Finitely forcible graphons

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Abstract

We investigate families of graphs and graphons (graph limits) that are defined by a finite number of prescribed subgraph densities. Our main focus is the case when the family contains only one element, i.e., a unique structure is forced by finitely many subgraph densities. Generalizing results of Turán, Erdős–Simonovits and Chung–Graham–Wilson, we construct numerous finitely forcible graphons. Most of these fall into two categories: one type has an algebraic structure and the other type has an iterated (fractal-like) structure. We also give some necessary conditions for forcibility, which imply that finitely forcible graphons are “rare”, and exhibit simple and explicit non-forcible graphons.

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1 Introduction

A classical theorem by Turán says that if a graph $G$ has edge density $1 - \frac{1}{k}$ and the density of complete $(k+1)$-graphs is 0, then $G$ is a complete $k$-partite graph with equal color classes. Here two subgraph densities force a unique structure on a graph $G$. Stability theorems (Erdős and Simonovits [17, 8]) imply that if the densities are “close” to the above values, then the structure of the graph is “close” to the complete $k$-partite graph.

Another interesting theorem of this type is by Chung, Graham and Wilson [6] asserting that if the edge density of $G$ is “close” to $1/2$ and the 4-cycle density is “close” to $1/16$, then $G$ is quasi-random, which means (among many other nice properties) that then the density of an arbitrary fixed graph $F$ is “close” to $2^{-|E(F)|}$.

The second theorem is different from the first one in two important ways. First, this pair of subgraph densities can never be attained by finite graphs (they can be approximated with arbitrary precision). Second, the structure forced by the two subgraph densities is not as well defined as in the first example. Motivated by their results, Chung, Graham and Wilson introduced the notion of a forcing family, which is any set of graphs that can be used to force quasi-randomness in a similar way. They ask which graph families are forcing families.

Our paper goes in a slightly different direction. Instead of asking which graph families can be used to force quasi-randomness, we ask which structures can be forced by prescribing the densities of finitely many subgraphs. For this reason we will define forcing families more generally.

Most of the time we consider finite simple graphs, i.e., graphs without loops and multiple edges; in the few cases where we allow multiple edges, we emphasize this by talking about multigraphs.

For two graphs $F$ and $G$ we denote by $t(F,G)$ the density of $F$ in $G$ (this can be defined as the probability that a random map $V(F) \to V(G)$ is a graph homomorphism, i.e., it preserves edges).
Definition 1.1 Let $F_1, F_2, \ldots, F_k$ be graphs and $t_1, t_2, \ldots, t_k$ be real numbers in $[0, 1]$. We say that the set $\{(F_i, t_i) : i = 1, \ldots, k\}$ is a forcing family if there is a sequence of simple graphs $\{G_i\}_{i=1}^{\infty}$ with $\lim_{j \to \infty} t(F_i, G_j) = t_i$ for $1 \leq i \leq k$, and every such graph sequence is convergent in the sense that $\lim_{j \to \infty} t(F, G_j)$ exists for every graph $F$.

This definition immediately implies that there is a graph parameter (a function $r : F \mapsto [0, 1]$ on the set $F$ of finite graphs) such that $\lim_{j \to \infty} t(F_i, G_j) = r(F)$ whenever $\{G_i\}_{i=1}^{\infty}$ satisfies $\lim_{j \to \infty} t(F_i, G_j) = t_i$ for $1 \leq i \leq k$. The graph parameter $r$ encodes the unique structure which is forced.

The result of Chung, Graham and Wilson mentioned above says in this language that $\{(K_2, 1/2), (C_4, 1/16)\}$ is a forcing family. The graph parameter describing the limit is $r(F) = 2^{-|E(F)|}$. Graph sequences satisfying the conditions in the definition are called quasi-random.

The forced structure in definition [14] is best described as the limit of a graph sequence, using the newly developed theory of convergent graph sequences.

Let $W$ denote the set of bounded symmetric measurable functions of the form $W : [0, 1]^2 \to \mathbb{R}$, and let $W_0 \subset W$ consist of those functions with range in $[0, 1]$. The elements of $W$ are called graphons. A graphon $W$ is a step function if there is a partition $\{S_1, \ldots, S_n\}$ of $[0, 1]$ into measurable sets such that $W$ is constant on each product set $S_i \times S_j$.

The notion of subgraph densities can be extended to graphons: for a graph $F = (V, E)$ and graphon $W \in W$, we define

$$t(F, W) = \int_{[0, 1]^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i.$$  \hspace{1cm} (1)

(This definition applies to multigraphs $F$ too.) These quantities were introduced in [11] and it was proved that graph sequences in which the densities of every fixed graph converge can be considered to converge to a graphon in $W_0$. If two graphons have the same subgraph densities, then they are called weakly isomorphic (cf. [12] [1]).

In this paper we reformulate the problem of forcing in terms of measurable functions. An immediate advantage of this can already be seen from the simpler definition of finite forcing:

Definition 1.2 Let $\mathcal{A} \subseteq W$. Let $F_1, F_2, \ldots, F_k$ be graphs and $t_1, t_2, \ldots, t_k \in [0, 1]$. We say that the set $\{(F_i, t_i) : i = 1 \ldots k\}$ is a forcing family in $\mathcal{A}$ if there is a unique (up to weak isomorphism) graphon $W \in \mathcal{A}$ with $t(F_i, W) = t_i$ for every $1 \leq i \leq k$. In this case say that $W$ is finitely forcible (in $\mathcal{A}$), and the family $\{F_i : i = 1 \ldots k\}$ is a forcing family for $W$ (in $\mathcal{A}$).

The two main choices for $\mathcal{A}$ will be $\mathcal{A} = W$ and $\mathcal{A} = W_0$. If a graphon $W \in W_0$ is finitely forcible in $W$, then it is also finitely forcible in $W_0$, but the reverse is open.

Besides the advantage of a simpler definition, the new language enables us to specify the structure which is forced and to use analytic methods together with algebraic ones. In this language the Chung-Graham-Wilson theorem says that if $t(P_2, W) = 1/2$ and $t(C_4, W) = 1/16$ then $W$ is the constant $1/2$ function. A generalization of this was proved by Lovász and Sós in [14]: every step function is finitely forcible in $W_0$.  

After the Lovász–Sós result it remained open for a few years whether only stepfunctions are finitely forcible. This is true in a one-variable analogue of the forcing problem: a bounded function \( f : [0, 1] \to \mathbb{R} \) is forced by finitely many moments if and only if it is a stepfunction. In this paper we show that in the 2-variable case more complicated structures can be forced. One family of these structures is the indicator function of a level set of a monotone symmetric 2-variable polynomial. Our other main example has an iterated (fractal like) structure.

So being a stepfunction is not a characterization of finitely forcible graphons. The examples mentioned above are, however, stepfunctions in a weaker sense, i.e., their range is finite. Even this is not necessary: In Section 3 we develop a class of operations on graphons that preserve finite forcibility in \( \mathcal{W} \), and applying these to the first type of examples, we construct finitely forcible graphons whose range is a continuum. In the finite language this implies the surprising fact that we can create an extremal problem involving the densities of finitely many subgraphs such that in the unique asymptotically optimal solution a continuous spectrum of probabilities for quasi-randomness appears.

Of course, instead of forcing a given graphon by a finite number of density conditions, we can also try to force various properties. Let \( \mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{W} \) be closed under weak isomorphism. We say that \( \mathcal{B} \) is finitely forcible in \( \mathcal{A} \) if there exists a family \( \{(F_i, t_i) : i = 1\ldots k\} \) (where the \( F_i \) are graphs and \( t_i \in [0, 1] \)) such that a graphon \( W \in \mathcal{A} \) satisfies the conditions \( t(F_i, W) = t_i \) (\( i = 1\ldots k \)) if and only if it is in \( \mathcal{B} \). While the study of this generalization is not the goal of this paper, we do need certain facts about forcing some simple properties, which are discussed in Section 4.

Since finitely forcible graphons can be described by finitely many real numbers, we believe that they are very special. However, a full characterization seems to be very difficult. In fact, it is not easy to find necessary conditions for finite forcibility. We present a few in this paper and formulate many open problems in this direction.

## 2 Preliminaries

### 2.1 Labeled, colored, and quantum graphs

Suppose that the edges of a graph \( F \) are partitioned into two sets \( E' \) and \( E'' \), called “blue” and “red”. The triple \( \hat{F} = (V, E', E'') \) will be called a 2-edge-colored graph. Then we define

\[
\int_{[0,1]^V} \prod_{ij \in E'} W(x_i, x_j) \prod_{ij \in E''} (1 - W(x_i, x_j)) \prod_{i \in V} dx_i .
\]  

If all edges are blue, then \( t(\hat{F}, W) = t(F, W) \). If all edges are red, then \( t(\hat{F}, W) = t(F, 1 - W) \). In general, \( t(\hat{F}, W) \) can be expressed as

\[
t(\hat{F}, W) = \sum_{\mathcal{Y} \subseteq E''} (-1)^{|\mathcal{Y}|} t((V, E' \cup \mathcal{Y}), W) .
\]  

For a graph \( F \), let \( \hat{F} \) denote the complete graph on \( V(F) \) in which the edges of \( F \) are colored blue and the other edges are colored red. Let \( G \) be a graph and let \( W_G \) denote the associated
graphon. Then $t(\hat{F}, W_G)$ is the probability that a random map $V(F) \to V(G)$ preserves both adjacency and nonadjacency. If $F$ is fixed and $|V(G)| \to \infty$, then this is asymptotically $t_{\text{ind}}(F, G)$, the probability that a random injection $V(F) \to V(G)$ is an embedding as an induced subgraph.

Let $F = (V, E)$ be a $k$-labeled graph, i.e., a graph with $k$ specified nodes labeled 1, \ldots, $k$ and any number of unlabeled nodes. Let $V_0 = V \setminus [k]$ be the set of unlabeled nodes. For $W \in \mathcal{W}$, we define a function $t_k^*(F, W) : [0, 1]^k \to \mathbb{R}$ by

$$t_k^*(F, W)(x_1, \ldots, x_k) = \int_{x \in [0, 1]^k} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V_0} dx_i.$$  

Note that $t^0 = t$. We can extend this notation to 2-edge-colored graphs $\hat{F}$ to get $t_k^*(\hat{F}, W)$ in the obvious way.

Let $F'$ arise from $F$ by unlabeling node $k$ (say), then

$$t_{k-1}(F', W)(x_1, \ldots, x_{k-1}) = \int_{[0,1]} t_k^*(F, W)(x_1, \ldots, x_k) \, dx_k.$$  

A $k$-labeled quantum graph is a formal finite linear combination (with real coefficients) of $k$-labeled graphs. A 0-labeled quantum graph will be called simply a quantum graph. We can extend the definition of the product to the product of two $k$-labeled quantum graphs by distributivity, and we can define $t_k^*(f, W) : [0, 1]^k \to \mathbb{R}$ for every $k$-labeled quantum graph $f$ so that it is linear in $f$.

**Observation 2.1** Let $F_1$ and $F_2$ be two $k$-labeled quantum graphs. Let $F_1 \ast F_2$ be the graph obtained from $F_1F_2$ by reducing multiple edges. Then

$$t_k^*(F_1F_2, W) = t_k^*(F_1, W)t_k^*(F_2, W) \quad (4)$$

and if $W$ is a 0-1 valued function then

$$t_k^*(F_1 \ast F_2, W) = t_k^*(F_1, W)t_k^*(F_2, W) \quad (5)$$

where the multiplication on the right hand side is just the product of two real functions with the same domain.

