Research Article

Stochastic Periodic Solution and Persistence of a Nonautonomous Impulsive System with Nonlinear Self-Interaction

Weili Kong and Yuanfu Shao

1College of Teacher Education, Qujing Normal University, Qujing, Yunnan 655011, China
2College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China

Correspondence should be addressed to Yuanfu Shao; shaoyf0829@163.com

Received 6 May 2019; Accepted 13 January 2020; Published 10 February 2020

Copyright © 2020 Weili Kong and Yuanfu Shao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In real world, the habitat space and food for species are relatively scare; hence, the interspecific competition phenomenon exists extensively. Usually, the competitive interaction is assumed to be linear, see [1, 2]. However, some experimental tests showed that the term of self-interaction may be nonlinear, so Ayala et al. [3] proposed the following model:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_1(t)(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1^2(t)), \\
\frac{dx_2(t)}{dt} &= x_2(t)(r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_2^2(t)).
\end{align*}
\]

(1)

In practice, the environmental white noise is almost everywhere and inevitably affects the growth of species. Thus, in the progress of mathematical modeling, random disturbance is introduced to reveal the effect of white noise [4–9]. For ecological system, some discrete effects often appear at some short time interval, such as periodic spraying pesticides, harvesting, and stocking, which affect the growth of species and are often modeled by impulsive parameters. In last decades, many impulsive systems have been proposed and many good results have been reported, see, e.g., [10–15] and references cited therein. For example, Tan et al. [15] investigated the existence of solution, stochastic permanence of the following impulsive model:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_1(t)(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_1(t)x_1^2(t)) + \sigma_1(t)x_1(t)d\omega_1(t), \quad t \neq t_k, \\
\frac{dx_2(t)}{dt} &= x_2(t)(r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_2(t)x_2^2(t)) + \sigma_2(t)x_2(t)d\omega_2(t), \quad t \neq t_k, \\
x_1(t_k) &= (1 + \lambda_{1k})x_1(t_k), \quad t = t_k, \\
x_2(t_k) &= (1 + \lambda_{2k})x_2(t_k), \quad t = t_k,
\end{align*}
\]

(2)
where \( \sigma_i^2 (i = 1, 2) \) represents the density of white noise and \( \omega_1 (t) \) and \( \omega_2 (t) \) are independent standard Brownian motions defined on the probability space \((\Omega, F, \{F_t \}_{t \in [0, \infty)}, P)\), where \( \{F_t \}_{t \in [0, \infty)} \) denotes a filtration, which is right continuous, and all \( \mathcal{P} \)-null set is contained in \( F_0 \). For biological meanings of other parameters, refer to [15]. By constructing suitable functional and using inequality techniques, the stochastic permanence of (2) and extinction of \( x_i (t) \) were studied.

In natural world, due to individual life cycle and seasonal variation, the carrying capacity of species, birth rate, and other parameters always present periodic changes for population systems [16–18]. For the determinate biological system, the existence of periodic solution is a very important dynamical behavior [10, 19–22]. Similarly, for stochastic system, it is very interesting to study the existence of stochastic periodic solution (periodic Markovian process). On the other hand, the extinction and permanence in the mean and stochastic persistency in probability are all very important dynamical behaviors (see [12, 13, 23, 24]), but all these are not investigated in [15]. Hence, it is necessary for us to further explore these dynamical behaviors of (2). For this purpose, we give the following assumptions.

**Assumption 1.** All coefficients \( a_i (t), a_j (t), b_i (t), b_j (t), a (t), \sigma_i (t), \) and \( \sigma_j (t) \) are bounded, continuous, and periodic functions with period \( P \).

**Assumption 2.** The impulsive points satisfy \( 0 < t_1 < t_2 < \cdots \), \( \lim_{t \rightarrow t_i^-} t_i = +\infty \), and there exists an integer \( q \) such that \( t_{k+q} = t_k + T \) and \( \lambda_{i,k+q} = \lambda_{ik}, i = 1, 2, k \in N \).

**Assumption 3.** By the biological meanings, we assume \( \lambda_{ik} > -1 \) for \( i = 1, 2, k \in N \).

For the following stochastic differential equation (see [21]),

\[
\frac{d}{dt} x(t) = f(x(t), t) dt + g(x(t), t) d\omega(t), \quad t \geq t_0,
\]

with initial data \( x(t_0) = x_0 \in \mathbb{R}^n \), we define the following differential operator:

\[
L = \frac{d}{dt} + \sum_{k=1}^n f_k (x, t) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,j=1}^n \left[ g^T(x, t) g(x, t) \right]_{kj} \frac{\partial^2}{\partial x_k \partial x_j}.
\]

For any bounded and continuous function \( f(t) \), we use the following notations:

\[
\begin{align*}
\sup_{t \geq 0} f(t) & = \sup \{ f(t), t \geq 0 \}, \\
\inf_{t \geq 0} f(t) & = \inf \{ f(t), t \geq 0 \}, \\
\lim_{t \rightarrow \infty} \sup f(t) & = f^*, \\
\lim_{t \rightarrow \infty} \inf f(t) & = f_*.
\end{align*}
\]

and \( \langle f \rangle = (1/T) \int_0^T f(s) ds \) and \( \langle f \rangle_T = (1/T) \int_0^T f(s) ds \) if \( f(t) \) is integrable.

The rest of this paper is organized as follows. Section 2 begins with some definitions and lemmas. Section 3 focuses on the existence and uniqueness of the periodic Markovian solution. Section 4 is devoted to the extinction and permanence in the mean of species. The stochastic persistence of (2) is studied in Section 5. Some numerical examples are showed in Section 6 to validate the main results. Finally, a brief conclusion and discussion are given to conclude the paper in Section 7.

## 2. Preliminaries

In this section, we introduce the definitions of the periodic Markovian process, the solution of the impulsive stochastic differential equation, and some auxiliary results of the existence of the periodic Markovian process.

**Definition 1** (see [21]). Let \( \Xi(t) = \xi(t, \omega)(-\infty < t < \infty) \) be a stochastic process and \( t_1 , t_2 , \ldots , t_n \) be a finite sequence of numbers. If the joint distribution of random variables \( \xi (t_1 + h), \ldots , \xi (t_n + h) \) is independent of \( h \), where \( h = kT, k = \pm 1, \pm 2, \ldots \), then \( \xi(t) \) is said to be \( T \)-periodic stochastic process.

**Definition 2** (see [13, 15]). The impulsive stochastic process is given as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t))dt + g(t, x(t))d\omega(t), \quad t \neq t_k, \\
x(t_k^+) &= (1 + \lambda_{ik}) x(t_k^-), \quad k \in N,
\end{align*}
\]

with \( x(0) = x_0 \). A stochastic process \( x(t) = (x_1(t), x_2(t), \ldots , x_n(t)) \), \( t \in R_+ = (0, +\infty) \), is said to be a solution of the above system, if

\[
\begin{align*}
(i) & \quad x(t) \text{ is continuous on } (0, t_1) \text{ and } (t_n, t_{n+1}) \subset R_+ (k \in N) \text{ and } F_t \text{-adapted}, \quad f(t, x(t)) \in L^2(R_+, \mathbb{R}^n) \text{ and } g(t, x(t)) \in L^2(R_+, \mathbb{R}^n), \\
(ii) & \quad x(t_k^+) = \lim_{t \rightarrow t_k^-} x(t) \text{ and } x(t_k^-) = \lim_{t \rightarrow t_k^+} x(t) \text{ exist, and } x(t_k^+) = x(t_k^-) + \lambda_{ik} x(t_k^-) \text{ with probability one, } k \in N, \\
(iii) & \quad x(t) \text{ satisfies the following integral equation:}
\end{align*}
\]

\[
\begin{align*}
x(t) &= x_0 + \int_0^t f(s, x(s)) ds + \int_0^t g(s, x(s)) d\omega(s), \\
&= \int_{t_k}^{t_{k+1}} f(s, x(s)) ds + \int_{t_k}^{t_{k+1}} g(s, x(s)) d\omega(s),
\end{align*}
\]

for \( t \in [0, t_1] \) and satisfies

\[
\begin{align*}
x(t) &= x(t_k^+) + \int_{t_k}^{t_{k+1}} f(s, x(s)) ds + \int_{t_k}^{t_{k+1}} g(s, x(s)) d\omega(s),
\end{align*}
\]

for \( t \in [t_k, t_{k+1}] \), \( k \in N \).

