Complex World-Sheets from $N = 2$ Strings

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We study some properties of target space strings constructed from (2,1) heterotic strings. We argue that world-sheet complexification is a general property of the bosonic sector of such target world-sheets. We give a target space interpretation of this fact and relate it to the non-gaussian nature of free String Field Theory. We provide several one loop calculations supporting the stringy construction of critical world-sheets in terms of (2,1) models. Using finite temperature boundary conditions in the underlying (2,1) string we obtain non-chiral target space spin structures, and point out some of the problems arising for chiral spin structures, such as the heterotic world-sheet. To this end, we study the torus partition function of the corresponding asymmetric orbifold of the (2,1) string.
1. Introduction

String Theory, in its present perturbative formulation, is defined in terms of its underlying world-sheet two dimensional field theory. However, recent developments in non-perturbative string dynamics (see for example [1]) seem to suggest that the world-sheet description plays no such fundamental rôle, and it is only valid in some corners of the moduli space of vacua. Then, any non-perturbative formulation of String Theory must reduce to perturbative world-sheet descriptions in some weak or strong coupling regimes. An interesting scenario, in which different world-sheet theories appear to be derived from a more fundamental principle, was introduced by Green [2], who proposed to obtain string world-sheets as target space theories of two-dimensional strings. In this way, space-time coordinates arise dynamically as the massless excitations of some underlying string theory living in a two dimensional target space. A serious difficulty one meets in trying to recover strings from strings is the fact that the massless modes of the underlying string theory have complicated interactions induced by the massive modes which have been integrated out. Another problem is that, in general, the field content corresponds to a non-critical rather than critical world-sheet.

One way of keeping interactions between massless modes under control is by considering $N = 2$ strings [3] as the underlying theory. $N = 2$ strings have a number of interesting features [4]. Their critical dimension is four but, because of the existence of $N = 2$ superconformal symmetry, the target manifold must have a complex structure so the target signature is either $4 + 0$ or $2 + 2$. The $N = 2$ superconformal algebra contains, in addition to the two fermionic generators $\hat{G}^{\pm}$, a $U(1)$ current $\hat{J}$. In heterotic constructions of the type (2,0) or (2,1) this current has to be balanced by a left-moving counterpart $\hat{J}$ whose gauging forces the introduction of a new set of ghosts raising the critical dimension in the left-moving sector by 2. In the case of the (2,1) models this means that we have a left-moving internal $N = 1$ SCFT with $\hat{c} = 8$. Absence of BRST anomalies further imposes the left-moving $U(1)$ current $\hat{J}$ has to lie in a null subspace. Finally, their spectrum is very simple, containing only a finite number of massless modes, and their $n$-point scattering amplitudes vanish for $n \geq 4$. For the (2,1) models with two-dimensional target space, even the on-shell three-point functions vanish.

Recently, Kutasov and Martinec [5] have rescued the idea of Green and proposed that both string world-sheets and membrane world-volumes can be obtained as different vacua of (2,1) heterotic strings. Taking $\hat{J}$ to have no component on the left-moving internal sector
the string theory lives effectively in 1+1 dimensions and one can recover the classic bosonic, type-IIB and heterotic world-sheet theories in a physical gauge, for different choices of the internal $\hat{c} = 8$ SCFT. By relaxing the condition that $J$ has to lie entirely in the $2 + 2$ non compact space one gets an effective $2 + 1$ theory that corresponds to the world-volume theory of two dimensional membranes. In fact, the $(2,1)$ string construction seems to provide a unified picture of all “M-brane” vacua [6].

The computation of the torus partition function in the target world-sheet theory involves now the evaluation of string vacuum amplitudes in the underlying theory. In fact, the results obtained at one loop order in the topological expansion of the $N = 2$ string suggest that there are no higher order corrections in $\lambda$, the $N = 2$ string coupling constant. There are however non-perturbative corrections of order $e^{-1/\lambda^2}$ which would account, for example, for global features of the target world-sheet fields. In this paper we will only consider one loop partition functions both in the $N = 2$ and in the target string sense, thereby summing embeddings of the world-sheet tori onto target two dimensional tori.

Since we are dealing with a string theory in a toroidal space-time we will have winding as well as momentum modes around, and the first impression is that we will get twice the number of states we would like to. This fact has been noticed in the case of $N = 2$ strings [4][7] from the following peculiar form of the one-loop free energy in a target torus with modular parameters $T$ and $U$:

$$\log Z(T,U)_{\text{target}} = -D_T \log \left( \sqrt{T_2}|\eta(T)|^2 \sqrt{U_2}|\eta(U)|^2 \right), \quad (1.1)$$

where $D_T$ is the effective number of transverse degrees of freedom (number of target space free bosons minus fermions). For the $(2,2)$ string $D_T = 2$, whereas for the $(2,1)$ string whose target space dynamics is the standard bosonic world-sheet, $D_T = 24$. In general, $D_T = 0$ for supersymmetric strings with vanishing two-dimensional cosmological constant. $T$ and $U$ are respectively the complex and the Kähler moduli of the target torus, defined in terms of the background values of the metric and the torsion as

$$T = \frac{g_{12} + i \sqrt{\det g}}{g_{22}} \quad \alpha' U = b_{12} + i \sqrt{\det g}.$$ 

One way of getting rid of the unwanted modes is just by taking the field theory limit in the underlying theory, decoupling all winding modes by sending the string tension to infinity or, equivalently, by stretching the target world-sheet to infinite area. However, as it stands,
the most striking feature of (1.1) is that the number of massless degrees of freedom in the target space appears doubled, as well as the target space moduli parameters. The complex structure moduli $T$ enters as a standard world-sheet moduli parameter and, remarkably, the Kähler parameter $U$ is reinterpreted as the complex structure of an identical decoupled world-sheet theory. Thus, the effective spacetime of the target theory is a complex world-sheet, with the “imaginary” component being provided by the mirror pair.

Complex world-sheets have been claimed to be involved in the description of the high-energy phase of String Theory, either from the study of strings at finite temperature [9] or from the behavior of string scattering amplitudes at high energies [10][11]. It is then natural to consider (1.1) as an indication that the target string construction based on underlying $N = 2$ models is in some sense related to the high energy phase of standard strings. The relation of the $N = 2$ construction to M-theory [6] makes this observation very suggestive.

The relation of mirror symmetry to the $T-U$ factorization phenomenon suggests an $N = 2$ world-sheet mechanism based on the properties of the corresponding chiral rings (see [5]). In this paper we will adopt a target space point of view and try to find the conditions for factorization in a general two-dimensional string background. The main conclusion is that the target space of two dimensional string theories, as defined by massless *bosonic* probes, is generally complexified. Interpreted as a world-sheet theory itself, we are led to a theory with complex world-sheets. Although it is clear that the doubling of degrees of freedom has to do with the winding modes of the underlying $(2,1)$ model, the precise space-time interpretation of the factorization between $T$ and $U$ dependence is not so obvious. By providing a physical interpretation of the calculation of (1.1) in [7] (along the lines of the second reference in [4]), we argue that such factorization is a rather general feature of the massless sector of two dimensional String Field Theory, and involves some simple, yet subtle properties of one-loop string path integrals. We also find that the universality of the factorization is lost when considering the fermionic sector of the target strings, even in the simplest $(2,1)$ models.

The plan of the paper is as follows. In section 2 we will study in detail the stringy mechanism behind the doubling of the target partition function of the $(2,1)$ models when all

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1 The $T-U$ symmetry in (1.1) is related to the stronger “triality” symmetry of the integrand of $N = 2$ partition functions. This symmetry is characteristic of the solitonic sector [8].
the target space fields are periodic. To this end we will analyze the correspondence between
the massless sector of two dimensional String Field Theory and one loop computations
in the first-quantized underlying string model. In section 3 we will be concerned with
the study of target space spin structures from the point of view of the underlying (2,1)
string. We will compute the target space partition function in the four sectors of non-
chiral boundary conditions of the target fermions of the type-IIB superstring and will
outline some ideas of how to deal with chiral target spin structures. As an application of
this proposal we will briefly study the case of the target SO(32) heterotic string. Finally
in section 4 we will summarize our conclusions. The details of the computations in section
3 are given in Appendix A, while Appendix B contains a review of β-duality.

2. Fields vs. Strings on the target world-sheet

One of the most interesting distinctive features of (2, 1) strings is the simplicity of the
corresponding String Field Theory (SFT). Indeed, since the spectrum consists of a finite
number of massless free fields, one is tempted to assume that (2, 1) SFT is more or less
equivalent to free Field Theory in the two-dimensional target space, once the conformal
gravitational factor and the dilaton have been gauged away. It is on this basis that one
identifies the different critical world-sheet theories as vacua of the (2, 1) models. However,
the doubling of degrees of freedom exhibited in (1.1) is an indication of certain “stringy”
features which make the target world-sheet rather exotic. In this section we present a
general discussion of the formula (1.1), emphasizing the differences between SFT and
standard Field Theory.

It will be instructive to keep the discussion as general as possible, considering arbitrary
string vacua with two dimensional space-time, and concentrate on the free approximation.
The conformal structure on left and right movers depends on the gauged world-sheet
gravity. In the $N = 0$ case we have a “longitudinal” $c_L = 2$ free CFT for the space-
time directions, with Minkowskian or Euclidean signature, and an internal or “transverse”
c_T = 24 CFT. With $N = 1$ worldsheet supergravity we have a $\hat{c}_L = 2$ longitudinal SCFT,
and a $\hat{c}_T = 8$ internal SCFT ($N = 1$ SCFTs with $\hat{c} = 8$ have been studied in [12]). Finally,
the $N = 2$ case has a $\hat{c} = 4$ free system with 2 + 2 signature, which is directly critical with
only one complex scalar degree of freedom. For the application to target string theories,
we can define models with two dimensional target space by gauging a convenient null
This procedure leaves no propagating degrees of freedom, apart from the zero modes. So, the \((2, \ast)\) strings have a purely holomorphic torus partition function, except from the contribution of the zero modes.

In a \(1 + 1\) Minkowski target space, the on-shell condition is given by

\[
(L_0 + \overline{L}_0)\Psi_f = \frac{\alpha'}{2} (p^2 + M^2) \Psi_f = 0,
\]

where

\[
M^2 = \frac{2}{\alpha'}(L'_0 + \overline{L}'_0) + \frac{2}{\alpha'}(L_0 + \overline{L}_0)T + \frac{2}{\alpha'}(L_0 + \overline{L}_0)_{\text{ghost}}
\]

is the mass operator, whose spectrum is constrained by the various physical state conditions, including appropriate GSO projections. In (2.2) we denote by \(L'_0\) the oscillator part of the “longitudinal” Virasoro operator. Apart from global (discrete) states, its excitations lead to longitudinal (gauge) degrees of freedom.

