Interaction and evolution of the magnetic moments are studied for different physical systems: ferromagnets [1, 2], ferrofluids [3], strongly interacting ultracold fermions [4], spin-1/2 quantum plasmas [5, 6], finite chains of spins [7], quantum ferrofluids of ultracold bosons [8–10] and this paper is focused on the weakly interacting spin-1/2 fermions.

In contrast with bosons, fermions have a large Fermi pressure at the zero temperature. This relict of the pressure caused by the distribution of particles on different energy levels exists due to the Pauli blocking. Microscopic derivation of the spin evolution equation shows that the spin current caused by the distribution of particles on different energy levels also exists [11–13]. At small temperatures it reduces to a spin current caused by the Pauli blocking. It is similar to the Fermi pressure in the Euler equation thus it is called the Fermi spin current [13].

The explicit form of the Fermi pressure is well known for a long time. It gives an equation of state for the pressure of degenerate spin-1/2 fermions with zero spin polarization. An account of the spin polarization in the equation of state of the degenerate fermion gas can be easily done [14]. However, the Fermi spin current has not been considered (spin currents caused by the interaction are not discussed here see for instance [15]). Recently, an equation of state has been suggested at the analysis of the spin imbalanced electron gas. It is based on a minimal coupling scheme which selfconsistently bind a non-linear Pauli equation, Euler equations for spin-up fermions.
and spin-down electrons and the spin evolution equation [13]. The account of the Fermi spin current dramatically modifies the spin wave spectrum [13]. Moreover, the importance of the account of the separate spin evolution at the analysis of the concentration and velocity field evolution follows from the appearance of the spin-electron acoustic wave [16].

This result can be applied to neutral spin-1/2 fermions considered in different aspects of the spinor quantum gases [17–20]. Spectrum of collective excitations in the spin imbalanced spin-1/2 fermions is calculated in this paper considering the short-range interaction between fermions with different spin projection in the first order by the interaction radius.

The external magnetic field creates a preferable direction in space. Therefore, we can expect anisotropy of the spectrum. Particularly, we have different spectrums for wave propagation parallel and perpendicular to external field.

Dynamics of a system with non-zero spin particles of a single species can be presented in form of the single fluid hydrodynamics [21]. These hydrodynamic equations include the spin evolution equation along with equations for evolution of concentration and velocity field of all particles. However, a multi-fluid description of the system can be applied either (see for instance [22] for a three-fluid hydrodynamics of spin-1 Bose–Einstein condensates).

Suggesting model is similar to the spinor BEC [23–25] (see reviews [26] and [27] as well). There are 2-types of the ground state in the spin-1 BEC. They are called ferromagnetic state (phase) and polar state. In ferromagnetic phase all spins have same direction. In polar or antiferromagnetic phase there are same numbers of spins with opposite directions or all spin-1 particles have zero projection of their spins for illustration see figure 2 in [28] and figure 1 in [29]. There are three branches of two types of the collective excitations in each ground state: a Bogoliubov (sound) mode and spin modes [23, 24]. Spin-1/2 fermions also show three branches of two types of the collective excitations. The spin-3/2 and spin-5/2 Fermi gases are also under consideration in literature [30].

If we have full equilibrium spin polarization of spin-1/2 fermions they show two branches of waves: one sound wave and spin wave. If there is no spin perturbation the sound wave only exists in the system. The partial spin polarization leads to the splitting of the sound wave on two sound waves (similar splitting exists for the zero sound [31]).

This paper is organized as follows. In section 2, the basic equations are presented in two forms: hydrodynamic form and in the form of non-linear Pauli equation. In section 3, the spectrum of waves propagating in ultracold neutral fermions is studied. Section 3 consists of several subsections. In section 3.1, the equilibrium state is considered. In section 3.2, linearized hydrodynamic equations are presented. In section 3.3, the spectrum of waves propagating parallel to the equilibrium spin polarization is obtained. In section 3.4, the spectrum of waves propagating perpendicular to the equilibrium spin polarization is found. In section 3.5, regime of zero external magnetic field is described. In section 4, a possibility of the spin acoustic solitons is discussed for the small amplitude non-linear waves propagating parallel to the equilibrium spin polarization. In section 5, a summary of the obtained results is presented.

Figure 1. The figure shows the dependence of the spin polarization $\eta$ on the normalized scattering length $\alpha = a n^{1/3} \eta_0$ at the zero external magnetic field and $\eta_0 = 10^{14}$ cm$^{-3}$. Other parameters do not affect this dependence as it can be seen from equation (10). Further increase of the scattering length $\alpha$ remains the spin polarization equals to 1.

2. Model

2.1. On the possibility of the hydrodynamic description of fermions

System of fermions can be described by the many-particle wave function (wave spinor) $\Psi(R, t) = \Psi(r_1, \ldots, r_N, t)$. It allows to construct collective variables as the quantum mechanical average of required operators. One of the simplest collective variables known from hydrodynamics is the concentration of particles which can be defined as averaging of operator $\sum_{i=1}^{N} \delta(r - r_i)$. It gives the following definition

$$n_f(r, t) = \int \Psi^*(R, t) \sum_{i=1}^{N} \delta(r - r_i) \Psi(R, t) dR.$$  (1)
Evolution of concentration is determined by the Hamiltonian via the time evolution of the wave function.

Next, we can go further and split wave spinor $\Psi(R,t)$ on two column $\Psi_u(R,t)$ and $\Psi_d(R,t)$ which are related to the spin-up and spin-down states of each particle. It allows us to introduce the partial concentrations for spin-1/2 fermions

$$n_p(R,t) = \int \Psi_u^+(R,t) \sum_{i=1}^N \delta(R-R_i) \Psi_u(R,t) dR.$$  \hspace{1cm} (2)

Evolution of these functions reveals in the hydrodynamic equations which can be represented in non-linear Pauli equation introduced above (3) in accordance with [13, 16, 31].

Truncation of the chain of the hydrodynamic equations gives limitations of the model. Simple hydrodynamic models do not show full agreement with the results of the kinetic theory. Particularly, it is essential for fermions. The fermions are always distributed on quantum states with different momentum while bosons in the Bose–Einstein condensate state avoid this problem.

Justification of the hydrodynamic equations for fermions and suitable truncation for the obtaining of the correct plasmon dispersion law are considered in [32, 33] in context of the electron gas. Authors start from the kinetic equation and obtain required extended set of the hydrodynamic equations. Authors stress on the area of applicability of the found equations. They find a wider range of parameters for the applicability of the hydrodynamic equations, so the found equations are valid beyond the local equilibrium condition.

In this paper we are focus on relatively simple hydrodynamic model of degenerate neutral fermions, but we focus on the independent description of fermions with different spin projections. Presented model can be generalized in accordance with [32, 33].

2.2. Basic equations

We consider partial spin polarized spin-1/2 fermions. Hence, we have the contribution of the short-range interaction between spin-up and spin-down fermions.