We can identify a 2-edge-colored graph $\hat{F} = (V, E', E'')$ with the quantum graph

$$\sum_{X \subseteq E''} (-1)^{|V|}(V, E' \cup X),$$

then the two possible definitions of $t(\hat{F}, W)$ give the same result by (3).
2.2 Graphons

Let $W_1, W_2, \ldots \in W$ be graphons (a finite or countable sequence), and let $a_1, a_2, \ldots$ be positive real numbers such that $\sum_i a_i = 1$. We use $\sum_i a_i W_i$ to denote the pointwise linear combination.

We also define the weighted direct sum $W = \bigoplus_i a_i W_i$ as the graphon on $[0, 1]$ as follows: we break the $[0, 1]$ interval into intervals $I_1, I_2, \ldots$ of length $a_1, a_2, \ldots$, take homothetical maps $\phi_i : J_i \to [0, 1]$ and define $W(x, y) = W_i(\phi_i(x), \phi_i(y))$ if $(x, z) \in J_i \times J_i$ for some $i$, and $W(x, y) = 0$ otherwise. See [10] for more on weighted direct sums and decomposing graphons into connected components.

Somewhat confusingly, we can introduce three “product” operations on graphons, and we will need all three of them. Let $U, W \in W$. We denote by $U W$ their product as functions, i.e.,

$$U W(x, y) = U(x, y) W(x, y).$$

We denote by $U \circ W$ the product of $U$ and $W$ as kernel operators, i.e.,

$$(U \circ W)(x, y) = \int_0^1 U(x, z) W(z, y) \, dz.$$ 

Finally, we denote by $U \otimes W$ their tensor product; this is defined as a function $[0, 1]^2 \times [0, 1]^2 \to [0, 1]$ by

$$(U \otimes W)(x_1, x_2, y_1, y_2) = U(x_1, y_1) W(x_2, y_2).$$

However, we can consider any measure preserving map $\phi : [0, 1] \to [0, 1]^2$, and define the graphon

$$(U \otimes W)^\phi(x, y) = (U \otimes W)(\phi(x), \phi(y)).$$

These graphons are weakly isomorphic for all $\phi$, and so we can call any of them the tensor product of $U$ and $W$.

Recall that a graphon $W$ is finitely forcible (in $\mathcal{A} \subseteq W$), if $W \in \mathcal{A}$ there is a finite number of graphs $F_1, \ldots, F_k$ so that whenever a graphon $U \in \mathcal{A}$ satisfies

$$t(F_i, U) = t(F_i, W) \quad (i = 1, \ldots, k)$$

then $W$ and $U$ are weakly isomorphic. We could be more general and allow quantum graphs in the forcing family, but this would not lead to more finitely forcible functions.

2.3 Moments of one-variable functions

Let $w = (w_1, \ldots, w_r)$, where $w_1, \ldots, w_r : [0, 1] \to \mathbb{R}$ are measurable functions. For $a \in \mathbb{Z}_+^r$, we define

$$M(w, a) = \int_{[0, 1]^r} w_1(x)^{a_1} \cdots w_r(x)^{a_r}$$

(if the integral exists).

**Theorem 2.2 (Doob)** Let $a_1, \ldots, a_m \in \mathbb{Z}_+^r$, and suppose that $M(w, a_j)$ exists for all $j = 1, \ldots, m$. Then there are stepfunctions $u_1, \ldots, u_r$ such that

$$M(u, a_j) = M(w, a_j) \quad (j = 1, \ldots, m).$$
A certain converse of this theorem is also true:

**Proposition 2.3** Let \( u_1, \ldots, u_r \) be stepfunctions. Then there is a finite set of vectors \( a_1, \ldots, a_m \in \mathbb{Z}_+^r \) such that the values \( M(u, a_j) \) \((j = 1, \ldots, m)\) uniquely determine the functions \( u_i \) up to a measure preserving transformation of \([0, 1]\).

### 2.4 Typical points of graphons

We need some technical results about 2-variable functions. In this section \( W \) denotes a measurable function \([0, 1]^2 \rightarrow [0, 1]\) (not necessarily symmetric). Let \( R(W) \) denote the set of 1-variable functions \( \{W(x, \cdot) : x \in [0, 1]\}\). Clearly \( R(W) \) inherits a topology from \( L_1[0, 1] \), and it also inherits a probability measure \( \pi \) from \([0, 1]\).

**Definition 2.4** Let \( T(W) \) be the set of functions \( f \in L_1[0, 1] \) such that every neighborhood of \( f \) intersects \( R(W) \) in a set with positive measure. A point \( x \in [0, 1] \) will be called **typical** if \( W(x, \cdot) \in T(W) \).

**Lemma 2.5** Let \( W : [0, 1]^2 \rightarrow [0, 1] \) be a measurable function. Then almost every point of \([0, 1]\) is typical.

**Proof.** For \( \varepsilon > 0 \) and \( f \in L_1[0, 1] \), let \( B_\varepsilon(f) = \{g \in L_1[0, 1] : \|f - g\|_1 \leq \varepsilon\} \). For \( S \subseteq L_1[0, 1] \), let \( B_\varepsilon(f) = \cup\{B_\varepsilon(f) : f \in S\} \). Let \( A_\varepsilon = \{f \in R(W) : \pi(B_\varepsilon(f) \cap R(W)) = 0\} \). Since a point \( x \) is non-typical if and only if \( W(x, \cdot) \in \cup_{k \in \mathbb{N}}A_{1/k} \), it suffices to prove that \( \pi(A_\varepsilon) = 0 \) for every \( \varepsilon > 0 \).

Let \( \delta > 0 \) and let \( U : [0, 1]^2 \rightarrow [0, 1] \) be a stepfunction such that \( \|U - W\|_1 < \varepsilon\delta/2 \). Since

\[
\|U - W\|_1 \geq \frac{\varepsilon}{2} \pi(R(W) \setminus B_{\varepsilon/2}(R(U))),
\]

this implies that \( \pi(R(W) \setminus B_{\varepsilon/2}(R(U))) < \delta \).

Since \( U \) is a stepfunction, \( R(U) \) is finite, and so the set \( S = \{f \in R(U) : \pi(B_{\varepsilon/2}(f) \cap R(W)) = 0\} \) is also finite. Hence \( \pi(B_{\varepsilon/2}(S) \cap R(W)) = 0 \).

Now every \( x \in A_\varepsilon \) is either in a ball \( B_\varepsilon(f) \) with \( f \in S \) or in \( R(W) \setminus B_{\varepsilon/2}(R(U)) \). This implies that \( \pi(A_\varepsilon) < \delta \). Since \( \delta > 0 \) was arbitrary, it follows that \( \pi(A_\varepsilon) = 0 \).

The following useful property of typical points is straightforward to prove:

**Lemma 2.6** Let \( W \) be a graphon, and let \( f \) be a \( k \)-labeled quantum graph such that the labeled nodes are independent in each graph constituting \( f \). Assume that \( t^k(f, W) = 0 \) almost everywhere. Then \( t^k(f, W)(x_1, \ldots, x_k) = 0 \) for every \( k \)-tuple of typical points.

**Proof.** We may assume that \( \|W\|_\infty \leq 1 \). Suppose that \( t^k(f, W)(x_1, \ldots, x_k) = \varepsilon > 0 \). Let

\[
f = \sum_{i} \alpha_i F_i, \quad c_f = \sum_{i} |\alpha_i| \cdot |E(F_i)| \quad \text{and} \quad \delta = \varepsilon/c_f.
\]

By the definition of typical points, there are sets \( Z_i \subseteq [0, 1] \) with positive measure such that \( \|W(x_i, \cdot) - W(z_i, \cdot)\|_1 \leq \delta \) for all \( z \in Z_i \). We claim that for every choice of points \( z_i \in Z_i \), we have

\[
|t^k(f, W)(x_1, \ldots, x_k) - t^k(f, W)(z_1, \ldots, z_k)| \leq c_f \delta.
\]
Clearly it suffices to verify this for the case when $f = F$ is a graph. Let $u_1 v_1, \ldots, u_q v_q$ be the edges of $F$ incident with the labeled nodes; say, $u_r$ is labeled but $v_r$ is not (here we use the assumption about $f$). Let $u_{q+1} v_{q+1}, \ldots, u_m v_m$ be the other edges of $F$, and $U = V \setminus \{1, \ldots, k\}$. Then

$$|t^k(f, W)(x_1, \ldots, x_k) - t^k(f, W)(z_1, \ldots, z_k)|$$

$$= \left| \int_{[0,1]^U} \sum_{j=1}^m \left( \prod_{i<j} W(x_{u_i}, x_{v_i}) \right) (W(x_{u_j}, x_{v_j}) - W(z_{u_j}, z_{v_j})) \left( \prod_{i>j} W(z_{u_i}, z_{v_i}) \right) \prod_{u \in U} dx_u \right|$$

$$\leq \sum_{j=1}^m \int_{[0,1]^U} \left| \prod_{i<j} W(x_{u_i}, x_{v_i}) \right| (W(x_{u_j}, x_{v_j}) - W(z_{u_j}, z_{v_j})) \left( \prod_{i>j} W(z_{u_i}, z_{v_i}) \right) \prod_{u \in U} dx_u$$

$$\leq \sum_{j=1}^m \int_{[0,1]^U} \left| W(x_{u_j}, x_{v_j}) - W(z_{u_j}, z_{v_j}) \right| \prod_{u \in U} dx_u$$

$$= \sum_{j=1}^m \left| W(x_{u_j}, z_{u_j}) - W(z_{u_j}, z_{u_j}) \right|_1 \leq m \delta.$$

This proves (7), which in turn implies that $t^k(f, W)(z_1, \ldots, z_k) \neq 0$. Since this holds for all $z_i \in Z_i$, we get that $t^k(f, W) = 0$ cannot hold almost everywhere, a contradiction. \hfill \Box

**Remark 2.7** We can use Lemma 2.6 to define a “normalization” of graphons: by modifying a graphon on a set of measure 0, we can obtain one in which every point is typical. Lemma 2.6 implies then that if $t^k(f, W) = 0$ almost everywhere, then it is identically 0.

## 3 Operations on graphs and graphons

We discuss various operations on graphs and graphons in connection with forcing.

### 3.1 Contraction

Let $F$ be a $k$-labeled graph, and let $\mathcal{P} = \{S_1, \ldots, S_m\}$ be a partition of $[k]$. We say that $\mathcal{P}$ is legitimate for $F$, if each set $S_i$ is independent in $F$. If this is the case, then we define the $m$-labeled multigraph $F/\mathcal{P}$ by identifying the nodes in each $S_i$, and labeling the obtained node with $i$.

For a $k$-labeled quantum graph $f$, we say that $\mathcal{P}$ is legitimate for $f$ if it is legitimate for every graph occurring in $f$ with nonzero coefficient. Then we can define $f/\mathcal{P}$ by linear extension.

**Lemma 3.1** Let $f$ be a $k$-labeled quantum graph and $\mathcal{P}$, a legitimate partition for $f$ with $r$ classes. Let $W \in \mathcal{W}$, and suppose that $t^k(f, W) = 0$ almost everywhere. Then $t^r(f/\mathcal{P}, W) = 0$ almost everywhere.