At the massless level one finds \(D_T\) effective (physical) degrees of freedom, counting bosons with a \(+1\) and fermions with a \(-1\). Naively, the corresponding SFT action can be written, in a somewhat symbolic form as,

\[
S_{\text{target}} = \frac{1}{\alpha' \lambda^2} \int_{\text{spacetime}} \sum_f \Psi_f (L_0 + \overline{L}_0) \Psi_f + \text{interactions},
\]

where the two dimensional fields \(\Psi_f\) have dimensions of length, and \(\lambda\) is the string coupling of the underlying string theory. If (2.3) is interpreted as a target world-sheet action, the Regge slope of the target string is given by \(\alpha'_{\text{target}} \sim \alpha' \lambda^2\). Accordingly, target world-sheet instanton effects \(\sim e^{-1/\alpha'_{\text{target}}}\) correspond to non-perturbative string effects in the underlying string model.

In an attempt to provide a target space explanation of (1.1), we could use the action (2.3) to compute the one-loop partition function in the massless sector, when the space-time is an euclidean torus or radii \(R_1\) and \(R_2\). The result is

\[
\log Z_{\text{massless}} \sim -\frac{1}{2} \text{Tr}_{M^2=0} (-1)^F \log \left(-\partial^2\right) \sim -D_T \log \left(\frac{R_1}{R_2} \left|\eta(iR_1/R_2)\right|^2\right) + \text{constant}.
\]

where the prime stands for zero mode subtraction. For a straight torus with no torsion background we have \(T = iR_1/R_2\) and \(U = iR_1R_2\). Clearly, we fall short in reproducing (1.1) since we should get a second factor with \(R_2\) replaced by \(1/R_2\). This suggests that
we need the winding modes which are absent in (2.3). The diagonal kinetic kernel for a
target torus is in fact (in units $\alpha' = 1$),

$$2(L_0 + T_0) \equiv K_0 + M^2 = \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \ell_1^2 R_1^2 + \ell_2^2 R_2^2 + M^2. \quad (2.5)$$

The integers $n_1, n_2$ label momentum modes and $\ell_1, \ell_2$ correspond to the winding modes.

In a standard interpretation, we may regard the integers $\ell_1, \ell_2$ as labels of additional
massive fields, with an effective mass $M_{\text{eff}}^2 = M^2 + \ell_1^2 R_1^2 + \ell_2^2 R_2^2$. However, it is well known
that, when writing (2.3), there is a very general ambiguity in the “definition” of space-
time, versus the discrete label of the tower of field excitations. For example, working
in position space, we could try to construct SFT as standard Field Theory in the
“stringy” spacetime $T^2_{(R_1, R_2)} \times T^2_{(\tilde{R}_1, \tilde{R}_2)} = S^1_{R_1} \times S^1_{R_2} \times S^1_{1/R_1} \times S^1_{1/R_2}$, since $K_0$ becomes
simply a Laplace operator in the doubled space-time,

$$K_0 = -\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial \tilde{x}_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial \tilde{x}_2^2} \right), \quad (2.6)$$

where $(x_1, x_2)$ parametrize the torus $T^2_{(R_1, R_2)}$ and $(\tilde{x}_1, \tilde{x}_2)$ parametrize the T-dual torus.

However, this is not quite correct, because the new winding modes enter the level matching
condition as

$$(n_1 \ell_1 + n_2 \ell_2) + (L'_0 - T'_0)_L + (L_0 - T_0)_T + (L_0 - T_0)_{\text{ghost}} = 0. \quad (2.7)$$

If we do not want to place additional constraints on the massless spectrum of $D_T$ fields
appearing in (2.3) in the decompactification limit, then we may saturate the constraint
within the zero mode sector: $n_1 \ell_1 + n_2 \ell_2 = 0$. That is, we look at vertex operators of the form

$$V_{\text{massless}} = V_{1+1} \, e^{ip_L X_L} e^{ip_R X_R}$$

where $V_{1+1}$ is a vertex operator for a massless state of the non-compact theory, satisfying
(2.3), and $p^i_{L(R)} = \frac{1}{2} \left( \frac{n_i}{R_i} \pm \ell_i R_i \right)$. In position space, we find that level matching is realized
at the massless level as an interesting “chirality” condition:

$$\left( \frac{\partial^2}{\partial x_1 \partial \tilde{x}_1} + \frac{\partial^2}{\partial x_2 \partial \tilde{x}_2} \right) \Psi_{\text{massless}} = 0. \quad (2.8)$$

\[ A \text{ good example is a } c = 1 \text{ model at the self-dual radius, which may be interpreted as a string in a circle, or as a string in a small } S^3. \]
Notice that, it is only for those fields saturating non-compact level matching that the condition on the longitudinal zero modes takes this suggestive form. In any case, it is clear that the determinant of the operator $K_0$, in the space of functions satisfying the constraint $(2.8)$, still fails to give the correct answer $(1.1)$. A direct inspection of $(1.1)$ reveals that we need to decouple the field degrees of freedom in the torus $S^1_{R_1} \times S^1_{R_2}$ from those propagating in $S^1_{R_1} \times S^1_{1/R_2}$, with T-duality acting only on the second circle. That is, we need to enforce the decomposition:

$$\Psi_{\text{massless}}(x_1, x_2, \tilde{x}_1, \tilde{x}_2) = \Psi(x_1, x_2) + \Psi(x_1, \tilde{x}_2).$$

$(2.9)$

This does not follow from $(2.8)$ unless the level matching condition is somehow further reduced to $\ell_2 n_2 = 0$, by separately setting $\ell_1 = 0$. Fortunately, there is a stringy mechanism for achieving just that. This is the subject of the next subsection.

2.1. Coset extensions

On general grounds, a (perhaps infinite) set of free fields has a one loop partition function given by a determinant of some kinetic kernel $K$. We can then write down a Schwinger representation of the form

$$\log Z_{\text{target}} = -\frac{1}{2} \text{Tr} (-1)^F \log (K) = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} (-1)^F e^{-s K}$$

$(2.10)$

conveniently regularized (for example, using a zeta-function procedure), so that we obtain an ultraviolet and infrared finite quantity which we may compare with the string computation. Indeed, eq. $(2.10)$ is reminiscent of a string partition function

$$\log Z_{\text{target}} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} Z(\tau, \bar{\tau}),$$

$(2.11)$

where, essentially $Z(\tau, \bar{\tau}) \sim \text{Tr} q^{L_0} \bar{q}^{\bar{L}_0}$, and the trace here may contain various GSO projections. There are, however, two important differences between $(2.10)$ and $(2.11)$. The CFT partition function is integrated over a fundamental domain $\mathcal{F}$: $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$, $|\tau| \geq 1$ of the genus one modular group and, in particular, there is no clear notion of ultraviolet region in such moduli space. The other difference is the presence of stringy states without particle analog, such as winding modes around compact dimensions, which nevertheless we may try to incorporate either as extra fields in the spectrum of $K$, as in the previous
paragraph, or as additional “small” dimensions, as in (2.6). Clearly, the first difference is the most important one, because it implies that the free SFT measure seems to be non-gaussian.

The resolution of this puzzle is quite interesting. By a well known mechanism [14][15], these two “stringy” features tend to cancel each other, at least partially. If the string theory is sufficiently regular, we can trade the sum over one set of winding modes by an extension of the integration region from the fundamental domain $F$ to the strip $S$: $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$, $\tau_2 \geq 0$. Technically, we just perform an extension of (2.11) by cosets of the translation group $\tau \rightarrow \tau + 1$. In this way we can tile the strip $S$ with a set of $SL(2, \mathbb{Z})$ transforms of $F$, labeled by two coprime integers $(c, d) = 1$, of the form $\gamma(c, d)(\tau) = \frac{a\tau + b}{c\tau + d}$, where $a$ and $b$ are determined by the condition $ad - bc = 1$. We then have $S = \bigcup_{\gamma(c, d)}(F)$, and the change of variables in the integrand absorbs one of the winding sums, because a pair of integers can be replaced by a single integer, together with a pair of coprime integers. The basic identity is the following

$$\int_{F} \frac{d^2\tau}{\tau_2^2} F(\tau, \bar{\tau}) \sum_{(\ell, \ell') \neq (0,0)} 2\pi R e^{-\frac{\pi b_2^2}{\tau_2} |\ell' \tau + \ell|^2} = \int_{S} \frac{d^2\tau}{\tau_2^2} F(\tau, \bar{\tau}) \sum_{\ell \neq 0} 2\pi R e^{-\frac{\pi b_2^2}{\tau_2} \ell^2}, \quad (2.12)$$

where $F(\tau, \bar{\tau})$ is modular invariant. The theta function on the left hand side of (2.12) is the classical partition function of harmonic maps from the world-sheet torus to the target circle $S^1_1$, while we find the particle analog on the right hand side. Accordingly, all integers $\ell, \ell'$ represent winding numbers. We can pass to a hamiltonian representation, based on the mixed momentum-winding labels in (2.3), by formally choosing a “timelike” homology cycle on the torus, and applying a Poisson resummation with respect to the corresponding winding numbers.

