Abstract

A quantum $sl(2,\mathbb{R})$ coalgebra (with deformation parameter $z$) is shown to underly the construction of a large class of superintegrable potentials on 3D curved spaces, that include the non-constant curvature analogues of the spherical, hyperbolic and (anti-)de Sitter spaces. The connection and curvature tensors for these “deformed” spaces are fully studied by working on two different phase spaces. The former directly comes from a 3D symplectic realization of the deformed coalgebra, while the latter is obtained through a map leading to a spherical-type phase space. In this framework, the non-deformed limit $z \to 0$ is identified with the flat contraction leading to the Euclidean and Minkowskian spaces/potentials. The resulting Hamiltonians always admit, at least, three functionally independent constants of motion coming from the coalgebra structure. Furthermore, the intrinsic oscillator and Kepler potentials on such Riemannian and Lorentzian spaces of non-constant curvature are identified, and several examples of them are explicitly presented.
1 Introduction

In the context of Hamiltonian systems with an arbitrary finite number of degrees of freedom, a deep connection between the coalgebra symmetry of a given system and its Liouville integrability was firmly established in [1]. Moreover, the intrinsic superintegrability properties of the coalgebra construction were further explored in [2]. Since then, this framework has lead to the coalgebra interpretation of the integrability properties of many well-known systems, as well as to the construction of many new superintegrable systems by using both Lie and $q$-Poisson coalgebras (see [1, 2, 3, 4] and references therein).

In particular, by making use of the Poisson coalgebra given by the non-standard quantum deformation of $sl(2, \mathbb{R})$, the construction of integrable 2D geodesic flows corresponding to 2D Riemannian and Lorentzian spaces with non-constant curvature was presented in [5]. Furthermore, these systems revealed a geometric interpretation of the quantum deformation, since the (in general, non-constant) curvature of these spaces was just a smooth function of the deformation parameter. Later, integrable potentials on such 2D “quantum deformed” spaces were introduced by preserving the underlying deformed coalgebra symmetry [6]. In this context, the search for the appropriate Kepler–Coulomb (KC) and oscillator potentials on such 2D curved spaces was posed as an interesting problem, and some hints were proposed.

In this contribution we present the generalization of all these results to the 3D case and we fully solve the question concerning the generic form of the intrinsic oscillator and KC potentials on all the corresponding “quantum deformed” 3D spaces, thus opening the path for the generalization of this construction to $N$ dimensions. In the next section we make use of the quantum $sl(2, \mathbb{R})$ coalgebra symmetry in order to construct the family of superintegrable 3D geodesic flows that define the spaces of hyperbolic type, whose sectional and scalar curvatures are also obtained. In section 3 an analytic continuation procedure is introduced through a set of appropriate spherical-type coordinates, thus leading to the Lorentzian counterparts of the previous spaces. Moreover, it is shown that these coordinates allows the separability of the geodesic flow Hamiltonians in all the cases. Finally, section 5 is devoted to the characterization of those potentials that will preserve the superintegrability properties of the free Hamiltonian. In particular, by applying the prescription given in [7, 8, 9, 10], the intrinsic oscillator and KC potentials on all the previous spaces are explicitly constructed, and some particular examples are analysed.

2 Superintegrable Hamiltonians

Let us consider the Poisson coalgebra version of the non-standard quantum deformation $sl(2, \mathbb{R})$, hereafter denoted $(sl_2(\mathbb{R}), \Delta) \equiv sl_z(2)$, where $z$ is a real deformation parameter ($q = e^z$). Its deformed Poisson brackets, coproduct $\Delta$ and Casimir $C$ are given by [3]:

\[
\{ J_3, J_+ \} = 2J_+ \cosh z J_-, \quad \{ J_3, J_- \} = -2 \frac{\sinh z J_-}{z}, \quad \{ J_-, J_+ \} = 4J_3, \quad \tag{1}
\]

\[
\Delta (J_+) = z J_+ \quad \text{and} \quad \Delta (J_-) = \frac{1}{z} J_-. \]
\[ \Delta(J_-) = J_- \otimes 1 + 1 \otimes J_-, \quad \Delta(J_l) = J_l \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_l, \quad l = +, 3, \]

\[ \mathcal{C} = \frac{\sinh zJ_-}{z}, \]

A one-particle symplectic realization of (1) reads

\[ J_{(1)}^- = q_1^2, \quad J_{(1)}^+ = \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2}, \quad J_{(1)}^3 = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1, \]

where \( b_1 \) is a real parameter that labels the representation through \( \mathcal{C}^{(1)} = b_1 \). Hence dimensions of the deformation parameter are \( |z| = [J_-]^{-1} = [q_1]^{-2} \).

Starting from (1), the coproduct (2) determines the corresponding two-particle realization of (1) defined on \( sl_z(2) \otimes sl_z(2) \) that depends on two real parameters \( b_1, b_2 \):

\[ J_{(2)}^- = q_1^2 + q_2^2, \quad J_{(2)}^+ = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 e^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 e^{-zq_1^2}, \]

\[ J_{(2)}^3 = \left( \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) e^{zq_2^2} + \left( \frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) e^{-zq_1^2}. \]

Then the two-particle Casimir is given by

\[ \mathcal{C}^{(2)} = \left( \frac{\sinh zq_1^2}{zq_1^2} \sinh zq_2^2 \right) \left( q_1 p_2 - q_2 p_1 \right)^2 + b_1 \frac{\sinh zq_2^2}{zq_2^2} + b_2 \frac{\sinh zq_1^2}{zq_1^2} \right) e^{-zq_1^2} e^{zq_2^2} + b_1 e^{2zq_2^2} + b_2 e^{-2zq_1^2}. \]