We suggest a minimal coupling model allowing the hydrodynamic description with a non-linear Schrodinger (Pauli) equation (NLSE). The NLSE arises as

$$i \hbar \partial_t \psi = \left(-\frac{\hbar^2 \Delta}{2m} + \tilde{\pi} + \tilde{\pi} \sigma B\right)\psi $$  \hspace{1cm} (3)

where $\psi$ is the spinor wave function which is a column with two elements $\{\psi_u, \psi_d\}$, subindexes $u$ and $d$ refer to the spin-up and spin-down states, $\sigma$ are the Pauli matrices, $\tilde{\pi} = \left(\begin{array}{cc} \pi_u & 0 \\ 0 & \pi_d \end{array}\right)$ is the spinor pressure term, which arises as a diagonal second rank spinor, where $\pi_s = (6\pi^2 n_s)^{1/3} \hbar^2 / 2m$ are the contribution of the Fermi pressure. It can be represented in term of the Pauli matrices $\tilde{\pi} = \pi_u (\hat{1} + \hat{\sigma})/2 + \pi_d (\hat{1} - \hat{\sigma})/2$, where $\hat{1}$ is the unit second rank spinor $\hat{1} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$. The short-range interaction is presented by $	ilde{\pi} = \left(\begin{array}{cc} gn_u & 0 \\ 0 & gn_d \end{array}\right) = g (n_u + n_d) \hat{1} + (n_d - n_u) \hat{\sigma}_z / 2$.

It gives an interspecies interaction [34–36]. Considering the short-range interaction we drop the contribution of terms arising in the third order on the interaction radius leading to non-local terms [37–39]. Notions spin-up and spin-down are defined relatively to the direction of equilibrium spin polarization.

We choose the coordinate frame with the $z$ axis directed along the equilibrium spin polarization. The magnetic field $B$ in equation (3) is the superposition of the external magnetic field $B_{ext} = B_0 e_z$, the magnetic field created by the magnetic moments of the system (the spin–spininteraction), and the effective magnetic field $B_{eff}$ representing the radio frequency field creating the equilibrium spin polarization.

The continuity equations corresponding to the minimal coupling model (3) appears as follows

$$\partial_t n_s + \nabla (n_s v_s) = \pm \frac{\mu}{\hbar} (S_s B_y - S_y B_s), $$  \hspace{1cm} (4)

where we have applied $S_s$ and $S_y$ for mixed combinations of $\psi_u$ and $\psi_d$, $\pm$ means $+ a t s = u$ and $- a t s = d$, $n_s = \psi^\dagger \sigma_{s} \psi$, $S_s = \psi^\dagger \sigma_{s} \psi$, with $^\dagger$ the complex conjugation and $^\sigma$ the hermitian conjugation.

The velocity fields presented in the continuity equation (4) satisfy the following Euler equations

$$mn_s (\partial_t + v_s \nabla) v_s + \nabla p_s \hspace{1cm}$$

where $F_{g,s}$ is the force field of short-range interaction: $F_{g,u} = - gn_u \nabla n_d$, $F_{g,d} = - gn_d \nabla n_u$, $F_{S,s}$ is the force field of spin–spininteraction, its explicit form arises as

$$F_{S,s} = \pm \mu \sigma (\Delta y B_z + \frac{\mu}{2}(S_s \partial_y B_y + S_y \partial_y B_y)$$

with the following explicit form of the substantial (material, convective) and quantum parts of spin currents

$$J^{\alpha \beta}_{(s)} = \frac{1}{2} \left(\frac{\nabla^2 + \nabla^2}{2} \right) S^{\alpha \beta} - e^{\alpha \gamma z} \frac{\hbar}{4m} \left(\partial^2 n_u - \partial^2 n_d \right) S^{\gamma \beta} \hspace{1cm}$$

and pressures

$$p_s = \frac{2}{5} \pi_s n_s.$$  \hspace{1cm} (7)

The generalized Bloch equation describes the spin evolution which is related to evolution of both species simultaneously:
\[ n(\partial_t + v \nabla) \frac{S}{n} + \frac{(6\pi^2)^2 \hbar}{m} (n_u^3 - n_d^3)[S, e_z] \]

\[-\frac{\hbar}{2m} \partial^2[S, \partial^2 \frac{S}{n}] = \frac{2\mu}{\hbar} [S, B] + \frac{g}{\hbar} (n_u - n_d)[S, e_z], \tag{9} \]

where [a, b] is the vector product of vectors a and b. \( n = n_u + n_d \) is the full concentration, and \( v = (n_u v_u + n_d v_d)/n \) is the velocity field of whole system. The Fermi spin current is presented by the second term.

The hydrodynamic equations (4)–(9) is coupled with the Maxwell magneto-static equations which are \( \nabla B = 0, \nabla \times B = 4\pi \mu \nabla \times S \), where \( S = \{S_x, S_y, n_u - n_d\} \) is the three vector of spin density.

It is well-known that hydrodynamic equations do not include the zero sound which can be included by a kinetic model \([40]\).

The hydrodynamic model presented by equations (4)–(9) is similar to the separate spin evolution quantum hydrodynamics developed for electron gas \([16]\).

It is well-known that the hydrodynamic form of equations (which sometimes can be represented as the non-linear Schrodinger equation) for fermions is justified for two phases: the superfluid phase and the normal phase for the strongly interacting regime. For instance, a phenomenological hydrodynamic model of one dimensional motion of cold fermions in the Boussinesque approximation with dissipation is considered in \([41]\) for the study of weakly nonlinear wave perturbations in such systems. However, in this paper, we consider fermions in the normal phase for weakly interacting regime. Hence, the basic equations require a justification. Disregarding the spin–spin interaction and the separate spin evolution, a hydrodynamic model of fermions with short range interaction is derived for weakly interacting regime \([37]\). More precisely, it is shown that a hydrodynamic form of equations of motion can be always found for fermions of a chosen species. However, the explicit forms of the pressure term and the force field require the specification of the regime and appear to be a problem in most of the cases. The pressure and the force field are approximately found in \([37]\) for weakly interacting fermions. The well-known Fermi pressure is considered as the equation of state in this regime. The short range interaction between the fermions having same spin spin projection is equal to zero in the first order on the interaction radius. It happens due to the antisymmetry of the wave function relatively the permutation of particles. However, a nonzero force field appears in the next order. The third order on the interaction radius approximation leads to the nonlocal terms describing the short range interaction presented by equations (53), (59) in \([37]\) and equation (7) in \([39]\). Interaction between spin-up and spin-down fermions can be considered as an interspecies interaction. It is also considered in \([37]\), where bosons and fermions are considered as an example of two species, but the result is relevant for other pairs of species. Hence, this result is used in equations (3) and (5).

3. Spectrum of collective excitations: linear regime

3.1. Equilibrium state

Consider an equilibrium state. To this end, the wave function is presented in the following form \( \psi_\pi = A_\chi e^{-\mu_\chi s/\hbar} \), where \( \mu_\chi \) is the chemical potential and \( A_\chi \) are constant which do not depend on \( r \) and \( t \). Hence, we have \( \Delta \psi_\pi = 0 \) in the equilibrium.

