Parabolic refined invariants and Macdonald polynomials

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Abstract

A string theoretic derivation is given for the conjecture of Hausel, Letellier and Rodriguez-Villegas on the cohomology of character varieties with marked points. Their formula is identified with a refined BPS expansion in the stable pair theory of a local root stack, generalizing previous work of the first two authors in collaboration with G. Pan. Haiman’s geometric construction for Macdonald polynomials is shown to emerge naturally in this context via geometric engineering. In particular this yields a new conjectural relation between Macdonald polynomials and refined local orbifold curve counting invariants. The string theoretic approach also leads to a new spectral cover construction for parabolic Higgs bundles in terms of holomorphic symplectic orbifolds.

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1 Introduction

The main goal of this paper is a string theoretic derivation of the conjecture of Hausel, Letellier and Rodriguez-Villegas [29] on the topology of character varieties of punctured Riemann surfaces. Analogous results have been obtained in [12, 11] in the absence of marked points, identifying the main conjecture of Hausel and Rodriguez-Villegas [30] with a refined Gopakumar-Vafa expansion. The same framework yields a recursion relation for Poincaré and Hodge polynomials of Higgs bundle moduli spaces using the wallcrossing formula of Kontsevich and Soibelman [42]. A motivic version of this recursion relation is derived by Mozgovoy in [49], and proved to be in agreement with the Hausel-Rodriguez-Villegas formula. The string theoretic construction also provides quantitative supporting evidence [11] for the $P = W$ conjecture formulated by de Cataldo, Hausel, and Migliorini in [14], and proven in loc. cit. for rank two Higgs bundles. The present paper carries out a similar program for character varieties with marked points, the starting point being the main conjecture formulated in [29], which is briefly reviewed below.

1.1 The Hausel-Letellier-Rodriguez-Villegas formula

Let $C$ be a smooth complex projective curve of genus $g \geq 0$, and $D = p_1 + \cdots + p_k$ a divisor of distinct reduced marked points on $C$. Let $\gamma_1, \ldots, \gamma_k$ denote the generators of the fundamental group $\pi_1(C \setminus D)$ corresponding to the marked points. For any nonempty partition $\mu = (\mu_1, \ldots, \mu^l)$ of $r \geq 1$, let $C_\mu$ be a semisimple conjugacy class in $GL(r, \mathbb{C})$ such that the eigenvalues of any matrix in $C_\mu$ have multiplicities $\{\mu_1, \ldots, \mu^l\}$.

Let $\mu = (\mu_1, \ldots, \mu_k)$ be a collection of partitions of an integer $r \geq 1$. Then the character variety $\mathcal{C}(C, D; \mu)$ is the moduli space of conjugacy classes of representations

$$f : \pi_1(C \setminus D) \to GL(r, \mathbb{C})$$

such that $f(\gamma_i) \in C_\mu_i$ for all $1 \leq i \leq k$. The character variety $\mathcal{C}(C, D; \mu)$ actually depends on the choice eigenvalues but we will suppress this dependence from the notation since the topological invariants we compute below are independent of this choice.
According to [29, Thm. 2.1.5], for sufficiently generic conjugacy classes \( C_i, C(C, D; \mu) \) is either empty or a smooth quasi-projective variety of dimension \( d = r^2(2g - 2 + k) - \sum_{i=1}^{k} \sum_{j=1}^{l_i}(\mu_i)^2 + 2 \), where \( l_i \) is the length of the partition \( \mu_i, 1 \leq i \leq k \), as above. The compactly supported cohomology \( H^*_c(C(C, D; \mu)) \) carries a weight filtration \( W \) and the mixed Poincaré polynomial is defined by

\[
P_c(C(C, D; \mu); u, t) = \sum_{i, k \geq 0} \dim \left( Gr^W_i H^k_c(C(C, D; \mu)) \right) u^{i/2} (-t)^k. \tag{1.1}
\]

A priori the right hand side of (1.1) takes values in \( \mathbb{Z}[u^{1/2}, t] \), but it was conjectured in [29] that it is in fact a polynomial in \( (u, t) \).

In order to formulate the main conjecture of [29], for any partition \( \lambda \) let

\[
\mathcal{H}^\lambda(z, w) = \prod_{(\square) \in \lambda} \frac{(z^{2a(\square)} + w^{2l(\square)}/1) \cdots (z^{2a(\square)} + w^{2l(\square)})}{(z^{2a(\square)}/1) \cdots (z^{2a(\square)} - w^{2l(\square)})}. \tag{1.2}
\]

where \( a(\square), l(\square) \) denote the arm, respectively leg length of \( \square \in \lambda \). Moreover, for each marked point \( p_i \), let \( x_i = (x_{i,1}, x_{i,2}, \ldots) \) be an infinite collection of formal variables, \( 1 \leq i \leq k \), and \( \tilde{H}_\lambda(z^2, w^2; x_i) \) be the modified MacDonald polynomial [24, 27] labelled by \( \lambda \). Then [29, Conjecture 1.2.1.(iii)] states that

\[
Z_{HLRV}(z, w, x_i) = \exp \left( \sum_{k=1}^{\infty} \sum_{\mu} \frac{1}{k} \frac{w^{-k\mu} P_c(C(C, D; \mu); z^{-2k}, -zw^k)}{(1 - z^{2k})(w^{2k} - 1)} \prod_{i=1}^{k} m_{\mu_i}(x_i^k) \right) \tag{1.3}
\]

where

\[
Z_{HLRV}(z, w, x_i) = \sum_{\lambda} \mathcal{H}^\lambda(z, w) \prod_{i=1}^{k} \tilde{H}_\lambda(z^2, w^2; x_i)
\]

and \( m_{\mu_i}(x_i) \) are the monomial symmetric functions. For ease of exposition equation (1.3) will be referred to as the HLRV formula.

Note also that the character variety \( C(C, D; \mu) \) is diffeomorphic to a moduli space of strongly parabolic Higgs bundles on \( C \). By analogy with the \( P = W \) conjecture formulated in [14], one expects the weight filtration on the compactly supported cohomology on the character variety to be identified with a perverse Leray filtration for the Hitchin map on the moduli of parabolic Higgs bundles. This conjectural identification plays an important role in this paper.
1.2 The main conjecture

In this paper we propose a program for verifying (1.3) by following a sequence of string-theoretic and geometric dualities providing identifications of various counting functions. Our main string theoretic construction relies on a conjectural identification of the generating function $Z_{HLRV}(z, w; x)$ with the stable pair theory of a Calabi-Yau orbifold $\tilde{Y}$. This orbifold is constructed in Section 4 using the results of [51, 26], which identify parabolic Higgs bundles on $C$ with Higgs bundles on a root stack. The root stack is an orbifold curve $\tilde{C}$ equipped with a natural projection to $C$, which makes $C$ its coarse moduli space. Its construction depends on the discrete invariants of the parabolic structure and is reviewed in detail in Section 3.1. In particular, note that the closed points of $\tilde{C}$ have generically trivial stabilizers, the orbifold points being in one-to-one correspondence with the marked points on $C$.

Given a line bundle $M$ on $C$, the three dimensional Calabi-Yau orbifold $\tilde{Y}_M$ is defined to be the total space of the rank two bundle $\tilde{Y}_M := \text{tot}(\nu^*M^{-1} \oplus (K_{\tilde{C}} \otimes \nu^*M))$ on $\tilde{C}$. In what follows we will call such three dimensional Calabi-Yau orbifolds local orbifold curves. Initially we focus on $\tilde{Y} := \tilde{Y}_O := \text{tot}(\mathcal{O}_{\tilde{C}} \oplus K_{\tilde{C}})$.

By analogy with Pandharipande and Thomas [55], the stable pair theory of a local orbifold curve $\tilde{Y}$ is defined in Section 4 as a counting theory for pairs $(F, s)$ with $F$ a pure dimension one sheaf on $\tilde{Y}$, and $s : \mathcal{O}_Y \to F$ a generically surjective section. The discrete invariants of $F$ are the Euler character $n = \chi(F)$ and a collection of integral vectors $\mathbf{m} = (m_i)_{1 \leq i \leq k}, \ m_i \in (\mathbb{Z}_{\geq 0})^{s_i}, \ s_i \geq 1$, encoding the $K$-theory class of $F$. In Section 4.1 we extend the analysis of [16]: given a line bundle $M$ on $C$, we reformulate the stable pair theory of the local orbifold curve $\tilde{Y}_M$ in terms of parabolic ADHM sheaves on the curve $C$. This yields an explicit construction of the perfect obstruction theory of the moduli space, and makes the relation with parabolic Higgs bundles more transparent.

Assuming the foundational aspects of motivic Donaldson-Thomas theory from [42], one obtains a series of refined Pandharipande-Thomas (PT) invariants for the orbifold $\tilde{Y}_M$:

$$Z^{ref}_{\tilde{Y}_M}(q, \mathbf{x}, y) = \sum_{n \in \mathbb{Z}} \sum_{\mathbf{m}} PT(\tilde{Y}_M, n, \mathbf{m}; y) q^n \prod_{i=1}^{k} \frac{x_i^{m_i}}{m_i} \tag{1.4}$$

for some formal variables $\mathbf{x} = (x_1, \ldots, x_k)$, $x_i = (x_{i,0}, \ldots, x_{i,s_i-1})$, $1 \leq i \leq k$. Then the main conjecture in this paper is the following identity:

**Conjecture.** After a change of variables, the counting function for the refined PT invariants on the orbifold Calabi-Yau $\tilde{Y} = \text{tot}(\mathcal{O}_{\tilde{C}} \oplus K_{\tilde{C}})$ is identified with the combinatorial HLRV
\[ Z^{ref}_{\gamma}(z^{-1}w, x, z^{-1}w^{-1}) = Z_{HLRV}(z, w, x). \] (1.5)

In Section 6 we explain how the relation (1.5) and the parabolic \( P = W \) conjecture imply that the same change of variables converts the HLRV formula (1.3) into a refined Gopakumar-Vafa expansion. Moreover, an application of the wallcrossing formula of Kontsevich and Soibelman yields a recursion relation for Poincaré polynomials analogous to [12]. As shown in Section 7, the main arguments of [49] apply to the present case as well, proving that the solution of this recursion formula is in agreement with the predictions of the HLRV formula.

A rigorous proof of identity (1.5) is one of the most important open problems emerging from this paper. Supporting evidence for this conjecture is provided in Sections 5 and 8, which are briefly summarized below.

1.3 Macdonald polynomials via geometric engineering

Geometric engineering is used in Section 5 to relate the stable pair generating function (1.4) to a D-brane quiver quantum mechanical partition function. Analogous results in the physics literature were obtained in [37, 43, 19, 53, 33, 34, 20, 32, 41, 44, 35] while a general mathematical theory of geometric engineering is currently being developed by Nekrasov and Okounkov in [52]. The treatment in Section 5 follows the usual approach in the physics literature via IIA/M-theory duality and D-brane dynamics. A detailed comparison with the formalism of [52] is left as an open problem, as briefly explained below.

For simplicity it is assumed that there is only one marked point on \( C \). The local orbifold curve is taken of the form form \( \tilde{Y}_M = \text{tot}(\nu^*M^{-1} \oplus K_{C_{\gamma}} \otimes \nu^*M) \) where \( M \) is a degree \( p \geq 0 \) line bundle on \( C \). As shown in Section 5.2, a two step chain of dualities relates the resulting stable pair theory to a series of equivariant \( K \)-theoretic invariants of nested Hilbert schemes of points in \( \mathbb{C}^2 \). The construction of the \( K \)-theoretic partition function is explained in Section 5.3. The final formula recorded in equation (5.6) is a generating function of the form

\[ Z_K(q_1, q_2; \gamma, \bar{x}) = \sum_{\gamma} \chi_T^{\gamma}(\mathcal{V}(\gamma)) m_{\mu(\gamma)}(\bar{x}) \] (1.6)
where $\chi^T_\gamma(V(\gamma))$ is the equivariant Hirzebruch genus of a vector bundle $V(\gamma)$ on the nested Hilbert scheme $N(\gamma)$. Here the sum is over all finite collections $\gamma = (\gamma_\ell)_{0 \leq \ell \leq \ell}$ of positive integers labelling discrete invariants of flags of ideal sheaves on $\mathbb{C}^2$, as explained in Section 5.2, above equation (5.5). For any $\gamma$, $\mu(\gamma)$ denotes the unordered partition of $|\gamma| = \sum_{\ell=0}^\ell \gamma_\ell$ determined by $\gamma$, and $m_\mu(\tilde{x})$ are the monomial symmetric functions in an infinite set of variables $(\tilde{x}_0, \tilde{x}_1, \ldots)$.

Note that the formalism of [52] relates the above stable pair theory with the equivariant K-theoretic stable pair theory of the product $\tilde{C} \times \mathbb{C}^2$. Then one expects the partition function (1.6) to follow from this theory by virtual localization computations. In particular the bundle of fermion zero modes derived in Appendix B is expected to be naturally determined by the induced perfect obstruction theory on the fixed loci. This computation will be left as an open problem.

The main result of section 5 is identity (5.18) expressing the generating function (1.6) in terms of modified Macdonald polynomials,

$$Z_K(q_1, q_2; \tilde{x}, \tilde{y}) = \sum_{\mu} \Omega^\mu_{\gamma, p}(q_1, q_2, \tilde{y}) \tilde{H}_\mu(q_2, q_1, \tilde{x}).$$

(1.7)

The $\Omega^\mu_{\gamma, p}(q_1, q_2, \tilde{y})$ are rational functions of the equivariant parameters $(q_1, q_2)$ and $\tilde{y}$ determined explicitly by a fixed point theorem, according to equation (5.17).

Formula (1.7) is proven in Section 5.5 using Haiman’s geometric construction of Macdonald polynomials in terms of isospectral Hilbert scheme [27, 28]. The proof also requires some geometric comparison results between nested and isospectral Hilbert schemes established in Section 5.4.

As supporting evidence for equation (1.5), it is shown in Section 6, equation (6.4), that a simple change of variables relates the right hand side of equation (1.7) to the HLRV generating function $Z_{HLRV}(z, w; x)$,

$$Z_{HLRV}(z, w; x) = Z_K(w^2, z^2; (zw)^{-1}, (-1)^{g-1}(zw)^g x).$$

(1.8)

Further supporting evidence is provided in Section 8, which is briefly summarized below.

### 1.4 Parabolic conifold invariants and the equivariant index

A direct computational test of conjecture (1.5) is carried out in Section 8 using the formalism developed by Nekrasov and Okounkov in [52]. The computations are carried out for the special case where $C$ is the projective line with one marked point $p$, and the local threefold
is $\widetilde{Y}_{\mathcal{O}_{C}^{(1)}}$ i.e. the total space of the rank two bundle $\nu^{*}\mathcal{O}_{C}(-1) \oplus K_{\widetilde{C}} \otimes_{\widetilde{C}} \nu^{*}\mathcal{O}_{C}(1)$ on $\widetilde{C}$. A conjectural relation between the equivariant index defined in [52] and orbifold refined stable pair invariants is formulated in Section 8.1, equation (1.5). This identification is checked by explicit virtual localization computations for low degree terms up to three box partitions in Section 8.2.

An important outcome of the string theoretic derivation is a new geometric construction of spectral data for parabolic Higgs bundles which lays the ground for a generalization of the HLRV formula. This is carried out in Section 3, a brief outline being provided below.

1.5 Outline of the program

For the convenience of the reader we now list all the ingredients in the physical derivation of the HLRV conjecture (1.3) in their logical sequence:

**Step 1.** Identify the combinatorial left hand side of the HLRV formula with the counting function for refined stable pair invariants on the three dimensional Calabi-Yau orbifold $\widetilde{Y}$. This identification is provided by the conjectural formula (1.5). The construction of the orbifold stable pair theory for this step is presented in Section 4.

**Step 2.** Identify the counting function for the refined stable pair invariants on $\widetilde{Y}$ with the generating function for the perverse Poincaré polynomials of the moduli of parabolic Higgs bundles. This identification is a combination of two components:

(i) A geometric isomorphism of the moduli of Bridgeland stable pure dimension one sheaves on $\widetilde{Y}$ and the product of the moduli space of parabolic Higgs bundles on $C$ with the affine line. This identification is based on the spectral cover construction explained in Section 3.1.

(ii) A conjectural refined Gopakumar-Vafa expansion of the stable pair theory of $\widetilde{Y}$ generalizing the unrefined conjecture formulated in [55]. Granting identity (1.5), the specialization of the HLRV formula to Poincaré polynomials follows recursively from the Kontsevich-Soibelman wall-crossing formula [42] for the variation of Bridgeland stabilities on the stable pair moduli by analogy with [12, 49]. The details are presented in Sections 6, 7.

**Step 3.** Identify the generating function for the perverse Poincaré polynomials of the moduli of parabolic Higgs bundles with the generating function for the weight-refined Poincaré...
polynomial of the character variety. This is a parabolic version of the $P = W$ conjecture of Hausel, de Cataldo, and Migliorini. A brief discussion is provided in Section 6.

Note that the refined Gopakumar-Vafa expansion needed here was conjectured for toric Calabi-Yau threefolds in [35] and also [10] building on previous work of [25, 38]. This conjecture was extended to higher genus local curves in [11]. Here it is further extended to local orbifold curves.

In the mathematical literature, a weak form of the unrefined Gopakumar-Vafa conjecture stable pair theory was proven in [5], [59], while the full unrefined conjecture was proven in [60]. These results prove the existence of a suitable integral expansion, but do not provide a cohomological interpretation of the resulting integral invariants. The latter is also needed in the string theory derivation of the HLRV formula.

The geometric framework developed in this program admits a generalization to parabolic Higgs bundles with nontrivial eigenvalues at the marked points. This yields in particular a new orbifold spectral cover presentation for such objects, generalizing the construction in Section 3.1. This is carried out in Sections 3.2, 3.3 and 3.4, which are summarized below.

### 1.6 Orbifold spectral data for nontrivial eigenvalues

The orbifold spectral cover construction applies to a particular flavor of parabolic Higgs bundles introduced in Section 2.3. These are Higgs bundles with simple poles at the marked points, whose residues are $\xi$-parabolic maps. This condition requires each graded component of the Higgs field residue at a marked point with respect to the flag to be a specified multiple of the identity. Parabolic Higgs bundles satisfying this condition are called *diagonally parabolic*, or, more specifically, $\xi$-parabolic, and form a closed substack of the moduli stack of semistable parabolic Higgs bundles.

The unordered eigenvalues of parabolic Higgs bundles are parameterized by the quotient $Q$ of the Hitchin base defined in equation (2.12). For each point $q \in Q$, the construction in Section 3.2 produces a holomorphic symplectic orbifold surface $\tilde{S}_q$. The moduli space of semistable $\xi$-parabolic Higgs bundles is conjecturally identified with a moduli space of semistable torsion sheaves on $\tilde{S}_q$ with fixed $K$-theory class. The precise statement of this conjecture is given in Section 3.3, a brief outline of the proof being provided in Section 3.4.

In this geometric framework string theory arguments predict a formula of the form (1.3), where the left hand side is given by the refined stable pair theory $Z_{PT}^{\text{ref}}(\tilde{S}_q \times \mathbb{C})$ up to a change of variables. The right hand side will be a similar generating function for perverse
Poincaré polynomials for moduli spaces of stable $\xi$-parabolic Higgs bundles. As pointed out by Emmanuel Letellier and Tamas Hausel, the latter is expected to be identical with the right hand side of equation (1.3), even away from the nilpotent locus. This leads to a rather surprising conjecture stating that the refined stable pair theory $Z_{\text{ref}}^\alpha (\widetilde{S}_\delta \times \mathbb{C})$ is independent on $\delta$. In particular, it should be identical with the stable pair theory of $\widetilde{Y}$ in equation (1.4).

1.7 Open problems

We conclude the introduction with a list of open problems emerging from this work. Several such questions have already been encountered above, including:

(a) the proof of (1.5),

(b) the derivation of an analogous formula for the stable pair theory of the orbifolds $\widetilde{S}_\delta \times \mathbb{C}$, confirming deformation invariance, and

(c) an explicit comparison with the equivariant $K$-theoretic stable pair theory of $\widetilde{C} \times \mathbb{C}^2$ in the context of geometric engineering.

Additional possible future directions include:

(d) Section 6 presents quantitative evidence for a parabolic version of the $P = W$ conjecture formulated in [14]. It would be very interesting if the parabolic $P = W$ can be proven by direct comparison methods in certain classes of examples.

(e) Another problem is to prove the crepant resolution conjecture for stable pair invariants formulated in [56, Conj. 4], for the orbifolds $\widetilde{S}_\delta \times \mathbb{C}$. Similar results in Donaldson-Thomas theorists have been proven in [7, 9].

(f) Elaborating on the same topic, a further question is whether $\xi$-parabolic Higgs bundles admit a spectral cover presentation in terms of torsion sheaves on the resolutions of the coarse moduli spaces. Again, the Fourier-Mukai transform should provide important input in finding the answer.

(g) Finally, a natural question is whether one can construct a TQFT formalism for (unrefined) curve counting invariants of local orbifold curves, by analogy with the results of Bryan and Pandharipande [8], and Okounkov and Pandharipande [54].

1.8 Notation and conventions

$C$ - a smooth complex projective curve of genus $g \geq 0$.

$D = p_1 + \cdots + p_k$ - a divisor of distinct reduced marked points on $C$. 
\[ \mu = (\mu_1, \ldots, \mu_k) \] - a collection of partitions of an integer \( r \geq 1 \).

\[ \mathcal{C}(C, D; \mu) \] - the character variety, i.e. the moduli space of conjugacy classes of representations of \( \pi_1(C \setminus D) \) with values in fixed conjugacy classes at the punctures.

\[ P_\xi(\mathcal{C}(C, D; \mu); u, t) \] - the mixed Poincaré polynomial for the weight filtration on the compactly supported cohomology of the character variety.

\[ \mathcal{H}_\lambda^g(z, w) \] - the HLRV \((z, w)\)-deformation of the \( 2g - 2 \) power of the hook polynomial given in equation (1.2).

\[ \tilde{H}_\mu(q_2, q_1, \tilde{x}) \] - the modified MacDonald polynomial [24, 27].

\[ Z_{HLRV}(z, w, x) \] - the combinatorial HLRV partition function appearing in the left hand side of the HLRV formula.

\( \tilde{C} \) - an orbifold curve equipped with a morphism \( \nu : \tilde{C} \to C \), which is an isomorphism outside \( D \).

\( \tilde{Y} \) - a three dimensional Calabi-Yau orbifold given as \( \tilde{Y} = \text{tot} (\mathcal{O}_{\tilde{C}} \oplus K_{\tilde{C}}) \).

\( \tilde{Y}_M \) - a three dimensional Calabi-Yau orbifold given as \( \tilde{Y}_M = \text{tot} (\nu^* M \oplus (K_{\tilde{C}} \otimes \nu^* M^{-1})) \) for some line bundle \( M \) on \( C \).

\[ Z_{Y_M}^{\text{ref}}(q, y, x) \] - the counting function of refined stable pair invariants on \( \tilde{Y}_M \).

\[ Z_K(q_1, q_2; \tilde{y}, \tilde{x}) \] - the counting function of equivariant K-theoretic invariants of nested Hilbert schemes of points in \( \mathbb{C}^2 \).

\[ \mathcal{H}_\xi^{ss}(C, D; m, e, \alpha) \] - the moduli stack of semistable \( \xi \)-parabolic Higgs bundles on \( C \).

\( \tilde{S}_\delta \) - a symplectic orbifold surface associated with a zero dimensional subscheme inside \( \text{tot}(K_C(D)) \).

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2 Parabolic Higgs bundles and spectral covers

The goal of this section is to provide some background on parabolic Higgs bundles, summarizing the main results used throughout the paper. Let $C$ be a smooth projective curve over $\mathbb{C}$ and $D = \sum_{i=1}^{k} p_i$ a reduced effective divisor on $C$. A meromorphic Higgs bundle is a pair $(E; \Phi)$ with $E$ a locally free sheaf and $\Phi : E \rightarrow E \otimes K_C(D)$ a morphism of sheaves on $C$. Parabolic Higgs bundles are a refinement of meromorphic ones defined by specifying a parabolic structure at each marked point, as discussed in detail below.

2.1 Parabolic structures

In order to fix notation, let $V$ be a finite dimensional vector space and $\mathbf{m} = (m_a)_{0 \leq a \leq s-1} \in \mathbb{Z}_{\geq 0}^s$ an ordered collection of non-negative integers such that

$$\sum_{a=0}^{s-1} m_a = \dim V. \quad (2.1)$$

A flag of type $\mathbf{m}$ in $V$ is a filtration

$$0 = V^s \subsetneq V^{s-1} \subsetneq \cdots \subsetneq V^{1} \subsetneq V^0 = V$$

by vector subspaces such that

$$\dim (V^a/V^{a+1}) = m_a, \quad 0 \leq a \leq s-1. \quad (2.2)$$

Note that degenerate flags are allowed i.e. the inclusions do not have to be strict, but the length of the filtration is fixed.

Suppose $V,W$ are finite dimensional vector spaces equipped with filtrations $V^\bullet$, $W^\bullet$ of the same length $s$. A linear map $f : V \rightarrow W$ will be called parabolic if $f(V^a) \subsetneq W^a$ for all $0 \leq a \leq s$. The map $f$ will be called strongly parabolic if $f(V^a) \subsetneq W^{a+1}$ for all $0 \leq a \leq s-1$. 

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A more refined compatibility condition can be defined when $W = V \otimes L$, with $L$ a one dimensional vector space, and $W^\bullet$ is the natural filtration determined by $V^\bullet$. Given a collection of linear maps $\xi = (\xi_a : C \to L)_{0 \leq a \leq s-1}$, a linear map $f : V \to W$ will be called $\xi$-parabolic if $f$ is parabolic and the induced maps $f_a : V^a/V^{a+1} \to V^a/V^{a+1} \otimes L$ are of the form

$$f_a = 1_{V^a/V^{a+1}} \otimes \xi_a.$$  

Following the notation introduced in [23], let $\text{PHom}(V, W)$, $\text{SPHom}(V, W)$ denote the linear space of parabolic, respectively strongly parabolic linear maps. Let also

$$\text{APHom}(V, W) = \text{Hom}(V, W)/\text{PHom}(V, W)$$

be the vector space parameterizing equivalence classes of morphisms not preserving the filtrations.

For any exact sequence

$$0 \to V' \to V \to V'' \to 0,$$

a flag $V^\bullet$ in $V$ of length $s$ induces canonical flags $V'^\bullet$, $V''^\bullet$ of $V'$, $V''$ of the same length. In fact vector spaces equipped with flags of fixed length $s$ form an abelian category.

Given a reduced effective divisor $D = \sum_{i=1}^k p_i$ on the curve $C$, for each $i \in \{1, \ldots, k\}$ let $m_i = (m_{i,a}), 0 \leq a \leq s_i, s_i \geq 1$, be an ordered collection of integers of length $s_i \geq 1$ satisfying conditions (2.1). A quasi-parabolic structure on a vector bundle $E$ on $C$ is a collection $(E_i^\bullet)_{1 \leq i \leq k}$ of flags of type $m_i$ in the fiber $E_{p_i}$ for each $i \in \{1, \ldots, k\}$. For ease of exposition, such a quasi-parabolic vector bundle will be denoted by $E^\bullet$, and its numerical type by $\underline{m} = (m_i)_{1 \leq i \leq k}$.

