On the Algebraic Structure of the Holomorphic Anomaly for $N = 2$ Topological Strings

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Abstract

The special geometry ($(t, \bar{t})$-equations) for twisted $N = 2$ strings are derived as consistency conditions of a new contact term algebra. The dilaton field appears in the contact terms of topological and antitopological operators. The holomorphic anomaly, which can be interpreted as measuring the background dependence, is obtained from the contact algebra relations.

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Introduction

For \( N = 2 \) twisted topological strings, the problem of background independence seems "ab initio" an almost trivial issue. In fact once we fix a background by choosing a particular point \((t_0, \bar{t}_0)\) in the moduli space \( \mathcal{M} \) of the underlying \( N = 2 \) SCFT, a change in the background turns out to be equivalent, for the twisted theory, to the coupling of a pure BRST state, which by standard procedures should be expected to vanish. The holomorphic anomaly recently discovered in reference\[1, 2\], clearly shows that this naive picture drastically changes after coupling to gravity. As we will prove in this letter what makes the holomorphic anomaly different from the standard BRST anomaly is its special nature as a string amplitude. In fact the holomorphic anomaly can be interpreted as defining a new type of string amplitudes where the topological and antitopological sectors are explicitly fused. In this letter we will reinterpret these \((t, \bar{t})\) amplitudes in terms of a contact term algebra, similar to the one defining pure topological gravity \[3, 4\]. The consistency conditions of this algebra are proved to be the special geometry relations \[3\] for the moduli space of the \( N = 2 \) SCFT \[2\]. In reference\[4\] an integrated version of the holomorphic anomaly is defined in terms of a particular set of Feynman rules. The most surprising fact about these rules is the appearance of a dilaton field, defined by the same recursion relations that in topological gravity \[3, 4\]. We will show that this dilaton field is crucial for defining a fused \((t, \bar{t})\) contact term algebra which completely determines the \((t, \bar{t})\) amplitudes.

\((t, \bar{t})\)-amplitudes and moduli measures

For \( N=2 \) twisted SCFT’s with \( \hat{c}=3 \), the coupling to gravity is obtained by defining the following measures \[3\] on moduli space of Riemann surfaces:

i) Partition function measures: The measures on the moduli space \( \mathcal{M}_g \) of Riemann surfaces with genus \( g \) are given by:

\[
\int_{\mathcal{M}_g} [dm] \langle \prod_{a,\bar{a}=1}^{3g-3} G^{-}(\chi_a)\bar{G}^{-}(\bar{\chi}_{\bar{a}}) \rangle_{\Sigma_g}
\]

(1)

where the \( \chi_a, \bar{\chi}_{\bar{a}} \) represent a basis of Beltrami differentials. For simplicity, we will adopt the notation referring to the A-twist, as already done in (1); the corresponding expressions for the B-twist can be immediately obtained replacing \( \bar{G}^{-} \) by \( G^{+} \).
ii) Amplitude measures for truly marginal operators:

\[
\int_{\mathcal{M}_{g,N}} [dm][dz] \left\langle \prod_{i=1}^{N} \oint_{C_{z_i}} G \bar{G} \phi_i(z_i) \prod_{a,\bar{a}=1}^{3g-3} G^{-}(\chi_a)\bar{G}^{-}(\bar{\chi}_{\bar{a}}) \right\rangle_{\Sigma_{g,N}}
\]  

which define measures on the moduli space \( \mathcal{M}_{g,N} \) of Riemann surfaces with genus \( g \) and \( N \) punctures. Notice that only for marginal fields, which have \( U(1) \) charge equal one, these measures are non vanishing. In fact the \( U(1) \) anomaly is completely saturated by the \((3g - 3)\) insertions of the supersymmetric \( G^{-} \) charges.

The philosophy underlying these definitions is the formal similarity between the twisted \( \mathcal{N}=2 \) theory and the bosonic string:

\[
2j_{BRST} \leftrightarrow G^{+} \quad b \leftrightarrow G^{-} \quad bc \leftrightarrow J
\]

For the particular case \( \hat{c}=3 \) the \( U(1) \) anomaly coincides with the dimension of the moduli space, making possible to use the supersymmetric charges \( G^{-} \) as the standard \( b \)-ghost for the definition of the measure on the moduli space.

