Equivalence transformations and differential invariants of a generalized nonlinear Schrödinger equation

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Abstract. By using the Lie’s invariance infinitesimal criterion we obtain the continuous equivalence transformations of a class of nonlinear Schrödinger equations with variable coefficients. Starting from the equivalence generators we construct the differential invariants of order one. We apply these latter ones to find the most general subclass of variable coefficient nonlinear Schrödinger equations which can be mapped, by means of an equivalence transformation, to the well known cubic Schrödinger equation. We also provide the explicit form of the transformation.
1. Introduction

In this paper we investigate the equivalence transformations and the differential invariants associated with the following \((1+1)\) dimensional generalized nonlinear Schrödinger equation

\[
iu_t + f(t, x)u_{xx} + g(t, x)u^2 + h(t, x)u = 0, \tag{1.1}
\]

where \(f(t, x), g(t, x)\) and \(h(t, x)\) are real functions of \(t\) and \(x\) and subscript denotes partial differentiation with respect to that variable. In the case \(f, g = 1\) and \(h = 0\), Eq. (1.1) reduces to the well known nonlinear Schrödinger equation (NLS)

\[
iu_t + u_{xx} + u^2 = 0. \tag{1.2}
\]

Eq. (1.2) arises in several branches of physics including Nonlinear Optics, Condensed Matter Physics, Hydrodynamics and so on (for the derivation and wide applications of Eq. (1.2), refer [1–5]). It has been shown that the NLS equation is one of the completely integrable nonlinear partial differential equations (PDEs) in \((1+1)\) dimensions and admits several interesting mathematical properties including infinite number of conservation laws, Lie-Bäcklund symmetries, \(N\)-soliton solutions and so on.

An equivalence transformation for the family (1.1) is a non-degenerate transformation of dependent and independent variables mapping (1.1) to another equation of the same family but with different functions, say \((\hat{f}, \hat{g}, \hat{h})\), from the original ones. Thus solutions of an equation can be transformed to the solutions of an equivalent equation. The main advantage of this procedure is that instead of solving individual equations one can develop schemes for complete equivalent classes. One may note that the classical Lie symmetry transformations are nothing but special subgroups of these equivalence groups of transformations since Lie symmetry transformations map an equation into itself. Recently, several works have been devoted to study the equivalence transformations of certain important nonlinear dynamical systems (see for a review Refs. [6–8] and bibliography therein).

One of the classical studies in the theory of differential equations is finding differential invariants. The origin dates back to Laplace who derived two invariants for linear hyperbolic equations. These invariants are now called Laplace invariants. The generalization of these invariants to elliptic and hyperbolic equations were derived by Cotton (for the historical notes and recent developments one may refer [9, 10]). Recently, renewal of interest has been initiated to study the differential invariants of the equivalence algebra of certain multidimensional linear PDEs as well as nonlinear PDEs [11–25].

In this paper we explore both equivalence transformations and some of their differential invariants for the eq. (1.1). More specifically by treating the functions \(f, g\) and \(h\) as arbitrary parameters we explore the most general nonlinear PDE of the form (1.1) which can be transformed to the standard NLS. Through this analysis we bring out a family of integrable variable coefficient NLS equations that can be mapped to
the standard NLS. We also construct the transformation that connects the variable coefficient NLS to the standard NLS.

In the form of these equations fall several cases of coefficient variable nonlinear Schrödinger equations, in particular we recall here that a wide subclass of equations considered in [27] belongs to the class \( \text{1.1} \), while this last one could be considered a specialization of the so-called VCNLSE considered by P. Winternitz and L.Gagnon in [28].

The plan of the paper is as follows. In the following Sec. 2, we study the equivalence transformations of Eqs. \( \text{1.1} \). In Sec. 3, we investigate the invariants associated with \( \text{1.1} \) and show that the latter admits first order differential invariants. As an application of these differential invariants, in Sec. 4, we derive the functional forms of \( f, g \), and \( h \) which characterize the subclass of Eqs. \( \text{1.1} \) that can be transformed to a standard NLS Eq. \( \text{1.2} \). We present our conclusions in Sec. 5.

