Self modulation and scattering instability of a relativistic short laser pulse in an underdense plasma

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Received 14 January 2019, revised 14 April 2019
Accepted for publication 5 June 2019
Published 25 June 2019

Abstract

In this paper, we present an analysis framework for description of nonlinear, self-consistent laser-plasma evolutions during propagation of a short intense laser pulse in a high-density sub-critical plasma (the pulse length exceeds the plasma wavelength). In this context, the pulse evolutions are attributed to the wakefield induced self-modulation and destabilization via parametric exponentiation of the initial noise content. The self-consistent plasma evolutions are formulated in terms of used-to-be motion constants in the absence of pulse evolutions. This proves very useful both in uncovering important plasma dynamics during self-modulation and also in facilitating the instability studies in the strongly nonlinear regime through refinement of unstable plasma perturbations. General analytical solutions, at arbitrary pulse conditions, are derived for self-modulation, indicating that the envelope evolutions are driven by the induced spatial frequency-chirp. Also, these results state that the envelope attains fine modulations which produce long wavelength low-frequency modes via beating the carrier mode. The plasma wave variations are found to convect and amplify away from the pulse front. Regarding parametric instability, we assess different scattering regimes at different pulse shapes and peak intensities, manifesting anomalous behaviors ranging from wild positioning of the Stokes wave in dispersion plane to broadening in the scattered spectrum and halting the instability. Our analyses are assisted and verified by numerous fluid and particle-in-cell simulations. Based on our results, we discuss phenomena like the pulse breakup and its different regimes, and assisted particle acceleration in the presence of pulse evolutions.

Keywords: relativistic laser-plasma interactions, self-modulation, parametric instabilities, Raman instability, nonlinear evolutions

(Some figures may appear in colour only in the online journal)

1. Introduction

Nowadays a unique opportunity is provided by modern high-intensity short pulse lasers, for experimental investigation of many fundamental phenomena [1], and realization of very attractive applied concepts like laser-plasma accelerator [2] and fast-ignition scheme of inertial confinement fusion (ICF) [3]. These investigations and applications most often imply or contain the propagation of a short intense laser pulse through a high-density transparent plasma [1–17]. This is the case for example in the self-modulated laser wakefield accelerator (SMLWFA) [9, 10], and in laser interactions with solid targets [11, 12] and ICF targets [3, 4, 11], which usually leads to production of coronal plasmas by pulse pedestal before arriving the main part of the pulse. We may also add, laser interaction with modern foam targets, which have attracted many recent attentions [14, 15].

The short pulse propagation inside the high-density plasma is characterized by the regime for which the pulse length, $L_p$, exceeds the initial plasma wavelength, $c/\omega_{pe}$...
where $\omega_p = \sqrt{n_e e^2/\varepsilon_0 m_e}$ is the plasma frequency and $c$ is the light speed (and $n_e$, $\varepsilon_0$, $e$, $m_e$ are respectively the initial plasma density, the vacuum permittivity, elementary charge and electron mass), viz. the condition $L_p > c/\omega_p$ implies $n_e > \varepsilon_0 m_e e^2/e^2L_p^2$. This regime has significant differences with the so-called ultra-short pulse regime (the area of the laser wakefield accelerator [2]) regarding the stability of the laser pulse propagation inside the plasma. Despite the latter, in the former the laser pulse is susceptible to scattering instabilities [16–20] and pulse breakup [17, 21–24]. In most applications, these effects on the laser pulse act as a double-edged sword; they both restrict the pulse penetration into the plasma [25, 26], and in the same time lead to enhanced particle acceleration [27–34]. Generally, scattering assisted electron acceleration is now considered as a generic mechanism for production of very-energetic/super-ponderomotive electron populations observed in high density plasma irradiation by intense pulses [13, 27, 30–34]. In SMLWFA, the electron acceleration is also enhanced via the wakefield amplification by the pulse breakup [27]. In the context of ICF, it is most often intended to control the instabilities and pulse breakup to enhance the laser penetration into the target [4], but in the same time, production of energetic electrons may be considered as a mechanism for attaining the electron ignition-beam [13]. Therefore, careful characterization of such pulse evolutions is crucial for their control and possibly their optimized utilization.

The pulse breakup phenomenon has been extensively studied in previous works [16, 17, 21–24], and has been subjected to many debates regarding its origins [17]. Antonson and Mora [16], and Sprangle et al [21] have described this phenomenon as an adiabatic sausaging process originated from the transverse plasma oscillations. Mori et al [17], on the other hand, have attributed this phenomenon to the direct forward Raman scattering (FRS), a quite longitudinal mechanism. Following Mori et al, Gordon et al [35] (see also Mima et al [36]), have proposed a two stage process; in the first phase, shortly after the laser entrance, the laser experiences Raman backward scattering (RBS) instability which seeds the plasma wave and the subsequent Raman forward scattering instability. In the second phase, the pulse breakup continues via seeded RFS, as described by Mori et al [17].

In past two decades, many efforts have also been made on parametric instabilities of intense and/or short laser pulses [16–20, 37–42]. It should be emphasized that in these studies the wake excitation has been usually ruled out by assuming infinite pulse lengths or low intensities. In this regard, we may mention the works by Barr et al [41, 42] which apparently considered the most relevant conditions to the present study, say, relativistic short pulse interaction. Even in these works, the wake excitation has been implicitly ignored via assumption of a homogenous pump wave in Lorentz boosted frame. For a real short pulse, though experiencing Lorentz elongation (inverse Fitzgerald-contraction), the pulse length remains of finite length in the boosted frame, and the wakefield may not be ignored. A treatment of short laser pulses in the boosted frame is presented by Yazdanpanah [43] which reveals the importance of induced wakefield on pulse evolutions.

In this paper, we present an analysis framework for description of self-consistent laser-plasma evolutions during a short intense laser pulse propagation in a high density plasma (pulse length exceeds the plasma wave length). The most distinguishing feature of the intense short pulse interactions, compared to the common cases of infinite pulse length and subrelativistic intensities, is the possible presence of initial strong wakefield. This in turns leads to faster development of self-modulation with respect to FRS. In this regard, the pulse evolutions are attributed to the wakefield induced self-modulation accompanied by destabilization via parametric exponential of the initial noise content (scattering instabilities). When the pulse shape is initially smooth (very weak longitudinal electric field), the wake excitation is found to be seeded by RBS instability. Especially, we describe the longitudinal pulse breakup in terms of self-modulation rather than FRS considered by Mori et al [17] and Gordon et al [35]. In addition, we recover the interplay between the initial wake excitation and parametric instabilities, which remained unresolved in previous studies [16–20, 37–42].

We formulate the plasma wave in terms of quantities which used to be motion constants in the absence of pulse evolutions, say electron energy and electron flux in the Galilean comoving window (not Lorentz boosted frame). This not only reveals the important dynamics of these quantities in the presence of pulse evolutions, but also, as will be seen, facilitates instability studies in the presence of strong wakefield via refinement of unstable field-plasma perturbations.

In the case of self-modulation, our basic equations for pulse evolutions are the generalized form of those used by Schroeder et al [44] for description of nonlinear evolutions of ultra-short pulses (pulse lengths shorter than plasma wave-length). Here, our generalization leads to results which remain valid for pulse-lengths exceeding the plasma wavelength. Moreover, by applying a technical mathematical treatment, the obtained equations are solved and reduce to closed results which well agree with presented simulations. In the case of instabilities, after assessment of different regimes, we employ numerous fluid and particle-in-cell (PIC) simulations to accomplish and verify our analytical arguments.

The organization of our paper is as follow; basic equations are summarized in section 2. In section 3, we describe pulse self-modulation and self-consistent plasma motion. In section 4, we describe parametric instabilities. In section 5, we summarize the numerical simulations and discuss our results. Finally, in section 6, we point out our conclusions and remarks.

2. Basic equations

A vast number of nonlinear phenomena in the intense short laser interaction with under-dense plasma are investigated by applying the well-known set of cold fluid plus Maxwell equations (see e.g. [2] and references therein). Here we consider these equations to examine the interaction of a plane
p-polarized laser pulse with a uniform plasma slab (one dimensional geometry), where the laser propagates along x axis and is polarized in y direction. Therefore, after transforming into Pulse Co-Moving Window (PCMW) ((t, x) → (ξ = x − v_p (t, t),)), our basic equations in terms of normalized quantities read as:

\[ \frac{\partial n_e}{\partial t} + c \frac{\partial}{\partial \xi} [n_e (v_{ex} - v_g)] = 0 \]  
(1a)

\[ \left( \frac{\partial}{\partial t} - c v_g \frac{\partial}{\partial \xi} \right) p_{\phi} = - \frac{\partial}{\partial \xi} (\gamma_e - \phi) \]  
(1b)

\[ R_y = A_y \]  
(1c)

\[ \frac{\partial^2 \phi}{\partial \xi^2} = \frac{\omega_p^2}{c^2} [n_e - 1] \]  
(1d)

\[ \left[ \frac{1}{\gamma_e^2} \frac{\partial^2}{\partial \xi^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right] A_y = \frac{\Omega_p^2}{c^2} A_y \]  
(1e)

Here, \( n_e, p_{\phi}, v_c = \beta / \gamma_e, \gamma_e = (1 - v_g / v_c)^{-1/2}, \phi, A, v_g = c k_0 / \omega_0 \) and \( \gamma_g = (1 - v_g^2 / v^2)^{-1/2} \) stand for electron density, electron momentum, electron velocity, electron gamma factor, scalar potential, vector potentials and linear group velocity and its relativistic gamma factor, respectively; \( k_0 \) and \( \omega_0 \) are carrier wave-number and frequency. We have also defined \( \Omega_p^2 \equiv \omega_p^2 n_e / \gamma_e \) where \( \omega_p = (n_e e^2 / \varepsilon_0 m_e)^{1/2} \) is the usual plasma frequency. All plasma and field quantities have been normalized in above equations but length and time remained unnormalized; we have set eA/m_e c → A, e/\( \omega_0^2 m_e c^2 \) → \( \phi \), \( n_e / n_0 \) → \( n_e \), \( p_{\phi} / m_e c \rightarrow p_{\phi}, v_c / c \rightarrow v_c \) and \( v_g / c \rightarrow v_g \).

