On extremal index of max-stable random fields

Enkelejd Hashorva

University of Lausanne, Chamberonne 1015 Lausanne, Switzerland
(e-mail: enkelejd.hashorva@unil.ch)

Received January 2, 2020; revised August 31, 2020

Abstract. For a given stationary max-stable random field $X(t)$, $t \in \mathbb{Z}^d$, the corresponding generalized Pickands constant coincides with the classical extremal index $\theta_X \in [0, 1]$. In this contribution, we discuss necessary and sufficient conditions for $\theta_X$ to be 0, positive, or equal to 1 and also show that $\theta_X$ is equal to the so-called block extremal index. Further, we consider some general functional indices of $X$ and prove that for a large class of functionals they coincide with $\theta_X$.

MSC: 60G15, 60G70

Keywords: max-stable random fields, Brown–Resnick random fields, Pickands constants, classical extremal index, block extremal index, functional index

1 Introduction

The connection between Pickands constant and extremal index of stationary max-stable Brown–Resnick random fields (r.f.) has been initially pointed out in [16]. Calculation of Pickands constants for a general stationary max-stable r.f. $X(t)$, $t \in \mathbb{Z}^d$, has been later dealt with in [24]. Previous investigations concerned with the calculation of extremal index in the context of max-stable processes are [8, 10, 20, 46]. Recent research in [2, 25, 44, 50] has shown, contrary to the prevailing intuitions, that there are certain subtleties (if $d > 1$) when dealing with stationary multivariate regularly varying r.f.s (see, e.g., [47] for the definition) and the calculation of their extremal indices. Influenced by the findings of [7], several formulas for extremal indices of stationary regularly varying time series have appeared in the literature; see, for example, [34] and the references therein. Various (less known) formulas have been discovered also for Pickands constants in contributions unrelated to time series modeling, for instance, in sequential analysis and statistical applications [41, 42] and extremes of random fields [28, 51], just to mention a few. For large classes of Gaussian r.f.s, extremal indices have been discussed in [12, 23, 43]; see also [4, 48] for non-Gaussian cases and related results.

Without loss of generality, we focus on the class of max-stable r.f.s with Fréchet marginals. Since these are limiting r.f.s (see, e.g., [18]), our formulas for their extremal indices are valid (with obvious modifications) also for the candidate extremal index of more general stationary regularly varying r.f.s (see [34] for recent findings). Studying max-stable r.f.s, instead of these more general r.f.s, is also justified by Lemma 2 stated in Section 2 and Remark 2(iii).

* Partially supported by SNSF grants 200021-175752/1 and 200021-196888.
In view of the well-known de Haan characterization given in [14], the r.f. $X$ with nondegenerate marginal distributions corresponds to some nonnegative spectral r.f. $Z(t), t \in \mathbb{Z}^d$, having the following representation (in distribution):
\[
X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Z_i(t), \quad t \in \mathbb{Z}^d,
\] (1.1)
where $\Gamma_i = \sum_{k=1}^i Q_k$ with $Q_k, k \geq 1$, unit exponential random variables (r.v.s) independent of $Z_i$, which are independent copies of $Z$.

Clearly, $Z$ is not unique since also $\tilde{Z}(t) = RZ(t), t \in \mathbb{Z}^d$, is a spectral r.f. for $X$, provided that $R$ is a nonnegative r.v. independent of $Z$ such that $E\{R^\alpha\} = 1$. Note that if for some $h \in \mathbb{Z}^d$, we have $Z(h) = 1$ almost surely, then in view of Balkema’s lemma [15, Lemma 4.1], any spectral r.f. $\tilde{Z}$ of $X$ has the same law as $Z$. We assume without loss of generality that for some $\alpha \in (0, \infty)$,
\[
P\left\{ \max_{t \in \mathbb{Z}^d} Z(t) > 0 \right\} = 1, \quad E\{Z^\alpha(t)\} = 1, \quad t \in \mathbb{Z}^d.
\] (1.2)

Lemma A1 in Appendix shows how to construct a spectral r.f. $Z$ such that the first assumption in (1.2) holds. Note that $E\{Z^\alpha(t)\} = 1$ implies that $X(t)$ has the $\alpha$-Fréchet distribution function $e^{-x^{-\alpha}}, x > 0$. This is no restriction since we are interested in stationary max-stable r.f.s. As in [24], define the Pickands constant (when the limit exists) with respect to the spectral r.f. $Z$ by
\[
\mathcal{H} = \lim_{n \to \infty} \frac{1}{n^d} E\left\{ \max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z^\alpha(t) \right\} \leq \lim_{n \to \infty} \frac{1}{n^d} \sum_{t \in [0,n]^d \cap \mathbb{Z}^d} E\{Z^\alpha(t)\} \leq 1.
\]
Since the finite-dimensional distributions (fidis) of $X$ can be calculated explicitly (see (6.1)) if $\mathcal{H}$ exists, we have
\[
P\left\{ \max_{t \in [0,n]^d \cap \mathbb{Z}^d} X(t) \leq n^d x \right\} = e^{-(1/n^d)}E\{\max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z^\alpha(t)/x^n\} \to e^{-\mathcal{H}/x^\alpha}
\] (1.3)
as $n \to \infty$ for all $x > 0$.

As argued in [16] and [11, 24], the subadditivity of maximum functional implies that $\mathcal{H}$ is well-defined and finite, provided that $X$ is stationary. Consequently, in view of (1.3), the extremal index (or the classical extremal index by the terminology of [50]) of the stationary max-stable r.f. $X$ (denoted further by $\theta_X$) always exists, does not depend on the particular spectral r.f. $Z$ but on the law of the r.f. $X$, and is given by
\[
\theta_X = \mathcal{H} \in [0, 1].
\] (1.4)
In the particular case
\[
X(t) = V_t, \quad t \in \mathbb{Z}^d,
\] (1.5)
where $V_t$ are independent $\alpha$-Fréchet r.v.s, we have $\theta_X = 1$. We will show that this is the only max-stable r.f. with unit Fréchet marginals satisfying $\theta_X = 1$. Using this fact and Lemma 2, we can construct a spectral r.f. $Z$ for $X$; see Remark 4(iii).

Hereafter we will assume for simplicity that the max-stable r.f. $X$ has unit Fréchet marginal distributions, that is, we will consider the case $\alpha = 1$.

If the spectral r.f. $Z$ is not easy to determine or $X(t), t \in \mathbb{Z}^d$, is stationary but not max-stable, then commonly the block extremal index (denoted further by $\tilde{\theta}_X$) is utilized in various applications related to extreme value analysis. Assuming for simplicity that $X$ has unit Fréchet marginals, it is defined as (see [22,50])
\[
\tilde{\theta}_X := \lim_{n \to \infty} \frac{P\left\{ \max_{0 \leq i \leq n^d} X(i) > n\tau \right\}}{\prod_{j=1}^d n^{\tau_j}} P\{X(0) > n\tau\}.
\] (1.6)
for \( \tau > 0 \) and any sequence \( r_n \in \mathbb{Z}^d, n \geq 1 \), with nondecreasing integer-valued components \( r_{n,j}, j \leq d \), such that \( \lim_{n \to \infty} r_{n,j} = \lim_{n \to \infty} n/r_{n,j} = \infty \) for all \( j \leq d \). In (1.6), \( i \leq r_n \) is interpreted componentwise, that is, \( i_j \leq r_{n,j} \) for all \( j \leq d \) components of \( i \) and \( r_n \).

Next, we define the functional indices \( \theta_{X,F} \) of \( X \) as

\[
\theta_{X,F} = E\{ Z(0) F(Z) \} \in [0, 1],
\]

where \( F : E \mapsto [0, 1] \) is a measurable functional with respect to the product \( \sigma \)-field \( \mathcal{E} \) on \( E := [0, \infty)^{\mathbb{Z}_d} \).

As mentioned before, different choices of \( Z \) for \( X \) are possible. To make the definition of \( \theta_{X,F} \) independent of the choice of \( Z \) and thus only dependent on the law of \( X \), we will also require that \( F \) is 0-homogeneous, that is, \( F(cf) = F(f) \) for any \( c > 0 \) and \( f \in E \). Indeed, under this assumption, we have that

\[
\theta_{X,F} = E\{ Z(0) F\left( \frac{Z}{Z(0)} \right) \} = E\{ F(\Theta_0) \},
\]

where the r.f. \( \Theta_h \) is defined by (hereafter \( I(\cdot) \) denotes the indicator function)

\[
P\{ \Theta_h \in A \} = E\left\{ Z(h) I\left( \frac{Z}{Z(h)} \in A \right) \right\}, \quad A \in \mathcal{E}.
\]

It is known that for any \( h \in \mathbb{Z}^d \), the law of \( \Theta_h \) does not depend on the particular choice of the spectral r.f. \( Z \) and can be directly determined by \( X \). In the case that for a spectral r.f. \( Z \) of \( X \), we have that \( Z(h) > 0 \) almost surely, which follows from Balkema’s lemma. The proof for the general case follows from [24, Lemma A.1] or from [49, Thm. 1.1] and [30, Thm. 2]. Consequently, the functional index \( \theta_{X,F} \) depends only on the law of \( X \). Note that for the definition of \( \theta_{X,F} \), no stationarity of \( X \) is assumed.

It is well known that a max-stable r.f. \( X \) with Fréchet marginals is a multivariate regularly varying r.f. For general multivariate regularly varying r.f.s, which are not max-stable, there is no spectral process \( Z \) as in our case of max-stable \( X \), and therefore the r.f.s \( \Theta_h, h \in \mathbb{Z}^d \), are defined via a conditional limit; see, e.g., [18, 39] and (2.1). The key advantage in the framework of max-stable r.f.s is that \( \Theta_h \) is directly obtained by tilting a given spectral r.f. \( Z \).

At this point, two natural questions arise for a given stationary max-stable r.f. \( X \):

**Question 1.** What is the relation between \( \theta_X \) and \( \tilde{\theta}_X \)?

**Question 2.** For what \( F \) the functional index \( \theta_{X,F} \) equals \( \theta_X \)?

In this contribution, we show that we simply have \( \theta_X = \tilde{\theta}_X \) and then describe a large class of functionals \( F \) such that \( \theta_X = \theta_{F,X} \). Further, we consider in some detail the cases \( \theta_X = 0 \) and \( \theta_X = 1 \).

Organization of the rest of the paper is as follows. In the next section, we discuss some basic properties of the r.f.s \( \Theta_h, h \in \mathbb{Z}^d \), and then show how to construct a stationary max-stable r.f. \( X \) from a given r.f. \( \Theta^* \), which is in turn necessarily equal to \( \Theta_0 \). In Section 3, we claim that \( \theta_X = \Theta_0 \) for any stationary max-stable r.f. \( X \). Additionally, we give equivalent conditions that guarantee \( \theta_X > 0 \) or \( \theta_X = 0 \) and then present several formulas for \( \theta_X \). Section 4 is concerned with the anticlustering condition, whereas in Section 5, we give some examples. All the proofs are relegated to Section 6, which is followed by the Appendix.

