On-line list coloring of matroids

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Abstract. A coloring of a matroid is proper if elements of the same color form an independent set. A theorem of Seymour asserts that a \( k \)-colorable matroid is also colorable from any lists of size \( k \). In this note we generalize this theorem to the on-line setting. We prove that a coloring of a matroid from lists of size \( k \) is possible even if appearances of colors in the lists are recovered color by color by an adversary, while our job is to assign a color immediately after it is recovered. We also prove a more general weighted version of our result with lists of varying sizes. In consequence we get a simple necessary and sufficient condition for matroid list colorability in general case. The main tool we use is the multiple basis exchange property, which we give a simple proof.

1. Introduction

Let \( M \) be a loopless matroid on a ground set \( E \). A coloring of the set \( E \) is proper if elements of the same color form an independent set of \( M \). The chromatic number of \( M \), denoted by \( \chi(M) \), is the minimum number of colors needed to color properly the set \( E \). In case of a graphic matroid \( M = M(G) \), the number \( \chi(M) \) is a well studied parameter known as the arboricity of the underlying graph \( G \).

In [5] Seymour considered the following list coloring problem for matroids, in analogy to the list coloring of graphs. By a simple application of the matroid union theorem he proved that matroidal version of the choice number stays the same as the chromatic number.

Theorem 1. (Seymour [5]) Suppose that every element \( e \in E \) of a matroid \( M \) is assigned a set of colors \( L(e) \) of size at least \( \chi(M) \). Then there is a proper coloring \( c \) of \( M \) satisfying \( c(e) \in L(e) \) for each \( e \in E \).

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In this paper we prove the on-line version of Seymour’s theorem. Consider the following game played by Alice and Bob on a matroid $M$, in analogy to the graph coloring game introduced by Schauz [6] (cf. [8]). Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of colors, and let $k$ be a fixed positive integer. In the first round Bob chooses arbitrary non-empty subset $B_1 \subseteq E$ and inserts color 1 to the lists of all elements of $B_1$. Then Alice chooses some independent set $A_1 \subseteq B_1$ and colors its elements by color 1. In the second round Bob picks arbitrarily a non-empty subset $B_2 \subseteq E$ and inserts color 2 to the lists of all elements of $B_2$. Then Alice chooses an independent subset $A_2 \subseteq B_2 \setminus A_1$ and colors its elements with color 2. And so on, until all lists will have exactly $k$ elements. If at the end of the play the whole matroid is colored, then Alice is the winner. In the opposite case, Bob is the winner. Let $\bar{\chi}(M)$ denote the minimum number $k$ guaranteeing a win for Alice.

Our main result reads as follows.

**Theorem 2.** Every matroid $M$ satisfies $\bar{\chi}(M) = \chi(M)$.

The proof relies on a multiple basis exchange property. Actually we prove a more general result, in which we allow for lists of varying sizes and coloring by sets of colors. This also gives the fractional version of the theorem (for fractional on-line list coloring of graphs see [1]).

2. The proof

We will need some notation. Let $\mathcal{P}(\mathbb{N})$ denote the family of all subsets of the set of positive integers $\mathbb{N}$ (we use $\mathbb{N}$ as the set of colors as well as the set of numbers). Let $w : E \rightarrow \mathbb{N}$ be an assignment of weights to the elements of a matroid $M$. A $w$-coloring of a matroid $M$ is a function $W : E \rightarrow \mathcal{P}(\mathbb{N})$ such that every coloring $c$ satisfying $c(e) \in W(e)$ is a proper coloring of $M$. Let $\ell : E \rightarrow \mathbb{N}$ be any function and let $L : E \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment of size $\ell$, that is, for each $e \in E$ we have $|L(e)| = \ell(e)$. We say that $M$ is $w$-colorable from lists $L$ if there is a $w$-coloring $W$ of $M$ satisfying condition $W(e) \subseteq L(e)$ for each $e \in E$.

Now we may consider a generalized game on a matroid $M$ with given functions $w$ and $\ell$, which goes in the same way as described in the introduction, except that the goal of Alice is a $w$-coloring of $M$ from lists of size $\ell$. If she has a winning strategy, then we say that $M$ is on-line $(w, \ell)$-colorable.

Our aim is to prove a sufficient condition for the above property. We need two simple lemmas. The first is a well-known generalized exchange property. We will prove this lemma for the sake of completeness.
Lemma 1. Let $I_1$ and $I_2$ be independent sets of a matroid $M$. Then for every $X \subseteq I_1$ there exists $Y \subseteq I_2$ such that both sets, $(I_1 \setminus X) \cup Y$ and $(I_2 \setminus Y) \cup X$, are independent.

Proof. Let $I = I_1 \cap I_2$. We can restrict to the case where $I = \emptyset$. Indeed, if $I \neq \emptyset$, then consider matroid $M$ with contracted set $I$ and two independent sets $I_1 \setminus I$, and $I_2 \setminus I$. For $X \setminus I_2$ we get $Y$, which is also good in the previous case.

