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Anomalous magnetohydrodynamics with constant anisotropic electric conductivities

Ren-jie Wang\textsuperscript{a}, Patrick Copinger\textsuperscript{b}, Shi Pu\textsuperscript{a}

\textsuperscript{a}Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China
\textsuperscript{b}Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Abstract

We study anomalous magnetohydrodynamics in a longitudinal boost invariant Bjorken flow with constant anisotropic electric conductivities as outlined in Ref. [1]. For simplicity, we consider a neutral fluid and a force-free magnetic field in the transverse direction. We derived analytic solutions of the electromagnetic fields in the laboratory frame, the chiral density, and the energy density as functions of proper time.

Keywords: Magnetohydrodynamics, chiral magnetic effect, anisotropic electric conductivities

1. Introduction

Recently, several novel quantum transport phenomena related to strong magnetic fields have been extensively studied. Some examples include the chiral magnetic effect and chiral separation effect [2, 3, 4]. Similarly, electric fields can also induce the chiral separation [5, 6, 7, 8] and chiral Hall separation effects [8]. There are also other non-linear chiral transport effects, such as are discussed in Refs. [9, 10, 11, 12] and the connection to Schwinger pair production [13] (also see the reference therein).

Anomalous magnetohydrodynamics (anomalous MHD), which is the relativistic magnetohydrodynamics in the presence of the chiral magnetic effect and chiral anomaly, is a framework in which to study the aforementioned chiral transport phenomena. In Refs. [14, 15] we derived analytic solutions for ideal MHD with longitudinal boost invariance and a transverse magnetic field. In the latter part of Ref. [14], we considered magnetization effects. Following this framework, the studies have been extended to 2+1 dimensional MHD with Bjorken flow [16, 17] and Gubser flow [18]. The numerical simulation of ideal MHD can be found in Ref. [19].

Very recently, we obtained analytic solutions for anomalous MHD with Bjorken flow [1]. In that work, we only consider anomalous MHD with a constant electric conductivity, $\sigma$. In a strong magnetic field, the electric conducting flow will be $j^\mu = \sigma^{\mu\nu}E^\nu$, where $\sigma^{\mu\nu}$ is the anisotropic electric conductivity tensor. The tensor $\sigma^{\mu\nu}$ includes the classical Hall conductivity, $\sigma_H$, and the electric conductivities parallel and perpendicular to the magnetic fields, i.e., $\sigma_\parallel$ and $\sigma_\perp$, respectively. $\sigma_\parallel$ and $\sigma_\perp$ have been computed by a
sum over all Landau levels \[20\]. Since the magnetic fields are extremely strong in relativistic heavy ion collisions, we also need to consider anomalous MHD with anisotropic electric conductivities.

In Ref. \[20\], the authors have found that both \(\sigma_T\) and \(\sigma_L\) depend on the temperature and magnetic field strength. For simplicity, in the present work, we assume that \(\sigma_H, \sigma_T,\) and \(\sigma_L\) are all constants. We will present the results for a temperature and magnetic field dependent \(\sigma^{ij}\) somewhere else.

The structure of this work is as follows. In Sec. 2, we derive an analytic solution for anomalous MHD with constant anisotropic electric conductivities. We then summarize in Sec. 3. Throughout this work, we will use the metric \(g_{\mu\nu} = \text{diag}\{+,-,-,-\}\), and choose Levi-Civita tensor satisfying \(\epsilon^{0123} = -6_{0123} = +1\).

2. Anomalous magnetohydrodynamics with constant anisotropic electric conductivities

The main equations for MHD are the energy-momentum and currents conservation equations coupled with Maxwell’s equations. (See e.g., Ref. \[14, 18, 16, 17, 21\] and references therein for details). Here, we will neglect other dissipative effects, such as the bulk viscous pressure, shear viscous tensor, and heat conducting flow. We follow the framework and assumptions as used in our previous work \[1\]. The energy-momentum conservation equation without viscous effects reads

\[ \partial_\mu T^{\mu\nu} = 0, \]

where \(T^{\mu\nu}\) is the full energy momentum tensor and can be decomposed as,

\[ T^{\mu\nu} = (\varepsilon + p + E^2 + B^2)u^\mu u^\nu - (p + \frac{1}{2}E^2 + \frac{1}{2}B^2)\delta^{\mu\nu} - E^\mu E^\nu - B^\mu \dot{B}^\nu - u^\mu \varepsilon e^{\lambda\mu\nu\beta} E_\lambda B_\beta - u^\nu \varepsilon e^{\lambda\mu\nu\beta} E_\lambda B_\beta , \]

where \(\varepsilon\) and \(p\) are energy density and pressure, respectively. Here, we have introduced the four-vector form of electromagnetic fields in a co-moving frame of a fluid cell,

\[ E^\mu = F^{\mu\nu}u_\nu, \quad B^\mu = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta}, \quad E'^\mu E_\mu = -E^2, \quad B'^\mu B_\mu = -B^2. \]

