HIGHER OPERATIONS IN STRING TOPOLOGY OF CLASSIFYING SPACES

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Abstract. Examples of non-trivial higher string topology operations have been regrettablly rare in the literature. In this paper, working in the context of string topology of classifying spaces, we provide explicit calculations of a wealth of non-trivial higher string topology operations associated to a number of different Lie groups. As an application of these calculations, we obtain an abundance of interesting homology classes in the twisted homology groups of automorphism groups of free groups, the ordinary homology groups of holomorphs of free groups, and the ordinary homology groups of affine groups over the integers and the field of two elements.

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1. Introduction

Since Chas and Sullivan’s original observation that the homology of the free loop space of a manifold admits the structure of a Batalin–Vilkovisky algebra [CS99], string topology has been extended in many ways, both by replacing Batalin–Vilkovisky algebras with more elaborate algebraic structures, and by constructing similar structures from objects other than manifolds. In modern formulations, string topology of a space $E$ is typically expressed as a field theory consisting (roughly speaking) of compatible maps

$$H_*(M) \otimes H_*(E^X) \longrightarrow H_*(E^Y)$$

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where $X$ and $Y$ are (typically) compact 1-manifolds, $E^X$ and $E^Y$ denote the spaces of maps from $X$ and $Y$ into $E$, respectively, and $M$ is some kind of space of cobordisms between $X$ and $Y$. The space $E$ could be, as in Chas and Sullivan’s original work, a closed oriented manifold [God07], or it could be the classifying space of a compact Lie group [CM12] or even a stack [BGNX12]. The space $M$, on the other hand, might be a model for the moduli space of Riemann surfaces modelled on a cobordism $\Sigma$: $X \to Y$ [God07, CM12, BGNX12] or a space of suitable Sullivan chord diagrams [Cha05]; see also [Kau08, PR11, WW11]. Typically, the degree 0 part of the structure, that is, the part corresponding to $H_0$ of the spaces $M$, is relatively accessible. On the other hand, higher string topology operations, that is, maps $H_*(E^X) \to H_*(E^Y)$ corresponding to positive-degree homology classes of $M$, have proved much harder to understand: while the Batalin–Vilkovisky $\Delta$-operator has been computed in a number of cases (see eg. [Tam06, Men09, Hep09, Hep10, Yan13, BB15]), examples of other non-trivial higher operations remain rare. Indeed, Tamańo [Tam09] has shown that string topology operations associated to the best-understood classes in the homology of moduli spaces of Riemann surfaces, namely those in the stable range, vanish. In the context of string topology of manifolds, Wahl [Wah12] has recently succeeded in constructing families of non-trivial higher operations not arising from the $\Delta$-operator, and it appears that these are the only examples of such operations so far (with the possible exception of the Goresky–Hingston $\ast$-product [GH09], which could be interpreted as a higher operation associated to a degree 1 homology class of a compactified moduli space). On the other hand, in the context of string topology of classifying spaces of compact Lie groups, it seems that the first and only examples of non-trivial higher operations are given by the computation in [HL13, section 9] for the group $\mathbb{Z}/2$.

The present paper can be viewed as an argument for the following three theses: first, that non-trivial higher string topology operations exist in abundance; second, that many examples of such operations can be computed explicitly; and third, that such computations have applications to mathematics outside string topology. To support the first two theses, we will provide a wealth of explicit calculations of non-trivial higher operations in the string topology of classifying spaces of a number of different compact Lie groups. We will work within the context of the recent extension [HL13] (in characteristic 2) of Chataur and Menichi’s string topology classifying spaces [CM12] into a novel kind of field theory called a Homological H-Graph Field Theory, or an HHGFT. Recall from [HL13] that an $h$-graph means any space homotopy equivalent to a finite graph, and that an $h$-graph cobordism $S: X \to Y$ from an $h$-graph $X$ to another $h$-graph $Y$ is a diagram $X \leftarrow S \leftrightarrow Y$ of $h$-graphs satisfying certain conditions [HL13, Definition 2.5]. A family of $h$-graph cobordisms $S/B: X \to Y$ consists of maps of $h$-graphs satisfying certain conditions [HL13, Definition 2.8] which in particular imply that for every $b \in B$, the restriction to fibres over $b$ in (1) gives an $h$-graph cobordism $S_b: X \to Y$. An $h$-graph cobordism $S: X \to Y$ is called positive if $X$ meets every component of $S$, and a family of $h$-graph cobordism $S/B: X \to Y$ is called positive if all its fibres are positive. Given a compact Lie group $G$, the HHGFT $\Phi^G$ constructed in [HL13] consists of maps

$$\Phi^G(S/B): H_{*+\dim(G)}(S,X)(B) \otimes H_*(BG^X) \to H_*(BG^Y),$$

one for every positive family of $h$-graph cobordisms $S/B: X \to Y$, satisfying various compatibility axioms [HL13, subsection 3.1]. Here $\chi(S,X)$ denotes the locally constant function $B \to \mathbb{Z}$ sending each point $b \in B$ to the Euler characteristic $\chi(S_b, X)$.
The string topology operations we compute are associated to families of h-graph cobordisms $S_n/B\Sigma_n$: pt $\mapsto$ pt from the one-point space to itself constructed as follows. Let $\hat{S}_n$: pt $\mapsto$ pt be the h-graph cobordism depicted below consisting of $n$ strings joining the two points.

\[
\hat{S}_n = \begin{array}{ccc}
& 1 & \\
2 & \text{pt} & \text{pt} \\
\vdots \\
& n & \\
\end{array}
\]

The symmetric group $\Sigma_n$ acts on $\hat{S}_n$ by permuting the $n$ strings, and performing the Borel construction gives us the family h-graph cobordisms

$S_n/B\Sigma_n = E\Sigma_n \times_{\Sigma_n} \hat{S}_n/B\Sigma_n$: pt $\mapsto$ pt.

We will compute explicitly the operations

$\Phi^G(S_n/B\Sigma_n): H_{\dim(G)(n-1)}(B\Sigma_n) \otimes H_*(BG) \rightarrow H_*(BG)$

for all $n \geq 1$ when $G$ is an elementary abelian 2-group or a dihedral group of order 2 mod 4, for $1 \leq n \leq 7$ when $G$ is a torus $T^l = (S^1)^l$, and for $1 \leq n \leq 3$ when $G = SU(2)$. The computation done in [HL13, section 9] amounts to the calculation of $\Phi^{\mathbb{Z}/2}(S_2/B\Sigma_2)$. It turns out that for any compact Lie group $G$, all higher operations associated to $S_n/B\Sigma_n$ vanish when $n$ is not a power of 2. On the other hand, when $n$ is a power of 2, our calculations provide a wealth of non-trivial operations for all of the groups $G$ above. In particular, these calculations give the first examples of non-trivial higher operations in string topology of classifying spaces where the group $G$ is non-abelian or a positive-dimensional compact Lie group.

To support the third thesis, we show that our calculations detect a wealth of non-trivial elements in the homology groups of certain highly interesting groups, namely in the twisted homology groups $H_*(B\text{Aut}(F_n); \mathbb{F}_2^n)$ of $\text{Aut}(F_n)$ and in the ordinary mod 2 homology groups of the holomorph $H_*(B\text{Hol}(F_n) = F_n \rtimes \text{Aut}(F_n)$ and the affine groups $\text{Aff}_n(\mathbb{Z}) = \mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ and $\text{Aff}_n(\mathbb{F}_2) = \mathbb{F}_2^n \rtimes GL_n(\mathbb{F}_2)$. Here $F_n$ denotes the free group on $n$ generators, and $\mathbb{F}_2^n$ denotes $\mathbb{F}_2^\infty$ equipped with the $\text{Aut}(F_n)$-action given by the composite $\text{Aut}(F_n) \rightarrow GL_n(\mathbb{Z}) \rightarrow GL_n(\mathbb{F}_2)$, where the first homomorphism is given by abelianization and the second one by reduction mod 2. The elements we detect are exhibited in Corollaries 8.11, 8.16, 8.18, 8.19, 8.22, and 8.23 below. For each of the aforementioned groups, the homology groups are known to stabilize in the sense that the homology groups for successive values of $n$ are connected by a map, the stabilization map, which is known to be an isomorphism in degrees small enough relative to $n$, the stable range. In each case, the homology groups are well understood in the stable range, but remain mysterious outside it; see section 8.1 and the discussion following Corollary 8.18. It is in this poorly understood unstable range where the elements we detect live. Indeed, many of the elements are unstable in the sense that they are annihilated by an iteration of the stabilization map; see Corollaries 8.19 and 8.23 as well as Corollary 8.18 together with the discussion following it.

**Example 1.1.** The reader may appreciate an example of the non-trivial homology classes we construct. Let $u$ be an integer of the form $u = \sum_{i=1}^k u_i$ where $u_1, \ldots, u_k$ are positive integers with the property that no two of them have a 1 in common in their binary expansions. Let $f: \{1, \ldots, k\} \rightarrow \{1, \ldots, r\}$ be a surjective function, and denote

$N = \sum_{i=1}^r (2^{|f^{-1}(i)|} - 1)$. 
Then, in view of Proposition 8.1, our computations for the group $G = \mathbb{Z}/2$ combine with:

1. Corollary 8.11 to produce a non-trivial element in $H_u(B\text{Hol}(F_N))$. This class is stable, but not in the image of the stabilization map $H_u(B\text{Hol}(F_{N-1})) \to H_u(B\text{Hol}(F_N))$. See Remark 8.13 and Corollary 8.16.
2. Corollary 8.18 to produce a non-trivial element in $H_u(B\text{Aut}(F_N); \tilde{\mathbb{F}}^N_2)$. Like all elements in $H_1(B\text{Aut}(F_N); \tilde{\mathbb{F}}^N_2)$, this class is unstable. See the discussion following Corollary 8.18.
3. Corollary 8.19 to produce a non-trivial unstable element in $H_u(B\text{Hol}(F_N))$ which is not in the image of the stabilization map $H_u(B\text{Hol}(F_{N-1})) \to H_u(B\text{Hol}(F_N))$.
4. Corollary 8.22 to produce a non-trivial element in $H_u(B\text{Aff}_N(\mathbb{Z}))$ and another one in $H_u(B\text{Aff}_N(\mathbb{F}_2))$. Like all positive-dimensional classes in the homology of $\text{Aff}_N(\mathbb{F}_2)$, the element in $H_u(B\text{Aff}_N(\mathbb{F}_2))$ is unstable. See Theorems 8.4 and 8.6.
5. Corollary 8.23 to produce a non-trivial unstable element in $H_u(B\text{Aff}_N(\mathbb{Z}))$ and another one in $H_u(B\text{Aff}_N(\mathbb{F}_2))$.

The paper is organized as follows. In section 2, we prove a general result, Theorem 2.1, which expresses $\Phi^{G_1 \times G_2}$ in terms of $\Phi^{G_1}$ and $\Phi^{G_2}$. This result is used to reduce the computation of $\Phi^{(\mathbb{Z}/2)^i}$ and $\Phi^{\mathbb{Z}^i}$ to the calculation of $\Phi^{(\mathbb{Z}/2)^i}$ and $\Phi^{\mathbb{Z}^i}$, respectively, and could be used to further extend the list of groups $G$ for which the operations $\Phi^G(S_n/B\Sigma_n)$ are known by taking products of the groups we consider. In section 3, we prove a vanishing result, Theorem 3.3, which implies the vanishing of the higher operations associated to $S_n/B\Sigma_n$ when $n$ is not a power of 2, and reduces the calculation of the operations $\Phi^G(S_{2k}/B\Sigma_{2k})$ to the computation of certain maps

$$\alpha_k^G: H_{s - \dim(G)(2^k - 1)}(BV_k) \otimes H_s(BG) \to H_s(BG),$$

where $V_k$ denotes the elementary abelian 2-group of dimension $k$. See Definition 3.7. In section 4, we develop descriptions of the maps $\alpha_k^G$ suitable for computations. Sections 5, 6 and 7 are then devoted to the computation of the maps $\alpha_k^G$ when $G$ is an elementary abelian 2-group or a dihedral group of order 2 mod 4, a torus, or $SU(2)$, respectively. Finally, in section 8 we discuss the aforementioned applications of our computations to the homology of $\text{Aut}(F_n)$, $\text{Hol}(F_n)$, $\text{Aff}_n(\mathbb{Z})$ and $\text{Aff}_n(\mathbb{F}_2)$.

Notation and conventions. Throughout the paper, we will work over the field $\mathbb{F}_2$ of two elements. Unless stated otherwise, homology and cohomology groups are taken with $\mathbb{F}_2$ coefficients. If $S/B: X \to Y$ is a positive family of h-graph cobordisms, we denote by $\Phi^G(S/B)^+$ the adjoint

$$\Phi^G(S/B)^+: H_{s + \dim(G)\chi(S,X)}(B) \to \text{Hom}_s(H_s(BG^X), H_s(BG^Y))$$

of the map

$$\Phi^G(S/B): H_{s + \dim(G)\chi(S,X)}(B) \otimes H_s(BG^X) \to H_s(BG^Y).$$

Here $\text{Hom}_s$ denotes the internal hom in the category of graded $\mathbb{F}_2$-vector spaces. If $S: X \to Y$ is an h-graph cobordism, we write $h\text{Aut}(S)$ for the topological monoid of self homotopy equivalences of $S$ fixing $X$ and $Y$ pointwise. The symbols $\circ$ and $\sqcup$ refer to external composition and external disjoint union of families of h-graph cobordisms, respectively; see [HL13, Definition 2.8]. As in [HL13], when working with topological spaces, we will work within the category of $k$-spaces, and when working with fibred spaces, the base spaces are in addition assumed to be weak Hausdorff spaces.

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2. The product theorem

Let $G_1$ and $G_2$ be compact Lie groups. For any space $X$, the projections of $G_1 \times G_2$ onto $G_1$ and $G_2$ induce a natural homeomorphism

$$B(G_1 \times G_2)^X \xrightarrow{\approx} B(G_1)^X \times B(G_2)^X.$$ Combining this with the homology cross product, we obtain a natural isomorphism

$$\kappa_X : H_*(B(G_1)^X) \otimes H_*(B(G_2)^X) \xrightarrow{\approx} H_*(B(G_1 \times G_2)^X).$$

The purpose of this section is to prove the following theorem expressing $\Phi_{G_1 \times G_2}$ in terms of $\Phi_{G_1}$ and $\Phi_{G_2}$.

**Theorem 2.1** (Product theorem). Suppose $S/B : X \to Y$ is a positive family of h-graph cobordisms, and let $d_1 = \dim G_1$ and $d_2 = \dim G_2$. Then the following diagram commutes.

$$H_{*+(d_1+d_2)}(S/X)(B) \otimes H_*(B(G_1 \times G_2)^X) \xrightarrow{\Phi_{G_1 \times G_2}(S/B)} H_*(B(G_1 \times G_2)^Y)$$

Here $\Delta$ denotes the diagonal map and $\tau$ denotes the twist map.

Recall from [HL13, Definition 5.2] that a base space $B$ is called good if it can be embedded as an open subset of some CW complex. Let $G$ be a compact Lie group and let $S/B : X \to Y$ be a positive family of h-graph cobordisms over a good base space. In [HL13, section 7.6 and 7.8], it is shown that for an h-graph with basepoints $(Z, R)$, there is a natural zigzag of homotopy equivalences

$$B(G^{\Pi_1(Z,R)}) \xrightarrow{\eta_{Z,R}^G} B G^{\Pi_1(Z,R)} \xrightarrow{\alpha_{Z,R}^G} B G^{\Pi_0(Z,R)} \xrightarrow{\beta_{Z,R}^G} B G^Z \xleftarrow{\approx} B G^Z.$$ (2)

Let $\eta_{Z,R}^G$ denotes the induced isomorphism

$$\eta_{Z,R}^G : H_* B(G^{\Pi_1(Z,R)}) \xrightarrow{\approx} H_*(B G^Z).$$
Proposition 2.2. Let $S/B : X 	o Y$ be a positive family of $h$-graph coherisms over a good base space. Choose basepoints $P \to X$ and $Q \to Y$ for $X$ and $Y$, respectively. Then $\Phi^G(S/B)$ is given by the composite

$$H^* + \dim(G) \chi(S,X)(B) \otimes H^*(BG^X) \xrightarrow{1 \otimes (\eta^G_{X,P})^{-1}} H^* + \dim(G) \chi(S,X)(B) \otimes H^*(G\Pi_1(X,P)) \xrightarrow{\times} H^* + \dim(G) \chi(S,X)(B \times B(G\Pi_1(X,P))) \approx H_* B(G\Pi_1(S,P)) \xrightarrow{(a)} H_* B(G\Pi_1(S,P \sqcup Q)) \approx H_* B(G\Pi_1(YQ)) \approx \eta^G_{Y,Q} \approx H_* (BG^Y)$$

where the map labeled by $!$ is the umkehr map [HL13, section 7.2] associated to the map

$$B \times B(G\Pi_1(X,P)) \xleftarrow{\text{pr}} B(G\Pi_1(S,P)) \xrightarrow{\text{pr}} B \times BG^P$$

of fibrewise manifolds induced by the map $B \times X \to S$; where the map $(a)$ is induced by the inverse of the homotopy equivalence

$$B(G\Pi_1(S,P \sqcup Q)) \xrightarrow{\sim} B(G\Pi_1(S,P))$$

induced by the inclusion $P \hookrightarrow P \sqcup Q$; and where the map $(b)$ is induced by the map $(S \to B, P \sqcup Q) \to (Y \to pt, Q)$ in the category $\widetilde{\mathcal{H}}^{\text{top}}$ of [HL13, subsection 7.3 and Definition 5.10] given by the diagram

$$(S, P \sqcup Q) \xleftarrow{(B \times Y, Q)} \xrightarrow{\text{pr}} (Y, Q)$$

where the top left map is induced by the inclusions $B \times Y \to S$ and $Q \to P \sqcup Q$. □

Proof of Theorem 2.1. By picking a CW approximation to $B$ and using the base change axiom of HHGFTs, we may assume that $B$ is a CW complex and hence a good base space. A further application of the base change axiom shows that without loss of generality we may restrict to the case where $B$ is connected. This simplifies notation, since in this case $\chi(S, X)$ is an integer rather than an integer-valued locally constant function on $B$. The theorem follows by a lengthy but straightforward diagram chase using the description for $\Phi^G(S/B)$ given in Proposition 2.2 and the following observations:
• For any family \((T, R)\) of h-graphs with basepoints over a good base space \(C\), the projections of \(G_1 \times G_2\) onto \(G_1\) and \(G_2\) induce a natural isomorphism

\[
B\left(\left(G_1 \times G_2\right)^{\Pi_1(T, R)}\right) \cong B\left(G_1^{\Pi_1(T, R)}\right) \times_B B\left(G_2^{\Pi_1(T, R)}\right)
\]

\[
C \times B\left(G_1 \times G_2\right)^R \cong C \times BG_1^R \times BG_2^R
\]

of fibrewise manifolds.

• The zigzag (2) is natural with respect to maps of Lie groups. It follows that the diagram

\[
\begin{array}{ccc}
H_* B\left(G_1 \times G_2\right)^{\Pi_1(X, P)} & \xrightarrow{\eta_{G_1} \times \eta_{G_2}} & H_* B\left(G_1 \times G_2\right)^X \\
\cong & & \cong \\
H_* \left(B\left(G_1^{\Pi_1(X, P)}\right) \times B\left(G_2^{\Pi_1(X, P)}\right)\right) & \cong & H_* \left(BG_1^X \times BG_2^X\right)
\end{array}
\]

\[
\begin{array}{ccc}
\eta_{X,P} \eta_{X,P} & \cong & \eta_{X,P} \eta_{X,P} \\
\cong & \cong & \cong \\
H_* B\left(G_1^{\Pi_1(X, P)}\right) \otimes H_* B\left(G_2^{\Pi_1(X, P)}\right) & \xrightarrow{\eta_{X,P} \otimes \eta_{X,P}} & H_* \left(BG_1^X \otimes H_* \left(BG_2^X\right)\right)
\end{array}
\]

commutes, and similarly for \((Y, Q)\). Here the middle horizontal arrow is

\[
(\tilde{\beta}^{G_1} \times \tilde{\beta}^{G_2})^{-1} \circ (\tilde{\alpha}^{G_1} \times \tilde{\alpha}^{G_2})_s \circ (\tilde{\theta}^{G_1} \times \tilde{\theta}^{G_2})_s
\]

and the vertical maps in the top square are induced by the projections of \(G_1 \times G_2\) onto \(G_1\) and \(G_2\).

• Consider the commutative diagram

\[
\begin{array}{cccc}
\begin{array}{ccc}
B \times B\left(G_1^{\Pi_1(X, P)}\right) & \times & B\left(G_1^{\Pi_1(S, P)}\right) \times_B B\left(G_2^{\Pi_1(S, P)}\right) \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
B \times BG_1^P \times BG_2^P \\
\downarrow \\
B \times BG_1^P \times B \times BG_2^P \\
\downarrow \\
B \times B\left(G_1^{\Pi_1(X, P)}\right) \times B \times B\left(G_2^{\Pi_1(X, P)}\right) \leftarrow B\left(G_1^{\Pi_1(S, P)}\right) \times B\left(G_2^{\Pi_1(S, P)}\right)
\end{array}
\]

(4)

where the horizontal arrows are induced by the maps (3) of fibrewise manifolds for \(G = G_1, G_2\); the right-hand vertical map is the inclusion; and the middle and left-hand vertical maps are induced by the diagonal map \(B \rightarrow B \times B\). Decorating the left-hand corners of the diagram with degree shift \((d_1 + d_2)\chi(S, X)\) and the right-hand corners with degree shift 0, we obtain a 2-cell in the double category \(G_{\text{sh}}(\mathcal{M})\) of [HL13, subsection 7.1]. Applying the umkehr functor \(U_{\text{man}}\) [HL13, subsection 7.2], we therefore obtain a commutative square

\[
H_{*+(d_1 + d_2)\chi(S, X)} \left( B \times B\left(G_1^{\Pi_1(X, P)}\right) \times B\left(G_2^{\Pi_1(X, P)}\right) \right) \xrightarrow{\text{1}} H_* \left( B\left(G_1^{\Pi_1(S, P)}\right) \times_B B\left(G_2^{\Pi_1(S, P)}\right) \right)
\]

\[
H_{*+(d_1 + d_2)\chi(S, X)} \left( B \times B\left(G_1^{\Pi_1(X, P)}\right) \times B \times B\left(G_2^{\Pi_1(X, P)}\right) \right) \xrightarrow{\text{1}} H_* \left( B\left(G_1^{\Pi_1(S, P)}\right) \times B\left(G_2^{\Pi_1(S, P)}\right) \right)
\]
where the horizontal maps are the umkehr maps associated with the respective horizontal maps in (4).

- The square

\[
\begin{array}{c}
H_{*+(d_1+d_2)\chi(S,X)}(B \times B(G_1^{\Pi_1(X,P)})) \\
H_{*+(d_1+d_2)\chi(S,X)}(B \times B(G_2^{\Pi_1(X,P)}))
\end{array}
\]

\[
\xrightarrow{\otimes} \xrightarrow{\approx} \xrightarrow{\otimes} \xrightarrow{\approx} \xrightarrow{\approx}
\]

\[
\begin{array}{c}
H_{*+(d_1+d_2)\chi(S,X)}(B \times B(G_1^{\Pi_1(X,P)})) \\
H_{*+(d_1+d_2)\chi(S,X)}(B \times B(G_2^{\Pi_1(X,P)}))
\end{array}
\]

commutes, as it is an instance of the monoidality constraint $U_{\text{man.} \otimes 1}$. Here the top horizontal map is the tensor product of the umkehr maps associated with the maps (3) of fibrewise manifolds for $G = G_1, G_2$, while the bottom horizontal map is the umkehr map associated with the direct product of these maps. □

### 3. A Vanishing Result

Our aim in this section is to prove a vanishing result, Theorem 3.3 below. Combined with a sufficient understanding of the homology of symmetric groups, this result in particular shows that the operations $\Phi^G(S_n/B\Sigma_n)$ vanish whenever $n$ is not a power of 2, and reduces the computation of the operations $\Phi^G(S_{2^k}/B\Sigma_{2^k})$ to the computation of certain maps

\[
\alpha_k^G: H_{*-(\dim(G)(2^k-1))}(BV_k) \otimes H_*(BG) \longrightarrow H_*(BG)
\]

where $V_k$ is an elementary abelian 2-group of dimension $k$. See Definition 3.7.