For any exact sequence of vector bundles

$$0 \to F \to E \to G \to 0,$$

a quasi-parabolic structure on $E$ at $D$ of type $\underline{m}^E$ induces quasi-parabolic structures of types $\underline{m}^F$, $\underline{m}^G$ on $F, G$ such that $\underline{m}^F + \underline{m}^G = \underline{m}^E$. Moreover, as explained in [23, Sect 2.2], for any two parabolic bundles $E^\bullet$, $F^\bullet$ there is a sheaf of parabolic morphisms $PHom_C(E^\bullet, F^\bullet)$ which fits in an exact sequence

$$0 \to PHom_C(E^\bullet, F^\bullet) \to \text{Hom}_C(E, F) \to \oplus_{i=1}^k \text{APHom}(E_{p_i}^\bullet, F_{p_i}^\bullet) \otimes \mathcal{O}_{p_i} \to 0 \quad (2.3)$$

$$0 \to \text{Hom}_C(E, F(-D)) \to PHom_C(E^\bullet, F^\bullet) \to \oplus_{i=1}^k PHom(E_{p_i}^\bullet, F_{p_i}^\bullet) \otimes \mathcal{O}_{p_i} \to 0 \quad (2.4)$$
Similarly, there is a sheaf of local strongly parabolic morphisms $\mathcal{SPHom}_C(E^\bullet, F^\bullet)$ which fits in analogous exact sequences. Note also that there is a natural duality relation $[23, \text{Prop. 2.3.i}]
\mathcal{PHom}_C(E^\bullet; F^\bullet) \cong \mathcal{SPHom}_C(F^\bullet; E^\bullet) \otimes C \mathcal{O}_C(D). \tag{2.5}$

A parabolic bundle on $C$ is a quasi-parabolic bundle $E^\bullet$ equipped in addition with collections of weights $\alpha_i = (\alpha_{i,a})_{0 \leq a \leq s_i - 1} \in \mathbb{R}^{s_i}$ for each $i \in \{1, \ldots, l\}$ such that

$$0 \leq \alpha_{i,0} < \cdots < \alpha_{i,s_i - 1} < 1. \tag{2.6}$$

The data $(\alpha_i)_{1 \leq i \leq k}$ will be denoted by $\alpha$ and parabolic bundles will be denoted by $(E^\bullet, \alpha)$.

There is a natural stability condition for parabolic bundles formulated in terms of parabolic slopes. The parabolic degree of $(E^\bullet, \alpha)$ is defined as

$$\deg(E^\bullet, \alpha) = \deg(E) + \sum_{i=1}^{k} \sum_{a=0}^{s_i - 1} m_{i,a} \alpha_{i,a}, \tag{2.7}$$

and the parabolic slope is given by

$$\mu(E^\bullet, \alpha) = \frac{\chi(E^\bullet, \alpha)}{\text{rk}(E)}. \tag{2.8}$$

with

$$\chi(E^\bullet, \alpha) = \deg(E^\bullet, \alpha) - \text{rk}(E)(g - 1) = \chi(E) + \sum_{i=1}^{k} \sum_{a=0}^{s_i - 1} m_{i,a} \alpha_{i,a}. \tag{2.9}$$

Any nontrivial proper saturated subsheaf $0 \subset E' \subset E$ inherits an induced parabolic structure $(E'^\bullet, \alpha')$ on $E'$. The parabolic bundle $(E^\bullet, \alpha)$ is (semi)stable if any such subsheaf satisfies the parabolic slope condition

$$\mu(E'^\bullet, \alpha') \leq \mu(E^\bullet, \alpha). \tag{2.10}$$

As shown in [47], this stability condition yields projective moduli spaces of $S$-equivalence classes of semistable objects. Moreover, for sufficiently generic weights these moduli spaces are smooth.

### 2.2 Higgs fields, spectral covers, and foliations

A quasi-parabolic Higgs bundle on $C$ is a quasi-parabolic vector bundle $E^\bullet$ equipped with a Higgs field $\Phi : E \to E \otimes_C K_C(D)$ such that the residue $\text{Res}_p(\Phi) : E_p \to E_p$ is a parabolic
map for each marked point. A parabolic Higgs bundle is defined by specifying in addition a collection of weights \( \alpha \) as in the previous section.

There is a natural notion of stability for parabolic Higgs bundles, defined by imposing the parabolic slope inequality (2.10) for all proper saturated subsheaves preserved by the Higgs field. The results of [62], imply that semistable parabolic Higgs bundles with fixed numerical invariants \( m, \deg(E) = e \) form an algebraic stack of finite type \( \mathcal{H}^{ss}_{\text{par}}(C; D; m, e, \alpha) \). The stable ones form an open substack \( \mathcal{H}^s_{\text{par}}(C; D; m, e, \alpha) \). Moreover there is a coarse moduli space \( \mathcal{H}^{ss}_{\text{par}}(C; D; m, e, \alpha) \) parameterizing \( S \)-equivalence classes of semistable objects which contains an open subspace \( \mathcal{H}^s_{\text{par}}(C; D; m, e, \alpha) \) parameterizing isomorphism classes of stable objects. According to [63], \( \mathcal{H}^{ss}_{\text{par}}(C; D; m, e, \alpha) \) is a normal quasi-projective variety while the stable open subspace is smooth. Note also that any semistable object must be stable for sufficiently generic weights and primitive numerical invariants.

Similar considerations apply to strongly parabolic Higgs bundles, in which case the moduli stacks/spaces will be labelled with a subscript \( \text{s-par} \) instead of \( \text{par} \). In addition, one can construct similarly moduli spaces of parabolic and strongly parabolic Higgs bundles where the Higgs field takes values in an arbitrary coefficient line bundle \( M \), that is \( : E \to E \otimes C M(D) \).

In this case, the line bundle \( M \) will be specified in the notation of the moduli space e.g. \( \mathcal{H}^{ss}_{\text{par}}(C, D, M; m, e, \alpha) \).

Taking polynomial invariants of the Higgs field yields the Hitchin map

\[
 h : \mathcal{H}^{ss}_{\text{par}}(C, D; m, e, \alpha) \to B(C, D; r), \quad B(C, D; r) = \oplus_{i=1}^r H^0(C, (K_C(D))^i). \tag{2.11}
\]

This is a surjective proper morphism, its generic fibers being disjoint unions of abelian varieties. As observed in [46, 45], the unordered eigenvalues of the Higgs field at the marked points are parameterized by the quotient \( B/B_0 \) where \( B_0 \subset B \) is the linear subspace

\[
 B_0(C, D; r) = \oplus_{i=1}^r H^0(C, (K_C(D))^i \otimes O_C(-D)) \subset B(C, D; r).
\]

The moduli space is foliated by the fibers of the resulting projection,

\[
 p : \mathcal{H}^{ss}_{\text{par}}(C, D; m, e, \alpha) \to B(C, D; r)/B_0(C, D; R). \tag{2.12}
\]

Parabolic Higgs bundles admit a spectral cover presentation as parabolic pure dimension one sheaves on the total space \( P \) of \( K_C(D) \). Let \( \pi : P \to C \) denote the canonical projection, \( P_i = \pi^{-1}(p_i) \) the fiber at the marked point \( p_i \), and \( D_P = \sum_{i=1}^k P_i \). A quasi-parabolic structure \( F^* \) on a pure dimension one sheaf \( F \) on \( P \) is defined by a sequence of surjective morphisms

\[
 F \otimes_P P_i \to F_{i-1}^{s_i} \to \cdots \to F_1^1 \tag{2.13}
\]
where \( F_i^n \) are sheaves on \( P_i \) for all \( 1 \leq i \leq k \). Moreover \( F \) is required to have compact support, which implies that the sheaves \( F \otimes_P P_i \) are zero dimensional. In this case \( \text{ch}_1(F) = d\sigma \) with \( \sigma \) the class of the zero section. A parabolic structure is defined by specifying in addition parabolic weights \( \underline{\alpha} = (\alpha_i^a) \) as above.

Any saturated sub sheaf \( F' \subset F \) inherits a natural induced parabolic structure. Then one defines a stability condition using the parabolic slope

\[
\mu(F^*, \underline{\alpha}) = \frac{1}{d} \left( \chi(F) + \sum_{i=1}^{k} \sum_{a=0}^{s_i-1} \alpha_{i,a} \left( \chi(F_i^{a+1}) - \chi(F_i^a) \right) \right).
\]

This yields an algebraic moduli stack of semistable objects which is isomorphic to the moduli stack of semistable parabolic Higgs bundles on \( C \) with numerical invariants

\[
m_i^n = d - \chi(F_i^n), \quad e = \chi(F) + d(g-1).
\]

This isomorphism assigns to any sheaf \( F \) the bundle \( E = \pi_* F \), the flags being determined by

\[
E_i^n = \text{Ker}(E_{p_i} \to \pi_* F_i^n).
\]

The Higgs field \( \Phi : E \to E \otimes_C K_C(D) \) is the pushforward \( \Phi = \pi_* y \) of the multiplication map \( F \to F \otimes_P \pi^* K_C(D) \) by the tautological section \( y \in H^0(P, \pi^* K_C(D)) \).

Again, a similar spectral construction applies to parabolic Higgs bundles with coefficients in a line bundle \( M \), as defined above (2.11). In that case, \( P \) will be the total space of the line bundle \( M(D) \).

### 2.3 Diagonally parabolic Higgs bundles

For further reference it will be convenient to note here that the moduli stack of semistable Higgs bundles contains a closed substack where the Higgs field has \( \xi_i \)-parabolic residues at each marked point \( p_i \), where \( \xi_i \) is a collection \( \xi_i = (\xi_i^0, \ldots, \xi_i^{s_i-1}) \in K_C(D)_{p_i}^{s_i} \). Using the notion introduced in Section 2.1, this means that \( \Phi|_{p_i} : E_{p_i} \to E_{p_i} \otimes K_C(D)_{p_i} \) is parabolic, and the induced maps

\[
E_{p_i}/E_{p_i}^{a+1} \to E_{p_i}^a/E_{p_i}^{a+1} \otimes K_C(D)_{p_i}
\]

are of the form \( 1 \otimes \xi_i^a \). Such objects will be called \( \underline{\xi} \)-parabolic, where \( \underline{\xi} = (\xi_i^a) \), \( 1 \leq i \leq k \), \( 0 \leq a \leq s_i \). The closed substack of such objects will be denoted by \( \mathcal{H}^a_{\underline{\xi} \text{-par}}(C, D; m, e, \underline{\alpha}) \).

For \( \xi_i = (0, \ldots, 0) \), \( 1 \leq i \leq k \), one recovers the moduli stack of strongly parabolic Higgs bundles.
It will be shown in Section 3 that moduli spaces of $\xi$-parabolic Higgs bundles occur naturally in string theory.

3 Spectral data via holomorphic symplectic orbifolds

The goal of this section is to formulate a variant of the spectral cover construction for parabolic Higgs bundles. In this variant the spectral data are torsion sheaves on holomorphic symplectic orbifold surfaces. This construction is different from the standard spectral construction from Section 2.2 in which the spectral data are parabolic dimension one sheaves on the total space $P$ of the line bundle $K_C(D)$. The main motivation for this alternative approach resides in string theory, where parabolic structures must arise naturally from D-brane moduli problems rather than being specified as additional data.

For a brief outline, suppose $C$ is a smooth projective curve equipped with a reduced divisor $D = \sum_{i=1}^{k} p_i$ of marked points. We want to describe orbifold spectral data for parabolic Higgs bundles on $(C;D)$. Consider the moduli stack $\mathcal{H}_{\xi-par}^{ss}(C,D;m,e,\alpha)$ of diagonally parabolic Higgs bundles introduced in Section 2.3. In this section we will show that any Higgs bundle $(E,\Phi)$ in $\mathcal{H}_{\xi-par}^{ss}(C,D;m,e,\alpha)$ can be represented by a spectral datum $\tilde{G}$ which is a Bridgeland semistable pure dimension one coherent sheaf on a certain orbifold symplectic surface $\tilde{S}_\delta$.

The particular $\tilde{S}_\delta$ depends on $C$, the divisor $D$, and a zero dimensional subscheme $\delta$ inside $\text{tot}(K_C(D))$. To describe it let $P$ denote the total space of $K_C(D)$, and let $P_i$ be the fiber over $p_i$, $1 \leq i \leq k$. Let $s_i \in \mathbb{Z}_{\geq 1}$, $1 \leq i \leq k$ be fixed positive integers and $\delta = (\delta_i)_{1 \leq i \leq k}$ be a fixed collection of degree $s_i$ divisors

$$\delta_i = \sum_{j=1}^{\ell_i} s_{i,j} \mathcal{O}_{p_i,j}, \quad \ell_i \geq 1, s_{i,j} \geq 1, 1 \leq j \leq \ell_i,$$

on $P_i$ for each $1 \leq i \leq k$. By convention, set $s_{i,0} = 0$ for each $1 \leq i \leq k$. In Section 3.2 we check that the weighted blowup of $P$ along $\delta$ produces a holomorphic symplectic orbifold surface $\tilde{S}_\delta$.

The particular $\delta$ needed for the spectral description of the Higgs bundles in $\mathcal{H}_{\xi-par}^{ss}(C,D;m,e,\alpha)$ is constructed out of the eigenvalues $\xi$ of the residues of the Higgs fields, and the flag types $m$. Specifically we take $s_i$ to be the number of steps in the parabolic filtration at the point $p_i$ and $\ell_i$ to be the number of distinct entries in the vector $\xi_i = (\xi_{i,0}, \dots, \xi_{i,s_i-1}) \in K_C(D)_{p_i}^{\otimes s_i}$.

We label the distinct entries of $\xi_i$ by $\varphi_{i,1}, \dots, \varphi_{i,\ell_i}$, and we write $s_{i,j}$ for the multiplicity
with which \( \varphi_{i,j} \) is repeated as a coordinate inside \( \xi_i \). In other words we choose a function
\[
j : \{0, \ldots, s_i - 1\} \to \{1, \ldots, \xi_i\}
\]
so that
\[
\xi_i^a = \varphi_{i,j(a)}, \quad 0 \leq a \leq s_i - 1.
\]
These choices define a zero dimensional subscheme \( \delta \subset P \) and an orbifold symplectic surface \( \tilde{S}_\delta \).

Then the main result of this section is the existence of an isomorphism
\[
\mathcal{H}^{ss}_{\xi_{-\text{par}}} (C, D; m, e, \alpha) \cong \mathcal{M}^{ss}_{\beta} \left( \tilde{S}_\delta, d \right)
\]
of the moduli stack \( \mathcal{H}^{ss}_{\xi_{-\text{par}}} (C, D; m, e, \alpha) \) between semistable \( \xi \)-parabolic bundles on \( C \) with the moduli stack \( \mathcal{M}^{ss}_{\beta} \left( \tilde{S}_\delta, d \right) \) of Bridgeland \( \beta \)-semistable pure dimension one sheaves on \( \tilde{S}_\delta \) with \( K \)-theory class \( d \in K_c^0(\tilde{S}_\delta) \).

To set up this isomorphism we first construct an identification of discrete invariants
\[
(r, m, e) \longleftrightarrow d
\]
and an identification
\[
\alpha \longleftrightarrow \beta
\]
of the parabolic weights on the Higgs side with the Bridgeland stability parameters \( \beta \) on the spectral data side. These identifications are based on an explicit computation of the compactly supported \( K \)-theory of \( \tilde{S}_\delta \). A precise statement is formulated in Section 3.3.

The simplest instance of this construction is \( \xi = 0 \), in which case \( \xi \)-parabolic bundles are the same as strongly parabolic bundles. In this case the construction of \( \tilde{S}_0 \) follows from standard root stack constructions in the literature, as explained below.

### 3.1 Root stacks and orbifold spectral covers

Using the construction of [2, 3], parabolic Higgs bundles have been identified with ordinary Higgs bundles on an orbicurve in [51, 26]. This section reviews the basics of this construction following the algebraic approach of [26].

Given the curve \( C \) with marked points \( p_i \), \( 1 \leq i \leq k \) one first constructs an orbicurve \( \tilde{C} \) as follows. Let \( U = C \setminus \{ p_1, \ldots, p_k \} \). For any point \( p_i \), let \( \mathbb{D}_{p_i} \), denote the formal disc centered at \( p_i \) and \( \mathbb{D}_{p_i}^o = \mathbb{D}_{p_i} \times_U U \) the punctured formal disc. Let \( \varphi_i : \mathbb{D}_{p_i} \to \mathbb{D}_{p_i} \) be the
$s_i : 1$ cover given by $z_i \mapsto z_i^{s_i}$. There is a natural $\mu_{s_i}$-action on $\tilde{D}_{p_i}$ sending $z_i \mapsto \omega_i z_i$, where $\omega_i = \exp(2\pi \sqrt{-1}/s_i)$. The quotient stacks $[\tilde{D}_{p_i}/\mu_{s_i}]$ are then glued to $U$ using the morphisms $\varphi_i$ to identify the open substacks $[\tilde{D}_{p_i}/\mu_{s_i}]$ with the punctured disks $D^o_{p_i}$. In characteristic zero this yields a smooth Deligne-Mumford stack $\tilde{C}$ equipped with a map $\nu : \tilde{C} \rightarrow C$ which identifies $C$ with its coarse moduli space.

Following [26, Sect 2.4], a Higgs bundle on $\tilde{C}$ is a vector bundle $\tilde{E}$ equipped with a Higgs field $\tilde{\Phi} : \tilde{E} \rightarrow \tilde{E} \otimes_{\tilde{C}} K_{\tilde{C}}$. This data determines a parabolic Higgs bundle on $C$ as follows. For each point $p_i$ there is a line bundle $\tilde{L}_i$ on $\tilde{C}$ such that $\tilde{L}_i^{s_i} = \nu^* \mathcal{O}_C(p_i)$. Locally, $\nu^* \mathcal{O}_C(p_i)$ corresponds to the rank one free $\mathbb{C}[[z_i]]$-module generated by $z_i^{-s_i}$, while $\tilde{L}_i$ corresponds to the $\mathbb{C}[[z_i]]$-module generated by $z_i^{s_i}$. Now let

$$E = \nu_* \tilde{E},$$

and

$$F^a_i = \nu_* \left( \tilde{E} \otimes_{\tilde{C}} \tilde{L}_i^{-a} \right)$$

for each $1 \leq i \leq k$, $0 \leq a \leq s_i - 1$. By the base change theorem, all direct images are locally free and the sheaves $F^a_i$, $0 \leq a \leq s_i - 1$, form a filtration

$$E(-p_i) \subseteq F^a_i \subseteq \cdots \subseteq F^0_i = E$$

for each $1 \leq i \leq k$.

For concreteness, note that any locally free sheaf $\tilde{E}$ is locally isomorphic to a sum of line bundles of the form $\oplus_{j=1}^r \tilde{L}_i^{n_i,j}$, corresponding to the $\mathbb{C}[[z_i]]$-module $\bigoplus_{j=1}^r z_i^{-n_i,j} \mathbb{C}[[z_i]]$. The morphism $\nu : \tilde{C} \rightarrow C$ is locally of the form $t_i = z_i^{s_i}$, where $t_i$ is a local coordinate on $C$ centered at $p_i$. The direct image $E = \nu_* \tilde{E}$ corresponds locally to the $\mathbb{C}[[t_i]]$-module obtained by multiplying the $\mu_{s_i}$-fixed part of $\bigoplus_{j=1}^r z_i^{-n_i,j} \mathbb{C}[[z_i]]$. The subsheaves $F^a_i$ are obtained similarly by taking the $\mu_{s_i}$-fixed part of $\bigoplus_{j=1}^r z_i^{-n_i,j+a} \mathbb{C}[[z_i]]$, $0 \leq a \leq s_i - 1$.

The filtration (3.4) determines a flag

$$E^a_i = \text{Ker}(E_{p_i} \otimes \mathcal{O}_{p_i} \rightarrow E/F^a_i)$$

in the fiber $E_{p_i}$, hence one obtains a quasi-parabolic bundle $E^\bullet$ on $C$. Note also that the snake lemma yields an isomorphism

$$E^a_i \otimes \mathcal{O}_{p_i} \simeq F^a_i / E(-p_i)$$

for all $0 \leq a \leq s_i$, $1 \leq i \leq k$. 

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According to [26, Prop. 2.16] assigning the quasi-parabolic bundle $E^\bullet$ to $\tilde{E}$ yields an equivalence of groupoids. Moreover, the degree of $\tilde{E}$ as an orbibundle equals the parabolic degree of $E^\bullet$ with weights $\alpha_i = a_i/s_i$.

Next consider a Higgs field $\tilde{\Phi} : \tilde{E} \to \tilde{E} \otimes_{\tilde{C}} K_{\tilde{C}}$ on the stack and note the isomorphisms

$$K_{\tilde{C}} \cong \bigotimes_{i=1}^k \tilde{L}_i^{(a_i-1)} \otimes_{\tilde{C}} \nu^* K_C \cong \bigotimes_{i=1}^k \tilde{L}_i^{-1} \otimes_{\tilde{C}} \nu^* K_C(D).$$

Then $\Phi = \nu_* \tilde{\Phi} : E \to E \otimes_C \Omega^1_C(D)$ is a Higgs field on $E$ and equations (3.3) imply that

$$\Phi(F_i^n) \subseteq F_i^{n+1} \otimes_C \Omega^1_C(D)$$

for all $0 \leq a \leq s_i - 1$, $1 \leq i \leq k$. Using isomorphisms (3.6), this implies that the residue $\text{Res}_{p_i}(\Phi)$ is strongly parabolic with respect to the flag (3.5) for each $i \in \{1, \ldots, k\}$.

According to [26, Prop. 20], the above construction yields a one-to-one correspondence between Higgs fields $\tilde{\Phi}$ on the orbifold $\tilde{C}$ and Higgs fields $\Phi$ on $C$ with strongly parabolic residues with respect to the flag $E^\bullet$ at each $p_i$, $1 \leq i \leq k$. Furthermore, the degree of the orbifold bundle $\tilde{E}$ equals the parabolic degree $\text{deg}(E, \alpha)$ for special values of the weights

$$\alpha_i = a_i/s_i, \quad 0 \leq a \leq s_i - 1, \quad 1 \leq i \leq k. \quad (3.7)$$

Based on this observation, it is straightforward to check that this correspondence maps (semi)stable orbibundles to (semi)stable parabolic Higgs bundles with weights (3.7). Since it works for flat families as well, it yields isomorphisms of moduli stacks.

For completeness, note that the fiber $\tilde{E}_{i,0}$ of $\tilde{E}$ at the closed point $0 \in [\tilde{D}_{p_i}/\mu_{s_i}]$ carries a natural action of the stabilizer group $\mu_{s_i}$. Hence it decomposes into irreducible representations,

$$\tilde{E}_{i,0} \cong \bigoplus_{a=0}^{s_i-1} R_{i,a}^{\mu_{n_i,a}}, \quad (3.8)$$

where $R_{i,a}$ denotes the one dimensional representation of $\mu_{s_i}$ with character $\omega_i^a$. Then [26, Lemma 2.19] proves that the discrete invariants $m_{i,a}$ of the flag (3.5) are given by $m_{i,a} = n_{i,a}$, for $0 \leq a \leq s_i - 1$. In string theoretic language this means that the flag type encodes the fractional charges with respect to the twisted sector Ramond-Ramond fields.

Using the standard spectral cover construction, an orbifold Higgs bundle $(\tilde{E}, \tilde{\Phi})$ corresponds to a pure dimension one sheaf $\tilde{F}$ on the total space $\tilde{S}$ of $K_{\tilde{C}}$, finite with respect to the natural projection $\pi : \tilde{S} \to \tilde{C}$. The bundle $\tilde{E}$ is obtained by push-forward, $\tilde{E} = \pi_* \tilde{F}$, and the Higgs field $\tilde{\Phi}$ is the push-forward of the multiplication map $\tilde{F} \to \tilde{F} \otimes \pi^* K_{\tilde{C}}$ by
the tautological section. This is a one-to-one correspondence which holds for flat families as well, preserving (semi)stability. Therefore it induces again isomorphisms of moduli stacks of semistable objects. As shown in detail in the next subsection, one can in fact obtain arbitrary values of the parabolic weights using Bridgeland stability conditions for pure dimension one sheaves on $\tilde{S}$.

Finally, all above statements generalize immediately to Higgs fields with values in an arbitrary line bundle $M$ on $C$. Namely, let

$$\tilde{M} = \bigotimes_{i=1}^{k} \tilde{L}_{i}^{s_{i}-1} \otimes_{\bar{C}} \nu^{*}M.$$  

Then there is again a one-to-one correspondence between strongly parabolic Higgs bundles $(E, \Phi)$, with $\Phi : E \to E \otimes_{C} M(D)$ and Higgs orbibundles $(\tilde{E}, \tilde{\Phi})$ with $\tilde{\Phi} : \tilde{E} \to \tilde{E} \otimes_{\bar{C}} \tilde{M}$. The latter admit again a spectral cover presentation, as pure dimension one sheaves on the total space of $\tilde{M}$.

At the same time analogous considerations hold for parabolic Higgs bundles $(E^{*}, \Phi)$ on $C$ with $\Phi : E \to E \otimes_{C} M$ a Higgs field preserving the flag $E_{i}^{*}$ at each point $p_{i}$, $1 \leq i \leq k$. Note that in this case $\Phi$ is regular at $p_{i}$ and $\Phi|_{p_{i}}$ is not necessarily strongly parabolic. Repeating the above arguments, such objects are in one-to-one correspondence with Higgs orbibundles $(\tilde{E}, \tilde{\Phi})$ on $\bar{C}$, with $\tilde{\Phi} : \tilde{E} \to \tilde{E} \otimes_{\bar{C}} \nu^{*}M$.

### 3.2 Orbifold spectral data for diagonally parabolic Higgs bundles

As in the second paragraph of Section 3, let $\delta = (\delta_{i})_{1 \leq i \leq k}$ be a collection of degree $s_{i}$ divisors

$$\delta_{i} = \sum_{j=1}^{\ell_{i}} s_{i,j} \varphi_{i,j}, \quad \ell_{i} \geq 1, s_{i,j} \geq 1, \ 1 \leq j \leq \ell_{i},$$

on $P_{i}$ for each $1 \leq i \leq k$. Note that one can choose an affine chart $(x_{i}, y_{i,j})$ on $P$ centered at each point $\varphi_{i,j}$, where $x_{i}$ is an affine coordinate on $C$ centered at $p_{i}$, and $y_{i,j}$ a linear coordinate on the fibers of $P$ centered at $\varphi_{i,j}$. Let $V_{i,j} = \text{Spec} \mathbb{C}[x_{i}, y_{i,j}] \subset P$ denote the domain of this affine coordinate chart. Then any collection $\delta = (\delta_{i})_{1 \leq i \leq k}$, determines a smooth orbifold surface $\tilde{P}_{\delta}$, the stack theoretic weighted projective blow-up of $P$ at the points $\varphi_{i,j}$ with weights $(s_{i,j}, 1)$ with respect to the affine chart $(x_{i}, y_{i,j})$.

