Soliton quantization and internal symmetry

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Abstract

We apply the method of collective coordinate quantization to a model of solitons in two spacetime dimensions with a global $U(1)$ symmetry. In particular we consider the dynamics of the charged states associated with rotational excitations of the soliton in the internal space and their interactions with the quanta of the background field (mesons). By solving a system of coupled saddle-point equations we effectively sum all tree-graphs contributing to the one-point Green’s function of the meson field in the background of a rotating soliton. We find that the resulting one-point function evaluated between soliton states of definite $U(1)$ charge exhibits a pole on the meson mass shell and we extract the corresponding S-matrix element for the decay of an excited state via the emission of a single meson using the standard LSZ reduction formula. This S-matrix element has a natural interpretation in terms of an effective Lagrangian for the charged soliton states with an explicit Yukawa coupling to the meson field. We calculate the leading-order semi-classical decay width of the excited soliton states discuss the consequences of these results for the hadronic decay of the $\Delta$ resonance in the Skyrme model.

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1 Introduction

Soliton solutions in classical field theory are parametrized by a finite number of collective coordinates. In general, each collective coordinate corresponds to a continuous symmetry of the theory which is broken by the soliton. In addition to the usual translational modes, the soliton can also possess degrees of freedom corresponding to a compact internal symmetry of the theory. In this case, a special feature arises when the theory is quantized and rotations in the internal space give rise to an infinite tower of excited states transforming according to successively higher representations of the internal symmetry group [1]. An important example of this phenomenon occurs for the Skyrme model [2] where the isorotational excitations of the skyrmion are identified with the baryon states of large-$N$ QCD, the two lowest states corresponding to the nucleon and the $\Delta$ multiplets [3, 4]. In the full quantum theory, the soliton states interact with the elementary quanta of the background field which we will refer to as ‘mesons’. These mesons carry the conserved charge associated with the global internal symmetry and transitions between consecutive soliton states in the tower can occur via the emission or absorption of a single meson. In the case of the Skyrme model, the quanta of the chiral field are naturally identified with the pions and a simple example of the above transitions is the hadronic decay of the $\Delta$ resonance; $\Delta \rightarrow N + \pi$. In this paper we present a method for calculating the leading semi-classical contribution to the S-matrix for these processes. For clarity we illustrate this method in the context of the simplest possible model, a scalar field theory in two spacetime dimensions having an unbroken $U(1)$ symmetry. In particular, we calculate the leading-order decay widths of the excited soliton states in this model. The extension of our results to the $SU(2)$ Skyrme model will be presented elsewhere [5].

Beyond the calculation of a specific decay amplitude, we intend this paper to be the first in a series clarifying what we perceive to be a confused state of the literature on meson-soliton interactions. A useful way to illustrate this confusion is to consider the division of the meson field into a classical part, corresponding to the soliton background, and a fluctuating part according to $\phi = \phi^{cl} + \delta\phi$. This division, which is a generic feature of soliton quantization, also serves as a convenient characterization of the different contributions to the meson-soliton S-matrix considered by various authors. In the Skyrme case, Diakanov, Petrov and Pobylitsa [6] considered the classical contribution to the two-point Green’s function of the meson field, which we denote schematically as $\langle \phi^{cl} \phi^{cl} \rangle$. They argued that it contributes a Born term to the amplitude for pion-nucleon scattering at the same order as the background scattering terms considered by several authors [7, 8]. The latter terms are extracted from the two-point function for the fluctuating field, which can be written as $\langle \delta\phi \delta\phi \rangle$. The argument for a classical contribution to the meson-soliton S-matrix is quite general and dates back to the early literature on soliton quantization [9, 10]. In particular, the characteristic exponentially decaying tail of the soliton background, $\phi^{cl}(x)$, as $|x| \rightarrow \infty$ dictates that the matrix element of the background field exhibits a pole in momentum space. The corresponding residue can be interpreted as a point-like coupling of the mesons to the soliton states which leads to a Born contribution to the meson-soliton scattering amplitudes [11].
In principle, the same point-like interaction should mediate the decay of an excited soliton state to a lower one via the emission of a single meson. However, the classical contribution to the matrix element of the meson field between initial and final states has the following general form as $|x| \to \infty$:

$$\langle f | \phi^{cl}(x) | i \rangle \sim \text{FT} \left[ \frac{\delta(k_0 - \Delta E)}{|k|^2 + m^2} \right]$$

(1.1)

where $\Delta E$ is the splitting between the states, $m$ is the meson mass and FT denotes Fourier transform. In order to contribute to the S-matrix element for the decay, the momentum-space one-point function should exhibit a pole at the real value of the spatial momentum given by the mass-shell condition: $|k|^2 = \Delta E^2 - m^2$. Clearly, the one-point function determined by (1.1), exhibits a pole at an imaginary value of the spatial momentum given by $|k|^2 = -m^2$. As a result, the classical contribution to the one-point Green’s function cannot, by itself, contribute to the physical decay amplitude. In contrast, several attempts to calculate the amplitude for $\Delta$-decay purely from the contribution of the fluctuating field, $\delta \phi$, have also appeared in the Skyrme model literature \[12\]. These authors identify a linear term in the effective action for the field fluctuations around the rotating skyrmion background. This term gives a contribution to the matrix element of the pion field between different baryon states, which we can write schematically as $\langle f | \delta \phi | i \rangle$. As usual, the field $\delta \phi$ is defined subject to functional constraints which render it orthogonal to the soliton zero-modes. The fact that the physical decay amplitude obtained in this way is highly dependent upon the precise choice of this constraint is a strong indication that the results of Refs \[12\] are incorrect.

In fact, the correct interpretation of the linear term in $\delta \phi$ is as a sign that one is expanding about the \textit{wrong} classical background field $\phi^{cl}$, namely the time-dependent configuration obtained by rotating the static skyrmion. More specifically, the linear term reflects the fact that this field configuration does not satisfy the full time-dependent equation of motion of the model. In this connection it is useful to note that the analogous configuration obtained by translating the soliton does not satisfy this equation either. For the case of uniform translation, this is simply a consequence of the relativistic invariance of the theory; a soliton is an extended object and hence as it translates with a constant velocity $v$, the appropriate solution of the classical equation of motion is Lorentz contracted by a factor $\sqrt{1 - v^2/c^2}$. In the two-dimensional model considered by Gervais, Jevicki and Sakita (GJS) \[13\], similar linear terms arise in the effective action for fluctuations of the field around the background of a translating kink. These terms are precisely due to the mismatch between the static kink solution and its Lorentz contracted counterpart. By solving a set of coupled saddle-point equations, GJS successfully recovered the Lorentz contracted kink as the true stationary point of the effective action. The saddle-point contribution to the action corresponds to the infinite sum of all vacuum tree diagrams in the weak-coupling perturbation theory for the fluctuating field.

In this paper we solve an analogous set of saddle-point equations for the case of an internally rotating soliton in the $U(1)$ model. Despite the fact that there is no symmetry of the model which plays a role analogous to that of Lorentz symmetry in the translational case, the resulting
saddle-point configuration, denoted $\bar{\phi}^{cl}$, has a simple form. Our main result is conveniently expressed in terms of the matrix element of this new classical background field between soliton states. As $|x| \to \infty$ we have,

$$\langle f | \bar{\phi}^{cl}(x) | i \rangle \sim \text{FT} \left[ \frac{\delta(k_0 - \Delta E)}{|k|^2 + m^2 - \Delta E^2} \right]$$

Comparing (1.1) and (1.2), we see that the sole modification of the classical one-point function due to the replacement $\phi^{cl} \to \bar{\phi}^{cl}$ is to shift the pole in the static soliton background to exactly the position required for a physical contribution to the S-matrix. The new one-point Green’s function (1.2) is equal to the old one (1.1) plus an infinite number of tree-diagrams in the perturbation theory for the fluctuating field.

The first part of the paper is a straightforward generalization of the methods of GJS to the case of an internal degree of freedom. After introducing the $U(1)$ model in Section 2, in Section 3 we perform a canonical transformation of the path integral variables which separates out a collective coordinate $\theta(t)$ corresponding to the $U(1)$ phase angle, from the remaining field degrees of freedom. The expansion of the field around the rotating soliton background contains linear terms analogous to those found in the Skyrme case and in Section 4 we solve the corresponding set of saddle-point equations to find the correct stationary point of the effective action. We show explicitly that the resulting field configuration is free of any ambiguities related to the choice of the functional constraint. After replacing the meson field by its saddle-point value in the path-integral for the one-point function, the remaining functional integral over the collective coordinate is equivalent to a problem in Hamiltonian quantum mechanics. In Section 5 we solve this problem exactly, paying careful attention to the problem of operator ordering; the S-matrix element for the decay of an excited soliton state and the corresponding decay width is given in Section 6. We find the S-matrix element is formally equivalent to that given at tree-level by an effective Lagrangian in which the charged soliton states are represented by fermionic fields with an explicit Yukawa coupling to the meson field. Finally, in Section 7, we discuss some consequences of these results for the Skyrme model.

