Time ‘betwins’

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Abstract: Discussions on the Langevin Twins ‘paradox’ are most often based on a “triangular” diagram which outlines the twins spacetime travels. It won’t be our way, avoiding what we think to be a problem at the basis of numerous controversies. Our approach relies on a fundamentally different Equivalence Principle, namely the so-called “Punctual Equivalence Principle” (GBP01), from which we think that a very conformal aspect proceeds.

We present a resolution of this paradox in the framework of the so-called “scale gravity”. This resolution hinges on a clear determinism of the Twins proper times, in some definite situations, and a fundamental under-determinism in some other particular ones, the physical discrimination of which being achieved out of a precise mathematical description of conformal geometry.

Moreover, we find that the time discrepancy between the twins, could somehow be at the root expression of the second fundamental law of thermodynamics.

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I. TRIANGULAR DIAGRAMS

Very often, discussions on the Langevin twins paradox are based on a “triangular” diagram, and we would like, first, just to recall a little problem of distance measurement, evaluation and/or conception in spacetime, which appeared in one of the lightest “triangle” approach given by H. Bondi (Bon57) and in fact historically due to Lord Halsbury (see H. Dingle’s work in (?)) and involving three unaccelerated observers.

Bondi considered diagrams (referred as the Fig. 2 and Fig. 3 in his 1957 paper) that can be assembled in a unique diagram as follows (Figure 1.) with the so-called apex at the crossing point Y:

![Figure 1](image1.png)  
Figure 1.

The ticks of all the clocks are represented by numbered marks on the three straight lines AA, BB and CC, and at each tick a light ray is emitted isotropically and is represented by a dashed line. In this diagram the inertial on-earth...
twin observer represents himself on the line AA. The traveller twin observer moves along the line BB from point X to point Y where he transmits his proper time to a third crossing twin traveller coming back towards the on-earth twin along the line CC from point Y to point Z. The speeds of each travelling twin along lines BB and CC are assumed to be obviously non-vanishing, the same and constant. Also, and that is the problem we focus on, Bondi implicitly assumed that the lengths between consecutive tick marks on the lines are all the same. Clearly from the latter figure, at point X, then each twin, comparing ticks intervals of his clock to the other twin one by measuring intervals between light rays arrivals on his own line, would get longer observed time intervals due to the well-known spacetime parallax, i.e. the time dilation which can be obtained applying Lorentz transformations. Hence each twin thinks the other has a slower time. That is a clock “paradox” which is not a logical one, but based on an incompleteness of the special or general relativities to determine, because of reciprocal viewpoints in formulations of each of these relativities, who is the oldest or younger twin. Also, how must be the spacetime axis orientation choice, as indicated in the figures, i.e. the so-called space-time splitting?

But, the same can be obtained for intervals between marks varying in a particular range in such a way that the equality of length between marks on each line, or passing from one line to another one, would be broken. Hence we could obtain a kind of figure as above (Figure 2.) with again a parallax (a time dilation) at point X, but an equal time for the twins when going on from X to Z! And then there is no twins paradox. That is no more than the classical acoustic Doppler effect!

What we want to point out in this example, is the following question:
• How to physically compare distances on different paths?

By the twins experiments! Indeed, we have no ubiquity property which would allow us to be “simultaneously” on two different paths to make comparisons, superpositions of extended objects! Also, we have a very different viewpoint about the concept of “absolute simultaneous events”, and we agree about the conventionality of simultaneity as T. A. Debs and M. L. G. Redhead described it (Pet87 DR96).

• Is acceleration a fundamental tool to solve the twin paradox and/or to reply to the previous question?

We know that A. Einstein (1918; See p. 43, §3 and §4 in (Pro67)), invoked 1) the space traveller twin acceleration at the “apex” of the triangular diagram where he comes back, 2) together with the “elevator metaphor”, i.e. the so-called Einstein equivalence principle (see a complete review of this principle in (GB01)), to justify the time dilation of the travelling twin proper time from the equivalence of the travelling twin acceleration with a gravitational field effect in a “static” situation.

Then to reply to this question, we must deal firstly with a general relativity description without accelerations. It would imply a kind of fundamental change of terminology as well as in the reasoning.

Let us remark that the Einstein’s explanation contradicts the given one proposed by H. Bondi in his 1957 paper, since he showed with the “three brothers paradox” that acceleration is not necessary to explain the latter time asymmetry. Hence, this paradox would appear at the special relativity ground level and would have to be solved, at this stage, without the use of acceleration but with a general relativity description.

That was the H. Bondi standpoint, and solving a special relativity effect with help of the general relativity would be a first contradiction only if and only if paths considerations are admitted in the framework of special relativity, what we reject as precised further and we would have to accept the Einstein arguments.

But in fact, there is another contradiction (!) in the Bondi solution quoting the deep H. Dingle remark (in spite of a large amount of contradictions in Dingle’s works!) based on the Einstein foundations of the special relativity (Din57) (See also quoted by G. Builder in (Bui58)):

“It should be obvious that if there is an absolute effect which is a function of velocity then velocity must be absolute. No manipulation of formulae or devising of ingenious experiments can alter that simple fact.”

But the problem coming after, was to expand the latter Dingle remark to acceleration, and thus to the Einstein arguments. Mendel Sachs did it, and in fact at any order of the “proper time derivation” (pp.94-95 in (Sac79)):

“Einstein tried to resolve this paradox by taking into account the periods of non-uniform motion during the round trip journey… ...it implies that while [velocity] is a relative dynamical variable, [acceleration] is an absolute dynamical variable (since it acts as the cause of an absolute physical effect). But this is not true, according to Einstein’s theory of relativity! If the spatial and temporal coordinates are all relative to the reference frame in which they are expressed (in contrast with the “absolute” temporal coordinate of classical physics), then [any] order derivative of any of these coordinates with respect to any other must also be relative…[That is], acceleration is as relative as velocity is.”
This has been clearly demonstrated by Unruh \cite{Unr81} in the framework of the twin paradox and in considering the acceleration at the apex of a triangular diagram. He shown that the latter acceleration is absolutely correlated with a sudden kinematical acceleration of the on-earth twin seen by the travelling twin. Hence, neither the Bondi nor the Einstein arguments can be used, and in fact Einstein would contradict himself because of the elevator metaphor and the relativity principle!

Moreover from the Einstein elevator metaphor, we can’t be able to distinguish in an acceleration the part due to gravitation from the one due to kinematics (\textdagger). Hence acceleration is not the main characteristic of gravitation. Roughly speaking, gravitation may be the efficient cause to phenomena correlated to relative accelerations, not the converse.

But then from (\textdagger), what is gravitation if acceleration of gravitation is not the main characteristic?

On the other hand, we consider that paths are truly and only described in general relativity and we make a difference between time and duration following the H. Bergson philosophy. And if there is an absolute duration, the latter can’t be associated to gravitation which must also be relative.

It seems that the Twins pseudo-paradox is only a problem coming from special relativity and it wouldn’t have to be solved in a general relativity framework. Indeed, it appears that it is only a pure chronogeometrical effect for which general relativity is of no relevance (at least in the standard exemple). The Minkowski spacetime would be the coherent and suitable framework to compute the proper time on every worldline, right or not (otherwise, it would be equivalent to forbid computation of lengths of curves in the Euclidean geometry !). In this latter framework the acceleration wouldn’t be relative since only frames in a relative uniform unaccelerated motion are equivalent. In fact, we can consider that in each frames, relative acceleration can be measured and defined. The acceleration doesn’t appear to state an equivalence between frames, thought it would be the case in the Einstein relativity rather than in the Galilean relativity. That is the meaning of the Einstein elevator metaphor. Moreover, the heart problem is that the proper time of the special relativity can only be evoked for parallax effects, i.e. effects coming from comparison of vectors and not curves. If the Twins pseudo-paradox would come from a parallax effect, then it would have both experimental and theoretical reciprocity, expressed from the reversibility of the Lorentz transformations applied to vectors (in a tangent space) of the special relativity. Then the theory would be complet.

In fact special relativity involves to consider only vectors with same base point and not curves. If this would be possible, we would be able to deduce lengths of curves from their projections on planes (the tangent spaces) with their origins being the measurement events. It would be as if we would deduce lengths of curves from their projections! We would make a lot of errors in our lengths evaluation, unless to know the projections. It would be as to deduce the length of a road from a plane picture of the road. In fact, this latter metaphor gives the way to solve the Twins pseudo-paradox: how to know the projections. Hence, in order to make the length of curves evaluation, we need to move along the worldlines and to be able to deduce how the displacement and the projections all along are. It is a kind of projective geometry effect. Then considering curves and not vectors, we need to be in the general relativity framework even if acceleration are not considered.