This is a standard construction in the orbifold literature employed for example in [39, 40]. Following [40, Sect. 2.1], the weighted projective blow-up of $P$ at $\varphi_{i,j}$ is a quotient stack $[X_{i,j}/C^{\times}]$, where $X_{i,j}$ is a scheme obtained by gluing $(\mathbb{C}^{2} \setminus \{0\}) \times \mathbb{C}$ to $(P \setminus \{\varphi_{i,j}\}) \times C^{\times}$
along \((V_{i,j} \setminus \{0\}) \times \mathbb{C}^\times\). In terms of linear coordinates \((u, v, t)\) on \(\mathbb{C}^2 \times \mathbb{C}\), the gluing map reads
\[
x_i = t^{s_{i,j}} u, \quad y_{i,j} = tv, \quad z = t^{-1}
\] (3.9)
where \(z\) is a linear coordinate on the target \(\mathbb{C}^\times\). The \(\mathbb{C}^\times\)-action on \(X\) is induced by the \(\mathbb{C}^\times\)-action
\[
(\zeta, (u, v, t)) \mapsto (\zeta^{s_{i,j}} u, \zeta^{1} v, \zeta^{-1} t)
\]
on \((\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}\) and the scaling action on the second factor of \((P \setminus \{\varnothing_{i,j}\}) \times \mathbb{C}^\times\).

Proceeding this way for each point \(\varnothing_{i,j}\), one obtains a smooth Deligne-Mumford stack \(\tilde{P}_\delta\) equipped with a natural projection \(\eta : \tilde{P}_\delta \to P\). In terms of the above affine coordinates, the projection map is given by
\[
x_i = ut^{s_{i,j}}, \quad y_{i,j} = vt.
\] (3.10)
For each pair \((i, j)\) the inverse image \(\eta^{-1}(V_{i,j})\) is an open substack of \(\tilde{P}_\delta\) isomorphic to the smooth toric stack \(X_{i,j} = [\text{Spec} \mathbb{C}[u, v, t]/\mathbb{C}^\times]\) with weights \((s_{i,j}, 1, 1)\). According to [39, Prop. 4.5], the open substack of \(X_{i,j}\) where \(u \neq 0\) is isomorphic to a quotient stack \([\text{Spec} \mathbb{C}[\tilde{x}, \tilde{y}]/\mu_{s_{i,j}}]\), where
\[
\tilde{x}^{s_{i,j}} = ut^{s_{i,j}}, \quad \tilde{y}^{s_{i,j}} = u^{-1}v^{s_{i,j}}.
\]
This shows that there is an orbifold point \(\tilde{\varnothing}_{i,j}\) with stabilizer \(\mu_{s_{i,j}}\) mapping to each blow-up center \(\varnothing_{i,j}\). All other closed points of \(\tilde{P}_\delta\) have trivial stabilizers. The reduced exceptional divisor \(\Xi_{i,j}\) corresponding to \(\varnothing_{i,j}\) is isomorphic to a weighted projective line \(\mathbb{P}[s_{i,j}, 1]\) passing through the orbifold point. Moreover, the total transform of the fiber \(P_i\) is
\[
\eta^* P_i = P_i' + \sum_{j=1}^{t_i} s_{i,j} \Xi_{i,j}
\] (3.11)
where \(P_i'\) is a line on \(\tilde{P}_\delta\) disjoint from the orbifold points.

In terms of the local coordinates \((\tilde{x}, \tilde{y})\) exceptional divisor is given by \(\tilde{x} = 0\), and the projection map (3.10) is given by
\[
x_i = \tilde{x}^{s_{i,j}}, \quad y_{i,j} = \tilde{x} \tilde{y}.
\] (3.12)
Then
\[
\eta^*(dx_i \wedge dy_{i,j}) = s_{i,j} \tilde{x}^{s_{i,j}} d\tilde{x} \wedge d\tilde{y},
\]
which implies that the canonical class $K_{\tilde{P}_\delta}$, is given by

$$K_{\tilde{P}_\delta} = \eta^* K_P + \sum_{i=1}^k \sum_{j=1}^{s_{i,j}} \epsilon_{i,j}.$$

Since $P$ has canonical class

$$K_P = -\sum_{i=1}^k P_i,$$

using equation (3.11), the canonical class of $\tilde{P}_\delta$ is

$$K_{\tilde{P}_\delta} = -\sum_{i=1}^k P'_i.$$

(3.13)

Therefore the complement

$$\tilde{S}_\delta = \tilde{P}_\delta \setminus \cup_{i=1}^k P'_i$$

is a holomorphic symplectic orbifold surface. This holomorphic symplectic surface will be used below in the construction of spectral data for diagonally parabolic Higgs bundles.

The first task is to classify the discrete invariants of compactly supported torsion sheaves on $\tilde{S}_\delta$. Consider the natural bilinear pairing

$$(\ , \ ) : K^0(\tilde{P}_\delta) \times K^0_{cpt}(\tilde{P}_\delta) \to \mathbb{Z}$$

(3.14)

defined by

$$(\tilde{F}, \tilde{G}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(\tilde{F}, \tilde{G})$$

for any two coherent sheaves $\tilde{F}, \tilde{G}$ on $\tilde{P}_\delta$, where $\tilde{G}$ has compact support. The discrete invariants of compactly supported coherent sheaves on $\tilde{P}_\delta$ will be numerical equivalence classes in

$$\Gamma_{cpt}(\tilde{P}_\delta) = K^0_{cpt}(\tilde{P}_\delta)/K^0(\tilde{P}_\delta)^\perp,$$

(3.15)

where

$$K^0(\tilde{P}_\delta)^\perp = \{ \kappa \in K^0_{cpt}(\tilde{P}_\delta) \mid (\kappa, \gamma) = 0, \forall \gamma \in K^0(\tilde{P}_\delta) \}.$$ 

For future reference, let

$$\Gamma(\tilde{P}_\delta) = K^0(\tilde{P}_\delta)/K^0_{cpt}(\tilde{P}_\delta)^\perp$$

be defined analogously. Then (3.14) descends to a nondegenerate bilinear pairing

$$\chi : \Gamma(\tilde{P}_\delta) \times \Gamma_{cpt}(\tilde{P}_\delta) \to \mathbb{Z}.$$  

(3.16)
Note that similar definitions apply equally well to $P$, resulting in a nondegenerate bilinear pairing
\[ \Gamma(P) \times \Gamma_{\text{cpt}}(P) \to \mathbb{Z}. \]
Since $\pi : P \to C$ is the total space of a line bundle over $C$, $\Gamma(P)$ is generated by the line bundle classes
\[ [\mathcal{O}_P], \quad [\mathcal{O}_P(f)] \]
where $f = \pi^{-1}(p)$ is a fiber of $\pi$ over a generic point $p \in C \setminus D$. The compactly supported lattice $\Gamma_{\text{cpt}}(P)$ is generated by the sheaf classes
\[ [\mathcal{O}_\sigma], \quad [\mathcal{O}_\varphi] \]
where $\varphi \in P$ is a generic point, $\pi(\varphi) \in C \setminus D$.

Then an explicit presentation of $\Gamma(\widetilde{P}_3)$, $\Gamma_{\text{cpt}}(\widetilde{P}_3)$ follows from [40, Thm. 2], which proves a structure result for the derived category $D^b(\widetilde{P}_3)$. According to loc. cit. $D^b(\widetilde{P}_3)$ admits a semiorthogonal decomposition
\[ D^b(\widetilde{P}_3) = \langle \eta^*(D^b(P)), T_{i,j}^l \rangle \quad (3.17) \]
where $T_{i,j}^l$ are extensions by zero of standard line bundles on the exceptional divisors $\Xi_{i,j} \simeq \mathbb{P}[s_{i,j}, 1]$:
\[ T_{i,j}^l = \mathcal{O}_{\Xi_{i,j}}(l), \quad 0 \leq l \leq s_{i,j} - 1. \]
Here $\mathcal{O}_{\Xi_{i,j}}(l)$ denotes the $l$-th power of the line bundle $\mathcal{O}_{\Xi_{i,j}}(1)$ on the weighted projective line $\mathbb{P}[s_{i,j}, 1]$. In the present context, $\mathcal{O}_{\Xi_{i,j}}(-1)$ is the restriction:
\[ \mathcal{O}_{\Xi_{i,j}}(-1) \simeq \mathcal{O}_{\widetilde{P}_3}(\Xi_{i,j})|_{\Xi_{i,j}} \quad (3.18) \]
on $\Xi_{i,j}$, as shown in the proof of [40, Prop. 3].

In the following it will be more convenient to work with the alternative $K$-theory generators
\[ Q_{i,j}^l = T_{i,j}^{s_{i,j}-l-1} \otimes \mathcal{O}_{\widetilde{P}_3}(K_{\widetilde{P}_3}). \quad (3.19) \]
Using (3.11), (3.13) and (3.18), one has an isomorphism
\[ Q_{i,j}^l \simeq \mathcal{O}_{\Xi_{i,j}}(-l - 1) \quad (3.20) \]
for all $(i, j)$ and all $0 \leq l \leq s_{i,j} - 1$, since $\eta^*P_i|_{\Xi_{i,j}} = 0$. The semiorthogonal decomposition (3.17), implies that $\Gamma(\widetilde{P}_3)$ is generated by the numerical equivalence classes of the sheaves
\[ \eta^*\mathcal{O}_P = \mathcal{O}_{\widetilde{P}_3}, \quad \eta^*\mathcal{O}_P(f), \quad Q_{i,j}^l. \]
Now suppose $\tilde{G}$ is a (nonzero) pure dimension one sheaf on $\tilde{P}_\delta$ with compact support. Then the discrete invariants of $\tilde{G}$ will be defined as

$$n(\tilde{G}) = \chi(\eta^* O_P, \tilde{G}), \quad r(\tilde{G}) = n(\tilde{G}) - \chi(\eta^* O_P(f), \tilde{G}), \quad d_{i,j}^l(\tilde{G}) = -\chi(Q_{i,j}^l, \tilde{G})$$ (3.21)

with $1 \leq i \leq k$, $1 \leq j \leq \ell_i$, $0 \leq l \leq s_{i,j} - 1$. Note that $\eta_\delta \tilde{G}$ is a compactly supported torsion sheaf on $P$, which is pure dimension one on the complement $P \setminus \cup_{i=1}^k P_i$. Then it is straightforward to check that $\text{ch}_1(\eta_\delta \tilde{G}) = r(\tilde{G})\sigma$, and

$$r(\tilde{G}) = \chi(O_P, \eta_\delta \tilde{G} \otimes P O_f) > 0,$$ (3.22)

for a generic fiber $f$ disjoint from $P_i$, $1 \leq i \leq k$.

Next suppose in addition that $\tilde{G}$ has compact support in $\tilde{S}_\delta \subset \tilde{P}_\delta$, therefore disjoint from the strict transforms $P'_i$, $1 \leq i \leq k$ of the marked fibers. Then, as shown below, there is a relation of the form

$$r(\tilde{G}) = \sum_{j=1}^{\ell_i} \sum_{l=0}^{s_{i,j}-1} d_{i,j}^l$$ (3.23)

for each $1 \leq i \leq k$. In conclusion the discrete invariants of such sheaves can be labelled by $(n(\tilde{G}), d_{i,j}^l(\tilde{G}))$, keeping in mind that they satisfy relations (3.23).

Using the isomorphism $O_{P_\delta}(l\Xi_{i,j})|_{\Xi_{i,j}} \simeq O_{\Xi_{i,j}}(-1)$ and equations (3.20), each $Q_{i,j}^l$ has a locally free resolution

$$0 \to O_{P_\delta}(l\Xi_{i,j}) \to O_{P_\delta}((l+1)\Xi_{i,j}) \to Q_{i,j}^l \to 0.$$

This yields

$$d_{i,j}^l = \chi(O_{P_\delta}(l\Xi_{i,j}), \tilde{G}) - \chi(O_{P_\delta}((l+1)\Xi_{i,j}), \tilde{G}).$$

Summing the above relations from $l = 0$ to $l = s_{i,j} - 1$ yields

$$\sum_{l=0}^{s_{i,j}-1} d_{i,j}^l = \chi(O_{P_\delta}, \tilde{G}) - \chi(O_{P_\delta}(s_{i,j}\Xi_{i,j}), \tilde{G}) = \chi(O_{P_\delta}, \tilde{G} \otimes P_\delta O_{s_{i,j}\Xi_{i,j}})$$

Since the support of $G$ is disjoint from the strict transforms $P'_i$ for each $1 \leq i \leq k$, summing the above relation from $j = 1$ to $j = \ell_i$ and using relation (3.11), one obtains

$$\sum_{j=1}^{\ell_i} \sum_{l=0}^{s_{i,j}-1} d_{i,j}^l = \chi(O_{P_\delta}, \tilde{G} \otimes P_\delta O_{\eta^*P_i}) = \chi(O_{P_\delta}, \tilde{G} \otimes P_\delta O_{\eta^*f}).$$
Since $\eta^*P_1$ is linearly equivalent with the $\eta^*f$, for $f$ an arbitrary fiber of $P$ over $C \setminus D$, it follows that
\[
\sum_{j=1}^{l_i} \sum_{l=0}^{s_{i,j}-1} d_{i,j} = \chi(\mathcal{O}_{\widetilde{P}_q}, \widetilde{G} \otimes_{\widetilde{P}_q} \mathcal{O}_{\eta^*f}).
\]
Using relations (3.22), this implies relation (3.23).

For the spectral cover construction one also needs a suitable stability condition for the torsion sheaves $\mathcal{G}$. This will be a twisted stability condition depending on a function $\beta : K_{\text{cusp}}(\tilde{S}_6) \to \mathbb{R}$, which should be regarded as the real part of a Bridgeland stability function on the $K$-theory of $\tilde{P}_3$. The resulting twisted stability condition should be regarded as the specialization of a Bridgeland stability condition to dimension one sheaves. In string theory, $\beta$ is the expectation value of the orbifold flat $B$-field on $\tilde{S}_6$.

For any nontrivial pure dimension sheaf $\mathcal{G}$ as above define the twisted slope
\[
\mu_\beta(\mathcal{G}) = \frac{\chi(\mathcal{G}) + \beta([\mathcal{G}])}{d(\mathcal{G})}.
\]
Such a sheaf will be called $(\omega, \beta)$-(semi)stable if
\[
\mu_{(\omega, \beta)}(\mathcal{G}') \leq \mu_{(\omega, \beta)}(\mathcal{G})
\]
for any proper nontrivial subsheaf $0 \subset \mathcal{G}' \subset \mathcal{G}$.

In order to conclude this section, note that the above construction is especially simple in the case where all $s_{i,j} = 1$. Then the surface $\tilde{P}_q$ is just the standard scheme theoretic blow-up of $P$ at the points $\varnothing_{i,j}$.

### 3.3 Equivalence of moduli stacks

We now summarize the constructions made up to this point and state the main claim.

#### 3.3.1 The parabolic moduli stack

The stack $\mathcal{H}_\xi^{ss}(C, D; \underline{m}, e, \underline{\alpha})$ was constructed in section 2.3. It is the moduli stack of semistable $\xi$-parabolic Higgs bundles on $C$ with poles on the divisor $D = \sum_{i=1}^{k} p_i$ and with discrete invariants $\underline{m} = (m_{i,a})$, weights $\underline{\alpha} = (\alpha_{i,a})$, $0 \leq a \leq s_i - 1$, $1 \leq i \leq k$ and degree $e$. Here $s_i$ is the number of steps in the parabolic filtration at point $p_i$, and the $\underline{m} = (m_{i,a})$ specify the type of the flag, as in section 2.1: the dimension of the flag spaces $V_i^a$ are determined by equation (2.2), namely: $\dim(V_i^a/V_i^{a+1}) = m_{i,a}, \ 0 \leq a \leq s_i - 1$. Finally,
\( \xi = (\xi_i) = (\xi^a_i) \) specifies the residues of the Higgs fields on the subquotients \( V^a_i / V^{a+1}_i \), with \( \xi_i = (\xi^0_i, \ldots, \xi^{n_i}_i) \in K_C(D)_{p_i}^{a+n_i} \).

### 3.3.2 Some combinatorics

For each \( i, 1 \leq i \leq k \), let \( \{ \varphi_{i,j} : j \in J_i \} \) be the set consisting of the \( \xi^a_i \) (ignoring multiplicities), and let \( \ell_i \) be the cardinality of the index set \( J_i \). The relation \( \varphi_{i,j(a)} = \xi^a_i \) determines a natural map \( j : \{0, \ldots, s_i - 1\} \to J_i \). Let \( s_{i,j} \) be the number of \( a \)'s that map to a given \( j \), and denote the set of such \( a \)'s, in increasing order:

\[
\{ a_{ij}^0, \ldots, a_{ij}^{n_i-1} \} = \{ a_{ij}^l \mid l \in S_{i,j} \},
\]

where \( S_{i,j} = \{ 0, \ldots, s_{i,j} - 1 \} \). There is a natural bijection \( a : \sqcup_{j \in J_i} S_{i,j} \to \{0, \ldots, s_i - 1\} \) sending \((j, l) \mapsto a_{ij}^l \). In particular \( \sum_{j \in J_i} s_{i,j} = s_i \). The composition \( j \circ a \) sends \( S_{i,j} \) to \( j \).

### 3.3.3 The symplectic orbifold surface

The orbi-surface \( S_\delta \) was constructed in the beginning of section 3.2. We start with the total space \( P \) of the line bundle \( K_C(D) \) on \( C \), and in the fiber \( P_i \) above each marked point \( p_i \in C \) we fix a divisor \( \delta_i = \sum_{j=1}^{\ell_i} s_{i,j} \varphi_{i,j} \) consisting of \( \ell_i \geq 1 \) points \( \varphi_{i,j} \), \( 1 \leq j \leq \ell_i \), with assigned multiplicities \( s_{i,j} \geq 1 \). We then consider the stack theoretic weighted projective blowup \( \widetilde{P}_\delta \) of \( P \) at the \( \varphi_{i,j} \) with weights \( (s_{i,j}, 1) \). It is a smooth orbifold surface containing a unique orbifold point above each \( \varphi_{i,j} \). Our symplectic orbifold surface \( \widetilde{S}_\delta \) is then obtained from this as the complement \( \widetilde{S}_\delta = \widetilde{P}_\delta \setminus \bigcup_{i=1}^k P'_i \), where \( P'_i \) is the proper transform of the fiber \( P_i \).

### 3.3.4 The orbifold spectral stack

The stack \( \mathcal{M}_{\beta}^a(S_\delta, \{ d_{i,j}^l \}, n) \) was also constructed in section 3.2. It is the moduli stack of compactly supported \( \beta \)-semistable pure dimension one sheaves \( G \) on \( S_\delta \) with discrete invariants

\[
n(\tilde{G}) = \chi(O_{\tilde{S}}, \tilde{G}) = n, \quad d_{i,j}^l(\tilde{G}) = \chi(O_{i,j}^l, \tilde{G}) = d_{i,j}^l
\]

for \( 1 \leq i \leq k, 1 \leq j \leq \ell_i \), \( 0 \leq l \leq s_{i,j} \). For the purpose of the spectral construction, the twisted stability condition will be specified by a function \( \beta : \Gamma_{cpt}(\widetilde{P}_\delta) \to \mathbb{R} \),

\[
\beta(\gamma) = \sum_{i,j,l} \beta^l_{i,j} \chi(O_{i,j}^l, \gamma) \tag{3.24}
\]

with \( \beta^l_{i,j} \in \mathbb{R} \).
3.3.5 Equivalence of moduli stacks.

Our main result is:
There is an isomorphism of stacks
\[ H^s_s(C; D; m, e, \alpha) \cong M^s_s(\tilde{S}_\delta, \{d^l_{i,j}\}, n), \]  
(3.25)
where the labels on the right hand side are determined by those on the left:

- The \( J_i \), the \( s_{i,j} \), and the map \( a : \sqcup_{j \in J_i} S_{i,j} \rightarrow \{0, \ldots, s_i - 1\} \) are determined by the combinatorics of the \( \xi_i \) as above.
- \( \beta \) is determined by: \( \beta^l_{i,j} := \alpha_{i,a(j,l)} \)
- \( \delta = (\delta_i) \), where \( \delta_i = \sum_{j \in J_i} s_{i,j} \varphi_{i,j} \)
- \( d^l_{i,j} = m_{i,a(j,l)} \)
- \( n = e + r(g - 1) \).

3.4 Outline of the proof

The construction of a natural morphism from pure dimension sheaves on \( \tilde{S}_\delta \) to \( \xi \)-parabolic Higgs bundles is sketched below. An inverse morphism can be constructed in principle using methods analogous to [3]. The details of this construction and a complete proof of the above stack isomorphism will appear elsewhere.

Let \( \beta : \Gamma_{\text{cpt}}(\tilde{S}_\delta) \rightarrow \mathbb{R} \) be a stability function of the form (3.24). Suppose \( \{\beta^l_{i,j} | 1 \leq j \leq \ell_i, 0 \leq l \leq s_{i,j} - 1\} \) are pairwise distinct for each \( 1 \leq i \leq k \), and satisfy the inequalities
\[ 0 < \beta^0_{i,j} < \beta^1_{i,j} < \cdots < \beta^{s_{i,j}-1} < 1 \]  
(3.26)

independently, for each fixed values \( 1 \leq i \leq k, 1 \leq j \leq \ell_i \).

Such a function \( \beta \) determines a collection of combinatorial data as defined in Section 3.3.2. Since the \( \beta^l_{i,j} \) are assumed pairwise distinct for each \( i \), there is a bijection \( a : \sqcup_{j \in J_i} S_{i,j} \rightarrow \{0, \ldots, s_i - 1\} \) defined by
\[ a(j,l) = |\{\beta^l_{i,j'} < \beta^l_{i,j}\}|. \]

Here \( |A| \) denotes the cardinality of the finite set \( A \). One also has the function
\[ j := p \circ a^{-1} : \{0, \ldots, s_i - 1\} \rightarrow J_i \]
where
\[ p : \bigsqcup_{j \in J} S_{ij} \to J_i \]
is the natural projection.

Suppose \( \tilde{G} \) is a \( \beta \)-stable pure dimension one sheaf on \( \tilde{P}_\delta \) with compact support in \( \tilde{S}_\delta \). Using the spectral correspondence in Section 2.2, it suffices to construct a stable parabolic pure dimension sheaf \( F \) on \( P \) with parabolic structure along the marked fibers \( P_i, 1 \leq i \leq k \).

By pushforward along \( \pi : P \to C \), one will then obtain a stable \( \mathfrak{L} \)-parabolic bundle on \( C \) with combinatorial data \( a, j \) at each marked point.

Let \( F = \eta_* \tilde{G} \), which is a pure dimension one sheaf with compact support on \( P \). For each \( 1 \leq j \leq \ell_i \) there is a canonical filtration
\[ 0 \subset \tilde{G}(-s_{ij} \Xi_{ij}) \subset \cdots \subset \tilde{G}(-\Xi_{ij}) \subset \tilde{G} \tag{3.27} \]
of \( \mathcal{O}_{\tilde{P}_\delta} \)-modules. Moreover, relations (3.11) yield an isomorphism
\[ F(-P_i) \cong \eta_*(\tilde{G} \otimes_{\tilde{P}_\delta} \eta^* \mathcal{O}_P(-P_i)) \cong \eta_* \tilde{G}(-\Xi_i) \]
where \( \Xi_i = \sum_{j=1}^{\ell_i} s_{ij} \Xi_{ij} \). Therefore for fixed \( (i, j) \) the filtration (3.27) yields a filtration
\[ 0 \subset F(-P_i) \subset \eta_* \tilde{G}(-(r_{ij} - 1) \Xi_{ij}) \subset \cdots \subset \eta_* \tilde{G}(-\Xi_{ij}) \subset F. \tag{3.28} \]

Taking quotients one obtains a sequence of surjective morphisms
\[ F|_{P_i} \twoheadrightarrow F_{i,j}^{s_{ij}-1} \twoheadrightarrow \cdots \twoheadrightarrow F_{i,j}^1 \tag{3.29} \]
for each pair \( (i, j) \).

The sequences (3.29) are then assembled into the following quasi-parabolic structure along \( P_i \):
\[ F|_{P_i} \twoheadrightarrow F_i^{s_i-1} \twoheadrightarrow \cdots \twoheadrightarrow F_i^1 \twoheadrightarrow F_i^0 = 0 \tag{3.30} \]
where
\[ F_i^a = \bigoplus_{j \in J_i, \ 0 \leq l \leq s_{ij} - 1 \atop a(j,l) \leq a} F_{i,j}^l. \]
The epimorphisms in (3.30) are canonically determined by those in (3.29). Using the weights \( \alpha_{i,a} = \beta_{i,j}^l \) with \( a(j,l) = a \), one obtains a parabolic pure dimension one sheaf \( F \) on \( P \), as claimed above.

As stated above, compatibility with stability conditions and the construction of the inverse morphism will be left for future work.
4 Orbifold stable pairs and parabolic ADHM sheaves

This section introduces stable pair invariants of local orbifold curves and explains their relation with parabolic ADHM sheaves on ordinary curves.

Let $\mathcal{C}$ be a smooth projective curve over $\mathbb{C}$, $D = \sum_{i=1}^{k} p_i$ a reduced effective divisor on $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ be a root stack as in Section 3.1 with stabilizers $\mu_{s_i}$, $1 \leq i \leq k$ at each marked point. Let $\tilde{X}$ be the total space of a rank two bundle $\tilde{M}_1 \oplus \tilde{M}_2$ on $\tilde{\mathcal{C}}$, where $\tilde{M}_1$, $\tilde{M}_2$ are line bundles such that $\tilde{M}_1 \otimes_{\tilde{\mathcal{C}}} \tilde{M}_2 \simeq K_{\tilde{\mathcal{C}}}$. Hence $\tilde{X}$ is a smooth Calabi-Yau three dimensional Deligne-Mumford stack with generically trivial stabilizers.

A stable pair on $\tilde{X}$ will be defined as a pair $(\tilde{F}, \tilde{s})$ where

- $\tilde{F}$ is a pure dimension one sheaf with proper support, finite-to-one over $\tilde{\mathcal{C}}$, and
- $\tilde{s} : \mathcal{O}_Y \to \tilde{F}$ is a section with zero dimensional cokernel.

The string theoretic derivation of the HLRV formula is based on a relation between orbifold stable pairs and parabolic Higgs bundles which generalizes the similar relation found in [12, 11] for ordinary Higgs bundles. In order to understand this relation in detail, note that stable pairs on $\tilde{X}$ admit a presentation in terms of ADHM sheaves on the orbicurve $\tilde{\mathcal{C}}$ by analogy with [16].

An ADHM sheaf on a curve $\mathcal{C}$ with coefficient line bundles $M_1, M_2$ was defined [16] as a collection $(E, \Phi_1, \Phi_2, \phi, \psi)$ where $E$ is a vector bundle on $\mathcal{C}$ and $\Phi_j : E \to E \otimes M_j$, $j = 1, 2$, $\phi : E \to M_1 \otimes_{\mathcal{C}} M_2$, $\psi : \mathcal{O}_\mathcal{C} \to E$ are morphisms of sheaves satisfying the ADHM relation

$$ (\Phi_1 \otimes 1_{M_2}) \circ (\Phi_2) - (\Phi_2 \otimes 1_{M_1}) \circ \Phi_1 + (\psi \otimes 1_{M_1 \otimes_{\mathcal{C}} M_2}) \circ \phi = 0. \quad (4.1) $$

Such an object $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ is called asymptotically stable if the restriction $\mathcal{E}|_x$ is a cyclic ADHM quiver representation for all but finitely many closed points $x \in \mathcal{C}$. Equivalently, the subsheaf $\text{Im}(\psi) \subset E$ generates $\mathcal{E}$ as a quiver sheaf at all closed points $x \in \mathcal{C}$ except a finite set.