## 2 Preliminaries

Unless otherwise specified, we further consider a max-stable r.f. \( X(t), t \in \mathbb{Z}^d \), as in the Introduction with spectral r.f. \( Z \) such that \( E\{ Z(t) \} = 1, t \in \mathbb{Z}^d \). Hence \( X(t) \) has unit Fréchet distribution \( e^{-1/x}, x > 0 \). We first discuss the case where \( X \) is nonstationary.
2.1 General max-stable $X$

The importance of the r.f.s $\Theta_h, h \in \mathbb{Z}^d$, defined in (1.7) relates to the following conditional convergence results. Namely, in view of [24, Lemma 2.1(A.1), Rem. 6.4] or [18, Lemma 3.5], we have the convergence in distribution

$$
\frac{X(t)}{X(h)} \bigg| (X(h) > u) \xrightarrow{d} \Theta_h(t), \quad t \in \mathbb{Z}^d,
$$

(2.1)

$$
u^{-1}X(t) \bigg| (X(h) > u) \xrightarrow{d} Y_h(t), \quad t \in \mathbb{Z}^d,
$$

(2.2)

as $u \to \infty$ in the product topology of $E = [0, \infty)^{2d}$, where $\Theta_h$ is defined in (1.7), and $Y_h(t) = R\Theta_h(t)$, $t \in \mathbb{Z}^d$.

with $R$ an $\alpha$-Pareto r.v. with survival function $x^{-\alpha}, x \geq 1$, independent of any other random element (recall that we consider $\alpha = 1$ for simplicity).

Remark 1. For the case of the stationary Brown–Resnick model, (2.4) is stated in [50, Prop. 6.1] for $h = 0$.

2.2 Stationary max-stable $X$

In view of [24, Thm. 6.9], the max-stable r.f. $X(t), t \in \mathbb{Z}^d$, with unit Fréchet marginals is stationary if and only if

$$
E\{Z(h)F(Z)\} = E\{Z(0)F(B^hZ)\}, \quad h \in \mathbb{Z}^d,
$$

(2.5)

for any measurable 0-homogeneous function $F : E \mapsto [0, \infty]$. Here $B$ is the shift-operator: $B^hZ(\cdot) = Z(\cdot - h), h \in \mathbb{Z}^d$. Note that for the stationary Brown–Resnick model, the claim in (2.5) is first formulated in [16, Lemma 5.2].

For notational simplicity, we will omit the subscript 0 and write simply $\Theta$ and $Y$ instead of $\Theta_0$ and $Y_0$, respectively; in our notation the origin of $\mathbb{R}^k, k \in \mathbb{N}$, is denoted by 0.

In view of [24, Thm. 4.3], condition (2.5) is equivalent to the equality in law $\Theta_h \xrightarrow{d} B^h\Theta$ for all $h \in \mathbb{Z}^d$.

Yet another equivalent formulation of condition (2.5) stated for the r.f. $\Theta$ is

$$
E\{\Theta(h)F(\Theta)\} = E\{F(B^h\Theta)I(B^h\Theta(0) \neq 0)\}, \quad h \in \mathbb{Z}^d,
$$

(2.6)

for all measurable functionals $F$ as before; see, for example, [2, 18].
We note in passing that with the same arguments as in [18], we can show that (2.6) is equivalent to the so-called time-change formula derived in [2] for multivariate regularly varying r.f.s.

Next, since for stationary \( X \), we have (2.2), in view of [2,18], \( X \) is a multivariate regularly varying r.f., and \( Y \) is the so-called tail r.f. of \( X \), whereas \( \Theta \) is the so-called spectral tail r.f. Therefore for a stationary max-stable r.f. \( X \), the r.f. \( \Theta \) defined in (1.7) is simply the spectral tail r.f. of \( X \).

Adopting the terminology of [27] for stationary max-stable r.f.s \( X \), we refer to their spectral r.f.s \( Z \) as Brown–Resnick stationary (abbreviated as BRs) r.f.s.

From \( Z \) we can easily define the spectral tail r.f. \( \Theta \). Moreover, as mentioned in (2.3), we simply have \( \Theta \overset{d}{=} Z \) if \( Z(0) = 1 \) almost surely. The key properties of BRs r.f.s \( Z \) and spectral tail r.f.s \( \Theta \) are the TSF (2.5) and identity (2.6), respectively. This is revealed by our next result, which shows how to construct a BRs r.f. \( Z \) from a given r.f. \( \Theta^* \) that satisfies (2.6) and \( \Theta^*(0) = 1 \) almost surely, extending thus [26, Thm. 4.2] to r.f.s.

Let further
\[
I_{fm}(p \cdot Y) = \min \left\{ i \in \mathbb{Z}^d : \max_{j \in \mathbb{Z}^d} |p_j Y(j)| = |p_i Y(i)| \right\},
\]
where \( p_j \)'s are nonnegative numbers such that \( \sum_{j \in \mathbb{Z}^d} p_j = 1 \) (recall that \( \alpha = 1 \) in our case).

Hereafter \( N \) is a r.v. independent of any other random element such that \( P\{N = j\} = p_j > 0, \ j \in \mathbb{Z}^d \). Further, both min and max are defined with respect to a translation-invariant order on \( \mathbb{Z}^d \); see [2] for the definition.

**Lemma 2.** If \( Y(t) = R\Theta^*(t), \ t \in \mathbb{Z}^d \), with \( R \) a unit Pareto r.v. independent of \( \Theta^* \) that satisfies (2.6) and \( \Theta^*(0) = 1 \) almost surely, then \( Z_N \) given by
\[
Z_N(t) = \frac{B^N Y(t)}{\max_{i \in \mathbb{Z}^d} p_i B^N Y(i)} I(I_{fm}(p \cdot B^N Y) = N), \quad t \in \mathbb{Z}^d,
\]
is a spectral r.f. of some stationary max-stable r.f. \( X(t), \ t \in \mathbb{Z}^d \), with unit Fréchet marginal. Moreover, the spectral tail r.f. \( \Theta \) of \( X \) has the same law as \( \Theta^* \).

**Remark 2.**

(i) When \( \alpha \neq 1 \), this construction is still valid if the denominator therein is replaced by \( (\max_{i \in \mathbb{Z}^d} p_i B^N Y(i))^{1/\alpha} \). In fact, (2.7) is a minor modification of the construction given in [18, Prop. 2.12]. The other known constructions in [18, 26, 34] can be easily extended for the case \( d > 1 \); we omit the details.

(ii) An \( \mathbb{R}^q \)-valued r.f. \( \Theta(t), \ t \in \mathbb{Z}^d \), is called a spectral tail r.f. if it satisfies (2.6) where \( \Theta(h) \) and \( \Theta(-h) \) are substituted by \( \|\Theta(h)\| \) and \( \|\Theta(-h)\| \) with \( \|\cdot\| \) a norm on \( \mathbb{R}^q \). \( F \) is redefined accordingly, and \( P\{\|\Theta(0)\| = 1\} = 1 \); see, for example, [2,3,34]. For such a r.f., a BRs r.f. \( Z_N \) can be determined as in (2.7) by changing \( \sum_{i \in \mathbb{Z}^d} p_i B^N Y(t) \) to \( \sum_{i \in \mathbb{Z}^d} p_i B^N \|Y(t)\| \) and instead of \( \max_{i \in \mathbb{Z}^d} p_i B^N Y(t) \) and \( p \cdot B^N \) putting \( \max_{i \in \mathbb{Z}^d} p_i B^N \|Y(t)\| \) and \( p \cdot B^N \|Y(t)\| \), respectively (with \( Y(t) = R\Theta(t) \) and \( R \) a unit Pareto r.v. independent of \( \Theta \)).

3 Classical, block, and functional indices

As mentioned in the Introduction, the classical extremal index \( \theta_X \) of a stationary max-stable r.f. \( X \) always exists. We first show that it is equal to the block extremal index \( \tilde{\theta}_X \) defined in (1.6) and then answer the question of when \( \theta_X = 0 \). This is already known for \( d = 1 \) [11]. Our main finding in Theorem 1 gives several formulas for \( \theta_X \). The next result is a minor generalization of the case \( d = 1 \) stated in [10].

**Lemma 3.** If \( X(t), \ t \in \mathbb{Z}^d \), is a stationary max-stable r.f., then \( \theta_X = \tilde{\theta}_X \).

Now we slightly modify the definition of anchoring maps introduced in [2]. Write next \( \mathbb{Z}^d \setminus \{\infty\} \) and recall that \( E = [0, \infty) \mathbb{Z}^d \) is equipped with the product \( \sigma \)-field \( \mathcal{E} \).
DEFINITION 1. We call a measurable map $I : E \mapsto \mathbb{Z}^d$ anchoring if for $O = \{ f \in E : I(f) \in \mathbb{Z}^d \}$, the following conditions are satisfied for all $f \in O, i \in \mathbb{Z}^d$:

(i) $I(f) = i$ implies $f(i) \geq \min(f(0), 1)$;
(ii) $I(f) = (B^1 f) - i$.

As in [2], we define two important anchoring maps, which are specified with respect to a translation-invariant order on $\mathbb{Z}^d$. In particular, the minimum and maximum below are with respect to such an order. An instance of a translation-invariant order is the lexicographical one. Hereafter $S(f) = \sum_{t \in \mathbb{Z}^d} f^\alpha(t)$ for any $f \in E$. Note that apart from Section 5.2, we have considered for simplicity only the case $\alpha = 1$.

Example 1. Let the nonempty set $O \in \mathcal{E}$ be given by

$$O = \{ f \in E : S(f) < \infty, \max_{i \in \mathbb{Z}^d} f(i) > 0 \}$$

and define the first maximum functional

$$I_{fm}(f) = \min\left( j \in \mathbb{Z}^d : f(j) = \max_{i \in \mathbb{Z}^d} f(i) \right), \quad f \in O,$$

where $I_{fm}(f) = \infty$ if $f \notin O$. Clearly, $I_{fm}(f)$ is finite for $f \in O$, and condition (i) holds by the definition, whereas condition (ii) follows by the invariance (in the sense of [50]) of the translation-invariant order.

The first and last maximum functionals are important since they are both anchoring and $0$-homogeneous. Moreover, for a stationary max-stable r.f. $X(t), t \in \mathbb{Z}^d$, with spectral r.f. $\Theta$ and Fréchet marginals $\Phi(x) = e^{-1/x^\alpha}, x > 0$, we have that the law of $X$ is specified by $I_{fm}$ and $\Theta$ as follows:

$$-\ln P\{ X(i) \leq x_i, i \in \mathbb{Z}^d \} = \sum_{i \in \mathbb{Z}^d} \frac{1}{x_i} P\{ I_{fm}\left( \frac{\Theta}{B^{-i} x} \right) = 0 \}$$

(3.1)

for any $x = (x_i)_{i \in \mathbb{Z}^d}$ with finitely many positive components and the rest equal to $\infty$; here $\Theta/(B^{-i} x) = (\Theta(j)/x_{j+i})_{j \in \mathbb{Z}^d}$. The proof of (3.1) is displayed in the Appendix; see also [24, Eq. (6.10)]. Note in passing that (3.1) shows that the law of $X$ is uniquely determined by $\Theta$.

Example 2. Define the first exceedance functional by

$$I_{fe}(f) = \min\left( j \in \mathbb{Z}^d : f(j) > 1 \right), \quad f \in O,$$

and set $I_{fe}(f) = \infty$ if $f \notin O$, where

$$O = \{ f \in E : S(f) < \infty, \max_{t \in \mathbb{Z}^d} f(t) > 1 \} \in \mathcal{E}.$$

Clearly, $I_{fe}(f)$ for $f \in O$ is finite, and (i) holds. Moreover, since $I_{fe}(f), f \in O$, is determined by a finite number of points in a neighborhood of 0, $I_{fe}$ is measurable. Again, condition (ii) is implied by the translation invariance of the chosen order on $\mathbb{Z}^d$.

We call a measurable map $F : E \mapsto [0, \infty]$ shift-invariant if $F(B^h f) = F(f), h \in \mathbb{Z}^d, f \in E$.