\section{II. OBSERVERS TIED WITH THEIR SPACETIME ENVIRONMENT: A PUNCTUAL PRINCIPLE OF EQUIVALENCE}

Hence, an observer would be tied with a spacetime environment. Which one and what does it mean exactly? What is at the core of a \textit{"state of motion"} concept? How to evaluate distances?

The coordinate maps, from the spacetime to $\mathbb{R}$ (vector) spaces are only defined on those subspaces of points at which serial discret physical measurements (labeling, counting, coding, etc.) are performed. Only distances could be defined on those subspaces. These maps can be defined on some neighborhoods and then on open subsets. Hence, most of the \textit{“time”} we can only deduce a local metric at a point, not a field of metric in a spacetime, meaning that we have only values of \textit{germs} of metric fields at a point, i.e. $0$-jets, not defined on $\mathcal{M}$ in full generality. In fact, more generally, we have a metric attached to each point of a line, a trajectory of a spacetime ship, and this metric would be built out of a local moving frame attached to material probes (i.e. objects).

The tangent spaces $T_{p(u)}\mathcal{M}$ of moving frames, will be viewed as the spaces for comparison between given objects and rulers, callipers and watches, \textquoteleft unfolded’’ on $\mathcal{M}$. These two separate categories, namely the rulers and those which are ruled, can be viewed as \textit{“twins”} categories, and each of these objects are associated to different \textit{jets} of sections of $T\mathcal{M}$ defined on each given open set $U \subset \mathcal{M}$. Hence, each measurement necessarily involves twins objects each one associated to a unique ascrisption of a velocity to a point in $\mathcal{M}$. We think that this relation between a velocity and a point, is thus what is defining a classical object.

Then the punctual equivalence principle, we think being compatible with the previous remarks, is the following. Let us assume the 4D-ocean $\mathcal{M}$ to be a manifold of class $C^2$, of dimension 4, locally arc-connected and paracompact
(i.e. Hausdorff and union of a countable set of compacts). Let \( p_0 \) be a particular point in \( \mathcal{M} \), \( U(p_0) \) an open neighborhood of \( p_0 \) in \( \mathcal{M} \), and \( T_{p_0} \mathcal{M} \) its tangent space. The “punctual” principle of equivalence we use, states it exists a local proper diffeomorphism \( \varphi_0 \) (assumed to be of class \( C^2 \)), that we call the equivalence map, attached to \( p_0 \) putting in a one-to-one correspondence the points \( p \in U(p_0) \) with some vectors \( \xi \in T_{p_0} \mathcal{M} \) in an open neighborhood of the origin of \( T_{p_0} \mathcal{M} \):

\[
\varphi_0 : p \in U(p_0) \subset \mathcal{M} \longrightarrow \xi \in \tilde{U}_0 \equiv \varphi_0(U(p_0)) \subset T_{p_0} \mathcal{M}, \quad \varphi_0(p_0) = 0.
\]

In fact, we consider that only shapes are kept, i.e. projectable on “board”, i.e. on \( T_{p_0} \mathcal{M} \), and that we can’t evaluate distance ratios passing from \( \mathcal{M} \) to \( T_{p_0} \mathcal{M} \) (somehow the standarts can’t be compared because of non-ubiquity). In some way, it would be a consequence of a kind of time travel inability and/or time non-ubiquity, since traveling in time and time ubiquity would be needed to make comparison of distances, “simultaneously” in \( \mathcal{M} \) and \( T_{p_0} \mathcal{M} \).

Here in the figure above (Figure 3.), the point \( p_0 \) (other points could be chosen) is the crossing point of two worldlines, which can be geodesic or not, and two particular points \( p \) and \( p' \) are taken respectively on the latter. The spacetime \( \mathcal{M} \) is assumed to be locally analytic, i.e. we can make local analytic charts, in particular in a neighborhood \( U(p_0) \) of \( p_0 \). Consequently, \( \varphi_0 \) is an analytic map. From this local description, we give, in the following subsection, the transformation laws related to conformal or projective geometry and from which special relativity can be deduced and generalized. It has to be noticed that conformal geometry involves angle measurements \emph{via} stereometry (telemetry) as W. Unruh clearly demonstrated in his approach of the Twins paradox [Unr81].

An important remark has to be done about significance of “path”, since what means “path” for an unique object in an empty spacetime ? If two objects, we can only speaks about a relative path and the two paths drawn on Figure 3 (paths \( \gamma \) and \( \gamma' \)) is a physical non-sense, unless a recording observer at a crossing event at \( p_0 \) exists, and from which the two relative paths are “assessed”.

We emphasize that all in our geometrical presentation is local. Then, we will only consider jets of maps defined on opens in \( \mathcal{M} \), and never maps on \( \mathcal{M} \) itself, meaning that the global geometrical or topological structures of \( \mathcal{M} \) won’t be considered, but its local \emph{étale} presheaf structures only. Hence by considering \( \mathcal{M} \), in fact we consider a given open subset \( U \) with the same local topological properties : Hausdorff, paracompact, etc... and by \( \mathcal{M} \) we have to consider the full set (which is an equivalence class) of manifolds with the same local topological structure on \( U \) and local rings of maps on \( U \). In other words we would denote \([\mathcal{M}](U) \equiv [\mathcal{M}]\) instead of \( \mathcal{M} \) to indicate the class (i.e. the germ of manifolds on \( U \) endowed with the map \( \varphi_{p_0} \)) of \( \mathcal{M} \). In order to lighten the notation we only denote this class \([\mathcal{M}]\) by one of its representant, namely \( \mathcal{M} \).

**III. CONFORMAL AFFINE TRANSFORMATION LAWS: SPECIAL RELATIVITY WITH ACCELERATION**

This chapter can be skiped at a first lecture, since it deals with the demonstration to find smooth paths linked by an isometric function \( f \).

Then, we consider a Lorentzian metric \( g \) (not a field of) at point \( p \in \mathcal{M} \), and its corresponding Lorentzian metric \( h \) at vector \( \eta \) such that \( \varphi_0(p) = \eta \in T_{p_0} \mathcal{M}, \varphi_0(p_0) = 0 \) and \( \varphi_0(h) = g \); The same for \( p' \) but with prime symbols adding up to the previous relations, but considering only one equivalence map \( \varphi_0 \) as shown on the figure above. We denote \( \mathcal{T} \equiv T_{p_0} \mathcal{M} \).

At a first step we consider paths as \( C^2 \) not-piecewise proper embeddings (no self intersections contrarily to immersions) and consequently without critical points (any inverse image of a compact is a compact), included in
\( U(p_0) \subset M: \)

\[
\gamma : s \in [0, 1] \mapsto p \in C \equiv \text{Im} \gamma \subset U(p_0), \quad \gamma' : s' \in [0, 1] \mapsto p' \in C' \equiv \text{Im} \gamma' \subset U(p_0), \quad \gamma(1) = \gamma'(1) \equiv p_i,
\]

with \( U(p_0) \) simply connected and \( \text{relatively compact} \), i.e. \( U(p_0) \) is compact (it will insure \( U(p_0) \) to be bounded), and assumed simply connected. Moreover we consider in the sequel, paths with only 2 intersections in \( U(p_0) \). We denote \( \xi \equiv \gamma(s) \equiv \varphi_0 \circ \gamma(s) \) and \( \xi' \equiv \gamma'(s') \equiv \varphi_0 \circ \gamma'(s') \) the corresponding paths on \( T_{p_0}M, \) with \( \gamma(0) = \gamma'(0) = 0, \gamma(1) = \gamma'(1) = \iota, \) \( C = \gamma([0, 1]) \) and \( C' = \gamma'([0, 1]) \). The conformal assumption we make, from the identification of \( T_\xi T \) (resp. \( T_\xi T' \)) with \( T \), and which lighten the discussions below and thus motivating to work on \( T \) rather than on \( U(p_0), \) states that only \( \text{"a priori"} \) if the vector \( \xi \) is an element of \( \hat{C} \) (resp. \( \xi' \) an element of \( \hat{C}' \)) then \( h = e^{2\alpha t} h_0 \) (resp. \( h' = e^{2\alpha' t} h_0 \) with \( \alpha \) (resp. \( \alpha' \)) being a value \((0\text{-jet})\) in \( \mathbb{R} \) at \( \xi \in \hat{C} \) of \textit{germs} of \( C^2 \) functions defined on \( \hat{C} \), and \( h_0 \) being a metric at 0 \((= \varphi_0(p_0))\). Also we easily deduce the following relation between the 0-jets of metrics \( h \) and \( h' \) (which are not metrics fields; See appendix), respectively at \( \xi \) and \( \xi' \):

\[
h'(\ldots) = e^{2(\alpha' - \alpha)} h(\ldots).
\]

Moreover we assume that the values for the \( a \)'s are bounded on \( \hat{U}_0 \) (which is relatively compact since \( U(p_0) \) is relatively compact and \( \varphi_0 \) is a proper map) and such that to \( p \in U(p_0) \) corresponds \( a_\xi \equiv a_p \) with \( \xi = \varphi_0(p) \) (the same with prime marks). Also, in full generality, at 0 = \( \varphi_0(p_0), a_\xi' \neq a_\xi \). Then \( a_\xi \equiv a_p \) and \( a_\xi' \equiv a_p' \) are 0-jets of germs of \( C^2 \) functions defined on \( C \) and \( C' \). Also, let us remark that \( \varphi_0 \) is not necessary a conformal map.