Next, the 3-sites coproduct, \( \Delta^{(3)} = (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \), gives rise to a three-particle symplectic realization of (1) in terms of three real parameters \( b_i \) which is defined on \( sl_z(2) \otimes sl_z(2) \otimes sl_z(2) \) [1, 3]; namely

\[ J_{(3)}^- = q_1^2 + q_2^2 + q_3^2 \equiv q^2, \]

\[ J_{(3)}^+ = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1 e^{zq_2^2} e^{zq_3^2} + \frac{\sinh zq_2^2}{zq_2^2} q_2 p_2 e^{zq_3^2} e^{-zq_1^2} + \frac{\sinh zq_3^2}{zq_3^2} q_3 p_3 e^{-zq_1^2} e^{-zq_2^2}, \]

\[ J_{(3)}^3 = \left( \frac{\sinh zq_1^2}{zq_1^2} p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) e^{zq_2^2} e^{zq_3^2} + \left( \frac{\sinh zq_2^2}{zq_2^2} p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) e^{-zq_1^2} e^{zq_3^2} + \left( \frac{\sinh zq_3^2}{zq_3^2} p_3^2 + \frac{zb_3}{\sinh zq_3^2} \right) e^{-zq_1^2} e^{-zq_2^2}. \]

Hence if we denote the three sites on \( sl_z(2) \otimes sl_z(2) \otimes sl_z(2) \) by \( 1 \otimes 2 \otimes 3 \) the coalgebra approach [1, 11] provides three “relevant” functions, coming from the two- and three-sites coproduct of the Casimir [3]: (i) the two-particle Casimir \( C^{(2)} \) which is defined on \( 1 \otimes 2 \); (ii) another two-particle Casimir \( C^{(2)} \) but defined on \( 2 \otimes 3 \); and (iii) the three-particle Casimir \( C^{(3)} \) defined on \( 1 \otimes 2 \otimes 3 \). These are given by (5) and

\[ C^{(2)} = \left( \frac{\sinh zq_2^2}{zq_2^2} \frac{\sinh zq_3^2}{zq_3^2} \left( q_2 p_3 - q_3 p_2 \right)^2 + b_2 \frac{\sinh zq_3^2}{zq_3^2} + b_3 \frac{\sinh zq_2^2}{zq_2^2} \right) e^{-zq_1^2} e^{zq_2^2} e^{zq_3^2} \]
perintegrable systems. Obviously, in 3D, quasi-maximally superintegrability is equivalent to weak superintegrability, but this is no longer true in higher dimensions. Therefore the 3D case can be considered as the cornerstone for the generalization of all the results we shall present here to arbitrary dimension \( N \).

\[ C^{(3)} = \left( \frac{\sinh z q_1^2}{z q_1^2} \sinh \frac{z q_2^2}{z q_2^2} (q_1 p_2 - q_2 p_1)^2 + b_1 \frac{\sinh z q_3^2}{z q_3^2} + b_2 \frac{\sinh z q_1^2}{z q_1^2} \right) e^{-2q_1^2} e^{2q_2^2} e^{2q_3^2} \]

\[ + \left( \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_3^2}{z q_3^2} (q_1 p_3 - q_3 p_1)^2 + b_1 \frac{\sinh z q_3^2}{z q_3^2} + b_3 \frac{\sinh z q_1^2}{z q_1^2} \right) e^{-2q_1^2} e^{-2q_2^2} e^{2q_3^2} \]

\[ + b_1 e^{2q_1^2} e^{2q_3^2} + b_2 e^{-2q_1^2} e^{2q_3^2} + b_3 e^{2q_1^2} e^{-2q_2^2} \]  

(7)

All the above expressions give rise to the (non-deformed) \( sl(2, \mathbb{R}) \) coalgebra \([1]\) under the limit \( z \to 0 \), that is, the Poisson brackets and Casimir are non-deformed, the coproduct is primitive, \( \Delta(X) = X \otimes 1 + 1 \otimes X \), and the symplectic realization reads \( J^{(3)}_i = q_i^2, \)

\( J^{(3)}_+ = p^2 + \sum_{i=1}^3 b_i / q_i^2, \) \( J^{(3)}_q = q \cdot p. \) If we set all the \( b_i = 0 \), the three Casimir functions reduce to the components of the angular momentum \( l_{ij} = q_i p_j - q_j p_i; \) \( C^{(2)} = l_{12}^2, C^{(2)} = l_{23}^2, \) and \( C^{(3)} = l_{12}^2 + l_{13}^2 + l_{23}^2. \)

In this way a large family of superintegrable Hamiltonians can be constructed through the following statement.

**Proposition 1.** (i) The three-particle generators \([6]\) fulfil the commutation rules \([6]\) with respect to the canonical Poisson bracket \( \{ q_i, p_j \} = \delta_{ij} \).

(ii) These generators Poisson commute with the three functions \( C^{(2)}, C^{(2)} \) and \( C^{(3)} \).

(iii) Any function defined on \([6]\), i.e.,

\[ H = H(J^{(3)}_-, J^{(3)}_+), \]  

provides a completely integrable Hamiltonian since either \( \{ C^{(2)}, C^{(3)}, H \} \) or \( \{ C^{(2)}, C^{(3)}, H \} \) are three functionally independent functions in involution.

(iv) The four functions \( \{ C^{(2)}, C^{(2)}, C^{(3)}, H \} \) are functionally independent.

