THE RELATIVE POWER AND ITS INVARIANCE

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Abstract. The relative power of actions in a Cauchy body suffering mutations due to defect evolution is introduced. It is shown that its invariance under the action of the Euclidean group over the ambient space and the material space allows one to obtain (i) the balance of standard and configurational actions and (ii) the identification of configurational ingredients from a unique source.

1.

Actions driving the evolution of defects in materials are commonly called configurational because they are associated with processes mutating the material structure of a body in a way that can be represented through alterations of the reference macroscopic configuration. It seems that the term ‘configurational’ should be attributed to Nabarro (see remarks in [7]). In a pioneering paper [8], Eshelby observed that, in simple elastic bodies undergoing large deformations, the equations obtained by means of horizontal variations of the bulk elastic energy – they are the variations generated by altering the reference place by using appropriate diffeomorphisms – are associated with the equilibrium of defects with non-vanishing volume. The irreversible evolution of these defects is also described through the introduction of peculiar driving forces. The analysis of the evolution of point, line and surface defects (vacancies, dislocations, interfaces, cracks) and the justification of the relevant balances of the actions governing the equilibrium and the evolution have been discussed largely in the subsequent literature. Various points of view generated a burning debate about the nature of the balances of configurational actions, governing equilibrium and possible evolution of defects in simple and complex bodies. On one side it has been claimed that the local configurational balance is just the projection through the inverse motion of the Cauchy balance in terms of Piola-Kirchhoff stress, in absence of dissipative driving forces [22, 20, 21]. On another side the fundamental independent nature of the balance of configurational forces has been supported: such a balance has been postulated a priori in an abstract way, then its (at the beginning) unspecified ingredients (Hamilton-Eshelby stress and configurational bulk forces) have been identified in terms of standard actions by means of an invariance requirement and the second law of thermodynamics, the use of which presumes the assignment of the free energy [13, 11, 12].

Further contributions to the debate are manifold[1].

Essential differences between the balance of forces involving the first Piola-Kirchhoff stress and the balance of configurational actions have been pointed out

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1Examples are [16, 23, 20, 27, 25, 29, 33, 2, 6, 11, 20, 31, 32] and references therein. Of course, I do not claim completeness of this list which collects possible choices.
indirectly by the results in an earlier paper, namely \cite{9} (see also further remarks in \cite{10}), in the case of elastic bodies undergoing large deformations. In fact, results in calculus of variations help once more in addressing the discussion. Consider only the balances in the bulk just for the sake of simplicity. For smooth minimizers it is obviously true that, in absence of evolution governed by a driving force, the configurational balance equations can be obtained by pulling back in the reference place by means of the inverse motion the relevant balances in terms of standard Piola-Kirchhoff stress. Different is the case of irregular minimizers. They are common because existence theorems place minimizers of the elastic energy in Sobolev spaces. Sobolev maps do not admit always tangential derivatives. For this reason one cannot compute the balance of forces in terms of Piola-Kirchhoff stress from the first variation of the energy functional. The so-called horizontal variations are admitted: they are variations which alter the reference place and lead to the balance of configurational forces (at least the one not accounting for driving force). Similar variations are also admissible on the actual place of the body: under appropriate bounds for the derivatives of the energy (or better of its policonvex representative) one finds the weak form of the balance of forces in terms of Cauchy stress and proves also that such a stress is locally $L^1$. Thus, for irregular minimizers the technique based on the inverse motion mentioned above cannot be applied. The technique based on horizontal variations has been later applied to various cases (see, e.g., \cite{22}).

In the ensuing sections, by restricting the attention to simple bodies, I present a procedure based on $\mathbb{R}^3 \ltimes SO(3)$ invariance of a certain power that I call the relative power. It allows one to obtain (i) the balance of both standard and configurational actions and (ii) the identification of configurational ingredients from a unique source. The idea is based on the definition of two virtual velocity fields $v$ and $w$ acting one over the ambient space and the other over the space in which the material configuration of the body is placed. The latter field is then pushed forward on the ambient space, along the motion and the power performed by the standard actions in the difference between $v$ and the image of $w$ is evaluated. Such a power is supplemented by a power of the energy flux generated by the possible disarrangements and permutation of defects that are determined by the action of $w$ in a material space, a space endowed with its own energy. The sum of all these contributions is exactly the functional that I call the relative power. Its definition is not exotic and is not different in essence from the one of standard power. It reduces to the standard expression of the power when the reference place is fixed once and for all as it happens in standard continuum mechanics.