We start by recalling facts about the homology of symmetric groups. The standard inclusions $\Sigma_n \times \Sigma_m \rightarrow \Sigma_{m+n}$ induce on $\bigsqcup_{n \geq 0} B\Sigma_n$ the structure of a topological monoid, making the homology groups $H_*(\bigsqcup_{n \geq 0} B\Sigma_n)$ into a ring bigraded by degree and weight, the homogeneous elements of degree $d$ and weight $n$ being precisely the elements of $H_d(B\Sigma_n)$. In fact, $\bigsqcup_{n \geq 0} B\Sigma_n$ is not just a monoid, but an $E_\infty$-space. The following description of its homology in terms of the concomitant Dyer–Lashof operations $Q^*$ can be found in [CLM76, §I.4].

**Theorem 3.1.** We have

\[
H_*(\bigsqcup_{n \geq 0} B\Sigma_n) = F_2[Q^I | I \text{ is admissible and } e(I) > 0].
\]

Here a multi-index $I = (s_1, \ldots, s_k)$ with $s_j \geq 0$ is called admissible if $s_j \leq 2s_{j+1}$ for all $1 \leq j < k$; is said to have excess $e(I) = s_1 - \sum_{j=2}^k s_j$; and has associated operation $Q^I = Q^{s_1} \cdots Q^{s_k}$. (By convention, the empty multi-index $I = \emptyset$ is admissible, has excess $\infty$, and has associated operation $Q^\emptyset = 1$.) The element $[1]$ is the generator of $H_0(B\Sigma_1)$, and $Q^I[1]$ for $I = (s_1, \ldots, s_k)$ has degree $s_1 + \cdots + s_k$ and weight $2^k$.

For our purposes, a slightly different description of $H_*(\bigsqcup_{n \geq 0} B\Sigma_n)$ is more convenient. A choice of bijection between the product $\{1, \ldots, n\} \times \{1, \ldots, m\}$ and $\{1, 2, \ldots, nm\}$ induces a homomorphism $\Sigma_n \times \Sigma_m \rightarrow \Sigma_{nm}$, with different choices leading to conjugate homomorphisms. Thus we obtain a map

\[
H_*(B\Sigma_n) \otimes H_*(B\Sigma_m) \longrightarrow H_*(B\Sigma_{nm})
\]

which does not depend on the bijection chosen. These maps for varying $n$ and $m$ fit together to equip $H_*(\bigsqcup_{n \geq 0} B\Sigma_n)$ with another commutative associative product which
we denote by \( \circ \). The element \([1]\) is the identity element for this product. For each \( i \geq 0 \), let \( E_i \) denote the non-trivial element of \( H_1(B\Sigma_2) \). From Theorem 3.1 and the computation of \( H_*(QS^0) \) given in [Tur97], we can easily read off the following result.

**Theorem 3.2.** We have

\[
H_*(\bigsqcup_{n \geq 0} B\Sigma_n) = F_2[E_{i_1} \circ E_{2i_2} \circ \cdots \circ E_{2k-1i_k} | k \geq 0, 1 \leq i_1 \leq \cdots \leq i_k]
\]

where the empty \( \circ \)-product is to be interpreted as the element \([1] \in H_0(B\Sigma_1)\).

Notice that the element \( E_{i_1} \circ E_{2i_2} \circ \cdots \circ E_{2k-1i_k} \) has degree \( i_1 + 2i_2 + \cdots + 2^{k-1}i_k \) and weight \( 2^k \).

**Proof of Theorem 3.2.** Let \( i: \bigsqcup_{n \geq 0} B\Sigma_n \to QS^0 \) be the group completion map afforded by the Barratt–Quillen–Priddy theorem, so that

\[
i_*: H_*(\bigsqcup_{n \geq 0} B\Sigma_n) \to H_*(QS^0)
\]

is a ring map identifying \( H_*(QS^0) \) with the localization \( H_*(\bigsqcup_{n \geq 0} B\Sigma_n)[[1]^{-1}] \). Since multiplication by \([1]\) is injective in \( H_*(\bigsqcup_{n \geq 0} B\Sigma_n) \) (see eg. [Nak60, Theorem 5.8]), the map \( i_* \) is injective. The map \( i_* \) sends each element \( E_i \in H_1(B\Sigma_2) \) to the element of \( H_*(QS^0) \) of the same name defined on p. 213 in [Tur97]; to see this, use the factorization of the transfer map \( tr^{(n)} \) given on p. 212 of [Tur97]. Moreover, [MM79, Theorem 3.10] implies that the map \( i_* \) preserves the \( \circ \)-product. Thus \( i_* \) takes each element

\[
E_{i_1} \circ E_{2i_2} \circ \cdots \circ E_{2k-1i_k} \in H_*(\bigsqcup_{n \geq 0} B\Sigma_n), \quad 1 \leq i_1 \leq \cdots \leq i_k
\]

(5)

to the element of the same name in \( H_*(QS^0) \). By [Tur97, Theorem 4.15], these elements of \( H_*(QS^0) \) are algebraically independent, whence so are the elements (5). It remains to show that the elements (5) generate all of \( H_*(\bigsqcup_{n \geq 0} B\Sigma_n) \). But this follows from the description of \( H_*(\bigsqcup_{n \geq 0} B\Sigma_n) \) given in Theorem 3.1 by comparison of bigraded Poincaré series using the observation that for each \( k \), setting \( i_t = s_t - \sum_{j=t+1}^{k} s_j \) for \( 1 \leq t \leq k \) defines a bijection between

\[
\{I = (s_1, \ldots, s_k) | I \text{ admissible, } \epsilon(I) > 0\} \quad \text{and} \quad \{(i_1, \ldots, i_k) | 1 \leq i_1 \leq \cdots \leq i_k\}
\]

with \( s_1 + \cdots + s_k = i_1 + 2i_2 + \cdots + 2^{k-1}i_k \).

**Theorem 3.3.** Let \( G \) be a positive-dimensional compact Lie group or a finite group of even order. Then the map

\[
\Phi^G(S_n/B\Sigma_n)^{\Sigma_2}(a): H_*(BG) \to H_{*+|\epsilon|+\dim(G)(n-1)}(BG)
\]

is zero whenever \( a \in H_*(B\Sigma_n) \) is decomposable in the ring \( H_*(\bigsqcup_{n \geq 0} B\Sigma_n) \).

**Remark 3.4.** The case where \( G \) is a finite group of odd order is uninteresting, since in that case the homology of \( BG \) is concentrated in degree 0.

Before proving Theorem 3.3, we need to establish an auxiliary vanishing result. Let \( I: pt \to pt \) and \( \mu: pt \sqcup pt \to pt \) and \( \delta: pt \to pt \sqcup pt \) and \( \varepsilon: pt \to \emptyset \) be the h-graph cobordisms depicted below.

\[
I = \quad \mu = \quad \delta = \quad \varepsilon =
\]

(6)

While in the context of the present discussion our interest in the following result is motivated by the role it plays in the proof of Theorem 3.3, the result is of some interest of its own, since it implies that the product on \( H_*(BG) \) induced by \( \mu \) is uninteresting.
Proposition 3.5. Let $G$ be a compact Lie group of positive dimension or a finite group of even order. Then the operation

$$\Phi^G(\mu/\text{pt}): H_{*-\dim(G)}(\text{pt}) \otimes H_*(BG \times BG) \to H_*(BG)$$

is zero.

Proof. The h-graph cobordism $\mu$ decomposes as $\mu \simeq (I \sqcup (\varepsilon \circ \mu)) \circ (\delta \sqcup I)$ as illustrated below.

When $G$ is positive dimensional, the claim now follows from the observation that the operation $\Phi^G((\varepsilon \circ \mu)/\text{pt}): H_{*-\dim(G)}(\text{pt}) \otimes H_*(BG \times BG) \to H_*(\text{pt})$ must vanish for degree reasons. Let us assume that $G$ is finite of even order. Let $P$ denote the two incoming points in $\mu$. Then by Proposition 2.2, the operation $\Phi^G(\mu/\text{pt})$ factors through the umkehr map associated with the map

$$B(G^{\Pi_1(P)}(\mu,\text{pt})) \to B(G^{\Pi_1(P,P)})$$

of fibrewise manifolds (or, in this case, simply covering spaces) over $BG^P$ induced by the inclusion $P \to \mu$. To prove the claim, it is therefore enough to show that this umkehr map is zero. By [HL13, Lemma 8.6], the umkehr map in this case is just the transfer map. Through the zigzag (2), the map (7) corresponds to the map $BG \to BG \times BG$ which, up to a homotopy equivalence of the source, is just the diagonal map $\Delta: BG \to BG \times BG$. By standard properties of the transfer, the composite

$$H_*(BG \times BG) \xrightarrow{\Delta^1} H_*(BG) \xrightarrow{\Delta^*} H_*(BG \times BG)$$

of the transfer map and the induced map associated with $\Delta$ is multiplication by the index $[G \times G : G] = |G|$, and hence zero by the assumption that the order of $G$ is even. Since $\Delta^*$ is injective, having a left inverse induced by the projection map $\text{pr}_1: BG \times BG \to BG$, it follows that the transfer map $\Delta^1$ is zero, giving the claim.

□

Theorem 3.3 now follows easily from Proposition 3.5.

Proof of Theorem 3.3. Suppose $a \in H_*(B\Sigma_n)$ decomposes as a product $a = bc$ for some $b \in H_*(B\Sigma_k)$ and $c \in H_*(B\Sigma_l)$ where $k, l \geq 1$, $k + l = n$. Then $a = (Bm_{k,l})_*(b \times c)$ where $m_{k,l}: \Sigma_k \times \Sigma_l \to \Sigma_n$ is the standard inclusion. By the base change axiom, we have

$$\Phi^G(S_n/B\Sigma_n)^\sharp(a) = \Phi^G((Bm_{k,l})^*(S_n/(B\Sigma_k \times B\Sigma_l))^\sharp(b \times c).$$

But $(Bm_{k,l})^*(S_n/(B\Sigma_k \times B\Sigma_l))$ decomposes (up to a 2-cell [HL13, Definition 2.10] which is a homeomorphism on base spaces) as the composite

$$(\mu/\text{pt}) \circ ((S_k/B\Sigma_k) \sqcup (S_l/B\Sigma_l)) \circ (\delta/\text{pt}).$$

Thus the claim follows from Proposition 3.5. □

Combining Theorem 3.3 with Theorem 3.2 (or Theorem 3.1), we obtain the following result.

Corollary 3.6. Let $G$ be a positive-dimensional compact Lie group or a finite group of even order. Then the operation $\Phi^G(S_n/B\Sigma_n)$ vanishes unless $n$ is a power of 2. □
Theorem 3.3 and Theorem 3.2 also imply that to understand \( \Phi^G(S_n/BΣ_n) \) for \( n = 2^k \), it is enough to understand its behaviour on classes in the image of the map \( H_*BG \rightarrow H_*BΣ_{2^k} \) induced by the map \( \iota: \Sigma_2^k \rightarrow Σ_{2^k} \) (well defined up to conjugacy) obtained by iterating the maps inducing the \( \circ \)-product on \( H_*(\bigsqcup_{n \geq 0} BΣ_n) \). In the following sections, we will therefore focus on computing the maps \( α_k \) defined below. Observing that \( Σ_2^k \) is simply the \( k \)-dimensional elementary abelian 2-group, we write \( V_k \) for this group and switch to additive notation for the group operation.

**Definition 3.7.** We define \( α_k^G \) to be the composite map

\[
α_k^G: H_{*-d(2^k-1)}(BV_k) \otimes H_*(BG) \xrightarrow{(B\iota)^{\otimes 1}} H_{*-d(2^k-1)}(BΣ_{2^k}) \otimes H_*(BG) \xrightarrow{\Phi^G(S_{2^k}/BΣ_{2^k})} H_*(BG)
\]

where \( d = \dim(G) \). We write \( (α_k^G)^\sharp \) for the adjoint map

\[
(α_k^G)^\sharp: H_{*-d(2^k-1)}(BV_k) \rightarrow \text{Hom}_*(H_*(BG), H_*(BG)).
\]

Notice that we may alternatively describe \( α_k^G \) as the operation

\[
α_k^G = \Phi^G((B\iota)^*S_{2^k}/BV_k).
\]

The identity axiom of HHGFTs [HL13, subsection 3.1] implies the following computation of the map \( α_0^G \).

**Proposition 3.8.** For any compact Lie group \( G \), the map \( (α_0^G)^\sharp \) sends the generator of \( H_*(BV_0) \approx F_2 \) to the identity map of \( H_*(BG) \). \( \square \)

To conclude the section, let us elaborate on the homology of \( V_k \) and the map \( \iota \). Recall that the homology \( H_*(BV_k) \) is isomorphic as a ring to the divided power algebra generated by \( V_k \) where \( V_k \) is identified with \( H_1(BV_k) \). We write \( v^{[n]} \) for the \( n \)-th divided power of \( v \in H_{>0}(BV_k) \), and recall the formula \( v^{[n]}v^{[m]} = \binom{n+m}{m}v^{[n+m]} \) and the convention \( v^{[0]} = 1 \). Let us write \( x_1, \ldots, x_k \) for the basis of \( V_k \) corresponding to the product decomposition \( V_k = Σ_2^k \). By the choice of \( \iota \), we then have the following pleasant formula for the map induced by \( \iota \) on homology. This formula makes it easy to read off the operation \( \Phi^G(S_{2^k}/BΣ_{2^k}) \) from formulas describing the map \( α_k^G \), and it is the main motivation for our preference of the description of \( H_*(\bigsqcup_{n \geq 0} BΣ_n) \) given in Theorem 3.2 over that given in Theorem 3.1.

**Proposition 3.9.** The map \( \iota_* \) is given by

\[
\iota_*: H_*(BV_k) \longrightarrow H_*(BΣ_{2^k}), \quad x_1^{[n_1]} \cdots x_k^{[n_k]} \longmapsto E_{n_1} \circ \cdots \circ E_{n_k}.
\]

**Remark 3.10.** Observe that the map \( \iota \) can alternatively be understood as the Cayley embedding obtained from some identification of \( V_k \) with the set \( \{1, 2, 3, \ldots, 2^k\} \). Since the composite of a Cayley embedding \( Γ \rightarrow Σ_{\lvert Γ\rvert} \) of a group \( Γ \) with an automorphism of \( Γ \) is conjugate to the original embedding, we see that the map \( \iota_* \) factors through \( H_*(BV_k)GL_k(F_2) \). Thus the formula for \( \iota_* \) given in Proposition 3.9 is in fact valid for any basis \( x_1, \ldots, x_k \) of \( V_k \). It can be shown that the map \( H_*(BV_k)GL_k(F_2) \rightarrow H_*(BΣ_{2^k}) \) induced by \( \iota_* \) is injective. See eg. [MM79, p. 61].

4. Preliminary calculations

The main purpose of this section is to develop a description of the maps \( α_k^G \) amenable for making calculations. The main results are Propositions 4.5 and 4.6.

For \( Γ \) a topological group and \( X \) a space with a \( Γ \)-action, we write \( X/Γ \) for the Borel construction \( ET \times_Γ X \). Equivalently, \( X/Γ \) can be described as the bar construction
B(pt, Γ, X) or the classifying space of the action groupoid of the Γ-action on X. Recall from [HL13, proof of Proposition 7.34] that if (T, R) is an h-graph with basepoints, then for any compact Lie group G there is a natural homeomorphism

$$B(G_{\Pi_1(T, R)}) \approx \text{fun}(\Pi_1(T, R), G)/G$$

(8)

where \(\text{fun}(\Pi_1(T, R), G)\) denotes the space of functors from \(\Pi_1(T, R)\) to \(G\) (thought of as a one-object topological category) and where the action of \(G\) on the space \(\text{fun}(\Pi_1(T, R), G)\) is given by the formula

$$\delta \cdot f)(\alpha) = \delta(y) \cdot f(\alpha) \cdot \delta(x)^{-1}$$

for \(f \in \text{fun}(\Pi_1(T, R), G)\), \(\delta \in G\), and \(\alpha : x \to y\) a morphism of \(\Pi_1(T, R)\). If a discrete group Γ acts on \(T\) in such a fashion that the image of the basepoint map \(R \to T\) is kept pointwise fixed, we obtain a family of h-graphs with basepoints \((\tilde{E}T \times_{\Gamma} T, R)\) over \(B\Gamma\), and have natural isomorphisms

$$B(G_{\Pi_1(ET \times_{\Gamma} T, R)}) \approx \tilde{E}T \times_{\Gamma} B(G_{\Pi_1(T, R)}) \approx \tilde{E}T \times_{\Gamma} (\text{fun}(\Pi_1(T, R), G)/G)$$

$$\approx \text{fun}(\Pi_1(T, R), G)/\Gamma \times G$$

(9)

of fibrewise manifolds over \(B\Gamma \times BG\) where the action of \(\Gamma \times G\) on \(\text{fun}(\Pi_1(T, R), G)\) is given by

$$((\gamma, \delta) \cdot f)(\alpha) = \gamma(y) \cdot \delta^{-1}(\alpha) \cdot \delta(x)^{-1}$$

for \(f \in \text{fun}(\Pi_1(T, R), G)\), \(\gamma \in \Gamma\), \(\delta \in G\), and \(\alpha : x \to y\) a morphism of \(\Pi_1(T, R)\).

**Definition 4.1.** Suppose \(\tilde{S} : X \to Y\) is an h-graph cobordism equipped with an action of a discrete group \(\Gamma\) in which \(X\) and \(Y\) are kept pointwise fixed. We say that a family of h-graph cobordisms \(S/B\Gamma : X \to Y\) is obtained from \(\tilde{S}\) by the Borel construction if \(S/B\Gamma\) is equipped with a homotopy equivalence \(S \to \tilde{E}T \times_{\Gamma} \tilde{S}\) which is a map over \(B\Gamma\) and under \((X \sqcup Y) \times B\Gamma\). Here \(\tilde{E}T \times_{\Gamma} \tilde{S}\) is made into a space under \((X \sqcup Y) \times B\Gamma\) by the map \((X \sqcup Y) \times B\Gamma \approx \tilde{E}T \times_{\Gamma} (X \sqcup Y) \to \tilde{E}T \times_{\Gamma} S\).

**Remark 4.2.** The motivation for Definition 4.1 comes from the fact that it is not always clear that the map \((X \sqcup Y) \times B\Gamma \to \tilde{E}T \times_{\Gamma} \tilde{S}\) is a closed fibrewise cofibration, as would be required for \(\tilde{E}T \times_{\Gamma} \tilde{S}\) itself to give a family of h-graph cobordisms from \(X\) to \(Y\) over \(B\Gamma\). However, this problem can always be fixed simply by replacing \(\tilde{E}T \times_{\Gamma} \tilde{S}\) by the mapping cylinder \((\tilde{E}T \times_{\Gamma} \tilde{S})'\) of the map \((X \sqcup Y) \times B\Gamma \to \tilde{E}T \times_{\Gamma} \tilde{S}\). Then \((\tilde{E}T \times_{\Gamma} \tilde{S})'/B\Gamma : X \to Y\) is obtained from \(\tilde{S}\) by the Borel construction in the above sense. (The map \((\tilde{E}T \times_{\Gamma} \tilde{S})' \to B\Gamma\) is a fibration by [Cla81, Proposition 1.3].) Moreover, it is easy to show that any family \(S/B\Gamma : X \to Y\) obtained from \(\tilde{S}\) by the Borel construction is homotopy equivalent over \(B\Gamma\) and under \((X \sqcup Y) \times B\Gamma\) to \((\tilde{E}T \times_{\Gamma} \tilde{S})'\).

In the case of the families \(S_n = E\Sigma_n \times_{\Sigma_n} \tilde{S}_n\) over \(B\Sigma_n\), the map \((pt \times pt) \times B\Sigma_n \to E\Sigma_n \times_{\Sigma_n} \tilde{S}_n\) is a closed fibrewise cofibration over \(B\Sigma_n\), as can be seen using the criterion [MS06, Lemma 5.2.4]. Thus there is no need to replace \(E\Sigma_n \times_{\Sigma_n} \tilde{S}_n\) by a mapping cylinder in this case.

Using the isomorphisms (9), we obtain the following reinterpretation of Proposition 2.2 in the case where the family of h-graph cobordisms \(S/B\Gamma\) is obtained by the Borel construction from an h-graph cobordism equipped with a group action.

**Proposition 4.3.** Suppose a discrete group \(\Gamma\) acts on a positive h-graph cobordism \(\tilde{S} : X \to Y\) in such a way that \(X\) and \(Y\) are kept pointwise fixed, and suppose \(S/B\Gamma : X \to Y\) is obtained from \(\tilde{S}\) by the Borel construction. Choose basepoints \(P \to X\) and \(Q \to Y\)
for $X$ and $Y$, respectively. Then for any compact Lie group $G$, the operation $\Phi^G(S/B\Gamma)$ agrees with the composite

$$H_{s+\dim(G)}(\chi(S,X)(B\Gamma)) \otimes H_s(BG^X)$$

$$\overset{1 \otimes (\eta_{X,P}^G)^{-1}}{\approx} \overset{\times}{\approx} \overset{(a)}{\approx} \overset{!}{\approx} \overset{(b)}{\approx} \overset{(c)}{\approx} \overset{\eta_{Y,Q}^G}{\approx} \overset{\approx}{\approx}$$

$$H_s(\fun(\Pi_1(X,P),G)/\Gamma \times G^P)$$

$$H_s(\fun(\Pi_1(\hat{\mathcal{S}},P),G)/\Gamma \times G^P)$$

$$\overset{\approx}{\approx}$$

$$H_s(\fun(\Pi_1(Y,Q),G)/G^Q)$$

$$\overset{\approx}{\approx}$$

$$H_s(BG^Y)$$

where $\eta_{X,P}^G$ and $\eta_{Y,Q}^G$ are obtained from the maps $\eta_{X,P}^G$ and $\eta_{Y,Q}^G$ of section 2 and the homeomorphism (8); where the map (a) is induced by the homotopy equivalence

$$(\text{pt}/\Gamma) \times (\fun(\Pi_1(X,P),G)/G^P) \approx \fun(\Pi_1(X,P),G)/\Gamma \times G^P;$$

where the map labeled by $!$ is the umkehr map [HL13, section 7.2] associated with the map

$$\fun(\Pi_1(X,P),G)/\Gamma \times G^P \overset{\approx}{\approx} \fun(\Pi_1(\hat{\mathcal{S}},P),G)/\Gamma \times G^P$$

(10)

of fibrewise manifolds induced by the inclusion $X \hookrightarrow \hat{\mathcal{S}}$; where the map (b) is induced by the inverse of the homotopy equivalence

$$\fun(\Pi_1(\hat{\mathcal{S}},P \sqcup Q),G)/\Gamma \times G^{P\cup Q} \overset{\approx}{\approx} \fun(\Pi_1(\hat{\mathcal{S}},P),G)/\Gamma \times G^P$$

(11)

induced by the inclusion $P \hookrightarrow P \sqcup Q$; and where the map (c) is induced by the map $(Y,Q) \rightarrow (\hat{\mathcal{S}},P \sqcup Q)$ of h-graphs with basepoints and the projection $\Gamma \times G^{P\cup Q} \rightarrow G^Q$. □

**Remark 4.4.** For any $f \in \fun(\Pi_1(\hat{\mathcal{S}},P \sqcup Q),G)$, the projection map $\Gamma \times G^{P\cup Q} \rightarrow \Gamma \times G^P$ maps the stabilizer of $f$ in the $\Gamma \times G^{P\cup Q}$-action injectively into $\Gamma \times G^P$. One way to see this is as follows. Choose for each $q \in Q$ a morphism $\alpha_q$ in $\Pi_1(\hat{\mathcal{S}},P \sqcup Q)$ from an element of $P$ to $q$, and observe that a functor $\Pi_1(\hat{\mathcal{S}},P \sqcup Q) \rightarrow G$ is then precisely the same data as a functor $\Pi_1(\hat{\mathcal{S}},P) \rightarrow G$ and the assignment of an element of $G$ to each $\alpha_q$, $q \in Q$. Thus we get a homeomorphism

$$\fun(\Pi_1(\hat{\mathcal{S}},P \sqcup Q),G) \approx \fun(\Pi_1(\hat{\mathcal{S}},P),G) \times G^Q.$$  

(12)

Under this homeomorphism, the restriction of the $\Gamma \times G^{P\cup Q}$-action to $G^Q$ on the left hand side corresponds to multiplication action on the $G^Q$-factor on the right hand side, showing that the $G^Q$-action on $\fun(\Pi_1(\hat{\mathcal{S}},P \sqcup Q),G)$ is free.

