STRUCTURAL CHARACTERIZATION OF COMPOUNDNESS

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Abstract: We recover the rays in the tensor product of Hilbert spaces within a larger class of so called ‘states of compoundness’, structured as a complete lattice with the ‘state of separation’ as its top element. At the base of the construction lies the assumption that the cause of actuality of a property of one of the (as individual considered) entities in the compound system can be actuality of a property of the other one.

Keywords: property lattice, tensor product of Hilbert spaces, separated system, Galois duality.

1. INTRODUCTION

Most approaches towards a realistic description of compound —quantum— systems are based on the recognition of subsystems, imposing some mathematical universal property as a structural criterion (Hellwig and Krauser, 1977; Zecca, 1977; Aerts and Daubechies, 1978; Aerts, 1982; Pulmannová, 1984; Ischi, 1999; Valckenborgh, 2000). In this paper we take a different point of view, essentially focusing on a structural characterization of the interaction between the individual entities, rather than on the compound system as a whole. More precisely, we structurize the concept of ‘mutual induction of actuality’ for the ‘individual entities’ in a compound system, inspired by the existence of an —essentially unique— representation for compound quantum systems when postulating that a state transition of one individual entity induces a state transition of the others —see Coecke (1998a). In particular will we consider ‘separation’ as one particular state of compoundness, and not as a type of ‘entity’ as in Aerts (1982) and Ischi (1999), as such avoiding some axiomatic drawbacks that emerge when taking the latter perspective. Formally, an essential ingredient of the reasoning can be borrowed from Faure et al. (1995) where propagation of states and properties in maximal deterministic evolutions is studied. As an application of our way of looking at compoundness we mention a representation for spin systems (Coecke 1995, 1998b). The mathematical preliminaries to this paper are basic notions on linear operators for which we refer to Weidmann (1981) and that of a Galois dual pair, i.e., a couple of isotope maps \( f : M \to N \) and \( f^* : N \to M \) satisfying \( \forall x \in M, y \in N : x \leq f^*(y) \iff f(x) \leq y \), where \( f \) preserves existing joins, \( f^* \) preserves existing meets, and we have existence and uniqueness of it for a meet (resp. join) preserving map between complete lattices (Birkhoff, 1940; Johnstone, 1982).

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2. COMPOUNDNESS AND ASSIGNMENT OF CAUSES

Let us first recall the general concept of an entity, along the lines of Jauch and Piron (1969), Piron (1976) and Aerts (1982)—we will not go into the details on this and refer for the most recent overview to Moore (1999). We consider an entity to be a physical system described by a collection of either potential or actual properties \( \mathcal{L} \), partially ordered by an —operationally motivatable— implication relation ‘\( \leq \)’, and which proves to be a complete lattice (Piron, 1976). As is discussed in Piron (1977) and Moore (1999), the meet ‘\( \wedge \)’ can be treated as a classical conjunction. A property is said to be ‘actual’ if we get true with certainty when it would be verified in any possible way; \( a \in \mathcal{L} \) is stronger (resp. weaker) than \( b \in \mathcal{L} \) iff \( a \leq b \) (resp. \( b \leq a \)); the top element 1 of the complete lattice can be seen as expressing ‘existence’ of the entity and the bottom element 0 expresses the ‘absurd’, i.e., what can never be true. As an example, the property lattice \( \mathcal{L}_\mathcal{H} \) of a quantum entity described in a Hilbert space \( \mathcal{H} \) is the set of closed subspaces ordered by inclusion with intersection as meet and closed linear span as join. We will also systematically use the term ‘individual entity’ when considering identifiable ‘parts’ in a larger system—as such to be seen as a compound system—since in general, these individual entities do not satisfy the general conception of what an entity is in the references mentioned above—for a discussion on this aspect see Coecke (1998a). In the presence of interaction between individual entities, actuality of a property \( a_2 \in \mathcal{L}_2 \) of individual entity \( S_2 \) might be due to the actuality of a property \( a_1 \in \mathcal{L}_1 \) of individual entity \( S_1 \). In particular, we will show that all interaction involved in quantum entanglement can be expressed in this way, and therefore we define a map:

\[
    f^* : \mathcal{L}_2 \to \mathcal{L}_1 : a_2 \mapsto \text{"the cause in } \mathcal{L}_1 \text{ of the actuality of } a_2\"
\]