The regularity conditions ensuring the validity of this formula are essentially the absence of infrared divergences driven by physical or unphysical tachyons, which would invalidate the exchange of sums and integrals necessary to prove (2.12) (see [16]). The distinction between the two kinds of tachyons is of technical nature. Physical tachyons satisfy the non-compact level matching condition:

$$(L_0 - \overline{L}_0)_L + (L_0 - \overline{L}_0)_T + (L_0 - \overline{L}_0)_{\text{ghost}} = 0, \quad (2.13)$$

and they produce infrared instabilities in the non-compact theory. On the other hand, unphysical tachyons are not present in the non-compact theory either because they do not
satisfy the non-compact condition (2.13), for example the tachyon in the bosonic sector in heterotic (0,\(\ast\)) models, or because they are projected out by GSO projections, as in the case of the NS tachyonic ground state in (1,\(\ast\)) heterotic strings. However these unphysical tachyons may satisfy the finite volume version (2.7) of the level matching condition with \(n_1\ell_1 + n_2\ell_2 \neq 0\). The state resulting from this coupling of the unphysical tachyon with a winding-momentum state has a \(M^2_{\text{eff}}\) which diverges in the decompactification limit but that can vanish or even become negative at some finite values of the compactification radii, rendering the integral (2.11) infrared divergent. Whenever there are unphysical tachyons (2.11) has to be defined with an an appropriate integration prescription, namely one should perform the \(\tau_1\) integral first. However, after the transformation to the strip (2.12), this prescription turns into a complicated and not very useful integration rule over \(S\). In practice, (2.12) holds only for sufficiently large \(R\), and singularities may develop at critical values of the radius, when the winding unphysical tachyon states develop a vanishing effective mass \(M^2_{\text{eff}} = \sum_i (n_i^2/R_i^2 + \ell_i^2 R_i^2) + M^2 = 0\), with \(M^2\) given by (2.2). A well known example of this phenomenon is the standard Hagedorn thermodynamic singularity, in which the corresponding state becomes tachyonic in a whole region of the compactification moduli. In other cases the state just becomes massless at one point and bounces back into the \(M^2_{\text{eff}} > 0\) region, producing singularities in the correlation functions or its derivatives with respect to the moduli [17]. We will encounter examples of this latter type in section 3, and a more detailed discussion can be found in Appendix B.

Another condition on (2.12) is the restriction to non-vanishing winding number in the sums. It corresponds to the subtraction of the non-compact expression (the formal limit \(R \to \infty\) or, in other words, the non-compact vacuum energy, with \(R\) interpreted as euclidean time). This is important for the right hand side of (2.12) to be well defined in the \((\tau_2 \sim 0)\) ultraviolet region.  

The important point is that the integrals over the strip are reminiscent of (2.10), with \(\tau_2\) being proportional to the proper time Schwinger parameter, and the integral over \(\tau_1\)

\[^3\] The coupling of the NS tachyon with winding-momentum modes is ubiquitous in toroidal compactifications breaking space-time supersymmetry.

\[^4\] The present subtraction is rather natural in the context of string thermodynamics. It is well known that the non-vanishing vacuum energy of closed strings cannot be written in field theoretical terms. Of course, this subtlety is trivial for supersymmetric strings with vanishing vacuum energy.
enforcing the level matching condition. We then conclude that, modulo appropriate regularizations and vacuum subtractions, we can write down a standard gaussian measure for free SFT, only after we have disentangled precisely one set of winding modes. Thus, there is a tension between modular invariance and gaussian measures, due to the “temporal” winding states. This mechanism provides an explanation of the doubling observed in (1.1), where only the relative T-duality between the two cycles (mirror symmetry) is relevant. We now understand that, in order to write a determinant (leading to the standard Dedekind function), we must use up one set of winding modes (say around $S_{R_1}^1$) in going from the $\mathcal{F}$ representation to the $\mathcal{S}$ representation.

2.2. Doubling of vacuum partition functions

In order to make these observations more precise, it is convenient to separate the “transverse” CFT from the longitudinal one and various ghost systems, which cancel each other in the partition function (up to discrete states of measure zero), and the contribution of the zero modes. We will suppose throughout this section that there is no coupling between the transverse partition function and the longitudinal zero modes. Such a coupling will appear in the next section when we allow non-trivial boundary conditions of the target fermions on the target torus. In what follows, we consider vacuum (periodic) boundary conditions for all space-time fields. The resulting structure is ($q = e^{2\pi i \tau}$)

$$\log Z_{\text{target}} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \Lambda_L(\tau, \bar{\tau}) \Lambda_{\text{ghost}}(\tau, \bar{\tau}) \Lambda_T(\tau, \bar{\tau})$$

$$= \frac{1}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \Lambda_T(\tau, \bar{\tau}) \tau_2 \sum_{n_1, \ell_1} q^{\frac{1}{4} (\frac{n_1}{R_1} + \ell_1 R_1)^2} \bar{q}^{\frac{1}{4} (\frac{n_1}{R_1} - \ell_1 R_1)^2} \sum_{n_2, \ell_2} q^{\frac{1}{4} (\frac{n_2}{R_2} + \ell_2 R_2)^2} \bar{q}^{\frac{1}{4} (\frac{n_2}{R_2} - \ell_2 R_2)^2}$$

(2.14)

where $\Lambda_T(\tau, \bar{\tau})$ is the modular invariant non-compact transverse partition function. It is constructed as a combination of the holomorphic traces $\text{Tr}_{\mathcal{H}_T} q^{L_0-a}$ and $\text{Tr}_{\mathcal{H}_T} (-1)^F q^{L_0-a}$, depending on the appropriate GSO projections, and similarly for the right moving CFT. In particular, for the case of the $(2, *)$ strings, the transverse partition function is purely holomorphic, and thus has the general form

$$\Lambda_T(q)_{N=2} = D_T + C J(q),$$

(2.15)

5 The normal ordering constant $a$ is the standard intercept, $a = 1$ for a purely bosonic sector, and $a = 0, 1/2$ in the Ramond and Neveu-Schwarz sectors respectively.
where $J(q)$ is the unique holomorphic modular invariant function with a single pole at the origin and no zero mode, and $D_T$ is the effective number of degrees of freedom. For example, for the (2,0) string both constants are non-trivial, whereas in the case of the (2,1) string leading to the critical bosonic world-sheet, $C = 0$ and $D_T = 24$. For supersymmetric target worldsheets, the non-compact transverse partition function vanishes by supersymmetry.

In order to use (2.12) we have to transform from the mixed winding-momentum representation of (2.14) to a complete winding representation, by an appropriate Poisson resummation. The result, after going back to a mixed representation in the strip is:

$$\log Z_{\text{target}} + C.T. = \frac{1}{2} \sum_{n_1,n_2,\ell_2} \int_0^\infty \frac{d\tau_2}{\tau_2} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Lambda_T(\tau, \bar{\tau}) e^{2\pi i n_1 \tau_1 n_2 \ell_2} \right) e^{-\pi \tau_2 \left(\frac{n_1^2}{\tau_1^2} + \frac{n_2^2}{\tau_2^2} + \ell_2^2 R_2^2\right)}$$

(2.16)

where $C.T.$ stands for various counterterms, whose structure we will detail below. In (2.16), the $\tau_1$ integral enforces the level matching condition (2.7), without the “temporal” winding ($\ell_1 = 0$), which has also dissapeared from the heat kernel. As a result, any space-time interpretation involves a single circle of radius $R_1$, in agreement with our comments above. In general, the partition function does not factorize in a natural way. If we want to make more precise statements, it is necessary to further truncate the partition sum in (2.16).

For example, if we restrict the loop trace to the states satisfying the non-compact level matching condition (2.13), then we are effectively projecting

$$\Lambda_T(\tau, \bar{\tau}) \rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Lambda_T(\tau, \bar{\tau}).$$

(2.17)

For (2,*) strings, with holomorphic partition function, this projection simply replaces $\Lambda_T(\tau)$ by the effective number of massless fields $D_T$. In more general cases we are essentially inserting the ultralocal kernel

$$\sum_f e^{-\pi \tau_2 M_f^2},$$

where the index $f$ labels the fields satisfying the non-compact level matching constraint.\footnote{Notice that the projection (2.17) eliminates off-shell tachyons from the partition sum.}

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With these truncations, the effect of the $\tau_1$ integral in (2.16) is to set $n_2 \ell_2 = 0$, which splits the partition function into three terms:

$$
\frac{1}{2} \sum f \int_0^\infty \frac{d\tau_2}{\tau_2} \left[ \sum_{n_1, n_2} e^{-\pi \tau_2 \left( \frac{n_2^2}{R_2^2} + \frac{n_2^2 + \ell_2^2 R_2^2}{M_f^2} \right)} \right] + \sum_{n_1, \ell_2} e^{-\pi \tau_2 \left( \frac{n_2^2}{R_2^2} + \ell_2^2 R_2^2 + M_f^2 \right)} - \sum_{n_1} e^{-\pi \tau_2 \left( \frac{n_2^2}{R_2^2} + M_f^2 \right)} \right].
$$

(2.18)

In particular, if we perform a truncation to the massless 1 + 1 fields, we finally obtain the desired result (1.1):

$$
\log Z_{\text{massless}} = -\frac{D_T}{2} \log \left[ \prod_{n_1, n_2} \left( \frac{n_2^2}{R_1^2} + \frac{n_2^2}{R_2^2} \right) \prod_{n_1, \ell_2} \left( \frac{n_2^2}{R_1^2} + \ell_2^2 R_2^2 \right) \prod_n \left( \frac{R_1^2}{n^2} \right) \right]
$$

(2.19)

where we have discarded the zero mode terms in the sums in (2.18), and defined the infinite products by zeta-function regularization. Note that the factorization exhibited in (2.19) is exact, in the sense that it involves no truncations, when the partition function is of the form (2.15) with $C = 0$. This is the case of the (2, 1) models corresponding to the bosonic and type II worldsheet in the vacuum (periodic) sector. In the next section we show that the exact factorization is generally lost when considering more general boundary conditions for the target fermions.