**Lemma 1** (see [13, 21]). For the following Itô’s differential equation,

\[
\frac{dx(t)}{dt} = b(t, x(t)) dt + s(t, x(t)) d\omega(t),
\]

all coefficients of (9) satisfy linear growing condition and the Lipschitz condition in every cylinder \( U_l \times R_+ (l > 0) \), \( U_l = \{ x : |x| \leq l \} \) and are \( T \)-periodic in \( t \). Furthermore, there exists a \( T \)-periodic and once continuously differentiable function \( v = v(t, x) \) in \( t \), which is twice continuously
differentiable with respect to $x$ and satisfies the following conditions:

$$
\inf_{(t,x)} v(t,x) \longrightarrow \infty, \quad \text{as} \; l \longrightarrow \infty,
$$

(10)

$$
L v(t,x) \leq -1, \quad \text{outside some compact set,}
$$

and then there exists a solution of (9) which is $T$-periodic Markov process.

**Lemma 2** (see [14]). Suppose that $Z(t) \in C[\Omega \times [0,\infty), R_+]$ and $\lim_{t \to \infty} (F(t)/t) = 0, \; a.s.$

(a) If there exist $t_0 > 0$ and $\lambda_0 > 0$ such that, for all $t > t_0$,

$$
\ln Z(t) \leq \lambda t - \lambda_0 \int_0^t z(s)ds + F(t), \quad a.s.
$$

then

(11)

(b) If there exist $t_0 > 0$, $\lambda_0 > 0$, and $\lambda$ such that, for all $t > t_0$,

$$
\ln Z(t) \geq \lambda t - \lambda_0 \int_0^t z(s)ds + F(t), \quad a.s.
$$

then

$$
\langle Z \rangle_s \geq \frac{\lambda}{\lambda_0}, \quad a.s.
$$

(14)

To investigate (2), we consider the following non-impulsive system:

$$
\begin{align*}
\frac{du_1(t)}{dt} &= u_1(t) \left( r_1(t) + \frac{1}{T} \sum_{0 \leq t_1 < t} \ln(1 + \lambda_{1k}) - a_{11}(t)A_1(t)u_1(t) - a_{12}(t)A_2(t)u_2(t) - b_1(t)A_1^2(t)u_1^2(t) \right) dt + \sigma_1(t)u_1(t)d\omega_1(t), \\
\frac{du_2(t)}{dt} &= u_2(t) \left( r_2(t) + \frac{1}{T} \sum_{0 \leq t_1 < t} \ln(1 + \lambda_{2k}) - a_{21}(t)A_1(t)u_1(t) - a_{22}(t)A_2(t)u_2(t) - b_2(t)A_2^2(t)u_2^2(t) \right) dt + \sigma_2(t)u_2(t)d\omega_2(t),
\end{align*}
$$

(15)

where

$$
A_1(t) = \left[ \prod_{j=1}^{d} (1 + \lambda_{1j}) \right]^{\frac{t(T)}{\omega}} \prod_{0 \leq t_1 < t} (1 + \lambda_{1k}),
$$

$$
A_2(t) = \left[ \prod_{j=1}^{d} (1 + \lambda_{2j}) \right]^{\frac{t(T)}{\omega}} \prod_{0 \leq t_1 < t} (1 + \lambda_{2k}).
$$

(16)

Assume that the product equals unity if the number of factors is zero. Obviously, $A_1(t)$ and $A_2(t)$ are both $T$-periodic (for details, see [20]).

For (2) and (15), we have the following.

**Lemma 3.** Let $u_1(t) = A_1(t)x_1(t)$ and $u_2(t) = A_2(t)x_2(t)$:

1. If $(x_1(t), x_2(t))$ is solution of (2), then $(u_1(t), u_2(t))$ is solution of (15).
2. If $(u_1(t), u_2(t))$ is solution of (15), then $(x_1(t), x_2(t))$ is solution of (2).

**Remark 1.** The proof is similar to that of Theorem 3.1 in [20] and is omitted. Lemma 3 reveals the equivalence of (2) and (15), and hence, we will consider (15) later.

**Lemma 4.** For any given initial value $(x_1(0), x_2(0)) \in R^2$, if $a_{ij}^l > b_1^l$, $a_{ij}^u > b_2^u$, then system (2) has a unique solution $(x_1(t), x_2(t))$ on $t \geq 0$ and the solution remains in $R_+$ with probability one.

**Remark 2.** By reference [6], the existence of solutions of (15) can be derived; then, by Lemma 3, the required assertion is obtained.

### 3. Existence of Stochastic Periodic Solution

**Theorem 1.** Suppose the following condition holds:

$$
(H_1)\eta_1\frac{a_{11}A_1^l}{2b_1^u(A_1)^2} + \eta_2\frac{a_{22}A_2^l}{2b_2^u(A_2)^2} > \max\{\theta, \chi\},
$$

(17)

where

$$
\eta_i = \langle r_i(t) - (a_i^l(t)/2) \rangle_T + (1/T)\sum_{0 \leq t_1} \ln(1 + \lambda_{ik}),
$$

$i = 1, 2$, and $\theta, \chi$ are defined later; then, there exists a periodic Markovian process for system (2).

**Proof.** Obviously, Lemma 4 implies the existence of positive solutions of (2); then, according to Lemma 3, it is only needed to prove the solution of (15) is a periodic Markovian process. By Lemma 1, it suffices to find a $C^2$-function
Let $V(t, u_1, u_2) = V_1(t, u_1) + V_2(t, u_2)$, then it is not difficult to verify that $V(t, u_1, u_2)$ is $T$-periodic function on $[0, \infty)$ and
\[
\liminf_{(u_1, u_2) \to (0, 0)} V(t, u_1, u_2) \to \infty, \quad \text{as } k \to \infty,
\]
where $U_k = \{(u_1, u_2) : (u_1, u_2) \in ((1/k), (k)) \times ((1/k), k) \}$. Hence, the first condition of Lemma 1 is satisfied. Now, we are in the progress of proving the second condition of Lemma 1. Using Itô’s formula on $V_1(t, u_1)$ and $V_2(t, u_2)$, respectively, we obtain

\[
LV_1 = \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} w_1(t) + \left[ 1 - \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \frac{1}{u_1} \right] u_1 \left[ r_1(t) + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_1) - a_{11}(t) A_1(t) u_1 - a_{12}(t) A_2(t) u_2 - b_1(t) A_1^2(t) u_1^2 \right] + \frac{1}{2} \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \sigma_1^2(t) = \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \left[ w_1(t) - r_1(t) - \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_1) + \frac{1}{2} \sigma_1^2(t) \right] + r_1(t) u_1 + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_1) u_1 - a_{11}(t) A_1(t) u_1^2 - a_{12}(t) A_2(t) u_2 - b_1(t) A_1^2(t) u_1^2 + \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \left[ a_{11}(t) A_1(t) u_1 + a_{12}(t) A_2(t) u_2 + b_1(t) A_1^2(t) u_1^2 \right] \leq -\eta_1 \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} u_1 + \left[ r_1 + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_1) + \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \frac{1}{u_1} \right] u_1 + \frac{a_{11}^{\prime} A_1}{2 b_1^{\prime} (A_1^0)^2} \frac{1}{u_1} u_2,
\]

