RIGIDITY OF THIN DISK CONFIGURATIONS, VIA FIXED-POINT INDEX

ANDREY M. MISHCHENKO

ABSTRACT. We prove some rigidity theorems for configurations of closed disks. First, fix two collections \( C \) and \( \tilde{C} \) of closed disks in the Riemann sphere \( \hat{\mathbb{C}} \), sharing a contact graph which (mostly-)triangulates \( \hat{\mathbb{C}} \), so that for all corresponding pairs of intersecting disks \( D_i, D_j \in C \) and \( \tilde{D}_i, \tilde{D}_j \in \tilde{C} \) we have that the overlap angle between \( D_i \) and \( D_j \) agrees with that between \( \tilde{D}_i \) and \( \tilde{D}_j \). We require the extra condition that the collections are thin, meaning that no pair of disks of \( C \) meet in the interior of a third, and similarly for \( \tilde{C} \). Then \( C \) and \( \tilde{C} \) differ by a Möbius or anti-Möbius transformation. We also prove the analogous statements for collections of closed disks in the complex plane \( \mathbb{C} \), and in the hyperbolic plane \( \mathbb{H}^2 \).

Our method of proof is elementary and self-contained, relying only on plane topology arguments and manipulations by Möbius transformations. In particular, we generalize a fixed-point argument which was previously applied by Schramm and He to prove the analogs of our theorems in the circle-packing setting, that is, where the disks in question are pairwise interiorwise disjoint. It was previously thought that these methods of proof depended too crucially on the pairwise interiorwise disjointness of the disks for there to be a hope for generalizing them to the setting of configurations of overlapping disks.

We end by stating some open problems and conjectures, including conjectured generalizations both of our main result and of our main technical theorem. Specifically, we conjecture that our thinness condition is unnecessary in the statements of our main theorems.

CONTENTS

1. Introduction 2
2. Related work 5
3. Fixed-point index preliminaries 9
4. Rigidity proofs in the circle packing case 12
5. Our main technical result, the Index Theorem 16
6. Subsumptive collections of disks 20
7. Proofs of our main rigidity theorems 25
8. Topological configurations 29
9. Preliminary topological lemmas 33
10. Torus parametrization 38
11. Proof of Propositions 11.1 and 11.5 42
12. Generalizations, open problems, and conjectures 49
References 50

Date: February 12, 2013.

The author was partially supported by NSF grants DMS-0456940, DMS-0555750, DMS-0801029, DMS-1101373. This article is adapted from the Ph.D. thesis Mis12 of the author. MSC2010 subject classification: 52C26.
1. Introduction

A circle packing is defined to be a collection of pairwise interiorwise disjoint metric closed disks in the Riemann sphere $\hat{\mathbb{C}}$. We will always consider $\hat{\mathbb{C}}$ to have the usual constant curvature $+1$ spherical metric. The contact graph of a circle packing $\mathcal{P}$ is the graph $G$ having a vertex for every disk of $\mathcal{P}$, so that two vertices of $G$ are connected by an edge if and only if the corresponding disks of $\mathcal{P}$ meet. If $\mathcal{P}$ is a locally finite circle packing in $\hat{\mathbb{C}}$, then clearly its contact graph is simple and planar. A graph is simple if it has no loops and no repeated edges. If a circle packing $\mathcal{P}$ has contact graph $G$ then we say that $\mathcal{P}$ realizes $G$. It turns out that the converse also holds: if $G$ is a simple planar graph, then there is a circle packing in $\hat{\mathbb{C}}$ having $G$ as its contact graph. This well-known result, known as the Circle Packing Theorem, is originally due to Koebe, first appearing in [Koe36].

The Circle Packing Theorem settles the question of existence of circle packings in $\hat{\mathbb{C}}$. It is then natural to ask for rigidity statements. In the same article, Koebe states a theorem equivalent to the following:

**Koebe–Andreev–Thurston Theorem 1.1.** Let $G$ be the 1-skeleton of a triangulation of the 2-sphere $S^2$. Then the circle packing realizing $G$ is unique, up to Möbius and anti-Möbius transformations.

A triangulation of a topological surface $S$ is a collection of triangular faces, each of which is a topological closed disk, so that two given faces are glued either along a single edge, or at a single vertex, or not at all, and so that there are no gluings along the boundary of any one fixed triangle, such that that the resulting object is homeomorphic to $S$. An anti-Möbius transformation is the composition of a Möbius transformation with $z \mapsto \bar{z}$. Möbius and anti-Möbius transformations send circles to circles and preserve contact graphs, so the rigidity given by Theorem 1.1 is the best possible.

After Koebe, the Circle Packing Theorem and Theorem 1.1 were for a long time forgotten. They were reintroduced to the mathematical community at large in the 1970s by Thurston. There he discussed his methods of proof based on Andreev’s characterization of finite-volume hyperbolic polyhedra given in [And70]. The best source we are aware of for Thurston’s original work on this topic is his widely circulated lecture notes, [Thu80, Section 13.6].

Thurston later conjectured that the Riemann mapping can be approximated by circle packings. The subsequent proof of this conjecture by Rodin and Sullivan in [RSS7] confirmed the importance of circle packing to complex analysis. A flurry of research in the area followed, and circle packing has since found applications in many other areas, for example, in combinatorics, hyperbolic 3-manifolds, probability, and geometric analysis. A list of references for successful applications of circle packing to other areas appears for example in [Roh11, Section 2.2].

It is natural to ask for rigidity statements in the spirit of Theorem 1.1 in geometries besides the spherical one, specifically in Euclidean and hyperbolic geometries. This line of investigation led Schramm, and later He, to the following theorem:

---

1At the International Congress of Mathematicians, Helsinki, 1978, according to [Sac94, p. 135].
2In his address at the International Symposium in Celebration of the Proof of the Bieberbach Conjecture, Purdue University, March 1985, according to [HS93, p. 371].
Discrete Uniformization Theorem 1.2. Let $G$ be the 1-skeleton of a triangulation of a topological open disk. Suppose that $P$ and $\tilde{P}$ are circle packings realizing $G$, so that $P$ is locally finite in $\mathbb{C}$ and $\tilde{P}$ is locally finite in $\hat{\mathbb{C}}$, where each of $G$ and $\hat{G}$ is equal to one of $\mathbb{C}$ and $\mathbb{H}^2$. Then $G = \hat{G}$. Furthermore, the packings $P$ and $\tilde{P}$ differ by a Euclidean similarity if $G = \hat{G} = \mathbb{C}$, and by a hyperbolic isometry if $G = \hat{G} = \mathbb{H}^2$.

From now on, we consider the hyperbolic plane $\mathbb{H}^2$ to be identified with the open unit disk $D \subset \mathbb{C}$ via the Poincaré embedding, and embed $\mathbb{C} \subset \hat{\mathbb{C}}$ via usual stereographic projection. Then $\mathbb{H}^2 \cong D \subset \mathbb{C} \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Furthermore, a metric closed disk in $\mathbb{H}^2$ embeds into a metric closed disk in $D \subset \mathbb{C}$ under the Poincaré embedding. Also, a metric closed disk in $\hat{\mathbb{C}}$ is identified with a metric closed disk in $\hat{\mathbb{C}}$ under stereographic projection. For clarity we remark that metric centers of disks are in general not preserved under the Poincaré embedding, nor under stereographic projection.

The first complete proof of Theorem 1.2 was given by Schramm in [Sch91], using only elementary plane topology arguments. Then, in [HS93], He and Schramm implicitly reinterpreted the method of [Sch91] as a fixed-point argument. This approach turned out to be quite powerful, and allowed them to prove much more general statements about domains in $\hat{\mathbb{C}}$ whose boundary components are circles and points. In particular, they prove the Koebe conjecture for domains having countably many boundary components. They also prove an existence statement for circle packings having countably many boundary components. They also prove an existence statement for circle packings in $\mathbb{C}$ and $\mathbb{H}^2$, to go along with the rigidity of Theorem 1.2. We discuss the results of [HS93] in more detail in Section 2 on related work. Other proofs of Theorem 1.2 have since been found, which we discuss briefly also in Section 2.

In this article, we generalize the fixed-point arguments used in [HS93], and implicitly in [Sch91], to prove generalizations of Theorems 1.1 and 1.2 to collections of disks whose interiors may overlap. It was previously thought that those arguments depended too crucially on the pairwise interiorwise disjointness of the disks for there to be hope of generalizing them in this direction. Specifically, we prove rigidity and uniformization theorems for so-called thin disk configurations:

Definition 1.3. A disk configuration is a collection of metric closed disks on the Riemann sphere $\hat{\mathbb{C}}$, so that no disk of the collection is contained in any other, but with no other conditions. A disk configuration is called thin if no three disks of the configuration have a common point. The contact graph of a disk configuration $C$ is a graph with a vertex for every disk of $C$, so that two vertices share an edge if and only if the corresponding disks meet.

Suppose that $G = (V,E)$ is a graph, with vertex set $V$ and edge set $E$, so that $G$ is the contact graph of the disk configuration $C = \{D_v\}_{v \in V}$. Let $\Theta : E \to [0, \pi)$ be so that if $\langle u,v \rangle$ is an edge of $G$, then $\angle(D_u, D_v) = \Theta \langle u,v \rangle$, with $\angle(\cdot, \cdot)$ defined as in Figure 1. Then $(G, \Theta)$ is called the incidence data of $C$, and $C$ is said to realize $(G, \Theta)$.

The main rigidity and uniformization result of this paper is the following theorem:

Main Uniformization Theorem 1.4. Let $G$ be the 1-skeleton of a triangulation of a topological open disk. Suppose that $C$ and $\hat{C}$ are thin disk configurations, locally finite in $\mathbb{C}$ and $\hat{\mathbb{C}}$, respectively, where each of $G$ and $\hat{G}$ is equal to one of $\mathbb{C}$ and $\mathbb{H}^2$, so that $C$ and $\hat{C}$ realize the same incidence data $(G, \Theta)$. Then $G = \hat{G}$, and $C$ and $\hat{C}$ differ by a Euclidean similarity if $G = \hat{G} = \mathbb{C}$, or by a hyperbolic isometry if $G = \hat{G} = \mathbb{H}^2$.

\footnote{For example, see comments in [He99, p. 3], made by one of the authors of [HS93]}
Figure 1. The definition of $\angle(A,B)$, the *external intersection angle* or *overlap angle* between two closed disks $A$ and $B$.

We also prove the following closely related theorem, using the same techniques:

**Main Rigidity Theorem 1.5.** Let $G$ be the 1-skeleton of a triangulation of the 2-sphere $S^2$. Suppose that $C$ and $\tilde{C}$ are thin disk configurations in $\hat{C}$ realizing the same incidence data $(G, \Theta)$. Then $C$ and $\tilde{C}$ differ by a Möbius or an anti-Möbius transformation.

Although its statement has never appeared in the literature, Theorem 1.5 is not new in the sense that it follows as a corollary of known results, for example Rivin’s characterization of ideal hyperbolic polyhedra, c.f. Section 2.3. We discuss this further in Section 2 on related work. No counterexamples are known to Theorems 1.4 and 1.5 if the thinness condition is omitted from their statements, and we conjecture that the theorems continue to hold in this case. More details are given in Section 12.

This article is organized as follows. First, we give a brief survey of related work in Section 2. Then, in Section 3 we introduce the so-called *fixed-point index*, which will be the essential technical tool in our proofs of the main rigidity results of this article, Theorems 1.3 and 1.4.

In Section 4 we apply fixed-point index to prove Theorems 1.1 and 1.2 on rigidity and uniformization of classical circle packings, via the arguments of [HS93]. We include Section 4 for the following reasons. First, He and Schramm in [HS93] work in a much more technical setting, and do not actually work out proofs of Theorems 1.1 and 1.2; rather, they describe how such proofs may be obtained by adapting, in a non-trivial way, their proof of the countably-connected case of the Koebe Conjecture. The hope is that the exposition given in Section 4 works to isolate the main ideas of [HS93], and to clarify what is required to generalize those ideas to our setting.

In Section 5 we state our main technical result, which we call the Index Theorem 5.3 and sketch its proof. The proofs of Theorems 1.3 and 1.4 require some elementary lemmas from plane geometry, and these are proved in Section 6. In Section 7 we prove our main rigidity results, Theorems 1.3 and 1.4 using the Index Theorem 5.3. Sections 8–11 are spent completing the proof of the Index Theorem 5.3. We conclude with a discussion of related open questions and generalizations of our results in Section 12.

**Acknowledgments.** Thanks to my Ph. D. advisor Jeff Lagarias, for helpful comments on many portions of my dissertation, from which this article is adapted. Thanks to Kai Rajala and Karen Smith for reading and commenting on an early version of the proofs in this article. Thanks to Jordan Watkins for many fruitful discussions, especially for pointing us strongly in the direction of Section 10, greatly simplifying that portion of the exposition. Thanks to Mario Bonk for helpful comments on this article.
2. Related work

2.1. Koebe uniformization. Circle packings are closely related to classical complex analysis. As we have already mentioned, it was conjectured by Thurston, and proved by Rodin and Sullivan in [RS87], that circle packings can be used to approximate the Riemann mapping, in some precise sense. Conversely, theorems in circle packing can sometimes be proved via applications of results of classical complex analysis. For example, Koebe first discovered circle packing while researching what is now known as the Koebe Conjecture, posed in [Koe08, p. 358]:

**Koebe Conjecture 2.1.** Every domain \( \Omega \subset \hat{\mathbb{C}} \) is biholomorphically equivalent to a circle domain.

A *circle domain* is a connected open subset of \( \hat{\mathbb{C}} \) all of whose boundary components are circles and points. In the same article, Koebe himself gave a construction, via iterative applications of the Riemann mapping, biholomorphically uniformizing an \( \Omega \) having finitely many boundary components to a circle domain. Later, in [Koe36], he used this uniformization to prove that any finite simple planar graph \( G \) admits a circle packing realizing it. His construction approximates the desired circle packing by first arranging disjoint not-necessarily-round compact sets roughly according to the contact pattern demanded by \( G \), then uniformizing the resulting complementary region to a circle domain. The desired circle packing is then obtained as a limit.

There is an existence statement associated to the rigidity statement of the Koebe–Andreev–Thurston Theorem 1.1: if \( G \) is the 1-skeleton of a triangulation of \( S^2 \), then \( G \) is finite, simple, and planar, so there exists a circle packing in \( \hat{\mathbb{C}} \) realizing \( G \). It is natural to ask for an analogous existence statement to go along with the Discrete Uniformization Theorem 1.2. In [HS93] He and Schramm prove that if \( G \) is the 1-skeleton of a triangulation of a topological open disk, then there exists a locally finite circle packing in one of \( \mathbb{C} \) and \( \mathbb{H}^2 \) which realizes \( G \). In the same article, they also prove the existence of the uniformizing map described in the Koebe Conjecture 2.1 for countably connected domains, that is, domains having countably many boundary components. The two existence proofs are closely intertwined, both appealing to fixed-point arguments at some crucial points.

Sometimes when the Koebe Conjecture 2.1 is stated, a statement of uniqueness of the uniformizing biholomorphism, up to postcomposition by Möbius transformations, is included as part of the conjecture. The article [HS93] establishes this rigidity portion of the conjecture as well, for countably connected domains. The main idea of this rigidity proof is visible, adapted to the setting of circle packings, in Section 4. A sketch of the proof is given in [Roh11, Theorem 2.11].

In the case of uncountably connected domains, the uniqueness part of the Koebe Conjecture 2.1 is well-known to be false. A counterexample can be obtained by “placing a nonzero Beltrami differential supported on [a Cantor set of non-zero area] and solving the Beltrami equation to obtain a quasiconformal map which is conformal outside the Cantor set,” as noted by [HS93, p. 370]. The existence portion of the Koebe Conjecture 2.1 is still open in this case.

2.2. Existence statements for collections of disks with overlaps. Given that there are existence statements to go along with both Theorems 1.1 and 1.2 it is also natural to ask
for analogous existence statements to go along with the main results of this paper, Theorems 1.5 and 1.4. No such existence results are presently available.

There are many non-trivial conditions on incidence data which are necessary for the existence of a disk configuration in the Riemann sphere realizing that data. For instance, it is not hard to show that given \( n \) disks \( D_i \), with \( i \in \mathbb{Z}/n\mathbb{Z} \), so that \( D_i \) and \( D_j \) meet if and only if \( i = j \pm 1 \), we have that \( \sum_{i=1}^{n} \angle(D_i, D_{i+1}) < (n-2)\pi \); see Figure 5 on p. 21. In general, conditions on \((G, \Theta)\) which force extraneous contacts are not well understood. An example of such a condition is when \( G \) contains a closed \( n \)-cycle consisting of distinct edges \( e_1, \ldots, e_n \) so that \( \sum_{i=1}^{n} \Theta(e_i) \geq (n-2)\pi \): in this case, by the earlier discussion, for there to be any hope of a positive answer to the existence question for the data \((G, \Theta)\), there must be at least one additional contact among the vertices which are the endpoints of the \( e_i \).

The general existence question for configurations of disks realizing certain given incidence data appears not to have been studied much. Presently, the main obstruction to obtaining theorems in this vein is not in finding proofs, but in finding the correct statements. For example: as we mentioned, the proof given in [HS93] of existence of circle packings having contact graphs triangulating a topological open disk relies at crucial points on fixed-point arguments. Our Main Index Theorem 5.3 would exactly fill the gaps in the fixed-point portions of what would be the generalizations of those arguments to our setting, that of thin disk configurations. In general, the methods used to prove existence of circle packings are quite robust and varied, and at least some of these methods are likely to generalize nicely to the setting of collections of disks with overlap, if the correct statements to be proved were known. For further discussion on the general question of existence of disk configurations realizing given incidence data \((G, \Theta)\), see [Mis12, Section 2.9]. The special case when \( \Theta \) is uniformly bounded above by \( \pi/2 \) is much simpler than the general situation, and is discussed further in Section 2.3 and especially Section 2.4.

2.3. Hyperbolic polyhedra. Configurations of disks on \( \hat{\mathbb{C}} \) are closely related to hyperbolic polyhedra. For example, given a collection of disks covering \( \hat{\mathbb{C}} \), we may construct a hyperbolic polyhedron by cutting out the half-spaces which are bounded at \( \partial \mathbb{H}^3 = \hat{\mathbb{C}} \) by the disks in our collection. This construction can be used to translate theorems on hyperbolic polyhedra to theorems about circle packings or disk configurations, and vice versa.

In [And70], Andreev gives a characterization of finite-volume hyperbolic polyhedra satisfying the condition that every two faces sharing an edge meet at an interior angle of at most \( \pi/2 \). In particular, the combinatorics and interior angles of such a polyhedron completely determine it, up to hyperbolic isometry of \( \mathbb{H}^3 \). From this one may deduce Theorem 1.1. This was the approach originally taken by Thurston. For the details of the construction, see [Thu80, Section 13.6].

Rivin has worked extensively on generalizations of Andreev’s characterization theorems. In particular, he has given a complete characterization of ideal hyperbolic polyhedra all of whose vertices lie on \( \hat{\mathbb{C}} = \partial \mathbb{H}^3 \), with no requirements on the incidence angles of the faces; see [Riv94, Theorem 14.1 (rigidity); Riv96, Theorem 0.1 (existence); Riv03, (generalizations)]. Our Theorem 1.5 can be obtained as a corollary of his. The full strength of our Theorem 1.5 cannot be obtained from Andreev’s results, because of the bound on the interior angles \( \Theta(e) \) at the edges \( \{e\} \) of the polyhedra in his hypotheses. Rivin remarks that in the setting of
hyperbolic polyhedra, the restriction $\Theta \leq \pi/2$ is a very strong one\textsuperscript{4}. However, interestingly, there are few places in the present article where a corresponding upper bound of $\pi/2$ on the overlap angles of our disks would simplify the arguments significantly.

No proofs of existence nor of rigidity statements for circle packings having contact graphs triangulating a topological open disk have been obtained directly via these or similar theorems on hyperbolic polyhedra. A major obstruction is that the “polyhedron” constructed via Thurston’s methods from an infinite circle packing in $\hat{\mathbb{C}}$ typically has infinite volume.

Rivin’s theorem characterizing ideal hyperbolic polyhedra can be directly translated into a statement about configurations of disks on $\hat{\mathbb{C}} = \partial \mathbb{H}^3$. One may then hope to generalize this translated statement to the higher-genus setting. Bobenko and Springborn have done exactly this in \cite[Theorem 4]{BS04}, where they prove an existence and uniqueness statement for disk configurations on positive-genus closed Riemann surfaces. Their proof uses variational principles.

One may hope that because Rivin’s characterization of ideal hyperbolic polyhedra includes an existence component, we may obtain an existence statement about disk configurations to go along with our Rigidity Theorem \ref{thm:rigidity} from the direct translation of Rivin’s results. However, the existence portion of this translation does not take as input the contact graph $G$ of the disk configuration $\mathcal{C}$ which is eventually realized, rather taking a certain planar subgraph of $G$. There is no known method for computing the eventual contact graph $G$ obtained this way from only the allowed input to the translated theorem, although by the rigidity portion we know that $G$ is completely determined by said input. This itself may be a difficult problem: for a heuristic argument explaining why, see \cite[Section 2.9.6]{Mis12}. This subtle issue also underlies Bobenko and Springborn’s results, although it is not directly addressed by those authors.

2.4. Vertex extremal length and modulus. A discretized version of classical conformal modulus, equivalently extremal length, of a curve family has been used extensively to prove circle packing theorems. One of the earliest such applications was by He and Schramm in \cite{HS95}. There, using so-called vertex extremal length, given a graph $G$ which is the 1-skeleton of a triangulation of a topological open disk, they reprove the existence of a locally finite circle packing realizing $G$ in exactly one of $\mathbb{C}$ and $\mathbb{H}^2$. They also give discrete-analytic conditions on $G$, for example the recurrence or transience of a simple random walk on $G$, which determine whether the circle packing realizing $G$ lives naturally in $\mathbb{C}$ or in $\mathbb{H}^2$.

These ideas have been generalized by He to the setting of disk configurations with overlaps. In \cite{He99} he proves a generalization of Theorem \ref{thm:he-schramm} using similar methods for configurations of disks whose overlap angles are bounded above by $\pi/2$. He also includes an existence statement. In the same paper, he wrote\textsuperscript{5} that he intended to generalize his techniques further to handle the case of arbitrary overlap angles, but he never published any work doing so.

2.5. Disk configurations in other Riemann surfaces. Given a triangulation $X$ of an open or closed oriented topological surface $S$ without boundary, it is possible to find a complete constant curvature Riemannian metric $d$ on $S$, and a circle packing $\mathcal{P}$ in $(S, d)$,
whose contact graph is the 1-skeleton of $X$. To see why, first lift $X$ to a triangulation of the universal cover of $S$, allowing us to obtain a periodic circle packing in a simply connected constant curvature surface, one of $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{H}^2$. Quotienting by this periodicity gives our desired circle packing realizing $X$ in some complete constant curvature Riemann surface $(S,d)$. Note that the metric $d$ and the packing $\mathcal{P}$ in $(S,d)$ are both essentially uniquely determined by $X$, by the rigidity of Theorem 1.2. This construction is well-known, appearing for example in [BS90], and later in modified form in [HS93, Section 8].

