Relaxed Leverage Sampling for Low-rank Matrix Completion

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Abstract

We consider the problem of exact recovery of any $m \times n$ matrix of rank $\rho$ from a small number of observed entries via the standard nuclear norm minimization framework in (2) (Candes and Recht [2009]). Such low-rank matrices have degrees of freedom $(m + n)\rho - \rho^2$. We show that such arbitrary low-rank matrices can be recovered exactly from as small as $\Theta\left(\frac{(m + n)\rho - \rho^2}{\log (m + n)}\right)$ randomly sampled entries, thus matching the lower bound on the required number of entries (in degrees of freedom), with an additional factor of $O(\log^2 (m + n))$. The above bound on sample size is achieved if each entry is observed according to probabilities proportional to the sum of corresponding row and column leverage scores, minus their product (see (4)). We show that this relaxation in sampling probabilities (as opposed to sum of leverage scores in Chen et al. [2014]) gives us an additive improvement on the (best known) sample size obtained by Chen et al. [2014] for the optimization problem in (2). Experiments on real data corroborate the theoretical improvement on sample size.

Further, exact recovery of (a) incoherent matrices (with restricted leverage scores), and (b) matrices with only one of the row or column spaces to be incoherent, can be performed using our relaxed leverage score sampling, via (2), without knowing the leverage scores a priori. In such settings also we achieve additive improvement on sample size.

Keywords: matrix completion, nuclear norm, leverage score, randomized algorithms

1 Introduction

Suppose we have a data matrix $M \in \mathbb{R}^{m \times n}$ with incomplete/missing entries, say, we have information about only a small number elements of $M$. The matrix completion problem (Candes and Recht [2009]) is to predict those missing entries as accurately as possible based on the observed entries. Such partially-observed data may appear in many application domains. For example, in a user-recommendation system (a.k.a collaborative filtering) we have incomplete user ratings for various products, and the goal is to make predictions about a user’s preferences for all the products (e.g., the Netflix problem). Also, the incomplete data could represent some partial distance matrix in a sensor network, or missing pixels in digital images because of occlusion or tracking failures in a video surveillance system (Candes and Tao [2010]).

More mathematically, we have information about the entries $M_{ij}, (i, j) \in \Omega$, where $\Omega \subset [m] \times [n]$ is a sampled subset of all entries, and $[n]$ denotes the list $\{1, ..., n\}$. The problem is to recover the unknown matrix $M$ in a computationally tractable way from as few observed entries as possible. However, without further assumption on $M$ it is impossible to predict the unobserved elements from a limited number of known entries. One popular assumption is that $M$ has low-rank, say rank $\rho$. Such matrices have degrees of freedom $(m + n)\rho - \rho^2$, i.e., the elements of such low-rank matrices are controlled by this many parameters. To see this, we consider the singular
value decomposition (SVD) of $M$,

$$M = \sum_{i \in [\rho]} \sigma_i u_i v_i^T,$$

(1)

where $\sigma_i$ is the $i$-th largest singular value, and the corresponding left and right singular vectors are $u_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^n$, respectively. Left and right singular vectors form two sets of orthonormal vectors. The first left singular vector has $m - 1$ degrees of freedom because of unit norm constraint. The second singular vector has $m - 2$ degrees of freedom as it has two constraints: the unit norm, and the orthogonality to the first vector. In this way, all the left singular vectors have total $(m - 1) + (m - 2) + \ldots + (m - \rho) = m\rho - \rho(\rho + 1)/2$ degrees of freedom. Similarly, the number is $n\rho - \rho(\rho + 1)/2$ for the right singular vectors. Considering additional degrees of freedom for the singular values, total number of degrees of freedom for $M$ is $(m + n)\rho - \rho^2$ (Candes and Tao 2010). This implies, if the number of observed entries $s = |\Omega| < (m + n)\rho - \rho^2$, there can be infinitely many matrices of rank at most $\rho$ with exactly the same entries in $\Omega$; therefore, exact recovery of unobserved entries is impossible. So, we need at least $(m + n)\rho - \rho^2$ many observed entries for exact matrix completion.

The matrix $M$, with the observed entries, can be interpreted as an element in $mn$-dimensional linear space, with available information about $O((m + n)\rho - \rho^2)$ coordinates. Remaining $mn - O((m + n)\rho - \rho^2)$ many coordinates are unknown. The set of matrices compatible with the observed entries forms a large affine space. Then, exact matrix completion problem is to specify an efficient algorithm which uniquely picks $M$ from this high-dimensional affine space (Gross 2011). Since, our target matrix $M$ is low-rank, a natural optimization problem for finding $M$ would be,

$$\min_{X \in \mathbb{R}^{m \times n}} \text{rank}(X)$$

$$\text{subject to } X_{ij} = M_{ij}, \quad (i, j) \in \Omega.$$  

However, minimizing rank over an affine space is known to be NP-hard. Candes and Recht 2009 proposed to solve the heuristic optimization in (2) to recover the low-rank matrix $M$.

$$\min_{X \in \mathbb{R}^{m \times n}} \|X\|_*$$

$$\text{subject to } X_{ij} = M_{ij}, \quad (i, j) \in \Omega,$$

(2)

where the nuclear norm $\|X\|_*$ of a matrix $X$ is defined as the sum of its singular values,

$$\|X\|_* = \sum_i \sigma_i(X).$$

(2) is a convex optimization problem that is efficiently solvable via semi-definite programming. Exact matrix completion thus becomes proving that the nuclear norm restricted to the affine space has a strict and global minima at $M$. That is, if $M + Z \neq M$ is a matrix in the affine space in (2), we need to show

$$\|M + Z\|_* > \|M\|_*.$$  

Candes and Recht 2009, Gross 2011, Recht 2011, Candes et al. 2011 developed the sufficient conditions and main probabilistic tools in order to recover $M$ as a unique solution to (2).

One natural question is: which elements of $M$ should we observe in (2), i.e., how should we construct the sample set $\Omega$? We want to define some probabilities on the entries of $M$. Most of the existing work focused on the case when $\Omega$ in (2) is constructed by observing the entries of $M$.
uniformly randomly \cite{Candes_Recht_2009, Gross_2011, Recht_2011, Candes_etal_2011}). However, this data-oblivious sampling scheme has a cost. To see this, let \( M \) be rank-1 \((\varrho = 1)\), and \( m = n, u_1 = v_1 = e_1, \sigma_1 \) is arbitrary in \([1]\), where \( e_i \) is the standard basis vector \((i\text{-th component 1, others are zeros})\). This \( M \) has only one non-zero entry, i.e., \( M_{1,1} = \sigma_1 \), and all other entries are 0. The probability that a sample set \( \Omega \) of size \( 2n - 1 \) (degrees of freedom), via uniform sampling, containing only zero entries of \( M \) is \( 1 - O(\frac{1}{n^3}) \approx 1 \). That is, this matrix cannot be recovered from sampling its entries uniformly unless we see almost all the entries. This is because by observing only zeros it is impossible to predict non-zeros of a matrix. This suggests that \( M \) cannot be in the null space of the sampling operator \((\text{to be defined later})\) extracting the values of a subset of the entries. Matrices similar to the above example can be characterized by the structure of their singular vectors. The singular vectors are \((\text{closely})\) ‘aligned’ with the standard basis \(\text{(i.e., having very high inner product with the basis vectors)}\). Therefore, the components of singular vectors should be sufficiently spread to reduce the number of observations needed to recover a low-rank matrix. Such restrictions on the row and column spaces of a low-rank matrix are called the \emph{incoherence} assumptions \((\text{to be defined later})\). \cite{Gross_2011, Recht_2011} showed that such restricted class of \( n \times n \) matrices of rank \( \varrho \) can be recovered exactly, with high probability, by observing \( \Theta(n \varrho \log^2 n) \) entries sampled uniformly.

Very recently, \cite{Chen_etal_2014} proposed non-uniform probabilities proportional to the sum of row and column leverage scores of \( M \) to observe its entries \((\text{leveraged sampling})\). They eliminated the need for those ‘incoherence’ assumptions, and showed that any arbitrary \( n \times n \) matrix of rank \( \varrho \) can be recovered exactly, with high probability, from as few as \( \Theta(n \varrho \log^2 n) \) observed elements via leveraged sampling.

Similar to \cite{Chen_etal_2014}, we also incorporate the row and column leverage scores of the reconstructing matrix \( M \) into our proposed probability of observing an entry. However, we use a relaxed notion of leverage score sampling. Specifically, we propose to observe an entry with probability proportional to the sum of the corresponding row and column leverage scores, \emph{minus their product}. Theorem \[\] shows that observing entries according to this relaxed leverage score sampling in \[\], we can recover any arbitrary \( m \times n \) matrix of rank-\( \varrho \) exactly, with high probability, from as few as \( \Theta(((m + n)\varrho - \varrho^2)\log^2 (m + n)) \) observed entries, via the standard nuclear norm minimization framework in \[\]. This bound on the sample size is optimal \((\text{up to } \log^2 (m + n) \text{ factor})\) in the number of degrees of freedom of a rank-\( \varrho \) matrix. Also, this gives us an \emph{additive improvement} on the sample size obtained by \cite{Chen_etal_2014}.

For an \( n \times n \) matrix \( M \) of rank-\( \varrho \) whose column space is incoherent and row space is arbitrarily coherent, \cite{Chen_etal_2014} give a provable sampling scheme \((\text{using leveraged sampling})\) which requires no prior knowledge of the leverage scores of \( M \). They show that this \( M \) can be recovered exactly, with high probability, using sample size as small as \( \Theta(n \varrho \log^2 n) \). We can incorporate our relaxed leverage scores in such setting, with no prior knowledge of leverage scores, to achieve additive improvement on the sample size obtained by \cite{Chen_etal_2014}, while recovering \( M \) exactly with high probability.

Finally, our notion of relaxation in sampling probabilities also achieves an additive improvement on the sample size even in case of uniform sampling for incoherent matrices.

### 1.1 Notations and preliminaries

We briefly describe the main notations used in this work. Natural number \( \{1, \ldots, n\} \) are denoted by \( [n] \). Natural logarithm of \( x \) is denoted by \( \log(x) \). Matrices are bold uppercase, vectors are bold lowercase, and scalars are not bold. We denote the \((i, j)\)-th entry of a matrix \( X \) by \( X_{ij} \). \( e_i \) denotes the \( i \)-th standard basis vector in \( \mathbb{R}^d \), with \( i \)-th component 1 and other entries zero. The dimension of \( e_i \) will be clear from the context. \( X^T \) and \( x^T \) denote the transpose of matrix \( X \) and
Then, \( M \) set \( \Omega \) in (2)) to reduce the sample size, such that  

\[
1 \Rightarrow \{ \Omega \}
\]

This suggests that the elements in important rows and columns, indicated by high leverage scores observed entries, according to (4), is

\[
\text{Probabilities in (4) are biased towards the leverage score structure of the reconstructing matrix. This suggests that the elements in important rows and columns, indicated by high leverage scores } \{ \mu_i \} \text{ and } \{ \nu_j \}, \text{ of a matrix should be observed more frequently in order to reduce the number of observations needed for exact matrix completion.} \quad \text{[Chen et al. 2014] also noticed this, and they proposed to sum up } \frac{\mu_i \theta}{m} \text{ and } \frac{\nu_j \theta}{n} \text{ in the sampling probabilities. However, our distribution in (4)}
\]

Linear operators acting on matrices are denoted by calligraphic letters. The spectral norm (largest singular value) of such operator \( A \) will be denoted by \( \| A \|_F \cdot \text{spectral norm is denoted by } \| A \|_2 \), and \( \| A \|_2 \cdot \text{Frobenius norm of } A \cdot \| A \|_F \cdot \text{Frobenius norm is denoted by } \| A \|_2 \). Also, we denote \( f(n) = \Theta(g(n)) \) when \( \alpha_1 \cdot g(n) \leq f(n) \leq \alpha_2 \cdot g(n) \), for some positive universal constants \( \alpha_1, \alpha_2 \).