Let us now specialize to the computation of the operations $\alpha_k^G$ of Definition 3.7. The family of h-graph cobordisms $(Bu)^*S_{2k}/BV_k$: $\text{pt} \rightarrow \text{pt}$ inducing the map $\alpha_k^G$ is isomorphic to the family $EV_k \times V_k \tilde{S}_{2k}/BV_k$: $\text{pt} \rightarrow \text{pt}$ obtained by the Borel construction from the h-graph cobordism $\tilde{S}_{2k}$: $\text{pt} \rightarrow \text{pt}$; the action of $V_k$ on $\tilde{S}_{2k}$ is the one obtained by restricting
the $\Sigma_{2k}$-action on $\hat{S}_{2k}$ along the inclusion $\iota: V_k \to \Sigma_{2k}$. Let $p$ and $q$ denote the incoming and outgoing points of $\hat{S}_{2k}$, respectively, and recall that we may interpret $\iota: V_k \to \Sigma_{2k}$ as the Cayley embedding associated to some bijection between $V_k$ and $\{1, 2, 3, \ldots, 2^k\}$. For $v \in V_k$, let $s_v$ be the path from $q$ to $p$ that traces the string of $\hat{S}_{2k}$ corresponding to $v$ under this bijection. Then the homotopy classes of the paths $s_v$ determine a basis for the finite free groupoid $\Pi_1(\hat{S}_{2k}, \{p, q\})$, and we obtain a homeomorphism
\[
\text{fun}(\Pi_1(\hat{S}_{2k}, \{p, q\}), G) \xrightarrow{\approx} G^{V_k}, \quad f \mapsto (f([s_v]))_{v \in V_k}
\]
under which the $V_k \times G^{(p,q)}$-action on $\text{fun}(\Pi_1(\hat{S}_{2k}, \{p, q\}), G)$ corresponds to the $V_k \times G^{(p,q)}$-action on $G^{V_k}$ given by
\[
(u, g_p, g_q) \cdot (g_v)_{v \in V_k} = (g_p g_{u+v} g_q^{-1})_{v \in V_k}.
\]
From the homeomorphism (12) of Remark 4.4, we can further deduce that (13) induces an homeomorphism
\[
\text{fun}(\Pi_1(\hat{S}_{2k}, \{p\}), G) \xrightarrow{\approx} G^{V_k}/\Delta G
\]
where $G^{V_k}/\Delta G$ denotes the space of left cosets of the diagonal subgroup $\Delta G$ of the product group $G^{V_k}$. Under this homeomorphism, the the $V_k \times G^{(p)}$-action on the space $\text{fun}(\Pi_1(\hat{S}_{2k}, \{p\}), G)$ corresponds to the $V_k \times G^{(p)}$-action on $G^{V_k}/\Delta G$ given by
\[
(u, g_p) \cdot (g_v)_{v \in V_k} \Delta G = (g_p g_{u+v})_{v \in V_k} \Delta G.
\]
Proposition 4.3 now implies that we may compute $\alpha_k^G$ by a push-pull construction in the diagram
\[
\begin{array}{ccc}
G^{V_k}/\Delta G//V_k \times G^{(p)} & \xrightarrow{\approx} & G^{V_k}/V_k \times G^{(p,q)} \\
\downarrow & & \downarrow \text{pt}//G^{(q)} \\
\text{pt}//V_k \times G^{(p)} & \xrightarrow{\approx} & \text{pt}//G^{(q)}
\end{array}
\]
where the arrows are induced by the evident quotient maps of spaces and projection homomorphisms of groups. More precisely, we have the following result.

**Proposition 4.5.** Let $G$ be a compact Lie group. Then the map $\alpha_k^G$ equals the composite
\[
H_*(BV_k) \otimes H_*(BG) \xrightarrow{\approx} H_*(BV_k \times BG) \xrightarrow{(a)} H_*(\text{pt}//V_k \times G^{(p)}) \xrightarrow{(b)} H_*(G^{V_k}/\Delta G//V_k \times G^{(p)}) \xrightarrow{(c)} H_*(\text{pt}//G^{(q)}) \xrightarrow{(d)} H_*(BG)
\]
where (a) and (d) are induced by the evident homeomorphisms
\[BV_k \times BG \approx \text{pt}//V_k \times G^{(p)} \quad \text{and} \quad \text{pt}//G^{(q)} \approx BG,
\]
respectively; where the map labeled by ! is the umkehr map [HL13, section 7.2] associated to the map ! of diagram (16) considered as a map of fibrewise manifolds over $\text{pt}//V_k \times G^{(p)}$; where the map (b) is induced by the homotopy inverse of the horizontal map in (16); and where the map (c) is induced by the right-hand diagonal map in (16). □
In the case of a finite group $G$, we may compute the composite of the maps $!$, (b) and (c) of Proposition 4.5 by decomposing $G^V_k$ into $V_k \times G^{(p,q)}$-orbits.

**Proposition 4.6.** Let $G$ be a finite group, and let $O \subset G^V_k$ be a set of orbit representatives for the $V_k \times G^{(p,q)}$-action on $G^V_k$. Then the composite of the maps $!$, (b) and (c) in Proposition 4.5 is equal to the sum over all $\tilde{g} \in O$ of the composite maps

$$H_*(pt/\Sigma V_k \times G^{(p)}) \xrightarrow{t} H_*(pt/((V_k \times G^{(p,q)}_{\tilde{g}}) \Sigma V_k \times G^{(q)})$$

where the first map is the transfer map associated with the map

$$pt/((V_k \times G^{(p,q)}_{\tilde{g}}) \Sigma V_k \times G^{(p)}) \rightarrow pt/\Sigma V_k \times G^{(p)}$$

induced by the inclusion of the stabilizer $(V_k \times G^{(p,q)}_{\tilde{g}})$ into $V_k \times G^{(p,q)}$ and the projection $V_k \times G^{(p,q)} \rightarrow V_k \times G^{(p)}$; and where the second map is induced by the map

$$pt/((V_k \times G^{(p,q)}_{\tilde{g}}) \Sigma V_k \times G^{(q)})$$

given by the composite of the inclusion $(V_k \times G^{(p,q)}_{\tilde{g}}) \hookrightarrow V_k \times G^{(p,q)}$ and the projection $V_k \times G^{(p,q)} \rightarrow G^{(q)}$.

**Remark 4.7.** From Remark 4.4 or a direct calculation using (14), we know that the composite map

$$(V_k \times G^{(p,q)}_{\tilde{g}}) \hookrightarrow V_k \times G^{(p,q)} \xrightarrow{pt/\Sigma V_k \times G^{(p)}} V_k \times G^{(p)}$$

is injective for every $\tilde{g} \in G^V_k$.

**Proof of Proposition 4.6.** When $G$ is finite, the map labeled by $!$ in Proposition 4.5 is just the transfer map; see [HL13, Lemma 8.6]. By the homotopy invariance of transfers, in this case the composite of the maps $!$ and (b) in Proposition 4.5 is simply the transfer map associated with the composite

$$G^V_k/\Sigma V_k \times G^{(p,q)} \rightarrow pt/\Sigma V_k \times G^{(p)}$$

of the left-hand diagonal map and the horizontal map of (16). Now observe that for any group $\Gamma$ and transitive $\Gamma$-set $X$ and element $x \in X$, the map

$$pt/\Gamma_x \rightarrow X/\Gamma$$

given by the map $pt \mapsto x$ and the inclusion $\Gamma_x \hookrightarrow \Gamma$ is a homotopy equivalence. Indeed, it suffices to consider the case where $X = \Gamma/H$ for a subgroup $H \leq \Gamma$ and $x = eH \in \Gamma/H$. It follows that we have a homotopy equivalence

$$\bigsqcup_{\tilde{g} \in O} pt/((V_k \times G^{(p,q)}_{\tilde{g}}) \Sigma V_k \times G^{(p,q)}) \xrightarrow{\sim} G^V_k/\Sigma V_k \times G^{(p,q)}$$

which on the summand corresponding to $\tilde{g} \in O$ is given by the map $pt \mapsto \tilde{g}$ and the inclusion $(V_k \times G^{(p,q)}_{\tilde{g}}) \hookrightarrow V_k \times G^{(p,q)}$. The claim now follows from standard properties of transfer maps. \qed

We conclude the section with the following simple observation which dramatically reduces the number of orbits we need to take into account in our applications of Proposition 4.6.

**Lemma 4.8.** Let $\Gamma$ be a group and let $H \leq \Gamma$ be an even-index subgroup such that the map $i_*: H_*(pt/\Gamma) \rightarrow H_*(pt/\Gamma)$ induced by the inclusion $i: H \hookrightarrow \Gamma$ is injective. Then the transfer map $i^*: H_*(pt/\Gamma) \rightarrow H_*(pt/\Gamma)$ is zero.

**Proof.** By standard properties of the transfer map, the composite $i_*i^*$ is multiplication by the index of $H$ in $\Gamma$, and hence zero as we are working with $\mathbb{F}_2$ coefficients. The claim now follows from the injectivity of $i_*$. \qed
5. Computations for elementary abelian 2-groups and dihedral groups

The purpose of this section is to compute the maps $\alpha^G_k$ when $G$ is an elementary abelian 2-group $E$ or a dihedral group $D_{4n+2}$.

We allow the case $n = 0$, in which case $D_{4n+2}$ reduces to a copy of $\mathbb{Z}/2$. Recall that the quotient homomorphism $D_{4n+2} \to D_{4n+2}/\langle r \rangle \approx \mathbb{Z}/2$ induces an isomorphism on mod 2 homology, as can be seen for example by considering the Hochschild–Serre spectral sequence of the short exact sequence

$$1 \to \langle r \rangle \to D_{4n+2} \to \mathbb{Z}/2 \to 1.$$ 

It follows that the inclusion $\langle s \rangle \to D_{4n+2}$ also induces an isomorphism on mod 2 homology, since the composite $\mathbb{Z}/2 \approx \langle s \rangle \to D_{4n+2}$ is a right inverse to the quotient map $D_{4n+2} \to \mathbb{Z}/2$. Our aim is to prove the following two theorems, the second one of which reduces the computation of $\alpha^D_{4n+2}k$ to the first.

**Theorem 5.1.** Let $E$ be an elementary abelian 2-group. Then the map

$$\alpha^E_k : H_*(BV_k) \otimes H_*(BE) \to H_*(BE)$$

is given by

$$a \otimes b \mapsto \sum_K K_*(a)b$$

where the sum is over all linear maps $K : V_k \to E$.

**Theorem 5.2.** The diagram

$$
\begin{array}{ccc}
H_*(BV_k) \otimes H_*(BD_{4n+2}) & \xrightarrow{\alpha^D_{4n+2}k} & H_*(BD_{4n+2}) \\
\approx & & \approx \\
H_*(BV_k) \otimes H_*(B\langle s \rangle) & \xrightarrow{\alpha^\langle s \rangle_k} & H_*(B\langle s \rangle)
\end{array}
$$

where the vertical maps are induced by the inclusion $\langle s \rangle \to D_{4n+2}$ commutes for all $k \geq 1$.

To prove Theorems 5.1 and 5.2, we will use Propositions 4.5 and 4.6, making use of Lemma 4.8 to limit the number of orbits we need to take into account when applying Proposition 4.6. When computing $\alpha^D_{4n+2}k$, Lemma 4.8 together with the following lemma show that we may restrict attention to the orbits of those elements $\tilde{g} \in D_{4n+2}$ with the property that the image of the stabilizer $(V_k \times D_{4n+2})^{[\nu,q]} \tilde{g}$ in $V_k \times D_{4n+2}^{[\nu]}$ is an odd-index subgroup.

**Lemma 5.3.** Let $H$ be a subgroup of $V_k \times D_{4n+2}$. Then the inclusion of $H$ into $V_k \times D_{4n+2}$ induces an injection on homology.

**Proof.** Let $W = H \cap V_k$, and let $Q = H/W$. We then have the commutative diagram

$$
\begin{array}{ccc}
1 & \to & W & \to & H & \to & Q & \to & 1 \\
\downarrow & & \downarrow i' & & \downarrow i & & \downarrow i'' & & \downarrow 1 \\
1 & \to & V_k & \to & V_k \times D_{4n+2} & \to & D_{4n+2} & \to & 1
\end{array}
$$

where the rows are exact and $i$ and $i'$ are the inclusions and $i''$ is the map induced by $i$. Notice that $i''$ is also injective. There results a map between the Hochschild–Serre
spectral sequences associated with the two rows. On the $E^2$-page, this map is given by the tensor product

$$i''_* \otimes i'_*: H_*(BQ) \otimes H_*(BW) \to H_*(BD^{4n+2}) \otimes H_*(BV_k).$$

Recalling every subgroup of $D_{4n+2}$ is cyclic of the form $\langle r^d \rangle$ for some $d|(2n+1)$ or dihedral of the form $\langle r^d, r^s \rangle$ for some $d|(2n+1)$, $0 \leq i < 2n+1$, and observing that in each case the map induced by the inclusion of the subgroup into $D_{4n+2}$ is injective on homology, we see that the map $i''_*: H_*(BQ) \to H_*(BD^{4n+2})$ is injective. Since the map $i'$ admits a left inverse, the map $i'_*: H_*(BW) \to H_*(BV_k)$ is injective as well. Thus the map between the two spectral sequences is injective on the $E^2$-page. Since the spectral sequence of the second row collapses on the $E^2$-page, it follows that the spectral sequence of the first row also does. Thus the map between spectral sequences is also injective on the $E^\infty$-page, which suffices to imply the claim.

The next two lemmas provide an analysis of the orbits containing an element $\bar{g} \in D_{4n+2}$ with the aforementioned property as well as the associated stabilizers.

**Lemma 5.4.** Suppose $K: V_k \to \langle s \rangle$ is a homomorphism. Then the stabilizer of the element $(K(v))_{v \in V_k} \in D_{4n+2}^V$ in the $V_k \times D_{4n+2}$ action on $D_{4n+2}^V$ is

$$\{(u, g_p, g_q) \in V_k \times D_{4n+2}^{p,q} \mid g_q = g_p\}$$

if $K$ is the trivial homomorphism and

$$\{(u, g_p, g_q) \in V_k \times D_{4n+2}^{p,q} \mid g_p \in \langle s \rangle, g_q = g_p K(u)\}$$

otherwise.

**Proof.** In both cases, it is trivial to check that the subgroup given is contained in the stabilizer. To prove the reverse containment, suppose $(u, g_p, g_q)$ is in the stabilizer of $(K(v))_{v \in V_k}$. Then we have

$$g_p K(u + v) g_q^{-1} = K(v)$$

(17) for all $v$. Setting $v = 0$, we see that $g_q = g_p K(u)$, which in the case of trivial $K$ reduces to the equation $g_q = g_q$. Substituting $g_q = g_p K(u)$ back to (17), we see that we must in addition have $g_p K(v) g_q^{-1} = K(v)$ for all $v$. If $K$ is trivial, this condition is satisfied for all $g_p \in D_{4n+2}$. On the other hand, if $K$ is non-trivial, that is, $s$ is in the image of $K$, then $g_p$ must belong to the centralizer of $s$ in $D_{4n+2}$, which is $\langle s \rangle$. The claim follows.

**Lemma 5.5.** The map

$$(K: V_k \to \langle s \rangle) \mapsto \text{the orbit of } (K(v))_{v \in V_k} \in D_{4n+2}^V$$

(18)

gives a bijection from the set of homomorphisms $K: V_k \to \langle s \rangle$ onto the set of orbits of the $(V_k \times D_{4n+2}^{p,q})$-action on $D_{4n+2}^V$ having the property that for some (and hence every) element $\bar{g}$ in the orbit, the image of the stabilizer $(V_k \times D_{4n+2}^{p,q})_{\bar{g}}$ under the projection $V_k \times D_{4n+2}^{p,q} \to V_k \times D_{4n+2}^{p,q}$ is an odd-index subgroup of $V_k \times D_{4n+2}^{p,q}$.

**Proof.** Lemma 5.4 implies that the map (18) takes values in the claimed subset of orbits. To see that the map is injective, suppose $K_1, K_2: V_k \to \langle s \rangle$ are two homomorphisms such that $(u, g_p, g_q) \cdot (K_1(v))_{v \in V_k} = (K_2(v))_{v \in V_k}$ for some $(u, g_p, g_q) \in V_k \times D_{4n+2}^{p,q}$. Then

$$g_p K_1(u + v) g_q^{-1} = K_2(v)$$

(19)

for all $v \in V_k$. Taking $v = 0$ gives $g_q = g_p K_1(u)$, and substituting this back to (19), we see that $K_2(v) = g_p K_1(v) g_p^{-1}$ for all $v \in V_k$. In particular, $K_1(v)$ and $K_2(v)$ are non-trivial for precisely the same $v \in V_k$. Since $K_1$ and $K_2$ both take values in the group $\langle s \rangle$ which has only one non-trivial element, it follows that $K_1 = K_2$. 

It remains to show that the map (18) is surjective. Suppose \( \tilde{g} = (g_v)_{v \in V_k} \in D_{4n+2}^{(p)} \) is an element such that the image of the stabilizer of \( \tilde{g} \) is an odd-index subgroup \( H \) of \( V_k \times D_{4n+2}^{(p)} \). By Remark 4.7, the stabilizer of \( \tilde{g} \) is of the form
\[
\{(u, g_p, \varphi(u, g_p)) \in V_k \times D_{4n+2}^{(p,q)} \mid (u, g_p) \in H\}
\]
for some homomorphism \( \varphi: H \to D_{4n+2}^{(q)} \). Since \( H \) has odd index in \( V_k \times D_{4n+2}^{(p)} \), we must have \( V_k \subset H \). Thus for every \( u \in V_k \), the element \((u, e, \varphi(u, e))\) is in the stabilizer of \( \tilde{g} \), so that \( g_{u+\varphi(u, e)^{-1}} = g_v \) for all \( v \in V_k \). Taking \( v = 0 \), we see that
\[
g_u = g_0 \varphi(u, e)
\]
for all \( u \in V_k \). The image of \( V_k \) under the homomorphism \( \varphi \) is a 2-subgroup of \( D_{4n+2} \), and hence conjugate by some \( \gamma \in D_{4n+2} \) to a subgroup of \( \langle s \rangle \). Now
\[
(0, \gamma g_0^{-1}, \gamma) \cdot \tilde{g} = (\gamma g_0^{-1} g_0 \gamma^{-1})_{v \in V_k} = (\gamma g_0^{-1} g_0 \varphi(v, e) \gamma^{-1})_{v \in V_k} = (\gamma \varphi(v, e) \gamma^{-1})_{v \in V_k}
\]
is a representative of the desired form for the orbit of \( \tilde{g} \).

We are now ready to prove Theorems 5.1 and 5.2.

**Proof of Theorems 5.1 and 5.2.** Our strategy is to prove Theorem 5.2 simultaneously with the case \( E = \mathbb{Z}/2 \) of Theorem 5.1, and then use Theorem 2.1 to complete the proof of Theorem 5.1.

By Proposition 4.5 together with Proposition 4.6 and Lemmas 4.8 and 5.3 and 5.5, the map \( \alpha_{D_{4n+2}} \) equals the composite
\[
H_s(BV_k) \otimes H_s(BD_{4n+2}) \xrightarrow{\times} H_s(BV_k \times BD_{4n+2}) \xrightarrow{(a)} H_s(pt//V_k \times D_{4n+2}^{(p)}) \xrightarrow{(b)} H_s(pt//D_{4n+2}^{(q)}) \xrightarrow{(c)} H_s(BD_{4n+2})
\]
where (a) and (c) are induced by the evident homeomorphisms and where (b) is the sum of the composite maps
\[
H_s(pt//V_k \times D_{4n+2}^{(p)}) \xrightarrow{i_K} H_s(pt//H_K) \xrightarrow{(p_K)_*} H_s(pt//D_{4n+2}^{(q)}) \tag{20}
\]
as \( K \) runs through all linear maps \( V_k \to \langle s \rangle \). Here \( H_K \leq V_k \times D_{4n+2}^{(p,q)} \) is the stabilizer of \( (K(v))_{v \in V_k} \in D_{4n+2}^{(q)} \) computed in Lemma 5.4, the first map is the transfer map associated to the injective homomorphism
\[
i_K: H_K \hookrightarrow V_k \times D_{4n+2}^{(p)}, \quad (v, \varepsilon, \varepsilon_q) \mapsto (v, \varepsilon_p),
\]
and the second map is induced by the map
\[
p_K: H_K \hookrightarrow D_{4n+2}^{(q)}, \quad (v, \varepsilon, \varepsilon_q) \mapsto \varepsilon_q.
\]
For every \( K: V_k \to \langle s \rangle \), the maps (20) fit into a commutative diagram
\[
\begin{array}{ccc}
H_s(pt//V_k \times \langle s \rangle) & \xrightarrow{(1 \times j)_*} & H_s(pt//H_K) \xrightarrow{(p_K)_*} H_s(pt//D_{4n+2}^{(q)}) \\
\downarrow_{(1 \times j)_*} & & \downarrow_{(p_K)_*} \\
H_s(pt//V_k \times D_{4n+2}^{(p)}) & \xrightarrow{i_K} & H_s(pt//H_K) \xrightarrow{(p_K)_*} H_s(pt//D_{4n+2}^{(q)}) \\
\end{array} \tag{21}
\]

where the vertical maps are induced by the inclusion \( j : \langle s \rangle \hookrightarrow D_{4n+2} \), the diagonal arrow is induced by the map
\[
d : V_K \times \langle s \rangle \longrightarrow H_K, \quad (v, g) \mapsto (v, gK(v)),
\]
and the bottom horizontal arrow is induced by the map
\[
q_K : V_K \times \langle s \rangle \longrightarrow \langle s \rangle, \quad (v, g) \mapsto gK(v).
\]
Indeed, the trapezoid in (21) commutes since \( p_Kd = jq_K \) on the level of homomorphisms. Furthermore, the maps \( i_K \) and
\[
(i_K)_* : H_* (\text{pt} / \text{pt} H_K) \longrightarrow H_* (\text{pt} / V_k \times D_{4n+2}^{[p]})
\]
are inverse isomorphisms, as follows from the observation that the composite map \( (i_K)_* \circ i_K' \) is the identity, being equal to multiplication by the index of \( i_K H_K \) in \( V_k \times D_{4n+2}^{[p]} \) and Lemma 5.3, which implies that the map \( (i_K)_* \) is injective. Thus the commutativity of the triangle in (21) follows from the identity \( i_K \circ d = 1 \times j \). We conclude that the map \( \alpha_k^{D_{4n+2}} \) fits into the commutative diagram
\[
\begin{array}{ccc}
H_* (BV_k) \otimes H_* (BD_{4n+2}) & \xrightarrow{\alpha_k^{D_{4n+2}}} & H_* (BD_{4n+2}) \\
1 \otimes j_* \approx & & \approx j_* \\
H_* (BV_k) \otimes H_* (B \langle s \rangle) & \longrightarrow & H_* (B \langle s \rangle)
\end{array}
\]
where the bottom horizontal map is given by \( a \otimes b \mapsto \sum_K K_* (a) b \), where the sum is over all homomorphisms \( K : V_k \rightarrow \langle s \rangle \).