If there is no short range interaction and external magnetic field, equation (3) reduces to \( \hbar \partial_t \psi_\pi = \pi \psi_\pi \). Substituting \( \psi_\pi \) explicitly, we find \( \mu_\chi \psi_\pi = \bar{\pi} \psi_\pi \). Consequently, we obtain \( \mu_\chi = \pi_\pi = \pi_d \) and \( n_u = n_d = n_d/2 \). If there is no external magnetic field at zero temperature all fermions occupy quantum states with lower energies. All states below the Fermi energy \( \varepsilon_F = (3\pi^2 n)^{3/2} \hbar^2/2m = \pi_s(n_s = n_0/2) \) are occupied due to the Pauli blocking. So, there is no spin polarization and we have the polar equilibrium state since we have equal numbers of fermions in states with opposite spin projections. It is similar to the polar state described in \([23, 29]\).

The external magnetic field creates the spin imbalance since it transfers energy to some spin-down fermions to change their direction to spin-up states (it is assumed that the Lande factor is positive). Obviously, the external magnetic field creates the spin polarization, but we consider the short range interaction with no external magnetic field. It gives the following equations \( \mu_\chi = \pi_u + gn_d \) and \( \mu_u = \pi_d + gn_u \). Consequently, we find \( \pi_u + gn_d = \pi_d + gn_u \). It leads to

\[ (n_u^3 - n_d^3)(n_u^3 + n_d^3 - \frac{8\pi a}{\hbar(\pi^2)^2}(n_u^2 + n_d^2 n_u^2 + n_d^2)) = 0. \tag{10} \]

There are two solutions. One of them describes the polar phase with \( n_u = n_d \). Figure 1 confirms that equation (10) has a non-trivial solution. This equation has the second solution for the repulsive short range interaction \( \alpha > 0 \). There is a narrow interval for the second solution. A solution with the experimentally achievable concentrations \( n_0 = n_{0u} + n_{0d} \) is \( 10^{22} \pm 10^{24} \) cm\(^{-3} \) exists at specific scattering length \( a \sim 10^{-9} \) cm, which are realistic scattering length. Therefore, the spatially spin polarized state can be realized for spin-1/2 fermions due to the repulsive short range interaction between fermions with different spin projections.

Presence of the magnetic field or the radio frequency field creating equilibrium spin polarization, so it can be described described as an effective equilibrium magnetic field \( B_{eff} \), changes both possible equilibrium states. It creates the polarization in case of the zero initial polarization or increases the spin polarization created by the repulsive short range interaction. The attractive short range interaction decreases the spin polarization created by the external or effective magnetic field. Consider it in more details. To this end, rewrite the Pauli equation (3) for this equilibrium state: \( \mu_\chi \psi_\pi = (\pi_u + gn_d - \mu B_F) \psi_u \) and \( \mu_\chi \psi_\pi = (\pi_d + gn_u + \mu B_F) \psi_d \). Where \( B_z \to B_0 = B_z \to B_0 = B_{ext} + B_{eff} \). Equating the different forms of the chemical potential \( \mu_\chi \) obtained from these two equations we find the following relation between \( n_u \) and \( n_d \):

\[ \pi_u - \pi_d = g(n_u - n_d) + 2\mu B_0 \tag{11} \]

containing two parameters \( g \) and \( B_0 \).
Equation (11) can be rewritten in terms of the spin polarization:

\[(1 + \eta)^{3/2} - (1 - \eta)^{3/2} - \eta \frac{8 \pi an_0^{3/2}}{(3\pi^2)^{3/2}} - \frac{4m\mu B_0}{(3\pi^2)^{3/2} n_0^3 \hbar^2} = 0. \quad (12)\]

Consider a characteristic concentration \(n_0 = 10^{14} \text{ cm}^{-3}\) and solve numerically equation (10) (regime of zero magnetic field). The solution is shown in figure 1. It demonstrates that change of the scattering length near \(a \approx 100 \text{ nm}\) allows to reach any value of spin polarization \(\eta \in [0, 1]\).

Include the magnetic field and solve equation (12) numerically for different regimes. A solution for zero short range interaction scattering length \(a = 0\), relatively small mass \(m = 7 \text{ u}, \mu = 2\mu_B\) is shown in figure 2, where \(u\) is the unified atomic mass unit (Dalton). It demonstrates that relatively small magnetic field \(B_0 \sim 0.05 \text{ G}\) can create the full spin polarization \(\eta = 1\). The full spin polarization can be reached at smaller magnetic field for atoms of larger masses and larger magnetic moments. Influence of small magnetic field \(B_0 = 0.001 \text{ G}\) on the dependence of the spin polarization on the scattering length is demonstrated in figure 3.

As it is mentioned above, the repulsive short range interaction make it easier to create the spin polarization by the magnetic field (see figure 4) and the attractive short range interaction require larger magnetic field to create the required spin polarization (see figure 5).

Relatively large mass of particles (in compare with the electron) and small concentrations (in compare with condense matter physics \(n_0 \sim 10^{18} \div 10^{22} \text{ cm}^{-3}\)) leads to small Fermi energy \(E_F = 0.62 \times 10^{-20} \text{ egr}\), where \(m = 1 \text{ u}, n_0 = 10^{14} \text{ cm}^{-3}\). Hence, the Zeeman energy \(\mu B_0 = 0.9 \times 10^{-20} \text{ egr}\) at \(B_0 = 1 \text{ G}\) overcome the Fermi energy creating conditions for the full spin polarization.

### 3.2. Small perturbations

Next, the small perturbations of an equilibrium state are considered. The equilibrium state is described by the concentrations \(n_{0\alpha}, n_{0\bar{\alpha}}\), the external magnetic field \(B_0 = B_0 \hat{e}_z\), and the spin density projection on the magnetic field direction \(S_{\alpha} = n_{0\alpha} - n_{0\bar{\alpha}}\). Other quantities like the velocity fields \(v_{\alpha}, v_{\bar{\alpha}},\) the transverse spin density projections \(S_{\alpha}, S_{\bar{\alpha}}\) are equal to zero. We consider the perturbations in form of plane monochromatic wave such as a perturbation of each quantity can be presented in the following form \(\delta f = F \exp(-i\omega t + ik \mathbf{r})\).