2. Equivalence transformations

In order to study the equivalence transformations of Eqs. \( \text{1.1} \), we rewrite the complex equation \( \text{1.1} \) as a system of real equations by introducing a transformation \( u = v + iw \), that is,

\[
\begin{align*}
  v_t + f(t,x)w_{xx} + g(t,x)(v^2 + w^2)w + h(t,x)w &= 0, \\
  w_t - f(t,x)v_{xx} - g(t,x)(v^2 + w^2)v - h(t,x)v &= 0.
\end{align*}
\]

An equivalence transformation of the system \( \text{2.1a}-\text{2.1b} \) is a non degenerate change of variables from \((t, x, v, w)\) to \((\hat{t}, \hat{x}, \hat{v}, \hat{w})\) and transforming the equation of the form \( \text{2.1a}-\text{2.1b} \) into another system of the same form but with different functions \( \hat{f}(\hat{t}, \hat{x}), \hat{g}(\hat{t}, \hat{x}) \) and \( \hat{h}(\hat{t}, \hat{x}) \). The equivalence transformations for our system \( \text{2.1a}-\text{2.1b} \) is obtained by making use of the Lie’s infinitesimal criterion [6]. However, in the case of the infinitesimal equivalence generator we demand not only the invariance of \( \text{2.1a}-\text{2.1b} \) but also the invariance of the so called auxiliary conditions in the augmented space \((t, x, v, w, f, g, h)\). In other words one needs to consider the following conditions also, in addition to the usual invariance conditions,

\[
\begin{align*}
  f_v = f_w = 0, & \quad g_v = g_w = 0, & \quad h_v = h_w = 0,
\end{align*}
\]

which characterize the functional dependence of the functions \( f, g \) and \( h \).

Let us consider the one-parameter group of equivalence transformations, \( G_\varepsilon \), in the augmented space \((t, x, v, w, f, g, h)\) given by,

\[
\begin{align*}
  \hat{t} &= t + \varepsilon \xi^1(t, x, v, w) + O(\varepsilon^2), \\
  \hat{x} &= x + \varepsilon \xi^2(t, x, v, w) + O(\varepsilon^2), \\
  \hat{v} &= v + \varepsilon \eta^1(t, x, v, w) + O(\varepsilon^2), \\
  \hat{w} &= w + \varepsilon \eta^2(t, x, v, w) + O(\varepsilon^2), \\
  \hat{f} &= f + \varepsilon \nu^1(t, x, v, w, f, g, h) + O(\varepsilon^2),
\end{align*}
\]
\[ \hat{g} = g + \varepsilon \nu^2(t, x, v, w, f, g, h) + \mathcal{O}(\varepsilon^2), \]
\[ \hat{h} = h + \varepsilon \nu^3(t, x, v, w, f, g, h) + \mathcal{O}(\varepsilon^2), \]
where \( \varepsilon \) is the infinitesimal group parameter. The vector field associated with the infinitesimal equivalence transformations (2.3a)-(2.3g) can be written as
\[ Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial v} + \eta^2 \frac{\partial}{\partial w} + \nu^1 \frac{\partial}{\partial f} + \nu^2 \frac{\partial}{\partial g} + \nu^3 \frac{\partial}{\partial h}. \]

Since (2.1a)-(2.1b) involve second derivatives, we need to consider second prolongation of the operator \( Y \). Before proceeding further we introduce the notion
\[ (f^1, f^2, f^3) \equiv (f, g, h), \quad (x^1, x^2) \equiv (t, x), \quad (y^1, y^2) \equiv (v, w), \]
\[ y_j^i = \frac{\partial y^i}{\partial x^j}, \quad y_{jk}^i = \frac{\partial^2 y^i}{\partial x^j \partial x^k}, \quad j, i, k = 1, 2. \]

With the above notations the second prolongation of the operator \( Y \) can be written as
\[ Y^{(2)} = Y + \zeta^i_j \frac{\partial}{\partial y^j} + \zeta^i_{jj} \frac{\partial^2}{\partial y^j \partial y^j} + \tilde{\omega}^i_j \frac{\partial}{\partial f^j} + \tilde{\omega}^i_{jr} \frac{\partial}{\partial f^r}, \quad r = 1, 2, 3, \]
where the coefficients \( \zeta^i_j \) and \( \zeta^i_{jj} \) are given by
\[ \zeta^i_j = D_j \eta^i - y_k^i D_j \xi^k, \quad \zeta^i_{jj} = D_j \zeta^i_j - y_{jk}^i D_j \xi^k, \]
with
\[ D_j = \frac{\partial}{\partial x^j} + y_j^i \frac{\partial}{\partial y^i} + y_{jk}^i \frac{\partial}{\partial y^j}. \]