**Proof.** If $k = 2$ and $\mathcal{P}$ identifies the two labels, then the Lemma follows from Lemma 2.6 since $t^2(f/\mathcal{P}, W)(x) = t^2(f, W)(x, x)$ is 0 whenever $x$ is a typical point, i.e., almost everywhere. (We could also invoke Theorem 1.6 in [13] here.)
Now for an arbitrary $k \geq 2$, it suffices to prove the case when $P$ identifies a single pair of labeled nodes, say 1 and 2. If $t^k(f, W) = 0$, then $t^k(f^2, W) = 0$ almost everywhere. Let $h$ be obtained from $f^2$ by unlabeling $3, \ldots, k$, then it follows that $t^2(h, W) = 0$ almost everywhere. By the above, we have $t(h', W) = 0$, where $h'$ is the unlabeled quantum graph obtained from $h$ by identifying nodes 1 and 2. But $h'$ can be obtained from $(f/P)^2$ by unlabeling all nodes, and hence it follows that $t(r(f/P), W) = 0$ almost everywhere. 

\section{Labeling and unlabeling}

The condition that $t^k(g, W) = 0$ for some $k$-labeled quantum graph $g$ (where $k > 0$) seems to carry much more information than a condition that $t(f, W) = 0$ for an unlabeled quantum graph $f$. However, there is a way to translate labeled conditions to unlabeled conditions. We start with a simple observation.

\textbf{Lemma 3.2} Let $f$ be a $k$-labeled quantum graph and $r$, a positive even integer. Let $g$ be the quantum graph obtained from $f^r$ by unlabeled the nodes (here $f^r$ is the $r$-th power of $f$ in the algebra of $k$-labeled quantum graphs). Then for any $W \in W$, $t^k(f, W) = 0$ if and only if $t(g, W) = 0$.

(Clearly, the same conclusion holds if only some of the nodes are unlabeled.)

\textbf{Proof.} We have

$$t(g, W) = \int_{[0,1]^k} t^k(f^r, W)(x) \, dx = \int_{[0,1]^k} t^k(f, W)(x)^r \, dx,$$

so this is 0 if and only if $t^k(f, W)(x) = 0$ for almost all $x \in [0,1]^k$. 

One drawback of this lemma is that $f^k$ may have multiple edges, even if $f$ does not. The following construction gets around this.

\textbf{Lemma 3.3} For every $k$-labeled quantum graph $f$ without multiple edges there is an unlabeled quantum graph $g$ such that for any $W \in W$, $t(g, W) = 0$ if and only if $t^k(f, W) = 0$ almost everywhere.

\textbf{Proof.} Take $2^k$ disjoint copies of $f$. It will be convenient to denote these by $f_X$, where $X \subseteq [k]$. Label the node of $f_X$ originally labeled $i$ by the pair $(X, i)$, to get a $k2^k$-labeled graph. Identify the nodes $(X, i)$ and $(Y, i)$ if and only if the strings $X \triangle Y = \{i\}$. This way we get a $k2^{k-1}$-labeled quantum graph $h$, where the labels can be thought of as pairs $(X, Y)$ where $X \subseteq Y \subseteq [k]$ and $|Y \setminus X| = 1$. Let $g$ be obtained from $h$ by unlabeled all nodes. Clearly $g$ is a quantum graph.

It is clear that $t^k(f, W) = 0$ implies that $t(g, W) = 0$. Suppose that $t(g, W) = 0$.

We define two auxiliary sequences of quantum graphs. For $1 \leq i \leq k$, let the $(k - i)$-labeled quantum graph $g_i$ be defined as follows: throw away from $h$ all the nodes labeled with $(X, Y)$ where $X \not\subseteq [i]$, and also all the unlabeled nodes of the $f_X$ with $X \not\subseteq [i]$. For $j > i$, identify all nodes $(X, Y)$ where $X \subseteq [i]$ and $Y = X \cup \{j\}$, and label this node with $j$. Unlabel all the nodes labeled with $(X, Y)$ where $Y \subseteq [i]$. 

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Furthermore, for $0 \leq i \leq k - 1$, let the $r_i := (2^i + k - i - 1)$-labeled quantum graph $h_i$ be defined as follows: throw away from $h$ all the nodes labeled with $(X, Y)$ where $X \not\subseteq \{i\}$, and also all the unlabeled nodes of the $f_X$ with $X \not\subseteq \{i\}$. For $j > i + 1$, identify all nodes $(X, Y)$ where $X \subseteq \{i\}$ and $Y = X \cup \{j\}$, and label this node with $j$. Keep all nodes labeled $(X, Y)$ where $X \subseteq \{i\}$ and $Y = X \cup \{i + 1\}$ separate and labeled. Unlabel all the nodes labeled with $(X, Y)$ where $Y \not\subseteq \{i\}$.

It is easy to check that $g_k = g$ and $g_0 = f$. Furthermore, partially unlabeled $h_i^2$ we get $g_{i+1}$, and also merging all nodes in $h_i$ labeled $(X, Y)$ where $X \subseteq \{i\}$ and $Y = X \cup \{i + 1\}$ (and labeling the new node with $i + 1$), we get $g_i$. Hence, using Lemmas 3.1 and 3.2

\[ t(g_k, W) = 0 \Rightarrow t^{k-1}(h_{k-1}, W) = 0 \Rightarrow t^1(g_{k-1}, W) = 0 \Rightarrow t^{k-2}(h_{k-2}, W) = 0 \]
\[ \Rightarrow \ldots \Rightarrow t(h_0, W) = 0 \Rightarrow t(g_0, W) = 0. \]

This completes the proof. $\square$

**Corollary 3.4** Suppose that for $W \in \mathcal{W}$ there is a family \( \{f_1, \ldots, f_m\} \), where $f_i$ is a $k_i$-labeled quantum graph, such that the conditions $t^{k_i}(f_i, U) = 0$, $U \in \mathcal{W}$ imply that $U$ is weakly isomorphic with $W$. Then $W$ is finitely forcible in $\mathcal{W}$. Similar assertion holds for forcing in $\mathcal{W}_0$.

### 3.3 The adjoint of an operator

Let $\mathcal{F}$ denote the set of graphs (up to isomorphism), and let $\mathcal{Q}$ be the linear space of quantum graphs.

**Definition 3.5** Let $F : \mathcal{W} \rightarrow \mathcal{W}$ be an operator (not necessarily linear) and let $F^* : \mathcal{F} \rightarrow \mathcal{Q}$ be a map. We say that the map $F^*$ is an **adjoint** of $F$ if

\[ t(F, F(W)) = t(F^*(F), W) \]

for every $F \in \mathcal{Q}$ and $W \in \mathcal{W}$. We denote the set of functionals which have an adjoint by $\mathcal{D}$.

It is clear from this definition that the elements of $\mathcal{D}$ form a semigroup with respect to composition.

**Example 3.6** Fix a real number $\lambda$ and let $F$ denote the functional defined by $F(W) = \lambda W$. It is easy to see that $F$ has an adjoint defined by $F^*(F) = \lambda |E(F)| F$.

**Example 3.7** Let $\lambda$ be a real number and $F(W) = W + \lambda$. Then $F$ has an adjoint defined by

\[ F^*(F) = \sum_{Z \subseteq E(F)} t(E, W) \lambda^{|E(F) \setminus Z|}. \]

**Example 3.8** Let $A \in \mathcal{W}$ be a fixed function and define $F(W)$ as the tensor product $A \otimes W$. Then $F^*(F) = t(F, A) F$ defines an adjoint of $F$. 

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Example 3.9 Let $\mathbf{F}(W)$ be the $k$-th tensor power of $W$ (for any fixed $k \geq 1$). Then an adjoint $\mathbf{F}^*$ can be defined by letting $\mathbf{F}^*(F)$ be the disjoint union of $k$ copies of $F$.

Example 3.10 Let $p(z) = \sum_{i=1}^n a_i z^i$ be a real valued polynomial. We define $\mathbf{F}(W)$ as $p(W)$ where $W$ is substituted into $p$ as an integral kernel operator. For any graph $F = (V,E)$, we define $\mathbf{F}^*(F) = \sum_{k \in [n]} a_k F^{(k)}$, where $a_k = \prod_{j \in E} a_{k(j)}$ and $F^{(k)}$ is the graph obtained from $F$ by subdividing each edge $e$ by $k(e) - 1$ nodes. It is easy to see that $\mathbf{F}^*$ is an adjoint of $\mathbf{F}$.

Example 3.11 Let $H$ be a 2-labeled graph which has an automorphism interchanging the labeled nodes. Then $\mathbf{F}(W)(x,y) = t^2(H,W)(x,y)$ is a symmetric 2-variable function in $x, y$. Let $\mathbf{F}^*(F)$ be the graph obtained from $F$ by replacing each edge by a copy of $H$ where the labeled nodes of $H$ are identified with the endpoints of the edge. Then $\mathbf{F}^*$ is an adjoint of $\mathbf{F}$.

As a special case, if $H$ is a triangle with two labeled nodes, then $\mathbf{F}(W) = (W \circ W)^W$.

Lemma 3.12 Let $W \in \mathcal{W}$ be finitely forcible in $\mathcal{W}$, and assume that $\mathbf{F}^{-1}(W)$ is a finite set for some $\mathbf{F} \in \mathcal{D}$. Then every element in $\mathbf{F}^{-1}(W)$ is finitely forcible in $\mathcal{W}$.

Proof. If $W$ can be forced by the conditions $t(F_i, W) = t_i$ ($i = 1, \ldots, k$), then obviously the set $\mathbf{F}^{-1}(W)$ can be forced by the conditions $t(\mathbf{F}^*(F_i), U) = t_i$. Since $\mathbf{F}^{-1}(W)$ is assumed to be finite, the equivalence class of each element can be distinguished from the others by at most $|\mathbf{F}^{-1}(W)| - 1$ further graph density conditions. □

Applying this lemma with examples 3.6 and 3.7 we get

Corollary 3.13 If $W \in \mathcal{W}$ is finitely forcible (in $\mathcal{W}$), then so is $\lambda W + \mu$ for $\lambda, \mu \in \mathbb{R}$.

Corollary 3.14 For every finitely forcible graphon $W \in \mathcal{W}$ there are numbers $\lambda \neq 0$ and $\mu$ such that $\lambda W + \mu$ is in $\mathcal{W}_0$ and is finitely forcible in $\mathcal{W}_0$.

For our next corollary, we need a simple lemma.

Lemma 3.15 Let $p$ be a polynomial which is a bijection on $\mathbb{R}$ with $p(0) = 0$. Then $W \mapsto p(W)$ is injective on $\mathcal{W}$.

Proof. Let $W \in \mathcal{W}$, and consider any function $U \in \mathcal{W}$ with $p(U) = W$. Let

$$U(x,y) \sim \sum_{i=1}^{\infty} \mu_i f_i(x)f_i(y).$$

be the spectral spectral decomposition of $U$, where $\{f_i\}_{i=1}^\infty$ is an orthonormal system of functions in $L_2[0,1]$ and $\sum_i \mu_i^2 < \infty$. Then

$$W(x,y) \sim \sum_{i=1}^{\infty} p(\mu_i) f_i(x)f_i(y),$$

be the spectral spectral decomposition of $W$, where $\{f_i\}_{i=1}^\infty$ is an orthonormal system of functions in $L_2[0,1]$ and $\sum_i \mu_i^2 < \infty$. Then
is a spectral decomposition of $W$, since clearly $\sum p(\mu_i)^2 < \infty$. Since the spectral decomposition of $W$ is unique (up to an orthogonal basis transformation in the eigensubspaces) and $p$ is injective, and we see that the $\mu_i$ and $f_i$ are determined by $W$ (again, up to an orthogonal basis transformation in the eigensubspaces), and so $U$ is uniquely determined. □

The next corollary is a direct consequence of Lemmas 3.12 and 3.15.

