WEIGHTED COMPOSITION–DIFFERENTIATION OPERATOR
ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. In this paper, we consider the sum of weighted composition operator $C_{\psi_0,\varphi_0}$ and the weighted composition–differentiation operator $D_{\psi_n,\varphi_n,n}$ on the Hardy and weighted Bergman spaces. We describe the spectrum of a compact operator $C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n}$ when the fixed point $w$ of $\varphi_0$ and $\varphi_n$ is inside the open unit disk and $\varphi_n$ has a zero at $w$ of order at least $n$. Also the lower estimate and the upper estimate on the norm of a weighted composition–differentiation operator on the Hardy space $H^2$ are obtained. Furthermore, we determine the norm of a composition–differentiation operator $D_{\varphi,n}$ acting on the Hardy space $H^2$, in the case where $\varphi(z) = bz$ for some complex number $b$ that $|b| < 1$.

1. Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane. The Hardy space $H^2$ is the set of all analytic functions $f$ on $\mathbb{D}$ such that

$$
\|f\| = \left( \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{1/2} < \infty.
$$

For $-1 < \alpha < \infty$, the weighted Bergman space $A^2_\alpha$ is the space of all analytic functions $f$ on $\mathbb{D}$ so that

$$
\|f\| = \left( \int_{\mathbb{D}} |f(z)|^2 (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z) \right)^{1/2} < \infty,
$$

where $dA$ is the normalized area measure on $\mathbb{D}$. The case when $\alpha = 0$, usually denoted $A^2$, is called the (unweighted) Bergman space. Throughout this paper, we will write $H_\alpha$ to denote the Hardy space $H^2$ for $\alpha = -1$ or the weighted Bergman space $A^2_\alpha$ for $\alpha > -1$.

The weighted Bergman spaces and the Hardy space are reproducing kernel Hilbert spaces. For every $w \in \mathbb{D}$ and each non-negative integer $n$, let $K^{[n]}_{w,\alpha}$ denote the unique function in $H_\alpha$ that $\langle f, K^{[n]}_{w,\alpha} \rangle = f^{(n)}(w)$ for each $f \in H_\alpha$, where $f^{(n)}$ is the $n$th derivative of $f$ (note that $f^{(0)} = f$); for convenience, we use the notation $K_{w,\alpha}$ when $n = 0$. The function $K^{[n]}_{w,\alpha}$ is called the reproducing kernel function. The reproducing kernel functions for evaluation at $w$ are given by $K_{w,\alpha}(z) = 1/(1-wz)^{\alpha+2}$ and

$$
K^{[n]}_{w,\alpha}(z) = \frac{(\alpha + 2)...(\alpha + n + 1)z^n}{(1-wz)^{n+\alpha+2}}
$$

for $z,w \in \mathbb{D}$ and $n > 1$. 

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For an operator $T$ on $H_\alpha$, we write $\|T\|_\alpha$ to denote the norm of $T$ acting on $H_\alpha$. Through this paper, the spectrum of $T$ and the point spectrum of $T$ and the spectral radius of $T$ are denoted by $\sigma_\alpha(T)$, $\sigma_{p,\alpha}(T)$, and $r_\alpha(T)$, respectively.

We write $H^\infty$ to denote the space of all bounded analytic functions on $\mathbb{D}$, with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$.

We say that an operator $T$ on a Hilbert space $H$ is hyponormal if $T^*T - TT^* \geq 0$, or equivalently if $\|T^*f\| \leq \|Tf\|$ for all $f \in H$. Moreover, the operator $T$ is said to be cohyponormal if $T^*T$ is hyponormal. Let $P$ denote the projection of $L^2(\partial \mathbb{D})$ onto $H^2$. For each $b \in L^2(\partial \mathbb{D})$, we define the Toeplitz operator $T_b$ on $H^2$ by $T_b(f) = P(bf)$. For $\varphi$ an analytic self-map of $\mathbb{D}$, let $C_\varphi$ be the composition operator such that $C_\varphi(f) = f \circ \varphi$ for any $f \in H_\alpha$. The composition operator $C_\varphi$ acts boundedly for every $\varphi$, with

$$\left(\frac{1}{1 - |\varphi(0)|^2}\right)^{(n+2)/2} \leq \|C_\varphi\|_\alpha \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{(n+2)/2}.$$

(See [2, Corollary 3.7] and [10, Lemma 2.3].) Let $\psi$ be an analytic function on $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$ for $f \in H_\alpha$.