We have been deliberately cavalier regarding regularizations in order to exhibit more clearly the important points. There are however several subtractions of infrared and ultraviolet type involved. These infinite counterterms relate our regularization and the ones implicit in the calculation of [7]. There is an infrared divergence already present in the stringy expression (2.14), when the string spectrum is purely massless, coming from the trivial term in the zero mode sums. This infinity is independent of the moduli $T, U$, and it is normally subtracted by the replacement\footnote{A somewhat disturbing feature of this regularization is its lack of modular invariance.} $\Lambda_T(\tau, \bar{\tau}) \to \Lambda_T(\tau, \bar{\tau}) - \tau_2$ in (2.13). There are analogous logarithmic divergences, in this case picking both infrared and ultraviolet components, in the expression to be integrated over the strip in (2.16). These terms are subtracted from (2.19) in order to define the determinants by the zeta function procedure:

$$
\text{Tr}' \log K = -\frac{d}{ds} \int_0^\infty dt \left. t^{s-1} \text{Tr}' e^{-t K} \right|_{s=0}.
$$
Finally, there is a vacuum energy subtraction in (2.12), in the complete winding representation. Altogether, the complete counterterm to be added to the zeta-function regularization is purely ultraviolet and has the form

\[
C.T. = \frac{1}{2} \int_{\mathcal{F}} d^2\tau \frac{d^2\bar{\tau}}{\tau_2^2} (R_1 f_{R_2}(\tau, \bar{\tau}) - \tau_2),
\]

(2.20)

where \(f_{R_2}(\tau, \bar{\tau})\) denotes the integrand of the partition function corresponding to the “hamiltonian” target space \(\mathbf{R} \times S^1_{R_2}\). The obvious asymmetry between \(R_1\) and \(R_2\) in the structure of (2.20) is related to our choice of \(R_1\) as “time”, whose winding modes have been disentangled in going from \(\mathcal{F}\) to \(\mathcal{S}\) using (2.12). The counterterm (2.20) ensures the symmetry of the finite piece (2.19). In the supersymmetric case, when \(D_T = 0\) and we have vacuum boundary conditions, \(f_{R_2}(\tau, \bar{\tau})\) is just the vanishing vacuum energy, taking \(R_1\) as the euclidean time direction. In general, when the target theory has \(D_T \neq 0\) or we consider non periodic boundary conditions in the \(S^1_{R_2}\) circle, as in the next section, then the subtraction is non trivial. A closely related quantity is the integral of the same function over the fundamental domain \(\mathcal{F}\), which can be directly extracted from the \(R_1 \to \infty\) limit of (2.19), with the result

\[
\lim_{R_1 \to \infty} \frac{1}{2\pi R_1} \log Z_{\text{target}} = D_T \frac{1}{12} \left(\frac{R_2}{R_2} + \frac{1}{R_2}\right).
\]

(2.21)

This has the appropriate form to be interpreted as a Casimir energy contribution \(-\frac{\pi}{6} \frac{1}{2\pi R_2}\) per degree of freedom in the space \(S^1_{R_2}\), plus the same contribution from \(S^1_{1/R_2}\). For \(D_T = 1\) (2.21) coincides with the vacuum energy of the \(c = 1\) matrix model [18].

3. Target space spin structures

We have shown in the previous section that world-sheet complexification in the sense of eq. (1.1) is an exact property of (2, *) models with \(C = 0\) in (2.15), and an approximate property, in the sense of the massless truncation (2.17) in all other cases. This conclusion holds under the condition that no correlation exists between the longitudinal zero mode labels and “transverse” quantum numbers is (2.14). This means that all space-time fields have the same vacuum boundary conditions as the bosons, namely periodic around both \(R_1\) and \(R_2\) target cycles. In what follows we examine the situation where target fermions have antiperiodic boundary conditions around one or both cycles. In other words, we want
to obtain the torus spin structures of the target space string. Because of fermion antiperiodicity, we are led to consider finite temperature boundary conditions in the underlying (2, 1) model. Obtaining in this way the correct target space partition function as a function of the target complex parameter $T$ is an interesting check of the stringy construction of world-sheets. It also provides a testing ground of the complexification phenomenon beyond the vacuum sector studied in the previous section. We will find that, in general, the factorization of $T$ and $U$ dependence is lost as an exact property, even in models with a non-compact partition function (2.13) with $C = 0$. A truncation to massless fields in the spirit of (2.17) is still possible, but the procedure is somewhat ambiguous, and the factorization depends on the particular projection.

In order to exhibit the details, let us consider the simplest fermionic target string; the type-II model, which is constructed as a (2, 1) background in terms of a left moving $\hat{c} = 8$ SCFT with eight fermions together with eight bosons compactified in the root lattice of $E_8$. In this theory, the target space statistics is completely determined by the world-sheet parity of the $N = 1$ left-moving sector. The oscillator partition function is then given by [3]:

$$\Lambda^{s'}_s (\tau) = C(s', s) \frac{E_4(\tau)}{\eta^{12}(\tau)} \frac{1}{2} \vartheta^4 \left[ \begin{array}{c} s' \\ s \end{array} \right]_4 (0 | \tau),$$

with $E_4(\tau)$ the weight 4 Eisenstein series, which equals the theta function for the $E_8$ root lattice [19], and $C(s', s)$ the phases giving the desired GSO projection ($C(\frac{1}{2}, \frac{1}{2}) = 0, C(0, 0) = -C(0, \frac{1}{2}) = -C(\frac{1}{2}, 0) = 1$).

A modular invariant prescription for finite temperature boundary conditions was introduced in ref. [3]. Target fermions become antiperiodic if the world-sheet spin structures $(s', s)$ are coupled to the winding numbers $(\ell, \ell')$ around the corresponding “thermal” circle by the phases

$$U^{s's'}_{\ell \ell'} = (-1)^{2s\ell + 2s'\ell' + \ell \ell'}.$$  

These phases can be motivated as modular invariant extensions of the standard phases in finite temperature Feynmann diagrams. It is important to notice that this procedure only yields non-chiral spin structures for the target fermions, namely both left and right movers in the target are rendered antiperiodic by the phases (3.2). We shall postpone for the moment the discussion of chiral spin structures.

The general form of the partition function integrand (2.11) is

$$Z(\tau, \bar{\tau}) = \sum_{\{s\}} \Lambda^{s'}_s (\tau) \sum_{\{\ell\}} U^{s_1 s'_1}_{\ell_1 \ell'_1} U^{s_2 s'_2}_{\ell_2 \ell'_2} e^{-S_{cl}(\ell_1, \ell'_1; \ell_2, \ell'_2)} = \sum_{\{s\}} Z^{cl}_{s, s'} (\tau, \bar{\tau}) \Lambda^{s'}_s (\tau),$$

where
where we have introduced a different set of phases for each target circle. In this formula, when the cycle $S_{k_{i}}$ is periodic, we set $U_{\ell_{i}\ell'_{i}} \rightarrow 1$ and $s_{j} = s$, $s'_{j} = s'$ for $j \neq i$. When both cycles are antiperiodic (the (NS,NS) in the target), we set $s_{1} = s_{2} = s$, $s'_{1} = s'_{2} = s'$.

Finally, the classical action $S_{cl}$ is just the contribution of the classical embeddings of the world-sheet torus onto the target torus with winding numbers $(\ell_{1}, \ell'_{1}; \ell_{2}, \ell'_{2})$

$$S_{cl} = \frac{\pi}{\tau_{2}} (\bar{\tau} \ell_{a} + \ell'_{a})(g + b)_{ab}(\tau \ell_{b} + \ell'_{b}).$$

When the target torus is characterized by the complex moduli $T$ and $U$ we have

$$e^{-S_{cl}} = U_{2} \exp \left[ -\frac{\pi U_{2}}{T_{2} \tau_{2}} \left[ |T|^2 |\ell_{1}\tau + \ell'_{1}|^2 + |\ell_{2}\tau + \ell'_{2}|^2 + 2T_{1} Re(\ell_{1}\tau + \ell'_{1})(\ell_{2}\tau + \ell'_{2}) \right] \right] (3.5)$$

where $U_{2}$, the volume of the target torus, arises from the integration of the zero modes in the path integral.

We have explained in the previous section how to go from the integral over the fundamental domain $F$ to the integral over the strip $S$ by effectively setting one winding number to zero. This is also going to work when the real parts of both $T$ and $U$ are switched on, as it is the case now. So if we set $\ell_{1} = 0$ in $(3.3)$ and substitute in $(3.3)$ we end up with the following expression for the classical part of the partition function in $S$

$$Z_{s_{1} s'_{1}, s_{2} s'_{2}}^{cl}(\tau, \bar{\tau}) = U_{2} \sum_{\{\ell\}} \exp \left[ -\frac{\pi U_{2}}{T_{2} \tau_{2}} \left[ |T|^2 |\ell'_{1} + \ell_{2}\tau|^{2} + 2T_{1} Re(\ell_{1}\tau + \ell'_{1})(\ell_{2}\tau + \ell'_{2}) \right] \right]$$

$$\times e^{2\pi i(\ell'_{1} s_{1} + \ell_{2} s_{2} + \ell'_{2} s'_{2} + \frac{1}{2} \ell_{2} \ell'_{2})}.$$ (3.5)

We can treat all the cases at the same time by remembering that whenever the target space fermions have periodic boundary conditions along the $i$-th cycle ($i = 1, 2$) we have to set $s_{i}, s'_{i} \rightarrow 0$ and $s_{i} + \frac{\ell_{i}}{2} \rightarrow 0$, while in the case (NS,NS) we have $s_{i} = s$, $s'_{i} = s'$. After performing a Poisson resummation in $\ell'_{i}$ to go back to the mixed winding-momentum representation we find

$$Z_{s_{1} s'_{1}, s_{2} s'_{2}}^{cl}(\tau, \bar{\tau}) = \tau_{2} \sum_{n_{1}, n_{2}, \ell_{2}} (-1)^{2\ell_{2} s_{2}} q^{-\ell_{2}(n_{2} + s'_{2} + \frac{1}{2} \ell_{2})} (q\bar{q})^{\frac{1}{2} T_{2} \tau_{2}} |n_{1} + s'_{1} - T(n_{2} + s'_{2} + \frac{1}{2} \ell_{2}) + U/\ell_{2}|^{2}$$

in terms of which we can write $Z_{target}$ as

$$\log Z_{target} = \frac{1}{2} \int_{S} \frac{d^{2}\tau}{\tau_{2}} \sum_{\{s\}} Z_{s_{1} s'_{1}, s_{2} s'_{2}}^{cl}(\tau, \bar{\tau}) S_{s_{1} s'_{1}}^{\prime}(\tau) + \text{counterterms}.$$
The vacuum energy counterterm depends on whether the \( S_{R_2}^1 \) cycle is supersymmetric or not. The density \( R_{1_2}(\tau, \bar{\tau}) \) in (2.20) must be replaced by

\[
\sum_{\{s\}} \Lambda_{s}^{s'}(\tau) \sum_{\ell_2, \ell_2'} U_{\ell_2 \ell_2'} e^{-S_{cl}(0,0;\ell_2,\ell_2')},
\]

if the second cycle is periodic, then \( U_{\ell_2 \ell_2'} \to 1 \) and the vacuum subtraction vanishes because of supersymmetry: \( \sum_{\{s\}} \Lambda_{s}^{s'}(\tau) = 0 \).

It will be useful to introduce the constants

\[
Q_{s}^{s'} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Lambda_{s}^{s'}(\tau),
\]

which measure the number of physical states in each spin structure for the non-compact theory; Jacobi’s *aequatio* translates into the identity

\[
Q_{0}^{0} + Q_{0}^{\frac{1}{2}} + Q_{\frac{1}{2}}^{0} = 0. \tag{3.8}
\]

The number of bosonic degrees of freedom for the type-II model is then just given by \( -Q_{0}^{\frac{1}{2}} = 8 \).

