ON THE NATURALTY OF THE MATHAI-QUILLEN
FORMULA

GYULA LAKOS

Abstract. We give an alternative proof for the Mathai-Quillen formula for a
Thom form using its natural behaviour with respect to fiberwise integration.
We also study this phenomenon in general context.

1. The Mathai-Quillen formula

Let \( \pi : V \to M \) be an oriented vector bundle. A smooth, closed differential form
\( \Phi \) on \( V \) is called a Thom-form for \( \pi \), if it is of vertically compact support, and its
vertical integral \( \pi^* \Phi = 1 \). (See Bott and Tu, [2] for more details.) Nevertheless,
it is enough to assume that \( \Phi \) is only vertically rapidly decreasing, because by
a simple push-forward operation one can bring it to vertically compact form. In
what follows we mean rapidly decreasing Thom forms.

If \( \pi : V \to M \) is an oriented bundle of rank \( n \) endowed with a metric then local
trivializations can be chosen so that \( x = [x_i] \) are the coordinates in the trivialized
bundle, \( dx_1 \wedge \ldots \wedge dx_n \) is the orientation form of the bundle, and \( x^2 = x^\top x = \sum x_i^2 \)
is the metric. Such trivializations are called compatible. If \( \nabla \) is a connection on
the bundle then \( \theta = [\theta_{ij}] \) are the connection 1-forms, assuming that \( \nabla \)
is of local form \( d + \theta \) as a covariant differential with respect to the coordinates \( x \); and
\( \Omega = d\theta + \theta \cdot \theta = [\Omega_{ij}] \) are the corresponding local curvature 2-forms. (If \( u \)
is a column vector then we write \( u_i = e_i^\top u \) for its \( i \)-th element, and if \( M \) is a matrix,
then we write \( M_{ij} = e_i^\top Me_j \) for its \( (i, j) \)-th entry.)

The famous result of Mathai and Quillen in its most naive form says the fol-
lowing:

Theorem 1 (Mathai, Quillen, [3]). If \( \pi : V \to M \) is an oriented bundle of rank \( n \)
endowed with a metric and a compatible connection \( \nabla \) then there is a well-defined
Thom form \( \text{TMQ}(V, \nabla) \) on \( V \) such that

\[
\text{TMQ}(V, \nabla) = \pi^{-n/2} e^{-x^2} \sum_{I \cup I' = [1, \ldots, n], |I'| \text{ is even}} (-1)^{(I, I')}(dx + \theta x)_{i_1} \ldots (dx + \theta x)_{i_{|I|}} \left(\prod_{\sigma \in \Sigma_{|I'|}} \Omega_{\sigma(1), \sigma(2)}^{i_1} \ldots \Omega_{\sigma(|I'|-1), \sigma(|I'|)}^{i_{|I'|}} \right)
\]

in any compatible trivialization. (Here the convention is that \( I = \{i_1, \ldots, i_{|I|}\} \) with
elements listed in increasing order; similarly for \( I' \); and \( (I, I') \) is the permutation
which puts the elements of \( I \) in front of the elements of \( I' \).)

The form \( \text{TMQ}(V, \nabla) \) is rather explicit; the main difficulty lies in proving that
it is well-defined and closed, even if it is a local computation. One quite simple

\[\begin{array}{c}
2000 \text{ Mathematics Subject Classification. Primary 58A10, Secondary 53C05, 57R20.}
\text{Key words and phrases. Mathai-Quillen formula, Thom form, Gaussian integrals, Wick}
\text{formula.}
\end{array}\]
proof can be found in the book \[1\] of Berline, Getzler, and Vergne using elementary supercalculus.

We give an alternative proof for Theorem \[1\] above, realizing a different strategy: One classical existence proof for the Thom form goes as follows: We embed \(V\) into a trivial bundle like

\[
\begin{array}{c}
V \\
\downarrow^\pi \\
M
\end{array}
\begin{array}{c}
\pi_1
\end{array}
\begin{array}{c}
V \oplus W \cong M \times \mathbb{R}^{n+m},
\end{array}
\]

we take a Thom-form \(\Phi_0\) on the trivial bundle, and then we can take the fiberwise integral \(\Phi = (\pi_1)_* \Phi_0\) as a Thom-form on \(V\).

