An arbitrary-order predefined-time exact differentiator for signals with exponential growth bound

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Abstract

Constructing differentiation algorithms with a fixed-time convergence and a predefined Upper Bound on their Settling Time (UBST), i.e., predefined-time differentiators, is attracting attention for solving estimation and control problems under time constraints. However, existing methods are limited to signals having an $n$-th Lipschitz derivative. Here, we introduce a general methodology to design $n$-th order predefined-time differentiators for a broader class of signals: for signals, whose $(n+1)$-th derivative is bounded by a function with bounded logarithmic derivative, i.e., whose $(n+1)$-th derivative grows at most exponentially. Our approach is based on a class of time-varying gains known as Time-Base Generators (TBG). The only assumption to construct the differentiator is that the class of signals to be differentiated $n$-times have a $(n+1)$-th derivative bounded by a known function with a known bound for its $(n+1)$-th logarithmic derivative. We show how our methodology achieves an UBST equal to the predefined time, better transient responses with smaller error peaks than autonomous predefined-time differentiators, and a TBG gain that is bounded at the settling time instant.

Key words: Fixed-time stability, predefined-time, unknown input observers, online differentiators

1 Introduction

The arbitrary-order exact differentiator problem is a relevant problem in control theory, that has recently received a great deal of attention (Levant, 1998, 2003; Levant and Livne, 2020; Sanchez et al., 2016; Reichhartinger and Spurgeon, 2018; Reichhartinger et al., 2017), as it can be applied to a wide range of control problems (Fridman et al., 2008, 2011; Rios and Teel, 2018; Rios et al., 2015; Alwi et al., 2011; Ferreira De Loza et al., 2015; Shtessel et al., 2014; Imine et al., 2011).

Differentiator algorithms where the user can prescribe the desired upper bound for the convergence time, i.e., predefined-time differentiators, are crucial for control systems under time constraints. For instance, predetermining the desired convergence time is of paramount importance in fault-detection to ensure that during the observer’s convergence, the system’s trajectory does not leave a given compact set, where the controller can be switched on based on theoretically exact knowledge of the states.

However, in (Levant, 2003; Levant and Livne, 2020), the knowledge of the set of possible initial conditions is

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required to set a priori the upper bound for the convergence (Cruz-Zavala and Moreno, 2019). To remove such restriction, fixed-time stability has been investigated, where there exists an Upper Bound of the Setting Time (UBST) that is valid for all initial conditions. Nonetheless, often the the fixed-time stability property is asserted using homogeneity theory (Andrieu et al., 2008) without explicit computation of the UBST (Angulo et al., 2013).

Predefined-time first-order differentiators have been proposed in (Cruz-Zavala et al., 2011; Seeber et al., 2020). However, these approaches yield a very conservative UBST. Satisfying the time constraints in such a conservative manner typically leads to differentiator errors that are larger than necessary. An alternative approach, based on a class of time-varying gains, known as time-base generators (TBG) (Morasso et al., 1997), was proposed in (Holloway and Krstic, 2019) for polynomial signals of $n$-th order, where the convergence is obtained precisely at the user-defined time, but with a TBG gain that tends to infinity at the settling time instant, which makes this application challenging under scenarios with measurement noise and/or limited numerical precision.

The above methods have been developed under the assumption that the $n$-th derivative is Lipschitz. This is a reasonable assumption for state estimation problems of chaotic systems with state-dependent disturbances (Gómez-Gutiérrez et al., 2017), since the evolution converges toward an invariant bounded set; it does not hold, however, in unstable systems' estimation problems with state-dependent disturbances (Fridman et al., 2011). However, since the states of an unstable linear or Lipschitz nonlinear system cannot grow faster than exponentially (Bejarano et al., 2011; Rodrigues and Oliveira, 2018), assuming disturbances with bounded logarithmic derivative has shown to reasonable in the evolutions of chaotic systems with state-dependent disturbances (Gómez-Gutiérrez et al., 2017), since the evolution behavior under noisy measurements compared with our previous result.

**Notation:** $\mathbb{R}$ is the set of real numbers, $\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+ = \mathbb{R}_+ \cup \{\infty\}$. For $x \in \mathbb{R}$, $|x|^\alpha = |x|^\alpha \text{sign}(x)$, if $\alpha \neq 0$ and $|x|^\alpha = \text{sign}(x)$ if $\alpha = 0$. For a function $\phi : I \to J$, its reciprocal $\phi(t)^{-1}$, $\tau \in I$, is such that $\phi(\tau)^{-1} \phi(\tau) = 1$ and its inverse function $\phi^{-1}(t)$, $t \in J$, is such that $\phi(\phi^{-1}(t)) = t$. For functions $\psi$, $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \circ \psi(t)$ denotes the composition $\phi(\psi(t))$. Given a matrix $A$, $A^T$ represents its transpose. Given a vector $v \in \mathbb{R}^{n+1}$, $\|v\| = \sqrt{v^T v}$. $\mathcal{U} := [u_{ij}] \in \mathbb{R}^{(n+1) \times (n+1)}$, where $u_{ij} = 1$ if $j = i + 1$ and $u_{ij} = 0$, otherwise. $D := \text{diag}(0, \ldots, n)$, $\mathcal{B} := [0, \ldots, 0, 1]^T \in \mathbb{R}^{n+1}$.