In general, the resulting integral after \( \tau_1 \) integration is still too complicated, and there is no obvious factorization of \( T \) and \( U \) dependence. We will nevertheless proceed as in the previous section, defining the massless truncation of the partition function by the projection \( \Lambda_{s}^{s'} \to Q_{s}^{s'} \) in (3.6), which is analogous to (2.17) within each spin structure. Doing so and integrating over \( \tau_1 \) leads to the analog of formula (2.18):

\[
\log Z_{\text{massless}} = \frac{1}{2} \int_{0}^{\infty} \frac{d\tau_2}{\tau_2} \sum_{\{s\}} \sum_{n_1, \ell_2} Q_{s}^{s'} (-1)^{2\ell_2 s_2} (q \bar{q})^{A \bar{A}} \delta_{\ell_2 (n_2 + s_2 + \ell_2/2),0} \tag{3.9}
\]

where \( A \) is given by

\[
A = \frac{1}{2\sqrt{T_2 U_2}} \left[ n_1 + s_1' - T (n_2 + s_2' + \frac{\ell_2}{2}) + U \ell_2 \right].
\]

The remaining integral can be reduced to a sum of logarithms of infinite products which can be computed using zeta-function regularization (the details of the computation are worked out in Appendix A). The final result in the four sectors can be written as

\[
\begin{align*}
Z_{(R,R)} &= 0 \times 1 \\
Z_{(NS,R)} &= [f_2(T)f_2(U)]^4 \\
Z_{(R,NS)} &= [f_4(T)f_4(2U)]^4 \tag{3.10} \\
Z_{(NS,NS)} &= (qTq_{2U})^{-\frac{1}{8}} [f_3(T)f_3(2U)]^4,
\end{align*}
\]

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where \( f_i(T) \) is given in terms of standard Jacobi theta functions:

\[
f_i(T) \equiv \frac{\theta_i(0|T)}{\sqrt{T_2\eta^3(T)}}.
\]

We have extracted the holomorphic square root of \( Z_{\text{massless}} \) in (3.10) in order to exhibit only the chiral structures. The Ramond sector zero mode has been included by hand in the first line of (3.10), since the (2, 1) string torus diagram only sums up the non-zero modes in target space, whose overall contribution is one, due to the supersymmetric boson-fermion cancellation, i.e. \( D_T = 0 \).

It is interesting that we get the correct \( T \)-dependent terms; the modular invariant combinations of Jacobi and Dedekind functions with the right multiplicity to be interpreted as building blocks of the target type-II worldsheet\(^8\). The method used to introduce antiperiodic boundary conditions in the target fermions always produces left-right symmetric structures in the target, i.e. \( |f_i(T)|^2 \). A fully stringy construction of more general asymmetric terms like \( f_i(T)f_j(T) \) is a more complicated issue on which we offer some comments in the next subsection. In general, additive structures in \( Z_{\text{target}} \) are beyond the reach of (2, 1) string perturbation theory, such as the rules for combining the different target spin structures (target GSO projection) or the global structure of the target bosons (effects due to compactness of the target bosons are of order \( e^{-1/\alpha'} \).

Regarding the \( U \) dependence, (3.10) has a second piece that only in the (R,R) and (NS,R) sectors can be interpreted as the corresponding partition function in the mirror torus. There is a simple rule to get these \( U \)-terms; in the sectors \((\ast,\text{R})\) they are obtained from the \( T \)-terms by a \( T \)-duality transformation in the second cycle, \( T \rightarrow U \). For the \((\ast,\text{NS})\) cases the second term is obtained instead by \( \beta \)-duality, \( T \rightarrow 2U \) (see Appendix B). In addition, it seems that whenever we have different boundary conditions in the two cycles, the result depends on what cycle we choose to disentangle the windings to pass from the fundamental domain to the strip. The condition for the choice to be irrelevant is that \( f_2(U) \) should be invariant under the Atkin-Lehner transformation \( [20], f_2(U) = f_2(-1/2U) \), condition which is not fulfilled in our case. As a result, the \( U \)-dependence seems to be sensitive to the order in which we truncate the theory.

---

\(^8\) Notice, however, the occurrence of a target modular anomaly in the (NS,NS) sector, \( (q_T q_{2U})^{-1/8} \), whose \( T \)-dependent term does not disappear in the field theory limit \( \alpha' \rightarrow 0 \).
These results may seem rather puzzling at first sight. It is not difficult to convince oneself that the $\mathcal{F}$-representation of the target partition function (3.3) is invariant under T-duality in the cycles with R boundary conditions and $\beta$-duality in those with NS. On the other hand, our expression (3.10) only has one of those invariances, while the one associated with the cycle used to disentangle the winding modes is broken. The origin of this breaking can actually be traced back to the projection $\Lambda_s^s' \rightarrow Q_s^s'$ made in order to compute the integral over the strip. This projection retains only the left-right symmetric states of the non-compact theory in the transverse partition function, and the final result is sensitive to the asymmetry between $R_1$ and $R_2$ cycles in the mixed spin structures. On the other hand, if we were to compute the full integral over $\mathcal{F}$ without any projection we would find expressions diverging logarithmically as the radius of any cycle with antiperiodic boundary conditions approaches the self-dual value under $\beta$-duality. Only the contribution of the (R,R) spin structure vanishes identically even before integration and therefore will be regular for all value of the moduli. The origin of these divergences lies in the existence of some winding state becoming massless at special values of the moduli (the off-shell tachyons mentioned in section 2). As we discuss in more detail in Appendix B, a careful analysis of the $\mathcal{F}$ and the projected $\mathcal{S}$ representation together with the type of singularities encountered in the modular invariant expression, implies that the result of performing the full integral over $\mathcal{F}$ of the partition function (3.3) does not factorize into two parts depending only on $T$ and $U$ respectively. We would obtain the factorized contribution (3.10) plus a collection of non-factorizable terms $F(T, U)$ which diverge logarithmically on some codimension one submanifolds of the target moduli space.

The situation is reminiscent of the one in (2,0) models. The computation of $\log Z_{\text{target}}$ in ref. [7] gives, in addition to the contribution (1.1) coming from the 1+1 massless states, a term $-\log |J(T) - J(U)|^2$ which spoils factorization of the integrated partition function. This latter term represents the contribution of states which do not satisfy the non-compact level matching condition (2.13). Here we also find a logarithmic divergence triggered by winding states becoming massless whenever one of the radii reaches the self-dual value under T-duality, $R_i = 1$ [17].

Incidentally, the result (3.10) in the (NS,NS) sector is manifestly $\beta$-duality invariant in both cycles except for the anomalous term $(q_T q_2 U)^{-1/8}$, which should be then interpreted as an artifact of the truncation.
3.1. Chiral spin structures and the target heterotic string

In this section we present some observations on the problem of constructing fully chiral spin structures for the target string. As we already pointed out previously, the standard finite-temperature phases (3.2) render both left and right moving target fermions antiperiodic around a given cycle. The problem of handling chiral spin structures is even more relevant in the case of the (2, 1) model which reproduces the world-sheet theory of the (1, 0) heterotic string. In this case both matter and gauge fermions have only one chirality, so there is no symmetric sector which could be obtained by using the phases $U_{\ell\ell'}^{ss'}$.

The correct construction must treat the zero modes chirally in the target space sense, and it is natural to expect that a more sophisticated asymmetric orbifold (cf. [21]) is needed in this case. We can try to guess some pieces of the full answer from the structure of eq. (3.9). Focusing on the field theory limit for simplicity, we set $\ell_2 \to 0$ and concentrate on the $T$-dependence (all $U$-dependence is reduced to an additive term $\log U_2$). Then, the generalization of (3.9) leading to chiral target space structures has the form

$$\log Z_{\text{massless}} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2} \sum_{\{s\}} \sum' n_i Q_{s'}(q\bar{q})^{A_LA_R}$$ (3.11)

where now $A_L$ and $A_R$ are not in general complex conjugate of each other, but depend on the boundary conditions for the target left and right moving fermions,

$$A_L = \frac{1}{2\sqrt{T_2 U_2}} [n_1 + s'_1 - T(n_2 + s'_2)]$$
$$A_R = \frac{1}{2\sqrt{T_2 U_2}} [n_1 + t'_1 - T(n_2 + t'_2)].$$ (3.12)

Now $s_i, (t_i) \to 0$ whenever the left (right) moving fermions are periodic along the $i$-th cycle whereas in the case in which the left (right) moving fermions are antiperiodic in both cycles we will have $s_i = s, s'_i = s'$ ($t_i = s, t'_i = s'$). It is easy to check, using the formulae of Appendix A, that the computation of $\log Z_{\text{massless}}$ gives the correct chiral terms $f_i(T)f_j(T)$.

The situation is much more complicated if we retain the dependence on $U$, or in other words, if we work at a finite value of $\alpha'$. The reason being that when $\ell_2 \neq 0$, (3.9) contains a phase $(-1)^{2\ell_2 s_2}$ and therefore there is an ambiguity in how to split this phase into contributions from left and right moving target fermions. A related question is how
to promote the delta function imposing the level matching in (3.3) to a fully modular invariant integrand.

Another aspect of the subtleties with chiral target fields arises when considering the oscillator partition function of the (2, 1) model leading to the target heterotic string. In [5] [6] this vacuum was constructed as an orbifold of the type-IIB background. The worldsheet degrees of freedom of the (2,1) string are the complex fields \( Z^\mu = x^\mu + ix^{\mu+1} \), \( \Psi^\mu = \psi^\mu + i\psi^{\mu+1} \) (\( \mu = 0, 2 \)) and their corresponding right moving components, together with the \( N = 1 \) left moving superfields \( \Phi^i = y^i + \theta \lambda^i \) (\( i = 1, \ldots, 8 \)) in the internal \( \hat{c} = 8 \) SCFT. Let us compactify all open dimensions except \( x^1 \). The orbifold construction leading to the target (1,0) heterotic string is defined by (see [6])

\[
\begin{align*}
Z^\mu &\rightarrow Z^{\mu*} \\
\Psi^\mu &\rightarrow \Psi^{\mu*} \\
\Phi^i &\rightarrow -\Phi^i
\end{align*}
\] (3.13)

and the analog for \( \overline{Z}^\mu, \overline{\Psi}^\mu \). If we gauge the left moving current \( J = \partial x^1 + \partial x^3 \) we get the fermionic version of the \( SO(32) \) heterotic string, whereas if we take \( J = \partial x^1 + \partial y^1 \) what we find is the \( E_8 \times E_8 \) heterotic string at finite coupling.

