ is invertible if and only if $X_{ii} \in U(S)$ for $1 \leq i \leq n$ and $X_{ij} \in V(S)$ for $1 \leq i < j \leq n$.

**Theorem 2.2.** Let $S$ be a commutative semiring and $n \in \mathbb{N}$. Let $\mathcal{X}$ be a minimal (and irredundant) semigroup generating set for the group of units of $UT_n(S)$, $\Omega \subseteq S$ be a minimal (and irredundant) set such that $U(S)(\Omega \cup V(S))$ generates $(S, +)$, and $Y \subseteq S$ be a minimal (and irredundant) set such that $Y \cup U(S)$ generates $(S, \cdot)$. Then, the monoid $UT_n(S)$ is minimally (and irredundantly) generated by $\mathcal{X} \cup E(\Omega) \cup A(Y)$ where

$E(\Omega) = \{E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n\}$, and

$A(Y) = \{A_i(y) : y \in Y, 1 \leq i \leq n\}$.

**Proof.** If $a \in U(S)$ then $A_i(a) \in \langle \mathcal{X} \rangle$ as $A_i(a)$ is invertible by Lemma 2.1. If $a \in S \setminus U(S)$, then we can write $a = x_1 \cdots x_s$ for some $x_1, \ldots, x_s \in Y \cup U(S)$. Thus, $A_i(a) = A_i(x_1) \cdots A_i(x_s)$ and hence $A_i(a)$ is generated by
the corresponding product of diagonal matrices each of which lies in $A(Y)$ or $\langle X \rangle$. Thus, for $a \in S$ and $1 \leq i \leq n$, we can generate $A_i(a)$ by the given set.

Fix $a \in S$. Since $U(S)(\Omega \cup V(S))$ generates $(S, +)$ we can write $a = \sum_{t=1}^{m} b_t l_t$ where $l_t \in U(S)$ and $b_t \in \Omega \cup V(S)$. For all $i < j$, it is straightforward to verify that

$$E_{ij}(a) = \prod_{t=1}^{m} A_t(l_t)E_{ij}(b_t)A_i(l_t^{-1})$$

and each of these factors is either in $E(\Omega)$ or $\langle X \rangle$ as $E_{ij}(b_t) \in \langle X \rangle$ if $b_t \in V(S)$ by Lemma 2.1. Further, define, for $1 \leq i \leq n$,

$$E_{ii}(a) = A_i(a).$$

Thus, $E_{ij}(a)$ for $a \in S$ and $1 \leq i \leq j \leq n$ can be expressed as a product of the given set. Hence, for any $A = (a_{ij}) \in UT_n(S)$,

$$A = \prod_{l=0}^{n-1} \prod_{k=0}^{n-1-l} E_{n-i-k,n-l}(a_{n-l-k,n-l}).$$

Therefore, every matrix in $UT_n(S)$ can be expressed as the product of matrices from the given set.

We show that this is a minimal generating set by contradiction. Suppose that there exists a generating set $\Gamma$ for $UT_n(S)$ such that $|\Gamma| < |\langle X \rangle \cup E(\Omega) \cup A(Y)|$. Let $\Gamma_1 \subset \Gamma$ be the set of all units in $\Gamma$. As any product containing a non-unit is a non-unit by Lemma 1.2, $\Gamma_1$ generates the group of units of $UT_n(S)$. Therefore, as $\langle X \rangle$ is a minimal generating set for the group of units, we have that $|\langle X \rangle| \leq |\Gamma_1|$, and hence $|\Gamma \setminus \Gamma_1| < |E(\Omega) \cup A(Y)|$.

Let $T = \langle X \cup E(\Omega) \rangle$ and $\Gamma_2 \subset \Gamma \setminus \Gamma_1$ be all the matrices of $\Gamma$ in $T$ but not in $\Gamma_1$. Now, as $T$ is exactly the set of matrices with diagonal entries in $U(S)$, we can see that $XY \in T$ if and only if $X \in T$ and $Y \in T$ by considering the diagonal entries with Lemma 1.1. Thus, $\langle \Gamma_1 \cup \Gamma_2 \rangle = T$. Suppose $|\Gamma_2| < |E(\Omega)|$. To get a contradiction, we show that in order to generate every $E_{ij}(x)$ with $x \in S \setminus V(S)$ and $i < j$, we need at least $|\Omega|$ elements and hence conclude $|\Gamma_2| \geq |E(\Omega)|$.

So, suppose $\prod_{i=1}^{m} N_i = E_{ij}(x)$ for some $x \in S \setminus V(S)$, $i < j$, and $N_1, \ldots, N_m \in UT_n(S)$. Let $k < l$ such that $(k, l) \neq (i, j)$. Then,

$$\prod_{t=1}^{m} N_{kt} = \sum_{(i_0, \ldots, i_m)} \prod_{s=1}^{m} (N_s)_{i_{s-1},i_s} = (E_{ij}(x))_{kl} = 0_{S}.$$

where the sum ranges over $k = i_0 \leq \cdots \leq i_m = l$. Thus, for all $1 \leq t \leq m$,

$$(N_t)_{k} \cdots (N_{l-1})_{k} (N_t)_{kl} (N_{l+1})_{kt} \cdots (N_m)_{lt} \in V(S)$$

and hence, $(N_t)_{kl} \in V(S)$ as $(N_t)_{hh} \in U(S)$ for all $1 \leq h \leq n$ by Lemma 1.1 as $\prod_{t=1}^{m} (N_t)_{hh} = (\prod_{t=1}^{m} N_t)_{hh} = (E_{ij}(x))_{hh} = 1_S$.

By considering the $(i, j)$ entry of $\prod_{t=1}^{m} N_t = E_{ij}(x)$, letting $t_1, \ldots, t_{m'}$ be all the distinct values such that $(N_{a_{ij}})_{ij} \in S \setminus V(S)$ where $1 \leq a \leq m'$, and recalling that for $a, b \in V(S)$ and $c \in S$, $a + b, ca, ac \in V(S)$, $x$ may be
expressed as
\[ x = v + \sum_{a=1}^{m'} g_{t_a} (N_{t_a})_{ij} \]
where \( v \in V(S) \), \( g_{t_a} \in U(S) \).

Therefore, to construct \( E_{ij}(x) \) for all \( x \in S \setminus V(S) \), it is necessary to find a set \( X \) such that for all \( x \in S \), there exists an \( m_x \in \mathbb{N}_0 \) such that there exist \( v \in V(S) \), \( g_1, \ldots, g_{m_x} \in U(S) \), and \( x_1, \ldots, x_{m_x} \in X \) such that
\[ x = v + \sum_{t=1}^{m_x} g_{t_i} x_t. \]

Thus, \( U(S)X \cup V(S) \) generates \( (S, +) \) and hence, by the definition of \( \Omega \), \( |X| \geq |\Omega| \), as \( U(S)(X \cup V(S)) = U(S)X \cup V(S) \). Moreover, as we have to construct \( E_{ij}(x) \) for all \( x \in S \setminus V(S) \) and \( i < j \), we get that \( |\Gamma_2| \geq \frac{n}{2}(n-1) \cdot |\Omega| = |E(\Omega)| \) giving a contradiction. Hence, \( |\Gamma_3| < |A(Y)| \), where \( \Gamma_3 = \Gamma \setminus (\Gamma_1 \cup \Gamma_2) \).

For each \( s \in S \setminus U(S) \) and \( 1 \leq i \leq n \), \( A_i(s) \notin (\Gamma_1 \cup \Gamma_2) \), so consider a product \( \prod_{t=1}^{m} N_{t_i} = A_i(s) \). Then
\[ (\prod_{t=1}^{m} N_{t_i})_{ij} = \prod_{t=1}^{m} (N_{t_i})_{ii} = s \quad \text{and} \quad (\prod_{t=1}^{m} N_{t_i})_{hh} = \prod_{t=1}^{m} (N_{t_i})_{hh} = 1_S \]
for all \( 1 \leq h \leq n \) with \( h \neq i \). Thus, \( (N_{t_i})_{hh} \in U(S) \) for all \( t \) and \( h \neq i \).