2 The $U(1)$ model

A simple example of solitons having a compact internal degree of freedom occurs for the case of a real two-component scalar field $\phi = (\phi_1, \phi_2)$ in two space-time dimensions,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) \cdot (\partial^\mu \phi) - \frac{m^2}{2} |\phi|^2 - W[\phi, \sigma]$$

where the potential $W$ is chosen so that the Lagrangian (2.3) has an unbroken global $U(1)$ symmetry, $\phi \to \mathcal{M}(\theta) \phi$, where

$$\mathcal{M}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$
and the resulting classical field equations admit static, finite-energy solutions. In general, for the theory to satisfy both these conditions, $W$ must also depend on at least one additional scalar field $\sigma$ and its derivatives, which couples to $\phi$. Models of this type were first considered by Rajaraman and Weinberg \[1\]. These authors investigated the specific case for which

$$W = \frac{a}{2}|\phi|^2(\sigma^2 - \mu^2/\lambda^2) + \frac{h}{4}|\phi|^4 - \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{\mu^2}{2}\sigma^2 + \frac{\lambda}{4}\sigma^4$$

(2.5)

and demonstrated the existence and energetic stability of soliton solutions of the required form for a certain range of the Lagrangian parameters. Although the additional field is necessary to stabilize the solution, it does not transform under the $U(1)$ symmetry and its presence does not affect the outcome of our analysis; hence we will suppress it in the following discussion.

We begin by assuming that the classical field equation,

$$\frac{\partial^2 \phi_i}{\partial t^2} - \frac{\partial^2 \phi_i}{\partial x^2} + m^2 \phi_i + \frac{\delta W}{\delta \phi_i} = 0$$

(2.6)

admits a static soliton solution, $\phi^{cl}(x)$. We will need to impose the additional technical requirement (see Appendix B) that the soliton points in the same internal direction, $\phi^{cl}/|\phi^{cl}| \equiv \hat{\phi}_S$ at each point in space. This condition is always true for Lagrangians such as (2.5) where the potential $W$ depends only on $|\phi|$. In this case a global $U(1)$ transformation suffices to rotate the soliton into the first component of the field vector as $\phi^{cl} = (\phi_S, 0)$, which is the convention we adopt from now on. In a theory with unbroken symmetry, the finite energy condition implies that the soliton profile, $\phi_S(x)$, must go to zero rapidly at spatial infinity. In this respect the static solutions considered here are different from the kink configurations found in other two-dimensional field theories which interpolate between different vacua at left and right spatial infinity. The asymptotic behavior of the profile can be found by linearizing the equation of motion; writing $\phi_S = \phi_S(x; m)$ to emphasize the parametric dependence of the solution on the meson mass $m$, we find $\phi_S(x; m) \to A \exp(-m|x|)$ as $|x| \to \infty$. The coefficient $A$ depends on any dimensionless ratios of the Lagrangian parameters and must be determined by solving the full non-linear equation of motion.

Starting from the particular solution $\phi^{cl}$, the general static solution is generated by spatial translations and $U(1)$ rotations,

$$\phi^{cl}(x; X, \theta) = M(\theta)\phi^{cl}(x - X)$$

(2.7)

In addition to these time-independent solutions, equation (2.6) also has solutions which rotate in the internal space with a constant angular velocity $\omega$. These are easily found by writing $\phi(x, t) = M(\omega t)\hat{\phi}(x)$ with $\hat{\phi} = (\phi_\omega, 0)$. The resulting equation for $\phi_\omega$ is,

$$-\frac{\partial^2 \phi_\omega}{\partial x^2} + \frac{\delta W}{\delta \phi_\omega} + (m^2 - \omega^2)\phi_\omega = 0$$

(2.8)
For $\omega = 0$, equation (2.8) reduces to the static equation of motion which is solved by the soliton profile $\phi_S(x; m)$. The effect of non-zero angular velocity is simply to shift the effective mass term in the field equation and the corresponding solution is just $\phi_\omega = \phi_S \left( x; \sqrt{m^2 - \omega^2} \right)$. For $\omega > m$, the asymptotic behavior of the solution is oscillatory and the solution is no longer localized in space. In the corresponding quantum theory, soliton states of sufficiently high internal angular momentum are unstable and can decay into a lower state via the emission of a single meson. As we will demonstrate below, the field configurations $\phi_\omega$ for suitable values of $\omega$, dominate the path-integral describing this process in the semi-classical limit. In this case, the oscillating tail of the configuration is naturally identified with the asymptotic meson emitted in the decay.

The zero-mode of the soliton associated with infinitesimal $U(1)$ rotations is obtained by differentiating the static solution (2.7) with respect to the symmetry parameter $\theta$. In the chosen coordinate system, this rotational zero-mode is given by $\psi_R = (0, \phi_S)$. The moment of inertia of the soliton with respect to internal rotations can be expressed in terms of the zero-mode as,

$$\Lambda = \int dx \psi_R(x) \cdot \dot{\psi}_R(x).$$

By analogy with the Skyrme model we will describe the semi-classical limit of the theory in terms of a dimensionless coupling constant $1/N$. The conventional semi-classical limit is obtained by setting $\phi_S \sim \sqrt{N}$ in which case $\Lambda \sim N$. For the model of Rajaraman and Weinberg, this behavior follows from taking both the dimensionless ratios $h/m^2$ and $\lambda/\mu^2$ to be $O(1/N)$. Expanding the field around the static soliton background in the $N \to \infty$ limit as $\phi(x, t) = \phi^{cl}(x) + \chi(x, t)$ we find,

$$\mathcal{L}[\phi] = \mathcal{L}[\phi^{cl}] + \frac{1}{2} \chi \cdot \hat{\mathcal{L}} \chi + V(\chi)$$

(2.10)

where the operator of quadratic fluctuations is

$$\hat{\mathcal{L}}_{ij} = \left( -\frac{\partial^2}{\partial x^2} + m^2 \right) \delta_{ij} + \frac{1}{2} \frac{\delta^2 W}{\delta \phi_i \delta \phi_j} [\phi^{cl}]$$

(2.11)

In the limit $|x| \to \infty$, the potential contribution to the fluctuation operator goes to zero rapidly, and $\hat{\mathcal{L}}_{ij} \to (-\partial^2_x + m^2) \delta_{ij}$ which is just the free inverse propagator for field quanta of mass $m$. However, as expected, the full operator $\hat{\mathcal{L}}$ has two vanishing eigenvalues and so cannot be inverted directly. Using the $U(1)$ invariance of the potential $W$, it can easily be shown that one of the zero eigenfunctions of $\hat{\mathcal{L}}$ coincides with the rotational zero-mode $\psi_R$ as expected. The other zero eigenfunction is associated with infinitesimal translations and can be obtained in a similar way by differentiation with respect to the soliton center of mass coordinate, $X$.

3 Collective coordinate quantization

The zero-modes discussed in the previous section lead directly to infra-red divergences in the naive perturbation expansion around the soliton background coming from the Lagrangian (2.10). In order to construct a finite perturbation theory it is necessary to remove these zero-modes.
from the spectrum by introducing explicit collective coordinate fields \( X(t) \) and \( \theta(t) \). In this paper we will adopt the path-integral quantization method of Gervais, Jevicki and Sakita (GJS) in which the collective coordinates are introduced by performing a canonical transformation of the dynamical variables in the phase-space path integral \([13, 14]\). This method was originally developed in the context of a kink configuration in two dimensions with a single translational degree of freedom and this section is a straightforward generalization of their method to the case of an internal collective coordinate. In the following we will focus exclusively on the dynamics of internal soliton rotations and will henceforth ignore the translational motion of the soliton. The collective coordinate quantization of solitons with a translational zero mode has been extensively discussed in the literature and offers no new features in the present case. Our analysis can easily be extended to include the translational degree of freedom using the methods described in \([13]\).

In particular, the inclusion of translations leads to recoil corrections to the S-matrix elements calculated below which are suppressed by factors of \( 1/N \) relative to the leading-order results.

The transition amplitude between initial and final soliton states, described by wavefunctions \( \Psi_i \) and \( \Psi_f \), at times \( t = -T \) and \( t = +T \) can be written as a phase-space path integral,

\[
T_{fi}[J] = \int D\phi D\pi \Psi_f^*[\phi(x, +T)] \Psi_i[\phi(x, -T)] \exp \left( i \int d^2x \pi \cdot \dot{\phi} - \mathcal{H}(\pi, \phi) + J \cdot \phi \right)
\]  

(3.12)

with the Hamiltonian density,

\[
\mathcal{H} = \frac{1}{2} \pi \cdot \pi + \frac{1}{2} \phi' \cdot \phi' + \frac{m^2}{2} |\phi|^2 + W(\phi)
\]  

(3.13)

where the conjugate field momenta, \( \pi = (\pi_1, \pi_2) \) are given by \( \pi_i = \delta \mathcal{L}/\delta \dot{\phi}_i \) and we have included an external source \( J \) which couples to \( \phi \). In Section 5 below we will identify soliton states of definite \( U(1) \) charge and specify the wavefunctions in \([3.12]\) accordingly. The Green’s functions describing the meson emission and absorption which accompanies transitions between these states can then be obtained by functional differentiation with respect to the source.

