ON THE BOUNDARY STRICHARTZ ESTIMATES FOR WAVE AND SCHRÖDINGER EQUATIONS

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Abstract. We consider the $L^2_t L^r_x$ estimates for the solutions to the wave and Schrödinger equations in high dimensions. For the homogeneous estimates, we show $L^2_t L^\infty_x$ estimates fail at the critical regularity in high dimensions by using stable Lévy process in $\mathbb{R}^d$. Moreover, we show that some spherically averaged $L^2_t L^\infty_x$ estimate holds at the critical regularity. As a by-product we obtain Strichartz estimates with angular smoothing effect. For the inhomogeneous estimates, we prove double $L^2_t$-type estimates.

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1. Introduction

In this paper, we study the space-time (Strichartz) estimates for solutions to the Schrödinger type dispersive equations

$$i\partial_t u + D^a u = g, \quad u(0, x) = f(x)$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is the unknown function, $D = \sqrt{-\Delta}$, $0 < a \leq 2$. Two typical examples of (1.1) are of particular interest: the wave equation ($a = 1$) and the Schrödinger equation ($a = 2$). The rest cases ($0 < a < 2$) are also known as fractional Schrödinger equation which has attracted many attentions recently and is a fundamental equation of fractional quantum mechanics, which was derived by Laskin (see [11, 12]) as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths.

The Strichartz estimates address the following space-time estimates for the solution $u$ to (1.1), e.g. when $g = 0$,

$$\|e^{itD^a} P_0 f\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L^2},$$

where $e^{itD^a}$, $P_0$ and $L^q_t L^r_x$ are defined in the end of this section. In a pioneering paper Strichartz [20] first proved (1.2) for the case $q = p$ by the Fourier restriction
method and then the estimates were substantially extended by many authors. It is now well-known (see [10] and references therein) that for $d \geq 1$ the estimate (1.2) holds if and only if $(q,p)$ satisfies the admissible conditions:

$$2 \leq q, p \leq \infty, \quad \frac{1}{q} \leq \frac{d - d_a + 2\left(\frac{1}{2} - \frac{1}{p}\right)}{2}, \quad (q,p,d) \neq (2, \infty, d_a),$$

where

$$d_a = \begin{cases} 2, & a \neq 1; \\ 3, & a = 1. \end{cases}$$

(1.4)

In particular, the endpoint estimates $(q,p) = (2, 2 + \frac{4}{d - d_a})$ for $d > d_a$ were proved in [10], and the failure of $(q,p,d) = (2, \infty, d_a)$ was proved in [14] for $a = 1, 2$.

By the scaling invariance of the equation (1.1): for $\lambda > 0$ $f(x) \to f(\lambda x)$, $u \to u(\lambda^a t, \lambda x)$, the Minkowski inequality and the Littlewood-Paley square function theorem, one can get from (1.2) the following frequency-global Strichartz estimates

$$\|e^{itD_a} f\|_{L_t^q L_x^p} \lesssim \|f\|_{\dot{H}^{s}},$$

(1.5)

if $(q,p)$ satisfies the admissible conditions (1.3), $p \neq \infty$ and the natural scaling condition

$$s = \left(\frac{1}{2} - \frac{1}{p}\right)d - \frac{a}{q}.$$ 

On the other hand, for $p = \infty$, the estimates (1.5), namely

$$\|e^{itD_a} f\|_{L_t^q L_x^\infty} \lesssim \|f\|_{\dot{H}^{s-c}},$$

(1.6)

need special treatment due to the failure of the Littlewood-Paley theory in $L^\infty$. For $q > 2$ and $d \geq d_a$ (or $q \geq 4$, $d = 1$ and $a \neq 1$), one can prove (1.6) by interpolations or directly by $TT^*$ method (see Section 2). For wave equation $a = 1$, (1.6) was studied in [2] and in particular the estimate (1.6) was shown to be false for $(a,q,d) = (1, 4, 2)$. Therefore, the only unknown estimates for (1.5) are the endpoints $(q,p) = (2, \infty)$ for $d > d_a$ which can not be handled by interpolations or $TT^*$ method as $(2, \infty)$ lies on the boundary of admissible conditions. When $d = d_a$, these endpoint estimates (even weaker version (1.2)) fail and it was known that the failure is logarithmic due to the $t$-integration on the whole line. Indeed, the following estimate was proved by Tao in [22]:

$$\|e^{it\Delta} P_0 f\|_{L_t^2 L_x^\infty(\mathbb{R}^2)} \lesssim \log(2 + |I|)^{1/2} \|f\|_{L^2(\mathbb{R}^2)}.$$ 

For $d > d_a$, (1.2) holds for $(q,p) = (2, \infty)$ from which we can get the following two estimates

$$\|e^{itD_a} f\|_{L_t^2 L_x^{\infty, \text{BMO}}} \lesssim \|f\|_{H^{s}}, \quad \|e^{itD_a} f\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{H^{s}},$$

where $s > s_c := \frac{d - a}{2}$, and BMO is the space of bounded mean oscillation. BMO is usually a good substitute for $L^\infty$ in harmonic analysis. Thus we see the $L_t^2 L_x^\infty$ estimate is logarithmically missing at the critical regularity.

The purpose of this paper is to study various $L_t^2 L_x^\infty$-type estimates for (1.1). Our first result is
**Theorem 1.1.** If $0 < a \leq 2, d > d_a$ with $d_a$ given by (1.4), then the following estimate fails:

$$\|e^{itD_x}f\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{H^{\frac{d-a}{2}}}.$$  \hfill (1.7)

Different from the case $d = d_a$, the logarithmic failure of the above estimate is due to the summation over the frequency. We will use the $a$-stable Lévy process to prove the above theorem. When $a = 2$, this process reduces to the Brownian motion. Our ideas are inspired by [14].

In spite of the results in [14], Tao (see [21]) showed in the radial case the $L_t^2 L_x^\infty$ estimate for the 2D Schrödinger equation holds. Actually, he proved a spherically averaged estimate:

$$\|e^{it\Delta}f\|_{L_t^2 L^\infty_x L^2_\rho(R \times R^2)} \lesssim \|f\|_{L^2_\rho(R^2)},$$

where $L^p_\rho L^\infty L^2_\rho$ is defined by (1.11). Moreover, he proved there is an $\epsilon$-angular smoothing effect: $\exists \epsilon > 0$ such that

$$\|\Lambda_\omega^\epsilon e^{it\Delta}f\|_{L_t^2 L_x^\infty L^2_\rho(R \times R^2)} \lesssim \|f\|_{L^2},$$

where $\Lambda_\omega$ is the angular derivative (see the end of this section). In Theorem 5.1 of [15], an upper bound on the smoothing effect $\epsilon \leq 1/3$ was shown. Our second result extends Tao’s result to the cases $0 < a < \infty, d \geq d_a$.

