On Higher-Order Cryptography
(Long Version)

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Abstract
Type-two constructions abound in cryptography: adversaries for encryption and authentication schemes, if active, are modelled as algorithms having access to oracles, i.e. as second-order algorithms. But how about making cryptographic schemes themselves higher-order? This paper gives an answer to this question, by first describing why higher-order cryptography is interesting as an object of study, then showing how the concept of probabilistic polynomial time algorithm can be generalised so as to encompass algorithms of order strictly higher than two, and finally proving some positive and negative results about the existence of higher-order cryptographic primitives, namely authentication schemes and pseudorandom functions.

1 Introduction
Higher-order computation generalizes classic first-order one by allowing algorithms and functions to not only take strings but also functions in input. It is well-known that this way of computing gives rise to an interesting computability and complexity theory [16, 15, 20], and that it also constitutes a conceptual basis for the functional programming paradigm, in which higher-order subroutines allow for a greater degree of modularity and conciseness in programs.

In cryptography [11, 18, 14], computation is necessarily randomized, and being able to restrict the time complexity of adversaries is itself crucial: most modern cryptographic schemes are unsecure against computationally unbounded adversaries. Noticeably, higher-order constructions are often considered in cryptography, in particular when modeling active adversaries, which have access to oracles for the underlying encryption, decription, or authentication functions, and can thus naturally be seen as second-order algorithms.

Another example of useful cryptographic constructions which can be spelled out at different type orders, are pseudorandom primitives. Indeed, pseudorandomness can be formulated on (families of) strings, giving rise to so-called pseudorandom generators [2], but also on (families of) first-order functions, giving rise to the so-called pseudorandom functions [12]. In the former case, adversaries (i.e., distinguishers) are ordinary polynomial time algorithms, while in the latter case, they are polytime oracle machines.

Given the above, it is natural to wonder whether standard primitives like encryption, authentication, hash functions, or pseudorandom functions, could be made higher-order. As discussed in Section 2 below, that would represent a way of dealing with code-manipulating programs and their security in a novel, fundamentally interactive, way. Before even looking at the feasibility of this goal, there is one obstacle we are facing, which is genuinely definitional: how could we even give notions of security (e.g. pseudorandomness, unforgeability, and the like) for second-order functions, given that those definitions would rely on a notion of third-order probabilistic polynomial time adversary, itself absent from the literature? Indeed, although different proposals exist for classes of feasible deterministic functions [5, 15], not much is known if the underlying algorithm has access to a source of randomness. Moreover, the notion of feasibility used in cryptography is based on the so-called security parameter, a global numerical parameter which controls the complexity of all the involved parties. In Section 3 we give a definition of higher-order probabilistic
polynomial time by way of concepts borrowed from game semantics [13, 1], and being inspired by recent work by Ferée [8]. We give evidence to the fact that the provided definition is general enough to capture a broad class of adversaries of order strictly higher than two.

After having introduced the model, we take a look at whether any concrete instance of a secure higher-order cryptographic primitive can be given. The results we provide are about pseudorandomness functions and their application to authentication. We prove on the one hand that those constructions are not possible, at least if one insists on them to have the expected type (see Section 4.2). On the other hand, we prove (in Section 4.3 below) that second-order pseudorandomness is possible if the argument function takes as input a string of logarithmic length.

2 The Why and How of Encrypting or Authenticating Functions

Encryption and authentication, arguably the two simplest cryptographic primitives, are often applied to programs rather than mere data. But when this is done, programs are treated as ordinary data, i.e., as strings of symbols. In particular, two different but equivalent programs are seen as different strings, and their encryptions or authentication tags can be completely different objects. It is natural to ask the following: would it be possible to deal with programs as ordinary data, i.e., as strings of symbols, and not as programs, seeing them as black boxes, thus without any access to their code?

For the sake of simplicity, suppose that the program \( P \) we deal with has a very simple IO behaviour, i.e., it takes an input a binary string of length \( n \) and returns a boolean. Authenticating it could in principle be done by querying \( P \) on some of its inputs and, based on the outputs to the queries, compute a tag for \( P \). As usual, such an authenticating scheme would be secure if no efficient adversary \( A \) could produce a tag for \( P \) without knowing the underlying secret key \( k \) (such that \( |k| = n \)) with non-negligible probability. Please notice that the adversary, contrarily to the scheme itself, will have access to the code of \( P \), even if that code has not been used during the authenticating process.

But how could security can be defined in a setting like the above? The three entities at hand have the following types, where \( \text{MAC} \) is the authentication algorithm being defined, \( S = \{0, 1\}^* \) is the set of binary strings, and \( B = \{0, 1\} \) is the set of boolean values:

\[
P : S \to B \quad \text{MAC} : S \to (S \to B) \to S \quad A : ((S \to B) \to S) \to (S \to B) \times S
\]

The first argument of \( \text{MAC} \) is the key \( k \), which is of course not passed to the adversary \( A \). The latter can query \( \text{MAC} \) and produce a tagged message, and whose type, as expected, is third-order. The above is not an accurate description of the input-output behaviour of the involved algorithm, and in particular of the fact that length of the input string to \( P \) might be in a certain relation to the length of \( k \), i.e., the underlying security parameter. Reflecting all this in the types is however possible by replacing occurrences of the type \( S \) with refinements of it, as follows:

\[
P : S[n] \to B
\]

\[
\text{MAC} : S[n] \to (S[r(n)] \to B) \to S[p(n)]
\]

\[
A : ((S[r(n)] \to B) \to S[p(n)]) \to (S[r(n)] \to B) \times S[p(n)]
\]

But how could the time complexity of the three algorithms above be defined? While polynomial time computability for the function \( P \) and the authentication \( \text{MAC} \) can be captured in a standard way using, e.g., oracle Turing machines, the same cannot be said about \( A \). How to, e.g., appropriately account for the time \( A \) needs to “cook” a function \( f \) in \( S[n] \to B \) to be passed to its argument functional? So, appealing as it is, our objective of studying higher-order forms of cryptography is bound to be nontrivial, even from a purely definitional point of view.

The contributions of this paper can be now described at a more refined level of detail, as follows:
• On the one hand, we give a definition of a polynomial-time higher-order probabilistic algorithm whose time complexity depends on a global security parameter and which is captured by games and strategies, in line with game semantics [13, 1]. This allows to discriminate satisfactorily between efficient and non-efficient adversaries, and accounts for the complexity of first-and-second-order algorithms consistently with standard complexity theory.

• On the other hand, we give some positive and negative results about the possibility of designing second-order cryptographic primitives, and in particular pseudorandom functions and authentication schemes. In particular we prove, by an essentially information-theoretic argument, that secure deterministic second-order authentication schemes cannot exist. A simple and direct reduction argument shows that a more restricted form of pseudorandom function exists under standard cryptographic assumptions. Noticeably, the adversaries we prove the existence of are of a very restricted form, while the ones which we prove impossible to build are quite general.