We remark that in the 2D case, the generic function \( H = H(J^{(2)}_-, J^{(2)}_+, J^{(2)}_q) \) determines, in principle, an integrable (but not superintegrable!) Hamiltonian as it is only endowed with a single constant of the motion \( C^{(2)} \) \([6]\). On the contrary, in the 3D case any Hamiltonian \( H \) \([8]\) is (at least) a weakly-superintegrable one \([12]\), since one more constant of the motion is lacking in order to ensure the maximal superintegrability of the system. It is well-known \([11, 22]\) that in the ND generic case the coalgebra approach would provide \( 2N - 2 \) constants of the motion, thus leading to the construction of quasi-maximally superintegrable systems. Obviously, in 3D, quasi-maximal superintegrability is equivalent to weak superintegrability, but this is no longer true in higher dimensions. Therefore the 3D case can be considered as the cornerstone for the generalization of all the results we shall present here to arbitrary dimension \( N \).
3 Curved spaces from geodesic flows

As a byproduct of proposition 1 we can obtain an infinite family of superintegrable free Hamiltonians by setting the three \( b_i = 0 \) and by choosing, amongst the family \( \mathcal{H} \), the following expression for \( \mathcal{H} \):

\[
\mathcal{H} = \frac{1}{2} J_+ f(z J_-),
\]

where \( f \) is any smooth function such that \( \lim_{z \to 0} f(z J_-) = 1 \); hence \( \lim_{z \to 0} \mathcal{H} = \frac{1}{2} \mathbf{p}^2 \) gives the kinetic energy on the 3D Euclidean space. For the sake of simplicity, from now on we drop the index “(3)” in the generators. Thus by writing the Hamiltonian \( \mathcal{H} \) as a free Lagrangian,

\[
\mathcal{H} = \frac{1}{2} \left( \frac{z q_i^2}{\sinh z q_i^3} e^{-z q_i^2} e^{-z q_i^3} q_i^2 + \frac{z q_j^2}{\sinh z q_j^2} e^{z q_j^3} e^{-z q_j^2} q_j^2 + \frac{z q_k^2}{\sinh z q_k^3} e^{z q_k^2} e^{-z q_k^2} q_k^2 \right) f(z \mathbf{q}^2),
\]

we find the geodesic flow on a 3D space with a definite positive metric given by

\[
ds^2 = \left( \frac{2 z q_i^2}{\sinh z q_i^3} e^{-z q_i^2} e^{-z q_i^3} dq_i^2 + \frac{2 z q_j^2}{\sinh z q_j^2} e^{z q_j^3} e^{-z q_j^2} dq_j^2 + \frac{2 z q_k^2}{\sinh z q_k^3} e^{z q_k^2} e^{-z q_k^2} dq_k^2 \right) \frac{1}{f(z \mathbf{q}^2)}.
\]

The connection \( \Gamma^i_{jk} \) \( (i, j, k = 1, 2, 3) \) can now be straightforwardly computed:

\[
\begin{align*}
\Gamma^i_{ii} &= \left( \frac{1}{q_i^3} - \frac{z}{\tanh z q_i^3} - \frac{f'(x)}{f(x)} \right) q_i, \quad \Gamma^i_{jk} = 0, \quad i, j, k \text{ different}, \\
\Gamma^i_{ij} &= -z q_j \left( 1 + \frac{f'(x)}{f(x)} \right), \quad \Gamma^j_{ij} = z q_i \left( 1 - \frac{f'(x)}{f(x)} \right), \quad i < j, \\
\Gamma^i_{jj} &= -z e^{-2 q_i^2} e^{-q_j^3} q_j^2 \sinh z q_j^3 q_i \sinh z q_j^3 \left( 1 - \frac{f'(x)}{f(x)} \right), \quad j = i + 1, \\
\Gamma^j_{ii} &= z e^{-2 q_i^2} e^{-q_j^3} q_i^2 \sinh z q_i^3 q_j \sinh z q_i^3 \left( 1 + \frac{f'(x)}{f(x)} \right), \quad j = i + 1, \\
\Gamma^1_{33} &= -z e^{2 q_1^2} e^{q_3^3} q_3^2 \sinh z q_3^3 q_1 \sinh z q_3^3 \left( 1 - \frac{f'(x)}{f(x)} \right), \\
\Gamma^3_{11} &= z e^{2 q_1^2} e^{-q_3^3} q_3^2 \sinh z q_3^3 q_1 \sinh z q_3^3 \left( 1 + \frac{f'(x)}{f(x)} \right),
\end{align*}
\]

where \( x \equiv z J_-^{(3)} = z \mathbf{q}^2 \), \( f'(x) = \frac{df(x)}{dx} \) and \( f''(x) = \frac{d^2 f(x)}{dx^2} \). Therefore, the three geodesic equations for \( q_i(s) \) read,

\[
\frac{d^2 q_i}{ds^2} + \sum_{j,k=1}^3 \Gamma^i_{jk} \frac{dq_j}{ds} \frac{dq_k}{ds} = 0,
\]

where \( s \) is the canonical parameter of the metric \( (10) \). Next the Riemann tensor, the sectional curvatures \( K_{ij} \) in the planes \( 12, 13 \) and \( 23 \), and the scalar curvature \( K \) can be
deduced \[3\]. The latter turn out to be, in general, non-constant and read

\[
K_{ij} = \frac{z}{4} e^{-\frac{zq^2}{4}} \left\{ \left(1 + e^{2zq_i^2} - 2e^{2zq_i^2} \right) \left(f(x) + f''(x)/f(x) \right) \\
+ 2 \left(1 + e^{2zq_i^2} \right) f'(x) - 2 \left(1 - e^{2zq_i^2} - e^{2zq_i^2} \right) f''(x) \right\},
\]

\[
K_{13} = \frac{z}{4} e^{-\frac{zq^2}{4}} \left\{ \left(2 - e^{2z(q_1^2 + q_2^2)} - 2e^{2zq^2} \right) \left(f(x) + f''(x)/f(x) \right) \\
+ 2 \left(1 + e^{2zq^2} \right) f'(x) - 2 \left(1 - e^{2z(q_1^2 + q_2^2)} - e^{2zq^2} \right) f''(x) \right\},
\]

\[
K_{23} = \frac{z}{4} e^{-\frac{zq^2}{4}} \left\{ \left(2 - e^{2z(q_1^2 + q_2^2)} - 2e^{2zq^2} \right) \left(f(x) + f''(x)/f(x) \right) \\
+ 2 \left(1 + e^{2zq^2} \right) f'(x) - 2 \left(1 - e^{2z(q_1^2 + q_2^2)} \right) f''(x) \right\},
\]

\[
K(x) = z \left(6f'(x) \cosh x + \left(4f''(x) - 5f(x) - 5f'(x)/f(x) \right) \sinh x \right).
\]

Notice that although the sectional curvatures are not symmetric with respect to the coordinates \(q_i\), the scalar curvature is actually a radial function, since it does only depend on \(x = zq^2\) and fulfills \(K = 2(K_{12} + K_{13} + K_{23})\).

