SPECTRAL DIMENSIONS OF KREĬN–FELLER OPERATORS IN HIGHER DIMENSIONS

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Abstract. We study the spectral dimensions of KreĬn–Feller operators for arbitrary finite Borel measures \( \nu \) on the \( d \)-dimensional unit cube \( (d \geq 2) \) via a form approach. We make use of the spectral partition function of \( \nu \) as introduced in [KN23] and, assuming that the lower \( \infty \)-dimension of \( \nu \) exceeds \( d - 2 \), we identify the upper Neumann spectral dimension as the unique zero of the spectral partition function, thus revealing the intrinsic connection of these spectral and fractal-geometric quantities. We show that if the lower \( \infty \)-dimension of \( \nu \) is strictly less than \( d - 2 \), the form approach breaks down. Examples are given for the critical case, that is the lower \( \infty \)-dimension of \( \nu \) equals \( d - 2 \), such that for one case the form approach breaks down, another case, where the operator is well defined but we have no discrete set of eigenvalues, and for the third case, where the spectral dimension exists. We provide additional regularity assumptions on the spectral partition function, guaranteeing that the Neumann spectral dimension exists and coincides with the Dirichlet spectral dimension. The significance of our new approach is illustrated by several prominent examples previously treated in the literature, namely absolutely continuous measures and more generally Ahlfors–David regular measures, and examples not previously treated in the literature, namely self-conformal measures with or without overlaps, for which we show that both the Dirichlet and Neumann spectral dimensions exist and how they can be obtained from the \( L^q \)-spectrum of the measures. We demonstrate how our approach can be used to obtain upper and lower asymptotic spectral bounds for the case of Ahlfors–David regular measures. Moreover, we provide sharp bounds for the upper Neumann spectral dimension in terms of the upper Minkowski dimension of the support of \( \nu \) and its lower \( \infty \)-dimension. Finally, we give an example for which the spectral dimension does not exist.

1. Introduction and statement of main results

1.1. Introduction and background. In this article we extend our work on the spectral dimensions of the KreĬn–Feller operators with respect to compactly supported finite Borel measures \( \nu \) to higher dimensions. KreĬn–Feller operators for the one-dimensional case were introduced in [Kre51; Fel57; KK58] and since the late 1950’s have been studied in some detail by various authors [Kac59; UH59; MR62; BS70; KW82; Fuj87; SV95; Vol05; Nga11; Fag12; Arz14; Arz15; DN15; FW17; NTX18; FM20; Min20; NX20; PS21]; more recently, in [KN22b; KN22c] the authors gave an almost complete picture of the relationships between the spectral dimension and the \( L^q \)-spectrum of \( \nu \). For dimension \( d \geq 2 \), however, the situation is quite different; in general, it is not even possible to define the KreĬn–Feller operator for a given finite Borel measure \( \nu \), since in general there is no continuous embedding of the Sobolev space of weakly differentiable functions into \( L^q \) (for example, when \( \nu \) has atoms). For Dirichlet boundary conditions, in [HLN06] a sufficient condition in terms of the maximal asymptotic direction of the \( L^q \)-spectrum of \( \nu \) has been established, as provided in (1.3), which ensures a compact embedding of the relevant Sobolev space into \( L^q \). We would like to note that Triebel already stated this condition implicitly in 1997 in the fundamental book [Tri97]. In 2003 (see [Tri03; Tri04]) he also indicated that there should be a subtle connection between the multifractal concept of the \( L^q \)-spectrum and analytic properties of the associated “fractal” operators, a conjecture that we can confirm with this work. The connection of fractal properties with spectral properties of reasonable associated operators is a long ongoing task and we refer the interested reader to [LF89; Lap89; Lap92; KL93; BH97].

In this paper we extend ideas for the one-dimensional case developed in [KN22b; KN22c] to higher dimensions \( d \geq 2 \) and in this way follow the line of investigation outlined in [BS67; NS95; NS01; HLN06; NX21]. We will introduce the new notion of partition functions, which is needed for higher dimensions and naturally generalises \( L^q \)-spectra (Section 3). This new construction sheds also...
light on certain optimal embedding constants for Sobolev spaces (see Section 5.2) as elaborated by Maz’ya and Preobrazenskii [Maz85; MP84] for $d = 2$ and Adams [Maz11, Section 1.4.1] for $d > 2$.

In contrast to the one-dimensional case, the spectral dimension of Kreĭn–Feller operators has recently been computed for self-similar measures under the open set condition (OSC), by Triebel [Tri97, Theorem 30.2] in particular in the setting of Ahlfors–David regular measures with a lot of interesting refinements and open questions (e.g., in the recent work [RS21; RT22], and also by Ngai and Xie [NX21] for a class of graph-directed self-similar measures satisfying the graph open set condition). In [NX21, Sec. 5] Ngai and Xie pointed out that it would also be interesting to study self-similar measures defined by IFSs with overlaps on $\mathbb{R}^d$ with $d \geq 2$. Indeed, as an application of our general results from Section 1.3, we are able to extend these achievements to self-conformal measures without any restriction on the separation conditions.

We prove that under the assumption (1.3) the spectral dimension for self-conformal measures can be identified as the unique intersection of the $L^q$-spectrum with the line of slope $2 - d$ through the origin (Theorem 1.16). This work is based on the dissertation [Nie21] by the second author.

1.2. Preliminaries. We now proceed to outline the theoretical preliminaries necessary to determine the spectral properties of the Kreĭn–Feller operator $\Delta^{\Omega/N}_0$ for a given finite non-zero Borel measure $\nu$ on the fixed $d$-dimensional unit cube $\Omega := \prod_{i=1}^d I_i$, $d \geq 2$ with $I_i$ unit intervals, for $f = 1, \ldots, d$, each of which can be chosen to be either half-open, open, or closed (that is we have $\nu(\Omega) = \nu(\mathbb{R}^d) \in (0, +\infty)$). In the following we fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, that is a bounded domain with Lipschitz boundary, for which we assume without loss of generality for notational convenience that $\Omega$ lies in the open unit cube. Let us define the Sobolev spaces $H^q(\Omega)$ as the completion of $C^\infty_0(\Omega) := C^\infty_0(\overline{\Omega})$ with respect to the metric $\| \cdot \|_{H^q(\Omega)}$ given by the inner product

$$\langle f, g \rangle_{H^q(\Omega)} := \int_{\Omega} f g \ d\Lambda + \int_{\Omega} \nabla f \nabla g \ d\Lambda,$$

and let $H^q(\Omega)$ be the respective completion of $C^\infty_0(\Omega) := C^\infty_0(\Omega)$. Here, $\Lambda$ denotes the $d$-dimensional Lebesgue measure, $C^\infty_0(\Omega)$ the vector space of smooth functions with compact support contained in $\Omega$, and $C^\infty_0(\Omega)$ the vector space of functions $f : \overline{\Omega} \to \mathbb{R}$ such that $f|_{\Omega} \in C^\infty(\Omega)$ for all $m \in \mathbb{N}$ with $D^m f|_{\Omega}$ uniformly continuous on $\Omega$ for all $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ (and therefore allowing a unique continuous extension to $\overline{\Omega}$).

For all $u \in H^q(\Omega)$, resp. $u \in \{ f \in H^q(\Omega) : \int_{\Omega} f \ d\Lambda = 0 \}$, the Poincaré inequality, resp. Poincaré–Wirtinger inequality (see [Rui12, Lemma 3, p. 500] and Lemma 2.1), reads, for some constant $c > 0$, as follows

$$\| u \|_{L^2(\Omega)} \leq c \| \nabla u \|_{L^2(\Omega)}.$$  \hfill (1.1)

As a consequence this gives rise to an equivalent metric $\| \cdot \|_{H^q(\Omega)}$ on $H^q(\Omega)$ given by the inner product

$$\langle f, g \rangle_{H^q(\Omega)} := \int_{\Omega} \nabla f \nabla g \ d\Lambda.$$

The space $H^q(\Omega)$ with the bilinear form $\langle \cdot, \cdot \rangle_{H^q(\Omega)}$ defines a Hilbert space and $H^q(\Omega)$ a closed subspace. Let us write $\Omega^\nu := \overline{\Omega}$ and $\Omega^0 := \Omega$. On the one hand, for a finite Borel measure $\nu$ with $\text{supp } \nu \subset \Omega^\nu$, we will see in Proposition 2.6 that the natural embedding

$$i : C^\infty_{\nu,0}(\Omega) \to L^2(\Omega^\nu)$$

is not continuous if the lower $\infty$-dimension of $\nu$,

$$\dim_{\infty}(\nu) := \liminf_{r \to 0} \frac{\sup_{x \in \Omega} \log \nu(B(x, r))}{-\log r}$$  \hfill (1.2)

lies under a certain threshold, namely, $\dim_{\infty}(\nu) < d - 2$. Here, $B(x, r)$ denotes the open euclidean ball with centre $x$ and radius $r$. Obviously, we always have $\dim_{\infty}(\nu) \leq d$, and the assumption $\dim_{\infty}(\nu) > 0$ excludes the possibility of $\nu$ having atoms.

On the other hand, if the measure $\nu$ fulfils the Hu–Lau–Ngai condition from [HLN06], i.e.

$$\dim_{\infty}(\nu) > d - 2,$$  \hfill (1.3)

then the $\nu$-Poincaré inequality holds, that is for some $c > 0$ we have

$$\| u \|_{L^2(\Omega)} \leq c \| \nu \|_{H^q(\Omega)} \quad \text{for all } u \in C^\infty_{\nu}(\Omega).$$  \hfill (1.4)
Since $C^0_{D,N}(\Omega)$ lies dense in $H^{D,N}(\Omega)$ the inequality gives rise to a continuous mapping
\[ \iota \colon H^{D,N}(\Omega) \to L^2(\Omega^{D,N}). \]
If $\iota$ is also injective, then we may regard $H^{D,N}(\Omega)$ as a subspace of $L^2(\Omega^{D,N})$. In case the map is not injective we consider the following closed subspace of $H^{D,N}(\Omega)$
\[ \mathcal{H}^{D,N} := \ker (\iota) = \{ f \in H^{D,N}(\Omega) : \|f\|_{L^2(\Omega^{D,N})} = 0 \} \]
and obtain a natural embedding of its orthogonal complement in $H^{D,N}(\Omega)$,
\[ \left( \mathcal{H}^{D,N} \right)^* := \{ f \in H^{D,N}(\Omega) : \forall g \in \mathcal{H}^{D,N} : (f,g)_{H^{D,N}(\Omega)} = 0 \} \hookrightarrow L^2(\Omega^{D,N}), \]
which is again denoted by $\iota$. Since $\iota$ maps $\left( \mathcal{H}^{D,N} \right)^*$ bijectively to $\mathcal{E}^{D,N}(\Omega)$, we may define the relevant corresponding forms with Dirichlet and Neumann boundary conditions by the push forward of the inner product in $H^{D,N}(\Omega)$, that is for $u, v \in \text{dom}(\mathcal{E}^{D,N})$,
\[ \mathcal{E}^{D,N}(u,v) := \left\langle \iota^{-1}u, \iota^{-1}v \right\rangle_{H^{D,N}(\Omega)}. \]
Assuming (1.4), we have that $\text{dom}(\mathcal{E}^{D,N})$ equipped with the inner product $\langle f,g \rangle + \mathcal{E}^{D,N}(f,g)$ defines a Hilbert space, i.e. $\mathcal{E}^{D,N}$ is a closed form with respect to $L^2(\Omega^{D,N})$. Hence, under the assumption (1.4) for $\mathcal{E}^{D,N}$, e.g. by [Kig01, Theorem B.1.6], there exists non-negative self-adjoint operator $\Delta^{D,N} := \Delta^{D,N}_{\lambda}$ on $L^2(\Omega^{D,N})$ such that
\[ f \in \text{dom}(\Delta^{D,N}) \iff \{ f \in \text{dom}(\mathcal{E}^{D,N}) \text{ and } \exists u \in L^2(\Omega^{D,N}) : \forall g \in \text{dom}(\mathcal{E}^{D,N}) : \mathcal{E}^{D,N}(f,g) = (u,g)_{L^2(\Omega^{D,N})} \}. \]
In this case we have $\Delta^{D,N} f = u$. Note that $\text{dom}(\Delta^{D,N}) \subset \text{dom}(\Delta^{D,N}_{\lambda}) = \text{dom}(\mathcal{E}^{D,N})$. For a vector space $V$ we will use the shorthand notation $V^* := V \setminus \{0\}$. We call $\Delta^{D,N}$ Kreĭn–Feller operator and $f \in \text{dom}(\Delta^{D,N})$ a (Dirichlet/Neumann) eigenfunction with eigenvalue $\lambda \in \mathbb{R}$ if
\[ \mathcal{E}^{D,N}(f,g) = \lambda (f,g)_{L^2(\Omega^{D,N})} \text{ for all } g \in \text{dom}(\mathcal{E}^{D,N}). \]
To deduce that the embedding $\text{dom}(\mathcal{E}^{D,N}) \hookrightarrow L^2(\Omega^{D,N})$ is compact under the assumption $\dim_p(\nu) > d - 2$, we need the following result due to Maz'ya [Maz85, Theorem 3, p. 386] and [Maz85, Theorem 4, p. 387]: Let $H^1(\mathbb{R}^d)$ denote the usual Sobolev space for $\mathbb{R}^d$ with corresponding norm $\| \cdot \|_{H^1(\mathbb{R}^d)}$. Then, for $a, b \in \mathbb{R}$, setting
\[ \xi_{a,b} := \begin{cases} \sup \{ \|g\|_{H^1(\mathbb{R}^d)} : x \in \mathbb{R}^d, \varphi \in (0,r) \}, & \text{for } a = 0, \\ \sup \{ \|g\|_{H^1(\mathbb{R}^d)} : x \in \mathbb{R}^d, \varphi \in (0,r) \}, & \text{for } a \neq 0, \end{cases} \tag{1.5} \]
the set $\{ u \in C^0_{\nu}(\mathbb{R}^d) : \|u\|_{H^1(\mathbb{R}^d)} \leq 1 \}$ is precompact in $L^2_{\nu}$ if and only if $\lim_{\nu \to 0} C_{\nu,2-d/2} = 0$. That this condition is guaranteed by our assumption (1.3) is demonstrated for the Dirichlet case in [HNL06] and follows along the same lines also for the Neumann case by appropriately using the continuity of the extension operator (see Section 2.1 and [Nie22] for details).
If the embedding $\iota$ is compact, then $\Delta^{D,N}$ admits a countable set of eigenfunctions spanning $L^2(\Omega^{D,N})$ with a non-negative and non-decreasing sequence of eigenvalues $\{ \lambda_{\nu} \}_{\nu \in \mathbb{N}}$ tending to infinity corresponding to the orthonormal systems of eigenfunctions $\{ \varphi_{\nu} \}_{\nu \in \mathbb{N}}$.

Since we mainly concentrate on the case where $\Omega$ is equal to the interior $\mathbb{B} = (0,1)^d$ of the unit cube $\mathbb{B}$, we write in this case $H^{D,N} := H^{D,N}(\mathbb{B})$, and $L^2_{\nu} := L^2_{\nu}(\mathbb{B})$.

As mentioned above, the Hu–Lau–Ngai condition already appeared implicitly in [Tri97, Theorem 30.2 (Isotropic fractal drum)] in the context of Ahlfors–David regular measures, for which we provide more details in Section 1.4.3 below) and for higher order operators an appropriately adapted version also appears in the recent work [RS21; RT22].

For the $\infty$-dimension of $\nu$ with supp($\nu$) $\subset \mathbb{B}$ we alternatively have (see e.g. [Str93]) $\dim_{\infty}(\nu) = \lim_{\nu \to \infty} \max_{Q \in \mathcal{D}(\nu)} \log v(Q) / \log (2^{-a})$, where $\mathcal{D}(\nu)$ denotes a partition of $\mathbb{B}$ by cubes of the form $Q := \{ \nu \}_{\nu \in \mathbb{N}} I$, with (half-open, open, or closed) intervals $I$, with endpoints in the dyadic grid of size $2^{-a}$, i.e. $(k-1)2^{-a}, k2^{-a}$ for some $k \in \mathbb{Z}$. Note that by our assumption on the intervals $I$, which are individually chosen for each $Q$, these cubes are not necessarily congruent, in that we allow that
In general, there exists a constant $W$ we will provide an example showing that the upper and lower spectral dimension in general do not necessarily impose spectral power law asymptotics (see [KN22c]). The spectral dimension also provides some essential information on the domains of the associated Dirichlet forms and the Krein–Feller operator, namely via the spectral representation given by

$$
\text{dom}(\mathcal{D}^{D/N}) = \left\{ \sum_{s \geq 0} x_s \varphi_s^{D/N} : \sum_{s \geq 0} \alpha_s \varphi_s^{D/N} < \infty \right\},
$$

$$
\text{dom}(\mathcal{D}^{L/N}) = \left\{ \sum_{s \geq 0} x_s \varphi_s^{L/N} : \sum_{s \geq 0} \alpha_s \varphi_s^{L/N} < \infty \right\}.
$$

Next, let us turn to the concept of partition functions, which in a certain extent is borrowed from the thermodynamic formalism. Following [KN23], for an arbitrary monotone set function $\mathcal{D} : \mathcal{D} \rightarrow \mathbb{R}_{\geq 0}$, we define the $\mathcal{D}$-partition function, for $q \in \mathbb{R}_{\geq 0}$,

$$
\tau^{\mathcal{D}/N}_{\mathcal{D}}(q) := \limsup_{n \to \infty} \tau^{\mathcal{D}/N}_{\mathcal{D}}(q) \quad \text{with} \quad \tau^{\mathcal{D}/N}_{\mathcal{D}}(q) := \frac{1}{2^n} \log \sum_{Q \in \mathcal{D}^{N}} \mathcal{D}(Q)^q (1.6)
$$

with $\mathcal{D}^{N} := \left\{ Q \in \mathcal{D}^{N} : \partial Q \cap \overline{Q} = \emptyset \right\}$. The reason why the definition of $\mathcal{D}^{N}$ is appropriate becomes apparent in constructing certain functions with compact support contained in $\mathcal{D}$ for the proof of the lower bounds in the Dirichlet case (see proof of Lemma 6.1). Note that we use the convention $0^0 = 0$, that is for $q = 0$ we neglect the summands with $\mathcal{D}(Q) = 0$ in the definition of $\tau^{\mathcal{D}/N}_{\mathcal{D}}$. We consider the critical exponent

$$
\kappa_3 := \inf \left\{ q \geq 0 : \sum_{Q \in \mathcal{D}^{N}} \mathcal{D}(Q)^q < \infty \right\}.
$$

An important special case is the $L^s$ spectrum $\beta^{\mathcal{D}/N}_{s} := \tau^{\mathcal{D}/N}_{s}$ of $\mathcal{D}$, which is the relevant quantity in the one-dimensional case and also in certain higher-dimensional cases. While in the higher dimensional, we will particularly be interested in the set function

$$
\mathcal{J}_{a,b}(Q) := \sup_{Q \in \mathcal{D}(Q)} \left\{ \nu \left( \mathcal{D}(Q)^{a} \right) \right\}, \quad \mathcal{J}_a(\mathcal{D}(Q)) := \mathcal{J}_{a,2} \mathcal{J}_{\mathcal{D},1} (Q) \quad a \geq 0,
$$

with $b \geq 0$ and $a \in \mathbb{R}$. For $t \geq 2$, we write $\mathcal{J}_t(\mathcal{D}(Q)) := \mathcal{J}_{\mathcal{D},1} \mathcal{J}_{\mathcal{D},-1} (Q)$. We note that the general parameter $a,b$ will also prove useful when considering polyharmonic operators in higher dimensions or approximation order with respect to Koltmogorov, Gel'fand, or linear widths as elaborated in [KN22a; KW23]. In these works, the deep connection to the original ideas of entropy numbers introduced by Koltmogorov also becomes apparent. In [KNZ23] we address the quantization problem, that is the speed of approximation of a compactly supported Borel probability measure by finitely supported measures (see [GL00] for an introduction), by adapting the methods from [KN23] presented in Section 4 to $\mathcal{J}_{a,b}$ with $a \in \mathbb{R}$ and identify the upper quantization dimension of $\nu$ with its Rényi dimension.