3 Higher-Order Probabilistic Polynomial Time Through Parametrized Games

In this section, we introduce a framework inspired by game semantics in which one can talk about the efficiency of probabilistic higher-order programs in presence of a global security parameter. While the capability of interpreting higher-order programs is a well-established feature of game semantics, dealing at the same time with probabilistic behavior and efficiency constraints has—to the best of the authors’ knowledge—not been considered so far. The two aspects have however been tackled independently. Several game models of probabilistic languages have been introduced: we can cite here for instance the fully abstract model of probabilistic Idealized Algol by Danos and Harmer [7], or the model of probabilistic PCF by Clairambault at al. [4]. About efficiency, we can cite the work by Férée [8] on higher-order complexity and game semantics, in which the cost of evaluating higher-order programs is measured parametrically on the size of all their inputs, including functions, thus in line with type-two complexity [5]. We are instead interested in the efficiency of higher-order definitions with respect to the security parameter. Unfortunately, existing probabilistic game models do not behave well when restricted to capture feasibility: polytime computable probabilistic strategies in the spirit of Danos and Harmer do not compose (see the Extended Version of this paper for more details).

Contrary to most works on game semantics, we do not aim at building a model of a particular programming language, but we take game semantics as our model of computation. As a consequence, we are not bound by requirements to model particular programming features or to reflect their discriminating power.

We present our game-based model of computation in three steps: first we define a category of deterministic games and strategies called $\mathcal{PD}$—for parametrized games—which capture computational agents whose behavior is parametrized by the security parameter. This model ensures that the agent are total: they always answer any request by the opponent. In a second step, we introduce $\mathcal{PPD}$, as a sub-category of $\mathcal{PD}$ designed to model those agents whose time complexity is polynomially bounded with respect to the security parameter. Finally, we deal with randomized agents by allowing them to interact with a probabilistic oracle, that outputs (a bounded amount of) random bits.

3.1 Parametrized Deterministic Games

Our game model has been designed so as to be able to deal with security properties that—as exemplified by computational indistinguishability—are expressed by looking at the behavior of adversaries at the limit, i.e., when the security parameter tends towards infinity. The agents we consider are actually families of functions, indexed by the security parameter. As such, our game model can be seen as a parametrized version of Wolverson’s simple games [21], where the set of
plays is replaced by a family of sets of plays, indexed by the natural numbers. Moreover, we require the total length of any interaction between the involved parties to be polynomially bounded in the security parameter.

We need a few preliminary definitions before delving into the . Given two sets \( X \) and \( Y \), we define \( \text{Alt}(X,Y) \) as \( \{(a_1,\ldots,a_n) \mid a_i \in X \text{ if } i \text{ is odd}, a_i \in Y \text{ if } i \text{ is even}\} \), i.e., as the set of finite alternating sequences whose first element is in \( X \). Given any set of sequences \( X \), \( \text{Odd}(X) \) (respectively, \( \text{Even}(X) \)) stands for the subset of \( X \) containing the sequences of odd (respectively, even) length. From now on, we implicitly assume that any (opponent or player) move \( m \) can be faithfully encoded as a string in an appropriate, fixed alphabet. This way, moves and plays implicitly have a length, that we will note \(|\cdot|\). We fix a set \( \text{Pol} \) of unary polynomially-bounded total functions on the natural numbers, which includes the identity \( \iota \), base-2 logarithm \( \lfloor \log_2 \cdot \rfloor \), addition, multiplication, and closed by composition. \( \text{Pol} \) can be equipped with the pointwise partial order: \( p \leq q \) when \( \forall n \in \mathbb{N}, p(n) \leq q(n) \).

**Definition 1 (Parametrized Games).** A parametrized game \( G = (O_G, P_G, L_G) \) consists of sets \( O_G \), \( P_G \) of opponent and player moves, respectively, together with a family of non-empty prefix-closed sets \( L_G = \{L^n_G\}_{n \in \mathbb{N}} \), where \( L^n_G \subset \text{Alt}(O_G, P_G) \), such that there is \( p \in \text{Pol} \) with \( \forall n \in \mathbb{N}, \forall s \in L_G^n, |s| \leq p(n) \). The union of \( O_G \) and \( P_G \) is indicated as \( M_G \).

For every \( n \in \mathbb{N}, L^n_G \) represents the set of legal plays, when \( n \) is the current value of the security parameter. Observe that the first move is always played by the opponent, and that for any fixed value of the security parameter \( n \), the length of legal plays is bounded by \( p(n) \), where \( p \in \text{Pol} \).

**Example 2 (Ground Games).** We present here some games designed to model data-types in our computation model. The simplest game is probably the unit game \( I = (\{?\}, \{\ast\}, \{L^n_I\}_{n \in \mathbb{N}}) \) with just one opponent move and one player move, where \( L^n_I = \{?,?,?\} \setminus 1 \) for every \( n \). Just slightly more complicated than the unit game is the boolean game \( B \) in which the two moves 0 and 1 take the place of \( \ast \). In the two games introduced so far, parameterisation is not really relevant, since \( L^n_B = L^n_I \) for every \( n \in \mathbb{N} \). The latter is not true in \( S[p] = (\{?\}, \{0,1\}, \{L^n_S[p]\}_{n \in \mathbb{N}}) \) with \( L^n_S[p] = \{?,?\} \cup \{|s| \mid s = p(n)\} \), which will be our way of capturing strings. A slight variation of \( S[p] \) is \( L[p] \) in which the returned string can have length smaller or equal to \( p(n) \).

**Example 3 (Oracle Games).** As another example, we describe how to construct polynomial boolean oracles as games. For every polynomial \( p \in \text{Pol} \) we define a game \( O^p \) as \( O^p = (\{?\}, \{0,1\}, \{L^n_{O^p}\}_{n \in \mathbb{N}}) \)

\[
L^n_{O^p} = \{?,?\} \cup \{?b_1?b_2\ldots?b_m \mid b_i \in \{0,1\} \land m \leq p(n)\} \cup \{?b_1?b_2\ldots?b_m? \mid b_i \in \{0,1\} \land m < p(n)\}.
\]

Our oracle games are actually a special case of a more general construction, that consists of taking any game \( G \), and a polynomial \( p \), and to build a game which consists in playing \( p(n) \) times with the game \( G \).