Therefore the coalgebra construction giving rise to the family of metrics \[10\] can be understood as the introduction of a variable curvature (controlled by the quantum deformation parameter \(z\)) on the formerly flat Euclidean space \[5, 14\]. This, in turn, means that the non-deformed limit \(z \to 0\) can then be identified with the flat contraction providing the proper 3D Euclidean space with metric \(ds^2 = \sum_{i=1}^3 dq_i^2\).

Let us present two specific instances for \(\mathcal{H} \[9\] which have been studied in \[5, 14\].

- The simplest free Hamiltonian arises by setting \(f(x) \equiv 1\). The scalar curvature yields

\[
K = -5z \sinh(zq^2),
\]

which, for any non-zero value of \(z\), determines a hyperbolic Riemannian space of non-constant negative curvature.

- When \(f(x) = e^x\) we find a very singular case as all the curvatures are constant: \(K_{ij} = z\) and \(K = 6z\). It turns out that the resulting Hamiltonian is a Stäckel system \[15\] which provides an additional constant of motion \[3\] (that cannot be derived from the underlying coalgebra symmetry):

\[
I = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} p_1^2.
\]

Since this additional integral is functionally independent with respect to the three constants of the motion given in proposition 1, the geodesic motion is maximally superintegrable, as it should be the case. This result encompasses the three classical Riemannian spaces of constant curvature, i.e., the spherical, hyperbolic and Euclidean spaces, provided that a positive, negative and zero quantum deformation parameter \(z\) is respectively considered. Notice that a fully equivalent situation arises when \(f(x) = e^{-x}\) is chosen, thus leading to sectional curvatures given by \(K_{ij} = -z\).
4 Spherical-type coordinates and Lorentzian spaces

It is also possible to obtain non-constant curved spaces of pseudo- and semi-Riemannian type (with Lorentzian and degenerate metrics) through a graded contraction (or analytic continuation) approach [5, 14]. Explicitly, let us consider the spherical-type coordinates \((r, \theta, \phi)\) defined by:

\[
\begin{align*}
\cos^2(\lambda_1 r) &= e^{-2zq^2}, \\
\tan^2(\lambda_1 r) \cos^2(\lambda_2 \theta) &= e^{2zq^2}e^{2zq^2}(e^{2zq^2} - 1), \\
\tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta) \cos^2 \phi &= e^{2zq^2}(e^{2zq^2} - 1), \\
\tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta) \sin^2 \phi &= e^{2zq^2} - 1,
\end{align*}
\]

(11)

where \(z = \lambda_2^2\) and \(\lambda_2\) is an additional parameter which can be either a real or a pure imaginary number [5]. Under this change of coordinates the metric (10) adopts a much more familiar expression:

\[
ds^2 = \frac{1}{g(\lambda_1 r) \cos(\lambda_1 r)} \left( dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \left( d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right) \right)
\]

\[
= \frac{1}{g(\lambda_1 r) \cos(\lambda_1 r)} ds_0^2.
\]

(12)

Thus we have obtained a family of metrics, parametrized by \(\lambda_1, \lambda_2\) and depending on the function \(g(\lambda_1 r) \equiv f(zq^2)\), which is just the metric \(ds_0\) of the spaces of constant curvature [16] multiplied by a global conformal factor \(1/(g(\lambda_1 r) \cos(\lambda_1 r))\). Hence in this approach the deformation parameter \(\lambda_1\) governs the curvature and \(\lambda_2\) provides the signature of the corresponding space.

Therefore, according to the pair \((\lambda_1, \lambda_2)\) (each non-zero \(\lambda_i\) can be scaled to \(\pm 1\) or \(\pm i\)) we obtain “deformed analogues” [14] of the 3D spherical (1, 1), hyperbolic (i, 1), anti-de Sitter (1, i) and de Sitter (i, i) spaces with non-constant sectional curvature. By one hand, these reduce to the proper flat Euclidean (0, 1) and Minkowskian (0, i) spaces under the contraction \(\lambda_1 \to 0\) (i.e. \(z \to 0\)):

\[
ds^2 \equiv ds_0^2 = dr^2 + \lambda_2^2 r^2 \left( d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right).
\]

On the other hand, the contraction \(\lambda_2 \to 0\) leads to “deformed” oscillating and expanding Newton–Hooke (NH) spacetimes \((\lambda_1 = 1, i)\), and whose limit \(z \to 0\) gives the flat Galilean spacetime with degenerate metric \(ds^2 \equiv ds_0^2 = dr^2\). Nevertheless, in the following we shall avoid the Newtonian cases with \(\lambda_2 = 0\) since their metric is degenerate and their direct relationship with a 3D Hamiltonian is lost. The specific metrics of these “deformed” spaces are displayed in table [1] whereas the connection and all the curvature tensors for the generic metric (12) are presented in table [2].