Therefore, in order to construct each \( A_i(s) \) for \( s \in S \setminus U(S) \) we need to find a set \( \Lambda \) such that for all \( s \) there exist \( \lambda_1, \ldots, \lambda_{m_s} \in \Lambda \) such that \( s = g_1 \lambda_1 \cdots \lambda_{m_s} \) for some \( g \in U(S) \).

However, \( Y \) is the minimal set such that \( Y \cup U(S) \) generates \( (S, \cdot) \), so \( |\Lambda| \geq |Y| \). Moreover, as we need to construct \( A_i(s) \) for all \( s \in S \setminus U(S) \) and for all \( 1 \leq i \leq n \), we get that \( |\Gamma_3| \geq n|Y| = |A(Y)| \), giving a contradiction. Thus, \( |\Gamma| \geq |X \cup E(\Omega) \cup A(Y)| \) and hence the given generating set minimally generates \( UT_n(S) \).

If we assume \( X, \Omega \) and \( Y \) are irredundant. Then, by Lemma 1.2 in \( UT_n(S) \) any product containing a non-unit is a non-unit. Thus, all the elements of \( X \) are irredundant in the generating set given.

Now, suppose that \( E_{ij}(\omega) \) is redundant for some \( i < j \) and \( \omega \in \Omega \). Then, in order to construct \( E_{ij}(x) \) for each \( x \in S \), we have that for each \( x \in S \), there exists an \( m_x \in \mathbb{N}_0 \) such that there exist \( v \in V(S) \), \( g_1, \ldots, g_{m_x} \in U(S) \), and \( x_1, \ldots, x_{m_x} \in \Omega \setminus \{\omega\} \) such that \( x = v + \sum_{t=1}^{m_x} g_{t_i} x_t \) by above. This gives a contradiction as \( \Omega \) is an irredundant set such that \( U(S)(\Omega \cup V(S)) \) generates \( (S, +) \).

Now, suppose that \( A_i(y) \) is redundant for some \( y \in Y \). Then, in order to construct \( A_i(s) \) for each \( s \in S \), we have that for each \( s \in S \), there exist \( g \in U(S) \) and \( \lambda_1, \ldots, \lambda_{m_s} \in Y \setminus \{y\} \) such that \( s = g_{l_1} \lambda_1 \cdots \lambda_{m_s} \) by above. This gives a contradiction as \( Y \) is an irredundant set such that \( Y \cup U(S) \) generates \( (S, \cdot) \). Thus, the given generating set minimally and irredundantly generates \( UT_n(S) \).

\[ \square \]

**Corollary 2.3.** Let \( S \) be a commutative anti-negative semiring and \( n \in \mathbb{N} \). Let \( X \) be a minimal (and irredundant) generating set for \( (U(S)^n, \cdot) \) and \( \Omega \subseteq S \) be a minimal (and irredundant) set such that \( U(S)\Omega \) generates \( (S^n, +) \), and \( Y \subseteq S \) be a minimal (and irredundant) set such that \( Y \cup U(S) \) generates \( (S, \cdot) \). The monoid \( UT_n(S) \) is minimally (and irredundantly) generated by
diag(X) ∪ E(Ω) ∪ A(Y) where
\[
\text{diag}(X) = \{ \text{diag}(\overline{x}) = A_1(x_1) \cdots A_n(x_n) : \overline{x} = (x_1, \ldots, x_n) \in X \}, \quad \text{and}
\]
\[
E(\Omega) = \{ E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n \}.
\]
\[
A(Y) = \{ A_i(y) : y \in Y, 1 \leq i \leq n \}.
\]

Proof. The invertible elements of UT_n(S) are the diagonal matrices with entries in U(S), by Lemma 2.1, as V(S) = \{0_S\}. Therefore the group of units of UT_n(S) is isomorphic to U(S)^n and hence minimally generated by diag(X). Moreover, as S is anti-negative, a set X generates (S^*, +) if and only if X ∪ \{0_S\} generates (S, +).

Corollary 2.4. Let S be an anti-negative semifield and \( n \in \mathbb{N} \). Let X be a minimal (and irredundant) generating set for \((S^*)^n, \cdot\). The monoid UT_n(S) is minimally (and irredundantly) generated by diag(X) ∪ E(1_S) ∪ A(0_S) where
\[
\text{diag}(X) = \{ \text{diag}(\overline{x}) = A_1(x_1) \cdots A_n(x_n) : \overline{x} = (x_1, \ldots, x_n) \in X \}, \quad \text{and}
\]
\[
E(1_S) = \{ E_{ij} : 1 \leq i < j \leq n \},
\]
\[
A(0_S) = \{ A_i(0_S) : 1 \leq i \leq n \}.
\]

Proof. As U(S) = S^*, we may take Ω = \{1_S\} and Y = \{0_S\} in Corollary 2.3.

Corollary 2.5. Let \( n \in \mathbb{N} \). The monoid UT_n(\mathbb{Z}_{\text{max}}) is minimally and irredundantly generated by A(1) ∪ −1 · I_n ∪ E(0) ∪ A(−∞) where
\[
A(1) = \{ A_i(1) : 1 \leq i \leq n \}, \quad E(0) = \{ E_{ij} : 1 \leq i < j \leq n \}, \quad \text{and}
\]
\[
A(−∞) = \{ A_i(−∞) : 1 \leq i \leq n \}.
\]
Recall that \( 1 \neq 1_{\mathbb{Z}_{\text{max}}} = 0 \neq 0_{\mathbb{Z}_{\text{max}}} = −∞ \quad \text{and that} \quad −1 \cdot I_n \quad \text{is the diagonal matrix with} \quad −1 \quad \text{on the diagonal and} \quad −∞ \quad \text{elsewhere.}

Proof. By Corollary 2.4, it suffices to show that A(1) ∪ −1 · I_n forms a minimal and irredundant generating set for the diagonal matrices with entries in \( \mathbb{Z} \). Observe that, A_i(1) for \( 1 \leq i \leq n \), and −1 · I_n forms a minimal generating set for \( U(UT_n(\mathbb{Z}_{\text{max}})) \), as \( U(UT_n(\mathbb{Z}_{\text{max}})) \cong \mathbb{Z}^n \) by Lemma 2.1, and \( \mathbb{Z}^n \) is minimally \( n + 1 \) generated as a semigroup [2, Corollary 4.3]. Moreover, as A(1) ∪ −1 · I_n is finite and minimal, it is irredundant. Thus, this generating set is minimal and irredundant.

3. Unitriangular Matrix Monoids

In this section, we construct a minimal generating set for the monoid of unitriangular matrices over a commutative semiring. Let U_n(V(S)) denote the submonoid of U_n(S) with entries off the diagonal from V(S); this is the group of units of U_n(S) by Lemma 2.1.

Theorem 3.1. Let S be a commutative semiring and \( n \in \mathbb{N} \). Let \( X \) be a minimal (and irredundant) semigroup generating set for the group of units of U_n(S). Let \( \Omega \) be a minimal (and irredundant) set such that \( \Omega \cup V(S) \) generates (S, +). The monoid U_n(S) is minimally (and irredundantly) generated by \( X \cup E(\Omega) \) where
\[
E(\Omega) = \{ E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n \}.
\]
Proof. Fix $a \in S$. Since $\Omega \cup V(S)$ generates $(S,+)$ there exists $b_1, \ldots, b_m \in \Omega \cup V(S)$ such that $a = b_1 + \cdots + b_m$, and hence, for $i < j$, $E_{ij}(a) = E_{ij}(b_1) \cdots E_{ij}(b_m)$ and each of these factors is either in $\langle X \rangle$ by Lemma 2.1 or in $E(\Omega)$. Thus, $E_{ij}(a)$ for any $a \in S$ and $i < j$, can be expressed as a product of the given set. Hence, for any $A = (a_{ij}) \in U_n(S)$,

$$A = \prod_{l=1}^{n-1} \prod_{i=1}^{n-l} E_{i,n+1-l}(a_{i,n+1-l}).$$

where $E_{ij}(0_S) = I_n$. Therefore, every $A \in U_n(S)$ can be expressed as a product of matrices from the given set.