We now compute the partition function for the orbifold associated with the \( SO(32) \) heterotic string. In the \( N = 2 \) right moving sector this was done by Mathur and Mukhi in [22]. What they found is that in each sector of boundary conditions the contribution from the matter fields is exactly canceled by the ghosts, giving a constant partition function. The computation on the right moving \( N = 1 \) part is more subtle because of the gauging of the null current \( J \). Here we have, in addition to the \( N = 1 \) ghost system \((b, c)\) and \((\beta, \gamma)\), a \((1,0)\) ghost associated with the \( U(1) \) gauge current \((t, \bar{t})\) and its \( N = 1 \) superpartner of spin \((1/2,1/2)\), \((f, \bar{f})\). Altogether, we have effectively two light-cones, and four sets of oscillators are canceled out. The zero modes in the \((x^1,x^3)\) plane are also eliminated against the integration over \( U(1) \) Wilson lines, and we are essentially left only with the transverse \( \hat{c} = 8 \) internal excitations. A somewhat delicate point is the determination the statistics of the target space fields in the space-time \((x^0, x^2)\). From the analysis of reference

\[\text{Our notation here follows that of [3]. } x^0, \ldots, x^3 \text{ are the coordinates in the dim=2C space-time with signature } (- - + +) \text{ and } y^i \text{ are the coordinates in the internal left moving } E_8 \text{ torus. In addition we have their fermionic partners } \psi^0, \ldots, \psi^3 \text{ and } \lambda^i.\]
it can be seen that in the untwisted sector the statistics is determined solely by the internal $\hat{c} = 8$ SCFT, whereas in the twisted sector the ground state in the $N = 2$ part is fermionic and therefore one has to take into account both left and right movers. After a rather simple computation one arrives at the following result:

$$Z_{SO(32)} = \left[ \frac{\theta_3^4 - \theta_4^4}{2\eta^{12}} \left( \frac{E_4 + \theta_3^4 \theta_4^4}{2} \right) - \frac{\theta_2^4}{2\eta^{12}} \left( \frac{E_4 - \theta_3^4 \theta_4^4}{2} \right) \right] - \frac{\theta_3^4 + \theta_4^4}{\eta^{12}} \left( \frac{\theta_3^4 \theta_2^4 + \theta_4^4 \theta_2^4}{2} \right) - \frac{1}{\eta^{12}} \left( \frac{\theta_3^4 \theta_2^4 - \theta_4^4 \theta_2^4}{2} \right). \quad (3.14)$$

The first two terms in the right-hand side correspond to the contribution from the untwisted sector while the last two come from the twisted sector. There is a relative factor of 2 in the twisted sector, from the total number of fixed points in the non-chiral part of the orbifold. The twisted internal bosons $y^i$ yield inverse powers of Jacobi theta functions, which can be transformed to the numerator using the identity $\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) = 2\eta(\tau)^3$. The corresponding factors of 2 cancel the chiral fixed point degeneracy ($\sqrt{2^8}$). Finally, we should multiply (3.14) by the solitonic partition function associated with the $(x^0, x^2)$ torus, $Z_{2,2}(T, U)$.

Although $Z_{2,2}(T, U)$ is apparently inert under (3.13), the structure of the massless states in (3.14) suggests that the chiral structure in target space has to be imposed on $Z_{2,2}(T, U)$ by hand, correlated with the twisting in (3.13). In the twisted sector it is easy to check that we have only 32 fermionic states, which correspond to the gauge fermions of the $SO(32)$ heterotic string. In fact, using some elementary properties of the Jacobi theta functions, it is possible to show that the whole twisted sector adds to a constant equal to $-32$. In the untwisted sector, however, the interpretation is not so straightforward. We find 8 bosons as one would expect (the standard $\theta_3^4 - \theta_4^4$ term), but there are no massless fermions (the term proportional to $\theta_2^4$ has no zero mode). This seems to be in conflict with the analysis of [6] in which it is found that after projecting onto the states invariant under the modding (3.13), we are left with 8 fermions with a definite chirality. A solution to

\[\text{It is interesting to note that a different sign convention is possible in the twisted sector, compatible with modular invariance, giving an overall positive constant which would be interpreted as the number of chiral bosons in the bosonic formulation of the heterotic world-sheet. For this to work, we have to consider non-compact $(x^1, x^3)$ directions, resulting in only one orbifold fixed point. This removes the relative factor of 2 in (3.14), thus producing a constant twisted partition function equal to +16.}\]
this apparent paradox would be that the oscillator part of the partition function (3.14) is not sensitive to the target space chirality of the fields, which should in turn be associated with selection rules in the $Z_{2,2}(T,U)$ solitonic terms. From this point of view, projecting out 8 chiral fermions in the untwisted sector results in the elimination of 8 non-chiral fermions from the partition function, the number we began with. This would be resolved by adjoining different solitonic factors to the two terms $E_4$ and $\theta_1^4 \theta_4^4$, thereby correlating the modding (3.13) in the directions $(x^1, x^3, y^i)$ with a “longitudinal” asymmetric orbifold in the $(x^0, x^2)$ target world-sheet. The question is to find a modular invariant procedure to do this.

With this in mind, we may put together a set of ad hoc rules, based on (3.12), to obtain the chiral terms in the target partition function. As we pointed out before, any determination of the relative signs between spin structures is beyond the perturbative analysis of the (2,1) heterotic string and can only be addressed by making use of non-perturbative information.

4. Concluding remarks

In the present paper we have studied some of the properties of the world-sheet theories emerging from the target space dynamics of (2,1) heterotic strings. In particular we have been specially concerned with the apparent doubling of the target world-sheet degrees of freedom. We have described the physical mechanism underlying this doubling in the vacuum sector of the partition function, and found it to be a rather generic feature of certain massless truncations of two-dimensional strings. The factorization is exact for the vacuum sector of the (2,1) models leading to the bosonic and type-II world-sheets. In the case of the (2,0) string such doubling happens only after performing the projection (2.17) onto the states that satisfy the non-compact level matching condition (2.13). We have also considered non-trivial boundary conditions for the target fermions of the target type-II model. In general, the $T-U$ factorization does not hold in this sector, whose behavior is similar to the (2,0) string. There are several “natural” massless projections which produce a $T-U$ factorized result, although it seems that the universality of the vacuum sector is lost. A partial discussion of the heterotic target string is offered, including the calculation of the oscillator partition function of the corresponding (2,1) asymmetric orbifold, and some remarks on the handling of chirality in target space.
Despite its limitations, the target world-sheet doubling found in the bosonic sector is rather significative. It is interesting to elaborate on its geometric interpretation. According to our target space analysis in section 2, the level matching condition is satisfied by propagating fields in the space $S^1_{R_1} \times S^1_{R_2} \times S^1_{1/R_2}$ with the additional chirality constraint,

$$\frac{\partial^2}{\partial x_2 \partial \tilde{x}_2} \Psi_{\text{massless}} = 0.$$  \hspace{1cm} (4.1)

So, we can only double space, and we still have to impose a chirality condition on the field excitations in this “stringy” space-time. Equation (4.1) is now solved by fields living in factorized Hilbert spaces

$$\mathcal{H} = \mathcal{H}(S^1_{R_2}) \otimes \mathcal{H}(S^1_{1/R_2}).$$  \hspace{1cm} (4.2)

This interpretation is also suggested by the Casimir energy in the “hamiltonian” manifold $\mathbb{R} \times S^1_{R_2}$, eq. (2.21). Because of conformal invariance in the target, the Hamiltonian evolution depends only on the combinations $T_2 = R_1/R_2$ and $U_2 = R_1R_2$. If we switch on the torsion, the complete “twisted” euclidean evolution operator on a cylindrical target space becomes

$$(q_T q_U)^{L_0 \text{target}} - D_T/24 \quad (\bar{q}_{\bar{T}} \bar{q}_{\bar{U}})^{\bar{L}_0 \text{target}} - D_T/24.$$  \hspace{1cm} (4.3)

We may use these operators as building blocks for an operator formalism sewing higher order target world-sheets, and also to insert external scattering states. We conclude that local operator insertions also have complexified moduli. Accordingly, the vertex operators of the target string must be integrated over the world-sheet moduli $z_i$ and the complex conjugated $\bar{z}_i$ independently. This is exactly the structure emerging from the analysis of generalized high-energy scattering saddle points, as explained in [11]!. Following [10], the high energy asymptotics of string scattering is dominated by saddle points in the complete world-sheet moduli space. In particular, there is a “dominant” location of the vertex operators on the world-sheet. It was pointed out by Witten in [11] that, by allowing both momentum and winding modes for the external particles, the most general saddle point configuration is obtained by varying independently the holomorphic positions of the vertex operators.

It is then natural to suspect that perhaps the (2, 1) construction gives directly the high energy phase of standard strings. It would be very interesting to make this correspondence more explicit. Notice that the high energy behavior is only determined by the bosonic sector of the colliding strings, because the saddle points only depend on the tachyonic part.
of the vertex operators $e^{ipX}$. Perhaps this explains why we find difficulties in implementing the complexification beyond the bosonic sector of the target space strings. A potentially interesting observation is the following: formally, the high energy limit of the target string world-sheet dynamics corresponds to the limit $\alpha'_\text{target} \sim \alpha'\lambda^2 \to \infty$, (see [10]), which naturally singles out the strong coupling limit of the $(2,1)$ model.

If we take the complexification idea seriously, we should understand how these results follow from a gauge fixing procedure of target space gravity. The required doubling of degrees of freedom has to do with the existence of a graviton-axion-dilaton system both in momentum variables, $\xi_{ab}\partial X^a\bar{\partial}X^b e^{ipX}$ and in winding variables, $\tilde{\xi}_{ab}\partial \tilde{X}^a\bar{\partial}\tilde{X}^b e^{i\tilde{p}\tilde{X}}$. In a weak field expansion, our previous remarks indicate that we only have one independent “winding” gravitational system, because the windings around one of the cycles have been used to write the target space dynamics as field theory dynamics. It would be interesting to make these remarks more precise. In addition, the complexification in (4.3) should be tested at higher orders in the target topological expansion.