The remaining coefficients in (2.6) are obtained through the following prolongation formula
\[ \tilde{\omega}^i_j = \tilde{D}_j (\nu^r) - f^r_{jk} \tilde{D}_j (\xi^k) - f^r_{yr} \tilde{D}_j (\eta^r), \]
where
\[ \tilde{D}_j = \frac{\partial}{\partial x^j} + f^r_{xj} \frac{\partial}{\partial f^r}, \quad \tilde{D}_j = \frac{\partial}{\partial y^j} + f^r_{yr} \frac{\partial}{\partial f^r}. \]

The invariance of Eqs. (2.1a)-(2.1b) under the one-parameter group of equivalence transformations (2.3a)-(2.3g) can be written as [6]
\[ Y^{(2)} (v_t + f(t, x)w_{xx} + g(t, x)(v^2 + w^2)w + h(t, x)w) = 0, \]
\[ Y^{(2)} (w_t - f(t, x)v_{xx} - g(t, x)(v^2 + w^2)v - h(t, x)v) = 0, \]
\[ Y^{(2)} (f_v) = Y^{(2)} (f_w) = Y^{(2)} (g_v) = Y^{(2)} (g_w) = Y^{(2)} (h_v) = Y^{(2)} (h_w) = 0, \]
under the constraints that the variables \( v, w, f, g \) and \( h \) have to satisfy the equations (2.1a)-(2.1b) and (2.2).

Substituting the second prolongation (2.6) into (2.10)-(2.12) and solving the resultant equations (determining system) we find
\[ \xi^1 = \varphi(t), \quad \xi^2 = \psi(x), \quad \eta^1 = \left( a + \frac{\psi_x}{2} \right) v, \quad \eta^2 = \left( a + \frac{\psi_x}{2} \right) w, \]
\[ \nu^1 = (2\psi_x - \varphi_t)f, \quad \nu^2 = -(\psi_x + \varphi_t + 2a)g, \quad \nu^3 = -(h \varphi_t + \frac{1}{2} f \psi_{xxx}), \]
where $a$ is an arbitrary constant and $\varphi(t)$ and $\psi(x)$ are the arbitrary functions of $t$ and $x$ and subscripts denote partial derivatives.

The associated equivalence algebra $\mathcal{E}$ is an infinite dimensional one and is generated by the operators

$$Y_a = v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2g \frac{\partial}{\partial g}, \quad (2.14)$$

$$Y_\varphi = \varphi \frac{\partial}{\partial t} - f \varphi \frac{\partial}{\partial f} - g \varphi \frac{\partial}{\partial g} - h \varphi \frac{\partial}{\partial h}, \quad (2.15)$$

$$Y_\psi = \psi \frac{\partial}{\partial x} + \frac{1}{2} \psi_x \frac{\partial}{\partial v} + \frac{1}{2} \psi_x \frac{\partial}{\partial w} + 2f \psi_x \frac{\partial}{\partial f} - g \psi_x \frac{\partial}{\partial g} - \frac{1}{2} f \psi_{xxx} \frac{\partial}{\partial h}. \quad (2.16)$$

3. Differential invariants of the equivalence algebra

Differential invariants of order $n$ of the equivalence algebra $\mathcal{E}$ are not only functions of the independent variables $t$ and $x$ but also functions of $f$, $g$, $h$ and their derivatives up to the maximal order $n$ and invariant with respect to the equivalence operator $Y$.

3.1. Differential invariants of order zero

First let us seek differential invariants of order zero of the form

$$J = J(t, x, f, g, h). \quad (3.1)$$

Applying the invariant test $Y(J) = 0$ to the operators $Y_a, Y_\varphi$ with $\varphi = 1$ and $Y_\psi$ with $\psi = 1$ we find $J = J(f, h)$. Since $\psi_x$ and $\psi_{xxx}$ are functionally independent the invariant test $Y_\psi(J) = 0$ provides us the following two equations

$$\frac{\partial J}{\partial f} = 0, \quad \frac{\partial J}{\partial h} = 0. \quad (3.2)$$

As a result Eqs. (2.1a)-(2.1b) do not admit any differential invariant of order zero. In the following we seek higher order differential invariants, if any.