We show that this is a minimal generating set by contradiction. Suppose that there exists a generating set $\Gamma$ for $U_n(S)$ such that $|\Gamma| < |X \cup E(\Omega)|$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ is the set of all units in $\Gamma$. By Lemma 1.2 in $U_n(S)$ any product containing a non-unit is a non-unit, so $\Gamma_1$ generates the group of units of $U_n(S)$. Therefore, as $X$ is a minimal generating set for the group of units, we have that $|\Gamma_1| \geq |X|$ and hence $|\Gamma_2| < |E(\Omega)|$.

Suppose $\prod_{i=1}^{m} N_t = E_{ij}(x)$ for some $x \in S \setminus V(S)$, $i < j$, and $N_1, \ldots, N_m \in UT_n(S)$. Let $k < l$ such that $(k,l) \neq (i,j)$. Then,

$$\prod_{t=1}^{m} N_t = \sum_{(i_0, \ldots, i_m)} \prod_{s=1}^{m} (N_s)_{i_s-1,i_s} = (E_{ij}(x))_{kl} = 0_S.$$

where $k = i_0 \leq \cdots \leq i_m = l$. Thus, for all $1 \leq t \leq m$,

$$(N_1)_{kk} \cdots (N_{i_0-1})_{kk} (N_{i_0+1})_{kl} \cdots (N_{i_1})_{kl} = (N_t)_{kl} \in V(S)$$

as $(N_t)_{hh} = 1_S$ for all $1 \leq h \leq n$.

By considering the $(i,j)$ entry of $\prod_{t=1}^{m} N_t = E_{ij}(x)$, letting $t_1, \ldots, t_m$ be all the distinct values such that $(N_{t_a})_{ij} \in S \setminus V(S)$ where $1 \leq a \leq m'$, and recalling that for $a,b \in V(S)$ and $c \in S$, $a + b, ca, ac \in V(S)$, $x$ may be expressed as

$$x = v + \sum_{a=1}^{m'} (N_{t_a})_{ij}$$

where $v \in V(S)$.

Therefore, to construct $E_{ij}(x)$ for all $x \in S \setminus V(S)$, it is necessary to find a set $X$ such that for all $x \in S$, there exists an $m_x \in \mathbb{N}_0$ such that there exist $v \in V(S)$ and $x_1, \ldots, x_{m_x} \in X$ such that $x = v + \sum_{t=1}^{m_x} x_t$.

Thus, $X \cup V(S)$ generates $(S,+)$ and hence, by the definition of $\Omega$, $|X| \geq |\Omega|$. Moreover, as we have to construct $E_{ij}(x)$ for all $x \in S \setminus V(S)$ and $i < j$, we get that $|\Gamma_2| \geq \frac{1}{2}(n-1) \cdot |\Omega| = |E(\Omega)|$, giving a contradiction. Thus, $|\Gamma| \geq |X \cup E(\Omega)|$ and hence the given generating set minimally generates $U_n(S)$.

If we assume $X$ and $\Omega$ are irredundant. Then, by Lemma 1.2 in $U_n(S)$ any product containing a non-unit is a non-unit. Thus, all the elements of $X$ are irredundant in the generating set given.

Now, suppose that $E_{ij}(\omega)$ is redundant for some $i < j$ and $\omega \in \Omega$. Then, in order to construct $E_{ij}(x)$ for each $x \in S$, we have that for each $x \in S$, there exists an $m_x \in \mathbb{N}_0$ such that there exist $v \in V(S)$ and $x_1, \ldots, x_{m_x} \in \Omega \setminus \{\omega\}$ such that $x = v + \sum_{t=1}^{m_x} x_t$ by above. This gives a contradiction as $\Omega$ is
Lemma 4.2. Let matrices with non-zero generating set for \( \mathbb{Z} \). Note that every matrix \( m \in M \) has at most \( \phi \) divisors. Then, the invertible matrices of \( M \) minimally and irredundantly generate \( U_n(S) \).

\[ \text{Proof.} \] Note that \( \phi \) \( \{ E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n \} \)

Corollary 3.2. Let \( S \) be a commutative anti-negative semiring and \( n \in \mathbb{N} \). Let \( \Omega \) be a minimal (and irredundant) generating set of \( (S^*, +) \). The monoid \( U_n(S) \) is minimally (and irredundantly) generated by \( I_n \cup E(\Omega) \) where

\[ E(\Omega) = \{ E_{ij}(\omega) : \omega \in \Omega, 1 \leq i < j \leq n \} \]

\[ \text{Proof.} \] The group of units of \( U_n(S) \) is \( \{ I_n \} \), since \( V(S) = \{ 0_S \} \). Moreover, as \( S \) is anti-negative, a set \( X \) generates \( (S^*, +) \) if and only if \( X \cup \{ 0_S \} \) generates \( (S, +) \).

Corollary 3.3. Let \( n \in \mathbb{N} \). The monoid \( U_n(\mathbb{Z}_{\text{max}}) \) is minimally and irredundantly generated \( I_n \cup E(\mathbb{Z}) \) where

\[ E(\mathbb{Z}) = \{ E_{ij}(z) : z \in \mathbb{Z}, 1 \leq i < j \leq n \} \]

\[ \text{Proof.} \] Note that \( \max \{ x, y \} \in \{ x, y \} \) for all \( x, y \in \mathbb{Z} \). Thus, the minimal and irredundant generating set for \( (\mathbb{Z}, \text{max}) \) is \( \mathbb{Z} \).

4. 2 × 2 Full Matrix Monoids

In this section, we construct minimal generating sets for \( 2 \times 2 \) matrices over anti-negative semifields \( S \) with the property that for all \( x, y \in S \), for some \( a \in \{ x, x^{-1} \} \) and \( b \in \{ y, y^{-1} \} \) there exists \( t \in S \) such that \( a + t = b \) or \( b + t = a \).

For a semiring \( S \), we say a matrix \( M \in M_n(S) \) is a monomial matrix if it has exactly one non-0 entry in each row and column, and say \( M \) has underlying permutation \( \sigma \in S_n \) if \( M_{ij} \neq 0_S \) if and only if \( j = \sigma(i) \) for all \( 1 \leq i \leq n \). Moreover, we say \( M \) is the permutation matrix of \( \sigma \) if \( M_{ij} = 1_S \) if and only if \( j = \sigma(i) \) for all \( 1 \leq i \leq n \). We denote the group of units of \( M_n(S) \) as \( GL_n(S) \).

We define two functions which we use throughout the rest of the paper. Let \( S \) be an anti-negative semifield with no zero-divisors and \( \mathbb{B} \) be the Boolean semiring, then define \( \psi : S \to \mathbb{B} \) to be the map that sends \( 0_S \) to 0 and \( S^+ \) to 1. Let \( \phi_n : M_n(S) \to M_n(\mathbb{B}) \) be the map that sends \( A \) to \( \phi_n(A) \) where \( \phi_n(A)_{ij} = \psi(A_{ij}) \). Then, \( \psi \) and \( \phi_n \) are surjective morphisms for all \( n \in \mathbb{N} \) and hence the cardinality of a minimal generating set for \( M_n(S) \) is at least the cardinality of a minimal generating set for \( M_n(\mathbb{B}) \).

The following lemma tells us when a matrix over a commutative anti-negative semiring with no zero divisors is invertible, this can be deduced from [16] Corollary 3.3.

Lemma 4.1. Let \( S \) be a commutative anti-negative semiring with no zero divisors. Then, the invertible matrices of \( M_n(S) \) are exactly the monomial matrices with non 0 entries in \( U(S) \).

