A pure geometric approach to stellar structure

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Abstract: The present work represents a step in dealing with stellar structure using a pure geometric approach. Geometric field theory is used to construct a model for a spherically symmetric configuration. In this case, two solutions have been obtained for the field equations. The first represents an interior solution which may be considered as a pure geometric one in the sense that the tensor describing the material distributions is not a phenomenological object, but a part of the geometric structure used. A general equation of state for a perfect fluid, is obtained from, and not imposed on, the model. The second solution gives rise to Schwarzschild exterior field in its isotropic form. The two solutions are matched, at a certain boundary, to evaluate the constants of integration. The interior solution obtained shows that there are different zones characterizing the configuration: a central radiation dominant zone, a probable convection zone as a physical interpretation of the singularity of the model, and a corona like zone. The model may represent a type of main sequence stars. The present work shows that Einstein’s geometerization scheme can be extended to gain more physical information within material distribution, with some advantages.

Keywords: stellar structure • geometric field theories • main sequence stars • absolute parallelism geometry

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1. Introduction

Gravity plays an important role in the structure and evolution of macroscopic objects. It affects many physical parameters of objects, such as density, pressure and temperature. In relativistic theories of gravity, especially General Relativity (GR), the relation between gravity and such parameters are represented by the field equations of the theory. In the case of GR, the field equations within the material distribution are given by [1],

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -\kappa T_{\mu \nu}, \quad (1) \]

where \( T_{\mu \nu} \) is a tensor giving the physical properties of the material distribution, while the L.H.S. of this equation gives the geometric description of the gravitational field. In orthodox GR, written in the context of Riemannian geometry, the L.H.S. of (1) is a pure geometric object while the R.H.S. is a phenomenological one.

It is well known that when using (1) to study problems concerning the structure and evolution of large scale struc-
tures, an extra condition (an equation of state) is imposed in order to solve the problem. This is done since the field equations (1) are, in general, not sufficient to derive the unknown functions of the problem. This scheme, of imposing an equation of state, leads in some cases to unsatisfactory results (e.g. Schwarzschild interior solution (cf. [2]). The solution, in this case, can be considered as a mathematical one, without any physical application. Reasons for this problem are implied by the R.H.S. of (1), which is not a part of the geometric structure used: The Riemannian geometry. It is preferable, if we use a theory in which the material-energy tensor is a part of the geometric structure used. Riemannian structure is just sufficient to describe gravitational fields, thus a much wider geometric structure is needed, in order to describe the material distribution. Among the advantages of using a pure geometric model are: (i) The parameter(s) of the equation of state, characterizing the material distribution within the model, is predicted by the model and not imposed from outside, as we show in the text. In this case one don’t need an extra condition (arbitrary) to solve the field equations. (ii) The scheme will show that matter can be induced by geometry. This would be similar to the case of Schwarzschild exterior solution, in which the source of the field (gravitational mass) is induced by geometry.

The aim of the present work is to use pure geometric field theory, i.e. a theory in which the tensor describing the material-energy distribution is a part of the geometric structure, in applications. For this reason we give, in Section 2, a brief review of a geometric structure, with simultaneously non-vanishing curvature and torsion. In Section 3 we briefly review the field equations of a pure geometric theory written in the context of the structure given in Section 2. In Section 4 we solve these field equations in the case of spherical symmetry, concentrating on special case of a perfect fluid. In Section 5, we discuss the model obtained. Some concluding remarks are given in section 6.

2. The underlying geometry

In this section we briefly review a geometric structure, that is much wider than the Riemannian one. This structure is a version of ‘Absolute Parallelism’ (AP)-geometry (cf. [3]), in which both curvature and torsion are simultaneously non-vanishing objects ([4, 5]). The structure of the conventional AP-geometry is defined completely (in 4-dimensions) by a tetrad vector field $\lambda_{\mu}^{\alpha}$ such that, its determinant $\lambda^{\mu}_{\mu} = \lambda_{\mu}^{\alpha}$ is non-vanishing (Latin indices are used for vector numbers while Greek indices are used for coordinate components) and,

$$\lambda_{\mu}^{\alpha} \lambda_{\mu}^{\nu} = \delta_{\nu}^{\nu},$$  (2)

where $\lambda_{\mu}^{\alpha}$ are the contravariant components of $\lambda_{\mu}$. Einstein’s summation convention is carried over repeated indices wherever they exist. Using these vectors, one can define the second order symmetric tensor,

$$g_{\mu\nu} \overset{\text{def}}{=} \lambda_{\mu}^{\alpha} \lambda_{\nu}^{\alpha},$$  (3)

which is clearly non-degenerate. This symmetric tensor can be used to play the role of the metric of Riemannian space, associated with the AP-space, when needed.

Connections, curvatures and torsion

The AP-space admits a non-symmetric affine connection $\Gamma_{\mu\nu}^{\alpha}$, which arises as a consequence of the AP-condition (cf. [3]), i.e.

$$\lambda_{\mu}^{\alpha} \lambda_{\nu}^{\nu} = \lambda_{\mu}^{\nu} - \Gamma_{\mu\nu}^{\alpha} \lambda_{\nu}^{\alpha} = 0,$$  (4)

where the stroke and the (+) sign denote tensor differentiation as defined\(^1\) above. Equation (4) can be solved to give,

$$\Gamma_{\mu\nu}^{\alpha} = \lambda_{\mu}^{\alpha} \lambda_{\nu}^{\alpha}.$$  (5)

Since $\Gamma_{\mu\nu}^{\alpha}$ is non-symmetric, as clear from (5), then one can define the dual connection (cf. [3]) as,

$$\tilde{\Gamma}_{\mu\nu}^{\alpha} \overset{\text{def}}{=} \Gamma_{\mu\nu}^{\alpha}.$$  (6)

Also, the Christoffel symbol $\{\Gamma_{\mu\nu}^{\alpha}\}$ can be defined using the metric tensor (3). So, in the AP-geometry at least four linear connections can be defined: The non-symmetric connection (5), its dual (6), Christoffel symbol defined using (3), and the symmetric part of (5), $\Gamma_{\mu\nu}^{\alpha}$.