Now, the resulting superintegrable geodesic flow Hamiltonian (9) on the above family of curved spaces together with its three constants of motion in the spherical-type phase
Table 1: Metric of six 3D spaces of non-constant curvature expressed in spherical-type coordinates \((r, \theta, \phi)\). The values of the parameters are \(z = \lambda_1^2 \in \{\pm 1\}\) and \(\lambda_2^2 \in \{\pm 1, 0\}\).

| Space Type | Metric | Parameters | \(z = \pm 1\) | \((\lambda_1, \lambda_2)\) |
|------------|--------|------------|----------------|--------------------------|
| Deformed spherical space \(S_z^2\) | \[ds^2 = \frac{1}{g(r) \cos r} (dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2))\] | \(\lambda_1, \lambda_2 = (1, 1)\) | \(+1\) | \((1, 1)\) |
| Deformed oscillating \(\text{NH}^2_{z, r}\) spacetime | \[ds^2 = \frac{1}{g(r) \cos r} dr^2\] | \(\lambda_1, \lambda_2 = (1, 0)\) | \(+1\) | \((1, 0)\) |
| Deformed anti-de Sitter spacetime \(\text{AdS}_z^{2+1}\) | \[ds^2 = \frac{1}{g(r) \cos r} (dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2))\] | \(\lambda_1, \lambda_2 = (1, i)\) | \(+1\) | \((1, i)\) |
| Deformed hyperbolic space \(H_z^3\) | \[ds^2 = \frac{1}{g(r) \cosh r} (dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2))\] | \(\lambda_1, \lambda_2 = (i, 1)\) | \(-1\) | \((i, 1)\) |
| Deformed expanding \(\text{NH}^{2+1}_{z, z}\) spacetime | \[ds^2 = \frac{1}{g(r) \cosh r} dr^2\] | \(\lambda_1, \lambda_2 = (i, 0)\) | \(-1\) | \((i, 0)\) |
| Deformed de Sitter spacetime \(\text{dS}_z^{2+1}\) | \[ds^2 = \frac{1}{g(r) \cosh r} (dr^2 - \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2))\] | \(\lambda_1, \lambda_2 = (i, i)\) | \(-1\) | \((i, i)\) |

Space variables turn out to be

\[
H = \frac{1}{2} g(\lambda_1 r) \cos(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left( p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2 \right) \right),
\]

\[
C^{(2)} = p_\phi^2, \quad C_{(2)} = \left( \cos \phi \ p_\theta - \lambda_2 \frac{\sin \phi \ p_\phi}{\tan(\lambda_2 \theta)} \right)^2, \quad C^{(3)} = p_\theta^2 + \frac{\lambda_2^2 p_\phi^2}{\sin^2(\lambda_2 \theta)},
\]

where \(H = 2 \mathcal{H}\), \(C^{(2)} = 4C^{(2)}\), \(C_{(2)} = 4\lambda_2^2 C_{(2)}\), \(C^{(3)} = 4\lambda_2^2 C^{(3)}\) and \((p_r, p_\theta, p_\phi)\) are the conjugate momenta of \((r, \theta, \phi)\). It is worth stressing that in this latter form the constants of motion \(\{C^{(2)}, C^{(3)}, H\}\) directly induce the separability of the system:

\[
C^{(2)}(\phi, p_\phi) = p_\phi^2, \quad C^{(3)}(\theta, p_\theta) = p_\theta^2 + \frac{\lambda_2^2 C^{(2)}}{\sin^2(\lambda_2 \theta)}, \quad H(r, p_r) = \frac{1}{2} g(\lambda_1 r) \cos(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2 C^{(3)}}{\lambda_2^2 \sin^2(\lambda_1 r)} \right).
\]

Notice that the expressions for the two specific examples previously commented, that is, the non-constant curved case with \(g = 1\) and the well-known spaces of constant curvature with \(g = 1/\cos(\lambda_1 r)\) studied in \([14]\) are directly recovered from these expressions.

5 Superintegrable potentials

The results of proposition 1 allows to construct many types of superintegrable potentials on 3D curved spaces through specific choices of the Hamiltonian function \(\mathcal{H}\) which
could be momenta-dependent potentials, central ones, centrifugal terms, etc. (see [4] for the 2D case). Our aim now is, firstly, to characterize in this $sl_2(2)$ coalgebra framework the superposition of central potentials with (up to) three centrifugal terms and, secondly, to single out which would be the corresponding intrinsic KC and oscillator potentials. Hereafter we shall mainly make use of the spherical-type phase space introduced in the previous section since these variables will allow us to deal with both curved Riemannian and Lorentzian spaces simultaneously.

5.1 Central potentials with centrifugal terms

If we consider the 3D symplectic realization of $sl_2(2)$ [6], with arbitrary $b_i$'s, and we add a smooth function $\mathcal{V}(zJ_-)$ to the free Hamiltonian [3],

$$\mathcal{H} = \frac{1}{2} J_+ f(zJ_-) + \mathcal{V}(zJ_-),$$

we obtain a system formed by the superposition of a central potential with three centrifugal $b_i$-terms. By taking into account that, in terms of the spherical-type variables, $e^{-zJ_-} = \cos(\lambda_1 r)$, $\mathcal{V}(q^2) \equiv U(\lambda_1 r)$ and

$$J_+ = \cos(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left( p_{\theta}^2 + \frac{\lambda_2^2 p_{\phi}^2}{\sin^2(\lambda_2 \theta)} \right) \right)$$

$$+ \frac{\lambda_1^2 \cos(\lambda_1 r)}{\sin^2(\lambda_1 r)} \left( \frac{b_1}{\cos^2(\lambda_2 \theta)} + \frac{b_2 \lambda_2^2}{\sin^2(\lambda_2 \theta) \cos^2 \theta} + \frac{b_3 \lambda_2^2}{\sin^2(\lambda_2 \theta) \sin^2 \phi} \right),$$

the Hamiltonian (15) turns out to be

$$H = \frac{1}{2} g(\lambda_1 r) \cos(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left( p_{\theta}^2 + \frac{\lambda_2^2 p_{\phi}^2}{\sin^2(\lambda_2 \theta)} \right) \right) + U(\lambda_1 r)$$

$$+ \frac{\lambda_1^2 g(\lambda_1 r) \cos(\lambda_1 r)}{2 \sin^2(\lambda_1 r)} \left( \frac{b_1}{\cos^2(\lambda_2 \theta)} + \frac{b_2 \lambda_2^2}{\sin^2(\lambda_2 \theta) \cos^2 \theta} + \frac{b_3 \lambda_2^2}{\sin^2(\lambda_2 \theta) \sin^2 \phi} \right),$$

which is, by construction, superintegrable. We stress that, for any choice of $f$ and $U$, they share the same set of three constants of the motion [7], which now depend on $\lambda_1$ and $\lambda_2$; these are explicitly given, up to some additive and multiplicative constants, by