The rest of the manuscript is organized as follows: In Section 2, we introduce the problem statement and some preliminaries. In Section 3 we present the main result. In Section 4, we discuss the main features of our approach and contrast against state-of-the-art algorithms. Finally, in Section 5, we present the conclusions and the future work. The proofs are collected in the Appendix.

## 2 Problem statement and preliminaries

### 2.1 Problem statement

**Definition 1** Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be a function and $M > 0$ be a positive constant satisfying

$$\frac{1}{L(t)} \left| \frac{dL(t)}{dt} \right| \leq M \text{ for all } t \geq 0, \quad (1)$$

i.e., a function with logarithmic derivative bounded by $M$. The set of admissible signals $\mathcal{Y}^{(n+1)}_{L(M)}$ is the set of n times differentiable signals $y(t) \in \mathbb{R}$ satisfying

$$\left| y^{(n+1)}(t) \right| \leq L(t), \text{ for all } t \geq 0.$$ 

**Problem 2** Let $y \in \mathcal{Y}^{(n+1)}_{L(M)}$ and consider a user-defined time $T_c > 0$. The problem consists in accurately obtaining, from measuring $y(t)$ and the knowledge of $L(t)$, the functions $y^{(i)}(t)$, $i = 0, \ldots, n$, for all time $t \geq T_c$. 

Compared to our previous results also based on a TBG gain: (Aldana-López et al., 2020), which was limited to signals having a Lipschitz $n$-th derivative, and (R. Aldana-López et al., 2020) which is limited to the first-order differentiation case. In both papers, a single type of TBG gains was used. Here, we present the arbitrary-order case and we extend the class of functions that can be differentiated as well as the class of TBG gains used. We will show numerically that these two features allow us to obtain better transient behavior and improved behavior under noisy measurements compared with our previous result.
Solving this problem enables the application to control problems with time constraints. Given a desired convergence time, to solve Problem 2, we propose to design functions \( H : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{n+1} \), together with the (differentiator) algorithm:

\[
\dot{z} = -H(e_0, t; T_c) + Uz, \tag{2}
\]

where \( z(t) = [z_0(t), \cdots, z_n(t)]^T \), \( e_0 = z_0(t) - y(t) \) and \( U \) is defined in the Notation. Here, the function \( H(e_0, t; T_c) \) is continuous in \( e_0 \) except at \( e_0 = 0 \), and continuous in \( t \) almost everywhere. The solutions of (2) are understood in the sense of Filippov [1988].

Consider the differentiation errors \( e_i(t) = z_i(t) - y^{(i)}(t) \) for \( i = 0, \ldots, n \). To analyze the convergence of the differentiators, we analyze the stability of the origin of the differentiation error dynamics given by

\[
\dot{e} = -H(e_0, t; T_c) + Ue - By^{(n+1)}(t), \tag{3}
\]

where \( e(t) = [e_0(t), \cdots, e_n(t)]^T \), \( |y^{(n+1)}(t)| \leq L(t) \), and \( U \) and \( B \) are matrices defined in the Notation.

Assume that \( H(e_0, t; T_c) \) is designed ensuring that the origin of (3) is asymptotically stable and, perhaps except at sets of measure zero, (3) has the properties of existence and uniqueness of solutions in forward-time on the interval \([0, \infty)\) Filippov [1988]. Moreover, assume that the origin is the unique equilibrium point of (3). For the system in Eq. (3), its \textit{settling-time function} \( T(e(0)) \) for the initial state \( e(0) \in \mathbb{R}^{n+1} \) is defined as:

\[
T(e(0)) = \inf \left\{ \xi \geq 0 : \forall y \in \mathcal{Y}^{(n+1)}_{(L,M)}, \lim_{t \to \xi} e(t; e(0), y) = 0 \right\},
\]

where \( e(t; e(0), y) \) is the solution of (3) for \( t \geq 0 \), with signal \( y(t) \) and initial condition \( e(0) \).

We say that the origin of system (3) is \textit{finite-time stable} if it is asymptotically stable Khalil and Grizzle [2002] and for every initial state \( e(0) \in \mathbb{R}^n \), the settling-time function \( T(e(0)) \) is finite. We say that the origin of system (3) is \textit{fixed-time stable} if it is asymptotically stable Khalil and Grizzle [2002] and there exists \( T_{\text{max}} < \infty \) such that \( T(e(0)) \leq T_{\text{max}} \), for all \( e(0) \in \mathbb{R}^n \). The quantity \( T_{\text{max}} \) is called an Upper Bound of the Settling Time (UBST) of the system (3).

With the above definitions, a differentiator is said to be \textit{exact} if the origin of its error dynamic is globally finite-time stable. Algorithm (2) is a predefined-time exact differentiator if the origin of its differentiation error dynamic (3) is fixed-time stable with a predefined UBST.

\[2.2 \text{ Levant’s differentiators with time-varying gains}\]

