TWO LINEAR, UNCONDITIONALLY STABLE, SECOND ORDER DECOUPLING METHODS FOR THE ALLEN–CAHN–NAVIER–STOKES PHASE FIELD MODEL

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Abstract. Hydrodynamics coupled phase field models have intricate difficulties to solve numerically as they feature high nonlinearity and great complexity in coupling. In this paper, we propose two second order, linear, unconditionally stable decoupling methods based on the Crank–Nicolson leap-frog time discretization for solving the Allen–Cahn–Navier–Stokes (ACNS) phase field model of two-phase incompressible flows. The ACNS system is decoupled via the artificial compression method and a splitting approach by introducing an exponential scalar auxiliary variable. We prove both algorithms are unconditionally long time stable. Numerical examples are provided to verify the convergence rate and unconditional stability.

Key words. phase field model, Allen–Cahn–Navier–Stokes, artificial compression, Crank–Nicolson leap-frog, scalar auxiliary variable, unconditional stability

1. Introduction. Phase field or diffuse interface models are widely used to study the interfacial dynamics in many scientific and engineering applications [1, 9, 10, 11]. This topic has received an increasing amount of attention from the research community in mathematical and numerical analysis in the past decade. In particular, the ACNS phase field model is popularly used to describe the motion of a mixture of two incompressible fluids [31, 46]. It features a nonlinear system consisting of the incompressible Navier–Stokes (NS) equations coupled with the Allen–Cahn (AC) equations. In the model, a continuous phase field function \( \phi \) is introduced to label different fluid components while their sharp interface is implicitly tracked by a thin but smooth transition layer, i.e. the diffuse interface. By studying the evolution of the phase field function, one can avoid explicit interface tracking and perform simulations on a fixed mesh grid, rendering a convenient numerical approach to simulate various interfacial problems. The dynamics of the phase field variable is described by the Allen–Cahn equation, obtained by the gradient flow method, namely, minimizing the total free energy in the space of \( L^2 \).

There exist many effective numerical schemes for each component of the ACNS system. For instance, the projection method [5, 32, 12, 14] and artificial compression method [7] for the incompressible Navier-Stokes equations, methods for phase-field type models including the convex splitting method [33, 2, 18], the Lagrange multiplier approach [15], the Invariant Energy Quadratization (IEQ) method [44, 11, 45], and the Scalar Auxiliary Variable approach (SAV) [35, 43], etc. For solving a hydrodynamics coupled phase field model, however, a simple combination of the methods from each component may not produce an accurate, efficient, and unconditionally long time
stable scheme. The main challenges lie in the high nonlinearity in the AC and NS equations, the coupling between the phase field variable $\phi$ and velocity $u$ through a phase induced stress term in the NS equations and a fluid induced transport term in the AC equations, and the coupling between the fluid velocity $u$ and pressure $p$ in the Navier–Stokes equations.

Due to the intricate complexity of the ACNS system, most of the schemes developed in the literature are either first-order time accurate [36, 37], or nonlinear and coupled schemes [16, 41], or fully decoupled but without provable energy stability analysis [8, 19], see an overview also in [16, 42]. The first method to have all the desired characteristics (i.e., linear, unconditionally energy stable, fully-decoupled, and second-order accurate in time) for solving the NS equations coupled with mass-conserved AC phase field model is based on the second-order backward differentiation formula (BDF2) for time stepping and the projection method for velocity and pressure decoupling [42]. In the scheme a nonlocal variable and a related ordinary differential equation are introduced to split the nonlinear coupling terms, and a stabilization term is added to maintain energy stability for large time steps. However, the projection method used in [42] is only first-order accurate for pressure due to the artificial boundary conditions applied to the pressure, and one needs to solve a Poisson equation in each time step to update the pressure. The discretization method [42] simply makes the NS equations explicit, this generally loses accuracy and feasibility for moderate Reynolds number, as observed in [24]. Moreover, the scheme demands inserting an extra stabilization term to maintain stability. Therefore, to develop simple, efficient, and unconditionally long time stable high order time stepping methods is still in great need.

While existing methods are based on either the Crank–Nicolson or the BDF2 time stepping, we focus on the Crank–Nicolson leap-frog time discretization (CNLF) [27, 20, 21, 22] to develop two efficient, linear, unconditionally long time stable decoupling schemes. The CNLF method is commonly used in the atmospheric and oceanic simulations for its high accuracy, but has been less theoretically studied in the literature, and used for other engineering applications. Herein we design two numerical schemes based on CNLF for the phase field model and relevant applications. In the first scheme, we adopt an idea of Lagrange multiplier of [15] to linearize the AC equations, then combine it with the artificial compression technique for decoupling the velocity and pressure in the NS equations. Unlike the most frequently employed projection type methods which were first introduced by Chorin [5] and Temam [40] in the late 1960s, the artificial compression methods which were also first studied in the sixties by Chorin [4], Temam [33, 40], Kuznetsov, Vladimirova and Yanenko [25], are less studied in the literature and have only recently received increasing attention. The artificial compression methods relax the incompressibility constraint in the NS equations by adding a perturbation, e.g., $\epsilon \partial_p$, to the mass conservation equation which facilitates decoupling the computation of velocity and pressure in time marching schemes, see [4, 8, 26, 13, 6] for recent developments. It is worth noting the outstanding feature of the artificial compression methods is that the pressure can be updated directly without solving a Poisson equation which avoids the spurious oscillations in the boundary layer of pressure due to artificial boundary conditions required in a projection method. We will derive a linear and unconditionally stable scheme that is partially decoupled. Despite the fact that the computation of the phase field variable is still coupled with that of the velocity, we only need to solve a linear system in reduced size without the use of Picard/Newton iterations, which significantly reduces
the computational cost.

The second idea is to incorporate the artificial compression technique with an SAV decoupling strategy for developing a highly efficient, fully decoupled scheme. The SAV approach was first studied in [34, 35] for gradient flows. It introduces a new scalar auxiliary variable that will be used to form a modified system of the underlying partial differential equation (PDE) system so that the nonlinear part can be canceled out in time discretization, leading to unconditionally stable methods for solving nonlinear systems [28, 30]. Following the SAV ideas in [23, 29], we find it is also possible to cancel out the coupling terms that usually lead to the time step constraints in a typical decoupling method. This is achieved by introducing a scalar auxiliary variable to handle the lagged coupling terms in the ACNS model, namely the phase induced stress term in the NS equations and the fluid induced transport term in the AC equations. A modified PDE system which is equivalent to the original ACNS model is then formulated, and an efficient, fully decoupled discretization method can be derived and proved to be long time stable without any time step constraints. The SAV decoupling strategy proposed here can be easily extended to other popular time stepping methods and produce a family of unconditionally stable numerical schemes.

The rest of this article is organized as follows. In Section 2 we briefly introduce the ACNS model. In Section 3 the CNLFAC numerical scheme is presented and proved to be unconditionally long time stable. In Section 4 the fully decoupled ACSAV scheme is proposed with proved unconditional stability and implementation details. In Section 5 we perform various numerical simulations to demonstrate the stability, accuracy and feasibility of the two proposed algorithms. Some conclusion remarks are made in Section 6.

2. The ACNS model. Consider the modeling of a mixture of two immiscible, incompressible fluids in a bounded Lipschitz domain Ω in $\mathbb{R}^d$ ($d = 2, 3$). To label the two different fluids, a phase-field variable (macroscopic labeling function) $\phi$ is introduced, i.e.

$$\phi(x, t) = \begin{cases} 1, & \text{for fluid 1,} \\ -1, & \text{for fluid 2,} \end{cases}$$

with the discontinuity of the function smoothed by a thin, smooth transition region of width $O(\eta)$ ($\eta \ll 1$). The total energy $W$ of the system is a sum of the kinetic energy and the Ginzburg-Landau type of Helmholtz free energy:

$$W = \int_{\Omega} \left( \frac{1}{2} |u|^2 + \lambda \left( \frac{|\nabla \phi|^2}{2} + F(\phi) \right) \right) dx,$$

where $u$ is the fluid velocity, $\lambda$ is related to the surface tension parameter. The second term in the expression of $W$ contributes to the tendency of mixing between the materials, while the third term, the double-well bulk energy $F(\phi) = \frac{\eta}{4\alpha} (\phi^2 - 1)^2$, represents the tendency of separation. As the consequence of the competition between the two types of interactions, a diffusive interface with thickness proportional to the parameter $\eta$ will form in equilibrium.