The above definitions admit a straightforward generalization to an orbicurve $\tilde{\mathcal{C}}$ equipped with two line bundles $\tilde{M}_1, \tilde{M}_2$. By analogy with [16, Sect. 7] one obtains an isomorphism between the moduli space of stable pairs on $\tilde{X}$ and the moduli space of asymptotically stable orbifold ADHM data on $\tilde{\mathcal{C}}$. Moreover, these moduli spaces carry natural perfect obstruction theories, which are also identified by this isomorphism. The details are not essential for the main goals of this paper and will be omitted. Instead it will be helpful to note that for certain choices of the line bundles $\tilde{M}_1, \tilde{M}_2$ orbifold stable pairs are further identified with
parabolic ADHM sheaves on the ordinary curve $C$. This second identification also provides an efficient construction of the perfect obstruction theory, as shown below.

### 4.1 ADHM parabolic structure

Suppose $\tilde{M}_1 = \nu^* M^{-1}$, $\tilde{M}_2 = K_C \otimes \nu^* M$ for a line bundle $M$ on $C$. In this case $\tilde{X}$ will be denoted by $\tilde{Y}_M$ as in Section 1.2.

Using the correspondence in Section 3.1, an ADHM sheaf $E$ on $\tilde{C}$ with coefficients $(\tilde{M}_1, \tilde{M}_2)$ corresponds to an ADHM sheaf $E$ on $C$ with coefficient line bundles $M_1 = M^{-1}$, $M_2 = K_C \otimes C M(D)$. Moreover, one obtains a flag $E_i^*$ in the fiber of $E$ at each marked point satisfying natural compatibility conditions with the two Higgs fields $\Phi_1 : E \to E \otimes C M^{-1}$, $\Phi_2 : E \to E \otimes C K_C \otimes C M(D)$. This yields the following definition for (quasi-)parabolic ADHM sheaves.

A quasi-parabolic ADHM sheaf on $C$ of type $\underline{m}$ is a collection $(E^*, \Phi_1, \Phi_2, \phi, \psi)$, where

(a) $E^*$ is a quasi-parabolic bundle on $C$ of type $\underline{s}$.

(b) $\Phi_1 : E \to E \otimes M_1$ and $\Phi_2 : E \to E \otimes M_2(D)$ are morphisms of sheaves such that $\Phi_1|_{p_i} : E|_{p_i} \to E|_{p_i}$ is parabolic and $\Phi_2|_{p_i} : E(D)|_{p_i} \to E \otimes C M_2(D)|_{p_i}$ is strongly parabolic for all $1 \leq i \leq k$.

(c) $\phi : E \to M_1 \otimes C M_2$, $\psi : O_C \to E$ are morphisms of sheaves such that the following relation is satisfied

$$ (\Phi_1 \otimes 1_{M_2(D)}) \circ (\Phi_2) - (\Phi_2 \otimes 1_{M_1}) \circ \Phi_1 + ((s_D \circ \psi) \otimes 1_{M_1 \otimes C M_2}) \circ \phi = 0. \quad (4.2) $$

A collection of data satisfying the above conditions will be denoted $\mathcal{E}^*$.

By analogy with the Higgs bundle case, a parabolic ADHM sheaf will be defined as a quasi-parabolic object $\mathcal{E}^*$ equipped with weights $\underline{a} = (a_i)_{1 \leq i \leq k}$ satisfying conditions (2.6).

Generalizing the results of [17], parabolic ADHM sheaves admit natural stability conditions depending on a stability parameter $\delta \in \mathbb{R}$. For a nonzero parabolic bundle $(E^*, \underline{a})$, define the $\delta$-parabolic slope by

$$ \mu_\delta(E^*, \underline{a}) = \frac{\delta + \chi(E^*, \underline{a})}{\text{rk}(E)}, $$

where $\chi(E^*, \underline{a})$ is given in (2.9). Recall that the parabolic slope $\mu(E^*, \underline{a})$ was defined in (2.8).

Then a parabolic ADHM sheaf $(\mathcal{E}^*, \underline{a})$ is $\delta$-(semi)stable if the following conditions hold.
(i) Any proper nontrivial saturated subsheaf $0 \subset E' \subset E$ preserved by $\Phi_1, \Phi_2$ and contained in $\text{Ker}(\phi)$ satisfies

$$\mu(E'^\bullet, \omega) \leq \frac{\delta + \chi(E'^\bullet, \omega)}{\text{rk}(E)}.$$ 

(ii) Any proper nontrivial saturated subsheaf $0 \subset E' \subset E$ preserved by $\Phi_1, \Phi_2$ containing $\text{Im}(\psi)$ satisfies

$$\frac{\delta + \chi(E'^\bullet, \omega)}{\text{rk}(E')} \leq \frac{\delta + \chi(E'^\bullet, \omega)}{\text{rk}(E)}.$$

There is also a natural duality transformation for parabolic ADHM sheaves, generalizing the one introduced in [17] for ADHM sheaves. Given a parabolic ADHM sheaf $(E^\bullet, \omega)$ let $\hat{E}$ denote its underlying ADHM sheaf, forgetting the parabolic structure. Let $\hat{E} = (\hat{E}, \hat{\Phi}_1, \hat{\Phi}_2, \hat{\phi}, \hat{\psi})$ denote the dual of $E$ defined in [17, Sect 2.1]. Note that $\hat{E} = E^\vee \otimes_C M_1 \otimes_C M_2$ and the morphism data $(\hat{\Phi}_1, \hat{\Phi}_2, \hat{\phi}, \hat{\psi})$ is naturally determined by $(\Phi_1, \Phi_2, \phi, \psi)$.

Next note that given a flag $E^a_i$ in the fiber $E_{p_i}$, one can define a dual flag in the fiber $E^\vee_{p_i}$. Each subspace $E^a_i \subset E_{p_i}$ determines a locally free sheaf $F^a_i = \text{Ker}(E \twoheadrightarrow E_{p_i} / E^a_i \otimes \mathcal{O}_{p_i})$ on $C$, which yields a filtration

$$0 \subset E(-p_i) = F^a_i \subset F^{a-1}_i \subset \cdots \subset F^1_i \subset F^0_i = E.$$ 

The dual filtration yields a flag

$$0 = (E^\vee)^a_i \subset (E^\vee)^{a-1}_i \subset \cdots \subset (E^\vee)^1_i \subset (E^\vee)^0_i = (E^\vee)_{p_i},$$

hence also a flag $\hat{E}^\bullet_i$ on $\hat{E}_{p_i}$ by taking tensor product with $M_1 \otimes_C M_2$. It is straightforward to check that successive quotients of the dual flag have dimensions

$$\hat{m}_{i,a} = m_{i,s_i-a-1}$$

for $0 \leq a \leq s_i - 1$. Moreover the morphisms $\hat{\Phi}_1, \hat{\Phi}_2$ and the filtrations $\hat{E}^\bullet_i$ satisfy naturally condition $(b)$ in Section 4.1.

In conclusion the data $(\hat{E}^\bullet, \hat{\Phi}_1, \hat{\Phi}_2, \hat{\phi}, \hat{\psi})$ determines a quasi-parabolic ADHM sheaf with numerical invariants $(\hat{m}_i, -e + r(\text{deg}(M_1) + \text{deg}(M_2))$. In addition, let $\hat{\alpha}_{i,a} = -\alpha_{i,s_i-1-a}$ for $1 \leq i \leq k, 0 \leq a \leq s_i - 1$. Then it easily follows that the parabolic ADHM sheaf $(\hat{E}^\bullet, \hat{\omega})$ is $\delta$-(semi)stable if and only if $(E^\bullet, \omega)$ is $(-\delta)$-(semi)stable. Note that the dual parabolic weights $\hat{\alpha}$ defined here differ from the usual conventions in the literature, where they are defined as $1 - \alpha_{i,s_i-1-a}$ in order to bring them in the range $[0, 1)$.
4.2 Moduli spaces and counting invariants

Moduli spaces of $\delta$-semistable parabolic ADHM sheaves are constructed by analogy with [16, 17], using the boundedness results for parabolic sheaves proven in [47]. Repeating the arguments of [16, 17] in the parabolic framework, one obtains a moduli an algebraic moduli stack $\mathcal{PM}^\text{ss}_\delta(C, M_1, M_2, D; m, e, \alpha)$ of $\delta$-semistable ADHM sheaves with discrete invariants $(m, e)$ and parabolic weights $\alpha$.

Varying $\delta \in \mathbb{R}$ for fixed $\alpha$ yields a finite chamber structure on the real axis analogous to the one studied in [17]. For sufficiently large stability parameter, $\delta$-stability reduces to asymptotic stability [17, Def. 4.5] of the underlying ADHM sheaf, forgetting the parabolic structure. This condition is completely independent of the parabolic weights, hence the moduli space in the asymptotic chamber will be denoted by $\mathcal{PM}_\infty(X, M_1, M_2, D; m, e)$.

Let also $\mathcal{M}_\infty(X, M_1, M_2(D); r, e)$ denote the moduli space of space of asymptotically stable ADHM sheaves without parabolic structure and with fixed numerical invariants $(r, e)$. Then note that there is a proper morphism

$$\mathcal{PM}_\infty(X, M_1, M_2, D; m, e) \to \mathcal{M}_\infty(X, M_1, M_2(D); r, e) \quad (4.3)$$

forgetting the flags at the marked points. Properness follows from the fact that for fixed morphisms $(\Phi_1, \Phi_2, \phi, \psi)$ the moduli space of collections of flags $(E_p^\bullet)_{1 \leq i \leq k}$ compatible with the ADHM data is proper.

Note also that the stability condition for parabolic ADHM takes a special form in the asymptotic chamber $\delta << 0$ as well. Keeping the numerical invariants $(m, e)$ parabolic weights $\alpha$ fixed a sheaf $(E^\bullet, \alpha)$ is $\delta$-semistable for $\delta << 0$ if and only its dual $(\tilde{E}^\bullet, \tilde{\alpha})$ is asymptotically stable.

In order to define parabolic virtual invariants, the moduli space must be equipped with a perfect obstruction theory. This is carried out in analogy with [16, Sect. 5] using known results on the deformation theory of parabolic Higgs bundles [58, 63]. A concise summary of such results is provided in [23, Sect. 2.2].

By analogy with the non-parabolic case, [16, Sect. 4.1], the deformation complex of a parabolic ADHM sheaf $E^\bullet$ is the three term complex of amplitude $[0, 2]$

$$0 \to \mathcal{C}^0(E^\bullet) \to \mathcal{C}^1(E^\bullet) \to \mathcal{C}^2(E^\bullet) \to 0 \quad (4.4)$$

where

$$\mathcal{C}^0(E^\bullet) = PEnd_{C}(E^\bullet)$$
\[ C^1(\mathcal{E}^\bullet) = P\text{End}_C(E^\bullet) \otimes M_1 \oplus S\text{PEnd}_C(E^\bullet) \otimes M_2(D) \]
\[ \oplus \text{Hom}_C(O_C, E) \oplus \text{Hom}_C(E, M_1 \otimes_C M_2) \]
\[ C^2(\mathcal{E}^\bullet) = S\text{PEnd}_C(E^\bullet) \otimes M_1 \otimes_C M_2(D). \]

The differentials are the same as in [16, Def. 4.3], but their explicit form will not be needed in the following. The main technical result needed in the construction of a perfect obstruction theory requires the hypercohomology groups \( H^i(\mathcal{C}(\mathcal{E}^\bullet)) \) to vanish for \( i \leq 0 \) as well as \( i \geq 3 \). This follows by analogy with [16, Lemma 4.10] using the duality relation (2.5). Then the existence of a perfect obstruction theory follows by the same formal arguments as in [16, Sect. 5]. Furthermore, it is important to note that the resulting perfect obstruction theory is symmetric provided that \( M_1 \otimes_C M_2 \simeq K_C \).

Finally, since the moduli spaces are noncompact, parabolic invariants will be defined by equivariant virtual integration with respect to a torus action with compact fixed locus. For asymptotically stable parabolic ADHM sheaves, such an action is again obtained by analogy with [16, Sect 3]. Namely, \( T = \mathbb{C}^\times \times \mathbb{C}^\times \) acts by
\[ (t_1, t_2) \times (E^\bullet, \Phi_1, \Phi_2, \phi, \psi) \mapsto (E^\bullet, t_1 \Phi_1, t_2 \Phi_2, t_1 t_2 \phi, \psi). \]
(4.5)

Compactness of the fixed locus follows from [16, Prop. 3.1] and the observation that the proper forgetful morphism (4.3) is equivariant.

Motivic and refined invariants will be defined using the theory of Kontsevich and Soibelman [42] and assuming all the required foundational results. The refined parabolic ADHM invariants will be denoted by \( A_3(m, e, \alpha; y) \). The asymptotic ones will be denoted by \( A_{\pm\infty}(m, e; y) \). The duality transformation introduced at the end of Section 4.1 yields relations of the form
\[ A_{-\delta}(m, e, \alpha; y) = A_3(\bar{m}, \bar{e}, \bar{\alpha}; y) \]
(4.6)
where
\[ \bar{m}_{i,a} = m_{i,s_i-1-a}, \quad \bar{\alpha}_{i,a} = -\alpha_{i,s_i-1-a}, \quad \bar{e} = -e + 2r(g - 1). \]
for all \( 1 \leq i \leq k, \ 0 \leq a \leq s_i - 1 \).

In conclusion, one obtains a series of asymptotic refined invariants
\[ Z_{\text{ADHM}}^{ref}(q; \underline{x}, y) = \sum_{n \in \mathbb{Z}} \sum_m A_\infty(m, e; y)q^{-r(g-1)} \prod_{i=1}^k \frac{m_i}{x_i} \]
(4.7)
for some formal variables \( \underline{x} = (x_1, \ldots, x_k) \), \( \underline{x}_i = (x_{i,0}, \ldots, x_{i,s_i-1}) \), \( 1 \leq i \leq k \). As explained in the paragraph preceding Section 4.1, generalizing the results of [16], the above series is identical with the left hand side \( Z_{\text{YM}}^{ref}(q, \underline{x}, y) \) of equation (1.4).
5 Geometric engineering, Hilbert schemes and Macdonald polynomials

A conjectural formula for the refined stable pair theory of the orbifold \( \tilde{Y} \) is derived in this section by geometric engineering. Using IIA/M-theory duality, stable pair invariants are related to degeneracies of BPS wavefunction in D-brane quiver quantum mechanics, which are counted by equivariant \( K \)-theoretic invariants as in [53]. Similar results have been obtained in [37, 43, 19, 53, 33, 20, 32, 41, 44, 35], where the resulting quantum mechanical system describes instanton particles in five dimensional gauge theories. As shown in Section 5.2, in the present case one obtains a moduli space of parabolic ADHM quiver representations which can be identified with a nested Hilbert scheme of points in the complex plane. The quantum mechanical partition function is the generating function for equivariant \( K \)-theoretic invariants given in equation (5.6). The main result of this section is formula (5.18) expressing this partition function in terms of Macdonald polynomials.

5.1 Orbifold stable pairs in string theory

In this section, \( \tilde{Y} \) will be the total space of a rank two bundle \( \nu^* M^{-1} \oplus K_{\tilde{C}} \otimes \mu^* M \) as in Sections 1.2, 4.1. The subscript \( M \) used in Sections 1.2, and 4.1 will be suppressed for brevity.

From a string theory perspective, the refined stable pair invariants of \( \tilde{Y} \) are identified with BPS degeneracies of D6-D2-D0 bound states in the IIA vacuum \( \tilde{Y} \times \mathbb{R}^{1,3} \). This theory will be called IIA\(^{(1)}\). As explained below, a chain of string duality transformations relates such BPS states with D6-D2-D0 bound states in a different type IIA background of the form \( T^* \tilde{C} \times TN_1 \times \mathbb{R}^{1,1} \), where \( TN_1 \) is the one center Taub-NUT manifold. This theory will be called IIA\(^{(2)}\).

Applying a Wick rotation in the IIA\(^{(1)}\) theory and making the time direction periodic results in a background geometry \( \tilde{Y} \times \mathbb{R}^3 \times S^1_T \). For configurations with a single D6-brane on \( \tilde{Y} \), lifting IIA\(^{(1)}\) to M-theory produces the eleven dimensional vacuum \( \tilde{Y} \times TN_1 \times S^1_T \). Note that there is a fiberwise circle action \( S^1_M \times \tilde{Y} \to \tilde{Y} \) which scales the two line bundles \( \tilde{L}_1, \tilde{L}_2 \) with opposite weights leaving the 0-section pointwise fixed. Reducing the theory along the orbits of this action yields the theory with underlying geometry \( T^* \tilde{C} \times TN_1 \times \mathbb{R} \times S^1_T \). Since the fixed locus of \( S^1_M \)-action on \( \tilde{Y} \) is the 0-section, in the new duality frame there is a D6-brane supported on the submanifold \( \tilde{C} \times TN_1 \times \{0\} \times S^1_T \), where \( \tilde{C} \) is embedded
in $T^*\widetilde{C}$ as the 0-section. Note also that the IIA$^{(2)}$ geometric background does not depend on the line bundle $M$ used in the construction of $\tilde{Y}$. By analogy with previous examples studied in the literature [33, 34, 20, 44, 57, 35] this dependence is expected to be encoded in the level $n$ of the Chern-Simons coupling in the effective five dimensional gauge theory on the D6-brane wrapped on $\widetilde{C} \times TN_1 \times S^1_L$, which is related by duality to the level of the five dimensional space-time Chern-Simons coupling in M-theory. In the IIA$^{(2)}$ string theory, $n$ is determined by the Ramond-Ramond flux $\int_{\widetilde{C}} G_2 = n$ via standard D-brane couplings. Inspecting the examples in loc. cit., for smooth local genus zero curves, the Chern-Simons level is given by $n = \deg(M) - 1$. This formula was extended to $n = g - 1 + \deg(M)$ for local genus $g$ curves in [12, 11]. Here one needs a further generalization for local orbifold curves, which have never been studied in this context before. The solution to this string duality puzzle follows from the observation that $n$ is invariant under continuous deformations of the $M$-theory vacuum $\tilde{Y} \times TN^1 \times S^1_L$. Such deformations connect the present orbifold vacuum to a smooth geometric vacuum obtained by resolving the quotient singularities of the coarse moduli space $Y$ of $\tilde{Y}$. The resolution $\tilde{Y} \rightarrow Y$ contains a local genus $g$ curve isomorphic to $C$ in addition with exceptional $(0, -2)$ rational curves which play no role in this argument. This leads to the conclusion that that the value of $n$ in the present orbifold vacuum must be the same as that found in [12, 11] for local genus $g$ curves i.e.

$$n = g - 1 + \deg(M). \quad (5.1)$$

The above derivation is not rigorous, hence it should be regarded as a conjecture. The explicit computations in Section 8 provide ample evidence for this formula.

As in [18], D6-D2-D0 configurations in IIA$^{(1)}$ theory lift to spinning M2-branes in the M-theory. Since such states carry no momentum along the orbits of the fiberwise $S^1_M$-action, they reduce to D2-brane supported on the zero section $\tilde{C}$ in the the IIA$^{(2)}$ vacuum with zero D0-brane charge. In conclusion one is left with a D6-D2 configuration in the IIA background $T^*\tilde{C} \times TN_1 \times \mathbb{R} \times S^1_L$ consisting of one non-compact D6-brane on $\tilde{C} \times TN_1 \times S^1_L$ and a stack of D2-branes with some multiplicity $r \geq 1$ supported on $\tilde{C} \times S^1_L$. The Chan-Paton bundle on the D2-branes is topological trivial since these configurations carry no D0-brane charge.

This chain of duality transformations relates orbifold stable pair invariants in the original IIA$^{(1)}$ theory with D6-D2 supersymmetric bound states in the new IIA vacuum. The latter are counted by the partition function of the D2-brane low energy theory, which is a topological gauge theory on $\tilde{C} \times S^1_L$. The topological twist is determined by the normal bundle of the zero section $\tilde{C}$ in $T^*\tilde{C}$, which is isomorphic to $K_{\tilde{C}}$. Using the topological symmetry, the
background manifold can be changed from $T^*\tilde{C} \times TN_1 \times S^1_T$ to $T^*\tilde{C} \times \mathbb{C}^2 \times S^1_T$, leaving the BPS degeneracies invariant. Moreover, in order to detect the spin quantum numbers in the $M$-theory framework, the D6-D2 theory must be placed in an Ω-background [53] determined by the natural $\mathbb{C}^\times \times \mathbb{C}^\times$ scaling action on $\mathbb{C}^2$.

5.2 From D-branes to nested Hilbert schemes

The next goal is to count BPS states of the above D6-D2 system on $T^*\tilde{C} \times \mathbb{C}^2 \times \mathbb{R}$. As explained above, the D2-brane low energy effective theory is a topologically twisted $\mathcal{N} = 4$ three dimensional gauge theory on $\tilde{C} \times S^1_T$. By analogy with the five dimensional situation, [53] the topological partition function counts BPS states of vortex particles on $\tilde{C}$. Mathematically, these are holomorphic sections of a certain bundle of fermionic zero modes on the vortex moduli space. Hence, taking into account the Ω-background along $\mathbb{C}^2$, the partition function will be generating function for equivariant $K$-theoretic invariants of the moduli space.

The field content of the D2-brane low energy theory consists of two adjoint chiral multiplets $\hat{A}_1, \hat{A}_2$ corresponding to fluctuations in the untwisted normal directions, and an adjoint valued one-form $\hat{A}_3$ on $\tilde{C}$ corresponding to fluctuations in the normal directions to $\tilde{C}$ in $T^*\tilde{C}$. In addition, the D2-D6 open string sector yields two extra chiral multiplets $\hat{I}, \hat{J}$ in the fundamental, and anti-fundamental representation of the gauge group. BPS vortex solutions are field configurations in this gauge theory satisfying $F$ and $D$-flatness constraints.

Using Hitchin-Kobayashi correspondence for quiver bundles [1], gauge equivalence classes of BPS vortex solutions are in one-to-one correspondence with isomorphism classes of holomorphic data $(\tilde{E}, \hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{I}, \hat{J})$ satisfying the $F$-term equations and a stability condition determined by the $D$-term constraints. Namely, $E$ is a holomorphic vector bundle on $\tilde{C}$ and

$$\hat{A}_1, \hat{A}_2 : \tilde{E} \to \tilde{E}, \quad \hat{A}_3 : \tilde{E} \to \tilde{E} \otimes_{\tilde{C}} K_{\tilde{C}}, \quad \hat{I} : \mathcal{O}_{\tilde{C}} \to \tilde{E}, \quad \hat{J} : \tilde{E} \to \mathcal{O}_{\tilde{C}}$$

are morphisms of sheaves on $\tilde{C}$. The $F$ term equations yield conditions of the form

$$[\hat{A}_1, \hat{A}_2] + \hat{I}\hat{J} = 0, \quad [\hat{A}_3, \hat{A}_1] = [\hat{A}_3, \hat{A}_2] = 0, \quad \hat{A}_3\hat{I} = 0, \quad \hat{J}\hat{A}_3 = 0. \quad (5.2)$$

Note that the data $\tilde{E} = (\tilde{E}, \hat{A}_1, \hat{A}_2, \hat{I}, \hat{J})$ determines an ADHM sheaf with trivial coefficient line bundles on the orbifold $\tilde{C}$, while $\hat{A}_3$ is an extra field. The $D$ term constraints yield a stability condition for $(\tilde{E}, \hat{A}_3)$ via Hitchin-Kobayashi correspondence, which depends on a Fayet-Iliopoulos parameter. Then a straightforward computation shows that for suitable
values of this parameter the stability condition requires $\tilde{E}$ to be an asymptotically stable ADHM sheaf on $\tilde{C}$ as defined in Section 4. In this case, it is also straightforward to prove that any field $\tilde{A}_3$ satisfying the $F$ term equations (5.2) must be identically 0. The details will be omitted for brevity.

In conclusion, with a suitable choice of FI parameters, the vortex moduli spaces in the quiver gauge theory on D2 branes are isomorphic to moduli spaces of asymptotically stable ADHM sheaves on the orbifold $\tilde{C}$ with trivial coefficient line bundles. According Section 4, such orbifold ADHM sheaves are identified with stable ADHM sheaves $\mathcal{E}^\bullet = (E, A_1, A_2, I, J)$ on $C$ where $E$ is equipped with a parabolic structure $E^\bullet$ along $D$. Since the coefficient line bundles are trivial, the Higgs fields $A_1, A_2 : E \to E$ are required to preserve the flag at each point $p_i, 1 \leq i \leq k$. Moreover the ADHM stability condition implies that $J$ is identically zero. In addition, as explained in the previous section, the bundle $E$ must be topologically trivial.

In appendix A it is shown that a degree 0 asymptotically stable ADHM sheaf $\mathcal{E}$ of arbitrary rank $r \geq 1$ on $C$ must have underlying bundle $E \simeq \mathcal{O}_C^{\oplus r}$. This implies that all morphisms $A_1, A_2, I$ are constant maps, hence the moduli space of such sheaves is isomorphic to the moduli space of stable ADHM data

$$(A_1, A_2, I, 0) \in \text{End}(\mathbb{C}^r)^{\oplus 2} \oplus \text{Hom}(\mathbb{C}, \mathbb{C}^r).$$

(5.3)

The ADHM stability condition forbids the existence of linear subspaces $0 \subsetneq V \subsetneq \mathbb{C}^r$ preserved by $A_1, A_2$ and containing the image of $I$. Two such data are equivalent if they are related by the natural action $GL(r, \mathbb{C})$ action.

Since asymptotic stability for parabolic ADHM sheaves reduces to asymptotic stability of the underlying ADHM sheaf, adding the parabolic structure yields a moduli space of data

$$(A_1, A_2, I, 0; V^\bullet_{i})_{1 \leq i \leq k}$$

where

- $(A_1, A_2, I, 0)$ is a stable ADHM data
- $V^\bullet_{i}$ is a flag in $\mathbb{C}^r$ of type $m_i$ preserved by $A_1, A_2$ for all $1 \leq i \leq k$.

Again two such data are equivalent if they are related by the natural $GL(r, \mathbb{C})$ action.

For the remaining part of this section suppose there is a single marked point $p \in C$, and the flag at $p$ is of the form

$$0 = E^r \subsetneq E^{r-1} \subsetneq \cdots \subsetneq E^0 = E_p,$$

(5.4)
where \( r = \text{rk}(E) \). Such flags are not necessarily full since the inclusions are not required to be strict. However let \( a_i \in 0, \ldots, r - 1 \), \( 0 \leq i \leq \ell \) be the values of \( 0 \leq a \leq r - 1 \) such that \( m_{a_i} > 0 \). Then one can canonically associate a flag

\[
0 \subsetneq E^t \subsetneq \cdots \subsetneq E^1 \subsetneq E^0 = E_p
\]

to the flag (5.4) such that all inclusions are strict and the discrete invariants are \( \dim E^t/E^{t+1} = m_{a_i} \) for \( 0 \leq i \leq \ell \). This will be called the minimal flag associated to \( E \). It is clear that the moduli space of asymptotically stable parabolic ADHM data depends only on the ordered sequence \( \gamma = (m_{a_i})_{0 \leq i \leq \ell} \), hence it will be denoted by \( \mathcal{M}(\gamma) \). The entries of \( \gamma \) will be denoted by \( \gamma_i = m_{a_i}, 0 \leq i \leq \ell \), in the following.