\textbf{Theorem 3 (Levant and Livne 2018)} Let \( y \in \mathcal{Y}^{(n+1)}_{(L,M)} \) and consider the algorithm:

\[
\dot{z} = -\Phi(e_0, t; M, L(t)) + Uz, \tag{4}
\]

where

\[
\Phi(e_0, t; M, L(t)) := \left[ \phi_0(e_0, t; M, L(t)), \cdots, \phi_n(e_0, t; M, L(t)) \right]^T
\]

and the functions \( \{ \phi_i \}_{i=0}^n \) are defined recursively as

\[
\phi_0(w; M, L(t)) := \chi_0(w; M, L(t))
\]

and

\[
\phi_i(w; M, L(t)) := \chi_i(\phi_{i-1}(w; M, L(t)); M, L(t)) \text{ for } i = 0, \ldots, n,
\]

with

\[
\chi_i(w; M, L(t)) := \lambda_{n-i} L(t) \pi^{-i+1} [w]^{n-i+1} \mu_{n-i} M w,
\]

wherein \( \lambda_i \) and \( \mu_i \) are constant parameters. Then, there exist positive constants \( \lambda_i, \mu_i (i = 0, \ldots, n) \) such that the algorithm of Eq. (4) is an exact differentiator in \( \mathcal{Y}^{(n+1)}_{(L,M)} \) or, in other words, such that the origin of the system

\[
\dot{e} = -\Phi(e_0, t; M, L(t)) + Ue - By^{(n+1)}(t), \tag{5}
\]

where \( e_i = z_i(t) - y^{(i)}(t), i = 0, \ldots, n, \) (i.e., system (3) with \( H(w, t; T_c) = \Phi(w, t; M, L(t)) \), is finite-time stable.

Notice that, if we relax the condition (1) to: M is such that there exist \( t^* \) such that \( t_0 \leq t^* < \infty \) and

\[
\frac{dL(t)}{dt} \leq M, \text{ for all } t \geq t^*, \text{ then the finite-time stability of (5) is maintained.}
\]

\[3 \text{ Main result}\]

To solve Problem 2 we take the correction function \( \Phi_0, t; \bullet, \bullet \) of the finite-time differentiator (4) from Levant and Livne [2018] to “redesign” it together with a TBG gain to obtain a new correction function \( H(e_0, t; T_c) \) such that Eq. (2) is a predefined-time differentiator and the desired UBST of the error dynamics (3) is given by the parameter \( T_c \).

To introduce our main result consider the following assumption on the functions \( \Omega \) that will be used to construct the TBG gain.

\textbf{Assumption 4} The function \( \Omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \setminus \{0\} \) is such that: \( \int_0^\infty L(t) dt = 1; \Omega(t) < \infty, \text{ for all } t > 0; \) it
is either non-increasing or locally Lipschitz on \(\mathbb{R}_+ \setminus \{0\}\); and \(\Omega(z) = -\frac{d\Omega(z)}{dz}\) is uniformly bounded with respect to time and satisfies
\[
\lim_{z \to \infty} \Omega(z) = -c
\]
for some finite constant \(c \geq 0\).

Notice that \(\Omega\) is a probability density function. An essential part of our approach is the TBG gain \(\kappa(t)\) defined as \(\kappa(t) := \frac{d\varphi(t)}{dt}\), where \(\varphi\) is defined by means of its inverse function as \(\varphi^{-1}(t) := T_c \int_0^t \Omega(\xi) d\xi\). As we will show, the TBG gain is related to the time-scale transformation \(\tau = \varphi(t)\) because \(\frac{d\tau}{dt} = \kappa(t)\).

**Theorem 5** Consider the class of signals \(Y_{(L,M)}^{(n+1)}\). Let \(T_c > 0\) be a user-defined time, \(\Omega(\bullet)\) and \(c \geq 0\) satisfy Assumption 4, and define \(M > (n + 1)c\) and \(Q(c) := [(\mathcal{U} - c\mathcal{D})^n \mathcal{B}; \ldots; (\mathcal{U} - c\mathcal{D}) \mathcal{B}; \mathcal{B}]\), where \(\mathcal{B}, \mathcal{D}\) and \(\mathcal{U}\) are defined in the Notation section. Furthermore, let \(\Phi(\varepsilon_0; \bullet, \bullet)\) be the error correction function of Levant’s differentiator recalled in Theorem 3, with parameters \(\lambda_i\) and \(\mu_i\) as in that theorem.

Then, using the algorithm (2) with \(H(w, t; T_e)\) as
\[
H(\varepsilon_0, t; T_e) = \Lambda(t) \left( Q(c) \Phi(\varepsilon_0; M, L(t) \kappa(t)^{-n+1}) + (\mathcal{U} - c\mathcal{D})^{n+1} \mathcal{B} \varepsilon_0 \right)
\]
for \(t \in [0, T_e]\), and \(H(\varepsilon_0, t; T_e) = \Phi(\varepsilon_0, t; M, L(t))\), otherwise, solves Problem 2, i.e., the origin of the differentiation error dynamics (3) is fixed-time stable with a predefined UBST given by \(T_e\), where \(\Lambda(t) = \text{diag}(\kappa(t)^{n+1})\) and \(\kappa(t)\) as described above. Additionally, if \(L(t)\) is such that the settling-time function of (5) satisfies
\[
\sup_{e(0) \in \mathbb{R}^{n+1}} T(e(0)) = \infty,
\]
then, \(T_e\) is the least UBST of (3).

**Remark 6** Note that (7) is usually satisfied, unless \(L(t)\) is chosen in a special way to achieve fixed-time convergence of the original differentiator, such as in [Moreno (2017)].