\[
LV_2 = \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} w_2(t) + \left[ 1 - \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \frac{1}{u_2} \right] u_2 \left[ r_2(t) + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_2) - a_{21}(t) A_1(t) u_1 - a_{22}(t) A_2(t) u_2 - b_2(t) A_2^2(t) u_2^2 \right] + \frac{1}{2} \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \sigma_2^2(t) = \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \left[ w_2(t) - r_2(t) - \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_2) + \frac{1}{2} \sigma_2^2(t) \right] + r_2(t) u_2 + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_2) u_2 - a_{21}(t) A_1(t) u_1 u_2 - a_{22}(t) A_2(t) u_2^2 - b_2(t) A_2^2(t) u_2^2 + \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \left[ a_{21}(t) A_1(t) u_1 + a_{22}(t) A_2(t) u_2 + b_2(t) A_2^2(t) u_2^2 \right] \leq -\eta_2 \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} u_2 + \left[ r_2 + \frac{1}{T} \sum_{0 \leq t < T} \ln(1 + \lambda_2) + \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \frac{1}{u_2} \right] u_2 + \frac{a_{12}^{\prime} A_2}{2 b_2^{\prime} (A_2^0)^2} \frac{1}{u_2} u_1.
\]
Therefore,

\[ LV(t, u_1, u_2) \leq -\eta_1 \frac{d_1^1, A_1^i}{2b_1^i (A_1^i)^2} - \eta_2 \frac{d_2^1, A_2^i}{2b_2^i (A_2^i)^2} - \frac{d_1^1, A_1^i}{2} u_1 - \frac{d_2^1, A_2^i}{2} u_2 \]

\[ + \left[ r_1^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{ik}) + \frac{d_1^1, A_1^i a_1^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_2^1, a_1^1}{2b_2^i (A_2^i)^2} \right] u_1 \]

\[ + \left[ r_2^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{3k}) + \frac{d_1^1, A_1^i a_1^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_2^1, a_1^1}{2b_2^i (A_2^i)^2} \right] u_2. \]

(22)

Denote

\[ f(u_1) = -\frac{d_1^1, A_1^i}{2} u_1^2 + \left[ r_1^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{ik}) + \frac{d_1^1, A_1^i a_1^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_1^1, a_2^1}{2b_2^i (A_2^i)^2} \right] u_1, \]

\[ f(u_2) = -\frac{d_2^1, A_2^i}{2} u_2^2 + \left[ r_2^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{3k}) + \frac{d_1^1, A_1^i a_2^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_2^1, a_2^1}{2b_2^i (A_2^i)^2} \right] u_2. \]

(23)

Thus,

\[ f^\mu(u_1) = \left[ r_1^\mu + (1/T) \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{ik}) + \frac{d_1^1, A_1^i a_1^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_1^1, a_1^1}{2b_2^i (A_2^i)^2} \right] u_1, \]

\[ f^\mu(u_2) = \left[ r_2^\mu + (1/T) \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{3k}) + \frac{d_1^1, A_1^i a_2^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_2^1, a_2^1}{2b_2^i (A_2^i)^2} \right] u_2. \]

(24)

For any small positive \( \epsilon < 1 \), define a closed set

\[ D_{\epsilon} = \{(u_1, u_2) : (u_1, u_2) \in \left[ \frac{1}{\epsilon}, 1 \right] \times \left[ \frac{1}{\epsilon}, 1 \right] \}, \]

(25)

where \( D_{\epsilon} \) is compact and its component \( D_{\epsilon}^i = (R_{\epsilon}^i; D_{\epsilon}^i) = \cup_{i=1}^3 D_{\epsilon}^i \), in which

\[ D_{\epsilon}^1 = \{(u_1, u_2) : 0 < u_1 < \epsilon \}, \]

\[ D_{\epsilon}^2 = \{(u_1, u_2) : 0 < u_2 < \epsilon \}, \]

\[ D_{\epsilon}^3 = \{(u_1, u_2) : u_1 > \frac{1}{\epsilon} \}, \]

\[ D_{\epsilon}^3 = \{(u_1, u_2) : u_2 > \frac{1}{\epsilon} \}. \]

(26)

Discuss \( LV \) as follows:

(i) If \( (u_1, u_2) \in D_{\epsilon}^1 \), by \((H_1)\), we choose \( \epsilon > 0 \) small enough such that

\[ \left[ r_1^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{ik}) + \frac{d_1^1, A_1^i a_1^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_1^1, a_1^1}{2b_2^i (A_2^i)^2} \right] u_1 + \chi \]

\[ \leq \eta_1 \frac{d_1^1, A_1^i}{2b_1^i (A_1^i)^2} + \eta_2 \frac{d_2^1, A_2^i}{2b_2^i (A_2^i)^2}, \]

(27)

and hence, \( LV < -1 \).

(ii) If \( (u_1, u_2) \in D_{\epsilon}^2 \), by \((H_1)\) again, for \( \epsilon > 0 \) small enough, we have

\[ \left[ r_1^\mu + \frac{1}{T} \sum_{t \leq t < \epsilon} \ln(1 + \lambda_{3k}) + \frac{d_1^1, A_1^i a_2^1}{2b_1^i A_1^i} + \frac{d_2^1, A_2^i a_2^1, a_2^1}{2b_2^i (A_2^i)^2} \right] u_2 + \chi \]

\[ \leq \eta_1 \frac{d_1^1, A_1^i}{2b_1^i (A_1^i)^2} + \eta_2 \frac{d_2^1, A_2^i}{2b_2^i (A_2^i)^2}. \]

(28)
Therefore, \( LV < -1 \).

(iii) If \((u_1, u_2) \in D_1^*\) or \((u_1, u_2) \in D_1^1\), by the monotonicity, obviously, \( LV \to -\infty \).

To summarize, the conditions of Lemma 1 are all satisfied and the required assertion is directly derived. This completes the proof. \( \square \)

Remark 3. Based on the existence theorem of periodic Markovian process and extreme-value theory of quadratic function, the sufficient conditions assuring the existence of stochastic periodic solution are established, which is not discussed in [15]. Theorem 1 shows that impulsive disturbance and stochastic disturbance affect the periodic behavior of (2), which is shown in Figures 1(g) and 1(h) in Section 6.

4. Extinction and Permanence in the Mean

Lemma 5. The solution \( x_i(t) \) of (2) with initial value \((x_1(0), x_2(0))^T \in R_+^2\) satisfies

\[
\limsup_{t \to \infty} E(x_i^p(t)) < H_i(p),
\]

for any \( p > 0 \), where \( H_i(p)(i = 1, 2) \) is a positive constant. Furthermore,

\[
\limsup_{t \to \infty} P(|x_i(t)| > \zeta_i) < \varepsilon, \quad i = 1, 2,
\]

that is, the solution of (2) is stochastically ultimately bounded.