Specializing to the case \( n = 0 \), we have obtained a proof of Theorem 5.1 in the special case \( E = \mathbb{Z}/2 \). In particular, the bottom horizontal arrow in (22) is equal to \( \alpha_k^{(s)} \), proving Theorem 5.2. Finally, Theorem 5.1 in the general case follows at once from the special case \( E = \mathbb{Z}/2 \) and Theorem 2.1.

In practice, it is useful to have a more explicit formula for the maps \( \alpha_k^E \) than the one given in Theorem 5.1. Let \( x_1, \ldots, x_k \) be a basis for \( V_k \) and let \( x \) be the generator of \( \mathbb{Z}/2 \). Then we have
\[
\alpha_k^E (\langle x_1^{[n_1]} \cdots x_k^{[n_k]} \rangle \otimes b) = \sum_L L_* (x_1^{[n_1]} \cdots x_k^{[n_k]}) b
\]
\[
= \sum_K K_* (x_1^{[n_1]} \cdots x_k^{[n_k]}) b
\]
\[
= \sum_K K_* (i_1)_* (x_1^{[n_1]}) \cdots K_* (i_k)_* (x_k^{[n_k]}) b
\]
\[
= \left( \sum_{L : \mathbb{Z}/2 \rightarrow E} L_* (x_1^{[n_1]}) \right) \cdots \left( \sum_{L : \mathbb{Z}/2 \rightarrow E} L_* (x_k^{[n_k]}) \right) b
\]
\[
= \left\{ \begin{array}{ll}
(x_1^{[n_1]} \cdots x_k^{[n_k]}) b & \text{if } n_1, \ldots, n_k > 0 \\
0 & \text{otherwise}
\end{array} \right.
\]
Here \( i_j : \mathbb{Z}/2 \rightarrow V_k \) is the map sending \( x \) to \( x_j \), and the indices \( L \) run over all homomorphisms \( \mathbb{Z}/2 \rightarrow E \). In particular, in the case \( E = \mathbb{Z}/2 \) we get the following formula.
\[
\alpha_k^{\mathbb{Z}/2} (\langle x_1^{[n_1]} \cdots x_k^{[n_k]} \rangle \otimes b) = \left\{ \begin{array}{ll}
x_1^{[n_1]} \cdots x_k^{[n_k]} b & \text{if } n_1, \ldots, n_k > 0 \\
0 & \text{otherwise}
\end{array} \right.
\]
Recalling that $x^{[n]} x^{[m]} = (n+m)^{x^{[n+m]}}$ in $H_* B(\mathbb{Z}/2)$ and that $(n+m)^{x^{[n+m]}}$ is odd if the binary expansions of $n$ and $m$ have no 1’s in common and even otherwise, we obtain the following result.

**Proposition 5.6.** The operation

$$(\alpha_k^{\mathbb{Z}/2})^I (x_1^{[n_1]} \cdots x_k^{[n_k]}) : H_* B(\mathbb{Z}/2) \longrightarrow H_{*+n_1+\cdots+n_k} B(\mathbb{Z}/2)$$

is non-trivial precisely when the numbers $n_1, \ldots, n_k$ are positive and no two of them have a 1 in common in their binary expansions. Moreover, in this case the operation amounts to multiplication by $x^{[n_1+\cdots+n_k]}$. □

Let us now consider the case of a general elementary abelian 2-group $E$ of dimension $l \geq 1$. Choose a basis $t_1, \ldots, t_l$ for $E$. Then

$$\sum_{\varepsilon \in E} \varepsilon^{[n]} = \sum_{t_1, \ldots, t_l \geq 1} t_1^{[i_1]} \cdots t_l^{[i_l]}$$

for every $n > 0$ as follows by induction from the equation

$$\sum_{\varepsilon \in E} \varepsilon^{[n]} = \sum_{\varepsilon \in \langle t_1, \ldots, t_{l-1} \rangle} (\varepsilon^{[n]} + (\varepsilon + t_l)^{[n]}) = \sum_{\varepsilon \in \langle t_1, \ldots, t_{l-1} \rangle} \sum_{i=1}^{n-1} \varepsilon^{[n-i]} t_i^{[i]}$$

where $\langle t_1, \ldots, t_{l-1} \rangle$ denotes the span of $t_1, \ldots, t_l$ and we have assumed that $l \geq 2$. Substituting (24) into (23), we see that for $n_1, \ldots, n_k > 0$, we have

$$\alpha_k^E ((x_1^{[n_1]} \cdots x_k^{[n_k]}) \otimes b)$$

$$= \left( t_1^{[i_1]} \cdots t_l^{[i_l]} \right) \left( \sum_{i_{k1}, \ldots, i_{kl} \geq 1 \atop i_{k1} + \cdots + i_{kl} = n_k} t_1^{[i_1]} \cdots t_l^{[i_l]} \right) b$$

$$= \sum_I \left( t_1^{[i_1]} \cdots t_l^{[i_l]} \right) \left( t_1^{[i_1]} \cdots t_l^{[i_l]} \right) b$$

$$= \sum_{e_1, \ldots, e_l} \alpha(n_1, \ldots, n_k; e_1, \ldots, e_l) t_1^{[e_1]} \cdots t_l^{[e_l]} b.$$

Here the index $I$ runs over all $(k \times l)$-matrices $I = (i_{cd})$ of positive integers with row sums $n_1, \ldots, n_k$, and $A(n_1, \ldots, n_k; e_1, \ldots, e_l)$ denotes the number of such matrices $I$ having column sums $e_1, \ldots, e_l$ and satisfying the additional property that no two numbers in the same column have a common 1 in their binary expansions. It is useful to think of $A(n_1, \ldots, n_k; e_1, \ldots, e_l)$ as counting the number of ways to distribute the powers of 2 occurring in the binary expansions of $e_1, \ldots, e_l$ among the rows of a $(k \times l)$-matrix in such a way that the result has the prescribed row sums and every entry of the matrix is positive.

For the operation

$$(\alpha_k^E)^I (x_1^{[n_1]} \cdots x_k^{[n_k]}) : H_* (BE) \longrightarrow H_{*+n_1+\cdots+n_k} (BE)$$

to be non-trivial, we must clearly have $n_i \geq l$ for all $i$. Moreover, all the $n_i$’s have to be different, since the square of any positive-degree element in $H_* (BE)$ is trivial. The problem of determining precisely when the operation $(\alpha_k^E)^I (x_1^{[n_1]} \cdots x_k^{[n_k]})$ is non-zero appears complicated, and we will not attempt a complete solution here. Instead, we content ourselves with observing that it is easy to find examples where the operation is non-zero. Coupled with the observation that $(\alpha_1)^I (x_1^{[n]})$ is non-trivial for all $n \geq l$, the following result suffices to construct infinitely many examples of non-trivial operations for every $k \geq 1$. 


Proposition 5.7. Suppose \((\alpha^E_k)^s (x_1^{[n_1]} \cdots x_k^{[n_k]})\) is non-trivial and let \(n_{k+1}\) be a number of the form \(n_{k+1} = 2^r\) where \(r \geq l\) and \(2^s > n_1 + \cdots + n_k\). Then the operation \((\alpha^E_{k+1})^s (x_1^{[n_1]} \cdots x_k^{[n_k+1]})\) is non-trivial.

Proof. Since \((\alpha^E_k)^s (x_1^{[n_1]} \cdots x_k^{[n_k]})\) is non-trivial, we can find exponents \(e_1, \ldots, e_l\) such that \(A(n_1, \ldots, n_k; e_1, \ldots, e_l)\) is odd. By the assumption on \(n_{k+1}\), we can write it as a sum \(n_{k+1} = r_1 + \cdots + r_j\) where each \(r_j\) is some positive multiple of \(2^s\). Let \(I\) be a matrix of the type counted by \(A(n_1, \ldots, n_{k+1}; e_1 + r_1, \ldots, e_l + r_1)\). Since \(r_j\) is divisible by \(2^s\) and

\[2^s > n_1 + \cdots + n_k = e_1 + \cdots + e_l \geq e_j\]

for all \(j\), we see that for each \(j\), the set of powers of \(2\) occurring in the binary expansion of \(e_j + r_j\) is the disjoint union of those occurring in the binary expansions of \(e_j\) and \(r_j\). Since the powers of \(2\) coming from \(r_j\) are all greater or equal to \(2^s\) and \(2^s > n_1 + \cdots + n_k \geq n_i\) for \(1 \leq i \leq k\), the powers of \(2\) coming from the \(r_j\)'s must all have been assigned to the last row of \(I\), for otherwise the row sum of some other row of \(I\) would be too large. On the other hand, no powers of \(2\) occurring in the binary expansions of the \(e_j\)'s may have been assigned to the last row, for then the row sum of the last row of \(I\) would exceed \(n_{k+1} = r_1 + \cdots + r_j\). Thus forgetting the last row provides a bijection from the set of matrices counted by \(A(n_1, \ldots, n_{k+1}; e_1 + r_1, \ldots, e_l + r_1)\) to the set of matrices counted by \(A(n_1, \ldots, n_k; e_1, \ldots, e_l)\). Therefore \(A(n_1, \ldots, n_{k+1}; e_1 + r_1, \ldots, e_l + r_1) = A(n_1, \ldots, n_k; e_1, \ldots, e_l)\), and the claim follows.

We also have the following result, which is an immediate corollary of the observation that \(A(n_1, \ldots, n_k; e_1, \ldots, e_l) = A(2n_1, \ldots, 2n_k; 2e_1, \ldots, 2e_l)\) for all \(n_1, \ldots, n_k\) and \(e_1, \ldots, e_l\).

Proposition 5.8. If the operation \((\alpha^E_k)^s (x_1^{[n_1]} \cdots x_k^{[n_k]})\) is non-trivial, so is the operation \((\alpha^E_k)^t (x_1^{[2n_1]} \cdots x_k^{[2n_k]})\).

\[\square\]

6. Computations for tori

The purpose of this section is to compute the operations \(\alpha_1^G\) and \(\alpha_2^G\) when \(G\) is a torus \(T^1 = (S^1)^l\). Recall that \(H_*(BT^1) \approx H_*(CP^\infty)\) is a ring, the divided power algebra on a single generator of degree 2. The inclusion \(\beta\) of \(V_1\) into \(T^1\) as \(\pm 1\) induces a ring homomorphism \(\beta_* : H_*BV_1 \to H_*BT^1\) which is zero in odd degrees and an isomorphism in even degrees, as can be seen for example by considering the Serre spectral sequence of the fibre sequence \(T^1 \to BV_1 \to BT^1\). Write \(x\) for the generator of \(V_1\), and let \(x_1\) and \(x_2\) form a basis of \(V_2\). We will prove the following results.

Theorem 6.1. The map

\[\alpha_1^{T^1} : H_{s-1}(BV_1) \otimes H_*(BT^1) \longrightarrow H_*(BT^1)\]

is given by

\[x^{[n]} \otimes b \mapsto \beta_*(x^{[n+1]})b.\]

Theorem 6.2. The map

\[\alpha_2^{T^1} : H_{s-3}(BV_2) \otimes H_*(BT^1) \longrightarrow H_*(BT^1)\]

is given by

\[x_1^{[n_1]} x_2^{[n_2]} \otimes b \mapsto \left( 1 + \binom{n_1 + n_2 + 2}{n_1 + 1} \right) \beta_*(x^{[n_1+n_2+3]})b.\]
Remembering that the maps $H_*(BV_1) \to H_*(BV_1 \times BV_1)$ and $H_*(BV_2) \to H_*(BV_2 \times BV_2)$ induced by the diagonal maps are given by
\[
x^{[n]} \mapsto \sum_{i=0}^n (x^{[i]} \times x^{[n-i]}) \quad \text{and} \quad x_1^{[n]} x_2^{[m]} \mapsto \sum_{i=0}^n \sum_{j=0}^m (x_1^{[i]} x_2^{[j]} \times x_1^{[n-i]} x_2^{[m-j]}),
\]
respectively, Theorem 2.1 allows us to read off the following formulas for $\alpha_1^\pi$ and $\alpha_2^\pi$ from Theorems 6.1 and 6.2. Observe that $H_*(BT^l) \approx H_*(BT^1)^{\otimes l}$ is also a ring, the divided power algebra on $l$ generators of degree 2.

**Theorem 6.3.** The map
\[
\alpha_1^\pi : H_{*-l}(BV_1) \otimes H_*(BT^l) \to H_*(BT^l)
\]
is given by
\[
x^{[n]} \otimes b \mapsto \sum_{i_1, \ldots, i_l \geq 0 \atop i_1 + \cdots + i_l = n} \left( \beta_*(x^{[i_1+1]}) \times \cdots \times \beta_*(x^{[i_l+1]}) \right) b
\]
while the map
\[
\alpha_2^\pi : H_{*-3l}(BV_2) \otimes H_*(BT^l) \to H_*(BT^l)
\]
is given by
\[
x_1^{[n_1]} x_2^{[n_2]} \otimes b \mapsto \sum_l \prod_{a=1}^l \left( 1 + \left( \frac{i_a + j_a + 2}{i_a + 1} \right) \right) \left( \beta_*(x^{[i_1+j_1+3]} \times \cdots \times \beta_*(x^{[i_l+j_l+3]}) \right) b
\]
where the sum is over all non-negative integers $i_1, \ldots, i_l$ and $j_1, \ldots, j_l$ such that $i_1 + \cdots + i_l = n_1$ and $j_1 + \cdots + j_l = n_2$.

In particular, the operation $(\alpha_1^\pi)^2(x^{[n]})$ is non-trivial precisely when $n = 2k - l$ for some $k \geq l$.

**Proof of Theorem 6.1.** Let us interpret $V_1 = \{ \pm 1 \}$ and write the group operation multiplicatively. From Proposition 4.5 we deduce that we may compute $\alpha_1^\pi$ by a push-pull construction in the diagram
\[
\begin{array}{ccc}
T^1 \slash V_1 \times (T^1)^{(p)} & \xrightarrow{\eta} & (T^1)^{(p)} \slash V_1 \times (T^1)^{(p,q)} \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
pt \slash V_1 \times (T^1)^{(p)} & & pt \slash (T^1)^{(p,q)}
\end{array}
\]
(25)
where the $V_1 \times (T^1)^{(p)}$-action on $T^1$ is given by
\[
(\varepsilon, z_p) \cdot z = z^\varepsilon,
\]
where the $V_1 \times (T^1)^{(p,q)}$-action on $(T^1)^{V_1}$ is given by
\[
(\varepsilon, z_p, z_q) \cdot (z_{-1}, z_1) = (z_p z_q^{-1} z_{-\varepsilon}, z_p z_q^{-1} z_\varepsilon),
\]
where $\eta$ is induced by the map
\[
(T^1)^{V_1} \to T^1, \quad (z_{-1}, z_1) \mapsto z_{-1} z_1^{-1}
\]
and the projection $V_1 \times (T^1)^{(p,q)} \to V_1 \times (T^1)^{(p)}$, and where $\pi_1$ and $\pi_2$ are induced by the evident projections of spaces and groups. Diagram (25) is simply diagram (16) with $G = T^1$, $k = 1$ and the space $(T^1)^{V_1} \slash \Delta T^1$ replaced with $T^1$ using the homeomorphism
\[
(T^1)^{V_1} / \Delta T^1 \xrightarrow{\approx} T^1, \quad (z_{-1}, z_1) \Delta T^1 \mapsto z_{-1} z_1^{-1}.
\]
Observe that the source of $\pi_1$ splits as a product
\[ T^1/V_1 \times (T^1)^{[p]} \approx (T^1/V_1) \times (pt/\!(T^1)^{[p]}) \]
where the $V_1$-action on $T^1$ is given by $\varepsilon \cdot z = z^\varepsilon$, while the target of $\pi_1$ splits as a product
\[ pt/V_1 \times (T^1)^{[p]} \approx (pt/V_1) \times (pt/\!(T^1)^{[p]}). \]
Under these homeomorphisms, the map $\pi_1$ corresponds to the map
\[ \pi'_1 \times 1: (T^1/V_1) \times (pt/\!(T^1)^{[p]}) \longrightarrow (pt/V_1) \times (pt/\!(T^1)^{[p]}), \]
where $\pi'_1$ is induced by the projection $T^1 \rightarrow pt$. By the compatibility of umkehr maps with direct products, under the isomorphisms induced by the above homeomorphisms and homology cross products, the umkehr map $\pi'_1$ corresponds to the map
\[ (\pi'_1)^1 \otimes 1: H_*(pt/V_1) \otimes H_*(pt/\!(T^1)^{[p]}) \longrightarrow H_{*+1}(T^1/V_1) \otimes H_*(pt/\!(T^1)^{[p]}). \]
By [HL13, Lemma 8.4], we may compute the map $(\pi'_1)^1$ using the Serre spectral sequence for the fibration $\pi'_1: T^1/V_1 \rightarrow pt/V_1$. More precisely, $(\pi'_1)^1$ is given by the composite
\[ H_n(pt/V_1) \xrightarrow{\pi'_1} H_n(pt/V_1; H_1T^1) = E^2_{n,1} \longrightarrow E^\infty_{n,1} \longrightarrow H_{n+1}(T^1/V_1) \] (26)
where $E^2$ and $E^\infty$ refer to pages in the Serre spectral sequence of $\pi'_1$, the first map is induced by the fundamental class of $T^1$, and the last two maps arise from the fact that we are working with the top row of the spectral sequence.

The assignments $pt \mapsto \pm 1 \in T^1$ define sections $s_{\pm 1}: pt/V_1 \rightarrow T^1/V_1$ of $\pi'_1$. Since the spectral sequence of $\pi'_1$ has only two rows, the existence of the sections $s_{\pm 1}$ implies that the spectral sequence collapses on the $E^2$-page. Thus the short exact sequence
\[ 0 \longrightarrow E^\infty_{n,1} \longrightarrow H_{n+1}(T^1/V_1) \longrightarrow E^\infty_{n+1,0} \longrightarrow 0 \]
together with the description (26) for $(\pi'_1)^1$ give a short exact sequence
\[ 0 \longrightarrow H_n(pt/V_1) \xrightarrow{(\pi'_1)^1} H_{n+1}(T^1/V_1) \xrightarrow{\pi'_1} H_{n+1}(pt/V_1) \longrightarrow 0. \] (27)

Let $U_{-1} = T^1 \setminus \{1\}$ and $U_1 = T^1 \setminus \{-1\}$, and observe that $V_1$ acts freely on the intersection $U_{-1} \cap U_1$ and that $U_1$ and $U_{-1}$ admit $V_1$-equivariant deformation retractions onto $\{1\} \subset T^1$ and $\{-1\} \subset T^1$, respectively. From the Mayer–Vietoris sequence for $U_{-1}/V_1$ and $U_1/V_1$ we now conclude that the map
\[ (s_{-1})_* + (s_1)_*: H_n(pt/V_1) \oplus H_n(pt/V_1) \longrightarrow H_n(T^1/V_1) \]
is an isomorphism for $n > 0$. Since the maps $s_{\pm 1}$ are sections of $\pi'$, from the short exact sequence (27) we can deduce that under this isomorphism, the map $(\pi'_1)^1$ corresponds to the map sending $x^{[n]} \in H_n(pt/V_1)$ to the diagonal element $(x^{[n+1]}, x^{[n+1]}) \in H_{n+1}(pt/V_1) \oplus H_{n+1}(pt/V_1)$.

Observe now that the two maps
\[ \tilde{s}_1, \tilde{s}_{-1}: pt/V_1 \times (T^1)^{[p]} \longrightarrow (T^1)^{V_1}/V_1 \times (T^1)^{[p,q]} \]
defined by the assignments $pt \mapsto (1, 1), (1, -1) \in (T^1)^{V_1}$ and the homomorphisms
\[ V_1 \times (T^1)^{[p]} \longrightarrow V_1 \times (T^1)^{[p,q]}, \quad (\varepsilon, z_p) \longmapsto (\varepsilon, z_p, z_p), (\varepsilon, z_p, \varepsilon z_p), \]
provide lifts of the respective maps
\[ s_1 \times 1, s_{-1} \times 1: (pt/V_1) \times (pt/\!(T^1)^{[p]}) \longrightarrow (T^1/V_1) \times (pt/\!(T^1)^{[p]}). \]
is the map induced by the projection \( pr: T \times V \to T \). From now on, for the sake of brevity, let us write \( \tilde{s}_{\pm 1} \) and understand the umkehr map (\( \tilde{s}_{\pm 1} \)) given by

\[ x[n] \otimes b \mapsto \text{pr}(x^{n+1} \times b) \quad \text{and} \quad x[n] \otimes b \mapsto \mu_*(\beta \times 1)_*(x^{n+1} \times b). \]

The claim follows.

The rest of this section is dedicated to the proof of Theorem 6.2. Again, from Proposition 4.5 we deduce that we may compute \( \alpha_2^{\pi_1} \) by a push-pull construction in the following special case of diagram (16):

\[
\begin{array}{ccc}
(T^1)^{V_1}/\Delta T^1/V_2 \times (T^1)^{\{p\}_{\times}} & \xrightarrow{\pi_1} & (T^1)^{V_2} \times (T^1)^{\{p,q\}} \\
\pi_2 \downarrow & & \downarrow \pi_2 \\
pt/V_2 \times (T^1)^{\{p\}} & \xrightarrow{\eta} & pt/(T^1)^{\{q\}} \\
\end{array}
\] (28)

From now on, for the sake of brevity, let us write \( T^3 \) for the space \( (T^1)^{V_2}/\Delta T^1 \) (which is a 3-dimensional torus). The map \( \pi_1 \) fits into the commutative square

\[
\begin{array}{ccc}
T^3/V_2 \times (T^1)^{\{p\}} & \xrightarrow{\pi_1} & (T^3/V_2) \times (pt/(T^1)^{\{p\}}) \\
\pi_2 \downarrow & & \downarrow \pi_2 \\
pt/V_2 \times (T^1)^{\{p\}} & \xrightarrow{\eta} & pt/(T^1)^{\{q\}} \\
\end{array}
\] (29)

where the \( V_2 \)-action on \( T^3 \) is given by the formula

\[ u \cdot (z_v)_{v \in V_2} \Delta T^1 = (z_{u+v})_{v \in V_2} \Delta T^1, \]

the map \( \pi'_1 \) is induced by the projection \( T^3 \to pt \), and the horizontal maps are given by the evident homeomorphisms. Thus to compute the umkehr map \( \pi'_1 \), it suffices to understand the umkehr map (\( \pi'_1 \)).