## 2 Main Results

Our focus is to define a probability distribution on the entries of \( M \) (i.e., to construct the sample set \( \Omega \) in (2)) to reduce the sample size, such that \( M \) becomes the unique optimal solution to (2). In this work our sampling follows the Bernoulli model (Candes et al. [2011], Chen et al. [2014]), where each entry \((i, j)\) is observed independently with some probability \( p_{ij} \). Before we state our main result and the distribution, we first need to define the normalized leverage scores (Candes and Recht [2009], Recht [2011], Chen et al. [2014]).

### Definition 1

Let \( M \in \mathbb{R}^{m \times n} \) be of rank \( q \) with SVD \( M = U \Sigma V^T \), where \( U \) and \( V \) are the left and right singular matrices, respectively, and \( \Sigma \) is the diagonal matrix of singular values. Normalized leverage scores for \( i \)-th row (denoted by \( \mu_i \)) and \( j \)-th column (denoted by \( \nu_j \)) are defined as follows:

\[
\mu_i = \frac{m}{q} \| U^T e_i \|_2^2, \quad i = 1, \ldots, m, \\
\nu_j = \frac{n}{q} \| V^T e_j \|_2^2, \quad j = 1, \ldots, n
\]

(3)

Normalized leverage scores are non-negative, and they depend on the structure of row and column spaces of the matrix. Also, we have \( \sum_i \frac{\mu_i \theta}{m} = \sum_j \frac{\nu_j \theta}{n} = \theta \), because \( U \) and \( V \) have orthonormal columns. We state our main result.

### Theorem 1

Let \( M \in \mathbb{R}^{m \times n} \) of rank \( q \). Suppose, we have a subset of observed entries \( \Omega \subset [m] \times [n] \), where each entry \((i, j)\) is observed independently with probability \( p_{ij} \), such that,

\[
p_{ij} = \max \left\{ \min \left\{ c_1 \left( \frac{\mu_i \theta}{m} + \frac{\nu_j \theta}{n} - \frac{\mu_i \theta}{m} \cdot \frac{\nu_j \theta}{n} \right) \log^2 (m + n), \ 1 \right\}, \ \frac{1}{(mn)^c} \right\}
\]

(4)

for some universal constant \( c_1 > 0 \). Then, \( M \) is the unique optimal solution to (2) with probability at least \( 1 - 3 \log^3 (m + n)(m + n)^{3-c} \), for sufficiently large \( c > 3 \). Moreover, if the number of observed entries, according to (4), is

\[
| \Omega | = \Theta \left( (m + n)^{c} - \theta^2 \right) \log^2 (m + n),
\]

then, \( M \) is the unique optimal solution to (2) with probability at least \( 1 - 6 \log^3 (m + n)(m + n)^{3-c} \), for sufficiently large \( c > 3 \).

Probabilities in (4) are biased towards the leverage score structure of the reconstructing matrix. This suggests that the elements in important rows and columns, indicated by high leverage scores \( \{ \mu_i \} \) and \( \{ \nu_j \} \), of a matrix should be observed more frequently in order to reduce the number of observations needed for exact matrix completion. [Chen et al. 2014] also noticed this, and they proposed to sum up \( \frac{\mu_i \theta}{m} \) and \( \frac{\nu_j \theta}{n} \) in the sampling probabilities. However, our distribution in (4)
reduces this bias by subtracting the term \( \mu_i \rho - \nu_j \rho \) while maintaining the leverage score pattern in \( p_{ij} \). This relaxation in sampling probabilities helps us to reduce number of observations comparing to Chen et al. [2014], in additive sense, to recover the low-rank matrix exactly, via (2).

As discussed earlier, we need a minimum of \( \Theta((m + n)^2 - \rho^2) \) elements to recover a matrix exactly, regardless of the choice of probabilities. Theorem 1 proves that if we observe elements according to our relaxed leverage scores, we match this lower bound, up to a factor of \( O(\log(2(m + n))) \).

Also, using the relaxed leverage score sampling we observe improvement on sample size even in case of uniform sampling for matrices with incoherence restrictions. Let \( M \in \mathbb{R}^{n \times n} \) be the rank-\( \rho \) reconstructing matrix with SVD \( U \Sigma V^T \). Candes and Recht [2009], Candes and Tao [2010], Recht [2011], Gross [2011] use two incoherence parameters, \( \mu_0 \) and \( \mu_1 \), for exact matrix completion using uniform sampling, where, (a) \( \max_{i,j} \{\mu_i, \nu_j\} \leq \mu_0 \), and (b) \( \|UV^T\|_{\infty} = \mu_1 \sqrt{\rho/n^2} \). A meaningful range of \( \mu_0 \) is \( 1 \leq \mu_0 \leq \min\{m, n\}/\rho \). Then, the best known result was that if the sampling probability is uniform, such that,

\[
p_{ij} = p \geq \frac{c_u \max\{\mu_0, \mu_1^2\} \rho \log^2 n}{n}, \quad \forall i, j,
\]

where \( c_u \) is a constant, then \( M \) is the unique optimal solution of (2) with high probability. Actually, the lower bound achieved on the sample size in a sample-with-replacement model was \( O(\max\{\mu_0, \mu_1^2\} n \rho \log^2 n) \) (Recht [2011]). Above, \( \mu_1 \leq \mu_0 \sqrt{\rho} \), and it could create a suboptimal dependence of sample size on \( \rho \), in the worst case. Chen et al. [2014] showed that observing entries with uniform probability satisfying,

\[
p \geq c_1 \left(\frac{2\mu_0 \rho}{n}\right) \log^2 n, \quad \forall i, j
\]

for some constant \( c_1 \), would recover the matrix exactly, with high probability. In this case, the bound on sample size is \( O(2\mu_0 n \rho \log^2 n) \). They eliminated the need for the parameter \( \mu_1 \), and consequently the suboptimal dependence on \( \rho \).

It follows from Theorem 1 that we can recover the matrix exactly, with high probability, if each entry is sampled uniformly with probability satisfying,

\[
p = \max \left\{ \min \left\{ c_1 \left(\frac{2\mu_0 \rho}{n} - \frac{\mu_0^2 \rho^2}{n^2}\right) \log^2 n, 1 \right\}, n^{-10} \right\}, \quad \forall i, j.
\]

This requires a sample size of \( O((2\mu_0 n \rho - \mu_0^2 \rho^2) \log^2 n) \), achieving an additive improvement on all the existing bounds above.

### 2.1 Column-Space-Incoherent Matrix Completion

Here we discuss exact completion of a low-rank matrix whose column space is incoherent, and we have control over the sampling of matrix entries. This setting is interesting in application domains like recommendation systems and gene expression data analysis (Krishnamurthy and Singh [2013]).

Algorithm 1, adapted from Chen et al. [2014], performs exact completion of a matrix \( M \) with incoherent column space, without a priori knowledge of leverage scores of \( M \). Step 3 of Algorithm 1 computes the column leverage scores of \( M \) exactly, with high probability, from only a small number of (uniformly) observed rows. We construct an additional sample set \( \Omega \) of observed entries using our relaxed leverage scores in Step 4. Step 5 solves the nuclear norm minimization problem in (2) with \( \Omega \) to recover \( M \) exactly, with high probability. Theorem 2 proves the correctness of Algorithm 1.
**Algorithm 1 Column-Space-Incoherent Matrix Completion**

1: **Input:** \( M \in \mathbb{R}^{m \times n} \), with \( \max_i \mu_i \leq \mu_0, \forall i \in [m] \), s.t. \( 1 \leq \mu_0 \leq m/\varrho \).

2: Observe all the entries of a row of \( M \) picked with probability

\[
p = \min \left\{ \frac{c_2 \mu_0 \log m}{m}, 1 \right\},
\]

where \( c_2 \) is a constant.

3: Compute the leverage scores, \( \{ \tilde{\nu}_j \} \forall j \in [n] \), of the space spanned by these rows, and use them as estimates for true \( \{ \nu_j \}, \forall j \in [n] \) of \( M \).

4: Construct a sample set \( \Omega \) of entries \( (i, j) \) of \( M \) observed with probabilities

\[
\hat{p}_{ij} = \min \left\{ c_1 \left( \frac{\mu_0 \varrho}{m} + \tilde{\nu}_j \varrho/n - \frac{\mu_0 \varrho}{m} \cdot \tilde{\nu}_j \varrho/n \right) \log^2 (m + n), 1 \right\}, \forall i, j.
\]

5: Solve (2) using sample set \( \Omega \), and let \( X^* \) be the unique optimal solution.

6: **Output:** \( X^* \).

---

**Theorem 2** Algorithm 1 computes the column leverage scores of \( M \) exactly (step 3), i.e., \( \tilde{\nu}_j = \nu_j, \forall j \in [n] \). Using the sample set \( \Omega \), Algorithm 1 recovers \( M \) as the unique optimal solution of (2). The total number of samples required by Algorithm 1 is

\[
\Theta(\mu_0((m+2n)\varrho - \varrho^2)\log^2 (m + n)).
\]

The above results hold with probability at least, \( 1 - 6 \log^3 (m + n)(m + n)^{3-c} \), for sufficiently large \( c > 3 \).

We compare the bound on sample size in Theorem 2 with a couple of existing results. Let us assume \( m = n \) for simplicity. Theorem 2 achieves an additive improvement \( O(\varrho^2 \log^2 n) \) on the sample size of Chen et al. [2014] while recovering \( M \) exactly, with high probability, via (2). Krishnamurthy and Singh [2013] proposed an adaptive sampling algorithm that recovers \( M \) exactly, with probability at least \( 1 - O(\varrho \delta) \), and a sample size \( \Theta(\mu_0 n \varrho^{3/2} \log (\varrho/\delta)) \). Assuming comparable failure probabilities, sample size in Theorem 2 is better when \( \varrho \) is not too small.