Our most powerful auxiliary object is the (Dirichlet/Neumann) spectral partition function with respect to $\nu$ given by the special choice $\mathcal{J} = \mathcal{J}_{a,b}$. As a consequence of Lemma 3.2 we know that the
spectral partition function does not depend on the specific choice of the collection of dyadic cubes $\mathcal{D}_N$. First, to obtain upper estimates of the spectral dimension, we construct optimal partitions using an adaptive approximation algorithm as worded out in [KN23] and presented in Section 4. Let us define the set of $3$-partitions $\Pi_3$ to be the set of finite collections of dyadic cubes such that for all $P \in \Pi_3$ there exists a partition $P$ of $\mathbb{Z}$ by dyadic cubes from $\mathcal{D}$ with $P = \{Q \in \tilde{P} : \mathcal{J}(Q) > 0\}$. We define

$$M_N(x) := \inf \left\{ \frac{\text{card}(P)}{\text{card}(P) : P \in \Pi_3, \max_{Q \in P} \mathcal{J}(Q) < 1/x} \right\}.$$ 

and

$$\tilde{M}_N := \limsup_{x \to \infty} \frac{\log M_N(x)}{\log x}, \quad \underline{h}_3 := \liminf_{x \to \infty} \frac{\log M_N(x)}{\log x}$$

will be called the upper, resp. lower, $3$-partition entropy. In Section 4 we will recall results from [KN23] to establish a connection between $J$ and set $\Pi_3 = \mathcal{J}$. Indeed, from (4.1) we infer

$$\|u\|_{L^2(\Omega)} \leq J(\Omega) \|\nabla u\|_{L^2(\Omega)}^2.$$ 

Then $N^3 \leq M_3$ and in particular, $\tilde{M}_3 \leq \underline{h}_3$ and $\underline{h}_3 \leq \bar{h}_3$.

For lower estimates of the spectral dimension we use certain disjoint families of dyadic cubes and borrow ideas from the coarse multifractal analysis (see [Fal14; Rie95]) which will be the topic of Section 4, which summarises results from [KN23]. In there we will also see how the dyadic partition approach and the optimal partition approach are related by ideas from large deviation theory. For all $n \in \mathbb{N}$ and $\alpha > 0$, we define

$$N_{3,n} := \frac{\text{card} B_{3,n}}{B_{3,n}}, \quad B_{3,n} := \left\{ Q \in \mathcal{D}_n : \mathcal{J}(Q) \geq 2^{-\alpha n} \right\},$$

and set

$$\overline{F}_3(n) := \inf_{n} \frac{\log^+ (N_{3,n})}{\log (2^n)} \quad \text{and} \quad \underline{F}_3(n) := \inf_{n} \frac{\log^+ (N_{3,n})}{\log (2^n)},$$

with $\log^+(x) := \max \{0, \log(x)\}$, $x \geq 0$. We refer to the quantities

$$\overline{F}_3 := \sup_{\alpha > 0} \frac{\overline{F}_3(\alpha)}{\alpha} \quad \text{and} \quad \underline{F}_3 := \sup_{\alpha > 0} \frac{\underline{F}_3(\alpha)}{\alpha}$$

as the upper, resp. lower, optimised (Dirichlet/Neumann) coarse multifractal dimension with respect to $\mathcal{J}$. The lower estimate of the spectral dimension is based on the following abstract observation which connects the optimised coarse multifractal dimension and the spectral dimension.

**Theorem 1.2.** Assume there exists a non-negative, monotone set function $\mathcal{J}$ on $\mathcal{D}$ with $\dim_{\mathcal{J}} \mathcal{J} > 0$ such that for every $Q \in \mathcal{D}_n$ with $\mathcal{J}(Q) > 0$ there exists a non-negative and non-zero function $\psi_Q \in C_0^\infty$ with support contained in $Q$ (the definition of $\mathcal{J}_Q$ is stated just above Lemma 2.5) such that

$$\|\psi_Q\|_{L^2}^2 \geq \mathcal{J}(Q) \|\nabla \psi_Q\|_{L^2(\hat{\mathbb{Z}^d})}^2.$$ 

Then we have $\overline{F}_3 \leq \overline{F}_3$ and $\underline{F}_3 \leq \underline{F}_3$.

We will see that in our setting, using the general results of [KN23], the upper and lower bounds are related to the partition entropy. Indeed, from (4.1) we infer

$$\underline{h}_3 \leq \underline{h}_3 = \underline{h}_3 = \underline{F}_3.$$ 

Also note, if $\mathcal{J}$ is uniformly vanishing and $0 < q^{1/\nu}_N < \infty$, then $q^{1/\nu}_N$ is the unique zero of $\overline{r}_3^N$ and $q^{1/\nu}_N = \underline{h}_3$; in general, we have $\underline{h}_3 \leq q^{1/\nu}_N$. Under the condition (1.3) and for any $r \in (2, 2 \dim_{\mathcal{J}}(\nu)/(d - 2))$ the set function $\mathcal{J}_{rs}$ is uniformly vanishing and using [Maz85, Corollary, p. 54], Theorem 1.1 is
applicable for 3, (see Corollary 5.4 and Proposition 4.3). For the critical case $\dim_v(v) = d - 2$ there is the possibility of no continuous embedding, a continuous but non-compact or a compact embedding. In Section 8.1 we give examples (for $d = 3$) of absolutely continuous measures with $\dim_v(v) = d - 2 = 1$ such that each possibility is realised. For the case of compact embedding, Theorem 1.1 can be employed to show that in our example $\nu^N = 3/2$ (Example 8.2).

We will see that Theorem 1.2 is applicable for $\lambda = 2$ in the case $d = 2$ and $\lambda = 3$, for $d > 2$. The following list of results give the main achievements of this paper. The proofs are postponed to Section 7. As an auxiliary quantity we need

$$\dim^N_{\nu,N} (v) := \lim inf_{n \to \infty} - \log \left( \max_{Q \in 2^\mathbb{Z}^d} \nu(Q) \right) / \log (2^n)$$

and we introduce the shorthand notation $q^{D,N} := q^{D,N}_0$, $T^{D,N} := T^{D,N}_0$, $E^{D,N} := E^{D,N}_0$, $\nu_0 := \nu_0$, $\overline{h} := \overline{h}_0$, and $\underline{h} := \min \{ \delta \in [\delta_{\nu} - 3, 3] : \}$. In the following we write $\dim_{\nu} (A)$ for the upper Minkowski dimension of the bounded set $A \subset \mathbb{R}^d$ and—slightly abusing notation—we also write $\dim_{\nu} (v) := \dim_{\nu} (\text{supp} (v))$ for the compactly support Borel measure $\nu$.

**Theorem 1.3.** Let $\nu$ be a finite Borel measure on $\mathbb{S}$ such that $\dim_{\nu}(\nu) > d - 2$.

1. Under Neumann boundary conditions we have

$$E^N / \rho^N / \leq \overline{h} \leq \overline{h} = \overline{h} = \nu^N = \nu^N.$$  (1.7)

2. Under Dirichlet boundary conditions and $\nu(\chi) > 0$ we have

$$E^D / \rho^D / \leq \overline{h} \leq \overline{h} = \overline{h} = \rho^D \leq \rho^D \leq \nu^D.$$  (1.8)

**Remark 1.4.** We will see in Corollary 1.11 that $q^D / \rho^N \geq \dim_{\nu} (v) / (\dim_{\nu} (v) - d + 2)$. Hence, we can replaces $q^D$ by $\dim_{\nu} (v) / (\dim_{\nu} (v) - d + 2)$ on the right hand side in (1.8) making this condition a bit weaker but independent of $q^D$. Moreover, (1.8) can easily be verified for particular measures $\nu$ such that

1. $\dim_{\nu} (\text{supp} (v) \cap \partial \mathbb{S}) < \dim_{\nu} (v) \dim_{\nu} (v) - d + 2$,

2. $\dim_{\nu} (v) > d - 1$ and $\dim_{\nu} (\text{supp} (v) \cap \partial \mathbb{S}) \leq \dim_{\nu} (v) / 2$,

3. $\dim_{\nu} (\text{supp} (v) \cap \partial \mathbb{S}) = 0$, particularly for supp (v) $\subset \chi$,

4. or $v$ is given by the $d$-dimensional Lebesgue measure $\lambda_{\mathbb{S}}$ restricted to $\mathbb{S}$ (then the left-hand side in (1.8) is equal to $(d - 1) / 2$).

Let us also remark that in Section 8.2 we present an example for which $\rho^N / \nu^N$ applies.

### 1.3.1. Regularity results.

**Definition 1.5.** We define two notions of regularity for $\nu$ assuming $\dim_{\nu} (\nu) > d - 2$.

1. We call $\nu$ Dirichlet/Neumann multifractal-regular (D/N-MF-regular) if $E^{D,N} = \nu^N$.

2. We call $\nu$ Dirichlet/Neumann partition function regular (D/N-PF-regular) if

- $r^{D,N} (q) := \lim sup_{\varepsilon \to 0} r^{D,N}_\varepsilon (q)$ for $q \in (q^{D,N} - \varepsilon, q^{D,N})$, for some $\varepsilon > 0$,

- $r^{D,N} (q^{D,N}) = \lim inf_{\varepsilon \to 0} r^{D,N}_\varepsilon (q^{D,N})$ and $r^{D,N}$ is differentiable at $q^{D,N}$.

**Remark 1.6.** The above theorem and the notion of regularity give rise to the following list of observations for measures $\nu$ with $\dim_{\nu} (\nu) > d - 2$:

1. An easy calculation shows that

$$E^N / \rho^N / \leq \inf \left\{ q > 0 : \lim inf_{n} r^{D,N}_n (q) < 0 \right\} \leq q^N = \nu^N.$$  

From this it follows that N-MF-regular implies that $r^N$ exists as a limit in $q^N$. 

(2) If the Neumann spectral dimension with respect to \( \nu \) exists, then it is given by purely measure-geometric data encoded in the \( \nu \)-partition entropy, namely we have \( \beta = \frac{\dim \tau}{2} \) and this value coincides with the spectral dimension.

(3) N-MF-regularity implies equality everywhere in the chain of inequalities (1.7) and in particular the Neumann spectral dimension exists. If \( \nu \) is D-MF-regular, then we have equality everywhere in all chains of inequalities above and in particular both Neumann and Dirichlet spectral dimensions exist.

(4) To the best of our knowledge, all measures examined in the literature, for which the spectral dimension is known, are PF-regular.

The following theorem shows that the spectral partition function is a valuable auxiliary concept to determine the spectral behaviour for a given measure \( \nu \).

**Theorem 1.7.** Under the assumption \( \dim_{M}(\nu) > d - 2 \) we have the following regularity result:

1. If \( \nu \) is N-PF-regular, then it is N-MF-regular and the Neumann spectral dimension \( s^N \) exists.

2. If \( \nu \) is D-PF-regular and \( q^D = 0 \), then both the Dirichlet and Neumann spectral dimension exist and coincide, i.e. \( \beta^D = \beta^N \).

This result is optimal in the sense that there is an example (derived from an similar example for \( d = 1 \) in [KN22b]) of a measure \( \nu \) which is not \( \tau \)-regular and for which \( \tau^N > \tau^D \). It should be noted that PF-regularity is easily accessible if the spectral partition function is essentially given by the \( L^1 \)-spectrum.

**Corollary 1.8.** For \( d = 2 \), \( \dim_{M}(\nu) > 0 \) and \( \beta^N \) is differentiable in 1, then \( s^N = 1 \). Additionally, if \( \nu(\mathcal{Z}) > 0 \), then also \( \beta^D \) is differentiable in 1 and in particular, \( s^D = s^N = 1 \).

1.3.2. General bounds in terms of fractal dimensions. In the following proposition we present lower bounds of the lower spectral dimension in terms of the subdifferential , defined as

\[ \partial \tau^D_{\nu} (q) = \left\{ a \in \mathbb{R} : \forall t \in \mathbb{R}, \tau^D_{\nu}(t) \geq a(t - q) + \tau^D_{\nu}(q) \right\} . \]

**Proposition 1.9.** Let us assume \( \dim_{M}(\nu) > d - 2 \). If for \( q \in [0, q^D] \), we have \( \tau^D_{\nu}(q) = \liminf_{x} \tau^D_{\nu}(q) \) and \( -\partial \tau^D_{\nu}(q) = [a, b] \), then

\[ \frac{aq + \tau^D_{\nu}(q)}{b} \leq \frac{q^D}{\nu} . \]

**Remark 1.10.** In the case that \( \tau^D_{\nu}(q^D) = \liminf_{\nu} \tau^D_{\nu}(q^D) \) and \( \tau^N \) is differentiable in \( q^D \), we infer \( q^D \leq s^N \) and hence obtain a direct proof of the regularity statement, namely, \( q^D = s^N = \tau^D_{\nu} \). Also, if \( \tau^D_{\nu}(0) = \liminf_{\nu} \tau^D_{\nu}(0) = d - 2 \), we have the lower bound

\[ -\frac{\partial \tau^D_{\nu}(0)}{\partial \tau^D_{\nu}(1)} \leq \frac{q^D}{\nu} , \]

where \( \partial f(x) \) denotes the left-sided, resp. right-sided, derivative of \( f : \mathbb{R} \to \mathbb{R} \) in \( x > 0 \).

We obtain general bounds for \( \tau^N \) in terms of the upper Minkowski dimension \( \dim_{M}(\nu) \) and the possibly smaller lower \( \infty \)-dimension \( \dim_{M}(\nu) \) of \( \nu \) (see also Figure 1.1 on page 8).

**Corollary 1.11.** Assume \( \dim_{M}(\nu) > d - 2 \). For the Neumann upper spectral dimension we have

\[ \frac{d}{2} \leq \frac{\dim \tau^N_{\nu}}{\dim_{M}(\nu) - d + 2} \leq \frac{\dim \tau^N_{\nu}}{\dim_{M}(\nu) - d + 2} . \]

In particular, for \( d = 2 \), we have \( \tau^N = 1 \), and assuming \( \nu(\mathcal{Z}) > 0 \), also \( \tau^D = 1 \).

**Remark 1.12.** Note that \( d \geq 3 \) and by choosing \( \nu \) with \( \dim_{M}(\nu) \) close to \( d - 2 \) we can easily find examples where \( \tau^N \) becomes arbitrarily large.

It is also worth mentioning that the analogous situation in dimension \( d = 1 \) is quite different (cf. [KN22b; KN22c]), namely the lower bound becomes an upper bound,

\[ \frac{1}{2} \leq \frac{\dim \tau^N_{\nu}}{\dim_{M}(\nu) + 1} \leq \frac{1}{2} . \]

The inequalities in Corollary 1.11 naturally link to the famous question by M. Kac [Kac66], ‘Can one hear the shape of a drum?’ This question has been modified by various authors e.g. in [Ber79; Ber80;
which in turn reflects many important fractal-geometric properties of

4 Proposition 1.13. Absolutely continuous measures first studied by H. Weyl [Weyl11] and in higher generality by Birman and Solomyak in [BS70; BS74]. On the one hand, our methods are in

Absolutely continuous measures. For all other dimensions, our results show

asymptotic bounds. The essence of Birman’s and Solomyak’s achievements, with contributions from

some respects a refinement of the methods developed by Birman and Solomyak, since we are able

to determine the exact upper spectral dimension for all relevant situations, many of which were

previously inaccessible. On the other hand, their methods often allow us to obtain upper spectral

asymptotic bounds. The essence of Birman’s and Solomyak’s achievements, with contributions from

Rozenblum, in this regard is contained in the following examples. In the following we write

In this case, Kac’s question regarding dimensional quantities must be answered in the affirmative.

1.4. Special examples and spectral asymptotic bounds. On the one hand, our methods are in

some respects a refinement of the methods developed by Birman and Solomyak, since we are able
to determine the exact upper spectral dimension for all relevant situations, many of which were

previously inaccessible. On the other hand, their methods often allow us to obtain upper spectral

asymptotic bounds. The essence of Birman’s and Solomyak’s achievements, with contributions from

Rozenblum, in this regard is contained in the following examples. In the following we write \( f(x) \ll g(x) \) if there is a constant \( C > 0 \) such that for all \( x \) large enough \( f(x) \leq C g(x) \); if \( f(x) \ll g(x) \) and

\( g(x) \ll f(x) \), then we write \( f(x) \asymp g(x) \). If \( g(x)/f(x) \to 1 \) as \( x \to \infty \), we write \( f(x) \sim g(x) \).

1.4.1. Absolutely continuous measures. As a first application of Theorem 1.3, we present the case of

absolutely continuous measures first studied by H. Weyl [Weyl11] and in higher generality by Birman

and Solomyak in [BS70; BS74].

Proposition 1.13. Let \( \nu \) be absolutely continuous with respect to the \( d \)-dimensional Lebesgue measure with density that is \( r \)-integrable for some \( r > d/2 \). Then the Dirichlet and Neumann spectral partition function exist as a limit with

\[
\tau^N(q) = \tau^D(q) = d - 2q, \quad \text{for } q \in [0, r),
\]
$v$ is $D(N)$-PF-regular, and the Dirichlet and Neumann spectral dimension exist, coincide and equal $s^D = s^N = d/2$.

Note that under the assumption of Proposition 1.13, it has been shown in [BS70] that indeed

$$N^{D/N}(x) \sim \frac{x^{d/2}}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \int \left( \frac{dv}{d\Lambda} \right)^{d/2} d\Lambda.$$ 

For related results on spectral properties of higher order elliptic differential operators with respect to absolutely continuous measures we refer the interested reader to [EK96].

In Section 8.1 we treat absolutely continuous measures in dimension $d = 3$ with densities which are $r$-integrable for $r \in (1, d/2)$ and not $r$-integrable for $r > d/2$. These are critical cases with respect to our condition (1.3) and the spectral properties can only be determined by a finer analysis. Such examples concerning the critical case, i.e. $\dim_\nu v = d - 2$, and those where the spectral dimension does not exist are deferred to the last section.

In the context of absolutely continuous measures the following rigidity result, which has been obtained for $d = 1$ in [KN22c, Cor. 1.4], is also of interest.

**Proposition 1.14.** For $d \geq 3$ and $\dim_\nu(v) > d - 2$, the following rigidity result holds:

1. If $\tau^D(q) = d - 2q$ for all $q \in [0, d/2)$,
2. If $\tau^N(q) = d - 2q$ for some $q > d/2$, then $\tau^N(q') = d - 2q'$ for all $q' \in [0, q]$ and $\tau^N = d/2$.

If additionally $\tau^N(d/2)$ exists as a limit, then $s^N/d^2$.}

1.4.2. Ahlfors–David regular measure. As a second application, we consider a class of measures with linear partition functions, namely we treat $\alpha$-Ahlfors–David regular measures $v$ on $\mathbb{C}$ for $\alpha > 0$. We call a measure $\alpha$-Ahlfors–David regular if for some $c > 0$, all $x \in \text{supp}(v)$ and $r \in (0, \text{diam}(\text{supp}(v))]$ we have

$$c^{-1} r^{\alpha} \leq v(B(x, r)) \leq cr^{\alpha}.$$ 

Note that for $\alpha$-Ahlfors–David regular measures $v$ we have $\tau^{D/N}_\alpha(q) = \mu^{D/N}(q) + (2 - d) q = (\alpha + 2 - d)q - \alpha$ and in particular, $\alpha = \dim_\mu(v) = \dim_\nu(v)$.

**Proposition 1.15.** Let $v$ be a finite $\alpha$-Ahlfors–David regular Borel measure with $\alpha \in (d - 2, d)$, $d > 2$ and such that $v(\mathbb{C}) > 0$, then

$$N^{D/N}(x) \asymp x^{\alpha/(d - 2)},$$ 

In particular, $s^D = s^N = \alpha/(\alpha - d + 2)$.

This proposition rediscover some of the major achievements on isotropic $\alpha$-sets $\Gamma$ (in our terms this means that the $\alpha$-dimensional Hausdorff measure restricted to $\Gamma$ is $\alpha$-Ahlfors–David regular) as investigated by Triebel in [Tr97]. For $d = 2$ our result is partially contained in Rozenblum et al. [RS21] where the upper asymptotic bound has been obtained, namely $N^{D/N}(x) \ll x$.

Moreover, we partially revisit some results of the recent publications [RS21; RT22] by Rozenblum et al. in which the eigenvalue asymptotics of Birman–Schwinger type operators is discussed in detail. In our setting, the inverse spectral problems of Krein–Feller operators are special cases of Birman–Schwinger type operators with respect to Sobolev space of order 1. The case $d > 2$ corresponds to the so-called subcritical case considered in [RT22, Theorem 3.3.3-4]; this case for differential order 1 corresponds to our result. More precisely, in [RT22, Theorem 3.3.3-4], using some clever covering arguments, it is shown that $N^{D}(x) \ll x^{\alpha/(d - 2)}$ is valid alone under the relaxed assumption that only the second inequality in (1.9) holds. Clearly, under this assumption we have $\dim_\nu(v) \geq \alpha > d - 2$, hence our general assumption (1.3) is satisfied. Therefore, using our general upper bound from Corollary 1.11 we obtain

$$s^{D/N} \leq \frac{\dim_\nu(v)}{2 - d + \dim_\nu(v)} \leq \frac{\alpha}{\alpha - d + 2}.$$ 

Note that this inequality is sharp; if $\dim_\mu(v) = \dim_\nu(v)$ – this holds in particular for Ahlfors–David regular measures (Section 1.4.2), then the $L^p$-spectrum of $v$ is linear, given by $q \mapsto \dim_\mu(v)(1/q)$ on $\mathbb{R}_{>0}$ and by Corollary 1.11 we have equality in (1.10) and therefore the upper asymptotics $N^{D/N}(x) \ll x^{\alpha/(d - 2)}$ is optimal. On the other hand, for all measures $v$ such that $\tau^{D/N}_\alpha$ is not affine-linear on $[0, q')$ the first inequality in (1.10) is strict, the polynomial upper bound $x^{\alpha/(d - 2)}$ is therefore far from optimal and our result improves the result of [RT22, Theorem 3.3.3-4], in that we determine the smallest exponent for an upper asymptotic which is strictly smaller than $\alpha/(2 - d + \alpha)$. 
1.4.3. Self-conformal measures. Finally, we give an example where the spectral partition function is essentially given by the $L^q$-spectrum of $\nu$ (see Section 3.4.3 and Section 3.4.1) and in this case we are able to provide the complete picture provided by our main theorem. In fact, we deal with self-conforming measures with possible overlaps, following up on the question explicitly posed in this context in [NX21, Sec. 5]. The existence and basic properties of such measures originate from the seminal work [Hut81].