**Theorem 1.2.** (1) If $0 < a < \infty, d \geq 3$ and $s < \frac{d-2}{2}$, then

$$\|\Lambda_\omega^s e^{itD_x}f\|_{L_t^2 L^\infty_x L^2_\rho} \lesssim \|f\|_{H^{\frac{d-a}{2}}}.$$  \hfill (1.8)

(2) If $1 < a < \infty, d \geq 2$ and $s < \frac{1}{2} + \frac{d-a}{2}$, then

$$\|\Lambda_\omega^s e^{itD_x}f\|_{L_t^2 L^\infty_x L^2_\rho} \lesssim \|f\|_{H^{\frac{d-a}{2}}}.$$  \hfill (1.9)

Besides its own interest, the $L_t^p L_x^\infty$ estimates play important roles in the study of the nonlinear problems. Especially, it is useful for fractional Schrödinger equations ($a < 2$) when the classical Strichartz estimates have a loss of derivatives, e.g. see [8]. In the appendix we apply the above theorem to study the cubic fractional Schrödinger equations.

The general spherically averaged estimates

$$\|e^{itD_x}P_0f\|_{L_t^2 L^\infty_x L^2_\rho} \lesssim \|f\|_{L^2}$$  \hfill (1.8)

were also studied. It was known that (1.8) allows a wider range of indices $(q, p)$ than (1.2). For the wave equation $a = 1$ and $d \geq 2$, the optimal range of $(q, p)$ for (1.8) is (see [9] and references therein, [17] for $d = 2$): $(q, p) = (\infty, 2)$ or

$$2 \leq q, p \leq \infty, \quad \frac{1}{q} < (d - \frac{d_a}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{p}).$$  \hfill (1.9)

For the case $a > 1$ and $d \geq 2$, (1.8) holds if $(q, p)$ satisfies either (1.9) or

$$2 \leq q, p \leq \infty, \quad \frac{1}{q} = (d - \frac{d_a}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{p}), \quad (q, p) \neq (2, \frac{4d-2}{2d-3}).$$

These conditions are also necessary except the endpoints $(2, \frac{4d-2}{2d-3})$ which are still open (see [3] and references therein). To apply these estimates to the nonlinear
problems, one needs inhomogeneous estimates
\[ \left\| \int_0^t e^{i(t-s)D^a} P_0 g(s) ds \right\|_{L^r_t L^q_x} \lesssim \|g\|_{L^r_t L^q_x} \]
which can be obtained by the standard Christ-Kiselev lemma. However, by Christ-
Kiselev lemma one misses the double \(L^q_t\) type estimates, namely \(q = \tilde{q} = 2\) in the
above estimates. Our last result is concerned with the generalized double endpoint
inhomogeneous Strichartz estimates.

**Theorem 1.3.** Let \(0 < a \leq 2\) and \(d > d_a\). Assume \(p, r > \frac{4d + 2 - 2d_a}{2d - d_a - 1}\). Then the
following estimate holds
\[ \left\| \int_0^t e^{i(t-s)D^a} P_0 g(s) ds \right\|_{L^r_t L^q_x} \lesssim \|g\|_{L^r_t L^q_x}. \] (1.10)

**Remark 1.4.** The condition \(d > d_a\) is necessary in our proof. We do not know
whether (1.10) holds for \(d = d_a\).

**Notations.** We use \(F(f)\) and \(\hat{f}\) to denote the Fourier transform of \(f\):
\(\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx\). For \(a > 0\), define \(S_a(t) = e^{itD^a} = F^{-1} e^{it\xi^a} F\).

Let \(\eta : \mathbb{R} \to [0, 1]\) be an even, smooth, non-negative and radially decreasing
function which is supported in \(\{\xi : |\xi| \leq \frac{2}{3}\}\) and \(\eta \equiv 1\) for \(|\xi| \leq \frac{1}{3}\). For \(k \in \mathbb{Z}\), let
\(\chi_k(\xi) = \eta(\frac{2}{3}\xi) - \eta(\frac{3}{16}\xi)\) and \(\chi_{\leq k}(\xi) = \eta(\frac{3}{16}\xi)\), and define Littlewood-Paley operators
\(P_k, P_{\leq k}\) on \(L^2(\mathbb{R}^d)\) by \(\hat{P}_k u(\xi) = \chi_k(|\xi|) \hat{u}(\xi), \quad \hat{P}_{\leq k} u(\xi) = \chi_{\leq k}(|\xi|) \hat{u}(\xi)\).

\(\Delta_\omega\) denotes the Laplace-Beltrami operator on the unite sphere \(S^{d-1}\) endloaded with
the standard metric \(g\) and with the standard measure \(d\omega\). Let \(\Lambda_\omega = \sqrt{1 - \Delta_\omega}\).

Denote \(L^p_\omega(S^{d-1}) = L^p(\mathbb{R}^d \times S^{d-1} : d\omega)\), \(\mathcal{H}^p_p = \mathcal{H}^p(S^{d-1}) = \Lambda_\omega^{-\frac{d}{2}} L^p_\omega\).

Let \(L^p(\mathbb{R}^d)\) denote the usual Lebesgue space, and \(L^p(\mathbb{R}^+ : t^{d-1} dt)\). \(L^p_\omega L^q_\omega\) are Banach spaces on \(\mathbb{R}^d\) defined by the following norms:
\[ \|f\|_{L^p_\omega L^q_\omega} = \left\| \|f(r\omega)\|_{L^q_\omega}\right\|_{L^p_\omega} \] (1.11)
with \(x = r\omega, \ \omega \in S^{d-1}\). Let \(X\) be a Banach space on \(\mathbb{R}^d\). \(L^p_\omega X\) denotes the space-
time function space on \(\mathbb{R} \times \mathbb{R}^d\) with the norm \(\|u\|_{L^p_\omega X} = \|\|u(t, \cdot)\|_{X}\|_{L^p_\omega}\). \(H^p_\omega\) (\(H^p_\omega\))
are the usual inhomogeneous (homogeneous) Sobolev spaces on \(\mathbb{R}^d\).

2. \(L^q_\omega L^\infty_x\) ESTIMATES FAIL AT THE CRITICAL REGULARITY

In this section, we consider the \(L^q_\omega L^\infty_x\) estimates. First we prove the following
proposition

**Proposition 2.1.** Let \(a > 0, \ d \geq 1\) and \(2 < q < \infty\). Then
\[ \|e^{itD^a} f\|_{L^r_\omega L^q_x} \lesssim \|f\|_{H^q_\omega}. \] (2.1)
holds if assuming either of the following conditions:

- \(a = 1, \ \frac{2}{q} < \frac{d-1}{2}\).
- \(a \neq 1, \ \frac{2}{q} \leq \frac{d}{2}\).

