An Efficient Iterative Algorithm for Convex Hull Pricing Problems on MISO Case Studies

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Abstract—To increase market transparency, independent system operators (ISOs) have been working on minimizing uplift payments based on convex hull pricing theorems. However, the large-scale complex systems for ISOs bring computational challenges to the existing convex hull pricing algorithms. In this paper, based on the analysis of specific generator features in the Midcontinent ISO (MISO) system, besides reviewing integral formulations for several special cases, we revise an integral formulation of a single generator that can capture time-variant capacity, time-variant ramp-up/down rate limit, flexible min-up/down time limit, and max up-time limit constraints, as well as time-dependent start-up cost. We then build a compact convex hull pricing formulation based on these integral formulations. Furthermore, to improve the computational efficiency, we propose iterative algorithms with convergence properties, plus a complementary algorithm, to provide near-optimal convex hull prices. The computational results on MISO instances with and without transmission constraints indicate that the optimal solutions can be obtained within 20 minutes.

Index Terms—Convex Hull Pricing, Iterative Algorithm, Integral Formulation.

NOMENCLATURE

A. Set and Dimension
G Index set of all generators.
T Set of operation time span.
S Set of start-up states, i.e., S = \{1(hot), 2(warm), 3(cold)\}.

B. Parameters
Dit Load at time t. (MW)
Fsit The start-up cost of generator i in state s ∈ S.
Gsit No load cost of generator i at time t.
Qsit Piecewise linear production cost interception coefficient of generator i in segment k at time t.
Hsit Piecewise linear production cost slope coefficient of generator i in segment k at time t.
Psit/Tsit Min/max generation amount of generator i at time t. (MW)
Lt/ℓi Min-up-down time limit of generator i. (h)
Ti Maximum up-time of generator i. (h)
Vuit/Vsit Ramp-up-down rate of generator i at time t. (MW/h)
Vuit/Vsit Start-up/shut-down ramp rate of generator i at time t. (MW/h)
Th/i The down-time limit for warm/cold start. The start-up cost is hot-start cost if the shut-down time is longer than Tth, warm-start cost if it is longer than Tth and less than Tth, and cold-start cost if it is longer than Tth. (h)
S'it/Sit Time-dependent start-up/shut-down cost for generator i.
S'/Sit Constant start-up/shut-down cost for generator i.

C. Decision Variables
uit On/Off status of unit i at time t.
 vit Start-up status of unit i at time t.
wit Start-up status of unit i at time t.
δit Indicator variable, which is 1 if generator i starts at time t in state s.
xit Generation amount of generator i at time t.
fit Production cost of generator i at time t.
git Indicator variable, which is 1 if a generator shuts down for the first time at time t + 1.
 yit Indicator variable, which is 1 if a generator starts up at time t and shuts down at time k + 1.
zit Indicator variable, which is 1 if a generator shuts down at time t + 1 and starts up at time k.
φit Production cost of a generator at time s when it starts up at time t and shuts down at time k + 1.
φit Production cost of a generator at time s when it starts up at time t and shuts down at time k + 1.

I. INTRODUCTION

In the current U.S. day-ahead electricity market operated by ISOs, unit commitment and economic dispatch (UCED) problems are solved to determine the generation amount of each generator and the market clearing price. The optimal dual values corresponding to the load balance constraints in the ED problem, in which the commitment decision variables are fixed at their optimal values, are referred as the local marginal prices (LMPs) for market clearance. Since the commitment statuses are fixed, the LMPs cannot cover the start-up and no-load costs of the generators. Accordingly, uplift payments are introduced to make up the possible loss and motive the generation side market participants to comply with the commitment schedule provided by the ISOs. Since the uplift payment cost is not transparent, the ISOs aim to minimize the uplift payment cost for their daily operations.

To minimize the uplift payments, a convex hull pricing approach has been recently introduced and received significant attention. This pricing approach aims to minimize the uplift payments over all possible uniform prices. This approach has an advantage to provide an optimal uniform price...
for the system. However, on the other hand, it requires to obtain an optimal dual multiplier for a mixed-integer UCED problem and has been computationally intractable for market implementation [1].

To overcome this challenge, researchers and practitioners have explored two different approaches to derive prices that could provide good approximations for the optimal convex hull. One efficient approach is to apply the Lagrangian relaxation framework to derive a high quality approximated convex hull price in a short time. For instance, in [2], the authors developed an extreme-point sub-differential method to strengthen the traditional Lagrangian relaxation approach to obtain the price. In [1], the authors developed an approximation of convex hull pricing based on Lagrangian relaxation for MISO, named extended local marginal prices (ELMPs), and further a single-hour approximation, named aELMP, to separate from the multi-hour model to reduce the computational burden. The other approach is called the primal formulation approach. For this approach, as shown in [3], the researchers derive better linear programming approximations for the original UCED formulation and use the dual variables associated with the load balance constraints in the corresponding formulation as an approximation of the optimal convex hull price.

It can be observed that both above approaches have provided effective approaches to tackle the problem, although no optimal solutions could be obtained from either approach. Recently, in our recent work in [4], by providing an integral formulation of a general single generator UCED problem that can capture ramping as well as capacity and min-up/-down time limit constraints, an optimal convex hull price can be obtained by solving a linear program. It is shown that the commercial optimization software under default setting can obtain an optimal solution in a short time for small systems, e.g., the IEEE-118 bus system.

