Quantum vortex tunneling: Microscopic theory and application to d-wave superconductors

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(Dated: March 22, 2022)

We present a microscopic approach to the quantum tunneling of vortices. The formalism characterizes the rate at which a many-body superconducting state with a vortex in one location makes a transition to a second many-body superconducting state with a vortex in a second location. The theory is utilized to study the effect of a d-wave order parameter upon the directionality of vortex tunneling.

PACS numbers: 74.25.Qt,74.50.+r,74.72.-h,03.65.Xp

For decades, researchers have studied the motion of vortices in type-II superconductors. This motion determines fundamental characteristics of superconductors, including critical current and mixed-state resistivity, which often dictate their technological usefulness. Because of the complicated nature of a vortex in a superconductor, however, many aspects of the theoretical understanding of vortex motion have remained incomplete. In particular, the theory of quantum tunneling of vortices has been phenomenological in character, treating a vortex as a point-particle governed by an effective action. In this paper, we approach the problem of vortex quantum tunneling from a fully microscopic quantum mechanical perspective. Our calculations are based on the Nambu-Gorkov equations of superconductivity; we do not require effective parameters and do not assume that the vortex can be treated as a point-particle. Because of its microscopic foundation, the theory (i) is capable of addressing qualitative phenomena that are beyond the reach of effective models and (ii) provides a foundation for quantitative, first principles analysis of vortex motion in real materials. We focus in this paper upon non-dissipative motion, a topic that has elicited sustained theoretical concern even without the additional complication of dissipation (dissipation would be incorporated into the theory in the spirit of [1]). As a first application, we study the effects of the d-wave nature of the order parameter in high-temperature superconductors. Our calculations predict that d-wave symmetry suppresses tunneling along the nodes of the order parameter. This effect cannot be captured even in principle within the existing point-particle theories.

We begin our formulation by introducing the state \( |v_N\rangle \) that describes a superconductor with \( N \) electrons forming a vortex at the origin. For simplicity, we assume that the electrons are confined to 2 dimensions, which is appropriate for a superconducting thin film or possibly for a layered superconductor. The extension to 3 dimensions is straightforward. The microscopic Hamiltonian of the system is \( H = H_o + V \) where \( H_o \) describes electrons in an ionic potential and a homogeneous magnetic field, undergoing an effective electron-electron interaction that gives rise to superconductivity. A separate pinning potential \( V \) results from microscopic impurities or dislocations.

We consider the dynamics of this Hamiltonian in a basis of states of the form \( |v_N(\rho)\rangle \equiv T(\rho) |v_N\rangle \), where the operator \( T(\rho) \) translates the vortex from the origin to the position \( \rho \). It is important to emphasize that we are not making a phenomenological point-particle assumption about the vortex coordinate \( \rho \) when we deal with states of the form \( |v_N(\rho)\rangle \); the vortex states are the lowest energy states of a superconductor in a magnetic field and simply compose the low-energy Hilbert space of the microscopic Hamiltonian. Physically, this basis is a natural choice for describing a superconductor with a moving vortex. To compute the behavior of a time-dependent vortex state \( |\Psi(t)\rangle = \sum_{\rho'} \Psi(\rho', t) |v_N(\rho')\rangle \), in the usual way we first compute the eigenstates \( |\psi\rangle = \sum_{\rho'} \psi(\rho') |v_N(\rho')\rangle \) of the time-independent Schrödinger equation \( H_o + V - E |\psi\rangle = 0 \). In our basis, this equation is

\[
\sum_{\rho'} \langle v_N(\rho')| H_o + V - E |v_N(\rho')\rangle \psi(\rho') = 0. \tag{1}
\]

Before we can contemplate solving this equation, the essential challenge is to evaluate the matrix elements \( \langle v_N(\rho)| H_o + V |v_N(\rho')\rangle \) and the overlap \( \langle v_N(\rho) |v_N(\rho')\rangle \), keeping in mind that the states in our basis are not orthogonal. These matrix elements seem to require \( 2N \) dimensional many-body integrals, which can become utterly intractable for even relatively small \( N \). In the following, we provide tractable forms for them by describing \( |v_N(\rho)\rangle \) using the mean-field, Nambu-Gorkov equations of superconductivity. Of course, a solution to (1) is a superposition \( |\psi\rangle = \sum_{\rho'} \psi(\rho') |v_N(\rho')\rangle \) of mean-field states, so that our analysis goes well beyond mean-field theory.