Proof. By Lemma 3, it is only needed to study the equivalent system (15). Define \( V(u_1, u_2) = e^t(u_1^p + u_2^p) \) for \((u_1, u_2) \in \mathbb{R}_+^2\) and \( p \geq 1 \). Applying Itō’s formula to \( V(u_1, u_2) \), we obtain

\[
LV(u_1, u_2) = e^t \left\{ u_1^p + u_2^p + pu_1^p \left( r_1(t) + \frac{1}{T} \sum_{0 \leq s < t} \ln(1 + \lambda_{ik}) + \frac{(p-1)\sigma_{ik}^2(t)}{2} - a_{11}(t)A_1(t)u_1(t) \right) 
\]

\[
- a_{12}(t)A_2(t)u_2(t) - b_1(t)A_1(t)u_1(t) - b_2(t)A_2(t)u_2(t)
\]

\[
+ pu_2^p \left( r_2(t) + \frac{1}{T} \sum_{0 \leq s < t} \ln(1 + \lambda_{ik}) + \frac{(p-1)\sigma_{ik}^2(t)}{2} \right)
\]

\[
+ pu_1^p \left( r_1(t) + \frac{1}{T} \sum_{0 \leq s < t} \ln(1 + \lambda_{ik}) + \frac{(p-1)\sigma_{ik}^2(t)}{2} \right)
\]

\[
= e^t \tilde{H}(p).
\]

Integrating both sides of the above inequality from 0 to \( t \) and taking expectation yields,

\[
e^t E(u_1^p + u_2^p) \leq u_1^p(0) + u_2^p(0) + E \int_0^t e^s \tilde{H}(p) ds = u_1^p(0) + u_2^p(0) + \tilde{H}(p)(e^t - 1).
\]

By comparison theorem,

\[
\limsup_{t \to \infty} E(u_1^p + u_2^p) \leq \tilde{H}(p) < \infty,
\]

which means the existence of constants \( \tilde{H}_1(p) \) and \( \tilde{H}_2(p) \) such that

\[
\limsup_{t \to \infty} E(u_1^p(t)) \leq \tilde{H}_1(p),
\]

\[
\limsup_{t \to \infty} E(u_2^p(t)) \leq \tilde{H}_2(p).
\]

Hence, there exists \( t_1 > 0 \); for any \( t \geq t_1 \), we have \( E(u_1^p(t)) \leq \tilde{H}_1(p) \) and \( E(u_2^p(t)) \leq \tilde{H}_2(p) \). Furthermore, by
Figure 1: Continued.
the continuity of $E(u_1^0(t))$ and $E(u_2^0(t))$, for any $t \leq t_1$, there exist $\hat{H}_1(p) > 0$ and $\hat{H}_2(p) > 0$ such that $E(u_1^0(t)) \leq \hat{H}_1(p)$ and $E(u_2^0(t)) \leq \hat{H}_2(p)$. Let $H_1(p) = \max \{ \hat{H}_1(p), \hat{H}_1(p) \}$ and $H_2(p) = \max \{ \hat{H}_2(p), \hat{H}_2(p) \}$, then
\begin{equation}
E(u_1^0(t)) \leq H_1(p), \quad E(u_2^0(t)) \leq H_2(p),
\end{equation}
hold for all $t \geq 0$. Applying Chebyshev inequality, we obtain
\begin{equation}
\limsup_{t \to \infty} P(\mid u_i(t) \mid > \zeta_i) < \epsilon, \quad i = 1, 2,
\end{equation}
which means (15) is stochastically ultimately bounded. This completes the proof. \[\square\]

**Theorem 2.** For system (2), let $\kappa_1 = \alpha_{12}^a A_{21}^a \zeta_2$ and $\kappa_2 = \alpha_{21}^a A_{11}^a \zeta_1$, then the following results hold:

(i) If $\eta_1 < 0$ and $\eta_2 < 0$, then all species are extinct, i.e., $\lim_{t \to \infty} x_1(t) = 0$ and $\lim_{t \to \infty} x_2(t) = 0$

(ii) If $\eta_1 > 0$ and $\eta_2 < 0$, then $x_1(t)$ is permanent in the mean and $x_2(t)$ is extinct, i.e., $\bar{\alpha}_1 \leq x_1(t)$ and $\lim_{t \to \infty} x_2(t) = 0$

(iii) If $\eta_1 < 0$ and $\eta_2 > 0$, then $x_1(t)$ and $x_2(t)$ is extinct and $x_2(t)$ is permanent in the mean, i.e., $\lim_{t \to \infty} x_1(t) = 0$ and $\bar{\alpha}_2 \leq (x_2(t))$.

(iv) If $\eta_1 > \kappa_1$ and $\eta_2 > \kappa_2$, then $x_1(t)$ and $x_2(t)$ are both permanent in the mean, i.e., $\alpha_i \leq (x_i(t)) \leq \beta_i$, $i = 1, 2$, where $\zeta_i$ and $\eta_i$ are defined as before and $\bar{\alpha}_i$, $\alpha_i$, and $\beta_i$ are all constants defined later in the proof, $i = 1, 2$.

**Proof.** By Lemma 3, it suffices to prove that there exist some positive constants $\bar{\alpha}_i, \alpha_i$, and $\beta_i$ such that these conclusions hold for system (15). Above all, applying Itô’s formula to $\ln u_i(t)$, we have

\begin{equation}
d \ln u_i = \left( r_i(t) + \frac{1}{T} \sum_{0 < t_k < t} \ln(1 + \lambda_{ik}) - \frac{\sigma_i(t)^2}{2} - a_{i1}(t)A_{11}^u(t)u_1(t) - a_{i2}(t)A_{21}^u(t)u_2(t) - b_i(t)A_i^u(t)u_i^2(t) \right) dt + \sigma_i(t) du_i(t).
\end{equation}
Integrating both sides of (37) from 0 to \( t \), we obtain

\[
\ln u_1 (t) = \ln u_1 (0) + \int_0^t \left( r_1 (s) + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik}) - \frac{\sigma_1 (s)^2}{2} \right) ds
\]

\[- \int_0^t a_{11} (s) A_1 (s) u_1 (s) ds - \int_0^t a_{12} (s) A_2 (s) u_2 (s) ds - \int_0^t b_1 (s) A_1^2 (s) u_1^2 (s) ds + \int_0^t \sigma_1 (s) d\omega_1 (s),
\]

\[
\leq \ln u_1 (0) + \int_0^t \left( r_1 (s) + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik}) - \frac{\sigma_1 (s)^2}{2} \right) ds - a_{11}^\mu A_1^\mu \int_0^t u_1 (s) ds + \int_0^t \sigma_1 (s) d\omega_1 (s).
\]

(38)

Since \( \lim_{t \to \infty} (1/T) \int_0^T \sigma_1 (s) d\omega_1 (s) = 0 \), under \( \eta_1 > 0 \), using comparison theorem of stochastic differential equation, we obtain

\[
\langle u_1 \rangle \leq \left( \frac{\langle r_1 (t) - (\sigma_1 (t)^2/2) \rangle_T + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik})}{a_{11}^\mu A_1^\mu} \right) = \beta_1^\prime.
\]

(39)

On the other hand, by (15), we have

\[
d \ln u_2 (t) = u_2 (t) \left( r_2 (t) + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{2k}) - \frac{\sigma_2 (t)^2}{2} - a_{21} (t) A_1 u_1 (t) - a_{22} (t) A_2 u_2 (t) - b_2 (t) A_2^2 (t) u_2^2 (t) \right) dt + \sigma_2 (t) d\omega_2 (t).
\]

(40)

In the same manner, under condition \( \eta_2 > 0 \), we have

\[
\langle u_2 \rangle \leq \left( \frac{\langle r_2 (t) - (\sigma_2 (t)^2/2) \rangle_T + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{2k})}{a_{22}^\mu A_2^\mu} \right) = \beta_2^\prime.
\]

(41)

(i) If \( \eta_1 < 0 \) and \( \eta_2 < 0 \), then it directly follows from (39) and (41) that \( \lim_{t \to \infty} u_i (t) = 0, i = 1, 2 \). That is, all species of (15) are extinct.

