Building Blocks of Physical States in a Non-Critical 3-Brane on $R \times S^3$

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Abstract

The physical states in a world-volume model of a non-critical 3-brane are systematically constructed using techniques of four-dimensional conformal field theories on $R \times S^3$ developed recently. Invariant combinations of creation modes under a special conformal transformation provide building blocks of physical states. Any state can be created by acting with such building blocks on a conformally invariant vacuum in an invariant way under the other conformal charges: the Hamiltonian and rotation generators on $S^3$. We explicitly construct building blocks for scalar, vector and gravitational fields, and classify them as finite types.
1 Introduction

Conformal invariance is one of the most important symmetry in statistical mechanics, strings and quantized gravity. Applications of conformally invariant quantum field theories to such physics are modern streams to study their non-perturbative effects. Especially, it is well-known that in two dimensions an infinite number of generators form the Virasoro algebra, and yield powerful constraints on the classification and the physical properties of two-dimensional critical points [1] and strings [2, 3]. In higher dimensions, the number of the generators becomes finite, but the conformal invariance is still powerful.

Recently, there have been remarkable developments in four-dimensional conformal field theories [4, 5, 6, 7, 8, 9, 10, 11]. As an advance on space-time physics [5, 6, 7, 8], a renormalizable world-volume model of a non-critical 3-brane [7, 8] was quantized on $R \times S^3$ [6, 8] in a strong gravity phase, in which world-volume fluctuations are dominated by the conformal field. This model possesses exact conformal symmetry as a realization of the background-metric independence. The conformal symmetry is generated by 15 conformal charges: the Hamiltonian, $H$, the rotation generators on $S^3$, $R_{MN}$, and the charges for the special conformal transformations, $Q_M$, and their conjugates, $Q_M^\dagger$. These charges satisfy the conformal algebra [6, 8]:

\[
\begin{align*}
\left[ Q_M, Q_N^\dagger \right] &= 2\delta_{MN}H + 2R_{MN}, \\
\left[ H, Q_M \right] &= -Q_M, \\
\left[ H, R_{MN} \right] &= \left[ Q_M, Q_N \right] = 0, \\
\left[ Q_M, R_{M_1 M_2} \right] &= \delta_{M_1 M_2} Q_{M_1} - \epsilon_{M_1} \epsilon_{M_2} \delta_{M_1 - M_2} Q_{M_1 - M_2}, \\
\left[ R_{M_1 M_2}, R_{M_3 M_4} \right] &= \delta_{M_1 M_4} R_{M_1 M_2} - \epsilon_{M_1} \epsilon_{M_2} \delta_{M_1 - M_2} R_{M_1 - M_2} \\
&\quad - \delta_{M_2 M_3} R_{M_1 M_4} + \epsilon_{M_1} \epsilon_{M_2} \delta_{M_1 M_3} R_{M_2 M_4}. 
\end{align*}
\] (1.1)

Explicit forms of the conformal charges for scalar [6], vector [8] and gravitational fields [8, 6] have been constructed.

The conformal invariance imposes strong constraints on the physical states. The physical states in a non-critical 3-brane must satisfy the conformal in-
variance conditions,
\[ Q_M \langle \text{phys} | = H \langle \text{phys} | = R_{MN} \langle \text{phys} | = 0. \]  

(1.2)

Such states have been constructed in Ref.[8]. In this paper, we further develop the arguments and give a systematic method to construct and classify the physical states.

The physical state is divided into sectors of matter fields and gravitational fields. Here, we consider the scalar and vector fields as matter fields. The gravitational fields are further decomposed into two sectors: the conformal mode and the traceless mode. Dynamical fields are mode-expanded in the spherical tensor harmonics on \( S^3 \). The standard Fock state created by acting with a creation mode on a vacuum is an eigenstate of the Hamiltonian belonging to a certain representation of the rotation group on \( S^3 \). However, the charge for the special conformal transformation, \( Q_M \), maps a creation mode to another creation mode belonging to a different representation. Therefore, this charge yields a strong constraint on the physical states. We seek a \( Q_M \)-invariant combination of creation modes in each sector. Such an operator provides a building block of physical states. We have found that apart from a few creation modes, such a building block is obtained by a particular combination of the products of two creation modes, and classified as finite types. Any state satisfying the first condition in (1.2) is created by acting with building blocks on a conformally invariant vacuum. The third condition in (1.2) can be easily satisfied by combining all sectors in a rotation-invariant way. The Hamiltonian condition is imposed last by adjusting the zero-mode momentum existing in the conformal field. In this way we can obtain an infinite number of physical states in a non-critical 3-brane.

This paper is organized as follows. In the next section we briefly review a world-volume model of a non-critical 3-brane on \( R \times S^3 \) developed in Ref.[8]. We here give mode expansions of dynamical fields, canonical commutation relations, and explicit forms of the conformal charges and so on. In Sect.3 we discuss conformally coupled scalar fields in detail. Basic ideas on how to construct and classify physical states in a non-critical 3-brane are given here. A \( Q_M \)-invariant creation operator with a scalar index is the building block in this sector. In order to help our intuitive understanding of the structure

\footnote{Here, these charges include the ghost sector in the radiation\(^+\) gauge discussed in Appendix F.}
of states, a graphical representation is introduced here. In Sect.4 we develop an argument for the case of vector fields. We find that there are three types of building blocks with scalar, vector and rank 2 tensor indices. In Sect.5 we construct building blocks for the traceless mode sector. For gravitational fields, both positive-metric and negative-metric modes are required to form a closed conformal algebra. However, no negative-metric modes commute with the charge $Q_M$, and thus these modes are not independent physical modes. We find that $Q_M$-invariant building blocks are given by particular combinations of the products of such positive-metric and negative-metric modes, apart from the lowest positive-metric mode. They are classified as seven types with tensor indices up to rank 4. The building blocks for the conformal mode sector are constructed in Sect.6. They are classified as two types with a scalar index. Because of the existence of the zero mode, the conformal mode sector is managed separately. Physical states are constructed in Sect.7 by acting with building blocks on a conformally invariant vacuum in such a way that the conditions for the Hamiltonian and the rotation generators in Eqs.(1.2) are satisfied. The physical states up to level 6 are constructed explicitly. Sect.8 is devoted to conclusions and a discussion.

2 A World-Volume Model of a Non-Critical 3-Brane

In this section we review recent developments on a world-volume model of a non-critical 3-brane [8].

2.1 Canonical Quantization on $R \times S^3$

The world-volume metric is decomposed to the conformal mode, $\phi$, and the traceless mode, $h^{\lambda\nu}$, as

$$g_{\mu\nu} = e^{2\phi} \hat{g}_{\mu\lambda}(\delta^{\lambda\nu} + th^\lambda_{\nu} + \cdots),$$

with $tr(h) = 0$, and $\hat{g}_{\mu\nu}$ is the background metric. In this paper we use the $R \times S^3$ background with the Lorentzian signature ($-1,1,1,1$). The traceless mode, whose field strength is given by the square of the Weyl tensor divided by $t^2$, is handled perturbatively in terms of the coupling $t$, while the conformal mode is treated exactly. Recently, it was shown that the perturbative
expansion in $t$ is renormalizable and asymptotically free [7]. The asymptotic freedom implied at very high energies above the Planck mass the Weyl tensor should vanish, and world-volume fluctuations are dominated by the conformal mode. The dynamics is described by a four-dimensional conformal field theory (CFT$_4$). In the following we consider the case of the vanishing coupling realized at very high energies where mass scales can be neglected.

The action for the conformal mode is induced from the measure, as in the case of a non-critical string [3]. The four-dimensional counteraction of the Polyakov-Liouville action in a non-critical string is the Riegert action [5], given by

$$S = -\frac{b_1}{(4\pi)^2} \int d^4x \sqrt{-\hat{g}} \left\{ 2\hat{\Delta}_4 \phi + \hat{E}_4 \phi \right\},$$

(2.2)

where $\sqrt{-\hat{g}}\Delta_4$ is a conformally invariant 4-th order operator, defined by $\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla_\mu R) \nabla_\mu$, and $E_4 = G_4 - \frac{2}{3} \Box R$ and $G_4$ is the Euler density. This action is related to the conformal anomaly proportional to the Euler density. The coefficient $b_1$ has been calculated as

$$b_1 = \frac{1}{360} \left( N_X + \frac{11N_W}{2} + 62N_A \right) + \frac{769}{180},$$

where $N_X$, $N_W$ and $N_A$ are the number of conformal scalar fields, Weyl fermions and gauge fields, respectively.

The kinetic term of the traceless mode is given by the linearized form of the Weyl action, which is invariant under a gauge transformation, $\delta_\xi h_{\mu\nu} = \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \frac{1}{2} \hat{g}_{\mu\nu} \hat{\nabla}^\lambda \xi_\lambda$. To quantize the traceless mode, we take the radiation$^+$ gauge [8],

$$h^0_0 = \hat{\nabla}_i h^i_0 = \hat{\nabla}_i h^i_j = 0.$$

$^2$More precisely, the diffeomorphism transformation, $\delta_\xi g_{\mu\nu} = g_{\nu\lambda} \nabla_\mu \xi^\lambda + g_{\mu\lambda} \nabla_\nu \xi^\lambda$, is decomposed into the transformation of the conformal mode, $\delta_\xi \phi = \xi^\lambda \partial_\lambda \phi + \frac{1}{2} \hat{\nabla}_\lambda \xi^\lambda$, and that of the traceless mode expanded by the coupling as

$$\delta_\xi h_{\mu\nu} = \frac{1}{t} \left( \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu - \frac{1}{2} \hat{g}_{\mu\nu} \hat{\nabla}^\lambda \xi_\lambda \right)$$

$$+ \xi^\lambda \hat{\nabla}_\lambda h_{\mu\nu} + \frac{1}{2} \hat{h}_{\mu\lambda} \left( \hat{\nabla}_\nu \xi^\lambda - \hat{\nabla}^\lambda \xi_\nu \right) + \frac{1}{2} \hat{h}_{\nu\lambda} \left( \hat{\nabla}_\mu \xi^\lambda - \hat{\nabla}^\lambda \xi_\mu \right) + o(t),$$

where $\xi_\mu = \hat{g}_{\mu\nu} \xi^\nu$. The linearized Weyl action is invariant under the lowest term of this transformation, $\delta_\xi h_{\mu\nu}$, and therefore we write here only this term, in which $\xi/t$ is replaced by $\xi$. It is worth commenting that in the case that $\xi^\mu$ are the conformal Killing vectors, the lowest term of the transformation vanishes, and thus the next order terms become effective, even in the linearized model. This transformation is the conformal transformation discussed in the text.
where \( h_{1/2}^0 \) is the lowest mode of \( h^0i \) in the spherical-harmonics expansion defined below. In this gauge, the space of the residual gauge symmetry becomes equivalent to that spanned by conformal Killing vectors on \( R \times S^3 \).

A world-volume model of a non-critical 3-brane coupled to the conformal scalar and vector fields is considered here. For the vector field, the radiation gauge, \( A^0 = \hat{\nabla}_i A^i = 0 \), is taken. The action on \( R \times S^3 \) in the radiation gauge is given by

\[
I_{\text{CFT}} = \int dt \int_{S^3} d^3x \sqrt{\gamma} \left\{ -\frac{2b_1}{(4\pi)^2} \phi \left( \partial_t^4 - 2\hat{\nabla}^2 \partial_t^2 + \hat{\nabla}^4 + 4\partial_t^2 \right) \phi \\
- \frac{1}{2} h^i_j \left( \partial_t^4 - 2\hat{\nabla}^2 \partial_t^2 + \hat{\nabla}^4 + 8\partial_t^2 - 4\hat{\nabla}^2 + 4 \right) h^j_i \\
+ h^0_i \left( \hat{\nabla}^2 + 2 \right) \left( -\partial_t^2 + \hat{\nabla}^2 - 2 \right) h^0i \\
+ \frac{1}{2} X \left( -\partial_t^2 + \hat{\nabla}^2 - 1 \right) X \\
+ \frac{1}{2} A_i \left( -\partial_t^2 + \hat{\nabla}^2 - 2 \right) A^i \right\},
\]

where \( \hat{\nabla}^2 = \hat{\nabla}^i \hat{\nabla}_i \) is the Laplacian on \( S^3 \). The background metric is parametrized as in (A.1), in which the radius of \( S^3 \) is taken to be unity. Here and henceforth, \( t \) denotes the time, not the traceless mode coupling.

Since the model is conformally invariant, we could take any conformal background. The advantages that we use the \( R \times S^3 \) background are that mode expansions of higher-derivative gravitational fields have quite simple forms, and then canonical commutation relations of these modes become diagonal. Furthermore, we can use the properties of Clebsch-Gordan coefficients because the isometry group of \( S^3 \) is \( SO(4) = SU(2) \times SU(2) \).

Dynamical fields are expanded in symmetric-traceless-transverse (ST\(^2\)) spherical tensor harmonics \([12, 10, 8]\). The ST\(^2\) tensor harmonics of rank \( n \) are classified using the \((J + \varepsilon_n, J - \varepsilon_n)\) representation of \( SU(2) \times SU(2) \) for each sign of \( \varepsilon_n = \pm \frac{n}{2} \). They are, denoted by \( Y_{J(M\varepsilon_n)}^{i_1 \cdots i_n} \), the eigenfunction of the Laplacian on \( S^3 \),

\[
\hat{\nabla}^2 Y_{J(M\varepsilon_n)}^{i_1 \cdots i_n} = \{-2J(2J + 2) + n\} Y_{J(M\varepsilon_n)}^{i_1 \cdots i_n},
\]
where $J \geq \frac{n}{2}$ takes integer or half-integer values, and $M = (m, m')$ represents the multiplicity for $\varepsilon_n$, which takes the following values:

\[
\begin{align*}
m &= -J - \varepsilon_n, \quad -J - \varepsilon_n + 1, \ldots, J + \varepsilon_n - 1, \quad J + \varepsilon_n, \\
m' &= -J + \varepsilon_n, \quad -J + \varepsilon_n + 1, \ldots, J - \varepsilon_n - 1, \quad J - \varepsilon_n.
\end{align*}
\] (2.6)

Thus, the multiplicity of the ST$^2$ tensor harmonic of rank $n$ is given by the product of the left and right $SU(2)$ multiplicities, $(2(2J + \varepsilon_n) + 1)(2(2J - \varepsilon_n) + 1)$, for each sign of $\varepsilon_n$, and thus it is totally $2(2J + n + 1)(2J - n + 1)$ for $n \geq 1$. For $n = 0$, the multiplicity is given by $(2J + 1)^2$. The explicit forms of the ST$^2$ tensor harmonics of any rank are given in Ref.[8]. They are normalized as

\[
\int_{S^3} d\Omega_3 Y^{i_1 \cdots i_n *}_{(M \varepsilon_n)} Y^{i_1 \cdots i_n}_{(M \varepsilon_n)} = \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{\varepsilon_1 \varepsilon_2},
\] (2.7)

where

\[
Y^{i_1 \cdots i_n *}_{(M \varepsilon_n)} = (-1)^n \epsilon_M Y^{i_1 \cdots i_n}_{(-M \varepsilon_n)}.
\] (2.8)

Below, we use the following parametrizations for the tensor indices up to rank 4:

\[
\begin{align*}
\varepsilon_0 &= 0, \quad \varepsilon_1 = y = \pm \frac{1}{2}, \quad \varepsilon_2 = x = \pm 1, \quad \varepsilon_3 = z = \pm \frac{3}{2}, \quad \varepsilon_4 = w = \pm 2.
\end{align*}
\] (2.9)

From the CFT$^4$ action (2.4), we can easily obtain the equations of motion and mode expansions of the dynamical fields. The scalar and vector fields are expanded as

\[
\begin{align*}
X &= \sum_{J \geq 0} \sum_M \frac{1}{\sqrt{2(2J + 1)}} \left\{ \varphi_{JM} e^{-i(2J + 1)t} Y_{JM} + \varphi^\dagger_{JM} e^{i(2J + 1)t} Y^*_{JM} \right\},
\end{align*}
\] (2.10)

\[
\begin{align*}
A^i &= \sum_{J \geq \frac{1}{2}} \sum_{M, y} \frac{1}{\sqrt{2(2J + 1)}} \left\{ q_{J(My)} e^{-i(2J + 1)t} Y^i_{J(My)} + q^\dagger_{J(My)} e^{i(2J + 1)t} Y^*_{J(My)} \right\}.
\end{align*}
\] (2.11)

These fields are normalized such that the canonical commutation relations become

\[
\begin{align*}
&\{ \varphi_{J_1 M_1}, \varphi^\dagger_{J_2 M_2} \} = \delta_{J_1 J_2} \delta_{M_1 M_2}, \\
&\{ q_{J_1 (M_1 y_1)}, q^\dagger_{J_2 (M_2 y_2)} \} = \delta_{J_1 J_2} \delta_{M_1 M_2} \delta_{y_1 y_2}.
\end{align*}
\] (2.12)
where \( \delta_{M_1M_2} = \delta_{m_1m_2}\delta_{m_1'm_2'} \).