Although for each positive integer $n$, the differentiation operator $D_\alpha(f) = f^{(n)}$ is unbounded on $H_\alpha$ (note that $\lim_{m \to \infty} \|D_\alpha(z^m)\|/\|z^m\| = \infty$), there are some analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$ such that the operator $C_{\varphi}D_\alpha$ is bounded. The bounded and compact operators $C_{\varphi}D_\alpha$ on $H_\alpha$ were determined in [5], [8], and [9]. Recently, the authors and Hammond [3] obtained the adjoint, norm, and spectrum of some operators $C_{\varphi}D_1$ on the Hardy space. For an analytic self-map $\varphi$ and a positive integer $n$, the composition–differentiation operator on $H_\alpha$ is defined by the rule $D_{\psi,\varphi,n}(f) = f^{(n)} \circ \varphi$; for convenience, we use the notation $D_{\varphi}$ when $n = 1$. For an analytic function $\psi$ on $\mathbb{D}$, the weighted composition–differentiation operator $D_{\psi,\varphi,n}$ on $H_\alpha$ is defined

$$D_{\psi,\varphi,n}f(z) = \psi(z)f^{(n)}(\varphi(z)).$$

Some properties of weighted composition–differentiation operators were considered in [11] and [5].

In this paper, we determine the spectra of a compact operator $C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n}$ when the fixed point $w$ of $\varphi_0$ and $\varphi_n$ is inside the open unit disk and the function $\psi_n$ has a zero at $w$ of order at least $n$ (Theorem 2.4). The spectral radius of a class of compact weighted composition–differentiation operators is obtained (Theorem 2.5). In addition, we find the lower estimate and the upper estimate for $\|D_{\psi,\varphi,n}\|_{-1}$ (Propositions 3.2 and 3.6). Moreover, the norm of a composition–differentiation operator $D_{\varphi,n}$, acting on the Hardy space $H^2$, is determined in the case where $\varphi(z) = bz$ for some complex number $b$ that $|b| < 1$ (Theorem 3.5).

2. Spectral Properties

To find the spectrum of $C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n}$ we need to obtain an invariant subspace of $(C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^*$. To do this, we consider the action of the adjoint of the operator $C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n}$ on the reproducing kernel functions.

Lemma 2.1. Let $m$ be a non-negative integer. Suppose that $C_{\psi_0,\varphi_0}$ and $D_{\psi_n,\varphi_n,n}$ are bounded operators on $H_\alpha$ and the fixed point $w$ of $\varphi_0$ and $\varphi_n$ is inside the open unit disk. Assume that the function $\psi_n$ has a zero at $w$ of order at least $n$. 

(i) If \( m > n \), then
\[
(C_{\psi_0, \varphi_0} + D_{\psi_n, \varphi_n, n})^* K_{w, \alpha}^{[m]} = \sum_{j=0}^{m-1} \alpha_j(w) K_{w, \alpha}^{[j]} + \sum_{i=n}^{m-1} \beta_{i-n}(w) K_{w, \alpha}^{[i]}
\]
\[
+ \left( \psi_0(w) (\varphi_0' (w))^m + \binom{m}{n} \psi_n(w) (\varphi_n'(w))^{m-n} \right) K_{w, \alpha}^{[n]},
\]

(ii) if \( m = n \), then
\[
(C_{\psi_0, \varphi_0} + D_{\psi_n, \varphi_n, n})^* K_{w, \alpha}^{[m]} = \sum_{j=0}^{n-1} \alpha_j(w) K_{w, \alpha}^{[j]} + \binom{n}{n} \psi_n(w) (\varphi_n'(w))^{n-n} K_{w, \alpha}^{[n]},
\]

(iii) if \( m < n \), then
\[
(C_{\psi_0, \varphi_0} + D_{\psi_n, \varphi_n, n})^* K_{w, \alpha}^{[m]} = \sum_{j=0}^{m-1} \alpha_j(w) K_{w, \alpha}^{[j]} + \psi_0(w) (\varphi_0'(w))^m K_{w, \alpha}^{[m]},
\]

where the functions \( \alpha_j \)'s and \( \beta_j \)'s consist of some products of the derivatives of \( \psi_0 \) and \( \varphi_0 \) and some products of the derivatives of \( \psi_n \) and \( \varphi_n \), respectively.