Consider the linearized set of separate spin evolution quantum hydrodynamic equations (4)–(9)

\[\partial_t \delta n_\alpha + n_{0\alpha} \nabla \delta v_\alpha = 0, \quad (13)\]

\[m \partial_t \delta v_\alpha + \frac{\nabla p_\alpha}{n_{0\alpha}} - \frac{\hbar^2}{4m} \nabla \Delta \delta n_\alpha = -g \nabla \delta n_\alpha \pm \mu \nabla \delta B_z, \quad (14)\]

and

\[\partial_t \delta \mathbf{S} + \frac{(6\pi^2)^{3/2}}{m} (\eta_{0\alpha} - \eta_{0\bar{\alpha}}) [\delta \mathbf{S}, \mathbf{e}_z] = \frac{\hbar}{2m} [\mathbf{S}_0, \nabla \frac{\delta \mathbf{S}}{n_0}] + \frac{\hbar}{\Lambda} [\mathbf{S}_0, \delta \mathbf{B}]. \quad (15)\]

which allows to find the spectrum of small perturbations, \(s' \neq s\). They are coupled with the following linearized equations of field

\[\nabla \times \delta \mathbf{B} = 4\pi \mu \nabla \times \delta \mathbf{S}, \quad \nabla \delta \mathbf{B} = 0. \quad (16)\]

Below we consider different types of waves. Let us discuss some general properties of the sound waves and the spin waves. Sound waves have linear spectrum \(\omega \sim k\) which can be modified by the quantum Bohm potential, so \(\omega^2 \sim k^2 + ak^4\). In both cases the frequency is equal to zero at the zero wave.
vector \( \omega(k = 0) = 0 \). Sound waves can exist as density wave if there is no spin density perturbations. Existence of two spin projections for spin-1/2 fermions leads to two concentrations for particles with spin-up \( n_u \) and particles with spin-down \( n_d \). Increase of the species number can increase the number of sound waves. In-phase evolution of \( n_u \) and \( n_d \) gives the density wave. Different evolution of \( n_u \) and \( n_d \) gives the evolution of the spin projection on the direction of external field \( \delta S_z = \delta n_u - \delta n_d \) with approximately linear spectrum, so we refer to it as the spin acoustic wave. In terms of the separate spin projection this wave is a density wave since \( n_u \) and \( n_d \) are independent variables while \( S_z \) is not an independent variable in the model.

The spin waves have nonzero frequency at the zero wave vector \( \omega(0) \neq 0 \). In the simplest case, the frequency \( \omega(0) \) is equal to the cyclotron frequency \( \Omega \) describing the precession of the single magnetic moment in the external magnetic field. Different many particle effects such as the Fermi spin current and the spin–spin interaction can change this frequency. The frequency dependence on the wave vector usually is quadratic \( \omega \sim 1 + \hbar k^2 \), where the term proportional to \( k^2 \) caused by the quantum Bohm potential.

3.3. Propagation parallel to the equilibrium spin polarization

At the propagation parallel to the anisotropy direction created by the equilibrium spin density and the external magnetic field we choose the wave vector in the following form \( k = \{0, 0, k\} \).

In this regime equations (13)–(16) are simplified to the following form:

\[
\omega \delta n_x = n_0 k \delta v_{xc},
\]

for the continuity equation,

\[
-\omega \delta v_{xc} + \frac{k \delta p_x}{m \omega} + \frac{\hbar^2 k^3 \delta n_x}{4m^2 n_0} = -\frac{g k}{m} \delta n_x,
\]

for the Euler equation,

\[
-\omega \delta S_x + \frac{(6\pi^2)^2 \hbar}{m} \left( \frac{n_{u0} - n_{d0}}{n_0} \right) \delta S_x - \frac{\hbar^2 S_0}{2m n_0} \delta S_y = \frac{8 \mu^2}{\hbar} S_0 \delta S_x,
\]

for the \( x \)-projection of the spin evolution equation,

\[
-\omega \delta S_y - \frac{(6\pi^2)^2 \hbar}{m} \left( \frac{n_{u0} - n_{d0}}{n_0} \right) \delta S_y + \frac{\hbar^2 S_0}{2m n_0} \delta S_x
\]

\[
+ \frac{2\mu}{\hbar} \delta S_x B_0 + \frac{g}{\hbar} \left( n_{u0} - n_{d0} \right) \delta S_y = -\frac{8 \mu^2}{\hbar} S_0 \delta S_x,
\]

for the \( y \)-projection of the spin evolution equation, and

\[
e_x (\delta B_x - 4 \mu \delta S_x) - e_y (\delta B_y - 4 \mu \delta S_y) = 0,
\]

for the Maxwell equations, with \( \delta v_{xx} = \delta v_{yy} = 0 \), \( \delta p_x = (6\pi^2 n_{u0})^{2/3} \delta n_x/3m \), where it is included that the last term in the Euler equation is equal to zero since \( \delta B_x = 0 \) in accordance with the Maxwell equation (21).

The right-hand side of the spin evolution equations (19) and (20) contain perturbations of the magnetic field expressed via the spin density in accordance with the Maxwell equation (21).

The continuation equation (17) and the Euler equation (18) do not contain the spin density \( \delta S_z \), \( \delta S_y \), and the magnetic field \( \delta B \). So, they are decoupled from the spin evolution equations and the equations of field. The continuity equation and the Euler equation for the spin-up fermions are coupled to the continuity equation and the Euler equation for the spin-down fermions via the short range interaction. They lead to the two sound waves.

Substituting the velocity perturbation from equations (17) into (18) we find a set of two equations for the perturbations of the partial concentrations. A non-trivial solition of this set is exists if the determinant of this set is equal to zero. It gives the following equation

\[
\left( \omega^2 - (6\pi^2 n_{u0})^{2/3} \frac{\hbar^2 k^2}{3m^2} + \frac{\hbar^2 k^4}{4m^2} \right) \times \left( \omega^2 - (6\pi^2 n_{d0})^{2/3} \frac{\hbar^2 k^2}{3m^2} + \frac{\hbar^2 k^4}{4m^2} \right) = \left( \frac{g k^2}{m} \right)^2 n_{u0} n_{d0}.
\]

For the zero equilibrium spin polarization equation (22) simplifies to

\[
\left( \omega^2 - (3\pi^2 n_0)^{2/3} \frac{\hbar^2 k^2}{3m^2} + \frac{\hbar^2 k^4}{4m^2} \right) \times \left( \omega^2 - (3\pi^2 n_0)^{2/3} \frac{\hbar^2 k^2}{3m^2} + \frac{\hbar^2 k^4}{4m^2} \right) = \pm \frac{g n_0 k^2}{2m},
\]

where we have included \( n_{u0} = n_{d0} = n_0/2 \).

The equations for the spin density are coupled to equations of field for \( \delta B_x \), \( \delta B_y \). They lead to the spin wave.
\( g = k/n_0^{1/3} \). The figure shows the dispersion dependence for different spin polarization. The upper (lower) curve is obtained for \( \eta = 0.3 \) (\( \eta = 0.1 \)). Other parameters are \( \eta_0 = 10^{14} \text{ cm}^{-3}, \mu = 2\mu_B, m = 6u, g = 4\pi\hbar^2a/m, \alpha = -0.4 \), where \( \mu_0 \) is the Bohr magneton. The figure (b) shows the dispersion dependence for different mass of particles. The upper (middle, lower) curve is obtained for \( m = 6u \) (\( m = 16u, m = 66u \)) at \( \eta = 0.1 \). Other parameters are \( \eta_0 = 10^{14} \text{ cm}^{-3}, \mu = 2\mu_B, \alpha = -0.4 \).

