Homotopical Aspects of Commutative Algebras II:  
Freeness Conditions for Quadratic Modules

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Abstract

This article investigates the homotopy theory of simplicial commutative algebras with a view to homological applications.

Introduction

Baues, [10], defined the notion of quadratic module of groups as an algebraic model for homotopy (connected) 3-types and gave a relation between quadratic modules and simplicial groups. This model is also a 2-crossed module structure defined by Conduché (cf. [16]) with additional nilpotent conditions.

In the series of articles [4, 5, 6] the first author with T. Porter have started to see how a study of the links between simplicial commutative algebras and classical constructions of homological algebra can be strengthened by interposing crossed algebraic models for the homotopy types of simplicial algebras. Particularly, in [5], they gave a construction of a free 2-crossed module of algebras in terms of the 2-skeleton of a free simplicial algebra. In the group case, a closely related structure to that of 2-crossed module is that of a quadratic module (cf. [7]). Although it seems intuitively clear that the results of the papers [5] and [7] can be extended to an algebra version of quadratic modules. Hence, in addition to this series of papers, in [8], the first and second authors defined the notion of quadratic module for commutative algebras and explored the relations among quadratic modules and simplicial commutative algebras in terms of hypercrossed complex pairings. In this work, we continue this process using these methods to give a construction of a (totally) free quadratic module of commutative algebras in terms of the 2-skeleton of a free simplicial algebra. We also give the homotopical significance of these constructions in Propositions 2.5 and 2.6.

Alternatively, by using tensors and coproducts in the category of crossed modules of commutative algebras, we give a neat description of a free quadratic module from a free quadratic modules, free simplicial algebras, free 2-crossed modules, free crossed squares.
crossed square of commutative algebras analogue to that given by Ellis (cf. [19]) in the group case. In [9], Arvasi and Ulualan gave an alternative description of the top algebra of the free crossed squares of algebras on 2-construction data in terms of tensors and coproducts of crossed modules of commutative algebras and gave a link between free crossed squares and Koszul complexes. By using this close relationship between crossed squares and quadratic modules, we shall give an alternative construction of the totally free quadratic module. In this context, our result has an advantage; Baues’s theory is also related via a quotient functor

\[ q: \text{(free crossed squares)} \to \text{(free quadratic modules)} \]

for the case of commutative algebras in terms of the 2-skeleton of a free simplicial algebra.

1 Simplicial Algebras, 2-Crossed Modules and Crossed Squares

In what follows ‘algebras’ will be commutative algebras over an unspecified commutative ring, \( k \), but for convenience are not required to have a multiplicative identity. The category of commutative algebras will be denoted by \( \text{Alg} \).

A simplicial (commutative) algebra \( E \) consists of a family of algebras \( \{ E_n \} \) together with face and degeneracy maps \( d_i = d^n_i : E_n \to E_{n-1}, \ 0 \leq i \leq n, \ (n \neq 0) \) and \( s_i = s^n_i : E_n \to E_{n+1}, \ 0 \leq i \leq n, \) satisfying the usual simplicial identities given in André [1] or Illusie [23] for example. It can be completely described as a functor \( E : \Delta^{op} \to \text{Alg} \) where \( \Delta \) is the category of finite ordinals \( \{ 0 < 1 < \cdots < n \} \) and increasing maps. We denote the category of simplicial algebras by \( \text{SimpAlg} \).

Given a simplicial algebra \( E \), the Moore complex \((NE, \partial)\) of \( E \) is the chain complex defined by

\[ NE_n = \ker d^n_0 \cap \ker d^n_1 \cap \cdots \cap \ker d^n_{n-1} \]

with \( \partial : NE_n \to NE_{n-1} \) induced from \( d^n_0 \) by restriction.

The \( n \)th homotopy module \( \pi_n(E) \) of \( E \) is the \( n \)th homology of the Moore complex of \( E \), i.e.

\[ \pi_n(E) \cong H_n(NE, \partial) = \bigcap_{i=0}^{n} \ker d^n_i / d^{n+1}_i \left( \bigcap_{i=0}^{n} \ker d^{n+1}_i \right). \]

We say that the Moore complex \( NE \) of a simplicial algebra is of length \( k \) if \( NE_n = 0 \) for all \( n \geq k + 1 \), so that a Moore complex of length \( k \) is also of length \( l \) for \( l \geq k \). We denote thus the category of simplicial algebras with Moore complex of length \( k \) by \( \text{SimpAlg} \leq k \).

Arvasi and Porter in [4] defined the notion of \( C_{\alpha, \beta} \) functions which is a variant of the Carrasco-Cegarra pairing operators (cf. [14]) called Peiffer Pairings and by defining an ideal
In the Moore complex of the simplicial algebra for \( n = 2, 3, 4 \) and they have shown that if \( E \) is a simplicial commutative algebra with Moore complex \( NE \), and for \( n > 0 \) the ideal generated by the degenerate elements in dimension \( n \) is \( E_n \), then

\[
\partial_n(NE_n) \supseteq \sum_{I,J} K_I K_J.
\]

This sum runs over those \( \emptyset \neq I, J \subset [n-1] = \{0, \ldots, n-1\} \) with \( I \cup J = [n-1] \), and \( K_I = \bigcap_{i \in I} \ker d_i \).

Castiglioni and Ladra \cite{15} gave a general proof for the inclusions partially proved by Arvasi and Porter in \cite{4}. Their approach to the problem is different from that of cited work. They have succeeded with a proof, for the case of algebras, over an operad by introducing a different description of the adjoint inverse of the normalization functor \( N : Ab^{\Delta^{op}} \to Ch_{\geq 0} \).

1.1 2-Crossed Modules from Simplicial Algebras

Crossed modules of groups were introduced by Whitehead in \cite{31}. Crossed modules techniques give a very efficient way of handling information about on a homotopy type. They model homotopy types with trivial homotopy groups in dimension bigger than 2.

The commutative algebra analogue of crossed modules has been given by Porter in \cite{28}. Throughout this paper we denote an action of \( r \in R \) on \( c \in C \) by \( r \cdot c \).

A pre-crossed module of algebras is a homomorphism of algebras \( \partial : C \to R \) together with an action of \( R \) on \( C \) written \( c \cdot r \) for \( r \in R \) and \( c \in C \) satisfying the condition \( \partial(c \cdot r) = \partial(c)r \) for all \( r \in R \) and \( c \in C \). A pre-crossed module \( \partial : C \to R \) is called a crossed module, if it satisfies the extra condition \( c \cdot \partial(c') = cc' \) for all \( c, c' \in C \). This condition is called the Peiffer identity. We will denote such a crossed module by \((C, R, \partial)\). A morphism of crossed modules from \((C, R, \partial)\) to \((C', R', \partial')\) is a pair of \( k \)-algebra homomorphisms, \( \phi : C \to C' \) and \( \varphi : R \to R' \) such that \( \phi(r \cdot c) = \varphi(r)\phi(c) \). We thus define the category of crossed modules of algebras denoting it by \( XMod \).

As stated earlier, the notion 2-crossed modules of groups was introduced by Conduché in \cite{16} as a model for connected 3-types. He showed that the category of 2-crossed modules is equivalent to the category of simplicial groups with Moore complex of length 2. Grandjean and Vale, \cite{21}, gave the notion of 2-crossed modules for algebras. Now, we recall from \cite{21} the definition of a 2-crossed module:

A 2-crossed module of \( k \)-algebras consists of a complex of \( k \)-algebras

\[
\begin{array}{ccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\end{array}
\]
together with (a) actions of $C_0$ on $C_1$ and $C_2$ so that $\partial_2$, $\partial_1$ morphisms of $C_0$-algebras, where the algebra $C_0$ acts on itself by multiplication, and (b) a $C_0$-bilinear function

$$\{ -, \cdot, -\} : C_1 \otimes C_0 \rightarrow C_2$$

called a Peiffer lifting. This data must satisfy the following axioms:

2CM1) \[ \partial_2 \{ y_0 \otimes y_1 \} = y_0 y_1 - y_0 \cdot \partial_1 y_1 \]

2CM2) \[ \{ \partial_2 (x_1) \otimes \partial_2 (x_2) \} = x_1 x_2 \]

2CM3) \[ \{ y_0 \otimes y_1 y_2 \} = \{ y_0 y_1 \otimes y_2 \} + \{ y_0 \otimes y_1 \} \cdot \partial_1 y_2 \]

2CM4) \[ a) \{ \partial_2 x \otimes y \} = x \cdot y - x \cdot \partial_1 y \]

\[ b) \{ y \otimes \partial_2 x \} = x \cdot y \]

2CM5) \[ \{ y_0 \otimes y_1 \} \cdot z = \{ y_0 \cdot z \otimes y_1 \} = \{ y_0 \otimes y_1 \cdot z \} \]

for all $x, x_1, x_2 \in C_2$, $y, y_0, y_1, y_2 \in C_1$ and $z \in C_0$.

We denote the category of 2-crossed module by $X_2\text{Mod}$.

Arvasi and Porter in [4] studied the truncated simplicial algebras and their properties. They turned to a simplicial algebra $E$ which is 2-truncated, i.e., its Moore complex looks like:

$$NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

and they then showed the following result:

**Theorem 1.1** The category $X_2\text{Mod}$ of 2-crossed modules is equivalent to the category $\text{SimpAlg}_{\leq 2}$ of simplicial algebras with Moore complex of length 2.

### 1.2 2-Crossed Modules from Crossed Squares

Crossed squares were initially defined by Guin-Waléry and Loday in [22]. The commutative algebra analogue of crossed squares has been studied by Ellis (cf. [18]).