In this companion paper, we study the generalization of the convex hull pricing problem with the focus on the real-world MISO system, which has more features than those described in the literature. In addition, MISO has the largest generation capacity among the U.S. wholesale markets. It is required for the market clearing process to solve large-scale problems with millions of decision variables [5]. Efficient algorithms are desired to achieve a high computational efficiency. The main contributions of this paper are as follows:

1) Based on the specific features of the MISO system including various types of thermal generators, we review and develop the integral formulations for each class of generators. For the class of generators with most physical and operational restrictions, we develop the corresponding integral formulation, which can capture time-variant capacity, time-variant ramp-up/-down limit, flexible min-up/-down time limit, and max up-time limit constraints, as well as time-dependent start-up cost, besides traditional physical restrictions. This integral formulation could lead to a perfect convex hull price for the MISO system. In addition, this integral formulation can also be customized and applied to solve the instances in other ISOs.

2) To solve the large-size MISO instances, we develop an iterative algorithm, as well as its variant, to speed up the process to solve the problem. We prove that the iterative algorithm converges as the number of iterations increases. Furthermore, we develop a complementary algorithm to utilize several processors to solve the problem together, which can provide MISO the flexibility to get a better solution within a given time limit.

3) We test the algorithms on MISO instances. The numerical experiment results show that our proposed algorithm can lead to an optimal convex hull price and a minimal uplift payment for all of the testing instances within an acceptable time limit for the cases with and without transmission constraints.

The remainder of this paper is organized as follows. First, in Section II we review the integral formulations in the literature that can be used in our convex hull pricing problem. Then, in Section III we refine a new integral formulation that can capture the special characteristics of the MISO instances. After that, in Section IV we describe the efficient iterative algorithm, its variant, the complementary algorithm, and the convergence proof of this algorithm. Finally, in Section V we report computational results on MISO instances.

II. FORMULATIONS FOR SEVERAL SPECIAL CASES

There are different types of generators in practice within MISO. We first present a traditional 3-bin UC formulation as a base for building our model as follows:

\[
Z_{op}^* = \min_{f, x, u, v, e, \delta} \sum_{i \in G} \sum_{t \in T} \left( \sum_{s \in S} F_{is}^s \delta_{it}^s + S_{it} e_{it} + G_{it} u_{it} + f_{it} \right) \tag{1a}
\]

subject to:

\[
\sum_{i \in G} x_{it} = D_t, \forall t \in T, \tag{1b}
\]

\[
f_{it} \geq H_{it} x_{it} - Q_{it}^k u_{it}, \forall k, \forall t \in T, \tag{2a}
\]

\[
\delta_{it}^s \leq \sum_{j=1}^{L_s} e_{it(j-t)}, \forall s \in S/\{3\}, \forall t \in T, \tag{2b}
\]

\[
\sum_{s \in S} \delta_{it}^s = v_{it}, \forall t \in T, \tag{2c}
\]

\[
u_{it} - u_{i(t-1)} = v_{it} - e_{it}, \forall t \in [2, T], \tag{2d}
\]

\[
\sum_{j=t-L_t+1}^{t} v_{ij} \leq u_{it}, \forall t \in [L_t+1, T], \tag{2e}
\]

\[
\sum_{j=t-L_t+1}^{t} v_{ij} \leq 1 - u_{i(t-\ell_t)}, \forall t \in [\ell_t + 1, T], \tag{2f}
\]

\[
x_{it} \geq p_{ui} u_{it}, \forall t \in T, \tag{2g}
\]

\[
x_{it} \leq \bar{p}_{ui} u_{it}, \forall t \in T, \tag{2h}
\]

\[
x_{it} - x_{i(t-1)} \leq V_h u_{i(t-1)} + \bar{v}_i e_{it}, \forall t \in T, \tag{2i}
\]

\[
x_{i(t-1)} - x_{it} \leq V_i u_{it} + \bar{v}_i e_{it}, \forall t \in T, \tag{2j}
\]

where the objective function is to minimize the total cost, including start-up, shut-down, no-load, and generation costs. Constraints (1b) represent the load balance restrictions and \(x_{it}\) represents the feasible region of generator \(i\):
where $P_i^e, P_{i-1}^e, V_i^e, V_{i-1}^e$ represent time-invariant parameters. Constraints (2a) represent the piecewise linear approximation of the convex cost functions. Constraints (2b) and (2c) use indicator variables to represent each start-up type, which depend on the number of time periods the generator has been off before it was started-up, where $T^e_i$ and $T_i^e$ represent the two end points of each interval, corresponding to $T^e_i$ and $T_i^e$. Constraints (2d) represent the logic relationships. Constraints (2e) and (2f) represent the min-up/down time restrictions (i.e., $2g$ - $2h$), generation lower and upper bounds (i.e., $2j$ - $2r$), and ramp-up/down rates (i.e., $2s$ - $2t$). Now we present convex hull results for two types of generators below.