(1)

This cause of actuality of \( a_2 \) is the weakest \( a_1 \in \mathcal{L}_1 \) that assures actuality of \( a_2 \); indeed, any \( b_1 \in \mathcal{L}_1 \) with \( b_1 \leq a_1 \) then automatically causes actuality of \( a_2 \) since it implies \( a_1 \). Note that existence of such a weakest \( a_1 \in \mathcal{L}_1 \) follows from the fact that ‘assuring actuality of \( a_2 \)’ precisely defines it as a property for \( S_1 \). As an example, when considering two separated individual entities it is the bottom element \( 0_1 \in \mathcal{L}_1 \) that assures actuality of any \( a_2 \in \mathcal{L}_2 \setminus \{1_2\} \), explicitly expressing that no state of \( \mathcal{L}_1 \) assures anything about \( \mathcal{L}_2 \). On the contrary, the property \( 1_1 \) assures actuality of \( 1_2 \) since we a priori assume the existence of both individual entities. When considering meets \( \wedge \cap a_{2,i} \) in \( \mathcal{L}_2 \), due to their significance as a classical conjunction, i.e. they can be read as ‘and’, \( \wedge i f^*(a_{2,i}) \) assures actuality of \( \wedge i a_{2,i} \), as such assuring \( f^* \) to preserve non-empty meets. Since \( \wedge \emptyset = 1 \) and \( f^*(1_2) = 1_1 \) by assumption of the existence of both individual entities, it also preserves the empty meet. Thus, there exists a unique join-preserving Galois dual for \( f^* \), namely:

\[
    f : \mathcal{L}_1 \to \mathcal{L}_2 : a_1 \mapsto \wedge \{ a_2 \in \mathcal{L}_2 | a_1 \leq f^*(a_2) \} = \min \{ a_2 \in \mathcal{L}_2 | a_1 \leq f^*(a_2) \}
\]

(2)

From eq.(2) it follows that \( f \) assigns to a property \( a_1 \in \mathcal{L}_1 \) the strongest property \( a_2 \in \mathcal{L}_2 \) of which it assures actuality —the minimum of all \( a_2 \in \mathcal{L}_2 \) such that \( a_1 \leq f^*(a_2) \)— implicitly implying actuality of all \( b_2 \geq a_2 \). Thus, \( f \) expresses exactly induction of actuality of \( \mathcal{L}_1 \) on \( \mathcal{L}_2 \).
Conclusion 1. A ‘state of compoundness for \(S_1\) on \(S_2\)’ is a join preserving map \(f : L_1 \to L_2\).

i) STRUCTURING STATES OF COMPOUNDNESS:

Denote by \(Q(L_1, L_2)\) the join preserving maps from \(L_1\) to \(L_2\) and set for all \(\{f_i\}_i \subseteq Q(L_1, L_2)\):

\[
\bigvee_i f_i := L_1 \to L_2 : a \mapsto \forall_i f_i(a)
\]  

Eq.\(\text{[3]}\) defines a complete internal operation on \(Q(L_1, L_2)\) since for \(\{f_i\}_i \subseteq Q(L_1, L_2)\): \((\bigvee_i f_i)(\bigvee_j a_j) = \bigvee_i(\bigvee_j f_i(a_j))\), and one easily verifies that \(\bigvee_i f_i\) is the least upper bound of \(\{f_i\}_i\) in \(Q(L_1, L_2)\).

Conclusion 2. The states of compoundness for \(S_1\) to \(S_2\) are described by a complete lattice \((Q(L_1, L_2), \bigvee)\), inheriting its join from the underlying property lattice \(L_2\) pointwisely.