To understand the map (\( \pi'_1 \)), we will study in detail the \( V_2 \)-space \( T^3 \). For a homomorphism \( \alpha: V_2 \to \mathbb{Z}^x \), write \( R(\alpha) \) for the associated 1-dimensional real representation. From representation theory we know that the maps

\[ \prod_{\alpha: V_2 \to \mathbb{Z}^x} R(\alpha), \quad v \mapsto \sum_{\alpha: V_2 \to \mathbb{Z}^x} \alpha(v) e_\alpha \]

and

\[ \prod_{\alpha: V_2 \to \mathbb{Z}^x} R(\alpha), \quad e_\alpha \mapsto \frac{1}{4} \sum_{v \in V_2} \alpha(v) v \]
Figure 1. The $V_2$-CW structure on $\prod_{\alpha \neq 1} R(\alpha)/L \approx T^3$. Acting by $x_1$, $x_2$ and $x_1 + x_2$ sends the cube in the middle to the cube on the left, on the right, and at the bottom, respectively. Faces and vertices are identified as indicated. Edges with the same label and orientation are also identified. Edges with the same label and opposite orientation are not identified, but belong to the same $V_2$-orbit. Vertices labeled by 1 correspond to elements in $L$, and vertices labeled by $\eta$, $\theta$ and $\zeta$ correspond to the respective basis vectors of $\prod_{\alpha \neq 1} R(\alpha)$.

are inverse isomorphisms of $V_2$-representations. Here $e_\alpha$ denotes the unit basis vector associated to the $R(\alpha)$-factor in $\prod_\alpha R(\alpha)$. These isomorphisms induce a $V_2$-equivariant homeomorphism

$$\prod_{\alpha \neq 1} R(\alpha)/L \overset{\approx}{\longrightarrow} (R[V_2]/\Delta R)/(Z[V_2]/\Delta Z), \quad e_\alpha + L \mapsto \left[ \frac{1}{4} \sum_{v \in V_2} \alpha(v)v \right]$$

where 1 denotes the trivial homomorphism $V_2 \to Z^\times$, $L \subset \prod_{\alpha \neq 1} R(\alpha)$ is the lattice generated the vectors $\sum_{\alpha \neq 1} \alpha(v)e_\alpha$ for $v \in V_2$, $v \neq 0$, and $\Delta R$ and $\Delta Z$ denote the diagonal subgroups of $R[V_2]$ and $Z[V_2]$, respectively. The target of the above is canonically $V_2$-homeomorphic to $(R[V_2]/Z[V_2]/\Delta R/\Delta Z)$, which in turn is in an evident way $V_2$-homeomorphic to the space $T^3 = (T^1)^2/\Delta T^1$. Thus we obtain a $V_2$-equivariant homeomorphism

$$\prod_{\alpha \neq 1} R(\alpha)/L \overset{\approx}{\longrightarrow} T^3.$$
Write $\eta$, $\theta$ and $\zeta$ for the unique surjective homomorphisms $V_2 \to \mathbb{Z}^\times$ sending $x_1$, $x_2$ and $x_1 + x_2$ to the identity element, respectively, and continue to write $1$ for the constant homomorphism $V_2 \to \mathbb{Z}^\times$. The lattice $\prod_{\alpha \neq 1} \mathbb{Z}(\alpha) \subset \prod_{\alpha \neq 1} \mathbb{R}(\alpha)$ induces a $V_2$-equivariant cubical complex structure on $\prod_{\alpha \neq 1} \mathbb{R}(\alpha)$, and this structure descends to the quotient $\prod_{\alpha \neq 1} \mathbb{R}(\alpha)/L$ to give it and hence, via (30), the space $T^3$ the $V_2$-CW structure depicted in Figure 1. Thus we obtain the cellular chain complex

$$C_* = C_*(T^3) = (0 \to C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0) \quad (31)$$

for $T^3$ where

$$C_3 = F_2[V_2]\text{cube}$$
$$C_2 = F_2[V_2]\{e_{\text{top}}, e_{\text{front}}, e_{\text{right}}\}$$
$$C_1 = F_2[V_2/\langle x_1 \rangle]\{e_{\bullet}, e_{\circ}\} \oplus F_2[V_2/\langle x_2 \rangle]\{e_{\bullet}, e_{\circ}\}$$
$$\oplus F_2[V_2/\langle x_1 + x_2 \rangle]\{e_{\bullet}, e_{\circ}\}$$
$$C_0 = F_2\{e_1, e_\eta, e_\theta, e_\zeta\}$$

and

$$d_3(e_{\text{cube}}) = (1 + [x_1 + x_2])e_{\text{top}} + (1 + [x_2])e_{\text{front}} + (1 + [x_1])e_{\text{right}}$$
$$d_2(e_{\text{top}}) = [x_2]e_{\bullet} + e_{\circ} + [x_1]e_{\bullet} + [x_1]e_{\circ}$$
$$d_2(e_{\text{front}}) = [x_2]e_{\bullet} + [x_2]e_{\circ} + [x_1]e_{\bullet} + [x_1]e_{\circ}$$
$$d_2(e_{\text{right}}) = [x_1]e_{\bullet} + e_{\circ} + [x_1]e_{\bullet} + e_{\circ}$$
$$d_1(e_{\bullet}) = e_1 + e_\eta$$
$$d_1(e_{\circ}) = e_\theta + e_\zeta$$
$$d_1(e_{\bullet}) = e_1 + e_\theta$$
$$d_1(e_{\circ}) = e_\eta + e_\zeta$$
$$d_1(e_{\bullet}) = e_1 + e_\zeta$$
$$d_1(e_{\circ}) = e_\eta + e_\theta$$

Here $e_{\text{cube}}$ refers to the middle cube in Figure 1, $e_{\bullet}$ etc. refer to the edge with the corresponding label whose orientation agrees with that of a coordinate axis, and $e_{\text{top}}, e_{\text{front}}, e_{\text{right}}$ and $e_1, e_\eta, e_\theta, e_\zeta$ refer to the faces and vertices with corresponding labels.

For clarity we have written $[u]$ for the element of $F_2[V_2]$ corresponding to $v \in V_2$.

The $0$-cells of $T^3$ induce sections

$$s_1, s_\eta, s_\theta, s_\zeta : \text{pt} \to T^3$$

of $\pi_1'$; alternatively, $s_\lambda$ can be described as the map defined by the assignment

$$\text{pt} \mapsto (\lambda(v))_{v \in V_2} \Delta T^1 \in T^3 = (\mathbb{T}^1)^2 / \Delta \mathbb{T}^1.$$

where we interpret $\mathbb{Z}^\times = \{\pm 1\} \subset \mathbb{T}$. The following lemma is the key ingredient in the proof of Theorem 6.2.

**Lemma 6.4.** The umkehr map $(\pi_1')^! : H_*(\text{pt} \to T^3) \to H_{*+3}(T^3 \to V_2)$ is given by

$$(\pi_1')^!(x_1^{[n_1]}x_2^{[n_2]}) = [(s_1)_* + (s_\eta)_* + (s_\theta)_* + (s_\zeta)_*] \left( \sum_{i=0}^{n_1+1} x_1^{[i]}x_2^{[n_1+n_2+3-i]} \right)$$

for all $n_1, n_2 \geq 0$.

**Proof.** Let $F_*$ be the divided power algebra over the ring $F_2[V_2]$ generated by variables $X_1$ and $X_2$ of degree 1, and equip $F_*$ with the differential

$$d(X_1^{[k_1]}X_2^{[k_2]}) = t_1 X_1^{[k_1-1]}X_2^{[k_2]} + t_2 X_1^{[k_1]}X_2^{[k_2-1]}.$$
Here $t_i = 1 + x_i$, $i = 1, 2$; notice that $F_2[V_2]$ is simply the exterior algebra $\Lambda(t_1, t_2)$ over $F_2$. Equipped with the augmentation
\[
\varepsilon: F_0 = \Lambda(t_1, t_2) \rightarrow F_2
\]
given by the $F_2$-algebra homomorphism sending $t_1$ and $t_2$ to 0, the chain complex $F_*$ becomes a free $F_2[V_2]$-resolution of $F_2$. The multiplication on $F_*$ is an admissible product in the sense of [Bro82, section V.5], and hence models the Pontryagin product on $H_*(pt//V_2) = H_*(F_* \otimes V_2 F_2)$. Concretely, the product $x_1^{[n_1]} x_2^{[n_2]}$ is represented by the cycle $X_1^{[n_1]} X_2^{[n_2]} \otimes_{V_2} 1$ of $F_* \otimes F_2$.

Consider now the double complex $F_* \otimes V_2 C_*$. Its total complex is a chain model for the space $T^3//V_2$, and the spectral sequence associated with $F_* \otimes V_2 C_*$ obtained by first taking homology in the $C_*$-direction agrees (from the $E^2$-page onwards) with the Serre spectral sequence of the fibration $\pi'_1: T^3//V_2 \rightarrow pt//V_2$. As before, by [HL13, Lemma 8.4], the umkehr map $(\pi'_1)^!$ is equal to the composite
\[
H_*(pt//V_2) \xrightarrow{\pi'_1^*} H_*(pt//V_2; H_3T^3) = E^2_{n,3} \xrightarrow{d_{n,3}} E^\infty_{n,3} \xrightarrow{\delta} H_{n+3}(T^3//V_2)
\]
where $E^2$ and $E^\infty$ refer to pages in the Serre spectral sequence of $\pi'_1$, the first map is induced by the fundamental class of $T^3$, and the last two maps arise from the fact that we are working with the top row of the spectral sequence. Observing that the fundamental class of $T^3$ is represented by the cycle $t_1 t_2 e_{\text{cube}} \in C_3$, we obtain the following recipe for computing $(\pi'_1)^!$: for any $n_1, n_2 \geq 0$, the element $(\pi'_1)^!(x_1^{[n_1]} x_2^{[n_2]}) \in H_*(T^3//V_2)$ is the homology class of the element $X_1^{[n_1]} X_2^{[n_2]} \otimes_{V_2} t_1 t_2 e_{\text{cube}}$ considered as a cycle in $\text{Tot}(F_* \otimes V_2 C_*)$.

Let
\[
c_1 = X_1^{[n_1+1]} X_2^{[n_2]} \otimes_{V_2} t_2 e_{\text{cube}}
\]
\[
c_2 = X_1^{[n_1+2]} X_2^{[n_2]} \otimes_{V_2} t_2 (e_{\text{top}} + e_{\text{right}})
\]
and
\[
c_3 = X_1^{[n_1+1]} X_2^{[n_2+1]} \otimes_{V_2} (e_e + e_{\perp}) + \sum_{i=0}^{n_1+2} X_1^{[i]} X_2^{[n_1+n_2+3-i]} \otimes_{V_2} (e_{\theta} + e_{\zeta}).
\]

Using the formulas (32) for the boundary map in $C_*$, it is easy to compute that
\[
d(c_1 + c_2 + c_3) = X_1^{[n_1]} X_2^{[n_2]} \otimes_{V_2} t_1 t_2 e_{\text{cube}} + \sum_{i=0}^{n_1+1} X_1^{[i]} X_2^{[n_1+n_2+3-i]} \otimes_{V_2} (e_1 + e_{\eta} + e_{\theta} + e_{\zeta})
\]
in $\text{Tot}(F_* \otimes V_2 C_*)$, showing that the two summands on the right hand side are homologous. The claim follows.

\begin{proof}[Proof of Theorem 6.2] For $\lambda: V_2 \rightarrow \{\pm 1\} \subset T^1$ a homomorphism, the assignment $pt \mapsto (\lambda(v))_{v \in V_2} \in (T^1)^{V_2}$ and the homomorphism
\[
V_2 \times (T^1)^{\{p\}} \rightarrow V_2 \times (T^1)^{\{p,q\}}, \quad (v, z_p) \mapsto (v, z_p, \lambda(v) z_p)
\]
define a map
\[
\tilde{s}_\lambda: pt//V_2 \times (T^1)^{\{p\}} \rightarrow (T^1)^{V_2} // V_2 \times (T^1)^{\{p,q\}}
\]
lifting \( s_\lambda \times 1 \) through the map \( \kappa \) of diagram (28) in the sense that the following diagram commutes:

\[
\begin{array}{c}
(pt//V_2) \times (pt//(T^1)^{[p]}) \\ \downarrow s_\lambda \times 1
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
(pt//V_2) \times (T^3//V_2) \\ \downarrow \tilde{s}_\lambda
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
(T^3//V_2) \times (pt//(T^1)^{[p]}) \\ \downarrow \kappa
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
(T^3//V_2) \times (pt//(T^1)^{[p,q]})
\end{array}
\]

Here the homeomorphisms on the left are as in diagram (29). Notice that the composite of \( \tilde{s}_\lambda \) with the map \( \pi_2 \) of (28) is the map \( pt//V_2 \times T^1 \rightarrow pt//T^1 \) induced by the composite homomorphism

\[
V_2 \times T^1 \xrightarrow{\lambda \times 1} V_1 \times T^1 \xrightarrow{\beta \times 1} T^1 \times T^1 \xrightarrow{\mu} T^1
\]

where we interpret \( \lambda \) as a homomorphism into \( V_1 \) and \( \mu \) denotes the multiplication map of \( T^1 \). Using Lemma 6.4, we now obtain the formula

\[
\alpha_{T^2}^\beta (x_1^{[n_1]} x_2^{[n_2]} \otimes b) = \beta_x \left( \sum_{\lambda: V_2 \rightarrow V_1} \lambda_x \left( \sum_{i=0}^{\frac{n_1+1}{2}} \lambda_x(x_1^{[n_1+n_2+3-i]} x_2^{[n_1+n_2+3-i]}) b \right) \right) 
\]

\[
= \beta_x \left( \sum_{\lambda: V_2 \rightarrow V_1} \lambda_x(x_1^{[n_1+n_2+3]} + \sum_{i=1}^{\frac{n_1+1}{2}} \lambda_x(x_1^{[i]} x_2^{[n_1+n_2+3-i]}) b \right) 
\]

\[
= \beta_x \left( \sum_{i=1}^{\frac{n_1+1}{2}} \left[ \binom{n_1+n_2+3}{i} x_2^{[n_1+n_2+3]} \right] b \right) 
\]

\[
= \beta_x \left( \sum_{i=1}^{\frac{n_1+1}{2}} \left( \binom{n_1+n_2+2}{i-1} + \binom{n_1+n_2+2}{i} \right) x_2^{[n_1+n_2+3]} \right) b 
\]

\[
= \left( 1 + \frac{n_1+n_2+2}{n_1+1} \right) \beta_x(x_2^{[n_1+n_2+3]} b).
\]

\[ \square \]

7. Computations for \( SU(2) \)

The purpose of this section is to compute the operation \( \alpha_{1}^{SU(2)} \). Interpret \( V_1 \) as \( \pm 1 \), and let \( \nu \) be the map

\[
\nu: V_1 \times SU(2) \rightarrow SU(2), \quad (\varepsilon, A) \mapsto \varepsilon A.
\]

It is easy to check that the map induced by \( \nu \) makes \( H_\ast(BSU(2)) \) into a module over the ring \( H_\ast(BV_1) \), which we continue to interpret as the divided power algebra over a single generator \( x \) of degree 1. Our aim is to prove the following theorem.

**Theorem 7.1.** The map

\[
\alpha_{1}^{SU(2)}: H_{\ast-3}(BV_1) \otimes H_\ast(BSU(2)) \rightarrow H_\ast(BSU(2))
\]

is given by

\[
x^{[n]} \otimes b \mapsto x^{[n+3]} \cdot b
\]

where \( \cdot \) refers to the module structure induced by \( \nu \).
Before proving the theorem, let us spend a moment analyzing the $H_*BV_1$-module structure of $H_*BSU(2)$. Recall that $H_*BSU(2)$ is isomorphic to $\mathbb{Z}/2$ in degrees divisible by 4 and vanishes in other degrees. Let $i$ denote the inclusion $i: V_1 \to SU(2)$, $\varepsilon \mapsto \varepsilon I$ where $I \in SU(2)$ is the identity matrix. We claim that the map $i_*: H_*BV_1 \to H_*BSU(2)$ is an isomorphism in degrees divisible by 4 and zero in other degrees. One way to see this is to recall that the cohomology of $BSU(2)$ is a polynomial algebra generated by the Euler class $e(\gamma_2^C)$ of the tautological complex 2-plane bundle $\gamma_2^C$ over $BSU(2)$, and observe that the pullback $(Bi)^*\gamma_2^C$ is isomorphic to the direct sum of four copies of the tautological real line bundle $\gamma_1^R$ over $BV_1$. Thus the map $i^*: H^*BSU(2) \to H^*BV_1$ sends the generator $e(\gamma_2^C) = w_4(\gamma_2^C) = w_4(\bigoplus^4 \gamma_1^R) = w_1(\gamma_1^R)^4 = x^4$, and the claim follows by passing to duals.

With the behaviour of the map $i_*$ understood, we can now read off the $H_*BV_1$-module structure of $H_*BSU(2)$ from the commutative diagram

$$
\begin{array}{c}
H_*(BV_1) \otimes H_*(BV_1) \xrightarrow{1 \otimes i_*} H_*(BV_1) \\
\downarrow{1 \otimes i_*} \quad \quad \downarrow{i_*} \\
H_*(BV_1) \otimes H_*(BSU(2)) \xrightarrow{\cdot i_*} H_*(BSU(2))
\end{array}
$$

where the upper $\cdot$ refers to the product on $H_*(BV_1)$ and the lower one to the module structure. Explicitly, $H_*BSU(2)$ is isomorphic as a $H_*BV_1$-module to the quotient of $H_*BV_1$ by the ideal generated by $x$ and $x^{[2]}$. In particular, we see that the operation $(\alpha_1^{SU(2)})^\sharp(x^{[n]})$ is non-trivial precisely when $n \equiv 1 \mod 4$.

**Proof of Theorem 7.1.** The proof shares many similarities with the slightly simpler proof of Theorem 6.1 above. Again, Proposition 4.5 implies that we may compute $\alpha_1^{SU(2)}$ by a push-pull construction in the diagram

$$
\begin{array}{ccc}
SU(2)//V_1 \times SU(2)^{[p]} & \xrightarrow{\eta} & SU(2)^{V_1} //V_1 \times SU(2)^{[p,q]} \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
pt//V_1 \times SU(2)^{[p]} & \xrightarrow{\eta} & pt//SU(2)^{[q]}
\end{array}
$$

where the $V_1 \times SU(2)^{[p]}$-action on $SU(2)$ is given by

$$(\varepsilon, g_p) \cdot g = g_p g^{\varepsilon g_p^{-1}},$$

where the $V_1 \times SU(2)^{[p,q]}$-action on $SU(2)^{V_1}$ is given by

$$(\varepsilon, g_p, g_q) \cdot (g_{-1}, g_1) = (g_p g_{-\varepsilon g_q^{-1}}, g_p g_q g_1^{-1}),$$

where $\eta$ is induced by the map

$$SU(2)^{V_1} \longrightarrow SU(2), \quad (g_{-1}, g_1) \longmapsto g_{-1} g_1^{-1}$$

and the projection $V_1 \times SU(2)^{[p,q]} \to V_1 \times SU(2)^{[p]}$, and where $\pi_1$ and $\pi_2$ are induced by the evident projections of spaces and groups. Again, diagram (25) is obtained from diagram (16) by taking $G = SU(2)$ and $k = 1$ and replacing the space $SU(2)^{V_1}/\Delta SU(2)$ by $SU(2)$ using the homeomorphism

$$SU(2)^{V_1}/\Delta SU(2) \xrightarrow{\approx} SU(2), \quad (g_{-1}, g_1) \Delta SU(2) \longmapsto g_{-1} g_1^{-1}.$$
By \cite[Lemma 8.4]{HL13}, we may compute the umkehr map $\pi_1^*$ using the Serre spectral sequence for $\pi_1$. More precisely, the map $\pi_1^*$ is given by the composite

$$H_n(\text{pt}/V_1 \times SU(2)\{p\}) \xrightarrow{\pi_1^*} H_n(\text{pt}/V_1 \times SU(2)\{p\}; H_3SU(2)) = E^2_{n,3}
\xrightarrow{d_3} E^n_{n,3} \hookrightarrow H_{n+3}(SU(2)/V_1 \times SU(2)\{p\}). \quad (35)$$

Here $E^2$ and $E^n$ refer to pages in the Serre spectral sequence of $\pi_1$, the first map is induced by the fundamental class of $SU(2)$, and the last two maps arise from the fact that we are working with the top row of the spectral sequence.

The assignments $pt \mapsto \pm I \in SU(2)$ define sections

$$s_{\pm 1}: \text{pt}/V_1 \times SU(2)\{p\} \rightarrow SU(2)/V_1 \times SU(2)\{p\}.$$ 

Since the spectral sequence has only two non-zero rows, the existence of these sections implies that the Serre spectral sequence for $\pi_1$ collapses at the $E^2$-page. Thus the short exact sequence

$$0 \rightarrow E^n_{n,3} \rightarrow H_{n+3}(SU(2)/V_1 \times SU(2)\{p\}) \rightarrow E^n_{n+3,0} \rightarrow 0$$

and the description (35) for $\pi_1^*$ give the following short exact sequence:

$$0 \rightarrow H_n(\text{pt}/V_1 \times SU(2)\{p\}) \xrightarrow{\pi_1^*} H_n(\text{pt}/V_1 \times SU(2)\{p\}) \rightarrow H_n(\text{pt}/V_1 \times SU(2)\{p\}) \rightarrow 0. \quad (36)$$

To proceed with our analysis of $\pi_1^*$, it is helpful to dualize and work with the dual map

$$(\pi_1)_*: H^*(SU(2)/V_1 \times SU(2)\{p\}) \rightarrow H^{*-3}(\text{pt}/V_1 \times SU(2)\{p\})$$

instead. In general, if $\pi: M \rightarrow B$ is a closed fibrewise manifold of fibre dimension $d$, then the dual $\pi_1: H^*M \rightarrow H^{*-d}B$ of the umkehr map $\pi^*: H_B \rightarrow H_{B+d}M$ is a map of $H^B$-modules where $H^*M$ is made into a $H^B$-module via the map $\pi^*$. This follows from the pullback diagram

$$\begin{array}{ccc}
M & \xrightarrow{\pi_1} & B \times M \\
\downarrow & & \downarrow 1 \times \pi \\
B & \rightarrow & B \times B
\end{array}$$

by using the compatibility of the umkehr maps with pullbacks and direct products. Dualizing the short exact sequence (36), we therefore obtain a short exact sequence

$$0 \rightarrow H^*(\text{pt}/V_1 \times SU(2)\{p\}) \xrightarrow{\pi_1^*} H^*(SU(2)/V_1 \times SU(2)\{p\}) \rightarrow H^*(\text{pt}/V_1 \times SU(2)\{p\}) \rightarrow 0. \quad (37)$$

of $H^*(\text{pt}/V_1 \times SU(2)\{p\})$-modules.

The map

$$s_1^* + s_{-1}^*: H^*(SU(2)/V_1 \times SU(2)\{p\}) \rightarrow H^*(\text{pt}/V_1 \times SU(2)\{p\}) \quad (38)$$

is a map of $H^*(\text{pt}/V_1 \times SU(2)\{p\})$-modules satisfying $(s_1^* + s_{-1}^*) \circ \pi_1^* = \text{id} + \text{id} = 0$, so from the short exact sequence (37) we can deduce that

$$s_1^* + s_{-1}^* = \phi \circ (\pi_1)_!$$

for some $H^*(\text{pt}/V_1 \times SU(2)\{p\})$-linear map

$$\phi: H^*(\text{pt}/V_1 \times SU(2)\{p\}) \rightarrow H^{*+3}(\text{pt}/V_1 \times SU(2)\{p\}).$$
Making use of the evident homeomorphism
\[ \kappa: BV_1 \times BSU(2) \xrightarrow{\cong} \text{pt}\!//V_1 \times SU(2) \{p\}, \]
we see that the cohomology \( H^*(\text{pt}\!//V_1 \times SU(2) \{p\}) \) is isomorphic to a polynomial algebra generated by a class \( w_1 \) of degree 1 coming from the \( BV_1 \) factor and a class \( u_4 \) of degree 4 coming from the \( BSU(2) \) factor. Since the map \( \phi \) is \( H^*(\text{pt}\!//V_1 \times SU(2) \{p\}) \)-linear, we have
\[ \phi(z) = z\phi(1) \]
for all \( z \in H^*(\text{pt}\!//V_1 \times SU(2) \{p\}) \). Lemma 7.2 below implies that the map \( \phi \) is non-trivial, so we must have \( \phi(1) = w_1^3 \), since \( w_1^3 \) is the only non-trivial element in \( H^3(\text{pt}\!//V_1 \times SU(2) \{p\}) \).