The curvature tensors, corresponding to the above mentioned connections, respectively, are [4]

$$M_{\mu\nu\sigma} \overset{\text{def}}{=} \Gamma_{\mu\nu,\sigma}^{\alpha} - \Gamma_{\mu\sigma,\nu}^{\alpha} + \Gamma_{\nu,\sigma}^{\epsilon} \Gamma_{\mu,\nu}^{\alpha} - \Gamma_{\nu,\nu}^{\epsilon} \Gamma_{\mu,\nu}^{\alpha} - \Gamma_{\mu,\sigma}^{\epsilon} \Gamma_{\nu,\nu}^{\alpha},$$  (7)

\(^1\) We are going to use () for ordinary partial differentiation and (\_) to represent covariant differentiation using the Christoffel symbol.
\[ M^\alpha_{\mu\nu} \equiv \delta^\alpha_{\mu\nu}, \]  
\[ R^\alpha_{\mu\nu} \equiv \{ \rho, \sigma \}_{\mu} - \{ \rho, \sigma \}_{\nu} + \{ \rho, \sigma \}_{\nu} - \{ \rho, \sigma \}_{\mu}, \]  
\[ \Lambda^\alpha_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = -\Lambda^\alpha_{\nu\mu}. \]  

This tensor is the torsion tensor of AP-space. One can define another third order tensor viz,
\[ \gamma^\alpha_{\mu\nu} \equiv \lambda^\alpha_{\mu\nu}. \]

This tensor is called the contortion of the space. Using (12), it can be shown that
\[ \Gamma^\alpha_{\mu\nu} = \gamma^\alpha_{\mu\nu} + \{ \rho, \sigma \}_{\rho\sigma}. \]

where \( \gamma_{\mu\rho\sigma} \) is skew-symmetric in its first two indices. A basic vector thus can be obtained by contraction of any one of the above third order tensors,
\[ C^\alpha_{\mu} \equiv \Lambda^\alpha_{\mu\nu} = \gamma^\alpha_{\mu\nu}. \]

Using the contortion, we can define the following symmetric third order tensor,
\[ \Delta^\alpha_{\mu\nu} \equiv \gamma^\alpha_{\mu\nu} + \gamma^\alpha_{\nu\mu}. \]

The skew-symmetric and symmetric parts of the tensor \( \gamma^\alpha_{\mu\nu} \) are, respectively
\[ \gamma^\alpha_{[\mu\nu]} \equiv \Gamma^\alpha_{[\mu\nu]} \equiv \frac{1}{2} \Lambda^\alpha_{\mu\nu}, \]
\[ \gamma^\alpha_{(\mu\nu)} = \frac{1}{2} \Delta^\alpha_{\mu\nu}, \]

where the brackets [ ] and the parenthesis ( ) are used for anti-symmetrization and symmetrization, of tensors, with respect to the enclosed indices, respectively.

**Tensor derivatives**

Using the connections mentioned above, one can define the following derivatives [4],
\[ A^\alpha_{\nu|\mu} = A^\alpha_{\nu\mu} + \Gamma^\alpha_{\rho\nu} A^\rho_{\mu}, \]
\[ A^\alpha_{|\mu\nu} = A^\alpha_{\nu\mu} + \Gamma^\alpha_{\rho\nu} A^\rho_{\mu}, \]
\[ A^\alpha_{\nu|\mu} = A^\alpha_{\nu\mu} + \Gamma^\alpha_{\rho\nu} A^\rho_{\mu}. \]

The following table is extracted from [3] and contains second order tensors that are used in most applications.

**Table 1. Second Order World Tensors [9].**

| Skew-Symmetric Tensors | Symmetric Tensors |
|------------------------|-------------------|
| \( \xi_{\mu\nu} \equiv \gamma_{\mu\nu} \) | \( C_{\mu\nu} = C_{\nu\mu} \) |
| \( \zeta_{\mu\nu} \equiv \Lambda_{\mu\nu} \) | \( \phi_{\mu\nu} \equiv \Delta_{\mu\nu} \) |
| \( \chi_{\mu\nu} \equiv \Lambda_{\mu\nu \gamma} \) | \( \psi_{\mu\nu} \equiv \Delta_{\mu\nu \gamma} \) |
| \( \epsilon_{\mu\nu} \equiv \Lambda_{\mu\nu \gamma} + \Lambda_{\nu\mu \gamma} \) | \( \theta_{\mu\nu} \equiv \Lambda_{\mu\nu \gamma} + \Lambda_{\nu\mu \gamma} \) |
| \( \kappa_{\mu\nu} \equiv \gamma_{\mu\nu \gamma} - \gamma_{\nu\mu \gamma} \) | \( \omega_{\mu\nu} \equiv \gamma_{\mu\nu \gamma} + \gamma_{\nu\mu \gamma} \) |
| \( R_{\mu\nu} \equiv \frac{1}{2} (\psi_{\mu\nu} - \phi_{\mu\nu} - \theta_{\mu\nu}) + \omega_{\mu\nu} \) |

where \( \Lambda^\alpha_{\mu\nu \gamma} \equiv \Lambda^\alpha_{\nu\mu \gamma} \). It can be easily shown that there exists an identity between 2nd order skew-tensors of Table 1 (cf. [3]):
\[ \eta_{\mu\nu} + \epsilon_{\mu\nu} - \chi_{\mu\nu} \equiv 0. \]

A useful relation between the torsion and the contortion is given by [6],
\[ \gamma_{\rho\alpha\nu} = \frac{1}{2} (\Lambda_{\rho\alpha\nu} - \Lambda_{\rho\nu\alpha} - \Lambda_{\alpha\rho\nu}). \]

We see from Table 1 that the torsion tensor plays an important role in the structure of the AP-space. All the tensors in Table 1 vanish if the torsion vanishes (see (23)).
The structure to be used in the present work is characterized by the dual linear connection (6), its corresponding torsion (11) and curvature (8) which are, in general, simultaneously non-vanishing objects. So, it is clear that this structure is wider than the Riemannian one and is of the Riemannian–Cartan type.