Spectrum of the spin waves arises as a result of evolution of \( \delta S_x \) and \( \delta S_y \) has the following form:

\[
\omega = \left| \Omega + \frac{g}{h} \eta \eta_0 - w - \frac{8\pi \mu^2}{\hbar} \eta \eta_0 + \frac{\eta \varepsilon_k}{\hbar} \right|, \tag{24}
\]

where

\[
w = \frac{(6\pi^2)^{\frac{3}{2}}}{m} (n_0^z - n_{\text{bd}}^z) \tag{25}
\]

is the characteristic frequency of the Fermi spin current, \( \varepsilon_k = \hbar k^2/2m \) is the kinetic energy appearing from the quantum Bohm potential, \( \Omega = 2\mu_B/(2\mu_0) \) is the cyclotron frequency, and \( \eta = (n_0 - n_{\text{bd}})/n_0 \in [-1,1] \) is the spin polarization. The first and last terms in equation (24) are similar to the transverse spin wave mode obtained in [24] formula (14). Presence of a small magnetic field \( B_0 = 10^{-3} \text{ G} \) makes the cyclotron frequency \( \Omega \) comparable with other terms in (24). At the larger magnetic field the cyclotron term has the major contribution.

If \( g > 0 \) and \( \Omega + \eta \eta_0 / \hbar \) dominates over \( w \) or \( g < 0 \) and \( \Omega \) dominates over \( w + \eta \mid g \mid m_0 / \hbar \), it gives us a positive constant \( \Omega + \frac{g}{\hbar} \eta \eta_0 - w - \frac{8\pi \mu^2}{\hbar} \eta \eta_0 \) at \( k = 0 \), we have increase of function \( \omega(k) \) with the increase of the wave vector \( k \) due to \( \eta \varepsilon_k / \hbar \). If we have a negative constant at \( k = 0 \) (it happens in the limits opposite to the described above) we have a decrease of the frequency \( \omega \) at the increase of the wave vector \( k \).

Solution (24) is presented in figure 6(a) for two different spin polarization. The frequency at the zero wave vector \( \omega(k = 0) \) is shifted from the cyclotron frequency by the short-range interaction (second term) and spin effects (third and fourth terms). The repulsive short-range interaction increases the frequency, while the spin effects decrease it. In the considering range of parameters, the spin effects (the Fermi spin current dominates, while the fourth terms describing the spin–spin interaction is rather small) are dominate over the short-range interaction and the cyclotron frequency. Increasing the spin polarization we increase the frequency of spin wave. The increase of the spin polarization increases all terms except the first term. However, the Fermi spin current grows faster since it is proportional to \([1 + (1 - \eta)^{2/3} - (1 - \eta)^{2/3}] \) while the other terms are proportional to \( \eta \).

Solution (24) is presented in figure 6(b) for different masses of particles for a fixed spin polarization. Fixed spin polarization for a fixed scattering length at different masses appears at different magnetic fields. Hence, the cyclotron frequency is different for all curves. Change of mass itself changes the short range interaction term, the Fermi spin current term and the quantum Bohm potential. Altogether, effect of mass increase and corresponding change of magnetic field lead to the frequency decrease of spin waves.

Spectrum of the waves of concentrations \( n_0, n_d \) (and the velocity fields \( v_u, v_d \)) appears from equations (4) and (5):

\[
\omega^2 = \frac{1}{2} \left[ U_u^2 + U_d^2 \pm \sqrt{(U_u^2 - U_d^2)^2 + 4g^2 (n_0 n_d / m^2)^2} \right], \tag{26}
\]

where \( U_u^2 = (6\pi^2 n_0)^{\frac{3}{2}} \hbar^2 / 3m^2 + \varepsilon_x / 2m \) and \( U_{\text{bd}} = U_{\text{bd}} \). Sign of the scattering length do not affect dispersion dependence (26).

If there is no spin polarization we have \( n_0 = n_{\text{bd}} = n_0 / 2 \) and \( U_{\text{bd}} = U_{\text{bd}} \). It can be reached by a combination of a magnetic field and corresponding attractive short range interaction. Consequently equation (26) simplifies to \( \omega^2 = \beta k^2 \hbar^2 + g n_0 k^2 / 2m \), with \( \beta = (3\pi^2 m_0)^{\frac{3}{2}} \hbar^2 / 3m^2 + \varepsilon_x / 2m \). The upper sign in \( \omega^2 \) corresponds to \( \delta n_a = -\delta n_d \) (the spin acoustic mode) and the lower sign corresponds to \( \delta n_a = \delta n_d \) (the usual sound mode).

Using \( \nabla \cdot B = 0 \) we find \( \delta B_x k_x = 0 \), which gives \( \delta B_x = 0 \). Thus, the magnetic field perturbation disappears from equations (4) and (5). Therefore, the sound waves and the spin waves are described by the independent equations (in linear regime).

Obtained solution (26) is similar to the spin–electron acoustical wave found in [16], and spin plasmon obtained for the two dimensional electron gas [42].

Equation (26) can be represented in the following form:

\[
\omega = \frac{\hbar n_0^z}{m} \kappa \left[ \frac{(3\pi^2)^{\frac{3}{2}}}{6} \left( (1 + \eta)^{\frac{3}{2}} + (1 - \eta)^{\frac{3}{2}} \right) \right] + \frac{\kappa^2}{4} \left[ \frac{\pi}{3} \left( (1 + \eta)^{\frac{3}{2}} - (1 - \eta)^{\frac{3}{2}} \right)^2 + 16(1 - \eta^2) \alpha^2 \right]. \tag{27}
\]
At the increase of the scattering length up to $\eta = 0.1$ the dashed lines present the sound waves at $\eta = 0.1$ (the black dashed line almost coincide with the black continuous line). The figure shows that the splitting between the dispersion dependencies of the sound waves increases with the increase of the spin polarization. Other parameters are the following: $m = 6\, u$, $n_0 = 10^{14}\, \text{cm}^{-3}$, $\mu = 2\mu e$, $g = 4\pi\hbar^2 a/m$, $|\alpha| = |a| n_0^{1/3} = 0.4$, $B_0 = 10^{-3}\, \text{G}$ for $\eta = 0.1$ and $B_0 = 3.1 \times 10^{-3}\, \text{G}$ for $\eta = 0.3$.

The sound waves described by equation (26) are presented in figures 7–13. Increasing the external magnetic field we increase the spin polarization. It increases the splitting of two branches of the sound waves figure 7. The upper wave is a modified sound wave existing even at the zero spin polarization. The lower branch is the spin acoustic wave. Further increase of the spin polarization increases the splitting of two branches figure 8. The spin acoustic wave spectrum is more affected by the spin polarization than the upper branch. The increase of the spin polarization decreases the frequency of the spin acoustic wave figures 7 and 8. Large spin polarizations leads to the modification of form of the spin acoustic wave spectrum. In this regime its almost linear spectrum starts at $k_s > 0$, so $\omega(k_s) = 0$, as it is demonstrated in figure 9.