$$C^{(2)} = p_\phi^2 + \frac{b_2 \lambda_2^4}{\sin^2(\lambda_2 \theta)} + \frac{b_3 \lambda_2^4}{\sin^2(\lambda_2 \phi)};$$

$$C^{(2)} = \left( \cos \phi p_\theta - \frac{\lambda_2 \sin \phi p_\phi}{\tan(\lambda_2 \theta)} \right)^2 + b_1 \lambda_2^2 \tan^2(\lambda_2 \theta) \cos^2 \phi + \frac{b_2 \lambda_2^4}{\sin^2(\lambda_2 \theta) \sin^2 \phi},$$

$$C^{(3)} = p_\phi^2 + \frac{\lambda_2^2 p_\phi^2}{\sin^2(\lambda_2 \theta)} + \frac{b_1 \lambda_2^4}{\cos^2(\lambda_2 \theta)} + \frac{b_2 \lambda_2^4}{\sin^2(\lambda_2 \theta) \cos^2 \phi} + \frac{b_3 \lambda_2^4}{\sin^2(\lambda_2 \theta) \sin^2 \phi}. \quad (17)$$
Hence the constants of motion \( C^{(2)} \) and \( C^{(3)} \) and the Hamiltonian can be written as:

\[
C^{(2)}(\phi, p_\phi) = p_\phi^2 + \frac{b_2 \lambda_2^2}{\cos^2 \phi} + \frac{b_3 \lambda_2^2}{\sin^2 \phi},
\]

\[
C^{(3)}(\theta, p_\theta) = p_\theta^2 + \frac{b_1 \lambda_2^2}{\cos^2(\lambda_2 \theta)} + \frac{\lambda_2 \lambda_4^2}{\sin^2(\lambda_2 \theta)} C^{(2)},
\]

\[
H(r, p_r) = \frac{1}{2} g(\lambda_1 r) \cos(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2 C^{(3)}}{\lambda_2^3 \sin^2(\lambda_1 r)} \right) + U(\lambda_1 r). \tag{18}
\]

Therefore, similarly to the free motion, the Hamiltonian (16) is separable and reduced to a 1D radial system.

### 5.2 Intrinsic Kepler–Coulomb and oscillator potentials

In order to define the “intrinsic” KC and oscillator potentials on the 3D curved Riemannian and Lorentzian spaces with metric (12) given in table 1 we shall apply the approach introduced in [7, 8, 9, 10] for 3D spherically symmetric spaces. This requires to transform the metric (12) by introducing

\[
\Theta := \lambda_2 \theta, \quad R := \lambda_1 r, \quad h(R) := \cos(\lambda_1 r) g(\lambda_1 r),
\]

which yields

\[
ds^2 = \frac{1}{\lambda_1^2 h(R)} \left( dR^2 + \sin^2 R \left( d\Theta^2 + \sin^2 \Theta d\phi^2 \right) \right).
\]

Next the radial symmetric Green function \( U(R) \) on the curved space (up to multiplicative and additive constants) is defined as the positive non-constant solution to the equation

\[
\Delta_{LB} U(R) = \frac{\lambda_1^2 h(R)^{3/2}}{\sin^2 R} \left( \frac{\sin^2 R}{\sqrt{h(R)}} \frac{dU(R)}{dR} \right) = 0,
\]

where \( \Delta_{LB} \) is the (intrinsic) Laplace–Beltrami operator in the above coordinates. Then

\[
U(\lambda_1 r) \equiv U(R) = \int^R \sqrt{\frac{h(R')}{\sin^2 R'}} dR'. \tag{19}
\]

And the intrinsic KC and oscillator potentials are defined by

\[
U_{KC}(\lambda_1 r) := \alpha U(\lambda_1 r), \quad U_{O}(\lambda_1 r) := \beta \frac{U^2(\lambda_1 r)}{U(\lambda_1 r)}, \tag{20}
\]

where \( \alpha \) and \( \beta \) are real constants.

In what follows we illustrate these results through some examples according to some particular choices of the function \( g \).
5.2.1 Constant curvature: \( g(\lambda_1 r) = 1 / \cos(\lambda_1 r) = \exp(zJ_-) \)

To start with let us consider the very singular case of constant curvature with \( h(R) = 1 \) so that the Green function turns to be

\[
U(R) = -\frac{1}{\tan R}.
\]

Consequently we recover the well-known KC and oscillator potentials [16, 17, 18, 19, 20, 21, 22, 23] for the spherical and (anti-)de Sitter spaces with \( \lambda_1 \) real, and for the hyperbolic and de Sitter spaces with \( \lambda_1 \) imaginary:

\[
U_{\text{KC}}(\lambda_1 r) = -\frac{\alpha \lambda_1}{\tan(\lambda_1 r)}, \quad U_{\text{O}}(\lambda_1 r) = \beta \frac{\tan^2(\lambda_1 r)}{\lambda_1^2}.
\]

The scalar curvature is \( K = 6\lambda_1^2 \). Obviously, the limit \( \lambda_1 \to 0 \) is well defined and leads to flat Euclidean/Minskowskian potentials: \(-\alpha/r \) and \( \beta r^2 \). In terms of the \( sl_2(2) \) generators the resulting Hamiltonians are expressed by

\[
H_{\text{KC}} = \frac{1}{2} J_+ e^{zJ_-} - \alpha \sqrt{\frac{z e^{-zJ_-}}{\sinh(zJ_-)}},
\]

\[
H_{\text{O}} = \frac{1}{2} J_+ e^{zJ_-} + \beta \frac{\sinh(zJ_-)}{z} e^{zJ_-},
\]

which were already given in [6] for the 2D case. Recall that the three centrifugal terms are directly obtained from \( J_+ \) by considering a symplectic realization with non-vanishing \( b_i \)'s. In this way the generalized KC Hamiltonian [12, 16, 24, 25, 26] and the curved Smorodinsky–Winternitz system [16, 18, 20, 26, 27, 28, 29, 30, 31, 32, 33, 34] are recovered.