Lemma 4.2. Let \( m \geq 1 \) and \( X = \{ x_1, \ldots, x_m \} \) be a minimal group generating set for \( \mathbb{Z}^m \). Then \( X \cup \{ x_0 \} \) where \( x_0 = x_1^{-1} \cdots x_m^{-1} \) is a minimal semigroup generating set for \( \mathbb{Z}^m \).

\[ \text{Proof.} \] Note that \( \mathbb{Z}^m \) is minimally generated as a semigroup by a set of cardinality \( m + 1 \) [23 Corollary 4.3], so it suffices to construct \( x_i^{-1} \) as products.
of $x_0, \ldots, x_m$ for $1 \leq i \leq m$. Observe, $x_i^{-1} = x_0 \cdots x_{i-1} x_{i+1} \cdots x_m$ for $1 \leq i \leq m$.

For a semiring $S$, we introduce the notation that for $x, y \in S$, $x \leq y$ if and only if there exists $t \in S$ such that $t + y = x$. We remark that this is the Green’s $J$-order in the additive monoid of $S$.

**Theorem 4.3.** Let $S$ be an anti-negative semifield such that for all $x, y \in S$, $a \leq b$ or $b \leq a$ for some $a \in \{x, x^{-1}\}$ and $b \in \{y, y^{-1}\}$. Let $X = \{x_0, x_1, \ldots\}$ be a minimal generating set for $(S^*, \cdot)$ and if $(S^*, \cdot)$ is non-trivial, choose $X$ such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. The monoid $M_2(S)$ is minimally generated by the matrices:

$$A = \begin{pmatrix} 0_S & x_0 \\ 1_S & 0_S \end{pmatrix}, \quad B(x_i) = \begin{pmatrix} x_i & 0_S \\ 0_S & 1_S \end{pmatrix} \text{ for all } x_i \in X \setminus \{x_0\},$$

$$C = \begin{pmatrix} 0_S & 0_S \\ 0_S & 1_S \end{pmatrix}, \quad \text{and } D = \begin{pmatrix} 1_S & 1_S \\ 1_S & 0_S \end{pmatrix}$$

**Proof.** First, we show that if $(S^*, \cdot)$ is non-trivial we can find a minimal generating set $X = \{x_0, x_1, \ldots\}$ for $(S^*, \cdot)$ such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. If $|X| = m$ for some $m \in \mathbb{N}$, then as $S$ is an anti-negative semifield, every element but $0_S$ and $1_S$ has infinite multiplicative order [7, Lemma 2.1(ii)]. Thus, $(S^*, \cdot)$ is a finitely generated torsion-free abelian group and therefore isomorphic to $\mathbb{Z}^{m-1}$, as $\mathbb{Z}^{m-1}$ is $m$-generated as a semigroup [2, Corollary 4.3]. For $m \geq 2$, by Lemma 4.2 we can choose $X$ to be a minimal generating set such that $x_0^{-1} \in \langle X \setminus \{x_0\} \rangle$. If $m = 1$, then $(S^*, \cdot)$ is trivial and if $|X| > m$ for all $m \in \mathbb{N}$, then for any minimal generating set $X' = \{x_1, x_2, \ldots\}$ for $(S^*, \cdot)$, $X = \{1_S, x_1, x_2, \ldots\}$ is a minimal generating set for $(S^*, \cdot)$ such that $1_S^{-1} = 1_S \in \langle X \setminus \{1_S\} \rangle$.

When $(S^*, \cdot)$ is non-trivial, there exists $y_1, \ldots, y_s \in (X \setminus \{x_0\})$ such that $y_1 \cdots y_s = x_0^{-1}$. Thus, $B(x_0^{-1}) = B(y_1) \cdots B(y_s)$ and when $(S^*, \cdot)$ is trivial, $A^2 = B(x_0^{-1}) = I_2$. Note that

$$F = \begin{pmatrix} 0_S & 1_S \\ 1_S & 0_S \end{pmatrix} = B(x_0^{-1})A, \text{ and } B(x_0) = \begin{pmatrix} x_0 & 0_S \\ 0_S & 1_S \end{pmatrix} = AB(x_0^{-1})A.$$

For all $x \in S^*$, there exist $z_1, \ldots, z_t \in X$ such that $x = z_1 \cdots z_t$ and hence $B(x) = B(z_1) \cdots B(z_t)$. Therefore, $B(x)$ can be expressed as the product of generators as each $B(z_i)$ is a generator or can be expressed as a product of generators. Moreover, pre-multiplying any matrix $X$ by $F$ swaps the rows and post-multiplying by $F$ swaps the columns, so in order the prove we can generate every matrices as a product of the given generators, it suffices to express each matrix up to rearranging rows and columns.

Observe that, for $x \in S^*$,

$$\begin{pmatrix} 0_S & 0_S \\ 0_S & 0_S \end{pmatrix} = CFC, \text{ and } \begin{pmatrix} 0_S & 0_S \\ 0_S & x \end{pmatrix} = CFB(x).$$

Thus, from the given matrices, we are able to construct any matrix containing three or four $0_S$ entries. For the case where a matrix contains one or two $0_S$ entries, let $x, y, z \in S^*$, then
$\begin{pmatrix} 0_S & 0_S \\ x & y \end{pmatrix} = CFB(y)DB(y^{-1}x)$,

$\begin{pmatrix} 0_S & x \\ 0_S & y \end{pmatrix} = B(x)FB(y)DFC$,

$\begin{pmatrix} 0_S & x \\ y & 0_S \end{pmatrix} = B(x)FB(y)$, and

$\begin{pmatrix} 0_S & x \\ y & z \end{pmatrix} = B(x)FB(z)DFB(z^{-1}y)$.

Hence, every matrix with at least one $0_S$ entry can be expressed as a product of matrices from the given matrices.

Finally, for $a, b, c, d \in S^*$,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1_S & 1_S \\ db^{-1} & ca^{-1} \end{pmatrix} B(b)FB(a)$,

$= \begin{pmatrix} bd^{-1} & 1_S \\ 1_S & ca^{-1} \end{pmatrix} B(d)FB(a)$.

So, it suffices to express, for all $x, y \in S^*$, $\begin{pmatrix} 1_S & 1_S \\ x & y \end{pmatrix}$ or $\begin{pmatrix} x & 1_S \\ 1_S & y \end{pmatrix}$ as a product of the given matrices. To do this, we must check each case of $a \leq b$ where $a \in \{x, x^{-1}\}$ and $b \in \{y, y^{-1}\}$, where the $b \leq a$ cases are given similarly. We may suppose without loss of generality that $x \leq y$; indeed if $y \leq x$ then we can post-multiply by $F$ to swap the columns. As $x \leq y$, there exists $t \in S$ such that $t + y = x$ and

$\begin{pmatrix} 1_S & 1_S \\ x & y \end{pmatrix} = \begin{pmatrix} 0_S & 1_S \\ y^{-1}t & 0_S \end{pmatrix}$.

Similarly, we may suppose that $x^{-1} \leq y^{-1}$ since if $y^{-1} \leq x^{-1}$ then we can post-multiply by $F$ to swap the columns. As $x^{-1} \leq y^{-1}$, there exists $t \in S$ such that $t + y^{-1} = x^{-1}$ and

$\begin{pmatrix} 1_S & 1_S \\ x & y \end{pmatrix} = \begin{pmatrix} y^{-1} & t \\ 1_S & 0_S \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}$.

Similarly, we may suppose $x \leq y^{-1}$ since if $y \leq x^{-1}$ then we can pre-multiply and post-multiply by $F$ to swap the entries. As $x \leq y^{-1}$, there exists $t \in S$ such that $t + y^{-1} = x$ and

$\begin{pmatrix} x & 1_S \\ 1_S & y \end{pmatrix} = \begin{pmatrix} 1_S & y^{-1} \\ 0_S & 1_S \end{pmatrix} \begin{pmatrix} t & 0_S \\ 0_S & y \end{pmatrix}$.

