Overlap with the Separable State and Phase Transition in the Dicke Model: Zero and Finite Temperature

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Overlap with the separable state is introduced in this paper for the purpose of characterizing the overall correlation in many-body systems. This definition has clear geometric and physical meaning, and moreover can be considered as the generalization of the concept-Anderson Orthogonality Catastrophe. As an exemplification, it is used to mark the phase transition in the Dicke model for zero and finite temperature. And our discussion shows that it can faithfully reflect the phase transition properties of this model whether for zero or finite temperature. Furthermore the overlap for ground state also indicates the appearance of multipartite entanglement in Dicke model.

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I. INTRODUCTION

Correlation in condensed matter systems predominates the understanding of many-body effects. Fundamentally one can define different correlation functions for describing the unusual connections in many-body systems. For instance it is general to introduce the order parameter for the description of phase transitions induced by local perturbation, and furthermore to classify the diverse phase transitions by scaling the singularity of correlation functions with the universal critical exponents. That is so called Landau-Ginzburg-Wilson(LGW) paradigm. However the situation becomes different for the strongly correlated electronic systems. The quantum Hall effect appearing in two-dimensional electron gas with high magnetic field shows the distinct features not captured by LGW paradigm. Instead topological order consequently is defined to describe the underlying symmetry in quantum Hall systems, which is distinct from the notion of spontaneously broken symmetry.

Recently the extensive researches of quantum entanglement in condensed matters show the potentiality that quantum entanglement would act the universal description for many-body effects. Especially some general conclusions have been obtained about the connection between quantum entanglement and quantum phase transition in many-body systems. The concurrence, a measurement of two-party entanglement, has been constructed exactly in one dimensional spin-chain systems, and the similar behavior for single-copy entanglement is also founded. Recently the entanglement spectrum has been defined to obtain the general information about the many-body systems. As for quantum Hall systems, it is shown that the scaling behavior of entanglement entropy is directly related to the quantum number, which is used to characterize the topological order. And entanglement spectrum can also be used to detect the non-local features of quantum Hall systems.

Although these important progresses have been made, there are a few exceptions that lead to the suspicion of the validity of quantum entanglement as an universal description for many-body effects. Entanglement entropy sometimes provides ambiguous information about the phase transitions in higher dimensional many-body systems. Even for one dimensional systems, it cannot present the complete information in some situations. As an example, the recent studies show that the block entanglement entropy for the Valence-Bond-Solid(VBS) state of integer spin seems unsensible to the degeneracy manifested by the underlying topological symmetry and also does not display the dependence on the parity of spin number $s$, which however both can be manifested clearly by introducing string order parameter. As for quantum Hall systems, the entanglement entropy and entanglement spectrum have also been shown the limited ability of identifying the topological orders.

In my point, this defect would attribute to the trace-out of the superfluous degrees of freedom when one obtains the reduced density matrix. And some information for the global features in many-body systems is inevitably lost. This point has been exemplified in a recent paper of our group, in which the geometric entanglement (GE) as a measurement of multipartite entanglement is calculated for VBS state. The interesting result in this paper is that GE displays two different scaling behaviors dependent on the parity of spin number $s$, and the global GE is divergent linearly with the particle number.

Through this short introduction, it seems promising to measure multipartite entanglement in order to obtain the
complete information for many-body effects. Recently some efforts have been made in this direction. The connection of multipartite entanglement and quantum phase transition has been discussed in some special models [14–17]. However the crucial obstacle for further development is the absence of the unified understanding of the multipartite entanglement [18,19]. Whereas the maximally entangled state can be defined unambiguously for bipartite systems, what is the maximally entangled state for multipartite systems is unclear until now [19]. Fortunately it is well accepted that the fully separable state can be defined as

$$\rho^{\text{sep}} = \sum_i p_i \rho_1^{(i)} \otimes \rho_2^{(i)} \otimes \cdots \rho_N^{(i)},$$

where $N$ is the particle number, and $p_i$ denotes the common probability with which the single-particle state $\rho_n^{(i)}$ $(n = 1,2,\ldots,N)$ happens. With respect of this point, geometric entanglement (GE) is introduced first by Shimony for pure bipartite state [20] and generalized to the multipartite case by Carteret et al. [21], Barnum and Linden [22], Wei and Goldbart [23], and to the mixed state by Cao and Wang [24]. GE is a genuine multipartite entanglement measurement. The main idea of GE is to minimize the distance $D$ between the state $|\Psi\rangle$ to be measured and the fully separable state $|\Phi\rangle$ in Hilbert space,

$$D = \min_{|\Phi\rangle} \{|||\Psi\rangle - |\Phi\rangle||^2\}. \quad (2)$$

For the normalized $|\Psi\rangle$ and $|\Phi\rangle$, the evaluation of $D$ is reduced to find the maximal overlap [23]

$$\Lambda(\Psi) = \max_{\{|\Phi\rangle\}} \langle\Phi|\Psi\rangle. \quad (3)$$

Geometrically $\Lambda(|\Psi\rangle)$ depicts the overlap angle between the vectors $|\Psi\rangle$ and $|\Phi\rangle$ in Hilbert space. Then the larger $\Lambda(\Psi)$ is, the shorter is the distance and the less entangled is $|\Psi\rangle$. But the optimum is in general a forbidden task, not spoken for mixed state, and the analytical results can be obtained only for some very special cases [16,17,24]. Recently many efforts are devoted to the reduction of the optimum and some interesting results are obtained [23].