In the absence of temporal pulse evolutions in the PCMW, as described for example in [45] and references therein, we may set time derivatives (\( \partial / \partial t \)) zero in equations (1a) and (1b), i.e. plasma profiles become static in PCMW. After these simplifications, the following motion-constraints (already given in classical literatures, see e.g. [45]) are respectively obtained by spatial integration of (1a) and (1b),

\[ J_x \equiv n_e (v_{ex} - v_g) = -v_g, \]  
(2a)

\[ H_x \equiv \gamma_e - v_g p_{ex} - \phi = 1. \]  
(2b)

Generally, the laser pulse undergoes temporal variations in PCMW, and \( J_x \) and \( H_x \) do not remain further as motion constants. In the so-called quasi-static approximation they are supposed to evolve very slowly in time, according to slow evolutions of the laser pulse (see e.g. [2, 45]). Quite generally, irrespective of \( J_x \) and \( H_x \) being constant or not, every plasma quantity may be expressed in terms these variables and vector and scalar potentials, i.e. we may make a transformation from the set of variables \( \{ n_e, p_{\phi}, \phi, A \} \) to \( \{ J_x, H_x, \phi, A \} \). This will turn very useful in description of the plasma motion in presence of pulse evolutions. For constant \( J_x \) and \( H_x \) (for \( J_x = -v_g \) and \( H_x = 1 \) [45]), the procedure of this transformation may be easily found in the well-known literatures, e.g. see [2] and references therein. For \( v_g \approx 1 \) the final results become very simplified as summarized e.g. in [45]. Here, we consider the general situation of variable \( J_x \) and \( H_x \), and \( v_g \approx 1 \) but in the limit of \( \gamma_g \approx 1 \) (initial plasma density well below the critical density), viz. \( v_g \approx 1 - 1/2 \gamma_g^2 \). Applying this approximation in (2a) and (2b), these equations respectively read as,

\[ J_x^* \equiv n_e (v_{ex} - 1) \]  
(2c)

\[ H_x^* \equiv \gamma_e - p_{ex} - \phi, \]  
(2d)

where \( J_x^* \equiv J_x - n_0^0 / \gamma_g^2 \) and \( H_x^* \equiv H_x - p_{ex}^0 / \gamma_g^2 \) in which \( (0) \) suffix indicates the solution when \( v_g = 1 \) be assumed (the leading order solution). By proper combination of equations (2c) and (2d), after some mathematical manipulations, we get for our frequently used quantities,

\[ v_{ex} = \gamma_g^2 - \left( \phi + H_x^* \right)^2 \]  
(3a)

\[ \gamma_e = \frac{1}{2} \left( \gamma_g^2 + \phi + H_x^* \right) \]  
(3b)

\[ n_e = -J_x^* \gamma_g^2 - \left( \phi + H_x^* \right)^2 \]  
(3c)

where \( \gamma \equiv \sqrt{1 + A^2} \).

In order to accomplish the proposed transformation from \( \{ n_e, p_{\phi}, \phi, A \} \) to \( \{ J_x, H_x, \phi, A \} \), after some mathematical manipulations, we transform equations (1a) and (1b) into the following equations for \( H_x \) and \( J_x \):

\[ \frac{dH_x}{dt} = - \frac{\partial \phi}{\partial \xi} + \frac{A_x}{\gamma_e} \frac{\partial A_x}{\partial t} \]  
(4)

\[ \frac{dJ_x}{dt} + \frac{n_e}{\gamma_e} \frac{\partial H_x}{\partial \xi} = \frac{n_e v_{ex}}{\gamma_e} \frac{\partial A_x}{\partial \xi} \]  
(5)

where the complete time derivative has its usual meaning \( (d/\partial t \equiv \partial / \partial t + c v_{ex} \partial / \partial x = \partial / \partial t + c (v_{ex} - v_g) \partial / \partial \xi) \) and \( \gamma_{ex} \) is defined as \( \gamma_{ex} = (1 - v_{ex}^2)^{-1/2} \). Now, equations (4) and (5) together with equations (1d) and (1e) (assisted by auxiliary relations (3a)–(3c)) form a complete set of equations for investigation of self-consistent laser-plasma evolutions in the fully nonlinear regime.

Using the obtained equation we examine both the laser pulse self-modulation via wake excitation, and its destabilization via exponentiation of the initial noise content. In section 3, the self modulation is treated as a slow deformation in the pulse envelope and the set of equations (4), (5), (1d) and (1e) are solved within a generalization of the slow envelope approximation (SEA). In section 4 these equations are used to investigate propagation destabilization. This process is considered as a fast phenomena produced by coupling between the strong, low-frequency (in PCMW) laser field as the pump wave, and a high frequency electromagnetic mode from noise content, similar to the usual Raman instability.

3. The pulse self modulation and self-consistent plasma motion

3.1. Pulse evolutions

We consider the wakefield induced pulse self-modulations within a generalization of the SEA [2, 21, 22, 43, 44, 46, 47].
In this regard, it should be mentioned that SEA has been frequently applied to study ultra-short [21, 43, 44, 46, 47] and short (longer than plasma wavelength) [21, 22] pulse interactions with plasma. Especially, we should mention the more recent work by Schroeder et al [44] on nonlinear evolutions of ultra-short laser pulses in plasma, which our approach in this subsection is a generalization of this work to include general pulse lengths (short and ultra-short) and pulse-shapes. We will find that laser frequency retains spatial chirp immediately after pulse entrance into the plasma, and that amplitude evolutions are driven dominantly by this frequency chirp.

The wave equation (1e) may be easily rewritten in terms of $A_0$ envelope,

$$
\frac{1}{\gamma_g^2} \frac{\partial^2}{\partial \xi^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + 2 \frac{v_g}{c} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{2i\omega_0}{c^2} \frac{\partial}{\partial t} \frac{\partial}{\partial \xi} \hat{A} = \frac{\Omega^2_p - \omega_p^2}{c^2} \hat{A},
$$

(6)

where $\hat{A}$ is the envelope function satisfying \( A_0(t, \xi) = (\hat{A}(\xi, t)/2)e^{i(\theta(x,t) - \omega(t)\xi)} + \text{c.c.} \). Further, by applying the linear dispersion relation, \( \omega^2(k) = c^2k^2 + \omega_p^2 \), the zero-order phase-factor \( \theta_0 \equiv k_0x - \omega_0t \) may be rewritten as \( \theta_0 = k_0\xi - (\omega_0 - c_k0v_0)t \equiv k_0\xi - \omega_{00}t \) which carrier frequency \( \omega_{00} \) in the commoving window;

$$
\omega_{00} = \omega_0 - c_k0v_0 = \omega_p^2 - \omega_0^2 = \frac{\omega_0^2}{\gamma_g^2}.
$$

(7)

The envelope function \( \hat{A}(\xi, t) \) contains not only amplitude modulations but also phase corrections produced by spatiotemporal frequency and wave-number modulations. Substituting the phasor presentation \( \hat{A}(\xi, t) = a_0(\xi, t)e^{i\theta(x,t)} \) for \( \hat{A} \) into (6), where \( a_0 = |\hat{A}| \) and \( \theta = \arg(\hat{A}) \), and decomposing the resulting equation into real and imaginary parts, we get,

$$
\begin{align*}
\frac{1}{\gamma_g^2} \left( \frac{\partial^2}{\partial \xi^2} \right) + 2 \frac{v_g}{c} \left( \frac{\partial}{\partial \xi} \right)^2 - \frac{2i\omega_0}{c^2} \frac{\partial}{\partial t} \frac{\partial}{\partial \xi} a_0 = & \left( \frac{\Omega^2_p - \omega_p^2}{c^2} - \frac{1}{\gamma_g^2} \right) a_0, \\
\left( \frac{1}{\gamma_g^2} \right) \left( \frac{\partial}{\partial \xi} \right)^2 + 2 \frac{v_g}{c} \left( \frac{\partial}{\partial \xi} \right) \left( \frac{\partial}{\partial t} \right) + \frac{1}{\gamma_g^2} \left( \frac{\partial}{\partial \xi} \right)^2 a_0 = & \left( \frac{1}{\gamma_g^2} \right) \left( \frac{\partial^2}{\partial \xi^2} \right) + 2 \frac{v_g}{c} \left( \frac{\partial}{\partial \xi} \right)^2 + \frac{1}{\gamma_g^2} \left( \frac{\partial}{\partial \xi} \right)^2 a_0, \\
& \frac{1}{\gamma_g^2} \left( \frac{\partial}{\partial \xi} \right)^2 - \frac{1}{\gamma_g^2} \left( \frac{\partial}{\partial \xi} \right)^2 + 2 \frac{v_g}{c} \left( \frac{\partial}{\partial \xi} \right)^2 a_0
\end{align*}
$$

(8a)

which present coupled equation for phase and amplitude evolutions. As, at initial time \( (t = 0) \), the pulse envelope is given by a real position-dependent function \( \hat{A}_0(\xi) = \hat{A}(\xi, t = 0) \), the variables \( \theta \) and \( a_0 \) should satisfy the following initial conditions,

$$
a_0(\xi, t = 0) = \hat{A}_0(\xi), \quad \theta(\xi, t = 0) = 0. \quad (9)
$$

The physical significance of the phase factor, \( \theta \), is that \( \varpi = -\partial^2 \theta / \partial t \) and \( \bar{\theta} = \partial \theta / \partial \xi \) respectively represent corrections to (modulations of) local frequency and wave-number in the PCMW, appeared due to strong density modulations in driven plasma wave.

In order to solve equations (8a) and (8b), we need to simplify these equations via relevant approximations. The common assumption in all envelope analyses, is that envelope parameters evolve rather slowly compared to the phase factor \( \theta_0 = k_0x - \omega_0t = k_0\xi - \omega_{00}t \). This assumption holds as long as the initial plasma density is well below the critical density (but still the high density condition \( n_{00} > \varepsilon_0m_e^2/c^2L_\rho^2 \), stated in introduction, may hold). The differences of our analysis with the previous slow analyses (SEA) arise from two points: (i) Proper assignment of the rate of the zero-order-phase factor \( \theta(t) \), which, according to equation (7), is the carrier frequency (CF) in PCMW \( \omega_{00} \) rather than CF in laboratory window \( \omega_{00} \).