**Lemma 4.** Let $\Theta(t), t \in \mathbb{R}^d$, be a real-valued r.f. satisfying (2.6) with $\Theta(0) = 1$ almost surely. If $R$ is a unit Pareto r.v. independent of $\Theta$, then for any two anchoring maps $I, I'$ and any shift-invariant map $F$, we have (setting $Y(t) = R\Theta(t), t \in \mathbb{Z}^d$)

$$P\{ I(Y) = 0, I'(Y) \in \mathbb{Z}^d, F(Y) < \infty \} = P\{ I'(Y) = 0, I(Y) \in \mathbb{Z}^d, F(Y) < \infty \}.$$  

(3.2)

Moreover, $P\{ I(Y) = 0, F(Y) < \infty \} = 0$ is equivalent to $P\{ I(Y) \in \mathbb{Z}^d, F(Y) \} < \infty = 0$. 

Remark 3. If $\mathcal{I}(Y), \mathcal{I}'(Y)$ are almost surely in $\mathbb{Z}^d$, then (3.2) boils down to $P\{\mathcal{I}'(Y) = 0\} = P\{\mathcal{I}(Y) = 0\}$, which is already shown in [2, Lemma 3.5]. In general, $\mathcal{I}(Y)$ might not be finite almost surely. Therefore the event $\{S(Y) < \infty\}$ enters in our calculations. In fact, under the conditions of Theorem 1 on both $\mathcal{I}, \mathcal{I}'$, we will show that

$$\theta_X = P\{\mathcal{I}'(Y) = 0, S(Y) < \infty\} = P\{\mathcal{I}(Y) = 0, S(Y) < \infty\}.$$ 

Hereafter we consider anchoring maps $\mathcal{I} : E \to \mathbb{Z}^d$ such that

$$P\{\mathcal{I}(Y) \in \mathbb{Z}^d, S(Y) < \infty\} = P\{S(Y) < \infty\},$$

(3.3)

which is in particular valid for both first (last) maximum and first (last) exceedance functionals.

Lemma 5. If $X(t), t \in \mathbb{Z}^d$, is a stationary max-stable r.f. with some spectral r.f. $Z$ and spectral tail r.f. $\Theta$, then $\theta_X = 0$ if and only if $P\{S(\Theta) = \infty\} = P\{S(Z) = \infty\} = 1$. Further, if the anchoring map $\mathcal{I}$ satisfies (3.3), then $\theta_X = 0$ is equivalent to $P\{\mathcal{I}(Y) = 0, S(Y) < \infty\} = 0$.

Since the first and last maximum functionals are 0-homogeneous and finite on the set $O = \{ f \in E : S(f) < \infty, \max_{i \in \mathbb{Z}^d} f(i) > 0\}$, we have that $P\{S(Z) = \infty\} = 1$ is equivalent to $P\{\mathcal{I}_{fm}(Z) \notin \mathbb{Z}^d\} = 1$, and the same also holds for the last maximum functional.

In view of Lemmas 5 and A2 and [19], $\theta_X = 0$ is equivalent to $P\{S(Z) = \infty\} = 1$. Further, we have the following equivalent statements (where $\|\cdot\|$ is a norm on $\mathbb{R}^d$):

(A1) $Z(t) \to 0$ almost surely as $\|t\| \to \infty$;

(A2) $\Theta(t) \to 0$ almost surely as $\|t\| \to \infty$;

(A3) $S(Z) < \infty$ almost surely;

(A4) $S(\Theta) < \infty$ almost surely.

The equivalence of (A1) and (A3) is shown in [19], whereas the equivalence of (A1) and (A2) is a direct consequence of Lemma A2, and similarly for the equivalence of (A3) and (A4). The equivalence of (A2) and (A4) follows from [26] and [50]. Note further that $Y(t) = \bar{R}(\Theta(t) \to 0$ almost surely as $\|t\| \to \infty$ is equivalent to (A2), and $S(Y) = \bar{R}S(\Theta) < \infty$ almost surely is equivalent to (A4).

We next state the main result of this section. Define $B(Y) = \sum_{t \in \mathbb{Z}^d} I(Y(t) > 1)$ and interpret $0 : 0$ and $\infty : \infty$ as 0.

Theorem 1. Let $\mathcal{I}, X$ be as in Lemma 5. If $\mathcal{I}$ satisfies (3.3) and $P\{S(\Theta) < \infty\} > 0$, then

$$\theta_X = P\{\mathcal{I}(Y) = 0, S(Y) < \infty\} = P\{\mathcal{I}_{fe}(Y) = 0\} = P\{\mathcal{I}_{fm}(\Theta) = 0\}$$

(3.4)

$$= E\left\{ \frac{\max_{t \in \mathbb{Z}^d} \Theta(t)}{\sum_{t \in \mathbb{Z}^d} \Theta(t)} \right\}$$

(3.5)

$$= E\left\{ \frac{1}{B(Y)} \right\},$$

(3.6)

where (3.4) holds if further $\mathcal{I}$ is 0-homogeneous. Moreover, $\{B(Y) < \infty\} = \{S(Y) < \infty\}$ almost surely, and, in particular, $\theta_X = 1$ if and only if $\Theta(i) = 0$ almost surely for all $i \in \mathbb{Z}^d$, $i \neq 0$.

Remark 4.

(i) Let $\Theta(t) = \Theta_1(t_1)\Theta_2(t_2)$, $t_1 \in \mathbb{Z}^k$, $t_2 \in \mathbb{Z}^m$, $t = (t_1, t_2) \in \mathbb{Z}^d$, with independent r.f.s $\Theta_1, \Theta_2$ satisfying (2.6) and $P\{\Theta(0) = 1\} = 1$, $i = 1, 2$. Then (3.5) implies that $\theta_X = \theta_{X_1}\theta_{X_2}$, where $X$ and $X_i, i = 1, 2$, are stationary max-stable r.f.s with spectral r.f.s $\Theta$ and $\Theta_i$, $i = 1, 2$, respectively.

Lith. Math. J., 61(2):217–238, 2021.
(ii) For \( d = 1 \) and \( \theta_X = 1 \), the claim that \( \Theta(i) = 0, i \neq 0 \), in Theorem 1 also follows from [29, Prop. 2.2(ii)].

(iii) Since \( \Theta \) uniquely defines \( X \), Theorem 1 implies that the only stationary max-stable r.f. \( X \) such that \( \theta_X = 1 \) is that given by (1.5). In view of (2.1), \( \Theta(i) = 0, i \neq 0 \), and hence by (2.7) \( Z_N(t) = (1/p_t)I(N = t), t \in \mathbb{Z}^d \), is a spectral r.f. for \( X \) specified in (1.5), where \( N \) is a discrete r.v. with positive probability mass function \( p_t > 0, t \in \mathbb{Z}^d \).

(iv) Taking \( F(f) = I(I(f) = 0, S(f) < \infty) \), (3.4) implies \( \theta_X = \theta_{X,F} \) under the further assumption that \( \mathcal{I} \) is a 0-homogeneous functional satisfying (3.3).

(v) From the proof of Theorem 1 it follows that (3.6) holds without the assumption \( \mathbb{P}\{S(\Theta) < \infty\} > 0 \). Hence \( \theta_X = 0 \) if and only if \( B(Y) = \infty \) almost surely. Further, from Theorem 1 we have that (A1), (A2), (A3), and (A4) are equivalent to (A5): \( B(Y) < \infty \) almost surely.

(vi) Formula (3.5) initially appears as the extremal index in [37, 38] and as the Pickands constant in [17].

### 4 The anticlustering condition

Since stationary max-stable r.f.s with Fréchet marginals are multivariate regularly varying (for more details, see [2]) the classical extremal index of those r.f.s can be calculated using the results of [2] and [50]. In the framework of stationary multivariate regularly varying r.f.s the anticlustering condition of [7] plays a crucial role for the calculation of the extremal index. Considering a stationary max-stable r.f. \( X(t), t \in \mathbb{Z}^d \), with unit Fréchet marginals, in view of [2], the aforementioned condition reads as follows.

**Condition C**. There exists a nondecreasing sequence of positive integers \( r_n \to \infty \) as \( n \to \infty \) such that \( \lim_{n \to \infty} r_n^d / n = 0 \) and for all \( s > 0 \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left\{ \max_{m < \|t\| < r_n, t \in \mathbb{Z}^d} X(t) > ns \mid X(0) > ns \right\} = 0.
\]

The equivalence of Condition C and \( \mathbb{P}\{S(\Theta) < \infty\} = 1 \) in the case \( d = 1 \) is known [18]. The case \( d \geq 1 \) of Brown–Resnick model is dealt with in [50, Prop. 6.2]. Next, we show that this equivalence holds for a general stationary max-stable r.f. \( X \) with spectral tail r.f. \( \Theta \) and spectral r.f. \( Z \).

**Lemma 6.** The anticlustering Condition C for \( X \) is equivalent to (Ai), \( i = 1, \ldots, 5 \).

If \( \mathbb{P}\{S(\Theta) < \infty\} = 1 \) or, equivalently, Condition C holds, then by [2, Lemmas 3, 6] and [2, Prop. 5.2] for any anchoring map \( \mathcal{I} \),

\[
\theta_X = \mathbb{P}\{\mathcal{I}(Y) = 0\} = \mathbb{P}\{\mathcal{I}_{fm}(Y) = 0\} = \mathbb{P}\{\mathcal{I}_{fm}(\Theta) = 0\} \in (0, 1],
\]

provided that \( \mathbb{P}\{\mathcal{I}(Y) \in \mathbb{Z}^d\} = 1 \). In the particular case \( \mathcal{I} = \mathcal{I}_{fs} \) (as shown already in [2]),

\[
\theta_X = \mathbb{P}\left\{ \max_{0 < t} Y(t) \leq 1 \right\}.
\]

Here \( \prec \) denotes a translation-invariant order on \( \mathbb{Z}^d \).

**Remark 5.** Equality (4.2) is a well-known formula in the Gaussian setup and has appeared in numerous papers inspired by [1]. This special formula for the Gaussian setup is also referred to as Albin’s constant; see [17]. In the context of stationary regularly varying time series the same formula has appeared in [3].

Next, consider the case that Condition C does not hold, that is, \( p = \mathbb{P}\{S(\Theta) < \infty\} \in (0, 1) \), and define the r.f.s \( \Theta_1 = \Theta \mid (S(\Theta) < \infty) \) and \( \Theta_2 = \Theta \mid (S(\Theta) = \infty) \). In view of [19, Thm. 9, Prop. 10], for two
independent stationary max-stable r.f.s \( \eta_i(t), t \in \mathbb{Z}^d, i = 1, 2 \), with unit Fréchet marginals and corresponding spectral tail r.f.s equal in law to \( \Theta_i, i = 1, 2 \), we have that \( X \) has the same law as

\[
\max(p\eta_1(t), (1-p)\eta_2(t)), \quad t \in \mathbb{Z}^d.
\]

(4.3)

Since \( \eta_1 \) satisfies Condition C, by [50, Prop. 5.2], Lemma 3, (4.1), and Theorem 1 we have

\( \theta_X = pP\{\mathcal{I}_{f_m}(\Theta_1) = 0\} = p, \quad \theta_{\eta_1} \in (0, 1] \).

Alternatively, since by the stationarity of \( X \) we have that \( \theta_X \) exists and moreover \( \theta_{\eta_2} = 0 \), Lemma A5 implies that \( \theta_X = p\theta_{\eta_1} \). Consequently, we conclude that Condition C, Lemma A5, representation (4.3), together with the findings of [2], establish the validity of the first four expressions in Theorem 1.