1) The set of generators with constant start-up costs and without ramping constraints (labelled as set $G_1$): Among various types of generators, low capacity fast-start gas generators are relatively easy to model in terms of physical constraints. They can start-up quickly and the ramping capability is larger than the gap between upper and lower generation limits. Also, the start-up costs are constant and not dependent on the off-time periods before start-up. Thus, constraints (2b), (2c), (2d), and (2f) are redundant and (2) can be simplified as follows:

$$Z_{up} = \min_{x, u, v, e} \sum_{t \in T} \left( S_i^e u_{it} + S_i e_{it} + G_i u_{it} + f_{it} \right)$$

s.t. (1a), (1b), $\left(f_i, x_i, u_i, v_i, e_i\right) \in \chi_i^1$, $\forall i \in G_1$.

$$\chi_i^1 = \left\{ f_i, x_i, u_i, v_i, e_i \in \mathbb{R}^+ | f_i \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \right\}$$

$$D_i^1 = \left\{ f_i, x_i, u_i, v_i, e_i \in \mathbb{R}^+ | f_i \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \right\}$$

where parameter $\kappa_t$ indicates whether the min-up time constraint is forced for the generator at time $t$ (yes if $\kappa_t = 1$ and no if $\kappa_t = 0$). Similarly, $\omega_t$ is a parameter indicating whether the min-down time constraint is forced for generator $i$ at time $t$.

For more complicated generators (besides $G_1$ and $G_2$) in which the start-up costs of the generator depend on the startup time before the start-up and also time-dependent start-up costs and ramping constraints are considered, we utilize the network flow integral formulation as described in [6]. In this formulation, we use $y_{tk}$ and $z_{tk}$ to keep track of the up and down time intervals. To incorporate the flexible min-up/down time limit constraint, we redefine the range of $k$ corresponding to the on-time indicator variable $y_{tk}$ as follows:

- If $\kappa_t = 1$, we define $y_{tk}, \forall k \in [t + L - 1, T]$, which indicates the generator must stay online for $L$ time periods after its start-up time $t$ based on the min-up time constraint.
- If $\kappa_t = 0$, we define $y_{tk}, \forall k \in [t, T]$, which indicates the min-up time is relaxed to be 1 and the generator can shut down any time after its start-up time $t$.

We can redefine the range of $z_{tk}$ and $\theta_t$ accordingly.

**B. Max up-time limit**

For some generators in the MISO market (labelled as $G_3$), there are also restrictions on maximum time periods that the generator can stay online because of machine deterioration. For these types of generators, the start-up cost of the generator is constant. The general ramping rates are not binding, except the start-up ramping.

**Theorem 1.** The convex hull description for the 3-bin model with max-on restriction can be described as follows:

$$D_i^3 = \left\{ f_i, x_i, u_i, v_i, e_i \in \mathbb{R}^+ | f_i \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \right\}$$

$$\sum_{j=t+1}^{t+T_i} v_{ij} \geq u_{i(t+T_i)} \forall t \in T \right\}$$

**III. FORMULATIONS FOR THE GENERAL MISO INSTANCES**

For the generators in MISO, besides special generators $G_1$ and $G_2$ as described in [11] there are several generator features which reflect the market needs and capture more details in practice. We study three features and derive the refined integral formulation.

**A. Flexible min-up/down time limit**

In MISO, the min-up/down time limit is set to be time variant to resolve offer data conflicts. For example, participants may submit must-run for hours 1 - 5 and hours 10 - 24 with the min-down time limit as 6 hours. This will force UCED to commit between hours 6 and 9 even if it is very expensive. So MISO developed a set of rules to ignore min-up/down-time limit if there are such conflicts. In this example, the min-down time limit is relaxed to be 1 between hours 6 and 9 so that the unit will not be committed if it is very expensive. It will force market participants to submit proper offers and prevent market manipulation. If they do want to run through, they should submit must-run for all hours. If they do want MISO to determine on/off in between, they should have 6 hours in between. To accommodate this, the refined min-up/down-time constraints in the 3-bin formulation (such as $G_1$ and $G_2$) can be described as follows (generator index $i$ is omitted for brevity):

$$\sum_{j=t-L+1}^{t} \kappa_t v_{ij} \leq u_t, \forall t \in [L + 1, T]$$

$$\sum_{j=t-\ell+1}^{t} \omega_t v_{ij} \leq 1 - u_{t-\ell}, \forall t \in [\ell + 1, T]$$

where parameter $\kappa_t$ indicates whether the min-up time constraint is forced for the generator at time $t$ (yes if $\kappa_t = 1$ and no if $\kappa_t = 0$). Similarly, $\omega_t$ is a parameter indicating whether the min-down time constraint is forced for generator $i$ at time $t$.
Proof. Based on Proposition 2 in [9], for the min-up/down time only polytope (without ramping constraints) with max up-time restrictions, the convex hull description of the feasible binary variables \((u, v, e)\) is \(\{u, v, e \in \mathbb{R}^T \times \mathbb{R}^T \times \mathbb{R}^T\}\) \(^{(2c)}\) \(-\((2h)\), start-up ramping constraints \((6)\), and linear function \((2a)\) does not affect integrality. Thus, the conclusion holds. \(\blacksquare\)

In the network flow integral formulation, we reconsider the upper bound of \(k\) for the on-period variable \(y_{tk}\) within the max up-time limit \(T_i\) to satisfy \(k - t + 1 \leq T_i\), by redefining the range of \(k\) for \(y_{tk}\) to be \([\min\{t+L_i-1, T\}, \min\{t+L_i-1, T\}]).\)

**C. Time-variant parameters**

In MISO, market participants are allowed to offer capacity and ramp rate varying by hour due to physical constraints. The time-variant parameters make the convex hull more complicated and have rarely studied in the literature. We observe that these time-variant parameters can be directly integrated into our proposed integral formulation.

