Electromagnetic proton-neutron mass difference

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We discuss the Cottingham formula and evaluate the proton-neutron electromagnetic mass difference exploiting the state-of-the-art phenomenological input. We decompose individual contributions to the mass splitting into Born, inelastic and subtraction terms. We evaluate the subtraction-function contribution from the experimental input only and the Born term accounting for the modern low-$Q^2$ data.

Two isospin-violating effects inside nucleons, the difference between the up and down quark masses as well as electromagnetic interaction, result in the shift between the proton $M_p$ and neutron $M_n$ masses $\delta M_{p-n}$ \cite{1}:

$$\delta M_{p-n} = M_p - M_n = -1.29333217(42) \text{ MeV.} \quad (1)$$

It is well known that the QED contributions enter Eq. (1) with a positive sign. The leading electromagnetic correction was related to the phenomenological input from the electron-proton scattering by Cottingham in Ref. \cite{2} and investigated in detail together with ideas about the negative sign contributions in a historical review of Ref. \cite{3}. The origin of the negative sign due to the difference between up and down quark masses was pointed in Ref. \cite{4}, where authors evaluated as well the electromagnetic contribution: $\delta M_{\gamma\gamma}^\gamma = 0.76 \pm 0.30 \text{ MeV.}$ In Ref. \cite{5}, the author has renormalized the Cottingham formula explicitly and pointed on the small correction from the high-energy counterterms. Recent studies of Ref. \cite{6} accounted for the modern experimental data on the inelastic proton structure and have corrected the elastic contribution of Ref. \cite{4}. The new result $\delta M_{\gamma\gamma\gamma}^\gamma = 1.30 \pm 0.47 \text{ MeV}$ \cite{6} is within uncertainties of Refs. \cite{4, 7}. However, the central values are quite different, which motivates to explore individual contributions to the Cottingham formula in detail. The electromagnetic effect was studied also in Refs. \cite{7–10}, while the QCD splitting was investigated in Refs. \cite{11–15}. Both contributions were evaluated on the lattice in Refs. \cite{16–19}. The dispersive estimate of Ref. \cite{8}: $\delta M_{\gamma\gamma}^\gamma = 1.04 \pm 0.11 \text{ MeV}$, gave smaller uncertainty due to optimistic assumptions about our knowledge of the subtraction function and of the isovector nucleon polarizability. The best lattice result with four nondegenerate quark flavours for the electromagnetic contribution is $\delta M_{\gamma\gamma\gamma}^\gamma = 1.00 \pm 0.16 \text{ MeV}$ \cite{18}. It is in a good agreement with phenomenological estimates and has smaller error. The four-flavor result is smaller than the three-flavor calculation of Ref. \cite{19}: $\delta M_{\gamma\gamma\gamma}^\gamma = 1.71 \pm 0.30 \text{ MeV}$, and of Ref. \cite{17}: $\delta M_{\gamma\gamma\gamma}^\gamma = 1.59 \pm 0.46 \text{ MeV}$ and larger than the three-flavor studies of Ref. \cite{16} with the shift $\delta M_{\gamma\gamma\gamma}^\gamma = 0.38 \pm 0.68 \text{ MeV}$ and of Ref. \cite{10} with results in the range: $\delta M_{\gamma\gamma\gamma}^\gamma = 0.53–0.84 \text{ MeV.}$ To put constraints on the up-down quark mass difference, the lattice result of Ref. \cite{18} requires an independent cross check within the dispersion calculation.

In this paper, we present the derivation of the Cottingham formula considering the decomposition into the Born, inelastic and subtraction contributions. We evaluate Born and subtraction terms from the modern experimental input.

The forward doubly virtual Compton scattering (VVCS) process on a nucleon (see Fig. 1 for kinematics): $\gamma^\mu (q) + N(p) \rightarrow \gamma^\nu (q') + N(p')$, is described by the amplitude $T$. The latter can be expressed in terms of the forward VVCS tensor $M_{\mu\nu}$ as

$$T = \epsilon_{\nu}^\mu (q) \bar{\epsilon}_{\nu}\mu (q') \bar{N}(p') (4\pi M_{\mu\nu}) N(p), \quad (2)$$

where $N, \bar{N}$ denote the nucleon spinors, $\epsilon_{\nu}^\mu, \bar{\epsilon}_{\nu}\mu$ are the initial and final virtual photon polarization vectors. The nucleon is at rest in the laboratory frame, i.e., $p = (M, 0)$, while the photon energy is given by $\nu_{\gamma} = (p \cdot q)/M$ and the virtuality is $Q^2 = -q^2$.