There is no obstruction to applying the same argument in the more general setting of disk configurations with overlaps. For example, applying our Theorem 1.4, if $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are thin disk configurations realizing the same incidence data, living in complete constant curvature Riemann surfaces $R$ and $\tilde{R}$ respectively, then $R$ and $\tilde{R}$ are conformally isomorphic. Thus it is generally sufficient to prove an existence or rigidity statement for disk configurations in $\mathcal{C}$ or $\mathbb{H}^2$ to get analogous statements in multiply connected surfaces.

Some authors have studied related theorems in higher genus surfaces directly. For example, Thurston himself proved an existence theorem for disk configurations with overlap angles bounded above by $\pi/2$ on closed finite-genus surfaces without boundary, in [Thu80, Theorem 13.7.1]. He did not give a rigidity statement. As was already mentioned, Bobenko and Springborn proved an existence and uniqueness theorem for circle patterns on closed finite-genus surfaces without boundary, in [BS04]. Both of these proofs use variational principles.

2.6. Further references. Today many proofs are known of the Circle Packing Theorem and of Theorem 1.1. For example, some proofs using variational principles appear in [CdV91, Bra92, Riv94]. Thurston describes how Theorem 1.1 may be obtained from Mostow–Prasad rigidity in [Thu80, Proof of 13.6.2]. There is a short, clean proof of Theorem 1.1 in [Roh11, Section 2.4.3], which is attributed to Oded Schramm. The earliest published version of this same argument that we are aware of appears in [He99, Section 2].

Many of these arguments generalize readily to prove our Theorem 1.5. However, few of them have been adapted to prove Theorem 1.2 and it therefore appears unlikely that they will work to prove our Theorem 1.4 either.

The fixed-point index techniques we use here have recently been used by Merenkov to prove rigidity statements for Sierpinski carpets, see [Mer12, Section 12].

Theorem 1.2 in the case where $G$ has uniformly bounded-above vertex degree is proved in [BS90], by a modification of an argument by Rodin and Sullivan given in [RS87, Appendix 1]. The proof uses quasi-conformal mapping theory. It appears unlikely that this method of proof can be generalized to the unbounded-valence case. However, it would likely prove the bounded-valence case of our Theorem 1.4 just as well.

A short survey of parts of the area of circle packing which nowadays are considered classical is given by Sachs in [Sac94]. It also gives a rough outline of the history of circle packing through 1994. The first half of [Roh11] is an excellent survey by Rohde focused on the contributions of Oded Schramm. Rohde also gives a long list of successful applications of circle packing to other areas of math in his Section 2.2. Stephenson’s book [Ste05] provides a more detailed, elementary, and mostly self-contained introduction to the area and could serve as a kind of “first course in circle packing.” Stephenson’s methods of proof in this book are essentially adapted from those of He and Schramm in [HS93]. Finally, the introduction
of the Ph. D. thesis [Mis12, Chapter 2] of the present author, from which this article is adapted, gives a fairly thorough survey of the area of circle packing.

3. Fixed-point index preliminaries

A Jordan curve is a homeomorphic image of a topological circle $S^1$ in the complex plane $\mathbb{C}$. A Jordan domain is a bounded open set in $\mathbb{C}$ with Jordan curve boundary. We use the term closed Jordan domain to refer to the closure of a Jordan domain. Suppose that a Jordan curve appears as the boundary of a set having non-empty interior. Then the positive orientation of $\partial X$ with respect to $X$ is defined as usual. That is, the interior of $X$ stays to the left as we traverse $\partial X$ in what we call the positive direction. As a rule, if we write a Jordan curve as $\partial X$, where $X$ is an open or closed Jordan domain or the complement thereof, then we will take that to mean that $\partial X$ is oriented positively with respect to $X$, unless otherwise noted. In particular, if $X$ is an open or closed Jordan domain, then the positive orientation induced on $\partial X$ is the counterclockwise one as usual.

We now define the fixed-point index, our main technical tool. The rest of this section will consist of the proofs of several fundamental lemmas on fixed-point index.

**Definition 3.1.** Let $\gamma$ and $\tilde{\gamma}$ be oriented Jordan curves. Let $\phi : \gamma \to \tilde{\gamma}$ be a homeomorphism which is fixed-point-free and orientation-preserving. We call such a homeomorphism indexable. Then $\{\phi(z) - z\}_{z \in \gamma}$ is a closed curve in $\mathbb{C}$ which misses the origin. It has a natural orientation induced by traversing $\gamma$ according to its orientation. We define the fixed-point index of $\phi$, denoted $\eta(\phi)$, to be the winding number of $\{\phi(z) - z\}_{z \in \gamma}$ around the origin.

Intuitively, fixed-point index counts the following. Suppose that $\Phi : K \to \tilde{K}$ is a homeomorphism of closed Jordan domains having only isolated fixed points, which restricts to an indexable $\phi : \partial K \to \partial \tilde{K}$. Then $\eta(\phi)$ counts the number of fixed points, with signed multiplicity, of $\Phi$. For more discussion on the history and broader relevance of fixed-point index, see [HS93, Section 2]. Every integer, positive or negative, occurs as a fixed-point index.

Our first lemma says that the fixed-point index between two (round) circles is always non-negative:

**Circle Index Lemma 3.2.** Let $K$ and $\tilde{K}$ be closed Jordan domains in $\mathbb{C}$, and let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism. Then the following hold.

1. The homeomorphism $\phi^{-1} : \partial \tilde{K} \to \partial K$ is indexable with $\eta(\phi) = \eta(\phi^{-1})$.
2. If $K \subset \tilde{K}$ or $\tilde{K} \subset K$, then $\eta(\phi) = 1$.
3. If $K$ and $\tilde{K}$ have disjoint interiors, then $\eta(\phi) = 0$.
4. If $\partial K$ and $\partial \tilde{K}$ intersect in exactly two points, then $\eta(\phi) \geq 0$.

As a consequence of the above, if $K$ and $\tilde{K}$ are metric closed disks in the plane, then $\eta(\phi) \geq 0$.

This lemma can be found in [HS93, Lemma 2.2]. There it is indicated that the same lemma appeared earlier in [Str51]. We sketch the proof of Lemma 3.2 given in [HS93, Lemma 2.2]:

“Proof.” (1) By definition $\eta(f^{-1})$ is the winding number of $\{f^{-1}(\tilde{z}) - \tilde{z}\}_{\tilde{z} \in \partial \tilde{K}}$ around the origin, which is equal to the winding number of $\{z - f(z)\}_{z \in \partial K}$ around the origin under the coordinate change $f(z) = \tilde{z}$. But the winding number around the origin of a closed curve $\{\gamma(t)\}_{t \in S^1}$ which misses 0 is equal to the winding number around the origin of $\{-\gamma(t)\}_{t \in S^1}$.
Part (2) is believable if we imagine $K$ to be “very small,” and contained in $\tilde{K}$. Then the endpoint $z$ of the vector $f(z) - z$ does not move very much as $z$ traverses $\partial K$, while the endpoint $f(z)$ of the same vector “winds once positively around $K$.” Part (3) is believable for similar reasons if we imagine $K$ and $\tilde{K}$ to be very far away from each other. These ideas can be made into proofs via simple homotopy arguments.

For part (4) we may assume without loss of generality by parts (2) and (3) that $\partial K$ and $\partial \tilde{K}$ meet transversely at both of their intersection points, so after applying an isotopy to $C$ we have that $K$ and $\tilde{K}$ are the square and circle depicted in Figure 2, c.f. Lemma 8.1. We ask ourselves when the vector $f(z) - z$ can possibly point in the positive real direction, as in Figure 2. If $z \in \partial K$ does not lie in the interior of $\tilde{K}$, the vector $f(z) - z$ has either an imaginary component, or a negative real component. Similarly if $f(z) \in \partial \tilde{K}$ does not lie inside of $K$, then $f(z) - z$ has a negative real component. We conclude that the only way that $f(z) - z$ can be real and positive is if $z$ lies along $\partial K$ in the interior of $\tilde{K}$ and $f(z)$ lies along $\partial \tilde{K}$ inside of $K$. But in this case because of the orientations on $\partial K$ and $\partial \tilde{K}$, the vector $f(z) - z$ is locally turning in the positive direction. Thus whenever the curve $\{f(z) - z\}_{z \in \partial K}$ crosses the positive real axis it is turning in the positive direction, so this curve’s total winding number around the origin cannot be negative. □

Our next lemma says essentially that fixed-point indices “add nicely”:

**Index Additivity Lemma 3.3.** Suppose that $K$ and $L$ are interiorwise disjoint closed Jordan domains which meet along a single positive-length Jordan arc $\partial K \cap \partial L$, similarly for $\tilde{K}$ and $\tilde{L}$. Then $K \cup L$ and $\tilde{K} \cup \tilde{L}$ are closed Jordan domains.

Let $\phi_K : \partial K \to \partial \tilde{K}$ and $\phi_L : \partial L \to \partial \tilde{L}$ be indexable homeomorphisms. Suppose that $\phi_K$ and $\phi_L$ agree on $\partial K \cap \partial L$. Let $\phi : \partial(K \cup L) \to \partial(\tilde{K} \cup \tilde{L})$ be induced via restriction to $\phi_K$ or $\phi_L$ as necessary. Then $\phi$ is an indexable homeomorphism and $\eta(\phi) = \eta(\phi_L) + \eta(\phi_K)$.

**Proof.** By the definition of the fixed-point index, we have that $\eta(\phi_L)$ is equal to $1/2\pi$ times the change in argument of the vector $f(z) - z$, as $z$ traverses $\partial K$ once in the positive direction, and similarly for $\eta(\phi_K)$ and $\eta(\phi)$. The orientation induced on $\partial K \cap \partial L$ by the positive orientation on $\partial K$ is opposite to the one induced by the positive orientation on $\partial L$, so that $\eta(\phi_K)$ is equal to $-1/2\pi$ times the change in argument of the vector $f(z) - z$, as $z$ traverses $\partial K \cap \partial L$ once in the positive direction. Therefore, $\eta(\phi) = \eta(\phi_L) + \eta(\phi_K)$. □
so as $z$ varies positively in $\partial K$ and in $\partial \tilde{K}$ the contributions to the sum $\eta(\phi_K) + \eta(\phi_L)$ along $\partial K \cap \partial L$ exactly cancel.

If we consider the alternative interpretation of the fixed-point index of $\phi$ to be counting the number of fixed points with signed multiplicity of a homeomorphic extension of $\phi$ to all of $K \cup L$, and similarly for $\eta(\phi_K)$ and $\eta(\phi_L)$, then the lemma is also clear. □

Moving on, we make a definition. Let $K$ and $\tilde{K}$ be closed Jordan domains. We say that $K$ and $\tilde{K}$ are in **transverse position** if $\partial K$ and $\partial \tilde{K}$ cross wherever they meet. More precisely, we say that $K$ and $\tilde{K}$ are in **transverse position** if for any $z \in \partial K \cap \partial \tilde{K}$, there is an open neighborhood $U$ of $z$ and a homeomorphism $\phi : U \to \mathbb{D}$ to the open unit disk sending $\partial K \cap U$ to $\mathbb{R} \cap \mathbb{D}$ and sending $\partial \tilde{K} \cap U$ to $i\mathbb{R} \cap \mathbb{D}$.

We now state our next fundamental lemma about fixed-point index. This lemma says essentially that we may almost always prescribe the images of three points on $\partial K$ in $\partial \tilde{K}$, and obtain an indexable homeomorphism $\partial K \to \partial \tilde{K}$ with non-negative fixed-point index, which respects this prescription.

**Three Point Prescription Lemma 3.4.** Let $K$ and $\tilde{K}$ be closed Jordan domains in transverse position. Let $z_1, z_2, z_3 \in \partial K \setminus \partial \tilde{K}$ appear in positively oriented order around $\partial K$, and similarly $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \partial \tilde{K} \setminus \partial K$. Then there is an indexable homeomorphism $\phi : \partial K \to \partial \tilde{K}$ sending $z_i \mapsto \tilde{z}_i$ for $i = 1, 2, 3$, so that $\eta(\phi) \geq 0$.

A version of this lemma is stated in [Ste05, Lemma 8.14], with the following heuristic argument: by Carathéodory’s theorem (see [Car13]), because $K$ and $\tilde{K}$ are closed Jordan domains, any Riemann mapping from the interior of $K$ to that of $\tilde{K}$ extends homeomorphically to a map $\Phi : K \to \tilde{K}$. Furthermore $\Phi$ may be chosen so that $\Phi : z_i \mapsto \tilde{z}_i$ for $i = 1, 2, 3$. Fix such a $\Phi$, and let $\phi : \partial K \to \partial \tilde{K}$ be defined by restriction.

Suppose that $\phi : \partial K \to \partial \tilde{K}$ does not have any fixed points. It is automatically orientation-preserving because a Riemann mapping always is. Suppose also that $\partial K$ and $\partial \tilde{K}$ are piecewise smooth. Then, using the standard complex analysis definition of winding number, we have that:

$$\eta(\phi) = \oint_{\{\phi(z) - z\}_{z \in \partial K}} \frac{dw}{w} = \oint_{\partial K} \frac{\Phi'(w) - 1}{\Phi(w) - w} dw$$

Then by the standard Argument Principle, the second integral counts the number of zeros minus the number of poles of $\Phi(z) - z$ in the interior of $K$, but $\Phi(z) - z$ is holomorphic there, thus has no poles there, so this integral is non-negative.

Actually $\Phi'$ is undefined on $\partial K$, because $\Phi$ is not holomorphic in a neighborhood of $K$, so the second integral does not quite make sense. There is another more serious issue, which is that in general $\phi : \partial K \to \partial \tilde{K}$ may have many fixed points, and it is not clear how to get rid of them. The argument given in [Ste05, Lemma 8.14] does not address these two issues.

We give an original elementary inductive proof of Lemma 3.4, using only plane topology arguments, in [Mis12, Section 3.5]. The proof is not hard, but is lengthy to include here. This lemma fails if we try to prescribe the images of four points. For a counterexample, see [HS93, Figure 2.2].
In this section we prove rigidity theorems for circle packings which are special cases of our main rigidity results on thin disk configurations. The arguments here are adapted from those in [HS93].

The following well-known and easy-to-check lemma will be implicit in our discussion below, although we will not refer to it directly:

**Lemma 4.1.** Let $G = (V,E)$ be a 3-cycle. Let $\mathcal{P} = \{D_v\}_{v \in V}$ and $\tilde{\mathcal{P}} = \{\tilde{D}_v\}_{v \in V}$ be circle packings in $\hat{\mathcal{C}}$ having contact graph $G$. Then any Möbius transformation sending the three tangency points of $\mathcal{P}$ to those of $\tilde{\mathcal{P}}$ in fact identifies $\mathcal{P}$ and $\tilde{\mathcal{P}}$.

(Let $C^*$ and $\tilde{C}^*$ be the circles passing through the tangency points of $\mathcal{P}$ and those of $\tilde{\mathcal{P}}$, respectively. Then one can show that $C^*$ meets every $\partial D_v$ orthogonally, similarly $\tilde{C}^*$ and the $\partial \tilde{D}_v$, and the lemma follows.)

Before moving on, it will help to have some definitions. First, suppose that $\mathcal{P}$ is a circle packing locally finite in $\mathcal{G}$, where $\mathcal{G}$ is equal to one of $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{H}^2$. Then $\mathcal{P}$ induces an embedding of its contact graph $G$ in $\mathcal{G}$ by placing every vertex $v$ of $G$ at the metric center of its associated disk $D_v \in \mathcal{P}$, and connecting the centers $u,v$ of touching disks $D_u, D_v$ with a geodesic arc passing through the point $D_u \cap D_v$. We call this the geodesic embedding of $G$ in $\mathcal{G}$ induced by $\mathcal{P}$. Next, two circle packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are in general position if the following hold:

- Every pair of disks $D \in \mathcal{P}$ and $\tilde{D} \in \tilde{\mathcal{P}}$ are in transverse position as closed Jordan domains. For a reminder of what this means see p. 11.
- If $z$ is an intersection point of two distinct disks of $\mathcal{P}$, then $z$ does not lie on $\partial \tilde{D}$ for any $\tilde{D} \in \tilde{\mathcal{P}}$. Similarly if $\tilde{z}$ is an intersection point of two distinct disks of $\tilde{\mathcal{P}}$, then $\tilde{z}$ does not lie on $\partial D$ for any $D \in \mathcal{P}$.

We now proceed to the proofs of our rigidity theorems on circle packings. First:

**Theorem 4.2.** Suppose that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are circle packings in $\hat{\mathbb{C}}$, sharing a contact graph $G$ which is the 1-skeleton of a triangulation of the 2-sphere $\mathbb{S}^2$. Then $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by a Möbius or an anti-Möbius transformation.

**Proof.** First, recall that if $G$ is the 1-skeleton of a triangulation of $\mathbb{S}^2$, then there are exactly two ways to embed $G$ in $\mathbb{S}^2$, up to orientation-preserving self-homeomorphism of $\mathbb{S}^2$. Therefore we may suppose without loss of generality, by applying $z \mapsto \tilde{z}$ to one of the two packings if necessary, that the geodesic embeddings of $G$ in $\hat{\mathbb{C}}$ induced by $\mathcal{P}$ and by $\tilde{\mathcal{P}}$ are images of one another by orientation-preserving self-homeomorphisms of $\hat{\mathbb{C}}$. In our next three proofs we note this preliminary procedure simply by saying that we may suppose without loss of generality that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ have the same orientation.

Then we proceed by contradiction, supposing that there is no Möbius transformation sending $\mathcal{P}$ to $\tilde{\mathcal{P}}$. The first step of the proof is to normalize $\mathcal{P}$ and $\tilde{\mathcal{P}}$ in a convenient way. In particular, we apply Möbius transformations so that the following holds:

**Normalization 4.3.** There are disks $D_a, D_b, D_c \in \mathcal{P}$ and $\tilde{D}_a, \tilde{D}_b, \tilde{D}_c \in \tilde{\mathcal{P}}$, where $a,b,c$ are distinct vertices of the common contact graph $G$ of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, so that the following hold:

- One of $D_v$ and $\tilde{D}_v$ is contained in the interior of the other, for all $v = a,b,c$. 

4. Rigidity proofs in the circle packing case
Figure 3. The packing $\mathcal{P}$, with $\infty$ in the interior of $D_a$; and the interaction between the disks $D_a, D_b, D_c$ and $\tilde{D}_a, \tilde{D}_b, \tilde{D}_c$ after our normalization.

- The point $\infty \in \hat{C}$ lies in the interior of $D_a \cap \tilde{D}_a$.
- The packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are in general position.

We do not prove here that Normalization 4.3 is possible, because it is a special case of the stronger normalization we construct in detail in the proof of our Main Rigidity Theorem 1.5. See Figure 3 for a model drawing of the present situation.

An interstice of the packing $\mathcal{P}$ is defined to be a connected component of $\hat{C} \setminus \cup_{D \in \mathcal{P}} D$. That is, the interstices of $\mathcal{P}$ are the curvilinear triangles which make up the complement of the packing. If we write $F$ to denote the set of faces of the triangulation of $S^2$ having $G$ as its 1-skeleton, then the interstices of $\mathcal{P}$ are in natural bijection with the faces $F$. (If we embed $G$ via the embedding induced by $\mathcal{P}$, then every face of the resulting triangulation of $\hat{C}$ contains precisely one interstice.) We write $T_f$ to denote the interstice of $\mathcal{P}$ corresponding to the face $f \in F$. The interstices $\tilde{T}_f$ of $\tilde{\mathcal{P}}$ are defined analogously. Note also that there is a natural correspondence from the corners of $T_f$ to those of $\tilde{T}_f$, for a given $f \in F$. For every $f \in F$, fix an indexable homeomorphism $\phi_f : \partial T_f \to \partial \tilde{T}_f$ which identifies corresponding corners with $\eta(\phi_f) \geq 0$. We may do so by the Three Point Prescription Lemma 3.4. We remark that it is also important at this step that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ have the same orientation, as per the first paragraph of this proof. Then the homeomorphisms $\phi_f$ induce, by restriction, indexable homeomorphisms $\phi_v : \partial D_v \to \partial \tilde{D}_v$ between the boundary circles of the disks $D_v \in \mathcal{P}$ and $\tilde{D}_v \in \tilde{\mathcal{P}}$. Here $v$ ranges over the vertex set $V$ of $G$.

Orient $\partial D_a$ and $\partial \tilde{D}_a$ positively with respect to the open disks they bound in $\hat{C}$. We remark for clarity that this is the opposite of the positive orientation on $\partial D_a$ and $\partial \tilde{D}_a$ with respect to the interiors of $D_a$ and $\tilde{D}_a$, in $\hat{C}$. Then $\eta(\phi_v) = 1$ for all $v = a, b, c$, by the Circle Index Lemma 3.2 because of our Normalization 4.3. On the other hand, by the Index Additivity Lemma 3.3, we have the following:

$$1 = \eta(\phi_a) = \sum_{f \in F} \eta(\phi_f) + \sum_{v \in V \setminus \{a\}} \eta(\phi_v)$$

Every $\eta(\phi_f)$ in the first sum is non-negative by construction, and every $\eta(\phi_v)$ in the second sum is non-negative by the Circle Index Lemma 3.2. Also, we have contributions from $\eta(\phi_b) = 1$ and $\eta(\phi_c) = 1$ to second sum, so it must be at least 2, giving us the desired contradiction. □
The proofs of our other rigidity and uniformization theorems for circle packings are adapted from the proof of Theorem 4.2 using similar ideas. We give these proofs now without further comment:

**Theorem 4.4.** There cannot be two circle packings $P$ and $\tilde{P}$ sharing a contact graph $G$ which is the 1-skeleton of a triangulation of a topological open disk, so that one of $P$ and $\tilde{P}$ is locally finite in $\mathbb{C}$ and the other is locally finite in the open unit disk $\mathbb{D}$, or equivalently the hyperbolic plane $\mathbb{H}^2 \cong \mathbb{D}$.

**Proof.** We again proceed by contradiction, supposing that $P$ is locally finite in $\mathbb{C}$ and $\tilde{P}$ is locally finite in the open unit disk $\mathbb{D}$. As before we apply $z \mapsto \bar{z}$ to one of the packings if necessary to ensure that the geodesic embeddings of $G$ in $\mathbb{C}$ and in $\mathbb{H}^2 \cong \mathbb{D} \subset \mathbb{C}$ induced by $P$ and $\tilde{P}$ respectively are identified via some orientation-preserving homeomorphism $\mathbb{C} \to \mathbb{D}$, ensuring that $P$ and $\tilde{P}$ have the same orientation. This time we normalize by applying orientation-preserving Euclidean similarities to $P$ so that the following holds:

**Normalization 4.5.** There are disks $D_a, D_b \in P$ and $\tilde{D}_a, \tilde{D}_b \in \tilde{P}$, where $a, b$ are distinct vertices of the common contact graph $G$ of $P$ and $\tilde{P}$, so that the following hold:

- One of $D_v$ and $\tilde{D}_v$ is contained in the interior of the other, for all $v = a, b$.
- The packings $P$ and $\tilde{P}$ are in general position.