To employ Nambu-Gorkov theory, we introduce the state \( |v\rangle \) of a superconductor that has a vortex at the origin and is in contact with a particle bath. This state can be well characterized using mean-field theory and the \( N \) electron state \( |v_N\rangle \) can be obtained from
it by projection $|v_N\rangle = P_N |v\rangle$. The mean-field description of $|v\rangle$ is contained in the Green's functions, $G_{s',s}(r',r,t) = -\langle v| T(d_{r,s}(t), d_{r',s'}(t' = 0)) |v\rangle$ which are defined using a particle-hole transformation from the usual electron destruction operators to $d_{r,s=1} = c_{r,s=1}$ and $d_{r,s=-1} = c_{r,s=-1}$. After Fourier transforming in time, one finds \[ G = i\hbar \delta (r - r') \begin{pmatrix} K - E & \Delta \\ -\Delta^* & -K^* - E \end{pmatrix} \]

where $K = \frac{1}{2m}(p + |e| A)^2 + U(r) - E_F$ and $G$ is a 2 by 2 matrix with elements $G_{s',s}(r',r,E)$. The crystal potential $U(r)$ includes the ionic potential and electron-electron Hartree interactions. The superconducting order parameter $\Delta$ is defined by $\Delta G_{s',s}(r',r,E) = \int d\tilde{r} \Delta(r,\tilde{r}) G_{s',s}(\tilde{r},r,E)$ and should be computed self-consistently. This equation can be solved numerically on a finite real space lattice, in which case the Green's function can be regarded as a matrix.

The vortex state $|v_N(\rho)\rangle$ is obtained by translating $|v_N\rangle$, which contains a vortex at the origin. Because of the presence of an external, homogeneous magnetic field in the Hamiltonian, one effects the translations using magnetic translation operators $c_{r,s} \rightarrow e^{i\pi B/r}c_{r,s}$. Magnetic translation operators are usually defined with respect to single-particle wavefunctions; in the Landau gauge $A(r) = -B y \hat{x}$, we have the definition $\tau(\rho) |\psi(r)\rangle = e^{i(xB/\Phi_o)\rho_y(x-\rho_z/2)} |\psi(r-\rho)\rangle$ where $\Phi_o = \hbar c/2|e|$ is the superconducting flux quantum. This operator must be generalized in order to translate a many-body vortex state properly. A suitable generalization is

$$ T(\rho) = \Pi_{R,S}(C_{R,S}C_{R,S}^\dagger + \frac{1}{2} c_{R,S}^\dagger c_{R,S}) + \Pi_{r,s}(C_{r,s} c_{r,s} + e^{i(xB/\Phi_o)\rho_y(x-\rho_z/2)} c_{r,s}^\dagger c_{r,s}) \]$$

where $N_o$ is the number of single-particle orbitals in the system and

$$ T(\rho)P_N = \int \frac{d\chi}{2\pi} e^{i(N_o/2-N)\chi} \det \mathcal{M}(\chi) \]$$

is an $N_o \times N_o$ matrix indexed by row $(r, s)$ and column $(r', s')$.

Note that equation \[ \mathcal{M}(\chi) = G_{s',s}(r - \rho', r' - \rho, t = 0^+) - G_{s,s'}(r - \rho', r' - \rho, t = 0^-) \]

is an $N_o \times N_o$ matrix indexed by row $(r, s)$ and column $(r', s')$. In this expression we have introduced fermionic operators of the form $c_{r,s}^\dagger$ that create electrons in an artificial, auxiliary space. Such operators are needed because “holding” space is required when exchanging a collection of objects – to exchange the balls in two boxes, a third “holding” box is required temporarily. In other words, one should not shift an electron from $r$ to $r + \rho$ without first placing an electron already in $r + \rho$ into a holding box. One can verify that $T(\rho)\Pi_{r,s}(C_{r,s}^\dagger c_{r,s} + \frac{1}{2} c_{r,s} c_{r,s}^\dagger) T(\rho')P_N |v\rangle$ that appears in equation \[ \ref{eq:vanishing} \] up to an unimportant phase factor that can be absorbed into the vortex state, the result is

$$ \langle v_N(\rho)| v_N(\rho') \rangle = \int \frac{d\chi}{2\pi} e^{i(N_o/2-N)\chi} \det \mathcal{M}(\chi), \]$$

where $N_o$ is the number of single-particle orbitals in the system.
We have a magnetic field and we also integrate over phase \( \chi \) to project \( |v> \) on to the state \( |v_N> \).