The background dependence of these amplitudes, i.e their derivatives with respect to \( \bar{t}_i \), are given by the following ”(\( t, \bar{t} \)) amplitude”:

\[
\int_{\mathcal{M}_{g,N}} [dm][dz] \int dw \left\langle \oint_{C_w} G^{+}\bar{G}^{+}\bar{\phi}_i(w) \prod_{i=1}^{N} \oint_{C_{z_i}} G^{-}\bar{G}^{-}\phi_i(z_i) \prod_{a,\bar{a}=1}^{3g-3} G^{-}(\chi_a)\bar{G}^{-}(\bar{\chi}_{\bar{a}}) \right\rangle_{\Sigma_{g,N}}
\]

The string interpretation of this amplitude as a measure on the corresponding moduli space \( \mathcal{M}_{g,n+1} \), is by no means direct. It is precisely at this point where the holomorphic anomaly differs from the standard BRST-anomaly of the bosonic string. In fact the integration of the moduli parameter associated with the insertion of the antitopological field is performed using the ”antitopological ghost”, i.e the supersymmetric current \( G^{+} \).

Searching for a standard string interpretation of the \((t,\bar{t})\) amplitude (4), we can formally try to represent it as follows:

\[
\int_{\mathcal{M}_{g,n+1}} [dm][dz] \left\langle \oint_{C_w} G^{-}\bar{G}^{-}\mathcal{O}_i(w) \prod_{i=1}^{N} \oint_{C_{z_i}} G^{-}\bar{G}^{-}\phi_i(z_i) \prod_{a,\bar{a}=1}^{3g-3} G^{-}(\chi_a)\bar{G}^{-}(\bar{\chi}_{\bar{a}}) \right\rangle_{\Sigma_{g,n+1}}
\]

where we have introduced new operators \( \mathcal{O}_i \) with positive ghost number equal one. In order to make explicit the ghost number counting we can use the following ”gravitational
We interpret the operators $\mathcal{O}_{\vec{i}(i)}$ as having implicitly a gravitational descendent index $n$, i.e $\mathcal{O}_{\vec{i}(i),n}$. By analogy with topological matter coupled to topological gravity models [8, 9], their ghost number will be defined by the rule:

$$gh(\mathcal{O}_{\vec{i}(i),n}) = n + q_{\vec{i}(i)}$$

(6)

with $q_{\vec{i}(i)}$ the $U(1)$ charge of the corresponding fields. Following this prescription, the operators $\mathcal{O}_{\vec{i}}$ have $n = 2$. With the same logic we shall associate with the truly marginal fields $\phi_i$, also of ghost number one, operators of the type $\mathcal{O}_{i,0}$. Summarizing the ”gravitational picture” for the $(t, \bar{t})$ amplitudes is defined by the following set of formal rules:

$$\oint G^- \bar{G}^- \phi_i \rightarrow \oint G^- \bar{G}^- \mathcal{O}_{i,0}$$

$$\oint G^+ \bar{G}^+ \bar{\phi}_i \rightarrow \oint G^- \bar{G}^- \mathcal{O}_{i,2}$$

(7)

The philosophy underlying (7.b) can be thought as a process in two steps. Inspired by the relation between gravitational descendants and pure BRST states in $\hat{c} < 1$ Landau-Ginzburg models coupled to gravity [11, 12], we first interpret the pure BRST insertion of the antitopological field as representing a gravitational descendant. Secondly, we integrate over the position of the corresponding puncture in the standard topological way, namely using $G^-$ as the b-ghost.

One more reason supporting the gravitational picture comes from the subleading divergences in the operator product:

$$\bar{\phi}_j(z) \phi_i(0) = \frac{G_{\vec{i}\vec{j}}}{|z|^2}$$

(8)

with $G_{\vec{i}\vec{j}}$ the Zamolodchikov metric associated with the truly marginal deformations [13]. These subleading singularities [14], which depend linearly on the curvature of the surface, determine the ”dilaton” contribution to the holomorphic anomaly [2]. The natural translation of them into the gravitational picture presented above, would be defined by the following contact term:

$$\int_D \mathcal{O}_{j,2}|\mathcal{O}_{i,0}⟩ = G_{\vec{i}\vec{j}} |\sigma_1⟩$$

(9)

with $\sigma_1$ representing a dilaton field, in the sense of topological gravity, and the domain of integration $D$ is an infinitesimal neighborhood of the point where the field $\mathcal{O}_{i,0}$ is inserted. The dynamics of the dilaton field will be later determined by its contact terms with the rest of the fields.
The previous arguments should be interpreted only as providing some heuristic support for the rules defined in equation (7). Our strategy now will be to find a contact term algebra for the operators in the gravitational picture. From this contact term algebra we will be able to derive the holomorphic anomaly and the special geometry of the moduli space of $N = 2$ SCFT ($(t, \bar{t})$- equations [15]), which will appear in this context as the consistency conditions of the algebra.