**D. The integral formulation for MISO**

After considering all the general aspects described above, we refine the integral formulation (referred as EUC formulation) as follows. Note here we assume the generator has been initially on for \(s_0\) time periods before time 1. Thus, the generator cannot shut down until time \(t_0 = 1\), with \(t_0 = [L - s_0]\), due to min-up constraints.

\[
\min_{t_k \in T_k, k \in [T]} \sum_{t_k \in T_k, k \in [T]} s(k + t_0)w_k + \sum_{t_k \in T_k, k \in [T]} s(k - t + 1)u_k(k + t_0)w_k + \sum_{t_k \in T_k, k \in [T]} s(k - t + 1)u_k
\]

\[
+ \sum_{k \in [T]} s'(t - k - 1)z_{kt} + \sum_{t_k \in T_k, s \in k} \phi_{tk}^s
\]

\[
\text{s.t.} \quad \sum_{t_0} w_{t_0} = 1, \quad w_{t_0} \geq 0 \quad (10 \text{a})
\]

\[
-w_{t_0} + \sum_{z_{t_0} - k \in k, t - t_0} y_{tk} + \theta_{t_0} = 0, \quad \forall t' \in [t_0, T - 1], \quad (10 \text{b})
\]

\[
\sum_{t_k \in T_k, t \in t'} y_{tk} - \sum_{z_{t_0} - k \in k, t \in t'} y_{tk} + \theta_{t_0} = 0, \quad \forall t' \in [t_0, t_0 + \ell_t - 1], \quad (10 \text{c})
\]

where \(T_k = T_k^1 \cup T_k^2\), in which \(T_k^1\) represents the set of all possible combinations of \(t = 1\) and each \(k \in [0 + \min\{L - s_0, T\}]\) to construct a time interval \([t, k]\), \(T_k^2\) represents the set of all possible combinations of each \(t \in [0 + \ell_t - 1, T - 1]\) and \(k \in [\min\{t+L_i-1, T\}, \min\{t+L_i-1, T\}]\) to construct a time interval \([t, k]\), \(\ell_t\) represents the set of all possible combinations of each \(t \in [0, T - \ell_t - 1]\) and \(t \in [k + \ell_t + 1, T]\) to construct a time interval \([t, k]\), \(\ell_t\) represents the union of \([t_0, T - \ell_t - 1]\) and \([t'\] where \(z_{t_0} = 0\).

**Theorem 2.** The convex hull description for the general single generator MISO UC is as follows:

\[
D^1_i = \{w, z, y, \theta, t_0\} \text{ subject to } \sum_{i \in G_i} \sum_{u \in U_i} s_{i, u} + \sum_{i \in U_i} s_{i, u}^q (10 \text{a})
\]

\[
\text{s.t.} \quad (11b), (f_i, x_j, u_{i, j}, v_{i, j}, e_{i, j}) \in D^1_i, \forall i \in G_i,
\]

\[
(f_i, x_j, u_{i, j}, v_{i, j}, e_{i, j}) \in D^3_i, \forall i \in G_i,
\]

\[
(w_i, z, y, \theta, t_0, t_0) \in D^1_i, \forall i \in G_i,
\]

where the cost function \(g_{it}\) of each generator \(i\) in \(G_i \cup G_2 \cup G_3\) can be expressed as

\[
g_{it} = S_iw_{it} + S_i e_{it} + G_i u_{it} + f_i
\]

and the cost function \(g_{it}\) of each generator \(i\) in \(G_1\) can be expressed as

\[
g_{it} = \sum_{t_k \in T_k^1, k \in [T]} S_i(k + t_0)w_k + \sum_{t_k \in T_k^2, k \in [T]} S_i(k - t + 1)u_k
\]

\[
+ \sum_{t_k \in T_k, s \in k} \phi_{tk}^s
\]

This formulation is sufficient to solve the convex hull pricing problem. However, a large number of variables and constraints in the integral formulation \(D^1_i\) (i.e., \(10\)) increase the computational complexity and lead to a long solving time.

**IV. CONVEX-HULL PRICING AND ITERATIVE ALGORITHMS FOR MISO INSTANCES**

Based on the above convex hull descriptions \((5), (6), (9),\) and \((11)\), we derive the general convex hull pricing formulation \((P)\) as follows. For notation brevity, we let \(G_4 = G_1 \cup G_2 \cup G_3\) represent all the unclassified generators using formulation in \(III-D\). Since we have the convex hull descriptions for each type of generators, based on Theorem 2 in \(9\), we can provide the optimal convex hull price and minimize the uplift payment by solving the following linear program:

\[
(P) : Z^*_m(f, x_j, u_{i, j}, v_{i, j}, e_{i, j}) \leq \min_{f, x_j, u_{i, j}, v_{i, j}, e_{i, j}} \sum_{i \in U_i} s_{i, u} + \sum_{i \in U_i} s_{i, u}^q (12 \text{a})
\]

\[
\text{s.t.} \quad (11b), (f_i, x_j, u_{i, j}, v_{i, j}, e_{i, j}) \in D^1_i, \forall i \in G_i,
\]

\[
(f_i, x_j, u_{i, j}, v_{i, j}, e_{i, j}) \in D^3_i, \forall i \in G_3,
\]

\[
(w_i, z, y, \theta, t_0, t_0) \in D^1_i, \forall i \in G_4,
\]

where the cost function \(g_{it}\) of each generator \(i\) in \(G_1 \cup G_2 \cup G_3\) can be expressed as
when the number of generators in $G_4$ is large. To solve the large-scale problem ($P$), we develop an iterative algorithm, in which a relaxation without adding the constraints in (10) is solved in the first step. Then the integral formulation constraints are added gradually when needed to tighten the relaxation. Our approach can provide a very tight approximation of the convex hull price with most of the cases converging at the optimal solution and meanwhile the algorithm terminates in a short time.