**Proof.** The proof can be found in Appendix A.2.

Examples of time-varying gains for predefined-time convergence are given in Table 1.

| \(\kappa(z)\) | \(c\) |
|-----------------|-------------|
| \((i)\) \(\alpha^{-1}(T_c - z)^{-1}\) | \(\alpha\) |
| \((ii)\) \(\frac{\pi}{2} \sec\left(\frac{\pi}{2n}\right)^2\) | 0 |
| \((iii)\) \(\frac{\pi}{T_c} \tan\left(\frac{\pi}{T_c} + \frac{\pi}{2} - \gamma\right)\) | 1 |
| \((iv)\) \(\frac{\pi}{T_c} \tan\left(\frac{\pi}{T_c} + \frac{\pi}{2} - \gamma\right)\) | \(\frac{\alpha}{1 + \beta}\) |

Table 1: Examples of time-varying gains for predefined-time convergence. Above, \(\alpha, \beta > 0\) and \(\gamma \in (0, \frac{\pi}{2})\).

**Remark 7** It is straightforward to extend our methodology to filtering differentiators [Levant and Livne 2020, Carujo-Rubio et al. 2020]. Specifically, let \(H(w, t; T_e) = [h_0(w, t; T_e), \ldots, h_n(w, t; T_e)]^T\) be selected as in Theorem 5 for signals in \(Y_{(L,M)}^{(n)}\). Then, the algorithm
\[
\dot{w}_i = -h_{i-1}(w_1, t; T_e) + w_{i+1},
\]
for \(i = 1, \ldots, n_f - 1\),
\[
\dot{w}_i = -h_{i-1}(w_1, t; T_e) + (z_0 - y),
\]
for \(i = n_f\),
\[
\dot{z}_{i-n_f-1} = -h_{i-1}(w_1, t; T_e) + z_{i-n_f},
\]
for \(i = n_f + 1, \ldots, n_d + n_f\),
\[
\dot{z}_{n_d} = -h_n(w_1, t; T_e),
\]
with \(n = n_d + n_f\), is a predefined-time exact differentiator but now for signals in \(Y_{(L,M)}^{(n+1)}\). However, in this case, we obtain that, for all \(t \geq T_e\) and every initial condition: \(w_i(t) = 0, i = 1, \ldots, n_f\), and \(z_i(t) = y^{(i)}(t)\) for all \(i = 0, \ldots, n_d\). For \(n_f = 0\), \(w_i(t)\) is defined as \(w_i(t) = z_0(t) - y(t)\). The filtering properties of this algorithm can be very useful in the presence of noise [Levant and Livne 2020].

4 Discussion on the contribution

In this section we highlight the contribution together with numerical examples to support our claims. The simulations below were created in OpenModelica using the Euler integration method with a step of \(2 \times 10^{-4}\). For the sake of simplicity we use the sequence
\[
\{\lambda_i, \mu_i\}_{i=0}^n = (1.1, 2), (1.5, 3), (2, 4), (3, 7), (5, 9), \ldots
\]
as suggested in [Levant and Livne 2018].

4.1 Comparison with arbitrary order exact differentiators for polynomial signals of \(n\)-th order

Notice that predefined-time higher-order exact differentiators are only provided in [Ménard et al. 2017, Holoway and Krstic 2019], but only for polynomial signals...
of \( n \)-th order. Whereas the UBST in \cite{Ménard2017} is very conservative, the convergence in the algorithm given in \cite{Holloway2019}, occurs precisely at the predefined-time.

**Remark 8** The TBG gain of Table 1-(i) was also used in \cite{Holloway2019} with \( \alpha = 1 \). However, notice that in such algorithm, for every nonzero \( e(0) \),

\[
\lim_{t \to T^*(e(0))} \kappa(t) = \infty.
\]

This drawback also appear in the controllers based on TBG gains \cite{Song2017,Chitour2020,Pal2020}.

Nonetheless, in our approach, although \( \lim_{t \to T_c} \kappa(t) = \infty \), the TBG gain is bounded at the settling time \( T(e(0)) \), i.e., for any initial condition \( e(0) \in \mathbb{R}^{n+1} \), \( \kappa(T(e(0))) < \infty \). Thus, in practice different methods can be used to maintain the TBG gain bounded for all time. A simple approach is to choose a constant \( T^* < T_c \) and to do the switching in \( H(e_0; t; T_c) \) at \( T^* \) instead that at \( T_c \).

With such workaround, the differentiation error of our algorithm is still finite-time convergent. Moreover, there exists a neighbourhood of initial conditions around the origin, whose settling time is bounded by \( T_c \). The size of such neighbourhood can be set arbitrarily large with a suitable selection of the \( \alpha \) and \( \mathcal{M} \) parameters.

The above workaround was suggested in \cite{Holloway2019}. However, with such workaround an exact differentiator is no longer obtained with the algorithm in \cite{Holloway2019} (recall that convergence occurs precisely at \( T_c \)).