3. The field theory used

In this section we briefly review the field equations of a pure geometric field theory constructed by [7], we follow a procedure similar to that used in constructing the field equations of GR, in the context of the geometric structure given in the previous section. Since this structure is wider than the Riemannian one, as mentioned in section 2, it will be shown that the tensor representing the material distribution is a part of the geometric structure, as expected. The method used to derive the field equations is that of [8], a variational method.

Field equations

The set of field equations, to be used, has the form [7],

\[ S^\nu_\nu = 0. \]  
(24)

This set can be written explicitly in the form,

\[ S^\mu_\nu = \left( -2G^\mu_\nu + N\delta^\mu_\nu - 2N^\nu_\mu + 2\gamma^{\nu\rho}_{\mu,\rho} + 2\gamma^\rho_{\mu,\nu} - 2\gamma^{\rho\sigma}_{\nu,\mu} \right) + \gamma^{\alpha\beta}_{\mu}y^{\alpha\beta}_{\nu,v} - 2\gamma^{\mu}_{\nu,v} = 0, \]  
(25)

where \( N \equiv g^{\mu\nu}N_{\mu\nu} \).

To discuss the physical consequences of the geometric set (25), it is convenient to write it in the covariant form as,

\[ S^\mu_\nu = \left( -2G^\mu_\nu + N\delta^\mu_\nu - 2N^\nu_\mu + 2\gamma^{\nu\rho}_{\mu,\rho} + 2\gamma^\rho_{\mu,\nu} - 2\gamma^{\rho\sigma}_{\nu,\mu} \right) + \gamma^{\alpha\beta}_{\mu}y^{\alpha\beta}_{\nu,v} - 2\gamma^{\mu}_{\nu,v} = 0, \]  
(26)

where \( S_{\nu\mu} = g^{\nu\sigma}S^\sigma_\mu \).

### The symmetric part of \( S_{\nu\mu} \)

The symmetric part of \( S_{\nu\mu} \) is defined as usual by,

\[ S_{[\nu\mu]} = \frac{1}{2}(S_{\nu\mu} + S_{\mu\nu}). \]

Substituting from (26) into the above definition and using the symmetric tensors of Table 1, we can write

\[ S_{[\nu\mu]} = -2G_{\nu\mu} - g_{\nu\mu}\omega + \psi_{\nu\mu} + 2\omega_{\nu} - \phi_{\nu\mu} = 0, \]  
(27)

which can be written in the, more convenient, form

\[ G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2}g_{\nu\mu}R = T_{\nu\mu}^0, \]  
(28)

where,

\[ T_{\nu\mu} = \frac{1}{2}\psi_{\nu\mu} - \frac{1}{2}\psi_{\mu\nu} + \omega_{\nu\mu} - \frac{1}{2}g_{\nu\mu}\omega. \]  
(29)

From (28) it is clear that,

\[ T^0_{\nu\mu} = 0. \]  
(30)

This implies conservation (since the vectorial divergence of the left hand side of (28) vanishes identically). The tensor (29) can then be used to represent the material distribution in the theory.

### The skew-symmetric part of \( S_{\nu\mu} \)

The skew part of the field equations (26) is given by

\[ S_{[\nu\mu]} = \frac{1}{2}(S_{\nu\mu} - S_{\mu\nu}) = 0 \]

which can be written, using the skew tensors of Table 1, as

\[ S_{[\nu\mu]} = \chi_{\nu\mu} - \eta_{\nu\mu} = 0. \]  
(31)

Now using the identity (22), the skew part of the field equations can be written in the form,

\[ \epsilon_{\nu\mu} = 0, \]  
(32)
4. Solutions with spherical symmetry

In what follows, we are going to solve the field equations (24) for the specific case of spherical symmetry. The tetrad vector field, which defines the structure of an AP-space with spherical symmetry (using the coordinate system \( x^1 \equiv t, x^2 \equiv r, x^2 \equiv \theta, x^3 \equiv \phi \)) can be written in the form [10],

\[
\lambda^\nu_i = \begin{pmatrix}
A & Dr & 0 & 0 \\
B \sin \theta \cos \varphi & \frac{\partial}{\partial r} \cos \theta \cos \varphi & -\partial \sin \varphi & \frac{\partial}{\partial \sin \varphi} \\
B \sin \theta \sin \varphi & \frac{\partial}{\partial r} \cos \theta \sin \varphi & \frac{\partial}{\partial \cos \varphi} & \frac{\partial}{\partial \cos \theta \sin \varphi} \\
0 & \partial \cos \theta & 0 & 0
\end{pmatrix} \tag{33}
\]

where \( A, B \) and \( D \) are functions of \( r \) only. We are going to take \( D = 0 \) which implies that all skew tensors that appear in the structure of the field equations (24) will vanish. This guarantees the absence of interactions, other than gravity, if any. In this case (33) will reduce to,

\[
\lambda^\nu_i = \begin{pmatrix}
A & 0 & 0 & 0 \\
B \sin \theta \cos \varphi & \frac{\partial}{\partial r} \cos \theta \cos \varphi & -\frac{\partial}{\partial \sin \varphi} & \frac{\partial}{\partial \sin \varphi} \\
B \sin \theta \sin \varphi & \frac{\partial}{\partial r} \cos \theta \sin \varphi & \frac{\partial}{\partial \partial \cos \varphi} & \frac{\partial}{\partial \cos \theta \sin \varphi} \\
0 & \partial \cos \theta & 0 & 0
\end{pmatrix} \tag{34}
\]