The moduli space \( \mathcal{M}(\gamma) \) is identified in [4] with a nested Hilbert scheme of points in \( \mathbb{C}^2 \) using the ADHM construction. Given an ordered sequence \( \gamma = (m_{a_i})_{0 \leq i \leq \ell} \) of positive integers as above, let \( \mathcal{N}(\gamma) \) denote the Hilbert scheme parameterizing flags of ideal sheaves

\[
0 \subset I_\ell \subset \cdots \subset I_0
\]

of zero dimensional subschemes \( Z_i \subset \mathbb{C}^2 \) with

\[
\chi(O_{Z_i}) = \sum_{j=0}^{i} \gamma_j
\]

for each \( 0 \leq i \leq \ell \). Then, according to [4], there is an isomorphism of moduli spaces \( \mathcal{M}(\gamma) \simeq \mathcal{N}(\gamma) \).

### 5.3 \( K \)-theoretic partition function

By analogy with [12, 11], the \( K \)-theoretic partition function for fixed numerical invariants \( \gamma \) will be the equivariant Hirzebruch genus of a bundle \( V(\gamma) \) on \( \mathcal{N}(\gamma) \) with respect to the \( T = \mathbb{C}^\times \times \mathbb{C}^\times \) action induced by the scaling action on \( \mathbb{C}^2 \). On general grounds, \( V(\gamma) \) is the bundle of fermion zero modes on the moduli space twisted by a line bundle determined by the space-time Chern-Simons coupling in M-theory [57]. As shown in Appendix B, the bundle of zero modes is simply the pullback \( \eta^*(T^*H^r)^{\otimes g} \) via the natural projection \( \eta : \mathcal{N}(\gamma) \to H^r \) to the Hilbert scheme of \( r \) points in \( \mathbb{C}^2 \). According to [57], the twisting line bundle is \( \eta^*\text{det}(V)^n \), where \( V \) is the tautological bundle on the Hilbert scheme and \( n \) is the Chern-Simons level given in equation (5.1). For completeness note that \( V \) is the pushforward of the structure sheaf of the universal subscheme \( Z \subset H^r \times \mathbb{C}^2 \to H^r \). In conclusion,

\[
V(\gamma) \simeq \eta^*V_{g,p}, \quad V_{g,p} = (T^*H^r)^{\otimes g} \otimes \text{det}(V)^{g-1+p},
\]

39
and the $K$-theoretic partition function is given by

$$Z_K^{(r)}(q_1, q_2; \tilde{y}, \tilde{x}) = \sum_{m=(m_0, \ldots, m_{r-1}) \in \mathbb{Z}_{\geq 0}^r, \sum m_a = (r_1-1)} \chi_g^T(\mathcal{V}((m))) \prod_{a=0}^{r-1} \tilde{x}_a^{m_a}$$  \hspace{1cm} (5.6)$$

where $\chi_g^T$ is the $T$-equivariant Hirzebruch genus. For any collection $m = (m_0, \ldots, m_{r-1})$ of nonnegative integers, $\gamma(m)$ denotes the sequence of distinct values in $m$ defined below equation (5.4).

The above partition function will be expressed in terms of Macdonald polynomials in Section 5.5. The next subsection summarizes the geometric results needed in that computation.

### 5.4 Nested and isospectral Hilbert schemes

This section explains the relation between the nested Hilbert scheme $\mathcal{N}(\gamma)$ and the isospectral Hilbert scheme employed in the work of Haiman [27, 28] on Macdonald polynomials. The main results are the pushforward formulas (5.11) and (5.15).

First consider the case of full flags of ideal sheaves, where $\ell = r - 1$ and $\gamma = (1, \ldots, r)$. Then it will be shown below that there exists a surjective birational projection with connected fibers mapping $\mathcal{N}(1, \ldots, 1)$ to the isospectral Hilbert scheme.

Each flag of ideal sheaves (5.5) determines a collection of nested zero-dimensional subschemes $Z_0 \subset Z_1 \subset \cdots \subset Z_{r-1}$ where $Z_a$ has length $a + 1$ for $0 \leq a \leq r - 1$. Hence there are exact sequences of sheaves on $\mathbb{C}^2$

$$0 \to \mathcal{O}_{p_{a+1}} \to \mathcal{O}_{Z_{a+1}} \to \mathcal{O}_{Z_a} \to 0$$  \hspace{1cm} (5.7)$$

for all $0 \leq a \leq r - 2$, where $p_a$ are closed points in $\mathbb{C}^2$. Moreover, $Z_0$ must also be a closed point $p_0$ since it has length 1. This determines a morphism $\mathcal{N}(1, \ldots, 1) \to (\mathbb{C}^2)^r$. Using also the natural projection onto the Hilbert scheme $\mathcal{H}^r$, one obtains a commutative diagram

$$\begin{align*}
\mathcal{N}(1, \ldots, 1) & \quad \xrightarrow{\rho} \quad \mathcal{H}^r \quad \xrightarrow{\pi} \quad (\mathbb{C}^2)^r \\
\mathcal{H}^r & \quad \xrightarrow{\pi} \quad S^r(\mathbb{C}^2)
\end{align*}$$  \hspace{1cm} (5.8)$$
where the square is Cartesian and the bottom horizontal arrow is the Hilbert-Chow morphism. The upper left corner of the square is by definition the isospectral Hilbert scheme \( \tilde{H}^r \). The above diagram determines a morphism \( \rho : N(1, \ldots, 1) \to \tilde{H}^r \) such that \( \eta = \pi \circ \rho \). Since both \( \pi \) and \( \eta \) are surjective and proper, so is \( \rho \). Moreover, according to Appendix C, \( N(1, \ldots, 1) \) is reduced, hence \( \rho \) factors through a morphism \( \rho_{\text{red}} : N(1, \ldots, 1) \to \tilde{H}_{\text{red}}^r \), which is surjective and proper as well.

The next step is to show that

\[
\rho_{\text{red}}* O_{N(1, \ldots, 1)} \simeq O_{\tilde{H}_{\text{red}}^r}.
\]  

(5.9)

The proof will rely on [28, Prop. 3.3.2] and [28, Thm. 3.1], which prove that the reduced scheme \( \tilde{H}_{\text{red}}^r \) is irreducible and normal. Then, by Stein factorization, \( \rho_{\text{red}} \) factors as

\[
N(\gamma) \xrightarrow{f} \tilde{H}' \xrightarrow{g} \tilde{H}_{\text{red}}^r,
\]

where \( \tilde{H}' = \text{Spec}_{\tilde{H}_{\text{red}}^r} \rho_{\text{red}}* O_{N(\gamma)} \) and \( g \) is a finite morphism. In particular

\[
f_* O_{N(1, \ldots, 1)} \simeq O_{\tilde{H}'}.
\]  

(5.10)

Since \( N(1, \ldots, 1) \) is reduced according to Appendix C, \( \tilde{H}' \) is reduced as well.

Next note that there is an open subset \( \tilde{U} \subset \tilde{H}^r \) such that the restriction of \( \rho \) to \( \rho^{-1}(\tilde{U}) \) is an isomorphism onto to \( \tilde{U} \). This open subset is the inverse image \( \pi^{-1}(U) \), where \( U \subset H^r \) is the open subset parameterizing subschemes \( Z \subset \mathbb{C}^2 \) consisting of \( r \) distinct points in \( \mathbb{C}^2 \).

By construction, the restriction of \( g \) to \( g^{-1}(\tilde{U}) \) is an isomorphism as well.

In order to conclude the proof of (5.9), it suffices to show that \( \rho_{\text{red}} \) has connected fibers. This implies that \( g \) is one-to-one on closed points, which further implies that \( \tilde{H}' \) is irreducible since \( \tilde{H}_{\text{red}}^r \) is irreducible. Therefore \( \tilde{H}' \) is reduced and irreducible. Since \( g \) is an isomorphism over the open subset \( g^{-1}(\tilde{U}) \), it follows that \( g \) is birational, hence an isomorphism. Therefore (5.9) follows from (5.10).

To prove that the fibers of \( \rho_{\text{red}} \) are connected, let \((p_0, \ldots, p_{r-1}; Z_{r-1})\) be a closed point of \( \tilde{H}_{\text{red}}^r \), with \((p_0, \ldots, p_{r-1}) \in (\mathbb{C}^2)^r \), and \( Z_{r-1} \subset \mathbb{C}^2 \) a closed zero-dimensional subscheme of length \( r \). In particular \((p_0, \ldots, p_{r-1}) \in (\mathbb{C}^2)^r \) and \( Z_{r-1} \) are mapped to the same point in \( S^r(\mathbb{C}^2) \) in diagram (5.8). The fiber \( N(p_0, \ldots, p_{r-1}; Z_{r-1}) = \rho_{\text{red}}^{-1}(p_0, \ldots, p_{r-1}; Z_{r-1}) \) parametrizes collections of length \( a+1 \) zero-dimensional subschemes \( Z_a \subset \mathbb{C}^2 \), \( 0 \leq a \leq r-2 \), such that their structure sheaves fit in exact sequences of the form (5.7). The inductive argument given below shows that this fiber is connected.

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Since $Z_{r-1}$ and $p_{r-1}$ are fixed, the moduli space of subschemes $Z_{r-2}$ such that $\mathcal{O}_{Z_{r-2}}$ fits in an exact sequence

$$0 \to \mathcal{O}_{p_{r-1}} \to \mathcal{O}_{Z_{r-1}} \to \mathcal{O}_{Z_{r-2}} \to 0$$

is isomorphic to the projective space $\mathbb{P}\text{Hom}(\mathcal{O}_{p_{r-1}}, \mathcal{O}_{Z_{r-1}})$. Therefore there is a natural projection

$$\pi_{r-1} : N(p_0, \ldots, p_{r-1}; Z_{r-1}) \to \mathbb{P}\text{Hom}(\mathcal{O}_{p_{r-1}}, \mathcal{O}_{Z_{r-1}}).$$

The fiber of $\pi_{r-1}$ over a point parameterized by a subscheme $Z_{r-2}$ is isomorphic to $N(p_0, \ldots, p_{r-2}; Z_{r-2})$. If all fibers $N(p_1, \ldots, p_{r-2}; Z_{r-2})$ are connected, it follows that the total space $N(p_0, \ldots, p_{r-1}; Z_{r-1})$ is also connected. Therefore it suffices to prove connectedness for $r = 2$. In that case the moduli space $N(p_0, p_1; Z_1)$ is a single point for any choice of $(p_0, p_1; Z_1)$ as above, hence the claim follows.

Now recall that the projection $\pi_{\text{red}} : \tilde{\mathcal{H}}_{\text{red}} \to \mathcal{H}_r$ is flat, according to [28, Thm 3.1], hence the pushforward $\pi_{\text{red}}_* \mathcal{O}_{\tilde{\mathcal{H}}_{\text{red}}}$ is a rank $r!$ vector bundle $\mathcal{P}$ on the Hilbert scheme. Equation (5.9) implies that

$$\eta_* \mathcal{O}_{N(1, \ldots, 1)} \simeq \mathcal{P}. \quad (5.11)$$

Next consider the case of arbitrary discrete invariants $\gamma = (\gamma_i)_{0 \leq i \leq \ell}$. Let $S_\gamma = S_{\gamma_\ell} \times \cdots \times S_{\gamma_0} \subset S_r$ be the stablizer of the ordered partition $\gamma$. The group action $S_r \times \tilde{\mathcal{H}}^r \to \tilde{\mathcal{H}}^r$ yields by restriction an action of $S_\gamma \times \tilde{\mathcal{H}}^r \to \tilde{\mathcal{H}}^r$. Let $\tilde{\mathcal{H}}^\gamma$ denote the quotient of $\tilde{\mathcal{H}}^r$ by $S_\gamma$, which is a quasi-projective scheme. Since $\tilde{\mathcal{H}}_{\text{red}}^r$ is normal and reduced and irreducible, so is $\tilde{\mathcal{H}}_{\text{red}}^\gamma$. Similarly, the quotient $S^\gamma(\mathbb{C}^2) = (\mathbb{C}^2)^r/S_\gamma$ is a quasi-projective variety and there is a commutative diagram

$$\begin{array}{ccc}
\tilde{\mathcal{H}} & \to & (\mathbb{C}^2)^r \\
\downarrow \kappa & & \downarrow \\
\tilde{\mathcal{H}}^\gamma & \to & S^\gamma(\mathbb{C}^2)
\end{array} \quad (5.12)
$$

where both squares are Cartesian.

Next note that there is a Hilbert-Chow morphism $\mathcal{N}(\gamma) \to S^\gamma(\mathbb{C}^2)$ defined as follows. Given a flag of zero dimensional subschemes

$$Z_0 \subset \cdots \subset Z_\ell \subset \mathbb{C}^2 \quad (5.13)$$

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there are exact sequences
\[ 0 \to K_i \to \mathcal{O}_{Z_i} \to \mathcal{O}_{Z_{i-1}} \to 0 \]
with \( K_i \) a zero dimensional sheaf on \( \mathbb{C}^2 \), for \( 1 \leq i \leq \ell \). The morphism \( \mathcal{N}(\gamma) \to S^\gamma(\mathbb{C}^2) \) sends a flag of subschemes of the form (5.13) to the cycle classes associated to the zero dimensional sheaves \( (\mathcal{O}_{Z_0}, K_1, \ldots, K_\ell) \) via the Hilbert-Chow morphism. Then the bottom Cartesian square in (5.12) yields a morphism \( \rho^\gamma : \mathcal{N}(\gamma) \to \tilde{\mathcal{H}}^\gamma \) which factors through a morphism \( \rho^\gamma_{\text{red}} : \mathcal{N}(\gamma) \to \tilde{\mathcal{H}}^\gamma_{\text{red}} \). The following generalization of (5.9) will be proven below:

\[ \rho^\gamma_{\text{red}} \mathcal{O}_{\mathcal{N}(\gamma)} = \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}}. \tag{5.14} \]

Let \( U^\gamma = (\pi^\gamma)^{-1} U \) be the inverse image of the open subset parameterizing subschemes of \( \mathbb{C}^2 \) supported at \( r \) distinct closed points. Then \( \eta^\gamma \) is an isomorphism over the open subset \( (\eta^\gamma)^{-1} U^\gamma \). Then equation (5.14) follows from the Zariski Main Theorem provided one can prove that \( \mathcal{N}(\gamma) \) is connected. This is shown in Appendix C.

For future reference, note that by construction the bundle \( \mathcal{P} = \pi_{\text{red}}^\gamma \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}} \) is equipped with a fiberwise action of the symmetric group \( S_r \) such that its fiber over any point \([I] \in \mathcal{H}^r\) is isomorphic to the regular representation. By construction,

\[ \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}} \simeq (\kappa_{\text{red}} \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}})^S \gamma \]

where \( \kappa_{\text{red}} : \tilde{\mathcal{H}}^\gamma_{\text{red}} \to \tilde{\mathcal{H}}^\gamma_{\text{red}} \) is the morphism of reduced schemes determined by \( \kappa \) in diagram (5.12). Pushing forward this identity to \( \mathcal{H}^r \) via \( \pi_{\text{red}}^\gamma \), one learns that

\[ \pi_{\text{red}}^\gamma \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}} \simeq (\pi_{\text{red}}^\gamma \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}})^S \gamma = \mathcal{P}^S \gamma. \tag{5.15} \]

In particular, since \( \mathcal{P} \) is locally free, so is \( \mathcal{P}^\gamma = \pi_{\text{red}}^\gamma \mathcal{O}_{\tilde{\mathcal{H}}^\gamma_{\text{red}}} \). Moreover \( \mathcal{P}^\gamma \) is equipped with a fiberwise action of \( S_r \) such that its fiber at any closed point \([I] \in \mathcal{H}^r\) is isomorphic to the permutation representation \( M_\gamma \) of \( S_r \) with stabilizer \( S_\gamma = \prod_{i=0}^{\ell} S_{\gamma_i} \subset S_r \).

### 5.5 Nested partition function and Macdonald polynomials

This section concludes the computation of the partition function (5.6) using the results of [28] and the previous subsection.

As a preliminary remark, note that \( \chi_T^{{\mathcal{Y}}}(\nu_{\mu,p}) \) on the Hilbert scheme \( \mathcal{H}^r \) can be easily computed by a fixed point theorem. The fixed points of the \( T \)-action on \( \mathcal{H}^r \) are monomial ideals \([I_\mu] \in \mathcal{H}^r\) in one-to-one correspondence with partitions \( \mu \) of \( r \). For any equivariant
bundle $F$ on $H^*$, let $F_{\mu}$ denote the fiber of $F$ at $[\mu]$. An exception will be made for the cotangent bundle $T^*H^*$, in which case the fiber at $[\mu]$ will be denoted by $T^*_\mu H^*$. Then equivariant localization yields

$$\chi^T_{\tilde{g}}(V_{g,p}) = \sum_\mu \Omega^g_\mu(q_1, q_2, \tilde{y}),$$

(5.16)

where

$$\Omega^g_\mu(q_1, q_2, \tilde{y}) = \frac{\text{ch}_T(\det V_{g-1+p}) \text{ch}_T A_{\tilde{g}}(T^*_\mu H^* \otimes g)}{\text{ch}_T A_{-1}(T^*_\mu H^*)}.$$ (5.17)

Then the main formula proven in this section reads

$$Z^{(r)}(q_1, q_2, \tilde{y}, \tilde{x}) = \sum_\mu \Omega^g_\mu(q_1, q_2, \tilde{y}) \tilde{H}_\mu(q_2, q_1, \tilde{x})$$

(5.18)

where $\tilde{H}_\mu(q_2, q_1, \tilde{x})$ are the modified MacDonald polynomials.

First note that the pushforward formulas (5.14), (5.15) are valid in $T$-equivariant setting, hence one obtains an identity

$$\chi^T_{\tilde{g}}(N(\gamma(m)), \eta^{\gamma*} V_{g,p}) = \chi^T_{\tilde{g}}(H^*, (P^{S_{\gamma}} \otimes_{H^*} V_{g,p})).$$ (5.19)

The right hand side of equation (5.19) can be evaluated again by equivariant localization:

$$\chi^T_{\tilde{g}}(H^*, (P^{S_{\gamma}} \otimes_{H^*} V_{g,p})) = \sum_\mu \Omega^g_\mu(q_1, q_2, \tilde{y}) \text{ch}_T(P_{\mu}^\gamma).$$ (5.20)

Now let $\tilde{\gamma}$ denote the unordered partition of $r$ determined by the sequence $\gamma = (\gamma_0, \ldots, \gamma_\ell)$. Then following formula

$$\text{ch}_T(P_{\mu}^\gamma) = \sum_\lambda K_{\lambda, \tilde{\gamma}} \tilde{K}_{\lambda, \mu}(q_1, q_2).$$ (5.21)

will be proven bellow, where the sum is over all partitions $\lambda$ of $r$, $K_{\lambda, \tilde{\gamma}}$ are the Kostka numbers, and $\tilde{K}_{\lambda, \mu}(q_2, q_1)$ are the modified Kostka–MacDonald coefficients.

Since the fiberwise $S_r$-action on $P$ is compatible with the $T$-equivariant structure, there is a direct sum decomposition

$$P_{\mu} \simeq \bigoplus_\lambda V_{\mu, \lambda} \otimes R_{\lambda}$$ (5.22)

$R_{\lambda}$ is the irreducible $S_r$-representation labelled by the partition $\lambda$ and $V_{\mu, \lambda}$ are finite dimensional representations of $T$. According to [28, Thm. 3.1, Prop. 3.7.3, Thm. 3.2], the $T$-character of $V_{\lambda, \mu}$ is given by the modified Kostka-MacDonald coefficients,

$$\text{ch}_T V_{\lambda, \mu} = \tilde{K}_{\lambda, \mu}(q_2, q_1).$$ (5.23)
The pushforward formula (5.15) shows that the fiber $P_{\mu}$ is the $S_\gamma$-fixed subspace of $P_\mu$. This yields
\[
\text{ch}_T P_{\mu}^\gamma = \frac{1}{|S_\gamma|} \sum_{g=(g_0,\ldots,g_\ell) \in S_\gamma} \sum_{\lambda} \chi_{R_\lambda} (g) \text{ch}_T V_{\mu,\lambda}.
\] (5.24)

Now recall the branching rule for representations of the symmetric group. Given a subgroup $S_{r_1} \times S_{r_2} \subset S_r$, with $r_1 + r_2 = r$, the irreducible $S_r$-representation $R_\lambda$ has a direct sum decomposition
\[
R_\lambda \simeq \bigoplus_{\nu_1,\nu_2} N_{\nu_1,\nu_2,\lambda} (R_{\nu_1} \boxtimes R_{\nu_2})
\] (5.25)
where $\nu_1, \nu_2$ are partitions of $r_1, r_2$ respectively, and $N_{\nu_1,\nu_2,\lambda}$ are the Littlewood-Richardson coefficients. Applying the rule (5.25) recursively one finds
\[
R_\lambda \simeq \bigoplus_{\nu_0,\ldots,\nu_\ell} N_{\nu_0,\ldots,\nu_\ell,\lambda} (R_{\nu_0} \boxtimes \cdots \boxtimes R_{\nu_\ell})
\] (5.26)
where $\nu_\ell$ is a partition of $\gamma_\ell$ for $0 \leq \ell \leq \ell$. Substitution in (5.24) yields
\[
\text{ch}_T P_{\mu}^\gamma = \sum_{\lambda} \sum_{\nu_0,\ldots,\nu_\ell} N_{\nu_0,\ldots,\nu_\ell,\lambda} \prod_{\ell=0}^\ell \left( \frac{1}{|S_{\gamma_\ell}|} \sum_{g_i \in S_{\gamma_\ell}} \chi_{R_{\nu_\ell}} (g_\ell) \right) \text{ch}_T V_{\mu,\lambda}.
\] (5.27)

Next note that
\[
\frac{1}{|S_{\gamma_\ell}|} \sum_{g_i \in S_{\gamma_\ell}} \chi_{R_{\nu_\ell}} (g_\ell) = \dim R_{\nu_\ell}^{S_{\gamma_\ell}}
\]
is the dimension of the $S_{\gamma_\ell}$-fixed subspace of $R_{\nu_\ell}$. Since $R_{\nu_\ell}$ is an irreducible $S_{\gamma_\ell}$-representation,
\[
\dim R_{\nu_\ell}^{S_{\gamma_\ell}} = 0
\]
unless $R_{\nu_\ell}$ is the trivial representation corresponding to the length one partition $\nu_\ell = (\gamma_\ell)$. In the latter case,
\[
\dim R_{(\gamma_\ell)}^{S_{\gamma_\ell}} = 1.
\]
Then equation (5.27) reduces to
\[
\text{ch}_T P_{\mu}^\gamma = \sum_{\lambda} N_{(\gamma_0),\ldots,(\gamma_\ell),\lambda} \text{ch}_T V_{\mu,\lambda}.
\]

formula (5.21) Using equation (5.23), formula (5.21) follows from the identity
\[
N_{(\gamma_0),\ldots,(\gamma_\ell),\lambda} = K_{\lambda,\gamma},
\] (5.28)

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The latter is proven in [22, Appendix 9]. More precisely, as shown in loc. cit., the Littlewood-Richardson coefficients occur in the decomposition of the product of two Schur functions:

\[ s_{\nu_1}(x)s_{\nu_2}(x) = \sum_{\lambda} N_{\nu_1,\nu_2,\lambda} s_{\lambda}(x). \]  

(5.29)

Furthermore, applying formula (5.29) recursively as in [22, Eqn. (A.9) pp. 456] yields

\[ s_{(r_1)}(x) \cdots s_{(r_k)}(x) = \sum_{\lambda} K_{\lambda,\rho} s_{\lambda}(x) \]  

(5.30)

where \( \rho \) is the partition of \( r \) determined by \( (r_1, \ldots, r_k) \), and \( K_{\lambda,\rho} \) are the Kostka numbers. This implies identity (5.28).

Using equations (5.20), (5.21) the contribution of a fixed point \([I_\mu]\) to the partition function (5.6) reduces to:

\[ \Omega^{(g,p)}(q_1, q_2, \tilde{y}) \sum_{\lambda} K_{\lambda,\mu}(q_2, q_1)K_{\lambda,\nu}m_{\nu}(\tilde{x}) = \]

\[ \Omega^{(g,p)}(q_1, q_2, \tilde{y}) \sum_{\lambda} K_{\lambda,\mu}(q_2, q_1)s_{\lambda}(\tilde{x}) = \Omega^{(g,p)}(q_1, q_2, \tilde{y})\tilde{H}_\mu(q_2, q_1; \tilde{x}). \]

where \( m_{\nu}(\tilde{x}) \) are the monomial symmetric functions, and \( s_{\lambda}(\tilde{x}) \) the Schur functions. This concludes the proof of equation (5.18).

6 BPS expansion and a parabolic \( P = W \) conjecture

Collecting the results of the previous two sections, here it is shown that geometric engineering yields a conjectural expression for the refined stable pair partition function (1.4), which agrees with the left hand side of the HLRV formula (1.3) by a change of variables. Furthermore, it will be checked that the same change of variables relates the right hand side of equation (1.3) with a refined Gopakumar-Vafa expansion, completing the physical derivation of the HLRV formula.

As in Section 5.4 it will be assumed that there is a single marked point on \( C \). The root stack \( \tilde{C} \) be the root stack has stabilizer \( \mu_s \) at the unique orbifold point, for some \( s \geq 1 \). The local threefold \( \tilde{Y}_M \) is the total space of the rank two bundle \( \nu^*M^{-1} \oplus K_{\tilde{C}} \otimes \nu^*M \) on the root stack \( \tilde{C} \), with \( M \) a degree \( p \) line bundle on \( C \). Let \( Z^{\text{ref}}_{Y_M}(q, x, y) \) be the refined stable pair partition function of \( Y \), where \( x = (x_0, \ldots, x_{s-1}, 0, \ldots) \) are the formal counting variables associated to the marked point. Physically, these are chemical potentials for twisted sector
Ramond-Ramond charges at the orbifold point. By analogy with [12, 11], the geometric engineering conjecture reads:

\[ Z_{\overline{\psi}}^{ref}(q, x, y) = 1 + \sum_{r \geq 1} Z_{K}^{(r)}(qy^{-1}, q^{-1}y^{-1}, y, (-1)^{(g-1)p})y^{-g}x). \]  

(6.1)

The terms in the right hand side are given by (5.18). Equation (5.17) yields

\[ \Omega_{\mu}^{g,p}(q_1, q_2, \overline{y}) = \prod_{\square \in \mu} (q_1^{l(\square)}q_2^{a(\square)})^{g-1+p}(1 - \overline{y}q_1^{l(\square)}q_2^{a(\square)+1})^{g}(1 - \overline{y}q_1^{l(\square)+1}q_2^{a(\square)})^{g} \]

where \( a(\square), l(\square) \) are the arm and leg length of a box \( \square \in \mu \). Making the change of variables in equation (6.1) yields

\[ Z_{\overline{\psi}}^{ref}(q, y, x) = 1 + \sum_{\mu \neq \emptyset} Z_{\mu}^{g,p}(q, y)\tilde{\mathcal{H}}_{\mu}(qy^{-1}, q^{-1}y^{-1}, x), \]  

(6.2)

where

\[ Z_{\mu}^{g,p}(q, y) = (-1)^{p|\mu|} \prod_{\square \in \mu} (q^{l(\square)-a(\square)}y^{1-h(\square)})^{p}(qy^{-1})^{(2l(\square)+1)(g-1)}(1 - y^{l(\square)-a(\square)}q^{-h(\square)})^{2g} \]

(6.3)

(1 - y^{l(\square)-a(\square)-1}q^{-h(\square)}) \]

with \( h(\square) = a(\square) + l(\square) + 1 \), and the sum is over all Young diagrams \( \mu \). This formula can be also written as

\[ Z_{\mu}^{g,p}(q, y) = (-1)^{p|\mu|} \prod_{\square \in \mu} (q^{l(\square)-a(\square)}y^{1-h(\square)})^{p}(qy^{-1})^{(2a(\square)+1)(g-1)}(1 - y^{a(\square)-l(\square)}q^{h(\square)})^{2g} \]

(6.3)

\[ \prod_{\square \in \mu} (1 - y^{a(\square)-l(\square)-1}q^{h(\square)}) \]

A further change of variables yields

\[ Z_{\overline{\psi}}^{ref}(z^{-1}w, z^{-1}w^{-1}, x) = 1 + \sum_{\mu \neq \emptyset} \mathcal{H}_{\mu}^{g,p}(z, w)\tilde{\mathcal{H}}_{\mu}(z^{2}, w^{2}, x), \]  

(6.4)

where

\[ \mathcal{H}_{\mu}^{g,p}(z, w) = \prod_{\square \in \mu} \left( z^{2a(\square)}w^{2l(\square)} \right)^{p} \left( z^{2a(\square)+1}w^{2l(\square)+1} \right)^{2g} \]

(6.4)

\[ \prod_{\square \in \mu} \left( z^{2a(\square)} - w^{2l(\square)} \right) \]

For \( p = 0 \) this is the left hand side of the HLRV formula evaluated at formal variables \( x = (x_0, \ldots, x_{s-1}, 0, 0, \ldots) \).