Next, consider two spin polarizations $\eta_1 = 0.1$ and $\eta_2 = 0.3$ at different scattering lengths $\alpha = an_0^{1/3}$. At the increase of the scattering length up to $\alpha = 0.5$ we see that the upper sound has almost same spectrum for different spin polarization.

However, the increase of the scattering length noticeable decreases spin acoustic wave spectrum. It can be seen from the comparison of the lower dashed lines in figures 7 and 10 found for $\eta = 0.1$. The nonlinearity of the spectrum becomes more noticeable for larger $\alpha$. Same increase of the scattering length for $\eta = 0.3$ gives more changes to the spectrum. In addition to the frequency decrease we find that spectrum starts at $k_s > 0$. Hence, the increase of $\alpha$ and increase of $\eta$ changes the spin acoustic wave spectrum in the same way. Next, focus on the decrease of $\alpha$ in compare with its value used in figure 7. It increases frequency of the spin acoustic wave and decreases frequency of the upper sound wave figure 11. Consequently, it decreases splitting between spectrums of sound waves figure 11. These effects become more prominent at the further decrease of the scattering length (see figures 12 and 13).

### 3.4. Propagation perpendicular to the equilibrium spin polarization

At the perpendicular propagation we choose the wave vector in the following form $\mathbf{k} = \{k, 0, 0\}$.

Let us present simplified form of equations (13)–(16) existing in this regime:

![Figure 7](image7.png)  
**Figure 7.** The figure shows two sound waves which are presented by equation (26). Each of them is presented for two different spin polarizations. Continuous lines show the dispersion dependencies for $\eta = 0.3$. The dashed lines present the sound waves at $\eta = 0.1$ (the black dashed line almost coincide with the black continuous line). The figure shows that the splitting between the dispersion dependencies of the sound waves increases with the increase of the spin polarization. Other parameters are the following: $m = 6\, u$, $n_0 = 10^{14}\, \text{cm}^{-3}$, $\mu = 2\mu e$, $g = 4\pi\hbar^2 a/m$, $|\alpha| = |a| n_0^{1/3} = 0.4$, $B_0 = 10^{-3}\, \text{G}$ for $\eta = 0.1$ and $B_0 = 3.1 \times 10^{-3}\, \text{G}$ for $\eta = 0.3$.

![Figure 9](image9.png)  
**Figure 9.** The figure shows two sound waves which are presented by equation (26) in another regime of the spin polarization. Each solution is presented for two different spin polarizations. Continuous lines show the dispersion dependencies for $\eta = 0.62$. The dashed lines present the sound waves at $\eta = 0.1$. Other parameters are the following $m = 6\, u$, $n_0 = 10^{14}\, \text{cm}^{-3}$, $|\alpha| = 0.4$ and coincide with parameters in figure 7.

![Figure 10](image10.png)  
**Figure 10.** The figure shows two sound waves which are presented by equation (26) for the following spin polarizations $\eta = 0.3$ (continuous lines) and $\eta = 0.1$ (dashed lines). This figure presents results for a larger scattering length $|\alpha| = |a| n_0^{1/3} = 0.5$. Other parameters are the following $m = 6\, u$, $n_0 = 10^{14}\, \text{cm}^{-3}$.

Within these figures we observe the following: at $k_s > 0$ the spectrum starts, so $\omega(k_s) = 0$, as it is demonstrated in figure 9. Further, focus on the decrease of $\alpha$ in compare with its value used in figure 7. It increases frequency of the spin acoustic wave and decreases frequency of the upper sound wave figure 11. Consequently, it decreases splitting between spectrums of sound waves figure 11. These effects become more prominent at the further decrease of the scattering length (see figures 12 and 13).
Figure 11. The figure shows two sound waves which are presented by equation (26) for the following spin polarizations $\eta = 0.3$ (continuous lines) and $\eta = 0.1$ (dashed lines). This figure presents results for the following scattering length $|a| = |a| n_0^{1/3} = 0.3$. Other parameters are the following $m = 6$ u, $n_0 = 10^{14}$ cm$^{-3}$.

Figure 12. The figure shows two sound waves which are presented by equation (26) for the following spin polarizations $\eta = 0.3$ (continuous lines) and $\eta = 0.1$ (dashed lines). This figure presents results for the following scattering length $|a| = |a| n_0^{1/3} = 0.1$. Other parameters are the following $m = 6$ u, $n_0 = 10^{14}$ cm$^{-3}$.

$$\omega \delta n_s = n_0 k \delta v_{sx}, \quad (28)$$

$$-m \omega \delta v_{sx} + \frac{k \delta p_s}{n_0} + \frac{\hbar^2 k^3}{4m} \delta n_s = -gk \delta n_p \pm \mu k \delta B_z, \quad (29)$$

$$-i \omega \delta S_x + \left(\frac{2\pi^2}{m} \right) \left(\frac{i}{n_0} - \frac{i}{n_0} \right) \delta S_y = -\frac{8\pi \mu^2}{\hbar} S_0 \delta S_y, \quad (30)$$

$$-i \omega \delta S_y - \left(\frac{2\pi^2}{m} \right) \left(\frac{i}{n_0} - \frac{i}{n_0} \right) \delta S_x + \frac{2\mu}{\hbar} \delta S_y B_0 + \frac{\hbar}{g} (n_{0u} - n_{0d}) \delta S_x = 0, \quad (31)$$

and

$$e_s (\delta B_z - 4\pi \mu (\delta n_u - \delta n_d)) - e_s (\delta B_y - 4\pi \mu \delta S_y) = 0, \quad (32)$$

Other projections of the velocity field are equal to zero: $\delta v_{sy} = \delta v_{sz} = 0$. The right-hand side of the spin evolution equations (30) and (31) contain perturbations of the magnetic field expressed via the spin density in accordance with the Maxwell equation (32).

The set of continuity and Euler equations (28) and (29) is coupled with equation of field for $\delta B_z$, via the last term in each Euler equation. As we can see $\delta B_z$ is proportional to $\delta n_u - \delta n_d$. Substituting the found form of $\delta B_z$ in the Euler equation (29) we find a closed set of equations for $\delta n_u$, $\delta n_d$, $\delta S_x$, $\delta S_y$ which is independent from the equations for the spin density $\delta S_x$, $\delta S_y$ (30) and (31). Equations for $\delta n_u$, $\delta n_d$ are coupled to equations for $\delta n_d$, $\delta B_y$ via two terms in the Euler equation (29): the short range interaction and the spin–spin interaction. Evolution of concentrations and velocity fields leads to two sound modes. Evolution of the spin density $\delta S_x$, $\delta S_y$ together with equations of field $\delta B_z = 0$ and $\delta B_y = 4\pi \mu \delta S_y$ leads to the spin wave.