(ii) Despite previous approaches, in evaluating the slowness, we dissociate the phase \( \theta \) and module \( (a_0) \) of the complex envelope \( \hat{A} = a_0e^{i\theta} \), and put the condition on \( \varpi \) instead of \( \theta \) itself. With these descriptions, our slowness conditions in the mathematical language read as:

$$
\frac{\partial \omega_0}{\partial \tilde{\varpi}} \ll \omega_0, \quad \frac{\partial a_0}{\partial \tilde{\varpi}} \ll a_0, \quad \frac{\partial \varpi}{\partial \tilde{\varpi}} \ll \varpi,
$$

(10)

where \( n = 0, 1, 2, \ldots \), saying that the time derivative satisfies \( \partial \omega_0/\partial \tilde{\varpi} \ll \omega_{00} \) when it operates on slow quantities including \( a_0 \) and \( \varpi \), and their time-derivatives. Note that, as will be seen, \( \partial \theta / \partial t = -\varpi \) may reach values comparable to \( \omega_{00} \), whence \( \theta \) could not be evaluated as a slow quantity. This is why we have put the slowness condition on \( \varpi \) instead of \( \theta \).

Conditions (10), as well be seen, allow for utilization of the series-expansion method [48] to solve equations (8a) and (8b). To apply this method, we substitute the following time-domain Taylor expansion (around \( t = 0 \)) of quantities into the differential equations, (8a) and (8b),

$$
a_0(\xi, t) = a_0(\xi, 0) + t\alpha_{10}(\xi) + t^2\alpha_{20}(\xi) + \ldots \quad (11a)
$$

$$
\varpi(\xi, t) = \varpi(\xi, 0) + t\alpha_{30}(\xi) + t^2\alpha_{40}(\xi) + \ldots \quad (11b)
$$

$$
\theta(\xi, t) = -\varpi(\xi, 0) - t^2\alpha_{50}(\xi) + \ldots \quad (11c)
$$

where, in all quantities, \( t \) indices show the time derivatives and 0 index show the initial value, e.g. \( a_0(\xi, 0) = a_0(\xi, t = 0) \), \( a_{10}(\xi) = ([\partial / \partial t]a_0(\xi, t = 0), a_{20}(\xi) = ([\partial^2 / \partial t^2]a_0(\xi, t = 0), and so on. Equation (11c) has been resulted from integration of
\[ \omega(\xi, t) = \omega(\xi) + i\omega_0(\xi) + \frac{t^2 \omega_0(\xi)}{2!} + \ldots \] by applying the initial condition \( \omega(\xi, 0) = \omega(\xi) \). After these substitutions, and ordering the resulting equations in powers of \( t \), from factorizing \( \ell^0 \) multipliers we get,

\[
[2[\omega_0 + \omega_0(\xi)]\hat{A}_0 = \left[ \Omega_0^2 - \omega_0^2 - \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2} \right] \hat{A}_0
+ 2c_0 \frac{dA_{\ell, 0}}{d\xi},
\]

\[
+ 2c_0 \frac{dA_{\ell, 0}}{d\xi} + \frac{\gamma_g}{c^2} \frac{d^2A_{\ell, 0}}{d\xi^2} - 2c_0 \frac{dA_{\ell, 0}}{d\xi} + \Omega_0^2 A_{\ell, 0}
+ A_{\ell, m0},
\] (12a)

From \( \ell^1 \) factorization we obtain,

\[
2[\omega_0 + \omega_0(\xi)]A_{\ell, 0} = \left[ \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2} + 2c_0 \frac{dA_{\ell, 0}}{d\xi} \right]
\frac{dA_{\ell, 0}}{d\xi} + \frac{\gamma_g}{c^2} \frac{d^2A_{\ell, 0}}{d\xi^2} - 2c_0 \frac{dA_{\ell, 0}}{d\xi} + \Omega_0^2 A_{\ell, 0},
\]

\[
+ A_{\ell, m0}.
\] (13a)

This states that very longer pulse lengths (envelope scale-lengths) are allowed within our approximations, with respect to the usual SEA which needs the condition \( L_p \sim c/\omega_p \) (ultrashort pulse length) \([43, 44, 46, 47]\). Therefore, our approximations are fulfilled about smooth pulses with lengths satisfying (14) as well as quickly rising pulses (with scale lengths \( \leq c/\omega_p \)), meaning their universality regarding the pulse shape.

After elimination of \( \lambda_0 \) and \( \omega_0 \) respectively in equations (13a) and (13b), from resulting equations we may express \( \omega_0 \) and \( \omega_0(\xi) \) in terms of \( \lambda_0 \) and \( \omega_0 \), and substitute the results into equations (12a) and (12b) to obtain the latter quantities in terms of \( \hat{A}_0 \). After this step, the initial conditions \( \lambda_0 \) and \( \omega_0(\xi) \) are determined and may be used into equations from the previous step to get \( \lambda_0 \) and \( \omega_0(\xi) \), and the solution procedure ends. One may notice the subtlety that, in this procedure, unidentified initial conditions, \( \lambda_0 \) and \( \omega_0(\xi) \), are compensated by approximations \( \lambda_0 \approx 0 \) and \( \omega_0(\xi) \approx 0 \), dictated by (10).

In order to get analytical results, we should further simplify equations (12) and (13) by considering conditions (10) and (14). In the case of (13a), after substitution of the second bracket in its left hand side via equation (12a), and keeping the most significant terms, we obtain,

\[
2c_0 \frac{dA_{\ell, 0}}{d\xi} + \left[ \omega^2_0 + \omega_0(\xi) - \Omega_0^2 - \omega_0^2 \right] A_{\ell, 0}
+ \frac{\gamma_g}{c^2} \frac{d^2A_{\ell, 0}}{d\xi^2} + 2c_0 \frac{dA_{\ell, 0}}{d\xi} + \Omega_0^2 A_{\ell, 0}
+ A_{\ell, m0},
\]

which states that this quantity depends not only to plasma-frequency variations, but also to the spatial chirp resulted from induced density modulations.

In the same way, we may ignore \( \lambda_0 \) and \( \omega_0(\xi) \) in the right hand sides of (12a) and (12b), respectively, that is taking the series solution up to \( \ell^0 \). After these simplifications, by combination of these two equations, we obtain,

\[
\frac{d}{d\xi} \left[ \frac{1}{\omega_0 + \omega_0(\xi)} \frac{d}{d\xi} \right] \hat{A}_0 = \frac{\hat{A}_0}{\Omega_0^2 - \omega_0^2 - \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2}}
+ 2\omega_0 \omega_0(\xi)
+ 2\omega_0 \omega_0(\xi)
\]

\[
+ \frac{d}{d\xi} \left[ \frac{1}{\omega_0 + \omega_0(\xi)} \frac{d}{d\xi} \right] \hat{A}_0 = \frac{\hat{A}_0}{\Omega_0^2 - \omega_0^2 - \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2}}
\]

\[
\frac{d}{d\xi} \left[ \frac{1}{\omega_0 + \omega_0(\xi)} \frac{d}{d\xi} \right] \hat{A}_0 = \frac{\hat{A}_0}{\Omega_0^2 - \omega_0^2 - \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2}}
\]

\[
\hat{A}_0.
\] (16a)

Now, the partial differential equations (8a) and (8b) has been transformed into a set of ordinary differential equations. We do not go further in higher powers of \( t \), as equations (12a), (12b), (13a) and (13b) are sufficient to lead the order evolutions of the most important plasma parameters. Having conditions (10), in equations (13a) and (13b), we may ignore \( \lambda_0 \) and \( \omega_0(\xi) \) versus \( \Omega_0 \) and \( \frac{dA_{\ell, 0}}{d\xi} \) and \( \frac{dA_{\ell, 0}}{d\xi} \), respectively. The sufficient criteria for this simplification is that \( \partial / \partial \xi \) behaves as \( \left| \partial \xi / \partial x \right| \geq \omega_0(\xi) \gamma_g \) where \( x = \{ \lambda_0, \omega_0(\xi) \} \), which in turn needs the pulse length satisfy,

\[
L_p \leq c/\omega_0 \gamma_g = \frac{\gamma_g}{c/\omega_p},
\]

(14)

Using the above equations we determine the local-amplitude evolution rate, \( \lambda_0 \), and the spatial frequency modulation \( \omega_0(\xi) \) which in turns gives the spatial frequency chirp \( \partial \omega_0 / \partial \xi \). The ordinary differential equation (16a) may be solved, either numerically or analytically. If we linearize this equation with respect to \( \omega_0(\xi) / \omega_0 \ll 1 \), it takes a familiar form, \( \frac{d^2}{d\xi^2}\left[ \lambda_0 \omega_0 \right] + \frac{\omega_0}{c^2 \gamma_g^2} \lambda_0 \omega_0 = \left( \lambda_0 + \omega_0 / 2c^2 \gamma_g \right)^2 \Omega_0^2 - \omega_0^2 - \frac{c^2}{\gamma_g} \frac{d^2}{d\xi^2} \lambda_0 \) having the following formal solution,
The above integral may be approximated to get the leading order solution, but a much easier way is simplifying the original equation (16a). This may be done by keeping only the highest order terms, including 2\(\omega_0\)\(v_0\) and \(\Omega_{\rho 0}^2 - \omega_p^2\), and finally we obtain,

\[
\omega_0 \approx \frac{\Omega_{\rho 0}^2 - \omega_p^2}{2\omega_0} = \frac{\omega_0^2(\Omega_{\rho 0}^2 - 1)}{2}. 
\]

(17)

\(a_{t,0}\) can be obtained by substituting (16c) or (17) into equation (16b).

Equation (17) states that, as noticed below conditions (10), frequency corrections may reach values comparable to \(\omega_0\) in the nonlinear regime where \(\Omega_{\rho 0}^2\) substantially differs from \(\omega_p^2\). Moreover, putting this equation together with (16b) and (15), we may easily verify that, \(a_{t,0} < \omega_0/\Omega_{\rho 0}\) and \(\omega_0 < \omega_0\omega_0\), whence consistency of our formulation regarding the assumptions (10).

Among our final results (15), (16b), (16c) and (17), only the former displays dependence on wakefield evolutions (through \(\Omega_{\rho 0}, \omega_0\)). Unfortunately, as will be fully discussed in the next sub-section, wakefield evolutions may not generally be reduced into a closed solution. The exception is ultra-short pulses for which approximations \(H_s \approx 1\) and \(J_s \approx -v_s\) will be verified to hold. However, this fact does not decrease the validity of SEA, as it will be shown that the phase factor is eliminated in the evolutions of \(\Omega_{\rho 0}^2\), therefore this quantity evolves as slow as the pulse envelope.