**Example 9** Consider the second-order differentiator problem. To contrast with \cite{Holloway2019} consider the quadratic polynomial \( y(t) = \frac{1}{2}T^2 + T + 1 \), and notice that the algorithm in \cite{Holloway2019} is given by Eq. (2) with \( h_i(e_0; t; T_c) = g_i(e_0; t; T_c) e_0 \), and

\[
\begin{align*}
\quad \quad g_0(e_0; t; T_c) &= l_1 + T_c \kappa(t) \left( \frac{p_{11}(m+3)}{T_c} - p_{21} \right), \\
\quad \quad g_1(e_0; t; T_c) &= l_2 + T_c^2 \kappa(t)^2 \left( \frac{p_{21}(m+4)}{T_c} - p_{31} \right) - p_{21} T_c \kappa(t) y_0(e_0; t; T_c) \\
\quad \quad g_2(e_0; t; T_c) &= l_3 + T_c^3 \kappa(t)^3 \frac{p_{31}(m+5)}{T_c} - p_{31} T_c^2 \kappa(t)^2 g_0(e_0; t; T_c) - p_{32} T_c \kappa(t) g_1(e_0; t; T_c)
\end{align*}
\]

where \( p_{11} = 1, p_{21} = -\frac{2(m+3)}{T_c}, p_{31} = \frac{(m+3)(m+4)}{T_c^2}, p_{32} = -\frac{m+3}{T_c} \), \( m \) is a design parameter, here chosen to be \( m = 1 \), and \( [l_1, l_2, l_3] = [6, 11, 6] \). Moreover, \( \kappa(t) \) is given in Fig. 1. Magnitude of the error at \( T^* = 0.9 < T_c \) for different initial conditions for the algorithm in \cite{Holloway2019} using the workaround described in Remark 8 to maintain bounded gains.

Table 1-(i) with \( \alpha = 1 \) and \( L = 1 \). Consider the workaround discussed in Remark 8 to maintain a bounded TBG gain. For illustration purposes we choose \( \kappa(t) \leq 10 \). Thus, \( T^* = 0.9 \). Notice that, zero differentiation error is no longer obtained. In fact, the magnitude \( \|e(T^*)\| \) grows linearly with the initial condition, as shown in Fig. 1.

For our algorithm consider \( \kappa(t) \) given by Table 1-(i) with \( \alpha = 1 \), \( L = 0.1 \exp(-5t) \), \( \mathcal{M} = 0.5 \), and \( \mathcal{M} = 20 \), whose convergence is shown in the second row of Fig. 2. Notice that, under the same workaround, even for an initial condition satisfying \( z_0(0) = z_1(0) = z_2(0) = 10000 \) zero differentiation error before \( T_c \) is still obtained. A significant advantage in contrast with the algorithm in \cite{Holloway2019}.

**4.2 Comparison with autonomous first-order differentiator**

As mentioned in the introduction, predefined-time autonomous exact differentiators for signals whose \( n \)-th derivative is Lipschitz have only been proposed in the literature for the case where \( n = 1 \) in \cite{CruzZavala2011,Seebere2020}. In the next example we compare against the algorithm in \cite{Seebere2020}.

**Example 10** Let \( y(t) = 0.75 \cos(t) + 0.0025 \sin(10t) + t \) with \( L(t) = 1 \). For comparison, consider the algorithm in \cite{Seebere2020}, i.e., algorithm (2) with \( h_i(w; t; T_c) = k_i \nu_i(w; T_c), i = 1, 2 \), where

\[
\begin{align*}
\quad \nu_1(w; T_c) &= |w|^{\frac{1}{2}} + k_2 |w|^{\frac{1}{2}} \\
\quad \nu_2(w; T_c) &= |w|^{2} + 4k_3^2 w + 3k_4^2 |w|^2
\end{align*}
\]

where \( k_1 = 4 \sqrt{T_c}, k_2 = 2L \) and \( k_3 = \frac{9.8}{T_c \sqrt{T_c}} \). The convergence of (9), under different initial conditions is shown in the first row of Fig. 3, where can be observed that the convergence occurs before 0.25s. Now consider our algorithm with \( \kappa(t) \) given in Table 1-(i) with \( \alpha = 1, L = 1, M = 0.1 \) and \( \mathcal{M} = 3.2 \). The convergence is observed in the second row of Fig. 3. Also consider our algorithm with \( \kappa(t) \) given in Table 1-(iv) with \( \alpha = 1, \beta = 0.1 L = 1, \)
It can be verified in Fig. 3 that in our algorithm the slack between the least UBST and the desired one given by $T_c$ is significantly reduced. In addition, the trajectory of the error, the integral of the magnitude of the error correction term $\mathcal{H}(e_0, t; T_c)$ is shown in the second column of Fig. 4. Notice that in the second row of Fig. 4 that with our approach, the quality of the transient is significantly improved. Moreover, notice in the third row of Fig. 4 that having the degree of freedom to select different classes of TBG gains results in allowing to further reduce the maximum error.

4.3 Robustness to Noisy measurements

Although we show simulations under noise, a formal analysis of our algorithm’s filtering properties is considered future work. Notice that after $T_c$, the results in (Levant and Livne, 2018) can be applied. In the next example, we show numerically the robustness to noise and the precision obtained before $T_c$.

Consider the redesign method proposed in (Aldana-López et al., 2020) using as a base differentiator the algorithm from Seeber et al. (2020). We highlight that, although the TBG gain remains bounded, in the resulting algorithm, the part associated with the homogeneous approximation at the origin (for details on
**Example 10** Let \( y(t) = 0.75 \cos(t) + 0.0025 \sin(10t) + t \) with \( L(t) = L = 1 \). Consider the differentiator in [Seeber et al., 2020], described in Example 10 with \( T_c = 1 \). The differentiator in [Aldana-López et al., 2020, Example 2] with \( T_f = T_c = 1 \) and \( \alpha = 3 \), for the sake of comparison we set the parameter \( \eta \) in [Aldana-López et al., 2020] to \( \eta = 1 \), but we switch from the redesigned differentiator to Levant’s differentiator at \( t = T_c(1 - \exp(-\alpha T_f)) \approx 0.95 \) which is an UBST for it. Now, consider our algorithm with \( \kappa(t) \) given in Table 1-(i) with \( \alpha = 3 \), and let \( L = 1 \), \( M = 0.1 \) and \( M = 3.2 \). To maintain the TBG bounded we use the workaround in Remark 8 with \( T^* = 0.95 \). Consider an initial condition \( z_0(0) = z_1(0) = 10 \) grea for each algorithm. In Fig. 5 we show the integral of the magnitude of the error correction term \( \int_{t_0}^{t} \|H(c_0, \xi; T_c)\| d\xi \) for each differentiator, where we can verify that our algorithm is the one with the lowest value. In Fig. 6, we show a benchmark under a Gaussian noise with zero mean and standard deviation \( \sigma \), where it can be verified that our algorithm has a significant better behavior under noise.