Similarly, we may suppose $x^{-1} \leq y$ since if $y^{-1} \leq x$ then we can pre-multiply and post-multiply by $F$ to swap the entries. As $x^{-1} \leq y$, there exists $t \in S$ such that $t + y = x^{-1}$ and

$\begin{pmatrix} x & 1_S \\ 1_S & y \end{pmatrix} = \begin{pmatrix} t & 1_S \\ 0_S & x^{-1} \end{pmatrix} \begin{pmatrix} 0_S & x \\ x & xy \end{pmatrix}$.

Thus, any matrix can be expressed as a product of the given matrices as all the matrices above contain at least one $0_S$ entry.
Now, we show that the given generating set is minimal. The invertible matrices are the monomial matrices with entries in $S^*$, by Lemma 4.1. Let $\text{perm} : GL_2(S) \rightarrow (S^*, \cdot)$ be the surjective morphism that maps $A$ to $\prod_{i=1}^n A_{i, \sigma(i)}$ where $\sigma$ is the underlying permutation of $A$. $(S^*, \cdot)$ is minimally generated by $|X|$ elements by assumption, so $GL_2(S)$ is minimally generated by at least $|X|$ matrices. However, $GL_2(S)$ is generated by the $|X|$ matrices $A$ and $B(x)$ for $x \in X \setminus \{x_0\}$, so these matrices minimally generate $GL_2(S)$. Moreover, in $M_2(S)$, any product containing a non-invertible matrix is not invertible by Lemma 1.2, hence any minimal generating set for $M_2(S)$ must contain a generating set for $GL_2(S)$.

If $X$ is infinite, then the generating set is minimal as every generating set has to contain at least $|X| = |X| + 2$ elements. If $X$ is finite, then for a contradiction, suppose there exists a generating set $\Gamma$ of size $|X| + 1$ for $M_2(S)$. By above $|X|$ elements of $\Gamma$ are in the $GL_2(S)$. Let $\Gamma'$ be all the elements of $\Gamma$ in $GL_2(S)$ and $\Gamma \setminus \Gamma' = \{\gamma\}$. Consider $\phi_2(\Gamma)$, this is a generating set for $M_n(\mathbb{R})$. Moreover, as $\phi_2(\Gamma') = GL_2(\mathbb{R})$ and $GL_2(\mathbb{R})$ is generated by $\phi_2(A)$, we can see that $\phi_2(A) \cup \phi_2(\gamma)$ is a generating set for $M_2(S)$. However, this gives a contradiction as $M_2(S)$ is minimally generated by 3 matrices $\mathfrak{g}$. Therefore, $M_2(S)$ is minimally generated by these $|X| + 2$ matrices. □

**Corollary 4.4.** The monoid $M_2(\mathbb{Z}_{\text{max}})$ is minimally generated by the matrices:

$$A = \begin{pmatrix} -\infty & -1 \\ 0 & -\infty \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -\infty \\ -\infty & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -\infty & -\infty \\ -\infty & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & -\infty \end{pmatrix}$$

5. **3 × 3 Full Matrix Monoids**

In this section, we show that $M_3(S)$ is not finally generated for any infinite anti-negative semifield $S$, and construct a minimal and irredundant generating set for $M_3(\mathbb{Z}_{\text{max}})$.

For matrices $X, Y \in M_n(S)$, we say that $X$ is a permutation of $Y$ if $X$ can be obtained by permuting the rows and permuting the columns of $Y$. Equivalently, $X = PYP'$ for some permutation matrices $P, P' \in M_n(S)$.

In order to show that $M_3(\mathbb{Z}_{\text{max}})$ is not finitely generated, we show that we can find an infinite set of prime matrices each in a different $\mathcal{J}$-class in $M_3(S)$ when $S$ is an infinite commutative anti-negative semiring with no zero divisors.

**Lemma 5.1.** Let $S$ be a commutative anti-negative semiring with no zero divisors. For $s \in S^*$, let

$$X_s = \begin{pmatrix} 0_S & 1_S & s \\ 1_S & 0_S & 1_S \\ 1_S & 1_S & 0_S \end{pmatrix}.$$

Then $X_s$ is prime in $M_3(S)$. Moreover, if $X_t \mathcal{J} X_s$ for some $t \in S^*$ then $t = s$ or $ts = 1_S$. 
Proof. Note that $\phi_3(X_s)$ is prime \[4\] Theorem 1, so if $AB = X_s$ then $\phi_3(A)$ or $\phi_3(B)$ is a unit. Without loss of generality suppose $\phi_3(A)$ is a unit. Hence, $\phi_3(A)$ is a permutation matrix and $A$ is a monomial matrix. Suppose that $A$ is not a unit, then $A$ has a non 0-unit, non-invertible entry by Lemma 1.1. Thus, some row of $AB$ is a scaling of a row of $B$ by a non-invertible element of $S$. However, each row of $X_s$ contains a 1 entry, giving a contradiction by Lemma 1.1. Hence, $A$ is a unit and therefore $X_s$ is prime.

Suppose $X_t \in X_s$ for some $s, t \in S$, then $UX_sV = X_t$ where $U, V \in GL_3(S)$ as $X_t$ is prime. We may write $U = PD$ and $V = D'P'$ for permutation matrices $P$ and $P'$ and diagonal matrices with entries in $U(S)$, $D$ and $D'$ by Lemma 1.1. Suppose $P$ and $P'$ are the permutation matrix of $\sigma$ and $\tau$ respectively then, for $1 \leq i \leq 3$, the $(\sigma^{-1}(i), \tau(i))$ entry of $UX_sV$ is given by

$$P_{\sigma^{-1}(i),i}D_{i,i}(X_s)_{i,i}D'_{i,i}P'_{\tau(i),i} = 0_S$$

as $(X_s)_{i,i} = 0_S$. Therefore, $(X_t)_{\sigma^{-1}(i),\tau(i)} = 0_S$, but $(X_t)_{i,j} = 0_S$ if and only if $i = j$, so $\sigma = \sigma'$ and $P' = P^{-1}$.

Thus, $PDX_sD'P^{-1} = X_t$. Moreover, for $i \neq j$ and $(i,j) \neq (1,3)$,

$$P_{i,\sigma(i)}D_{\sigma(i),\sigma(i)}(X_s)_{\sigma(i),\sigma(j)}D'_{\sigma(j),\sigma(j)}P^{-1}_{\sigma(j),j} = (X_t)_{ij} = 1_S$$

and for $(i,j) = (1,3)$,

$$P_{1,\sigma(1)}D_{\sigma(1),\sigma(1)}(X_s)_{\sigma(1),\sigma(3)}D'_{\sigma(3),\sigma(3)}P^{-1}_{\sigma(3),3} = (X_t)_{13} = t.$$  \hfill (2)

Suppose $\sigma = 1_S$, then $D_{11}sD'_{33} = t$. In order the satisfy the equalities (1), we have that

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = D_{22} = (D')^{-1}_{33}.$$  

Thus, $s = t$.

If $\sigma \neq 1_S$, then $D_{11}sD'_{33} = 1_S$ and $D_{\sigma(1),\sigma(1)}D'_{\sigma(3),\sigma(3)} = t$, so $s, t \in U(S)$. Suppose that $\sigma$ is an even permutation, then

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = t^{-1}D_{22} = t^{-1}(D')^{-1}_{33}$$

if $\sigma = (1,2,3)$ as $D_{22}D'_{11} = t$ by (2), and

$$D_{11} = (D')^{-1}_{22} = t^{-1}D_{33} = t^{-1}(D')^{-1}_{11} = t^{-1}D_{22} = t^{-1}(D')^{-1}_{33}$$

if $\sigma = (1,3,2)$ as $D_{33}D'_{22} = t$ by (2). Thus, $D_{11}sD'_{33} = t^{-1}s = 1_S$, so $s = t$.