The $(t, \bar{t})$ equations as consistency conditions of a contact term algebra

Let us consider the algebra of operators generated by: $\mathcal{O}_{i,0}$, $\mathcal{O}_{i,2}$ and the dilaton field $\sigma_1$ with $i = 1, \ldots, n$ for $n$ the number of truly marginal deformations. We define the following contact term algebra:

\[ \int_D \mathcal{O}_{i,0} |\mathcal{O}_{j,0}\rangle = \Gamma_{ij}^k |\mathcal{O}_{k,0}\rangle , \quad \int_D \mathcal{O}_{i,2} |\mathcal{O}_{j,2}\rangle = \tilde{\Gamma}_{ij}^k |\mathcal{O}_{k,2}\rangle \]
\[ \int_D \mathcal{O}_{i,2} |\mathcal{O}_{j,0}\rangle = G_{ji} |\sigma_1\rangle , \quad \int_D \mathcal{O}_{i,0} |\mathcal{O}_{j,2}\rangle = \tilde{G}_{ij} |\sigma_1\rangle \]
\[ \int_D |\mathcal{O}_{i,0}\rangle = a |\mathcal{O}_{i,0}\rangle , \quad \int_D |\mathcal{O}_{i,0}\rangle |\sigma_1\rangle = b |\mathcal{O}_{i,0}\rangle \]
\[ \int_D |\mathcal{O}_{i,2}\rangle = c |\mathcal{O}_{i,2}\rangle , \quad \int_D |\mathcal{O}_{i,2}\rangle |\sigma_1\rangle = d |\mathcal{O}_{i,2}\rangle \]
\[ \int_D |\sigma_1\rangle = e |\sigma_1\rangle \]

In order to take into account the contribution of the curvature we introduce the operator $e^{\frac{2}{\hbar} \phi(z)}$, the exponential of the bosonized $U(1)$ current. The contact term algebra for this operator is defined as follows:

\[ \int_D \mathcal{O}_{i,0} e^{\frac{2}{\hbar} \phi(z)} = A_i |e^{\frac{2}{\hbar} \phi(z)}\rangle , \quad \int_D \mathcal{O}_{i,2} e^{\frac{2}{\hbar} \phi(z)} = 0 \]
\[ \int_D |\sigma_1\rangle e^{\frac{2}{\hbar} \phi(z)} = a |e^{\frac{2}{\hbar} \phi(z)}\rangle \]

The undetermined constants appearing in (10) and (11) will be now fixed by imposing the following consistency conditions:

\[ \int_D a \int_D b |c\rangle = \int_D b \int_D a |c\rangle \]

for three arbitrary operators. These conditions simply imply that the string amplitudes
are independent of the order in which the fields are integrated\footnote{Notice that there are two contributions to each side of (12): the successive contact terms of the operators $a$ and $b$ with the bracketed $c$, and the contact term between $a$ and $b$ first, then carried over $|c|$.}. To solve the consistency conditions we will assume:

i) That $G_{i,j}$ is invertible.

ii) The value of $a$ equal -1. This condition is based on the way the dilaton field measures the curvature.

iii) The following derivation rules:

\[
\mathcal{O}_{i,0} \Gamma^\gamma_{\alpha\beta}(t, \bar{t}) = \partial_t \Gamma^\gamma_{\alpha\beta}(t, \bar{t})
\]

\[
\mathcal{O}_{i,2} \Gamma^\gamma_{\alpha\beta}(t, \bar{t}) = (-1)^{F(\Gamma^\gamma_{\alpha\beta})} \partial_{\bar{t}} \Gamma^\gamma_{\alpha\beta}(t, \bar{t})
\]