We remark that from the above arguments, by (4.2) and Lemma 1 we obtain

\[
\theta_X = E\left\{ \max_{0 \leq t} \Theta(t) - \max_{0 \leq t} \Theta(t); \ S(\Theta) < \infty \right\} = E\left\{ \max_{0 \leq t} Z(t) - \max_{0 \leq t} Z(t); \ S(Z) < \infty \right\}.
\]

The first formula is already obtained for the Brown–Resnick model (see Section 5) in [50, Cor. 6.3] and for the case \( d = 1 \) in [10, Thm. 2.1].

## 5 Examples

We present some examples, starting first with the Brown–Resnick model. The second example and Lemma 2 show in particular how to construct stationary max-stable r.f.s starting from any \( \alpha \)-summable deterministic sequence. Then we discuss how to construct from some given r.f. a stationary max-stable r.f. \( X \) such that \( \theta_X \) equals a given constant.

### 5.1 Brown–Resnick model

Consider \( Z(t) = e^{W(t) - \sigma^2(t)/2}, t \in \mathbb{Z}^d \), with \( W(t), t \in \mathbb{Z}^d \), a centered Gaussian r.f. with variance function \( \sigma^2 \), which is not identically 0, and \( \sigma(0) = 0 \). Let \( X(t), t \in \mathbb{Z}^d \), denote a max-stable r.f. with spectral r.f. \( Z \). The case where \( W \) is a standard Brownian motion and \( d = 1 \) is investigated in [6], and therefore this construction is referred to as the Brown–Resnick model.

For any fixed \( h \in \mathbb{Z}^d \), the Gaussian r.f. (set \( \gamma(s,t) = \text{Var}(W(t) - W(s)), s,t \in \mathbb{Z}^d \))

\[
S_h(t) = W(t) - W(h) - \frac{\gamma(h,t)}{2}, \quad t \in \mathbb{Z}^d,
\]

is such that \( S_h(h) = 0 \) almost surely and has the variance function \( \sigma_h^2(t) = \gamma(h,t) \).

With the same arguments as in [24], it follows that \( Z_h(t) = e^{S_h(t)}, t \in \mathbb{Z}^d \), is also a spectral r.f. for \( X \) for any \( h \in \mathbb{Z}^d \). Since \( S_h(t), t \in \mathbb{Z}^d \), is a Gaussian r.f. with variance \( \text{Var}(W(t) - W(h)) = \gamma(t,h) \), the law of \( X \) depends only on \( \gamma(h,t) \) and not on \( \sigma^2 \). If we assume that \( W \) has stationary increments, then (2.5) implies that \( X \) is a stationary max-stable r.f. Since \( Z_h(h) = 1 \) for any \( h \in \mathbb{Z}^d \) almost surely, \( \Theta := \Theta_0 \) defined in (1.7) is simply given by \( \Theta(t) = Z(t), t \in \mathbb{Z}^d \), and hence, recalling that \( Y = R\Theta \),

\[
Y(t) = e^{\tilde{W}(t) + Q}, \quad \tilde{W}(t) = W(t) - \frac{\sigma^2(t)}{2}, \quad t \in \mathbb{Z}^d,
\]

where \( Q = \ln R \) is a unit exponential r.v. independent of \( W \).
For an $N(0, 1)$ r.v. $V$ with distribution $\Phi$ independent of $Q$ and all $c > 0$, $x \in \mathbb{R}$ (set $\Phi = 1 - \Phi$, $V_c = cV - c^2/2$)

$$
\mathbb{P}\{V_c + Q > x\} = \mathbb{P}\{V_c + Q > x, V_c > x\} + \mathbb{P}\{V_c + Q > x, V_c \leq x\}
= \mathbb{P}\{V_c > x\} + e^{-x} \mathbb{E}V^2 \mathbb{I}(V_c \leq x)
= \mathbb{P}\{V_c > x\} + e^{-x} \mathbb{P}\{cV \leq x - \frac{c^2}{2}\},
$$

where we used that the exponentially tilted r.v. $U$ defined by $\mathbb{P}\{U \leq x\} = \mathbb{E}V^2 \mathbb{I}(V_c \leq x)$, $x \in \mathbb{R}$, has the distribution $N(c^2/2, c^2)$; see, for example, [24, Lemma 7.1]. Consequently, for all $t \in \mathbb{Z}^d$ such that $c := \sigma(t) > 0$ and all $y > 0$,

$$
\mathbb{P}\{Y(t) \leq y\} = \Phi(c^{-1} \ln y + \frac{c}{2} - e^{-1/y} \Phi(c^{-1} \ln y - \frac{c}{2})
$$

which agrees with the claim of [50, Prop. 6.1], where the stationary case is considered.

Next, under the assumption that $W$ has stationary increments, by (3.5) and (3.6) we have

$$
\theta_X = E \left\{ \frac{1}{\sum_{t \in \mathbb{Z}^d} \mathbb{I}(\tilde{W}(t) + Q > 0)} \right\} = E \left\{ \frac{\max_{t \in \mathbb{Z}^d} e^{\tilde{W}(t)}}{\sum_{t \in \mathbb{Z}^d} e^{\tilde{W}(t)}} \right\},
$$

which yields the following lower bound:

$$
\theta_X = E \left\{ \frac{1}{\sum_{t \in \mathbb{Z}^d} \mathbb{I}(W(t) + Q > 0)} \right\} \geq \frac{1}{E\{\sum_{t \in \mathbb{Z}^d} \mathbb{I}(W(t) + Q > 0)\}}
= \frac{1}{\sum_{t \in \mathbb{Z}^d} \mathbb{P}\{W(t) + Q > 0\}} = \frac{1}{\sum_{t \in \mathbb{Z}^d} \Phi(\sigma^2(t)/2)},
$$

where we used the Fubini theorem for the first equality, and (5.1) implies (5.3). The lower bound is strictly positive under some growth conditions on $\sigma$; see [13] for similar calculations in the continuous case. Derivation of a tight positive lower bound is of general interest, since in most cases direct evaluation of $\theta_X$ is not feasible.

It is of some interest to compare two different extremal indices of stationary max-stable Brown–Resnick r.f.s for different variance functions. With similar arguments as in [11, Thm. 3.1], we can prove the following result.

**Lemma 7.** Let $X_1(t), t \in \mathbb{Z}^d$, and $X_2(t), t \in \mathbb{Z}^d$, be two stationary max-stable Brown–Resnick r.f.s corresponding to two centered Gaussian processes $W_1$ and $W_2$ with stationary increments, continuous trajectories, and variance functions $\sigma_1^2$ and $\sigma_2^2$ vanishing at the origin. If $\sigma_1(t) \geq \sigma_2(t)$ for all $t \in \mathbb{Z}^d$, then $\theta_{X_1} \geq \theta_{X_2}$.

**Remark 6.**

(i) Under the conditions of Lemma 7,

$$
E \left\{ \frac{1}{\sum_{t \in \mathbb{Z}^d} \mathbb{I}(W_1(t) + Q > 0)} \right\} \geq E \left\{ \frac{1}{\sum_{t \in \mathbb{Z}^d} \mathbb{I}(W_2(t) + Q > 0)} \right\}.
$$

(ii) The calculation of $\theta_X$ and different expressions for it have appeared in the literature in various contexts: the most prominent one concerns extremes of Gaussian r.f.s, where, in fact, $\theta_X$ has been originally calculated; see, for example, [9, 28, 33]. The first expression in (5.2) for the continuous setup, $d = 1$, and the fractional Brownian motion case is obtained in [5, Thm. 10.5.1]. Applications to sequential analysis and statistics have given rise to various forms of formula (5.2); see, for example, [31, 40]. As already shown in [17], (5.2) is useful for simulations of $\theta_X$. 
5.2 \( \Theta \) generated by summable sequences

Let \( c_i, i \in \mathbb{Z}^d \), be nonnegative constants satisfying \( \sum_{i \in \mathbb{Z}^d} c_i^\alpha = C \in (0, \infty) \) for some \( \alpha > 0 \) and define

\[
\Theta(i) = \frac{c_i+S}{c_S}, \quad i \in \mathbb{Z}^d,
\]

for a given r.v. \( S \) with values in \( \mathbb{Z}^d \) satisfying

\[
P\{S = i\} = \frac{c_i^\alpha}{C}, \quad i \in \mathbb{Z}^d.
\]

Clearly, \( \Theta(0) = 1 \) almost surely, and, moreover, \( \Theta \) satisfies (2.6) stated for the case \( \alpha > 0 \) as follows: for any \( h \in \mathbb{Z}^d \),

\[
E\{\Theta^\alpha(h) F(\Theta)\} = E\left\{ \frac{c_{h+S}}{c_S} I(c_S \neq 0) F(c_{h+S}) \right\} = \frac{1}{C} \sum_{i \in \mathbb{Z}^d} c_i^\alpha I(c_i \neq 0) F(c_{i+1})
\]

\[
= E\{F(B^h \Theta) I(\Theta(-h) \neq 0)\}
\]

for any 0-homogeneous measurable functional \( F : E \mapsto [0, \infty] \).

Clearly, \( S(\Theta) = \sum_{t \in \mathbb{Z}^d} \Theta^\alpha(t) \) is finite almost surely, and hence

\[
\theta_X = E\left\{ \frac{\max_{t \in \mathbb{Z}^d} c_t^\alpha}{\sum_{t \in \mathbb{Z}^d} c_t^\alpha} \right\} = \frac{1}{C} \max_{t \in \mathbb{Z}^d} c_t^\alpha \in (0, 1]. \quad (5.4)
\]

Note that \( \theta_X \) given in (5.4) is the extremal index of a large class of stationary r.f.s, see, for example, [4, 44].

5.3 Constructions of \( X \) with given extremal index

From the previous example we conclude that for any \( a \in (0, 1) \), we can construct a stationary max-stable r.f. \( X \) such that \( \theta_X = a \). We further present examples of r.f. \( X \) satisfying \( \theta_X = 0 \) and then construct stationary max-stable r.f.s \( X^{(p)} \) indexed by \( p \in (0, 1) \) and calculate their extremal indices.

Next, consider independent nonnegative r.f.s \( \Theta_k(t), t \in \mathbb{Z}, k \leq d \), that satisfy (2.6) such that

\[
P\{\Theta_k(0) = 1\} = 1, \quad k \in \mathbb{Z}.
\]

It follows that the r.f. \( \Theta(t) = \prod_{1 \leq k \leq d} \Theta_k(t_k), t = (t_1, \ldots, t_k) \in \mathbb{Z}^d \), also satisfies (2.6). In view of Lemma 2, we can construct stationary max-stable r.f.s \( X, X_k, k \leq d \), corresponding to \( \Theta, \Theta_k \), \( k \in \mathbb{Z} \). As already mentioned in Remark 4(ii), we have \( \theta_X = \prod_{k \leq d} \theta_{X_k} \), and therefore \( \theta_X = 0 \) if some \( \theta_{X_k} \) equals zero. If we define \( \Theta_k(j) = 1 \) for all even integers \( j \) and \( \Theta_k(j) = 0 \) for all odd integers \( j \), then \( \Theta \) satisfies (2.6). Since \( S(\Theta_k) = \infty \) almost surely, it follows that \( \theta_{X_k} = 0 \), and hence also \( \theta_X = 0 \).

In view of our examples, we can construct two independent stationary max-stable r.f.s \( \eta_1(t), \eta_2(t), t \in \mathbb{Z}^d \), with unit Fréchet marginals and spectral tail r.f.s \( Z_1 \) and \( Z_2 \), respectively, satisfying

\[
P\{S(Z_1) < \infty\} = P\{S(Z_2) = \infty\} = 1.
\]

The r.f. \( X^{(p)}(t) = \max(p\eta_1(t), (1-p)\eta_2(t)) \), \( t \in \mathbb{Z}^d \), for any given \( p \in (0, 1) \) is stationary and max-stable with unit Fréchet marginals. As already shown in the previous section, we have \( \theta_{X^{(p)}} = p\theta_{\eta_1} \).

