NEW PARAMETERIZATION OF THE RESONANT PRODUCTION AMPLITUDES NEAR AN INELASTIC THRESHOLD

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New formulae for the resonant scattering and the production amplitudes near an inelastic threshold are derived. It is shown that the Flatté formula, frequently used in experimental analyses, is not sufficiently accurate. Its application to data analysis can lead to a substantial distortion of the effective mass spectra and of the resonance pole positions.

A unitary parameterization, satisfying a generalized Watson theorem for the production amplitudes, is proposed. It can be easily applied to study production processes, multichannel meson-meson interactions and the resonance properties, including among others the scalar resonances $a_0(980)$ and $f_0(980)$.

Keywords: multichannel scattering, approximations, meson-meson interactions

PACS numbers: 11.80.Gw, 11.80.Fw, 13.75.Lb

1. Introduction

Resonant scattering amplitudes near inelastic thresholds cannot be accurately described by the Breit-Wigner formula. About 30 years ago S. M. Flatté proposed the following parameterization of the $S$-wave $\pi\eta$ and the $K\bar{K}$ production amplitudes which are dominated by the $a_0(980)$ resonance [1]

$$A_j \sim \frac{M_R \sqrt{\Gamma_0 \Gamma_j}}{M_R^2 - E^2 - iM_R(\Gamma_1 + \Gamma_2)}, \quad j = 1, 2.$$  \hfill (1)

Here $M_R$ is the resonance mass, $E$ is the effective mass (c.m. energy) and $\Gamma_j = g_j k_j$ are the channel widths, $k_1$ is the pion (or $\eta$) c.m. momentum, $k_2$ is the kaon c.m. momentum and $g_i$ are the channel coupling constants. Below the $K\bar{K}$ threshold $\Gamma_2$ is imaginary: $\Gamma_2 = ig_2 p_2$, where $p_2 = \sqrt{m_K^2 - E^2}$, $m_K$ being the kaon mass. At the $K\bar{K}$ threshold the energy $E_0 = 2m_K$, $k_1 = q$ and $\Gamma_0 = g_1 q$. Above the threshold $E = 2(k_2^2 + m_K^2)^{1/2}$.

From (1) we see that apart of the normalization factors the Flatté production amplitudes (1) depend on three real parameters: the resonance mass $M_R$ and two coupling constants $g_1$ and $g_2$. One can, however, easily demonstrate that in presence of an inelastic coupling to the $\pi\eta$ channel the three parameters are not sufficient to...
describe a behaviour of the $K\bar{K}$ elastic scattering amplitude $T_{22}$ near its threshold. If this coupling is switched off then the following threshold formula, called the effective range approximation, can be used:

$$T_{22} = \frac{1}{k_2 \cot \delta_2 - i k_2}, \quad k_2 \cot \delta_2 \approx \frac{1}{a} + \frac{1}{2} r k_2^2. \quad (2)$$

Here $\delta_2$ is the $K\bar{K}$ phase shift and the two real parameters $a$ and $r$ denote the $K\bar{K}$ scattering length and the effective range, respectively. However, if the interchannel coupling is nonzero then one has to introduce the inelasticity parameter $\eta$ and to replace the real parameters $a$ and $r$ by the complex ones, $A$ and $R$:

$$T_{22} = \frac{1}{2 i k_2} (\eta e^{2i\delta_2} - 1) \approx \frac{1}{A - i k_2 + \frac{1}{2} R k_2^2}. \quad (3)$$

It means that one needs at least four real parameters to describe the $K\bar{K}$ scattering near its threshold. These parameters should also appear in the formulae for the production amplitudes $A_1$ and $A_2$. This can be achieved by introduction of a new complex constant $N$ in the denominator $W$ of the production amplitudes $A_i$:

$$W = M_R^2 - E^2 - i M_R g_1 q - i M_R g_2 k_2 + N k_2^2. \quad (4)$$

Dividing $W$ by the product $M_R g_2$ we get

$$\frac{W}{M_R g_2} = \frac{1}{A - i k_2 + \frac{1}{2} R k_2^2}, \quad (5)$$

where the inverse of the scattering length $A$ and the effective range $R$ are written in terms of the three Flatté parameters $M_R, g_1, g_2$ and of the new parameter $N$:

$$\frac{1}{A} = \frac{M_R^2 - E_0^2}{M_R g_2} - i g_2 q, \quad R = \frac{2N - 8}{M_R g_2}. \quad (6)$$

In the Flatté approximation $N = 0$, hence $ReR = -8/M_R g_2$ and $ImR = 0$. The zero value of the imaginary part of the effective range is an essential limitation of the Flatté formula.

2. Scattering Amplitudes

The first channel elastic scattering amplitude $T_{11}$ can be parameterized in terms of five parameters. The four of them: $ReA, ImA, ReR$ and $ImR$ are the same as those defined in (3) for the second channel amplitude. The fifth parameter is the first channel phase shift $\delta_0$ determined at the $K\bar{K}$ threshold. Using the two-channel unitarity one can derive the following approximate formula for $T_{11}$:

$$T_{11} \approx \frac{e^{i\delta_0}}{k_1} \sin \delta_0 + i \frac{Im \left( e^{-i\delta_0} A \right) k_2}{1 - i A k_2 + \frac{1}{2} A R k_2^2} \cdot (7)$$

Below the $K\bar{K}$ threshold one has to replace $k_2$ by $ip_2$. In the Flatté approximation $\delta_0$ equals to the phase of the complex scattering length $A$. 