(We give a detailed construction of a stronger normalization in the proof of Theorem 7.3.)

Let $X = (V, E, F)$ be a triangulation of a topological open disk with vertices $V$, edges $E$, and faces $F$, considered only up to its combinatorics, so that the 1-skeleton $(V, E)$ of $X$ is $G$. We define the interstices $T_f$ and $\tilde{T}_f$ as before, and again fix $\phi_f : \partial T_f \to \partial \tilde{T}_f$ having $\eta(\phi_f) \geq 0$. For every $v \in V$ we again write $\phi_v : \partial D_v \to \partial \tilde{D}_v$ for the indexable homeomorphism induced by restriction to the $\phi_f$.

Let $(V_0, E_0, F_0) = X_0 \subset X$ be a subtriangulation of $X$, so that $X_0$ is a triangulation of a topological closed disk, and so that $\mathbb{D} \subset \bigcup_{v \in V_0} D_v \cup \bigcup_{f \in F_0} T_f$. Call this total union $K$, and define $\tilde{K}$ analogously as $\tilde{K} = \bigcup_{v \in V_0} \tilde{D}_v \cup \bigcup_{f \in F_0} \tilde{T}_f$. Let $\phi_K : \partial K \to \partial \tilde{K}$ be the indexable homeomorphism induced by restriction to the $\phi_v$. Then $\eta(\phi_K) = 1$ by the Circle Index Lemma 3.2 because $\tilde{K} \subset \mathbb{D} \subset K$. On the other hand, by the Index Additivity Lemma 3.3:

$$1 = \eta(\phi_K) = \sum_{f \in F_0} \eta(\phi_f) + \sum_{v \in V_0} \eta(\phi_v)$$

As before, in the first sum every $\eta(\phi_f) \geq 0$ by construction and in the second sum every $\eta(\phi_v) \geq 0$ by the Circle Index Lemma 3.2. Also the second sum has contributions from $\eta(\phi_a) = 1$ and $\eta(\phi_b) = 1$, again giving us a contradiction as desired. $\square$

**Theorem 4.6.** Suppose that $P$ and $\tilde{P}$ are circle packings locally finite in the open unit disk $\mathbb{D}$, equivalently the hyperbolic plane $\mathbb{H}^2 \cong \mathbb{D}$, so that $P$ and $\tilde{P}$ share a contact graph $G$ which is the 1-skeleton of a triangulation of a topological open disk. Then $P$ and $\tilde{P}$ differ by a hyperbolic isometry, that is, a Möbius or anti-Möbius transformation fixing $\mathbb{D} \cong \mathbb{H}^2$ set-wise.

**Proof.** As usual, we may suppose that $P$ and $\tilde{P}$ have the same orientation. Proceeding by contradiction, we then normalize by orientation-preserving Euclidean similarities so that the following holds:
Normalisation 4.7. There are disks $D_a, D_b \in \mathcal{P}$ and $\tilde{D}_a, \tilde{D}_b \in \tilde{\mathcal{P}}$, where $a, b$ are distinct vertices of the common contact graph $G$ of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, so that the following hold:

- One of $D_v$ and $\tilde{D}_v$ is contained in the interior of the other, for all $v = a, b$.
- Letting $D$ and $\tilde{D}$ denote the images of the open unit disk $\mathbb{D}$ under the normalizations applied to $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively, we have that one of $D$ and $\tilde{D}$ is contained in the interior of the other.
- The packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are in general position.

(We give a detailed construction of a stronger normalization in the proof of Theorem 4.4.)

Let $X = (V, E, F)$ be a triangulation of a topological open disk, considered up to its combinatorics, having 1-skeleton $G = (V, E)$. We define all of $T_f, \tilde{T}_f, \phi_f, \phi_{\tilde{f}}$ as before.

Suppose without loss of generality, by interchanging the roles of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ if necessary, that $\tilde{D}$ is contained in the interior of $D$. Let $(V_0, E_0, F_0) = X_0 \subset X$ be a subtriangulation of $X$, so that $X_0$ is a triangulation of a topological closed disk, and so that $\tilde{D} \subset \bigcup_{v \in V_0} D_v \cup \bigcup_{f \in F_0} T_f$. Call this total union $K$, and define $\tilde{K}$ analogously as before, again getting $\tilde{K} \subset \tilde{D} \subset K$. We obtain the desired contradiction as in the proof of Theorem 4.4. □

Theorem 4.8. Suppose that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are circle packings locally finite in $\mathbb{C}$, sharing a contact graph $\tilde{G}$ which is the 1-skeleton of a triangulation of a topological open disk. Then $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by a Euclidean similarity.

Proof. As usual, we may suppose that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ have the same orientation. We proceed by contradiction, and begin by normalizing $\mathcal{P}$ and $\tilde{\mathcal{P}}$ by Moebius transformations so that the following holds:

Normalisation 4.9. There are disks $D_a, D_b, D_c \in \mathcal{P}$ and $\tilde{D}_a, \tilde{D}_b, \tilde{D}_c \in \tilde{\mathcal{P}}$, where $a, b, c$ are distinct vertices of the common contact graph $G$ of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, so that the following hold:

- One of $D_v$ and $\tilde{D}_v$ is contained in the interior of the other, for all $v = a, b, c$.
- The point $\infty \in \hat{\mathbb{C}}$ lies in the interior of $D_a \cap \tilde{D}_a$.
- Letting $\tilde{z}_\infty$ and $z_\infty$ denote the images of $\infty$ under the normalizations applied to $\mathcal{P}$ and $\tilde{\mathcal{P}}$ respectively, we have that $z_\infty \neq \tilde{z}_\infty$.
- The packings $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are in general position.

(We give a detailed construction of a stronger normalization in the proof of Theorem 4.4.)

We define all of $X = (V, E, F), T_f, \tilde{T}_f, \phi_f, \phi_{\tilde{f}}$ as before. Let $U$ and $\tilde{U}$ be small disjoint open neighborhoods of $z_\infty$ and $\tilde{z}_\infty$ respectively, and let $(V_0, E_0, F_0) = X_0 \subset X$ be a subtriangulation of $X$ so that the following hold:

- We have that $X_0$ is a triangulation of a topological closed disk.
- Setting $L = \bigcup_{v \in V_0} D_v \cup \bigcup_{f \in F_0} T_f$, and defining $\tilde{L}$ analogously, we have that $L \subset U$ and $\tilde{L} \subset \tilde{U}$.

Then the $\phi_v$ induce, via restriction, an indexable homeomorphism $\phi_L : \partial L \to \partial \tilde{L}$, with $\eta(\phi_L) = 0$ by the Circle Index Lemma 3.2, because $U \supset L$ and $\tilde{U} \supset \tilde{L}$ are disjoint.

We orient $\partial D_a$ and $\partial \tilde{D}_a$ positively with respect to the open disks they bound in $\mathbb{C}$, as in the proof of Theorem 4.2. Then by the Index Additivity Lemma 3.3 we have:

$$1 = \eta(\phi_a) = \sum_{f \in F_0} \eta(\phi_f) + \sum_{v \in V_0} \eta(\phi_v) + \eta(\phi_L)$$
5. Our main technical result, the Index Theorem

Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ and $\mathcal{\tilde{K}} = \{\tilde{K}_1, \ldots, \tilde{K}_n\}$ be collections of closed Jordan domains. We denote $\partial \mathcal{K} = \partial \bigcup_{i=1}^n K_i$, similarly $\partial \mathcal{\tilde{K}} = \bigcup_{i=1}^n \tilde{K}_i$. A homeomorphism $\phi : \partial \mathcal{K} \to \partial \mathcal{\tilde{K}}$ is called \textit{faithful} if whenever we restrict $\phi$ to $K_j \cap \partial \mathcal{K}$ we get a homeomorphism $K_j \cap \partial \mathcal{K} \to \tilde{K}_j \cap \partial \mathcal{\tilde{K}}$. The particular choice of indices of $K_i$ and $\tilde{K}_i$ is important in determining whether a given homeomorphism is faithful, so we consider the labeling to be part of the information of the collections. Note that in general $\partial \mathcal{K}$ and $\partial \mathcal{\tilde{K}}$ need not be homeomorphic, and even if they are homeomorphic there may still be no faithful homeomorphism between them.

We now give a weak form of our main technical result, to illustrate the manner in which we generalize the Circle Index Lemma 3.2. It may be helpful to recall Definition 1.3 on p. 3.

Main Index Theorem (weak form). Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ and $\mathcal{\tilde{D}} = \{\tilde{D}_1, \ldots, \tilde{D}_n\}$ be finite thin disk configurations in the plane $\mathbb{C}$ realizing the same incidence data $(G, \Theta)$. Let $\phi : \partial \mathcal{D} \to \partial \mathcal{\tilde{D}}$ be a faithful indexable homeomorphism. Then $\eta(\phi) \geq 0$.

This follows from the full statement of the Main Index Theorem 5.3. To get rid of the general position hypothesis from the statement of Theorem 5.3 we need a lemma, which we do not prove in this article, which says that the fixed-point index of a homeomorphism is invariant under a small perturbation of its domain or range, see [Mis12, Lemma 3.3; Ste05, Lemma 8.11].

Note that our current Definition 3.1 of fixed-point index is not strong enough to accommodate the theorem statement we just gave. This is because $\bigcup_{D \in \mathcal{D}} D$ need not be a closed Jordan domain. In light of the Index Additivity Lemma 3.3 it is clear how to adapt Definition 3.1 to suit our needs. In particular:

Definition 5.1. Suppose that $K$ is a union of finitely many closed disks in $\mathbb{C}$, some of which may intersect. Suppose also that $\partial K$ is oriented positively with respect to $K$, meaning as usual that the interior of $K$ stays to the left as we traverse $\partial K$ in what we call the positive direction. Then $\partial K$ decomposes, possibly in more than one way, as a union of finitely many oriented Jordan curves $\gamma_1, \ldots, \gamma_n$ any two of which meet only at finitely many points. Some of the $\gamma_i$ will be oriented positively with respect to the finite Jordan domains they bound in $\mathbb{C}$, some negatively. Suppose $\tilde{K}$ is another finite union of closed disks, with $\partial \tilde{K}$ similarly decomposing as $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$, and that $\phi : \partial K \to \partial \tilde{K}$ is a fixed-point-free orientation-preserving homeomorphism which extends to a homeomorphism $K \to \tilde{K}$, and which identifies $\gamma_i$ with $\tilde{\gamma}_i$ for $1 \leq i \leq n$. Then we define $\eta(\phi) = \sum_{i=1}^n \eta(\gamma_i \overset{\phi}{\to} \tilde{\gamma}_i)$.

Here we write $\gamma_i \overset{\phi}{\to} \tilde{\gamma}_i$ to denote the restriction of $\phi$ to $\gamma_i \to \tilde{\gamma}_i$. We will continue to use this notational convention in the future. We remark that in the definition, the decomposition of $\partial K$ into $\gamma_1, \ldots, \gamma_n$ may not be unique, similarly for $\partial \tilde{K}$. We leave it as an exercise for the reader to verify that the same value for the fixed-point index is obtained regardless of which decomposition is chosen. We remark also that the natural generalization of our Index Additivity Lemma 3.3 continues to hold, and leave this as an exercise as well. Definition 5.1 will be general enough to completely accommodate the statement of our Main Index Theorem 5.3.

The first sum is non-negative and the second sum is at least 2 as usual, and $\eta(\phi_L) = 0$, so we get our desired contradiction.
To give the full statement of our Main Index Theorem 5.3, we need one more definition:

**Definition 5.2.** Let \( \mathcal{D} = \{D_1, \ldots, D_n\} \) and \( \tilde{\mathcal{D}} = \{\tilde{D}_1, \ldots, \tilde{D}_n\} \) be finite collections of closed disks in the complex plane \( \mathbb{C} \), sharing a contact graph \( G \), with \( D_i \in \mathcal{D} \) corresponding to \( \tilde{D}_i \in \tilde{\mathcal{D}} \) for all \( 1 \leq i \leq n \). A subset \( I \subset \{1, \ldots, n\} \) is called subsumptive if

- either \( D_i \subset \tilde{D}_i \) for every \( i \in I \), or \( \tilde{D}_i \subset D_i \) for every \( i \in I \), and
- the set \( \cup_{i \in I} D_i \) is connected, equivalently the set \( \cup_{i \in I} \tilde{D}_i \) is connected.

Let \( I \) be a subsumptive subset of \( \{1, \ldots, n\} \). Then \( I \) is called isolated if there is no \( i \in I \) and \( j \in \{1, \ldots, n\} \setminus I \) so that one of \( D_i \cap D_j \) and \( \tilde{D}_i \cap \tilde{D}_j \) contains the other. The collections \( \{D_i\}_{i \in I} \) and \( \{\tilde{D}_i\}_{i \in I} \) together are called a pair of subsumptive subconfigurations of \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \). The pair is called isolated if \( I \) is isolated.

The main technical result of this article is the following theorem:

**Main Index Theorem 5.3.** Let \( \mathcal{D} = \{D_1, \ldots, D_n\} \) and \( \tilde{\mathcal{D}} = \{\tilde{D}_1, \ldots, \tilde{D}_n\} \) be finite thin disk configurations in the complex plane \( \mathbb{C} \), in general position, realizing the same incidence data \( (G, \Theta) \), with \( D_i \in \mathcal{D} \) corresponding to \( \tilde{D}_i \in \tilde{\mathcal{D}} \) for all \( 1 \leq i \leq n \). Let \( \phi : \partial \mathcal{D} \to \partial \tilde{\mathcal{D}} \) be a faithful indexable homeomorphism. Then \( \eta(\phi) \) is at least the number of maximal isolated subsumptive subsets of \( \{1, \ldots, n\} \). In particular \( \eta(\phi) \geq 0 \).

For an example, look ahead to Figure 8 on p. 23. There we know that \( \eta(\phi) \geq 1 \) for \( \phi \) satisfying the hypotheses of Theorem 5.3. We discuss possible generalizations of our Main Index Theorem 5.3 at the end of Section 12.

We will now prove Theorem 5.3 modulo four propositions. We give the complete statements of these propositions in the running text of the proof, and number them according to where they are found with their own proofs in this article.

**Proof of Theorem 5.3.** We first need to make some preliminary definitions and observations. We say that two closed disks overlap if their interiors meet. Suppose that \( D_i \neq D_j \) overlap. Then the eye between them is defined to be \( E_{ij} = E_{ji} = D_i \cap D_j \). When we quantify over the eyes \( E_{ij} \) of \( \mathcal{D} \), we keep in mind that \( E_{ij} = E_{ji} \) and treat this as a single case. The eyes of \( \tilde{\mathcal{D}} \) are defined analogously. A homeomorphism \( \epsilon_{ij} : \partial E_{ij} \to \partial \tilde{E}_{ij} \) is called faithful if it restricts to homeomorphisms \( D_i \cap \partial E_{ij} \to \tilde{D}_i \cap \partial \tilde{E}_{ij} \) and \( \tilde{D}_j \cap \partial E_{ij} \to \tilde{D}_j \cap \partial \tilde{E}_{ij} \).

We first note that for every eye \( E_{ij} \) there exists a faithful indexable homeomorphism \( \epsilon_{ij} : \partial E_{ij} \to \partial \tilde{E}_{ij} \). The only way that there could fail to exist any faithful fixed-point-free homeomorphisms \( \partial E_{ij} \to \partial \tilde{E}_{ij} \) is if a pair of corresponding points in \( \partial D_i \cap \partial D_j \) and \( \partial \tilde{D}_i \cap \partial \tilde{D}_j \) coincide, which cannot happen by the general position hypothesis on \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \). Furthermore, however they are chosen, the homeomorphisms \( \epsilon_{ij} \) agree with one another because their domains are disjoint, and every \( \epsilon_{ij} \) agrees with \( \phi \) on \( \partial E_{ij} \cap \partial \mathcal{D} \) because of the faithfulness conditions on \( \phi \) and on the \( \epsilon_{ij} \).

For every \( E_{ij} \) pick a faithful indexable \( \epsilon_{ij} \). For \( i \in \{1, \ldots, n\} \) let \( \delta_i : \partial D_i \to \partial \tilde{D}_i \) be the function induced by restricting to \( \phi \) or to the \( \epsilon_{ij} \), as necessary. It is routine to check that \( \delta_i \) defined this way is an indexable homeomorphism. The following observation serves as a good intuition builder, and will be appealed to later in our proof:

**Observation 5.4.** \( \eta(\phi) = \sum_{i=1}^{n} \eta(\delta_i) - \sum_{E_{ij}} \eta(\epsilon_{ij}) \)
The second sum is taken over all eyes $E_{ij}$ of $\mathcal{D}$. This observation follows from the Index Additivity Lemma \ref{lem:3.3} notice that $\eta(\epsilon_{ij})$ is exactly double-counted in the sum $\eta(\delta_i) + \eta(\delta_j)$. We remark now that as we hope to prove in particular that $\eta(\phi) \geq 0$, one of our main strategies will be to try to get $\epsilon_{ij}$ so that $\eta(\epsilon_{ij}) = 0$. Recall that we always have $\eta(\delta_i) \geq 0$ by the Circle Index Lemma \ref{lem:3.2}.

If $I \subset \{1, \ldots, n\}$ then let $\mathcal{D}_I = \{D_i : i \in I\}$, similarly $\tilde{\mathcal{D}}_I$. We denote by $\phi_I : \partial \mathcal{D}_I \to \partial \tilde{\mathcal{D}}_I$ the function obtained by restriction to $\phi$ or to the $\epsilon_{ij}$, as necessary. Then $\phi_I$ is a faithful indexable homeomorphism. We make another observation.

**Observation 5.5.** Let $I, J \subset \{1, \ldots, n\}$ be disjoint and non-empty, satisfying $I \cup J = \{1, \ldots, n\}$. Then by the Index Additivity Lemma \ref{lem:3.3} we get

$$\eta(\phi) = \eta(\phi_I) + \eta(\phi_J) - \sum \eta(\epsilon_{ij})$$

where the sum is taken over all $E_{ij}$ so that $i \in I, j \in J$.

We now proceed to the main portion of our proof. The proof is by induction on $n$ the number of disks in each of our configurations $\mathcal{D}$ and $\tilde{\mathcal{D}}$. The base case $n = 1$ follows from the Circle Index Lemma \ref{lem:3.2} so we suppose from now on that $n \geq 2$. We begin with a simplifying observation that gives us access to our main propositions:

**Observation 5.6.** Suppose that $D_j \setminus \cup_{i \neq j} D_i =: d_j$ and $\tilde{D}_j \setminus \cup_{i \neq j} \tilde{D}_i =: \tilde{d}_j$ are disjoint for some $j$. Then we are done by induction.

To see why, observe the following. First, if neither of $D_j$ and $\tilde{D}_j$ contains the other, then $j$ does not belong to any subsumptive subset of $\{1, \ldots, n\}$, so letting $I = \{1, \ldots, n\} \setminus \{j\}$, we observe that the lower bound we wish to prove on $\eta(\phi)$ is the same as the lower bound we get on $\eta(\phi_I)$ by our induction hypothesis. Then by the Index Additivity Lemma \ref{lem:3.3} we get $\eta(\phi) = \eta(\phi_I) + \eta(\partial d_j \phi_{\epsilon_{ij}} \tilde{d}_j) = \eta(\phi_I)$. Here $\partial d_j \phi_{\epsilon_{ij}} \tilde{d}_j$ denotes the indexable homeomorphism induced by restriction to $\phi$ and to the $\epsilon_{ij}$, as necessary. The fixed-point index of this homeomorphism is 0 because $d_j$ and $\tilde{d}_j$ are disjoint.

On the other hand, suppose that one of $D_j$ and $\tilde{D}_j$ contains the other. We will be done by the same argument if we can show that the number of maximal isolated subsumptive subsets of $\{1, \ldots, n\}$ is the same as the number of maximal isolated subsumptive subsets of $\{1, \ldots, n\} \setminus \{j\}$. Suppose without loss of generality that $\tilde{D}_j \subset D_j$. Because $d_j$ and $\tilde{d}_j$ are disjoint, it follows that there must be an $i \neq j$ so that $\tilde{D}_j \subset D_i$. It is also not hard to see that if $E_{jk}$ and $\tilde{E}_{jk}$ are eyes one of which contains the other, then we must have $k = i$ and $E_{ij} \subset E_{ij}$. Let $J$ be the maximal subsumptive subset of $\{1, \ldots, n\}$ containing $j$. If $\tilde{D}_i \not\subset D_i$, then $i \not\in J$, but $E_{ij} \subset E_{ij}$, so $J$ is not isolated. In fact $J = \{j\}$, so we are done by induction. To see why note that if $k \in J$ is different from $j$ then $\tilde{D}_k \subset D_k$, so $\tilde{E}_{jk} \subset E_{jk}$, so $k = i$ by the earlier discussion, contradicting $\tilde{D}_i \not\subset D_i$. So, finally, suppose that $\tilde{D}_i \subset D_i$, so $i \in J$. If $J$ fails to be isolated, then it does so at a $k \in J$ different from $j$, and $J \setminus \{j\}$ is a maximal subsumptive subset of $\{1, \ldots, n\} \setminus \{j\}$, both by the preceding argument. Thus $J \setminus \{j\}$ is isolated in $\{1, \ldots, n\} \setminus \{j\}$ if and only if $J$ is isolated in $\{1, \ldots, n\}$. An element of $\{1, \ldots, n\}$ may belong to at most one maximal subsumptive subset of $\{1, \ldots, n\}$, so once again we are done by induction. This completes the proof of Observation \ref{obs:5.6}.
We therefore assume without loss of generality via Observation 5.6 for the remainder of the proof the weaker statement that $D_j \setminus D_i$ and $\tilde{D}_j \setminus \tilde{D}_i$ meet, for all $i, j$.

The following proposition will be key in our induction step:

**Proposition 11.1.** Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of overlapping closed disks in the complex plane $\mathbb{C}$, in general position. Suppose that neither of $E = A \cap B$ and $\tilde{E} = \tilde{A} \cap \tilde{B}$ contains the other. Suppose further that $A \setminus B$ and $A \setminus \tilde{B}$ meet, and that $B \setminus A$ and $B \setminus \tilde{A}$ meet. Then there is a faithful indexable homeomorphism $\epsilon : \partial E \to \partial \tilde{E}$ satisfying $\eta(\epsilon) = 0$.