Consequently, using (2), we get

\[
\lambda^\nu_i = \begin{pmatrix}
\frac{1}{r^2} & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2}{\partial \sin \theta \cos \varphi}
\end{pmatrix} \tag{35}
\]

Using definition (3) and the tetrad (35) we get,

\[
g_{\nu\nu} = \begin{pmatrix}
\frac{1}{r^2} & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2}{\partial \sin \theta \cos \varphi}
\end{pmatrix} \tag{36}
\]

and then,

\[
g^{\nu\nu} = \begin{pmatrix}
A^2 & 0 & 0 & 0 \\
0 & B^2 & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
0 & 0 & 0 & \frac{1}{r^2 \sin \theta \cos \varphi}
\end{pmatrix} \tag{37}
\]

From the above tetrad and metric we can evaluate second order tensors of Table 1, necessary for solving the field equations (26), in the present case.

**Second order symmetric tensors**

The symmetric tensors necessary for the field equations (26), using the definitions of the tensors given in Table 1, have the following non-vanishing components,

\[
\psi_{00} = 2 \left[ -\frac{BB' A'}{A^2} - B^2 A'' + B^2 A^2 - 2B^2 A' \frac{r}{r^2} \right],
\]

\[
\psi_{11} = -4 \frac{B' B'}{B^2},
\]

\[
\psi_{22} = 2 \left[ -\frac{t^2 B''}{B} - \frac{B'}{B} \right], \quad \psi_{33} = \psi_{22} \sin^2 \theta, \quad (38)
\]

and,

\[
\varphi_{00} = B^2 \left[ -\frac{4A' B'}{A^2 B} - 2A'' \right],
\]

\[
\varphi_{22} = B^2 \left[ -\frac{4B'^2 \theta^2}{AB'} - 2A' B^2 \frac{r}{AB'} \right], \quad \varphi_{33} = \varphi_{22} \sin^2 \theta \quad (39)
\]

Also,

\[
\omega_{11} = \frac{A'^2}{A} + \frac{2B'^2}{B}, \quad (40)
\]

and,

\[
R_{00} = \frac{B' A' B}{A^2} - B^2 A'' + 2B' A' + 2B^2 A' - 2B^2 A' \frac{r}{r^2},
\]

\[
R_{11} = -\frac{B' B'}{B^2} - \frac{A''}{A} + 2A'^2 + 2B^2 A' - \frac{B' A'}{B},
\]

\[
R_{22} = -\frac{B''}{B} - 3r \frac{B'}{B} - \frac{A'}{A} + 2r \frac{B'^2}{B^2} + r^2 \frac{B'A'}{B},
\]

\[
R_{33} = R_{22} \sin^2 \theta. \quad (41)
\]

where \( A' \overset{\text{def}}{=} \frac{dA}{dr}, \quad B' \overset{\text{def}}{=} \frac{dB}{dr} \).

**Second order skew tensors**

Using the tetrad (34), (35), the metric (36), (37) and using the definitions given in Table 1, we found that all components of the 2nd order skew tensor \( \varepsilon_{\alpha \beta} \) vanish identically, as expected. Consequently, the skew part of the field equations, (32), is satisfied identically.

**Scalars**

The scalars which are necessary for the field equations (3.3) can be obtained from the above components of the second order symmetric tensors. These are,

\[
\omega = B^2 \left( \frac{A'^2}{A} + \frac{2B'^2}{B} \right), \quad (42)
\]
and
\[ R = 2A'B'B - 2B'A' + 4B^2/A - 4A'B^2 + 6B^2 - 4B'B - \frac{8}{7}B'B. \] 

(43)

4.1. An interior solution

The field equations

Using the above calculated tensors and the field equations (24) in its mixed form, we obtain the following set of non-linear differential equations,

\[ B^2 \left[ 4B'' - 4B^2 - 8B' - 2A'B + 2A' - 3A^2 + 4A'/Ar \right] = 0, \]

(44)

\[ B^2 \left[ -4B^2 - 8B' - 4B'A + A^2 + 4A'/Ar \right] = 0, \]

(45)

\[ B^2 \left[ 4B'' - 4B^2 + 4B' - 2A'B + 2A' - 3A^2 + 2A'/Ar \right] = 0. \]

(46)

Assuming that B is non-vanishing, to prevent singularities in (35) and (36), then the above set of differential equations may be written in the form,

\[ 4B'' - 4B^2 + 8B' + 2A'B + 2A' - 3A^2 + 4A'/Ar = 0, \]

(47)

\[ -4B^2 - 8B' - 4B'A - A^2 + 4A'/Ar = 0, \]

(48)

\[ 4B'' - 4B^2 + 4B' - 2A'B + 2A' - 3A^2 + 2A'/Ar = 0. \]

(49)

From (44) and (46) we get

\[ \frac{A'}{A} = -2B', \]

(50)

which gives, by integration

\[ A = \frac{C}{B'}, \]

(51)

where \( C \) is the constant of integration. The solution (48) satisfies equation (45) without any further condition. Thus the set of differential equations (44) – (46) is not sufficient to determine the explicit forms of the two function A and B. This will be discussed in Section 5.

In what follows we are going to obtain some physical information from the geometric model given above.