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### 2.2 Coherent Matrix Completion using Two-Phase-Sampling

We have so far seen that any arbitrary \( m \times n \) matrix \( M \) of rank \( \varrho \) can be recovered exactly using \( \Theta((m + n)\varrho - \varrho^2)\log^2 (m + n)) \) observed entries sampled according to relaxed leverage scores of \( M \). However, in reality, we do not have knowledge about the leverage scores of \( M \), i.e., \( \{ \mu_i \} \) and \( \{ \nu_j \} \), even when we have control over how to choose entries. Chen et al. [2014] proposed a heuristic two-phase sampling procedure (Algorithm 1 of Chen et al. [2014]) for exact matrix completion with no a priori knowledge about the leverage scores. Here is an informal description of it.

Let, the total budget of samples be \( s \), and \( \beta \in [0, 1] \) be a parameter. First, construct an initial set \( \Omega_1 \) by sampling entries uniformly (without replacement), such that, \( |\Omega_1| = \beta s \). Let \( \hat{M} \) be the matrix with \( \hat{M}_{ij} = M_{ij} \) if \( (i,j) \in \Omega_1 \), and \( \hat{M}_{ij} = 0 \) if \( (i,j) \notin \Omega_1 \). Let the rank-\( \varrho \) SVD of \( \hat{M} \) be \( \hat{U} \hat{S} \hat{V}^T \). Compute the leverage scores of \( \hat{M} \) and use them as estimates for the leverage scores of \( M \), i.e., use \( \hat{\mu}_i = \frac{\varrho}{\varrho} \| \hat{U}^T e_i \|_2^2 \) as \( \mu_i \) for \( i \in [m] \), and \( \hat{\nu}_j = \frac{\varrho}{\varrho} \| \hat{V}^T e_j \|_2^2 \) as \( \nu_j \) for \( j \in [n] \). In the second phase, use these estimates to sample (without replacement) remaining \( (1 - \beta)s \) entries of
with probabilities proportional to $(\tilde{\mu}_i \tilde{\nu}/m + \tilde{\nu}_j \tilde{\nu}/n) \log^2 (m + n)$, to form the sample set $\Omega_2$. Then perform matrix completion using sample set $\Omega = \Omega_1 \cup \Omega_2$ in [2].

This heuristic is shown to work well on synthetic data that are less coherent (Chen et al. [2014]). For highly coherent data, e.g., only few entries are non-zeros and others are zeros, it works poorly, as expected.

We can incorporate our notion of relaxed leverage scores into the second phase of the above procedure by observing (without replacement) the remaining $(1 - \beta)s$ entries of $M$ with probabilities

$$p_{ij} \propto \left( \frac{\tilde{\mu}_i \tilde{\nu}/m + \tilde{\nu}_j \tilde{\nu}/n}{m} \cdot \frac{\tilde{\nu}_j \tilde{\nu}/n}{n} - \frac{\tilde{\mu}_i \tilde{\nu}/m}{m} \cdot \frac{\tilde{\nu}_j \tilde{\nu}/n}{n} \right) \log^2 (m + n)$$

to form sample set $\hat{\Omega}_2$, and perform nuclear norm minimization in (2) using $\Omega = \Omega_1 \cup \hat{\Omega}_2$.

We expect our relaxed leverage score sampling to follow similar trend as above, although we do not evaluate this heuristic numerically.

Rest of the content is organized as follows. Section 3 shows experimental results on real datasets to support the theoretical gain on the sample size using relaxed leverage score sampling. We give proofs of Theorem 1 and Theorem 2 in Section 4 and Section 5, respectively. Section 6 contains a proof of the sufficient conditions for the optimization problem in (2). Finally, Section 7 contains proofs of intermediate lemmas.

3 Experiments

We show experimental performance of the exact recovery of real data matrices via nuclear norm minimization in (2) using our relaxed leverage score sampling. We use the software ‘TFOCS’ v1.2, written by Stephen Becker, Emmanuel Candes, and Michael Grant, to solve (2).

3.1 Experimental Design

Let $M$ be the rank-$\rho$ data matrix. We construct the sample set $\Omega_{\text{relax}}$ by observing $(i, j)$-th entry of $M$ according to the relaxed leverage score probabilities in (6):

$$p_{ij}^{[\text{relax}]} = \min \left\{ c_r \cdot \left( \frac{\mu_i \theta}{m} + \frac{\nu_j \theta}{n} - \frac{\mu_i \theta}{m} \cdot \frac{\nu_j \theta}{n} \right), 1 \right\}, \quad \forall i,j$$

where $c_r$ is a universal constant. Similarly, we construct the sample set $\Omega_{\text{lev}}$ by observing $(i, j)$-th entry of $M$ according to the leverage score probabilities in (7):

$$p_{ij}^{[\text{lev}]} = \min \left\{ c_l \cdot \left( \frac{\mu_i \theta}{m} + \frac{\nu_j \theta}{n} \right), 1 \right\}, \quad \forall i,j$$

where $c_l$ is a universal constant. We use $\Omega_{\text{relax}}$ and $\Omega_{\text{lev}}$ in the optimization problem (2), separately, to recover $M$. Let $X^*$ be the optimal solution to (2) using a sample set $\Omega$. We say $X^*$ recovers $M$ exactly if $\|M - X^*\|_F / \|M\|_F < \varepsilon$, where $\varepsilon$ is a tiny fraction. We set $\varepsilon = 0.001$. We perform 10 independent trails (sampling and recovery) and declare success if $M$ is recovered exactly at least 9 times (i.e., at least 90% success rate). Let $s_r$ and $s_l$ be the average sample size for successful recovery of $M$ using $\Omega_{\text{relax}}$ and $\Omega_{\text{lev}}$, respectively. We expect $c_r \approx c_l$, and the gain in sample size $(s_l - s_r)$ for $\Omega_{\text{relax}}$ to be strictly positive, as suggested by the theory. Further, we investigate how $(s_l - s_r)$ behaves with respect to the rank $\rho$. For this, we define

$$\text{Normalized Gain (} \Delta_s \text{)} = \sqrt{\frac{s_l - s_r}{c_r}}.$$  

We expect $\Delta_s$ to be close to $\rho$ as the theory suggests $(s_l - s_r) \propto O(\rho^2)$. For fairness of comparison, we use the same random seed for both the sampling methods in (6) and (7).
3.2 Datasets

MovieLens: This collaborative filtering dataset was collected through the MovieLens web site (movielens.umn.edu). It contains 100,000 ratings between 1 and 5 by 943 users on 1682 movies. Each user has rated at least 20 movies. We note that this dataset is numerically not low-rank. We perform rank truncation to create an explicit low-rank matrix to apply the theory in (2). We observe the singular value spectrum of this data to heuristically choose two values for rank: $\rho = 10$ and $\rho = 20$.

TechTC: We use a dataset from the Technion Repository of Text Categorization Database (TechTC) (Gabrilovich and Markovitch [2004]). Here each row is a document describing a topic, and words (columns) are the features for the topics. The $(i, j)$-th entry of this matrix is the frequency of $j$-th word appearing in $i$-th document. We choose a dataset containing the topics with IDs 11346 and 22294. We preprocessed the data by removing all words of length four or less. Then, each row is normalized to have unit norm. Also, we observe the singular value spectrum of this preprocessed $125 \times 14392$ data to heuristically choose two values for rank: $\rho = 10$ and $\rho = 20$, to make the data explicitly low-rank.

3.3 Results

Figures 1 and 2 plot the singular values and the normalized leverage scores for rank-10 approximation for MovieLens and TechTC data, respectively. Normalized leverage scores are close to 1 when they are incoherent in nature. MovieLens dataset is reasonably coherent, and TechTC dataset has extremely high coherence.

Table 1 shows the constants $c_l$ and $c_r$, and the normalized gain $\Delta_s$ for exact recovery of MovieLens data. We see $c_l = c_r$ and $\Delta_s \approx \rho$, as expected. We observe similar results for TechTC data in Table 2.

Overall, these results support the accuracy of the theoretical analysis on the gain in sample size using the relaxed leverage score sampling for exact recovery of a low-rank matrix via (2).
| $c_l/c_r$ | $\Delta_s$ |
|---------|--------|
| 11/11  | 9.7    |
| 7/7    | 18.4   |

Table 1: [MovieLens] Gain in sample size for exact recovery using relaxed leverage score sampling.

| $c_l/c_r$ | $\Delta_s$ |
|---------|--------|
| 4/4    | 6.6    |
| 3/3    | 15.2   |

Table 2: [TechTC] Gain in sample size for exact recovery using relaxed leverage score sampling.

4 Proof of Theorem 1

The main proof strategy was outlined by Candes and Recht [2009], Recht [2011], Gross [2011]: it is sufficient to construct a dual certificate $Y$ obeying specific sub-gradient inequalities in order to show that $M$ is the unique optimal solution to (2) (see Section 6 for more detail). We give a proof of Theorem 1 closely following the proof strategy of Recht [2011], Chen et al. [2014]. Before stating the optimality conditions we need additional notations.

Recall, $U$ and $V$ are the left and right singular matrices of $M$, respectively. Let $u_k$ (respectively $v_k$) denote the $k$-th column of $U$ (respectively $V$). Let $T$ be a linear space spanned by elements of the form $u_k y^T$ and $x v_k^T$, $1 \leq k \leq \rho$, for arbitrary $x, y$, and $T^\perp$ be its orthogonal complement, i.e., $T^\perp$ is spanned by the family $(x y^T)$, where $x$ (respectively $y$) is any vector orthogonal to the space spanned by the left singular vectors (respectively right singular vectors). Then, orthogonal projection onto $T$ is given by the linear operator $P_T : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$, defined as

$$P_T(X) = U U^T X + X V V^T - U U^T X V V^T.$$  

Similarly, orthogonal projection onto $T^\perp$ is

$$P_{T^\perp}(X) = X - P_T(X) = U^\perp U^T X V V^T V^T.$$  

Note that any $m \times n$ matrix $X$ can be expressed as a sum of rank-one matrices as follows:

$$X = \sum_{i,j=1}^{m,n} \langle e_i e_j^T, X \rangle e_i e_j^T.$$  

(9)

We define the sampling operator $R_\Omega : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ as,

$$R_\Omega(X) = \sum_{i,j=1}^{m,n} \frac{1}{p_{ij}} \delta_{ij} \langle e_i e_j^T, X \rangle e_i e_j^T.$$  

(10)

where, $\delta_{ij} = \mathbb{I}((i, j) \in \Omega)$, $\mathbb{I}(\cdot)$ being the indicator function. That is, $R_\Omega$ extracts the terms, corresponding to the indices $(i, j) \in \Omega$, from (9) to form a partial sum in (10). Let $P_\Omega(X)$ be the matrix with $(P_\Omega(X))_{ij} = X_{ij}$ if $(i, j) \in \Omega$, and zero otherwise.
4.1 Optimality Conditions

Following the proof road map of [Recht 2011, Chen et al. 2014], we restate the sufficient conditions for \( M \) to be the unique optimal solution to (2) (Section 6 contains a proof of sufficiency).

**Proposition 1** The rank-\( q \) matrix \( M \in \mathbb{R}^{m \times n} \) with SVD \( M = U \Sigma V^T \) is the unique optimal solution to (2) if the following conditions hold:

1. \( \|P_T R_\Omega P_T - P_T\|_{op} \leq 1/2 \).
2. There exists a dual certificate \( Y \) which satisfies \( P_\Omega(Y) = Y \), and
   
   \[ \|P_T(Y) - UV^T\|_F \leq \sqrt{q(m + n)^{-15}} \]
   
   \( (b) \|P_{T^\perp}(Y)\|_2 \leq 1/2 \).