Given this difficulty, we introduce another different quantity in this paper to capture the overall correlation in condensed matter systems, i.e. the overlap with a special fully separable state. The starting point is still to find the minimal distance between the state to be measured and a special fully separable state defined in the next section. In contrast to GE the optimum can be reduced by utilizing the geometric property of the overlap, and this overlap has very clear physical meaning, whether for pure or mixed state. In Sec.II, the definition of this overlap is introduced, and the differences with several known similar definitions are clarified. Furthermore we point out that our definition is connected intimately with the concept of Anderson Orthogonality Catastrophe (AOC) [26,27]. As an illustration of the validity of our definition, the collective phase transition appeared in Dicke model is discussed by this quantity in Sec.III. Multipartite entanglement in this model is also studied for displaying the potential connection between this overlap and multipartite entanglement. Finally conclusions and further discussion are presented in Sec.IV.

**II. OVERLAP WITH FULLY SEPARABLE STATE**

Similar to the introduction of GE, our starting point is also to find the minimal distance $D$ between the fully separable state $\rho^{\text{sep}}$ and the state $\rho$ to be measured.

$$D = \min_{\{\rho^{\text{sep}}\}} \{||\rho - \rho^{\text{sep}}||^2\}. \quad (4)$$

Generally this minimal distance is still decided mainly by the maximal overlap

$$\Lambda = \max_{\{\rho^{\text{sep}}\}} \text{Tr}[\rho\rho^{\text{sep}}]. \quad (5)$$

The density matrix can also be written as the Bloch form

$$\rho = (I + \sum_{i=1}^{d^2-1} r_i \lambda_i)/d, \quad (6)$$

where $d$ denotes the dimension, $\lambda_i$ is the generator of SU($d$) group and $\{r_i\}$ is so called Bloch vector [30]. Thus $\Lambda$ has clear geometric meaning which depicts the minimal overlap angle $\theta$ between the Bloch vectors $\{r_i\}$ and $\{r_i\}^{\text{sep}}$ in the Bloch-vector space, i.e.,

$$\max_{\{\rho^{\text{sep}}\}} \text{Tr}[\rho\rho^{\text{sep}}] = \frac{1}{d}(1 + \langle r_i\rangle\langle r_i\rangle^{\text{sep}}\cos\min\theta). \quad (7)$$

Two limit cases are beneficial to the understanding of the physical meaning of $\theta$. For $\cos\theta = 1$, the overlap is maximal and $\rho$ and $\rho^{\text{sep}}$ share the same physical characters since Bloch vector $\{r_i\}$ is the reflection of the intrinsic symmetry in the systems [30]. While for $\cos\theta = -1$ one has minimal overlap, and $\rho^{\text{sep}}$ shows distinct properties from that of $\rho$.

In contrast to the Bures fidelity [31], the overlap $\Lambda$ have clear geometric meaning whether for pure or mixed state, as shown above. Furthermore by this geometric meaning, the optimal procession can be reduced to find the fully separable state $\rho^{\text{sep}}$ sharing the same physical properties with $\rho$ (see Appendix A for a proof). Moreover this definition is more popular than Eq.(3). First $\Lambda$ comes back to the form Eq.(3) for pure states. Second Eq.(5) includes the case when one state is pure and the other is mixed. This situation always happens as exemplified in Ref. [24], but is not covered in the original discussion [23]. Thirdly for mixed state the geometric characters of GE becomes
ambiguous because of the *convex roof* construction \[23\], while the geometric meaning of \( \Lambda \) is clear whenever for pure or mixed state.

With these advantages, the evaluation of \( \Lambda \) however is difficult for mixed state \( \rho^{sep} \) since there are infinite possibilities of the decomposition for \( \rho^{sep} \). Recently we note a popular concept in condensed matter physics-Anderson’s Orthogonality Catastrophe(AOC)\[26, 27\], which refers to the vanishing of the overlap between the many-body ground states with and without the potential as a power law in the number of particles in the systems. AOC is defined as

\[
\Delta = |\langle \Phi | \Phi^p \rangle|^2, \tag{8}
\]

where \( |\Phi^p \rangle \) and \( |\Phi \rangle \) correspond respectively to the many-body states with the potential and the state described entirely in terms of free plane waves, including the ground state of the unperturbed system\[26\]. Anderson proved entirely in terms of free plane waves, including the ground state with the potential as a power law in the number of particles in the systems. AOC is defined as

\[
\Delta = \max_{\{\rho^s\}} \text{Tr}[\rho \rho^s], \tag{10}
\]

This definition is the main contribution in this paper, and has some distinct advantages, summarized as following:

1. \( \Delta \) has clear geometric meaning, which depicts the minimal overlap angle between the Bloch vectors \( \{r_i\} \) and \( \{r_i\}_{\ast} \). And for pure states, it returns to the original definition Eq.(3) of GE.

2. By this geometric meaning, the optimal process in \( \Delta \) can be reduced to find the fully separable state \( \rho^s \) sharing the same physical features with \( \rho \).