We will see that the Mathai-Quillen form occurs as the result of such a process. More specifically: A splitting of the trivial bundle \(M \times \mathbb{R}^{n+m}\) can be realized by an involution valued \((n + m) \times (n + m)\) matrix function \(Q\) on \(M\), where \(V\) is the \((-1)\)-eigenspace of \(Q\) and \(W\) is the \((+1)\)-eigenspace. Let \(\pi_1\) be the projection to the first factor. Let \(y = [y_i]\) denote the column vector of the coordinates of the trivial bundle \(M \times \mathbb{R}^{n+m}\). Now, there is a natural connection \(\nabla_0\) on the trivial bundle, such that in the coordinates \(y\) its covariant derivative is given as \(d + \theta(y) = d + \frac{1}{2} Q dQ\). This connection leaves the bundles \(V, W\) invariant. Furthermore, if we assume that \(Q\) is orthogonal then it is compatible to the metric which is the restriction of the metric \(y^2\) from the trivial bundle. We denote the restriction of the connection above to \(V\) by \(\nabla_0|_V\). Let us take the Thom form \(\Phi_0 = \pi^{-\frac{(n+m)}{2}} e^{-y^2} dy_1 \wedge \ldots \wedge dy_{n+m}\). Then the key statement is:

**Theorem 2.** If \(V\) is oriented then

\[
\text{TMQ}(V, \nabla_0|_V) = (\pi_1)_* \Phi_0,
\]

where \((\pi_1)_*\) is according to the complementary orientation on \(W\).

(The statement includes the well-definedness of the left side.)

One can ask that how general the pairs \((V, \nabla_0|_V)\) are among arbitrary metric bundle and connection pairs. The answer is that they are completely general: According to a theorem of Narasimhan and Ramanan \[1\], every metric pair can be realized like that. (Their formalism is a bit different.) As a corollary we have Theorem \[1\]. In fact, we are interested in a local computation, so we do not even have to use \[1\] in its full power.

**Proof of Theorem \[2\].** Locally, we can always take an \(\text{SO}(n+m)\)-valued function \(A\) such that \(Q = AQ_0A^{-1}\), where \(Q_0 = \begin{bmatrix} -1_{n \times n} & 0_{m \times n} \\ 0_{m \times m} & 1_{m \times m} \end{bmatrix}\) is a fixed matrix. Let us then define \(x = A^{-1} y\) as the new set of coordinates. We see that \(y\) is in the \(\pm 1\)-eigenspace of \(Q\) if and only if \(x\) is in the \(\pm 1\)-eigenspace of \(Q_0\). From this it is follows that the coordinates \(x_1, \ldots, x_n\) trivialize \(V\). Actually more is true: \(x_{n+1}, \ldots, x_{n+m}\) trivialize \(W\). This is more than we expect (we want to trivialize \(V\) only), nevertheless we can note that these trivializations account for all possible compatible trivializations of \(V\) with respect to the metric, because such a local trivialization of \(V\) can always be extended to a well-oriented orthogonal trivialization of \(V \oplus W\). It is practical to incorporate the variables \(x_1, \ldots, x_n\) into a column vector \(x_o\), and the variables \(x_{n+1}, \ldots, x_{n+m}\) into the column vector \(x_h\). With some abuse of notation \(x = x_o + x_h\).

Now, we have a connection on \(M \times \mathbb{R}^{n+m}\), whose covariant derivative can be written as \(d + \theta(y) = d + \frac{1}{2} Q dQ\) with respect to the coordinate functions \(y\). Then
the covariant derivative is given as

\(d + \theta^{(x)} = (A^{-1})(d + \frac{1}{2} QdQ)(A) = d + \frac{1}{2} (A^{-1}dA + Q_0 A^{-1}dAQ_0)\).

with respect to the coordinate functions \(x\). The curvature form is \(\Omega^{(y)} = \frac{1}{4} dQ dQ\) with respect to \(y\). Similarly, with respect to \(x\) it is given as

\(\Omega^{(x)} = -\frac{1}{4} (A^{-1}dA - Q_0 A^{-1}dAQ_0) (A^{-1}dA - Q_0 A^{-1}dAQ_0)\).

Using the block decomposition \(A^{-1}dA = \begin{bmatrix} (A^{-1}dA)_{oo} & (A^{-1}dA)_{oh} \\ (A^{-1}dA)_{ho} & (A^{-1}dA)_{hh} \end{bmatrix}\) we find

\(\theta^{(x)} = \begin{bmatrix} (A^{-1}dA)_{oo} \\ (A^{-1}dA)_{hh} \end{bmatrix}\),

\(\Omega^{(x)} = -\begin{bmatrix} (A^{-1}dA)_{oh} (A^{-1}dA)_{ho} \\ (A^{-1}dA)_{ho} (A^{-1}dA)_{oh} \end{bmatrix}\).