The transition amplitude $T_{12}$ from the first to the second channel in the new approach is given by

$$T_{12} \approx \frac{1}{\sqrt{k_1}} e^{i\delta_0} \sqrt{\frac{\text{Im} A - \frac{1}{2} |A|^2 \text{Im} R k_2^2}{1 - i A k_2 + \frac{1}{2} A R k_2^2}}.$$  \hspace{1cm} (8)

Let us remark that all the three amplitudes, given by Eqs. (3), (7) and (8), have a common denominator

$$D(k_2) = 1 - i A k_2 + \frac{1}{2} A R k_2^2.$$  \hspace{1cm} (9)

The complex zeroes $z_{1,2}$ of $D(k_2)$ are related to $A$ and $R$ by

$$z_{1,2} = \frac{i}{R} \pm \sqrt{\frac{-1}{R^2} - \frac{2}{AR}}.$$  \hspace{1cm} (10)

In the Flatté approximation $\text{Re} \ z_1 = -\text{Re} \ z_2$ which limits possible positions of the amplitude poles at the energies $E_{1,2} = \sqrt{E_0^2 + 4z_{1,2}^2}$. These pole positions are the most essential quantities characterizing the resonances.

3. Phenomenological Studies of the $a_0(980)$ Threshold Parameters

It is instructive to show numerical results concerning the threshold parameters of the $a_0(980)$ resonance, situated close to the $KK$ threshold. One can use a coupled channel formalism for the separable meson-meson interactions in two or three channels. It has been developed in Ref. [2] and further applied to study the $a_0$ resonances in the $\pi\eta$ and the $KK$ channels [3]. The model parameters were fixed using the data of the Crystal Barrel and of the E852 Collaborations. The following threshold parameters have been presently calculated: $\text{Re} \ A = 0.17 \text{ fm}$, $\text{Im} \ A = 0.41 \text{ fm}$, $\text{Re} \ R = -11.32 \text{ fm}$, and $\text{Im} \ R = -3.18 \text{ fm}$. Let us stress here that the imaginary part of the effective range is nonzero and cannot be neglected. In Fig. 1 we see differences between the phase shifts and inelasticities calculated using the theoretical model of Ref. [3] and the Flatté approximation in which $\text{Im} R \equiv 0$. They are sufficiently large to create.

![Fig. 1. Comparison of phase shifts and inelasticities of the scattering amplitudes](image)
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deviations of the order of hundred percent between the squares of the moduli of the
scattering amplitudes $|T_{11}|^2$ or $|T_{22}|^2$ already at the distance of 50 MeV away from
the $K\bar{K}$ threshold (see Fig. 1 of Ref. 4). Another discrepancy between the Flatté
formula and the above model is a shift of the pole complex energies $E_1$ and $E_2$.
For example, the $ReE_1$ shift is larger than 10 MeV and exceeds the experimental
energy resolution of many present experiments (see Fig. 2 of Ref. 4).

4. New Formulae for the Production Amplitudes

Let us assume that there are no initial state strong interactions. Then one can
generalize the Watson theorem, which is satisfied below the inelastic threshold
$$\text{Im } A_1 = k_1 T_{11} A_1^*, \quad (11)$$
to a form valid for the two coupled channels above this threshold:
$$\text{Im } A_1 = k_1 T_{11} A_1^* + k_2 T_{12} A_2^*, \quad (12)$$
$$\text{Im } A_2 = k_2 T_{22} A_2^* + k_1 T_{21} A_1^*. \quad (13)$$

One can propose the following new parameterization of the production amplitudes:
$$A_1 = f_1 T_{11} + f_2 T_{12}, \quad A_2 = f_1 T_{12} + f_2 T_{22}. \quad (14)$$

Here $f_1, f_2$ are real functions of energy (or momentum $k_2$) which near the threshold
can be approximated by
$$f_1 \approx n_1 + \beta_1 k_2^2, \quad f_2 \approx n_2 + \beta_2 k_2^2. \quad (15)$$

In (15) $n_1$ and $n_2$ are normalization constants, $\beta_1$ and $\beta_2$ are additional real coef-
ficients. The formula given by (1) is finally replaced by (14). The parameters to be
fitted from experiments are: complex $A, R$ and real $\delta_0, n_1, n_2, \beta_1, \beta_2$.

A generalization of the above formulae to a case where the particle masses
in the second channel are different ($m_a$ and $m_b$) is simple. Above the inelastic
threshold one defines the momentum $k_2 = \frac{1}{2p_2} \left( [E^2 - (m_a + m_b)^2] [E^2 - (m_a - m_b)^2] \right)^{\frac{1}{2}}$. Below the threshold (for $E < E_0 = m_a + m_b$) $k_2$ is replaced by $ip_2$, where $p_2 = \frac{1}{2} \left( [(m_a + m_b)^2 - E^2] [E^2 - (m_a - m_b)^2] \right)^{\frac{1}{2}}$.

The new formulae can be applied in numerous analyses of present and future ex-
periments (for example Belle, BaBar, CLEO, BES, KLOE, COSY, Tevatron, LHCh, JLab, PANDA ...) and also to reanalyse older experiments in order to update our
information on meson spectroscopy and on reaction mechanisms.

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