Assuming a generalized Ficks law holds and the fluid is incompressible, the governing equations of the Allen-Cahn-Navier-Stokes model are derived by minimizing
the total energy in the space of $L^2$. It writes as

\[
\begin{cases}
\partial_t \phi + u \cdot \nabla \phi = -M\mu, \\
\mu = \frac{\delta W}{\delta \phi} = \lambda(-\Delta \phi + f(\phi)), \\
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = \mu \nabla \phi, \\
\nabla \cdot u = 0,
\end{cases}
\]

where $f(\phi) = F'(\phi) = \frac{(\phi^2 - 1)}{\eta^2}(\phi^3 - \phi)$, $\mu = \frac{\delta W}{\delta \phi}$ is the variational derivative or chemical potential, $p$ is the pressure, $M$ is the relaxation or mobility parameter of the phase function, and $\nu$ is the viscosity parameter.

We assume the following boundary conditions for the simplicity of the presentation of the analysis:

\[u|_{\partial \Omega} = 0, \quad \partial_n \phi|_{\partial \Omega} = 0, \quad \partial_n \mu|_{\partial \Omega} = 0.\]

The analysis can extended to other boundary conditions with minor modification.

Throughout this paper the $L^2(\Omega)$ norm of scalars, vectors, and tensors will be denoted by $\| \cdot \|$ with the usual $L^2$ inner product denoted by $(\cdot, \cdot)$. By taking the $L^2$ inner product of (2.1) with $\mu$, (2.2) with $-\partial_t \phi$, (2.3) with $u$, and then using (2.4) and summing up the resulted identities we can easily get

\[
\frac{dW}{dt} = -M\|\mu\|^2 - \nu\|\nabla u\|^2.
\]

This means the total energy of the ACNS system is dissipative.

3. CNLFAC Method for the ACNS model. In this section we introduce the Crank–Nicolson leap-frog artificial compression method (CNLFAC) in the semi-discrete form and prove it is unconditionally long time stable.

In the ACNS equations, the function $f(\phi) = \frac{1}{\eta}((\phi^3 - \phi)$ is nonlinear and usually results in nonlinear semi-discrete schemes that require Picard/Newton iterations for computation. This makes the computation expensive and takes more simulation time. For differential equations that involve nonlinearity, linear schemes are usually desirable but difficult to design due to stability and convergence issues. For ACNS, only a few fully linear time stepping schemes are available and most of them are conditionally stable. A recent paper by Han etc [16] proposed an unconditionally stable, second order, linear scheme for the ACNS equations adopting an idea of Lagrange multiplier of [15]. Here we adopt the same idea and combine it with an artificial compression method based on the Crank–Nicolson leap-frog time stepping method.

Let $q = \frac{1}{\eta}((\phi^3 - \phi)$ and thus $f(\phi) = \phi q$. Taking derivative of $q$ with respect to $t$ gives $\partial_t q = \frac{2}{\eta^2} \phi \partial_t \phi$, which can be discretized by the Crank–Nicolson leap-frog scheme as

\[
\frac{q^{n+1} - q^{n-1}}{2\Delta t} = \frac{2}{\eta^2} \phi^n \phi^{n+1} - \phi^{n-1}. \]

By introducing the variable $q$, the total energy can be written as

\[
W = \int_{\Omega} \left( \frac{1}{2} |u|^2 + \lambda \left( \frac{1}{2} |\nabla \phi|^2 + \frac{\eta^2}{4} q^2 \right) \right) dx. \quad (3.1)
\]
We will use \( q \) as an intermediate variable in our algorithm so that a fully linear scheme could be devised. Let \( t_n = n\Delta t, n = 0, 1, 2, \cdots, N \), where \( N = T/\Delta t \), denote a uniform partition of the interval \([0, T]\). We now propose an unconditionally stable, second order, linear, artificial compression method given by

**Algorithm 3.1.** Given \( u^{n-1}, u^n, p^{n-1}, p^n, \phi^{n-1}, \phi^n, q^{n-1}, q^n \), find \( u^{n+1}, p^{n+1}, \phi^{n+1} \) and \( q^{n+1} \) satisfying

\[
\begin{align*}
\dot{\phi}^{n+1} - \lambda M \left( \Delta \phi^{n+1} + \phi^{n-1} \right) &= 0, \\
q^{n+1} - q^{n-1} &= \frac{2}{\eta} \phi^n (\phi^{n+1} - \phi^{n-1}), \\
u^{n+1} - \nu^{n-1} &= \frac{2}{\Delta t} \nabla \cdot \left( u^{n+1} + u^{n-1} \right) - \beta \Delta t^{-1} \nabla \left( \nabla \cdot u^{n+1} - \nabla \cdot u^{n-1} \right) + (u^n \cdot \nabla) \left( \frac{u^{n+1} + u^{n-1}}{2} \right) + \frac{1}{2} \nabla \cdot \left( \frac{u^{n+1} + u^{n-1}}{2} \right) \\
&\quad - \nu \Delta \left( \frac{u^{n+1} + u^{n-1}}{2} \right) + \nabla p^n + \frac{1}{M} \phi^{n+1} \nabla \phi^n = 0,
\end{align*}
\]

where

\[
\dot{\phi}^{n+1} = \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} + \left( \frac{u^{n+1} + u^{n-1}}{2} \right) \cdot \nabla \phi^n.
\]

Note that we could rewrite (3.3) as

\[
q^{n+1} = q^{n-1} + \frac{2}{\eta} \phi^n (\phi^{n+1} - \phi^{n-1}),
\]

and replace \( q^{n+1} \) in (3.2) so that (3.2) becomes

\[
\dot{\phi}^{n+1} - \lambda M \left( \Delta \phi^{n+1} + \phi^{n-1} \right) + \lambda M \phi^n \left( \frac{1}{\eta} \phi^n (\phi^{n+1} - \phi^{n-1}) + q^{n-1} \right) = 0.
\]

Therefore adding the intermediate variable \( q \) does not increase the computational cost. One still only needs to solve for the primary variables \( u, p, \text{and } \phi \) while \( q \) is updated directly using the formula (3.6) in the procedure.

In this algorithm, the AC method is incorporated by adding a small perturbation of \( \partial_t p \) to the mass conservation equation, and discretized as \( 2\alpha \Delta t^2 \cdot \frac{p^{n+1} - p^{n-1}}{2\Delta t} \), see (3.6). So the scheme has \( O(\Delta t^2) \) consistency error. The pressure \( p^{n+1} \) can then be updated directly without solving an additional Poisson equation which is required by the projection methods.

**Theorem 3.2 (Stability of Algorithm 3.1).** Taking \( \alpha \) and \( \beta \) such that \( \alpha \beta \geq \frac{1}{4} \), then for any \( N \geq 2 \)

\[
\begin{align*}
\Delta t \sum_{n=1}^{N-1} \frac{1}{M} \||\phi^{n+1}\||^2 + \left( \frac{\lambda}{4} \||\nabla \phi^N||^2 + \frac{\lambda}{4} \||\nabla \phi^{N-1}||^2 \right) + \frac{\lambda \eta^2}{8} \left( ||q^N||^2 + ||q^{N-1}||^2 \right) \\
+ \Delta t \sum_{n=1}^{N-1} \nu \left( \||\nabla u^{n+1} + u^{n-1}||^2 + \frac{1}{2} \left( ||u^{n}||^2 + ||u^{n-1}||^2 \right) \right)
\end{align*}
\]
\[
\begin{align*}
\leq \left( \frac{\lambda}{4} \| \nabla \phi^1 \|^2 + \frac{\lambda}{4} \| \nabla \phi^0 \|^2 \right) + \frac{\lambda \eta^2}{8} (\| q^1 \|^2 + \| q^0 \|^2) \\
+ \frac{1}{4} \left( \| u^1 \|^2 + 2 \beta \| \nabla \cdot u^1 \|^2 \right) + \frac{1}{4} \left( \| u^0 \|^2 + 2 \beta \| \nabla \cdot u^0 \|^2 \right) \\
+ \frac{1}{2} \alpha \Delta t^2 \left( \| p^1 \|^2 + \| p^0 \|^2 \right) + \frac{1}{2} \Delta t (p^1, \nabla \cdot u^0) - \frac{1}{2} \Delta t (p^0, \nabla \cdot u^1).
\end{align*}
\]