Following the standard method of collective coordinate quantization we introduce \( \delta \)-function constraints orthogonalizing the fluctuations of the field around the soliton background to the rotational zero mode by inserting the following identity into the integrand of the transition amplitude,

\[
\int D\theta(t) D\phi(t) J_\theta J_P \delta(F_1[\theta; \phi]) \delta(F_2[\theta, P; \phi, \pi]) = 1
\]  

(3.14)

where \( J_\theta = \delta F_1/\delta \theta \) and \( J_P = \delta F_2/\delta P \) and the angular collective coordinate \( \theta(t) \) is restricted to lie in the range \([0, 2\pi]\). In the following, \( P(t) \) will be interpreted as the internal angular momentum conjugate to \( \theta(t) \) and we will later find that it is restricted to a discrete set of values which label the physical soliton states. The constraints \( F_1 \) and \( F_2 \) are chosen as

\[
F_1(t) = \int dx \left[ \mathcal{M}^T(\theta(t)) \bar{\phi}(x, t) \cdot \phi^cl(x) \right]
\]  

(3.15)

\[
F_2(t) = \Lambda^{-1} \int dx \left[ \mathcal{M}^T(\theta(t)) \bar{\pi}(x, t) \cdot \pi^cl(x) \right]
\]  

(3.16)

\( \Lambda \) is a taste breaking parameter which we will discuss in more detail later.
where \( \mathbf{F}^{cl} \) will be defined below. The introduction of a \( \delta \)-function constraint in this context is somewhat analogous to the Faddeev-Popov gauge-fixing procedure in Yang-Mills theory and the precise choice of the function \( f(x) \) which appears in the constraint is frequently referred to as the choice of a ‘gauge’. The analogy is a good one to the extent that all physical quantities such as S-matrix elements should be ‘gauge-invariant’ at each order in the semi-classical expansion. In general, any choice for the gauge function which has non-zero functional overlap with the rotational zero-mode is sufficient to remove the corresponding infra-red divergences occurring in the naive perturbation theory around the soliton background. The simplest choice is \( f = \psi_R \) which corresponds to the so-called ‘rigid gauge’. We will not choose a specific gauge for this calculation as we seek to demonstrate the gauge-invariance of the resulting S-matrix elements. For notational convenience we will restrict our attention to gauges for which the vector \( f \) is parallel to \( \psi_R \) at each point in space and we also fix the overall normalization of these functions to be the same. We therefore set \( f = (0, f(x)) \) and impose

\[
\int dx \, f(x) \phi_S(x) \neq 0 \quad \int dx \left[ f(x) \right]^2 = \Lambda \tag{3.17}
\]

However, it can easily be checked that our conclusions are not altered if these restrictions are relaxed.

We now make a non-linear change of variables in the path-integral introducing the rotated field \( \tilde{\Phi} = \mathcal{M}^T(\theta(t))\Phi \). This change of variables can be thought of as a transformation from the laboratory frame to the body-fixed frame of the rotating soliton. In general, given any quantity \( A \) in the laboratory frame, we will denote the corresponding quantity in the body-fixed frame as \( \tilde{A} \). In terms of the new variables the Jacobian factor \( J_\theta \) is given by:

\[
J_\theta = \int dx \, f \times \tilde{\phi} \tag{3.18}
\]

The corresponding transformation of the field momentum is, \( \mathbf{p}(x, t) = \mathcal{M}(\theta(t))\left(\mathbf{p}^{cl} + \mathbf{\pi} \right) \), where the ‘classical’ piece of the momentum, \( \mathbf{p}^{cl} \), which also appears in the constraint (3.16), must be chosen carefully to ensure that the above change of variables constitutes a canonical transformation. More precisely, we demand that the Legendre term in the phase-space path integral retain its canonical form in terms of the new variables,

\[
\int dx \, \mathbf{p} \cdot \dot{\mathbf{\hat{\phi}}} = P\dot{\theta} + \int dx \, \mathbf{\pi} \cdot \dot{\mathbf{\hat{\phi}}} \tag{3.19}
\]

and also that the two Jacobian factors associated with the change of variables cancel in the integrand of (3.14): \( J_\theta J_P = 1 \). In Appendix A we show that these two requirements specify a unique choice for \( \mathbf{p} \). In terms of the body-fixed frame fields we have,

\[
\mathbf{p}^{cl} = \frac{1}{J_\theta} \left( P - \int dx \, \mathbf{\pi} \times \tilde{\phi} \right) f(x) \tag{3.20}
\]

\[3\]In the following we define a scalar cross product of a pair of two-dimensional vectors by \( \mathbf{a} \times \mathbf{b} = \epsilon^{ij} a_i b_j \).
The resulting transition amplitude can then be expressed as an integral over all paths in the collective coordinate phase space \((P(t), \theta(t))\),

\[
T_{fi}[J] = \int \mathcal{D}\theta(t)\mathcal{D}P(t)\Psi^*[\theta(+T)]\Psi_i[\theta(-T)] \exp \left( i \int_{-T}^{+T} dt P\dot{\theta} \right) \exp iS[\theta, P; J] \tag{3.21}
\]

where we have anticipated the fact that the wave-functions of the soliton states of fixed \(U(1)\) charge are functions of the collective coordinate only and do not depend on the remaining field degrees of freedom. The effective action for a given path, \(S[\theta, P; J]\), is given in terms of a constrained path-integral over the body-fixed frame fields,

\[
\exp iS[\theta, P; J] = \int \mathcal{D}\tilde{\phi}(x,t)\mathcal{D}\tilde{\pi}(x,t) \delta \left( \int dx \, \tilde{f} \cdot \tilde{\phi} \right) \delta \left( \int dx \, \tilde{f} \cdot \tilde{\pi} \right) \times \exp \left( i \int d^2x \, \tilde{\pi} \cdot \dot{\tilde{\phi}} - \mathcal{H}[\tilde{\pi}, \tilde{\phi}] + J \cdot \tilde{\phi} \right) \tag{3.22}
\]

where the Hamiltonian density for the body-fixed frame fields is found to be,

\[
\mathcal{H} = \frac{1}{2} \tilde{\pi} \cdot \tilde{\pi} + \frac{1}{2} \tilde{\phi} \cdot \tilde{\phi} + \frac{m^2}{2} |\tilde{\phi}|^2 + W(\tilde{\phi}) + \frac{\Lambda}{2J^2} \left( P - \int dx \, \tilde{\pi} \times \tilde{\phi} \right)^2 + \Delta V \tag{3.23}
\]

As with any non-linear point canonical transformation of the path integral, additional terms in the effective potential arise beyond those obtained by naive substitution in the action. We have denoted these terms as \(\Delta V\) in (3.23). In the corresponding canonical quantization of the system \([[15]]\), the additional terms are associated with operator ordering ambiguity presented by a Hamiltonian such as (3.23) which involves arbitrarily complicated products of fields and their conjugate momenta. In the functional integral approach adopted here, the equivalent ambiguity arises in choosing the appropriate discretized definition of the path-integral \([[16]]\). The additional terms in the potential are independent of the collective coordinate and its conjugate angular momentum and contribute to the renormalization of the soliton profile function, \(\phi_S\), at the two-loop level. These terms do not affect our results for the S-matrix at leading order and we will disregard them in the following.

### 4 Perturbation theory and the saddle-point equations

The first step in obtaining the leading-order contribution to the transition amplitude (3.21) is to evaluate the inner path integral over the body-fixed frame fields (3.22) in the semi-classical limit. In this section we will accomplish this directly by applying the powerful saddle-point method used by GJS in the translational case. Before applying this method we will briefly discuss its relation to the more familiar weak-coupling perturbation theory which was also introduced by GJS in Ref \([[13]]\). Expanding the body-fixed frame field around the static soliton background as \(\tilde{\phi} = \phi^{cl} + \chi\), the transition amplitude becomes,

\[
T_{fi}[J] = \int \mathcal{D}\theta(t)\mathcal{D}P(t)\Psi^*[\theta(+T)]\Psi_i \int \mathcal{D}\chi(x,t)\mathcal{D}\tilde{\pi}(x,t) \delta \left( \int dx \, \tilde{f} \cdot \chi \right) \delta \left( \int dx \, \tilde{f} \cdot \tilde{\pi} \right) \times \exp \left( i \int dt L_0 + L_{\text{int}} + \int dx \, J \cdot \tilde{\phi} \right) \tag{4.24}
\]
where,

\[
L_0 = \dot{P} \dot{\theta} + \int dx \, \tilde{\pi} \cdot \dot{\chi} - H_0 \tag{4.25}
\]

\[
H_0 = M + \int dx \left[ \frac{1}{2} \tilde{\pi} \cdot \tilde{\pi} + \frac{1}{2} \chi T \hat{L} \chi \right] \tag{4.26}
\]

\[
L_{int} = \frac{\Lambda}{2 J_0} \left( P - \int dx \, \tilde{\pi} \times \dot{\phi} \right)^2 + \int dx \, V(\chi) \tag{4.27}
\]