Neither surface and line defects, nor material complexity are accounted for. They are matter of future work. Here the attention is focused only on the basic skeletal idea.

Peculiarities of the ensuing developments are summarized below.

1. Use of the inverse motion is not required.
2. No integral configurational balance is postulated.
3. The integral balance of configurational forces and a configurational balance of torques are derived and correspond to Killing fields of the metric in the material space.
4. The existence of a free energy density is postulated but the list of state variables entering its constitutive structure is not specified to a certain extent.
(5) No use is made of the mechanical dissipation inequality to identify the purely mechanical part of configurational forces. In fact, the identification follows directly from the procedure.

(6) The procedure does not require a variational structure and holds in dissipative setting.

(7) A balance arising by the requirement of invariance of the relative power under changes in observers corresponds in purely conservative case to an integral version of Noether theorem.

Differences and analogies with the two different points of view analyzed in [21] and [11] (and developed in subsequent papers) are further discussed in the last section.

2.

The description of the standard kinematics of simple continuous deformable bodies is so well known that it barely needs to be retold. The setting is the classical space-time. A fit region $B$ (more simply, an open, connected set with Lipschitz boundary) of the standard ambient space $\mathbb{R}^3$ receives a body in its reference place. Each ensuing configuration is reached in an isomorphic copy of $\mathbb{R}^3$, indicated by $\hat{\mathbb{R}}^3$, by means of a transplacement, an orientation preserving diffeomorphism $x \mapsto y : y(x) \in \hat{\mathbb{R}}^3$. The set $B_a := y(B)$ is then the actual configuration (placement) of the body. The spatial derivative of $x \mapsto y$ is indicated by $F := Dy(x) \in \text{Hom}(T_x B, T_y(x)\hat{B}_a)$. The positivity of the determinant of $F$ at each $x$ from $B$, i.e. $\det F > 0$, is implied by the assumption that the generic transplacement be orientation preserving. The additional requirement

$$\int_B \hat{f}(x, y(x)) \det Dy(x) \, dx \leq \int_{\hat{\mathbb{R}}^3} \sup_{x \in B} \hat{f}(x, z) \, dz$$

for all $\hat{f} \in C_0^\infty(\hat{B} \times \hat{\mathbb{R}}^3)$ is a global one-to-one condition allowing frictionless self-contact of the boundary while still preventing self-penetration (see [10]).

In representing motions, time come into play and one has

$$(x, t) \mapsto y := y(x, t) \in \hat{\mathbb{R}}^3, \quad x \in B, \quad t \in [0, \bar{t}],$$

with a presumption of sufficient smoothness in time, so that the velocity field is defined by

$$(x, t) \mapsto \dot{y} = \frac{d}{dt} y(x, t) \in \hat{\mathbb{R}}^3,$$

in the reference configuration.

Every subset $b$ from $B$ with non-vanishing ‘volume’ measure and the same regularity of $B$ itself is called a part. The set $\mathfrak{P}(B)$ of all parts of $B$ is an algebra with respect to the operations of meet and join (see [4]).

Virtual velocity fields are defined over the ambient space and the reference places:

$$x \in B, \quad t \in [0, \bar{t}], \quad (x, t) \mapsto v := v(x, t) \in \hat{\mathbb{R}}^3, \quad (x, t) \mapsto w := w(x, t) \in \mathbb{R}^3,$$

They are assumed to be differentiable in space at every instant. The symbols $V_v$ and $V_w$ denote the functional spaces containing them. Elements from $V_v$ and $V_w$ can be considered as virtual velocity fields over the body.

In the previous picture, the generic material element is collapsed just in a point which is its sole morphological descriptor. I use to call Cauchy bodies those bodies for which the minimalist approach summarized above is sufficient to represent the
main essential peculiarities of their morphology, the representation of actions is then conjugated in terms of power. Different is the case of complex bodies for which descriptors of the material substructure selected in a differentiable manifold are included in the representation of the morphology of the generic material element.

3.

An observer is a representation of all geometrical environments that are necessary to describe the morphology of a body and its motion.

