An Eigen Expansion Method for Elastic Boundary Conditions

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Abstract—With the aid of symplectic system, the mechanical problems related to elastic plates are studied. Using this method, displacement and stress components are taken as basic variables, so it is no longer necessary to make assumptions like the semi inverse method, and thus a set of concrete methods for dealing with non-homogeneous equations and boundary conditions are given. The mechanical response of elastic plates under external forces is discussed.

1. INTRODUCTION

In the study of traditional elastic mechanics problems, various computational methods have been usually used to eliminate the basic equations, so that a higher-order partial differential equations can be obtained after elimination, and then the unknown function can be solved. Such solution methods belong to the classical Lagrange system, and although it can reduce a certain amount of unknown variables through elimination, the order of the equations is also increased [1, 2]. Therefore, the difficulty of the problem will be correspondingly increased. Moreover, the semi inverse method is a kind of single variable method, which depends on the specific problems and can only find some special solutions. This method can not give an accurate answer to whether there are other solutions and how to find them. So there are some limitations in this method. With the deepening of research, this method is not very reasonable [3, 4].

In recent years, there has been a great breakthrough in solving the problem of elasticity. Zhong took the lead in introducing Hamilton system and symplectic mathematics into elastic materials and structures, and solved the problem of poor accuracy, incomplete analytical solution and small range of analytical solution brought by traditional methods [5, 6]. Due to the existence of conjugate symplectic orthogonal relationship between eigenvectors in symplectic system, the symplectic orthogonal relation can be used to expand the eigenfunctions in the study of elasticity, thus making up for the traditional separation of variables method. From the point of view of solution method, the problem in Lagrange system has been successfully transferred to symplectic system, which makes the research on the related problems of elastic plates rise a lot.

2. SOLUTION METHOD

Consider the cantilever shown in Fig. 1
Equilibrium equation and the geometric equation are

$$\sigma_{y,j} = 0$$  \hspace{1cm} (1)

$$\varepsilon_y = \frac{1}{2}(u_{jj} + u_{ij})$$  \hspace{1cm} (2)

The boundary condition of free end ($z = 0$) of cantilever beam can be written as

$$\int_{-h}^{h} x\sigma_x dx = M, \quad (z = 0)$$

$$\int_{-h}^{h} \tau_{xz} dx = P$$  \hspace{1cm} (3)

The displacement condition of the fixed end is

$$w = 0, \quad u = 0 \quad (z = l)$$  \hspace{1cm} (4)

Stress-strain relationship of viscoelastic materials can be expressed in integral form

$$\sigma_y(t) = \int_{-\infty}^{t} \frac{E(t-\tau)}{1+v} \partial_{\tau} \varepsilon_y d\tau$$

$$+ \delta_y \int_{-\infty}^{t} \frac{vE(t-\tau)}{(1+v)(1-2v)} \partial_{\tau} \varepsilon_{kl} d\tau$$  \hspace{1cm} (5)

The strain energy density function in phase space can be expressed as
According to the correspondence principle, the above stress-strain relationship can be described in phase space as

\[
\varepsilon_p = \frac{E}{2(1-v^2)} \left[ (\partial_y \bar{u})^2 + \bar{w}^2 + 2v \hat{w} \partial_y \bar{u} \right] + \frac{E}{4(1-v)} (\hat{u} + \partial_y \bar{w})
\] (6)

According to the correspondence principle, the above stress-strain relationship can be described in phase space as

\[
\sigma_y (s) = \frac{E(s)}{1+v} \varepsilon_y (s) + \frac{E(s)\nu}{(1-2\nu)(1+v)} \varepsilon_{kk} (s) \delta_y
\] (7)

If the constitutive relation obeys Maxwell model (as shown in Fig.2), the Young's modulus of phase space is

\[
E = \frac{\eta E_s}{(E + \eta s)}
\] (8)

In symplectic system, the governing equation of elastic solids is

\[
\dot{\psi} = H\psi + f
\] (9)

According to the properties of symplectic system, the eigen equation is expressed as

\[
\varphi = \varphi (r, \theta) e^{\mu z}
\] (10)

The corresponding lateral conditions are

\[
\begin{align*}
E \frac{\partial \nu}{\partial y} + \nu \sigma &= \sigma_0^z \\
\tau &= \tau_0^z
\end{align*}
\] (11)

The nonhomogeneous lateral boundary conditions can be reduced to homogeneous form. Therefore, we need to introduce a new set of dual variables

\[
\begin{align*}
u^* &= u \\
y^* &= \nu \\
\sigma^* &= \sigma - \frac{1}{2} \left[ \sigma_0^z (1 + \nu) + \tau_0^z (1 - \nu) \right] \\
\tau^* &= \tau - \frac{1}{2} \tau_0^z \left[ (1 + \nu) + \tau_0^z (1 - \nu) \right]
\end{align*}
\] (12)