A *crossed square* of commutative algebras is a commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\downarrow{\chi} & & \downarrow{\mu} \\
N & \xrightarrow{\nu} & R
\end{array}
$$

together with the actions of $R$ on $L, M$ and $N$. There are thus commutative actions of $M$ on $L$ and $N$ via $\mu$ and a function

$$h : M \times N \rightarrow L$$

the $h$-map. This data must satisfy the following axioms:

1. The $\lambda, \lambda', \mu, \nu$ and $\mu \lambda = \nu \lambda'$ are crossed modules;
2. The maps $\lambda, \lambda'$ preserve the action of $R$;
3. $kh(m,n) = h(km,n) = h(m, kn)$;
4. $h(m + m', n) = h(m, n) + h(m', n)$;
5. $h(m, n + n') = h(m, n) + h(m, n')$;
6. $r \cdot h(m, n) = h(r \cdot m, n) = h(m, r \cdot n)$;
7. $\lambda h(m, n) = m \cdot n$;
8. $\lambda' h(m, n) = n \cdot m$;
9. $h(m, \lambda' l) = m \cdot l$;
10. $h(\lambda l, n) = n \cdot l$

for all $l \in L, m, m' \in M$ and $n, n' \in N, r \in R, k \in k$.

We denote such a crossed square by $(L, M, N, R)$. A morphism of crossed squares $\Phi : (L, M, N, R) \rightarrow (L', M', N', R')$ consists of homomorphisms

\[
\Phi_L : L \rightarrow L' \quad \Phi_M : M \rightarrow M' \quad \Phi_N : N \rightarrow N' \quad \Phi_R : R \rightarrow R'
\]

such that the cube of homomorphisms is commutative

\[
\Phi_L h(m, n) = h(\Phi_M m, \Phi_N n)
\]

with $m \in M$ and $n \in N$, and the homomorphisms $\Phi_L, \Phi_M, \Phi_N$ are $\Phi_R$-equivariant. The category of crossed squares will be denoted by $\text{Crs}^2$.

Conduché (in a private communication with Brown and (see also published version [17])) gives a construction of a 2-crossed module from a crossed square of groups. On the other hand, the first author gave a neat description of the passage from a crossed square to a 2-crossed module of algebras in [3]. He constructed a 2-crossed module from a crossed square of commutative algebras

\[
\begin{array}{ccc}
L & \xrightarrow{\lambda} & M \\
\downarrow{\lambda'} & & \downarrow{\mu} \\
N & \xrightarrow{\nu} & R
\end{array}
\]

as

\[
\begin{array}{ccc}
L & \xrightarrow{(-\lambda, \lambda')} & M \rtimes N \\
\downarrow{\nu} & & \downarrow{\mu + \nu} \\
N & & R
\end{array}
\]

analogue to that given by Conduché in the group case (cf. [17]). Note that the construction given by Arvasi preserves the homotopy modules. In fact, Ellis in [19] proved that the homotopies of the crossed square (A) are the homologies of the complex (B).

1.3 Crossed Squares from Simplicial Algebras

In 1991, Porter, [26], described a functor from the category of simplicial groups to that of crossed $n$-cubes defined by Ellis and Steiner in [20], based on ideas of Loday (cf. [24]). Crossed
n-cubes in algebraic settings such as commutative algebras, Jordan algebras, Lie algebras have been defined by Ellis in [18]. In this paper, we use only the case \( n = 2 \), that is for the crossed squares. From [18] we can say that, in low dimension, a crossed 1-cube is the same as a crossed module and a crossed 2-cube is the same as a crossed square.

In [3], Arvasi adapt that description to give an obvious analogue of the functor given by Porter, [26], for the commutative algebra case. The following result is the 2-dimensional case of a general construction of a crossed \( n \)-cube of algebras from a simplicial algebra given by Arvasi, [3], analogue to that given by Porter, [26], in the group case.

Let \( E \) be a simplicial algebra. Then the following diagram

\[
\begin{array}{ccc}
NE_2/\partial_3NE_3 & \overset{\partial_2}{\longrightarrow} & NE_1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
NE_1 & \overset{\mu}{\longrightarrow} & E_1
\end{array}
\]

is the underlying square of a crossed square. The extra structure is given as follows;

\( NE_1 = \ker d_0 \) and \( \overline{NE}_1 = \ker d_1 \). Since \( E_1 \) acts on \( NE_2/\partial_3NE_3, \overline{NE}_1 \) and \( NE_1 \), there are actions of \( \overline{NE}_1 \) on \( NE_2/\partial_3NE_3 \) and \( NE_1 \) via \( \mu' \), and \( NE_1 \) acts on \( NE_2/\partial_3NE_3 \) and \( \overline{NE}_1 \) via \( \mu \). Both \( \mu \) and \( \mu' \) are inclusions, and all actions are given by multiplication. The \( h \)-map is

\[
h : NE_1 \times \overline{NE}_1 \longrightarrow NE_2/\partial_3NE_3
\]

\[
(x, \overline{y}) \mapsto h(x, \overline{y}) = s_1x(s_1y - s_0y) + \partial_3NE_3.
\]

Here \( x \) and \( y \) are in \( NE_1 \) as there is a natural bijection between \( NE_1 \) and \( \overline{NE}_1 \). This is clearly functorial and we denote it by

\[
M(-, 2) : \text{SimpAlg} \longrightarrow \text{Crs}^2.
\]

We shall use this functor to give a construction of a totally free quadratic module.

## 2 Quadratic Modules of Algebras

### 2.1 Quadratic Modules from 2-Crossed Modules

As we mentioned in introduction, Baues in [10] defined the quadratic module of groups as an algebraic model of connected 3-types. In [8, Section 5], the first and second authors defined the notion of quadratic module for commutative algebras and constructed a functor from 2-crossed modules to quadratic modules of algebras. In this section, we will show that this functor preserves the homotopy modules as an analogue to that given in [7] for the group case. That is, we will prove that the homotopy modules of the 2-crossed module are isomorphic to
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that of its associated quadratic module of algebras. Before giving this description, we briefly
recall some basic definition and the construction of a quadratic module from a 2-crossed
module of algebras from [8].

For an algebra \( C \), \( C/C^2 \) is the quotient of the algebra \( C \) by its ideal of squares. Then, there
is a functor from the category of \( \mathbb{k} \)-algebras to the category of \( \mathbb{k} \)-modules. This functor goes
from \( C \) to \( C/C^2 \), plays the role of abelianization in the category of \( \mathbb{k} \)-algebras. As modules
are often called singular algebras (e.g., in the theory of singular extensions) we shall call this
functor “singularisation”.

Now, we generalise these notions to pre-crossed modules. Let \( \partial : C \to R \) be a pre-crossed
module, \( P_1(\partial) = C \), let \( P_2(\partial) \) be the Peiffer ideal of \( C \) generated by elements of the form
\[
\langle x, y \rangle = xy - x \cdot \partial y
\]
which is called the Peiffer element for \( x, y \in C \).

A nil(2)-module is a pre-crossed module \( \partial : C \to R \) with an additional “nilpotency”
condition. This condition is \( P_3(\partial) = 0 \), where \( P_3(\partial) \) is the ideal of \( C \) generated by Peiffer
elements \( \langle x_1, x_2, x_3 \rangle \) of length 3, (cf. [8]).

The homomorphism
\[
\partial^r : C^r = C/P_2(\partial) \to R
\]
is the crossed module associated to the pre-crossed module \( \partial : C \to R \).

Now, we can give the definition of a quadratic module of algebras from [8].

**Definition 2.1** A quadratic module \((\omega, \delta, \partial)\) is a diagram

\[
\begin{array}{ccc}
C \otimes C & \xleftarrow{\omega} & C \\
\downarrow{w} & & \downarrow{\partial} \\
M & \xrightarrow{\delta} & N
\end{array}
\]

of homomorphisms of algebras such that the following axioms are satisfied.

**QM1)** The homomorphism \( \partial : M \to N \) is a nil(2)-module and the quotient map \( M \to C =
M^r/(M^r)^2 \) is given by \( x \mapsto [x] \), where \([x] \in C \) denotes the class represented by \( x \in M \). The
map \( w \) is defined by Peiffer multiplication, i.e., \( w([x] \otimes [y]) = xy - x \cdot \partial(y) \).

**QM2)** The homomorphisms \( \delta \) and \( \partial \) satisfy \( \delta \partial = 0 \) and the quadratic map \( \omega \) is a lift of the
map \( w \), that is \( \delta \omega = w \) or equivalently
\[
\delta \omega([x] \otimes [y]) = w([x] \otimes [y]) = xy - x \cdot \partial(y)
\]
for \( x, y \in M \).

**QM3)** \( L \) is a \( N \)-algebra and all homomorphisms of the diagram are equivariant with respect
to the action of $N$. Moreover, the action of $N$ on $L$ satisfies the following equality
\[ a \cdot \partial(x) = \omega([\delta a] \otimes [x] + [x] \otimes [\delta a]) \]
for $a \in L, x \in N$.

QM4). For $a, b \in L$,
\[ \omega([\delta a] \otimes [\delta b]) = ab. \]

Now, we give a brief construction of a quadratic module from a 2-crossed module of algebras from [8, Section 5] to use it in the proof of Proposition 2.5.

Let
\[ \begin{CD}
C_2 @> \partial_2 >> C_1 @> \partial_1 >> C_0
\end{CD} \]
be a 2-crossed module of algebras. Let $P_3$ be the ideal of $C_1$ generated by elements of the form $\langle \langle x, y \rangle, z \rangle$ and $\langle x, \langle y, z \rangle \rangle$ for $x, y, z \in C_1$. Let
\[ q_1 : C_1 \rightarrow C_1/P_3 \]
be the quotient map and let $M = C_1/P_3$. Thus, the map $\partial : M \rightarrow C_0$ given by $\partial(x + P_3) = \partial_1(x)$, for all $x \in C_1$, is a well-defined homomorphism. Let $P'_3$ be the ideal of $C_2$ generated by elements of the form $\{x \otimes \langle y, z \rangle\}$ and $\{\langle x, y \rangle \otimes z\}$, where $\{- \otimes -\}$ is the Peiffer lifting map. Let $L = C_2/P'_3$. We can write $\partial_2(P'_3) = P_3$. Then, $\delta : L \rightarrow M$ given by $\delta(l + P'_3) = \partial_2 l + P_3$ is a well-defined homomorphism. Let $C = \frac{M^{\text{cr}}}{(M^{\text{cr}})^2}$. Thus, in [8], we get that the diagram
\[ \begin{CD}
C \otimes C @> \omega >> C \\
L @> \delta >> M @> \partial >> N
\end{CD} \]
is a quadratic module of algebras.