Spectrum of the spin waves appears from equations (30)–(32) as follows:

$$\omega^2 = \left( \Omega + \frac{g}{\hbar} \eta n_0 - w - \frac{\eta \varepsilon_k}{\hbar} \right) \times \left( \Omega + \frac{g}{\hbar} \eta n_0 - w - \frac{8\pi \mu^2}{\hbar} \eta n_0 + \frac{\eta \varepsilon_k}{\hbar} \right). \quad (33)$$

The spin wave dispersion dependence is different for parallel and perpendicular propagation. This difference appears due to the single term $\frac{\varepsilon_p^2}{\hbar^2} \eta n_0$. This term $\frac{\varepsilon_p^2}{\hbar^2} \eta n_0$ exists in one of two multipliers in formula (33). While it exists in both multipliers at the parallel propagation (24). This difference is presented in figure 14.

Considering $\eta \sim 0.1$, $B_0 \sim 10^{-3}$ G and larger, $n_0 \sim 10^{14}$ cm$^{-3}$, $g \sim 10$ nm, we see that term $8\pi \mu^2 n_0 / \hbar$ is the smallest term in equations (24) and (33) (two or three order smaller then others). Hence, the spectrum of spin waves is almost isotropic. The difference of spin wave spectrum for waves propagating in different directions is demonstrated in figure 14.
The spin evolution is proportional to the magnetic moment evolution (33), where we shall put the analytical form of the sound wave solutions (26) and (34), in equations obtained above. We see that it does not change the preferable direction 

3.5. Perturbations at zero external magnetic field

It is demonstrated in section 3.1 that the repulsive short range interaction between fermions with different spin projections can create an equilibrium spin polarization at the zero external magnetic field. However, this regime is unstable to the Cooper pair formation. It is possible to consider the Fermi gas with an attractive short-range interaction and effective magnetic field $B_{eg}$ leading to the equilibrium spin polarization $\eta \neq 0$. Hence, we have equilibrium spin polarization creating the preferable direction $S_0 = S_0 e_\eta$. Consider perturbations of this equilibrium state. To do this we need to substitute $B_0 = 0$ in equations obtained above. We see that it does not change the analytical form of the sound wave solutions (26) and (34), but it decreases the spin polarization existing in this equations. Substituting $B_0 = 0$ affects the spin wave solutions (24) and (33), where we shall put $\Omega = 0$. Removing $\Omega$ changes the spin wave frequency as it is shown in figure 15. It increases frequency of spin waves, since we do not need to subtract the cyclotron frequency from the Fermi spin current term $w$.

Experimental realization of the described excitations is similar to the experimental realization of the collective excitations in spinor Bose–Einstein condensates [23, 24, 29].

4. On a possibility of spin acoustic solitons

If equilibrium spin polarization is caused by the radio-frequency we have equilibrium effective magnetic field $B_0$. Next, the spin evolution is proportional to the magnetic moment evolution. Hence, it causes the appearance of the magnetic field. 

\[ \omega^2 = \frac{k^2}{2} \left[ U^2_x + U^2_y - \Lambda^2 \right] \pm \sqrt{\left( U^2_x - U^2_y - \eta \Lambda^2 \right)^2 + \left( g + 4\pi \mu^2 \right)^2 \left( 4n_0 n_0 \right. \left. \frac{m^2}{\mu^2} \right) - \left( \frac{\mu}{\hbar} \right)^2} \] (34)

where $\Lambda^2 = 4\pi \mu^2 n_0 / m$. The effects of magnetic moment are relatively small in formula (34) for the considering parameters range. Therefore, the sound wave spectrum is almost isotropic.

The paper is focused on the nonlinear waves propagating parallel to the direction of the equilibrium spin polarization $S_0 = S_0 e_\eta$. This direction of wave propagation leads to the following structure of the gradient operator: $\nabla = \{0, 0, \partial_t \}$.

Consider perturbations of the magnetic field via the analysis of the magneto-static Maxwell equations. Equation $\nabla \cdot B = 0$ is the first step of our analysis. Substituting the gradient operator presented above in this equation find the following differential equation $\partial_t B_z = 0$ which can be easily solved $B_z = c_1 = B_0$.

The second step is the analysis of the equation $\nabla \times B = 4\pi \mu \nabla \times S$. It gives $-\partial_t (B_x - 4\pi \mu S_x) e_y + \partial_t (B_y - 4\pi \mu S_y) e_x + 0e_z = 0$. It leads to $B_x = 4\pi \mu S_x = c_2$ and $B_y = 4\pi \mu S_y = c_3$. In equilibrium state we have $B_x = B_0 = 0$ and $S_x = S_0 = 0$. Consequently, $c_2 = c_3 = 0$. Therefore, the magnetic field is found $B_x = 4\pi \mu S_x$ and $B_y = 4\pi \mu S_y$. These relations between projections of the spin density and the magnetic field lead to the zero $z$-projection of the spin-torque entering the continuity equations: $T_z = [B_x, S_x] = B_x S_y - B_y S_x = 0$.

Present the modified set of hydrodynamic equations for the regime of perturbations propagating parallel to the equilibrium spin polarization:

\[ \partial_t n_s + \partial_z (n_s v_{sz}) = 0, \] (35)

and

\[ mn_s (\partial_t + v_{sz}\partial_z) v_{sz} + \delta_{sz} \partial_t p_s \]

\[ - \frac{\hbar^2 n_s}{4m} \delta_{sz} \partial_z \left( \frac{\partial^2 n_s}{n_s} - \frac{(\partial n_s)^2}{2n_s^2} \right) = F_{sz} + F_{sz}, \] (36)

where $F_{sz}$ is the force field of short-range interaction: $F_{sz} = \{0, 0, -gn_{sz}\partial_z n_{sz}\}$, $F_{sz} = \{0, 0, -g_{sz}\partial_z n_{sz}\}$, $F_{sz}$ is the force field of spin–spin interaction, its explicit form arises as

\[ F_{sz}^2 = \pm \mu n_s \delta_{sz} \partial_z B_z + \frac{\mu}{2} \delta_{sz} (S_z \partial_z B_z + S_z \partial_z B_z), \]

\[ \pm 4\pi \mu n^2 \frac{\mu}{\hbar} \left( J_{sz}^\alpha S_z - J_{sz}^\alpha S_z \right). \] (37)
with the following explicit form of the spin currents
\[ J^\beta_{(M)} = \frac{g^\beta}{2} (v_\alpha^\beta + v_\alpha^\beta - e^{\alpha \gamma z} \frac{h \delta \beta \gamma}{4m} (\frac{\partial n_a}{n_a} - \frac{\partial n_d}{n_d}) S^\gamma). \] (38)

Moreover, the first line in \( F_{2s}^p \) is equal to zero:
\[ F_{2s}^p = \pm 4\pi \frac{m \mu^2}{\hbar} (J^\alpha_{(M)} S_y - J^\alpha_{(M)} S_z). \] (39)

Next, using explicit form of the spin-currents find
\[ J^\alpha_{(M)} S_y - J^\alpha_{(M)} S_z = -\frac{\hbar}{4m} \delta \alpha \gamma \left( \frac{\partial n_a}{n_a} - \frac{\partial n_d}{n_d} \right) (S_z^2 + S_y^2). \] (40)