**Definition 4 (Bounded Exponentials).** Let \( G = (O_G, P_G, L_G) \) be a parametrized game. For every \( p \in \text{Pol} \), we define a new parametrized game \( \downarrow_p G := (O_{\downarrow_p G}, P_{\downarrow_p G}, L_{\downarrow_p G}) \) as follows:

- \( O_{\downarrow_p G} = \mathbb{N} \times O_G \), and \( P_{\downarrow_p G} = \mathbb{N} \times P_G \);
- For \( n \in \mathbb{N} \), \( L^n_{\downarrow_p G} \) is the set of those plays \( s \in \text{Alt}(O_{\downarrow_p G}, P_{\downarrow_p G}) \) such that:
  - for every \( i \), the \( i \)-th projection \( s_i \) of \( s \) is in \( L^n_G \).
  - if a move \( (i+1,?) \) appears in \( s \), then a move \( (i,?) \) appears at some earlier point of \( s \), and \( i + 1 \leq p(n) \).

Observe that the game \( O^p \) is isomorphic to the game \( \downarrow_p B \)—we can build a bijection between legals plays having the same length. Games specify how agents could play in a certain interactive scenario. As such, they do not represent one such agent, this role being the one of strategies. Indeed, a strategy on a game is
precisely a way of specifying a deterministic behavior of the player, i.e. how the player is going to answer to any possible behavior of an adversary. We moreover ask our strategies to be total, in the sense that the player cannot refuse to play when it is their turn.

**Definition 5 (Strategies).** A strategy on a parametrized game \( G = (O_G, P_G, L_G) \) consists of a family \( f = \{ f_n \}_{n \in \mathbb{N}} \), where \( f_n \) is a partial function from \( \text{Odd}(L_G^n) \) to \( P_G \) such that:

- for every \( s \in \text{Odd}(L_G^n) \), if \( f_n(s) = x \) is defined, then \( s \times x \in L_G^n \);
- \( sxy \in \text{Dom}(f_n) \) implies that \( x = f_n(s) \);
- \( s \in \tilde{T}_n \wedge s \times x \in L_G^n \Rightarrow s \times x \in \text{Dom}(f_n) \).

If \( f \) is a strategy, we define the set of plays characterising \( f \), that we note \( \tilde{T} \), as the family \( \{ \tilde{T}_n \} \) where:

\[
\tilde{T}_n = \{ \varepsilon \} \cup \{ sf_n(s) \mid s \in \text{Dom}(f) \} \subseteq L_G^n.
\]

Any strategy \( f \) is entirely characterized by its set of plays \( \tilde{T} \). Intuitively, a strategy stands for a deterministic recipe that informs the player on how to respond to any behavior of the opponent. As such, it does not need to be effective.

Up to now, the games we have described are such that their strategies are meant to represent concrete data: think about how a strategy for, e.g., \( B \) or even \( O^n \) could look like. It is now time to build games whose strategies are functions, this being the following construction on games:

**Definition 6 (Constructing Functional Games).** The game \( G \in H \) is given as \( G \in H = P_G + O_H \), \( P_G \in H = O_G + P_H \), and \( L_G \in H = \{ s \in \text{Alt}(O_G \rightarrow H, P_G \rightarrow H) \mid s_0 \in L_G, s_1 \in L_H \} \).

**Example 7.** As an example, we look at the game \( O^n \rightarrow S[p] \), which captures functions returning a string of size \( p(n) \) after having queried a binary oracle at most \( p(n) \) times. First, observe that:

\[
O^n \rightarrow S[p] = (\{ S[p], 0, 1 \}, \{ O^n \} \cup \{ 0, 1 \}^*, (L_n^{O^n \rightarrow S[p]}_{n \in \mathbb{N}}),
\]

where \( L_n^{O^n \rightarrow S[p]} \) is generated by the following grammar:

\[
q = L_n^{O^n \rightarrow S[p]} ::= S[p] \mid S[p] \circ \mid S[p]s \quad s \in \{ 0, 1 \}^* \text{ with } |s| \leq p(n)
\]

\[
e ::= \varepsilon \mid O^n b_1 \ldots O^n b_m \quad o ::= O^n \quad b_i \in \{ 0, 1 \}, m \leq p(n)
\]

Of course there are many strategies for this game, and we just describe two of them here which both make use of the oracle: the first one—that we will call once asks for a random boolean to the oracle, and when this boolean is 0, returns the string \( 0^n \), and returns the string \( 1^n \) otherwise. It is represented in Figure 12. The second strategy—denoted mult, and represented in Figure 18—generates a random key of length \( p(n) \) by making \( p(n) \) calls to the probabilistic oracle.

**Example 8.** In this example, we look at the way of weakening our probabilistic oracle: more precisely, if we have an oracle that makes available \( p(n) \) booleans for each value of the security parameter \( n \)—then we can build from it an oracle that gives only \( q(n) \) booleans, when \( q \leq p \), by taking only the \( q(n) \) first iteration of our original oracle. We do this construction formally in a slightly more general setting: for any game \( G \in SG \), we build a canonical strategy \( w^n_G \) on the game \( L_pG \in \varepsilon^n G \).

We now look at how to compose strategies: given a strategy on \( G \in H \), and \( H \in K \), we want to build a strategy on \( G \in K \) that combines them. We define composition as in [21], except that we need to take also in count the security parameter \( n \).

**Definition 9 (Composition of Strategies).** Let \( G, H, K \) be parametrized games, and let \( f, g \) be two strategies on \( G \in H \) and \( H \in K \) respectively. We first define the set of interaction sequences of \( f \) and \( g \) as:

\[
(f \parallel g)_n = \{ s \in (M_G + M_H + M_K)^* \mid s \in \tilde{T}_n \wedge s \in \tilde{K}_n \}.
\]

From there, we define the composition of \( f \) and \( g \) as the unique strategy \( f \parallel g \) such that:

\[
\tilde{f}(\tilde{g})_n = \{ s \mid \tilde{f} \parallel \tilde{g} \in (f \parallel g)_n \}.
\]
We can check that \( f; g \) is indeed a strategy on \( A \to C \), and that moreover the composition seen as an operation on strategies is associative and admits an identity. We can thus define \( PG \) as the category whose objects are parametrized games, and whose set of morphisms \( PG(G, H) \) consists of the parametrized strategies on the game \( G \to H \).

### 3.2 Polynomially Bounded Parametrized Games

Parametrized games have been defined so as to have polynomially bounded length. However, there is no guarantee on the effectiveness of strategies, i.e., that the next player move, can be computed algorithmically from the history, uniformly in the security parameter. This can be however tackled by considering a subcategory of \( PG \) in which strategies are not merely functions, but can be computed efficiently:

**Definition 10** (Polytime Computable Strategies). Let \( G \) a parametrized game in \( PG \), and \( f \) be a strategy on \( G \). We say that \( f \) is polytime computable when there exists a polynomial time Turing machine which on the entry \((1^n, s)\) returns \( f(n)(s) \).

All strategies we have given as examples in the previous section are polytime computable. For example, the two strategies from Example 7 are both computable in linear time.