Finally, suppose that $\sigma$ is an odd permutation, then

$$D_{11} = (D')^{-1}_{22} = D_{33} = (D')^{-1}_{11} = D_{22} = t(D')^{-1}_{33}$$

if $\sigma = (1,2)$ as $D_{22}D'_{33} = t$ by (2),

$$D_{11} = (D')^{-1}_{22} = D_{33} = t(D')^{-1}_{11} = tD_{22} = t(D')^{-1}_{33}$$

if $\sigma = (1,3)$ as $D_{33}D'_{11} = t$ by (2), and

$$D_{11} = t(D')^{-1}_{22} = tD_{33} = t(D')^{-1}_{11} = tD_{22} = t(D')^{-1}_{33}$$

if $\sigma = (2,3)$ as $D_{11}D'_{22} = t$ by (2). Thus, $D_{11}sD'_{33} = ts = 1_S$. Therefore, if $X_t \in X_s$, then $t = s$ or $ts = 1_S$. \hfill $\Box$

**Theorem 5.2.** Let $S$ be an infinite commutative anti-negative semiring with no zero divisors. Then, the monoid $M_3(S)$ is not finitely generated.
Proof. Let $\tilde{S}$ be an infinite subset of $S$ such that if $x \in \tilde{S}$ and $x^{-1}$ exists and $x^{-1} \notin \tilde{S}$, then $x^{-1} \notin \tilde{S}$. Consider the matrices $X_s$ for $s \in \tilde{S}$ where

$$X_s = \begin{pmatrix} 0_s & 1_s & s \\ 1_s & 0_s & 1_s \\ 1_s & 1_s & 0_s \end{pmatrix}.$$

By Lemma 5.1, we have that $X_s$ is prime for all $s \in \tilde{S}$. Moreover, if $s^{-1}$ exists and $s \neq s^{-1}$, then $s^{-1} \notin \tilde{S}$, so $J_{X_s} \cap J_{X_t} = \emptyset$ for any $s \neq t \in \tilde{S}$. Thus, any generating set for $M_3(S)$ must contain a matrix $J$-related to $X_s$ for each $s \in \tilde{S}$ and is therefore not finitely generated.

Corollary 5.3. The monoid $M_3(\mathbb{Z}_{\text{max}})$ is not finitely generated.

Lemma 5.4. Let $S$ be a commutative anti-negative semiring with no zero divisors, and $X = \{x_1, x_1^{-1}, x_2, \ldots, x_m\}$ be a generating set for $(U(S), \cdot)$. Then, for $n \geq 2$, $GL_n(S)$ is generated by the following matrices:

$$A = A_1(x_1) \cdot P_{(1,\ldots,n-1)}; B = A_1(x_1^{-1}) \cdot P_{(1,\ldots,n)}$$

and $A_1(x)$ for $x \in X \setminus \{x_1, x_1^{-1}\}$

where $P_{s \sigma}$ for $\sigma \in S_n$, is the permutation matrix of $\sigma$.

Recall $A_1(x)$ is the diagonal matrix where the $(1,1)$ entry is $x$ and all other diagonal entries are $1_s$.

Proof. Clearly $A^{n-1} = A_1(x_1) \cdots A_{n-1}(x_1)$. Hence

$$B^{n-2}A_{n-1}B = (A_1(x_1^{-1})P_{(1,\ldots,n)})^{n-2}A_1(x_1) \cdots A_{n-1}(x_1)B$$

$$= P_{(1,\ldots,n)}^{n-2}A_{n-1}(x_1^{-1}) \cdots A_2(x_1^{-1}) \cdot A_1(x_1) \cdots A_{n-1}(x_1)B$$

$$= P_{(1,\ldots,n)}^{n-2}A_1(x_1)B$$

$$= P_{(1,\ldots,n)}^{n-2}A_1(x_1)A_1(x_1^{-1})P_{(1,\ldots,n)}$$

$$= P_{(1,\ldots,n)}^{n-1}$$

as $A_i(x_1^{-1})P_{(1,\ldots,n)} = P_{(1,\ldots,n)}A_{i+1}(x_1^{-1})$ for $1 \leq i \leq n-1$. Therefore,

$$(B^{n-2}A_{n-1})B = P_{(1,\ldots,n)}^{n-1} = P_{(1,\ldots,n)}.$$ Moreover,

$$B(B^{n-2}A_{n-1}B)A = A_1(x_1^{-1})P_{(1,\ldots,n)}P_{(1,\ldots,n)}^{n-1}A_1(x_1)P_{(1,\ldots,n-1)}$$

$$= A_1(x_1^{-1})A_1(x_1)P_{(1,\ldots,n-1)}$$

$$= P_{(1,\ldots,n-1)}.$$ Finally, note that

$$P_{(1,\ldots,n)}^{n-2}P_{(1,\ldots,n-1)}P_{(1,\ldots,n)} = P_{(1,2)}.$$ Thus, $S_n$ can be generated by the permutations $(1,2)$ and $(1,\ldots,n)$ [15].

Exercise 2.9(iii)], every permutation matrix can be expressed as a product consisting of $A$ and $B$. Furthermore,

$$A_i(x_1) = P_{(1,i)}A P_{(1,\ldots,n-1)}^{n-2}P_{(1,i)}; A_i(x_1^{-1}) = P_{(1,i)}B P_{(1,\ldots,n)}^{n-1}P_{(1,i)}$$

and $A_i(x) = P_{(1,i)}A_1(x)P_{(1,i)}$ for $x \in X \setminus \{x_1, x_1^{-1}\}$. Thus, $A_i(x)$ for $x \in X$ and $1 \leq i \leq n$ can be expressed as the product of the given set and hence every diagonal matrix with entries in $U(S)$ can
be expressed as a product of the given set, as they can be expressed as a product using matrices $A_i(x)$ for $x \in X$ where $1 \leq i \leq n$.

Finally, as every invertible matrix in a commutative anti-negative semiring with no zero divisors is a monomial matrix by Lemma 4.1, they can be expressed as diagonal matrix with $U(S)$ entries multiplied by a permutation matrix. Thus, by observing that all the matrices in the generating set are invertible, we have that, $GL_n(S)$ is generated by the given set. □

**Corollary 5.5.** Let $n \geq 2$. $GL_n(\mathbb{Z}_{\text{max}})$ is minimally generated by the following matrices:

$$A = A_1(1) \cdot P_{(1,\ldots,n-1)} \text{ and } B = A_1(-1) \cdot P_{(1,\ldots,n)},$$

where $P_{\sigma}$, for $\sigma \in S_n$, is the permutation matrix of $\sigma$.

**Proof.** By Lemma 5.4, $A$ and $B$ generate $GL_n(\mathbb{Z}_{\text{max}})$. To show minimality, observe that $GL_n(\mathbb{Z}_{\text{max}})$ is non-abelian and hence not 1-generated. Thus, $A$ and $B$ minimally generate $GL_n(\mathbb{Z}_{\text{max}})$. □

**Lemma 5.6.** Let $n \geq 2$. The submonoid $M \subseteq M_n(\mathbb{Z}_{\text{max}})$ generated by the following matrices contains $UT_n(\mathbb{Z}_{\text{max}})$:

$$A = A_1(1) \cdot P_{(1,\ldots,n-1)} \text{ and } B = A_1(-1) \cdot P_{(1,\ldots,n)},$$

$E_{12}$, and $A_1(-\infty)$.

where $P_{\sigma}$, for $\sigma \in S_n$, is the permutation matrix of $\sigma$.