A method for fixing the unknown function

In this theory, it is clear that the material-energy tensor is a pure geometric object \( T_{\nu\sigma} \), which has the definition (29), viz

\[ T_{\nu\sigma} = \frac{1}{2} \Psi_{\nu\sigma} - \frac{1}{2} \Psi_{\nu\sigma} + \omega_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} \omega. \]

Using the second order symmetric tensors calculated above, we can get the following non-vanishing components of the “geometric material-energy tensor” (29),

\[ T_{00} = \frac{B^2}{A^2} - \frac{A'^2}{A^2} + \frac{3A'^2}{2A^2} - 2A'' - \frac{B'^2}{B^2}, \]

(52)

\[ T_{11} = \frac{B^2}{B^2} - 2A' + \frac{A'^2}{2A^2}, \]

(53)

\[ T_{22} = r^2 - \frac{B^2}{B^2} - \frac{B'^2}{B^2} + \frac{A'^2}{AB} - \frac{A'^2}{A^2}, \]

(54)

and

\[ T_{33} = T_{33} \sin^2 \theta. \]

(55)

Using the definition \( T^{\mu\nu} = g^{\mu\nu} T_{\nu\sigma} \), the solution (47) and the metric (37) we can write the above components in the form,

\[ T^{00} = \frac{B^2}{A^2} - \frac{2A'^2}{A^2} + 4A'' + \frac{B'^2}{B^2}, \]

(56)

\[ T^{11} = 3B'^2 + 2B'' + \frac{B'^2}{B^2}, \]

(57)

\[ T^{22} = -B^2 - B' - 3B'^2, \]

(58)

and

\[ T^{33} = T^{22}. \]

(59)

As we have seen, the field equations (24), in the present case, do not fix the functions A and B. The resulting differential equations (44), (45) and (46) gave rise only to the relation between A and B (48). So, in order to obtain explicit forms of the functions A and B, we need a further condition. Let us assume that the components of
the geometric material energy tensor satisfy the following condition,
\[ T^{1}_{1} = T^{2}_{2} = T^{3}_{3}. \] (55)

This will lead to the condition obtained by equating (53) and (54), viz:
\[ \frac{B''}{B} + \frac{B''}{B^2} - \frac{B'}{Br} = 0, \]
which is a second order differential equation in (B) only. Integrating this differential equation twice we get,
\[ B = \left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{1}{2}}, \] (56)
where \( C \) and \( C_1 \) are constants of integration. So, from the relation (48) we can write
\[ A = \frac{C^*}{B^2} = \frac{C^*}{\left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{1}{2}}}. \] (57)

This shows that the field equations (26) can fix the unknown functions of the model, only if we use an extra condition (55). The physical consequences of using this condition will be discussed in Section 5.

The metric of the associated Riemannian space

In order to obtain some physical information about the solution obtained, (56) and (57), we write the metric of Riemannian space, associated with the AP-space (35), which is in general, given by
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \] (58)

Substituting the solution obtained, (56) and (57), into (36) we can write,
\[ ds^2 = \frac{1}{\left(C^*\right)^2} \left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{1}{2}} dt^2 + \frac{1}{\left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{1}{2}}} \left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right). \] (59)

In order to facilitate comparison with the results of GR, we are going to write (59) assuming that \( (C^*)^2 = -C_2 \) (where \( C_2 \) is another arbitrary constant) and \( d\tau = ids \), then we obtain,
\[ d\tau^2 = \frac{1}{C^2} \left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{4}{5}} dt^2 - \frac{1}{\left(7C\frac{r^2}{2} + 7C_1\right)^{\frac{1}{2}}} \left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \] (60)
which is the metric of the pseudo-Riemannian space associated with the AP-space. It has the same signature usually used in such applications of GR.

4.2. The exterior solution

Field equations

We are looking for the solution outside the material distribution. In this case, we take the material energy tensor (29).

\[ T^*_{\nu\nu} \equiv 0. \] (61)

Substituting into the symmetric part of the field equations (26), we get
\[ g_{\mu\nu} = 0. \] (62)

Further substituting the tetrad vector field (34) (from Table 1) into the equations (62), we obtain the following set of differential equations:
\[ \frac{4BB'}{rA'} - \frac{3B'^2}{A'} + \frac{2BB''}{A'^2} = 0, \] (i)
\[ \frac{2AB'}{rB} - \frac{A'B'}{B} - \frac{2A'B}{B} + \frac{A''}{r} = 0, \] (ii)
\[ \frac{B'r}{B} + \frac{B''}{B} r^2 - \left(\frac{B'}{B}\right)^2 r^2 + \frac{A'}{A} r - 2 \left(\frac{A'}{A}\right)^2 r^2 + \frac{A''}{A} r^2 = 0. \] (iii)

Integrating equation (i) twice, we get
\[ B = \frac{1}{\left(\frac{C_1}{2r} - c_4\right)^2}, \] (iv)
where \( c_1, c_4 \) are constants of integration. In order to facilitate comparison with GR, we set \( c_3 = m, c_4 = 1 \), then (iv) can be written as
\[ B = \frac{1}{\left(1 - \frac{m}{2r}\right)^2}. \] (63)
Substituting from (63) into (ii) we obtain, after integration,
\[ A = c_5 \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}, \tag{64} \]
where \( c_5 \) is a constant of integration. The solution given by (63), (64) satisfy equation (iii) without any further conditions.

The metric of associated Riemannian space

Now substituting (63), (64) into the metric (58) of the Riemannian space associated with the AP space (35) we can write (using definition (3)),
\[ ds^2 = c_5^2 \left( \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^4 d\tau^2 + \left( 1 + \frac{m}{2r} \right)^4 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \tag{65} \]
As explained in the previous subsection, in order to facilitate comparison with the results of GR, we write (65) taking \( c_5^2 = -1 \) and \( d\tau = ds \). Thus
\[ dr^2 = \left( \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^4 d\tau^2 - \left( 1 + \frac{m}{2r} \right)^4 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \tag{66} \]
This metric is identical to the Schwarzschild exterior solution, of GR, in its isotropic form. In other words, the only solution with spherical symmetry of the field equations (26), outside the material distribution, give rise to the Schwarzschild exterior solution.