**Proposition 11** (Stability of Polytime Computable Strategies). Let \( G, H, K \) be polynomially bounded games. If \( f, g \) are polytime computable strategies, respectively on \( G \to H \), and \( H \to K \), then \( f; g \) is itself a polytime computable strategy.

We can thus write \( PPG \) for the sub-category of \( PG \) whose objects are parametrized games, and whose morphisms are polytime computable strategies.

Let us now consider \( MAC \) from Section 2. Its type can be turned into the parametrized game \( S[i] \to \mathsf{\neg e} G[S[i] \to B] \to S[j] \). The bounded exponential \( \mathsf{\neg e} \) serves to model the fact that the argument function can be accessed a number of times which is polynomially bounded on \( n \). As a consequence, \( MAC \) can only query the argument function a number of times which is negligibly smaller than the number of possible queries, itself exponential in \( n \). As we will see in Section 4, this is the crux in proving security of such a message authentication code to be unattainable.

### 3.3 Probabilistic Strategies

Both in \( PG \) and in \( PPG \), strategies on any game \( G \) are purely functions, and the way they react to stimuli from the environment is completely deterministic. How could we reconcile all this with our claim that the framework we are introducing adequately model probabilistic higher-order
computation? Actually, one could be tempted to define a notion of probabilistic strategy in which the underlying function family \( \{f_n\}_{n \in \mathbb{N}} \) is such that \( f_n \) returns a probability distribution \( f_n(s) \) of player moves when fed with the history \( s \). This, however, would lead to some technical problems when composing strategies: it would be hard to keep the composition of two efficient strategies itself efficient.

It turns out that a much more convenient route consists, instead, in defining a probabilistic strategy on \( G \) simply as a deterministic polytime strategy on \( O^p \rightarrow G \), namely as an ordinary strategy having access to polynomially many random bits. Actually, we have already encountered strategies of this kind, namely once and for all \( \text{mult} \) from Example \( \ref{example-mult} \). This will be our way of modeling higher-order probabilistic computations.

**Definition 12.** Let \( f \) be a strategy on the game \( O^p \rightarrow B \). For every \( b \in B \), we define the probability of observing \( b \) when executing \( f \) as follows:

\[
\Pr(f \upharpoonright^n b) = \sum_{(b_1, \ldots, b_k) \in \mathbb{B}^k} \frac{1}{2^k} \quad \text{with} \quad (\tau^n_b \Gamma_{O^p}^b \cdots \Gamma_{O^p}^{b_k} b) \in (\Gamma)^n
\]

Given a probabilistic strategy \( f \) on \( G \) (i.e. a strategy on \( O^q \rightarrow G \) and \( p \in \text{Pol} \), we indicate as \( !_p f \) the strategy in which \( p(n) \) copies of \( f \) are played, but in which randomness is resolved just once and for all, i.e. \( !_p f \) is a strategy on \( O^q \rightarrow !_p G \).

We want now to be able to observe the result of a computation. Similarly to what is done for instance in \( \ref{example-mult} \), we observe only at ground types: for instance for the type \( \text{Bool} \), the observables will be a pair \( (p_0, p_1) \), where \( p_0 \) is the probability that the program returns false, and \( p_1 \) the probability that the program returns true. In our model, the ground types are those games where the only legal plays are the initial question by the opponent, followed by a terminal answer of the player.

**Definition 13.** Let \( G \) a game in \( SG \). We say that \( G \) is observable when for every \( n \in \mathbb{N} \):

\[
L^n_G \subseteq \{?_G\} \cup \{?_G r \mid r \in P_G\}.
\]

Observe that the games \( B \) and \( S[p] \) that we considered in Example \( \ref{example-mult} \) are indeed observable games. However, it is not the case of the game build using the \( \rightarrow \) construct, for instance \( S[p] \rightarrow !_p S[p] \).

Recall that in our framework, a probabilistic program of some ground type \( G \) is actually represented by a strategy \( f \) on the game \( O^p \rightarrow G \), for some polynomials \( p \). At this point, it should be noted that this polynomial \( p \) is somehow arbitrary, since if we take a polynomial \( q \geq p \), we can easily transform \( f \) in a strategy on \( O^q \rightarrow G \), using the strategy on \( O^q \rightarrow O^p \) highlighted in Example \( \ref{example-mult} \). In order to recover information about the probabilistic behavior of the program, we compute the probability that the return value will be \( r \): to do that we look at all the possible sequences of booleans outputted by the oracle that lead to the player returnin \( r \) under the strategy \( f \), and then we suppose that those sequences are uniformly distributed, in a probabilistic sense.

**Definition 14.** Let \( G \) be an observable game, and \( f \) a strategy on the game \( O^p \rightarrow G \). For any \( r \in P_G \), we define the probability of observing \( r \) when executing \( f \) as the function \( \text{Prob}_f(r) : \mathbb{N} \rightarrow [0,1] \) such that

\[
n \mapsto \sum_{k \leq p(n)} \sum_{(b_1, \ldots, b_k) \in \{0,1\}^k} \left( \frac{1}{2} \right)^k \quad \text{with} \quad (\tau^n_{b_1} \cdots \tau^n_{b_k} b) \in (\Gamma)^n
\]

Observe that if \( f \) is a strategy on \( O^p \rightarrow f \), and \( w_{q,p} \) the canonical strategy on \( O^q \rightarrow O^p \) as defined in Example \( \ref{example-mult} \), then for every value \( r \), \( \text{Prob}_f(r) = \text{Prob}_{w_{q,p}}(r) \): it means that the probability of observing \( r \) is independant of the choice of the polynomial oracle—as soon as it produces at least as many random booleans as asked by the agent.
4 The (In)feasibility of Higher-Order Cryptography

In this section, we will give both negative and positive results about the possibility of defining a deterministic polytime strategy for the game $S[i] \rightarrow q(S[r] \rightarrow B) \rightarrow S[p]$ which could be used to authenticate functions. When $r$ is linear, this is provably impossible, as proved in Section 4.2 below. When, instead, $r$ is logarithmic (and $q$ is at least linear), a positive result can be given, see Section 4.3.

But how would a strategy for the game $l_2^q(S[r] \rightarrow B) \rightarrow S[p]$ look like? Plays for this game are as in Figure 2. A strategy for such a game is required to determine the value of the query $s_{i+1} \in S[r(n)]$ based on $t_1, \ldots, t_i \in B$. Moreover, based on $t_1, \ldots, t_m$ (where $m \leq q(n)$), the strategy should be able to produce the value $v \in S[p(n)]$. Strictly speaking, the strategy should also be able to respond to a situation in which the opponent directly replies to a move $(i, ?_B)$ by way of a truth value $(i, t_i)$, without querying the argument. This is however a signal that the agent with which the strategy is interacting represents a constant function, and we will not consider it in the following.