In particular, restricted to \(V\), with respect to the coordinate functions \(x_o\), it yields

\((\dagger)\) \(\theta = (A^{-1}dA)_{oo}\), and \(\Omega = -(A^{-1}dA)_{oh}(A^{-1}dA)_{ho}\).

Let us note that \(\Phi_0\) can be written as \(\pi^{-(n+m)/2} e^{-y^2 (A^{-1}dy)_1 \wedge \ldots \wedge (A^{-1}dy)_{n+m}}\), because \(A\) is SO\((n + m)\)-valued. Applying \(y = Ax\) we find that

\(\Phi_0 = \pi^{-(n+m)/2} e^{-x^2 (dx + A^{-1}dAx)_1 \wedge \ldots \wedge (dx + A^{-1}dAx)_{n+m}}\).

In this local coordinate system the fiberwise integration \((\pi_1)_s\) is just integration in the variables \(x_{n+1}, \ldots, x_{n+m}\) whose infinitesimal forms appear only in the last \(m\) terms in the integrand. So, it yields

\((\pi_1)_s \Phi = \pi^{-(n+m)/2} \int e^{-x^2 (dx + A^{-1}dAx)_1 \wedge \ldots \wedge (dx + A^{-1}dAx)_{n+m}} dx_{n+1} \ldots dx_{n+m}\).

Let us decompose the column vector \(x\) as \(x_o + x_h\). Then \(x_o\) contains the non-integration variables \(x_1, \ldots, x_n\), while \(x_h\) contains the integration variables \(x_{n+1}, \ldots, x_{n+m}\). The term \((dx + A^{-1}dAx)_i\) decomposes as \(dx_i + ((A^{-1}dA)_{oo}x_o)_i + ((A^{-1}dA)_{oh}x_h)_i\). After reordering we find that \((\pi_1)_s \Phi\)

\[\pi^{-(n+m)/2} e^{-x^2} \sum_{I \cup I' = \{1, \ldots, n\}} (dx_o + (A^{-1}dA)_{oo}x_o)_i \ldots (dx_o + (A^{-1}dA)_{oo}x_o)_{i_{|I|}} \]

\[(-1)^{|I \cap I'|} e^{-x^2} ((A^{-1}dA)_{oh}x_h)_i' \ldots ((A^{-1}dA)_{oh}x_h)_{i'_{|I'|}} dx_{n+1} \ldots dx_{n+m}\]

(the sign change is due to the fact that 1-forms anticommute).

Let us remind ourselves to the Wick formula of Gaussian integrals. It says that if we are in \(\mathbb{R}^l\), \(z = [z_k]\) is the column vector of the variables, \(b_1, \ldots, b_s\) are scalar-valued fixed column vectors then

\[\int e^{-x^2} (b_1^\top z) \ldots (b_s^\top z) = \frac{\pi^{l/2}}{2^s(s/2)!} \sum_{\sigma \in \Sigma_s} (b_{\sigma(1)}^\top b_{\sigma(2)}) \ldots (b_{\sigma(l-1)}^\top b_{\sigma(l)})\],

if \(s\) is even, and \(= 0\) if \(s\) is odd. If the column vectors \(b_i\) are 1-form valued then the expression remains the same except that \((-1)^\sigma\) must be inserted after \(\sum\), because interchanging the order of the coefficients yields some sign changes.
Applying the formula with respect to \( m, x_h, e^T_{b_j} (A^{-1} dA)_{oh} \) in the place of \( l, z, b^T_j \), respectively, we find

\[
(\pi_1)_* \Phi_0 = \pi^{-n/2} e^{-x^2} \sum_{I \cup I' = \{1, \ldots, n\}, |I'| \text{ even}} \frac{1}{2^{|I'|}(|I'|/2)!} \sum_{\sigma \in \Sigma_{|I'|}} (-1)^\sigma \\
\cdots (dx_o + (A^{-1} dA)_{oh} x_o)_{i_1} \cdots \frac{1}{2^{|I'|}(|I'|/2)!} \sum_{\sigma \in \Sigma_{|I'|}} (-1)^\sigma \Omega_{\sigma^{(1)}}^x dx_1 \cdots \Omega_{\sigma^{(|I'|-1)}}^x dx_{|I'|-1}.
\]