The analysis given in the previous Section, which leads to the above form for the transition amplitude is completely analogous to the quantization of the kink system given by GJS. In particular, (4.24) is the natural generalization of corresponding expression for the kink transition amplitude given as Eqn (2.14) in Ref [13] and the resulting perturbation theory has a similar structure. In both cases, the \( \delta \)-function constraints in the path-integral effectively eliminate the zero-eigenvalue of the quadratic fluctuation operator and the propagator for the fluctuating field can be obtained by inverting this operator on the functional subspace orthogonal to the zero-modes. The expansion of the interaction Lagrangian in powers of \( 1/N \) gives rise to an infinite series of vertices for the fluctuating field some of which depend on the collective coordinates. In general, these Feynman rules are cumbersome and highly gauge-dependent although, as we have already stressed, this dependence must eventually cancel in the resulting S-matrix elements. In the translational case, GJS give explicit expressions for the propagators and the first few vertices in the "rigid gauge" where the constraint function is chosen to coincide with the translational zero mode. The lowest-order vertex which depends on the kink momentum is a one-point vertex which corresponds to a linear term in the effective action for the fluctuating field (see Figure 3a of Ref [13]). Expanding the interaction Lagrangian in the present case we find,

\[
L_{int} = \frac{\Lambda P^2}{2 \int dx \, f \times \phi d} \left[ 1 - 2 \frac{\int dx \, f \times \chi}{\int dx \, f \times \phi d} + O(|\chi|^2) \right] \tag{4.28}
\]

The second term in the brackets is linear in the fluctuating field \( \chi \) and yields an analogous one-point vertex. When combined with the propagators and the higher-order vertices, this vertex leads directly to an infinite number of connected tree-diagrams which contribute to the transition amplitude.

As discussed in Section 1, the presence of a linear term indicates that the background field configuration around which we are expanding is not a stationary point of the effective action. For the case of a uniformly translating kink, GJS solved the saddle-point equations for the effective action and demonstrated that the true stationary point is given by the appropriately Lorentz contracted kink configuration which also solves the full time-dependent classical field equation of the model. As usual, the saddle-point method corresponds to the reorganization of the perturbation theory as a loop expansion and, in this context, the Lorentz contracted kink is equal to the the original kink configuration plus the sum of all tree diagrams with one external leg. Similarly, in the case of a rotating soliton considered here, the simple static background \( \phi_d \) is not a stationary point of the exponent of (4.24) for \( P \neq 0 \). By analogy with the translational
case we might expect that the true stationary configuration would be given by a suitable time-dependent solution of the original classical field equation for the $U(1)$ model, Eqn (2.6). In the following we will demonstrate that this is indeed the case and give an explicit expression for the sum of all tree-diagrams contributing to the one-point function of the meson field.

In order to apply the saddle-point method of Ref [13] to the constrained path integral (3.22), it is first necessary to exponentiate the constraints by introducing Lagrange multiplier fields $\lambda(t)$ and $\nu(t)$. In the absence of the source $J$, we have,

$$\exp iS[\theta, P; 0] = \int D\tilde{\phi}(x,t) D\tilde{\pi}(x,t) D\lambda(t) D\nu(t) \exp \left( i \int d^2x \tilde{\pi} \cdot \dot{\tilde{\phi}} - \mathcal{H} + \lambda f \cdot \tilde{\phi} + \nu f \cdot \pi \right)$$

(4.29)

The saddle-point field configuration which provides the dominant contribution to $S[\theta, P; 0]$ in the semi-classical limit will be a stationary point of the effective Lagrangian for the body-fixed frame fields which appears in the exponent of (4.29). Returning to the full expression for the transition amplitude (3.21) and noticing that $S[\theta, P; 0]$ depends only on $P(t)$ and not on $\theta(t)$, we see that the angular momentum is necessarily a constant of the collective coordinate motion for $J = 0$. For this reason, we will restrict our attention to the case $P(t) \equiv P$ for which the corresponding field configuration is time-independent. The static saddle-point equation which follows from varying the exponent of (4.29) with respect to $\tilde{\phi}_i$ is,

$$\frac{\partial^2 \tilde{\phi}_i}{\partial x^2} - m^2 \tilde{\phi}_i - \frac{\delta W}{\delta \tilde{\phi}_i} + \frac{\Lambda}{J^2} \varepsilon^{ij} \tilde{\pi}_j \left( P - \int dx \tilde{\pi} \times \tilde{\phi} \right) - \frac{\Lambda}{J^2} \varepsilon^{ij} f_j \left( P - \int dx \tilde{\pi} \times \tilde{\phi} \right)^2 + \lambda f_i = 0$$

(4.30)

where we have used $\delta J / \delta \tilde{\phi}_i = -\varepsilon^{ij} f_j$ which follows from (3.18). The corresponding equation for the conjugate field momentum $\tilde{\pi}_i$ is given by

$$\tilde{\pi}_i + \frac{\Lambda}{J^2} \varepsilon^{ij} \tilde{\phi}_j \left( P - \int dx \tilde{\pi} \times \tilde{\phi} \right) - \nu f_i = 0$$

(4.31)

The saddle-point field configurations are found by solving equations (4.30) and (4.31) simultaneously with the constraint equations,

$$\int dx f \cdot \tilde{\phi} = 0 \quad \int dx f \cdot \tilde{\pi} = 0$$

(4.32)

which follow from varying the Lagrangian with respect to $\lambda$ and $\nu$.

The solution of the above system of equations can be obtained by a series of manipulations which are completely analogous to those used by GJS in the translational case; details are given in Appendix B. The main result of this analysis is that both the momentum field, $\tilde{\pi}$, and the Lagrange multipliers, $\lambda$ and $\nu$, can be eliminated leaving the single gauge-invariant equation,

$$- \frac{\partial^2 \tilde{\phi}_S}{\partial x^2} + m^2 \tilde{\phi}_S + \frac{\delta W}{\delta \tilde{\phi}_S} - \frac{P^2 \tilde{\phi}_S}{\left( \int dx \tilde{\phi}_S^2 \right)^2} = 0$$

(4.33)
where \( \tilde{\phi} = (\tilde{\phi}_S, 0) \). Identifying the angular velocity \( \omega \) in terms of the collective coordinate angular momentum by \( \omega = P/\tilde{\Lambda} \) where the field-dependent moment of inertia \( \tilde{\Lambda} \) is given by,

\[
\tilde{\Lambda} = \int dx \tilde{\phi}_S^2
\]

we see that equation (4.33) coincides exactly with equation (2.8) obeyed by \( \phi_\omega \), the uniformly rotating solution of the original time-dependent equation of motion (2.6). It follows that the corresponding solution for \( \tilde{\phi}_S \) can be expressed in terms of the unrotated soliton profile \( \phi_S \) as,

\[
\tilde{\phi}_S(x; m) = \phi_S \left( x; \sqrt{m^2 - P^2/\tilde{\Lambda}} \right)
\]

where the effective moment of inertia, \( \tilde{\Lambda} = \tilde{\Lambda}(P) \) satisfies the self-consistency condition,

\[
\tilde{\Lambda} = \int dx \left[ \phi_S \left( x; \sqrt{m^2 - P^2/\tilde{\Lambda}} \right) \right]^2
\]

For any given potential \( W \), solving this self-consistency equation requires the full numerical solution for the soliton profile for a range of values of \( m \). Fortunately, with some mild assumptions about the general behavior of the soliton solutions as the meson mass is varied, we are able to demonstrate the existence of a solution for sufficiently small internal angular momentum. In particular, we assume only that the static soliton moment of inertia is a continuous function of the meson mass with a non-zero limit, \( \Lambda_0 \), as the meson mass is taken to zero. We note that the corresponding assumptions are certainly true in the case of the Skyrme model where static equation of motion has been solved numerically both in the case of massless pions [3] and in the massive case [4]. In the Appendix we give a graphical proof that (4.36) necessarily has a real solution for \( P/m < \Lambda_0 \) as long as these assumptions are satisfied. Expanding both sides of (4.36) and recalling that \( \phi_S \sim \sqrt{N} \) we see that \( \tilde{\Lambda}(P) = \Lambda(1 + O(1/N)) \). In the following we will also assume that this series gives the correct analytic continuation of \( \tilde{\Lambda} \) to the regime \( P/m > \Lambda_0 \).