where $\Gamma^\gamma_{\alpha\beta}$ stands for a generic contact term tensor, $q_\alpha$ for the $U(1)$ charge associated to the field $\mathcal{O}_\alpha$, and which defines the way the operators act on the coefficients appearing in the contact term algebra. Notice that in general these coefficients will depend on the moduli parameters $(t, \bar{t})$. The logic for this rule is the equivalence between the insertion of a marginal field and the derivation with respect to the corresponding moduli parameter. For this reason we will not associate any derivative with the dilaton field. The derivation rule (13.b) is forced by the gravitational picture we are using. Once we decide to work with gravitational descendant and to define the measure using only $G^-$ insertions, we must accommodate to this picture the coupling of the spin connection to the $U(1)$ current. Since the derivation $\partial_t$ correspond to the insertion of an antitopological field, in order to pass to the gravitational picture, we need to change, in the neighborhood of the insertion, the sign of the coupling of the $U(1)$ current to the background gauge field defined by the spin connection. This fact gives raise to the factor $(-1)^{F(\Gamma)}$ in (13.b).

We will pass now, using i), ii) and iii), to solve the consistency conditions (12). Let us start analyzing the following relation:

\[
\int_D \sigma_1 \int_D \mathcal{O}_{i,0} \mathcal{O}_{j,0} = \int_D \mathcal{O}_{i,0} \int_D \sigma_1 |\mathcal{O}_{j,0}\rangle
\]

Applying the contact term algebra (10), we get:

\[
b \Gamma^k_{ij} |\mathcal{O}_{k,0}\rangle - \Gamma^k_{ij} |\mathcal{O}_{k,0}\rangle = -2 \Gamma^k_{ij} |\mathcal{O}_{k,0}\rangle
\]

which, for a non vanishing $\Gamma^k_{ij}$, implies that:

\[
b = -1
\]
From the condition:
\[ \int_{D} \sigma_{1} \int_{D} \mathcal{O}_{i,0} |\sigma_{1}\rangle = \int_{D} \mathcal{O}_{i,0} \int_{D} \sigma_{1} |\sigma_{1}\rangle \]  
(17)
together with equation (16) and the derivation rules (13), we obtain:
\[ \partial_{i} e |\sigma_{1}\rangle - e |\mathcal{O}_{i,0}\rangle = |\mathcal{O}_{i,0}\rangle \]  
(18)
being solved by:
\[ e = -1 \]  
(19)
To continue the study, we take the condition:
\[ \int_{D} \mathcal{O}_{i,2} \int_{D} \mathcal{O}_{j,2} |\mathcal{O}_{k,0}\rangle = \int_{D} \mathcal{O}_{j,2} \int_{D} \mathcal{O}_{i,2} |\mathcal{O}_{k,0}\rangle \]  
(20)
which leads to:
\[ (\tilde{\Gamma}_{ji}^{k} G_{kl} + \partial_{i} G_{kj}) |\sigma_{1}\rangle + d G_{kj} |\mathcal{O}_{i,2}\rangle = (\tilde{\Gamma}_{ij}^{k} G_{kl} + \partial_{j} G_{ki}) |\sigma_{1}\rangle + d G_{ki} |\mathcal{O}_{j,2}\rangle \]  
(21)
Using that \( G_{ij} \) is invertible, and for a general number of truly marginal deformations, we get from the above equation:
\[ d = 0 \]  
(22)
Moreover, the consistency condition:
\[ \int_{D} \sigma_{1} \int_{D} \mathcal{O}_{i,2} |\mathcal{O}_{i,0}\rangle = \int_{D} \mathcal{O}_{i,2} \int_{D} \sigma_{1} |\mathcal{O}_{i,0}\rangle \]  
(23)
and equation (22) imply that:
\[ c = 0 \]  
(24)
From (16), (19), (22) and the consistency condition:
\[ \int_{D} \mathcal{O}_{i,0} \int_{D} \mathcal{O}_{j,2} |\sigma_{1}\rangle = \int_{D} \mathcal{O}_{j,2} \int_{D} \mathcal{O}_{i,0} |\sigma_{1}\rangle \]  
(25)
we get easily:
\[ \tilde{G}_{ij} = 0 \]  
(26)
The next conditions we will analyze involve the curvature operator \( e^{\frac{1}{2} \phi(z)} \):
\[ \int_{D} \mathcal{O}_{i,0} \int_{D} \mathcal{O}_{j,2} |e^{\frac{1}{2} \phi(z)}\rangle = \int_{D} \mathcal{O}_{j,2} \int_{D} \mathcal{O}_{i,0} |e^{\frac{1}{2} \phi(z)}\rangle \]  
(27)
\[ \int_{D} \mathcal{O}_{i,0} \int_{D} \mathcal{O}_{j,0} |e^{\frac{1}{2} \phi(z)}\rangle = \int_{D} \mathcal{O}_{j,0} \int_{D} \mathcal{O}_{i,0} |e^{\frac{1}{2} \phi(z)}\rangle \]
from which we get, assuming that $\Gamma^k_{ij}$ is symmetric in the lower indices:

\[ G_{ij} = \partial_j A_i \]
\[ \partial_i A_j = \partial_j A_i \]