Condition 1 of Proposition 1 suggests \( R_\Omega \) should be nearly the identity operator on the subspace \( T \). Next we discuss the construction of a dual certificate \( Y \).

4.1.1 Constructing the Dual Certificate

We follow the so-called golfing scheme [Gross 2011, Candès et al. 2011, Chen et al. 2014] to construct a matrix \( Y \) (the dual certificate) that satisfies Condition 2 in Proposition 1. Recall, we assume that the set of observed elements \( \Omega \) follows the Bernoulli model for \( \Omega \). Note that, \( q_{ij} \geq p_{ij}/k_0 \) because of overlapping of \( \Omega_k \)’s. We set \( k_0 = 11 \cdot \log(m + n) \). Then,

\[
q_{ij} \geq \min \left\{ c_0 \cdot \log(m + n) \cdot \frac{\mu_1 \theta}{m} + \frac{\mu_2 \theta}{n} \right\} \cdot \frac{\nu_1 \theta}{m} - \frac{\nu_2 \theta}{n}, 1 \right\}, \quad (11)
\]

where \( c_0 = c_1/11 \). Starting with \( W_0 = 0 \) and for each \( k = 1, \ldots, k_0 \), we recursively define

\[
W_k = W_{k-1} + R_{\Omega_k} P_T(UV^T - P_T(W_{k-1})) \quad (12)
\]

where the sampling operator \( R_{\Omega_k} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \) is defined as

\[
R_{\Omega_k}(X) = \sum_{i,j} \frac{1}{q_{ij}} \mathbb{I}((i,j) \in \Omega_k) (e_i e_j^T, X) e_i e_j^T.
\]

We set \( Y = W_{k_0} \). This \( Y \) is supported on \( \Omega \), i.e., \( P_\Omega(Y) = Y \).

Let the sample set \( \tilde{\Omega} \) be such that

\[
\tilde{\Omega} \in \{ \Omega_k : \Omega = \cup_{k=1}^{k_0} \Omega_k, \Omega_k \sim \text{Bernoulli}(q_{ij}) \} \quad (13)
\]

Since \( \Omega_k \sim \text{Bernoulli}(q_{ij}) \) implies \( \Omega \sim \text{Bernoulli}(p_{ij}) \), for each \( k = 1, \ldots, k_0 \), we prove (in Lemma 1) Condition 1 of Proposition 1 using sample set \( \tilde{\Omega} \) in (13).

**Lemma 1** Let \( \tilde{\Omega} \) be a sample set in (13). Then, for any universal constant \( c > 1 \), we have

\[
\|P_T R_{\tilde{\Omega}} P_T - P_T\|_{op} \leq \frac{1}{2} \quad (14)
\]

holding with probability at least

\[
1 - (m + n)^{1-c}.
\]
Before we validate Condition 2 in Proposition 1 using the $Y$ constructed above, we claim the following results to hold with high probability. First, we borrow the following definitions of weighted infinity norms for a matrix $Z \in \mathbb{R}^{m \times n}$ from Chen et al. [2014].

$$
\|Z\|_{\mu(\infty,2)} := \max \left\{ \max_i \sqrt{\frac{m}{\mu_i \varrho}} \|Z_{i,*}\|_2, \max_j \sqrt{\frac{n}{\nu_j \varrho}} \|Z_{*,j}\|_2 \right\}
$$

$$
\|Z\|_{\mu(\infty)} := \max_{i,j} |Z_{ij}| \sqrt{\frac{m}{\mu_i \varrho}} \sqrt{\frac{n}{\nu_j \varrho}}
$$

where $Z_{i,*}$ and $Z_{*,j}$ denote the $i$-th row and $j$-th column of $Z$, respectively.

Lemma 2 bounds the spectral norm of the matrix $(R_{\tilde{\Omega}} - I)(Z)$ using the sample set $\tilde{\Omega}$.

**Lemma 2** Let $Z \in \mathbb{R}^{m \times n}$ be a fixed matrix. Let $\tilde{\Omega}$ be a sample set in (13). Then, for any universal constant $c > 1$, we have

$$
\left\| (R_{\tilde{\Omega}} - I) Z \right\|_2 \leq 2 \sqrt{\frac{c}{c_0} \|Z\|_{\mu(\infty,2)} + \frac{c}{c_0} \|Z\|_{\mu(\infty)}}
$$

holding with probability at least

$$
1 - (m + n)^{1-c}.
$$

Next two results control the $\mu(\infty,2)$ and $\mu(\infty)$ norms of the projection of a matrix after random sampling.

**Lemma 3** Let $Z \in \mathbb{R}^{m \times n}$ be a fixed matrix. Let $\tilde{\Omega}$ be a sample set in (13). Then, for any universal constant $c > 2$, we have

$$
\left\| (P_T R_{\tilde{\Omega}} - P_T) Z \right\|_{\mu(\infty,2)} \leq \frac{1}{2} \left( \|Z\|_{\mu(\infty,2)} + \|Z\|_{\mu(\infty)} \right)
$$

holding with probability at least

$$
1 - (m + n)^{2-c}.
$$

**Lemma 4** Let $Z \in \mathbb{R}^{m \times n}$ be a fixed matrix. Let $\tilde{\Omega}$ be a sample set in (13). Then, for any universal constant $c > 3$, we have

$$
\left\| (P_T R_{\tilde{\Omega}} - P_T) Z \right\|_{\mu(\infty)} \leq \frac{1}{2} \|Z\|_{\mu(\infty)}
$$

holding with probability at least

$$
1 - (m + n)^{3-c}.
$$

We now validate Condition 2 in Proposition 1 using the $Y$ constructed above.

**Bounding** $\|UV^T - P_T(Y)\|_F$

We set $\Delta_k = UV^T - P_T(W_k)$, for $k = 1, \ldots, k_0$. Then, from definition of $W_k$ we have

$$
\Delta_k = (P_T - P_T R_{\Omega_k} P_T) \Delta_{k-1}.
$$

We used $P_T(UV^T) = UV^T$ and $P_T P_T(X) = P_T(X)$. Using the independence of $\Delta_{k-1}$ and $\Omega_k$,

$$
\|\Delta_k\|_F = \|(P_T - P_T R_{\Omega_k} P_T) \Delta_{k-1}\|_F \leq \|P_T - P_T R_{\Omega_k} P_T\|_{op} \|\Delta_{k-1}\|_F.
$$
We can bound this by recursively applying Lemma 2 with $\Omega_k$, for all $k$. Thus,

$$\|P_T(Y) - UV^T\|_F = \|\Delta_{k_0}\|_F = \left(\frac{1}{2}\right)^{k_0} \|UV^T\|_F \leq \sqrt{\frac{q}{(m + n)^{15}}}$$

The above result fails with probability at most $(m + n)^{1-c}$ for each $k$; thus, total probability of failure is at most $11(m + n)^{1-c} \log(m + n)$.

Bounding $\|P_T(Y)\|_2$

By definition, $Y$ can be written as

$$Y = \sum_{k=1}^{k_0} R_{\Omega_k} P_T (UV^T - P_T(W_{k-1})) = \sum_{k=1}^{k_0} R_{\Omega_k} P_T(\Delta_{k-1})$$

It follows that,

$$\|P_T(Y)\|_2 = \left\| \sum_{k=1}^{k_0} (R_{\Omega_k} P_T - P_T) (\Delta_{k-1}) \right\|_2 \leq \sum_{k=1}^{k_0} \| (R_{\Omega_k} - I)(\Delta_{k-1}) \|_2$$

We use

$$P_T(\Delta_k) = P_T(UV^T - P_T(W_k)) = UV^T - P_T(W_k) = \Delta_k,$$

for all $k$. We apply Lemma 3 to each summand in the above inequality, with corresponding $\Omega_k$, to obtain

$$\|P_T(Y)\|_2 \leq 2 \sqrt{\frac{c}{c_0} \sum_{k=1}^{k_0} \|\Delta_{k-1}\|_{\mu(\infty,2)} + \frac{c}{c_0} \sum_{k=1}^{k_0} \|\Delta_{k-1}\|_{\mu(\infty)}}$$

(15)

holding with probability at least $1 - (m + n)^{1-c}$, for each $k$. We can derive the following, applying Lemma 4 $k$ times, with $\Omega_k$,

$$\|\Delta_k\|_{\mu(\infty)} = \|(P_T - P_T R_{\Omega_k}) \Delta_{k-1}\|_{\mu(\infty)} \leq \left(\frac{1}{2}\right)^i \|\Delta_{k-i}\|_{\mu(\infty)} \leq \left(\frac{1}{2}\right)^k \|UV^T\|_{\mu(\infty)}$$

(16)

For each $k$, the above fails with probability at most $k \cdot (m + n)^{3-c}$.

Similarly, applying Lemma 3 and Lemma 4 recursively, with $\Omega_k$, we can derive,

$$\|\Delta_k\|_{\mu(\infty,2)} = \|(P_T - P_T R_{\Omega_k} P_T) \Delta_{k-1}\|_{\mu(\infty,2)} \leq \frac{1}{2} \|\Delta_{k-1}\|_{\mu(\infty)} + \frac{1}{2} \|\Delta_{k-1}\|_{\mu(\infty,2)}$$

(17)
Above, we apply Lemma 3 \( k \) times, and Lemma 4 up to \( \sum_{i=1}^k i = k(k+1)/2 \) times. Thus, the above relation holds with failure probability at most \( k \cdot (m + n)^{2-c} + k(k+1) \cdot (m + n)^{3-c}/2 \), for each \( k \). Using (16) and (17), it follows from (15),

\[
\|P_{\perp}(Y)\|_2 \leq 2 \sqrt{\frac{c}{c_0} \sum_{k=1}^{k_0} (k-1) \left( \frac{1}{2} \right)^{k-1} \|UV\|^T \|_{\mu(\infty)} + 2 \sqrt{\frac{c}{c_0} \sum_{k=1}^{k_0} \left( \frac{1}{2} \right)^{k-1} \|UV\|^T \|_{\mu(\infty,2)}} \]

\[+ \frac{c}{c_0} \sum_{k=1}^{k_0} \left( \frac{1}{2} \right)^{k-1} \|UV\|^T \|_{\mu(\infty,2)} \]

We note that, for all \((i, j)\),

\[
|(UV^T)_{ij}| = |e^T_i UV e_j| \leq \sqrt{\frac{\mu_i \varrho}{m}} \sqrt{\frac{\nu_j \varrho}{n}} \leq 1,
\]

\[
\|(UV^T)_{i,\star}\|_2 = \|e^T_i UV^T\|_2 = \sqrt{\frac{\mu_i \varrho}{m}}, \quad \|(UV^T)_{\star,j}\|_2 = \|UV e_j\|_2 = \sqrt{\frac{\nu_j \varrho}{n}}
\]

Thus,

\[
\|UV\|^T \|_{\mu(\infty,2)} = \max \left\{ \max_i \sqrt{\frac{m}{\mu_i \varrho}} \|(UV^T)_{i,\star}\|_2, \max_j \sqrt{\frac{n}{\nu_j \varrho}} \|(UV^T)_{\star,j}\|_2 \right\} = 1
\]

Therefore,

\[
\|P_{\perp}(Y)\|_2 \leq 2 \sqrt{\frac{c}{c_0} \sum_{k=1}^{k_0} \left( \frac{1}{2} \right)^{k-1} + \left( \frac{1}{2} \right)^{k-1}} + \frac{c}{c_0} \sum_{k=1}^{k_0} \left( \frac{1}{2} \right)^{k-1}
\]

\[
< 2 \sqrt{\frac{c}{c_0} \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{k-1}} + \frac{c}{c_0} \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{k-1} = 8 \sqrt{\frac{c}{c_0} + \frac{2c}{c_0}} \leq 1/2,
\]

by setting \( c_0 \geq 264c \).