We denote by a dot \( \cdot' \) the derivatives with respect to \( s \) or \( s' \). Then we assume that (the two worldlines \( \hat{C} \) and \( \hat{C}' \) are supposed to be in the futur cones, i.e. \( h > 0 \) and \( h' > 0 \) when on the two worldlines tangent spaces)

\[
h_0(\hat{\xi}, \hat{\xi}) = 1, \quad h_0(\hat{\xi}', \hat{\xi}') = 1,
\]

regardless of the \( s \) and \( s' \) values, where we denote

\[
\hat{\xi} \equiv \frac{d\gamma}{ds} = \frac{d\xi}{ds} = T_\xi \gamma, \quad \hat{\xi}' \equiv \frac{d\gamma'}{ds'} = \frac{d\xi'}{ds'} = T_{s'} \gamma'.
\]

Then the conditions necessarily involves that there are no critical points justifying the previous assumption in the definitions of \( \gamma \) and \( \gamma' \) as embeddings. Moreover we just need there are of class \( C^2 \) and, as a consequence, the angular discontinuities are cancelled out from the not-piecewise path embeddings assumption.

We define the 4-velocity tangent vectors \( \xi^1 \in T_\xi \hat{C} \) and \( \xi'^1 \in T_{\xi'} \hat{C}' \) by the relations

\[
h(\xi^1, \xi'^1) = 1, \quad \text{and} \quad h'(\xi'^1, \xi'^1) = 1,
\]

denoting:

\[
\xi^1 \equiv \frac{d\xi}{d\tau}, \quad \xi'^1 \equiv \frac{d\xi'}{d\tau'},
\]

where \( \tau \in [0, T] \) and \( \tau' \in [0, T'] \) are the proper times on \( \hat{C} \) and \( \hat{C}' \). Then, since \( \tau \) and \( \tau' \) are new curvilinear coordinates, then the velocity vectors are colinear and we find easily that:

\[
\hat{\xi} = e^{\alpha \xi} \xi^1, \quad \hat{\xi}' = e^{\alpha' \xi'} \xi'^1,
\]

and consequently

\[
d\tau = e^{\alpha \xi} ds, \quad d\tau' = e^{\alpha' \xi'} ds'.
\]

Let us remark that \( \xi^1 \) and \( \xi'^1 \) are vector values of \textit{germs}, i.e. 0-jets at \( \xi \) and \( \xi' \), of velocity 4-vectors fields since the \( a \)'s are values of germs of scalar fields themselves. Also we denote by \( || \cdot || \) (resp. \( || \cdot' || \) and \( || \cdot_0 || \)) the norm associated to \( h \) (resp. \( h' \) and \( h_0 \)).

Since \( \Gamma \) is simply connected, it always exists a local homeomorphism \(^1\) (assumed to be of class \( C^2 \)) on \( \hat{C} \), denoted by \( f \) such as \( f(\hat{C}) = \hat{C}' \), \( f(0) = 0 \) and \( f(\iota) = \iota \). This application \( f \) is a homotopy map assumed to keep paths

\(^1\) The 1935 Whitney’s Theorems assert that any proper continuous map from a \( k \)-dimensional manifold to a \( n = 2k + 1 \) one, is homotopic to an embedding, and two embeddings from a \( k \)-dimensional to a \( n = 2k + 2 \) one, are isotopic. Hence, the case \( n = 4 \) is the limit case for our discussion about isotopic embedded paths in manifolds.
orientations. More exactly it exists a proper differential map (an homotopy) $F : [0, 1] \times \hat{C} \rightarrow T$ such as $F(0, \hat{C}) = \hat{C}$ and $F(1, \hat{C}) = \hat{C}' \equiv f(\hat{C})$. Hence, there exists a local diffeomorphism between the two projected (on $T$) worldlines $\hat{C}$ and $\hat{C}'$, as well as between the two corresponding primary ones on $M$. Then, we easily deduce that $\forall s \in [0, 1], \exists s' \in [0, 1]$ such as $f \circ \gamma'(s) = \gamma'(s')$.

Then from $T_{\xi}f \circ T_{\xi}\gamma' ds = T_{\xi}\gamma' ds' \iff T_{\xi}f \circ \dot{\xi} ds = \dot{\xi}' ds'$ and \(2\), we deduce

$$ds' = |\det(T_{\xi}f)|^{1/4} ds.$$

Then, if $|\det(T_{\xi}f)| = 1$, meaning $f$ is an isometry, i.e. an element of the so-called Poincaré pseudogroup and therefore up to an additive constant we can set $s = s'$. Moreover we deduce in that case:

$$\dot{\xi}' = T_{\xi}f \cdot \dot{\xi}, \quad \xi'^1 = e^{(a_{\xi}-a_{\xi'})} T_{\xi}f \cdot \xi \equiv \Omega_{\xi'\cdot \xi} \cdot \xi^1.$$  \tag{7}

Then $(\xi, \xi' = f(\xi), \Omega_{\xi'\cdot \xi})$ is an element of a conformal groupoid associated to $h$ (or $h'$):

$$h'(\Omega_{\xi'\cdot \xi}, \Omega_{\xi'\cdot \xi}) = h(\xi, \xi) \Rightarrow h(\Omega_{\xi'\cdot \xi}, \Omega_{\xi'\cdot \xi}, \xi^1) = e^{2(a_{\xi}-a_{\xi'})} h(\xi^1, \xi^1) e^{2(a_{\xi}-a_{\xi'})},$$

from which we also deduce that $T_{\xi}f$ is a Lorentz transformation for the metric $h$.

In fact, in order to modify the determinant of $f$, considering the couple $(C, C')$ being fixed and given, we have in fact a class of admissible couples $(\varphi_0, f)$ which are equivalent, i.e. meaning that $(\varphi_0', f') \simeq (\varphi_0, f)$ if and only if $\varphi_0^{-1} \circ f \circ \varphi_0(C) = C'$ and $\varphi_0^{-1} \circ f' \circ \varphi_0'(C) = C'$, or equivalently $\varphi_0^{-1} \circ f \circ \varphi_0(C) = \varphi_0^{-1} \circ f' \circ \varphi_0'(C)$, taking care that the latter is not an equality between functions but image sets. Then, $f$ can be modified to be an isometry, when $\varphi_0$ is “adequately” deformed to $\varphi_0'$.

In fact, we can set the relations : $\varphi_0(C) = \hat{C}$ and $\varphi_0'(C) = \hat{C}'$, and we also define $f'(\hat{C}) \equiv \hat{C}''$ (see Figure 4.). Consequently $\varphi_0' \circ \varphi_0^{-1}(\hat{C}') = \hat{C}''$. Thenceforth, from the relation \(2\), we deduce after integration that there exists a function $\alpha_f$ such that $s' = \alpha_f(s)$, with $\alpha_f(1) = 1$ and $\alpha_f(0) = 0$. Then, we can redefine the curvilinear coordinate on $\hat{C}'$ with $s$ such that $\gamma'(s') = \gamma' \circ \alpha_f(s) \equiv \gamma_f(s) = \xi'$. From the latter we deduce that

$$h_0 \left( \frac{d\xi'}{ds}, \frac{d\xi'}{ds} \right) = |\det(T_{\xi}f)|^{1/2} \neq 1.$$  \tag{8}

On the other hand

$$\frac{d\xi''}{ds} = T(\varphi_0' \circ \varphi_0^{-1}) \frac{d\xi'}{ds}.$$  \tag{8}

And, if we consider that

$$h_0 \left( \frac{d\xi''}{ds}, \frac{d\xi''}{ds} \right) = 1,$$  \tag{8}

finally one deduces the relation which allows us to obtain $\varphi_0'$ from $\varphi_0$ in a conformal way :

$$|\det T(\varphi_0' \circ \varphi_0^{-1})| = |\det T_{\xi}f|^{-1},$$  \tag{8}

both with $\hat{C}''$ being parametrized by $s$ instead of $s''$ (see Figure 4.), and which involves that the absolute value of the determinant of $T_{\xi}f'$ equals 1. Then, we consider hereafter $f$ to be an isometry.