(ii) If \( \eta_2 < 0 \), then \( \lim_{t \to \infty} u_2 (t) = 0 \). Using (37) again, we have

\[
\ln u_1 (t) = \ln u_1 (0) + \int_0^t \left( r_1 (s) + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik}) - \frac{\sigma_1 (s)^2}{2} \right) ds - \int_0^t a_{11} (s) A_1 (s) u_1 (s) ds
\]

\[- \int_0^t a_{12} (s) A_2 (s) u_2 (s) ds - \int_0^t b_1 (s) A_1^2 (s) u_1^2 (s) ds + \int_0^t \sigma_1 (s) d\omega_1 (s),
\]

\[
\geq \ln u_1 (0) + \int_0^t \left( r_1 (s) + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik}) - \frac{\sigma_1 (s)^2}{2} \right) ds
\]

\[- \left( a_{11}^\mu A_1^\mu + b_1^\mu (A_1^2) \right) \int_0^t u_1 (s) ds + \int_0^t \sigma_1 (s) d\omega_1 (s).
\]

(42)

Applying Lemma 2 yields

\[
\langle u_1 \rangle \geq \left( \frac{\langle r_1 (t) - (\sigma_1 (t)^2/2) \rangle_T + \frac{1}{T} \sum_{0 \leq s < T} \ln (1 + \lambda_{ik})}{a_{11}^\mu A_1^\mu + b_1^\mu (A_1^2) \right) = \bar{\alpha}^\prime.
\]

(43)
Similarly, we derive from (41) that Lemma 2 implies

\( (i) \) If \( \eta \leq 0 \), then \( \lim_{t \to \infty} u(t) = 0 \). From (40), we have

\[
\ln u_1(t) \geq \ln u_1(0) + \int_0^t \left( r_1(s) + \frac{1}{T} \sum_{0 \leq i < s} \ln(1 + \lambda_{ik}) - \frac{\sigma_1(s)^2}{2} \right) ds - a_{11} A_1^u{s} + \left( a_{12} A_2^u + b_2 (A_1^u)^2 \zeta_1 \right) \int_0^t u_1(s) ds + \int_0^t \sigma_1(s) d\omega_1(s). \tag{46}
\]

(iii) If \( \eta > 0 \), then \( \lim_{t \to \infty} u_1(t) = 0 \). From (40), we have

\[
\ln u_1(t) = \ln u_1(0) + \int_0^t \left( r_1(s) + \frac{1}{T} \sum_{0 \leq i < s} \ln(1 + \lambda_{ik}) - \frac{\sigma_1(s)^2}{2} \right) ds - \int_0^t a_{21} A_1^u{s} + \left( a_{22} A_2^u + b_2 (A_2^u)^2 \zeta_2 \right) \int_0^t u_2(s) ds + \int_0^t \sigma_2(s) d\omega_1(s). \tag{47}
\]

Similarly, we derive from (41) that

\[
\ln u_2(t) = \ln u_2(0) + \int_0^t \left( r_2(s) + \frac{1}{T} \sum_{0 \leq i < s} \ln(1 + \lambda_{ik}) - \frac{\sigma_2(s)^2}{2} \right) ds - \int_0^t a_{22} A_2^u{s} + \left( a_{22} A_2^u + b_2 (A_2^u)^2 \zeta_2 \right) \int_0^t u_2(s) ds + \int_0^t \sigma_2(s) d\omega_1(s). \tag{48}
\]
Using Lemma 2 yields

\[
\langle u_2 \rangle_* \geq \frac{r_2(t) - \left(\sigma_2(t)^2/2\right)}{T} + \frac{(1/T) \sum_{0 < t_k < t} \ln (1 + \lambda_{2k}) - a_{21}^u A_1^u \zeta_1}{a_{22}^u A_2^u + b_{2}^u (A_2^u)^2 \zeta_2} \quad = \alpha_2'.
\]

(49)

Combining (39) and (41), \( a_i' \leq u_i(t) \leq \beta_i' \) holds for \( i = 1, 2 \).

To summarize the above discussion and combine Lemma 3, the required results are obtained. The proof is completed.

\[ \square \]

**Remark 4.** The result of extinction for all species (case (i)) is in accordance with Theorem 3.2 of [15], but other dynamics such as permanence in the mean for all species (case (ii)–case (iv)) is not studied in [15], which reveals richer dynamical behaviors of this system.

## 5. Stochastic Persistence in Probability

Firstly, we give the following lemmas.

**Lemma 6.** The solutions of (2) are uniformly continuous.

**Proof.** Using the property of expectation and Lemma 5, we have

\[
\mathbb{E} \left[ u_1(t) \left( r_1(t) + \frac{1}{T} \sum_{0 < t_k < t} \ln (1 + \lambda_{1k}) - a_{11} (t) A_1 (t) u_1 (t) - a_{12} (t) A_2 (t) \right) \right]^p \leq \frac{1}{2} \mathbb{E} [ u_1 ]^{2p} + \mathbb{E} \left[ r_1 (t) + \frac{1}{T} \sum_{0 < t_k < t} \ln (1 + \lambda_{1k}) - a_{11} (t) A_1 (t) u_1 (t) - a_{12} (t) A_2 (t) \right]^{2p} \leq \frac{1}{2} \left[ H_1^p + 4^{p-1} \left( r_1^u + \frac{1}{T} \sum_{0 < t_k < t} \ln (1 + \lambda_{1k}) \right)^{2p} + a_{11}^u A_1^u H_1^u (2p) + a_{12}^u A_2^u H_2^u (2p) + b_{1}^u (A_1^u)^2 (t) H_1^u (2p) \right] \leq G_1 (p).
\]

(50)

By stochastic integral inequality, for \( 0 < t_1 < t_2 \) and \( p > 2 \), we obtain

\[
\mathbb{E} \left[ \int_{t_1}^{t_2} \sigma_1 (s) u_1 (s) d\omega_1 (s) \right]^p \leq (\sigma_1^u)^p \left( \frac{p(p - 1)}{2} \right)^{p/2} \left( t_2 - t_1 \right)^{(p-2)/2} \int_{t_1}^{t_2} |u_1|^p \, ds \leq (\sigma_1^u)^p \left( \frac{p(p - 1)}{2} \right)^{p/2} \left( t_2 - t_1 \right)^{p/2} H_1 (p).
\]

(51)
Hence, when $0 < t_1 < t_2 < \infty$, $t_2 - t_1 \leq 1$, and $(\frac{1}{p} + \frac{1}{q}) = 1$, we have

$$
\mathbb{E} \left[ |u_1(t_2) - u_1(t_1)|^p \right] = \mathbb{E} \left[ u_1(t) \left( r_1(t) + \frac{1}{T} \sum_{0 \leq t' \leq t} \ln(1 + \lambda_{1k}) - a_{11}(t)A_1(t)u_1(t) - a_{12}(t)A_2(t)u_2(t) - b_1(t)A_1^2(t)u_1^2(t) \right) ds 
+ \int_{t_1}^{t_2} \sigma_1(s)u_1(s) d\omega_1(s) \right)^p
\leq 2^{p-1} \mathbb{E} \left[ u_1(t) \left( r_1(t) + \frac{1}{T} \sum_{0 \leq t' \leq t} \ln(1 + \lambda_{1k}) - a_{11}(t)A_1(t)u_1(t) - a_{12}(t)A_2(t)u_2(t) - b_1(t)A_1^2(t)u_1^2(t) \right) ds 
+ 2^{p-1} \sigma_1^p(t) u_1(t) d\omega_1(t) \right]^p
\leq 2^{p-1} (t_2 - t_1)^p \mathbb{E} \left[ u_1(t) \left( r_1(t) + \frac{1}{T} \sum_{0 \leq t' \leq t} \ln(1 + \lambda_{1k}) - a_{11}(t)A_1(t)u_1(t) - a_{12}(t)A_2(t)u_2(t) - b_1(t)A_1^2(t)u_1^2(t) \right) ds 
+ 2^{p-1} \sigma_1^p(t) \int_{t_1}^{t_2} \gamma(s) d\omega_1(s) \right]^p
\leq 2^{p-1} (t_2 - t_1)^{p(\rho+1)} G_1(p) + 2^{p-1} \sigma_1^p(t) \int_{t_1}^{t_2} \gamma(s) d\omega_1(s) \right]^p
\leq 2^{p-1} (t_2 - t_1)^{p(\rho+1)} G_2(p),
$$

(52)

where $G_2(p) = \max\{G_1(p), \sigma_1^p[H_1(p)]\}$. Therefore, the solution $u_1(t)$ of (15) is uniformly continuous. Similarly, we can obtain the uniform continuity of $u_2(t)$. Lemma 3 implies the required assertion. This completes the proof. □