![FIG. 1: Forward VVCS process.](image)

The nucleon self-energy correction is determined by the symmetric part of the forward VVCS tensor $M_{\mu\nu}^S$:

$$M_{\mu\nu}^S = \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1(\nu_{\gamma}, Q^2) + \left(p_{\nu} - \frac{p \cdot q}{q^2} q_{\nu}\right) \left(p_{\nu} - \frac{p \cdot q}{q^2} q_{\nu}\right) \frac{T_2(\nu_{\gamma}, Q^2)}{M^2}, \quad (3)$$

with the unpolarized forward Compton amplitudes $T_1$ and $T_2$, which enter Eq. (3) in a gauge-invariant way,
i.e., $q_0M^{\mu\nu} = q_0M^{\nu\mu} = 0$. The imaginary parts of the forward VVCS amplitudes $T_1$ and $T_2$ are related to the unpolarized proton structure functions $F_1$ and $F_2$ by

$$
\Im T_1(\nu, Q^2) = \frac{e^2}{4M} F_1(\nu, Q^2),
$$

$$
\Im T_2(\nu, Q^2) = \frac{e^2}{4\nu} F_2(\nu, Q^2),
$$

where $e$ denotes the electric charge.

The real part of the even amplitude $T_1$ is related to the imaginary part through the subtracted dispersion relation:

$$
\Re T_1(\nu, Q^2) = T_{\text{subt}}(0, Q^2) + \Re T_{\text{Born}}(\nu, Q^2) + \frac{e^2\nu^2}{2\pi} \int_0^\infty \frac{F_1(\nu, Q^2)}{M^2 - \nu^2 - \nu_\text{thr}^2} \, \text{d}\nu,
$$

with the pion-nucleon production threshold: $
u_\text{thr}^\text{inel} = m_x + (m_x^2 + Q^2)/(2M)$, where $m_x$ denotes the pion mass, $T_{\text{subt}}(0, Q^2)$ is the subtraction function at zero photon energy $\nu = 0$, and $F_1$ contains only the inelastic contributions since we have separated the Born piece $T_{\text{Born}}(\nu, Q^2)$

The real part of the unpolarized amplitude $T_2$ can be obtained from the unsubtracted dispersion relation:

$$
\Re T_2(\nu, Q^2) = \Re T_{2,\text{Born}}(\nu, Q^2) + \frac{e^2\nu^2}{2\pi} \int_0^\infty \frac{F_2(\nu, Q^2)}{\nu^2 - \nu_\text{thr}^2} \, \text{d}\nu.
$$

resulting into the electromagnetic mass shift $\delta M^\gamma$:

$$
\delta M^\gamma = \int \frac{i\text{d}^4q}{(2\pi)^3} \frac{M^\mu_\nu}{q^2}.
$$

To relate it to the experimental input, we perform the Wick rotation first: $q_0 \to i\nu, \gamma$, and introduce the space-like virtuality $Q^2 = -q^2$. The mass shift is given by

$$
\delta M^\gamma = \int \frac{\text{d}\nu \text{d}^3Q}{(2\pi)^3} \frac{\sqrt{Q^2 - \nu^2} M^\mu_\nu}{Q^2}.
$$

Changing the integration order and accounting for the crossing properties of the Compton scattering, the Cotttingham formula [2] gives:

$$
\delta M^\gamma = \int_0^1 \frac{\text{d}Q^2}{(2\pi)^2} \int_0^\tau \frac{\sqrt{1 - \tau^2} \text{d}\hat{\tau}}{\sqrt{\tau}} M^\mu_\nu,
$$

with $\hat{\tau} = \nu^2/Q^2$ and the trace of the forward VVCS tensor:

$$
M^\nu_\mu = -3T_1(\nu\nu, Q^2) + (1 - \tau) T_2(\nu\nu, Q^2).
$$

Following the decomposition of the forward VVCS amplitudes of Eqs. (6) and (7), we introduce the Born contribution $\delta \text{M}_{\text{Born}}$, the inelastic correction $\delta \text{M}_{\text{inel}}$ and the subtraction term $\delta \text{M}_{\text{subt}}$:

$$
\delta M^\gamma = \delta \text{M}_{\text{Born}} + \delta \text{M}_{\text{inel}} + \delta \text{M}_{\text{subt}}.
$$