The mode expansions of gravitational fields are given by

\[
\phi = \frac{\pi}{2\sqrt{b_1}} \left[ 2(\hat{q} + \hat{p}t)Y_{00} + \sum_{J \geq \frac{1}{2}} \sum_{M} \frac{1}{\sqrt{J(2J + 1)}} \left\{ a_{JM} e^{-itJ}Y_{JM} + a_{JM}^\dagger e^{itJ}Y_{JM}^* \right\} + \sum_{J \geq 0} \sum_{M} \frac{1}{\sqrt{(J + 1)(2J + 1)}} \left\{ b_{JM} e^{-i(2J+2)t}Y_{JM} + b_{JM}^\dagger e^{i(2J+2)t}Y_{JM}^* \right\} \right],
\]

(2.13)

\[
h_{ij} = \frac{1}{4} \sum_{J \geq 1} \sum_{M,x} \frac{1}{\sqrt{J(2J + 1)}} \left\{ c_{J(M,x)} e^{-i2Jt}Y_{iJ}^{(x)} + c_{J(M,x)}^\dagger e^{i2Jt}Y_{iJ}^{(x)*} \right\} + \frac{1}{4} \sum_{J \geq 1} \sum_{M,x} \frac{1}{\sqrt{(J + 1)(2J + 1)}} \left\{ d_{J(M,x)} e^{-i(2J+2)t}Y_{iJ}^{(x)} + d_{J(M,x)}^\dagger e^{i(2J+2)t}Y_{iJ}^{(x)*} \right\},
\]

(2.14)

\[
h_{0i} = \frac{1}{2} \sum_{J \geq 1} \sum_{M,y} \frac{1}{\sqrt{(2J - 1)(2J + 1)(2J + 3)}} \left\{ e_{J(M,y)} e^{-i(2J+1)t}Y_{iJ}^{(y)} + e_{J(M,y)}^\dagger e^{i(2J+1)t}Y_{iJ}^{(y)*} \right\},
\]

(2.15)

where \( Y_{00} = \frac{1}{\sqrt{\text{Vol}(S^3)}} = \frac{1}{\sqrt{2\pi}} \). The radiation+ gauge (2.3) implies that in the mode expansion of \( h_{0i} \) the lowest mode with \( J = \frac{1}{2} \) is removed, because this mode satisfies the equation \( (\nabla^2 + 2)h_{1/2} = 0 \), and therefore there is no kinetic term of this mode. The canonical commutation relations of the gravitational modes are given by

\[
[\hat{q}, \hat{p}] = i,
\]

\[
\left[ a_{J_1M_1}, a_{J_2M_2}^\dagger \right] = -\left[ b_{J_1M_1}, b_{J_2M_2}^\dagger \right] = \delta_{J_1J_2}\delta_{M_1M_2},
\]

\[
\left[ c_{J_1(M_1x_1)}, c_{J_2(M_2x_2)}^\dagger \right] = -\left[ d_{J_1(M_1x_1)}, d_{J_2(M_2x_2)}^\dagger \right] = \delta_{J_1J_2}\delta_{M_1M_2}\delta_{x_1x_2},
\]

\[
\left[ e_{J_1(M_1y_1)}, e_{J_2(M_2y_2)}^\dagger \right] = -\delta_{J_1J_2}\delta_{M_1M_2}\delta_{y_1y_2}.
\]

(2.16)

Thus, the \( a_{JM} \) and \( c_{J(M,x)} \) are positive-metric modes, and the \( b_{JM} \), \( d_{J(M,x)} \) and \( e_{J(M,y)} \) are negative-metric modes.
2.2 Conformal charges on $R \times S^3$

Because the conformal field, $\phi$, is quantized exactly without introducing the coupling constant concerning this field, the model possesses exact conformal invariance. This conformal symmetry is generated by 15 charges: the Hamiltonian, the 6 rotation generators on $S^3$, and the 8 charges for the special conformal transformations.

The Hamiltonian is given by

$$
H = \frac{1}{2}p^2 + b_1 + \sum_{J \geq 0} \sum_M \left\{ 2Ja^\dagger_{JM}a_{JM} - (2J + 2)b^\dagger_{JM}b_{JM} \right\} \\
+ \sum_{J \geq 1} \sum_{M,x} \left\{ 2Je^\dagger_{J(Mx)}e_{J(Mx)} - (2J + 2)d^\dagger_{J(Mx)}d_{J(Mx)} \right\} \\
- \sum_{J \geq 1} \sum_{M,y} (2J + 1)e^\dagger_{J(My)}e_{J(My)} \\
+ \sum_{J \geq 0} \sum_M (2J + 1)\varphi^\dagger_{JM}\varphi_{JM} \\
+ \sum_{J \geq \frac{1}{2}} \sum_{M,y} (2J + 1)q^\dagger_{J(My)}q_{J(My)}. 
$$

(2.17)

The rotation generators on $S^3$, $R_{MN}$, satisfy the relations

$$
R_{MN} = -\epsilon_M\epsilon_NR_{-N-M}, \quad R^\dagger_{MN} = R_{NM},
$$

(2.18)

where $\epsilon_M = (-1)^{m-m'}$, and the indices, $M$ and $N$, are the 4 vectors on $SU(2) \times SU(2)$. From these relations only 6 of these generators are independent. If we parametrize the 4 representation of $SU(2) \times SU(2)$ as $\{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})\} = (1,2,3,4)$, and identify $A_+ = R_{31}$, $A_- = R_{13}^\dagger$, $A_3 = \frac{1}{2}(R_{11} + R_{22})$, $B_+ = R_{21}$, $B_- = R_{21}^\dagger$ and $B_3 = \frac{1}{2}(R_{11} - R_{22})$, the closed algebra of $R_{MN}$ in (1.1) can be expressed by the standard $SU(2) \times SU(2)$ algebra, i.e.,

$$
[A_+, A_-] = 2A_3, \quad [A_3, A_\pm] = \pm A_\pm, \\
[B_+, B_-] = 2B_3, \quad [B_3, B_\pm] = \pm B_\pm,
$$

(2.19)

and $A_{\pm,3}$ and $B_{\pm,3}$ commute. The generators $A_{\pm,3}$ ($B_{\pm,3}$) act on the left (right) $SU(2)$ index of $M = (m,m')$ in each mode. Explicit forms of $R_{MN}$ are given in Ref.[8].
The most important conformal charges to determine the physical states are the charges for the special conformal transformations, denoted by $Q_M$, and their hermite conjugates. For the scalar and vector fields, they are given by

\begin{equation}
Q_M = \sum_{J \geq 0} \sum_{M_1, M_2} \mathcal{C}^{\frac{1}{2}M}_{JM, J+\frac{1}{2}M_2} \sqrt{(2J + 1)(2J + 2)} \tilde{\varphi}^J_{JM_1} \varphi_{J+\frac{1}{2}M_2} - \sum_{J \geq \frac{1}{2}} \sum_{M_1, y_1, M_2, y_2} \mathcal{D}_{J(M_1 y_1), J+\frac{1}{2}(M_2 y_2)}^{\frac{1}{2}M} \times \sqrt{(2J + 1)(2J + 2)} \tilde{q}^J_{J(M_1 y_1)} q_{J+\frac{1}{2}(M_2 y_2)}.
\end{equation}

(2.20)

For the gravitational fields, they are given by

\begin{equation}
Q_M = \left( \sqrt{2b_1 - i\hat{p}} \right) a^{\frac{1}{2}M}_J + \sum_{J \geq 0} \sum_{M_1, M_2} \mathcal{C}^{\frac{1}{2}M}_{JM, J+\frac{1}{2}M_2} \left\{ \alpha(J) \tilde{a}^J_{JM} a_{J+\frac{1}{2}M_2} + \beta(J) \tilde{b}^J_{JM} b_{J+\frac{1}{2}M_2} + \gamma(J) \tilde{a}^J_{J+\frac{1}{2}M_2} b_{JM} \right\}
+ \sum_{J \geq 1} \sum_{M_1, x_1, M_2, x_2} \mathcal{E}^{\frac{1}{2}M}_{J(M_1 x_1), J+\frac{1}{2}(M_2 x_2)} \left\{ \alpha(J) \tilde{c}^J_{J(M_1 x_1)} c_{J+\frac{1}{2}(M_2 x_2)} + \beta(J) \tilde{d}^J_{J(M_1 x_1)} d_{J+\frac{1}{2}(M_2 x_2)} + \gamma(J) \tilde{c}^J_{J+\frac{1}{2}(M_2 x_2)} d_{J(M_1 x_1)} \right\}
+ \sum_{J \geq 1} \sum_{M_1, x_1, M_2, y_2} \mathcal{H}^{\frac{1}{2}M}_{J(M_1 x_1), J(M_2 y_2)} \left\{ \alpha(J) \tilde{e}^J_{J(M_1 x_1)} e_{J(M_2 y_2)} + \beta(J) \tilde{f}^J_{J(M_1 x_1)} f_{J(M_2 y_2)} + \gamma(J) \tilde{e}^J_{J(M_2 y_2)} f_{J(M_1 x_1)} \right\}
+ \sum_{J \geq 1} \sum_{M_1, y_1, M_2, y_2} \mathcal{D}^{\frac{1}{2}M}_{J(M_1 y_1), J+\frac{1}{2}(M_2 y_2)} \left\{ \alpha(J) \tilde{q}^J_{J(M_1 y_1)} q_{J+\frac{1}{2}(M_2 y_2)} + \beta(J) \tilde{r}^J_{J(M_1 y_1)} r_{J+\frac{1}{2}(M_2 y_2)} + \gamma(J) \tilde{q}^J_{J+\frac{1}{2}(M_2 y_2)} r_{J(M_1 y_1)} \right\}.
\end{equation}

(2.21)

Here, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{H}$ are the $SU(2) \times SU(2)$ Clebsch-Gordan coefficients, defined in Appendix B. The modes with the tilde are defined by:

\begin{align*}
\tilde{\varphi}^J_M &= \epsilon_M \varphi^{J-M}, \\
\tilde{q}^J_{J(M)} &= \epsilon_M q^{J(-M)}.
\end{align*}
\[ \tilde{a}_{JM} = \epsilon_M a_{J-M}, \quad \tilde{b}_{JM} = \epsilon_M b_{J-M}, \]
\[ \tilde{c}_{J(Mx)} = \epsilon_M c_{J(-Mx)}, \quad \tilde{d}_{J(Mx)} = \epsilon_M d_{J(-Mx)}, \quad \tilde{e}_{J(My)} = \epsilon_M e_{J(-My)} \]

(2.22)

where \( \epsilon_M = (-1)^{m-m'} \). The coefficients are given by:

\[
\begin{align*}
\alpha(J) &= \sqrt{2J(2J+2)}, \\
\beta(J) &= -\sqrt{(2J+1)(2J+3)}, \\
\gamma(J) &= 1, \\
A(J) &= \frac{4J}{\sqrt{(2J-1)(2J+3)}}, \\
B(J) &= \frac{2(2J+2)}{\sqrt{(2J-1)(2J+3)}}, \\
C(J) &= \frac{(2J-1)(2J+1)(2J+2)(2J+4)}{2J(2J+3)}.
\end{align*}
\]

(2.23)

These conformal charges satisfy the closed algebra (1.1).

3 Building Blocks for Scalar Fields

We first consider the scalar field sector in the physical states. Basic ideas on how to construct building blocks of physical states in a non-critical 3-brane are given here.

The standard Fock state created by acting with a creation mode on a vacuum is an eigenstate of the Hamiltonian. It belongs to a representation of the rotation group on \( S^3, SU(2) \times SU(2) \). However, the charges for the special conformal transformations, \( Q_M \), map a creation mode to another creation mode belonging to a different representation. Therefore, the \( Q_M \) conditions in (1.2) are non-trivial, and we must impose them in each sector, while the \( H \) and \( R_{MN} \) conditions are imposed last after combining all sectors.

Let us seek creation operators that commute with \( Q_M \). The commutator of \( Q_M \) and the creation mode \( \varphi_{JM}^\dagger \) is calculated as

\[ [Q_M, \varphi_{JM_1}^\dagger] = \sqrt{2J(2J+1)} \sum_{M_2} \epsilon_{M_1} C_{JM_1,J-M_2,M_2}^M \varphi_{JM_2}^\dagger. \]

(3.1)
Thus, only the lowest mode, $\varphi_{00}^\dagger$, commutes with $Q_M$.

Consider creation operators with the scalar index $JN$, constructed from the products of two creation modes. The general operator with level $H = 2L + 2$ is given by

$$\tilde{\Phi}_{JN}^{[2L+2]} = \sum_{K=0}^{L} \sum_{M_1, M_2} \tilde{f}(L, K) C_{L-K M_1, K M_2}^{JN} \varphi_{L-K M_1}^{\dagger} \varphi_{K M_2}^{\dagger}. \quad (3.2)$$

An operator without the tilde is defined by $\Phi_{JN}^{[2L+2]} = \epsilon_N \tilde{\Phi}_{J-N}^{[2L+2]}$. Because the property of the $SU(2) \times SU(2)$ Clebsch-Gordan coefficient, $C_{J_1 M_1, J_2 M_2}^{J-N} = C_{J_1-M_1, J_2-M_2}$, and $N = M_1 + M_2$, such that $\epsilon_N = \epsilon_{M_1} \epsilon_{M_2}$, $\Phi_{JN}^{[2L+2]}$ is expressed by (3.2) without the tildes on $\varphi^\dagger$s. The function $\tilde{f}$ is defined by

$$\tilde{f}(L, K) = \frac{f(L, K)}{\sqrt{(2L-2K+1)(2K+1)}}, \quad (3.3)$$

where $f$ satisfies the symmetric condition

$$f(L, K) = f(L, L - K). \quad (3.4)$$

Because of the triangular conditions for the Clebsch-Gordan coefficient of type $C$ (B.2), this operator exists for $J \leq L$.

Here, a graphical representation is introduced to help our intuitive understanding of the structure of the physical states. The creation and annihilation modes are described as

\[
\begin{array}{cccc}
J & J & J & J \\
\varphi_{JM}^\dagger & \varphi_{JM}^\dagger & \varphi_{JM} & \varphi_{JM} \\
\varnothing & \varnothing & \times & \times \\
\end{array}
\]

The $SU(2) \times SU(2)$ Clebsch-Gordan coefficients of type $C$ are
Using these graphs, the creation operator (3.2) is expressed as

\[ \tilde{\Phi}^{[2L+2]\dagger} = \sum_K \tilde{f}(L, K) \]

Also, the conformal charge, \( Q_M \), is expressed as

\[ Q_M = \sum_J \rho(J) \]

where \( \rho(J) = \sqrt{(2J + 1)(2J + 2)} \).

The commutator of \( Q_M \) and the creation operator (3.2) is calculated as

\[ [Q_M, \tilde{\Phi}^{[2L+2]\dagger}] = \sum_{K=0}^{L} \sum_{M_1, M_2} \tilde{\phi}^{\dagger}_{L-K-K/2-M_1} \tilde{\phi}^{\dagger}_{K-M_2} \times \sum_S \left\{ f(L, K) \sqrt{\frac{2L - 2K}{2K + 1}} \epsilon_S C^{1M}_{L-K-K/2-M_1, L-K-S} C^{JN}_{L-KS, K-M_2} + f(L, K + \frac{1}{2}) \sqrt{\frac{2K + 1}{2L - 2K}} \epsilon_S C^{4M}_{K-M_2, K+1-S} C^{JN}_{K+1S, L-K-K/2-M_1} \right\}. \]
This equation is graphically expressed in Fig.1.

\[ [Q_M, \tilde{\Phi}^{[2L+2]}_{JN}] \]

\[
= \sum_K \rho(L - K - \frac{1}{2}) \tilde{f}(L, K) + \sum_K \rho(K - \frac{1}{2}) \tilde{f}(L, K)
\]

\[
= \sum_K \left\{ \rho(L - K - \frac{1}{2}) \tilde{f}(L, K) + \rho(K) \tilde{f}(L, K + \frac{1}{2}) \right\}
\]

Figure 1: Commutator of the charge, \( Q_M \), and the operator, \( \tilde{\Phi}^{[2L+2]}_{JN} \).

Let us seek a function \( f \) that makes the r.h.s. of Eq.(3.5) vanish. To find such a \( f \), crossing relations among the \( SU(2) \times SU(2) \) Clebsch-Gordan coefficients of type \( C \) are useful. Here, we consider the integral of the product of four scalar harmonics,

\[
\int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y_{J_1 M_1} Y_{J_2 M_2} Y^*_{J_N}.
\]  (3.6)

Using the product expansion

\[
Y^*_{\frac{1}{2}M} Y_{J_1 M_1} = \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I = J_1 + \frac{1}{2}} \sum_S \epsilon_S C^M_{J_1, M_1, I - S} Y_{I S},
\]  (3.7)

where the product is taken at the same point, and \( S = (s, s') \), we obtain the crossing relation (Fig.2)

\[
\sum_{I = J_1 + \frac{1}{2}} \sum_{S} \epsilon_S C^M_{J_1 M_1, I - S} C^{J_N}_{I S, J_2 M_2} = \sum_{I = J_2 + \frac{1}{2}} \sum_{S} \epsilon_S C^M_{J_2 M_2, I - S} C^{J_N}_{I S, J_1 M_1}.
\]  (3.8)
Consider relation (3.8) with the values $J_1 = L - K - \frac{1}{2}$ and $J_2 = K$. In this case, the intermediate values, $I$, are $L - K - 1$ and $L - K$ for the l.h.s., and $K - \frac{1}{2}$ and $K + \frac{1}{2}$ for the r.h.s. of this equation. To make the commutator (3.5) vanish, crossing relations with the intermediate values, $L - K$ and $K + \frac{1}{2}$, are required. For the general value of $J$, this condition is not satisfied. The crossing relation that we seek is obtained if we take $J = L$ because of the triangular conditions for the Clebsch-Gordan coefficients of type C (B.2).

Using this crossing relation we find that if $f$ satisfies the recursion relation,

$$f \left( L, K + \frac{1}{2} \right) = - \frac{2L - 2K}{2K + 1} f(L, K), \quad (3.9)$$

the commutator (3.5) vanishes. By solving this recursion relation, we obtain

$$f(L, K) = (-1)^{2K} \left( \frac{2L}{2K} \right) \quad (3.10)$$

up to the $L$-dependent normalization. Note that this solution satisfies the equation $f(L, L - K) = (-1)^{2L} f(L, K)$. However, $f$ must satisfy the symmetric condition (3.4). Hence, $f$ is given by Eq.(3.10) for integer $L$, while $f$ vanishes for half integer $L$.

Thus, we find that the creation operators (3.2) commute with $Q_M$ only when $J = L$, and $f$ is given by (3.10), where $L$ is a zero or positive integer. Hereafter, we express these operators as

$$\Phi_{L}^{\dagger} \equiv \tilde{\Phi}_{L}^{[2L+2]\dagger} = \sum_{K=0, M_1, M_2}^{L} \frac{(-1)^{2K}}{\sqrt{(2L - 2K + 1)(2K + 1)}} \left( \frac{2L}{2K} \right) \times C^{L}_{L-KM_1, K M_2} \tilde{\varphi}_{L-KM_1}^{\dagger} \tilde{\varphi}_{K M_2}^{\dagger}. \quad (3.11)$$

If we impose $Z_2$ symmetry, $X \leftrightarrow -X$, the operator $(\varphi_{00}^{\dagger})^n$ with odd $n$ is excluded, while that with even $n$ is generated from $\Phi_{00}^{\dagger} = (\varphi_{00}^{\dagger})^2$. 