**Proof.** By TT* method, the estimate (2.1) is equivalent to
\[ \left\| \int_0^t e^{i(t-s)D^a} D^{\frac{d}{q}-\frac{d}{2}} g(s) ds \right\|_{L^r_\omega L^q_x} \lesssim \|g\|_{L^r_\omega L^q_x}. \]
By the dispersive estimates given in the lemma below and the Hardy-Littlewood-Sobolev inequality we get
\[
\left\| \int e^{it(s)} D^\frac{2a}{q} g(s) ds \right\|_{L_t^q L_x^\infty} \lesssim \left\| \int \left| t-s \right|^{-2/q} g(s) ds \right\|_{L_t^q} \lesssim \left\| g \right\|_{L_t^q L_x^1}.
\]
Therefore we complete the proof. □

**Lemma 2.2.** Let \( a > 0, d \geq 1 \) and \( 0 < q < \infty \). Then
\[
\left\| D^\frac{2a}{q} e^{itD^a} \phi \right\|_{L_x^q} \leq C \left| t \right|^{-\frac{a}{q}} \left\| \phi \right\|_{L_x^1}
\]
holds if assuming either of the following conditions:
- \( a = 1, \frac{2}{q} < \frac{d-1}{2} \).
- \( a \neq 1, \frac{2}{q} \leq \frac{d}{2} \).

**Proof.** By Theorem 1 in [6] we get for \( 0 \leq \theta \leq \frac{d-2a+2}{2} \)
\[
\left\| e^{itD^a} P_\theta \phi \right\|_{L_x^q} \leq C \left| t \right|^{-\theta} 2^{j(d-\theta)} \left\| \phi \right\|_{L_x^1}.
\]
If \( a = 1 \) and \( \frac{2}{q} < \frac{d-1}{2} \), or if \( a \neq 1 \) and \( \frac{2}{q} < \frac{d}{2} \), then we get
\[
\left\| D^\frac{2a}{q} e^{itD^a} \phi \right\|_{L_x^q} \leq C \sum_j \inf_{\theta} (2j)^{\frac{2a}{q} - \theta} \left| t \right|^{-\theta} \left\| \phi \right\|_{L_x^1}
\]
\[
\lesssim \sum_{2^j \leq \left| t \right|^{-1/a}} 2^{ja/q} \left\| \phi \right\|_{L_x^1} + \sum_{2^j \geq \left| t \right|^{-1/a}} 2^{ja(\frac{2}{q} - \frac{d-2a+2}{2})} \left| t \right|^{-\frac{d-2a+4}{2}} \left\| \phi \right\|_{L_x^1}
\]
\[
\lesssim \left| t \right|^{-2/q} \left\| \phi \right\|_{L_x^1}.
\]
Thus it remains to show the case: \( a \neq 1, q = \frac{4}{d} \).

By the Young inequality it suffices to show
\[
\left| \int e^{i\xi^a} \chi_{\epsilon \leq k}(\xi) \xi^{\frac{2a}{q} - d} d\xi \right| \leq C \left| t \right|^{-\frac{2}{q}}, \quad \forall k \in \mathbb{N}, x \in \mathbb{R}^d.
\]
Without loss of generality, we may assume \( t > 0 \). By a change of variable \( \xi = t^{-1/a} \eta \), it suffices to show
\[
\sum_{j \leq k} \int e^{i\eta^a} \chi_{J_j}(\xi) \xi^{\frac{2a}{q} - d} d\xi \leq C, \quad \forall k \in \mathbb{N}, x \in \mathbb{R}^d.
\]
Fix \( x \in \mathbb{R}^d, k \in \mathbb{N} \). Denote \( I_j(x) = \int e^{i\eta^a} \chi_{J_j}(\xi) \xi^{\frac{2a}{q} - d} d\xi \). By the Fourier-Bessel formula (see [18]), we have
\[
I_j(x) = \int e^{ir^a} \chi_{J_j}(r) r^{\frac{2a}{q} - 1} \left( r \left| x \right| \right)^{-\frac{d-2}{2}} J_{d-2} \left( r \left| x \right| \right) dr, \quad d \geq 2,
\]
\[
I_j(x) = \int e^{i\eta^a} \chi_{J_j}(\xi) \xi^{\frac{2a}{q} - 2} d\xi, \quad d = 1.
\]
Here \( J_\nu(r) \) is the Bessel function defined by
\[
J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+1/2) \pi^{1/2}} \int_{-1}^{1} e^{ir\theta} (1 - \theta^2)^{\nu-1/2} d\theta, \quad \nu > -1/2.
\]
Case 1: \( 2^j \leq |x|^{-1} \).
First we have the trivial bound \( |I_j(x)| \lesssim 2^\frac{2a}{q} \). On the other hand, when \( d \geq 2 \), by the fact that
\[
\left| \frac{d^k}{dr^k}(r^{-q}J_r(r)) \right| \lesssim 1
\]
and using integration by part \( n \) times we get
\[
|I_j(x)| = \left| \int \left[ \left( iar^{\frac{a-1}{q}} - \partial_r \right)^n e^{ir\alpha} \right] \chi_j(r)r^{\frac{2a}{q}-1}(r|x|) \frac{d^k}{dr^k}(r^{-q}J_{\frac{d}{2}a}(r|x|))dr \right| 
\lesssim 2^\frac{2a}{q} 2^{-jna}
\]
for any \( n \in \mathbb{N} \). Then we have \( \sum 2^j \leq |x|^{-1} |I_j(x)| \lesssim \sum 2^j \leq |x|^{-1} \min(2^\frac{2a}{q}, 2^{-jna}, 2^\frac{2a}{q}) \lesssim 1 \).
Similarly, the same holds for \( d = 1 \).

Case 2: \( 2^j \gg |x|^{-1} \).
Using the fact \( r^{-\frac{a}{2}} J_{\frac{d}{2}a}(r) = c_d R(e^{r\alpha}h(r)) \) where \( h \) satisfies \( |\partial_r^n h| \lesssim (1+r)^{-\frac{a}{2}-m} \) (see Section 1.4, Chapter VIII of [19]), it suffices to show
\[
\sum_{|x|^{-1} \ll 2^j \leq 2^k} |\tilde{I}_j(x)| \lesssim 1
\]
with
\[
\tilde{I}_j(x) := \int e^{i(r\alpha - |x|)} \chi_j(r)|r|^{\frac{2a}{q}-1}h(r|x|) dr, \quad d \geq 2,
\]
\[
\tilde{I}_j(x) := \int e^{i\xi x} e^{i\xi x} \chi_j(\xi)|\xi|^{\frac{2a}{q}-1} d\xi, \quad d = 1.
\]
Hence the 1D case is the same as the higher dimensions with \( h(r) \equiv 1 \).