Lemma 7 (see [25]). Let $f$ be a nonnegative function defined on $\mathbb{R}$ such that $f$ is integrated and uniformly continuous, then

$$
\lim_{t \to -\infty} f(t) = 0.
$$

Theorem 3. Suppose

$$(H_2)t_1 = a_{11}(t) - a_{21}(t) > 0,$n
$$
(53)

$$
d^*V = \text{sgn}(u_1 - \bar{u}_1) \left\{ -a_{11}(t) A_1(t)(u_1 - \bar{u}_1) - a_{12}(t) A_2(t)(u_2 - \bar{u}_2) - b_1(t) A_1^2(t)(u_1 - \bar{u}_1)(u_1 + \bar{u}_1)(t) \right\} dt
+ \text{sgn}(u_2 - \bar{u}_2) \left\{ -a_{21}(t) A_1(t)(u_1 - \bar{u}_1) - a_{22}(t) A_2(t)(u_2 - \bar{u}_2) - b_2(t) A_2^2(t)(u_2 - \bar{u}_2)(u_2 + \bar{u}_2)(t) \right\} dt
\leq -A_1(t)(a_{11}(t) + 2b_1(t)A_1(t)\zeta - a_{21}(t))|u_1 - \bar{u}_1| - A_2(t)(a_{22}(t) + 2b_2(t)A_2(t)\zeta - a_{12}(t))|u_2 - \bar{u}_2|
\leq -A_1(t)(a_{11}(t) - a_{21}(t))|u_1 - \bar{u}_1| - A_2(t)(a_{22}(t) - a_{12}(t))|u_2 - \bar{u}_2|.
$$

(55)

then the solution $(x_1(t), x_2(t))$ of (2) is globally attractive. Furthermore, for any other solution $(\bar{x}_1(t), \bar{x}_2(t))$, we have

$$
\lim_{t \to -\infty} \mathbb{E} |x_1 - \bar{x}_1| = 0,
$$

$$
\lim_{t \to -\infty} \mathbb{E} |x_2 - \bar{x}_2| = 0.
$$

Proof. Similarly, we only prove the conclusion for (15). Define $V(u_1, u_2) = |\ln u_1 - \ln \bar{u}_1| + |\ln u_2 - \ln \bar{u}_2|$. Applying Itô’s formula and computing the right derivative of function $V(u_1, u_2)$ along the solution of (15) yield
where \( \zeta_i \) is the sup of \( u_i \) and \( u_2 \), i.e., \( u_i \leq \zeta_i (i = 1, 2) \). According to \((H_2)\), there exist \( \rho > 0 \) and \( t_0 > 0 \) such that \( a_{11}(t) - a_{12}(t) > \rho \) and \( a_{22}(t) - a_{12}(t) > \rho \) for \( t \geq t_0 \); hence, \( D^*V_t \leq -\rho \left( |u_1(s) - \bar{u}_1(s)| + |u_2(s) - \bar{u}_2(s)| \right) \) for \( t \geq t_0 \). Integrating both sides of it from \( t_0 \) to \( t \) leads to

\[
V(t) + \rho \int_{t_0}^{t} \left( |u_1(s) - \bar{u}_1(s)| + |u_2(s) - \bar{u}_2(s)| \right) ds \leq V(t_0) < +\infty.
\]

That is,

\[
\begin{align*}
|u_1(t) - \bar{u}_1(t)| & \in L^1[0, +\infty), \\
|u_2(t) - \bar{u}_2(t)| & \in L^1[0, +\infty).
\end{align*}
\]

and hence, \( E(u_t) \) is uniformly continuous. The uniform continuity of \( E(u_t) \) can be similarly obtained. According to (58) and Barbalat’s conclusion [25],

\[
\lim_{t \to \infty} E|u_1(t) - \bar{u}_1(t)| = 0,
\]

\[
\lim_{t \to \infty} E|u_2(t) - \bar{u}_2(t)| = 0.
\]

Therefore, the global attractiveness of (15) is obtained by Lemmas 6 and 7. On the other hand,

\[
\delta_{x(s)} \text{ represents a Dirac measure at } x(s).
\]

**Remark 5.** The definition of \( S_\eta \) shows one or more populations have a density less than \( \eta \), then, stochastic persistence means that all populations spend an arbitrarily small fraction of time at arbitrarily low densities. It is more appropriate than permanence in the mean and Definition 2 of [15], see [27].

**Lemma 8** (see [26, 27]). Let \( A_0 = \{ a \in R^m_+ | a_i = 0 \text{ for some } i, 1 \leq i \leq m \} \). System (61) is said to be stochastically persistent in probability if there exists a unique invariant probability measure \( \mu \) such that \( \mu(A_0) = 0 \) and the distribution of \( x(t) \) converges to \( \mu \) as \( t \to \infty \) whenever \( x(0) \in R^m_+ \).

**Theorem 4.** Suppose \((H_2)\) and \( \eta_i > \kappa_i \), then (2) is stochastically persistent in probability, where \( \eta_i \) and \( \kappa_i (i = 1, 2) \) are defined in Theorem 2.

**Proof.** Firstly, the proof of stochastic persistence in probability is motivated by [20]. Let \((u_1(t), u_2(t)) (u_1(0), u_2(0)) \) be the solution of (15) with initial data \((u_1(0), u_2(0)) \) in \( R^2_+ \), \( P(t, u(0), dy) \) be the transition probability of \( u(t, u(0)) \), and \( P(t, u(0), \Gamma) \) be the probability of events \( u(t, u(0)) \) in \( \Gamma \). The stochastic boundedness of \((u,t,(u(0)))\) implies there exists a compact subset \( \mathcal{K} \in R^2_+ \) such that \( P(t, \phi, \mathcal{K}) \geq 1 - \epsilon \) for any given \( \epsilon > 0 \), that is, \( \{ P(t, u(0), dy) : t \geq 0 \} \) is tight. \( \mathcal{P}(R^2_+) \) represents the probability measure on \( R^2_+ \), then, we prove the series \( \{ P(t, u(0), R^2_+) \} \) is Cauchy in equation.
For any $P_1, P_2 \in \mathcal{P}(R^2_+)$, define the metric of $P_1$ and $P_2$ as follows:

$$d_F(P_1, P_2) = \sup_{f \in F} \int_{R^2_+} |f(u)P_1(du) - \int_{R^2_+} f(u)P_2(du)|,$$

where

$$F = \{f: R^2_+ \rightarrow R \mid \|f(u_1) - f(u_2)\| \leq \|u_1 - u_2\|, f(\cdot) \leq 1\}.$$  \hspace{1cm} (63)

For any $f \in F$ and $s, t > 0$,

$$|Ef(u(t + s, u(0))) - Ef(u(t, u(0)))|$$

$$\leq \int_{R^2_+} |Ef(u(t, u(0)))p(s, u(0), du(0)) - Ef(u(t, u(0)))|$$

$$\leq \int_{R^2_+} \|Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))\|p(s, u(0), d\tilde{u}(0))$$

$$\leq \int_{\mathcal{U}_K} \|Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))\|p(s, u(0), d\tilde{u}(0)) + \int_{R^2_+ \setminus \mathcal{U}_K} \|Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))\|p(s, u(0), d\tilde{u}(0)),$$  \hspace{1cm} (64)

where $\mathcal{U}_K = \{u \in R^2_+: |u| \leq K\}$. Because of the tightness of $\{p(t, u(0), dy)\}$, one can find sufficiently large $K$ such that $P(s, u(0), (R^2_+/\mathcal{U}_K)) < \epsilon$ for any $s > 0$. Then,

$$\int_{R^2_+ \setminus \mathcal{U}_K} |Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))|p(s, u(0), d\tilde{u}(0)) \leq 2P(s, u(0), R^2_+/\mathcal{U}_K) \leq 2\epsilon.$$  \hspace{1cm} (65)