6 Proofs

Proof of Lemma 1. For a given nonnegative spectral r.f. \( Z \) of a max-stable r.f. \( X \) with unit Fréchet marginals, by the de Haan representation of \( X \) for any \( t_i \in \mathbb{Z}^d, x_i \in (0, \infty), i \leq n, \)

\[
- \ln P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\} = E\left\{ \max_{1 \leq i \leq n} \frac{Z(t_i)}{x_i} \right\}.
\]
Consequently, with \( t_0 = h \in \mathbb{Z}^d \) and \( x_0 = 1 \), we obtain, as \( u \to \infty \),

\[
P\{ u^{-1} X(t_i) \leq x_i, \ i = 1, \ldots, n \mid X(t_0) > u \}
\sim u \left( P\{ u^{-1} X(t_i) \leq x_i, \ i = 1, \ldots, n \mid u^{-1} X(t_0) > x_0 \} \right)
= u \left( P\{ u^{-1} X(t_i) \leq x_i, \ i = 1, \ldots, n \} - P\{ u^{-1} X(t_i) \leq x_i, \ i = 0, \ldots, n \} \right)
\to E \left\{ \max_{i=0, \ldots, n} \frac{Z(t_i)}{x_i} - \max_{i=1, \ldots, n} \frac{Z(t_i)}{x_i} \right\}, \quad u \to \infty,
\]

\[
= E \left\{ I(Z(t_0) > 0) \left[ \max_{i=0, \ldots, n} \frac{Z(t_i)}{x_i} - \max_{i=1, \ldots, n} \frac{Z(t_i)}{x_i} \right] \right\}
= E \left\{ Z(t_0) I(Z(t_0) > 0) \left[ \max_{i=0, \ldots, n} \frac{Z(t_i)}{Z(t_0)x_i} - \max_{i=1, \ldots, n} \frac{Z(t_i)}{Z(t_0)x_i} \right] \right\}
= E \left\{ \max_{i=0, \ldots, n} \frac{\Theta_h(t_i)}{x_i} - \max_{i=1, \ldots, n} \frac{\Theta_h(t_i)}{x_i} \right\},
\]

where the last line follows by the definition of \( \Theta_h \) in (1.7). Hence in view of (2.2) and the fact that \( \Theta_h(h) = 1 \) almost surely, the proof is complete. \( \square \)

**Proof of Lemma 2.** By the assumption \( \sum_{j \in \mathbb{Z}^d} p_j = 1 \) and nonnegativity of \( \Theta^* \) we have that for any \( j \in \mathbb{Z}^d \),

\[
E \left\{ \sum_{i \in \mathbb{Z}^d} p_i \Theta^*(i - j) \right\} = \sum_{i \in \mathbb{Z}^d} p_i E\{ \Theta^*(i - j) \} = \sum_{i \in \mathbb{Z}^d} p_i P\{ \Theta^*(j - i) > 0 \} \leq 1,
\]

which, together with the nonnegativity of \( \Theta^* \), implies that for some norm \( \| \cdot \| \) on \( \mathbb{R}^d \),

\[
\lim_{\|t\| \to \infty} \frac{p_t \Theta^*(t - j)}{\|t\|} = \lim_{\|t\| \to \infty} \frac{p_t Y(t - j)}{\|t\|} = 0 \tag{6.2}
\]

almost surely. Consequently, since

\[
P\{ p_N > 0 \} = P\{ Y(0) > 1 \} = 1,
\]

\( \max_{i \in \mathbb{Z}^d} p_t B^N Y(t) \in (0, \infty) \) almost surely, and thus \( Z_N \) in (2.7) is well defined.

Next, for any \( a, h \in \mathbb{Z}^d \) and any 0-homogeneous measurable functional \( F : E \mapsto [0, \infty] \), by the independence of \( N \) and \( Y \), applying the Fubini theorem, we obtain

\[
E\left\{ Z_N(h) F(B^a Z_N) \right\}
= E\left\{ \frac{B^N Y(h)}{\max_{s \in \mathbb{Z}^d} p_s B^N Y(s)} I(\mathcal{I}_{fm}(p \cdot B^N Y) = N) F(B^{a+N} Y) \right\}
= \sum_{j \in \mathbb{Z}^d} E\left\{ \frac{B^j \Theta^*(h)}{\max_{s \in \mathbb{Z}^d} p_s \Theta^*(s-j)} I(\mathcal{I}_{fm}(p \cdot B^j \Theta^*) = j) F(B^{a+j} \Theta^*) \right\}
= \sum_{j \in \mathbb{Z}^d} E\left\{ B^j \Theta^*(h) I(\mathcal{I}_{fm}(p \cdot B^j \Theta^*) = j) F(B^{a+j} \Theta^*) \right\}
\]


\[= \sum_{j \in \mathbb{Z}^d} \mathbb{E}\{I(T_{fm}(p \cdot B^h \Theta^*) = j, \Theta^*(j - h) > 0) F(B^{a+h} \Theta^*)\}\]

\[= \mathbb{E}\{F(B^{a+h} \Theta^*) \sum_{j \in \mathbb{Z}^d} I(T_{fm}(p \cdot B^h \Theta^*) = j, \Theta^*(j - h) > 0)\}\]

\[= \mathbb{E}\{F(B^{a+h} \Theta^*)\} = \mathbb{E}\{Z_N(a) F(B^h Z_N)\},\]

where the third equality follows since \(T_{fm}(p \cdot B^i \Theta^*) = j\) implies

\[\max_{s \in \mathbb{Z}^d} p_s \Theta^*(s - j) = p_j B^j \Theta^*(j) = p_j \Theta^*(0) = p_j > 0\]

almost surely, the fourth equality follows from (2.6) and the assumption that \(\mathbb{P}\{\Theta^*(0) = 1\} = 1\), and the sixth one is a consequence of the equality

\[\sum_{j \in \mathbb{Z}^d} I(T_{fm}(p \cdot B^h \Theta^*) = j) = I(T_{fm}(p \cdot B^h \Theta^*) \in \mathbb{Z}^d) = 1\]

almost surely (which follows from (6.2)) and the fact that \(T_{fm}(p \cdot B^h \Theta^*) = j\) implies, for any \(h \in \mathbb{Z}^d\),

\[p_j \Theta^*(j - h) \geq p_h \Theta^*(0) \geq p_h > 0\]

almost surely and consequently \(\Theta^*(j - h) > 0\) almost surely. Finally, the last claimed equality is established by repeating the calculations for \(\mathbb{E}Z_N(a) F(B^h Z_N)\). Hence the proof follows by (2.5) and the definition of the spectral tail r.f. \(\Theta\) via the spectral r.f. \(Z\). \(\square\)

**Proof of Lemma 3.** Let \(r_n \in \mathbb{Z}^d, n \geq 1\), be nonnegative integers with components \(r_{nj}, j \leq d\), such that \(\lim_{n \to \infty} n/r_{nj} = \lim_{n \to \infty} r_{nj} = \infty\). The stationarity of \(X\) further yields

\[C(A) = \mathbb{E}\{\max_{i \in A} Z(i)\} = C(A')\]

for any finite set of indices \(A \subset \mathbb{Z}^d\) and any \(A' \subset \mathbb{Z}^d\) that is a shift/translation of \(A\). Moreover, by the subadditivity of the maximum

\[C(A \cup B) \leq C(A) + C(B)\]

Hence the growth of \(C(A)\) is as that of the counting measure of \(A\); see [16] for this argument and [32]. Consequently,

\[\lim_{n \to \infty} \frac{\mathbb{E}\{\max_{0 \leq i \leq r_n, i \in \mathbb{Z}^d} Z(i)\}}{\prod_{j=1}^d r_{nj}} = \lim_{n \to \infty} n^{-d} \frac{\mathbb{E}\{\max_{i \in [0,n]^d, i \in \mathbb{Z}^d} Z(i)\}}{\prod_{j=1}^d r_{nj}} = \mathcal{H}.\]

The assumption on \(r_n\) and (6.1) imply that

\[\tilde{\theta}_X \sim \frac{\mathbb{P}\{\max_{0 \leq i \leq r_n, i \in \mathbb{Z}^d} X(i) > n\}}{\prod_{j=1}^d r_{nj} \mathbb{P}\{X(0) > n\}} \sim \frac{\mathbb{E}\{\max_{0 \leq i \leq r_n, i \in \mathbb{Z}^d} Z(i)\}}{\prod_{j=1}^d r_{nj}}, \quad n \to \infty.\]

Hence \(\mathcal{H} = \theta_X\) establishes the proof. \(\square\)
Proof of Lemma 4. We give first a key characterization of tail r.f.s initially proved in [34] and also stated for r.f.s in [2]. Namely, for any measurable map $F : E \mapsto [0, \infty]$,

$$E \left\{ F(Y) I \left( Y(i) > \frac{1}{t} \right) \right\} = t E \left\{ F(B^i Y) I \left( Y(-i) > t \right) \right\} \quad (6.3)$$

for all $i \in \mathbb{Z}^d, t > 0$. If $I, I'$ are two anchoring maps, then since $Y(0) = R > 1$ almost surely and $I(Y) = i$ implies $Y(i) > 1$ almost surely, by (6.3) we have

$$P \{ I(Y) \in \mathbb{Z}^d, I'(Y) = 0, F(Y) < \infty \}$$

$$= \sum_{i \in \mathbb{Z}^d} P \{ I(Y) = i, I'(Y) = 0, F(Y) < \infty \}$$

$$= \sum_{i \in \mathbb{Z}^d} P \{ I(Y) = i, Y(i) > 1, I'(Y) = 0, F(Y) < \infty \}$$

$$= \sum_{i \in \mathbb{Z}^d} P \{ I(B^i Y) = i, Y(-i) > 1, I'(B^i Y) = 0, F(Y) < \infty \}$$

$$= \sum_{i \in \mathbb{Z}^d} P \{ I(Y) = 0, F(Y) < \infty, Y(-i) > 1, I'(Y) = -i \}$$

$$= \sum_{i \in \mathbb{Z}^d} P \{ I(Y) = 0, F(Y) < \infty, I'(Y) = -i \}$$

$$= P \{ I'(Y) \in \mathbb{Z}^d, I(Y) = 0, F(Y) < \infty \}.$$

With similar arguments we obtain

$$P \{ I(Y) \in \mathbb{Z}^d, F(Y) < \infty \} = \sum_{i \in \mathbb{Z}^d} P \{ I(Y) = 0, F(Y) < \infty, Y(-i) > 1 \}.$$ 

Consequently, $P \{ I(Y) = 0, F(Y) < \infty \} = 0$ is equivalent to

$$P \{ I(Y) \in \mathbb{Z}^d, F(Y) < \infty \} = 0,$$

establishing the proof. □

Proof of Lemma 5. As shown in [19], the condition $P \{ S(Z) = \infty \} = 1$ is equivalent to $X$ being generated by a nonsingular conservative flow. The latter is equivalent to $\theta_X = 0$, see [20] (which follows by [37] for $d = 1$ and by [36] for $d > 1$). In view of Lemma 4 and (3.3), $P \{ I(Y) = 0, S(Y) < \infty \} = 0$ is equivalent to $P \{ S(Y) < \infty \} = 0$. By Lemma A2 in Appendix the latter is equivalent to $P \{ S(Z) < \infty \} = 0$. This establishes the proof since the latter is equivalent to $\theta_X = 0$. □