Analogously, the collection of meet preserving maps \(f^* : L_2 \to L_1\) denoted by \(Q^*(L_1, L_2)\) is a meet complete lattice with respect to pointwise meet, denoted as \(\land\). Note here that the significance of the lattice meet as a classical conjunction is lifted by the pointwise computed meets to the level of assignment of temporal causes, as such giving to \(f^* \land g^*\) the significance of ‘\(f^*\) and \(g^*\)’. Set \(L^*(f, a_2) = \{a_1 \in L_1| f(a_1) \leq a_2\}\) and \(L(f^*, a_1) = \{a_2 \in L_2| a_1 \leq f^*(a_2)\}\). If \(\forall a_1 \in L_1 : f(a_1) \leq g(a_1)\), then \([g(a_1) \leq a_2 \Rightarrow f(a_1) \leq a_2]\) and \(\forall a_2 \in L_2 : L^*(g, a_2) \subseteq L^*(f, a_2)\). Thus, \(\forall L^*(g, a_2) \subseteq L^*(f, a_2)\) yielding \(\forall a_2 \in L_2 : g^*(a_2) \leq f^*(a_2)\) and \(g^* \leq f^*\). Conversely, \(\forall a_2 \in L_2 : g^*(a_2) \leq f^*(a_2)\) implies \([g^*(a_2) \geq a_1 \Rightarrow f^*(a_2) \geq a_1]\), \(L(g^*, a_1) \subseteq L(f^*, a_1)\), \(\land L(g^*, a_1) \supseteq \land L(f^*, a_1)\), and thus \(f \leq g\). As such \(f \leq g\) iff \(g^* \leq f^*\). It follows that \((Q^*(L_1, L_2), \land)^{op}\) and \((Q(L_1, L_2), \bigvee)\) are isomorphic complete lattices —‘\((-)^{op}\)’ stands for reversal of partial order.

Conclusion 3. The map \(* : Q(L_1, L_2) \to Q^*(L_1, L_2) : f \mapsto f^*\) ‘interprets’ \(\bigvee_i f_i\), the join of a set of states of compoundness, as \(\land_i f_i^*\) a conjunction of the corresponding assignments of temporal causes.

ii) SEPARATION AS THE TOP STATE OF COMPOUNDNESS:

Since we have \(f^* : \{L_2 \setminus \{1_1\} \to L_1 : a_2 \mapsto 0_1 ; 1_2 \mapsto 1_1\}\) in case of separation, the ‘state of separation’ for \(S_1\) on \(S_2\) is given by \(f : \{L_1 \setminus \{0_2\} \to L_2 : a_1 \mapsto 1_2 ; 0_1 \mapsto 0_2\}\) and this is exactly the top element \(1_{1,2}\) of \(Q(L_1, L_2)\) —note that \(0_1 \mapsto 0_2\) is required for any join preserving map, the bottom being the empty join. Remark that the bottom element \(0_{1,2}\) of \(Q(L_1, L_2)\) stands for the ‘absurd state of compoundness’. Indeed, since \(f : L_1 \to L_2 : a_1 \mapsto 0_1\) we have \(f^* : L_2 \to L_1 : a_2 \mapsto 1_1\), i.e., existence of \(S_1\) causes actuality of all properties of \(S_2\), and as such actuality of the absurd property \(\land L_2 = 0_2\).

iii) QUANTUM ENTANGLEMENT AS ATOMIC STATES OF COMPOUNDNESS:

We will now consider those \(f \in Q(L_{\mathcal{H}_1}, L_{\mathcal{H}_2})\) that send atoms to atoms or \(0_2\) for \(L_{\mathcal{H}_1}\) and \(L_{\mathcal{H}_2}\) the lattices of closed subspaces of Hilbert spaces. The following result can be found in Faure et al. (1995) and is essentially
Proposition 1. If \( f \in Q(\mathcal{L}_{\mathcal{H}_1}, \mathcal{L}_{\mathcal{H}_2}) \) sends atoms to atoms or \( 0_2 \) then \( f \) is itself an atom or \( 0_{1,2} \).

Proof: Let \( K_f \) and \( K_g \) be the respective kernels of the linear maps \( F,G : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) induced by \( f,g \in Q(\mathcal{L}_{\mathcal{H}_1}, \mathcal{L}_{\mathcal{H}_2}) \) with \( f < g \), and thus, \( K_g \subset K_f \) and \( K_f^\perp \subset K_g^\perp \). For \( \psi \in K_f^\perp \setminus \{0_1\} \) and \( \phi \in (K_f \cap K_g^\perp) \setminus \{0_1\} \) we have \( F(\psi + \phi) = F(\psi) = G(\psi) \) whereas \( G(\psi + \phi) = G(\psi) + G(\phi) \), forcing \( G(\psi) = kG(\phi) \) for \( k \) non-zero. However, then \( G(k\psi - \phi) = 0 \) although \( k\psi - \phi \in K_g^\perp \), yielding contradiction except when \( K_f^\perp = 0_1 \), i.e., \( [f : a_1 \mapsto 0_2] = 0_{1,2} \).