3. \( \Delta \) can be regarded as the generalization of AOC to mixed state, and can faithfully reflect the overall correlation in many-body systems.

It should emphasize that this definition does not try to present a complete measurement of the multipartite entanglement. Instead our purpose is to find an universal method to characterize the overall correlation in many-body systems, whether quantum or classical. However this definition is also meaningful to find the unified understanding of multipartite entanglement in many-body systems. As shown in the next section, \( \Delta \) indeed presents the interesting information for the phase transition in Dicke model. And moreover the connection between \( \Delta \) and multipartite entanglement in Dicke model has also been discussed in Sec.III. Additionally in contrast to the recent interest in the fidelity for many-body systems\[32\], \( \Delta \) does not serve for the state discrimination.

### III. EXEMPLIFICATION: PHASE TRANSITION IN THE DICKE MODEL

In order to demonstrate the generality of this definition, the phase transition in Dicke model is discussed by \( \Delta \) in this section. Dicke model describes the dynamics of \( N \) independently identical two-level atoms coupling to the same quantized electromagnetic field\[28\]. Due to the presence of dipole-dipole force between atoms, Dicke model shows the normal-superradiant transition\[34\].

Dicke model is related to many fundamental issues in quantum optics, quantum mechanics and condensed matter physics, such as the coherent spontaneous radiation\[34\], the dissipation of quantum system\[35\], quantum chaos\[36\] and atomic self-organization in a cavity\[37\]. The normal-superradiant transition have been first observed with Rydberg atoms\[33\], and recently in a superfluid gas coupled to an optical cavity\[37\] and nuclear spin ensemble surrounding a single photon emitter\[38\]. Quantum entanglement in Dicke model has also been discussed extensively in\[39, 40\]. Furthermore Dicke model is also related to the issues of how the opened multipartite systems is affected by the environment and the robustness of multipartite entanglement\[42\].

The Hamiltonian for single-model Dicke model reads

\[
H = \omega a^\dagger a + \frac{\omega_0}{2} \sum_{i=1}^{N} \sigma_i^z + \frac{\lambda}{\sqrt{N}} \sum_{i=1}^{N} (\sigma_i^+ \sigma_i^-)(a^\dagger + a) = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}} (a^\dagger + a)(J_+ + J_-), \tag{11}
\]
where \( J_z = \sum_{i=1}^{N} \sigma_i^z / 2 \) and \( J_\pm = \sum_{i=1}^{N} \sigma_i^\pm \) are the collective angular momentum operators. At zero temperature, the normal-superradiant transition happens when \( \lambda = \lambda_c = \sqrt{\omega \omega_0} / 2 \). For finite temperature, the critical temperature is decided by the relation

\[
\beta_c = \frac{\omega_0 \tanh(\beta \omega / 2)}{2 \lambda^2 \tanh(\beta \omega_0 / 2)}.
\]  

(12)

An intrinsic property of Dicke model is the parity symmetry,

\[
[H, \Pi] = 0, \\
\Pi = e^{i \pi (a^\dagger a + J_z + 2)}. 
\]  

(13)

Moreover Eq.(11) is obviously permutation invariant by exchanging any two atoms.

With these information, the overlap \( \Delta \) for the Dicke model is studied explicitly for zero and finite temperature in the following two subsections. Some interesting features of \( \Delta \) are displayed.

\[ H^{(1)} = \omega a^\dagger a + \omega_0 b^\dagger b + \lambda (a^\dagger + a) (b^\dagger + b) - \frac{N}{2} \omega_0, \quad \lambda < \lambda_c; \]

\[ H^{(2)} = \omega a^\dagger a + \left[ \omega_0 + \frac{2}{\omega} (\lambda^2 - \lambda_c^2) \right] b^\dagger b + \frac{(\lambda^2 - \lambda_c^2)(3\lambda^2 + \lambda_c^2)}{2\omega (\lambda^2 + \lambda_c^2)} (b^\dagger + b)^2 \]

\[ + \frac{\sqrt{2}\lambda^2}{\sqrt{\lambda^2 + \lambda_c^2}} (a^\dagger + a) (b^\dagger + b) + \text{const.}, \quad \lambda > \lambda_c. \]  

(15)

A. Zero Temperature

With respect of the permutation invariance of atoms in Eq. (11), it is convenient to introduce the Holstein-Primakoff(HP) transformation

\[
J_z = b^\dagger b - \frac{N}{2} \\
J_+ = b^\dagger \sqrt{N - b^\dagger b} \\
J_- = \sqrt{N - b^\dagger b}. \]  

(14)

with bosonic operator \( b^{(1)} \). Semiclassically there is a ground state with \( J_z = -N/2 \) for Dicke model under thermodynamic limit \( N \to \infty \). Hence it is reasonable to adopt the low-excitation approximation at zero temperature, and then one obtains two effective Hamiltonians for different regions of \( \lambda \) (refer to [44] for details).