- If \( 2^{(a-1)} \sim |x| \gg 2^{-j} \), then \( 2^j|x| \sim 2^{ja} \) and by the van der Corput lemma (see [19]) we get
\[
|\tilde{I}_j(x)| \lesssim 2^{-\frac{d(a-2)}{2}a} 2^{j(\frac{2a}{q}-1)}(2^j|x|)^{-\frac{d-1}{2}} a \lesssim 2^{ja(-\frac{d}{2}+\frac{1}{2})} \lesssim 1.
\]
- If \( 2^{(a-1)} \ll |x| \), integrating by parts \( n \) times, we have for any \( n \in \mathbb{N} \)
\[
|\tilde{I}_j(x)| = \left| \int \left[ -i\left( ar^{\frac{a-1}{q}} - |x|\right)^{-1}\partial_r \right]^n e^{i(r\alpha - |x|)} \chi_j(r)|r|^{\frac{2a}{q}-1}h(r|x|) dr \right| 
\lesssim (2^j|x|)^{-\frac{d-1}{2} a 2^{ja} + \frac{d}{2} a}.
\]
- If \( 2^{(a-1)} \gg |x| \), then \( 2^j \geq 2^{j(1-a)} \) and hence \( j \geq 0 \). Integrating by parts \( n \) times, we have for any \( n \in \mathbb{N} \)
\[
|\tilde{I}_j(x)| = \left| \int \left[ -i\left( ar^{\frac{a-1}{q}} - |x|\right)^{-1}\partial_r \right]^n e^{i(r\alpha - |x|)} \chi_j(r)|r|^{\frac{2a}{q}-1}h(r|x|) dr \right| 
\lesssim (2^j|x|)^{-\frac{d-1}{2} 2^{-jan} 2^{j\frac{2a}{q}}} \leq 2^{-jan} 2^{j\frac{2a}{q}}.
\]
Therefore, we get
\[
\sum_{|x|^{-1} \ll 2^j \leq 2^k} |\tilde{I}_j(x)| \lesssim \sum_{2^{(a-1)} \sim |x|} \sum_{\min(2^j, 2^{j(1-a)}) \gg |x|^{-1}} (2^j|x|)^{-\frac{d-1}{2} - n 2^{ja} + \frac{d}{2} a} + \sum_{2^j \gg |x|^{-1} \gg 2^{j(1-a)}} 2^{-jan} 2^{j\frac{2a}{q}} \lesssim 1.
\]
We complete the proof of the lemma. □

The failure of the estimate (2.1) for \((a,q,d) = (1,4,2)\) was shown in [2]. In the rest of this section we prove Theorem 1.1 by using \(a\)-stable Lévy processes\(^1\). First we collect some properties of these processes. For \(a \in (0,2]\), let

\[
f_a(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} e^{-|\xi|^a} \, d\xi.
\]

Then \(f_a\) is a smooth strictly positive radial function on \(\mathbb{R}^d\) satisfying \(\int_{\mathbb{R}^d} f_a(x) \, dx = 1\). In particular, we have

\[
f_1(x) = C_1(1 + |x|^2)^{-\frac{d+1}{2}}, \quad f_2(x) = C_2 e^{-\frac{|x|^2}{4}}, \quad 0 < a < 2 \implies f_a(x) \sim (1 + |x|)^{-\frac{d+1}{2}},
\]

see [1]. It is well-known that Random variables with distributions given by the density \(f_a(0 < a \leq 2)\) are stable. For \(t > 0\), let \(f_a(t, x) = t^{-d/a} f_a(t^{-1/a}x)\).

Let \(\{Y(t) : t \geq 0\}\) be the independent symmetric \(a\)-stable Lévy process in \(\mathbb{R}^d\) with \(Y(0) = 0\), that is, a process with the stationary independent increments, and the increment \(Y_t - Y_s\) has a distribution given by the density \(f_a(|t-s|, x)\). Let \(\{\widetilde{Y}(t) : t \geq 0\}\) be another independent copy of \(Y(t)\). The existence of these processes and their properties were well-understood [13, 1]. We construct a process \(X_t := X(t)\) on the whole line \(\mathbb{R}\) by defining

\[
X(t) = \begin{cases} 
Y(t), & t \geq 0; \\
\widetilde{Y}(-t), & t < 0.
\end{cases}
\]

By this construction we know \(X_t - X_s\) has a distribution given by the density \(f_a(|t-s|, x)\) for \(t \neq s\), and hence

\[
\mathbb{E}e^{i\eta(X_t-X_s)} = \int_{\mathbb{R}^d} e^{i\eta x} f_a(|t-s|, x) \, dx = e^{-|t-s|\eta^a}
\]

where \(\mathbb{E}\) is the expectation.

Now we prove Theorem 1.1. The estimate (1.7) is equivalent to

\[
\left\| e^{itD^a} \langle D \rangle^{-\frac{d-a}{2}} f \right\|_{L^2_t L^\infty_x} \leq C \|f\|_{L^2}.
\]

By \(TT^*\) method, we see it is further equivalent to

\[
\left\| \int e^{i(t-s)D^a} \langle D \rangle^{a-d} g(s, \cdot) \right\|_{L^2_t L^\infty_x} \leq C^2 \|g\|_{L^2_t L^1_x}.
\]

Assume \(g(t, x) = \alpha(t) h(x - X(t))\), where \(\alpha \in L^2(\mathbb{R})\), and \(h\) is a Schwartz function. Then the above inequality implies

\[
\left\| \int e^{i(t-s)D^a} \langle D \rangle^{a-d} g(s, \cdot) \right\|_{L^2_t L^\infty_x} \leq C^2 \|\alpha\|_{L^2_t} \|h\|_{L^1_t}
\]

which further implies

\[
\left\| \int K(t, s) \alpha(s) \, ds \right\|_{L^2_t} \leq C^2 \|\alpha\|_{L^2_t} \|h\|_{L^1_t}
\]

\(^1\)The authors would like to thank Kais Hamza for the discussion on the Lévy processes.
where
\[
K(t, s) = \int_{\mathbb{R}^d} e^{i(t-s)|\xi|^a} (1 + |\xi|^2) \frac{2-a}{2} e^{i|X(t)-X(s)|\xi} \hat{h}(\xi) d\xi.
\]
By Minkowski’s inequality the above inequality implies
\[
\left\| \int \mathbb{E}K(t, s)\alpha(s) ds \right\|_{L^1_t} \leq C^2 \|\alpha\|_{L^1_t} \|h\|_{L^1_t}
\]
where
\[
\mathbb{E}K(t, s) = \mathbb{E} \int_{\mathbb{R}^d} e^{i(t-s)|\xi|^a} (1 + |\xi|^2)^{\frac{2-a}{2}} e^{i|X(t)-X(s)|\xi} \hat{h}(\xi) d\xi
\]
\[
= \int_{\mathbb{R}^d} e^{i(t-s)|\xi|^a} (1 + |\xi|^2)^{\frac{2-a}{2}} e^{-|t-s||\xi|^a} \hat{h}(\xi) d\xi
\]
\[=: K_h(t - s).
\]
In the second equality above we used (2.4). For a fixed function \(h\), the operator given by the kernel \(K_h\) is a convolution operator, and thus its \(L^2\) boundedness implies
\[
\|K_h(\tau)\|_{L^\infty_{\tau}} \leq C \|h\|_{L^1_t}, \quad (2.5)
\]
for any Schwartz function \(h\).