For the sake of transparency of the argument we will from now on only consider finite dimensional Hilbert spaces. Let \( \mathcal{H}_1' \) be the Hilbert space of continuous linear functionals on \( \mathcal{H}_1 \), connected to it by the correspondence \( \mathcal{H}_1 \rightarrow \mathcal{H}_1' : \psi \mapsto \langle \psi | - \rangle \). Then the tensor product \( \mathcal{H}_1' \otimes \mathcal{H}_2 \) is isomorphic to the space of linear operators with indicated domain and codomain —denoted as \( B(\mathcal{H}_1, \mathcal{H}_2) \) — by the isomorphism:

\[
\mathcal{H}_1' \otimes \mathcal{H}_2 \rightarrow B(\mathcal{H}_1, \mathcal{H}_2) : \left[ \sum_{i=1}^m c_i \langle \psi_i | - \rangle \otimes \phi_i \right] \mapsto \left[ F_{\{c_i\}_i} : \mathcal{H}_1 \rightarrow \mathcal{H}_2 : \psi \mapsto \sum_{i=1}^m c_i \langle \psi_i | \psi \rangle \phi_i \right] \tag{4}
\]

with \( \{\psi_i\}_i \) and \( \{\phi_i\}_i \) fixed orthonormal bases —note that \( ||F_{\{c_i\}_i}||_{HS} = ||\sum_{i=1}^m c_i \langle \psi_i | - \rangle \otimes \phi_i||_{\mathcal{H}_1' \otimes \mathcal{H}_2} \) for \( ||-||_{\mathcal{H}_1' \otimes \mathcal{H}_2} \) the Hilbert space metric and \( ||-||_{HS} \) the Hilbert-Schmidt norm. Considering anti-linear maps \( \sum_{i=1}^m c_i \langle - | \psi_i \rangle \phi_i \)—say \( B'(\mathcal{H}_1, \mathcal{H}_2) \)— with a reasoning along the same lines, we obtain \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). As such, we recover the rays of the tensor product of Hilbert spaces \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) as a special case of our more general class of atomic states of compoundness, besides the rays \( \mathcal{H}_1' \otimes \mathcal{H}_2 \). Indeed, although \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) and \( \mathcal{H}_1' \otimes \mathcal{H}_2 \) are isomorphic as Hilbert spaces, there rays represent different states of compoundness.

3. COMPOUNDNESS AS EVOLUTION — AND VICE VERSA

Although we were able to use much material from Faure et al. (1995) —that deals with evolution of an entity— we should note that an essential difference between the two formal developments is due to the fact that existence of the entity at a certain instance of time in general does not assure existence in the future when considering evolution. However, imposing a condition on the ‘type’ of evolution that we consider by ‘requiring preservation’ makes an illustrative comparison possible, and as such we will proceed. To fix ideas, identify \( \mathcal{L}_1 \) with the property lattice of a fixed entity at time \( t_1 \) and \( \mathcal{L}_2 \) as its property lattice at time \( t_2 \). We can again consider a map \( f^* : \mathcal{L}_2 \rightarrow \mathcal{L}_1 \) that assigns to a property that is actual at time \( t_2 \) the cause of its actuality at time \( t_1 \), and the corresponding Galois dual \( f : \mathcal{L}_1 \rightarrow \mathcal{L}_2 \) that now expresses temporal propagation of properties, all of them again structured in \( Q(\mathcal{L}_1, \mathcal{L}_2) \).

\( ^4 \)I.e., with at least three non-collinear elements in its range. Note that if the image is spanned by either one or two atoms there is an obvious representation as a linear map, extending the collection of representations in a natural way, which motivates us to assume a linear representation on the underlying Hilbert space for any state of compoundness.
Example 1: A ‘maximal deterministic evolution’ is defined as one where atoms propagate to atoms or 0— the latter in order to express a domain for initial states— mirroring atomic states of compoundness.