By the global attractivity (Theorem 3), for arbitrarily given $\epsilon > 0$ and sufficiently large $t$, we have

$$\int_{\mathcal{U}_K} |Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))|p(s, u(0), d\tilde{u}(0))$$

$$\leq \int_{\mathcal{U}_K} \|Ef(u(t, \tilde{u}(0))) - Ef(u(t, u(0)))\|p(s, u(0), d\tilde{u}(0))$$

$$\leq \int_{\mathcal{U}_K} \epsilon p(s, u(0), d\tilde{u}(0))$$

$$= \epsilon P(s, u(0), \mathcal{U}_K)$$

$$\leq \epsilon.$$  \hspace{1cm} (66)

Consequently, for any $s > 0$ and sufficiently large $t$,

$$d_F(p(t + s, u(0), \cdot), p(t, u(0), \cdot)) = \sup_{f \in F} |Ef(u(t + s, u(0))) - Ef(u(t, u(0)))| \leq 3\epsilon.$$  \hspace{1cm} (67)
Hence, for any initial data, \( u(0) = (u_1(0), u_2(0)) \in \mathbb{R}^2_+ \), \( \{ p(t, u(0), \cdot) : t \geq 0 \} \) is Cauchy in \( \mathbb{P}(\mathbb{R}^2_+) \) with the metric \( d_F \) and there exists a probability measure \( \nu(\cdot) \) such that, for fixed initial data \( \bar{u}(0) \in \mathbb{R}^2_+ \),

\[
\lim_{t \to \infty} d_F(p(t, u(0), \cdot), \nu(\cdot)) = 0. \tag{68}
\]

Finally, by the triangle inequality, we have

\[
\begin{align*}
\int_0^t d_F( p(t, u(0), \cdot), \nu(\cdot)) & \leq \int_0^t d_F( p(t, \bar{u}(0), \cdot), \nu(\cdot)) + d_F( p(t, \bar{u}(0), \cdot), \nu(\cdot)).
\end{align*}
\tag{69}
\]

In view of the global attractivity, we can deduce

\[
\begin{align*}
d_F( p(t, u(0), \cdot), \nu(\cdot)) & = \sup_{f \in F} \mathbb{E}( f(u(t, u(0))) - \mathbb{E}( f(u(t, \bar{u}(0)))) \\
& \leq \sup_{f \in F} \mathbb{E}( f(u(t, u(0))) - f(u(t, \bar{u}(0)))) + d_F( p(t, \bar{u}(0), \cdot), \nu(\cdot)).
\end{align*}
\tag{70}
\]

Therefore, for any \( u(0) \in \mathbb{R}^2_+ \), \( \lim_{t \to \infty} d_F( p(t, u(0), \cdot), \nu(\cdot)) = 0 \), that is, for any \( u(0) \in \mathbb{R}^2_+ \), there exists unique probability measure \( \nu(\cdot) \) such that the transition probability \( p(t, u(0), \cdot) \) of \( u(t) \) converges weakly to \( \nu(\cdot) \) as \( t \to \infty \). Therefore, (15) is asymptotically stable in distribution. Meanwhile, Theorem 2 implies

\[
\frac{1}{t} \int_0^t u_1(s) ds > 0,
\]

\[
\frac{1}{t} \int_0^t u_2(s) ds > 0,
\tag{71}
\]

a.s.,

and hence, (15) is stochastically persistent in probability by Lemma 8. Applying Lemma 3 yields the required assertion. This completes the proof. \( \square \)

### 6. Examples and Simulations

In this section, by use of the numerical method [28], we give some simulations.

For system (2), with no special mention, we always take \( r_1(t) = 5 + 2 \sin t \), \( a_{11}(t) = 4 + 2 \cos t \), \( a_{12}(t) = 1 + 0.2 \sin t \), \( b_1(t) = 1 + 0.3 \sin t \), \( r_2(t) = 5 + 3 \sin t \), \( a_{21}(t) = 0.4 + 0.2 \cos t \), \( a_{22}(t) = 2 + 0.1 \sin t \), \( b_2(t) = 1 + 0.1 \sin t \), \( \sigma_1(t) = 0.2 + 0.02 \sin t \), \( \sigma_2(t) = 0.1 + 0.01 \sin t \), \( \lambda_{1k} = 0.1 + 0.1 \sin t_k \), \( \lambda_{2k} = 0.1 + 0.1 \sin t_k \), and \( t_k = 0.1k, k \in \mathbb{N} \).

An easy computation shows \( (H_1) \) holds. Theorem 1 implies that the solution of (2) is a periodic Markovian process. Using the Milstein method and writing MATLAB code, we get the simulation result of periodic solution of (2), see Figure 1. Under \( (H_1) \), Figures 1(a) and 1(b) are the time series graphs of \( x_1(t) \) and \( x_2(t) \) for impulsive and stochastic case, respectively. Figures 1(c)–1(f) are the phase graphs of periodic system (2). Let \( \sigma_1(t) = 3 + 0.02 \sin t \) and \( \sigma_2(t) = 3 + 0.01 \sin t \), then the stochastic disturbance is large, then the periodic process does not appear, illustrated in Figure 1(g). If \( \lambda_{1k} = -0.4 + 0.1 \sin t_k \), \( \lambda_{2k} = -0.4 + 0.1 \sin t_k \), and \( t_k = 0.1k, k \in \mathbb{N} \), then there exists no periodic process, see Figure 1(h). Figures 1(g) and 1(h) show that impulse and stochastic perturbation affects the existence of stochastic periodic solution.

Next, we simulate the extinction and permanence in the mean of solution of (2). Let \( \sigma_1(t) = 3.5 + 0.02 \sin t \) and \( \sigma_2(t) = 4.01 \sin t \) or \( \lambda_{1k} = -0.4 + 0.1 \sin t \) and \( \lambda_{2k} = -0.6 + 0.1 \sin t \), then \( \eta_1 < 0 \) and \( \eta_2 > 0 \). Hence, it follows from Theorem 2 that \( x_1(t) \) and \( x_2(t) \) are both extinct, illustrated in Figures 2(a) and 2(b). Let \( \sigma_1(t) = 4 + 0.02 \sin t \) and \( \sigma_2(t) = 2 + 0.01 \sin t \), then \( \eta_1 < 0 \) and \( \eta_2 > 0 \). By Theorem 2 again, \( x_1(t) \) is extinct and \( x_2(t) \) is permanent in the mean, illustrated in Figure 2(c). Let \( \sigma_1(t) = 1 + 0.02 \sin t \) and \( \sigma_2(t) = 5 + 0.01 \sin t \), then \( \eta_1 > 0 \) and \( \eta_2 < 0 \). Using Theorem 2 yields that \( x_1(t) \) is permanent in the mean and \( x_2(t) \) is extinct, illustrated in Figure 2(d). Let \( \sigma_1(t) = 1 + 0.02 \sin t \) and \( \sigma_2(t) = 1 + 0.01 \sin t \), then \( \eta_1 > k_1 \) and \( \eta_2 > k_2 \). Similarly, Theorem 2 implies \( x_1(t) \) and \( x_2(t) \) are both permanent in the mean, see Figures 2(e) and 2(f). Figure 2 shows that the impulse disturbance and stochastic disturbance bring some important influence to the dynamics of all species.
Figure 2: Continued.
Finally, we simulate the global attractivity and stochastic persistence in probability of solution of (2). By computation, all parameters meet the conditions of Theorem 4; therefore, the solution of (2) is globally attractive (see Figures 3(a) and 3(b)) and the distribution of all species are asymptotic stability (see Figure 4(a)). The stochastic persistence is simulated in Figures 4(b)–4(d), which are the time series graphs of $x_1(t)$, $x_2(t)$, and $z(t) = (x_1(t) + x_2(t))^{1/2}$, respectively.
7. Conclusions and Discussions

In this paper, for the stochastic competitive system with impulsive and nonlinear term of self-interaction proposed in [15], we further study such dynamics as the existence of periodic solution, the extinction and permanence in the mean, the global attractivity of solutions, and stochastic persistence of this system. Theorem 1 gives the sufficient conditions of the existence of periodic Markovian process. Theorem 2 gives the conditions of extinction and permanence in the mean of both species. Theorem 3 establishes the condition assuring the global attractivity. Theorem 4 establishes the condition of stochastic persistence in probability of this system. Finally, simulations (Figures 1–4) are given to verify the obtained results. Our main results are new and different from [15], which is presented by giving Remarks 3, Remark 4, and Remark 5 in detail.