Here an observer is then a triple of atlas, one over the ambient space $\hat{\mathbb{R}}^3$, one over the material space containing $\mathcal{B}$ and the last one over the time interval. Changes in observers are then changes in these atlas, governed by the relevant groups of diffeomorphisms. In particular I consider synchronous isometric changes in observers. Synchronicity means that the representation of the time interval is left invariant.

By indicating by $v^*$ the pull back in the first observer of the rate measured by the second observer, the action of the semi-direct product $\hat{\mathbb{R}}^3 \ltimes SO(3)$ over the ambient space $\hat{\mathbb{R}}^3$ gives rise to the standard formula

$$v^* = \hat{c}(t) + \hat{q}(t) \times (y - y_0) + v,$$

where $y_0$ is an arbitrary point in $\hat{\mathbb{R}}^3$, $\hat{c}(t) \in \hat{\mathbb{R}}^3$ and $\hat{q}(t) \in so(3)$, with $so(3)$ the Lie algebra of $SO(3)$. In standard approaches, it is then assumed that all observers evaluate the same $\mathcal{B}$.

Here the assumption is removed and the independent action of the semi-direct product $\hat{\mathbb{R}}^3 \ltimes SO_\diamond(3)$, with $SO_\diamond(3)$ a copy isomorphic to $SO(3)$, over $\hat{\mathbb{R}}^3$ is considered. It leads to the formula

$$w^* = c(t) + q(t) \wedge (x - x_0) + w,$$

where $x_0$ is an arbitrary point in $\mathbb{R}^3$, $c(t) \in \mathbb{R}^3$ and $q(t) \in so_\diamond(3)$, with $so_\diamond(3)$ the Lie algebra of $SO_{\diamond}(3)$. The transformation $w \mapsto -w*$ can be also considered as an isometric shift superposed to a generic relabeling in the ‘material space’ with infinitesimal generator $w$.

4.

Surface and bulk actions are associated with (generated by) relative changes of places between neighboring material elements: at every $x \in \mathcal{B}$ they are represented respectively by the first Piola-Kirchhoff stress $P \in Hom(T_x^* \mathcal{B}, T_{y(x)}^* \mathcal{B}) \simeq \hat{\mathbb{R}}^{3*} \otimes \mathbb{R}^3$ and the vector of bulk forces $b \in \hat{\mathbb{R}}^{3*}$ which includes inertial actions when they are present.

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2See \[3, 11, 18\] and references therein.

3Such a definition has non-trivial consequences above all in the mechanics of complex bodies, rather than in the one of simple bodies (see references in footnote 2).

4Such a point of view has been recently also discussed in \[24\] (see also reference therein) for different purposes. It is also used in \[12\] with strict reference to the derivation of configurational balances. In fact, a requirement of invariance of an expression of a power with respect to such changes in observers is called upon. The power selected involves a number of configurational actions (stresses, internal and external bulk forces and couples) in an abstract way, without discussing at that stage their possible expression in terms of standard actions. This point is further analized in the last section.
The standard power of external actions over a generic part $b$ is given by the expression

$$P_{b}^{\text{ext}}(\dot{y}) := \int_{b} b \cdot \dot{y} \, dx + \int_{\partial b} Pn \cdot \dot{y} \, d\mathcal{H}^2.$$  

Note that this expression is usually written by imagining that the reference place does not undergo mutations. The requirement of invariance of $P_{b}^{\text{ext}}(\dot{y})$, under changes in observers leaving invariant $B$ and altering isometrically the ambient space, furnishes integral and then pointwise balance equations [25].

Here the point of view is different: the body can mutate its material structure. The world ‘mutation’ needs mechanical definition. I do not consider any specific mechanism of mutation. Rather, I account for the indirect effects of classes of mutations: energy fluxes in the material, bulk driving forces and configurational couples. All these ingredients are pictured in $B$. They can be considered as due to the rearrangements of possible inhomogeneities, their possible evolution and/or to more general alterations of the material structure that can be pictured through mutations of the reference placement $B$. An extended notion of power is then required. I call it a relative power: it is the power of standard actions evaluated on the velocity relative to the rates of mutations in the reference place, supplemented by the energy fluxes and the power of driving forces and configurational couples. The definition of the relative power is presented after necessary ensuing preliminaries.