At the end, it is worth mentioning that by substitution of our approximate equation (17) into (16b) we find \(a_{t,0} \equiv (c\omega_0/2\omega_0\omega_0^2)(d/d\xi)(\Omega_{\rho 0}^2\lambda_0) - (c\omega_0/2\omega_0^2)\lambda_0\lambda_0/d\xi\). This result is approximately equal to the result given by Schroeder et al [44] below their equation (5), which in terms of our variables reads as \(a_{t,0}^* \equiv (c\omega_0/2\omega_0\omega_0^2)(d/d\xi)(\Omega_{\rho 0}^2\lambda_0)\) where \(a_{t,0}^*\) is measured on the light speed comoving window (not group-velocity comoving window); the relation between \(a_{t,0}^*\) and \(a_{t,0}\) is \(a_{t,0} = a_{t,0}^* + c(1 - v_s)\lambda_0/d\xi\) which verifies coincidence of two results. It should be also mentioned that formulas for spatial frequency chirp (16c) and temporal frequency chirp (15), and explicit relation between frequency-chirp and envelope evolutions are obtained here for the first time. Moreover, fine oscillations at wave number \(\omega_0/c\) are predicted to appear in the pulse envelope according to equations (16c) and (16b). These oscillations lead to production of very low-frequency long-wavelength (\(k \approx 0, \omega_0 \approx 0\)) modes via beating the carrier wave-number and frequency.

3.2. The plasma motion

In the presence of pulse evolutions, the plasma motion is governed by the set of equations (4), (5) and the Poisson equation (1d), viz., every plasma quantity may be expressed in terms of \(\phi, H_s\) and \(J_s\). An important property of these equations is that each of their solution is decomposed into two parts, the dominant part with slow spatial variations (we term as secular part) and another small part with fine spatial oscillations at multiples of the pulse wave-number (we term as oscillatory part). Moreover, the effect of temporal oscillations is completely isolated in the latter part. Therefore, we examine the following trial solutions into the set of equations (1d), (4) and (5),

\[
\phi = \phi_s(\xi, t) + \phi_o(\xi, t) \tag{18a}
\]

\[
H_s = H_{so}(\xi, t) + H_{so}(\xi, t) \tag{18b}
\]

\[
J_s = J_{s0}(\xi, t) + J_{s0}(\xi, t) \tag{18c}
\]

where ‘s’ and ‘o’ respectively show secular and oscillatory parts used to satisfy \(|X_s| \gg |X_o|\).

Let first consider the Poisson equation (1d); after substitution of \(n_0\) using equation (3c) and expanding it up to the first order in small oscillatory quantities, the resulting equation may be decomposed into two equations for \(\phi_s\) and \(\phi_o\) as follows,

\[
\frac{\partial^2 \phi_s}{\partial \xi^2} = -\frac{\omega_p^2}{c^2} \left\{ f_{\phi_s} \frac{1 + a_{t,0}^*}{2(\phi_s + H_{so}^*)} + \frac{J_{s0}^*}{2} \right\}, \tag{19a}
\]

\[
\frac{\partial^2 \phi_o}{\partial \xi^2} = -\frac{\omega_p^2}{c^2} \left\{ f_{\phi_o} \frac{1 + a_{t,0}^*}{2(\phi_o + H_{so}^*)} + \frac{J_{s0}^*}{2} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \rt
with respect to $\phi_\gamma$, $H_\omega$, and $J_\omega$. In this way, for $1/\gamma_e$ we obtain,

$$\frac{1}{\gamma_e} = \frac{1}{\Gamma_0 + \Gamma_\omega} + \Delta_\gamma,$$

(20a)

where $\Gamma_\omega \equiv (\gamma^2 / K^*_\omega + K^*_\gamma / 2)$, $\Gamma_0 \equiv a_2^3 \cos(2\theta_0 + 2\theta) / 4K^*_\omega$ and

$\Delta_\gamma \equiv -(\phi_\gamma + H_\omega / 2(1 - \gamma^2 / K^*_\gamma / 2)) \Gamma_\omega$. $K^*_\omega \equiv \phi_\gamma + H_\omega \Gamma_\omega$, $K_\omega \equiv \phi_\omega + H_\omega$ and $a_2^3 = \sqrt{1 + a_2^6 / 2}$. For $\nu_e$, we notice that (30a) may be simplified into $\nu_e = 1 - (\phi + H_\omega^\gamma / \gamma_e)$. Therefore, with the aid of (20a) we obtain,

$$\nu_e = \nu_0 + \Delta_e,$$

(20b)

where $\nu_0 \equiv 1 - K_\omega^\gamma / (\Gamma_0 + \Gamma_\omega)$ and $\Delta_e \equiv - K_e \Delta_\gamma - (\phi_\gamma + H_\omega) / (\Gamma_0 + \Gamma_\omega)$. It should be noticed that the oscillatory quantity, $\Gamma_\omega$, and the mixed quantity, $\nu_\omega$, are not small quantities in spite of $\phi_\gamma$, $H_\omega$ and $J_\omega$. We substitute (20a) and (20b) into (4) and linearize the result with respect to $\phi_\gamma$, $H_\omega$, and $J_\omega$ and finally decompose the resulting equation into secular and oscillatory parts. After these parts, and performing some tedious mathematical manipulations with applicable approximations, summarized in the appendix, we get the following results for $H_\omega$ and $H_\omega^\gamma$,

$$c(V_\omega - \nu_0) \frac{\partial H_\omega}{\partial \xi} = (\omega_{\nu_0 + \nu_e}) \frac{a_2^3 \sin(2\theta_0 + 2\theta)}{2} \left( \Gamma_0 + \Gamma_\omega \right),$$

(21a)

$$\frac{\partial H_\omega}{\partial t} + c \left( 1 - \frac{\phi_\omega + H_\omega}{\Gamma_0 \sqrt{1 - b^2}} - \nu_0 \right) \frac{\partial H_\omega}{\partial \xi} = - \frac{\partial \phi_\omega}{\partial t} + \frac{1}{\Gamma_\omega} \frac{1}{b \sqrt{1 + b^2}} \frac{a_1 \partial a_1}{\partial t},$$

(21b)

where $b$ has been defined as $b \equiv (a_2^4 / 2) / (1 + a_2^6 / 2 + (H_\omega + \phi_\gamma)^2)$. Equation (21a), is readily solved via two-scale analysis (as for $\phi$ in (19c)) to give,

$$H_\omega \approx \frac{\omega_{\nu_0 + \nu_e}}{4 \kappa_0 c (\phi_\gamma + H_\omega)} \frac{a_2^3 \cos(2\theta_0 + 2\theta)}{2},$$

(22)

Moreover, at these conditions from $\phi_\gamma + H_\omega \approx \gamma_\omega$, we get $\partial(\phi_\gamma + H_\omega) / \partial t \approx (a_2^3 / 2) \partial a_1 / \partial t$. Substituting this result into (23) and applying the condition $b^2 \ll 1$, we get $| \partial H_\omega / \partial \xi | \ll (a_2^3 / 2) | \partial a_1 / \partial t |$ stating that $H_\omega$ may be well approximated by unity at adiabatic conditions.

Equation (23), is not only valid at adiabatic conditions, but also it is a very good approximation even for non smooth pulse shapes as long as the intensity is not very high, letting $S[V_\omega] < \nu_e$. By recalling the relation between time derivatives in transformation from $(x, t)$ to $(\xi, t)$ (from laboratory to pulse commoving window (PCMW)), we have $\partial H_\omega(\xi, t) / \partial t = \partial H_\omega(\xi, t) / \partial t - \nu_\nu H_\omega(\xi, t) / \partial \xi$, therefore, equation (23) is

$$\frac{\partial H_\omega(\xi, t)}{\partial t} = - \frac{\partial \phi_\gamma(\xi, t)}{\partial t} + \frac{1}{\Gamma_\omega(\xi, t)} \frac{\partial a_1(\xi, t) \partial a_1(\xi, t)}{2 \partial \nu},$$

(23a)

where $\xi = x - \nu_e t$. In the right hand side, should be evaluated after time derivatives being taken ($\xi$ and $t$ are independent variables).

For non-smooth pulse shapes $\phi_\gamma(\xi, t)$, appeared in the right hand side of equation (24a), depends on $H_\omega(\xi, t)$ itself through the Poisson equation (19a). Therefore, equation (24a) is, in fact, a complex integro-differential equation which is very hard to obtain its full solution. Nevertheless, we may describe the qualitative behavior of $H_\omega$ through the formal solution of this equation, say,

$$H_\omega(x, t) - 1 = \int_0^t d t' \left\{ - \frac{\partial \phi(\xi', t')}{\partial t'} + \frac{1}{\Gamma(\xi', t') b(\xi', t')} \left( 1 - \frac{1 - b(\xi', t')}{1 + b(\xi', t')} \right) \times \frac{a_1(\xi', t') \partial a_1(\xi', t')}{2 \partial t'} \right\},$$

(24b)

where $\xi' = x - \nu_\nu t$ should be evaluated after time derivatives being taken. This equation states that, variation in $H_\omega$ at a given point $x$ is resulted from cumulative effects of the pulse evolutions during its full passage through that point. Moreover, as $\phi(\xi', t')$ satisfies Poisson equation, even after the pulse leaving, effects of pulse evolutions on the front region is transformed back to that point via $\phi(\xi', t')$. Therefore, we generally expect that $H_\omega - 1$ increases across the pulse, from its front to back, a behavior which will be seen in our simulations.

For future uses, we describe the behavior of $\Omega_p^2 = \omega_p^2 n_e / n_0, \gamma_e$ (appears in pulse evolutions) in terms of $H_\omega, \xi, \phi$. To do this, by substitution of (18a)-(18c) into (3b) and (3a), and making linear expansions in results, we get,

$$\Omega_p^2 = \frac{J_0}{H_\omega + \phi_\gamma} \approx \Omega_{p,s}^2 = \frac{J_0}{H_\omega + \phi_\gamma},$$

(25)
where \( \Omega_{p,q}^2 = -j_{q} / (H_{eq} + \phi_{q}) \). This equation shows that despite \( n_{p} \) and \( \gamma_{eq} \) themselves, \( \Omega_{p,q}^2 \) contains only very small oscillatory content in the form \( \cos(2\theta_{p} + 2\phi_{q}) \). Finally, as an important result verifying our assumptions made in section 3.1, equation (25) states that \( \Omega_{p,q}^2 \) evolves as slow as pulse envelope, as it mimics \( H_{eq} \) behavior.