A. Methodology background

Before describing the detailed algorithms, we first introduce Lemma 1 and Theorem 3 to provide a theoretical foundation to show that the uplift payment amount will decrease and converge under our algorithm. For notation brevity, we use $x \in \mathbb{R}^n$ to represent all the variables in (1), which contains binary variable set $x_1$ and continuous variable set $x_2$. The traditional 3-bin MILP UC formulation (1) can be abstracted as follows:

$$Z_{\text{op}} = \min \{c^T x | Dx = d, x \in \mathcal{X} \},$$

where

$$\mathcal{X} = \{x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} | Ax \leq b \}.$$  

**Lemma 1.** For $A \in \mathbb{R}^{m \times n}$, $A' \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^m$, $b' \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ and $c \in \mathbb{R}^n$, we consider the integer optimization problem (15), its tightened linear programming relaxation problem ($P_C$) in which $A'x \leq b'$ dominates $Ax \leq b$, and the Lagrangian relaxation $D_C$ corresponding to a dual value $\gamma$ as follows:

$$(P_C) : Z_c = \min \{c^T x | A'x \leq b', Dx = d, x \in \mathbb{R}^n \},$$

$$(D_C) : Z_c(\gamma) = \min \{c^T x + \gamma (d - Dx) | A'x \leq b', x \in \mathbb{R}^n \}.$$  

*Given an optimal dual value $\gamma$ for constraints $Dx = d$ in (16), if*  

(i) $x^* = \{x_1^*, x_2^* \}$ is an optimal solution to problem $Z_c(\gamma)$,  

(ii) $x_1^*$ are all binaries,  

*then the uplift payment given price $\bar{\gamma}$ can be calculated as $U = Z_{\text{op}} - Z_c(\gamma)$.*

**Proof.** Based on the definition of uplift payment, we know for a given price $\bar{\gamma}$ the uplift payment is the difference between the maximum profit given $\bar{\gamma}$ (defined as $P_{\text{Self}}(\bar{\gamma})$) and the profit following the ISO’s schedule (defined as $P_{\text{ISO}}$). The profit following the ISO’s schedule can be calculated as $P_{\text{ISO}} = \bar{\gamma} T - Z_{\text{op}}$, and $P_{\text{Self}}(\bar{\gamma})$ can be obtained by solving the following problem:

$$P_{\text{Self}}(\bar{\gamma}) = \min \{c^T x - \bar{\gamma} T | A'x \leq b', x_1 \in \mathbb{R}^{n_1} \}. \tag{18}$$

It is clear that $Z_c(\gamma) - \bar{\gamma} T \leq -P_{\text{Self}}(\gamma)$, since $x_1$ is relaxed to be continuous between 0 and 1 in (17). Based on (i) and (ii), $x^*$ is a feasible solution to (18) and so $Z_c(\gamma) - \bar{\gamma} T \geq -P_{\text{Self}}(\gamma)$. Thus, $Z_c(\gamma) - \bar{\gamma} T = -P_{\text{Self}}(\gamma)$.

Therefore, $U = P_{\text{Self}}(\bar{\gamma}) - P_{\text{ISO}} = -(Z_c(\gamma) - \bar{\gamma} T) - \bar{\gamma} T = Z_{\text{op}} - Z_c(\gamma)$.

**Theorem 3.** For $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$, $D \in \mathbb{R}^{p \times n}$, $b_1 \in \mathbb{R}^m$, $b_2 \in \mathbb{R}^m$, $d \in \mathbb{R}^p$, and $c \in \mathbb{R}^n$, considering the integer optimization problem (15), its linear programming relaxation $z = \min \{c^T x | Ax \leq b, Dx = d, x \in \mathbb{R}^n \}$ (for notation brevity, we define $X_L = \{x \in \mathbb{R}^n | Ax \leq b \}$, and two tightened linear programming relaxation problems ($P_1$, $P_2$) shown below,

$$\begin{align*}
(P_1) : Z_{c1} &= \min \{c^T x | Dx = d, x \in \mathcal{X} \}, \\
(P_2) : Z_{c2} &= \min \{c^T x | Dx = d, x \in \mathcal{X} \},
\end{align*}$$

in which $X_1 \subseteq X_2 \subseteq X_L$, then the uplift payments under $\gamma_1$ and $\gamma_2$ will be no larger than that under $\gamma_1$ if conditions (i) and (iii) in Lemma 1 hold.

**Remark 1.** If $X_2 = \text{conv}(X)$, then $\bar{\gamma}_2$ is a convex hull price and $Z_{\text{op}} - Z_c(\gamma_2)$ is the smallest uplift payment as shown in (16).

**Remark 2.** When the transmission constraints are considered, the above framework still holds and the convex hull price can be computed as described in (10). We omit it here for brevity.