The step is to decide the fully separable state \( \rho^s \) for atom system. As mentioned in Sec.II and proved in Appendix A, the optimum process in Eq.(4) can be reduced to find \( \rho^s \) sharing the same global features with the ground state Eq.(17). First with the requirement of the permutation invariance of atoms in Dicke model, the single atomic state should have the same form in \( \rho^s \), i.e. \( \rho_i = \varrho, i = 1, 2, \ldots, N \), and then

\[ \rho^s = \varrho^{\otimes N}. \]  

(19)

Second the parity symmetry for Dicke model is reduced for single atom state \( \rho \) as

\[ [e^{i \pi J_z}, \rho^s] = 0 \]

\[ \Rightarrow [e^{i \pi \sigma_z}, \varrho] = 0. \]  

(20)

Thus one has under \( \sigma_z \) representation

\[ \varrho = \begin{pmatrix} a & 0 \\ 0 & 1 - a \end{pmatrix}. \]  

(21)
for this plotting, and the critical point is $\lambda_c = 0.5$ in this case. The inset is a plotting for the purity of the reduced density of atomic freedom under $N \to \infty$.

Finally with requirement of Eq. (18), $a = 1/2 + \frac{(J_z)}{N}$. Thus $\rho^s$ can be uniquely determined as

$$\rho^s = \left( \frac{1}{2} + \frac{(J_z)}{N}, 0 \right)^{\otimes N}.$$  

(22)

It should point out that the procedure for the determination of $\rho^s$ is popular whether for zero or finite temperature.

For the evaluation of the overlap $\Delta$, it should note that $\rho^s$ can be rewritten as the following contract form under the $J_z$ representation,

$$\rho^s = \sum_{n=1}^{N} N C_n a^n (1 - a)^{N - n} |n - \frac{N}{2} \rangle \langle n - \frac{N}{2}|,$$

(23)

where $N C_k$ denotes the binomial function, and $|n - \frac{N}{2}\rangle$ presents the state for which $n$ particles is spin-up and the other is spin-down. Together with the HP transformation Eq. (13), it is obvious

$$b^\dagger b |n - \frac{N}{2}\rangle = n |n - \frac{N}{2}\rangle.$$  

(24)

Then $\Delta$ can be evaluated easily under this representation.

In Fig. 1 the overlap $\Delta$ with $\rho^s$ is plotted. At normal phase $(\lambda < \lambda_c)$, one has $\langle J_z \rangle / N = -0.5$, $a = 0$, and then $\rho^s$ is the fully separable pure state. $\Delta$ is determined mainly by the first diagonal element of the reduced density matrix of atom system in this special case. While for $\lambda > \lambda_c$, $\Delta$ shows a sudden rising and then decreases with the increment of $\lambda$, and tends to be steady with $\lambda \to \infty$. Moreover under $N \to \infty$, $\Delta$ tends to be vanishing. Then two different phases can be clearly identified by evaluating $\Delta$.

Some intricate features of the phase transition can be disclosed by $\Delta$. For normal phase $\lambda < \lambda_c$, it is known that the atom system becomes entangled with the electromagnetic field, and attains the maximal value at the critical point $[10]$. The entanglement leads the state of atom system to be mixed, and the purity of its reduced density is decreased as shown by the inset in Fig. 1. At the same time the pairwise entanglement between two any atoms is also raised mediated by their couplings to the electromagnetic field, and the atoms become correlated with each other $[10]$. These intrinsic properties can be captured by $\Delta$ at the same time. For normal phase $\rho^s$ is pure and fully separable. Thus the decrement of $\Delta$ reflects the fact that the atoms become correlated with each other, and attains the maximal correlation at the critical point, at which $\Delta$ has minimal value. Furthermore since there is no interaction among atoms, the only reason for the construction of correlation in atoms is the couplings to the same electromagnetic field, which just induces the state for atom system to be mixed. This feature can also manifested by the decrement of $\Delta$ with respect that $\rho^s$ is pure.

For superradiant phase $\lambda > \lambda_c$, it is known that the entanglement between the atoms and electromagnetic field decreases monotonously to a steady value with the increment of $\lambda$, while the pairwise entanglement in atoms disappears asymptotically $[10]$. Contrastly the purity for the state of atom system has a sudden increasing closed to $\lambda_c$, and then decreases to a steady value, as shown by the inset of Fig. 1. The two different behaviors can also be captured by $\Delta$. Similar to the behavior of the purity of the state for atom system, $\Delta$ has also a sudden arising closed to $\lambda_c$ and then decreases to a steady value with the increment of $\lambda$. With respect that $\rho^s$ is mixed in this case and its purity is monotonically decreased with the increment of $\lambda$, the abrupt increment of $\Delta$ means that the sudden recovery of the purity of atomic system is at the expense of the reduction of correlation in atoms. It is obvious from Fig. 1 that $\Delta$ tends to be zero with the increment of $N$ for large $\lambda$. However the vanishing of $\Delta$ cannot attribute to the mixedness of $\rho^s$ since the steady value of $\Delta$ for finite $N$ is always bigger than the maximal mixedness $1/N$, manifested by Fig. 1. This feature means that the correlation in atoms still exists. Since the pairwise entanglement of atoms is known to be vanished in this limit $[10]$, the correlation in atoms must be global.