**Proof.** Taking the inner product of (3.2) with \( \frac{2^{n+1} - 2^{n-1}}{2 M \Delta t} \) gives

\[
\begin{align*}
\frac{1}{M} \| \phi^{n+1} \|^2 - \frac{1}{M} \left( \frac{\phi^{n+1} + u^{n-1}}{2}, \nabla \phi^n \right) \\
+ \frac{\lambda}{\Delta t} \left( \frac{1}{4} \| \nabla \phi^{n+1} \|^2 - \frac{1}{4} \| \nabla \phi^{n-1} \|^2 \right) \\
+ \frac{\lambda}{2 \Delta t} \left( \frac{\phi^n q^{n+1} + q^{n-1}}{2}, \phi^{n+1} - \phi^{n-1} \right) = 0.
\end{align*}
\]

Taking inner product of (3.3) with \( \frac{\lambda \eta^2 \| q^{n+1} \|^2 - \| q^{n-1} \|^2}{8 \Delta t} \) gives

\[
\left( \frac{\lambda \eta^2 \| q^{n+1} \|^2 - \| q^{n-1} \|^2}{8 \Delta t} \right) = \frac{\lambda}{2 \Delta t} \left( \phi^n (\phi^{n+1} - \phi^{n-1}), \frac{q^{n+1} + q^{n-1}}{2} \right). \tag{3.8}
\]

Taking inner product of (3.4) with \( \frac{u^{n+1} + u^{n-1}}{2} \) gives

\[
\begin{align*}
\frac{1}{4 \Delta t} \left( \| u^{n+1} \|^2 + 2 \beta \| \nabla \cdot u^{n+1} \|^2 \right) - \frac{1}{4 \Delta t} \left( \| u^{n-1} \|^2 + 2 \beta \| \nabla \cdot u^{n-1} \|^2 \right) \\
+ \nu \| \nabla \frac{u^{n+1} + u^{n-1}}{2} \|^2 - \frac{1}{2} (p_n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1}) \\
+ \frac{1}{M} \left( \phi^{n+1} \nabla \phi^n, \frac{u^{n+1} + u^{n-1}}{2} \right) = 0.
\end{align*}
\]

Taking inner product of (3.5) with \( \frac{u^{n+1} + u^{n-1}}{2} \) yields

\[
\frac{1}{2} \alpha \Delta t \left( \| p^{n+1} \|^2 - \| p^{n-1} \|^2 \right) + \frac{1}{2} (\nabla \cdot u^n, p^{n+1} + p^{n-1}) = 0. \tag{3.10}
\]

By adding (3.7), (3.8), (3.9) and (3.10) we have

\[
\begin{align*}
\frac{1}{M} \| \phi^{n+1} \|^2 + \frac{\lambda}{\Delta t} \left( \frac{1}{4} \| \nabla \phi^{n+1} \|^2 + \frac{1}{4} \| \nabla \phi^n \|^2 \right) - \frac{\lambda \eta^2 \| q^{n+1} \|^2 - \| q^{n-1} \|^2}{8 \Delta t} \\
+ \nu \| \nabla \frac{u^{n+1} + u^{n-1}}{2} \|^2 + \frac{1}{4 \Delta t} \left( \| u^{n+1} \|^2 + 2 \beta \| \nabla \cdot u^{n+1} \|^2 \right) \\
- \frac{1}{4 \Delta t} \left( \| u^{n-1} \|^2 + 2 \beta \| \nabla \cdot u^{n-1} \|^2 \right) \\
+ \frac{1}{4 \Delta t} \left( \| u^n \|^2 + 2 \beta \| \nabla \cdot u^n \|^2 \right) - \frac{1}{4 \Delta t} \left( \| u^{n-1} \|^2 + 2 \beta \| \nabla \cdot u^{n-1} \|^2 \right) \\
+ \frac{1}{2} \alpha \Delta t \left( \| p^{n+1} \|^2 - \| p^{n-1} \|^2 \right) + \frac{1}{2} \alpha \Delta t \left( \| p^n \|^2 - \| p^{n-1} \|^2 \right) \\
+ \frac{1}{2} (\nabla \cdot u^n, p^{n+1} + p^{n-1}) - \frac{1}{2} (p^n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1}) = 0.
\end{align*}
\]
The last two terms of (3.11) can be rewritten as

\[
\frac{1}{2} (\nabla \cdot u^n, p^{n+1} + p^{n-1}) - \frac{1}{2} (p^n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1})
\]

\[
= \frac{1}{2} [ (p^{n+1}, \nabla \cdot u^n) - (p^n, \nabla \cdot u^{n-1}) ] - \frac{1}{2} [ (p^n, \nabla \cdot u^{n+1}) - (p^{n-1}, \nabla \cdot u^n) ].
\]

Then summing up (3.11) from \( n = 1 \) to \( n = N - 1 \) and multiplying through by \( \Delta t \) gives

\[
\Delta t \sum_{n=1}^{N-1} \frac{1}{M} ||\phi^{n+1}||^2 + \left( \frac{\lambda}{4} ||\nabla \phi^N||^2 + \frac{\lambda}{4} ||\nabla \phi^{N-1}||^2 \right) + \frac{\lambda \eta^2}{8} (||q^N||^2 + ||q^{N-1}||^2)
\]

\[
+ \Delta t \sum_{n=1}^{N-1} \nu \frac{||\nabla u^{n+1} + u^{n-1}||^2}{2} + \frac{1}{4} (||u^N||^2 + 2\beta ||\nabla \cdot u^N||^2)
\]

\[
+ \frac{1}{4} (||u^{N-1}||^2 + 2\beta ||\nabla \cdot u^{N-1}||^2) + \frac{1}{4} \alpha t^2 (||p^N||^2 + ||p^{N-1}||^2)
\]

\[
+ \frac{1}{2} \Delta t (p^N, \nabla \cdot u^N) - \frac{1}{2} \Delta t (p^{N-1}, \nabla \cdot u^{N-1}) \quad (3.12)
\]

\[
= \left( \frac{\lambda}{4} ||\nabla \phi^1||^2 + \frac{\lambda}{4} ||\nabla \phi^0||^2 \right) + \frac{\lambda \eta^2}{8} (||q^1||^2 + ||q^0||^2)
\]

\[
+ \frac{1}{4} (||u^1||^2 + 2\beta ||\nabla \cdot u^1||^2) + \frac{1}{4} (||u^0||^2 + 2\beta ||\nabla \cdot u^0||^2)
\]

\[
+ \frac{1}{2} \alpha t^2 (||p^1||^2 + ||p^0||^2) + \frac{1}{2} \Delta t (p^1, \nabla \cdot u^0) - \frac{1}{2} \Delta t (p^0, \nabla \cdot u^1).
\]

The last two terms on the left hand side of (3.12) can be bounded as

\[
\left| \frac{1}{2} \Delta t (p^N, \nabla \cdot u^{N-1}) - \frac{1}{2} \Delta t (p^{N-1}, \nabla \cdot u^N) \right|
\]

\[
\leq \frac{\beta}{2} ||\nabla \cdot u^{N-1}||^2 + \frac{1}{8\beta} \Delta t^2 ||p^N||^2 + \frac{\beta}{2} ||\nabla \cdot u^N||^2 + \frac{1}{8\beta} \Delta t^2 ||p^{N-1}||^2.
\]

So if \( \alpha \geq \frac{1}{4\beta} \), (3.12) reduces to

\[
\Delta t \sum_{n=1}^{N-1} \frac{1}{M} ||\phi^{n+1}||^2 + \left( \frac{\lambda}{4} ||\nabla \phi^N||^2 + \frac{\lambda}{4} ||\nabla \phi^{N-1}||^2 \right) + \frac{\lambda \eta^2}{8} (||q^N||^2 + ||q^{N-1}||^2)
\]

\[
+ \Delta t \sum_{n=1}^{N-1} \nu \frac{||\nabla u^{n+1} + u^{n-1}||^2}{2} + \frac{1}{4} (||u^N||^2 + ||u^{N-1}||^2)
\]

\[
\leq \left( \frac{\lambda}{4} ||\nabla \phi^1||^2 + \frac{\lambda}{4} ||\nabla \phi^0||^2 \right) + \frac{\lambda \eta^2}{8} (||q^1||^2 + ||q^0||^2)
\]

\[
+ \frac{1}{4} (||u^1||^2 + 2\beta ||\nabla \cdot u^1||^2) + \frac{1}{4} (||u^0||^2 + 2\beta ||\nabla \cdot u^0||^2)
\]

\[
+ \frac{1}{2} \alpha t^2 (||p^1||^2 + ||p^0||^2) + \frac{1}{2} \Delta t (p^1, \nabla \cdot u^0) - \frac{1}{2} \Delta t (p^0, \nabla \cdot u^1).
\]
4. ACSAV Method for the ACNS model. Inspired by the SAV ideas in [23, 29], we now proceed to incorporate the artificial compression technique with an SAV decoupling strategy for developing a highly efficient, fully decoupled scheme. This is achieved by introducing a scalar auxiliary variable to handle the lagged coupling terms in the ACNS model, namely the phase induced stress term in the NS equations and the fluid induced transport term in the AC equations. An artificial compression based scalar auxiliary variable scheme (ACSAV) is then derived and proved to be long time stable without any time step constraints.