For the remaining part of this section, let \( Y := \overline{Y}_{OC} \) be the product \( \mathbb{A}^{4} \times \widetilde{S} \), with \( \widetilde{S} = \text{tot}(K_{\overline{Y}}) \). Then it easy to check that any moduli stack of compactly supported Bridgeland stable pure dimension one sheaves on \( \overline{Y} \) with fixed numerical class is isomorphic to a product
\[ A^1 \times \mathcal{M}_\beta(\bar{S}, \gamma), \] where \( \mathcal{M} \) is a moduli stack of \( \beta \)-stable pure dimension one sheaves on \( \bar{S} \) with fixed numerical equivalence class \( \gamma \). The notation used here is the same as in Sections 3.2, equation (3.21), and 3.3.4. Since in this particular case there is a single marked point, and the eigenvalues \( \lambda \) are trivial, \( \gamma \) will be labelled by integers \( d^l \geq 1, 1 \leq l \leq s, \) and \( n \in \mathbb{Z} \). According to Section 3.1, the moduli stack \( \mathcal{M}_\beta(\bar{S}, \gamma) \) is isomorphic to a moduli stack of stable strongly parabolic Higgs bundles on \( C \).

From a string theoretic perspective this chain of isomorphisms identifies parabolic Higgs bundles on \( C \) with supersymmetric D2-D0 configurations on the Calabi-Yau threefold \( \bar{Y} \). Then the HLRV formula is identified with a refined Gopakumar-Vafa expansion \([25, 38, 35, 10]\) provided that one assumes a parabolic variant of the \( P = W \) conjecture \([14]\). Some details are provided below for completeness.

For a precise formulation of the parabolic \( P = W \) conjecture, consider a smooth projective curve \( C \) with two marked points \( p, \infty \in C \) and let \( \gamma_p, \gamma_\infty \in \pi_1(C \setminus \{p, \infty\}) \) be the generators associated to the marked points. Let \( (r, e) \in \mathbb{Z}_{>0} \times \mathbb{Z} \) be coprime integers and let \( C_\lambda \) denote the \( GL(r, \mathbb{C}) \) conjugacy class of a diagonal matrix with (ordered) eigenvalues

\[ \lambda = (\lambda_1, \ldots, \lambda_r). \]

Let also \( \mu = (\mu^1, \ldots, \mu^l) \) denote the partition of \( r \) determined by the multiplicities of the above eigenvalues.

Now let \( C_\lambda(C, p, \infty) \) be the character variety with conjugacy classes \( C_\lambda; \exp(2e^{\pi \sqrt{-1}/r}) \) at the marked points \( p, \infty \). According to \([29, \text{Thm. 2.1.5}]\), for sufficiently generic \( \lambda \), \( C_\lambda(C, p, \infty) \) is either empty or a smooth quasi-projective variety of complex dimension

\[ d_\mu = r^2(2g - 2 + 1) - \sum_{j=1}^{l} (\mu^j)^2 + 2. \] (6.5)

Note that \( d_\mu \) is even; using the identity \( r = \sum_{j=1}^{l} \mu^j \),

\[ d_\mu = 2b_\mu, \quad b_\mu = r^2(g - 1) + 1 + \sum_{1 \leq j_1, j_2 \leq l, j_1 < j_2} \mu^{j_1} \mu^{j_2}. \] (6.6)

Since the marked curve \((C, p, \infty)\) is fixed throughout this section, the character variety will be denoted simply by \( C_\lambda^{\circ} \) in the following.

Next consider the specialization of the HLRV formula (1.3) to the present case taking \( x_\infty = (x_\infty, 0, 0, \ldots) \). Since \( \mu_\infty \) is the length one partition \((r)\), the variable \( x_{\infty,0} \) can be scaled
off by a redefinition of the formal variable $x$ associated to $p$. Moreover, as observed in [29], the mixed Poincaré polynomial $P_c(C_e; u, t)$ depends only on $\mu$ as long as $e$ is coprime with $r$. Therefore equation (1.3) yields a formula of the form

$$\sum_{\mu} h_{\mu}^{q,0}(z, w) \tilde{H}_{\mu}(z^2, w^2, x) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} w^{-kd_{\mu}} P_{c,\mu}(z^{-2k}, -(zw)^k) \right)$$

(6.7)

where $P_{c,\mu}(u, t) = P_c(C_e; u, t)$.

By analogy with [14], the parabolic $P = W$ conjecture identifies the weight filtration $W^*H_{cpt}(C_e^r)$ with the perverse sheaf filtration on the compactly supported cohomology of a moduli space of stable strongly parabolic Higgs bundles. As a first step, note that Conjecture 1.2.1(ii) in [29] yields the identifications

$$W_{2p}H_{cpt}(C_e^r) = W_{2p+1}H_{cpt}(C_e^r)$$

for all $p$, just as in the unmarked case studied in [30].

Next, let $H^e_m$ denote the moduli space of rank $r \geq 1$, degree $e$ stable parabolic Higgs bundles $(E, \Phi)$ on the marked curve $(C, p)$ with parabolic structure of type $m$ at $p$. The Higgs field $\Phi : E \to E \otimes K_C(p)$ has nilpotent residue at $p$ with respect to the. Let $\mu$ be the partition of $r$ determined by $m$. For primitive discrete invariants $(m, e)$ and sufficiently generic parabolic weights there are no strictly semistable objects, and the moduli space is a smooth quasi-projective variety of dimension $d_{\mu}$. Furthermore, $H^e_m$ is diffeomorphic in this case with the character variety $C^e_m$ provided the eigenvalues $\lambda_i$ are related to the parabolic weights by $\lambda_i = e^{2i\pi \alpha_i}$, $1 \leq i \leq r$. There is also a Hitchin map

$$h : H^e_m \to B_m$$

with $B_m \subset \bigoplus_{i=1}^{r} H^0(K_C(p) \otimes i)$ a linear subspace of dimension $b_{\mu}$. The generic fibers of $h$ are smooth abelian varieties of dimension $b_{\mu}$, and the total space $H^e_m$ is an algebraically complete integrable system. By analogy with [15, 14], this yields a perverse sheaf filtration $P^*H(H^e_m)$. The parabolic $P = W$ conjecture states that

$$W_{2p}H(C_e^r) = P_pH(H^e_m)$$

(6.8)

for all values of $p$.

Equation (6.8) leads to an identification of the HLRV formula (1.3) with a refined BPS expansion in close analogy with [11, Sect. 4]. Very briefly, using the methods in [15] one
can prove a hard Lefschetz theorem for the parabolic Hitchin map and also choose a (non-canonical) splitting of the perverse sheaf filtration as in [14, Sect. 1.4.2, 1.4.3]. This yields an $SL(2, \mathbb{C}) \times \mathbb{C}^\times$ action on the cohomology $H(\mathcal{H}_p^e)$, which splits as a direct sum

$$H(\mathcal{H}_p^e) \simeq \oplus_{p=0}^{b_p} R_{p-\dim(Q^p, 0)}^{\text{dim}(Q^p, 0)}$$

(6.9)

where $R_{j_L}$ is the irreducible $SL(2, \mathbb{C})$-representation of spin $j_L \in \mathbb{1/2}\mathbb{Z}$. In the above formula $p$ is the perverse degree and $Q^p, 0$ the primitive cohomology of perverse degree $p$. The cohomological degree is encoded in the $\mathbb{C}^\times$ action, which scales the quotient $Gr^F_p H^k(\mathcal{H}_p^e)$ with weight $l = k - p - b_p$. Then specializing $x$ to $x = (x_0, \ldots, x_{s-1}, 0, 0, \ldots)$ and making the same change of variables

$$(z, w) = ((qy)^{-1/2}, (qy^{-1})^{1/2})$$

as in equation (6.4) converts equation (6.7) into a refined Gopakumar-Vafa expansion. This computation is completely analogous with [11, Sect. 4], hence the details are omitted.

### 7 Recursion via wallcrossing

The recursion relation conjectured in [12] for the Poincaré polynomial of the moduli space of Hitchin pairs admits a natural generalization to parabolic Higgs bundles. The derivation of this formula is completely analogous to loc. cit. assuming again all the foundational aspects of motivic Donaldson-Thomas theory [42]. The final result will be recorded below, omitting most intermediary steps.

For simplicity it will be assumed again that the curve $C$ has only one marked point $p$. To fix notation, the discrete invariants of a parabolic rank bundle $E^\bullet$ on $C$ are the degree $e \in \mathbb{Z}$ and the flag type $m = (m_a)_{0 \leq a \leq s-1} \in (\mathbb{Z}_{\geq 0})^{s \times s}$. Let

$$|m| = \sum_{a=0}^{s-1} m_a, \quad \chi(m, e) = e - |m|(g - 1).$$

For any weights $\alpha = (\alpha_a)_{0 \leq a \leq s-1}$ let

$$m \cdot \alpha = \sum_{a=0}^{s-1} m_a \alpha_a.$$

The parabolic slope and the parabolic $\delta$-slope are defined respectively by

$$\mu(m, e, \alpha) = \frac{\chi(m, e) + m \cdot \alpha}{|m|}, \quad \mu_\delta(m, e, \alpha) = \frac{\chi(m, e) + m \cdot \alpha + \delta}{|m|},$$

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and the ordinary slopes are given by
\[ \mu_\delta(m, e) = \frac{\chi(m, e) + \delta}{|m|}, \quad \mu(m, e) = \frac{\chi(m, e)}{|m|}. \]

### 7.1 Generic parabolic weights

The recursion formula will be derived from wallcrossing with respect to variations of the stability parameter \( \delta \) introduced in Section 4.1. The refined parabolic ADHM invariants will be denoted by \( A_\delta(m, e, \alpha; y) \) while the refined parabolic Higgs bundle invariants by \( H(m, e, \alpha; y) \). Note that \( H(m, e, \alpha; y) \in \mathbb{Q}(y) \) are the rational refined invariants obtained directly from the motivic integration map in [42], not the integral refined invariants \( \mathcal{I}(m, e, \alpha; y) \in \mathbb{Z}[y, y^{-1}] \).

The relation between the two sets of invariants for sufficiently generic weights is given by the refined multicover formula
\[
H(m, e, \alpha; y) = \sum_{k \geq 1, (m, e) = k(m', e')} \frac{1}{k[k]_y} \mathcal{I}(m', e', \alpha; y^k).
\] (7.1)

For fixed numerical invariants and fixed parabolic weights, there are finitely many critical values \( \delta_c \in \mathbb{R} \), where strictly semistable objects can exist. Using the formalism of [42], the wallcrossing formula at such a critical value \( \delta_c \neq 0 \) is
\[
A_{\delta_c+}(m, e, \alpha; y) - A_{\delta_c-}(m, e, \alpha; y) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{\Delta_l(\delta_c, m, e, \alpha)} A_{\delta_c-}(m_1, e_1, \alpha; y) \prod_{i=2}^{l} [\chi(m_i, e_i)]_y H(m_i, e_i, \alpha; y).
\] (7.2)

where
\[
\Delta_l(\delta_c, m, e, \alpha) = \left\{ (m_1, \ldots, m_l), (e_1, \ldots, e_l) \mid m_i \in (\mathbb{Z}_{\geq 0})^r, e_i \in \mathbb{Z}, |m_i| > 0, 1 \leq i \leq l, (m_1, e_1) + \cdots + (m_l, e_l) = (m, e), \mu(m_1, e_1, \alpha) = \mu_{\delta_c}(m_1, e_1, \alpha), 2 \leq i \leq l \right\}
\]

and
\[
[n]_y = \frac{y^n - y^{-n}}{y - y^{-1}}
\]
for any integer \( n \in \mathbb{Z} \). This is the same wallcrossing formula as [12, Eqn. 1.3], except the Higgs invariants differ by a sign \((-1)^{\chi(m, e)}\) from the used in loc. cit. The present normalization is more natural in this context. There is a similar formula for \( \delta_c = 0 \), including an extra term with \( m_1 = 0 \) as in [12, Eqn. 1.4].
Applying equation (7.2) iteratively from $\delta > 0$ to $\delta < 0$, and using the duality relations (4.6), one obtains a wallcrossing formula of the form

$$\left[\chi(m, e)\right]yH(m, e; y) = A_{+\infty}(m, e; y) - A_{+\infty}(\tilde{m}, \tilde{e}; y)$$

$$+ \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\Delta^{(\times)}_{i}(m, e, \alpha)} A_{+\infty}(m_1, e_1; y) \prod_{i=2}^{l} [\chi(m_i, e_i)]yH(m_i, e_i, \alpha; y)$$

$$- \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\Delta^{(\times)}_{i}(m, e, \alpha)} A_{+\infty}(m_1, e_1; y) \prod_{i=2}^{l} [\chi(m_i, e_i)]yH(m_i, e_i, \alpha; y)$$

(7.3)

$$- \sum_{l \geq 2} \frac{1}{l!} \sum_{\Delta^{(\times)}_{i}(m, e, \alpha)} \prod_{i=1}^{l} [\chi(m_i, e_i)]yH(m_i, e_i, \alpha; y)$$

where

$$\Delta^{(\times)}_{i}(m, e, \alpha) =$$

$$\{(m_1, \ldots, m_l), (e_1, \ldots, e_l) \mid m_i \in (\mathbb{Z}_{\geq 0})^x, e_i \in \mathbb{Z}, |m_i| > 0, 1 \leq i \leq l, (m_1, e_1) + \cdots + (m_l, e_l) = (m, e), \mu(m_i, e_i, \alpha) \triangleq \mu(m, e, \alpha), 2 \leq i \leq l\}$$

(7.4)

the symbol $\triangle$ taking values $>, \geq, =$ respectively. Note that for any discrete invariants $n$ there exists a lower bound $d_0 \in \mathbb{Z}$ such that $A_{+\infty}(n, d; y) = 0$ for all $d < d_0$. This can be proven by standard bounding arguments, or, alternatively, it follows easily from the conjectural formula (6.2). Therefore the number of terms in the right hand side of equation (7.3) is finite and bounded above by a constant independent of the parabolic weights $\alpha$.

The recursion formula (7.3) together with the geometric engineering conjecture (6.1) completely determines the parabolic refined invariants $H(m, e, \alpha; y)$. Using the arguments employed by Mozgovoy in [49], it will be shown below that the resulting invariants are compatible with those determined by the HLRV formula (6.7).

For simplicity, consider local curves on type $(0, 2g - 2)$ in the following. Using the same notation as [49], the refined partition function (4.7) will be denoted by $A_{+\infty}(q, y, x)$. Hence

$$A_{+\infty}(q, y, x) = \sum_{m, e}[A_{+\infty}(m, e; y)q^{\chi(m, e)}x^m].$$

Since there is a single marked point, the formal variable $x = (x_0, x_1, \ldots)$ does not carry an extra index.
As shown in Section 6, geometric engineering predicts that $A_\infty(q, y, x)$ is determined by equation (6.1)

$$A_\infty(q, y, x) = 1 + \sum_{\mu \neq \emptyset} Z^{q,0}_\mu(q, y) \tilde{H}_\mu(qy^{-1}, qy^{-1}; x).$$

Following [49], let $\tilde{P}_m(q, y)$ be defined by the formula

$$A_\infty(q, y, x) = \exp \left[ \sum_{k \geq 1} \sum_m \frac{x^k m k f(q^k, y^k) \tilde{P}_m(q^k, y^k)}{k} \right]$$

where

$$f(q, y) = \frac{q}{(1 - qy)(y - q)}.$$  

Note that $\tilde{P}_m(q, y)$ is related to the mixed Poincaré polynomial of the character variety $C^\lambda_\chi$ defined in Section 6, where $\lambda$ is the partition of $r = |m|$ determined by $m$. Using the change of variables $(z^2, w^2) = (q^{-1}y^{-1}, qy^{-1})$ in equation (6.7), one obtains

$$\tilde{P}_m(q, y) = y^{b_\lambda + 2} q^{-b_\lambda} P_e(C^\lambda_\chi, qy, -y^{-1})$$

where $b_\lambda = d_\lambda / 2$ is half the complex dimension of the character variety.

Now let

$$\Omega(m, e; y) = y \tilde{H}(m, e, y; y)$$

for any discrete invariants $(m, e)$. Assuming that the invariants $\Omega(m, e; y)$ are independent of the degree $e \in \mathbb{Z}$ for any $m$, it will be shown below that

$$\Omega(m, e; y) = \tilde{P}_m(1, y)$$

for all $(m, e)$. The proof is entirely analogous to the proof of [49, Thm. 4.6], some details being presented below for completeness. Note that the assumption that $\Omega(m, e; y)$ are independent of $e$ is a standard conjecture [36] for Donaldson-Thomas invariants of pure dimension one sheaves on Calabi-Yau threefolds.

Following [49], for any series

$$I = \sum_{(m, e)} I(m, e) q^{\chi(m, e)} x^m \in \mathbb{Q}(y)[[q^{\pm 1}, x]]$$

and any $\mu \in \mathbb{R}$, let

$$I_{\omega \mu} = \sum_{(m, e, \omega) \omega \mu} I(m, e) q^{\chi(m, e)} x^m.$$
where $\diamond \in \{ =, \geq, >, \leq, < \}$. Furthermore, for any $\mu \in \mathbb{R}$ define
\[
C_{\mu}(q, y, x) = \exp \left[ \sum_{k \geq 1} \sum_{\mu(m, e, x) = \mu} \frac{x^{km}}{k(y^{2k} - 1)} \Omega(m, e; y^k)((qy)^{kx(m, e)} - (qy^{-1})^{kx(m, e)}) \right]
\]
and
\[
C_\diamond \mu(q, y, x) = \prod_{\eta \diamond \mu} C_\eta(q, y, x)
\]
Then the recursion relation (7.3) can be recast in the form
\[
C_{\mu}(q, y, x) = (\overline{A}_\infty(q, y, x)C_{>\mu}(q, y, x))_\mu - (\overline{A}_\infty(q^{-1}, y, x)C_{\geq -\mu}(q^{-1}, y, x))_{-\mu}
\]
by analogy with [49, Remark. 4.5], where
\[
\overline{A}_\infty(q, y, x) = A_\infty(q, y, x) - 1.
\]
In order to prove equation (7.6) it suffices to show that the statement of [49, Thm 4.7] holds in the present context. Namely, it suffices to prove the identity
\[
A_\infty(q, y, x)C_{>\mu}(q, y, x) = A_\infty(q^{-1}, y, x)C_{\geq -\mu}(q^{-1}, y, x)
\]
in $\mathbb{Q}(y)[[q^{\pm 1}, x]]$. The proof given in [49, Sect 5] is based on several essential facts.

First note that independence of $\Omega(m, e; y)$ of degree yields a factorization of the form
\[
\sum_{(m, e)} \Omega(m, e; y)q^{\chi(m, e)}(y^{\chi(m, e)} - y^{-\chi(m, e)})x^m =
\]
\[
\left( \sum_{m} \Omega(m; y)x^m \right) \sum_{n \in \mathbb{Z}} ((qy)^n - (qy^{-1})^n),
\]
where $\Omega(m; y)$ denotes the common value of $\Omega(m, e; y)$. Moreover, note that the function $f(q, y)$ defined below equation (7.5) satisfies
\[
f(q, y) = f(q^{-1}, y).
\]
These two facts imply that completely analogous statements to [49, Lemma 5.1], [49, Lemma 5.4] and [49, Prop. 5.5] hold in the present context.

The next important observation is that
\[
A_\infty(q, y, x) = A_\infty(q^{-1}, y, x).
\]
In the present context, this follows from equation (6.3), which shows that
\[ Z_{\gamma}^0(q^{-1}, y) = Z_{\gamma}^0(q, y), \]
and the standard property of MacDonald polynomials
\[ \tilde{H}_{\lambda}(t, s; x) = \tilde{H}_{\lambda'}(s, t; x), \]
which yields
\[ \tilde{H}_{\lambda}(qy^{-1}, q^{-1}y^{-1}; x) = \tilde{H}_{\lambda'}(q^{-1}y^{-1}, qy^{-1}; x). \]
Since \( f(q, y) \) is invariant under \( q \mapsto q^{-1} \), equation (7.9) implies that
\[ \tilde{P}_{m}(q, y) = \tilde{P}_{m}(q^{-1}, y) \]
which is analogous to [49, Lemma 5.7].

From this point on, the proof of identity (7.6) is identical with the proof of [49, Thm. 4.7] given in Section 5 of loc. cit.

### 7.2 Trivial weights

An alternative recursion formula may be derived along the same lines, working with trivial weights, \( \alpha_a = 0 \) for all \( 0 \leq a \leq r - 1 \), rather than generic weights. This is usually a very degenerate limit in the theory of parabolic Higgs bundles. However, the theory of motivic Donaldson-Thomas invariants [42] works equally well for trivial weights. Moreover, the wall-crossing formula of [42] shows that the refined Donaldson-Thomas invariants \( H(m, e, \alpha; y) \) are in fact independent of the weights, as long as the weights are sufficiently generic. In fact, if one is willing to grant the refined generalization of [36, Thm. 6.16], even more is expected to be true. That is, the integral refined invariants \( \overline{H}(m, e, \alpha; y) \) are expected to be independent of the weights \( \alpha \), for all possible values, including non-generic ones.

Moreover, note that for coprime numerical invariants \( (r, e) = 1 \), the moduli space of stable parabolic bundles with fixed flag type \( m \) is independent of the parabolic weights as long as they are sufficiently small, including non-generic values. This can be proven using a boundedness argument. Therefore the strong form of the conjecture in the previous paragraph holds at least for \( (|m|, e) \) coprime.

In the following we will simply write down the \( \alpha = 0 \) version of the recursion formula (7.3). As explained above, it will yield the same results for the integral refined invariants as
(7.3) at least for \((|m|, e)\) coprime. If one is willing to grant the strong form of the refined non-dependence conjecture, it will yield the same values even for non-coprime pairs \((|m|, e)\).

Setting \(a = 0\), equation (7.4) specializes to

\[
\Delta_l^{(0)}(m, e, 0) = \left\{ (m_1, \ldots, m_l), (e_1, \ldots, e_l) \mid m_i \in \mathbb{Z}_{\geq 0}^r, \ e_i \in \mathbb{Z}, \ |m_i| > 0, \ 1 \leq i \leq l, \ (7.10) \right. \\
(m_1, e_1) + \cdots + (m_l, e_l) = (m, e), \ \mu(|m|, e_i) \triangleq \mu(|m|, e), \ 2 \leq i \leq l \right\}
\]

with \(\triangle \in \{>, \geq, =\}\). Note that the slope inequalities in the right hand side of equation (7.10) depend only on \(|m|, |e|\), hence they are invariant under permutations of the entries of \(m_i, m\), \(1 \leq i \leq l\). Moreover, the conjectural formula (6.2), and the parabolic \(P = W\) conjecture in Section 6 imply that the invariants \(A_\infty(m, e; y), H(m, e; y)\) are also invariant under permutations of the entries of \(m\). Therefore they depend only on the partition \(\lambda\) of \(|m| = r\) determined by \(m\). Abusing notation, they will be denoted by \(A_\infty(\lambda, e; y), H(\lambda, e; y)\). Moreover, for each partition \(\lambda\) of \(r \geq 1\), let

\[
\Delta_l^{(0)}(\lambda, e) = \left\{ (\lambda_1, \ldots, \lambda_l), (e_1, \ldots, e_l) \mid \lambda_i \neq \emptyset, \ e_i \in \mathbb{Z}, \ 1 \leq i \leq l, \ (7.11) \right. \\
(\{|\lambda_1|, e_1\} + \cdots + (|\lambda_l|, e_l) = (|\lambda|, e), \ \mu(|\lambda_i|, e_i) \triangleq \mu(|\lambda|, e), \ 2 \leq i \leq l \right\}
\]

with \(\triangle \in \{>, \geq, =\}\). Then a straightforward computation shows that the zero weight specialization of the recursion formula (7.3) can be set in the form:

\[
[e - |\lambda|(g - 1)]yH(\lambda, e; y) = A_\infty(\lambda, e; y) - A_\infty(\lambda, \bar{e}; y)
\]

\[
+ \left[ \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\Delta_l^{(\geq)}(\lambda, e)} A_\infty(\lambda_1, e_1; y)m_{\lambda_1}(x) \prod_{i=2}^{l}[e_i - |\lambda_i|(g - 1)]yH(\lambda_i, e_i; y)m_{\lambda_i}(x) \right]
\]

\[
- \left[ \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{\Delta_l^{(\leq)}(\lambda, e)} A_\infty(\lambda_1, e_1; y)m_{\lambda_1}(x) \prod_{i=2}^{l}[e_i - |\lambda_i|(g - 1)]yH(\lambda_i, e_i; y)m_{\lambda_i}(x) \right]
\]

\[
- \sum_{l \geq 2} \frac{1}{l!} \sum_{\Delta_l^{(\leq)}(\lambda, e)} \prod_{i=1}^{l}[e_i - |\lambda_i|(g - 1)]yH(\lambda_i, e_i; y)m_{\lambda_i}(x) \right]_\lambda
\]

where \([f(x)]_\lambda\) is the coefficient of \(m_\lambda(x)\) in the expansion of the symmetric function \(f(x)\) in the monomial symmetric basis.

Proceeding as in the Section 7.1, it is straightforward to show that the solution to the recursion relation (7.12) is also in agreement with the predictions of the HLRV formula.
This confirms the weight independence conjecture stated at the beginning of the current subsection.

8 A conifold experiment

The goal of this section is to present numerical evidence for the geometric engineering conjecture (6.1) for refined parabolic invariants on a resolved conifold. Therefore the curve $C$ will be the projective line $\mathbb{P}^1$ and the line bundles $L_1, L_2$ will be isomorphic to $\mathcal{O}_C(-1)$. Choosing homogeneous coordinates $[z_0, z_1]$ on $C$, the marked point $p$ will be $z_1 = 0$. Parabolic refined invariants will be computed by virtual localization, using the equivariant $K$-theoretic index defined by Nekrasov and Okounkov in [52]. According to [48] this definition agrees under certain conditions with the motivic construction of Kontsevich and Soibelman [42]. The equivariant index has been also employed in [10] for similar computations of refined stable pair invariants of toric Calabi-Yau threefolds.