To prove Theorem 1.1, it suffices to disprove (2.5). By direct calculation we get
\[
\hat{K}_h(\tau) = C \int_{\mathbb{R}^d} \frac{|\xi|^a(1 + |\xi|^2)^{\frac{2-a}{2}}}{(\tau - |\xi|^a)^2 + |\xi|^{2a}} \hat{h}(\xi) d\xi.
\]
Thus \(\hat{K}_h(0) = \int_{\mathbb{R}^d} |\xi|^{-a}(1 + |\xi|^2)^{\frac{2-a}{2}} \hat{h}(\xi) d\xi\). The integrand for large \(\xi\) is essentially \(|\xi|^{-d} \hat{h}(\xi)\) which makes the integral a logarithmic infinity. For example, one can take \(h\) to be an approximating sequence of \(\delta(x)\). Then clearly, (2.5) fails.

Remark 2.3. The above proof only use the fact that \(X(t)\) satisfies (2.4). For the wave equation (namely when \(a = 1\)), we can take \(X(t) = tZ\), where \(Z\) is a random variable with characteristic function \(e^{-|\xi|^2}\).

3. Strichartz estimates with angular smoothing effect

In this section, we prove Theorem 1.2. It is equivalent to show
\[
\|\Lambda_\omega^a (T_a f)\|_{L^4_t L^6_x} \lesssim \|f\|_{L^2_x}, \quad (3.1)
\]
where
\[
T_a f(t, x) = \int_{\mathbb{R}^d} e^{i(x \xi + t|\xi|^a)} |\xi|^{\frac{2-d}{2}} f(\xi) d\xi.
\]
Now we apply the spherical-radius decomposition to \(f\) (see [18])
\[
f(\xi) = f(\rho \omega) = \sum_{k \geq 0} \sum_{1 \leq l \leq n(k)} \hat{d}_k(\rho) Y^l_k(\omega), \quad \rho = |\xi|, \ \omega = \frac{\xi}{|\xi|} \in S^{d-1},
\]
where \(k \geq 0, 1 \leq l \leq n(k), \ n(k) = C^k_{d+k-1} - C^k_{d+k-3}\). \(\{Y^l_k\}\) is the standard orthonormal basis (spherical harmonics of degree \(k\)) in \(L^2(S^{d-1})\). We have (see [18])
\[
T_a f(t, x) = \sum_{k, l} \hat{c}_{d, k} T^l_a (\rho \frac{d+1}{2} \hat{a}_k(t, |x|) Y^l_k(x/|x|),
\]
Then we have
\[ T^\nu_a(h)(t, r) = r^{-\frac{d-2}{2}} \int_0^\infty e^{it\rho^a} J_\nu(r\rho) \rho^{-\frac{d+1+a}{2}} h(\rho) d\rho. \]

Here \( J_\nu(r) \) is the Bessel function given by (2.2). Thus (3.1) is equivalent to
\[ (1 + |k|)^s T^\nu_a(a_k^l) \| L^2_t L^\infty_x \|_{a_k^l}^2 \lesssim \| \{ a_k^l(\rho) \} \|_{L^2_x}^2. \] (3.2)

To prove (3.2), it is equivalent to show
\[ \| T^\nu_a(h) \|_{L^2_t L^\infty_x} \lesssim C(1 + \nu)^{-s} \| h \|_{L^2}, \] (3.3)

with constant \( C \) independent of \( \nu \).

Decompose \( T^\nu_a(h) = \sum_{j \in \mathbb{Z}} T^\nu_{a, j}(h) \), where
\[ T^\nu_{a, j}(h)(t, r) = \chi_j(r)r^{-\frac{d-2}{2}} \int_0^\infty e^{it\rho^a} J_\nu(r\rho) \rho^{-\frac{d+1+a}{2}} h(\rho) d\rho. \]

Then obviously
\[ \| T^\nu_a(h) \|_{L^2_t L^\infty_x} \lesssim \| T^\nu_{a, j}(h) \|_{L^2_t L^\infty_x}. \]

We decompose further \( T^\nu_{a, j}(h) = T^\nu_{a, j, +}(h) + \sum_{j' \geq -j - 4} T^\nu_{a, j, j'}(h) \), where
\[ T^\nu_{a, j, +}(h)(t, r) = \chi_j(r)r^{-\frac{d-2}{2}} \int_0^\infty e^{it\rho^a} J_\nu(r\rho) \rho^{-\frac{d+1+a}{2}} \chi_{\leq -j - 5}(\rho) h(\rho) d\rho, \]
\[ T^\nu_{a, j, j'}(h)(t, r) = \chi_j(r)r^{-\frac{d-2}{2}} \int_0^\infty e^{it\rho^a} J_\nu(r\rho) \rho^{-\frac{d+1+a}{2}} \chi_{j'}(\rho) h(\rho) d\rho. \]

Then we have
\[ \| T^\nu_{a, j}(h) \|_{L^2_t L^\infty_x} \leq \| T^\nu_{a, j, +}(h) \|_{L^2_t L^\infty_x} + \sum_{j' \geq -j - 4} \| T^\nu_{a, j, j'}(h) \|_{L^2_t L^\infty_x}. \]

First we control \( T^\nu_{a, j, +} \). We will use the vanishing properties of \( J_\nu(r) \) near \( r = 0 \). Using the formula (2.2) and Taylor’s expansion for \( e^{irt} \) we get
\[ T^\nu_{a, j, +}(h) = \chi_j(r)r^{-\frac{d-2}{2}} \int_0^\infty \frac{e^{it\rho^a}(r\rho/2)^\nu}{\Gamma(\nu + 1/2)\pi^{1/2}} \int_{-1}^1 \sum_{n=0}^{\infty} \frac{(i\theta r)^n}{n!} (1 - \theta^2)^{\nu - 1/2} d\theta \]
\[ \times \rho^{-\frac{d+1+a}{2}} \chi_{\leq -j - 5}(\rho) h(\rho) d\rho. \]