The resulting \( U(1) \) family of saddle-point configurations is generated by rotations of the particular solution \( \tilde{\phi}^d = (\tilde{\phi}_S, 0) \). It is easy to check that these configurations together with the corresponding solutions for the field momentum, \( \tilde{\pi}^d \), and the Lagrange multipliers, \( \tilde{\lambda}^d \) and \( \tilde{\nu}^d \), provide a consistent solution to the full system of saddle-point equations (4.30-4.32). The energy of this solution is conveniently expressed as,

\[
E(P) = \int dx \tilde{\mathcal{H}}[\tilde{\phi}^d, \tilde{\pi}^d] = \frac{P^2}{2\tilde{\Lambda}} + \int dx \mathcal{H}[-\tilde{\phi}, \tilde{\pi}^d]
\]

By expanding (4.37) in powers of \( 1/N \), we obtain,

\[
E(P) = M + \frac{P^2}{2\Lambda} + O \left( \frac{1}{N^2} \right)
\]

Thus, as expected, at leading order the energy of the soliton due to its rotation is exactly that of a rigid body with moment of inertia \( \Lambda \). The corresponding action is given by \( S[\theta, P; 0] = 2E(P)T \).
and represents the infinite sum of all connected vacuum tree-diagrams in the weak-coupling perturbation theory outlined above. We note that it is manifestly independent of the gauge function $f$. In the analysis of Section 3, the source term was introduced only as convenient device to obtain the Green’s functions of laboratory frame field. The saddle-point contribution to these Green’s functions is given by replacing the full field by the same configuration which dominates the the source-free transition amplitude. For this reason we disregard any modification of the saddle-point field caused by the reintroduction of source and write,

$$S[\theta, P; J] = \int_{-T}^{+T} dt \left( - E(P) + \int dx J \cdot \varphi \right)$$  \hspace{1cm} (4.39)$$

where $\varphi(x, t) = M(\theta(t)) \tilde{\phi}^d(x)$ is the corresponding saddle-point configuration of the laboratory field $\tilde{\phi}$.

The higher-order corrections to this result can be obtained by expanding the body-fixed frame field around the saddle-point configuration as $\tilde{\phi} = \tilde{\phi}^d + \delta \tilde{\phi}$. Because we are now expanding about the true stationary point, there will be no linear term for the fluctuating field $\delta \tilde{\phi}$ in the effective action and therefore the corresponding Feynman rules will not contain a one-point vertex. It follows that the corrections to the one-point function in this perturbation theory necessarily involve at least one loop. As expected, the new perturbation theory is simply a reorganization of the old perturbation theory as a loop expansion. In this case, the loop expansion is an improvement on the old perturbation theory in two ways. First, we have demonstrated that the zeroth-order contribution in the loop expansion is manifestly gauge invariant. Second, as we discussed in Section 1, single terms in the old perturbation theory do not contribute to the S-matrix directly. In the following we will demonstrate that the zero-th-order Green’s functions obtained by replacing the full field by its saddle-point value have physical poles in momentum space which contribute Born terms to the meson-soliton S-matrix. However, we should point out that even the loop expansion by itself does not always provide a systematic expansion for the S-matrix elements in powers of $1/N$. In the case of the two-point function, an exact cancellation of the leading order occurs and the first non-vanishing contribution contains not only the subleading part of the Born terms but also background scattering terms which come from the next order in the expansion around the saddle-point $\tilde{\phi}$. Fortunately the one-point function which we will calculate below is free from any such cancellations and unambiguously gives the leading order contribution to the S-matrix for soliton decay.

## 5 The transition amplitude and Green’s functions

Applying the static saddle-point analysis of the previous section, the leading semi-classical contribution to the source-free transition amplitude given by (3.21) can be written as,

$$T_{fi}[0] = \int D\theta(t) D P(t) \Psi^*_i[\theta(+T)] \Psi_i[\theta(-T)] \exp \left( i \int_{-T}^{+T} dt P \dot{\theta} - E(P) \right)$$  \hspace{1cm} (5.40)$$
This phase-space path integral given above describes the quantum mechanics of a single degree of freedom governed by the Hamiltonian \( H = E(P) \). As indicated in the previous section, this Hamiltonian coincides with that of a non-relativistic rigid top at leading order in \( 1/N \). Using the standard equivalence between the path integral and canonical quantization, we introduce a collective coordinate operator \( \hat{\theta} \) and its conjugate momentum \( \hat{P} \) with \([\hat{\theta}, \hat{P}] = i\). The momentum operator \( \hat{P} \) commutes with the corresponding operator Hamiltonian \( \hat{H} \) and so the stationary states of the system are labelled by a conserved charge \( p \) with \( \hat{P}|p\rangle = p|p\rangle \). Because the collective coordinate is an angular variable, the charge \( p \) is constrained to be an integer; the orthonormalized wavefunctions are

\[
\psi_p(\theta) = \frac{1}{(2\pi)^{1/2}} \exp(ip\theta)
\]  
(5.41)

The corresponding energy levels are given by \( \hat{H}|p\rangle = E(p)|p\rangle \).

The stationary states of \( \hat{H} \) correspond to excited states of the soliton of definite \( U(1) \) charge and are directly analogous to the baryon states of the Skyrme model. The transition amplitude between the states \( |p_i\rangle \) and \( |p_f\rangle \) can be obtained by setting \( \Psi_i = \psi_{p_i} \) and \( \Psi_f = \psi_{p_f} \) in the integrand of (3.21). In the absence of the source \( J \) which couples the collective coordinates to the background field, these states are necessarily stable and the resulting transition amplitude is diagonal. Choosing an appropriate overall normalization for the path integral (5.40) we find,

\[
T_{fi}[0] = \delta_{p_f,p_i} \exp[-2iTE(p_i)]
\]  
(5.42)

However, the field \( \phi \) interpolates mesons of unit charge and, once its coupling to the quantum mechanical degrees of freedom is restored, we expect that transitions between adjacent soliton states will be mediated by the absorption and emission of these quanta. These processes are described the Green’s functions obtained from the corresponding transition amplitude in the presence of the source \( J \). Using (4.39), the saddle-point contribution to the transition amplitude is given by,

\[
T_{fi}[J] = \int D\theta(t)DP(t)\psi^*_{p_f}[\theta(+T)]\psi_{p_i}[\theta(-T)] \exp \left( i \int_{-T}^{+T} dt \hat{P}\dot{\theta} - E(P) + \int dx J \cdot \varphi \right)
\]  
(5.43)

It is convenient to write the laboratory-frame saddle-point configuration \( \varphi \) in terms of components \( \varphi_\pm = \varphi_1 \mp i\varphi_2 \) corresponding to field quanta of definite \( U(1) \) charge. Using the explicit form of the saddle-point solution we have;

\[
\varphi_\pm(x; P(t), \theta(t)) = \exp(\pm i\theta(t))\phi_S \left( x; \sqrt{m^2 - \omega(P(t))^2} \right)
\]  
(5.44)

where \( \omega(P) = P/\tilde{\Lambda}(P) \) is the effective angular velocity. Arbitrary Green’s functions for the fields \( \phi_\pm \) which interpolate asymptotic quanta of charge \( p = \pm 1 \) are obtained by differentiating the transition amplitude (5.43) with respect to the source components \( J_\pm = (J_1 \mp iJ_2)/2 \). The general \( n \)-point function obtained in this way is written as,

\[
\langle p_i, -T| \prod_{a=1}^n \phi_{sa}(x_a, t_a)|p_f, +T\rangle = \frac{\delta^n T_{fi}[J]}{\delta J_{sa}(x_n, t_n)\ldots \delta J_{sa}(x_1, t_1)} \bigg|_{J=0}
\]  
(5.45)
where \( s_a = \text{sign}(r_a) \) with \( r_a = \pm 1 \) specifies a particular choice of positive and negative field components. Using (5.43), the above Green’s function can be related to a corresponding \( n \)-point function for the meson field between initial and final soliton states of definite orientations, \( \theta_i \) and \( \theta_f \),

\[
\langle p_f, +T | \prod_{a=1}^{n} \phi_{s_a}(x_a, t_a) | p_i, -T \rangle = \int d\theta_{i} d\theta_{f} \psi_{p_{f}}^{*}(\theta_{f}) \psi_{p_{i}}(\theta_{i}) \langle \theta_{f}, +T | \prod_{a=1}^{n} \phi_{s_a}(x_a, t_a) | \theta_{i}, -T \rangle \tag{5.46}
\]

which is defined as a path integral with fixed boundary conditions at \( t = \pm T \):

\[
\langle \theta_{f}, +T | \prod_{a=1}^{n} \phi_{s_a}(x_a, t_a) | \theta_{i}, -T \rangle = \int \mathcal{D}P(t) \int_{\theta(-T)=\theta_{i}}^{\theta(+T)=\theta_{f}} \mathcal{D}\theta(t) \prod_{a=1}^{n} \varphi_{s}(x_a, t_a) \exp \left[ i \int_{-T}^{+T} dt P \dot{\theta} - H \right] \tag{5.47}
\]

We see from the above expression that the momentum \( P(t) \) is still conserved except at the \( n \) points \( t_a \) and we have assumed that the static saddle-point configuration of the previous section is also applicable, to leading order, at these isolated points.