For the remainder of the proof, suppose that for every eye $E_{ij}$ of $D$, we have chosen our faithful indexable $\epsilon_{ij}$ so that $\eta(\epsilon_{ij}) = 0$ whenever neither of $E_{ij}$ and $\tilde{E}_{ij}$ contains the other, and necessarily so that $\eta(\epsilon_{ij}) = 1$ otherwise. Then for example if for no $i, j$ is it the case that one of $E_{ij}$ and $\tilde{E}_{ij}$ contains the other, then we are done by Observation 5.4. Alternatively, if there exist disjoint non-empty $I, J \subset \{1, \ldots, n\}$ so that $I \cup J = \{1, \ldots, n\}$, and so that for every $i \in I, j \in J$ we have that neither of $E_{ij}$ and $\tilde{E}_{ij}$ contains the other, then we are done by induction and Observation 5.5.

Our next key proposition is the following:

**Proposition 6.6.** Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ and $\tilde{\mathcal{D}} = \{\tilde{D}_1, \ldots, \tilde{D}_n\}$ be as in the statement of Theorem 5.3. Let $I$ be a maximal nonempty subsumptive subset of $\{1, \ldots, n\}$. Then there is at most one pair $i \in I, j \in \{1, \ldots, n\} \setminus I$ so that one of $E_{ij} = D_i \cap D_j$ and $\tilde{E}_{ij} = \tilde{D}_i \cap \tilde{D}_j$ contains the other.

This proposition says that maximal subsumptive configurations are always at least “almost” isolated, and, together with Proposition 6.5 will allow us to excise maximal subsumptive configurations from $\mathcal{D}$ and $\tilde{\mathcal{D}}$ in the style of Observation 5.5 to complete our proof by induction. We explain how in more detail shortly.

There is a potential problem: we would like to say that if $I \subset \{1, \ldots, n\}$ is subsumptive, implying that one of $\cup_{i \in I} D_i$ and $\cup_{i \in I} \tilde{D}_i$ contains the other, then $\eta(\phi_I) = 1$. However, a priori, this may fail, for example see Figure 4. Our next proposition addresses this issue:

**Proposition 6.5.** Let $n \geq 3$ be an integer. Let $\{D_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and $\{\tilde{D}_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ be thin collections of closed disks in the plane $\mathbb{C}$, in general position, so that the following conditions hold.

- We have that $\tilde{D}_i$ is contained in the interior of $D_i$ for all $i$.
- The disk $D_i$ overlaps with $D_{i+1}$, and the disk $\tilde{D}_i$ overlaps with $\tilde{D}_{i+1}$, for all $i$.
- If $D_i$ and $D_j$ meet, then $i = j \pm 1$.

Then $\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \angle(\tilde{D}_i, \tilde{D}_{i+1}) < \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \angle(D_i, D_{i+1})$. In particular, for some $i$ we must have $\angle(D_i, D_{i+1}) \neq \angle(\tilde{D}_i, \tilde{D}_{i+1})$.

Thus suppose that $I$ is a maximal nonempty subsumptive subset of $\{1, \ldots, n\}$. Then by Proposition 6.5 and the Circle Index Lemma 3.2, we have that $\cup_{i \in I} D_i$ and $\cup_{i \in I} \tilde{D}_i$ are closed Jordan domains, so $\eta(\phi_I) = 1$. Let $J = \{1, \ldots, n\} \setminus I$. If $I$ is isolated in $\{1, \ldots, n\}$, then every $\epsilon_{ij} = 0$, with $i \in I$ and $j \in J$. Also $J$ has one fewer maximal isolated subsumptive subset than does $\{1, \ldots, n\}$. Thus we are done by induction and Observation 5.5. On the other hand, suppose that $I$ is not isolated. Then it is not hard to see that $J$ has as many maximal isolated subsumptive subsets as does $\{1, \ldots, n\}$. Also, by Proposition 6.6, there is
Figure 4. Two closed chains of disks with $\bar{D}_i \subsetneq D_i$ for all $i$. The $D_i$ are drawn solid and the $\bar{D}_i$ dashed. Proposition 6.5 implies that $\angle(\bar{D}_i, \bar{D}_{i+1}) \neq \angle(D_i, D_{i+1})$ for some $i$.

exactly one eye $E_{ij}$ with $i \in I$ and $j \in J$ so that $\eta(e_{ij}) = 1$, and for all the others we have $\eta(e_{ij}) = 0$. Again, we are done by induction and Observation 5.5.

We now state our final key proposition in the proof of Theorem 5.3:

**Proposition 11.5.** Let $D = \{D_1, \ldots, D_n\}$ and $\bar{D} = \{\bar{D}_1, \ldots, \bar{D}_n\}$ be as in the statement of Theorem 5.3, and so that for all $i, j$ the sets $D_i \setminus D_j$ and $\bar{D}_i \setminus \bar{D}_j$ meet. Suppose that there is no $i$ so that one of $D_i$ and $\bar{D}_i$ contains the other. Suppose that for every pair of disjoint non-empty subsets $I, J \subset \{1, \ldots, n\}$ so that $I \cup J = \{1, \ldots, n\}$, there exists an eye $E_{ij}$, with $i \in I$ and $j \in J$, so that one of $E_{ij}$ and $\bar{E}_{ij}$ contains the other. Then for every $i$ we have that any faithful indexable homeomorphism $\delta_i : \partial D_i \to \partial \bar{D}_i$ satisfies $\eta(\delta_i) \geq 1$. Furthermore there is a $k$ so that $D_i$ and $D_k$ overlap for all $i$, and so that one of $E_{ij}$ and $\bar{E}_{ij}$ contains the other if and only if either $i = k$ or $j = k$.

Unless one of our earlier propositions has already finished off the proof of Theorem 5.3 by induction, the hypotheses of Proposition 11.5 hold, and we are done by Observation 5.4.

We need to establish Propositions 6.5, 6.6, 11.1, and 11.5. We establish Propositions 6.5 and 6.6 next, in Section 6. Their proofs are quick and elementary, and some ingredients of their proofs are used in the proofs of our main rigidity theorems. We then prove our main rigidity theorems in Section 7 using Theorem 5.3. The proofs of Propositions 11.1 and 11.5 take up most of the rest of the article.

6. Subsumptive collections of disks

In this section we prove some lemmas, and Propositions 6.5 and 6.6, having to do with subsumptive configurations of disks.

First, we establish some geometric facts, starting with the following important observation, which is illustrated in Figure 5.
Observation 6.1. Suppose that $D_1, D_2, D_3, D_4$ are metric closed disks in $\mathbb{C}$, so that there is a bounded connected component $U$ of $\mathbb{C} \setminus \bigcup_{i=1}^4 D_i$ which is a curvilinear quadrilateral, whose boundary decomposes as the union of four circular arcs, one taken from each of $\partial D_1, \partial D_2, \partial D_3, \partial D_4$. Suppose that as we traverse $\partial U$ positively, we arrive at $\partial D_1, \partial D_2, \partial D_3, \partial D_4$ in that order. Then $\sum_{i=1}^4 \angle(D_i, D_{i+1}) < 2\pi$, where we consider $D_5 = D_1$.

We use Observation 6.1 to prove the following key lemma, illustrated in Figure 6.

Lemma 6.2. Let $d_{-1}, d_{+1}, D, \tilde{D}$ be closed disks in $\mathbb{C}$, so that $\tilde{D}$ is contained in the interior of $D$, so that both of $D$ and $\tilde{D}$ meet both of $d_{-1}$ and $d_{+1}$, and so that $d_{-1} \cap d_{+1} \cap D$ is empty. Suppose that neither of $d_{-1}$ and $d_{+1}$ is contained in $D$. We denote $\theta_{-1} = \angle(D, d_{-1})$ and $\tilde{\theta}_{-1} = \angle(\tilde{D}, d_{-1})$, defining $\theta_{+1}$ and $\tilde{\theta}_{+1}$ analogously. Then $\tilde{\theta}_{-1} + \theta_{+1} < \theta_{-1} + \theta_{+1}$.

Proof. Suppose first that $d_{-1}$ and $d_{+1}$ are disjoint, as in Figure 6. Let $z$ be a point in the interior of $D \setminus (\tilde{D} \cup d_{-1} \cup d_{+1})$, and let $m$ be a Möbius transformation sending $z$ to $\infty$. Then $m$ inverts the disk $D$ but none of the disks $\tilde{D}, d_{-1}, d_{+1}$. Because $m$ preserves angles we get $(\pi - \theta_{-1}) + (\pi - \theta_{+1}) + \tilde{\theta}_{-1} + \theta_{+1} < 2\pi$ by Observation 6.1 and the desired inequality follows. The case where $d_{-1}$ and $d_{+1}$ meet outside of $D$ is proved identically.

The following follows as a corollary of Lemma 6.2 by applying a suitable Möbius transformation:

Lemma 6.3. Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of overlapping closed disks in the plane $\mathbb{C}$, in general position, so that $\angle(A, B) = \angle(\tilde{A}, \tilde{B})$. Suppose that $\tilde{A}$ is contained in the interior of $\tilde{B}$.

Figure 5. A complementary component of the union of four disks as in Observation 6.1. The sum of the angles inside of the dashed honest quadrilateral is exactly $2\pi$. This sum is greater than the sum of the external intersection angles of the disks.

Figure 6. A Möbius transformation chosen to prove Lemma 6.2.
A and that $\tilde{B}$ is contained in the interior of $B$. Suppose also that neither $\tilde{A} \subset B$ nor $\tilde{B} \subset A$. Then $2\angle(A, B) = 2\angle(\tilde{A}, \tilde{B}) < \angle(A, B) + \angle(A, \tilde{B})$.

In particular, it works to apply a Möbius transformation sending a point in the interior of $B \setminus (A \cup \tilde{A} \cup \tilde{B})$ to $\infty$.

We proceed to our final preliminary geometric lemma, illustrated in Figure 7:

**Lemma 6.4.** Let $A$, $B$, $C$ be closed disks, none of which is contained in any other. Suppose that $A$ and $C$ overlap, with $A \cap C \subset B$. Then $A$ and $B$ overlap, with $\angle(A, C) < \angle(A, B)$.

**Proof.** Let $z \in \partial A \setminus B$. Because of the hypothesis that $A \cap C \subset B$, we have that $z \not\in C$. Apply a Möbius transformation sending $z \mapsto \infty$ so that $A$ becomes the left half-plane. Because $z \not\in B$, we have that $B$ and $C$ remain closed disks after this transformation. Let $C'$ be the closed disk so that $\angle(A, C') = \angle(A, B)$, and so that $C$ and $C'$ have the same Euclidean radius and the same vertical Euclidean coordinate. Then $C' \subset A$. Also, notice that $C$ is obtained from $C'$ by a translation to the right or to the left. In fact it must be a translation to the right, because the points $\partial B \cap \partial C$ must lie in the complement of $A$, which is the right-half plane. But we see that $\angle(A, C')$ is monotone decreasing as $C'$ slides to the right. □

We now proceed to the proofs of Propositions 6.5 and 6.6. We restate them here for the convenience of the reader. Our first proposition was illustrated in Figure 4 on p. 20:

**Proposition 6.5.** Let $n \geq 3$ be an integer. Let $\{D_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and $\{\tilde{D}_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ be thin collections of closed disks in the plane $\mathbb{C}$, in general position, so that the following conditions hold.

- We have that $\tilde{D}_i$ is contained in the interior of $D_i$ for all $i$.
- The disk $D_i$ overlaps with $D_{i \pm 1}$, and the disk $\tilde{D}_i$ overlaps with $\tilde{D}_{i \pm 1}$, for all $i$.
- If $D_i$ and $D_j$ meet, then $i = j \pm 1$.

Then $\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \angle(\tilde{D}_i, \tilde{D}_{i+1}) < \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \angle(D_i, D_{i+1})$. In particular, for some $i$ we must have $\angle(D_i, D_{i+1}) \neq \angle(\tilde{D}_i, \tilde{D}_{i+1})$.

**Proof.** Note first that for $\angle(D_i, D_{i+1})$ to be well-defined, we still need to show that neither $D_i \subset D_{i+1}$ nor $D_{i+1} \subset D_i$. The same is true for $\angle(\tilde{D}_i, \tilde{D}_{i+1})$. Suppose for contradiction that $D_i \subset D_{i+1}$. Then $D_{i-1} \cap D_i \subset D_{i+1}$, contradicting our hypotheses. By symmetry we get that $D_{i+1} \not\subset D_i$. The proof that $\angle(\tilde{D}_i, \tilde{D}_{i+1})$ is well-defined is identical.
Figure 8. The directed graph $H$ associated to a maximal subsumptive subconfigurations. The solid disks are the $D_i$ and the dashed disks are the $\tilde{D}_i$. The graph $H_u$ can be obtained by undirecting every edge.

To finish off the proof, we apply Lemma 6.2 twice. In both cases we will let $D = D_i$ and $\tilde{D} = \tilde{D}_i$. First let $d_{-1} = D_{i-1}$ and $d_{+1} = D_{i+1}$. This gives:

\[(1) \quad \angle(D_{i-1}, \tilde{D}_i) + \angle(D_{i+1}, \tilde{D}_i) < \angle(D_{i-1}, D_i) + \angle(D_{i+1}, D_i)\]

Next let $d_{-1} = \tilde{D}_{i-1}$ and $d_{+1} = \tilde{D}_{i+1}$. This gives:

\[(2) \quad \angle(\tilde{D}_{i-1}, \tilde{D}_i) + \angle(\tilde{D}_{i+1}, \tilde{D}_i) < \angle(D_i, \tilde{D}_{i-1}) + \angle(D_i, \tilde{D}_{i+1})\]

If we let $i$ range over $\mathbb{Z}/n\mathbb{Z}$, the sum of the terms on the left side of equation 1 is equal to the sum of the terms on the right side of equation 2. The desired inequality follows. \[\square\]

**Proposition 6.6.** Let $D = \{D_1, \ldots, D_n\}$ and $\tilde{D} = \{\tilde{D}_1, \ldots, \tilde{D}_n\}$ be as in the statement of Theorem 5.3, configurations in $\mathbb{C}$ which are thin and in general position, realizing the same incidence data. Suppose there is some pair $D_i$ and $\tilde{D}_i$, one of which contains the other. Let $I$ be a maximal nonempty subsumptive subset of $\{1, \ldots, n\}$. Then there is at most one pair $i, j \in \{1, \ldots, n\} \setminus I$ so that $D_i$ and $D_j$ overlap and one of $E_{ij} = D_i \cap D_j$ and $\tilde{E}_{ij} = \tilde{D}_i \cap \tilde{D}_j$ contains the other.

**Proof.** Suppose from now on, without loss of generality, that $\tilde{D}_i \subset D_i$ for all $i \in I$.

First, let $H_u$ be the undirected simple graph defined as follows: the vertex set is $I$, and there is an edge between $i$ and $j$ if and only if $D_i$ and $D_j$ overlap. Observe:

**Observation 6.7.** The graph $H_u$ is connected and is a tree.

This follows from Proposition 5.3 and the general position hypothesis.

Next, let $H$ be the directed graph so that $\langle i \to j \rangle$ is an edge of $H$ if and only if:

- we have that $\langle i, j \rangle$ is an edge of $H_u$, and
- either $\angle(\tilde{D}_i, D_j) > \angle(D_i, D_j)$ or $\tilde{D}_i \subset D_j$.

If $\langle i \to j \rangle$ is an edge of $H$ then we call $\langle i \to j \rangle$ an edge pointing away from $i$ in $H$. The idea is that if $\langle i \to j \rangle$ is an edge in $H$ then the disk $\tilde{D}_i \subset D_i$ is “shifted towards $D_j$ in $D_i$.” See Figure 8 for an example.

We now make a series of observations about $H$ and $H_u$. First:

**Observation 6.8.** If $\langle i, j \rangle$ is an edge in $H_u$ then at least one of $\langle i \to j \rangle$ and $\langle j \to i \rangle$ is an edge in $H$, and possibly both are.
This follows from Lemma 6.3.

**Observation 6.9.** For every \( i \in I \), there is at most one edge \( \langle i \to j \rangle \) in \( H \) pointing away from \( i \).

This follows from Lemma 6.2 with \( D = D_i, \tilde{D} = \tilde{D}_i, d_{-1} = D_j, d_{+1} = D_k \), for \( j, k \in I \) so that \( D_i \) overlaps with both \( D_j \) and \( D_k \).

**Observation 6.10.** Let \( \langle i_1, i_2, \ldots, i_m \rangle \) be a simple path in \( H_u \), meaning that \( \langle i_\ell, i_{\ell+1} \rangle \) is an edge in \( H_u \) for all \( 1 \leq \ell < m \) and that \( i_\ell \) and \( i_{\ell'} \) are distinct for \( \ell \neq \ell' \). Suppose that \( \langle i_m-1 \to i_m \rangle \) is an edge in \( H \). Then \( \langle i_\ell \to i_{\ell+1} \rangle \) is an edge in \( H \) for \( 1 \leq \ell < m \).

This follows from Observations 6.8 and 6.9 and induction.

**Observation 6.11.** There is at most one \( i \in I \) so that there is no edge pointing away from \( i \) in \( H \).

This follows from Observations 6.8, 6.9, and 6.10 because \( H_u \) is connected. If there is an \( i \) as in the statement of Observation 6.11 then we call this \( i \) the sink of the subsumptive subset \( I \subset \{1, \ldots, n\} \).

Having established all we need to about \( H \), we are ready to make two final observations which will complete the proof of Proposition 6.6. First:

**Observation 6.12.** Let \( i \in I \). Then there is at most one \( 1 \leq j \leq n \) different from \( i \) so that \( D_i \) and \( D_j \) overlap and either \( \tilde{D}_i \subset D_j \) or \( \angle(D_i, D_j) < \angle(\tilde{D}_i, D_j) \).

This follows from Lemma 6.2 in the same way as does Observation 6.9. Next:

**Observation 6.13.** Suppose that \( i \) and \( j \) are as in the last sentence of the statement of Proposition 6.6. Then we have that \( i \in I \) and \( j \in \{1, \ldots, n\} \setminus I \) so that \( D_i \) and \( D_j \) overlap and \( \tilde{E}_{ij} \subset E_{ij} \). Then \( \angle(D_i, D_j) = \angle(\tilde{D}_i, \tilde{D}_j) < \angle(\tilde{D}_i, D_j) \).

This follows by an application of Lemma 6.4 with \( \tilde{D}_i = A, D_j = B, \) and \( \tilde{D}_j = C \). Thus if \( i \) and \( j \) are as in the statement of Proposition 6.6 then \( i \) is the unique sink of \( H \). Furthermore by Observations 6.12 and 6.13 there is no \( k \in \{1, \ldots, n\} \setminus I \) different from \( j \) so that \( D_i \) and \( D_k \) overlap and so that one of \( E_{ik} \) and \( \tilde{E}_{ik} \) contains the other. Proposition 6.6 follows. □

The following lemmas will be helpful in the next section, and it is best to get them out of the way now:

**Lemma 6.14.** Let \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \) be as in the statement of Proposition 6.6. Suppose the disks \( D_i \in \mathcal{D} \) and \( \tilde{D}_i \in \tilde{\mathcal{D}} \) are so that \( \tilde{D}_i \) is contained in the interior of \( D_i \). Suppose finally that for every \( j \in \{1, \ldots, n\} \) different from \( i \), so that \( D_i \) and \( D_j \) meet, we have either that \( \tilde{D}_i \) is disjoint from \( D_j \), or that \( \angle(\tilde{D}_i, D_j) < \angle(D_i, D_j) \). Then \( i \) is the unique sink of some maximal isolated subsumptive subset of \( \{1, \ldots, n\} \).

Lemma 6.14 is really just an observation. Let \( I \) be the subsumptive subset of \( \{1, \ldots, n\} \) containing \( i \). Define the directed graph \( H \) as in the proof of Proposition 6.6. Then by definition of \( H \) there is no edge pointing away from \( i \) in \( H \).

The next lemma is an easy corollary of Lemma 6.14.
Lemma 6.15. Let $D$ and $\tilde{D}$ be as in the statement of Proposition 6.6. Suppose that the disks $D_i \in D$ and $\tilde{D}_i \in \tilde{D}$ have coinciding Euclidean centers. Then $i$ is the unique sink of some maximal isolated subsumptive subset of $\{1, \ldots, n\}$.

Proof. Note that by the general position hypothesis the disks $D_i$ and $\tilde{D}_i$ cannot be equal. We may suppose without loss of generality in our proof that $\tilde{D}_i \subset D_i$. Then the lemma follows from Lemma 6.14 because it is easy to see that if closed disks $A$ and $B$ in $\hat{\mathbb{C}}$ overlap, so that neither is contained in the other, then $\angle(A, B)$ is monotone decreasing as we shrink $B$ by a contraction about its Euclidean center. □

7. Proofs of our main rigidity theorems

In this section we prove our main rigidity results using our Main Index Theorem 5.3. The main idea of the proofs we will see here is similar to the main idea of the proofs of the circle packing rigidity theorems given in Section 4. It may be helpful to review those now. It may also be helpful to recall Definition 1.3 on p. 3. The normalizations we construct were inspired by those of Merenkov, given in [Mer12, Section 12].

The following lemma will be implicit in much of our discussion below:

Lemma 7.1. Let $G = (V, E)$ be a 3-cycle, and $\Theta : G \to [0, \pi)$. Then there is precisely one triple $\{D_v\}_{v \in V}$ of disks in $\hat{\mathbb{C}}$, up to action by Möbius transformations, realizing the incidence data $(G, \Theta)$.

This is not hard to prove, and we leave it as an exercise.

The first rigidity theorem we prove here is Theorem 1.5, restated here for the reader’s convenience:

Theorem 1.5. Let $C$ and $\tilde{C}$ be thin disk configurations in $\hat{\mathbb{C}}$ realizing the same incidence data $(G, \Theta)$, where $G$ is the 1-skeleton of a triangulation of the 2-sphere $S^2$. Then $C$ and $\tilde{C}$ differ by a Möbius or an anti-Möbius transformation.

Proof. We begin by applying $z \mapsto \bar{z}$ to one of the configurations, if necessary, to ensure that the geodesic embeddings of $G$ in $\hat{\mathbb{C}}$ induced by $P$ and $\tilde{P}$ differ by an orientation-preserving self-homeomorphism of $\hat{\mathbb{C}}$. For a reminder of the meaning of geodesic embedding, see the proof of Theorem 4.2 on p. 12. The proof then proceeds by contradiction, supposing that there is no Möbius transformation identifying $C$ and $\tilde{C}$.