Let \( \psi \) be the “division by \( w_1 \) map”
\[ \psi: H^*(\text{pt}\!//V_1 \times SU(2) \{p\}) \rightarrow H^{* -3}(\text{pt}\!//V_1 \times SU(2) \{p\}), \]
\[ w_1^k u_4 \rightarrow \begin{cases} w_1^{k-3} u_4 & \text{if } k \geq 3 \\ 0 & \text{otherwise}. \end{cases} \]
Then \( \psi\phi = \text{id} \), and we obtain the formula \( (\pi_1)_! = \psi\phi(\pi_1)_! = \psi(s_1^* + s_{-1}^*) \). Dualizing, we see that the composite map
\[ H_*(BV_1 \times BSU(2)) \xrightarrow{\kappa_*} H_*(\text{pt}\!//V_1 \times SU(2) \{p\}) \xrightarrow{\pi_1^*} H_{*+3}(SU(2) \!//V_1 \times SU(2) \{p\}) \]
is given by
\[ \pi_1^*\kappa_*(x[n] \times b) = ((s_1)_* + (s_{-1})_*)\kappa_*(x^{[n+3]} \times b). \quad (39) \]

The maps
\[ \bar{s}_1, \bar{s}_{-1}: \text{pt}\!//V_1 \times SU(2) \{p\} \rightarrow SU(2) \!//V_1 \times SU(2) \{p+q\} \]
induced by the assignments \( \text{pt} \mapsto (I, I), (I, -I) \in SU(2)^V_1 \) and the homomorphisms
\[ V_1 \times SU(2) \{p\} \rightarrow V_1 \times SU(2) \{p+q\}, \quad (\varepsilon, g_p) \mapsto (\varepsilon, g_p, g_p, (\varepsilon, g_p, \varepsilon g_p) \]
give lifts of the respective maps \( s_1 \) and \( s_{-1} \) through the map \( \eta \). The composite
\[ \pi_2 \circ \bar{s}_{\pm 1}: \text{pt}\!//V_1 \times SU(2) \{p\} \rightarrow \text{pt}\!//SU(2) \{q\} \]
is the map induced by the projection \( pr: V_1 \times SU(2) \rightarrow SU(2) \) in the case of \( \bar{s}_1 \) and the map induced by \( \nu \) in the case of \( \bar{s}_{-1} \). Using the formula \( (39) \) for \( \pi_1^* \), we conclude that the operation \( a_1^{SU(2)} \) is the sum of the maps
\[ H_*(BV_1) \otimes H_*(BSU(2)) \rightarrow H_{*+3}(BSU(2)) \]
given by
\[ x^{[n]} \otimes b \mapsto pr_*(x^{[n+3]} \times b) \quad \text{and} \quad x^{[n]} \otimes b \mapsto \nu_*(x^{[n+3]} \times b). \]
The claim follows. \( \square \)

In the proof of Theorem 7.1, we made use of the following lemma.

**Lemma 7.2.** The map
\[ s_1^* + s_{-1}^*: H^*(SU(2) \!//V_1 \times SU(2) \{p\}) \rightarrow H^*(\text{pt}\!//V_1 \times SU(2) \{p\}) \]
in the proof of Theorem 7.1 is non-zero.
Proof. Let us interpret $SU(2)$ as the group of unit quaternions, and define
\[ D^3_+ = \{ q \in SU(2) \mid \text{Re } q > -1/2 \} \]
\[ D^3_- = \{ q \in SU(2) \mid \text{Re } q < 1/2 \}. \]
Then $D^3_+ \cup D^3_- = SU(2)$, the spaces $D^3_+\mid_{\{1\}}$ are invariant under the $V_1 \times SU(2)^{[p]}$-action on $SU(2)$, and there exist $V_1 \times SU(2)^{[p]}$-equivariant deformation retractions of $D^3_+\mid_{\{1\}}$ onto $\{1\} \subset SU(2)$, $D^3_-\mid_{\{-1\}}$ onto $\{-1\} \subset SU(2)$ and $D^3_+ \cap D^3_- \cap \text{onto the subspace}$
\[ S^3_0 = \{ q \in SU(2) \mid \text{Re } q = 0 \} \approx S^2 \]
of $SU(2)$. From the Mayer–Vietoris sequence of the subspaces $D^3_+\mid_{\{1\}} \cup V_1 \times SU(2)^{[p]}$ and $D^3_-\mid_{\{-1\}} \cup V_1 \times SU(2)^{[p]}$ of $SU(2)/V_1 \times SU(2)^{[p]}$ we therefore obtain an exact sequence
\[ H^3(SU(2)/V_1 \times SU(2)^{[p]}) \xrightarrow{(s^*_1,s^*_{-1})} H^3(pt\mid V_1 \times SU(2)^{[p]};\mathbb{Z}) \xrightarrow{\eta} H^3(S^3_0/V_1 \times SU(2)^{[p]}). \]
Observe that
\[ S^3_0/V_1 \times SU(2)^{[p]} \approx (S^3_0/V_1)/SU(2)^{[p]}. \]
The $V_1$-action on $S^3_0/V_1$ is given by the map $q \mapsto -q$, so we have $S^3_0/V_1 \approx \mathbb{R}P^2$. From the Serre spectral sequence of the fibration $(S^3_0/V_1)/SU(2)^{[p]} \to \text{pt}/SU(2)^{[p]}$ we now deduce that
\[ H^3(S^3_0/V_1 \times SU(2)^{[p]}) \approx H^3((S^3_0/V_1)/SU(2)^{[p]}). \]
Thus the map $(s^*_1,s^*_{-1})$ in the exact sequence (40) is an epimorphism. Consequently the map
\[ s^*_1 + s^*_{-1}: H^3(SU(2)/V_1 \times SU(2)^{[p]}) \to H^3(\text{pt}/V_1 \times SU(2)^{[p]}) \]
is also an epimorphism. But the group $H^3(\text{pt}/V_1 \times SU(2)^{[p]})$ is non-trivial: in the notation of the proof of Theorem 7.1, it contains the non-trivial element $w_1^3$. Thus the claim follows.

8. Applications

The computations made in the preceding sections show the existence of a large number of classes $a \in H_*(BSG)$ for which the operation
\[ \Phi^G(S_n/BS_{\Sigma_2})^a: \to H_{*+[a]+\dim(G)(n-1)}(BG) \]
is non-trivial for some compact Lie group $G$. More generally, the computations provide lots of examples of classes $a = a_1 \times \cdots \times a_r \in H_*(BS_{\Sigma_{n_1}} \times \cdots \times BS_{\Sigma_{n_r}})$ for which the operation
\[ \Phi^G(S_{n_1,\ldots,n_r}/BS_{\Sigma_{n_1,\ldots,n_r}})^a: \to H_{*+[\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}]}(BG) \]
is non-trivial for some compact Lie group $G$. Here $\Sigma_{n_1,\ldots,n_r} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$ and $S_{n_1,\ldots,n_r} = S_{n_1} \circ \cdots \circ S_{n_r}$. For example, we have the following result for $G = \mathbb{Z}/2$.

Proposition 8.1. Let $u_1, \ldots, u_k$ be positive integers no two of which have a 1 in common in their binary expansions, and let $f: \{1, \ldots, k\} \to \{1, \ldots, r\}$ be a surjective function. Let
\[ n_i = 2^{f^{-1}(i)} \quad \text{and} \quad a_i = \bigcirc_{j \in f^{-1}(i)} E_{u_j} \in H_*(BS_{\Sigma_{n_i}}) \]
for $i = 1, \ldots, r$, and let $a = a_1 \times \cdots \times a_r$. Then the operation (41) is non-trivial for $G = \mathbb{Z}/2$.

Proof. Let $x$ be the generator of $H_1(B\mathbb{Z}/2)$. Recalling Definition 3.7 and using Propositions 3.9 and 5.6, we see that the operation (41) is simply the map $H_*(B\mathbb{Z}/2) \to H_{*+[u_1+\cdots+u_k]}(B\mathbb{Z}/2)$ given by multiplication by $x^{[u_1+\cdots+u_k]}$. □
The purpose of this section is to show that classes \( a \) for which the operation (41) is non-trivial for a suitable compact Lie group \( G \) give rise to non-trivial classes in the homology of certain highly interesting groups: first, the ordinary homology of the holomorph \( \text{Hol}(F_N) = F_N \rtimes \text{Aut}(F_N) \); second, the twisted homology \( H_n(B\text{Aut}(F_N); \mathbb{F}_2^N) \) of \( \text{Aut}(F_N) \) where \( \mathbb{F}_2^N \) denotes \( \mathbb{F}_2 \) equipped with the tautological \( \text{Aut}(F_N) \)-action; and third, the ordinary homology of the affine groups \( \text{Aff}_N(\mathbb{Z}) = \mathbb{Z}^N \rtimes \text{GL}_N(\mathbb{Z}) \) and \( \text{Aff}_N(\mathbb{F}_2) = \mathbb{F}_2^N \rtimes \text{GL}_N(\mathbb{F}_2) \). Here and for the rest of the section \( N = \sum_{i=1}^r (n_i - 1) \). The families of non-trivial homology classes we construct are exhibited in Corollaries 8.11, 8.16, 8.18, 8.19, 8.22, and 8.23 below.

8.1. Stability results. In preparation for the discussion of the applications of our computations to the homology of \( \text{Aut}(F_n) \), \( \text{Hol}(F_n) \) and \( \text{Aff}_n(R) = R^n \rtimes \text{GL}_n(R) \) for \( R = \mathbb{Z} \) and \( \mathbb{F}_2 \), we will now recall stability results concerning the homology of these groups. We will start with the result relevant to the groups \( \text{Aut}(F_n) \) and \( \text{Hol}(F_n) \), which we will consider in the context of a larger family of groups \( A(\Gamma, L) \) defined below.

Let \( \Gamma \) be a connected finite graph, by which we mean a connected finite CW complex of dimension \( \leq 1 \), and let \( u : L \to \Gamma \) be an injective map from a finite set \( L \). We think of \( u \) as providing labels for a finite set of distinguished points on \( \Gamma \), namely the points in the image of \( u \). Omitting \( u \) from notation, we denote by \( H(\Gamma, L) \) the space of self homotopy equivalences of \( \Gamma \) fixing the distinguished points, and set \( \pi_0(\Gamma, L) = \pi_0 H(\Gamma, L) \). Composition of maps makes \( A(\Gamma, L) \) into a group. The group \( \text{Aut}(F_n) \) is realized in this way as \( A(\mathcal{V}^n S^1, \text{pt}) \) where the basepoint of \( \mathcal{V}^n S^1 \) is the sole distinguished point; explicitly, an isomorphism

\[
\text{Aut}(F_n) \xrightarrow{\simeq} A(\mathcal{V}^n S^1, \text{pt})
\]

is given by the map sending an automorphism \( \theta \) to the component of the map \( \mathcal{V}^n S^1 \to \mathcal{V}^n S^1 \) which on the \( i \)-th wedge summand is given by \( \theta(x_i)|_{l_1, \ldots, l_n} \). Here \( x_1, \ldots, x_n \) denotes the basis of \( F_n \), \( l_j \) denotes the inclusion of the \( j \)-th wedge summand into \( \mathcal{V}^n S^1 \), and \( v|_{l_1, \ldots, l_n} \) for \( v \in F_n \) denotes the result of substituting \( l_j \) for \( x_j \) in \( v \) for all \( 1 \leq j \leq n \). Similarly, the group \( \text{Hol}(F_n) \) is realized as \( A(T_n, \{p, q\}) \) where \( T_n \) is the graph depicted below.

\[
T_n = \begin{pmatrix}
  l_1 & \cdots & l_n \\
  \cdot & \cdot & \cdot \\
  p & c & q
\end{pmatrix}
\]

(42)

We will often regard \( T_n \) as an h-graph cobordism \( T_n : \text{pt} \to \text{pt} \), where the incoming point is given by \( p \) and the outgoing point by \( q \). An isomorphism

\[
\psi : \text{Hol}(F_n) \xrightarrow{\simeq} A(T_n, \{p, q\}) = \pi_0 h\text{Aut}(T_n)
\]

(43)

is given by the map sending an element \( (w, \theta) \in F_n \rtimes \text{Aut}(F_n) \) to the component of the map \( T_n \to T_n \) whose restriction to the edge \( l_i \) in (42) is the path \( \theta(x_i)|_{l_1, \ldots, l_n} \) and whose restriction to the edge \( c \) in (42) is the path \( c \cdot w|_{l_1, \ldots, l_n} \).

By the rank of \( \Gamma \) we mean the first Betti number of \( \Gamma \). If \( \Gamma' \) is another connected finite graph equipped with and injection \( L \to \Gamma' \) and \( \text{rank}(\Gamma) = \text{rank}(\Gamma') \), then \( \Gamma \) and \( \Gamma' \) are homotopy equivalent under \( L \). Conjugation by a fixed homotopy equivalence then yields an isomorphism \( A(\Gamma, L) \approx A(\Gamma', \text{pt}) \) which up to conjugacy is independent of the homotopy equivalence chosen. In particular, we obtain a well-defined isomorphism \( H_n B\text{Aut}(\Gamma, L) \approx H_n B\text{Aut}(\Gamma', \text{pt}) \) which is independent of any choice.

We will consider the following three types of maps between the groups \( A(\Gamma, L) \).
(A) Given \( l \in L \), we have a map \( A(\Gamma, L) \to A(\Gamma, L \setminus \{l\}) \) which forgets the distinguished point labeled by \( l \). If \( l \) labels a leaf, we will also consider a variant \( (A') \) which follows the above map with the isomorphism given by collapsing the edge corresponding to \( l \).

(B) Given \( l \in L \), we have a map \( A(\Gamma, L) \to A(\Gamma \cup \{l\}, L \sqcup pt) \) which attaches a whisker to \( \Gamma \) at the point labeled by \( l \). The free endpoint of the whisker provides a new distinguished point.

(C) Given distinct elements \( l_1, l_2 \in L \), we have a map \( A(\Gamma, L) \to A(\Gamma/\sim, L/\sim) \) which identifies the points of \( \Gamma \) labeled by \( l_1 \) and \( l_2 \) and identifies these two labels.

Maps of the aforementioned types are known to give isomorphisms on homology in a range of degrees, the stable range, which depends on the rank of \( \Gamma \). The formulation of the stability result given below is close to the one given in [Gal11, Theorem 1.4].

**Theorem 8.2** ([HV04, HVW06]). Let \( A \) be an abelian group, and let \( n \) be the rank of \( \Gamma \). Then maps of type \( (B) \) and \( (C) \) induce isomorphisms upon application of \( H_k(-; A) \) whenever \( n > 2k + 1 \), while maps of type \( (A) \) induce isomorphisms whenever \( n > 2k + 1 \) if \( |L| > 1 \) and whenever \( n > 2k + 3 \) if \( |L| = 1 \).

In particular, the preceding theorem implies that the inclusion \( \text{Aut}(F_n) \hookrightarrow \text{Aut}(F_{n+1}) \) induces an isomorphism on \( k \)-th homology group if \( n > 2k + 1 \): in terms of graphs, this inclusion is realized by the map \( A(V^nS^1, pt) \to A(V^{n+1}S^1, pt) \) given by attaching a new copy of \( S^1 \), which amounts to the composite of a map of type \( (B) \) and a map of type \( (C) \). Similarly, the inclusion \( \text{Hol}(F_n) \hookrightarrow \text{Hol}(F_{n+1}) \) (realized similarly as a composite of maps of type \( (B) \) and \( (C) \)) induces isomorphisms on \( H_k \) when \( n > 2k + 1 \), as do the inclusion \( \text{Aut}(F_n) \hookrightarrow \text{Hol}(F_n) \) (realized by a map of type \( (B) \)) and the quotient map \( \text{Hol}(F_n) \to \text{Aut}(F_n) \) (realized by a map of type \( (A') \)). We call the maps \( H_*B\text{Aut}(F_n) \to H_*B\text{Aut}(F_{n+1}) \) and \( H_*B\text{Hol}(F_n) \to H_*B\text{Hol}(F_{n+1}) \) induced by the inclusions stabilization maps.

In the stable range \( n > 2k + 1 \), the homology of \( \text{Aut}(F_n) \) and \( \text{Hol}(F_n) \) is completely understood:

**Theorem 8.3** ([Gal11, Theorem 1.1]). For any abelian group \( A \), the inclusion of \( \Sigma_n \) into \( \text{Aut}(F_n) \) as the automorphisms permuting the basis elements of \( F_n \) induces an isomorphism upon application of \( H_k(-; A) \) for \( n > 2k + 1 \).

On the other hand, outside the above stable range, the homology of \( \text{Aut}(F_n) \) and \( \text{Hol}(F_n) \) remains poorly understood. For an up-to-date summary of what is known about the homology of \( \text{Aut}(F_n) \) with rational coefficients, we refer the reader to the introduction of [CHKV15]. Some information about torsion in the homology of \( \text{Aut}(F_n) \) is available via computations of the \( p \)-torsion in the integral Farrell cohomology of \( \text{Aut}(F_n) \) for \( n \in \{p - 1, p, 2(p - 1)\} \) for primes \( p \geq 3 \) and for \( n \in \{p + 1, p + 2\} \) for primes \( p \geq 5 \). See [GMV98, Che97, Jen01]. The \( k \)-th homology groups of the holomorphs \( \text{Hol}(F_n) \) have been computed by Jensen [Jen04] with mod \( p \) coefficients for \( p \) an odd prime for \( k \leq 2 \) and with rational coefficients for \( k \leq 5 \). Jensen also supplies a computation of the twisted homology groups \( H_k(B\text{Aut}(F_n); \hat{R}^n) \) for \( R = \mathbb{Q} \) and \( k \leq 4 \), while Satoh [Sat06, Sat07] has computed these groups when \( R = \mathbb{Z} \), \( k = 1 \) and \( n \geq 2 \) and when \( R = \mathbb{Z}[1/2], \ k = 2 \) and \( n \geq 6 \). Here \( \hat{R}^n \) for a ring \( R \) denotes \( R^n \) with the action of \( \text{Aut}(F_n) \) given by the homomorphism \( \text{Aut}(F_n) \to GL_n(\mathbb{Z}) \to GL_n(R) \).

We now turn to the stability result relevant to the homology of affine groups. The following result can be read off from [vdK80, Theorem 4.8]; in the notation of [vdK80], we have \( \text{Aff}_n(R) = G_{n+1}^{F_p} \) where \( P = \emptyset \) and \( Q = \{1\} \).
Theorem 8.4. Let $A$ be an abelian group, and let $R$ be a principal ideal domain. Then the inclusions

$$GL_n(R) \hookrightarrow GL_{n+1}(R), \quad GL_n(R) \hookrightarrow \text{Aff}_n(R) \quad \text{and} \quad \text{Aff}_n(R) \hookrightarrow \text{Aff}_{n+1}(R)$$

induce isomorphisms upon application of $H_*(-; A)$ whenever $n \geq 2k+1$. \hfill \Box

In particular, as $n$ tends to infinity, the homology of $\text{Aff}_n(Z)$ and the homology of $GL_n(R)$ stabilize to a common value, namely the homology of the infinite general linear group $GL(R) = \text{colim}_n GL_n(R)$. Again, we call the maps induced by the inclusions $\text{Aff}_n(R) \hookrightarrow \text{Aff}_{n+1}(R)$ and $GL_n(R) \hookrightarrow GL_{n+1}(R)$ on homology stabilization maps. The following two theorems describe the stable homology in the cases $A = \mathbb{F}_2$ and $R = \mathbb{Z}$ or $\mathbb{F}_2$.

Theorem 8.5 ([Mit92], [AMNY99, Theorem 1]). The homology $H_*(BGL(\mathbb{Z}); \mathbb{F}_2)$ is isomorphic to $H_*(BO; \mathbb{F}_2) \otimes H_*(SU; \mathbb{F}_2)$ as a Hopf algebra. \hfill \Box

Theorem 8.6 ([Qui72, section 11, Corollary 2]). $H_*(BGL(\mathbb{F}_2); \mathbb{F}_2) = \mathbb{F}_2$. \hfill \Box

Again, in contrast with the complete information we have concerning the stable values, outside the stable range the mod 2 homology of $\text{Aff}_n(Z)$ and $\text{Aff}_n(\mathbb{F}_2)$ remains poorly understood. Indeed, the same is true of the closely related groups $GL_n(Z)$ and $GL_n(\mathbb{F}_2)$. Quillen’s computations [Qui72] give complete information on the cohomology of $GL_n(k)$ with mod $p$ coefficients when $k$ is a finite field of characteristic prime to $p$, but yield little information when $k = \mathbb{F}_2$ and $p = 2$. For computations of the integral cohomology of $GL_n(Z)$ and $SL_n(Z)$ for small $n$ and modulo small primes, see [Bro82, Exercise II.7.3], [Sou78], [LS78], and [EVGS13]. The mod $p$ Farrell cohomology of $GL(n; Z)$ for an odd prime $p$ has been computed for $p - 1 \leq n \leq 2p - 3$ by Ash [Ash89] and studied for $n = 2p - 2$ by Manjrekar [Man96], who has also computed the mod 3 cohomology of $GL(4; \mathbb{Z})$ in dimensions $> 3$ [Man95]. For an in-depth discussion of the homology of linear groups in general, we refer the reader to the monograph [Knu01].

8.2. Other preliminaries. Our aim in this subsection is to relate the family $S_{n_1,\ldots,n_r}$ of h-graph cobordisms over $B\Sigma_{n_1,\ldots,n_r}$ to a certain family $U_N$ of h-graph cobordisms over $B\text{Hol}(F_N)$ to obtain a useful description of the operation $\Phi^G(U_N/B\text{Hol}(F_N))$.

Definition 8.7. We define $U_n/B\text{Hol}(F_n) : pt \to pt$ to be the family of h-graph cobordisms obtained by performing the mapping cylinder construction of Remark 4.2 to the Borel construction $E\text{Hol}(F_n) \times_{\text{Hol}(F_n)} B\Pi_1(T_n, \{p,q\})$, where $T_n$ is as depicted in (42) and $\text{Hol}(F_n)$ acts on the h-graph cobordism $B\Pi_1(T_n, \{p,q\}) : \{p\} \to \{q\}$ via the isomorphism $\psi$ of (43). Observe that the family $U_n/B\text{Hol}(F_n)$ is then obtained from $B\Pi_1(T_n, \{p,q\})$ by the Borel construction in the sense of Definition 4.1.

The h-graph cobordism

$$\tilde{S}_{n_1,\ldots,n_r} = \tilde{S}_{n_1} \circ \cdots \circ \tilde{S}_{n_r} : pt \to pt \quad (44)$$

is homotopy equivalent (relative to the two endpoints) to $T_N$. Conjugation with a fixed homotopy equivalence $\tilde{S}_{n_1,\ldots,n_r} \to T_N$ therefore gives us an isomorphism

$$\phi : \pi_0\text{hAut}(\tilde{S}_{n_1,\ldots,n_r}) \xrightarrow{\simeq} \pi_0\text{hAut}(T_N) \quad (45)$$

which up to conjugacy is independent of the homotopy equivalence chosen. Let

$$U'_N/B\pi_0\text{hAut}(\tilde{S}_{n_1,\ldots,n_r}) : pt \to pt \quad (46)$$

be the pullback of the family $U_N/B\text{Hol}(F_N)$ along the homeomorphism

$$B\pi_0\text{hAut}(\tilde{S}_{n_1,\ldots,n_r}) \xrightarrow{\simeq} B\text{Hol}(F_N)$$
induced by \( \phi \) and the isomorphism \( \psi \) of (43) We then have a 2-cell \([HL13, \text{Definition } 2.10]\)

\[
U'_N/B\pi_0\text{hAut}(\hat{S}_{n_1,\ldots,n_r}) \xrightarrow{\sim} U_N/B\text{Hol}(F_N)
\]

which on base spaces is given by the homeomorphism (46). Observe that the family \( S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r} \) is obtained by the Borel construction from the h-graph cobordism \( \hat{S}_{n_1,\ldots,n_r} \) of (44) equipped with the evident \( \Sigma_{n_1,\ldots,n_r} \)-action derived from the \( \Sigma_{n_i} \)-actions on the factors \( \hat{S}_{n_i} \). Furthermore, observe that the family of h-graph cobordisms \( U'_N/B\pi_0\text{hAut}(\hat{S}_{n_1,\ldots,n_r}) \) is obtained from the h-graph cobordism

\[
\text{BPI}_1(\hat{S}_{n_1,\ldots,n_r},\{p,q\}): \{p\} \rightarrow \{q\}
\]

by the Borel construction with respect to the evident action of \( \pi_0\text{hAut}(\hat{S}_{n_1,\ldots,n_r}) \) on \( \text{BPI}_1(\hat{S}_{n_1,\ldots,n_r},\{p,q\}) \). Here \( p \) and \( q \) denote the endpoints of \( \hat{S}_{n_1,\ldots,n_r} \). Thus the following proposition provides a 2-cell

\[
S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r} \xrightarrow{\sim} U'_N/B\pi_0\text{hAut}(\hat{S}_{n_1,\ldots,n_r})
\]

which on base spaces is induced by the homomorphism \( \Sigma_{n_1,\ldots,n_r} \rightarrow \pi_0\text{hAut}(\hat{S}_{n_1,\ldots,n_r}) \) obtained from the \( \Sigma_{n_1,\ldots,n_r} \)-action on \( \hat{S}_{n_1,\ldots,n_r} \).