Let \( \delta = (m + n)^{3-c} \). The above holds with failure probability at most \( \delta + (k-1)\delta + \delta \cdot (k-1)(k+2)/2 \), for each \( k \). Summing over all \( k = 1, \ldots, k_0 \), total failure probability does not exceed \( \delta \cdot k_0^2 \). Here we upper bound the failure probability by simply counting the number of times the intermediate lemmas are called. A better bookkeeping could give us tighter bound. Also, we note that the constant \( c_1 \) in Theorem 1 depends on \( c_0 \) and \( k_0 \), where \( c_0 \) depends on \( c \) (related to failure probability). It is easy to verify that the final bound on \( c_0 \) remains the same for both the leverage score sampling \( \text{(Chen et al., 2014)} \) and the relaxed leverage score sampling in (4). Consequently, the final global constant \( c_1 \) is the same for both the sampling methods, for a fixed \( k_0 \) (we can set the same \( k_0 \) for both the sampling methods). Also, the failure probabilities are the same for both the sampling methods. Therefore, the sample size needed to solve (2) is strictly smaller for the relaxed leverage score sampling \( \text{(4)} \), comparing to the leveraged sampling in \( \text{Chen et al., 2014} \). Experimental results on real datasets in Section 3 is in compliance with this theoretical analysis.

We sample each \((i, j)\)-th entry independently with probability \( p_{ij} \) to form the set of observations \( \Omega \). That is, total number of sampled entries, denoted by \( s \), is a random variable. Expected number of observed entries required to solve (2) is

\[
\mathbb{E}(s) = \sum_{i,j} p_{ij} = O((m + n)g - g^2)\log^2(m + n)).
\]
Condition 2a and 2b in Proposition 1 fail with probabilities at most $k_0 \cdot (m + n)^{1-c}$ and $k_0^3 \cdot (m + n)^{3-c}$, respectively. Also, Lemma 1 used in Lemma 6 fails with probability at most $(m + n)^{1-c}$. Summing all of them, failure probability never exceeds $3 \log^3(m + n)(m + n)^{3-c}$ (a tighter bound could be calculated), for sufficiently large $c > 3$.

Finally, we can apply Hoeffding’s inequality to show that $s$ is sharply concentrated around its expectation, i.e., $s = \Theta(((m + n)\varrho - \varrho^2)\log^2(m + n))$ with probability at least $1 - 6 \log^3(m + n)(m + n)^{3-c}$, for sufficiently large $c > 3$.

This completes the proof of Theorem 1.

5 Proof of Theorem 2

We closely follow the proof given by Chen et al. [2014]. We pick each row of $M$ with some probability $p$ and observe all the entries of this sampled row. Let $\Gamma \subseteq [m]$ be the set of indices of the row picked, and $S_\Gamma(X)$ be a matrix obtained from $X$ by zeroing out the rows outside $\Gamma$. Recall, SVD of $M$ is $U\Sigma V^T$. We use the following lemma (Lemma 14 of Chen et al. [2014]).

**Lemma 5** Let $\mu_i \leq \max_{i \in [m]} \|Ue_i\|_2 \leq \mu_0$, $\forall i \in [m]$, and $p \geq c_2 \frac{\mu_0}{m} \varrho \log m$ for some universal constant $c_2$. Then, for any universal constant $c > 1$, and $c_2 \geq 20c$, $\|U^T S_\Gamma(U) - I_\varrho\|_2 \leq 1/2$, holds with probability at least $1 - (m + n)^{1-c}$, where $I_\varrho$ is the identity matrix in $\mathbb{R}^{\varrho \times \varrho}$.

Now, $\|U^T S_\Gamma(U) - I_\varrho\|_2 \leq 1/2$ implies that $U^T S_\Gamma(U)$ is invertible and $S_\Gamma(U) \in \mathbb{R}^{m \times \varrho}$ has rank-$\varrho$. Using SVD of $M$, we can write $S_\Gamma(M) = S_\Gamma(U)\Sigma V^T$, and this has full rank-$\varrho$. Therefore, $S_\Gamma(M)$ and $M$ have the same row space, and we conclude that $\hat{\nu}_j = \nu_j$, $\forall j \in [n]$. Thus, using the sample set $\Omega$ in Algorithm 1 we can recover $M$ exactly via nuclear norm minimization in (2), with high probability. Expected number of entries observed in Algorithm 1 is

$$pmn + \sum_{i,j} p_{ij} = O(\mu_0((m + 2n)\varrho - \varrho^2)\log^2(m + n)),$$

where, $p_{ij}$ as in (5). We apply standard Hoeffding inequality to bound the actual sample size, and Theorem 2 follows as a corollary of Theorem 1.

6 Proof of Optimality Conditions in Proposition 1

Let $M$ be the low-rank target matrix with rank-$\varrho$ SVD $M = U\Sigma V^T$. Let $Z$ be any matrix such that $R_{\Omega}(Z) = 0$, e.g., $Z$ is in the null space of $R_{\Omega}$ operator. We can choose $U_\perp$ and $V_\perp$ such that $[U, U_\perp]$ and $[V, V_\perp]$ are unitary matrices for which $\langle U_\perp V_\perp^T, P_{T_\perp}(Z) \rangle = \|P_{T_\perp}(Z)\|_*$. Then
it follows from standard inequality of trace norm, for some $Y$ in the range of $\mathcal{R}_\Omega$,

$$
\|M + Z\|_* \geq \langle UV^T + U \perp V_\perp^T, M + Z \rangle \\
= \|M\|_* + \langle UV^T + U \perp V_\perp^T, Z \rangle \\
= \|M\|_* + \langle UV^T - P_T(Y), P_T(Z) \rangle + \langle U \perp V_\perp^T - P_T(\perp Y), P_T(\perp Z) \rangle \\
\geq \|M\|_* - \|UV^T - P_T(Y)\|_F \cdot \|P_T(Z)\|_F + \|P_T(\perp Z)\|_F - \langle P_T(\perp Y), P_T(\perp Z) \rangle \\
\geq \|M\|_* - \|UV^T - P_T(Y)\|_F \cdot \|P_T(Z)\|_F + (1 - \|P_T(\perp Y)\|_2) \|P_T(\perp Z)\|_F \\
\geq \|M\|_* + \left(1 - \|P_T(\perp Y)\|_2\right) - \frac{\|UV^T - P_T(Y)\|_F \left(\max_{i,j} \frac{1}{\sqrt{p_{ij}}} \right)}{(1 - \|P_T \mathcal{R}_\Omega P_T - P_T\|_{op})^2} \|P_T(\perp Z)\|_F \\
> \|M\|_*
$$

Above, (a) follows from Von-Neumann trace inequality, and (b) follows from Lemma 6. Using $\max_{i,j} \frac{1}{\sqrt{p_{ij}}} \leq (mn)^{5/2} \leq (m+n)^3$, and the conditions in Proposition 1 we derive the final inequality. Thus, if there exists a dual certificate $Y$ satisfying Condition 2a and 2b in Proposition 1, we have for any $X$ obeying $\mathcal{R}_\Omega(M - X) = 0$, i.e., $X_{ab} = M_{ab}$, for all $(a, b) \in \Omega$, $\|X\|_* > \|M\|_*$. Hence, $M$ is the unique minimizer of $\mathcal{Z}$.

The following lemma is similar to Lemma 13 of [Chen et al. 2014].

**Lemma 6** For any $Z \in \mathbb{R}^{m \times n}$, s.t., $P_\Omega(Z) = 0$,

$$
\|P_T(Z)\|_F \leq \left(1 - \|P_T \mathcal{R}_\Omega P_T - P_T\|_{op}\right)^{-\frac{1}{2}} \left(\max_{i,j} \frac{1}{\sqrt{p_{ij}}} \right) \|P_T(\perp Z)\|_*
$$

**Proof:** Let us define the operator $\mathcal{R}_\Omega^{1/2} : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ as

$$
\mathcal{R}_\Omega^{1/2}(Z) := \sum_{i,j} \frac{1}{\sqrt{p_{ij}}} \delta_{ij} \langle e_i e_j^T, Z \rangle e_i e_j^T
$$

Note that $\mathcal{R}_\Omega^{1/2}$ is self-adjoint, and $\mathcal{R}_\Omega^{1/2} \mathcal{R}_\Omega^{1/2} = \mathcal{R}_\Omega$. Therefore, we have

$$
\left\|\mathcal{R}_\Omega^{1/2} P_T(Z)\right\|_F^2 = \langle \mathcal{R}_\Omega P_T(Z), P_T(Z) \rangle \\
= \langle P_T \mathcal{R}_\Omega P_T(Z), P_T(Z) \rangle \\
= \langle P_T \mathcal{R}_\Omega P_T(Z) - P_T(\perp Z), P_T(Z) \rangle + \langle P_T(\perp Z), P_T(Z) \rangle \\
\geq (1 - \|P_T \mathcal{R}_\Omega P_T - P_T\|_{op}) \cdot \|P_T(\perp Z)\|_F^2
$$

(18)

Also, we have $\left\|\mathcal{R}_\Omega^{1/2}(Z)\right\|_F = 0$ for any $Z$ s.t. $P_\Omega(Z) = 0$. It follows,

$$
0 = \left\|\mathcal{R}_\Omega^{1/2}(Z)\right\|_F \geq \left\|\mathcal{R}_\Omega^{1/2} P_T(Z)\right\|_F - \left\|\mathcal{R}_\Omega^{1/2} P_T(\perp Z)\right\|_F \\
\left\|\mathcal{R}_\Omega^{1/2} P_T(Z)\right\|_F \leq \left\|\mathcal{R}_\Omega^{1/2} P_T(\perp Z)\right\|_F \leq \left(\max_{i,j} \frac{1}{\sqrt{p_{ij}}} \right) \|P_T(\perp Z)\|_F,
$$

(19)

where we use

$$
\left\|\mathcal{R}_\Omega^{1/2} P_T(\perp Z)\right\|_F \leq \max_{i,j} \frac{1}{\sqrt{p_{ij}}} \left\|\sum_{i,j} \delta_{ij} \langle e_i e_j^T, P_T(\perp Z) \rangle e_i e_j^T\right\|_F \leq \max_{i,j} \frac{1}{\sqrt{p_{ij}}} \|P_T(\perp Z)\|_F
$$
Combining (18) and (19), and using \( \|X\|_F \leq \|X\|_s \),
\[
\sqrt{(1 - \|P_T \mathcal{R}_\Omega P_T - P_T\|_{op}) \cdot \|P_T(Z)\|_F} \leq \left( \max_{i,j} \frac{1}{\sqrt{p_{ij}}} \right) \|P_T(\mathcal{Z})\|_F \leq \left( \max_{i,j} \frac{1}{\sqrt{p_{ij}}} \right) \|P_T(\mathcal{Z})\|_s
\]
The result follows.