**Corollary 3.16** Let $p$ be a polynomial which is a bijection on $\mathbb{R}$ with $p(0) = 0$. If $p(W)$ is finitely forcible for some $W \in \mathcal{W}$, then so is $W$.

## 4 Finitely forcible properties

As mentioned in the introduction, instead of forcing specific graphons by a finite number of subgraph densities, we can more generally ask which properties of graphons can be forced this way. Clearly, every such property is invariant under weak isomorphism, and also closed under convergence. (More generally, it is closed in the cut-norm [11],[3], but we don’t need this in this paper.)

Some important properties are finitely forcible, but some others are not. It is sometimes the case, however, that in the presence of some other condition, such properties become finitely forcible. The property that $W$ is 0-1 valued is an example (to be discussed below).

### 4.1 Regularity

We call a graphon $d$-regular, or regular of degree $d$ ($0 \leq d \leq 1$), if

$$\int_0^1 W(x, y)\,dy = d$$

for almost all $0 \leq x \leq 1$. These graphons can be forced by two subgraph density conditions:

$$t(K_2, W) = d, \quad t(P_3, W) = d^2.$$

Regular graphons (without specifying the degree $d$) can be forced by the condition $t(P_3, W) = t(K_2, W)^2$.

### 4.2 Zero-one valued functions

Trivially, $W \in \mathcal{W}$ is 0-1 valued almost everywhere if and only if $t^2(\hat{C}_2, W) = 0$, where $\hat{C}_2$ is the 2-labeled 2-edge-colored graph on 2 nodes with 2 parallel edges, one blue and one green. By Lemma 3.3 this is equivalent to the single numerical equation $t(\hat{B}_4, W) = 0$, where $\hat{B}_4$ is the unlabeled 2-edge-colored graph on 2 nodes with 4 parallel edges, 2 blue and 2 green.

So we can “force” the property of being 0-1 valued using multigraphs, but we cannot express it in terms of simple graphs. This follows from the observation that if $G(n, 1/2)$ is the Erdős–Rényi random graph with $n$ nodes and edge density $1/2$, and $W_n = W_{G(n,1/2)}$, then with probability 1, $W_n$ tends to the identically 1/2 function $U_{1/2}$ in the $\|\|\square$ norm, which implies that
$t(F,W_n) \rightarrow t(F,U_{1/2})$ for every simple graph $F$. So every condition of the form $t(F,W_n) = 0$, where $F$ is a simple graph, is inherited by $U_{1/2}$, which is not 0-1 valued.

It makes sense to formulate sufficient conditions for being 0-1 valued. Here is a useful one.

**Lemma 4.1** Let $\hat{F}$ be a 2-edge-colored bipartite graph on $n$ nodes, all labeled, and let $\hat{F}'$ be obtained by unlabeling all the nodes. Suppose that for some $W \in \mathcal{W}$ we have

$$t^n(\hat{F}, W) = 0.$$  \hspace{1cm} (8)

Then $W(x, y) \in \{0, 1\}$ almost everywhere. If $W \in \mathcal{W}_0$, then it suffices to assume that

$$t(\hat{F}', W) = 0.$$  \hspace{1cm} (9)

**Proof.** By Lemma 3.1, (8) implies that for the 2-labeled 2-edge-colored multigraph $J$ obtained by identifying each color class of $F$, we have $t^2(J, W) = 0$. This clearly implies that $W$ is 0-1 valued.

If $W \in \mathcal{W}_0$, then (9) implies that in the integral

$$t(\hat{F}', W) = \int_{[0,1]^{V(F)}} t^n(\hat{F}, W)(x) \, dx$$

the integrand is 0 almost everywhere. This means that (8) holds. \hfill \square

### 4.3 Monotonicity

Let $\mathcal{M}'$ denote the set of monotone decreasing 0-1 valued graphons. Let $\mathcal{M}$ be the set of graphons which are weakly isomorphic to some function in $\mathcal{M}_0$. In this section we show that the set $\mathcal{M}$ is finitely forcible in $\mathcal{W}$.

Let $\hat{C}_4$ denote a 2-edge-colored 4-labeled 4-cycle, with two opposite edges colored red, the other two colored blue. Let $\hat{C}_4'$ be obtained from $\hat{C}_4$ by unlabeling all its nodes.

**Lemma 4.2** Let $W \in \mathcal{W}$, then $W \in \mathcal{M}$ if and only if

$$t^4(\hat{C}_4, W) = 0.$$  \hspace{1cm} (10)

If $W \in \mathcal{W}_0$, then it is enough to assume that

$$t(\hat{C}_4', W) = 0.$$  \hspace{1cm} (11)

**Proof.** It is easy to see that every $W \in \mathcal{M}'$, and (10) and (11).

Next we prove if that $W \in \mathcal{W}_0$ satisfies (11) then $W \in \mathcal{M}$. By Lemma 4.1 $W$ is 0-1 valued almost everywhere, and we may assume that it is 0-1 valued. Let $N(x)$ denote the support of the function $W(x, \cdot)$ ($x \in [0,1]$).

By the Monotone Reordering Theorem, there is a monotone decreasing function $f : [0,1] \rightarrow [0,1]$ and a measure preserving map $\varphi : [0,1] \rightarrow [0,1]$ such that $\lambda(N(x)) = f(\varphi(x))$ almost everywhere. The function $U(x,y) = 1_{y \leq f(x)}$ is clearly monotone decreasing in both variables.
We claim that $W = U^\varphi$ almost everywhere. This will show that $W$ is weakly isomorphic to a graphon in $\mathcal{M}'$.

We can change $W$ on a set of measure 0 so that $\lambda(N(x)) = f(\varphi(x))$ for all $x$. Since $f$ is monotone decreasing, it follows that if $\varphi(x) \leq \varphi(y)$, then $\lambda(N(x)) \geq \lambda(N(y))$, and so $\lambda(N(y) \setminus N(x)) = 0$.

Let

$$A = \{(x, y, u) \in [0, 1]^3 : W(x, u) = 0, W(y, u) = 1, \varphi(x) \leq \varphi(y)\}$$

and $A(u) = \{(x, y) : (x, y, u) \in A\}$. We claim that $\lambda_3(A) = 0$. First, note that for almost all $x, y \in [0, 1]$, either $\lambda(N(x) \setminus N(y)) = 0$ or $\lambda(N(y) \setminus N(x)) = 0$. Indeed, for every $u \in N(x) \setminus N(y)$ and $v \in N(y) \setminus N(x)$, the 4-tuple $(x, u, y, z)$ satisfies $W(x, u)(1 - W(u, y))W(y, v)(1 - W(v, x)) > 0$, and so $t(C_4^\lambda, W) = 0$ implies that the measure of pairs $(x, y)$ for which there is a positive measure of such pairs $(u, v)$ must be 0. Hence for almost all pairs $(x, y)$, $\lambda\{u : (x, y, u) \in A\} = 0$, which implies that $\lambda_3(A) = 0$. This also implies that for almost all $u \in [0, 1]$, $\lambda_2(A(u)) = 0$.

Let $K(u) = \varphi^{-1}[0, f(\varphi(u))]$. For every point $u$ with $\lambda_2(A(u)) = 0$ we must have $\lambda(N(u) \setminus K(u)) = 0$; indeed, we have $\lambda(N(u)) = f(\varphi(u)) = \lambda(K(u))$, and so if $\lambda(N(u) \Delta K(u)) > 0$, then $\lambda(N(u) \setminus K(u)) > 0$ and $\lambda(K(u) \setminus N(u)) > 0$. But every $x \in K(u) \setminus N(u)$ and $y \in N(u) \setminus K(u)$ satisfies $(x, y) \in A(u)$, a contradiction.

So we know that for almost all $u$, $\lambda(N(u) \Delta K(u)) = 0$. Hence for almost all pairs $(x, u)$, $W(x, u) = 1$ if and only if $x \in K(u)$, i.e., $\varphi(x) \leq f(\varphi(u))$. This proves that $W = U^\varphi$ almost everywhere, and so $W \in \mathcal{M}$.

Finally, assume that $W \in \mathcal{W}$ satisfies (10). By Lemma 4.1, $W$ is 0-1 valued almost everywhere, so $W \in \mathcal{W}_0$. Since it trivially satisfies (11), it follows that $W \in \mathcal{M}$. $\square$

## 5 Forcible graphons I: polynomials

### 5.1 Positive supports of polynomials

**Theorem 5.1** Let $p$ be a real symmetric polynomial in two variables, which is monotone decreasing on $[0, 1]^2$. Then the function $W(x, y) = 1_{p(x, y) \geq 0}$ is finitely forcible in $\mathcal{W}$.

**Proof.** We in fact prove that the equations

$$t^4(C_4^\lambda, U) = 0 \quad (12)$$

and

$$t(K_{a, b}, U) = t(K_{a, b}, W) \quad (0 \leq a, b \leq \deg(p) + 1) \quad (13)$$

form a forcing family for $W$ in $\mathcal{W}$. The condition on the monotonicity of $p$ implies that $U = W$ satisfies (12). It is trivial that the other equations are satisfied by $U = W$.

Let $U \in \mathcal{W}$ be any graphon satisfying (12)-13. By Lemma 4.1 we may assume that $U$ is 0-1 valued and monotone decreasing. Let $S_U = \{(x, y) : U(x, y) = 1\}$.

We have

$$t(K_{a, b}, U) = \int_{[0, 1]^a} \int_{[0, 1]^b} \prod_{i=1}^{a} \prod_{j=1}^{b} U(x_i, y_j) dy dx.$$

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Split this integral according to which \( x_i \) and which \( y_j \) is the largest. Restricting the integral to, say, the domain where \( x_1 \) and \( y_1 \) are the largest, we have that whenever \( U(x_1, y_1) = 1 \) then also \( U(x_i, y_j) = 1 \) for all \( i \) and \( j \), and hence

\[
\int_{x_1 \in [0, 1]} \int_{x_2, \ldots, x_a \leq x_1} \int_{y_1, \ldots, y_b \leq y_1} \prod_{i=1}^{a} \prod_{j=1}^{b} U(x_i, y_j) \, dy \, dx
\]

\[
= \int_{x_1 \in [0, 1]} \int_{y_1 \in [0, 1]} U(x_1, y_1) x_1^{a-1} y_1^{b-1} \, dy_1 \, dx_1 = \int_{(x, y) \in S_U} x^{a-1} y^{b-1} \, dy \, dx.
\]

Hence

\[
t(K_{a,b}, U) = ab \int_{(x, y) \in S_U} x^{a-1} y^{b-1} \, dy \, dx.
\]

By Stokes’ Theorem, we can rewrite this as

\[
t(K_{a,b}, U) = b \int_{\partial S_U} x^a y^{b-1} n_1(x, y) \, ds,
\]

where \( ds \) is the arc length of \( \partial S \) and \( n = (n_1, n_2) \) is the outward normal of \( \partial S \). (Since \( \partial S \) is the graph of a monotone function, this normal exists almost everywhere.) Interchanging the roles of \( x \) and \( y \), and adding, we get

\[
\int_{\partial S_U} x^a y^b (n_1(x, y) + n_2(x, y)) \, ds = \frac{1}{a+1} t(K_{a+1, b}, U) + \frac{1}{b+1} t(K_{a, b+1}, U).
\]

(14)

Now consider the following integral:

\[
I(U) = \int_{\partial S_U} x^2 y^2 p(x, y)^2 (n_1(x, y) + n_2(x, y)) \, ds.
\]

By (14), this can be expressed as a linear combination of the values \( t(K_{a,b}, U) \), where \( a, b \leq \deg(p) + 1 \) and the coefficients depend only on \( a, b \) and \( p \). Hence it follows that \( I(U) = I(W) = 0 \).