Example 2: We could define a ‘maximal indeterministic—or minimal deterministic— evolution’ as one that strictly assures existence at time $t_2$, yielding the mirror of the state of separation.

For a more elaborated formal discussion on the connection between compoundness and evolution we refer to Coecke and Stubbe (1999a) and Coecke and Moore (2000).

4. ON ASSIGNMENT OF PROPER STATES TO INDIVIDUALIZED ENTITIES

The discussion on evolution was not merely illustrative but constitutes an essential ingredient in this section, where we discuss state transitions of individual entities due to a state of compoundness: in Section 2 we characterized entanglement between individual entities in a compound system, but no consideration on a characterization of individual entities themselves has been made. Indeed, a description of a compound system consisting of two individual entities $S_1$ and $S_2$ requires a characterization of these individual entities besides the states of compoundness $f_{1,2}$ of $S_1$ on $S_2$ and $f_{2,1}$ of $S_2$ on $S_1$. However, this proves to requires a more general concept of state than in Piron (1976), Aerts (1982) and Moore (1995), and a different name proper state has been introduced in Coecke (1998a) to stress this difference. Therefore consider for each individual entity an a priori set of proper states $\Sigma$ and denote by $C : \mathcal{P}(\Sigma) \to \mathcal{L}$ the map that assigns to any $T$ in $\mathcal{P}(\Sigma)$, the powerset of $\Sigma$, the strongest property in $\mathcal{L}$ that is implied by every $p \in T$. As shown in Coecke and Stubbe (1999a, 1999b), $C$ canonically induces a pre-order on $\Sigma$ by $p \leq_C q \iff C(\{p\}) \leq C(\{q\})$.

Moreover, the above discussed requirement that properties propagate with preservation of join (see also Pool, 1968) restricts the —not-necessarily deterministic— state transitions to be described by a map in:

$$Q^\#(\Sigma) = \{ f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f(\cup T) = \cup f(T), f(C(T)) \subseteq C(f(T)) \}$$

that proves to be a quantale $(Q^\#(\Sigma), \cup, \circ)$—i.e., a join-complete lattice equipped with an additional operation $\circ$ that distributes over arbitrary joins— where $\cup$ is computed pointwise, and which is in epimorphic—quantale— correspondence with the ‘quantale’ $(Q(\mathcal{L}, \mathcal{L}), \vee, \circ)$—see above— by:

$$\forall [-] : Q^\#(\Sigma) \to Q(\mathcal{L}, \mathcal{L}) : f \mapsto [f_L : C(T) \mapsto C(f(T))]$$

Note that this quantale epimorphism indeed exactly expresses that with any state transition there corresponds a join preserving propagation of properties. We apply all this to the context of this paper:

For each individual entity $S$, let $\Psi : \mathcal{L} \to Q^\#(\Sigma)$ be the map that assigns to any property $a \in \mathcal{L}$ the —not-necessarily deterministic— state transition $\Psi(a) : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that $S$ undergoes when actuality of property $a$ is induced on it by interaction with another individual entity.

Clearly this implicitly determines the propagation of properties $\Psi_L : \mathcal{L} \to Q(\mathcal{L}, \mathcal{L})$ by $\forall \Psi(-) = \Psi_L(-)$. We will now discuss $\Psi$ and $\Psi_L$ for $\mathcal{L}$ orthomodular, and more specific, for the Hilbert space case.
Proposition 2. If the following conditions are satisfied:

(i) $\Psi(a)$ does not alter proper states of which the strongest actual property is stronger than $a$;
(ii) all properties compatible to the induced one that are actual beforehand remain actual;
(iii) the assignment $\text{im}(\Psi_L) \rightarrow \text{im}(\Psi) : \Psi_L(a) \mapsto \Psi(a)$ preserves composition — with $\text{im}(-) = \text{image}$, then, $p \leq q \iff \exists a \leq C(\{q\}) : p \in \Psi(a)(\{q\})$ defines a poset $(\Sigma, \leq)$ that embeds in $(\Sigma, \leq_C)$.