**Theorem 1.16.** If $\nu$ is a self-conformal measure on the closed unit cube $\mathcal{Q}$ with $\nu(\partial \mathcal{Q}) = 0$, contractions $(S_i : \mathcal{Q} \to \mathcal{Q})_{i=1,\ldots,\ell}$ (with possible overlaps) and probability vector $(p_i)_{i=1,\ldots,\ell} \geq 2$ (see Section 3.4.3 for precise definitions), then the spectral partition function exists as a limit and is given by

$$\tau^{DN}(q) = \beta^N_q(q) + (d - 2)q,$$

where $\beta^N_q$ denotes the Neumann-$L^q$-spectrum of $\nu$ (see Section 3.2 for definition). Assuming $\dim_{\nu}(\nu) > d - 2$, then $\nu$ is $DN$-PF-regular and the Dirichlet and Neumann spectral dimension exist and equal $s^D = s^N = q^N$. In particular, in the case $d = 2$, we always have $s^D = s^N = 1$.

**Remark 1.17.** We remark that in the situation of Theorem 1.16 under OSC the spectral dimension can be expressed in terms of an associated pressure function, i.e. $q^D$ is the unique zero of $q \mapsto P(q; \psi + (2 - d)\varphi)$, where $\psi : \{1, \ldots, \ell\}^\mathbb{Z} \to \mathbb{R}$ and $\varphi : \{1, \ldots, \ell\}^\mathbb{Z} \to \mathbb{R}$ such that for all cubes $Q$ and $\omega \mapsto \log \|S_{\omega}(\varphi_c)\|$. Moreover, if the open set condition is satisfied and the measure is given either by affine contractions (see [Sol94]) or as a Gibbs measures constructed by a one-dimensional conformal IFS (see [KN22b]), the following asymptotics hold (see Remark 5.2 for a short proof of the upper spectral asymptotics in the self-similar case)

$$N^{DN}(x) \asymp x^{s^D}.$$

**Remark 1.18.** In general, it can be difficult to verify the condition $\dim_{\nu} (\nu) > d - 2$, but in the case $d = 2$ a sufficient condition is that the measure $\nu$ is invariant with respect to an IFS given by a system of bi-Lipschitz contractions such that the attractor is not a singleton ([HLN06, Lemma 5.1]). This carries over to self-similar measures provided that the contractive similitudes do not share the same fixed point, so that $\dim_{\nu}(\nu) > 0$ and the spectral dimension is then given by $s^D = s^N = 1$.

2. Some technical prerequisites

In this section we provide some technical details needed in the proofs of our main theorems. Throughout, we assume $\text{card}(\text{supp}(\nu)) = \infty$, or equivalently, $L^1_1(\Omega)$ is an infinite dimensional vector space.

2.1. Stein extensions. We will use the fact, going back to Stein [Ste70, Sec. 3.2 and 3.3], that any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ permits a Stein extension in the sense that there exists a bounded linear operator

$$\mathfrak{E}_\Omega : C^\infty_0(\Omega) \to C^\infty_0(\mathbb{R}^d)$$

such that for all $f \in C^\infty_0(\Omega)$ we have $\mathfrak{E}_\Omega(f) \big|_{\Omega} = f$. Clearly, by continuation, this operator gives rise to a continuous linear operator $\mathfrak{E}_\Omega : H^2(\Omega) \to H^2(\mathbb{R}^d)$ such that $\mathfrak{E}_\Omega(f) \big|_{\Omega} = f$ a.e. for all $f \in H^2(\Omega)$, which in the literature is also called a Stein extension. The existence of the operator on $C^\infty_0(\Omega)$ is not stated explicitly in [Ste70, Sec. 3.2 and 3.3], but in Stein’s proof we observe that the auxiliary functions $\Lambda_\ast$ and $\Lambda$ defined therein have compact support provided $\Omega$ is bounded. Since the extension is then constructed as the product of smooth functions with compact support with a finite sum of smooth functions, our requirements are met. Now, $\mathfrak{E}$ as a bounded convex open set is a bounded Lipschitz domain (see e.g. [Gri85, Corollary 1.2.2.3] or [Ste70, Example 2, p. 189]), the Stein extension $\mathfrak{E}_\Omega$ with the above properties exists.

**Lemma 2.1.** There exists a constant $D_\Omega > 0$ such that for all cubes $Q \subset \mathcal{Q}$ with edges parallel to the coordinate axes and $u \in H^2(\Omega)$,

$$D_\Omega \|u\|^2_{H^2(\Omega)} \leq \|\nabla u\|^2_{L^2_\Omega(Q)} + \frac{1}{\Lambda(\Omega)} \left| \int_Q u \, d\Lambda \right|^2 \leq \|u\|^2_{H^2(\Omega)}.$$

Further, let $T : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto x_0 + hx$, with $h \in (0,1)$, $x_0 \in \mathcal{Q}$, such that $\mathcal{Q} = T(\mathcal{Q})$. Then $\mathfrak{E}_{\mathcal{Q}} : H^2(\mathcal{Q}) \to H^2(\mathbb{R}^d)$, $u \mapsto \mathfrak{E}_{\mathcal{Q}}(u \circ T) \circ T^{-1}$ defines a Stein extension and with $N_{\Lambda}(\mathcal{Q}) :=$
where we write \( G \geq u \) for some unique decomposition inequality we consider an \( \{ u \} \). Clearly, by the Cauchy-Schwarz inequality, for all \( u \in H^N(Q) \), we have \( \| u \|^2_{L^2(Q)} / \| u \|^2_{L^2(Q)} \)). Hence, using the inclusion from (dom (\( \Omega \))), we obtain the following proposition will be crucial for the proof of the upper bound of the spectral dimension as stated in Corollary 5.5.

### 2.2. Min-Max principle.
Let \( E \) be a closed form with domain dom (\( E \)) densely defined on \( L^2_0 \), in particular dom (\( E \)) defines a Hilbert space with respect to \( (f, g)_{E} := (f, g) + E(f, g) \), and assume that the inclusion from (dom (\( E \)), \( (\cdot, \cdot)_{E} \)) into \( L^2_0 \) is compact. Then the Poincaré–Courant–Fischer–Weyl min-max principle is applicable, for the i-th eigenvalue \( \lambda_i(E) \) of \( E \), i \( \in \mathbb{N} \), we have (see also [Kig01, Theorem B.1.14] or [Dav95; KL93])

\[
\lambda_i(E) = \inf \{ R(G^*) : G < i \} \in \text{dom}(E), \langle \cdot, \cdot \rangle_{E}
\]

where we write \( G < i \) (\( H, \langle \cdot, \cdot \rangle \)) if \( G \) is a linear subspace of the Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \) and the vector space dimension of \( G \) is equal to \( i \) \( \in \mathbb{N} \); for \( \psi \in \text{dom}(E) \) the Rayleigh–Ritz quotient is given by \( R(\psi) := E(\psi, \psi) / \langle \psi, \psi \rangle \), and if \( \mathcal{T} \subset \text{dom}(E) \) we write \( R(\mathcal{T}) := \sup \{ r(\psi) : \psi \in \mathcal{T} \} \).

The following proposition will be crucial for the proof of the upper bound of the spectral dimension as stated in Corollary 5.5.

### Proposition 2.2.
For all \( i \in \mathbb{N} \), we have

\[
\lambda_{D^{\Omega}}^{(i)} = \inf \left\{ R_{D^{\Omega}}^{(i)}(G^*) : G < i \left( \left( R_{D^{\Omega}}^{(i)} \right)^-, \langle \cdot, \cdot \rangle_{D^{\Omega}(\Omega)} \right) \right\}
\]

where the relevant Rayleigh–Ritz quotient is given by \( R_{D^{\Omega}}^{(i)}(\psi) := \langle \psi, \psi \rangle_{D^{\Omega}(\Omega)} / \langle \psi, \psi \rangle \).

### Proof.
The first equality follows by the min-max principle and the fact that dom \( \{ E^{(i)} \} \subset H^{D^{\Omega}(\Omega)} \). The part ‘\( \geq \)’ for the second equality follows from the inclusion \( \left( R_{D^{\Omega}}^{(i)} \right)^- \subset H^{D^{\Omega}(\Omega)} \). For the reverse inequality we consider an i-dimensional subspace \( G = \text{span}(f_1, \ldots, f_i) \subset H^{D^{\Omega}(\Omega)} \). There exists a unique decomposition \( f_i = f_i \parallel f_i \parallel + f_i \perp \), with \( f_i \parallel \in \left( R_{D^{\Omega}}^{(i)} \right)^- \) and \( f_i \perp \in \left( R_{D^{\Omega}}^{(i)} \right)^- \), \( j = 1, \ldots, i \). Suppose that
Lemma 2.5. Let $Q$ be a cube with side length $mr$. For $(f_1, f_2) \in \mathbb{R}^{D/N}$, we have $\langle f_1, f_2 \rangle = 0$. Then
\[
\frac{\lambda_1(f_1 + f_2) + \cdots + \lambda_i(f_i + f_{i+1})}{\varepsilon^{D/N}_+} = \langle f_1, f_2 \rangle = 0.
\]
Using $E^{D/N}(g, g) > 0$, we obtain in this case $R_{H^{D/N}}(G^*) = \infty$. Otherwise, using the assumption $f_1, f_2 \in \mathbb{R}^{D/N}$ and particularly $\langle f_1, f_2 \rangle = 0$, we have for every vector $(a_i) \in \mathbb{R} \setminus \{0\}$
\[
R_{H^{D/N}} \left( \sum_i a_i f_{1,i} + \sum_j a_j f_{2,j} \right) = \left\langle \sum_i a_i f_{1,i}, \sum_i a_i f_{1,i} \right\rangle_{H^{D/N}(Q)} + \left\langle \sum_i a_i f_{2,j}, \sum_i a_i f_{2,j} \right\rangle_{H^{D/N}(Q)}
\]
\[
\geq R_{H^{D/N}} \left( \sum_i a_i f_{1,i} \right).
\]
Note that span$\{f_1, \ldots, f_i\} \subset \mathbb{R}^{D/N}$ is also $i$-dimensional subspace in $H^{D/N}$. Hence, in any case the reverse inequality follows.

Proposition 2.3. For all $i \in \mathbb{N}$, we have $A_{e_i} \ll A_{e_i}$.

Proof. Using Poincaré inequality (1.1) and $c > 0$ as defined therein, we obtain for all $u \in H^0$ that $\langle u, u \rangle_{H^0} \leq (c^2 + 1) \langle u, u \rangle_{H^0}$. Since $H^0 \subset H^1$, the claim follows from Proposition 2.2.

The leading idea to obtain lower bounds on $N_{D/N}$ is to construct appropriate finite dimensional subspaces of $H^{D/N}$. This will be subject of the following corollary, which is an immediate consequence of the min-max principle.

Corollary 2.4. For a finite orthogonal family $F \subset H^{D/N}(Q)^*$ we have $N_{D/N}(R_{H^{D/N}}(F)) \geq \text{card}(F)$.

2.3. Smoothing methods. For $m > 1$ and $r > 0$, let $Q$ be a cube with side length $mr$ and $Q' \subset Q$ a central and parallel sub-cube with side length $r$. Then using the standard smoothing methods by normalised Friedrichs’ mollifier one easily checks that there exists
\[
\varphi_{Q,m} \in C^\infty_c(\mathbb{R}^d)
\]
(2.1)
with the following properties:

(1) $0 \leq \varphi_{Q,m}(x) \leq 1$ for all $x \in \mathbb{R}^d$,
(2) supp$\{\varphi_{Q,m}\} \subset \bar{Q}$,
(3) $\varphi_{Q,m}(x) = 1$ for all $x \in Q$,
(4) there exists a constant $C_1 > 0$ (depending only on $d$) such that $\left|\partial_{i} \varphi_{Q,m}(x)\right| \leq C_1 / (r \nu(m - 1))$
for all $i = 1, \ldots, d$ and $x \in Q$.

For $s > 0$ let $(Q)_s \equiv T_s(Q) = (1 - s)x_0$ with $T_s : x \mapsto sx, x \in \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$ is the centre of $Q$. Note that we have $\left\langle (Q)_{1/1}\right\rangle = Q$.

Lemma 2.5. Let $Q$ be a cube with side length $mr > 0$, $m > 1$, $r > 0$. Then there exists a constant $C > 0$ depending on $m > 1$ and $d$ such that for $\varphi_{Q,m}$ as defined in (2.1) we have
\[
\int |\nabla \varphi_{Q,m}|^2 \frac{d\Lambda}{\int \varphi_{Q,m}^2 dv} \leq C \Lambda \left( (Q)_{1/m} \right)^{1-2/d}.
\]

Proof. Using (2.1) with $r = \Lambda(Q)^{1/d}$ and $\Lambda(Q)m^{-d} = \Lambda \left( (Q)_{1/m} \right)$, we estimate with $C_1 > 0$ as in (2.1)
\[
\int |\nabla \varphi_{Q,m}|^2 \frac{d\Lambda}{\int \varphi_{Q,m}^2 dv} \leq C_2 \Lambda \left( (Q)_{1/m} \right)^{1-2/d} \left( (m - 1)^r r^2 \right) \frac{dC_2 m^{-2} \Lambda \left( (Q)_{1/m} \right)^{1-2/d}}{(m - 1)^r r^2} \nu\left( (Q)_{1/m} \right).
\]
The next proposition applies only in the case $d > 2$. We will make use of the following definition
\[ \dim^0_n := \dim_n \quad \text{and} \]
\[ \dim^0_n(v) := \liminf_{n \to \infty} -\log \left( \max_{Q \in D^0_n} v(Q) / \log (2^n) \right). \]
Clearly, $\dim^0_n \leq \dim^0_n$.

**Proposition 2.6.** If $\dim^0_n(v) < d - 2$, then $I^0_n(C_{D^0_n}/\Omega) \to L^2_{\nu}(C_{D^0_n})$ is not continuous.

**Proof.** First note that $\dim^0_n(v) = \liminf_{n \to \infty} \max_{Q \in D^0_n} \log v(Q) / \log 2^d < d - 2$ implies that there exists a sequence of cubes $(Q_n) \in \left(D^0_n\right)^\infty$ with strictly decreasing diameters such that $v(Q_n) \geq \Lambda(Q_n)^{d/(d-2)}$, $n \in \mathbb{N}$, for some $a \in (\dim_n(v), d - 2)$. Now we have for $u_n := \Lambda ((Q_n)^{1/(d-2)} \psi_{(Q_n);2})$ with $C > 0$ given in (2.1)
\[ \|u_n\|_{L^2_n}^2 \geq \Lambda ((Q_n)^{1/(d-2)} v(Q_n) \geq 2^{d-a} \Lambda (Q_n)^{a + 2 - d/d} \to \infty. \]

3. Partition functions and $L^\nu$-spectra

As set up in the introduction we consider the $d$-dimensional unit cube $\Sigma$, for $d \geq 2$ and the semiring of dyadic cubes $D$. Even though, the particular choice of the set of dyadic cubes is not unique, we will see that this does not affect our results (see Lemma 3.2, Lemma 3.4 and Lemma 3.7).

3.1. Partition functions. Recall the definition in (1.6) of the partition function $\tau_3^{D^0_n}$ with respect to $\chi$ as well as the critical values $q_3^{D^0_n}$ and $k_3$. We will assume that $\chi : D \to \mathbb{R}_{>0}$ is locally non-vanishing, that is, if $\chi(Q) > 0$ for $Q \in D$, then there exists $Q' \subseteq Q, Q' \in D$ with $\chi(Q') > 0$. Note that this assumption is satisfied for the specific choice $\chi = \chi_{0,a,b}$. We start with some general observations for which we need the following objects:
\[ \supp(\chi) := \bigcup_{k \in \mathbb{Z}} \bigcup_{n, k} \{ Q : n \in D_n, \chi(Q) > 0 \} \quad \text{and} \quad \dim_0(\chi) := \lim_{n \to \infty} \max_{Q \in D_n} \log \chi(Q) / \log (2^n). \]

We call $\dim_0(\chi)$ the $\infty$-dimension of $\chi$ which generalises the lower $\infty$-dimension for $\nu$ defined in (1.2). By [KN23, Lemma 2.3] we have that $\dim_0(\chi) > 0$ implies that $\chi$ is uniformly vanishing, i.e. $\lim_{n \to \infty} \sup_{Q \in D_n} \sup_{\chi(Q) > 0} \chi(Q) / \log (2^n) = 0$.

We also make use of the following observation ([KN23, Lemma 2.4]), where we use the convention $-\infty \cdot 0 = 0$ and, as for measures, we write $\dim_{\mu}(\chi) := \dim_{\mu}(\supp(\chi))$. For $q \geq 0$, we have
\[ -\dim_0(\chi) q \leq \tau_3^{\chi}(q) \leq \dim_{\mu}(\chi) - \dim_0(\chi) q \]
and we have $\dim_0(\chi) > 0$ if and only if $q_3^{\chi} < \infty$. In particular, $q_3^{\chi} \leq \dim_{\mu}(\chi) / \dim_0(\chi)$ and we have that $q_3^{\chi} < \infty$ implies $k_3 = q_3^{\chi}$. Note that in the case $\dim_0(\chi) \leq 0$, we also deduce from the above inequality that $\tau_3^{\chi}(q)$ is non-negative for $q \geq 0$, hence $q_3^{\chi} = \infty$. However, it is possible that $k_3 < \infty$. Indeed, in Example 8.2 we provide a measure $\nu$, where $k_3$ gives the upper spectral dimension, while $k_3 < q_3^{\nu} = \infty$.

**Definition 3.1.** We say that a non-negative, monotone set function $\chi$ defined on all possible dyadic sub-cubes of $\Sigma$ (with respect to the choice of their faces) is locally almost subadditive if for any two sets of dyadic partitions $D, D$ of $\Sigma$ by dyadic cubes there exists a constant $C > 0$ such that for every $Q \in D_n$ we have
Slightly abusing notation, we write $\tau$ on all possible dyadic sub-cubes of $\mathcal{Q}$, we have that the definition of $r_{D}^{\mathbb{D}}$ does not depend on the particular choice of the dyadic partition.

Proof. Let $\{D_{n}^{\mathbb{D}}\}$ and $\{\hat{D}_{n}^{\mathbb{D}}\}$ be two sequences of partitions of $\mathcal{Q}$. Then for all $Q \in D_{n}^{\mathbb{D}}$ we have $\text{card}\{Q' \in \hat{D}_{n}^{\mathbb{D}} : Q \cap Q' \neq \emptyset\} \leq 6^d$. Hence,

$$\sum_{Q \in D_{n}^{\mathbb{D}}} \zeta(Q) \leq \sum_{Q \in \hat{D}_{n}^{\mathbb{D}}} \left(C \sum_{Q' \in \hat{D}_{n+1}^{\mathbb{D}}} \sum_{Q \subset Q'} \zeta(Q') \right) \leq 6^d C^d \sum_{Q \in \hat{D}_{n}^{\mathbb{D}}} \max_{Q' \in \hat{D}_{n}^{\mathbb{D}}} \zeta(Q') \leq 6^d C^d \sum_{Q \in \hat{D}_{n}^{\mathbb{D}}} \zeta(Q'),$$

using in the last inequality the fact that each $Q' \in \hat{D}_{n}^{\mathbb{D}}$ intersects at most $2^d$ cubes in $\hat{D}_{n}^{\mathbb{D}}$. Exchanging the role of $\mathcal{D}$ and $\hat{D}$ proves the lemma. \hfill \square

We now summarise the above and mention a few more basic characteristics (see also [KN23]).

Fact 3.3. We make the following elementary observations under the assumption $\dim_{m}(\zeta) \in (0, \infty)$:

1. $\tau_{\zeta}^{N}$ is convex and strictly decreasing on $[N, \infty)$. In particular, if $q_{\zeta}^N > 0$, then $q_{\zeta}^N$ is the unique zero of $\tau_{\zeta}^{N}$.