**Proof.** If we are able to construct the generators from Corollary 2.5 from the given matrices, we are done. First, $-1 \cdot I_n, A_1(1), \ldots, A_n(1)$ can be generated as they are units in $M_n(\mathbb{Z}_{\text{max}})$ and $A$ and $B$ generate $GL_n(\mathbb{Z}_{\text{max}})$ by Corollary 5.5. So, it suffices to show that we can generate $A_i(-\infty)$ for $1 \leq i \leq n$ and $E_{ij}$ for $1 \leq i < j$. Observe that,

$$A_i(-\infty) = P_{(i)}A_1(-\infty)P_{(i)}, \text{ and } E_{ij} = \begin{cases} P_{(j)2}E_{12}P_{(2)j} & \text{if } i = 1, \\ P_{(2)j}E_{12}P_{(2)i} & \text{if } i = 2, \\ P_{(i)j2}E_{12}P_{(2)ji} & \text{if } 2 < i < j. \end{cases}$$

Thus, as $P_{\sigma} \in GL_n(\mathbb{Z}_{\text{max}})$ for all $\sigma \in S_n$, $UT_n(\mathbb{Z}_{\text{max}}) \subseteq M$. □

**Theorem 5.7.** The monoid $M_3(\mathbb{Z}_{\text{max}})$ is minimally and irredundantly generated by the following matrices:

$$A = A_1(1) \cdot P_{(1,2)}, B = A_1(-1) \cdot P_{(1,2,3)}, E_{12}, A_1(-\infty), \text{ and }$$

$$X_i = \begin{pmatrix} -\infty & 0 & i \\ 0 & -\infty & 0 \\ 0 & 0 & -\infty \end{pmatrix} \text{ for } i \in \mathbb{N}_0,$$

where $P_{\sigma}$, for $\sigma \in S_3$, is the permutation matrix of $\sigma$.

**Proof.** Consider the following matrices:

$$A' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & -\infty & -1 \\ -\infty & 0 & -\infty \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 1 & -\infty \\ -\infty & -\infty & 0 \end{pmatrix},$$
By Lemma 5.4, \( A \) and \( B \) generate \( GL_3(\mathbb{Z}_{\text{max}}) \). So, as \( A', B' \in GL_3(\mathbb{Z}_{\text{max}}) \), they can be generated by the given matrices. \( C' \) and \( D' \) can also be generated as \( C' = P_{(1,2)}A_1(-\infty)P_{(1,2)} \) and \( D' = P_{(1,3,2)}E_{12}P_{(1,3)} \). By restricting \( A', B', C', \) and \( D' \) to the second and third column and rows, we get a generating set for \( M_2(\mathbb{Z}_{\text{max}}) \) by Lemma 4.4, we may do this as these matrices are block diagonal, and hence the restriction is a morphism. Hence, by multiplying by \( A \times \) upper triangular matrix, which can be generated by Lemma 5.6, or a block generating set for \( M_2(\mathbb{Z}_{\text{max}}) \) by Lemma 4.4, we may do this as these matrices are block diagonal, and hence the restriction is a morphism. Hence, by multiplying by \( A_1(x) \) for \( x \in \mathbb{Z}_{\text{max}} \), we can construct any block diagonal matrix with any \((1,1)\) entry and a \( 2 \times 2 \) block in the second and third row and column. Moreover, as permutation matrices are in \( GL(3, \mathbb{Z}_{\text{max}}) \), we can generate any permutation of this matrix.

Every matrix with at least four \(-\infty\) entries is either a permutation of an upper triangular matrix, which can be generated by Lemma 5.6 or a block diagonal matrix with a \( 2 \times 2 \) block, which can be generated by above.

We now show that we can construct all matrices with at most three \(-\infty\) entries. It suffices to check up to permutation, that is, permuting the rows and permuting the columns. Moreover, as \( A^T, B^T \) and \( C^T \) have more that four \(-\infty\) entries and \( X_i^T = P_{(1,3)}X_iP_{(1,3)} \), we can see that the transposes of the generators can be generated and hence, we only have to check we can generate all matrices up to transposition.

Note that, for \( a, b, c, d, e \in \mathbb{Z} \) and \( x, y \in \mathbb{Z}_{\text{max}} \),

\[
\begin{pmatrix} a & b & c \\ d & e & x \\ -\infty & -\infty & y \end{pmatrix} = \begin{pmatrix} 0 & -\infty & c \\ -\infty & 0 & x \\ -\infty & -\infty & y \end{pmatrix} \begin{pmatrix} a & b & -\infty \\ d & e & -\infty \\ -\infty & -\infty & 0 \end{pmatrix}
\]

Thus, as the above matrix can be expressed as a product of matrices with at least four \(-\infty\) entries, these can be generated. Moreover, note that for \( f \in \mathbb{Z} \) the matrix

\[
\begin{pmatrix} a & b & c \\ -\infty & d & e \\ -\infty & -\infty & f \end{pmatrix}
\]

is already an upper triangular matrix and can be generated by Lemma 5.6.

For the final matrix with three \(-\infty\) entries, we can assume some entries are 0 by multiplying by a diagonal matrix.

\[
\begin{pmatrix} -\infty & a & b \\ c & -\infty & d \\ e & f & -\infty \end{pmatrix} = \begin{pmatrix} a & -\infty & -\infty \\ -\infty & d & -\infty \\ -\infty & -\infty & e \end{pmatrix} \begin{pmatrix} -\infty & 0 & b-a \\ c-d & -\infty & 0 \\ 0 & f-e & -\infty \end{pmatrix}.
\]

We split this matrix into two cases, first if \( x+y+z = i \geq 0 \), then

\[
\begin{pmatrix} -\infty & 0 & x \\ y & -\infty & 0 \\ 0 & z & -\infty \end{pmatrix} = \begin{pmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \end{pmatrix} X_i \begin{pmatrix} -z & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & x-i \end{pmatrix},
\]

and if \( x+y+z = -i \leq 0 \), then

\[
\begin{pmatrix} -\infty & 0 & x \\ y & -\infty & 0 \\ 0 & z & -\infty \end{pmatrix} = \begin{pmatrix} -\infty & -\infty & 0 \\ -\infty & -\infty & -\infty \\ z & -\infty & -\infty \end{pmatrix} X_i \begin{pmatrix} -\infty & -\infty & x \\ -\infty & 0 & -\infty \\ -z-i & -\infty & -\infty \end{pmatrix}.
\]
Thus, we have now shown we can generate all matrices with at least three \(-\infty\) entries.

Finally, let \(a, b, c, d, e, f, g \in \mathbb{Z}\) and \(x \in \mathbb{Z}_{\text{max}}\) and split into two cases. If \(a + e \geq b + d\), then
\[
\begin{pmatrix}
  a & b & c \\
  d & e & -\infty \\
  f & x & g
\end{pmatrix}
= \begin{pmatrix}
  0 & b - e & -\infty \\
  -\infty & 0 & -\infty \\
  -\infty & -\infty & 0
\end{pmatrix}
\begin{pmatrix}
  a & -\infty & c \\
  d & e & -\infty \\
  f & x & g
\end{pmatrix},
\]
and if \(b + d \geq a + e\), then
\[
\begin{pmatrix}
  a & b & c \\
  d & e & -\infty \\
  x & f & g
\end{pmatrix}
= \begin{pmatrix}
  0 & a - d & -\infty \\
  -\infty & 0 & -\infty \\
  -\infty & -\infty & 0
\end{pmatrix}
\begin{pmatrix}
  -\infty & b & c \\
  d & e & -\infty \\
  x & f & g
\end{pmatrix}.
\]

Thus, to show we can generate all matrices with two \(-\infty\) entries, take \(x = -\infty\), and note that it is expressible as product of matrices with at least three \(-\infty\) entries. Then, taking \(x \in \mathbb{Z}\), we have shown that we can express any matrix with at least one \(-\infty\) entry as the product matrices with at least two \(-\infty\) entries and therefore a product of the given matrices.