Proof. Let \( f \) be an arbitrary function in \( \mathcal{H}_\alpha \). For each non-negative integer \( m \), we have
\[
\langle f, C_{\psi_0, \varphi_0}^* K_{w, \alpha}^{[m]} \rangle = \langle C_{\psi_0, \varphi_0} f, K_{w, \alpha}^{[m]} \rangle = \sum_{j=0}^{m} \binom{m}{j} \psi_0^{(m-j)}(w) \left( f \circ \varphi_0 \right)^{(j)} (w)
\]
\[
= \langle f, \sum_{j=0}^{m-1} \alpha_j(w) K_{w, \alpha}^{[j]} + \psi_0(w) (\varphi_0'(w))^m K_{w, \alpha}^{[m]} \rangle.
\]

Since \( f \) is an arbitrary function in \( \mathcal{H}_\alpha \), we conclude that
\[
(2.1) \quad C_{\psi_0, \varphi_0}^* K_{w, \alpha}^{[m]} = \sum_{j=0}^{m-1} \alpha_j(w) K_{w, \alpha}^{[j]} + \psi_0(w) (\varphi_0'(w))^m K_{w, \alpha}^{[m]}.
\]

Let \( m < n \). Since \( \psi_n \) has a zero at \( w \) of order at least \( n \), we have
\[
\langle f, D_{\psi_n, \varphi_n, n}^* K_{w, \alpha}^{[m]} \rangle = \left( \psi_n \cdot (f^{(n)} \circ \varphi_n) \right)^{(m)} (w)
\]
\[
= \sum_{i=0}^{m} \binom{m}{i} \psi_n^{(m-i)}(w) (f^{(n)} \circ \varphi_n)^{(i)} (w)
\]
\[
= 0.
\]

It shows that \( D_{\psi_n, \varphi_n, n}^* K_{w, \alpha}^{[m]} = 0 \).
Now assume that $m \geq n$. We obtain
\[
\langle f, D^*_\psi,\varphi_n \cdot K^{[m]} \rangle = \sum_{i=0}^{m} \binom{m}{i} \psi^{(m-i)}(w)(f^{(m)} \circ \varphi_n)^{(i)}(w)
\]
\[
= \sum_{i=0}^{m-n} \binom{m}{i} \psi^{(m-i)}(w)(f^{(m)} \circ \varphi_n)^{(i)}(w)
\]
\[
+ \sum_{i=m-n+1}^{m} \binom{m}{i} \psi^{(m-i)}(w)(f^{(m)} \circ \varphi_n)^{(i)}(w)
\]
(2.2)
\[
= \sum_{i=0}^{m-n} \binom{m}{i} \psi^{(m-i)}(w)(f^{(m)} \circ \varphi_n)^{(i)}(w).
\]
If $m > n$, then by (2.2), we get
\[
\langle f, D^*_\psi,\varphi_n \cdot K^{[m]} \rangle = \langle f, \sum_{i=0}^{m-n-1} \beta_i(w)K^{[i+n]} + \binom{m}{m-n} \psi_n^{(m-n)}(w)(\varphi_n(w))^{m-n} \rangle K^{[m]},
\]
so
\[
D^*_\psi,\varphi_n \cdot K^{[m]} = \sum_{i=n}^{m-1} \beta_i(w)K^{[i]} + \binom{m}{n} \psi_n^{(m-n)}(w)(\varphi_n(w))^{m-n} K^{[m]}.
\]
If $m = n$, then by (2.2), we see that
\[
\langle f, D^*_\psi,\varphi_n \cdot K^{[m]} \rangle = \psi_n^{(n)}(w)f^{(n)}(w) = \langle f, \psi_n^{(n)}(w)K^{[m]} \rangle.
\]
Hence the result follows. \[\square\]

In the next proposition, we identify all possible eigenvalues of $C_{\psi,\varphi_n} + D_{\psi,\varphi_n}$. 

**Proposition 2.2.** Suppose that $C_{\psi,\varphi_n}$ and $D_{\psi,\varphi_n}$ are bounded operators on $H_\alpha$ and the fixed point $w$ of $\varphi_0$ and $\varphi_n$ is inside the open unit disk. If the function $\psi_n$ has a zero at $w$ of order at least $n$, then
\[
\left\{ \psi_0(w), \psi_0(w)(\varphi_0(w))^l : l \in \mathbb{N}_{\geq n} \right\} \bigcup \left\{ \psi_0(w)(\varphi_0(w))^l + \binom{l}{n} \psi_n(w)(\varphi_n(w))^{l-n} : l \in \mathbb{N}_{\geq n} \right\}
\]
contains the point spectrum of $C_{\psi,\varphi_0} + D_{\psi,\varphi_n}$. 