A free energy density $e$ is defined over $B$; it is function of the state $\varsigma$, the place $x$ and the time $t$, namely

$$e := e(x, t; \varsigma).$$

The state $\varsigma$ of a material element is not specified here. The explicit (direct) dependence on $x$ underlines the assumption that the material is not homogeneous. The explicit dependence on time may describes only some aspects of possible mutations, for example aging. In fact, for elastic bodies with time-dependent moduli, the Clausius-Duhem inequality in its isothermal version implies $\partial_t e \leq 0$ which corresponds exactly to aging in time. In what follows the derivative $\partial_x e$ can be considered as the explicit derivative with respect to $x$, holding fixed the state.

For the sake of simplicity, I do not consider below the explicit dependence on time, so that from now on the free energy depends on the place $x$ and the state $\varsigma$.

Standard tractions and bulk forces arise during a generic motion. They are power-conjugated with the rate of changes of (relative) places of material elements. They contribute to the equilibrium of defects and their evolution. In presence of evolving structural mutations in the bulk, annihilation and creation of material bonds occur. Bulk actions are then power-conjugated with mechanisms of annihilation and creation (or restoration) of material bonds. A bulk force $f$ is then associated with evolving mutations: it is the so-called driving force. Effects of anisotropy in the distribution of mutations and anisotropies induced by the ‘permutations’ of defects in the bulk are accounted for by body couples $\mu$. By definition $\mu = 0$ when anisotropies are absent in the material space. Driving forces are introduced and justified variously in the literature (see [1]). Configurational bulk couples have been introduced in [12] with the same meaning adopted here. Both $f$ and $\mu$ are described by co-vectors over $B$ because they are associated with mechanisms mutating $B$ itself. No configurational traction associated with a primitive configurational stress is presumed a priori: it is found later as a derived ‘object’.
Inertia is neglected here for the sake of simplicity. It can be included by considering the bulk forces decomposed additively into inertial and non-inertial parts and ‘adding’ to the kinetic energy.

**Definition 1.** For Cauchy bodies a linear functional $P_{rel} : \mathcal{P} (\mathcal{B}) \times V_v \times V_w \to \mathbb{R}$ is called a relative power when it is additive over disjoint parts, is linear over the space of rates and admits the explicit expression

$$P_{rel} (v, w) := P_{rel-a} (v, w) + P_{dis} (v, w)$$

with

$$P_{rel-a} (v, w) := \int_b b \cdot (v - Fw) \, dx + \int_{\partial b} Pn \cdot (v - Fw) \, dH^2,$$

$$P_{dis} (v, w) := \int_{\partial b} (n \cdot w) e \, dH^2 + \int_b (\partial x e - f) \cdot (w - \text{curl} \, w \times (x - x_0)) \, dx + \int_b \mu \cdot \text{curl} \, w \, dx.$$

I call $P_{rel-a} (v, w)$ the relative power of actions and $P_{dis} (v, w)$ the power due to disarrangements.

1. The power of actions is said to be relative because it is developed along the difference between the actual velocity and the push forward of the material velocity $w$ in $\mathcal{B}$.
2. More difficult is the interpretation of the terms in $P_{dis} (v, w)$. Recall that the velocity field $(x, t) \mapsto v$ moves just points in space where no material elements are necessarily placed. The one-parameter group of diffeomorphism associated with the field $(x, t) \mapsto w$ alters the distribution of the material elements, even permuting them in a virtual way (it has the same role of the relabeling in calculus of variation). A flux of energy through the boundary $\partial b$ appears. Moreover, the distribution of energy in space can be in principle inhomogeneous. As mentioned above, both $x \mapsto f (x)$ and $x \mapsto \mu (x)$ are co-vectors fields over $\mathcal{B}$, thus material interactions power conjugated with the rate of change of the inhomogeneities. Contrary, all standard forces are co-vectors over $\mathcal{B}$. Remind that couples $\mu$ are associated with the anisotropy induced by the material mutations, including the permutation of defects. The presence of mutations allow also one to include in the scenario the driving force $f$. All effects associated with anisotropies both in the evolving mutations and in the distribution of the energy as a consequence of the relabeling are all included in $\mu$. Such a remark justifies the term $(w - \text{curl} \, w \times (x - x_0))$. The negative sign before $f$ is there only for the sake of convenience.