**Proposition 8.8.** Suppose \( \hat{S} \): \( P \rightarrow Q \) is a positive h-graph cobordism such that \( \hat{S} \) is a finite CW complex of dimension \( \leq 1 \) and \( P \) and \( Q \) consist of 0-cells of \( \hat{S} \). Assume further that each 1-cell of \( \hat{S} \) is equipped with the choice of a characteristic map \( I \rightarrow \hat{S} \). Let \( \Gamma \) be a discrete group acting on \( \hat{S} \) by permuting the cells. Suppose this action respects the chosen characteristic maps and keeps \( P \) and \( Q \) pointwise fixed. Assume \( S/\Gamma : P \rightarrow Q \) is obtained from \( \hat{S} \) by the Borel construction in the sense of Definition 4.1. Let \( U/B\pi_0\text{hAut}(\hat{S}) : P \rightarrow Q \) be a family of h-graph cobordisms obtained by the Borel construction from the h-graph cobordism \( \text{BPI}_1(\hat{S},P \sqcup Q) : P \rightarrow Q \) equipped with the evident \( \pi_0\text{hAut}(\hat{S}) \)-action. Then there exists a 2-cell \([HL13, \text{Definition } 2.10]\) \( S/\Gamma \Rightarrow U/B\pi_0\text{hAut}(\hat{S}) \) which is the identity on \( P \) and \( Q \) and which on base spaces is the map induced by the homomorphism \( \Gamma \rightarrow \pi_0\text{hAut}(\hat{S}) \) obtained from the \( \Gamma \)-action on \( \hat{S} \).

**Proof.** Suppose \( X \) is a finite CW complex of dimension \( \leq 1 \), and let \( X^{(0)} \) denote the 0-skeleton of \( X \). Choosing characteristic maps for the 1-cells of \( X \), we obtain a map \( \beta_X : X \rightarrow \text{BPI}_1(X,X^{(0)}) \) sending the 0-cells of \( X \) to the corresponding 0-cells in \( \text{BPI}_1(X,X^{(0)}) \) and sending each 1-cell \( c \) to the 1-cell of \( \text{BPI}_1(X,X^{(0)}) \) corresponding to the morphism of \( \Pi_1(X,X^{(0)}) \) represented by the characteristic map of \( c \). The finite free groupoid \( \Pi_1(X,X^{(0)}) \) has a basis consisting of the homotopy classes of the characteristic maps for the 1-cells of \( X \), as can be shown by induction on the number of 1-cells in \( X \) using the compatibility of \( \Pi_1 \) with pushouts \([Hig05, \text{Theorem } 17']\). Thus it follows from \([HL13, \text{Lemma } 7.38]\) that the map \( \beta_X \) is a homotopy equivalence. Clearly \( \beta_X \) is natural with respect to cellular maps preserving characteristic maps for 1-cells.

In the context of the h-graph cobordism \( \hat{S} \), we obtain the zigzag

\[
\hat{S} \xrightarrow{\beta_{\hat{S}}} \text{BPI}_1(\hat{S},\hat{S}^{(0)}) \xleftarrow{\sim} \text{BPI}_1(\hat{S},P \sqcup Q)
\]

of homotopy equivalences which are \( \Gamma \)-equivariant maps under \( P \sqcup Q \). Here the second map is induced by the inclusion of \( P \sqcup Q \) into \( \hat{S}^{(0)} \). Performing the Borel construction, we obtain the zigzag

\[
ET \times_\Gamma \hat{S} \xrightarrow{\sim} ET \times_\Gamma \text{BPI}_1(\hat{S},\hat{S}^{(0)}) \xleftarrow{\sim} ET \times_\Gamma \text{BPI}_1(\hat{S},P \sqcup Q)
\]

(49)
of maps over $B\Gamma$ and under $(P \sqcup Q) \times B\Gamma$ which restrict to homotopy equivalences on fibres.

For a space $Z$ over $B$ and under $(P \sqcup Q) \times B$, let us denote by $Z'$ the mapping cylinder of the map of $(P \sqcup Q) \times B$ into $Z$. Then the inclusion of $(P \sqcup Q) \times B$ into $Z'$ is a closed fibrewise cofibration over $B$. Moreover, by [Cla81, Proposition 1.3] the map $Z' \to B$ is a fibration if the map $Z \to B$ is. By [HL13, Remark B.3], the zigzag

$$(ET \times_{\Gamma} \hat{S})' \longrightarrow (ET \times_{\Gamma} B\Pi_1(\hat{S}, \hat{S}^{(0)}))' \longrightarrow (ET \times_{\Gamma} B\Pi_1(\hat{S}, P \sqcup Q))'$$

obtained from (49) consists of homotopy equivalences over $B\Gamma$ and under $(P \sqcup Q) \times B\Gamma$. Choosing a homotopy inverse over a fibration if the map $Z$ fibrewise cofibration over $B$ obtained from (49) consists of homotopy equivalences over $B\Gamma$ and under $(P \sqcup Q) \times B\Gamma$ for the second map, we obtain a 2-cell $(ET \times_{\Gamma} \hat{S})'/B\Gamma \Rightarrow (ET \times_{\Gamma} B\Pi_1(\hat{S}, P \sqcup Q))'/B\Gamma$ which is the identity on $P$ and $Q$ and base spaces. Furthermore, the map $\Gamma \to \pi_0h\text{Aut}(\hat{S})$ given by the $\Gamma$-action induces a 2-cell

$$(ET \times_{\Gamma} B\Pi_1(\hat{S}, P \sqcup Q))'/B\Gamma \longrightarrow (E\pi_0H \times_{\pi_0H} B\Pi_1(\hat{S}, P \sqcup Q))'/B\pi_0H$$

which is the identity on $P$ and $Q$. Here $H$ denotes $h\text{Aut}(\hat{S})$. As observed in Remark 4.2, $S$ is homotopy equivalent over $B\Gamma$ and under $(P \sqcup Q) \times B\Gamma$ to $(ET \times_{\Gamma} \hat{S})'$, and similarly for $U$ and $(E\pi_0H \times_{\pi_0H} B\Pi_1(\hat{S}, P \sqcup Q))'$, so the claim follows. \qed

Composing the 2-cells (48) and (47), we obtain a 2-cell

$$S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r} \longrightarrow U_N/B\text{Hol}(F_N)$$

which on base spaces is induced by the composite homomorphism

$$\zeta: \Sigma_{n_1,\ldots,n_r} \longrightarrow \pi_0h\text{Aut}(\hat{S}_{n_1,\ldots,n_r}) \overset{\phi}{\longrightarrow} \pi_0h\text{Aut}(T_N) \overset{\psi^{-1}}{\longrightarrow} \text{Hol}(F_N)$$

where the first map is given by the action of $\Sigma_{n_1,\ldots,n_r}$ on $\hat{S}_{n_1,\ldots,n_r}$. Thus the base change axiom of HHGFT's implies the following result.

**Theorem 8.9.** For any compact Lie group $G$, the map

$$\Phi^G(S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r})^\natural: H_\asthshift(B\Sigma_{n_1,\ldots,n_r}) \longrightarrow \text{Hom}_\ast(H_\astBG, H_\astBG)$$

factorizes as the composite

$$H_\asthshift(B\Sigma_{n_1,\ldots,n_r}) \longrightarrow \hat{\zeta}, \quad \Phi^G(U_N/B\text{Hol}(F_N))^\flat \longrightarrow \text{Hom}_\ast(H_\astBG, H_\astBG).$$

Here ‘shift’ equals $\dim(G)(N - 1)$. \qed

Observe that the homomorphism $\zeta$ is, up to conjugacy, independent of the choice of a homotopy equivalence $\hat{S}_{n_1,\ldots,n_r} \to T_N$ involved in the construction of $\phi$, and hence in particular the induced map $\zeta_\ast$ on homology is independent of this choice.

In the remainder of this subsection, our goal is to use Proposition 4.3 to develop a description of the operation $\Phi^G(U_N/B\text{Hol}(F_N))$. To this end, we would like to obtain more concrete descriptions of the spaces of functors

$$\text{fun}(\Pi_1(B\Pi_1(T_N, \{p, q\}), \{p, q\}), G) \quad \text{and} \quad \text{fun}(\Pi_1(B\Pi_1(T_N, \{p, q\}), \{p\}), G).$$

By [HL13, Lemma 7.27], there is a natural isomorphism of finite free groupoids

$$\Pi_1(T_N, \{p, q\}) \overset{\sim}{\longrightarrow} \Pi_1(B\Pi_1(T_N, \{p, q\}), \{p, q\})$$

(50)
which is the identity on objects and sends each morphism to the homotopy class of the path defined by the corresponding 1-simplex of $B\Pi_1(T_n, \{p, q\})$. This isomorphism restricts to give an isomorphism

$$\Pi_1(T_n, \{p\}) \xrightarrow{\sim} \Pi_1(B\Pi_1(T_n, \{p, q\}), \{p\}).$$

(51)

Moreover, observe that the loops $l_1, \ldots, l_n$ in (42) give a basis for the finite free groupoid $\Pi_1(T_n, \{p\})$ and that $l_1, \ldots, l_n$ together with the arc $c$ give a basis for $\Pi_1(T_n, \{p, q\})$. Evaluation against these basis elements gives isomorphisms

$$\text{fun}(\Pi_1(T_n, \{p\}), G) \xrightarrow{\sim} G^n \quad \text{and} \quad \text{fun}(\Pi_1(T_n, \{p, q\}), G) \xrightarrow{\sim} G^n \times G.$$  

(52)

Under these isomorphisms, the $\text{Hol}(F_n) \times G^{(p)}$-action on $\text{fun}(\Pi_1(T_n, \{p\}), G)$ corresponds to the $\text{Hol}(F_n) \times G^{(p)}$-action on $G^n$ given by

$$((w, \theta), g_p) \cdot ((g_i)_{1 \leq i \leq n}, g_c) = ((g_p \theta^{-1}(x_i)|_{g_1 \cdots g_n g_p^{-1}})_{1 \leq i \leq n}, g_q g_c g_p^{-1} (w^{-1})|_{g_1 \cdots g_n g_p^{-1}}).$$

(53)

and the $\text{Hol}(F_n) \times G^{(p,q)}$-action on the space $\text{fun}(\Pi_1(T_n, \{p, q\}), G)$ corresponds to the $\text{Hol}(F_n) \times G^{(p,q)}$-action on $G^n \times G$ given by

$$(w, \theta, g_p, g_q) \cdot ((g_i)_{1 \leq i \leq n}, g_c) = ((g_p \theta^{-1}(x_i)|_{g_1 \cdots g_n g_p^{-1}})_{1 \leq i \leq n}, g_q g_c g_p^{-1} (w^{-1})|_{g_1 \cdots g_n g_p^{-1}}).$$

(54)

Here $(w, \theta) \in \text{Hol}(F_n) = F_n \rtimes \text{Aut}(F_n)$, we use $x_i$ to denote the $i$-th basis element of $F_n$, and $v|_{g_1 \cdots g_n}$ for $v \in F_n$ denotes the result of substituting $g_i$ for $x_i$ in $v$ for every $i = 1, \ldots, n$.

Making use of the isomorphisms (50), (51) and (52) to simplify the spaces

$$\text{fun}(\Pi_1(B\Pi_1(T_n, \{p, q\}), \{p, q\}), G) \quad \text{and} \quad \text{fun}(\Pi_1(B\Pi_1(T_n, \{p, q\}), \{p\}), G),$$

Proposition 4.3 implies that we may compute the operation $\Phi^G(U_n / B\text{Hol}(F_n))$ by a push-pull construction in the diagram

$$\begin{array}{ccc}
G^n / \text{Hol}(F_n) \times G^{(p)} & \xrightarrow{\sim} & G^n / G / \text{Hol}(F_n) \times G^{(p,q)} \\
\downarrow & & \downarrow \\
\text{pt} / \text{Hol}(F_n) \times G^{(p)} & \xrightarrow{\sim} & \text{pt} / G^{(q)}
\end{array}$$

(55)

where the left-hand diagonal map is induced by the projection $G^n \to \text{pt}$, the horizontal map is induced by the projections $G^n \times G \to G^n$ and $\text{Hol}(F_n) \times G^{(p,q)} \to \text{Hol}(F_n) \times G^{(p)}$, and the right-hand diagonal map is induced by the projection $\text{Hol}(F_n) \times G^{(p,q)} \to G^{(q)}$. More precisely, we obtain the following result

**Lemma 8.10.** The operation $\Phi^G(U_n / B\text{Hol}(F_n))$ agrees with the composite

$$\begin{array}{ccc}
H_\ast(B\text{Hol}(F_n)) \otimes H_\ast(BG) & \xrightarrow{\sim} & H_\ast(B\text{Hol}(F_n) \times BG) \\
\downarrow \text{ (a)} & & \downarrow \text{ (a)} \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
H_\ast(\text{pt} / \text{Hol}(F_n) \times G^{(p)}) & \xrightarrow{\sim} & H_\ast(\text{pt} / G^{(q)}) \\
\downarrow \text{ (c)} & & \downarrow \text{ (c)} \\
H_\ast(G^n / \text{Hol}(F_n) \times G^{(p)}) & \xrightarrow{\sim} & H_\ast(G^n \times G / \text{Hol}(F_n) \times G^{(p,q)}) \\
\downarrow \text{ (d)} & & \downarrow \text{ (d)} \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
H_\ast(\text{pt} / G^{(q)}) & \xrightarrow{\sim} & H_\ast(BG)
\end{array}$$
where (a) and (e) are induced by the evident homeomorphisms; where the map labeled by ! is the umkehr map [HL13, section 7.2] associated to the map ! of (55) considered as a map of fibrewise manifolds over pt//Hol(F_n) \times G^{(p)}; where the isomorphism (c) is induced by the homotopy inverse of the horizontal map in (55); and where the map (d) is induced by the right-hand diagonal map in (55).

8.3. Applications to holomorphs of free groups. We will consider two families of non-trivial elements in the homology of Hol(F_N). In this section, we will focus on the first one of these families, which consists of the non-trivial elements produced by the following immediate corollary of Theorem 8.9.

**Corollary 8.11.** Let a ∈ H_s(BΣ_{n_1,...,n_r}) be such that Φ^G(S_{n_1,...,n_r}/BΣ_{n_1,...,n_r})^*(a) is non-trivial for some compact Lie group G. Then the element ζ(a) ∈ H_sBhol(F_N) is non-zero.

It turns out that the elements in this first family are stable in the sense that they survive under arbitrary iterations of the stabilization map H_sBhol(F_n) → H_sBhol(F_{n+1}).

Our main results about these elements are Theorem 8.12 below, which determines their images in the stable range under iteration of the stabilization map, and Corollary 8.16, which strengthens the conclusion of Corollary 8.11 to the statement that the class ζ(a) is not in the image of the stabilization map H_sBhol(F_{N-1}) → H_sBhol(F_N). The second family of elements of H_sBhol(F_N), constructed in Corollary 8.19 in the next subsection, consists of unstable classes.

**Theorem 8.12.** The following diagram commutes when N + r + L > 2k + 1.

\[
\begin{array}{ccc}
H_k(BΣ_{n_1,...,n_r}; A) & \overset{H_k(BΣ_{N+r}; A)}{\longrightarrow} & H_k(BΣ_{N+r+L}; A) \\
\downarrow \ ζ* & & \downarrow \approx \\
H_k(Bhol(F_N); A) & \longrightarrow & H_k(Bhol(F_{N+r+L}); A)
\end{array}
\]

Here A is an arbitrary abelian group and all maps except ζ* are induced by the standard inclusions of groups. In particular, μ denotes the inclusion

\[μ: Σ_{n_1,...,n_r} \hookrightarrow Σ_{n_1+...+n_r} = Σ_{N+r}.
\]

**Proof.** For the duration of the proof, we will work with homology with coefficients in A, and will omit the coefficients from the notation. After the proof is complete, we will resume our convention of defaulting to \(\mathbb{F}_2\)-coefficients.

Diagram (56) embeds as the top left-hand rectangle into the diagram

\[
\begin{array}{ccc}
H_k(BΣ_{n_1,...,n_r}) & \overset{H_k(BΣ_{N+r})}{\longrightarrow} & H_k(BΣ_{N+r+L}) \\
\downarrow \ ζ* & & \downarrow \approx \\
H_k(Bhol(F_N)) & \longrightarrow & H_k(Bhol(F_{N+r+L})) \\
\downarrow \approx & & \downarrow \approx \\
H_k(BAut(F_N)) & \longrightarrow & H_k(BAut(F_{N+r+L}))
\end{array}
\]

\[\text{(57)}\]
Here the vertical maps in the bottom left-hand square are induced by the quotient maps from the holomorphs to the automorphism groups, and all remaining maps except for $\zeta_*$ are induced by inclusions. That the maps so indicated are isomorphisms follows from Theorems 8.2 and 8.3. The bottom left-hand square and the right-hand rectangle commute, so to prove the claim, it is enough to show that the outer rectangle commutes.

Let us denote $R_n = \bigvee^n S^1 \vee \bigvee^r I$, where the basepoint of $I$ is one of the endpoints. We label the basepoint of $R_n$ with $p_0$ and the free endpoints of the $r$ intervals with $p_1, \ldots, p_r$. We also use $p_0, \ldots, p_r$ to label the 0-cells of $S_{n_1, \ldots, n_r}$, with $p_0$ corresponding to the incoming point and $p_r$ to the outgoing point, and label the point $p$ of $T_N$ in (42) by $p_0$ and the point $q$ with $p_r$. Fix a homotopy equivalence $h: \hat{S}_{n_1, \ldots, n_r} \to R_N$ under $\{p_0, \ldots, p_r\}$, and let $h': \hat{S}_{n_1, \ldots, n_r} \to T_N$ be the composite of $h$ with the homotopy equivalence that collapses the edges of $R_N$ corresponding to $p_1, \ldots, p_{r-1}$. We then have the following diagram.

$$
\begin{array}{cccc}
\Sigma_{n_1, \ldots, n_r} & \overset{\mu}{\longrightarrow} & \Sigma_{N+r} & \overset{\text{incl}}{\longrightarrow} & \Sigma_{N+2r+L} \\
\downarrow \text{act} & & \downarrow \text{perm} & & \downarrow \text{perm} \\
A(\hat{S}_{n_1, \ldots, n_r}, \{p_0, \ldots, p_r\}) & \overset{(C)}{\longrightarrow} & A(\bigvee^{N+r} S^1, \text{pt}) & \overset{\text{stab}}{\longrightarrow} & A(\bigvee^{N+2r+L} S^1, \text{pt}) \\
\downarrow h_\ast & & \downarrow (C) & & \downarrow (C) \\
A(S_{n_1, \ldots, n_r}, \{p_0, p_r\}) & \overset{(A)}{\cong} & A(N, \{p_0, \ldots, p_r\}) & \overset{\text{stab}}{\longrightarrow} & A(N+r+L, \{p_0, \ldots, p_r\}) \\
\downarrow h'_\ast & & \downarrow (A') & & \downarrow (A') \\
A(T_N, \{p_0, p_r\}) & \overset{(A)}{\cong} & A(\bigvee^N S^1, \{p_0\}) & \overset{\text{stab}}{\longrightarrow} & A(\bigvee^{N+r+L} S^1, \{p_0\})
\end{array}
$$

Here the map ‘incl’ is the inclusion; the map ‘act’ is given by the $\Sigma_{n_1, \ldots, n_r}$-action on $\hat{S}_{n_1, \ldots, n_r}$; the maps labeled ‘perm’ are inclusions of the symmetric groups as permutations of wedge summands; the maps labeled ‘act’ are given by attaching copies of $S^1$; $h_\ast$ and $h'_\ast$ are the maps induced by $h$ and $h'$, respectively; and the maps labeled by $(A)$, $(A')$ and $(C)$ are iterates of maps of the respective types defined on page 33.

All squares and the triangle in the above diagram commute up to conjugacy, and hence they yield a strictly commutative diagram upon application of $H_k$. By assumption on $N + r + L$, the right-hand vertical map labeled by $(A')$ induces an isomorphism on $H_k$. The inverse to this isomorphism is induced by an iterate of maps of type $(B)$, and the composite of this inverse with the right-hand vertical map labeled $(C)$ amounts to an $r$-fold iteration of the map that attaches a new copy of $S^1$. Now, on $H_k$, starting at $\Sigma_{n_1, \ldots, n_r}$ and proceeding counterclockwise along the outer edge of the diagram, the composite of ‘act’, $(A)$, and $h'_\ast$ yields the map $\zeta_*$; the map $(A')$ gives the lower left-hand vertical map in (57); and the composite of ‘act’, inverse of $(A')$, and $(C)$ gives the bottom row in (57). On the other hand, the composite along the top row yields the composite along the top row in (57), and the right-hand map labeled ‘perm’ induces the right-hand vertical map in (57). Thus the outer rectangle in diagram (57) commutes, as desired.

Remark 8.13. If $A$ is a ring, the map $\mu_\ast$ in (56) amounts to iterated multiplication in the ring $H_*\left(\bigcup_{n \geq 0} B\Sigma_n; A\right)$. By [Nak60, Theorem 5.8]], the top map on the left in (56) is an injection, so Theorem 8.12 implies that $\zeta_*(a) \in H_\ast(B\text{Hol}(F_N); A)$ is a stable class whenever $a \in H_\ast(B\Sigma_{n_1, \ldots, n_k}; A)$ multiplies to a non-trivial element in the ring
$H_*(\bigsqcup_{n \geq 0} B\Sigma_n; A)$. In the case $A = F_2$, it follows easily from Theorems 3.2 and 3.3 that this in particular is the case for all positive-dimensional $a \in H_*(B\Sigma_{n_1},...,n_k)$ for which $\Phi^G(S_{n_1},...,n_k/B\Sigma_{n_1},...,n_k)^2(a)$ is non-trivial for some $G$. (Notice that such a $G$ cannot be finite of odd order, so Theorem 3.3 applies.)

**Remark 8.14.** Similarly, using diagram (57) in place of (56), we obtain a non-trivial stable element in $H_*(\text{Aut}(F_N); A)$ for every $a \in H_*(B\Sigma_{n_1},...,n_k; A)$ multiplying to a non-trivial element in the ring $H_*(\bigsqcup_{n \geq 0} B\Sigma_n; A)$.

**Theorem 8.15.** Suppose $b \in H_*B\text{Hol}(F_N)$ is a positive-dimensional class which is in the image of the stabilization map $H_*(B\text{Hol}(F_{N-1}) \to H_*(B\text{Hol}(F_N))$. Then the operation $\Phi^G(U_N/B\text{Hol}(F_N))^2(b)$ vanishes for every compact Lie group $G$.

**Proof.** If $G$ is a finite group of odd order, then $\Phi^G(U_N/B\text{Hol}(F_N))^2(b) = 0$ since $b$ is positive-dimensional and $H_*BG$ is concentrated in degree zero. For the remainder of the proof, let us assume that $G$ is a positive-dimensional compact Lie group or a finite group of even order.

Let $i : \text{Hol}(F_{N-1}) \hookrightarrow \text{Hol}(F_N)$ be the inclusion, and let $i^*U_N/B\text{Hol}(F_{N-1})$ denote the pullback of $U_N/B\text{Hol}(F_N)$ along the map $B\text{Hol}(F_{N-1}) \to B\text{Hol}(F_N)$ induced by $i$. To prove the claim, it is enough to show that the operation $\Phi^G(i^*U_N/B\text{Hol}(F_{N-1}))$ is zero. We have

$$i^*U_N = (E\text{Hol}(F_{N-1}) \times_{\text{Hol}(F_{N-1})} B\Pi_1(T_N, \{p, q\}))'$$

where $(-)'$ denotes the mapping cylinder construction of Remark 4.2. Here the group $\text{Hol}(F_{N-1})$ acts on $B\Pi_1(T_N, \{p, q\})$ via the composite

$$\text{Hol}(F_{N-1}) \xrightarrow{i} \text{Hol}(F_N) \xrightarrow{\psi} \pi_0h\text{Aut}(T_N)$$

where $\psi$ is the isomorphism (43). Notice that the above composite is the same as the composite

$$\text{Hol}(F_{N-1}) \xrightarrow{\psi} \pi_0h\text{Aut}(T_{N-1}) \xrightarrow{} \pi_0h\text{Aut}(T_N)$$

where the latter map is given by attaching a copy of $S^1$ and extending by identity. Our strategy is to use this observation to obtain a decomposition of the family $i^*U_N$ which features the h-graph cobordism $\mu$ of (6), and then deduce the claim from Proposition 3.5.