2.2 Examples of Quadratic Modules of Algebras

By using the connections between 2-crossed modules, crossed squares and simplicial groups, Porter in [29] has illustrated some examples of 2-crossed modules. Since there is a close relationship between 2-crossed modules and quadratic modules of algebras described briefly above, we have analogous examples for the quadratic modules of algebras as follows.

**Example 2.2** Any nil(2)-module $\partial_1 : M \rightarrow N$ gives a quadratic module, since if $(M, N, \partial_1)$ is a nil(2)-module, we need add a trivial $L = 0$, and the resulting diagram
\[ \begin{CD}
C \otimes C @> \omega >> C \\
0 @> \delta >> M @> \partial_1 >> N
\end{CD} \]
with the obvious action is a quadratic module. The quadratic map \( \omega : C \otimes C \to 0 \) is trivial. This is, of course, functorial and the category of nil(2)-modules can be considered to be a full subcategory of the category of quadratic modules.

**Example 2.3** If

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L \\
\delta & \searrow & \downarrow \\
M & \xrightarrow{\partial_1} & N
\end{array}
\]

is a quadratic module of algebras, then there is an induced crossed module structure on \( \overline{\partial_1} : M/\delta(\omega C \otimes C) \to N \) given by

\[
\overline{\partial_1}(m + \delta(\omega(C \otimes C))) = \partial_1(m) \quad \text{for} \quad m \in M.
\]

Of course, since \( \delta(\omega(C \otimes C)) = w(C \otimes C) \) and \( \partial_1(w(C \otimes C)) = 0 \), the map \( \overline{\partial_1} \) is well defined. The action of \( n \in N \) on \( m + w(C \otimes C) \) is given by

\[
m + w(C \otimes C) \cdot n = m \cdot n + w(C \otimes C).
\]

Using this action, for \( m, m' \in M \), we have

\[
(m + w(C \otimes C)) \cdot (\overline{\partial_1}(m' + w(C \otimes C))) = m \cdot \partial_1(m') + w(C \otimes C)
\]

\[
= mm' + w(C \otimes C)
\]

since \( m \cdot \partial_1(m') - mm' \in w(C \otimes C) \). Thus \( \overline{\partial_1} \) is a crossed module.

**Example 2.4** Any nil(2)-module complex of length 2, that is one of form

\[
0 \to C_0 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_2} C_2 \to 0
\]

in which \( \partial_1 \) is a nil(2)-module gives us a quadratic module; \( C = (C_1)^{\sigma}/(C_1^{\sigma})^2 \), \( L = C_2 \), \( M = C_1 \) and \( N = C_0 \) with the quadratic map \( \omega : C \otimes C \to L; \omega([x] \otimes [y]) = 0 \) for all \( x, y \in M \). In this way; we have \( \partial_2(\omega([x] \otimes [y])) = \partial_2(0) = 0 = w([x] \otimes [y]) = xy - x \cdot \partial_1 y \) for all \( x, y \in M \). That is, we have that \( \partial_1 \) is a crossed module.

### 2.2.1 Exploration of trivial quadratic map

The obvious thing to do is to see what each of defining properties of a quadratic module of algebras give in this case.

Now, suppose we have a quadratic module of algebras

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L \\
\delta_2 & \searrow & \downarrow \\
M & \xrightarrow{\partial_1} & N
\end{array}
\]
with extra condition that \( \omega([x] \otimes [y]) = 0 \) for all \( x, y \in M \).

(i) From the axiom (QM2), since \( \partial_2(\omega([x] \otimes [y])) = w([x] \otimes [y]) \) for \( x, y \in M \), if \( \omega([x] \otimes [y]) = 0 \), we have \( x \cdot \partial_1 y = xy \) for all \( x, y \in M \). Thus, \( \partial_1 \) becomes a crossed \( N \)-module of algebras as explained in the above example. Further, \( M^{cr} = M/P_2(\partial_1) \) is isomorph to \( M \), since \( P_2(\partial_1) = w(C \otimes C) = 0 \) and \( C \cong M/M^2 \).

(ii) If \( m \in M \) and \( l \in L \), from the axiom (QM3), we have
\[
\partial_1(m) \cdot l = \omega([\partial_2 l] \otimes [m] + [m] \otimes [\partial_2 l]) = 0,
\]
so \( \partial_1(M) \) has trivial action on \( L \).

(iii) If \( l_1, l_2 \in L \), from the axiom (QM4), we have
\[
\omega([\partial_2 l_1] \otimes [\partial_2 l_2]) = l_1 l_2 = 0,
\]
so \( L \) is an \( N \)-algebra with the zero multiplication.

We can give the following proposition related to the above construction analogously to that given in [7] for the group case.

**Proposition 2.5** Let
\[
\begin{array}{ccc}
C_2 & \xrightarrow{\partial_2} & C_1 \\
& \downarrow & \downarrow \\
& \partial_1 & \downarrow \\
& & C_0 \\
\end{array}
\]
be the 2-crossed module and \( \pi_i \) be its homotopy modules for all \( i \geq 0 \). Let \( \pi'_i \) be the homotopy modules of its associated quadratic module
\[
\begin{array}{ccc}
\omega & \downarrow \\
L & \xrightarrow{\delta} & M \\
& \downarrow \partial \\
& \xrightarrow{\omega} & N. \\
\end{array}
\]
Then, \( \pi_i \cong \pi'_i \) for all \( i \geq 0 \).

**Proof:** The homotopy modules of the 2-crossed module are
\[
\pi_i = \begin{cases} 
C_0/\partial_1(C_1) & i = 1, \\
\ker \partial_1/\text{Im} \partial_2 & i = 2, \\
\ker \partial_2 & i = 3, \\
0 & i > 3 \end{cases}
\]
and the homotopy modules of its associated quadratic module are
\[
\pi'_i = \begin{cases} 
N/\partial(M) & i = 1, \\
\ker \partial/\text{Im} \delta & i = 2, \\
\ker \delta & i = 3, \\
0 & i = 0, i > 3 \end{cases}
\]
Now, we show that $\pi_i \cong \pi'_i$ for all $i \geq 0$. Since $\partial_1(P_3) = 0$, $M = C_1/P_3$, $N = C_0$ and $\partial(M) \cong \partial_1(C_1)$, clearly we have

$$\pi_1 = C_0/\partial_1(C_1) \cong N/\partial(M) = \pi'_1.$$ 

Since $\ker \partial \cong \ker \partial_1/P_3$ and $\text{Im} \delta \cong \text{Im} \partial_2/P_3$ so that we have

$$\pi'_2 = \frac{\ker \partial}{\text{Im} \delta} \cong \frac{\ker \partial_1/P_3}{\text{Im} \partial_2/P_3} \cong \frac{\ker \partial_1}{\text{Im} \partial_2} = \pi_2.$$ 

Now, we show that $\pi_3 \cong \pi'_3$. Consider that

$$\pi'_3 = \{ x + P'_3 : \partial_2(x) \in P_3 \}.$$ 

For an element $x + P'_3$ of $\pi'_3$, we show that there is an element $x' + P'_3$ of $\pi'_3$ such that $x + P'_3 = x' + P'_3$ and $x' \in \ker \partial_2$. In fact, we observe from 2CM1) that $\partial_2\{\langle x, y \rangle \otimes z \} = \langle \langle x, y \rangle, z \rangle$ and $\partial_2\{x \otimes \langle y, z \rangle\} = \langle x, \langle y, z \rangle \rangle$, and we have $\partial_2(P'_3) = P_3$. Hence $\partial_2(x) \in P_3$ implies $\partial_2(x) = \partial_2(w)$, $w \in P'_3$; thus $\partial_2(x - w) = 0$; then take $x' = x - w$, so that $x + P'_3 = x' + P'_3$ and $\partial_2(x') = 0$. Define $\mu : \pi'_3 \rightarrow \pi_3$ by $(x' + P'_3) \mapsto x'$ and $\nu : \pi_3 \rightarrow \pi'_3$ by $x \mapsto x + P_3$. Then $\mu$ and $\nu$ are inverse bijections, that is, we have $\pi_3 \cong \pi'_3$. Thus, the homotopy modules of the 2-crossed module are isomorphic to that of its associated quadratic module. 

\[\square\]

2.3 Quadratic Modules from Simplicial Algebras

Baues, [10], defined a functor from simplicial groups to quadratic modules. In the construction of this functor, to define the quadratic map $\omega$, Baues in [10] used the Conduché’s Peiffer lifting map $\{- , - \}$ given for the construction of a 2-crossed module from a simplicial group (cf. [16]).

In [8] Section 4], Arvasi and Ulualan constructed a functor from simplicial algebras to quadratic modules by using $C_{\alpha, \beta}$ functions given by the first author and Porter in [4]. In this construction, to define the quadratic map $\omega$, it was used the Peiffer lifting map given by the first author and Porter in [4] Proposition 5.2. In this section, we shall show that the homotopy type is preserved by this construction. We shall use this functor to construct of a totally free quadratic module of algebras from a simplicial resolution in the next section.

Now, we give a brief construction of a quadratic module from a simplicial algebra in terms of hypercrossed complex pairings from [8] Section 4].