Define a scalar auxiliary variable \( r(t) \) by

\[
r(t) = \exp\left(-\frac{t}{T}\right).
\]

Then we have

\[
\frac{dr}{dt} = \frac{1}{T}r + \frac{1}{\exp\left(-\frac{t}{T}\right)} \int_{\Omega} \left( (u \cdot \nabla \phi)\mu - (u \cdot \nabla \phi) \mu \right) dx.
\] (4.1)

To decouple the phase field variable \( \phi \) and fluid velocity \( u \), one usually needs to lag the coupling terms \( u \cdot \nabla \phi \) in (4.1) and \( \mu \nabla \phi \) in (2.3) to the previous time steps and this inevitably results in a time step condition to ensure long time stability. The introduction of the zero term \( \int_{\Omega} \left( (u \cdot \nabla \phi)\mu - (u \cdot \nabla \phi) \mu \right) dx \) in (4.1) makes it possible to cancel out the same term in the partitioned scheme with lagged coupling terms.

The governing equations of the Allen-Cahn-Navier-Stokes (ACNS) system are then equivalent to:

\[
\begin{aligned}
\partial_t \phi + \frac{r(t)}{\exp\left(-\frac{t}{T}\right)} u \cdot \nabla \phi &= -M \mu, \\
\mu &= \lambda (-\Delta \phi + q \phi), \\
\partial_t q &= \frac{2}{\eta} \phi \partial_t \phi, \\
\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= \frac{r(t)}{\exp\left(-\frac{t}{T}\right)} \mu \nabla \phi, \\
\nabla \cdot u &= 0, \\
\frac{dr}{dt} &= -\frac{1}{T} r + \frac{1}{\exp\left(-\frac{t}{T}\right)} \int_{\Omega} \left( (u \cdot \nabla \phi)\mu - (u \cdot \nabla \phi) \mu \right) dx.
\end{aligned}
\]

We now propose an unconditionally stable, second order, linear, fully decoupled ACSAV method given by

**Algorithm 4.1.** Given \( u^{n-1}, u^n, p^{n-1}, p^n, \phi^{n-2}, \phi^n, q^{n-1}, q^n, r^n, \) and \( r^{n+1}, p^{n+1}, q^{n+1}, \) and \( r^{n+1} \) satisfying \( r^n, \phi^{n-1}, \phi^n, q^{n-1}, q^n, \phi^{n+1}, \) and \( q^{n+1} \), find \( u^{n+1}, p^{n+1}, p^{n+1}, q^{n+1}, \phi^{n+1}, \) and \( \phi^{n+1} \) satisfying

\[
\begin{aligned}
\frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} + \frac{r^{n+1} + r^{n-1}}{2\exp\left(-\frac{t}{T}\right)} u^n \cdot \nabla \phi^n &= -M \bar{\mu}^n, \\
\bar{\mu}^n &= \lambda (-\Delta \phi^{n+1} + \phi^{n+1}) + \frac{q^{n+1} + q^{n-1}}{2} \phi^n, \\
\frac{q^{n+1} - q^{n-1}}{2\Delta t} &= \frac{2}{\eta^2} \phi^n \phi^{n+1} - \phi^{n-1}, \\
\frac{u^{n+1} - u^{n-1}}{2\Delta t} &= (u^n \cdot \nabla) \left( \frac{u^{n+1} + u^{n-1}}{2} \right).
\end{aligned}
\] (4.2) (4.3) (4.4)
\[ + \frac{1}{2} (\nabla \cdot u^n) \left( \frac{u^{n+1} + u^{n-1}}{2} - \beta \Delta t^{-1} \nabla (\nabla \cdot u^{n+1} - \nabla \cdot u^{n-1}) \right) \] (4.5)

\[ - \nu \Delta \left( \frac{u^{n+1} + u^{n-1}}{2} \right) + \nabla p^n = \frac{r^{n+1} + r^{n-1}}{2 \exp\left(-\frac{r}{T}\right)} \mu^n \nabla \phi^n, \]

\[ \mu^n = -\frac{1}{M} \left( \frac{3\phi^n - 4\phi^{n-1} + \phi^{n-2}}{2\Delta t} + \frac{r^n}{\exp\left(-\frac{r}{T}\right)} u^n \cdot \nabla \phi^n \right), \] (4.6)

\[ \alpha \Delta t \left( p^{n+1} - p^{n-1} \right) + \nabla \cdot u^n = 0, \] (4.7)

\[ \frac{r^{n+1} - r^{n-1}}{2 \Delta t} = -\frac{1}{T} \frac{r^{n+1} + r^{n-1}}{2} + \frac{1}{\exp\left(-\frac{r}{T}\right)} \int_\Omega (u^n \cdot \nabla \phi^n) \tilde{\mu}^n \, dx \] (4.8)

\[ - \frac{1}{\exp\left(-\frac{r}{T}\right)} \int_\Omega \left( \frac{u^{n+1} - u^{n-1}}{2} \cdot \nabla \phi^n \right) \cdot \mu^n \, dx. \]

Note that we could rewrite (4.4) as (3.0) and replace \( q^{n+1} \) in (4.3) so that (4.3) becomes

\[ \tilde{\mu}^n = \lambda \left( - \frac{\phi^{n+1} + \phi^{n-1}}{2} \right) + \lambda \left( q^{n-1} + \frac{1}{\eta^2} \phi^n \left( \phi^{n+1} - \phi^{n-1} \right) \right) \phi^n. \] (4.9)

Therefore, as in the CNLFAC scheme, adding the intermediate variable \( q \) does not increase the computational cost. Though the variables \( \phi^{n+1} \) and \( u^{n+1} \) are coupled with \( r^{n+1} \), we will state later that an efficient splitting procedure can be employed to separate the computation of \( \phi^{n+1} \) from \( r^{n+1} \) and \( u^{n+1} \) from \( r^{n+1} \). At last, the calculation of \( \phi^{n+1}, u^{n+1}, r^{n+1} \) are truly completely decoupled.

**Theorem 4.2 (Stability of Algorithm (4.1)).** Taking \( \alpha \) and \( \beta \) such that \( \alpha \beta \geq \frac{1}{4} \), then for \( N \geq 3 \)

\[ \Delta t M \sum_{n=2}^{N-1} \||\tilde{\mu}^n||^2 + \left( \frac{\lambda}{4} ||\nabla \phi^N||^2 + \frac{\lambda}{4} ||\nabla \phi^{N-1}||^2 \right) + \frac{\lambda \eta^2}{8} (||q^N||^2 + ||q^{N-1}||^2) \]

\[ + \frac{1}{4} ||u^N||^2 + \frac{1}{4} ||u^{N-1}||^2 + \Delta t \sum_{n=2}^{N-1} \nu ||\nabla \frac{u^{n+1} + u^{n-1}}{2}||^2 \]

\[ + \frac{1}{4} (||r^N||^2 + ||r^{N-1}||^2) + \frac{\Delta t}{T} \sum_{n=2}^{N-1} \frac{r^{n+1} + r^{n-1}}{2} ||2^2 \]

\[ \leq \left( \frac{\lambda}{4} ||\nabla \phi^2||^2 + \frac{\lambda}{4} ||\nabla \phi^1||^2 \right) + \frac{\lambda \eta^2}{8} (||q^2||^2 + ||q^1||^2) \]

\[ + \frac{1}{4} (||u^2||^2 + 2\beta||\nabla \cdot u^2||^2) + \frac{1}{4} (||u^1||^2 + 2\beta||\nabla \cdot u^1||^2) \]

\[ + \frac{1}{2} \alpha \Delta t^2 (||p^2||^2 + ||p^1||^2) + \frac{1}{4} (||r^2||^2 + ||r^1||^2) \]

\[ + \frac{1}{2} \Delta t (p^2, \nabla \cdot u^1) - \frac{1}{2} \Delta t (p^1, \nabla \cdot u^2). \]