Proof of Theorem 1. We have that $P \{ S(Z) < \infty \} = 0$ is equivalent to $X$ being generated by a nonsingular conservative flow, which, in view of [35, 36, 37], is equivalent to $\theta_X = 0$. Applying Lemma A3 in Appendix to BRs spectral r.f. $Z$, we have that $ZF(Z)$ is also a BRs spectral r.f. for any measurable functional $F : E \mapsto [0, \infty]$ that is 0-homogeneous and shift-invariant. Since both $I(S(f) = \infty)$ and $I(S(f) < \infty)$, $f \in E$, are measurable 0-homogeneous and shift-invariant functionals and by the above

$$\lim_{n \to \infty} \frac{1}{n^d} E \left\{ \max_{t \in [0, n]^d \cap \mathbb{Z}^d} Z(t) I(S(Z) = \infty) \right\} = 0,$$
we have using further (1.4)
\[
\theta_x = \mathcal{H} = \lim_{n \to \infty} \frac{1}{n^d} \mathbb{E} \left\{ \max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z(t) \right\} = \lim_{n \to \infty} \frac{1}{n^d} \mathbb{E} \left\{ \max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z(t) \mathbb{I} (S(Z) < \infty) \right\}.
\]

Next, assuming that \( P \{ S(Z) < \infty \} > 0 \), by Lemma A2 we have \( P \{ S(\Theta) < \infty \} > 0 \), and the converse also holds. Setting \( Z_s(t) = Z(t) \mathbb{I} (S(Z) < \infty) \), by Lemma A3 it is BRs, and further \( S(Z_s) < \infty \) almost surely. In view of Lemma A1, we can assume that \( S(Z_s) > 0 \) almost surely. Applying (2.5) and using the equivalence of (A1) and (A3), we further obtain
\[
\theta_x = \lim_{n \to \infty} \frac{1}{n^d} \sum_{h \in [0,1]^{d \cap \mathbb{Z}^d}} \mathbb{E} \left\{ Z_s(h) \max_{t \in [0,n]^d \cap \mathbb{Z}^d} \frac{Z_s(t)}{S(\Theta)} \mathbb{I} (S(\Theta) < \infty) \right\}
\]
\[
= \lim_{n \to \infty} \frac{1}{n^d} \sum_{h \in [0,1]^{d \cap \mathbb{Z}^d}} \mathbb{E} \left\{ Z_s(h) \frac{\max_{t \in [0,n]^d \cap \mathbb{Z}^d} B^h Z_s(t)}{\sum_{t \in [0,n]^d \cap \mathbb{Z}^d} B^h Z_s(t)} \right\}
\]
\[
= \lim_{n \to \infty} \lim_{d \to \infty} \frac{1}{n^d} \sum_{h \in [0,1]^{d \cap \mathbb{Z}^d}} \mathbb{E} \left\{ Z_s(h) \frac{\max_{t \in [0,n]^d \cap \mathbb{Z}^d} B^h Z_s(t)}{\sum_{t \in [0,n]^d \cap \mathbb{Z}^d} B^h Z_s(t)} \right\}
\]
\[
= \mathbb{E} \left\{ \max_{I \in \mathbb{Z}^d} Z_s(I) \sum_{t \in \mathbb{Z}^d} \frac{Z_s(t)}{S(\Theta)} \mathbb{I} (S(\Theta) < \infty) \right\}.
\]

Since by definition the events \( \{ I_{fm}(\Theta) \in \mathbb{Z}^d \} \) and \( \{ S(\Theta) < \infty \} \) are almost surely the same, the 0-homogeneity of \( I_{fm}(\cdot) \) implies (recalling that \( \theta(0) = 1 \) almost surely)
\[
\theta_x = \mathbb{E} \left\{ \frac{\max_{I \in \mathbb{Z}^d} \Theta(I)}{S(\Theta)} \mathbb{I} (I_{fm}(\Theta) \in \mathbb{Z}^d) \right\} = \sum_{j \in \mathbb{Z}^d} \mathbb{E} \left\{ \frac{\max_{I \in \mathbb{Z}^d} \Theta(I)}{S(\Theta)} \mathbb{I} (I_{fm}(\Theta) = j) \right\}
\]
\[
= \sum_{j \in \mathbb{Z}^d} \mathbb{E} \left\{ \frac{\max_{I \in \mathbb{Z}^d} \Theta(I)}{S(\Theta)} \mathbb{I} (I_{fm}(\Theta) = j) \right\} = \mathbb{P} \{ I_{fm}(\Theta) = 0 \}
\]
\[
= \mathbb{P} \{ I_{fm}(\Theta) = 0, S(\Theta) < \infty \},
\]
where we applied (2.6) in the last third line combined with condition (ii) in the definition of anchoring maps and also used that \( S(f), f \in E, \) is a shift-invariant functional. Clearly, the last two formulas hold also for the last maximum functional. Since (3.3) implies
\[
P \{ I(Y) \notin \mathbb{Z}^d, S(Y) < \infty \} = 0,
\]
using Lemma 4 to obtain the second equality below, we have
\[
P \{ I_{fm}(\Theta) = 0, S(\Theta) < \infty \} = P \{ I_{fm}(Y) = 0, S(Y) < \infty, I(Y) \in \mathbb{Z}^d \}
\]
\[
+ P \{ I_{fm}(Y) = 0, S(Y) < \infty, I(Y) \notin \mathbb{Z}^d \}
\]
\[
= P \{ I_{fm}(Y) \in \mathbb{Z}^d, S(Y) < \infty, I(Y) = 0 \}
\]
\[
= P \{ I(Y) = 0, S(Y) < \infty \},
\]
Lith. Math. J., 61(2):217–238, 2021.
and hence $\theta_X = \mathbb{P}\{\mathcal{I}_{\tau^c}(Y) = 0\}$, and the same is also true for the last exeedance function. In view of the equivalence (A2) and (A4), we have

$$\{S(Y) < \infty\} \subset \{B(Y) < \infty\} \quad (6.5)$$

with $B(Y) := \sum_{t \in \mathbb{Z}^d} \mathbb{I}(Y(t) > 1)$. Hence since $Y(0) = R\Theta(0) = R > 1$ almost surely implies $B(Y) \geq 1$ almost surely,

$$\mathbb{E}\left\{\frac{B(Y)}{B(Y)} \mathbb{I}(\mathcal{I}(Y) = 0, S(Y) < \infty)\right\}$$

$$= \sum_{t \in \mathbb{Z}^d} \mathbb{E}\left\{\frac{1}{B(Y)} \mathbb{I}(\mathcal{I}(Y) = 0, Y(t) > 1, S(Y) < \infty)\right\}$$

$$= \mathbb{E}\left\{\frac{1}{B(Y)} \sum_{t \in \mathbb{Z}^d} \mathbb{I}(\mathcal{I}(Y) = -t, Y(-t) > 1, S(Y) < \infty)\right\}$$

$$= \mathbb{E}\left\{\frac{1}{B(Y)} \mathbb{I}(\mathcal{I}(Y) \in \mathbb{Z}^d, S(Y) < \infty)\right\}$$

$$= \mathbb{E}\left\{\frac{1}{B(Y)} \mathbb{I}(S(Y) < \infty)\right\} = \mathbb{E}\left\{\frac{1}{B(Y)} \mathbb{I}(\Theta < \infty)\right\},$$

where we used (6.3) to derive the last fourth line, and the last second equality follows from (6.4). With the same arguments as in the proof of [45, Lemma 2.5] considering the discrete setup as in [18], for any $n > 0$, we have

$$\mathbb{E}\left\{\max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z(t)\right\} = \sum_{t \in [0,n]^d \cap \mathbb{Z}^d} \mathbb{E}\left\{\frac{1}{\sum_{s \in [0,n]^d \cap \mathbb{Z}^d} \mathbb{I}(Y(s - t) > 1)}\right\}.$$  

Since $Y(0) > 1$ almost surely and thus the denominator in the expectation is greater that or equal to 1 and converges as $n \to \infty$ almost surely to $B(Y)$, by the dominated convergence theorem it follows that

$$\theta_X = \lim_{n \to \infty} n^{-d} \mathbb{E}\left\{\max_{t \in [0,n]^d \cap \mathbb{Z}^d} Z(t)\right\} = \mathbb{E}\left\{\frac{1}{B(Y)}\right\} \leq 1,$$

and hence (3.6) holds. From the last two expressions of $\theta_X$ we conclude that $\mathbb{E}\{(1/B(Y))\mathbb{I}(S(Y) = \infty)\} = 0$. Consequently, almost surely, $\{B(Y) < \infty\} \subset \{S(Y) < \infty\}$, which, together with (6.5), implies that almost surely

$$\{B(Y) < \infty\} = \{S(Y) < \infty\}. $$

Next, if $\mathbb{P}\{\Theta(i) = 0\} = 1$ for all $i \neq 0, i \in \mathbb{Z}^d$, then

$$\theta_X = \mathbb{E}\left\{\frac{\max_{t \in \mathbb{Z}^d} \Theta(t)}{\sum_{t \in \mathbb{Z}^d} \Theta(t)} \mathbb{I}(S(\Theta) < \infty)\right\} = 1.$$

Conversely, if $\theta_X = 1$, then necessarily $\mathbb{P}\{S(\Theta) < \infty\} = 1$, and thus

$$\theta_X = 1 = \mathbb{E}\left\{\frac{\max_{t \in \mathbb{Z}^d} \Theta(t)}{\sum_{t \in \mathbb{Z}^d} \Theta(t)}\right\},$$
implying that \( \max_{t \in \mathbb{Z}^d} \Theta(t) = \sum_{t \in \mathbb{Z}^d} \Theta(t) \) almost surely. Taking \( \mathcal{I}(f) = \mathcal{I}_{fm}(f) \) we have that \( \theta_X = \mathbb{P}\{\mathcal{I}(\Theta) = 0\} = 1 \) implies that \( \max_{t \in \mathbb{Z}^d} \Theta(t) = \Theta(0) = 1 \) almost surely, and therefore
\[
\sum_{t \in \mathbb{Z}^d} \Theta(t) = 1 + \sum_{t \in \mathbb{Z}^d, t \neq 0} \Theta(t) = 1
\]
almost surely. Consequently (recall that \( \Theta(i) \) are nonnegative), \( \mathbb{P}\{\Theta(i) = 0\} = 1 \) for all \( i \neq 0, i \in \mathbb{Z}^d \), establishing the proof. \( \square \)

**Proof of Lemma 6.** For any \( s > 0 \) and any nondecreasing sequence of integers \( r_n, n \in \mathbb{N} \), tending to infinity such that \( \lim_{n \to \infty} r_n/n = 0 \), for any positive integer \( m \) (recalling that \( \mathbb{E}\{Z(t)\} = 1 \) for \( t \in \mathbb{Z}^d \)), we have
\[
n^{-1}\mathbb{E}\left\{ \max_{m < \|t\| < r_n, t \in \mathbb{Z}^d} Z(t) \right\} \leq n^{-1} \sum_{m < \|t\| < r_n, t \in \mathbb{Z}^d} \mathbb{E}\{Z(t)\} \to 0, \quad n \to \infty,
\]
and hence, by (6.1) and the dominated convergence theorem,
\[
1 - \lim_{n \to \infty} \mathbb{P}\left\{ \max_{m < \|t\| < r_n, t \in \mathbb{Z}^d} X(t) > ns \bigg| X(0) > ns \right\}
= s \lim_{n \to \infty} n \mathbb{P}\left\{ \max_{m < \|t\| < r_n, t \in \mathbb{Z}^d} X(t) \leq ns, X(0) > ns \right\}
= \mathbb{E}\left\{ \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} X(t) - \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} Z(t) \right\}
= \mathbb{E}\left\{ \mathbb{I}(Z(0) > 0) \left[ \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d, t = 0} Z(t) - \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} Z(t) \right] \right\}
= \mathbb{E}\left\{ \left(1 - \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} \Theta(t) \right) \right\}
\]
for any positive integer \( m \) (recall that \( \Theta(0) = 1 \) almost surely). If (A1) holds, then by the dominated convergence theorem we have
\[
\lim_{m \to \infty} \mathbb{E}\left\{ \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d, t = 0} Z(t) - \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} Z(t) \right\} = \mathbb{E}\{Z(0)\} = 1,
\]
and hence Condition C is satisfied.