Let $E$ be a simplicial algebra with Moore complex $NE$. We will obtain a quadratic module by using the following commutative diagram

$$\begin{array}{ccc}
NE_1 \times NE_1 & \xrightarrow{\omega'} & NE_1 \\
\downarrow{w} & & \downarrow{\partial_1} \\
NE_2/\partial_3(NE_3) & \xrightarrow{\partial_2} & NE_1
\end{array}$$
where the map $w$ is given by $w(x, y) = xy - x \cdot \partial_1 y$ for $x, y \in NE_1$ and the map $\omega'$ is given by $\omega'(x, y) = s_1 x(s_1 y - s_0 y) = s_1 x(y - s_0 y) + \partial_3(NE_3)$ for $x, y \in NE_1$. Let $P_3(\partial_1)$ be the ideal of $NE_1$ generated by elements of the form $\langle x, \langle y, z \rangle \rangle$ for $x, y, z \in NE_1$. The map $\partial : NE_1/P_3(\partial_1) \rightarrow NE_0$ given by $\partial(x + P_3(\partial_1)) = \partial_1 x$ for $x \in NE_1$ is a well-defined homomorphism. Let $P'_3(\partial_1)$ be the ideal of $NE_1$ generated by elements of the form $\langle x, \langle y, z \rangle \rangle$ and $\langle \langle x, y \rangle, z \rangle$ for $x, y, z \in NE_1$. Let $P_3(\partial_1)$ be the ideal of $NE_2/\partial_3(NE_3)$ generated by the formal Peiffer elements of $x, y, z \in NE_1$ as given in [8]. Let $M = NE_1/P_3(\partial_1)$ and $L = (NE_2/\partial_3(NE_3))/P'_3(\partial_1)$ be the quotient algebras. The map $\delta : (NE_2/\partial_3(NE_3))/P'_3(\partial_1) \rightarrow NE_1/P_3(\partial_1)$ can be given by $\delta(a + P'_3(\partial_1)) = \partial_2(a) + P_3(\partial_1)$. Thus in [8], we get that the diagram

$$
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega'} & M \\
\downarrow \quad \downarrow \delta & & \downarrow \partial \\
L & \xrightarrow{\partial} & N
\end{array}
$$

is a quadratic module of algebras. It was proven in [8] that all the axioms of quadratic module are verified by using the images of the $C_{\alpha, \beta}$ functions.

Alternatively, this proposition can be reproved differently, by making use of the 2-crossed module constructed from a simplicial algebra by Arvasi and Porter (cf. [1]). We now give a sketch of the argument. In [4, Proposition 5.2], it is shown that given a simplicial algebra $E$, one can construct a 2-crossed module

$$
NE_2/\partial_3(NE_3 \cap D_3) \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0
$$

where $\{x \otimes y\} = s_1 x(s_1 y - s_0 y) + \partial_3(NE_3 \cap D_3)$ for $x, y \in NE_1$.

Clearly we have a commutative diagram

$$
\begin{array}{ccc}
NE_2/\partial_3(NE_3 \cap D_3) & \xrightarrow{\partial_2} & NE_1 \\
\downarrow j & & \downarrow \partial_1 \\
NE_2/\partial_3(NE_3) & \xrightarrow{\partial_3} & NE_0
\end{array}
$$

Consider now the quadratic module associated to the 2-crossed module (1), as explained in Section 2.1 of this paper.
Then one can see that $L' = A/\partial_3(NE_3 \cap D_3)$, where $A$ is the ideal of $NE_2$ generated by elements of the form

$$s_1(x, y)(s_1z - s_0z) \text{ and } s_1(x)(s_1(y, z)) - s_0(y, z)).$$

On the other hand we have, from Section 2.1, $L = A/\partial_3(NE_3)$. Hence there is a map $i : L' \to L$ with

$$\omega = i\omega', \quad \delta' = \delta i.$$ (2)

Since $C \otimes C \omega' \to L' \delta' \to M \delta \to N$ is, by construction a quadratic module, it is straightforward to check, using (2), that $C \otimes C \omega \to L \delta \to M \delta \to N$ is also a quadratic module. Therefore, about the homotopy modules of this construction, we can give the following proposition analogously to that given in [7] for the group case.

**Proposition 2.6** Let $E$ be a simplicial algebra, let $\pi_i$ be the homotopy modules of the classifying space of $E$ and let $\pi'_i$ be the homotopy modules of its associated quadratic module of algebras; then $\pi_i \cong \pi'_i$ for $i = 0, 1, 2, 3$.

**Proof:** Let $E$ be a simplicial algebra. The $n$th homotopy module of $E$ is isomorphic to the $n$th homology of the Moore complex of $E$, i.e., $\pi_n(E) \cong H_n(NE)$. Thus the homotopy modules $\pi_n(E) = \pi_n$ of $E$ are

$$\pi_n = \begin{cases} NE_0/d_1(NE_1) & n = 1, \\ \ker d_1 \cap NE_1 / d_2(NE_2) & n = 2, \\ \ker d_2 \cap NE_2 / d_3(NE_3) & n = 3, \\ 0 & n = 0, \text{ or } n > 3 \end{cases}$$

and the homotopy modules $\pi'_n$ of its associated quadratic module are

$$\pi'_n = \begin{cases} NE_0/\partial(M) & n = 1, \\ \ker \partial/\ker \delta & n = 2, \\ \ker \delta & n = 3, \\ 0 & n = 0, \text{ or } n > 3. \end{cases}$$
We claim that $\pi_n' \cong \pi_n$ for $n = 1, 2, 3$. Since $M = NE_1/P_3(\partial_1)$ and $\partial_1(P_3(\partial_1)) = 0$, we have
\[
\partial(M) = \partial(NE_1/P_3(\partial_1)) = d_1(NE_1)
\]
and then
\[
\pi_1' = NE_0/\partial(M) \cong NE_0/d_1(NE_1) = \pi_1.
\]
Also $\ker \partial = \frac{\ker d_1 \cap NE_1}{P_3(\partial_1)}$ and $\text{Im} \delta = d_2(NE_2)/P_3(\partial_1)$ so that we have
\[
\pi_2' = \frac{\ker \partial}{\text{Im} \delta} = \frac{(\ker d_1 \cap NE_1)/P_3(\partial_1)}{d_2(NE_2)/P_3(\partial_1)} \cong \frac{\ker d_1 \cap NE_1}{d_2(NE_2)} = \pi_2.
\]
The isomorphism between $\pi_3'$ and $\pi_3$ can be proved similarly to the proof of Proposition 2.5.

2.4 Quadratic Modules from Crossed Squares

In [8, Section 6], we gave a construction of a quadratic module from a crossed square. In this section, we will give briefly this construction to use it in the next sections. We shall use this functor when giving an alternative description of a totally free quadratic module of algebras from a free crossed square.

Recall that the first author in [3] constructed a 2-crossed module from a crossed square of commutative algebras
\[
\begin{array}{ccc}
L & \overset{\lambda}{\longrightarrow} & M \\
\chi & \downarrow & \mu \\
N & \overset{\nu}{\longrightarrow} & R
\end{array}
\]
as
\[
\begin{array}{ccc}
L & \overset{(-\lambda,\lambda')}{\longrightarrow} & M \times N & \overset{\mu + \nu}{\longrightarrow} & R
\end{array}
\]
(3)
analogue to that given by Conduché in the group case (cf. [17]), as explained in Section 1.2 of this paper.

Now let
\[
\begin{array}{ccc}
L & \overset{\lambda}{\longrightarrow} & M \\
\chi' & \downarrow & \mu \\
N & \overset{\nu}{\longrightarrow} & R
\end{array}
\]
be a crossed square of algebras. Consider its associated 2-crossed module (4). From this 2-crossed module, we can construct a quadratic module as explained in Section 2.1.
where $N = R$, $M' = (M \times N)/P_3$, $L' = L/P'_3$, $C = (M')^{cr}/((M')^{cr})^2$.

The Peiffer elements in $M \times N$ are given by

$$\langle (m, n), (c, a) \rangle = (c \cdot \nu(n), -n \cdot \mu(c)).$$

The quadratic map $\omega : C \otimes C \rightarrow L'$ can be given by $\omega([q_1(m, n)] \otimes [q_1(c, a)]) = q_2(h(c, na))$ for all $(m, n), (c, a) \in M \times N$, $q_1(m, n), q_1(c, a) \in M'$ and $[q_1(m, n)] \otimes [q_1(c, a)] \in C \otimes C$, and where $h$ is the $h$-map of the crossed square. For the axioms see [8].

### 3 Free Quadratic Modules of Algebras

In this section, we will define the notion of a totally free quadratic module of algebras and we will construct it by using 2-skeleton of a free simplicial algebra. André [1] used simplicial methods to investigate homological properties of commutative algebras and introduced ‘step-by-step’ construction of a resolution of a commutative algebra. The reader is referred to the book of André [1] and to the article of Arvasi and Porter [6] for full details and more references.

**Free Simplicial Algebras**

We recall briefly the ‘step-by-step’ construction of a free simplicial algebra from the article [6]. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \rightarrow [n]$ as given in [1], [6] and [25], where $[n] = \{0 < 1 \cdots < n\}$ is an ordered set.

Let $E$ be a simplicial algebra and $k \geq 1$ be fixed. Suppose we are given a set $\Omega$ of elements $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(E)$; then we can choose a corresponding set of elements $w_\lambda \in NE_{k-1}$ so that $x_\lambda = w_\lambda + \partial_\lambda(NE_k)$. We want to ‘kill’ the elements in $\Omega$. It is formed a new simplicial algebra $F_n$ where $F_n$ is a free $E_n$-algebra, $F_n = E_n[y_{\lambda,t}]$ with $\lambda \in \Lambda$ and $t \in \{n, k\}$, and for $0 \leq i \leq n$, the algebra homomorphisms $s^n_i : F_n \rightarrow F_{n+1}$ and $d^n_i : F_n \rightarrow F_{n-1}$ are obtained from the homomorphism $s^n_i : E_n \rightarrow E_{n+1}$ and $d^n_i : E_n \rightarrow E_{n-1}$ respectively together with the relations given in [6].

