Maxwell Fisheye Lens Based Retrodirective Array

Muhammad Ali Babar Abbasi & Vincent F. Fusco

A Maxwell fisheye lens using parallel plate index grading is presented in this study to develop a passive retrodirective antenna array. As a proof-of-concept a design frequency of 10 GHz was selected for fabrication and experiment. The design principles of the lens are discussed, which enables 85% energy flow at the drain probe (also referred to as image point) of the lens. It is shown that the image in the Maxwell fisheye lens has a point symmetry with a reverse phase, which makes it possible to realize passive retrodirective action using the lens. This arrangement is significantly more practical than previous passive retrodirective topologies due to the un-constrained number of connections to radiating elements that it can support without the need for multi-layer technology. In the realization described here, a cross-polarized microstrip patch antenna array is connected to the source and drain probes of the lens structure in order to form the retrodirective array. The strategy for selecting the optimal transmission line lengths required to connect the antennas to the lens for maximum re-radiation power is described and implemented. Experimental results for a prototype high efficiency passive retrodirective array based on the theoretical design considerations presented in this paper are reported.

Maxwell Fisheye Lens Design Principles

Let \( x, y, z \) represent a point in a Cartesian coordinate system within which a ray trajectory can be written as \( r(t) = x(t), y(t), z(t) \) where \( t \) is the parametric representation throughout the trajectory. A set of rays in a spherically symmetric optical system can be projected into a plane without losing generality. In our case, the lens is cylindrical and lies in a disk in the \( xy \)-plane with the center of the lens located at \( z = 0 \). Here ray trajectories \( r(t) \) can be calculated using conformal mapping of the points on a sphere onto the aforementioned \( xy \)-plane. For this case, the path length differential on the sphere is mapped as an optical length differential in the \( xy \)-plane. In this way, the geodesic points on the great circle of the sphere can be mapped directly to a fisheye refractive index profile. For a spherical coordinate system with \( \theta = 0^\circ \) defining the \( xy \)-plane the required mapping takes the form,

\[
x = \left( \frac{1 - \sin \theta}{\cos \theta} \right)^2 \cos 2\phi \quad \text{and} \quad y = \left( \frac{1 - \sin \theta}{\cos \theta} \right)^2 \sin 2\phi
\]

(1)

This mapping is illustrated in Fig. 1(a). This is known as stereographic projection in which southern hemisphere of a sphere is mapped to the interior, while the northern hemisphere is mapped to the exterior of a circle in \( xy \)-plane having radius equal to the radius of the sphere. The line element segments \( dx \) and \( dy \) can be expressed in terms of spherical coordinates, and for a cylindrical lens \( \theta = 0^\circ \) as

\[
n^2(x, y, 0)[dx^2 + dy^2] = n_s^2[db^2 + \sin^2 \theta d\phi^2]
\]

(2)
such that at the equator, \( n \) is equal to the index \( n_0 \) of the reference sphere. Typical lens profiles which emulate the fisheye lens\(^3\) in a 2D plane can be described by:

\[
n = \sqrt{\Delta r} = \frac{2n_0}{1 + r^2/R^2}, \quad \text{when } r \in [0, \infty)
\]

(3)

where, \( 2n_0 \) is the refractive index at the center of the lens, and \( R \) is the characteristic radius of the reference sphere.

After translating 3D fisheye characteristics into 2D, the next step is its practical realization. Generally, one can implement a variety of technologies in order to create a refractive index profile that performs fisheye focusing e.g. thin plates\(^4\), holy parallel plates\(^5\). In this paper we take a different approach in which fisheye principle is realized using the mode theory of a parallel-plate waveguide\(^6\). Here between ideally conducting parallel plates with spacing \( d \) we introduce, a medium characterized by dielectric \( \varepsilon_r(\omega) \) given in Fig. 1(b). The general form for the propagation constant of the fields propagating parallel to the plates is\(^7\),

\[
k_m = \sqrt{k^2 - m^2\left(\frac{\pi}{d}\right)^2}
\]