The reasoning leading to equation (4) also permits us to compute the matrix elements of the pinning potential

\[
⟨v_N(ρ)|V|v_N(ρ')⟩ = ⟨v| T^I(ρ)VT(ρ')P_N|v⟩
\]

\[
e = \sum_{r_o,s_o} V_{s_o}(r_o) \int \frac{dχ}{2π} e^{i(N_o/2-N)χ} det L(χ, r_o, s_o)
\]

where

Equations (1), (3), and (4) provide the ingredients for a microscopic calculation of quantum vortex tunneling. We stress that, although formulae (3) and (4) are complicated in appearance, they are attractive and manageable from a computational standpoint. No formula is needed for the remaining matrix element \( 〈v_N(ρ)|H_o|v_N(ρ')⟩ \) if we are willing to approximate that \( H_o \) has \( |v_N(ρ)> \) as an eigenstate. We then just get \( 〈v_N(ρ)|H_o|v_N(ρ')⟩ = E_v 〈v_N(ρ)|v_N(ρ')⟩ \) and the constant energy \( E_v \) merely shifts the eigenvalue \( E \) in (1).

This formalism makes it possible to study, for instance, how \( d_{x^2-y^2} \)-wave order parameter symmetry influences vortex quantum tunneling in a high temperature superconductor. We model a high temperature superconductor with the Hamiltonian described in reference [14]. We follow the methodology of [14] to obtain self-consistent quasiparticle amplitudes \( (u, v) \) and then sum these amplitudes to produce the self-consistent Nambu-Gorkov Green’s function appearing in (3) and (4) – this is equivalent to solving the Nambu-Gorkov equations directly. Calculations are made on a 14 × 28 real-space lattice \( (N_o = 2 × 14 × 28 = 784 \) single particle states), where the number \( N \approx 0.8(N_o/2) = 312 \) of electrons in the superconductor is set near optimal doping. Figure (1) plots the magnitude of the calculated overlap \( |〈v_N(ρ = X x + Y y)|v_N(0)⟩| \) as a function of displacement \( X x + Y y \) between the states. We model quantum vortex tunneling between two pinning sites by inserting impurities into our self-consistent calculation of the Green’s function. Of course, if our aim were to guide technological efforts to raise critical currents, we would seek quantitative results from the formalism by computing a realistic pinning potential \( V \). To explore generic behavior, though, we take the pinning potential to be a double well of well spacing \( ρ_o \), with depth given by 1/7 the Fermi energy, a reasonable magnitude. We further assume that the effect of this potential on the system’s electrons is to lower the energy of \( |v_N(0)⟩ \) and \( |v_N(ρ_o)⟩ \) below the energy of all other states \( |v_N(ρ)⟩ \). Then the Schrödinger equation (1) is solved, in the two dimensional Hilbert space spanned

FIG. 1: Magnitude of the overlap \( |〈v_N(ρ = X x + Y y)|v_N(0)⟩| \) of two vortex states as a function of displacement \( X x + Y y \) between the states.
by these states. If the superconductor begins in a time-dependent state $|\Psi(t)\rangle$ satisfying $|\Psi(t = 0)\rangle = |\nu_N(\rho_o)\rangle$, the probability that tunneling has occurred by time $t$ is $P(t) = 1 - |\langle \nu_N(\rho_o) | \Psi(t) \rangle|^2$. This probability is found to be periodic in our double well model, and we therefore set the tunneling rate $\Gamma(\rho_o)$ equal to the reciprocal of the period. Although the model produces tunneling back to the initial well, it is assumed, as in other macroscopic quantum tunneling models [16], that the environment intercedes to prevent this from occurring. Since the dissipative effects of the environment are not explicitly included in the calculation, we should obtain an upper bound on the tunneling rate [17]. One could include the effects of dissipation in future work by recasting the dynamics of $\Gamma$ in a functional integral as in [7].