### 4.4 Comparison for high-order predefined-time exact differentiators

To our best knowledge the only arbitrary-order predefined-time exact differentiator that has been proposed in the literature is our previous work (Aldana-López et al., 2020), which is only applied to signals whose \((n + 1)\)-th derivative is bounded by a known constant. We show in the next example that this restriction is not presented in this paper.

**Example 12** Consider the differentiator algorithm (2) with \( n = 2 \) and let \( y(t) = 2 \sin \left( \frac{1}{2} t^2 \right) \), which satisfies

\[
|\dddot{y}(t)| \leq L(t)
\]

with \( L(t) = \sqrt{1 + 4t^6 + 36t^2} \), \( \frac{1}{L(t)} \left| \frac{dL(t)}{dt} \right| \leq M = 3.5 \). Consider our algorithm with \( \kappa(t) \) given in Table 1-(iii) with \( \gamma = 0.01 \), \( \beta = 6 \) and \( M = 8 \). The convergence of this algorithm for different initial conditions is shown in the first row of Fig. 7, where \( T_c = 5 \). To maintain the \( \kappa(t) \) bounded, we select \( T^* = 4.5 \).

Now consider the filtering differentiator (8) with \( n_f = 1 \) and \( n_d = 1 \), it is easy to verify that \( y \) satisfies

\[
|\dot{y}(t)| \leq L(t)
\]

with \( L(t) = \sqrt{1 + 4t^6} \), \( \frac{1}{L(t)} \left| \frac{dL(t)}{dt} \right| \leq M = 3.5 \). Consider our algorithm with \( \kappa(t) \) given in Table 1-(iii) with \( \gamma = 0.01 \), \( \beta = 6 \) and \( M = 8 \); and let the function \( y(t) \) be affected by a white Gaussian noise signal \( \eta(t) \) with zero mean and standard deviation \( \sigma = 0.5 \). To maintain the \( \kappa(t) \) bounded, we select \( T^* = 4.5 \). The convergence of this algorithm for different initial conditions is shown in the second row of Fig. 7.
5 Conclusion

This paper introduced a predefined-time arbitrary-order exact differentiation algorithm for signals whose \((n+1)\)-th derivative has an exponential growth bound. It was shown that by suitable design of the parameters, the achieved Upper Bound of the Settling Time (UBST) is equal to the predefined time.

For the narrower class of signals having a Lipschitz continuous \(n\)-th derivative, we show that, compared to autonomous differentiator algorithms, in our approach, the maximum differentiation error is several orders of magnitude lower. Compared to other differentiators based on TBG gains, simulations show that in our approach, zero differentiation error is obtained before the time moment when the singularity in the TBG gain occurs. We show in numerical examples that our has better filtering properties than state-of-the-art predefined-time differentiators.

As future work we consider the discretization of our algorithm. Discretization of differentiators is an active area of research (Livne and Levant, 2014; Carvajal-Rubio et al., 2019; Barbot et al., 2020; Wetzlinger et al., 2019). Of particular interest is consistent discretization maintaining the convergence properties (Koch and Reichhartinger, 2019; Polyakov et al., 2019; Jiménez-Rodríguez et al., 2020).

A Appendix

A.1 Preliminaries on time-scale transformations

The trajectories corresponding to the system solutions of (3) are interpreted, in the sense of differential geo-
try (Kühnel, 2015), as regular parametrized curves (Picó et al., 2013). Since we apply regular parameter transformations over the time variable, this reparametrization is referred to as time-scale transformation.

**Definition 13 (Regular parametrized curve [Kühnel, 2013, Definition 2.1])** A regular parametrized curve with parameter $t$, is a $C^{1}(I)$ immersion $c : I \rightarrow \mathbb{R}$, defined on a real interval $I \subseteq \mathbb{R}$. This means that $\frac{dc}{dt} \neq 0$ holds everywhere.

**Definition 14 (Regular curve [Kühnel, 2013, Pg. 8])** A regular curve is an equivalence class of regular parametrized curves, where the equivalence relation is given by regular (orientation preserving) parameter transformations $\varphi$, where $\varphi : I \rightarrow I'$ is $C^{1}(I)$, bijective and $\frac{dc}{d\tau} > 0$. Therefore, if $c : I \rightarrow \mathbb{R}$ is a regular parametrized curve and $\varphi : I \rightarrow I'$ is a regular parameter transformation, then $c$ and $c \circ \varphi : I' \rightarrow \mathbb{R}$ are considered to be equivalent.