Figure 2: Crossing relation (3.8).
These operators, $\Phi_{LN}^\dagger$, provide building blocks of physical states in the scalar field sector. Any state satisfying the $Q_M$-invariance condition is created by these operators. It is a Hamiltonian eigenstate belonging to a certain representation of the rotation group on $S^3$. The rotation invariant state is obtained by contracting out all scalar indices, $N$, using the $SU(2) \times SU(2)$ Clebsch-Gordan coefficients. For example, such invariant combinations are constructed as (Fig.3)

$$\sum_N \Phi_{LN}^\dagger \Phi_{LN} |0\rangle, \quad \sum_{N_1,N_2,N_3} C_{L_1 N_1,L_2 N_2}^{L_3 N_3} \Phi_{L_1 N_1}^\dagger \Phi_{L_2 N_2}^\dagger \Phi_{L_3 N_3}^\dagger |0\rangle, \quad (3.12)$$

where $|0\rangle$ indicates the standard Fock vacuum.

Figure 3: Examples for the $Q_M$ and $R_{MN}$ invariant states in the scalar field sector. The circles with $\Phi^\dagger$ inside denote the creation operators, $\Phi^\dagger$. The arrow is defined as in the case of creation modes $\varphi^\dagger$.

In this way, an infinite number of states can be constructed from the building blocks, $\Phi_{LN}^\dagger$. These states are graphically represented by tree diagrams in which the operators are connected using the Clebsch-Gordan coefficients. Using crossing relations, we can deform any type of tree diagram to a comb-type tree diagram in Fig.4. Loop diagrams also reduce to tree diagrams due to the properties of the Clebsch-Gordan coefficients [13]. Thus, we can deform any type of connected diagram to the comb-type tree diagram. Therefore, all $Q_M$-invariant states will be classified by the comb-type tree diagrams constructed from the building blocks, $\Phi_{LN}^\dagger$, with integer $L$.

Here, we consider states restricted within the scalar field sector. In general, the $R_{MN}$ condition should be imposed last after all sectors are combined.
4 Building Blocks for Vector Fields

Next, we consider the vector field sector. The commutator of the charge \( Q_M \) and the creation mode \( \tilde{q}^\dagger_{J(M_y)} \) is given by

\[
[Q_M, \tilde{q}^\dagger_{J(M_y)}] = -\sqrt{2J(2J + 1)} \sum_{M_2, y_2} \epsilon_{M_1} D^{JM}_{J(M_y), J - \frac{1}{2}(M_2 y_2)} \tilde{q}^\dagger_{J - \frac{1}{2}(M_2 y_2)}.
\]

(4.1)

Thus, the lowest mode, \( \tilde{q}^\dagger_{\frac{1}{2}(M_y)} \), is the only creation mode that commutes with \( Q_M \).

As in the case of the scalar field, we seek the \( Q_M \)-invariant creation operators constructed from the product of two creation modes. For the vector field, we must consider creation operators with scalar, vector and rank 2 tensor indices. The general forms of such creation operators with level \( 2L + 2 \) are given by

\[
\tilde{\Psi}^{[2L+2] \dagger}_{JN} = - \sum_{K=\frac{1}{2} M_1, y_1, M_2, y_2} \tilde{f}_0(L, K) D^{IJ}_{L-K(M_1 y_1), K(M_2 y_2)} \tilde{q}^\dagger_{L-K(M_1 y_1)} \tilde{q}^\dagger_{K(M_2 y_2)},
\]

(4.2)

\[
S, A \tilde{\xi}^{[2L+2] \dagger}_{J(Ny)} = - \sum_{K=\frac{1}{2} M_1, y_1, M_2, y_2} \tilde{f}_1(L, K) S, A V^{J(Ny)}_{L-K(M_1 y_1), K(M_2 y_2)} \tilde{q}^\dagger_{L-K(M_1 y_1)} \tilde{q}^\dagger_{K(M_2 y_2)},
\]

(4.3)
These operators without the tilde are defined by the relations

\[ \tilde{\Psi}^{[2L+2]}_{J(N_x)} = \epsilon_N \Psi^{[2L+2]}_{J_{-N}} , \tilde{\Xi}^{[2L+2]}_{J(N_y)} = \epsilon_N \Xi^{[2L+2]}_{J_{-N_y}} \text{ and } \tilde{\Upsilon}^{[2L+2]}_{J(N_z)} = \epsilon_N \Upsilon^{[2L+2]}_{J_{-N_z}} . \]

The functions \( \bar{f}_n \), with \( n = 0, 1, 2 \), are defined by

\[ \bar{f}_n(L, K) = \frac{f_n(L, K)}{\sqrt{(2L - 2K + 1)(2K + 1)}} , \]

where \( f_n \) satisfy the symmetric conditions

\[ f_n(L, L - K) = f_n(L, K) . \]

The new \( SU(2) \times SU(2) \) Clebsch-Gordan coefficients of types \( S, A, V \) and \( F \) are defined by

\[ S^{J(M_y)}_{J_1(M_1y_1), J_2(M_2y_2)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} \hat{\nabla}^{(i} Y^{j)}_{J(M_y)} Y_{iJ_1(M_1y_1)} Y_{jJ_2(M_2y_2)} , \]

\[ A^{V^{J(M_y)}}_{J_1(M_1y_1), J_2(M_2y_2)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} \hat{\nabla}^{[i} Y^{j]}_{J(M_y)} Y_{iJ_1(M_1y_1)} Y_{jJ_2(M_2y_2)} , \]

\[ F^{J(M_x)}_{J_1(M_1y_1), J_2(M_2y_2)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} Y^{ij*}_{J(M_x)} Y_{iJ_1(M_1y_1)} Y_{jJ_2(M_2y_2)} . \]

Graphically, these operators are expressed as

\[ \Psi^{[2L+2]}_{J(N_x)} = \sum_K \bar{f}_0(L, K) \]

\[ \Xi^{[2L+2]}_{J(N_y)} = \sum_K \bar{f}_1(L, K) \]

\[ \Upsilon^{[2L+2]}_{J(N_z)} = \sum_K \bar{f}_2(L, K) \]
The wavy and spiral lines indicate the vector and rank 2 tensor indices, respectively. The vertices denote the $SU(2) \times SU(2)$ Clebsch-Gordan coefficients of types, $D$, $V$ and $F$, respectively. Each line has an arrow, but it is omitted here. It can be easily recovered.

First, consider the creation operator with scalar index (4.2). Because of the triangular conditions of type $D$ (B.4), this operator exists for $J \leq L$. The commutator of this operator and $Q_M$ is calculated as

$$\left[ Q_M, \tilde{\Psi}^{[2L+2] \dagger}_{JN} \right] = -\sum_K \tilde{f}_2(L, K) \sum_{\frac{L-K}{2}} \sum_{\frac{M_1y_1}{2}} \sum_{\frac{M_2y_2}{2}} \tilde{q}^\dagger_{L-K, \frac{L-K}{2}(M_1y_1)} \tilde{q}^\dagger_{M_2y_2}
\times \sum_{V,y} \left[ f_0(L, K) \sqrt{\frac{2L - 2K}{2K + 1}} \epsilon_V D^M_{L-K-\frac{1}{2}(M_1y_1), L-K(-V)y} D^{JN}_{L-K(V)y, K(M_2y_2)}
+ f_0 \left( L, K + \frac{1}{2} \right) \sqrt{\frac{2K + 1}{2L - 2K}} \epsilon_V D^M_{(M_2y_2), K+\frac{1}{2}(-V)y} D^{JN}_{K+\frac{1}{2}(V)y, L-K-\frac{1}{2}(M_1y_1)} \right].$$

Therefore, according to the procedure developed in Sect.3, we seek a crossing relation that consists of only the type $D$ coefficients in order to find a function $f_0$ that makes the r.h.s. of this commutator vanish.

Consider the integral of the product of two scalar and two vector harmonics,

$$\int_{S^3} d\Omega_3 Y^*_M Y^i_{J_1(M_1y_1)} Y^j_{J_2(M_2y_2)} Y^*_N.$$  (4.11)

From the product expansion,

$$Y^*_M Y^i_{J_1(M_1y_1)}$$
\[
\begin{align*}
&= -\frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I = J_1 \pm \frac{1}{2}} \sum_{V:y} \epsilon_V D^{M}_{J_1(M_1y_1), I(-V_y) J_1(V_y)} \\
&\quad + \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I = J_1} \sum_{S} \frac{1}{2I(2I + 2)} \epsilon_S G^{M}_{J_1(M_1y_1); I-S} \hat{Y}^i Y IS, \tag{4.12}
\end{align*}
\]
we obtain the crossing relation (Fig.5)

\[
\begin{align*}
&\sum_{I = J_1 \pm \frac{1}{2}} \sum_{V:y} \epsilon_V D^{M}_{J_1(M_1y_1), I(-V_y) D^{JN}_{I(V_y), J_2(M_2y_2)}} \\
&\quad - \sum_{I = J_1} \sum_{S} \frac{1}{2I(2I + 2)} \epsilon_S G^{M}_{J_1(M_1y_1); I-S} G^{JN}_{J_2(M_2y_2); IS} \\
&= [J_1(M_1y_1) \leftrightarrow J_2(M_2y_2)]. \tag{4.13}
\end{align*}
\]

Figure 5: The crossing relation (4.13).

Substituting the values \( J_1 = L - K - \frac{1}{2} \) and \( J_2 = K \) into this crossing relation, we find that the intermediate values, \( I \), are restricted to be \( L - K - 1 \) and \( L - K \ (K + \frac{1}{2}) \) for the \( \mathbf{D} \cdot \mathbf{D} \) part in the l.h.s. (r.h.s.), and \( L - K - \frac{1}{2} \ (K) \) for the \( \mathbf{G} \cdot \mathbf{G} \) part in the l.h.s. (r.h.s.), respectively. The crossing relation that we seek is that in which the \( \mathbf{D} \cdot \mathbf{D} \) part in the l.h.s. (r.h.s.) has intermediate values of \( L - K \ (K + \frac{1}{2}) \), and the \( \mathbf{G} \cdot \mathbf{G} \) parts vanish. These conditions are satisfied only when \( J = L \), because the triangular conditions of type \( \mathbf{D} \) (B.4) and type \( \mathbf{G} \) (B.8) read

\[
\begin{align*}
&D^{LN}_{L-K-1(V_y), K(M_2y_2)} = D^{LN}_{K-\frac{1}{2}(V_y), L-K-\frac{1}{2}(M_1y_1)} = 0, \tag{4.14} \\
&G^{LN}_{K(M_2y_2); L-K-\frac{1}{2}S} = G^{LN}_{L-K-\frac{1}{2}(M_1y_1); KS} = 0. \tag{4.15}
\end{align*}
\]

Using this crossing relation, we find that the commutator (4.10) vanishes if the function \( f_0 \) satisfies the same recursion relation to \( f \) (3.9) in the scalar field sector, and hence

\[
f_0(L, K) = f(L, K), \tag{4.16}
\]
where $f$ is given by Eq.(3.10), and $L$ is an integer in order to satisfy the symmetric condition (4.6).

Thus, we obtain a creation operator with scalar index that commutes with $Q_M$ as

$$\tilde{\Psi}^\dagger_{LM} \equiv (2L+2)\tilde{\Psi}^\dagger_{LM} = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1,y_1,M_2,y_2} \sqrt{(2L-2K+1)(2K+1)} \left( \frac{2L}{2K} \right) \times \mathbf{D}^{LM}_{L-K(M_1y_1),M_2y_2} \tilde{q}^\dagger_{L-K(M_1y_1)} \tilde{q}^\dagger_{L-K(M_2y_2)};$$  \hspace{1cm} (4.17)

with integer $L(\geq 2)$. Here, $L = 1$ is trivial because $[Q_M, q^\dagger_{\frac{1}{2}(My)}] = 0$, and therefore it is removed.

In the same way, we can obtain a $Q_M$-invariant creation operator with rank 2 tensor index. The triangular conditions of the type $F$ coefficient (4.9) can be obtained from the expression

$$F^{J_{(Mx)}}_{J_1(M_1y_1),J_2(M_2y_2)} \propto C^{J_{+x}m}_{J_1+y_1m_1,J_2+y_2m_2} C^{J_{-x}m'}_{J_1-y_1m_1',J_2-y_2m_2'};$$  \hspace{1cm} (4.18)

and thus this coefficient is non-vanishing for

$$J \leq J_1 + J_2,$$  \hspace{1cm} (4.19)

with integer $J + J_1 + J_2$, where the equality is saturated only when $x, y_1, y_2$ have the same sign. Therefore, the creation operator (4.4) exists for $J \leq L$. The commutator of this operator and $Q_M$ is calculated as

$$[Q_M, \tilde{\Psi}^{[2L+2]}_{LM}] = -\sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1,y_1,M_2,y_2} \tilde{q}^\dagger_{L-K-\frac{1}{2}(M_1y_1)} \tilde{q}^\dagger_{L-K-\frac{1}{2}(M_2y_2)} \times \sum_{V,y} \left\{ f_2(L, K) \sqrt{\frac{2L-2K}{2K+1}} \frac{c_{V}}{D^{\frac{1}{2}}_{L-K-\frac{1}{2}(M_1y_1),L-K(-V)y}} \mathbf{F}^{J_{(My)}}_{L-K(-V)y,K(M_2y_2)} \right. \left. + f_2 \left( L, K + \frac{1}{2} \right) \sqrt{\frac{2K+1}{2L-2K}} \frac{c_{V}}{D^{\frac{1}{2}}_{L-K-\frac{1}{2}(M_2y_2),L-K+\frac{1}{2}(V)y}} \mathbf{F}^{J_{(My)}}_{K+\frac{1}{2}(V)y,L-K-\frac{1}{2}(M_1y_1)} \right\}.$$  \hspace{1cm} (4.20)

The crossing relation that we need in this case is obtained from the integral

$$\int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^*_{j_1(M_1y_1)} Y^*_{j_2(M_2y_2)} Y^*_{ij,Nx}$$  \hspace{1cm} (4.21)
as (Fig.6)

\[ \sum_{I=J_1+\frac{1}{2}}^{J} \sum_{V:y} \epsilon_{V} D_{J_1(M_1y_1),I(-Vy)}^{\pm M} F_{I(Vy),J_2(M_2y_2)}^{I(Nx)} + \sum_{I=J_1}^{L} \sum_{S} \frac{1}{2I(2I+2)} G_{J_1(M_1y_1),IS}^{\pm M} H_{I(-N_x);J_2(M_2y_2)}^{IS} \]

\[ = [J_1(M_1y_1) \leftrightarrow J_2(M_2y_2)], \quad (4.22) \]

where the product expansion (4.12) is used.

\[ \frac{1}{2} \quad J_1 \pm \frac{1}{2} \quad J \quad + \quad \frac{1}{2} \quad J_1 \quad \xrightarrow{\leftrightarrow} \quad J_2 \]

\[ J_2 \]

Figure 6: Crossing relation (4.22).

The necessary condition for the commutator (4.20) to vanish is now that there is a crossing relation with \( J_1 = L - K - \frac{1}{2} \) and \( J_2 = K \) that consists of only the \( D \cdot F \) part with the intermediate value \( I = L - K (K + \frac{1}{2}) \) in the l.h.s. (r.h.s.) of (4.22). If we take \( J = L \), we obtain the required relation, because the triangular conditions of type \( F \) (4.19) and type \( H \) (B.10) read

\[ F_{L-K-1(Vy),K(M_2y_2)}^{L(Nx)} = F_{K-\frac{1}{2}(Vy),L-K-\frac{1}{2}(M_1y_1)}^{L(Nx)} = 0, \quad (4.23) \]

\[ H_{L(Nx);K(M_2y_2)}^{L-K-\frac{1}{2}S} = H_{L(Nx);L-K-\frac{1}{2}(M_1y_1)}^{KS} = 0. \quad (4.24) \]

Using this crossing relation, we find that \( f_2 \) must also satisfy the same recursion relation to \( f \) in order that the commutator (4.20) vanishes. Thus, we obtain

\[ f_2(L, K) = f(L, K), \quad (4.25) \]

where \( L \) is an integer in order to satisfy the symmetric condition (4.6).

Thus, the \( Q_M \)-invariant creation operator with rank 2 tensor index is given by

\[ \tilde{\Upsilon}_{L(Nx)}^{[2L+2]t} = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1,y_1} \sum_{M_2,y_2} \frac{(-1)^{2K+1}}{\sqrt{(2L - 2K + 1)(2K + 1)}} \left( \begin{array}{c} 2L \\ 2K \end{array} \right). \]
\[ \times \mathcal{F}^{L(M_2y_2)}_{L-K(M_1y_1),K(M_2y_2)} \hat{q}^\dagger_{L-K(M_1y_1)} \hat{q}^\dagger_{K(M_2y_2)}, \]  
(4.26)

with integer \( L(\geq 2) \).

Next, consider the creation operators with vector index (4.3). The commutators of \( Q_M^J \) and these operator are given by expression (4.10) with the quantities \( f_0, D^J_{L-K,K} \) and \( s.A V^J_{K+\frac{1}{2},L-K-\frac{1}{2}} \) replaced by \( f_1, s.A V^J_{L-K,K} \) and \( s.A V^J_{K+\frac{1}{2},L-K-\frac{1}{2}} \), respectively. Thus, the necessary conditions for these commutators to vanish are that there are crossing relations that consist of only the \( D, s.A V \) parts.