The way we will prove authentication impossible when $r$ is linear consists in showing that since $q$ is polynomially bounded (thus negligibly smaller than the number of possible queries of type $S[r]$ any function is allowed to make to its argument), there are many argument functions $S[r(n)] \rightarrow B$ which are indistinguishable, and would thus receive the same tag. In Section 4.2 we prove that the (relatively few) influential coordinates can be efficiently determined.

### 4.1 Efficiently Determining Influential Variables

A key step towards proving our negative result comes from the theory of influential variables in decision trees. In this section, we are going to give some preliminary definitions about it, without any aim at being comprehensive (see, e.g., [19]).

The theory of influential variables is concerned with boolean functions, and with their statistical properties. The first definitions we need, then, are those of variance and influence (from now on, metavariables like $M, N, L$ stand for natural number unrelated to the security parameter, unless otherwise specified). Given a natural number $N \in \mathbb{N}$, $[N]$ denotes the set $\{1, \ldots, N\}$. Whenever $j \in [N], e_j \in S[N]$ is the sequence which is everywhere 0 except on the $j$-th component, in which it is 1.

**Definition 15** (Variance and Influence). For every distribution $D$ over $S[N]$, and $F : S[N] \rightarrow B$, we write $\text{Var}_D(F)$ for the value $\mathbb{E}(F(D)^2) - \mathbb{E}(F(D))^2 = \Pr_{x \sim D}[F(x) \neq F(y)]$, called the variance of $F$ under $D$. For every distribution $D$ over $S[N]$, $F : S[N] \rightarrow B$, and $j \in [N]$, we define the influence of $j$ on $F$ under $D$, written $\text{Inf}_j^D(F)$, as $\Pr_{x \sim D}[F(x) \neq F(x \oplus e_j)]$.

The quantity $\text{Inf}_j^D(F)$ measures how much, on the average, changing the $j$-th input to $F$ influences its output. If $F$ does not depend too much on the $j$-th input, then $\text{Inf}_j^D(F)$ is close to 0, while it is close to 1 when switching the $j$-th input has a strong effect on the output.
Example 16. Let $\text{PARITY}_N : S[N] \to \mathbb{B}$ be the parity function on $N$ bits. It holds that

$$\text{Inf}_D^\text{L}(\text{PARITY}_N) = \Pr_{x \sim D}[\text{PARITY}_N(x) \neq \text{PARITY}_N(x \oplus e_j)]$$

$$= \sum_x D(x) \cdot |\text{PARITY}_N(x) - \text{PARITY}_N(x \oplus e_j)| = \sum_x D(x) = 1$$

If $F : A \to S[L]$, and $t \in [L]$, we define $F_t : A \to \mathbb{B}$ to be the function that on input $x \in A$ outputs the $t$-th bit of $F(x)$.

The kind of distributions over $S[N]$ we will be mainly interested are the so-called semi-uniform ones, namely those in which some of the $N$ bits have a fixed value, while the others take all possible values with equal probability. It is thus natural to deal with them by way of partial functions. For every partial function $g : [N] \to \mathbb{B}$ we define $\text{Dom}(g) \subseteq [N]$ to be the set of inputs on which $g$ is defined, and $U_g$ to be the uniform distribution of $x$ over $S[N]$ conditioned on $x_j = g(j)$ for every $j \in \text{Dom}(g)$, i.e., the distribution defined as follows:

$$U_g(x) = \begin{cases} \frac{1}{2^{N-|\text{Dom}(g)|}} & \text{if } x_j = g(j) \text{ for every } j \in \text{Dom}(g); \\ 0 & \text{otherwise.} \end{cases}$$

If a distribution $D$ can be written as $U_g$, where $g : [N] \to \mathbb{B}$, we say that $D$ is an $\text{Dom}(g)$-distribution, or a semi-uniform distribution. Given a distribution $D : S[N] \to \mathbb{R}_{[0,1]}$, some index $j \in [N]$ and a bit $b \in \mathbb{B}$, the expression $D[j \leftarrow b]$ stands for the conditioning of $D$ to the fact that the $j$-th boolean argument is $b$. Note that if $D$ is an $S$-distribution and $j \in [N] \setminus S$, then $D[j \leftarrow b]$ is an $S \cup \{j\}$-distribution.

A crucial concept in the following is that of a decision tree, which is a model of computation for boolean functions in which the interactive aspects are put upfront, while the merely computational aspects are somehow abstracted away.

Definition 17 (Decision Tree). Given a function $F$, a decision tree $T$ for $R$ is a finite ordered binary tree:

- Whose internal nodes are labelled with an index $i \in [N]$.
- Whose leaves are labelled with a bit $b \in \mathbb{B}$.
- Such that whenever a path ending in a leaf labelled with $b$ is consistent with $x \in S[N]$, it holds that $F(x) = b$.

The depth of any decision tree $T$ is defined as for any tree.

Example 18. An example of a decision tree that computes the function $\text{PARITY}_3 : S[3] \to \mathbb{B}$ defined in Example 16 can be found in Figure 3.

The following result, which is an easy corollary of some well-known result in the literature, allows to put the variance and the influence in relation whenever the underlying function can be computed by way of a decision tree of limited depth.

Lemma 19. Suppose that $F$ is computable by a decision tree of depth at most $q$ and $g : [N] \to \mathbb{B}$ is a partial function. Then there exists $j \in [N] \setminus \text{Dom}(g)$ such that

$$\text{Inf}_{U_g}^\text{L}(F) \geq \frac{\text{Var}[U_g](F)}{q}$$

Proof. This is basically corollary of Corollary 1.2 from [19].
Every decision tree $T$ makes on any input a certain number of queries, which of course can be different for different inputs. If $D$ is a distribution, $S$ is a subset of $[N]$ and $T$ is a decision tree, we define $\Delta_{D,S}(T)$ as the expectation over $x \sim D$ of the number of queries that $T$ makes on input $x$ outside of $S$, which is said to be the average query complexity of $T$ on $D$ and $S$. The following result relates the query complexity before and after the underlying semi-uniform distribution is updated: if we fix the value of a variable, then the average interactive complexity goes down (on the average) by at least the influence of the variable:

**Lemma 20.** For every decision tree $T$ computing a function $F$, $S \subseteq [N]$, $j \in [N] \setminus S$, and $S$-distribution $D$, it holds that

$$\frac{1}{2} \Delta_{D[j \leftarrow 0],S \cup \{j\}}(T) + \frac{1}{2} \Delta_{D[j \leftarrow 1],S \cup \{j\}}(T) \leq \Delta_{D,S}(T) - \text{Inf}^j_D(F)$$

**Proof.** We start by showing that

$$\Delta_{D,S \cup \{j\}}(T) \leq \Delta_{D,S}(T) - \text{Inf}^j_D(F) \quad (1)$$