**Lemma 15 (Aldana-López et al., 2019)** Let $\Omega(\bullet)$ satisfy Assumption 4 and let $\varphi(t)$ be such $\varphi^{-1}(\tau) := T_{\tau}^{0} \Omega(\xi)d\xi$. Then, $t = \varphi^{-1}(\tau)$ is a parameter transformation (time-scaling).

### A.2 Proofs of the Main Result

Before giving a proof for the main result, consider the following auxiliary result.

**Lemma 16** Let $e := [e_0, \ldots, e_n]^T$ and let $\Omega(\bullet)$ and $\Phi(\bullet, \bullet, \bullet)$ satisfy the conditions of Theorem 5 and let $\delta(\tau)$ be a disturbance satisfying $|\delta(\tau)| \leq \mathcal{L}(\tau)$, and $\mathcal{M}$ such that $\exists \tau^* > 0$, $\frac{1}{\mathcal{L}(\tau)} \frac{d\mathcal{L}(\tau)}{d\tau} \leq \mathcal{M}$ for all $\tau > \tau^*$. Then, the origin of the system

$$
\frac{de}{dt} = -\Phi(e_0; \mathcal{M}, \mathcal{L}(\tau)) + \mathcal{U}e + \mathcal{B}\delta(\tau) + \left(\Omega(\tau)^{-1}\frac{d\Omega(\tau)}{d\tau} + c\right)\mathcal{Q}(c)^{-1}\mathcal{D}\mathcal{Q}(c)e
$$

is globally finite-time stable.

**PROOF.** Rewriting system (A.1) using the coordinates $\psi(\tau) = \frac{e(\tau)}{\mathcal{L}(\tau)}$ yields

$$
\frac{d\psi}{dt} = -\Phi(\psi_0; \mathcal{M}, 1) - \frac{1}{\mathcal{L}(\tau)} \frac{d\mathcal{L}(\tau)}{d\tau} \psi + \mathcal{U}\psi + \mathcal{B}\delta(\tau) + \left(\Omega(\tau)^{-1}\frac{d\Omega(\tau)}{d\tau} + c\right)\mathcal{Q}(c)^{-1}\mathcal{D}\mathcal{Q}(c)e
$$

Consider first the nominal (unperturbed) part

$$
\frac{d\psi}{dt} = -\Phi(\psi_0; \mathcal{M}, 1) - \frac{1}{\mathcal{L}(\tau)} \frac{d\mathcal{L}(\tau)}{d\tau} \psi + \mathcal{U}\psi + \mathcal{B}\delta(\tau) + \left(\Omega(\tau)^{-1}\frac{d\Omega(\tau)}{d\tau} + c\right)\mathcal{Q}(c)^{-1}\mathcal{D}\mathcal{Q}(c)e.
$$

Due to $|\frac{1}{\mathcal{L}(\tau)} \frac{d\mathcal{L}(\tau)}{d\tau}| \leq \mathcal{M}$ and $|\delta(\tau)| \leq 1$, its trajectories are solutions of the time-invariant inclusion

$$
\frac{d\psi}{dt} \in -\Phi(\psi_0; \mathcal{M}, 1) + \mathcal{U}\psi + [-\mathcal{M}, \mathcal{M}]\psi + \mathcal{B}[-1,1].
$$

(A.2)

According to Theorem 3, this inclusion is finite-time stable. Furthermore, it follows from (Levant and Livne, 2018, Lemma 2) that it is also uniformly exponentially stable, because the constants $\mathcal{Q}_n$, $\Delta T_n$ in that lemma do not depend on the initial time instant nor on the initial state. Hence, according to (Clarke et al., 1998) there exists a smooth, strong Lyapunov function $V_{\mathcal{F}} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Using standard arguments, see (Khalil and Grizzle, 2002, Lemma 9.1), convergence of all trajectories of the perturbed system (A.1) to the origin is hence guaranteed, because the perturbation is linear in $\epsilon$ and its coefficient is eventually bounded by a sufficiently small constant due to Eq. (6).

To show that also finite-time stability is maintained, consider the homogeneous approximation at the origin of the time-invariant inclusion (A.2) that is obtained by setting $\mathcal{M} = 0$. Being a special case of (A.2), this system is still finite-time stable in addition to having homogeneity degree minus one with respect to the weights $(n+1, \ldots, 1)$. It is straightforward to verify that the matrix $\mathcal{Q}(c)$ and consequently also the matrix $\mathcal{Q}(c)^{-1}\mathcal{D}\mathcal{Q}(c)$ are lower-triangular by construction. Hence, the lowest-degree homogeneous approximation of the perturbation term $\mathcal{Q}(c)^{-1}\mathcal{D}\mathcal{Q}(c)e$ has a degree of at least zero. Finite-time stability is then concluded by applying (Bhat and Bernstein, 2003, Theorem 7.4) and noting that the theorem (and its proof) stay valid in the time-varying case, as long as the lowest-degree homogeneous approximation (with degree minus one) is time invariant and the time-varying part is uniformly bounded with respect to time.

**PROOF.** [Proof of Theorem 5] Let $e_i(t) = z_i - \frac{d^i y(t)}{dt^i}$, $0, \ldots, n$. The proof is divided in two parts. First we will show that $e_0(t) = 0$, $i = 0, \ldots, n$, for $t \in [\hat{t}, T_e)$ for some time $\hat{t}$. Afterwards, we show that the condition $e_i(t) = 0$, $i = 0, \ldots, n$ is maintained for all $t > T_e$.