From orthogonality \((A^{-1} dA)^T_{oh} = -(A^{-1} dA)_{ho}\), and applying (3) we obtain

\[
(\pi_1)_* \Phi_0 = \pi^{-n/2} e^{-x^2} \sum_{I \cup I' = \{1, \ldots, n\}, |I'| \text{ even}} \frac{1}{2^{|I'|}(|I'|/2)!} \sum_{\sigma \in \Sigma_{|I'|}} (-1)^\sigma \Omega_{\sigma^{(1)}}^x dx_1 \cdots \Omega_{\sigma^{(|I'|-1)}}^x dx_{|I'|-1}.
\]

The formula applies to every compatible trivialization \( x_o \) of \( V \) locally. This implies that the Mathai-Quillen form is well-defined on \( V \), and Theorem 2 holds. \( \Box \)

2. Constrained connections and fiberwise integrals

We can put Theorem 2 into more general context. In general, assume that \( \nabla \) is a connection on a bundle \( V \), and \( Q \) is an involution-valued function on the bundle, the subbundles \( V \) and \( W \) are the \((-1)\)- and \((+1)\)-eigenspaces of \( Q \). Then one can define the constrained connection

\[
\nabla_Q = \frac{1}{2} (\nabla + (Q \cdot) \nabla (Q \cdot)) = \nabla + \frac{1}{2} Q (\nabla Q) 
\]

This connection leaves the bundles \( V \) and \( W \) invariant; in particular, \( \nabla_Q |_V \) can be taken. Now, if \( V \) is a metric bundle, and \( V \) is a subbundle then it is reasonable to define the constrained connection \( \nabla |_V \) as \( \nabla_Q |_V \), where \( Q \) is that unique orthogonal involution-valued function whose \((-1)\)-eigenspace is \( V \). In this case one can simply describe the constrained connection \( \nabla |_V \) locally as follows: Assume that the variables \( x_1, \ldots, x_n \) trivialize the bundle \( V \), and \( x_1, \ldots, x_k \) trivialize \( V \), and the metric is given by \( x^2 \). Assume that the connection \( \nabla \) as a covariant derivative in the coordinates \( x \) is given by the connection form \( \theta \). Then we can just take that covariant derivative on \( W \) whose connection form is just \( \theta \mid_{k \times k} \), i.e. \( \theta \) restricted to the variables \( x_1, \ldots, x_k \). This constrain operation has many natural properties, which are immediate to see, for example:

**Lemma 1.** (a) If \( V_2 \subset V_1 \subset V \) then \((\nabla |_{V_1}) |_{V_2} = \nabla |_{V_2} \).
(b) If \( V_1 \subset V, V_2 \subset V \) then \((\nabla_1 \oplus \nabla_2 |_{V_1 \oplus V_2}) = \nabla_1 |_{V_1 \oplus V_2} \oplus \nabla_2 |_{V_2} \). \( \Box \)

Let us take the orthogonal decomposition \( V \oplus V^\perp = V \), and the projection \( \pi_1 \) to the first factor.

**Theorem 3.** If \( V \) and \( V \) are oriented then

\[
\text{TMQ}(V, \nabla |_V) = (\pi_1)_* \text{TMQ}(V, \nabla)
\]

where \((\pi_1)_* \) is according to the complementary orientation on \( V^\perp \).

Clearly, Theorem 2 is the special case when the pair \((V, \nabla)\) is trivial.
Proof. According to [4] locally we can embed \((\mathcal{V}, \nabla)\) into the trivial pair \((M \times \mathbb{R}^{n+m}, \nabla_0)\) such that \(\nabla = \nabla_0|_{\mathcal{V}}\). Then Lemma 1a, and the fact that fiberwise integration has the similar property implies our statement through Theorem 2. \(\square\)

We remark that Theorem 3 is not hard to prove with simple supercalculus directly; the calculations are similar to the ones in the proof of Theorem 2.

Using elementary properties of the constrained connections we can also recover the behaviour of the Mathai-Quillen form with respect to patchings.