Then we have
\[ |T^\nu_{a, j, +}(h)| \lesssim \sum_{n=0}^\infty \chi_j(r)2^{-j\frac{d-2}{2}} \int_{-1}^1 \frac{(i\theta r)^n}{n!} (1 - \theta^2)^{\nu - 1/2} d\theta \rho^{-\frac{d+1+a}{2}} \chi_{\leq -j - 5}(\rho) h(\rho) d\rho. \]
Making a change of variable $\eta = \rho^\alpha$, and then using Plancherel’s equality, we get
\[
\| T_{a,j}^\nu(h) \|_{L^2_j L^{\infty}_t} \lesssim \frac{1}{\Gamma(\nu + 1/2)} \sum_{n=0}^{\infty} C^n \frac{1}{n!} \left\| \sup_j \left( \int_0^\infty e^{it\rho^\nu} (2^j \rho)^{\nu + \frac{d-2}{2}} \chi_{\leq -j-5}(\rho) \rho^{\frac{\nu-1}{2}} h(\rho) d\rho \right) \right\|_{L^2_t}
\]
\[
\lesssim \frac{1}{\Gamma(\nu + 1/2)} \sum_{n=0}^{\infty} C^n \frac{1}{n!} \left\| \sup_j \left( \int_0^\infty e^{it\rho^\nu} (2^j \rho)^{1/2} \chi_{\leq -j-5}(\rho) h(\rho) \rho^{\frac{\nu-1}{2}} d\rho \right) \right\|_{L^2_t}
\]
\[
=: \frac{1}{\Gamma(\nu + 1/2)} \sum_{n=0}^{\infty} C^n \frac{1}{n!} A_n.
\]

Let $\hat{h}(\rho) = h(\rho^{1/a}) \rho^{-\frac{4a}{2a+1}} 1_{[0,\infty)}(\rho)$. Then $\| \hat{h} \|_2 \sim \| h \|_2$. If $k + n = 0$, then
\[
A_n \lesssim \left\| \sup_j \left( \int e^{it\rho^\nu} \chi_{\leq -j-5}(\rho^{1/a}) \hat{h}(\rho) d\rho \right) \right\|_{L^2_t}
\]
\[
\lesssim \| M(\hat{h})(t) \|_{L^2_t} \lesssim \| h \|_2,
\]
where $M$ is the Hardy-Littlewood maximal operator since $\chi_{\leq 1}(\rho^{1/a})$ is a Schwartz function. If $k + n \geq 1$, then by Plancherel’s equality we get
\[
A_n \lesssim \left\| \int e^{it\rho^\nu} (2^j \rho^{1/a})^{k+n} \chi_{\leq -j-5}(\rho^{1/a}) \hat{h}(\rho) d\rho \right\|_{L^2_t}
\]
\[
\lesssim \left\| (2^j \rho^{1/a})^{k+n} \chi_{\leq -j-5}(\rho^{1/a}) \hat{h}(\rho) \right\|_{L^2_t}
\]
\[
\lesssim \left\| \sum_{j+j' \leq -5} 2^{(j+j')k+n} \chi_{j+j'}(\rho^{1/a}) \hat{h}(\rho) \right\|_{L^2_t} \lesssim \| \hat{h} \|_2 \lesssim \| h \|_2.
\]

So we get
\[
\| T_{a,j}^\nu(h) \|_{L^2_j L^{\infty}_t} \lesssim (1 + \nu)^{-K} \| h \|_2, \quad \forall K \in \mathbb{N}.
\]

Next we control $T_{a,j,j'}^\nu$. We will use the asymptotic behaviour of $J_\nu(r)$ for $r \to \infty$. We have
\[
\| T_{a,j,j'}^\nu(h) \|_{L^2_j L^{\infty}_t} \lesssim 2^{-j \frac{d+2}{2} - j' \frac{d-2}{2} + j' \frac{d+2}{2} + \frac{d+1+a}{a}} \left\| \chi_{j+j'}(r) \int_0^\infty e^{it\rho^\nu} J_\nu(\rho) \rho^{\frac{4a}{2a+1}} \chi_0(\rho) h(2^j \rho) 2^{j'/2} d\rho \right\|_{L^2_t L^{\infty}_t}
\]
\[
= 2^{-(j+j') \frac{d+2}{2}} \| S_{j+j'}^\nu(h,j') \|_{L^2_j L^{\infty}_t},
\]
where we denote $h_{j'} := h(2^{j'} \rho) 2^{j'/2}$ and the operator $S_{j'}^\nu$ is defined by
\[
S_{j'}^\nu(h) := \chi_{j'}(r) \int_0^\infty e^{it\rho^\nu} J_\nu(\rho) \rho^{\frac{4a}{2a+1}} \chi_0(\rho) h(\rho) d\rho.
\]

It is the same operator as $S_{R}^{\nu,a}$ with $R = 2^j$ that was studied in [3], or the operator $T_{j}^\nu$ with $\omega(\rho) = \rho^2$, $k = 0$ studied in [4].

**Case 1**: $\nu \lesssim 2^{j+j'}$.

In this case, we can use the result in Lemma 3.6 in [4] for $T_{j,k}^\nu$, and then get
\[
\| S_{j+j'}^\nu(h) \|_{L^2_j L^{\infty}_t} \lesssim \| h \|_2,
\]
and thus
\[ \|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x} \lesssim 2^{-(j+j')\frac{d}{2}} \|\chi_{j'}(\rho)h(\rho)\|_2. \]

Therefore, we get
\[ \left\| \sum_{j'+j \geq -4} \|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x}\right\|_{l^2_j} \lesssim \nu^{-\frac{d+2}{2}} \|h\|_2. \]

This suffices to show Part (1) of Theorem 1.2.

**Case 2:** $\nu \gg 2^{j+j'}$.

In this case, we use the Stirling formula for the Gamma function and get
\[ \Gamma(\nu + 1) \geq C\nu^{1/2}(\nu/\epsilon)^\nu \]
from which we get better decay
\[ |J_\nu(r)| + |J'_\nu(r)| \leq C\nu^{-K}, \quad \forall K \in \mathbb{N}. \]

By the above bound and the Sobolev embedding we get
\[ \|S_{j,j'}^\nu(h)\|_{L^2_t L^\infty_x} \lesssim \|S_{j,j'}^\nu(h)\|_{L^2_t L^2_x} + \||\partial_x S_{j,j'}^\nu(h)\|_{L^2_t L^2_x} \lesssim \nu^{-K} \|h\|_2, \quad \forall K \in \mathbb{N}. \quad (3.4) \]

and thus
\[ \|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x} \lesssim \nu^{-K} 2^{-(j+j')\frac{d-4}{4}} \|\chi_{j'}(\rho)h(\rho)\|_2. \]

Therefore, we get
\[ \left\| \sum_{j'+j \geq -4} \|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x}\right\|_{l^2_j} \lesssim \nu^{-K} \|h\|_2, \quad \forall K \in \mathbb{N} \]

which suffices for our purpose.

When $1 < a < \infty$, we can improve Case 1. Applying the results in Lemma 3.10 in [4] by taking $\lambda = R^{3/7}$ we get
\[ \|S_{j,j'}^\nu(h)\|_{L^2_t L^\infty_x} \lesssim 2^{-(j+j')/7} \|h\|_2. \]

Therefore, for the case $1 < a < \infty$ we get
\[ \left\| \sum_{j'+j \geq -4} \|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x}\right\|_{l^2_j} \lesssim \nu^{-\frac{7d-12}{14}} \|h\|_2. \]

This suffices to show Part (2) of Theorem 1.2.