Exploiting the integral:

$$
\int_0^1 \frac{\sqrt{1 - \tau} \text{d}\hat{\tau}}{\sqrt{\tau}} = \frac{\pi}{2},
$$

the contribution of the subtraction function $T_{1,\text{subt}}(0, Q^2)$ to the proton-neutron mass difference $\delta M_{p-n}$ can be easily expressed as [4, 6, 7]

$$
\delta M_{p-n} = -\frac{3}{8\pi} \int_0^\infty \text{d}Q^2 T_{1,\text{subt}}(0, Q^2).
$$

Instead of evaluation of the isovector magnetic polarizability from the derivative of the longitudinal to transverse cross sections ratio at origin relying on four data points at relatively large virtuality $Q^2 \gtrsim 0.75$ GeV$^2$ [21, 22] with assumption of energy independence and isospin symmetry [7], we take the difference between the proton $\beta^p_M$ and neutron $\beta^n_M$ magnetic polarizabilities:

$$
\beta^p_{M-p-n} = \beta^p_M - \beta^n_M = (-1.2 \pm 1.3) \times 10^{-4} \text{ fm}^3,
$$

from p.d.g. [1] and estimate the subtraction function at higher $Q^2$ evaluating the unsubtracted dispersion relation for the amplitude free from the Regge high-energy effects.
behavior, see Refs. [23, 24], with an input from Refs. [25–28]. We estimate the uncertainty of the proton structure functions at 3% level, double the error for the neutron structure functions and assign a 30% uncertainty to a Reggeon pole residue [7]. We connect the experimental isovector magnetic polarizability and higher-$Q^2$ region on the level of $\beta_{M}^{p-n}(Q^2) = T_{1,p-n}^{\text{subt}}(0, Q^2) / Q^2$, with the p.d.g. value at zero virtuality $\beta_{M}^{p-n}(0) = \beta_{M}^{p-n}$, see Fig. 3 for details. The subtraction term contributes:

\[
\delta M_{p-n}^{\text{subt}} = 0.54 \pm 0.46 \text{ MeV},
\]

where we have chosen the upper integration limit as 2 GeV$^2$ [6]. We have added uncertainties of the subtraction-function contribution: 0.44 MeV,\(^1\) and due to the variation of the upper integration limit over the range 1.5–2.5 GeV$^2$: 0.13 MeV, in quadrature. The saturation of this term from the empirically estimated subtraction function is better than from the dipole form of the subtraction function from Ref. [6] (with the final result $\delta M_{p-n}^{\text{subt}} = 0.47 \pm 0.47$ MeV) but worse than from the suppressed by one or two additional powers of $Q^2$ function from Ref. [8] (with the final result $\delta M_{p-n}^{\text{subt}} = 0.21 \pm 0.11$ MeV). In future, it would be interesting to compare the high-energy behavior of the data-based isovector evaluation to the operator product expansion of Ref. [29], where applications to the proton case were discussed in detail. Our central value is determined by the isovector nucleon magnetic polarizability and can change with the forthcoming Compton scattering data on the proton and deuteron targets [30–33]. In order to compete with the lattice calculation of Ref. [18], besides the necessary improvement of the structure functions in the resonance and DIS regions the uncertainty on the isovector magnetic polarizability has to be reduced to $(0.3–0.4) \times 10^{-3} \text{ fm}^3$, at least. Moreover, additional studies within the framework of low-energy effective field theories [34, 35] could shed more light on the most uncertain low-$Q^2$ region, and the operator product expansion [29] could constrain uncertainties from high-$Q^2$ region.

We obtain the Born contribution substituting the corresponding unpolarized Compton amplitudes $T_1^{\text{Born}}$ and $T_2^{\text{Born}}$:

\[
T_1^{\text{Born}}(\hat{\tau}, Q^2) = \frac{\alpha}{M} \left( \frac{G_E^M(Q^2)}{1 - \frac{i}{\tau_P} - i\varepsilon} - F_E^p(Q^2) \right),
\]

\[
T_2^{\text{Born}}(\hat{\tau}, Q^2) = \frac{\alpha}{M} \left( \frac{G_E^M(Q^2) + \tau_P G_M^E(Q^2)}{1 + \tau_P} \left( 1 - \frac{i}{\tau_P} - i\varepsilon \right) \right),
\]

