Bifurcation of Periodic Solutions and Its Maximum Number in a Circular Mesh Antenna System with 1:2 Internal Resonance

Lishuang Jiang¹, Jing Li†⁰ and Wei Zhang²

¹ College of Applied Sciences, Beijing University of Technology, Beijing, P. R. China
² College of Mechanical Engineering, Beijing University of Technology, Beijing, P. R. China
†Corresponding author E-mail: leejing@bjut.edu.cn

Abstract. In this paper, we study the bifurcation of periodic solutions for a four-dimensional deployable circular mesh antenna system. The tools for proving these results are the averaging theory and Brouwer degree theory. Based on constructing displacement maps, we study the bifurcation of the periodic solutions of linear center, and to discuss the maximum number of periodic solutions in certain parameter control conditions. The results in this paper are helpful to the study of nonlinear dynamic characteristics and vibration control of deployable circular mesh antenna model.

1. Introduction
The research and its application of bifurcation theory of periodic solutions of high-dimensional nonlinear differential system are the frontier topics in the field of nonlinear dynamics and control. As early as 1900, the second part of the Hilbert’s 16th problem [1] was to study the maximum number and relative position of periodic solutions for nth plane polynomial vector fields. With the development of modern science and technology, the research of planar polynomial system is far from satisfying the actual engineering needs. Therefore, the existence and number of periodic solutions, and their distribution configurations of high-dimensional polynomial systems have aroused great interest of scholars at home and abroad.

For the bifurcation of periodic solutions of polynomial systems, many scholars did a lot of researches and got some results [2-9]. Chen et al. [4] proposed the C-L method to reveal the relationship between the bifurcation periodic solutions and the structural parameters of nonlinear system. Teixeira et al. [5] studied the conditions of bifurcation of periodic solutions by the averaging theory and obtained the number of periodic solutions of a four-dimensional perturbed system. Han el at. [6] applied the Poincaré map and integral manifold theory to study the existence conditions of periodic solutions and invariant torus of four-dimensional nonlinear dynamical systems. Li et al. [7] used the dynamic method and Cosgrove theory to study the exact parameter expressions of heteroclinic orbits and periodic solutions in three-dimensional Hamilton system. Li et al. [9] used the subharmonic Melnikov function method to study the bifurcation behavior of periodic solutions of a four-dimensional nonlinear dynamic system, and obtained the existence conditions and number of subharmonic periodic solutions. In the process of dynamical analysis of engineering application model, it is necessary to study the nonlinear dynamical behavior of high-dimensional perturbed polynomial system in depth.
Large-scale deployable space spacecraft structures have a very wide range of applications in the field of spaceflight, such as developable circular antenna (see figure 1), solar panels and mechanical arms. Recently, many scholars have done a lot of researches on the structure, dynamical model, control and experimental simulations of deployable antenna [10, 11]. But there are few theoretical studies on the bifurcation of periodic solutions in deployable antenna model. In this paper, we apply the theory of high-dimensional nonlinear dynamic systems to the circular mesh antenna model, which has a certain guiding role in the analysis of vibration characteristics, flutter control and fault diagnosis of antenna systems.

Figure 1. The deployable circular mesh antenna.

2. Deployable Circular Mesh Antenna System

The model presented in this paper is the four-dimensional deployable circular mesh antenna model which is influenced by multiple excitation factors such as the amplitude of thermal excitation $f_T$ and the damping coefficients $\mu_i$ (i = 1, 2) in space as follows

$$
\ddot{w}_1 + \mu_1 \dot{w}_1 + \omega_1^2 w_1 + \gamma_{11} f_T \cos(\Omega t) w_1 + \gamma_{12} w_2 + \gamma_{13} w_1^3 + \gamma_{14} w_2^3 w_2
$$

$$
\quad + \gamma_{15} w_1^2 w_2 + \gamma_{16} w_2^3 = \gamma_{17} + \gamma_{18} f_T \cos(\Omega t)
$$

$$
\ddot{w}_2 + \mu_2 \dot{w}_2 + \omega_2^2 w_2 + \gamma_{21} f_T \cos(\Omega t) w_2 + \gamma_{22} w_1 + \gamma_{23} w_1^3 + \gamma_{24} w_2^3 w_1
$$