As a measure of the directionality of the tunneling, we compute the ratio $\Gamma(a\hat{x} + a\hat{y})/\Gamma(a\hat{y})$, using formulae [11, 3], and [4] to evaluate $\Gamma(\rho_o)$. In the $s$-wave case for a $14 \times 28$ lattice, we obtain a ratio of 50%. In the $d_{x^2-y^2}$-wave case for a $14 \times 28$ lattice, the ratio is 2%. In comparison to the $s$-wave case, the $d_{x^2-y^2}$-wave order parameter symmetry strongly suppresses the relative tunneling rate along the order parameter nodes. When the high potential region between the two wells is obstructing motion along the nodes of the order parameter, it is more effective in confining the superconducting state.

The effect could be probed by magnetization measurements of the kind reviewed in [19]. A potential experimental arrangement involves the fabrication of two superconducting cylindrical tubes of square cross-section. One cylinder has the nodes of the $d$-wave order parameter parallel to its walls; the other cylinder has the nodes angled at 45° to its walls. A vortex trapped in the hollow interior of a cylinder by field cooling will tunnel out at a rate influenced by the order parameter orientation.

The author thanks A. Auerbach, who initially suggested that the author study anisotropic dynamics of $d$-wave vortices, J. Jain, G. Koren, M. Wolraich and Y. Yeshurun. The author is grateful for the hospitality of The Institute for Theoretical Physics at Technion during the early stages of this work and for funding provided by a Packard Foundation.

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Space constraints make it impossible to present detailed derivations of our results within the body of our manuscript. This appendix supplies explicit mathematical arguments for the convenience of interested readers.

**PARTICLE HOLE TRANSFORMATION**

To evaluate the overlap matrix element \( \langle v_N(\rho) | v_N(\rho') \rangle \), one inserts the appropriate translation operators
\[
\langle v_N(\rho) | v_N(\rho') \rangle = \langle v | T(\rho') P_N | v \rangle.
\]
Note that there is no operator \( P_N \) next to \( T(\rho) \); a second projection operator would not change the matrix element.

Now, the definition of the many-body translation operator is
\[
T(\rho) = \Pi_{R,S} (C_{R,S}c_R^\dagger c_s + c_R^\dagger C_{R,S}) \Pi_{r,s}(c_{\tau-R}^\dagger \rho_{\tau-R} c_{\tau-R} + e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} \Pi'_{r,s} C_{r,s}^\dagger c_{\tau-R}),
\]
and it is convenient introduce a particle-hole transformation. We write
\[
\begin{align*}
\Pi_R(C_{R,-1} C_{R,-1}^\dagger + c_{R,-1}^\dagger C_{R,-1}) & = \Pi_R c_{\tau-R}^\dagger \Pi_R \left( c_{\tau-R}^\dagger c_{\tau-R} + e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} C_{R,-1} \right) \\
& = \Pi_R c_{\tau-R}^\dagger (C_{R,-1} C_{R,-1}^\dagger + c_{R,-1}^\dagger C_{R,-1}) \\
& = \Pi_R c_{\tau-R}^\dagger (C_{R,-1}^\dagger c_{\tau-R}^\dagger + c_{R,-1}^\dagger C_{R,-1}) \\
& = \Pi_R c_{\tau-R}^\dagger \Pi_R \left( c_{\tau-R}^\dagger c_{\tau-R} + e^{-i(\pi B/\Phi_s) \rho (x-\rho_x/2)} C_{R,-1} c_{\tau-R}^\dagger \right)
\end{align*}
\]
The phase factor at the end arises since magnetic boundary conditions imply \( \Pi_R c_{\tau-R}^\dagger e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} = \Pi_R c_{\tau-R}^\dagger e^{-i(\pi B/\Phi_s) \rho (x-\rho_x/2)} \). Defining “holding-space” operators \( D_{R,1} = C_{R,1} \) and \( D_{R,-1} = C_{R,-1}^\dagger \), we therefore find
\[
T(\rho) = \Pi_R e^{i(\pi B/\Phi_s) \rho \rho_x} \Pi_{R,S}(D_{R,S} D_{R,S}^\dagger + d_{R,S}^\dagger d_{R,S}) \Pi_{r,s}(d_{\tau-R}^\dagger d_{\tau-R} - e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} d_{\tau-R}^\dagger d_{\tau-R}) \Pi_R d_{\tau-R}^\dagger (\Pi_R d_{\tau-R})^-1.
\]
The leading phase factor can be absorbed into the definition of the vortex state if desired, as \( \langle v_N(\rho) \rangle \equiv \Pi_R e^{-i(\pi B/\Phi_s) \rho \rho_x/2} T(\rho) \langle v_N \rangle \), and can therefore be omitted from \( \Pi_R \).