In the canonical language, the path integral (5.47) is equivalent to a product of expectation values of the operators corresponding to the saddle point fields; \( \varphi_{\pm} = \exp(\pm i\dot{\theta}) \varphi_{S}(x; \sqrt{m^2 - \omega(\dot{P})^2}) \). These composite operators involve products of arbitrary powers of \( \dot{P} \) and \( \dot{\theta} \) and the correct ordering convention for these non-commuting factors needs to be specified. In contrast to the ordering problem which arises from the non-linear form of the canonical transformation discussed in Section 3, the resulting correction terms in this case explicitly involve the collective coordinates and must be included in order to correctly determine the positions of the momentum-space poles in the tree-level Green’s functions. In the path-integral approach, the operator ordering problem is replaced by an equivalent ambiguity, namely the need to specify an appropriate discretized definition of the functional integral [17]. The appropriate resolution of this ambiguity in the context of soliton quantization was given by Gervais and Jevicki [16]. Following their analysis we choose the symmetric midpoint formula for the discretized path integral. The time interval \([-T, T]\) is divided into \( N + 1 \) equal steps, \( t_{(k)} = -T + k\epsilon \) for \( k = 0, 1, \ldots, N \) where the step-size \( \epsilon \) is equal to \( 2T/N \). Writing \( \theta(k) = \theta(t_{(k)}) \), \( P(k) = P(t_{(k)}) \) and \( J(x, k) = J(x, t_{(k)}) \), the transition amplitude is defined as,

\[
T_{fi}[\mathcal{J}] \sim \lim_{\epsilon \to 0} \int \prod_{k=0}^{N-1} \frac{dP(k)}{(2\pi)} \prod_{k=0}^{N} d\theta(k) \psi_{p_{f}}^{*}(\theta(N)) \psi_{p_{i}}(\theta(0)) \exp \left( i \sum_{k=0}^{N-1} A(k + 1, k; \mathcal{J}) \right) \tag{5.48}
\]

where the overall normalization of this expression is fixed by equation (5.42). The discretized action is given by,

\[
A(k + 1, k; \mathcal{J}) = P(k) \Delta \theta(k) - \epsilon \left[ E(P(k)) - \int dx J(x, k) \cdot \varphi(x; P(k), \theta(k)) \right] \tag{5.49}
\]

where the midpoint and difference fields \( \tilde{\theta}(k) \) and \( \Delta \theta(k) \), are defined by

\[
\tilde{\theta}(k) = \frac{1}{2} (\theta(k + 1) + \theta(k)) \quad \Delta \theta(k) = \theta(k + 1) - \theta(k) \tag{5.50}
\]

respectively.
Differentiating the expression (5.48) with respect to the sources $J_{\pm}(x, q_a)$, where the grid points labelled by $q_a$ are defined so that $t_a = -T + q_a \epsilon$, yields a discretized definition of the Green’s functions (5.47) in which each insertion of the saddle-point field configuration is evaluated at the midpoint value of the collective coordinate as $\varphi_\pm(x, q_a) = \exp(\pm i\bar{\theta}(q_a))\phi_S\left(x; \sqrt{m^2 - \omega(P(q_a))^2}\right)$. Each insertion depends on the collective coordinate value only at two consecutive grid points; $\theta(q_a)$ and $\theta(q_a + 1)$ and therefore only contributes to the integral over these two variables. The integrations at remaining grid points are unaffected by these terms and thus, in the limit $\epsilon \to 0$, the resulting path integral is divided into segments of the form

\[
D(\theta, t; \theta', t') = \int \mathcal{D}P \int_{\theta(t)=\theta}^{\theta(t')=\theta'} \mathcal{D}\theta \exp\left[i \int_t^{t'} dt'' P\dot{\theta} - H\right] (5.51)
\]

describing free propagation between initial and final orientations $\theta$ and $\theta'$ in the time interval $[t, t']$. In the simplest case of the one-point function between states of definite orientation the resulting expression involves one insertion of the mid-point field sandwiched between two such propagators,

\[
\langle \theta_f, +T | \varphi_\pm(x, t) | \theta_i, -T \rangle \sim \int d\theta_- d\theta_+ D(\theta_i, -T; \theta_-, t) D(\theta_+, t; \theta_f, +T) \times \left(\int \frac{dP}{(2\pi)} \exp(iP\Delta\theta)\varphi_\pm(x; P, \bar{\theta})\right) (5.52)
\]

where $2\bar{\theta} = (\theta_+ + \theta_-)$ and $\Delta\theta = \theta_+ - \theta_-$. Using a standard identity for operator products (see the Appendix of Ref [16]), we recognize the field insertion term in the brackets as an expectation value of the Weyl ordered form of the field operator; $\{\varphi_\pm\}_W$,

\[
\langle \theta_+ | \{\varphi_\pm(x; \hat{P}, \bar{\theta})\}_W | \theta_- \rangle = \int \frac{dP}{(2\pi)} \exp(iP\Delta\theta)\varphi_\pm(x; P, \bar{\theta}) (5.53)
\]

As usual, the midpoint definition for the path integral yields the same results as the Weyl ordering prescription in the corresponding operator approach [18].

Using the standard representation for the free propagator (5.51) in terms of the exact eigenfunctions,

\[
D(\theta, t; \theta', t') = \sum_{p=\pm\infty}^{+\infty} \psi^*_p(\theta')\psi_p(\theta) \exp it'(t - t)E(p) (5.54)
\]

it is straightforward to evaluate the three residual integrals in (5.52). After projecting onto initial and final states of definite charge, $p_i$ and $p_f$, the integrals over $\theta_+$ and $\theta_-$ yield $\delta$-functions which impose $U(1)$ charge conservation $p_f = p_i + 1$ and also pick out a single value $\bar{p} = (p_i + p_f)/2$ in the integration over the intermediate angular momentum $P$. The final result for the one-point function is,

\[
\langle p_f, +T | \varphi_\pm(x, t) | p_i, -T \rangle = \delta_{p_f, p_i \pm 1} n_i n_f \exp(it\Delta E)\phi_S\left(x; \sqrt{m^2 - \omega(P)^2}\right) (5.55)
\]
where $\Delta E = E(p_i) - E(p_f)$ and we define the initial and final state normalization factors $n_i(T) = \sqrt{T_i[0]} = \exp iE(p_i)T$ and $n_f(T) = \sqrt{T_f[0]} = \exp iE(p_f)T$ respectively. The only non-trivial feature of this result is the charge dependent shift in the effective meson mass which results from using the correct saddle-point configuration in the path-integral. As we will illustrate in the next section, this factor is essential in order to get a physical contribution to the S-matrix.

It is straightforward to obtain similar expressions for the saddle-point contribution to multi-point Green’s functions directly from the discretized definition of the transition amplitude. For completeness, we will give the corresponding formula for the $n$-point function (5.45). Choosing the time-ordering $-T < t_1 < ... < t_n < +T$, we define the intermediate charges and energies $p_a$ and $E_a$ at times $t_a$ by,

$$p_a = p_i + \sum_{b=1}^a r_b \quad \quad E_a = E(p_a) \quad (5.56)$$

and the corresponding mid-point and difference variables as

$$\bar{p}_a = \frac{1}{2} (p_i + p_{i-1}), \quad \Delta E_a = E_a - E_{a-1} \quad (5.57)$$

The result is,

$$\langle p_f, +T | \prod_{a=1}^n \phi_{s_a}(x_a, t_a) | p_i, -T \rangle = \delta_{p_f, p_i} n_i n_f \prod_{a=1}^n \exp(it_a \Delta E_a) \phi_S \left( x_a; \sqrt{m^2 - \omega(\bar{p}_a)^2} \right) \quad (5.58)$$

As for the one point function, the $\delta$-function imposes the conservation of the total charge, $p_f - p_i = \sum_{a=1}^n r_a$, while the time dependence of the expression is the appropriate one for energy conservation at each vertex.

### 6 The soliton S-matrix

In the previous section we calculated the saddle-point contribution to Green’s functions which describe the interaction of the asymptotic quanta with the charged soliton states $|p\rangle$. The S-matrix elements describing any physical process involving one soliton and an arbitrary number of mesons can, in principle, be extracted from these Green’s functions by the usual LSZ reduction formula. The simplest such process is the decay of the charged soliton state $|p\rangle$ into the state $|p-1\rangle$ by the emission of a single meson. In this Section we will calculate the decay width of the initial soliton state to leading order in $1/N$, directly from the appropriate semi-classical one-point function,

$$\langle p - 1, +T | \phi_+(x, t) | p, -T \rangle = n_i n_f \exp(i \Delta E) \phi_S \left( x; \sqrt{m^2 - \omega(\bar{p})^2} \right) \quad (6.59)$$

where $\bar{p} = (p - 1/2)$ and $\Delta E = E(p) - E(p - 1)$. 
It is convenient to express the soliton profile function \( \phi_S(x; \mu) \), for arbitrary mass \( \mu \), in terms of a momentum space residue function \( A(k, \mu) \) according to,

\[
\phi_S(x; \mu) = \int \frac{dk}{2\pi} \exp(ikx) \frac{A(k; \mu)}{k^2 + \mu^2}
\]  