7 Proof of Technical Lemmas

Here we prove Lemmas 1 through 5 using the matrix Bernstein inequality of Lemma 8 as the main tool. Also, we frequently use the fact in (21) and the result in Lemma 7. Note that \( P_T \) is self-adjoint linear operator. Thus we can write the following for any \( X \in \mathbb{R}^{m \times n} \):

\[
P_T(X) = \sum_{i,j} \langle P_T(X), e_i e_j^T \rangle e_i e_j^T = \sum_{i,j} \langle P_T(X), P_T(e_i e_j^T) \rangle e_i e_j^T = \sum_{i,j} \langle X, P_T(e_i e_j^T) \rangle e_i e_j^T
\]

(20)

We can derive, for all \( i \) and \( j \),
\[
\|P_T(e_i e_j^T)\|_F^2 = \langle P_T(e_i e_j^T), e_i e_j^T \rangle = \frac{\mu_i \theta}{m} + \frac{\nu_j \theta}{n} - \frac{\mu_i \theta}{m} \cdot \frac{\nu_j \theta}{n}
\]

(21)

Also, we know for all \( i, j \),
\[
0 \leq \frac{\mu_i \theta}{m} \leq \sqrt{\frac{\mu_i \theta}{m}} \leq 1, \quad 0 \leq \frac{\nu_j \theta}{n} \leq \sqrt{\frac{\nu_j \theta}{n}} \leq 1.
\]

(22)

Lemma 7 Using our notations, for all \( i, j \),
\[
\frac{\mu_i \theta}{m} + \frac{\nu_j \theta}{n} - \frac{\mu_i \theta}{m} \cdot \frac{\nu_j \theta}{n} \geq \frac{\mu_i \theta}{m} \cdot \frac{\nu_j \theta}{n}
\]

Proof: Let, \( x = \frac{\mu_i \theta}{m} \) and \( y = \frac{\nu_j \theta}{n} \). Then,
\[
(x + y - xy)^2 = xy + (x^2 - x^2 y) + (y^2 - xy^2) + x^2 y^2 + xy - x^2 y - xy^2 = xy + x^2 (1 - y) + y^2 (1 - x) + xy(1 - x)(1 - y) \geq xy \text{ using (22)}
\]

Also, \( x + y - xy \geq 0 \). Thus, \( x + y - xy \geq \sqrt{xy} \).

Lemma 8 [Tropp [2012], (Theorem 16] of Chen et al. [2014])

Let \( X_1, ..., X_N \in \mathbb{R}^{m \times n} \) be independent, zero-mean random matrices. Suppose
\[
\max \left\{ \left\| \sum_{t=1}^N \mathbb{E} \left[ X_t X_t^T \right] \right\|_2, \left\| \sum_{t=1}^N \mathbb{E} \left[ X_t^T X_t \right] \right\|_2 \right\} \leq \sigma^2
\]

and \( \|X_t\|_2 \leq \gamma \) almost surely for all \( t \). Then for any \( c > 0 \), we have
\[
\left\| \sum_{t=1}^N X_t \right\|_2 \leq 2\sqrt{\sigma^2 \log(m + n) + c \gamma \log(m + n)}
\]

with probability at least \( 1 - (m + n)^{-c/2} \).

We consider sampling probabilities \( \{q_{ij}\} \) of the form (11) to prove Lemmas 1 through 4.

Notation Overloading: For simplicity, we reuse some of the notations in Section 7.1 through 7.4. Specifically, we replace \( \Omega \) by \( \Omega \) to denote a sample set in (13), and, \( \delta_{ij} = I((i, j) \in \Omega) \).
7.1 Proof of Lemma 1

For any matrix \( Z \in \mathbb{R}^{m \times n} \), we can write

\[
(P_T R_{\Omega} P_T - P_T) (Z) = \sum_{i,j} \left( \frac{1}{q_{ij}} \delta_{ij} - 1 \right) \langle P_T (e_i e_j^T), Z \rangle P_T (e_i e_j^T) = \sum_{i,j} S_{ij}(Z).
\]

Using \( \mathbb{E} [\delta_{ij}] = q_{ij} \), we have \( \mathbb{E} [S_{ij}(Z)] = 0 \) for any \( Z \). Thus, we conclude that \( \mathbb{E} [S_{ij}] = 0 \). Also, \( S_{ij} \)'s are independent of each other. Using probabilities in (11) (\( S_{ij} \)'s vanish when \( q_{ij} = 1 \), for all \( Z \) and \( (i, j) \)), and (21), we derive

\[
\|S_{ij}(Z)\|_F \leq \frac{1}{q_{ij}} \|P_T (e_i e_j^T)\|_F^2 \|Z\|_F \leq \frac{\|Z\|_F}{c_0 \cdot \log(m + n)}.
\]

From definition of operator norm, \( \|S_{ij}\|_{op} \leq \frac{1}{c_0 \cdot \log(m + n)} \). Also, we derive

\[
\mathbb{E} \left[ S_{ij}^2(Z) \right] = \mathbb{E} \left[ \left( \frac{1}{q_{ij}} \delta_{ij} - 1 \right)^2 \right] \langle e_i e_j^T, P_T(Z) \rangle \langle e_i e_j^T, P_T(e_i e_j^T) \rangle P_T (e_i e_j^T)
\]

\[
= \frac{1 - q_{ij}}{q_{ij}} \langle e_i e_j^T, P_T(Z) \rangle \langle e_i e_j^T, P_T(e_i e_j^T) \rangle P_T (e_i e_j^T)
\]

\[
\left\| \sum_{i,j} \mathbb{E} \left[ S_{ij}^2(Z) \right] \right\|_F \leq \left( \max_{i,j} \frac{1 - q_{ij}}{q_{ij}} \|P_T(e_i e_j^T)\|_F^2 \right) \left\| \sum_{i,j} \langle e_i e_j^T, P_T(Z) \rangle \|P_T (e_i e_j^T)\|_F \right\|_F
\]

\[
= \left( \max_{i,j} \frac{1 - q_{ij}}{q_{ij}} \|P_T(e_i e_j^T)\|_F^2 \right) \|P_T \left( \sum_{i,j} \langle e_i e_j^T, P_T(Z) \rangle (e_i e_j^T) \right) \|_F \]

\[
\leq \left( \max_{i,j} \frac{1 - q_{ij}}{q_{ij}} \|P_T(e_i e_j^T)\|_F^2 \right) \|P_T(Z)\|_F
\]

\[
\left\| \sum_{i,j} \mathbb{E} \left[ S_{ij}^2 \right] \right\|_{op} \leq \max_{i,j} \frac{1 - q_{ij}}{q_{ij}} \|P_T(e_i e_j^T)\|_F^2 \leq \frac{1}{c_0 \cdot \log(m + n)}
\]

We apply Matrix Bernstein inequality in Lemma 5 using

\[
\sigma^2 = \frac{1}{c_0 \cdot \log(m + n)}, \quad \gamma = \frac{1}{c_0 \cdot \log(m + n)},
\]

to obtain, for any \( c > 1, c_0 \geq 20c \),

\[
\|P_T R_{\Omega} P_T - P_T\|_{op} \leq 1/2
\]

holding with probability at least

\[
1 - (m + n)^{(1-c)}.
\]
7.2 Proof of Lemma 2

We can write the matrix \((R_\Omega - I)Z\) as sum of independent matrices:

\[
(R_\Omega - I)Z = \sum_{i,j} \left( \frac{1}{q_{ij}} \delta_{ij} - 1 \right) Z_{ij}e_ie_j^T = \sum_{i,j} S_{ij}.
\]

We note that, \(E[S_{ij}] = 0\), and \(S_{ij}\)'s are zero matrix when \(q_{ij} = 1\), for all \((i,j)\). We have \(\|S_{ij}\|_2 \leq \frac{|Z_{ij}|}{q_{ij}}\). Moreover,

\[
\sum_{i,j} E \left[ S_{ij}S_{ij}^T \right] = \sum_{i,j} Z_{ij}^2 e_ie_i^T \left[ \left( \frac{1}{q_{ij}} \delta_{ij} - 1 \right)^2 \right] = \sum_i \left( \sum_j Z_{ij}^2 \frac{1 - q_{ij}}{q_{ij}} \right) e_ie_i^T
\]

Thus,

\[
\left\| \sum_{i,j} E \left[ S_{ij}S_{ij}^T \right] \right\|_2 \leq \max_i \sum_{j=1}^n \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2
\]

Similarly,

\[
\left\| \sum_{i,j} E \left[ S_{ij}^T S_{ij} \right] \right\|_2 \leq \max_j \sum_{i=1}^m \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2
\]

Clearly, when \(q_{ij} = 1\) the above quantities are zero. Using \(q_{ij}\) in (11), and Lemma 7, we have

\[
\|S_{ij}\|_2 \leq \frac{1}{c_0 \cdot \log(m+n)} |Z_{ij}| \sqrt{\frac{m}{\mu_0}} \sqrt{\frac{n}{\nu_0}} \leq \frac{\|Z\|_{\mu(\infty)}}{c_0 \cdot \log(m+n)}.
\]

Using \(q_{ij}\) in (11), and noting that \((\frac{\nu_0}{m} \frac{\nu_0}{n} - \frac{\mu_0}{m} \frac{\nu_0}{n}) \geq \frac{\mu_0}{m}\), we have

\[
\sum_{j=1}^n \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2 \leq \frac{1}{c_0 \cdot \log(m+n)} \cdot \frac{m}{\mu_0} \sum_{j=1}^n Z_{ij}^2 \leq \frac{\|Z\|_{\mu(\infty,2)}^2}{c_0 \cdot \log(m+n)}.
\]

Similarly,

\[
\sum_{i=1}^m \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2 \leq \frac{1}{c_0 \cdot \log(m+n)} \cdot \frac{n}{\nu_0} \sum_{i=1}^m Z_{ij}^2 \leq \frac{\|Z\|_{\mu(\infty,2)}^2}{c_0 \cdot \log(m+n)}.
\]

The lemma follows from Matrix Bernstein inequality in Lemma 8 with

\[
\gamma \log(m+n) \leq \frac{1}{c_0} \|Z\|_{\mu(\infty)}, \quad \sigma^2 \log(m+n) \leq \frac{1}{c_0} \|Z\|_{\mu(\infty,2)}^2.
\]