The conservation equations for the currents are

\[ \partial_\mu j_\mu^e = 0, \quad \partial_\mu j_5^e = -e^2 CE \cdot B, \]

where \(j_\mu^e\) and \(j_5^e\) are the electric and chiral (axial) current, respectively, and \(C = 1/(2\pi^2)\) refers to the chiral anomaly. These currents can be decomposed as

\[ j_\mu^e = n_e u^\mu + \sigma^{e\mu} E^\nu + \xi B^\nu, \quad j_5^e = n_5 u^\mu + \sigma_5 E^\mu + \xi_5 B^\mu, \]

where \(n_e\) and \(n_5\) are the electric and chiral charge densities, respectively, and \(\sigma_5\) is the chiral electric conductivity \[5, 8, 8\]. The coefficients \(\xi = e\mu_5\) and \(\xi_5 = e\mu_5\) are associated with the chiral magnetic and separation effects \[8, 22, 23\], with \(\mu_e\) and \(\mu_5\) being the electric and chiral chemical potential, respectively.

The anisotropic electric conductivity tensor can be written as \[20\],

\[ \sigma^{e\mu\nu} = \sigma_H \varepsilon^{\mu\nu\beta\alpha} u_\alpha \frac{B_\beta}{B} - \sigma_T \frac{B^\mu B^\nu}{B^2} + \sigma_L \left( \varepsilon^{\mu\nu\beta\alpha} \frac{B_\alpha}{B} + \frac{B^\mu B^\nu}{B^2} \right), \]

where \(\sigma_H, \sigma_T,\) and \(\sigma_L\) denote the classical Hall, longitudinal, and transverse conductivity, respectively. We use the covariant form for Maxwell’s equations,

\[ \partial_\mu F^{\mu\nu} = j_\nu, \quad \partial_\mu (\varepsilon^{\mu\nu\beta\alpha} F_{\alpha\beta}) = 0. \]

We also choose the equation of state in a hot fireball limit,

\[ \varepsilon = c_s^2 p, \quad n_e = a_{\mu_5} T^2, \quad n_5 = a_{\mu_5} T^2, \]

where \(a_{\mu_5} \approx 0.25\).
where $a$ is again a dimensionless constant and $T$ is the temperature. For an ideal fluid, we have $a = 1/3$ \cite{22,24}.

In our previous work \cite{1}, we derived a self-consistent analytic solution for anomalous MHD in a Bjorken flow. For simplicity, we assume the fluid is charge neutral, i.e., $\mu_e = n_e = 0$. This is applicable since the chiral electric conductivity, $\sigma_5$, is proportional to $\mu_e$ as $\sigma_5 \propto \mu_e \mu_5$ in the small $\mu_e$ and $\mu_5$ limits \cite{3,4,5}. Therefore, in this case, $\sigma_5 \approx 0$. Similarly, $\xi_5 \propto \mu_e$ also vanishes. We also assume that the system is longitudinally boost invariant and that the electromagnetic fields in the longitudinal direction are negligible. The fluid velocity in a Bjorken flow reads,

$$u^\rho = (\cos \eta, 0, 0, \sinh \eta) = \gamma (1, 0, 0, z/t),$$

with $\gamma = \sqrt{1 - z^2}$ and $\eta = \frac{1}{2} \ln [(t + z)/(t - z)]$ being the proper time and the space-time rapidity, respectively.

Usually, the electromagnetic fields can accelerate the fluid velocity through the Lorentz force. In our case, however, we have found a special field configuration fields that keeps the fluid force-free,

$$E^\rho = (0, 0, \chi E(\tau), 0), \quad B^\rho = (0, 0, B(\tau), 0),$$

where $\chi = \pm 1$, and without loss of generality, we only consider fields in the $\tau$ direction. From Eq. (1), we have checked that $(g_{\mu\nu} - u_\mu u_\nu) (\partial_\tau T^{\mu \nu}_{\text{Matter}} - J^\mu J^\nu) = 0$, where $T^{\mu \nu}_{\text{Matter}} = T^{\mu \nu}_{\text{Elec}} = 0$, is automatically satisfied according our assumptions, i.e., the electromagnetic fields will not accelerate the fluid.