(4)

when \( m \in \{1, 2, \cdots\} \) and \( k = \omega/\sqrt{\varepsilon_r} \). The fields will propagate as long as \( k_m \) is real, however when \( \pi/\sqrt{\varepsilon_r}/d > k \), \( k_m \) becomes imaginary the propagating wave decays exponentially along the direction of propagation. The parallel plate waveguide then acts as a high pass filter with cut-off frequency

\[
\omega_c = \frac{m\pi c}{d\sqrt{\varepsilon_r(\omega)}}, \quad \text{when } m \in \{1, 2, \cdots\}
\]

(5)

Here \( m \) is considered the mode number. For the propagating wave with \( \omega > \omega_c \), the phase velocity will be \( v = \omega/k_m \) and the energy will transport at a group velocity \( v_g = \partial \omega/\partial E_m \). We can control this group velocity by using a graded index profile following the fisheye Eq. (3). In this paper this is achieved by partially filling the parallel plate waveguide with dielectric material whose permittivity \( \varepsilon_r \) is fixed but whose thickness varies across the lens we can create an effective dielectric \( \varepsilon_r' \) such that

\[
\varepsilon_r' = 1 - t(r)\left(\frac{1 + \varepsilon_r}{d}\right)
\]

(6)

where \( r \) is the radius of the parallel plates and \( t \) is the dielectric material thickness. Note that the notations \( \varepsilon_r \) in this paper represents general relative permittivity of a material, whereas \( \varepsilon_r' \) represents the achieved effective permittivity inside partial dielectric filled metallic parallel plates. The method of controlling the permittivity in such a case is given in\(^8\). Considering (3), (5) and (6) we formulated the dielectric profile between parallel plates.

**Fisheye Lens Development**

The fisheye lens schematic presented in Fig. 3(a) comprises of two perfectly electrically conducting, PEC, plates spaced with \( d = 5 \text{ mm} \). The substrate profile used to fill in the parallel plate waveguide is Rexolite\(^10\), \( \varepsilon_r = 2.53 \), dispersion factor = 0.00066, and low coefficient of linear thermal expansion = \( 3.8 \times 10^{-5} \text{ /°F} \) in inches. A circular band of copper of diameter = 100 mm is used to define a closed boundary for the lens, see Fig. 2. A coaxial probe...
with $h = 4.3$ mm (selected for 50Ω match) is used to excite the lens while the same probe length is used to form the drain point. A Triumph Dulex milling machine is used to machine the lens profile from a cylindrical Rexolite stock, and then a Tech-Gen precision finisher is deployed in order to polish the surface, making sure to have a surface roughness of $<\lambda/8$ across the lens profile. Full-wave EM simulations using CST Microwave Studio\textsuperscript{11} show the maximum electric fields within this confined space. Energy focusing at the image point is illustrated via the field strength plot in Fig. 3(b) confirming the operation of the device.

Figure 4(a) shows the focusing capabilities of the device, here $\sim85\%$ simulated $E$-field can be extracted from the drain point. Since the lens structure is confined, we believe the remaining field loss is associated with imperfect imaging in the active drain located at the image point of the lens, as argued in\textsuperscript{12,13}. A wave leaving the source in the fisheye lens follows a circular path and focuses at the image point. The real component of the electric field in time lapse images are shown in Fig. 4(b), here the difference in the propagating EM waves along the edges and centre of the lens can be clearly seen. In simulation we noted that if we locate the mirror very close to the probe location, energy loss to impedance mismatch occurs, which is undesirable for the device operation. The waves propagating between the outer edge of the lens (mirror) and the start of the Rexolite dielectric are found to have the same $v_g$ which aids in image formation at the drain probe through phase alignment. Due to multiple transitions, from mirrored parallel plate, to dielectric filled parallel plate and coax-type probe (drain) it is not straightforward to identify which modes are propagating in these portions of the fisheye lens structure. The wave equation solution for two parallel plates are characterized by $m$ (Eq. (5)) and $\text{TE}_m$ and $\text{TM}_m$ modes are defined on discrete frequencies with specific cut-off frequencies. It is important to note that a solution of wave equation with no magnetic-field in the direction of propagation ($\text{TM}_0$) has a fundamental mode ($\text{TM}_0$) that has no cut-off frequency. Considering an infinite parallel plate waveguide where wave is propagating along $z$-direction, at the $\text{TM}_0$ mode, the propagation constant is equal to $k$. This means that the $\text{TM}_0$ mode has neither electric nor magnetic field along the direction of propagation, making it a transverse electro-magnetic (quasi-TEM) mode, also supported by the lens structure.
The length of a transmission line in (7) is deconstructed in two parts. First part is an integer multiple of wavelength, i.e. \( n \lambda \) and second is an additional section of a transmission line along the direction of reflected waves (in this case about 330°). We also observed that for the microstrip patch antenna array, maximum re-radiation occurs when \( q = 0.6 \lambda \) which is close to the conclusions in [27] for a classical Van Atta configuration. Optimum transmission line length governed by \( q \) has an impact of approximately 1.5 dB in the re-radiated field strength along the DoA. With an optimal \( q = 0.6 \lambda \), retrodirective array performance for several representative DoAs is shown in Fig. 7. As the DoA is skewed away from the broadside direction of the array, the re-radiated field strength weakens such that the difference between re-radiated field at DoA = 60° is ~13 dB below compared to the re-radiated field at DoA = 0°. To isolate the retro and scattered patterns we used orthogonal polarization diversity such that the