Equations (28) imply that the metric $G_{ij}$ is Kähler, for a certain potential $K(t, \bar{t})$:

\[ G_{ij} = \partial_i \partial_j K \]

With this information, we can return to (21) and deduce that the tensor $\tilde{\Gamma}^k_{ij}$ is symmetric in the lower indices:

\[ \tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji} \]

Using now:

\[ \int_D \mathcal{O}_{i,0} \int_D \mathcal{O}_{j,2}|\mathcal{O}_{k,2}\rangle = \int_D \mathcal{O}_{j,2} \int_D \mathcal{O}_{i,0}|\mathcal{O}_{k,2}\rangle \]

we obtain that $\tilde{\Gamma}^l_{jk}$ is only function of the antitopological variables:

\[ \partial_i \tilde{\Gamma}^l_{jk} = 0 \]

Condition (32), together with $\int_D \mathcal{O}_{i,2} \int_D \mathcal{O}_{j,2}|\mathcal{O}_{k,2}\rangle = \int_D \mathcal{O}_{j,2} \int_D \mathcal{O}_{i,2}|\mathcal{O}_{k,2}\rangle$ allow to impose a vanishing contact term for antitopological operators.

To conclude the study of the consistency conditions we will consider now the relation:

\[ \int_D \mathcal{O}_{i,0} \int_D \mathcal{O}_{j,2}|\mathcal{O}_{k,0}\rangle = \int_D \mathcal{O}_{j,2} \int_D \mathcal{O}_{i,0}|\mathcal{O}_{k,0}\rangle \]

Using equations (16) and (26), we obtain:

\[ \int_D \mathcal{O}_{i,0} \int_D \mathcal{O}_{j,2}|\mathcal{O}_{k,0}\rangle = \partial_i G_{k\bar{j}} |\sigma_1\rangle - G_{k\bar{j}} |\mathcal{O}_{i,0}\rangle - G_{ij} |\mathcal{O}_{k,0}\rangle + \text{fact terms} \]
\[ \int_D \mathcal{O}_{j,2} \int_D \mathcal{O}_{i,0}|\mathcal{O}_{k,0}\rangle = -\partial_j \Gamma^l_{ik} |\mathcal{O}_{i,0}\rangle + \Gamma^l_{ik} G_{ij} |\sigma_1\rangle \]

In topological gravity, the gravitational descendent index required to factorize the surface, is $n \geq 2$. Therefore, and due to the non vanishing correlation function $C^k_{ij}$ at genus zero for three marginal fields, we should consider the existence of factorization terms associated to the $\mathcal{O}_{j,2}$ insertions. We can write generically the factorization term as follows:

\[ \text{fact terms} = B_{\bar{j}}^{ln} C_{ikn} |\mathcal{O}_{i,0}\rangle \]

\[ ^2 \text{The symmetry of } \Gamma^k_{ij} \text{ will assure that } \int_D \mathcal{O}_{i,0} \int_D \mathcal{O}_{j,0}|\mathcal{O}_{k,0}\rangle = \int_D \mathcal{O}_{j,0} \int_D \mathcal{O}_{i,0}|\mathcal{O}_{k,0}\rangle \text{ is satisfied.} \]
From equations (33)-(35), we obtain that the coefficient $\Gamma^k_{ij}$ is the connection for the metric $G_{i\bar{j}}$, which we already know that is Kähler:

$$\Gamma^k_{ij} = (\partial_i G_{j\bar{k}}) G_{\bar{k}\bar{l}}$$

(36)

and a $(t, \bar{t})$ type equation:

$$\partial_t \Gamma^k_{ij} = G_{i\bar{n}} \delta^k_j + G_{j\bar{n}} \delta^k_i - B_{\bar{m}k} C_{ijm}$$