The scaling behavior of $\Delta$ near the critical point show some interesting features. At the normal phase ($\lambda < \lambda_c$), one has for $\omega = \omega_0 = 1$

$$\Delta = \frac{q^{2/3} (1 - 4\lambda^2)^{1/4}}{[1 + 3\sqrt{1 - 4\lambda^2} + 0.5(\sqrt{1 + 2\lambda} + \sqrt{1 - 2\lambda})^3]}.$$  

(25)

Similar to the method in Ref. [17], one can define the globe overlap $-\ln \Delta$ to measure the atomic correlation in Dicke model. It is obvious that the globe overlap is mainly determined by $(1 - 4\lambda^2)^{1/4}$ near $\lambda_c = 1/2$, and
then
\[-\ln \Delta \sim -\frac{1}{4} \ln(1 - \frac{\lambda}{\lambda_c}),\] (26)
which is same to the scaling behavior of multipartite entanglement in the Lipkin-Meshkov-Glick (LMG) model\[17\]. This result is not strange since Dicke model and LMG model belong to the same universality class. However it strongly implies that \(\Delta\) could be correlated directly to the multipartite entanglement in Dicke model. As shown in Sec.IV, the atom system indeed displays the multipartite entanglement in this case.

**B. Finite Temperature**

At finite temperature the phase transition is induced by thermal fluctuation. In order to determine the critical temperature, the general method is to evaluate the partition function \(z\). In Ref.\[43\], \(z\) has been obtained analytically
\[
z = \frac{\sqrt{1/2\pi}}{1 - e^{-\beta \Delta}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} \times \left\{ 2 \cosh[\beta \sqrt{\frac{\omega_0^2}{4} + \frac{x^2 \lambda^2}{N} \coth \frac{\beta \omega}{2}}] \right\}^N (27)
\]
and the critical temperature is determined by Eq.(12). For \(\omega = \omega_0\), it is reduced to \(T_c = 2\lambda^2/k_B\omega_0\). With the same trick used in \[43\], the overlap \(\Delta\) can also be written analytically as (see Appendix B for the details of calculation)
\[
\Delta = \frac{1}{z} \frac{\sqrt{1/2\pi}}{1 - e^{-\beta \Delta}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} \times \left\{ 2 \cosh \left[ \beta \sqrt{\frac{\omega_0^2}{4} + \frac{x^2 \lambda^2}{N} \coth \frac{\beta \omega}{2}} \right] + \frac{\omega_0 (1 - 2a) / 2}{\sqrt{\frac{\omega_0^2}{4} + \frac{x^2 \lambda^2}{N} \coth \frac{\beta \omega}{2}}} \sinh \left[ \beta \sqrt{\frac{\omega_0^2}{4} + \frac{x^2 \lambda^2}{N} \coth \frac{\beta \omega}{2}} \right] \right\}^N (28)
\]
As shown in Fig.2 \(\Delta\) can clearly detect the phase transition by its abrupt variance closed to the critical line. With respect that \(\rho^s\) is mixed and fully separable in this case, \(\Delta\) reflects that the correlation in atoms exists even for finite temperature. However this type of correlation is obviously induced by the thermal fluctuation, and thus is incoherent in contrast to that for zero temperature. This difference will become clear if one focuses on the multipartite entanglement of atoms in the next section.

**IV. MULTIPARTITE ENTANGLEMENT IN DICKE MODEL**

Another interesting aspect for Dicke model is the multipartite entanglement in atoms. Since all atoms simultaneously couple isotropically to the same electromagnetic field, then it is expected that the multipartite correlation of atoms could be readily constructed in this case.

However the measure of multipartite entanglement is a difficult task in general, especially for mixed state. An indirect way of resolving this difficulty is to find the characters uniquely belonging to the fully separable state Eq.(1), and the violation of these properties implies the appearance of multipartite entanglement. Spin squeeze-
ing is one of the most successful approaches to the multipartite entanglement in this way \[45\]. Recently G. Tóth, et al. provides a series of inequalities about spin squeezing to identify the multipartite entanglement in collective models \[10\],

\[
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4}, \quad (29a)
\]

\[
\Delta^2 J_x + \Delta^2 J_y + \Delta^2 J_z \geq \frac{N}{4}, \quad (29b)
\]

\[
\langle J_\alpha^2 \rangle + \langle J_\beta^2 \rangle - \frac{N}{2} \leq (N-1)\Delta^2 J_\gamma, \quad (29c)
\]

\[
(N-1) \left[ \Delta^2 J_\alpha + \Delta^2 J_\beta \right] \geq \langle J_\gamma^4 \rangle + \frac{N(N-2)}{4}, \quad (29d)
\]

where \(\alpha, \beta, \gamma\) adopt the all permutation of \(x, y, z\), and \(\Delta^2 J_\alpha = \langle J_\alpha^2 \rangle - \langle J_\alpha \rangle^2\). The violation of any one of these inequalities implies the appearance of entanglement \[10\]. With respect of the limit large \(N\), these inequalities can be rewritten as

\[
\frac{1}{N^2} \langle J_\alpha^2 \rangle + \frac{1}{N^2} \langle J_\beta^2 \rangle + \frac{1}{N^2} \langle J_\gamma^2 \rangle \leq \frac{1}{4}, \quad (30a)
\]