B. The detailed algorithm

We consider the following (P1) as the starting point:

$$(P1) : Z_{ip} = \min_{f, x, u, v, e} \sum_{i \in G} \sum_{i \in T} g_{it}$$  

s.t.  

$$Z_{\text{op}} - Z_c(\gamma) \geq -P_{\text{Self}}(\gamma).$$

In this formulation, we keep the convex hull descriptions for generators in $G_1 \cup G_2 \cup G_3$ in problem (P), and relax the convex hull set $D_i^1$ for each generator in $G_4$ to be $D_i^4$. As stated above, adding EUC formulations (10) to all of the generators in $G_4$ will lead to the perfect formulation, but it will increase the computational complexity significantly. The iterative algorithm is designed to select “necessary” generators in $G_4$ and only add (19) of “necessary” generators into (P1), which can effectively tighten the formulation and improve $Z_{ip}$ with less computational burden. The detailed algorithm is shown in Algorithm 1. Here we provide the explanation.

Following the conditions in Lemma 1 we first solve (P1) and get the dual value $\bar{\gamma}$ of equation (19). Then we solve the following (P2) which is the profit maximization problem for generators based on the given price $\bar{\gamma}$:

$$(P2) : Z(\bar{\gamma}) = \bar{\gamma} T + \min_{f, x, u, v, e} \sum_{i \in G} \sum_{i \in T} (g_{it} - \bar{\gamma} x_{it})$$  

s.t.  

$$P_1, P_2, P_3, P_4.$$
To satisfy conditions (i) and (ii) in Lemma 3, we need to get an integral solution for (P2). In (P2), since there are no coupling constraints among generators, this problem is equivalent to maximizing profit for each generator independently and summing them together. For each generator \( i \in G_1 \cup G_2 \cup G_3 \), the optimal solution of \( u_i, v_i, e_i \) will be binary for any given price \( \bar{\gamma} \), since \( D_i^1, D_i^2, \) and \( D_i^3 \) provide the convex hull descriptions. For each generator \( i \in G_4 \), we may get fractional values in \( u_i, v_i, e_i \) since \( D_i^3 \) is a relaxation of \( D_i^1 \). In this algorithm, we consider the constraints in \( D_i^1 \) as a cutting plane group. When a fractional solution for a particular generator \( i \in G_4 \) is obtained in (P2), we add this cutting plane group to (P2) and (P1) and replace the objective function \( g_{it}^* \) with \( g_{it}^{\prime*} \) to ensure that this generator will have integral solutions with any given price \( \bar{\gamma} \). The set \( \Gamma \) (representing the set of generators in \( G_4 \) using the constraints in \( D_i^1 \)) is updated accordingly by adding this generator. Since the dual value \( \bar{\gamma} \) could change after update the formulation for certain generators in \( G_4 \), we need to solve (P1) again and get an updated \( \bar{\gamma} \). We thus create an inner loop to repeat this process until we could not find an updated \( \bar{\gamma} \).

Next, we further decrease the uplift payment by tightening (P1) utilizing Theorem 3. The optimal solution to the updated problem (P1) could be infeasible for constraints (10) for some generator \( i \in G_4/\Gamma \) since \( D_i^1 \) is relaxed to \( D_i^3 \). Given the optimal solution \( (f_j^*, x_j^*, u_j^*, v_j^*, e_j^*) \) of each generator \( i \) in (P1), we can identify infeasible generators by solving the following (P3) for each generator \( i \in G_4/\Gamma \) with fractional \( u_i^*, v_i^*, \) or \( e_i^* \):

\[
(P3): \min_{u_i, z_i, \theta_i, \phi_i} \quad 1 \quad \text{s.t.} \quad (23a)
\]

\[
\sum_{k \in K, t \leq s \leq k} q_{k\gamma}^i f_s^i = \sum_{k \in K, t \leq s \leq k} \phi_{k\gamma}^i, \forall s \in [1, T], \quad (23b)
\]

\[
\sum_{k \in K, t = s} z_{kt} e_s^i = \sum_{k \in K, t = s} z_{kt} + \theta_s, \forall s \in [1, T], \quad (23c)
\]

\[
\sum_{k \in K, t \leq s \leq k} y_{kt} e_s^i = \sum_{k \in K, t \leq s \leq k} y_{kt} + \phi_s, \forall s \in [1, T], \quad (23d)
\]

where constraints (23b)-(23d) build the linear mapping between the optimal solution of EUC formulation (10) and that of the traditional 3-bin MILP UC formulation (2) following Proposition 2 in [6]. If the above (P3) is infeasible for generator \( i \), then this generator \( i \) will be included in set \( \Gamma \). We update problems (P1) and (P2) by adding the constraints in \( D_i^1 \) and replacing \( g_{it}^* \) with \( g_{it}^{\prime*} \), which tightens (P1) by cutting off the infeasible solution and meanwhile ensure integral solutions in (P2) for those generators. Note here that the generators with integral solution to \( u_i^*, v_i^*, \) and \( e_i^* \) are always feasible to (P3), since the convex hull \( D_i^1 \) contains all of the feasible integral solutions.