An interesting problem that we have not dealt with here is the three-dimensional generalization (membrane world-volume). The complexification phenomenon in the sense of (4.3) is essentially two-dimensional. The reason being that the Hilbert space decoupling (4.2) only occurs in this case. For a higher dimensional toroidal compactification, the level matching condition saturated over zero modes becomes

$$\sum_{i=2}^{d} \frac{\partial^2}{\partial x_i \partial \tilde{x}_i} \Psi_{\text{massless}} = 0,$$

where we have already disentangled the “temporal” winding modes $\ell_1 = 0$. So, there is a certain chirality condition imposed over the spatial dependence of the fields, but the geometric interpretation is certainly more complicated than the two dimensional case.

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Appendix A. Computation of determinants

In this appendix we will detail the computation of the integral \((3.9)\). The first important thing to notice is that the phase \((-1)^{2\ell_2 s_2}\) is trivial; the Kronecker delta function enforces either \(\ell_2 = 0\) or \(\ell_2 = 2(n_2 + s'_2)\), so that \(2\ell_2 s_2 = 4s_2 s'_2 \mod 2\), which can only be odd if \(s_2 = s'_2 = \frac{1}{2}\). But this corresponds to the contribution of the odd spin structure which vanishes \((Q_{1/2}^1 = 0)\). Then we can forget about this phase in the subsequent computation.

The integral \((3.9)\) can then be easily written in terms of infinite products as

\[
\log Z_{\text{massless}} = -\frac{1}{2} \sum_{s,s'} Q_{s'}^s \log \left[ \prod_{n_1,n_2} \left| \frac{n_1 + s'_1 - T(n_2 + s'_2)}{T_2 U_2} \right|^2 \prod_{n_1,\ell_2} \left| \frac{n_1 + s'_1 - U\ell_2}{T_2 U_2} \right|^2 \right] + \frac{1}{2} \sum_{s,s'} Q_{s'}^s \delta_{s'_2,0} \log \prod_{n_1} \left( \frac{(n_1 + s'_1)^2}{T_2 U_2} \right) + \text{constants}
\]

where

\[
\ell_2^c = \begin{cases} \ell_2 & \text{in the (*,R) case} \\ 2(n_2 + s'_2) & \text{in the (*,NS) case} \end{cases}
\]

and the primes in the products indicate that we omit the zero mode whenever there is one.

In order to compute the previous expression using zeta-function regularization we start with the well-known identities

\[
\prod_{n=1}^\infty \left( A + \frac{n^2}{B} \right) = \frac{2}{\sqrt{A}} \sinh \pi \sqrt{AB}
\]

\[
\prod_{n=0}^\infty \left( A + \frac{(n + \frac{1}{2})^2}{B} \right) = 2 \cosh \pi \sqrt{AB}
\]

and using them we write the relevant infinite products in terms of standard theta-functions

\[
\prod_{m,n} |n - mT|^2 = \left| \vartheta' \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (0|T) \eta(T) \right|^2 = 4\pi^2 |\eta(T)|^4
\]

\[
\prod_{m,n} |n + a - (m + b) T|^2 = (qT \bar{q} T)^{-1/8ab} \left| \vartheta \left[ \begin{array}{c} b + 1/2 \\ a + 1/2 \end{array} \right] (0|T) \eta(T) \right|^2
\]

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where \( a, b = 0, \frac{1}{2} \) but not simultaneously zero.

Now we can proceed with the computation in the four different sectors. The easiest one is the \((R,R)\)

\[
\log |Z_{(R,R)}|^2 = -\frac{1}{2} \left( \sum_{s,s'} Q^s_{s'} \right) \log \left[ \prod_{m,n} |n - m - nT|^2 \prod_{m,n} |n + m| \prod_n T_U \right] = - \sum_{s,s'} Q^s_{s'} \log \left[ \sqrt{T_2} |\eta(T)|^2 \sqrt{U_2} |\eta(U)|^2 \right] = 0
\]

because of \((3.8)\). The other cases are less trivial; for example, in the \((NS,R)\) and \((R,NS)\) sectors we find using again \((3.8)\)

\[
\log |Z_{(NS,R)}|^2 = \frac{8}{2} \log \left[ \frac{1}{T_2 U_2} \prod_{m,n} |m + \frac{1}{2} - n \left( \frac{1}{7} \right) |^2 \prod_{m,n} |m + \frac{1}{2} + n \left( \frac{1}{2U} \right) |^2 \right] = 4 \log \left[ \frac{\theta_2(0|T)}{\sqrt{T_2 \eta^3(T)}} \right] \left[ \frac{\theta_2(0|U)}{\sqrt{U_2 \eta^3(U)}} \right] ^2
\]

and

\[
\log |Z_{(R,NS)}|^2 = \frac{8}{2} \log \left[ \frac{|T|^2 |2U|^2}{T_2 U_2} \prod_{m,n} \left| \frac{m + \frac{1}{2} - n \left( \frac{1}{7} \right) }{m - n \left( \frac{1}{7} \right) } \right|^2 \prod_{m,n} \left| \frac{m + \frac{1}{2} + n \left( \frac{1}{2U} \right) }{m + n \left( \frac{1}{2U} \right) } \right|^2 \right] = 4 \log \left[ \frac{\theta_4(0|T)}{\sqrt{T_2 \eta^3(T)}} \right] \left[ \frac{\theta_4(0|2U)}{\sqrt{2U_2 \eta^3(2U)}} \right] ^2
\]

Finally we have to go to the \((NS,NS)\) sector. The procedure is exactly the same and the final result is

\[
\log |Z_{(NS,NS)}|^2 = \frac{8}{2} \log \left[ \prod_{m,n} \left| m + \frac{1}{2} - (n + \frac{1}{2}) T \right|^2 \prod_{m,n} \left| m + \frac{1}{2} + (n + \frac{1}{2}) (2U) \right|^2 \right] = 4 \log \left[ q_T \frac{\theta_3(0|T)}{\sqrt{T_2 \eta^3(T)}} \right] \left[ q_T \frac{\theta_3(0|2U)}{\sqrt{2U_2 \eta^3(2U)}} \right] ^2
\]

All these formulae hold up to finite additive numerical constants.
Appendix B. β-duality vs. T-duality

T-duality is a well known symmetry of the toroidal compactifications of both bosonic and heterotic strings. The theory is symmetric under the following combined transformation of the compactification radius and the string coupling constant [23]

\[ R \rightarrow \frac{\alpha'}{R}, \quad \lambda \rightarrow \frac{\sqrt{\alpha'}}{R} \lambda. \]  
(B.1)

In the case of the type-II superstring the previous transformation is not a symmetry of the theory, but rather maps the type-IIA into the type-IIB and vice versa [24].

In the case of fermionic strings one can think of more general toroidal compactifications in which the boundary conditions of the target space fermions are correlated with those of the world-sheet fermions. A particular class of such compactifications is that in which space-time fermions are taken to be antiperiodic along the compactified dimension [25][9]. These boundary conditions are implemented in the correlation functions by the insertion of a set of phases in the sum over spin structures and over classical vacua of the two-dimensional world-sheet theory. At genus \( g \) these phases are [9]

\[ U^\bar{s},\bar{s}'_{\bar{\ell},\bar{\ell}'} = (-1)^{2\bar{\ell} \cdot \bar{s} + 2\bar{\ell}' \cdot \bar{s}' + \bar{\ell} \cdot \bar{\ell}'} \]

where \( \bar{\ell}, \bar{\ell}' \) are the winding numbers of the \( 2g \) homology cycles \( \{a_i,b_i\} \) \( (i = 1,\ldots,g) \) into the target circle \( S^1_R \) and \( \{\bar{s},\bar{s}'\} \) are the corresponding spin structures. In the case of a heterotic string theory, in which space-time fermions arise only from the left moving sector, we have to introduce only one set of phases. Then the genus-\( g \) contribution to the logarithm of the partition function can be written as

\[ \left[ \log Z(R) \right]_g = \lambda^{2g-2} \int_{\mathcal{F}_g} d\mu(\tau) \sum_{\bar{s},\bar{s}'} \Lambda^\bar{s}'_{\bar{s}}(\tau,\bar{\tau}) \times \sum_{\ell,\ell'} (-1)^{2\bar{\ell} \cdot \bar{s} + 2\bar{\ell}' \cdot \bar{s}' + \bar{\ell} \cdot \bar{\ell}'} e^{-\frac{2\beta R^2}{\alpha'} \sum (\ell_k \tau_{k1} + \ell'_j (\text{Im} \tau)_{ij}^{-1} (\ell_k \tau_{jk} + \ell'_j))} \]  
(B.2)

where \( \Lambda^\bar{s}'_{\bar{s}}(\tau,\bar{\tau}) \) is the genus-\( g \) partition function for the uncompactified theory. By using Poisson resummation in \( \ell_k \) and summing over all genera it is possible to prove that (B.2) is invariant under the \( \beta \)-duality transformation (see last reference in [23])

\[ R \rightarrow \frac{\alpha'}{2R}, \quad \lambda \rightarrow \frac{\sqrt{\alpha'}}{\sqrt{2R}} \lambda. \]  
(B.3)
A remarkable fact about this transformation is that it differs from T-duality (B.1) by a numerical factor; in fact (B.3) is equivalent to perform a T-duality transformation on the theory with radius $R' = \sqrt{2}R$. In general $\beta$-duality will be a symmetry of any heterotic string for which only even spin structures contribute to the genus-$g$ cosmological constant.

When we are dealing with a type-II theory, in which target space fermions arise from both the left and right movers, we need to introduce one phase for each world-sheet chirality. In that case $\beta$-duality might not be a symmetry of the theory. This is what happens, for example, in the type-II superstring [9], which under $\beta$-duality is mapped at one loop into itself plus some twisted bosonic theory [26].

Because of the antiperiodic boundary conditions of the space-time fermions, (B.2) can be interpreted as the genus-$g$ contribution to the canonical free energy of a gas of strings at inverse temperature $\beta = 2\pi R$. Alternatively, (B.2) can be viewed as the genus-$g$ cosmological constant of an asymmetric orbifold compactification which breaks space-time supersymmetry at zero temperature [25][13][27]. In any case, for string theories with an infinite tower of massive states these compactifications are only stable in the large radius limit; when the compactifications radius is of the order of the string scale, tachyons may appear in the spectrum rendering correlation functions infrared divergent. From the thermal viewpoint this is nothing but the old problem of the Hagedorn temperature caused by the exponential growth of the number of on-shell states per mass level. These kind of difficulties are not present when dealing with theories with a finite number of propagating degrees of freedom, such as the (2,1) string backgrounds considered in this paper.