**Remark 3.1.** For $0 < a \leq 1$ and $d = 2$, by the similar arguments we can get the following estimate: if $q > 2$
\[ \|\Lambda_{\omega}^{\frac{1}{2} - \frac{d}{2}} e^{itD^a} f\|_{L^q_t L^\infty_x L^2_{\mathbb{R}^2}} \lesssim \|f\|_{H^{\frac{d}{2} - \frac{d}{4}}}^{2}, \quad \forall \epsilon > 0. \quad (3.5) \]

Indeed, to prove (3.5), we just interpolate the estimates $\|T_{a,j,j'}^\nu(h)\|_{L^2_t L^\infty_x} \lesssim \|h\|_{L^2}$ with the following estimate
\[ \|T_{a,j,j'}^\nu(h)\|_{L^\infty_t L^\infty_x} \lesssim 2^{-(j+j')/2} \|h\|_{L^2}. \]

Then we use the $\epsilon$-room to do the summation.
4. Inhomogeneous Strichartz estimates

In this section we prove Theorem 1.3. These double endpoint inhomogeneous Strichartz estimates have useful applications of controlling potential terms. These estimates cannot be deduced directly by Christ-Kiselev lemma from homogeneous estimates.

Proof of Theorem 1.3. We may assume $q \geq r$, since otherwise we consider the adjoint estimate. Thus we may further assume $q = r$ by Bernstein’s inequality. It suffices to prove

$$|T(F, G)| \lesssim \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x},$$

where

$$T(F, G) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \langle e^{-iD^a} P_0 F(\cdot, s), e^{-itD^a} P_0 G(\cdot, t) \rangle_{L^r_x} ds dt.$$

Decompose $T$ dyadically $T(F, G) = \sum_j T_j(F, G)$ where

$$T_j(F, G) = \int_{-\infty}^{\infty} \int_{j-2^{j+1} < s \leq t < j} \langle e^{-iD^a} P_0 F(\cdot, s), e^{-itD^a} P_0 G(\cdot, t) \rangle_{L^r_x} ds dt.$$

It suffices to prove

$$\sum_{j \in \mathbb{Z}} |T_j(F, G)| \lesssim \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x}.$$

Now we consider the following estimates for $T_j(F, G)$:

$$|T_j(F, G)| \lesssim C(j) \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x}.$$

We may assume $F, G$ both have compact supports in $t$ on an interval of length $O(2^j)$. First we have the trivial estimates

$$|T_j(F, G)| \lesssim \|P_0 F\|_{L^2_t L^r_x} \|P_0 G\|_{L^2_t L^r_x} \lesssim 2^j \|P_0 F\|_{L^2_t L^r_x} \|P_0 G\|_{L^2_t L^r_x}$$

$$\lesssim 2^j \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x} \lesssim 2^j \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x}.$$

This estimate suffices to sum over $j \leq 0$. It remains to consider $j > 0$. By the dispersive estimate we have

$$|T_j(F, G)| \lesssim 2^{-\frac{(d+2-\frac{d}{2})}{2} j} \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x} \lesssim 2^{-\frac{(d+2-\frac{d}{2})}{2} j} \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x}. \quad (4.1)$$

Fix $a \in \left(\frac{4d+2-2d_0}{2d-d_0-1}, r\right)$. By the Christ-Kiselev lemma we can get for any $\varepsilon > 0$

$$\left\| \int_{|t-s|<2^j} e^{i(t-s)D^a} P_0 f(s) ds \right\|_{L^2_{t,s} L^{r\varepsilon}_x} \lesssim \left\| f \right\|_{L^2_t L^r_x} \quad (4.2)$$

where

$$\frac{1}{2(\varepsilon)} = \frac{1}{2} - \varepsilon.$$

Then we get

$$|T_j(F, G)| \lesssim \left\| \int_{|t-s|<2^j} e^{i(t-s)D^a} P_0 F(s) ds \right\|_{L^2_{t,s} L^{r\varepsilon}_x} \|G\|_{L^2_{t,s} L^{r\varepsilon}_x} \lesssim 2^{\varepsilon j} \|F\|_{L^2_t L^r_x} \|G\|_{L^2_t L^r_x} \quad (4.2)$$
Interpolating (4.1) and (4.2), and choosing \( \varepsilon \) sufficiently small, we get that for some \( \theta > 0 \)
\[
|T_j(F,G)| \lesssim 2^{-\theta j} \|F\|_{L^2_t L^\infty_x} \|G\|_{L^2_t L^\infty_x}.
\]
Thus we can sum over \( j \geq 0 \) and hence complete the proof. \( \square \)

In the above proof, we see the dispersive estimates of rate \(|t|^{-\theta}\) with some \( \theta > 1 \) is crucial. If the decay rate \( \theta = 1 \), the estimates may fail. Indeed, in [21] Tao showed that
\[
\left\| \int_0^t e^{-i(t-s)\Delta} f(s) ds \right\|_{L^\infty_t L^\infty_x} \lesssim \|f\|_{L^2_t L^2_x}
\]
fails even when \( f \) is radial in \( x \). Finally we observe its extensions to general dimensions. Consider a general form of Strichartz estimate
\[
\left\| \int_{-\infty}^t e^{i(t-s)D^\alpha} f(s) ds \right\|_{L^q_t L^r_x} \lesssim \|f\|_{L^q_t L^r_x}
\]
where \( X, Y \) are two Banach spaces embedded into \( S'(\mathbb{R}^d) \). By duality, it is equivalent to
\[
\int \int_{s<t} \left| (e^{i(t-s)D^\alpha} f(s)) g(t) dt ds \right| \lesssim \|f\|_{L^q_t X} \|g\|_{L^q_t Y}.
\]
Restriction to the functions with separated variables yields
\[
K_{\varphi, \psi}(t) := (e^{iD^\alpha} \varphi|\psi)
\]
\[
\implies \int \int_{s<t} K_{\varphi, \psi}(t-s) f(s) g(t) ds dt \lesssim \|f\|_{L^q_t X} \|g\|_{L^q_t Y} \|\varphi\|_X \|\psi\|_Y.
\]
A simple case of \( K(t) \) is when, with a parameter \( \sigma > 0 \),
\[
\hat{\varphi} = |\xi|^\alpha \sigma^\gamma e^{-\sigma |\xi|^2/2}, \quad \hat{\psi} = |\xi|^\beta \sigma^\gamma e^{-\sigma |\xi|^2/2}, \quad \alpha + \beta = a - d.
\]
We can explicitly compute
\[
c_d K(t) = \int_0^\infty e^{it\sigma^\alpha - \sigma^\alpha} ar^{a-1} dr = \int_0^\infty e^{is - \sigma s} ds = \frac{1}{\sigma - it} = \frac{\sigma + it}{\sigma^2 + t^2}.
\]
If we have a uniform bound of the above estimate for such \( \varphi, \psi \) with \( \sigma \to 0^+ \), then the limit after taking the imaginary part is
\[
\int \int_{s<t} \frac{1}{t - s} f(s) g(t) ds dt,
\]
which is clearly divergent, for any \( p, q \in [1, \infty] \). Thus we have obtained the following criterion for the Strichartz estimate, not necessarily at the endpoint.