**Proof.** Taking the inner product of (4.2) with \( \tilde{\mu}^n \), the inner product of (4.3) with \( -\frac{\phi^{n+1} - \phi^{n-1}}{2 \Delta t} \), the inner product of (4.4) with \( \frac{\lambda \eta^2}{2} q^{n+1} + q^{n-1} \) and then adding the equations together gives

\[ \frac{r^{n+1} + r^{n-1}}{2 \exp\left(-\frac{r}{T}\right)} (u^n \cdot \nabla \phi^n, \tilde{\mu}^n) + M ||\tilde{\mu}^n||^2 + \frac{\lambda}{4 \Delta t} (||\nabla \phi^{n+1}||^2 - ||\nabla \phi^{n-1}||^2) \]

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\[ \frac{\lambda \eta^2}{8 \Delta t} (\|q^{n+1}\|^2 - \|q^{n-1}\|^2) = 0. \] (4.10)

Taking inner product of (4.5) with \( u^{n+1} + u^{n-1} \) gives
\[ \frac{1}{4 \Delta t} \left( \|u^{n+1}\|^2 + 2 \beta \|\nabla \cdot u^{n+1}\|^2 \right) - \frac{1}{4 \Delta t} \left( \|u^{n-1}\|^2 + 2 \beta \|\nabla \cdot u^{n-1}\|^2 \right) + \nu \|\nabla u^{n+1} + u^{n-1}\|^2 - \frac{1}{2} \left( p_n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1} \right) = \frac{r_{n+1} + r_{n-1}}{2 \exp(-\frac{t_m}{\Delta t})} \left( \mu \gamma \phi^{n+1} + u^{n+1} + u^{n-1} \right). \] (4.11)

Taking inner product of (4.7) with \( u^{n+1} + p^{n-1} \) yields
\[ \frac{1}{2} \Delta t \left( \|p^{n+1}\|^2 - \|p^{n-1}\|^2 \right) + \frac{1}{2} \left( \nabla \cdot u^n, p^{n+1} + p^{n-1} \right) = 0. \] (4.12)

Taking product of (4.8) with \( u^{n+1} + p^{n-1} \) yields
\[ \frac{1}{4 \Delta t} \left( \|u^{n+1}\|^2 - |r^{n-1}|^2 \right) - \frac{r_{n+1} + r_{n-1}}{2 \exp(-\frac{t_m}{\Delta t})} \int_\Omega (u^n \cdot \nabla \phi^n) \mu^n dx \]
\[ - \frac{r_{n+1} + r_{n-1}}{2 \exp(-\frac{t_m}{\Delta t})} \int_\Omega \left( \frac{|u^{n+1} + u^{n-1}|}{2} \cdot \nabla \phi^n \right) \cdot \mu^n dx. \] (4.13)

By adding (4.10), (4.11), (4.12) and (4.13) we have
\[ M \|\bar{\mu}^n\|^2 + \frac{\lambda}{4 \Delta t} \left( \|\nabla \phi^{n+1}\|^2 - \|\nabla \phi^{n-1}\|^2 \right) + \frac{\lambda \eta^2}{8 \Delta t} \left( \|q^{n+1}\|^2 - \|q^{n-1}\|^2 \right) + \frac{1}{4 \Delta t} \left( \|u^{n+1}\|^2 + 2 \beta \|\nabla \cdot u^{n+1}\|^2 \right) - \frac{1}{4 \Delta t} \left( \|u^{n-1}\|^2 + 2 \beta \|\nabla \cdot u^{n-1}\|^2 \right) + \nu \|\nabla u^{n+1} + u^{n-1}\|^2 + \frac{1}{2} \Delta t \left( \|p^{n+1}\|^2 - \|p^{n-1}\|^2 \right) + \frac{1}{4 \Delta t} \left( |r^{n+1}|^2 - |r^{n-1}|^2 \right) + \frac{1}{2} \left( p^n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1} \right) - \frac{1}{2} \left( p^n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1} \right) = 0. \] (4.14)

The last two terms on the left hand side of (4.14) can be rewritten as
\[ \frac{1}{2} \left( \nabla \cdot u^n, p^{n+1} + p^{n-1} \right) - \frac{1}{2} \left( p^n, \nabla \cdot u^{n+1} + \nabla \cdot u^{n-1} \right) = \frac{1}{2} \left[ (p^{n+1}, \nabla \cdot u^n) - (p^n, \nabla \cdot u^{n+1}) \right] - \frac{1}{2} \left[ (p^n, \nabla \cdot u^{n+1}) - (p^{n-1}, \nabla \cdot u^n) \right]. \]

Then summing up (4.13) from \( n = 2 \) to \( n = N-1 \) and multiplying through by \( \Delta t \) gives
\[ \Delta t M \sum_{n=2}^{N-1} \|\bar{\mu}^n\|^2 + \left( \frac{\lambda}{4} \|\nabla \phi^N\|^2 + \frac{\lambda}{4} \|\nabla \phi^{N-1}\|^2 \right) + \frac{\lambda \eta^2}{8} \left( \|q^N\|^2 + \|q^{N-1}\|^2 \right) \]
Here we present an efficient splitting procedure to separate the computation of $\alpha$ from $\beta$ original form of Algorithm 4.1, the variables $\alpha$ So if $\alpha \geq n$, (4.15) reduces to
\[
\frac{\Delta t}{T} \sum_{n=2}^{N-1} \nu \frac{\|u^{n+1} + u^{n-1}\|^2}{2} + \frac{1}{2} \alpha \Delta t^2 \left( \|p^N\|^2 + \|p^{N-1}\|^2 \right)
+ \frac{1}{4} (|r^N|^2 + |r^{N-1}|^2) + \frac{\Delta t}{T} \sum_{n=2}^{N-1} \frac{r^{n+1} + r^{n-1}}{2}^2
\]
\[
(4.15)
\]
\[
\frac{1}{2} \Delta t (p^N, \nabla \cdot u^{N-1}) - \frac{1}{2} \Delta t (p^{N-1}, \nabla \cdot u^N)
= \left( \frac{\lambda}{4} \|\nabla \phi^2\|^2 + \frac{\lambda}{4} \|\nabla \phi^1\|^2 \right) + \frac{\lambda \eta^2}{8} (\|q^N\|^2 + \|q^{N-1}\|^2)
+ \frac{1}{4} (\|u^2\|^2 + 2\beta \|\nabla \cdot u^2\|^2) + \frac{1}{4} (\|u^1\|^2 + 2\beta \|\nabla \cdot u^1\|^2)
+ \frac{1}{2} \alpha \Delta t^2 \left( \|p^2\|^2 + \|p^1\|^2 \right) + \frac{1}{4} (|r^2|^2 + |r^1|^2)
+ \frac{1}{2} \Delta t (p^2, \nabla \cdot u^1) - \frac{1}{2} \Delta t (p^1, \nabla \cdot u^2).
\]

The last two terms on the left hand side of (4.15) can be bounded as
\[
\frac{1}{2} \Delta t (p^N, \nabla \cdot u^{N-1}) - \frac{1}{2} \Delta t (p^{N-1}, \nabla \cdot u^N)
\leq \frac{\beta}{2} \|\nabla \cdot u^{N-1}\|^2 + \frac{1}{8\beta} \Delta t^2 \|p^N\|^2 + \frac{\beta}{2} \|\nabla \cdot u^N\|^2 + \frac{1}{8\beta} \Delta t^2 \|p^{N-1}\|^2.
\]