First, note that we may suppose without loss of generality that there is a vertex $a$ of $G$ so that no disk of $C \setminus \{D_a\}$ overlaps with $D_a$, and that every contact between $D_a$ and another disk $D_b \in C$ is a tangency. Then necessarily the same holds for $\tilde{D}_a$. Every face $f$ of the triangulation of $\tilde{C}$ coming from the geodesic embedding of $G$ induced by $C$ contains exactly one interstice. Index these faces by $F$, and write $T_f$ to denote the interstice of $C$ contained in the face corresponding to $f \in F$. We define the interstices $\tilde{T}_f$ of $\tilde{C}$ analogously. Pick an interstice $T_f$ of $C$, and let $D$ be the metric closed disk of largest spherical radius whose interior fits inside of $T_f$. Let $\tilde{D}$ be constructed analogously for the corresponding interstice $\tilde{T}_f$ of $\tilde{C}$. Each of the disks $D$ and $\tilde{D}$ is internally tangent to all three sides of its respective interstice $T_f$ or $\tilde{T}_f$. Then it is not hard to show using Lemma 7.1 that any Möbius transformation sending $C$ to $\tilde{C}$ will send $T_f$ to $\tilde{T}_f$, thus also $D$ to $\tilde{D}$, so $C$ and $\tilde{C}$ are Möbius
equivalent if and only if $\mathcal{C} \cup \{D\}$ and $\tilde{\mathcal{C}}\cup \{\tilde{D}\}$ are. It is therefore harmless to add $D$ and $\tilde{D}$ to our configurations if necessary.

If there is no disk $D_b \in \mathcal{C}$ which does not meet $D_a$, then $G$ is the 1-skeleton of a tetrahedron and it is easy to check Theorem 1.5 by hand using Lemma 7.1. Thus suppose that $D_b \in \mathcal{C}$ is disjoint from $D_a$.

We now apply a series of Möbius transformations, to explicitly describe a normalization on $\mathcal{C}$ and $\tilde{\mathcal{C}}$ in terms of one non-negative real parameter $\varepsilon \geq 0$:

1. First ensure that $\infty$ lies in the interiors of $D_a$ and of $D_a$, so that the circles $\partial D_a$ and $\partial D_b$ are concentric when considered in $\mathbb{C}$, and so that $\partial \tilde{D}_a$ and $\partial \tilde{D}_b$ are concentric when considered in $\mathbb{C}$. Apply orientation-preserving Euclidean similarities so that $D_b$ and $\tilde{D}_b$ are both equal to the closed unit disk $\bar{D}$. Then the Euclidean centers in $\mathbb{C}$ of the circles $\partial D_a, \partial D_b, \partial \tilde{D}_a, \partial \tilde{D}_b$ all coincide. The disks $D_b$ and $\tilde{D}_b$ are equal, and the disks $D_a$ and $\tilde{D}_a$ may be equal or unequal.

2. Pick a vertex $c$ of $G$, so that the disks $D_c$ and $\tilde{D}_c$ differ either in Euclidean radii or in the distances of their Euclidean centers from the origin, or both. Such a $c$ must certainly exist: for instance, if $\tilde{D}_a \subset D_a$, then we may pick $\tilde{D}_c$ meeting $D_a$. If $D_a = \tilde{D}_a$ and such a $c$ did not exist then we could argue via Lemma 7.1 that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ differ by a rotation, after all of the normalizations applied thus far.

3. Apply a rotation to both packings so that the Euclidean centers of $D_c$ and $\tilde{D}_c$ both lie on the positive real axis. Then apply a positive non-trivial dilation about the origin to one of the two packings, so that the Euclidean centers of $D_c$ and $\tilde{D}_c$ coincide.

4. At this point, the Euclidean centers of $\partial D_a$ and $\partial \tilde{D}_a$ coincide, for all $v = a, b, c$. Because we applied a non-trivial positive dilation to one of the packings in the previous step, we have that $\partial D_b \neq \partial \tilde{D}_b$. For either $v = a, c$, the disks $D_v$ and $\tilde{D}_v$ may be equal or unequal. Regardless, our final step is to apply a dilation to $\mathcal{P}$ by $1 + \varepsilon$ about the common Euclidean center of $\partial D_c$ and $\partial \tilde{D}_c$. Call the resulting normalization $N(\varepsilon)$.

Note that there clearly is an open interval $(0, \ldots)$, having one of its endpoints at 0, of positive values that $\varepsilon$ may take so that after applying $N(\varepsilon)$, we have that one of $D_v$ and $\tilde{D}_v$ is contained in the interior of the other, for all $v = a, b, c$. For only finitely many of these values is it the case that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ fail to end up in general position.

Denote $\mathcal{D} = \mathcal{C} \setminus \{D_a\}$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{C}} \setminus \{\tilde{D}_a\}$. Then $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are thin disk configurations in $\mathbb{C}$ realizing the same incidence data. Let $G_\mathcal{D}$ denote their common contact graph, having vertex set $V_\mathcal{D}$. Then we have the following:

**Observation 7.2.** If $\varepsilon > 0$ is sufficiently small, then after applying $N(\varepsilon)$, we have that each of $b$ and $c$ belongs to a different maximal isolated subsumptive subset of the common index set $V_\mathcal{D}$ of $\mathcal{D}$ and $\tilde{\mathcal{D}}$.

To see why, first note that $D_b$ and $\tilde{D}_b$ are unequal and concentric in $\mathcal{C}$ under the normalization $N(0)$. Thus by the argument we used to prove Lemma 6.15 for any $v \in V_\mathcal{D}$ so that $D_v$ and $\tilde{D}_b$ meet, we have that either $D_b$ is disjoint from $D_v$ or $\angle(D_b, D_v) < \angle(D_b, \tilde{D}_v)$. If we consider all of the disks to vary under $N(\varepsilon)$, then these angles are continuous in the variable $\varepsilon$, so for some small interval $[0, \ldots)$ they continue to hold. For all $\varepsilon$ in this interval $b$ will be the unique sink of a maximal isolated subsumptive subset of $V_\mathcal{D}$ by Lemma 6.14. Next, recall that $D_c$ and $\tilde{D}_c$ are concentric under any $N(\varepsilon)$, and are unequal for all but one value $\varepsilon$.  

Then \( c \) is the unique sink of a maximal isolated subsumptive subset of \( V_D \) by Lemma 6.15 and Observation 7.2 follows.

We are now ready to obtain the desired contradiction to complete the proof of Theorem 1.5. Pick \( \varepsilon > 0 \) sufficiently small as per Claim 7.2 so that in addition \( C \) and \( \tilde{C} \) are in general position, and so that one of \( D_v \) and \( \tilde{D}_v \) is contained in the interior of the other for all \( v = a, b, c \). For every pair of corresponding interstices \( T_f \) and \( \tilde{T}_f \) of the packings \( C \) and \( \tilde{C} \), let \( \phi_f : \partial T_f \to \partial \tilde{T}_f \) be an indexable homeomorphism identifying corresponding corners, satisfying \( \eta(\phi_f) \geq 0 \). We may do so by the Three Point Prescription Lemma 3.3. Then the \( \phi_f \) induce a faithful indexable homeomorphism \( \phi_D : \partial D \to \partial \tilde{D} \). By our choice of \( \varepsilon \) and Claim 7.2 and by our Main Index Theorem 5.3, we have that \( \eta(\phi_D) \geq 2 \). On the other hand, orient \( \partial D_a \) and \( \partial \tilde{D}_a \) positively with respect to the open disks they bound in \( \mathbb{C} \). This is the opposite of the positive orientation on them with respect to \( D_a \) and \( \tilde{D}_a \). Then \( \eta(\phi_a) = 1 \) by the Circle Index Lemma 5.2. Then we get a contradiction, because \( \eta(\phi_a) = \eta(\phi_D) + \sum_{f \in \mathcal{F}} \eta(\phi_f) \) by the Index Additivity Lemma 3.3 and \( \eta(\phi_f) \geq 0 \) for all \( f \) by construction. \( \square \)

We next prove our Main Uniformization Theorem 1.4. We break the statement of Theorem 1.4 into three theorems, and prove each of these separately. The proofs are adapted from the proof of the constituent theorems of Theorem 1.2 were adapted from the proof of Theorem 1.1, so we will not give the full details. Instead, we will construct in detail the appropriate normalization to start the proof, and omit the last part of each proof, where the contradiction is obtained.

**Theorem 7.3.** There do not exist thin disk configurations \( C \) and \( \tilde{C} \) realizing the same incidence data \( (G, \Theta) \), where \( G \) is the 1-skeleton of a triangulation of a topological open disk, so that \( C \) is locally finite in \( \mathbb{C} \) and \( \tilde{C} \) is locally finite in the open unit disk \( \mathbb{D} \), equivalently the hyperbolic plane \( \mathbb{H}^2 \cong \mathbb{D} \).

**Proof.** This proof proceeds by contradiction, supposing that \( C \) is locally finite in \( \mathbb{C} \) and \( \tilde{C} \) is locally finite in \( \mathbb{D} \). Apply \( z \mapsto \bar{z} \) to one of the configurations, if necessary, to ensure that the geodesic embeddings of \( G \) in \( \mathbb{C} \) and \( \mathbb{D} \) induced by \( P \) and \( \tilde{P} \) respectively differ by an orientation-preserving homeomorphism \( \mathbb{C} \to \mathbb{D} \). We now apply a series of orientation-preserving Euclidean similarities, to explicitly describe a normalization on \( C \) and \( \tilde{C} \) in terms of one non-negative real parameter \( \varepsilon \geq 0 \):

1. First, pick \( D_a \in C \) and \( \tilde{D}_a \in \tilde{C} \), and apply translations to both configurations, and a scaling to \( C \), so that \( D_a \) and \( \tilde{D}_a \) coincide, and are centered at the origin.
2. Pick disks \( D_b \in C \) and \( \tilde{D}_b \in \tilde{C} \) which differ either in their Euclidean radii or in the distances of their Euclidean centers from the origin, or both. We may obviously do so. Apply a rotation about the origin to both configurations so that the Euclidean centers of \( D_b \) and \( \tilde{D}_b \) both lie on the positive real axis, and then apply a non-trivial dilation about the origin to one of the configurations so that the Euclidean centers of \( D_b \) and \( \tilde{D}_b \) coincide.
3. At this point \( D_a \) and \( \tilde{D}_a \) are unequal, but are concentric in \( C \), and \( D_b \) and \( \tilde{D}_b \) are concentric in \( \mathbb{C} \), and may be equal or unequal. As our last step, we dilate \( P \) by a factor of \( 1+\varepsilon \) about the common Euclidean center of \( D_b \) and \( \tilde{D}_b \). Denote the resulting normalization \( N(\varepsilon) \).

The rest of the proof proceeds in the same way as did the proof of Theorem 1.4. \( \square \)
**Theorem 7.4.** Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be thin disk configurations realizing the same incidence data $(G, \Theta)$, where $G$ is the 1-skeleton of a triangulation of a topological open disk, so that both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are locally finite in $\mathbb{C}$. Then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ differ by a Euclidean similarity.

Proof. Apply $z \mapsto \bar{z}$ to one of the configurations, if necessary, to ensure that the geodesic embeddings of $G$ in $\mathbb{C}$ induced by $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by an orientation-preserving self-homeomorphism of $\mathbb{C}$. Suppose for contradiction that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ do not differ by any orientation-preserving Euclidean similarity. They therefore do not differ by any Möbius transformation. We now apply a series of Möbius transformations, to explicitly describe a normalization on $\mathcal{C}$ and $\tilde{\mathcal{C}}$ in terms of one non-negative real parameter $\varepsilon \geq 0$:

1. We first argue as in the proof of Theorem 7.5 that we may assume without loss of generality that we may take $D_a \in \mathcal{C}$ and $\tilde{D}_a \in \tilde{\mathcal{C}}$ so that no other disk $D_v \in \mathcal{C} \setminus \{D_a\}$ overlaps with $D_a$.

2. Pick $b \in V \setminus \{a\}$ so that $b$ does not share an edge with $a$ in $G$. Apply Möbius transformations so that $\infty$ lies in the interiors of both $D_a$ and $\tilde{D}_a$, and so that all of the circles $\partial D_a, \partial D_b, \partial \tilde{D}_a, \partial \tilde{D}_b$ have their Euclidean centers at the origin. Apply a Euclidean scaling to $\mathcal{C}$ so that $D_b$ and $\tilde{D}_b$ coincide. At this point, the disks $D_a$ and $\tilde{D}_a$ may be equal or unequal.

3. We argue as before that we may pick $c \in V \setminus \{a, b\}$ so that $D_c$ and $\tilde{D}_c$ differ in their Euclidean radii, the distances of their Euclidean centers from the origin, or both. Apply rotations about the origin so that the Euclidean centers of $D_c$ and $\tilde{D}_c$ lie on the positive real axis, and apply a scaling to $\mathcal{C}$ so that the Euclidean centers of $D_c$ and $\tilde{D}_c$ coincide.

4. At this point the disks $D_a$ and $\tilde{D}_a$ may be equal or unequal, the disks $D_b$ and $\tilde{D}_b$ are unequal, and the disks $D_c$ and $\tilde{D}_c$ may be equal or unequal. All of $\partial D_a, \partial D_b, \partial \tilde{D}_a, \partial \tilde{D}_b$ are centered at the origin, and $D_a$ and $\tilde{D}_a$ are concentric in $\mathbb{C}$. As the last step of our normalization, apply a dilation by a factor of $1 + \varepsilon$ about the common Euclidean center of $D_b$ and $\tilde{D}_b$ to $\mathcal{C}$. Denote the resulting normalization $N(\varepsilon)$.

The rest of the proof proceeds in the same way as the proof of Theorem 7.4. □

**Theorem 7.5.** Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be thin disk configurations realizing the same incidence data $(G, \Theta)$, where $G$ is the 1-skeleton of a triangulation of a topological open disk, so that both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are locally finite in the open unit disk $\mathbb{D}$, equivalently in the hyperbolic plane $\mathbb{H}^2 \cong \mathbb{D}$. Then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ differ by a hyperbolic isometry.

Proof. As always, apply $z \mapsto \bar{z}$ to one of the configurations if necessary, to ensure that the geodesic embeddings of $G$ in $\mathbb{D}$ induced by $\mathcal{P}$ and $\tilde{\mathcal{P}}$ differ by an orientation-preserving self-homeomorphism of $\mathbb{D}$. Suppose for contradiction that $\mathcal{C}$ and $\tilde{\mathcal{C}}$ do not differ by any orientation-preserving hyperbolic isometry. They therefore do not differ by any Möbius transformation. We now apply a series of Möbius transformations, to explicitly describe a normalization on $\mathcal{C}$ and $\tilde{\mathcal{C}}$ in terms of one non-negative real parameter $\varepsilon \geq 0$:

1. First, there must be disks $D_a \in \mathcal{C}$ and $\tilde{D}_a \in \tilde{\mathcal{C}}$ which have different hyperbolic radii in $\mathbb{D} \cong \mathbb{H}^2$, otherwise the two configurations coincide by elementary arguments. Apply hyperbolic isometries to both configurations so that $D_a$ and $\tilde{D}_a$ are centered at the origin, and apply a Euclidean scaling centered at the origin to $\mathcal{C}$, so that $D_a$ and $\tilde{D}_a$ coincide.
Pick disks $D_b \in C$ and $\tilde{D}_b \in \tilde{C}$ which differ in their Euclidean radii, or the distances of their Euclidean centers from the origin, or both. Apply rotations centered at the origin so that the Euclidean centers of $D_b$ and $\tilde{D}_b$ lie on the positive real axis, and apply a Euclidean scaling centered at the origin to one configuration so that the Euclidean centers of $D_b$ and $\tilde{D}_b$ coincide.

At this point the disks $D_a$ and $\tilde{D}_a$ are unequal, and are concentric in $C$, the disks $D_b$ and $\tilde{D}_b$ are concentric in $\tilde{C}$, and may be equal or unequal. Also, denoting by $D$ and $\tilde{D}$ the images of $D$ under the normalizations applied thus far to $C$ and $\tilde{C}$ respectively, we have that $D$ and $\tilde{D}$ are centered at the origin, and may be equal or unequal. As the last step of our normalization, apply a dilation by a factor of $1 + \varepsilon$ about the common Euclidean center of $D_b$ and $\tilde{D}_b$ to $C$. Denote the resulting normalization $N(\varepsilon)$.

The rest of the proof proceeds in the same way as the proof of Theorem 4.6.

8. Topological configurations

Suppose that $X_1, \ldots, X_n$ and $X'_1, \ldots, X'_n$ are all subsets of $C$. Then we say that the collections \{X_1, \ldots, X_n\} and \{X'_1, \ldots, X'_n\} are in the same topological configuration if there is an orientation-preserving homeomorphism $\varphi : C \to C$ so that $\varphi(X_i) = X'_i$ for all $1 \leq i \leq n$. In practice the collections of objects under consideration will not be labeled $X_i$ and $X'_i$, but there will be some natural bijection between the collections. Then our requirement is that $\varphi$ respects this natural bijection.

The following lemma says that when working with fixed-point index, we need to consider our Jordan domains only “up to topological configuration.”

**Lemma 8.1.** Suppose $K$ and $\tilde{K}$ are closed Jordan domains. Let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism. Suppose that $K'$ and $\tilde{K}'$ are also closed Jordan domains, so that \{K, $\tilde{K}$\} and \{K', $\tilde{K}'$\} are in the same topological configuration, via the homeomorphism $\psi : C \to C$. Let $\phi' : \partial K' \to \partial \tilde{K}'$ be induced in the natural way, explicitly as $\phi' = \psi|_{\partial K} \circ \phi \circ \psi^{-1}|_{\partial K'}$. Then $\phi'$ is indexable and $\eta(\phi) = \eta(\phi')$.

This follows via homotopy arguments from the well-known fact that every orientation-preserving homeomorphism $C \to C$ is homotopic to the identity map via homeomorphisms.

The following proposition limits the relevant topological configurations that two disks may be in to finitely many possibilities, which by Lemma 8.1 reduces every subsequent proof to at worst a case-by-case analysis:

**Proposition 8.2.** Suppose that $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ are pairs of overlapping closed disks in the plane $C$ in general position. Suppose that $A \setminus B$ meets $\tilde{A} \setminus \tilde{B}$, that $A \cap B$ meets $\tilde{A} \cap \tilde{B}$, and that $B \setminus A$ meets $\tilde{B} \setminus \tilde{A}$. Then given any three of the disks $A, B, \tilde{A}, \tilde{B}$, the topological configuration of those three disks is one of those depicted in Figures 9 and 10.

We will often make reference to the configurations depicted in Figures 9 and 10. If the appropriate three-disk subset of $\{A, B, \tilde{A}, \tilde{B}\}$ is in a topological configuration depicted in one of these figures, we will indicate this simply by saying that the corresponding configuration occurs, for example that $\Box\alpha$ occurs.
Proof of Proposition 8.2. Note that by the symmetries involved, it suffices to prove that \{A, B, \tilde{A}\} must be in one of the topological configurations on the ♦ side of Figure 9. Therefore we restrict our attention to this case from now on.

The following observation, which is an easy exercise, will be the key to our proof:

**Observation 8.3.** Fix \(\ell_1\) and \(\ell_2\) to be unequal straight lines in \(\mathbb{C}\) both of which pass through the origin. The lines \(\ell_1\) and \(\ell_2\) divide the plane into four regions, which we loosely refer to as quasi-quadrants. If \(C\) is a variable metric circle in \(\mathbb{C}\) which is not allowed to pass through the origin, nor to be tangent to either of \(\ell_1\) and \(\ell_2\), then the topological configuration of \(\{C, \ell_1, \ell_2\}\)
Figure 11. The definitions of \( u \) and \( v \) in terms of the orientations on \( \partial A \) and \( \partial B \).

Figure 12. Two different topological configurations of three disks \( \{ A, B, \tilde{A} \} \), where \( \partial \tilde{A} \) passes through the same components of \( C \setminus (\partial A \cup \partial B) \) in both cases. We see that \( \tilde{A} \) and \( A \cap B \) do not meet in either case, so this example should not worry us too much in light of the hypotheses of Proposition 8.2.

is uniquely determined by which of the four quasi-quadrants the circle \( C \) passes through. Note also that \( C \) cannot pass through two diagonally opposite quasi-quadrants without passing through at least one of the two remaining quasi-quadrants.

Then the idea of the proof of Proposition 8.2 is to apply a Möbius transformation sending one of the two points of \( \partial A \cap \partial B \) to \( \infty \). The images of the circles \( \partial A \) and \( \partial B \) will act as the lines \( \ell_1 \) and \( \ell_2 \) of Observation 8.3, and \( \partial \tilde{A} \) will act as \( C \).

We make one preliminary notational convention. First, orient \( \partial A \) and \( \partial B \) positively as usual, and let \( \{ u, v \} = \partial A \cap \partial B \). Label \( u \) and \( v \) so that \( u \) is the point of \( \partial A \cap \partial B \) where \( \partial A \) enters \( B \), and \( v \) is the point of \( \partial A \cap \partial B \) where \( \partial B \) enters \( A \). See Figure 11 for an example.

Ultimately, we would like to say that if we fix overlapping \( A \) and \( B \), letting \( \tilde{A} \) vary under the constraint that the general position hypothesis is not violated, then the topological configuration of \( \{ A, B, \tilde{A} \} \) is uniquely determined by two pieces of information:

- whether or not \( v \in \tilde{A} \), and
- which of the four regions \( A \cap B, A \setminus B, B \setminus A, C \setminus (A \cup B) \) the circle \( \partial \tilde{A} \) passes through.

Unfortunately, this is not completely true. There is a minor obstruction, illustrated in Figure 12. However, we will see that this is the only possible obstruction, and the nice classification described in this paragraph otherwise holds:

Lemma 8.4. Let \( \{ A, B \} \) and \( \{ \tilde{A}, \tilde{B} \} \) be pairs of overlapping metric closed disks in general position, similarly \( \{ A', B' \} \) and \( \{ \tilde{A}', \tilde{B}' \} \). Define \( u' \) and \( v' \) for \( A' \) and \( B' \) as we defined \( u \) and \( v \) for \( A \) and \( B \) in Figure 11. Suppose that \( v \in \tilde{A} \) if and only if \( v' \in \tilde{A}' \), and that the subset of the four regions \( A' \cap B', A' \setminus B', B' \setminus A', C \setminus (A' \cup B') \) through which \( \partial \tilde{A}' \) passes corresponds to the subset of the regions \( A \cap B, A \setminus B, B \setminus A, C \setminus (A \cup B) \) through which \( \partial \tilde{A} \) passes, in the natural way. Then \( \{ A, B, \tilde{A} \} \) and \( \{ A', B', \tilde{A}' \} \) are in the same topological configuration, unless one triple is arranged as in the left side of Figure 12 and the other is arranged as in the right side of the same figure.
Proof. Let \( m \) be a Möbius transformation sending \( v \) to \( \infty \) and \( u \) to the origin, letting \( \ell_1 = m(\partial A) \) and \( \ell_2 = m(\partial B) \). By the general position hypothesis, letting \( C = m(\partial \tilde{A}) \) we have that \( C \) is a circle as in the statement of Observation 8.3. Similarly, let \( m' \) be a Möbius transformation sending \( v' \) to \( \infty \) and \( u' \) to the origin. Then by Observation 8.3 we may apply an orientation-preserving homeomorphism \( \psi : \mathbb{C} \to \mathbb{C} \) so that \( \ell_1 = m(\partial A) = \psi \circ m'(\partial A') \), similarly \( \ell_2 = m(\partial B) = \psi \circ m'(\partial B') \) and \( C = m(\partial \tilde{A}) = \psi \circ m'(\partial \tilde{A'}) \).