On the other hand, the integrand in \( I(U) \) is clearly 0 on the axes, and it is nonnegative on the rest of the boundary (on the curve \( y = f(x) \)). Hence it must be identically 0, which means that \( \partial(S_U) \) must be contained in the union of the axes and the curve \( p = 0 \). But this clearly implies that \( U = W \) except perhaps on the boundary. \( \square \)

The following special case is perhaps the simplest. Define the **half-graphon** by \( W_h(x, y) = 1_{x+y \leq 1} \).

**Corollary 5.2** The half-graphon is finitely forcible in \( W \).

In fact, carrying our the above proof carefully, we get that the following equations force the half-graphon:

\[
t(\tilde{C}_4, W) = 0,
\]

(15)

and

\[
t(K_2, W) - t(P_3, W) + 1/6 = 0.
\]

(16)
Clearly, the left hand side of (15) is always nonnegative. It is easy to show that in (16), the left hand side is nonnegative, provided equality holds in (15).

Half-graphons are natural limits of half-graphs, defined by \( V(G) = [n] \) and \( E(G) = \{ij : i + j \leq n\} \). Corollary 5.2 implies the following graph-theoretic extremal result.

**Corollary 5.3** Among all graphs with no induced matching with 2 edges, the difference \( t(P_3, .) - t(K_2, .) \) is asymptotically minimal for half-graphs.

### 5.2 Continuous range

Applying the results of Section 3 to the half-graphon, we get:

**Proposition 5.4** Let \( U(x, y) = 1_{x+y>1} \). Then the element \( W \in W \) defined by \( W \circ W \circ W + W = U \) is finitely forcible in \( W \), and its range consists of two nontrivial intervals.

**Proof.** Let \( \lambda_1, \lambda_2, \lambda_3, \ldots \) be the non-zero eigenvalues of \( U \) and let \( f_1, f_2, \ldots \) be the corresponding eigenfunctions. It is clear that \( f_1, f_2, \ldots \) are all continuous in \([0, 1]\) because

\[
\lambda_i f_i(x) = \int_{y=1-x}^{1} f_i(y) \, dy.
\]

(In fact, from this integral equation one can calculate that the eigenvalues and eigenfunctions are

\[
\lambda_k = \frac{2}{\pi (4k + 1)}, \quad f_k(x) = \cos \left( \frac{1}{2} (4k + 1) \pi x \right), \quad k \in \mathbb{Z},
\]

but we don’t use this explicit formula).

Let us consider the difference \( P = U - W \). We have

\[
P(x, y) = \sum_{i=1}^{\infty} f_i(x)f_i(y)(\lambda_i - z(\lambda_i))
\]

where \( z \) is the inverse function of \( x \mapsto x^3 + x \). The function \( z_2(x) = x - z(x) \) has the property that \( z_2(0) = z'_2(0) = 0 \), and it is analytic in the unit circle. This implies that \( z_2(t) = O(t^2) \) as \( t \to 0 \). Since the series \( \sum \lambda_i^2 \) is absolute convergent, so is \( \sum z_2(\lambda_i) \). This implies that \( P \) is a continuous function on \([0, 1]^2\). It is clear that \( P \) is not constant. We obtain that \( W \) is equal to \( G \) plus a non-constant continuous function on \([0, 1]^2\). This completes the proof. \( \square \)

Now Corollary 3.14 implies that we can transform \( W \) into an element from \( \mathcal{W}_0 \). This implies the following.

**Corollary 5.5** There is finitely forcible function in \( \mathcal{W}_0 \) whose range is of continuum cardinality.

### 6 Forcible graphons II: complement reducible graphons

#### 6.1 The finite case

A finite graph is called complement reducible, or for short, a CR-graph, if it can be constructed starting from single nodes, by repeated application of disjoint union and complementation.
(These graphs are often called, in a somewhat unfortunate way, cographs). One of the many characterizations of these graphs is the following (see e.g. [4] for more on these graphs).

**Proposition 6.1** A graph is a complement reducible if and only if it does not contain a path on 4 nodes as an induced subgraph.

Let \( \hat{P}_4 \) denote the graph \( K_4 \) in which the edges of a path of length 3 are colored red, and the remaining edges are colored blue. Then the condition in the proposition can be rephrased as

\[
t(\hat{P}_4, G) = 0.
\]

Every CR-graph \( G \) can be described by a rooted tree \( T \) in a natural way [5]: each node of \( T \) represents a CR-graph; the leaves represent single nodes, the root represents \( G \), and the descendants of each internal node represent the connected components of the graph represented by the node, complemented. Another way of describing this connection is that \( G \) is defined on the leaves of \( T \), and two nodes of \( G \) are connected by an edge if and only if their last common ancestor is at an odd distance from the root.

### 6.2 CR graphons and trees

We define a **CR-graphon** as any graphon \( W \) with \( t(\hat{P}_4, W) = 0 \). Also we extend the notion of CR-graphs to infinite graphs by the requirement that no four nodes induce a path of length four.

One can construct CR-graphons from trees, but (unlike in the finite case) not all CR-graphons arise this way (we shall see later that all regular CR-graphons do). In this section, we describe a simple construction of CR-graphons from trees; see Appendix 3 for a more general construction.

Let \( T \) be any (possibly infinite) rooted tree, in which every non-leaf node has at least two and at most countably many children, except possibly the root, which may have only one child. Let \( \Omega = \Omega_T \) be the set of maximal paths starting at the root \( r \) (they are either infinite or end at a leaf). For each node \( v \), let \( C_v \) be the set of its children, and let \( \Omega_v \) denote the set of paths in \( \Omega \) passing through \( v \). The sets \( \Omega_v \) generate a \( \sigma \)-algebra \( A_T \).

We define a graph on node set \( \Omega \) by connecting two nodes if the last common node of the corresponding paths is at an odd depth (where the depth of the root \( r \) is 0). We also define the adjacency function \( U_T : \Omega \times \Omega \rightarrow \{0, 1\} \) by letting \( U_T(x, y) = 1 \) if and only if \( x \) and \( y \) are adjacent. It is clear that \( U_T \) is measurable with respect to \( A \times A \).

If we choose any probability measure \( \pi \) on \( (\Omega_T, \mathcal{A}_T) \), this completes the construction of a graphon \( (\Omega_T, \mathcal{A}_T, \pi, U_T) \). Note that such a measure can be specified through the values

\[
f(v) = \pi(\Omega_v).
\]

It is clear that these values satisfy

\[
f(r) = 1, \quad f(u) \geq 0, \text{ and } f(u) = \sum_{v \in C_u} f(v). \tag{17}
\]

Conversely, every function satisfying (17) defines a probability measure on \( (\Omega_T, \mathcal{A}_T) \).
6.3 Regular CR-graphons

Of special interest for us will be regular CR-graphons. Our first goal is to prove:

**Theorem 6.2** Every regular CR-graphon $W$ can be represented (up to weak isomorphism) as $(\Omega_T, A_T, \pi, U_T)$ where $T$ is a locally finite tree and $\pi$ is a probability measure on $A_T$.

The proof of this theorem will need several lemmas. Recall that for every graphon $W : [0,1]^2 \to [0,1]$ there is a random graph model $G(W,n)$ on node set $\{1,2,\ldots,n\}$, created as follows: We pick independent random points $x_1, x_2, \ldots, x_n \in [0,1]$ and connect two distinct nodes $i$ and $j$ with probability $W(x_i, x_j)$. Let $d_i(G(W,n))$ denote the degree of $i$ in the resulting graph.

**Lemma 6.3** If a graphon $W$ is $d$-regular, then with probability at least $1 - 2n e^{-(n-1)\varepsilon^2}$ we have

$$\left| d_i(G(W,n)) - d \right| < \varepsilon$$

simultaneously for all $1 \leq i \leq n$.

**Proof.** It is easy to see that for every $1 \leq i \leq n$ the value $d_i(G(W,n))$ is the sum of $n-1$ independent random variables all taking $1$ with probability $d$ and $0$ with probability $1 - d$. This implies by Azuma’s inequality that

$$P\left( \left| d_i(G(W,n)) - d \right| \geq \varepsilon \right) \leq 2e^{-(n-1)\varepsilon^2}.$$  

This means that the probability that there exist at least one number $1 \leq i \leq n$ with $|d_i(G(W,n))/n - 1 - d| \geq \varepsilon$ is at most $2ne^{-(n-1)\varepsilon^2}$. □

**Definition 6.4** We say that $\{G_i\}_{i=1}^\infty$ is a degree-uniformly convergent sequence of finite graphs with limiting degree $0 \leq d \leq 1$ if it is convergent, $\lim_{i \to \infty} |V(G_i)| = \infty$ and

$$\lim_{i \to \infty} \frac{d_{\text{max}}(G_i)}{|V(G_i)|} = \lim_{i \to \infty} \frac{d_{\text{min}}(G_i)}{|V(G_i)|} = d.$$ 

**Lemma 6.5** If a CR-graphon $W$ is $d$-regular then there is a sequence of CR-graphs $\{G_n\}_{n=1}^\infty$ that degree-uniformly converges to $W$.

**Proof.** The Borel-Cantelli lemma together with lemma 6.3 implies that with probability one

$$\lim_{i \to \infty} \frac{d_{\text{max}}(G(W,n))}{n - 1} = \lim_{i \to \infty} \frac{d_{\text{min}}(G(W,n))}{n - 1} = d.$$ 

We also now that with probability one $\{G(W,n)\}_{n=1}^\infty$ converges to $W$ and furthermore for every $n$ with probability one $G(W,n)$ is a CR-graph. These together imply that the sequence $\{G(W,n)\}_{n=1}^\infty$ satisfies the conditions with probability one. □
Lemma 6.6 Let $G$ be a finite disconnected CR-graph on $n$ vertices such that $|d(v)/n - d| \leq \varepsilon$ for every $v \in V(G)$ with some $d \in [0, 1]$. Then the connected components of $G$ have size at most $n(\frac{2}{3} + \frac{4}{3}\varepsilon)$.

Proof. Let $G'$ be a connected component of $G$ of maximal size. Let $n = |V(G)|$, $a = |V(G')|$ and let $v$ be a node in $V(G) \setminus V(G')$. We have that $d(v) \leq 1 - a$. Furthermore, since $G'$ is a connected CR-graph there is a vertex $w \in V(G')$ with degree at least $a/2$. Assume $n - a \leq a/2$. Then by our assumption

$$2\varepsilon n \geq |d(w) - d(v)| \geq \left|\frac{2}{3}a - n\right|.$$  

This implies that $\frac{3}{2}a - n \leq 2\varepsilon n$ and thus $a \leq n(\frac{2}{3} + \frac{4}{3}\varepsilon)$.

\[\square\]

Lemma 6.7 Let $W$ be a $d$-regular CR-graphon. Then either $W$ or $1 - W$ is can be decomposed as a weighted direct sum of at least two graphons, all of them with weight at least $d$.