Assume that \((\mathcal{V}_1, \nabla_1), \ldots, (\mathcal{V}_s, \nabla_s)\) are metric bundle and connection pairs. Let \((\mathcal{V}, \nabla)\) be the direct sum of these pairs. Assume that \(V\) is a metric vector bundle and \(\iota_s : V \to \mathcal{V}_s\) are metric inclusions of bundles. Assume that \((\xi_1, \ldots, \xi_s)\) is a quadratic partition of unity, i.e., \(\sum \xi^2_s = 1\). Let \(\iota : V \to \mathcal{V}\) be the inclusion given by

\[
\iota(x) = \xi_1 \iota_1(x) + \cdots + \xi_s \iota_s(x).
\]

Then we claim:

**Lemma 2.**

\[
\iota^*(\nabla|_{\iota(V)}) = \xi^2_1 \iota^1_1(\nabla|_{\iota_1(V)}) + \cdots + \xi^2_s \iota^1_s(\nabla|_{\iota_s(V)}).
\]

**Proof.** By [4], Lemma 1 and locality, it is enough to prove the statement when \(V\) is \(M \times \mathbb{R}^n\), and \(\mathcal{V}_i\) are copies of \(M \times \mathbb{R}^{n+m}\) with trivial connections. Locally we can extend the inclusions \(\iota_s : M \times \mathbb{R}^n \to M \times \mathbb{R}^{n+m}\) to metric isomorphisms \(A_k : M \times \mathbb{R}^{n+m} \to M \times \mathbb{R}^{n+m}\). Let us consider the following block matrix, where every block is an \((n+m) \times (n+m)\) matrix:

\[
A = \begin{bmatrix}
0 & -\xi_1 \text{Id} & -\xi_2 \text{Id} & \cdots & -\xi_s \text{Id} \\
\xi_1 A_1 & (1 - \xi_1^2) A_1 & -\xi_1 \xi_2 A_1 & \cdots & -\xi_1 \xi_s A_1 \\
\xi_2 A_2 & -\xi_2 \xi_1 A_2 & (1 - \xi_2^2) A_2 & \cdots & -\xi_2 \xi_s A_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\xi_s A_s & -\xi_s \xi_1 A_s & -\xi_s \xi_2 A_s & \cdots & (1 - \xi_s^2) A_s
\end{bmatrix}.
\]

One can see that \(A\) is an orthogonal matrix. Let \(Q_0\) be that matrix which is the \((n+m)(s+1) \times (n+m)(s+1)\) identity matrix, except that its first \(n\) diagonal entries are \(-1\). Let \(X_{oo}\) denote the restriction of a matrix \(X\) to its top left \(n \times n\) submatrix (for block matrices they fall into the first block). Then a straightforward matrix computation yields:

\[
(A^{-1} dA)_{oo} = \sum_{i=1}^s \xi^2_s \sum_{i=1}^s (A^{-1}_s dA_i)_{oo}.
\]

According to the argument in the proof of Theorem 2, which culminated in [4], we see that the meaning of this equation is that the \((-1)\)-eigenspace bundle of \(Q_0\) pushed forward by \(A\) obtains a connection through the constrain operation such that its pull-back through \(A\) is the linear combination of the similar connections obtained through \(A_i\), which fact in the present situation is equivalent to the formula of the statement. \(\square\)

In fact, we may notice that the explicit part of the calculation above is essentially the same one which makes possible the result of [4]. As a corollary we obtain:

**Theorem 4.** Let \(V\) be a metric bundle and \(\nabla_1, \ldots, \nabla_s\) be various compatible connections. Let \(\xi_1, \ldots, \xi_s\) be a quadratic partition of unity. Let the inclusion \(\iota : V \to V \oplus \cdots \oplus V\) be defined as before. Then

\[
\text{TMQ}(V, \xi^2_1 \nabla_1 + \cdots + \xi^2_s \nabla_s) = \iota^*(\pi_\xi)_* \text{TMQ}(V \oplus \cdots \oplus V, \nabla_1 \oplus \cdots \oplus \nabla_s),
\]

where \(\pi_\xi\) is the orthogonal projection to \(\iota(V)\) given by the matrix \([\xi \xi_j \text{Id}_V]\).
Proof. This follows form Lemma 2 and Theorem 3 when the inclusions $\iota_k$ are isomorphisms.

References

[1] N. Berline, E. Getzler, M. Vergne: *Heat kernels and Dirac operators*. 2nd ed. Grundlehren der mathematischen Wissenschaften 298, Springer-Verlag, Berlin, 1998.

[2] R. Bott, L. W. Tu: *Differential forms in algebraic topology*. Graduate Texts in Mathematics 82, Springer-Verlag, Berlin, 1982.

[3] V. Mathai, D. Quillen: Superconnections, Thom classes and equivariant differential forms. *Topology* 25 (1986), 85–110.

[4] M. S. Narasimhan, S. Ramanan: Existence of universal connections. *Amer. J. Math.* 83 (1961), 583–572.

Department of Geometry, Eötvös University, Pázmány Péter s. 1/C, Budapest, H–1117, Hungary

E-mail address: lakos@cs.elte.hu