with the Dirac ($G_D$), Sachs electric ($G_E$) and magnetic ($G_M$) form factors, the electromagnetic coupling constant $\alpha \equiv e^2/(4\pi)$ and the notation $\tau_P = Q^2/(4M^2)$. Introducing the additional notation $\rho(\tau)$:

\[\rho(\tau) = 2 \left( \tau - \sqrt{\tau(1 + \tau)} \right),\]

and exploiting the integral:

\[\int_0^1 \frac{\sqrt{1 - \tau d\hat{\tau}}}{\sqrt{\tau \left( 1 + \frac{\hat{\tau}}{\tau} \right)}} = -\frac{\pi}{2}\rho(\tau),\]

we express the Born contribution to the proton-neutron mass difference $\delta M_{p-n}^{\text{Born}}$ as

\[
\delta M_{p-n}^{\text{Born}} = \frac{3\alpha}{8\pi M} \int_0^\infty dQ^2 \left( F_D^p(Q^2)^2 + \rho(\tau_P) G_M^E(Q^2) \right)
\]

\[\times \left( 1 + \frac{\tau_P}{\rho(\tau_P)} \right) \times \frac{G_E^M(Q^2) + \tau_P G_M^E(Q^2)}{1 + \tau_P}.\]

For the numerical evaluation, we take the up-to-date proton form factors with uncertainties from Refs. [36, 37] and the neutron form factors from Refs. [38–42]. For the neutron, we obtain the central value averaging over the form factor parametrizations and estimate the uncertainty as a difference between the largest and smallest results. The resulting Born contribution is given by

\[
\delta M_{p-n}^{\text{Born}} = 0.74 \pm 0.01 \text{ MeV},
\]

where we integrate over the same regions as for the subtraction term. The corrections to the proton mass $\delta M_{p}^{\text{Born}}$:

\[
\delta M_{p}^{\text{Born}} = 0.54 \pm 0.01 \text{ MeV},
\]

and neutron mass $\delta M_{n}^{\text{Born}}$:

\[
\delta M_{n}^{\text{Born}} = -0.20 \pm 0.01 \text{ MeV},
\]

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\(^1\)Note that without the uncertainty of the Reggeon pole residue, the related error reduces to 0.36 MeV. Without the uncertainty of the structure functions, it reduces to 0.33 MeV.
have an opposite sign enhancing the electromagnetic mass difference. Note that the analytical expression of Eq. (24) has no analogous in Ref. [6], the difference is in the $G_2^n$ contribution to the subtraction term [6, 20]. Apparently, this mismatch was accounted in the numerical evaluation, since the result of Ref. [6] for the whole elastic contribution: 0.77 MeV, is quite close to ours.

With the same integrals of Eqs. (16) and (23), the inelastic contribution is expressed in terms of the unpolarized structure functions as

$$\delta M_{p-n}^{\text{inel}} = -\frac{\alpha}{4\pi} \int_0^\infty dQ^2 \int_0^{\nu^\text{inel}} \frac{d\nu_\gamma}{\nu_\gamma} \left\{ \rho(\hat{\tau}) \frac{F_2(\nu_\gamma, Q^2)}{\nu_\gamma} - (1 + \rho(\hat{\tau})) \left( \frac{3F_1(\nu_\gamma, Q^2)}{M} - \nu_\gamma F_2(\nu_\gamma, Q^2) \right) \right\},$$

(28)

which is exactly the result of Refs. [4, 6, 7].

Accounting for the inelastic correction of Refs. [6, 21, 27, 28, 43]:

$$\delta M_{p-n}^{\text{inel}} = 0.057 \pm 0.016 \text{ MeV},$$

(29)

and neglecting the counterterms contribution [5, 6], the resulting mass difference $\delta M_{p-n}^\gamma$ is given by

$$\delta M_{p-n}^\gamma = 1.33 \pm 0.46 \text{ MeV}. \quad (30)$$

We have presented the Cottingham formula in terms of the phenomenological input. We have updated the Born correction and estimated the subtraction term based on the experimental input. Our total result is within errors of the previous estimates [4, 6, 8] due to the large uncertainty of the correction from the subtraction function. However, the knowledge of the Born contribution and of the subtraction term is improved. Precise studies of the proton and neutron magnetic polarizabilities, inelastic structure functions and Regge trajectories will be able to improve the dispersive evaluation further.

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