Now, if we can choose \( \psi \) so that \( m(\infty) = \psi \circ m'(\infty) \), then we will be done, because \( m^{-1} \circ \psi \circ m' \) will be an orientation-preserving homeomorphism \( \mathbb{C} \to \mathbb{C} \) with \( \{A', B', \tilde{A}'\} \). Clearly there is such a \( \psi \) so long as \( \mathbb{C} \setminus m(A \cup B \cup \tilde{A}) \) equivalently \( \mathbb{C} \setminus (A' \cup B' \cup \tilde{A}') \) are connected. This happens if and only if \( \mathbb{C} \setminus (A \cup B \cup \tilde{A}) \) equivalently \( \mathbb{C} \setminus (A' \cup B' \cup \tilde{A'}) \) are connected, and it is easy to show that this fails only for the two configurations shown in Figure 12. The lemma follows.

We can now complete the proof by exhaustion. We will break the proof into two major cases, depending on whether \( v \in \tilde{A} \) or \( v \notin \tilde{A} \). We will not make reference to Observation 8.3 again, so we overload terminology, using the term quasi-quadrants from now on to refer to the four regions \( A \cap B, A \setminus B, B \setminus A, C \setminus (A \cup B) \).

The following observation will be our source of contradictions to the hypotheses of Proposition 8.2.

Observation 8.5. Suppose that the hypotheses of Proposition 8.2 hold. Then we have

- that \( \tilde{A} \) meets both \( A \setminus B \) and \( A \cap B \), and
- that \( B \setminus A \) is not contained in \( \tilde{A} \).

To see why, note that if \( \tilde{A} \) does not meet \( A \setminus B \), then \( \tilde{A} \) cannot possibly meet \( \tilde{A} \setminus B \). We must have that \( \tilde{A} \) meets \( A \cap B \) for a similar reason. Also, if \( B \setminus A \) is contained in \( \tilde{A} \), then \( B \setminus A \) cannot possibly meet \( \tilde{B} \setminus \tilde{A} \).

In the forthcoming case analysis, we will rely on the reader to supply his own drawings of the cases which we throw out by Observation 8.5. In general it is not hard to draw an example of a configuration \( \{A, B, \tilde{A}\} \) given the type of hypotheses we write down below, and once a single example is drawn Lemma 8.4 ensures that it is typically the only one, up to topological configuration.

Case 1. \( v \in \tilde{A} \)

We now consider the possibilities depending on how many of the quasi-quadrants \( \partial A \) passes through. If it passes through only one, then it is easy to see that it must be \( \mathbb{C} \setminus (A \cup B) \), otherwise we would violate \( v \in \tilde{A} \). Then \( A \cup B \subset \tilde{A} \), in particular \( B \setminus A \subset \tilde{A} \), so we may ignore this possibility by Observation 8.5.

Next, suppose that \( \partial A \) passes through exactly two (necessarily adjacent) quasi-quadrants. Then which two it hits is exactly determined by which one of the four arcs \( \partial A \cap B, \partial A \setminus B, \partial B \cap A, \partial B \setminus A \) it hits. It cannot be \( \partial A \cap B \) nor \( \partial B \cap A \) without violating \( v \in \tilde{A} \). If it is \( \partial A \setminus B \) then \( B \subset \tilde{A} \), so we may ignore this possibility by Observation 8.5. The remaining possibility is represented as \( \triangleright b \).

Now suppose that \( \partial A \) passes through exactly three quasi-quadrants. For brevity we will indicate which three it hits by saying instead which one it misses. If it misses \( B \setminus A \), then
\(B \setminus A \subset \tilde{A}\), so we throw this case out Observation 8.5. Next, it cannot miss \(C \setminus (A \cup B)\) without violating \(v \in \tilde{A}\). The two remaining cases are represented in ♦a and ♦c.

Finally, the case where \(\partial \tilde{A}\) passes through all four quasi-quadrants is drawn in ♦d.

**Case 2.** \(v \not\in \tilde{A}\)

Suppose first that \(\partial \tilde{A}\) hits exactly one of the quasi-quadrants. Then \(\tilde{A}\) is contained in that quasi-quadrant. But \(\tilde{A}\) must meet at least two quasi-quadrants, by Observation 8.5 if the hypotheses of Proposition 8.2 hold.

Next, suppose that \(\partial \tilde{A}\) meets exactly two quasi-quadrants. Again we indicate which two by saying which one of the four arcs \(\partial A \cap B, \partial A \setminus B, \partial B \cap A, \partial B \setminus A\) it hits. If it is \(\partial B \setminus A\) or \(\partial A \setminus B\), then \(\tilde{A}\) is disjoint from \(A \setminus B\), and if it is \(\partial A \setminus B\), then \(\tilde{A}\) is disjoint from \(A \cap B\).

Thus we throw these cases out by Observation 8.5. The final possibility is represented in ♦f.

Suppose now that \(\partial \tilde{A}\) meets exactly three quasi-quadrants. As before we indicate which three by indicating which one it misses. If it misses \(A \setminus B\), then we will get that \(A \setminus B\) and \(\tilde{A}\) are disjoint. If it misses \(A \cap B\), then the disks are in one of the configurations of Figure 12, in which case \(\tilde{A}\) and \(A \cap B\) are disjoint. We throw these cases out by Observation 8.5.

The remaining two cases are depicted in ♦g and ♦h.

Last, if \(\partial \tilde{A}\) passes through all four quasi-quadrants, and \(v \not\in \tilde{A}\), then the disks are configured as in ♦e.

This completes the proof of Proposition 8.2.

9. Preliminary topological lemmas

In the section after this one, we will introduce a tool, called *torus parametrization*, for working with fixed-point index. This tool will handle most of our cases for us relatively painlessly, but for some special cases we will need extra lemmas. This section is devoted to the statements and proofs of those lemmas. We also state and prove some simplifying facts that greatly cut down the number of cases we will eventually need to check.

First:

**Lemma 9.1.** Suppose \(K\) and \(\tilde{K}\) are closed Jordan domains in transverse position. Then \(\partial K\) and \(\partial \tilde{K}\) meet a finite, even number of times, by compactness and the transverse position hypothesis. In particular:

Suppose that \(z \in \partial K \cap \partial \tilde{K}\). Orient \(\partial K\) and \(\partial \tilde{K}\) positively with respect to \(K\) and \(\tilde{K}\) as usual. Then one of the following two mutually exclusive possibilities holds at the point \(z\).

1. The curve \(\partial \tilde{K}\) is entering \(K\), and the curve \(\partial K\) is exiting \(\tilde{K}\).
2. The curve \(\partial K\) is entering \(\tilde{K}\), and the curve \(\partial \tilde{K}\) is exiting \(K\).

Thus as we traverse \(\partial K\), we alternate arriving at points of \(\partial K \cap \partial \tilde{K}\) where (1) occurs and those where (2) occurs, and the same holds as we traverse \(\partial \tilde{K}\).

This is easy to check with a simple drawing.

Our next lemma characterizes the ways in which two convex closed Jordan domains may intersect:
Figure 13. Two convex closed Jordan domains $K$ and $\tilde{K}$ in transverse position, with boundaries meeting at six points. As $\theta$ varies positively, the ray $R_\theta$ scans around the boundaries of both $K$ and $\tilde{K}$ positively.

Lemma 9.2. Let $K$ and $\tilde{K}$ be convex closed Jordan domains in transverse position, so that $\partial K$ and $\partial \tilde{K}$ meet $2M > 0$ times. Suppose that $K'$ and $\tilde{K}'$ are also convex closed Jordan domains in transverse position so that $\partial K$ and $\partial \tilde{K}$ meet $2M > 0$ times. Then $\{K, \tilde{K}\}$ and $\{K', \tilde{K}'\}$ are in the same topological configuration.

Proof. For the following construction, see Figure 13. Let $w$ be a common interior point of $K$ and $\tilde{K}$. Let $R_\theta$ be the ray emanating from the point $w$ at an angle of $\theta$ from the positive real direction. Let $P_i$ be the points of $\partial K \cap \partial \tilde{K}$ where $\partial K$ is entering $\tilde{K}$, and let $\tilde{P}_i$ be those where $\partial \tilde{K}$ is entering $K$. Define $w', R'_\theta, P'_i, \tilde{P}'_i$ analogously for $K'$ and $\tilde{K}'$. Identify $S^1$ with the interval $[0, 2\pi]$ with its endpoints identified, and define a homeomorphism $S^1 \to S^1$, denoting the image of $\theta \in [0, 2\pi]$ by $\theta'$, so that $R_\theta$ hits a point $P_i$ if and only if $R'_{\theta'}$ hits a point $P'_i$, similarly for $\tilde{P}_i$ and $\tilde{P}'_i$. Define homeomorphisms $R_\theta \to R'_{\theta'}$ piecewise linearly on the components $R_\theta \setminus (\partial K \cup \partial \tilde{K}) \to R'_{\theta'} \setminus (\partial K' \cup \partial \tilde{K}')$. Then these homeomorphisms glue to an orientation-preserving homeomorphism $\mathbb{C} \to \mathbb{C}$ sending $\{K, \tilde{K}\}$ to $\{K', \tilde{K}'\}$. \quad \square

Lemma 9.2 is very much false if we omit the condition that the Jordan domains are convex. Which a priori topological configurations can occur for two Jordan curves in transverse position is a poorly understood question, and is known as the study of meanders. 6 We are fortunate that our setting is nice enough that a statement like Lemma 9.2 is possible. The clean construction we use in our proof is due to Nic Ford and Jordan Watkins.

We now take a moment to introduce some notation we use throughout the rest of the article. Let $\gamma$ be an oriented Jordan curve. Let $a, b \in \gamma$ be distinct. Then $[a \to b]_\gamma$ is the oriented closed sub-arc of $\gamma$ starting at $a$ and ending at $b$. Then for example $[a \to b]_\gamma \cap [b \to a]_\gamma = \{a, b\}$ and $[a \to b]_\gamma \cup [b \to a]_\gamma = \gamma$.

Throughout the rest of this section, let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of overlapping closed disks in general position. We label $\{u, v\} = \partial A \cap \partial B$ as in the preceding section, see

6 Thanks to Thomas Lam for informing us of the topic of meander theory
Figure 14. The possible relevant topological configurations for two generally positioned eyes whose boundaries meet at six points. There are two remaining possibilities, not depicted, obtained by simultaneously swapping \( u \) with \( v \) and \( \tilde{u} \) with \( \tilde{v} \), which are irrelevant by symmetry.

Figure 11 on p. 31 for a reminder. Label \( \tilde{u} \) and \( \tilde{v} \) analogously. We denote \( E = A \cap B \) and \( \tilde{E} = \tilde{A} \cap \tilde{B} \), and loosely refer to these as eyes. The rest of this section consists of the proofs of an assortment of lemmas about these disks, which we give without further comment.

Lemma 9.3. The Jordan curves \( \partial E \) and \( \partial \tilde{E} \) meet exactly 0, 2, 4, or 6 times.

Proof. That they meet an even number of times is a consequence of the general position hypothesis. There is an immediate upper bound of 8 meeting points because each of \( \partial E \) and \( \partial \tilde{E} \) is the union of two circular arcs. Suppose for contradiction that \( \partial E \) and \( \partial \tilde{E} \) meet 8 times. Thus every meeting point of one of the circles \( \partial A \) and \( \partial B \) with one of \( \partial \tilde{A} \) and \( \partial \tilde{B} \) lies in \( \partial E \cap \partial \tilde{E} \). It follows that \( \partial (A \cup B) \) does not meet \( \partial (\tilde{A} \cup \tilde{B}) \). But these are Jordan curves, so then we have either that one of \( A \cup B \) and \( \tilde{A} \cup \tilde{B} \) contains the other, or that they are disjoint. They cannot be disjoint because \( \partial E \) and \( \partial \tilde{E} \) meet (8 times) by hypothesis, so suppose without loss of generality that \( \tilde{A} \cup \tilde{B} \) is contained in \( A \cup B \), in particular in its interior by the general position hypothesis. Then the sub-arc \( \partial E \cap \partial A \) must enter the region \( \tilde{A} \cup \tilde{B} \) somewhere, so that it may intersect \( \partial \tilde{E} \), a contradiction. \( \square \)

Lemma 9.4. Suppose that \( \partial E \) and \( \partial \tilde{E} \) meet 6 times, that \( A \setminus B \) and \( \tilde{A} \setminus \tilde{B} \) meet, and that \( B \setminus A \) and \( \tilde{B} \setminus \tilde{A} \) meet. Then \( \{E, u, v, \tilde{E}, \tilde{u}, \tilde{v}\} \) are in one of the two topological configurations represented in Figure 14, up to possibly simultaneously swapping \( u \) with \( v \) and \( \tilde{u} \) with \( \tilde{v} \).

Proof. By Lemma 9.2 if \( \partial E \) and \( \partial \tilde{E} \) meet 6 times then they are in the topological configuration shown in Figure 15. We denote by \( \epsilon_i \) the connected components of \( \partial E \setminus \partial \tilde{E} \), and by \( \tilde{\epsilon}_i \) the connected components of \( \partial \tilde{E} \setminus \partial E \), labeled as in Figure 15. We consider the indices of the \( \epsilon_i \) and \( \tilde{\epsilon}_i \) only modulo 6. For example, we write \( \epsilon_{2+5} = \epsilon_1 \).

Proposition 8.2 allows us to make the following observation:

Observation 9.5. Neither \( \tilde{u} \) nor \( \tilde{v} \) may lie in \( E \), and neither \( u \) nor \( v \) may lie in \( \tilde{E} \).

To see why, note that if \( \partial E \) and \( \partial \tilde{E} \) meet six times, then by the pigeonhole principle at least one of \( \partial A \) and \( \partial B \) must meet \( \partial \tilde{E} \) at least three times. Thus at least one of \( \Delta \) and \( \Delta \tilde{g} \) must occur. Thus \( \tilde{u} \) and \( \tilde{v} \) lie outside of at least one of \( A \) and \( B \), but \( E = A \cap B \), thus neither \( \tilde{u} \) nor \( \tilde{v} \) lies in \( E \). The other part follows identically.

35
Thus we may assume that \( u \in \epsilon_1 \). Then \( v \) lies along \( \epsilon_3 \) or \( \epsilon_5 \). By relabeling the \( \epsilon_i \) and switching the roles of \( u \) and \( v \) as necessary, we may assume that \( v \in \epsilon_3 \). Our proof will be done once we show that neither \( \tilde{u} \) nor \( \tilde{v} \) may lie along \( \tilde{\epsilon}_2 \). Suppose for contradiction that \( \tilde{u} \) lies along \( \tilde{\epsilon}_2 \). Then \( \tilde{v} \) lies along either \( \tilde{\epsilon}_4 \) or \( \tilde{\epsilon}_6 \). If \( \tilde{v} \in \tilde{\epsilon}_4 \), then the circular arc \( \tilde{v} \) \( \to \tilde{u} \) \( \partial E \) meets the circular arc \( \tilde{v} \) \( \to \tilde{u} \) \( \partial E \) three times, a contradiction. Similarly, if \( \tilde{v} \in \tilde{\epsilon}_6 \), then the circular arc \( \tilde{v} \) \( \to \tilde{u} \) \( \partial E \) meets the circular arc \( \tilde{u} \) \( \to \tilde{v} \) \( \partial E \) three times, also a contradiction. Thus \( \tilde{u} \notin \tilde{\epsilon}_2 \). The argument is the same if we had initially let \( \tilde{v} \in \tilde{\epsilon}_2 \).

**Lemma 9.6.** The following four statements hold.

1. If \([\tilde{u} \to \tilde{v}]_{\partial E} \subset A\) and \([\tilde{v} \to \tilde{u}]_{\partial E} \) meets \( \partial A \), then \( B \setminus A \) and \( \tilde{B} \setminus \tilde{A} \) are disjoint.
2. If \([\tilde{v} \to \tilde{u}]_{\partial E} \subset B\) and \([\tilde{u} \to \tilde{v}]_{\partial E} \) meets \( \partial B \), then \( A \setminus B \) and \( \tilde{A} \setminus \tilde{B} \) are disjoint.
3. If \([u \to v]_{\partial E} \subset A\) and \([v \to u]_{\partial E} \) meets \( \partial A \), then \( B \setminus A \) and \( A \setminus B \) are disjoint.
4. If \([v \to u]_{\partial E} \subset B\) and \([u \to v]_{\partial E} \) meets \( \partial B \), then \( A \setminus B \) and \( \tilde{A} \setminus \tilde{B} \) are disjoint.

**Proof.** We prove only (1), as (2), (3), (4) are symmetric restatements of it. Suppose the hypotheses of (1) hold. Then both \( \tilde{u} \) and \( \tilde{v} \) lie in \( A \). Thus the circular arc \([\tilde{v} \to \tilde{u}]_{\partial E} \) meets \( \partial A \) either exactly twice or not at all, in fact exactly twice because of the hypotheses. But \([\tilde{v} \to \tilde{u}]_{\partial E} = [\tilde{v} \to \tilde{u}]_{\partial B} \). Thus \([\tilde{u} \to \tilde{v}]_{\partial B} \) does not meet \( \partial A \), and has its endpoints lying in \( A \), so \([\tilde{u} \to \tilde{v}]_{\partial B} \subset A \).

From our definitions of \( \tilde{u} \) and \( \tilde{v} \), it is easy to check that \( \partial(\tilde{B} \setminus \tilde{A}) \) is the union of the arcs \([\tilde{u} \to \tilde{v}]_{\partial E} \) and \([\tilde{u} \to \tilde{v}]_{\partial E} \). It follows that \( \partial(\tilde{B} \setminus \tilde{A}) \) is contained in \( A \). Thus \( \tilde{A} \setminus \tilde{B} \) is contained in \( A \), and so is disjoint from \( B \setminus A \).

**Lemma 9.7.** Suppose \( \{E, u, v, \tilde{E}, \tilde{u}, \tilde{v}\} \) are in the topological configuration depicted in Figure 16. Then \( A \setminus B \) and \( A \setminus B \) do not meet, and \( A \setminus A \) and \( B \setminus A \) do not meet.
Proof. The curves $\partial \tilde{A} \setminus \partial \tilde{E}$ and $\partial \tilde{B} \setminus \partial \tilde{E}$ both have $\tilde{u}$ and $\tilde{v}$ as their endpoints and otherwise avoid $\tilde{E}$. Thus each must cross $\partial E$ twice. These four crossings together with the points $\partial E \cap \partial \tilde{E}$ accounts for all eight possible intersection points between $\partial A \cup \partial B$ and $\partial \tilde{A} \cup \partial \tilde{B}$. Thus the arc $[\tilde{v} \to \tilde{u}]_{\partial \tilde{E}}$ does not meet $\partial B$. Because this arc meets $B \supset E$, we conclude that $[\tilde{v} \to \tilde{u}]_{\partial \tilde{E}}$ is contained in $B$. Note that $[\tilde{u} \to \tilde{v}]_{\partial \tilde{E}}$ meets $\partial B$. Thus by part (1) of Lemma 9.7 we get that $A \setminus B$ and $\tilde{A} \setminus \tilde{B}$ are disjoint. That $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ are disjoint follows by symmetry.

Lemma 9.8. Suppose $u \in \tilde{E}$ and $\tilde{u} \in E$. Then $A \setminus B$ and $\tilde{A} \setminus \tilde{B}$ do not meet, or $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ do not meet.

Proof. Suppose for contradiction that $u \in \tilde{E}$ and $\tilde{u} \in E$, but that $A \setminus B$ and $\tilde{A} \setminus \tilde{B}$ meet, and that $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ meet.

Observation 9.9. Neither of $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ contains the other.

To see why this is true, note that $u$ is in fact an interior point of $\tilde{E}$ by the general position hypothesis, and that $u \in \partial (B \setminus A)$. Thus $B \setminus A$ meets the exterior of $\tilde{B} \setminus \tilde{A}$. A similar argument gives that $\tilde{B} \setminus \tilde{A}$ meets the exterior of $B \setminus A$.

We are supposing for contradiction that $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ meet, and by Observation 9.9 neither of them contains the other. Thus if we can show that $\partial (B \setminus A)$ and $\partial (\tilde{B} \setminus \tilde{A})$ do not meet we will have derived a contradiction, as desired.

Note that Proposition 8.2 applies. This allows us to make the following observation.

Observation 9.10. Either $\Diamond a$ or $\Diamond e$ occurs, either $\nabla a$ or $\nabla a$ occurs, either $\blacklozenge a$ or $\blacklozenge c$ occurs, and either $\blacksquare a$ or $\blacksquare d$ occurs.

We prove that either $\Diamond a$ or $\Diamond e$ occurs, and the other parts of the observation follow similarly. Because $\tilde{E} \subset \tilde{A}$, we may eliminate any candidate topological configurations where $u \notin \tilde{A}$. This eliminates $\Diamond d$, $\Diamond g$, and $\Diamond h$. Next, because $\tilde{u} \in \partial \tilde{A}$, we may eliminate any candidate topological configurations where $\partial A$ does not meet $E$, as this would preclude $\tilde{u} \in E$. This eliminates $\blacklozenge b$ and $\blacklozenge c$ leaving us with only the two claimed possibilities. Thus the remainder of our proof breaks into cases as follows.

Case 1. Suppose that both $\Diamond a$ and $\Diamond a$ occur.

Then $\partial (A \setminus B)$ is contained in $\tilde{A}$, and $\partial (B \setminus A)$ is contained in $\tilde{B}$. Thus $\partial (A \setminus B) \cup \partial (B \setminus A)$ is contained in $\tilde{A} \cup \tilde{B}$. But $\partial (A \cup B)$ is contained in $\partial (A \setminus B) \cup \partial (B \setminus A)$, thus in $\tilde{A} \cup \tilde{B}$. We conclude that $A \cup B \subset \tilde{A} \cup \tilde{B}$. Now $\tilde{u} \in \partial (\tilde{A} \cup \tilde{B})$ and $E \subset A \cup B$, so by the general position hypothesis we get a contradiction.

Case 2. Suppose that both $\Diamond a$ and $\Diamond a$ occur.

Then $u \in \tilde{E}$ and $v \in \tilde{A} \setminus \tilde{B}$. One of the following two sub-cases occurs.

Sub-case 2.1. Suppose that $\blacksquare a$ occurs.

Then $\partial A$ does not meet $\tilde{A} \setminus \tilde{B}$. But $v$ lies on $\partial A$, contradicting $v \in \tilde{A} \setminus \tilde{B}$.