2. $\lim_{q \to \infty} \tau_{\zeta}^{N}(q)/q = -\dim_{m}(\zeta)$.

3. $\tau_{\zeta}^{N}(q) > -\infty$ for all $q \geq 0$.

4. $\tau_{\zeta}^{N}(0) = \dim_{M}(\zeta) \leq d$, where $\dim_{M}(\zeta)$ denotes the upper Minkowski dimension of $\text{supp}(\zeta)$ given by

$$\dim_{M}(\zeta) := \limsup_{n \to \infty} \frac{\log (\text{card}\{Q \in D_{n}^{\mathbb{D}} : Q \cap \text{supp}(\zeta) \neq \emptyset\})}{\log (2^n)}.$$

5. If $q_{\zeta}^N \geq 1$ hold, then $\frac{\dim_{M}(\zeta)}{\dim_{M}(\zeta) - \tau_{\zeta}^{N}(1)} \leq q_{\zeta}^N \leq \frac{\dim_{M}(\zeta) + \tau_{\zeta}^{N}(1)}{\dim_{M}(\zeta)}$.

6. If $q_{\zeta}^N < 1$, then $\frac{\dim_{M}(\zeta) + \tau_{\zeta}^{N}(1)}{\dim_{M}(\zeta)} \leq q_{\zeta}^N \leq \frac{\dim_{M}(\zeta)}{\dim_{M}(\zeta) - \tau_{\zeta}^{N}(1)}$.

7. If $\text{supp}(\zeta) \subset \hat{\mathcal{Q}}$, then we have $\tau_{\zeta}^{N}(q) = \tau_{\zeta}^{N}(q)$.

8. The partition function is scale invariant, i.e. for $c > 0$, we have $\tau_{c\zeta}^{D_{n}^{\mathbb{D}}} = \tau_{\zeta}^{D_{n}^{\mathbb{D}}}$.

3.2. $L^{s}$-spectra. In this section we collect some important facts about the $L^{s}$-spectrum $\nu$, which is defined by $\beta_{\nu}^{N} := \tau_{\nu}^{D_{n}^{\mathbb{D}}}$ with approximations $\beta_{\nu}^{N} := \tau_{\nu}^{D_{n}^{\mathbb{D}}}$, $n \in \mathbb{N}$. We will assume $\nu(\hat{\mathcal{Q}}) > 0$, implying that there exists a sub-cube $Q \in \mathcal{D}$ with $\overline{Q} \subset \hat{\mathcal{Q}}$, $\nu(Q) > 0$ and hence $-\infty < \beta_{\nu}^{D_{n}^{\mathbb{D}}} \leq \beta_{\nu}^{D_{n}^{\mathbb{D}}}$.

Slightly abusing notation, we write $\tau_{\nu}^{D_{n}^{\mathbb{D}}} = \tau_{\nu}^{D_{n}^{\mathbb{D}}}$, $\tau_{\nu}^{D_{n}^{\mathbb{D}}}$.

Since $\nu$ is locally almost subadditive and $\lim_{n} \beta_{\nu}(q)/q = -\dim_{m}(\nu)$, we obtain from Lemma 3.2 the following lemma.

Lemma 3.4. The definition of $\beta_{\nu}^{D_{n}^{\mathbb{D}}}$ and $\dim_{m}(\nu)$ does not depend on the particular choice of the dyadic partition.

Fact 3.5. We make the following elementary observations:

1. $\beta_{\nu}^{D_{n}^{\mathbb{D}}}(0) = \dim_{m}(\nu)$.

2. $\dim_{m}(\nu) \leq d$.

3. $\beta_{\nu}(1) = 0$ and if $\nu(\hat{\mathcal{Q}}) > 0$, then also $\beta_{\nu}(1) = 0$.

4. For the Dirichlet $L^{s}$-spectrum we have $\beta_{\nu}^{D_{n}^{\mathbb{D}}} = \beta_{\nu}^{D_{n}^{\mathbb{D}}}$.

5. For $q \geq 0$, we have $-qd \leq \beta_{\nu}^{D_{n}^{\mathbb{D}}}(q)$ and if $\nu(\hat{\mathcal{Q}}) > 0$, also $-qd \leq \beta_{\nu}^{D_{n}^{\mathbb{D}}}(q)$.

6. If $\text{supp}(\nu) \subset \hat{\mathcal{Q}}$, then we have $\beta_{\nu}^{D_{n}^{\mathbb{D}}} = \beta_{\nu}^{D_{n}^{\mathbb{D}}}$.
Proof. \( \text{Proposition 3.8.} \)\n
Let \( a \in \mathbb{R} \). We only consider the case \( \nu \geq 0 \) and \( a \in \mathbb{R} \), as defined in Section 1.1 and recall the special notation for the spectral partition function of \( v \) given by \( \tau_{3,\nu}^{D/N} = \tau_{3,\nu; b; 1-2}^{D/N} \) and \( \tau_{3,\nu}^{N} = \tau_{3,\nu}^{D/N} \). For the Dirichlet case we always assume \( v(L^2) > 0 \).

We first investigate under which condition the Definition 3.1 for \( \tilde{\mathcal{J}}_{\nu,ab} \) is fulfilled.

\textbf{Lemma 3.7.} The set function \( \tilde{\mathcal{J}}_{\nu,ab} \) with \( b \in \mathbb{R}_{>0} \), \( a \in \mathbb{R} \) is non-negative, monotone, uniformly \( \text{vanishing and locally almost subadditive} \), provided \( b \text{dim}_s(v) + ad > 0 \). In particular, if \( \text{dim}_s(v) > d - 2 \), then Definition 3.1 is fulfilled for \( \tilde{\mathcal{J}}_{\nu,ab} \) with \( t \in (0, 2 \text{dim}_s(v)/(d - 2)) \) and \( C = 2^{|a|b} b^d \).

\textbf{Proof.} We only consider the case \( a \neq 0 \). The case \( a = 0 \) follows in a similar way. Let \( s \in \mathbb{R} \) such that \( -ad/b < s < \text{dim}_s(v) \). Hence, we have for \( n \) sufficiently large \( v(Q) \leq 2^{-n} \) for all \( Q \in D_s^N \). This gives \( \sup_{\nu \in [a,b]} \nu \left( \frac{\nu}{\nu} \right)^q \leq 2^{-n} b^d \). Hence, \( \tilde{\mathcal{J}}_{\nu,ab} \) is non-negative, monotone and uniformly vanishing. To show that \( \tilde{\mathcal{J}}_{\nu,ab} \) is locally almost subadditive we fix \( Q \in D_s^N \), for which we have \( \sup_{\nu \in [a,b]} \nu \left( \frac{\nu}{\nu} \right)^q A(Q)^q = \nu \left( \frac{\nu}{\nu} \right)^q \left( \nu \left( \frac{\nu}{\nu} \right)^q A(Q)^q \right) \) for some \( \nu \in D_s \) with \( m \geq n \). Consequently,

\[
\tilde{\mathcal{J}}_{\nu,ab}(Q) = \nu \left( \frac{\nu}{\nu} \right)^q A(Q)^q \leq 2^{-n} b^d \max_{Q' \in D_{n+1}^N} \nu \left( \frac{\nu}{\nu} \right)^q A(Q')^q \
\leq 2^{-n} b^d \max_{Q' \in D_{n+1}^N} \nu \left( \frac{\nu}{\nu} \right)^q A(Q')^q \leq 2^{-n} b^d \max_{Q' \in D_{n+1}^N} \nu \left( \frac{\nu}{\nu} \right)^q \tilde{\mathcal{J}}_{\nu,ab}(Q') 
\]

\( \square \)

We now elaborate some connections between the \( L^q \)-spectrum and the spectral partition function.

\textbf{Proposition 3.8.} Fix \( a \in \mathbb{R}, b \in \mathbb{R}_{>0} \) with \( b \text{dim}_s(v) + ad > 0 \).

1. If \( a \geq 0 \), then \( \tau_{3,\nu}^{D/N}(q) = \beta_v^{D/N}(bq) - adq \) for \( q \geq 0 \).

2. If \( a < 0 \), then \( \beta_v^{D/N}(bq) - adq \leq \tau_{3,\nu}^{D/N}(q) \leq \beta_v^{D/N}(q(b + ad/\text{dim}_s(v))) \) for \( q \geq 0 \), and in particular, \( \tau_{3,\nu}^{D/N}(0) = \beta_v^{D/N}(0) \).

\textbf{Proof.} We only consider the case \( a < 0 \). Let \( q \geq 0 \). We have for every \( -ad/b < s < \text{dim}_s(v) \) and \( n \) large enough that \( v(Q) \leq 2^{-n} \) for all \( Q \in D_s^N \). This leads to \( n \leq -\log_2(v(Q))/s \). Hence, we obtain

\[
v(Q)^q A(Q)^q = v(Q)^q 2^{-adq} \leq v(Q)^q 2^{-adq} \log_2(v(Q)/s) = v(Q)^q (b + ad/s). \]

We get \( v(Q)^q A(Q)^q \leq \tilde{\mathcal{J}}_{\nu,ab}(Q)^q \leq v(Q)^q (b + ad/s) \) and \( \tau_{3,\nu}^{D/N}(q) \leq \beta_v^{D/N}(q(b + ad/s)) \). Finally, the continuity of \( \beta_v^{D/N} \) gives \( \tau_{3,\nu}^{D/N}(q) \leq \beta_v^{D/N}(q(b + ad/\text{dim}_s(v))) \).

\textbf{Corollary 3.9.} Let \( a \neq 0 \). Assume \( b \text{dim}_s(v) + ad > 0 \) and \( \beta_v^N \) is linear on \([0, \infty)\). Then, for all \( q \geq 0 \), we have

\[
\tau_{3,\nu}^{D/N}(q) = \beta_v^{D/N}(bq) - adq = \text{dim}_s(v) - q(\text{dim}_s(v) + ad). 
\]

\textbf{Proposition 3.10.} Assume \( \text{dim}_s(v) > 0 \). Then for all \( b > 0 \) and \( q \geq 0 \), we have

\[
\beta_v^{D/N}(bq) = \lim_{\nu \to \infty} \tau_{3,\nu; b}^{D/N}(q). 
\]

Furthermore, if \( \beta_v^{D/N}(bq) \) exists as limit, then \( \beta_v^{D/N}(bq) = \lim_{\nu \to \infty} \tau_{3,\nu; b; 0}^{D/N}(q) \).
Proof. Let \( q > 0 \). For \( \dim_{\omega}(v) > \varepsilon > 0 \), we have for \( n \) large enough and all \( Q \in \mathcal{D}^\infty_n \)
\[
v(Q) \leq 2^{-\varepsilon n}
\]
or, alternatively, \( v(Q)^{d/e} \leq \Lambda(Q) \). For \( n \) large, \( v(Q) \) becomes uniformly small for all \( Q \in \mathcal{D}^\infty_n \). Thus, for every \( 0 < \delta < b \) and \( n \) large, we obtain
\[
(\log(1/v(Q)))^q \leq v(Q)^{-q}
\]
This leads to
\[
(\log(2)d)^q \sum_{Q \in \mathcal{D}^\infty_n} v(Q)^{q \delta} \leq \sum_{Q \in \mathcal{D}^\infty_n} \sum_{0\ll d} \sup_{Q \subseteq E^D} \|\log(v(Q))\|^q v(Q)^{q \delta} \leq (d\varphi)^q \sum_{Q \in \mathcal{D}^\infty_n} v(Q)^{q(\theta-\delta)}
\]
Hence, \( \beta^{D/N}_N(qb) \leq \tau^{D/N}_N(q) \leq \beta^{D/N}_N(q(b-\delta)) \) and for \( \delta \searrow 0 \), the continuity of \( \beta^{D/N}_N \) gives \( \beta^{D/N}_N(qb) = \tau^{D/N}_N(q) \). In the same way, assuming that \( \beta^{D/N}_N \) exists as limit, it follows \( \beta^{D/N}_N(bq) = \lim_{n \to \infty} \tau^{D/N}_{N(bq)}(q) \).

Corollary 3.11. If \( d = 2 \) and \( \dim_{\omega}(v) > 0 \), then \( \tau^{N}_{\gamma}(1) = \beta^{N}(1) = 0 \), or equivalently, \( \tau^{N}_{\gamma} = 1 \). If additionally \( v(\mathcal{Z}) > 0 \), then \( \tau^{N}_{\gamma}(1) = \beta^{N}(1) = 0 \), or equivalently, \( \tau^{N}_{\gamma} = 1 \).

By virtue of Proposition 3.8 and Proposition 3.10 we arrive at the following list of facts.

Fact 3.12. Assuming \( b \dim_{\omega}(v) + ad > 0 \), the following list of properties of the spectral partition function applies:

1. \( \sup(\mathcal{N}_{\omega,b}) = \sup(\mathcal{V}) \).
2. \( \dim_{\omega}(\mathcal{N}_{\omega,b}) = b \dim_{\omega}(v) + ad > 0 \).
3. \( \tau^{N}_{\mathcal{N}_{\omega,b}} \) is the unique zero of \( \tau^{N}_{\gamma} \) and, by Proposition 3.8, for \( a \leq 0 \)
\[
\frac{\dim_{\omega}(v)(v) + ad}{b \dim_{\omega}(v) + ad} \leq \tau^{N}_{\gamma} \leq \frac{\dim_{\omega}(v)}{b \dim_{\omega}(v) + ad}
\]
and for \( a > 0 \)
\[
\frac{\dim_{\omega}(v)}{b \dim_{\omega}(v) + ad} \leq \tau^{N}_{\gamma} \leq \frac{\dim_{\omega}(v)}{b \dim_{\omega}(v) + ad}
\]
4. We always have \( \dim_{\omega}(v) \leq \dim_{\omega}(v) \).
5. We have \( \frac{d}{2} \leq \frac{\dim_{\omega}(v)}{\dim_{\omega}(v) - d + 2} \leq \frac{\beta^{N}(1)_{\mathcal{N}_{\omega,b}}}{\beta^{N}(1)_{\mathcal{N}_{\omega,b}}} \equiv \tau^{N}_{\mathcal{N}_{\omega,b}} \) and if additionally, \( \dim_{\omega}(v) = \dim_{\omega}(v) \), then
\[
\frac{\dim_{\omega}(v)}{\dim_{\omega}(v) - d + 2} = \frac{\dim_{\omega}(v)}{\dim_{\omega}(v) - d + 2}.
\]
6. If \( v \) is absolutely continuous with density \( h \in L^r_\gamma \) for some \( r > d/2 \), then \( \tau^{N}_{\mathcal{N}_{\omega,b}}(q) = \tau^{N}_{\mathcal{N}_{\omega,b}}(q) = \beta^{N}(q) + (d - 2)q \), for all \( q \in [0, r] \).
7. For the Dirichlet spectral partition function we have \( \tau^{0}_{\mathcal{N}_{\omega,b}} = \tau^{0}_{\mathcal{N}_{\omega,b}} \).
8. For \( c > 0 \), we have \( \tau^{D/N}_{\mathcal{N}_{\omega,b}} = \tau^{D/N}_{\mathcal{N}_{\omega,b}} \) and we can assume without loss of generality that \( v \) is a probability measure.

3.3.1. Relations between Neumann and Dirichlet spectral partition function. In this section we investigate under which condition, we can guarantee that for given \( q \geq 0 \), we have \( \tau^{N}_{\mathcal{N}_{\omega,b}}(q) = \tau^{N}_{\mathcal{N}_{\omega,b}}(q) \). We assume \( \dim_{\omega}(v) > d - 2 \). As auxiliary quantities we need
\[
\dim_{\omega}^{N,D}(v) := \liminf_{n \to \infty} \frac{\log \max_{Q \subseteq E^D} v(Q)}{-\log(2^n)} \quad \text{ and } \quad \dim_{\omega}^{D}(v) := \liminf_{n \to \infty} \frac{\log \max_{Q \subseteq E^D} v(Q)}{-\log(2^n)}
\]

Lemma 3.13. For any \( q \geq 0 \) such that
\[
\dim_{\omega}(v) + \delta \mathcal{Z} = v(\dim_{\omega}^{N,D}(v) - d + 2) < \tau^{N}(q),
\]
we have \( \tau^{0}(q) = \tau^{N}(q) \). This implication holds in particular if
\[
\dim_{\omega}(v) + \delta \mathcal{Z} = v(\dim_{\omega}(v) - d + 2) < \tau^{N}(q).
\]
Remark 3.14. Using $\dim_{M}(v)/\dim_{M}(v) - d + 2 \leq q_N^{\beta}$, the assumption in Lemma 3.13 give

$$\dim_{M}(\text{supp}(v) \cap \partial \Xi) < \dim_{M}(v) - d + 2$$

implying $\tau_{\nu}^{N}(q_{N}^{\beta}) = \tau^{D}(q_{N}^{\beta}) = 0$.

Proof. First, we consider the case $d > 2$. Note that

$$\sum_{Q \in D_{t}^{\nu}} \beta_{Q}^{\nu}(Q) \leq \sum_{Q \in D_{t}^{\nu}} \beta_{Q}^{\nu}(Q) + \sum_{Q \in D_{t+2}^{\nu}} \beta_{Q}^{\nu}(Q).$$

Set $\tau^{N,D}(q) := \lim sup_{Q_{n} \to \infty} \left(\log(2^{n}) \log \sum_{Q \in D_{t}^{\nu}} \beta_{Q}^{\nu}(Q)\right)$. Then for $q \geq 0$

$$\tau_{\nu}^{N}(q) \leq \tau_{\nu}^{D}(q), \quad \tau_{\nu}^{D}(q) = \max\{\tau_{\nu}^{N,D}(q), \tau_{\nu}^{D,N}(q)\}.$$

Further, we always have

$$0 \leq \dim_{\nu}(v) - d + 2 \leq \frac{\log \max_{Q \in D_{t}^{\nu}} \beta_{Q}^{\nu}(Q)}{-n \log 2} = \lim_{q \to \infty} \frac{\tau^{N,D}(q)}{-q}.$$

By definition of $\beta_{Q}$, we have $\dim_{\nu}(v) \leq d + 2 \geq A$ and $\dim_{\nu}(v) \leq d + 2 - 2 \geq \dim_{\nu}(v) - d + 2 > 0$.

Fix $0 < s < \dim_{\nu}(v)$, then we obtain for all $n$ large and $Q \notin D_{s}^{\nu} \setminus D_{t}^{\nu}$,

$$\nu(\varepsilon) \Lambda(\varepsilon) (\varepsilon)^{2(d-1)} \leq 2^{n(d-2-s)}.$$

Therefore, $A \geq s - d + 2$, which yields $A = \dim_{\nu}(v) - d + 2$. By the definition of $\tau^{N,D}$, we have

$$\tau^{N,D}(q) \leq \dim_{\nu}(\text{supp}(v) \cap \partial \Xi) - qA.$$

Hence, by our assumption $\dim_{\nu}(\text{supp}(v) \cap \partial \Xi) - qA < \tau_{\nu}^{D}(q)$, we obtain $\tau^{N,D}(q) < \tau_{\nu}^{D}(q)$.

For the case $d = 2$, notice that by Proposition 3.10, we have $\tau_{\nu}^{D,N} = \beta_{\nu}^{D,N}$. Hence, this case follows in a similar way. The second claim follows from the fact that $\dim_{\nu}(v) \leq \dim_{\nu}^{N}(v)$.

In the next section, we will see that all examples studied so far in the literature ([NS95; Tri97; NX21]), fulfil $\tau^{\nu} = \tau^{D}$.

3.4. Special cases. In this section we show that for some particular cases (absolutely continuous measures, Ahlfors–David regular measures, and self-conformal measures) the spectral partition function is completely determined by the $L^{s}$-spectrum assuming $\dim_{\nu}(v) > d - 2$. Furthermore, for these classes of measures we investigate under which conditions the Dirichlet and the Neumann $L^{s}$-spectra coincide. Later, we will use the results to calculate the spectral dimension for these classes of measures.

3.4.1. Absolutely continuous measures.

Lemma 3.15. Let $\nu$ be a non-zero absolutely continuous measure with Lebesgue density $f \in L_{r}(\mathbb{R})$ for some $r \geq 1$. Then, for all $q \in [0, r]$, $\lim inf_{n \to \infty} \beta^{D,N}_{\nu}(q) = \beta^{D,N}(q) = d(1 - q)$.

Proof. First, we remark that, since $\nu(\partial \Xi) = 0$, there exists an open set $O \subset \mathbb{R}$ with $\nu(O) > 0$.