**Proof.** Let $\lambda$ be an arbitrary eigenvalue for $C_{\psi,\varphi_0} + D_{\psi,\varphi_n}$ with corresponding eigenvector $f$. Note that
\[
(2.3) \quad \lambda f(z) = \psi_0(z)f(\varphi_0(z)) + \psi_n(z)f^{(n)}(\varphi_n(z))
\]
for each $z \in \mathbb{D}$. If $f(w) \neq 0$, then $\lambda = \psi_0(w)$. Let $f$ have a zero at $w$ of order $l \geq 1$. Differentiate (2.3) $l$ times and evaluate it at the point $z = w$ to obtain
\[
(2.4) \quad \lambda f^{(l)}(w) = \sum_{j=0}^{l} \binom{l}{j} \psi_0^{(l-j)}(w)(f \circ \varphi_0)^{(j)}(w)
\]
\[
+ \sum_{j=0}^{l} \binom{l}{j} \psi_n^{(l-j)}(w)(f^{(n)} \circ \varphi_n)^{(j)}(w).
\]
First assume that $l < n$. Since $\psi_n$ has a zero at $w$ of order at least $n$, we have $\lambda = \psi_0(w)(\varphi_0(w))^l$ by (2.4).
Now assume that \( l \geq n \). Then \( \psi_n^{(l-j)}(w) = 0 \) for each \( j > l - n \). Hence \( (2.3) \) implies that

\[
\lambda f^{(l)}(w) = \sum_{j=0}^{l} \binom{l}{j} \psi_0^{(l-j)}(w) (f \circ \varphi_0)^{(j)}(w) + \sum_{j=0}^{l-n} \binom{l}{j} \psi_n^{(l-j)}(w) (f^{(n)} \circ \varphi_n)^{(j)}(w)
\]

and so

\[
\lambda f^{(l)}(w) = \psi_0(w) f^{(l)}(w) (\varphi_0'(w))^l + \binom{l}{l-n} \psi_n(w) f^{(l)}(w) (\varphi_n'(w))^{l-n}.
\]

(Note that in case of \( \varphi_n'(w) = 0 \) and \( l = n \), we set \( (\varphi_n'(w))^{l-n} = 1 \).) Therefore, in this case, any eigenvalue must have the form

\[
\psi_0(w) (\varphi_0'(w))^l + \binom{l}{n} \psi_n(w) (\varphi_n'(w))^{l-n}
\]

for a natural number \( l \) with \( l \geq n \).

\( \square \)

**Proposition 2.3.** Suppose that the hypotheses of Proposition 2.2 hold. Then the point spectrum of \( (C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^* \) contains

\[
\{ \overline{\psi_0(w), \psi_0(w)(\varphi_0'(w))^n + \psi_n(w)} : l \in \mathbb{N}_{\geq n} \} \bigcup \{ \psi_0(w)(\varphi_0'(w))^l + \binom{l}{n} \psi_n(w) (\varphi_n'(w))^{l-n} : l \in \mathbb{N}_{\geq n} \}.
\]

**Proof.** Let \( l \) be a positive integer with \( l \geq n \) and \( K_l \) denote the span of \( \{K_{w,\alpha}, K_{w,\alpha}^{[1]}, ..., K_{w,\alpha}^{[l]} \} \). Note that this spanning set is linearly independent and so is a basis. Let \( A_l \) be the matrix of the operator \( (C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^* \) restricted to \( K_l \) with respect to this basis. We infer from Lemma 2.1 that

\[
A_l = \begin{bmatrix}
B_n & \ast & \ast & \ldots & \ast \\
0 & \psi_0(w)(\varphi_0'(w))^n + \psi_n(w) & \ldots & \ast & \ast \\
0 & 0 & \ldots & \ast & \ast \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \psi_0(w)(\varphi_0'(w))^l + \binom{l}{n} \psi_n(w) (\varphi_n'(w))^{l-n} & \ast
\end{bmatrix},
\]

where \( B_n \) is an \( n \times n \) upper triangular matrix that its main diagonal entries are \( \psi_0(w), \psi_0(w)(\varphi_0'(w))^n, ..., \psi_0(w)(\varphi_0'(w))^{n-1} \). Then \( A_l \) is an upper triangular matrix too. Since the subspace \( K_l \) is finite dimensional, it is closed and so the space \( \mathcal{H}_n \) can be decomposed as

\[
\mathcal{H}_n = K_l \oplus K_l^\perp.
\]