Then, there are two main classes of physical assumptions, closely related to the elevator metaphor: the case for which physical considerations are made somehow in reference to the paths intersection event at $p_0$, and the other considering the viewpoints of travelling observers on $C$ or $C'$. Also, we will show that the terms $\alpha_f$ are, up to a constant for units, potential of accelerations and thus, that their variations are related to accelerations vectors. It will follow, as shown below, the first case corresponds to what we call the “all-motion/all-gravitation” assumption, whereas the second corresponds to the “intermediate” one.

**Preliminaries:** We denote by $\eta$ any vector in $T$, and then from the 0-jet feature of $a_{\xi}$, we set below the definition of $b_{\xi}$ as the “gradient-like” vector (evaluated from germs of scalar fields with value $a_{\xi}$ at $\xi$), with respect to the 0-jet of metric $h$ at $\xi$ :

$$da_{\xi} \equiv da_{\eta}/\xi \equiv h(b_{\xi}, d\eta/\xi),$$  \tag{8}
(taking care that between values of germs at \( \xi \), we get the general situation: \( b_\xi \not= \vec{\nabla} a \), since \( a \) is a 0-jet at \( \xi \) and not a function: Nevertheless we can have the equality along the worldline since \( a_\xi \) can be considered as a function \( a_\xi \equiv \alpha(\tau) \) and \( b_\xi \equiv \beta(\tau) \) the given gradient of \( a_\xi \) along this same worldline) and we denote

\[
b^0_\xi \equiv e^{2a_\xi} b_\xi ,
\]

(the same with prime marks). From (4) and denoting \( \xi^2 = d\xi^1/d\tau \), we obtain also in addition the relation:

\[
\xi^2 = e^{-2a_\xi} \left\{ \xi - h_0(b^0_\xi, \xi) \xi \right\} .
\]

We have to take care of that \( h_0(\xi, \xi) \equiv 0 \) whereas \( h(\xi^2, \xi^1) \not= 0 \) in general.

We denote by \( \nabla \) (resp. \( \nabla^0 \)) the Levi-Civita covariant derivative at any \( \eta \in \mathcal{U}_0 \) (resp. at 0) associated to any given representative metric field of germs with the 0-jet metric \( h \) at \( \xi \) (resp. \( h_0 \) at 0). Then, we deduce from the definition of \( \nabla \), and when restricted on the paths and not at angular discontinuities, \( \forall \rho, \zeta \) being vector values (0-jets) at \( \xi \) of germs of vector fields on \( \mathcal{C} \):

\[
\nabla_\rho \zeta = \nabla^0_\rho \zeta + h(b_\xi, \rho) \zeta - h(\zeta, \rho) b_\xi + h(b_\xi, \zeta) \rho .
\]

And also from the definition of the Levi-Civita covariant derivative

\[
h(\nabla \zeta, \zeta) = \zeta, h(\zeta, \zeta) .
\]

In particular, we deduce all along \( \mathcal{C} \):

\[
\nabla_\xi \xi^1 = e^{-2a_\xi} \left\{ \nabla^0_\xi \xi^1 + h_0(b^0_\xi, \xi) \xi^1 - b^0_\xi \right\} .
\]

This is a Haantjes vector (see the line “b”) in formula (15) in \( \text{(Haad41)} \), which is conformally equivariant, i.e. \( \nabla_\xi \xi^1 \) is transformed to \( \nabla^\prime_\xi \xi^1 \) by \( \Omega_\xi^\prime \xi \). Let us remark the important fact that if we set the so-called meshing assumption of Ghins and Budden \( \text{(GB01)} \):

\[
\nabla^0_\xi \equiv \hat{\xi} ,
\]

a colinearity which is possible from (2) and \( h_0(\nabla^0_\xi \hat{\xi}, \xi) = \hat{\xi}, h^0(\hat{\xi}, \hat{\xi}) = \hat{\xi}(1) = 0 \). Then from (11) we find on \( \mathcal{C} \):

\[
\nabla_\xi \xi^1 = \xi^2 + 2h(b_\xi, \xi^1) \xi^1 - b_\xi .
\]

We will see that \( \nabla_\xi \xi^1 \) can be attributed to an acceleration given in the proper frame by mechanic gauges like our human bodies when “feeling” accelerations... whereas the \( \xi^2 \) alone is a measured acceleration of motion with respect to another external point in the vicinity of \( \xi \), and thus having a conventional status as it will be precisaed in the sequel in recalling that the covariant derivatives are also conformally equivariant vectors on the contrary to the \( \xi^2 \)’s. We can also notice that:

\[
h(\nabla \xi, \xi^1, \xi^1) = 0 ,
\]

from which we deduce with the relation \( \text{(11)} \) that

\[
h(\xi^2, \xi^1) = -h(b_\xi, \xi^1) \quad (\not= 0 \text{ in full generality}) .
\]

We call \( \nabla_\xi \xi^1 \) the “gauged acceleration”, i.e. the acceleration gauged by its velocity \( \xi^1 \).

Case 1 : (“all-motion/all-gravitation” assumption) Then, we have the formula \( (\|\xi^1\| > 0 \implies \text{the signature of } h_0, h \text{ and } h' \text{ is } (+,-,-,-)) \):

\[
\ln \|\xi^1\| = a_\xi + \ln \|\xi^1\|_0 = 0 ,
\]

meaning that at \( \xi \) any potential of acceleration with value \( a_\xi \) assumed to be due to gravitation, can also be considered as, or equivalently as, having a \( \xi^1 \) velocity origin. It involves also from \( \text{(17)} \) that scaling might be due to velocity, i.e. a length contraction phenomenon in special relativity.
Also from the Haantjes vector (14), if $\nabla_{\xi_1} \xi^1$ is set to 0 (we consider a geodesic, i.e. special relativity), then we obtain:

$$\xi^2 = b_\xi - 2h(b_\xi, \xi^1) \xi^1.$$  \hspace{1cm} (18)

And, if the motion, i.e. $\xi^1$, is normal to the acceleration vector of gravity $b_\xi$, then $\xi^2 \equiv b_\xi$ and $h(\xi^2, \xi^1) = 0$. In other words, there is no gravitation but only motion, or no motion and gravitation only: All is motion or exclusively all is gravitation. But also it shows that vanishing of $\nabla_{\xi_1} \xi^1$ is the indicator for free-falling (geodesic) motions, inducing a local special relativity, as demonstrated again by Ghins and Budden (GB01).

Also we add (and it is important for the sake of latter physical interpretations) that $\xi^1$ (or $\xi^1$ anyway) has the status of a “transported relative velocity” between two travellers at $\xi'$ and $\xi$. Indeed, at first, it is a relative velocity as it suffices to verify at $s = s' = 0$. For that, denoting by $\eta^1$ (resp. $\eta^1$) the vector $\xi^1$ (resp. $\xi^1$) at $0 (= \varphi_0(p_0))$ with $||\eta^1||_0 = ||\eta^1||_0 = 1$, and $T_\xi f$ being the tangent map of $f$ at $\xi$, then we easily deduce if $a_0 = a_0$ that $T_{b_\xi} f$ is a matrice of isometry (i.e. a Lorentzian one in the pseudo-Euclidean case) and that $\eta^1 = T_{b_\xi} f \cdot \eta^1$ is a relative velocity vector since $\eta^1$ would be the time-like tangent vector attached to $\gamma$ at $p_0$. And, secondly, this vector $\eta^1$ is transported along $\gamma'$ when considering $s' \neq 0$. Hence, the $f$ isometric homotopy involves that the Lorentz group is an holonomy group in the present case. Then, we consider hereafter and throughout the paper that $a_0 = a_0$.

**Case 2 : (intermediate assumption)** Coming back to the formula (14), we ask for how an observer, in his own proper frame, can evaluate a vector such as $\xi^1$ ? Indeed, in writing “$\xi_1$”, we mean a 4-velocity at $\xi$ but with respect to which observable point in the closed vicinity of $\xi$ ?! This would be the case with respect to the initial crossing point 0 if still “observable” in the future meanwhile by definition this event doesn’t exist any more. In fact from an experimental point of view, since an absolute reference point doesn’t exist, $\xi^1$ appears to be arbitrary while it is not (Ber93, Ber96, Ber00, see appendix II) ! The frames “appear” to become relative themselves as well as the vectors (14), which is the “essence” of the relativity theory.