Moreover, we have $\beta^{D,N}_{\nu}(0) = 0$ and $\beta^{D,N}_{\nu}(1) \leq d$. Hence, the convexity of $\beta^{D,N}_{\nu}$ implies $\beta^{D,N}_{\nu}(q) \leq d(1 - q)$ for all $q \in [0, 1]$. Furthermore, for $n$ large, we have $\beta^{D,N}_{\nu}(\varepsilon)(1) = 0$ and $\beta^{D,N}_{\nu}(\varepsilon)(0) \leq d$. Consequently, for all $q \in [1, \infty)$, the convexity of $\beta^{D,N}_{\nu}(\varepsilon)$ gives $\beta^{D,N}_{\nu}(\varepsilon)(q) \geq d(1 - q)$. This implies

$$d(1 - q) \leq \lim inf_{n \to \infty} \beta^{D,N}_{\nu}(\varepsilon)(q) = \lim inf_{n \to \infty} \beta^{D,N}_{\nu}(\varepsilon)(q) \leq \lim inf_{n \to \infty} \beta^{D,N}(q).$$

Moreover, by Jensen’s inequality, for all $q \in [0, 1]$ and $n$ large, we have

$$\sum_{Q \in D_{t}^{\nu}} \nu(Q) \leq \sum_{Q \in D_{t}^{\nu}} \frac{\int_{Q} f \, d\Lambda}{\Lambda(Q)} \nu(Q) \geq \sum_{Q \in D_{t}^{\nu}} \frac{\nu(Q) \Lambda(Q)}{\Lambda(Q)^{r-1}} \int_{Q} f \, d\Lambda \geq \Lambda(Q)^{r-1} \int_{Q} f \, d\Lambda,$$

implying $\lim inf_{n \to \infty} \beta^{D,N}(q) \geq d(1 - q)$. Further, Jensen’s inequality, for all $q \in [1, r]$, yields

$$\sum_{Q \in D_{t}^{\nu}} \nu(Q) \leq \sum_{Q \in D_{t}^{\nu}} \frac{\int_{Q} f \, d\Lambda}{\Lambda(Q)} \Lambda(Q)^{r} \leq \Lambda(Q)^{r-1} \sum_{Q \in D_{t}^{\nu}} \int_{Q} f \, d\Lambda \leq \Lambda(Q)^{r} \int_{Q} f \, d\Lambda.$$
Hence, we obtain \( \lim_{n \to \infty} \beta_{D,n}^\nu(q) \leq d(1-q) \).

**Proposition 3.16** (Absolutely continuous measures). Let \( d > 2 \) and \( \nu \) be a non-zero absolutely continuous measure with Lebesgue density \( f \in L_1^\nu \) for some \( r \geq d/2 \). Then, for all \( q \in [0, r] \),
\[
\lim_{n \to \infty} \tau_{D,n}^\nu(q) = \tau_{D,n}^\nu(q) = \beta_{D,n}^\nu(q) - (2 - d)q = d - 2q.
\]

**Proof.** First, we note that there exists an open set \( O \), with \( \overline{O} \subset \hat{\mathbb{C}} \) such that \( \int f \, d\Lambda > 0 \). This implies
\[
\tau_{D,n}^\nu(q) \geq \beta_{D,n}^\nu(q) - (2 - d)q = d - 2q.
\]

By Jensen’s inequality, for \( d/2 \leq q \leq r \) and \( Q \in \mathcal{D}_{D,n}^\nu \), we have
\[
\nu(Q)^\nu = \left( \int_Q f \Lambda(Q)^{-1} \, d\Lambda \right)^\nu \leq \left( \int_Q f^\nu \, d\Lambda \right) \Lambda(Q)^{\nu - 1}.
\]

This shows that \( \nu(Q)^\nu \Lambda(Q)^{2\nu - \nu} \leq \left( \int_Q f^\nu \, d\Lambda \right) \Lambda(Q)^{2\nu - 1} \), and since \( 0 \leq 2q/d - 1 \), we notice that the right-hand side is monotonic in \( Q \). Therefore we get the following upper bound
\[
\sum_{Q \in \mathcal{D}_{D,n}^\nu} \nu(Q)^\nu \Lambda(Q)^{2\nu - \nu} \leq \sum_{Q \in \mathcal{D}_{D,n}^\nu} \left( \int_Q f^\nu \, d\Lambda \right) \Lambda(Q)^{2\nu - 1} \leq 2^{-m(2q/d - 1)} \|f\|_{\Lambda, \nu}^\nu.
\]

Combining this with Lemma 3.15, we obtain
\[
d - 2q = \lim_{n \to \infty} \beta_{D,n}^\nu(q) + (2 - d)q \leq \lim inf_{n \to \infty} \tau_{D,n}^\nu(q) \leq \tau_{D,n}^\nu(q) \leq d - 2q.
\]

For the remaining case, we use the convexity of \( \tau_{D,n}^\nu \), the lower bound obtained above, and the fact that \( \tau_{D,n}^\nu(0) = d \) and \( \tau_{D,n}^\nu(r) = d - 2r \), to obtain for \( q \in [0, r] \),
\[
d - 2q \geq \tau_{D,n}^\nu(q) \geq \lim inf_{n \to \infty} \tau_{3n}^\nu(q) \geq \lim inf_{n \to \infty} \beta_{D,n}^\nu(q) + (2 - d)q = d - 2q. \]

**3.4.2. Product measures.** The following special case will be used to give an example for the non-existence of the spectral dimension (see Section 8.2). First we will need the following observation for the one-dimensional situation.

**Lemma 3.17.** For \( d = 1 \) and \( \nu((0,1]) = 0 \) we have \( \beta^D = \beta^\nu \) on \( \mathbb{R}_{\geq 0} \).

**Proof.** First we show that \( \dim_{\nu,D}^\nu(v) \geq \dim_{\nu,D}^\nu(v) = \dim_{\nu,D}^\nu(v) \). We start with the case \( \dim_{\nu,D}^\nu(v) > 0 \). For \( \dim_{\nu,D}^\nu(v) > s > 0 \) and \( n \in \mathbb{N} \) large, we have \( \nu(Q) \leq 2^{-m} \) for all \( Q \in \mathcal{D}_{D,n}^\nu \). Hence, using \( \nu((0,1]) = 0 \), it follows
\[
\nu((0,2^{-m})) = \sum_{k=0}^m \nu\left(\left(2^{-m-k+1}, 2^{-m-k}\right]\right) \leq \sum_{k=0}^m 2^{-d-k} x^{+k+1} \leq 2^{-m} \sum_{k=0}^m 2^{-k}
\]
and
\[
\nu\left(\left(2^{k} - 1, 2^{k}\right]\right) = \sum_{k=0}^m \nu\left(\left(2^{k+1} - 1, 2^{k+1}\right]\right) \leq 2^{-m} \sum_{k=0}^m 2^{-k}.
\]

Hence, we obtain \( \dim_{\nu,D}^\nu(v) \geq \dim_{\nu,D}^\nu(v) \). Now, observe
\[
\frac{\log \left( \max_{Q \in \mathcal{D}_{D,n}^\nu} \nu(Q) \right)}{- \log (2^n)} = \min_{k \geq d(\nu,D,n)} \frac{\log \left( \max_{Q \in \mathcal{D}_{D,n}^\nu} \nu(Q) \right)}{- \log (2^n)} = \min_{k \geq d(\nu,D,n)} \frac{\log \left( \max_{Q \in \mathcal{D}_{D,n}^\nu} \nu(Q) \right)}{- \log (2^n)}.
\]

Leading to
\[
\dim_{\nu,D}^\nu(v) \geq \min \left\{ \dim_{\nu,D}^\nu(v), \dim_{\nu,D}^\nu(v) \right\} = \dim_{\nu,D}^\nu(v) \geq \dim_{\nu,D}^\nu(v).
\]

If \( \dim_{\nu,D}^\nu(v) = 0 \), then clearly \( \dim_{\nu,D}^\nu(v) = 0 \). Thus, in any cases, we obtain \( \dim_{\nu,D}^\nu(v) = \dim_{\nu,D}^\nu(v) \).

To conclude the proof, we note that if for some \( q > 0 \) we have \( -q \dim_{\nu,D}^\nu(v) \leq -q \dim_{\nu,D}^\nu(v) < \beta^\nu(q) \), then \( \beta^\nu(q) = \beta^\nu(q) \). Setting \( \alpha := \inf \{ s > 0 : \beta^\nu(s) > -s \dim_{\nu,D}^\nu(v) \geq 1 \} \), we have \( \beta^\nu(q) = \beta^\nu(q) \) for all \( q < \alpha \). Note that \( \beta^\nu(q) \geq 0 \) for all \( q \in [0,1] \), implying \( \alpha > 1 \). If \( \alpha = \infty \) we are finished.
Otherwise, the convexity of $\beta^N_\nu$ and $\beta^N(\nu) \geq -q \dim_w(v)$ impose the identity $\beta^N(\nu) = -q \dim_w(v)$ for all $q \geq \alpha$. This gives for $q \geq \alpha$

$$-q \dim_w(v) = -q \dim_w(v) \leq \beta^D_q(\nu) \leq \beta^N(\nu) = -q \dim_w(v).$$

\[\square\]

Fix $d \geq 3$ and a non-zero finite Borel measure $\nu_d$ on $(0, 1)$, and let $A^1$ denote the one-dimensional Lebesgue measure on $(0, 1)$. For $v := A^1 \otimes \ldots \otimes A^1 \otimes \nu_d$ we have for every $Q \in D$ $d$-times

$$\exists_v(Q) = \sup_{Q' \prod \in Q} \nu(Q' \land (Q')^2 \| d) = 2^{-n} \nu_d(\pi_d(Q)),
$$

where $\pi_d$ is projection onto the $d$-th component. Hence, for all $q \geq 0$, we have

$$r^N_{x,v}(q) = 2^{(d-1)n-2q} \sum_{Q \in \nu_d(Q)} \nu(Q)^q \quad \text{and} \quad r^D_{x,v}(q) = (2^n - 2)^{(d-1)n-2q} \sum_{Q \in \nu_d(Q)} \nu(Q)^q.$$

It follows from Lemma 3.17 that

$$r^N_{x,v}(q) = d = 1-q + \beta^N_\nu(q) = d = 1-q + \beta^D_\nu(q) = r^D_{x,v}(q).$$

3.4.3. Conformal iterated function systems. Let $U \subset \mathbb{R}^d$ be an open set. We say a $C^1$-map $S : U \to \mathbb{R}^d$ is conformal if for every $x \in U$ the matrix $S'(x)$, giving the total derivative of $S$ in $x$, satisfies $|S'(x) \cdot y| = |S'(x)| \cdot |y|$ for all $y \in \mathbb{R}^d$ and $|S'(x)| \geq 1$. Let us assume that $\mathcal{Q}$ is closed. A family of conformal mappings $(S_i : \mathcal{Q} \to \mathcal{Q})_{i \in I}$, with $I = \{1, \ldots, \ell\}$, $\ell \geq 2$ is a conformal iterated function system if for each $i \in I$, the contraction $S_i$ extends to an injective conformal map $S_i : U \to U$ on an open set $U \supset \mathcal{Q}$ such that sup $|S_i'(x)| : x \in U | < 1$. Taking into account that $d \geq 2$, we note that from the previous assumptions the following bounded distortion property holds (see [MU03, Theorem 4.1.3]): there exists a constant $D \geq 1$ such that for all $n \in \mathbb{N}$ and $u \in \{1, \ldots, \ell\}^+$

$$\frac{D^{-1}}{D} \leq \frac{|S_i'(x)|}{|S_i'(y)|} \leq D$$

for $x, y \in U$ with $S_i = S_{i_1} \circ \cdots \circ S_{i_n}$, where $|u|$ denotes the length of $u$. Further, we suppose that the contractions $S_i$, $i \in \{1, \ldots, \ell\}$, do not share the same fixed point. For a conformal iterated function system $(S_i : \mathcal{Q} \to \mathcal{Q})_{i \in I}$ there exists a unique compact set $\mathcal{K} \subset \mathcal{Q}$ such that $\mathcal{K} = \bigcup_{i \in I} S_i(\mathcal{K})$.

Let $(p_\nu)_\mathcal{Q}$ be the associated positive probability vector and define $p_a := \prod_{i \in I} p_{a_i} \nu_a$. Then there is a unique Borel probability measure $\nu$ with support $\mathcal{K}$ such that

$$\nu(A) = \sum_{i \in I} p_a \nu(S_i^{-1}(A))$$

for $A \in \mathcal{B}(\mathbb{R}^d)$. We refer to $\nu$ as the self-conformal measure.

Finally, we need the following result from [PS00, Theorem 1.1]: for a self-conformal measure $\nu$, the $L^\nu$-spectrum $\beta^\nu$ exists as a limit on $\mathbb{R}_{\geq 0}$.

**Lemma 3.18.** For $d \geq 2$ and any self-conformal measure $\nu$ on the closed cube $\mathcal{Q}$ with $\dim_w(v) > d - 2$ we have for $q \geq 0$,

$$\beta^N_q(\nu) + (d - 2)q = \liminf_{n \to \infty} \beta^N_q(\nu) + (d - 2)q = r^N_{x,v}(q) = \liminf_{n \to \infty} r^N_{x,v}(q).$$

**Proof.** Note that $a := 2 - d > \dim_w(v)$ implies $\sup_{Q \in \mathcal{Q}^d} \nu(Q) \land (Q)^{\| d} = C < \infty$. For $n \in \mathbb{N}$, as in [PS00], we let

$$W_n := \{w \in \mathcal{I}^d \land \dim(w(\mathcal{Q})) \leq 2^{-n} \land \dim(w(\mathcal{Q}))< \infty\}.$$
Then we have
\[ v(Q) \wedge (Q') = 2^{-\dim_{s,m}} \sum_{u \in W_s} p_u v(S^{-1}_u(Q')) = 2^{-\dim_{s,m}} \sum_{u \in D} p_u 2^{-m} v(S^{-1}_u(Q')) \]
\[ \leq 2^{-\dim_{s,m}} \sum_{u \in D} p_u 2^{-k} \sum_{Q \in D^\infty} 2^{-\dim_{s,m} - k} v(Q) \]
\[ \leq 2^{-d - 3d} \max_{Q \in D^\infty} \left( v(Q) \wedge (Q') \right) \sum_{u \in D} p_u \]
\[ \leq 2^{-d - 3d} \max_{Q \in D^\infty} \left( v(Q) \wedge (Q') \right) 2^{-\dim_{s,m}} \left( \bigcup_{u \in D} S_u(Q) \right) \leq 2^{-d - 3d} C v(Q') 2^{-m}. \]

Since in the above inequality \( Q' \in D(Q) \) was arbitrary, we deduce for \( q > 0 \),
\[ \sum_{Q \in D^\infty} \exists_0(Q) \leq 2^{d - 2d} 3^d C 2^{-m} \sum_{Q \in D^\infty} \nu(Q^c) \]
\[ \leq 2^{d - 2d} 3^d C 2^{-m} \sum_{Q \in D^\infty} \left( \sum_{Q' \in D} v(Q') \right)^{q} \]
\[ \leq 2^{d - 2d} 3^d C 2^{-m} 3^{dq} \sum_{Q \in D^\infty} \max_{Q' \in D^\infty} v(Q')^{q}. \]

This gives \( \beta^0(q) - aq \geq \tau^0_{s,m}(q) \). Furthermore, observe that \( \beta^0(0) = \dim_{s,m}(v) = \tau^0_{s,m}(0) \). To complete the proof, observe that \( \sum_{Q \in D^\infty} v(Q^c) \wedge (Q') \leq \sum_{Q \in D^\infty} \exists_0(Q) \). Finally, \( \beta^0(q) - aq \leq \lim inf_{n \to \infty} \tau^0_{s,m}(q) \) for \( q > 0 \).

**Lemma 3.19.** Let \( v \) denote a self-conformal measure on \( \mathcal{C} \) and suppose \( \dim_{s,m}(v) > d - 2 \). Then, for all \( q > 0 \),
\[ \beta^V_s(q) = \beta^0(q) = \lim inf_{n \to \infty} \beta^0_{s,m}(q) = \lim inf_{n \to \infty} \beta^V_{s,m}(q). \]

**Proof.** We use the same notation as in the proof of Lemma 3.18. By our assumption there exists \( n \in \mathbb{N} \) such that \( S_n(\mathcal{C}) \subset \mathcal{C} \) for some \( u \in W_u \). Indeed assume for all \( n \in \mathbb{N} \) and \( u \in W_u \), we have \( S_n(\mathcal{C}) \cap \partial \mathcal{C} \neq \emptyset \). Further, using \( \sup_{u \in W_u} \text{diam}(S_n(\mathcal{C})) \leq 2^n \to 0 \) for \( n \to \infty \) and \( \mathcal{K} \subset \bigcup_{u \in W_u} S_n(\mathcal{C}) \), we deduce that \( \mathcal{K} \subset \partial \mathcal{C} \). This gives \( v(\partial \mathcal{C}) = 0 \) contradicting our assumption.

Let us assume that the distance of \( S_n(\mathcal{C}) \) to the boundary of \( \mathcal{C} \) is at least \( 2^{-m} \) for some \( m \in \mathbb{N} \). Then all cubes \( Q \in D^\infty \) intersecting \( S_n(\mathcal{C}) \) lie in \( D^\infty_{m} \) for all \( m > m_0 \). Therefore, using the self-similarity and \( \beta^0(q) = \beta^0(q) = \lim inf_{n \to \infty} \beta^0_{s,m}(q) = \lim inf_{n \to \infty} \beta^V_{s,m}(q) \) for \( q > 0 \). The reverse inequalities are obvious. Hence, the claim follows from [PS00, Theorem 1.1].

**Proposition 3.20.** Let \( v \) denote a self-conformal measure on \( \mathcal{C} \) and suppose \( \dim_{s,m}(v) > d - 2 \). Then, for all \( q > 0 \), we have
\[ \beta^V_s(q) + (d - 2)q = \tau^V_{s,m}(q) = \lim inf_{n \to \infty} \tau^V_{s,m}(q) = \lim inf_{n \to \infty} \tau^0_{s,m}(q). \]

**Proof.** The case \( d = 2 \) follows immediately from Proposition 3.10 and Lemma 3.19. For \( d > 2 \), we obtain from Lemma 3.19 and Lemma 3.18 the following chain of inequalities
\[ \beta^V_s(q) + (d - 2)q = \lim inf_{n \to \infty} \beta^V_{s,m}(q) + (d - 2)q \leq \lim inf_{n \to \infty} \tau^0_{s,m}(q) \leq \lim inf_{n \to \infty} \tau^V_{s,m}(q) = \tau^V_{s,m}(q) = \beta^0(q) + (d - 2)q. \]
The inequalities

Then by our main result in [KN23, Theorem 1.3 and Theorem 1.7] we have

We additionally assume that \( \text{dim}_{\mu}(\mathcal{Z}) > 0 \) and that there exists \( a > 0 \) and \( b \in \mathbb{R} \) such that \( r^{bn}_{\mathcal{Z}}(a) \geq b \) for all \( n \in \mathbb{N} \) large enough. Note that this second condition is naturally fulfilled for \( \mathcal{Z} = \mathcal{Z}_{r,a,b} \).

For \( x > 1/\mathcal{Z}(\mathbb{Q}) \), we define \( M_{\mathcal{Z}}(x) \) as in Section 1.2 and recall the definition of exponential growth rate \( \beta_\nu \) and \( \beta^\langle \nu \rangle \) referred to as the upper, resp. lower, \( \mathcal{Z} \)-partition entropy as stated at the end of Section 1.2. We additionally assume that \( \text{dim}_{\mu}(\mathcal{Z}) > 0 \), and we note that the assumption in [KN23], namely that there is \( a > 0 \) and \( b \in \mathbb{R} \) such that \( r^{bn}_{\mathcal{Z}}(a) \geq b \) for all \( n \in \mathbb{N} \) large enough, is always satisfied for \( \mathcal{Z} = \mathcal{Z}_{r,a,b} \).

For completeness let us also include the dual problem as worked out in [KN23]: For \( n \in \mathbb{N} \) we let

and define the upper and lower exponents of convergence of \( \gamma_\nu \) by

Then by our main result in [KN23, Theorem 1.3 and Theorem 1.7] we have

The inequalities \( F_{\nu}^{\langle \nu \rangle} \leq F_{\nu}^{D} \) and \( F_{\nu}^{D} \leq F_{\nu}^{J} \) hold generally by definition.

By [KN23, Theorem 1.11] we also have the following general regularity results: If \( \nu \) is D/N-PF-regular, then

Next we study the special case of the \( \mathcal{Z}_{r,a,b} \)-partition entropy, which is ultimately associated with the spectral dimension for a certain choice of parameters \( a, b \). Let us introduce the following notation:

The following theorem deals with \( \mathcal{Z}_{r,a,b} \) for the special case \( a = 0 \), which we need to handle the spectral problem for \( d = 2 \).

**Proposition 4.1.** If \( \text{dim}_{\mu}(\nu) > 0 \), then

**Proof.** On the one hand, by Proposition 3.10 we have \( r^{\nu}_{\mathcal{Z}}(q) = r^{\nu}_{\mathcal{Z}_{r,a,b}}(q) \) for \( q \geq 0 \). On the other hand, \( \nu(Q)^{1/\log(2)} \leq \mathcal{Z}_{r,a,b}(Q) \) implies \( \mathcal{Z}_{r,a} \leq \mathcal{Z}_{r,a,b} \). Hence, the result follows from [KN23, Cor. 1.5].