Sub-case 2.2. Suppose that $\blacksquare c$ occurs. Then one of $\blacksquare a$ and $\blacksquare d$ occurs.
From ♠e and that \( u \in \tilde{E} \) and \( v \in \tilde{A} \setminus \tilde{B} \), it follows that \( \partial (B \setminus A) \cap \partial A = [u \to v]_{\partial E} \) does not meet \( \partial (\tilde{B} \setminus \tilde{A}) \). If ♣e occurs, then \( \partial B \supset \partial (B \setminus A) \cap \partial B \) does not meet \( \partial (\tilde{B} \setminus \tilde{A}) \). If ♦e occurs, then via \( u \in \tilde{E} \) and \( v \in \tilde{A} \setminus \tilde{B} \) we get that \( \partial (B \setminus A) \cap \partial B = [u \to v]_{\partial B} \) does not meet \( \partial (\tilde{B} \setminus \tilde{A}) \). In either case \( \partial (B \setminus A) \) and \( \partial (\tilde{B} \setminus \tilde{A}) \) do not meet, giving us a contradiction.

Cases (1) and (2) together rule out ♢a ♠a and ♣a by symmetry, so the only remaining case is the following.

**Case 3.** Suppose that ♣c ♦d ♠e and ♣d occur.

By ♣c and ♦d we have that \( u \in \tilde{E} \) and \( v \in \mathcal{C} \setminus (\tilde{A} \cup \tilde{B}) \). Then from ♠e and ♣d we get that neither \( \partial (B \setminus A) \cap \partial A = [u \to v]_{\partial A} \) nor \( \partial (B \setminus A) \cap \partial B = [u \to v]_{\partial B} \) meets \( \partial (B \setminus \tilde{A}) \), again giving us the desired contradiction. \( \square \)

### 10. Torus parametrization

In this section we introduce a tool, called *torus parametrization* that allows us to work with fixed-point index combinatorially. This will allow us to systematically and relatively painlessly handle the remaining case analysis.

Let \( K \) and \( \tilde{K} \) be closed Jordan domains in transverse position, so that \( \partial K \) and \( \partial \tilde{K} \) meet at \( 2M \geq 0 \) points, with boundaries oriented as usual. Let \( \partial K \cap \partial \tilde{K} = \{P_1, \ldots, P_M, \tilde{P}_1, \ldots, \tilde{P}_M\} \), where \( P_i \) and \( \tilde{P}_i \) are labeled so that at every \( P_i \) we have that \( \partial K \) is entering \( K \), and at every \( \tilde{P}_i \) we have that \( \partial \tilde{K} \) is entering \( \tilde{K} \). Imbue \( S^1 \) with an orientation and let \( \kappa : \partial K \to S^1 \) and \( \tilde{\kappa} : \partial \tilde{K} \to S^1 \) be orientation-preserving homeomorphisms. We refer to this as fixing a *torus parametrization* for \( K \) and \( \tilde{K} \).

We consider a point \((x, y)\) on the 2-torus \( T = S^1 \times S^1 \) to be parametrizing simultaneously a point \( \kappa^{-1}(x) \in \partial K \) and a point \( \tilde{\kappa}^{-1}(y) \in \partial \tilde{K} \). We denote by \( p_i \in T \) be the unique point \((x, y) \in T \) satisfying \( \kappa^{-1}(x) = \tilde{\kappa}^{-1}(y) = P_i \), similarly \( \tilde{p}_i \in T \). Note that by the transverse position hypothesis no pair of points in \( \{P_1, \ldots, P_M, \tilde{P}_1, \ldots, \tilde{P}_M\} \) share a first coordinate, nor a second coordinate.

Suppose we pick \((x_0, y_0) \in S^1 \times S^1 \). Then we may draw an image of \( T = S^1 \times S^1 \) by letting \( \{x_0\} \times S^1 \) be the vertical axis and letting \( S^1 \times \{y_0\} \) be the horizontal axis. Then we call \((x_0, y_0)\) a *base point* for the drawing. See Figure 17 for an example.

Suppose that \( \phi : \partial K \to \tilde{\partial K} \) is an orientation-preserving homeomorphism. Then \( \phi \) determines an oriented curve \( \gamma \) in \( T \) for us, namely its graph \( \gamma = \{(\kappa(z), \tilde{\kappa}(f(z))) \in \partial K \}, \) with orientation obtained by traversing \( \partial K \) positively. Note that \( \phi \) is fixed-point-free if and only if its associated curve \( \gamma \) misses all of the \( p_i \) and \( \tilde{p}_i \). Pick \( u \in \partial K \) and denote \( \tilde{u} = \phi(u) \). Then if we draw the torus parametrization for \( K \) and \( \tilde{K} \) using the base point \((\kappa(u), \tilde{\kappa}(\tilde{u}))\), the curve \( \gamma \) associated to \( \phi \) “looks like the graph of a strictly increasing function.” The converse is also true: given any such \( \gamma \), it determines for us an orientation-preserving homeomorphism \( \partial K \to \tilde{\partial K} \) sending \( u \) to \( \tilde{u} \), which is fixed-point-free if and only if \( \gamma \) misses all of the \( p_i \) and \( \tilde{p}_i \).

The nicest thing about torus parametrization is that it allows us to compute \( \eta(\phi) \) easily by looking at the curve \( \gamma \) associated to \( \phi \). In particular, suppose that \( \phi(u) = \tilde{u} \), equivalently that \( (\kappa(u), \tilde{\kappa}(\tilde{u})) \in \gamma \). The curve \( \gamma \) and the horizontal and vertical axes \( \{\tilde{\kappa}(\tilde{u})\} \times S^1 \) and \( S^1 \times \{\kappa(u)\} \) divide \( T \) into two simply connected open sets \( \Delta_\uparrow(u, \gamma) \) and \( \Delta_\downarrow(u, \gamma) \) as shown in
Figure 17. A pair of closed Jordan domains $K$ and $\tilde{K}$ and a torus parametrization for them, drawn with base point $(\kappa(u), \tilde{\kappa}(\tilde{u}))$. The key points to check are that as we vary the first coordinate of $\mathbb{T}$ positively starting at $u$, we arrive at $\kappa(P_1)$, $\kappa(\tilde{P}_1)$, $\kappa(P_2)$, and $\kappa(\tilde{P}_2)$ in that order, and as we vary the second coordinate of $\mathbb{T}$ positively starting at $\tilde{\kappa}(\tilde{u})$, we arrive at $\tilde{\kappa}(P_1)$, $\tilde{\kappa}(\tilde{P}_1)$, $\tilde{\kappa}(P_2)$, and $\tilde{\kappa}(\tilde{P}_1)$ in that order.

Figure 18. A homotopy from $\partial \Delta_\downarrow(u, \gamma)$ to $\Gamma$. Here the orientation shown on $\gamma$ is the opposite of the orientation induced by traversing $\partial K$ positively.

Figure 18. We suppress the dependence on $\tilde{u}$ in the notation because $\tilde{u} = \phi(u)$. If neither $u \in \partial \tilde{K}$ nor $\tilde{u} \in \partial K$ then every $p_i$ and every $\tilde{p}_i$ lies in either $\Delta_\downarrow(u, \gamma)$ or $\Delta_\uparrow(u, \gamma)$. In this case we write $\#p_i(u, \gamma)$ to denote $|\{p_1, \ldots, p_M\} \cap \Delta_\downarrow(u, \gamma)|$ the number of points $p_i$ which lie in $\Delta_\downarrow(u, \gamma)$, and we define $\#p_i(u, \gamma)$, $\#\tilde{p}_i(u, \gamma)$, and $\#\tilde{p}_i(u, \gamma)$ in the analogous way. Denote by $\omega(\alpha, z)$ the winding number of the closed curve $\alpha \subset \mathbb{C}$ around the point $z \not\in \alpha$. Then:

**Lemma 10.1.** Let $K$ and $\tilde{K}$ be closed Jordan domains. Fix a torus parametrization of $K$ and $\tilde{K}$ via $\kappa$ and $\tilde{\kappa}$. Let $\phi : \partial K \to \partial \tilde{K}$ be an indexable homeomorphism, with graph $\gamma$ in $\mathbb{T}$. Suppose that $\phi(u) = \tilde{u}$, where $u \not\in \partial K$ and $\tilde{u} \not\in \partial K$. Then:

\begin{align*}
\eta(\phi) &= w(\gamma) = \omega(\partial K, \tilde{u}) + \omega(\partial \tilde{K}, u) - \#p_\downarrow(u, \gamma) + \#\tilde{p}_\downarrow(u, \gamma) \\
&= \omega(\partial K, \tilde{u}) + \omega(\partial \tilde{K}, u) + \#p_\uparrow(u, \gamma) - \#\tilde{p}_\uparrow(u, \gamma)
\end{align*}
Proof. Suppose $\gamma_0$ is any oriented closed curve in $\mathbb{T}\setminus\{p_1,\ldots,p_M,\tilde{p}_1,\ldots,\tilde{p}_M\}$. Then the closed curve $\{\tilde{\kappa}^{-1}(y) - \kappa^{-1}(x)\}_{(x,y)\in\gamma_0}$ misses the origin, and has a natural orientation obtained by traversing $\gamma_0$ positively. We denote by $w(\gamma_0)$ the winding number around the origin of $\{\tilde{\kappa}^{-1}(y) - \kappa^{-1}(x)\}_{(x,y)\in\gamma_0}$. First:

**Observation 10.2.** If $\gamma_1$ and $\gamma_2$ are homotopic in $\mathbb{T}\setminus\{p_1,\ldots,p_M,\tilde{p}_1,\ldots,\tilde{p}_M\}$ then $w(\gamma_1) = w(\gamma_2)$.

This is because the homotopy between $\gamma_1$ and $\gamma_2$ in $\mathbb{T}\setminus\{p_2,\ldots,p_M,\tilde{p}_1,\ldots,\tilde{p}_M\}$ induces a homotopy between the closed curves $\{\tilde{\kappa}^{-1}(y) - \kappa^{-1}(x)\}_{(x,y)\in\gamma_1}$ and $\{\tilde{\kappa}^{-1}(y) - \kappa^{-1}(x)\}_{(x,y)\in\gamma_2}$ in the punctured plane $\mathbb{C}\setminus\{0\}$.

If $\gamma$ has orientation induced by traversing $\partial K$ and $\partial \tilde{K}$ positively, then the following is a tautology.

**Observation 10.3.** $\eta(\phi) = w(\gamma)$

Orient $\partial \Delta_4(u,\gamma)$ as shown in Figure 18. Then $\partial \Delta_4(u,\gamma)$ is the concatenation of the curve $\gamma$ traversed backwards with $\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}$ and $\{\kappa(u)\} \times \mathbb{S}^1$, where the two latter curves are oriented according to the positive orientation on $\mathbb{S}^1$.

**Observation 10.4.** If $\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}$ and $\{\kappa(u)\} \times \mathbb{S}^1$ are oriented according to the positive orientation on $\mathbb{S}^1$, then $w(\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}) = \omega(\partial \tilde{K}, \tilde{u})$ and $w(\{\kappa(u)\} \times \mathbb{S}^1) = \omega(\partial K, u)$.

It is also easy to see that if we concatenate two closed curves $\gamma_1$ and $\gamma_2$ that meet at a point, we get $w(\gamma_1 \circ \gamma_2) = w(\gamma_1) + w(\gamma_2)$. Thus in light of the orientations on $\partial \Delta_4(u,\gamma)$ and all other curves concerned we get:

$$w(\partial \Delta_4(u,\gamma)) = w(\mathbb{S}^1 \times \{\tilde{\kappa}(\tilde{u})\}) + w(\{\kappa(u)\} \times \mathbb{S}^1) - w(\gamma)$$

$$= \omega(\partial K, \tilde{u}) + \omega(\partial \tilde{K}, u) - \eta(\phi)$$

For every $i$ let $\zeta(p_i)$ and $\zeta(\tilde{p}_i)$ be small squares around $p_i$ and $\tilde{p}_i$ respectively in $\mathbb{T}$, oriented as shown in Figure 18. By square we mean a simple closed curve which decomposes into
four “sides,” so that on a given side one of the two coordinates of $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}$ is constant. Pick the rectangles small enough so that the closed boxes they bound are pairwise disjoint and do not meet $\partial \Delta_i(u, \gamma)$.

Let $\Gamma$ be the closed curve in $\Delta_i(u, \gamma)$ obtained in the following way. First, start with every loop $\zeta(p_i)$ and $\zeta(\tilde{p}_i)$ for those $p_i$ and $\tilde{p}_i$ lying in $\Delta_i(u, \gamma)$. Let $\delta_0$ be an arc contained in the interior of $\Delta_i(u, \gamma)$ which meets each $\zeta(p_i)$ and $\zeta(\tilde{p}_i)$ contained in $\Delta_i(u, \gamma)$ at exactly one point. It is easy to prove inductively that such an arc exists. Let $\delta$ be the closed curve obtained by traversing $\delta_0$ first in one direction, then in the other. Then let $\Gamma$ be obtained by concatenating $\delta$ with every $\zeta(p_i)$ and $\zeta(\tilde{p}_i)$ contained in $\Delta_i(u, \gamma)$.

**Observation 10.5.** The curves $\Gamma$ and $\partial \Delta_i(u, \gamma)$ are homotopic in $\mathbb{T} \setminus \{p_1, \ldots, p_M, \tilde{p}_1, \ldots, \tilde{p}_M\}$. Also $w(\delta) = 0$. It follows that:

$$w(\partial \Delta_i(u, \gamma)) = w(\Gamma) = \sum_{p_i \in \Delta_i(u, \gamma)} w(\zeta(p_i)) + \sum_{\tilde{p}_i \in \Delta_i(u, \gamma)} w(\zeta(\tilde{p}_i)).$$

See Figure 18 for an example. On the other hand, the following holds.

**Observation 10.6.** $w(\zeta(p_i)) = 1$, $w(\zeta(\tilde{p}_i)) = -1$

To see why, suppose that $\zeta(p_i) = \partial([x_0 \to x_1]_{\mathbb{S}^1 \times [y_0 \to y_1]_{\mathbb{S}^1}})$. Then up to orientation-preserving homeomorphism the picture near $P_i$ is as in Figure 19. We let $(x, y)$ traverse $\zeta(p_i)$ positively starting at $(x_0, y_0)$, keeping track of the vector $\tilde{k}^{-1}(y) - \tilde{k}^{-1}(x)$ as we do so. The vector $\tilde{k}^{-1}(y_0) - \tilde{k}^{-1}(x_0)$ points to the right. As $x$ varies from $x_0$ to $x_1$, the vector $\tilde{k}^{-1}(y) - \tilde{k}^{-1}(x)$ rotates in the positive direction, that is, counter-clockwise, until it arrives at $\tilde{k}^{-1}(y_0) - \kappa^{-1}(x_1)$, which points upward. Continuing in this fashion, we see that $\tilde{k}^{-1}(y) - \tilde{k}^{-1}(x)$ makes one full counter-clockwise rotation as we traverse $\zeta(p_i)$. The proof that $w(\zeta(\tilde{p}_i)) = -1$ is similar. Combining all of our observations establishes equation 3. The proof that equation 4 holds is similar. \qed
Figure 21. Graphs of homeomorphisms $\epsilon$ giving $\eta(\epsilon) = 0$ for a pair of eyes whose boundaries meet twice. The torus parametrizations are drawn using the indicated choice of base point.

11. Proof of Propositions 11.1 and 11.5

In this section, we prove the two remaining outstanding propositions to complete the proof of our Main Index Theorem 5.3 and thus our main rigidity results.

Proposition 11.1. Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of overlapping closed disks in the plane $\mathbb{C}$ in general position. Suppose that neither of $E = A \cap B$ and $\tilde{E} = \tilde{A} \cap \tilde{B}$ contains the other. Suppose further that $A \setminus \tilde{B}$ and $\tilde{A} \setminus B$ meet, and that $B \setminus A$ and $\tilde{B} \setminus \tilde{A}$ meet. Then there is a faithful indexable homeomorphism $\epsilon : \partial E \to \partial \tilde{E}$ satisfying $\eta(\epsilon) = 0$.

Proof. If $\partial E$ and $\partial \tilde{E}$ do not meet, we get that $E$ and $\tilde{E}$ are disjoint. Then any indexable homeomorphism $\epsilon : \partial E \to \partial \tilde{E}$ satisfies $\eta(\epsilon) = 0$. Thus suppose that $\partial E$ and $\partial \tilde{E}$ meet. Fix a torus parametrization for $E$ and $\tilde{E}$ via $\kappa : \partial E \to S^1$ and $\tilde{\kappa} : \partial \tilde{E} \to S^1$. As before denote by $p_i$ the points of $\partial E \cap \partial \tilde{E}$ where $\partial E$ is entering $\tilde{E}$, and by $\tilde{p}_i$ those where $\partial \tilde{E}$ is entering $E$. Note that $\partial E$ and $\partial \tilde{E}$ meet at exactly 2, 4, or 6 points by Lemma 9.3. The proof breaks into these three cases.

Case 1. Suppose that $\partial E$ and $\partial \tilde{E}$ meet at exactly two points.

Then with an appropriate choice of base point, the torus parametrization for $E$ and $\tilde{E}$ is as shown in Figure 20a. The points $s_1, s_2 \in S^1$ in Figure 20a are exactly the topologically distinct places where $\kappa(u)$ may be, similarly $\tilde{s}_1, \tilde{s}_2 \in S^1$ for $\tilde{\kappa}(\tilde{u})$. A choice of $(s_j, \tilde{s}_j) = (\kappa(u), \tilde{\kappa}(\tilde{u}))$ completely determines the topological configuration of $\{E, \tilde{E}, u, \tilde{u}\}$, and conversely every possible topological configuration of those sets is achieved via this procedure. By Lemma 9.8 we may suppose without loss of generality that $u \not\in \tilde{E}$, thus that $\kappa(u) = s_1$.

Suppose first that $\tilde{\kappa}(\tilde{u}) = \tilde{s}_2$. In Figure 20b we redraw the torus parametrization for $E$ and $\tilde{E}$ using the base point $(s_1, \tilde{s}_2) = (\kappa(u), \tilde{\kappa}(\tilde{u}))$. Then the points $t_1, t_2, t_3 \in S^1$ are exactly the topologically distinct places where $\kappa(v)$ may be, similarly $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3 \in S^1$ for $\tilde{\kappa}(\tilde{v})$. 


Figure 22. The situation if two eyes’ boundaries meet four times. Figure 22a shows the torus parametrization for $E$ and $\tilde{E}$ with some suitable choice of base point. Figures 22b–22d give graphs of homeomorphisms $\epsilon$ giving $\eta(\epsilon) = 0$, with torus parametrizations drawn using base point $(\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_j, \tilde{s}_j)$ as indicated.

Observation 11.2. A choice of $(s_j, \tilde{s}_j) = (\kappa(u), \tilde{\kappa}(\tilde{u}))$ and a subsequent choice of $(t_k, \tilde{t}_k) = (\kappa(v), \tilde{\kappa}(\tilde{v}))$ together completely determine the topological configuration of $\{E, \tilde{E}, u, \tilde{u}, v, \tilde{v}\}$. Conversely every possible topological configuration of $\{E, \tilde{E}, u, \tilde{u}, v, \tilde{v}\}$ is achieved by some choice of $(s_j, \tilde{s}_j)$, and then a subsequent choice of $(t_k, \tilde{t}_k)$, for $(\kappa(u), \tilde{\kappa}(\tilde{u}))$ and $(\kappa(v), \tilde{\kappa}(\tilde{v}))$ respectively.

We are currently working under the assumption that $(\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_1, \tilde{s}_2)$. For every choice of $(t_k, \tilde{t}_k) = (\kappa(v), \tilde{\kappa}(\tilde{v}))$ we hope to find a faithful indexable homeomorphism $\epsilon : \partial E \to \partial \tilde{E}$ so that $\eta(\epsilon) = 0$. 

43
Observation 11.3. Suppose we have drawn the parametrization for \(E\) and \(\tilde{E}\) using \((s_j, \tilde{s}_j) = (\kappa(u), \tilde{\kappa}(\tilde{u}))\) as the base point. Then finding a faithful indexable homeomorphism \(\epsilon : \partial E \to \partial \tilde{E}\) amounts to finding a curve \(\gamma\) in \(T \setminus \{p_1, \ldots, \tilde{p}_M, \bar{p}_1, \ldots, \bar{p}_M\}\) which “looks like the graph of a strictly increasing function,” from the lower-left-hand corner \((\kappa(\tilde{u}), \tilde{\kappa}(\tilde{v}))\) to the upper-right-hand corner, passing through \((\kappa(v), \tilde{\kappa}(\tilde{v})) = (t_k, \tilde{t}_k)\). Having fixed such a curve \(\gamma\), we may compute \(\eta(\epsilon)\), where \(\epsilon\) is the homeomorphism associated to \(\gamma\), using Lemma 10.4.

In our current situation \(\kappa(u) = s_1\) implies that \(u \not\in \tilde{K}\) and \(\tilde{\kappa}(\tilde{u}) = \tilde{s}_2\) implies that \(\tilde{u} \not\in K\). Thus by Lemma 10.1 we wish to find curves \(\gamma\) so that both \(p_1\) and \(\tilde{p}_1\) lie in the upper diagonal \(\Delta_1(u, \gamma)\), or both lie in the lower diagonal \(\Delta_1(u, \gamma)\). Figure 21b depicts such a \(\gamma\) for every \((t_k, \tilde{t}_k)\) except for \((t_2, \tilde{t}_2)\). Suppose \((t_2, \tilde{t}_2) = (\kappa(v), \tilde{\kappa}(\tilde{v}))\). Then \(v \in \tilde{K}\) and \(\tilde{v} \in K\), so we get a contradiction by Lemma 9.8. From now on points \((t_k, \tilde{t}_k)\) which are handled via Lemma 9.8 will be labeled with an asterisk, as in Figure 21b.

Next suppose that \((\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_1, \tilde{s}_1)\). The situation is depicted in Figure 21a. Then \(u \not\in \tilde{K}\) and \(\tilde{u} \in K\), so to achieve \(\eta(\epsilon) = 0\) we wish to find curves \(\gamma\) so that \(p_1 \in \Delta_1(u, \gamma)\) and \(\tilde{p}_1 \in \Delta_1(u, \gamma)\). This time there are four \((t_k, \tilde{t}_k)\) for which this is not possible. For \((\kappa(v), \tilde{\kappa}(\tilde{v})) = (t_2, \tilde{t}_1), (t_2, \tilde{t}_3)\) we again get contradictions via Lemma 9.8. The following observation will be helpful for \((\kappa(v), \tilde{\kappa}(\tilde{v})) = (t_1, \tilde{t}_3), (t_3, \tilde{t}_1)\).