5.2.2 \( g(\lambda_1 r) = 1 \)

The case \( g \equiv 1 \), which was commented in the previous sections, leads to \( h(R) = \cos R \). Now the Green function involves an elliptic integral of the second kind, \( E(x|m) \):

\[
U(\lambda_1 r) = -\frac{\lambda_1}{\tan(\lambda_1 r)} \sqrt{\cos(\lambda_1 r)} - \lambda_1 E\left(\frac{1}{2} \lambda_1 r | 2\right).
\]

Note that a multiplicative constant \( \lambda_1 \) has been introduced in order to ensure a well defined non-deformed limit (flat contraction): \( \lim_{\lambda_1 \to 0} U(\lambda_1 r) = -1/r \).

This can be considered as the “simplest” case from the \( sl_2(2) \) viewpoint, since the kinetic term is just \( \frac{1}{2} J_+ \). In spite of this fact, the scalar curvature (which can be obtained from the general expression given in table [2]) reads

\[
K = -\frac{5}{2} \lambda_1^2 \sin(\lambda_1 r) \tan(\lambda_1 r),
\]

and the associated KC and oscillator potentials are obtained from [20]. In this respect, we remark that these potentials turn out to be much more complicated than the ones coming from the Ansatz proposed in [6].
5.2.3 \( g(\lambda_1 r) = (\cos(\lambda_1 r))^{4k-1} = \exp(-z(4k - 1)J_-) \) for a real constant \( k \neq 1 \)

The two aforementioned examples can be included in this more general class of systems as they are reproduced for \( k = 0 \) and \( k = 1/4 \), respectively. Starting from \( h(R) = (\cos R)^{4k} \) it is found that the resulting Green function depend on a hypergeometric function \( {}_2F_1(a, b, c, x) \) that can also be expressed by means of an incomplete beta function \( B(x, a, b) \) in the form

\[
U(\lambda_1 r) = \frac{(\sin(\lambda_1 r))^{2(k-1)}}{2(k-1)} {}_2F_1 \left( 1 - k, \frac{1}{2} - k, 2 - k, \frac{1}{\sin^2(\lambda_1 r)} \right)
\]

\[
= -\frac{1}{2} B \left( \frac{1}{\sin^2(\lambda_1 r)}, 1 - k, \frac{1}{2} + k \right)
\]

provided that we have dropped a multiplicative constant \( i(-1)^{-k} \). The scalar curvature of the underlying spaces is

\[
K = \lambda_1^2 (\cos(\lambda_1 r))^{4k-2} \left( 3 - 4k(k + 4) + (3 + 4k(k - 2) \cos(2\lambda_1 r) \right).
\]

5.2.4 \( g(\lambda_1 r) = \cos^3(\lambda_1 r) = \exp(-3zJ_-) \)

To end with, we study the forbidden value \( k = 1 \) in the previous example, which corresponds to take \( h(R) = \cos^4 R \). Then we obtain the following closed expression for the Green function

\[
U(\lambda_1 r) = -\lambda_1^2 r - \frac{\lambda_1}{\tan(\lambda_1 r)},
\]

where we have introduced a multiplicative constant \( \lambda_1 \) such that \( \lim_{\lambda_1 \to 0} U(\lambda_1 r) = -1/r \). This yields the corresponding KC and oscillator potentials

\[
U_{KC}(\lambda_1 r) = -\alpha \left( \lambda_1^2 r + \frac{\lambda_1}{\tan(\lambda_1 r)} \right), \quad U_O(\lambda_1 r) = \beta \frac{\tan^2(\lambda_1 r)}{\lambda_1^2 (1 + \lambda_1 r \tan(\lambda_1 r))^2},
\]

which are worth to be compared with the corresponding expressions of the constant curvature case. Finally, the scalar curvature for this space is:

\[
K = -\lambda_1^2 \cos^2(\lambda_1 r)(17 + \cos(2\lambda_1 r)).
\]