(37)

The tensor $B_{\bar{j}n}^l$ can be derived from the contact term algebra by the following argument. Let’s consider the consistency condition on a general string amplitude:

$$\langle O_{i,2} O_{j,2} \prod_{l=1}^{N} O_{l,0} \rangle_g = \langle O_{j,2} O_{i,2} \prod_{l=1}^{N} O_{l,0} \rangle_g$$

(38)

from (10) we get:

$$(\Gamma^k_{ij} - \Gamma^k_{ji}) \langle O_{i,2} \prod_{l=1}^{N} O_{l,0} \rangle_g = \sum_{l=1}^{N} \mathcal{R}_{D_l} + \sum_{\text{nodes}} \mathcal{R}_\Delta$$

(39)

where $\mathcal{R}_{D_l}$ denotes the commutator of the contact terms of $O_{i,2}$ and $O_{j,2}$ with $O_{l,0}$, and $\mathcal{R}_\Delta$ the commutator of those at the nodes. Using now the symmetry of $\Gamma^k_{ij}$ in the lower indices (27), we can conclude:

$$\sum_{l=1}^{N} \mathcal{R}_{D_l} = \sum_{\text{nodes}} \mathcal{R}_\Delta = 0$$

(40)

The contribution at a node associated with the factorization of the surface, will be defined by the tensor $B_{\bar{j}n}^l$ as follows [4]:

$$\langle O_{i,2} O_{j,2} \prod_{l \in S} O_{l,0} \rangle_{g,\Delta} = \sum_{r=0}^{g} \sum_{X \cup Y = S} \left[ B_{\bar{j}n}^{\alpha\beta} G_{\alpha\bar{i}} \langle \sigma_1 \prod_{l \in X} O_{l,0} \rangle_r \langle \sigma_1 \prod_{n \in Y} O_{n,0} \rangle_{g-r} + \partial_{\bar{t}} B_{\bar{j}n}^{\alpha\beta} \langle O_{\alpha,0} \prod_{l \in X} O_{l,0} \rangle_r \langle O_{\beta,0} \prod_{n \in Y} O_{n,0} \rangle_{g-r} \right]$$

(41)

where $S$ refers to the set of all punctures, $X$ and $Y$ is a partition of it, and the tensor $B$ can be chosen symmetric in the upper indices. Using now (40) we get:

$$B_{\bar{j}n}^{\alpha\beta} G_{\alpha\bar{i}} = B_{\bar{j}n}^{\alpha\beta} G_{\alpha\bar{i}}$$

(42)

$$\partial_{\bar{t}} B_{\bar{j}n}^{\alpha\beta} = \partial_{\bar{t}} B_{\bar{j}n}^{\alpha\beta}$$
By an analogous argument, we find from condition (33) and for a general string amplitude:

$$\partial_{\alpha} B_{\beta}^{\gamma} + B_{\beta}^{\delta} \Gamma_{\gamma}^{\alpha} + B_{\gamma}^{\delta} \Gamma_{\alpha}^{\alpha} - 2 \partial_{i} K B_{\beta}^{\gamma} = 0$$  \hspace{1cm} (43)$$

Let’s define $$B_{\beta}^{\delta} = B_{\delta}^{\beta} e^{2K} e^{\alpha} G_{\beta}^{\alpha}.$$ Then, equations (42) and (43) imply that $$B_{\beta}^{\delta}$$ is proportional to the three point correlation function for the antitopological fields. Substituting this information into equation (37), we obtain the $$(t, \bar{t})$$-equation \[15\]:

$$\partial_{\bar{t}} \Gamma_{ij}^{k} = G_{in} \delta_{j}^{k} + G_{jn} \delta_{i}^{k} - \bar{C}_{mn} C_{ijm}$$  \hspace{1cm} (44)$$

Notice that in order to get the special geometry relation (44) from the contact term algebra, it was necessary to make use of the derivation rule (13.b). From (44) we can conclude that the metric $$G_{ij}$$ is the Zamolodchikov metric for the marginal deformations. With this result we finish the derivation of the $$(t, \bar{t})$$ equations as consistency conditions for the contact term algebra (10). Our next objective will be the derivation of the holomorphic anomaly.