Thus from the ‘step-by-step’ construction we can give the definition of a free simplicial algebra as follows.

Let $E$ be a simplicial algebra and $k \geq 1$, $k$-skeletal be fixed. A simplicial algebra $F$ is called a free if

(i) $F_n = E_n$ for $n < k$,

(ii) $F_k = a$ free $E_k$-algebra over a set of non- degenerate indeterminates, all of whose faces are zero except the $k$th,

(iii) $F_n$ is a free $E_n$-algebra over the degenerate elements for $n > k$. 

A variant of the ‘step-by-step’ construction gives: if $A$ is a simplicial algebra, then there exists a free simplicial algebra $E$ and an epimorphism $E \to A$ which induces isomorphisms on all homotopy modules. The details are omitted as they are well-known.

**Definition 3.1** Let 

$$
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & C_0 \\
\downarrow{\partial_2} & & \downarrow{\partial_1} \\
C_2 & & C_1 \\
\end{array}
$$

be a quadratic module and let $\vartheta : Y \to C_2$ be the function. Then $\left(\omega, \partial_2, \partial_1\right)$ is said to be a free quadratic module with basis $\vartheta$, or alternatively on the function $\partial_2 \vartheta : Y \to C_1$, for any quadratic module $\left(\omega', \partial'_2, \partial'_1\right)$ and $\vartheta' : Y' \to C'_2$ such that $\partial'_2 \vartheta' = \partial_2 \vartheta$, there is a unique morphism $\Phi : C_2 \to C'_2$ such that $\partial'_2 \Phi = \partial_2$. We say that a free quadratic module $\left(\omega, \partial_2, \partial_1\right)$ is a totally free quadratic module if $\partial_1$ is a free nil(2)-module.

Now, we construct the totally free quadratic module of algebras. For this, we need to recall the 2-skeleton of a free simplicial algebra given as

$$E^{(2)} : \cdots R[s_0 X, s_1 X][Y] \xrightarrow{\varphi} R[X] \xrightarrow{\psi} R$$

with the simplicial structure defined as in Section 3 of [6].

Analysis of this 2-dimensional construction data shows that it consists of some 1-dimensional data, namely the function $\vartheta : X \to R$, that is used to induce $d_1 : R[X] \to R$, together with strictly 2-dimensional construction data consisting of the function $\psi : Y \to R^+[X]$ and this function is used to induce $d_2 : R[s_0 X, s_1 X][Y] \to R[X]$. We will denote this 2-dimensional construction data by $(\vartheta, \psi, R)$.

**Theorem 3.2** A totally free quadratic module of algebras exists on the 2-dimensional construction data $(\vartheta, \psi, R)$.

**Proof:** Suppose that given a 2-dimensional construction data $(\vartheta, \psi, R)$, i.e. given a function $\vartheta : X \to R$ and $\psi : Y \to R^+[X]$, with the obvious action of $R$ on $R^+[X]$, we have a free pre-crossed module $\partial : R^+[X] \to R$ with basis $\vartheta : X \to R$. Here $R^+[X] = NE^{(1)}_1 = \ker d_0$ is the positively degree part of $R[X]$. Now let $P_3$ be the ideal of $R^+[X]$ generated by the Peiffer elements of length 3 in $R^+[X]$. We can define the quotient morphism $q_1 : R^+[X] \to R^+[X]/P_3 = M$ and then, we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & R^+[X] \\
\downarrow{\vartheta} & & \downarrow{q_1} \\
R & \xrightarrow{\partial_1} & R^+[X]/P_3
\end{array}
$$
where \( i \) is the inclusion.

Since \( \partial : R^+[X] \to R \) is a pre-crossed module we have \( \partial(P_3) = 0 \) and then the map \( \partial_1 : R^+[X]/P_3 \to R \) given by \( x + P_3 \mapsto \partial(x) \) for \( x \in R^+[X] \) is a well-defined homomorphism. Therefore we have a free nil(2)-module \( \partial_1 : M = R^+[X]/P_3 \to R \) with basis \( \vartheta : X \to R \), or on the function \( \partial_1 q_1 \), since the triple Peiffer elements are trivial in \( M \).

Now, take \( D = NE^{(2)}_2 = R[s_0 X]^+ [s_1 X, Y] \cap ((s_0 - s_1)(X)) \). Then, as 2-dimensional construction data, the function

\[
q_1 \varphi : Y \to M = R^+[X]/P_3
\]

induces a morphism of algebras

\[
\theta : R[s_0 X]^+ [s_1 X, Y] \cap ((s_0 - s_1)(X)) \to R^+[X]/P_3
\]

given by \( \theta(y) = q_1 \varphi(y) \). Let \( P' = \partial_3(N E^{(2)}_3) \subset D \) be the Peiffer ideal. Then, \( q_1 \varphi(P') = 0 \) as all generator elements of \( P' \) in \( \ker d_2 \). By taking the factor algebra \( L = D/P' \) then there is a morphism \( \varphi' : L \to M \) such that the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{q} & L \\
\downarrow{\theta} & & \downarrow{\psi'} \\
M & & \end{array}
\]

commutes, that is, \( \varphi'q = q_1 \varphi = \theta \). The Peiffer elements in \( R^+[X] \) are given by

\[
\langle X_i, X_j \rangle = X_i X_j - \partial(X_i) X_j
\]

for \( X_i, X_j \in R^+[X] \). Let \( P'_3 \) be the ideal of \( L \) generated by elements of the form

\[
s_1(\langle X_i, X_j \rangle)s_1(X_k) - s_1(\langle X_i, X_j \rangle)s_0(X_k)
\]

and

\[
s_1(X_i)s_1(\langle X_j, X_k \rangle) - s_1(X_i)s_0(\langle X_j, X_k \rangle)
\]

for all \( X_i, X_j, X_k \in R^+[X] \). Let \( L' = L/P'_3 \) and let \( q_2 : L \to L' = L/P'_3 \) be the quotient morphism. We have \( \varphi(P'_3) = P_3 \). Hence, we define a morphism \( \varphi'' \) from \( L' \) to \( M \) such that \( \varphi'' q_2 = q_1 \varphi \). Then, the diagram

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & L' \\
\downarrow{\varphi''} & & \downarrow{\psi} \\
M & \xrightarrow{\partial_1} & R
\end{array}
\]
becomes a totally free quadratic module of algebras, where \( C = M^{cr}/(M^{cr})^2 \) and the quadratic map \( \omega \) can be given by

\[
\omega(\{q_1 X_i\} \otimes \{q_1 X_j\}) = q_2(s_1 X_i(s_1 X_j - s_0 X_j) + P')
\]

for \( X_i \in R^+[X], \ q_1(X_i) \in M = R^+[X]/P_3 \) and \( \{q_1 X_i\} \in C = M^{cr}/(M^{cr})^2 \).

Let \( C \otimes C \xrightarrow{\omega'} A \xrightarrow{\partial_2} M \xrightarrow{\partial_1} R \) be any quadratic module of algebras and let \( \partial' : Y \to A \) be the function. Then there exists a unique morphism \( \Phi : L' \to A \) given by \( \Phi(q_2(y + P')) = \partial'(y) \) such that \( \partial'_2 \Phi = \psi'' \). Thus \( (\omega, \psi'', \partial_1) \) is the required totally free quadratic module with basis \( q_1 \psi : Y \to M \).

### 4 Alternative Constructions of Free Quadratic Modules

In this section, we give alternative constructions of the totally free quadratic module of algebras by using the universality of tensor product and coproduct in the category of crossed modules of algebras and the functor \( M(-, 2) \).

Ellis, [19], presented the notion of a free crossed square for the case of groups, and in the context of CW-complexes gave a neat description of the top term \( L \) in a free crossed square of groups. He also constructed a free quadratic module of groups from a free crossed square by using the quotient functor \( q \). We gave in [9] a construction of the top algebra \( L \) in a free crossed square of algebras on the 2-dimensional construction data in terms of tensor and coproduct in the category of crossed modules of algebras. In this construction, we have used the functor \( M(-, 2) \) from simplicial algebras to crossed squares.

Now, we recall from [9] the definition of a free crossed square of algebras on a pair of functions \( (f_2, f_3) \):

Let \( S_1, S_2 \) and \( S_3 \) be sets and \( f_2 : S_2 \to R \) is a function from the set \( S_2 \) to the free algebra \( R \) on \( S_1 \). Let \( \partial : M \to R \) be the free pre-crossed module on the function \( f_2 \). Consider the semi-direct product \( M \rtimes R \) and the inclusion \( \mu : M \to M \rtimes R \) and the ideals \( M \) and \( \overline{M} \) of \( M \rtimes R \) where \( \overline{M} = \{(m, -\partial m) : m \in M\} \) with the inclusion \( \mu' : \overline{M} \to M \rtimes R \). Assume given a function \( f_3 : S_3 \to M \), which is to satisfy \( \partial f_3 = 0 \). Then there is a corresponding function \( \overline{f}_3 : S_3 \to \overline{M} \) given by \( y \mapsto (f_3(y), 0) \).

We say a crossed square

\[
\begin{array}{ccc}
L & \xrightarrow{\partial_1} & M \\
\downarrow \partial_2 & & \downarrow \mu' \\
M & \xrightarrow{\mu} & M \rtimes R
\end{array}
\]

is totally free on the pair of functions \((f_2, f_3)\) if

(i) \((M, R, \partial)\) is the free pre-crossed module on \(f_2\),

(ii) \(S_3\) is a subset of \(L\) with \(f_3\) and \(f_3'\) the restrictions of \(\partial_2\) and \(\partial_2'\) respectively,

(iii) given any crossed square \((L', M, \overline{M}, M \rtimes R)\) and function \(\nu : S_3 \rightarrow L'\) there is a unique morphism \(\phi : L \rightarrow L'\) such that \(\phi \nu' = \nu\) where \(\nu' : S_3 \rightarrow L\) is the inclusion.