4.3. The boundary condition

The situation is now as follows. Applying the field equations (26) of the theory to a spherically symmetric tetrad vector field given by (34), we obtain two solutions:

1. The first solution, given by (56) and (57), is an interior solution, characterized by the non-vanishing of some components of the geometric material-energy components given by (52-54). The pseudo Riemannian space associated with the AP space, in this case is given by (60).

2. The second solution, given by (63) and (64) of the previous subsection, is the exterior solution, characterized by the vanishing of the material tensor, i.e. the solution outside the material distribution. This gives rise to the pseudo Riemannian space whose metric is given by the expression (66) of the previous Section. This metric is found to be identical to the Schwarzschild exterior solution of GR (cf. [2]).

Now, the two solutions obtained belong to the same theory using the same tetrad field (34). So in order to fix the constants of integration \( C, C_1 \) and \( C_2 \), we need to match the metric (60) and the Schwarzschild exterior metric (66) at the boundary \( r = a \). Now, let us define the geometric density \( \rho_0^g \) and the geometric pressure \( \rho_0^p \) (these quantities are defined in relativistic units), as
\[ \rho_0^g(r) \overset{\text{def}}{=} - T^{00}_\text{0}, \quad \rho_0^p(r) \overset{\text{def}}{=} T^{11}_\text{1} = T^{22}_\text{2} = T^{33}_\text{3}. \tag{67} \]
Then we can determine \( \rho_0^p \) from equation (53), (67) using the solution (56) and (57). This can be written in the form,
\[ \rho_0^p(r) = \frac{4C^2r^2 + 14CC_1}{7C_2^2 + 27C_1^2}. \tag{68} \]
Then, assuming that \( \rho_0^p(r) = 0 \) at the boundary \( r = a \) we get
\[ C = -\frac{7C_1}{2a^2}, \]
and from matching the metric (60) with the Schwarzschild exterior solution (66) at \( r = a \) we get,
\[ 7C_1 = \left( -\frac{4}{3} \right) \frac{1}{\left( 1 + \frac{a}{2r} \right)^{\frac{1}{3}}}, \]
and
\[ C_2 = \frac{1}{\left( 1 - \frac{a}{2r} \right)^2 \left( 1 + \frac{a}{2r} \right)^{\frac{5}{2}}}. \]
Consequently we can write the solution (56) and (57) in the form,
\[ B = \left( \frac{4}{3} \right)^{\frac{1}{2}} \left( 1 - \frac{2r}{4a^2} \right)^{\frac{1}{2}} \left( 1 + \frac{a}{2r} \right)^{\frac{1}{2}}, \tag{69} \]
and
\[ A = \frac{i \left( 1 + \frac{a}{2r} \right)}{\left( 1 - \frac{a}{2r} \right)^{\frac{1}{2}} \left( 1 - \frac{2r}{4a^2} \right)^{\frac{1}{2}}}. \tag{70} \]
Then the metric (60) can now be written in the form,
\[ d\tau^2 = b_1 \left( 1 - \frac{7}{4a^2} r^2 \right)^{\frac{1}{2}} d\tau^2 - \frac{b_2}{\left( 1 - \frac{2r}{4a^2} \right)^{\frac{1}{2}}} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \tag{71} \]
Substitute the solution from (69) into (52) and (53), using where

\[ b_1 \overset{\text{def}}{=} \left( -\frac{4}{3} \right) \frac{1}{2} \left( 1 - \frac{a}{r} \right)^2 \left( 1 + \frac{a}{r} \right)^2, \quad (i) \]

and

\[ b_2 \overset{\text{def}}{=} \left( 1 + \frac{a}{r} \right)^4 \left( -\frac{1}{4} \right)^2, \quad (ii) \]

Substitute the solution from (69) into (52) and (53), using (i) and (ii), we can evaluate \( \rho_0^* (r) \) and \( \rho_1^* (r) \) in the forms,

\[ \rho_0^* (r) = \frac{1}{a^2 b_2} \frac{1}{(1 + \frac{a}{r} r^2)^{\frac{7}{2}}} \quad (72) \]

and

\[ \rho_1^* (r) = 3 \frac{1}{a^2 b_2} \frac{1}{(1 - \frac{a}{r} r^2)^{\frac{7}{2}}} \quad (73) \]

These two equations give the values of the geometric pressure and density as functions of the radial distance \( r \).

### 5. Discussion

In the present work, we have applied a pure geometric field theory, in which the tensor representing the material-energy distribution, given by (29), is a part of the geometric structure used. The field equations of the theory are applied to a geometric structure having spherical symmetry. It appears that the field equations are not sufficient to fix the unknown functions in this case, since spherical symmetry represents a spectra of physical solutions characterizing the fluid. The non-vanishing components of the geometric material-energy tensor, as clear from equations (49), (50) and (51), show that the model represents the field within a material distribution. It is shown that, the unknown functions of the model can be fixed by assuming, in addition to the field equations, the condition (55). Certain components of the material-energy tensors (61) are chosen to represent geometric density and pressure. Some boundary conditions are used to fix the constants of integration. This is done by assuming, at the boundary \( r = a \), that:

(i) The geometric pressure \( \rho_0^* \) vanishes.

(ii) The metric (60) associated with the interior solution and the metric (66) associated with the exterior solution, should be the same at this boundary.