Three or more species often coexist in the real world, and time delays often appear in biological system, then how to deal with the effects of time delays on the stochastic behaviors of three-species biological system is very interesting to be further investigated. On the other hand, regime switching is another common random perturbation, e.g., stochastic hybrid phytoplankton-zooplankton model with toxin-producing phytoplankton, stochastic tumor-immune model with regime switching, and impulsive perturbations. All these are necessary and very interesting for us to study in the future.

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

WK carried out all studies and drafted the manuscript. YS conceived the study, performed the simulation, and helped to draft the manuscript.

Acknowledgments

This work was supported by the Natural Science Foundation of China (11861027).

References

[1] A. J. Lotka, Elements of Mathematical Biology, Dover, New York, NY, USA, 1924.
[2] V. Volterra, Leçons sur la Théorie Mathématique de la Lutte pour la Vie, Gauthier-Villars, Paris, France, 1931.
[3] F. J. Ayala, M. E. Gilpin, and J. G. Ehrenfeld, “Competition between species: theoretical models and experimental tests,” Theoretical Population Biology, vol. 4, no. 3, pp. 331–356, 1973.
[4] X. Liu and L. Chen, “Global dynamics of the periodic logistic system with periodic impulsive perturbations,” Journal of Mathematical Analysis and Applications, vol. 289, no. 1, pp. 279–291, 2004.
[5] Y. Shao, Y. Chen, and B. X. Dai, “Dynamical analysis and optimal harvesting of a stochastic three-species cooperative system with delays and Lévy jumps,” Advances in Difference Equations, vol. 2018, p. 423, 2018.
[6] N. H. Du and V. H. Sam, “Dynamics of a stochastic Lotka-Volterra model perturbed by white noise,” Journal of Mathematical Analysis and Applications, vol. 324, no. 1, pp. 82–97, 2006.
[7] X. Li and X. Mao, “Population dynamical behaviour of non-autonomous Lotka-Volterra competitive system with random perturbation,” Discrete & Continuous Dynamical Systems—A, vol. 24, no. 2, pp. 523–545, 2009.
[8] W. Zuo, D. Jiang, X. Sun, T. Hayat, and A. Alsaeedi, “Long-time behaviors of a stochastic cooperative Lotka-Volterra system with distributed delay,” Physica A: Statistical Mechanics and Its Applications, vol. 506, pp. 542–559, 2018.
[9] K. Tran and G. Yin, “Stochastic competitive Lotka-Volterra ecosystems under partial observation: feedback controls for permanence and extinction,” Journal of the Franklin Institute, vol. 351, no. 8, pp. 4039–4064, 2014.
[10] Y. Shao and Y. Li, “Dynamical analysis of a stage structured predator-prey system with impulsive diffusion and generic functional response,” Applied Mathematics and Computation, vol. 220, pp. 472–481, 2013.
[11] Y. Xie, L. Wang, Q. Deng, and Z. Wu, “The dynamics of an impulsive predator-prey model with communicable disease in the prey species only,” Applied Mathematics and Computation, vol. 292, pp. 320–335, 2017.
[12] S. Zhang and D. Tan, “Dynamics of a stochastic predator-prey system in a polluted environment with pulse toxicant input and impulsive perturbations,” Applied Mathematical Modelling, vol. 39, no. 20, pp. 6319–6331, 2015.
[13] S. Zhang, X. Meng, T. Feng, and T. Zhang, “Dynamics analysis and numerical simulations of a stochastic non-autonomous predator-prey system with impulsive effects,” Nonlinear Analysis: Hybrid Systems, vol. 26, pp. 19–37, 2017.
[14] M. Liu, C. Du, and M. Deng, “Persistence and extinction of a modified Leslie-Gower Holling-type II stochastic predator-prey model with impulsive toxicant input in polluted environments,” Nonlinear Analysis: Hybrid Systems, vol. 27, pp. 177–190, 2018.
[15] R. Tan, Z. Liu, S. Guo, and H. Xiang, “On a nonautonomous competitive system subject to stochastic and impulsive perturbations,” Applied Mathematics and Computation, vol. 256, pp. 702–714, 2015.
[16] L. Chen and F. Chen, “Dynamic behaviors of the periodic predator-prey system with distributed time delays and impulsive effect,” Nonlinear Analysis: Real World Applications, vol. 12, no. 4, pp. 2467–2473, 2011.
[17] R. J. Sacker and H. F. von Bremen, “A conjecture on the stability of the periodic solutions of Ricker’s equation with periodic parameters,” Applied Mathematics and Computation, vol. 217, no. 3, pp. 1213–1219, 2010.
[18] C. Zhang, X.-j. Han, and Q. Bi, “Complicated behaviors and bifurcation mechanism of the periodic parameter-switching Chen system,” Optik, vol. 130, pp. 568–575, 2017.
[19] Y. Niu and X. Li, “Periodic solutions of semilinear Duffing equations with impulsive effects,” Journal of Mathematical Analysis and Applications, vol. 467, no. 1, pp. 349–370, 2018.
[20] Y. Zhang, S. Chen, S. Gao, and X. Wei, “Stochastic periodic solution for a perturbed non-autonomous predator-prey model with generalized nonlinear harvesting and impulses,” Physica A: Statistical Mechanics and Its Applications, vol. 486, pp. 347–366, 2017.
[21] D. Jiang, Q. Zhang, T. Hayat, and A. Alsaeedi, “Periodic solution for a stochastic non-autonomous competitive Lotka-Volterra model in a polluted environment,” Physica A: Statistical Mechanics and Its Applications, vol. 471, pp. 276–287, 2017.
[22] H. Qi, S. Zhang, X. Meng, and H. Dong, “Periodic solution and ergodic stationary distribution of two stochastic SIQS epidemic systems,” Physica A: Statistical Mechanics and Its Applications, vol. 508, pp. 223–241, 2018.
[23] R. Wu, X. Zou, and K. Wang, “Asymptotic behavior of a stochastic non-autonomous predator-prey model with impulsive perturbations,” Communications in Nonlinear Science and Numerical Simulation, vol. 20, no. 3, pp. 965–974, 2015.
[24] C. Ji and D. Jiang, “Dynamics of a stochastic density dependent predator-prey system with Beddington-DeAngelis functional response,” Journal of Mathematical Analysis and Applications, vol. 381, no. 1, pp. 441–453, 2011.
[25] I. Barbalat, “Systemes d’équations différentielles d’osci d’oscillations nonlinéaires,” Revue Roumaine de Mathématiques Pures et Appliqués, vol. 4, pp. 267–270, 1959.
[26] S. J. Schreiber, M. Benaïm, and K. A. S. Atchadé, “Persistence in fluctuating environments,” Journal of Mathematical Biology, vol. 62, no. 5, pp. 655–683, 2011.
[27] M. Liu and C. Bai, “Analysis of a stochastic tri-trophic food-chain model with harvesting,” Journal of Mathematical Biology, vol. 73, no. 3, pp. 597–625, 2016.
[28] D. J. Higham, “An algorithmic introduction to numerical simulation of stochastic differential equations,” SIAM Review, vol. 43, no. 3, pp. 525–546, 2001.