It gives that force field \( F_{2s, x} \) has only z-projection
\[ F_{2s, x} = \pi \mu^2 \delta \alpha \gamma \left( \frac{\partial n_a}{n_a} - \frac{\partial n_d}{n_d} \right) (S_z^2 + S_y^2). \] (41)

Consequently, we obtain \( n_1 \left( \partial_t + v_{xf} \partial_x \right) v_{sx} = 0 \) and \( n_1 \left( \partial_t + v_{x} \partial_x \right) v_{xy} = 0 \) while
\[ m n_1 \left( \partial_t + v_{x} \partial_x \right) v_{sx} + \partial_x p_x \]
\[ = -\frac{\hbar^2}{4m_n} \partial_x \left( \frac{\partial_n}{n} \right) - \frac{(\partial \cdot n^2)}{2n^2} = F_{2s, x} + F_{2s, z}. \] (42)

It will be shown below that the spin–spin interaction force given by equation (41) is the term of third order on the small parameter while our weakly nonlinear analysis includes terms up to the second order on the small parameter. Hence, it can be neglected. Therefore, there is no need to include the spin evolution equation in the following analysis.

4.1 Small perturbations

Equations (35) and (42) are solved by the perturbative method [43–45].

We consider \( \xi = \varepsilon^{1/2} (x - Ut) \) and \( \tau = \varepsilon^{3/2} Ut \), where \( U \) is the phase velocity of the wave. The decomposition of the concentration and velocity field involves a small parameter \( \varepsilon \) in the following form: \( n_s = n_{0s} + \varepsilon n_{1s} + \varepsilon^2 n_{2s} + ... \), and \( v_x = \varepsilon v_{1x} + \varepsilon^2 v_{2x} + ... \).

Moreover, the spin density projections have similar decompositions \( S_x = \varepsilon S_{1x} + ... \) and \( S_y = \varepsilon S_{1y} + ... \). These decompositions of the spin density make the spin–spin interaction force proportional to \( \varepsilon^{7/2} \). So, it does not contribute in the two lowest orders \( \varepsilon^{3/2} \) and \( \varepsilon^{5/2} \) involved in our analysis.

From the lowest order on the parameter \( \varepsilon \) of the hydrodynamic equations we find the phase velocity:
\[ U^2 = \frac{1}{6} \left( v_{Fa}^2 + v_{Fd}^2 \right) \pm \sqrt{\left( v_{Fd}^2 - v_{Fa}^2 \right)^2 + 36 \frac{\gamma^2}{m^2} n_{0a} n_{0d}}. \] (43)

Two phase velocities are found. It corresponds to the possibility of the existence of two solitons.

Considering terms proportional to \( \varepsilon^{5/2} \), we find corresponding set of equations. Representing all variables via \( n_{1a} \), we find that \( n_{1a} \) satisfies the Korteweg–de Vries equation:
\[ \partial_t n_{1a} + p_{1a} \partial_x n_{1a} + q \partial_x^2 n_{1a} = 0. \] (44)

where
\[ q = -\frac{\hbar^2}{8m^2 U^2}, \]

and
\[ p = \frac{(3U^2 - v_{Fa}^2/9)(U^2 - v_{Ca}^2/3)^2 + \frac{4}{3} n_{0a} (3U^2 - v_{Ca}^2/9)(U^2 - v_{Ca}^2/3)}{2U^2 m v_{Fa} n_{0d} (2U^2 - (v_{Fa}^2 + v_{Fd}^2)/3)}. \] (46)

Concentration \( n_{1a} \) appearing as the solution of equation (48) can be presented in the following form
\[ n = n_0 + \frac{2V \varepsilon}{p} \cdot sech^2 \left( \sqrt{\frac{V \eta}{q^2}} \right), \] (47)

where \( \eta = \xi - V \tau \).

Parameter \( q \) defines the width of the soliton. Hence, it should be positive for a soliton to exist. However, parameter \( q \) given by equation (45) is negative. Hence, there is no soliton solution in this regime.

Let us mention that if there is no interaction \( g = 0 \) then there is no soliton solution:
\[ \frac{2}{3} \delta_0 m v_{Fa}^2 \partial_x n_{1a} + \frac{8}{3} \delta_0 v_{Fa} v_{Fd} \partial_x n_{1a} - \frac{\hbar^2}{4m^2} \partial_x^2 n_{1a} = 0. \] (48)

It is correct for each species of fermions. As it is demonstrated above, the interspecies interaction does not change this picture.

We conclude that there is an analog of linear spin-electron acoustic wave (existing in the electron gas) in the gas of neutral fermions (this wave is called the spin acoustic wave). However, there is no analog of the spin-electron acoustic soliton (existing in the electron gas) for neutral fermions with the short range interaction in the first order by the interaction radius.

5. Conclusion

A minimal coupling model of neutral weakly interacting spin-1/2 fermions with the short-range and spin–spin interactions has been presented in form of a non-linear Pauli equation with the spinor pressure term and two-fluid hydrodynamic equations. The presentation of the model in the hydrodynamic form explicitly shows the Fermi spin current (a part of the spin current caused by the Pauli blocking). Being in the partially polarized phase the spin-1/2 fermions show three collective excitations, two of them are sound waves, and the third wave is a spin wave.

The repulsive short range interaction leads to the spin polarization and the external magnetic field increases it. Hence, the system shows an anisotropic behavior. Described structure of the spectrum exist for both limit regimes of the wave propagation: parallel and perpendicular to the equilibrium spin polarization. It has been shown that the Fermi spin current contributes in the spin wave spectrum only. The Fermi spin current gives main contribution in the spin wave spectrum at the small magnetic field \( B_0 < 0.01 \). The cyclotron
frequency gives the main contribution at larger magnetic field. Spectrum of the spin waves differ at propagation parallel and perpendicular to the equilibrium spin polarization. This difference by the spin–spin interaction. However, the difference is rather small, so the spectrum is almost isotropic. The increase of the spin polarization increases the frequency of the spin waves.

As it is mentioned above, there are two sound waves. The upper sound wave exist due to the full Fermi pressure of fermions with both spin projections. It can be found even in the single fluid model of spin-1/2 fermions. The lower sound wave is the spin acoustic wave. It exists due to relative motion of spin-up and spin-down fermions.

The upper sound wave increase its frequency at the increase of the spin polarization. This effect is more pronounced at the small positive scattering length. In this regime, the upper sound wave increase its frequency at the change of the spin polarization and large scattering length. In this regime, the wave does not exist at small $k < k_\ast$, where $\omega(k_\ast) = 0$. The spin acoustic wave frequency increases almost linearly at $k > k_\ast$. The external magnetic field $B_0 \geq 0.01$ G creates full spin polarization of repulsive fermions. In this regime, we have only one subspecies of fermions. Hence, there is only one sound wave.

Overall, the equilibrium states and spectrum of small perturbations of weakly interacting spin-1/2 fermions are studied.

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