Now, we direct the attention to the following points:

(1) **The field theory** used can be classified as a pure geometric one. In other words, all physical quantities are represented by geometric objects that are constructed from the building blocks of the geometric structure used, an AP-structure of the Riemann–Cartan type. This is done to avoid the use of a phenomenological material-energy tensor and an equation of state.

The theory proposed reduces to GR outside the material distribution, i.e. when \( T_{\mu \nu} = 0 \). So, it will give all the results of GR in the Solar system. In other words, the theory is in good agreement with the results of laboratory and Solar system tests. We stress here on the fact that the type of AP-geometry used in the present work is different from that used in the literature. In the case of \( T_{\mu \nu} \neq 0 \), there is no necessity to obtain GR results.

(2) **The geometric structure** with spherical symmetry, given in Section 3, used in the present work, is that usually used in such applications (cf. [11–14]). The non-vanishing of 2nd-order skew tensors (see Table 1) may give rise to interactions other than gravity. So, to restrict the model to gravitational interactions only, we have assumed the vanishing of the function \( D \) which leads to the vanishing of all second order skew tensors, relevant to the theory used.

(3) It is shown that the field equations (24) are not sufficient to determine the unknown functions \( A (r) \), \( B (r) \) of the model. Instead, it gives a relation (48) between \( A \) and \( B \). Any pair of functions satisfying this relation represents a solution within a material distribution. This means that relation (48) implies a family of solutions. On the other hand, the spherically symmetric geometric structure, used in the present study, can be considered as admitting representation of several physical situations. Once a further condition (55) is added, the functions can be fixed as shown in (63), (64). This is to be expected, since the components of the material-energy tensor (52), (53) and (54), are pure geometric objects, before any restrictions, expect for spherical symmetry. In other words, the relation (48) represents several situations. Once a certain situation is fixed, the field equations can be solved completely.

(4) The **boundary condition** used in the previous section, \( r = a \), can be considered as a boundary between two regions: The first is characterized by \( \rho_0^* \neq 0 \) while the second is characterized by \( \rho_0^* = 0 \). We can consider the value \( (a) \) as representing the radius of the spherical configuration (a static or a slowly rotating star). This boundary condition can be applied to Sun-like-stars. The pressure near the center of the Sun likes stars is much higher than that in the Corona, so, this pressure can be
neglected at the boundary relative to its value at the center. The radial distance \( r \), measured from the center of the star, can be written in terms of the dimensionless quantity,

\[
q = \frac{r}{a}.
\]

In this case the solution (69) and (70), the geometric pressure (72) and the geometric density (73) can be written in terms of \( q \) and the parameters \( b_1 \) and \( b_2 \), of the previous section, as

\[
B = \frac{(1 - \frac{7}{4}q^2)^{\frac{3}{2}}}{\sqrt{b_2}}, \quad \text{and} \quad A = \frac{i}{\sqrt{b_1(1 - \frac{7}{4}q^2)^{\frac{3}{2}}}},
\]

and

\[
p_0^*(r) = \frac{1}{a^2 b_2} \frac{(1 - q^2)}{(1 - \frac{7}{4}q^2)^{\frac{3}{2}}}, \quad \text{and} \quad p_0^*(r) = \frac{3}{a^2 b_2} \frac{(1 - q^2)}{(1 - \frac{7}{4}q^2)^{\frac{3}{2}}}.
\]

(5) **Equation of state:** Using the values of the geometric pressure (76) and density (77), we can write the following relation between \((p_0^*)\) and \((p_0^*)\), as

\[
p_0^* = \frac{(1 - q^2)}{3(1 - \frac{7}{4} q^2)} p_0^*.
\]

If we accept the definitions (67), of the density and pressure of the geometric origin, then (78) represents a general equation of state. It has a form similar to that of a perfect fluid, \( p_0 = \omega \rho_0 \), where \( p_0 \) and \( \rho_0 \) are the phenomenological pressure and density of the material distribution, respectively. The main advantages of (78) is that \( \omega \) is not a parameter but it is a function of position \( q \)

\[
\omega(q) = \frac{1 - q^2}{3(1 - \frac{7}{4} q^2)}.
\]

In other words, in the conventional treatment we assume a certain value of \( \omega \) for different regions within a star. The main advantages of using (78) is that it is more appropriate for a geometric treatments, since we can get physical information about the material distribution within a star once we know the position of the material. The function \( \omega(q) \) is obtained from and not imposed on, the model. Equation (78) is an advantage of using a pure geometric theory of gravity. In fact equation (78) represents a continues spectra of characteristics of the geometry induced material distribution within the star. This spectra can not be obtained if one uses the conventional approach, since the parameter of the equation of state is imposed on the model, in that case.

Fig. 1 gives the relation between the two dimensionless quantities \((q)\) and \((\frac{p_0^*}{\rho_0^*})\).

The relation between the geometric material-energy tensor and the phenomenological one (written in cgs unit) is obtained by comparing the R.H.S. of (28) and the R.H.S. of (1). So, we can write

\[
T^\mu_\nu = -\kappa T^\mu_\nu.
\]

Thus,

\[
\frac{p_0^*}{\rho_0^*} = \frac{p_0}{\rho_0},
\]

The above relations, (78) and (79), and Figure 1 show how \((\frac{p_0^*}{\rho_0^*})\) varies from the center to the surface of the star.
(6) **Corona**: If we evaluate the geometric density (67) at \( r = a \), or \( q = 1 \), we obtain

\[
\rho^0_0 = \frac{0.7771717}{a^2 b^2}, \tag{80}
\]

If we consider the model obtained as representing a certain type of main sequence stars, then imposing the condition for vanishing density \( (\rho^0_0 = 0) \) on (77), we get

\[
q = \sqrt{2}. \tag{81}
\]

So, we have a region (spherical shell), outside the star, within which \( \rho^0_0 \Rightarrow 0 \) while \( \rho^0_0 \neq 0 \). The width of this region is in the range \( \sqrt{2} > q \geq 1 \). This region can be considered as a corona of the star. However, the corona of the Sun is not an exact spherical shell. This is due to the magnetic activity of the Sun. Recall that the model in the present treatment is a pure gravity model, i.e. without considering any electromagnetic influence. This may give an interpretation for the difference between the shape of the corona, given in the present work, and that of an actual star.