\[
\frac{1}{N^2} \left[ \Delta^2 J_\alpha \right] + \frac{1}{N^2} \left[ \Delta^2 J_\beta \right] + \frac{1}{N^2} \left[ \Delta^2 J_\gamma \right] - \frac{1}{N^2} \geq 0, \quad (30b)
\]

\[
\frac{1}{N} \left[ \langle J_\alpha^2 \rangle + \langle J_\beta^2 \rangle \right] - \frac{1}{N} \langle J_\gamma^2 \rangle \geq 0, \quad (30c)
\]

\[
\frac{1}{N} \left[ \Delta^2 J_\alpha + \Delta^2 J_\beta \right] - \frac{1}{N} \langle J_\gamma^2 \rangle \geq \frac{1}{4}, \quad (30d)
\]

in which \(\frac{1}{N^2} \langle J_\alpha^2 \rangle\) and \(\frac{1}{N} \Delta^2 J_\alpha\) are equivalent to evaluate the average \(\langle J_\alpha/N \rangle^2\) and \(\Delta^2 (J_\alpha/N) = \langle J_\alpha/N \rangle^2 - \langle J_\alpha/N \rangle^2\). For large \(N\), these inequalities have nontrivial result since the average magnetization per particle and its fluctuation still have nonvanishing values. It should point that Eq. (30a) is obviously satisfied for arbitrary state. So the following discussion is mainly about Eqs. (30b)-(d).

A. Zero Temperature

The evaluations of \(\langle J_\alpha/N \rangle\) and \(\langle (J_\alpha/N)^2 \rangle\) can be implemented readily through Bogoliubov transformation \[44\]. Our calculations show that Eq. (30b) is always satisfied at both normal and superradiant phases. In Fig. 4 several situations for Eqs. (30) have been plotted with limit \(N \to \infty\), and the others can be proved to be bigger than zero. The violation implies that the atoms should be entangled. Moreover since the pairwise entanglement between atoms is known to be vanished with increment of \(\lambda \[40\], this entanglement is sure to be multipartite. Furthermore there is also a sudden increment closed to the critical point, similar to the behavior of \(\Delta\) shown in Fig. 1. This feature means that there is a sudden reduction of the correlation of atoms, and \(\Delta\) can also be used to detect the entanglement of atoms in Dicke model at zero temperature.

B. Finite Temperature

At finite temperature, the evaluations of \(\langle J_\alpha/N \rangle\) and \(\langle (J_\alpha/N)^2 \rangle\) can adopt the same trick used in Ref. \[43\] (also shown in Appendix B). In Figs. 3 Eqs. (30b)-(d) have been plotted with all possible permutation of \(x, y, z\). It is obvious that all inequalities are satisfied simultaneously, and then one can conclude that there is no quantum entanglement of atoms in this case. This result is not surprising since the thermal fluctuation is dominant at finite temperature, and is considered to be incoherent. Although the absence of quantum correlation, the correlation induce by thermal fluctuation predominates, as shown in Figs. 2 by \(\Delta\), which means that \(\Delta\) can also be used to detect the thermal correlation.

V. CONCLUSIONS AND FURTHER DISCUSSION

In this paper, the overlap \(\Delta\) with a special fully separable state defined in Eq. (10) is introduced, in order to capture the overall correlation in many-body systems, whether quantum or classical. \(\Delta\) has clear geometric and physical meaning shown in Sec.II. With these features, the optimum process in the definition of \(\Delta\) can be reduced to find the fully separable state \(\rho^s\) defined in Eq. (9), which shares the same physical properties with the state to be measured. Importantly \(\Delta\) can be considered as the generalization of the concept Anderson’s Orthogonality Catastrophe \[20, 27\], which is critical for the understanding of some effects in condensed matter physics. This important connection displays the popularity of \(\Delta\) to detect the global correlation in many-body systems. And as an exemplification the phase transition in Dicke model has been discussed by \(\Delta\).

As shown in Sec.III, \(\Delta\) unambiguously depicts the phase transition features and the global correlation in Dicke model, whether for zero or finite temperature. At zero temperature, \(\Delta\) displays the distinct behaviors across the critical point. Furthermore with the information of \(\rho^s\), \(\Delta\) predicts the appearance of the multipartite entanglement in atom system, as verified in Sec.IVA.
implies that the correlation in atoms would exist even under high temperature. Moreover under the $J_z$ representation, the dimension is proportional to the atomic number $N$, and the value of the overlap shown in Figs. 2 has exceeded greatly the limit by $N$. Thus this phenomenon cannot attribute to the mixedness of the state for atom system. Unfortunately we do not know how to understand these two different features.

Although $\Delta$ cannot present a complete measurement of multipartite entanglement, it has been shown the intimate connection to the quantum entanglement in some special cases, such as the discussion for Dicke model at zero temperature in this paper. From this discussion $\Delta$ would presents a complete description for the global correlation in many-body systems, whether quantum or classical. So it is not surprising that $\Delta$ can be used to identify the quantum entanglement in some special cases. However it is difficult to answer the question what the general relation between $\Delta$ and quantum entanglement is since it is difficult to answer the question what the general relation between $\Delta$ and quantum entanglement is since the absence of the unified understanding of multipartite entanglement. This point will be studied in the future publication.