For matrices without \(-\infty\) entries, we may scale the columns so that the top row only contains 0 entries. So, we only need to consider the matrices
\[
\begin{pmatrix}
  0 & 0 & 0 \\
  a & b & c \\
  d & e & f
\end{pmatrix}
\]
where \(a, b, c, d, e, f \in \mathbb{Z}\). Further, we may rearrange the columns to assume \(a \leq b \leq c\). Now, observe that if \(d \leq e, f\), then
\[
\begin{pmatrix}
  0 & 0 & 0 \\
  a & b & c \\
  d & e & f
\end{pmatrix}
= \begin{pmatrix}
  0 & -\infty & -\infty \\
  a & b & c \\
  d & e & f
\end{pmatrix}
\begin{pmatrix}
  0 & 0 & 0 \\
  -\infty & 0 & -\infty \\
  -\infty & -\infty & 0
\end{pmatrix}
\]
as \(b, c \geq a\), and \(e, f \geq d\). If \(d \geq e\), then
\[
\begin{pmatrix}
  0 & 0 & 0 \\
  a & b & c \\
  d & e & f
\end{pmatrix}
= \begin{pmatrix}
  0 & -b & -d \\
  c & 0 & -\infty \\
  f & -\infty & 0
\end{pmatrix}
\begin{pmatrix}
  -\infty & -\infty & 0 \\
  a & b & -\infty \\
  d & e & -\infty
\end{pmatrix}
\]
as \(0 \geq a - b\) and \(0 \geq e - d\). Finally, if \(d \geq f\), then
\[
\begin{pmatrix}
  0 & 0 & 0 \\
  a & b & c \\
  d & e & f
\end{pmatrix}
= \begin{pmatrix}
  0 & -c & -d \\
  b & 0 & -\infty \\
  e & -\infty & 0
\end{pmatrix}
\begin{pmatrix}
  -\infty & 0 & -\infty \\
  a & -\infty & c \\
  d & -\infty & f
\end{pmatrix}
\]
as \(0 \geq a - c\) and \(0 \geq f - d\). Thus, as every product above contains at least one \(-\infty\) entry we can generate every matrix with no \(-\infty\) entries.

Therefore, every matrix can be expressed as a product of the given matrices. To show that this generating set is irredundant, note that each \(X_i\) is in a different prime \(J\)-class by Lemma 5.1 and as any generating set for \(M_3(\mathbb{Z}_{\text{max}})\) must require each at least one representative from each, all the \(X_i\) matrices are irredundant.

By Corollary 5.5, \(A\) and \(B\) minimally generate \(GL_3(\mathbb{Z}_{\text{max}})\). Moreover, as \(A\) and \(B\) are the only invertible elements in the generating set, \(A\) and \(B\) are irredundant by Lemma 1.2.
As every matrix in the generating set apart from \( A_1(−∞) \) has at least one finite entry in each row and column, so must any product of these matrices, hence \( A_1(−∞) \) cannot be expressed as a product of the other generators and is irredundant. Similarly, as each matrix in the generating set apart from \( E_{12} \) has at most one finite entry in each row and column, so must any product of these matrices, so \( E_{12} \) is irredundant.

Therefore, all matrices in the generating set are irredundant, so the generating set is irredundant. The generating set is minimal by Corollary 5.3. □

For a semigroup \( S \), we say \( x \in S \) is regular if there exists \( y \in S \) such that \( xyx = x \). In 1968, Devadze [5] showed that the size of minimal generating sets for \( M_n(\mathbb{B}) \) grows exponentially. However, Kim and Roush [12] showed that there exists a semigroup generated by four matrices from \( M_n(\mathbb{B}) \) which contains all regular matrices in \( M_n(\mathbb{B}) \). Lemma 5.6 and the following Corollary, shows that a similar result holds for \( M_n(\mathbb{Z}_{\text{max}}) \) when \( n \leq 3 \).

**Corollary 5.8.** The submonoid \( X \subset M_3(\mathbb{Z}_{\text{max}}) \) generated by the following matrices:

\[
A = A_1(1) \cdot P_{(1,2)}, \quad B = A_1(−1) \cdot P_{(1,2,3)}, \quad E_{12}, \quad \text{and} \quad A_1(−∞)
\]

contains all regular matrices in \( M_3(\mathbb{Z}_{\text{max}}) \).

**Proof.** By the proof of Theorem 5.7 we can use the matrices above to generate every matrix not \( J \)-related to \( X_i \) for some \( i \in \mathbb{N}_0 \). We will show that every matrix \( J \)-related to \( X_i \) is not regular. By Lemma 5.4, each \( X_i \) is prime in \( M_3(\mathbb{Z}_{\text{max}}) \), hence every matrix \( J \)-related to \( X_i \) is prime. Let \( M \not\in X_i \) for some \( i \in \mathbb{N}_0 \), so in particular \( M \not\in M_3(\mathbb{Z}_{\text{max}}) \) such that \( M = AX_iB \). Now, suppose \( M \) is regular. Then there exists \( Y \) such that \( MYM = M \). However, this implies that \( YM \) is an idempotent, and moreover \( YM \) is a unit as \( M \) is prime. Thus, \( Y = M^{-1} \), giving a contradiction as \( M \not\in \text{GL}_3(\mathbb{Z}_{\text{max}}) \). Therefore, \( X \) contains all regular matrices in \( M_3(\mathbb{Z}_{\text{max}}) \). □

We now pose the question whether the theorem proved by Kim and Roush [12] is again true when applied to \( M_n(\mathbb{Z}_{\text{max}}) \) rather than \( M_n(\mathbb{B}) \).

**Question 5.9.** Is it the case that the matrices in the statement of Lemma 5.6 generate all regular matrices of \( M_n(\mathbb{Z}_{\text{max}}) \)?

**References**

[1] G. Ayik, H. Ayik, L. Bugay, and O. Kelkeci. Generating sets of finite singular transformation semigroups. In *Semigroup Forum*, volume 86, pages 59–66. Springer, 2013.

[2] M. Branco, G. Gomes, and P. Silva. On the semigroup rank of a group. *Semigroup Forum*, 99, 12 2019.

[3] M. J. J. Branco, G. M. S. Gomes, and P. V. Silva. On the semigroup rank of a group. *Semigroup Forum*, 99(3):568–578, 2018.

[4] D. De Caen and D. A. Gregory. Prime boolean matrices. In Robert W. Robinson, George W. Southerm, and Walter D. Wallis, editors, *Combinatorial Mathematics VII*, pages 76–82, Berlin, Heidelberg, 1980. Springer Berlin Heidelberg.

[5] H. M. Devadze. Generating sets of the semigroup of all binary relations in a finite set, (Russian). In *Dokl. Akad. Nauk BSSR*, volume 12, pages 765–768, 1968.

[6] J. East, J. Jonušas, and J. D. Mitchell. Generating the monoid of \( 2 \times 2 \) matrices over max-plus and min-plus semirings, 2020. arXiv:2009.10372 [math.RA].
[7] V. Gould, M. Johnson, and M. Naz. Matrix semigroups over semirings. *International Journal of Algebra and Computation*, 30(02):267–337, 2020.
[8] R. Gray and N. Ruškuc. Generating sets of completely 0-simple semigroups. *Communications in Algebra*, 33(12):4657–4678, 2005.
[9] F. Hivert, J. D. Mitchell, F. L. Smith, and W. A. Wilson. Minimal generating sets for matrix monoids, 2021. arXiv:2012.10323 [math.RA].
[10] P. Huisheng. On the rank of the semigroup $T_E(X)$. *Semigroup Forum*, 70:107–117, 2005.
[11] Z. Izhakian and S. Margolis. Semigroup identities in the monoid of two-by-two tropical matrices. arXiv preprint arXiv:0902.1174, 2009.
[12] K. Kim and F. Roush. On generating regular elements in the semigroup of binary relations. In *Semigroup Forum*, volume 14, pages 29–32. Springer, 1977.
[13] Y-L. Liao and X-P. Wang. Note on invertible matrices over commutative semirings. *Linear and Multilinear Algebra*, 64(3):477–483, 2016.
[14] C. Reutenauer and H. Straubing. Inversion of matrices over a commutative semiring. *Journal of Algebra*, 88:350–360, 1984.
[15] J. J. Rotman. *An introduction to the theory of groups*, volume 148. Springer Science & Business Media, 2012.
[16] Y-J. Tan. On invertible matrices over commutative semirings. *Linear and Multilinear Algebra*, 61(6):710–724, 2013.