4. Stability/Instability against high frequency noise

Here, we consider the presence of a weak background of high frequency (in PCMW) radiations in the plasma and its effect in the overall system dynamics, that is following the general ansatz is examined through the wave equation (1e),

\[
A_{y}(\xi, t) = A_{yq}(\xi, t) + A_{yf}(\xi, t),
\]

(26a)

where \( A_{yq} \) is the quasi-static solution, discussed in the previous sections, and \( A_{yf} \) is a weak, quickly evolving (in PCMW) perturbation satisfying \( |A_{yf}| \ll |A_{yq}| \). Associated with \( A_{yf} \), quickly evolving perturbations are produced in plasma quantities;

\[
H_{y} = H_{yq} + H_{yf}, \quad j_{y} = j_{yq} + j_{yf}, \quad \phi = \phi_{q} + \phi_{f}.
\]

(26b)

It should be emphasized that analyses in the above sections have been about dynamics of quasi-static quantities \( (A_{yq}, H_{yq}, J_{yq} \text{ and } \phi_{q}) \) without using \( \text{’q’ indices on quantities.} \) When expansions (26a) and (26b) are substituted back to the fundamental equations (1e), (4), (5) and (1d) subjected to linearization, after some simplifications to be mentioned, and subtraction of zero order equations (equations for \( H_{yq}, J_{yq} \text{ and } \phi_{q} \), which discussed in the previous section), it is fairly straightforward to obtain the following equations,

\[
\left[ 1 \frac{\partial^2}{\xi^2} - 1 \frac{\partial^2}{\xi^2} + \frac{2v_{q}}{c} \frac{\partial^2}{\xi^2} - \frac{\Omega_{p,q}^2}{c^2} \right] A_{yf} = - \frac{\Omega_{p,q}^2 A_{yq}}{\omega_{p}} \left\{ \frac{\Omega_{p,q}^2}{\omega_{p}} (H_{yf} + \phi_{f}) + J_{yf} \right\}
\]

(27a)

\[
\frac{\partial H_{yf}}{\partial t} + c (v_{ex,q} - v_{f}) \frac{\partial H_{yf}}{\partial \xi} = - \frac{\partial \phi_{f}}{\partial t} + \frac{A_{yq} \partial A_{yf}}{\gamma_{q} \partial t}
\]

(27b)

\[
\frac{\partial J_{yf}}{\partial t} + c (v_{ex,q} - v_{f}) \frac{\partial J_{yf}}{\partial \xi} = - \frac{\Omega_{p,q}^2}{\omega_{p}^2 \gamma_{q} \gamma_{n}} \frac{\partial H_{yf}}{\partial \xi}
\]

\[
- \frac{\Omega_{p,q}^2 v_{ex,q} A_{yq}}{\omega_{p} \gamma_{n} \partial t} + \frac{A_{yq} \partial A_{yf}}{\gamma_{n} \partial t}
\]

(27c)

\[
\frac{\partial^2 \phi_{f}}{\partial \xi^2} = - \frac{\Omega_{p,q}^2}{\beta_{p} \omega_{p}^2} \left\{ \frac{\Omega_{p,q}^2 \gamma_{n}}{\omega_{p}} (H_{yf} + \phi_{f}) \right\} - \frac{A_{yq} A_{yf}}{\gamma_{n} \gamma_{q} \partial t} - \frac{\omega_{p}^2}{2c^2} \left\{ \frac{\Omega_{p,q}^2 \gamma_{n}}{\beta_{p} \omega_{p}^2} \right\} J_{yf}.
\]

(27d)

where we have used the definition \( \Omega_{p}^2 \equiv \omega_{p}^2 n_{p} / \gamma_{eq} \) and equation (25). Moreover, in deriving equations (27b) and (27c), it has been noticed that the dominant, secular parts of \( H_{yq}, J_{yq} \) very vary smoothly over the space, according to the analyses presented in the previous section, viz. according to (21b) \( \partial H_{yq} / \partial \xi \sim \partial j_{yq} / \partial t \). On the other hand, rapidly-varying (over the space) oscillatory parts of \( H_{yq}, J_{yq} \) are very small, (see (22)) and may be ignored when multiplied by fast perturbations. In this regard, for example in equation (27b), we have ignored \( v_{ex,q} (\partial / \partial \xi) H_{yq} \) versus \( v_{ex,q} (\partial / \partial \xi) H_{yf} \). In addition, we have ignored \( (\gamma_{q}^{-1} A_{yq} (\partial / \partial t) A_{yq}) \) and \( (\gamma_{q}^{-1} A_{yq} (\partial / \partial t) A_{yq}) \) versus \( \gamma_{q}^{-1} A_{yq} (\partial / \partial t) A_{yq} \) because \( A_{yq} \) evolves much more slower than \( A_{yf} \). The same simplifications have been made in equation (27c). Furthermore, as high rate evolutions are intended here, we may totally ignore the ignored spatio-temporal evolutions of \( H_{yq} \) and \( J_{yq} \), and set \( H_{yq} \equiv 1 \) and \( J_{yq} \equiv - \beta_{p} \).

By properly combining the set of equations (27b)–(27d), we may recover the simple oscillator form for equations governing \( H_{yf}, J_{yf} \text{ and } \phi_{f} \). To do this, we firstly take the time derivative of equation (27d) and substitute \( \partial (H_{yq} + \phi_{q}) / \partial t \) and \( \partial J_{yf} / \partial t \) respectively from equations (27b) and (27c). Doing so, after some mathematical manipulations we end up with \( (\partial^2 / \partial \xi^2)(\partial \phi_{f} / \partial t) = - (\omega_{p}^2 / c) (\partial J_{yf} / \partial \xi) \) which after integration over the space gives,

\[
\frac{\partial^2 \phi_{f}}{\partial \xi^2} = - \frac{\omega_{p}^2}{c} J_{yf}.
\]

(28a)

Now, we take \( \partial / \partial \xi \) of equation (27b) and use the above equation in the result to obtain,

\[
\frac{\partial^2 H_{yf}}{\partial \xi^2} + c \frac{\partial}{\partial \xi} [(v_{ex,q} - v_{f}) \frac{\partial H_{yf}}{\partial \xi}] = \frac{\omega_{p}^2}{c} J_{yf} + c \frac{\partial}{\partial \xi} \left[ \frac{A_{yq} \partial A_{yf}}{\gamma_{eq} \partial t} \right].
\]

(28b)

which its left hand side is entirely in terms of \( H_{yf} \). Using the relation \( \partial v_{ex,q} / \partial t \approx 0 \) (because quasi-static quantities evolve much slower than fast quantities), we may rewrite the above equation in a much more compact and interpretable form.

Next, we take once the time derivative of the above equation and another time its space derivative, being left by two equations respectively for \( \partial J_{yf} / \partial t \) and \( \partial J_{yf} / \partial \xi \). These two equations are substituted into equation (27c), and finally we get,

\[
\frac{\partial^2 H_{yf}}{\partial \xi^2} + c \frac{\partial^2}{\partial \xi^2} [(v_{ex,q} - v_{f}) \frac{\partial H_{yf}}{\partial \xi}] + c (v_{ex,q} - v_{f}) \frac{\partial^2}{\partial \xi^2} [\frac{A_{yq} \partial A_{yf}}{\gamma_{eq} \partial t}]
\]

\[
- v_{f} \frac{\partial^2 H_{yf}}{\partial \xi^2} + c^2 (v_{ex,q} - v_{f}) \frac{\partial^2}{\partial \xi^2} [(v_{ex,q} - v_{f}) \frac{\partial H_{yf}}{\partial \xi}] - v_{f} \frac{\partial^2 H_{yf}}{\partial \xi^2} \frac{\partial H_{yf}}{\partial \xi} = - \frac{\Omega_{p,q}^2}{\gamma_{eq} \partial t} \frac{\partial H_{yf}}{\partial \xi} + \frac{\partial^2}{\partial \xi^2} \left[ \frac{A_{yq} \partial A_{yf}}{\gamma_{eq} \partial t} \right].
\]

(28c)
\[
\begin{align*}
\left( \frac{d^2}{dt^2} + \nu_f \frac{d}{dt} + \Omega_f^2 \right) \frac{\partial H_f}{\partial \xi} &= 0, \\
\left( \frac{d^2}{dt^2} + \nu_f \frac{d}{dt} + \Omega_f^2 \right) \frac{\partial A_f}{\partial \xi} &= -\frac{\Omega_{pq}^2}{\gamma_{pq}} \frac{\partial A_f}{\partial t} + \frac{d}{dt} \frac{\partial}{\partial \xi} \left[ \Omega_{pq}^2 \frac{\partial A_f}{\partial t} \right], \\
\nu_{fl} \equiv c \frac{\partial v_{ex,q}}{\partial \xi}, \quad \Omega_f^2 \equiv \Omega_{pq}^2 / \gamma_{pq}^2 + (v_{ex,q} - v_f) \frac{\partial^2 \nu_{ex,q}}{\partial \xi^2},
\end{align*}
\]

which is the equation of the damped driven oscillator with position dependent parameters. The complete time derivative is \( d/dt = \partial / \partial t + (v_{ex,q} - v_f) \partial / \partial \xi \) as used before, stating that variations are measured along the fluid element trajectory in the Galilean PCMW, i.e. along \( \xi = \xi_0 + \int_0^t dt' [v_{ex,q} - v_f] \).