$$
\quad + \gamma_{25} w_2^2 + \gamma_{26} w_1^3 = \gamma_{27} + \gamma_{28} f_T \cos(\Omega t), \tag{1}
$$

where $w_1$, $w_2$ are the first and second modes of transverse movement, respectively, and $\omega_1$, $\omega_2$ are the natural frequencies, $\gamma_{ij}$ is curvature displacement relation. The method of multiple scales is one of the most effective methods in singular perturbation theory, which is widely used in the study of nonlinear problems in natural science and engineering technology. We focus on the case of primary parametric resonance

$$
\omega_1^2 = \frac{1}{4} \Omega^2 + \varepsilon \sigma_1, \quad \omega_2^2 = \Omega^2 + \varepsilon \sigma_2,
$$

where $\sigma_1$, $\sigma_2$ are tuning parameters, $\Omega = 1$ and $\varepsilon$ is a small parameter. With the scale transformations

$$
\gamma_{ij} \rightarrow \varepsilon \gamma_{ij}, \quad (i = 1, 2, 1 \leq j \leq 8, \quad j \in N),
$$

system (1) will be in the form

$$
\ddot{w}_1 + \mu_1 \dot{w}_1 + \omega_1^2 w_1 + \varepsilon \gamma_{11} f_T \cos(\Omega t) w_1 + \varepsilon \gamma_{12} w_2 + \varepsilon \gamma_{13} w_1^3 + \varepsilon \gamma_{14} w_2^3 w_2
$$

$$
\quad + \varepsilon \gamma_{15} w_1^2 w_2 + \varepsilon \gamma_{16} w_2^3 = \varepsilon \gamma_{17} + \varepsilon \gamma_{18} f_T \cos(\Omega t),
$$
\[
\begin{align*}
\dot{w}_i &= \mu_i w_i + \omega_i^2 w_i + \varepsilon \gamma_{21} f_r \cos(\Omega t) w_i + \varepsilon \gamma_{22} w_i + \varepsilon \gamma_{23} w_i^3 + \varepsilon \gamma_{24} w_i^5 w_i^3 \\
+ \varepsilon \gamma_{25} w_i^7 w_i^5 &= \varepsilon \gamma_{26} f_r \cos(\Omega t),
\end{align*}
\]

Assume that the solutions of system (2) has the following form

\[
w_i = \sum_{k=0}^{\infty} \varepsilon^k w_{ik}(T_0, T_1, T_2) + O(\varepsilon^3),
\]

where \( T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon^2 t, i = 1, 2 \). The differential operator of equation (3) is

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots,
\]

where \( D_k = \frac{\partial}{\partial T_k}, k = 0, 1 \). By using the method of multiple scales, we obtain the averaged equation (4) of system (2)

\[
\begin{align*}
\dot{x}_1 &= (a_{11} + a_{12}) x_1 + (a_{12} - a_{13} - a_{14}) x_1 x_2 + (a_{13} + a_{14}) x_2 + a_{15} x_3 (x_1^2 + x_2^2) \\
&+ a_{16} x_4 (x_1^2 + x_2^2) + \varepsilon (a_{17} - 6a_{18}) (x_1^2 + x_2^2) x_i + (a_{19} + a_{20} - a_{21}) (x_i^2 + x_3^2) x_2 \\
&+ x_2 (7a_{18} x_2^2 + (-6a_{20} + 2a_{21}) x_2^2) + (6a_{20} - 2a_{21}) x_1 x_2 x_4,
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 &= -(a_{11} + a_{22}) x_1 + (a_{12} - a_{13} + a_{14}) x_1 x_2 + (a_{14} - a_{15}) x_2 + a_{16} x_3 (x_1^2 + x_2^2) \\
&+ a_{17} x_4 (x_1^2 + x_2^2) + (a_{18} - a_{17}) (x_1^2 + x_2^2) x_1 + (-a_{19} + a_{20} - a_{21}) (x_1^2 + x_3^2) x_1 \\
&+ x_1 (7a_{18} x_2^2 + (-6a_{20} + 2a_{21}) x_2^2) - (6a_{20} - 2a_{21}) x_1 x_2 x_4,
\end{align*}
\]