Then the block matrix of \( (C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^* \) with respect to the above decomposition must be of the form

\[
\begin{bmatrix}
A_l & C_l \\
0 & El
\end{bmatrix}
\]

(note that \( K_l \) is invariant under \( (C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^* \) by Lemma 2.1 and so the lower left corner of the above matrix is 0). Since the spectrum of \( (C_{\psi_0,\varphi_0} + D_{\psi_n,\varphi_n,n})^* \) is the union of the spectrum of \( A_l \) and the spectrum of \( El \) (see [2, p. 270]), we conclude that the union of \( \{ \overline{\psi_0(w), \psi_0(w)(\varphi_0'(w))^t} : t \in \mathbb{N} \text{ and } t < n \} \) and
\[
\left\{ \psi_0(w)(\varphi_0'(w))^t + \binom{l}{n}\psi_n(w)(\varphi_n'(w))^{t-n} : t \in \mathbb{N} \text{ and } n \leq t \leq l \right\}
\]
is the subset of \(\sigma_p,\alpha\left((C_{\psi,\varphi_0} + D_{\psi,\varphi,n})^\ast\right)\). Since \(l\) is arbitrary, the result follows. \(\square\)

In the following theorem, we characterize the spectrum of an operator \(D_{\psi,\varphi,n}\) under the conditions of Proposition 2.2. The spectrum of an operator \(D_{\psi,\varphi,n}\) which was obtained in \([5, \text{Theorem 3.1}]\) is an example for Theorem 2.4.

**Theorem 2.4.** Suppose that the hypotheses of Proposition 2.2 hold. If \(C_{\psi,\varphi_0} + D_{\psi,n,\varphi,n}\) is compact on \(\mathcal{H}_\alpha\), then

\[
\sigma_\alpha(C_{\psi,\varphi_0} + D_{\psi,n,\varphi,n}) = \{0\}
\]

\[
\bigcup \left\{ \psi_0(w), \psi_0(w)(\varphi_0'(w))^l : l \in \mathbb{N}_{<n} \right\}
\]

\[
\bigcup \left\{ \psi_0(w)(\varphi_0'(w))^l + \binom{l}{n}\psi_n(w)(\varphi_n'(w))^{l-n} : l \in \mathbb{N}_{\geq n} \right\}
\]

In particular, if \(\psi_n(w) = 0\), then the operator \(D_{\psi,n,\varphi,n}\) is quasinilpotent; that is, its spectrum is \(\{0\}\).

In the next theorem, we obtain the spectral radius of a compact operator \(D_{\psi,\varphi,n}\).

**Theorem 2.5.** Suppose that \(D_{\psi,\varphi,n}\) is a compact operator on \(\mathcal{H}_\alpha\). Assume that the fixed point \(w\) of \(\varphi\) is inside the open unit disk and the function \(\psi\) has a zero at \(w\) of order \(n\). Then

\[
r_\alpha(D_{\psi,\varphi,n}) = \left(1 - \frac{\varphi'(w)}{n}\right)\left|\psi(n)(w)\right|\left|\varphi'(w)\right|^{l-n},
\]

where \([\cdot]\) denotes the greatest integer function.

**Proof.** Theorem 2.4 implies that

\[
\sigma_\alpha(D_{\psi,\varphi,n}) = \left\{ \binom{l}{n}\psi_n(w)(\varphi'(w))^{l-n} : l \in \mathbb{N}_{\geq n} \right\}
\]

and so

\[
r_\alpha(D_{\psi,\varphi,n}) = \sup \left\{ \binom{l}{n}\left|\psi(n)(w)\right|\left|\varphi'(w)\right|^{l-n} : l \in \mathbb{N}_{\geq n} \right\}.
\]