(7) **Radiation Zone**: It is clear from the equation of state (5.7) (or from Fig. 1) that as \( q \Rightarrow 0 \), \( p_0 \Rightarrow \frac{1}{2} \). This value characterizes the situation of a radiation dominant central zone. This situation washes out gradually as we move towards the surface of the star \( (q = 1) \), at which \( p_0 = 0 \), which characterizes energy transfer in a class of main sequence stars (cf. [15]), according to Schwarzschild classification. A radiation zone is usually assumed in many of the models of stars.

(8) **Singularities and Convection Zone**: It is clear that the solution given (75) is singular at \( q = \sqrt{2} \approx 0.755928 \). It is also clear from (76)) and (77) that the geometric pressure and density are also singular at the same value of \( q \), while the equation of state (78) is still regular. The appearance of singularity, may indicate that the model is applied outside its domain of applicability, in this region. In other words, the model may include one or more assumptions that is not applicable at the singularity. Let us recall the case of the Sun. There is a convection zone at a depth about 28\% of the radius of the Sun, i.e. 72\% of the radius as measured from the center of the Sun (cf. [16]). The perfect fluid assumption within the convection zone is no longer valid. The presence of singularity in the present work may indicate the existence of a zone in which perfect fluid assumption is violated, a convection zone. Fig. 2 shows the variation of the density and pressure as function of \( q \).

(9) The model obtained, in the present work, is far from representing a compact object. This is because the equation of state for such objects is the adiabatic one (cf. [17]), given by

\[
p_0 = K \rho_0^{1+\frac{1}{n}}, \tag{82}
\]

where \( (K) \) is constant and \( (n) \) is the polytropic index. This index varies from \( \frac{1}{2} \) for non-relativistic particles to \( 3 \) for relativistic particles. It is clear that the equation of state obtained, (78), has a polytropic index tending to infinity, which characterizes an isothermal configuration (cf. [15]).

(10) The use of the tensor \( T^\mu^\nu \) (3.6) as a geometric representative of the material–energy tensor can be justified by some of its properties, in particular:

- It has the same properties of the phenomenological material–energy tensor, i.e. it is a symmetric second order tensor which has been shown to satisfy the relations (30)

\[
T^{\mu^\nu^\nu^\nu} = 0,
\]

which is considered as a generalization of conservation [7].

![Figure 2](https://example.com/figure2.png)

**Figure 2.** The relation between \( q \) and \( \rho^0_0 \) is represented by the dashed line, while the relation between \( q \) and \( \rho^0_0 \) is represented by the solid line.
• In view of the field equation (28) $T^{\mu\nu}$ can be considered as the source of the gravitational field represented by the L.H.S. of this equation.

• It is shown, in the present article (Subsection 4.2), that the vanishing of $T^{\mu\nu}$ gives rise to the Schwarzschild exterior metric which represent a gravitational field outside a spherical symmetry material distribution.

• In the present application the equation of state (78) associated with the interior solution is similar to that of a perfect fluid. Special cases obtained ($p_0^* = \frac{1}{3} \rho_0^*$, radiation; $p_0^* = 0$, dust) are physical situations, obtained from pure geometric considerations, already exist in some regions within real stars.

6. Summary and concluding remarks

Many of the suggested gravity theories have satisfactory PN-parameters, but are enable to cover the results of recent observation, e.g. SN-type Ia, the velocity curves of spiral galaxies, ... The reason may be the use of a phenomenological material-energy tensor. The present treatment throws some light on this problem.

In the present work, a pure geometric field theory, in which the material-energy tensor is a part of the geometric structure, is used for application in the case of spherical symmetry. Two solutions have been obtained: the 1st is an interior solution with a non vanishing material-energy tensor and the 2nd is the exterior solution characterized by the vanishing of all components of the material-energy tensor. It is shown that while the interior solution is a special one, the exterior solution is the general one, which gives rise to the Schwarzschild exterior solution of GR. One can conclude that the theory used reduces to GR in the case of spherical symmetry outside the material distribution. This theory is qualitatively superior than GR within the material distribution, at least in the case of spherical symmetry. Motivation for using a certain geometric object to represent the material distribution is given and the results obtained justify the scheme used. This scheme enables determination of the physical properties, of the fluid within the material distribution, from pure geometric consideration. For example the geometric equation of state obtained shows that the fluid present within the spherical configuration is a perfect one with well defined geometric pressure and density.

The situation, concerning the interior solution, is similar to the Schwarzschild exterior solution, of GR, in which pure geometric considerations led to many successful physical interpretations and predictions. The present work shows that Einstein geometerization scheme can be extended to cover domains within material distributions.

We may thus conclude that the model obtained in the present work represents a lower main sequence star according to the Schwarzschild classification (cf. [15]) with a corona, radiation zone and convection zone. As far as we know, this is the first attempt to obtain the above mentioned physical results from pure geometric considerations. This deserves consideration, since the model obtained in the present treatment deals with a region (stellar interior) for which matter has no direct observation.

Finally, it is worthy of mentioning that the model we obtained, in the present work, is far from being a complete one for a realistic star. This is just an attempt to construct a permeative model from pure geometric consideration. It may be considered as a first step in a series, which needs more efforts towards other steps in this direction. The importance of this approach is that stellar interiors have not been accessible by direct observation, so far.

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