So if $\alpha \geq \frac{1}{8\beta}$, (4.15) reduces to
\[
\Delta t M \sum_{n=2}^{N-1} \|\tilde{p}^n\|^2 + \left( \frac{\lambda}{4} \|\nabla \phi^N\|^2 + \frac{\lambda}{4} \|\nabla \phi^N-1\|^2 \right) + \frac{\lambda \eta^2}{8} (\|q^N\|^2 + \|q^{N-1}\|^2)
+ \frac{1}{4} \|u^N\|^2 + \frac{1}{4} \|u^{N-1}\|^2 + \Delta t \sum_{n=2}^{N-1} \nu \frac{\|u^{n+1} + u^{n-1}\|^2}{2}
+ \frac{1}{4} (|r^N|^2 + |r^{N-1}|^2) + \frac{\Delta t}{T} \sum_{n=2}^{N-1} \frac{r^{n+1} + r^{n-1}}{2}^2
\leq \left( \frac{\lambda}{4} \|\nabla \phi^2\|^2 + \frac{\lambda}{4} \|\nabla \phi^1\|^2 \right) + \frac{\lambda \eta^2}{8} (\|q^2\|^2 + \|q^1\|^2)
+ \frac{1}{4} (\|u^2\|^2 + 2\beta \|\nabla \cdot u^2\|^2) + \frac{1}{4} (\|u^1\|^2 + 2\beta \|\nabla \cdot u^1\|^2)
+ \frac{1}{2} \alpha \Delta t^2 \left( \|p^2\|^2 + \|p^1\|^2 \right) + \frac{1}{4} (|r^2|^2 + |r^1|^2)
+ \frac{1}{2} \Delta t (p^2, \nabla \cdot u^1) - \frac{1}{2} \Delta t (p^1, \nabla \cdot u^2).
\]

\[\Box\]

**Implementation of the ACSAV Algorithm.** As mentioned before, in the original form of Algorithm \ref{alg:acsv}, the variables $\phi^{n+1}$ and $u^{n+1}$ are coupled with $r^{n+1}$. Here we present an efficient splitting procedure to separate the computation of $\phi^{n+1}$ from $r^{n+1}$ and $u^{n+1}$ from $r^{n+1}$. We will introduce a new scalar $V^{n+1}$ to decompose
\( \phi^{n+1} \) into two parts yielding two subproblems for the two components \( \hat{\phi}^{n+1}, \check{\phi}^{n+1} \) respectively, which do not contain \( r^{n+1} \). Similarly, we do a decomposition for \( u^{n+1} \). A separate equation for updating \( V^{n+1} \) (hence \( r^{n+1} \)) will be derived then.

Let

\[
V^{n+1} = \frac{r^{n+1} + r^{n-1}}{2\exp\left(\frac{-t_n}{T}\right)}, \quad \phi^{n+1} = \hat{\phi}^{n+1} + V^{n+1}\check{\phi}^{n+1}, \quad u^{n+1} = \hat{u}^{n+1} + V^{n+1}\check{u}^{n+1}.
\]

(4.16)

Then instead of solving (4.2), (4.9), and (4.8), we solve the following two subproblems for \( \hat{\phi}^{n+1}, \check{\phi}^{n+1} \) respectively.

\[
\hat{\phi}^{n+1} - \phi^{n-1} - \frac{1}{M} \Delta \left( \frac{\hat{\phi}^{n+1} + \phi^{n-1}}{2} \right) + \lambda M \phi^n \left( \frac{1}{\eta^2} \phi^n (\hat{\phi}^{n+1} - \phi^{n-1}) + q^{n-1} \right) = 0.
\]

(ACSAV subproblem 1)

\[
\check{\phi}^{n+1} = \frac{1}{2\Delta t} + u^n \cdot \nabla \phi^n - \lambda M \left( \frac{\Delta \check{\phi}^{n+1}}{2} \right) + \lambda M \phi^n \left( \frac{1}{\eta^2} \phi^n \check{\phi}^{n+1} \right) = 0.
\]

(ACSAV subproblem 2)

Instead of solving (4.5), (4.6), and (4.8), we solve the following two subproblems for \( \hat{u}^{n+1}, \check{u}^{n+1} \) respectively.

\[
\hat{u}^{n+1} - u^{n-1} + (u^n \cdot \nabla) \left( \frac{\hat{u}^{n+1} + u^{n-1}}{2} \right) + \frac{1}{2} (\nabla \cdot u^n) \left( \frac{\hat{u}^{n+1} + u^{n-1}}{2} \right) - \beta \Delta t^{-1} \nabla \left( \nabla \cdot \hat{u}^{n+1} - \nabla \cdot u^{n-1} \right) - \nu \Delta \left( \frac{\hat{u}^{n+1} + u^{n-1}}{2} \right) + \nabla p^n = 0.
\]

(ACSAV subproblem 3)

\[
\check{u}^{n+1} = \frac{1}{2\Delta t} + (u^n \cdot \nabla) \frac{\check{u}^{n+1}}{2} + \frac{1}{2} (\nabla \cdot u^n) \frac{\check{u}^{n+1}}{2} - \beta \Delta t^{-1} \nabla \left( \nabla \cdot \check{u}^{n+1} \right) - \nu \Delta \left( \frac{\check{u}^{n+1}}{2} \right) = \mu^n \nabla \phi^n.
\]

(ACSAV subproblem 4)

Now we can derive equation \( V^{n+1} \). From (4.16), we have

\[
r^{n+1} = 2\exp\left(-\frac{t_n}{T}\right)V^{n+1} - r^{n-1}.
\]

(4.17)

From the decomposition of \( \phi^{n+1} \) and equation (4.2) we have the splitting of \( \mu^n \) as

\[
\tilde{\mu}^n = \hat{\mu}^{n+1} + V^{n+1}\tilde{\mu}^{n+1},
\]

where

\[
\hat{\mu}^{n+1} = -\frac{1}{M} \left( \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} \right),
\]

(4.18)

\[
\tilde{\mu}^{n+1} = -\frac{1}{M} \left( \frac{\check{\phi}^{n+1}}{2\Delta t} + u^n \cdot \nabla \phi^n \right).
\]

(4.19)
Plugging the expression (4.17) of \( r^{n+1} \) into (4.13) gives
\[
(\frac{1}{\Delta t} + \frac{1}{T} \exp(\frac{2\mu}{T}) (V^{n+1})^2 - \frac{1}{\Delta t} \exp(\frac{t}{T}) \mu^{n-1} V^{n+1}) = (V^{n+1})^2 \int_{\Omega} (u^n \cdot \nabla \phi^n) \mu^{n+1} dx + V^{n+1} \int_{\Omega} (u^n \cdot \nabla \phi^n) \mu^{n+1} dx
\]
\[- (V^{n+1})^2 \int_{\Omega} (\frac{\tilde{u}^{n+1}}{2} \cdot \nabla \phi^n) \cdot \mu^n dx - V^{n+1} \int_{\Omega} (\frac{\tilde{u}^{n+1} + u^{n-1}}{2} \cdot \nabla \phi^n) \cdot \mu^n dx.
\]
Then we can compute \( V^{n+1} \) by solving
\[
A^{n+1} V^{n+1} + B^{n+1} = 0,
\]
where
\[
A^{n+1} = (\frac{1}{\Delta t} + \frac{1}{T} \exp(-\frac{2\mu}{T}) - \int_{\Omega} (u^n \cdot \nabla \phi^n) \tilde{\mu}^{n+1} dx + \int_{\Omega} (\frac{\tilde{u}^{n+1}}{2} \cdot \nabla \phi^n) \cdot \mu^n dx,
\]
\[
B^{n+1} = -\frac{\mu^{n-1}}{\Delta t} \exp(-\frac{t}{T}) - \int_{\Omega} (u^n \cdot \nabla \phi^n) \tilde{\mu}^{n+1} dx + \int_{\Omega} (\frac{\tilde{u}^{n+1} + u^{n-1}}{2} \cdot \nabla \phi^n) \cdot \mu^n dx.
\]

**Theorem 4.3.** The scalar equation (4.20) admits a solution.

**Proof.** Taking the inner product of (4.19) with \( \tilde{\mu}^{n+1} \), we get
\[
- \int_{\Omega} (u^n \cdot \nabla \phi^n) \tilde{\mu}^{n+1} dx = M \|\tilde{\mu}^{n+1}\|^2 + \frac{1}{2\Delta t} (\tilde{\phi}^{n+1}, \tilde{u}^{n+1}).
\]
Plugging (4.19) into ACSAV subproblem 2, and taking its inner product with \( \tilde{\phi}^{n+1} \), we obtain
\[
(\tilde{\phi}^{n+1}, \tilde{\mu}^{n+1}) = \frac{\lambda}{2} \|\nabla \tilde{\phi}^{n+1}\|^2 + \frac{\lambda}{2\eta^2} \|\phi^n \tilde{\phi}^{n+1}\|^2.
\]
Adding (4.23) and (4.24), we have
\[
- \int_{\Omega} (u^n \cdot \nabla \phi^n) \tilde{\mu}^{n+1} dx = M \|\tilde{\mu}^{n+1}\|^2 + \frac{\lambda}{4\Delta t} \|\nabla \phi^n \tilde{\phi}^{n+1}\|^2 + \frac{\lambda}{2\eta^2} \|\phi^n \tilde{\phi}^{n+1}\|^2 \geq 0.
\]
Taking the inner product of ACSAV subproblem 4 with \( \tilde{\phi}^{n+1} \), we get
\[
\int_{\Omega} (\frac{\tilde{u}^{n+1}}{2} \cdot \nabla \phi^n) \cdot \mu^n dx = \frac{1}{4\Delta t} \|\tilde{u}^{n+1}\|^2 + \frac{\beta}{2\Delta t} \|\nabla \tilde{u}^{n+1}\|^2 + \frac{\nu}{4} \|\nabla \tilde{u}^{n+1}\|^2 \geq 0.
\]
From (4.25), (4.26) and the expression of \( A^{n+1} \) in (4.21), we conclude that
\[
A^{n+1} > 0.
\]

Therefore, the scalar equation \( A^{n+1} V^{n+1} + B^{n+1} = 0 \) admits a solution.