Proof: Following Piron (1976), p.69, Theorem 4.3, if properties described by an orthomodular property lattice change in such a way that a property $a \in \mathcal{L}$ becomes actual, and such that all properties compatible to $a$ that where actual beforehand, are still actual afterwards, then the corresponding transition of properties is exactly described by the Sasaki projection $\varphi_a : \mathcal{L} \rightarrow \mathcal{L} : b \mapsto a \wedge (b \vee a^\perp)$. Thus, (ii) assures that $\Psi_L(a) = \varphi_a$.

If $p \in \Psi(a)(\{q\})$ for $a \leq C(\{q\})$, then $C(\{p\}) \leq C(\Psi(a)(\{q\})) = \varphi_a(C(\{q\})) \leq a$ — the equality follows from eq.($\mathbb{B}$) — we have $C(\{p\}) \leq C(\{q\})$ and thus $p \leq_C q$. Note that (i) is equivalent to $\Psi(a)$ being identical on all $p \in \Sigma$ such that $C(\{p\}) \leq a$, and thus yields reflexivity since it forces the restriction of $\Psi(a)$ to $\{T \subseteq \Sigma | C(T) \leq a\}$ — which is the only part of the domain involved in defining $\leq_C$ — to be idempotent. We can’t have $p \prec q$ and $q \prec p$ since any transition $\Psi(a)(\{q\}) = \{p\}$ requires again by (i) that $C(\{p\}) \leq a$ where $C(\{q\}) \leq a$. Following Foulis (1960), the maps $\{\varphi_a | a \in \mathcal{L}\}$ are structured in a complete lattice isomorphic to $\mathcal{L}$ itself when ordered by $\varphi_a' \leq \varphi_a \iff \varphi_a' = \varphi_a \varphi_a$. Now consider $a \geq a'$, then $\varphi_a \geq \varphi_a' = \varphi_a' \varphi_a$, i.e., $\varphi_a \varphi_a$ only depends on $a'$ and not on $a$. Thus, (iii) yields transitivity since it forces $\Psi(a')(\Psi(a)(\{q\})) = \Psi(a')(\{q\})$, and this independent of $a$ provided that $a \geq a'$.

Note that condition (ii) says that actual properties are only altered in a minimal way. Now consider — under the assumptions of the above proposition — a chain $a_1 \geq a_1' \geq \ldots \in \mathcal{L}_1$ of consecutive strongest actual properties of $S_1$. By isotonicity of $f_{1,2}$ we have $f_{1,2}(a_1) \geq f_{1,2}(a_1') \geq \ldots$ and thus for $\{p_2\} = \Psi(f_{1,2}(a_1'))(\{p_2\})$ we have $f_{1,2}(a_1) = C(\{p_2\}) \geq C(\{p_2\}) \geq \ldots$ for $p_2$ initial, yielding $p_2 \geq p_2 \geq \ldots$. As such, the relation $\leq_C$ expresses evolution of $S_2$ due to mutual induction of actuality as a descending chain of proper states in $\Sigma_2$, with descending strongest actual properties in $\mathcal{L}_2$.

ii) PROPER STATES FOR COMPOUND QUANTUM SYSTEMS

At this point it is required to propose a candidate for the sets of proper states $\Sigma_1$ and $\Sigma_2$ in the Hilbert space case that allows us to fully recover the description of compound quantum systems. Let $P_{a_i} : \mathcal{H}_i \rightarrow \mathcal{H}_i$ be the orthogonal projector corresponding to $\varphi_{a_i}$ for $i \in \{1, 2\}$ with $a_i$ a closed subspace of $\mathcal{H}_i$ and set:

$$\begin{align*}
\left\{ \langle \Sigma_i, C_i \rangle = \left( \{\rho_i : \mathcal{H}_i \rightarrow \mathcal{H}_i | \rho_i \text{ is a density operator} \} , \mathcal{P}(\Sigma_i) \rightarrow \mathcal{L}_{\mathcal{H}_i} : \{\rho_i\} \mapsto \{\rho_i(\phi) | \phi \in \mathcal{H}_i \} \right) \right\} \\
\Psi_i(a_i) = \left[ C_i(\rho_i) \perp a_i : \{\rho_i\} \mapsto \emptyset ; C_i(\rho_i) \perp a_i : \{\rho_i\} \mapsto \left\{ \frac{1}{\text{tr}(P_{a_i} P_{a_i}')} P_{a_i} \rho_i P_{a_i} \} \right\} \right]
\end{align*}
$$

(7)