3.2. Differential invariants of first order

To obtain differential invariants of first order

$$J = J(t, x, f, g, h, f_t, f_x, g_t, g_x, h_t, h_x), \quad (3.3)$$

in which the invariant includes both the spatial and time derivatives of the functions $f, g$ and $h$, we consider the first prolongation of the operator $Y$ given by

$$Y^{(1)} = Y + \tilde{\omega}^r_j \frac{\partial}{\partial f^r_j}, \quad (3.4)$$

where the coefficients $\tilde{\omega}^r_j$ can be constructed from Eq. (2.8). The explicit forms of the first prolongations of the generators $Y_{a}^{(1)}, Y_{\varphi}^{(1)}$ and $Y_{\psi}^{(1)}$ are given by,

$$Y_{a}^{(1)} = v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} - 2g \frac{\partial}{\partial g} - 2g_x \frac{\partial}{\partial g_x} - 2g_t \frac{\partial}{\partial g_t}. \quad (3.5)$$
\[ Y_{\varphi}^{(1)} = \varphi \frac{\partial}{\partial t} - f \varphi_t \frac{\partial}{\partial f} - g \varphi_t \frac{\partial}{\partial g} - h \varphi_t \frac{\partial}{\partial h} - f_x \varphi_t \frac{\partial}{\partial f_x} - (2f_t \varphi_t + f \varphi_{tt}) \frac{\partial}{\partial f_t}, \]
\[ - g_x \varphi_t \frac{\partial}{\partial g_x} - (2g_t \varphi_t + g \varphi_{tt}) \frac{\partial}{\partial g_t} - h_x \varphi_t \frac{\partial}{\partial h_x} - (2h_t \varphi_t + h \varphi_{tt}) \frac{\partial}{\partial h_t}, \]  
\[ (3.6) \]

\[ Y_{\psi}^{(1)} = \psi \frac{\partial}{\partial x} + \frac{\psi_x}{2} \frac{\partial}{\partial \psi_x} + \frac{\psi_w}{2} \frac{\partial}{\partial \psi_w} + 2f \psi_x \frac{\partial}{\partial f} - g \psi_x \frac{\partial}{\partial g} \]
\[ - \psi_x \frac{\partial}{\partial \psi_x} - \frac{1}{2} \psi_{xxx} \frac{\partial}{\partial \psi_{xxx}}. \]  
\[ (3.7) \]

The differential invariant test concerned with the function \( J \) given by (3.3) reads
\[ Y_{\varphi}^{(1)}(J) = 0, \quad Y_{\psi}^{(1)}(J) = 0, \quad Y_{a}^{(1)}(J) = 0. \]  
\[ (3.8) \]

Since the arbitrary functions \( \varphi \) and \( \psi \) and their derivatives are to be treated functionally independent, Eqs. (3.8) can be split into the following conditions:
\[ \frac{\partial J}{\partial t} = 0, \quad \frac{\partial J}{\partial x} = 0, \quad \frac{\partial J}{\partial h_x} = 0, \]  
\[ (3.9) \]
\[ f \frac{\partial J}{\partial h} + f_t \frac{\partial J}{\partial f_t} = 0, \]  
\[ (3.10) \]
\[ 2f \frac{\partial J}{\partial f_x} - g \frac{\partial J}{\partial g_x} = 0, \]  
\[ (3.11) \]
\[ f \frac{\partial J}{\partial f_t} + g \frac{\partial J}{\partial g_t} + h \frac{\partial J}{\partial h_t} = 0, \]  
\[ (3.12) \]
\[ g \frac{\partial J}{\partial g} + g_x \frac{\partial J}{\partial g_x} + g_t \frac{\partial J}{\partial g_t} = 0, \]  
\[ (3.13) \]
\[ 4f \frac{\partial J}{\partial f} - 2g \frac{\partial J}{\partial g} + 2f_x \frac{\partial J}{\partial f_x} + 4f_t \frac{\partial J}{\partial f_t} - 4g_x \frac{\partial J}{\partial g_x} - 2g_t \frac{\partial J}{\partial g_t} = 0, \]  
\[ (3.14) \]
\[ f \frac{\partial J}{\partial f} + g \frac{\partial J}{\partial g} + h \frac{\partial J}{\partial h} + f_x \frac{\partial J}{\partial f_x} + 2f_t \frac{\partial J}{\partial f_t} + g_x \frac{\partial J}{\partial g_x} + 2g_t \frac{\partial J}{\partial g_t} + 2h_t \frac{\partial J}{\partial h_t} = 0. \]  
\[ (3.15) \]