As a consequence, the observer can split the gravitational part and the velocity one in such a way that the relation (17) is always maintained. That means he could consider $\bar{a}_\xi \equiv a_\xi + v_\xi$, possibly at an other point $\xi$ with velocity $\xi^1$, and $\ln ||\xi^1||_0 \equiv \ln ||\xi^1||_0 - v_\xi$ as other admissible potentials and velocities with an other potential of gravity $v_\xi \equiv v_\xi(\xi)$ being a priori an arbitrary value. Hence what would be really assessed as an intermediate situation, could be only the relative discrepancy, i.e. $v_\xi \equiv \bar{a}_\xi - a_\xi$, or $v_\xi \equiv \ln(||\xi^1||_0/||\xi^1||_0)$, or (up to an arbitrary multiplicative constant) :

$$\frac{1}{3} v_\xi \equiv (\bar{a}_\xi - \ln ||\xi^1||_0) - (a_\xi - \ln ||\xi^1||_0) = \bar{a}_\xi - a_\xi - \ln(||\xi^1||_0/||\xi^1||_0),$$  \hspace{1cm} (19)

definition depending on the experimental protocol for evaluation and/or the convention of splitting we adopt in the latter protocol to discriminate between velocity and potential of gravity origins (thanks to the Einstein elevator metaphor translated in the potentials of acceleration case rather than accelerations). Let us remark that the last definition for $v_\xi$ is analogous to (14) with the difference between $\xi^2$ and $b_\xi$. Roughly speaking, the normalization of $\xi^1$ expressed by the relation (17), won’t be experimentaly obtained, but the relative scalar $v_\xi$ only.

It matters that if the measurements of the variations of $v_\xi$ reflect those of $\bar{a}_\xi$ and $\ln ||\xi^1||_0$ satisfying an analogous relation (14), then a increasing of $\bar{a}_\xi$ to 0 (considering a negative attractive potential at $\xi$) when $\xi - \xi$ tends toward infinity, involves a decreasing of $||\xi^1||_0$ and thus, an increasing of the modulus of its 3-vector part, thanks to the signatures of the metrics. The latter modulus increasing could make objetcs at $\xi$ and $\xi'$ look further away. This could be related to the leak of galaxies and the Hubble constant would have to be taken into account in the definitions of the potentials $a$. On the other hand, this modulus variation could be related to object moves bringing them closer together. To discriminate between these two options, we can evoke the Unruh paper (Unr81) and his discussion on the twins paradox based on principles of stereometric measurement in experimental protocols. As he clearly demonstrated, a change of acceleration is relative, i.e. if a twin takes the decision to accelerate and stop to be in an inertial frame, or is suddenly embedded in a gravitational field (thanks to the Einstein elevator metaphor !), then his vectorial acceleration “feel” is exactly the vectorial opposite of the kinematical acceleration of the other twin, as in a kind of action-reaction principle. In other words, and in our formalism, if $b_\xi$ becomes suddenly non-vanishing whereas $\nabla_{\xi_1} \xi^1 = 0$ and $h(b_\xi, \xi^1) = 0$ all along this variation (geodesic motion and/or special relativity), then $b_\xi = \xi^2 = -\xi^2$, i.e. each twin “sees” the other to be suddenly moving away if $b_\xi$ is directed in the $\xi$ to $\xi$ direction (let us recall that the vectors $\xi^1$ are evaluated with respect to a point, which could be $\xi$, but different from $\xi$). Hence, a variation $\Delta b_\xi$ is the opposite of a variation $\Delta \xi^2$. Then, passing from $\xi$ to $\xi$ with $b_\xi$ decreasing and directed all along in the $\xi$ to $\xi$ direction, means $\Delta b_\xi$ at $\xi$ is directed in the $\xi$ to $\xi$ direction, and then $\Delta \xi^2$ at $\xi$ is directed in the $\xi$ to $\xi$ direction, meaning we have a leak.
But in fact, without any other point of reference such as $\hat{\xi}$, the velocity $\xi^1$ could be only deduced by integration from the vector $\nabla_{\hat{\xi}} \xi^1$ which can always be obtained from local mechanic gauges without need of distance measurements. But a constant of integration would remain and since it would be deduced by integration, from recorded measurements of the mechanic gauges states, it would refer to a kind of “mean” frame, and not to the moving frame at which the recordings finish, which is the one containing the right velocity vector $\xi^1$. Hence the result would be purely formal or virtual though less arbitrary or relative, leading to “virtual velocities $V^{1\gamma}$”. This has to be linked strongly to the concept of “phantasmal or virtual viewpoint” defined by Bergson [Ber68, Ber88, Ber99, see appendix III), and also to the convention involved in defining spacetime coordinates in the special relativity framework as W. Unruh demonstrated again [Unr81].

Also from (11) and (20), we deduce

$$\xi^2 + 2h'(b_{\xi'}, \xi^1)\xi^1 - b_{\xi'} = \Omega_{\xi', \xi}\{\xi^2 + 2h(b_{\xi}, \xi^1)\xi^1 - b_{\xi}\}.$$  

(20)

Also, clearly, this shows the affine feature of the acceleration of gravitation vectors when passing from a path to another path. This is again close to the Einsteinian relativity and to the Einstein’s principle of equivalence. Indeed the elevator metaphor in this principle, points out the affine feature of the acceleration which involves equivalence between relative uniformly accelerated frames when the latter have a relative velocity $\xi^1 (\tau' = 0)$ and relative acceleration $\xi^2 (\tau' = 0)$ at their crossing point $p_0$.

A. Consequences on physical interpretations

1. Integrations along worldlines and a conformal Langevin twins paradox solution

In full generality, the time discrepancy at $\iota$ is defined by the relation (where $\hat{\mathcal{L}} \equiv \hat{C} \circ \hat{C}^{-1}$):

$$\Delta_{\iota}(\hat{\mathcal{L}}) = \int_0^1 h(\xi'^1(s), \xi^1(s)) \, ds - \int_0^1 h(\xi'^1(s), \xi^1(s)) \, ds = \int_0^1 h(\xi'^1(s), \xi^1(s)) \left( e^{3(a_{\xi'(s)} - a_{\xi(s)})} - 1 \right) \, ds,$$

(21)

where $a_{\xi'(s)}$ and $a_{\xi(s)}$ are scalar sections from $[0, 1] \ni s \mapsto \mathbb{R}$ which are not “factorizable”, meaning it doesn’t exist scalar fields $A' \circ A$ from $\mathcal{T}$ to $\mathbb{R}$ such as $a_{\xi'(s)} \equiv A' \circ \gamma'(s)$ and $a_{\xi(s)} \equiv A \circ \gamma(s)$ (the same for the velocity vectors). This remark forbids, by pull-back by $\gamma$ (and $\gamma'$), to consider equalities such as

$$\int_0^1 h(\xi'^1(s), \xi^1(s)) \, ds \equiv \int_{C' \times C} h(\xi'^1(\xi'), \xi^1(\xi)) \, d\tau(\xi),$$

and then pulled back 1-forms (fields of co-vectors) $h(\xi'^1(\xi'), \xi^1(\xi)) \, d\tau(\xi)$ on $\mathcal{T} \times \mathcal{T} \ni (\xi, \xi')$.

Or equivalently, it forbids to write $(\hat{\mathcal{L}}^{-1} \equiv \hat{C} \circ \hat{C}^{-1})$:

$$\Delta_{\iota}(\hat{\mathcal{L}}) \equiv \int_{\hat{C} \times \hat{C}^{-1}} \omega,$$

(22)

where $\omega$ is a 1-forms along the loop equals to $h(\xi'^1(\xi'), \xi^1(\xi)) \, d\tau(\xi)$ on $\hat{C} \times \hat{C}'$ and $h(\xi'^1(\xi'), \xi^1(\xi)) \, d\tau'(\xi')$ on $\hat{C}' \times \hat{C}$. A few comments are needed to explain our definition of $\Delta_{\iota}(\hat{\mathcal{L}})$, and precisely the definitions of the two integrals in (21). Let us define $\delta \tau \equiv \gamma_0 \, e^{\gamma_2} \, ds$ where $\gamma_0 \equiv h_0(\xi'^1, \xi^1)$. We can consider $\gamma_0$ as the “gamma” coefficient usually used in special relativity, and then, at a first approximation, we write $\gamma_0 \simeq 1 - \frac{1}{2} \frac{\vec{u}^2}{c^2}$ where, at first approximation again, $\vec{u}$ is the modulus of the relative 3-vector of velocity associated to $\xi'^1$, and $c$ the speed of light. Then we deduce the following approximation :

$$\delta \tau - ds \simeq \left( a_{\xi} - \frac{\vec{u}^2}{2c^2} \right) \, ds.$$

It follows that considering the following ascriptions : $\delta \tau \leftrightarrow \tau$ (“the time recorded by the flight”), $ds \leftrightarrow \tau_0$ (“the ground reference time”) and at a first approximation : $a_{\xi} \leftrightarrow \text{the potential of gravity}$ [HK72], we obtain exactly the formula given by J. C. Hafele and R. Keating when they computed the theoretical infinitesimal time discrepancy between two flying atomic clocks [HK72, formula (2)]:

$$\tau - \tau_0 = [gh/c^2 - (2R\Omega v + v^2)/2c^2] \, \tau_0,$$
where $R$ is the earth radius, $\Omega$ the angular velocity of the earth and $v$ the velocity of the flight with respect to the ground frame on earth, $h$ the altitude of the flight and $g$ the acceleration of gravity constant.