Consider the following integrals:

\[ \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^j_{j_1(M_1 y_1)} Y^j_{j_2(M_2 y_2)} \hat{\nabla}^*(i) Y^*_{j_2(M_2 y_2)}, \]  
(4.27)

\[ \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^j_{j_1(M_1 y_1)} Y^j_{j_2(M_2 y_2)} \hat{\nabla}^i (i) Y^*_{j_2(M_2 y_2)}. \]  
(4.28)

These integrals give the crossing relations (Fig.7)

\[ \sum_{I=J_1 \pm \frac{1}{2}} \sum_{V,y'} \epsilon_V D^\frac{1}{2} M_{j_1(M_1 y_1),I(-V' y')} s.A V^J_{I(V'y'),j_2(M_2 y_2)} \]

\[ - \sum_{I=J_1 t} \sum_S \frac{1}{2I(2I+2)} \epsilon_S G^\frac{1}{2} M_{j_1(M_1 y_1);I-S} s.A U^J_{J_2(M_2 y_2);I S} \]

\[ = [J_1(M_1 y_1) \leftrightarrow J_2(M_2 y_2)], \]  
(4.29)

where

\[ s.U^J_{J_1(M_1 y_1);J_2 M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 \hat{\nabla}^*(i) Y^j_{J_2(M_2 y_2)} Y_{j_1(M_1 y_1)} \hat{\nabla}^i j_2(M_2 y_2) \]  
(4.30)

\[ a.U^J_{J_1(M_1 y_1);J_2 M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 \hat{\nabla}^i (i) Y^j_{J_2(M_2 y_2)} Y_{j_1(M_1 y_1)} \hat{\nabla}^*(i) j_2(M_2 y_2). \]  
(4.31)

Figure 7: Crossing relations (4.29).
Here, the triangular conditions for the Clebsch-Gordan coefficients of types $S^AV$ and $S^AU$ are obtained from expression (C.1). From the expression

$$S^AV^{J(My)}_{J_1(M_1y_1),J_2(M_2y_2)} \propto C^{J+y m}_{J_1+y_1 m_1,J_2+y_2 m_2} C^{J-y m'}_{J_1-y_1 m_1',J_2-y_2 m_2'},$$

(4.32)

this coefficient is non-vanishing for

$$J \leq J_1 + J_2 - \frac{1}{2},$$

(4.33)

with half integer $J + J_1 + J_2$. Also, from the expression

$$S^AU^{J(My)}_{J_1(M_1y_1):J_2M_2} \propto C^{J+y m}_{J_1+y_1 m_1,J_2 m_2} C^{J-y m'}_{J_1-y_1 m_1',J_2 m_2'},$$

(4.34)

we obtain the non-vanishing condition for this coefficient as

$$J \leq J_1 + J_2,$$

(4.35)

with integer $J + J_1 + J_2$. The equality in (4.35) is saturated only at $y = y_1$.

In order that the $Q_M$ invariant operator of type (4.3) exists, it is required that there is a crossing relation with $J_1 = L - K - \frac{1}{2}$ and $J_2 = K$ that consists of only the $D \cdot S^AV$ parts with the intermediate values $L - K$ and $K + \frac{1}{2}$ in the l.h.s. and r.h.s. of (4.29), respectively. The problem of finding such a crossing relation is to find a quantity $J$ satisfying the following conditions:

$$S^AV^{J(Ny)}_{L-K(Vy'),K(M_2y_2)} \neq 0, \quad S^AV^{J(Ny)}_{L-K-1(Vy'),K(M_2y_2)} = 0,$$

(4.36)

and

$$S^AU^{J(Ny)}_{K(M_2y_2):L-K-\frac{1}{2}s} = 0.$$

(4.37)

From inequality (4.33), the conditions (4.36) are satisfied only when $J = L - \frac{1}{2}$. However, because of the triangular condition for $S^AU$ (4.35), this value does not satisfy the second condition (4.37). Thus, we cannot make the $G \cdot U$ parts in the relation (4.29) vanish. Consequently, we find that there is no $Q_M$-invariant creation operator with vector index of type (4.3). Therefore, the only building block with vector index is given by the lowest vector mode, $q_{\frac{1}{2}(Ny)}^\dagger$.

Finally, we briefly show that there is no other $Q_M$-invariant creation operators with tensor index of higher rank, $n$. Such a operator is obtained by
replacing the Clebsch-Gordan coefficients of type $D$ in expression (4.2) with the generalized one, $^n\mathbf{D}$, defined in Appendix C. As in the previous argument, the necessary condition for the commutator of this operator and $Q_M$ to vanish is that there is a crossing relation that consists of the only $D \cdot ^n\mathbf{D}$ parts. The crossing relation with such $D \cdot ^n\mathbf{D}$ terms is given by the type III relation (E.8) derived in Appendix E. However, for $n \geq 3$, because of the triangular conditions of type $^n\mathbf{H}$ (C.6) and type $^n\mathbf{G}$ (C.9), this relation does not close only within the $D \cdot ^n\mathbf{D}$ parts. Thus, there is no $Q_M$-invariant creation operator with index of rank $n \geq 3$.

The two types of creation operators, $\Psi^\dagger_{LN}$ (4.17) and $\Upsilon^\dagger_{L(Nx)}$ (4.26), and the lowest creation mode, $q^\dagger_{\mathfrak{g}(Ny)}$, provide the building blocks of physical states in the vector field sector. They are summarized in Table 2. Any $Q_M$-invariant Hamiltonian eigenstate belonging to a certain representation of $R_{MN}$ will be constructed from these operators, using the $SU(2) \times SU(2)$ Clebsch-Gordan coefficients. As in the case of the scalar field sector, such a state will be classified by the comb-type tree diagram.

| rank of tensor index | 0   | 1   | 2   |
|----------------------|-----|-----|-----|
| creation op.         | $\Psi^\dagger_{LN}$ | $q^\dagger_{\mathfrak{g}(Ny)}$ | $\Upsilon^\dagger_{L(Nx)}$ |
| level ($L \in \mathbb{Z}_{\geq 2}$) | $2L + 2$ | $2$ | $2L + 2$ |

Table 2: Building blocks in the vector field sector.

5 Building Blocks for Gravitational Traceless Fields

In this section, we construct and classify building blocks for the traceless mode sector. The commutators of $Q_M$ and the traceless modes are given by:

\[
[Q_M, \tilde{c}^\dagger_{J(M_1x_1)}] = \alpha \left( J - \frac{1}{2} \right) \sum_{M_2,x_2} \epsilon_{M_1} E^M_{J(-M_1x_1),J-\frac{1}{2}(M_2x_2)} \tilde{c}^\dagger_{J-\frac{1}{2}(M_2x_2)} (5.1)
\]

\[
[Q_M, \tilde{d}^\dagger_{J(M_1x_1)}] = -\gamma (J) \sum_{M_2,x_2} \epsilon_{M_1} E^M_{J(-M_1x_1),J+\frac{1}{2}(M_2x_2)} \tilde{c}^\dagger_{J+\frac{1}{2}(M_2x_2)} - \beta \left( J - \frac{1}{2} \right) \sum_{M_2,x_2} \epsilon_{M_1} E^M_{J(-M_1x_1),J-\frac{1}{2}(M_2x_2)} \tilde{d}^\dagger_{J-\frac{1}{2}(M_2x_2)}
\]

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\[-B(J) \sum_{M_2,y_2} \epsilon_{M_1} \mathbf{H}^{M}_{J(-M_1x_1);J(M_2x_2)} \tilde{c}^{\dagger}_{J(M_2y_2)}, \quad (5.2)\]

\[[Q_M, \tilde{c}^{\dagger}_{J(M_1y_1)}] = -A(J) \sum_{M_2,x_2} \epsilon_{M_1} \mathbf{H}^{M}_{J(M_2x_2);J(-M_1y_1)} \tilde{c}^{\dagger}_{J(M_2x_2)}
- C(J - \frac{1}{2}) \sum_{M_2,y_2} \epsilon_{M_1} \mathbf{D}^{M}_{J(-M_1y_1),J(-\frac{1}{2}(M_2y_2))} \tilde{c}^{\dagger}_{J(-\frac{1}{2}(M_2y_2))}, \quad (5.3)\]

The only \(Q_M\)-invariant mode is the lowest rank 2 tensor creation mode with a positive metric, \(\tilde{c}^{\dagger}_{1(M_2)}\). No negative-metric creation modes, \(\tilde{d}^{\dagger}\) and \(\tilde{e}^{\dagger}\), commute with \(Q_M\).

Let us consider creation operators with tensor index \((N\varepsilon_n)\) of rank \(n\). We here need operators with index up to rank 4. The general form of such a creation operator with level \(H = 2L\) constructed from the products of two creation modes is given by

\[
\tilde{O}^{[2L]}_{J(N\varepsilon_n)} = \sum_{K=1}^{L-1} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{x}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K(M_1x_1),K(M_2x_2)} \tilde{c}^{\dagger}_{L-K(M_1x_1)} \tilde{c}^{\dagger}_{K(M_2x_2)}
+ \sum_{K=1}^{L-2} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{y}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K-1(M_1x_1),K(M_2x_2)} \tilde{d}^{\dagger}_{L-K-1(M_1x_1)} \tilde{c}^{\dagger}_{K(M_2x_2)}
+ \sum_{K=1}^{L-3} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{z}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K-2(M_1x_1),K(M_2x_2)} \tilde{d}^{\dagger}_{L-K-2(M_1x_1)} \tilde{d}^{\dagger}_{K(M_2x_2)}
+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{w}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K-\frac{3}{2}(M_1x_1),K(M_2y_2)} \tilde{e}^{\dagger}_{L-K-\frac{3}{2}(M_1x_1)} \tilde{c}^{\dagger}_{K(M_2y_2)}
+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{u}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K-\frac{3}{2}(M_1x_1),K(M_2y_2)} \tilde{d}^{\dagger}_{L-K-\frac{3}{2}(M_1x_1)} \tilde{e}^{\dagger}_{K(M_2y_2)}
+ \sum_{K=1}^{L-2} \sum_{M_1,y_1} \sum_{M_2,y_2} \tilde{u}_n(L, K) \tilde{y}^{J(N\varepsilon_n)}_{L-K-1(M_1y_1),K(M_2y_2)} \tilde{d}^{\dagger}_{L-K-1(M_1y_1)} \tilde{c}^{\dagger}_{K(M_2y_2)}, \quad (5.4)\]

where new Clebsch-Gordan coefficients, \(\tilde{y}^{n}, \tilde{y}^{n}, \tilde{y}^{n}, \tilde{y}^{n}\), are defined in Appendix C, and also their non-vanishing conditions used in this section are summa-
The unknown functions are defined as

\[
\begin{align*}
\bar{x}_n(L, K) &= \frac{x_n(L, K)}{\sqrt{(2L - 2K + 1)(2K + 1)}}, \\
\bar{y}_n(L, K) &= \frac{y_n(L, K)}{\sqrt{(2L - 2K - 1)(2K + 1)}}, \\
\bar{z}_n(L, K) &= \frac{z_n(L, K)}{\sqrt{(2L - 2K - 3)(2K + 1)}}, \\
\bar{w}_n(L, K) &= \frac{w_n(L, K)}{\sqrt{(2L - 2K)(2K + 1)}}, \\
\bar{u}_n(L, K) &= \frac{u_n(L, K)}{\sqrt{(2L - 2K - 2)(2K + 1)}}, \\
\bar{v}_n(L, K) &= \frac{v_n(L, K)}{\sqrt{(2L - 2K - 1)(2K + 1)}}.
\end{align*}
\]

(5.5) \quad (5.6) \quad (5.7) \quad (5.8) \quad (5.9) \quad (5.10)

Here, \( x_n, z_n \) and \( v_n \) satisfy the following symmetric conditions:

\[
\begin{align*}
x_n(L, K) &= x_n(L, L - K), \\
z_n(L, K) &= z_n(L, L - K - 2), \\
v_n(L, K) &= v_n(L, L - K - 1).
\end{align*}
\]

(5.11) \quad (5.12) \quad (5.13)

The commutator of \( Q_M \) and this creation operator is computed as

\[
[Q_M, \tilde{O}^{[2L]^\dagger}_{J(N_{\epsilon n})}] = \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{c}_{L-K-\frac{1}{2}(M_1x_1)} \tilde{c}_{K(M_2x_2)} \\
\times 2\bar{x}_n(L, K) \alpha \left( L - K - \frac{1}{2} \right) \sum_{T,x} \epsilon_T E^{J+M}_{L-K-\frac{1}{2}(M_1x_1), L-K(-T)x} \times n \bar{E}^{J(N_{\epsilon n})}_{L-K(Tx), K(M_2x_2)} \\
-\bar{y}_n(L, K) \gamma (L - K - 1) \sum_{T,x} \epsilon_T E^{J+M}_{L-K-\frac{1}{2}(M_1x_1), L-K-1(-Tx)} \times n \bar{E}^{J(N_{\epsilon n})}_{L-K-1(Tx), K(M_2x_2)}
\]

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\[-\bar{w}_n(L, K) A(K) \sum_{V,y} \epsilon_T \mathbf{H}_{K(M_2 x_2); K(-V_y)}^{\frac{1}{M}} \mathbf{H}_{L-K-\frac{3}{2}(M_1 x_1); K(V_y)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1, x_1} \sum_{M_2, x_2} d_{L-K-\frac{3}{2}(M_1 x_1)}^{T} \tilde{d}_{K(M_2 x_2)}^{T} \]

\[\times \left[ -2 \tilde{v}_n(L, K) \beta (L - K - \frac{3}{2}) \sum_{T,x} \epsilon_T \mathbf{E}_{L-K-\frac{3}{2}(M_1 x_1), L-K-2(-T x)}^{\frac{1}{M}} \right. \]

\[\times \left. \mathbf{E}_{L-K-2(T x), K(M_2 x_2)}^{\frac{1}{M}(N_{\varepsilon_n})} \right] \]

\[+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1, y_1} \sum_{M_2, y_2} \bar{e}_{L-K-\frac{3}{2}(M_1 y_1)}^{T} \bar{e}_{K(M_2 y_2)}^{T} \]

\[\times \left[ -\bar{u}_n(L, K) B (L - K - \frac{3}{2}) \sum_{T,x} \epsilon_T \mathbf{H}_{L-K-\frac{3}{2}(-T x); L-K-\frac{3}{2}(M_1 y_1)}^{\frac{1}{M}} \right. \]

\[\times \mathbf{H}_{L-K-\frac{3}{2}(T x); K(M_2 y_2)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[+ \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1, x_1} \sum_{M_2, x_2} \tilde{d}_{L-K-\frac{3}{2}(M_1 x_1)}^{T} \tilde{e}_{K(M_2 x_2)}^{T} \]

\[\times \left[ -\bar{g}_n(L, K) \beta (L - K - \frac{3}{2}) \sum_{T,x} \epsilon_T \mathbf{E}_{L-K-\frac{3}{2}(M_1 x_1), L-K-1(-T x)}^{\frac{1}{M}} \right. \]

\[\times \mathbf{E}_{L-K-1(T x), K(M_2 x_2)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[+ \bar{g}_n \left( L, K + \frac{1}{2} \right) \alpha (K) \sum_{T,x} \epsilon_T \mathbf{E}_{K(M_2 x_2), K+\frac{1}{2}(-T x)}^{\frac{1}{M}} \]

\[\times \mathbf{E}_{K+\frac{1}{2}(T x), L-K-\frac{3}{2}(M_1 x_1)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[+ \bar{g}_n \left( L, K + \frac{1}{2} \right) \beta (L - K - \frac{3}{2}) \sum_{T,x} \epsilon_T \mathbf{E}_{K(M_2 x_2), K+\frac{1}{2}(-T x)}^{\frac{1}{M}} \]

\[\times \mathbf{E}_{K+\frac{1}{2}(T x), L-K-\frac{3}{2}(M_1 x_1)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[+ \bar{g}_n \left( L, K + \frac{1}{2} \right) \gamma (K - \frac{1}{2}) \sum_{T,x} \epsilon_T \mathbf{E}_{K(M_2 x_2), K-\frac{1}{2}(-T x)}^{\frac{1}{M}} \]

\[\times \mathbf{E}_{K-\frac{3}{2}(T x), L-K-\frac{3}{2}(M_1 x_1)}^{\frac{1}{M}(N_{\varepsilon_n})} \]

\[27\]
\[-\bar{u}_n(L, K) A(K) \sum_{V,y} \epsilon_V \mathbf{H}^{2M}_{K(M_2x_2); K(-V_y)} n^{n^{I(N \xi_n)}}_{L-K-\frac{3}{2}(M_1x_1); K(V_y)} \]

\[+ \sum_{K=1}^{L-2} \sum_{M_1,x_1} \sum_{M_2,y_2} c^1_{L-K-1(M_1x_1)} \bar{c}^1_{K(M_2y_2)} \times \left[ -\bar{y}_n(L, L - K - 1) B(K) \sum_{T,x} \epsilon_T \mathbf{H}^{2M}_{K(-T_x); K(M_2y_2)} \right. \]

\[\times n^{n^{I(N \xi_n)}}_{K(Tx), L-K-1(M_1x_1)} \]

\[+ \bar{w}_n(L, K) \alpha(L - K - 1) \sum_{T,x} \epsilon_T \mathbf{E}^{2M}_{L-K-1(M_1x_1), L-K-\frac{1}{2}(-T_x)} \]

\[\times n^{n^{I(N \xi_n)}}_{L-K-\frac{1}{2}(Tx); K(M_2y_2)} \]

\[-\bar{w}_n \left( L, K + \frac{1}{2} \right) C(K) \sum_{V,y} \epsilon_V \mathbf{D}^{2M}_{K(M_2y_2), K+\frac{1}{2}(-V_y)} \]

\[\times n^{n^{I(N \xi_n)}}_{L-K-1(M_1x_1); K+\frac{1}{2}(V_y)} \]

\[-\bar{u}_n(L, K) \gamma \left( L - K - \frac{3}{2} \right) \sum_{T,x} \epsilon_T \mathbf{E}^{2M}_{L-K-1(M_1x_1), L-K-\frac{1}{2}(-T_x)} \]

\[\times n^{n^{I(N \xi_n)}}_{L-K-\frac{1}{2}(Tx); K(M_2y_2)} \]

\[-2 \bar{u}_n(L, K) A(L - K - 1) \sum_{V,y} \epsilon_V \mathbf{H}^{2M}_{L-K-1(M_1x_1); L-K-1(-V_y)} \]

\[\times n^{n^{I(N \xi_n)}}_{L-K-1(V_y), K(M_2y_2)} \]

\[+ \sum_{K=1}^{L-3} \sum_{M_1,x_1} \sum_{M_2,y_2} \bar{d}^1_{L-K-2(M_1x_1)} \bar{c}^1_{K(M_2y_2)} \times \left[ -2 \bar{z}_n(L, K) B(K) \sum_{T,x} \epsilon_T \mathbf{H}^{2M}_{K(-T_x); K(M_2y_2)} \right. \]

\[\times n^{n^{I(N \xi_n)}}_{K(Tx), L-K-2(M_1x_1)} \]

\[-\bar{u}_n(L, K) \beta(L - K - 2) \sum_{T,x} \epsilon_T \mathbf{E}^{2M}_{L-K-2(M_1x_1), L-K-\frac{1}{2}(-T_x)} \]

\[\times n^{n^{I(N \xi_n)}}_{L-K-\frac{1}{2}(Tx); K(M_2y_2)} \]

\[-\bar{u}_n \left( L, K + \frac{1}{2} \right) C(K) \sum_{V,y} \epsilon_V \mathbf{D}^{2M}_{K(M_2y_2), K+\frac{1}{2}(-V_y)} \]

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In the following, for each \( n \), we seek the functions \( (x_n, y_n, z_n, w_n, u_n \text{ and } v_n) \) that make this commutator vanish.