We note that Eq. (3.14) can be simplified, with the use of (3.13), to
\[ 2f \frac{\partial J}{\partial f} + f_x \frac{\partial J}{\partial f_x} + 2f_t \frac{\partial J}{\partial f_t} - g_x \frac{\partial J}{\partial g_x} = 0. \]  
\[ (3.16) \]

As a result it is sufficient to solve Eqs. (3.9)-(3.13), (3.15) and (3.16) instead of Eqs. (3.9)-(3.15).

Eq. (3.9) simplify the differential invariant (3.3) to the form
\[ J = J(f, g, h, f_t, f_x, g_t, g_x, h_t). \]  
\[ (3.17) \]

Solving the characteristic equation associated with (3.10),
\[ \frac{dh}{f} = \frac{dh_t}{f_t}, \]  
\[ (3.18) \]
we get 
\[ \lambda_1 = \frac{hf_t}{f} - h_t. \] (3.19)

Hence the function \( J \) becomes \( J = J(f, g, f_t, f_x, g_t, g_x, \lambda_1) \).

Similarly from (3.11) we obtain 
\[ \lambda_2 = \frac{g}{f} f_x + 2g_x \] (3.20)

and so the differential invariant reduces to
\[ J = J(f, g, f_t, g_t, \lambda_1, \lambda_2). \] (3.21)

Substituting (3.21) into (3.12) and simplifying the resultant equation we arrive at
\[ f \frac{\partial J}{\partial f_t} + g \frac{\partial J}{\partial g_t} = 0, \] (3.22)

from which we get
\[ \lambda_3 = \frac{g}{f} f_t - g_t. \] (3.23)

As a consequence (3.21) can be further reduced to \( J = J(f, g, \lambda_1, \lambda_2, \lambda_3) \). Substituting the latter into (3.16) we obtain
\[ 2f \frac{\partial J}{\partial f} - \lambda_2 \frac{\partial J}{\partial \lambda_2} = 0, \] (3.24)

which upon integration yields the invariant
\[ \beta = f \lambda_2^2. \] (3.25)

Therefore, the differential invariant \( J \) assumes the form
\[ J = J(g, \lambda_1, \lambda_3, \beta). \] (3.26)

Now substituting (3.20) into (3.13) we get
\[ g \frac{\partial J}{\partial g} + \lambda_3 \frac{\partial J}{\partial \lambda_3} + 2\beta \frac{\partial J}{\partial \beta} = 0. \] (3.27)

The characteristic equations associated with the PDE (3.27) can be written as
\[ \frac{dg}{g} = \frac{d\lambda_3}{\lambda_3} = \frac{d\beta}{2\beta}. \] (3.28)

The invariant associated with the Eq. (3.28) can be easily found to be of the form
\[ \alpha_1 = \frac{\lambda_3}{g}, \quad \alpha_2 = \frac{\beta}{g^2}. \] (3.29)

At this point, we have
\[ J = J(\lambda_1, \alpha_1, \alpha_2) \] (3.30)

and Eq. (3.15) left unsolved. Substituting (3.30) into (3.15) and simplifying it we arrive at
\[ 2\lambda_1 \frac{\partial J}{\partial \lambda_1} + \alpha_1 \frac{\partial J}{\partial \alpha_1} + \alpha_2 \frac{\partial J}{\partial \alpha_2} = 0. \] (3.31)
By integrating the characteristic equations associated with the PDE (3.31), we arrive at the following result.