Now let $I_1 \cap I_2 = \emptyset$. Let $M_1$ be matroid $M$ restricted to the set $X \cup I_2$, and let $M_2$ be matroid $M$ restricted to the set $(I_1 \setminus X) \cup I_2$. Let $I_1 \cup I_2$ be their common ground set, and denote their rank functions by $r_1$, $r_2$ respectively. Observe that for each $A \subseteq I_1 \cup I_2$ we have:

$$r_1(A) + r_2(A) = r(A \cap (X \cup I_2)) + r(A \cap ((I_1 \setminus X) \cup I_2)) \geq r(A \cap (I_1 \cup I_2)) + r(A \cap I_2) \geq |A \cap I_1| + |A \cap I_2| = |A|,$$

where the first inequality is just a submodularity of a rank function.

From the matroid union theorem (see [4]) it follows that $I_1 \cup I_2$ can be covered by sets $I'_1$, $I'_2$ independent in $M_1$ and $M_2$ respectively, so also in $M$. Now $Y = I_2 \cap I'_2$ is a good choice, since $(I_1 \setminus X) \cup Y = I'_2$ and $(I_2 \setminus Y) \cup X = I'_1$.

□

As a corollary we get the multiple basis exchange property (see [2, 7]).

Corollary 1. (Multiple basis exchange property) Let $B_1$ and $B_2$ be two bases of a matroid $M$. Then for every $X \subseteq B_1$ there exists $Y \subseteq B_2$, such that $(B_1 \setminus X) \cup Y$ and $(B_2 \setminus Y) \cup X$ are also bases.

We say that a collection of sets $I_1, \ldots, I_k$ is a $w$-cover of a set $E$ if for each $e \in E$ we have $|\{i : e \in I_i\}| = w(e)$. For a given subset $U \subseteq E$, let $c_U$ denote the characteristic function of $U$, that is, $c_U(e) = 1$ if $e \in U$ and $c_U(e) = 0$, otherwise. Now we prove the following inductive step lemma.

Lemma 2. Let $I_1, \ldots, I_k$ be a collection of independent sets in a matroid $M$ forming a $w$-cover of its ground set $E$. Then for every set $V \subseteq E$ there exists an independent set $I \subseteq V$ and independent sets $I'_1, \ldots, I'_k$ satisfying the following conditions.

1. The sets $I'_1, \ldots, I'_k$ form a $(w - c_I)$-cover of $E$.
2. For each $e \in E$, if $e \in I'_t$ then $e \in I_t$ for some $t \geq s + c_V(e)$.

Proof. Let $X_1 = (V \cap I_1) \setminus (I_1 \cap I_2)$. By Lemma 1 there exists $Y_2 \subseteq I_2$ such that $I'_1 = (I_1 \setminus X_1) \cup Y_2$ and $I'_2 = (I_2 \setminus Y_2) \cup X_1$ are independent. In general let $X_i = (V \cap I''_i) \setminus (I''_i \cap I_{i+1})$. So again by Lemma 1 there exists $Y_{i+1} \subseteq I_{i+1}$, such that $I'_i = (I''_i \setminus X_i) \cup Y_{i+1}$ and
$I_{i+1}^n := (I_{i+1} \setminus Y_{i+1}) \cup X_i$ are independent. Let $I = X_k$. It is easy to see that conditions (1) and (2) are satisfied.

We are ready to prove the following generalization of the theorem of Seymour.

**Theorem 3.** Let $\ell$ be a given list-size function on the ground set $E$ of a matroid $M$. If $M$ is $w$-colorable from lists of the form $L(e) = \{1, 2, \ldots, \ell(e)\}$, $e \in E$, then $M$ is on-line $(w, \ell)$-colorable.

**Proof.** We prove it by the induction on the number $w(E) = \sum_{e \in E} w(e)$. If $w(E) = 0$, then $w$ is the zero vector and the assertion holds trivially. Suppose now that $w(E) \geq 1$ and the assertion of the theorem holds for all $w'$ with $w'(E) < w(E)$. Let $V \subseteq E$ be the set of elements picked by Bob in the first round of the game. So, all elements of $V$ have color 1 in their lists. Let $I_1, \ldots, I_k$ be a $w$-coloring of $M$ which exists by the assumption. By Lemma 2, there exist independent sets $I \subseteq V$ and $I'_1, \ldots, I'_k$, such that $I'_1, \ldots, I'_k$ is a $(w - c_I)$-cover of $E$. Now Alice colors all elements from $I$ with color 1. By condition (2) of Lemma 2 matroid $M$ is $(w - c_I)$-colorable from lists $L'(e) = \{1, 2, \ldots, \ell(e) - c_V(e)\}$. The assertion of the theorem follows by induction. 

Observe that the condition from the assumption of Theorem 3 is not only sufficient, but also a necessary for a matroid to be on-line $(w, \ell)$-colorable.

Taking $w = (1, 1, \ldots, 1)$ and $l = (k, k, \ldots, k)$, with $k = \chi(M)$, we get immediately Theorem 2. Theorem 3 is an on-line generalization of Theorem 3 from [3]. Let us mention one of this off-line consequences.

**Corollary 2.** If $M$ is colorable from lists of the form $L(e) = \{1, 2, \ldots, \ell(e)\}$, $e \in E$, then $M$ is colorable from any lists of size $\ell$.

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