The rest of this section deals with the case \( a \neq 0 \). Recalling the definition of \( q^{\nu}_{\mathcal{Z}_{r,a,b}} \), we find \( q^{\nu}_{\mathcal{Z}_{r,a,b}} \leq q^{\nu}_{\mathcal{Z}_{r,a}} \) with equality for the case \( a > 0 \). We need the following elementary lemma.

**Lemma 4.2.** For \( c, d \in \mathbb{R} \) with \( c < d \), let \( (f_n : [c, d] \rightarrow \mathbb{R})_{n \in \mathbb{N}} \) be a sequence of decreasing functions converging pointwise to a function \( f \). We assume that \( f_n \) has a unique zero in \( x_n \), for all \( n \in \mathbb{N} \) and \( f \) has a unique zero in \( x \). Then \( x = \lim_{n \rightarrow \infty} x_n \).

**Proposition 4.3.** Suppose \( b \dim_{\mu}(\nu) + ad > 0 \). If \( a < 0 \), then

If \( a > 0 \), then

If \( a > 0 \), then

\[ \mathcal{Z}_{r,a,b} \leq q^{\nu}_{\mathcal{Z}_{r,a,b}} = \inf \{ q > 0 : \beta^\langle \nu \rangle(bq) < ad \} \leq \frac{\mathcal{Z}_{r,a,b}}{b \dim_{\mu}(\nu) + ad} \leq \frac{1}{b \dim_{\mu}(\nu) + ad}. \]
In particular, if \( \dim_\infty (\nu) > d - 2 \) and \( d > 2 \), then for all \( t \in I := (0, 2 \dim_\infty (\nu) / (d - 2)) \) we have

\[
\tilde{\gamma}_{2d-2-(d-1)/2} = \frac{t}{2} \tilde{\gamma}_{2(2d-2-(d-1)/2)} \leq \frac{t}{2} \frac{\dim_\infty (\nu)}{2 \dim_\infty (\nu) / (d - 2)} \frac{d}{d - 2}.
\]

Moreover, \( \lim_{t \to 2} q_{2d-2-(d-1)/2}^N = d^N \).

**Proof.** Since \( b \dim_\infty (\nu) + ad > 0 \), we obtain from Fact 3.3 that \( \dim_\infty (\tilde{\Theta}_{ad}) = \dim_\infty (\nu) + ad/b > 0 \) and that \( q_{2d-2-(d-1)/2}^N \) is the unique zero of \( \tau^N_{2d-2-(d-1)/2} \). Using the definition of \( M_{2d} (x) \) and (4.1) applied to \( \tilde{\Theta} = \tilde{\Theta}_{ad} \), we obtain

\[
\lim_{t \to 2} \frac{\log (M_{2d} (x))}{\log (x)} = \lim_{t \to 2} \frac{\log (M_{2d} (\nu))}{\log (\nu)} \leq \frac{q_{2d-2-(d-1)/2}^N}{\nu}.
\]

The estimate of \( q_{2d-2-(d-1)/2}^N \) for the case \( a > 0 \) follows from \( \beta_b (bq) \leq \dim_\infty (\nu) (1 - bq) \) for all \( 0 \leq q \leq 1/b \). For the case \( a < 0 \), Fact 3.12 implies \( q_{2d-2-(d-1)/2}^N \leq \dim_\infty (\nu) / (\dim_\infty (\nu) + ad/b) \).

Now, for \( \dim_\infty (\nu) > d - 2 \) and \( t \in I \) we have \( \dim_\infty (\tilde{\Theta}_{ad}) = \dim_\infty (\nu) + dt(2/d - 1)/2 > 0 \). Hence, the third claim follows from the first part.

The rest of the proof is devoted to prove \( \lim_{t \to 2} q_{2d-2-(d-1)/2}^N = q_{2d-2-(d-1)/2}^N \). For \( a = 2/d - 1 \) and all \( t \in I \), \( q_{2d-2-(d-1)/2}^N \), is the unique zero of \( \tau_{2d-2-(d-1)/2}^N \). Observe that for \( s \in ((d - 2)/2, \dim_\infty (\nu)) \), \( n \) large and all \( Q \in D^n \), we have \( \nu (Q) \leq 2^{-n} \). For fixed \( q \geq 0 \), consider

\[
t \mapsto \tau_{2d-2-(d-1)/2}^N (q) = \lim_{t \to 2} \frac{\log (\sum_{Q \in D^n} \max_{Q \in D^n} v (Q)^y (\Lambda (Q)^y)^{-1/2})}{\log (2^n)}.
\]

Since \( f_2 \) is \( \nu (Q)^y (\Lambda (Q)^y)^{-1/2} \mapsto \nu (Q)^y (\Lambda (Q)^y)^{-1/2} \) is log-convex, it follows that the mapping \( t \mapsto \max_{Q \in D^n} v (Q)^y (\Lambda (Q)^y)^{-1/2} \) is also log-convex (the existence of the maximum is ensured by \( \tilde{\Theta}_{ad} \)). Therefore, by the Hölder inequality, we get that \( t \mapsto \tau_{2d-2-(d-1)/2}^N (q) \) is convex, which carries over to the limit superior of convex functions \( t \mapsto \tau_{2d-2-(d-1)/2}^N (q) \), which is therefore continuous implying \( \lim_{t \to 2} \tau_{2d-2-(d-1)/2}^N (q) = \tau_{2d-2-(d-1)/2}^N (q) \). The claim follows therefore by Lemma 4.2.

**Proposition 4.4 ([KN23]).** For a subsequence \( (n_k) \) define the convex function on \( \mathbb{R} \) by \( B := \lim \sup \tau_{2d-2-(d-1)/2}^N \), and for some \( q \geq 0 \), we assume \( B (q) = \lim \tau_{2d-2-(d-1)/2}^N \) and set \( \mu' = -B (q) \). Then we have \( a' \geq \dim_\infty (\tilde{\Theta}) \) and

\[
a \frac{a' \cdot B (q)}{b'} \leq \sup_{a \geq \dim_\infty (\tilde{\Theta})} \lim_{k \to 2} \frac{\log N_{a' \cdot B (q)} (n_k)}{\log (2^n)}.
\]

Moreover, if \( B (q) = \tau_{2d-2-(d-1)/2}^N (q) \), then \( \mu' (a, b') = -\mu' \tau_{2d-2-(d-1)/2}^N (q) \) and if additionally \( 0 \leq q \leq \tau_{2d-2-(d-1)/2}^N \), then

\[
\frac{a \cdot q + \tau_{2d-2-(d-1)/2}^N (q)}{b} \leq \frac{a' \cdot q + B (q)}{b'}.
\]

The following corollary shows that our result in Proposition 4.4 covers the corresponding statement for the one-dimensional case in [KN22c, Prop. 4.17].

**Corollary 4.5.** Let \( \tilde{\Theta} (Q) \mapsto \nu (Q) \Lambda (Q)^y \) with \( y > 0 \), \( Q \in D \). Then \( \tau_{2d-2-(d-1)/2}^D (Q) = \mu' \nu (Q)^y - dq > 0 \) and \( \dim_\infty (\tilde{\Theta}) \neq \dim_\infty (\nu) + dy > 0 \). Suppose there exists a subsequence \( (n_k) \) and \( q \in (0, 1] \) such that \( \tau_{2d-2-(d-1)/2}^D (q) = \lim \tau_{2d-2-(d-1)/2}^D (q) \). Then for \( B := \lim \sup \tau_{2d-2-(d-1)/2}^D \), we have \( -\mu' \tau_{2d-2-(d-1)/2}^D (q) \in (0, 1] \) and

\[
a \frac{a \cdot q + \tau_{2d-2-(d-1)/2}^D (q)}{b} \leq \sup_{a \geq \dim_\infty (\tilde{\Theta}) + dy} \lim_{k \to 2} \frac{\log N_{a' \cdot B (q)} (n_k)}{\log (2^n)}.
\]

**Proof.** The first claim is obvious since \( y > 0 \). The second inequality follows immediately from Proposition 4.4 and \( \dim_\infty (\tilde{\Theta}) = \dim_\infty (\nu) + dy \). To prove the first inequality observe that \( -\mu' \tau_{2d-2-(d-1)/2}^D (q) \in (a, b] \) with \( -\mu' \tau_{2d-2-(d-1)/2}^D (q) = (a, b] \). Using \( [a', b'] \subset [a + dy, b + dy] \), \( \tau_{2d-2-(d-1)/2}^D (q) = \nu (Q)^y - dq \) and \( \mu' \tau_{2d-2-(d-1)/2}^D (q) = 0 \), we obtain

\[
\frac{(a + dy)}{b + dy} \frac{\tau_{2d-2-(d-1)/2}^D (q)}{b} \leq \frac{a \cdot q + \tau_{2d-2-(d-1)/2}^D (q)}{b} - dy < \frac{a' \cdot q + \tau_{2d-2-(d-1)/2}^D (q)}{b'}.
\]

\( \square \)
In this section we obtain upper bounds for the spectral dimension with respect to a finite Borel measure \( \nu \) on \( \mathcal{Q} \).

### 5. Upper Bounds

#### 5.1. Embedding constants and upper bounds for spectral dimensions

This section establishes an upper bound for the spectral dimension in terms of the embedding constants on sub-cubes.

**Proof of Theorem 1.1.** For a partition \( \mathcal{Z} \in \Pi_1 \), let us define the following closed linear subspace of \( H^\nu \)

\[
\mathcal{F}_\mathcal{Z} = \left\{ u \in H^\nu : \int_Q u \, d\Lambda = 0, \ Q \in \mathcal{Z} \right\}.
\]

We define an equivalence relation \( \sim \) on \( H^\nu \) induced by \( \mathcal{F}_\mathcal{Z} \) as follows \( u \sim v \) if and only if \( u - v \in \mathcal{F}_\mathcal{Z} \). Note that we have \( \dim H^\nu / \mathcal{F}_\mathcal{Z} = \text{card} (\mathcal{Z}) \). Further, by our assumption, we have for all \( u \in \mathcal{C}_c^\nu (\mathcal{Q}) \cap \mathcal{F}_\mathcal{Z} \)

\[
\|u\|_{L_2}^2 = \sum_{\mathcal{Q} \in \mathcal{Z}} \int_Q u^2 \, dv \leq \sum_{\mathcal{Q} \in \mathcal{Z}} \mathcal{H}(\mathcal{Q}) \|\nabla u\|_{L_2}^2 \leq \max_{\mathcal{Q} \in \mathcal{Z}} \mathcal{H}(\mathcal{Q}) \sum_{\mathcal{Q} \in \mathcal{Z}} \|\nabla u\|_{L_2}^2 \leq \max_{\mathcal{Q} \in \mathcal{Z}} \mathcal{H}(\mathcal{Q}) \|\nabla u\|_{L_2}^2.
\]

Next we show that \( \mathcal{C}_c^\nu (\mathcal{Q}) \cap \mathcal{F}_\mathcal{Z} \) lies dense in \( \mathcal{F}_\mathcal{Z} \) with respect to \( H^\nu \). Since \( \mathcal{Q} \) has the extension property we readily see that \( \mathcal{C}_c^\nu (\mathcal{Q}) \) lies dense in \( H^\nu \). Hence, for every \( u \in \mathcal{F}_\mathcal{Z} \), there exists a sequence \( u_n \) in \( \mathcal{C}_c^\nu (\mathcal{Q}) \) such that \( u_n \to u \) in \( H^\nu \). The Cauchy-Schwarz inequality gives for all \( Q \in \mathcal{Z} \)

\[
\left| \int_Q u_n \, d\Lambda - \int_Q u \, d\Lambda \right| \leq \left( \int_Q (u_n - u)^2 \, d\Lambda \right)^{1/2} \to 0.
\]

It follows that \( \int_Q u_n \, d\Lambda \to 0 \). Furthermore, for every \( Q \in \mathcal{Z} \) there exists \( u_Q \in \mathcal{C}_c^\nu (\mathcal{Q}) \) such that \( u_Q \mathcal{Q} = 0 \) and \( \int_Q u_Q \, d\Lambda = 1 \). Then for \( u'_n := u_n - \sum_{\mathcal{Q} \in \mathcal{Z}} 1_{\mathcal{Q}} u_Q \mathcal{Q} \in \mathcal{C}_c^\nu (\mathcal{Q}) \cap \mathcal{F}_\mathcal{Z} \) with \( u'_n \to u \) in \( H^\nu \). Thus, for \( u \in \mathcal{F}_\mathcal{Z} \), we obtain

\[
\int_Q |u|^2 \, d\nu \leq \max_{\mathcal{Q} \in \mathcal{Z}} \mathcal{H}(\mathcal{Q}) \|\nabla u\|_{L_2(\mathcal{Q})}^2.
\]

Define for \( i \in \mathbb{N} \)

\[
\lambda_{\mathcal{F}_\mathcal{Z},i} := \inf \left\{ \text{sup} \{ R_{\mathcal{F}_\mathcal{Z}} (\phi) \ ; \ \phi \in \mathcal{G}^* : \ G \in \mathcal{F}_\mathcal{Z}, \langle \phi, \cdot \rangle_{\mathcal{F}_\mathcal{Z}} \} \right\}
\]

\[ R_{\mathcal{F}_\mathcal{Z}} (\psi) := \langle \psi, \phi \rangle_{\mathcal{F}_\mathcal{Z}} / \| \phi \|, \] and \( N^\nu(y, \mathcal{F}_\mathcal{Z}) := \text{card} \left\{ i \in \mathbb{N} : \lambda_{\mathcal{F}_\mathcal{Z},i} \leq y \right\}, \ y > 0 \). Hence, \( \text{max}_{Q \in \mathcal{Z}} \mathcal{H}(Q) < 1/\text{ixs} \), implies \( \lambda_{\mathcal{F}_\mathcal{Z},i} > x \).

In view of the min-max principle as stated in Proposition 2.2 (see also [Kig01, proof of Theorem 4.1.7]), we deduce

\[ N^\nu(x) \leq N^\nu(x, \mathcal{F}_\mathcal{Z}) + \text{card}(\mathcal{Z}) = \text{card}(\mathcal{Q}), \]

implying \( N^\nu(x) \leq M^\nu(x) \) and hence \( \mathcal{T}^\nu \leq \mathcal{B}_\mathcal{Z} \) and \( N^\nu \leq \mathcal{B}^\nu \). \( \square \)

**Remark 5.1.** Note that in the one dimensional case the assumption of Theorem 1.1 is always valid. Indeed, there exists \( C > 0 \) such that for all intervals \( I \) contained in \([0, 1] \) and \( u \in \mathcal{C}_c^\nu (\mathcal{I}) \) with \( \int_I u \, d\Lambda = 0 \), we have

\[
\|u\|_{L_2(\mathcal{I})}^2 \leq C_{\nu} (I) \lambda(I) \|\nabla u\|_{L_2(\mathcal{I})}^2 = C_{\nu, 1, 1}(I) \|\nabla u\|_{L_2(\mathcal{I})}^2,
\]

(see for instance the proof of [BS67, Theorem 3.3.1]). With this observation our general results reproduce the upper bounds for spectral dimension in \( d = 1 \) in terms of the fixed point of the \( L^2 \)-spectrum ([KN22c]).

**Remark 5.2.** The ideas underlying in Theorem 1.1 correspond to some extent to those developed in [NS95; Sol94],[NS01, Chapter 5], that is, reducing the problem of estimating the spectral dimension to an auxiliary counting problem. To illustrate the parallel, we present an alternative proof of the upper estimate of the eigenvalue counting function for self-similar measures under OSC ([Sol94, Theorem 1]). As in the setting in [Sol94] we let \( \nu \) denote a self-similar measure under OSC with contractive similitudes \( \lambda_1, \ldots, \lambda_m \) and corresponding contraction ratios \( h_i \in (0, 1) \) and probability weights \( p_i \in (0, 1) \), for \( i = 1, \ldots, m \) (see [Hut81]). We assume \( \nu(\mathcal{Q}) = \nu(\mathcal{Q}) \) and \( \dim_{\text{H}}(\nu) = d - 2 \), which is in this case equivalent to \( \text{max} \, p_i h_i^{d-2} < 1 \). For simplicity we assume the feasible set is
given by $\tilde{C}$, i.e. $S(\tilde{C}) \subset \tilde{C}$. Instead of $D$ we will consider a symbolic partition by the cylinder sets $\tilde{D} := \left\{ T_\omega(\tilde{C}) : \omega \in I \right\}$ with $I := \{1, \ldots, m\}$. Then $\tilde{3}$ will be replaced by $\tilde{3} : \tilde{D} \to \mathbb{R}_{>0}$ with $\tilde{3}(T_\omega(\tilde{C})) := p_\omega h_{2,2}^{\tilde{C}}$, $\omega \in I$. Now, observe that for $0 < t < \min_{i=1, \ldots, n} p_i h_{2,2}^{\tilde{D}}$, we have

$$\tilde{p}_i := \left\{ \omega \in I : p_\omega h_{2,2}^{\tilde{D}} < t \leq p_\omega h_{2,2}^{\tilde{D}} \right\},$$

is a partition of $I^\iota$. Further, $\delta$ is the unique solution of $\sum_{i=1}^m (p_i h_{2,2}^{\tilde{D}})^2 = 1$. Then there exists $K > 0$ such that for all $u \in H^N$ with $\int_{\tilde{D}} \chi_{\tilde{D}} u \, d\Lambda = 0$, $\omega \in \tilde{p}_i$

$$\int_{\tilde{D}} \|u\|^2 \, d\nu \leq K \max_{\omega \in \tilde{p}_i} \tilde{3}(T_\omega(\tilde{C})) \int_{\tilde{D}} \|u\|^2 \, d\Lambda < t K \int_{\tilde{D}} \|u\|^2 \, d\Lambda$$

(see [NS01, p. 502]). Then a simple computation gives the two-sided estimate

$$\Gamma^\delta \leq \text{card} \left( \tilde{p}_i \right) \leq \frac{\Gamma^\delta}{\min_{i=1, \ldots, m} p_i h_{2,2}^{\tilde{D}}},$$

The variational principle gives

$$N^N \left( t K^{-1} \right) \leq \text{card} \left( \tilde{p}_i \right) \leq \frac{\Gamma^\delta}{\min_{i=1, \ldots, m} p_i h_{2,2}^{\tilde{D}}},$$

hence the results of [NS01; NS95, Theorem 1.] follow from this simple counting argument without the need for renewal theory. Using the specific structure of self-similar measures as in [NS95; Sol94; NS01, Chapter 5] or in the above argument leads nicely to the asymptotic spectral bounds, but at the same time this approach does not provide room for generalisations to study arbitrary finite and finitely supported Borel measures as was our concern in this paper.

### 5.2. Upper bounds on the embedding constants

In this section, up to multiplicative uniform constants, we make use of best embedding constants for the embedding $C^a_c(\mathbb{R}^d)$ into $L_\infty$, $t > 2$, to estimate the spectral dimension from above. Let us recall the definition (1.5) of $\zeta_{r, a, b}$ from the introduction and, for ease of notation, set $\zeta_{r, a, b}(Q) = \zeta_{r, a, b}(Q)$, with $r = 1/2$ for $a = 0$ and $r = \infty$ for $a \neq 0$. Then the best constant $C$ in

$$\|u\|_{L_2(Q \cap \mathbb{R}^d)} \leq C \|u\|_{L^2(Q \cap \mathbb{R}^d)}$$

(5.1) is equivalent to $\zeta_{r, a, b}(Q)$ in the sense that there exist $c_1, c_2 > 0$ only depending on $d$ and $t$ such that $c_1 C \leq \zeta_{r, a, b}(Q) \leq c_2 C$. This result for the case $d > 2$ is a corollary of Adams’ Theorem on Riesz potentials (see e.g. [Maz11, p. 67]) and the case $d = 2$ is due to Maz’ya and Preobrazenskii and can be found in [Maz11, p. 83] or [MP84]. The following lemma establishes an alternative representation of the best equivalent constant in terms of dyadic cubes.