**Theorem 1.** The general form of the first order differential invariants of Eqs. (2.1a)-(2.1b) (or Eq. (1.1)), when \( \lambda_1 \neq 0 \), is

\[
J = J(\gamma_1, \gamma_2),
\]

(3.32)

where \( \gamma_1 \) and \( \gamma_2 \) are two independent invariants and their explicit forms read

\[
\gamma_1 = \frac{\alpha_1^2}{\lambda_1} = \frac{(gf_t - g t f)^2}{fg^2(hf_t - h tf)}, \quad \gamma_2 = \frac{\alpha_2^2}{\lambda_1} = \frac{(g f_x + 2 g x f)^4}{fg^4(hf_t - h tf)}.
\]

(3.33)

If \( \lambda_1 = 0 \), the corresponding form of Eqs. (2.1a)-(2.1b) (or Eq. (1.1)) should be considered separately. It is now easy to show that the cases \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \) are also exceptional. Therefore, we can state that the equations \( \lambda_1 = 0, \alpha_1 = 0 \) and \( \alpha_2 = 0 \) are invariant with respect to the equivalence algebra \( \mathcal{E} \).

### 4. Application of differential invariants

As stated earlier, our motivation is to map an equation of the form (1.1) to the standard NLS equation through equivalence transformations. To do this we make use of the differential invariant (or the invariant equations) which we derived in the previous section. In particular, we consider the following equation as target one

\[
i \hat{u}_t + k_1 \hat{u}_{xx} + k_2 |\hat{u}|^2 \hat{u} = 0, \quad k_1, k_2 \neq 0,
\]

(4.1)

where \( k_1 \) and \( k_2 \) are real constants. One can easily check that the Eq. (4.1) can be transformed to (1.2) easily.

We observe that Eq. (4.1) does not have any first order differential invariant, but admits invariant equations \( \lambda_1 = 0, \alpha_1 = 0 \) and \( \alpha_2 = 0 \). This observation leads us to formulate a necessary condition for an equation belonging to the class (1.1) that can be mapped through an equivalence transformation of \( \mathcal{G}_\mathcal{E} \) into (4.1). The condition is that the functions \( f, g \) and \( h \) must satisfy the following three equations

\[
g f_t - f g_t = 0, \quad h f_t - f h_t = 0, \quad g f_x + 2f g_x = 0.
\]

(4.2)

The most general forms of \( f, g \) and \( h \) satisfying (4.2) are

\[
f = f_0 \frac{n(t)}{l^2(x)}, \quad g = g_0 n(t) l(x), \quad h = n(t) m(x), \quad f_0, g_0, n(t), l(x) \neq 0,
\]

(4.3)

where \( l(x), m(x) \) and \( n(t) \) are real functions, while \( f_0 \) and \( g_0 \) are real constants. The above forms fix Eq. (1.1) of the form

\[
iu_t + f_0 \frac{n(t)}{l^2(x)} u_{xx} + g_0 n(t) l(x) |u|^2 u + n(t) m(x) u = 0.
\]

(4.4)

Eq. (4.4) can be further transformed into

\[
iu_t + \frac{1}{l^2(x)} u_{xx} + g_0 l(x) |u|^2 u + m(x) u = 0,
\]

(4.5)
by means of the equivalence transformation of $G_\xi$

$$
\hat{t} = \int^t \varphi(s) \, ds, \quad \hat{x} = x, \quad \hat{u} = u,
$$

with $\varphi(t) = n(t)$.

To transform (4.5) further into (4.1) we need to calculate the second order differential invariants of (4.5) or its associated real system

$$
v_\hat{t} + f_0 \frac{1}{l^3(x)} w_{xx} + g_0 l(x) (v^2 + w^2) w + m(x) w = 0,
$$

$$
w_\hat{t} - f_0 \frac{1}{l^3(x)} v_{xx} - g_0 l(x) (v^2 + w^2) v - m(x) v = 0.
$$

We look for the functions of the form

$$
J = J(\hat{t}, x, v, w, l, m, l_x, m_x, l_{xx}, m_{xx}),
$$

which are invariant with respect to the infinitesimal equivalence generators of (4.7)-(4.8), with the auxiliary conditions $l_i = m_i = 0$,

$$
\Upsilon = \xi^1 \partial_\xi + \xi^2 \partial_\nu + \eta^1 \partial_\nu + \eta^2 \partial_\nu + \mu^1 \partial_\nu + \mu^2 \partial_\mu
$$

and its appropriate prolongations.