(6.60)

where, as we will see below, \( A(k, \mu) \) is regular and non-zero at \( k^2 = -\mu^2 \). The one-point function for a meson of energy \( E \) and momentum \( k \) is given by the Fourier transform of Eqn (6.59). The Fourier transform with respect to \( t \) yields an energy-conserving \( \delta \)-function while the transform with respect to \( x \) follows from the inverse of eqn (6.60);

\[
\langle p \rightarrow p-1, +T | \phi_+(k, E) | p, -T \rangle = n_i n_f \delta(E - \Delta E) \frac{A(k; \sqrt{m^2 - \omega(\bar{p})^2})}{k^2 + m^2 - \omega(\bar{p})^2}
\]  

(6.61)

In order for this one-point function to contribute to the S-matrix element for the physical process \( |p \rangle \rightarrow |p-1 \rangle + \text{meson} \), it must have a pole on the meson mass shell, \( E^2 = k^2 + m^2 \). Clearly, the RHS of (6.61) exhibits such a pole if and only if \( \Delta E = \omega(\bar{p}) \). For the present purposes we will be content to verify this relation at leading non-trivial order in \( 1/N \), in any case loop corrections which we have so far ignored will start to contribute both to \( \omega \) and to \( \Delta E \) at the next order. Expanding both these quantities in powers of \( 1/N \) using (4.38) we find \( \Delta E = \omega(\bar{p}) = (p-1/2)/\Lambda + O(1/N^2) \) as required. Hence, as advertised in Section 1, the static pole in the soliton background has been shifted to exactly the position required by energy conservation. We can now apply the standard LSZ reduction formula to the Green’s function (6.61). The normalization factors \( n_i = \exp(iTE(p)) \) and \( n_f = \exp(iTE(p-1)) \) correspond to non-relativistic propagators for the initial and final soliton states and it is appropriate to amputate these external legs in the usual way. The resulting formula for the momentum space S-matrix element is,

\[
S_p = \lim_{E^2 \rightarrow k^2 + m^2} A(k; \sqrt{m^2 - E^2})
\]  

(6.62)

The behavior of the residue function \( A(k; \mu) \) in the limit \( k^2 \rightarrow -\mu^2 \) is determined by the asymptotic form of the soliton profile at large distance. As \( |x| \rightarrow \infty \) we have, \( \phi_S(x, \mu) \rightarrow A \exp(-\mu|x|) \) where the dimensionless constant \( A \) can still depend on \( \mu \) through its ratio with some other mass parameter occurring the original Lagrangian (2.3). For convenience, we will ignore this model-dependent complication and assume that \( A \) is independent of \( \mu \) at leading order in \( 1/N \). By taking the inverse Fourier transform of (6.60) we find, \( A(i\mu, \mu) = 2A\mu \). The S-matrix element (6.62) therefore has the simple form \( S_p = 2Ak \).

The decay width of the soliton state \( |p \rangle \) is calculated from the S-matrix element by integrating over the final-state phase space. As we have neglected the translational mode of the soliton throughout, we have not taken account of the \( O(1/N) \) corrections arising from the spatial recoil of the soliton. In this approximation the differential decay rate is just,

\[
d\Gamma_p = |S_p|^2 \frac{dk}{(2\pi)^2 E^2} 2\pi \delta(E - \Delta E)
\]  

(6.63)
At leading order in $1/N$ the resulting width is given by,

$$
\Gamma_p = 2A^2 \sqrt{(p - 1/2)^2/\Lambda^2 - m^2}
$$

This expression depends only on the meson mass, the soliton moment of inertia and the asymptotic behavior of the profile function, $\phi_S$, and we stress again that this result is completely gauge independent. For large values of $p$, the width of the excited states $|p\rangle$ grows linearly with the charge simply reflecting the increased phase space available for the decay. In general, we expect the state $|p\rangle$ to appear as a resonance of width $\Gamma_p$ in the scattering cross-section for a single meson off a soliton in the state $|p - 1\rangle$. Contributions to the corresponding S-matrix elements for scattering processes involving an arbitrary number of mesons can be obtained by applying the LSZ reduction formula to the general Green’s function (5.58). In particular, the Born contribution to the amplitude for two-body resonant scattering comes from the two-point Green’s function $\langle p - 1|\phi_+\phi_-|p - 1\rangle$ in which $\Gamma_p$ is self-consistently included as an imaginary part for the energy of the intermediate soliton state.

7 Conclusion

In this paper we have applied the semi-classical method systematically to the interactions of mesons with the charged soliton states associated with an internal $U(1)$ collective coordinate. In particular we have calculated the complete tree-level contribution to the one-point Green’s functions describing the emission of a single meson and demonstrated that it is gauge-invariant. The one-point function has a pole on the meson mass shell which is consistent with the kinematics of the physical decay process. We calculated the leading semi-classical contribution to the decay widths of the charged states. Our results can be interpreted as the zeroth-order in a semi-classical loop expansion and indicate that it this expansion rather than the standard weak-coupling perturbation theory which is appropriate for the systematic calculation of the meson-soliton S-matrix. However, in this connection, it is important to note that exact cancellations of the leading-order contributions can lead to further subtleties in the case of multi-point Green’s functions. There are several important omissions in our discussion of the $U(1)$ model. First, we have not explicitly formulated the perturbation theory in fluctuations around the saddle-point which is necessary to calculate the loop corrections. Second, it is necessary to introduce an explicit collective coordinate to describe the translational motion of the soliton. As we have already discussed, the corresponding resummation of all tree graphs in the translational case naturally leads to an appropriately Lorentz contracted saddle-point configuration. By carefully accounting for the effects of this Lorentz contraction in the analysis given above, it should be possible to compute the corrections to the leading-order decay width due to the spatial recoil of the soliton. Finally, taking into account both the Born terms and the background scattering terms, all of the above could be generalized to the two-meson S-matrix element to give a systematic semi-classical expansion of the cross-section for elastic meson-soliton scattering in this model.

More generally, we expect any quantized theory of solitons to correspond to some effective point-like theory at least for sufficiently low energies. A well known example of such a co-
correspondence is the mapping between the sine-Gordon model and the Thirring model provided by bosonization [19]. In this connection it is interesting to note that the one-meson S-matrix element, \( S_p \), calculated in Section 6 has a simple interpretation in terms of an effective theory in which the charged soliton states \( |p\rangle \) are represented by explicit spinor fields \( \psi_p \) with Dirac masses \( M(p) = E(p) \). We consider the following effective Lagrangian for the interactions of the fields \( \psi_p \) and the charged mesons

\[
L_{\text{eff}} = L_{\phi} + L_{\psi} + L_I
\]

(7.65)

where,

\[
L_{\phi} = \frac{1}{2}(\partial_{\mu}\phi_+)(\partial^{\mu}\phi_+) + \frac{1}{2}(\partial_{\mu}\phi_-)(\partial^{\mu}\phi_-) + \frac{m^2}{2}(\phi_+^2 + \phi_-^2)
\]

(7.66)

\[
L_{\psi} = \sum_{p=\infty}^{\infty} \bar{\psi}_p(i\gamma^5 - M(p))\psi_p
\]

(7.67)

\[
L_I = G \sum_{p=\infty}^{\infty} (\partial_x\phi_+\bar{\psi}_p\gamma^5\psi_{p-1} + \partial_x\bar{\psi}_p\gamma^5\psi_{p-1})
\]

(7.68)

where, in two space-time dimensions, the Yukawa coupling \( G \) has the dimensions of mass. In the limit \( N \to \infty \) the fermion masses become large and the Yukawa interaction \( G\phi\bar{\psi}\gamma^5\psi \) can be replaced by its standard non-relativistic reduction \( (G/2M)(\partial_x\phi)\bar{\psi}\psi \). In this limit the interaction Lagrangian becomes,

\[
L_I \to \frac{G}{2M} \sum_{p=\infty}^{\infty} (\partial_x\phi_+\bar{\psi}_p\psi_{p-1} + \partial_x\bar{\psi}_p\psi_{p-1})
\]

(7.69)

The standard Feynman rules derived from this Lagrangian yield a tree-level amplitude for the decay process \( \psi_p \to \psi_{p-1} + \phi_+ \) given by \( \tilde{S}_p = (G/2M)k \) where \( k \) is the momentum of the emitted meson in the center of mass frame. Clearly \( \tilde{S}_p = S_p \) for all values of \( p \) if we make the identification \( G = 4MA \). It is natural to expect that this correspondence will remain valid when recoil corrections are included and will also hold for the multi-particle scattering amplitudes.

The method presented here is directly applicable to the problem of \( \Delta \)-decay in the \( SU(2) \) Skyrme model [3]. We expect that, as for the \( U(1) \) soliton, the net result of systematically accounting for the isorotational motion of the skyrmion in the semi-classical expansion will be to move the static pole in the Fourier transform of the skyrmion background field to exactly the position required for a physical contribution to the decay amplitude. Assuming this is the case, the resulting S-matrix element for \( \Delta \)-decay would be equal to the one originally given by Adkins, Nappi and Witten [3] as a result of more general arguments. In particular the proposed model-independent relation between the effective coupling constants, \( g_{\pi\Delta} = (2/3)g_{\pi NN} \), would be obeyed exactly. In the \( U(1) \) model, we discovered that the widths of the excited soliton states increase with the charge. Assuming a similar result holds for the Skyrme model and the widths of the excited baryon states grow rapidly with increasing \( SU(2) \) quantum numbers, it is likely that the predicted \( I = J = 5/2 \) resonance would be too broad to be distinguished from the background in pion-nucleon scattering. This suggests a natural explanation for the fact that, of
the infinite tower of large-$N$ baryons, only the two states of lowest isospin, the nucleon and the $\Delta$, are observed in nature. Finally, the possible correspondence between the soliton S-matrix elements and an effective point-like theory is of particular interest in the Skyrme model where the latter would give an effective Lagrangian description for the interactions of baryons and mesons in large-$N$ QCD. We hope to discuss these interesting possibilities in more detail in the near future.