Let $e := [e_0, \ldots, e_n]^T$; $c := [e_0, \ldots, e_n]^T$, and define

$$
\mathcal{A}(c) := -\mathcal{Q}^{-1}(\mathcal{U} - \alpha D)^{n+1}\mathcal{B}[1,0, \ldots, 0].
$$

Then, the dynamic for the differentiation error can be
Then, the dynamic of the $\epsilon$ for $t< Q_d\epsilon$ is given by

$$\dot{\epsilon} = -\Lambda(t)Q(c)[\Phi(e_0; \mathcal{M}, \mathcal{L}(\varphi(t))) - A(c)e] + \U e - B \eta^{(n+1)}(t).$$

With the coordinate change $\epsilon = \kappa(t)Q(c)^{-1}\Lambda(t)^{-1}e$, and considering that

$$\frac{d\kappa(t)}{dt} = -i\kappa(t)^{-1}\frac{d\kappa(t)}{dt},$$

$$\Lambda(t)^{-1}U\Lambda(t) = \kappa(t)U\kappa(t)^{-1}B = \kappa(t)^{-n}B.$$

Then, the dynamic of the $\epsilon$ variable, is given by

$$\dot{\epsilon} = \kappa(t)\left(Q(c)^{-1}[U - \kappa(t)^{-2}\frac{d\kappa(t)}{dt}\mathcal{D}]Q(c)e - \Phi(e_0; \mathcal{M}, \mathcal{L}(\varphi(t))) + A(c)e - \kappa(t)^{-1}\eta^{(n+1)}B\eta^{(n+1)}(t)\right).$$

(A.3)

Now, consider the time-scaling given in Lemma 15 and notice that $\mathcal{L}(\tau) := L(\varphi^{-1}(\tau))\rho(\tau)^{-n+1}$, where

$$\rho(\tau) = \frac{1}{T_c}\Omega(\tau)^{-1} = \kappa(t)|_{\tau=\varphi^{-1}(\tau)}$$

and $L(\varphi^{-1}(\tau)) = L(t)|_{t=\varphi^{-1}(\tau)}$. Notice that,

$$\frac{1}{\mathcal{L}(\tau)} \left| \frac{d\mathcal{L}(\tau)}{d\tau} \right| \leq \left| M\rho(\tau)^{-1} - (n+1)\rho(\tau)^{-1}\frac{d\rho(\tau)}{d\tau} \right|.$$ 

Thus, if $\mathcal{M} > (n+1)c$, there exists $\tau^*$ such that $\frac{1}{\mathcal{L}(\tau)} \left| \frac{d\mathcal{L}(\tau)}{d\tau} \right| \leq \mathcal{M}$, since $\rho(\tau)^{-1}$ tends to zero and $\rho(\tau)^{-1}\frac{d\rho(\tau)}{d\tau}$ tends to $c$ for $\tau \to \infty$.

Then, $\frac{d\tau}{dt} = \frac{d\tau}{dt}|_{t=\varphi^{-1}(\tau)}$. Since $\frac{d\tau}{dt}|_{t=\varphi^{-1}(\tau)} = \kappa(t)^{-1}$, for $t \in [0, T_c)$, then the dynamics of (A.3) in the $\tau$-time is given by

$$\frac{d\epsilon}{d\tau} = Q(c)^{-1}[U + \left(\Omega(\tau)^{-1}\frac{d\Omega(\tau)}{d\tau} + c - c\right)\mathcal{D}]Q(c)e - \Phi(e_0; \mathcal{M}, \mathcal{L}(\varphi)) + A(c)e - \mathcal{B}\delta(\tau),$$

Since, $Q(c)\mathcal{U} = (\mathcal{U} - c\mathcal{D})Q(c) + Q(c)A(c)$, then $Q(c)^{-1}[\mathcal{U} - c\mathcal{D}]Q(c) = \mathcal{U} - A(c)$, and

$$\frac{d\epsilon}{d\tau} = -\Phi(e_0; \mathcal{M}, \mathcal{L}(\varphi)) + \U e + \mathcal{B}\delta(\tau) + \left(\Omega(\tau)^{-1}\frac{d\Omega(\tau)}{d\tau} + c\right)Q(c)^{-1}\mathcal{D}Q(c)e, \quad (A.4)$$

which according to Lemma 16, system (A.4) is finite-time stable and has a settling time function $T(\epsilon(0))$. Using Lemma 15, we can conclude that the settling-time function of (3) is

$$T(\epsilon(0)) = \lim_{\tau \to \infty} \varphi(\tau) = T_c \int_0^{T_c(\epsilon(0))} \Omega(\xi)d\xi.$$

(A.5)

Thus,

$$\sup_{\epsilon(0) \in \mathbb{R}^{n+1}} T(\epsilon(0)) \leq T_c. \quad (A.6)$$

Then, $e_i(t) = 0, i = 1, \ldots, n$ for $t \in [\bar{i}, T_c)$, where $\bar{i} = T_c(1 - \exp(-\alpha T(\epsilon(0))))$. Moreover, it follows from (A.5), that the equality in (A.6) holds when $\sup_{\epsilon(0) \in \mathbb{R}^{n+1}} T(\epsilon(0)) = \infty$.

The second part of the proof follows trivially from Theorem 3, because, for all $t \geq T_c$, the differentiation error dynamics is given by system (5).

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