Introducing additional change of variables

$$
f = \frac{f_0}{l^2}, \quad g = g_0 l, \quad h = m
$$

and following the procedure adopted in [26] for the change of variables from the old coordinates $\xi^1, \xi^2, \eta^1, \eta^2, \nu^1, \nu^2, \nu^3$ of the generator $Y$ to the new coordinates $\hat{\xi}^1, \hat{\xi}^2, \hat{\eta}^1, \hat{\eta}^2, \mu^1, \mu^2$ of $\Upsilon$, we obtain

$$
\hat{\xi}^1 = -\frac{4}{3} a \hat{t} + a_0, \quad \hat{\xi}^2 = \xi^2 = \psi(x),
$$

$$
\hat{\eta}^1 = \eta^1 = \left(a + \frac{\psi_x}{2}\right) v, \quad \hat{\eta}^2 = \eta^2 = \left(a + \frac{\psi_x}{2}\right) w,
$$

$$
\mu^1 = - \left(\psi_x + \frac{2}{3} a\right) l, \quad \mu^2 = - \frac{f_0}{2 l^2} \psi_{xxx} + \frac{4}{3} a m,
$$

with $a_0$ an arbitrary constant.

Considering the prolongation formula

$$
\Upsilon^{(2)} = \Upsilon + \omega^1_x \partial_\xi + \omega^2_x \partial_\nu + \omega^1_x \partial_\nu + \omega^2_x \partial_\nu,
$$

where

$$
\omega^1_x = \frac{\partial \mu^1}{\partial x} + l_x \frac{\partial \mu^1}{\partial l} + m_x \frac{\partial \mu^1}{\partial m} - l_x \frac{\partial \xi^2}{\partial x},
$$

$$
\omega^2_x = \frac{\partial \mu^2}{\partial x} + l_x \frac{\partial \mu^2}{\partial l} + m_x \frac{\partial \mu^2}{\partial m} - m_x \frac{\partial \xi^2}{\partial x},
$$

$$
\omega^1_{xx} = \frac{\partial \omega^1_x}{\partial x} + l_x \frac{\partial \omega^1_x}{\partial l} + m_x \frac{\partial \omega^1_x}{\partial m} + l_{xx} \frac{\partial \omega^1_x}{\partial l} + m_{xx} \frac{\partial \omega^1_x}{\partial m} - l_{xx} \frac{\partial \xi^2}{\partial x},
$$

$$
\omega^2_{xx} = \frac{\partial \omega^2_x}{\partial x} + l_x \frac{\partial \omega^2_x}{\partial l} + m_x \frac{\partial \omega^2_x}{\partial m} + l_{xx} \frac{\partial \omega^2_x}{\partial l} + m_{xx} \frac{\partial \omega^2_x}{\partial m} - m_{xx} \frac{\partial \xi^2}{\partial x}.$$
and repeating the same procedure followed in Sec. 4 we find that the system (4.7) - (4.8) (or Eq. (4.5)) does not possess any second order differential invariants but admits the following invariant equation

\[ m + \frac{1}{2} f_0 \left( \frac{3 l_x^2}{2 l^4} - \frac{l_{xx}}{l^3} \right) = 0. \]  (4.13)

By observing that (4.13) is also an invariant equation of (4.1), we can conclude that the necessary condition for the Eq. (1.1) to be mapped into (4.1) through the equivalence transformation of \( G_E \) is that the functions \( f, g \) and \( h \) must be related by the following relations

\[ f = f_0 \frac{n(t)}{l^2(x)}, \quad g = g_0 n(t) l(x), \quad h = \frac{1}{2} f_0 n(t) \left( \frac{l_{xx}}{l^3} - \frac{3 l_x^2}{2 l^4} \right). \]  (4.14)

Now, we verify whether the conditions (4.14) are also sufficient or not.

To do this, let us consider the equivalence transformation of \( G_E \), that is,

\[ \hat{t} = \int^t \varphi(s) ds, \quad \hat{x} = \int^x \psi(s) ds, \quad \hat{u}(\hat{t}, \hat{x}) = u(t, x) \sqrt{\psi(x)}, \quad \psi(x) > 0 \]  (4.15)

and substituting it into the equation

\[ iu_t + f_0 \frac{n(t)}{l^2(x)} u_{xx} + g_0 n(t) l(x) |u|^2 u + \frac{1}{2} f_0 n(t) \left( \frac{l_{xx}}{l^3} - \frac{3 l_x^2}{2 l^4} \right) u = 0, \]  (4.16)

we get

\[ \varphi(t) i \hat{u}_t + f_0 \frac{n(t)}{l^2(x)} \hat{u}_{\hat{x}\hat{x}} + g_0 \frac{n(t)}{\psi(x)} l(x) |\hat{u}|^2 \hat{u} \]

\[ + \frac{1}{2} f_0 n(t) \left( -\frac{\psi_{xx}}{\psi^3} + \frac{3 \psi_x^2}{2 \psi^4} + \frac{l_{xx}}{l^3} - \frac{3 l_x^2}{2 l^4} \right) \hat{u} = 0. \]  (4.17)