To summarize, Algorithm 4.1 can be implemented in the following way:

- **Step 1:** Solve \( \tilde{\phi}^{n+1} \) and \( \phi^{n+1} \) from ACSAV subproblem 1 and 2 separately.
- **Step 2:** Compute \( \tilde{\mu}^{n+1} \) and \( \mu^{n+1} \) by (4.18) and (4.19).
- **Step 3:** Solve \( \tilde{u}^{n+1} \) and \( u^{n+1} \) from ACSAV subproblem 3 and 4 separately.
- **Step 4:** Calculate \( V^{n+1} \) from (4.20).
- **Step 5:** Calculate \( \phi^{n+1} \) and \( u^{n+1} \) from (4.19).
- **Step 6:** Calculate \( p^{n+1} \) from (4.7).
5. Numerical Experiments. In this section, we first verify the convergence order of the proposed CNLFAC scheme (Algorithm 3.1) and ACSAV scheme (Algorithm 4.1) with numerical examples. Then several standard numerical tests of the phase field fluid model are performed to verify the stability and effectiveness of these schemes.

5.1. Convergence Test. The studied phase field model does not have a natural forcing term, which makes it difficult to construct a solution to be used in a convergence test. Herein we verify the convergence rate of the algorithms by computing the Cauchy difference, as is done in [16]. Specifically, we run the simulations on four successively refined meshes with mesh size \( h = \frac{\sqrt{2}}{2^{l-1}}, \ l = 1, \ldots, 4 \). We also take a constant ratio of the time step and mesh size so that \( \Delta t = 0.05 \sqrt{2^l} h \). Then we compute the Cauchy difference of the solutions at two successive levels, namely \( v_{n+1}^{l+1} - v_n^{l+1} \), at time \( t^{n+1} = (n+1)\Delta t = 0.1 \), for \( v = \phi, u_1, u_2, p, \ l = 1, 2, 3, 4 \).

The initial condition for the phase variable \( \phi \) is set as \( \phi^0 = 0.24 \cos(2\pi x) \cos(2\pi y) + 0.4 \cos(\pi x) \cos(3\pi y) \).

For velocity and pressure, we solve the following Stokes problem with the Taylor-Hood element pair \( P_2 - P_1 \) and use the solution as the initial condition \((u^0, p^0)\):

\[
\begin{align*}
-\nu \Delta u + \nabla p &= \mu^0 \nabla \phi^0 \\
\nabla \cdot u &= 0
\end{align*}
\]

where \( \mu^0 = \lambda(-\Delta \phi^0 + f(\phi^0)) \). Note that the initial condition \((u^0, p^0)\) obtained from solving this problem satisfies the discrete inf-sup condition, which guarantees stability of the simulation in the first step.

The proposed CNLFAC algorithm is 3-level and the ACSAV algorithm is 4-level, so a 2-level method is needed to initialize the first two or three steps in time. We use the following first order, implicit algorithm to carry out this computation, which is studied for the Cahn-Hilliard-Navier-Stokes system in [10].

**Algorithm 5.1** (2-level, first order scheme for ACNS). Given \( u^n \) and \( \phi^n \), find \( u^{n+1}, p^{n+1}, \) and \( \phi^{n+1} \) satisfying

\[
\begin{align*}
\dot{\phi}^{n+1} - \lambda M (\Delta \phi^{n+1} - f_0(\phi^{n+1}, \phi^n)) &= 0, \quad (5.1) \\
\frac{u^{n+1} - u^n}{\Delta t} + u^n \cdot \nabla u^{n+1} - \nu \Delta u^{n+1} + \nabla p^{n+1} + \frac{1}{M} \dot{\phi}^{n+1} \cdot \nabla \phi^n &= 0, \quad (5.2) \\
\nabla \cdot u^{n+1} &= 0, \quad (5.3)
\end{align*}
\]

where

\[
\begin{align*}
\dot{\phi}^{n+1} &= \frac{\phi^{n+1} - \phi^n}{\Delta t} + u^{n+1} \cdot \nabla \phi^n, \\
f_0(\phi^{n+1}, \phi^n) &= 1 \frac{(\phi^{n+1})^2 + (\phi^n)^2 \phi^{n+1} + \phi^n}{2} - \frac{1}{2} \phi^n.
\end{align*}
\]

Algorithm 5.1 contains implicit nonlinear term \( f_0(\phi^{n+1}, \phi^n) \), which only appears in the first equation. So we use a Picard iteration algorithm to decouple the computation of \( \phi \) and \( u \) and apply Newton’s method to the first nonlinear equation for the update of \( \phi \). This procedure is listed in Algorithm 1.
Table 5.1
Cauchy differences of numerical solutions computed by the CNLFAC scheme with inputs \( \eta = 0.1, M = 10, \lambda = 0.0001, \nu = 0.8 \).

| \( l \) | \( \| \nabla \phi_{l+1}^{n+1} - \nabla \phi_{l+1}^n \| \) | rate | \( \| \nabla u_{l+1}^{n+1} - \nabla u_{l+1}^n \| \) | rate | \( \| p_{l+1}^{n+1} - p_{l+1}^n \| \) | rate |
|-------|------------------|------|------------------|------|------------------|------|
| 1     | 3.7409e-01       |      | 2.5467e-05       |      | 5.4504e-05       |      |
| 2     | 9.7100e-02       | 1.95 | 6.7258e-06       | 1.92 | 1.2777e-05       | 2.09 |
| 3     | 2.4751e-02       | 1.97 | 1.4333e-06       | 2.23 | 1.7790e-06       | 2.84 |
| 4     | 6.2438e-03       | 1.99 | 2.1241e-07       | 2.75 | 3.6667e-07       | 2.28 |

Table 5.2
Cauchy differences of numerical solutions computed by the ACSAV scheme with inputs \( \eta = 0.1, M = 10, \lambda = 0.0001, \nu = 0.8 \).

| \( l \) | \( \| \nabla \phi_{l+1}^{n+1} - \nabla \phi_{l+1}^n \| \) | rate | \( \| \nabla u_{l+1}^{n+1} - \nabla u_{l+1}^n \| \) | rate | \( \| p_{l+1}^{n+1} - p_{l+1}^n \| \) | rate |
|-------|------------------|------|------------------|------|------------------|------|
| 1     | 3.7412e-01       |      | 2.9741e-05       |      | 2.4582e-05       |      |
| 2     | 9.7102e-02       | 1.95 | 7.0300e-06       | 2.08 | 7.5678e-06       | 1.70 |
| 3     | 2.4751e-02       | 1.97 | 1.4496e-06       | 2.28 | 1.7154e-06       | 2.14 |
| 4     | 6.2438e-03       | 1.99 | 2.1416e-07       | 2.76 | 3.6739e-07       | 2.22 |

Algorithm 1 Picard iteration for Algorithm 5.1

1: \textbf{procedure} \( \{ \text{Given } u^n, \phi^n, \text{ Find } u^{n+1}, p^{n+1} \text{ and } \phi^{n+1} \} \)
2: \( u^{n+1} \leftarrow 0 \)
3: \( u_{n+1}^{\text{temp}} \leftarrow u^n \)
4: \textbf{while } \| u^{n+1} - u_{n+1}^{\text{temp}} \| \geq \text{tolerance} \textbf{do}
5: \quad \text{Solve (5.1) with } u^{n+1} = u_{n+1}^{\text{temp}} \text{ using Newton’s method and get } \phi^{n+1} \)
6: \quad \text{Then solve (5.2)-(5.3) with } \phi^{n+1} \text{ and get } u^{n+1} \)
7: \( u_{n+1}^{\text{temp}} \leftarrow u^{n+1} \)

We now test the convergence rate of the CNLFAC and ACSAV schemes. The parameters in this problem are \( \eta = 0.1, M = 10, \lambda = 0.0001, \nu = 0.8 \). The stabilization parameters are set as

\[
\alpha = 1, \quad \beta = 0.25.
\]

Here we use the P2 finite element space for variable \( \phi \), and the P2-P1 finite element spaces for \( u \) and \( p \). From the consistency error, one would expect second order convergence rate for \( \phi \) in \( H^1 \) semi-norm, \( u \) in \( H^1 \) semi-norm, and \( p \) in \( L^2 \) norm, since the numerical error is \( O(h^2 + \Delta t^2) = O(\Delta t^2) \).