Our main equations are $u_\nu \partial_\nu T^{\mu \nu} = 0$ from Eq. (1), coupled with Maxwell’s equations, (2), and their corresponding constitution equations, (3, 5, 6, 10). After some calculations, those coupled equations reduce to

$$\frac{d}{d\tau} E + \frac{1}{\tau} E + \sigma_\parallel E + \chi \xi B = 0, \quad \frac{d}{d\tau} B + \frac{B}{\tau} = 0,$$

$$\frac{d}{d\tau} \epsilon + (\epsilon + \rho) \frac{1}{\tau} - \sigma_\parallel E^2 - \chi \xi EB = 0, \quad \frac{d}{d\tau} n_5 + \frac{n_5}{\tau} = e^2 C_\chi EB.$$

We notice that Eqs. (12) are similar to the Eqs. (23, 24, 25, 27) in Ref. \cite{1} by replacing $\sigma$ with $\sigma_0$, a reasonable replacement: For the electromagnetic field configuration in Eq. (10), the electric field is parallel to the magnetic field. Therefore, from Eq. (6), only $\sigma_0$ will contribute to our final result.

Using the non-conserved charges method \cite{25,26}, we obtain analytic solutions for Eqs. (12) with the EoS, Eq. (8), in a small $\mu_5 / T$ limit,

$$E(\tau) = E_0 \left( \frac{T_0}{\tau} \right)^{1/2} \left\{ e^{-\sigma_0(\tau - \tau_0)} - a_1 e^{-\sigma_0 \tau} \left[ E_{1-2\xi / \tau_0}(-\sigma_0 \tau) - \left( \frac{\tau}{\tau_0} \right)^{1/2} E_{1-2\xi / \tau_0}(-\sigma_0 \tau) \right] + O(a^2) \right\},$$

$$n_5(\tau) = n_{5,0} \left( \frac{T_0}{\tau} \right)^{1/2} \left\{ 1 + a_2 e^{\sigma_0 \tau} \left[ E_{1-2\xi / \tau_0}(\sigma_0 \tau) - E_{1-2\xi / \tau_0}(\sigma_0 \tau) \right] + O(a^2) \right\},$$

$$\epsilon(\tau) = \epsilon_0 \left( \frac{T_0}{\tau} \right)^{1/2} \left\{ 1 + \sigma_\parallel E_0 \left[ e^{2\sigma_0 \tau} \left[ E_{0+2\xi / \tau_0}(2\sigma_0 \tau) - \left( \frac{\tau}{\tau_0} \right)^{1/2} E_{0+2\xi / \tau_0}(2\sigma_0 \tau) \right] \right] \right\} + a_3$$

$$\times \left[ e^{2\sigma_0 \tau} \left[ E_{0-2\xi / \tau_0}(\sigma_0 \tau) - \tau \left( \frac{\tau}{\tau_0} \right)^{1/2} E_{0-2\xi / \tau_0}(\sigma_0 \tau) \right] \right] + O(a^2, a E_0 / \epsilon_0),$$

where $E_\alpha(\xi) \equiv \int_0^\infty dt e^{-t} e^{-\tau t}$ is the generated exponential integral. The coefficients $a_1$, $a_2$ and $a_3$ are dimensionless constants determined by the initial conditions,

$$a_1 = e C_\chi B_0 n_{5,0} E_0, \quad a_2 = e^2 C_\chi E_0 B_0, \quad a_3 = e C_\chi n_{5,0} E_0 B_0 / \epsilon_0 T_0 \tau_0,$$

where the lower index 0 denotes the quantity at the initial proper time, $\tau_0$.

The electromagnetic fields in the laboratory frame are given by,

$$E_{\text{lab}} = (\gamma \gamma^2 B(\tau), \chi \gamma E(\tau), 0), \quad B_{\text{lab}} = (-\gamma \gamma^2 \chi E(\tau), \gamma B(\tau), 0),$$

where $B(\tau) = B_0(\tau_0 / \tau)$, $\chi = \pm 1$, and $E(\tau)$ is given by Eq. (13). We find that $B^+_{\text{lab}} \propto 1/\tau$ and $B^-_{\text{lab}} \propto \exp(-\sigma_0 \tau) / \tau$, i.e., the magnetic field decays much slower in this case than in the vacuum \cite{3}.
3. Summary

In this work, we have studied anomalous MHD with anisotropic electric conductivities. The fluid expands along the longitudinal direction with the Bjorken boost invariant. For simplicity, we consider a charge neutral fluid and a particular force-free electromagnetic field configuration, i.e., one that does not accelerate the fluid velocity. We have derived an analytic solution for anomalous MHD in our cases in a small neutral fluid and a particular force-free electromagnetic field configuration, i.e., one that does not accelerate. In the future, we plan to extend the discussion to cases with temperature and magnetic field dependent anisotropic electric conductivities.

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