Conversely, if Condition C is satisfied for some sequence \( r_n, n \geq 1 \), of nonnegative increasing integers, then by the previous calculations
\[
1 - \lim_{m \to \infty} \lim_{n \to \infty} \mathbb{P}\left\{ \max_{m < \|t\| < r_n, t \in \mathbb{Z}^d} X(t) > ns \bigg| X(0) > ns \right\}
= \lim_{m \to \infty} \mathbb{E}\left\{ \left(1 - \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} \Theta(t) \right) \right\} = 1,
\]
and thus almost surely \( \max_{m < \|t\| < \infty, t \in \mathbb{Z}^d} \Theta(t) \to 0 \) as \( m \to \infty \). Consequently, condition (A2) holds by Lemma A4 in Appendix, and hence the proof follows from Remark 4. \( \square \)
Appendix

For notational simplicity, we only consider the case \( \alpha = 1 \). The results for \( \alpha > 0 \) can be formulated with obvious modifications.

**Lemma A1.** If \( X(t), t \in \mathbb{Z}^d \), is a max-stable r.f. with de Haan representation (1.1) and some spectral r.f. \( Z \) satisfying \( \mathbf{E}Z(t) \in (0, \infty) \) for all \( t \in \mathbb{Z}^d \), then there is a spectral r.f. \( Z_* \) for \( X \) such that \( \max_{t \in \mathbb{Z}^d} Z_*(t) > 0 \) almost surely.

**Proof.** Let \( w_i, i \in \mathbb{Z}^d \), be positive constants such that

\[
\mathbf{E} \left\{ \sum_{i \in \mathbb{Z}^d} w_i Z(i) \right\} \in (0, \infty).
\]

Such \( w_i \) exist since \( \mathbf{E} Z(i) \in (0, \infty) \) for \( i \in \mathbb{Z}^d \). By the choice of \( w_i \) we have that

\[
M = \max_{i \in \mathbb{Z}^d} w_i Z(i)
\]

is a nonnegative r.v. and \( a = \mathbf{E} M \in (0, \infty) \). Let \( Z_*(t), t \in \mathbb{Z}^d \) be a r.f. defined by

\[
\mathbf{P}\{Z_* \in A\} = \mathbf{E} \left\{ \frac{M I(aZ/M \in A)}{a} \right\}
\]

for any measurable set \( A \subset \mathbb{E} \). Since by the above definition

\[
\mathbf{P}\left\{ \max_{i \in \mathbb{Z}^d} w_i Z_*(i) = 0 \right\} = \mathbf{E} \left\{ \frac{M I(\max_{i \in \mathbb{Z}^d} w_i Z(i)/M = 0)}{a} \right\} = 0,
\]

it follows that \( \mathbf{P}\{\max_{i \in \mathbb{Z}^d} Z_*(i) = 0\} = 0 \). Moreover, for any \( x_i \in (0, \infty), t_i \in \mathbb{Z}^d, i \leq n, \)

\[
- \ln \mathbf{P}\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\} = \mathbf{E} \left\{ \max_{1 \leq i \leq n} \frac{Z(t_i)}{x_i} \right\} = \mathbf{E} \left\{ \frac{M I(M > 0)}{a} \mathbf{I}(\max_{1 \leq i \leq n} Z(t_i) > 0) \max_{1 \leq i \leq n} \frac{a Z(t_i)}{M x_i} \right\}
\]

where the third equality is valid since \( \max_{1 \leq i \leq n} Z(t_i) > 0 \) implies \( M > 0 \). Hence \( Z_* \) is a spectral r.f. for \( X \).

The calculations above show that we can alternatively define \( Z_*(t) = \mathbf{P}\{\max_{s \in \mathbb{Z}^d} Z(s) > 0\} Z(t) \) conditioned on \( \max_{s \in \mathbb{Z}^d} Z(s) > 0 \), as suggested by the reviewer. \( \square \)

**Proof of (3.1).** As in the proof of Lemma A1, without loss of generality, we can assume that \( Z \) is such that \( \max_{t \in \mathbb{Z}^d} (Z(t)/x_i) > 0 \) almost surely for any positive sequence \( x = (x_j)_{j \in \mathbb{Z}^d} \). Suppose for simplicity that \( \alpha = 1 \) and let \( x \) be a sequence with finite number of positive elements and the rest equal to \( \infty \) (we interpret \( a/\infty \) as \( 0 \)). Since \( Z/x \) consists of zeros and finitely many positive numbers, we have that \( \mathcal{I}_{fm}(Z/x) \in \mathbb{Z}^d \) almost surely. Consequently, by (6.1), the Fubini theorem, and the fact that \( \mathcal{I}_{fm}(Z/x) = j \)
implies $\max_{i \in \mathbb{Z}^d} (Z(t_i)/x_i) = Z(j)/x_j$ almost surely we have

$$-\ln P \{ X(i) \leq x_i, \ i \in \mathbb{Z}^d \}$$

$$= E \left\{ \max_{i \in \mathbb{Z}^d} \frac{Z(t_i)}{x_i} I \left( I_{fm} \left( \frac{Z}{x} \right) \in \mathbb{Z}^d \right) \right\} = \sum_{j \in \mathbb{Z}^d} E \left\{ \max_{i \in \mathbb{Z}^d} \frac{Z(t_i)}{x_i} I \left( I_{fm} \left( \frac{Z}{x} \right) = j \right) \right\}$$

$$= \sum_{j \in \mathbb{Z}^d} \frac{1}{x_j} E \left\{ Z(j) I_{fm} \left( \frac{Z}{x} \right) = j \right\} = \sum_{j \in \mathbb{Z}^d} \frac{1}{x_j} E \left\{ Z(0) I_{fm} \left( B^j Z/x \right) = j \right\}$$

$$= \sum_{j \in \mathbb{Z}^d} \frac{1}{x_j} \left\{ I_{fm} \left( \frac{\Theta}{B^{-j} x} \right) = 0 \right\},$$

where the fourth first equality follows from (2.5), and the last equality follows since $I_{fm}$ is an anchoring map. □

**Lemma A2.** Let $Z(t), \ t \in \mathbb{Z}^d$, be a BRs r.f. satisfying (1.2). If $F : E \mapsto [0, \infty]$ is a shift-invariant $0$-homogeneous measurable map, then $E\{F(Z)\} = 0$ is equivalent to $E\{F(\Theta)\} = 0$. Moreover, if $F$ is bounded by $1$, then $E\{F(Z)\} = 1$ is equivalent to $E\{F(\Theta)\} = 1$.

**Proof.** By the shift-invariance of $F$ and (2.5) we have

$$0 = E \{ F(\Theta) \} = E \left\{ Z(0) F \left( \frac{Z}{Z(0)} \right) \right\} = \sum_{i \in \mathbb{Z}^d} E \{ Z(i) F(Z) \} \geq E \left\{ \left( \max_{i \in \mathbb{Z}^d} Z(i) \right) F(Z) \right\}. $$

Since $Z$ is chosen such that $\max_{i \in \mathbb{Z}^d} Z(i) > 0$ almost surely, it follows that $E\{F(Z)\} = 0$. If $E\{F(Z)\} = 0$, then $F(Z) = 0$ almost surely, and thus

$$0 = E \{ Z(0) F(Z) \} = E \{ F(\Theta) \} = 0.$$

Next, $E\{F(\Theta)\} = 1$ is the same as $E\{1 - F(\Theta)\} = 0$, which is equivalent to $E\{1 - F(Z)\} = 0$ as shown above, establishing the proof. □

**Lemma A3.** If $F : E \mapsto [0, \infty]$ is a $0$-homogeneous measurable functional and $Z(t), \ t \in \mathbb{Z}^d$, is a BRs r.f., then $Z_s = ZF(Z)$ is also a BRs r.f., provided that $E Z_s(t_0) \in (0, \infty)$ for some $t_0 \in \mathbb{Z}^d$.

**Proof.** Using (2.5), we have that $E Z_s(t) = E Z_s(t_0) \in (0, \infty)$ for any $t \in \mathbb{Z}^d$ and, in particular, $P\{F(Z) = 0\} < 1$ and $P\{F(Z) = \infty\} = 0$. Since $F$ is $0$-homogeneous, we have that $Z_s$ satisfies (2.5), which is an equivalent condition for a spectral r.f. to be a BRs r.f.; see [24]. □

**Lemma A4.** If $V(t), \ t \in \mathbb{Z}^d$, is a nonnegative r.f., then $P\{\lim_{t \to \infty} V(t) = 0\} = 1$ if and only if there exists a nondecreasing sequence of integers $r_n, \ n \geq 1$, that converges to infinity as $n \to \infty$ such that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P \left\{ \max_{m \leq \|t\| \leq r_n} V(t) > \delta \right\} = 0 \quad (A.1)$$

for any $\delta > 0$.  

Lith. Math. J., 61(2):217–238, 2021.
Proof. It is well known that (see, e.g., [21, A1.3]) \( \mathbb{P}\{\lim_{n \to \infty} V(t) = 0\} = 1 \) if and only if for all large \( m \) and any positive \( \delta \) and \( \varepsilon \), \( \mathbb{P}\{\max_{|t| \geq m} V(t) > \delta\} < \varepsilon \), which clearly implies (A.1). Assuming that the latter condition holds for given positive \( \delta \) and \( \varepsilon \), there exists \( N \) such that for all \( m, n > N \), we have \( \mathbb{P}\{\max_{m \leq |t| \leq n} V(t) > \delta\} \leq \varepsilon \). Since \( \lim_{n \to \infty} r_n = \infty \), we have that \( \mathbb{P}\{\max_{m \leq |t|} V(t) > \delta\} \leq \varepsilon \), and hence the claim follows. \( \square \)

Lemma A5. Let \( \eta_i(t), i = 1, 2, t \in \mathbb{Z}^d \), be two independent stationary r.f.s with unit Fréchet marginal distributions. If the extremal indices of both \( \eta_1 \) and \( \eta_2 \) exist, then the r.f. \( X(t) = \max(p\eta_1(t), (1-p)\eta_2(t)), t \in \mathbb{Z}^d \), has for any \( p \in (0, 1) \) the extremal index \( \theta_X = p\theta_{\eta_1} + (1-p)\theta_{\eta_2} \in [0, 1] \).