We may obtain a similar equation as (28d) for \( J_f \), via the same procedure. We give the following result, only.

\[
\begin{align*}
\left( \frac{d^2}{dt^2} + \nu_f \frac{d}{dt} + \Omega_f^2 \right) J_f &= -\nu_f \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial A_f}{\partial t} - \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial A_f}{\partial t} \\
&+ \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial}{\partial \xi} \left[ \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial A_f}{\partial t} \right], \\
\nu_{jfl} \equiv c \frac{\partial v_{ex,q}}{\partial \xi} - \left( \frac{\gamma_{pq}^2}{\Omega_{pq}^2} \right)(v_{ex,q} - v_f) \frac{\partial}{\partial \xi} \left( \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \right), \quad \Omega_f^2 \equiv \Omega_{pq}^2 / \gamma_{pq}^2 \frac{\partial}{\partial \xi} \left( \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \right),
\end{align*}
\]

which according to (28a) gives also,

\[
\begin{align*}
\left( \frac{d^2}{dt^2} + \nu_f \frac{d}{dt} + \Omega_f^2 \right) \frac{\partial^2 \phi_f}{\partial \xi^2} &= \nu_f \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial A_f}{\partial t} + \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial}{\partial \xi} \left[ \frac{\Omega_{pq}^2}{\gamma_{pq}^2} \frac{\partial A_f}{\partial t} \right].
\end{align*}
\]

The set of equations (27a), (28d), (29a) and (29b) present a system of coupled spatiotemporal oscillators, which fully describe the laser plasma evolutions beyond the quasi-static regime. At very low laser intensities, these equations may be easily combined to recover the well-known quasi-linear equations for momentum and density, suited in common studies of Raman scatterings, viz. at very low intensities we would have \( \gamma_{pq} \approx 1, \quad v_{ex,q} \approx 0, \quad \{ \nu_f, \nu_{fl} \} \approx 0, \quad \Omega_{pq}^2 \approx \Omega_f^2 \approx \omega_f^2, \quad H_f \approx \gamma_{pq}^2 - \beta_f R_{x,f} - \phi_f \approx A_{pq} A_f - \beta_f R_{x,f} - \phi_f \) and \( J_f \approx R_{x,f} - \beta_f R_{x,f} \). The outcome of plasma wave formulation in terms of driven simple oscillators is an important benefit of our approach over the previous approaches, which produces a manifest analogy with the quasi-linear regime. This simplicity attained because, even in the presence of the strong plasma wave, the unperturbed profiles of \( H_f \) and \( J_f \) \((H_{x,q} \text{ and } J_{x,q})\) remain approximately unmodulated (spatial constants); if we have used the density presentation for plasma wave (plasma wave in terms of \( n_e \)), as in the previous studies, the unperturbed density profile \( n_{ex,q} \) would be modulated in the presence of wakefield, and the form of the plasma wave would become very complicated. This is while within our system of coupled oscillator equations, (27a), (28d), (29a) and (29b), we may immediately deduce the possibility of parametric self-amplification according to analogy with the quasi-linear regime; in the linear regime this self-amplification leads to parametric instabilities (exponentiations). Furthermore, as their formulations are similar in nature, we may even understand the main distinguishing points and signatures of quasi-linear and nonlinear regimes.

In the so-called adiabatic regime of smooth pulse interactions, according to the analyses presented in the previous section (note that in that section all quantities have been quasi-static without being indexed by 'q') we have for the secular part of \( v_{ex,q} \ll 1 \) (see arguments above equation (23)), and the secular parts of plasma wave parameters reduce to,

\[
\begin{align*}
\Omega_{pq}^2 \approx \Omega_f^2 \approx \Omega_{pq}^2 / \gamma_{pq}^2 \approx \frac{\omega_p^2}{1 + \alpha_n^2}, \quad \nu_f \approx 0, \\
\nu_{jfl} \approx 0.
\end{align*}
\]

At these conditions equations (28d), (29a) and (29b) become very simplified, and quite analogous to the quasi-linear regime. The only superior effect is the well-known growth rate reduction by the relativistic mass increase [39–42].

Despite the above simplicity attained at smooth pulse shapes, the situation may become complicated when the pulse shape initially has a quickly rising part. At these conditions, due to the wakefield excitation, the local frequencies \( \Omega_{pq}^2 \), \( \Omega_f^2 \) and \( \Omega_{pq}^2 \) (see equations (28d), (29a) and (29b)) becomes both highly modulated and displaced with respect to each other, and dissipation factors \( \nu_f \) and \( \nu_{fl} \) becomes non-zero. Moreover, as is seen in (28e), the trajectory of each fluid element deviates highly from \( \xi = \xi_0 - v_f t \). In these regards, and in terms of quantities \( H_f \) and \( J_f \), one may imagine the plasma disturbance as a collection of transverse oscillators whose centers perform longitudinal oscillations and whose frequencies change with time. As a result a complex phase relation is produced between the oscillators and both the pump and the scattered electromagnetic waves. These behaviors leads at most into two consequences; (i) broadening in the scattered wave spectrum (ii), possible losing of the plasma wave resonance with driving electromagnetic pump and scattered waves. Moreover, if an resonance be pertained at all, the Stokes (anti-Stokes) wave may display anomalous shifts in \( \omega - k \) plan with respect to the pump wave \((k_0, \omega_0)\), depending on the values of frequencies \( \Omega_{pq}^2 \), \( \Omega_f^2 \) at the most effective parts of the plasma wave. All these behaviors will be observed in our simulations.

5. Numerical and simulation investigations

5.1. General considerations on simulation methods and setups

We use both direct numerical solution of our basic fluid-Maxwell equations (1a)–(1e) (will be termed as fluid simulations) and PIC simulations to validate our obtained results, and further investigate the problem under consideration. In our fluid simulations, we use a second order upwind algorithm [50], very similar to what has been implemented in [51].

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In these fluid simulations we have a limited range of numerical stability over our physical parameter-space. This discrepancy is not exclusive to our work, but rather it is a global and well-known problem which arises during fluid solution of nonlinear equation-systems (see e.g. [50, 52]). In PIC simulations, on the other hand, it is impossible to examine a noise-free or zero-temperature plasma (see [53] and references therein) which is desired to verify the impact of noise content in producing instabilities, as proposed in section 4. In this regards, we use fluid simulations to demonstrate the importance of initial noise content. On the other hand, we perform our parametrical investigations mainly via PIC simulations. Moreover, using both methods in conjunction, we may go beyond our fluid model (equations (1a)–(1e)) and investigate the kinetic effects (non-zero temperature effects), by comparing results from cold fluid and PIC simulations at non-zero temperatures.

We have conducted several instances of PIC (for simulation code see [53, 54]) and cold-fluid simulations, at different pulse intensities, lengths and shapes, and plasma densities –some of them are presented here. For all presented simulations, the laser pulse is launched into a slab quiescent plasma from its left boundary, and we invariably set \( \lambda = 1 \, \mu \text{m}, n_e/n_0 = 0.01, k_B T_0 = 48.5 \, \text{eV} \) for PIC simulations and \( k_B T_0 = 0 \) for fluid simulations where \( \lambda, n_0, n_e, T_0 \) and \( k_B \) are respectively the laser wavelength, the initial plasma density, the critical density \( \left( n_c = \varepsilon_0 m_e \omega_0^2 / e^2 \right) \), the initial electron temperature (typical in laser plasma interactions) and the Boltzmann constant. This is while the dimensionless pulse amplitude \( (a_0) \), duration (denoted by \( \tau_L \)) and pulse shape (denoted by \( SF \)) vary among different runs. The pulse shape comprises three rise, flat and fall parts into a trapezoidal shape whose straight sides are replaced by sinusoidal curves. For each value of this factor, we give an ordered triple array as \( SF = \{ \text{rise time, flat time, fall time} \} \). Totally, we discuss three values of \( a_0, a_0 = \{0.5, 1, 2\} \), two values of \( \tau_L, \tau_L = \{200, 300\} \, \text{fs} \), and four values of \( SF, SF = \{100, 0, 100, 150, 0, 150\}, \{30, 140, 30\}, \{30, 240, 30\} \) fs.

In cold fluid simulations we are able to introduce an initial small disturbance in the density of the quiescent plasma (denoted by \( \delta n_{\text{noise}} \)), acting as an artificial noise which its level (denoted \( \delta n_{\text{noise}} \)) may be controlled from an input parameter. Here we set either \( \delta n_{\text{noise}}/n_0 = 0 \) or the typical value \( \delta n_{\text{noise}}/n_0 \sim 10^{-4} \), and respectively refer to the corresponding run by ’without noise’ and ’with noise’. Though, in the absence of noise, the fluid model is stable over the mentioned physical parameters, this stability may be lost for some pulse shapes (as will be mentioned in the next subsection) in the presence of the noise. As will be seen, this important nonlinear behavior is due to interference between physical and numerical instabilities.

6. Results and discussions

First of all, in figure 1, we plot the pulse envelope evolutions together with frequency modulations as predicted by our analytical equations, (11a), (16b) and (16c), for pulse parameters \( a_0 = 2, \tau_L = 200 \, \text{fs} \) and \( SF = [30, 140, 30] \). In plotting the envelope profiles, we have substituted \( \tau_0 \) from (16c) into (16b) and then substituted the result for \( a_{0,\text{fl}} \) into (11a) while the terms beyond \( t^2 \) have been ignored in this equation, i.e. \( a_0 (\xi, t) = a_0 (\xi) + i a_{0,\text{fl}} (\xi) \). It is clearly seen that the modulated frequency acts as an effective potential in driving amplitude evolutions as suggested by equation (16b). Also, appearance of fine oscillation in envelope is predicted by equation (16c), as mentioned at the end of section 3.1.

Next, we make comparison between results of two simulation methods with each other and with presented analyses. In this line, in figure 2, we have plotted the fundamental quantities \( H_z, J_z \), and \( A_z \) versus \( \xi \) at different interaction times \( t = 300 \, \text{fs} \) and \( t = 400 \, \text{fs} \), from both fluid ‘with noise’ and PIC simulations, at pulse parameters \( a_0 = 2, \tau_L = 200 \, \text{fs} \) and \( SF = [30, 140, 30] \). In addition, to be verified, we have plotted our analytical results for \( A_z \), envelope in the manner described just above figure 1. In the case of \( H_z \), PIC results are \( x - h_x \), snapshots of electrons where \( h_x \) in correspondence with \( H_z \), is defined as \( h_x = \gamma - \beta_p x - \phi (x) \) for each individual electron with position \( x \) and longitudinal momentum \( p_x \).

As it is seen in figure 2, for considered pulse parameters, both PIC and fluid models are in very good agreement about the secular variations of \( H_z \) and \( J_z \) profiles inside the pulse. This is while, the kinetic model, which definitely includes non-zero temperature effects, predicts amplification in fine oscillations of these profiles across the pulse, an effect which eventually leads to formation of a chaotic pattern in the phase snapshots at sufficiently large distances from the pulse front. Despite these superior kinetic effects in plasma motion, pulse evolutions quite matched in both kinetic and fluid models. Moreover, in this case, an excellent agreement is observed among simulations and our analytical results described on figure 1.

Figure 2, also, presents an example of spatiotemporal behavior of \( H_z \), which clearly shows the convective nature of evolutions in this quantity, as described by equation (24b). Also fine oscillations are observed in fluid results for this quantity as predicted by equation (22).