**The Holomorphic Anomaly**

The holomorphic anomaly \[1, 2\], i.e dependence on the background point, is determined by the $$(t, \bar{t})$$-amplitudes \(4\). In this section we will compute these amplitudes by using the gravity representation defined by the formal rules \(7\) and the machinery of the contact term algebra. In the gravitational picture, the amplitude \(4\) becomes:

$$\partial_{t} \ C_{1 \ldots i N}^{g} =$$

$$= \int_{\mathcal{M}_{g,N+1}} \langle \bar{t} \rangle \int \bar{G} \bar{G} \bar{O}(z) \prod_{i=1}^{N} \int \bar{G} ar{G} \bar{\phi}_{i}(z_{i}) \prod_{a=1}^{3g-3} \prod_{\bar{a}=1}^{3g-3} G^{-}(\chi_{a}) G^{-}(\bar{\chi}_{\bar{a}}) \rangle_{\Sigma_{g,N+1}} =$$

$$= \langle \bar{O}_{i,2} \prod_{i=1}^{N} \bar{O}_{i,0} \rangle_{g}$$  \hspace{1cm} (45)$$

where we have introduced the last equality to simplify the notation. The contributions to (45) can be written:

$$\langle \bar{O}_{i,2} \prod_{i=1}^{N} \bar{O}_{i,0} \rangle_{g} = \sum_{i=1}^{N} R_{D_{i}} + \sum_{\text{nodes}} \sum_{\text{nodes}} R_{\Delta}$$  \hspace{1cm} (46)$$

where $$R_{D_{i}}$$ is the contact term of $$\bar{O}_{i,2}$$ with the $$\bar{O}_{i,0}$$ insertion, and $$R_{\Delta}$$ the contact term contribution that factorize the surface through a node. Let’s start by analyzing the $$R_{D_{i}}$$. 
boundaries:

\[ \sum_{i=1}^{N} R_{D_i} = \sum_{i=1}^{N} \langle O_{t,2} \prod_{j=1}^{N} O_{j,0} \rangle_{D_i} = \sum_{i=1}^{N} G_{it} \langle \sigma_i \prod_{j \neq i} O_{j,0} \rangle = \]

\[ = \sum_{i=1}^{N} G_{it} (2 - 2g - n + 1) \langle \prod_{j \neq i} O_{j,0} \rangle \]  

(47)

The internal nodes \( \Delta \) are associated to the two types of boundaries of a Riemann surface of genus \( g \) and \( N \) punctures. The first one, we will note it as \( \Delta_1 \), comes from pinching a handle, leading to a surface of genus \( g - 1 \):

\[ \langle O_{t,2} \prod_{i=1}^{N} O_{i,0} \rangle_{g, \Delta_1} = \frac{1}{2} B_{t}^{\alpha \beta} \langle O_{\alpha,0} O_{\beta,0} \prod_{i=1}^{N} O_{i,0} \rangle_{g-1} \]  

(48)

where the factor \( \frac{1}{2} \) should be added to reflect the equivalency between the order in which the two new insertions \( O_{\alpha,0} \) are integrated. The factorization tensor \( B' \) satisfies the same set of equations (42) and (43) that the tensor \( B \), thus it is also proportional to the three point correlation function. With an appropriate choice of normalization of the string amplitudes, the proportionality constant between both factorization tensors can be set equal to one \([10]\).

The second ones, noted \( \Delta_2 \), come from the factorization of the surface into two surfaces of genus \( r \) and \( s \) punctures, and genus \( g - r \) and \( N - s \) punctures respectively:

\[ \langle O_{t,2} \prod_{i \in S} O_{i,0} \rangle_{g, \Delta_2} = \frac{1}{2} \sum_{r=0}^{g} \sum_{X \cup Y = S} B_{t}^{\alpha \beta} \langle O_{\alpha,0} \prod_{j \in X} O_{j,0} \rangle_r \langle O_{\beta,0} \prod_{k \in Y} O_{k,0} \rangle_{g-r} \]  

(49)

Collecting now equations (47), (48) and (49), we obtain the equation for the \( \bar{t} \)-dependence of any string amplitude:

\[ \partial_t \langle \prod_{i \in S} O_{i,0} \rangle_g = \frac{1}{2} B_{t}^{\alpha \beta} \langle O_{\alpha,0} O_{\beta,0} \prod_{i \in S} O_{i,0} \rangle_{g-1} + \]

\[ + \frac{1}{2} \sum_{r=0}^{g} \sum_{X \cup Y = S} B_{t}^{\alpha \beta} \langle O_{\alpha,0} \prod_{j \in X} O_{j,0} \rangle_r \langle O_{\beta,0} \prod_{k \in Y} O_{k,0} \rangle_{g-r} + \]

\[ + \sum_{i \in S} G_{it} (2 - 2g - n + 1) \langle \prod_{j \neq i} O_{j,0} \rangle_g \]  