**Lemma 5.3.** Let $Q \in D$ and $\nu$ a finite Borel measure on $\mathbb{R}^d$ and $a \leq 0$ and $b > 0$. Then there exists a constant $C > 0$, depending only on $a, b, d, \nu$, such that

$$C^{-1} \zeta_{r, a, b}(Q) \leq \zeta_{r, a, b}(Q) \leq C \zeta_{r, a, b}(Q).$$

**Proof.** Let $Q \in D^\nu_d$. Since $a < 0$ we assume with out loss of generality that $0 < \nu \leq (\sqrt{d} - a)^{-1}$. Then for $m \geq n - 1$ with $\sqrt{d} - a \leq 2^{-m - 1}$, and $x \in \mathbb{R}^d$,

$$\nu(Q \cap B(x, \varrho))^e \leq 2^{-2d} \sum_{Q < D^\nu_m \cap Q \cap B(x, \varrho), \varrho > 0} \nu(Q \cap Q') 2^{-m_a} \leq \left(3 \sqrt{d} \right)^{2 - a} \sup_{Q < D^\nu_m} \nu(Q) \Lambda(Q)^{r/d} \leq \left(3 \sqrt{d} \right)^{2 - a} \nu(Q) \Lambda(Q)^{r/d} = \left(3 \sqrt{d} \right)^{2 - a} \zeta_{r, a, b}(Q),$$

where we used the fact that $B(x, \varrho) \cap Q$ can be covered by at most $\left(3 \sqrt{d} \right)^d$ elements of $D^\nu_m$ and if $Q' \cap Q \neq \varnothing$, then $Q' \subset Q$ for $m \geq n$, and since $Q \in D^\nu_m$, $\max_{Q' < D^\nu_m} \nu(Q \cap Q') \Lambda(Q')^{r/d} \leq \nu(Q) \Lambda(Q)^{r/d} = \max_{Q' < D^\nu_m} \nu(Q \cap Q') \Lambda(Q')^{r/d}$. Since $x \in \mathbb{R}^d$ and $\nu > 0$ were arbitrary, the second inequality follows.
On the other hand, for $Q' \in D_0^m$ with $Q' \subset Q$ and $Q := \sqrt{d-1}$ we find $x \in \mathbb{R}^d$ such that $Q' \subset B(x, \delta)$. Then
\[
v(Q')^a \Lambda(Q')^{a\beta} \leq \nu(Q \cap B(x, \delta))^a 2^{-m \nu} \leq \left( \sqrt{d-1} \right)^{a\nu} \nu(Q \cap B(x, \delta))^a \leq \left( \sqrt{d-1} \right)^{-\nu} \sup_{x \in \mathbb{R}^d, \rho \in 0} \delta^a \nu(Q \cap B(x, \delta))^a.
\]
For case $a = 0$, we have for any $2^{-m \nu} \leq \rho < 2^{-m \nu}, m \in \mathbb{N}$ and $x \in \mathbb{R}^d$,
\[
\log(Q) \nu(Q \cap B(x, \delta))^a \leq \log(2)(m + 1) \nu(Q \cap B(x, 2^{-m \nu})).
\]
Using (5.1) in combination with Hölder's inequality we obtain the following corollary.

**Proposition 4.1 and Proposition 4.3** we obtain
\[
\nu(Q')^a \nu(Q)^b \leq \nu(Q \cap B(x, \delta))^a \nu(Q)^b.
\]

**Corollary 5.4.** For $d \geq 2, \tau > 2$, there exists a constant $D > 0$ such that for all finite Borel measure $\nu$ on $\mathbb{R}^d$, $Q \in D$ and $u \in C_c^\infty(\mathbb{R}^d)$ with $\int u \mathrm{d}\Lambda = 0$ we have
\[
\|u\|_{L^2(Q)} \leq D \nu(Q)^{1/2-1/2} \sqrt{\mathbb{E}_{\mathbb{R}^d}}(Q) \|\nabla u\|_{L^2(Q)} \leq D \nu(Q)^{1/2-1/2} \sqrt{\mathbb{E}_{\mathbb{R}^d}}(Q) \|\nabla u\|_{L^2(Q)}.
\]

**Proof.** Using [Maz85, Corollary, p. 54] or [Maz85, Theorem, p. 381–382] for $d > 2$, [Maz85, Corollary 1, p. 382] for $d = 2$ (note there is a typo, the constant $C_3$ has to be replaced by $C_3^{1/2}$; see also [Maz85, p. 83] for the correct version) and the Hölder inequality, we find a constant $C_3 > 0$ independent of $Q \in D$ and $\nu$ such that all $u \in C_c^\infty(\mathbb{R}^d)$
\[
\|u\|_{L^2(Q)} \leq \nu(Q)^{1/2-1/2} \|\nabla u\|_{L^2(Q)} \leq \nu(Q)^{1/2-1/2} C_3 \mathbb{E}_{\mathbb{R}^d}(Q) \|\nabla u\|_{L^2(Q)}.\]

Therefore, for $D := C_3 C \mathbb{E}_{\mathbb{R}^d}(Q)$, where $C > 0$ is chosen according to Lemma 5.3 for $a = 2 - d$ and $b = 2/\tau, 2$, combined with Lemma 2.1, we have for all $u \in C_c^\infty(\mathbb{R}^d)$
\[
\|u\|_{L^2(Q)} = \|\nabla^\alpha u\|_{L^2(Q)} \leq CD \nu(Q)^{1/2-1/2} \sqrt{\mathbb{E}_{\mathbb{R}^d}}(Q) \|\nabla^\alpha u\|_{L^2(Q)} \leq \nu(Q)^{1/2-1/2} C_3 \mathbb{E}_{\mathbb{R}^d}(Q) \|\nabla^\alpha u\|_{L^2(Q)}.
\]

**Corollary 5.5.** Let $\nu$ be a finite Borel measure on $\mathbb{R}^d$ with $\dim_\nu(\nu) > d - 2$. Then
\[
\mathfrak{h}^D \leq \mathfrak{h}^\nu \leq \lim_{\tau \downarrow 2} \mathfrak{h}^{3_2, d-2, \nu} \leq \mathfrak{h}^\nu \leq \lim_{\tau \downarrow 2} \mathfrak{h}^{3_2, d-2, \nu}.
\]

**Proof.** Note that $\dim_\nu(\nu) > d - 2$ implies that for all $t \in (2, 2 \dim_\nu(\nu)/(d - 2))$, $3_2, d-2, t$ is non-negative, monotone and uniformly vanishing on $D$. Combining Corollary 5.4, Theorem 1.1, Proposition 4.1 and Proposition 4.3 we obtain $\mathfrak{h}^D \leq \mathfrak{h}^{3_2, d-2, \nu}$ and $\mathfrak{h}^\nu \leq \mathfrak{h}^{3_2, d-2, \nu}$. In particular, in the case $d = 2$ we have $\mathfrak{h}^D \leq \mathfrak{h}^\nu \leq \mathfrak{h}^{3_2, d-2, \nu}$.
6. Lower bounds

This section provides the necessary estimates for the lower bounds.

6.1. Lower bound on the spectral dimension. In the following we always assume \( \dim_w (\nu) > d - 2 \). Let \( n \in \mathbb{N} \) and \( \alpha > 0, \)

\[
N_{n, \alpha}^{D_0} (n) = \text{card} \left( B_{n, \alpha}^{D_0} (n) \right) \quad \text{with} \quad B_{n, \alpha}^{D_0} (n) = \{ Q \in D_{n, \alpha}^{D_0} : \exists (Q) \geq 2^{-n} \}.
\]

**Lemma 6.1.** Assume the conditions of Theorem 1.2 are fulfilled. Then for fixed \( \alpha > 0 \), for all \( x > 0 \) large, and with \( n_{\alpha,x} := \lfloor \log_2 (x / \alpha) \rfloor \), we have

\[
N_{n, \alpha}^{D_0} (n_{\alpha,x}) S^d - 1 \leq N^D (x) \quad \text{and} \quad N_{n, \alpha}^{D_0} (n_{\alpha,x}) S^d / 2 - 1 \leq N^D (x / D_{\alpha}).
\]

**Proof.** For \( n \in \mathbb{N} \) large enough, i.e. \( B_{n, \alpha}^{D_0} (n) \neq \emptyset \), via a finite induction, we construct a subset \( E_n \) of \( B_{n, \alpha}^{D_0} (n) \) of cardinality \( e_n \), such that for all cubes \( Q, Q' \in E_n \) with \( Q \neq Q' \) we have \( (Q) \cap (Q') = \emptyset \), where the definition of \( (Q) \) is given just before Lemma 2.5. At the initial step of the induction we set \( D^{(0)} := B_{n, \alpha}^{D_0} (n) \). Assume we have constructed \( D^{(0)} \supset D^{(1)} \supset \cdots \supset D^{(j-1)} \) such that the following condition holds: There exists \( Q, Q_j \in D^{(j-1)} \) with \( Q \neq Q_j \) and \( (Q_j) \cap Q = \emptyset \). Then we set

\[
D^{(j)} := \left\{ Q' \in D^{(j-1)} : Q' \cap (Q_j) = \emptyset \right\} \cup \{ Q \}.
\]

By this construction, we have \( \text{card} (D^{(j)}) < \text{card} (D^{(j-1)}) \), since \( Q \cap (Q_j) = \emptyset \). If \( (Q_j) \cap (\bar{Q}) = \emptyset \), for all \( Q, Q_j \in D^{(j)} \) with \( Q \neq Q_j \), then we set \( E_n = D^{(j+1)} \). In each inductive step, we remove at most \( S^d - 1 \) elements of \( D^{(j+1)} \), while one element, namely \( Q_j \), is kept. This implies \( \text{card} (E_n) \geq N_{n, \alpha}^{D_0} (n / S^d) \).

Let us first consider the Dirichlet case. Since for each \( Q \in D \) with \( \partial Q \cap \mathbb{C} = \emptyset \), it follows that \( (\mathbb{C}) \subset \mathbb{C} \) and therefore \( \Phi_0 \subset C^0 (\mathbb{C}) \). Now, with \( n_{\alpha,x} := \lfloor \log_2 (x / \alpha) \rfloor \) for each \( Q \in E_{n,\alpha} \) we have

\[
\int \left| \nabla \psi_0 \right|^2 \, d\lambda / \int \left| \psi_0 \right|^2 \, dv \leq 1 / \lambda (Q) \leq x.
\]

Hence, the \( \left\{ \psi_0 : Q \in E_{n,\alpha} \right\} = \left\{ f_i : i = 1, \ldots, e_{n,\alpha} \right\} \) are mutually orthogonal both in \( L^2 \) and in \( H^D \), and we obtain that \( \text{span} \left( f_i : i = 1, \ldots, e_{n,\alpha} \right) \) is an \( e_{n,\alpha} \)-dimensional subspace of \( H^D \). Hence, we deduce from Corollary 2.4 that \( N_{n, \alpha}^{D_0} (n_{\alpha,x}) S^d - 1 \leq e^D \leq N^D (x) \).

In the Neumann case, we proceed similarly. For fixed \( \alpha > 0 \) set \( n_{\alpha,x} := \lfloor \log_2 (x / \alpha) \rfloor \) and write \( E_{n,\alpha} = \{ Q_1, \ldots, Q_{\text{card}(E_{n,\alpha})} \} \). For each \( i = 1, \ldots, e_{n,\alpha} / 2 = N_{n,\alpha} \), we have \( f_i := \mathcal{R} (a_{2,1,1,\alpha} \psi_{Q_{0,1,1,\alpha}} + a_{2,2,1,\alpha} \psi_{Q_{0,2,1,\alpha}}) \in C^0 (\mathbb{C}) \), where we choose \( (a_{2,1,1,\alpha}, a_{2,2,1,\alpha}) \in \mathbb{R}^2 \setminus \{ (0, 0) \} \) such that \( \int f_i \, d\lambda = 0 \). Using \( (\mathbb{C}) \subset \mathbb{C} \), and the properties of mediants and Lemma 2.1, we obtain

\[
\frac{\left( f_i, f_j \right)_{H^D}}{\left( f_i, f_j \right)_{L^2}} \leq \frac{\int \left| \nabla \psi_{Q_{0,1,1,\alpha}} \right|^2 \, d\lambda}{\int \left| \psi_{Q_{0,1,1,\alpha}} \right|^2 \, dv} \leq \frac{\alpha_1}{D_{\alpha}} \frac{\int \left| \nabla \psi_{Q_{0,1,1,\alpha}} \right|^2 \, d\lambda}{\int \left| \psi_{Q_{0,1,1,\alpha}} \right|^2 \, dv} \leq \frac{1}{D_{\alpha}} \max \left\{ \frac{\int \left| \nabla \psi_{Q_{0,1,1,\alpha}} \right|^2 \, d\lambda}{\int \left| \psi_{Q_{0,1,1,\alpha}} \right|^2 \, dv}, \frac{\int \left| \nabla \psi_{Q_{0,2,1,\alpha}} \right|^2 \, d\lambda}{\int \left| \psi_{Q_{0,2,1,\alpha}} \right|^2 \, dv} \right\} \leq \frac{1}{D_{\alpha}} \left( \frac{1}{\lambda (Q_{0,1,1,\alpha})} + \frac{1}{\lambda (Q_{0,2,1,\alpha})} \right) \leq \frac{x}{D_{\alpha}}.
\]

Hence, the \( f_i \) are mutually orthogonal in \( H^N \), and also in \( L^2 \), we obtain that \( \text{span} \left( f_1, \ldots, f_{n_{\alpha,x}} \right) \) is a \( N_{n,\alpha,x} \)-dimensional subspace of \( H^D \). Again, an application of Corollary 2.4 gives the second inequality.

**Proof of Theorem 1.2.** From the above lemma, we have \( N_{n, \alpha}^{D_0} (n_{\alpha,x}) / S^d - 1 \leq N^D (x) \). Consequently, we conclude

\[
\liminf_{x \to \infty} \frac{\log \left( N^D (x) \right)}{\log (x)} = \liminf_{n \to \infty} \frac{\log \left( N_{n, \alpha}^{D_0} (n) \right)}{\log (2^n)} = \frac{F^D (\alpha)}{\alpha},
\]

taking the supremum over all \( \alpha > 0 \) gives \( F^D \leq s^D \). Furthermore, for \( x_{n,x} := 2^n \) with \( n \in \mathbb{N} \), we see that

\[
F^D \geq \limsup_{n \to \infty} \frac{\log \left( N^D (x_{n,x}) \right)}{\log (x_{n,x})} \geq \limsup_{n \to \infty} \frac{\log \left( N_{n, \alpha}^{D_0} (n) \right)}{\log (2^n) \alpha}.
\]
implying $F_\nu^D \leq 2^\nu$. In the Neumann case, using $N^N_{\Delta,\alpha}(x_{n,\alpha}) / (2 \cdot 2^d) - 2 \leq N^N(x/D_\nu)$, we obtain in the same ways as in the Dirichlet case that $F_\nu^N \leq 2^\nu$ and $F_\nu^D \leq 2^\nu$. □

6.2. Lower bound on the embedding constant. In the following we assume (1.3), that is $\dim_{\alpha}(v) > d-2$. We need a slight modification of $\Delta$, for the case $d = 2$. We define $\tilde{\Delta}_n(Q) := \sup_{Q \in D(Q)} \nu(Q) \Lambda(Q)^{2/d-1}$ for $Q \in D$. Hence, in the case $d = 2$, we have $\tilde{\Delta}_2(Q) = \nu(Q)$. Clearly, we again have $\dim_{\alpha}(\tilde{\Delta}_n) > d-2$, $\tilde{\Delta}_n^D = \tilde{\Delta}_n^N$ by Proposition 3.10 and for $d > 2$, $\tilde{\Delta}_n^D = \tilde{\Delta}_n^N$ and $\tilde{\Delta}_n^D = \tilde{\Delta}_n^N$. The case $d = 2$ is covered by the following lemma.

**Lemma 6.2.** In the case $d = 2$, we have $F_2^D = F_2^N$ and $F_2^D = F_2^N$.

**Proof.** We always have

$$\left\{ Q \in D^D_n : \sup_{Q \in D(Q)} \nu(Q) \left| \log(\Lambda(Q)) \right| \geq 2^{-\alpha n} \right\} \subset \left\{ Q \in D^D_n : \nu(Q) \geq 2^{-\alpha n} \right\}$$

and, using $\dim_{\alpha}(\nu) > d-2$, we obtain for every $1 < \delta$ and $n \in \mathbb{N}$ large enough

$$\nu(Q) \left| \log(\Lambda(Q)) \right| \leq \nu(Q)^{1/\delta}, \quad Q \in D^D_n.$$  

Indeed, for $d-2 < s < \dim_{\alpha}(v)$, we have for all $n$ large and $Q \in D^D_n$ with $\nu(Q) > 0$ that $\nu(Q) \leq 2^{-m}$. Further, for fixed $0 < \varepsilon < 1$ large and $Q \in D^D_n$, we obtain $\nu(Q) \left| \log(\Lambda(Q)) \right| \leq \nu(Q)^{1-\varepsilon}$. This leads to

$$\left\{ Q \in D^D_n : \sup_{Q \in D(Q)} \nu(Q) \left| \log(\Lambda(Q)) \right| \geq 2^{-\alpha n} \right\} \subset \left\{ Q \in D^D_n : \nu(Q) \geq 2^{-\alpha \delta n} \right\}.$$  

Hence, the claim follows. □

**Proposition 6.3.** There exists a constant $K > 0$ such that for every $Q \in \mathcal{D}$ with $\tilde{\Delta}_n(Q) > 0$ there exists a function $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ with support contained in $Q$ such that

$$\left\| \phi_0 \right\|_{L^2}^2 \geq K \tilde{\Delta}_n(Q) \left\| \nabla \phi_0 \right\|_{L^2(Q)}^2.$$  

**Proof.** Since $\dim_{\alpha}(\tilde{\Delta}_n) > 0$, it follows that for each $Q \in \mathcal{D}$ there exists $Q_0 \in \mathcal{D}(Q)$ such that $\tilde{\Delta}_n(Q) = \nu(Q_0) \Lambda(Q_0)^{2/d-1}$. Now, choose $\phi_0 := \hat{\varphi}_{(0)} \cdot 1_{Q_0}$ as in Lemma 2.5. Then $\phi_0 \cdot 1_{Q_0} = 1_{Q_0}$, supp(\phi_0) $\subset \langle Q_0 \rangle \subset \langle Q \rangle$, and

$$\frac{\int |\nabla \phi_0|^2 \, dv}{\int |\phi_0|^2 \, dv} \leq C \frac{1}{2^{1-\alpha n}} \frac{\Lambda((Q_0) \cdot 1_{Q_0})^{1-2/d} \nu((Q_0) \cdot 1_{Q_0})}{\nu(Q_0)} = C \frac{1}{2^{1-\alpha n}} \frac{\Lambda^{1-2/d} \nu^{1-2/d}}{\nu(Q_0)} = C \frac{1}{2^{1-\alpha n}} \tilde{\Delta}_n(Q).$$  

□

**Proposition 6.4.** For fixed $\alpha > 0$ and for $x > 0$ large, we have

$$N^D_{\Delta,\alpha}(n_{\alpha,x}) 2^{-d} - 1 \leq N^D(x/K)$$

with $n_{\alpha,x} := \lfloor \log_2(x) / \alpha \rfloor$. In particular $F_x^D \leq 2^D$ and $F_\nu^D \leq 2^\nu$.

**Proof.** This follows from Proposition 6.3, Lemma 6.1, and Lemma 6.2. □

In the same way we obtain the following proposition for the Neumann case.

**Proposition 6.5.** For fixed $\alpha > 0$, we have for $x > 0$ large

$$N^N_{\Delta,\alpha}(n_{\alpha,x}) 2^{-d/2} - 1 \leq N^N(x/K/D)$$

with $n_{\alpha,x} := \lfloor \log_2(x) / \alpha \rfloor$. In particular $F_x^N \leq 2^N$ and $F_\nu^N \leq 2^\nu$. □
7. Proof of the remaining main results

This chapter is devoted to the proofs of the other main results. To break up the results of Theorem 1.3, we start with the following proposition and its proof.

**Proposition 7.1.** We have $\mathcal{T}_d^d \leq q^N_{\beta_0} = \mathcal{F}^N_{\beta_0} = \mathcal{F}_d^d$ and $\mathcal{T}_d^d \leq s^N_d \leq \mathcal{F}_d^d$.

**Proof.** From Proposition 6.5 and (4.1) applied to $\beta_0 = \beta_0$, we obtain $q^N_{\beta_0} = \mathcal{F}^N_{\beta_0} \leq s^N_d$. Corollary 5.5 yields $s^N_d \leq s^N \leq \lim_{t \to 0} \mathcal{F}^N_{\beta_0} \leq q^N_{\beta_0}$ and $\mathcal{F}^N_{\beta_0} = \mathcal{F}_d^d = q^N_{\beta_0}$ which proves the claimed inequalities.

**Proof of Theorem 1.3.** The first and second claim follow Proposition 7.1, Proposition 6.5 and Proposition 6.4. To prove the third claim, we note that (1.8) fulfills the assumption of Lemma 3.13, which implies $q^N_{\beta_0} = q^N_{\beta_0}$.

**Proof of Theorem 1.7.** Under assumption that $\nu$ is D/N-PF regular, we obtain from (4.2), Proposition 6.5 and Proposition 6.4 that $\mathcal{L}^{(N)} \geq \mathcal{F}^{(N)} = q^{(N)}$ and $\mathcal{H}^d \leq s^N_d$. Together with (4.1) and Proposition 2.3 the claim follows.