Let $T'_{N-1}$ and $C$ be the h-graph cobordisms pictured below.

$$T'_{N-1} = \left( \begin{array}{c} l_1, \ldots, l_{N-1} \\ p, q_1, q_2, q_0 \end{array} \right) : \text{pt} \rightarrow \bigsqcup^3 \text{pt}$$

$$C = \left( \begin{array}{c} q_2, q_1, q_0, q \end{array} \right) : \bigsqcup^3 \text{pt} \rightarrow \text{pt}$$

We have an evident homeomorphism $T_N \approx C \circ T'_{N-1}$, and this homeomorphism induces a homotopy equivalence

$$B\Pi_1(T_N, \{p, q\}) \xrightarrow{\simeq} B\Pi_1(C \circ T'_{N-1}, \{p, q_0, q_1, q_2, q\})$$

(58)
under \{p, q\}. From $T'_{N-1}$ and $C$ we obtain h-graph cobordisms

$$B\Pi_1(T'_{N-1}, \{p, q_0, q_1, q_2\}) : \text{pt} \rightarrow \bigsqcup^3 \text{pt} \quad \text{and} \quad (59)$$

$$B\Pi_1(C, \{q_0, q_1, q_2, q\}) : \bigsqcup^3 \text{pt} \rightarrow \text{pt}, \quad (60)$$

and we would like to compare the composite of these h-graph cobordisms to the target in (58).

By [HL13, Proposition 7.26], the underlying pushout square of spaces of the following square of h-graphs with basepoints

$$\begin{array}{c}
\{q_0, q_1, q_2, \} \rightarrow \{q_0, q_1, q_2, \} \\
\downarrow \\
\{p, q_0, q_1, q_2\} \rightarrow \{p, q_0, q_1, q_2\}
\end{array} \quad (61)$$

and the square

$$\begin{array}{c}
B\Pi_1(\{q_0, q_1, q_2\}) \rightarrow B\Pi_1(\{q_0, q_1, q_2\}) \\
\downarrow \\
B\Pi_1(T'_{N-1}, \{p, q_0, q_1, q_2\}) \rightarrow B\Pi_1(C \circ T'_{N-1}, \{p, q_0, q_1, q_2\})
\end{array} \quad (62)$$

obtained from (61) by applying $B\Pi_1$ are connected by a zigzag of natural homotopy equivalences. Since the underlying square of spaces of the diagram (61) is a homotopy cofibre square, it follows that square (62) also is. Observing that space in the top left corner of (62) is just the three-point space \{q_0, q_1, q_2\}, we see that the pushout of the top and the left arrows in (62) is the composite of (59) and (60). Since the maps out of the top left-hand corner in (62) are cofibrations, it follows that the map

$$B\Pi_1(C, \{q_0, q_1, q_2, q\}) \circ B\Pi_1(T'_{N-1}, \{p, q_0, q_1, q_2\}) \approx \sim \quad (63)$$

induced by the square (62) is a homotopy equivalence.

We let $\text{Hol}(F_{N-1})$ act on the sources and targets of (58) and (63) as follows. On $B\Pi_1(T'_{N}, \{p, q\})$ the action has already been specified. Notice that $T'_{N-1}$ is obtained from $T_{N-1}$ by attaching the two edges joining $p$ to $q_1$ and $q_2$, and write $j'$ for the resulting map

$$j : \pi_0\text{hAut}(T_{N-1}) \longrightarrow \pi_0\text{hAut}(T'_{N-1})$$

given by extension by identity along the new edges. The action on the space $B\Pi_1(C \circ T'_{N-1}, \{p, q_0, q_1, q_2\})$ is then given by the composite

$$\begin{array}{c}
\text{Hol}(F_{N-1}) \overset{\psi}{\longrightarrow} \pi_0\text{hAut}(T_{N-1}) \overset{j}{\longrightarrow} \pi_0\text{hAut}(T'_{N-1}) \longrightarrow \pi_0\text{hAut}(C \circ T'_{N-1}),
\end{array}$$

where the last map is given by attaching $C$ and extending by identity. Finally, the action on $B\Pi_1(C, \{q_0, q_1, q_2\}) \circ B\Pi_1(T'_{N-1}, \{p, q_0, q_1, q_2\})$ is induced by the trivial action on the $B\Pi_1(C, \{q_0, q_1, q_2\})$-factor and the action given by the composite

$$\begin{array}{c}
\text{Hol}(F_{N-1}) \overset{\psi}{\longrightarrow} \pi_0\text{hAut}(T_{N-1}) \overset{j}{\longrightarrow} \pi_0\text{hAut}(T'_{N-1}) \longrightarrow \pi_0\text{hAut}(C \circ T'_{N-1}),
\end{array}$$

on the $B\Pi_1(T'_{N-1}, \{p, q_0, q_1, q_2\})$-factor.

The maps (58) and (63) are $\text{Hol}(F_{N-1})$-equivariant with respect to the aforementioned actions, and induce a zigzag of 2-cells, all given by homeomorphisms on base spaces,
connecting $i^*\text{U}/B\text{Hol}(\text{F}_{N-1})$ to the composite of the families

$$E\text{Hol}(\text{F}_{N-1}) \times_{\text{Hol}(\text{F}_{N-1})} \text{BII}_1(T_{\text{F}_{N-1}}; \{p, q_0, q_1, q_2\})' / \text{BII}(\text{F}_{N-1}): \text{pt} \hookrightarrow \bigcup^3 \text{pt}$$

and

$$\text{BII}_1(C, \{q_0, q_1, q_2, q\}) / \text{pt}: \bigcup^3 \text{pt} \to \text{pt}.$$ 

Here again $(-)'$ denotes the mapping cylinder construction of Remark 4.2. The $h$-graph cobordism $\text{BII}_1(C, \{q_0, q_1, q_2, q\})$ is homotopy equivalent relative to the incoming and outgoing points to the $h$-graph cobordism $C$, which, using the notation of (6), decomposes as

$$C \simeq (\varepsilon \circ \mu) \sqcup I.$$ 

Thus the claim follows from Proposition 3.5. \qed

Theorems 8.9 and 8.15 have the following corollary.

**Corollary 8.16.** Suppose $a \in H_*\left( B\Sigma_{n_1, \ldots, n_r} \right)$ is a positive-dimensional class such that $\Phi^G(S_{n_1, \ldots, n_r}/BS_{n_1, \ldots, n_r})^2(a)$ is non-trivial for some compact Lie group $G$. Then the element $\zeta(a) \in H_*\text{BII}(\text{F}_N)$ is not in the image of the stabilization map $H_*\text{BII}(\text{F}_{N-1}) \to H_*\text{BII}(\text{F}_N)$. \qed

### 8.4. Applications to automorphism groups of free groups.

We now proceed to discuss the promised applications to the twisted homology $H_*\left( B\text{Aut}(\text{F}_N); \mathbb{F}_2^N \right)$. Here $\mathbb{F}_2^N$ denotes the vector space $\mathbb{F}_2^N$ equipped with the $\text{Aut}(\text{F}_N)$-action given by the composite homomorphism $\text{Aut}(\text{F}_N) \to \text{GL}_N(\mathbb{Z}) \to \text{GL}_N(\mathbb{F}_2)$ where the first map is induced by abelianization of $\text{F}_N$ and the second map is given by reduction mod 2. Considering the Hochschild–Serre spectral sequence associated to the split short exact sequence

$$1 \longrightarrow \text{F}_N \longrightarrow \text{Hol}(\text{F}_N) \longrightarrow \pi \longrightarrow \text{Aut}(\text{F}_N) \longrightarrow 1,$$

we see that there is a short exact sequence

$$0 \longrightarrow H_{*-1}(\text{BII}(\text{F}_N); \mathbb{F}_2^N) \overset{j}{\longrightarrow} H_*\text{BII}(\text{F}_N) \overset{\pi_*}{\longrightarrow} H_*\text{BII}(\text{F}_N) \longrightarrow 0. \quad (64)$$

Using the inclusion $\text{Aut}(\text{F}_N) \to \text{Hol}(\text{F}_N)$ to split this short exact sequence, we obtain a decomposition of the homology of $\text{Hol}(\text{F}_N)$ as a direct sum

$$H_*\text{BII}(\text{F}_N) \approx H_{*-1}(\text{BII}(\text{F}_N); \mathbb{F}_2^N) \oplus H_*\text{BII}(\text{F}_N). \quad (65)$$

Let $\rho: H_*\text{BII}(\text{F}_N) \to H_{*-1}(\text{BII}(\text{F}_N); \mathbb{F}_2^N)$ be the projection onto the first factor in this decomposition. Our aim is to prove the following theorem.

**Theorem 8.17.** Let $G$ be a positive-dimensional compact Lie group or a finite group of even order. Then the map

$$\Phi^G(\text{U}/\text{BII}(\text{F}_N))^2: H_*\text{shift}(\text{BII}(\text{F}_N)) \longrightarrow \text{Hom}_*(H_*\text{BG}, H_*\text{BG})$$

factors through the map

$$\rho: H_*\text{shift}(\text{BII}(\text{F}_N)) \longrightarrow H_*\text{shift}_{-1}(\text{BII}(\text{F}_N); \mathbb{F}_2^N).$$

Here ‘shift’ equals $\text{dim}(G)(N-1)$.

Theorems 8.9 and 8.17 have the following corollary.

**Corollary 8.18.** If $a \in H_*\left( B\Sigma_{n_1, \ldots, n_r} \right)$ is such that $\Phi^G(S_{n_1, \ldots, n_r}/BS_{n_1, \ldots, n_r})^2(a)$ is non-trivial for some compact Lie group $G$ which is either positive-dimensional or finite of even order, then the element $\rho\zeta(a) \in H_*\left( B\text{Aut}(\text{F}_N); \mathbb{F}_2^N \right)$ is non-zero. \qed
The inclusions $\text{Aut}(F_n) \hookrightarrow \text{Aut}(F_{n+1})$ and $F_2^n \hookrightarrow F_2^{n+1}$ induce a stabilization map

$$H_*(B\text{Aut}(F_n); \hat{F}_2^n) \rightarrow H_*(B\text{Aut}(F_{n+1}); \hat{F}_2^{n+1}).$$

It follows from Theorem 8.2 that the map $\pi_*$ in (64) is an isomorphism on $H_k$ for $N > 2k + 1$, so we see that $H_k(B\text{Aut}(F_n); \hat{F}_2^n) = 0$ for $n > 2k + 3$. Thus all elements in $H_*(B\text{Aut}(F_N); \hat{F}_2^n)$ are unstable in the sense that they are annihilated by an iteration of the stabilization map. As the decomposition (65) is compatible with the stabilization maps on both sides, it follows that all elements in the image of the map $j$ of (64) are also unstable. Thus Theorems 8.9, 8.17 and 8.15 imply the following corollary.

**Corollary 8.19.** Let $a \in H_*(BS_{n_1,...,n_r})$ be such that $\Phi^G(S_{n_1,...,n_r}/BS_{n_1,...,n_r})^2(a)$ is non-trivial for some compact Lie group $G$ which is either positive-dimensional or finite of even order. Then $j\rho\zeta(a) \in H_*(B\text{Hol}(F_N))$ is an unstable element which is not in the image of the stabilization map $H_*(B\text{Hol}(F_{N-1}) \rightarrow H_*(B\text{Hol}(F_N))$.

Notice that the element $j\rho\zeta(a)$ in Corollary 8.19 can alternatively be written as $\zeta(a) + i_*\pi_*\zeta(a)$, where $i$ denotes the inclusion $i: \text{Aut}(F_N) \hookrightarrow \text{Hol}(F_N)$.

Before proving Theorem 8.17, we need to establish an auxiliary result.

**Lemma 8.20.** Suppose $\pi: M \rightarrow B$ is a fibrewise closed manifold in the sense of [HL13, Definition 7.1] and suppose the fibres of $\pi$ are either positive-dimensional or finite of even cardinality. Then the composite

$$H_*(B) \xrightarrow{\pi^!} H_{*+d}(M) \xrightarrow{\pi_*} H_{*+d}(B)$$

is zero. Here $\pi^!$ denotes the umkehr map of [HL13, section 7.2] associated to the map $\pi$ (considered as a map of fibrewise manifolds over $B$) and $d$ denotes the fibre dimension of $M$.

**Proof.** If the fibres of $\pi$ are finite, the umkehr map $\pi^!$ is just the transfer map [HL13, Lemma 8.6], and the composite $\pi_*\pi^!$ amounts to multiplication by the cardinality of the fibre. By assumption this is an even number, so $\pi_*\pi^!$ vanishes since we are working with $\mathbb{F}_2$ coefficients. Let us assume that the fibres of $\pi$ are positive-dimensional. Then we have the following diagram where $r_B$ and $r_M$ denote the unique maps $B \rightarrow \text{pt}$ and $M \rightarrow \text{pt}$, respectively.

\[
\begin{array}{ccc}
H_*(B) & \xrightarrow{\pi^!} & H_{*+d}(M) \\
\downarrow \Delta_* & & \downarrow (1,\pi)_* \\
H_*(B \times B) & \xrightarrow{(\pi \times 1)^!} & H_{*+d}(M \times B) \\
\downarrow \times & & \downarrow \times \\
H_*(B \otimes H_*(B)) & \xrightarrow{\pi^! \otimes 1} & H_{*+d}(M) \otimes H_*(B) \\
& & \downarrow (r_M \otimes 1)_* \\
& & H_{*+d}(\text{pt} \otimes B)
\end{array}
\]

The top left square of the diagram commutes by the compatibility of umkehr maps with pullbacks, while the lower left square commutes by the compatibility of umkehr maps with direct products. The top right square commutes since the underlying square of spaces does, and finally the lower right square commutes by the naturality of the cross product. The claim now follows from the observation that the composite of the maps $\pi^!$ and $(r_M)_*$ in the bottom row vanishes for degree reasons.

**Proof of Theorem 8.17.** It is enough to show that the map $\Phi^G(U_N/B\text{Hol}(F_N))^2$ vanishes on classes which are in the image of the map

$$i_*: H_2B\text{Aut}(F_N) \rightarrow H_2B\text{Hol}(F_N)$$
induced by the inclusion \( i: \text{Aut}(F_N) \hookrightarrow \text{Hol}(F_N) \). The diagram (55) for computing \( \Phi^G(U_N/B\text{Hol}(F_N)) \) fits into a commutative square on homology after taking umkehr maps of the maps labeled

\[
\begin{array}{ccc}
G^N/\text{Hol}(F_N) \times G^{(p)} & \xrightarrow{\cong} & G^N \times G/\text{Hol}(F_N) \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
\text{pt}/\text{Hol}(F_N) \times G^{(p)} & \xrightarrow{\cong} & \text{pt}/G^{(q)}
\end{array}
\]

We deduce that the composite

\[
\pi
\]

denotes the result of substituting \( x \) spaces. Here \( \pi \) and the maps \( \eta \) are given by

\[
(\theta, g_p) \cdot (g_i)_{1 \leq i \leq N} = (g_p\theta^{-1}(x_i)|_{g_1, \ldots, g_N, g_p^{-1}})_{1 \leq i \leq N},
\]

(67)

the \( \text{Aut}(F_N) \times G^{(p,q)} \)-action on \( G^N \times G \) is given by

\[
(\theta, g_p, g_q) \cdot ((g_i)_{1 \leq i \leq N}, g_c) = ((g_p\theta^{-1}(x_i)|_{g_1, \ldots, g_N, g_p^{-1}})_{1 \leq i \leq N}, g_qg_cg_p^{-1}),
\]

(68)

and the maps \( \pi_1, \pi_2 \) and \( \eta \) are induced by the evident projection maps of groups and spaces. Here \( x_i \) again denotes the \( i \)-th basis element of \( F_N \), and \( v|_{g_1, \ldots, g_N} \) for \( v \in F_N \) denotes the result of substituting \( g_i \) for \( x_i \) in \( v \) for every \( i = 1, \ldots, N \).

Observe that the left-hand parallelogram in (66) is a pullback square, and gives rise to a commutative square on homology after taking umkehr maps of the maps labeled by \(!\) and induced maps of the vertical maps. To obtain the umkehr map \( \pi_1 \), we should interpret \( \pi_1 \) as a map of fibrewise manifolds over \( \text{pt}/\text{Aut}(F_N) \times G^{(p)} \). Using Lemma 8.10, we deduce that the composite

\[
H_\ast \text{-shift}(B\text{Aut}(F_N)) \otimes H_\ast(BG) \xrightarrow{i_\ast \otimes \mathbb{1}} H_\ast \text{-shift}(B\text{Hol}(F_N)) \otimes H_\ast(BG)
\]

\[
\Phi^G(U_N/B\text{Hol}(F_N)) \otimes H_\ast(BG)
\]

factors through the composite \( (\pi_2)_\ast \circ \eta^{-1}_p \circ \pi_1 \). The map

\[
\eta': G^N/\text{Aut}(F_N) \times G^{(p)} \to G^N \times G/\text{Aut}(F_N) \times G^{(p,q)}
\]

induced by the maps

\[
G^N \to G^N \times G, \quad (g_i)_{1 \leq i \leq N} \mapsto ((g_i)_{1 \leq i \leq N}, e)
\]

and

\[
\text{Aut}(F_N) \times G^{(p)} \to \text{Aut}(F_N) \times G^{(p,q)}, \quad (\theta, g_p) \mapsto (\theta, g_p, g_p)
\]

is a right inverse to \( \eta \). It follows that \( \eta^{-1}_p = \eta'_p \). But the composite \( \pi_2 \circ \eta' \) equals \( \pi_3 \circ \pi_1 \), where \( \pi_3: \text{pt}/\text{Aut}(F_N) \times G^{(p)} \to \text{pt}/G^{(q)} \) is induced by the projection away from \( \text{Aut}(F_N) \). Thus \( (\pi_2)_\ast \circ \eta^{-1}_p \circ \pi_1 = (\pi_3)_\ast \circ (\pi_1)_\ast \circ \pi_1 \), and the claim follows from Lemma 8.20. \( \square \)
8.5. **Applications to affine groups.** We now turn to the applications to the homology of the affine groups \( \text{Aff}_N(Z) \) and \( \text{Aff}_N(F_2) \). For any \( n \), the abelianization \( F_n \rightarrow Z^n \) induces a homomorphism

\[
\alpha_Z : \text{Hol}(F_n) \rightarrow \text{Aff}_n(Z),
\]

which we may compose with the homomorphism \( \text{Aff}_n(Z) \rightarrow \text{Aff}_n(F_2) \) given by reduction mod 2 to obtain a homomorphism \( \alpha_{F_2} : \text{Hol}(F_n) \rightarrow \text{Aff}_n(F_2) \). Our aim is to prove the following result.

**Theorem 8.21.** Let \( G \) be an abelian compact Lie group. Then the map

\[
\Phi^G(U_N/B\text{Hol}(F_N))^2 : H_*-\text{shift}(B\text{Hol}(F_N)) \rightarrow \text{Hom}_*(H_*(BG), H_*(BG))
\]

factors through the map

\[
(\alpha_Z)_* : H_*-\text{shift}(B\text{Hol}(F_N)) \rightarrow H_*-\text{shift}(B\text{Aff}_N(Z)).
\]

If \( G \) is an elementary abelian 2-group, then \( \Phi^G(U_N/B\text{Hol}(F_N))^2 \) factors through the map

\[
(\alpha_{F_2})_* : H_*-\text{shift}(B\text{Hol}(F_N)) \rightarrow H_*-\text{shift}(B\text{Aff}_N(F_2)).
\]

Here ‘\text{shift}’ equals \( \text{dim}(G)(N-1) \).

Theorems 8.9 and 8.21 together imply the following companion to Corollary 8.11.

**Corollary 8.22.** If \( a \in H_*(B\Sigma_{n_1,\ldots,n_r}) \) is such that \( \Phi^G(S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r})^2(a) \) is non-trivial for some abelian compact Lie group \( G \), then \( (\alpha_Z)_*\zeta(a) \in H_*(B\text{Aff}_N(Z)) \) is non-zero. If \( G \) can be chosen to be an elementary abelian 2-group, then \( (\alpha_{F_2})_*\zeta(a) \in H_*(B\text{Aff}_N(F_2)) \) is non-zero.

By Theorem 8.6, all positive-dimensional elements in the homology of \( \text{Aff}_N(F_2) \) obtained in this way are unstable. The following companion to Corollary 8.19 exhibits unstable elements in the homology of \( \text{Aff}_N(Z) \) as well. It follows from Theorems 8.9, 8.17 and 8.21, Corollary 8.19, and the observation that the maps \( \alpha_Z \) (resp. \( \alpha_{F_2} \)) are compatible with the inclusions \( \text{Hol}(F_n) \hookrightarrow \text{Hol}(F_{n+1}) \) and \( \text{Aff}_n(Z) \hookrightarrow \text{Aff}_{n+1}(Z) \) (resp. \( \text{Aff}_n(F_2) \hookrightarrow \text{Aff}_{n+1}(F_2) \)).

**Corollary 8.23.** Let \( a \in H_*(B\Sigma_{n_1,\ldots,n_r}) \) be such that \( \Phi^G(S_{n_1,\ldots,n_r}/B\Sigma_{n_1,\ldots,n_r})^2(a) \) is non-trivial for some abelian compact Lie group \( G \) which is positive dimensional or finite of even order. Then \( (\alpha_Z)_*\rho\zeta(a) \in H_*(B\text{Aff}_N(Z)) \) is a non-trivial unstable element. If \( G \) can be chosen to be an elementary abelian 2-group, then the element \( (\alpha_{F_2})_*\rho\zeta(a) \in H_*(B\text{Aff}_N(F_2)) \) is a non-trivial unstable element. 

**Proof of Theorem 8.21.** Observe that for abelian \( G \), the actions of \( \text{Hol}(F_N) \times G^{(p)} \) on \( G^N \) and \( \text{Hol}(F_N) \times G^{(p,q)} \) on \( G^N \times G \) given by (53) and (54) factor through the homomorphisms

\[
\alpha_Z \times 1 : \text{Hol}(F_N) \times G^{(p)} \rightarrow \text{Aff}_N(Z) \times G^{(p)}
\]

and

\[
\alpha_Z \times 1 : \text{Hol}(F_N) \times G^{(p,q)} \rightarrow \text{Aff}_N(Z) \times G^{(p,q)},
\]

respectively, inducing an action of \( \text{Aff}_N(Z) \times G^{(p)} \) on \( G^N \) and an action of \( \text{Aff}_N(Z) \times G^{(p,q)} \) on \( G^N \times G \). The diagram (55) for computing \( \Phi^G(U_N/B\text{Hol}(F_N)) \) now fits into a
where the maps in the bottom part of the diagram are induced by the evident projection maps of groups and spaces and where the unlabeled vertical arrows are induced by \( \alpha Z \).

The left-hand parallelogram is a pullback square, and induces a commutative square after passing to homology and taking umkehr maps of the maps labeled by \(!\) and ordinary induced maps of the vertical maps. When taking the umkehr maps, the lower map labeled by \(!\) should be interpreted as a map of fibrewise manifolds over \( pt/\text{Aff}_N(\mathbb{Z}) \times G\{p\} \).

The claim for abelian compact Lie groups now follows from Lemma 8.10. The claim for elementary abelian 2-groups follows in a similar way from the observation that the actions (53) and (54) in this case factor through the homomorphisms

\[
\alpha_{F_2} \times 1: \text{Hol}(F_N) \times G^{\{p\}} \rightarrow \text{Aff}_N(F_2) \times G^{\{p\}}
\]

and

\[
\alpha_{F_2} \times 1: \text{Hol}(F_N) \times G^{\{p,q\}} \rightarrow \text{Aff}_N(F_2) \times G^{\{p,q\}},
\]

respectively.

\[\square\]
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