\[
\begin{align*}
\dot{x}_3 &= (b_{12} + b_{13}) x_1 + b_{13} x_2 (x_1^2 + x_2^2) + b_{14} x_3 (x_1^2 + x_2^2) + (b_{15} + b_{16} - b_{17}) (x_1^2 + x_3^2) x_4 \\
&+ b_{18} x_4 (x_1^2 + x_2^2) + 6(b_{17} - b_{18}) x_1 x_2 x_3 + 2(b_{17} - b_{18}) x_1^2 x_4 + b_{19} x_4 x_2 + b_{20} x_4 x_3 + b_{21} x_4 x_4,
\end{align*}
\]

\[
\begin{align*}
\dot{x}_4 &= (5b_{12} - b_{13}) x_1 + b_{13} x_2 (x_1^2 + x_2^2) + b_{14} x_3 (x_1^2 + x_2^2) + (b_{15} - b_{16} + 5b_{17}) (x_1^2 + x_3^2) x_4 \\
&+ x_1^2 x_3 + b_{18} x_3 (x_1^2 + x_2^2) + (b_{17} - b_{18}) x_1 x_2 x_3 + 2(b_{17} - 5b_{17}) x_1^2 x_3 \\
&+ (\frac{1}{2} b_{18} - b_{19}) (x_1^2 + x_2^2) + b_{24} x_4^2 + (b_{22} - 2b_{23}) x_1^2 x_3 + (b_{23} + 2b_{24}) x_1^2 + b_{25},
\end{align*}
\]

where \( a_{ij}, b_{uv} \) are the combined coefficients shown in appendix, \( 11 \leq 10 i + j \leq 22, 11 \leq 10 u + v \leq 24, i > 0 \) and \( i, j, u, v \in N \).

3. Bifurcation of Periodic Solutions of the Antenna System

3.1. The Maximum Number of Periodic Solutions

In this section, we investigate the existence and the maximum number of periodic solutions of the four-dimensional deployable circular mesh antenna system. The tools for proving these results are the averaging theory and Brouwer degree theory [2, 3]. Firstly, we introduce the scale transformations for system (4)

\[
a_{ij} \rightarrow \varepsilon a_{ij}, b_{uv} \rightarrow \varepsilon b_{uv},
\]
where $0<\varepsilon<1$, $12\leq 10i+j\leq 22$, $12\leq 10u+v\leq 24$, $i>0$ and $i, j, u, v \in \mathbb{N}$. Then, system (4) is transformed into the following perturbed system
\begin{align*}
\dot{x}_1 &= a_{11} x_1 + \varepsilon F_1(x_1, x_2, x_3, x_4), \\
\dot{x}_2 &= -a_{11} x_1 + \varepsilon F_2(x_1, x_2, x_3, x_4), \\
\dot{x}_3 &= b_{11} x_1 + \varepsilon F_3(x_1, x_2, x_3, x_4), \\
\dot{x}_4 &= -b_{11} x_1 + \varepsilon F_4(x_1, x_2, x_3, x_4),
\end{align*}
(5)
where $F_1(x_1, x_2, x_3, x_4)$ and the coefficients are defined in system (4).

For $\varepsilon = 0$, system (5) becomes
\[ x = Ax, \]
which consists of two planar isochronous center systems with $\frac{2\pi}{a_{11}}$-period and $\frac{2\pi}{b_{11}}$-period respectively.

When $a_{11} = b_{11}$, the origin is a global isochronous center for system (5), that means all orbits different from the origin are periodic with the same period.

For $\varepsilon \neq 0$, in order to analyze the existence and number of periodic solutions, we transform system (5) into a perturbed system which is topologically equivalent to the system (5). Assuming $b_{11} = 3a_{11}$ and making the change of variables
\begin{align*}
x_1 &= r \sin(a_{11} \theta), \\
x_2 &= r \cos(a_{11} \theta), \\
x_3 &= \rho \sin(b_{11}(\theta + s)), \\
x_4 &= \rho \cos(b_{11}(\theta + s)),
\end{align*}