Appendix A: Find the nearest $\rho^*$ for a definite $\rho$

For two arbitrary density matrixes $\rho_1$ and $\rho_2$, they always has the following decompositions simultaneously

$$\rho_1 = \sum_n p_n^{(1)} |n\rangle_1 \langle n|$$
$$\rho_2 = \sum_m p_m^{(2)} |m\rangle_2 \langle m|,$$

(31)

where $p_n^{(m)}$ denotes the probability that the system is being in the state $|n(m)\rangle_{1(2)}$. It should emphasize that it is unnecessary for the states labeled by different $n$ or $m$ to be orthogonal with each other. And thus the decompositions above can always be realized at the same time. Then the overlap between $\rho_1$ and $\rho_2$ reads

$$\text{Tr}[\rho_1 \rho_2] = \sum_{m,n} p_n^{(1)} p_m^{(2)} |\langle m|n\rangle_1|^2.$$

(32)

Obviously the maximization of overlap is dependent on the inner product $|\langle m|n\rangle_1|^2$. It is well known that for two different states $|v\rangle$ and $|w\rangle$ their inner product is bounded by Cachy-Schwartz(CS) inequality, i.e.,

$$|\langle v|w\rangle|^2 \leq \langle v|v\rangle \langle w|w\rangle,$$

(33)

where the equality occurs if and only if the two vectors $|v\rangle$ and $|w\rangle$ in Hilbert state are linearly related, i.e. $|v\rangle = c|w\rangle$ for some scalar $c$. The important point for this condition is that $c$ is not necessary a constant, for which $|v\rangle$ and $|w\rangle$ become physically identical, and CS inequality has trivial consequence. Thus

$$\text{Tr}[\rho_1 \rho_2] \leq \sum_{m,n} p_n^{(1)} p_m^{(2)} |\langle m|n\rangle_1 |\langle m|n\rangle_2|,$$

(34)
where the equality occurs if and only if for arbitrary \( |n\rangle_1 \) and \( |m\rangle_2 \) they are still linearly related. But in this case the scalar \( c \) has to be dependent on both \( n \) and \( m \), i.e. \( c = c_{mn} \), which means that any \( |n\rangle_1 \) have to be linearly related to all \( |m\rangle_2 \). An interesting consequence for this condition is \( [\rho_1, \rho_2] = 0 \), which means that \( \rho_1 \) and \( \rho_2 \) share the same set of eigenvectors, and thus they share the same global symmetry and belong to the same space.

This conclusion is not strange if one notes that the overlap between two matrixes is mainly determined by the inclusion relation of the spaces decided by the matrixes. As an example let consider two matrixes belong to two completely different spaces. And then the overlap must be zero since mathematically the intersection of the two spaces is null and there is no crossing items between the two matrixes. Comparably if one space is the subspace or equivalent to the other space, the overlap then is nontrivial generally since the two matrixes belong to the same space. Hence in order to find the maximal overlap between two matrixes, it is also necessary for the two matrixes to be in the same space. From physical point, it means that the two operator is necessary commutative. Furthermore it is easy to understand why the maximal GE for pure entangled state always happens for purely separable state.

As for the determination of \( \rho^s \) in Eq.\( (\text{[10]} \) \), it is required for \( \rho^s \) to be commutative to \( \rho \), i.e. \( [\rho, \rho^s] = 0 \), which means that \( \rho^s \) shares the same global symmetry with \( \rho \). With this point one can determine \( \rho \) as Eq.\( (\text{[21]} \). Furthermore since \( \rho^s \) is diagonal under the collective basis \( \{|n - \frac{N}{2}\rangle, n = 0, 1, \ldots, N\} \),

\[
\text{Tr}[\rho^s] = \sum_n \rho_{nn}^s \leq \sum_n \rho_{nn}^2 + (\rho_{nn}^s)^2/2 \; (35)
\]

where \( \rho_{nn} \) and \( \rho_{nn}^s \) denote the diagonal elements of \( \rho \) and \( \rho^s \) respectively. Obviously the second equality occurs if and only if \( \rho_{nn} = \rho^s_{nn} \), which means that \( (J_z) \) has same value for both \( \rho \) and \( \rho^s \). And then \( a \) can be determined in Eq.\( (\text{[21]} \). As for the state

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|1100\rangle + |1010\rangle) \; (36)
\]

our discussion above is also applicable. We should emphasize our point clearly in this place that \( |\psi\rangle \) is not really translational invariance. Actually when one talks of the translational invariance about a systems, it means

\[
DH D^\dagger = H \; (37)
\]

in which \( D \) is the translation operator, and \( H \) is the Hamiltonian for this system. Hence that one speaks of the translational invariance for a state is meaningless without specifying the Hamiltonian. Our discussion about Dicke model manifests clearly this point. So the crucial point is to find the Hamiltonian for which \( |\psi\rangle \) is the eigenvector. It seems that one can construct the following Hamiltonian

\[
H = \sum_i \sigma_i^x \sigma_{i+1}^x \; (38)
\]

for which \( \psi \) is seemingly one of the degenerate ground states. If the translational invariance is required for this system, one must have the periodic boundary condition \( \sigma_{N+1}^x = \sigma_1^x \), where \( N \) is the total particle number. However \( |\psi\rangle \) tells us that for one particle, its neighbored particles always has opposite state to its state, which obviously does not satisfy this periodic boundary condition. So we argue in this place that the translational invariance for \( |\psi\rangle \) is only occasional because of its special form.