(50)

Notice that in our derivation of the holomorphic anomaly from the contact term algebra we have only considered the contact terms of the antitopological operator \( O_{t,2} \) with the rest of the operators \( O_{i,0} \) but not the contact terms among the operators \( O_{i,0} \) themselves. This
is equivalent to define the correlators $\langle \prod \mathcal{O}_{i,0} \rangle$ by covariant derivatives of the generating functional. There are however some aspects of the previous derivation that should be stressed at this point.

1) The correlators $\langle \prod \mathcal{O}_{i,0} \rangle$ for topological operators can not be determined by the contact term algebra, by contrast to what happen in topological gravity. In fact from the contact term algebra we can only get relations of the type:

\[
(\Gamma^k_{ij} - \Gamma^k_{ji}) \langle \mathcal{O}_{k,0} \prod_{l=1}^{N} \mathcal{O}_{l,0} \rangle = \sum_{l=1}^{N} R_{D_l} + \sum_{\text{nodes}} R_{\Delta}
\]

(51)

which does not imply $(\Gamma^k_{ij} - \Gamma^k_{ji} = 0)$ anything on the surface contribution. Moreover they are compatible with making all contact terms $R_{D_i}$ equal to zero by covariantization.

2) If in the computation of $\langle \mathcal{O}_{\bar{t},2} \prod_{i=1}^{N} \mathcal{O}_{i,0} \rangle$ we take into account all contact terms, i.e contact terms between the $\mathcal{O}_{i,0}$ operators, we will find, as a consequence of the derivation rules (13) and the $(t, \bar{t})$ equations (44), that the holomorphic anomaly is cancelled, reflecting the commutativity of ordinary derivatives $[\partial_t, \partial_{\bar{t}}] = 0$.

3) We should say that from the contact term algebra we can not prove, at least directly, that the correlators $\langle \mathcal{O}_{\bar{t},2} \prod_{i=1}^{N} \mathcal{O}_{i,0} \rangle$ are saturated by contact terms. The fact we have proved is that the contact term contribution dictated by the contact term algebra (10) is precisely the holomorphic anomaly.

4) The curvature of the initial surface is augmented by two units in both processes of pinching a handle or factorizing the surface. In order to take this into account, the two insertions $\mathcal{O}_{\alpha,0}, \mathcal{O}_{\beta,0}$ generated in these processes should include, in addition, an extra unit of curvature. Therefore, the total balance of curvature for the new insertions is zero. This can be seen as the reason for the zero contact term between the dilaton field $\sigma_1$ and the antitopological operators $\mathcal{O}_{\bar{t},2}$ (see equations (28) and (30)).

Final Comments

The main result of this letter was the derivation of the $(t, \bar{t})$-equations as consistency conditions of a contact term algebra. Using this point of view we get a more direct understanding on the connections between the $(t, \bar{t})$-equations, the holomorphic anomaly and the dynamical role of the dilaton field. Moreover the relation between the $(t, \bar{t})$-equations and contact term algebras shed some light on its very topological meaning.

Some open problems are suggested by our analysis that deserve a more careful study. Next we briefly mention some of them. A more mathematically understanding of the
gravitational picture we are using in this letter, will require to study the equivariant cohomology \([11, 12]\) which characterize the physical states of the topological string theory obtained by coupling to gravity \(\hat{c} = 3\), \(N = 2\) SCFT’s, in the way prescribed by (1) and (2).

It is well known that the contact term algebra of pure topological gravity is equivalent to the Virasoro constraints for matrix models \([16]\), thus it is natural to ask what would be the analog for the contact term algebras introduced in this letter. More interesting will be to generalize our derivation of the \((t, \bar{t})\)-equations to the massive case.

From a more fundamental point of view and following the line of thought initiated in references \([2, 9]\), we can conceive the holomorphic anomaly as a way to define background independence in string field theory. In this spirit the topological theory defined by the contact term algebra seems to indicate the type of dynamics would be necessary to add in order to get background independence.

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