Proof. By Lemma 6.5, there is a sequence of CR-graphs $\{G_n\}_{n=1}^\infty$ that degree-uniform converges to $W$. For each $n$, either $G_n$ or $\overline{G_n}$ is disconnected, since $G_n$ is a CR-graph. We may assume, by restricting ourselves to a subsequence, that either $G_n$ is disconnected for all $n$, or $\overline{G_n}$ is disconnected for all $n$. By complementing if necessary, we may assume that $G_n$ is disconnected for all $n$.

Let $H_{n,1}, \ldots, H_{n,k_n}$ be the connected components of $G_n$. Since the convergence is degree-uniform, it follows that for any $0 < d' < d$, all degrees of $G_n$ are larger than $d'|V(G_n)|$ if $n$ is large enough, and then trivially $|V(H_{n,i})| \geq d'|V(G_n)|$. This implies that $k_n$ remains bounded, and so by going to a subsequence again, we may assume that $k_n = k$ is independent of $n$. By the same token, we may assume that $|V(H_{n,i})|/|V(G_n)|$ has a limit $a_i$ as $n \to \infty$. Clearly $a_i \geq d$ and $\sum_i a_i = 1$. We may also assume that for each $1 \leq i \leq k$, the sequence of graphs $(H_{n,i})_{n=1}^\infty$ is convergent. Let $W_i$ denote its limit graphon. It is straightforward to check that $W_i$ is a regular CR-graphon. Furthermore, the weighted direct sum $\bigoplus_i a_i W_i$ is the limit of the graphs $G_n$. By the uniqueness of the limit, $W$ is weakly isomorphic to $\bigoplus_i a_i W_i$. \[\square\]

Now we are able to prove Theorem 6.2.

Proof. By Lemma 6.7, either $W$ or $1 - W$ can be written as a weighted direct sum of at least two and at most $1/d$ regular CR-graphons.

Assume that $W = \bigoplus_{i=1}^k a_i W_i$. We may assume that the $W_i$ cannot be written as weighted direct sums in a nontrivial way. We build a tree by starting with a root corresponding to $W$, having $k$ children corresponding to $1 - W_1, \ldots, 1 - W_k$. If any of these functions is almost everywhere 0, then this node will be a leaf. Else, we continue building the tree from this node as root.

If $W$ cannot be written as a weighted direct sum of at least two and at most $1/d$ regular CR-graphons, then $1 - W$ can, and we start the tree with a root with a single child, corresponding to $1 - W$.

This way we obtain a tree $T$, where each node $v$ is labeled by a regular CR-graphon $W_v$. For each node of the tree constructed this way, we define $f(v)$ as the product of the weights of the
graphons along the path from the root to \( v \). It is straightforward to check that the \( W \) is weakly isomorphic to the graphon \( U_T \) with the probability distribution defined by \( f \).

Let \( W \) be a regular CR-graphon represented by the tree \( T \) with a measure \( \mu \) on \( \Omega_T \). For a node \( v \in V(T) \) let \( f(v) := \mu(\Omega_v) \). We observe that for every \( v \in V(T) \), the subtree \( T_v \) of \( T \) rooted at \( v \), with the same local distributions (i.e., with node weights \( f(u)/f(v) \)), defines another regular CR-graphon. This implies that the value

\[
c(v) = \int_{\Omega_v} U_T(x,y) \, dy
\]

is the same for all \( x \in \Omega_v \). The degree of the graphon on \( T_v \) is \( d(v) = c(v)/f(v) \). (Note however that not \( U_{T_v} \) but \( 1 - U_{T_v} \) is an induced subgraphon of \( U_T \).) For \( u \in V(T) \) and \( v \in C_u \) we have

\[
c(u) + c(v) = f(v),
\]

and for every leaf \( u \) (if any)

\[
c(u) = 0.
\]

Lemma 6.8 Let \( T \) be a locally finite tree and \( c, f : V(T) \rightarrow \mathbb{R}_+ \), two functions satisfying \( \text{(17)}, \text{(18)} \) and \( \text{(19)} \). Then the probability measure defined by \( f \) gives a regular CR-graphon on \( T \).

**Proof.** By \( \text{(17)} \), the function \( f \) defines a probability measure \( \pi \) on \( \Omega, A \). Let \( x = (v_0, v_1, v_2, \ldots) \) be a maximal path starting at the root \( r = v_0 \). Then \( x \) is connected to all paths \( y \) that branch off from \( x \) at \( v_1, v_3, v_5 \), etc. The \( \pi \)-measure of these paths is \( (f(v_1) - f(v_2)) + (f(v_3) - f(v_4)) + \cdots \), which by \( \text{(18)} \) can be written as

\[
(c(v_0) + c(v_1)) - (c(v_1) + c(v_2)) + (c(v_2) + c(v_3)) - \cdots = c(v_0)
\]

(if the path ends at a leaf, then we use \( \text{(19)} \)). This is indeed independent of the path. \( \square \)

The following simple lemma gives some conditions that \( f \) and \( c \) satisfy.

Lemma 6.9 Let \( W_T \) be a regular CR-graphon.

(a) If \( u \in V(T) \) has \( r \) children, then \( c(u) \leq \frac{1}{r} f(u) \).

(b) If \( u \in V(T), v \in C_u \) and \( v \) has \( r \) children, then \( f(u) \geq (2 - \frac{1}{r}) f(v) \).

**Proof.** (a) Let \( v_1, v_2, \ldots, v_r \) be the children of \( u \). By \( \text{(18)} \), \( f(v_i) = c(u) + c(v_i) \geq c(u) \), and summing this over \( i \), we get \( f(u) \geq r c(u) \).

(b) Let \( v' \) be a sibling of \( v \). Using \( \text{(18)} \) and (a),

\[
f(v') = c(u) + c(v') \geq c(u) = f(v) - c(v) \geq (1 - \frac{1}{r}) f(v),
\]

and so

\[
f(u) \geq f(v) + f(v') \geq (2 - \frac{1}{r}) f(v).
\]

Now we are able to prove the second main result in this section:
Theorem 6.10  For every locally finite rooted tree $T$ there is a unique regular CR-graphon on $T$.

Proof.  Existence. First we prove this for a finite tree, by induction on the depth. For a single node, the function $U \equiv 0$ is a regular CR-graphon.

Suppose that the tree has more than one node, and let $u_1, \ldots, u_k$ be the children of the root. By induction, we find regular CR-graphons on $T_{u_1}, \ldots, T_{u_k}$, with degrees $d_1, \ldots, d_k$. Note that since $u_i$ is either a leaf or has at least two children, we must have $d_i < 1$. Let

$$d = \frac{1}{\sum_{i=1}^{k} 1/d_i}, \quad a_i = \frac{d}{1-d_i},$$

then scaling the measure of $T_{u_i}$ by $a_i$, complementing each $T_{u_i}$ and taking their disjoint union, we get a $d$-regular CR-graphon on $T$.

Now suppose that $T$ is infinite, and let $T_k$ denote the tree obtained by deleting all nodes farther than $k$ from the root. By the above, there is a regular CR-graphon on $T_k$, which yields two functions $f^k$ and $c^k$ on $V(T_k)$ satisfying (17), (18) and (19). We can select a subsequence of the indices $k$ such that $f^k(v)$ tends to some $f(v)$ and $c^k(v)$ tends to some $c(v)$ as $k$ ranges through this subsequence. Clearly, the functions $f$ and $c$ also satisfy (17), (18) and (19), and so by Lemma 6.8 they yield a regular CR-graphon on $T$.

Uniqueness. Suppose that there are two weightings $f, f'$ of the nodes of $T$ such that they both define a regular CR-graphon.

Let $P = (v_0, v_1, v_2, \ldots)$ be an infinite path starting from any node $v = v_0$, then we get

$$f(v_0) - f(v_1) + f(v_2) - \cdots + (-1)^k f(v_k) = c(v_0) + (-1)^k c(v_k)$$

Note that the sequence $f(v_k)$ is monotone decreasing and it tends to 0. Hence $c(v_k) \to 0$, and

$$c(v_0) = \sum_{k=0}^{\infty} (-1)^k f(v_k).$$

In the other weighting,

$$c'(v_0) = \sum_{k=0}^{\infty} (-1)^k f'(v_k).$$

Let

$$z_i = \frac{f'(v_i)}{f'(v_{i-1})} \frac{f(v_i)}{f(v_{i-1})},$$

then $f'(v_k) = z_1 \ldots z_k f(v_k)$. Thus

$$c'(v_0) = \frac{f'(v_0)}{f(v_0)} \sum_{k=0}^{\infty} (-1)^k z_1 \cdots z_k f(v_k) = \frac{f'(v_0)}{f(v_0)} \left( f(v_0) - z_1 (f(v_1) - z_2 (f(v_2) - \ldots)) \right). \quad (20)$$

Choose the path $v_0, v_1, \ldots$ as follows. Given $v_i$, choose $v_{i+1} \in C_{v_i}$, so that $z_i \leq 1$ if $i$ is even, and $z_i \geq 1$ if $i$ is odd. This is clearly possible. Lowering $z_1$ to 1, then raising $z_2$ to 1, then lowering $z_3$ to 1 etc. increases the expression in (20), and hence

$$c'(v_0) \leq \frac{f'(v_0)}{f(v_0)} c(v_0).$$
Since the reverse inequality follows similarly, we get that 
\[
\frac{c'(v_0)}{c(v_0)} = \frac{f'(v_0)}{f(v_0)},
\]
and from the fact that equality holds, we also get that 
\[
\frac{f'(v_0)}{f(v_0)} = \frac{f'(v_1)}{f(v_1)}
\]
for any child \( v_1 \) of \( v_0 \). So if \( f(v) = f'(v) \) then this also holds for the children of \( v \), which proves that \( f = f' \).

We conclude with an easy observation that, however, will be important for us.

**Lemma 6.11** If a \( d \)-regular CR-graphon is a stepfunction, then \( d \) is rational.

**Proof.** By induction on the depth of the tree.

\[\square\]

### 6.4 Forcible regular CR-graphons with irrational edge densities

In this section we prove that for every \( \alpha \in [0, 1] \) there is a regular CR-graphon of degree \( \alpha \) which is finitely forcible. As lemma 6.11 shows, for irrational \( \alpha \) such a graphon is not a step function.

Let \( Z \) be the set of those regular CR-graphons that don’t contain any induced copy of \( C_4 \) and \( \overline{C_4} \). This is equivalent to saying that in the tree representing such a graphon, every node has at most one child that is not a leaf. This means that the tree is one (possibly infinite) path with additional leaves hanging from its nodes. Thus the structure is determined by the integer sequence \( n_1, n_2, \ldots \) where \( n_k \) is the number of leaves at the \( k \)-th level.

We start with a simple example. Let \( \alpha = (3 - \sqrt{5})/2 \). Note that \( \alpha = (1 - \alpha)^2 \). There exists a unique graphon \( W \) which is the disjoint union of a clique of size \( \alpha \) and a version of the complement of \( W \) scaled to the size \( 1 - \alpha \). The graphon \( W \) has an iterated structure. The choice of \( \alpha \) guarantees that \( W \) is \( \alpha \)-regular. We show that \( W \) is finitely forcible.