Observation 11.4. Choose \((s_j, \tilde{s}_j) = (\kappa(u), \tilde{\kappa}(\tilde{u}))\) and draw the torus parametrization for \(E\) and \(\tilde{E}\) using \((\kappa(u), \tilde{\kappa}(\tilde{u}))\) as the base point. Then a choice of \((t_k, \tilde{t}_k) = (\kappa(v), \tilde{\kappa}(\tilde{v}))\) defines for us four “quadrants,” namely \([\kappa(u) \to \kappa(v)]_{\tilde{S}^1} \times [\tilde{\kappa}(\tilde{u}) \to \tilde{\kappa}(\tilde{v})]_{\tilde{S}^1}\) the points “below and to the left of” \((t_k, \tilde{t}_k)\), etc. Then which of the two arcs \(\partial A \cap \partial E\) and \(\partial B \cap \partial E\), and which of \(\partial A \cap \partial \tilde{E}\) and \(\partial B \cap \partial \tilde{E}\), a point \(P_i\) or \(\tilde{P}_i\) lies on is determined by which quadrant \(p_i\) or \(\tilde{p}_i\) lies in.

For example, suppose \((\kappa(v), \tilde{\kappa}(\tilde{v})) = (t_1, \tilde{t}_3)\). Then \(p_1\) and \(\tilde{p}_1\) lie in the lower-right-hand quadrant \([\kappa(v) \to \kappa(u)]_{\tilde{S}^1} \times [\tilde{\kappa}(\tilde{u}) \to \tilde{\kappa}(\tilde{v})]_{\tilde{S}^1}\), so both \(P_1\) and \(\tilde{P}_1\) lie on \(\partial E \cap \partial B = [v \to u]_{\partial E}\) and on \(\partial \tilde{E} \cap \partial \tilde{A} = [\tilde{u} \to \tilde{v}]_{\partial \tilde{E}}\). Also \(\tilde{v} \to \tilde{u}\) is contained in \(E\), because both \(\tilde{v}\) and \(\tilde{u}\) are, and no \(p_i\) nor any \(\tilde{p}_i\) lies the two upper quadrants \([v \to u]_{\tilde{S}^1} \times \tilde{S}^1\). Then we get a contradiction via Lemma 9.6. A similar argument gives us a contradiction via Lemma 9.6 for \((\kappa(u), \tilde{\kappa}(\tilde{u})) = (t_1, \tilde{t}_1)\). From now on points \((t_k, \tilde{t}_k)\) which are handled via Lemma 9.6 in this way will be labeled with a diamond, as in Figure 21a. This completes the proof of Proposition 11.1 when \(\partial E\) and \(\partial \tilde{E}\) meet at exactly two points.

Case 2. Suppose that \(\partial E\) and \(\partial \tilde{E}\) meet at exactly four points.

Lemma 9.2 guarantees that with a correct choice of base point, the torus parametrization for \(E\) and \(\tilde{E}\) as in Figure 22a. As before, we may suppose without loss of generality that \(u \not\in \tilde{K}\), thus \(\kappa(u) = s_1\), by Lemma 9.8 and relabeling the \(s_i\) if necessary. Thus we have the possibilities \(\tilde{\kappa}(\tilde{u}) = \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\) to consider. The cases \((\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_1, \tilde{s}_2)\) and \((\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_1, \tilde{s}_4)\) are symmetric by Figure 23. Figures 22a–22d give the solutions for \(\tilde{\kappa}(\tilde{u}) = \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\) modulo some remaining special cases.

Points \((t_k, \tilde{t}_k)\) labeled with an asterisk or a diamond are handled via Lemma 9.8 or 9.6 respectively as before. Suppose \((\kappa(u), \tilde{\kappa}(\tilde{u})) = (s_1, \tilde{s}_1)\), and \((\kappa(v), \tilde{\kappa}(\tilde{v})) = (t_1, \tilde{t}_1)\) in Figure 22a. Then the upper-right-hand quadrant defined for us by \((t_1, \tilde{t}_1)\) contains all four points \(p_1, p_2, \tilde{p}_1, \tilde{p}_2\) thus the circular arcs \([v \to u]_{\partial E}\) and \([\tilde{v} \to \tilde{u}]_{\partial \tilde{E}}\) meet four times, a contradiction. All points that are handled in this way are labeled with a small circle. Finally, if
\[ u = \kappa^{-1}(s_1) \]
\[ \tilde{u} = \tilde{\kappa}^{-1}(\tilde{s}_2) \]

Figure 23. The topological configurations of \( \{E, u, \tilde{u}\} \) leading to the cases \((\kappa(u), \tilde{\kappa}(u)) = (s_1, \tilde{s}_2), (s_1, \tilde{s}_4)\). We see that these are equivalent via a rotation, because \( \eta(\epsilon) = \eta(\epsilon^{-1}) \).

\[ \tilde{u} = \tilde{\kappa}^{-1}(\tilde{s}_4) \]
\[ u = \kappa^{-1}(s_1) \]

(\(\kappa(u), \tilde{\kappa}(\tilde{u})\)) = \((s_1, \tilde{s}_3)\) and \((\kappa(v), \tilde{\kappa}(\tilde{v})\)) = \((t_3, \tilde{t}_3)\) in Figure 22d, we get a contradiction via Lemma 9.7.

**Case 3.** Suppose that \( \partial E \) and \( \partial \tilde{E} \) meet at exactly six points.

Then Lemma 9.4 restricts us to two cases to consider. These are handled in Figure 24. This completes the proof of Proposition 11.1.

**Proposition 11.5.** Let \( D = \{D_1, \ldots, D_n\} \) and \( \tilde{D} = \{\tilde{D}_1, \ldots, \tilde{D}_n\} \) be as in the statement of our Main Index Theorem 5.3. That is, they are thin disk configurations in the plane \( \mathbb{C} \) in general position, realizing the same pair \((G, \Theta)\) where \( G = (V, E) \) is a graph and \( \Theta : E \to [0, \pi) \). In addition, suppose that for all \( i, j \) the sets \( D_i \setminus D_j \) and \( \tilde{D}_i \setminus \tilde{D}_j \) meet. Suppose that there is no \( i \) so that one of \( D_i \) and \( \tilde{D}_i \) contains the other. Suppose that for every disjoint non-empty \( I, J \subset \{1, \ldots, n\} \) so that \( I \cup J = \{1, \ldots, n\} \), there exists an eye \( E_{ij} \) with \( i \in I \) and \( j \in J \) so that one of \( E_{ij} \) and \( \tilde{E}_{ij} \) contains the other. Then for every \( i \) we have that any faithful indexable homeomorphism \( \delta_i : \partial D_i \to \partial \tilde{D}_i \) satisfies \( \eta(\delta_i) \geq 1 \). Furthermore there is a \( k \) so that \( D_i \) and \( D_k \) overlap for all \( i \), and so that one of \( E_{ij} \) and \( \tilde{E}_{ij} \) contains the other if and only if either \( i = k \) or \( j = k \).
Figure 25. The image of the Möbius transformation described in the proof of Lemma 11.6.

Figure 26. A Möbius transformation chosen to prove Lemma 11.7. Here $\partial m(D) = \mathbb{R}$, and $m(D)$ is the lower half-plane.

For the proof of Proposition 11.5, we need to establish three geometric lemmas:

**Lemma 11.6.** Suppose that $D, d_{-1}, d_{+1}$ are closed disks in the plane $\mathbb{C}$ in the topological configuration depicted in Figure 27a. Then $\pi + \angle(d_{-1}, d_{+1}) < \angle(d_{-1}, D) + \angle(d_{+1}, D)$.

*Proof.* Let $m$ be a Möbius transformation sending a point on the bottom arc of $\partial D \setminus d_{-1} \cup d_{+1}$ to $\infty$, so that $m(D)$ is the lower half plane. Then the images of the disks under $m$ are as depicted in Figure 25. We see that $(\pi - \angle(d_{-1}, D)) + (\pi - \angle(d_{+1}, D)) + \angle(d_{-1}, d_{+1}) < \pi$ and the desired inequality follows. \hfill \square

**Lemma 11.7.** Suppose that $D, d_{-1}, d_{+1}$ are closed disks in the plane $\mathbb{C}$ in the topological configuration depicted in Figure 26. Then $\angle(d_{-1}, D) + \angle(d_{+1}, D) < \pi + \angle(d_{-1}, d_{+1})$.

*Proof.* This is proved similarly to Lemma 11.6, see Figure 26. \hfill \square

**Lemma 11.8.** Suppose that $D, d_{-1}, d_{+1}$ are closed disks in the plane $\mathbb{C}$ in one of the two topological configurations depicted in Figure 27. In either case, we get that both $\angle(d_{-1}, D)$ and $\angle(d_{+1}, D)$ are strictly greater than $\angle(d_{-1}, d_{+1})$.

*Proof.* Suppose that the disks are in the configuration depicted in Figure 27a. Let $m$ be a Möbius transformation sending a point on $\partial d_{+1} \setminus D$ to $\infty$. We may suppose without loss
of generality that $m(d_{+1})$ is the lower half-plane. Then the image of our disks under $m$ is as in Figure 28 where $\theta_1 = \angle(d_{+1}, D)$ and $\theta_2 = \angle(d_{-1}, d_{+1})$. It is then an easy exercise to show that $\theta_2 < \theta_1$ because the two circles $\partial m(d_{-1})$ and $\partial m(D)$ meet in the upper half-plane. The other inequality follows by symmetry. The case where the disks are in the configuration depicted in Figure 27 follows from the first case after applying a Möbius transformation sending a point in the interior of $D \cap d_{-1} \cap d_{+1}$ to $\infty$. \hfill \square

**Proof of Proposition 11.5** Recalling notation from before, if $D_i$ and $D_j$ overlap then $E_{ij} = D_i \cap D_j$, similarly $\tilde{E}_{ij}$, and a homeomorphism $\delta_i : \partial D_i \to \partial \tilde{D}_i$ is called *faithful* if it restricts to homeomorphisms $D_j \cap \partial D_i \to \tilde{D}_j \cap \partial \tilde{D}_i$ for all $j$.

**Claim 11.9.** Let $i, j$ be so that $\tilde{E}_{ij} \subset E_{ij}$. Denote $A = D_i$, $B = D_j$, $\tilde{A} = \tilde{D}_i$, $\tilde{B} = \tilde{D}_j$. Then both $\spadesuit c$ and $\spadesuit e$ occur. Also one of $\clubsuit d$, $\spadesuit e$, $\spadesuit g$ occurs, and one of $\heartsuit a$, $\spadesuit e$, $\spadesuit g$ occurs. Furthermore at least one of $\spadesuit g$ and $\spadesuit g$ occurs.

To see why, note first that both $\spadesuit c$ and $\spadesuit e$ occur, because these are the only candidates in Figure 10 where $\tilde{A} \cap \tilde{B}$ is contained in the respective one of $A$ and $B$. Note the following by Lemma 11.8
\begin{equation}
\angle(A, B) = \angle(\tilde{A}, \tilde{B}) < \angle(\tilde{A}, \tilde{B})
\end{equation}
and the following by Lemma 11.6
\begin{equation}
\pi + \angle(\tilde{A}, \tilde{B}) < \angle(A, \tilde{A}) + \angle(\tilde{B}, A), \quad \pi + \angle(\tilde{A}, \tilde{B}) < \angle(\tilde{A}, \tilde{B}) + \angle(\tilde{B}, \tilde{B})
\end{equation}

Next, because $\tilde{A} \cap \tilde{B}$ contains part of $\partial \tilde{A}$ and part of $\partial \tilde{B}$, both of these circles must pass through $A \cap B$. Noting that $\heartsuit e$ cannot occur because $\tilde{A} \not\subset A$, we conclude that one of $\clubsuit a$, $\clubsuit d$, $\clubsuit e$, $\clubsuit g$, and $\heartsuit h$ occurs. If either of $\heartsuit a$ and $\heartsuit h$ occurs, then Lemma 11.8 implies that $\angle(A, B) < \angle(A, B)$, contradicting $\clubsuit a$. This leaves us with only the claimed possibilities $\clubsuit d$, $\clubsuit e$, and $\clubsuit g$. By symmetry we also get that one of $\heartsuit d$, $\heartsuit e$, and $\heartsuit g$ occurs.

Finally, note by Lemma 11.7 that if $\heartsuit d$ or $\heartsuit e$ occurs then we get $\angle(\tilde{A}, A) + \angle(\tilde{A}, B) < \pi + \angle(A, B)$, and if $\heartsuit d$ or $\heartsuit e$ occurs then we get $\angle(\tilde{B}, A) + \angle(\tilde{B}, B) < \pi + \angle(A, B)$. We get that if neither of $\heartsuit g$ and $\heartsuit g$ occurs, then we may combine these two inequalities with $\heartsuit a$ to arrive at a contradiction, establishing Claim 11.9.

Moving on, pick $1 \leq i \leq n$. By the hypotheses of Proposition 11.5 there is a $j$ so that one of $E_{ij}$ and $\tilde{E}_{ij}$ contains the other, without loss of generality so that $\tilde{E}_{ij} \subset E_{ij}$. Let $\delta_i : \partial D_i \to \partial \tilde{D}_i$ be a faithful indexable homeomorphism. Continuing with the notation of Claim 11.9 regardless of which of $\clubsuit d$, $\clubsuit e$, and $\clubsuit g$ occurs, there is a point $z \in \partial A \cap \partial E$ so that $z$ lies in the interior of $\tilde{A}$. Furthermore note that $\delta_i(z) \in \partial \tilde{E}$ by the faithfulness.
condition, and that $\tilde{E} \subset A$ by our hypotheses, so $\delta_i(z)$ lies in the interior of $A$. Thus if we draw a torus parametrization for $A$ and $\tilde{A}$ using $(\kappa(z), \tilde{\kappa}(\delta_i(z)))$ as the base point, Lemma 10.1 implies that $\eta(\delta_i) \geq 1$, because $\partial A$ and $\partial \tilde{A}$ meet exactly twice. This establishes the first part of Proposition 11.5.

Next, let $H_u$ be the undirected simple graph having $\{1, \ldots, n\}$ as its vertex set, so that $\langle i,j \rangle$ is an edge in $H_u$ if and only if $D_i$ and $D_j$ overlap and one of $E_{ij}$ and $\tilde{E}_{ij}$ contains the other. Note that $H_u$ is connected, otherwise we could pick $I$ to be the vertex set of one connected component of $H_u$ and $J$ to be $\{1, \ldots, n\} \setminus I$ to contradict the hypotheses of Proposition 11.5.

Let $H$ be the directed graph obtained from $H_u$ in the following way. Suppose $\langle i,j \rangle$ is an edge in $H_u$. Denote $A = D_i$, $B = D_j$, $\tilde{A} = \tilde{D}_i$, $\tilde{B} = \tilde{D}_j$. Then $\langle i \to j \rangle$ is an edge in $H$ if and only if one of ♦ and ♠ occurs. In particular Claim 11.9 implies that if $\langle i,j \rangle$ is an edge in $H_u$ then at least one of $\langle i \to j \rangle$ and $\langle j \to i \rangle$ is an edge in $H$, and possibly both are.

Claim 11.10. Suppose that $\langle i \to j \rangle$ is an edge in $H$. Then $\langle i,j \rangle$ is the only edge in $H_u$ having $i$ as a vertex.

To see why, observe first that if ♦ or ♦ occurs then one intersection point $\partial A \cap \partial \tilde{A}$ lies in the interior of $B$, and if ♦ occurs then both do. Suppose without loss of generality that $\tilde{D}_i \cap \tilde{D}_j \subset D_i \cap D_j$. Then both intersection points $\partial D_i \cap \partial \tilde{D}_i$ lie in the interior of $D_j$. For contradiction let $k \neq j$ so that $\langle i,k \rangle$ is an edge in $H_u$. There are two cases.

Case 1. Suppose that $\tilde{D}_i \cap \tilde{D}_k \subset D_i \cap D_k$.

Then one or both points $\partial D_i \cap \partial \tilde{D}_i$ lie in the interior of $D_k$. Then there is a point in the interior of $D_i$ which lies in the interiors of both $D_j$ and $D_k$, a contradiction.

Case 2. Suppose that $D_i \cap D_k \subset \tilde{D}_i \cap \tilde{D}_k$.

Then by a symmetric restatement of Claim 11.9 we get that both points $\partial D_i \cap \partial D_k$ lie in $\tilde{D}_i$. On the other hand $\tilde{D}_i \cap \partial D_i$ is contained in the interior of $D_j$ by ♦. Thus there are points interior to all of $D_i, D_j, D_k$, a contradiction.

This establishes Claim 11.10.

Thus $H$ is either the graph on two vertices $\{i,j\}$ having one or both of $\langle i \to j \rangle$ and $\langle j \to i \rangle$ as edges, or is a graph having $\{k,i_1, \ldots, i_{n-1}\}$ as vertices and exactly the edges $\langle i_\ell \to k \rangle$ for $1 \leq \ell < n$. The last part of Proposition 11.5 follows.

This completes the proofs of the main results of this article.
12. Generalizations, open problems, and conjectures

We conclude the article with some general conjectures which are directly related to the new results of this article.

First, we discuss eliminating the thinness condition from the hypotheses of our theorem statements. Most simply, it seems likely that our Main Theorems 1.5 and 1.4 should continue to hold with the thinness condition completely omitted. In this direction, we conjecture the following fixed-point index statement for non-thin configurations of disks:

**Conjecture 12.1.** Suppose that $D$ and $\tilde{D}$ are disk configurations in $\mathbb{C}$ realizing the same incidence data. Then any faithful indexable homeomorphism $\phi: \partial D \to \partial \tilde{D}$ satisfies $\eta(\phi) \geq 0$.

Note that something stronger than Conjecture 12.1 would be required to prove the corresponding generalizations of our main results on disk configurations using the methods of this article: in particular, we would probably need to generalize the notion of an isolated subsumptive subset of the common index set of $D$ and $\tilde{D}$. However, it is plausible that this is a workable approach.

We remark at this point that in the present author’s thesis, see [Mis12], the definition of thin used in the statements of our Main Theorems 1.5 and 1.4 is slightly weaker than the one we have given here: there we call a disk configuration thin if given three disks from it, the intersection of their interiors is empty. The proofs there are essentially the same, without any interesting new ideas, but there are technically annoying degenerate situations to deal with, so we do not work at this level of generality here.

More strongly, we make two conjectures which together would subsume all other currently known rigidity and uniformization statements on disk configurations. First:

**Conjecture 12.2.** Suppose that $C$ and $\tilde{C}$ are disk configurations, locally finite in $G$ and $\tilde{G}$ respectively, where each of $G$ and $\tilde{G}$ is equal to one of $\mathbb{C}$ and $\mathbb{H}^2$, with the a priori possibility that $G \neq \tilde{G}$. Suppose that $C$ and $\tilde{C}$ share a contact graph $G = (V, E)$. Suppose further that $\text{cal} C$ and $\tilde{C}$ fill their respective spaces, in sense that every connected component of $G \cup D_{\in C} D$ or of $\tilde{G} \cup D_{\in \tilde{C}} \tilde{D}$ is bounded. Then $G = \tilde{G}$.

Second:

**Conjecture 12.3.** Suppose that $C$ and $\tilde{C}$ are disk configurations, both locally finite in $G$, where $G$ is equal to one of $\mathbb{C}$ and $\mathbb{H}^2$. Suppose that $C$ and $\tilde{C}$ realize the same incidence data $(G, \Theta)$. Suppose further that some maximal planar subgraph of $G$ is the 1-skeleton of a triangulation of a topological open disk. Then $P$ and $\tilde{P}$ differ by a Euclidean similarity if $G = \mathbb{C}$ or by a hyperbolic isometry if $G = \mathbb{H}^2$.

We also make the natural conjecture analogous to Conjecture 12.3 for disk configurations on the Riemann sphere. It seems plausible that a fixed-point index approach could work to prove Conjectures 12.2 and 12.3. An alternative approach to try to prove Conjecture 12.2 is via vertex extremal length arguments, along the lines of [He99, Uniformization Theorem 1.3] and [HS95].

Finally, we conjecture that Conjecture 12.3 is the best possible uniqueness statement of its type, in the following precise sense:
Figure 29. A counterexample to Theorem 5.3 if we allow $\angle(D_1, D_2) \neq \angle(\tilde{D}_1, \tilde{D}_2)$. Any indexable $\phi : \partial(D_1 \cup D_2) \to \partial(\tilde{D}_1 \cup \tilde{D}_2)$ making the shown identifications gives $\eta(f) = -1$.

**Conjecture 12.4.** Let $C$ be a disk configuration which is locally finite in $G$, where $G$ is one of $\hat{C}$, $\mathbb{C}$, or $\mathbb{H}^2$. Let $(G, \Theta)$ be the incidence data of $C$. Suppose that no maximal planar subgraph of $G$ is the 1-skeleton of a triangulation of a topological open disk. Then there are other locally finite disk configurations in $G$ realizing $(G, \Theta)$ which are not images of $C$ under any conformal or anti-conformal automorphism of $G$.

The most promising tool to prove Conjecture 12.4 would be a good existence statement taking incidence data $(G, \Theta)$ as input.

Finally, we consider other directions in which our Main Index Theorem 5.3 could be generalized. First, one may hope to weaken the condition that $D$ and $\tilde{D}$ realize the same incidence data, insisting only that they share a contact graph. Figure 29 provides an explicit small-scale counterexample. Alternatively, we may hope to prove a theorem analogous to Theorem 5.3 for collections of shapes other than metric closed disks. For example, if $K$ and $\tilde{K}$ are compact convex sets in $\mathbb{C}$ having smooth boundaries, one of which is the image of the other by translation and scaling, then $\partial K$ and $\partial \tilde{K}$ meet at most twice, so the Circle Index Lemma 3.2 applies. This gives hope for a generalization of Theorem 5.3 in this direction. Schramm has proved rigidity theorems for packings by shapes other than circles using related ideas, for example in [Sch91].

**References**

[And70] E. M. Andreev, *Convex polyhedra of finite volume in Lobachevskii space*, Mat. Sb. (N.S.) 83 (125) (1970), 256–260 (Russian). MR0273510 (42 #8388)

[BS90] Alan F. Beardon and Kenneth Stephenson, *The uniformization theorem for circle packings*, Indiana Univ. Math. J. 39 (1990), no. 4, 1383–1425, DOI 10.1512/iumj.1990.39.39062. MR1087197 (92b:52038)

[BS04] Alexander I. Bobenko and Boris A. Springborn, *Variational principles for circle patterns and Koebe’s theorem*, Trans. Amer. Math. Soc. 356 (2004), no. 2, 659–689, DOI 10.1090/S0002-9947-03-03239-2. MR2022715 (2005b:52054)

[Brä92] Walter Brägger, *Kreispackungen und Triangulierungen*, Enseign. Math. (2) 38 (1992), no. 3-4, 201–217 (German). MR1189006 (94b:52032)

[Car13] C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis*, Math. Ann. 73 (1913), no. 2, 305–320, DOI 10.1007/BF01456720 (German). MR1511735

[CdV91] Yves Colin de Verdière, *Un principe variationnel pour les empilements de cercles*, Invent. Math. 104 (1991), no. 3, 655–669, DOI 10.1007/BF01245096 (French). MR1106755 (92h:57020)