In fact the Hafele and Keating experiment and the latter ascription would require to assume and to consider a third trajectory denoted $\tilde{C}_0$ from 0 to $\iota$ for the “fixed” (on earth) ground frame with the metric $h_0$ all along, i.e. a third embedding $\gamma_G \equiv \xi_G$, with $\bar{u}$ being the relative velocity with respect to this ground frame (see Figure 5. below).

Then we would consider two time discrepancies : one between $\tilde{C}$ and $\tilde{C}_0$, and the other between $\tilde{C}'$ and $\tilde{C}_0$. Then we would substract them in such way that we would recover our times discrepancy definition for which just two paths have to be taken into account, namely $\tilde{C}$ and $\tilde{C}'$.

Another argument would be that such formula expresses a “kind” of transport (from 0 to $\xi$ and $\xi'$) of the parallax effects of the special relativity such as the time dilatation.

Also, a very fundamental consequence of (21) is that proper times in general relativity do not have the same definition as in special relativity. They must be also “relative”. Indeed, if the $\tau$'s are proper times coming up and defined from special relativity, the $\tilde{\tau}$'s are those times which would have to be considered and defined as the relative proper times of the general and special relativity; “Relative” because of the terms $\gamma$ and $\gamma'$ depending on three vectors ($\xi^1$, $\xi'^1$, and $\xi'$ or $\xi$) whereas the $\tau$'s depend on one vector only ($\xi$ or $\xi'$). With such definition, we insure a continuity between general and special relativity. But also, and as very fundamental point, general relativity must deals with metrics such as $d\tilde{\tau}^2$, defined in the case of factorizable embeddings on $T \times T$ or $\tilde{T} \times T$ in the case of given sections; And consequently, spacetime would be defined by metrics on $T \times T \simeq M \times M$ which would the Spacetime of the general relativity and not $M$.

As a result of our definition for the time discrepancy, $\Delta_1(\tilde{L})$ is not necessarily vanishing, unless the relative variation of 0-jets of potential $\Delta a \equiv a_{\gamma'} - a_{\gamma}$ equals zero all along the paths. Let us remark that the difference of the two integrals can be merged into one only because $f$ is an isometry, i.e. a Lorentz transformation. It matters to notice that, although $U(p_0)$ is simply connected, the loop integral which could define $\Delta_1(\tilde{L})$ will not necessarily vanish (!) even if in all cases the loop $\tilde{L}$ is contractible (homotopic to the point 0). In fact, we can’t do this contraction in the loop integral, since it would require the knowledge and the unicity of a particular field of a continuous 1-form to allow the loop contraction in the whole of $U_0$ whereas we only have co-vector values $\delta\tilde{\tau}$ of germs of 1-forms, i.e. a local ring of 1-forms. Nevertheless, if we have possibly 1-forms $\delta\tilde{\tau}$, the latter would be “dilatated” by the pull-back defined from the loop contraction and it could not necessarily reduce the time discrepancy to 0. Then a variation of $\omega$ on the loop involves a discrepancy of time “betwins”. Thus, it rules out the necessity of introducing a peculiar given global topology (homotopy) of $M$ since the problem is solved locally on $U(p_0)$.

The case for which $\Delta a \equiv 0$ constantly along the path (which doesn’t need that the $a$’s are potential scalar fields depending on the $\xi$’s, i.e. factorizable), means there is no relative potential of acceleration (there may have non-vanishing difference of potentials of gravity since the $a$’s are not potentials of gravity but of acceleration only); And then, it involves necessarily that $\Delta_1(\tilde{L}) = 0$. That was the result given by H. Bergson in appendix III of “Durée et simultanéité” in 1922. But the definition of $\Delta_1(\tilde{L})$ involves also that a relative potential of acceleration ($\Delta a$) can produce a time discrepancy as A. Einstein noticed it about the twins paradox! Roughly speaking, we can consider the potentials of acceleration as kind of “catalysts” for the special relativistic contribution to the time discrepancy $\Delta_1(\tilde{L})$.

But, as mentioned previously in the case 2 (i.e. the intermediate assumption case with $\tilde{a}_{\gamma} \equiv a_{\gamma'}$), this integral can’t be computed by any observer because of generally unknowable 0-jets of potentials $a_{\gamma}$ and $a_{\gamma'}$. We must take care in the Hafele and Keating experiment that $a_{\gamma}$ is the potential of gravity at a first approximation, since we must consider that there is a neglected dependency with respect to the norm $\|\xi'\|_0 \simeq 1$ coming from the relation (17). Then, we can write from (17) (again we consider the $\xi^{(1)}$’s as non-factorizable proper embeddings from $[0, 1]$ to the

![Figure 5.](image-url)
Two Worldlines” experimental protocols

a. The tools

The Hafele and Keating experiment is what we call a three worldlines experimental protocol and this protocol allows us to avoid the integrals merging in the definition of the time discrepancy. In fact, we can consider this protocol as two “two worldlines” protocols, each one associated to a couple “aircraft-ground base”. In merging the integrals, we consider only one two worldlines protocol, and we can obtain the time discrepancy formula between this protocol as two “two worldlines” protocols, each one associated to a couple “aircraft-ground base”. In merging protocols gives a deduced non-approximated result in accordance with the time discrepancy observation at any point undeterminism which can’t be cancelled out expect for very specific experimental situations. But the goal is to find main difficulty results from the time delay between what each twin observes from the other. It is the first cause of undeterminism which can’t consider relative velocities as special relativity does. On contrary, our formula includes in the same time, positions (usual general relativity) and velocity (special relativity) in a unique formalism. The Hafele and Keating computations shows an incompletness of the general relativity to include special relative effects, and as a consequence, we would have two separate theories! This successful experiment shows successfully how the special and general relativities could be viewed as different theories. It is not the case as our time discrepancy formula exhibits it.

2. “Two Worldlines” experimental protocols

a. The tools

The Hafele and Keating experiment is what we call a three worldlines experimental protocol and this protocol allows us to avoid the integrals merging in the definition of the time discrepancy. In fact, we can consider this protocol as two “two worldlines” protocols, each one associated to a couple “aircraft-ground base”. In merging the integrals, we consider only one two worldlines protocol, and we can obtain the time discrepancy formula between one aircraft and the ground base by substituting $\tau$ by $s$, $\tau'$ by $\tau$, $h'$ by $h$ and $h_0$ by the formula (23). But the great difficulty comes from evaluations of non-approximated values of the fields of gravity and the velocities. The experimental accord with the theory as been obtained at first approximation, and the Hafele and Keating experiment can’t be used, because of the approximations, in cases of high velocities and fields of gravity. Then, we give, as examples, the following two worldlines protocols to emphasize the experimental difficulties which can be encountered in obtaining non-approximated experimental values and the resulting undeterminism it could involve. Of course, the main difficulty results from the time delay between what each twin observes from the other. It is the first cause of undeterminism which can’t be cancelled out expect for very specific experimental situations. But the goal is to find protocols giving a deduced non-approximated result in accordance with the time discrepancy observation at any point on the worldlines and especially at the ending crossing point $\iota$, which is in fact the meaning of the twins pseudo-paradox resolution. Moreover we consider a peculiar and may be the worth protocol in order to exhibit at least all kind of experimental difficulties. It is not the simplest one but the one which could reveal the full set of experimental problems.

Thus, we consider an initial time synchronization of some processes at the same spatial position, i.e. at the initial crossing point. And also, constant known time delay or ahead of time of “computation” between each algorithmic steps involved in any embarked travelling or on-earth processes. We would say that we would have clock processes, but one of the main advantage of the protocols presented below, would be rather to give precise correspondances between spatial positions and time positions, i.e. a precise evaluation of the positions events in the spacetime or equivalently a precise deduction of the two worldlines.

Hence, these protocols must give for any values of $\tau$ and $\tau'$, i.e. all along the two worldlines:

- The exact values of the spacetime positions $\xi$ and $\xi'$ as well as the velocities $\xi^1$ and $\xi'^1$.
- The values of $a_\xi$ and $a_{\xi'}$.
- Processes of transmission of the latter values between each twin, as well as their corresponding proper times $\tau$ and $\tau'$.