The new dual equation can be expressed as

\[
\dot{\psi}^* = H\psi^* + f^*
\] (13)
Another result of homogenization is that the nonhomogeneous term in the dual equation becomes complex

$$f^* = \left\{ \begin{array}{l}
\frac{1}{2E} \left[ \sigma^+(1+y) + \sigma^-(1-y) \right] \\
\frac{1}{2E} \left[ r^+(1+y) + r^-(1-y) \right] \\
\frac{1}{2} (r^- - r^+) - \frac{1}{2} \left[ \sigma^+(1+y) + \sigma^-(1-y) \right] \\
\frac{1}{2} (\sigma^- - \sigma^+) - \frac{1}{2} \left[ r^+(1+y) + r^-(1-y) \right]
\end{array} \right\} + f \quad (14)$$

Expanding the nonhomogeneous term the special solution according to the eigensolution, we have

$$f = \sum_n \left[ d^*_n(x)\psi_n^{(a)}(y) + g^*_n(x)\psi_n^{(b)}(y) \right]$$

$$\psi_n = \sum_n \left[ D_n(x)\psi_n^{(a)}(y) + G_n(x)\psi_n^{(b)}(y) \right] \quad (15)$$

Substituting Eq.9 into Eq.2, we get

$$\hat{D}_n(x) = \mu_n D_n(x) + d^*_n(x)$$

$$\hat{G}_n(x) = \mu_n G_n(x) + g^*_n(x) \quad (16)$$

The solution is

$$D_n(x) = \int_0^1 d^*_n(\xi)e^{\mu_n(x-\xi)}d\xi;$$

$$G_n(x) = \int_0^1 g^*_n(\xi)e^{-\mu_n(x-\xi)}d\xi \quad (17)$$

3. BOUNDARY CONDITIONS

As we all know, the end condition can be either displacement or stress condition, or the mixed boundary condition given by displacement and stress at the same time. In symplectic system, the end conditions can be expressed by primaries and dual variables, which makes it convenient to solve the problem. As an example, we discuss the case of the stress at the left end and the displacement at the right end:

$$p_{x-1}(y) = p_{-1}$$

$$q_{x-1}(y) = q_1 \quad (18)$$

We express the solution of the problem as a linear combination of eigensolutions:

$$\psi = \sum a_n e^{\mu_n y} \psi_n^{(a)}(y) + b_n e^{-\mu_n y} \psi_n^{(b)}(y) \quad (19)$$

The coefficient satisfy
Thus, the end condition can be expanded to
\[
\sum_n a_n p_n^{(\alpha)} e^{-\mu_{nl}} + \sum_n b_n q_n^{(\beta)} e^{-\mu_{nl}} = p_i(y) \tag{21}
\]
\[
\sum_n a_n q_n^{(\alpha)} e^{\mu_{nl}} + \sum_n b_n q_n^{(\beta)} e^{\mu_{nl}} = q_i(y) \tag{21}
\]

The above equation can also be described in the following form
\[
\begin{bmatrix}
\psi^{(\alpha)}_n, J, \psi^{(\beta)}_i \\
\psi^{(\alpha)}_n, J, \psi^{(\beta)}_i \\
\end{bmatrix}
= 0
\]
\[
\begin{bmatrix}
\psi^{(\alpha)}_n, J, \psi^{(\beta)}_i \\
\psi^{(\alpha)}_n, J, \psi^{(\beta)}_i \\
\end{bmatrix}
= \delta(t) \delta_{nk} \tag{22}
\]

Through the above derivation, we get independent linear equations about coefficients
\[
\begin{bmatrix}
\int \int a_n p_n^{(\alpha)} q_j^{(\beta)} dy = \int \int \sum_n (a_n p_n^{(\alpha)} + b_n p_n^{(\beta)}) q_j^{(\beta)} dy \\
\int \int a_n q_n^{(\alpha)} p_j^{(\beta)} dy = \int \int \sum_n (a_n q_n^{(\alpha)} + b_n q_n^{(\beta)}) p_j^{(\beta)} dy \\
\end{bmatrix} \tag{23}
\]

4. NUMERICAL RESULTS

A cantilever beam in the rectangular coordinate system is considered in this section. The free end is under the action of a concentrated force. Dimensions of cantilever beams are taken as
\[
I / h = 8, M = 4KN \tag{24}
\]

Figure 3. Normal stress distribution
Fig. 3 and Fig. 4 are normal stress and shear stress distribution curve, respectively. In the figures, the stress far away from the fixed end hardly changes, indicating that in these areas, the zero eigenvector plays a leading role, and the influence of non-zero eigenvector is so small that it can be ignored. This result verifies Saint Venant's principle, that is, when the effect of local action is not considered, the approximate calculation with equivalent load can get enough accuracy. Fig. 5 describes the longitudinal displacement of the neutral layer of the beam as a function of time.

5. CONCLUSION
In this paper, the Hamiltonian system method is applied to the study of elastic materials. Using this method, an arbitrary boundary condition is expanded into a linear combination form of eigenvectors, and the combination can be determined according to the specific boundary conditions. The results show that the longitudinal displacement of the neutral layer increases with time, and there is an obvious stress concentration at the fixed end. These conclusions have certain guiding significance for the analysis and design of specific engineering projects.

ACKNOWLEDGMENTS
This research was financially supported by Major Natural Science Research Projects in Colleges and Universities of Jiangsu Province (17KJA430012).
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