We suppose that \(\mu : M \rightarrow R\) and \(\nu : N \rightarrow R\) are crossed modules of algebras. The algebras \(M\) and \(N\) act on each other, and themselves, via the action of \(R\) on \(M\) and \(N\).

The tensor product \(M \otimes N\) is the algebra generated by the symbols \(m \otimes n\), for \(m \in M\), \(n \in N\), subject to the relations given in \([9]\).

There are morphisms \(\lambda : M \otimes N \rightarrow M\), \(m \otimes n \mapsto m \cdot n = \mu(m) \cdot n\) and \(\lambda' : M \otimes N \rightarrow N\), \(m \otimes n \mapsto n \cdot m = \mu(m) \cdot n\). The algebra \(R\) acts on \(M \otimes N\) by \(r \cdot (m \otimes n) = r \cdot m \otimes n = m \otimes r \cdot n\).

Thus

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\lambda} & N \\
\downarrow{\lambda'} & & \downarrow{\nu} \\
M & \xrightarrow{\mu} & R
\end{array}
\]

is a crossed square with the universality of tensor product.

The coproduct of \(\mu : M \rightarrow R\) and \(\nu : N \rightarrow R\) (in the category of crossed modules with codomain \(R\)) is a crossed module \(\partial : M \sqcup N \rightarrow R\). It is the crossed module associated to the pre-crossed module \(\delta : M \rtimes N \rightarrow R\), \((m, n) \mapsto \mu(m) + \nu(n)\). That is \(M \sqcup N = (M \rtimes N)^{cr} = (M \rtimes N) / \langle M \rtimes N, M \rtimes N \rangle\) and \(\partial = \delta^{cr}\) and where \(\langle M \rtimes N, M \rtimes N \rangle\) is the ideal of \(M \rtimes N\) generated by elements of the form \(\langle (m, n), (m', n') \rangle\) for \(m, m' \in M\) and \(n, n' \in N\). Thus, we have a commutative diagram

\[
\begin{array}{ccc}
M \rtimes N & \xrightarrow{q} & M \sqcup N \\
\downarrow{\delta} & & \downarrow{\partial} \\
R & & \end{array}
\]

where \(q\) is the quotient map.

The canonical inclusions \(i' : M \rightarrow M \rtimes N\) and \(j' : N \rightarrow M \rtimes N\) induce homomorphisms \(i = q i' : M \rightarrow M \sqcup N\) and \(j = q j' : N \rightarrow M \sqcup N\).

**Proposition 4.1** \([9]\) Let

\[
\begin{array}{ccc}
L & \xrightarrow{\overline{M}} & M \\
\downarrow{M} & & \downarrow{M \rtimes R} \\
M & & \end{array}
\]

be the free crossed square of algebras. Let \(\partial : C \rightarrow M \rtimes R\) be the free crossed module on \(S_3 \rightarrow M \rtimes R\). Form the crossed module \(M \otimes \overline{M} \rightarrow M \rtimes R\). Then

\[
L \cong \{(M \otimes \overline{M}) \sqcup C\} / \sim
\]
where \( \sim \) corresponds to the relations

(i) \( i_{M \otimes \overline{M}}(\partial c \otimes \overline{m}) \sim j(c) - j(\overline{m} \cdot c) \),

(ii) \( i_{M \otimes \overline{M}}(m \otimes \partial c) \sim j(m \cdot c) - j(c) \)

for \( c \in C, m \in M \) and \( \overline{m} \in \overline{M} \).

The homomorphisms \( L \to M \) and \( L \to \overline{M} \) are given by the homomorphisms \( \lambda : M \otimes M \to M, (m \otimes n) \mapsto mn - \partial(n)m \) and \( \lambda' : M \otimes M \to \overline{M}, (m \otimes \overline{n}) \mapsto m\overline{n} - \partial(m)\overline{n} \) and \( \partial : C \to M \cap \overline{M} \). The \( h \)-map is given by \( h(m, \overline{n}) = i(m \otimes \overline{n}) \).

As we mentioned above, Ellis \[19\] gave a construction of a free quadratic module of groups from a free crossed square. Now, we adapt the Ellis’s construction of a free quadratic module of groups to the algebra context:

Suppose that \( L \xrightarrow{\lambda'} \overline{M} \xleftarrow{\lambda} M \xrightarrow{\mu'} M \rtimes R \)

is a free crossed square of algebras. We denote by \( h(M, M, M) \) the ideal of \( L \) generated by elements of the form \( h(m, \langle m', m'' \rangle) \) and \( h(\langle m, m' \rangle, \overline{m}'') \) for \( m, m', m'' \in M \). (We are using the notation \( \overline{m} = (-m, \delta m) \) for \( m \in M \).) Consider the quotient algebra \( L/h(M, M, M) \) and the quotient morphism \( q_2 : L \to L/h(M, M, M) \).

Let \( \langle M, M, M \rangle \) be the ideal of \( M \) generated by elements of the form \( \langle \langle m, m' \rangle, m'' \rangle \) or \( \langle m, \langle m', m'' \rangle \rangle \) for \( m, m', m'' \in M \). Consider the quotient algebra \( M/\langle M, M, M \rangle \) and the quotient morphism \( q_1 : M \to M/\langle M, M, M \rangle \).

From the \( h \)-map axioms, we have \( \lambda h(M, M, M) = \langle M, M, M \rangle \) and then the map \( \overline{X} : L/h(M, M, M) \to M/\langle M, M, M \rangle \) given by \( \overline{X}(l + h(M, M, M)) = \lambda(l) + \langle M, M, M \rangle \) is a well-defined homomorphism. Similarly, since \( \partial : M \to R \) is a pre-crossed module, we have \( \partial(\langle M, M, M \rangle) = 0 \) and then the map \( \partial' : M/\langle M, M, M \rangle \to R \) given by \( \partial'(m + \langle M, M, M \rangle) = \partial(m) \) is well defined. Let \( M' = M/\langle M, M, M \rangle \) and \( C = (M^{\text{cr}})/(M^{\text{cr}})^2 \). Then

\[
\sigma : \begin{array}{c}
\begin{array}{c}
L \\
\mapright{h(M, M, M)} L/h(M, M, M) \\
\mapright{\overline{X}} \overline{M} \\
\mapdown{w} \\
\mapright{\partial'} M/\langle M, M, M \rangle \\
\mapdown{\varphi} \\
R
\end{array}
\end{array}
\]

is a free quadratic module of algebras. The \( h \)-map \( h : M \times \overline{M} \to L \) induces the quadratic map \( \omega : C \otimes C \to L/h(M, M, M) \).
It was showed in [19] that the homotopy groups of the crossed square

\[
\begin{array}{c}
L \\
\lambda \\
\downarrow \\
M \\
\lambda' \\
\downarrow \\
M' \\
\mu' \\
\downarrow \\
M \rtimes R
\end{array}
\]

are the homology groups of the complex

\[
L \xrightarrow{\lambda} M \xrightarrow{\partial} R \rightarrow 0.
\]

Thus the homotopy modules \( \pi_n(X) \) of the crossed square \( X \) are

\[
\begin{align*}
\pi_1(X) &= R/\text{Im}(\partial), \\
\pi_2(X) &= \ker \partial/\langle M, M, M \rangle, \\
\pi_3(X) &= \ker \lambda \text{ and } \pi_n(X) = 0 \text{ for } n \geq 4.
\end{align*}
\]

Consider the quadratic module \( \sigma \) associated to the crossed square \( X \). The homotopy modules \( \pi'_n(\sigma) \) of the quadratic module \( \sigma \) are the homology modules of the horizontal complex. More precisely,

\[
\begin{align*}
\pi'_1(\sigma) &= R/\text{Im}(\partial'), \\
\pi'_2(\sigma) &= \ker \partial'/\langle M, M, M \rangle, \pi'_3(\sigma) = \ker \lambda \text{ and } \pi'_n(\sigma) = 0 \text{ for } n \geq 4.
\end{align*}
\]

It is clear that since \( \partial((M, M, M)) = 0 \), we have

\[
\pi'_1(\sigma) = R/\text{Im}(\partial') \cong R/\text{Im}(\partial) = \pi_1(X).
\]

Also since \( \lambda h(M, M, M) = \langle M, M, M \rangle \), we have \( \ker \partial' = \ker \partial/\langle M, M, M \rangle \) and \( \text{Im}(X) = \text{Im}(\lambda)/\langle M, M, M \rangle \)

and then we have

\[
\pi'_2(\sigma) = \frac{\ker \partial'/\langle M, M, M \rangle}{\text{Im}(\lambda)/\langle M, M, M \rangle} \cong \frac{\ker \partial}{\text{Im}(\lambda)} = \pi_2(X).
\]

For \( n = 3 \), the isomorphism between \( \pi_3 \) and \( \pi'_3 \) can be proved similarly to the proof of Proposition 2.3.

Alternatively we can give another construction of a free quadratic module of algebras from a free crossed square by using the results of Section 2.3 of this paper.