Take note that $v$ may coincide with the true velocity $\dot{y}$ at $x$ and $t$.

Further physical justifications of Definition 1 are presented later.

**Axiom 1.** $P_{rel} (v, w)$ is invariant under isometric changes in observers.

All observers ‘measure’ the same value of the power which is a scalar. The axiom is not different in intrinsic meaning from the axiom of invariance of the standard power [25]. Differences in the expression of the power are dictated only by the situation under scrutiny. Consequences are summarized in the ensuing theorem which is the main result of this paper.
**Theorem 1.** (i) If the fields \( x \mapsto b := b(x) \) and \( x \mapsto P := P(x) \) are integrable over \( B \), then for every part \( b \) the following integral balances hold:

\[
\int_b b \, dx + \int_{\partial b} P n \, d\mathcal{H}^2 = 0, \\
\int_b (y - y_0) \times b \, dx + \int_{\partial b} (y - y_0) \times P n \, d\mathcal{H}^2 = 0, \\
\int_{\partial b} \mathcal{P} n \, d\mathcal{H}^2 - \int_b F^* b \, dx + \int_b (\partial_x e - f) \, dx = 0, \\
\int_{\partial b} (x - x_0) \times \mathcal{P} n \, d\mathcal{H}^2 - \int_b (x - x_0) \times F^* b \, dx + \int_b \mu \, dx = 0.
\]

where, with \( \mathcal{I} \) the second order unit tensor, 
\[ P := eI - F^* P. \]

(ii) If the fields \( x \mapsto P \) and \( x \mapsto \mathcal{P} \) are of class \( C^1(B) \cap C^0(\overline{B}) \) then 
\[ \text{Div} P + b = 0, \]
\[ \text{Skw} P F^* = 0, \]
\[ \text{Div} \mathcal{P} - F^* b + \partial_x e = f. \]

\[ 2\text{Skw} \mathcal{P} = \mu \times \]

(iii) If the material is homogeneous and no driving force is present, when the material distribution of defects and energy is isotropic in \( B \), then \( \mathcal{P} \) is symmetric and, in absence of body forces, the integral 
\[ \int_{\partial b} \mathcal{P} n \, d\mathcal{H}^2 \]

is ‘surface’ independent. (iv) An extended version of the virtual power principle holds:

\[ \mathcal{P}^\text{rel}_b (v, w) = \mathcal{P}^\text{rel-inn}_b (v, w), \]

where 
\[ \mathcal{P}^\text{rel-inn}_b (v, w) := \int_b (P \cdot \nabla v + \mathcal{P} \cdot \nabla w - (x - x_0) \otimes (\partial_x e - f) \cdot \text{Skw} \nabla w + \mu \cdot \text{curl} w) \, dx; \]

it reduces to 
\[
\int_b (v - Fw) \, dx + \int_{\partial b} Pn \cdot (v - Fw) \, d\mathcal{H}^2 + \int_{\partial b} (n \cdot w) \, e \, d\mathcal{H}^2 + \int_b (\partial_x e - f) \cdot w \, dx = \\
= \int_b P \cdot (\nabla v - F \nabla w) \, dx.
\]

**Proof.** The axiom of invariance and some elementary algebra impose that 
\[
d \cdot (\int_b b \, dx + \int_{\partial b} P n \, d\mathcal{H}^2) + \\
+ \hat{q} \cdot (\int_b (y - y_0) \times b \, dx + \int_{\partial b} (y - y_0) \times P n \, d\mathcal{H}^2) + \\
+ c \cdot (\int_{\partial b} (eI - F^* P) n \, d\mathcal{H}^2 - \int_b F^* b \, dx + \int_b (\partial_x e - f) \, dx) + \\
+ q \cdot (\int_{\partial b} (x - x_0) \times \mathcal{P} n \, d\mathcal{H}^2 - \int_b (x - x_0) \times F^* b \, dx + \int_b \mu \, dx) = 0.
\]
The arbitrariness of \( c, q, \hat{c} \) and \( \hat{q} \) implies the integral balances in Theorem 1, once one defines \( \mathbb{P} := eI - F^*P \). The point-wise balances follow by the application of Gauss theorem. They imply the equality between \( \mathcal{P}_{rel}^b(v, w) \) and \( \mathcal{P}_{rel}^{rel-inn} (v, w) \). The last statement of the theorem then follows straight away. Remind that \( \mu = 0 \) when anisotropies are absent in \( \mathcal{B} \).