\[
l = p\lambda + q\lambda
\]
The co-polarization component of the antenna is connected to the source probe while the cross-polarization component is attached to drain probe. It is important to mention that mutual coupling will have an impact on the results, and in the results below this is taken into consideration. The impact is reduced by designing the antennas with low return loss ($< -10$ dB), high port isolation between co- and cross-polarization ports of the same antenna (i.e. $< -20$ dB) and minimum achievable coupling between neighbouring array elements. Fabricated microstrip patch antenna unit cell, co- and cross-polarization feed, and the 6-element array is presented in Fig. 8(a). Note that we used phase aligned coax cables with SMA type RF connectors as transmission line of length $l$. Simulated
and measured results of the array are shown in Fig. 8(b). Measured $|S_{11}| < -10$ dB bandwidth of the antenna unit cell is from 9.88 GHz to 10.11 GHz. Although the proposed fisheye lens structure is frequency dependent but since our application demanded the bandwidth limitation to <300 MHz, we consider the same limit for the fisheye lens design. Results for the wave with a DoA = 15° are presented in Fig. 8(c,d) when $E_\theta$ and $E_\phi$ represent the cross- and co-polarization components of the fields. Undesirable scattered fields are shown to be nullified by the proposed approach. Results evaluated by experiments shown in Fig. 9 (E_\theta component in far-field) verifies the theoretical prediction.

The antenna unit cell number in passive retrodirective array in our work is 6, while this number is different in other retrodirective array examples in literature. Uniform linear array (ULA) configuration, for instance, are shown to have 2,4, 4 and 25,30,31 16 units. In addition to azimuth direction, elevation direction are also captured using uniform rectangular array (URA) configurations, shown to have 25,32,33,34, even 40 units. This makes it very difficult to directly compare our approach with different retrodirective mechanisms. However, the measured $E_\theta$-field results in Fig. 9 show a normalized comparison with the simulated scenario, proving the proposed system to be a practically viable solution.

**Conclusion**

A design approach for a cylindrically bounded Maxwell fisheye lens connected to a microstrip antenna array for retrodirective operation is presented. The operation of the lens connected to a linear array is shown to produce retrodirective action, wherein the re-radiated retrodirected beam has high isolation from un-wanted specular reflections by using polarisation diversity. Engineering details including the optimum fisheye lens design and transmission line lengths connecting the lens to the array are discussed. The approach suggests significant enhancement in the degrees-of-freedom for the implementation of high efficiency passive retrodirective array.

**Data availability**

The dataset based on measurements is available at go.qub.ac.uk/fisheye-lens.

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Author contributions

M.A.B.A. and V.F. contributed to the methods and simulation setups. M.A.B.A. ducted and processed the resulting data from simulations and measurements. V.F. supervised the project. Both authors wrote the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to M.A.B.A. or V.F.F.

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