**Lemma 6.12** The graphon \( W \) is the only element of \( Z \) with degree \( \alpha \) and thus it is finitely forcible.

**Proof.** Let \( W' \) be a graphon with the above property. Since \( 1/3 < \alpha < 1/2 \) we have that \( W' = \alpha \cdot 1 \oplus W'' \), where \( W'' \) has density \( 1 - \alpha \). This shows that we can inductively continue the process with the complement \( W'' \) and finally obtain the desired iterated structure.

\[\square\]

The previous argument can be easily generalized to other irrational values of \( \alpha \).

**Proposition 6.13** For every irrational number \( 0 < \alpha < 1 \) there is exactly one graphon in \( Z \) with edge density \( \alpha \) and so this graphon is finitely forcible.
**Proof.** Let $\alpha$ be a number between 0 and $1/2$ (the case $\alpha > 1/2$ is similar). Let $W$ be a graphon from $Z$ with edge density $\alpha$. Since $\alpha < 1/2$, the graphon $W$ is weighted direct sum of $n_1$ cliques, all with weight $\alpha$ and a connected element $W'$ of $Z$ with weight $0 < s < 1$ and with edge density between $1/2$ and 1. To guarantee edge density $\alpha$ in this component, $s$ has to be between $\alpha$ and $2\alpha$. Consequently $n_1$ is the unique natural number such that $\alpha \leq 1 - n_1\alpha < 2\alpha$. The complement $1 - W'$ is an element from $Z$ with edge density smaller than $1/2$ and we can continue the process. □

The reader can see that the number $n_1, n_2, \ldots$ are uniquely determined by $\alpha$ so it is a natural question to ask what these numbers are. An elementary calculation shows that these numbers are basically the numbers occurring in the (unique) continued fraction expansion of $\alpha$. The only exception is the first number, which is shifted by one:

$$\alpha = \frac{1}{n_1 + 1 + \frac{1}{n_2 + \frac{1}{n_3 + \ddots}}}.$$ 

This shows that one can force graphons that encode an arbitrary sequence of natural numbers in a very structural way.

### 6.5 Forcing the binary tree

In this section we prove that the regular CR-graphon $U_B$ defined by the complete binary tree $B$ of infinite depth is finitely forcible. We note that $U_B$ has the following alternative description: Consider the space $V(C_4)^\mathbb{N}$ with the uniform probability measure. We connect two nodes $x$ and $y$ of $V(C_4)^\mathbb{N}$ if for the first coordinate where they differ, say $i \in \mathbb{N}$, $x_i$ and $y_i$ are connected in $C_4$. The graphon $U_B$ can be called the **infinite lexicographic power** of $C_4$.

Let us define the following labeled graphs: $A$ is $K_2$ with one endnode labeled 1; $B$ is $P_3$ with the middle node labeled 1; and $C$ and $D$ are obtained from $A$ and $C$, respectively, by adding a new isolated node labeled 2. Consider the 2-colored graphs $\hat{B}, \hat{C}$ and $\hat{D}$, and also the graphs $\overline{B}, \overline{C}$ and $\overline{D}$ obtained from $B, C$ and $D$ by switching the red and blue edges.

**Proposition 6.14** Let $W$ be a graphon satisfying

$$t(\hat{P}_4, W) = 0, \quad t^1(A, W) = \frac{2}{3},$$

$$t^1(\hat{B}, W) = \frac{8}{45}, \quad t^1(\overline{B}, W) = \frac{2}{45}$$

and

$$2t^2(\hat{C}, W)^2 = 5t^2(\hat{D}, W), \quad 2t^2(\overline{C}, W)^2 = 5t^2(\overline{D}, W).$$

Then $W$ is weakly isomorphic with $U_B$ where $B$ is the binary tree.
Proof. It is straightforward to check that the graphon $U_B$ satisfies these identities.

The first two identities mean that $W$ is a regular CR-graphon with degree $2/3$. By Theorem 6.2 we know that $W$ can be represented by a locally finite tree $T$ and so we can assume that $W = U_T$. The edge density $2/3$ guarantees that the root of $T$ must have at least 2 children, i.e. $U_T$ is a connected graphon and $1 - U_T$ has at least 2 components.

Let $v$ and $w$ be two different children of $T$. We also know that $v$ has at least two children $v_1, v_2$. Let $x \in \Omega_{v_1}$ and $y \in \Omega_{v_2}$. Then

$$t_{xy}(\hat{C}, U_T) = \int_{\Omega} U_T(x, z)(1 - U_T(z, y)) \, dz = \int_{\Omega_{v_1}} U_T(x, z) \, dz$$

$$= \int_{\Omega} U_T(x, z) \, dz - \mu(\Omega_w) = t_x(A, W) - \mu(\Omega_w) = \frac{2}{3} - \mu(\Omega_w). \quad (21)$$

Similarly,

$$t_{xy}(\hat{D}, U_T) = \int_{\Omega \times \Omega} U_T(x, z)(1 - U_T(z, y))U_T(x, u)(1 - U_T(u, y))(1 - U_T(z, u)) \, du \, dz$$

$$= \int_{\Omega_{v_1} \times \Omega_{v_1}} U_T(x, z)U_T(x, u)(1 - U_T(z, u)) \, du \, dz$$

$$= \int_{\Omega \times \Omega} (1 - U_T(z, u)) \, du \, dz - \int_{\Omega \times \Omega} (1 - U_T(z, u)) \, du \, dz$$

$$= t_x(\hat{B}, U_T) - \frac{1}{3}\mu(\Omega_w) = \frac{8}{45} - \frac{1}{3}\mu(\Omega_w). \quad (22)$$

Using our conditions, we get from (21) and (22)

$$2\left(\frac{2}{3} - \mu(\Omega_w)\right)^2 = 2t^2(\hat{C}, W)^2 = 5t(\hat{D}, W) = \frac{8}{9} - \frac{5}{3}\mu(\Omega_w),$$

from where $\mu(\Omega_w) = 1/2$. This is true for every child of the root, and hence there are exactly two children. Furthermore, it is easy to check that the complement of the graphon $U_T$, satisfies the same identities as listed in the statement. Iterating the argument, we get that $T$ is a complete binary tree. $\square$

Using Lemma 3.3 we get

**Corollary 6.15** The graphon $U_B$ is finitely forcible.

7 Necessary conditions for finite forcing

7.1 Weak homogeneity

For every graph $F = (V, E)$, and node $i \in V$, let $F^i$ denote the 1-labeled quantum graph obtained by labeling $i$ by 1, and for every edge $ij \in E$, let $F^{ij}$ denote the 2-labeled quantum graph obtained from $F$ by deleting the edge $ij$, and labeling $i$ by 1 and $j$ by 2. Let $F^1 = \sum_{i \in V} F_i$ and $F^\perp = \sum_{i,j: ij \in E} F^{ij}$. We extend the operators $F \to F^1$ and $F \to F^\perp$ linearly to all quantum graphs.
These operations were introduced by Razborov [15,16] in the proof of conjectures about the minimum number of triangles in graphs with given edge density. For us, their significance is in the following formulas.

Let $U_t$, $0 \leq t \leq 1$ be a family of functions in $\mathcal{W}$. We say that $U_t$ is differentiable if for every $t \in [0,1]$ there exists a function $\dot{U}_t \in \mathcal{W}$ such that
\[
\left\| \frac{1}{s-t} (U_s - U_t) - \dot{U}_t \right\|_{\mathcal{W}} \to 0 \quad (s \in [0,1], \ s \to t).
\]

Let $U_t$, $0 \leq t \leq 1$ be a uniformly bounded differentiable family of functions in $\mathcal{W}$. A routine calculation familiar from variational calculus gives that for every graph $F = (V,E)$, the function $t(F,U_t)$ is differentiable as a function of $t$, and
\[
\frac{d}{dt} t(F,U_t) = \int_{[0,1]^2} \dot{U}_t(x_1,x_2) t^2(F^\dagger,U_t)(x_1,x_2) \, dx_1 \, dx_2.
\]
We can write the right hand side as $\langle \dot{U}_t, t^2(F^\dagger,U_t) \rangle$, where the inner product is in $L_2([0,1]^2)$.

We use this formula to derive a necessary condition for finite forcibility.

**Lemma 7.1** Suppose that $W \in \mathcal{W}$ is forced (in $\mathcal{W}$) by the graphs $F_1, \ldots, F_m$. Also suppose that there exists an $L_2$-neighborhood $U \subseteq \mathcal{W}$ of $W$ and a continuous map $\Phi : U \to \mathcal{W}$ such that for all $U \in \mathcal{U}$, $\langle \Phi(U), t^2(F^\dagger_i,U) \rangle = 0$ ($i = 1, \ldots, m$). Then $\langle \Phi(W), t^2(F^\dagger,W) \rangle = 0$ for every finite graph $F$.

**Proof.** Consider a 1-parameter family $\{U_s : s \in [0,c]\}$ of functions in $\mathcal{U}$ satisfying the differential equation
\[
\frac{d}{ds} U_s = \Phi(U_s), \quad U_0 = W.
\]
Equation (23) shows that for every graph $F$
\[
\frac{d}{ds} t(F,U_s) = \langle \frac{d}{ds} U_s, t^2(F^\dagger,U_s) \rangle = \langle \Phi(U_s), t^2(F^\dagger,U_s) \rangle.
\]
In particular, we have
\[
\frac{d}{ds} t(F_i,U_s) = \langle \Phi(U_s), t^2(F^\dagger_i,U_s) \rangle = 0
\]
for $i = 1, \ldots, m$, and hence $t(F_i,U_s) = t(F_i,U_0) = t(F_i,W)$. Since the $F_i$ force $W$, it follows that the $U_s$ are weakly isomorphic with $W$, and so $t(F,U_s) = t(F,W)$ for every $F$. But then $\langle \Phi(W), t(F^\dagger,W) \rangle = \frac{d}{ds} t(F,U_s) = 0$. \hfill \Box

The following proposition tells us that finitely forcible graphons are in a sense “homogeneous”.

**Proposition 7.2** Let $W \in \mathcal{W}$ be finitely forcible. Then there is a 2-labeled quantum graph $f \neq 0$ such that $t^2(f,W) = 0$.

**Proof.** Suppose that no such quantum graph exists. Then the functions $t^2(F^\dagger,W)$ (where $F$ ranges over all graphs) are linearly independent.

Let $W$ be forced by $t(F_1,W) = \alpha_1, \ldots, t(F_m,W) = \alpha_m$. Let $F$ be a further graph. For every $U \in \mathcal{W}$, let $\Phi(U)$ denote the component of $t^2(F^\dagger,U)$ orthogonal to the subspace spanned
by \(t^2(F_i^1, U), \ldots, t^2(F_m^1, U)\). Since the functions \(t^2(F_i^1, W), \ldots, t^2(F_m^1, W)\) and \(t^2(F_i, W)\) are linearly independent, so are the functions \(t^2(F_i^1, U), \ldots, t^2(F_m^1, U), t^2(F_i, U)\) if \(U\) is in some small \(L_2\)-neighborhood of \(W\), and hence the functional \(\Phi\) is well defined, continuous and nonzero in this neighborhood. But by Lemma 7.1 we have \(\langle \Phi(W), \Phi(W) \rangle = \langle \Phi(W), t(F_i, W) \rangle = 0\), which is a contradiction.