A nearly identical calculation yields the operator \( T(\rho) P_N \), although an additional phase factor \( e^{i(N_z/2) \chi} \) arises. Knowing the translation operator, we can compute the overlap and translation matrix elements.

**OVERLAP MATRIX ELEMENT**

The overlap matrix element is
\[
\langle v_N(\rho) | v_N(\rho') \rangle = \int \frac{dx}{2\pi} e^{i(N_z/2-N) \chi} \langle v | (\Pi_{r,s}(d_{r-R}^\dagger d_{r-R} - e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} d_{r-R}^\dagger d_{r-R}) \\
\Pi_{R,S}(D_{R,S} D_{R,S}^\dagger + d_{R,S}^\dagger d_{R,S}) \Pi_{r,s}(d_{r-R}^\dagger d_{r-R} - e^{i(\pi B/\Phi_s) \rho (x-\rho_x/2)} d_{r-R}^\dagger d_{r-R}) \\
\Pi_{r',s'}(d_{r'-R'}^\dagger d_{r'-R'} - e^{i(\pi B/\Phi_s) \rho (x'-\rho_x/2)} d_{r'-R'}^\dagger d_{r'-R'}) \Pi_{R',S'}(D_{R',S'} D_{R',S'}^\dagger + d_{R',S'}^\dagger d_{R',S'}) \rangle \langle v \rangle}
\]
\[
= \int \frac{dx}{2\pi} e^{i(N_z/2-N) \chi} \det \left[ G_{s,s'}(r - \rho', r' - \rho, t = 0^+) e^{i(\pi B/\Phi_s) \rho (x'-\rho_x/2) - \rho_y (x'-\rho_x/2))} \\
- G_{s,s'}(r - \rho', r' - \rho, t = 0^+) \right] \]
\[
= \int \frac{dx}{2\pi} e^{i(N_z/2-N) \chi} \det \mathcal{M}(\chi).
\]

The equality of (2) and (3) is demonstrated as follows. We start by noticing that the middle factors \( \Pi_{R} D_{R,-1}(\Pi R D_{R,-1})^-1 \) give unity when acting to the left or the right. With these operators eliminated, we consider the new middle factors
\[
\begin{align*}
\Pi_{R,S}(D_{R,S} D_{R,S}^\dagger + d_{R,S}^\dagger d_{R,S}) & \Pi_{R',S'}(D_{R',S'} D_{R',S'}^\dagger + d_{R',S'}^\dagger d_{R',S'}) \rangle = \\
\Pi_{R,S}(D_{R,S} d_{R,S}^\dagger + d_{R,S}^\dagger d_{R,S} + D_{R,S}^\dagger d_{R,S} d_{R,S} d_{R,S}^\dagger) & \to 1
\end{align*}
\]
since the operator $d_{r,s}^d$ yields zero when it acts to the right, the operator $d_{r,s}^d$ yields zero when it acts to the left, and the combination $d_{r,s}^d d_{r,s}^d$ yields unity. This leaves

$$\langle v_N(\rho) | v_N(\rho') \rangle = \int \frac{d\chi}{2\pi} e^{i(N_{\rho}/2-N) \chi} \langle \Pi_r \Pi_s (d_{r,\rho}^d d_{r,\rho}^d + e^{-i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r,\rho}^d d_{r,\rho}^d) D_{r,s} \rangle$$

$$\Pi_{r',s'} (d_{r',\rho'}^d d_{r',\rho'}^d + e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r',\rho'}^d d_{r',\rho'}^d) \langle \Pi_r \Pi_s D_{r,s} \rangle$$

Expanding the product out as a sum, one finds

$$= \int \frac{d\chi}{2\pi} e^{i(N_{\rho}/2-N) \chi} \sum_{n=0}^{N_{\rho}} \sum_{r_1,s_1} \ldots \sum_{r_n,s_n} \langle \Pi_r \Pi_s \rangle (e^{-i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d \ldots e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d \ldots e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d) \langle \Pi_r \Pi_s D_{r,s} \rangle$$

Invoking Wick’s theorem, this leaves us with a sum over permutations $P$

$$= \int \frac{d\chi}{2\pi} e^{i(N_{\rho}/2-N) \chi} \sum_{P} (-1)^P \sum_{n=0}^{N_{\rho}} \sum_{r_1,s_1} \ldots \sum_{r_n,s_n} \langle \Pi_r \Pi_s \rangle (e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d \ldots e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d \ldots e^{i\chi(\pi B/\Phi_0)\rho_0(\rho_0 - \rho_0/2)} d_{r_1,\rho_1}^d d_{r_1,\rho_1}^d) \langle \Pi_r \Pi_s D_{r,s} \rangle$$

The symbol $(-1)^P$ takes the value $+1$ for even permutations and $-1$ for odd permutations. In the second to last equality, we recognize that a sum over permutations can be written as a determinant.