Instead \( |\psi\rangle \) is the true ground state for Hamiltonian

\[
H = - \sum_i \sigma_i^x \sigma_{i+1}^x \; (39)
\]

with the boundary condition \( \sigma_{N+1}^x = \sigma_1^x \). It means that the particle always has the same state to its next neighbored particle. Thus it could explain naturally the reason that the maximal overlap with \( |\psi\rangle \) happens for the fully separable states \( |1010\rangle \) or \( |0101\rangle \), which obviously satisfy this boundary condition and also are the ground states for this Hamiltonian.

In another point, one can also find a state which seemingly satisfies the requirement of the ”translational invariance” defined by \( |\psi\rangle \), i.e.,

\[
\rho' = \frac{1}{2}(|0101\rangle\langle 0101| + |1010\rangle\langle 1010|) \; (40)
\]

which obviously maximize the overlap with \( |\psi\rangle \). This features demonstrate again that \( |\psi\rangle \) is not truly translational invariance since \( \rho' \) is the incoherent superposition of the two degenerate ground states for Eq.\( (\text{[39]} \).

Appendix B: Derivation of Eq.\( (\text{[25]} \)

Set

\[
H_0 = \omega a^\dagger a; \quad H_I = \omega_0 J_z + \frac{2\lambda}{\sqrt{N}}(a^\dagger + a) \; (41)
\]

Under \( \beta = \frac{1}{\beta H_I} \ll 1 \), the partition function can be approximated as

\[
\begin{align*}
&z = \text{Tr}[e^{-\beta(H_0 + H_I)}] \\
&= \text{Tr}[e^{-\beta H_0/2}e^{-\beta H_I/2}e^{-\beta H_0/2} + O(\beta^3)] \\
&\approx \text{Tr}[e^{-\beta H_0}e^{-\beta H_I}],
\end{align*}
\]

(42)

With respect of

\[
\rho^s = \sum_{n=1}^{N} NC_n a^n(1 - a)^{N-n}|n - \frac{N}{2}\rangle\langle n - \frac{N}{2}| \; (43)
\]
\[
\Delta = \frac{1}{z} \text{Tr}[\rho \rho^*] = \frac{1}{z} \text{Tr}\left[ \sum_{k=1}^{N} NC_n a^n (1 - a)^{N-n} \right] \\
= \frac{1}{z} \sum_{k=1}^{N} \langle n - \frac{N}{2} | e^{-\beta H_0} e^{-\beta H_1} | n - \frac{N}{2} \rangle.
\]

for which \(|\rho^*, H_0\rangle = 0\) is applied. The tricky for the tracing in the equation above is noting that \(|\frac{N}{2} - n\rangle\) denotes the state in which \(n\) particles are spin-up, and the others are spin-down. And then

\[
\langle n - \frac{N}{2} | e^{-\beta H_0} e^{-\beta H_1} | n - \frac{N}{2} \rangle = e^{-\beta \omega_n \alpha} \prod_{i=1}^{N} e^{-\beta (\frac{\omega_0^2}{4} + \frac{\lambda^2 N}{4} (a_i^+ a_i^0 + 1)^2)} | n - \frac{N}{2} \rangle
\]

\[
= e^{-\beta \omega_n \alpha} \left\{ \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \frac{\omega_0^2}{4} + \frac{\lambda^2 N}{4} (a_i^+ a_i^0 + 1)^2 \right\} (1 - \frac{\beta}{2k + 1} \frac{\omega_0}{2})^n
\]

Thus
\[
\Delta = \frac{1}{z} \text{Tr}[e^{-\beta \omega_n \alpha}] \left\{ \sum_{k=0}^{\infty} \frac{\beta^{2k}}{(2k)!} \frac{\omega_0^2}{4} + \frac{\lambda^2 N}{4} (a_i^+ a_i^0 + 1)^2 \right\} (1 - \frac{\beta}{2k + 1} \frac{\omega_0}{2})^n
\]

Expand the item in the corbel bracket
\[
\Rightarrow \sum_{k_1=0; k_2=0; \ldots; k_N=0}^{\infty} \left( \prod_{i=1}^{N} \frac{\beta^{2k_i}}{(2k_i)!} \frac{\omega_0^2}{4} + \frac{\lambda^2 N}{4} (a_i^+ a_i^0 + 1)^2 \right) \left[ 1 + \frac{\beta}{2k + 1} \frac{\omega_0}{2} (1 - 2a) \right]^n
\]

With the relations
\[
\frac{\partial^2 q}{\partial \eta^2} e^{\frac{1}{2} \coth \frac{q}{2}} \bigg|_{\eta=0} = (2q - 1)! \coth^q \frac{q}{2} (2q - 1)!
\]

Finally inverse the procedure from Eq.(48) to Eq.(47) for the sum item and apply relations \(\cosh x = \frac{e^x + e^{-x}}{2}\) and \(\sinh x = \frac{e^x - e^{-x}}{2}\), one then obtains the Eq.(28).
[47] R.K. Pathria, *Statistical Mechanics* (second edition), Butterworth-Heinemann (1996).