8.1 A parabolic conifold conjecture

Since the data $(C, L_1, L_2, p)$ will be fixed throughout this section, the moduli space of asymptotically stable parabolic ADHM sheaves with numerical invariants $(m, e)$ will be denoted by $\mathcal{P}M_\infty(m, e)$. As explained in Section 4.2, this moduli space is equipped with a symmetric perfect obstruction theory $E^\bullet$ and a natural $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ action given in equation (4.5). The symmetric obstruction theory is equivariant, but not equivariantly symmetric with respect to the $\mathbb{T}$-action. The the $\mathbb{T}$-fixed locus has been shown to be proper using the forgetful morphism (4.3). Since the curve $C$ is the projective line in this section, there is an enhanced $G = \mathbb{T} \times \mathbb{C}^*$ action on the moduli space where the action of the third factor $\mathbb{C}^*$ is induced by the scaling action on $C = \mathbb{P}^1$,

$$(s \times [z_0, z_1]) \mapsto [z_0, sz_1].$$

Using again the proper morphism (4.3) it can be easily shown that the $G$-fixed locus is finite. Again, the perfect obstruction theory is $G$-equivariant but not $G$-equivariantly symmetric. However using the deformation complex (4.4) it is straightforward to check that

$$E^\bullet = Z^{-1}Q_1Q_2(E^\bullet)^\vee[1],$$

where $(Q_1, Q_2, Z)$ denote the canonical generators of the representation ring of $G$. In particular $E^\bullet$ is equivariantly symmetric with respect to the action of the subtorus $G_0 = \{s^{-1}t_1t_2 = \}$
1 \subset G$. Note moreover, that in this example, one can easily prove that the moduli space of asymptotically stable parabolic ADHM sheaves is proper. The details are provided in Appendix D for completeness.

In this context, following [52], note that the virtual canonical bundle of the moduli space admits a square root $K^{1/2}$, which is equivariant with respect to the action of the double cover $\widetilde{G} \to G$ determined by the commutative diagram

\[
\begin{array}{ccc}
1 & \to & G_0 \\
\downarrow & & \downarrow \\
\to \tilde{G} & \to & \mathbb{C}^\times \\
\downarrow & & \downarrow \\
1 & \to & G_0 \\
& & \downarrow \\
& & \to G \\
& & \downarrow \\
& & \to \mathbb{C}^\times \\
& & \downarrow \\
& & 1.
\end{array}
\]

Then the equivariant index defined in [52] is the equivariant holomorphic Euler character of $K^{1/2}$,

\[
I_{\mu,e} = \chi_{\tilde{G}}(K^{1/2}).
\]

(8.2)

According to [52], relation (8.1) implies that $I_{\mu,e}$ is a Laurent polynomial in the element $R = (Z^{-1}Q_1Q_2)^{1/2}$ of the representation ring of $G$.

Specializing equation (6.1) to a local rational curve of type $(-1, -1)$ yields

\[
Z_{ref}^{ref}(q, y, x) = 1 + \sum_{\mu \neq \emptyset} Z_{0,1}^{0,1}(q, y) \tilde{\mathcal{H}}_{\mu}(q^{-1}y^{-1}, qy, x),
\]

(8.3)

where

\[
Z_{0,1}^{0,1}(q, y) = (-1)^{|\mu|} \prod_{[\square] \in \mu} \frac{q^{h([\square])}y^{a([\square])}l([\square])}{1 - q^{h([\square])}y^{a([\square])}l([\square])}.
\]

The expansion of the right hand side of equation (8.3) in the monomial symmetric basis can be written as

\[
Z_{ref}^{ref}(q, y, x) = 1 + \sum_{\mu \neq \emptyset} W_{\mu}(q, y)(-qy)^{|\mu|}m_{\mu}(x)
\]

(8.4)

with

\[
W_{\mu}(q, y) = \sum_{e \in \mathbb{Z}} W_{\mu,e}(y)q^e.
\]

Then the relation between the equivariant index (8.2) and formula (8.3) is the following conjectural identity

\[
(-1)^e I_{m,e}(R)_{R=y} = W_{\mu,e}(y)
\]

(8.5)

for any discrete invariants $(m, e)$. In the right hand side, $\mu$ is the partition of $r = |m|$ determined by $m$, and $l(\mu)$ is the length of $\mu$. This conjecture will be verified by explicit computations in Section 8.3.
8.2 Virtual localization and fixed points

The index (8.2) can be computed explicitly by virtual localization, using the virtual Riemann-Roch theorem proven in [21, 13]. Suppose \( m \) is an isolated \( G \)-fixed point in the moduli space, and let \( E^*_m \) be the restriction of the two term perfect obstruction complex to \( m \). Let

\[
[E^*_m] = V_2 - V_1
\]

be the virtual \( G \)-representation determined by the restriction of the perfect obstruction complex to \( m \). For parabolic ADHM sheaves, the \( K \)-theory class \( [E^*_m] \) is determined by the deformation complex (4.4),

\[
[E^*_m] = \sum_{i=0}^{1} \sum_{j=0}^{2} (-1)^{i+j} [H^i(C, C^j(E))] \tag{8.6}
\]

where \( E^* \) is the asymptotically stable parabolic ADHM sheaf on \( C \) corresponding to the fixed point \( m \). Since the obstruction theory is symmetric, there is an isomorphism \( V_2 \cong V_1^\vee \) of complex vector spaces, but not of \( G \)-representations. In particular \( \dim(V_2) = \dim(V_1) = v \).

Then note that

\[
V_2 = R^2 V_1^\vee \quad \text{and} \quad K_m^{1/2} = R^v \det(V_1)^{-1}
\]

in the representation ring of \( G \). The contribution of \( m \) to the virtual \( K \)-theoretic localization formula is

\[
K_m^{1/2} \Lambda^{-1}[([E^*_m]^\vee)] = R^v \det(V_1)^{-1} \frac{\Lambda^{-1}(V_2^\vee)}{\Lambda^{-1}(V_1^\vee)}. \tag{8.7}
\]

The virtual representation \( ([E^*_m]^\vee) \) is the equivariant \( K \)-theoretic Euler characteristic of the deformation complex (4.4) for the parabolic ADHM sheaf \( E^* \) on \( C \) corresponding to the fixed point \( m \) in the moduli space.

The next task is to classify the fixed loci and compute their local contribution to the fixed point theorem. For concreteness suppose \( L_1 = \mathcal{O}_C(-\infty) \) and \( L_2 = \mathcal{O}_C(-\infty) \) as \( \mathcal{G} \)-equivariant line bundles on \( C \), where \( \infty \in \mathbb{P}^1 \) is the point \( z_0 = 0 \) (as opposed to the marked point \( p \in C \), which is given by \( z_1 = 0 \).) Therefore the \( \mathcal{G} \)-equivariant canonical line bundle will be \( K_C = \mathcal{O}_C(-2\infty) \).

Using [16, Prop. 3.1], asymptotically stable parabolic ADHM sheaves \( E^* \) fixed by \( \mathcal{G} \) up to isomorphism are classified as follows. Forgetting the parabolic structure, the data \( E = (E, \Phi_1, \Phi_2, \phi, \psi) \) is an asymptotically stable ADHM sheaf on \( C \) with coefficient line bundles \( (L_1, L_2(p)) \). There is a one-to-one correspondence between such sheaves \( E \) and data.

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$(\Delta, d, k)$, where $\Delta \subset (\mathbb{Z}_{\geq 0})^2$ is a Young diagram, and $d : \Delta \to \mathbb{Z}$, $k : \Delta \to \mathbb{Z}$ two $\mathbb{Z}$-valued functions satisfying the inequalities:

\[ 0 \leq k(0, 0) \leq d(0, 0), \]
\[ 0 \leq k(i + 1, j) - k(i, j) \leq d(i + 1, j) - d(i, j) - 1, \]

for any $(i, j) \in \Delta$ such that $(i + 1, j) \in \Delta$, and

\[ -1 \leq k(i, j + 1) - k(i, j) \leq d(i + 1, j) - d(i, j) - 1, \]

for any $(i, j) \in \Delta$ such that $(i, j + 1) \in \Delta$.

Given a collection $(\Delta, d, k)$ as above, the underlying vector bundle of $\mathcal{E}$ is of the form

\[ E \cong \bigoplus_{(i, j) \in \Delta} E(i, j), \quad E(i, j) = Q_1^i Q_2^j \mathbb{Z}^{-k(i, j)} \mathcal{O}_C(d(i, j) \infty) \]

as a $\tilde{G}$-equivariant bundle on $C$. The nonzero components of the morphisms $(\Phi_1, \Phi_2, \phi, \psi)$ are

\[ \Phi_1(i, j) : \mathcal{O}_C(d(i, j) \infty) \to \mathcal{O}_C((d(i + 1, j) - 1) \infty) \]
\[ 1 \mapsto z_1^{k(i+1,j)-k(i,j)} z_0^{d(i+1,j)-d(i,j)-k(i+1,j)+k(i,j)-1} \]
\[ \Phi_2(i, j) : \mathcal{O}_C(d(i, j) \infty) \to \mathcal{O}_C((d(i, j + 1) - 1) \infty + p) \]
\[ 1 \mapsto z_1^{k(i,j+1)-k(i,j)+1} z_0^{d(i,j+1)-d(i,j)-k(i,j+1)+k(i,j)-1} \]
\[ \psi : \mathcal{O}_C \to \mathcal{O}_C(d(0, 0) \infty) \]
\[ 1 \mapsto z_1^{k(0,0)} z_0^{d(0,0)-k(0,0)} \]

All other components are identically zero.

In order to simplify the computations, it will be convenient to choose specific generators for the cohomology of equivariant line bundles on $C = \mathbb{P}^1$ using a standard Čech cohomology computation. Let $z = z_1/z_0$ be an affine coordinate centered at the marked point $p$. Then one can easily show that

\[ H^0(\mathcal{O}_C(d \infty + ap)) \cong \mathbb{C}\langle z^{-a}, z^{1-a}, \ldots, z^d \rangle \]

for any $a, d \in \mathbb{Z}_{\geq 0}$, and

\[ H^1(\mathcal{O}_C(-d \infty)) \cong \mathbb{C}\langle z^{-1}, \ldots, z^{1-d} \rangle \]
for any $d \in \mathbb{Z}_{\geq 2}$. In this basis the nontrivial components (8.12)-(8.14) read

$$\Phi_1(i, j) = z^{k(i+1,j)-k(i,j)}, \quad \Phi_2(i, j) = z^{k(i,j+1)-k(i,j)}, \quad \psi(0, 0) = z^{k(0,0)}. \quad (8.17)$$

For a $\tilde{G}$-fixed asymptotically stable parabolic ADHM sheaves $\mathcal{E}^\bullet$, one has to specify in addition a flag

$$0 = E_p^a \subseteq E_p^{a-1} \subseteq \cdots \subseteq E_p^0 = E_p \quad (8.18)$$
in the fiber at $p$ preserved by the $\tilde{G}$ such that:

(a) $\Phi_1|_p(E_p^a) \subseteq E_p^a$ for any $0 \leq a \leq s$, and

(b) $\operatorname{res}_p(\Phi_2)(E_p^a) \subseteq E_p^{a+1}$ for any $0 \leq a \leq s-1$.

Note that for any such flag the subspaces $E_p^a$, $0 \leq a \leq s - 1$ must be specified by a third function $\varphi : \Delta \to \{0, \cdots, s-1\}$ such that

$$E_p^a = \bigoplus_{(i,j) \in \varphi^{-1}(a)} Q_1^{-i}Q_2^{-j}Z^{-k(i,j)}\mathcal{O}_C(d(i,j)\infty)_p. \quad (8.19)$$

In conclusion the $\tilde{G}$-fixed points in the moduli space of asymptotically stable parabolic ADHM sheaves are in one-to-one correspondence with data $(\Delta, d, k, \varphi)$ satisfying inequalities (8.8)-(8.10) and the compatibility conditions (a), (b) above (8.18). A complete enumeration of such data for fixed numerical invariants $(m, e)$ is fairly tedious. This is done in detail in Section 8.1 for low rank examples.

To conclude this subsection, note that the contribution of a fixed point $m = (\Delta, d, k, \varphi)$ to the fixed point formula is determined by the deformation complex (4.4), using the exact sequences (2.3), (2.4) and their strongly parabolic analogues.

Recall that for any filtered vector space $V^\bullet$, $\operatorname{PEnd}(V^\bullet)$, $\operatorname{SPEnd}(V^\bullet)$ denote the linear spaces of parabolic, respectively strongly parabolic morphisms with respect to the flag. The space $\operatorname{APEnd}(V^\bullet)$ was defined in Section 2.1 as the quotient

$$\operatorname{APEnd}(V^\bullet) = \operatorname{End}(V)/\operatorname{PEnd}(V^\bullet).$$

Moreover, for any two $\tilde{G}$-equivariant bundles $E, F$ on $C$, let

$$\chi(E, F) = \operatorname{Ext}_C^0(E, F) - \operatorname{Ext}_C^1(E, F)$$
in the representation ring of $\tilde{G}$. Then a straightforward computation yields

$$V_2 - V_1 = T + P \quad (8.20)$$

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where
\[ T = \chi(E, E) - Q_1 \chi(E, E \otimes C L_1) - Q_2 \chi(E, E \otimes C L_2) \]
\[ + Q_1 Q_2 \chi(E, E \otimes C L_1 \otimes C L_2) - \chi(O_C, E) - Q_1 Q_2 \chi(E, L_1 \otimes C L_2) \] 
\[ \text{(8.21)} \]

and
\[ P = (1 - Q_1) \text{APEnd}(E_p^\bullet) + Q_2 (Q_1 - 1) \text{SPEnd}(E_p^\bullet) \otimes O_C(p). \] 
\[ \text{(8.22)} \]

Using the canonical exact sequence
\[ 0 \to O_C \to O_C(p) \to O_C(p) \to 0, \]
one finds that \( O_C(p)_p = Z^{-1} \) in the representation ring of \( \tilde{G} \). Moreover, there is a natural \( \tilde{G} \)-equivariant isomorphism
\[ \text{SPEnd}(E_p^\bullet) \simeq \text{APEnd}(E_p^\bullet)^\vee. \]

Therefore equation (8.22) yields
\[ P = (1 - Q_1) \text{APEnd}(E_p^\bullet) + Q_2 (Q_1 - 1) Z^{-1} \text{APEnd}(E_p^\bullet)^\vee. \] 
\[ \text{(8.23)} \]

### 8.3 Experimental evidence

The goal of this section is to provide some supporting evidence for the conjectural formula (8.5).

First note that
\[ W_{(r)}(q, y) = (-1)^r Z^{0,1}_{(r)}(q, y) \]
for length one partitions \( \mu = (r) \). In this case the conjectural formula (8.3) reduces to the case without marked points discussed in detail in [12, 11]. Then identity (8.5) is already verified by the computations of [10], both sides being in agreement with the geometric engineering predictions. Therefore only partitions of length \( l \geq 2 \) will be considered in the following. A
straightforward computation yields the following expressions

\[
W_{(11)}(q, y) = \frac{2y^{12} + 2y^{14}}{y^{13}} q + \frac{3y^{11} + 3y^{15} + 4y^{13}}{y^{13}} q^2 + \frac{4y^{10} + 6y^{12} + 4y^{16} + 6y^{14}}{y^{13}} q^3 + \ldots
\]

\[
W_{(21)}(q, y) = \frac{y^{13} + y^{15}}{y^{14}} q + \frac{2y^{16} + 2y^{12} + 4y^{14}}{y^{14}} q^2 + \frac{8y^{13} + 4y^{11} + 4y^{17} + 8y^{15}}{y^{14}} q^3 + \ldots
\]

\[
W_{(111)}(q, y) = \frac{3y^{14} + 3y^{16}}{y^{15}} q + \frac{6y^{13} + 6y^{17} + 9y^{15}}{y^{15}} q^2 + \frac{10y^{12} + 18y^{14} + 10y^{18} + 18y^{16}}{y^{15}} q^3 + \ldots
\]

A sample computation will be displayed below for \(\mu = (1, 1, 1)\) and \(e = 1\). Employing the results of Section 8.2 the fixed loci are in this case classified as follows. Taking into account inequalities (8.8)-(8.10), there are six fixed points:

1) \(E = \mathcal{O}_C \oplus Q_2^{-1} \mathcal{Z} \mathcal{O}_C + Q_2^{-2} Z^2 \mathcal{O}_C(\infty)\)

\[\Phi_2(0, 0) = z^{-1}, \quad \Phi_2(0, 1) = z^{-1}, \quad \psi(0, 0) = 1,\]

\[E_2^p = E(0, 2), \quad E_1^p = E(0, 2) \oplus E(0, 1).\]

2.a) \(E = \mathcal{O}_C \oplus Q_2^{-1} \mathcal{Z} \mathcal{O}_C + Q_2^{-2} Z^2 \mathcal{O}_C(\infty)\)

\[\Phi_2(0, 0) = z^{-1}, \quad \Phi_2(0, 1) = 1, \quad \psi(0, 0) = 1,\]

\[E_2^p = E(0, 2), \quad E_1^p = E(0, 2) \oplus E(0, 1).\]

2.b) \(E = \mathcal{O}_C \oplus Q_2^{-1} \mathcal{Z} \mathcal{O}_C + Q_2^{-2} Z^2 \mathcal{O}_C(\infty)\)

\[\Phi_2(0, 0) = z^{-1}, \quad \Phi_2(0, 1) = 1, \quad \psi(0, 0) = 1,\]

\[E_2^p = E(0, 1), \quad E_1^p = E(0, 2) \oplus E(0, 1).\]

2.c) \(E = \mathcal{O}_C \oplus Q_2^{-1} \mathcal{Z} \mathcal{O}_C + Q_2^{-2} Z^2 \mathcal{O}_C(\infty)\)

\[\Phi_2(0, 0) = z^{-1}, \quad \Phi_2(0, 1) = 1, \quad \psi(0, 0) = 1,\]

\[E_2^p = E(0, 1), \quad E_1^p = E(0, 0) \oplus E(0, 1).\]

3.a) \(E = \mathcal{O}_C \oplus Q^{-1} \mathcal{O}_C(\infty) + Q_2^{-1} Z^2 \mathcal{O}_C\)

\[\Phi_1(0, 0) = 1, \quad \Phi_2(0, 0) = z^{-1}, \quad \psi(0, 0) = 1,\]

\[E_2^p = E(1, 0), \quad E_1^p = E(1, 0) \oplus E(0, 1).\]
3.b) \[ E = \mathcal{O}_C \oplus Q^{-1}\mathcal{O}_C(\infty) + Q_2^{-1}Z^2\mathcal{O}_C \]
\( \Phi_1(0,0) = 1, \quad \Phi_2(0,0) = z^{-1}, \quad \psi(0,0) = 1, \)
\[ E_p^2 = E(0,1), \quad E_p^1 = E(1,0) \oplus E(0,1). \]

The underlying vector bundle \( E \) is encoded in a decorated Young diagram of the form

```
1
0
0
```

for cases (1) – (2.c), respectively

```
0
0
1
```

for cases (3.a) – (3.b). The expression (8.21) takes the form

\[ T_1 = 2 + Z^{-1}Q_2 + ZQ_2^{-1} - Z^2Q_1Q_2^2 - ZQ_1Q_2^{-1} + Z^{-2}Q_1Q_2 - Z + Z^{-3}Q_2^3 \]
\[ - 2Z^{-1}Q_1Q_2 - Z^{-2}Q_1Q_2^2 - Q_1 \]

for case (1),

\[ T_2 = 1 + Z^{-1}Q_2 + Q_2^{-1} - ZQ_1Q_2^2 - Q_1Q_2^{-1} - Z^{-1}Q_1Q_2 + Z^{-2}Q_2^3 \]
\[ + Z^{-1}Q_2^2 - Q_1 - Z^{-1}Q_1Q_2^2 \]

for cases (2.a)-(2.c), respectively

\[ T_3 = 1 + Z^{-1}Q_1^{-1}Q_2 + Q_1^2Q_2^{-1} - Z^{-1}Q_1^{-1}Q_2^2 - Z^{-1}Q_1Q_2 - Q_1^2 \]

for cases (3.a) and (3.b).

The expression (8.22) specializes to, respectively,

\[ P_1 = -2 + ZQ_1Q_2^{-1} + 2Q_1 - ZQ_2^{-1} + 2Z^{-1}Q_1Q_2 + Z^{-2}Q_1Q_2^2 - 2Z^{-1}Q_2 - Z^2Q_2^2 \]
\[ P_{2,a} = -1 + Q_1Q_2^{-1} + Z^{-1}Q_1 + Q_1 - Z^{-1} - Q_2^{-1} + Z^{-1}Q_1Q_2 \]
\[ + Q_1Q_2 + Z^{-1}Q_1Q_2^2 - Z^{-1}Q_2 - Q_2 - Z^{-1}Q_2^2 \]
\[ P_{2,b} = -1 + Q_1 + Z^{-1}Q_1Q_2 + 2Q_1Q_2^{-1} + 2Z^{-1}Q_1Q_2^2 - Z^{-1}Q_2 - 2Q_2^{-1} - 2Z^{-1}Q_2^2 \]
\[ P_{2,c} = -1 - Z^{-1}Q_2 - Q_2^{-1} - ZQ_2^2 + Z^{-1}Q_1Q_2 + Q_1Q_2^{-1} + ZQ_1Q_2^2 - Z^2Q_2^3 \]
\[ - Z^{-1}Q_2^2 + Z^{-2}Q_1Q_2^3 + Z^{-1}Q_1Q_2^2 + Q_1 \]

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\[ P_{3.a} = -1 - ZQ_1Q_2^{-1} + Q_1^2 + ZQ_1^2Q_2^{-1} + Z^{-1}Q_1Q_2 - Z^{-1}Q_1^{-1}Q_2 \]
- \[ Z^{-2}Q_1^{-1}Q_2^2 + Z^{-2}Q_2^2 \]
\[ P_{3.b} = -1 - Q_1 - 2Z^{-1}Q_1^{-1}Q_2 + Z^{-1}Q_1 + 2Q_1^2 + Z^{-1}Q_1Q_2 \]

Let \( F_m \) denote the right hand side of equation (8.7). Then, using the above computations, one obtains
\[ F_1 = R^3Z^{-2}Q_1^{-1}Q_2 \frac{(1 - Q_1^{-1})(1 - ZQ_1^{-1}Q_2^{-1})(1 - Z^3Q_2^{-3})}{(1 - Z^{-1})(1 - ZQ_2^{-1})(1 - Z^{-2}Q_1^{-1}Q_2^2)} \]
\[ F_{2.a} = R^3Q_1^{-1}Q_2 \frac{(1 - Q_1^{-1}Q_2^{-1})(1 - ZQ_1^{-1})(1 - Z^2Q_2^{-3})}{(1 - Q_2^{-1})(1 - Z)(1 - Z^{-1}Q_1^{-1}Q_2^2)} \]
\[ F_{2.b} = R^3Q_1^{-1}Q_2 \frac{(1 - ZQ_1^{-1}Q_2^{-2})(1 - Q_1^{-1}Q_2)(1 - Z^2Q_2^{-3})}{(1 - Q_2)(1 - ZQ_2^{-2})(1 - Z^{-1}Q_1^{-1}Q_2^2)} \]
\[ F_{2.c} = RZ^{-1}Q_2^{-1} - Z^2Q_1^{-1}Q_2^{-3} \]
\[ F_{3.a} = R^3Z^2Q_1Q_2^{-3} \frac{(1 - Q_1^{-2}Q_2^2)(1 - Z^2Q_2^{-2})(1 - Z^{-1}Q_1^{-2}Q_2)}{(1 - Z^{-1}Q_1^{-1}Q_2)(1 - Z^2Q_1Q_2^{-2})(1 - ZQ_1Q_2^{-2})} \]
\[ F_{3.b} = R^3Z^2Q_1Q_2^{-3} \frac{(1 - Q_1^{-2}Q_2^2)(1 - Q_1^{-2})(1 - ZQ_2^{-1})}{(1 - Q_1^{-1})(1 - ZQ_1Q_2^{-2})(1 - ZQ_1Q_2^{-1})} \]

where \( R = Z^{-1}Q_1Q_2 \). Adding all local contributions yields
\[ I_{(1,1,1),1} = -3(R^{1/2} + R^{-1/2}) \]
confirming conjecture (8.5) in this case. Similar computations confirm the conjecture for
\( (\mu, e) = ((1, 1), 1), ((1, 1), 2), ((1, 1), 3), ((2, 1), 1), ((2, 1), 2), ((2, 1), 3), ((1, 1, 1), 2), ((1, 1, 1), 3) \).

### A Degree zero ADHM sheaves

This section proves a result used in the main text stating that the underlying vector bundle of any rank \( r \), degree 0 asymptotically flat ADHM sheaf \( \mathcal{E} \) must be trivial, \( \mathcal{E} \simeq \mathcal{O}_C^{\oplus r} \).

If \( r = 1 \), the claim is obvious since \( \deg(E) = 0 \) and there is a nonzero morphism \( \psi : \mathcal{O}_C \to E \).
Suppose \( r \geq 2 \) and \( E \) is slope semistable. For any \((n_1, n_2) \in \mathbb{Z}_{\geq 0}^2\), let \( E(n_1, n_2) = \Phi_{n_1}^* \Phi_{n_2}^* \psi(O_C) \subset E \). Note that \( E(n_1, n_2) \) is either the zero sheaf or isomorphic to \( O_C \) since it is a locally free quotient of \( O_C \). Let
\[
E' = \sum_{(n_1, n_2) \in \mathbb{Z}_{\geq 0}^2} E(n_1, n_2) \subseteq E.
\]
The asymptotic stability condition implies that \( E/E' \) is a zero dimensional sheaf on \( C \). By construction there there exists a finite set \( \Delta \subset (\mathbb{Z}_{\geq 0})^2 \) and a surjective morphism
\[
V_\Delta = \bigoplus_{(n_1, n_2) \in \Delta} E(n_1, n_2) \twoheadrightarrow E'.
\]
Since \( E \) is semistable of degree 0, it follows that the resulting morphism \( V_\Delta \to E \) must be surjective as well, hence \( E \simeq E' \).

Now let
\[
0 = JE_0 \subset JE_1 \subset \cdots \subset JE_n = E
\]
be a Jordan-Hölder filtration of \( E \). Obviously, there is a commutative triangle of surjective morphisms
\[
\begin{array}{ccc}
V_\Delta & \rightarrow & E \\
\downarrow & & \downarrow \\
E/JE_{n-1} & \rightarrow & E
\end{array}
\]
This implies that there is at least one direct summand \( E(m_1, m_2) \subset V_\Delta \) which fits into a commutative triangle
\[
\begin{array}{ccc}
E(m_1, m_2) & \rightarrow & E \\
\downarrow & & \downarrow \\
E/JE_{n-1} & \rightarrow & E
\end{array}
\]
with all maps nontrivial. Moreover, the horizontal map must be in fact injective. Since \( E/JE_{n-1} \) is stable of degree 0, and \( E(m_1, m_2) \simeq O_C \), it follows that \( E/JE_{n-1} \simeq O_C \), and the map \( E(m_1, m_2) \rightarrow E/JE_{n-1} \) is an isomorphism. This implies that there is a splitting
\[
E \simeq E/JE_{n-1} \oplus JE_{n-1} \simeq O_C \oplus JE_{n-1}
\]
By construction, \( JE_{n-1} \) is degree 0 slope semistable and there is a surjective morphism
\[
\bigoplus_{(n_1, n_2) \in \Delta \setminus \{(m_1, m_2)\}} E(n_1, n_2) \twoheadrightarrow JE_{n-1}.
\]
Repeating the above argument shows that $JE_{n-1}/JE_{n-2} \simeq \mathcal{O}_C$ and there is a splitting
\[ JE_{n-1} \simeq \mathcal{O}_C \oplus JE_{n-2} \]
Proceeding recursively, one finds that $E \simeq \mathcal{O}_C^{\oplus r}$ in a finite number of steps.