In the earlier theorem, \( \mathbb{P} \) is called the Hamilton-Eshelby stress. As recalled in the preamble, the word ‘configurational’ is attributed to balances involving it. Besides its immediateness, the earlier theorem has some stringent theoretical consequences, as anticipated in the preamble.

1. To obtain the balance of configurational forces it is not necessary to make use of the procedure exploiting the inverse motion.
2. The Hamilton-Eshelby stress \( \mathbb{P} := eI - F^*P \) and the bulk actions \(-F^*b\) and \(\partial_x e\) are not introduced a priori as unknown objects and then identified with a procedure discussed further in the last section.
3. A version of the principle of virtual power different from usual arises, it includes the standard one when the reference place is considered invariant, invariance intended in the sense of absence of evolving defects.

The result can be extended to the case of complex bodies and to the case in which structured discontinuity surfaces and line defects are present. In the process, appropriate additional measures of interactions need to be introduced as objects power conjugated with the rate of change of the morphological descriptors of the substructure in the material elements (in the case of complex bodies) and/or deformations and evolution of surface and line defects. Even in that cases the procedure avoids the specification of the constitutive structure of the local state and the use to the mechanical dissipation inequality to identify the expression of the configurational forces in terms of standard measures of interaction.

5.

To explain further on the nature of the relative power, one may notice that in purely conservative case the equation

\[ \mathcal{P}_{rel}^b(v, w) = \mathcal{P}_{rel}^{rel-inn} (v, w) \]

reduces to an integral version of the pointwise balance appearing in Nöther theorem.

To prove such a statement consider a non-linear elastic inhomogeneous body with total energy given by

\[ e(x, F) + u(y), \]

with \( e \) the elastic potential – a function which is polyconvex in the gradient of deformation – and \( u \) the potential of body forces. Both \( e \) and \( u \) are assumed to be differentiable with respect to their arguments. The essential ingredients preparing Nöther theorem need also to be recalled briefly.

Consider smooth curves \( s \mapsto f_s \) on the group of diffeomorphisms \( Diff(\mathbb{R}^3, \mathbb{R}^3) \) such that \( f_0 = identity \) and at every point in \( \mathbb{R}^3 \) one gets \( v = \frac{df_s}{ds} |_{s=0} \), where the field \( y \mapsto v(y) \) coincides with the virtual velocity field introduced above over the ambient space.

The usual relabeling of the reference place is accounted for in \( \mathbb{R}^3 \). From a physical point of view it reduces just to the permutation of inhomogeneities over \( \mathcal{B} \). The relabeling is induced by the action of the special group of diffeomorphisms
$SDiff\left(\mathbb{R}^3,\mathbb{R}^3\right)$, a group on which one selects smooth curves $s_1 \mapsto f^*_1$, such that $f^*_1 = identity$ and at every $x$ one gets $w = \frac{d}{ds_1} f_{s_1} \big|_{s_1=0}$, where the field $x \mapsto w(x)$ is a special case of the virtual velocity field $w$ introduced earlier, special in the sense that it is isochoric.

Equivariance means that $SDiff\left(\mathbb{R}^3,\mathbb{R}^3\right)$, if one defines the vector density

$$Equivariance \ means \ that \ SDiff\left(\mathbb{R}^3,\mathbb{R}^3\right)$$

and may prove also that $y \mapsto \sigma$ belongs to $L^1_{loc}$ (see [10]).

By focusing the attention for the sake of simplicity on the Euler-Lagrange equations above, if one defines the vector density

$$\mathcal{E}(y) := \int_B \left( e(x, F) + u(y) \right) \, dx$$

if the total energy is equivariant with respect to the action of $Diff(\mathbb{R}^3, \mathbb{R}^3)$ and $SDiff\left(\mathbb{R}^3,\mathbb{R}^3\right)$, then (Noether theorem, see e.g. [15, 19])

$$Div\mathcal{E} = 0.$$
in terms of Piola-Kirchhoff stress, one realizes (after some algebra) that the last relation in Theorem 1 reduces to the integral version of Nöther theorem

\[ \int_{b} \mathbf{F} \cdot \mathbf{n} = 0 \]

on some arbitrary part \( b \) of \( B \).