We have to focus on the fact that these values don’t correspond to the ones observed by the twins when no transmission processes of these latter values exist. Hence, we consider the “observed values” we denote by brackets: $[\ldots]_{Ob(\xi)}$ or $[\ldots]_{Ob(\xi')}$.
The fundamental tools, we will consider, are the covariant derivatives of the velocities:

\[ \nabla_{\xi^1} \xi^1, \quad \nabla_{\xi^1} \xi^1. \] (24)

These are accelerations “felt” by the twins, we call the “travelling” twin and the “on-earth” twin. These accelerations can be experimentally evaluated from mechanical gauges in accelerometers, embarked on board with the twins. This is a kind of “blind” experiment since no knowledge of the spatial (i.e. geometrical) environment is required by the twins. Also, these accelerations avoid to define a convention in the splitting of the accelerations, because of the elevator metaphor, between parts due to kinematical accelerations and parts due to accelerations of gravity.

b. A “two worldlines” protocol with acceleration measurements. In this first protocol, which is the most complex, we consider a process of transmission of the data given by the accelerometers, from the travelling twin to the on-earth twin. The latter, receiving these data, geometrically interprets the associated covariant derivative of \( \xi^1 \) as being the same 4-vector but “relative” to his own proper frame at \( \xi \). We call this 4-vector the “observed covariant derivative at \( \tau \)”: \n
\[ [\nabla'_{\xi^1} \xi^1]_{Ob(\tau)}, \]

and which differs from the original one as outlined in the figure below (Figure 6.; \( \tau'(\tau - c \Delta(\tau)) \)) being the \textit{a priori} unknown value of \( \tau' \) at which the data are transmitted to and received at \( \xi(\tau) \):

Roughly speaking, the observed covariant derivative is oriented with the same angle with the curve \( \hat{C} \) than the “original” covariant derivative does with the curve \( \hat{C}' \). Then, from the definition of the time delayed transformation \( \Omega_{\xi',\xi} \) at \( \tau - c \Delta(\tau) \), with \( \Delta(\tau) \equiv [\nabla(\tau)]_{Ob(\xi)} \) being the \textit{observed} spatial distance between the twins evaluated by the on-earth twin at \( \tau \) (or \( \xi \)), thanks to his clock, and that we denote by \( \Omega^1_{\xi',\xi} \), we deduce that the covariant derivative at \( \xi'(\tau') \) is:

\[ \nabla'_{\xi^1} \xi^1 = \Omega^1_{\xi',\xi} \cdot L(\tau, \tau - c \Delta(\tau)) \cdot [\nabla'_{\xi^1} \xi^1]_{Ob(\tau)}, \] (25)

where \( L(\tau, \tau - c \Delta(\tau)) \) is the parallel transport from \( \tau - c \Delta(\tau) \) to \( \tau \).

Then, by integration of this acceleration with respect to the proper time of the on-earth twin, we can deduce the velocity \( \xi^1 \) at \( \xi'(\tau') \) (corresponding to \( \xi \) at \( \tau - c \Delta(\tau) \)):

\[ \xi^1 = \xi^1_0 + \int_0^{\tau} \Omega^1_{\xi',\xi} \cdot L(u, u - c \Delta(u)) \cdot [\nabla'_{\xi^1} \xi^1]_{Ob(u)} \, du, \] (26)

with \( \xi^1_0 \) being the velocity at 0. But, from the definition of \( \Omega^1_{\xi',\xi} \), we see that the latter formula is in fact an integral equation with respect to \( \xi^1 \):

\[ \xi^1 = \xi^1_0 + \int_0^{\tau} \left\{ \frac{e^{-\nu_t^2/3}}{\|\xi^1\|} T_{\xi^1} f \right\}_{u-c \Delta(u)} \cdot L(u, u - c \Delta(u)) \cdot [\nabla'_{\xi^1} \xi^1]_{Ob(u)} \, du, \] (27)

and which has to be solved to provide a deduced time discrepancy at the ending crossing point \( \epsilon \), but also for any given value of \( \tau \). In this integral equation, the tangent map \( T_{\xi^1} f \) is also depending on and defined univocally by the velocity \( \xi^1 \) from (4), but \( \nu_t(\xi') \) must be experimentally evaluated or deduced. This latter can be experimentally evaluated by the on-earth twin, and simultaneously with the covariant derivative data reception, in computing the potential field of gravity \( \nu_t(\xi') \) deduced from the known masses distributions or deduced by subtracting to frequency shifts of spectral light rays (for instance) those due to relativistic Doppler effects and then from evaluating \( \nu_t(\xi') \) from the resulting frequency shifts at rest only due to gravity fields due to the masses distributions.
As Unruh showed in his paper on the twins paradox, the latter geometrical evaluations of the observed positions \([\xi'_0(\Omega)]\) and the observed velocities \([\xi'^{1}_{0}(\Omega)]\) which are necessary to deduce the relativistic Doppler effects for the evaluation of \(v_\xi(\xi')(\neq v_\xi([\xi'_0(\Omega)])\), can be done by telemetry for instance, but other meanings exist, and that it involves spatial position and velocity measurements with respect to a “virtual” or “conventional” system of coordinates, as if the spacetime domain between the on-earth and the travelling twins would be a flat spacetime; And then without deviations of the light rays due to a curvature. Hence, the computed \(\xi'^{1}\) would give a vector with respect to the latter convention of flatness. In fact, it is not a theoretical difficulty as well as an experimental one. Indeed, it is coherent with the definition of the observed covariant derivative, since the latter will be interpreted by the on-earth twin precisely as an acceleration of the travelling twin in a flat spacetime; The latter convention being the simplest since the contrary would need complex or impossible spacetime curvature measurements all along the transmission of data worldline and moreover simultaneously during the transmission! In order words, the resulting computed velocity \(\xi'^{1}\) will be considered by the on-earth twin as a vector in the same frame as the initial velocity \(\xi'_0\), i.e. in the on-earth twin frame at the initial crossing point 0. Then, the on-earth twin would consider his proper frame relatively to the transmitted data and the telemetric (for instance) evaluations as a Galilean frame, i.e. in abstractio of his accelerations. But then, relatively with respect to this initial frame, it will correspond exactly to the true velocity vector \(\xi'^{1}\) at \(\tau - c \triangle(\tau)\). Then the integral \(\int T\) can be computed at \(\iota\) or at any point on the worldline and the result compared with the observed time discrepancy (if \(\tau'(\tau - c \triangle(\tau))\) is also transmitted and received at \(\tau\), giving the same result: \(\tau' = \tau - c \triangle(\tau)\).

Hence we see all the complexity of such protocols.

IV. CONCLUSION AND SELECTED VIEWPOINTS

A. Determinism and Temperature

Nevertheless, this computation, before the experimental observation of the time discrepancy, only and only if \(\xi'^{1}\) is a vector field and not a set of vector values at \(\xi\) of germs of vector fields on \(T_{p_0}\mathcal{M}\), i.e. the system is deterministic, meaning that \(\xi'^{1}\) is given by the flow map \(\Xi\) such that \(\xi'^{1} = \Xi((\eta'^{1}, s)\) where \(\eta'^{1}\) is the value of \(\xi'^{1}\) at \(s = 0\), i.e. at the origin 0. On contrary, if \(\xi'^{1}\) is a value of germs which are not determined by the local geometry of \(\mathcal{M}\) (striated by the trajectory lines of the vector fields) but by external forces with an exogenous or endogenous origin, acting along the trajectory (we think about a rocket producing external forces driven by a human being at will; Nevertheless, we could also consider “programmable” routes defining the paths, and then the system would remain deterministic in that particular case) the time discrepancy is undeterminable before the point \(\iota\).

Moreover, we have to consider the paths in \(TT\) can be computed of the non-factorizable embeddings used to compute the time discrepancy, and we can represent the lifted paths \(\hat{\mathcal{C}}^{1}\) and \(\hat{\mathcal{C}}^{n}\) as in the Figure 5 below which could be viewed as a Clapeyron cycle! It would suggest a link between the positivity of the entropy via the Clausius inequalities relative to the time discrepancy ascribing the \(\xi's\) as the so-called “temperature 4-vectors”.

Indeed, from \(\Pi\) and \(\Pi\) we deduce

\[
\|e^{a\xi} \xi^{1}\| = \|e^{a''} \xi^{1}\|
\]

where the norms \(\|\ldots\|\) are irrespectively taken from one of the three metrics \(h, h'\) or \(h_0\). Then the vectors \(u\) of the “form” \(u \equiv e^a \xi^1\) are invariant with respect to “transported” Lorentz transformations, i.e. equivariant with respect to diffeomorphism keeping equivariant the metrics; The latter being consequently elements of the so-called Poincaré pseudogroup. Obviously \(u\) is the “absolute” velocity vector \(\xi\). Setting

\[
T \equiv e^{a\xi}
\]

as the absolute temperature at \(\xi\), then we obtain the so-called well-known “temperature” 4-vector \(u\) and, as a matter of fact and a result, the parameter \(a_{\xi}\) is an entropy. Then it appears that falling in a potential of acceleration of gravitation for instance, is as equivalent as an increasing of entropy which then acquires the status usually ascribed to interactions, i.e. an origin to a motion (or the reverse), and not only from a configuration. It also means somehow that a 4D-ocean anisotropy is worked out from \(a_{\xi}\) and the entropy arrow is “an” anisotropy non-geometrical (not in a 4 dimensional manifold) arrow in \(\mathcal{M}\). Moreover at \(p_0\), then \(T = T\), i.e. the absolute temperature is a relativistic local invariant. Also it follows that

\[
T_{\xi}f(u) = e^{(a_{\xi} - a_{\xi'})}u'
\]

i.e. it is not a conformally equivariant vector, as well as the absolute temperature.
Also the time dilatation at 0 and ι due to a parallax effect, related to the Lorentz transformations \( T_0 f \) and \( T_ι f \), is kept, both at 0 and ι. Indeed, we have always a Lorentz transformation between two 4-vectors with same base point \( p_0 \), and these effects are completely included in the time discrepancy formula (21).