The last proposition can be used to show that “most” graphons are not finitely forcible.

**Theorem 7.3** The set of finitely forcible graphons is of first category in \(L_2([0, 1]^2)\).

**Proof.** Let \(f = \sum_{i=1}^{k} a_i F_i\) be a 2-labeled quantum graph. We claim that for fixed \(F_1, \ldots, F_k\), the set of graphons \(W\) for which there is a quantum graph composed of these \(F_i\) satisfying an equation \(t^2(f, W) = 0\) is nowhere dense. We may assume that every component of each \(F_i\) contains a labeled node. Let us fix a \(W\), we want to show that an arbitrary small neighborhood of \(W\) contains a graphon \(W'\) such that \(t^2(f, W') \neq 0\). This will be enough, since \(t^2\) is continuous and so there is an open set \(U\) in the neighborhood such that \(t^2(f, U) \neq 0\) for \(U \in U\).

Lemma 5 of [7] implies that there are graphons \(U_1, \ldots, U_k\) such that the matrix \((t(F_i, U_j))_{i,j=1}^{k}\) is nonsingular. We may assume that \(\|W\|_\infty, \|U_1\|_\infty, \ldots, \|U_k\|_\infty \leq 1\). For \(0 < \varepsilon < 1/k\), define \(W^\varepsilon = (1-k\varepsilon)W + \varepsilon U_1 + \cdots + \varepsilon U_k\) (so the components of \(W^\varepsilon\) are \(W, U_1, \ldots, U_k\), scaled by \(1-k\varepsilon, \varepsilon, \ldots, \varepsilon\)).

First we show that \(W^\varepsilon \to W\) in \(L_2([0, 1])\) if \(\varepsilon \to 0\). Let \(W_\varepsilon = (1-k\varepsilon)W + k\varepsilon 0\). Then
\[
\|W_\varepsilon - W\|_2 = \varepsilon^2(\|U_1\|_2 + \cdots + \|U_k\|_2) \to 0 \quad (\varepsilon \to 0),
\]
so it suffices to show that \(\|W - W_\varepsilon\|_2 \to 0\). This is easy if \(W\) is a stepfunction with interval steps, and it follows for general \(W\) as these can be approximated by such stepfunctions in \(L_2\).

Now suppose that
\[
t^2(f, W^\varepsilon) = \sum_{i=1}^{k} a_i t^2(F_i, W^\varepsilon) = 0.
\]
If we integrate only over the points in \(U_j\), we get that
\[
\sum_{i=1}^{k} a_i e^{i(F_i)}|t(F_i, U_j)| = 0 \quad (j = 1, \ldots, k).
\]
But this contradicts the nonsingularity of the matrix \((t(F_i, U_j))_{i,j=1}^{k}\).

If \(W\) is only finitely forced in \(W_0\), then we get a weaker condition, which we state without proof.

**Lemma 7.4** Suppose that \(W \in W_0\) is forced in \(W_0\) by the graphs \(F_1, \ldots, F_m\). Also suppose that there exists a neighborhood \(U \subseteq L_2[0, 1]\) of the all-1 function and a continuous map \(\Phi : U \to L_2[0, 1]\) such that for all \(U \in U\), \(\langle \Phi(U), t(F_i^1, U) \rangle = 0 \quad (i = 1, \ldots, m)\). Then \(\langle \Phi(W), t(F_i^1, W) \rangle = 0\) for every finite graph \(F\).

**Corollary 7.5** Let \(W \in W_0\) be finitely forcible in \(W_0\). Then there is a simple 1-labeled quantum graph \(f \neq 0\) such that \(t^1(f, W) = 0\).
7.2 Infinite rank

We define the rank of a graphon \( W \) as its rank as a kernel operator. In other words, the rank of \( W \) is the least nonnegative integer \( r \) such that there are measurable functions \( w_i : [0,1] \to \mathbb{R} \) and reals \( \lambda_i \) (\( i = 1, \ldots, r \)) such that

\[
W(x, y) = \sum_{k=1}^{r} \lambda_k w_k(x)w_k(y)
\]  

(24)

almost everywhere. If no such integer \( r \) exists, then we say that \( W \) has infinite rank.

**Theorem 7.6** If \( W \) has finite rank, then for every finite list \( F_1, \ldots, F_m \) of simple graphs there is a stepfunction \( U \) such that

\[
t(F_i, U) = t(F_i, W) \quad (i = 1, \ldots, m).
\]

**Proof.** We know that \( W \) has a decomposition \([23]\). Fix a simple graph \( F = (V, E) \). For a map \( \varphi : E \to [r] \), \( i \in V \), and \( e \in E \), let \( d_i(\varphi, i) \) denote the number of edges \( e \in E \) incident with \( i \) for which \( \varphi(e) = t \), and set \( \lambda_{\varphi} = \prod_{ij \in E} \lambda_{\varphi(ij)} \). Then

\[
t(F, W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j) \, dx = \int_{[0,1]^V} \prod_{ij \in E} \left( \sum_{k=1}^{r} \lambda_k w_k(x_i)w_k(x_j) \right) \, dx
\]

\[
= \int_{[0,1]^V} \lambda_{\varphi} \prod_{i \in E} w_{\varphi(ij)}(x_i)w_{\varphi(ij)}(x_j) \, dx
\]

\[
= \int_{[0,1]^V} \sum_{\varphi \in [r]^E} \lambda_{\varphi} \prod_{i \in V} \prod_{t \in [r]} w_t(x_i)^{d_t(\varphi, i)} \, dx = \sum_{\varphi \in [r]^E} \lambda_{\varphi} \prod_{i \in V} \int_0^1 \prod_{t \in [r]} w_t(y)^{d_t(\varphi, i)} \, dy
\]

\[
= \sum_{\varphi \in [r]^E} \lambda_{\varphi} \prod_{i \in V} M(w, d(\varphi, i)).
\]

So if \( (u_1, \ldots, u_r) \) is another set of functions that satisfy

\[
M(u, d(\varphi, i)) = M(w, d(\varphi, i))
\]  

(25)

for every \( 1 \leq j \leq m \), \( i \in V(F_j) \) and \( \varphi : V(F_j) \to [r] \), then the function

\[
U = \sum_{t=1}^{r} \lambda_t u_t(x)u_t(y)
\]

satisfies \( t(F_j, U) = t(F_j, W) \) for all \( j = 1, \ldots, m \). By Theorem \([22]\) there is a system of functions \( u \) satisfying \([27]\) which are stepfunctions, and then \( U \) is also a stepfunction. \( \square \)

**Corollary 7.7** Every finitely forcible graphon is either a stepfunction or has infinite rank.

In view of Theorem \([5,1]\) the following corollary of this theorem may be surprising:

**Corollary 7.8** Assume that \( W \in \mathcal{W}_0 \) can be expressed as a non-constant polynomial in \( x \) and \( y \). Then \( W \) is not finitely forcible.
8 Open Problems and further directions

It does not seem easy to characterize finitely forcible functions. Let us offer a few conjectures. The next question might be easy but the examples and theorems in the present paper don’t answer it.

**Question 1** Is there a non-constant continuous (or smooth) function on $[0,1]^2$ which is finitely forcible?

As we have seen, the simplest candidates, namely polynomial functions don’t work.

We believe that in Theorem 5.1, the assumption that $p$ is monotone can be omitted:

**Conjecture 2** For every symmetric 2-variable polynomial $p$, the function $1_{p(x,y) \geq 0}$ is finitely forcible in $\mathcal{W}$.

We note that using ad hoc tricks, the proof given in Section 5.1 can be extended to some non-monotone polynomials, for example, to $(1/2 - x - y)(3/2 - x - y)$.

We can try to generalize the results of Section 5.1 to more variables. Here is an interesting special case:

**Question 3** Is the following graphon finitely forcible: the underlying probability space is the uniform distribution on the surface of the unit sphere $S^2$, and $W(x,y) = 1$ if $x$ and $y$ are closer than $90^\circ$, and $W(x,y) = 0$ otherwise?

It is not clear whether the two notions of forcibility we have considered are really different.

**Question 4** If a function $W \in \mathcal{W}_0$ is finitely forcible in $\mathcal{W}_0$, is it also finitely forcible in $\mathcal{W}$?

We don’t know too much about algebraic operations which generate new forcible functions. For example, it is unreasonable to expect that the sum of two forcible functions is forcible, since the sum depends on the concrete representation of the graphons (not just on their weak isomorphism types). However the next question is natural.

**Question 5** Is the tensor product $U \otimes W$ of two finitely forcible graphons $U$ and $W$ forcible?

Corollary 6.15 suggests the following problem:

**Question 6** For which finite graphs $G$ is the infinite lexicographic power of $G$ finitely forcible?

Our motivation for the study of finitely forcible graphons was to understand the structure of extremal graphs. This would be fully justified by the following conjecture:

**Conjecture 7** If a finite set of constraints of the form $t(F_i, W) = a_i$ ($i = 1, \ldots, k$) is satisfied by some graphon, then it is satisfied by a finitely forcible graphon. This conjecture would imply the (imprecise) fact that *every extremal graph problem has a finitely forcible solution.*
The topology of the set $T(W)$ introduced in Section 2.4 gives rise to some interesting problems. It is easy to see that $R(W) \cap T(W)$ is dense in $T(W)$ (in the topology of $L^1[0,1]$, and if two graphons $W$ and $U$ are weakly isomorphic then $T(U)$ is homeomorphic to $T(W)$.

Surprisingly, in each of the finitely forcible examples of this paper $T(W)$ is a compact topological space. For positive supports of polynomials, $T(W)$ is equivalent with the interval $[0,1]$. The topology of the regular CR-graphon corresponding to the binary tree is the Cantor set $\{0,1\}^\infty$. The examples constructed in Section 6.4 correspond to the one point compactification of the natural numbers.

**Conjecture 8** If $W$ is finitely forcible in $\mathcal{W}_0$ then $T(W)$ is a compact space. (We can’t even prove that $T(W)$ is locally compact.)

It seems that for forcible graphons are in some sense “finite dimensional”.

**Conjecture 9** If $W$ is finitely forcible then $T(W)$ is finite dimensional. (We intentionally do not specify which notion of dimension is meant here—a result concerning any variant would be interesting.) Note that Corollary 4 implies that the linear hull of $T(W)$ is infinite dimensional unless $T(W)$ is a finite set.

In our examples $T(W)$ is either 0-dimensional or 1-dimensional. This is probably due to the fact that we have only found very simple examples.

**Question 10** Is there a finitely forcible graphon $W$ such that $T(W)$ is homeomorphic with $[0,1]^2$? (A positive answer would follow from Conjecture 9.)

One can also consider a more direct notion of dimension. We define the *dimension of the graphon* $W$ as the infimum of all $c > 0$ such that for every $\varepsilon > 0$ there is a stepfunction $W_\varepsilon$ with $O((1/\varepsilon)^c)$ steps such that $\|W - W_\varepsilon\|_\square \leq \varepsilon$. It was shown in [12] that the dimensions $W$ and $T(W \circ W)$ are related.

The dimension of $W$ can be described in terms of the number of classes in weak Szemerédi partitions (introduced by Freeze and Kannan [9]). So a positive answer to Conjectures 7 and 9 would imply that extremal graph problems have solution with efficient (polynomial-size) weak Szemerédi partitions. This could explain (in a weak sense) why Szemerédi partitions are so important in extremal graph theory.

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