**Building blocks with scalar index** \((n = 0)\): Let us first consider the creation operator with scalar index. Because of the triangular conditions for the \( SU(2) \times SU(2) \) Clebsch-Gordan coefficients, this operator vanishes for \( J > L \). If we take \( J = L \), the only terms with the function \( \bar{x}_0 \) in the operator (5.4) survives, because

\[
0\mathcal{E}_{L-K-1(M_1x_1),K(M_2x_2)} = 0, \quad 0\mathcal{E}_{L-K-2(M_1x_1),K(M_2x_2)} = 0, \quad (5.15)
\]

\[
0\mathcal{H}_{L-K-\frac{1}{2}(M_1x_1),K(M_2y_2)} = 0, \quad 0\mathcal{H}_{L-K-\frac{3}{2}(M_1x_1),K(M_2y_2)} = 0, \quad (5.16)
\]

\[
0\mathcal{D}_{L-K-1(M_1y_1),K(M_2y_2)} = 0. \quad (5.17)
\]

Thus, the only non-vanishing component in the commutator (5.14) is the first term in the \( c^\dagger c^\dagger \) part.

To find a function \( x_0 \) that make this term vanish, the crossing relation of type I (E.4) for \( n = 0 \) derived in the Appendix E is used. Consider the case of \( J_1 = L - K - \frac{1}{2} \) and \( J_2 = K \) in this relation. If we take \( J = L \), we obtain the crossing relation that consists of only the \( \mathbf{E} \cdot \mathbf{E} \) part with the intermediate value \( I = L - K \left( K + \frac{1}{2} \right) \) for the l.h.s (r.h.s.). Using this crossing relation and the symmetric condition (5.11), we find that the commutator vanishes if the function satisfies the recursion relation,

\[
x_0 \left( L, K + \frac{1}{2} \right) = -\sqrt{\frac{(2L - 2K - 1)(2L - 2K)}{2K(2K + 1)}} x_0(L, K). \quad (5.18)
\]

Solving this equation, we obtain

\[
x_0(L, K) = x(L, K) \quad (5.19)
\]

with

\[
x(L, K) = (-1)^{2K} \sqrt{\binom{2L}{2K} \left( \frac{2L - 2}{2K - 1} \right)}, \quad (5.20)
\]
up to the $L$-dependent normalization. To satisfy the symmetric condition (5.11), $L$ must be an integer. For a half-integer $L$, this function vanishes.

Thus, we obtain the $Q_M$-invariant creation operator with scalar index, denoted by $\tilde{A}_L^\dagger$, as

$$\tilde{A}_L^\dagger = \sum_{K=1}^{\frac{L}{2}} \sum_{M_1,x_1} \sum_{M_2,x_2} \tilde{x}(L,K) E_{L-K(M_1,x_1),K(M_2,x_2)}^L \tilde{c}_L^\dagger(K(M_1,x_1))^\dagger K(M_2,x_2), \quad (5.21)$$

with integer $L \geq 3$. Here, $L = 2$ is trivial because $[Q_M,c_1^\dagger] = 0$, and therefore it is removed. The function $\tilde{x}$ is defined as in (5.5), and the definition $\theta E = E$ is used.

Next, we consider the case of $J = L - 1$. Because

$$\theta L_{-1}^{L_{-1}} E_{L-K-2(M_1,x_1),K(M_2,x_2)} = 0, \quad (5.22)$$

$$\theta H_{L-1}^{L_{-1}} E_{L-K-2(M_1,x_1);K(M_2,y_2)} = 0, \quad (5.23)$$

the terms with the functions $\tilde{z}_0$ and $\tilde{u}_0$ vanish. Therefore, the $d^\dagger d^\dagger$ and $d^\dagger e^\dagger$ terms in the commutator (5.14) trivially vanish, and four terms with the functions $\tilde{x}_0$, $\tilde{y}_0$, $\tilde{w}_0$ and $\tilde{v}_0$, i.e. $c^\dagger c^\dagger$, $e^\dagger e^\dagger$, $d^\dagger e^\dagger$ and $c^\dagger e^\dagger$, survive. We first consider the $c^\dagger c^\dagger$ term. To find a solution for which this term vanishes, we use the crossing relation of type I (E.4) for $n = 0$ with the values $J = L - 1$, $J_1 = L - K - \frac{1}{2}$ and $J_2 = K$, and the symmetric conditions of $x_0$ (5.11). We then find that this term vanishes when $x_0$ satisfies the same recursion relation to Eq.(5.18), so that $x_0(L,K) = x(L,K)$, and $y_0$ and $w_0$ satisfy the equations,

$$y_0(L,K) = y(L,K), \quad (5.24)$$

$$w_0(L,K) = w(L,K), \quad (5.25)$$

with

$$y(L,K) = -2(2L - 2K - 1)x(L,K), \quad (5.26)$$

$$w(L,K) = -\sqrt{\frac{8(2L - 2K - 1)(2L - 2K)}{(2K - 1)2K(2K + 3)}}x(L,K). \quad (5.27)$$

The condition that the $c^\dagger e^\dagger$ term vanishes determines the function $v_0$. Using the already derived functions, $y_0 = y$ (5.26) and $w_0 = w$ (5.27), and
the crossing relation of type II (E.6) for \( n = 0 \) with \( J = L - 1, J_1 = L - K - 1 \) and \( J_2 = K \), in which the triangular conditions,

\[
0 \mathbf{T}^{L-1N}_{L-K-1(M_1x_1);KM_2} = 0
\]  

(5.28)

and so on, are taken into account, we find that \( v_0(L, K) = v(L, K) \) with

\[
v(L, K) = 2 \sqrt{\frac{(2K + 1)(2L - 2K - 1)}{(2K - 1)(2K + 3)(2L - 2K - 3)(2L - 2K + 1)}} x \left( L, K + \frac{1}{2} \right).
\]  

(5.29)

The \( e^\dagger e^\dagger \) term now consists of only the \( \mathbf{D} \cdot \hat{\mathbf{D}} \) part. Here, we consider the crossing relation of type III (E.8) for \( n = 0 \) with \( J = L - 1, J_1 = L - K - \frac{3}{2} \) and \( J_2 = K \). We then obtain the relation that consists of only the \( \mathbf{D} \) and \( \hat{\mathbf{D}} \) coefficients, because

\[
0 \mathbf{H}^{L-1N}_{L-K-\frac{3}{2}(M_1x_1);K(M_2y_2)} = 0, \quad \hat{\mathbf{H}}^{L-1N}_{L-K-\frac{3}{2}(M_1x_1);K(M_2y_2)} = 0,
\]  

(5.30)

\[
0 \mathbf{G}^{L-1N}_{L-K-\frac{3}{2}(M_1y_1);KM_2} = 0.
\]  

(5.32)

Using this crossing relation and the function \( v_0 = v \) (5.29), we find that the \( e^\dagger e^\dagger \) term vanishes.

The \( d^\dagger c^\dagger \) term consists of only the \( \mathbf{E} \cdot \hat{\mathbf{E}} \) part. Using the crossing relation of type I (E.4) for \( n = 0 \) and \( y_0 = y \) (5.26), we can show that this term vanishes.

Thus, we obtain another type of \( Q_M \) invariant operator with scalar index, denoted by \( \tilde{A}^{L-1N}_{L} \), as

\[
\tilde{A}^{L-1N}_{L} = \sum_{K=1}^{L-1} \sum_{M_1x_1} \sum_{M_2x_2} \bar{x}(L, K) \mathbf{E}^{L-1N}_{L-K(M_1x_1),K(M_2x_2)} c^\dagger_{L-K-1(M_1x_1)} \hat{c}^\dagger_{K(M_2x_2)} c^\dagger_{L-K-1(M_1x_1)} \hat{c}^\dagger_{K(M_2x_2)}
\]  

(5.33)
with integer $L(\geq 3)$. Here, the definitions $^0E = E$ and $^0H = H$, and the relation between $D$ and $^0D$ (D.10) are used. The function $v'$ is then given by

$$v'(L, K) = -\frac{1}{2}(2K - 1)(2L - 2K - 3)v(L, K). \quad (5.34)$$

The functions with the bar ($\bar{y}$, $\bar{w}$, and $\bar{v}'$) are defined as in (5.6), (5.8) and (5.10), respectively.

For $J = L - 2$, all components in the operator (5.4) contribute. In this case we find that we cannot make some terms in the commutator (5.14) vanish. For example, the $e^\dagger e^\dagger$ term is now given by a combination of the $H \cdot ^0H$ and $D \cdot ^0D$ parts. The crossing relation required in this case is of type III (E.8) for $n = 0$ with $J = L - 2$, $J_1 = L - K - \frac{3}{2}$ and $J_2 = K$. However, this relation also includes the non-vanishing $G \cdot ^0G$ part, so that we cannot make this term vanish unless the functions $u_0$ and $v_0$ vanish. From the other terms in the commutator, these functions must be non-vanishing, and thus we can show that for $J \leq L - 2$, there is no operator that commutes with the charge $Q_M$. Thus, the building blocks with scalar index are given by two types of the operators, $A^\dagger$ and $A^\dagger$.

**Building blocks with vector index ($n = 1$):** From the triangular conditions of the Clebsch-Gordan coefficients, the operator with vector index is non-vanishing for $J \leq L - \frac{1}{2}$. Let us consider the case of $J = L - \frac{1}{2}$. Because

$$\begin{align*}
\nu E^L_{L-\frac{1}{2}(Ny)}_{L-K-1(M_1x_1), K(M_2x_2)}^L &= \nu E^L_{L-K-2(M_1x_1), K(M_2x_2)}^L = 0, \quad (5.35) \\
\nu H^L_{L-K-\frac{1}{2}(Ny)}_{L-K-\frac{1}{2}(M_1x_1); K(M_2x_2)} &= 0, \\
\nu D^L_{L-\frac{1}{2}(Ny)}_{L-K-1(M_1y_1), K(M_2y_2)} &= 0, \quad (5.37)
\end{align*}$$

only the terms including the functions $x_1$ and $w_1$ survive. Therefore, the four terms ($d^\dagger d^\dagger$, $e^\dagger e^\dagger$, $d^\dagger c^\dagger$ and $d^\dagger e^\dagger$) in the commutator (5.14) vanish trivially. Using the crossing relations of type I (E.4) and type II (E.6) for $n = 1$, we find that the $c^\dagger c^\dagger$ and $c^\dagger e^\dagger$ terms vanish when

$$\begin{align*}
x_1(L, K) &= x(L, K), \\
w_1(L, K) &= w(L, K), \quad (5.38, 5.39)
\end{align*}$$
where $x$ and $w$ are given by Eqs. (5.20) and (5.27). Thus, we obtain a building block with vector index, denoted by $\tilde{B}_{L-\frac{3}{2}(Ny)}^{\dagger}$, as

\[
\tilde{B}_{L-\frac{3}{2}(Ny)}^{\dagger} = \sum_{K=1}^{L-1} \sum_{M_1,x_1} \sum_{M_2,x_2} \bar{x}(L,K) \frac{1}{L-K} \mathcal{E}_{L-K}(Ny)_{M_1,x_1,K(M_2,x_2)} \tilde{c}_{L-K}(M_1,x_1) \tilde{c}_{L-K}(M_2,x_2) + \sum_{K=1}^{L-\frac{3}{2}} \sum_{M_1,x_1} \sum_{M_2,y_2} \bar{w}(L,K) \frac{1}{L-K-\frac{3}{2}(Ny)}_{M_1,x_1,M_2,y_2} \tilde{c}_{L-K-\frac{3}{2}(Ny)}(M_1,x_1) \tilde{c}_{L-K-\frac{3}{2}(Ny)}(M_2,y_2),
\]

(5.40)

with integer $L(\geq 3)$.

For $J \leq L - \frac{3}{2}$, we find that the operator does not commute with $Q_M$, as discussed before.

**Building blocks with rank 2 tensor index** ($n = 2$): The operator with rank 2 tensor index has non-vanishing components for $J \leq L - 1$. For $J = L - 1$, because of

\[
2 \mathcal{E}_{L-K-1(M_1,x_1),K(M_2,x_2)}^{L-1(Ny)} = 2 \mathcal{E}_{L-K-2(M_1,x_1),K(M_2,x_2)}^{L-1(Ny)} = 0,
\]

(5.41)

\[
2 \mathcal{H}_{L-K-\frac{3}{2}(M_1,x_1),K(M_2,y_2)}^{L-1(Ny)} = 0,
\]

(5.42)

the terms with the functions $\bar{x}_2$, $\bar{w}_2$ and $\bar{v}_2$ survive. Therefore, we consider the three terms $\tilde{c}_{L}^{\dagger} \tilde{c}^{\dagger}$, $\tilde{e}_{L}^{\dagger} \tilde{e}^{\dagger}$ and $\tilde{c}_{L}^{\dagger} \tilde{e}^{\dagger}$ in the commutator (5.14). As in the case of the operator $A^{\dagger}$, using the crossing relations of type I (E.4) and type III (E.8) for $n = 2$, we can make the $c_{L}^{\dagger} c^{\dagger}$ and $e_{L}^{\dagger} e^{\dagger}$ terms vanish when the functions $x_2$, $w_2$ and $v_2$ are given by $x$ (5.20), $w$ (5.27) and $v$ (5.29), respectively. However, we cannot make the $c_{L}^{\dagger} \tilde{e}^{\dagger}$ term vanish because the corresponding type II relation (E.6) for $n = 2$ has an extra term including the non-vanishing coefficient $2 \mathcal{H}_{L-K-1(K)}^{L-1}$ contrary to (5.28). Thus, the operator with $J = L - 1$ does not commute with $Q_M$.

In the same way, we can show that there is no $Q_M$-invariant operator with $J \leq L - 2$. Thus, the building block with rank 2 tensor index is the only lowest positive-metric creation mode,

\[
c_{1(Nx)}^{\dagger}.
\]

(5.43)
Building blocks with rank 3 tensor index \((n = 3)\): The operator with rank 3 tensor index has non-vanishing components for \(J \leq L - \frac{1}{2}\). For \(J = L - \frac{1}{2}\), because of

\[
3\mathbf{E}^{L - \frac{1}{2}(Nz)}_{L - K - 1(M_1x_1), K(M_2x_2)} = 3\mathbf{E}^{L - \frac{1}{2}(Nz)}_{L - K - 2(M_1x_1), K(M_2x_2)} = 0,
\]

\[
3\mathbf{H}^{L - \frac{1}{2}(Nz)}_{L - K - \frac{1}{2}(M_1x_1); K(M_2y_2)} = 0,
\]

\[
3\mathbf{D}^{L - \frac{1}{2}(Nz)}_{L - K - 1(M_1y_1), K(M_2y_2)} = 0,
\]

the terms with the functions \(\bar{x}_3\) and \(\bar{w}_3\) survive. Therefore, we consider only the \(c^\dagger c^\dagger\) and \(c^\dagger e^\dagger\) terms in the commutator (5.14). As in the case of the operator \(B^\dagger\), using the crossing relation of type I (E.4) for \(n = 3\), we find that the \(c^\dagger c^\dagger\) term vanishes when

\[
x_3(L, K) = x(L, K),
\]

\[
w_3(L, K) = w(L, K),
\]

where \(x\) and \(w\) are given by Eqs.(5.20) and (5.27). Also, using this \(w_3\), we find that the \(c^\dagger e^\dagger\) term vanishes due to the crossing relation of type II (E.6) for \(n = 3\).

Thus, we obtain a building block with rank 3 tensor index, denoted by \(\tilde{D}^\dagger_{L - \frac{1}{2}(Nz)}\), as

\[
\tilde{D}^\dagger_{L - \frac{1}{2}(Nz)} = \sum_{K=1}^{L-1} \sum_{M_1x_1} \sum_{M_2x_2} \bar{x}(L, K) 3\mathbf{E}^{L - \frac{1}{2}(Nz)}_{L - K - 1(M_1x_1), K(M_2x_2)} \bar{c}^\dagger_{L - K - 1(M_1x_1), K(M_2x_2)}
\]

\[
+ \sum_{K=1}^{L-1} \sum_{M_1x_1} \sum_{M_2y_2} \bar{w}(L, K) 3\mathbf{H}^{L - \frac{1}{2}(Nz)}_{L - K - \frac{1}{2}(M_1x_1); K(M_2y_2)} \bar{c}^\dagger_{L - K - \frac{1}{2}(M_1x_1); K(M_2y_2)} \bar{e}^\dagger_{K(M_2y_2)},
\]

with integer \(L(\geq 3)\).

For \(J < L - \frac{1}{2}\), the operator does not commute with \(Q_M\).

Building blocks with rank 4 tensor index \((n = 4)\): The operator with rank 4 tensor index has non-vanishing components for \(J \leq L\). For \(J = L\),
because of
\[ 4 E_{L-K-1}(M_{1x1}, K(M_{2x2})} = 4 E_{L-K-2}(M_{1x1}, K(M_{2x2})} = 0, \quad (5.50) \]
\[ 4 H_{L-K-\frac{1}{2}}(M_{1x1}; K(M_{2y2})} = 4 H_{L-K-\frac{3}{2}}(M_{1x1}; K(M_{2y2})} = 0, \quad (5.51) \]
\[ 4 D_{L-K-1}(M_{1y1}); K(M_{2y2})} = 0, \quad (5.52) \]
the terms with the function \( \bar{x}_4 \) survive, and therefore we consider only the \( c^\dagger c^\dagger \) term in the commutator (5.14). We easily find that, using the crossing relation of type I (E.4) for \( n = 4 \), this term vanishes when
\[ x_4(L, K) = x(L, K). \quad (5.53) \]

Thus, we obtain a building block with rank 4 tensor index, denoted by \( \tilde{E}_{L(Nw)} \), as
\[ \tilde{E}_{L(Nw)} = \sum_{K=1}^{L-1} \sum_{M_{1x1}} \sum_{M_{2x2}} \bar{x}(L, K) \quad 4 E_{L-K)(M_{1x1}, K(M_{2x2})} c^\dagger_{L-K}(M_{1x1}) \tilde{c}^\dagger_{K(M_{2x2})}, \quad (5.54) \]
with integer \( L(\geq 3) \).