For every $L \in [N]$, let $I_L$ be the random variable that is equal to one if $L$ is queried by $T$ on $x \sim D$ and equal to 0 otherwise. Therefore we can rewrite (1) as

$$\sum_{L \not\subseteq S \cup \{j\}} \mathbb{E}(I_L) \leq \sum_{L \not\subseteq S} \mathbb{E}(I_L) - \text{Inf}^j_D(F)$$

or

$$\mathbb{E}(I_j) \geq \text{Inf}^j_D(F) \quad (2)$$

But for every $x$ on which $j$ is not queried by the tree $T$, $T(x) = T(x \oplus e_j)$. Hence if $j$ is influential w.r.t. $x$ then certainly $j$ is queried, and (2) follows. For obvious reasons $D = \frac{1}{2} D[j \leftarrow 0] + \frac{1}{2} D[j \leftarrow 1]$ and so in particular

$$\Delta_{D,S \cup \{j\}}(T) = \frac{1}{2} \Delta_{D[j \leftarrow 0],S \cup \{j\}}(T) + \frac{1}{2} \Delta_{D[j \leftarrow 1],S \cup \{j\}}(T)$$

from which the thesis easily follows, given (1). \hfill \Box

By somehow iterating over Lemma 20, we can get the following result, which states that fixing enough coordinates, the variance can be made arbitrarily low, and that those coordinates can be efficiently determined:

**Theorem 21.** For every $F : S[N] \to S[L]$ such that for every $t \in [L]$, $F_t$ is computable by a decision tree of depth $\leq q$, and every $\epsilon > 0$, there exist a natural number $m \leq Lq^2/\epsilon$ and a partial function $g : [N] \to \mathbb{B}$ where $|\text{Dom}(g)| \leq m$ such that

$$\text{Var}_{U_g}(F_t) \leq \epsilon \quad (3)$$

for every $t \in \{1, \ldots, L\}$. Moreover, there is an polytime randomized algorithm $A$ that on input $F$, $\delta > 0$, and $\epsilon > 0$, makes at most $O(LN) \cdot \text{poly}(q/(\delta\epsilon))$ queries to $F$ and outputs such a partial function $g$ with $|\text{Dom}(g)| \leq O((Lq^2)/((\epsilon\delta)))$ with probability at least $1 - \delta$.

**Proof.** For every $t \in [L]$ let $T_t$ be a tree computing $F_t$ and having depth at most $q$. The algorithm $A$ works as follows:

**Algorithm $A(F, \epsilon, \delta$):**

1. $i \leftarrow 0$;
2. $g_0 \leftarrow \emptyset$;
3. $D_0 \leftarrow U_{g_0}$;
4. $\gamma \leftarrow \delta/(NLq^2/(\delta\epsilon))$
5. For every $t \in [L]$, perform $v_t \leftarrow \text{EstimateVar}(D_t, F_t, \varepsilon/3, \gamma)$;
6. If for every $t \in [L]$, it holds that $v_t \leq (2/3)\varepsilon$, then return $g_t$, otherwise continue.
7. Let $t_1$ be a $t$ such that $v_t \geq 2\varepsilon/3$;
8. For every $j \in [N]$, perform $n_j \leftarrow \text{EstimateInf}(D_t, F_{t+1}, \varepsilon/(10q), \gamma)$;
9. Let $j$ be such that $n_j \geq 0.2 \cdot \varepsilon/q$;
10. $b \leftarrow \{0, 1\}$;
11. $g_{i+1} \leftarrow g_i \cup \{j, b\}$;
12. $D_{i+1} \leftarrow U_{g_{i+1}}$;
13. $i \leftarrow i + 1$;
14. Go back to 5.

The algorithm can be studied in two steps:
• We first of all analyse the complexity of the algorithm $A$. An analysis of the runtime gives us the following result:

**Proposition 22.** The number of times Algorithm $A$ executes its main loop is, in expectation, at most $O(L\varepsilon^2/\varepsilon)$.

**Proof.** We define the potential function

$$\varphi(i) = \sum_{t=1}^{L} \Delta_{D_t, s(g_t)}(T_t)$$

where $T_t$ is the tree computing $F_t$. The algorithm does not of course have access to this tree but this potential function is only used in the analysis. Initially, we are guaranteed that $\varphi(0) \leq L\varepsilon$ and by definition $\varphi(i) \geq 0$ for every $i$. Hence to show that we end in expected number of steps $O(L\varepsilon^2/\varepsilon)$ it is enough to show that in expectation

$$\mathbb{E}[\varphi(i) - \varphi(i + 1)] \geq \Omega(\varepsilon/q)$$

for every step $i$ in which we do not stop. Indeed, let $i$ be such a step and let $t_0$ be the index such that our estimate for $\text{Var}(F_{t_0}(D_{t_0})) \geq (2/3)\varepsilon$ and $j_0$ the influential variable for $F_{t_0}$ as we find. Recall that $g_{i+1}$ is obtained by extending $g_i$ to satisfy $g_{i+1}(j_0) = b$ for a random $b \in \mathbb{B}$. For every $t \neq t_0$, the expectation over $b$ of $\Delta_{D_{i+1}, s(g_{i+1})}(T_t)$ is equal to the expected number of queries that $T_t$ makes on $D_t$ that do not touch the set $s(g_t) \cup \{j\}$. This number can only be smaller than $\Delta_{D_t, s(g_t)}(T_t)$. For $t = t_0$, by Lemma 20, $\mathbb{E}[\Delta_{D_{i+1}, s(g_{i+1})}(F_{t_0})] \leq \mathbb{E}[\Delta_{D_{i+1}, s(g_{i+1})}(F_t)] + \Omega(\varepsilon/q)$. Hence, in expectation $\varphi(i + 1)$ is smaller than $\varphi(i)$ by an additive factor of $\Omega(\varepsilon/q)$ which is what we wanted to prove.

• A further discussion is needed to ensure that the desired level of accuracy is actually attained by the algorithm. The reason why $\gamma$ is set to $\delta/T$ (where $T = O(NLq^2/(\delta\varepsilon))$ is our bound on the number of times we need to use these estimates) is to ensure $\gamma$ to be small enough so that we can take a union bound over the chances that any of our estimates fail. Thus the algorithm will only stop when it reach $D_i = U_{g_i}$ for which the variance of $F_t$ is smaller than $\varepsilon$ for every $t \in [L]$.