The Cauchy differences of numerical solutions computed by the CNLFAC scheme are listed in Table 5.1 for the phase field \( \phi \), the fluid velocity \( u \), and the fluid pressure \( p \), illustrating that the CNLFAC algorithm is second order in time convergent. For the ACSAV scheme, the Cauchy differences are listed in Table 5.2 also showing second order in time convergent. Note that for the pressure projection scheme, which requires artificial boundary conditions for the pressure, the rate of convergence for pressure \( p \) is first order, whereas, the artificial compression method we employ here avoids the requirement of artificial boundary conditions for \( p \).

5.2. Stability on simulating spinodal decomposition. In this section, we perform several tests on simulating the spinodal decomposition of binary fluids by referring to [17] and [42] and show that our schemes satisfy the energy law, which in turn illustrates the long-time stability of our schemes.
The initial condition for the velocity and pressure is zero in the computational domain $\Omega = [0, 2\pi]^2$. The phase field variable $\phi$ is initially treated as a random field $\phi^0 = \bar{\phi} + r(x, y)$ with a mean component $\bar{\phi} = 0.0$ and random $r \in [-0.001, 0.001]$. The parameter values are set as $\eta = 0.1$, $\lambda = 0.01$, $M = 100$, $\nu = 1.0$. Under the AC dynamics, the mixture undergoes a rapid phase separation, in which phases of the same composition rapidly aggregate, then develops to a slower coarsening process in which smaller droplets are gradually absorbed by larger ones.

In order to verify that the CNLFAC and ACSAV algorithms maintain energy stability without any time step condition, we plot the evolution chart of the total free energy (3.1) computed with various time step sizes, as shown in Fig. 5.1. In Fig. 5.1(a), all the energy curves have a trend of monotonic decay, which confirms the unconditional stability of CNLFAC algorithm. Similarly, Fig. 5.1(b) shows the monotonic evolution of the energy for all time step sizes, so the unconditional stability of the ACSAV algorithm is illustrated. We set the time step to be $\Delta t = 0.01$, we plots snapshots of the curves of $\phi$ at different times using both CNLFAC and ACSAV algorithms, respectively. In Fig. 5.2(b) and Fig. 5.2(a), we observe that the final equilibrium solutions in both algorithm simulations are circular.

In addition, Fig. 5.3 plots the evolutions of the auxiliary variable $V^{n+1}$ computed in the ACSAV algorithm with different time step sizes ranging from 0.01 to 0.00125. The magnitude of $V^{n+1}$ is always close to 1, which implies the ACSAV algorithm is stable and convergent.

5.3. Shrinking circular bubble. We refer to [42] to simulate the shrinking process of a circular bubble in the computational domain $\Omega = [0, 2\pi]^2$. The initial conditions are given as follows:

$$
\begin{align*}
\phi^0(x, y) &= 1 + \sum_{i=1}^{2} \tanh(\frac{r_i - \sqrt{(x - x_i)^2 + (y - y_i)^2}}{1.5\eta}), \\
u^0(x, y) &= 0, \quad p^0(x, y) = 0
\end{align*}
$$

where $r_1 = 1.4$, $r_2 = 0.5$, $x_1 = \pi - 0.8$, $x_2 = \pi + 1.7$, $y_1 = y_2 = \pi$. The configuration profile of the initial condition of $\phi$ is given in the first pictures of Fig. 5.4 and Fig. 5.5.
(a) Snapshots are taken by CNLFAC algorithm at $t = 0.15$, $t = 0.3$, $t = 0.75$, $t = 1.0$, $t = 2.0$ and $t = 3.0$.

(b) Snapshots are taken by ACSAV algorithm at $t = 0.15$, $t = 0.3$, $t = 0.75$, $t = 1.0$, $t = 2.0$ and $t = 3.0$.

Fig. 5.2. The 2D dynamical evolution of the profile $\phi$ of the spin decomposition instances under the CNLFAC and ACSAV algorithms.

Fig. 5.3. The evolution of $V^{n+1}$ computed in the ACSAV algorithm.

It shows two circles with different radii in the initial time step. The parameters of the problem are $\eta = 0.04$, $\lambda = 0.01$, $M = 10$, $\nu = 1$.

We use CNLFAC and ACSAV algorithms to solve the ACNS system with time step size $\Delta t = 0.025$. Fig. 5.4 and Fig. 5.5 depict the profile state of the phase field variable $\phi$ at different times until steady state occurs. Due to the influence of roughening, the small circle is absorbed by the large circle, and the volume of the small circle gradually decreases. At $t = 1.25$, the small circle disappears.

5.4. Shape Relaxation. In this section, we refer to [10] to simulate the merging process of two circular bubbles using the ACNS system.

In the numerical experiment, the domain is set as $\Omega = [0, 1.5] \times [0, 1.5]$ and the
initial conditions are given by

\[
\begin{aligned}
\varphi^0(x, y) &= 1 + \sum_{i=1}^{2} \tanh \left( \frac{r_i - \sqrt{(x - x_i)^2 + (y - y_i)^2}}{\eta} \right), \\
u^0(x, y) &= 0, \quad p^0(x, y) = 0
\end{aligned}
\]

where \( r_1 = 0.25, r_2 = 0.25, x_1 = 0.5, x_2 = 1, y_1 = y_2 = 0.75 \). In this simulation, the parameters are \( \eta = 0.02, \lambda = 0.01, M = 10, \nu = 1, \Delta t = 0.005 \).

Fig. 5.6 shows the merging process of two circular bubbles simulated by the CNLFAC algorithm. We find that, because of the surface tension, two adjacent circles quickly join together, gradually merge, and relax into a circle with the least surface energy. Fig. 5.7 illustrates a similar process simulated by the ACSAV scheme. Apparently, simulations by the two schemes are almost identical.

6. Conclusion Remarks. In this paper, we have proposed two linear, unconditionally long time stable, second order decoupling methods for solving the ACNS system. The first method, namely the CNLFAC scheme, is based on the Crank–Nicolson leap-frog time discretization, the Lagrange multiplier method for linearizing the AC equations, and an artificial compression technique for decoupling the velocity and pressure in the NS equations. The second scheme, namely ACSAV, is formulated by incorporating an SAV decoupling strategy into the CNLFAC method so that the AC and NS equations are numerically decoupled and a highly efficient, fully decoupled scheme is built. We prove that both schemes are unconditionally stable without any time step conditions. Numerical experiments are performed to verify that our schemes are of second order accuracy in time and unconditionally stable.

It is worth noting that there are very few second order, unconditionally stable, fully decoupled numerical schemes for solving hydrodynamics coupled phase field models. The idea here (CNLF+AC+SAV) serves as a template for designing unconditionally stable and fully decoupled schemes for solving other related phase field models, such as the Cahn–Hilliard–Navier–Stokes equations, the Cahn–Hilliard–Hele–Shaw system, and phase field fluid models of variable densities. Other improvement
such as designing a fully explicit (not only for the NS component, but also the AC part) and also accurate scheme is in progress.

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The process of merging and relaxing of two kissing circles obtained by the ACSAV algorithm. From left to right, $t = 0$, $t = 0.1$, $t = 0.2$, $t = 0.3$, $t = 0.5$ and $t = 0.6$, $\eta = 0.02$, $\lambda = 0.01$, $M = 10$, $\nu = 1$, $\Delta t = 0.005$. 

**Fig. 5.7**

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