7.3 Proof of Lemma 3

Let,

\[
X = (P_T R_\Omega - P_T)Z = \sum_{i,j} \left( \frac{\delta_{ij}}{q_{ij}} - 1 \right) Z_{ij} P_T(e_ie_j^T)
\]

Weighted \(b\)-th column of \(X\) can be written as sum of independent, zero-mean column vectors.

\[
\sqrt{\frac{n}{\nu_b \rho}} X_{s,b} = \sum_{i,j} \left( \frac{\delta_{ij}}{q_{ij}} - 1 \right) Z_{ij} (P_T(e_ie_j^T)e_b) \sqrt{\frac{n}{\nu_b \rho}} = \sum_{i,j} s_{ij}
\]

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Clearly, \(E[s_{ij}] = 0\). We need bounds on \(\|s_{ij}\|_2\) and \(\left\| \sum_{i,j} E\left[s_{ij}^T s_{ij}\right] \right\|_2\) to apply Matrix Bernstein inequality. First, we need to bound \(\|P_T(e_i e_j^T) e_b\|_2\).

\[
\|P_T(e_i e_j^T) e_b\|_2 = \|UU^T(e_i e_j^T) e_b + (e_i e_j^T) VV^T e_b - UU^T(e_i e_j^T) VV^T e_b\|_2
\]

\[
= \begin{cases}
\|UU^T e_i + (I - UU^T) e_i\| VV^T e_b\|_2 \leq \sqrt{\frac{\mu_i q}{m}} + \frac{\nu_i q}{n} & j = b, \\
\| (I - UU^T) e_i e_j^T VV^T e_b\|_2 \leq \|e_j^T VV^T e_b\| & j \neq b,
\end{cases}
\]

(A) Above we use triangle inequality and definition of \(\mu_i\) and \(\nu_i\). Note that, \(s_{ij}\) is a zero vector when \(q_{ij} = 1\), for all \((i, j)\). Otherwise, for \(q_{ij} \neq 1\), we consider two cases. Using bounds in (23), we have for \(j = b\),

\[
\|s_{ij}\|_2 \leq \frac{1}{q_{ib}} \|Z\| \sqrt{\frac{n}{\nu_i q}} \left( \frac{\mu_i q}{m} + \frac{\nu_i q}{n} \right)
\]

Using \(q_{ij}\) in (11), \(q_{ib} \geq c_0 \log(m+n)\sqrt{\frac{\mu_i q}{m}}\sqrt{\frac{\nu_i q}{n}}\) and \(q_{ib} \geq c_0 \log(m+n)\cdot \frac{\mu_i q}{m}\). Combining these two inequalities, we have

\[
\|s_{ij}\|_2 \log(m+n) \leq \frac{2}{c_0} \|Z\| \sqrt{\frac{m}{\mu_i q}} \cdot \sqrt{\frac{n}{\nu_i q}} \left( \frac{\mu_i q}{m} + \frac{\nu_i q}{n} \right) \leq \frac{2}{c_0} \|Z\|_{\mu(\infty)}
\]

For \(j \neq b\), using \(q_{ib} \geq c_0 \log(m+n)\sqrt{\frac{\mu_i q}{m}}\sqrt{\frac{\nu_i q}{n}}\) (Lemma 1) and \(e_j^T VV^T e_b\) \(\leq \frac{\nu_i q}{n} + \frac{\nu_i q}{n}\),

\[
\|s_{ij}\|_2 \leq \frac{1}{q_{ij}} \|Z|j| \sqrt{\frac{n}{\nu_i q}} \cdot \sqrt{\frac{\nu_i q}{n}} \leq \frac{2}{c_0 \log(m+n)} \|Z\|_{\mu(\infty)}
\]

Therefore, for all \((i, j)\), we have \(\|s_{ij}\|_2 \leq \frac{2}{c_0 \log(m+n)} \|Z\|_{\mu(\infty)}\).

On the other hand,

\[
\sum_{i,j} E\left[s_{ij}^T s_{ij}\right] = \left( \sum_{j=b,i} + \sum_{j \neq b,i} \right) \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2 \frac{\|P_T(e_i e_j^T) e_b\|_2}{2} \cdot \frac{n}{\nu_i q}
\]

The above quantity is zero for \(q_{ij} = 1\). Otherwise, for \(q_{ij} \neq 1\), we consider two cases.

For \(j = b\), using (23) we have, \(\|P_T(e_i e_j^T) e_b\|_2 \leq \left( \sqrt{\frac{\mu_i q}{m}} + \sqrt{\frac{\nu_i q}{n}} \right) \leq 2 \left( \frac{\mu_i q}{m} + \frac{\nu_i q}{n} \right)\).

Using \(q_{ij}\) in (11), we have,

\[
\sum_{j=b,i} \leq 2 \sum_{i} \frac{1 - q_{ib}}{q_{ib}} Z_{ib}^2 \frac{\mu_i q}{m} \cdot \frac{n}{\nu_i q} \leq \frac{4}{c_0 \log(m+n)} \|Z\|_{\mu(\infty, 2)}^2,
\]

where we use the following bound in the second inequality. For all \((i, j)\), \(q_{ij} \neq 0\),

\[
\frac{\mu_i q}{m} + \frac{\nu_i q}{n} \leq \frac{\mu_i q}{m} + \frac{\nu_i q}{n} \leq 1 + \frac{\mu_i q}{m} \cdot \frac{\nu_i q}{n} \leq 1 + \frac{\mu_i q}{m} \cdot \frac{\nu_i q}{n} \leq 2.
\]
For \( j \neq b \), using \( q_{ij} \geq c_0 \log(m + n) \cdot \frac{\nu_{b,q}}{n} \) and \([23]\),

\[
\sum_{j \neq b,i} \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2 \left| e_j^T V V^T e_b \right|^2 \leq \frac{n}{\nu_{b,q}} \sum_{j \neq b} \left| e_j^T V V^T e_b \right|^2 \sum_{i} \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2
\]

\[
= \frac{n}{\nu_{b,q}} \sum_{j \neq b} \left| e_j^T V V^T e_b \right|^2 \left( \frac{1}{c_0 \log(m + n)} \cdot \frac{n}{\nu_{b,q}} \sum_{i} Z_{ij}^2 \right)
\]

\[
\leq \left( \frac{\|Z\|_{\mu(\infty, 2)}^2}{c_0 \log(m + n)} \right) \frac{n}{\nu_{b,q}} \sum_{j \neq b} \left| e_j^T V V^T e_b \right|^2
\]

\[
\leq \frac{\|Z\|_{\mu(\infty, 2)}^2}{c_0 \log(m + n)},
\]

where the last inequality follows from, \( \sum_{j \neq b} \left| e_j^T V V^T e_b \right|^2 \leq \|V V^T e_b\|_2^2 \leq \frac{\nu_{b,q}}{n} \).

Combining the two summations,

\[
\left\| \sum_{i,j} E \left[ s_i^T s_{ij} \right] \right\|_2 \leq \frac{5}{c_0 \log(m + n)} \|Z\|_{\mu(\infty, 2)}^2
\]

We can bound \( \left\| E \left[ \sum_{i,j} s_i s_{ij}^T \right] \right\|_2 \) in a similar way.

We apply Matrix Bernstein inequality in Lemma [S] with

\[
\gamma = \frac{2}{c_0 \log(m + n)} \|Z\|_{\mu(\infty)}, \quad \sigma^2 = \frac{5}{c_0 \log(m + n)} \|Z\|_{\mu(\infty, 2)}^2,
\]

to obtain

\[
\left\| \sum_{i,j} s_{ij} \right\|_2 \leq \sqrt{\frac{20c}{c_0}} \|Z\|_{\mu(\infty, 2)} + \frac{2c}{c_0} \|Z\|_{\mu(\infty)}.
\]

We set \( c_0 \geq 80c \) to derive

\[
\left\| \sqrt{\frac{n}{\nu_{b,q}}} X_{*,b} \right\|_2 = \left\| \sum_{i,j} s_{ij} \right\|_2 \leq \frac{1}{2} \left( \|Z\|_{\mu(\infty, 2)} + \|Z\|_{\mu(\infty)} \right).
\]

Similarly, we can bound \( \left\| \sqrt{\frac{n}{\mu_{a,q}}} X_{a,*} \right\|_2 \) by the same quantity. We take a union bound over all rows \( a \) and all columns \( b \) (i.e., total \((m + n)\) events) to obtain, for any \( c > 2 \),

\[
\| (P_T R_{\Omega} - P_T) (Z) \|_{\mu(\infty, 2)} \leq \frac{1}{2} \left( \|Z\|_{\mu(\infty, 2)} + \|Z\|_{\mu(\infty)} \right)
\]

holding with probability at least \( 1 - (m + n)^2 - c \).
7.4 Proof of Lemma 4

Let, $X = (\mathcal{P}_T \mathcal{R}_\Omega - \mathcal{P}_T)Z = \sum_{i,j} \left(\frac{\delta_{ij}}{Q_{ij}} - 1\right)Z_{ij} \left(\mathcal{P}_T(e_ie_j^T)\right)$. We write rescaled $(a,b)$-th element of $X$ as

$$\left[X\right]_{ab} = \sum_{i,j} \left(\frac{\delta_{ij}}{Q_{ij}} - 1\right)Z_{ij} \left(\mathcal{P}_T(e_ie_j^T)\right)_{ab} = \sum_{i,j} s_{ij}$$

This is a sum of independent, zero-mean random variables. We seek to bound $|s_{ij}|$ and $\sum_{i,j} \mathbb{E} \left[ s_{ij}^2 \right]$. First, we need to bound $|\langle e_a e_b^T, \mathcal{P}_T(e_i e_j^T) \rangle|$.

$$= \left| \frac{e_a^T UU^T \langle e_i e_j^T \rangle e_b + e_b^T \langle e_i e_j^T \rangle VV^T e_b - e_a^T UU^T \langle e_i e_j^T \rangle VV^T e_b}{\mathcal{P}_T(e_a e_b^T)} \right|^2_F = \frac{a a a + \nu \nu \nu}{m m m - \mu \mu \mu}$$

$$= \left\{ \begin{array}{ll}
\langle e_i^T (I - UU^T) e_a e_j^T \rangle VV^T e_b & \leq \langle e_i^T \rangle VV^T e_b & i = a, j \neq b, \\
\langle e_a^T UU^T e_i e_j^T \rangle VV^T e_b & \leq \langle e_a^T UU^T e_i \rangle \rangle VV^T e_b & i \neq a, j = b, \\
\langle e_b^T UU^T e_i e_j^T \rangle VV^T e_b & \leq \langle e_b^T UU^T e_i \rangle \rangle VV^T e_b & i \neq a, j \neq b \end{array} \right. \quad (24)$$

where we use $\|I - UU^T\|_2 \leq 1$ and $\|I - VV^T\|_2 \leq 1$.

Note that, $s_{ij} = 0$ when $q_{ij} = 1$. Otherwise, for $q_{ij} \neq 1$,

$$|s_{ij}| \leq \frac{1}{q_{ij}} |Z_{ij}| \left| \langle e_a e_b^T, \mathcal{P}_T(e_i e_j^T) \rangle \right| \sqrt{\frac{m}{\mu_{q}} \sqrt{n}} \frac{n}{v_{q}} \nu$$

We consider four cases.