If \(\varphi'(w) = 0\), then \(r_\alpha(D_{\psi,\varphi,n}) = \left|\psi(n)(w)\right|\). Now suppose that \(\varphi'(w) \neq 0\). Let the function \(h(x) = x(x-1)...(x-n+1)|\varphi'(w)|^{x-n}\) on \([n,+\infty)\). Since \(|\varphi'(w)| < 1\) (see the Grand Iteration Theorem), we conclude that \(\lim_{x \to +\infty} h(x) = 0\). Then \(h\) is a bounded function on \([n,+\infty)\) and so it obtains an absolute maximum point. If \(h'(t) = 0\) for some \(t \in [n,+\infty)\), then \(g(t) = -\ln |\varphi'(w)|\), where \(g(x) = \frac{1}{x} + \frac{1}{x-1} + ... + \frac{1}{x-n+1}\) for each \(x \in [n,+\infty)\). We can easily see that \(g\) is strictly decreasing and so the function \(h\) has at most one local extremum on \([n,+\infty)\), which must be its absolute maximum (note that if \(h'(t) \neq 0\) for all \(t\), then \(h\) has an absolute maximum of \(n!\) at \(n\)). Therefore, for obtaining \(r_\alpha(D_{\psi,\varphi,n})\), we must find the greatest natural number \(l\) such that \(l \geq n\) and

\[
(l-1)...(l-n)|\varphi'(w)|^{l-n-1} \leq l...(l-n+1)|\varphi'(w)|^{l-n}
\]
or equivalently

\[
l \leq \frac{n}{1 - |\varphi'(w)|}.
\]
We begin this section with an example which is a starting point for estimating a lower bound for \( \| D_{\psi, \varphi, n} \|_{-1} \).

**Example 3.1.** Suppose that \( \varphi(z) = bz^2 + az^3 \) with \( \frac{1}{2} < |a| < 1 \) and \( |a| + |b| < 1 \). We can see that \( \varphi(0) = \varphi'(0) = 0 \) and \( \varphi''(0) = 2a \). By the paragraph after Theorem 2.5 we have \( r_\alpha(D_{\varphi}) = 2|a| \) and so \( \| D_{\varphi} \|_\alpha \geq 2|a| > 1 \). Compare \( 2|a| \) with the lower bound for \( \| D_{\varphi} \|_{-1} \) which was found in [3, Proposition 4] (note that [3, Proposition 4] implies that \( \| D_{\varphi} \|_{-1} \geq 1 \)).

The preceding example leads to the lower estimate on the norm of \( D_{\psi, \varphi, n} \) on the Hardy space by using the spectrum of a weighted composition–differentiation operator which was obtained in Proposition 2.8.

**Proposition 3.2.** Suppose that \( D_{\psi, \varphi, n} \) is a bounded operator on \( H^2 \). Assume that the fixed point \( w \) of \( \varphi \) is inside the open unit disk.

(i) If \( \varphi'(w) \neq 0 \), then

\[
\| D_{\psi, \varphi, n} \|_{-1} \geq |\psi^{(n)}(w)| \left( \frac{1}{n} \right) |\varphi'(w)|^{-n}.
\]
(ii) if \( \varphi'(w) = 0 \), then
\[
\|D_{\psi,\varphi,n}\|_{-1} \geq |\phi^{(n)}(w)|;
\]

(iii) if \( \varphi'(w) = 0 \), \( \psi''(w) = 0 \) and \( n = 1 \), then
\[
\|D_{\psi,\varphi,1}\|_{-1} \geq \max \left\{ |\varphi'(w)|, |\psi(w)\varphi''(w)| \right\},
\]
where
\[
\phi(z) = \begin{cases} 
\psi(z), & \psi(0)(w) = \ldots = \psi(n-1)(w) = 0, \\
\psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^{n-m}, & \psi(0)(w) = \ldots = \psi(m-1)(w) = 0, \psi(m)(w) \neq 0 \text{ and } 1 \leq m < n, \\
\psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^n, & \psi(w) \neq 0.
\end{cases}
\]

Proof. First suppose that \( \psi(0)(w) = \ldots = \psi(n-1)(w) = 0 \). Proposition 2.3 and the idea which was used in the proof of Theorem 2.5 imply that
\[
(3.1) \quad \|D_{\psi,\varphi,n}\|_{-1} \geq |\psi(n)(w)|\left(1 - |\varphi'(w)|^2\right)^{\frac{n}{2}} |\varphi'(w)|^{\frac{n}{2}}. \tag{3.1}
\]
(Note that in case of \( \varphi'(w) = 0 \), we set \( |\varphi'(w)|\left(1 - |\varphi'(w)|^2\right)^{\frac{n}{2}} = 1 \).)