Now, before going further in pulse evolution analyses, we aim to demonstrate that systems with smooth pulse shapes are extremely sensitive to the presence of an initial noise content. In figure 3, we compare the action of fluid
simulations, after interaction time of 300 fs, in the absence and presence of the initial noise content at two different pulse shapes 1-sharp rise shape $SF = [30, 140, 30]$ (panels (a)–(c)) and 2-smooth rise shape $SF = [100, 0, 100]$ (panels (d)–(f)); other pulse parameters are kept identical, i.e. peak amplitude at $a_0 = 2$, and total duration at $\tau_c = 200$ fs. The initial noise content is introduced according to descriptions in the previous subsection with amplitude $\delta n_{\text{init}} \sim 10^{-4}n_0$, as is shown on the insets of panels (b) and (e). Surprisingly, despite absolute neutrality of the first setup ($SF = [30, 140, 30]$) with respect to the imposed small disturbance, the second setup develops very large oscillations in response, in both $H_x$ and $n_e$ quantities. This behavior confirms the instability of smooth pulses via the presence of noise content. It could not be deduced from PIC simulations, because the noise content could not be totally eliminated in this method. However, at latter times, the destabilized oscillations are subjected to numerical instabilities and become further and further amplified in unphysical manner. This makes (our) fluid simulations unsuitable for long term studies of smooth pulses. In the following, we

![Figure 2](image1.png)

**Figure 2.** $H_x, J_x$ and $A_y$ at $t = 100$ fs (a)–(c) and $t = 200$ fs (d)–(f), for pulse parameters given in the text. Black, green and blue data respectively correspond to PIC simulation, fluid ‘with noise’ simulation and analytical solution (see text).

![Figure 3](image2.png)

**Figure 3.** Profiles for $H_x, n_e$ and $A_y$ in the absence (green curves) and presence (blue curves) of initial noise content, for pulse shape $SF = [30, 140, 30]$ (left column) and pulse shape $SF = [100, 0, 100]$ (right column). Inset in panel (d) shows the same data as outset but with full range of $H_x$ axis. Insets in (b) and (e) represent the density profile at upstream of plasma.
utilize PIC simulation to study the physical instability of pulse propagation.

Next, in figures 4 – 10, we reveal the essential differences produced by the pulse shape factor in the system behavior in developing parametric instabilities, as viewed in plasma dynamics. In figures 4 and 5, from PIC simulations, we respectively summarize the spatiotemporal behaviors of \( H_x \) and \( J_x \) at laser intensity \( a_0 = 2 \) and duration \( \tau_L = 300 \text{ fs} \), and for two different pulse shapes \( SF = [150, 0, 150] \) (left columns) and \( SF = [30, 240, 30] \) (right columns). In these figures, we observe that while the smooth pulse shape \( SF = [150, 0, 150] \) develops large amplitude quasi-coherent oscillations in both \( H_x \) and \( J_x \) in the course of time, the quickly rising pulse \( SF = [30, 240, 30] \) produces irregular and much weaker perturbations. In figures 6 and 7, we respectively present the same plots as figures 4 and 5 at the same pulse shapes but a different intensity \( a_0 = 1 \). It is seen that compared to \( a_0 = 2 \), at this lower intensity \( a_0 = 1 \) more stronger oscillations appear in both \( H_x \) and \( J_x \) when the quickly rising pulse shape is applied, while the plasma behavior at smooth pulse shape remains qualitatively similar among both used intensities. Further, by setting \( a_0 = 0.5 \), we continue to further lowering the laser intensity to reach the sub-relativistic regime. In this regard, in figures 8 and 9, we present additional plots respectively for \( H_x \) and \( J_x \) at this sub-relativistic (quasi-linear) intensity, for same pulse shapes as in the previous cases, \( a_0 = 2 \) and \( a_0 = 1 \). It is seen that, in the covered region \(-50 \leq \xi \leq -20\), by reducing the intensity down to the sub-relativistic value \( a_0 = 0.5 \) the plasma behavior is quite altered with respect to relativistic case \( a_0 = 2 \), i.e. oscillations are more stronger in the case of quickly rising pulse shape.

In figure 10 we show snapshots of \( H_x \) over the interaction regions that, for the purpose of proper resolution of oscillations, remained uncovered in figures 4 and 8. Here, we plot \( H_x \) both over the full interaction region and also magnified uncovered region for considered lower \( a_0 = 0.5 \) and higher intensities \( a_0 = 2 \), and pulse shapes \( SF = [150, 0, 150] \) and \( SF = [30, 240, 30] \), at interaction time \( t = 300 \text{ fs} \). One may observe that, at relativistic intensities, oscillations finally evolve into chaotic structures, while remain regular at sub-relativistic intensities. Moreover, as the instantaneous laser field is also plotted inside each panel, one may find local correlations between laser and plasma variations.

The development of large oscillations in \( H_x \) and \( J_x \) observed both on fluid simulations of figure 3 and in present PIC simulations of figures 4 – 10, under the action of smooth and/or lower intensity pulses, may be described in attribution to development of Raman instability as summarized in section 4. In order to verify this idea, we should consider the self-consistent pulse \( (A_i) \) evolutions in the mode space \((k - \omega)\) plane. To this aim, in figures 11 – 14 we summarize \( k - \omega \) maps obtained via the time-space Fourier transform of \( A_i \) at desired times for different pulse parameters. These plots include both forward (positive \( \omega \), positive \( k \)) and backward (negative \( \omega \), positive \( k \)) going radiation components. At each of our figures, we have three column of panels which correspond to different interaction times, in order from left to right,
Figure 5. $J_x$ profiles for same parameters and settings as figure 4.

Figure 6. $x$–$h_x$ snapshots of electrons for intensity $a_0 = 1$, $\tau_L = 300\text{fs}$ and smooth $SF = [150, 0, 150]$ (left column) and quickly rising $SF = [30, 240, 30]$ (right column) pulse shapes.
Figure 7. $J_x$ profiles for same parameters and settings as figure 6.

Figure 8. $x$–$h_x$ snapshots of electrons for intensity $a_0 = 0.5$, $\tau_x = 300\text{fs}$ and smooth $SF = [150, 0, 150]$ (left column) and quickly rising $SF = [30, 240, 30]$ (right column) pulse shapes.
Figure 9. $J_x$ profiles for same parameters and settings as figure 8.

Figure 10. $x$–$h_x$ snapshots of electrons (blue dots) and instantaneous laser field (green curves) at interaction time $t = 300$ fs for $\tau_L = 300$ fs and different intensities $a_0 = 2$ (upper two rows) and $a_0 = 0.5$ (lower two rows), and different smooth $SF = [150, 0, 150]$ (left column) and quickly rising $SF = [30, 240, 30]$ (right column) pulse shapes. The second and fourth rows only magnify the data over the selected position range.
At sub-relativistic intensities (see figures 8, 9 and 13 all at $a_0 = 0.5$) the Raman scattering behaves ordinarily, as given by the quasi-linear theory (note $\omega_p = 0.1\omega_0$ for our parameters; see arrows set on figures). As the instability rate is proportional to pump wave amplitude $a_0$, it develops faster when the pulse reaches its peak amplitude more quickly, that is for quickly rising pulse. Therefore, the observed behavior of plasma oscillations (figures 8 and 9) as well as the mode space (figure 13) is ordinary and quite within the ordinary quasi-linear theory. This is while, when we enter the relativistic regime, as described at the end of section 4, the quick pulse rise leads to new factors like broadening and lose of resonance, which compete with exponentiation. This is why the effect of pulse quick rise-up becomes eventually reversed at high intensities ($a_0 = 2$, see figures 4, 5 and 11) and destruct Raman instability formation. Note this is not only due to the well known effect of relativistic mass increase (see equation (30) for smooth pulse), as these effect cannot produce order of magnitude or higher differences appeared for $t = 100$ ns, $t = 200$ ns and $t = 300$ ns. In addition, each figure comprises four rows of panels, the upper two rows (panels (a)-(f)) correspond to the smooth pulse shape and the lower two rows (panels (g)-(l)) correspond to the quickly rising pulse shape; out of these two pairs of rows, the second (d)-(f) and fourth (j)-(l) rows isolate negative-$\omega$ (backward) modes, in order to magnify the amplitude of these modes versus the strong forward-going pump wave. In figures 11–13 the same parameters as in figures 4, 6 and 8, are used respectively, say, they are given for different intensities $a_0 = 2$ (figure 11), $a_0 = 1$ (figure 12) and $a_0 = 0.5$ (figure 13), the identical pulse duration $T_L = 300$ ns, the smooth shape $SF = [150, 0, 150]$ (panels (a)-(f) in each figure) and the quickly rising shape $SF = [30, 240, 30]$ (in panels (g)-(l) in each figure). Figure 14 uses the same intensity as figure 11 but different pulse duration $T_L = 200$ fs (same as figure 3) and pulse shapes $SF = [100, 0, 100]$ (as smooth) and $SF = [30, 140, 30]$ (as quickly rising). As, is seen, the results presented in figures 10–14 confirm the occurrence and different behavior of Raman scattering at different pulse parameters, in a close correlation with observed plasma behavior (figures 4–10), and in agreement with analyses in presented section 4.
between different pulse shapes in figure 10. When the intensity is relativistic but moderate \((a_0 = 1\), see figures 6, 7, and 12) the Raman instability appears but at an anomalous Stokes (indexed by ‘s’) shift in \(k - \omega\) plane. On figure 12, for \(a_0 = 1\), we observe a shift twice the quasi-linear prediction \(k_s - k_0 \approx 2\omega_p/c\) and \(\omega_s + \omega_0 \approx 2\omega_p\), again in agreement with predictions at the end of section 4. This phenomenon may be understood by noting the electron density structure shown on figure 6(f). When we shorten the pulse length at intensity \(a_0 = 2\) from \(\tau_L = 300\) fs to \(\tau_L = 200\) fs, as is seen in figure 14, we observe increase in the spectrum broadening around the Stokes position, again in agreement with analyses in the end of section 4.

At the end of this section, we may notify appearance of very low-frequency long wavelength \((k \approx 0, \omega \approx 0)\) modes in \(k - \omega\) plane at high intensity quickly rising pulses, in agreement with descriptions in the end of section 3.1. Also, the radiation mode \((k = k_0, \omega = -\omega_0)\), observed in all simulation cases, may be attributed to back-reflection from the plasma boundary.