Similarly, a new dual value \( \bar{\gamma} \) could be obtained after solving the updated (P1), and it may be able to separate infeasible solutions. Thus we create an outer loop to repeat this process iteratively to ensure that we have an optimal solution feasible to the perfect formulation (P). The iterative algorithm terminates when we obtain an optimal solution feasible to (P) and an integral solution under the corresponding \( \bar{\gamma} \) is obtained in (P2). The resulting uplift payment under price \( \bar{\gamma} \) is \( Z^* = Z_{opt}^* - Z_c(\bar{\gamma}) \).

We can observe that this algorithm satisfies the conditions in Theorem 3 in each iteration and thus the uplift payment mostly decreases in each iteration and converges.

Algorithm 1: An iterative algorithm (IA1)

**Data:** The parameters of the system

**Result:** Convex hull price \( \bar{\gamma} \) and uplift payments \( Z^* \)

1. Initialization: initialize (P1) and construct set \( \Gamma = \emptyset \)
2. Solve MILP (1) with the optimal objective value \( Z_{opt}^* \)
3. Solve (P1) and get the dual price \( \bar{\gamma} \) and the corresponding \( Z_c(\bar{\gamma}) \)
4. do
5. Set count number \( m = 0 \)
6. for \( j \in G_4 \) do
7. Given dual price \( \bar{\gamma} \), solve (P2) and get the optimal solution \( (f_j^*, x_j^*, u_j^*, v_j^*, e_j^*) \)
8. if at least one of \( u_j^*, v_j^*, e_j^* \) is fractional then
9. \( \Gamma = \Gamma \cup \{ j \} \)
10. Replace the objective function \( g_{it} \) in (P1) and (P2) with \( g_{it}^{\prime*} \)
11. Replace the corresponding constraint set in (P1) and (P2) with \( D_i^3 \)
12. Solve the updated (P1) and get the updated optimal solution for each \( j \in \Gamma \), dual price \( \bar{\gamma} \), and the corresponding \( Z_c(\bar{\gamma}) \)
13. while \( m != 0 \)
14. Set count number \( n = 0 \)
15. for \( i \in G_4/\Gamma \) do
16. if at least one of \( u_i^*, v_i^*, e_i^* \) is fractional then
17. Fix \( u_i, v_i, e_i \) in (P3) with \( u_i^*, v_i^*, e_i^* \) and check the feasibility
18. if Infeasible then
19. \( \Gamma = \Gamma \cup \{ j \} \)
20. Replace the objective function \( g_{it} \) in (P1) and (P2) with \( g_{it}^{\prime*} \)
21. Replace the corresponding constraint set in (P1) and (P2) with \( D_i^3 \)
22. The estimated convex hull price is \( \bar{\gamma} \) and the uplift payment is \( Z^* = Z_{opt}^* - Z_c(\bar{\gamma}) \)
23. while \( n != 0 \)
24. C. Complementary algorithm

It can be first observed that the outer loop which tightens (P1) and the inner loop which ensures conditions in Lemma 3 held in (P2) can be exchanged, which may result in a different converging path and lead to a different result. We denote these two as IA1 and IA2. To further tighten the results from IA1/IA2, by taking advantage of the independence of (P3) to the other parts of the algorithm, we develop a complementary
algorithm, which can be implemented parallel to check and select the potential candidates from $G_i/\Gamma$ in the resulting (P1) after IA1/IA2 to add the corresponding $D_i^t$, which could further tighten the formulation.

Assuming there are $N+1$ parallel computing nodes with one master and $N$ slave nodes, the steps of the complementary algorithm, denoted as IAC1 and IAC2, are as follows:

Step 1: Run Algorithm IA1/IA2 on the master node. After it finishes, record the resulting (P1), price $\gamma$, set $\Gamma$, and $Z_c(\gamma)$.

Step 2: Divide the generators in $G_i/\Gamma$ into $N$ groups: $G'_1, \ldots, G'_N$. The resulting (P1) and $G'_i$ are distributed to each slave node $i$, $\forall i \in \{1, \ldots, N\}$.

Step 3: For each slave node $i \in \{1, \ldots, N\}$, each generator $j \in G'_i$ is updated in the resulting (P1) in sequence following Steps 12-13 in IA1/IA2. If the optimal objective value is improved by adding a generator $j$, the resulting (P1) in the master node will be updated accordingly and solved to get improved uplift payment and dual price $\gamma$.

Step 4: The stopping rules are flexible. The algorithm will stop if one of the following rules is satisfied: i) the master node receives $n$ updated generators from the slave nodes, ii) the time limit $t_{\text{limit}}$ is reached, or iii) all of the slave nodes terminate.

The final updated convex hull price $\gamma'$ and $Z_c(\gamma')$ can be used to calculate the uplift payment as $Z^* = Z_{\text{qep}} - Z_c(\gamma')$.

V. COMPUTATIONAL EXPERIMENTS ON MISO INSTANCES

In this section, we report the performance of our proposed models and algorithms for the MISO system. More specifically, we test the performance of the proposed tight linear programming formulation, the iterative algorithms, and their complementary parts. Each MISO instance includes over 1100 generators with their corresponding bidding information provided. We randomly select 12 instances each with 36-hour operation planning horizon and compute the corresponding convex hull pricing and uplift payment. All test instances were run on a 32-processor Intel(R) Xeon(R) CPU E5-2667 v4 @ 3.20GHz 528GB with Gurobi 8.0.1 as the optimization solver. The default MIP optimality gap is set to be $1e$-6.