For $i = a, j = b$, using $q_{ab} \geq c_0 \log(m + n) \left(\frac{\mu_a}{m} + \frac{\nu_a}{n} - \mu_a \frac{\nu_a}{m \cdot n}\right)$

$$|s_{ij}| \leq \frac{1}{q_{ab}} |Z_{ab}| \left| \mathcal{P}_T(e_a e_b^T) \right|^2_F \sqrt{\frac{m}{\mu_{q}} \sqrt{n}} \frac{n}{v_{q}} \nu$$

$$\leq \frac{c_0 \log(m + n)}{\mu_a \nu_a} \sqrt{\frac{m}{\mu_{q}} \sqrt{n}} \frac{n}{v_{q}} \nu \leq \frac{|Z|_{\mu(\infty)}}{c_0 \log(m + n)}$$

For $i = a, j \neq b$, using $q_{aj} \geq c_0 \log(m + n) \left(\frac{\mu_a}{m} + \frac{\nu_j}{n} - \mu_a \frac{\nu_j}{m \cdot n}\right) \geq c_0 \log(m + n) \frac{\nu_j}{n}$,

$$|s_{ij}| \leq \frac{|Z_{aj}|}{q_{aj}} \left| e_j^T VV^T e_b \right| \sqrt{\frac{m}{\mu_{q}} \sqrt{n}} \frac{n}{v_{q}} \nu \leq \frac{|Z_{aj}|}{c_0 \log(m + n)} \sqrt{\frac{m}{\mu_a \nu_j} \sqrt{n}} \frac{n}{v_{q}} \nu \leq \frac{|Z|_{\mu(\infty)}}{c_0 \log(m + n)}$$

Similarly, for $i \neq a, j = b$, using $q_{ib} \geq c_0 \log(m + n) \frac{\mu_b}{m}$

$$|s_{ij}| \leq \frac{|Z|_{\mu(\infty)}}{c_0 \log(m + n)}$$

For $i \neq a, j \neq b$, using $q_{ij} \geq c_0 \log(m + n) \sqrt{\frac{\mu_a}{m} \cdot \frac{\nu_j}{n}}$

$$|s_{ij}| \leq \frac{1}{q_{ij}} |Z_{ij}| \left| e_a^T UU^T e_i \right| \left| e_j^T VV^T e_b \right| \sqrt{\frac{m}{\mu_{q}} \sqrt{n}} \frac{n}{v_{q}} \nu$$

$$\leq \frac{1}{q_{ij}} \left| Z_{ij} \right| \sqrt{\frac{\mu_a}{m} \sqrt{n}} \frac{n}{v_{q}} \nu \leq \frac{1}{c_0 \log(m + n)} \left| Z_{ij} \right| \sqrt{\frac{m}{\mu_a \nu_j} \sqrt{n}} \frac{n}{v_{q}} \nu \leq \frac{1}{c_0 \log(m + n)} |Z|_{\mu(\infty)}.$$
Above we use $\sqrt{\frac{\mu_\alpha}{m}} \leq 1$, $\sqrt{\frac{\nu_\beta}{n}} \leq 1$, for all $i, j$. We conclude, for all $(i, j)$,

$$|s_{ij}| \leq \frac{1}{c_0 \log(m+n)} \|Z\|_{\mu(\infty)}.$$

On the other hand,

$$\left| \sum_{i,j} \mathbb{E} \left[ s_{ij}^2 \right] \right| = \sum_{i,j} \mathbb{E} \left[ \left( \frac{\delta_{ij} - 1}{q_{ij}} \right)^2 \right] Z_{ij}^2 \langle e_a e_b^T, \mathcal{P}_T(e_a e_b^T) \rangle \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta}$$

$$= \sum_{i,j} \frac{1 - q_{ij}}{q_{ij}} Z_{ij}^2 \langle e_a e_b^T, \mathcal{P}_T(e_a e_b^T) \rangle \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta}$$

$$= \sum_{i=a,j=b} + \sum_{i=a,j \neq b} + \sum_{i \neq a,j=b} + \sum_{i \neq a,j \neq b}$$

The above quantity is zero for $q_{ij} = 1$. We bound the above considering four cases for $q_{ij} \neq 1$.

For $i = a, j = b$, using $q_{ab} \geq c_0 \log(m+n) \left( \frac{\mu_\alpha}{m} + \frac{\nu_\beta}{n} - \frac{\mu_\alpha}{m} \cdot \frac{\nu_\beta}{n} \right)$,

$$\sum_{i=a,j=b} \leq Z_{ab}^2 \left( \frac{\mu_\alpha}{m} + \frac{\nu_\beta}{n} - \frac{\mu_\alpha}{m} \cdot \frac{\nu_\beta}{n} \right)^2 \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta} \leq \frac{\|Z\|_{\mu(\infty)}^2}{c_0 \log(m+n)}.$$

Above we use $\left( \frac{\mu_\alpha}{m} + \frac{\nu_\beta}{n} - \frac{\mu_\alpha}{m} \cdot \frac{\nu_\beta}{n} \right) \leq 1$, for all $i$ and $j$.

For $i = a, j \neq b$, using $q_{aj} \geq c_0 \log(m+n) \frac{\nu_\beta}{n}$,

$$\sum_{j \neq b} \leq \sum_{j \neq b} q_{aj} Z_{aj}^2 \|e_j^T V V^T e_b\|^2 \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta} \leq \frac{1}{c_0 \log(m+n)} \sum_{j \neq b} Z_{aj}^2 \left( \frac{n}{\nu_\beta} \frac{m}{\mu_\alpha} \right) \|e_j^T V V^T e_b\|^2 \frac{n}{\nu_\beta}$$

$$\leq \frac{1}{c_0 \log(m+n)} \|Z\|_{\mu(\infty)}^2.$$

Above we use,

$$\sum_{j \neq b} |e_j^T V V^T e_b|^2 \leq \|VV^T e_b\|_2^2 \leq \frac{\nu_\beta}{n}.$$

Similarly, we can derive identical bound for $\sum_{i \neq a, j = b}$.

We use $q_{ij} \geq c_0 \log(m+n) \sqrt{\frac{\mu_\alpha}{m} \cdot \frac{\nu_\beta}{n}} \geq c_0 \log(m+n) \frac{\mu_\alpha}{m} \cdot \frac{\nu_\beta}{n}$ to bound

$$\sum_{i \neq a, j \neq b} \leq \sum_{i \neq a, j \neq b} \frac{1}{q_{ij}} Z_{ij}^2 \|e_a^T U U^T e_i\|^2 \|e_j^T V V^T e_b\|^2 \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta} \leq \frac{\|Z\|_{\mu(\infty)}^2}{c_0 \log(m+n)} \sum_{i \neq a, j \neq b} \|e_a^T U U^T e_i\|^2 \|e_j^T V V^T e_b\|^2 \frac{m}{\mu_\alpha} \cdot \frac{n}{\nu_\beta}$$

$$= \frac{\|Z\|_{\mu(\infty)}^2}{c_0 \log(m+n)} \sum_{i \neq a} \|e_a^T U U^T e_i\|^2 \frac{m}{\mu_\alpha} \sum_{j \neq b} \|e_j^T V V^T e_b\|^2 \frac{n}{\nu_\beta}$$

$$\leq \frac{\|Z\|_{\mu(\infty)}^2}{c_0 \log(m+n)}.$$
Combining the summations, we derive

\[
\left| \sum_{i,j} E [s_{ij}^2] \right| \leq \frac{4 \|Z\|_{\mu(\infty)}^2}{c_0 \log(m + n)}.
\]

We now apply Bernstein inequality in Lemma 8 to obtain, for any \( c > 3, c_0 \geq 68c \)

\[
\| (\mathcal{P}_T \mathcal{R}_\Omega - \mathcal{P}_T)(Z) \|_{\mu(\infty)} \leq \frac{1}{2} \|Z\|_{\mu(\infty)}
\]

We take union bound over all \((a, b)\) (i.e., total \( mn \leq (m + n)^2 \) events) to conclude that the above result holds with probability at least

\[
1 - (m + n)^{\left(3 - c\right)}.
\]

### 7.5 Proof of Lemma 5

Let \( \delta_i = \mathbb{I}(i \in \Gamma) \), where \( \mathbb{I}(\cdot) \) is the indicator function. We can write,

\[
\mathbf{U}^T \mathcal{S}_\Gamma(U) - \mathbf{I}_g = \mathbf{U}^T \mathcal{S}_\Gamma(U) - \mathbf{U}^T \mathbf{U} = \sum_{i=1}^{m} \left( \frac{1}{p} \delta_i - 1 \right) \mathbf{U}^T \mathbf{e}_i \mathbf{e}_i^T \mathbf{U} = \sum_{i=1}^{m} \mathbf{S}_i,
\]

where \( \mathbf{S}_i \)'s are independent of each other. Clearly, \( E[\mathbf{S}_i] = 0_{m \times n} \).

Note that, \( \|\mathbf{S}_i\|_2^2 \leq \frac{1}{p} \|\mathbf{U}^T \mathbf{e}_i\|_2^2 \leq \frac{w_m}{pm} \). Also,

\[
\left\| E \left[ \sum_{i=1}^{m} \mathbf{S}_i \mathbf{S}_i^T \right] \right\|_2 = \left\| E \left[ \sum_{i=1}^{m} \mathbf{S}_i^T \mathbf{S}_i \right] \right\|_2 = \frac{1-p}{p} \left\| \sum_{i=1}^{m} \mathbf{U}^T \mathbf{e}_i \mathbf{e}_i^T \mathbf{U} \right\|_2
\]

\[
= \frac{1-p}{p} \left\| \mathbf{U}^T \left( \sum_{i=1}^{m} \mathbf{e}_i \mathbf{e}_i^T \mathbf{U} \mathbf{U}^T \right) \mathbf{U} \right\|_2
\]

\[
\leq \frac{1}{p} \left\| \sum_{i=1}^{m} \mathbf{e}_i \mathbf{e}_i^T \mathbf{U} \mathbf{U}^T \right\|_2
\]

\[
\leq \frac{1}{p} \max_i \|\mathbf{U}^T \mathbf{e}_i\|_2^2 \leq \frac{\mu_0 \phi}{pm}.
\]

We apply the matrix Bernstein inequality in Lemma 8 to derive the result.

### 8 Conclusion

It is possible to recover any arbitrary low-rank data matrix exactly via the optimization problem in (2) using the relaxed leverage score sampling proposed in this work. This notion of relaxation in leverage scores requires a strictly smaller sample size comparing to the best-known result of Chen et al. [2014]. Experimental results on real data sets corroborate the theoretical analysis.

It would be an interesting problem to reduce the bound on the sample size by a logarithmic factor to \( \Theta(((m + n)^g - \varphi^2)\log(m + n)) \). This is a theoretical lower bound established by Candes and Tao [2010], and further reduction is not possible.

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