For \( J = L - 1 \), because of
\[ 4 E_{L-K-2}(M_{1x1}, K(M_{2x2})} = 0, \quad (5.55) \]
\[ 4 H_{L-K-\frac{1}{2}}(M_{1x1}; K(M_{2y2})} = 0, \quad (5.56) \]
\[ 4 D_{L-K-1}(M_{1y1}); K(M_{2y2})} = 0, \quad (5.57) \]
the terms with the functions \( \bar{x}_4, \bar{y}_4 \) and \( \bar{w}_4 \) survive. Therefore, we consider the three terms \( c^\dagger c^\dagger, d^\dagger c^\dagger \) and \( c^\dagger c^\dagger \) in the commutator (5.14). As in the case of the operator \( A^\dagger \), using the crossing relations of type I (E.4) and type II (E.6) for \( n = 4 \) in company with the triangular condition,
\[ 4 I_{L-K-1}(M_{1x1}; KM_2) = 0, \quad (5.58) \]
we find that the commutator vanishes when
\[ x_4(L, K) = x(L, K), \quad (5.59) \]
\[ y_4(L, K) = y(L, K), \quad (5.60) \]
\[ w_4(L, K) = w(L, K). \quad (5.61) \]
Thus, we obtain another type of building block with rank 4 tensor index, denoted by $\tilde{\mathcal{E}}_{L-1(Nw)}$, as

$$
\tilde{\mathcal{E}}_{L-1(Nw)}^\dagger = \sum_{K=1}^{L-1} \sum_{M_1,x_1} \sum_{M_2,x_2} \bar{x}(L,K) \, 4\mathcal{E}^L_{L-K-1(M_1,x_1)} c^\dagger_{L-K-1(M_1,x_1)^4} c^\dagger_{K(M_2,x_2)} 
$$

$$
+ \sum_{K=1}^{L-2} \sum_{M_1,x_1} \sum_{M_2,x_2} \bar{y}(L,K) \, 4\mathcal{E}^L_{L-K-1(M_1,x_1)} d^\dagger_{L-K-1(M_1,x_1)^4} \tilde{c}^\dagger_{K(M_2,x_2)} 
$$

$$
+ \sum_{K=1}^{L-3} \sum_{M_1,x_1} \sum_{M_2,y_2} \bar{w}(L,K) \, 4\mathcal{H}^L_{L-K-\frac{1}{2}(M_1,x_1)} c^\dagger_{L-K-\frac{1}{2}(M_1,x_1)^4} \tilde{c}^\dagger_{K(M_2,y_2)},
$$

(5.62)

with integer $L(\geq 3)$.

For $J \leq L - 3$, there is no $Q_M$-invariant operator with rank 4 tensor index.

The building blocks in the traceless mode sector are summarized in Table 3. The operators without the tilde are defined by $O_{L(N\varepsilon_n)} = \epsilon_N \tilde{O}_{L(-N\varepsilon_n)}$. Any $Q_M$-invariant state will be constructed from these building blocks.

| rank of tensor index | 0  | 1  | 2  | 3  | 4  |
|---------------------|----|----|----|----|----|
| creation op.        | $A^\dagger_{LN}$ | $B^\dagger_{L-\frac{1}{2}(Ny)}$ | $c^\dagger_{1(Nx)}$ | $D^\dagger_{L-\frac{1}{2}(Nz)}$ | $E^\dagger_{L(Nw)}$ |
| level ($L \in \mathbb{Z}_{\geq 3}$) | $2L$ | $2L$ | $2$ | $2L$ | $2L$ |

Table 3: Building blocks in the traceless mode sector.

6 Building Blocks for the Conformal Field

There is an essential difference between the conformal field and the other fields, which is that the conformal field has zero modes. The commutators of $Q_M$ and the zero modes are given by

$$
[Q_M, \hat{q}] = -a^\dagger_{2M},
$$

(6.1)

$$
[Q_M, \hat{p}] = 0.
$$

(6.2)

The commutators with the conformal modes $\tilde{a}^\dagger_{JM}$ are calculated as

$$
[Q_M, \tilde{a}^\dagger_{JM_1}] = \left( \sqrt{2b_1 - i\tilde{p}} \right) \epsilon_M \delta_{M,-M_1}
$$

(6.3)
and

$$[Q_M, \tilde{a}_{JM}] = \alpha \left( J - \frac{1}{2} \right) \sum_{M_2} \epsilon_{M_1} C_{J-M_1, J-(J+1)M_2}^M \tilde{a}_{J+1/2M_2}$$

(6.4)

for \( J \geq 1 \). Also, the commutators with \( \tilde{b}_{JM} \) are given by

$$[Q_M, \tilde{b}_{JM}] = -\gamma(J) \sum_{M_2} \epsilon_{M_1} C_{J-M_1, J+1/2M_2}^M \tilde{a}_{J+1/2M_2}$$

$$-\beta \left( J - \frac{1}{2} \right) \sum_{M_2} \epsilon_{M_1} C_{J-M_1, J-(J+1)M_2}^M \tilde{b}_{J+1/2M_2}$$

(6.5)

for \( J \geq 0 \).

The building blocks are constructed as done in the case of the traceless mode. The differences are that we here use the Clebsch-Gordan coefficient of type \( C \), and we take care on the zero-mode. Then, we find two types of the building blocks with level \( H = 2L \),

$$\tilde{S}_{LN}^\dagger = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1, M_2} \bar{x}(L, K) C_{L-KM_1, KM_2}^{LN} \tilde{a}_{L-KM_1}^\dagger \tilde{a}_{KM_2}^\dagger$$

$$+ \chi(\tilde{p}) \tilde{a}_{LN}^\dagger$$

(6.6)

and

$$\tilde{S}_{L-1N}^\dagger = \sum_{K=\frac{1}{2}}^{L-\frac{1}{2}} \sum_{M_1, M_2} \bar{x}(L, K) C_{L-K-1M_1, KM_2}^{L-1N} \tilde{a}_{L-KM_1}^\dagger \tilde{a}_{KM_2}^\dagger$$

$$+ \sum_{K=\frac{1}{2}}^{L-1} \sum_{M_1, M_2} \tilde{y}(L, K) C_{L-K-1M_1, KM_2}^{L-1N} \tilde{b}_{L-K-1M_1}^\dagger \tilde{a}_{KM_2}^\dagger$$

$$+ \psi(\tilde{p}) \tilde{b}_{L-1N}^\dagger$$

(6.7)

with integer \( L(\geq 1) \). Here, \( \bar{x} \) and \( \tilde{y} \) are the same to the functions defined by Eqs.(5.20) and (5.26) in the traceless-mode sector. The zero-mode dependent operators, \( \chi \) and \( \psi \), are given by

$$\chi(\tilde{p}) = \frac{1}{\sqrt{2(2L-1)(2L+1)}} \left( \sqrt{2b_{1}} - i\tilde{p} \right)$$

(6.8)

$$\psi(\tilde{p}) = -\sqrt{2} \left( \sqrt{2b_{1}} - i\tilde{p} \right)$$

(6.9)
The building blocks in the conformal mode sector are summarized in Table 4. The operators without the tilde are defined by $S_{LN} = \epsilon_N \tilde{S}_{L-N}$ and $S_{LN} = \epsilon_N \tilde{S}_{L-N}$.

| rank of tensor index | 0 |
|----------------------|---|
| creation op.         | $S^\dagger_{LN}$ |
|                      | $S^\dagger_{L-1N}$ |
| level ($L \in \mathbb{Z}_{\geq 1}$) | $2L$ |

Table 4: Building blocks in the conformal mode sector.

7 Physical States in a Non-critical 3-brane

The physical state annihilated by all conformal charges is a conformally invariant vacuum, which is uniquely given by

$$|\Omega\rangle = e^{-\sqrt{2b_1} \phi_0}|0\rangle = e^{-2b_1 \phi_0}|0\rangle,$$

(7.1)

where $\phi_0$ indicates the zero mode of the conformal field (2.13), and $|0\rangle$ is the standard Fock vacuum with zero eigenvalue of $\hat{p}$ that vanishes when annihilation modes act. The physical states are spanned by the Fock space generated on the conformally invariant vacuum. They must satisfy the conformal invariance conditions,

$$Q_M|\text{phys}\rangle = 0$$

(7.2)

and

$$(H - 4)|\text{phys}\rangle = R_{MN}|\text{phys}\rangle = 0$$

(7.3)

where $-4$ comes from the ghost sector discussed in Appendix F, which indicates the number of the dimensions of the world-volume. As in the Gupta-Bleuler procedure, we do not impose a condition concerning $Q^\dagger_M$.

The physical state is now decomposed into four sectors: scalar fields, vector fields, the traceless mode and the conformal mode. Each sector consists of the Hamiltonian eigenstates satisfying the condition (7.2). Such states are constructed from the building blocks derived in the previous sections. Conditions (7.3) are imposed last after combining all sectors.

First, consider the states that depends only on the zero mode of the conformal field. Such a state satisfying the $Q_M$ invariance condition (7.2) is given by $|p, \Omega\rangle = e^{ip\hat{q}}|\Omega\rangle = e^{ip\sqrt{2b_1} \phi_0}|\Omega\rangle$. This is the eigenstate of $\hat{p}$ with
eigenvalue $p + i\sqrt{2b_1}$. The Hamiltonian condition in (7.3) gives the equation
\[ \frac{1}{2} \left( p + i\sqrt{2b_1} \right)^2 + b_1 = 4, \]
so that $p$ has a purely imaginary value, $-i\frac{\alpha_0}{\sqrt{2b_1}}$, with $\alpha_0 = 2b_1 \left( 1 - \sqrt{1 - \frac{4}{b_1}} \right)$. Here, the fact that $b_1 > 4$ is used, and the solution that $\alpha$ approaches the canonical value, 4, in the classical limit, $b_1 \to \infty$, is selected. This state is, expressed by
\[ e^{\alpha_0 \phi_0} |\Omega\rangle \quad (7.4) \]
identified with the cosmological constant.

The general state satisfying the conditions of $Q_M$ (7.2) and $R_{MN}$ in (7.3) is constructed by acting with the building blocks on the state $|p, \Omega\rangle$, in which all tensor indices are contracted out using the $SU(2) \times SU(2)$ Clebsch Gordan coefficients. This state is denoted by $|n, p\rangle = F_n(\Phi^\dagger, \cdots)|p, \Omega\rangle$, where $n$ is the level of $F_n$. The Hamiltonian condition gives the equation
\[ \frac{1}{2} \left( p + i\sqrt{2b_1} \right)^2 + b_1 + n = 4. \]
Solving this equation, we obtain the physical state
\[ F_n(\Phi^\dagger, \cdots) e^{\alpha_n \phi_0} |\Omega\rangle \quad (7.5) \]
with the conformal charge
\[ \alpha_n = 2b_1 \left( 1 - \sqrt{1 - \frac{4 - n}{b_1}} \right). \quad (7.6) \]

Now, we construct the lower level states up to the level 6. For $n = 2$, there are two physical states,
\[ \Phi^\dagger_{00} e^{\alpha_2 \phi_0} |\Omega\rangle \quad (7.7) \]
and purely gravitational state
\[ S^\dagger_{00} e^{\alpha_2 \phi_0} |\Omega\rangle. \quad (7.8) \]
The former corresponds to the diffeomorphism invariant field, $\int d^4x \sqrt{-g} X^2$, and the latter is the scalar curvature, $\int d^4x \sqrt{-g} R$.

For $n = 4$, the physical states coupled to the matter fields are given by
\[ (\Phi^\dagger_{00})^2 |\Omega\rangle, \quad \Phi^\dagger_{00} S^\dagger_{00} |\Omega\rangle, \quad q^\dagger_{4(Ny)} q^\dagger_{4(Ny)} |\Omega\rangle, \quad (7.9) \]
where \( \alpha_4 = 0 \) is taken into account. Here and below, the sums of the tensor indices are omitted. These states correspond to the diffeomorphism invariant fields \( \int d^4x \sqrt{-g} X^4 \), \( \int d^4x \sqrt{-g} RX^2 \) and the square of the field strength of the vector field, \( \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \), respectively. Furthermore, there are purely gravitational states,

\[
\begin{align*}
&c_{1(N_2)}^\dagger c_{1(N_2)}^\dagger |\Omega\rangle, \quad (S_{00}^\dagger)^2 |\Omega\rangle, \quad \tilde{S}_{1N} S_{1N}^\dagger |\Omega\rangle. \quad (7.10)
\end{align*}
\]

The first state corresponds to the square of the Weyl tensor, \( \int d^4x \sqrt{-g} C_{\mu\nu\lambda\sigma}^2 \), and the second is the square of the scalar curvature, \( \int d^4x \sqrt{-g} R^2 \). The third is an independent diffeomorphism invariant field other than the first two fields.

For the level \( n = 6 \), we obtain

\[
\begin{align*}
&\Phi_{00}^\dagger (S_{00}^\dagger)^2 e^{\alpha_6\phi_0} |\Omega\rangle, \quad (\Phi_{00}^\dagger)^2 S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&\bar{\Phi}_{00} S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad \Phi_{00} S_{00}^\dagger S_{1N}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&\Phi_{1N} S_{1N}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad \Phi_{00} q_{1/2(N_y)}^\dagger q_{1/2(N_y)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&\Phi_{00} q_{1/2(N_x)}^\dagger c_{1(N_x)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad \bar{q}_{1/2(N_y)}^\dagger q_{1/2(N_y)}^\dagger S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&D_{1N}^{(N_1y_1,1/2(N_y);1/2(N_y))} q_{1/2(N_y)}^\dagger q_{1/2(N_y)}^\dagger S_{1N}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&F_{1N}^{(N_x,1/2(N_y);1/2(N_y))} q_{1/2(N_y)}^\dagger q_{1/2(N_y)}^\dagger c_{1(N_x)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&G_{1N}^{(N_2);1N_1} q_{1/2(N_y)}^\dagger S_{1N_1}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&H_{1N}^{(N_3);1N_1} q_{1/2(N_y)}^\dagger c_{1(N_x)}^\dagger S_{1N_3}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&I_{1N}^{(N_y);1N_1} q_{1/2(N_y)}^\dagger c_{1(N_x)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle. \quad (7.11)
\end{align*}
\]

and purely gravitational states

\[
\begin{align*}
&(S_{00}^\dagger)^3 e^{\alpha_6\phi_0} |\Omega\rangle, \quad S_{00}^\dagger \tilde{S}_{1N} S_{1N}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&\tilde{S}_{1N} S_{1N}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad c_{1(N_x)}^\dagger c_{1(N_x)}^\dagger S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&C_{1N_1} S_{1N_2}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad \tilde{c}_{1(N_x)}^\dagger c_{1(N_x)}^\dagger S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&T_{1N_2} S_{1N_2}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad \tilde{c}_{1(N_x)}^\dagger c_{1(N_x)}^\dagger S_{00}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \\
&E_{1N}^{(N_1x_1);1N_2} c_{1(N_1x_1)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle, \quad 2F_{1N_1}^{(N_1x_1);1N_3} c_{1(N_1x_1)}^\dagger c_{1(N_2x_2)}^\dagger e^{\alpha_6\phi_0} |\Omega\rangle. \quad (7.12)
\end{align*}
\]
In the same way we can also construct higher level states, but the classification of them becomes complicated.

8 Conclusions and Discussion

In this paper we systematically constructed and classified the physical states in a world-volume model of a non-critical 3-brane on $R \times S^3$ at very high energies beyond the Planck mass scale. At this energy, the conformal invariance, reflecting the background-metric independence, becomes exact, and thus the dynamics is described by CFT$_4$. Therefore, the physical states must satisfy the conformal invariance conditions, (7.2) and (7.3).

The physical states are decomposed into four sectors: scalar fields, vector fields, the traceless mode and the conformal mode. We discussed four sectors separately, and then combined them last. Each sector consists of the Hamiltonian eigenstates invariant under the special conformal transformations, $Q_M$, which belong to certain representations of the rotation group on $S^3$. These eigenstates further factorize into the $Q_M$-invariant building blocks classified in finite types. The physical state was constructed by combining such eigenstates, and contracting out all of their tensor indices appropriately in a rotation invariant way using the $SU(2) \times SU(2)$ Clebsch-Gordan coefficients. The Hamiltonian condition was imposed by adjusting the zero-mode momentum of the conformal field with purely imaginary eigenvalue.

There is an essential difference between the conformal mode sector and the other three sectors: scalar fields, vector fields and the traceless mode. The conformal mode sector is not normalizable because of the purely imaginary eigenvalue of the zero-mode momentum, while the other three sectors are normalizable. This world-volume model seems to be in the same universality class as the four-dimensional simplicial quantum gravity, namely the dynamical triangulation approach to four-dimensional random surfaces [14]. The partition function of this lattice model is given by a grand canonical ensemble in the number of 4-simplices. This fact will be related to the non-normalizability of the conformal mode sector. The other normalizable sectors are regarded as canonical ensembles on random surfaces.

As an impact to spacetime physics, this world-volume model gives a dynamical scenario of inflation consistent with observations of the cosmic microwave background anisotropies [15].

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Appendix

A  The Metric on $R \times S^3$

The metric on $R \times S^3$ is parametrized as

$$ds^2_{R \times S^3} = \hat{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \hat{\gamma}_{ij}dx^i dx^j = -dt^2 + \frac{1}{4}(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma), \quad (A.1)$$

where $t$ is the time and $x^i = (\alpha, \beta, \gamma)$, with $i = 1, 2, 3$, are the Euler’s angles. Then, $\hat{R}_{\mu\nu\lambda} = \hat{R}_{\mu\nu} = 0$, $\hat{R}_{ijkl} = (\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \hat{\gamma}_{il}\hat{\gamma}_{jk})$, $\hat{R}_{ij} = 2\hat{\gamma}_{ij}$ and $\hat{R} = 6$.

The volume element on unit $S^3$ is

$$d\Omega_3 = d^3x \sqrt{\hat{\gamma}} = \frac{1}{8}\sin\beta d\alpha d\beta d\gamma, \quad (A.2)$$

and the volume is $\text{Vol}(S^3) = 2\pi^2$.

B  $SU(2) \times SU(2)$ Clebsch-Gordan Coefficients of Types, C, D, E, G and H

The $SU(2) \times SU(2)$ Clebsch-Gordan coefficients are defined by the integrals of three products of ST$^2$ tensor harmonics. Here, we give the basic coefficients of types, (C, D, E, G and H), calculated in Ref.[8].