Conversely, one can say that the presence of a principle of relative power is hidden in Nöther theorem. In fact, by starting from Nöther theorem, I have already introduced in earlier papers, namely \([5, 17, 18]\), a version of the relative power including constitutive issues but also surfaces and lines of discontinuity, without being conscious at that time of its generality in non-conservative setting. This note, in fact, has primarily the aim to stress this point.

6.

The approach to configurational forces presented in \([20, 21, 22]\) relies on constitutive assumptions. They are called upon only partially in this paper: the sole assumption \( e := e(x, t; \varsigma) \) is invoked without specifying the nature of the state \( \varsigma \).

The comparison with the approach proposed in \([13, 11]\) (see also \([12]\)) requires a rather extended preliminary description. That approach is based on two steps: (1) The balance of configurational forces is postulated first. Such a postulate can be expressed through the statement of an independent integral balance (like in \([11]\)) or by requiring the invariance of a certain power (a power which is different from the one used here) with respect to the transformation \( w \mapsto w^* \) (like in \([12]\); see also \([27]\)). In \([11]\) and \([12]\), independently of its origin, the balance of configurational forces involves a configurational stress, say \( \mathcal{P} \), and external and internal configurational bulk forces, say \( \tilde{g} \) and \( \tilde{e} \) respectively – they are different from the driving force \( f \) and the configurational couple \( \mu \) which play also a role in the treatment. The bulk configurational forces \( \tilde{g} \) and \( \tilde{e} \) are assumed to perform work only after a Galilean change in observer (i.e. \( w \mapsto w + c \)) at a first glance (see \([12]\), page 36). The point of view is then changed (\([12]\), page 39) by saying that only \( \tilde{g} \) does not perform power under time-dependent changes in reference is involved. Whether \( \mathcal{P}, \tilde{g} \) and \( \tilde{e} \) can be expressed in terms of standard actions and energy is not known at this stage. The identification of \( \mathcal{P}, \tilde{g} \) and \( \tilde{e} \) is matter of the second step. (2) It is essentially based on the exploitation of the second law of thermodynamics written in terms of a mechanical dissipation inequality in which only the power of the configurational traction \( \mathcal{P}n \) is added to the one of standard actions. The mechanical dissipation inequality is written with respect to control volumes with boundaries evolving in time. Invariance with respect to the reparametrization of such boundaries leads to the identification \( \mathcal{P} := eI - F^*P \). Note that in the mechanical dissipation inequality the energy is introduced. It is assumed also that \( e \) is differentiable with respect to time. The identification of \( \tilde{e} \) with \( -F^*b \) follows directly from the insertion of the expression \( eI - F^*P \) in the balance of configurational forces. The mechanical dissipation inequality has to be exploited to recognize that \( \tilde{g} \) coincides with \( -\nabla e + P : \nabla F \). The additional specification of the constitutive structure of the energy shows that \( \tilde{g} \) reduces finally to \( -\partial_x e \) (see \([11]\), page 78).

Comparison of the results in \([11]\) with the point of view in the previous sections has to be done at the end of the identification procedure (so that after step 2) because not only Theorem 1 collects the balance of configurational forces but also
it includes the results of the identification recalled above, at least in the setting discussed here.

To obtain Theorem 1 – I stress once more – no use is made of the mechanical dissipation inequality. No use is made of an additional requirement of invariance of the power with respect to the reparametrization of $\partial b$. Although the energy is also introduced here, no assumption of differentiability in time is necessary.

The state here is not specified: for example it can include $F$, the history of deformation, a number of internal variables conjugated with affinities, their histories and gradients. The sole restriction is that the state be compatible with the relative power of actions. In fact, higher-order Cauchy bodies (like, e.g., second-grade elastic bodies) or complex bodies require expressions of $\mathcal{P}_{rel-a}$ involving hyperstresses or microstresses and self-actions respectively.

I do not claim that the treatment proposed here is better than the ones discussed in this section. My approach is just parallel in some sense. The reader interested in foundational issues can find by himself/herself the right position of this thin note, written by using elementary mathematics.

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