**Proof of Proposition 1.9.** This follows from Proposition 6.5, Proposition 6.4 and Proposition 4.4.

**Proof of Corollary 1.11.** For $d > 2$, Theorem 1.3 gives $s^N = q^N_{\beta_0}$, hence the claim follows from the estimates of $q^N_{\beta_0}$ obtained in Fact 3.12. In the case $d = 2$ and $\nu(\mathcal{C}) > 0$, there exists an open dyadic cube $Q$ such that $\overline{Q} \subset \mathcal{C}$, $\nu(Q) > 0$ and $\dim_{\nu}(\nu_0) > d - 2 = 0$. Hence, we obtain

$$1 = d^N_{\beta_0} \leq \mathcal{F}^N_{\beta_0} = \mathcal{F}_d^d \leq \mathcal{F}_d^d = q^N_{\beta_0} \leq s^N_d = 1.$$

**Proof of Proposition 1.13.** We immediately obtain from Theorem 1.7 and Proposition 3.16 that $s^N = s^N = d/2$.

**Proof of Proposition 1.14.** Suppose $s^N = d/2$. Then by Theorem 1.3, we have $q^N_{\beta_0} = d/2$. Moreover, for all $0 \leq q \leq 1$, we have $d - 2q = \beta^0(q) + (d - 2)q \leq s^N_d$. The convexity of $s^N_d$ yields $s^N(q) \leq d - 2q$ for all $q \in [0, d/2]$. Further, the convexity of $\beta^0$ gives for all $q \geq 1$ that $d - 2q = \beta^0(q) + (d - 2)q$. This proves the first claim. For the second claim, assume $s^N(q) = d - 2q$ for some $q > d/2$. Again, for all $q' \in [0, q]$, we deduce $d - 2q' = \beta^0(q') + (d - 2)\tau(q') \leq \tau^N(q) \leq d - 2q$. In particular, $\tau^N(d/2) = 0$ implying $\mathcal{F}^N_\beta = d/2$. The final assertion follows from the regularity result Theorem 1.7.

**Proof of Proposition 1.15.** Let $\nu$ be a finite $\alpha$-Ahlfors–David regular Borel measure with $\alpha \in (d - 2, d]$, $d > 2$ and $\nu(\mathcal{C}) > 0$. Then for some fixed $t < 2\alpha/(d - 2)$, appropriate $c > 0$ and every $Q \in \mathcal{D}_\nu$ with $\nu(Q) > 0$ we have $\nu((Q)_t) \geq c^{-1}2^{-nt}$, $\nu(Q) \leq c^{-2nt}$ and therefore

$$\nu((Q)_t) \geq c^{-1}2^{-nt(d-t)}$$

and $\nu((Q)_{t/2}) \geq c^{-2}2^{-nt(d-t)/2} < c^{-2}2^{-nt(d-2)}$ with $c^{t} > c^{1}$. Indeed, since $\nu(\mathcal{C}) > 0$ we find an element $E \in \mathcal{D}$ with $\overline{E} \subset \mathcal{C}$ and $\nu(E) > 0$. Then, on the one hand, for $n \in \mathbb{N}$ large enough, we have

$$\text{card} \left[ Q \in \mathcal{D}_\nu^N : \tau^N_\nu((Q)_t) \geq c^{-1}2^{-nt(d-t)} \right] \geq \text{card} \left[ Q \in \mathcal{D}_\nu^N : \nu(Q) > 0 \right] \geq \frac{c}{2n} \sum_{Q \in \mathcal{D}_\nu^N : \nu(Q) > 0} \frac{c}{2n} \nu(Q) = \frac{c}{2n} \nu(E) \geq \frac{c}{2n} \nu(E).$$

This estimate combined with Lemma 6.1 (adopted with $7^d$ instead of $5^d$) proves the lower asymptotic bound. On the other hand,

$$\text{card} \left[ Q \in \mathcal{D}_\nu^N : 0 < \nu((Q)_{t/2}) < c^{-2}2^{-nt(d-2)/2} > \right] \leq \text{card} \left[ Q \in \mathcal{D}_\nu^N : \nu(Q) > 0 \right] \leq c^{-2nt} \sum_{Q \in \mathcal{D}_\nu^N : \nu(Q) > 0} c^{-1}2^{-nt}$$

$$\leq c^{-2nt} \sum_{Q \in \mathcal{D}_\nu^N : \nu(Q) > 0} \nu((Q)_t) \leq c^{2}2^{-2nt},$$

which together with Corollary 5.4 and Theorem 1.1 proves the upper asymptotic bound. □
Proof of Theorem 1.16. Let $\nu$ be a self-conformal measure with $\dim_{\nu} (v) > d - 2$. Then it follows from Proposition 3.20 that $\nu$ is D/N-PF-regular and $r_D^0 (q_{\nu}) = r_N^0 (q_{\nu}^*) = 0$. Now, Theorem 1.7 and Theorem 1.3 gives $s^0 = s^N = q^N$. \hfill \Box

8. Further Examples

In this last section we present some examples illuminating the critical case with $\dim_{\nu} (v) = d - 2$ and the possibility of non-existing spectral dimension.

8.1. Critical cases. We give three examples of measures $\nu_i$ in dimension $d = 3$ for the critical case, i.e. $\dim_{\nu_i} (v_i) = d - 2 = 1, i = 1, 2, 3$. In the first example the Krein–Feller operator exists but has no compact resolvent and we therefore have no orthonormal basis of eigenvectors. In the second example the operator we have a compact resolvent and we are able to determine the spectral dimension $s^0 = 3/2$. In the third example the operator cannot be defined via our form approach as there is no continuous embedding of the Sobolev space into $L^2$.

For the following three examples we assume that $\Omega \subset \mathbb{R}^3$ is aligned to the coordinate axis and the left front lower corner is the origin. For each example we consider the density functions on $\Omega$ given by $f_i (x, y, z) = z^{-2}$, $f_2 (x, y, z) = z^{-2} (\log (1/|z|))^{-\beta}$ and $f_3 (x, y, z) = z^{-2} \log (1/|z|)$, resp., for $x, y \in [0, \ell], 0 < z < 1/2$ and 0 otherwise (see Figure 8.1 on page 29). Then for $\nu_i := f_i \, d\lambda_{|\Omega}$, $i = 1, 2, 3$, we have for $Q := [0, 2]$, $Q_i := [0, 2^i]$

$$\dim_{\nu_i} (v_i) = \liminf_{Q_i \to Q} \left( \frac{\log \nu_i (Q_i)}{\log 3^{i}} \right) = \liminf_{Q_i \to Q} \left( \frac{\log \int_{Q_i} z f_i \, dz}{\log 3^{i}} \right) = 1.$$ Since $f_i \in L_\lambda^1$ if and only if $r \leq 2/3$, it follows that $\beta_i^N (q) = \begin{cases} 3 (1 - q) & 0 \leq q \leq 3/2, \\ -q & q \geq 3/2. \end{cases}$

By Proposition 3.16 $r_i^0$ is determined by $\beta_i^N$.

To determine in which case one have continuous or even compact embedding, we make the following observation which is crucial in the concrete calculation below. For a rectangular domain $R \subset \Omega$, by H"older’s inequality, we have

$$\|d|_{L^2_\lambda (R)} = \int_R |d| \, d\lambda \leq \|f |_{L^2_\lambda (R)} \| |d|_{L^1_\lambda (R)} \leq \|f \|_{L^2_\lambda (R)} \| |d|_{L^1_\lambda (R)}.$$ (8.1)

According to the Sobolev-Poincaré inequality, if the corresponding norms are finite, this leads to the following continuous embeddings $H^0 (R) \hookrightarrow L^2_\lambda (R) \hookrightarrow L^1_\lambda (R)$, where the embedding constant of the first embedding, denoted by $C_1 > 0$, is independent of $R$ (see [Ada75, Lemma 5.10]).
Example 8.1. For \( i = 1 \) the embedding is continuous but not compact. We observe that \( \|f_{u_n}\|_{L^2}^2 = (\frac{2^{2n}}{\Lambda} \varepsilon^{-1} d\lambda)^{2/3} \). This observation and (8.1) combined give

\[
\int |u|^2 \, dv_1 = \sum_n \int_{R_{u_n}} |u|^2 \, dv_1 \leq \sum_n \|f_{u_n}\|_{L^2}^2 \, \|u_{u_n}\|_{L^2}^2 = (\log 2)^{2/3} \sum_n \|u_{u_n}\|_{L^2}^2 \\
\leq (\log 2)^{2/3} C_1 \sum_n \|u_{u_n}\|_{L^2}^2 = (\log 2)^{2/3} C_1 \|u\|_{H^2}^2
\]

showing that we have a continuous embedding \( H^\infty \hookrightarrow L^2_1 \), and the self-adjoint Krein–Feller operator is well defined. Nevertheless, in this case the embedding \( C_0^\infty \hookrightarrow L^2_1 \) is not compact. Indeed, for \( Q_n := [0, 2^{2n}] \times [2^{-2n}, 2^{-2n+1}] \), let us consider the smooth functions \( u_n := \Lambda (Q_n)^{-1/6} \varphi_{(0,2^{n+1})/2} \) for \( n \in \mathbb{N} \). Then the sequence \( (u_n) \) is bounded in \( H^\infty \), since by (2.1), we have

\[
\int_{[0,2^{2n+1}]} |u|^2 \, d\lambda + \int_{[0,2^{2n}]} |\nabla u|^2 \, d\lambda \leq \frac{\Lambda (Q_n)^{1/2} + C_1 (\frac{1}{2})^{1/3} \Lambda (Q_n)^{1/3}}{\Lambda (Q_n)^{1/3}} \leq 1 + C_1 (\frac{3}{2})^{1-2d}. \]

Further,

\[
\nu_1 (Q_n) = \int_{[2^{-2n}, 2^{-2n+1}]} \int_{[0,2^{2n}]} \langle \varphi_{(0,2^{n+1})/2} (y) \rangle \, d\lambda \, d\nu_1 (y) \geq 2^{2n-2} \Lambda (Q_n) = \frac{1}{4} \Lambda (Q_n)^{1/3}.
\]

Since for \( n \neq m \), \( (Q_n)_{j/2} \cap (Q_m)_{j/2} = \emptyset \) we deduce

\[
\int |u_n - u_m|^2 \, dv_1 = \int |u|^2 + |u_n|^2 - |u_m|^2 \, dv_1 \geq \Lambda (Q_n)^{-1/3} \nu_1 (Q_n) + \Lambda (Q_m)^{-1/3} \nu_1 (Q_m) \\
\geq \left( \Lambda (Q_n)^{-1/3} \Lambda (Q_m)^{1/3} + \Lambda (Q_n)^{-1/3} \Lambda (Q_m)^{1/3} \right)^{1/3} = 1/2
\]

and convergence in \( L^2_1 \) is therefore excluded for any subsequence.

Example 8.2. For \( i = 2 \), we prove compact embedding. Using (8.1), we have for \( Q \in \mathcal{D} \) the following continuous embeddings \( H^0 (Q) \hookrightarrow L^2_2 (Q) \) with embedding constant \( C \|f_{u}\|_{L^2_2} \), with \( C \) independent of \( Q \). To show that \( B_1 := \{ u \in C_0^\infty (\mathbb{R}) : \|u\|_{H^2} \leq 1 \} \) is precompact in \( L^2_2 \), we first observe that with \( C_1 := [0, 2^{-1}] \)

\[
\sup_{Q \in \mathcal{D}} \|f_{u}\|_{L^2_2} = \|f_{u}\|_{H^2} = \int_0^{2^{-1}} (\log \varepsilon)^{-1} \, d\lambda = 1/ \log (2^2) \to 0 \text{ for } \ell \to \infty.
\]

For \( u \in B_1 \) we have

\[
1 \geq \|u\|_{H^2}^2 = \int_{\mathbb{R}} \|\nabla u\|^2 \, d\lambda + \int_{\mathbb{R}} u^2 \, d\lambda \geq \int_{Q_1} \|\nabla u\|^2 \, d\lambda + \int_{Q_2} u^2 \, d\lambda \geq C^{-1} \int_{Q_1} \|\nabla u\|^2 \, d\lambda
\]

for every sequence \( (u_n) \), in \( B_1 \) we find by the Poincaré inequality applied to \( \mathbb{R} \setminus Q \), and a diagonal argument a subsequence that is Cauchy with respect to \( L^2_2 \) for every \( \ell \). Accordingly, for every \( n, m > \ell \),

\[
\int |u_n - u_m|^2 \, dv_2 = \int_{Q_1} |u_n - u_m|^2 \, dv_2 + \int_{Q_2} |u_n - u_m|^2 \, dv_2 \leq \int_{Q_1} |u_n - u_m|^2 \, dv_2 + 4C \|f_{Q_2}\|_{L^2_2}
\]

Now consider the limit superior as \( m, n \) are tending to infinity and then let \( \ell \to \infty \). This proves that \( (u_n) \) is Cauchy in \( L^2_2 \) and convergent there as well. This shows that the embedding \( H^0 \hookrightarrow L^2_2 \) is compact. To finally determine the spectral dimension in this case, in view of Theorem 1.1 we choose \( \delta (Q) := \|f_{Q}\|_{L^2_2} \) with \( Q \in \mathcal{D} \) and prove \( \nu_0 \). For every cube \( Q \in \mathcal{D}_k \), for \( n > 1 \) lying in \( R_{n,k} := \{(x, y, z) \in \mathbb{Q} : k2^{-n} < z < (k + 1)2^{-n}\} \) with \( 2^{n-1} \geq k \geq 1 \) (these are \( (k + 1)^2 \times k^2 \) many with
Theorem 1.1 with regard to (8.2). Moreover, note that there exists an open sub-
cube and on the other hand, we have

\[ \frac{1}{k} \left( -\log(\beta) + \log(\log k) - \left( -\log(\log(\log k)) \right) \right) \leq k^{\frac{1}{2}} \left( \log(\log(k + 1)) \right)^{3/2}. \]

Using this estimate together with (8.2),

\[
\sum_{k \geq 2} \left\| f_{\chi_k} \right\|_{L^2_{\nu}}^2 \leq \left( \frac{2q}{3} \right) \sum_{k \geq 2} \sum_{n \geq 1} \frac{\text{card}(Q \in \mathcal{D}^N : Q \cap R_{\nu} \neq \emptyset, \nu(Q) > 0)k^{-2q}}{(\log(2^r/(k + 1)))^{3q/3}}
\]

\[
\leq \left( \frac{2q}{3} \right) \sum_{k \geq 2} \frac{1}{(\log(2^r/(k + 1)))^{3q/3}} + \sum_{n \geq 1} \frac{k^{-2q+2}}{(\log(2^r/(k + 1)))^{3q/3}}
\]

\[
= \left( \frac{2q}{3} \right) \sum_{k \geq 2} k^{-2q+2} \left( \log(2^r/(k + 1)))^{3q/3} \right)
\]

\[
\leq \left( \frac{2q}{3} \right) \sum_{n \geq 1} \left( n - \lfloor \log(2^r/(k + 1)) \rfloor \right) \log(2^r/(k + 1))^{3q/3}
\]

where we used \( \log(k + 1) \leq \log(k + 1) + 1 \) for all \( k \geq 1 \) and \( \zeta \) denotes the Riemann \( \zeta \)-function. Since for all \( q > 3/2 \) the right-hand side is finite, we find \( \mathcal{D}_\nu \leq \mathcal{N} \leq \mathcal{S} \leq \mathcal{S}_\nu \leq 3/2 \) as a consequence of Theorem 1.1 with regard to (8.2). Moreover, note that there exists an open sub-cube \( \Omega \subset \mathcal{C} \) such that \( \mathcal{C} \subset \mathcal{C} \) with \( \nu(Q) > 0 \). Since \( f_{\chi_k} \) is bounded, an application of the min-max principle gives \( 3/2 = \mathcal{S}_\nu \leq \mathcal{S}_\nu \) and we have \( 3/2 = \mathcal{S}_\nu = \mathcal{S}_\nu \).

**Example 8.3.** In the case \( \ell = 3 \), the embedding \( C^\infty_0 \hookrightarrow L^2 \) is not continuous. Indeed, for \( Q_\ell := \left[ 2^{-n}, 2^{-n+1} \right)^3 \times \left[ 2^{-n}, 2^{-n+1} \right] \), let us consider the smooth functions \( u_\nu : \mathcal{L}((Q_\ell)_{\nu}) \rightarrow \mathbb{C} \). The claim follows, since on the one hand, with \( C_1 > 0 \) as given in (2.1), we have for every \( n \in \mathbb{N} \)

\[
\left\| u_\nu \right\|_{H^1_{\nu}}^2 \leq \mathcal{L}((Q_\ell)_{\nu})^{-1/2} \mathcal{L}((Q_\ell)_{\nu})^{1/2} \mathcal{L}((Q_\ell)_{\nu}) \leq C_1 + 1
\]

and on the other hand, we have

\[
\left\| u_\nu \right\|_{L^2_{\nu}}^2 \leq \mathcal{L}((Q_\ell)_{\nu})^{-1/2} \mathcal{L}((Q_\ell)_{\nu}) \leq C_1 + 1
\]

8.2. Non-existence of the spectral dimension. Here, we present an example for which upper and lower spectral dimension differ.

**Example 8.4.** Let us consider the homogeneous Cantor measure \( \mu \) on \( (0, 1) \) from [KN22, Example 5.5 with probability \((1/2, 1/2)\)] with non-converging \( L^r \)-spectrum, for which we have \( \beta^\mu_{\nu} = 3/13 < 3/11 = \beta^\mu_{\nu} \).

\[
\beta^\mu_{\nu}(q) = \begin{cases} \frac{1}{2}(1 - q), & q \leq 0, 1, \\ \frac{1}{10}(1 - q), & q > 1 \end{cases} \quad \text{and} \quad \beta^\mu_{\nu}(q) = \lim \inf \beta^\mu_{\nu}(q) = \begin{cases} \frac{1}{10}(1 - q), & q \in [0, 1], \\ \frac{1}{10}(1 - q), & q > 1, \end{cases}
\]

Take the one-dimensional Lebesgue-measure \( \Lambda^1 \) restricted to \([0, 1]\) and define the product measure on \( \mathcal{C} \) by \( \nu := \mu \otimes \Lambda^1 \otimes \Lambda^1 \). Due to the product structure, we have for the \( L^r \)-spectrum of \( \nu \)

\[
\beta^\nu(q) = \beta^\mu_{\nu}(q) + \beta^\Lambda_{\nu}(q) = \beta^\mu_{\nu}(q) + 2(1 - q), q \geq 0,
\]
and hence $\dim_v(\nu) = 2 + 3/10 > 1$. Let $\pi_t$ denote the projection onto the first coordinate. Then for $t \in [2, 4)$ and $\tau_n = \tau_{3n^{-1/2;\alpha}}^N$ we have

$$\tau_{3n^{-1/2;\alpha}}^N(q) = \frac{1}{\log(2^t)} \log \sum_{Q \subset \mathbb{Q}^0, \nu \leq Q} \sup_{Q \subset \mathbb{Q}^0} \left( \nu(Q)^{2/t} \wedge (Q')^{-1/3} \right)^q$$

$$= \frac{1}{\log(2^t)} \log \sum_{Q \subset \mathbb{Q}^0, \nu \leq Q} \sup_{Q \subset \mathbb{Q}^0} \left( \mu(\pi_t Q)^{2/t} \wedge 2^{n(1-4/t)} \right)^q$$

$$= \frac{1}{\log(2^t)} \log \sum_{Q \subset \mathbb{Q}^0, \nu \leq Q} \mu(Q)^{2/q} \wedge 2^{n(1-4/t)} \pi^{2n} = \beta_{\nu,2}(2^q/t) - q(4/t - 1) + 2$$

and the spectral partition function $\tau(q) = \tau_{3n^{-1/2;\alpha}}^N(q) = \beta_{\nu}^N(q) - q(t-1) + 2$ for $q \in \mathbb{R}_{\geq 0}$ \{1\}. This gives $\tau(q) = q_
u = 23/13$ for the upper spectral dimension. From [KN23, Prop. 3.3] we know that if there exists a subsequence $(\eta_{k\nu})_{k\in \mathbb{N}}$ and $K > 0$ such that for all $k \in \mathbb{N}$, $\eta_{k\nu} \geq Q_n^1 \leq K2^{-m[n_n^1(\eta_{k\nu})]}$, where $\eta_{k\nu}$ is the unique zero of $\tau_{3n^{-1/2;\alpha}}^N$, then $\eta_{k\nu} \leq \liminf_{k \to \infty} \eta_{k\nu}$. Combining this with the result of Section 3.4.2, a similar calculation as in [KN22c, Example 5.5] shows $\delta^0 \leq \delta^N \leq \lim_{t \downarrow 0} 19/(8(4/t - 1) + 6)/t) = 19/11 < \delta^N = \delta^0$ where $19/11$ is the unique zero of $\liminf_n \tau_{3n^{-1/2;\alpha}}^N$.

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