In spite of having no Hagedorn-like interpretation in target space, the partition function of (2,1) models can be afflicted from similar world-sheet instabilities due to some states becoming massless at some point of the moduli space of the toroidal compactification (the off-shell tachyons). The only source for such a behavior in this case is the coupling of the left-moving NS ground state with the winding/momentum states. By going from the winding representation (3.3) to the mixed momentum-winding representation it is possible to write the whole partition function as the sum of contributions coming from the four conjugacy classes of $SO(8)$. In the (R,R) sector the phases are equal to one and the compactification preserves supersymmetry, so we will only have the contributions from the vector and spinorial conjugacy classes. In the remaining sectors, on the contrary, supersymmetry is broken and all the four conjugacy classes will contribute, in particular the scalar one containing the NS tachyonic ground state. Following the analysis of ref. [27]
we find that, whenever we have antiperiodic boundary conditions in the cycle with radius $R_i$, we will have a potentially divergent term which, for example, in the (NS,*) sector is of the form

\[ I_{\text{div}} = 2 \int_1^\infty \frac{d\tau_2}{\tau_2} \sum_{n_i, \ell_i} \exp \left\{ -\pi \tau_2 \left[ \left( \ell_1 + \frac{1}{2} \right)^2 (2R_1)^2 + \frac{(n_1 + \frac{1}{2})^2}{R_1^2} + \ell_2^2 R_2^2 + \frac{n_2^2}{R_2^2} - 1 \right] \right\} \times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \exp \left\{ 2\pi i \tau_1 \left[ 2 \left( \ell_1 + \frac{1}{2} \right) \left( n_1 + \frac{1}{2} \right) - \ell_2 n_2 - \frac{1}{2} \right] \right\}. \]

By integrating over $\tau_1$ we get a constraint over the integers $n_i, \ell_i$. It is easy to solve this constraint and find

\[ I_{\text{div}} = 4 \int_1^\infty \frac{d\tau_2}{\tau_2} e^{-\frac{\pi}{4} \left( 2R_1 - \frac{1}{\sqrt{2}} \right)^2} = 4 E_1 \left( \frac{\pi}{4} \left| 2R_1 - \frac{1}{R_1} \right|^2 \right), \tag{B.4} \]

plus other terms that are regular as functions of $R_1$. $E_1(z)$ is the first exponential integral function whose power expansion around $z = 0$ is

\[ E_1(z) = -\gamma_E - \log z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!}, \]

and we get a logarithmic singularity for $I_{\text{div}}$ in the limit $R_1 \to 1/\sqrt{2}$. Moreover, since $E_1(z)$ is not singled valued as a function of $z$, we have to introduce the absolute value in (B.4) in order to preserve the invariance of the original integral under $R_1 \leftrightarrow 1/2R_1$. In the (NS,NS) contribution we will have a term (B.4) for each radius, so divergences will appear when either $R_1$ or $R_2$ reach the self-dual value under $\beta$-duality, $R = 1/\sqrt{2}$.

Let us close this Appendix with some remarks about the equivalence between the integral over $\mathcal{F}$ of the partition function and the $S$ integral in which we have the projection $\Lambda_{s'} \to Q_{s'}$. If we were only compactifying one dimension with NS boundary conditions (i.e. the standard finite temperature situation), it would be easy to prove that both the $\mathcal{F}$ and the projected $S$ representation are equal whenever the compactification radius satisfy $R > 1/\sqrt{2}$. The reason is the following; in including the number of propagating fields in a Schwinger representation of the partition function we can write the number of bosons and fermions either as a couple of integers $(Q_0^0 + Q_0^\frac{1}{2}) \int_{-\frac{1}{2}}^{1/2} d\tau_1$ and $-Q_0^\frac{1}{2} \int_{-\frac{1}{2}}^{1/2} d\tau_1$ or in terms of their integral representations (3.7). This will provide us with two different integrals over $S$, $I_{\text{proj}}(R)_S$ and $I(R)_S$ giving the same function of $R$ if we integrate first in the real part of $\tau$. If we decide to rewrite $I_{\text{proj}}(R)_S$ and $I(R)_S$ as integrals over the fundamental
domain $F$ we will have to be very careful with the manipulations involved in the coset extension (formula (2.12)). The result will be that both integrals will coincide only when $R > 1/\sqrt{2}$; in particular the extension of $I_{\text{proj}}(R)_S$ will be a regular function for all values of $R$, whereas the modular invariant version of $I(R)_S$ will have $\beta$-duality and a finite discontinuity in its first derivative at the self-dual radius, $R = 1/\sqrt{2}$ (cf. [17]).

This situation changes when we have two compactified dimensions. To clarify the discussion let us focus in the (NS,$R$) sector in which we go to the $S$ representation by disentangling the windings in the cycle with NS boundary conditions. Now, because of the presence of winding modes around the second cycle, the projected and unprojected integrals over the strip will not be equivalent, since

\[
Q_s' \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \sum_{\{\ell\}} e^{-\frac{\pi R^2}{2} |\ell_2 \tau + \ell'_2|^2} \neq \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \Lambda_s' (\tau) \sum_{\{\ell\}} e^{-\frac{\pi R^2}{2} |\ell_2 \tau + \ell'_2|^2}. \tag{B.5}
\]

Therefore, the corresponding modular invariant extensions of both results will only be equivalent in the limit in which we decompactify the second cycle ($R_2 \to \infty$), and keep the first radius above $1/\sqrt{2}$. It is however important to notice that the left-hand side of (B.3) is recovered inside the right-hand side as the zero mode part in the expansion of $\Lambda_s'$ in powers of $q$ and $\bar{q}$. This suggests that the full integral over the fundamental domain $F$ contains the projected integrals (B.10), plus other terms that are singular at some codimension one regions of the moduli space of $T$ and $U$. The nature of these logarithmic singularities, analyzed above, supports the view that it is only the projected piece of the integral that factorizes into a $T$ and a $U$-dependent part, whereas this would not be the case for the terms that we are dropping in the projection $\Lambda_s' \to Q_s'$. 

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References

[1] E. Witten, *Nucl. Phys.* **B443** (1995) 85; J. Polchinski, Preprint NSF-ITP-95-157, [hep-th/9511157](https://arxiv.org/abs/hep-th/9511157).

[2] M. B. Green, *Nucl. Phys.* **B293** (1987) 593.

[3] M. Ademolo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, J.H. Schwarz, *Nucl. Phys.* **B111** (1976) 77.

[4] H. Ooguri and C. Vafa, *Mod. Phys. Lett.* **A5** (1990) 1389; *Nucl. Phys.* **B361** (1991) 469; *Nucl. Phys.* **B367** (1991) 83.

[5] D. Kutasov and E. Martinec, Preprint EFI-96-04, [hep-th/9510182](https://arxiv.org/abs/hep-th/9510182).

[6] D. Kutasov, E. Martinec and M. O'Loughlin, Preprint EFI-96-07, [hep-th/9603116](https://arxiv.org/abs/hep-th/9603116).

[7] L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B355** (1991) 649; J. Harvey and G. Moore, Preprint EFI-95-64, [hep-th/9510182](https://arxiv.org/abs/hep-th/9510182).

[8] R. Dijkgraaf, E. Verlinde and H. Verlinde in: *Perspectives in String Theory*, Proc. Copenhagen 1987, World Scientific, Singapore 1988; A. Giveon, N. Malkin and E. Rabinovici. *Phys. Lett.* **B220** (1989) 551.

[9] J. Atick and E. Witten, *Nucl. Phys.* **B310** (1988) 291.

[10] D.J. Gross and P.F. Mende, *Phys. Lett.* **B197** (1987) 129; *Nucl. Phys.* **B303** (1988) 407.

[11] E. Witten, *Phys. Rev. Lett.* **61** (1988) 670.

[12] M.A.R. Osorio and M.A. Vázquez-Mozo, Preprint IASSNS-HEP-95-96; [hep-th/9511157](https://arxiv.org/abs/hep-th/9511157) (*Nucl. Phys. B*, to appear); D. Pierce, Preprint IFP-604-UNC, [hep-th/9601125](https://arxiv.org/abs/hep-th/9601125).

[13] R.H. Brandenberger and C. Vafa, *Nucl. Phys.* **B316** (1988) 391.

[14] J. Polchinski, *Commun. Math. Phys.* **104** (1986) 37; K.H. O'Brien and C.I. Tan, *Phys. Rev.* **D36** (1987) 1184; B. McClain and B.D.B Roth, *Commun. Math. Phys.* **111** (1987) 539.

[15] E. Alvarez and M.A.R. Osorio, *Nucl. Phys.* **B304** (1988) 327.

[16] D. Kutasov and N. Seiberg, *Nucl. Phys.* **B358** (1991) 600.

[17] M.A.R. Osorio and M.A. Vázquez-Mozo, *Phys. Lett.* **B280** (1992) 21.

[18] D.J. Gross and I.R. Klebanov, *Nucl. Phys.* **B344** (1990) 345; M. Bershadsky and I.R. Klebanov, *Phys. Rev. Lett.* **65** (1990) 3088.
[19] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer, Berlin 1984.

[20] G. Moore, *Nucl. Phys.* B293 (1987) 139.

[21] K.S. Narain, M.H. Sarmadi and C. Vafa, *Nucl. Phys.* B288 (1987) 551.

[22] S.D. Mathur and S. Mukhi, *Nucl. Phys.* B302 (1988) 130.

[23] K. Kikkawa and M. Yamasaki, *Phys. Lett.* B149 (1984) 257; N. Sakai and I. Senda, *Prog. Theor. Phys.* 75 (1984) 692; P. Ginsparg and C. Vafa, *Nucl. Phys.* B289 (1987) 914; E. Alvarez and M.A.R. Osorio, *Phys. Rev.* D40 (1989) 1150.

[24] J. Dai, R.G. Leigh and J. Polchinski, *Mod. Phys. Lett.* A4 (1989) 2073; M. Dine, P. Huet and N. Seiberg, *Nucl. Phys.* B322 (1989) 301.

[25] R. Rohm, *Nucl. Phys.* B237 (1984) 553.

[26] M.A.R. Osorio, *Int. J. Mod. Phys.* A7 (1992) 4275.

[27] M.A.R. Osorio and M.A. Vázquez-Mozo, *Phys. Rev.* D47 (1993) 3411.