• Once we have Proposition 22, the theorem follows by noting that if the expected number of iterations of the main loop is $m$, the probability that we make more than $m/\delta$ iterations is at most $\delta$. (Since the potential function undergoes a biased random walk, a more refined analysis can be used to show that $O(m\log(1/\delta))$ iterations will suffice, but this does not matter much for our final application and so we use the cruder bound of $m/\delta$.)
4.2 On the Impossibility of Authenticating Functions

Theorem 21 tells that for every first-order boolean function which can be computed by a decision tree of low depth, there exist relatively few of its coordinates that, once fixed, determine the function’s output with very high probability. If \( N \) is exponentially larger than \( q \), in particular there is no hope for such a function to be a secure message authentication code. In this section, we aim at exploiting all this to prove that higher-order authentication is not possible if the argument function has too large a domain. In order to do it, we build a third-order randomized algorithm, which will be shown to fit into the our game-theoretic framework.

More specifically, we are concerned with the cryptographic properties of strategies for the parametrized game \( \text{SOF}_{g,r,p} = |q(S[r] \to B) \sim S[p]| \) and, in particular, with the case in which \( r \) is the identity, i.e., we are considering the game \( \text{LINSOF}_{g,p} = \text{SOF}_{g,r,p} \). Any such strategy, when deterministic, can be seen as computing a family of functions \( \{F_n\}_{n \in \mathbb{N}} \) where \( F_n : (\mathbb{S}[n] \to B) \to \mathbb{S}[p(n)] \). How could we fit all this into the hypotheses of Theorem 21?

The definitions of variance, influence, and decision tree can be easily generalised to functions in \( \mathbb{S}[n] \rightarrow \mathbb{B} \). For every \( x \in \mathbb{S}[n] \), we define the extension of \( x \), denoted by \( f_x \), as the function \( f_x : \mathbb{S}[n] \rightarrow \mathbb{B} \) such that for every \( i \in [2^n] \) (identifying \( \mathbb{S}[n] \) with the numbers \( \{0, \ldots, 2^n - 1\} \) in the natural way), it holds that \( f_x(i) = x_{i/N} + 1 \). That is, \( f_x \) is the function that outputs \( x_1 \) on the first \( n \) inputs, outputs \( x_2 \) on the second \( n \) inputs, and so on and so forth. If \( F : (\mathbb{S}[n] \to \mathbb{B}) \to \mathbb{B} \) is a function, then we define \( \tilde{F} : \mathbb{S}[n] \to \mathbb{B} \) as follows: for every \( x \in \mathbb{S}[n] \), \( \tilde{F}(x) = F(f_x) \). Given a distribution \( \mathcal{D} \) over \( \mathbb{S}[n] \), a distribution over functions \( \mathbb{S}[n] \rightarrow \mathbb{B} \) can be formed in the natural way as \( f_\mathcal{D} \).

We will also make use of the following slight variation on the classic notion of Hamming distance: define \( H(\cdot, \cdot) \) to be the so-called normalized Hamming distance. In fact, we overload the symbol \( H \) and use it for both strings in \( \mathbb{S}[n] \) and functions in \( \mathbb{S}[n] \rightarrow X \) for some set \( X \). That is, if \( x, y \in \{0, 1\}^N \) then \( H(x, y) = \Pr_{i \in [N]}[x_i \neq y_i] \) while if \( f, g \in \mathbb{S}[n] \rightarrow X \) then \( H(f, g) = \Pr_{x \in \mathbb{S}[n]}[f(i) \neq g(i)] \). We use the following simple lemma.

**Lemma 24.** For every \( x, y \in \mathbb{S}[n] \), \( H(f_x, f_y) = H(x, y) \)

**Proof.** If we choose \( i \in \mathbb{S}[n] \) then (identifying \( \mathbb{S}[n] \) with \( \{0, \ldots, 2^n - 1\} \), since \( N \) divides \( 2^n \), the distribution \( [i/N] \) is uniform over \( \{0, \ldots, N - 1\} \) and so the two probabilities in the definition of the Hamming distance are the same. More formally:

\[
H(f_x, f_y) = \Pr_{i \sim [N]}[f_x(i) \neq f_y(i)] = \sum_{i \in \mathbb{S}[n]} \frac{1}{2^n} |f_x(i) - f_y(i)|
\]

\[
= \sum_{i \in \mathbb{S}[n]} \frac{1}{2^n} |x_{i/N} + 1 - y_{i/N} + 1| = 2^{n-p} \sum_{j \in [N]} \frac{1}{2^n} |x_j - y_j|
\]

\[
= \sum_{j \in [N]} \frac{1}{2^p} |x_j - y_j| = \Pr_{j \sim [N]}[x_j \neq y_j] = H(x, y).
\]

The game \( T[q] \) is a slight variation on \( S[q] \) in which the returned string is in a ternary alphabet \( \{0, 1, \bot\} \). Any strategy for \( T[q] \) can be seen as representing a (family of) partial functions from \( [q(n)] \) to \( \mathbb{B} \).
Theorem 25. For every $\varepsilon, \delta > 0$, there is a polytime probabilistic strategy $\text{infvar}_{\varepsilon, \delta}$ on the game $\mathcal{I}_s(\text{LINSOF}_{q,p}) \rightarrow \mathbb{T}[t]$ such that for every deterministic strategy $f$ on $\text{LINSOF}_{q,p}$ computing \{${F}_{n}$\}$_{n \in \mathbb{N}}$, the composition $f ; \text{infvar}_{\varepsilon, \delta}$, with probability at least $1 - \delta$, computes some functions $g_n : [t(n)] \rightarrow \mathbb{B}$ such that $\text{Var}_{g_n}(F_n) \leq \varepsilon$ and $|\text{Dom}(g_n)| \leq O(p(n)q^2(n) / (\delta\varepsilon))$.

Proof. Please observe that $\tilde{F}$ is a function from $\mathbb{S}[N]$ to $\mathbb{S}[L]$ which, since $F$ is computable by a decision tree of depth $q$, can be computed itself by a decision tree of the same depth. The result follows by just looking at the function $\tilde{F}$ and applying to it Theorem 21. Actually, the queries (algorithm $A$ makes to $\tilde{F}$) correctly translate into queries of the form $f_i = f_{x_i}$.

Remarkably, the strategy $\text{infvar}_{\varepsilon, \delta}$ is that it infers the “influential variables” of the strategy with which it interacts without looking at how the latter queries its argument function, something which would anyway be available in the history of the interaction.

We can now obtain the main result of this section.

Theorem 26. For every $\delta$ there is a polytime probabilistic strategy $\text{coll}_\delta$ on a game $\mathcal{I}_s(\text{LINSOF}_{q,p}) \rightarrow (\mathbb{S}[t] \rightarrow \mathbb{B}) \otimes (\mathbb{S}[t] \rightarrow \mathbb{B})$ such that for every deterministic strategy $f$ on $\text{LINSOF}_{q,p}$ computing \{${F}_{n}$\}$_{n \in \mathbb{N}}$, the composition $f ; \text{coll}_\delta$, with probability at least $1 - \delta$, computes two function families $g, h$ with $g_n, h_n : \mathbb{S}[n] \rightarrow \mathbb{B}$ such that

1. $H(g_n, h_n) \geq 0.1$
2. $F_n(g_n) = F_n(h_n)$.
3. For every function $f$ on which $\text{coll}_\delta$ queries its argument, it holds that $H(f, g_n) \geq 0.1$ and $H(f, h_n) \geq 0.1$.