7. Summary and outlook

We have presented an integrated (including both slow and fast variations of the pulse, and plasma motion) analysis of non-linear evolutions of a relativistic short laser pulse propagating in an under-dense plasma, when the pulse-length exceeds the plasma wavelength. An important feature of our presented formulation is its universality over the pulse-shapes, i.e. our final equations \((16b), (16c), (24a), (24b), (27a), (28d), (29a)\) and \((29b)\) remain valid at arbitrary pulse shapes. This allows for description of the diverse behaviors of different pulses in the same physical grounds. This is despite the previous works which distinguish between smooth and quickly rising pulses (see e.g. [46, 47]), applying different instability and slow analyses to these types, respectively.

We have firstly obtained new general slow-scale solutions (see equations \((16b)\) and \((16c)\), figures 1, 2) for pulse evolutions (self-modulation or breakup) and related self-consistent plasma motions (see equations \((23)\) and \((24)\)) at general pulse-shapes (see descriptions below equation \((14)\)
and intensities. Afterward, we have issued instability (fast-scale) analyses of the modulating pulse utilizing our novel formulation of plasma-wave in terms of of $H_x$ and $I_x$ (see equations (27a), (28d), (29a) and (29b)). Using this formulation, it has become possible to refine the instability induced plasma perturbations from the large amplitude plasma wave excited by the laser ponderomotive force. In this way, for the first time, direct evidences of plasma instability in the presence of initial strong wavefield are presented in terms of easily-interpretable plasma perturbations (growing oscillatory perturbations in constant backgrounds of $H_x$ and $I_x$, see equations (28d), (29a) and (29b), and figures 4–10). Moreover, we have uncovered the self-consistent plasma motion in the self-modulating laser field (see equations (24a) and (24b)), which turns very important in describing electron acceleration in the SMLWFA (see below). Our results have been well verified by simulation data.

Our universal model recovers the interplay between initially induced, strong plasma-wave (longitudinal wavefield) and instabilities. The diverse behaviors appeared by this interplay at different pulse-shapes and intensities may be described within our framework (see descriptions at the end of section 4 and descriptions on figures 4–14), as has been verified through extensive simulation results presented in section 5. A summary of results is as follows:

According to section 4 and related simulation results in section 5, smooth pulse shapes (with very weak longitudinal field) behave ordinarily in relation with Raman scatterings, i.e. the scattered Stokes wave (indexed by ‘s’) appears as usual in the position $k_s \approx -k_0 + \omega_p/c$ and $\omega_s \approx \omega_0 - \omega_p$ (see figures 11–14) and amplifies in time. This is while quickly rising pulses at high intensities behave anomalously and show diversity in this respect: The Stokes (sidebands) may be highly displaced with respect to the usual position, e.g. $k_s = -k_0 + 2\omega_p/c$ and $\omega_s \approx \omega_0 - 2\omega_p$ for parameters of figure 12, and the instability becomes almost halted (beyond the relativistic mass increase effect) at sufficiently high intensities (here $a_0 = 2$, figures 11 and 14).

Reduction of $H_x$ across the pulse region (see figures 2, 4, 10 and equation (24b)) can play a significant role in electron injection into the acceleration phase (trapped population) in SMLWFA, as it causes the trajectory of wave-body electrons to get close to the separatrix between trapped and untrapped trajectories. In other words, it causes reduction of wave-break.
amplitude across the pulse, viz. the cold-fluid wave-break threshold, \( f_{WB} \), is obtained by setting \( v_{e x} = v_{g} \) at the maxima of the wake potential (see e.g. [2]), therefore, we would have \( f_{WB} = \frac{v_{e x}}{v_{g}} \). In the same way, Oscillatory and chaotic patterns in \( x - h_{c} \) plots of figures 6, 8 and 10, produced by interplay between kinetic effects and light scattering, lead to enhanced electron trapping.

Generalization of equations (16b) and (16c) to 3D is straightforward; 3D version of the wave equation (6) is obtained by adding \( \nabla_{\perp}^{2} \hat{A} \) to the left hand side of the present equation where \( \nabla_{\perp}^{2} = \frac{\partial^{2}}{\partial y^{2}} + \frac{\partial^{2}}{\partial z^{2}} \) and \( \hat{A} \) is now \( \hat{A}(\xi, y, z, t) \). It is fairly straightforward to show that this correction changes equation (16c) as:

\[
\omega_{0} = \frac{1}{4A_{0}c\nu_{y}} e^{-\kappa_{y}^{2}} \left\{ \sin \left( \frac{\omega_{0}}{c\nu_{y}} (\xi - \xi') \right) \right\} \Omega_{p0}^{2} \\
\times (\xi', y, z) - \omega_{p}^{2} - c^{2} \nabla_{\perp}^{2} - \frac{c^{2}}{\gamma_{s}^{2}} \frac{\partial^{2}}{\partial \xi'^{2}} \\
\times \hat{A}_{0}(\xi', y, z) \right\},
\]

and leaves equation (16b) unchanged. Also the useful formulation of the plasma wave in terms of \( J_{x} \) and \( H_{x} \) may be generalized to 3D geometry by finding non-evolving-pulse constants in this geometry.

**Appendix. Derivation of equations (21a), (21b) and (22)**

The decoupled equations for secular and oscillatory parts of \( H_{c} \) first appear as follow:

\[
\frac{\partial H_{a}}{\partial t} + c_{0} (S_{V_{a}} + \Delta_{1}) - v_{g} \frac{\partial H_{a}}{\partial \xi} = - \frac{\partial \phi_{a}}{\partial t} + c \frac{a_{a}}{2} \frac{\partial H_{a}}{\partial \xi} \left( \frac{1}{\Gamma_{a} + 1} + \Delta_{2} \right) \cos(2\theta_{0} + 2\vartheta) + 2\vartheta) ) + c_{S} S_{V_{a} + \Delta_{2}} \left( \frac{\partial H_{a}}{\partial \xi} + \omega_{m0} \right) + \omega_{0} \frac{\partial \Delta_{2}}{\partial \xi} \right]
\]

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+ (\omega_{m0}) \frac{\partial \Delta_{2}}{\partial \xi} \right]
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\[
\begin{align*}
\frac{\partial H_{ss}}{\partial t} + cO & \left( (V_{os} + \Delta) - \nu \right) = \frac{\partial H_{ss}}{\partial \xi} \\
+ cO[\Delta_{e}] & \frac{\partial H_{ss}}{\partial \xi} = -\frac{\partial \phi_{c}}{\partial t} + (\omega_{n0}) \\
+ \nu & \frac{\alpha_{0}^{2}}{2} \sin(2 \theta_{0} + 2 \nu) \\
+ cO[V_{os}] & \frac{\partial H_{ss}}{\partial \xi} \\
& = \left( \frac{1}{\Gamma_{n} + \Gamma_{o}} + \Delta_{s} \right) (1 + \cos(2 \theta_{0} + 2 \nu)) \\
& \times \frac{\alpha_{0}}{2} \frac{\partial \alpha_{0}}{\partial t} + (\omega_{n0} + \nu) \frac{\alpha_{0}^{2}}{2} [O[\Delta_{s}] \sin(2 \theta_{0} + 2 \nu)],
\end{align*}
\] (A.2)

where \( S[.] \) and \( O[.] \) respectively mean secular and oscillatory values of included expressions. In the right hand sides of above equations, we have also used the fact that \( S[\sin(2 \theta_{0} + 2 \nu) / (\Gamma_{n} + \Gamma_{o})] = 0 \) (multiplication of sin and functional of cos does not produce secular parts). In addition we have used the identity \( A_{e}(\partial \phi_{c} / \partial t) = \omega_{n} + \nu (\alpha_{0}^{2}/2) \sin(2 \theta_{0} + 2 \nu) + \left( (\alpha_{0}/2) (\partial \alpha_{0}/\partial t) [1 + \cos(2 \theta_{0} + 2 \nu)] \right) \).

Equation (A.2) may be dramatically simplified when we consider the order of magnitude of its different terms; \( \frac{\partial H_{ss}}{\partial t} \sim \omega_{n0} H_{ss}, \frac{\partial H_{ss}}{\partial \xi} \sim \frac{\partial \phi_{c}}{\partial t} \), and the approximation \( 1 - \nu \approx 1/2 \frac{\gamma_{s}^{2}}{2} \). Upon substitution of these equations into (21a) and truncating terms proportional to \( \gamma_{s}^{2} \), the integration of equation (21a) (in the same manner as done for \( \phi_{c} \)) ultimately gives (22). Afterward, we may use (22) to simplify equation (A.1), as we have,

\[
\begin{align*}
S [V_{os} + \Delta_{s}] \frac{\partial H_{ss}}{\partial \xi} &= 0, \\
S [\Delta_{s}, \sin(2 \theta_{0} + 2 \nu)] &= 0,
\end{align*}
\] (A.3)

therefore, we obtain,

\[
\begin{align*}
\frac{\partial H_{ss}}{\partial t} + cO[V_{os}] \frac{\partial H_{ss}}{\partial \xi} &= \frac{\partial \phi_{c}}{\partial t} \\
+ \frac{\alpha_{0}}{2} \frac{\partial \alpha_{0}}{\partial t} S & \left( \frac{1}{\Gamma_{n} + \Gamma_{o}} + \Delta_{s} \right) (1 + \cos(2 \theta_{0}) \\
+ 2 \nu) \right).
\end{align*}
\] (A.4)

To the leading order we may ignore \( \Delta_{s} \) versus \( 1/(\Gamma_{n} + \Gamma_{o}) \) and \( \Delta_{n} \) versus \( \nu \) in above equation, and we are left with calculation of secular parts \( (S[\ldots]) \). This is done as usual by integral averaging over the spatial interval \( 0 \leq \xi \leq \pi / (k_{0} + k) \) which eliminates oscillatory harmonics of \( 2 \theta_{0} + 2 \nu \) in all orders, viz. \( S[\ldots] = \frac{k_{0}}{\pi} \int_{0}^{\pi / (k_{0} + k)} \frac{\sin(2 \theta_{0} + 2 \nu)}{1 + \frac{1}{b}} d \xi [\ldots] \). The final results are,

\[
S \left[ 1 + \cos(2 \theta_{0} + 2 \nu) \right] = \frac{1}{\Gamma_{n} b} \left( 1 - \frac{1 - b}{1 + b} \right).
\] (A.5)

where \( b \equiv (\alpha_{0}^{2}/2)/(1 + \alpha_{0}^{2}/2 + (H_{ss} + \phi_{c})^{2}) \). Upon substitution of (A.5) and (A.6) into (A.4), we find equation (21b).

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