we obtain the averaged system associated to system (5) as
\begin{align*}
\dot{r} &= \varepsilon h_1(r, \rho, s), \\
\dot{\rho} &= \varepsilon h_2(r, \rho, s), \\
\dot{s} &= \varepsilon h_3(r, \rho, s),
\end{align*}
(6)
by applying the averaging theory, where
\begin{align*}
h_1(r, \rho, s) &= \frac{3}{2\pi} (3a_{11} - a_{21}) \rho r^2 \cos(2a_{11}s) + a_{14} r^4 + \frac{1}{\pi} (\pi a_{15} - 9a_{20} + 3a_{21}) \rho^2 r, \\
h_2(r, \rho, s) &= \rho (b_{14} r^2 + b_{13} \rho^2), \\
h_3(r, \rho, s) &= -\frac{r^2}{12a_{11}} (12a_{17} + 21a_{18} - 4b_{15}) - \frac{\rho^2}{3} (3a_{19} - b_{16}) - \frac{1}{3} (3a_{22} + 2b_{12}).
\end{align*}
(7)
are called averaged functions associated to system (5).

Suppose that $(r_0, \rho_0, s_0)$ is an isolated zero of $(h_1, h_2, h_3) = (0, 0, 0)$ in averaged system (6) and its corresponding Jacobian matrix is not equal to zero. There is no periodic solution in system (5) when $r_0 \rho_0 = 0$. Assume that $y_{21} \mu_1 \neq 0$ and $f_T y_{13} (4y_{21} - y_{11}) \neq 0$ hold. From $h_2(r, \rho, s) = 0$, we obtain
\begin{align*}
r_0^2 &= \frac{-h_{14}}{b_{13}} \rho_0^2. 
\end{align*}
(8)
Substituting equation (8) into $h_1(r, \rho, s) = 0$, we have
\begin{align*}
\rho_0^2 &= \frac{-4b_{13}(3a_{22} + 2b_{12})}{12a_{11} b_{14} + 21a_{18} b_{14} - 12a_{19} b_{13} + 4b_{13} b_{18} - 4b_{14} b_{15}}.
\end{align*}
(9)
Substituting equations (8) and (9) into \( h(r, \rho, s) = 0 \), we get
\[
\cos(2a_1 s_0) = -\frac{2(\pi a_i b_{13} - \pi a_i b_{14} - 9a_2 b_{13} + 3a_2 b_{14})}{3b_{13}(3a_{20} - a_{21})}.
\]

We get at most one isolated zeros of \( (h_1, h_2, h_3) = (0, 0, 0) \). Based on the above discussion and the averaging theory, system (5) has at most two periodic solution bifurcating from the periodic orbits of four-dimensional linear center provided by the displacement function of first order in \( \varepsilon \).

3.2. Numerical Simulations
Numerical calculations are carried out in order to complete and validate theoretical results obtained in previous sections. In this section, we present numerical simulations of system (5) for several values of the parameters. Let system (5) satisfy the parameter condition
\[
P = \{\mu_1, \mu_2, \sigma_1, \sigma_2, f_f, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \gamma_{26}, \gamma_{27}, \gamma_{28}\}.
\]
\[
= \{16.25, 40, 1, 6, 7, 3, 75, 0.01, 2, 12, 3, 1, 2, 20, 0.01, 2, 1.82, 16.2, 3.9, 4, 4.3\}.
\]

Then system (8) has one solution and the corresponding Jacobian matrix is not equal to zero. According to the theoretical analysis, system (5) has at most two periodic solution bifurcating from the periodic orbits of linear center. The periodic solution, relative position and time traces for system (5) under the parameter condition \( P \) are obtained, as shown in figure 2 and figure 3.

![Figure 2](image1.png)
![Figure 3](image2.png)

**Figure 2.** Two-dimensional phase portraits and time traces of one periodic solutions of system (5).

![Figure 3](image3.png)

**Figure 3.** Three-dimensional phase portraits of two periodic solutions of system (5).
4. Conclusions
In this paper, we study the bifurcation of periodic solutions and its maximum number in a deployable circular mesh antenna system with 1:2 internal resonance. Applying the method of multiple scales, we obtain the averaged equation of antenna system (2). Based on the averaging theory and Brouwer degree theory, we obtain the existence conditions and the maximum number of periodic solutions bifurcating from the periodic orbits of linear center. For parameters $\gamma_{25}\mu_{i} \neq 0$ and $f_{r}\gamma_{13}(4\gamma_{21} - \gamma_{11}) \neq 0$, it is verified that system (5) has at most two periodic solutions.