Now assume that \( \psi(z) = (w-z)^mg(z) \), where \( 1 \leq m < n \) and \( g(w) \neq 0 \). Let \( \phi(z) = \psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^{n-m} \). Since \( T_{\frac{w-z}{1-\overline{w}z}} \) is an isometry on \( H^2 \) and the \( n \)th derivative of \( \psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^{n-m} \) at the point \( w \) is \( \frac{(-1)^{n-1}g(w)}{(1-|w|^2)^{\frac{n}{2}}} \), by replacing \( \phi \) with \( \psi \) in (3.1), we obtain
\[
\|D_{\psi,\varphi,n}\|_{-1} = \|D_{\phi,\psi,n}\|_{-1} \geq \frac{n!|g(w)|}{(1-|w|^2)^{\frac{n}{2}}} \left(1 - |\varphi'(w)|^2\right)^{\frac{n}{2}} |\varphi'(w)|^{\frac{n}{2}}. \tag{3.1}
\]

Now suppose that \( \psi(w) \neq 0 \) and \( \phi(z) = \psi(z)\left(\frac{w-z}{1-\overline{w}z}\right)^n \). By replacing \( \phi \) with \( \psi \) in (3.1), we have
\[
\|D_{\psi,\varphi,n}\|_{-1} = \|D_{\psi,\varphi,n}\|_{-1} \geq \frac{n!|\psi(w)|}{(1-|w|^2)^{\frac{n}{2}}} \left(1 - |\varphi'(w)|^2\right)^{\frac{n}{2}} |\varphi'(w)|^{\frac{n}{2}}. \tag{3.1}
\]

Note that if \( \varphi'(w) = 0 \) and \( \psi''(w) = 0 \), then \( D_{\psi,\varphi,1}^nK_{w,-1}^{[2]} = \psi(w)\varphi''(w)K_{w,-1}^{[2]} \) by [4] Lemma 1]. Therefore, we conclude that \( \|D_{\psi,\varphi,1}\|_{-1} \geq |\psi(w)\varphi''(w)| \). Hence the result follows.

In the next example, we show that for some operators \( D_{\varphi} \), Proposition 3.2 is more useful than [3] Proposition 4] for estimating the lower bound for \( \|D_{\varphi}\|_{-1} \).

Example 3.3. Suppose that \( \varphi(z) = az^n + bz \), where \( \frac{1}{2} < |b| < 1 - |a| \) and \( n \) is a positive integer that \( n \geq 2 \). Proposition 3.2 implies that
\[
\|D_{\varphi}\|_{-1} \geq \frac{1}{1 - |b|} |b|^{1/(1 - |b|)} - 1 > 1
\]
and so this lower bound is greater than the lower bound for \( \|D_{\varphi}\|_{-1} \) which was estimated in [3] Proposition 4].

In the following proposition, we obtain \( \|D_{\psi,\varphi,n}\|_{\alpha} \), when \( D_{\psi,\varphi,n} \) is a compact hyponormal (or cohyponormal) operator which satisfies the hypotheses of Proposition 2.2.
Proposition 3.4. Suppose that \( \psi \) is not identically zero and \( \varphi \) is a nonconstant analytic self-map of \( \mathbb{D} \) so that \( D_{\psi, \varphi, n} \) is compact on \( H_\alpha \). Assume that \( w \) is the fixed point of \( \varphi \) and \( \psi \) has a zero at \( w \) of order at least \( n \). Then \( D_{\psi, \varphi, n} \) is hyponormal or cohyponormal on \( H_\alpha \) if and only if \( \psi(z) = az^n \) and \( \varphi(z) = bz \), where \( a \in \mathbb{C} \setminus \{0\} \) and \( b \in \mathbb{D} \setminus \{0\} \); moreover, in this case

\[
\|D_{\psi, \varphi, n}\|_\alpha = n!|a|^{1+\frac{1}{n}}|b|^{1-\frac{1}{n}} - n.
\]

Proof. Suppose that \( D_{\psi, \varphi, n} \) is hyponormal (or cohyponormal). If \( \psi^{(n)}(w) = 0 \), then \( r_\alpha(D_{\psi, \varphi, n}) = 0 \) by Theorem 2.4 and so \( D_{\psi, \varphi, n} \equiv 0 \) by [1] Proposition 4.6, p. 47. It follows that \( \psi \equiv 0 \) or \( \varphi \equiv 0 \) (note that \( D_{\psi, \varphi, n}(z^{n+1}) = (n+1)!\psi(z)\varphi(z) \)) which is a contradiction. Hence we assume that \( \psi^{(n)}(w) \neq 0 \). Since \( \psi \) has a zero at \( w \) of order \( n \), Lemma 2.1 shows that \( D_{\psi, \varphi, n}K_{\psi, \varphi, n}(z) = 0 \) and so \( w = 0 \). Lemma 2.1 implies that

\[
D_{\psi, \varphi, n}^* K_{\psi, \varphi, n}(z) = \frac{\psi^{(n)}(0)}{n!}(\psi(z) + (\alpha + n + 1)z^n).
\]