Type C

$$C_{J_{1M_{1}},J_{2M_{2}}} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^*_{J_{1M_{1}}} Y_{J_{1M_{1}}} Y_{J_{2M_{2}}} = \sqrt{(2J_{1} + 1)(2J_{2} + 1)} \frac{C_{J_{1m_{1}},J_{2m_{2}}} C_{J_{1m_{1}'}J_{2m_{2}'}}}{2J + 1}, \quad (B.1)$$

where $M = M_1 + M_2$ and

$$|J_{1} - J_{2}| \leq J \leq J_{1} + J_{2}, \quad (B.2)$$

with integer $J + J_1 + J_2$. 

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Type D \((y = \pm \frac{1}{2})\)

\[
D_{J_1(M_1y_1),J_2(M_2y_2)}^{JM} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y_{JM}^* Y_{J_1(M_1y_1)} Y_{J_2(M_2y_2)}
\]

\[
= -\sqrt{\frac{2J_1(2J_1 + 1)(2J_1 + 2)2J_2(2J_2 + 1)(2J_2 + 2)}{2J + 1}}
\times \left\{ \begin{array}{ccc} J & J_1 & J_2 \\ \frac{1}{2} & J_2 + y_2 & J_1 + y_1 \end{array} \right\} \left\{ \begin{array}{ccc} J & J_1 & J_2 \\ \frac{1}{2} & J_2 - y_2 & J_1 - y_1 \end{array} \right\}
\times C_{J_1+y_1m_1,J_2+y_2m_2}^{Jm,Jm'} C_{J_1-y_1m_1',J_2-y_2m_2'}^{Jm',Jm'},
\]

(B.3)

where \(M = M_1 + M_2\) and

\[
|J_1 - J_2| \leq J \leq J_1 + J_2,
\]

(B.4)

with integer \(J + J_1 + J_2\). The lower (upper) equality is satulated at \(y_1 = y_2\) \((y_1 \neq y_2)\).

Type E \((x = \pm 1)\)

\[
E_{J_1(M_1x_1),J_2(M_2x_2)}^{JM} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y_{JM}^* Y_{J_1(M_1x_1)} Y_{J_2(M_2x_2)}
\]

\[
= \sqrt{\frac{(2J_1 - 1)(2J_1 + 1)(2J_1 + 3)(2J_2 - 1)(2J_2 + 1)(2J_2 + 3)}{2J + 1}}
\times \left\{ \begin{array}{ccc} J & J_1 & J_2 \\ 1 & J_2 + x_2 & J_1 + x_1 \end{array} \right\} \left\{ \begin{array}{ccc} J & J_1 & J_2 \\ 1 & J_2 - x_2 & J_1 - x_1 \end{array} \right\}
\times C_{J_1+x_1m_1,J_2+x_2m_2}^{Jm,Jm'} C_{J_1-x_1m_1',J_2-x_2m_2'},
\]

(B.5)

where \(M = M_1 + M_2\) and

\[
|J_1 - J_2| \leq J \leq J_1 + J_2,
\]

(B.6)

with integer \(J + J_1 + J_2\). The lower (upper) equality is satulated at \(x_1 = x_2\) \((x_1 \neq x_2)\).

Type G \((y = \pm \frac{1}{2})\)

\[
G_{J_1(M_1y_1),J_2M_2}^{JM} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y_{JM}^* Y_{J_1(M_1y_1)} \hat{\nabla}_i Y_{J_2M_2}^i
\]

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\[ = - \frac{1}{2\sqrt{2}} \sum_{K=J_2 \pm \frac{1}{2}}^{2J_1 + 1} \frac{2J_1(2J_1 + 1)(2J_1 + 2)(2J_2 + 1)}{2J + 1} \sum_{K=J_2 \pm \frac{1}{2}}^{2K(2K + 1)(2K + 2)} \]

\[ \times \left\{ \begin{array}{ccc} J & J_1 & K \\ \frac{1}{2} & J_2 & J_1 + \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} J & J_1 & K \\ \frac{1}{2} & J_2 & J_1 - \frac{1}{2} \end{array} \right\} C_{J_1+y_1m_1,J_2m_2}^{J,J'} C_{J_1-x_1m_1',J_2m_2'}^{J,J'}; \]

(B.7)

where \( M = M_1 + M_2 \) and

\[ |J_1 - J_2| + \frac{1}{2} \leq J \leq J_1 + J_2 - \frac{1}{2}, \]  

(B.8)

with half integer \( J + J_1 + J_2 \).

**Type H \((x = \pm 1, y = \pm \frac{1}{2})\)**

\[ H_{J_1(M_1x_1);J_2(M_2y_2)}^{JM} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y_{JM}^{*} Y_{J_1(M_1x_1)}^{i} \nabla_{i} Y_{J_2(M_2y_2)}^{j} \]

\[ = - \frac{3}{2\sqrt{2}} \sum_{K=J_2 \pm \frac{1}{2}}^{2J_1 + 1} \frac{(2J_1 - 1)(2J_1 + 1)(2J_1 + 3)2J_2(2J_2 + 1)(2J_2 + 2)}{2J + 1} \]

\[ \times \sum_{K=J_2 \pm \frac{1}{2}}^{2K(2K + 1)(2K + 2)} \]

\[ \times \left\{ \begin{array}{ccc} K & 1 & J_2 + y_2 \\ \frac{1}{2} & J_2 & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} K & 1 & J_2 - y_2 \\ \frac{1}{2} & J_2 & \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{ccc} J & J_1 + x_1 & J_2 + y_2 \\ 1 & K & J_1 \end{array} \right\} \]

\[ \times \left\{ \begin{array}{ccc} J & J_1 - x_1 & J_2 - y_2 \\ K & J_1 & \end{array} \right\} C_{J_1+x_1m_1,J_2+y_2m_2}^{Jm} C_{J_1-x_1m_1',J_2-y_2m_2'}^{Jm'}; \]

(B.9)

where \( M = M_1 + M_2 \) and

\[ |J_1 - J_2| + \frac{1}{2} \leq J \leq J_1 + J_2 - \frac{1}{2}, \]  

(B.10)

with half integer \( J + J_1 + J_2 \). The lower (upper) equality is satulated at \( x_1 = 2y_2 \) \((x_1 \neq 2y_2)\).

C **Generalized Forms of SU(2) × SU(2) Clebsch-Gordan Coefficients**

General forms of \( SU(2) \times SU(2) \) Clebsch-Gordan coefficients are defined by the integrals of scalar quantities constructed from three products of \( ST^2 \)
tensor harmonics,
\[ CG_{J_1(M_1 \varepsilon_{n_1}), J_2(M_2 \varepsilon_{n_2})}^{J(M \varepsilon_n)} \sim \int_{S^3} d\Omega_{3} \mathcal{S}c \left\{ Y_{i_1 \cdots i_n}^{j_1 \cdots j_n} \cdot Y_{j_1(M \varepsilon_{n_1}), J_2(M_2 \varepsilon_{n_2})}^{k_1 \cdots k_n} \right\} \]
\[ \propto C_{J+\varepsilon_{m_1}, J_2+\varepsilon_{m_2}}^{J_1-\varepsilon_{m'_1}, J_2-\varepsilon_{m'_2}} \]

where \( \mathcal{S}c \{ \} \) indicates the operation making the product a scalar quantity by contracting out tensor indices, \( i, j \) and \( k \), and, if necessary, inserting derivatives appropriately. This coefficient has a non-vanishing value when \( M = M_1 + M_2 \) and the triangular conditions among \( J, J_1 \) and \( J_2 \) are satisfied, which are obtained from the conditions that two triangular conditions with respect to the left and right standard \( SU(2) \) Clebsch-Gordan coefficients must be satisfied simultaneously.

Here, we define the generalized \( SU(2) \times SU(2) \) Clebsch-Gordan coefficients, \( ^n\mathcal{E}, ^n\mathcal{H}, ^n\mathcal{D}, ^n\mathcal{G} \) and \( ^n\mathcal{T} \), used in the text to classify \( Q_M \)-invariant operators. The generalized coefficients with the rank 4 tensor index in the \( J \) component are defined by

\[ ^4\mathcal{E}_{J_1(M_1 x_1), J_2(M_2 x_2)}^{J(M w)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijjij}^{ijkl} Y_{ijkl}^{J(M w)} \cdot Y_{ijkl}^{J_1(M_1 x_1) J_2(M_2 x_2)} \]
\[ ^4\mathcal{H}_{J_1(M_1 y_1), J_2(M_2 y_2)}^{J(M w)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijjij}^{ijkl} Y_{ijkl}^{J(M w)} \cdot \nabla (k Y_{1}) J_1(M_1 y_1) J_2(M_2 y_2) \]
\[ ^4\mathcal{D}_{J_1(M_1 y_1), J_2(M_2 y_2)}^{J(M w)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijjij}^{ijkl} \nabla (i Y_{j}) J_1(M_1 y_1) \nabla (k Y_{2}) J_2(M_2 y_2) \]
\[ ^4\mathcal{G}_{J_1(M_1 y_1), J_2(M_2 y_2)}^{J(M w)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijjij}^{ijkl} \nabla (i Y_{j}) J_1(M_1 y_1) \nabla (k Y_{2}) J_2(M_2 y_2) \]

The generalized coefficients with the rank 3 tensor index in the \( J \) component, \( ^3\mathcal{E}_{J_1(M_2 x_1), J_2(M_2 x_2)}^{J(M_2 w)} \), \( ^3\mathcal{H}_{J_1(M_2 y_1), J_2(M_2 y_2)}^{J(M_2 w)} \), \( ^3\mathcal{D}_{J_1(M_2 y_1), J_2(M_2 y_2)}^{J(M_2 w)} \) and \( ^3\mathcal{G}_{J_1(M_2 y_1), J_2(M_2 y_2)}^{J(M_2 w)} \), are defined by replacing \( Y_{ijkl}^{J(M w)} \) in (C.2) with \( \nabla (i Y_{j}) J_{jkl}^{(M_2 w)} \).

The generalized coefficients with the rank 2 tensor index in the \( J \) component are defined by

\[ ^2\mathcal{E}_{J_1(M_2 x_1), J_2(M_2 x_2)}^{J(M x)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijkl}^{ijjij} Y_{ijkl}^{J(M x)} \cdot Y_{ijkl}^{J_1(M_2 x_1) Y_{ijkl}^{J_2(M_2 x_2)}} \]
\[ ^2\mathcal{H}_{J_1(M_2 x_1), J_2(M_2 x_2)}^{J(M x)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_{3} Y_{ijkl}^{ijjij} \nabla (k Y_{2}) J_1(M_2 x_1) \nabla (k Y_{2}) J_2(M_2 x_2) \]
\[ 2 \mathcal{D}^{(M_x)}_{J_1(M_1y_1), J_2(M_2y_2)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{2*}_{iJ(M_x)} \hat{\nabla}^{(k)Y^{j*}}_{iJ_1(M_1y_1)} \hat{\nabla}^{(k)Y_j}_{J_2(M_2y_2)}, \]
\[ 2 \mathcal{I}^{(M_x)}_{J_1(M_1x_1): J_2M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{2*}_{iJ(M_x)} Y^{*ki}_{J_1(M_1x_1)} \times \left( \hat{\nabla}^{(k)\nabla_j} - \frac{1}{3} \hat{\gamma}_{kJ} \hat{\nabla^2} \right) Y_{J_2M_2}, \]
\[ 2 \mathcal{G}^{(M_x)}_{J_1(M_1y_1): J_2M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{2*}_{iJ(M_x)} \hat{\nabla}^{(k)Y^{j*}}_{iJ_1(M_1y_1)} \times \left( \hat{\nabla}^{(k)\nabla_j} - \frac{1}{3} \hat{\gamma}_{kJ} \hat{\nabla^2} \right) Y_{J_2M_2}. \quad (C.3) \]

The generalized coefficients with the vector index in the \( J \) component, \( ^{1}E^{(M_y)}_{J_1(M_1x_1), J_2(M_2x_2)}, ^{1}H^{(M_y)}_{J_1(M_1x_1), J_2(M_2x_2)}, ^{1}\mathcal{D}^{(M_y)}_{J_1(M_1y_1), J_2(M_2y_2)}, ^{1}\mathcal{I}^{(M_y)}_{J_1(M_1y_1), J_2(M_2y_2)} \) and \( ^{1}\mathcal{G}^{(M_y)}_{J_1(M_1y_1), J_2M_2} \), are defined by replacing \( Y^{ij}_{J(M_x)} \) in (C.3) with \( \hat{\nabla}^{(i)Y^{j*}}_{J(M_y)} \).

The generalized coefficients with the scalar index in the \( J \) component are defined as
\[ ^{\nu}E^{JM}_{J_1(M_1x_1), J_2(M_2x_2)} = E^{JM}_{J_1(M_1x_1), J_2(M_2x_2)}, \]
\[ ^{\nu}H^{JM}_{J_1(M_1x_1): J_2(M_2y_2)} = H^{JM}_{J_1(M_1x_1): J_2(M_2y_2)}, \]
\[ ^{\nu}\mathcal{D}^{JM}_{J_1(M_1y_1), J_2(M_2y_2)} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{*}_{jJ(M_y)} \hat{\nabla}^{(i)Y^{j}}_{J_1(M_1y_1)} \hat{\nabla}^{(i)Y_j}_{J_2(M_2y_2)}, \]
\[ ^{\nu}\mathcal{I}^{JM}_{J_1(M_1x_1): J_2M_2} = I^{JM}_{J_1(M_1x_1): J_2M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{*}_{jJ(M_y)} Y^{ij}_{J_1(M_1x_1)} \hat{\nabla}^{(i)\nabla_j} Y_{J_2M_2}, \]
\[ ^{\nu}\mathcal{G}^{JM}_{J_1(M_1y_1): J_2M_2} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 Y^{*}_{jJ(M_y)} \hat{\nabla}^{(i)Y^{j}}_{J_1(M_1y_1)} \hat{\nabla}^{(i)\nabla_j} Y_{J_2M_2}. \quad (C.4) \]

The triangular conditions of these generalized Clebsch-Gordan coefficients are obtained from expression (C.1). The non-vanishing condition used in the text are summarized as
\[ ^{n}E^{JM}_{J_1, J_2} : \quad J \leq J_1 + J_2 - \frac{n}{2} \quad (n \leq 2), \quad J \leq J_1 + J_2 - \left| \frac{n}{2} - 2 \right| \quad (n \geq 2), \quad (C.5) \]
\[ ^{n}H^{JM}_{J_1, J_2} : \quad J \leq J_1 + J_2 - \frac{n}{2} - \frac{1}{2} \quad (n \leq 2), \quad J \leq J_1 + J_2 - \left| \frac{n}{2} - \frac{3}{2} \right| \quad (n \geq 2). \quad (C.6) \]
\[
^n \mathbf{D}_{J_1, J_2}^J : J \leq J_1 + J_2 - \frac{n}{2} \quad (n \leq 1),
\]
\[
J \leq J_1 + J_2 - \left| \frac{n}{2} - 1 \right| \quad (n \geq 1),
\]
(C.7)
\[
^n \mathbf{T}_{J_1; J_2}^J : J \leq J_1 + J_2 - \left| \frac{n}{2} - 1 \right|,
\]
(C.8)
\[
^n \mathbf{G}_{J_1; J_2}^J : J \leq J_1 + J_2 - \left| \frac{n}{2} - \frac{1}{2} \right|.
\]
(C.9)

The generalization of these coefficients to the cases with the higher rank index \(n > 4\) is straightforward, and then these conditions will be effective for such cases.

The coefficient \(\mathbf{0D}\) can be expressed as

\[
\mathbf{0D}_{J_1(M_1y_1), J_2(M_2y_2)}^{JM} = \left\{ J_1(J_1 + 1) + J_2(J_2 + 1) - J(J + 1) - \frac{3}{2} \right\} D_{J_1(M_1y_1), J_2(M_2y_2)}^{JM} + \frac{1}{2} \tilde{D}_{J_1(M_1y_1), J_2(M_2y_2)}^{JM},
\]
(C.10)

where

\[
\tilde{D}_{J_1(M_1y_1), J_2(M_2y_2)}^{JM} = \sqrt{\text{Vol}(S^3)} \int_{S^3} d\Omega_3 \left( \hat{\nabla}_i \hat{\nabla}_j Y_{JM}^* \right) Y_{J_1(M_1y_1)}^i Y_{J_2(M_2y_2)}^j.
\]
(C.11)

The coefficient \(\mathbf{0G}\) can be simplified in the form

\[
\mathbf{0G}_{J_1(M_1y_1), J_2(M_2y_2)}^{JM} = 2 \left\{ J_1(J_1 + 1) + J_2(J_2 + 1) - J(J + 1) - \frac{3}{4} \right\} G_{J_1(M_1y_1), J_2(M_2y_2)}^{JM},
\]
(C.12)

\section{D Relations between D and \(\mathbf{0D}\)}

The relations between the \(D_{J_1,J_2}^J\) and \(\mathbf{0D}_{J_1,J_2}^J\) coefficients used in the text are derived here. The \(\mathbf{0D}\) coefficient is expressed using \(\mathbf{D}\) in (C.10). In the case of \(J = \frac{1}{2}\), because of \(\hat{\nabla}_i \hat{\nabla}_j Y_{\frac{1}{2}M} = -\hat{\gamma}_{ij} Y_{\frac{1}{2}M}\), we obtain the relation

\[
\mathbf{D}_{J_1(M_1y_1), J_2(M_2y_2)}^{\frac{1}{2}M} = -\mathbf{D}_{J_1(M_1y_1), J_2(M_2y_2)}^{\frac{1}{2}M}.
\]
(D.1)
From this we obtain
\[ 0 \mathcal{D}^\frac{1}{2} M_{1(M_1y_1),J_2(M_2y_2)} = \left\{ J_1(J_1 + 1) + J_2(J_2 + 1) - \frac{11}{4} \right\} D^\frac{1}{2} M_{1(M_1y_1),J_2(M_2y_2)}. \]  
(D.2)

To discuss more general cases used in Sect.5, we consider two types of crossing relations constructed from the $D$ and $\tilde{D}$ coefficients. The one is the relation used to obtain the building blocks for the vector field in Sect.4,
\[ \sum_{V,y} \epsilon_V D^\frac{1}{2} M_{J-K-\frac{1}{2}(M_1y_1),J-K(-V)y} D^{JN}_{J-K(V_1),K(M_2y_2)} = \sum_{V,y} \epsilon_V D^\frac{1}{2} M_{K(M_2y_2),K+\frac{1}{2}(-V)y} D^{JN}_{K+\frac{1}{2}(V_1),J-K-\frac{1}{2}(M_1y_1)}. \]  
(D.3)