Let \( \partial : M \rightarrow R \) be a free pre-crossed module and let

\[
\begin{array}{c}
L \\
\lambda \\
\downarrow \\
M \\
\lambda' \\
\downarrow \\
M' \\
\mu' \\
\downarrow \\
M \rtimes R
\end{array}
\]

be a free crossed square. Consider the crossed modules \( \mu : M \rightarrow M \rtimes R \) and \( \mu' : M' \rightarrow M \rtimes R \) and the semidirect product \( M \rtimes M' \) and the pre-crossed module \( \partial_1 : M \rtimes M' \rightarrow M \rtimes R \) given by \( (m, m') \mapsto (\mu(m) + \mu'(m')) \). Let \( P_3 \) be the ideal of \( M \rtimes M' \) generated by the Peiffer elements of length 3 and let \( M' = (M \rtimes M')/P_3 \) be the quotient algebra. The map \( \partial'_1 : M' \rightarrow M \rtimes R \) given by \( (m, m') + P_3 \mapsto \partial_1((m, m')) \) is well-defined, since \( \partial_1(P_3) = 0 \). Consider the function

\[
\Phi : (M \rtimes M') \times (M \rtimes M') \rightarrow L
\]
given by \((m, \overline{n}), (m', \overline{n}')\) \(\mapsto h(m, \overline{mn'})\) for \(m, m' \in M\) and \(\overline{n}, \overline{n}' \in \overline{M}\). Let \(P'_3\) be the ideal of \(L\) generated by elements of the form

\[\Phi((\langle (m, \overline{n}), (m', \overline{n}') \rangle, (m'', \overline{n''})), \langle (m', \overline{n}), (m'', \overline{n''}) \rangle). \]

Consider the quotient algebra \(L' = L/P'_3\) and the crossed module \(\delta : L \rightarrow M \ltimes \overline{M}\) given by \(l \mapsto (-\lambda(l), \lambda'(l))\) for all \(l \in L\). We have \(\delta(P'_3) = P_3\) and then the map \(\delta' : L' \rightarrow M'\) given by \(l + P'_3 \mapsto (-\lambda(l), \lambda'(l)) + P_3\) is a well defined homomorphism. Let \(C = (M^{lcr}/((M^{lcr})^2). Then

\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\omega} & \text{w} \\
L' \xrightarrow{\delta'} M' \xrightarrow{\partial'_1} M \ltimes R
\end{array}
\]

becomes a quadratic module.

**Note:** As we mentioned earlier, Ellis in [19] showed that the homotopy modules of the crossed square \(X\) are isomorphic to the homology modules of the complex

\[
\begin{array}{ccc}
L & \rightarrow & M \ltimes \overline{M} \rightarrow M \ltimes R \rightarrow 0.
\end{array}
\]

This complex is the mapping cone complex, in the sense of Loday, of the crossed square \(X\), and it has a 2-crossed module structure of algebras. This result for the general case was proved by the first author in [3]. In Section 2.1 of this paper, we proved that the functor from 2-crossed modules to quadratic modules of algebras preserves the homotopy modules. Thus, the homotopy type is clearly preserved, as it is preserved at each step. That is, the homotopy modules of the crossed square \(X\) are isomorphic to that of its associated quadratic module (*) according to the above construction.

Using the functor \(M(-, 2)\) and the 2-skeleton of a free simplicial algebra in [9], we have constructed a totally free crossed square.

Now, consider the free pre-crossed module \(\partial : R^+[S_2] \rightarrow R\) on the function \(f\) from the set \(S_2\) to the free algebra \(R\) on the set \(S_1\) and recall the 2-skeleton of a free simplicial algebra;

\[
\begin{array}{ccc}
E^{(2)} : \cdots & R[s_0(S_2), s_1S_2][S_3] & \xrightarrow{R[S_2]} R
\end{array}
\]

with the simplicial morphisms as given in [6]. Here, \(S_2 = \{S_1, S_2, \ldots, S_n\}\) and \(S_3 = \{S'_1, S'_2, \ldots, S'_n\}\) are finite sets.
Thus, we proved in [9] that $M(E^{(2)}, 2)$ is a totally free crossed square. This totally free crossed square is

$$
\begin{array}{c}
\frac{R[s_0(S_2)]^+[s_1(S_2), S_3] \cap ((s_0 - s_1)(S_2))}{P_2} \\
\frac{\partial_2}{\lambda} \\
\frac{R^+[S_2]}{\lambda'}
\end{array}
\xrightarrow{\partial_2} R[S_2]
$$

where $P_2$ is the second order Peiffer ideal which is in fact just $\partial_3(NE^{(2)}_3)$. The $h$-map is given by

$$h(s_i, \overline{s_j}) = s_1(s_i)(s_1(s_j) - s_0(s_j)) + \partial_3(NE^{(2)}_3)$$

for $s_i \in R^+[S_2]$ and $\overline{s_j} \in R^+[S_2]$.  

Now, we apply the Ellis’s construction to this totally free crossed square to get a totally free quadratic module for the case of commutative algebras.

Let

$$L = \frac{R[s_0(S_2)]^+[s_1(S_2), S_3] \cap ((s_0 - s_1)(S_2))}{P_2}$$

and let $P_3 = \langle R^+[S_2], R^+[S_2], R^+[S_2] \rangle$ be the ideal of $R^+[S_2]$ generated by elements of the form $(s_i, (s_j, S_k))$ or $(S_i, s_j, S_k)$ for $s_i, S_j, S_k \in R^+[S_2]$. Consider the quotient algebra $R^+[S_2]/P_3$ and the quotient map $q_1 : R^+[S_2] \to R^+[S_2]/P_3$.

Let $P_3' = h(R^+[S_2], R^+[S_2], R^+[S_2])$ be the ideal of $L$ generated by elements of the form

$$s_1((S_i, s_j))(s_1(S_k) - s_0(S_k)) + \partial_3(NE^{(2)}_3)$$

and

$$s_1(S_i)(s_1((S_j, S_k)) - s_0((S_j, S_k))) + \partial_3(NE^{(2)}_3)$$

for $s_i, S_j, S_k \in R^+[S_2]$.

Consider the quotient algebra $L/P_3'$ and the quotient map $q_2 : L \to L/P_3'$. Let $C = M^c/(M^c)^2$, where $M = R^+[S_2]/P_3$. Then

$$
\begin{array}{c}
\frac{C \otimes C}{\omega} \\
\frac{L/P_3'}{\omega} \\
\frac{R^+[S_2]/P_3}{\overline{\partial}} \\
\frac{R}{\overline{\partial}}
\end{array}
\xrightarrow{w} R
$$

becomes a totally free quadratic module of algebras, where the map $\overline{\partial}_3 : L/P_3' \to R^+[S_2]/P_3$ is given by $\overline{\partial}_3(q_2(l)) = q_1(\overline{\partial}_2(l))$ and the map $\overline{\partial} : R^+[S_2]/P_3 \to R$ is given by $\overline{\partial}(q_1(s_i)) = \partial(S_i)$ for $s_i \in R^+[S_2]$. The quadratic map $\omega : C \otimes C \to L/P_3'$ is given by

$$\omega(\{q_1S_i \otimes \{q_1S_j\}) = q_2(s_1(S_i)(s_1(S_j) - s_0(S_j)) + \partial_3(NE^{(2)}_3))$$
for $S_i, S_j \in R^+[S_2]$ and $q_1(S_i), q_1(S_j) \in R^+[S_2]/P_3$ and $\{q_1S_i \otimes q_1S_j\} \in C \otimes C$.

**Remark:** We can briefly explain the construction described above as follows: Consider the 2-skeleton of a free simplicial algebra $E^{(2)}$ and apply the functor $M(-, 2)$ to this 2-skeleton and then we get a totally free crossed square $M(E^{(2)}, 2)$ from [9]. Apply the Ellis’s construction to this totally free crossed square, we get a totally free quadratic module $E^{(3)}$. On the other hand, in Theorem 5.2 we have constructed a totally free quadratic module of algebras by using the 2-skeleton of a free simplicial algebra. It can be easily seen that up to isomorphism the totally free quadratic module $E^{(3)}$ as described above is identical with the totally free quadratic module constructed in Theorem 5.2.

At this stage, it is important to note that nowhere in the argument was use made of the freeness of the 1-skeleton. Now, recall that the 1-skeleton $E^{(1)}$ of a free simplicial algebra is

$$E^{(1)} : \cdots R[s_0S_2, s_1S_2] \xrightarrow{\lambda} R[S_2] \xrightarrow{f} R/I.$$  

We showed in [9] that $\ker d_0^1 = R^+[S_2]$, $\ker d_1^1 = R^+[S_2]$ and

$$NE_2^{(1)}/\partial_3(NE_3^{(1)}) = \frac{R[s_0S_2]^+[s_1S_2] \cap (s_0(S_2) - s_1(S_2))}{P_2}$$

and by applying the functor $M(-, 2)$ to $E^{(1)}$, we have a free crossed square $M(E^{(1)}, 2)$:

$$\begin{array}{c}
R[s_0S_2]^+[s_1S_2] \cap (s_0(S_2) - s_1(S_2)) \\
\downarrow \quad \downarrow \quad \downarrow \\
R^+[S_2] \\
\end{array} \xrightarrow{\lambda'} R^+[S_2]$$

From [9] Corollary 5.2, we can say that there is the following isomorphism

$$\frac{R[s_0S_2]^+[s_1S_2] \cap (s_0(S_2) - s_1(S_2))}{P_2} \cong \frac{R^+[S_2] \otimes R^+[S_2]}{P_2}.$$  

Thus the free crossed square becomes

$$\begin{array}{c}
R^+[S_2] \otimes R^+[S_2] \\
\downarrow \quad \downarrow \\
R^+[S_2] \\
\end{array} \xrightarrow{\lambda'} R^+[S_2]$$