Since \( D_{\psi, \varphi, n} \) is hyponormal (or cohyponormal), it follows that

\[
D_{\psi, \varphi, n}K_{\psi, \varphi, n}(z) = \psi^{(n)}(0)(\alpha + 1)...(\alpha + n + 1)z^n.
\]

Because \( D_{\psi, \varphi, n}K_{\psi, \varphi, n}(z) = n!(\alpha + 1)...(\alpha + n + 1)\psi(z) \), we conclude that \( \psi(z) = \frac{\psi^{(n)}(0)}{n!}z^n \), where \( \psi^{(n)}(0) \neq 0 \). Then \( \psi^{(m)}(0) = 0 \) for each \( m \neq n \). Hence Lemma 2.1 shows that

\[
D_{\psi, \varphi, n}^* K_{\psi, \varphi, n}(n+1) = (n+1)!\psi^{(n)}(0)\varphi'(0)K_{\psi, \varphi, n}(n+1).
\]

Therefore, we have

\[
D_{\psi, \varphi, n}K_{\psi, \varphi, n}(n+1) = n!\psi^{(n)}(0)\varphi'(0)K_{\psi, \varphi, n}(n+1).
\]

On the other hand, we obtain

\[
D_{\psi, \varphi, n}K_{\psi, \varphi, n}(n+1)^+ = (n+1)!((\alpha + 1)...(\alpha + n + 2)\psi(z)\varphi(z)
\]

\[
= (n+1)!((\alpha + 1)...(\alpha + n + 2)\frac{\psi^{(n)}(0)}{n!}z^n\varphi(z)
\]

for each \( z \in \mathbb{D} \). Since \( D_{\psi, \varphi, n} \) is hyponormal (or cohyponormal), \( (3.2) \) and \( (3.3) \) imply that \( \varphi(z) = \varphi'(0)z \).

Conversely is obvious by [1] Proposition 3.2 (note that an analogue of [1] Proposition 3.2 holds in \( H^2 \) by the similar idea).

Due to the hyponormality (or cohyponormality) of \( D_{\psi, \varphi, n} \), invoking Theorem 2.5 it follows that

\[
\|D_{\psi, \varphi, n}\|_\alpha = r_\alpha(D_{\psi, \varphi, n}) = n!|a|^{1+\frac{1}{n}}|b|^{1-\frac{1}{n}} - n.
\]

\[\square\]

In the next theorem, we extend [8] Theorem 2.

Theorem 3.5. If \( \varphi(z) = bz \) for some \( b \in \mathbb{D} \setminus \{0\} \), then

\[
\|D_{\psi, \varphi, n}\|_1 = n!\left(\frac{n}{n-1}\right)|b|^n|z^n|^{-n}.
\]
Proof. The result follows immediately from Proposition 3.4 and the fact that $T_n$ is an isometry on $H^2$.

Let $\|\varphi\|_\infty \leq b < 1$ and $\psi \in H^\infty$. We define $\varphi_b = (1/b)\varphi$ and $\rho(z) = bz$ (see [3, p. 2898]). Observe that $D_{\varphi,n} = C_{\varphi_b} D_{\rho,n}$. Since $\|D_{\psi,\varphi,n}\|_{-1} \leq \|\psi\|_\infty \|C_{\varphi_b}\|^{-1} \|D_{\rho,n}\|_{-1}$, we can estimate the upper bound for $\|D_{\psi,\varphi,n}\|_{-1}$ by the same idea as stated for the proof of [3, Proposition 4].

**Proposition 3.6.** If $\varphi$ is a nonconstant analytic self-map of $D$ with $\|\varphi\|_\infty < 1$ and the function $\psi$ belongs to $H^\infty$, then

$$\|D_{\psi,\varphi,n}\|_{-1} \leq n! \|\psi\|_\infty \left(\frac{b + |\varphi(0)|}{b - |\varphi(0)|}\right) \left(\frac{n}{b + |\varphi(0)|}\right)^n$$

whenever $\|\varphi\|_\infty \leq b < 1$. In particular, $\|D_{\varphi,n}\|_{-1} = n!$ whenever both $\|\varphi\|_\infty \leq \frac{1}{n+1}$ and $\varphi(0) = 0$.

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