This shows that $\text{coll}$ finds a collision for $F_n$ as a pair of functions that are different from each other (and in fact significantly different in Hamming distance) but for which $F_n$ outputs the same value, and hence $F$ cannot be a collision-resistant hash function. Moreover, because the functions are far from those queried, this means that $F_n$ cannot be a secure message authentication code either, since by querying $F_n$ on $g_n$, the adversary can predict the value of the tag on $h_n$.

Proof. We run the algorithm of Theorem 25 and obtain a partial function $g$ such that $|S(g)| \leq 10Lq^2 / (\delta^2 \varepsilon)$ and such that the variance of every one of the functions $F_1, F_2, \ldots, F_L$ is smaller than $\delta \varepsilon$. We choose $N$ to be large enough so that the bound $10Lq^2 / (\delta^2 \varepsilon)$ on $|S(g)|$ is smaller than $\delta N / 100$. Once we achieve this, we sample $x'$ and $x''$ independently from $U_g$ and set $g = f_{x'}$ and $h = f_{x''}$. Since we fixed only a 0.01$\delta$ fraction of the coordinates, and $x'$ and $x''$ are random over the remaining ones, using the standard Chernoff bounds with high probability $g$ and $h$ will differ on at least $1 - 2^{-L} - \delta / 10$ fraction of the remaining coordinates from each other and also from all other previous queries. (Specifically, the probability of the difference being smaller than this is exponentially small in $\delta N$ and we can make $N$ big enough so that this is much smaller than $\delta / (Lm)$ and so we can take a union bound on all queries.) On the other hand, because of the variance, the probability that $F_i(g) \neq F_i(h)$ for every $i$ is less than $\delta / L$ and so we can take a union bound over all $L$ ‘i’s to conclude that $G(g) = G(h)$ with probability at least $1 - \delta$.

4.3 A Positive Result on Higher-Order Pseudorandomness

We conclude this paper by giving a positive result. More specifically, we prove that pseudorandomness can indeed be attained at second order, but at an high price, namely by switching to the type $\text{LOGSOF}_{p} = \text{SOF}_{\text{LINSOF}_p}$, $|S[1]|, p$. This indeed has the same structure of $\text{LINSOF}_{q,p}$, but the argument function takes in input strings of logarithmic size rather than linear size. Moreover, the argument function can be accessed in a linear number of times, which is enough to query it on every possible coordinate.

The fact that that a strategy on $\text{LOGSOF}_{p}$ can query its argument on every possible coordinate renders the attacks described in the previous section unfeasible. Actually, $\text{LOGSOF}_{p}$ can be seen
as an *interactive* variation of the game $S[i] \rightarrow S[p]$, for which pseudorandomness is well known to be attainable starting from standard cryptographic assumptions \cite{11}; simply, instead of taking in input the whole string at once, it queries it *bit by bit*, in a certain order. A random strategy of that type, then, would be one that, using the notation from Figure 2,

- Given $t_1, \ldots, t_q \in \mathbb{B}$, returns a string $s_{i+1}$ uniformly chosen at random from $S[r(n)] - \{s_1, \ldots, s_i\}$, for every $i < q(n)$.

- Moreover, based on $t_1, \ldots, t_{r(n)}$, the strategy produces a string $v$ chosen uniformly at random from $S[p(n)]$.

Please notice that this random strategy can be considered as a random functionals in $(S[r(n)] \rightarrow \mathbb{B}) \rightarrow S[p(n)]$ only if $r(n)$ is logarithmic, because this way the final result $v$ is allowed to depend on the value of the input function in *all possible coordinates*. The process of generating such a random strategy uniformly can be seen\footnote{the strategy at hand would, strictly speaking, need exponentially many random bits, which are not allowed in our model.} as a probabilistic strategy randsof.

We are now ready to formally define pseudorandom functions:

**Definition 27** (Second-Order Pseudorandom Function). A deterministic polytime strategy $f$ on $S[i] \rightarrow \text{LOGSOF}_p$ is said to be pseudorandom if for every probabilistic polytime strategy $A$ on $1\text{LOGSOF}_p \rightarrow \mathbb{B}$ there is a negligible function $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}_{[0,1]}$ such that

$$\Pr(!A(\cdot; f); A \downarrow^n 1) - \Pr(!, \text{randsof}; A \downarrow^n 1) \leq \varepsilon(n)$$

The way we will build a pseudorandom function consists in constructing a deterministic polytime strategy first2second for the following game $1_{i+1}(L[i] \rightarrow S[w]) \rightarrow \text{LOGSOF}_p$, where $w \in \textbf{Pol}$ is such that $w \geq \lfloor lg \rfloor$ and $w \geq p$. The way the strategy is defined is in Figure 4.

The function $\alpha$ interprets its first argument (a string in $S[w(n)]$), as an element of $S[[\lfloor lg \rfloor(n)]$ different from those it takes as second argument, and distributing the probabilities uniformly. The function $\beta$, instead, possibly discards some bits of the input and produces a possibly shorter string.

The way the strategy first2second is defined so as to be such that $f; first2second$ is statistically very close to the random strategy whenever $f$ is chosen uniformly at random among the strategies for the parametric game $L[i] \rightarrow L[w]$. This allows us to prove the following:

**Theorem 28.** Let $F : \{0,1\}^n \times \{0,1\}^{w(n)} \rightarrow \{0,1\}$ be a pseudorandom function and let $f_F$ be the deterministic polytime strategy for the game $S[i] \rightarrow 1_{i+1}(L[i] \rightarrow \text{S}[w])$ obtained from $F$. Then, $f_F; first2second$ is second-order pseudorandom.

## 5 Related Work

Game semantics and geometry of interaction are among the best-studied program semantic frameworks (see, e.g. \cite{11,13}), and can also be seen as computational models, given their operational
flavor. This is particularly apparent in the work on abstract machines [6, 9], but also on the so-called geometry of synthesis [10]. In this paper, we are particularly interested in the latter use of game semantics, and take it as the underlying computational model. Our game model is definitionally strongly inspired by Wolverson [21]: the main novelty with respect to the latter is the treatment of randomized strategies, and the bounded exponential construction, which together allows us to account for efficient randomized higher-order computation.

This is certainly not the first paper in which cryptography is generalized to computational models beyond the one of first-order functions. One should of course cite Canetti’s universally composable security [3], but also Mitchell et al.’s framework, the latter based on process algebras [17]. None of them deals with the security of higher-order functions, though.

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