The $h$-map

$$h : R^+[S_2] \times R^+[S_2] \longrightarrow \frac{R[s_0S_2]^+[s_1S_2] \cap (s_0(S_2) - s_1(S_2))}{P_2}$$

can be given by $h(x, \overline{y}) = s_1(x)(s_1(y) - s_0(y)) + \partial_3(NE_3^{(1)})$. But this is also $h(x, \overline{y}) = x \otimes \overline{y}$. Thus $x \otimes \overline{y} = s_1(x)(s_1(y) - s_0(y)) + \partial_3(NE_3^{(1)})$ under the identification via the isomorphism given above.
In above calculations, we have constructed a quadratic module (*) from a free crossed square as a result of section 2.4. If we apply this construction method to this free crossed square, we have a quadratic module on the 1-dimensional construction data:

\[
\begin{array}{c}
\xymatrix{
C \otimes C \\
R^+[S_2] \otimes R^+[S_2] \ar[r]^{\partial_3} & R^+[S_2] \times R^+[S_2] \ar[r]^{\partial_2} & R[S_2]
}
\end{array}
\]

where

\[
\partial_3(x \otimes y + P_3') = (-\lambda(x \otimes y), \lambda(x \otimes y)) + P_3
\]

and

\[
\partial_2((x, y) + P_3) = \mu(x) + \mu'(y).
\]

The quadratic map

\[
\omega : C \otimes C \rightarrow \frac{R^+[S_2] \otimes R^+[S_2]}{P_3'}
\]

can be given by \(\omega((q_1(x, y)) \otimes \{q_1(x', y')\}) = q_2(h(x, yy'))\), where \((x, y) \in R^+[S_2] \times \overline{R^+[S_2]}\), \(q_1(x, y) \in \frac{R^+[S_2] \rtimes \overline{R^+[S_2]}}{P_3'}\) and \(\{q_1(x, y)\} \in C\) and \(q_2\) is the quotient map from \(\frac{R^+[S_2] \otimes \overline{R^+[S_2]}}{P_3'}\) to \(\frac{R^+[S_2] \otimes \overline{R^+[S_2]}}{P_3'}\).

In fact, in the above construction, we used the complex

\[
\begin{array}{c}
\xymatrix{
R^+[S_2] \otimes R^+[S_2] \\
\ar[r]^{\partial_3} & R^+[S_2] \times R^+[S_2] \\
\ar[r]^{\partial_2} & R[S_2]
}
\end{array}
\]

which is the mapping cone complex, in the sense of Loday, of the crossed square \(M(E^{(1)}, 2)\), and we applied the functor from 2-crossed modules to quadratic modules as explained in section 2.1 of this paper (cf. [8]) to this mapping cone, to get the quadratic module \(X\).

If we apply the Ellis construction to the free crossed square \(M(E^{(1)}, 2)\), we get the quadratic module

\[
\begin{array}{c}
\xymatrix{
C \otimes C \\
\overline{h(R^+[S_2], R^+[S_2], R^+[S_2])} \ar[r]^{\chi} & R^+[S_2] \times R^+[S_2] \\
\ar[r]^{\partial_2} & R[S_2]
}
\end{array}
\]

and it is important to compare the two constructions. In fact, there is a split epimorphism from \(X\) to \(Y\) with kernel

\[
\begin{array}{c}
0 \rightarrow R^+[S_2] \rightarrow R^+[S_2]
\end{array}
\]
which has, of course, trivial homotopy. Thus $X$ and $Y$ encode the same information about the presentation of $R/I$, $I = \text{Im} \partial'$.

Consequently, for the 0-skeleton $E^{(0)}$ is

$$\cdots \longrightarrow R \longrightarrow R \longrightarrow R \xrightarrow{f} B$$

with the $d^n_i = s^n_j = \text{identity homomorphisms}$, we have the trivial crossed square $M(E^{(0)}, 2)$:

$$
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & R \\
\end{array}
$$

and its associated trivial quadratic module

$$0 \longrightarrow 0 \longrightarrow R.$$

For the 1-skeleton $E^{(1)}$ is

$$E^{(1)} : \cdots \longrightarrow R[s_0S_2, s_1S_2] \xrightarrow{\omega} R[S_2] \xrightarrow{\partial_1} R/I$$

we have the crossed square $M(E^{(1)}, 2)$ (cf. [9]) as given above and its associated quadratic modules $X$ and $Y$ on the 1-dimensional construction data. Here, the quadratic module $X$ is obtained by using the mapping cone of the crossed square and the quadratic module $Y$ is obtained from the crossed square $M(E^{(1)}, 2)$ by applying the Ellis construction method.

For the 2-skeleton $E^{(2)}$, by using the free simplicial algebra, we have the free quadratic module

$$
\begin{array}{ccc}
L' & \xrightarrow{w'} & M \\
\omega' & & \downarrow \partial_1 \\
C & \otimes & C
\end{array}
$$

as given in Theorem 3.2 and alternatively from the crossed square $M(E^{(2)}, 2)$ by applying the Ellis construction method, we have the totally free quadratic module

$$
\begin{array}{ccc}
L/P_3' & \xrightarrow{\omega} & R^+[S_2]/P_3 \\
\partial_2 & & \downarrow \partial \\
C & \otimes & C
\end{array}
$$

labelled by [9].
References

[1] M. André, Homologie des algèbres commutatives. Die Grundlehren der Mathematischen Wissenschaften, 206 Springer-Verlag (1970).

[2] M. Artin and B. Mazur, On the van Kampen theorem, Topology, 5, 179-189, (1966).

[3] Z. Arvasi, Crossed squares and 2-crossed modules of commutative algebras, Theory and Applications of Categories, Vol. 3, No. 7, pp 160-181, (1997).

[4] Z. Arvasi and T. Porter, Higher dimensional Peiffer elements in simplicial commutative algebras, Theory and Applications of Categories, Vol. 3, No. 1, pp 1-23, (1997).

[5] Z. Arvasi and T. Porter, Freeness conditions for 2-crossed module of commutative algebras, Applied Categorical Structures, 6, pp 455-477, (1998).

[6] Z. Arvasi and T. Porter, Simplicial and crossed resolution of commutative algebras, Journal of Algebra, 181, 426-448, (1996).

[7] Z. Arvasi and E. Ulualan, On algebraic models for homotopy 3-types, Journal of Homotopy and Related Structures Vol.1, No 1, pp.1-27, (2006).

[8] Z. Arvasi and E. Ulualan, Quadratic and 2-crossed modules of algebras, Algebra Colloquium 14 : 4 669-686 (2007).

[9] Z. Arvasi and E. Ulualan, Homotopical aspects of commutative algebras I: freeness conditions for crossed squares, arXiv:0911.4050v1 [math.AC] 20 Nov 2009.

[10] H.J. Baues, Combinatorial homotopy and 4-dimensional complexes, Walter de Gruyter, 15, 380 pages, (1991).

[11] C. Berger, Double loop spaces, braided monoidal categories and algebraic 3-types of spaces, Contemporary Mathematics, 227, 46-66, (1999).

[12] R. Brown and N.D. Gilbert, Algebraic models of 3-types and automorphism structures for crossed modules, Proc. London Math. Soc. (3) 59, pp 51-73, (1989).

[13] P. Carrasco, Complejos hiper cruzados, cohomologia y extensiones, Ph.D. Thesis, Univ. de Granada, (1987).

[14] P. Carrasco and A.M. Cegarra, Group-theoretic algebraic models for homotopy types, Journal of Pure and Applied Algebra, 75, pp 195-235, (1991).
[15] J.L. Castiglioni and M. Ladra, Peiffer elements in simplicial groups and algebras, *Journal of Pure and Applied Algebra* Vol. 212, Issue 9, 2115-2128, (2008).

[16] D. Conduché, Modules croisés gé néralisés de longueur 2, *Journal of Pure and Applied Algebra*, 34, pp 155-178, (1984).

[17] D. Conduché, Simplicial crossed modules and mapping cones, *Georgian Mathematical Journal*, 10, No. 4, 623-636, (2003).

[18] G.J. Ellis, Higher dimensional crossed modules of algebras, *Journal of Pure and Applied Algebra*, 52, 277-282, (1988).

[19] G.J. Ellis, Crossed squares and combinatorial homotopy , *Math.Z.*, 214, 93-110, (1993).

[20] G.J. Ellis and R. Steiner, Higher dimensional crossed modules and the homotopy groups of (n+1)-ads., *Journal of Pure and Applied Algebra*, 46, pp 117-136, (1987).

[21] A.R. Grandjeán and M.J. Vale, 2-modulos cruzados an la cohomologia de André-Quillen, *Memorias de la Reai Academia de Ciencias*, 22, (1986).

[22] D. Guin-W aléry and J-L. Loday, Obstruction à l’excision en K-theories algébrique, In: Friedlander, E.M.,Stein, M.R.(eds.) Evanston conf. on algebraic K-Theory 1980, (Lect. Notes Math., vol.854, pp 179-216), Berlin Heidelberg New York: Springer (1981).

[23] L.Illusie, Complex cotangent et deformations I, II, *Springer Lecture Notes in Math.*, 239 (1971) II 283 (1972).

[24] J-L. Loday, Spaces with finitely many non-trivial homotopy groups, *Journal of Pure and Applied Algebra*, 24, pp 179-202, (1982).

[25] C. Morgenegg, Sur les invariants d’un anneau local á corps résiduel de caractéristique 2, *Ph.D. thesis*, Lausanne EPFL (1979).

[26] T. Porter, n-type of simplicial groups and crossed n-Cubes, *Topology*, 32, pp 5-24, (1993).

[27] T. Porter, Some categorical results in the theory of crossed modules in commutative algebras, *Journal of Algebra*, 109, pp 415-429, (1987).

[28] T. Porter, Homology of commutative algebras and an invariant of Simis and Vasconceles, *Journal of Algebra*, 99, pp 458-465, (1986).

[29] T. Porter, The crossed menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology, [http://ncatlab.org/timporter/show/crossed-menagerie](http://ncatlab.org/timporter/show/crossed-menagerie)
[30] D. Quillen, On the homology of commutative rings, *Porc. Sympos. Pure Math.*, 17, (1970).

[31] J.H.C. Whitehead, Combinatorial homotopy II, *Bull. Amer. Math. Soc.*, 55, pp 453–496, (1949).

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