To finish the proof, suppose $E$ is not slope semistable. Then it will be shown below that this leads to a contradiction. By assumption, $E$ has a Harder-Narasimhan filtration
\[ 0 = HE_0 \subset HE_1 \subset \cdots \subset HE_l = E \]
with $l \geq 2$.

The first observation is that $\Phi_j(E_k) \subseteq E_k$ for all $1 \leq j \leq 2$ and $1 \leq k \leq l$. Suppose this fails for some $1 \leq k \leq l - 1$ and some $1 \leq j \leq 2$, and let $k$ be minimal with this property i.e. $\Phi_j(E_{k'}) \subseteq E_{k'}$ for all $k' < k$. Then let $k'' > k$ be minimal such that $\Phi_j(E_k) \subseteq E_{k''}$ for all $j \in \{1, 2\}$ and $\Phi_j(E_k) \not\subseteq E_{k''-1}$ for at least one value of $j \in \{1, 2\}$. Then $\Phi_j$ yields a nontrivial morphism $\Phi_j : E_k/E_{k-1} \to E_{k''}/E_{k''-1}$ contradicting the defining property of the Harder-Narasimhan filtration.

Since $\Phi_1, \Phi_2$ preserve the Harder-Narasimhan filtration, the asymptotic stability condition for ADHM sheaves implies that $\psi(\mathcal{O}_X) \not\subseteq E_{l-1}$. Hence $\psi$ yields a nontrivial morphism $\mathcal{O}_X \to E/E_{l-1}$. Since $E/E_{l-1}$ is semistable, this implies $\mu(E/E_{l-1}) \geq 0$, again contradicting the properties of the Harder-Narasimhan filtration which imply that $\mu(E/E_{l-1}) < \mu(E) = 0$.

In conclusion, the underlying bundle of an asymptotically stable degree 0 ADHM sheaf must be indeed isomorphic to $\mathcal{O}_C^{\oplus r}$.

**B Fermion zero modes**

The goal of this section is to determine the bundle of fermion zero modes on the moduli space of supersymmetric D2-D6 configurations found in Section 5. As proven in Section 5.2, supersymmetry constraints require the Chan-Paton bundle on $r$ such D2-branes to be isomorphic to the trivial rank $r$ bundle, and all field configurations to be constant. This shows that the low energy effective action of such a configuration is reduced to supersymmetric quantum mechanics. The detailed action of a similar system has been written in [6, Sect 2.2] as the dimensional reduction of a two dimensional $(0, 2)$ gauged linear sigma model. Analogous considerations will yield the action in the present case by dimensional reduction of a two dimensional $(0, 4)$ gauged linear sigma model. Omitting the details, note the resulting
quantum mechanical system will have a moduli space of flat directions isomorphic to $N(\gamma)$, as expected. Using standard $(0,2)$ sigma model technology [61], the bundle of fermion zero modes is isomorphic to the middle cohomology of a monad complex, as shown below.

In absence of the orbifold point $p$, the D2-D6 moduli space is isomorphic to the Hilbert scheme of points $\mathcal{H}^r$. For any stable ADHM data $(A_1, A_2, I)$, the space of fermion zero modes is isomorphic to the middle cohomology group of the complex $\mathcal{F}_{(A_1, A_2, I)}$

\[
H^1(\text{End}_C(E))^\oplus 2
\]

\[
0 \to H^1(\text{End}_C(E)) \overset{d_1}{\to} H^1(E) \overset{d_2}{\to} H^1(\text{End}_C(E)) \to 0
\]

(B.1)

where

\[
d_1(\alpha) = ([\alpha, A_1], [\alpha, A_2], \alpha I)
\]

\[
d_2(\beta_1, \beta_2, \gamma, \delta) = [\beta_1, A_2] + [A_1, \beta_2] + I\delta.
\]

Since $E \simeq \mathcal{O}_C^\oplus r$, one can easily prove using Serre duality that $\mathcal{F}_{(A_1, A_2, I, J)}$ is left and right exact while its middle cohomology is isomorphic to $(T^*_{(A_1, A_2, I, J)} \mathcal{H}^r)^\oplus g$, where $T^*_{(A_1, A_2, I, J)} \mathcal{H}^r$ is the fiber of the cotangent bundle to the Hilbert scheme $\mathcal{H}^r$ at the point $(A_1, A_2)$. Using the same argument in flat families of stable ADHM data, it follows that the bundle of fermion zero modes is isomorphic to the direct sum $(T^* \mathcal{H}^r)^\oplus g$.

Now suppose there is an orbifold point, in which case the supersymmetric configurations are in one-to-one correspondence with stable parabolic ADHM data $(A_1, A_2, I, J, V^*)$ as shown in Section 5.2. The space of fermion zero modes will then given by the middle cohomology of a complex $\mathcal{F}_{(A_1, A_2, I, J, V^*)}$ of the form (B.1), where $E$ is replaced with an orbibundle $\tilde{E}$ on $\tilde{C}$. Using the correspondence described in Section 4, this complex can be written in terms of parabolic data as follows.

Recall that there is a root line bundle $\tilde{L}$ on $\tilde{C}$ such that $\tilde{L}^s \simeq \nu^* \mathcal{O}_C(p)$, where $\nu : \tilde{C} \to C$ is the natural projection. Moreover, the canonical class of $\tilde{C}$ is given by

\[
K_{\tilde{C}} \simeq \nu^* K_C \otimes_{\tilde{C}} \tilde{L}^{(s-1)} \simeq \nu^* K_C(p) \otimes_{\tilde{C}} \tilde{L}^{-1}.
\]

Then Serre duality on the stack $\tilde{C}$ yields the following isomorphisms

\[
H^1(\text{End}_{\tilde{C}}(\tilde{E})) \simeq H^0(\text{End}_{\tilde{C}}(\tilde{E}) \otimes_{\tilde{C}} \nu^* K_C(p) \otimes_{\tilde{C}} \tilde{L}^{-1})^\vee
\]

\[
H^1(\tilde{E}^\vee) \simeq H^0(\tilde{E} \otimes_{\tilde{C}} \nu^* K_C(p) \otimes_{\tilde{C}} \tilde{L}^{-1})^\vee.
\]

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Now recall that the pushforward $E = \nu_*(\tilde{E})$ is a vector bundle on $C$ equipped with a filtration by subsheaves $F_a = \nu_*(\tilde{E} \otimes_{\tilde{C}} \tilde{L}^{-1})$, $a \geq 1$. This filtration determines a flag $E_p^*$ in the fiber $E_p$, hence a parabolic structure on $E$ at $p$. Moreover the higher direct images $R^k \nu_* \tilde{E}$ are trivial and there is a one-to-one correspondence between morphisms $\tilde{\Phi} : \tilde{E} \to \tilde{E}$ and parabolic morphisms $\Phi : E^* \to E^*$. Therefore one obtains isomorphisms of the form

$$H^1(\text{End}_{\tilde{C}}(\tilde{E})) \simeq H^0(S\text{PEnd}_C(E) \otimes_C \nu^* K_C(p))^\vee$$
$$H^1(\tilde{E}) \simeq H^0(F_1 \otimes_C \nu^* K_C(p))^\vee$$
$$H^1(E) \simeq H^1(E) \simeq H^0(E^* \otimes_C K_C)$$

Then dual complex is isomorphic to

$$H^0(S\text{PEnd}_C(E^*) \otimes C K_C(p)) \oplus^\oplus H^0(\nu^* \otimes_C K_C(p)) \oplus H^0(F_1 \otimes_C K_C(p)) \xrightarrow{d_1} H^0(S\text{PEnd}_C(E^*) \otimes C K_C(p)) \to 0$$

where $S\text{PEnd}_C(E^*)$ denotes the sheaf of strongly parabolic endomorphisms of $E^*$. The expressions of the differentials are formally identical with the ones given in (B.2).

Next note that by construction there is an exact sequence

$$0 \to \text{End}_C(E) \otimes_C K_C \to S\text{PEnd}_C(E^*) \otimes K_C(p) \to S\text{PEnd}(E^*_p) \otimes C \mathcal{O}_p(p) \to 0$$

(B.3)

of sheaves on $C$. Moreover, the inclusions

$$0 \subset E(-p) \subset F_1 \subset E$$

yield inclusions of vector spaces

$$0 \subset H^0(E \otimes_C K_C) \subset H^0(F_1 \otimes_C K_C(p)) \subset H^0(E \otimes_C K_C).$$

However since $E \simeq \mathcal{O}_C^{\oplus r}$, there is an isomorphism

$$H^0(E \otimes_C K_C) \simeq H^0(E \otimes_C K_C(p)).$$

Therefore there is an isomorphism

$$H^0(E \otimes_C K_C) \simeq H^0(F_1 \otimes_C K_C(p)).$$

(B.5)
Using the exact sequence (B.4) and isomorphism (B.5) a straightforward computation shows that there is an exact sequence of complexes

\[ 0 \to \mathcal{F}_{(A_1, A_2, I)} \to \mathcal{F}_{(A_1, A_2, I; V^\bullet)} \to \mathcal{D}_{(A_1, A_2; V^\bullet)} \to 0 \]  

where \( \mathcal{D}_{(A_1, A_2; V^\bullet)} \) is the three term complex

\[ 0 \to \text{SPEnd}(V^\bullet) \xrightarrow{\delta_1} \text{SPEnd}(V^\bullet)^{\otimes 2} \xrightarrow{\delta_2} \text{SPEnd}(V^\bullet) \to 0. \]

The differentials \( \delta_1, \delta_2 \) are given by

\[
\delta_1(f) = (\{f, A_1\}, \{f, A_2\})
\]

\[
\delta_2(g_1, g_2) = [g_1, A_2] + [A_1, g_2].
\]

Now note that under the current assumptions \( \delta_1 \) is injective and \( \delta_2 \) is surjective, hence the complex \( \mathcal{D}_{(A_1, A_2; V^\bullet)} \) has trivial cohomology.

To prove this claim, recall that \((A_1, A_2)\) is by assumption a cyclic commuting pair preserving the flag \( V^\bullet \). In particular \((A_1, A_2)\) is regular i.e. the subspace \( f \in \text{End}(\mathbb{C}^r) \) such that \([f, A_1] = [f, A_2]\) is isomorphic to a Cartan subalgebra of \( \text{End}(\mathbb{C}^r) \). On the other hand if \( f \in \text{SPEnd}(V^\bullet) \), it follows that \( f \) is nilpotent, hence it must be trivial. This shows that \( \text{Ker}(\delta_1) = 0 \). Surjectivity of \( \delta_2 \) follows by an analogous argument for the dual morphism

\[ \delta_2^\vee : \text{SPEnd}(V^\bullet)^\vee \to \text{SPEnd}(V^\bullet)^{\vee} \oplus \text{SPEnd}(V^\bullet)^{\vee} \]

The dual vector space \( \text{SPEnd}(V^\bullet)^\vee \) is isomorphic to a space of strongly parabolic maps on the dual vector space \( V^\vee \) equipped with the dual flag

\[ V^\vee_{s-a} = \text{Ker}(V^\vee \to (V^a)^\vee), \quad 0 \leq a \leq s. \]

That is \( \text{SPEnd}(V^\bullet)^\vee \simeq \text{SPEnd}(V^\bullet^\vee) \). Moreover,

\[
\delta_2^\vee(\xi) = ([\xi, A_1^\vee], [A_2^\vee, \xi]) = ([A_1, \xi^\vee], [\xi^\vee, A_2])^\vee.
\]

Then the same argument shows that \( \text{Ker}(\delta_2^\vee) = 0 \), hence \( \delta_2 \) is surjective.

In conclusion, the exact sequence (B.6) implies that the complexes \( \mathcal{F}_{(A_1, A_2, I; V^\bullet)} \) and \( \mathcal{F}_{(A_1, A_2, I)} \) are quasi-isomorphic.
C Some basic facts on nested Hilbert schemes

The goal of this section is to prove that the nested Hilbert scheme $\mathcal{N}(r)$ used in Section 5.2 is reduced and connected. The proof relies on an alternative presentation of $\mathcal{N}(r)$ given in [4] as a moduli space of stable framed quiver representations. Namely, consider the moduli space of stable framed quiver representations of the form:

$$
\begin{align*}
\begin{array}{c}
A_{\ell,1} \\
\mathbb{C}^{r_\ell}
\end{array} & \xrightarrow{f_{\ell-1,\ell}} \\
\begin{array}{c}
A_{\ell,2} \\
\mathbb{C}^{r_{\ell-1}}
\end{array} & \xrightarrow{f_{\ell-2,\ell-1}} \\
\vdots & \vdots \\
\begin{array}{c}
A_{0,1} \\
\mathbb{C}^{r_0}
\end{array} & \xleftarrow{f_{1,0}} \\
\begin{array}{c}
A_{0,2} \\
\mathbb{C}^{r_1}
\end{array}
\end{align*}
$$

with quadratic relations

$$
[A_{0,1}, A_{0,2}] = 0, \quad A_{i,1}f_{i,i+1} - f_{i,i+1}A_{i,1} = 0, \quad A_{i,2}f_{i,i+1} - f_{i,i+1}A_{i,2} = 0. \quad \text{(C.2)}
$$

The discrete invariants $r_i, 0 \leq i \leq \ell$, are given by

$$
r_i = \sum_{j=1}^\ell \gamma_j.
$$

For generic King stability parameters $(\theta_i, \theta_\infty) \in \mathbb{R}^{\ell+2}$ satisfying

$$
\theta_\infty = -\sum_{i=0}^\ell n_i \theta_i, \quad \theta_i > 0, \quad 0 \leq i \leq \ell.
$$

a representation of the above quiver is semistable if and only if the ADHM data $(A_{0,1}, A_{0,2}, I)$ is stable and the linear maps $f_{i,i+1}$ are injective for all $0 \leq i \leq \ell - 1$. Here $\infty$ denotes the framing node corresponding to the tail of the arrow $I$ in the above diagram. In particular for $(\theta_i, \theta_\infty)$ sufficiently generic there are no strictly semistable objects and the stabilizer group of any stable framed representation is trivial.

Let $\mathbb{A}(r)$ denote the linear space of all linear maps of the form (C.1), not subject to any stability condition or relations. Then the subset of stable quiver representations is an open subspace $U(r) \subset \mathbb{A}(r)$. Let $V(r) \subset \mathbb{A}(r)$ be the closed subscheme determined by the quadratic equations (C.2), and $V_U(r)$ its restriction to $U(r)$. Since all stabilizers are trivial, $V_U(r)$ is a principal $G(r)$ bundle over $\mathcal{N}(r)$, where $G(r) = \times_{i=0}^\ell GL(n_i, \mathbb{C})$. Therefore in order to conclude that $\mathcal{N}(r)$ is reduced it suffices to prove that $V(r)$ is reduced. Now recall that any ideal $I \in \mathbb{C}[x_1, \ldots, x_N]$ generated by irreducible polynomials is a radical ideal. This
statement can be easily proven by induction on \( N \). Then it suffices to prove that all quadrics in equation (C.2) are irreducible. A straightforward computation shows that any quadric in (C.2) is of the form
\[
\sum_{i=1}^{s} x_i y_i
\]
with \( s \geq 2 \), which is indeed irreducible.

In order to prove \( \mathcal{N}(\gamma) \) is connected, recall that the morphism \( \rho_{\text{red}} : \mathcal{N}(\gamma) \to \tilde{\mathcal{H}}_{\text{red}}^{r} \) constructed in diagram (5.8) was shown there to have connected fibers for \( \gamma = (1, \ldots, 1) \). This implies that \( \mathcal{N}(\gamma) \) is connected since \( \rho_{\text{red}} \) is also surjective and \( \tilde{\mathcal{H}}_{\text{red}}^{r} \) is connected. The above quiver moduli space yields a natural morphism \( \mathcal{N}(1, \ldots, 1) \to \mathcal{N}(\gamma) \) for any ordered partition \( \gamma \) of \( r \). Using the Jordan normal for the linear maps \( A_{i,1}, A_{i,2}, 0 \leq i \leq \ell \), it is straightforward to show that this morphism is surjective. Therefore \( \mathcal{N}(\gamma) \) must also be connected, as required in the proof of equation (5.14).

D A compactness result

This section proves that the moduli spaces of asymptotically stable parabolic ADHM sheaves in the example considered in Section 8 are proper. In that case \( C \simeq \mathbb{P}^1 \) and there is a single orbifold point \( p \), which is one of the fixed points of the canonical torus action on \( C \). The second fixed point is denoted by \( \infty \). The orbifold \( \tilde{Y} \) is the total space of the rank two bundle \( K_{\tilde{C}} \otimes \mathbb{C} \nu^* \mathcal{O}_C(\infty) \oplus \nu^* \mathcal{O}_C(-\infty) \). Therefore one has a moduli space of asymptotically stable parabolic ADHM sheaves on \( C \) with coefficient line bundles \( \mathcal{O}_C(-\infty), K_{C} \otimes_{C} \mathcal{O}(-\infty) \otimes_{C} \mathcal{O}_C(p) \). The underlying vector bundle \( E \) of any such ADHM sheaf \( \mathcal{E} \) splits as a direct sum
\[
E \simeq \bigoplus_{j=1}^{f} \mathcal{O}_C(e_j \infty)^{\oplus r_j}
\]
with
\[
0 \leq d_1 < d_2 < \cdots < d_f.
\]
Positivity follows from asymptotic ADHM stability, which requires \( \mathcal{E} \) to be generically generated by the image of the section \( \psi : \mathcal{O}_C \to E \) as a quiver sheaf. The Higgs fields \( \Phi_1, \Phi_2 \) have components
\[
\Phi_1(j, j') : \mathcal{O}_C(e_j \infty)^{\oplus r_j} \to \mathcal{O}_C((e_{j'} - 1) \infty)^{\oplus r_{j'}}
\]
\[
\Phi_2(j, j') : \mathcal{O}_C(e_j \infty)^{\oplus r_j} \to \mathcal{O}_C((e_{j'} - 1) \infty)^{\oplus r_{j'}} \otimes_C \mathcal{O}_C(p)
\]
For degree reasons $\Phi_1(j, j') = 0$ for all $j' \leq j$, and $\Phi_2(j, j') = 0$ for all $j' < j$. Moreover, note that the diagonal components $\Phi_2(j, j)$ must be constant maps. Since the residue $\text{Res}_p\Phi_2$ must be nilpotent, it follows that the components $\Phi_2(j, j)$ must vanish as well. This implies that all polynomial invariants of the quiver sheaf $\mathcal{E}$ are identically zero since $\phi : E \to \mathcal{O}_C$ is identically zero. Since the generalized Hitchin map determined by the polynomial invariants is proper, it follows that the moduli space is proper.

References

[1] L. Álvarez-Cónsul and O. García-Prada. Hitchin-Kobayashi correspondence, quivers, and vortices. Comm. Math. Phys., 238(1-2):1–33, 2003.

[2] I. Biswas. Parabolic bundles as orbifold bundles. Duke Math. J., 88(2):305–325, 1997.

[3] N. Borne. Fibrés paraboliques et champ des racines. Int. Math. Res. Not. IMRN, (16):Art. ID rnm049, 38, 2007.

[4] P. Borgas dos Santos and M. Jardim. ADHM description of flag Hilbert Schemes. in preparation.

[5] T. Bridgeland. Hall algebras and curve-counting invariants. J. Amer. Math. Soc. 24(4):969-998, 2011.

[6] U. Bruzzo, W.-y. Chuang, D.-E. Diaconescu, M. Jardim, G. Pan, et al. D-branes, surface operators, and ADHM quiver representations. Adv. Theor. Math. Phys., 15:849–911, 2011. arXiv:1012.1826.

[7] J. Bryan and T. Graber. The crepant resolution conjecture. Algebraic geometrySeattle 2005. Part 1, 2342, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009. arXiv:0610129.

[8] J. Bryan and R. Pandharipande. The local Gromov-Witten theory of curves. J. Amer. Math. Soc., 21(1):101–136 (electronic), 2008. With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande.

[9] B. Young. Generating functions for colored 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds. with an appendix by J. Bryan Duke Math. J. 152 (2010), no. 1, 115153. arXiv:0802.3948.
[10] J. Choi, S. Katz, and A. Klemm. The refined BPS index from stable pair invariants. 2012. arXiv:1210.4403.

[11] W. Chuang, D. Diaconescu, and G. Pan. BPS states and the P=W conjecture. arXiv:1202.2039, to appear in Moduli Spaces, Cambridge Univ. Press.

[12] W.-y. Chuang, D.-E. Diaconescu, and G. Pan. Wallcrossing and Cohomology of The Moduli Space of Hitchin Pairs. Commun.Num.Theor.Phys., 5:1–56, 2011.

[13] I. Ciocan-Fontanine and M. Kapranov. Virtual fundamental classes via dg-manifolds. Geom. Topol., 13(3):1779–1804, 2009.

[14] M. A. A. de Cataldo, T. Hausel, and L. Migliorini. Topology of Hitchin systems and Hodge theory of character varieties: the case $A_1$. Ann. of Math. (2), 175(3):1329–1407, 2012.

[15] M. A. A. de Cataldo and L. Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. Bull. Amer. Math. Soc. (N.S.), 46(4):535–633, 2009.

[16] D. E. Diaconescu. Moduli of ADHM sheaves and local Donaldson-Thomas theory. J. Geom. Phys., (62):763–799.

[17] D.-E. Diaconescu. Chamber structure and wallcrossing in the ADHM theory of curves, I. J. Geom. Phys., 62(2):523–547, 2012.

[18] R. Dijkgraaf, C. Vafa, and E. Verlinde. M-theory and a topological string duality. 2006. hep-th/0602087.

[19] T. Eguchi and H. Kanno. Five-dimensional gauge theories and local mirror symmetry. Nucl. Phys., B586:331–345, 2000.

[20] T. Eguchi and H. Kanno. Topological strings and Nekrasov’s formulas. JHEP, 12:006, 2003.

[21] B. Fantechi and L. Göttsche. Riemann-Roch theorems and elliptic genus for virtually smooth schemes. Geom. Topol., 14(1):83–115, 2010.

[22] W. Fulton, J. Harris. Representation theory. A first course. Graduate Texts in Mathematics, 129. Springer-Verlag, Berlin, New York, 1991. xvi+551 pp.
[23] O. García-Prada, P. B. Gothen, and V. Muñoz. Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Mem. Amer. Math. Soc.*, 187(879):viii+80, 2007.

[24] A.M. Garsia, M. Haiman. A graded representation model for Macdonald’s polynomials. *Proc. Nat. Acad. Sci. U.S.A.*, 90(8):3607-3610, 1993.

[25] R. Gopakumar, C. Vafa. M theory and topological strings II. arXiv:9812127.

[26] M. Groechenig. Hilbert schemes as moduli of Higgs bundles and local systems. arXiv:1206.5116.

[27] M. Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[28] M. Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001.

[29] T. Hausel, E. Letellier, and F. Rodriguez-Villegas. Arithmetic harmonic analysis on character and quiver varieties. *Duke Math. J.*, 160(2):323–400, 2011.

[30] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. Math.*, 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[31] N. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[32] T. J. Hollowood, A. Iqbal, and C. Vafa. Matrix Models, Geometric Engineering and Elliptic Genera. *JHEP*, 03:069, 2008.

[33] A. Iqbal and A.-K. Kashani-Poor. Instanton counting and Chern-Simons theory. *Adv. Theor. Math. Phys.*, 7:457–497, 2004.

[34] A. Iqbal and A.-K. Kashani-Poor. SU(N) geometries and topological string amplitudes. *Adv. Theor. Math. Phys.*, 10:1–32, 2006.

[35] A. Iqbal, C. Kozcaz, and C. Vafa. The refined topological vertex. *JHEP*, 10:069, 2009.

[36] D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. *Mem. Amer. Math. Soc.*, 217 (2012), no. 1020, iv+199 pp. arxiv.org:0810.5645.
[37] S.H. Katz, A. Klemm and C. Vafa. Geometric engineering of quantum field theories. *Nucl. Phys.* B497:173-195, 1997. hep-th/9609239.

[38] S.H. Katz, A. Klemm and C. Vafa. M theory, topological strings and spinning black holes. *Adv. Theor. Math. Phys.* 3:1445-1537, 1999.

[39] Y. Kawamata. Francia’s flip and derived categories. In *Algebraic geometry*, pages 197–215. de Gruyter, Berlin, 2002.

[40] G. Kerr. Weighted blowups and mirror symmetry for toric surfaces. *Adv. Math.*, 219(1):199–250, 2008.

[41] Y. Konishi. Topological strings, instantons and asymptotic forms of Gopakumar-Vafa invariants. hep-th/0312090.

[42] M. Kontsevich and Y. Soibelman. Stability structures, Donaldson-Thomas invariants and cluster transformations. arXiv.org:0811.2435.

[43] A. E. Lawrence and N. Nekrasov. Instanton sums and five-dimensional gauge theories. *Nucl. Phys.*, B513:239–265, 1998.

[44] J. Li, K. Liu, and J. Zhou. Topological string partition functions as equivariant indices. *Asian J. Math.*, 10(1):81–114, 2006.

[45] M. Logares and J. Martens. Moduli of parabolic Higgs bundles and Atiyah algebroids. *J. Reine Angew. Math.*, 649:89–116, 2010.

[46] E. Markman. Spectral curves and integrable systems. *Compositio Math.*, 93(3):255–290, 1994.

[47] M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. *Math. Ann.*, 293(1):77–99, 1992.

[48] D. Maulik. Motivic Residues and Donaldson-Thomas theory. to appear.

[49] S. Mozgovoy. Solutions of the motivic ADHM recursion formula. *Int. Math. Res. Not. IMRN*, (18):4218–4244, 2012.

[50] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.
[51] B. Nasatyr and B. Steer. Orbifold Riemann surfaces and the Yang-Mills-Higgs equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(4):595–643, 1995.

[52] N. Nekrasov and A. Okounkov. The index of M-theory. in preparation.

[53] N. A. Nekrasov. Seiberg-Witten Prepotential From Instanton Counting. *Adv. Theor. Math. Phys.*, 7:831–864, 2004.

[54] A. Okounkov and R. Pandharipande. The local Donaldson-Thomas theory of curves. *Geom. Topol.*, (14):1503–1567, 2010.

[55] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.

[56] D. Steinberg. Curve-counting invariants for crepant resolutions. arXiv:1208.0884.

[57] Y. Tachikawa. Five-dimensional Chern-Simons terms and Nekrasov’s instanton counting. *JHEP*, 02:050, 2004.

[58] M. Thaddeus. Variation of moduli of parabolic Higgs bundles. *J. Reine Angew. Math.*, 547:1–14, 2002.

[59] Y. Toda. Generating functions of stable pair invariants via wall-crossings in derived categories. *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*. *Adv. Stud. Pure Math.* 59:389-434, 2010.

[60] Y. Toda. Stability conditions and curve counting invariants on Calabi-Yau 3-folds. arXiv:1103.4229.

[61] E. Witten. Phases of $N = 2$ gauge theories in two-dimensions. *Nucl. Phys.* B403:159-222, 1993.

[62] K. Yokogawa. Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.*, 33(2):451–504, 1993.

[63] K. Yokogawa. Infinitesimal deformation of parabolic Higgs sheaves. *Internat. J. Math.*, 6(1):125–148, 1995.