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References

[1] R. Rajaraman and E. Weinberg, *Phys. Rev.* D11 (1975) 2950.

[2] T. H. R. Skyrme, *Proc. Roy. Soc.* A260 (1961) 127.

[3] G. Adkins, C. Nappi and E. Witten, *Nucl. Phys.* B228 (1983) 552.

[4] G. Adkins and C. Nappi, *Nucl. Phys.* B233 (1984) 109.

[5] N. Dorey, J. Hughes and M. P. Mattis, work in progress.

[6] D. I. Dyakonov, V. Yu. Petrov, and P. B. Pobylitsa, *Phys. Lett.* B205 (1988) 372.

[7] A. Hayashi, G. Eckart, G. Holzworth and H. Walliser, *Phys. Lett.* B147 (1984) 5.

[8] M. Mattis and M. Karliner *Phys. Rev.* D31 (1985) 2833.

M. Mattis and M. Peskin, *Phys. Rev.* D32 (1985) 58.

[9] J. Goldstone and R. Jackiw, *Phys. Rev.* D11 (1975) 1486.

[10] C. Callan and D. Gross, *Nucl. Phys.* B 93 (1975) 29.

[11] J. Gervais and B. Sakita, *Phys. Rev.* D30 (1984) 1795.

[12] H. Verschelde, *Phys. Lett.* B209 (1988) 34.

G. Holswarth, A. Hayashi and B. Schwesinger, *Phys. Lett.* B191 (1987) 27.

S. Saito, *Prog. Theor. Phys.* 78 (1987) 746.

[13] J. Gervais, A. Jevicki and B. Sakita *Phys. Rev.* D12 (1975) 1038.

[14] J. Gervais and B. Sakita, *Phys. Rev.* D11 (1975) 2943.

[15] E. Tomboulis, *Phys. Rev.* D12 (1975) 1678.

[16] J. Gervais and A. Jevicki *Nucl. Phys.* B110 (1976) 93.
Appendix A

In this Appendix we derive the choice for $\pi^{cl}$ given in Eqn (3.20) from the requirement that the change of phase space variables from $(\phi, \pi)$ to $(\theta, P)$ and $(\bar{\phi}, \bar{\pi})$ be a canonical transformation. Precisely, we demand that the canonical form of the Legendre term be preserved in the new variables. This condition is expressed in (3.19), the LHS of which becomes,

$$\int dx \pi \cdot \dot{\phi} = \int dx \left( \mathcal{M}(\theta)(\pi^{cl} + \bar{\pi}) \right) \cdot \frac{d}{dt} \left( \mathcal{M}(\theta)\dot{\phi} \right)$$

$$= \dot{\theta} \int dx (\pi^{cl} + \bar{\pi}) \cdot \left( \mathcal{M}^{T}(\theta) \frac{\partial \mathcal{M}(\theta)}{\partial \theta} \right) \cdot \dot{\phi} + \int dx \left( \pi^{cl} + \bar{\pi} \right) \cdot \ddot{\phi}$$

$$= \dot{\theta} \int dx (\pi^{cl} + \bar{\pi}) \times \dot{\phi} + \int dx \pi \times \dot{\phi} + \left[ \frac{d}{dt} \int dx \pi^{cl} \cdot \dot{\phi} + \int dx \bar{\pi}^{cl} \cdot \dot{\phi} \right] \quad (7.70)$$

Clearly Eqn (3.19) is satisfied if and only if the term in square brackets in the above equation vanishes and also,

$$P = \int dx (\pi^{cl} + \bar{\pi}) \times \dot{\phi} \quad (7.71)$$

Remembering the constraint equation (4.32), we see that these criteria are uniquely satisfied by the choice,

$$\pi^{cl} = \frac{1}{J_{\theta}} \left( P \int dx \pi \cdot \dot{\phi} \right) f(x) \quad (7.72)$$

It also follows that

$$J_{P} = \Lambda^{-1} \int dx f \frac{\delta \pi^{cl}}{\delta P} = J_{\theta}^{-1} \quad (7.73)$$

and therefore the Jacobian factors $J_{\theta}$ and $J_{P}$ cancel exactly in the integrand of (3.14).

Appendix B

The purpose of this Appendix is to find the simultaneous solution of the full system of saddle-point equations (4.30-4.32). The first step is to eliminate all the variables other than the saddle-point field itself. This can be accomplished by a direct generalization of the corresponding manipulations of equations (4.20-4.22) in Ref [13]. Taking the scalar product of equation (4.31) with $\bar{f}$ and integrating over space yields,

$$\bar{\nu} = \frac{1}{J_{\theta}} \left( P \int dx \pi \times \dot{\phi} \right) \quad (7.74)$$
We can now eliminate the Lagrange multiplier for the momentum constraint from equation (4.31). Contracting the resulting equation with $\varepsilon_{ik}\tilde{\phi}^k$ and integrating with respect to $x$ we find,

$$\frac{1}{J_0^2} \left( P - \int dx \tilde{\pi} \times \tilde{\phi} \right) = \frac{P}{\Lambda \left( \int dx |\tilde{\phi}|^2 \right)} \quad (7.75)$$

This relation allows us to eliminate $\tilde{\pi}$ from equation (4.30). Assuming for the moment that $\bar{\lambda} = 0$ in (4.30), the saddle-point equation for $\tilde{\phi}$ becomes,

$$- \frac{\partial^2 \tilde{\phi}_S}{\partial x^2} + m^2 \tilde{\phi}_S + \frac{\delta W}{\delta \phi_S} - \frac{P^2 \tilde{\phi}_S}{\left( \int dx |\tilde{\phi}_S|^2 \right)^2} = 0 \quad (7.76)$$

where $\tilde{\phi} = (\tilde{\phi}_S, 0)$. Clearly the above equation coincides with the static classical equation of motion with a shifted value of the meson mass parameter and is therefore solved by

$$\tilde{\phi}_S(x; m) = \phi_S \left( x; \sqrt{m^2 - P^2/\tilde{\Lambda}^2} \right) \quad (7.77)$$

where the effective moment of inertia, $\tilde{\Lambda} = \tilde{\Lambda}(P)$ must satisfy the self-consistency condition,

$$\tilde{\Lambda} = \int dx \left[ \phi_S \left( x; \sqrt{m^2 - P^2/\tilde{\Lambda}^2} \right) \right]^2 \quad (7.78)$$

Assuming such a value for $\tilde{\Lambda}$ exists, it is easy to show that $\tilde{\phi}^{cl} = (\tilde{\phi}_S, 0)$, and $\tilde{\chi}^{cl} = 0$ together with the corresponding configurations, $\tilde{\pi}^{cl}$ and $\bar{\nu}^{cl}$, which are determined by equations (4.31) and (7.74), form a consistent solution of the complete set of saddle-point equations (4.30-4.32). In particular, because the saddle-point configuration $\tilde{\phi}^{cl}$ lives only in the first component of the field vector, the first constraint equation in (4.32) is trivially satisfied and so the choice $\tilde{\chi}^{cl} = 0$ is justified a posteriori.

In fact it can easily be demonstrated that (7.78) has a real solution for sufficiently small values of the collective coordinate momentum $P$. We define a generalized moment of inertia $\lambda$ as a function of a variable mass parameter $\mu$ according to,

$$\lambda(\mu) = \int dx \left[ \phi_S(x; \mu) \right]^2 \quad (7.79)$$

From the definition, $\lambda(m) = \Lambda$, and we will further assume that $\lambda(\mu)$ is a continuous function of $\mu$ in the range $0 < \mu \leq m$ with a finite non-zero limit $\lambda(0) = \Lambda_0$ as $\mu \to 0$. The self-consistency equation is given by $\tilde{\Lambda} = \lambda(\sqrt{m^2 - P^2/\tilde{\Lambda}^2})$ and the right and left-hand sides of this equation are plotted as functions of $\tilde{\Lambda}$ in Figure 1. Noting that $\lambda(0) > 0$, we see that for for $P/m$ less than a critical value equal to $\lambda(0)$ the two curves must intersect. Hence for $P/m < \Lambda_0$, the existence of a real solution to the self-consistency equation is guaranteed by the continuity of the moment of inertia as the meson mass is varied.

### Figures

Figure 1. A graphical solution of the self-consistency equation (4.36) for the effective moment of inertia $\tilde{\Lambda}$. 

22