Eq. (4.17), with \( f_0 = k_1 \) and \( g_0 = k_2 \), reduces to (4.1) when the transformation (4.15) satisfies the following conditions:

\[ \varphi(t) = n(t), \quad \psi(x) = l(x). \]  (4.18)

As a result we demonstrated the following statement.

**Theorem 2.** An equation belonging to (1.1) can be transformed into the nonlinear Schrödinger equation (4.1) by an equivalence transformation of \( G_E \) if and only if the functions \( f, g \) and \( h \) are given by (4.14).

**Example.** Let us consider the equation

\[ iu_t + k_1 t^{\frac{1}{2}} x^2 u_{xx} + k_2 t^{\frac{3}{2}} x^{-1} |u|^2 u + \frac{1}{4} k_1 t^{\frac{1}{2}} u = 0. \]  (4.19)

Taking into account that \( \varphi(t) = t^{\frac{1}{2}} \) and \( \psi(x) = x^{-1} \). Using the change of variables (4.15) we get

\[ \hat{t} = \frac{2}{3} t^\sqrt{t}, \quad \hat{x} = \log x \quad \hat{u}(\hat{t}, \hat{x}) = \frac{u(t, x)}{\sqrt{x}}. \]  (4.20)
and rewriting \((t, x, u)\) in terms of \((\hat{t}, \hat{x}, \hat{u})\) we obtain
\[
t = \left(\frac{3}{2} \hat{t}\right)^{\frac{3}{2}}, \quad x = e^{\hat{x}}, \quad u(t, x) = e^{\hat{u}} \hat{u}(\hat{t}, \hat{x}).
\] (4.21)

Using (4.21) one can transform (4.19) into (4.1). Eq. (4.1), with \(k_1 = k_2 = 1\) admits the following solution
\[
\hat{u}(\hat{t}, \hat{x}) = \sqrt{2a} \exp\left[i \frac{V_e}{2} \hat{x} + i \left(a^2 - \frac{V_e^2}{4}\right) \hat{t}\right] \text{sech}\left[a(\hat{x} - V_e \hat{t} - x_0)\right].
\] (4.22)

Rewriting (4.22) in terms of the old variables (vide Eq. (4.20)) one gets
\[
u(t, x) = \sqrt{2a} \sqrt{x} \exp\left[i \frac{V_e}{2} \log x + i \left(a^2 - \frac{V_e^2}{4}\right) \frac{2t\sqrt{t}}{3}\right] \text{sech}\left[a(\log x - \frac{2}{3} V_e t \sqrt{t} - x_0)\right].
\] (4.23)

which is a solution of (4.19) with \(k_1 = k_2 = 1\).

5. Conclusions

In this paper we have constructed the equivalence transformations for the family of variable coefficient nonlinear Schrödinger equations (1.1). We have shown that the equivalence algebra is an infinite dimensional one. From the infinitesimal equivalence generators, by using the invariant test, we have shown that the family (1.1) does not admit zero order differential invariants but first order ones. As an application of the invariant relations we have characterized the most general NLS family of equations (1.1) that can be mapped to the standard NLS equation by an equivalence transformation. The explicit form of this latter one has also been given explicitly.

Acknowledgements

One of us (MT) wish to thank the members of the Centre for Nonlinear Dynamics, Bharathidasan University, for their warm hospitality. The work of MS forms part of a Department of Science and Technology, Government of India, sponsored research project. The work of MT and AV is supported by MIUR-COFIN 2003/05 through the project Nonlinear Mathematical Problems of Wave Propagation and Stability in Models of Continuous Media by the University of Catania through Progetti di Ricerca di Ateneo and by INdAM through G.N.F.M..

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