**Proposition 4.1.** Let \( d \in \mathbb{N}, a > 0, \alpha, \beta \in (-d, \infty) \) such that \( \alpha + \beta = a - d \). Let \( X, Y \) be two Banach spaces of functions embedded into \( S'(\mathbb{R}^d) \). Suppose that the pair of functions with a parameter \( \sigma > 0 \)
\[
\mathcal{F}^{-1}(|\xi|^\alpha \sigma^\gamma e^{-\sigma |\xi|^2/2}), \quad \mathcal{F}^{-1}(|\xi|^\beta \sigma^\gamma e^{-\sigma |\xi|^2/2})
\]
are bounded as \( \sigma \to +0 \) respectively in \( X \) and in \( Y \). Then for any \( p, q \in [1, \infty] \), the following estimate is false.

\[
\left\| \int_{-\infty}^{t} e^{i(t-s)D^{\alpha}} f(s)\,ds \right\|_{L_{t}^{p}X} \lesssim \| f \|_{L_{t}^{p}Y}.
\]

The most typical scaling (including Tao’s case \( a = 2 \)) is

\[
\alpha = \gamma = 0, \quad \beta = a - d, \quad X = \mathcal{L}_{c}^{1}L_{\theta}^{\infty}, \quad Y = D^{a-d}X.
\]

Then we have, using (2.3),

\[
\mathcal{F}^{-1}(|\xi|^a \sigma^\gamma e^{-\sigma|\xi|^a/2}) = D^{d-a}\mathcal{F}^{-1}(|\xi|^3 \sigma^{-\gamma} e^{-\sigma|\xi|^a/2}) = \mathcal{F}^{-1}e^{-\sigma|\xi|^a/2} \in X,
\]

so that we can apply the above proposition. Explicitly, the following inequality

\[
\left\| \int_{0}^{t} e^{i(t-s)D^{\alpha}} f(s)\,ds \right\|_{L_{t}^{p}L_{x}^{\infty}} \lesssim \| D^{d-a}f \|_{L_{t}^{p}L_{x}^{\infty}}
\]

fails, even when \( f \) is radial in \( x \). Note however we can not replace the norms by

\[
\mathcal{L}_{c}^{\infty} \to \dot{\mathcal{B}}_{1,\infty}^{\frac{d}{2}} \quad \text{or} \quad \mathcal{L}_{c}^{1} \to \dot{\mathcal{B}}_{1,\infty}^{0}
\]

for any \( p > 1 \), since \( \dot{\mathcal{B}}_{1,\infty}^{0} \) with \( q < \infty \) does not include the Gauss functions. In fact, \( f \in \dot{\mathcal{B}}_{1,\infty}^{0} \) with \( q < \infty \) implies that

\[
\| \hat{f} \|_{L^{\infty}(|\xi| \sim N)} \lesssim \| f_{N} \|_{L^{1}} \to 0 \quad (N \to +0).
\]

**Appendix A. Cubic fractional Schrödinger equations**

In the appendix, we consider the Cauchy problem to the fractional Schrödinger equation

\[
ivu_t + D^{\alpha}u = |u|^2u, \quad u(0, x) = \phi. \tag{A.1}
\]

By scaling invariance: for \( \lambda > 0 \),

\[
u(t, x) \to u_{\lambda} = \lambda^{\alpha/2}u(\lambda^{-a}t, \lambda x), \quad \phi \to \lambda^{\alpha/2}\phi(\lambda x).
\]

The critical Sobolev space in the sense of scaling is \( \dot{H}^{s_{c}} \) with \( s_{c} = \frac{d-a}{2} \) since \( \| \lambda^{\alpha/2}\phi(\lambda x) \|_{H^{s_{c}}} = \| \phi \|_{H^{s_{c}}} \). We prove the following results.

**Theorem A.1.** Assume \( d = 2 \), \( 0 < a < 2 \) and \( a \neq 1 \), \( \phi \in \dot{H}^{\frac{2-a}{2},-1} \). Then the Cauchy problem (A.1) is locally well-posed. Moreover, if \( \| \phi \|_{\dot{H}^{\frac{2-a}{2},-1}} \) is sufficiently small, then we have global well-posedness and scattering.

The space \( \dot{H}^{\frac{2-a}{2},1} \) is the Sobolev space \( \dot{H}^{\frac{2-a}{2}} \) with additional one order angular regularity and

\[
\| \phi \|_{\dot{H}^{\frac{2-a}{2},1}} = \| \phi \|_{\dot{H}^{\frac{2-a}{2}}} + \| \partial_{\theta}\phi \|_{\dot{H}^{\frac{2-a}{2}}}. \]

Unfortunately, we can’t cover the interesting case \( a = 1 \) which is the energy-critical half wave equation since the crucial \( L_{t}^{2}L_{x}^{\infty} \) estimate fails at \( d = 2 \) in the radial case (see [7]).

We prove Theorem A.1 by the standard iteration arguments using the \( L^{\infty}_{x} \) type estimate as in [8]. We only consider the radial case. By Duhamel’s principle, we get

\[
u = \Phi_{\phi}(u) := e^{iD^{\alpha}}\phi - i \int_{0}^{t} e^{i(t-s)D^{\alpha}}|u(s)|^2u(s)\,ds.
\]

For an interval \( I \), define the resolution space \( X_{I} \):

\[
X_{I} = \{ u : \text{radial}, \| u \|_{L_{t}^{2}L_{x}^{\infty}} \leq \eta, \| u \|_{L_{t}^{\infty}H^{\frac{2-a}{2}}_{x}} \leq M \}
\]
endowed with a distance $d(u,v) = \|u-v\|_{L^\infty_t L^2_x \cap L^\infty_t H^{\frac{1}{2} - a}_x}$, where $\eta, M$ will be determined later to make $\Phi_\phi: (X_I, d) \to (X_I, d)$ a contraction mapping. Then by fractional Leibniz rule we get

$$\|\Phi_\phi(u)\|_{L^2_t L^\infty_x} \lesssim \|e^{itD^a} \phi\|_{L^2_t L^\infty_x} + C\|u\|_{L^2_t H^{\frac{1}{2} - a}_x}^2.$$ \[ \lesssim \|e^{itD^a} \phi\|_{L^2_t L^\infty_x} + \|u\|_{L^2_t L^\infty_x}^2 \|u\|_{L^\infty_t H^{\frac{1}{2} - a}_x}.$$ 

Thus we can choose suitable $\eta, M$ to close the iteration arguments.

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