**MATRIX ELEMENT OF THE POTENTIAL**

The analysis that enable us to compute $\langle v_N(\rho) | v_N(\rho') \rangle$ can be extended to obtain matrix elements of the pinning potential $\langle v_N(\rho) | V | v_N(\rho') \rangle = \langle v | T'(\rho) V T'(\rho') P_N \rangle | v \rangle$.

First, the potential is expressed in terms of the $d$ operators

$$V = \sum_{r_0,s_0} V_{s_0}(r_0) d_{r_0,s_0}^d d_{r_0,s_0}^d = \sum_{r_0} V_{-1}(r_0) + \sum_{s_0} s_0 V_{s_0}(r_0) d_{r_0,s_0}^d d_{r_0,s_0}^d$$

(4)
Given the form (4), it is necessary to compute \( \langle v_N(\rho) | d_{r_{t,s_o}} d_{r_{t,s}} v_N(\rho') \rangle \). The result is

\[
\langle v_N(\rho) | d_{r_{t,s_o}} d_{r_{t,s}} v_N(\rho') \rangle = \int \frac{dx}{2\pi} e^{i(N_o/2-N) \chi} \langle v | (\Pi_{r,s}D_{r,s}^{\dagger}) \Pi_{r,s} (d_{r-\rho,s} d_{r-\rho,s} + e^{-i\pi B/\Phi_o}(x-\rho_x/2) d_{r-\rho,s} D_{r,s})
\Pi_{r,s} D_{r,s}^{\dagger} D_{r,s} + D_{r,s}^{\dagger} D_{r,s}) \Pi_{r,s} (d_{r-\rho,s} d_{r-\rho,s} + e^{-i\pi B/\Phi_o}(x-\rho_x/2) D_{r,s}^{\dagger})
\Pi_{r,s} D_{r,s}^{\dagger} D_{r,s} + D_{r,s}^{\dagger} D_{r,s}) \Pi_{r,s} (d_{r-\rho,s} d_{r-\rho,s} + e^{-i\pi B/\Phi_o}(x-\rho_x/2) D_{r,s}^{\dagger})\rangle \Pi_{r,s} D_{r,s}^{\dagger} | v \rangle
\]

From this calculation, we find that

\[
\langle v_N(\rho) | V | v_N(\rho') \rangle = \sum_{r_{s_o}} \frac{dx}{2\pi} e^{i(N_o/2-N) \chi} \sum_{P} (-1)^P
\]

\[
V_{-1}(r_o) \Pi_{r,s} \left( e^{i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle + \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
- V_{-1}(r_o) \left( e^{-i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
\Pi_{r,s} \left( e^{i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle + \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
\Pi_{r,s} \left( e^{-i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle + \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
= \sum_{r_{s_o}} \frac{dx}{2\pi} e^{i(N_o/2-N) \chi} \sum_{P} (-1)^P
\]

\[
V_{-1}(r_o) \left( \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle \right)
\]

\[
\Pi_{r,s} \left( e^{i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle + \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
\Pi_{r,s} \left( e^{-i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} \right) \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle + \langle v | d_{r-\rho,s} d_{r-\rho,s} | v \rangle
\]

\[
= \sum_{r_{s_o}} V_{s_o}(r_o) \int \frac{dx}{2\pi} e^{i(N_o/2-N) \chi} \text{det}\mathcal{L}(x, r_o, s_o)
\]

where

\[
\mathcal{L}(x, r_o, s_o) = G_{s_o}(r - \rho', r' - \rho, t = 0^-) e^{i\pi (x+\pi B/\Phi_o)(\rho_o'(x_o-\rho_x'/2)-\rho_o(x_o-\rho_x/2))} (1 - \delta(r', s') \delta(s_o, -1) - G_{s_o}(r - \rho', r' - \rho, t = 0^+) (1 - \delta(r', s') \delta(s_o, 1)).
\]