This is the special case of the crossing relation (4.13) with $J_1 = J - K - \frac{1}{2}$ and $J_2 = K$, where $J \geq 1$. The other is the crossing relation derived from the integral
\[ \int_{S^3} d\Omega_3 Y^*_M \hat{\nabla}^{[i} Y^{j]}_{J_1(M_1y_1)} \hat{\nabla}^{[i} Y^{j]}_{J_2(M_2y_2)} Y_{JN}. \]  
(D.4)

Because of the anti-symmetric property, the product expansion has the form
\[ Y^*_M \hat{\nabla}^{[i} Y^{j]}_{J_1(M_1y_1)} = -\frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I=J_1+\frac{1}{2}} \sum_{V,y} \frac{2}{(2I+1)^2} \left\{ I(I+1) + J_1(J_1+1) + \frac{1}{4} \right\} \times \epsilon_V D^\frac{1}{2} M_{J_1(M_1y_1),J(-V)y} \hat{\nabla}^{[i} Y^{j]}_{I(V_1)}. \]  
(D.5)

where Eq.(D.1) is used. From this, we can obtain the crossing relation only with the $D$ and $\tilde{D}$ coefficients. For the case of $J_1 = J - K - \frac{1}{2}$ and $J_2 = K$, it has the form
\[ \frac{2J - 2K}{2J - 2K + 1} \sum_{V,y} \epsilon_V D^\frac{1}{2} M_{J-K-\frac{1}{2}(M_1y_1),J-K(-V)y} \times \left\{ -2K(2J - 2K + 1) D^{JN}_{J-K(V_1),K(M_2y_2)} - \tilde{D}^{JN}_{J-K(V_1),K(M_2y_2)} \right\} = \frac{2K + 1}{2K + 2} \sum_{V,y} \epsilon_V D^\frac{1}{2} M_{K(M_2y_2),K+\frac{1}{2}(-V)y} \times \left\{ -(2K + 1)(2J - 2K - 1) + 1 \right\} D^{JN}_{K+\frac{1}{2}(V_1),J-K-\frac{1}{2}(M_1y_1)} \tilde{D}^{JN}_{K+\frac{1}{2}(V_1),J-K-\frac{1}{2}(M_1y_1)}. \]  
(D.6)
These two crossing relations, (D.3) and (D.6), should be equivalent. We here assume
\[ \tilde{D}_{JM}^{J-K(M_1y_1),K(M_2y_2)} = A(J, K)D_{JM}^{J-K(M_1y_1),K(M_2y_2)} \] (D.7)
with \( A(J, K) = A(J, J - K) \) and \( J \geq 1 \). Substituting this relation into Eq.(D.6) and comparing with Eq.(D.3), we obtain the recursion relation
\[ \frac{2J - 2K}{2J - 2K + 1} B(J, K) = \frac{2K + 1}{2K + 2} B \left( J, K + \frac{1}{2} \right), \] (D.8)
where \( B(J, K) = -2K(2J - 2K) + 1 - A(J, K) \). Using the initial condition \( A(J, \frac{1}{2}) = 2J + 2 \) easily calculated from the definition, we can solve the recursion relation, and thus we obtain the \( K \) independent value,
\[ A(J, K) = 2J + 2. \] (D.9)
From equation (C.10) and this result, we obtain the relation
\[ 0_{D_{JM}^{J-K(M_1y_1),K(M_2y_2)}} = -\frac{1}{2} (2K - 1)(2J - 2K - 1)D_{JM}^{J-K(M_1y_1),K(M_2y_2)}. \] (D.10)

E Crossing Relations of Types, I, II, III

The crossing relations used in Sect.5 and partialy in Sect.4 are derived here. We use two product expansions:
\[ Y_{\frac{1}{2}M}^{ij} Y_{J_1(M_1x_1)}^{ij} \]
\[ = \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I=J_1 \pm \frac{1}{2}} \sum_{T,x} \epsilon_T E_{J_1(M_1x_1),I(-T)x}^{\frac{1}{2}M} Y_{I(Tx)}^{ij} \]
\[ - \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I=J_1} \sum_{V,y} \frac{2}{(2I - 1)(2I + 3)} \epsilon_V H_{J_1(M_1x_1);I(-V)y}^{\frac{1}{2}M} \hat{\nabla}^{(i} Y_{j)}_{I(Vy)} \] (E.1)
and
\[ Y_{\frac{1}{2}M}^{ij} \hat{\nabla}^{(i} Y_{j)}_{J_1(M_1y_1)} \]
(D.2) and (C.12), respectively. The sum of three product expansions (E.1), we obtain the crossing relation

\[ \sqrt{\text{Vol}(S^3)} \left( \sum_{I=J_1+1/2}^{J_2} \sum_{T,x} \epsilon_T H^{1M}_{I(-Tx);J_1(M_1x_1)} Y^{ij}_{I(Tx)} \right) \]

\[ - \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I=J_1+1/2}^{J_2} \sum_{V,y} \left( \frac{2}{(2I-1)(2I+3)} \epsilon_V 0D^{1M}_{J_1(M_1y_1);I(-V_y)} \nabla^{(i} Y^{j)}_{I(V_y)} \right) \]

\[ + \frac{1}{\sqrt{\text{Vol}(S^3)}} \sum_{I=J_1}^{J_2} \sum_{S} \left( \frac{3}{2(2I-1)2I(2I+2)(2I+3)} \right) \]

\[ \times \epsilon_S 0G^{1M}_{J_1(M_1y_1);I-S} \left( \nabla^{(i} \nabla^{j)} - \frac{1}{3} \epsilon^{ij} \nabla^2 \right) Y_{IS}. \]  

(E.2)

Here, note that $0D^{1}_{J_1,J_2}$ and $0G^{1}_{J_1,J_2}$ can be expressed by $D^{1}_{J_1,J_2}$ and $G^{1}_{J_1,J_2}$ as (D.2) and (C.12), respectively. The sum of $I$ in each line of r.h.s. is fixed by the triangular conditions of the Clebsch-Gordan coefficients. Especially, $H^{1}_{J_1;J_2} \propto \delta_{J_1,J_2}$ and $G^{1}_{J_1;J_2} \propto \delta_{J_1,J_2}$ are taken into account.

**Crossing relations of type I** We first consider the following series of integrals:

\[ n = 0 \quad \int_{S^3} d\Omega_3 Y^{*}_{J_1(M_1x_1)} Y_{J_2(M_2x_2)} Y^{*}_{J_M}; \]

\[ n = 1 \quad \int_{S^3} d\Omega_3 Y^{*}_{J_1(M_1x_1)} Y^{ij}_{J_2(M_2x_2)} \nabla (i Y^{*}_{J})_{M_y}; \]

\[ n = 2 \quad \int_{S^3} d\Omega_3 Y^{*}_{J_1(M_1x_1)} Y^{ij}_{J_2(M_2x_2)} Y^{*}_{skM_x}; \]

\[ n = 3 \quad \int_{S^3} d\Omega_3 Y^{*}_{J_1(M_1x_1)} Y^{ij}_{J_2(M_2x_2)} \nabla (ijkl Y^{*})_{M_z}; \]

\[ n = 4 \quad \int_{S^3} d\Omega_3 Y^{*}_{J_1(M_1x_1)} Y^{ij}_{J_2(M_2x_2)} Y^{*}_{ijklM_w}; \]  

(E.3)

where $n$ denotes the rank of the last harmonics in each integrand. Using the product expansion (E.1), we obtain the crossing relation

\[ \sum_{I=J_1+1/2}^{J_2} \sum_{T,x} \epsilon_T E^{1M}_{J_1(M_1x_1);I(-Tx)} nE^{(M_e)}_{I(Tx);J_2(M_2x_2)} \]

\[ - \sum_{I=J_1}^{J_2} \sum_{V,y} \left( \frac{2}{(2I-1)(2I+3)} \epsilon_V H^{1M}_{J_1(M_1y_1);I(-V_y)} nH^{(M_e)}_{J_2(M_2x_2);I(V_y)} \right) \]

\[ = [(J_1, M_1, x_1) \leftrightarrow (J_2, M_2, x_2)], \]  

(E.4)

for each $n$. These equations refer to the crossing relations of type I.
Crossing relations of type II  Next, we consider the following series of integrals:

\[
\begin{align*}
n = 0 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y_{j_1(M_{1x_1})} \nabla (iY_{j}) J_2(M_{2y_2}) Y^*_{JM}, \\
n = 1 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^i_{j_1(M_{1x_1})} \nabla (kY_{j}) J_2(M_{2y_2}) Y^*_{Jy}, \\
n = 2 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^i_{j_1(M_{1x_1})} \nabla (kY_{j}) J_2(M_{2y_2}) Y^*_{ikJy}, \\
n = 3 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^{ij}_{j_1(M_{1x_1})} \nabla (kY_{j}) J_2(M_{2y_2}) Y^*_{ijklJy}, \\
n = 4 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} Y^{ij}_{j_1(M_{1x_1})} \nabla (kY_{j}) J_2(M_{2y_2}) Y^*_{ijklJy}. 
\end{align*}
\]

The only difference from the case of type I is that \(Y_{j_2(M_{2x_2})}^{ij}\) is replaced by \(\nabla (iY_{j}) J_2(M_{2y_2})\). Using the product expansions (E.1) and (E.2), we obtain the crossing relation

\[
\sum_{I=J_1+\frac{1}{2}} J_2 \sum_{T,x} \sum_{V,y} \frac{2}{(2I-1)(2I+3)} \epsilon_T E^{\frac{1}{2}M}_{J_1(M_{1x_1}), I(-Tx) J_2(M_{2y_2})}^{(M_{en})} - \sum_{I=J_1} \sum_{V,y} \frac{2}{(2I-1)(2I+3)} \epsilon_V H^{\frac{1}{2}M}_{J_1(M_{1x_1}), I(-Vy) J_2(M_{2y_2})}^{(M_{en})} \epsilon_T E^{(M_{en})}_{I(Tx), J_1(M_{1x_1})} = \sum_{I=J_2+\frac{1}{2}} J_1 \sum_{T,x} \sum_{V,y} \frac{2}{(2I-1)(2I+3)} \left\{ I(I+1) + J_2(J_2+1) - \frac{11}{4} \right\} \times \epsilon_V D^{\frac{1}{2}M}_{J_2(M_{2y_2}), I(-Vy) J_1(M_{1x_1})}^{(M_{en})} \epsilon_T H^{(M_{en})}_{J_1(M_{1x_1}), I(Vy)} + \sum_{I=J_2} \sum_{S} \frac{3}{(2I-1)2I(2I+2)(2I+3)} \left\{ I(I+1) + J_2(J_2+1) - \frac{3}{2} \right\} \times \epsilon_S G^{\frac{1}{2}M}_{J_2(M_{2y_2}), I-S}^{(M_{en})} T^{(M_{en})}_{J_1(M_{1x_1}); IS},
\]

for each \(n\). These equation refer to the crossing relations of type II.

Crossing relations of type III  Finally, we consider the following series of integrals:

\[
\begin{align*}
n = 0 & \quad \int_{S^3} d\Omega_3 Y^*_{\frac{1}{2}M} \nabla (iY_{j}) J_1(M_{1y_1}) \nabla (iY_{j}) J_2(M_{2y_2}) Y^*_{JM},
\end{align*}
\]
Killing vectors. The 15 ghosts to fix it are described as discussed in Ref. [6]. As discussed in the previous paper [8], in the radiation\textsuperscript{+} sector, indicate the \((2,2)\) representation of \(SU(2) \times SU(2)\), and \(c_{MN}\) satisfies the equations \(c_{MN}^\dagger = c_{NM}\) and \(c_{MN} = -\epsilon_M\epsilon_N c_{-N-M}\). We also introduce associate anti-ghost fields, denoted by \(b, b_M, b_M^\dagger\), and \(b_{MN}\), where \(b_{MN}^\dagger = b_{NM}\) and \(b_{MN} = -\epsilon_M\epsilon_N b_{-N-M}\). The

\begin{align*}
  n = 1 & \quad \int_{\Sigma^n} d\Omega_3 Y_M^{\dagger} \nabla^{(i} Y_{J_1(M_1 y_1)}^{k)} \nabla^{(j} Y_{k_2(p_2 y_2)}^{l)} \nabla^{(i} Y_{j(M y)}^{k)} \\
  n = 2 & \quad \int_{\Sigma^n} d\Omega_3 Y_M^{\dagger} \nabla^{(i} Y_{J_1(M_1 y_1)}^{k)} \nabla^{(j} Y_{k_2(p_2 y_2)}^{l)} \nabla^{(i} Y_{j(M x)}^{k)} \\
  n = 3 & \quad \int_{\Sigma^n} d\Omega_3 Y_M^{\dagger} \nabla^{(i} Y_{J_1(M_1 y_1)}^{j)} \nabla^{(k} Y_{J_2(M_2 y_2)}^{l)} \nabla^{(i} Y_{j(M z)}^{k)} \\
  n = 4 & \quad \int_{\Sigma^n} d\Omega_3 Y_M^{\dagger} \nabla^{(i} Y_{J_1(M_1 y_1)}^{j)} \nabla^{(k} Y_{J_2(M_2 y_2)}^{l)} \nabla^{(i} Y_{j(M w)}^{k)} 
\end{align*}

The difference from the cases of type I is that \(Y_{J_1(M_1 x_1)}^{ij}\) and \(Y_{J_2(M_2 x_2)}^{ij}\) are replaced by \(\nabla(i\gamma_j^{(i}) J_1(M_1 y_1)\gamma_j^{j)})\) and \(\nabla(i\gamma_j^{(i}) J_2(M_2 y_2)\gamma_j^{j)})\), respectively. Using the product expansion (E.2), we obtain the crossing relation

\begin{align*}
  & \sum_{I=J_1 T,x} \sum_{I=J_1 T,x} \epsilon_T H_{I(-T x):J_1(M_1 y_1)}^{\dagger M} H_{I(T x):J_2(M_2 y_2)}^{(M e n)} \\
  & \quad - \sum_{I=J_1 \pm \frac{1}{2} y y} \frac{2}{(2I-1)(2I+3)} \left\{ I(I+1) + J_1(J_1+1) - \frac{11}{4} \right\} \\
  & \quad \quad \times \epsilon_Y D_{J_1(M_1 y_1):I(-Y y)}^{\dagger M} D_{I(Y y):J_2(M_2 y_2)}^{(M e n)} \\
  & \quad + \sum_{I=J_1} \sum_{S} \frac{3}{(2I-1)(2I+3)(2I+2)} \left\{ I(I+1) + J_1(J_1+1) - \frac{3}{2} \right\} \\
  & \quad \quad \times \epsilon_S G_{J_1(M_1 y_1):I(-S)}^{\dagger M} G_{J_2(M_2 y_2):I S}^{(M e n)} \\
  & \quad = \left[ (J_1, M_1, y_1) \leftrightarrow (J_2, M_2, y_2) \right],
\end{align*}

for each \(n\). These equations refer to the crossing relations of type III.

**F Conformal Algebra in Ghost Sector**

In this section we reinvestigate conformal algebra in the ghost sector discussed in Ref. [6]. As discussed in the previous paper [8], in the radiation\textsuperscript{+} gauge, there is the residual gauge symmetry generated by the conformal Killing vectors. The 15 ghosts to fix it are described as \(c, c_M, c_M^\dagger\) and \(c_{MN}\), where the indices, \(M\) and \(N\), indicate the \((2,2)\) representation of \(SU(2) \times SU(2)\), and \(c_{MN}\) satisfies the equations \(c_{MN}^\dagger = c_{NM}\) and \(c_{MN} = -\epsilon_M\epsilon_N c_{-N-M}\). We also introduce associate anti-ghost fields, denoted by \(b, b_M, b_M^\dagger\), and \(b_{MN}\), where \(b_{MN}^\dagger = b_{NM}\) and \(b_{MN} = -\epsilon_M\epsilon_N b_{-N-M}\).
commutators of these fields are defined as
\[
\{b, c\} = 1, \\
\{b^\dagger_M, c_N\} = \{b_M, c_N^\dagger\} = \delta_{MN}, \\
\{b_{M_1N_1}, c_{M_2N_2}\} = \delta_{M_1M_2}\delta_{N_1N_2} - \epsilon_{M_1N_1}\delta_{-M_1N_2}\delta_{-N_1M_2},
\] (F.1)

The ghosts related to the time translation, c and b, and the rotations on $S^3$, $c_{MN}$ and $b_{MN}$, have the level 0. The ghosts related to the special conformal transformations, $c_M$ and $b_M$, have the level $-1$, and their conjugates, $c_M^\dagger$ and $b_M^\dagger$, have the level 1. Thus, the hamiltonian has the form
\[
H = \sum_R \left( b_R^\dagger c_R + c_R^\dagger b_R \right) + \text{constant}. \tag{F.2}
\]

Since the conformal charge $Q_M$ has the level $-1$ and belongs to the $(2, 2)$ representation of $SU(2) \times SU(2)$, the general form of this charge is given by
\[
Q_M = \lambda_1 b c_M + \lambda_2 b_M c + \sum_R \left( \kappa_1 b_R c_{RM} + \kappa_2 b_{MR} c_R \right). \tag{F.3}
\]

We here require that these charges form the conformal algebra (1.1). This requirement is satisfied if $\lambda_1 \lambda_2 = 2$ and $\kappa_1 \kappa_2 = 2$, and thus we obtain the following 15 conformal charges:
\[
Q_M = \lambda b c_M + \frac{2}{\lambda} b_M c + \sum_R \left( \kappa b_R c_{RM} + \frac{2}{\kappa} b_{MR} c_R \right), \\
H = \sum_R \left( b_R^\dagger c_R + c_R^\dagger b_R \right) - 4, \\
R_{MN} = b_N^\dagger c_M + c_N^\dagger b_M - \epsilon_M \epsilon_N \left( b_{-M}^\dagger c_{-N} + c_{-M}^\dagger b_{-N} \right) \\
- \frac{1}{2} \sum_R \left( b_{MR} c_{NR} + c_{RM} b_{RN} - b_{RN} c_{RM} - c_{NR} b_{MR} \right), \tag{F.4}
\]
where $\lambda$ and $\kappa$ are arbitrary constants. Note that the constant term in the Hamiltonian is fixed to be $-4$. This constant has a relationship to the world-volume dimensions.

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