B. Entropy

The diagram above suggest a link with the thermodynamics. Then we wonder if entropy could be exhibited out of the mathematical treatment we made using jets, germs and local rings of germs. As we demonstrated, it is rather impossible, in case of 0-jets \( a_ξ \) or \( a'_ξ \) of functions, to compute as well as predict consequently the time discrepancy \( \Delta ι(\mathcal{L}) \). Hence, we can defined a density of probability \( p_ι(δ) \) for \( \Delta ι(\mathcal{L}) \) to be equal to \( δ \in \mathbb{R} \) at the crossing ending point ι. Then, it is easy to define the variation of entropy \( \Delta S \) occuring between 0 and ι by the usual formula coming from statistics (\( k \) being the Boltzman constant):

\[
\Delta S = S_ι - S_0 \equiv -k \int_\mathbb{R} p_ι(δ) \ln(p_ι(δ)) \, dδ \geq 0.
\]

As a matter of fact, we didn’t consider there exists a potential (a field of scalars) \( \alpha \) on \( T_0 \mathcal{M} \) such that \( a_ξ = \alpha(ξ) \), \( a'_ξ = \alpha(ξ') = \alpha \circ f(ξ) \), \( a'_0 = a_0 \) and \( a'_ι = a_ι \) which would define somehow “geodesic valleys” \( \mathcal{C} \) and \( \mathcal{C}' \) with moving particles or massive objects streaming through. Then \( \Delta S \) would be vanishing if \( \alpha \circ f(ξ) = \alpha(ξ) \) (i.e. we would have geodesic trajectories with same starting and ending points in a gravitational field, and fields of 4-velocities), as well as the time discrepancy, since the density of probability \( p_ι(δ) \) would be equal to 1 for \( δ = 0 \) only. But (!), we have to take care, in order to be rigourous also, that a function \( \alpha \) does not necessary physically exist on \( T_0 \mathcal{M} \) or equivalently on \( \mathcal{M} \) (i.e. as a potential of acceleration), since it is also determined by the travellers themselves. Indeed they can vary their own acceleration as they wish and their own scalar (not potential on \( \mathcal{M} \)) of acceleration “\( a \)” by using rocket motors for instance. It is a fundamental point which consists in considering scalars fields, vectors fields or tensors fields not on \( \mathcal{M} \) but on the paths only. In other words, there are no mathematically pre-determined acceleration scalars on \( \mathcal{M} \) and consequently pre-determined manifold \( \mathcal{M} \) (!) since the \( a \)’s can also be viewed as scalars of deformations of \( \mathcal{M} \). This doesn’t mean that determined potentials of acceleration \( \alpha \) don’t exist in full generality. But they can for instance, as a result of deterministic mass distributions. That is a kind of direct consequence of the elevator metaphor defining the Einstein equivalence principle in such a way that we could create by rocket motors . . . a relative acceleration of gravity.

It is a first source of indeterminism, unless we consider the wishes of astronauts satisfy some equations defining also what would be a determined spacetime manifold \( \mathcal{M} \) (!). It would only be an idealized spacetime, and our brain would only simulate a predicted spacetime from a memorized observed initial one, satisfying the Einstein equations! The masses distribution can’t be determined by the Einstein equations, and consequently the latter can’t determine
a spacetime topology associated to a spacetime evolution or history via contingency. But then, it follows that the
genre and the spacetime manifold is not the solution obtained from the Einstein equation. Hence they can’t
bind a genuine spacetime geometry and a masses distribution! It could only be a simulated local “static” spacetime
manifold, which have to be revised each time, time after time from new experimental data. But the second is that
no one (as far as we know...) is ubiquitous and henceforth can compute $\Delta a$ because of a lack of data (except in very
particular cases) being “simultaneously” at $\xi$ and $\xi'$.

C. The light cones

Also there is an important problem: we considered an unique metric $h_0$ at $p_0$. This situation points out an other
problem in general relativity: if $\mathcal{M}$ is not foliated by a unique absolute time parameter, as we assume, and which is
equivalent to say that $\mathcal{M}$ is not endowed with global metric fields, then two crossing paths such as in Figure 6 below,
could be defined with two different light cones at the apex $p_0$ in full generality. We avoid this situation in assuming
that $h$ and $h'$ are related to the unique metric $h_0$ in a unique way. But the unicity of $h_0$ has to be explained as well
as the local spacetime anisotropy at $p_0$ it involves.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{light_cones.png}
\caption{Light cones}
\end{figure}

V. APPENDIX

In this appendix we recall the definitions of germs and local rings (see Atiyah, M. F. and Macdonald, I. G. :Introduction to Commutative Algebra. Reading, MA: Addison-Wesley, 1969).

Let $f, f': M \to N$ be two differentiable maps of class $C^r$ ($r \in \mathbb{N} \cup \{+\infty\}$ or $r = \omega$ for analytic maps), with $M$ and $N$ being differentiable manifolds of same class.

Definition 1 The two differentiable maps $f$ and $f'$ defined respectively in two neighborhoods $U$ and $U'$ of a same
point $p \in M$, have the same “common germ at $p$” if and only if there are equal in a third neighborhood of $p$ (this third
neighborhood can be included in the intersection $U \cap U'$). This a relation of equivalence that we denote by $f \sim_p f'$.

Definition 2 We call “germ” of a differentiable map $f$, the class of differentiable maps $f'$ such that $f \sim_p f'$, and we
use the notation $[f]_p$ for the germ of $f$.

Let $k$ be a field, and $V$ and $W$ two $k$-vector spaces. Moreover, to each point $p$, element of a topological space $U$ (resp.
$V$), is associated a ring (resp. a $k$-module) $\mathcal{O}_{U,p}$ (resp. $\mathcal{M}_{V,W,p}$) of the germs of functions (resp. $k$-morphisms from
$V$ to $W$) defined at $p$.

Definition 3 Let $\mathcal{M}_p$ be the ring (resp. $k$-module) of the non-invertible elements in $\mathcal{O}_{U,p}$ (resp. $\mathcal{M}_{V,W,p}$). If $\mathcal{M}_p$
is the unique maximal ideal in $\mathcal{O}_{U,p}$ (resp. $\mathcal{M}_{V,W,p}$), then $\mathcal{O}_{U,p}$ (resp. $\mathcal{M}_{V,W,p}$) is said to be a “local ring” (“non-
commutative local ring”).

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We can outline these definitions with the very schematic drawing below:

![Diagram of Local Ring](image)

**Definition 4** Let $\nabla$ be a covariant derivative defined on an open set $U$ of a differentiable manifold $M$. We call “jet of order $k$” or “$k$-jet” at a point $x_0 \in M$ of a germ $[f]_{x_0}$ at $x_0$ of a differentiable application $f$ defined on an open neighborhood $U$ of $x_0$, the full set of the $i$-th covariant derivatives with respect to the vector $\eta$

$$\nabla^i_\eta [f] \equiv \nabla_\eta \circ \cdots \circ \nabla_\eta [f] \text{ i times}$$

of $[f]_{x_0}$ at $x_0$ with $0 \leq i \leq k$. Usually, the derivatives are used in this definition instead of the covariant derivatives, but this can be extended to the covariant derivatives case without modifications, but taking care that the covariant derivative could be not defined on the whole of $M$, but at least on a neighborhood of $x_0$ only. Hence, we would have to consider a germ of covariant derivatives.

For instance, from the definition above, the value $f(x_0)$ is a 0-jet of a “large” set of germs. In fact, it matters to notice that a given $k$-jet at $x_0$ of a germ of an application $f$ defines the local ring of those differentiable applications with the same $k$-jet at $x_0$. Roughly speaking, a $k$-jet at a given point could be viewed as a set of germs, i.e. a local ring. And we consider this ascription in the whole of the present paper, since it prohibits to consider values of maps or fields at a given point to be associated to a unique given (determined) map or field; The latter which would be exhibited out of deterministic, *a priori*, given physical and/or geometrical global structures.

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