We report the results for the following models and algorithms:

- MIP: the traditional 3-bin MIP UC model [1];
- LMP: energy price is equal to the dual price corresponding to the load balance constraint in the UCED problem when commitment statuses are fixed at their optimal values;
- TLP: use the approximated convex hull pricing formulation (P1) to obtain the price;
- IA1, IA2: approximated convex hull pricing formulation using the algorithms described in Section IV to obtain the price;
- IAC1, IAC2: approximated convex hull pricing formulation using the iterative algorithm plus its complementary algorithm, implemented in a single processor. The setting for the complementary algorithm is as follows: 1) $N = |G_i/\Gamma|$, 2) $n = 2$, and 3) $t_{\text{limit}} = 200s$;
- OPT: use the exact convex hull pricing formulation (P) to obtain the optimal price.

We tested the cases with and without transmission constraints, respectively. The results for the cases without transmission constraints are reported in Table III. In the table, column “Solution ($)” represents the optimal cost for the system optimization model, with “MIP” representing the optimal objective for the MIP UC model and others representing the optimal objective for the corresponding linear programming relaxation, column “Uplift payment ($)” represents the total uplift payment generated from each approach, column “Time (s)” represents the computational time in seconds for each approach. Since IAC1 and IAC2 are complementary to IA1 and IA2, we only report additional time required to reach the optimal solution. That is, we use “$\circ$” to indicate that IAC1 or IAC2 stops after the stopping rule is satisfied, and “(+$$\delta$$)” after “$\circ$” to indicate that the complementary algorithm takes $\delta$ extra seconds after the iterative algorithm terminates to reach the optimal solution. “(+$$\oslash$$)” shows that the result of IA1 or IA2 is already optimal. Column “Save ($)” represents the uplift payment savings as compared to the “LMP” approach. The savings of the TLP approach are calculated as the difference of uplift payments between the TLP approach and the LMP approach. Since IA1, IA2, IAC1, and IAC2 tighten the TLP model, the savings of these four methods are represented as the extra savings beyond the TLP approach. For example, in case “C1”, the TLP approach can save $\$3,093$ as compared to the LMP approach. The IA1 approach can save extra $\$299$, which means the IA1 approach can save $\$3,093 + $\$299 = $\$3,392$ in total as compared to the LMP approach. The “OPT” provides a benchmark to indicate the efficiency of each approach. Finally, column “Optimal” indicates if an optimal convex hull price is achieved by each approach.

From Table III we can first observe that the TLP approach can effectively reduce the uplift payments with a smaller computational time as compared to the LMP approach. The price can be obtained without solving the UC problem. Second, we can observe that our proposed alternative algorithms IA1 and IA2 are very effective. They provide extra savings based on the TLP approach. Both IA1 and IA2 can solve eight out of the ten total cases into optimality. For the remaining two cases, at least one of IA1 and IA2 can achieve the optimal convex hull price. In practice, ISOs can run both IA1 and IA2 to improve the chance to obtain the optimal convex hull price. More importantly, the algorithm ran very fast and can terminate within 13 minutes for all cases. Third, for the cases whose optimal values are not reached by IA1 (or IA2), the complementary algorithm can effectively find out the “key” generators and add them into IA1 (or IA2) to ensure the optimality. Among the ten cases at most two generators are added after terminating IA1 (or IA2), which also verified the compactness of IA1 and IA2. What is more, the stopping rules for IAC1 and IAC2 can be flexible in terms of adding extra generators into the formulation, which can provide the ISOs the flexibility to get a proper price within the time limit.

The results for the cases with transmission constraints are reported in Table IV. We use the formula as described in [10] to compute the corresponding values. The results are also very promising and similar results are obtained. From the table for the two reported cases, both IA1 and IA2 achieve the optimal
### Table I

| Case | Model | Solution ($) | Uplift Payment ($) | Time (s) | Save ($) | Optimal |
|------|-------|--------------|--------------------|----------|----------|---------|
| C1   | MIP   | 3998653      |                    |          |          |         |
|      | LMP   | 3521         | 36                 |          |          | N       |
|      | TLP   | 39980331     | 428                | 15       | 3090     | N       |
|      | IA1   | 39986726     | 129                | 241      | +299     | Y       |
|      | IA2   | 39986726     | 129                | 247      | +299     | Y       |
|      | IAC1  | 39986726     | 129                | (o+)299  | +299     | Y       |
|      | IAC2  | 39986726     | 129                | (o+)299  | +299     | Y       |
|      | OPT   | 39986726     | 129                |          | +299     | *       |

### Table II

| Case | Model | Solution ($) | Uplift Payment ($) | Time (s) | Save ($) | Optimal |
|------|-------|--------------|--------------------|----------|----------|---------|
| C11  | MIP   | 61717153     |                    |          | 584      | -       |
|      | LMP   | 1667967      |                    |          | 68       | N       |
|      | TLP   | 6159521      | 92341              | 69       | 1575426  | N       |
|      | IA1   | 6160290      | 87824              | 1182     | +4717    | Y       |
|      | IA2   | 6160290      | 87824              | 1240     | +4717    | Y       |
|      | IAC1  | 6160290      | 87824              | (o+)4717 | +4717    | Y       |
|      | IAC2  | 6160290      | 87824              | (o+)4717 | +4717    | Y       |
|      | OPT   | 6160290      | 87824              |          | +4717    | *       |

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