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References

[1] A. Ballesteros and O. Ragnisco, J. Phys. A 31, 3791 (1998).
[2] A. Ballesteros, F.J. Herranz, F. Musso, and O. Ragnisco, in: Superintegrability in Classical and Quantum Systems, CRM Proc. and Lecture Notes vol. 37, ed. P Tempesta et al., (Providence, R.I., AMS, 2004), p. 1 [arXiv:math-ph/0412067].
[3] A. Ballesteros and F.J. Herranz, J. Phys. A 32, 8851 (1999).
[4] A. Ballesteros and F.J. Herranz, J. Phys. A 40, F51 (2007).
[5] A. Ballesteros, F.J. Herranz, and O. Ragnisco, Phys. Lett. B 610, 107 (2005).
[6] A. Ballesteros, F.J. Herranz, and O. Ragnisco, J. Phys. A 38, 7129 (2005).
[7] P. Li and L.F. Tam, Amer. J. Math. 109, 1129 (1987).
[8] P. Li and L.F. Tam, J. Differential Geom. 41, 277 (1995).
[9] A. Enciso and D. Peralta-Salas, J. Geom. Phys. 57, 1679 (2007).
[10] A. Ballesteros, A. Enciso, F.J. Herranz, and O. Ragnisco, Class. Quantum Grav. 25, 165005 (2008).
[11] A. Ballesteros, A. Enciso, F.J. Herranz, and O. Ragnisco, [arXiv:0812.1882].
[12] N.W. Evans, Phys. Rev. A 41, 5666 (1990).
[13] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. 38, 5416 (1997).
[14] A. Ballesteros, F.J. Herranz, O. Ragnisco, and M. Santander, Int. J. Theor. Phys. 47, 649 (2008).
[15] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie algebras (Birkhäuser, Berlin, 1990).
[16] A. Ballesteros and Herranz, SIGMA 2, 010 (2006).
[17] P.W. Higgs, J. Phys. A 12, 309 (1979).
[18] M.F. Rañada and M. Santander, J. Math. Phys. 40, 5026 (1999).
[19] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. 41, 2629 (2000).
[20] E.G. Kalnins, J.M. Kress, G.S. Pogosyan, and W. Miller, J. Phys. A 34, 4705 (2001).
[21] A. Nersessian and G.S. Pogosyan, Phys. Rev. A 63, 020103 (2001).
[22] J.F. Cariñena, M.F. Rañada, and M. Santander J. Math. Phys. 46, 052702 (2005).
[23] A.V. Shchepeletilov, J. Math. Phys. 46, 114101 (2005).
[24] M.A. Rodríguez and P. Winternitz, J. Math. Phys. 43, 1309 (2002).
[25] E.G. Kalnins, G.C. Williams, W. Miller, and G.S. Pogosyan, J. Phys. A 35, 4755 (2002).
[26] F.J. Herranz and A. Ballesteros, Phys. At. Nuclei 71, 905 (2008).
[27] J. Fris, V. Mandroslov, Y.A. Smorodinsky, M. Uhlir, and P. Winternitz, Phys. Lett. 16, 354 (1965).
[28] N.W. Evans, Phys. Lett. A 147, 483 (1990).
[29] N.W. Evans, J. Math. Phys. 32, 3369 (1991).
[30] C. Grosche, G.S. Pogosyan, and A.N. Sissakian, Fortschr. Phys. 43, 453 (1995).
[31] C. Grosche, G.S. Pogosyan, and A.N. Sissakian, Fortschr. Phys. 43, 523 (1995).
[32] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. 38, 5416 (1997).
[33] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Phys. A 33, 6791 (2000).
[34] A. Ballesteros, F.J. Herranz, M. Santander, and T. Sanz-Gil, J. Phys. A 36, L93 (2003).
Table 2: Non-zero components of the connection, Riemann and Ricci tensors, sectional and scalar curvatures of the metric \((\ref{eqn:metric})\) written in spherical-type coordinates \((r, \theta, \phi)\) for the 3D spaces with \(sl_2(2)\) symmetry. We denote \(g' = \frac{dg(y)}{dy}\) and \(g'' = \frac{d^2g(y)}{dy^2}\) with \(y = \lambda_1 r\).

| Component | Formula |
|-----------|---------|
| Connection | \(\Gamma^r_{rr} = \frac{\lambda_1}{2} \left( \tan(\lambda_1 r) - \frac{g'}{g} \right)\) \(\Gamma^\theta_{r\phi} = -\frac{\sin(2\lambda_2 \theta)}{2\lambda_2}\) \(\Gamma^\phi_{\phi\theta} = \frac{\lambda_2}{\tan(\lambda_2 \theta)}\) |
| | \(\Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = \lambda_1 \left( \frac{1 + \cos^2(\lambda_1 r)}{\sin(2\lambda_1 r)} - \frac{g'}{2g} \right)\) \(\Gamma^r_{\theta\theta} = -\frac{\lambda_2^2 \sin^2(\lambda_1 r)}{\lambda_1^2} \Gamma^\phi_{\phi r}\) |
| | \(\Gamma^r_{\phi\phi} = -\frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \sin^2(\lambda_2 \theta) \Gamma^\phi_{\phi r}\) |
| Riemann tensor | \(R^\theta_{\theta r} = \frac{1}{2} \lambda_1^2 \left( \frac{1}{\tan(\lambda_1 r)} \frac{g'}{g} + \frac{g''}{g} - \left( \frac{g'}{g} \right)^2 - \tan^2(\lambda_1 r) \right)\) \(R^r_{r\phi r} = \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \sin^2(\lambda_2 \theta) R^\phi_{\phi r}\) |
| | \(R^\phi_{\phi \phi} = \lambda_2^2 \left( 1 - \sin^2(\lambda_1 r) \left( \frac{1 + \cos^2(\lambda_1 r)}{\sin(2\lambda_1 r)} - \frac{g'}{2g} \right)^2 \right)\) \(R^\phi_{\phi \theta} = \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} R^\phi_{\phi \phi}\) |
| Ricci tensor | \(R_{rr} = \lambda_1^2 \left( \frac{1}{\tan(\lambda_1 r)} \frac{g'}{g} + \frac{g''}{g} - \left( \frac{g'}{g} \right)^2 - \tan^2(\lambda_1 r) \right)\) \(R_{\phi \phi} = \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} R_{\theta \theta}\) |
| | \(R_{\theta \theta} = \lambda_2^2 \sin^2(\lambda_1 r) \left( \frac{1 + 2 \cos^2(\lambda_1 r)}{\sin(2\lambda_1 r)} \frac{g'}{g} + \frac{g''}{2g} - 3 \frac{1}{4} \left( \frac{g'}{g} \right)^2 - 3 \frac{3}{4} \tan^2(\lambda_1 r) \right)\) |
| Sectional curvatures | \(K_{\phi r} = K_{r \phi} = \frac{1}{2} \lambda_1^2 \cos(\lambda_1 r) \left( \frac{1}{\tan(\lambda_1 r)} g' + g'' - \frac{g'}{g} \right) + \tan^2(\lambda_1 r) g\) |
| | \(K_{\theta \phi} = \lambda_1^2 \cos(\lambda_1 r) g \left( \frac{1}{\sin^2(\lambda_1 r)} - \left( \frac{1 + \cos^2(\lambda_1 r)}{\sin(2\lambda_1 r)} - \frac{g'}{2g} \right)^2 \right)\) |
| Scalar curvature | \(K = 2\lambda_1^2 \cos(\lambda_1 r) \left( \frac{1 + 3 \cos^2(\lambda_1 r)}{\sin(2\lambda_1 r)} g' + g'' - \frac{5}{4} \frac{g'}{g} - \frac{5}{4} \frac{g''}{g} - \frac{5}{4} \tan^2(\lambda_1 r) g \right)\) |