Proof. By the independence of \( \eta_1 \) and \( \eta_2 \) we have that \( X \) is stationary with unit Fréchet marginal distributions. To show the claim, it suffices to prove that \( \max_{t \in [0,n]^d} X(t)/n^d \) converges in distribution as \( n \to \infty \) to \( (p\theta_{\eta_1} + (1-p)\theta_{\eta_2})\xi \), where \( \xi \) is a unit Fréchet r.v. As \( n \to \infty \), by the assumptions \( \max_{t \in [0,n]^d} \eta_i(t)/n^d \) converge for \( i = 1, 2 \) in distribution to \( p_i\theta_{\eta_1}\xi_i \) with two independent unit Fréchet r.v.s \( \xi_1, \xi_2 \) and \( p_1 = 1 - p_2 = p \). Since \( \max(p_1\theta_{\eta_1}\xi_1, p_2\theta_{\eta_2}\xi_2) \) has the same d.f. as \( (p_1\theta_{\eta_1} + p_2\theta_{\eta_2})\xi \), the claim follows by the independence of \( \eta_1 \) and \( \eta_2 \) and Slutsky’s lemma. \( \square \)

Acknowledgment. I am in debt to the reviewer and the Editor for several suggestions, comments, and corrections, which significantly improved the manuscript.

References
1. J.M.P. Albin, On extremal theory for stationary processes, \textit{Ann. Probab.}, 18(1):92–128, 1990, https://doi.org/10.1214/aop/1176990940.
2. B. Basrak and H. Planinić, Compound Poisson approximation for random fields with application to sequence alignment, 2018, arXiv:1809.00723.
3. B. Basrak and J. Segers, Regularly varying multivariate time series, \textit{Stochastic Processes Appl.}, 119(4):1055–1080, 2009, https://doi.org/10.1016/j.spa.2008.05.004.
4. B. Basrak and A. Tafro, Extremes of moving averages and moving maxima on a regular lattice, \textit{Probab. Math. Stat.}, 34(1):61–79, 2014.
5. S.M. Berman, \textit{Sojourns and Extremes of Stochastic Processes}, The Wadsworth & Brooks/Cole Statistics/Probability Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.
6. B.M. Brown and S.I. Resnick, Extreme values of independent stochastic processes, \textit{J. Appl. Probab.}, 14:732–739, 1977, https://doi.org/10.2307/3213346.
7. R.A. Davis and T. Hsing, Point process and partial sum convergence for weakly dependent random variables with infinite variance, \textit{Ann. Probab.}, 23(2):879–917, 1995, https://doi.org/10.1214/aop/1176988294.
8. R.A. Davis, T. Mikosch, and Y. Zhao, Measures of serial extremal dependence and their estimation, \textit{Stochastic Processes Appl.}, 123(7):2575–2602, 2013, https://doi.org/10.1016/j.spa.2013.03.014.
9. K. Dębicki, Ruin probability for Gaussian integrated processes, \textit{Stochastic Processes Appl.}, 98(1):151–174, 2002, https://doi.org/10.1016/S0304-4149(01)00143-0.
10. K. Dębicki and E. Hashorva, On extremal index of max-stable stationary processes, \textit{Probab. Math. Stat.}, 37(2):299–317, 2017.
11. K. Dębicki and E. Hashorva, Approximation of supremum of max-stable stationary processes & Pickands constants, \textit{J. Theor. Probab.}, 33(1):444–464, 2020, https://doi.org/10.1007/s10959-018-00876-8.
12. K. Dębicki, E. Hashorva, and N. Soja-Kukielka, Extremes of homogeneous Gaussian random fields, \textit{J. Appl. Probab.}, 52(1):55–67, 2015, https://doi.org/10.1239/jap/1429282606.
13. K. Dębicki, Z. Michna, and X. Peng, Approximation of sojourn times of Gaussian processes, Methodol. Comput. Appl. Probab., 21(4):1183–1213, 2019, https://doi.org/10.1007/s11009-018-9667-7.

14. L. de Haan, A spectral representation for max-stable processes, Ann. Probab., 12(4):1194–1204, 1984, https://doi.org/10.1214/aop/1176993148.

15. L. de Haan and J. Pickands, III, Stationary min-stable stochastic processes, Probab. Theory Relat. Fields, 72(4):477–492, 1986, https://doi.org/10.1007/BF00344716.

16. A.B. Dieker and T. Mikosch, Exact simulation of Brown–Resnick random fields at a finite number of locations, Extremes, 18:301–314, 2015, https://doi.org/10.1007/s10687-015-0214-4.

17. A.B. Dieker and B. Yakir, On asymptotic constants in the theory of extremes for Gaussian processes, Bernoulli, 20(3):1600–1619, 2014, https://doi.org/10.3150/13-BEJ534.

18. C. Dombry, E. Hashorva, and P. Soulier, Tail measure and spectral tail process of regularly varying time series, Ann. Appl. Probab., 28(6):3884–3921, 2018, https://doi.org/10.1214/18-AAP1410.

19. C. Dombry and Z. Kabluchko, Ergodic decompositions of stationary max-stable processes in terms of their spectral functions, Stochastic Processes Appl., 127(6):1763–1784, 2017, https://doi.org/10.1016/j.spa.2016.10.001.

20. A. Ehlert and M. Schlather, Capturing the multivariate extremal index: Bounds and interconnections, Extremes, 11(4):353–377, 2008, https://doi.org/10.1007/s10687-008-0062-6.

21. P. Embrechts, C. Klüppelberg, and T. Mikosch, Modelling Extremal Events for Insurance and Finance, Stoch. Model. Appl. Probab., Vol. 33, Springer, Berlin, Heidelberg, 1997, https://doi.org/10.1007/978-3-642-33483-2.

22. H. Ferreira and L. Pereira, How to compute the extremal index of stationary random fields, Stat. Probab. Lett., 78(11):1301–1304, 2008, https://doi.org/10.1016/j.spl.2007.11.025.

23. J.P. French and R.A. Davis, The asymptotic distribution of the maxima of a Gaussian random field on a lattice, Extremes, 16(1):1–26, 2013, https://doi.org/10.1007/s10687-012-0149-y.

24. E. Hashorva, Representations of max-stable processes via exponential tilting, Stochastic Processes Appl., 128(9):2952–2978, 2018, https://doi.org/10.1016/j.spa.2017.10.003.

25. A. Jakubowski and N. Soja-Kukieła, Managing local dependencies in asymptotic theory for maxima of stationary random fields, Extremes, 22(2):293–315, 2019, https://doi.org/10.1007/s10687-018-0336-6.

26. A. Janßen, Spectral tail processes and max-stable approximations of multivariate regularly varying time series, Stochastic Processes Appl., 129(6):1993–2009, 2019, https://doi.org/10.1016/j.spa.2018.06.010.

27. Z. Kabluchko, M. Schlather, and L. de Haan, Stationary max-stable fields associated to negative definite functions, Ann. Probab., 37(5):2042–2065, 2009, https://doi.org/10.1214/09-AOP455.

28. S.G. Kobelkov and V.I. Piterbarg, On maximum of Gaussian random field having unique maximum point of its variance, Extremes, 22(3):413–432, 2019, https://doi.org/10.1007/s10687-019-00346-2.

29. D. Krizmanić, Functional weak convergence of partial maxima processes, Extremes, 19(1):7–23, 2016, https://doi.org/10.1007/s10687-015-0236-y.

30. I. Molchanov, M. Schmutz, and K. Stucki, Invariance properties of random vectors and stochastic processes based on the zonoid concept, Bernoulli, 20(3):1210–1233, 2014, https://doi.org/10.3150/13-BEJ519.

31. Y. Nardi, D.O. Siegmund, and B. Yakir, The distribution of maxima of approximately Gaussian random fields, Ann. Stat., 36(3):1375–1403, 2008, https://doi.org/10.1214/07-AOS511.

32. X. Nguyen, Ergodic theorems for subadditive spatial processes, Z. Wahrscheinlichkeitstheor. Verw. Geb., 48(2):159–176, 1979, https://doi.org/10.1007/BF01886870.

Lith. Math. J., 61(2):217–238, 2021.
33. J. Pickands, III, Asymptotic properties of the maximum in a stationary Gaussian process, *Trans. Am. Math. Soc.*, **145**:75–86, 1969, https://doi.org/10.2307/1995059.

34. H. Planinić and P. Soulier, The tail process revisited, *Extremes*, **21**(4):551–579, 2018, https://doi.org/10.1007/s10687-018-0312-1.

35. P. Roy, Nonsingular group actions and stationary $S\alpha S$ random fields, *Proc. Am. Math. Soc.*, **138**(6):2195–2202, 2010, https://doi.org/10.1090/S0002-9939-10-10250-0.

36. P. Roy and G. Samorodnitsky, Stationary symmetric $\alpha$-stable discrete parameter random fields, *J. Theor. Probab.*, **21**(1):212–233, 2008, https://doi.org/10.1007/s10959-007-0107-9.

37. G. Samorodnitsky, Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes, *Ann. Probab.*, **32**(2):1438–1468, 2004, https://doi.org/10.1214/aop/1093962235.

38. G. Samorodnitsky, Maxima of continuous-time stationary stable processes, *Adv. Appl. Probab.*, **36**(3):805–823, 2004, https://doi.org/10.1239/aap/1093962235.

39. J. Segers, Y. Zhao, and T. Meinguet, Polar decomposition of regularly varying time series in star-shaped metric spaces, *Extremes*, **20**(3):539–566, 2017, https://doi.org/10.1007/s10687-017-0287-3.

40. D. Siegmund and B. Yakir, Tail probabilities for the null distribution of scanning statistics, *Bernoulli*, **6**(2):191–213, 2000, https://doi.org/10.2307/3318574.

41. D. Siegmund, B. Yakir, and N.R. Zhang, Tail approximations for maxima of random fields by likelihood ratio transformations, *Sequential Anal.*, **29**(3):245–262, 2010, https://doi.org/10.1080/07474946.2010.487428.

42. D. Siegmund, B. Yakir, and N.R. Zhang, Detecting simultaneous variant intervals in aligned sequences, *Ann. Appl. Stat.*, **5**(2A):645–668, 2011, https://doi.org/10.1214/10-AOAS400.

43. N. Soja-Kukieła, Extremes of multidimensional stationary Gaussian random fields, *Probab. Math. Stat.*, **38**(1):191–207, 2018.

44. N. Soja-Kukieła, On maxima of stationary fields, *J. Appl. Probab.*, **56**(4):1217–1230, 2019, doi:10.1017/jpr.2019.69, https://doi.org/10.1017/jpr.2019.69.

45. P. Soulier, The tail process and tail measure of continuous time regularly varying stochastic processes, 2020, arXiv:2004.00325.

46. S.A. Stoev, Max–stable processes: Representations, ergodic properties and statistical applications, in P. Doukhan, G. Lang, D. Surgailis, and G. Teyssière (Eds.), *Dependence in Probability and Statistics*, Lect. Notes Stat., Vol. 200, Springer, Berlin, Heidelberg, 2010, pp. 21–42, https://doi.org/10.1007/978-3-642-14104-1_2.

47. C. Tillier and O. Wintenberger, Regular variation of a random length sequence of random variables and application to risk assessment, *Extremes*, **21**(1):27–56, 2018, https://doi.org/10.1007/s10687-017-0297-1.

48. K.F. Turkman, A note on the extremal index for space-time processes, *J. Appl. Probab.*, **43**(1):114–126, 2006, https://doi.org/10.1239/jap/1143936247.

49. Y. Wang and S.A. Stoev, On the association of sum- and max-stable processes, *Stat. Probab. Lett.*, **80**(5–6):480–488, 2010, https://doi.org/10.1016/j.spl.2009.12.001.

50. L. Wu and G. Samorodnitsky, Regularly varying random fields, *Stochastic Processes Appl.*, **130**(7):4470–4492, 2020, https://doi.org/10.1016/j.spa.2020.01.005.

51. B. Yakir, *Extremes in Random Fields: A Theory and Its Applications*, Wiley Ser. Probab. Stat., Higher Education Press, Beijing, 2013, https://doi.org/10.1002/9781118720608.