One easily verifies that $\Psi_i$ defines state transitions for $\langle \Sigma_i, C_i \rangle$ that fulfill Proposition 2 and that the one dimensional projectors are minimal in $\langle \Sigma_i, \leq \rangle$. Moreover, the restriction of $\Psi_i(a_i)$ to $\{T \subseteq \Sigma_i \mid C_i(T) \leq a_i \}$
maps a singleton on a singleton or \( \emptyset \) for all \( a_i \in \mathcal{L}_i \), i.e., the transitions due to induction of properties less than the strongest actual one are maximally deterministic. Now, given \( F \in B(\mathcal{H}_1, \mathcal{H}_2) \) representing \( f_{1,2} \) with \( F^\dagger \) as its adjoint, then:

\[
F \mapsto \left( F : \mathcal{H}_1 \rightarrow \mathcal{H}_2, \frac{1}{\text{Tr}(F^\dagger F)} F^\dagger F : \mathcal{H}_1 \rightarrow \mathcal{H}_1, \frac{1}{\text{Tr}(F^\dagger F)} FF^\dagger : \mathcal{H}_1 \rightarrow \mathcal{H}_1, F^\dagger : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \right)
\]

(8)

uniquely defines a quadruple \((f_{1,2}, \rho_1, \rho_2, f_{2,1})\) which exactly yields the quantum probability structure in the following way: (i) The transition probability for \( \{\rho_i\} \mapsto \{\rho'_i\} = \Psi_i(a_i)(\{\rho_i\}) \) with \( a_i \notin \mathcal{C}_i(\{\rho_i\}) \) in a measurement that verifies the property \( a_i \) is \( \text{Tr}(P_{a_i}\rho_iP_{a_i}) \); (ii) when this happens, say \( \{\rho_1\} \mapsto \{\rho'_1\} \), this transition causes \( a'_1 = \mathcal{C}_1(\{\rho'_1\}) \) to become actual, and consequently, causes \( a'_2 = f_{1,2}(a'_1) \) to become actual, having a transition \( \{\rho_2\} \mapsto \{\rho'_2\} = \Psi_2(a'_2)(\{\rho_2\}) \) as a consequence; (iii) this reasoning can be proceeded inductively, and stops once we reach the minimal elements of \((\Sigma_2, \leq)\). It can then be verified that the probability for the chains \( \rho_1 \geq \rho'_1 \geq \ldots \geq P_\psi \) and \( \rho_2 \geq \rho'_2 \geq \ldots \geq P_\psi \) to have a couple \((\mathcal{C}_1(P_\psi), \mathcal{C}_2(P_\psi))\) as respective outcome ‘states’ —which are indeed represented as atoms of the property lattice— is given by:

\[
\sum_{i=1}^m c_i \left( |\psi_i\rangle \phi_i \right) \frac{1}{||\psi||^2_{\mathcal{H}_1} ||\phi||^2_{\mathcal{H}_2} ||F^\dagger F||_{HS}} \langle \psi \otimes \phi | \sum_i c_i \psi_i \otimes \phi_i \rangle
\]

(9)

for \( F = \sum_{i=1}^m c_i (-|\psi_i\rangle \phi_i) \), and this is indeed the quantum transition probability in a measurement on a compound system described by \( \sum_i c_i \psi_i \otimes \phi_i \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). We end by stressing that due to the assignment in eq(8), the initial proper states of the compound quantum system are fully encoded in the states of compoundness, and thus encoded in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) via \( B'(\mathcal{H}_1, \mathcal{H}_2) \).

5. CONCLUSION

In this paper we proposed an alternative approach towards an understanding of the description of compound quantum systems, by essentially focusing on the interaction of the individual entities within the compound system. Obviously, a lot more investigation could be done on a more accurate characterization of quantum entanglement as a special case of the primal considerations made in this paper. Also an elaboration on the description of compound systems consisting of more than two entities would be worthwhile.

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\[A \text{ proof follows from identification of this construction with the representation for compound quantum systems in Coecke (1998a): our choice of } (\Sigma_i, \mathcal{C}_i) \text{ and the corresponding quadruples } (f_{1,2}, \rho_1, \rho_2, f_{2,1}) \text{ for linear maps } F \text{ coincide exactly.}\]
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