In addition, the numerical simulations give the detailed phase portraits and time traces of periodic solutions under parameter condition P. The upper bound of the number of periodic solutions in theoretical results is proved to be achievable. It has a certain theoretical significance and application value for the vibration reduction design of the circular mesh antenna structure and the parameters control of the antenna model in space.

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6. Compliance with Ethical Standards
Conflict of interest the authors declare that they have no conflict of interest concerning the publication of this manuscript.

7. Appendices

$$a_{11} = \sigma_{2}^{2} + \frac{1}{4} \mu_{2}^{2} + \frac{4}{3} \gamma_{22} \gamma_{12}, \quad a_{12} = \gamma_{13} \gamma_{28} f_{r}, \quad a_{13} = \frac{4}{3} \gamma_{14} \gamma_{18} f_{r}, \quad a_{14} = 8 \gamma_{14} \gamma_{17} + 2 \gamma_{15} \gamma_{27},$$

$$a_{14} = 8 \gamma_{14} \gamma_{17} + 2 \gamma_{15} \gamma_{27}, a_{15} = 2 \gamma_{15} \mu_{2}, a_{16} = 3 \gamma_{13} \mu_{4}, a_{17} = 6 \sigma_{1} \gamma_{13} + \frac{8}{3} \gamma_{14} \gamma_{22} + 4 \gamma_{12} \gamma_{26}, a_{18} = \gamma_{11} \gamma_{13} f_{r},$$

$$a_{19} = 8 \gamma_{16} \gamma_{22} + \frac{8}{3} \gamma_{12} \gamma_{24} + 4 \gamma_{16} \sigma_{1} - \frac{8}{3} \gamma_{12} \gamma_{14}, a_{20} = \frac{1}{3} f_{r} \gamma_{13} \gamma_{21}, a_{21} = \frac{1}{4} \gamma_{11} \gamma_{15} f_{r}, a_{22} = \frac{3}{8} f_{r}^{2} \gamma_{11},$$

$$b_{11} = -\frac{1}{24} \gamma_{21} f_{r}^{2}, b_{12} = \frac{1}{8} \gamma_{13}^{2} + \frac{1}{8} \mu_{2}^{2} - \frac{2}{3} \gamma_{12} \gamma_{22}, b_{13} = \frac{1}{2} \gamma_{25} \mu_{4}, b_{14} = \frac{3}{4} \gamma_{23} \mu_{2},$$

$$b_{15} = \frac{1}{2} \gamma_{25} \sigma_{2} - \frac{4}{3} \gamma_{14} \gamma_{22} + \frac{4}{3} \gamma_{22} \gamma_{24} - 4 \gamma_{12} \gamma_{26}, b_{16} = \gamma_{11} \gamma_{23} f_{r}, b_{17} = \frac{1}{6} \gamma_{21} \gamma_{25} f_{r},$$

$$b_{18} = \frac{3}{4} \gamma_{23} \sigma_{2} - \frac{4}{3} \gamma_{12} \gamma_{24} - 2 \gamma_{16} \gamma_{22}, b_{21} = \frac{3}{8} \gamma_{23} \gamma_{28} f_{r}, b_{22} = \frac{1}{8} \gamma_{25} \gamma_{28} f_{r} + \frac{1}{16} \gamma_{28} f_{r}, b_{23} = -12 \gamma_{17} \gamma_{26} - \gamma_{25} \gamma_{27}, b_{24} = \frac{1}{16} \mu_{2} \gamma_{28} f_{r} + \frac{1}{8} \gamma_{28} f_{r},$$

$$b_{25} = \frac{2}{3} \gamma_{18} \gamma_{24} f_{r}, b_{22} = \frac{1}{8} \gamma_{25} \gamma_{28} f_{r} - 2 \gamma_{26} \gamma_{18} f_{r}, b_{23} = -\frac{1}{3} \gamma_{18} \gamma_{22} f_{r} + \frac{1}{4} \gamma_{21} \gamma_{27} f_{r} + \frac{1}{16} \gamma_{28} f_{r} \sigma_{2}.$$

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